ON THE BILINEAR STRUCTURE ASSOCIATED TO THE SKEW BEZOUTIAN

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Abstract. Let \((p, q)\) be a couple of reciprocal (resp. \(q\) is reciprocal and \(p\) is skew-reciprocal) coprime polynomials of degree \(d\) over a field with characteristic not 2. To such a couple we can associate a non-singular antisymmetric (resp. symmetric) matrix of size \(d \times d\), which we call the skew Bezoutian. In this paper we study some properties of the corresponding symplectic (resp. quadratic) space. Using the skew Bezoutian we construct explicit isometries of bilinear spaces with given invariants (such as the characteristic polynomial or Jordan form and, in the quadratic case, the spinor norm).

1. The skew Bezoutian of two polynomials

1.1. Preliminaries. Fix once and for all a field \(k\) of characteristic different from 2. For a polynomial \(q \in k[T]\) we let \(q^*\) be the polynomial \(q^*\) with its coefficients reversed, i.e.,

\[ q^*(T) := T^{\deg(q)}q(1/T), \quad q \in k[T], \]

where \(\deg(q)\) is the degree of \(q\).

For further reference let us note a few simple observations about the operation \(*\). In general, \(*\) is not additive but we have

\[ (p + q)^* = p^* + q^*, \quad \text{if } \deg(p) = \deg(q) \]

and \((pq)^* = p^*q^*\) always. We will say that \(p \in k[T]\) is \(\text{reciprocal}\) if \(p^* = p\), \(\text{skew-reciprocal}\) if \(p^* = -p\) and in general \(\varepsilon\text{-reciprocal}\) if \(p^* = \varepsilon p\) with \(\varepsilon = \pm 1\).

To shorten the notation we let \(v_\varepsilon\), for \(\varepsilon = \pm 1\), denote the valuation on \(k[T]\) at \(T - (\pm 1)\).

If \(p \in k[T]\) is skew-reciprocal then \(p(1) = 0\). It follows that in fact \(v_+(p)\) must be odd since otherwise \(p(T)/(T - 1)^{v_+(p)}\) would be a skew-reciprocal polynomial not vanishing at \(T = 1\).

Similarly, if \(p \in k[T]\) is \(\varepsilon\text{-reciprocal}\) with \(\varepsilon = -(-1)^{\deg(p)}\) then \(p(-1) = 0\) and again, \(v_-(p)\) must be odd. In particular, if \(p(1) \neq 0\) then \(\deg(p)\) must be even.

1.2. Definition. Let \(p, q \in k[T]\) be two monic polynomials of degree \(d \geq 1\). Assume that \(q\) is reciprocal and \(p\) is \((-\varepsilon)\)-reciprocal for a fixed \(\varepsilon = \pm 1\). To this data we associate a \(d\)-dimensional \(k\)-vector space \(V\) equipped with a bilinear form \(\Psi\), which satisfies

\[ \Psi(v, u) = \varepsilon \Psi(u, v). \]

We will say that \((V, \Psi)\) is an \(\varepsilon\text{-symmetric}\) bilinear space over \(k\).

The construction goes as follows. Let \(R := k[T, T^{-1}]\) be the ring of Laurent polynomials in the variable \(T\) with coefficients in \(k\). Let \(\iota\) be the involution on \(R\) sending \(T\) to \(T^{-1}\). As our vector space we take \(V := R/(q)\). Consider the rational function \(w := -\varepsilon p/q\). Note that

\[ w(T^{-1}) = -\varepsilon w(T). \]
It follows from (1.2) that \( w(0) = 1 \) since it equals \(-\varepsilon w(\infty)\) and both \( p \) and \( q \) are assumed monic of the same degree. In particular, \( w \) admits a power series expansion around 0 of the form
\[
(1.3) \quad w = 1 + c_1 T + \cdots + c_n T^n + \cdots \quad c_n \in k. 
\]

Define a \( k \)-linear form \( c : R \to k \) by setting
\[
(1.4) \quad c(1) = 1 + \varepsilon \quad c(T^n) = \varepsilon c_n \quad c(T^{-n}) = c_n, 
\]
for any \( n \geq 1 \).

It will be convenient to represent elements in the dual space \( R^* \), such as \( c \), as formal infinite series of the form
\[
c := \sum_{n \in \mathbb{Z}} c(T^n) T^n. 
\]
Following the usual rules of multiplication of series gives \( R^* \) the structure of an \( R \)-module. With this notation we have
\[
c(u) = [u \cdot c]_0, \quad u \in R, 
\]
where \([\cdot]_0\) denotes constant term.

With \( c \in R^* \) as in (1.4) we have
\[
(1.5) \quad q \cdot c = q(T) \sum_{n \geq 0} c_n T^{-n} + \varepsilon q(T) \sum_{n \geq 0} c_n T^n = -\varepsilon(p^*(T) + \varepsilon p(T)) = 0 
\]
and therefore \( c \) vanishes on the ideal \( (q) \subseteq R \).

We now define the bilinear form on \( V \) as
\[
(1.6) \quad \Psi(u, v) := c(\tilde{u} \cdot \tilde{v}^t), \quad u, v \in V, 
\]
and call the resulting bilinear space \( (V, \Psi) \) the skew Bezoutian of \( p \) and \( q \) (we learned of this construction in [HA]). On the right hand side, \( \tilde{u}, \tilde{v} \) stand for representatives in \( R \) of the respective classes \( u, v \in V \); by (1.5) the result is independent of the choice. It is clear from (1.4) that it satisfies the symmetry relation (1.1).

To simplify the notation from now on we will not distinguish between elements of \( R \) and their classes in \( V \) as long as it does not lead to confusion.

