Physical observables of the Ising spin glass in $6 - \epsilon$ dimensions: asymptotical behavior around the critical fixed point

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The asymptotical behavior of physical quantities, like the order parameter, the replicon and longitudinal masses, is studied around the zero-field spin glass transition point when a small external magnetic field is applied. An effective field theory to model this asymptotics contains a small perturbation in its Lagrangian which breaks the zero-field symmetry. A first order renormalization group supplemented by perturbational results provides the scaling functions. The perturbative zero of the scaling function for the replicon mass defines a generic Almeida-Thouless surface stemming from the zero-field fixed point.

I. INTRODUCTION

Since the invention of the renormalization group (RG) by Wilson\cite{1}, replacing a statistical system (which is close to its critical state) by an effective field theory has become the basic analytical tool to calculate the asymptotical behavior of physical quantities around a critical point. Such an effective theory is defined by its Lagrangian $\mathcal{L}$, usually called the Landau-Ginzburg-Wilson (LGW) Lagrangian, which depends on the fluctuating order parameter components, the “fields”, the statistical weight of a configuration being $\sim e^{-\mathcal{L}}$. This formalism has been set up in the seventies of the last century for the prototype spin glass model of Edwards and Anderson (EA)\cite{2}, with an immediate application of the renormalization group\cite{3}. The EA model for $N$ Ising spins on a $d$-dimensional hypercubic lattice is defined by the Hamiltonian

$$\mathcal{H} = - \sum_{(ij)} J_{ij} s_i s_j - H \sum_i s_i$$

where the $J_{ij}$’s are independent, Gaussian distributed random variables with zero mean and variance $J^2$, and a homogeneous external magnetic field $H$ was also included. Summations are over nearest neighbour pairs $(ij)$ of lattice sites in the first sum, while over the $N$ lattice sites in the second one. Averages over the quenched disorder of the EA model are managed by the replica trick, and, as a result, the effective theory representing the lattice system close to criticality is a cubic replicated field theory with the fluctuating fields (in momentum space) $\phi^\alpha_p = \phi^\alpha_{p1}$ and $\phi^\beta_p = 0$ for $\alpha, \beta = 1 \ldots n$, with the replica number $n$ going to zero at the end of a calculation. Harris et. al\cite{4}, and later Refs.\cite{5} too, deduced the following LGW Lagrangian for the zero-external-field case, i.e. for $H = 0$:

$$\mathcal{L}_{\text{zero-field}} = \frac{1}{2} \sum_p \left( \frac{1}{2} \partial_p^2 + m \right) \sum_{\alpha\beta} \phi^\alpha_p \phi^\beta_{-p} - \frac{1}{6\sqrt{N}} \sum'_{p_1p_2p_3} w \sum_{\alpha\beta\gamma} \phi^\alpha_{p_1} \phi^\beta_{p_2} \phi^\gamma_{p_3} \tag{2}$$

Momentum conservation is indicated by the primed sum, and a continuum of $p$’s, cutoff at some $\Lambda$, results in the thermodynamic limit $N \to \infty$. Replica summations above and in the followings are unrestricted. For a nonzero magnetic field $H$ which is not necessarily small, the Lagrangian $\mathcal{L}$ gets additional replica symmetric (RS) invariants (i.e. homogeneous polynomials built up of the fields $\phi^\beta_p$’s which are invariant under any permutation of the $n$ replicas), see Ref\cite{5}, and the theory becomes the generic cubic RS field theory with $\mathcal{L} = \mathcal{L}_{\text{zero-field}} + \delta \mathcal{L}$, with $m$ and $w$ in $\mathcal{L}_{\text{zero-field}}$ replaced by $m_1 = m + \delta m_1$ and $w_1 = w + \delta w_1$, respectively, and

$$\delta \mathcal{L} = - \frac{1}{2} N^2 \hbar^2 \sum_{\alpha\beta} \phi^\alpha_0 + \frac{1}{2} \sum_p \left[ m_2 \sum_{\alpha\beta\gamma} \phi^\alpha_{p1} \phi^\beta_{p2} \phi^\gamma_{p3} + m_3 \sum_{\alpha\beta\gamma\delta} \phi^\alpha_{p1} \phi^\beta_{p2} \phi^\gamma_{p3} \phi^\delta_{p4} \right]$$

$$- \frac{1}{6\sqrt{N}} \sum'_{p_1p_2p_3} \left[ w_2 \sum_{\alpha\beta\gamma} \phi^\alpha_{p1} \phi^\beta_{p2} \phi^\gamma_{p3} + w_3 \sum_{\alpha\beta\gamma} \phi^\alpha_{p1} \phi^\beta_{p2} \phi^\gamma_{p3} + w_4 \sum_{\alpha\beta\gamma} \phi^\alpha_{p1} \phi^\beta_{p2} \phi^\gamma_{p3} \phi^\delta_{p4} + w_5 \sum_{\alpha\beta\gamma} \phi^\alpha_{p1} \phi^\beta_{p2} \phi^\gamma_{p3} + w_6 \sum_{\alpha\beta\gamma} \phi^\alpha_{p1} \phi^\beta_{p2} \phi^\gamma_{p3} \phi^\delta_{p4} \phi^\epsilon_{p5} \sum_{\alpha\beta\gamma} \phi^\alpha_{p1} \phi^\beta_{p2} \phi^\gamma_{p3} \phi^\delta_{p4} \phi^\epsilon_{p5} \phi^\sigma_{p6} \right]. \tag{3}$$

The zero-field Lagrangian of Eq. (2) contains RS invariants with all the replica indices occurring an even number of times, thus reflecting the spin inversion symmetry of the EA model without an external magnetic field. Although the
insertion of a small magnetic field breaks the spin inversion symmetry, and consequently the higher symmetry of the field theory with $\mathcal{L}_{\text{zero-field}}$, the generic RS field theory with all the coupling constants nonzero in (3) is redundant when the magnetic field is small. Accordingly, the first study of the Almeida-Thouless (AT) instability below 8 dimensions considered the simplest model with $h^2$ the only nonzero coupling in (3).

