ON FACTORS WITH PRESCRIBED DEGREES IN BIPARTITE GRAPHS

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Abstract. We establish a new criterion for a bigraph to have a subgraph with prescribed degree conditions. We show that the bigraph $G[X,Y]$ has a spanning subgraph $F$ such that $g(x) \leq \deg_F(x) \leq f(x)$ for $x \in X$ and $\deg_F(y) \leq f(y)$ for $y \in Y$ if and only if $\sum_{y \in Y} f(y) \geq \sum_{x \in X} \max \{0, g(a) - \deg_{G-B}(a)\}$ for $A \subseteq X, B \subseteq Y$. Using Folkman-Fulkerson’s Theorem, Cymer and Kano found a different criterion for the existence of such a subgraph (Graphs Combin. 32 (2016), 2315–2322). Our proof is self-contained and relies on alternating path technique. As an application, we prove the following extension of Hall’s theorem. A bigraph $G[X,Y]$ in which each edge has multiplicity at least $m$ has a subgraph $F$ with $g(x) \leq \deg_F(x) \leq f(x)$ for $x \in X$, $\deg_F(y) \leq m$ for $y \in Y$ if and only if $\sum_{y \in N_C(S)} f(y) \geq \sum_{x \in S} g(x)$ for $S \subseteq X$.

1. Introduction

Factor theory is one of the oldest and most active areas of graph theory [1], that started in the 19th century when Petersen showed that every even regular graph is 2-factorable. In this note, we are primarily concerned with factors with prescribed degree conditions in bigraphs.

A bigraph $G$ with bipartition $\{X, Y\}$ will be denoted by $G[X,Y]$, and for $S \subseteq X$, $\overline{S}$ means $X \setminus S$. For a real-valued function $f$ on a domain $D$ and $A \subseteq D$, $f(A) := \sum_{a \in A} f(a)$. For a graph $G = (V,E)$, $u \in V$ and $A \subseteq V$, $\deg_G(u)$ and $e_G(uA)$ denote the number of edges incident with $u$, and the number of edges between $u$ and $A$, respectively. Let $f, g$ be integer functions on the vertex set of a graph $G$ such that $0 \leq g(x) \leq f(x)$ for all $x$. A $(g,f)$-factor is a spanning subgraph $F$ of $G$ with the property that $g(x) \leq \deg_F(x) \leq f(x)$ for each $x$. An $f$-factor is an $(f,f)$-factor. Ore [7, 8] showed that $G[X,Y]$ has an $f$-factor if and only if $f(X) = f(Y)$ and

$$f(A) \leq \sum_{y \in Y} \min \{f(y), e_G(yA)\} \quad \forall A \subseteq X.$$ 

Folkman and Fulkerson proved a $(g,f)$-factor theorem for bigraphs [5] which was simplified by Heinrich et al. (Here, $x \div y$ means $\max\{0, x - y\}$).

Theorem 1.1. [6, Theorem 1] The bigraph $G[X,Y]$ has a $(g,f)$-factor if and only if

$$f(A) \geq \sum_{u \in A} \left(g(u) - \deg_{G-A}(u)\right) \quad \forall A \subseteq X \cup Y.$$ 

Recently Cymer and Kano found another simple criteria.

Theorem 1.2. [4, Theorem 5] The bigraph $G[X,Y]$ has a $(g,f)$-factor if and only if the following conditions hold.

$$g(A) \leq \sum_{y \in Y} \min \{f(y), e_G(yA)\} \quad \forall A \subseteq X,$$

$$g(B) \leq \sum_{x \in X} \min \{f(x), e_G(xB)\} \quad \forall B \subseteq Y.$$
Theorem 1.2 has been particularly useful in solving various generalized Sudoku puzzles [2, 3]; Solving some of these puzzles can be reduced to finding \((g, f)\)-factors with the additional property that \(g(y) = 0\) for \(y \in Y\) in a bigraph \(G[X, Y]\). Motivated by solving such problems, we establish the following new criterion for a bigraph to have a factor with prescribed degrees.

**Theorem 1.3.** A bigraph \(G[X, Y]\) has a \((g, f)\)-factor with \(g(y) = 0\) for \(y \in Y\) if and only if

\[
f(B) \geq \sum_{x \in A} \left( g(x) - e_G(xB) \right) \quad \forall A \subseteq X, B \subseteq Y.
\]

While Theorem 1.2 relies on Folkman-Fulkerson’s \((g, f)\)-factor theorem, our proof is self-contained and relies on alternating path technique [6]. Before we prove our main result, we provide the following corollary. Here, \(N_G(S)\) is the neighborhood of \(S\) in \(G\).

