In general quantum systems there are two kinds of spacetime modes, those that fluctuate and those that do not. Fluctuating modes have normalizable wavefunctions. In the context of 2D gravity and “non-critical” string theory these are called macroscopic states. The theory is independent of the initial Euclidean background values of these modes. Non-fluctuating modes have non-normalizable wavefunctions and correspond to microscopic states. The theory depends on the background value of these non-fluctuating modes, at least to all orders in perturbation theory. They are superselection parameters and should not be minimized over. Such superselection parameters are well known in field theory. Examples in string theory include the couplings \( t_k \) (including the cosmological constant) in the matrix models and the mass of the two-dimensional Euclidean black hole. We use our analysis to argue for the finiteness of the string perturbation expansion around these backgrounds.
Introduction and General Discussion

Many of the important questions in string theory circle around the issue of background independence. String theory, as a theory of quantum gravity, should dynamically pick its own spacetime background. We would expect that this choice would be independent of the classical solution around which the theory is initially defined. We may hope that the theory finds a unique ground state which describes our world.

We can try to draw lessons that bear on these questions from the exactly solvable matrix models of low dimensional string theory \[1\] \[2\] \[3\]. The physics of these models depends on a variety of parameters – to all orders in perturbation theory in every case, and in those models that are well defined, nonperturbatively as well. Example of these parameters (which can be considered superselection sector labels) include the KdV times \(t_k\) in the \(c < 1\) systems and the cosmological constant and radius of the \(c = 1\) system\[4\]. One might think that the presence of such superselection sectors is a special property, related in some way to the integrability of the matrix models. It will be one goal of this paper to argue that this is not the case, and that, at least to all orders in perturbation theory, it is a rather generic phenomenon. Much of our exposition is elementary but because these questions have been the source of a certain amount of confusion both for ourselves and others we thought it worthwhile to present a full discussion. We try to provide simple and intuitive pictures of some known results. Our general discussion applies both to field theory (where these phenomena are well known) and to string theory. We will present the arguments in the language of string theory.

We will be discussing strings embedded in a Euclidean target space. In such a situation we often restrict attention to operators in the world-sheet conformal field theory whose dimensions are bounded from below. These correspond to (delta-function) normalizable wavefunctions in the conformal field theory Hilbert space. In a space-time description the kinetic term in a target space “string field theory” Lagrangian is

\[
\Psi K \Psi
\]

where, essentially, \(K = L_0 - 1\). Therefore the eigenfunctions of \(K\) are the same as the eigenfunctions of \(L_0\). It is a basic principle of quantum mechanics that in the space-time functional integral we sum over (delta-function) normalizable eigenfunctions of \(K\). In a first quantized formalism (in field theory or in string theory) this fact follows from sewing amplitudes by inserting a complete set of normalizable states in intermediate channels.
Therefore, the quantum fluctuations and the states which flow in loops have normalizable wave functions and can be expanded in the normalizable eigenfunctions of $\mathcal{K}$.

Clearly, for a well defined Euclidean functional integral over the string field $\Psi$ the spectrum of $\mathcal{K}$ should not only be bounded from below but there should be no negative eigenvalues. Let us examine the different possible normalizable eigenmodes of $\mathcal{K}$:

1. Positive eigenmodes of $\mathcal{K}$ – “stable modes.” The integral over them is damped and leads to finite effects in perturbation theory.

2. Negative eigenmodes of $\mathcal{K}$ – “unstable modes.” The integral over them is divergent. In a proper time description of the amplitudes, there are exponential divergences. From a space-time point of view, these correspond to an instability of the system and render the expansion inconsistent. If the system has a non-compact direction and the spectrum of $\mathcal{K}$ is continuous, the existence of such negative modes implies the existence of associated (dressed) zero modes which are on-shell in Euclidean space.

3. Zero eigenmodes of $\mathcal{K}$ – “on-shell modes.” The integral over these modes depends on the details of the interaction terms in the Lagrangian. The most interesting case is when there is no potential for these modes. Then they should be treated carefully. The moduli of the known vacua of the critical string are of this kind. The existence of such modes can lead to mild infrared divergences (power law in proper time) in one loop diagrams. Their consequences at higher orders in perturbation theory depends on their couplings and will be discussed below.

