Automated Generation of Arbitrarily Many Kochen-Specker and Other Contextual Sets in Odd Dimensional Hilbert Spaces

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Development of quantum computation and communication, recently shown to be supported by contextuality, argues for a requisite supply of contextual sets. While that has been achieved in even dimensional spaces, only a dozen contextual critical Kochen-Specker (KS) sets have been found so far. In this paper we give three methods for automated generation of arbitrarily many contextual KS and non-KS sets in any dimension for possible future application and implementation and we employ them to obtain millions of KS and other contextual sets in dimensions 3, 5, 7, and 9 where previously only a handful of sets have been found. Also, no explicit vectors for the original Kochen-Specker set were known so far, while we now generate them from 24 vector components.

Keywords: quantum contextuality, MMP hypergraphs, Kochen-Specker sets, NBMMPH

Contextuality is paving the road for applications in quantum computation \cite{1,2}, quantum steering \cite{3}, and quantum communication \cite{4}, as proved by the methods of processing, preparing, and measuring of qudits (quantum dits). Qudits are the units of quantum information carrying a quantum system whose number of states (dimension) is an integer greater than one. Qubits (quantum bits) are two-dimensional qudits (d = 2) and their tensor products build the corresponding even-dimensional Hilbert spaces. The smallest contextual sets from these spaces have been implemented in a series of experiments, while in \cite{5–9} billions of contextual sets in 4-, 6-, 8-, 16-, and 32-dim Hilbert spaces, predominantly related to qubits, were generated by means of vector component algorithms (called method M3 below), particular symmetries, geometries, polytope correlations, parity filtering, Pauli operators, qubit states, and dimensional upscaling.

In contrast, far fewer contextual sets based on qudits in odd-dim spaces have been obtained so far. In particular, of Kochen-Specker contextual sets only four in the 3-dim space \cite{8,10}, four in the 5-dim space \cite{11,12}, five in the 7-dim space \cite{11,13,12}, supp. material, two in the 9-dim space \cite{12}, supp. material, and two in the 11-dim space \cite{12}, supp. material. General methods for automated generation of sets of a chosen structure did not exist.

Since the quantum communication and computation are supported by contextuality \cite{1,2}, the actual potential use of a large supply of contextual sets is twofold. First, quantum computation algorithms which would rely on contextual sets would arguably rely on a variety of such sets and on their automated generation. Second, structural properties of contextual sets differ according to their coordinatization, parities, dimensions, sizes, etc., and that can lead us to better understanding and applications of the sets.

In this paper, we offer universal and general algorithms for automated generation of arbitrarily many contextual sets in any dimension. In contrast to them, the programs we wrote to implement them are computationally demanding and therefore, here, we use them to generate sets that have not been generated so far: billions of KS and contextual non-KS sets in n = 3, 5, 7, 9-dim spaces. The programs themselves are freely available from our repository and technical details of their previous versions are given in \cite{8,9}.

To describe and handle contextual sets we make use of McKay-Megill-Pavičić-hypergraph (MMPH) language \cite{10,14}. An MMPH is a connected n-dim hypergraph k-l with k vertices and l hyperedges in which (i) every vertex belongs to at least one hyperedge; (ii) every hyperedge contains at least 2 and at most n vertices; (iii) no hyperedge shares only one vertex with another hyperedge; (iv) hyperedges may intersect each other in at most n−2 vertices. Graphically, vertices are represented as dots and hyperedges as (curved) lines passing through them.

We encode MMPHs by means of the following 90 ASCII characters: 1 2 ... 9 A B ... Z a b ... z ! " # $ % & ' ( ) * / : ; < = > ? @ [ \ ] ^ _ ` { | } ~ 10. When all 90 characters are exhausted, we reuse them prefixed by ‘+’ (91st character), then again by ‘++’, and so on. So encoded single ASCII characters (possibly prefixed by ‘+’) represent vertices; e.g., 1 or ++4A. Put together they represent hyperedges; e.g., 123 or 1++4+1Bd. To represent an MMPH, hyperedges are organized in a string in which they are separated by commas; each string ends with a period; e.g., the string 123,345,567,789,9A1. represents a noncontextual 3-dim MMPH pentagon.
limit on the size of an MMPH. An MMPH is a special kind of a general hypergraph in the sense that none of the aforementioned points (i-iv) holds for it.

Of course, instead of ASCII characters we could have used Unicode characters or 16-bit integers but 20 years ago we decided to proceed with the ASCII characters to encode MMPH strings and design our algorithms and programs which in turn yield all properties and features of MMPHs as well as their figures within the MMPH language. All our papers since 2000 [15] make use of the ASCII characters for the purpose.

The MMPHs above are defined without a coordinatization. The meaning of coordinatization is that a vector is assigned to each vertex and that all vectors assigned to vertices belonging to a common hyperedge are orthogonal to each other. Thus, in the MMPHs above, neither their vertices nor their hyperedges are related to either vectors or operators. We say that an MMPH is in an n-dim space, called MMPH space, when either all its hyperedges contain n vertices or when we might add vertices to hyperedges so that each contains n vertices. Orthogonality between vertices in an MMPH space just means that they are contained in common hyperedges. Our programs may handle MMPHs without any reference to either vectors or projectors. In an MMPH with a coordinatization, i.e. with vectors assigned to vertices, an n-dim MMPH space becomes an n-dim Hilbert space spanned by a maximal number of (possibly added) vectors within hyperedges. Whether we speak about an MMPH without or with a coordinatization will be clear from the context.

A non-binary MMPH (NBMMPH) is an n-dim \( (n \geq 3) \) k-l MMPH, whose i-th hyperedge contains \( \kappa(i) \) vertices \((2 \leq \kappa(i) \leq n, \ i = 1, \ldots, l)\), to which it is impossible to assign 1s and 0s in such a way that the following rules hold: (I) no two vertices in any hyperedge are both assigned the value 1 and (II) in any hyperedge, not all of the vertices are assigned the value 0. A binary MMPH (BMMPH) is an MMPH for which such an assignment is possible. NBMMPHs are nonclassical and contextual since they do not allow assignments of predefined 0s and 1s to their vertices. BMMPHs are classical and noncontextual since they do allow such an assignment. A KS MMPH is an NBMMPH with \( \kappa(i) = n \), \( \forall i \). The assignments of 0s and 1s do not require a coordinatization but an implementation of (N)BMMPHs does require it as well as their filled MMPHs, i.e. those in which to all hyperedges with \( \kappa(i) \leq n \) we add \( n - \kappa(i) \) vertices so as to satisfy the mutual orthogonalities. An example of a non-KS NBMMPH without a coordinatization is the 33-27 in Fig. 1(d) in the Supplemental Material (SM).

