MEROMORPHIC FUNCTIONS OF FINITE $\varphi$-ORDER AND LINEAR ASKEY-WILSON DIVIDED DIFFERENCE EQUATIONS

HUI YU, JANNE HEITTKANGAS*, JUN WANG, AND ZHI-TAO WEN

ABSTRACT. The growth of meromorphic solutions of linear difference equations containing Askey-Wilson divided difference operators is estimated. The $\varphi$-order is used as a general growth indicator, which covers the growth spectrum between the logarithmic order $\rho_{\log}(f)$ and the classical order $\rho(f)$ of a meromorphic function $f$.

KEY WORDS: Askey-Wilson divided difference operator, Askey-Wilson divided difference equation, lemma on the logarithmic difference, meromorphic function, $\varphi$-order.

MSC 2020: Primary 39A13; Secondary 30D35.

1. INTRODUCTION

Suppose that $q$ is a complex number satisfying $0 < |q| < 1$. In 1985, Askey and Wilson evaluated a $q$-beta integral [1, Theorem 2.1], which allowed them to construct a family of orthogonal polynomials [1, Theorems 2.2–2.5]. These polynomials are eigensolutions of a second order difference equation [1, p. 36] that involves a divided difference operator $D_q$ currently known as the Askey-Wilson operator. We will define $D_q$ below and call it the AW-operator for brevity. In general, any three consecutive orthogonal polynomials satisfy a certain three term recurrence relation, see [1, p. 4] or [8, p. 42].

Recently, Chiang and Feng [3] have obtained a full-fledged Nevanlinna theory for meromorphic functions of finite logarithmic order with respect to the AW-operator on the complex plane $\mathbb{C}$. The concluding remarks in [3] admit that the logarithmic order of growth appears to be restrictive, even though this class contains a large family of important meromorphic functions. This encourages us to generalize some of the results in [3] in such a way that the associated results for finite logarithmic order follow as special cases.

Let $\varphi : (R_0, \infty) \to (0, \infty)$ be a non-decreasing unbounded function. The $\varphi$-order of a meromorphic function $f$ in $\mathbb{C}$ was introduced in [5] as the quantity

$$\rho_\varphi(f) = \limsup_{r \to \infty} \log T(r, f) \frac{\log \rho(f)}{\log \varphi(r)}.$$ 

Prior to [5], the $\varphi$-order was used as a growth indicator for meromorphic functions in the unit disc in [4]. In the plane case, the logarithmic order $\rho_{\log}(f)$ and the classical order $\rho(f)$ of $f$ follow as special cases when choosing

*Corresponding author.
\( \varphi(r) = \log r \) and \( \varphi(r) = r \), respectively. This leads us to impose a global growth restriction

\[
\log r \leq \varphi(r) \leq r, \quad r \geq R_0.
\]

(1.1)

Here and from now on, the notation \( r \geq R_0 \) is being used to express that the associated inequality is valid ”for all \( r \) large enough”.

For an entire function \( f \), the Nevanlinna characteristic \( T(r, f) \) can be replaced with the logarithmic maximum modulus \( \log M(r, f) \) in the quantities \( \rho(f) \) and \( \rho_{\log}(f) \) by using a well-known relation between \( T(r, f) \) and \( \log M(r, f) \), see [7, p. 23]. The same is true for the \( \varphi \)-order, namely

\[
\rho_{\varphi}(f) = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log \varphi(r)},
\]

(1.2)

provided that \( \varphi \) is subadditive, that is, \( \varphi(a+b) \leq \varphi(a) + \varphi(b) \) for all \( a, b \geq R_0 \). In particular, this gives \( \varphi(2r) \leq 2\varphi(r) \), which yields (1.2). Moreover, up to a normalization, subadditivity is implied by concavity, see [5] for details.

Following the notation in [1] (see [3] and [6, p. 300] for an alternative notation), we suppose that \( f(x) \) is a meromorphic function in \( \mathbb{C} \), and let \( x = \cos \theta \) and \( z = e^{i \theta} \), where \( \theta \in \mathbb{C} \). Then, for \( x \neq \pm 1 \), the AW-operator is defined by

\[
(D_q f)(x) := \frac{\tilde{f}(q^{\frac{1}{2}} z e^{i \theta}) - \tilde{f}(q^{-\frac{1}{2}} z e^{i \theta})}{e(q^{\frac{1}{2}} z e^{i \theta}) - e(q^{-\frac{1}{2}} z e^{i \theta})} = \frac{\tilde{f}(q^{\frac{1}{2}} z e^{i \theta}) - \tilde{f}(q^{-\frac{1}{2}} z e^{i \theta})}{(q^{\frac{1}{2}} z^{-1/2} - q^{-\frac{1}{2}} z^{1/2})(z - 1/z)/2},
\]

(1.3)

where \( x = (z + 1/z)/2 = \cos \theta \), \( z = e^{i \theta} \), \( e(x) = x \) and

\[
\tilde{f}(z) = f((z + 1/z)/2) = f(x) = f(\cos \theta).
\]

In the exceptional cases \( x = \pm 1 \), we define

\[
(D_q f)(\pm 1) = \lim_{x \to \pm 1} (D_q f)(x) = f'(\pm (q^{\frac{1}{2}} + q^{-\frac{1}{2}})/2).
\]

The branch of the square root in \( z = x + \sqrt{x^2 - 1} \) can be fixed in such a way that for each \( x \in \mathbb{C} \) there corresponds a unique \( z \in \mathbb{C} \), see [3] and [6, p. 300]. It is known that \( D_q f \) is meromorphic for a meromorphic function \( f \) and entire for an entire function \( f \) [3, Theorem 2.1]. The AW-operator in (1.3) can be written in the alternative form

\[
(D_q f)(x) = \frac{f(\hat{x}) - f(\tilde{x})}{\hat{x} - \tilde{x}},
\]

where \( x = (z + 1/z)/2 = \cos \theta \) and

\[
\hat{x} = q^{\frac{1}{2}} z + q^{-\frac{1}{2}} z^{-1}, \quad \tilde{x} = q^{\frac{1}{2}} z^{-1} + q^{-\frac{1}{2}} z.
\]

Finally, AW-operators of arbitrary order are defined by \( D_q^0 f = f \) and \( D_q^n f = D_q(D_q^{n-1} f) \), where \( n \in \mathbb{N} \).