It is also important for our purposes to examine non-normalizable eigenfunctions of $\mathcal{K}$ although such wavefunctions are typically ignored. Allowing exponentially growing wavefunctions in Euclidean space, one finds many new solutions of the linearized equations of motion

$$\mathcal{K}\Psi = 0 \quad .$$

From the world-sheet point of view they correspond to $(1,1)$ operators. For example, one can construct operators like $e^{kX}$ out of the free field $X$ corresponding to a non-compact space coordinate. Their dimensions, $\Delta = -\frac{1}{2}k^2$, are not bounded from below for $k$ real. Using such negative dimension operators, other operators in the theory can be “dressed” and be turned into $(1,1)$ operators $O_k$. At least infinitesimally, one can add such on-shell operators with small coefficients $t_k$ to the world-sheet Lagrangian and construct new backgrounds. Such backgrounds can be thought of as the analytic continuations of oscillating Minkowski time dependent solutions to Euclidean space.
It is central to the point we are trying to make that such backgrounds do not fluctuate. As we discussed above, only the normalizable eigenmodes of $K$ are subject to quantum fluctuations. The “state” corresponding to the operator $O_k$ has non-normalizable (not even delta-function normalizable) wavefunction, is out of the conformal field theory Hilbert space, and therefore does not propagate in loops. Therefore, the corresponding background shifting parameter $t_k$ does not fluctuate. It labels a superselection sector and cannot be changed to all orders in perturbation theory. Clearly, the vacuum amplitude depends on $t_k$ but because it does not fluctuate, it will not relax to a minimum – every value is allowed and corresponds to a different background. Since the field $\Psi$ is arbitrarily large at infinity, it is intuitively clear why it is impossible to change the value of $t_k$. Regardless of how small the perturbation $t_k$ is, its effect at infinity is arbitrarily large.

In the Minkowski time picture of the non-compact coordinate, the parameter $t_k$ corresponds to an initial condition labeling the classical solution. Since Minkowskian evolution is oscillatory, the initial conditions have effects even at long time.

In the rest of this note we will discuss several examples of these phenomena in low dimensional string theory. A common characteristic of all these examples is the presence of a string coupling constant that varies rapidly in space.

**Examples: Theories with rapidly varying coupling constant**

An important class of backgrounds have a non-compact dimension $\phi$ and a dilaton field expectation value linear in $\phi$. The quadratic part of the target space effective Lagrangian is

$$\mathcal{L} = e^{-Q\phi} \left( \frac{1}{2} (\partial_\phi \Phi)^2 + (\Delta - 1) \Phi^2 \right)$$

where $\Phi$ is a generic target space field associated with an operator of dimension $\Delta$ in the conformal field theory of all the other coordinates. In general, there are many such fields,

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1 We should mention a more familiar example of superselection sectors, distinct from the one discussed in the text. It occurs both in field theory and in string theory with more than two non-compact asymptotically flat directions. Whenever there is a set of truly degenerate ground states (whether related by a symmetry or not), the Hilbert space breaks into separate Hilbert spaces (superselection sectors) built on top of every such ground state. Examples of such superselection sectors occur in the critical string and are labeled by the moduli of the conformal field theory. It is important that when the non-compact directions are compactified these moduli fluctuate. This distinguishes this situation from the one described in the text. We thank J. Distler and D. Kutasov for discussions of these points.
and if there are some more non-compact dimensions, even a continuum of such fields. In this case we should also include a graviton field and a dynamical dilaton field. However, in order to keep the notation simple we pretend that there is only one \( \Phi \). Clearly, the string coupling \( g_{st} = e^{\frac{1}{2}Q\phi} \) depends on \( \phi \). The theory is weakly coupled for \( \phi \to -\infty \) and strongly coupled for \( \phi \to \infty \). In order to apply our analysis above we should redefine the fields such that the kinetic term is canonical as in (1). Defining

\[
\Psi = g_{st}^{-1} \Phi = e^{-\frac{1}{2}Q\phi} \Phi ,
\]

and integrating by parts in \( \phi \), the kinetic term (3) becomes

\[
\frac{1}{2} (\partial_\phi \Psi)^2 + \frac{1}{2} m^2 \Psi^2
\]

with

\[
\frac{1}{2} m^2 = \Delta + \frac{Q^2}{8} - 1
\]

and we recognize \( m \) as the mass of the field \( \Psi \). The interaction terms have roughly the form

\[
\frac{1}{3} e^{-Q\phi} \Phi^3 = \frac{1}{3} e^{\frac{1}{2}Q\phi} \Psi^3 .
\]

