ON THE LARGE N LIMIT OF THE
ITZYKSON – ZUBER INTEGRAL

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ABSTRACT

We study the large N limit of the Itzykson – Zuber integral and show that the leading term is given by the exponent of an action functional for the complex inviscid Burgers (Hopf) equation evaluated on its particular classical solution; the eigenvalue densities that enter in the IZ integral being the imaginary parts of the boundary values of this solution. We show how this result can be applied to “induced QCD” with an arbitrary potential $U(x)$. We find that for a nonsingular $U(x)$ in one dimension the eigenvalue density $\rho(x)$ at the saddle point is the solution of the functional equation $G_+(G_-(x)) = G_-(G_+(x)) = x$, where $G_{\pm}(x) \equiv \frac{1}{2}U'(x) \pm i\pi\rho(x)$. As an illustration we present a number of new particular solutions of the $c = 1$ matrix model on a discrete real line.
1. Introduction

The analysis of most any multi-matrix model is based on the Itzykson – Zuber formula \[1\]

\[ I(\phi, \chi) \equiv \int \mathcal{D}U e^{N\text{Tr}[\phi U\chi U^\dagger]} = \frac{\det[e^{N\phi_i\chi_j}]}{\Delta(\phi)\Delta(\chi)}, \tag{1.1} \]

where \(\phi\) and \(\chi\) are arbitrary \(N \times N\) Hermitian matrices, \(\Delta(\phi)\) denotes the Van der Monde determinant constructed out of the eigenvalues \(\phi_i\) of these matrices, \(\Delta(\phi) = \prod_{i<j}(\phi_i - \phi_j)\), and \(\mathcal{D}U\) is the Haar measure on the unitary group \(U(N)\). The way to apply it is to diagonalize the matrices entering the model and to carry out the integration over the diagonalizing unitary matrices using (1.1). Then, as \(N \to \infty\), the residual integrations (over the eigenvalues of the matrices) can be treated by the saddle point method. In this limit the eigenvalues at the saddle point can be described by their distribution on the real axis: if \(di\) is the number of eigenvalues between \(\phi\) and \(\phi + d\phi\), then there is a smooth function \(\rho(\phi)\) such that \(di/N = \rho(\phi)d\phi\). If we keep the distribution \(\rho(\phi)\) of the eigenvalues \(\phi_i\) entering (1.1) (as well as \(\sigma(\chi)\) of \(\chi_i\) fixed), the integral (1.1) possesses the \(N \to \infty\) asymptotic expansion,

\[ I(\phi, \chi) \sim \exp\{N^2F_0(\phi, \chi) + F_1(\phi, \chi) + N^{-2}F_2(\phi, \chi) + ...\}. \tag{1.2} \]

Nevertheless, although there is an exact explicit formula (1.1) for the Itzykson – Zuber integral for any finite \(N\), the structure of the coefficients \(F_n\) is not known yet. For many applications it would be important to understand it.

An interesting example of such an application is the so-called “induced QCD” \[2\] which is the lattice gauge theory of a scalar matrix field \(\Phi(x)\) defined by the action

\[ S = \sum_x N\text{Tr}\left[-U(\Phi(x)) + \sum_\mu \Phi(x)U_\mu(x)\Phi(x + \mu a)U_\mu^\dagger(x)\right]. \tag{1.3} \]

(\(\mu\) runs over the \(2D\) lattice vectors on a hypercubic lattice of dimension \(D\), the field \(\Phi(x)\) transforms according to the adjoint representation of the gauge group \(U(N)\)). It describes the high temperature limit of \(D+1\) dimensional QCD and, in addition, can be related to the matrix models of \(c = D\) strings. The Itzykson – Zuber integral turns out to be the main computational tool necessary to analyze this theory.
In this paper we claim that large N limit of the IZ integral is given by

\[ I(\phi, \chi) \sim \exp(N^2 F_0(\phi, \chi)), \]

where

\[ F_0(\phi, \chi) = S[\rho, \sigma] + \frac{1}{2} \left\{ \int \rho(x)x^2 dx + \int \sigma(y)y^2 dy \right\} \]

\[- \frac{1}{2} \left\{ \int \rho(x_1)\rho(x_2) \ln |x_1 - x_2| dx_1 dx_2 \right\} \]

\[ + \int \sigma(y_1)\sigma(y_2) \ln |y_1 - y_2| dy_1 dy_2 \]

(1.4)

where \( S[\rho, \sigma] \) is the classical action corresponding to the Hopf equation

\[ \frac{\partial f}{\partial t} + f \frac{\partial}{\partial x} f = 0, \] (1.5)

calculated on its solution satisfying the boundary conditions

\[ \text{Im} f(x, t = 0) = \pi \rho(x) \]

\[ \text{Im} f(x, t = 1) = \pi \sigma(x). \] (1.6)

