MANY FORCING AXIOMS FOR ALL REGULAR UNCOUNTABLE CARDINALS

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Abstract. Our original aim was, in Abelian group theory to prove the consistency of: \( \lambda \) is strong limit singular and for some properties of abelian groups which are relatives of being free, the compactness in singular fails. In fact this should work for \( R \)-modules, etc. As in earlier cases part of the work is analyzing how to move between the set theory and the algebra.

Set theoretically we try to force a universe which satisfies G.C.H. and diamond holds for many stationary sets but, for every regular uncountable \( \lambda \), in some sense anything which “may” hold for some stationary set, does hold for some stationary set. More specifically we try to get a universe satisfying GCH such that e.g. for regular \( \kappa < \lambda \) there are pairs \( (S, B) \), \( S \subseteq S^\kappa_\lambda \) stationary, \( B \subseteq H(\lambda) \), which satisfies some pregiven forcing axiom related to \( (S, B) \), (so \( (\lambda \setminus S) \)-complete, i.e. “trivial outside \( S \)”) but no more, i.e. slightly stronger versions fail. So set theoretically we try to get a universe satisfying G.C.H. but still satisfies “many”, even for a maximal family in some sense, of forcing axioms of the form “for some stationary” while preserving GCH. As completion of the work lag for long, here we deal only with the set theory.
§0 Introduction, pg.3

§1 The iteration on all cardinals, pg.8

[We define when $t$ is a $\lambda$-task template, $(<\alpha)$-strategically $S$-complete, $\lambda$-tasks and cases of $\lambda$-task templates and when a $\lambda$-task is satisfied, i.e. each instance is (see 1.1, 1.4). We prove that if $\lambda = \lambda^{<\lambda} > \aleph_0$ and $2^\lambda = \lambda^+$ then there is a suitable forcing $P$ taking care of all $\lambda$-tasks (1.10) and that we can do it for all cardinals, (1.12).]

§2 An example: Relatives of diamonds, pg.13

§3 Parameters for completeness of forcing, pg.17

[Also this section is set theoretic but fine tuned to uniformization and hence Abelian groups (or modules) problems. We define when $(D, T)$ is a $((\mu, \kappa, \theta)$-special pair, (3.2) and when $p = (D, T, \lambda, S, \overline{\eta})$ is a $(\mu, \kappa, \theta)$-parameter (3.4(1)) and prove iteration claims for $p$ (in 3.6). We then look for suitable filters (see Definition 3.7), and prove existence (3.9, 3.10). We point out the case of stationary $S \subseteq \delta \geq (\alpha)^{<\mu}$, $\delta$-fat $S \subseteq \lambda$ (i.e., $S = \delta \geq S$) and $S$ complete, though actually our main forcing are strategically complete with the $p$ being forced. With those notions we may define uniformization properties and $p$-completeness of forcing notions (3.5(2)). This will allow us to iterate 3.6(1). We note that there is $(\mu, \mu, \aleph_0)$-special pairs if $\diamondsuit_{\mu}$ (3.9), and that we can combine $(\mu_\delta, \kappa, \theta)$-special pairs for enough $\delta$’s to a $(\mu, \kappa, \theta)$-special pair (see 3.11).]
§ 0. Introduction

The analysis of those questions lead to having many uniformation properties to many regular cardinals smaller then the singulars.

§ 0(A). An Abelian group theory motivation.

Compactness in singular (provable in ZFC) play a crucial role in the solution of Whitehead problem, see [Sh:52], [EM02].

There are some “compactness in singular cardinals” theorems for Abelian groups when we assume \( V = L \) (or something in this direction), of Eklof and more recently of Struengman. A natural question is whether the \( V = L \) is needed. In fact the proofs of each of those theorems can be decomposed to two parts: one part which is compactness in singular for another condition (say being a free Abelian group); another part is the equivalence of the two conditions using \( V = L \). Anyhow this stands behind the following questions:

**Question 0.1.** Compactness in singulars for \( \{ G : \text{Ext}(G, \mathbb{Z}) = 0 \} \), which arise from an old work of Eklof, [Ekl80, Theorem 8.5], see more in [EFSh:352].

**Question 0.2.** Does compactness in singulars for \( \{ G : \text{Ext}(G, T) = 0 \} \) holds (for \( T \) any torsion group)?

The question above, 0.2, was asked by Struengmann following the paper [Str02, Proposition 2.6] and was the immediate reason for this work on 0.1, ?? and more. Later Eklof has asked me on 0.2. We shall show the consistency of the negation in both cases.

**Notation 0.3.** For \( C \) a set of ordinals,

(a) let \( (\forall^{\ast} \alpha \in C) \varphi \) means \( \{ \alpha \in C : \varphi(\alpha) \} \) contains a co-bounded subset of \( C \)

(b) let \( (\forall^{D} \alpha \in C) \varphi(\alpha) \) means \( \{ \alpha \in C : \varphi(\alpha) \} \in D \), for \( D \) a filter on \( C \)

(c) using \( \forall^{I} \), \( I \) an ideal on \( C \) means \( \forall^{D} \), \( D \) the dual filter.

§ 0(B). The set theoretic view.

As said above our original motivation concerns Abelian groups and consistency of incompactness in singulars for algebraic problem we explained above. Explain the set theoretic side, starting from uniformization (see [Sh:587] and history there). Consider \( \lambda \) regular uncountable, stationary \( S \subseteq \lambda \) consisting of limit ordinals; ladder system \( \bar{C} = \langle C_{\delta} : \delta \in S \rangle \), i.e., \( C_{\delta} \subseteq \delta = \sup(C_{\delta}) \) (with \( C_{\delta} \) not necessarily a closed subset of \( \delta \)) and \( h : \lambda \to \lambda \), (if \( h \) is constantly 2 we may omit it; we do not consider here the case \( f_{\delta} \in \langle C_{\delta} \rangle \delta \)). Here and later we may replace \( (\forall \delta \in S) \) by \( (\forall^{D} \lambda \delta \in S) \) where \( D_{\lambda} \) is e.g. the club filter on \( S \), see 0.3 presently this does not cause a great difference, but the property is weaker.

\( \circ \) we say that \( \bar{C} \) has \( h \)-uniformization when for every sequence \( \langle f_{\delta} : \delta \in S \rangle \) satisfying \( f_{\delta} \in \prod_{\alpha \in C_{\delta}} h(\delta) \) for \( \delta \in S \) there is \( f \in \lambda^{\lambda} \) such that \( (\forall \delta \in S)(\forall^{\ast} \alpha \in C_{\delta})(f(\alpha)) = f_{\delta}(\alpha) \).

We may add \( \bar{D} = \langle D_{\delta} : \delta \in S \rangle \), \( D_{\delta} \) a filter on \( C_{\delta} \) and consider
we say that $\mathcal{C}$ has $(h, \bar{D})$-uniformization if for every $\bar{f} = \langle f_\delta : \delta \in S \rangle$, $f_\delta = \prod_{\alpha \in C_\delta} h(\alpha)$ there is $f \in \prod_{\alpha < \lambda} h(\alpha)$ such that $(\forall \delta \in S)(\forall D \alpha \in C_\delta) f(\alpha) = f_\delta(\alpha)$, i.e. $\{ \alpha \in C_\delta : f(\alpha) = f_\delta(\alpha) \} \in D_\delta$ for $\delta \in S$.

We use $S(\mathcal{C}) = S$.

We may consider questions close to Abelian group theory.

§ 3 (group-uniformity) for a sequence $\mathcal{K} = \{ K_\alpha : \alpha < \lambda \}$ of groups and $\mathcal{C}$ as above let $K_\delta = \prod_{\alpha \in C_\delta} K_\alpha$ and let ps-ext($\mathcal{K}, \mathcal{C}, \bar{D}$) be $K^*/\text{unf}(\mathcal{K}, \mathcal{C}, \bar{D})$ where $K_* = \prod_{\delta \in S} K_\delta$ and unf($\mathcal{K}, \mathcal{C}, \bar{D}$) = $\{ f \in S(\mathcal{C}) :$ there is $h \in \prod_{\alpha \in \lambda} K_\alpha$ such that $(\forall \delta \in S)(\forall \alpha \in C_\delta)[f_\delta(\alpha) = f(\alpha)]$, i.e. $\{ \alpha \in C_\delta : f(\alpha) = f_\delta(\alpha) \} \in D_\delta$ for $\delta \in S$.

We may vary more: for some function $F$ replace $f(\alpha) = f_\delta(\alpha)$ by $f_\delta(\alpha) = F(f \upharpoonright C_\delta \cap (\alpha + 1))$, this is closer to Ext.

We concentrate on the case that G.C.H. holds and we have unification for enough stationary sets $S$ for some appropriate $\mathcal{C}$. But here we try to do it for every regular uncountable $\lambda$ and for all “tasks” of such forms. On earlier works forcing uniformization see Eklof-Mekler [EM02].

So usually we have to assume:

§ 4 $\lambda = \mu^+$ and $\delta \in S \Rightarrow \text{cf}(\delta) = \text{cf}(\mu)$.

We can force (using relatives of pseudo-completeness, see § 2)

§ 5 (a) if a stationary $S \subseteq \lambda$ satisfies § 4 then for some stationary $S' \subseteq S$ some ladder system $\langle C_\delta : \delta \in S' \rangle$ has uniformization

(b) even if $\mathcal{C} = \langle C_\delta : \delta \in S \rangle$ is a ladder system then for some stationary $S' \subseteq S, \mathcal{C} \upharpoonright S$ has uniformization.

Here we shall be interested in getting distinction between quite close relatives of this, so we have to force “less” and the problem is to phrase exactly what we like to have but not to get more uniformization than we intend. In [Sh:587] we consider, for other reasons the case $\lambda = \mu^+, \mu = \text{cf}(\mu), D_\delta$ a somewhat regular filter on $C_\delta$ (so $|C_\delta| = \mu$). The aim there was to get some uniformization on $S^\lambda_\mu$; this could have been done also in [Sh:677] which concentrates on successor of singulars but there was no point gain by it. The iteration in § 1 preserves G.C.H. and for every successor $\lambda = \text{cf}(\lambda) > \aleph_0$ and stationary $S \subseteq \lambda$ (mainly satisfying § 4), add a stationary subset $S' \subseteq S$ and on it for something which is similar enough to a case of uniformization; so called $\lambda$-task, which may be relevant for Abelian groups. This is arranged such that there is little interaction between forcing for the different tasks. For each regular uncountable $\lambda$ there is an iterated forcing which adds enough “tasks” and mainly add solutions for each case of each such task. There is no problem in the forcing because if the task is “too hard” (e.g. gives a provably impossible situation), the forcing will, e.g. make $S'$ not be stationary (for abelian group - the group we add becomes free or examples intended to show Ext($G, Z) \neq 0$ are no longer so). So as in many cases “we solve our problems by putting them on someone else’s shoulders”.

The central case here is with $D_\delta$ a regular filter; then some relevant finitary linear combinations become critical. A major part in the proof is trying to prove that
the iterated forcing gives “good” limit. As we arrange it, not collapsing cardinals holds trivially (as we just ask “for some stationary \( S' \subseteq S \)) . So the difficulty is preserving stationarity of relevant sets, and/or showing that undesirable objects are not added by building inverse system of trees of forcing conditions, (a central method in such consistency proofs). In the main case, inside the forcing proof we have to consider only finitely many coordinates, using the regularity of the relevant filter. So trying to imitate the argument in [Sh:125], we can do it in higher cardinals, if we carry with us strong enough induction hypothesis. Should we in the forcing in §1 add non-reflecting stationary sets? We may use such \( \lambda \)-tasks, but we may like to preserve supercompactness, so we do not like to, but then we have to use reflection.

As finishing the full work large for too long, we delay the results on Ent. This work and the application to Ent were presented at the meeting in honor of Eklof in Summer 2008.

§ 0(C). Preliminaries.

**Definition 0.4.** 1) For \( \lambda \) regular uncountable and \( S \subseteq \mathcal{P}(\lambda) \) let nor \(-\text{id}(S) \) be the minimal normal ideal which includes \( S \).

