Matrix KP hierarchy and spin generalization of trigonometric Calogero-Moser hierarchy

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Abstract

We consider solutions of the matrix KP hierarchy that are trigonometric functions of the first hierarchical time \(t_1 = x\) and establish the correspondence with the spin generalization of the trigonometric Calogero-Moser system on the level of hierarchies. Namely, the evolution of poles \(x_i\) and matrix residues at the poles \(a_i^\alpha b_i^\beta\) of the solutions with respect to the \(k\)-th hierarchical time of the matrix KP hierarchy is shown to be given by the Hamiltonian flow with the Hamiltonian which is a linear combination of the first \(k\) higher Hamiltonians of the spin trigonometric Calogero-Moser system with coordinates \(x_i\) and with spin degrees of freedom \(a_i^\alpha, b_i^\beta\). By considering evolution of poles according to the discrete time matrix KP hierarchy we also introduce the integrable discrete time version of the trigonometric spin Calogero-Moser system.

1 Introduction

The matrix generalization of the Kadomtsev-Petviashvili (KP) hierarchy is an infinite set of compatible nonlinear differential equations with infinitely many independent (time) variables \(t = \{t_1, t_2, t_3, \ldots\}\) and matrix dependent variables. It is a subhierarchy of the
multi-component KP hierarchy \[1, 2, 3, 4\]. Among all solutions to these equations, of special interest are solutions which have a finite number of poles in the variable \(x = t_1\) in a fundamental domain of the complex plane. In particular, one can consider solutions which are trigonometric or hyperbolic functions of \(x\) with poles depending on the times \(t_2, t_3, \ldots\).

The dynamics of poles of singular solutions to nonlinear integrable equations is a well known subject in mathematical physics \[5, 6, 7, 8\]. It was shown that the poles of solutions to the KP equation as functions of the time \(t_2\) move as particles of the integrable Calogero-Moser many-body system \[9, 10, 11, 12\]. Rational, trigonometric and elliptic solutions correspond respectively to rational, trigonometric or elliptic Calogero-Moser systems.

The further progress was achieved in \[13\], where it was shown that the correspondence between rational solutions to the KP equation and the Calogero-Moser system with rational potential can be extended to the level of hierarchies. Namely, the evolution of poles with respect to the higher time \(t_m\) of the KP hierarchy was shown to be given by the higher Hamiltonian flow of the integrable Calogero-Moser system with the Hamiltonian \(H_m = \text{tr} L^m\), where \(L\) is the Lax matrix. Later this correspondence was generalized to trigonometric solutions of the KP hierarchy (see \[14, 15\]). It was shown that the dynamics of poles with respect to the higher time \(t_m\) is given by the Hamiltonian flow with the Hamiltonian

\[
H_m = \frac{1}{2(m+1)\gamma} \text{tr} \left( (L + \gamma I)^{m+1} - (L - \gamma I)^{m+1} \right),
\]

(1.1)

where \(I\) is the unity matrix and \(\gamma\) is a parameter such that \(\pi i/\gamma\) is the period of the trigonometric or hyperbolic functions. Clearly, the Hamiltonian \(H_m\) is a linear combination of the Hamiltonians \(H_k = \text{tr} L^k\).

In this paper we generalize this result to trigonometric solutions of the matrix KP hierarchy. The singular (in general, elliptic) solutions to the matrix KP equation were investigated in \[16\]. It was shown that the evolution of data of such solutions (positions of poles and some internal degrees of freedom) with respect to the time \(t_2\) is isomorphic to the dynamics of a spin generalization of the Calogero-Moser system (the Gibbons-Hermsen system \[17\]). It is a system of \(N\) particles with coordinates \(x_i\) with internal degrees of freedom given by \(N\)-dimensional column vectors \(a_i, b_i\) which pairwise interact with each other. The Hamiltonian is

\[
H = \sum_{i=1}^{N} p_i^2 - \gamma^2 \sum_{i \neq k} \frac{(b_i^T a_k)(b_k^T a_i)}{\sinh^2(\gamma(x_i - x_k))},
\]

(1.2)

(here \(b_i^T\) is the transposed row-vector) with the non-vanishing Poisson brackets \(\{x_i, p_k\} = \delta_{ik}, \{a_i^\alpha, b_k^\beta\} = \delta_{\alpha\beta} \delta_{ik}\). The model is known to be integrable, with the higher Hamiltonians (integrals of motion in involution) \(H_k = \text{tr} L^k\), where \(L\) is the Lax matrix of the model given by

\[
L_{jk} = -p_j \delta_{jk} - (1 - \delta_{jk}) \frac{\gamma b_j^T a_k}{\sinh(\gamma(x_j - x_k))}.
\]

(1.3)

Our main result in this paper is that the dynamics of poles \(x_i\) and vectors \(a_i, b_i\) (which parametrize matrix residues at the poles) with respect to the higher time \(t_m\) is given by
the Hamiltonian flow with the Hamiltonian (1.1) and with the Lax matrix (1.3). The corresponding result for rational solutions \((\gamma = 0)\) was established in [18].

We use the method suggested by Krichever [7] for elliptic solutions of the KP equation. It consists in substituting the solution not in the KP equation itself but in the auxiliary linear problem for it (this implies a suitable pole ansatz for the wave function). This method allows one to obtain the equations of motion together with the Lax representation for them.

Another result of this paper is the time discretization of the trigonometric spin Calogero-Moser (Gibbons-Hermsen) model. (The time discretization of the rational spin Calogero-Moser system within the same approach was suggested in [19].) Because of the precise correspondence between the trigonometric solutions of the matrix KP hierarchy and the trigonometric spin Calogero-Moser hierarchy, the integrable time discretization of the Calogero-Moser system and its spin generalization can be obtained from dynamics of poles of trigonometric solutions to semi-discrete soliton equations. (“Semi” means that the time becomes discrete while the space variable \(x\), with respect to which one considers pole solutions, remains continuous.) At the same time, it is known that integrable discretizations of soliton equations can be regarded as belonging to the same hierarchy as their continuous counterparts. Namely, the discrete time step is equivalent to a special shift of infinitely many continuous hierarchical times. This fact lies in the basis of the method of generating discrete soliton equations developed in [20]. For integrable time discretization of many-body systems see [21, 22, 23, 24, 25]. In this paper, we derive equations of motion in discrete time \(p\) for the spin generalization of the trigonometric Calogero-Moser model:

\[
\sum_j \coth(\gamma(x_i(p) - x_j(p + 1))(b_i^T(p)a_j(p + 1))(b_j^T(p + 1)a_i(p)) + \sum_j \coth(\gamma(x_i(p) - x_j(p - 1))(b_i^T(p)a_j(p - 1))(b_j^T(p - 1)a_i(p)) = 2 \sum_{j \neq i} \coth(\gamma(x_i(p) - x_j(p))(b_i^T(p)a_j(p))(b_j^T(p)a_i(p)),
\]

where \(a_i(p), b_i(p)\) are spin variables. In the limit \(\gamma \to 0\) the result of [19] is reproduced.