We will show that the action \( S[\rho, \sigma] \) defines the field theory with the Hamiltonian

\[ H[\rho, \Pi] = \frac{1}{2} \int \rho(x) \left\{ \left( \frac{\partial \Pi(x)}{\partial x} \right)^2 - \frac{\pi^2}{3} \rho^2(x) \right\} dx \] (1.7)

where \( \Pi(x) \) is the canonical momentum conjugate to \( \rho(x) \). In this notation

\[ f(x, t) \equiv \frac{\partial \Pi(x, t)}{\partial x} + i\pi \rho(x, t). \] (1.8)

Quite remarkably, (1.7) is the nonunitary version of the Hamiltonian, arising in the collective field formulism of \( c = 1 \) string theory [8].
The plan of this paper is as follows. In sections 2 and 3 we establish the relation between the IZ integral and the Hopf equation, and prove the boundary conditions (1.6). In section 4 we apply our results to analyze “induced QCD” with an arbitrary nonsingular potential $U(x)$. We obtain that in the one-dimensional case the eigenvalue density $\rho(x)$ as a solution of the master field equation must satisfy the functional constraint $G_+(G_-(x)) = G_-(G_+(x)) = x$, where $G_\pm(x) \equiv \frac{1}{2}U'(x) \pm i\pi\rho(x)$. Although we were not able to find an explicit general solution to this equation, we show how one can generate a number of particular solutions of it. We use them to analyze the $c = 1$ matrix model on a lattice for finite lattice spacings.

2. Reduction of the Itzykson – Zuber integral to the Hopf problem

One of the ways to approach the problem of the large $N$ expansion is to write the differential equation for the IZ integral and specify the corresponding boundary conditions. So, first we will derive the differential equation and then identify the relevant boundary conditions.

It is convenient to consider the one parameter family of integrals:

$$I(\phi, \chi|t) \equiv \frac{1}{t^{N/2}} \int \mathcal{D}U e^{-\frac{N}{2t} \text{Tr}(\phi - U\chi U^\dagger)^2}, \quad (2.1)$$

so that $I(\phi, \chi|t) = \exp\left(-\frac{N}{2t} \text{Tr}(\phi^2 + \chi^2)\right)I(\phi/\sqrt{t}, \chi/\sqrt{t})$ and the Itzykson – Zuber integral per se corresponds to $t = 1$. Since $I(\phi, \chi|t)$ depends only on the eigenvalues of $\phi$ and $\chi$, one can show [1] that (2.1) satisfies the partial differential equation

$$2N \frac{\partial I(\phi, \chi|t)}{\partial t} = \frac{1}{\Delta(\phi)} \sum_{i=1}^{N} \frac{\partial^2}{\partial \phi_i^2} \Delta(\phi) I(\phi, \chi|t), \quad (2.2)$$

where $\Delta(\phi) = \prod_{i<j}(\phi_i - \phi_j)$.

This equation is the linear heat equation for the quantity $\tilde{I}(\phi, \chi|t) = \Delta(\phi) I(\phi, \chi|t)$ and can be solved exactly to obtain the Itzykson – Zuber formula (1.1). However, unlike $I(\phi, \chi|t)$, $\tilde{I}(\phi, \chi|t)$ does not have any regular large $N$ asymptotics. Therefore we have to study the equation for $\tilde{I}$. Let us use the ansatz
\[ I(\phi, \chi|t) = \exp(N^2 W[\rho, \sigma|t]) \]

so that from (2.2) follows

\[
2 \frac{\partial W}{\partial t} = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial^2 W}{\partial \phi_i^2} + N \sum_{i=1}^{N} \left( \frac{\partial W}{\partial \phi_i} \right)^2 + 2 \sum_{i=1}^{N} V(\phi_i) \frac{\partial W}{\partial \phi_i},
\]

(2.3)

where

\[
V(\phi_i) = \frac{1}{N} \sum_{j \neq i} \frac{1}{\phi_i - \phi_j}.
\]

The equations above are exact. Now we assume that \( W[\rho, \sigma|t] \) has a smooth large N limit, \( i.e. \) it is a smooth functional of the eigenvalue densities \( \rho \) and \( \sigma \). This implies that \( \frac{\partial W}{\partial \phi_i} \sim \mathcal{O}(1/N) \) and

\[
\frac{\partial W}{\partial \phi_i} = \frac{1}{N} \left( \frac{\partial}{\partial x} \delta(x) \right) \bigg|_{x=\phi_i}. \tag{2.4}
\]