Lemma A below is a pointwise AW-type lemma on the logarithmic difference proved in [3, Lemma 4.2], and it is used in [3] to study the growth of meromorphic solutions of Askey-Wilson divided difference equations. We note that finite logarithmic order implies finite \( \varphi \)-order because of the growth restriction (1.1).
\textbf{Lemma A.} Let $f(x)$ be a meromorphic function of finite logarithmic order such that $\mathcal{D}_q f \not\equiv 0$, and let $\alpha_1 \in (0, 1)$ be arbitrary. Then there exists a constant $C_{\alpha_1} > 0$ such that for $2(|q^{1/2}| + |q^{-1/2}|)|x| < R$, we have

\[
\log^+ \left| \frac{\mathcal{D}_q f(x)}{f(x)} \right| \leq \frac{4R(|q^{1/2} - 1| + |q^{-1/2} - 1|)|x|}{(R - |x|)(R - 2(|q^{1/2} | + |q^{-1/2}|)|x|)} \left( m(R, f) + m(R, 1/f) \right) \\
+ 2(|q^{1/2} - 1| + |q^{-1/2} - 1|)|x| \left( \frac{1}{R - |x|} + \frac{1}{R - 2(|q^{1/2} | + |q^{-1/2}|)|x|} \right) \\
\times (n(R, f) + n(R, 1/f)) \\
+ 2C_{\alpha_1}(|q^{1/2} - 1|^\alpha_1 + |q^{-1/2} - 1|^\alpha_1)|x|^\alpha_1 \sum_{|c_n| < R} \frac{1}{|x - c_n|^\alpha_1} \\
+ 2C_{\alpha_1}|q^{-1/2} - 1|^\alpha_1 |x|^\alpha_1 \sum_{|c_n| < R} \frac{1}{|x + c(q)q^{1/2}z^{-1} - q^{-1/2}c_n|^\alpha_1} \\
+ 2C_{\alpha_1}|q^{1/2} - 1|^\alpha_1 |x|^\alpha_1 \sum_{|c_n| < R} \frac{1}{|x - c(q)q^{1/2}z^{-1} - q^{1/2}c_n|^\alpha_1} + \log 2,
\]

(1.4)

where $c(q) = (q^{-1/2} - q^{1/2})/2$ and $\{c_n\}$ is the combined sequence of zeros and poles of $f$.

The choice $R = r \log r$ in Lemma A is made in proving \cite[Theorem 3.1]{3}, which is an AW-type lemma on the logarithmic difference equation

\[
m \left( r, \frac{\mathcal{D}_q f(x)}{f(x)} \right) = O \left( (\log r)^{\rho_{\log}(f) - 1 + \varepsilon} \right),
\]

(1.5)

where $\varepsilon > 0$ is arbitrary and $f$ is a meromorphic function of finite logarithmic order $\rho_{\log}(f)$ such that $\mathcal{D}_q f \not\equiv 0$. The estimate (1.5) in turn is used to prove a growth estimate \cite[Theorem 12.4]{3} for meromorphic solutions of AW-divided difference equations, stated as follows.

\textbf{Theorem B.} Let $a_0(x), a_1(x), \ldots, a_{n-1}(x)$ be entire functions such that

$\rho_{\log}(a_0) > \max_{1 \leq j \leq n} \{\rho_{\log}(a_j)\}$.

Suppose that $f$ is an entire solution of the AW-divided difference equation

\[
\sum_{j=0}^{n} a_j(x)\mathcal{D}_q^j f(x) = 0,
\]

where $a_n(x) = 1$. Then $\rho_{\log}(f) \geq \rho_{\log}(a_0) + 1$.

Our main objectives are to find $\varphi$-order analogues of the estimate (1.5) and of Theorem B. A non-decreasing function $s : (R_0, \infty) \to (0, \infty)$ satisfying a global growth restriction

\[
r < s(r) \leq r^2, \quad r \geq R_0,
\]

(1.6)

will take the role of $R$ in Lemma A. Suitable test functions for $\varphi$ and $s$ then are, for example,

$\varphi(r) = \log^\alpha r, \quad \varphi(r) = \exp(\log^\beta r), \quad \varphi(r) = r^\beta$, 

\[\]
along with \( s(r) = r \log r \) and \( s(r) = r^\alpha \), where \( \alpha \in (1,2] \) and \( \beta \in (0,1] \).

This paper is organized as follows. A generalization of Theorem B for meromorphic solutions in terms of the \( \varphi \)-order is given in Section 2. Two AW-type lemmas on the logarithmic difference in terms of the \( \varphi \)-order are given in Section 3. One of them will be among the most important individual tools later on. Section 4 consists of lemmas on AW-type counting functions as well as on the Nevanlinna characteristic of \( \mathcal{D}_qf \). These lemmas are crucial in proving the main results, which are Theorem 2.1 and 2.2 below. The details of the proofs are given in Section 5.

2. Results on Askey-Wilson divided difference equations

We consider the growth of meromorphic solutions of AW-divided difference equations

\[
\sum_{j=0}^{n} a_j(x) \mathcal{D}_q^j f(x) = 0 \tag{2.1}
\]

and of the corresponding non-homogeneous AW-divided difference equations

\[
\sum_{j=0}^{n} a_j(x) \mathcal{D}_q^j f(x) = a_{n+1}(x), \tag{2.2}
\]

where \( a_0, \ldots, a_{n+1} \) are meromorphic functions, and \( a_0a_n \neq 0 \). The results that follow depend on growth parameters introduced in [5] and defined by

\[
\alpha_{\varphi,s} = \lim \inf_{r \to \infty} \frac{\log \varphi(r)}{\log \varphi(s(r))}\quad \text{and} \quad \gamma_{\varphi,s} = \lim \inf_{r \to \infty} \frac{\log \log s(r)}{\log \varphi(r)}. \tag{2.3}
\]

Due to the assumptions (1.1) and (1.6), we always have \( \alpha_{\varphi,s} \in [0,1] \) and \( \gamma_{\varphi,s} \in [-\infty,1] \). From now on, we make a global assumption

\[
\lim \inf_{r \to \infty} \frac{s(r)}{r} > 1,
\]

which ensures that \( \gamma_{\varphi,s} \in [0,1] \). Further properties and relations related to the growth parameters \( \alpha_{\varphi,s} \) and \( \gamma_{\varphi,s} \) can be found in [5].

Theorem 2.1 below reduces to Theorem B when choosing \( \varphi(r) = \log r \) and \( s(r) = r^2 \) and when the coefficients and solutions are entire functions.