In the basis \( 1, T, \cdots, T^{d-1} \) of \( V \) the bilinear form has Gram matrix
\[
(1.7) \quad B(p, q) := \begin{pmatrix}
1 + \varepsilon & c_1 & \cdots & c_{d-1} \\
\varepsilon c_1 & 1 + \varepsilon & \cdots & c_{d-2} \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon c_{d-1} & \varepsilon c_{d-2} & \cdots & 1 + \varepsilon
\end{pmatrix},
\]
which we will also call the skew Bezoutian of \( p \) and \( q \) at times. Indeed, by definition we have
\[
(1.8) \quad \Psi(T^i, T^j) = c(T^{i-j}), \quad i, j \in \mathbb{Z}, 
\]
so that, in particular, the skew Bezoutian \( B(p, q) \) is a Toeplitz matrix (a matrix with constant entries along diagonals).
1.3. **Properties.** A remarkable property of $B(p, q)$ is the following (for the usual Bezoutian at least this goes back to Bezout’s original paper concerning his well-known theorem about the intersection of curves in the plane).

**Proposition 1.1.** The determinant of the skew Bezoutian is the resultant of $p$ and $q$.

(1.9) \[ \det B(p, q) = \text{Res}(p, q), \]

In particular, $(V, \Psi)$ is non-degenerate if and only if $p$ and $q$ are relatively prime.

A proof of a generalization of (1.9) to subresultants can be found in [BSP, Prop. 11]. As it is quite short, we reproduce the proof of loc. cit. in the special case which is of interest for us.

**Proof.** Recall the power series expansion of $-\varepsilon p/q$ is

\[ w = 1 + c_1 T + c_2 T^2 + \cdots \]

Because of (1.2) we have

\[ -\varepsilon w(1/T) = w = -\varepsilon - \varepsilon c_1 T^{-1} - \varepsilon c_2 T^{-2} + \cdots \]

Writing $q = \sum_{k=0}^{d} q_k T^k$ and $p = \sum_{k=0}^{d} p_k T^k$ we deduce from the two equalities above:

\[ -\varepsilon p_0 = -\varepsilon q_0 - \varepsilon c_1 q_1 - \cdots - \varepsilon c_d q_d, \]

\[ -\varepsilon p_1 = -\varepsilon c_1 q_1 - \varepsilon c_2 q_1 - \cdots - \varepsilon c_{d-1} q_d, \]

\[ \vdots \]

\[ -\varepsilon p_d = -\varepsilon q_d. \]

Let us define the matrices

\[ \tilde{Q}_d := \begin{pmatrix} q_d & 0 & 0 \\ \vdots & \ddots & \vdots \\ q_1 & \cdots & 0 \end{pmatrix}, \quad \tilde{P}_d := \begin{pmatrix} -\varepsilon p_d & 0 & 0 \\ -\varepsilon p_{d-1} & -\varepsilon p_d \\ \vdots & \ddots & \ddots & 0 \end{pmatrix}, \]

\[ \tilde{E}_d := \begin{pmatrix} \varepsilon & 0 & 0 \\ \varepsilon c_1 & \varepsilon \\ \vdots & \ddots & \ddots & 0 \\ \varepsilon c_{d-1} & \cdots & \cdots & \varepsilon \end{pmatrix}. \]

The above relation between power series implies

\[ \tilde{P}_d = -\tilde{E}_d \cdot \tilde{Q}_d. \]

Similarly let us define

\[ Q_d := \begin{pmatrix} q_0 & \cdots & q_{d-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad P_d := \begin{pmatrix} -\varepsilon p_0 & \cdots & -\varepsilon p_{d-1} \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \end{pmatrix}, \]

\[ 3 \]

\[ 3 \]
\[ E_d := \begin{pmatrix} 1 & c_1 & \cdots & c_{d-1} \\ 0 & 1 & c_{d-2} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \]

We have
\[ P_d = E_d \cdot Q_d. \]

Putting those two matrix identities together we get the following equality of block matrices
\[ \left( \begin{array}{cc} \text{Id}_d & \text{Id}_d \\ E_d & -E_d \end{array} \right) \cdot \left( \begin{array}{cc} Q_d & 0 \\ 0 & \tilde{Q}_d \end{array} \right) = \left( \begin{array}{cc} Q_d & \tilde{Q}_d \\ P_d & \tilde{P}_d \end{array} \right). \]

Now we compute the determinant of both sides. For the leftmost matrix we subtract the \( i \)-th column from the \((i + d)\)-th column, \( 1 \leq i \leq d \). We deduce
\[ q_0^d q_d^d \det(-B(p, q)) = \text{Res}(-\varepsilon p, q), \]
since the rightmost matrix is the usual Sylvester matrix whose determinant is the resultant. The result now follows since \( q_0 = q_d = 1 \) and if \( \varepsilon = -1 \) then either \( d \) is even or \( p \) and \( q \) vanish at \( T = -1 \) and \( \text{Res}(p, q) = 0 \). \( \square \)

In addition to the bilinear form \( \Psi \) the Bezoutian carries some extra structures. It has a distinguished vector \( v_0 \), the class of the polynomial \( 1 \in R \), with \( \Psi(v_0, v_0) = 1 + \varepsilon \) and an isometry \( \gamma \), given by multiplication by \( T \) (the fact that it is an isometry is clearly seen in (1.8), for example). Note that by construction \( \gamma \) has characteristic polynomial \( q \). Moreover, the translates \( v_0, \gamma(v_0), \gamma^2(v_0), \cdots \) generate the whole space \( V \). In fact, these properties characterize the skew Bezoutian as we now show.

Given \( v_0 \in V \) with \( \Psi(v_0, v_0) = 1 + \varepsilon \) we define its associated \( \varepsilon \)-reflection to be the isometry given by
\[ \sigma(v) := v - \Psi(v_0, v) v_0. \]
(In the skew-symmetric case \( \sigma \) is usually called a transvection.) Note that \( \det(\sigma) = -\varepsilon \) and that \( \sigma \) is of order two if \( \varepsilon = +1 \) but of infinite order if \( \varepsilon = -1 \).