Thus we can neglect the first term on the left-hand side of (2.3)(it is \( \mathcal{O}(1/N) \) while the two other terms are \( \mathcal{O}(1) \)) to get

\[
2 \frac{\partial W}{\partial t} = N \sum_{i=1}^{N} \left( \frac{\partial W}{\partial \phi_i} \right)^2 + 2 \sum_{i=1}^{N} V(\phi_i) \frac{\partial W}{\partial \phi_i}, \tag{2.5}
\]

Substituting

\[
W = S - \frac{1}{2N^2} \sum_{i \neq j} \ln|\phi_i - \phi_j| - \frac{1}{2N^2} \sum_{i \neq j} \ln|\chi_i - \chi_j| \tag{2.6}
\]

we find

* If we would have been looking for \( \tilde{W} \) such that \( \tilde{I} = \exp(N^2 \tilde{W}) \), we would not have been able to neglect the first term.
\[
2 \frac{\partial S}{\partial t} = N \sum_{i=1}^{N} \left( \frac{\partial S}{\partial \phi_i} \right)^2 - \frac{1}{N} \sum_{i=1}^{N} V^2(\phi_i).
\]

Finally, we use the identity \( \frac{1}{N} \sum_{i=1}^{N} V^2(\phi_i) = \frac{1}{N^3} \sum_{i \neq j} \frac{1}{(\phi_i - \phi_j)^2} \) to obtain

\[
2 \frac{\partial S}{\partial t} = N \sum_{i=1}^{N} \left( \frac{\partial S}{\partial \phi_i} \right)^2 - \frac{1}{N^3} \sum_{i \neq j} \frac{1}{(\phi_i - \phi_j)^2}.
\]

(2.7)

It is very useful to rewrite this equation in terms of the eigenvalue distributions. To this end we replace all the sums by the integrals according to the rule

\[
\frac{1}{N} \sum_{i=1}^{N} \rightarrow \int \rho(x) dx.
\]

However, the sum \( \frac{1}{N^3} \sum_{i \neq j} \frac{1}{(\phi_i - \phi_j)^2} \) needs to be treated specially, since the relevant integral \( \int \int \frac{\rho(x) dx \rho(y) dy}{(x-y)^2} \) diverges. In fact the \( O(1) \) contribution to this sum comes from the pairs of the eigenvalues \( \phi_i \) and \( \phi_j \) such that \( |i - j| \ll N \). Thus \( \phi_i - \phi_j \approx |i - j|/(N \rho(\phi_i)) \) and

\[
\frac{1}{N} \sum_{j} \frac{1}{(\phi_i - \phi_j)^2} \sim N \rho^2(\phi_i) \sum_{j} \frac{1}{(i - j)^2} = \frac{\pi^2}{3} N \rho^2(\phi_i).
\]

Replacing sums by integrals and using (2.4) we see that the continuum version of (2.7) is

\[
2 \frac{\partial S}{\partial t} = \int \rho(x) dx \left\{ \left( \frac{\partial}{\partial x} \delta S/\delta \rho(x) \right)^2 - \frac{\pi^2}{3} \rho^2(x) \right\}.
\]

(2.8)

This is a nonlinear differential equation for the functional \( S[\rho(x), \sigma(y)|t] \). It would be extremely hard, if possible at all, to find its general solution. Fortunately, we have to find only a particular solution - the one which corresponds to the large \( N \) limit of the IZ integral. To do so we will guess what the desired solution is and then prove the conjecture.
First note that (2.8) can be represented in the Hamilton - Jacobi form
\[
\frac{\partial S}{\partial \tau} + H\left( \frac{\delta S}{\delta \rho(x)}, \rho(x) \right) = 0 \tag{2.9}
\]
where \( \tau \equiv -t \),
\[
H[\rho(x), \Pi(x)] = \frac{1}{2} \int \rho(x) \left\{ \left( \frac{\partial \Pi(x)}{\partial x} \right)^2 - \frac{\pi^2}{3} \rho^2(x) \right\} dx \tag{2.10}
\]
is the Hamiltonian*, and \( \Pi(x) \) - the canonical momentum conjugate to \( \rho(x) \). Our conjecture is that \( S[\rho(x), \sigma(y)|\tau] \) is given by the action of the dynamical system defined by (2.10) calculated along the trajectory which connects the two points \( \rho(x) \) and \( \sigma(y) \) in the configuration space of this system. The trajectory has to satisfy the equations of motion and take \( \rho(x) \) to \( \sigma(y) \) within the time interval \( \tau \).

By construction the so defined \( S[\rho(x), \sigma(y)|\tau] \) solves (2.9). One still has to show that it gives the right large \( N \) asymptotics of the IZ integral. This will be done in the next section. In the meantime we will investigate the properties of this solution in more detail.