**Theorem 2.1.** Suppose that \( \varphi(r) \) is subadditive, and let \( \alpha_{\varphi,s} \) and \( \gamma_{\varphi,s} \) be the constants in (2.3). Let \( a_0, \ldots, a_n \) be meromorphic functions of finite \( \varphi \)-order such that

\[
\rho_{\varphi}(a_0) > \max_{1 \leq j \leq n} \{\rho_{\varphi}(a_j)\}.
\]

(a) Suppose that \( \limsup_{r \to \infty} \frac{s(r)}{r} = \infty \) and that \( s(r) \) is convex and differentiable. If \( f \) is a non-constant meromorphic solution of (2.1), then

\[
\rho_{\varphi}(f) \geq \alpha_{\varphi,s}^n \rho_{\varphi}(a_0). \tag{2.4}
\]

Moreover, if the coefficients \( a_0, \ldots, a_n \) are entire, then

\[
\rho_{\varphi}(f) \geq \alpha_{\varphi,s}^n \rho_{\varphi}(a_0) + \alpha_{\varphi,s}^n \gamma_{\varphi,s}. \tag{2.5}
\]
(b) Suppose that \( \limsup_{r \to \infty} \frac{s(r)}{r} < \infty \). If \( f \) is a non-constant meromorphic solution of (2.1), then \( \rho_\varphi(f) \geq \alpha_{\varphi,s}^{n-1}\rho_\varphi(a_0) \).

**Remark 1.** For certain \( \varphi(r) \), for example, for \( \varphi(r) = \log^q r \), where \( q \in (1, 2] \), the conclusion of Theorem 2.1(a) is stronger than that of Theorem 2.1(b) due to different choices of \( s(r) \). If the coefficients \( a_0, \ldots, a_n \) are entire, then it follows from (2.3) and (2.5) that \( \rho_\varphi(f) \geq \rho_\varphi(a_0) + 1/\alpha \) in Theorem 2.1(a) when choosing \( s(r) = r^2 \), which is stronger than the conclusion \( \rho_\varphi(f) \geq \rho_\varphi(a_0) \) in Theorem 2.1(b) when choosing \( s(r) = 2r \).

On the other hand, the opposite is true for some suitable \( \varphi(r) \). For instance, choose \( \varphi(r) = r^\beta \), where \( \beta \in (0, 1] \), along with \( s(r) = 2r \) and \( s(r) = r^2 \), respectively. Then we get \( \rho_\varphi(f) \geq \rho_\varphi(a_0) \) from Theorem 2.1(b), which is stronger than the conclusion \( \rho_\varphi(f) \geq (1/2)^n \rho_\varphi(a_0) \) in Theorem 2.1(a), which in turn follows from (2.3) and (2.4).

The following result is a growth estimate for meromorphic solutions of the non-homogeneous equations (2.2).

**Theorem 2.2.** Suppose that \( \varphi(r) \) is subadditive. Let \( a_0, \ldots, a_n \) be meromorphic functions of finite \( \varphi \)-order such that

\[
\rho_\varphi(a_0) > \max_{1 \leq j \leq n+1} \{ \rho_\varphi(a_j) \}.
\]

If \( f \) is a non-constant meromorphic solution of (2.2), then \( \rho_\varphi(f) \geq \alpha_{\varphi,s}^{n-1}\rho_\varphi(a_0) \).

The proofs of Theorems 2.1 and 2.2 in Section 5 are based on an AW-type lemma on the logarithmic difference discussed in Section 3 as well as on estimates for AW-type counting functions discussed in Section 4.

### 3. Estimates for the Askey-Wilson Type Logarithmic Difference

Lemma 3.1 below is an AW-type lemma on the logarithmic difference, which reduces to [3, Theorem 3.1] when choosing \( \varphi(r) = \log r \) and \( s(r) = r^2 \). The proof uses the notation \( g(r) \lesssim h(r) \) to express that there exists a constant \( C \geq 1 \) such that \( g(r) \leq Ch(r) \) for all \( r \geq R_0 \).

**Lemma 3.1.** Let \( f \) be a meromorphic function of finite \( \varphi \)-order \( \rho_\varphi(f) \) such that \( D_q f \neq 0 \). Let \( \alpha_{\varphi,s} \) and \( \gamma_{\varphi,s} \) be the constants in (2.3), let \( \varepsilon > 0 \), and denote \( |x| = r \).

(a) If \( \limsup_{r \to \infty} \frac{s(r)}{r} = \infty \) and if \( s(r) \) is convex and differentiable, then

\[
m\left( r, \frac{D_q f(x)}{f(x)} \right) = O \left( \frac{\varphi(s(r))^{\rho_\varphi(f)+\frac{\varepsilon}{\gamma_{\varphi,s}}} + 1}{\log \frac{s(r)}{r}} \right) = O \left( \varphi(s(r))^{\rho_\varphi(f) - \alpha_{\varphi,s} \gamma_{\varphi,s} + \varepsilon} \right).
\]

(b) If \( \limsup_{r \to \infty} \frac{s(r)}{r} < \infty \) and if \( \varphi(r) \) is subadditive, then

\[
m\left( r, \frac{D_q f(x)}{f(x)} \right) = O \left( \varphi(r)^{\rho_\varphi(f) + \varepsilon} \right).
\]
Proof. (a) By the proof of [5, Lemma 3.1(a)], there exist non-decreasing functions \( u, v : [1, \infty) \to (0, \infty) \) with the following properties:

1. \( r < u(r) < s(r) \) and \( r < v(r) < s(r) \) for all \( r \geq R_0 \),
2. \( u(r)/r \to \infty \) and \( v(r)/r \to \infty \) as \( r \to \infty \),
3. \( 2^{-1}s(r) \leq v(u(r)) \leq s(r) \) for all \( r \geq R_0 \),
4. \( 2\log(u(r))/r \leq \log(s(r))/r \leq 2u(r)/r \) for all \( r \geq R_0 \).