**Theorem 1.2.** Let \( (V, \Psi) \) be a non-degenerate, finite dimensional, \( \varepsilon \)-symmetric bilinear space over \( k \). Suppose there exists an isometry \( \gamma \) of this space with reciprocal characteristic polynomial and a vector \( v_0 \in V \) such that
\[ \begin{align*} 
(i) & \quad \Psi(v_0, v_0) = 1 + \varepsilon \\
(ii) & \quad V \text{ is generated by } v_0, \gamma v_0, \gamma^2 v_0, \cdots .
\end{align*} \]

Then \( (V, \Psi) \) is the skew Bezoutian \( B(p, q) \), where \( q \) is the characteristic polynomial of \( \gamma \) and \( p \) is the characteristic polynomial of \( \gamma \sigma \) with \( \sigma \) the \( \varepsilon \)-reflection associated to \( v_0 \).

**Proof.** Let \( d \) be the dimension of \( V \). Note that \( V := \{v_0, \gamma v_0, \ldots, \gamma^{d-1} v_0\} \) is a basis of \( V \). Indeed, by the Cayley–Hamilton theorem \( \gamma^n v_0 \) is in the span of \( V \) and hence so is every \( v \) in \( V \) by hypothesis (ii). Again by hypothesis (ii), \( V \) is linearly independent.

Define for every \( n \in \mathbb{Z} \)
\[ c_n := \Psi(\gamma^n v_0, v_0). \]
Note that $c_n = \varepsilon c_n$. We claim that

$$1 + \sum_{n \geq 1} c_n T^n$$

is the power series expansion of a rational function of denominator $q$. Write $q = \sum_{k \geq 0} q_k T^k$. By assumption $q_{d-k} = q_k$ for $k = 0, \ldots, d$ and $q_0 = q_d = 1$. Then

$$q(T)(1 + \sum_{n \geq 1} c_n T^n) = \sum_{n \geq 0} r_n T^n = 1 + \sum_{n \geq 1} r_n T^n,$$

where $r_n = q_n + \sum_{k=1}^n c_k q_{n-k}$ for $n \geq 1$. Since $q_n = 0$ for $n > d$ we have

$$r_n = \sum_{k=0}^d c_{n-k} q_k = \sum_{k=0}^d c_{n-d+k} q_{d-k} = \sum_{k=0}^d c_{n-d+k} q_k, \quad n > d.$$ 

Hence

$$r_n = \Psi(\gamma^{n-d} q(\gamma)v_0, v_0) = 0, \quad n > d.$$ 

We now show that $r_{d-n} = -\varepsilon r_n$ for $n = 0, \ldots, d$. Since $q(\gamma) = 0$ we have for $n < d$

$$r_{d-n} = q_n + \sum_{k=1}^{d-n} q_{n+k} c_k = q_n - \Psi(\gamma^n v_0, v_0),$$

where $s_n := \sum_{k=0}^n q_k T^k$. Hence

$$r_{d-n} = q_n - \sum_{k=0}^n q_k c_{k-n} = q_n - (1 + \varepsilon)q_n - \varepsilon \sum_{k=0}^{n-1} q_k c_{n-k} = -\varepsilon r_n.$$ 

We have shown then that $p(T) := -\varepsilon \sum_{n=0}^d r_n T^n$ is $(-\varepsilon)$-reciprocal; since $r_0 = 1$ it is also monic. In other words, we have that $(V, \Psi)$ is isometric to the skew Bezoutian $B(p, q)$.

It remains to show that $p$ is the characteristic polynomial of $\delta := \gamma \sigma$. For every $n \in \mathbb{Z}$ let $\sigma_n$ be the $\varepsilon$-reflection associated to $v_n := \gamma^n v_0$. Note that $\Psi(v_n, v_n) = 1 + \varepsilon$. We have

$$\sigma_n = \gamma^n \sigma \gamma^{-n}$$

and hence by induction

$$\delta^n = \sigma_1 \cdots \sigma_n \gamma^n.$$ 

Let $u_0 := v_0$ and $u_n := \sigma_1^{-1} \cdots \sigma_{n-1}^{-1} v_0$ for $n > 0$. Let also $e_n := \varepsilon \Psi(\delta^n v_0, v_0)$ for $n \in \mathbb{Z}$. Then

$$e_{n+1} = \Psi(v_0, \sigma_1 \cdots \sigma_n v_{n+1}) = \Psi(u_{n+1}, v_{n+1}).$$

Since

$$u_{n+1} = \sigma_{n+1}^{-1} u_n = u_n - \varepsilon \Psi(u_n, v_{n+1}) v_{n+1}, \quad n \geq 0$$

we get

$$e_{n+1} = \Psi(u_n, v_{n+1}) - \varepsilon \Psi(u_n, v_{n+1}) \Psi(v_{n+1}, v_{n+1}) = -\varepsilon \Psi(u_n, v_{n+1}).$$

Therefore $u_{n+1} = u_n + e_{n+1} v_{n+1}$ and by induction

$$u_n = v_0 + \sum_{k=1}^n e_k v_k.$$
Finally,

\[-e_{n+1} = \Psi(v_{n+1}, u_n) = c_{n+1} + \sum_{k=1}^{n} e_k c_{n+1-k}\]

and

\[(1 + \sum_{n \geq 1} c_n T^n)(1 + \sum_{n \geq 1} e_n T^n) = 1.\]

Combined with our previous calculation we see that

\[p(T)(1 + \sum_{n \geq 1} e_n T^n) = -\varepsilon q(T).\]

So if \(p(T) = \sum_{n=0}^{d} p_n T^n\) then

\[0 = \sum_{k=0}^{d} e_{n-d+k} p_{d-k} = -\varepsilon \sum_{k=0}^{d} e_{n-d+k} p_k, \quad n > d\]

and

\[(1.12) \quad \Psi(p(\delta)v_0, \delta^{d-n}v_0) = 0, \quad n > d.\]

It is not hard to see that \(\delta^n v_0 = -\varepsilon v_n + \sum_{j=1}^{n-1} \alpha_{n,j} v_j\) for \(n = 1, \ldots, d-1\), for some \(\alpha_{n,j} \in k\)
(note that for \(n = 1\) the equality is \(\delta v_0 = -\varepsilon v_1\)). It follows that the \(\delta^n v_0\) with \(n \in \mathbb{Z} \text{ span } V\) and by (1.12) \(p(\delta) = 0.\) \(\square\)

**Remark 1.3.** The assumption that \(\gamma\) has reciprocal characteristic polynomial is not really a restriction and we only include it to simplify the exposition. If the characteristic polynomial is skew-reciprocal we replace \(\gamma\) with \(\delta := \gamma \sigma\) and reverse the roles of \(p\) and \(q\).