To find \( S[\phi, \chi] \) we have to solve the Hamilton equations following from (2.10). These are
\[
\frac{\partial \rho(x, t)}{\partial t} = -\frac{\partial}{\partial x} \left( \rho(x, t) \frac{\partial \Pi(x, t)}{\partial x} \right)
\]
\[
\frac{\partial \Pi(x, t)}{\partial t} = -\frac{1}{2} \left( \frac{\partial \Pi(x, t)}{\partial x} \right)^2 + \frac{\pi^2}{2} \rho^2(x, t).
\]

Introducing
\[
v(x, t) \equiv \frac{\partial \Pi(x, t)}{\partial x} \tag{2.11}
\]
we get

* The discrete version of this Hamiltonian, the one corresponding to (2.7), describes the completely integrable Calogero theory [9].
\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0 \tag{2.12}
\]
\[
\frac{\partial v}{\partial t} + v \frac{\partial}{\partial x} v = \frac{\pi^2}{2} \frac{\partial}{\partial x} \rho^2 .
\]

These are the Euler equations for a one dimensional liquid obeying the equation of state \( P = -\frac{\pi^2}{2} \rho^2(x) \). They can be easily solved. Indeed, let us introduce a new unknown function

\[
f(x, t) \equiv v(x, t) + i\pi \rho(x, t).
\]

Then \( f \) satisfies the Hopf equation

\[
\frac{\partial f}{\partial t} + f \frac{\partial}{\partial x} f = 0. \tag{2.13}
\]

Now we have to specify the boundary conditions for (2.12), (2.13) that lead uniquely to the original problem (1.1), (1.2). Our conjecture is to identify the endpoints of the classical solution we are looking for with the eigenvalue densities of the matrices \( \phi \) and \( \chi \). Therefore we demand that

\[
\left\{ \begin{array}{l}
\rho(x, t = 0) = \rho(x) \\
\rho(x, t = 1) = \sigma(x)
\end{array} \right.
\]

or, equivalently,

\[
\left\{ \begin{array}{l}
\text{Im} f(x, t = 0) = \pi \rho(x) \\
\text{Im} f(x, t = 1) = \pi \sigma(x)
\end{array} \right. \tag{2.14}
\]

Finally, taking into account that due to (2.1) \( F_0(\phi, \chi) = W[\rho, \sigma|t] + \frac{1}{2N} \text{Tr}(\phi^2 + \chi^2) \) and remembering the definition of \( S[\rho, \sigma] \), (2.6), we obtain our final result (1.4), (1.5), (1.6).

Is it possible to find an explicit formula for \( F_0(\phi, \chi) \)? To do this we have to find the trajectory obeying (2.13), (2.14). The Hopf equation is, in principle, exactly soluble. For a real function \( f \) the general solution of (1.5) can be written down in the parametrized form
\[ x = R(\alpha) + F(\alpha)t \]
\[ f(x, t) = F(\alpha) \tag{2.15} \]

where \( R(\alpha) \) and \( F(\alpha) \) are the two functions to be determined from the initial conditions supplementing (2.13). In fact, because of the invariance with respect to reparametrizations of \( \alpha \), (2.15) depends on only one function, consistent with the fact that the Hopf equation is of first order.

Our problem is a little more complicated, because \( f \) is complex valued. One can generalize the parametric solution to describe this situation provided that the initial condition \( f_0(x) = f(x, t = 0) \) can be analytically continued to the whole complex plane as a function of \( x \). This is indeed true in many applications.

In this case \( F(\alpha) \) and \( R(\alpha) \) can be viewed as the analytic functions of the complex parameter \( \alpha \). To impose the boundary conditions we introduce another pair of functions,

\[ G_+(x) = x + f(x, t = 0), \quad G_-(x) = x - f(x, t = 1) \]

and use (2.15) to get

\[ G_+(x) = x + [F \circ R^{-1}](x) = [(F + R) \circ R^{-1}](x) \]

as well as

\[ G_-(x) = x - [F \circ (F + R)^{-1}](x) = [R \circ (F + R)^{-1}](x) \]

so that

\[
\begin{align*}
G_+(G_-(x)) &= G_-(G_+(x)) = x \\
\text{Im}G_+(x) &= \pi \rho(x) \\
\text{Im}G_-(x) &= -\pi \sigma(x)
\end{align*}
\tag{2.16}
\]

where \( \circ \) denotes functional composition. Given \( \rho(x) \) and \( \sigma(x) \) we must solve (2.16) for, say, \( G_+(x) \), then the full initial condition for the Hopf equations (2.13) will be given by \( f(x, t = 0) = G_+(x) - x \). This allows one, in principle, to calculate the trajectory connecting \( \rho(x) \) and \( \sigma(x) \) and the action along it, which, in turn, gives the large \( N \) asymptotics of the Itzykson–Zuber integral.