**Definition 0.5.** Let \( \mathcal{D} \) be a filter on \( \lambda^+ \mathcal{P}(\lambda) \).
1) We call \( \mathcal{D} \) normal when for every \( \chi > \lambda \) and \( x \in H(\chi) \) there is \( \mathcal{Y} \in \mathcal{D} \) such that:
\[ \langle \mathcal{N}_i \mathcal{\cap} \lambda : i \leq \delta \rangle \in \mathcal{Y} \] where \( \bar{\mathcal{N}} = \langle \mathcal{N}_i : i \leq \delta \rangle \) obeys \( (\mathcal{D}, \chi, \lambda, x) \) which means (omitting \( \mathcal{Y} \) means for some \( \mathcal{Y} \))

\( a \) \( \mathcal{N}_i \prec (H(\chi), \in) \)
\( b \) \( \lambda, \mathcal{D} \in \mathcal{N}_i \)
\( c \) \( x \in \mathcal{N}_i \)
\( d \) \( \mathcal{N}_i \) is increasing continuous
\( e \) \( \bar{\mathcal{N}} \mathcal{\upharpoonright} (i + 1) \in \mathcal{N}_{i+1} \).

2) We say \( \mathcal{D} \) is a fat normal (filter) when for every \( \alpha < \lambda \) we have \( \mathcal{Y} \in \mathcal{D} \land \alpha < \lambda \Rightarrow \mathcal{Y}_{\geq \alpha} := \{ \bar{u} \in \mathcal{Y} : \ell g(\bar{u}) > \alpha \} \neq 0 \) mod \( \mathcal{D} \).

3) We say a forcing notion \( \mathbb{P} \) is \( \mathcal{D} \)-complete when:

\( a \) forcing by \( \mathbb{P} \) adds no new sequence of ordinals of length \( < \lambda \)
\( b \) for every \( \chi \) for some \( \mathcal{Y} \in \mathcal{D} \) and \( x \in H(\chi) \), for every \( \tilde{\mathcal{N}} \) obeying \( (\mathcal{D}, \chi, \lambda, x) \) we have:

- if \( j \leq \ell g(\tilde{\mathcal{N}}) \) is a limit ordinals, \( \tilde{p} = \langle p_i : i < j \rangle \in \prod_{i<j} (\mathcal{N}_i \cap \mathbb{P}) \) is increasing \( i < j \Rightarrow p((i + 1) \in \mathcal{N}_i \) and weakly generic for \( \tilde{\mathcal{N}} \) (if \( i < j \) then for some \( n \), for every \( \mathbb{P} \)-name of an ordinal \( \tau \in \mathcal{N}_i, p_{i+n} \models " \tau \in \mathbb{N}_{i+n} \cap \text{Ord}" \), as in [Sh:587] then \( \tilde{p} \) has a \( \leq_p \)-upper bound (hence preserve \( \lambda^+ \mathcal{V} \)).

4) For stationary \( \mathcal{w} \subseteq \lambda = \text{cf}(\lambda) \) let \( \mathbb{D}_{\lambda, \mathcal{w}} \) be the minimal normal filter on \( \lambda^+ \mathcal{P}(\lambda) \) to which the following set belongs \( \{ \bar{\alpha} : \bar{\alpha} \) is an increasing continuous sequence of ordinals from \( \mathcal{w} \} \).
5) If $\mathbb{D}$ is normal, let $\text{prog}_{\mathbb{D}}(\chi) = \{\mathcal{Y} : \mathcal{Y} \subseteq \{N : N \prec \mathcal{H}(\chi), \varepsilon\} \text{ and } \|N\| < \lambda\}$ and for some $x \in \mathcal{H}(\chi)$: the set $\mathcal{Y}$ includes $\{N_i : i \leq \delta\}$ which obeys $(\mathbb{D}, \chi, \lambda, x)\}$. 

**Definition 0.6.** Let $\text{Fun}(G, H)$ be the set of functions from $G$ to $H$, not necessarily homomorphisms.

* * *

Concerning strategic completeness

**Definition 0.7.** 1) For a regular uncountable $\lambda$ and ordinal $\xi \leq \lambda$ we say a forcing notion $\mathbb{Q}$ is $(\xi, \mathcal{S}_1, \mathcal{S}_2)$ - stg-complete when:

(a) $\mathcal{S}_2$ is a family of subsets of $\lambda$
(b) $\mathcal{S}_2$ is a $\mathbb{Q}$-name of a family of subsets of $\lambda$
(c) in the game $\mathcal{G} = \mathcal{G}_{\alpha}(\mathcal{S}_1, \mathcal{S}_2, \mathbb{Q})$ the completeness player has a winning strategy

(\alpha) a play last $\alpha_*$ moves
(\beta) in the $\alpha$-th move a quadruple $(p_\alpha, S_{1,\alpha}, S_{2,\alpha}, \varepsilon_\alpha)$ is chosen such that

- $p_\alpha \in \mathbb{Q}$ such that $\beta < \alpha \Rightarrow p_\beta \leq \mathbb{Q} p_\alpha$
- $\varepsilon_\alpha < \lambda$ such that $\langle \varepsilon_\beta : \beta \leq \alpha\rangle$ is increasing continuous
- $S_{1,\alpha} \in \text{nor} - \text{id}_\lambda(\mathcal{S}_1)$ such that $\langle S_{1,\beta} : \beta \leq \alpha\rangle$ is \(\subseteq\)-increasing continuous
- $S_{2,\alpha} \in \text{nor} - \text{id}_\lambda(\mathcal{S}_2)$ such that $\langle S_{2,\beta} : \beta \leq \alpha\rangle$ is $\subseteq$-increasing continuous
- $p_\alpha \forces \langle \varepsilon_\beta \notin S_{1,\alpha} \cup S_{2,\alpha}\rangle$

(\gamma) if $\alpha = 2n$ or $\alpha = \omega(1 + \beta) + 2n + 1$ then the incompleteness player chooses the quadruple, otherwise the completeness player chooses

(\delta) the completeness player wins a play if he has a legal move for every limit $\alpha < \alpha_*$ (the only problematic point is choosing).

We have natural iteration claims.

**Claim 0.8.** If (A) then (B) where:

(A) $q$ is a $(\lambda, \xi)$ - stg-iteration which means

(a) $\lambda = \lambda^\mathbb{P} > \aleph_0$ and $\xi \leq \lambda$
(b) $q = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta, \mathcal{S}_\beta : \alpha \leq \alpha_*, \beta < \alpha_*\rangle$
(c) $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha_*, \beta < \alpha_*\rangle$ is $(\mathbb{L})$-support iteration with limit $\mathbb{P}_\alpha$.
(d) $\mathcal{S}_\alpha$ is a $\mathbb{P}_\alpha$-name of a $\subseteq$-increasing family of subsets of $\lambda$
(e) $\forces_{\mathbb{P}_\alpha} \langle \lambda \notin \text{nor} - \text{id}_\lambda(\mathcal{S}_\alpha)\rangle$
(f) $\forces_{\mathbb{P}_\alpha} \langle \mathcal{Q}_\alpha \text{ is } (\xi, \mathcal{S}_\alpha, \mathcal{S}_{\alpha+1}) \text{ - stg-complete}\rangle$

(B) (a) $\mathbb{P}_{\alpha_0}$ is $\langle \xi, \mathcal{S}_0, \mathcal{S}_{\alpha_0}\rangle$ - stg-complete
(b) even $\mathbb{P}_\beta / \mathbb{P}_\alpha$ is when $\alpha < \beta \leq \alpha_*$.

Proof. Straight.

**Claim 0.9.** We have $q$ is a $(\lambda, \xi)$ - stg-iteration when:
\( (A)' \) clauses \((a),(c),(d)\) of \([U_\subseteq A]\)

\((\alpha)\) \(\lambda \notin \text{nor} - \text{id}_\lambda(S_0)\)

\((\beta)\) if \(\alpha < \alpha_{q_\alpha}\) and \(\Vdash_{P_\alpha}\) “if \(\lambda \notin \text{nor} - \text{id}_\lambda(S_\alpha)\) then the forcing notion \(Q_\alpha = (\xi, S_\alpha, S_{\alpha+1})\) is \(-\text{stg-complete}\)

\((\gamma)\) if \(\alpha\) is a limit ordinal \(\leq \alpha_q\) then \(S_\alpha = \cup\{S_\beta : \beta < \alpha\}\) or \(\lambda \notin \text{nor} - \text{id}_\lambda(S_\beta)\).

Remark 0.10. We may weaken \((\gamma)\) in \((A)'\) of \([U_\subseteq A]\), but no use for now.
§ 1. The Iteration on All Cardinals

This section is purely set theoretical, this is continued in § 2 but there we deal with iteration, whereas there it deals with “one forcing”.

Our program is as follows, we start with $V$ satisfying G.C.H. and preserve it. By the character of our aims we have to have relevant existence in all cardinals. The freedom we have is to try for each regular uncountable $\mu$ to add a stationary subset $S$ of $\mu$ for each task, and try to fulfill the task. Now if the task is “too hard”, say it contradicted by ZFC + GCH, the iteration may make $S$ not stationary (or “harm” it in other ways). So in our frame this causes no problem here, it just passes the burden to proofs about the specific tasks.

We may in the forcing axioms (related to specific stationary $S \subseteq M$) replace “of cardinality $\mu$” by “of cardinality $\mu^+$” satisfying a strong form of $\mu^+$-c.c. but this is not central here. The forcing for Abelian groups in $\lambda = \mu^+$ require we have such abelian groups in $\mu^+$ and only stationary $S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \text{cf}(\mu) \}$ are used, there are differences according to $\lambda$ being successor of regular, successor of singular or inaccessible.

Now § 1.1 defines a $\lambda$-task template. We intend for each $\lambda$-task to force an example, then later in the iteration we force a solution to each relevant case; we hope the rest of the iteration do preserve some desired properties; this means that for any $B \subseteq \mathcal{H}(\lambda)$ we have to deal with each specific $\lambda$-tasks derived from it (see Definition § 1.4).

**Definition 1.1.** For $\lambda$ regular uncountable we say $t$ is a $\lambda$-task template iff $t$ consists of:

(a) $S^1$, a subset of $\lambda$, normally stationary
(b) $\mathbb{Q}_t = \mathbb{Q}_1 = \mathbb{Q}[t]$, a forcing notion of cardinality $\lambda$ which is $(< \lambda)$-strategically complete, see § 1.3, [not a serious difference if we ask $\lambda$-strategically complete but see (c)(β)]
(c) a function $p \mapsto S_p = S^1_p$ from $\mathbb{Q}_t$ to $\{ S : S$ a bounded subset of $S^1$ presented as a characteristic function from some ordinal $< \lambda$ to $\{ 0, 1 \} \}$ and let $\delta_1(p) := \text{Dom}(S^1_p)$ so an ordinal $< \lambda$, such that:

(a) $\mathbb{Q}_t \models p \equiv q \Rightarrow S_p = S_q \cap \delta_1(p)$ so $p \leq \mathbb{Q}_t q \Rightarrow \delta_1(p) \leq \delta_1(q)$ and

(b) if $\ell \in \{ 0, 1 \}$ we have $s = \langle p_i : i < \alpha \rangle$ is increasing by $\leq \mathbb{Q}_t$ and $\langle \text{Dom}(S_p) : i < \alpha \rangle$ is increasing too, $\alpha < \lambda$ a limit ordinal and $s = \cup \{ \text{Dom}(S_{p_i}) : i < \alpha \}$

(c) for some $\delta \in S^1$ and $S^1_p(\delta) = \ell$; moreover for $\ell = 0$ there is a $\leq \mathbb{Q}_t$ minimal such $p$ (a canonical one suffices) with $\delta(p) = \delta + 1$

(d) if $p \in \mathbb{Q}_t$ then for some $q \in \mathbb{Q}_t$ we have $p \leq \mathbb{Q}_t q$ and $\delta(p) < \delta(q)$ and $S^1_q(S_1(p)) = 0$

(δ) so $S_1 = \cup \{ S^1_p : p$ $\in G \}$ is a $\mathbb{Q}_1$-name of a function from $\lambda$ to $\{ 0, 1 \}$

