A degenerate kernel method for eigenvalue problems of a class of non-compact operators

H. Majidian\textsuperscript{a} E. Babolian\textsuperscript{b,*}

\textsuperscript{a}Department of Mathematics, Tarbiat Modares University, PO Box 14115-175 Tehran, Iran
\textsuperscript{b}Department of Mathematics, Tarbiat Moallem University, Tehran 1561836314, Iran

Abstract

We consider the eigenvalue problem of certain kind of non-compact linear operators given as the sum of a multiplication and a kernel operator. A degenerate kernel method is used to approximate isolated eigenvalues. It is shown that entries of the corresponding matrix of this method can be evaluated exactly. The convergence of the method is proved; it is proved that the convergence rate is $O(h)$. By some numerical examples, we confirm the results.

Key words: Eigenvalue problem; Non-compact operators; Multiplicative operators; Degenerate kernel method

MSC: 45C05

1 Introduction

Consider the following non-compact operator defined on $X = \mathcal{L}^\infty[a,b]$ by

$$\mathcal{A} f(x) = \alpha \int_a^b k(x, u) f(u) \, du - x^2 f(x), \quad \forall x \in [a,b].$$

(1)

Here, $[a, b]$ is a compact interval, $k(.,.) \in \mathcal{C}^0([a, b] \times [a, b])$, and $\alpha$ is a real constant. The non-compactness of the operator $\mathcal{A}$ is due to the involved multiplication operator with the corresponding function $x^2$ [6].

* Corresponding author. Tel: +98 21 7750 7722 Fax: +98 21 7760 2988

Email addresses: majid@modares.ac.ir (H. Majidian), babolian@saba.tmu.ac.ir (E. Babolian).
We are concerned with the numerical approximation of isolated eigenvalues of the following eigenvalue problem:

\[ \lambda f = \mathcal{A} f. \]  

(2)

Eigenproblem (2) has some applications in electromagnetism [9]. Also, a more general form problem (2) describes the COA model in population genetics [2]. In [7], this model has been treated numerically by Nyström and Galerkin methods in the presence of some extra assumptions. A degenerate kernel method with piecewise linear interpolation with respect to the variable \( u \) has been developed for problem (2) in [5]. In this method, isolated eigenvalues of problem (2) are approximated by eigenvalues of an operator \( \mathcal{A}_n \) with rank \( n \); but the entries of the matrix eigenvalue problem associated with \( \mathcal{A}_n \) can not be evaluated exactly. Hence, we need to replace various integrals by numerical quadratures. This perturbation in entries causes unacceptable approximations for ill-conditioned eigenvalues.

In [4] a degenerate kernel method is proposed for the numerical solutions of integral equations of the second kind with a smooth kernel which does not need any numerical quadrature, and the entries of its corresponding matrix are evaluated exactly if we ignore round-off errors. Genaneshwar [3] has been extended this method to the corresponding eigenvalue problem with the same benefit. In this paper we extend this method for the numerical solutions of eigenproblem (2).

The rest of this paper is organized as follows: In Section 2 we describe the degenerate kernel method for numerical solutions of problem (2). Convergence of the method and its rate are discussed in Section 2. In Section 4 we give some numerical results to illustrate the accuracy of the method.

We recall that \( X = \mathcal{L}^\infty[a,b] \) throughout this paper.

2 Degenerate kernel method

We begin this section by some preliminaries.