Using the standard estimate

\[
N(v(r), f) - N(r, f) = \int_r^{v(r)} \frac{n(t, f)}{t} dt \geq n(r, f) \log \frac{v(r)}{r}
\]

and the properties (3) and (4), we deduce that

\[
n(u(r), f) \leq \frac{T(s(r), f)}{\log \frac{s(r)}{2r} - \log \frac{u(r)}{r}} \leq \frac{T(s(r), f)}{\log \frac{s(r)}{r}} \leq \frac{\varphi(s(r))^{\rho(\varphi) + \frac{\gamma}{r}}}{\log \frac{s(r)}{r}},
\]

and similarly for \( n(u(r), 1/f) \). Choose \( R = u(r) \). We integrate (1.4) from 0 to \( 2\pi \), and we make use of the properties (1) and (4) together with (3.1) and formulas (63)–(64) in [3], and obtain

\[
m\left(r, \frac{D_q f(x)}{f(x)}\right) \leq \frac{T(u(r), f)}{u(r)/r} + n(u(r), f) + n(u(r), 1/f) + 1
\]

\[
\leq \frac{\varphi(s(r))^{\rho(\varphi) + \frac{\gamma}{r}}}{\log \frac{s(r)}{r}} + 1.
\]

This proves the first identity in Case (a).

From (2.3), we get

\[
\alpha_{\varphi, s_{\varphi}, s} \leq \liminf_{r \to \infty} \left( \frac{\log \varphi(r)}{\log \varphi(s(r))} \cdot \frac{\log \frac{s(r)}{r}}{\log \varphi(r)} \right) = \liminf_{r \to \infty} \frac{\log \varphi(s)}{\log \varphi(s(r))},
\]

and so

\[
\log \frac{s(r)}{r} \geq \varphi(s(r))^{\alpha_{\varphi, s_{\varphi}, s}} - \frac{\gamma}{r}, \quad r \geq R_0.
\]

Recall from [5, Corollary 4.3] that, for a non-constant meromorphic function \( f \) of finite \( \varphi \)-order \( \rho_{\varphi}(f) \), we have \( \rho_{\varphi}(f) \geq \alpha_{\varphi, s_{\varphi}, s} \). Thus

\[
\frac{\varphi(s(r))^{\rho_{\varphi}(f) + \frac{\gamma}{r}}}{\log \frac{s(r)}{r}} \leq \varphi(s(r))^{\rho_{\varphi}(f) - \alpha_{\varphi, s_{\varphi}, s} + \epsilon}, \quad r \geq R_0,
\]

where the right-hand side tends to infinity as \( r \to \infty \). This proves the second identity in Case (a).

(b) By the assumptions on \( s(r) \), there exists a constant \( C \in (1, \infty) \) such that \( r < s(r) < Cr \) for all \( r \geq R_0 \). We choose \( R = Br \), where

\[
B = \max\{[C], [2(|q^{1/2}| + |q^{-1/2}|)]\} + 1
\]

is an integer. Integrating (1.4) from 0 to \( 2\pi \) and making use of formulas (63)–(64) in [3] together with

\[
T(2r, f) \geq \int_r^{2r} \frac{n(t, f)}{t} dt \geq n(r, f) \log 2,
\]

(3.4)
we obtain
\[ m \left( r, \frac{D_qf(x)}{f(x)} \right) \lesssim T(Br, f) + n(Br, f) + n(Br, 1/f) + 1 \tag{3.5} \]
\[ \lesssim \varphi(2Br)^{\rho_f(j + \varepsilon)} + 1. \]
Since the subadditivity of \( \varphi \) yields \( \varphi(2Br) \lesssim 2B\varphi(r) \), the assertion follows from (3.5). This completes the proof. \( \square \)

Lemma 3.2 below is a pointwise estimate for the AW-type logarithmic difference that holds outside of an exceptional set. The result reduces to [3, Theorem 3.2] when choosing \( \varphi(r) = \log r \) and \( s(r) = r^2 \).

**Lemma 3.2.** Let \( f \) be a meromorphic function of finite \( \varphi \)-order \( \rho_f(r) \) such that \( D_qf \neq 0 \). Let \( \alpha_{\varphi,s} > 0 \) and \( \gamma_{\varphi,s} \) be the constants in (2.3), let \( \varepsilon > 0 \), and denote \( |x| = r \). Suppose that \( \varphi(r) \) is continuous and satisfies
\[ \limsup_{r \to \infty} \frac{\log \varphi(r)}{\log r} = 0. \tag{3.6} \]

(a) If \( \limsup_{r \to \infty} \frac{s(r)}{r} = \infty \) and if \( s(r) \) is convex and differentiable, then
\[ \log^+ \left| \frac{D_qf(x)}{f(x)} \right| = O \left( \frac{\varphi(s(r))^{\rho_f(j + \varepsilon)}}{\log s(r)} + 1 \right) = O \left( \varphi(s(r))^{\rho_f(j - \alpha_{\varphi,s} \gamma_{\varphi,s} + \varepsilon)} \right) \]
holds outside of an exceptional set of finite logarithmic measure.

(b) If \( \limsup_{r \to \infty} \frac{s(r)}{r} < \infty \) and if \( \varphi(r) \) is subadditive, then
\[ \log^+ \left| \frac{D_qf(x)}{f(x)} \right| = O \left( \varphi(r)^{\rho_f(j + \varepsilon)} \right) \]
holds outside of an exceptional set of finite logarithmic measure.

**Proof.** We modify the proof of [3, Theorem 3.2] as follows. \( (a) \) Denote
\[ \{d_n\} := \{c_n\} \cup \{q^{1/2}c_n\} \cup \{q^{-1/2}c_n\}, \tag{3.7} \]
where \( \{c_n\} \) is the combined sequence of zeros and poles of \( f \). Let
\[ E_n = \left\{ r : r \in \left( \frac{|d_n|}{\varphi(|d_n| + 3)^{\rho_f(j + \varepsilon)}} \right), \left| d_n \right| + \frac{|d_n|}{\varphi(|d_n| + 3)^{\rho_f(j + \varepsilon)}} \right\} \]
and \( E = \cup_n E_n \), where \( \alpha_{\varphi,s} \in (0,1] \) is defined in (2.3). In what follows, we consider \( r \notin E \). We proceed to prove that
\[ |x - d_n| \geq \frac{|x|}{2\varphi(|x| + 3)^{\rho_f(j + \varepsilon)}}, \quad |x| = r \geq R_0. \tag{3.8} \]
The proof is divided into three cases in each of which \( |x| \geq R_0 \).
(1) Suppose that $|x| < |d_n| - \frac{|d_n|_{\varphi((|x|+3)^{-\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}})}}}{\varphi(|x|+3)}$. From (3.6), the function
\[
\frac{|x|}{\varphi(|x|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}}
\]
is increasing, and so
\[
|x - d_n| \geq ||x| - |d_n|| \geq \frac{|d_n|}{\varphi(|d_n|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}} \geq \frac{|x|}{2\varphi(|x|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}}.
\]
(2) Suppose that $|d_n| + \frac{|d_n|}{\varphi(|d_n|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}} \leq |x| - \frac{|x|}{\varphi(|x|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}}$. Clearly,
\[
|x - d_n| \geq \frac{|d_n|}{\varphi(|d_n|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}} + \frac{|x|}{\varphi(|x|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}} \geq \frac{|x|}{2\varphi(|x|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}}.
\]
(3) Suppose that $|d_n| + \frac{|d_n|}{\varphi(|d_n|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}} < |x|$ and
\[
|x - d_n| \geq \frac{|d_n|}{\varphi(|d_n|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}} \leq |d_n| + \frac{|d_n|}{\varphi(|d_n|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}}.
\]