When either state-dependent or state-independent tests of operators defined on vertices of an NPMMPH with \( \kappa(i) \leq n \) confirm the contextuality, e.g. [14-18], then the NBMMPH turns out to be contextual in all considered cases so far.

A critical NBMMPH is an NBMMPH which after removing any of its hyperedges becomes a BMMPH.

To generate (N)BMMPHs in the odd-dim spaces we make use of three methods—M1-M3.

M1 consists in an automated dropping of vertices contained in single hyperedges (multiplicity \( m = 1 \)) of MMPHs and a possible subsequent stripping of their hyperedges. The obtained smaller MMPHs are often NBMMPH although never KS.

M2 consists in an automated random addition of hyperedges to MMPHs so as to obtain larger ones which then serve us to generate smaller KS MMPHs by stripping hyperedges randomly again.

M3 consists in combining simple vector components so as to exhaust all possible collections of \( n \) mutually orthogonal \( n \)-dim vectors. These form big master MMPHs which consist of single or multiple MMPHs of different sizes. Master MMPHs may or may not be NBMMPH, what we find out by applying filters to them. NBMMPHs serve us to massively generate a class of smaller MMPHs via our algorithms and programs.

We carry out methods M1-M3 and by means of our programs MPMPSSTRIP (for stripping and adding hyperedges), STATES01 (for verifying the contextuality), MPMSSHUFFLE (for reorganizing MMPHs), and ONE (for evaluating the structural properties of MMPHs [14]).

We combine all three methods to obtain a large number of NBMMPHs in odd-dim spaces.

3-dim case. So far there have been only four known KS MMPHs, and as we show in [8] none of their varieties with stripped \( m = 1 \) vertices is critical. Via M1, i.e. by stripping their edges, we can obtain thousands of smaller non-KS NBMMPHs and BMMPHs down to a pentagon [8]. But this means that they are limited in size by the size of four original MMPHs. To get larger MMPHs we have to apply M2,3. The final distribution of critical KS MMPHs we generated is given in Fig. 1.

![FIG. 1. The distribution of critical 3-dim KS MMPHs obtained via M2,3 with the given vector components. Abscissa is \( l \) (number of hyperedges); ordinate is \( k \) (number of vertices). Dots represent \( (k,l) \). Consecutive dots (same \( l \)) are shown as strips. The same applies to Figs. 2 and 3. See text.](image-url)

MMPH strings, figures, and vectors are given in SM.
M2 consists in adding hyperedges to Bub, Conway-Kochen, and Peres’ KS MMPHs (for citations and figures see [4]) using our program MMPStrip, filtering out KS MMPHs, and stripping them to the critical KS MMPHs by STATES01. The latter critical MMPHs build their coordinatization from the two vectors component sets of the original three MMPHs.

These vector components also serve us to obtain the same critical KS MMPHs by employing M3 so as to exhaust all possible collections of 3 mutually orthogonal vectors representing hyperedges interwoven in MMPHs.

When applying M3 we obtain that Bub’s is the only MMPH and that there are no smaller KS ones for the considered vector components. There are no other critical KS MMPHs between 49-36 and 51-37; between 51-37 and Peres’ 57-40 we obtained the following non-isomorphic MMPHs: one 53-38, eight 54-39, and one 55-40. Peres’ 57-40 is generated from the components \(\{0, \pm 1, \pm \sqrt{2}, 3\}\) and all the smaller ones from \(\{0, \pm 1, \pm 2, 5\}\). When we apply M2 so as to add sufficiently many hyperedges to any of the three original MMPHs (Bub, Conway-Kochen, or Peres’) and then strip them back down in a search for smaller critical MMPHs, we always obtain the other two among the generated critical KS MMPHs.

The more components we use with M3, the larger the master files and the more critical KS MMPHs we obtain. For instance with the help of \(\{0, \pm 1, \sqrt{2}, 3\}\) (Peres’) components we obtain the master 81-52 which contains just one single critical set—Peres’ 57-40; \(\{0, \pm 1, \pm 2, 5\}\) yield the master 97-64 which generates 20 critical KS MMPHs from 49-36 to 55-40; in contrast, \(\{0, \pm 1, \sqrt{2}, \pm 2, \pm 3\}\) yield the master 301-184 which generates 81 critical KS MMPHs from 49-36 to 92-66; \(\{0, \pm \omega, 2\omega, \pm \omega^2, 2\omega^2\}\), where \(\omega = e^{2\pi i/3}\), yield 514 criticals from 69-50 to 106-79, etc. Several smallest MMPHs from these classes are shown in Table I and SM.

We did not find simple real vector components which would yield KS MMPHs smaller than 49-36, although we are able to generate many smaller KS MMPHs down to 19-13 or 39-27 shown in SM, although without a coordinatization based on components shown in Fig. 1.

The path taken in [19] is intractable for hundreds of small KS MMPHs we checked on our supercomputer since the number of free variables is too high. See Fig. 1(a) in SM.

Another path was taken in 2021 by Jean-Pierre Merlet who applied the interval analysis method of solving nonlinear equations to the 19-13 MMPH. The equations turned out not to have a real solution and complex ones were not calculable even on a supercomputer.

As for possible coordinatizations of smaller KS MMPHs, we draw a parallel with the original KS set 192-118 [20]. Its trigonometric formula in [20] looks simple, but its coordinatization is so complicated that a random search for them is infeasible. More specifically, the aforementioned trigonometric formula is not sufficient to provide a coordinatization. We, therefore, constructed apparently the first known coordinatization of the original Kochen-Specker set from 24 components in SM.