2 The matrix KP hierarchy

Here we briefly review the main facts about the multi-component and matrix KP hierarchies following [3, 4]. We start from the more general multi-component KP hierarchy. The independent variables are \(N\) infinite sets of continuous “times”

\[
t = \{t_1, t_2, \ldots, t_N\}, \quad t_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \ldots\}, \quad \alpha = 1, \ldots, N
\]

and \(N\) discrete integer variables \(s = \{s_1, s_2, \ldots, s_N\}\) (“charges”) constrained by the condition \(\sum_{\alpha=1}^{N} s_\alpha = 0\). In what follows, we will mostly put \(s_\alpha = 0\) since we are interested in the dynamics in the continuous times.
In the bilinear formalism, the dependent variable is the tau-function \( \tau(s; t) \). We also introduce the tau-functions
\[
\tau_{\alpha\beta}(t) = \tau(e_{\alpha} - e_{\beta}; t),
\]
(2.1)
where \( e_{\alpha} \) is the vector whose \( \alpha \)-th component is 1 and all other entries are equal to zero. The \( N \)-component KP hierarchy is the infinite set of bilinear equations for the tau-functions which are encoded in the basic bilinear relation
\[
\sum_{\nu=1}^{N} \epsilon_{\alpha\nu} \epsilon_{\beta\nu} \int_{C_{\infty}} dz \ z^{\delta_{\alpha\nu}+\delta_{\beta\nu}-2} e^{\xi(t_{\nu} - t'_{\nu}, z)} \tau_{\alpha\nu}(t - [z^{-1}]_{\nu}) \tau_{\nu\beta}(t' + [z^{-1}]_{\nu}) = 0
\]
(2.2)
valid for any \( t, t' \). Here \( \epsilon_{\alpha\beta} \) is a sign factor: \( \epsilon_{\alpha\beta} = 1 \) if \( \alpha \leq \beta \), \( \epsilon_{\alpha\beta} = -1 \) if \( \alpha > \beta \). In (2.2) we use the following standard notation:
\[
\xi(t_{\gamma}, z) = \sum_{k \geq 1} t_{\gamma,k} z^{k},
\]
\[
(t \pm [z^{-1}]_{\gamma})_{\alpha k} = t_{\alpha,k} \pm \delta_{\alpha\gamma} \frac{z^{-k}}{k}.
\]
The integration contour \( C_{\infty} \) is a big circle around \( \infty \). Hereafter, we omit the variables \( s \) in the notation for the tau-functions.

An important role in the theory of integrable hierarchies is played by the wave function. In the multi-component KP hierarchy, the wave function \( \Psi(t; z) \) and its adjoint \( \Psi^\dagger(t; z) \) are \( N \times N \) matrices with the components
\[
\Psi_{\alpha\beta}(t; z) = \epsilon_{\alpha\beta} \frac{\tau_{\alpha\beta}(t - [z^{-1}]_{\beta})}{\tau(t)} z^{\delta_{\alpha\beta}-1} e^{\xi(t_{\beta}, z)},
\]
\[
\Psi_{\alpha\beta}^\dagger(t; z) = \epsilon_{\beta\alpha} \frac{\tau_{\alpha\beta}(t + [z^{-1}]_{\alpha})}{\tau(t)} z^{\delta_{\alpha\beta}-1} e^{-\xi(t_{\alpha}, z)}
\]
(2.3)
(here and below \( \dagger \) does not mean the Hermitian conjugation). The complex variable \( z \) is called the spectral parameter. Around \( z = \infty \), the wave function \( \Psi \) can be represented in the form of the series
\[
\Psi_{\alpha\beta}(t; z) = \left( \delta_{\alpha\beta} + \sum_{k \geq 1} \frac{w^{(k)}_{\alpha\beta}(t)}{z^{k}} \right) e^{\xi(t_{\beta}, z)},
\]
(2.4)
where \( w^{(k)}(t) \) are some matrix functions. In terms of the wave functions, the bilinear relation (2.2) can be written as
\[
\int_{C_{\infty}} dz \ \Psi(t; z) \Psi^\dagger(t'; z) = 0.
\]
(2.5)

Another (equivalent) approach to the multi-component KP hierarchy is based on matrix pseudo-differential operators. The hierarchy can be understood as an infinite set of evolution equations in the times \( t \) for matrix functions of a variable \( x \). For example, the coefficients \( w^{(k)} \) of the wave function can be taken as such matrix functions, the evolution being \( w^{(k)}(x) \rightarrow w^{(k)}(x, t) \). In what follows we denote \( \tau(x, t), w^{(k)}(x, t) \) simply as \( \tau(t) \),
$w^{(k)}(t)$, suppressing the dependence on $x$. Let us introduce the matrix pseudo-differential “wave operator” $\mathcal{W}$ with matrix elements

$$\mathcal{W}_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{k \geq 1} w^{(k)}_{\alpha\beta}(t) \partial_x^{-k}, \quad (2.6)$$

where $w^{(k)}_{\alpha\beta}(t)$ are the same matrix functions as in (2.4). The wave function is a result of action of the wave operator to the exponential function:

$$\Psi(t; z) = \mathcal{W} \exp(x z I + \sum_{\alpha=1}^{N} E_{\alpha}(t, z)), \quad (2.7)$$

where $E_{\alpha}$ is the $N \times N$ matrix with the components $(E_{\alpha})_{\beta\gamma} = \delta_{\alpha\beta}\delta_{\alpha\gamma}$. The adjoint wave function can be written as

$$\Psi^\dagger(t; z) = \exp(-x z I - \sum_{\alpha=1}^{N} E_{\alpha}(t, z)) \mathcal{W}^{-1}. \quad (2.8)$$

Here the operators $\partial_x$ which enter $\mathcal{W}^{-1}$ act to the left (the left action is defined as $f \partial_x \equiv -\partial_x f$).

It is proved in [4] that the wave function and its adjoint satisfy the linear equations

$$\partial_{t_{\alpha,m}} \Psi(t; z) = B_{\alpha m} \Psi(t; z), \quad -\partial_{t_{\alpha,m}} \Psi^\dagger(t; z) = \Psi^\dagger(t; z) B_{\alpha m}, \quad (2.9)$$

where $B_{\alpha m}$ is the differential operator $B_{\alpha m} = \left(WE_{\alpha} \partial_x^{-m} \mathcal{W}^{-1}\right)^+$. The notation $(...)_+$ means the differential part of a pseudo-differential operator, i.e. the sum of all terms with $\partial_x^k$, where $k \geq 0$. Again, the operator $B_{\alpha m}$ in the second equation in (2.9) acts to the left. In particular, it follows from (2.9) at $m = 1$ that

$$\sum_{\alpha=1}^{N} \partial_{t_{\alpha,1}} \Psi(t; z) = \partial_x \Psi(t; z), \quad \sum_{\alpha=1}^{N} \partial_{t_{\alpha,1}} \Psi^\dagger(t; z) = \partial_x \Psi^\dagger(t; z), \quad (2.10)$$

so the vector field $\partial_x$ can be identified with the vector field $\sum_{\alpha} \partial_{t_{\alpha,1}}$.