Then we have $|x - d_n| \geq \frac{|d_n|}{\varphi(|d_n|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}}$ and $|x| = |d_n|(1 + o(1))$ as $|x| \to \infty$ (or as $n \to \infty$). This yields (3.8) by the continuity of $\varphi(r)$.

Keeping in mind that $r \not\in E$, this completes the proof of (3.8). Let $\alpha_1 \in (0, 1)$. From (3.8),
\[
\sum_{|c_n| < R} \frac{1}{|x - c_n|^{\alpha_1}} \leq 2^{\alpha_1} \varphi(|x|+3)^{\frac{\alpha_1}{\alpha_{\varphi,s}}} \frac{\alpha_1(3\varphi(f)+\varepsilon)}{\alpha_{\varphi,s}}(n(R, f) + n(R, 1/f)).
\]

From (3.6)–(3.8), we have, for all $|x|$ sufficiently large and hence for all $|z|$ sufficiently large,
\[
|x + c(q)q^{-1/2}z^{-1} - q^{-1/2}c_n| \geq |x - q^{-1/2}c_n| - |c(q)q^{-1/2}z^{-1}| \geq \frac{|x|}{3\varphi(|x|+3)}^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}},
\]
and similarly for $|x - c(q)q^{1/2}z^{1/2} - q^{1/2}c_n|$, where $c(q) = (q^{-1/2} - q^{1/2})/2$.

Therefore,
\[
\sum_{|c_n| < R} \frac{1}{|x + c(q)q^{-1/2}z^{-1} - q^{-1/2}c_n|^{\alpha_1}} + \sum_{|c_n| < R} \frac{1}{|x - c(q)q^{1/2}z^{1/2} - q^{1/2}c_n|^{\alpha_1}} \\
\leq 2 \cdot 3^{\alpha_1} \varphi(|x|+3)^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}} \frac{\alpha_1}{\alpha_{\varphi,s}}(n(R, f) + n(R, 1/f)).
\]

We make use of the proof of Lemma 3.1, according to which there exist non-decreasing functions $u, v : [1, \infty) \to (0, \infty)$ satisfying the aforementioned properties (1)–(4). Choose $R = u(r)$ and $\alpha_1 = \frac{\alpha_{\varphi,s}}{4(3\varphi(f)+\varepsilon)} \in (0, 1)$. Since $\varepsilon > 0$ is arbitrary, it follows from (3.1) that
\[
n(u(r), f) \leq \frac{\varphi(s(r))^{\frac{3\varphi(f)+\varepsilon}{\alpha_{\varphi,s}}}}{\log \frac{4R}{r}}.
\]
By substituting (3.9)–(3.11) into (1.4), and by using (3.2), we have
\[
\log^+ \left| \frac{D_q f(x)}{f(x)} \right| \lesssim \frac{T(u(r), f)}{u(r)/r} + \frac{n(u(r), f) + n(u(r), 1/f)}{u(r)/r} \\
+ \varphi(r + 3) \frac{\alpha_{\varphi,s}(\rho_\varphi(f)+\varepsilon)}{\rho_\varphi(f)+\varepsilon} \cdot \frac{\varphi(s(r))^{\rho_\varphi(f)+\varepsilon}}{\log \frac{s(r)}{r}} + 1 \\
\lesssim \frac{\varphi(s(r))^{\rho_\varphi(f)+\varepsilon}}{\log \frac{s(r)}{r}} + 1 \lesssim \varphi(s(r))^{\rho_\varphi(f) - \alpha_{\varphi,s} + \varepsilon}, \quad r \notin E.
\]

(3.12)

By (3.12), it suffices to prove that the logarithmic measure of the exceptional set \( E \) is finite. We recall from [2, p. 249] that, for a meromorphic function \( h(x) \),
\[
n(r, h(cx)) = n(|c| r, h(x)), \quad c \in \mathbb{C} \setminus \{0\}.
\]
We apply this formula to the functions \( f(q^{-1/2} x) \) and \( f(q^{1/2} x) \) and make use of [5, Lemmas 4.1–4.2] to get
\[
\lambda_\varphi = \rho_\varphi(n(A r, f) + n(A r, 1/f)) \leq \frac{\rho_\varphi(f)}{\alpha_{\varphi,s}} < \infty,
\]
where \( \lambda_\varphi \) is the \( \varphi \)-exponent of convergence of the sequence \( \{d_n\} \) defined in (3.7), and \( A = \max \{1, |q|^{-1/2}, |q|^{1/2} \} \). For \( N \geq R_0 \) and a given sufficiently small \( \delta > 0 \), we have
\[
\frac{\varphi(|d_N|)^{\rho_\varphi(f)+\varepsilon}}{\varphi(|d_N|)^{\alpha_{\varphi,s}}} < \delta.
\]
Using the fact that \( \log(1 + |x|) \leq |x| \)
for all \( |x| \geq 0 \), the constant \( C_\delta = \frac{2}{1 - \delta} > 0 \) satisfies
\[
\log \frac{1 + \frac{1}{\varphi(|d_N|)^{\rho_\varphi(f)+\varepsilon}}}{1 - \frac{1}{\varphi(|d_N|)^{\rho_\varphi(f)+\varepsilon}}} \leq C_\delta \cdot \frac{1}{\varphi(|d_N|)^{\rho_\varphi(f)+\varepsilon}}, \quad N \geq R_0.
\]
Therefore,
\[
\log \text{meas} (E) = \left( \int_{E \cap [1, |d_N|]} + \int_{E \cap [|d_N|, \infty)} \right) \frac{dt}{t} \leq \log |d_N| + \sum_{n=N}^\infty \int_{E_n} \frac{dt}{t} = \log |d_N| + \sum_{n=N}^\infty \log \left( 1 + \frac{1}{\varphi(|d_n|)^{\rho_\varphi(f)+\varepsilon}} \right)
\]
\[
\leq \log |d_N| + C_\delta \sum_{n=N}^\infty \frac{1}{\varphi(|d_n|)^{\alpha_{\varphi,s}}} \lambda_\varphi^{\alpha_{\varphi,s} + \varepsilon} < \infty,
\]
which yields the assertion.