We were not able to obtain an explicit expression for \( F_0 \) in (1.2) as a functional of \( \rho \) and \( \sigma \). However, in some cases (2.16) already provides enough information. This is the case in “induced QCD”. We will return to this question in section 4.
3. Boundary Conditions

In this section we will prove our conjecture about the boundary conditions for the Hopf equation. We will show that the quantity $S[\rho, \sigma]$ entering (1.4) is indeed the classical action along the trajectory of (1.7) with the endpoints

$$\rho(x, t = 0) = \rho(x); \quad \rho(x, t = 1) = \sigma(x). \quad (3.1)$$

To this end we note that the Itzykson – Zuber integral

$$I(\phi, \chi) \equiv \int \mathcal{D}U e^{N \text{Tr}[\phi U \chi U^\dagger]} \quad (3.2)$$

satisfies the set of identities with higher order differential operators $H^q_N \equiv \text{Tr}(\frac{1}{N} \frac{\partial}{\partial \phi})^q$

$$H^q_N I(\phi, \chi) = [\text{Tr}(\chi^q)] I(\phi, \chi), \quad k = 1, ..., N. \quad (3.3)$$

Since $I(\phi, \chi)$ depends only on the eigenvalues of the matrices $\phi$ and $\chi$, these identities reduce to

$$\mathcal{D}_q I(\phi, \chi) = [\text{Tr}(\chi^q)] I(\phi, \chi) \quad (3.4)$$

where

$$\mathcal{D}_q \equiv \frac{1}{\Delta(\phi)} \sum_{i=1}^{N} \frac{\partial_q}{\partial \phi_i^q} \Delta(\phi). \quad (3.5)$$

To take the large N limit of the differential operator in (3.5) we expand it explicitly to obtain

$$\mathcal{D}_q = \sum_{k_1, ..., k_s} \frac{q!(-) (k_1 - 1)_{n_1} + ... + (k_s - 1)_{n_s}}{\lambda! n_1 \cdots k_s \cdot n_1! \cdots n_s! (q - \lambda - n_1 k_1 - ... - n_s k_s)!} \cdot \left(\tilde{V}(\phi_n)\right)^\lambda \left(Z_{k_1}(\phi_n)\right)^{n_1} \cdots \left(Z_{k_s}(\phi_n)\right)^{n_s} \left(\frac{\partial}{\partial \phi_n}\right)^q (\lambda + n_1 k_1 + ... + n_s k_s)^q \quad (3.6)$$

where
\[ \tilde{V}(\phi_n) = \sum_{k \neq n} \frac{1}{\phi_n - \phi_k}, \]
\[ Z_m(\phi_n) = \sum_{k \neq n} \frac{1}{(\phi_n - \phi_k)^m}. \]

In the large N limit
\[ \tilde{V}(\phi_n) \to N \mathcal{P} \int \frac{\rho(x)dx}{\phi_n - x} = \mathcal{O}(N) \]

where \( \mathcal{P} \) denotes the principal value prescription, and, naively, \( Z_m(\phi_n) \) would go like \( \mathcal{O}(N) \) as well (since it includes one summation over \( k \)). However, it is not true because of the singularity near \( \phi_n \sim \phi_k \) where \( \phi_n - \phi_k \approx (n - k)/(N\rho(\phi_n)) \) so that
\[ Z_m(\phi_n) = N^m \rho^m(\phi_n) \sum_{k \neq n} \frac{1}{(n - k)^m} + \mathcal{O}(N^{m-1}) = \begin{cases} 2N^m \zeta(m) \rho^m(\phi_n) + \mathcal{O}(N^{m-1}), & \text{if } m \text{ is even} \\ \mathcal{O}(N^{m-1}), & \text{if } m \text{ is odd} \end{cases} \] (3.7)

where
\[ \zeta(m) = \sum_{k=1}^{\infty} \frac{1}{k^m} \]

is the Riemann zeta function.

