ARTIFACTS IN THE INVERSION OF THE BROKEN RAY TRANSFORM IN THE PLANE

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Abstract. We study the integral transform over a general family of broken rays in \( \mathbb{R}^2 \). One example of the broken rays is the family of rays reflected from a curved boundary once. There is a natural notion of conjugate points for broken rays. If there are conjugate points, we show that the singularities conormal to the broken rays cannot be recovered from local data and therefore artifacts arise in the reconstruction. As for global data, more singularities might be recoverable. We apply these conclusions to two examples, the V-line transform and the parallel ray transform. In each example, a detailed discussion of the local and global recovery of singularities is given and we perform numerical experiments to illustrate the results.

1. Introduction. The purpose of this work is to study the integral transform over a general family of broken rays in the plane. A broken ray in the Euclidean space is usually defined as a linear path reflecting from the boundary once, which will be an important example of the broken rays we define, see Section 5. In fact, one motivation of this work is the reconstruction of an unknown function from the integral transform over such broken rays in medical imaging. This integral transform is called the V-line transform. It is related to the Single Photon Emission Computed Tomography (SPECT) with Compton cameras in two dimensions, and has been studied in [10, 26].

We define a more general family of broken rays. Suppose \( f \) is a distribution with compact support. Roughly speaking, a broken ray \( \nu \) is the union of two rays \( l_1 \) and \( l_2 \) that are related by a diffeomorphism, as in Figure 1. For more details of the definition, see Section 2. The broken ray transform

\[
Bf(\nu) = \int a(\nu(t), \dot{\nu}(t))f(\nu(t))dt
\]

is a weighted integral of \( f \) along \( \nu \), where \( a \) is a smooth function. One way to think about this is to imagine that there is a curve smoothly connecting \( l_1 \) and \( l_2 \), then \( B \) becomes an X-ray type of transform over smooth curves. The connecting curve plays no role in the analysis, if we always assume that \( f \) is compactly supported away from it.

The goal is to understand which singularities of \( f \) can be recovered from the transform \( Bf \), i.e., whether we can recover \( f \) up to a smooth error. More specifically,
what part of the wave front set $WF(f)$ can be recovered. Conjugate points naturally exist for broken rays, see Section 3. One would expect and we confirm that recovery of singularities are affected by the existence of conjugate points on $\nu$. Much work has been done for the class of X-ray type transform with conjugate points [34, 33, 24, 11]. In the case of the transform for a generic family of smooth curves [8], if there are no conjugate points, the localized normal operator is an elliptic pseudodifferential operator (ΨDO) of order $-1$. Injectivity and the stability estimates are established, which in particular implies that we can recover the singularities uniquely. When conjugate points exist, however, artifacts may arise, and in some situations they cannot be resolved. A similar situation occurs in synthetic aperture radar imaging [34]; it is impossible to recover $WF(f)$ if the singularities hit the trajectory of the plane only once, because of the existence of mirror points. On the other hand, if the trajectory is the boundary of a strictly convex domain and we know a priori that $f$ has singularities in a compact set, then we can recover $WF(f)$ from the global data. However, as shown in [34] this is a global procedure and there is no local reconstruction. In the case of X-ray transforms over geodesic-like families of curves with conjugate points of fold type, a detailed description of the normal operator is given in [33]. Analysis of the normal operator for general conjugate points is done in [12]. Further, [24] shows that regardless of the type of the conjugate points, the geodesic ray transform on Riemannian surfaces is always unstable and we have loss of all derivatives, which leads to the artifacts in the reconstruction near pairs of conjugate points. It is also proved that the attenuated geodesic ray transform is well posed under certain conditions. Most recently, [11] provides a thorough analysis of the stability of attenuated geodesic ray transform and shows what artifacts we can expect when using the Landweber iterative reconstruction for unattenuated problems.

One important example of this setting is the V-line transform. As is shown in Figure 1, the diffeomorphism is given by the law of reflection. As mentioned above, we are motivated by the SPECT with Compton cameras in two dimensions. SPECT based on Anger camera is a widely used technique for functional imaging in medical diagnosis and biological research. The using of Compton camera in SPECT is proposed to greatly improve the sensitivity and resolution [38, 6, 31]. The gamma photons are emitted proportionally to markers density and then are scattered by two detectors. Photons can be traced back to broken lines. The mathematical model is the cone transform (or conical Radon transform) of an unknown density. Various inversion approaches for certain cases are proposed in [2, 29, 39, 32, 23, 9, 36, 22, 20, 30, 25, 37, 26]. The V-line transform can be considered as a special case in two dimensions [10, 26], where each vertex is restricted on a curve and is associated with a single axis. There are also some injectivity and stability results when we allow the rays to reflect from the boundary more than once [16, 14, 17, 18, 15, 19]. These
reconstructions are from full data and most of them assume specific boundaries at least for the reflection part, for example a flat one or a circle. It also should be mentioned that the broken ray transform or the V-line transform sometimes refers to a different transform from the one we consider in this work, see [27, 7, 1, 21]. In their settings, the V-line vertices are inside the object with a fixed axis direction. The integral near the vertices in the support of $f$ makes it possible to recover singularities there. In this work, however, the vertices are always away from support of $f$, which make the recovery more difficult.

Another motivation is the application of parallel ray transform in X-ray luminescence computed tomography (XLCT). A multiple pinhole collimator based on XLCT is proposed in [41] to promote photon utilization efficiency in a single pinhole collimator. In this method, multiple X-ray beams are generated to scan a sample at multiple positions simultaneously, which we mathematically model by the parallel ray transform, see Section 6. In fact, we can regard the parallel ray as a ray reflecting off a boundary at infinity.

We are also motivated by the scattering problem for the equation $(-\Delta - \lambda^2 + V)u = 0$ in $\mathbb{R}^2/\Omega$ with Dirichlet or Neumann boundary conditions, where $\Omega$ is a domain with smooth boundary. The recovery of the potential $V$ from the boundary data is related to recover its integral over rays reflected from the boundary in the high frequency limit.

We are inspired by [34, 24, 11] and the main results are

1. The local problem is locally ill-posed if there are conjugate points, i.e., singularities conormal to the broken rays cannot be recovered uniquely. We describe the microlocal kernel in Theorem 4.3.

2. For the V-line transform and the parallel ray transform, the global problem might be well-posed in some cases for most singularities, because singularities can be probed by more than one broken ray. The recovery depends on a discrete dynamical system (a sequence of conjugate covectors) inside the domain, see (25). This is a discrete analogue of propagation of singularities as in [34]. If this sequence goes out of the domain, then we can resolve the corresponding singularity.

The paper is structured as follows. In Section 2, we define the broken ray and introduce some notation and assumptions. In Section 3 and 4, we introduce conjugate points and conjugate covectors along broken rays and give a characterization of them. Then we consider the local problem, i.e., the data $Bf$ is known in a small neighborhood of a fixed broken ray. We show that $B$ is an FIO and the image of two conjugate covectors under its canonical relation are identical. Singularities can be canceled by these conjugate covectors. This implies that we can only reconstruct $f$ up to an error in the microlocal kernel. We also provide a similar analysis for the numerical result as in [11], if the Landweber iteration is used to reconstruct $f$. In Section 5 and 6, we apply these conclusions to two cases, the V-line transform and the parallel ray transform, as mentioned above. The conjugate points appearing in the V-line transform coincide with the caustics in geometrical optics, see [3]. Additionally, when the boundary is a circle, we show that there exists conjugate points of fold type as well as cusps. Geometrically, the caustics inside a circle are an interesting problem itself, which can be traced back to the middle of the 19th century [4]. As for the parallel ray transform, the conjugate covectors have simple forms and the sequence of them is given by translation, see (28). In both cases, we discuss the local and global recovery of singularities and we perform numerical
experiments to illustrate the results. In particular, for rays reflected from a circle, we connect our analysis with the inversion formula derived in [26].

2. Preliminaries. Throughout this work, we assume \( f \) is a distribution supported in a compact set and we use angular brackets to denote the inner product of vectors in \( \mathbb{R}^2 \). We say a singularity \((x, \xi)\) is recoverable from the broken ray transform if that \( Bf \) is smooth implies \((x, \xi) \notin \text{WF}(f)\).

Let \( v(\alpha) = (\cos \alpha, \sin \alpha) \) and \( w(\alpha) = (\sin \alpha, \cos \alpha) \). We use \((s, \alpha)\) to parameterize a directed line \( \{x \in \mathbb{R}^2 \mid x \cdot w(\alpha) = s\} \) with the direction \( v(\alpha) \) and the unit normal \( w(\alpha) \). Note that \((s, \alpha)\) and \((-s, \alpha + \pi)\) belong to the same line but have opposite directions.

Let \( \chi : (s_1, \alpha_1) \mapsto (s_2, \alpha_2) \) be a given diffeomorphism. Suppose \( l_1 \) is a portion of the line \((s_1, \alpha_1)\), which starts from infinity and ends at a point. Suppose \( l_2 \) is a portion of the line \((s_2, \alpha_2)\), which starts at a point and ends at infinity. A broken ray \( \nu \) is defined as the union of \( l_1 \) and \( l_2 \) if they are related by \((s_2, \alpha_2) = \chi(s_1, \alpha_1)\).

We call \( l_1 \) the incoming part and \( l_2 \) the outgoing part of \( \nu \). We say \( \nu \) is regular if these two parts do not lie in the same line. In this case, the part \( l_1 \) and \( l_2 \) might intersect in the support of \( f \) but their conormal bundles are always separated.

We say \( \Gamma \) is a smooth family of broken rays associated with \( \chi \), if

1. \( \Gamma \) is open and each broken ray in \( \Gamma \) can be parameterized by its incoming part \((s_1, \alpha_1)\).

2. There exists a smooth function \( q_0(s, \alpha) \) such that the starting point of the outgoing part of each broken ray \((s_1, \alpha_1)\) is given by \( q_0(s_1, \alpha_1) \) and satisfies \((q_0(s_1, \alpha_1), w(\alpha_2)) = s_2\); the similar is true for the endpoint of the incoming part.

We always assume we are given a smooth family of broken rays.

3. Conjugate points. In Riemannian geometry, the conjugate vector of a fixed point \( p \) is a vector \( v \) such that the differential of the exponential map \( d_x \exp_x(v) \) is not an isomorphism. The conjugate point is the image of \( v \) under the exponential map, for more details see [40, 33]. Conjugate points also exist in the case of broken ray transform, for example, the caustics in geometrical optics, see [4, 3]. The light reflected or refracted by a curved surface forms an envelope, which is the conjugate locus of the source. In this section, we define the exponential map and compute the conjugate points for broken rays. We show below that conjugate points on \( l_1 \) and \( l_2 \) do not depend on what kind of connecting curves we choose.