**Remark 1.4.** The subgroup \(\Gamma \subseteq \text{GL}(V)\) generated by \(\gamma, \delta, \sigma\) is a hypergeometric group in the sense of \([BH]\) with parameters the multisets of roots of \(p\) and \(q\). Our discussion shows that \(B(p, q)\) gives a non-degenerate bilinear form invariant under \(\Gamma\) (compare with \([BH, \text{Thm 4.3}]\)). This was mentioned in \([RV]\).

### 1.4. Examples.

We end this section with some examples. The skew Bezoutian construction can be done over a commutative ring (details will appear in a later publication). Here we work over \(\mathbb{Z}\).

1) Let

\[p = \Phi_1 \Phi_2 \Phi_3 \Phi_5 = x^8 + 2x^7 + 2x^6 + x^5 - x^3 - 2x^2 - 2x - 1, \quad q = \Phi_{30} = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1,\]

where \(\Phi_n\) is the \(n\)-th cyclotomic polynomial. Then

\[w = -p/q = 1 + x + x^2 + x^3 + x^4 + x^5 - x^10 + O(x^{11})\]

and

\[B(p, q) = \begin{pmatrix}
2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 2
\end{pmatrix}_{6}\]
The lattice \( \mathbb{Z}[x]/(q) \) with this quadratic form is the well-known \( E_8 \) lattice and \( \gamma \) is a Coxeter element of the corresponding Weyl group.

2) Similarly the \( A_n \) lattice with Cartan matrix

\[
C_n := \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
& & & \vdots & & \\
0 & & \cdots & -1 & 2 & -1 \\
0 & 0 & \cdots & -1 & 2
\end{pmatrix}
\]

arises as the skew Bezoutian \( B(p, q) \), where

\[
p = x^n - 1 \quad q = x^n + x^{n-1} + \cdots + x + 1
\]

and \( \gamma \) represents an \( n \)-cycle in \( S_n \).

3) Let \( q = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 \) be the Lehmer polynomial (the integer polynomial of smallest known Mahler measure bigger than 1). For \( p \), we search among the polynomials of degree 10 which are products of cyclotomics. We find eight such that \( I \) is isometric to the unimodular lattice \( I_{9,1} \) of signature \( (9,1) \). These are tabulated below.

\[
\begin{align*}
\Phi_1^3\Phi_2\Phi_3\Phi_5 & \quad x^{10} - x^8 - x^7 + x^3 + x^2 - 1 \\
\Phi_1\Phi_2^3\Phi_3^5 & \quad x^{10} + 4x^9 + 7x^8 + 7x^7 + 4x^6 - 4x^4 - 7x^3 - 7x^2 - 4x - 1 \\
\Phi_1\Phi_2^3\Phi_3\Phi_6 & \quad x^{10} + x^9 + x^8 + x^7 + x^6 - x^4 - x^3 - x^2 - x - 1 \\
\Phi_1\Phi_2\Phi_3\Phi_7 & \quad x^{10} + 2x^9 + 2x^8 + x^7 - x^3 - 2x^2 - 2x - 1 \\
\Phi_1\Phi_2\Phi_3\Phi_9 & \quad x^{10} + x^9 - x - 1 \\
\Phi_1\Phi_2\Phi_3\Phi_{18} & \quad x^{10} + x^9 - 2x^7 - 2x^6 + 2x^4 + 2x^3 - x - 1 \\
\Phi_1\Phi_2\Phi_5\Phi_8 & \quad x^{10} + x^9 + x^6 - x^4 - x - 1 \\
\Phi_1\Phi_2\Phi_5\Phi_{10} & \quad x^{10} - 1
\end{align*}
\]

We do not know if these isometries are in the same conjugacy class.

4) In the paper [MPV] the authors consider the modification \( q(x) := p(x) \pm x^m \) of a monic reciprocal polynomial \( p \) of even degree \( 2m \) consisting of adding a single monomial \( \pm x^m \). The Bezoutian \( B(p, q) \) then yields a skew-symmetric form of determinant \( \text{Res}(p, q) = 1 \) and a symplectic transformation of characteristic polynomial \( q \). For example, if we again take \( q \) to be the Lehmer polynomial we see that it is also the characteristic polynomial of a symplectic transformation. As pointed out in [MPV] it is remarkable that \( q(x) + x^5 \) is actually a product of cyclotomic polynomials.

In light of Theorem 1.2, the modification used in [MPV] can be seen as an example of modifying a symplectic transformation by multiplying it by a single transvection. It would be interesting to extend their results and study how this modification affects the Mahler measure of the characteristic polynomial.

2. Isometries with Given Characteristic Polynomial

The skew Bezoutian enables us to answer the following question.

