Shot noise of interference between independent atomic systems

A. Polkovnikov

Department of Physics, Boston University - Boston, MA 02215, USA

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Abstract – We study shot (counting) noise of the amplitude of interference between independent atomic systems. In particular, for the two interfering systems the variance of the fringe amplitude decreases as the inverse power of the number of particles per system with the coefficient being a non-universal number. This number depends on the details of the initial state of each system so that the shot noise measurements can be used to distinguish between such states. We explicitly evaluate this coefficient for the two cases of the interference between bosons in number states and in broken symmetry states. We generalize our analysis to the interference of multiple independent atomic systems. We show that the variance of the interference contrast vanishes as the inverse power of the number of the interfering systems. This result, implying high signal to noise ratio in the interference experiments, holds both for bosons and for fermions.

Interference between independent systems is an interesting phenomenon, which goes back to the discovery of the Hanbury-Brown-Twiss (HBT) effect [1]. The very possibility that particles from independent condensates can interfere, i.e. have a certain relative phase, is quite intriguing and is not entirely obvious [2]. The origin of this phenomenon is the quantum indistinguishability of identical particles and the superposition principle. Recent experimental advances in cold atom systems opened new possibilities to detect the interference both in equilibrium [3–6] and in nonequilibrium [7] situations. Such experiments even allow probing various properties of interacting systems. For example, an analysis of the scaling of the interference signal with the imaging size lead to the direct observation of the Kosterlitz-Thouless phase transition [6]. The problem of interference also attracted a lot of theoretical attention. In ref. [8] Javanainen and Yoo numerically studied the interference between two Fock states for a given set of detectors. It was shown that the resulting count distribution of atoms was similar to the one arising from the interference of two condensates with randomly broken phases. Later Castin and Dalibard analyzing another Gedanken experiment came to the same conclusion and showed that these random phases are induced by the detectors [9]. Emerging interference between two number states was later interpreted as an indication of the presence of hidden variables [10]. In refs. [11,12] the authors studied the interference between independent fluctuating condensates and reached a similar conclusion that for the large ideal condensates the amplitude of interference does not fluctuate. Recently, Altman et al. suggested using noise interferometry as a new probe of interacting atomic systems [13]. These ideas were later experimentally implemented [14,15].

In fact the reason why two independent condensates with fixed number of particles in each must interfere follows from the basic principles of quantum mechanics. Indeed, because of the number-phase uncertainty each of the condensates does not have a well-defined phase. However, according to quantum mechanics any measurement probe sensitive to the phase difference (whether it is a time-of-flight image or something else) will project the condensates to the state with a well-defined relative phase. As a result, the condensates will interfere but the relative phase will be random for each experimental run in agreement with refs. [9,11] and this work.

The main purpose of this paper is to study shot noise of the interference between independent atomic systems. For the most part we will consider bosonic systems (condensates for short). However, our approach is independent of the atom statistics and where necessary we will give the explicit expressions for fermions as well.

We point that if the systems are independent then the interference between them can be considered as a noise, similarly to the HBT effect. Indeed, the average
of the atom density over many experimental runs does not result in any oscillating component. However, for the two condensates with large number of particles, each run yields well-defined interference fringes [8–11]. The fringe amplitude is thus a quantity which does not average to zero. Moreover, as we mentioned above, this amplitude is expected to have vanishing fluctuations. Conversely, if the number of particles in each of the two systems is small, we expect to see significant shot noise and large fluctuations of the fringe amplitude. The analysis of these fluctuations is the subject of the present investigation.

On passing on, we mention that shot noise experiments are a very powerful tool in condensed matter physics (see, ref. [16] for a review) and in quantum optics (see, e.g., ref. [1]). Such experiments can be used to detect charge of quasi-particles, transmission properties of small conductors, entanglement between electrons, allow to distinguish quantum and classical light sources, etc.