There are two different ways to parameterize a line. We can use the Radon parametrization \((s, \alpha)\) as mentioned above, or we can parametrize it by an initial point and an angle. We use the latter one to define the exponential map. Consider a broken ray \( \nu_{p, \alpha_1}(t) \)

\[
\begin{align*}
l_1(t) &= p + tv(\alpha_1), & -\infty \leq t \leq t_1, \\
l_2(t) &= q_0 + (t - t_2)v(\alpha_2), & t \geq t_2 \geq t_1,
\end{align*}
\]

whose incoming part \( l_1 \) passes \( p \) and outgoing part \( l_2 \) starts from \( q_0 \). Recall that \( q_0 \) satisfying \((q_0, w(\alpha_2)) = s_0\) is chosen to depend on \((s_1, \alpha_1)\) in a smooth way. The time \( t_2 \) depends on the connecting curve and its parametrization. The analysis below shows that \( t_2 \) does not influence the conjugate point of \( p \) on \( l_2 \). Observe that the parameterization \((p, \alpha_1)\) gives us a unique Radon parameterization \((s_1, \alpha_1)\) by \( s_1 = (p, w(\alpha_1)) \). If we fix \( p \), then for each \( \alpha_1 \) the diffeomorphism \( \chi \) gives a unique
Inverse Problems and Imaging Volume 14, No. 1 (2020), 1–26

(s_2, \alpha_2), i.e., we have s_2, \alpha_2, and q_0 are all smooth functions of \alpha_1 itself. In the following, we use \frac{d}{d\alpha_1} to denote the derivative with respect to \alpha_1, when p is fixed and s_1 is given by s_1 = (p, w(\alpha_1)).

Now define the exponential map as \exp_p(t, \alpha_1) = \nu_{p, \alpha_1}(t), for t \in \mathbb{R}, \alpha_1 \in [0, 2\pi). We say q \in l_2 is the conjugate point of p if there is some (t, \alpha_1) such that the exponential map is not a diffeomorphism for q = \nu_{p, \alpha_1}(t). When t \geq t_2, the differential of the exponential map in polar coordinates is represented by the Jacobi matrix

\frac{\partial^2 \nu(t, \alpha_1)}{\partial \alpha_1 \partial t}, where

\frac{\partial t_2}{\partial \alpha_1} = (t - t_2) \frac{d
u(\alpha_2)}{d\alpha_1} - \frac{dt_2}{d\alpha_1} v(\alpha_2) + \frac{dq_0}{d\alpha_1}

= (t - t_2) \left( \frac{d\alpha_2}{d\alpha_1} + \langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle \right) w(\alpha_2) + \left( \langle \frac{dq_0}{d\alpha_1}, v(\alpha_2) \rangle - \frac{dt_2}{d\alpha_1} \right) v(\alpha_2).

Then it has the determinant

\det(\exp_p(tv(\alpha_1))) = \det \left[ \begin{array}{cc} \frac{\partial^2 \nu}{\partial t^2} & \frac{\partial^2 \nu}{\partial t \partial \alpha_1} \\ \frac{\partial^2 \nu}{\partial \alpha_1 \partial t} & \frac{\partial^2 \nu}{\partial \alpha_1^2} \end{array} \right] = (t - t_2) \left( \frac{d\alpha_2}{d\alpha_1} + \langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle \right) w(\alpha_2) + \left( \langle \frac{dq_0}{d\alpha_1}, v(\alpha_2) \rangle - \frac{dt_2}{d\alpha_1} \right) v(\alpha_2).

The last equality comes from the observation that \det[v(\alpha_2) w(\alpha_2)] = 1. Thus, the determinant vanishes if and only if

(t - t_2) \frac{d\alpha_2}{d\alpha_1} = -\langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle.

We are finding a solution to equation (3) satisfying t \geq t_2. There are two cases. If \frac{d\alpha_2}{d\alpha_1} = 0, then \langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle is zero as well. Otherwise, we must have \left( \frac{d\alpha_2}{d\alpha_1} \right)^{-1} \langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle \leq 0. On the other hand, differentiating \langle q_0, w(\alpha_2) \rangle = s_2 with respect to \alpha_1 shows

\langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle + \langle q_0, -v(\alpha_2) \rangle \frac{d\alpha_2}{d\alpha_1} = \frac{ds_2}{d\alpha_1}.

With the assumption that \chi is a diffeomorphism, \frac{d\alpha_2}{d\alpha_1} and \frac{dq_0}{d\alpha_1} cannot vanish at the same time. This excludes the first case.

Suppose q is the point on l_2 at time t such that \exp_p(tv_1) is not an isomorphism. We have t - t_2 = \langle q - q_0, v(\alpha_2) \rangle. By (3)(4), q should satisfy the equality

\langle q, v(\alpha_2) \rangle \frac{d\alpha_2}{d\alpha_1} = -\frac{ds_2}{d\alpha_1}.

Observe that the projection of q on v(\alpha_2) together with its projection on w(\alpha_2) determines q uniquely. On the contrary, if there exists q on l_2 such that the equation (5) is true, then the determinant of \exp_p(tv(\alpha_1)) will be zero.

**Proposition 1.** Suppose l_1, l_2, and q_0 as mentioned above. Let p be a fixed point on l_1. Then

(a) p has a conjugate point q on l_2 if and only if

\langle \frac{d\alpha_2}{d\alpha_1}, \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle \leq 0.

(b) If this occurs, q is uniquely determined by \langle q, v(\alpha_2) \rangle = -\left( \frac{d\alpha_2}{d\alpha_1} \right)^{-1} \frac{ds_2}{d\alpha_1}.

Here we use \frac{d}{d\alpha_1} to denote the derivative with respect to \alpha_1 with p fixed and s_1 given by s_1 = (p, w(\alpha_1)).
Microlocal analysis of the local problem.

4. Ray, then we can always find one and only one conjugate point $\nu$ transform. Recall the definition of a broken ray in Section 2. Suppose geodesic ray transform singularities can be canceled by conjugate points. In this remark, we consider the whole straight line where $q$ we define. Additionally, if we perturb $q_0$ a little bit, that is, let $q_0' = q_0 + \epsilon(\alpha_1)v(\alpha_2)$. Then $\frac{dq_0}{d\alpha_1} = \frac{dq_0}{d\alpha_1} + \epsilon(\alpha_1)w(\alpha_2) + \frac{d\epsilon(\alpha_1)}{d\alpha_1}v(\alpha_2)$. We have

$$\langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle = \langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle + \epsilon(\alpha_1).$$

It shows a small enough perturbation of $q_0$ doesn’t change the sign of $\langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle$. Therefore the existence of conjugated points is not affected by the choice of $q_0$ in a small neighborhood.

Remark 2. Suppose $p$ and $q$ belong to the incoming part $l_1$ and the outgoing part $l_2$ of $\nu$ respectively. If $q$ is the conjugate point of $p$, then we can show $p$ is the conjugate point of $q$ in some sense. Indeed, $q$ is conjugate to $p$ if and only if

$$\langle q, v(\alpha_2) \rangle = -\left( \frac{d\alpha_2}{d\alpha_1} \right)^{-1} \frac{ds_2}{ds_1} = -\frac{\partial s_2}{\partial s_1} - \frac{\partial s_2}{\partial s_1} (p, v(\alpha_1)) = -\frac{\partial s_2}{\partial s_1} (q, v(\alpha_1)).$$

Solving $\langle p, v(\alpha_1) \rangle$ out, we have

$$\langle p, v(\alpha_1) \rangle = -\frac{\partial s_2}{\partial s_1} - \frac{\partial s_2}{\partial s_1} (q, v(\alpha_2)).$$

Now let $\nu'$ be a broken ray passing $p$ and $q$ but with incoming part $(s_2, \alpha_2)$ and associated with $\chi^{-1}$. We list the Jacobian matrix in the following

$$d\chi = \begin{bmatrix} \frac{\partial s_2}{\partial s_1} & \frac{\partial s_2}{\partial s_1} \\ \frac{\partial s_2}{\partial s_1} & \frac{\partial s_2}{\partial s_1} \end{bmatrix}, \quad d(\chi^{-1}) = (d\chi)^{-1} = \frac{1}{\det(d\chi)} \begin{bmatrix} \frac{\partial s_2}{\partial s_1} & -\frac{\partial s_2}{\partial s_1} \\ -\frac{\partial s_2}{\partial s_1} & \frac{\partial s_2}{\partial s_1} \end{bmatrix}.$$  

Notice equation (6) exactly means $p$ is the conjugate point of $q$ along $\nu'$.

4. Microlocal analysis of the local problem. In [24], it is shown that in the geodesic ray transform singularities can be canceled by conjugate points. In this section, we prove the analogous results in Theorem 4.2 and 4.3 for the broken ray transform. Recall the definition of a broken ray in Section 2. Suppose $\nu$ is a broken ray represented by $(s, \alpha)$. We define the broken ray transform $Bf$ as

$$Bf(s, \alpha) = \int_\nu a(y, s, \alpha)f(y)dy,$$

where $a(y, s, \alpha)$ is smooth and nonvanishing. Comparing it with (1), here we use different parameterization for the weight but still denote it by $a$.

Suppose $f$ has support in a compact subset away from the connecting part. The support of $f$ implies the transform can be interpreted as a sum of Radon transforms over two lines. We can only expect to recover the singularities in their conormal bundles. In the following, we suppose $\nu_0$ is a regular broken ray. For fixed $\nu_0$, we consider $(x_1, \xi^1)$ and $(x_2, \xi^2)$ on its incoming and outgoing part respectively, with $\xi^1$ and $\xi^2$ conormal to them. Let $\Gamma(\nu_0)$ be a small neighborhood of $\nu_0$ and $V^k$ be disjoint small conic neighborhood of $(x_k, \xi^k)$, for $k = 1, 2$. We choose these neighborhoods small enough, such that $V^1$ is disjoint from the conormal bundles of all outgoing parts and $V^2$ is disjoint from that of all incoming parts of broken rays in $\Gamma(\nu_0)$. We project $V^k$ onto $\mathbb{R}^2$ to get the neighborhood $U_k$ of $x_k$. The set $U_k$
might intersect but $V^k$ are always disjoint, for $k = 1, 2$. Figure 2 shows a special case when we have disjoint $U_k$.

![Figure 2](image_url)

**Figure 2.** The small neighborhood $U_k$ and $(x_k, \xi^k)$, for $k = 1, 2$.

To further localize the problem, we suppose $\text{WF}(f) \subset V^1 \cup V^2$. For convenience, we simply assume $\text{supp } f \subset U_1 \cup U_2$. Let $f_k$ be $f$ restricted to $U_k$ and $B_k$ be $B$ restricted to distributions with wavefront set supported in $V^k$, for $k = 1, 2$. It follows that

$$Bf = B_1f_1 + B_2f_2. \quad (8)$$

In a small neighborhood, $B_1f_1$ can be regarded as the Radon transform of $f_1$ and $B_2f_2$ as the Radon transform performing along the line $(s_2, \alpha_2)$. More precisely, the restricted operators $B_1$ and $B_2$ have the following form up to some smoothing operators

$$B_1 = \phi R\varphi_1, \quad B_2 = \phi\chi^* R\varphi_2,$$

where $R$ is the Radon transform; $\phi(s, \alpha)$ is a smooth cutoff function with $\text{supp } \phi \subset \Gamma(v_0)$; $\varphi_k$ are cutoff $\Psi$DOs with essential support in $V^k$, for $k = 1, 2$; the pull back $\chi^*g(s, \alpha) = g(\chi(s, \alpha))$ is induced by the diffeomorphism $\chi$. We should note that outside $\Gamma(v_0)$, there might be another broken ray which carries the singularities $(x_1, \xi^1)$ but with it in the outgoing part. Thus, we actually multiply $\phi$ to $B$ itself as well to make equation (8) valid.