The matrix KP hierarchy is a subhierarchy of the multi-component KP one which is obtained by a restriction of the time variables in the following manner: $t_{\alpha,m} = t_m$ for each $\alpha$ and $m$. The corresponding vector fields are related as $\partial_{t_m} = \sum_{\alpha=1}^{N} \partial_{t_{\alpha,m}}$. The wave function for the matrix KP hierarchy has the expansion

$$\Psi_{\alpha\beta}(t; z) = \left(\delta_{\alpha\beta} + w^{(1)}_{\alpha\beta}(t) z^{-1} + O(z^{-2})\right) e^{xz + \xi(t, z)}, \quad (2.11)$$

where $\xi(t, z) = \sum_{k \geq 1} t_k z^k$. The coefficient $w^{(1)}_{\alpha\beta}(t)$ plays an important role in what follows. Equations (2.9) imply that the wave function of the matrix KP hierarchy and its adjoint satisfy the linear equations

$$\partial_{t_m} \Psi(t; z) = B_m \Psi(t; z), \quad -\partial_{t_m} \Psi^\dagger(t; z) = \Psi^\dagger(t; z) B_m, \quad m \geq 1, \quad (2.12)$$

where $B_m$ is the differential operator $B_m = \left(\mathcal{W} \partial_x^{-m} \mathcal{W}^{-1}\right)_+$. At $m = 1$ we have $\partial_{t_1} \Psi = \partial_x \Psi$, so we can identify $\partial_x = \partial_{t_1} = \sum_{\alpha=1}^{N} \partial_{t_{\alpha,1}}$ and the evolution in $t_1$ is simply a shift of the
variable $x$: $w^{(k)}(x, t_1, t_2, \ldots) = w^{(k)}(x + t_1, t_2, \ldots)$. At $m = 2$ equations (2.12) turn into the linear problems
\begin{align}
\partial_{t_2} \Psi &= \partial_x^2 \Psi + V(t) \Psi, \\
-\partial_{t_2} \Psi^\dagger &= \partial_x^2 \Psi^\dagger + \Psi^\dagger V(t)
\end{align}
which have the form of the matrix non-stationary Schrödinger equations with
\begin{equation}
V(t) = -2\partial_x w^{(1)}(t).
\end{equation}

Let us derive a useful corollary of the bilinear relation (2.2). Differentiating it with respect to $t_m$ and putting $t' = t$ after this, we obtain:
\begin{equation}
\frac{1}{2\pi i} \sum_{\nu=1}^N \oint_{C_{\infty}} dz \, z^m \Psi_{\alpha\nu}(t; z) \Psi^\dagger_{\nu\beta}(t; z) = -\partial_{t_m} w_{\alpha\beta}^{(1)}(t)
\end{equation}
or, equivalently,
\begin{equation}
\text{res}_{\infty}(z^m \Psi_{\alpha\nu} \Psi^\dagger_{\nu\beta}) = -\partial_{t_m} w_{\alpha\beta}^{(1)}.
\end{equation}
Here and below the summation from 1 to $N$ over repeated Greek indices is implied. The residue at infinity is defined according to $\text{res}_{\infty}(z^{-n}) = \delta_{n1}$.

At the end of this section let us make some remarks on the discrete time version of the matrix KP hierarchy. The discrete time evolution is defined as a special shift of the infinite number of continuous time variables according to the rule [20]
\begin{equation}
\tau^p = \tau \left( t - p \sum_{\alpha=1}^N [\mu^{-1}]_\alpha \right), \quad \Psi^p = \Psi \left( t - p \sum_{\alpha=1}^N [\mu^{-1}]_\alpha; z \right).
\end{equation}
Here $p$ is the discrete time variable and $\mu$ is a continuous parameter. Each $\mu$ corresponds to its own discrete time flow. The limit $\mu \to \infty$ is the continuous limit. One can show, using the explicit expressions of the wave functions through the tau-function and some corollaries of the bilinear relation (see [19]) that the corresponding linear problems have the form
\begin{align}
\mu \Psi^p_{\alpha\beta} - \mu \Psi^{p+1}_{\alpha\beta} &= \partial_x \Psi^p_{\alpha\beta} + \left( w^{(1)}_{\alpha\nu}(p + 1) - w^{(1)}_{\alpha\nu}(p) \right) \Psi^p_{\nu\beta}, \\
\mu \Psi^{\dagger p}_{\alpha\beta} - \mu \Psi^{\dagger p+1}_{\alpha\beta} &= -\partial_x \Psi^{\dagger p}_{\alpha\beta} + \Psi^{\dagger p}_{\alpha\nu} \left( w^{(1)}_{\nu\beta}(p) - w^{(1)}_{\nu\beta}(p - 1) \right).
\end{align}

3 Trigonometric solutions of the matrix KP hierarchy: dynamics of poles in $t_2$

Our aim is to study solutions to the matrix KP hierarchy which are trigonometric functions of the variable $x$ (and, therefore, $t_1$). For the trigonometric solutions, the tau-function has the form
\begin{equation}
\tau = C \prod_{i=1}^N (e^{2\gamma x} - e^{2\gamma x_i}),
\end{equation}
where $\gamma$ is a parameter. The period of the function is $\pi i/\gamma$. Real (respectively, imaginary) $\gamma$ corresponds to hyperbolic (respectively, trigonometric) functions. In the limit $\gamma \to 0$
one obtains rational solutions. The $N$ roots $x_i$ (assumed to be distinct) depend on the times $t$. It is convenient to pass to the exponentiated variables

$$w = e^{2\gamma t}, \quad w_i = e^{2\gamma x_i},$$

(3.2)

then the tau-function becomes a polynomial with the roots $w_i$: \( \tau = C \prod_i (w - w_i) \). Clearly, we have \( \partial_x = 2\gamma \partial_w, \partial_x^2 = 4\gamma^2(w^2\partial_w^2 + w\partial_w) \).

It is clear from (2.3) that the wave functions $\Psi$, $\Psi^\dagger$ (and thus the coefficient $w^{(1)}$), as functions of $x$, have simple poles at $x = x_i$. It is shown in [18] that the residues at these poles are matrices of rank 1. We parametrize them through the column vectors \( \mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})^T \), \( \mathbf{b}_i = (b_{i1}, b_{i2}, \ldots, b_{iN})^T \), \( \mathbf{c}_i = (c_{i1}, c_{i2}, \ldots, c_{iN})^T \) ($T$ means transposition) and the row vector \( \mathbf{c}_i^* = (c_{i1}^*, c_{i2}^*, \ldots, c_{iN}^*) \):

$$\Psi_{\alpha\beta} = e^{xz + \xi(t,z)} \left( C_{\alpha\beta} + \sum_i \frac{2\gamma w_i^{1/2} a_i^\alpha c_i^\beta}{w - w_i} \right),$$

(3.3)

$$\Psi^{\dagger}_{\alpha\beta} = e^{-xz - \xi(t,z)} \left( C^{-1}_{\alpha\beta} + \sum_i \frac{2\gamma w_i^{1/2} c_i^* a_i^\alpha}{w - w_i} \right),$$

(3.4)

where the matrix $C_{\alpha\beta}$ does not depend on $x$. Note that the constant term in the adjoint wave function is the inverse matrix $C^{-1}_{\alpha\beta}$. This follows from (2.8). For the matrices $w^{(1)}$ and $V = -2\partial_x w^{(1)}$ we have

$$w^{(1)}_{\alpha\beta} = S_{\alpha\beta} - \sum_i \frac{2\gamma w_i a_i^\alpha b_i^\beta}{w - w_i}, \quad V_{\alpha\beta} = -8\gamma^2 \sum_i \frac{ww_i a_i^\alpha b_i^\beta}{(w - w_i)^2},$$

(3.5)

where the matrix $S_{\alpha\beta}$ does not depend on $x$. Tending $w \to \infty$ in (2.17), one concludes that $\partial_{tm} S_{\alpha\beta} = 0$ for all $m \geq 1$, so the matrix $S_{\alpha\beta}$ does not depend on all the times. The components of the vectors $\mathbf{a}_i$, $\mathbf{b}_i$ are going to be spin variables of the Gibbons-Hermsen model.