Let \( Y \) be any Banach space over the complex field \( \mathbb{C} \). We denote the space of bounded linear operators from \( Y \) into \( Y \) by \( BL(Y) \). Let \( T \in BL(Y) \). The resolvent set of \( T \) is given by

\[ \rho(T) = \{ z \in \mathbb{C} : (T - zI)^{-1} \in BL(Y) \}. \]

The spectrum of \( T \), denoted by \( \sigma(T) \), is defined as \( \sigma(T) = \mathbb{C} \setminus \rho(T) \). The point spectrum of \( T \) consists of all \( \lambda \in \sigma(T) \) such that \( T - \lambda I \) is not one-to-one.
In this case, \( \lambda \) is called an eigenvalue of \( T \). If \( \lambda \) is an eigenvalue of \( T \), then the smallest positive integer \( l \) such that \( \ker(T - \lambda I)^l = \ker(T - \lambda I)^{l+1} \) is called the ascent of \( \lambda \). The dimensions of \( \ker(T - \lambda I) \) and \( \ker(T - \lambda I)^l \) are called geometric multiplicity and algebraic multiplicity of \( \lambda \), respectively. If the algebraic multiplicity of \( \lambda \) equals one, then it is called a simple eigenvalue of \( T \).

We now describe the degenerate kernel method. Define the operators \( \mathcal{K} \) and \( \mathcal{M} \) on \( X \) as follows:

\[
\mathcal{K} f(x) = \alpha \int_a^b k(x, u) f(u) \, du,
\]

and

\[
\mathcal{M} f(x) = x^2 f(x).
\]

Then \( \mathcal{A} = \mathcal{K} - \mathcal{M} \).

For an integer \( n > 1 \), consider the following partition of \([a, b]\):

\[
a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.
\]

Let \( I_j = [x_{j-1}, x_j] \), \( h_j = x_j - x_{j-1} \) for \( j = 1, 2, \ldots, n \). Also, let \( h = \max_{1 \leq j \leq n} h_j \) denotes the norm of the partition. We assume that \( h \to 0 \) as \( n \to \infty \). For a positive integer \( r \), let \( B_r = \{\tau_1, \ldots, \tau_r\} \) be the set of \( r \) Gauss points, i.e., the zeros of the Legendre polynomial \( \frac{d^r}{dt^r}(t^2 - 1)^r \) in the interval \([-1, 1]\). Define \( f_j : [-1, 1] \to I_j \) as follows:

\[
f_j(t) = \frac{1-t}{2} x_{j-1} + \frac{1+t}{2} x_j, \quad t \in [-1, 1].
\]

Then \( S = \bigcup_{j=1}^n f_j(B_r) = \{\xi_{ij} = f_j(\tau_i) : i = 1, \ldots, r, j = 1, 2, \ldots, n\} \) is the set of \( nh = nr \) Gauss points on \([a, b]\). For each \( i = 1, \ldots, r \), let

\[
\ell_i(x) = \frac{(x - \tau_1) \cdots (x - \tau_{i-1})(x - \tau_{i+1}) \cdots (x - \tau_r)}{(\tau_i - \tau_1) \cdots (\tau_i - \tau_{i-1})(\tau_i - \tau_{i+1}) \cdots (\tau_i - \tau_r)}, \quad x \in [-1, 1],
\]

be the Lagrange polynomial of degree \( r - 1 \) on \([-1, 1]\), which satisfies \( \ell_i(\tau_j) = \delta_{ij} \). It is notable that for \( r = 1 \), \( \ell_1(x) = 1 \) on \([-1, 1]\). Define

\[
\phi_{pq}(x) = \begin{cases} 
\ell_p(f_q^{-1}(x)), & x \in I_q, \\
0, & \text{otherwise}.
\end{cases}
\]

Then \( \phi_{pq}(\xi_{ij}) = \delta_{pq}\delta_{jq} \), for \( p, i = 1, \ldots, r \) and \( j, q = 1, 2, \ldots, n \).