To analyze the canonical relation of $B_1$ and $B_2$, we need that of Radon transform. The weighted Radon transform is defined as

$$Rf(s, \alpha) = \int_{(w(\alpha), y) = s} \omega(y, \alpha)f(y)dy, \quad (9)$$

where $w(\alpha) = (-\sin \alpha, \cos \alpha)$, and $\omega(y, \alpha)$ is a smooth function.

**Proposition 2.** The Radon transform $R$ is an FIO of order $-\frac{1}{2}$ associated with the canonical relation

$$C_R = \{(y, w(\alpha)), \alpha, \lambda, (\lambda v(\alpha), y), y, \lambda w) \}. \quad (10)$$

where $v(\alpha) = (\cos \alpha, \sin \alpha)$ and $w(\alpha)$ as before. Specifically, $C_R$ has two components, corresponding to the choice of the sign of $\lambda$. Each component is a local diffeomorphism. The inverse is also a local diffeomorphism.

$$C_R^{-1} : (s, \alpha, \hat{s}, \hat{\alpha}) \mapsto (y, \eta) \quad y = \frac{\hat{s}}{s} v(\alpha) + s w(\alpha), \quad \eta = \hat{s} w(\alpha) \quad (11)$$

$$C_R^{-1} : (s, \alpha, \hat{s}, \hat{\alpha}) \mapsto (y, \eta) \quad y = \frac{\hat{s}}{s} v(\alpha) + s w(\alpha), \quad \eta = \hat{s} w(\alpha)$$

**Inverse Problems and Imaging Volume 14, No. 1 (2020), 1–26**
Proof. We write the Radon transform as

\[ Rf(s, \alpha) = (2\pi)^{-1} \int e^{i\lambda(s - \langle w(\alpha), y \rangle)}\omega(y, \alpha)f(y)d\lambda dy. \]

The characteristic manifold is \( Z = \{(s, \alpha, y)|\Phi(s, \alpha, y) = \lambda(s - \langle w(\alpha), y \rangle) = 0\}. \) Then the Lagrangian \( \Lambda \) is given by

\[ \Lambda = N^*Z = \{(s, \alpha, y, \lambda \in \Phi_s, \lambda\langle v(\alpha), y \rangle - \lambda w(\alpha), \langle w(\alpha), y \rangle = s\}. \]

Therefore, the Radon transform is an FIO associated with \( \Lambda \) and the canonical relation \( C_R \) is obtained by twisting the Lagrangian. The sign of \( \lambda \) is chosen corresponding to the orientation of \( \eta \) with respect to \( w(\alpha) \). It is elliptic at \((y, \eta)\) if and only if \( \omega(y, \alpha) \neq 0 \) for \( \alpha \) such that \( w(\alpha) \) is colinear with \( \eta \).

**Lemma 4.1.** Suppose \( \chi : (s_1, \alpha_1) \mapsto (s_2, \alpha_2) \) is a diffeomorphism. Then \( \chi^* : g(s_2, \alpha_2) \mapsto \chi^*g(s_1, \alpha_1) = g(\chi(s_1, \alpha_1)) \) for \( g \in \mathcal{D}' \) is an FIO whose canonical relation is a diffeomorphism

\[ (12) \quad C_{\chi^*} = \{(s_1, \alpha_1, \tilde{\omega}_1, s_2, \alpha_2, (s_2, \alpha_2), (\tilde{\omega}_1 \hat{\chi})^{-1})(d\hat{\chi})^{-1}, (s_2, \alpha_2) = \chi(s_1, \alpha_1)\}. \]

**Proof.** The proof is similar to what we did in last proposition. The induced map \( \chi^* \) can be written as the following integral

\[
\chi^*g(s_1, \alpha_1) = \int \delta((s, \alpha) - \chi(s_1, \alpha_1))g(s, \alpha)d\alpha ds\alpha
= (2\pi)^{-2} \int e^{i\lambda(s - s_2 + \alpha_2 - \alpha_1)}g(s, \alpha)d\lambda d\lambda_1 d\lambda_2 ds\alpha,
\]

where \( \chi(s_1, \alpha_1) = (s_2, \alpha_2) \). The characteristic manifold is \( Z_{\chi^*} = \{(s, \alpha, s_1, \alpha_1)|\phi = (s_2 - s) + \lambda_2(\alpha_2 - \alpha) = 0\}. \) The Lagrangian is given by

\[ \Lambda_{\chi^*} = \{(s_1, \alpha_1, \phi, (\lambda_1, \lambda_2), (\Phi_{s_1, \alpha_1}^{-1}))(d\hat{\chi})^{-1}, (s, \alpha) = \chi(s_1, \alpha_1)\}. \]

Let \( (\lambda_1, \lambda_2)(d\hat{\chi}) = (\tilde{\omega}_1, \tilde{\alpha}_1) \) and replace \( (s, \alpha) \) by \((s_2, \alpha_2)\), we get the canonical relation as is shown above (12).

**Theorem 4.2.** We assume \((x, \xi)\) and \((y, \eta)\) are not conormal to the line joining \( x \) and \( y \). Suppose \( V^1 \) is a small enough conical neighborhood of \((x, \xi)\) and \( V^2 \) is that of \((y, \eta)\). Let \( B_k \) be \( B \) restricted to distributions with wavefront set supported in \( V^k \), for \( k = 1, 2 \). Suppose \( C_k \) is the canonical relation of \( B_k \). Then \( C_1(x, \xi) = C_2(y, \eta) \) if and only if there is a regular broken ray \( \nu \) joining \( x \) and \( y \) such that

(a) \( x \) and \( y \) are conjugate points along \( \nu \).
(b) \( \xi \) and \( \eta \) satisfy \( \lambda w(\alpha_1), \eta = \frac{\lambda}{\sin(\alpha_1)} \frac{d\alpha_2}{\alpha_1} w(\alpha_2) \) for some \( \lambda \neq 0 \), where \( \alpha_1 \) is the angle of the incoming part and \( \alpha_2 \) is that of the outgoing part of \( \nu \).

**Proof.** The assumption \((x, \xi)\) and \((y, \eta)\) are not conormal to the line joining \( x \) and \( y \) is to guarantee that if there is a broken ray that has \((x, \xi)\) and \((y, \eta)\) in its incoming part and outgoing part respectively, then this broken ray is regular. In this way, we can always assume \( V^1 \) is disjoint from the conormal bundles of all outgoing parts and \( V^2 \) is disjoint from that of all incoming parts of broken rays in the small neighborhood of the fixed broken ray, which simplifies the problem. Observe that the composition \( \chi^* R \) is also an FIO, whose canonical relation \( C_{\chi^* R} = C_{\chi^*} \circ C_R \) is a
local diffeomorphism. Additionally, the multiplication of cutoff functions does not influence the Lagrangian. Suppose the canonical relation of the restricted operator \( B_k \) is called \( C_k : (x, \xi^k) \mapsto (s_1, \alpha_1, \hat{s}_1, \hat{\alpha}_1) \), for \( k = 1, 2 \). As a result, \( C_1 \) is same as \( C_R \) and \( C_2 \) is same as \( C_{\lambda^*}R \). Suppose \((s_1, \alpha_1, \hat{s}_1, \hat{\alpha}_1)\) is the image of \((x, \xi^1)\) under \( C_R \) and \((s_2, \alpha_2, \hat{s}_2, \hat{\alpha}_2)\) is that of \((x, \xi^2)\). That is, with \( s_k \) and \( \alpha_k \) given by (10), for \( k = 1, 2 \), we have

\[
(s_1, \hat{\alpha}_1) = \lambda_1(1, \langle x, v(\alpha_1) \rangle), \quad (s_2, \hat{\alpha}_2) = \lambda_2(1, \langle x, v(\alpha_2) \rangle).
\]

Then from the analysis above, \( C_1(x, \xi^1) = C_2(x, \xi^2) \) if and only if

\[
(s_2, \alpha_2) = \chi(s_1, \alpha_1), \quad (s_2, \hat{\alpha}_2) = (s_1, \hat{\alpha}_1)(d\chi)^{-1}.
\]

The first equality says there is a regular broken ray \( \nu \) of which \((s_1, \alpha_1)\) and \((s_2, \alpha_2)\) are the incoming and outgoing part. The second condition is equivalent to

(13)

\[
\lambda_2(1, \langle x, v(\alpha_2) \rangle) = \frac{\lambda_1}{\det(d\chi)} \left( \frac{\partial \alpha_2}{\partial \alpha_1} - \langle x, v(\alpha_1) \rangle \frac{\partial \alpha_2}{\partial s_1}, \frac{\partial s_2}{\partial \alpha_1} - \langle x, v(\alpha_1) \rangle \frac{\partial s_2}{\partial s_1} \right).
\]

Notice \( \frac{\partial \alpha_2}{\partial \alpha_1} - \langle x, v(\alpha_1) \rangle \frac{\partial \alpha_2}{\partial s_1} \) and \( \frac{\partial s_2}{\partial \alpha_1} - \langle x, v(\alpha_1) \rangle \frac{\partial s_2}{\partial s_1} \) are exactly \( \frac{\partial \nu_1}{\partial \alpha_1} \) and \( \frac{\partial \nu_2}{\partial \alpha_1} \) if we fixed \( x_1 \) and consider \( s_2, \alpha_2 \) as functions of one variable \( \alpha_1 \). Therefore (13) can be written as

\[
\lambda_2(1, \langle x, v(\alpha_2) \rangle) = \frac{\lambda_1}{\det(d\chi)} \left( \frac{d\alpha_2}{d\alpha_1} - \frac{ds_2}{d\alpha_1} \right).
\]

This implies \( C_1(x_1, \xi^1) = C_2(x_2, \xi^2) \) if and only if

(a) \( \langle x_2, v(\alpha_2) \rangle = -\left( \frac{d\alpha_2}{d\alpha_1} \right)^{-1} \frac{ds_2}{d\alpha_1} \), i.e. \( x_1 \) and \( x_2 \) are conjugate points along \( \nu \), by Proposition 1.

(b) \( \lambda_2 = \frac{\lambda_1}{\det(d\chi)} \frac{d\alpha_2}{d\alpha_1} \), with \( \xi_1 = \lambda_1 \nu(\alpha_1) \) and \( \xi_2 = \lambda_2 \nu(\alpha_2) \).

\[\square\]

**Remark 3.** For \( \nu_0 \) that is not regular, if \((x, \xi)\) and \((y, \eta)\) are not identical, then we can perform a similar analysis by applying a cutoff \( \Psi DO \) instead of the cutoff function \( \phi \) to \( B \).

For \((y, \eta)\) satisfying this theorem, we call it the continuous covector of \((x, \xi)\). Since \( C_1 \) is a local diffeomorphism, it maps a small neighborhood of \((x, \xi)\) to a small neighborhood of \((s_1, \alpha_1, \hat{s}_1, \hat{\alpha}_1)\). The similar is true with \( C_2 \). Then by shrinking \( V^1 \) and \( V^2 \) a bit, we can assume \( C_1(V^1) = C_2(V^2) = V \).

**Theorem 4.3.** Suppose \((x, \xi)\) and \((y, \eta)\) are conjugate covectors along the broken ray \( \nu \). Suppose \( f_j \in \mathcal{E}'(U_j) \) with \( \text{WF}(f_j) \subset V^j \), for \( j = 1, 2 \). Then the local data, i.e. the broken ray transform in a small neighborhood of \( \nu \)

\[
B(f_1 + f_2) \in H^s(V)
\]

if and only if

\[
f_1 + F_{12}f_2 \in H^{s-\frac{1}{2}}(V^1) \iff F_{21}f_1 + f_2 \in H^{s-\frac{1}{2}}(V^2),
\]

where \( F_{12} \equiv B_1^{-1}B_2 \) and \( F_{21} \equiv B_2^{-1}B_1 \).