We first consider the dynamics of poles with respect to the time $t_2$. The procedure is similar to the rational case [18]. Following Krichever’s approach, we consider the linear problems (2.13), (2.14),

$$\partial_{t_2} \Psi_{\alpha\beta} = \partial_x^2 \Psi_{\alpha\beta} - 8\gamma^2 \sum_{i=1}^N \frac{ww_i a_i^\alpha b_i'}{(w - w_i)^2} \Psi_{\nu\beta},$$

$$-\partial_{t_2} \Psi^\dagger_{\alpha\beta} = \partial_x^2 \Psi^\dagger_{\alpha\beta} - 8\gamma^2 \sum_{i=1}^N \frac{ww_i a_i^{\nu'} b_i'}{(w - w_i)^2},$$

and substitute here the pole ansatz (3.3), (3.4) for the wave functions. Consider first the equation for $\Psi$. First of all, comparing the behavior of both sides as $w \to \infty$, we conclude that $\partial_{t_2} C_{\alpha\beta} = 0$, so $C_{\alpha\beta}$ does not depend on $t_2$ (in a similar way, from the higher linear problems one can see that $C_{\alpha\beta}$ does not depend on all the times $t_m$). After the substitution, we see that the expression has poles at $w = w_i$ up to the third order. Equating coefficients at the poles of different orders at $w = w_i$, we get the conditions:

- At $\frac{1}{(w - w_i)^3}$: $b_i' a_i^{\nu'} = 1$;
are $N$-dimensional vectors, 

where $\tilde{b}_i^\beta = b_i^\nu C_{\nu\beta}$, and $\dot{x}_i = \partial_{t_2} x_i$. Similar calculations for the linear problem for $\Psi^\dagger$ lead to the conditions

At $\frac{1}{(w-w_i)^3}$:

$$b_i^\nu a_i^\nu = 1;$$

At $\frac{1}{(w-w_i)^2}$:

$$\frac{1}{2} \dot{x}_i c_i^\beta - 2\gamma \sum_{k \neq i} \frac{w_{i}^{1/2} w_{k}^{1/2} b_i^\nu a_k^\nu c_k^\beta}{w_i - w_k} - (z - \gamma) c_i^\beta = w_i^{1/2} b_i^\beta;$$

At $\frac{1}{w-w_i}$:

$$\partial_{t_2} (w_i^{1/2} a_i^\alpha c_i^\beta) = 2\gamma w_i^{1/2} \dot{x}_i a_i^\alpha c_i^\beta + 8\gamma^2 \sum_{k \neq i} \frac{w_i^{1/2} w_k^{1/2} a_i^\alpha b_k^\nu a_k^\nu c_k^\beta}{(w_i - w_k)^2} - 8\gamma^2 \sum_{k \neq i} \frac{w_i^{3/2} w_k^{1/2} b_i^\nu a_k^\nu c_k^\beta}{(w_i - w_k)^2},$$

where $\tilde{a}_i^\alpha = C_{\alpha\nu}^{-1} a_i^\nu$.

The conditions coming from the third order poles are constraints on the vectors $a_i$, $b_i$. The other conditions can be written in the matrix form

$$\begin{cases}
(zI - (L + \gamma I))c^\beta = -W^{1/2}b^\beta \\
\dot{c}^\beta = Mc^\beta,
\end{cases} \quad (3.6)$$

$$\begin{cases}
c'^{\alpha}(zI - (L - \gamma I)) = \tilde{a}^\alpha W^{1/2} \\
\dot{c}'^{\alpha} = C^{\alpha\alpha} \tilde{M},
\end{cases} \quad (3.7)$$

where $c^\beta = (c_1^\beta, \ldots, c_N^\beta)^T$, $c'^{\alpha} = (c'^{\alpha}_1, \ldots, c'^{\alpha}_N)$, $\tilde{b}^\beta = (\tilde{b}_1^\beta, \ldots, \tilde{b}_N^\beta)^T$, $\tilde{a}^\alpha = (\tilde{a}_1^\alpha, \ldots, \tilde{a}_N^\alpha)$ are $N$-dimensional vectors, $I$ is the unity matrix, $W = \text{diag}(w_1, w_2, \ldots, w_N)$ and $L$, $M$, $\tilde{M}$ are $N \times N$ matrices of the form

$$L_{ik} = -\frac{1}{2} \dot{x}_i \delta_{ik} - 2\gamma (1 - \delta_{ik}) \frac{w_i^{1/2} w_k^{1/2} b_i^\nu a_k^\nu}{w_i - w_k}; \quad (3.8)$$

$$M_{ik} = \gamma \dot{x}_i - \Lambda_i \delta_{ik} + 8\gamma^2 (1 - \delta_{ik}) \frac{w_i^{3/2} w_k^{1/2} b_i^\nu a_k^\nu}{(w_i - w_k)^2}; \quad (3.9)$$

$$\tilde{M}_{ik} = \gamma \dot{x}_i + \Lambda_i^* \delta_{ik} - 8\gamma^2 (1 - \delta_{ik}) \frac{w_i^{1/2} w_k^{3/2} b_i^\nu a_k^\nu}{(w_i - w_k)^2}. \quad (3.10)$$

Here

$$\Lambda_i = \frac{\dot{a}_i^\alpha}{a_i^\alpha} + 8\gamma^2 \sum_{k \neq i} \frac{w_i w_k a_i^\alpha b_k^\nu a_k^\nu}{a_i^\alpha (w_i - w_k)^2}, \quad -\Lambda_i^* = \frac{\dot{b}_i^\alpha}{b_i^\alpha} - 8\gamma^2 \sum_{k \neq i} \frac{w_i w_k b_i^\alpha a_k^\nu b_k^\nu}{b_i^\alpha (w_i - w_k)^2} \quad (3.11)$$
do not depend on the index $\alpha$. In fact one can see that $\Lambda_i = \Lambda_i^\ast$. Indeed, multiplying equations (3.11) by $a_i^\alpha b_i^\alpha$ (no summation here!), summing over $\alpha$ and summing the two equations, we get $\Lambda_i - \Lambda_i^\ast = \partial_{t_2}(a_i^\alpha b_i^\alpha) = 0$ by virtue of the constraint $a_i^\alpha b_i^\alpha = 1$.