The target space objects \( \Phi \) and \( \Psi \) play important roles in the first quantized formalism. \( \Psi \) is the wavefunction of the first quantized state and \( \Phi = e^{\frac{1}{2}Q\phi} \Psi \) is related to the corresponding vertex operator \( V \). The extra factor of the string coupling \( e^{\frac{1}{2}Q\phi} \) can be understood from a change of variables between the sphere and the cylinder \([5]\) \([6]\). The change of variable is singular at the location of the operator and induces a delta function contribution to the two curvature \( R^{(2)} \) which is consistent with the change of the Euler character between these two topologies. This has the effect of shifting the world-sheet action by \( \frac{1}{2}Q\phi \) or, equivalently, multiplying the operator by \( e^{\frac{1}{2}Q\phi} \). Because of this factor, the identity operator in the world-sheet theory is associated with the wavefunction \( e^{-\frac{1}{2}Q\phi} \) which is not normalizable at \( \phi \to -\infty \) and is not in the Hilbert space of the conformal field theory. Since the \( SL(2,C) \) invariant state is not in the Hilbert space, many of the standard properties of a conformal field theory are not satisfied. In particular, the correspondence between states and operators breaks down \([5]\) \([2]\).

\[ Pointing out an analogy to a similar situation in mathematics G. Zuckerman has suggested the term “non-amenable quantum field theories” for these theories.
Without modifying the background the theory is ill defined. There is a region in the target space $\phi \to +\infty$ where the interactions (7) are infinitely strong. A consistent background should prevent the string from reaching this region. There should be a “wall” at $\phi$ of order one. Two such walls have been discussed in the literature. One possibility is to turn on an expectation value of some field $\Phi$ which modifies the naive kinetic term

$$K_0 = -\frac{1}{2} \partial_\phi^2 + \frac{1}{2} m^2$$

(8)

to

$$K = -\frac{1}{2} \partial_\phi^2 + \frac{1}{2} m^2 + \langle \Phi \rangle$$

(9)

with $\lim_{\phi \to +\infty} \langle \Phi \rangle = +\infty$. Of course, one should make sure that this modification is a solution of the equations of motion. An example of such a wall is the cosmological constant in Liouville theory \[7\][8][9][5][6]. Alternatively, since the target space metric is one of the fields, it can be chosen so that the region of large positive $\phi$ is not in the target space, as in the two dimensional Euclidean black hole \[10\]. We will discuss these two special cases in detail below.

We now examine the eigenmodes of the kinetic operator $K$ in this situation. The wall which suppresses the region of large $\phi$ affects the functional form of the eigenfunctions. The modes which fluctuate are the (delta-function) normalizable eigenmodes. With a soft wall as in (8) the wavefunctions vanish rapidly as $\phi \to \infty$ and with a hard wall as in the black hole they should satisfy the proper boundary conditions at the boundary of $\phi$. In most interesting cases the modification of the naive kinetic term (8) is small as $\phi \to -\infty$, and the analysis there is simple. A wavefunction with $K$ eigenvalue $\lambda > \frac{1}{2} m^2$ behaves there as $\sin k(\phi + \phi_0)$ with $k^2 + m^2 = 2\lambda$ and is delta function normalizable. A wavefunction with eigenvalue $\lambda < \frac{1}{2} m^2$ behaves there as $e^{k(\phi + \phi_0)}$ with $k = \sqrt{m^2 - 2\lambda}$ and has finite norm (“bound state”). Depending on the details of the theory there might or might not be negative $\lambda$ normalizable eigenmodes. If there are, the background is unstable. Otherwise, all the normalizable eigenmodes have positive $\lambda$ and the theory is finite. If there is a $\lambda = 0$ eigenmode or if there is a continuum $\lambda \in (0, \infty)$ a more careful analysis is necessary (see below).

As we mentioned above, we should also consider the non-normalizable zero modes of $K$. These can be turned on as perturbations. Since they do not fluctuate, they label superselection sectors. Just as for the normalizable modes, to ensure a sensible perturbation theory we should impose proper boundary conditions in the strong coupling region $\phi \to \infty$
– the wavefunction should decay. Then the lack of normalizability can come only from a divergence in the weak coupling region $\phi \to -\infty$.