5-dim case. In contrast to the 3-dim case, the 5-dim KS MMPHs can be obtained from just three vector components \(\{0, \pm 1\}\), which by method M3 generate the 105-136 master set and its 105-136 class of KS MMPHs. These include critical ones from the smallest 29-16 to the biggest 64-41, altogether 27,829,399 non-isomorphic MMPHs. The distribution of the MMPHs within the class is shown in Fig. 2(a) and the 29-16 in Fig. 2(b).

| Smallst Non.- Isom. | Smallest Vector Components | | |
|---|---|---|
| 5-5 | 1 | M1 | 40-23 | \{0, \pm 1, 2\} |
| 19-13 | 1 | M2 | 20-14 | none |
| 39-27 | 1 | M2 | 39-30 | ? |
| 49-36 | 1 | | | |
| 51-37 | 9 | M2-3 | 97-64 | \{0, \pm 1, \pm 2, 5\} |
| 53-38 | 1 | | | |
| 57-40 | 1 | M2-3 | 81-52 | \{0, \pm 1, \pm \sqrt{2}, \pm 3\} |
| 69-50 | 3 | M3 | 169-120* | \{0, \pm \omega, 2\omega, \pm \omega^2, 2\omega^2\} |
| 29-16 | 2 | M3 | 105-136 | \{0, \pm 1\} |
| 13-4 | 1 | M1,2 | 24-6* | ? |
| 28-14-1 | 1 | M2,3 | 805-9936 | \{0, \pm 1\} |
| 34-14 | 1 | M3 | 805-9936 | \{0, \pm 1\} |
| 207-97 | 1 | M3 | 805-9936 | \{0, \pm 1\} |
| 19-5 | 2 | M2 | 20-6* | ? |
| 47-16 | 2 | M3 | 9586-12068704 | \{0, \pm 1\} |
| 7-6 | 1 | M1,3 | 9586-12068704 | \{0, \pm 1\} |

TABLE I. Some of smaller NBMMPHs obtained by methods M1-3; ‘†’ indicates that the MMPH is a non-KS NBMMPH—all the others are KS NBMMPHs; ‘?’ indicates that obtaining the coordinatization is too demanding and that we were not able to carry it out on our supercomputers or that it might not exist at all.

We also obtained the master (1185-3596) and a number of elements from its class from the five components \(\{0, \pm 1, \sqrt{2}, 3\}\) and all the smaller ones from \(\{0, \pm 1, \pm 2, 5\}\). When we apply M2 so as to add sufficiently many hyperedges to any of the three original MMPHs (Bub, Conway-Kochen, or Peres’) and then strip them back down in a search for smaller critical MMPHs, we always obtain the other two among the generated critical KS MMPHs.
MMPHs as proved by the 34-28 MMPH (see the top of Fig. 3 a)) obtained in [11]. Particular targeted runs might give us particular smaller KS MMPHs, e.g. the critical 34-14 (Fig. 3 (a,e) and SM); it required stripping the master down to the MMPHs with 34 vertices via MMPStrip and then filtering them for the KS feature via STATES01; since such a procedure is too CPU-time demanding, a search for further smaller MMPHs is out of the scope of this paper. Instead, M1 can serve for a massive automated generation of smaller non-KS NBMMPHs with the help of MMPShuffled and MMPStrip. An outcome (28-14) is shown in Fig. 3(c), as obtained from the 207-97 Fig. 3(a,b). The procedure is analogous to stripping original 3-dim MMPHs [3].

Yet another way of automated generation of 7-dim MMPHs is via M2 by means of MMPStrip, an example of which is 13-6 shown in Fig. 3(d): we were not able to find its coordinatization and we conjecture that it does not have any. We confirmed that it is not determined by the vector components \(\{0, \pm1\}\) and we work on a program which would calculate coordinatization for bigger instances of such MMPHs we generated.

9-dim case. Two entangled qutrits live in a 9-dim space and we generated the MMPH master from \(\{0, \pm1\}\) components. It consists of 9,586 vertices and 12,068,705 hyperedges and that proved to be too huge for a direct generation of critical MMPHs (via stripping and filtering) from the master MMPH although the KS 47-16 given in SM proves that the master is a KS MMPH.

However, smaller critical KS MMPH can be obtained from simple BMMPHs via M2, in particular via an automated procedure of adding hyperedges and then generating critical KS MMPHs by stripping hyperedges via M1 and filtering them for the critical KS property. The critical KS MMPH 19-5 obtained in this way (Fig. 4(a)) has no coordinatization from \(\{0, \pm1\}\) and we conjecture that it does not have any, but it represents the proof of principle of how M2 works.

Also, a great many of MMPHs stripped of \(m = 1\) vertices exhibit contextuality. Smaller ones can easily be implemented. The smallest one we obtained by M3.1 is shown in Fig. 2(c) of SM and referred to in Table IV. Its filled MMPH is shown in Fig. 2(c) of SM. Their differences are discussed in SM.

To summarize, in this paper we give methods for generating KS as well as non-KS NBMMPHs in odd-dimensional Hilbert spaces. Our goal is not to find “record” smallest MMPHs but to establish general methods for automated generation of NBMMPHs in any di-
mension for any possible future application and implementation, e.g., in quantum computation and communication. The methods are especially needed in odd dimensional Hilbert spaces since, in contrast to even dimensional ones, we cannot make use of polytopes, Pauli operators, qubit states, parities, and other approaches specific to qubit spaces. We propose three such methods: \textbf{M1} which consists in dropping vertices contained in single hyperedges, \textbf{M2} consists in random addition of hyperedges to MMPHs, and \textbf{M3} which consists in combining simple components so as to exhaust all possible collections of mutually orthogonal vectors. Automated generation is achieved by means of our algorithms and programs presented above.

In the 3-dim space we generated roughly a million and a half non-isomorphic KS ones, ranging from MMPHs with 19 vertices and 13 hyperedges (19-13) without a coordinatization (via \textbf{M3}), over eleven 51-37s, up to a 232-172, all with coordinatizations, distributed as shown in Fig. 4. Special cases are given in Table III and SM. In SM we also give, for the first time, an explicit coordinatization of the original KS set.

In the 5-dim space, from the vector components \(a, b, c, d, e \in \mathbb{R}^5\), we generated roughly 28 million KS MMPHs whose distribution is shown in Fig. 5. We also explain how a combination of methods \textbf{M1-M3} can be employed to generate targeted smaller classes of non-KS NBMMMPHs. This approach is important because the operator and projector based contextual sets, which are recently being used in the literature, are often built by means of such NBMMMPHs.