**Question.** Given a monic reciprocal polynomial \( q \in k[T] \) of degree \( d \), is there a \( d \)-dimensional \( k \)-vector space \( V \) equipped with a symmetric (resp. antisymmetric) non-degenerate pairing \( \Psi \) such that \( q \) can be realized as the characteristic polynomial of some isometry \( \gamma \) of \( (V, \Psi) \)?
The case where $\varepsilon = 1$, $d$ is even, and $q$ has no double root in an algebraic closure of $k$ is studied e.g. by Edwards in [E] and in greater generality by Baeza in [Ba]. Notably [Ba, Th. 3.7] gives a complete classification of the non-degenerate $d$-dimensional quadratic spaces over $k$ for which an isometry with characteristic polynomial $q$ exists. This is also discussed by Gross [McG]. We discuss related work of Baeza and Edwards in the next section.

In the case where $k$ is finite [Ba, Th. 3.7] has the following nice interpretation. The quadratic space for which there is an isometry $\gamma$ having $q$ as its characteristic polynomial is entirely determined up to isomorphism: this is the so-called non-split quadratic form

$$\Psi : (x, y) \in k_{d} \times k_{d} \mapsto \frac{1}{2} \text{Tr}_{k_{d}/k}(xy)^{d/(2)},$$

where $k_{d}$ is an extension of $k$ with even degree $d$ (see, e.g., [Ka-L, Section 6]). In particular, in such a setting, it is not possible to impose both a characteristic polynomial and a quadratic structure.

The case where $k$ is a finite field and $\varepsilon = -1$ is investigated by Chavdarov in [Chav]. However his method is not constructive and there does not seem to be any way to adapt it to the case $\varepsilon = 1$. Indeed Chavdarov’s proof relies on the existence of Lang’s isogeny (see [La]) on the connected centralizer (under $Sp(d, \bar{k})$) of a semisimple matrix (over $\bar{k}$) having $q$ as its characteristic polynomial. The connectivity of the centralizer follows from theorem of Steinberg and uses the simple connectivity of the algebraic group $Sp(d)$ in a crucial way. As the orthogonal group $O(d)$ is not connected and the special orthogonal group $SO(d)$ is not simply connected, Chavdarov’s argument cannot be directly adapted to the case $\varepsilon = 1$.

The case where $q$ has no double root in an algebraic closure of $k$ and $\varepsilon = +1$ is also considered in [McG, Appendix A]. In what follows we will consider the general case.

We keep the notation of §1. In the proof of the following result we give an explicit answer to our question.

**Theorem 2.1.** Let $q \in k[T]$ be a monic reciprocal polynomial of degree $d \geq 1$. Then
1) There exists a non-degenerate symmetric bilinear space over $k$ of dimension $d$ with an isometry $\gamma$ of characteristic polynomial $q$.

2) If, in addition, $d$ is even there exists a non-degenerate skew-symmetric bilinear space over $k$ of dimension $d$ and determinant 1 with an isometry $\gamma$ of characteristic polynomial $q$.

**Proof.** The main idea is to use the skew Bezoutian. If we can find a polynomial $p \in k[T]$, which is $(-\varepsilon)$-reciprocal and coprime to $q$ then the skew Bezoutian $B(p, q)$ provides an explicit answer to what we are looking for. As discussed above, the skew Bezoutian comes equipped with an isometry of characteristic polynomial $q$ and is non-degenerate if $p$ and $q$ are coprime. Knowledge of $\text{Res}(p, q)$ will further determine the discriminant of the space by Proposition 1.1.

In the skew-symmetric case, where $\varepsilon = -1$ and $d$ is assumed even, we may always find such a $p$. Indeed, the polynomial

$$p(T) := q(T) + T^{m},$$

where $m := d/2$ satisfies all the requirements we need: $p$ is clearly reciprocal and coprime to $q$. Moreover, $\text{Res}(p, q) = (\prod_{b}b)^{m}$, where $b$ runs over the roots of $q$ counted with multiplicity,
and this equals \( q(0)^m \). Since \( q \) is monic and reciprocal \( q(0) = 1 \). Hence, by Proposition 1.1 the skew Bezoutian has determinant 1. This proves 2).

Now we turn to the case where \( \varepsilon = 1 \). Let us write
\[
q(T) = (T - 1)^{v_+}(T + 1)^{v_-}q_0(T), \quad q_0(\pm 1) \neq 0.
\]

As \( q \) is reciprocal, by the observations of §1.1, its order of vanishing \( v_+ \) at 1 is even. Hence, \( q_0 \) is also reciprocal; let \( d_0 \) be its degree. Assume for the moment that \( d_0 > 0 \) and set
\[
p_0(T) := (T - 1)^e(T + 1)^{d_0 - e},
\]
for some odd integer \( 0 \leq e \leq d_0 \). Note that \( d_0 - e \) is odd also since \( d_0 \) is even as \( q_0 \) is a reciprocal polynomial not vanishing at \(-1 \) (see §1.1).

By construction \( p_0 \) is monic, skew-reciprocal, of degree \( d_0 \) and coprime to \( q_0 \). The skew Bezoutian \((V_0, \Psi_0)\) of \( p_0, q_0 \) is then a non-degenerate symmetric bilinear space over \( k \) of dimension \( d_0 \). The corresponding isometry \( \gamma_0 \) has characteristic polynomial \( q_0 \). To obtain the space \( V \) we are after we consider
\[
V := V_0 \perp V_+ \perp V_- \]
where \( V_\pm \) is a vector space over \( k \) of dimension \( v_\pm \). We put on \( V_\pm \) an arbitrary non-degenerate symmetric bilinear form \( \Psi_\pm \) and consider \( \Psi := \Psi_0 \perp \Psi_+ \perp \Psi_- \) and \( \gamma := \gamma_0 \perp \text{id}_{V_+} \perp (-\text{id}_{V_-}) \). It should now be clear that \((V, \Psi)\) and \( \gamma \) fulfill the requirements.

The same construction works if \( d_0 = 0 \); just ignore \( V_0 \) altogether. This completes the proof of 1).

It seems natural to try and compute other invariants attached to the bilinear space constructed in terms of the polynomials \( p \) and \( q \). In the following section we focus on the case \( \varepsilon = 1 \) and we investigate how the spinor norm of the isometry \( \gamma \) constructed in the proof of Proposition 2.1 can be expressed in terms of the polynomial \( q \).