Differentiating the first equation in (3.6) by $t_2$, we get, after some calculations, the compatibility condition of equations (3.6):

$$\dot{L} + [L, M] = 0.$$ (3.12)

One can see, taking into account equations (3.11), which we write here in the form

$$\dot{a}_i^\alpha = \Lambda_i a_i^\alpha - 2\gamma^2 \sum_{k \neq i} \frac{a_k^\beta a_i^\gamma}{\sinh^2(\gamma(x_i - x_k))},$$ (3.13)

$$\dot{b}_i^\alpha = -\Lambda_i b_i^\alpha + 2\gamma^2 \sum_{k \neq i} \frac{b_k^\beta a_i^\gamma}{\sinh^2(\gamma(x_i - x_k))},$$ (3.14)

(in this form they are equations of motion for the spin degrees of freedom) that the off-diagonal elements of the matrix $\dot{L} + [L, M]$ are equal to zero. Vanishing of the diagonal elements yields equations of motion for the poles $x_i$:

$$\ddot{x}_i = -8\gamma^3 \sum_{k \neq i} \frac{\cosh(\gamma(x_i - x_k))}{\sinh^3(\gamma(x_i - x_k))} b_k^\beta a_i^\gamma a_i^\gamma b_k^\gamma.$$ (3.15)

The gauge transformation $a_i^\alpha \to a_i^\alpha q_i$, $b_i^\alpha \to b_i^\alpha q_i^{-1}$ with $q_i = \exp\left(\int_{t_2}^{t_2} \Lambda_i dt\right)$ eliminates the terms with $\Lambda_i$ in (3.13), (3.14), so we can put $\Lambda_i = 0$. This gives the equations of motion

$$\dot{a}_i^\alpha = -2\gamma^2 \sum_{k \neq i} \frac{a_k^\beta b_k^\gamma a_i^\gamma}{\sinh^2(\gamma(x_i - x_k))}, \quad \dot{b}_i^\alpha = 2\gamma^2 \sum_{k \neq i} \frac{b_k^\beta a_i^\gamma b_k^\gamma}{\sinh^2(\gamma(x_i - x_k))}.$$ (3.16)

Together with (3.15) they are equations of motion of the trigonometric Gibbons-Hermsen model. Their Lax representation is given by the matrix equation $\dot{L} + [L, M]$. It states that the time evolution of the Lax matrix is an isospectral transformation. It follows that the quantities $H_k = \text{tr} L^k$ are integrals of motion. In particular,

$$H_2 = \sum_{i=1}^{N} p_i^2 - \gamma^2 \sum_{i \neq k} \frac{b_k^\beta a_k^\gamma b_k^\gamma a_i^\gamma}{\sinh^2(\gamma(x_i - x_k))} = \text{tr} L^2$$ (3.17)

is the Hamiltonian of the Gibbons-Hermsen model. Equations of motion (3.16), (3.15) are equivalent to the Hamiltonian equations

$$\dot{x}_i = \frac{\partial H_2}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_2}{\partial x_i}, \quad \dot{a}_i^\alpha = \frac{\partial H_2}{\partial b_i^\alpha}, \quad \dot{b}_i^\alpha = -\frac{\partial H_2}{\partial a_i^\alpha}.$$ (3.18)
4 Dynamics of poles in the higher times

The main tool for the analysis of the dynamics in the higher times is the relation (2.17) which, after substitution of (3.3), (3.4) and (3.5) takes the form

\[
\begin{align*}
\text{res}_\infty \left[ z^m \left( C_{\alpha\nu} + \sum_i \frac{2\gamma w_i^{1/2} a_i^{\alpha\nu}}{w - w_i} (C_{\nu\beta}^{-1} + \sum_k \frac{2\gamma w_k^{1/2} e_k^{\nu\beta}}{w - w_k}) \right) \right] \\
= 2\gamma \sum_i \frac{\partial_t x_i w_i^2 a_i^{\nu}}{w - w_i} + 4\gamma^2 \sum_i \frac{\partial_t x_i w_i^2 a_i^{\nu}}{(w - w_i)^2}.
\end{align*}
\tag{4.1}
\]

The both sides are rational functions of \( w \) with poles at \( w = w_i \) vanishing at infinity. Identifying the coefficients in front of the second order poles, we obtain

\[
\partial_t x_i = \text{res}_\infty \left( z^m c_i c_i^{\nu} w_i^{1/2} e_i^{\nu} \right). \tag{4.2}
\]

Solving the linear equations (3.6), (3.7), we get

\[
c_i^{\nu} = -\sum_k (zI - (L + \gamma I))^{-1} w_k^{1/2} b_k^{\nu}, \quad c_i^{\nu} = \sum_k \tilde{a}_k^{\nu} w_k^{1/2} (zI - (L - \gamma I))^{-1} \, \tag{4.3}
\]

and, therefore, (4.2) reads

\[
\partial_t x_i = -\text{res}_\infty \sum_{k,k'} \left( z^m \tilde{a}_k^{\nu} \tilde{a}_k^{\nu} w_k^{1/2} \left( \frac{1}{zI - (L - \gamma I)} \right)_{ki}^{-1} \left( \frac{1}{zI - (L + \gamma I)} \right)_{ik'}^{-1} w_k^{1/2} \right)
= -\text{res}_\infty \text{tr} \left( z^m W^{1/2} R W^{1/2} \left( \frac{1}{zI - (L - \gamma I)} \right) W^{-1} E_i \left( \frac{1}{zI - (L + \gamma I)} \right) \right),
\]

where \( E_i \) is the diagonal matrix with matrix elements \((E_i)_{jk} = \delta_{ij}\delta_{ik}\) and \( R \) is the \( \mathcal{N} \times \mathcal{N} \) matrix

\[
R_{ik} = \tilde{b}_i^{\nu} \tilde{a}_k^{\nu} = b_i^{\nu} a_k^{\nu}. \tag{4.4}
\]

The following commutation relation can be checked directly:

\[
[L, W] = 2\gamma (W^{1/2} R W^{1/2} - W). \tag{4.5}
\]

Note that \( E_i = -\partial L / \partial p_i \). The rest of the calculation is similar to the one done in [15].

We have, using (4.5):

\[
\begin{align*}
\partial_t x_i &= \frac{1}{2\gamma} \text{res}_\infty \text{tr} \left( z^m (LW - WL + 2\gamma W) \frac{1}{zI - (L - \gamma I)} \frac{1}{zI - (L + \gamma I)} W^{-1} \frac{\partial L}{\partial p_i} zI - (L + \gamma I) \right) \\
&= \frac{1}{2\gamma} \text{res}_\infty \text{tr} \left( z^m \left( \frac{\partial L}{\partial p_i} zI - (L + \gamma I) - \frac{1}{\partial p_i} zI - (L - \gamma I) \right) \right) \\
&= \frac{1}{2\gamma} \text{tr} \left( \frac{\partial L}{\partial p_i} (L + \gamma I)^m - \frac{\partial L}{\partial p_i} (L - \gamma I)^m \right) \\
&= \frac{1}{2(m + 1)\gamma} \frac{\partial}{\partial p_i} \text{tr} \left( (L + \gamma I)^{m+1} - (L - \gamma I)^{m+1} \right) = \frac{\partial H_m}{\partial p_i},
\end{align*}
\]

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where $H_m$ is given by (1.1). Note that $H_2 = H_2 + \text{const}$. We have obtained one part of the Hamiltonian equations for the higher time flows. In the case $\gamma \to 0$ (rational solutions) the result of the paper [18] is reproduced.