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\textbf{SUPPLEMENTAL MATERIAL}

\textit{3-dim}

\textit{One of eight 51-37 MMPHs generated by \{0, \pm 1, \pm 2, 5\}, nonisomorphic to Conway-Kochen's MMPH}

\begin{align*}
51 &- 37: 213, 35b, bYb, bR, bYQ, bQ, cBD, DaF, FEG, GLH, IDP, PTU, UpK, KJL, LV8, 89A, AkI, 1cf, fde, eZA, XWn, nm2, 456, 783, MNL, STR, WDA, XXY, Zav, Z16, cG2, cVC, WT5, cVT, iJP, oSF, jba. \\
51 &- 37: 213, 35b, bYb, bR, bYQ, bQ, cBD, DaF, FEG, GLH, IDP, PTU, UpK, KJL, LV8, 89A, AkI, 1cf, fde, eZA, XWn, nm2, 456, 783, MNL, STR, WDA, XXY, Zav, Z16, cG2, cVC, WT5, cVT, iJP, oSF, jba. \\
51 &- 37: 213, 35b, bYb, bR, bYQ, bQ, cBD, DaF, FEG, GLH, IDP, PTU, UpK, KJL, LV8, 89A, AkI, 1cf, fde, eZA, XWn, nm2, 456, 783, MNL, STR, WDA, XXY, Zav, Z16, cG2, cVC, WT5, cVT, iJP, oSF, jba. \\
51 &- 37: 213, 35b, bYb, bR, bYQ, bQ, cBD, DaF, FEG, GLH, IDP, PTU, UpK, KJL, LV8, 89A, AkI, 1cf, fde, eZA, XWn, nm2, 456, 783, MNL, STR, WDA, XXY, Zav, Z16, cG2, cVC, WT5, cVT, iJP, oSF, jba.
\end{align*}

\textit{39-27 critical KS MMPH — possible coordinatization is an open problem}

\begin{align*}
39 &- 27: \text{1st 17 triads build a 17-gon: 837, 7NF, FEG, GLH, K4c, cbY, Y2S, SRT, TWX, X6H, HJI, IQV, Vd5, SUA, A9B, BCD, CZ8, 123, 456, MND, OPQ, UJ8, aVL, E62, dW3, P74, cW9.}
\end{align*}

\begin{align*}
\text{Other MMPHs generated by \{0, \pm 1, \pm 2, 5\}}
\end{align*}

\begin{align*}
53 &- 38: 213, 39A, AFG, GPb, BNX, XXY, YdK, KVF, fE5, 546, 678, 8ED, DIn, rqO, oLP, Pkl, lCH, HMa, aZb, bhJ, JSj, ji2, BC5, H12, JEC, KIG, L63, MNL, LKJ, QRS, TUV, cd8, ghA, mmP, op0, nFE, UND, RMP. \\
53 &- 38: 213, 39A, AFG, GPb, BNX, XXY, YdK, KVF, fE5, 546, 678, 8ED, DIn, rqO, oLP, Pkl, lCH, HMa, aZb, bhJ, JSj, ji2, BC5, H12, JEC, KIG, L63, MNL, LKJ, QRS, TUV, cd8, ghA, mmP, op0, nFE, UND, RMP.
\end{align*}
FIG. 5. (a) BMMPH obtained by Voráček and Navara [19]: the 18-33 MMPPH which one obtains by removing the vertices
o, r, u is a NBMMPPH; (b) KS MMPH which does not have a real coordinatization and most probably also not a complex one;
(c) One of nine 51-37 KS MMPHs we generated; it is a 22-gon non-isomorphic to Conway-Kochen’s 51-37; (d) KS NBMMPH
39-27 does not have a known coordinatization—non-KS NBMMPH 33-27 obtained by removal of all gray vertices from the
39-27 therefore also does not have a known coordinatization.

E=(1,0,0); F=(1,-1,0); G=(1,0,-1); H=(1,1,1); I=(0,0,1); J=(0,1,0); K=(0,1,1); L=(1,0,0); M=(0,1,1); N=(0,1,-1);
O=(0,1,2); P=(0,2,1); Q=(2,1,5); R=(2,1,1); S=(1,2,0); T=(2,5,1); U=(2,1,1); V=(1,0,2); W=(2,5,1); X=(1,2,1);
Y=(1,0,2); Z=(-1,2,5); a=(2,1,0); b=(1,2,0); c=(1,5,2); d=(2,0,1); e=(-1,5,2); f=(2,0,1); g=(-1,-2,5); h=(-2,1,0);
I=(1,-2,5); j=(2,1,0); k=(5,-1,2); l=(1,1,2); m=(5,1,2); n=(-1,1,1); o=(5,2,1); p=(1,-2,1); q=(5,-2,1); r=(1,2,-1).

One of the eight 54-39 MMPHs:

54-39 546, 6DE, EmW, WRV, VJ, JH, Ipq, qT, srG, FC, CAB, B38, 879, 9ZL, LMN, NOP, PbY, Yci, ihj, jdg, gF, fxa, a25, 123, KLJ, QRP, STN, XYG, bi3, ce9, ct2, dC6, dbZ, XV8, dUT, klR, nSB, oU5, laZ. 1=(-1,1,2); 2=(1,1,0); 3=(1,1,-1);
4=(1,1,2); 5=(1,1,2); 6=(0,2,1); 7=(1,2,1); 8=(1,0,1); 9=(1,1,1); A=(5,-2,1); B=(2,1,1); C=(0,1,1); D=(5,1,2);
E=(1,1,2); F=(5,2,1); G=(-1,2,1); H=(-2,5,1); I=(2,1,1); J=(0,1,2); K=(2,5,1); L=(-2,1,1); M=(2,1,5); N=(1,2,0);
O=(-2,1,5); P=(2,1,1); Q=(2,5,1); R=(1,0,2); S=(2,1,0); T=(0,0,1); U=(2,0,1); V=(0,1,0); W=(2,0,1); X=(1,0,1);
Y=(1,1,1); Z=(0,1,1); a=(-1,1,1); b=(0,1,1); c=(-1,0,1); d=(1,0,0); e=(5,2,1); f=(1,2,1); g=(0,1,2); h=(5,2,1);
i=(1,1,2); j=(0,2,1); k=(-2,5,1); l=(2,1,1); m=(-5,1,2); n=(-1,2,5); o=(1,5,2); p=(2,1,5); q=(1,2,0); r=(1,-2,5);
s=(2,1,0).

55-40 123, 456, 789, AB6, CD9, EF3, GHB, IJL, KFL, MN9, OP6, QR3, SNH, TRJ, UPL, Wv5, XV8, Za2, bCB, dCd, eCF, eAN, dYP, bWR, RN, fgS, hiU, jkT, LMH, NOJ, pqL, rM5, sQ8, tU2, mW, qaV, oZY, gDv, hEx, ibZ. 1=(5,-2,1); 2=(1,1,-2); 3=(0,2,1);
4=(2,5,1); 5=(2,1,1); 6=(1,0,2); 7=(1,-2,5); 8=(1,2,1); 9=(2,1,0); A=(5,-2,1); B=(2,1,1); C=(1,2,5); D=(1,2,1);
E=(5,1,2); F=(1,1,2); G=(2,-1,5); H=(1,2,0); I=(5,-1,2); J=(0,1,2); K=(1,5,2); L=(2,0,1); M=(1,2,0); N=(0,0,1);
O=(2,2,1); P=(0,1,0); Q=(0,1,-2); R=(2,1,0); S=(2,1,0); T=(0,2,1); U=(1,0,2); V=(1,1,1); W=(0,1,1); X=(1,1,1);
Y=(1,1,1); Z=(1,1,1); a=(-1,0,1); b=(0,1,1); c=(-1,1,1); d=(1,0,1); e=(1,1,0); f=(1,-2,5); g=(1,2,1); h=(2,5,1);
i=(-2,1,1); j=(4,2,1); k=(-1,1,2); l=(2,1,1); m=(5,2,1); n=(5,2,-1); o=(-1,2,1); p=(-1,-2,5); q=(1,1,2); r=(-2,1,5);
s=(5,-2,1); t=(-1,5,2).