3. Spinor norm of an isometry with prescribed characteristic polynomial

Recall that if \((V, \Psi)\) is a non-degenerate quadratic space, the spinor norm of an isometry \((V, \Psi)\) can be defined as follows: first let \( v \) be a non isotropic vector of \( V \) and let \( r_v \) be the reflection with respect to the hyperplane \( v^\perp \). We define the spinor norm \( N_{\text{spin}}(r_v) \) to be the class in \( k^*/(k^*)^2 \) of \( \Psi(v, v) \). Now any isometry \( \sigma \) of \( V \) is a product \( \prod_v r_v \), where \( v \) runs over a finite set of non-isotropic vectors of \( V \). It is known that \( \sigma \mapsto \prod_v \Psi(v, v) \) gives a well-defined spinor norm homomorphism
\[
N_{\text{spin}} : O(V, \Psi) \rightarrow k^*/(k^*)^2,
\]
which is onto as soon as \( n := \dim V \geq 2 \). Note in particular that \( N_{\text{spin}}(-\text{id}_V) = \det(V, \Psi) \), where \( \det(V, \Psi) := \det(\Psi(v_i, v_j)) \) for any basis \( v_1, \ldots, v_n \) of \( V \). (If \( v_1, \ldots, v_n \) is an orthogonal basis of \( V \) then \( -\text{id}_V = \prod_{i=1}^n r_{v_i} \) and \( \det(V) = \prod_{i=1}^n \Psi(v_i, v_i) \)).

We recall the following formula due to Zassenhaus (see [Za, p. 444]) which gives a useful way to compute the spinor norm of an isometry. Note that Zassenhaus first defines a morphism \( \text{sn} \) via (a generalization of) the formula given below and then shows that \( \text{sn} \) coincides with the definition of the spinor norm as given above (see [Za, p. 446]).
Theorem 3.1 (Zassenhaus). Let $\gamma$ be an isometry of a non-degenerate quadratic space $(V, \Psi)$ over $k$. Assume that the characteristic polynomial $q$ of $\gamma$ satisfies $q(-1) \neq 0$. Then
\[
N_{\text{spin}}(\gamma) = (-2)^{-\dim V}q(-1),
\]
in $k^*/(k^*)^2$.

Corollary 3.2. Let $\gamma$ be an isometry of a non-degenerate quadratic space $(V, \Psi)$ over $k$. Then the eigenspace $V_-$ (with dimension $v_-$) of $\gamma$ associated to the eigenvalue $-1$ is non-degenerate and, if we denote by $q_-$ the polynomial such that
\[
q(T) = (T + 1)^{v_-}q_-(T), \quad q_-(1) \neq 0,
\]
then
\[
N_{\text{spin}}(\gamma) = \det(V_-)(-2)^{-(\dim V - v_-)}q_-(1).
\]

Proof. The subspace $V_-$ is the eigenspace associated to the eigenvalue $1$ for the isometry $-\gamma$. Thus it is a non-degenerate subspace (see [Mas, Lemma 2.3]) and we have
\[
V = V_- \perp V_-^\perp,
\]
where $V_-^\perp$ is the orthogonal complement of $V_-$ in $V$. Thus
\[
N_{\text{spin}}(\gamma) = N_{\text{spin}}(\gamma | V_-)N_{\text{spin}}(\gamma | V_-^\perp) = \det(V_-)N_{\text{spin}}(\gamma | V_-^\perp).
\]
The restriction of $\gamma$ to $V_-^\perp$ has characteristic polynomial $q_-$ and therefore Zassenhaus’s Theorem applies to that restriction:
\[
N_{\text{spin}}(\gamma | V_-^\perp) = (-2)^{-(\dim V - v_-)}q_-(1),
\]
which completes the proof.

Proposition 3.3. Let the notation be as above. The spinor norm of an isometry $\gamma$ with characteristic polynomial $q$ is given by
\[
N_{\text{spin}}(\gamma) = q_0(1)\det(V_-).
\]

Proof. Recall
\[
q(T) = (T - 1)^{v_+}(T + 1)^{v_-}q_0(T), \quad q_0(\pm 1) \neq 0,
\]
thus, with notation of Corollary 3.2,
\[
q_-(T) = (T - 1)^{v_+}q_0(T).
\]
We deduce
\[
N_{\text{spin}}(\gamma) = \det(V_-)(-2)^{-(\dim V - v_-)}(-2)^{v_+}q_0(-1).
\]
Therefore:
\[
N_{\text{spin}}(\gamma) = \det(V_-)(-2)^{-(\dim V - v_- + v_+)}q_0(1) = \det(V_-)(-2)^{d_0}q_0(-1),
\]
modulo nonzero squares. That is the desired formula since $d_0$ is even.

The above proposition enables us to decide when we can prescribe the spinor norm and the characteristic polynomial of an isometry.

We will say that a polynomial $q \in k[x]$ of degree $d$ is separable if it has $d$ distinct roots in an algebraic closure of $k$. 

10
Corollary 3.4. (i) Let \( q \in k[T] \) be a monic reciprocal polynomial of degree \( d \geq 1 \). If \( v_-(q) > 0 \) then there exists a non-degenerate symmetric bilinear space over \( k \) of dimension \( d \) with an isometry \( \gamma \) of characteristic polynomial \( q \) and arbitrary spinor norm \( N_{\text{spin}}(\gamma) \). In particular, this is true if \( d \) is odd.

(ii) If \( v_-(q) = 0 \) and \( \gamma \) is an isometry with characteristic polynomial \( q \) then its spinor norm equals \( q_0(-1) \) (modulo nonzero squares). In particular, this is the case if \( q \) is separable and \( d \) is even.