(d) first order formulas $\psi_1(X,Y,B), \varphi_1(x,y,Y,B)$ with $B$ a $\mathbb{Q}_1$-name of a subset of $\mathcal{H}(\lambda), x, y$ individual variables and $X, Y$ monadic variable, the formulas are in the vocabulary of $(\mathcal{H}(\lambda), \in, <, \lambda)$
(e) in any universe $V' \supseteq V$ but with $(^{\lambda>\text{Ord}})V' = (^{\lambda>\text{Ord}})V$ (suffice if for some fat stationary $S \subseteq \lambda, V'$ is gotten from $V$ by forcing by some strategically $(\lambda, S)$-complete forcing), if $G \in V'$ and $\mathcal{G} \subseteq \mathcal{Q}_1$ is $(< \lambda)$-directed and $\mathcal{S}_1(G) := \cup\{S_\alpha^p : p \in \mathcal{G}\}$ is a function with domain $\lambda$, and $B = B_t[\mathcal{G}]$ and $A \subseteq \mathcal{H}(\lambda)$ from $V'$ and lastly the formula $(\forall \alpha \subseteq \mathcal{H}(\lambda)) \psi_t(X, A, B)$ is satisfied in $V'$ then:

\begin{itemize}
  \item the formula $\varphi_1 = \varphi_1(x, y, A, B)$ defines in the structure $(\mathcal{H}(\lambda), \in, ^*\alpha, \mathcal{G})$ a forcing notion $\mathcal{Q}_{t, A}$ such that
    \begin{itemize}
      \item $\delta = \delta_{t, A}$ is a function from $\mathcal{Q}_{t, A}$ to $\lambda$ such that if $\bar{p} = (p_\varepsilon : \varepsilon < \zeta)$ is an increasing sequence and $\cup\{\delta(p_\varepsilon) : \varepsilon < \zeta\} \in \lambda \setminus \mathcal{S}_1(G)$ then $\bar{p}$ has a lub.
    \end{itemize}
\end{itemize}

**Definition 1.3.** 1) For regular uncountable $\lambda$, stationary $S \subseteq \lambda$ and ordinal $\alpha \leq \lambda$ we say that a forcing notion $\mathcal{Q}$ is $\alpha$-strategically $S$-complete when for every $p^* \in \mathcal{Q}$ in the following game the completeness player has a winning strategy. A play lasts up to $\alpha$ moves, in the $\beta$-th move the completeness player chooses a condition $p_\beta$ and ordinal $\varepsilon_\beta$ such that $p_\beta$ is an upper bound of $\{p^*\} \cup \{q_\gamma : \gamma < \beta\}$ and $\varepsilon_\beta$ is an ordinal in $\bigcup_{\gamma < \beta} \zeta_\gamma \cup \alpha, \lambda$ satisfying $\beta \lim \varepsilon_\beta = \bigcup_{\gamma < \beta} \zeta_\gamma$ and the incompleteness player choose $q_\beta$ such that $p_\beta \leq q_\beta \in \mathcal{Q}$ and $\zeta_\beta \in (\varepsilon_\beta, \lambda)$. The completeness player wins if he always has a legal move or he does not for $\beta$, so $\beta$ is necessarily a limit ordinal but $\varepsilon_\beta \notin S$.

1A) In part (1) and (2), if $S = \lambda$ and understood from the context we may omit $S$. 2) Let “$(< \alpha)$-strategically $S$-complete” mean $\beta$-strategically $S$-complete for every $\beta < \alpha$. If $S = \lambda$ and $\lambda$ clear from the content we may omit it. If we omit “$(< \alpha)$” we mean $(< \lambda)$.

3) If $S \subseteq \lambda$ is fat, $\mathcal{Q}$ is $(< \lambda)$-strategically $S$-complete then forcing with $\mathcal{Q}$ adds no new sequence from $\omega^\mathcal{V}$.

**Definition 1.4.** 1) We call $s$ a (specific) $\lambda$-task when a triple of the form $(S_s, \psi_s(X, Y, B_s), \varphi_s(x, y, Y, B_s))$ such that:

\begin{itemize}
  \item $S_s \subseteq \lambda$
  \item $\psi_s, \varphi_s$ are first order formulas as in clause (d) of 1.3
  \item like clause (e) of Definition 1.1 except that we omit $\mathcal{G}$ and use $S$ instead of $\mathcal{S}_1(G), B_s$ instead of $B_t[\mathcal{G}]$ and replace $A \mapsto \mathcal{Q}_{s, A}$ defined by $\varphi_1$ by the function $A \mapsto \mathcal{Q}_{s, A}$ or $\mathcal{Q}_{s, A}$ defined by $\varphi_{gs}$.
2) We say the $\lambda$-task $s$ is a case of the $\lambda$-task template $t$, in fact is $s[t,G]$ for some $(< \lambda)$-directed $G \subseteq Q_1^t$ when we have: $s = t < G >$ which means:

(a) $S_s = \{S_p^t : p \in G\}$ is the characteristic function of $S$ (as a subset of $\lambda$)

(b) $G_s = \varphi_t, \psi_t = \psi_t, B(s) = B_s = B_t[G]$. 

3) We say that the $\lambda$-task $s$ is satisfied (in a universe $V$) when: for every $A \subseteq \mathcal{H}(\lambda)$, the $A$-instance of $s$ is satisfied, or $s$ is satisfied for $A$ which means that:

if $(\mathcal{H}(\lambda), \in, < \cdot \cdot \cdot)$ $\vdash (\forall X \subseteq \mathcal{H}(\lambda)) \varphi_t(X, A, B_s)$ and $\psi_t(x, y, A, B_s)$ define in $(\mathcal{H}(\lambda), \in, < \cdot \cdot \cdot)$ a $( < \lambda )$-strategically $\varphi_t(X, A, B_s)$-complete (see Definition 1.4) forcing notion which we call $\mathbb{Q}^s_A$ and $I_\alpha \subseteq \mathbb{Q}^s_A$ is a predense subset of $\mathbb{Q}^s_A$ for every $\alpha < \lambda$ then $\mathbb{Q}^s_A$ has a directed subset not disjoint to $I_\alpha$ for every $\alpha < \lambda$ (see more in $(d)^{-1}(\delta)$).

**Remark 1.5.** 1) Concerning Definition 1.4 we really are interested in finer versions of being satisfied. To begin with, $S_s$ being stationary, and then preserving further non-existence. Maybe add to $\lambda$-task templates (and their cases) a first order formula $\varphi(X, Y, B)$ and define strong satisfaction (in part (3)) to mean that $(\mathcal{H}(\lambda), \in) \models (\forall X \subseteq H(\lambda))[\varphi(X, G, B)]$. The idea is that in the forcing below, for any case of $s$ of $\lambda$-task templates, we do not force more than all the $\mathbb{Q}^s_A$'s.

2) So we actually will investigate universes which we get by forcing as in Definition 1.6(1)(2) below where we iterate the forcing from 1.1 on all regular cardinals.

* * *

**Definition 1.6.** Assume $\lambda = \lambda^{< \lambda} > \aleph_0$ and $2^\lambda = \lambda^+$.

Let $\mathbb{R}_1^\lambda$ be the class of $p$ which consists of the following (so $\mathbb{P}_p = \mathbb{P}_\lambda^+, \mathbb{P}_p, A = \mathbb{P}_\alpha, \mathbb{Q}_p, A = \mathbb{Q}_\alpha$, etc. [similarly $\mathbb{R}_1^\lambda, \xi$ for $\xi \leq \lambda^+$ when we replace $\lambda^+$ by $\xi$ and omit clause $\mathbb{C}(\xi)$ and also $\mathbb{R}_1^\lambda, <, \lambda^+$])

(A) (a) $(\mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \lambda^+)$ an iterated forcing with $(< \lambda^+)$ support (i.e. full)

(b) $\mathbb{P} = \mathbb{P}_\lambda^+ = \cup\{\mathbb{P}_\alpha : \alpha < \lambda^+\}$

(c) $\mathbb{P}_\alpha$ is $(< \lambda)$-strategically complete for $\alpha \leq \lambda^+$

(B) (a) $W_\alpha = \{W_\alpha : \alpha < \lambda^+\}$

(b) $W_\alpha \subseteq \lambda$ is increasing modulo $D_\lambda$

(c) for limit $\delta$, $\forall\delta \mathcal{V}_\delta / D_\lambda : \alpha < \delta$

(d) $f : \lambda^+ \rightarrow \lambda^+$ satisfies $f(\alpha) \leq \alpha$ and $f(f(\alpha)) = f(\alpha)$ for every $\alpha < \lambda^+$

(e) $\|\text{P}_p \text{ "Q}_\alpha \text{ is } ( < \lambda ) \text{-strategically } (\lambda|S_\alpha)\text{-complete}"$, see below

(C) (a) if $f(\alpha) = \alpha$ then $\mathbb{Q}_\alpha = \mathbb{Q}[t_\alpha]$ so in $V^{-\alpha}$ where $t_\alpha$ is a $\mathbb{P}_\alpha$-name of a $\lambda$-task template and $\mathcal{S}_\alpha$ is $\mathcal{S}_{t_\alpha}$, so a $\mathbb{P}_{\alpha+1}$-name of a subset of $\lambda$ disjoint to $W_\alpha$ and let $\mathcal{G}_\alpha = \mathcal{G}[t_\alpha, G_{\mathbb{Q}_\alpha}]$, is a case of $t_\alpha$ and $W_{\alpha+1} = W_\alpha \cup S_{\mathbb{G}_\alpha}$

(b) if $f(\alpha) < \alpha$ then $\mathbb{Q}_\alpha = \mathbb{Q}_{\lambda|S_\alpha}, A_\alpha$, a $\mathbb{P}_\alpha$-name of a subset of $\mathcal{H}(\lambda) [G_{\mathbb{P}_p}]$, so a $\mathbb{P}_{\alpha}$-instance of $\mathcal{G}_\alpha$ which is a case of $\mathcal{G}_{f(\alpha)}$

(c) if $f(\alpha) = \alpha \leq \beta < \lambda$ and $A$ is a $\mathbb{P}_\beta$-name of a subset of $H(\lambda)$ then for unboundedly many $\gamma \in (\beta, \lambda^+)$ if possible, $\mathcal{Q}_\gamma = \mathcal{Q}_{\lambda, A}$. 

* * *
Remark 1.7. 1) Why $\mathcal{W}_\alpha$? Because for $\alpha < \beta \leq \lambda^+$, $\mathbb{P}_\beta/\mathbb{P}_\alpha$ is not necessarily $(< \lambda)$-strategically complete but it is $(< \lambda)$-strategically $(\lambda\mathcal{W}_\alpha)$-complete and $\lambda \setminus \mathcal{W}_\alpha$ is fat.

2) We may consider task $t$ being dealt with in $\beta = f(\beta)$, but $S^t$ being a subset of $S_{\alpha,\alpha}$ for some $\alpha = f(\alpha)$. This is quite reasonable but:

(a) to see if $t$ is satisfied non-trivially say $\Vdash_{\mathbb{P}} S_{p,\beta}$ is stationary” depends also on $t_\alpha$

(b) as for the satisfaction of $t_\alpha$, it may be subtly changed as on $S_{p,\beta}$ we are doing something more.

3) We may even iterate (2), having a tree structure on $\{\alpha < \lambda^+ : \alpha = f(\alpha)\}$, see \[\text{0.7, 0.8, but we leave it for the time being.}\]

Definition 1.8. For $p \in R^1_\lambda$, i.e. as in \[\text{1.6}\]

1) We say $p$ is full when: if $t$ is a $\mathbb{P}_\lambda$-name of a $\lambda$-task template with $S_1 \setminus \bigwedge_{\varepsilon < \lambda} S_{\alpha(\varepsilon)}$ stationary for every sequence $\langle \alpha(\varepsilon) : \varepsilon < \lambda \rangle$ of members of $\text{Rang}(f)$, then for unboundedly many $\alpha < \lambda^+$, $t_\alpha = \check{t}(\lambda, \mathcal{W}_\alpha)$.

2) We say $p$ is $\lambda$-strategically complete when in clauses (A)(c),(B)(e) we replace $(< \lambda)$-strategically by $\lambda$-strategically.