When the operators \( D_q \) act on \( I(\phi, \chi) = \exp(N^2 F_0[\rho, \sigma]) \), the leading \( N \to \infty \) contribution comes from
\[ \left( \frac{\partial}{\partial \phi_n} \right)^p \exp(N^2 F_0) = \left[ N^p \left( N \frac{\partial F_0}{\partial \phi_n} \right)^p + \mathcal{O}(N^{p-1}) \right] \exp(N^2 F_0) \]

and is given by
\[ \mathcal{D}_q \exp(N^2 F_0) = \left\{ N^q \sum_{k_1, \ldots, k_s \text{even}}^{k_1, \ldots, k_s \text{all different, all } \geq 2} \frac{q!(-)^{(k_1-1)n_1+\ldots+(k_s-1)n_s}}{\lambda!k_1^{n_1} \ldots k_s^{n_s} n_1! \ldots n_s!} \right\} \exp(N^2 F_0) \]

Performing the summation over \( k_1, \ldots, k_s; n_1, \ldots, n_s \) and \( s \); using

\[ \frac{1}{N} \sum_{i=1}^{N} \to \int \rho(x) dx, \]

as \( N \to \infty \) we obtain *

\[
\mathcal{D}_q \exp(N^2 F_0[\rho, \sigma]) = \left[ \int d\phi \exp \left[ -2 \sum_{k=2,4,6,\ldots} \zeta(k) \frac{(iz \rho(\phi))^k}{k} \right] \right] \exp(N^2 F_0[\rho, \sigma])
\]

\[ \left[ \int \sigma(y) y^q dy \right] \exp(N^2 F_0[\rho, \sigma]) \]

and thus the large \( N \) limit of the identity (3.3) is given by (3.9). If we

* To do the sum it is convenient to introduce an operator \( \mathcal{D}_\infty = \sum_{q=0}^{\infty} \frac{(iz)^q}{q!} \mathcal{D}_q \). Then the summation can be performed using the identity

\[ \exp \left[ -2 \sum_{k=2,4,6,\ldots} \frac{\zeta(k)}{k} (iz \rho(\phi))^k \right] = \frac{\sinh(\pi z \rho(\phi))}{\pi z \rho(\phi)} \]

with the result

\[ \mathcal{D}_\infty \exp(N^2 F_0) = \frac{1}{\pi z} \int d\phi \exp \left[ iz (\tilde{V}(\phi) + \frac{\partial}{\partial \phi} \frac{\delta F_0}{\delta \rho(\phi)}) \right] \sinh(\pi z \rho(\phi)) \exp(N^2 F_0) \]
now substitute ansatz (1.4) into (3.9) we will get that the quantity

\[ \Pi(x) = -\frac{\delta S[\rho, \sigma]}{\delta \rho(x)} \]

should satisfy

\[
\frac{1}{2\pi i} \frac{1}{q+1} \int dx \left[ \left( -\frac{\partial}{\partial x} \Pi(x) + \tilde{V}(x) + i\pi \rho(x) \right)^{q+1} - \left( -\frac{\partial}{\partial x} \Pi(x) + \tilde{V}(x) - i\pi \rho(x) \right)^{q+1} \right] = \int \sigma(y) y^q dy . \tag{3.10}
\]

To prove our conjecture we have to show that the action \( S[\rho, \sigma] \) on the trajectory satisfying the boundary conditions (3.1) does have the property (3.10) as a consequence of the equations of motion (2.12). To see that this is indeed the case, we introduce the set of time dependent functions on the phase space (here \( v(x) = \frac{\partial}{\partial x} \Pi(x) \))

\[
H_q(\Pi, \rho, t) = \frac{1}{2\pi i t(q+1)} \int dx \left\{ \left( x - (v(x) - i\pi \rho(x)) t \right)^{q+1} - \left( x - (v(x) + i\pi \rho(x)) t \right)^{q+1} \right\} . \tag{3.11}
\]

Using (2.12), it is easy to check that

\[
\frac{dH_q}{dt} = 0
\]

on the equations of motion of our original Hamiltonian (1.7). So we have found an infinite number of conservation laws corresponding to the problem. Moreover,

\[
H_q(\Pi, \rho, t = 1) = \frac{1}{2\pi i (q+1)} \int dx \left\{ \left( x - (v(x) - i\pi \rho(x)) \right)^{q+1} - \left( x - (v(x) + i\pi \rho(x)) \right)^{q+1} \right\},
\]

and
\[ H_q(\Pi, \rho, t = 0) = \int \sigma(y) y^q dy. \]

Now, since \( H_q \) are the constants of motion, we conclude that (3.10) does hold, and, therefore, (1.4) is proved.