**Proof.** We follow the arguments in [24]. Notice \( B_1 \) is a FIO of order \( -\frac{1}{2} \) elliptic at \((x, \xi)\). An application of the parametrix \( B_1^{-1} \) to \( B(f_1 + f_2) \) shows

\[
B_1^{-1}B(f_1 + f_2) = f_1 + F_{12}f_2.
\]
Then $F_{12} = B_1^{-1}B_2$ is an FIO with canonical relation $C_{12} = C_1^{-1} \circ C_2 : V^2 \to V^1$; and $F_{21} = B_2^{-1}B_1$ is an FIO with canonical relation $C_{21} = C_2^{-1} \circ C_1 : V^1 \to V^2$. \hfill \Box

Thus, given a distribution $f_1$ singular in $V^1$, there exists a distribution $f_2$ singular in $V^2$ such that $B(f_1 + f_2)$ is smooth. One possible choice is $f_2 = -F_{21}f_1$. It is also the only choice up to smooth functions. We can introduce the concept of the microlocal kernel as in [11], which is defined as the space of distributions, modulo smooth functions, whose images by $B$ are smooth functions. Then for any $h$ with $\text{WF}(h) \subset V^1$, we have $h - F_{21}h$ in the microlocal kernel and this describes the later. Therefore the reconstruction for $f = f_1$ always has some error in form of $h - F_{21}h$, for some $h$. In other words, we can recover the singularities of $f$ only up to an error term of the form and therefore they cannot be resolved from the singularities of $Bf$.

On the other hand, suppose $(x, \xi) \in \text{WF}(f)$ is conormal to a regular broken ray $\nu_0$ and has no conjugate covectors along it. In the recovery of singularities of $f$ from local data, the covector $(x, \xi)$ is recoverable according to this theorem.

With the notation above, we are going to find out the artifacts arising when we use the backprojection $B^*B$ to reconstruct $f$, if there are conjugate covectors. Without loss of generality, we assume the weight $a(y, s, \alpha) = 1$ in the following. Suppose $\nu_0$ is the broken ray in Theorem 4.2. In a small neighborhood of $\nu_0$, we have

$$(14) \quad B^*Bf = B_1^*B_1f_1 + B_1^*B_2f_2 + B_2^*B_1f_1 + B_2^*B_2f_2.$$ 

Recall $B_1$ and $B_2$ are defined microlocally. Indeed, on the one hand, the assumption on $\text{supp} f_1$ plays the same role as restricting the operator on $U_k$, for $k = 1, 2$. For simplification, we just ignore them. On the other hand, if we concentrate on the small neighborhood of $\nu_0$, then we exclude the broken ray which has $(x, \xi)$ on its outgoing part. Microlocally $B_1$ is equivalent to the Radon transform operator, which indicates $B_1^*B_1$ is an elliptic $\Psi$DO of order $-1$. Especially, it has the principal symbol $4\pi/|\xi|$. The similar is true for $B_2^*B_2$. As for $B_1^*B_2$ and $B_2^*B_1$, since $(x, \xi)$ and $(y, \eta)$ are conjugate covectors, these two operators are FIOs of order $-1$ associated with canonical relation $C_1^{-1} \circ C_2$ and $C_2^{-1} \circ C_1$ respectively; if there are no conjugate covectors, they are smoothing operators since the canonical relations are empty.

One can follow the same argument in [24, 11] to show some properties of the normal operators. In addition, similarly to Radon transform, we can apply a filter to before the backprojection to get a zero order operator. We have

$$B^*ABf = B_1^*AB_1f_1 + B_1^*AB_2f_2 + B_2^*AB_1f_1 + B_2^*AB_2f_2,$$

where $\Lambda = \frac{1}{4\pi} \sqrt{-\partial_x^2}$.

The canonical relation of $B_1^*AB_1$ is the inverse of that of $B_k$. Therefore by Egorov’s theorem [13], $B_1^*AB_k$ is a pseudodifferential operator of order zero. We denote the principal symbol of a pseudodifferential operator $P$ by $\sigma_p(P)$. Recall Proposition 2, we have $\sigma_p(\Lambda) \circ C_1 = 1/(4\pi|\xi|)$ and $\sigma_p(\Lambda) \circ C_2 = 1/(4\pi|\eta|)$, where $C_k$ is the conical transformation corresponding to $B_k$ for $k = 1, 2$. Thus, the principal symbol of $B_1^*AB_k$ equals to $\sigma(B_1^*B_k)(\sigma(\Lambda) \circ C_k) = 1$, which implies

$$B_1^*AB_k \equiv I \mod \Psi^{-1}, \quad k = 1, 2.$$

This also coincides with the inversion formula for Radon transform. Then with the observation $B_1^*AB_2F_{21} = B_1^*AB_1$ and $F_{21}B_1^*AB_2 \equiv I$, we have $B_1^*AB_2 \equiv F_{12}$ up to a lower order. The same is true with $B_2^*AB_1$. Notice that the calculations are all microlocal and up to order $-1$. 

Inverse Problems and Imaging Volume 14, No. 1 (2020), 1–26
As a result,
\begin{equation}
B^*\Lambda B \equiv \begin{bmatrix}
\text{Id} & F_{12} \\
F_{12}^{-1} & \text{Id}
\end{bmatrix} := M, 
\end{equation}
where we follow the convention to think $f = f_1 + f_2$ as vector functions. It implies when performing the filtered backprojection, the reconstruction has two parts of artifacts, $F_{12}f_2$ in $V^1$ and $F_{21}f_1$ in $V^2$. As in [11], one can show that $F_{12}$ and $F_{21}$ are principally unitary in $H^{-\frac{1}{2}}$, and the artifacts have the same strength as the original distributions.

Next, we consider the numerical reconstruction by using the Landweber iteration as in [11]. For more details of the method, see [35]. We still focus on the local problem, that is, we consider $Bf$ in the small neighborhood of fixed $\nu_0$. With the notation above, we use a slightly different Landweber iteration to solve the equation $Bf = g$, where $g$ denotes the local data and it is assumed be in the range of $B$. We set $L = \Lambda_{1/2}B$ to have
\begin{equation}
(Id - (Id - \gamma L^* L))f = \gamma L^* \Lambda_{1/2} g.
\end{equation}
Then with a small enough and suitable $\gamma > 0$, it can be solved by the Neumann series
\begin{equation}
f = \sum_{k=0}^{+\infty} (Id - \gamma L^* L)^k \gamma L^* \Lambda_{1/2} g.
\end{equation}
The series converge to the minimal norm solution to $L f = \Lambda_{1/2} g$. Suppose the original function is $f = f_1 + f_2$. We track the terms of highest order, that is, order zero, to have the approximation sequence
\begin{equation}
(f^{(n)}) = \sum_{k=0}^{n} (Id - \gamma M)^k \gamma M f \Rightarrow (f^{(n)}) = \gamma (\sum_{k=0}^{n} (1 - 2\gamma)^k) M f.
\end{equation}
The second equality is from the observation $M^k = 2^{k-1} M$ for $k \geq 1$. The numerical solution is
\begin{equation}
f^{(n)} \to \frac{1}{2} \begin{bmatrix}
f_1 + F_{12}f_2 \\
F_{21}f_1 + f_2
\end{bmatrix}, \text{ as } n \to \infty.
\end{equation}
Therefore, the error equals to $\frac{1}{2} (f_1 - F_{12}f_2) + \frac{1}{2} (f_2 - F_{21}f_1)$, which belongs to the microlocal kernel.

5. The V-line transform. In this section, we apply the conclusions above to the V-line transform. Except in subsection 5.4, we suppose the weight function $a(y, s, \alpha) = 1$. First we verify the reflection operator is a diffeomorphism. Then we have the potential cancellation of singularities due to the existence of conjugates points. Especially, we derive an explicit formula to illustrate in which case the conjugate points exist.

5.1. The diffeomorphism. Suppose $\Omega$ is a bounded domain with a smooth boundary that can be parameterized by a regular curve $\gamma$. Suppose $\gamma$ is negatively oriented and we choose its arc length parameterization $\gamma(\tau) = (x(\tau), y(\tau))$. By negatively oriented, we mean when we travel on the curve we always have the curve interior to the right side. The unit tangent vector is $\dot{\gamma}(\tau) = (\dot{x}(\tau), \dot{y}(\tau))$ and unit outward normal is $n(\tau) = (-\dot{y}(\tau), \dot{x}(\tau))$, where $\dot{f}(\tau) \text{ refers to } \frac{d}{d\tau} f$. We still consider the local problems, and $\gamma$ could be just part of $\partial \Omega$. The signed curvature of $\gamma$ is defined as the scalar function $\kappa(\tau)$ such that $\dot{\gamma} = \kappa(\tau)n$. 
Suppose a ray \((s, \alpha)\) transversally hits \(\gamma\) at point \(\gamma(\tau_0) = (x(\tau_0), y(\tau_0))\) and then reflects, as is shown in Figure 3. In a small neighborhood of such a ray, we have \(\tau_0\) is a smooth function of \(s\) and \(\alpha\). The proof is simply an application of implicit function theorem. Indeed, since \(F(\tau_0, s, \alpha) = \langle w(\alpha), \gamma(\tau_0) \rangle - s = 0\) with \(\frac{\partial F}{\partial \tau} = \langle w(\alpha), \gamma'(\tau_0) \rangle \neq 0\), it follows that \(\tau_0\) could be written as a smooth function, say \(\tau_0(s, \alpha)\).

![Figure 3. A sketch of a broken ray reflected on a smooth boundary and the notation.](image)

Differentiating \(F(\tau_0, s, \alpha) = 0\) w.r.t. \(s\) and \(\alpha\), we get equations of \(\frac{\partial \tau_0}{\partial s}\) and \(\frac{\partial \tau_0}{\partial \alpha}\).

To distinguish \((s, \alpha)\) from the one we use for \(l_2\), we replace them by \((s_1, \alpha_1)\) in the following

\[
\frac{\partial \tau_0}{\partial s_1} = \frac{1}{\langle w(\alpha_1), \gamma \rangle} = k_s, \quad \frac{\partial \tau_0}{\partial \alpha_1} = \frac{\langle v(\alpha_1), \gamma(\tau_0) \rangle}{\langle w(\alpha_1), \gamma \rangle} = k_\alpha.
\]

**Claim.** The reflection operator \(\chi: (s_1, \alpha_1) \mapsto (s_2, \alpha_2)\) is a local diffeomorphism for \((s_1, \alpha_1)\) that hits the boundary transversally.

**Proof.** As is shown in Figure 3, the reflection \(\chi\) follows the rules:

\[
\begin{align*}
\alpha_2 &= \alpha_1 + 2\beta + \pi \\
\beta &= \langle v(\alpha_1), \gamma(\tau_0) \rangle
\end{align*}
\]

where \(\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})\) is the incident angle. We use \(\beta < 0\) to represent the case when \(v(\alpha_1)\) has negative projection along \(\gamma\).