FIG. 6. (a-c) Critical 3D MMPHs generated by the components \{0, \pm 1, \pm 2, 5\}: (a) the only 53-38; 22-gon; (b) one of the eight
54-39s; 23-gon; (c) the only 55-40; 22-gon; (d) the only 57-41—the smallest MMPH generated by \{0, \pm 1, 2, \pm 5, \pm \omega, 2\omega\}; 21-gon.
The smallest MMPH generated by \{0, \pm 1, 2, \pm 5, \pm \omega, 2 \omega\}, shown in Fig. 2(d).

57-41 213, 398, 876, 645, 5cd, dba, abG, GVU, TVU, UAR, RSO, GPR0, OHP, Pmn, nJD, D1F, FJe, Epi, ItY, YCF, fe2, ABC, DC6, EB3, FGH, IJK, LKS, MK2, NH8, KHA, XSG, YZG, g2L, hBM, ijL, kLM, oPp, qTO, stN, uvN, vaJ. 1 = (1,2,5); 2 = (2,-1,0); 3 = (1,-1,1); 4 = (1,2,-5); 5 = (2,1,0); 6 = (0,1,2); 7 = (5,2,-1); 8 = (0,1,2); 9 = (5,-2,1); A = (0,1,0); B = (1,0,1); C = (1,0,1); D = (1,1,1); E = (0,1,1); \( \Gamma \) = (1,0,1); F = (1,0,1); G = (0,1,1); H = (1,0,1); I = (1,-1,0); J = (-1,0,1); K = (0,0,1); L = (-1,0,1); M = (1,2,0); N = (0,2,1); P = (0,1,2); Q = (1,0,1); R = (1,0,1); S = (-1,0,1); T = (2,2,2); U = (2,2,2); V = (2,2,2); W = (2,2,2); X = (2,2,2); Y = (2,2,2); Z = (2,2,2); a = (1,1,1); b = (1,1,1); c = (1,1,1); d = (1,1,1); e = (1,1,1); f = (1,1,1); g = (1,1,1); h = (1,1,1); i = (1,1,1); j = (1,1,1); k = (1,1,1); l = (1,1,1); m = (1,1,1); n = (1,1,1); o = (1,1,1); p = (1,1,1); q = (1,1,1); r = (1,1,1); s = (1,1,1); t = (1,1,1).

Original KS set

The original 192-118 KS set found by Kochen and Specker [20] was not accompanied by an explicit coordinatization. Instead, they give the equation sin(\( \pi /10 \)) = \( 1 + \sqrt{5} \)/2, which is the golden ratio. When we compute for just the 117 black vertices on hypergraph \( \Gamma_2 \), the resulting 192 vectors have 73 different components. By using unnormalized vectors, i.e. rays, we reduced the number of different components to 24.

When normalized, the 192 vectors have 73 different components. By using unnormalized vectors, i.e. rays, we reduced the number of different components to 24.

\[
c_{15} = f(\ln(9/2 - 9/\sqrt{5})/\ln 2, 0, 0) = 3\sqrt{(5 - 2\sqrt{5})/10};
\]
\[
c_{16} = f(\ln(17/4 - 31/(4\sqrt{5})))/\ln 2, 0, 0) = 3\sqrt{(5 - 2\sqrt{5})/10}.
\]

FIG. 7. KS MMPH of the original Kochen-Specker 192-118 set presented according to [10, Fig. 6]; note that the figure in [10, Fig. 6] and the one-to-one correspondence to [20, \( \Gamma_2 \)] presented there is obtained exclusively from the ASCII string of the MMPH without any reference to its coordinatization; note also that one can use the string below to assign its vertices to the vertices of the figure by hand.

\[
c_{15} = f(\ln(9/2 - 9/\sqrt{5})/\ln 2, 0, 0) = 3\sqrt{(5 - 2\sqrt{5})/10};
\]
\[
c_{16} = f(\ln(17/4 - 31/(4\sqrt{5})))/\ln 2, 0, 0) = 3\sqrt{(5 - 2\sqrt{5})/10}.
\]
The components may be used to generate a master MMPH whose class contains the MMPH 192-118. The master has 2416 vertices and 1432 hyperedges.
J' + K + H (1,0,-1,0,1);
++S = (1,0,1,0,0,0,1,0);
++T = (1,0,0,1,0,0,0,0);  

\[ \text{S. D. Bartlett, M. Howard, J. Wallman, V. Veitech, and J. Emerson,} \]

\[ \text{Nature} 510, 351 (2014). \]
APPENDIX 1: PHYSICAL REVIEW LETTER EDITOR’S 1ST DECISION, 1ST REFEREE REPORT, AND MY RESPONSE TO IT

[Appendix 3 below contains Physical Review Letter Editor’s 2nd decision, 2nd referee report, and my response to it; Appendices 4 and 5 at the end contains Physical Review Letter Editor’s 3rd decision and my comment on it.]

Mladen Pavičić

Editor’s 1st decision

Date: Thu, 16 Jun 2022 14:00:15 +0000
Re: LP17301 Automated generation of arbitrarily many Kochen-Specker and other contextual sets in odd-dimensional Hilbert spaces by Mladen Pavičić and Norman D. Megill
Dear Prof. Pavicic,

This manuscript has been reviewed by our referee. A critique from the report appears below. Based on this we judge that the work probably warrants publication in some form, but does not meet the Physical Review Letters criteria of impact, innovation, and interest.

The paper, with revision as appropriate, might be suitable for publication in one of the topical Physical Review journals (e.g., Physical Review A) or Physical Review Research. The editors of that journal will make the decision on publication, and may seek further review. However, our complete file is available.

Yours sincerely,
Robert Garisto (he/him/his) Editor Physical Review Letters

P.S. Another referee we consulted was not able to review your manuscript.