We set the following notations \( t_{(j-1)r+i} = \xi_{ij}, \psi_{(j-1)r+i} = \phi_{ij} \), for \( i = 1, \ldots, r \) and \( j = 1, 2, \ldots, n \). Then \( S = \{t_i : i = 1, 2, \ldots, nh\} \) is the set of \( nr \) Gauss
points in \([a, b]\). Now define the degenerate kernel by
\[
k_{N_h}(x, u) = \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} k(t_i, t_j) \psi_i(x) \psi_j(u), \quad x, u \in [a, b].
\]
The approximated operator \(K_{N_h}\) is defined by
\[
K_{N_h} f(x) = \alpha \int_a^b k_{N_h}(x, u) f(u) \, du. \tag{5}
\]
For the approximation of the multiplication operator \(M\), define the piecewise constant functions \(s_{N_h}(x)\) on \([a, b]\) as follows:

If \(x \in I_i\), then
\[
s_{N_h}(x) = m_i = \left(\frac{x_{i-1} + x_i}{2}\right)^2, \quad i = 1, 2, \ldots, N_h.
\]

Now define the approximate operators \(K_{N_h}\) and \(M_{N_h}\) on \(X\) as follows:
\[
M_{N_h} f(x) = s_{N_h}(x) f(x). \tag{6}
\]
Consider now the following approximate eigenvalue problem on \(X\):
\[
\lambda_{N_h} f_{N_h}(x) = A_{N_h} f_{N_h}(x), \tag{7}
\]
where \(A_{N_h} = K_{N_h} - M_{N_h}\). The rank of the operator \(A_{N_h}\) is finite, so the eigenvalue problem (7) is equivalent to a matrix eigenvalue problem as we show in the following:

Eigenproblem (7) expands into
\[
\lambda_{N_h} f_{N_h}(x) = \alpha \sum_{p=1}^{N_h} \sum_{j=1}^{N_h} k(t_p, t_j) \psi_p(x) c_j - s_{N_h}(x) f_{N_h}(x), \tag{8}
\]
where
\[
c_j = \int_a^b \psi_j(u) f_{N_h}(u) \, du.
\]
For each \(i = 1, 2, \ldots, N_h\), we multiply both sides of (8) by \(\psi_i(x)\) and integrate over the interval \([a, b]\) with respect to the variable \(x\) to obtain
\[
\lambda_{N_h} c_i = \alpha \sum_{j=1}^{N_h} k_{i,j} c_j - m_i c_i, \quad i = 0, 1, \ldots, n, \tag{9}
\]
where
\[
k_{i,j} = \sum_{p=1}^{N_h} k(t_p, t_j) \int_a^b \psi_i(x) \psi_p(x) \, dx.
\]
Therefore, eigenvalues of the approximate problem (7) are the eigenvalues of the matrix
\[ A = (a_{i,j}) \text{ of order } N_h, \]
defined as
\[ a_{i,j} = \alpha k_{i,j} - m_i \delta_{i,j}, \quad i, j = 1, 2, \ldots, N_h. \] (10)

It is easy to see that
\[ \int_a^b \psi_i(x) \psi_p(x) \, dx = 0, \quad \text{if } i \neq p. \]

Therefore, \( k_{i,j} \) reduces to
\[ k_{i,j} = k(t_i, t_j) \int_a^b \psi_i^2(x) \, dx. \] (11)

Since \( \psi_i \) for \( i = 1, 2, \ldots, N_h \) are basis functions having small support on \([a, b]\) with the simple structure, the integrals \( \int_a^b \psi_i^2(x) \, dx \) can be evaluated exactly. Values of these integrals depend on the length of their support. Thus, if \( x_i \) for \( i = 1, 2, \ldots, n \) are equidistance points and the kernel \( k(.,.) \) is Hermitian, then the matrix \( A \) is normal and as a result, each eigenvalue of \( A \) is well-conditioned (see [8]).

### 3 Convergence and the rate of convergence

**Theorem 1** [1] Let \( Y \) be a Banach space. Let also \( T \) and \( \{T_n\}_{n=0}^{\infty} \) be operators in \( BL(Y) \). Assume that the sequence \( \{T_n\} \) converges in norm to \( T \), i.e., \( \|T_n - T\| \to 0 \) as \( n \to \infty \). Let \( \lambda \) be an isolated point of \( \sigma(T) \). For each positive \( \epsilon < \text{dist}(\lambda, \sigma(T) \setminus \{\lambda\}) \), define
\[ \Lambda_n := \{\lambda_n \in \sigma(T_n) : |\lambda_n - \lambda| < \epsilon\}. \]

Then for \( n \) large enough, \( \Lambda_n \neq \emptyset \) and if \( \lambda_n \in \Lambda_n \), the sequence \( \{\lambda_n\} \) converges to \( \lambda \).