The SK model has the Hamiltonian (1) on the complete graph, i.e. $\sum_{(ij)}$ means summation over all the pairs, and the variance of the $J_{ij}$’s is $J^2/N$. With the notation of $E[\ldots]$ for the average over the $J_{ij}$’s, the quenched averaged replicated partition function of the SK model can be put into the form

$$E[Z_{\text{SK}}] \sim \int \left[ \prod_{(\alpha\beta)} dq_{\alpha\beta} \right] e^{-\mathcal{L}_{\text{SK}}}$$

with

$$\frac{1}{N} \mathcal{L}_{\text{SK}} = -\frac{1}{2} \bar{H}^2 \sum_{\alpha\beta} q_{\alpha\beta} + \frac{1}{2} (\bar{\tau} + \bar{H}^2) \sum_{\alpha\beta} q_{\alpha\beta}^2 - \frac{1}{2} \bar{H}^2 \sum_{\alpha\beta\gamma} q_{\alpha\beta\gamma}^2 - \frac{1}{6} (1 - 3 \bar{H}^2) \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta}^2 - \frac{1}{3} \bar{H}^2 \sum_{\alpha\beta} q_{\alpha\beta}^3 + \bar{H}^2 \sum_{\alpha\beta} q_{\alpha\beta}^2 q_{\beta\gamma}^2 - \frac{1}{2} \bar{H}^2 \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta} q_{\beta\delta} q_{\alpha\gamma} + O(\bar{H}^4, q^4)$$

(4)

where $\bar{\tau} \equiv \frac{1}{2} [1 - (J/kT)^{-2}]$ and $\bar{H} \equiv H/kT$. Stationarity of $\mathcal{L}_{\text{SK}}$ with respect to $q_{\alpha\beta}$ yields the order parameter in the thermodynamic limit.

In the case of the field theory, it is the Legendre-transformed free energy $F(q_{\alpha\beta})$ that is stationary in the equilibrium state. It is defined by the common rules of the Legendre transformation, namely

$$F(q_{\alpha\beta}) = -\ln Z(H_{\alpha\beta}) + N \sum_{(\alpha\beta)} H_{\alpha\beta} q_{\alpha\beta} \quad \text{and} \quad \frac{\partial \ln Z(H_{\alpha\beta})}{\partial H_{\alpha\beta}} = N q_{\alpha\beta}$$

where the partition function $Z(H_{\alpha\beta}) = \int D\phi e^{-\mathcal{L}}$ acquires its dependence on the $H_{\alpha\beta}$’s by adding a source term $-N^{\frac{1}{2}} \sum_{(\alpha\beta)} H_{\alpha\beta} \phi^\alpha_{\beta} = 0$ to the RS Lagrangian $\mathcal{L}_{\text{zero-field}} + \delta \mathcal{L}$. ($\sum_{(\alpha\beta)}$ in these formulas means summation over the $n(n-1)/2$ pairs of replicas.) Neglecting fluctuations of the fields (tree approximation), i.e. replacing $\langle \phi^\alpha_{\beta} \rangle = \delta_{\alpha\beta} / \sqrt{N}$, provides the mean field, or Landau, free energy of the model:

$$\frac{1}{N} F(q_{\alpha\beta}) = -\frac{1}{2} h^2 \sum_{\alpha\beta} q_{\alpha\beta}^2 + \frac{1}{2} \left[ (m + \delta m_1) \sum_{\alpha\beta} q_{\alpha\beta}^2 + m_2 \sum_{\alpha\beta\gamma} q_{\alpha\beta\gamma}^2 + m_3 \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta} \right]$$

$$- \frac{1}{6} \left[ (w + \delta w_1) \sum_{\alpha\beta\gamma} q_{\alpha\beta\gamma}^3 q_{\gamma\alpha} + w_2 \sum_{\alpha\beta\gamma} q_{\alpha\beta\gamma}^2 q_{\beta\gamma}^2 + w_3 \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta}^2 q_{\beta\delta}^2 + w_4 \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta} q_{\beta\delta} q_{\alpha\gamma} + w_5 \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta} q_{\beta\delta} q_{\alpha\gamma} q_{\beta\gamma} + w_6 \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta} q_{\alpha\gamma} q_{\beta\delta} + w_7 \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta} q_{\beta\delta} q_{\alpha\gamma} q_{\beta\gamma} + w_8 \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta} q_{\beta\delta} q_{\alpha\gamma} q_{\beta\gamma} \right] + O(q^4).$$

(5)

Comparing Eqs. (4) and (5), one can conclude for the bare couplings of the field theory:

• Writing $m = m_c - \tau$ with $\tau = 0$ at the critical point of the field theory, $m_c \sim (T_c^2 - T_{c\text{mf}}^2)$ results where $T_c$ and $T_{c\text{mf}}$ are the critical temperatures of the field theory and its mean field approximation, respectively. This shows that $m_c$ is one-loop order.

• The couplings $h^2, \delta m_1, m_2, \delta w_1, w_2, w_3,$ and $w_5$ are of order $\bar{H}^2$, whereas all the other couplings are of order $H^4$.
A simple three-parameter model was used in Ref. 11 to study, among other things, the AT instability for $6 < d < 8$, $d = 6$, and $d \lesssim 6$, the nonzero bare parameters were $m_1 = m = m_c - \tau$, $w_1 = w$, and $h^2$. It was found in Ref. 11 that the critical field $h^2_{\text{AT}}$ behaves continuously while crossing the upper critical dimension 6 for fixed values of the reduced-temperature-like parameter $\tau$ and cubic coupling $w$, and the AT line takes the simple form

$$h^2_{\text{AT}} \approx \frac{4}{(1 - w^2 \ln \tau)^4} w \tau^2,$$

valid if $\tau \ll 1$ and $w^2 \ll 1$, in exactly the upper critical dimension. As the main motivation of the present paper, we want to check whether a suitable extension of this simple model in such a way that, beside $h^2$, the bare couplings $m_2$, $w_2$, $w_3$, and $w_5$ are small but nonzero too, will or will not modify the results of Ref. 11 about the AT instability around the zero-field critical point, and for $d \lesssim 6$. In the dimensional regime $6 < d < 8$ where a standard perturbational method is applicable, an extended parameter space with all the couplings which are of order $H^2$ seems to be a convenient extension. Below six dimensions, however, where the simple perturbative method breaks down (due to the more and more infrared divergent graphs as the number of loops increases), it becomes inevitable to apply the RG for the calculation of the asymptotical behavior of physical quantities. In this case, however, it is difficult to define the model by the set of bare couplings (by those, for instance, which are at least of order $H$), as new couplings will be generated by the RG flow.

In the present paper, we propose to define the model by the set of nonlinear scaling fields: this ensures the closeness of the model under RG flow. The simple three-parameter model of Ref. 11 can be formulated in this way, and its extension will be done by introducing a new (mass-like) nonlinear scaling field which, on the level of the bare couplings, leads to a more complicated model. In this more complicated field theory, one can calculate in $6 - \epsilon$ dimensions the RS order parameter, the replicon and longitudinal masses; all in the framework of first order RG combined with perturbational analysis. We focus on the asymptotical behavior close to the zero-field critical fixed point. The perturbative zero of the replicon mass defines the onset of the instability of the RS phase (AT surface). The problem of the runaway RG flows along this partially massless, i.e. massless in the replicon sector, manifold (caused by the repulsion of the critical fixed point) is also discussed.