### 4. Induced QCD With an Arbitrary Potential

Induced QCD is the model defined by the partition function [2]

\[ Z = \int D\Phi(x) D\mu(x) \exp \sum_x N Tr \left[ -U(\Phi(x)) + \sum_\mu \Phi(x) U_\mu(x) \Phi(x + \mu a) U_\mu^\dagger(x) \right]. \]

The link fields \( U_\mu(x) \) can be integrated out giving the effective theory of the field \( \Phi(x) \), with the action

\[ S = \sum_{x,i} \left[ -NU(\phi_i(x)) \right] + \sum_{x,i \neq j} \ln |\phi_i(x) - \phi_j(x)| \]
\[ + \sum_{x,i} \ln \left[ I(\phi(x), \phi(x + \mu a)) \right]. \tag{4.1} \]

In the large \( N \) limit the partition function of this effective theory is dominated by a translationally invariant saddle point which obeys the equation

\[ \frac{1}{2} U'(\phi_i) = \frac{1}{N} \sum_{j \neq i} \frac{1}{\phi_i - \phi_j} + DN \frac{\partial F_0(\phi, \chi)}{\partial \phi_i} \bigg|_{\phi=\chi}. \tag{4.2} \]

Using (1.4) and \( N \frac{\partial S}{\partial \phi_i} = (\frac{\partial}{\partial x} \frac{\delta S}{\delta \rho(x)}) \big|_{x=\phi_i} = (\frac{\partial}{\partial x} \Pi(x)) \big|_{x=\phi_i} = v(\phi_i) \), one can rewrite it in the form

\[ v(\phi_i) = \frac{1}{2D} U'(\phi_i) - \phi_i + \frac{D-1}{D} \sum_{j \neq i} \frac{1}{\phi_i - \phi_j} \]

or, equivalently,
Migdal [3] has converted this into a nonlinear integral equation for \( \rho(x) \). On the other hand, from the results of section 2 it follows that the problem can be reduced to a differential equation. Indeed,

\[
v(x) = \text{Re} f(x, t = 0) = -\text{Re} f(x, t = 1)
\]

where \( f(x, t) \) is the solution of the boundary problem

\[
\begin{align*}
\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial x} &= 0 \\
\text{Im} f(x, t = 0) &= \text{Im} f(x, t = 1) = \pi \rho(x)
\end{align*}
\]

Equations (4.3) and (4.4) provide a description of induced QCD at large \( N \).

Let us study the simplest case \( D = 1 \). Then for a nonsingular potential \( U(x) \)

\[
G_+(x) = x + f(x) = x + v(x) + i\pi \rho(x) = \frac{1}{2} U'(x) + i\pi \rho(x),
\]

and we obtain the constraints determining the saddlepoint of one-dimensional induced QCD in the form

\[
\begin{align*}
G_+(G_-(x)) &= G_-(G_+(x)) = x \\
\text{Re} G_\pm(x) &= \frac{1}{2} U'(x) \\
\text{Im} G_\pm(x) &= \pm \pi \rho(x)
\end{align*}
\]

Note that in this case \( G_+(x) = \overline{G_-(x)} \). Moreover, since \( D = 1 \) induced QCD is equivalent to the Hermitian matrix model on a discretized line, one can use (4.6) to investigate the properties of this model.

* Let us, however, notice that for some singular potentials (say, those considered in [4]) the analyticity hypothesis (2.15) does not hold and one has to study the general equations (4.3) and (4.4).
In fact, it is easy to construct a number of exact solutions to (4.6) using the following device. Consider an arbitrary real analytic symmetric function \( P(z, w) = P(w, z) \) (say, a polynomial) of two variables, \( z \) and \( w \). If we define \( G_+ \) and \( G_- \) as the appropriate roots of

\[
P(x, G_+(x)) = P(G_-(x), x) = 0,
\]

then, by construction, \( G_+(G_-(x)) = x \), and the roots \( G_+ \), \( G_- \) can be chosen to be complex conjugate, i.e. be the solutions of (4.6). To clarify this point, let us consider an example

\[
P(x, y) = x^2 + y^2 - m^2 xy + \mu. \tag{4.7}
\]

For this \( P(x, y) \)

\[
G_\pm(x) = \frac{m^2}{2} x \pm i \sqrt{\mu - \frac{\mu^2 x^2}{4}}
\]

so that \( \pi \rho(x) = \sqrt{\mu - \frac{\mu^2 x^2}{4}} \) (where \( \mu \) is fixed by \( \int \rho(x) dx = 1 \) to be \( \mu = \sqrt{m^4 - 4} \) -the semicircular law, which is known to solve induced QCD with the quadratic potential \( U(x) = m^2 x^2 / 2 \) [5]. One can verify explicitly that (4.6) indeed holds for these \( G_\pm \).

