NONCOMMUTATIVE GEOMETRY AND DUAL COALGEBRAS

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ABSTRACT. In arXiv:math/0606241v2 M. Kontsevich and Y. Soibelman argue that the category of noncommutative (thin) schemes is equivalent to the category of coalgebras. We propose that under this correspondence the affine scheme $\text{rep}(A)$ of a $k$-algebra $A$ is the dual coalgebra $A^o$ and draw some consequences. In particular, we describe the dual coalgebra $A^o$ of $A$ in terms of the $A_{\infty}$-structure on the Yoneda-space of all the simple finite dimensional $A$-representations.

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1. $\text{rep}(A) = A^o$

Throughout, $k$ will be a (commutative) field with separable closure $\bar{k}$. In [3, §I.2] Maxim Kontsevich and Yan Soibelman define a noncommutative thin scheme to be a covariant functor commuting with finite projective limits

$$X : \text{alg}_{fd}^k \longrightarrow \text{sets}$$

from the category $\text{alg}_{fd}^k$ of all finite dimensional $k$-algebras (associative with unit) to the category $\text{sets}$ of all sets. They prove [3, Thm. 2.1.1] that every noncommutative thin scheme is represented by a $k$-coalgebra.

Recall that a $k$-coalgebra is a $k$-vectorspace $C$ equipped with linear structural morphisms : a comultiplication $\Delta : C \longrightarrow C \otimes C$ and a counit $\epsilon : C \longrightarrow k$ satisfying the coassociativity $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$ and counitary property $(id \otimes \epsilon)\Delta = (\epsilon \otimes id)\Delta = id$.

By being representable they mean that every noncommutative thin scheme $X$ has associated to it a $k$-coalgebra $C_X$ with the property that for any finite dimensional $k$-algebra $B$ there is a natural one-to-one correspondence

$$X(B) = \text{alg}_k^k(B, C_X^*)$$

Here, for a $k$-coalgebra $C$ we denote by $C^*$ the space of linear functionals $\text{Hom}_k(C, k)$ which acquires a $k$-algebra structure by dualizing the structural coalgebra morphisms.

They call $C_X$ the coalgebra of distributions on $X$ and define the noncommutative algebra of functions on $X$ to be the dual $k$-algebra $k[X] = C_X^*$.

Whereas the dual $C^*$ of a $k$-coalgebra is always a $k$-algebra, for a $k$-algebra $A$ it is not true in general that the dual vectorspace $A^*$ is a coalgebra, because $(A \otimes A)^* \neq A^* \otimes A^*$. Still, one can define the subspace

$$A^o = \{ f \in A^* = \text{Hom}_k(A, k) \mid \ker(f) \text{ contains a twosided ideal of finite codimension} \}$$

and show that the duals of the structural morphisms on $A$ determine a $k$-coalgebra structure on this dual coalgebra $A^o$, see for example [5, Prop. 6.0.2].
With these definitions, Kostant duality asserts that the functors

\[ \text{alg}_k \overset{o}{\longrightarrow} \text{coalg}_k \]

are adjoint, [5 Thm. 6.0.5]. That is, for any \( k \)-algebra \( A \) and any \( k \)-coalgebra \( C \), there is a natural one-to-one correspondence between the homomorphisms

\[ \text{alg}_k(A, C^*) = \text{coalg}_k(C, A^o) \]

Moreover, we have [5 Lemma 6.0.1] that if \( f \in \text{alg}_k(A, B) \), the dual map \( f^* \) determines a \( k \)-coalgebra morphism \( f^* \in \text{coalg}_k(B^*, A^o) \).

For a \( k \)-algebra \( A \) one can define the contravariant functor \( \text{rep}(A) \) describing its finite dimensional representations [3 Example 2.1.9]

\[ \text{rep}(A) : \text{coalg}_k^{fd} \to \text{sets} \quad C \mapsto \text{alg}_k(A, C^*) \]

from finite dimensional \( k \)-coalgebras \( \text{coalg}_k^{fd} \) to \( \text{sets} \), which commutes with finite direct limits. As on finite dimensional \( k \)-(co)algebras Kostant duality is an anti-equivalence of categories

\[ \text{alg}_k^{fd} \overset{s}{\longrightarrow} \text{coalg}_k^{fd} \]

we might as well describe \( \text{rep}(A) \) as the noncommutative thin scheme represented by \( A^o \)

\[ \text{rep}(A) : \text{alg}_k^{fd} \to \text{sets} \quad B = C^* \mapsto \text{alg}_k(A, B = C^*) = \text{coalg}_k(C = B^*, A^o) \]

the latter equality follows again from Kostant duality. Therefore, we propose

**Definition 1.** The noncommutative affine scheme \( \text{rep}(A) \) is the noncommutative (thin) scheme corresponding to the dual \( k \)-coalgebra \( A^o \) of \( A \).