The outline of the paper is as follows: The method of using nonlinear scaling fields for the calculation of physical quantities below 6 dimensions is discussed in Section II. The results in this section are equivalently valid below and above the critical temperature. The free propagators (replicon and longitudinal) are constructed in Section III. The central part of the paper is Section IV where the critical asymptotics of physical quantities, such as the order parameter, the replicon and longitudinal masses, are elaborated. The more interesting case of $T < T_c$ is presented in subsection IV A whereas results for $T > T_c$ are also displayed for the sake of completeness and comparison in IV B. The limitations of the various approximations used to achieve our results are discussed in some details in the next section. Zeros of the replicon mass are found in a region of the parameter space around the critical fixed point which belongs to the range of applicability of our approximations. There is also a discussion of this Almeida-Thouless critical manifold in Section V. Some conclusive remarks and a paragraph about the applied perturbative method are left to Section VI. The basic perturbative formulas are displayed in the Appendix.

Many results in this paper, especially the connection between bare parameters and nonlinear scaling fields in Section II, are built upon the first order RG equations of Ref. 12.

II. BELOW 6 DIMENSIONS

The RG equations for the generic cubic field theory defined in Eqs. 2 and 3 can be obtained by integrating out degrees of freedom in a momentum shell at the cutoff $\Lambda$. The structure of these flow equations in the one-loop approximation, and for $n = 0$, can be written as:

$$\dot{h}^2 = \left[ 4 - \frac{\epsilon}{2} - H^{(2)}(m_1, m_2, m_3; w_1, \ldots, w_8) \right] h^2 + H^{(1)}(m_1, m_2, m_3; w_1, \ldots, w_8);$$

$$\dot{m}_i = 2 m_i + M_i^{(2)}(m_1, m_2, m_3; w_1, \ldots, w_8), \quad i = 1, 2, 3;$$

$$\dot{w}_i = \frac{\epsilon}{2} w_i + W_i^{(3)}(m_1, m_2, m_3; w_1, \ldots, w_8), \quad i = 1, \ldots, 8.$$  

The functions $F^{(k)}(m_1, m_2, m_3; w_1, \ldots, w_8)$ above (with $F = H$, $M_i$, or $W_i$) are homogeneous polynomials of degree $k$ in the $w$’s, while analytic in the masses with a nonzero value for $m_1 = m_2 = m_3 = 0$. All but the first equations in
has been published in Ref.\textsuperscript{12}, although the set of bare couplings was chosen differently there.\textsuperscript{1} The flow equation for the magnetic field in the generic case, however, has not been published before:

$$h^2 = \left( 4 - \frac{\epsilon}{2} - \frac{1}{2} \eta_L \right) h^2 + (3g_3 + 3g_6 + 2\bar{g}_7) \frac{1}{1 + 2m_1} + (3g_6 + 2\bar{g}_7) \frac{2m_2}{(1 + 2m_1)(1 + 2m_1 - 2m_2)} - 2g_6 \frac{m_2 - 2m_3}{(1 + 2m_1 - 2m_2)^2},$$

with

$$\eta_L = 2g_3^2 \frac{1 + 6m_1}{(1 + 2m_1)^4} - \frac{8}{3} \left( g_6^2 + g_6\bar{g}_7 \right) \frac{1 + 6m_1 - 6m_2}{(1 + 2m_1 - 2m_2)^4} + \frac{4}{3} g_6^2 \frac{m_2 - 2m_3}{(1 + 2m_1 - 2m_2)^5} \left( 1 + 18m_1 - 18m_2 \right) \tag{8}$$

where we adopted the notations from Ref.\textsuperscript{12} 2:

$$g_3 \equiv -w_1 + w_2 - \frac{1}{3} w_3,$$

$$g_6 \equiv 2w_1 - w_2 + w_3 - w_5 - w_6,$$

$$\bar{g}_7 \equiv -\frac{3}{2} w_1 + \frac{1}{2} w_2 - \frac{5}{6} w_3 + \frac{2}{3} w_4 + \frac{4}{3} w_5 + w_6 - \frac{2}{3} w_7.$$

One can benefit the following information from the RG equations \textsuperscript{7}:

- The zeros of the right-hand-side provide the fixed points. In this paper, we are interested in the vicinity of the zero-field critical fixed point: $w^{*2} = \frac{1}{2} \epsilon$, $m^{*} = -\frac{1}{2} w^{*2} = -\frac{1}{4} \epsilon$, and all the other couplings being zero. We prefer using $2w^{*2}$, instead of $\epsilon$, in the remainder part of the paper.

- All the eigenmodes of the linearized RG equations, with the only exception of that belonging to $h^2$, were published in Ref.\textsuperscript{12}. In this paper we restrict ourself to a model with the following four modes:

$$g_{h^2} \quad \text{with} \quad \lambda_{h^2} = 4 - \frac{2}{3} w^{*2}, \quad g_{m_1} \quad \text{with} \quad \lambda_{m_1} = 2 - \frac{10}{3} w^{*2}, \quad g_{m_2} \quad \text{with} \quad \lambda_{m_2} = 2 - \frac{4}{3} w^{*2},$$

and $g_w$ with $\lambda_w = -2w^{*2}$.\textsuperscript{10}

- The $g$’s above, with subscripts $h^2$, $m_1$, $m_2$, and $w$ referring the modes they belong to, are nonlinear scaling fields\textsuperscript{3} which satisfy exactly, by definition, the equations $\dot{g} = \lambda g$ and are zero at the fixed point. By means of the RG equations \textsuperscript{7} above, one can express the original bare couplings in terms of the $g$’s. Keeping the fields which break the zero-field symmetry (i.e. $g_{h^2}$ and $g_{m_2}$) linear in these expressions (which is sufficient for a small

\textsuperscript{1} For the sake of easing the reader, we give here the precise citations where the linear connection between the two sets of couplings can be found: For the masses (i.e. $m_R$, $m_A$, and $m_L$ in Ref.\textsuperscript{12} versus $m_1$, $m_2$, and $m_3$ here) see Eqs. (32) $\alpha$ Ref.\textsuperscript{12} and Eqs. (22-24) $\alpha$ Ref.\textsuperscript{12}, whereas for the cubic couplings (i.e. $g_i$, $i = 1\ldots8$ in Ref.\textsuperscript{12} versus $w_i$, $i = 1\ldots8$ here) see Eqs. (49a-h) $\alpha$ Ref.\textsuperscript{12}.

\textsuperscript{2} The last term in $\eta_L = \eta_A$ is wrongly missing in Eq. (87) of Ref.\textsuperscript{12}. A similar term proportional to $m_2 - 2m_3$ was also left out from the expression Eq. (86) for $\eta_R$.