Report of the 1st Referee – LP17301/Pavicic

*OVERALL EVALUATION*

This article aims to present new methods to generate large numbers of examples of Kochen-Specker sets in odd-dimensional Hilbert spaces. I found the manuscript lacking in a number of aspects, which I now highlight. (In the “Detailed Comments” section below, I go into more detail on some of these issues, illustrating them with concrete passages, among other minor comments.)
1. The manuscript does not provide enough motivation nor does it try to make a case for the relevance of this work and its appeal to a broad audience such as PRL’s. It claims that "development of quantum computation and communication, recently shown to be supported by contextuality, arguably asks for an abundant supply of contextual sets", but the authors do not bother to explain why such sets might be necessary. I’m not claiming that there isn’t a case to be made. There likely is, but the point is that the manuscript does this very weakly.

2. The presentation falls short of the standard expected in a research article, especially in a journal as good as PRL, in terms of clarity and rigor. Various passages are written in a confusing fashion. For example, it is often unclear from the text that a passage is to be taken as the definition of a new concept (e.g. when the notion of "coordinatization" is introduced). On the other hand, e.g. in the definition of MMPH hypergraph, some extra commentary is written as part of the definition when it does not belong there, namely item (v). Overall, the presentation is often imprecise (see detailed comments below for more examples).

3. The choice of material included in the text is odd, and it is inadequate for a journal article. At points, the authors choose to give too much information on irrelevant implementation details (e.g., the internal ASCII encoding of the graphs being manipulated, or the name of functions in their computer code). In contrast, and more worryingly, the text is quite vague in the description of the actual methods/algorithms being implemented (especially M3 which seems to be the less obvious of them). This is problematic for two reasons. First, it detracts from the interest that the manuscript may have to a journal’s audience: most readers would read the paper to learn the main ideas underlying the methods so they could replicate and adapt them (anyone keen on learning implementation details would refer to the code’s documentation and/or to the code itself). Moreover, this vagueness makes it hard for referees to judge the authors’ contributions in terms of correctness, novelty, etc.

For the above reasons, I am unfortunately not able to recommend this article for publication in PRL. I believe that the results may be of interest to a more specialized audience. I would therefore encourage the authors to submit a carefully *revised* version, with a more detailed presentation of the main novel technical contributions, to a journal such as PRA.

*DETAILED COMMENTS*

(some more important than others, in order of appearance in the manuscript)
- Abstract, line 1-2:
  "Development of quantum computation and communication, recently shown to be supported by contextuality, *arguably asks for an abundant supply of contextual sets*"

  Contextuality has been shown to underlie quantum advantage in various computation and communication tasks and protocols. However, the second part of the sentence does not follow directly from this. That is, why does the development of quantum computation and communication ask for an *abundant supply of contextual sets*? This may well be the case, but (1) it is not an obvious consequence of the results mentioned from the literature, and (2) the authors fail to argue for this. Given that this is presented as the main motivation for this work and for its broad appeal, I would expect to see a better case made for this.

- Page 1, col. 1, l. 7:
  "Qubits (quantum bits) are two-dimensional qudits (d = 2) and their tensor products build the corresponding even-dimensional Hilbert spaces."

  While the Hilbert space corresponding to an n-qubit system (i.e., an n-fold tensor of $C^2$) is even dimensional, not all even-dimensional Hilbert spaces are of this form. The Hilbert space of multiple qubits has dimension a power of two. One does not get, for example, a 6-dimensional Hilbert space in this fashion. Some of the discussion around even vs odd dimension in the article seems to misleadingly conflate even dimensionality with being realizable with qubits.

- Page 1, col. 1, l. 21:
  "and *several* in 5- to 11-dim spaces"

  This is a rather vague statement: what is meant by "several"?

- Page 1, col. 1, l. 22:
  "[...]of Kochen-Specker contextual sets only four [...] were explicitly provided for particular sets"

  It is unclear what the authors mean by "were explicitly provided for particular sets"?

- Page 1, col. 1, l. 38:
  "An MMPH is a connected hypergraph k-l with k vertices and l hyperedges in which [...] every hyperedge contains at least 2 and *at most n* vertices"

  The number "n" here appears out of nowhere. Only "k" and "l" are bound by the definition of "hypergraph k-l", but "n" means nothing in this context and therefore the requirement that every hyperedge contain at most n vertices seems to be an empty one.

- Page 1, col.2, l. 1:
  "(v) graphically, vertices are represented as dots and hyperedges as (curved) lines passing through them."
This is given as one of the conditions in the definition of MMPH graphs, but in reality it is not a condition on hypergraphs. It is just an explanation of the convention used to draw hypergraphs to ease the readability of Figure 2, for example. It should be separated from the definition of MMPH hypergraph.

- Page 1, col. 2, l. 4-10:

"We encode MMPHs by means of the following 90 ASCII characters: [list of ASCII characters] [...]"

This is the kind of implementation detail that is completely irrelevant to the potential readers of this paper. Its inclusion is a waste of space that could be used to convey much more interesting information about the methods implemented. This does not concern any substantial/significant algorithmic choice, but simply a matter of convention chosen for the internal representation. It’s the kind of detail that is typically hidden from the user even in the documentation of a well-written library, and it’s even less relevant in the context of a journal article, which ought to focus on explaining the methods instead.

- Page 1, col. 2, l. 14-16:

"An MMPH is a special kind of a general hypergraph in the sense that none of the aforementioned points (i-v) holds for it."

The meaning of this sentence is unclear. What is the referent of the pronoun "it" at the end of the sentence? (By the way, there is also a minor typo: "sens" should be spelt "sense").

- Page 1, col. 2, l. 23-25:

"Orthogonality between vertices in an MMPH space just means that the vertices are contained in their hyperedges".

This is yet another sentence whose meaning isn’t clear. What is meant by "the vertices are contained in *their* hyperedges"? In fact, this whole paragraph about "coordination" is written in a confusing fashion. It seems that the intention is for it to be a definition of MMPH with coordination, but the definition proper appears as seemingly an afterthought, in the clause starting with "i.e."

- Page 1, col. 2, l. 34 (and after):

The acronym "MMPH" is long and awkward enough as it is. The strings "NBMMPH" and "(N)BMMPH", and later even "KS NBMMPH", are beyond what is reasonable to expect a reader to parse. It would be much better to keep the full phrase "non-binary". I would even go further to suggest finding an alternative to MMPH. Also because two of the initials in this acronym refer to the authors of the present paper...