3) Assume $\mathcal{T}$ is a definition of a set of $\lambda$-task templates. We say $p$ is $\mathcal{T}$-full when:

- $p \in R^1_\lambda$
- if $f_p(\alpha) = \alpha$ then $\Vdash_{\mathbb{P}} \langle t_\alpha[p] \in \mathcal{T} \rangle$
- if $t$ is a $\mathbb{P}_\lambda$-name and $\alpha < \lambda^+$ then for unboundedly many $\beta \in (\alpha, \lambda^+]$ we have $\Vdash_{\mathbb{P}_\beta} \langle \check{t}[\beta][p] \in \mathcal{T} \rangle$

Definition 1.9. 1) We can above use $\mathcal{T}$, a set of $2^\lambda$ functions, $f$ with domain $\subseteq R^1_{\lambda, < \lambda^+}$, such that for $p \in R^1_{\lambda, < \lambda^+}$, $f(p)$ is a $\mathbb{P}_p$-name of a $\lambda$-task template if defined.

2) So $p$ is $\mathcal{T}$-full when:

- $p \in R^1_\lambda$
- if $f_p(\alpha) = \alpha$ then $t_\alpha[p] = f(\alpha, \mathbb{P}, \bar{q}[\alpha])$
- if $t \in \mathcal{T}$ and $\alpha < \lambda^+$ then either unboundedly many $\beta \in (\alpha, \lambda^+]$, $t_\beta[p] = f(\beta, [\beta])$ so the latter is well defined on $p[\beta] \notin \text{Dom}(f)$ for every $\beta < \lambda^+$ large enough.

Claim 1.10. 1) If $\lambda = \lambda^{< \lambda} > \aleph_0$ and $2^\lambda = \lambda^+$ then there is a full $p \in R^1_\lambda$.

2) Assume in addition that $\mathcal{T}$ is as in \[\text{1.7, 2}\] or as in \[\text{1.8}\] then there is a $\mathcal{T}$-full $p \in R^1_\lambda$.

3) If $p \in R^1_{\lambda, \xi}$ then the subset $\mathbb{P}' p$ is a dense subset of $\mathbb{P}_p : p \in \mathbb{P}_p$ if, for some limit $\delta = \delta_p(p)$ we have:

(a) $p \in \mathbb{P}_p$
(b) if $\alpha \in \text{Dom}(p)$ then $f_p(\alpha) \in \text{Dom}(p)$
(c) if $\alpha = f_p(\alpha) \in \text{Dom}(p)$ then $p[\alpha]$ forces $(\Vdash_{\mathbb{P}_p, \alpha})$ that $\text{Dom}(p(\alpha)) = \delta \notin S_{t_\alpha}$ or $\text{Dom}(p(\alpha)) = \delta + 1 \wedge p(\alpha)(\delta) = 0 \wedge \delta \in S_{t_\alpha}$ (O.K. even if $t_\alpha$ is a $\mathbb{P}_\alpha$-name)

(d) if $\alpha \in \text{Dom}(p)$ and $f_p(\alpha) < \alpha$ then $p[\alpha] \Vdash_{\mathbb{P}_\alpha} \langle \check{\delta}_{f(\alpha), \tilde{A}_\alpha}(p(\alpha)) = \delta \rangle$. 

We turn to another version of 1.8(4).

**Claim 1.11.** Let \( \xi \leq \lambda^+ \) and \( p \in R_{\lambda, \xi} \), see 1.9 and \( P = \mathbb{P}_p \), etc.

1) \( P_p \) is \((<\lambda^+)-strategically closed.
2) If \( \alpha < \xi \) then \( P_p/P_{\alpha} \) is \((<\lambda)-strategically \) \( W^p_{\alpha} \)-closed.
3) If \( p \in R_{\lambda, 1} \) and \( f_p(\alpha) = \alpha \) then in \( V[G] \), every case of the \( \lambda \)-task \( s[\xi_p[p], G^{\xi_p[a]}] \) is satisfied.
4) In part (3) if \( \models P_{\alpha} \) “\( S_{\alpha} \subseteq \lambda \) is stationary” and \( \in \check{\mathcal{I}}_{\lambda} \), which usually follows then \( \models P_{\alpha+1} \) “\( \lambda \in \check{I}[\lambda] \)”, hence no real loss assuming \( (\forall \lambda)(\lambda = \text{cf}(\lambda) > \aleph_0 \Rightarrow \lambda \in \check{I}[\lambda]) \).

**Proof.** Straightforward. □

**Claim 1.12.** Assume \( V \) satisfies G.C.H. and \( \Sigma \) is as in 1.8(3) or as in 1.9. Then there is a class forcing notion \( \mathbb{P} \) such that:

(a) forcing with \( \mathbb{P} \) preserves cardinality, cofinality and G.C.H.
(b) \( \mathbb{P} \) is \( \bigcup P_{\lambda} \), where \( (P_{\lambda}, Q_{\lambda} : \lambda \text{ regular uncountable}) \), is an iteration, with set support or Easton support
(c) in \( V^{P_{\lambda}} \), the forcing notion \( Q_{\lambda} \) is as in 1.10(1), or every as in 1.10(2) when we have \( \Sigma \)
(d) for each regular uncountable \( \lambda \) the set \( H(\lambda)^{V[\mathbb{P}]} \) is equal to \( H(\lambda)^{V[P_{\lambda}]} \)
(e) if \( \lambda \) is regular uncountable (in \( V \)), \( V^{P_{\lambda}} \models “S \subseteq \lambda \text{ is stationary” (} \in \check{I}[\lambda] \) for simplicity) then \( V^{P_{\lambda}} \models “S \text{ is stationary and } Q_{\lambda}” \).

**Proof.** Straight noting that \( P/P_{\alpha} \) is \((<\lambda^+)-strategically closed for any \( \lambda \) because of 1.11; i.e. of clause (A)(c) of Definition 1.8 □

**Discussion 1.13.** 1) Claims 1.10, 1.12 may seem too easy; particularly, if we compare them to more specific cases from [Sh:587]. The reason is that the relevant \( S \)'s are names, so the “too ambitious” \( \lambda \)-tasks template \( t \) will not cause the collapse of cardinals but just, e.g. having \( S_{\alpha}[p] \) non-stationary. Also the various tasks have little interaction except that forcing for one \( \lambda \)-task, create more instances (of \( A \)'s) for which we have to force for another \( \lambda \)-task, and even create new \( \lambda \)-tasks templates.
§ 2. AN EXAMPLE: RELATIVES OF DIAMONDS

We give an example of a $\lambda$-task template, see 2.2 on background.

Definition 2.1. Let $\lambda$ be a regular uncountable and $S \subseteq \lambda$ be stationary and $c_1, c_2$ are functions from $S$ to the set of cardinals $\leq \lambda^+$ such that $0 < c_1(\delta) < c_2(\delta)$; if $c_\ell$ is constantly $\kappa_\ell$ we may write $\kappa_\ell$ instead of $c_\ell$; if $\delta \in S \Rightarrow c_2(\delta) = (c_1(\delta))^+$ then we may omit $c_2$.

We define a $\lambda$-task template $t = t_1(\lambda, S, c_1, c_2)$ by:

$\exists S^t = S$ and $Q_t$ is defined by:

(A) \( p \in Q_t \) iff

(a) \( p = (\alpha, f, \bar{P}) = (\alpha_p, f_p, \bar{P}_p) \)

(b) \( \alpha < \lambda \)

(c) \( f : \alpha \to \{0, 1\} \) such that \( S^t_1 := f^{-1}(\{1\}) \subseteq S \)

(d) \( \bar{P} = (P_\delta : \delta \in S^t_1) \)

(e) \( P_\delta = \bar{P}_{p, \delta} \subseteq \delta \)

(f) \( c_1(\delta) \leq |P_\delta| < c_2(\delta) \) if $\delta \in S^t_1$.

(g) \( P_\delta = \{f_{\delta, i} : i < c_p(\delta)\} \) with no repetitions

(B) order: natural

(C) \( a ) \ S_1 = \{\alpha < \lambda : f_p(\alpha) = 1 \text{ for some } p \in G_{Q_t}\} \),

(b) \( B \) is \( (P_\delta : \delta \in S_1) \), i.e. the set \( \{\delta, P_\delta\} : \text{for some } p \in G_{Q_t} \text{ we have } f_p(\delta) = 1 \text{ and } P_\delta = \bar{P}_{p, \delta}\)

(c) \( \psi_1(X, Y, B) = \psi_1(Y, B) \) says that $Y \in \prod_{\delta \in S_1} P_\delta$

(d) \( \varphi(x, y, A, B) \) says that for some ordinal $\alpha < \lambda$:

(\alpha) $x$ has the form \((c_x, f_x)\) with \( c_x \in \omega^2 \) such that \( c_x^{-1}(\{1\}) \) is a closed subset of $\alpha$ and $f_x \in \omega^\lambda$ and $\delta \in S_1 \cap c_x^{-1}(\{1\})$.

\( \Rightarrow f_x \upharpoonright \delta \notin P_\delta \setminus \{Y(\delta)\} \)

(\beta) similarly $y = (c_y, f_y)$

(\gamma) $c_x \subseteq c_y \land f_x \subseteq f_y$.

Remark 2.2. 1) So the intention is that:

(i) \( S_1 \) is a stationary subset of $S^t$

(ii) \( (P_\delta : \delta \in S_1) \) is a diamond sequence, i.e. \( (\forall f \in \omega^\lambda)(\exists \text{stat} \delta \in S_1)(\exists \delta \in P_\delta) \) where \( c_1(\delta) \leq |P_\delta| < c_2(\delta), \) alternatively \( P_\delta = \{f_{\delta, i} : i < c(\delta)\} \) where \( \delta \in S_1 \Rightarrow c_1(\delta) \leq c(\delta) < c_2(\delta) \)

(iii) if we omit from each \( P_\delta \) one function for each $\delta \in S_1$ then (ii) stops to hold.

2) Fleissner proved (Fleissner diamond): assuming $\mathbf{V} = \mathbf{L}$ we have: if \( (P_\delta : \delta \in S) \) satisfies (ii) then for some \( \langle f_\delta : \delta \in S \rangle \in \prod_{\delta \in S} P_\delta \) the sequence \( \langle f_\delta : \delta \in S \rangle \) is a diamond sequence.

3) In [Sh:122] we prove that the assumption $\mathbf{V} = \mathbf{L}$ is necessary and more that is consistently we have a counterexample (with $|P_\delta| > 1$)
Claim 2.3. Let $\lambda$ be regular uncountable $S \subseteq \lambda$ stationary set $\in \mathcal{I}^V[\lambda]$ and $c_1, c_2$ are as in Definition \ref{definition:2.4}.

1) Then $t = t_1(\lambda, S, c_1, c_2)$ is a $\lambda$-task template, see Definition \ref{definition:2.1}, this holds also for $t|W$ when $W \subseteq S$ is stationary, satisfies the demands \ref{definition:2.2}.

2) If $S \subseteq S_\kappa$ is stationary for some $\kappa = \operatorname{cf}(\kappa) < \lambda$ is $c_1$ as in Definition \ref{definition:2.1} and $t = t_1(\lambda, S, c)$ and $c_1(\delta) \leq |\delta|$ all this in $V = V^{2^\lambda}$ where $q \in R^S_\lambda$ is from \ref{definition:1.7}(1) (or just $\mathfrak{T}$-full, $t_1(\lambda, S, c_1) \in \mathfrak{T}$) then for some directed $\mathcal{G} \subseteq \mathcal{Q}_1$:

   \begin{itemize}
   \item[(a)] $S_* = S_{1|\mathcal{G}}$ is a stationary subset of $\lambda$
   \item[(b)] $(\mathcal{P}_\delta|\mathcal{G}): \delta \in S_{1|\mathcal{G}}$ is a diamond sequence, see \ref{definition:2.2}(1)(ii)
   \item[(c)] $\mathcal{P}_\delta^* := \mathcal{P}_\delta|\mathcal{G}$ a family of subsets of $\delta$ of cardinality $\geq c_1(\delta)$ but $< c_2(\delta)$
   \item[(d)] if $Y \in \prod_{\delta \in S_*} \mathcal{P}_\delta^*$ then for some $f \subseteq ^\lambda \lambda$ and club $E$ of $\lambda$ we have $\delta \in S_* \Rightarrow f|\delta \notin \mathcal{P}_\delta^* \setminus \{Y(\delta)\}$.
   \end{itemize}

Proof. 1) Easy.

2) So $q \in R^S_\lambda$, in particular $\mathcal{Q} = (\mathbb{P}_\alpha, \mathcal{Q}_\beta : \alpha \leq \lambda^+, \beta < \lambda^+)$ are as in \ref{definition:1.6} and we work in $V$.