We emphasize that when one analyzes the interference between extended systems, one encounters at least two different sources of noise (apart from the noise associated with the probing beam). One of them originates from the phase fluctuations within each system, which can have either quantum or thermal origin. This type of noise was analyzed in detail in refs. [11,12]. In particular, it was shown that the scaling of the fringe amplitude with the system size and its distribution function contain information about phase correlation functions within each system. The second source of noise, which is studied in the present paper, has a purely quantum nature originating from the commutation relations between identical particles. This counting or shot noise exists even in ideal noninteracting systems and as we show below it contains important information about the nature of the interfering states.

Assume that all bosons in each of the two interfering systems (1 and 2) occupy the same quantum state. We consider the standard time of flight measurement setup (see fig. 1). At a certain moment in time these particles are allowed to expand. For simplicity we assume that the particles expand only in one direction \( x \). For the two-dimensional systems with strong transverse confinement like in, e.g., ref. [6], this assumption is usually well justified since the transverse kinetic energy is very large.

For the system of two one-dimensional condensates [7] the expansion for each condensate occurs in the radial direction. In this case one can always analyze the image in the plane of the two condensates and the present analysis remains intact. In a more general case one has to compute the overlap between the expanding Wannier functions corresponding to the two different condensates. However, we emphasize that this overlap brings a more complicated geometrical factor characterizing the interference, which enters only as the prefactor into the interference amplitude and does not affect the main conclusions of this paper.

After waiting until the size of the clouds greatly exceeds the original separation between the condensates \( d \), the density of these atoms is measured using probing beams. We assume that there is no photon shot noise and the atoms are detected with a 100% probability. For simplicity we also assume that \( d \) is much larger than the initial size of each condensate \( w_0 \).

As we already argued, there is no average interference contrast between the two condensates. However, there is a well-defined observable \( A_2 \), equal to the square of the interference amplitude [11]. Let us define the fluctuating variable \( A \), the average of which over many experimental runs gives \( A_2 \), in the following way:

\[
A = \int \int dx' dx \, p(x,t) p(x',t) e^{iQ(x-x')} - \int dx \, p(x,t). \tag{1}
\]

Here \( p(x,t) \) is the number of the absorbed photons at the time \( t \) at the position \( x \) and \( Q = md/\hbar t \), where \( m \) is the atom’s mass. The integral over \( x \) and \( x' \) should be understood as the sum, where the discretization step is determined by the detector and cannot be smaller than the photon wavelength. Apart from the second term in eq. (1), which as we will see below has a quantum origin, the expression for \( A \) is just a square of the Fourier transform of the absorption image taken at the wave vector \( Q \). Note that since by assumption all the detectors have 100% efficiency, the number of the absorbed photons coincides with the number of atoms at the detection point: \( p(x,t) = n(x,t) = a^\dagger(x,t) a(x,t) \), where \( a^\dagger(x,t) \) and \( a(x,t) \) are the time-dependent creation and annihilation operators of the atoms. We understand the equality sign here and in expressions below in the sense that the statistical properties of \( p(x,t) \) and the corresponding quantum operator are equivalent.

Using the bosonic commutation relations \( [a(x,t), a^\dagger(x',t)] = \delta(x-x') \) it is easy to see that eq. (1) can be rewritten as

\[
A = \int \int dx' dx \, a^\dagger(x,t) a^\dagger(x',t) a(x,t) a(x',t) e^{iQ(x-x')}, \tag{2}
\]

We can further simplify eq. (2) using that in the long time limit [4,17]

\[
a(x,t) \approx a_1 u(x,t) e^{Q_1(t)x} + a_2 u(x,t) e^{Q_2(t)x}, \tag{3}
\]

where \( Q_{1,2} = m(x \pm d/2)/\hbar t \), \( a_{1,2} \) are the bosonic operators in the Schrödinger representation corresponding to