\textsuperscript{3} The general theory of the application of nonlinear scaling fields was briefly summarized in Sec. 5.1 of Ref.\textsuperscript{12}. The concept of nonlinear scaling fields was introduced by Wegner\textsuperscript{12}.}
external field), only the following couplings in $\delta \mathcal{L}$ are generated:

$$w^* h^2 = \left(1 - \frac{1}{3} g_w - \frac{1}{3} w^* g_{m_1} \right) g h^2 + \left(-w^* - \frac{7}{3} w^* g_w + 2 g_{m_2} \right) g_{m_2},$$

$$m_2 = \left(1 + \frac{4}{3} g_w + 5 w^* g_{m_1} \right) g_{m_2},$$

$$w_2/w^* = \left(-12 w^* - 52 w^* g_w + 48 w^* g_{m_1} \right) g_{m_2},$$

$$w_3/w^* = \left(\frac{49}{2} w^* + \frac{637}{6} w^* g_w - 94 w^* g_{m_1} \right) g_{m_2},$$

$$w_4/w^* = \left(-\frac{9}{2} w^* - \frac{39}{2} w^* g_w + 18 w^* g_{m_1} \right) g_{m_2},$$

$$w_5/w^* = \left(\frac{1}{2} w^* - \frac{13}{6} w^* g_w - 2 w^* g_{m_1} \right) g_{m_2};$$

whereas the symmetric couplings $m_1$ and $w_1$ are

$$m_1 - m^* = \left[g_{m_1} - w^* g_w + \frac{10}{3} g_{m_1} g_w - 2 w^* g_w^2 + \frac{16}{3} w^* g_{m_1}^2 \right] + \left(-1 - \frac{4}{3} g_w - 5 w^* g_{m_1} \right) g_{m_2},$$

$$w_1/w^* - 1 = \left[5 w^* g_{m_1} + g_w + \frac{190}{6} w^* g_{m_1} g_w + \frac{3}{2} g_w - 14 w^* g_{m_1} \right] + \left(1 + \frac{1}{2} w^* + \frac{13}{6} w^* g_w + 2 w^* g_{m_1} \right) g_{m_2}.$$

(The zero-field-symmetric part above has been written up to quadratic order in $g_{m_1}$ and $g_w$.)

The three-parameter model of Ref. 11 corresponds to the three scaling fields: $g_{m_1}$ and $g_w$ span the symmetric (zero-field) system, whereas $g_{h^2}$ breaks this symmetry. Having a look at Eqs. (11) and (12), one can realize that $h^2$ is the only coupling of the symmetry breaking part $\delta \mathcal{L}$ which is generated. Therefore, this model can be equivalently defined by the bare couplings $m_1 = m$, $w_1 = w$, and $h^2$.

In the present paper, we supplement the model by $g_{m_2}$, whose introduction considerably complicates the model when it is written as in Eqs. (2) and (3). (One cannot avoid using this representation when, for instance, a scaling function is to be calculated.)

The following couplings enter for a small $g_{m_2}$, according to Eqs. (11) and (12): $\delta_{m_1}$, $\delta_{w_1}$, $m_2$, $w_2$, $w_3$, $w_4$, and $w_5$.

Any observable $\mathcal{O}$ can now be considered as depending on the four scaling fields, and according to the generic theory in Sec. 5.1 of Ref. 11, one can write the following asymptotically exact expression around the fixed point:

$$\mathcal{O}(g_{m_1}, g_w; g_{h^2}, g_{m_2}) = |g_{m_1}|^{\frac{\lambda_{m_1}}{\lambda_{m_1}}} \hat{\mathcal{O}}(x, y, z)$$

$$\times \left[1 + \frac{k_{m_1}}{\lambda_{m_1}} g_{m_1} + \frac{k_w}{\lambda_w} |g_{m_1}|^{\frac{\lambda_w}{\lambda_{m_1}}} x + \frac{k_{h^2}}{\lambda_{h^2}} |g_{m_1}|^{\frac{\lambda_{h^2}}{\lambda_{m_1}}} y + \frac{k_{m_2}}{\lambda_{m_2}} |g_{m_1}|^{\frac{\lambda_{m_2}}{\lambda_{m_1}}} z + \ldots \right],$$

where the RG invariants are defined as

$$x \equiv g_w |g_{m_1}|^{\frac{\lambda_w}{\lambda_{m_1}}}, \quad y \equiv g_{h^2} |g_{m_1}|^{\frac{\lambda_{h^2}}{\lambda_{m_1}}}, \quad z \equiv g_{m_2} |g_{m_1}|^{\frac{\lambda_{m_2}}{\lambda_{m_1}}}.$$

The $\ldots$ symbol means neglected terms, namely higher powers of the temperature-like field $g_{m_1}$ and/or quadratic or higher order monomials of the RG invariants. The $k$'s above are defined for a given $\mathcal{O}$ by the RG flow of it as

$$\hat{\mathcal{O}} = (k + k_{m_1} g_{m_1} + k_w g_w + k_{h^2} g_{h^2} + k_{m_2} g_{m_2} + \ldots) \mathcal{O}.\tag{15}$$

The scaling function $\hat{\mathcal{O}}$ is not determined by the renormalization group, but auxiliary information is needed (perturbative method, for instance) to compute it. Hereinafter we study three observables: the RS order parameter $q$, the replicon and longitudinal masses, i.e. $\Gamma_R$ and $\Gamma_L$. 
III. FREE PROPAGATORS OF THE MODEL

When the order parameter $q$ is nonzero, a reorganization of the perturbational series by the shift $\phi^{\alpha\beta}_p \to \phi^{\alpha\beta}_p - \sqrt{N} q \delta^{Kr}_{p=0}$ of the fluctuating fields is useful, as one gets then rid of “tadpole” insertions. As a result, the bare magnetic field and the masses suffer similar shifts:

\[ h^2 \to \tilde{h}^2 = h^2 + (-2m_1 + 2m_2) q + (-2w_1 + w_2 - w_3 + w_5) q^2, \]
\[ m_1 \to \tilde{m}_1 = m_1 + \left( w_1 - w_2 + \frac{1}{3} w_3 \right) q, \]
\[ m_2 \to \tilde{m}_2 = m_2 + \left( -w_1 - \frac{2}{3} w_3 + w_5 \right) q, \]
\[ m_3 \to \tilde{m}_3 = m_3 + \left( -\frac{2}{3} w_4 - \frac{1}{3} w_5 \right) q. \]

In the $n \to 0$ limit, two free propagators emerge in the generic RS theory, namely

\[ \tilde{G}_R = \frac{1}{p^2 + 2\tilde{m}_1}, \text{ the replicon propagator, and } \tilde{G}_L = \frac{1}{p^2 + 2\tilde{m}_1 - 2\tilde{m}_2}, \text{ the longitudinal propagator.} \]

Any perturbative contribution for some observable will, therefore, depend on $q$ which must be computed from the equation of state, i.e. from the condition $\langle \phi^{Kr}_p \rangle = 0$. For the free propagators, we need the tree (zero-loop) approximation of this equation:

\[ 2w^* q = h^2 q^{-1} - 2m_1 + 2m_2 + [ -2(w_1 - w^*) + w_2 - w_3 + w_5 ] q. \]

Using Eqs. (11) and (12) together with the definitions of the RG invariants in (14), the zero-loop order parameter follows, up to first order in $x$, $y$, and $z$, as:

\[ w^* q = |g_{m_1}| \times \begin{cases} 1 + \frac{7}{3} x + z + \frac{1}{2} y & \text{if } g_{m_1} < 0, \text{ i.e. } T < T_c \\ z + \frac{1}{2} y & \text{if } g_{m_1} > 0, \text{ i.e. } T > T_c. \end{cases} \]