So for some $\alpha(0)$, $\mathbb{P}_\delta$ is a $\mathbb{P}_{\alpha(0)}$-name. Hence by part (1) and \ref{definition:1.10} for some $\alpha(1) \in (\alpha(0), \lambda^+)$, we have $f^q_\delta(\alpha) = \alpha$ and $\mathbb{P}_\alpha = t$ and we choose $\mathcal{G}$ the generic for $\mathcal{Q}_\alpha^q$.

Let $S_* = S_{1|\mathcal{G}}$, $\mathcal{P}_\delta^*$ be from the generic of $\mathcal{Q}_\alpha^q$. The problem is to prove that:

\begin{itemize}
   \item[(*)] $\mathbb{P}_\delta \models \langle \mathcal{P}_\delta : \delta \in S \rangle$ is a diamond sequence”.
\end{itemize}

The proof is as in \cite{Sh:122} or \cite{Sh:587}, but we give details.

Toward contradiction assume:

\begin{itemize}
   \item[(*)1] $p_* \models \mathbb{P}_\delta \not\models g \in \check{\lambda} \lambda$ is a counterexample”.
\end{itemize}

Without loss of generality

\begin{itemize}
   \item[(*)] $p_* \not\models g \notin (\check{\lambda} \lambda)^{\mathbb{P}_\alpha(1)+1}$.
\end{itemize}

[Why? Otherwise without loss of generality $p_* \not\models \langle g \in (\check{\lambda} \lambda)^{\mathbb{P}_\alpha(1)+2} \rangle$ and clearly $Q_\alpha^q$ naturally forces a $\diamond^*$-sequence.]

Moreover for some $\alpha(3)$ and $p_{\alpha(3)}$ we have

\begin{itemize}
   \item[(*)] (a) $\alpha(1) < \alpha(2) \leq \alpha(3) < \lambda^+$
   \item[(b)] $p_* \leq p_{\alpha(3)} \in \mathbb{P}_p$
   \item[(c)] if $\alpha(2) \neq \alpha(3)$ then $f_p(\alpha(2)) = \alpha(1)$ and $p_{\alpha(3)} \not\models \mathbb{P}_\alpha(2) \quad \langle g = f_p(\alpha(2)) \rangle$
   \item[(β)] if $\alpha(2) = \alpha(3)$ then for every $\beta \in (\alpha(1), \alpha(3))$ we have $p_{\alpha(3)} \not\models \mathbb{P}_\alpha(2) \quad \langle g \neq f_p(\beta) \rangle$.
\end{itemize}

Let

\begin{itemize}
   \item[(*)] $(c_\alpha : \alpha < \lambda)$ witness $S \in \check{I}[\lambda]$
   \item[(a)] $c_\alpha \subseteq \alpha$
   \item[(b)] $\beta \in c_\alpha \Rightarrow c_\beta = c_\alpha \cap \beta$
   \item[(c)] otp($c_\alpha$) $\leq \kappa$
   \item[(d)] $\{\delta \in S : \delta > \sup(c_\delta)\}$ is not stationary
   \item[(e)] let $\iota(\alpha) = \text{otp}(c_\alpha)$.
\end{itemize}
We define the tree $T_i$ as $\{<\} \cup \{(j) : j < 1 + i\}$. Now we choose $9\bar{p}_i, \gamma_i$ by induction on $i < \lambda$ such that:

\[\mathbb{E}(A)\]

(a) $\bar{p}_i = \langle p_i \eta : \eta \in T_{i(\alpha)} \rangle$

(b) $p_{i,<>} \in \mathbb{P}_{\alpha(2)}$

(c) $p_{i,<>} \in \mathbb{P}_{\alpha(3)}$ is above $p_{**}$

(d) $p_{i,<>} | \alpha(2) = p_{i,<>$

(e) $p_{i,<>} | \alpha(2) = p_{i,<>$

(f) $p_{i,<>} \in \mathbb{P}_{\alpha(3)}$ moreover

(g) $\delta_p(p_{i,<>}) = \delta_p(p_{i,<>})$ is $\gamma_i$

\[B\]

(a) $p_{i,<>}$ forces a value to $g_i|\gamma_i$ call it $g_{i,<>}$

(b) $\langle g_{i,<>} : j < 1 + i \rangle$ is with no repetitions

(c) if $\beta \in \text{Dom}(p_{i,<>})$ and $f_p(\beta) = \alpha(1) < \beta$ then $p_i|\beta$ forces a value to $p(\beta)$ and so its $\in (\gamma_i)^s$

(d) if $\beta_i \in \text{Dom}(p_{i,<>})$, $f_p(\beta_i) = \alpha(1)$ for $\ell = 1, 2$ then $p_{i,<>}((\beta_1) \neq p_{i,<>}((\beta_2)$.

There is no problem to carry the definition and so $E = \{\delta < \lambda : \delta$ a limit ordinal and $i < \delta \Rightarrow \gamma_i < \delta\}$ is a club of $\lambda$.

Let $G_{\alpha(1)} \subseteq \mathbb{P}_{\alpha(2)}$ be generic over $V_\mu$ in $V[G_{\alpha(1)}]$ choose $\delta(*) \in S \cap E$, so $\text{otp}(C_\delta) = \kappa$ and we know $\kappa_1 = c_1(\delta), \kappa_2 = c_2(\delta)$ and $\kappa_1 \subseteq |\delta|$. We choose $r_1 \in G_{\alpha(1)}$ above $\langle p_{i,<>}, i \in C_\delta \rangle$ forcing the relevant condition. For $\varepsilon < \kappa, g\varepsilon^* \in \delta^s$ is $\cup \langle g_{i,<>} : i \in C_\delta \rangle$.

Note that $\langle g\varepsilon^* : \varepsilon \in \kappa_1 \rangle$ is a sequence of members of $\delta^s$ with no repetitions. If $\alpha(2) > \alpha(3), then for any $\varepsilon < \kappa, h_\varepsilon \notin \Lambda_\varepsilon := \{\cup \langle p_{i,<>} : i \in C_\delta \rangle \beta \in C_\delta \text{ and } \beta \in \text{Dom}(p_{i,<>}) \text{ and } f(\beta) = \alpha(1) \not< \beta\}$ and we can choose $r_2 \in \mathbb{P}_{\alpha(3)+1}$ above $r_1$ and above $p_i(\alpha(1)+1)$ for $i \in C_\delta$ such that $r_2(\alpha(1)+1)$ is a set $\mathcal{P}_\delta$ disjoint to $\Lambda_\varepsilon$ to which $h_\varepsilon$ belongs. Easily $\{r_2\} \cup \langle p_{i,<>} : i \in C_\delta \rangle$ has a common upper bound and we are done.

So assume $\alpha(2) = \alpha(3)$ hence $f_p(\alpha(2)) = \alpha(1)$ and we have $h_\varepsilon = \cup \langle p_{i,<>} \alpha(2) : i \in C_\delta \rangle$. Now find $r_2 \in \mathbb{P}_{\alpha(3)+1}$ above $r_1$ and $p_i(\alpha(1)+1)$ for $i \in C_\delta$ such that $r_2(\alpha(1)+1) = \{h_\varepsilon : \varepsilon < \kappa_1\}$. Let $r_3 + \mathbb{P}_{\alpha(2)}$ be above $\{r_2\} \cup \langle p_{i,<>} \alpha(4) : i \in C_\delta \rangle$ clearly exist and without loss of generality $r_3$ forces a value to $\gamma_{\alpha(4)}(\delta)$ say $h_{\varepsilon(*)}$ and now $\{r_3\} \cup \langle p_{i,<>} : i \in C_\delta \rangle$ has a common upper bound say $r_4$ and it is as required.

**Discussion 2.4.**

1) What occurs if in (2) we waive “$S \subseteq S^\lambda_\kappa$” and $c_1(\delta) \leq \delta$?

We can assume instead:

\[(*)\]

(a) $S \subseteq \lambda$ is stationary

(b) we have (a) or (b) where

\[(a)\]

(a) $\kappa = \lambda, S$ is a set of stronger inaccessible cardinals and $c_1(\delta) \leq 2^{[\delta]}$

for $\delta \in S$

(b) $S \subseteq S^\lambda_\kappa, S \in \hat{I}[\lambda], \kappa = \text{cf}(\kappa) < \kappa$ and $c_1(\delta) \leq |\delta|^\kappa$, moreover there is a tree $T$ with $\kappa$ levels, $< \lambda$ nodes and $\geq \sup \{c_1(\delta) : \delta \in S\}, \kappa$-branches.
The proof of 2.3 works, with some changes. First, $\kappa = \lambda$ then $\delta \in S \Rightarrow \operatorname{otp}(C_{\delta}) = \delta$. Second, having chosen $e = (e_{\alpha} : \alpha < \lambda)$, we also choose $(T_{\alpha} : \alpha < \lambda)$ such that $T_{\alpha}$ is a tree with $\operatorname{otp}(C_{\alpha}) + 1$ levels, $< \lambda$ nodes for transparency a sub-tree of $\operatorname{otp}(C_{\alpha}) \geq \delta$ such that $\alpha \in e_{\beta} \Rightarrow T_{\alpha} = T_{\beta} \cap \operatorname{otp}(C_{\alpha}) \geq \lambda$ and $\delta \in S \Rightarrow |\max(T_{\delta})| \geq c_{1}(\delta)$.

Clearly possible in both cases (and we can allow $S$ to be a set of weakly inaccessible cardinals (so $(\exists \mu < \lambda)(2^{\mu} = \lambda)$) and if $\kappa < \lambda$ waive the existence of $T$ when there is such $T$).

Now $\tilde{p}_{\alpha}$ is $\langle p_{\alpha, \eta} : \eta \in \max(T_{\alpha}) \rangle$ and $\beta \in C_{\alpha} \wedge \eta \in \max(T_{\alpha}) \Rightarrow p_{\beta, \eta} \in \max(T_{\beta}) \leq p_{\alpha, \eta}$.

In the end having chosen $\delta \in S \cap E$ we choose a sequence $\langle \eta_{\varepsilon} : \varepsilon < c_{1}(\delta) \rangle$ of pairwise distinct members of $\max(T_{\delta})$ and continue as there replacing $p_{\varepsilon, < \varepsilon}$ by $p_{\varepsilon, \varepsilon, \eta_{\varepsilon}}$.

Discussion 2.5. 1) Is it true that in 2.3(2), in $V[G], G \subseteq \mathbb{P}_{\mathcal{Q}}$ generic over $V$, we get that for every stationary $S \subseteq \lambda$ there are a stationary $S_{\ast} \subseteq S$ such that the conclusion there holds? This is almost true as if $\alpha(1) < \lambda$, $t_{\alpha}^{\ast} = t_{\alpha[1]}[\mathcal{Q}][G]$ is well defined but $\neq t$ and $S_{\alpha[1]} \subseteq S_{\ast}$, then it complicates the forcing argument; moreover it may be one which “promises” $\Diamond_{\alpha[1]}$.

2) However, if we use 1.11(2) for $\mathfrak{T}$ consisting of $t$ only, the conclusion above surely holds.

3) What if $\mathfrak{T}$ consists of all $t$‘s of this form? In this case our framework, i.e. Definition 1.6 demand that $\langle S_{\varepsilon, \alpha} : \alpha < \lambda^{+}, f_{\varepsilon}(\alpha) = \alpha \rangle$ have pairwise non-stationary intersections. We can waive this here and then seems O.K.

4) So why not generally allow this in 1.6? It is reasonable but it complicates things considerably and we do not have an urgent need.