Unlike the standard backgrounds of the critical string ($Q = 0$) here the mass $m^2$ is not equal to $2(\Delta - 1)$. Since $\Delta$ is the dimension of an operator in the conformal field theory, the standard identification of relevant operators with tachyons, irrelevant ones with massive and marginal operators with massless states breaks down [5]. In particular, truly marginal operators with $\Delta = 1$ are massive. The solution of the equation of motion $\mathcal{K}\Psi = 0$ depends on $\phi$ and is not normalizable. Hence, unlike the $Q = 0$ case, here the moduli do not fluctuate (again, to all orders in perturbation theory).

There is a simple spacetime picture of this lack of fluctuation. Imagine doing the functional integral over $\Phi$ with Lagrangian (3) as weight. In the region $\phi \to -\infty$ the string coupling vanishes and so the quantum fluctuations of $\Phi$ are strongly suppressed. Any classical background set in this region will remain locked. We should emphasize that this locking is field theoretic in character – there is nothing intrinsically “stringy” about it.

A more familiar (and closely related) example of this locking phenomenon occurs in field theories defined on a noncompact base space of constant negative curvature [11], e.g., the Poincaré disk. Here the constant zero eigenmode of the Laplacian is exponentially non-normalizable because of the exponentially large amount of volume at large distance. The normalizable spectrum is massive. The constant mode does not fluctuate, to all orders in perturbation theory, and functions as a superselection parameter. Here the locking is most naturally thought of as arising from the huge volume at infinity rather than the small coupling constant of the above discussion, but the distinction is primarily semantic. We should remind the reader that nonperturbative effects in these theories can be much larger than in flat space [11] and can in fact cause the zero mode to fluctuate.

2D gravity:

An important subclass of the theories with a rapidly varying coupling constant are the “non-critical strings.” These can also be interpreted as two-dimensional gravity coupled to matter [7][12]. The field $\phi$ is the conformal factor of the two-dimensional metric [13][14]. They have a non-perturbative realization as matrix models. The wall is provided by the world sheet cosmological constant. Thinking about these theories as theories of gravity provides us with a complementary interpretation of the discussion above.
The insertion of a state into the functional integral is obtained by cutting a little hole and gluing in the appropriate wavefunction. The length of the boundary in the fluctuating metric $\ell = \oint e^{\gamma\phi/2}$ is one of the degrees of freedom in the wave function. Surprisingly, it turns out that the Liouville parts of the wave functions $\Psi(\phi = \frac{2}{\gamma} \log \ell)$ satisfy the minisuperspace Wheeler-de-Witt equation

$$
\left( -\frac{1}{2} \partial^2_\phi + \frac{1}{2} \gamma^2 \mu \ell^2 + \frac{\gamma^2}{8} \nu^2 \right) \Psi_\nu(\phi) = 0
$$

which is the equation for an eigenfunction of the kinetic term (9). Demanding that the wave function suppresses infinite size holes leads to the IR boundary condition $\lim_{\phi \to \infty} \Psi(\phi) = 0$ which implies

$$
\Psi_\nu = \frac{1}{\pi} \sqrt{\nu \sin(\pi \nu)} K_\nu(2\sqrt{\mu\ell}) .
$$

In the general discussion above this boundary condition was motivated by suppressing the wave function in the region of infinitely strong coupling.

The delta function normalizable wave functions with imaginary $\nu$ propagate in loops and those with real $\nu$ diverge in the UV ($\phi \to -\infty$) and do not fluctuate. The latter label different backgrounds for the string. The gravitational interpretation of this fact is as follows. The states which propagate in handles have finite size. Hence their wave functions are localized in $\phi$ space and are spanned by the delta function normalizable states with imaginary $\nu$. Since these states have finite size they are called macroscopic. The states associated with real $\nu$ are not normalizable. Their wave functions are peaked at short distance and therefore they describe microscopic holes. Clearly only they can be associated with local operators. In this context world-sheet locality provides a natural reason to examine non-normalizable states.