This is the point where the calculations above and below $T_c$ separate. Writing the free propagators as

\[ \tilde{G}_R = \frac{1}{p^2 + |g_{m_1}| \times R}, \text{ and } \tilde{G}_L = \frac{1}{p^2 + |g_{m_1}| \times L}, \] (16)

it is obtained in the two respective regimes:

- $T < T_c$:
  \[ R = y \quad \text{and} \quad L = 2 + \frac{20}{3} x + 2y, \quad g_{m_1} < 0; \] (17)

- $T > T_c$:
  \[ R = 2 + \frac{20}{3} x + y \quad \text{and} \quad L = 2 + \frac{20}{3} x + 2y, \quad g_{m_1} > 0. \] (18)

Higher than first order terms in $x$, $y$, and $z$ are again neglected in the above formulas, in accordance with the smallness of these RG invariants.

IV. ASYMPTOTICAL BEHAVIOR AROUND $T_c$

A. Below $T_c$ ($g_{m_1} < 0$)

1. The order parameter $q$

The RG flow for $q$ is simply $\dot{q} = (2 - w^*^2 + \eta_L / 2) q$ with $\eta_L$ in (9). Inserting the nonlinear scaling fields by the help of (11) and (12), the $k$ coefficients for $q$ can be read off by the general definition in (13): $k = 2 - \frac{4}{3} w^*^2$, $k_{m_1} = \frac{2}{3} w^*^2$, $k_{m_2} = \frac{4}{3} w^*^2$.  


\(k_w = -\frac{2}{3} w^{*2}, k_{h^2} = 0, \) and \(k_{m_2} = -\frac{4}{3} w^{*2}.\) Using the eigenvalues of the various modes from (10), the generic scaling form in (13) becomes

\[
q = |g_{m_1}|^{1+w^{*2}} \hat{q}(x,y,z) \times \left[ 1 + \frac{1}{3} w^{*2} g_{m_1} + \frac{1}{3} |g_{m_1}|^{-w^{*2}} x - \frac{1}{3} w^{*2} |g_{m_1}|^{1+w^{*2}} z + \ldots \right].
\]  

(19)

Comparing this RG formula with its perturbative counterpart\(^4\)

\[
w^*q = |g_{m_1}| \left\{ \left[ 1 + (2 + \ln 2) w^{*2} \right] + \frac{1}{2} \left[ 1 + (4 - 2 \ln 2) w^{*2} \right] y + \frac{7}{3} \left[ 1 + O(w^{*2}) \right] x + \left[ 1 + O(w^{*2}) \right] z \right\}
\]

\[
+ w^{*2} (|g_{m_1}| \ln |g_{m_1}|) \left( 1 + \frac{1}{2} y + 2x + z \right)
\]

(20)

[which follows from Eq. (A1) by using of (11), (12), (10), (14), (A2), and (17)], has a double use: Firstly, the scaling function can be derived as

\[
w^*\hat{q} = \left[ 1 + (2 + \ln 2) w^{*2} \right] + \frac{1}{2} \left[ 1 + (4 - 2 \ln 2) w^{*2} \right] y + \left[ 2 + O(w^{*2}) \right] x + \left[ 1 + O(w^{*2}) \right] z.
\]  

(21)

Secondly, the logarithm in (20) should correctly exponentiate in accordance with the asymptotic scaling above: this property is easily checked by comparison.

2. The replicon mass

The replicon mass satisfies the equation \(\Gamma_R = (2 - \eta_R) \Gamma_R,\) with \(\eta_R\) computed in Ref.\(^{12}\) (See also footnote 2.) Instead of providing the complete formula for \(\eta_R\) here again, we show it expressed and linearly truncated in terms of the nonlinear scaling fields:

\[
\eta_R = -\frac{2}{3} w^{*2} (1 - 2g_{m_1} + g_{m_2} + 2g_w).
\]

The \(k\) coefficients (of \(\Gamma_R\)) follow then by (15): \(k = 2 + \frac{4}{3} w^{*2}, k_{m_1} = -\frac{4}{3} w^{*2}, k_w = \frac{4}{3} w^{*2}, k_{h^2} = 0, \) and \(k_{m_2} = \frac{4}{3} w^{*2}.\)

The generic result (13) can then be translated to the case of the replicon mass, see also (10), as

\[
\Gamma_R = |g_{m_1}|^{1+2w^{*2}} \hat{\Gamma}_R(x,y,z) \times \left[ 1 - \frac{2}{3} w^{*2} g_{m_1} - \frac{2}{3} |g_{m_1}|^{-w^{*2}} x + \frac{1}{3} w^{*2} |g_{m_1}|^{1+w^{*2}} z + \ldots \right].
\]  

(22)

The corresponding perturbative formula follows from (A3) and the use of Eqs. (11), (12), (10), (14), (A4), and (17):

\[
\Gamma_R = |g_{m_1}| \left\{ -4w^{*2} + \left[ 1 + (-8 + 3 \ln 2 - 4 \ln y) w^{*2} \right] y + O(w^{*2}) x + O(w^{*2}) z \right\} + w^{*2} (|g_{m_1}| \ln |g_{m_1}|) 2y.
\]  

(23)

Matching these two expressions of the replicon mass provides the scaling function:

\[
\hat{\Gamma}_R(x,y,z) = -4w^{*2} + \left[ 1 + (-8 + 3 \ln 2 - 4 \ln y) w^{*2} \right] y + O(w^{*2}) x + O(w^{*2}) z,
\]  

(24)

and it is easy to check that the criterion of proper exponentiation is satisfied.

3. The longitudinal mass

The \(k\) coefficients, defined in (15), for \(\Gamma_L\) follow from its RG equation \(\hat{\Gamma}_L = (2 - \eta_L) \Gamma_L\) and Eqs. (9), (11), and (12): \(k = 2 + \frac{4}{3} w^{*2}, k_{m_1} = -\frac{4}{3} w^{*2}, k_w = \frac{4}{3} w^{*2}, k_{h^2} = 0, \) and \(k_{m_2} = \frac{4}{3} w^{*2}.\) Just as for the replicon case, one can easily conclude the scaling form of the longitudinal mass as

\[
\Gamma_L = |g_{m_1}|^{1+2w^{*2}} \hat{\Gamma}_L(x,y,z) \times \left[ 1 - \frac{2}{3} w^{*2} g_{m_1} - \frac{2}{3} |g_{m_1}|^{-w^{*2}} x + \frac{2}{3} w^{*2} |g_{m_1}|^{1+w^{*2}} z + \ldots \right]
\]  

(25)

\(^4\) The notation \(O(w^{*2})\) will be consistently used in this section whenever the corresponding correction is not available in this first order RG calculation.
which can be confronted with (A5) of the Appendix:

\[
\Gamma_L = |g_{m_1}| \left\{ \left[2 + (-8 + 4 \ln 2) w^* z \right] + \left[2 + (1 + 4 \ln 2 - 6 \ln y) w^* z \right] y + \frac{20}{3} \left[1 + O(w^* z) \right] x + O(w^* z) \right\} \\
+ 4 w^* (|g_{m_1}| \ln |g_{m_1}|) \left(1 + y + \frac{11}{3} x \right). 
\]

(26)

[Use of Eqs. (11), (12), (10), (14), (A6), and (17) is necessary to put (A5) into this form.] The scaling function can now be read off as

\[
\hat{\Gamma}_L(x, y, z) = \left[2 + (-8 + 4 \ln 2) w^* z \right] + \left[2 + (1 + 4 \ln 2 - 6 \ln y) w^* z \right] y + \left[8 + O(w^* z) \right] x + O(w^* z) z, 
\]

(27)

and exponentiation can be checked.