2. \( \text{simp}(A) = \text{corad}(A^o) \)

The dual \( k \)-coalgebra \( A^o \) is usually a huge object and hence contains a lot of information about the \( k \)-algebra \( A \). Let us begin by recalling how the geometry of a commutative affine \( k \)-scheme \( X \) is contained in the dual coalgebra \( A^o \) of its coordinate ring \( A = k[X] \).

Recall that a coalgebra \( D \) is said to be simple if it has no proper nontrivial subcoalgebras. In particular, a simple coalgebra \( D \) is finite dimensional over \( k \) and by duality is such that \( D^* \) is a simple \( k \)-algebra, that is, \( D^* \) is a central simple \( L \)-algebra where \( L \) is a finite separable extension of \( k \).

Hence, in case \( A = k[X] \) (and \( k \) is separably closed) we have that all simple subcoalgebras of \( A^o \) are one-dimensional (and hence are spanned by a group-like element), because they correspond to simple representations of \( A \).

That is, \( A^o \) is pointed and by [5 Prop. 8.0.7] we know that any cocommutative pointed coalgebra is the direct sum of its pointed irreducible components (at the algebra level, this says that a semi-local commutative algebra is the direct sum of locals). Therefore,

\[ A^o = \bigoplus_{x \in X} C_x \]

where each \( C_x \) is pointed irreducible and cocommutative. As such, each \( C_x \) is a subcoalgebra of the enveloping coalgebra of the abelian Lie algebra on the tangent space \( T_x(X) \). That is, we recover the points of \( X \) as well as tangent information from the dual coalgebra \( A^o \).

But then, the dual algebra of \( A^o \), that is the 'noncommutative' algebra of functions \( A^{o*} \) decomposes as

\[ A^{o*} = \bigoplus_{x \in X} \hat{O}_{x,X} \]

the direct sum of the completions of the local algebras at points. The diagonal embedding \( A = k[X] \to A^{o*} \) inevitably leads to the structure sheaf \( \hat{O}_X \).

We will now associate a topological space associated to any \( k \)-algebra \( A \), generalizing the space of points equipped with the Zariski topology when \( A \) is a commutative affine
In the next section we will describe the dual coalgebra $A^\circ$ when $A$ is a noncommutative affine $k$-algebra.

The coradical $corad(C)$ of a $k$-coalgebra $C$ is the (direct) sum of all simple subcoalgebras of $C$. It is also the direct sum of all simple subcomodules of $C$, when $C$ is viewed as a left (or right) $C$-comodule.

In the example above, when $A = k[X]$, we have that $corad(A^\circ) = \bigoplus_{x \in X} k ev_x$ where the group-like element $ev_x$ is evaluation in the point $x$. This motivates:

**Definition 2.** For a $k$-algebra $A$ we define the space of points simp$(A)$ to be the set of direct summands of $corad(rep(A)) = corad(A^\circ)$. That is, simp$(A)$ is the set of simple subcoalgebras of $rep(A)$.

By Kostant duality it follows that simp$(A)$ is the set of all finite dimensional simple algebra quotients of the $k$-algebra $A$, or equivalently, the set of all isomorphism classes of finite dimensional simple $A$-representations, explaining the notation.

We can equip this set with a Zariski topology in the usual way, using the evaluation map

$$A^\circ \times A \xrightarrow{ev} k \quad (f,a) \mapsto f(a)$$

when restricted to the subcoalgebra $corad(A^\circ)$. Note that the evaluation map actually defines a measuring of $A$ to $k$ [5 Prop. 7.0.3], that is, $A^\circ \otimes A \xrightarrow{ev} k$ satisfies

$$ev(f \otimes aa') = \sum_{(f)} f_{(1)}(a)f_{(2)}(a') \quad \text{and} \quad ev(f \otimes 1) = \epsilon(f)1_k$$

**Definition 3.** The Zariski topology of a $k$-algebra $A$ is the set simp$(A)$ equipped with the topology generated by the basic closed sets

$$\mathcal{V}(a) = \{S \in \text{simp}(A) \mid ev(S \otimes a) = 0, \text{ that is } f(a) = 0, \forall f \in S \}$$

Having associated a topological space to a $k$-algebra, one might ask when this is a functor. Functoriality has always been a problem in noncommutative geometry. Indeed, a simple $B$-representation does not have to remain a simple $A$-representation under restriction of scalars via $\phi : A \xrightarrow{} B$.