In the general discussion above we motivated non-normalizable deformations by comparing them with an analytic continuation of backgrounds which oscillate in Minkowski time. This Euclidean time picture is also natural in gravity. Here we can define the theory

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3 We would like to note in passing a peculiarity of two-dimensional gravity with a negative cosmological constant. Clearly, the Euclidean functional integral is unstable. However, this theory might be a consistent Minkowskian world-sheet theory without topology change so that the tachyons are not excited. Unlike the positive cosmological constant theory, the gravitational wave functions of all matter states are normalizable. The two linearly independent solutions of (10) behave like $\ell^{-\frac{1}{4}} e^{\pm 2i\sqrt{\mu}\ell}$ as $\phi \to +\infty$. Any linear combination of them is normalizable there. This freedom can be used to pick a normalizable solution as $\phi \to -\infty$ even for real $\nu$ in (10).
at some distance scale \[ \phi \] and let it evolve by the renormalization group into the IR. In this picture, as many workers have pointed out, \( \phi \) plays the role of the renormalization group time. The values of the fields at some scale are the initial conditions for this renormalization group evolution and they affect the final answer. Therefore, one should not average over them but study the answers as a function of these initial conditions. This is the two-dimensional gravity version of the background dependence and superselection sectors we discussed above.

The \( c = 1 \) model:

All the physical operators in the \( c < 1 \) minimal models are dressed with real \( \nu > 0 \) Liouville wave functions and are massive. The \( c = 1 \) system has a massless operator – the identity operator. Its on-shell wave function \( K_0(2\sqrt{\mu \ell}) \) is proportional to \( \phi \) at \( \phi \to -\infty \). Hence it is not normalizable and the local vertex operator is \( \phi e^{\gamma \phi} = \phi e^{\gamma \phi} \) rather than \( e^{\gamma \phi} \) (the latter decouples \[ 16 \]). By the general discussion above, its coefficient in the action, \( \mu \), does not fluctuate and corresponds to a superselection sector.

The cosmological constant is almost a macroscopic state; it is at the bottom of the continuum of normalizable states with wave functions \( \Psi_\nu = \frac{1}{\pi} \nu \sin(\pi \nu) K_\nu(2\sqrt{\mu \ell}) \) with \( \nu \) imaginary. The fact that it is not macroscopic – it cannot quite propagate in intermediate channels – should have a signature in correlation functions. To examine this in more detail, consider the four point function of four microscopic states with \( X \) momenta \( p_1, p_2, p_3, p_4 \). Since \( \nu \) is not conserved, the singular part of the amplitude is given by

\[
\int_0^{i\infty} d\nu \frac{f_{p_1,p_2}(\nu)f_{p_3,p_4}(\nu)}{\nu^2 - (p_1 + p_2)^2} \tag{12}
\]

with the integral along the imaginary \( \nu \) axis \[ 5 \]. The function \( f_{p_1,p_2}(\nu) \) is a form factor for the on-shell microscopic tachyons of momenta \( p_1, p_2 \) to couple to the macroscopic state with imaginary \( \nu \). It can also be thought of as an operator product coefficient (except that there is no local operator with imaginary \( \nu \)). The expression for the four point function \[ 18 \] \[ 19 \] \[ 17 \] shows that the singularity of the amplitude as \( p_1 + p_2 \to 0 \) is proportional to \( |p_1 + p_2| \). This translates into \( f(\nu) \sim \nu \) as \( \nu \to 0 \) thus exhibiting the decoupling of the vanishing \( \nu \) state. We therefore expect that a general vertex of three macroscopic states \( f(\nu_1, \nu_2, \nu_3) \sim \nu_i \) as any \( \nu_i \to 0 \).

The following simple heuristic argument explains this result. For small imaginary \( \nu_i \) we expect the form factor to be given by