![Fig. 1: Schematic view of the interference experiment between two independent systems.](image_url)
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\[ A_4 = \int \int \int dx_1 dx_2 dx_1' dx_2' \left( n(x_1, t)n(x_2, t)n(x_1', t)n(x_2', t) : e^{iQ(x_1 + x_2 - x_1' - x_2')} \right) \]
+ \(2 \int \int dx_1 dx_2 dx_1' \left( n(x_1, t)n(x_2, t)n(x_1', t) : \right) e^{iQ(x_2 - x_1')} + \int dx_1 dx_2 \left( n(x_1, t)n(x_2, t) : \right), \quad (5)\)

\[ A_4 = (a_1^{+2} a_2^{+2} a_1^2 + 2 \{a_1^{+2} a_2^{+2} a_1^2\} + 2 \{a_1^{+2} a_2^{+2} a_1\} + \{a_1^{+2} a_2^2\} + \{a_2^{+2} a_2^2\} + 2\{a_1^{+2} a_2 a_1\} \]. \quad (6)\)

\[ \langle a_1^{+2} a_2^{+2} a_1^2 \rangle \rightarrow \int \int \int dz_1 dz_2 dz_3 dz_4 \left( a_1(z_1)a_2(z_2)a_3(z_3)a_4(z_4)a_2(z_1)a_2(z_2)a_1(z_3)a_1(z_4) \right), \quad (11)\)

\[ \langle a_1^{+2} a_2^{+2} a_1^2 \rangle \rightarrow \int \int \int dz_1 dz_2 \left( a_1(z_1)a_1(z_2)a_1(z_3)a_2(z_2)a_1(z_3)a_1(z_4) \right), \quad (12)\)

\[ \langle a_1^{+2} a_2^2 \rangle \rightarrow \int \int dz_1 dz_2 \left( a_1(z_1)a_2(z_2)a_1(z_2)a_1(z_1) \right), \quad (13)\)

\[ \langle a_1^{+2} a_2 a_1 \rangle \rightarrow \int \int dz_1 dz_2 \left( a_1(z_1)a_2(z_2)a_2(z_2)a_1(z_2) \right), \quad (14)\]

where the semicolons imply the normal-ordered form, i.e., that the creation operators appear on the left of the annihilation operators. Substituting (3) into the integrals above yields see eq. (6).

Let us define the relative width of the distribution \( w = \sqrt{A_4 - A_2^2}/A_2 \). We explicitly look into two different initial states. First we consider the interference between two independent coherent states with on average \( N \) atoms in each state. In this case from eqs. (4) and (6) we find:

\[ A_2 = N^2, \quad A_4 = N^4 + 4N^3 + 4N^2 \Rightarrow \]

\[ w = \frac{2\sqrt{N+1}}{N} \]. \quad (8)\)

As expected, \( w \) vanishes as \( N \to \infty \).

Next we consider the interference between the two number states with \( N \) atoms in each of them. Then

\[ A_2 = N^2, \quad A_4 = N^4 + 2N^3 + N^2 - 2N \Rightarrow \]

\[ w = \frac{\sqrt{2}}{\sqrt{N}} \sqrt{1 + \frac{1}{2N} - \frac{1}{N^2}}. \quad (10)\)

Asymptotically \( w \) also decreases as the inverse square root of the number of particles at large \( N \). However, the coefficient appears to be smaller by a factor of \( \sqrt{2} \) than in the case of the two coherent states.

We note that eq. (6) is also valid for fermions. However, in this case all the terms except the last one identically vanish and thus \( A_2 = 2A_2^2 \) independent of the details of the fermionic state. This result comes from the fact that in this simple setup there are at most two interfering particles.