(b) By making use of the proof of Lemma 3.1(b) and following the same method as in Case (a) above, we obtain (3.9) and (3.10). Choose \( R = B r \) and \( \alpha_1 = \frac{\alpha_{\varphi,s} + \varepsilon}{2(\rho_\varphi(f)+\varepsilon)} \in (0, 1) \), where \( B \) is defined in (3.3). Then by substituting
(3.4), (3.9) and (3.10) into (1.4), we have
\[
\log^{+} \left| \frac{D_{q}f(x)}{f(x)} \right| \lesssim T(Br, f) + n(Br, f) + n(Br, 1/f) \\
+ \varphi(r + 3) \frac{1}{\alpha_{\varphi,s}(\rho_{\varphi}(f) + \varepsilon)} \cdot \varphi(2Br)^{\rho_{\varphi}(f) + \varepsilon} + 1 \\
\leq \varphi(2Br)^{\rho_{\varphi}(f) + \varepsilon}, \quad r \notin E.
\]
Then the assertion follows from the subadditivity of \( \varphi \), that is, \( \varphi(2Br) \leq 2B\varphi(r) \). Similarly as in Case (a) above, we deduce that the logarithmic measure of the exceptional set \( E \) is finite. This completes the proof. \( \square \)

4. Askey-Wilson Type Counting Functions and Characteristic Functions

In this section we state three lemmas, whose proofs are just minor modifications of the corresponding results in [3]. For a non-constant meromorphic function \( f \), it follows from [5, Lemmas 4.1–4.2] that \( \rho_{\varphi}(f) \geq \alpha_{\varphi,s}\lambda_{\varphi} + \alpha_{\varphi,s}\gamma_{\varphi,s} \) and, if \( \alpha_{\varphi,s} > 0 \), then
\[
n(r, a, f) = O(\varphi(r)^{\lambda_{\varphi} + \varepsilon}) \leq O\left(\varphi(r)^{\frac{\rho_{\varphi}(f) - \gamma_{\varphi,s} + \varepsilon}{\alpha_{\varphi,s}}}\right),
\]
where \( \lambda_{\varphi} \) is the \( \varphi \)-exponent of convergence of the \( a \)-points of \( f \).

Lemma 4.1 below is essential in proving Lemma 4.2, and it reduces to [3, Theorem 5.1] when choosing \( \varphi(r) = \log r \) and \( s(r) = r^2 \).

**Lemma 4.1.** Let \( f \) be a non-constant meromorphic function of finite \( \varphi \)-order \( \rho_{\varphi}(f) \). Suppose that \( \varphi(r) \) is subadditive. Let \( \alpha_{\varphi,s} > 0 \) and \( \gamma_{\varphi,s} \) be the constants in (2.3), and let \( \varepsilon > 0 \) and \( a \in \mathbb{C} \).

(a) If \( \limsup_{r \to \infty} \frac{s(r)}{r} = \infty \) and if \( s(r) \) is convex and differentiable, then
\[
N(r, a, f(x)) = N(r, a, f(x)) + O\left(\varphi(r)^{\frac{\rho_{\varphi}(f)}{\alpha_{\varphi,s}} - \gamma_{\varphi,s} + \varepsilon}\right) + O(\log r),
\]
\[
N(r, a, f(x)) = N(r, a, f(x)) + O\left(\varphi(r)^{\frac{\rho_{\varphi}(f)}{\alpha_{\varphi,s}} - \gamma_{\varphi,s} + \varepsilon}\right) + O(\log r).
\]

(b) If \( \limsup_{r \to \infty} \frac{s(r)}{r} < \infty \), then
\[
N(r, a, f(x)) = N(r, a, f(x)) + O\left(\varphi(r)^{\rho_{\varphi}(f) + \varepsilon}\right) + O(\log r),
\]
\[
N(r, a, f(x)) = N(r, a, f(x)) + O\left(\varphi(r)^{\rho_{\varphi}(f) + \varepsilon}\right) + O(\log r).
\]

Lemma 4.2 below is a direct consequence of Lemma 4.1 and the definition of the AW-operator \( D_{q}f \), and it reduces to [3, Theorem 3.3] when choosing \( \varphi(r) = \log r \) and \( s(r) = r^2 \).

**Lemma 4.2.** Let \( f \) be a non-constant meromorphic function of finite \( \varphi \)-order \( \rho_{\varphi}(f) \). Suppose that \( \varphi(r) \) is subadditive. Let \( \alpha_{\varphi,s} > 0 \) and \( \gamma_{\varphi,s} \) be the constants in (2.3), and let \( \varepsilon > 0 \).
Consequently, we deduce from (\(\alpha\)) we replace \(\varepsilon\) in Lemma 4.3. Let \(f\) be a non-constant meromorphic function of finite \(\varphi\)-order \(\rho_\varphi(f)\). Suppose that \(\varphi(r)\) is subadditive. Let \(\alpha_{\varphi,s} > 0\) and \(\gamma_{\varphi,s}\) be the constants in (2.3), and let \(\varepsilon \in (0, 1)\).

(a) If \(\limsup_{r \to \infty} \frac{s(r)}{r} = \infty\) and if \(s(r)\) is convex and differentiable, then

\[
N (r, D_q f) \leq 2N(r, f) + O \left( \varphi(r) \frac{\rho_\varphi(f)}{\alpha_{\varphi,s} - \gamma_{\varphi,s} + \varepsilon} \right) + O(\log r).
\]

(b) If \(\limsup_{r \to \infty} \frac{s(r)}{r} < \infty\), then

\[
N (r, D_q f) \leq 2N(r, f) + O \left( \varphi(r)^{\rho_\varphi(f) + \varepsilon} \right) + O(\log r).
\]

The following result reduces to [3, Theorem 3.4] when choosing \(\varphi(r) = \log r\) and \(s(r) = r^2\).