Discussion 2.6. A variant of 2.1 - 2.3 is:

$(\alpha)$ $(\ast)_{1}$ $c_{1}(\delta)$ a cardinal $\leq \lambda^{+}$

$(\ast)_{2}$ $c_{2}(\delta)$ a family of subsets of $c_{1}(\delta)$ closed under subsets

$(\beta)$ in clause (A) of 2.1 clauses (d),(f) are replaced by

$(\beta)$ $P = \langle (P_{\delta}, J_{\delta}) : \delta \in S_{\mathcal{T}}^{\mathcal{V}} \rangle$

$(f)$ $P_{\delta}$ is non-empty, $J_{\delta} \subseteq P(P_{\delta})$ and $(P_{\delta}, J_{\delta})$ is isomorphic to some pair from $c_{2}(\delta)$

$(\gamma)$ in 2.1B(c)(\beta), $Y$ a function with domain $S_{\mathcal{Q}t}$ such that $Y(\delta) \subseteq P_{\delta}, Y(\delta) \notin J_{\delta}$

$(\delta)$ in (B)(c), $f_{x} \notin P_{\delta} \setminus Y(\delta)$

Older version:

$(\epsilon)$ in the proof of 2.3(2) replace $(\ast)$ by

$(\ast)$ and in the stronger version if $\langle P_{\delta}^{-} : \delta \in \mathcal{F} \rangle$ “if $Y \in \prod_{\delta \in S}(P_{\delta} \setminus J_{\delta})$ a $\mathbb{P}_{\alpha(i)}$-

name $\alpha(1) \leq \alpha(2) < \lambda^{+}$ then $\langle P_{\delta}^{-} : \delta \in S \rangle$ is a diamond sequence.

Discussion 2.7. For any $\kappa = \operatorname{cf}(\kappa) < \lambda$ then is a $\lambda$-task guaranteeing $\Diamond_{S_{1}}^{\mathfrak{S}}$ for some stationary $S_{1} \subseteq S$ for any stationary subset of $S$ from $V^{P_{\kappa}}$. 
§ 3. Parameters for completeness of forcing

This section is purely set theoretic. Trees of conditions continue to play major roles, see [Sh:587]; here see the proof of 2.3 but whereas in the proof of 2.3 we use "degenerate simple" tree $T_i = \{ \langle i, j \rangle \} \cup \{ \langle i \rangle : i < \kappa \}$ here we use larger trees and extra structure on them. We concentrate on successor $\lambda$, for inaccessibles we have to phrase it differently, see 3.3.

Recall

Definition 3.1. A filter $D$ on a set $A$ is $(\kappa, \theta)$-regular when we can find $A_\alpha \in D$ for $\alpha < \kappa$ such that $i < \mu \Rightarrow |\{ \alpha : i \in A_\alpha \}| < \theta$.

Definition 3.2. 1) For cardinals $\kappa \geq \theta = \text{cf}(\theta)$ and $\delta \geq \kappa$ a limit ordinal, we say $(D, T)$ is a $(\delta, \kappa, \theta)$-special pair when:

   (a) $D$ is a $(\kappa, \theta)$-regular filter on $\delta$
   (b) $T \subseteq \delta \setminus \delta$ and no $\eta \in T$ is $\prec$-maximal
   (c) $\langle \delta \rangle \in T$, $T$ is closed under initial segments
   (d) $T$ is closed, that is, if $\eta \in T \setminus \delta$ and $(\forall \alpha < \ell g(\eta))(\eta \uparrow (\alpha + 1) \in T)$ then $\eta \in T$
   (e) for every $\eta \in \text{lim}(T)$ the set $\{ \alpha < \delta : \text{Suc}_T(\eta \uparrow \alpha) \text{ is not a singleton} \}$ belongs to $D$.

1A) We say $T$ or $(T, D)$ has $\bar{c}$-successor; if $\text{\bigwedge}_\eta c_\eta = 0$ we may omit it.

   (b′) $T$ is $\subseteq \delta \setminus \delta$ ordered by $\prec$ with $\eta \in T \Rightarrow \eta^\uparrow \langle c_\eta \rangle \in T$ and we call a $c_\eta$ the default value (for $\eta$) and $\bar{c} = \langle c_\eta : \eta \in T \rangle$ is called the default sequence.

1B) Let "$T$ is a $\delta$-special tree" mean that clause (b),(c),(d) of part (1).

2) We say that $T \subseteq \delta \setminus \delta$ is $\bar{c}$-lean if (b),(c), (d) of part (1):

   (f) for every $\alpha < \delta$ the set $\{ \eta \in T : \ell g(\eta) = \alpha$ and $\text{Suc}_T(\eta)$ is not a singleton} has $< \delta$ members.

2A) Saying "$(D, T)$ is a $(\delta, \kappa, \theta)$-special lean pair" means $(D, T)$ is a $(\delta, \kappa, \theta)$-special pair and $T$ is $\theta$-lean.

2B) Saying $T$ is lean means $\theta$-lean when $\theta$ is clear from the context.

3) For a $(\delta, \kappa, \theta)$-special pair $(D, T)$, let

   (a) $\text{Sub}(D, T) = \{ T' : T' \subseteq T$ satisfies clauses (b)-(e) of part (1) and $\eta \in T' \land |\text{Suc}_{T'}(\eta)| > 1 \Rightarrow \text{Suc}_{T'}(\eta) = \text{Suc}_T(\eta) \}$
   (b) $\text{Ln} - \text{Sub}_0(D, T) = \{ T' \in \text{Sub}(D, T) : T'$ is $\bar{c}$-lean $\}$
   (c) in clause (b) if $\bar{c}$ is missing and $(\kappa, \theta)$, or $\theta$ is clear from the context we mean $\bar{c} = \theta$, so we may write $\text{Ln} - \text{Sub}(D, T)$.

4) For a subtree $T \subseteq \delta \setminus \delta$, let $n\ell(T) = \{ \eta \in T : \ell g(\eta) \text{ is not a limit ordinal} \}$.

5) We say that $(D, T, E)$ is a $(\delta, \kappa, \theta)$-special triple or $\delta$-special triple when clauses (a)-(e) of part (1) or clauses (a)-(e) part (1A) hold and

   (g) $E = \langle E_\eta : \eta \in T \rangle$
   (h) $E_\eta$ is a filter on $\text{Suc}_T(\eta)$ or just a non-empty family of non-empty subsets of $\text{Suc}_T(\eta)$ closed under supersets, that is $u \subseteq v \in \text{Suc}_T(\eta) \land u \in E_\eta \Rightarrow v \in E_\eta$. 

6) Let \( \text{Sub}(D, T, \bar{E}) \) be the set of \( T' \) such that \( T' \subseteq T \) satisfies clauses (b)-(d) of part (1) and

\[
(i) \text{ for every } \eta \in \lim_3(T') \text{ the set } \{ \alpha < \mu : \text{Suc}_T(\eta|\alpha) \in E_{\eta|\alpha} \} \text{ belongs to } D
\]

where \( \text{Suc}_T(\eta) := \{ \zeta : (\eta | \alpha) \prec \zeta \in T \} \).

7) \( \text{Ln} - \text{Sub}_\theta(D, T, \bar{E}) = \{ T' \in \text{Sub}(D, T, \bar{E}) : (D, T') \text{ is } \theta\text{-lean} \} \).

7A) We omit \( \delta \) in part (7) when \( \delta = \theta \) and \( \theta \) is clear from the context.

8) We may replace \( \delta \) by another set. If \( \kappa = |\delta| \) we may omit \( \kappa \) so write \( (\delta, \theta) \); usually \( \delta \) is a cardinal and then we tend to use \( \mu \).

Remark 3.3. 1) For simplifying we usually do not deal with the \( (D, T, \bar{E}) \)-version in this section, but may need it later.

2) We may replace \( \text{Suc}_T(\eta) \in E_\eta \) by \( \text{Suc}_T(\eta) \neq \emptyset \mod E_\eta \), i.e. use \( E_\eta^+ \).

Definition 3.4. 1) We say \( \mathbf{p} = (D, T, \lambda, S, W, \bar{\eta}, \bar{E}) \) is a \( (\mu, \kappa, \theta) \)-parameter when

\( (\text{if } E_\eta = \{ \text{Suc}_T(\eta) \} \text{ for } \eta \in T \text{ then we may omit } \bar{E} : ) \):

\[ (a) \quad (D, T, \bar{E}) \text{ is a } (\mu, \kappa, \theta)\text{-special pair} \]

\[ (b) \quad (\alpha) \lambda \text{ is regular uncountable} \]

\[ (\beta) \quad S, W \text{ are stationary subsets of } \lambda \]

\[ (\gamma) \quad \lambda \geq \mu \text{ [? } \lambda = \mu \text{ possible?]} \]

\[ (\delta) \quad S \subseteq S^\lambda_{\text{cl}(\mu)} \text{ when } \lambda \geq \mu^+ \]

\[ (\varepsilon) \quad D_\mathbf{p} := \bigcup_{\eta} \text{ is a fat normal filter on } \lambda^{>\mathcal{P}(\lambda)}, \text{ see Definition } 0.5 \]

\[ (c) \quad (\alpha) \quad \bar{\eta} = \langle \eta_\delta : \delta \in S \rangle \]

\[ (\beta) \quad \eta_\delta \text{ is an increasing sequence of ordinals } \delta \text{ from } W \text{ of limit length} \]

\[ (\gamma) \quad \ellg(\eta_\delta) = \mu \text{ if } \mu < \lambda \text{ and } \eta_\delta \in \delta \geq \delta \text{ otherwise} \]

\[ (\delta) \quad \cup \{ \eta_\delta(i) : i < \gamma \} \subseteq S \text{ for any limit } \gamma < \ellg(\eta_\delta), \delta \in S \]

\[ (d) \quad \text{if } \chi \text{ is large enough, } x \in \mathcal{H}(\chi) \text{ and } \mathcal{Y} = \text{ proj}_{\lambda, \omega}(\chi), \text{ see Definition } 0.5 \]

\[ \text{then we can find } N = \langle N_i : i \leq \ellg(\eta) \rangle \text{ and } \delta \in S \text{ such that} \]

\[ (\alpha) \quad N_i \text{ is increasing continuous} \]

\[ (\beta) \quad N_{i+1} \cap \lambda = \eta_\delta(i) \]

\[ (\gamma) \quad N_i = N_{i+1} \text{ if } i \text{ is a limit ordinal and } \eta(i) = \cup \{ \eta(j) : j < i \} \]

\[ (\delta) \quad \text{if } N_i \neq N_{i+1} \text{ then } \overline{N_i} \upharpoonright (i+1) \in N_{i+1} \text{ for } i < \mu \]

\[ (\varepsilon) \quad x \in N_i \times (\mathcal{H}(\chi), \in, <_{\chi}^*) \]

\[ (\zeta) \quad N_{i+1} \in \mathcal{Y} \text{ and } [N_{i+1}]^{<\theta} \subseteq N_{i+1} \text{ [2010/4/12 was } [N_i]^{<\theta} \subseteq N_i \text{ if } i = 0 \]

\[ \text{or } i = j+1 \text{ and } N_i \neq N_j \text{.} \]

2) If \( \kappa = \mu \) we may omit \( \kappa \), i.e. say \( \mathbf{p} \) is a \( (\mu, \theta) \)-parameter, if \( \kappa = \mu, \theta = \aleph_0 \) we may just write \( \mu \)-parameter.

Definition 3.5. Assume \( \mathbf{p} = (D, T, \lambda, S, W, \bar{\eta}, \bar{E}) \) is a \( (\mu, \kappa, \theta) \)-parameter.

1) We say that a forcing notion \( \mathbb{Q} \) is \( \mathbf{p} \)-complete when:

\[ (a) \quad \mathbf{p} \text{ is a } (\mu, \kappa, \theta)\text{-parameter} \]

\[ (b) \quad \mathbb{Q} \text{ adds no new sequences of ordinals of length } < \lambda, \text{ moreover is } D_\mathbf{p} \]

\[ \text{-complete, see } 3.3.1(b)(\varepsilon) \text{ and Definition } 0.5 \]
(c) there is a function $F$ (called a witness for the $p$-completeness of $Q$) such that for some $x$:

$\oplus$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when :

(□) (α) $\chi$ is large enough

$\otimes$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when $\alpha < \mu$.

$\diamondsuit$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when $\alpha < \mu$.

$\Box$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when $\alpha < \mu$.

$\Diamond$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when $\alpha < \mu$.

The following is not directly useful, as we shall use iterations as in 1.6 but used as a warmup, so there $S = S_1$ and $B$ includes $(\eta_\delta : \delta \in S)$.

**Claim 3.6.** 1) Assume

(a) $\mathbf{p} = (D, \mathcal{T}, \lambda, S, W, \bar{\eta}, \bar{E})$ is a $(\mu, \mu, \theta)$-parameter

(b) $\mu = \mu^<$ and $\alpha < \mu \Rightarrow |\alpha|^< \theta < \mu$ and $\theta$ is regular

(c) $S \subseteq \lambda$ is stationary.