In order to show that the degenerate kernel method proposed in previous section is convergent, it is enough to show that the operators \( A \) and \( \{A_{N_h}\} \) satisfy conditions of Theorem 1 i.e., they are in \( BL(X) \) and \( A_{N_h} \) converges in norm to \( A \) as \( N_h \to \infty \). The linearity of these operators is obvious. In [5], we have shown that the operator \( A \) is bounded. By a similar discussion it is seen that the operator \( A_{N_h} \) is bounded, too.

If we assume that \( k(.,.) \in C^r([a, b] \times [a, b]) \), then \( \|K - K_{N_h}\| = O(h^r) \) [3]. Also, from [5] we have \( \|M - M_{N_h}\| = O(h) \). Therefore,
\[ \|A - A_{N_h}\| = O(h). \] (12)
Table 1

| $n$ | $|\lambda_n(1) - \lambda(1)|$ | ratio(1) | $|\lambda_n(2) - \lambda(2)|$ | ratio(2) |
|-----|------------------------------|----------|------------------------------|----------|
| 10  | 0.0763                       |          | 0.0916                       |          |
| 20  | 0.0398                       | 1.92     | 0.0460                       | 1.99     |
| 40  | 0.0202                       | 1.97     | 0.0229                       | 2.01     |
| 80  | 0.0100                       | 2.02     | 0.0113                       | 2.03     |
| 160 | 0.0048                       | 2.08     | 0.0054                       | 2.09     |

On the other hand, with the assumptions of Theorem [1] if $\lambda$ is a simple eigenvalue of $A$, then there exists a constant $c$ such that [1] Page 201]

$$|\lambda_{N_h} - \lambda| \leq c \|A_{N_h} - A\|,$$  \hspace{1em} \text{for all large } N_h. \hspace{1em} (13)

From (12) and (13) we conclude that the proposed degenerate kernel method of previous section is convergent with the rate $O(h)$ for simple eigenvalues.

4 Numerical results

In this section, we solve two samples of problem (2) using the degenerate kernel method, proposed in Section 2. In order to show the convergence, the problems are solved for different numbers of interpolation nodes. Two arbitrary positive eigenvalues of this problem, denoted by $\lambda(1)$ and $\lambda(2)$, are approximated by $\lambda_n(1)$ and $\lambda_n(2)$, respectively. In both examples, we take $r = 2$ and compute the absolute error and the ratio of each eigenvalue.

4.1 Example 1

Consider problem (2) with the kernel

$$k(x, u) = \exp(-(u - x)^2),$$

and the constant $\alpha = 1$ on the interval $[-2, 2]$. Computational results are shown in Table 1.
Table 2

Computational results for sample problem 2.

| n  | $|\lambda_n(1) - \lambda(1)|$ | ratio(1) | $|\lambda_n(2) - \lambda(2)|$ | ratio(2) |
|----|-----------------------------|----------|-----------------------------|----------|
| 10 | 0.1607                      |          | 0.0712                      |          |
| 20 | 0.0793                      | 2.03     | 0.0287                      | 2.48     |
| 40 | 0.0396                      | 2.00     | 0.0124                      | 2.31     |
| 80 | 0.0196                      | 2.02     | 0.0055                      | 2.25     |
| 160| 0.0095                      | 2.06     | 0.0025                      | 2.20     |

4.2 Example 2

Consider problem (2) with the kernel

$$k(x, u) = \frac{1}{1 + (u - x)^2}$$

and the constant $\alpha = 1$ on the interval $[-4, 4]$. Computational results are shown in Table 2.

In both examples, convergence of solutions is seen. The ratio of error in each step also confirms our theoretical discussion in Section 3.

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