**Corollary 1.4.** A bigraph \(G[X, Y]\) in which the mutiplicity of each edge is at least \(m\), has a \((g, f)\)-factor with \(f(y) \leq m, g(y) = 0\) for \(y \in Y\) if and only if

\[
f(N_G(S)) \geq g(S) \quad \forall S \subseteq X.
\]

**Proof.** By Theorem 1.3, \(G[X, Y]\) has a \((g, f)\)-factor with \(g(y) = 0\) for \(y \in Y\) if and only if

\[
f(B) \geq \sum_{x \in A} \left( g(x) - e_G(xB) \right) \quad \forall A \subseteq X, B \subseteq Y.
\]

To complete the proof, we show that (2) and (3) are equivalent. First, let us assume that (3) holds, and let \(A = S \subseteq X, B = N_G(S) \subseteq Y\). We have

\[
f(N_G(S)) \geq \sum_{x \in S} \left( g(x) - e_G(xN_G(S)) \right) = \sum_{x \in S} g(x) = g(S),
\]

and so (2) is satisfied. Conversely, assume that (2) holds, and let \(A \subseteq X, B \subseteq Y\). Let

\[
S = \{x \in A \mid g(x) \geq e_G(xB)\}.
\]

If we show that \(g(S) \leq f(B) + e_G(SB)\), then we are done. We have

\[
g(S) \leq f(N_G(S)) = f(N_G(S) \cap B) + f(N_G(S) \setminus B) \leq f(B) + m|N_G(S) \setminus B| \leq f(B) + e_G(SB).
\]

\(\square\)

**Remark 1.5.** The case \(m = 1\) of Corollary 1.4 was previously settled in [4, Theorem 7]. Observe that the case \(m = 1, f(x) = g(x) = 1\) for \(x \in X\) of Corollary 1.4 corresponds to the famous Hall’s marriage theorem.

2. **Proof of Theorem 1.3**

To prove the necessity, suppose that \(G\) has a \((g, f)\)-factor \(F\), and let \(A \subseteq X, B \subseteq Y\). Define \(C = \{x \in A \mid g(x) > e_G(xB)\}\). If \(C = \emptyset\), then (1) is trivial. Otherwise, let \(x \in C\). There must be at least \(g(x) - e_G(xB)\) edges in \(F\) joining \(x\) to vertices in \(B\). Hence,

\[
\sum_{x \in C} \left( g(x) - e_G(xB) \right) \leq e_F(CB) \leq e_F(AB) \leq \sum_{y \in B} \deg_F(y) \leq f(B).
\]

To prove the sufficiency, suppose that (1) holds. Let \(F\) be a \((0, f)\)-factor that minimizes \(\delta := \sum_{x \in X} \left( g(x) - \deg_F(x) \right)\). If \(\delta = 0\), then \(F\) is a \((g, f)\)-factor and we are done. So let us assume that \(\delta > 0\), and so

\[
R := \{x \in X \mid g(x) > \deg_F(x)\} \neq \emptyset.
\]

To complete the proof, we find sets \(A \subseteq X, B \subseteq Y\) such that (1) fails. A path (possibly of length zero) is nice if it starts with a vertex in \(R\) and an edge in \(E(G) \setminus E(F)\) and whose edges are alternately in \(G - F\) and \(F\). Let \(W\) be the set of terminal vertices of all nice paths. Let \(A = R \cup S\) where \(S := (X \setminus R) \cap W\), and let \(B = Y \cap W\). We claim that
(a) If \( e \in E(F) \) with \( e = xy \) and \( y \in B \), then \( x \in A \).
(b) If \( e \in E(G) \setminus E(F) \) with \( e = xy \) and \( x \in A \), then \( y \in B \).
(c) \( e_G(x\delta) = \deg_F(x) - e_G(xB) \) for \( x \in A \).
(d) \( \deg_F(y) = f(y) \) for \( y \in B \).
(e) \( \deg_F(x) = g(x) \) for \( x \in S \).
(f) \( \deg_{G-B}(x) \leq g(x) \) for \( x \in A \).

Observe that (c) is an immediate consequence of (b), and (6) and (c) imply (f). To prove (a) and (b), let \( e = xy \). If \( y \in B \), there is a nice path \( P \) ending at \( y \) (whose last edge is not in \( F \)), and so if \( e \in E(F) \) and \( x \notin R \), then \( P + ex \) is a nice path ending at \( x \), and consequently, \( x \in S \). Similarly, if \( x \in A \), there is a nice path \( P \) ending at \( x \) (whose last edge is in \( F \)), and so if \( e \in E(G) \setminus E(F) \), then \( P + ey \) is a nice path ending at \( y \), and consequently, \( y \in B \). To prove (d), let \( y \in B \). There is a nice path \( P \) ending at \( y \). If \( \deg_F(y) < f(y) \), then since the last edge of \( P \) is in \( E(G) \setminus E(F) \), the \((0,f)\)-factor \( F' \) with \( E(F') = E(F) \Delta E(P) \) contradicts the minimality of \( \delta \) (We use \( \Delta \) for the symmetric difference).

Similarly, to prove (e), let \( x \in S \). There is a nice path \( P \) ending at \( x \). If \( \deg_F(x) > g(x) \), then since the last edge of \( P \) is in \( E(F) \), the \((0,f)\)-factor \( F' \) with \( E(F') = E(F) \Delta E(P) \) contradicts the minimality of \( \delta \). The following completes the proof.

\[
\sum_{x \in A} \big( g(x) - e_G(x\delta) \big) \overset{(f)}{=} \sum_{x \in A} \big( g(x) - e_G(xB) \big) \\
\overset{(c)}{=} \sum_{x \in A} \big( g(x) - \deg_F(x) + e_G(xB) \big) \\
\overset{(e)}{=} \sum_{x \in R} \big( g(x) - \deg_F(x) \big) + e_G(AB) \\
\overset{(6)}{>} e_G(AB) \\
\overset{(a)}{=} \sum_{y \in B} \deg_F(y) \\
\overset{(d)}{=} f(B).
\]

This completes the proof.

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