**Lemma 4.3.** Let \(f\) be a non-constant meromorphic function of finite \(\varphi\)-order \(\rho_\varphi(f)\). Suppose that \(\varphi(r)\) is subadditive. Let \(\alpha_{\varphi,s} > 0\) and \(\gamma_{\varphi,s}\) be the constants in (2.3), and let \(\varepsilon \in (0, 1)\).

(a) If \(\limsup_{r \to \infty} \frac{s(r)}{r} = \infty\) and if \(s(r)\) is convex and differentiable, then

\[
T (r, D_q f) \leq 2T(r, f) + O \left( \varphi(r) \frac{\rho_\varphi(f)}{\alpha_{\varphi,s} - \gamma_{\varphi,s} + \varepsilon} \right) + O(\log r).
\]

(b) If \(\limsup_{r \to \infty} \frac{s(r)}{r} < \infty\), then

\[
T (r, D_q f) \leq 2T(r, f) + O \left( \varphi(r)^{\rho_\varphi(f) + \varepsilon} \right) + O(\log r).
\]

**Proof.** Choose \(\varepsilon^* = \frac{\alpha_{\varphi,s} \varepsilon^2}{2(\rho_\varphi(f) + \alpha_{\varphi,s} \varepsilon)} \in (0, \frac{\alpha_{\varphi,s} \varepsilon}{2})\). By the definition of the constant \(\alpha_{\varphi,s}\) in (2.3), it follows that

\[
\varphi(s(r)) \leq \varphi(r)^{\frac{1}{\alpha_{\varphi,s} - \varepsilon^*}}, \quad r \geq R_0.
\]

We replace \(\varepsilon\) in Lemma 3.1(a) with \(\varepsilon' = \frac{\alpha_{\varphi,s} \varepsilon^*}{2} + \gamma_{\varphi,s} \varepsilon^* = \left(\frac{\alpha_{\varphi,s} \varepsilon^*}{2} + \frac{\alpha_{\varphi,s} \varepsilon^2}{2(\rho_\varphi(f) + \alpha_{\varphi,s} \varepsilon)}\right) \varepsilon\), which we are allowed to do since \(0 < \frac{\alpha_{\varphi,s} \varepsilon^*}{2} \leq \frac{\alpha_{\varphi,s} \varepsilon}{2} + \frac{\alpha_{\varphi,s} \varepsilon^2}{2(\rho_\varphi(f) + \alpha_{\varphi,s} \varepsilon)} < \alpha_{\varphi,s} \leq 1\). Consequently, we deduce from (4.1) that

\[
\frac{\varphi(s(r))^{\rho_\varphi(f) - \alpha_{\varphi,s} \gamma_{\varphi,s} + \varepsilon'}}{\varphi(r)} \leq \frac{\varphi(r)^{\rho_\varphi(f) - \alpha_{\varphi,s} \gamma_{\varphi,s} + \varepsilon'}}{\varphi(r) \frac{\rho_\varphi(f)}{\alpha_{\varphi,s} - \gamma_{\varphi,s} + \varepsilon}} \leq \varphi(r)^{\rho_\varphi(f) + \varepsilon}, \quad r \geq R_0.
\]

Case (a) now follows directly from (4.2) and Lemmas 3.1(a) and 4.2(a). Case (b) is more straightforward. \(\square\)

**Remark 2.** If \(\alpha_{\varphi,s} > 0\), it is easy to see that

\[
\rho_\varphi(D_q f) \leq \max \left\{ \rho_\varphi(f), \frac{\rho_\varphi(f)}{\alpha_{\varphi,s}} - \gamma_{\varphi,s} \right\}.
\]
5. Proofs of theorems

Proof of Theorem 2.1. All assertions are true if \( \rho_\psi(f) = \infty \) or if \( \alpha_{\psi,s} = 0 \), so we may suppose that \( \rho_\psi(f) < \infty \) and \( \alpha_{\psi,s} > 0 \).

(a) We begin by proving for every \( k \in \mathbb{N} \) that

\[
\rho_\psi(D^k_q f) \leq \max \left\{ \rho_\psi(f), \max_{1 \leq l \leq k} \left\{ \frac{\rho_\psi(f)}{\alpha_{\psi,s}^l} - \gamma_{\psi,s} \sum_{j=0}^{l-1} \frac{1}{\alpha_{\psi,s}^j} \right\} \right\} =: \rho_\psi,k. \quad (5.1)
\]

The case \( k = 1 \) is obvious by Remark 2. We suppose that (5.1) holds for \( k \), and we aim to prove (5.1) for \( k + 1 \). Applying Remark 2 to the meromorphic function \( D^k_q f \) yields

\[
\rho_\psi(D^{k+1}_q f) = \rho_\psi(D_q(D^k_q f)) \leq \max \left\{ \rho_\psi(D^k_q f), \frac{\rho_\psi(D^k_q f)}{\alpha_{\psi,s}^k} - \gamma_{\psi,s} \right\}
\]

\[
\leq \max \left\{ \rho_\psi(f), \max_{1 \leq l \leq k+1} \left\{ \frac{\rho_\psi(f)}{\alpha_{\psi,s}^l} - \gamma_{\psi,s} \sum_{j=0}^{l-1} \frac{1}{\alpha_{\psi,s}^j} \right\} \right\} =: \rho_\psi,k+1.
\]

The assertion (5.1) is now proved. Moreover, it is easy to see that \( \rho_\psi(f) \leq \rho_\psi,k \leq \rho_\psi,k+1 \) for \( k \in \mathbb{N} \).

Suppose first that the coefficients \( a_0(x), \ldots, a_n(x) \) are entire. We divide (2.1) by \( f(x) \) and make use of (4.2), (5.1) and Lemma 3.1(a) to obtain

\[
m(r, a_0) \leq \max_{1 \leq j \leq n} \{ m(r, a_j) \} + \sum_{1 \leq j \leq n} m \left( r, \frac{D^j_q f}{f} \right)
\]

\[
\lesssim \max_{1 \leq j \leq n} \{ m(r, a_j) \} + \max_{1 \leq j \leq n} \left\{ m \left( r, \frac{D^j_q f}{D^j_q f} \right) \right\}
\]

\[
\lesssim \varphi(r)^{\rho_\psi(a_0)-\varepsilon} + \varphi(r)^{\frac{\rho_{\psi,n-1}}{\alpha_{\psi,s}} -\gamma_{\psi,s}+\varepsilon}, \quad r \geq R_0.
\]