In order to obtain another part of the Hamiltonian equations, let us differentiate (4.2) with respect to $t_2$:

$$
\partial_{t_m} \dot{x}_i = -2\gamma \text{res}_\infty \left( z^m c_i \nu w_i^{-1} c_i \right) + \text{res}_\infty \left( z^m (c_i \nu w_i^{-1} \partial_{t_2} c_i \nu + \partial_{t_2} c_i \nu w_i^{-1} c_i) \right)
$$

$$
= \text{res}_\infty \sum_k \left( z^m (c_i \nu w_i^{-1} B_{ik} c_k - c_k \nu w_k^{-1} B_{ki} c_k) \right),
$$

where

$$
B_{jk} = 8\gamma^2 (1 - \delta_{jk}) \frac{w_j^{3/2} w_k^{1/2} b_j a_k}{(w_i - w_k)^2}.
$$

Therefore, we have, using (4.3):

$$
\partial_{t_m} p_i = \frac{1}{2} \partial_{t_m} \dot{x}_i = -\text{res}_\infty \left[ z^m \text{tr} \left( W^{1/2} RW^{1/2} \frac{1}{zI - (L - \gamma I)} G^{(i)} \frac{1}{zI - (L + \gamma I)} \right) \right],
$$

where the matrix $G^{(i)}$ is given by

$$
G^{(i)}_{jk} = 4\gamma^2 (\delta_{ij} - \delta_{ik}) \frac{w_j^{1/2} w_k^{1/2} b_j a_k}{(w_i - w_k)^2}.
$$

It is straightforward to check the identities

$$
(W G^{(i)} - G^{(i)} W)_{jk} = -2\gamma L_{jk} (\delta_{ij} - \delta_{ik}),
$$

$$
WG^{(i)} + G^{(i)} W = 2 \frac{\partial L}{\partial x_i}.
$$

A direct calculation which literally repeats the one done in [15] shows that

$$
\partial_{t_m} p_i = -\frac{1}{2\gamma} \text{res}_\infty \left[ z^m \text{tr} \left( (LW - WL + 2\gamma W) \frac{1}{zI - (L + \gamma I)} G^{(i)} \frac{1}{zI - (L + \gamma I)} \right) \right]
$$

$$
= -\frac{1}{2\gamma} \text{res}_\infty \left[ z^m \text{tr} \left( W G^{(i)} \frac{1}{zI - (L + \gamma I)} - G^{(i)} W \frac{1}{zI - (L - \gamma I)} \right) \right]
$$

$$
= -\frac{1}{2\gamma} \text{res}_\infty \left[ z^m \text{tr} \left( \frac{\partial L}{\partial x_i} \left( \frac{1}{zI - (L + \gamma I)} - \frac{1}{zI - (L - \gamma I)} \right) \right) \right]
$$

$$
= -\frac{1}{2\gamma} \text{tr} \left( \frac{\partial L}{\partial x_i} (L + \gamma I)^m - \frac{\partial L}{\partial x_i} (L - \gamma I)^m \right) = -\frac{\partial H_m}{\partial x_i}.
$$

We have established the remaining part of the Hamiltonian equations for the higher time dynamics of the $x_i$'s.
5 Dynamics of spin variables in the higher times

Comparison of the first order poles in (4.1) gives the following relation:

\[
\partial_{t_m}(w_i a_i^\alpha b_i^\beta) = \text{res}_\infty \left[ z^m \left( w_i^{1/2} C_{\alpha\nu} c_i^\nu b_i^\beta + w_i^{1/2} a_i^\alpha c_i^\nu C_{\nu\beta}^{-1} \right) \right]
\]

\[
+ 2\gamma \sum_{k \neq i} \frac{w_i^{1/2} w_k^{1/2}}{w_i - w_k} \left( a_k^\alpha b_k^\beta c_k^\nu c_k^\nu + a_k^\alpha b_k^\beta c_k^\nu c_k^\nu \right) \right] .
\]

Using (4.3), we can rewrite it in the form

\[
b_i^\beta = -\partial_{t_m} a_i^\alpha + \text{res}_\infty \left( z^m \left( \sum_k a_k^\alpha w_i^{-1/2} w_k^{1/2} \left( \frac{1}{z I - (L - \gamma I)} \right)_{ki} \right) \right)
\]

\[
- 2\gamma \sum_{k \neq i} \frac{w_i^{-1/2} w_k^{1/2}}{w_i - w_k} a_k^\alpha a_i^\nu w_i^{1/2} \left( \frac{1}{z I - (L - \gamma I)} \right)_{li} \left( \frac{1}{z I - (L + \gamma I)} \right)_{kn} \frac{w_i^{1/2} b_n^{\nu}}{w_n^{1/2} b_n^{\nu}} \right] .
\]

\[
- a_i^\alpha \left[ \partial_{t_m} b_i^\beta + \text{res}_\infty \left( z^m \left( \sum_k b_k^\beta w_i^{1/2} w_k^{1/2} \left( \frac{1}{z I - (L + \gamma I)} \right)_{lk} \right) \right) \right.
\]

\[
+ 2\gamma \sum_{k \neq i} \frac{w_i^{-1/2} w_k^{1/2}}{w_i - w_k} b_k^\beta a_i^\nu w_i^{1/2} \left( \frac{1}{z I - (L - \gamma I)} \right)_{lk} \left( \frac{1}{z I - (L + \gamma I)} \right)_{m} \frac{w_i^{1/2} b_m^{\nu}}{w_m^{1/2} b_m^{\nu}} \right]
\]

\[
= 2\gamma \partial_{t_m} x_i a_i^\alpha b_i^\beta .
\]

Separating the terms at \( k = i \) in the sums over \( k \) in the first and the third lines, and taking into account that

\[
2\gamma \partial_{t_m} x_i = \text{res}_\infty \text{tr} \left[ z^m \frac{1}{z I - (L - \gamma I)} - \frac{1}{z I - (L + \gamma I)} \right]
\]

\[
= \text{res}_\infty \left[ z^m \left( \frac{1}{z I - (L - \gamma I)} \right)_{ii} - z^m \left( \frac{1}{z I - (L + \gamma I)} \right)_{ii} \right] ,
\]

we represent this equation as follows:

\[
b_i^\beta P_i^\alpha - a_i^\alpha Q_i^\beta = 0 ,
\]

where

\[
P_i^\alpha = -\partial_{t_m} a_i^\alpha + \text{res}_\infty \left[ z^m \left( \sum_{k \neq i} a_k^\alpha w_i^{-1/2} w_k^{1/2} \left( \frac{1}{z I - (L - \gamma I)} \right)_{ki} \right) \right.
\]

\[
+ \text{tr} \left( W^{1/2} R W^{1/2} \frac{1}{z I - (L - \gamma I)} W^{-1} \frac{\partial L}{\partial b_i^\alpha} \frac{1}{z I - (L + \gamma I)} \right) \right] ,
\]

\[
Q_i^\beta = \partial_{t_m} b_i^\beta + \text{res}_\infty \left[ z^m \left( \sum_{k \neq i} b_k^\beta w_i^{1/2} w_k^{1/2} \left( \frac{1}{z I - (L + \gamma I)} \right)_{ik} \right) \right.
\]

\[
+ \text{tr} \left( W^{1/2} R W^{1/2} \frac{1}{z I - (L - \gamma I)} \frac{\partial L}{\partial a_i^\beta} W^{-1} \frac{1}{z I - (L + \gamma I)} \right) \right] ,
\]

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Here we took into account that
\[ \frac{\partial L_{jk}}{\partial b^a_i} = -2\gamma \delta_{ij} (1 - \delta_{jk}) \frac{w_i^{1/2} w_k^{1/2} a_k^a}{w_i - w_k}, \quad \frac{\partial L_{jk}}{\partial a_i^a} = -2\gamma \delta_{ik} (1 - \delta_{jk}) \frac{w_j^{1/2} w_i^{1/2} b_j^a}{w_j - w_i}. \]

It then follows from (5.1) that
\[ P^a_i = \frac{Q^a_i}{b_i^a} = -\Lambda_i^{(m)} \] (5.2)
does not depend on the indices \( \alpha, \beta \).