Proof. (i) Fix a representative \( s \) for a class in \( k^*/(k^*)^2 \). If \( v_- > 0 \) we can always choose \( V_- \) to have \( \det(V_-) \equiv s q_0(-1) \mod (k^*)^2 \). The result now follows from Proposition 3.3. If \( d \) is odd by the observations of §1.1 \( v_- \) is odd and hence positive.

(ii) The first statement follows from Proposition 3.3. Assume \( q \) to be separable; if \( v_-(q) > 0 \) then the quotient \( q(T)/(T+1) \) is a reciprocal polynomial of odd degree. So \(-1\) is also a root of the quotient which contradicts the separability of \( q \).

We now prove a generalization of the following result of Baeza, which in turn extends an earlier result of Edwards. If \( (V, \Psi) \) is a quadratic space over \( k \) we let its discriminant be \( \text{disc}(V, \Psi) := (-1)^{d(d-1)/2} \det(V, \Psi) \) where \( d := \dim V \).

Theorem 3.5 (Baeza–Edwards). Let \( q \) be a separable reciprocal polynomial in \( k[T] \) of even degree. If there exists an isometry \( q \) of a non-degenerate quadratic \( k \)-space \( (V, \Psi) \) of characteristic polynomial \( q \) then \( \text{disc}(V, \Psi) \equiv \text{disc}(q) \mod (k^*)^2 \).

We first generalize a formula due to Edwards.

Theorem 3.6. Let \( (V, \Psi) \) be a non-degenerate quadratic space over \( k \) and let \( q \) be a reciprocal polynomial which is the characteristic polynomial of an isometry of \( (V, \Psi) \). Then with the above notation

\[
\text{det}(V, \Psi) \equiv \text{det}(V_-) \text{det}(V_+) q_0(-1) q_0(1) \mod (k^*)^2.
\]

Proof. Let \( \gamma' = -\gamma^{-1} \) and \( q_0', V_+' \), etc. be the corresponding objects associated to \( q' \). We have \( N_{\text{spin}}(\gamma\gamma') = N_{\text{spin}}(- \text{id}_V) = \text{det}(V, \Psi) \). Applying Proposition 3.3 to both \( \gamma \) and \( \gamma' \), we get

\[
N_{\text{spin}}(\gamma) = \text{det}(V_-) q_0(-1), \quad N_{\text{spin}}(\gamma') = q_0'(1) \text{det}(V_+') .
\]

Now the eigenspace \( V_+' \) coincides with \( V_+ \). Indeed \( \gamma'(v) = -v \) if and only if \( \gamma\gamma'(v) = -\gamma(v) \), i.e. \( \gamma(v) = v \). We have likewise \( V_- = V_-' \). In particular the polynomials \( q_0 \) and \( q_0' \) have the same (even) degree and, because the set of roots of \( q_0 \) is invariant by inversion, the roots of \( q_0' \) are the opposites of those of \( q_0 \). We deduce \( q_0'(1) = q_0(1) \) which yields in turn the formula we wanted.

Corollary 3.7. Let \( q \in k[x] \) be a monic reciprocal polynomial with \( q(\pm 1) \neq 0 \). Then the discriminant \( \text{disc}(V, \Psi) \) of a non-degenerate quadratic space \( (V, \Psi) \) over \( k \) with an isometry of characteristic polynomial \( q \) is uniquely determined. More precisely, for any such space we have

\[
\text{det}(V, \Psi) \equiv q(-1)q(1) \mod (k^*)^2.
\]

Proof. The assumption \( q(\pm 1) \neq 0 \) precisely means that \( V_\pm \) is zero and the claim follows from (3.1).

(Proof of Theorem 3.5). It follows from Corollary 3.7 and the following lemma.
Lemma 3.8. Let \( q \in k[x] \) be a monic separable reciprocal polynomial of even degree \( 2m \). Then

\[
\text{disc } q \equiv (-1)^m q(-1)q(1) \mod (k^*)^2.
\]

Proof. The hypothesis on \( q \) guarantees that \( q(\pm 1) \neq 0 \), i.e., \( q = q_0 \). Indeed, if \( q \) is reciprocal then \( v_+ \) must be even. If in addition \( q \) is separable then \( v_+(q) = 0 \). As we argued in the proof of Corollary 3.4 (ii) we also have \( v_+(q) = 0 \).

We may assume without loss of generality that \( q \) is irreducible. Let \( K := k[x]/(q) \). The extension \( K/k \) is separable and \( \text{disc } q \) is the discriminant of the quadratic space \((K, \Psi)\), where \( \Psi(a, b) := \text{tr}_{K/k}(ab) \). A calculation like that in [McG, Prop. A.3] (see also the discussion at the beginning of section 2 in [Ba]) finishes the proof. (Let \( L \subseteq K \) be the subfield fixed by the involution \( x \mapsto x^{-1} \). Then \( K = L(x - x^{-1}) \). The subspaces \( L \) and \((x - x^{-1})L \) are orthogonal hence \( \det K = N_{L/k}(x - x^{-1}) \det L^2 \) and \( N_{L/k}(x - x^{-1}) = q(-1)q(1). \) \( \square \)

For an alternate proof of the lemma see [E, proof of Th. 2].

4. ISOMETRIES WITH GIVEN JORDAN FORM

We end with a characterization of the Jordan form of isometries of non-degenerate quadratic spaces. The main result goes back to (at least) Wall [Wa] (see also [HM], [Mil, section 3] and [SpSt, IV, 2.15 (ii)]). We include a proof for the reader’s convenience using the skew Bezoutian to construct the isometries.

We assume our field \( k \) is now algebraically closed (and of characteristic different from 2 as before). Fix a vector space \( V \) of dimension \( r \) over \( k \). For an \( M \in \text{End}(V) \), \( \lambda \in k^* \) and \( m \in \mathbb{N} \), let \( \mu(M; \lambda, m) \) be the number of Jordan blocks of \( M \) of size \( m \) and eigenvalue \( \lambda \).