\[
f(\nu_1, \nu_2, \nu_3) = \langle \Psi_{\nu_1} | \Phi_{\nu_2} | \Psi_{\nu_3} \rangle = \int d\phi e^{\frac{Q}{2} \phi} \Phi_{\nu_1}(\phi) \Psi_{\nu_2}(\phi) \Psi_{\nu_3}(\phi) . \tag{13}\]
The wall causes $\Psi$ to decay quickly as $\phi \rightarrow \infty$ and $g_{st} = e^{\frac{2}{\phi}}$ falls rapidly as $\phi \rightarrow -\infty$ so the integral in (13) is dominated by a finite region around $\phi = 0$. For $\phi$ finite and $\nu \rightarrow 0$, the wave function $\Psi_\nu(\phi) = N_\nu \Psi_0(\phi)$ where $N_\nu$ is a normalization factor. For $\nu$ fixed as $\phi \rightarrow -\infty$, $\Psi_\nu(\phi)$ must become a linear combination of solutions of the free WdW equation, $\sin(\nu \phi)$ and $\cos(\nu \phi)$. We demand smooth matching and delta function normalizability. Using the fact that $\Psi_0$ has nonvanishing derivative in the matching region ($\Psi_0 \sim \phi$ as $\phi \rightarrow -\infty$) we see that $\sin(\nu \phi)$ is the dominant solution and that $N_\nu \sim \nu$. For small $\nu$ the wavefunction is small near the wall. (Of course, one can derive this result from the known properties of the Bessel functions.) This is a signature of the $\nu = 0$ state not being normalizable. Therefore, as any $\nu_i$ approaches zero, the form factor is proportional to $\nu_i$. If all $\nu_i$ are small,

$$f(\nu_1, \nu_2, \nu_3) \sim \nu_1 \nu_2 \nu_3 .$$

(14)

We have thus recovered the familiar derivative coupling of the tachyon field [20][19][18][17][21][22][23].

This low $\nu$ behavior is important for the finiteness of the theory. Divergences in string theory are IR divergences associated with on-shell states in intermediate lines. Since the on-shell zero momentum tachyon is a massless state and is almost normalizable, there could be such IR divergences. More explicitly, the sum over intermediate states involves integrals similar to (12) along the imaginary $\nu$ axis which could diverge from the region near $\nu = 0$. The form factor $f$ suppresses this region and renders these integrals finite.  

As an aside let us make a few remarks about the high momentum behavior of the amplitudes. In a beautiful calculation Moore [19] has shown that the high momentum behavior of the $c = 1$ theory is dominated by non-perturbative effects. There is a signal of this in the high momentum behavior of the fixed genus amplitude – it grows like a power of the momentum. This can be explained qualitatively as follows. The tachyon background in (9) is needed in order to prevent the theory from sliding into the strong coupling region $\phi \rightarrow \infty$. However, since this wall is soft, the more energy the system has the deeper it

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4 For finiteness we really need to know that the form factor for one tachyon with any two other states $\sim \nu$ for small $\nu$. This follows by a very similar argument.

5 This point was understood in discussions with T. Banks, D. Kutasov and G. Moore.

6 The minisuperspace wave functions in the $c = 1$ system satisfy the Wheeler-de-Witt equation (10) with $< \Phi > = \mu e^{\gamma \phi}$ [19][15] rather than with $< \Phi > = \mu \phi e^{\gamma \phi}$. We do not understand how this is consistent with the fact that the cosmological constant operator is $\phi e^{\gamma \phi}$. 

9
can probe into the strong coupling region and the more important the quantum effects are. To make this picture somewhat more quantitative, consider the two point function of tachyons with momentum \( p \) and study it in the semiclassical approximation. Following the semiclassical analysis in [5], the amplitude is dominated by a constant positive curvature surface representing the reflection off the wall. The classical motion in the Liouville potential has energy \( p^2 \), and therefore the turning point of the classical trajectory \( \phi_t \) is determined by \( \mu e^{\gamma \phi_t} = p^2 \). The string coupling at that point is \( g_{st} = g_0 e^{\frac{\gamma}{2} Q \phi_t} = g_0 \frac{p^2}{\mu} \). We therefore expect the effective string coupling at high momentum to behave like \( \frac{p^2}{\mu} \). This is precisely the behavior found in [19]. Note that the string coupling gets strong in the space-time UV and in world-sheet IR. Such an inverse relationship between world-sheet and space-time scales has been noted in critical string theory [25].