Since \(\sin \beta = \langle v(\alpha_1), \gamma(\tau_0) \rangle\), it follows that \(\beta\) is a smooth function of \(s_1\) and \(\alpha_1\), which has the derivative

\[
\frac{\partial \beta}{\partial s} = \kappa k_s, \quad \frac{\partial \beta}{\partial \alpha_1} = \kappa k_\alpha - 1,
\]

where \(\kappa\) is the signed curvature. Then

\[
\frac{\partial \alpha_2}{\partial s_1} = 2\kappa k_s, \quad \frac{\partial \alpha_2}{\partial \alpha_1} = 2\kappa k_\alpha - 1.
\]

Consequently,

\[
\begin{align*}
\frac{\partial s_2}{\partial s_1} &= \langle \frac{\partial w(\alpha_2)}{\partial s_1}, \gamma(\tau_0) \rangle + \langle w(\alpha_2), \frac{\partial \gamma}{\partial s_1} \rangle = -\langle v(\alpha_2), \gamma(\tau_0) \rangle \frac{\partial \alpha_2}{\partial s_1} + k_s \langle w(\alpha_2), \gamma \rangle, \\
\frac{\partial s_2}{\partial \alpha_1} &= \langle \frac{\partial w(\alpha_2)}{\partial \alpha_1}, \gamma(\tau_0) \rangle + \langle w(\alpha_2), \frac{\partial \gamma}{\partial \alpha_1} \rangle = -\langle v(\alpha_2), \gamma(\tau_0) \rangle \frac{\partial \alpha_2}{\partial \alpha_1} + k_\alpha \langle w(\alpha_2), \gamma \rangle.
\end{align*}
\]
By row reduction, we have
\[ \det(d\chi) = \det \begin{bmatrix} \frac{\partial x_1}{\partial s_1} & \frac{\partial x_2}{\partial s_1} \\ \frac{\partial x_1}{\partial s_2} & \frac{\partial x_2}{\partial s_2} \end{bmatrix} = \det \begin{bmatrix} k_s\langle w(\alpha_2), \gamma \rangle & k_\alpha \langle w(\alpha_2), \gamma \rangle \\ 2\kappa k_s & 2\kappa k_\alpha - 1 \end{bmatrix}. \]

Thus,
\[ \det(d\chi) = \langle w(\alpha_2), \gamma \rangle \det \begin{bmatrix} k_s & k_\alpha \\ 2\kappa k_s & 2\kappa k_\alpha - 1 \end{bmatrix} = -\langle w(\alpha_1), \gamma \rangle (-k_s) = 1. \]

The determinant of \( d\chi \) is nonzero and therefore \( \chi \) is a local diffeomorphism. \( \Box \)

5.2. Conjugate points. The incoming ray \( l_1(t) \) and reflected ray \( l_2(t) \) are given in the following
\[
\begin{align*}
l_1(t) &= p + tv(\alpha_1), & 0 \leq t \leq t_1, \\
l_2(t) &= \gamma(\tau_0) + (t - t_1)v(\alpha_2), & t \geq t_1,
\end{align*}
\]
where \( q_0 = \gamma(\tau_0) \) is the intersection point on the boundary. Compared with (3), now \( t_1 = t_2 = q_0 \) connects \( l_1 \) and \( l_2 \). From now on, we use \( t_1 \) instead of \( t_2 \). Recall that \( \frac{d}{d\alpha_1} \) denotes the derivative with respect to \( \alpha_1 \) with \( p \) fixed and \( s_1 \) given by \( s_1 = \langle p, w(\alpha_1) \rangle \). By equation (17)(19), a straightforward calculation shows
\[
\frac{d\alpha_2}{d\alpha_1} = \frac{2\kappa t_1}{\langle w(\alpha_1), \gamma \rangle} - 1, \quad \frac{dq_0}{d\alpha_1} = \frac{t_1}{\langle w(\alpha_1), \gamma \rangle} \gamma_1,
\]
where \( t_1 = \langle q_0 - p, v(\alpha_1) \rangle \) is the time or length from \( p \) to \( q_0 \). Plugging these back into (2), we have
\[ \det(d_s \exp_p(v)) = (t - t_1)\left( \frac{d\alpha_2}{d\alpha_1} \right) - t_1. \]

Especially, the matrix is in the following
\[ d\exp_p(tv(\alpha_1)) = \left[ \begin{array}{c} v(\alpha_2), \\ (t - t_1)\left( \frac{d\alpha_2}{d\alpha_1} \right) - t_1w(\alpha_2) \end{array} \right]. \]

Corollary 1. Suppose an incoming ray \( l_1 \) hits the boundary \( \gamma \) transversally at point \( \gamma(\tau_0) \) and then reflects. Let \( p \) be a fixed point on \( l_1 \). Then
(a) \( p \) has a conjugate point \( q \) on \( l_2 \) if and only if \( \frac{d\alpha_2}{d\alpha_1} > 0 \), more specifically, if and only if
\[ \kappa(\tau_0) < \frac{\langle w(\alpha_1), \gamma(\tau_0) \rangle}{2t_1}. \]
(b) If this happens, \( q \) is uniquely determined by \( \Delta t_2 = \left( \frac{d\alpha_2}{d\alpha_1} \right)^{-1}\Delta t_1 \), where \( \Delta t_1 = t_1 \) is the time or length from \( p \) to \( \gamma(\tau_0) \) and \( \Delta t_2 = \langle q - \gamma(\tau_0), v(\alpha_2) \rangle \) is that from \( \gamma(\tau_0) \) to \( q \).

The statement (a) comes from the observation that the other factor \( \langle \frac{dq_0}{d\alpha_1}, w(\alpha_2) \rangle \) in Proposition 1(a) is always negative, as shown in Figure 3. This statement has a straightforward geometrical explanation, see Figure 4. It says there are conjugate points if and only if \( \alpha_2 \) increases as \( \alpha_1 \) increases.

For negatively oriented smooth curve that is the boundary of a convex set, the curvature \( \kappa < 0 \) and the inner product \( \langle w(\alpha_1), \gamma \rangle < 0 \). The inequality actually says \( |\kappa(\tau_0)| > \frac{||\langle w(\alpha_1), \gamma \rangle||}{2t_1} \). Additionally, observe that \( \langle w(\alpha_1), \gamma \rangle = -\cos \beta \), where \( \beta \) is the incident and the reflected angle. Each component involved in this criterion (21) is geometrical and therefore is invariant regardless of what kind of parameterization.
we choose for the boundary. We should mention that the equation in (b) coincides with the Generalized Mirror Equation in [3], but is in different form and is derived from the perspective of the exponential map.

**Remark 4** (of Theorem 4.2). For the V-line transform, a broken ray is regular if and only if its incoming part does not hit the boundary perpendicularly.

**Example 1.** Consider a parabolic mirror $-4ay = x^2$, which has the focus at $(0, -a)$. Suppose there is a light source located at the point $p = (0, -d)$. Here $a$ and $d$ are positive constants we are going to choose later. We would like to know in which directions of the light from $p$ there are conjugate points. This example will illustrate the criterion for conjugate points.

Let $\gamma(x) = (x, -\frac{x^2}{4a})$ be the boundary curve. The intersection point is $q_0 = \gamma(x_0)$. Then the incoming ray has the direction along $\overrightarrow{pq_0}$, and $w(\alpha_1), \gamma(x_0), \kappa(x_0), t_1$ are calculated directly by definition. After simplification, the criterion (21) is equivalent to

$$(a - d)(\frac{3}{4}x_0^2 - ad) > 0.$$ 

We have the following three cases.

**Case 1.** If $d > a$, then $p$ has conjugate points if and only if the incoming ray hits the boundary at the region $x^2 < \frac{1}{4}ad$, as is shown in Figure 5(a).

**Case 2.** If $d < a$, then $p$ has conjugate points if and only if the incoming ray hits the boundary at the region $x^2 > \frac{1}{4}ad$, as is shown in Figure 5(b).

**Case 3.** If $d = a$, then $p$ has no conjugate points for all directions, which coincides with the fact that all rays of light emitting from the focus reflect and travel parallel to the $y$-axis, as is shown in Figure 5(c).

**Example 2.** The second example is to illustrate that we have different types of conjugate points, specifically fold and cusps, if we have a circular mirror with a light source inside. We assume the mirror is centered at the origin $O$ and the source is not there. Suppose the mirror has radius 1. We follow some notations in the paper [24]. With $p$ fixed, the tangent conjugate locus $S(p)$ is the set of all vectors $v$ such that the differential of the exponential map $d_v \exp_p(v)$ is not an isomorphism. By calculations in Section 5.2,

$$S(p) = \{t(\cos \alpha_1, \sin \alpha_1), \text{s.t. } F(t, \alpha_1) = \frac{2\kappa t_1}{w(\alpha_1), \gamma(\tau_0)} - 1)(t - t_1) - t_1 = 0\},$$

where $t_1$ and $\tau_0$ are smooth functions of $\alpha_1$. Now for fixed $v$, we denote the kernel of $d_v \exp_p(v)$ by $N_p(v)$. According to equation (20), the differential $d_v \exp_p(v)$ has the matrix form $[v(\alpha_2), 0]$, which indicates that $N_p(v)$ is spanned by $\frac{\partial}{\partial \alpha_1}$. If $N_p(v)$ is transversal to $S(p)$, then we say $v$ is of fold type. In this case, $v$ is of fold type.
Figure 5. In (a) and (b), the bold part is the intersection region where the incoming rays hit there and reflect with conjugate points.

When $\frac{\partial F}{\partial \alpha_1} \neq 0$ for all $(t, \alpha_1) \in S(p)$. Otherwise, when there is some $(t, \alpha_1)$ such that $\frac{\partial F}{\partial \alpha_1} = 0$ and it is a simple zero, we have a cusp. We show in the following that the cusp exists. A straightforward calculation shows

$$F = (\frac{2t_1}{\cos \beta} - 1)t - 2 \frac{t_1^2}{\cos \beta} \Rightarrow \frac{\partial F}{\partial \alpha_1} = 6t_1^2 \sin \beta (t_1 - \cos \beta) / \cos^2 \beta (2t_1 - \cos \beta),$$

where $\cos \beta = -\langle w(\alpha_1), \dot{\gamma}(t_0) \rangle$ is a smooth function of $\alpha_1$. If there are conjugate points, we must have $2t_1 - \cos \beta > 0$. The incidence angle $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ so we at most have three zeros for $\frac{\partial F}{\partial \alpha_1}$

- $\beta = 0$, which means the incoming ray and reflected ray coincide. This is a simple zero, because $\frac{\partial}{\partial \alpha_1} \sin \beta = t_1 - \cos \beta = t_1 - 1 \neq 0$.
- $\cos \beta - t_1 = 0$ is true for some $\alpha_0$. This happens when $pO$ is perpendicular to the incoming ray. We check $\frac{d}{d\alpha_1} (\cos \beta - t_1) = \sin \beta \neq 0$. This is also a simple zero.

As a result, we have three cusps.

5.3. Numerical examples. This subsection aims to illustrate the artifacts arising in the reconstruction by numerical experiments. We say $(x, \xi)$ is visible if there is a broken ray $\gamma$ in the family of tomography, such that $(x, \xi)$ is in the conormal bundle of $\gamma$ excluding the connecting part. The fact that $(x, \xi)$ is visible does not necessarily imply that $(x, \xi)$ is recoverable.