- Page 2, col. 1, l. 2-6:

"NBMMPHs are nonclassical since they do not allow assignments of predefined 0s and 1s to their vertices; therefore, we consider them to be contextual. BMMPHs are classical since they do allow such an assignment; therefore, we consider them to be noncontextual."

This sentence reads awkwardly. What is meant by "*we consider* them to be (non)contextual". Is this supposed to be a definition?

- Page 2, col. 1, l. 11-12:

"When conditional or unconditional contextual operators [...]"

The terms conditionally contextual, unconditionally contextual, and conditionally noncontextual are never defined or even explained informally.

- Page 2, col. 1, l. 23-43:

"To generate (N)BMMPHs in the odd-dim spaces we make use of three methods [...]"

These short paragraphs are what is included as a description of the methods used. The descriptions of each of M1 to M3 provided are extremely brief and lacking in detail. A more detailed exposition of the methods and some examples illustrating them, especially for M3, would be useful and make the paper more interesting to potential readers. Also, it would be good to see more details on how the code deals with / finds / manipulates coordinatizations (i.e., realizations with quantum states and measurements) for the graphs in question.

- Page 2, col. 1, l. 57 - col. 2, l. 1:

"using our program "MMPSstrip" [...] "stripping them to the critical KS MMPHs by "States01"

The name of the program or routine that performs a given task (such as "MMPSstrip" or "States01") is totally irrelevant here. On the other hand, it would be more interesting to convey what these programs are actually doing: how are the edges added? randomly?, how are they stripped to critical sets?, etc.

- Page 2, col. 2, Fig. 1:

This figure is hard to interpret. The caption does not adequately explain, e.g., what do the axes represent, or what are the shaded black stripes. The same applies to Fig 3 and Fig 4 later on.

- Page 2, col. 2, l. 13-14:

"When applying M3 we obtain that Bub’s is the only 49-36 NBMMPH and that there are no smaller KS ones for the considered vector components."

Again, this statement is unnecessarily vague. What is meant by "for the considered vector components"? Which vector components are being considered which allow the authors to make this statement? This goes back, I believe, to the overly vague explanation given of M3.
"The methods are especially needed in odd dimensional Hilbert spaces since, in contrast to even dimensional ones, we cannot make use of polytopes, Pauli operators, qubit states, parities, and other approaches specific to qubit spaces.

Are analogous methods not available for qudits? Is there a reason why this is the case?

**Author’s response to the Report of the 1st Referee – LP17301/Pavicic**

The referee is not at home with the theory of hypergraphs on which our article is based; none of her/his comments holds water. So, an acceptance of her/his report as the only ground for a rejection of the present article would likely be detrimental for the reputation of the *Physical Review Letters* in this field.

In the revised version of the article (available at https://arxiv.org/abs/2202.08197) I made some changes that clarify the points the referee misinterpreted, misunderstood, or twisted. In my response below, I indicate such changes by “******************”. Referee’s comments I indicate by “>”.

> **OVERALL EVALUATION**

> This article aims to present new methods to generate large numbers of examples of Kochen-Specker sets in odd-dimensional Hilbert spaces.

> Definitely not. In the summary we clearly stated our aim: “Our goal is . . . to establish general methods for automated generation of NBMMMPHs [of which the Kochen-Specker sets are the subsets] in any dimension for any possible future application and implementation, e.g., in quantum computation and communication.” In the present article we focus on the odd-dimensional spaces since we obtained analogous results for the even-dimensional spaces in the previous publications of ours. Also, in the even-dimensional spaces there are numerous other methods by which one can achieve automated generation of NBMMMPHs, while in the odd-dimensional spaces apparently only the ones presented in this article are available.

> 1. The manuscript does not provide enough motivation nor does it try to make a case for the relevance of this work and its appeal to a broad audience such as PRL’s. It claims that “development of quantum computation and communication, recently shown to be supported by contextuality, arguably asks for an abundant supply of contextual sets,” but the authors do not bother to explain why such sets might be necessary. I’m not claiming that there isn’t a case to be made. There likely is, but the point is that the manuscript does this very weakly.

> On the contrary, the motivation is clearly expressed in the second paragraph of the paper: “none of the methods [previously employed in the even-dimensional spaces] is available in the odd-dimensional spaces. In this paper, we offer several methods for automated generation of arbitrarily many KS sets as well as contextual non-KS ones in \( n = 3, 5, 7, 9 \) dimensional spaces.”

> The sentence: “quantum computation and communication . . . arguably asks for an abundant supply of contextual sets” is taken from the abstract which is limited in size. In the very body of the article, it is substantiated by the motivation sentences cited above.

> In the present version of the paper I expanded the Abstract so as to make the first and the last sentences read: ****

> ************** Development of quantum computation and communication, recently shown to be supported by contextuality, arguably asks for a requisite supply of contextual sets. **************

> In this paper, we offer three methods for automated generation of arbitrarily many contextual KS and non-KS sets in any dimension for possible future application and implementation, e.g., in quantum computation and communication, which can be applied in both even and odd dimensional spaces and we employ them to obtain millions of KS and other contextual sets in dimensions 3, 5, 7, and 9. **************

> 2. The presentation falls short of the standard expected in a research article, especially in a journal as good as PRL, in terms of clarity and rigor. Various passages are written in a confusing fashion.

> After so many articles we published in *Phys. Rev. Lett.*, *Scientific Reports*, *Phys. Rev. A & D, Optics Express, J. Math. Phys.*, *J. Phys. A, J. Opt. Soc. Am. B, Entropy, Phys. Lett. A*, *Found. Phys.*, *Ann. H. Poincare* . . . over almost half a century, enters a referee to announce that we do not know how to write an article. The article is written up to the standards of the PRL and nothing is written in a confusing fashion, only the referee does not grasp the meaning of the presented material since she/he is not at home with the theory of hypergraphs.

> For example, it is often unclear from the text that a passage is to be taken as the definition of a new concept (e.g. when the notion of “coordinatization” is introduced).

> Wiktionary has a single item for: coordinatization (plural coordinatizations): (mathematics) “The application of coordinates to a system,” and in the 3rd paragraph of the right column on p. 1 we additionally clarify: “The MMPHs above are defined without a coordinatization, i.e., neither their vertices nor their hyperedges are related to either
vectors or operators." So, no “new concept” is introduced. The “coordinatization” is a word from a dictionary the referee should have consulted.

> On the other hand, e.g. in the definition of MMPH hypergraph, some extra commentary is written as a part of
> the definition when it does not belong there, namely item (v).