### B. Results for \(T_c > 0 \ (g_{m_1} > 0)\)

For the sake of completeness and a possible comparison with the \(T_c < 0\) case, results for the three observables above the critical temperature (in a small but finite magnetic field) are presented in this subsection. Their scaling forms in Eqs. (19), (22), and (25) are equally valid in this high temperature asymptotical regime, the scaling functions, however, are different. Due to the change of the free propagators according to (16) and (18), the one-loop perturbative results are now [instead of (20), (23), and (26)]:

\[
w^* q = g_{m_1} \left\{ \frac{1}{2} \left[1 - (1 + 2 \ln 2) w^* z \right] y + \left[1 + O(w^* z) \right] z \right\} + w^* (g_{m_1} \ln g_{m_1}) \left(1 + y + z \right), 
\]

\[
\Gamma_R = g_{m_1} \left\{ 2 \left[1 + (1 + 2 \ln 2) w^* z \right] + \frac{1}{2} \left[2 + (1 - 2 \ln 2) w^* z \right] y + \left[\frac{20}{3} + O(w^* z) \right] x + O(w^* z) \right\} \\
+ w^* (g_{m_1} \ln g_{m_1}) \left[4 + 2y + \frac{44}{3} x \right], 
\]

\[
\Gamma_L = g_{m_1} \left\{ 2 \left[1 + (1 + 2 \ln 2) w^* z \right] + \frac{1}{2} \left[4 + (5 + 2 \ln 2) w^* z \right] y + \left[\frac{20}{3} + O(w^* z) \right] x + O(w^* z) \right\} \\
+ w^* (g_{m_1} \ln g_{m_1}) \left[4 + 4y + \frac{44}{3} x \right]. 
\]

Comparing with the scaling forms in Eqs. (19), (22), and (25), the scaling functions above the critical temperature can be concluded:

\[
w^* \hat{q} = \frac{1}{2} \left[1 - (1 + 2 \ln 2) w^* z \right] y + \left[1 + O(w^* z) \right] z, 
\]

(28)

\[
\hat{\Gamma}_R(x, y, z) = 2 \left[1 + (1 + 2 \ln 2) w^* z \right] + \frac{1}{2} \left[1 - (1 - 2 \ln 2) w^* z \right] y + \left[8 + O(w^* z) \right] x + O(w^* z) z, 
\]

(29)

\[
\hat{\Gamma}_L(x, y, z) = 2 \left[1 + (1 + 2 \ln 2) w^* z \right] + \frac{1}{2} \left[5 + (5 + 2 \ln 2) w^* z \right] y + \left[8 + O(w^* z) \right] x + O(w^* z) z. 
\]

(30)

One can make the following observations about the behavior of the three quantities around the critical point:

- The high-temperature \((g_{m_1} > 0)\) and zero-external-magnetic-field \((y = z = 0)\) phase possesses a higher symmetry with zero order parameter [see (23)] and a single mass [due to the degeneration between the replicon and longitudinal masses, see Eqs. (29) and (30)].

- In zero-external-magnetic-field \((y = z = 0)\) below the critical temperature \((g_{m_1} < 0)\), the order parameter is nonzero [Eq. (21)]: this is the RS spin glass phase invented by Edwards and Anderson. However, according to Eq. (24), the replicon mass gets negative due to the one-loop term, showing that this phase is unstable, just as in mean field theory, and replica symmetry must be broken.
• There is a slight splitting between the replicon and longitudinal masses in a small magnetic field above $T_c$, Eqs. [29] and [30], whereas the longitudinal mass is definitely massive below $T_c$, Eq. (27), and therefore separates from the replicon one.

• It is obvious from Eq. (24) that stability of the RS phase is restored for $y > y_0 \sim O(w^2)$ and $g_{m_1} < 0$.

V. DISCUSSION: RANGE OF APPLICABILITY AND ASYMPTOTICALLY DETECTED ALMEIDA-THOULESS INSTABILITY

In deriving our basic results for the scaling forms and scaling functions of the three observables ($q$, $\Gamma_R$, and $\Gamma_L$), several approximations were applied in the previous section. For seeing clearly the limits of these approximations, it might be useful to give an overall list of them here:

• The RG equations and the auxiliary perturbative calculations have the one-loop character, and therefore $w^* = \epsilon/2 \ll 1$.

• The multiplicative factor (which is analytic in the fields $g_{m_1}$, $g_w$, $g_{h^2}$, and $g_{m_2}$) in the, in principle exact, scaling formula of Eq. (13) was truncated to linear order in the nonlinear scaling fields. We must have, therefore, $|g_{m_1}|$, $|g_w|$, $|g_{m_2}|$, and $|g_{h^2}|$ much smaller than unity. In fact, the normalization of the nonlinear scaling fields (which is not fixed originally) was chosen in such a way that their asymptotic regime around the fixed point be independent of $\epsilon$.

• Quadratic and higher order terms in the RG invariants were neglected in the scaling functions, i.e. $|x| \ll 1$, $|z| \ll 1$, and $y \ll 1$. The first of them is automatically fulfilled if $|g_w| \ll 1$, since $g_w$ is an irrelevant field. The other two fields are relevant and, therefore, we have the stronger conditions $|g_{m_2}| \ll |g_{m_1}|^{\lambda_{h^2}^{1/\Delta_1}}$ and $g_{h^2} \ll |g_{m_1}|^{\lambda_{h^2}^{1/\Delta_1}}$.