Example 3. In this example, we use filtered backprojection to recover $f$, which usually serves as the first attempt of reconstruction. Suppose the domain is a disk with radius $R$ and the boundary is negatively oriented. The family of broken rays $\Gamma$ contains any broken ray whose incoming part hits the boundary transversally and has positive projection on it. We choose $f_1$ to be a Gaussian concentrated near a single point, as an approximation of a delta function and $f_2$ to be zero. The support of $f = f_1 + f_2$ is in this disk.

In the code, $Bf$ is parameterized in the coordinate $(x_p, \alpha) \in [-R, R] \times [0, 2\pi)$. Here $(x_p, \alpha)$ refers to the incoming part and we use it to represent the broken ray. This parameterization follows the convention in Radon transform in MATLAB. The radial coordinate $x_p$ is the value along the $x'$-axis, which is oriented at $\alpha$ degree counterclockwise from the $x$-axis. We use the function radon to numerically
construct our operator $B$ by the following formula

$$Bf(x_p, \alpha) = Rf(x_p, \alpha) + Rf(x'_p, \alpha'),$$

where $(x'_p, \alpha')$ is given by the reflection. Since numerically $Rf$ is known on discrete values of $(x_p, \alpha)$, we use interpolation methods to approximate $Rf(x'_p, \alpha')$. Similarly, $B^*$ is numerically constructed by the function iradon and interpolation methods. To better recover $f$, we apply the filter $\Lambda$ to the data before applying the adjoint operator. The plots are shown in the Figure 6. We can clearly see the artifacts appear exactly in the location of conjugate points, compared with the caustics caused by a light source. Furthermore, they are explained by equation (14).

**Example 4.** This example is to illustrate the reconstruction from local data by Landweber iteration. Assume each $(x, \xi)$ in $\text{WF}(f)$, is visible and is perceived by only one broken ray. Then it has at most one conjugate point. To make it true, we use part of the circle as the reflection boundary. The tomography family $\Gamma$ is the set of all broken rays which comes from the left side with vertices on the boundary.

By [11], we choose $f$ to be a modified Gaussian with singularities located both in certain space and in direction, that is, a coherent state, as is shown in Figure 7(a). We use the Landweber iteration to reconstruct $f$. The artifacts are still there after 100 iterations and the error becomes stable. Then we rotate $f$ or move it to see what happens to the artifacts. Specifically, in (c) and (d), $f$ remains in the location but is rotated by some angles. In (e) and (f), we move $f$ closer to the center and rotate it a bit. As the wave front set of $f$ changes, the artifacts changes and always appear in the location of their conjugate vectors.

**5.4. The local problem with non-even weights.** Suppose $\nu$ is a regular broken ray parameterized by the incoming part $(s_1, \alpha_1)$. It has the reflected part $(s_2, \alpha_2)$. There is another broken ray $\nu'$ that has the same linear path as $\nu$ but is in opposite direction, which is parametrized by the incoming part $(-s_2, \alpha_2 + \pi)$. In this subsection, instead of working on a neighborhood of $\nu$, we consider the recovery of $f$ from the knowledge of $Bf$ near both $\nu$ and $\nu'$. The conjugate covectors along $\nu$ could also be probed by $\nu'$, which intuitively helps us to recover singularities.

We consider a pair of conjugate covectors $(p_1, \xi^1)$ and $(p_2, \xi^2)$ on the incoming and reflected part of $\nu$ respectively. Suppose they satisfy Theorem 4.2. A straightforward calculation similar to Remark 2 shows that $(p_2, \xi^2)$ is conjugate to $(p_1, \xi^1)$ along $\nu'$. As proved in [24], we form a similar theorem in the following.

**Theorem 5.1.** Suppose $V^k$ are small enough conical neighborhoods of conjugate covectors $(p_k, \xi^k)$, for $k = 1, 2$. Let $f = f_1 + f_2$ with $\text{WF}(f) \subset V^k$. If the weight
function for the V-line transform satisfies
\[
\det \begin{bmatrix}
a(p_1, s_1, \alpha_1) & a(p_2, s_1, \alpha_1) \\
a(p_1, -s_2, \alpha_2 + \pi) & a(p_2, -s_2, \alpha_2 + \pi)
\end{bmatrix} \neq 0,
\]
then \(B(f_1 + f_2) \in H^s(V)\) implies \(f_k \in H^{s-1/2}(V^k)\), for \(k = 1, 2\).

Proof. Let \(B_+\) be the broken ray transform restricted in a small neighborhood of \((s_1, \alpha_1)\) and \(B_-\) be that in a neighborhood of \((-s_2, \alpha_2 + \pi)\). Now Theorem 4.2 becomes a 2 \times 2 system of equations

\[
\begin{align*}
g_+ &= B_+ f = B_{+,1} f_1 + B_{+,2} f_2 \in H^s(V), \\
g_- &= B_- f = B_{-,1} f_1 + B_{-,2} f_2 \in H^s(V).
\end{align*}
\]

The assumption that the weight function is always nonzero implies that \(B_{\pm,1}\) and \(B_{\pm,2}\) are elliptic. Applying \(B_{+,2} B_{-,1}^{-1}\) to (23) and subtracting it from (22), we have

\[
(Id - Q) f_1 = B_{+,1}^{-1} (g_+ - B_{+,2} B_{-,2}^{-1} g_-),
\]
where

\[
Q = B_{+,1}^{-1} B_{+,2} B_{-,2}^{-1} B_{-,1}.
\]

We prove in the following that \(Q\) is a \(\Psi\)DO of order zero with principal symbol not equal to 1. As a result, \(Id - Q\) is invertible and we can recover \(f_1\) and \(f_2\) microlocally from this system.

First we define the operator \(N : (s, \alpha) \mapsto (-s, \alpha + \pi)\), which induces a diffeomorphism \(N^*\). Recall that \(\chi\) is the reflection operator. It is straightforward to check that \(\chi N \chi N = Id\). Then we write \(Q\) as

\[
Q = B_{-,1}^{-1} (B_{-,2} B_{+,2}^{-1} \chi^* N^*) (\chi^* N^* B_{+,2} B_{-,2}^{-1}) B_{-,1} \equiv B_{-,1}^{-1} Q_1 Q_2 B_{-,1},
\]
where \(Q_1\) is the composition inside the first parentheses and \(Q_2\) is that inside the second.

Figure 7. Local reconstruction by Landweber iteration.
Claim. As defined above, $Q_1$ is a $\Psi$DO of order zero with principal symbol $\sigma_1 \circ C_{B_{-1}}^{-1}$, where $\sigma_1 = a(p_1, -s_2, \alpha_2 + \pi) / a(p_1, s_1, \alpha_1)$. Additionally, $Q_2$ is a $\Psi$DO of order zero with principal symbol $\sigma_2 \circ C_{\chi^* N^* R_{+2}}^{-1}$, where $\sigma_2 = a(p_2, s_1, \alpha_1) / a(p_2, -s_2, \alpha_2 + \pi)$.

We will prove this claim below. Assuming it for the moment, by Egorov’s theorem, $Q$ is a $\Psi$DO with principal symbol
\[
(\sigma_1 \circ C_{B_{-1}}^{-1})(\sigma_2 \circ C_{\chi^* N^* R_{+2}}^{-1}) \circ C_{B_{-1}}^{-1} = \sigma_1(\sigma_2 \circ C_{B_{-2}}^{-1} C_{B_{-1}}^{-1})
\]
which implies that $Id - Q$ is an elliptic $\Psi$DO. Recall that $B_{\pm,i}$ is an FIO of order $-1/2$ for $i = 1, 2$, and therefore $B_{\pm,1}^{-1} B_{+2} B_{-2}^{-1}$ is of order $1/2$. As a result, $f = f_1 + f_2$ can be recovered microlocally by
\[
f_1 = (Id - Q)^{-1} B_{+1}^{-1}(g_+ - B_{+2} B_{-2}^{-1} g_-), \quad f_2 = B_{-2}^{-1}(g_+ - B_{-1} f_1).
\]

The proof of the claim. We connect $B_{\pm,k}$ with the Radon transform restricted to distributions singular in $V^k$ near a certain ray, for $k = 1, 2$. Let $R_{+,k}$ be the Radon transform in $V^k$ near the ray $(s_k, \alpha_k)$. Let $R_{-,k}$ be the Radon transform in $V^k$ near the ray $(-s_k, \alpha_k + \pi)$. We emphasize that the weights of Radon transform here comes from that of the V-line transform as defined in (7), which might conflict with the convention. Especially, $R_{+,2}$ is the Radon transform near $(s_2, \alpha_2)$ but has the weight $a(x, s_1, \alpha_1)$ and $R_{-,1}$ is that near $(-s_1, \alpha_1 + \pi)$ but has the weight $a(x, -s_2, \alpha_2 + \pi)$. It follows that
\[
B_{+,1} = R_{+,1}, \quad B_{+,2} = \chi^* R_{+,2}, \quad B_{-,1} = \chi^* R_{-,1}, \quad B_{-,2} = R_{-,2}.
\]
Observe that $R_{+,1}^{-1} N^* R_{-,1}$ is a $\Psi$DO with principal symbol
\[
\sigma_1 = a(x, -s_2, \alpha_2 + \pi) / a(x, s_1, \alpha_1).
\]
By Egorov’s theorem, $N^* R_{-,1} R_{+,1}^{-1} = R_{+,1} (R_{+,1}^{-1} N^* R_{-,1}) R_{+,1}^{-1}$ is a $\Psi$DO with principal symbol $\tau_1 \circ C_{R_{+,1}}^{-1}$. A similar argument shows that $N^* R_{+,2} R_{-,2}^{-1}$ is a $\Psi$DO with principal symbol $\sigma_2 \circ C_{N^* R_{+,2}}^{-1}$.

Consequently, writing $Q_1$ and $Q_2$ as
\[
Q_1 = B_{-,1} B_{+,1}^{-1} \chi^* N^* = \chi^* R_{-,1} R_{+,1}^{-1} \chi^* N^* = \chi^* N^* (N^* R_{-,1} R_{+,1}^{-1}) \chi^* N^*,
\]
\[
Q_2 = \chi^* N^* B_{+,2} B_{-,2}^{-1} = N^* R_{+,2} R_{-,2}^{-1}.
\]
Applying Egorov’s theorem to the first equation, we have
\[
\sigma_p(Q_1) = \sigma_1 \circ C_{R_{+,1}}^{-1} \circ C_{\chi^* N^*} = \sigma_1 \circ C_{B_{-,1}}^{-1}, \quad \sigma_p(Q_2) = \sigma_2 \circ C_{N^* R_{+,2}}^{-1} = \sigma_2 \circ C_{B_{-,2}}^{-1}.
\]
The second equality comes from the observation that $B_{-,2}^{-1} N^* R_{+,2}$ is a $\Psi$DO.

Remark 5. This condition fails for the attenuated V-line transform that comes from the setting of Compton camera in two dimensions. In that setting, the direction of a broken ray is fixed and we do not have two different directed rays.
5.5. **Global problems.** In this subsection, suppose \( \Omega \) is a strictly convex domain with smooth negatively oriented boundary. We consider the V-line transform over all broken rays whose incoming part hits the boundary transversally and has non-negative projection on it. These rays may reflect from the boundary more than once but here we only consider the one-reflection situation, since we are motivated by the SPECT with Compton camera. We consider the reconstruction of the V-line transform from full data.