Still, if we define $rep(A) = A^\circ$, we get functionality for free. If $A \xrightarrow{\phi} B$ is an algebra morphism, we have seen that the dual map maps $B^\circ$ to $A^\circ$, so we have a morphism

$$B^\circ = rep(B) \xrightarrow{\phi^*} rep(A) = A^\circ$$

A coalgebra is the direct limit of its finite dimensional coalgebras, and they correspond under duality to finite dimensional algebras. Hence, $\phi^*$ is the natural map on finite dimensional representations by restriction of scalars.

The observed failure of functoriality on the level of points translates on the coalgebra-level to the fact that for a coalgebra map $B^\circ \longrightarrow A^\circ$ the coradical $corad(B^\circ)$ does not have to be mapped to $corad(A^\circ)$, in general.

However, when $corad(B^\circ)$ is cocommutative, we do have that $\phi^*(corad(B^\circ)) \subset corad(A^\circ)$ by [5 Thm. 9.1.4]. In particular, we recover the functor of points in commutative algebraic geometry.

Clearly, we still have $corad(B^\circ) \longrightarrow A^\circ$ in general. This corresponds to the fact that there is always a map simp$(B) \longrightarrow rep(A)$.

Next, let us turn to the algebra of functions on $rep(A)$. By definition we have

$$k[rep(A)] = A^{\text{op}}$$

and we can ask how this algebra relates to the algebra $A$.

In general, it is not true that $A \longrightarrow A^{\text{op}}$. This only holds when $A^{\text{op}}$ is dense in $A^{\text{op}}$ in which case the $k$-algebra is said to be proper, see [5 §6.1].

In the commutative case, when $A$ is a finitely generated $k$-algebra, then $A$ is indeed proper and this is a consequence of the Hilbert Nullstellensatz and the Krull intersection theorem.
When $A$ is noncommutative, this is no longer the case. For example, if $A = A_n(k)$ the Weyl algebra over a field of characteristic zero $k$, then $A$ is simple whence has no two-sided ideals of finite codimension. As a result $A^o = 0$! As our proposal for the noncommutative affine scheme $\text{rep}(A)$ is based on finite dimensional representations of $A$, it will not be suitable for $k$-algebras having few such representations.

3. THE DUAL COALGEBRA $A^o$

In general though, $A^o$ is a huge object, so it is very difficult to describe explicitly. In this section, we will begin to tame $A^o$ even when $A$ is noncommutative.

In order not to add extra problems, we will assume that $k$ is separably closed in this section. The general case can be recovered by taking $\text{Gal}(k/k)$-invariants (replacing quivers by species in the sequel).

Over a separably closed field $k$ all simple subcoalgebras are full matrix coalgebras $M_n(k)^*$, that is, $M_n(k)^* = \oplus_{1 \leq i,j \leq n} k e_{ij}$ with 
\[
\Delta(e_{ij}) = \sum_{k=1}^n e_{ik} \otimes e_{kj} \quad \text{and} \quad \epsilon(e_{ij}) = \delta_{ij}.
\]
Hence, $\text{corad}(A^o) = \oplus S M_{n_S}(k)^*$ where the sum is taken over all finite dimensional simple $A$-representations $S$, each having dimension $n_S$.

In algebra, one can resize idempotents by Morita-theory and hence replace full matrices by the basefield. In coalgebra-theory there is an analogous duality known as Takeuchi equivalence, see [6].

The isotypical decomposition of $\text{corad}(A^o)$ as an $A^o$-comodule is of the form $\oplus_S C_S^\otimes$, the sum again taken over all simple $A$-representations. Take the $A^o$-comodule $E = \oplus_S C_S$ and its coendomorphism coalgebra
\[
A^\dagger = \text{coend}^{A^o}(E)
\]
then Takeuchi-equivalence (see for example [1] §4, [5] and the references contained in this paper for more details) asserts that $A^o$ is Takeuchi-equivalent to the coalgebra $A^\dagger$ which is pointed, that is, $\text{corad}(A^\dagger) = k \text{simp}(A) = \oplus_S k g_S$ with one group-like element $g_S$ for every simple $A$-representation. Remains to describe the structure of the full basic coalgebra $A^\dagger$.