• Up to this point, we have conditions for the parameters of the effective field theory representing the physical spin glass. Translating the above results as a requirement between temperature and magnetic field, we observe that $|g_{m_1}|$ is proportional to the reduced temperature, $g_{h^2} \approx w^* h^2 \sim H^2$, and $|g_{m_2}| \approx |m_2| \sim H^2$; see Eqs. (1) and (5). As $\lambda_{h^2}$ is the leading relevant eigenvalue, we arrive at

$$H^2 \ll |g_{m_1}|^{\lambda_{h^2}^{1/\Delta_1}} \sim \frac{T - T_c}{T_c} \frac{\lambda_{h^2}^{1/\Delta_1}}{\lambda_{m_1}^{1/\Delta_1}}.$$ 

An important consequence of the above analysis is that the ratio $z/y$ is independent of $H^2$ and $z \ll y$: this justifies the simple three parameter model in with the fields $|g_{m_1}|$, $g_w$, and $g_{h^2}$ (or equivalently $m_1$, $w_1$, and $h^2$). Anyway, $z$ entered only the scaling function for $q$ in (24).

The scaling functions in Eqs. (24), (21), and (27) for $T < T_c$ [and also Eqs. (28), (29), and (30) for $T < T_c$] constitute our basic result: they are the leading part of a perturbative series, and one could calculate, in principle, any higher order terms in $\epsilon$ and/or in the invariants (say $y$). These series belong completely to the critical fixed point, in other words: they are characteristics of the zero-magnetic-field fixed point. Their validity is, therefore, independent of the fate of the relevant couplings (like $h^2$, $m_2$, and $w_i$, $i = 2, \ldots, 5$) under the iteration of the renormalization group, i.e. whether they approach an other fixed point (perturbative or nonperturbative) or flow away to infinity.

As a matter of fact, the question is that what information can you extract from these perturbative series. Let’s make this point clearer by the case of the longitudinal mass in (27). (For the sake of simplicity, invariants other than $y$ are neglected in the following discussion.) $\tilde{\Gamma}_L$ is positive for $y = 0$, i.e. the zero-field spin glass phase is longitudinally massive. Although, it is physically plausible that $\Gamma_L$ remains massive in an external field too, this cannot be verified by (27) (or from a longer series), as a nonperturbative zero of $\tilde{\Gamma}_L$ is not available from such a series. The situation is fundamentally different for the replicon mass $\tilde{\Gamma}_R(y)$ below $T_c$, as it has a perturbative zero: $y_0 = \frac{4w^*}{4w^*} + \ldots$, whereas the longitudinal mode is massive, $\Gamma_L = 2 + 4 \ln 2 w^* + \ldots$, along this Almeida-Thouless instability surface. $\tilde{\Gamma}_R(y)$ will probably be singular at this zero:

$$\tilde{\Gamma}_R(y) \sim (y - y_0)^{\tilde{\gamma}},$$

with some exponent. This asymptotic form, however, cannot be verified from the series (24) due to the lack of proper exponentiation. The exponent $\tilde{\gamma}$ cannot be extracted from (24), as it does not belong to the critical fixed point, but possibly to some, at this moment unknown, zero-temperature fixed point. (The scenario drafted above follows closely the crossover behavior at a bicritical point presented in Ref. [14].)
VI. FINAL REMARKS

It has been shown in the preceding sections how one can detect the critical surface with zero replicon mass (the Almeida-Thouless critical manifold) asymptotically in the close vicinity of the zero-magnetic-field fixed point perturbatively just above the upper critical dimension. Nevertheless, this AT critical surface is spanned by relevant scaling fields like \( g_{h2} \) and \( g_{m2} \), which break the symmetry of the critical zero-magnetic-field fixed point, and runaway RG flows toward infinite couplings follow\(^{10} \). The lack of an attractive perturbative fixed point governing the AT instability surface\(^{12} \) and the runaway flows can be understood by the schematic phase diagrams from Refs.\(^{11,16} \): RG flows along the AT line terminate into a zero-temperature fixed point, and the effective cubic field theory (fitted to the asymptotics around the zero-magnetic-field critical transition) is, in fact, not appropriate for representing the zero-temperature spin glass. A field theory for the low-temperature spin glass is obviously sorely needed for the understanding of the critical asymptotics along the AT line.

What is claimed above, namely that the existence of a spin glass transition in an external magnetic field may be possible even if the RG trajectories run away from the critical fixed point without terminating into a perturbative novel fixed point, has been demonstrated in a simpler model where the interaction depends on the hierarchical distance between the Ising spins: i.e. in the Hierarchical Edwards-Anderson (HEA) model. A first order RG analysis of the generic replica symmetric phase\(^5 \) in Ref.\(^{12} \) found no relevant fixed point governing the transition in a field: the couplings renormalize toward infinite values. Notwithstanding that, a careful Monte Carlo simulation on a modified version of the HEA\(^{18} \) provided evidence for a transition in nonzero external field by a study of the spin glass susceptibility and the correlation function associated with it. Most importantly, Ref.\(^{22} \) found transition in nonzero field in the non-mean-field region \( \sigma \geq 2/3 \) where \( \sigma \), the parameter of the HEA analogous to the spacial dimension \( d \) of the short-ranged model in Euclidean space, was within 2% from the upper critical value \( \sigma = 2/3 \). This clearly shows that the AT instability persists while traversing the analogue of the upper critical dimension from the mean field to the non-mean-field region, inspite of the absence of a perturbative fixed point governing the AT critical surface\(^{22} \).

One point is still lacking here, namely the observation of the transition perturbatively by computing the asymptotical behavior of the spin glass susceptibility (or, equivalently, the replicon mass) around the critical fixed point. This is left to a subsequent work.

As for the short-ranged model, it has been advocated for some time past\(^{19,20} \) that the lower critical dimension for the AT line should be \( d = 6 \), i.e. that the spin glass transition in an external field disappears just at the upper critical dimension of the zero-field model. The fault in the arguments of Ref.\(^{19} \) about the behavior of the AT line (computed perturbatively for \( d > 6 \)), namely that it disappears while approaching \( d = 6 \) from above, was pointed out in\(^{18} \). The issue was reconsidered in Ref.\(^{21} \), admitting now that the six-dimensional AT line cannot be derived by a limiting process from the perturbative result in \( d > 6 \). Yeo and Moore\(^{21} \), however, incorrectly claimed that the calculation of the six-dimensional AT line in\(^{11} \) was performed by just this wrong limiting process. In fact, the \( d = 6 \) case was studied separately in Ref.\(^{11} \), as it must be, by the special one-loop perturbative RG at the upper critical dimension where the scaling exponent of the cubic coupling constant is zero. (See also Refs.\(^{22,23} \) and references therein.

From the discussion of the last two paragraphs, and also from the results of the present paper, it follows that the lower critical dimension for the spin glass transition of the Ising spin glass in an external magnetic field is probably less than \( d = 6 \). One must, however, emphasize that the perturbative RG is not able to make predictions about the existence of the AT line far below \( d = 6 \). Numerical simulation results in \( d = 3 \) and \( d = 4 \) (or in the corresponding long-ranged one-dimensional model as a “proxy” for the short-ranged system) in this regard are controversial; see\(^{24,25} \) and references therein.