To obtain a more general solution, we can perturb (4.7) by monomials of higher order. Then the relation \( G_+(G_-(x)) = G_-(G_+(x)) = x \) will not be violated, at least for small enough coefficients in front of these monomials. The simplest generalization is

\[
P(x, y) = x^2 + y^2 - bx^2 y^2 - m^2 xy + \xi(x + y) + \alpha.
\]

The corresponding potential and density are given by

\[
U'(x) = \frac{m^2 x - \xi}{1 - bx^2}; \quad \pi \rho(x) = \frac{1}{1 - bx^2} \sqrt{(1 - bx^2)(x^2 + \xi x + \alpha) - \frac{1}{4}(m^2 x - \xi)^2}. \tag{4.8}
\]

To see the physical meaning of this solution, let us consider its continuum limit. In the continuum limit we must scale...
\[ \Phi \rightarrow \frac{\varphi(t)}{\sqrt{a}}; \quad m^2 = 2 + a^2 M^2; \quad b = B a^3; \]

\[ \xi = \Xi a^{3/2}; \quad U(\Phi) = \Phi^2 + a W(\varphi), \]

where \( a \) is the lattice spacing. As \( a \rightarrow 0 \), the partition function of induced QCD reduces to

\[ Z = \int \mathcal{D}\varphi(t) e^{-N \int dt \left[ \text{Tr} \left( \frac{M^2}{2} \varphi^2(t) + W(\varphi(t)) + \frac{1}{2} (\partial_t \varphi(t))^2 \right) \right]} \quad (4.9) \]

In our example

\[ W'(\varphi) = \frac{M^2 \varphi + 2 B \varphi^3 - \Xi}{1 - B a^2 \varphi^2}. \]

We see that as \( a \rightarrow 0 \),

\[ W(\varphi) \rightarrow \frac{M^2 \varphi^2}{2} + \frac{B \varphi^4}{2} - \Xi \varphi \]

which is a general quartic potential. Note that our solution applies not only in the continuum limit, but for finite lattice spacings as well.

Therefore, it is possible to find nontrivial potentials, which have a well-defined continuum limit and, at the same time, admit an exact solution of the model.
5. Conclusions

We have shown that the large N behaviour of the Itzykson – Zuber integral, which, by itself, is not dominated by any of its $N!$ saddlepoints on the group manifold $U(N)$, is nevertheless controlled by a well defined classical trajectory in the space of eigenvalues. We obtained the representation of its asymptotics in terms of that trajectory and reduced the problem to a set of two algebraic constraints. It turned out that the analysis of induced QCD is the problem, dual to the computation of the Itzykson – Zuber integral: to solve induced QCD we have to find $\rho(x)$, given $U(x)$ in (4.6), while in order to compute the integral we must solve for $U(x)$, when $\rho(x)$ is known. We demonstrated how some particular solutions of this problem can be obtained.

Finally, it is necessary to make the following remark. Makeenko [6] claimed that induced QCD in $D$ dimensions with an even potential $U(x)$ has the same saddle point as the one-matrix model with the potential $\tilde{U}(x)$ related to $U(x)$ by

$$\tilde{U}'(x) = \frac{1}{2D-1} \left( (D-1)U'(x) + D\sqrt{(U'(x))^2 + 4(1-2D)x^2} \right). \quad (5.1)$$

However, it appears that there are some counterexamples to this statement. Let us, say, consider the $D = 1$ case. Choose

$$U(x) = \frac{m^2 x^2}{2} + \frac{gx^4}{4}.$$  

It is easy to find the solution of this model up to the first order in $g$. One can do it using (4.8) with $\xi = 0$ and $b = \frac{g}{m^2} \ll 1$. Then the eigenvalue density $\rho(x)$ reduces to

$$\rho(x) = \frac{1}{\pi} (cx^2 + d) \sqrt{a-x^2},$$

where

$$c = \frac{g}{2\mu m^2} (m^4 - 2) + O(g^2), \quad d = \frac{\mu}{2} + g \frac{m^2}{\mu^2} + O(g^2) \quad (5.2)$$

and $\mu = \sqrt{m^4 - 4}$. Then the constant $a$ is fixed by the requirement $\int \rho(x) dx = 1$. It can be checked that the same result follows independently from the perturbation theory for Migdal’s equation [3]. However,
the results following from (5.1) do not agree with that. Indeed, the corresponding one-matrix potential would be

\[ \tilde{U}(x) = \frac{\mu x^2}{2} + \frac{gm^2}{4\mu} x^4 + O(g^2). \]

This matrix model is easy to solve exactly [7]. Expanding the result in \( g \) one gets

\[ \rho(x) = \frac{1}{\pi} (\tilde{c} x^2 + \tilde{d}) \sqrt{\tilde{a} - x^2}, \]

with

\[ \tilde{c} = \frac{gm^2}{2\mu} + O(g^2), \quad \tilde{d} = \frac{\mu}{2} + g \frac{m^2}{\mu^2} + O(g^2). \]

We see that \( \tilde{c} \neq c \) already in the first order of perturbation theory, indicating problems with (5.1).

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