Let us transform the expressions for \( P^a_i \), \( Q^a_i \) using the commutation relation (4.5), i.e., substituting
\[ W^{1/2} R W^{-1/2} = \frac{1}{2\gamma} (LW - WL + 2\gamma W). \]

We have:
\[ P^a_i = -\partial_t a_i^a + \frac{1}{2\gamma} \text{Res} \left[ z^m \left( \text{tr} \left( \frac{\partial L}{\partial b_i^a} zI - \frac{1}{zI - (L + \gamma I)} - \frac{\partial L}{\partial b_i^a} zI - (L - \gamma I) \right) \right) \right. \\
\left. + 2\gamma \sum_{k \neq i} a_k^a w_i^{-1/2} w_k^{1/2} \left( \frac{1}{zI - (L - \gamma I)} \right)_{ki} + \text{tr} \left( \frac{\partial L}{\partial b_i^a} - W^{-1} \frac{\partial L}{\partial b_i^a} W \right) \frac{1}{zI - (L - \gamma I)} \right]. \]

But
\[ \left( \frac{\partial L}{\partial b_i^a} - W^{-1} \frac{\partial L}{\partial b_i^a} W \right)_{jk} = -2\gamma \delta_{ij} (1 - \delta_{jk}) w_i^{-1/2} w_k^{1/2} a_k^a, \]
ans so the second line is equal to zero. We are left with
\[ P^a_i = -\partial_t a_i^a + \frac{\partial H_m}{\partial b_i^a}. \] (5.3)

A similar calculation for \( Q^a_i \) yields
\[ Q^a_i = \partial_t b_i^a + \frac{\partial H_m}{\partial a_i^a}. \] (5.4)

Therefore, from (5.2) we have the equations of motion
\[ \partial_t a_i^a = \frac{\partial H_m}{\partial b_i^a} + \Lambda_i^{(m)} a_i^a, \quad \partial_t b_i^a = -\frac{\partial H_m}{\partial a_i^a} - \Lambda_i^{(m)} b_i^a. \]

The gauge transformation \( a_i^a \rightarrow a_i^a q_i^{(m)}, b_i^a \rightarrow b_i^a (q_i^{(m)})^{-1} \) with \( q_i^{(m)} = \exp \left( \int_{t_m}^{t_m} \Lambda_i^{(m)} dt \right) \) eliminates the terms with \( \Lambda_i^{(m)} \) and so we can put \( \Lambda_i^{(m)} = 0 \). In this way we obtain the Hamiltonian equations of motion for spin variables in the higher times:
\[ \partial_t a_i^a = \frac{\partial H_m}{\partial b_i^a}, \quad \partial_t b_i^a = -\frac{\partial H_m}{\partial a_i^a}. \] (5.5)

with \( H_m \) given by (1.1).
6 Time discretization of the trigonometric Gibbons-Hermsen model

Our strategy is to substitute the pole ansatz for the discrete time wave functions

\[ \Psi^{p}_{\alpha\beta} = \left(1 - \frac{z}{\mu}\right)^pe^{xz}\left(C_{\alpha\beta} + \sum_{i} \frac{2\gamma w_{i}^{1/2}(p)\alpha_{i}^{\alpha}(p)c_{i}^{\beta}(p)}{w - w_{i}(p)}\right), \]  

(6.1)

\[ \Psi^{\dagger p}_{\alpha\beta} = \left(1 - \frac{z}{\mu}\right)^{-p}e^{-xz}\left(C_{\alpha\beta}^{-1} + \sum_{i} \frac{2\gamma w_{i}^{1/2}(p)\alpha_{i}^{\alpha}(p)c_{i}^{\beta}(p)}{w - w_{i}(p)}\right), \]  

(6.2)

and \( w_{\alpha\beta}^{(1)} \) (see (3.5)) into the linear problems (2.18), (2.19) and identify the coefficients in front of the poles \((w - w_{i}(p))^{-2}\), \((w - w_{i}(p \pm 1))^{-1}\) and \((w - w_{i}(p))^{-1}\). (Note that the constant term \( S_{\alpha\beta} \) in \( w_{\alpha\beta}^{(1)}(p) \) cancels in the combination \( w_{\alpha\beta}^{(1)}(p + 1) - w_{\alpha\beta}^{(1)}(p) \) because the shift \( p \to p + 1 \) is equivalent to a shift of times and \( S_{\alpha\beta} \) does not depend on the times.) We begin with the linear problem (2.18) for \( \Psi \). From cancellation of different poles we have the following conditions:

- At \( \frac{1}{(w - w_{i}(p))^{2}} \):
  \[ b_{i}^{\nu}(p)a_{i}^{\nu}(p) = 1; \]

- At \( \frac{1}{w - w_{i}(p + 1)} \):
  \[ (z - \mu)c_{i}^{\beta}(p + 1) = -w_{i}^{1/2}(p)b_{i}^{\beta}(p + 1) - 2\gamma \sum_{j} \frac{w_{j}^{1/2}(p)w_{j}^{1/2}(p)b_{j}^{\nu}(p + 1)\alpha_{j}^{\nu}(p)c_{j}^{\beta}(p)}{w_{i}(p + 1) - w_{j}(p)}; \]

- At \( \frac{1}{w - w_{i}(p)} \):
  \begin{align*}
  (z - \mu - 2\gamma)a_{i}^{\alpha}(p)c_{i}^{\beta}(p) + \frac{w_{j}(p + 1)\alpha_{j}^{\alpha}(p + 1)b_{j}^{\nu}(p + 1)\alpha_{j}^{\nu}(p)c_{j}^{\beta}(p)}{w_{i}(p) - w_{j}(p + 1)} \]
  \begin{align*}
  &- 2\gamma \sum_{j} \frac{w_{j}(p + 1)\alpha_{j}^{\alpha}(p + 1)b_{j}^{\nu}(p + 1)\alpha_{j}^{\nu}(p)c_{j}^{\beta}(p)}{w_{i}(p) - w_{j}(p + 1)} \\
  &+ 2\gamma \sum_{j \neq i} \frac{w_{i}^{1/2}(p)w_{j}^{1/2}(p)\alpha_{i}^{\alpha}(p)b_{i}^{\nu}(p)\alpha_{j}^{\nu}(p)c_{j}^{\beta}(p)}{w_{i}(p) - w_{j}(p)} \]
  &+ 2\gamma \sum_{j \neq i} \frac{w_{j}(p)\alpha_{j}^{\alpha}(p)b_{j}^{\nu}(p)\alpha_{j}^{\nu}(p)c_{j}^{\beta}(p)}{w_{i}(p) - w_{j}(p)} = 0. \]

Introduce the matrices

\[ L_{ij}(p) = -\delta_{ij} \frac{\dot{x}_{i}(p)}{2} - 2\gamma(1 - \delta_{ij}) \frac{w_{i}^{1/2}(p)w_{j}^{1/2}(p)b_{i}^{\nu}(p)\alpha_{j}^{\nu}(p)}{w_{i}(p) - w_{j}(p)} \]  

(6.3)