Finally some notes about the perturbative method: The calculations of physical quantities were performed in the present paper by the combined use of the renormalization group and a series expansion in terms of the coupling constants. This method is absolutely conventional around a perturbative fixed point: a perturbative result like (20), for instance, is interpreted by the RG ansatz in (19), and the scaling function can be identified as in (21). In the meantime, a consistency check is available by the proper exponentiation of the logarithms of the temperature-like scaling field. Two peculiarities, however, occur: The first one is due to the quadratic symmetry breaking caused by the nonzero RS order parameter which leads to the two distinct free propagators, with the replicon mode almost massless in a small magnetic field below \( T_c \). The other one is related to the replicated nature of the field theory which may cause problems in the \( n \to 0 \) spin glass limit. Although this limit proved to be quite smooth in our model with the four scaling fields, the behavior and physical meaning of the remaining modes, like the third mass mode for instance, are not clear.

\(^5\) An analogous study for the short-ranged model in Euclidean \( d \)-dimensional space can be found in Ref.\(^{12} \), where the eigenmodes of the linearized RG around the critical fixed point are also presented.
Appendix A: Summary of some one-loop results for the generic replica symmetric theory

In this Appendix, we provide results which are equally valid in the high and low temperature regimes, assuming that the proper value of $R$ and $L$, see Eqs. (15) and (17), must be inserted.

1. The equation of state:

The order parameter $q$ satisfies the implicit equation

$$2w^* q = h^2 q^{-1} - 2m_1 + 2m_2 + [-2(w_1 - w^*) + w_2 - w_3 + w_5] q + q^{-1} \frac{1}{N} \sum_{\vec{p}} Y(p),$$

(A1)

with the one-loop graph

$$Y(p) = \left( w_2 + \frac{1}{3} w_3 + \frac{4}{3} w_4 - \frac{1}{3} w_5 - w_6 - \frac{4}{3} w_7 \right) G_R$$

and

$$+ \left( 3w_1 - 2w_2 + \frac{4}{3} w_3 + \frac{4}{3} w_4 - \frac{1}{3} w_5 - w_6 - \frac{4}{3} w_7 \right) 2m_2 G_R G_L + (4w_1 - 2w_2 + 2w_3 - 2w_5 - 2w_6) (-m_2 + 2m_3) G^2_L.$$

This one-loop integral can be computed, and one gets

$$w^* \frac{1}{N} \sum_{\vec{p}} Y(p) = w^{*2} |g_{m_1}|^2 \times$$

$$(1 - 2x)^{-1/2} \left[ \frac{1}{2} \ln |g_{m_1}|^{-1} + \frac{1}{2} (L - R) \ln |g_{m_1}| + \frac{3}{2} R^2 \ln R + \frac{1}{2} L(L - 4R) \ln L + L(L - R) \right] + O(w^4).$$

(A2)

2. The replicon mass

The one-loop formula for the replicon mass has been published in9; see Eqs. (49a-h) and (62). Here we reproduce it in terms of the set of bare parameters used throughout the present paper and for $n = 0$:

$$\Gamma_R = 2m_1 + 2w^* q + 2 \left[ (w_1 - w^*) - w_2 + \frac{1}{3} w_3 \right] q - \frac{1}{N} \sum_{\vec{p}} \sigma_R$$

(A3)

with the replicon self-energy:

$$\sigma_R = \left( -2w_1^2 - \frac{4}{3} w_1 w_3 - \frac{16}{3} w_1 w_4 - \frac{8}{3} w_1 w_5 + 2w_2^2 + \frac{8}{3} w_2 w_3 + \frac{16}{3} w_2 w_4 + \frac{4}{3} w_2 w_5 + \frac{2}{9} w_3^2 - \frac{16}{9} w_3 w_4 - \frac{8}{9} w_3 w_5 \right)$$

$$+ \frac{4}{9} w_5^2) \times \tilde{G}_R^2 + \left( -2w_1^2 + 12w_1 w_2 + \frac{4}{3} w_1 w_3 - \frac{16}{3} w_1 w_4 - \frac{16}{3} w_1 w_5 - 8w_2^2 + \frac{4}{3} w_2 w_3 + \frac{16}{3} w_2 w_4 + \frac{8}{3} w_2 w_5 + \frac{6}{9} w_3^2 \right)$$

$$- \frac{16}{9} w_3 w_4 - \frac{16}{9} w_3 w_5 + \frac{8}{9} w_5^2) \times 2m_2 G^2_R G_L + \left( -8w_1^2 + 16w_1 w_2 - \frac{16}{3} w_1 w_3 - 8w_2^2 + \frac{16}{3} w_2 w_3 - \frac{8}{9} w_3^2 \right) \times (-m_2 + 2m_3) G^2_R G^2_L$$

$$+ \frac{1}{9} (6w_1 - 3w_2 + 2w_3 - 2w_5)^2 \times 4m_2^2 G^2_R G^2_L.$$

Performing the momentum integral provides

$$\frac{1}{N} \sum_{\vec{p}} \sigma_R = w^{*2} |g_{m_1}| \times$$

$$(1 - 2x)^{-1} \left[ -|g_{m_1}|^{-1} + (L - 3R) \ln |g_{m_1}| + \frac{R(4L + 3R)}{L - R} \ln R + \frac{L(L - 8R)}{L - R} \ln L + 2(R + 2L) \right] + O(w^4).$$

(A4)
3. The longitudinal mass

The first order expression for the longitudinal mass takes the form (for $n = 0$)

$$\Gamma_L = 2m_1 - 2m_2 + 4w^* q + 2 \left[ 2(w_1 - w^*) - w_2 + w_3 - w_5 \right] q - \frac{1}{N} \sum_{\vec{F}} \sigma_L$$

(A5)

with the longitudinal self energy (which is identical with the anomalous one when $n = 0$); see Eqs. (49a-h), (63), and (64) of:

$$\sigma_L = \left[ 6 (w_1 - w_2 + \frac{1}{3} w_3)^2 - \frac{4}{3} (2w_1 - w_2 + w_3 - w_5 - w_6) (3w_1 - 3w_2 + w_3 + 4w_4 + 2w_5 - 4w_7) \right] \times \bar{G}_R^2$$

$$- \frac{8}{3} (2w_1 - w_2 + w_3 - w_5 - w_6) (3w_1 - 3w_2 + w_3 + 4w_4 + 2w_5 - 4w_7) \times (2\bar{m}_2 \bar{G}_R^2 \bar{G}_L + 2\bar{m}_2^2 \bar{G}_R^2 \bar{G}_L^2)$$

$$- 8(-2w_1 + w_2 - w_3 + w_5 + w_6)^2 \times (-\bar{m}_2 + 2\bar{m}_3) \bar{G}_L^3.$$

After integration it becomes:

$$\frac{1}{N} \sum_{\vec{F}} \sigma_L = w^* |g_{m_1}| \times (1 - 2x)^{-1} \left[ -|g_{m_1}|^{-1} - 2R \ln |g_{m_1}| + 6R \ln R - 8R \ln L + (8L - 9R) \right] + O(w^4).$$

(A6)

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