Since there exists a sequence \( \{r_n\} \) of positive real numbers tending to infinity such that \( m(r_n, a_0) \geq \varphi(r_n)^{\rho_\psi(a_0)-\frac{\varepsilon}{2}} \), we have

\[
\rho_\psi(a_0) - \frac{\varepsilon}{2} \leq \frac{\rho_{\psi,n-1}}{\alpha_{\psi,s}} - \gamma_{\psi,s} + \varepsilon,
\]

where we may let \( \varepsilon \to 0^+ \). This gives us

\[
\rho_\psi(a_0) \leq \max \left\{ \frac{\rho_\psi(f)}{\alpha_{\psi,s}} - \gamma_{\psi,s}, \max_{1 \leq l \leq n} \left\{ \frac{\rho_\psi(f)}{\alpha_{\psi,s}^{l+1}} - \gamma_{\psi,s} \sum_{j=0}^{l-1} \frac{1}{\alpha_{\psi,s}^j} \right\} \right\}
\]

\[
= \max_{1 \leq l \leq n} \left\{ \frac{\rho_\psi(f)}{\alpha_{\psi,s}^l} - \gamma_{\psi,s} \sum_{j=0}^{l-1} \frac{1}{\alpha_{\psi,s}^j} \right\},
\]

and so

\[
\alpha_{\psi,s}^n \rho_\psi(a_0) \leq \max_{1 \leq l \leq n} \left\{ \alpha_{\psi,s}^{n-l} \rho_\psi(f) - \gamma_{\psi,s} \sum_{j=0}^{l-1} \alpha_{\psi,s}^{n-j} \right\} \leq \rho_\psi(f) - \alpha_{\psi,s}^n \gamma_{\psi,s}.
\]

Then the assertion (2.5) follows.
Suppose then that some of the coefficients \(a_0(x), \ldots, a_n(x)\) have poles. We divide (2.1) by \(f(x)\) and make use of (5.1) and Lemma 4.3(a) to obtain

\[
N(r, a_0) \lesssim \max_{1 \leq j \leq n} \{T(r, a_j)\} + \sum_{j=0}^{n} T(r, D_q^j f)
\]

and thus, similarly as above,

\[
\rho_{\varphi}(a_0) \leq \max \left\{ \rho_{\varphi}(f), \max_{1 \leq j \leq n} \left( \frac{\rho_{\varphi}(f)}{\alpha_{\varphi,s}^n} - \gamma_{\varphi,s} \sum_{j=0}^{l-1} \frac{1}{\alpha_{\varphi,s}^j} \right) \right\} = \rho_{\varphi,n}.
\]

Hence the assertion (2.4) follows.

(b) Similarly as in Case (a) above, we make use of (5.1) and Lemmas 3.1(b) and 4.3(b) to obtain

\[
T(r, a_0) = m(r, a_0) + N(r, a_0)
\]

\[
\lesssim \max_{1 \leq j \leq n} \{T(r, a_j)\} + \sum_{1 \leq j \leq n} m \left( r, \frac{D_q^j f}{f} \right) + \sum_{j=0}^{n} T(r, D_q^j f)
\]

\[
\lesssim \varphi(r)^{\rho_{\varphi}(a_0) - \varepsilon} + \varphi(r)^{\rho_{\varphi,n-1} + \varepsilon} + \log r, \quad r \geq R_0.
\]

This together with the fact that \(f\) is non-constant, we deduce \(\rho_{\varphi}(a_0) \leq \rho_{\varphi,n-1}\), and so \(\alpha_{\varphi,s}^n \rho_{\varphi}(a_0) \leq \rho_{\varphi}(f)\). This completes the proof. \(\square\)

**Proof of Theorem 2.2.** Choose \(s(r)\) satisfying the assumptions of Theorem 2.1(b). We divide (2.2) by \(f(x)\) and make use of (5.1) and Lemmas 3.1(b) and 4.3(b) to obtain

\[
T(r, a_0) \lesssim \max_{1 \leq j \leq n+1} \{T(r, a_j)\} + \sum_{1 \leq j \leq n} m \left( r, \frac{D_q^j f}{f} \right) + m \left( r, \frac{1}{f} \right) + \sum_{j=0}^{n} T(r, D_q^j f)
\]

\[
\lesssim \varphi(r)^{\rho_{\varphi}(a_0) - \varepsilon} + \varphi(r)^{\rho_{\varphi,n-1} + \varepsilon} + \log r, \quad r \geq R_0.
\]

Similarly as in the proof of Theorem 2.1(b), the assertion follows. \(\square\)

**Acknowledgements**

The first author would like to thank the support of the China Scholarship Council (No. 201806330120). The third author was supported by National Natural Science Foundation of China (No. 11771090). The fourth author was supported by the National Natural Science Foundation of China (No. 11971288 and No. 11771090) and Shantou University SRFT (NTF18029).
References

[1] Askey R. and J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. Mem. Amer. Math. Soc. 54 (1985), no. 319, iv+55 pp.

[2] Bergweiler W., K. Ishizaki and N. Yanagihara, Meromorphic solutions of some functional equations. Methods Appl. Anal. 5 (1998), no. 3, 248–258.

[3] Chiang Y. M. and S. J. Feng, Nevanlinna theory of the Askey-Wilson divided difference operator. Adv. Math. 329 (2018), 217–272.

[4] Chyzhykov I., J. Heittokangas and J. Rättyä, Finiteness of $\varphi$-order of solutions of linear differential equations in the unit disc. J. Anal. Math. 109 (2009), 163–198.

[5] Heittokangas J., J. Wang, Z. T. Wen and H. Yu, Meromorphic functions of finite $\varphi$-order and linear q-difference equations. https://arxiv.org/abs/2010.12356, 28 pp.

[6] Ismail M. E. H., Classical and Quantum Orthogonal Polynomials in One Variable. Encyclo. Math. Appl., vol. 98, Camb. Univ. Press, 2005.

[7] Rubel L. A., Entire and Meromorphic Functions. Springer-Verlag, New York, 1996.

[8] Szegő G., Orthogonal Polynomials. Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., xiii+432 pp, 1975.

(Yu, Heittokangas) Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, 80101 Joensuu, Finland
Email address: huiy@uef.fi, janne.heittokangas@uef.fi

(Wang) School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China
Email address: majwang@fudan.edu.cn

(Wen) Department of Mathematics, Shantou University, Shantou 515063, Guangdong, P.R. China
Email address: zhtwen@stu.edu.cn