In the case of the interference of extended condensates, which expand only in the transverse directions (see refs. [6,11,12]), the expressions above are easily generalized. For example in eq. (6) one has to do the following substitutions: see eqs. (11), (12), (13), (14)
Other substitutions can be obtained from these expressions by simple permutations. In the equations above $z_j$ denote coordinates along the condensates and the integration is taken over the area or the length of the condensates. For the systems with long-range correlations the terms containing the largest number of bosonic operators give the leading contribution to $A_4$ in agreement with the results of refs. [11,12]. Indeed, for example, for the interference between two 1D condensates at zero temperature eqs. (6) and (11)-(14) give

$$A_4 = AL^{4-2/K} (1 + B(\rho L)^{1/K - 1}),$$

(15)

where $A$ and $B$ are the non-universal constants, $\rho$ is the mean particle density, $L$ is the system size or the imaging length (see refs. [11,12] for more details). The Luttinger parameter $K$ above characterizes the strength of the interactions in the condensates. For the repulsive bosons with point-like interactions $K$ interpolates between 1 in the impenetrable Tonks-Girardeau regime and $\infty$ in the non-interacting system [18]. It is clear that shot noise given by the second term in the brackets of eq. (15) is subdominant for large systems as long as $K > 1$. We emphasize that even if shot noise is negligible still $A_4 > A_2$ as long as $K$ is finite, i.e. there is no true long range order in the interfering systems [11]. This inequality implies that $w$ is finite and the amplitude of the interference fringes still fluctuates because of nontrivial particle-particle correlation functions in each system. On the contrary, in systems with short range correlations the situation changes and one cannot simply ignore shot noise. For example, in one-dimensional condensates at finite temperature $T$ the particle-particle correlation functions decay exponentially at long distances [18]:

$$\langle a^\dagger(x,t) a(x',t) \rangle \sim e^{-|x|/\xi_T},$$

(16)

where $\xi_T \sim 1/T$ is the correlation length. Then instead of eq. (15) we find

$$A_4 = \tilde{A}L^2\xi_T^{2-2/K} \left[ 1 + \tilde{B}(\rho\xi_T)^{1/K - 1} \right],$$

(17)

with some other nonuniversal coefficients $\tilde{A}$ and $\tilde{B}$. Note that unlike the zero temperature case, the second term which corresponds to the shot noise remains finite in the thermodynamic limit. At low temperatures $\rho\xi_T \gg 1$ its contribution remains small but the shot noise becomes increasingly important as the temperature grows and $\xi_T$ decreases. This result can be readily understood since in systems with finite correlation length only particles within this length (or the correlation volume in higher dimensions) coherently contribute into the interference amplitude [11]. The shorter this length the smaller the number of such coherent atoms and thus the stronger the effects of the shot noise.

Next we turn to the interference of multiple independent condensates. For simplicity we assume that all the condensates are identical and that they form a one-dimensional array with a distance $d$ separating the nearest neighbors. Then for $M$ condensates the density operator $n(x,t)$ assumes the following form in the long time limit:

$$n(x,t) \approx u^2(x,t) \sum_{j,k=1}^{M-1} a_j^\dagger a_k e^{i(Q_j-Q_k)x},$$

(18)

where $Q_j = m(x + dj)/\hbar$ and as before we use the fact that the Wannier functions corresponding to different condensates are identical. We note that using eq. (1) as the measure of the interference is not very efficient if the number of condensates is large. In particular, one can show that the relative width of the distribution of $A_2$ for large $M$ is $w \approx 1 + 1/N$, where $N$ is the average number of particles per condensate. This width remains finite even in the limit of large $N$. Physically this happens because each condensate comes with its own random phase and thus there is no constructive interference between them. Instead one can define a different interference measure:

$$\tilde{A} = \sum_q \int \int dx dx' p(x,t)p(x',t) e^{iQ_q(x-x')} - \int dx p(x,t),$$

(19)

where $Q_q = q Q$ with $q = 1,2,\ldots M-1$. We note that in the case of multiple condensates there are alternative ways of defining such measure. For example in ref. [15] Fölling et al. used the quantity $C(b) = \int dx p(x,t)p(x+b,t)$. The fluctuations of $C(b)$ can be analyzed by methods similar to those described in this paper. The evaluation of $\tilde{A}_2 \equiv \overline{\tilde{A}}$ is analogous to that of $A_2$ so we only present the final answer:

$$\tilde{A}_2 = \frac{M(M-1)}{2} N^2.$$  

(20)