Theorem 4.1. Let \( M \in \text{End}(V) \). Then \( M \) preserves a non-degenerate \( \varepsilon \)-symmetric bilinear form on \( V \) if and only if

(i) \[ \mu(M, \lambda, m) = \mu(M, \lambda^{-1}, m), \quad \lambda \neq \pm 1, \quad m \in \mathbb{N}, \]

and

(ii) \[ (m - \delta)\mu(M, \pm 1, m) \equiv 0 \mod 2, \quad m \in \mathbb{N}, \]

where \( \delta := \frac{1}{2}(1 - \varepsilon) \).

Proof. We give details for the orthogonal case \( \varepsilon = 1 \) the symplectic case \( \varepsilon = -1 \) is completely analogous. For \( m \geq 1 \) let \( J_m(\lambda) \) denote the Jordan block with size \( m \) and eigenvalue \( \lambda \).

First we exhibit an isometry with a prescribed Jordan form satisfying the hypothesis (i) and (ii). Identify \( V \) with \( k^r \). If \( M = J_m(\lambda) \oplus J_m(\lambda^{-1}) \) with \( \lambda \neq \lambda^{-1} \) consider \( q = (T - \lambda)^m(T - \lambda^{-1})^m \). By Theorem 2.1 there exists a skew-reciprocal polynomial \( p \in k[T] \) such that \( q \) is the characteristic polynomial of an isometry of the non-degenerate quadratic space determined by \( B(p, q) \), which by Theorem 1.2 has Jordan form \( M \). A similar argument applies to \( J_m(\pm 1) \) for \( m \) odd taking \( q = (T - 1)^m \) and \( p = (T + 1)^m \).

Finally, let \( m \) be even and set again \( p := (T + 1)^m \) and \( q := (T - 1)^m \). Now \( U := B(p, q) \), however, is skew-symmetric. Consider instead the symmetric matrix

\[
A = \begin{pmatrix} 0 & U \\ -U & 0 \end{pmatrix}.
\]
Since $p$ and $q$ are relatively prime $U$ and hence also $A$ yield non-degenerate bilinear pairings. By Theorem 1.2 and Theorem 2.1 there exists $\gamma^\pm$ with Jordan form $J_m(\pm 1)$ preserving $U$. The map $M := \gamma^+ \oplus \gamma^-$ then preserves $A$ giving our desired isometry.

We now show that the conditions on the multiplicities of the Jordan blocks are necessary. Suppose then that $M \in \text{End}(V)$ preserves a non-degenerate, symmetric bilinear pairing $\langle \cdot, \cdot \rangle$ on $V$. It follows that as $k[x, x^{-1}]$-modules $V^* \simeq V$. This implies (i).

For any polynomial $h \in k[x]$ we have
\begin{equation}
\langle h(M)u, v \rangle = \langle u, h(M^{-1})v \rangle, \quad u, v \in V.
\end{equation}

For $\lambda \in k^*$ let $V_\lambda \subseteq V$ be the subspace annihilated by some power of $M - \lambda$ and let $W_\lambda := V_\lambda \oplus V_{\lambda^{-1}}$ if $\lambda \neq \lambda^{-1}$ and $W_{\pm 1} := V_{\pm 1}$. Taking $h(x) = (x - \lambda)(x - \lambda^{-1})$ or $x - (\pm 1)$ in (4.1) we see that the distinct non-zero $W_\lambda$'s are mutually orthogonal with orthogonal sum $V$ and, in particular, they are non-degenerate. To prove (ii) we may hence assume without loss of generality that $M$ is unipotent so $V = V_1$.

We have

$$V = \bigoplus_{m \geq 1} V^{(m)},$$

where $M$ acts on $V^{(m)}$ as a sum of $\mu(M, 1, m)$ Jordan blocks $J_1(m)$. Let $n$ be the largest index $m$ with $V^{(m)} \neq 0$. We claim that $V^{(n)}$ is non-degenerate. For a quadratic space $W$ let $\text{rad}W := W^\perp$ be its radical.

Since $M$ preserves $\text{rad} V^{(n)}$ we have $\text{rad} V^{(n)} \subseteq \ker(M - \text{id}_V) = \Im ((M - \text{id}_V)^{n-1})$. Taking $h(x) = (x - 1)^{n-1}$ in (4.1) it follows that $\text{rad} V^{(n)} \subseteq \text{rad} V$ proving our claim.

Finally then we may assume that $V = V^{(m)}$. Define the nilpotent endomorphisms $N_\pm \in \text{End}(V)$ by $M^{\pm 1} = \text{id}_V + N_{\pm}$. Since $M$ is an isometry we have
\begin{equation}
\langle N_+ u, v \rangle = \langle u, N_- v \rangle.
\end{equation}

Define the bilinear pairing on $V$

$$(u, v) := \langle N_+ u, v \rangle - \langle N_- u, v \rangle,$$

which is skew-symmetric since it equals $\langle N_+ u, v \rangle - \langle N_+ v, u \rangle$ by (4.2). Note that $N_- = M^{-1} - \text{id}_V = -M^{-1}(M - \text{id}_V) = -M^{-1}N_+$. Hence $\ker N_+ \subseteq \ker N_-$ and by symmetry $\ker N_+ = \ker N_-$. It follows that the pairing descends to $W := V/\ker N_+$ and, moreover, it is non-degenerate on $W$. To see this, suppose that $(u, v) = 0$ for all $v \in V$. Then $\langle N_+ u, v \rangle = \langle N_- u, v \rangle$ for all $v$. Since $\langle \cdot, \cdot \rangle$ is non-degenerate this is equivalent to $N_+ u = N_- u$ or $(M^2 - \text{id}_V)u = 0$, which in turn is equivalent to $u \in \ker N_+$. This proves our claim.

As a consequence, the dimension of $W = \mu(M, 1, m)(m - 1)$ is even, which is what we wanted to prove. 

\[\square\]

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