Suppose \( Bf \) is smooth. We would like to find out whether a given covector \((x_0, \xi^0)\) is in the wave front set of \( f \). Assume \((x_0, \xi^0)\) in \( \text{WF}(f) \). There are two broken rays in \( \Gamma \) that could carry this singularity. One broken ray \( \nu_0 \) represented by \((s_0, \alpha_0)\) has it in the incoming part, and the other one \( \nu_{-1} \) represented by \((s_{-1}, \alpha_{-1})\) has it in the reflected part. Suppose \((x_1, \xi^1)\) and \((x_{-1}, \xi^{-1})\) are its conjugate covectors along \( \nu_0 \) and \( \nu_{-1} \), if they exist. When both \( \nu_0 \) and \( \nu_{-1} \) are regular, we have the following cases.

If at least one of \((x_1, \xi^1)\) and \((x_{-1}, \xi^{-1})\) does not exist, for example \((x_1, \xi^1)\), then the singularity caused by \((x_0, \xi^0)\) in \( V^0 \) cannot be canceled via \( \nu_0 \). With the assumption that \( Bf \) is smooth, this indicates \((x_0, \xi^0) \in \text{WF}(f) \) impossible.

If both \((x_1, \xi^1)\) and \((x_{-1}, \xi^{-1})\) exist, then the singularities might be canceled by them. We continue to consider \( \nu_1, \nu_{-2} \) and so on. As a result, we get a sequence of broken rays (we assume they are all regular at this stage) and conjugate covectors. We define

\[
\mathcal{M}(x_0, \xi^0) = \{(x_k, \xi^k) \text{ if it exists and is conjugate to } (x_k', \xi^{k'})\},
\]

where \( k' = k - \text{sgn } k \), for \( k = \pm 1, \pm 2, \ldots \) as the set of all conjugate covectors related to \((x_0, \xi^0)\). If \( \mathcal{M}(x_0, \xi^0) \) contains finitely many \((x_k, \xi^k)\) whose index \( k \) is positive or negative, we say it is nontrapping in positive or negative direction. Otherwise, we say it is trapping.

Next, let \( V^k \) be a small conic neighborhoods of \((x_k, \xi^k) \in \mathcal{M}(x_0, \xi^0) \) and \( U_k = \pi(V^k) \). Let \( f_k \) be the restriction of \( f \) on \( U_k \). Now we suppose \( \mathcal{M}(x_0, \xi^0) \) is nontrapping, for example, in the positive direction. That is, there exists a maximal integer \( k_0 \) such that \((x_{k_0}, \xi^{k_0}) \in \mathcal{M}(x_0, \xi^0) \). From analysis above, we assume \( k_0 \geq 1 \). For \( k = 1, \ldots, k_0 \), by shrinking \( V^k \) carefully, we have \( C(V^k) = V^{k-1} \). Then the cancellation of singularities shows

\[
B_{k-1}f_{k-1} + B_kf_k = 0 \quad \text{mod } C^\infty, \quad k = 1, \ldots, k_0.
\]

Finally we have

\[
B_{k_0}f_{k_0} = 0 \quad \text{mod } C^\infty.
\]

By applying the diffeomorphism \( \chi^* \) and forward substitution, we can show that all \( f_k \) must be smooth, for \( k = 0, \ldots, k_0 \). It is similar if \( \mathcal{M}(x_0, \xi^0) \) is nontrapping in negative direction.

The above analysis holds when each \((x_k, \xi^k) \in \mathcal{M}(x_0, \xi^0) \) is carried by a regular broken ray. If it is not true, we can still define the sequence of conjugate covectors \( \mathcal{M}(x_0, \xi^0) \). If the sequence is nontrapping, then by considering \( B \) microlocally and by performing the similar arguments we can show \( f \) is smooth. This proves when \( \mathcal{M}(x_0, \xi^0) \) is nontrapping, \((x_0, \xi^0)\) is a recoverable singularity.

**Theorem 5.2.** Suppose \( \Omega \) is a strictly convex domain with smooth boundary. Let \( f \) be a distribution supported in \( \Omega \). Then \((x_0, \xi^0)\) is recoverable if \( \mathcal{M}(x_0, \xi^0) \) is nontrapping. In other words, when \( Bf \in C^\infty \), we must have \((x_0, \xi^0) \notin \text{WF}(f) \).
Example 5. As is shown in Figure 8, we use the same domain and family of tomography as in Example 2. Especially, we suppose the disk is centered at the origin for simplification. Considering a point \((p_0, \xi_0)\), we have a sequence of broken rays...

![Figure 8](image-url)

**Figure 8.** Inside a circular mirror, a sequence of broken rays and conjugate points on them.

...as well as the set \(M(p_0, \xi_0)\).

Proposition 3. In Example 5, we say \((p_0, \xi_0)\) is radial if \(p_0\) is the midpoint of a chord such that \(\xi_0\) is in its conormal. Then \(M(p_0, \xi_0)\) is trapping if and only if \((p_0, \xi_0)\) is radial.

Proof. Fix a point \(p_k\). It might have a conjugate point \(p_{k+1}\) along \(b_{k-1}b_kb_{k+1}\) or \(p_{k-1}\) along \(b_{k-2}b_{k-1}b_k\). Let \(d_i = |b_kp_k|\) be the distance along the ray from \(p_k\) to the boundary point \(b_k\). Notice all incidence and reflection angles are equal (call them \(\beta\)). Then \(|b_kb_{k+1}| = 2 \cos \beta\) for all integer \(k\).

Recall Corollary 1. In this case, we have \(\Delta t_1 = d_1, \Delta t_2 = 2 \cos \beta - d_2,\) and \(\frac{d_{k+1}}{d_k} = \frac{2d_k}{\cos \beta} - 1\). Then \(p_k\) has a conjugate point \(p_{k+1}\) inside the domain if and only if \(d_{k+1}\) given by

\[
\frac{1}{d_i} + \frac{1}{2 \cos \beta - d_{k+1}} = \frac{2}{\cos \beta}
\]

has a solution in \((0, 2 \cos \beta)\). To simplify, we change the variable that \(d_k = \cos \beta(a_k + 1)\). Thus,

\[
\frac{1}{1 + a_k} + \frac{1}{1 - a_{k+1}} = 2 \implies 2a_i a_{k+1} + a_{k+1} - a_k = 0.
\]

(27)

The requirement that \(p_k\) is inside the domain means we are finding solutions for \(a_k \in (-1, 1)\).

Case 1. \(a_0 = 0\), which implies by \(a_k = 0\) for any integer \(k\). This is the case when we have \(p_0\) at the midpoint of some chord and \(\xi_0\) is the conormal of the chord. The same is true with all \((p_k, \xi^0)\). We have a trapping \(M(p_0, \xi^0)\).

Case 2. \(a_k \neq 0\). Then (27) can be reduced to the following iteration formula

\[
\frac{1}{a_{k+1}} = \frac{1}{a_k} + 2.
\]

Suppose we start from some \(a_0\). Each time, the next \(\frac{1}{a_k}\) increases or decreases by 2. With \(\frac{1}{a_0} \in (-\infty, -1) \cup (1, \infty)\), finally we must have some \(\frac{1}{a_k}\) belonging to the...
interval \((-1, 1)\), which mean \(p_k\) goes out of the domain. In this case, \(M(p_0, \xi^0)\) is always nontrapping.

**Corollary 2.** Suppose everything as in Example 5. Then \((x_0, \xi^0)\) is recoverable if \((x_0, \xi^0)\) is not radial.

**Example 6.** With the same set up as above, we first choose \(f_1\) to be a modified Gaussian of coherent state whose singularities are not radial. To compare, next we choose \(f_2\) to be with radial singularities. As is shown in Figure 9, after performing

![Figure 9.](image-url)

Landweber iteration of 100 steps, all artifacts fade out and the reconstruction has a small error if \(f\) has non-radial singularities. On the contrary, if \(f\) has radial singularities, the error still decreases as the iteration but in a much slower speed. In these two cases, since \(f\) is only supported in a small set, the artifacts arising in the reconstruction may seem not so obvious. However, when \(f\) is more complicated, the artifacts might be unignorable. In the following we choose \(f_3\) to be a Modified Shepp-Logan phantom.

The error plots of these three cases are in Figure 11 to better illustrate the difference between radial and non-radial singularities. They also show where the artifacts appear (for more details, see 5.6). It is clear to see the error of reconstruction is much smaller when we have non-radial singularities than radial ones.

**Example 7.** In this example we consider the reconstruction of the V-line transform in an elliptical domain \(Q\) from global data. By [28], the billiard trajectory in an elliptical table has the following cases. If the trajectory crosses one of the focal points, then it converges to the major axis of \(Q\). If the trajectory crosses the line segment between the two focal points, then it is tangent to a unique hyperbola, which is determined by the trajectory and shares the same focal points with \(Q\). If
it does not cross the line segment between the two focal points, then it is tangent to a unique ellipse, which shares the same focal points with $Q$.

In the following numerical experiments, we choose $f$ as a coherent state. It is located and rotated such that the trajectory carrying its singularities falls into the last two cases above. We use Landweber iterations to reconstruct $f$ by iterating 100 steps. As in Figure 12, in the reconstruction of the first coherent state, the artifacts disappear as we iterate, since some conjugate points are outside the domain. On the contrary, with conjugate points staying in the domain at least for the first reflection, there is a relative larger error in the reconstruction of the second one. A more complete analysis of the ellipse case is behind the scope of this work.

5.6. Comparison with previous results for a circular domain. This subsection is to connect our analysis to the results in [26]. By expanding $f$ and the data $Bf$ as Fourier series with respect to the angular variable, [26] gives an inversion formula (2.8) for V-line transform with vertices on a circle. The denominator inside the integral has zeros for certain radius $r$ and with noises it could be very unstable. This indicates we can expect certain patterns of the artifacts in the reconstruction. We show these artifacts predicted by (2.8) coincides with the conjugate covectors of radial singularities in the following.

When $(x_0, \xi_0)$ is radial, $M(x_0, \xi_0)$ is trapping and we have two cases. One is the case that $M(x_0, \xi_0)$ is a periodic set with period $m$. That is, the broken rays that carry $(x_0, \xi_0)$ after several reflections form a regular polygon of $m$ edges, a convex or star one. The set $P$ of all possible regular polygons can be described by the Schlafli symbol $[5],

$$P = \{(m/n), \; p, q \in \mathbb{N}, \; 2 \leq 2n < m, \; \gcd(m, n) = 1\}.$$

Figure 10. Reconstruction from global data for Modified Shepp-Logan phantom $f_3$, where $e = \frac{\|f - f^{(100)}\|}{\|f\|_2}$ is the relative error.

Figure 11. The error plot for the reconstruction of $f_1, f_2, f_3$ in order. The first two has the same range of color bar.
Figure 12. Reconstruction of two coherent states. Left to right: true $f$, the envelopes (caused by trajectories that carry singularities and are reflected only once), $f^{(100)}$ (where $e = \frac{\|f-f^{(100)}\|_2}{\|f\|_2}$), the error.