Let us now return to our main theme. Typically the linearized equation of motion (2) does not have normalizable solutions. There are however some familiar examples of such zero modes of \( \mathcal{K} \) in the critical string. The zero momentum mode of every massless field is a (delta-function) normalizable zero mode of \( \mathcal{K} \). Its wave function is independent of the non-compact directions. In the situation with a \( \phi \) dependent coupling constant we impose the boundary condition that the wave function vanishes in the strong coupling region \( \phi \to \infty \). With such a boundary condition at one end it is non-generic to find solutions of (2) which do not diverge at the other end, \( \phi \to -\infty \). There is however a notable example of such a “miraculous solution.” It occurs in the Ramond sector of the “non-critical” fermionic string. The supersymmetric matter ground state has \( \Delta = \frac{c}{16} \) and is always massless. Unlike the non-supersymmetric minimal models which have only massive operators, half of the supersymmetric minimal models have such \( \Delta = \frac{c}{16} \) states which are massless. Furthermore, the minisuperspace Wheeler-de-Witt equation is

\[
\left( -\frac{1}{2} \partial_\phi^2 + \frac{1}{2} |W'(\phi)|^2 - \frac{1}{2} W''(\phi) + \frac{\gamma^2}{8} \nu^2 \right) \Psi_\nu(\phi) = 0 \tag{15}
\]

where the superpotential \( W \) is

\[
W(\phi) = \mu e^{\gamma \phi} . \tag{16}
\]

This problem was first analyzed in [26]. The \( \nu = 0 \) eigenfunction is, as always in supersymmetric theories,

\[
\Psi_0 = e^{-W} = e^{-\mu e^{\gamma \phi}} . \tag{17}
\]

\[ ^7 \] Note that this argument gives the conceptual basis for the difference between the high momentum behavior in the \( c = 1 \) model and in the \( D = 26 \) bosonic string [24].
It satisfies the boundary conditions in the strong coupling region $\phi \to \infty$, and approaches a constant in the weak coupling region $\phi \to -\infty$. Note that unlike the zero momentum tachyon in the $c = 1$ system which is also massless, the wave function of this state is delta function normalizable. Hence, the corresponding state is macroscopic and it fluctuates. It is not clear to us whether this state decouples (as similar states decouple in the critical string), or if not, whether it leads to IR divergences in perturbation theory.

2D Euclidean black hole:

As another example of background dependence, we now discuss the 2D Euclidean black hole [10]. The world-sheet Lagrangian is characterized by the cigar metric and dilaton

$$ds^2 = dr^2 + \tanh^2 r d\theta^2$$

$$D = D_0 + \log \cosh^2 r .$$

The comparison with the $c = 1$ system is obtained by writing

$$\phi = -\frac{1}{Q} (\log M + \log \cosh^2 r)$$

in terms of the parameter $M$ which plays the role of the mass of the black hole. Here $Q = 2\sqrt{2}$ corresponds to $k = \frac{9}{4}$ in the coset construction. In these coordinates the dilaton is linear in $\phi$ and the metric is

$$ds^2 = \frac{Q^2}{4(1 - Me^{Q\phi})} d\phi^2 + (1 - Me^{Q\phi}) d\theta^2 .$$

Note that unlike the $c = 1$ system, here $\phi$ has a maximum allowed value, $-\frac{1}{Q} \log M$, which is obtained at the tip of the cigar, $r = 0$. Instead of the cosmological constant wall, here the region of strong coupling is not accessible at all. In the weak coupling region, $\phi \to -\infty$ ($r \to \infty$) the metric is approximately

$$ds^2 \approx \frac{Q^2}{4}(1 + Me^{Q\phi}) d\phi^2 + (1 - Me^{Q\phi}) d\theta^2 .$$

The first signal of the wall as we approach it from this region is the existence of the world-sheet operator $e^{Q\phi}(\frac{Q^2}{4} \partial \phi \bar{\partial} \phi - \partial \theta \bar{\partial} \theta)$. This operator is $(1, 1)$ because $\partial \theta \bar{\partial} \theta$ and $\partial \phi \bar{\partial} \phi$ are $(1, 1)$ and $e^{Q\phi}$ is $(0, 0)$. However, it seems to violate the bound on the exponent $\alpha < \frac{Q}{2}$. Furthermore, one might think that since $\alpha = Q$, the corresponding wave function of this operator behaves like $e^{\frac{1}{2}Q\phi}$ as $\phi \to -\infty$ and hence it might be normalizable. If so, the mass of the black hole $M$ could fluctuate.
In fact the situation is more subtle. It is misleading to view this system as a perturbation of the problem without the wall. The topology of the non-zero $M$ target space is different from flat space with $M = 0$. We should define the mass operator as the operator which induces infinitesimal changes of $M$ around non-zero $M$. It is obtained by varying the world-sheet action with respect to $M$