If $\mathbb{Q}$ is $(\leq \lambda)$-support iteration of $\mathbf{p}$-complete strategically $(\lambda \setminus S)$-complete (or just $\mathcal{D}_{\lambda, W}$-complete) forcing notions then $\text{Lim}(\mathbb{Q})$ is $\mathbf{p}$-complete and strategically $(\lambda \setminus S)$-complete.

2) If $\mathbb{Q}$ is $\mathbf{p}$-complete and $\mathbf{p}$ is $(\mu, \kappa)$-parameter, then $\mathbb{Q}$ does not add new $\mu$-sequences of ordinals and preserve the stationarity of $S^\mathbf{p}$.

**Proof.** As in [Sh:587], FILL?

We now look for “interesting” filters $D$.

**Definition 3.7.** 1) We say that $D$ is a $(\mu, S, \kappa, \theta, \mathbf{f})$-1-special filter iff

(a) $S$ is a subset of the uncountable cardinal $\mu$ of cardinality $\mu$

(b) $D$ is a $(\kappa, \theta)$-regular filter on the cardinal $\mu$ and $\kappa \geq \theta = \text{cf}(\theta)$

(c) $D$ is uniform (so every co-bounded subset of $\mu$ belongs to $D$) and $S \in D$

(d) $D$ is $\theta$-complete

(e) $\mathbf{f} : \mu \rightarrow \mu \setminus \{0\}$

(f) there is a witness $\bar{F}$ which means

$\otimes$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when $\eta < \nu \in n\ell(\mathcal{T}^\prime)$ then $p_{\eta} \leq_Q p_{\nu}$ (if defined)

$\diamondsuit$ for every $f \in \prod_{i < \mu} \mathbf{f}(i)$ the set $\{\alpha < \mu : f \upharpoonright \alpha \in F_\alpha\}$ belongs to $D$. 

$\Box$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when $\alpha < \mu$.

$\Diamond$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when $\alpha < \mu$.

$\Square$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when $\alpha < \mu$.

$\lozenge$ for some $\eta \in \text{lim}(\mathcal{T}^\prime)$, the set $\{p_{\eta|i} : i < \mu$ non-limit} has an upper bound in $Q$ when $\alpha < \mu$.
1A) Omitting \( f \) we mean \( i < \mu \Rightarrow f(i) = \theta \).

2) We say \((D, T)\) is \( (\mu, S, \kappa, \theta, f) \)-2-special pair when:

   (A) \( \mu \geq \kappa > \theta, \theta \) regular, \( S \subseteq \mu = \sup(S) \) and \( f: \mu \to \mu + 1 \)
   
   (B) (a) \( D \) is a \( (\mu, S, \kappa, \theta, f) \)-1-filter, older: \( (\kappa, \theta) \)-regular uniform filter on \( \mu \)
        (b) \( D \) is \( \theta \)-complete and \( S \in D \)
        (c) \( T \) is a subtree of \( \bigcup_{\alpha < \delta} \prod_{\beta < \alpha} f(\beta) \subseteq \mu^+ \mu \) of cardinality \( \leq \mu \)
        (d) \( T \) is a \( \theta \)-lean, see Definition 3.2(2) and \( T \subseteq \mu^+ \mu \)
        (e) \( (D, T) \) is a \( (\mu, \kappa, \theta) \)-special pair
        (f) if \( \eta \in T \) and \( \text{Suc}_T(\eta) \) is not a singleton then \( \{ \varepsilon: \eta^+(\varepsilon) \in T \} = f(\ell g(\eta)) \) (if we have \( E \) as in 3.2(5) then it is natural to demand just that the set to \( \in E_\eta \)).

3) We say \((D, \bar{u}, f, \bar{v})\) is \( (\mu, S, \kappa, \theta, f) \)-3-special when \( T, T_*, f_* \) satisfies: (omitting \( \bar{v} \)
means for some \( \bar{v} \), omitting \( \bar{u}, \bar{v} \)
means for some \( \bar{u}, \bar{v} \))

   (A) (a) \( \mu \geq \kappa > \theta = \text{cf}(\theta) \)
        (b) \( S \subseteq \mu \)
   
   (B) (a) \( D \) is a \( (\kappa, \theta) \)-regular
        (b) \( D \) is \( \theta \)-complete
        (c) \( T \subseteq \mu^+ \mu \) is as in part (2), hence \( \theta \)-lean
        (d) \( (D, T) \) is a \( (\mu, \kappa, \theta) \)-special pair
        (e) \( S \in D \)
   
   (C) (a) \( \bar{u} = \langle u_\alpha: \alpha < \mu \rangle \) is \( \subseteq \)-increasing with union \( \mu \)
        (b) \( \text{Dom}(f_*) = \mu \) and \( f_*(\alpha) = \langle u_\alpha \rangle f(\alpha) \)
        (c) \( \bar{v} = \langle v_\alpha: \alpha < \mu \rangle, v_\alpha \subseteq \mu, |v_\alpha| < \theta \)
        (d) \( \bar{F} = \langle \langle F_\alpha, <_\alpha: \alpha \in S \rangle \rangle \)
        (e) \( F_\alpha \subseteq \{ \bar{\eta}: \bar{\eta} = \langle \eta_i: i \in v_\alpha \rangle \text{ and } \eta_i \in T \text{ and } \ell g(\eta_i) = \alpha \text{ for } i \in v_\alpha \} \)
        (f) \( |F_\alpha| < \theta \)
        (g) \( <_\alpha \)
        (h) if \( \eta_i \in \text{lim}_{\mu}(T) \) for \( i < \mu \) then \( \{ \alpha < \mu: \langle \eta_i^* \rangle: i \in v_\alpha \} \in D \)
        (i) if \( <_* \) is a well ordering of \( \nu_\alpha \) then \( \{ \alpha: <_* \nu_\alpha \} \neq \emptyset \text{ mod } D \)
        (j) \( T_* \) is defined by: \( \eta \in T_* \) iff:

   \( \eta \) is a sequence
   
   (a) \( \eta \) is a sequence
   
   (b) \( \ell g(\eta) < \mu \)
   
   (c) \( \eta(\alpha) = \langle \eta(\alpha, \varepsilon): \varepsilon \in v_\alpha \rangle \) and \( \eta(\alpha, \varepsilon) \in T \) has length \( \alpha + 1 \)
   
   (d) if \( \alpha < \beta < \ell g(\eta) \) and \( \varepsilon \in u \) then \( \eta(\alpha, \varepsilon) < \eta(\beta, \varepsilon) \)
   
   (e) \( \eta(\alpha) < \ell g(\eta) \) and \( \varepsilon \in u_\beta \setminus u_\alpha \) then \( \eta(\beta, \varepsilon)(\alpha) \) is 0
   
   (f) if \( \eta(\alpha) < \ell g(\eta) \), \( \varepsilon \in u_\alpha \setminus v_\alpha \) then \( \eta(\alpha, \varepsilon)(\alpha) \) is 0 or just the default.

4) We can above allow \( \mu = \aleph_0 = \kappa = \theta \).

5) If we omit \( \kappa \) we mean \( \mu = \kappa \).
Claim 3.8. 1) Assume \( D \) is a \( (\mu, S, \kappa, \theta, f) \) \(-1\)-special filter, \( \kappa = \mu \) and \( \alpha < \mu \Rightarrow |\prod_{\beta < \alpha} f(\beta)| < \mu \). Then for some \( T \), the pair \((D, T)\) is \((\mu, S, \kappa, \theta, f)\)-2-special.

2) Assume \( f: \mu \to \mu + 1 \) and \( g, f_* \) are as in 3.7(3)(C),(a),(b). If \( D \) is \((\mu, S, \kappa, \theta, f)\)\(-1\)-special then \( D \) is \((\mu, S, \kappa, \theta, f)\)-3-special.

Proof. 1) Let \( \tilde{F} \) be a witness for \( D \) being \((\mu, S, \kappa, \theta, f)\)-1-special filter, i.e. as in 3.7(2). We define \( T \) as the set of \( \eta \) such that:

\[(*)_1 (a) \; \eta \in \prod_{i<\alpha} f(i) \text{ for some } \alpha < \mu \]

\[ (b) \; \text{if } \alpha < \ell g(\eta) \text{ and } \eta|\alpha \notin F_\alpha (\text{which holds trivially if } \alpha \notin S) \text{ then } \eta(\alpha) = 0. \]

2) Should be clear. \( \square \)

For \( \mu \) regular below we construct such filters.

Claim 3.9. If \( \mu > \theta \) are regular, \( S \subseteq \mu \) is stationary such that \( \Diamond_S \) holds and \( f: \mu \to \mu \) satisfies \( i < \mu \Rightarrow \theta \Rightarrow f(i) \leq i \) then there some \( D \) is a \((\mu, S, \mu, \theta, f)\)-1-special filter.

Proof. As \( \Diamond_S \) clearly \( \mu = \mu^{<\mu} \) hence \( \mu = \mu^{<\theta} \). Let \( cd:\theta^+ \to \mu \) be one to one onto and \( cd(\langle \alpha_i : i < \eta \rangle) = \sup\{\alpha_i : i < \eta\} \) and let \( g: \mu \to \theta \) and \( cd_i: \mu \to \mu \) be such that \( \eta \in \theta^+ \Rightarrow g(\ell g(\eta)) = \ell g(\eta) \) and \( \bigwedge_{i<\ell g(\eta)} \eta(i) = cd_i(\ell g(\eta)) \).

As \( \Diamond_S \) holds, there is a sequence \( \langle f_\delta : \delta \in S \rangle \) such that:

\[(i) \; f_\delta \in \delta \delta \]

\[(ii) \; \text{if } f \in \mu^\mu \text{ then the set } \{ \delta \in S : f_\delta = f : \delta \} \text{ is a stationary subset of } \mu. \]

Now for \( \delta \in S \) we define \( F_\delta \subseteq [\delta^{\delta}]^{<\theta} \) as follows:

\[ F_\delta = \{ f \in \delta : \text{ for some } j < i < \theta \text{ for every } \alpha < \delta \text{ we have } g(f_\delta(\alpha)) = i \text{ and } f(\alpha) = cd_j(f_\delta(\alpha)) \}. \]

By the choice of \( g \) we have \( g(f_\delta(0)) < \theta \) so clearly \( F_\delta \subseteq [\delta^{\delta}]^{<\theta} \). Now if \( g(f_\delta(\alpha)) : \alpha < \delta \) is not constant then \( F_\delta = \emptyset \), so we can ignore such \( \delta \)'s.

Now for every \( h \in \mu^\mu \) let

\[ A_h = \{ \delta \in S : h \circ \delta \in F_\delta \} \]

and let \( D \) be the \( \theta \)-complete filter on \( \mu \) generated by \( \{ A_h : h \in \mu^\mu \} \cup \{ C : C \text{ a club of } \mu \} \). Now

\[ (\alpha) \; \emptyset \notin D. \]

[Why? It suffices to prove that \( \cap\{ A_h : \varepsilon < \zeta_1 \} \cap \{ C_\varepsilon : \varepsilon < \zeta_2 \} \) is non-empty when \( \zeta_1, \zeta_2 < \theta, h_\varepsilon \in \mu^\mu \) for \( \varepsilon < \zeta_1 \) and \( C_\varepsilon \) is a club of \( \mu \) for \( \varepsilon < \zeta_2 \). Let \( f \in \mu^\mu \) be defined by \( f(\alpha) = cd((h_\varepsilon(\alpha) : \varepsilon < \zeta_1) \in \mu \), hence \( S_f = \{ \delta \in S : f \circ \delta = f_\delta \text{ (hence } \delta \in \delta) \}, \) is a stationary subset of \( \mu \). Clearly

- \( \alpha < \delta \in S_f \Rightarrow g(f_\delta(\alpha)) = \zeta_1 \) and \( \bigwedge_{\varepsilon < \zeta_1} cd_\varepsilon(f_\delta(\alpha) = h_\varepsilon(\alpha) \) and
- \( \cap\{ C_\varepsilon : \varepsilon < \zeta_2 \} \) is a club of \( \mu \).]
So there is $\delta \in S_{\delta} \cap \{C_\varepsilon : \varepsilon < \zeta_2\}$ and clearly $\varepsilon < \zeta_1 \Rightarrow h_\varepsilon | \delta \in F_\delta \Rightarrow \delta \in A_{\delta}$, so
$\delta \in \bigcap \langle \varepsilon < \zeta_1 \rangle C_\varepsilon \text{ hence we are done.}
$

(\beta) D \text{ is } (\mu, \theta)\text{-regular.}

[Why? For } \varepsilon < \mu \text{ let } h_\varepsilon \in \mu \text{ be constantly } \varepsilon; \text{ so } \{A_{\delta_1} : \varepsilon < \mu \} \subseteq D \text{ and no } \alpha < \mu \text{ belongs to } \geq \theta \text{ of them.]