(the same Lax matrix as in (3.8)) and

\[ M_{ij}(p) = 2\gamma \frac{w_{i}^{1/2}(p + 1)w_{j}^{1/2}(p)b_{i}^{\nu}(p + 1)\alpha_{j}^{\nu}(p)}{w_{i}(p + 1) - w_{j}(p)}, \]  

(6.4)
then the above conditions can be written as

\[
\begin{aligned}
(z - \mu)c_i^\beta(p + 1) &= -w_i^{1/2}(p + 1)\tilde{b}_i^\beta(p + 1) - \sum_j M_{ij}(p)c_j^\beta(p) \\
\sum_j \left((z - \gamma)\delta_{ij} - L_{ij}(p)\right)c_j^\beta(p) + w_i^{1/2}(p)\tilde{b}_i^\beta(p) &= 0 \\
+ c_i^\beta(p) \left[\sum_j a_j^\alpha(p + 1)(W^{1/2}(p + 1)M(p)W^{-1/2}(p))_{ji}\right] \\
+ \sum_j a_j^\alpha(p)(W^{1/2}(p)L(p)W^{-1/2}(p))_{ji} - (\mu + \gamma)a_i^\alpha(p) &= 0.
\end{aligned}
\]

The first line in the second equation vanishes by virtue of (3.6). Therefore, we have the following equations:

\[
(z - \mu)c^\beta(p + 1) = -W^{1/2}(p + 1)\tilde{b}^\beta(p + 1) - M(p)c^\beta(p),
\]

\[
a^\alpha_T(p + 1)W^{1/2}(p + 1)M(p)W^{-1/2}(p) + a^\alpha_T(p)W^{1/2}(p)L(p)W^{-1/2}(p) = (\mu + \gamma)a^\alpha_T(p).
\]

A similar solution of the linear problem (2.19) for $\Psi^\dagger$ gives the equations

\[
(z - \mu)c^\alpha(p - 1) = \tilde{a}^\alpha_T(p - 1)W^{1/2}(p - 1) - c^\alpha(p)M(p - 1),
\]

\[
W^{-1/2}(p)M(p - 1)W^{1/2}(p - 1)b^\beta(p - 1) + W^{-1/2}(p)L(p)W^{1/2}(p)b^\beta(p) = (\mu - \gamma)b^\beta(p).
\]

A simple calculation, similar to the one done in [19], shows that the compatibility condition of equations (3.6), (6.6) is the discrete Lax equation

\[
L(p + 1)M(p) = M(p)L(p)
\]

which holds true provided equations (6.7), (6.9) are satisfied.

Equations (5.7), (6.9) are equations of motion of the discrete time trigonometric Gibbons-Hermsen model. Let us consider equation (6.7) and represent it in a somewhat better form. In order to do this, write it in the form

\[
2\gamma \sum_k \frac{w_i(p)(p + 1)b_{ki}^\gamma(p + 1)a_{ki}^\alpha(p)}{w_k(p + 1) - w_i(p)} + 2\gamma \sum_{k \neq i} \frac{w_i(p)a_{ki}^\alpha(p)b_{ki}^\gamma(p)a_{ki}^\alpha(p)}{w_i(p) - w_k(p)} \\
+ 2\gamma \sum_k a_k^\alpha(p + 1)b_k^\gamma(p + 1)a_{ki}^\alpha(p) - 2\gamma \sum_{k \neq i} a_k^\alpha(p)b_{ki}^\gamma(p)a_{ki}^\alpha(p) - (\mu + \gamma)a_i^\alpha(p) - \frac{\dot{x}_i(p)}{2} a_i^\alpha(p) = 0
\]

and add it to the original equation taking into account that

\[
\sum_k a_k^\alpha(p + 1)b_k^\gamma(p + 1) = \sum_k a_k^\alpha(p)b_k^\gamma(p).
\]
This follows from the fact that $\sum a_i^\alpha b_i^\beta$ is an integral of motion, i.e., $\partial_t (\sum a_i^\alpha b_i^\beta) = 0$ for all $m$. Indeed, we have

$$\partial_t (\sum a_i^\alpha b_i^\beta) = \sum \left( b_i^\beta \frac{\partial H_m}{\partial b_i^\alpha} - a_i^\alpha \frac{\partial H_m}{\partial a_i^\beta} \right)$$

and this is zero because $H_m$ is a linear combination of $H_k = \text{tr} L^k$ and

$$\sum_i \left( b_i^\beta \text{tr} \left( \frac{\partial L}{\partial b_i^\alpha} L^{m-1} \right) - a_i^\alpha \text{tr} \left( \frac{\partial L}{\partial a_i^\beta} L^{m-1} \right) \right) = \sum_i \sum_{j,k} \left( b_i^\beta \frac{\partial L_{jk}}{\partial b_i^\alpha} L_{kj}^{m-1} - a_i^\alpha \frac{\partial L_{jik}}{\partial a_i^\beta} L_{kj}^{m-1} \right) = 2\gamma \sum_{i \neq j} \delta_{ij} \frac{w_i^{1/2} w_j^{1/2} b_j^\beta a_k^\alpha}{w_i - w_j} L_{kj}^{m-1} = 0.$$

As a result, we obtain the equation

$$\gamma \sum_k \text{coth}(\gamma (x_k(p+1) - x_i(p))a_k^\alpha(p+1)b_k^\nu(p+1)a_i^\nu(p)) = \gamma \sum_{k \neq i} \text{coth}(\gamma (x_k(p) - x_i(p))a_k^\alpha(p)b_k^\nu(p)a_i^\nu(p)) + \frac{\dot{x}_i(p)}{2} a_i^\alpha(p) + \mu a_i^\alpha(p). \quad (6.11)$$

A similar transformation of equation (6.9) leads to the equation

$$\gamma \sum_k \text{coth}(\gamma (x_i(p) - x_k(p-1))b_k^\nu(p-1)b_i^\nu(p)a_k^\nu(p-1)) = \gamma \sum_{k \neq i} \text{coth}(\gamma (x_i(p) - x_k(p))b_k^\nu(p)b_i^\nu(p)a_k^\nu(p)) + \frac{\dot{x}_i(p)}{2} b_i^\nu(p) + \mu b_i^\alpha(p). \quad (6.12)$$

Multiply the first equation by $b_i^\nu(p)$ and sum over $\alpha$, then multiply the second equation by $a_i^\nu(p)$, sum over $\alpha$ and take into account the constraint $b_i^\nu a_i^\nu = 1$. Subtracting the resulting equations, we eliminate $\dot{x}_i(p)$ and obtain the equations of motion (1.4):

$$\sum_j \text{coth}(\gamma (x_i(p) - x_j(p+1))b_j^\nu(p)a_j^\nu(p+1)b_i^\nu(p+1)a_i^\nu(p)) + \sum_j \text{coth}(\gamma (x_i(p) - x_j(p-1))b_j^\nu(p)a_j^\nu(p-1)b_i^\nu(p-1)a_i^\nu(p)) = 2 \sum_{j \neq i} \text{coth}(\gamma (x_i(p) - x_j(p))b_j^\nu(p)a_j^\nu(p))b_i^\nu(p)a_i^\nu(p) \quad (6.13)$$

These equations of motion generalize the ones for the rational Gibbons-Hermsen model obtained in [19]. They look like the Bethe ansatz equations for the quantum trigonometric Gaudin model “dressed” by the spin variables. In the continuum limit the equations of motion (3.15) are reproduced.
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