$$V_M = \int \frac{Q^2 e^{Q\phi}}{4(1 - Me^{Q\phi})^2} \partial\phi \bar{\partial}\phi - e^{Q\phi} \partial\theta \bar{\partial}\theta$$

whose wave function seems normalizable at $\phi \to -\infty$. However, a properly computed norm of this metric deformation

$$|\delta g|^2 = \frac{1}{2} \int \sqrt{g}(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})\delta g_{\alpha\beta}\delta g_{\gamma\delta}$$

diverges at the tip of the cigar. Hence, this deformation is not normalizable and the parameter $M$ cannot fluctuate\(^8\). Equivalently, a shift of $M$ is equivalent to a shift of $D_0$ which determines the string coupling at the tip of the cigar. The operator corresponding to such a deformation is $\partial \bar{\partial}\phi$ whose wave function is proportional to $e^{-\frac{1}{2}Q\phi}$. It is not normalizable as $\phi \to -\infty$. Hence the string coupling and $D_0$ (which is related to the mass $M$) cannot fluctuate. We conclude that there is a one parameter family of backgrounds labeled by $M$ or $D_0$ which do not explore each other to all orders in perturbation theory.

To further examine the stability of the black hole and the finiteness of the perturbation expansion, we should look for negative or zero modes of the small fluctuation operator $K$ which are normalizable. We restrict our attention to the $Q = 2\sqrt{2}$, $k = \frac{9}{4}$ case. The genus one calculation of \cite{29} shows that there are no localized negative modes (which would have led to instabilities) and that the entire light spectrum comes from the massless “tachyon” field\(^9\). First we examine the zero momentum tachyon. Its wavefunction is \cite{31}\cite{29}

$$\Psi_0(r) = \frac{1}{g_{st}(r)} \int d\theta (\cosh 2r + \cos \theta \sinh 2r)^{-\frac{1}{2}}.$$ \hspace{1cm} (24)

It is proportional to $r$ (the generic behavior) as $r \to \infty$ and is not normalizable.

Repeating the analysis of the form factor in the $c = 1$ system, it is easy to show that form factors here also vanish like the $r$ momentum, $\nu$, as $\nu \to 0$. Again this is a signature

\(^8\) A similar calculation leads to a similar conclusion in four dimensions \cite{27}. Other (unstable) modes are important there, though \cite{28}\cite{27}. We thank M. Douglas for pointing this out to us.

\(^9\) But note that the alternate quantization of \cite{30} predicts an instability for $k < 3$. 


of the non-normalizability of the zero momentum state. Therefore, even though there are normalizable eigenmodes of \( K \) with arbitrarily small eigenvalues, these modes decouple at low momentum and there will be no IR divergences in the genus expansion. We thus argue that the vacuum amplitude genus expansion of the 2D Euclidean black hole is finite to all orders in the genus expansion (except for a trivial volume divergence at genus one).

The implications of these results for the Minkowski space evolution of the black hole are not clear to us\(^{10} \).

The nature of the wall in this theory is different from that of the \( c = 1 \) theory. It is a hard wall completely preventing the string from reaching the strong coupling region. Therefore, we speculate that the high momentum behavior of the amplitudes is softer here than in the \( c = 1 \) system. The effective coupling constant as a function of the momentum either does not grow indefinitely, or grows slower than in the \( c = 1 \) theory.

In summary, we have argued that the presence of the kind of superselection parameters observed in the matrix models is rather generic, at least to all orders in perturbation theory. It is due to the presence of non-normalizable deformations, a known phenomenon in field theory. The persistence of such behavior nonperturbatively is an interesting and for the most part open question. Since such superselection sectors occur in the critical string, we should understand why our world is not “stuck” in such a state and is rather in its known background with four non-compact flat dimensions. In particular, why is the world not in a “non-critical” string background?

*Note added.* E. Witten has also argued for the stability of the 2D black hole.

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\(^{10}\) See [32] for a discussion of related issues.
Acknowledgements

It is a pleasure to thank T. Banks, J. Distler, M. Douglas, D. Friedan, K. Gawedzki, A. Kupiainen, E. Martinec, G. Moore, L. Susskind, E. Witten, A.B. Zamolodchikov and especially D. Kutasov for useful discussions. This work was supported in part by DOE grant DE-FG05-90ER40559.
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