For a (possibly infinite) quiver $\vec{Q}$ we define the path coalgebra $k\vec{Q}$ to be the vectorspace $\oplus_p k p$ with basis all oriented paths $p$ in the quiver $\vec{Q}$ (including those of length zero, corresponding to the vertices) and with structural maps induced by
\[
\Delta(p) = \sum_{p = p' \circ p''} p' \otimes p'' \quad \text{and} \quad \epsilon(p) = \delta_{\lambda(p)}(p)
\]
where $p' \circ p''$ denotes the concatenation of the oriented paths $p'$ and $p''$ and where $\lambda(p)$ denotes the length of the path $p$. Hence, every vertex $v$ is a group-like element and for an arrow $\overrightarrow{a}$ we have $\Delta(a) = v \otimes a + a \otimes w$ and $\epsilon(a) = 0$, that is, arrows are skew-primitive elements.

For every natural number $i$, we define the $\text{ext}^i$-quiver $\text{ext}^i_A$ to have one vertex $v_S$ for every $S \in \text{simp}(A)$ and such that the number of arrows from $v_S$ to $v_T$ is equal to the dimension of the space $\text{Ext}^i_A(S,T)$. With $\text{ext}^i_A$ we denote the $k$-vectorspace on the arrows of $\text{ext}^i_A$.

The Yoneda-space $\text{ext}^1_A = \oplus \text{ext}^1_A$ is endowed with a natural $A_\infty$-structure [2], defining a linear map (the homotopy Maurer-Cartan map, [4])
\[
\mu = \oplus_i m_i : k \text{ext}^1_A \longrightarrow \text{ext}^2_A
\]
from the path coalgebra $k \text{ext}^1_A$ of the $\text{ext}^1$-quiver to the vectorspace $\text{ext}^2_A$, see [2] §2.2 and [4].

**Theorem 1.** The dual coalgebra $A^o$ is Takeuchi-equivalent to the pointed coalgebra $A^\dagger$ which is the sum of all subcoalgebras contained in the kernel of the linear map
\[
\mu = \oplus_i m_i : k \text{ext}^1_A \longrightarrow \text{ext}^2_A
\]
determined by the $A_\infty$-structure on the Yoneda-space $\text{ext}^1_A$. 

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We can reduce to finite subquivers as any subcoalgebra is the limit of finite dimensional subcoalgebras and because any finite dimensional $A$-representation involves only finitely many simples. Hence, the statement is a global version of the result on finite dimensional algebras due to B. Keller [2, §2.2].

Alternatively, we can use the results of E. Segal [4]. Let $S_1, \ldots, S_r$ be distinct simple finite dimensional $A$-representations and consider the semi-simple module $M = S_1 \oplus \ldots \oplus S_r$ which determines an algebra epimorphism

$$\pi_M : A \longrightarrow M_{n_1}(k) \oplus \ldots \oplus M_{n_r}(k) = B$$

If $m = \text{Ker}(\pi_M)$, then the $m$-adic completion $\hat{A}_m = \varprojlim A/m^n$ is an augmented $B$-algebra and we are done if we can describe its finite dimensional (nilpotent) representations. Again, consider the $A_\infty$-structure on the Yoneda-algebra $Ext^*_A(M, M)$ and the quiver on $r$-vertices $\text{Ext}^1_A(M, M)$ and the homotopy Mauer-Cartan map

$$\mu_M = \oplus_i m_i : \text{Ext}^1_A(M, M) \longrightarrow \text{Ext}^2_A(M, M)$$

Dualizing we get a subspace $\text{Im}(\mu_M^*)$ in the path-algebra $\text{Ext}^1_A(M, M)^*$ of the dual quiver. Ed Segal’s main result [4, Thm. 2.12] now asserts that $\hat{A}_m$ is Morita-equivalent to

$$\hat{A}_m \sim \frac{\text{Ext}^1_A(M, M)^*}{\text{Im}(\mu_M^*)}$$

where $(\text{Ext}^1_A(M, M)^*)$ is the completion of the path-algebra at the ideals generated by the paths of positive length. The statement above is the dual coalgebra version of this.

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