So clearly clauses (a) – (f) of Definition 3.7 hold. □

Also we can combine such filters.

Claim 3.10. $D$ is a $(\mu, S, \kappa, \theta, f)$-1-special filter when for some $S, \bar{D}, \bar{\mu}, \bar{\kappa}, \bar{S}, \bar{\kappa}, \bar{f}$ we have:

(a) $\mu$ is uncountable and $\theta = cf(\theta) \leq \kappa \leq \mu$
(b) $S_\kappa \subseteq \mu$ is unbounded
(c) $\bar{\mu} = \langle \mu_\delta : \delta \in S_\kappa \rangle$ such that for every $\delta \in S_\kappa$ we have:
   (i) $\delta \leq \mu_\delta < \mu$
   (ii) $\alpha \in S_\kappa \Rightarrow \kappa_\alpha \geq \prod \alpha < \mu_\delta f_\delta(\alpha)$
   (iii) $\langle \delta, \delta + \mu_\delta \rangle \cap S_\kappa = \emptyset$ and $\text{otp}(\delta) = \text{otp}(\delta \setminus [\delta', \delta' + \mu_\delta] : \delta' \in S \cap \delta)$
   (iv) $\delta \in S_\kappa$
   (v) $D_\delta$ is a $(\mu_\delta, S_\kappa, \kappa, \theta, f_\delta)$-1-special filter, see 3.7(1) and $\theta_\delta \leq \theta, \kappa_\delta < \kappa$
   (vi) $D_\mu$ is a $\theta$-complete (cf($\mu$))-regular filter on $\mu$ such that $S_\kappa \in D$ containing
   the co-bounded subsets of $\mu$
   (vii) if $\theta_\kappa < \theta$ and $\kappa_\kappa < \kappa$ then $\alpha \in S_\kappa : \theta \geq \theta_\alpha > \theta_\kappa$ and $\kappa_\alpha > \kappa_\kappa$ and
   $\kappa_\alpha < \kappa_\kappa$
   (viii) $\exists D \subseteq \mu$ such that $\bar{\delta} : \alpha < \mu_\delta \in D_\delta$ and $f$ is 1 otherwise
   (ix) $D = \{A \subseteq \mu : \text{ the set } \{\alpha < \mu_\delta : \delta + \alpha \in A \} \in D_\delta \} \text{ belongs to } D_\mu$
   (x) $S = \cup \{\delta + \alpha : \delta \in S_\kappa \text{ and } \alpha \in S_\delta\}$

Proof. Clause (a) of 3.7 By clause (b) + (c) we have $\mu = \sup \{\mu_\delta : \delta \in S_\kappa\}$. By clause (d), if $\delta \in S_\kappa$ then $S_\delta$ is an unbounded $\gamma$ subset of $\mu_\delta$. Hence by clauses (b) + (i) clearly $S$ is an unbounded subset of $\mu$ and even of cardinality $\mu$.

Clause (b) of 3.7 First, $D$ is a filter on $\mu$ by its definition (in clause (h) here) as $D_\mu$ is a filter on $\mu$ to which $S_\kappa$ belongs (see clause (e) of our assumptions) and $D_\delta$ is a filter on $\mu_\delta$ for $\delta \in S_\kappa$. Second, why is $D$ $(\kappa, \theta)$-regular? Recall $D_\delta$ is $(\kappa_\delta, \theta_\delta)$-regular by clause (d) of the assumption and let $\langle A_{\delta, i} : i < \kappa_\delta \rangle$ witness it.

Let $A_\varepsilon = \{\delta + \alpha : \kappa_\delta \in \kappa_\delta \text{ and } \alpha < \mu_\delta \text{ and } \alpha \in A_{\delta, \varepsilon}\} \subseteq S_\kappa$ so it suffices to show that $A = \{A_\varepsilon : i < \kappa_\delta\}$ witness $D$ is $(\kappa, \theta)$-regular.

First, if $\varepsilon < \kappa$ then
- $A^1_\varepsilon = \{\delta \in S_\kappa : \kappa_\delta < \kappa_\delta\} \in D_\mu$ by clause (f) of the assumption
- if $\delta \in A^1_\varepsilon$ then $\alpha < \mu_\delta : \delta + \alpha \in A_\varepsilon = A_{\delta, \varepsilon} \in D_\delta$
so together $A_i \in D$.

Next let $v \subseteq \kappa, |v| \geq \theta$ and we should prove that $\cap \{A_i : i \in v\} = \emptyset$. Toward contradiction assume $\delta + \alpha \in \cap \{A_i : \varepsilon \in v\}$ where $\delta \in S_\alpha, \alpha < \mu_\delta, \varepsilon \in v \Rightarrow \varepsilon < \kappa_\delta$ and $\varepsilon \in v \Rightarrow \alpha \in A_{\delta,\varepsilon}$ hence $\{\varepsilon < \kappa_\delta : \alpha \in A_{\delta,\varepsilon}\} \supseteq v$ has cardinality $\geq \theta$, contradiction to the choice of $\langle A_{\delta,\varepsilon} : \varepsilon < \kappa_\delta\rangle$, so indeed $\cap \{A_i : i \in v\} = \emptyset$ so $A$ exemplifies $D$ is $(\kappa, \theta)$-regular.

Clause (c) of 3.7 Why $D$ is a uniform filter on $\mu$?

Assume $\chi < \mu$ and $A \in D$ and we shall prove $|A| \geq \tau$. Let $A_1 : \{\delta \in S_\alpha : A_{2,\delta} \in D_\mu\}$ where $A_{2,\delta} = \{\alpha < \mu_\delta : \delta + \alpha \in A_i\}$. Now $D_\mu$ contains the co-bounded subsets of $\mu$, hence $\mu = \sup(A_1)$ so choose $\delta \in A_1$ such that $\delta > \chi$. Now $|A_{2,\delta}| = \mu_\delta$ by 3.7(1)(b), the uniformity and clause (d) of the assumption. Also $|A| \geq |A \cap [\delta, \mu_\delta + \delta]| = |A_{2,\delta}|$ and $\mu_\delta \geq \delta$ by clause (c) of the assumption. Together $|A| \geq |A \cap [\delta, \mu_\delta + \delta]| = |A_{2,\delta}| \geq \mu_\delta \geq \delta \geq \chi$ so we are done.

Also $S \in D$ by the choice of $D$ in clause (h) and the choice of $S$ in clause (i).

Clause (d) of 3.7 $D$ is $\theta$-complete as

- $D_\mu$ is $\theta$-complete by clause (e) of the assumption
- $D_\delta$ is $\theta_\delta$-complete by clause (d) of the assumption
- $\theta = \text{cf}(\theta)$ by clause (a)
- By clause (f), $\theta_\alpha < \theta \Rightarrow \{\delta \in S_\alpha : \theta_\alpha < \theta\} \in D$

and the choice of $D$ in clause (h).

Clause (e) of 3.7 By the choice of the function $f$ in clause (g), indeed $f : \mu \rightarrow \mu - \{0\}$.

Clause (f) of 3.7 For each $\delta \in S_\alpha$ let $\langle g_{\delta,\varepsilon} : \varepsilon < \kappa_\delta\rangle$ list $\prod_{\alpha \in S_\delta, \varepsilon < \mu_\alpha} f_{\alpha}(i)$, possible by clause (c)(\beta) of the assumption. For each $\delta \in S_\alpha$ by the assumption "$D_\delta$ is $(\mu_0, S_\kappa, \kappa_\delta, g_{\delta,\varepsilon})$-1-special filter", there is a sequence $\bar{A}_{\delta} = \langle A_{\delta,\varepsilon} : \varepsilon < \kappa_\delta\rangle$ exemplifying "$D_\delta$ is $(\kappa_\delta, \theta_\delta)$-regular", and a sequence $\mathcal{F}_{\delta} = \langle \mathcal{F}_{\delta,\alpha} : \alpha \in S_\delta\rangle$ witnessing clause (f) of 3.7(1) for this tuple and let $\mathcal{F}_{\delta,\alpha} = \{f : \text{Dom}(f) = [\delta, \delta + \alpha) \text{ and } \langle f(\delta + \alpha) : \alpha < \mu_\delta \rangle \in \mathcal{F}_{\delta,\alpha}\}$ is a subset of $\Pi\{f(\beta) : \beta \in [\delta, \delta + \alpha)\}$ of cardinality $< \theta_\alpha$.

We define $\mathcal{F} = \langle \mathcal{F}_{\beta} : \beta \in S\rangle$ by

\[ (*) \] $f(\beta) = \delta + \alpha, \delta \in S_\alpha$ and $\alpha \in S_\delta$ then $\mathcal{F}_\beta = \{f \cup g : f \in \mathcal{F}_{\delta,\alpha} \text{ and } g \in \{g_{\delta,\varepsilon} : \alpha \in A_{\delta,\varepsilon}\}\}$.

We should check the four subclause of clause (f) of 3.7

Subclause (a): holds; trivially by the choice of $\mathcal{F}$.

Subclause (\beta):

Recall the choice of $f$ and $\mathcal{F}_{\alpha}$.

Subclause (\gamma): As $\delta \in S \Rightarrow \theta_\delta \leq \theta$ it follows that if $\beta = \delta + \alpha \in S, \delta \in S_\alpha, \alpha \in S_\delta$ then $|\mathcal{F}_\beta| = |\mathcal{F}_{\delta,\alpha}| \times |\{\varepsilon : \alpha \in A_{\delta,\varepsilon}\}|$ a product of two cardinals $< \theta_\delta \leq \theta$, as required in subclause (\gamma).

Subclause (\delta): Let $f \in \prod_{\alpha < \mu_\delta} f_{\alpha}(\alpha)$, for $\delta \in S$, let $f_\delta \in \prod_{\alpha < \mu} f_{\delta}(\alpha)$ be defined by $f_{\delta}(\alpha) = f(\delta + \alpha)$ for $\alpha \in S_\delta$. So for every $\delta \in S$, the set $B_{\delta} = \{\alpha < \mu_\delta : f_\delta(\alpha) \in \mathcal{F}_{\delta,\alpha}\} = \{\alpha < \mu_\delta : f(\mu_\delta, \mu_\delta + \alpha) \in \mathcal{F}_{\delta,\alpha}\}$ belongs to $D_\delta$. Also $f_\delta(\delta) \in \{g_{\delta,\varepsilon} : \varepsilon < \kappa_\delta\}$, so we can choose $\varepsilon(\delta) < \kappa_\delta$ such that $g_{\delta,\varepsilon} = f_\delta|\delta$. Hence if $\alpha < \mu_\delta$
belongs to $B_\delta \cap A_{\delta,\varepsilon}(\delta)$ then $f|\delta,\delta + \alpha \in F_{\delta,\alpha}^\delta, f|\delta \in \{g_{\delta,\varepsilon} : \alpha \in A_{\delta,\varepsilon}\}$ together
$f|(\delta + \alpha) \in F_{\delta + \alpha}^\delta$.

By the definition of $D$ it follows that \(\{\alpha \in S : f|\alpha \in F_{\alpha}\}\) belongs to $D$ as required. \(\square\)

3.10 Conclusion

Assume $\mu > \sigma, f \in M\{\sigma\}$ and $(\forall \kappa < \mu)(2^\kappa = \kappa^+)$. 

1) There is a $(\mu, \mu, \mu, \text{\aleph}_0, f) - 1$-special filter.
2) Moreover, there is $(\mu, \mu, \mu, \text{\aleph}_0, f) - 2$-special pair $(D, I)$.

Proof. 1) By 3.9 and 3.10
2) By part (1) and 3.8(1). \(\square\)
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