Next we can compute $\tilde{A}_4$, which describes the fluctuations of $A_2$. The procedure is again quite similar to the one described above for the case of two condensates. However, the actual computations and the resulting expression for $\tilde{A}_4$ are quite cumbersome, so we present only the two leading terms in the number of the condensates $M$:

$$\tilde{A}_4 \approx \frac{M^4 N^4}{4} + M^3 \left( \langle n^2 \rangle N^2 - \frac{5}{6} N^4 + 3 N^3 + \frac{5}{3} N^2 \right).$$

(21)

This expression yields the following asymptotic form of the relative width of the distribution of the interference contrast:

$$\tilde{w} \approx \frac{2}{N \sqrt{M}} \sqrt{\frac{N^2}{3} + \langle n^2 \rangle} + 3 N + \frac{5}{3}.$$  

(22)

Note that the width $\tilde{w}$ vanishes with the number of the condensates even for the small number of particles per condensate. The coefficient multiplying $1/\sqrt{M}$ in eq. (22) weakly depends on the initial state of each condensate. Thus for the interference between bosons in number states this coefficient gradually decreases from $4.2$ for $N = 1$ to $1.6$ for $N \to \infty$.  

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A very similar calculation can be done for the case of interfering fermions. The only difference between bosons and fermions is that all bosonic contributions to $\tilde{A}_4$ come with a positive sign and fermionic contributions can bear both negative and positive signs. While eq. (20) for $\tilde{A}_2$ remains intact and holds for fermions as well, the corresponding expression for $\tilde{A}_4$ gets modified:

$$\tilde{A}_4^{(f)} \approx \frac{M^4 N^4}{4} + M^3 \left( -\frac{3}{2} N^4 + N^3 + \frac{1}{3} N^2 \right),$$

(23)

where the superscript $(f)$ stands for fermions. Correspondingly the relative width of the distribution becomes:

$$w^{(f)} \approx \frac{2}{N \sqrt{M}} \sqrt{-N^2 + N + \frac{1}{3}}.$$

(24)

Note that both eqs. (22) and (24) have similar scaling with the number of interfering systems. This observation leads us to an interesting conclusion that a high interference contrast is possible for the case of multiple incoherent fermionic sources. This is opposite to the situation of two interfering systems, where the fringe contrast always remains low.

We would like to mention that in real experiments dependence of the interference contrast on $M$ can saturate at large $M$. The reason is that the derivation of eqs. (22) and (24) relied on taking into account all the Fourier components appearing in eq. (19). However, at large wave vectors $q_0$ this can become problematic because of finite resolution of the apparatus, finite width of Wannier functions, not entirely free expansion of the atoms, etc. Thus realistically the sum over $q$ in eq. (19) is always bounded by some value $q_{\text{max}}$. Then obviously increasing the number of interfering systems beyond $q_{\text{max}}$ will not improve the interference contrast. We also point out that eq. (18) is valid only in the long time limit provided that $\hbar^2 t/m \gg (Md)^2$, or equivalently $w \gg (Md)^2/w_0$, where $w$ is the width of the condensate after the expansion and $w_0 < d$ is the initial width of each condensate. This condition is hard to achieve if the number of the interfering systems is large. Alternatively one can view the relation $M^2 = \sqrt{w w_0}/d$ as determining the maximum number of the interfering systems beyond which the contrast saturates.

In conclusion, we analyzed fluctuations of the amplitude of interference of independent atomic systems. We showed that for the interference of two bosonic systems the fluctuations of the fringe amplitude are inversely proportional to the square root of the number of particles in each system. The coefficient of proportionality is non-universal; it depends on the details of the wave functions (or more generally density matrices) of each system. In particular, we found that the fluctuations are larger for the case of two interfering condensates with broken phases than for the case of two Fock states. We generalized our analysis to the interference of multiple systems and showed that fluctuations of the fringe contrast vanish as the number of interfering systems (bosonic or fermionic) becomes large.

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