Here $(m/n)$ refers to a regular polygon with $m$ sides which winds $n$ times around its center. When $n = 1$, it is a convex regular one; otherwise it is a star one. For the polygon $(m/n)$, the internal angle equals to $\frac{\pi(m-2n)}{m}$. This implies $|x| = \cos \frac{n\pi}{m}$, where $x$ is the midpoint of one edge. Suppose $Bf$ is smooth. We have

$$B_{i-1}f_{i-1} + B_if_i = 0 \mod C^\infty, \quad i = 1, \ldots, p-2$$

$$B_{m-1}f_{m-1} + B_0f_0 = 0 \mod C^\infty.$$  

By forward substitution, we get

$$ (1 + (-1)^{m-1})R_0f_0 = 0 \mod C^\infty.$$  

When $m$ is odd, $f_0$ must be smooth, which implies $f$ is smooth and therefore $(x_0, \xi^0)$ is recoverable. When $m$ is even, it possibly causes artifacts. These artifacts are located at radius $|x| = \cos \frac{(2k+1)\pi}{2m}$, where $m = 2l$ and $n = 2k+1$ with $0 \leq 2n < m$. These radius are exactly the positive solution of $s$ such that $\cos(n(\arcsin(s) - \pi/2)) = 0$ in Formula (2.8) in [26].

In the following example, we use the same function as in Figure 9 but move them closer to the origin. The plot of error shows the artifacts are centered at the midpoint of each edges of regular stars.

Figure 13. Another case of radial singularities. Left to right: true $f$, reconstruction $f^{(100)}$, error for $f$ with radial singularities after 100 iterations. The relative error $e$ is defined as before.
We should mention that in the numerical reconstruction in [26], the regularization (2.12) is used to remove the instabilities caused by these zeros. Therefore the artifacts are removed but on the other hand some true singularities are removed as well. In [30], the regularization is also used in the numerical reconstruction of a smiley phantom but we can still see some artifacts caused by the radial singularities (see Figure 2 in [30]).

6. The parallel ray transform. We define the parallel ray transform as an integral transform over two or more equidistant parallel rays. The simplest case is the one over two parallel rays and is defined in the following

\[ P f(s, \alpha) = \int_{x \cdot \omega(\alpha) = s} f(x) dx + \int_{x \cdot \omega(\alpha) = s + d} f(x) dx. \]

This transform can be regarded as over a general family of broken rays that we defined in Section 2, if we suppose the two rays are connected by a smooth curve outside the support of \( f \) or simply at the infinity. Additionally, the diffeomorphism \( \chi \) is the translation which maps \((s, \alpha)\) to \((s + d, \alpha)\). Following the previous notations and calculations, we have

\[ \frac{d\alpha_2}{d\alpha_1} = 1, \quad \frac{ds_2}{ds_1} = -\langle p, v(\alpha_1) \rangle. \]

Suppose \( p \) is on the ray \((s_1, \alpha_1)\). By Proposition 1, if \( p \) has a conjugate point \( q \) on the ray \((s_2, \alpha_2)\), then \( q \) is determined by \( q = p + w(\alpha_1) d \). By Theorem 4.2, a singularities \((x, \xi)\) can be canceled by \((y, \eta)\) if and only if \( x \) and \( y \) are conjugate points and \( \xi = \eta \). It is shown in Figure 3 that the artifacts arising when we use the backprojection as the first attempt to recover \( f \).

Now we consider the reconstruction by iteration process. Suppose \((x_0, \xi^0)\) \(\in\) \(\text{WF}(f)\) belongs to the ray \((s_1, \alpha_1)\). It can be canceled by two conjugate covectors \((x_0 \pm \frac{d\xi^0}{|\xi^0|}, \xi^0)\). We follow the same analysis as in the previous section to have

\[ M(x_0, \xi^0) = \{ (x_0 \pm j d \frac{\xi^0}{|\xi^0|}, \xi^0), \ j = \pm 1, \pm 2, \ldots \}. \]

The typology of \( M(x_0, \xi^0) \) is quite clear. It is a discrete set of points which has equal distance. Assume \( Pf \) is smooth. Then \((x_0, \xi^0)\) \(\in\) \(\text{WF}(f)\) implies \( M(x_0, \xi^0) \subseteq \text{WF}(f) \) by the same argument as before. Thus, we have the following proposition, see also [34].

**Proposition 4.** Suppose \( f \in \mathcal{D}'(\mathbb{R}^2) \) and assume \( Pf \) is smooth. Then for any \((x, \xi)\), either \( M(x, \xi) \subset \text{WF}(f) \) or \( M(x, \xi) \cap \text{WF}(f) = \emptyset \).

In particular, with a prior knowledge that \( \text{WF}(f) \) is in a compact set, the singularities are recoverable.

**Corollary 3.** Suppose \( f \in \mathcal{E}'(\mathbb{R}^2) \) and assume \( Pf \) is smooth. Then \( f \) is smooth.

In the numerical experiment, we use the Landweber iteration to reconstruct \( f \). With the assumption that \( f \in \mathcal{E}'(\mathbb{R}^2) \), a cutoff operator is performed at every step. After 100 iterations, we get a quite good reconstruction (with \( \|f^{(100)} - f\|_\infty = 0.003 \)).

It should be mentioned that Corollary 3 shows \( f \) with singularities in a compact set could be recovered from the global data. This implies when performing the transform, we move the parallel rays around until all of them leave the compact set.
In fact, from our analysis above, the condition that the rays leaving at least one side the compact set is enough. On the other hand, the local problem (illumination of a region of interest only) could create artifacts.

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REFERENCES

[1] G. Ambartsoumian, Inversion of the V-line Radon transform in a disc and its applications in imaging, Comput. Math. Appl., 64 (2012), 260–265.
[2] R. Basko, G. L. Zeng and G. T. Gullberg, Application of spherical harmonics to image reconstruction for the Compton camera, Phys. Med. Biol., 43 (1998).
[3] J. A. Boyle, Using rolling circles to generate caustic envelopes resulting from reflected light, Amer. Math. Monthly, 122 (2015), 452–466.
[4] A. Cayley, A memoir upon caustics, Philos. Trans. Royal Soc. London, 147 (1857), 273–312.
[5] H. S. M. Coxeter, Introduction to Geometry, John Wiley & Sons, Inc., New York-London, 1961.
[6] D. B. Everett, J. S. Fleming, R. W. Todd and J. M. Nightingale, Gamma-radiation imaging system based on the Compton effect, Proceedings of the Institution of Electrical Engineers, 124, 1977.
[7] L. Florescu, V. A. Markel and J. C. Schotland, Inversion formulas for the broken-ray Radon transform, Inverse Problems, 27 (2011), 13pp.
[8] B. Frigyik, P. Stefanov and G. Uhlmann, The X-ray transform for a generic family of curves and weights, J. Geom. Anal., 18 (2008), 89–108.
[9] R. Gouia-Zarrad and G. Ambartsoumian, Exact inversion of the conical Radon transform with a fixed opening angle, Inverse Problems, 30 (2014), 12pp.
[10] M. Haltmeier, S. Moon and D. Schiefeneder, Inversion of the attenuated V-line transform with vertices on the circle, IEEE Trans. Comput. Imaging, 3 (2017), 853–863.
[11] S. Holman and G. Uhlmann, On the microlocal analysis of the geodesic X-ray transform with conjugate points, J. Differential Geom., 108 (2018), 459–494.
[12] L. Hörmander, The Analysis of Linear Partial Differential Operators, Classics in Mathematics, Springer, Berlin, 2007.
[13] M. Hubenthal, The broken ray transform on the square, J. Fourier Anal. Appl., 20 (2014), 1050–1082.
[14] M. Hubenthal, The broken ray transform in n dimensions with flat reflecting boundary, Inverse Probl. Imaging, 9 (2015), 143–161.
[15] J. Ilmavirta, Broken ray tomography in the disc, Inverse Problems, 29 (2013), 17pp.
[16] J. Ilmavirta, On the broken ray transform, preprint, arXiv:1409.7500.
[17] J. Ilmavirta, A reflection approach to the broken ray transform, Math. Scand., 117 (2015), 231–257.
[19] J. Ilmavirta and M. Salo, Broken ray transform on a Riemann surface with a convex obstacle, Comm. Anal. Geom., 24 (2016), 379–408.
[20] C.-Y. Jung and S. Moon, Inversion formulas for cone transforms arising in application of Compton cameras, Inverse Problems, 31 (2015), 20pp.
[21] R. Krylov and A. Katsevich, Inversion of the broken ray transform in the case of energy-dependent attenuation, Phys. Med. Biol., 60 (2015), 4313–4334.
[22] P. Kuchment and F. Terzioglu, Three-dimensional image reconstruction from Compton camera data, SIAM J. Imaging Sci., 9 (2016), 1708–1725.
[23] V. Maxim, M. Frande and R. Prost, Analytical inversion of the Compton transform using the full set of available projections, Inverse Problems, 25 (2009), 21pp.
[24] F. Monard, P. Stefanov and G. Uhlmann, The geodesic ray transform on Riemannian surfaces with conjugate points.
[25] S. Moon, On the determination of a function from its conical Radon transform with a fixed central axis, SIAM J. Math. Anal., 48 (2010), 1833–1847.
[26] S. Moon and M. Haltmeier, Analytic inversion of a conical Radon transform arising in application of Compton cameras on the cylinder, SIAM J. Imaging Sci., 10 (2017), 535–557.
[27] M. Morvidone, M. K. Nguyen, T. Truong and H. Zaidi, On the V-line Radon transform and its imaging applications, IEEE International Conference on Image Processing, Hong Kong, 2010.
[28] S. Park, An introduction to dynamical billiards.
[29] L. C. Parra, Reconstruction of cone-beam projections from Compton scattered data, IEEE Trans. Nuclear Science, 47 (2000), 1543–1550.
[30] D. Schiefeneder and M. Haltmeier, The Radon transform over cones with vertices on the sphere and orthogonal axes, SIAM J. Appl. Math., 77 (2017), 1335–1351.
[31] M. Singh, An electronically collimated gamma camera for single photon emission computed tomography. Part I: Theoretical considerations and design criteria, J. Comput. Assisted Tomography, 7 (1983), 421–427.
[32] B. Smith, Reconstruction methods and completeness conditions for two Compton data models, JOSA A, 22 (2005), 445–459.
[33] P. Stefanov and G. Uhlmann, The geodesic X-ray transform with fold caustics, Anal. PDE, 5 (2012), 219–260.
[34] P. Stefanov and G. Uhlmann, Is a curved flight path in SAR better than a straight one?, SIAM J. Appl. Math., 73 (2013), 1596–1612.
[35] P. Stefanov and Y. Yang, Multiwave tomography with reflectors: Landweber’s iteration, Inverse Probl. Imaging, 11 (2017), 373–401.
[36] F. Terzioglu, Some inversion formulas for the cone transform, Inverse Problems, 31 (2015), 21pp.
[37] F. Terzioglu and P. Kuchment, Inversion of weighted divergent beam and cone transforms, Inverse Probl. Imaging, 11 (2017), 1071–1090.
[38] R. W. Todd, J. M. Nightingale and D. B. Everett, A proposed γ camera, Nature, 251 (1974), 132–134.
[39] T. Tomitani and M. Hirasawa, Image reconstruction from limited angle Compton camera data, Phys. Med. Biol., 47 (2002).
[40] F. W. Warner, The conjugate locus of a Riemannian manifold, Amer. J. Math., 87 (1965), 575–604.
[41] W. Zhang, D. Zhu, M. Lun and C. Li, Multiple pinhole collimator based X-ray luminescence computed tomography, Biomed. Opt. Express, 7 (2016), 2506–2523.

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