The referee is obviously not at home with the theory of hypergraphs. E.g., Berge, *Hypergraphs: Combinatorics of
Finite Sets*, North-Holland (1989); Ch. 1, Sec. 1, p. 1, 4th paragraph: “A hypergraph may be drawn as a set of points
representing the vertices [and a hyper]edge is represented . . . by a single closed curve enclosing the vertices”—as a
part of his definition. There is no particular kind of a “definition” of ours which would differ from Berge’s kind except
that we introduce, in point (v), a curve passing through vertices instead of Berge’s curve which encloses them.

> Overall, the presentation is often imprecise (see detailed comments below for more examples).

The referee fails to substantiate her/his claim in any of her/his comments.

> 3. The choice of material included in the text is odd, and it is inadequate for a journal article. At points, the
> authors choose to give too much information on irrelevant implementation details (e.g., the internal ASCII
> encoding of the graphs being manipulated, or the name of functions in their computer code).

Again, the referee clearly shows that she/he is not at home with the theory of hypergraphs. Berge, Ch. 1, Sec. 1,
p.1, 5th paragraph: “One may also define a hypergraph by its incidence matrix, with columns representing the
[hyperf]edges and rows representing the vertices (cf. Figure 1, p. 2).” We, instead of an incidence matrix, make use of
MMPH ASCII strings. Some kind of codification is unavoidable and our MMPH string codification is an integral part
of the MMPH language and these strings are generated by our algorithms and programs and they then serve us as
further direct inputs to our algorithms and programs whose outputs are all MMPH properties and features including
graphical representations of the obtained MMPHs. The difference between incidence matrices and MMPHs is obvious;
for instance, the MMPH string of Fig. 2(c) can be written in a single line, while the incidence matrix takes 37 columns
and 51 lines. Also an MMPH string can be put in any of our programs while no such programs for arbitrary incidence
matrices are known to us. And then enters the referee to proclaim the MMPH codification “irrelevant.”

***** In the present version of the paper I add the following sentences to the right column, p. 1: *****

************** We stress here that MMPH strings are generated by our algorithms and programs and then
processed to yield all properties and features of MMPHs, including their graphical representations.**************

> In contrast, and more worryingly, the text is quite vague in the description of the actual methods/algorithms
> being implemented (especially M3 which seems to be the less obvious of them).

I'm flabbergasted. What can be simpler than, p. 2: “M3 consists in combining simple vector components so as to
exhaust all possible collections of $n$ mutually orthogonal $n$-dim vectors.” For instance 3-dim components $(0, 1, 0)$ yield:

$123.1=(0, 0, 1), 2=(0, 1, 0), 3=(1, 0, 0); 4$-dim $(0, ±1)$ yield Peres' set 24-24; etc.

> This is problematic for two reasons. First, it detracts from the interest that the manuscript may have to a
> journal’s audience: most readers would read the paper to learn the main ideas underlying the methods so they
> could replicate and adapt them (anyone keen on learning implementation details would refer to the code's
> documentation and/or to the code itself). Moreover, this vagueness makes it hard for referees to judge the
> authors' contributions in terms of correctness, novelty, etc.

What is problematic here are the motives for such twists on the part of the referee. The first sentence seems to
contradict itself: “most readers would read the paper to learn the main ideas underlying the methods so they could
replicate and adapt them,” but they would not like “to learn the code itself.” Note that without the MMPH language
the results obtained in the article are inaccessible as they were in the last half of century.

In the second sentence, “this vagueness”—read MMPH language—makes it hard for [malevolent] referees to judge
the authors’ contributions in terms of correctness, novelty, etc. As if it is not transparent that the correctness and
novelty of all results can be verified directly at the results presented in the article and its appendices or at our
repository [http://puh.srce.hr/s/Qegixzz2BdjYwFL](http://puh.srce.hr/s/Qegixzz2BdjYwFL) where all algorithms and programs are freely available.

> For the above reasons, I am unfortunately not able to recommend this article for publication in PRL.

> I believe that the results may be of interest to a more specialized audience.

The researchers were striving to obtain these results for at least quarter of century and it is regretful that the
editor bought such a recommendation.

> I would therefore encourage the authors to submit a carefully *revised* version, with a more detailed
presentation of the main novel technical contributions, to a journal such as PRA.

Our algorithms do require sophisticated programs but that does not make the contributions “technical,” because there is no other way to obtain them.

*DETAILED COMMENTS*

– Abstract, line 1-2:

”Development of quantum computation and communication, recently shown to be supported by contextuality, arguably asks for an abundant supply of contextual sets” Contextuality has been shown to underlie quantum advantage in various computation and communication tasks and protocols. However, the second part of the sentence does not follow directly from this. That is, why does the development of quantum computation and communication ask for an abundant supply of contextual sets? This may well be the case, but (1) it is not an obvious consequence of the results mentioned from the literature, and (2) the authors fail to argue for this. Given that this is presented as the main motivation for this work and for its broad appeal, I would expect to see a better case made for this.

Common scientific sense tells us that when “what gives quantum computers that extra oomph over their classical digital counterparts . . . is an intrinsic, measurable aspect of quantum mechanics called contextuality” [1,2] we should have more than a few sets per dimension. Those recent result is so well-known that we considered it far outside the scope of the present paper to elaborate on it or to even add references.

[*DETAILED COMMENTS*

“Qubits (quantum bits) are two-dimensional qudits \(d = 2\) and their tensor products build the corresponding even-dimensional Hilbert spaces.” While the Hilbert space corresponding to an \(n\)-qubit system (i.e., an \(n\)-fold tensor of \(\mathbb{C}^2\)) is even dimensional, not all even-dimensional Hilbert spaces are of this form. The Hilbert space of multiple qubits has dimension a power of two. One does not get, for example, a 6-dimensional Hilbert space in this fashion. Some of the discussion around even vs odd dimension in the article seems to misleadingly conflate even dimensionality with being realizable with qubits.

This is ridiculous. We first write: “[Qubit] tensor product build the corresponding even-dimensional Hilbert spaces.” Every undergraduate student understands that that means: \(2^n\), \(n = 2, 3, 4, \ldots\). Then we write: “in [5–9] billions of contextual sets in 4-, 6-, 8-, 16-, and 32-dim Hilbert spaces, predominantly related to qubits, were generated.” And again every undergraduate student would immediately recognize that “predominantly” refers to 4, 8, 16, and 32, but not to 6. It seems to be a failed weaponized pedantry?

– Page 1, col. 1, l. 21:

“and *several* in 5- to 11-dim spaces” This is a rather vague statement: what is meant by “several”?

Again, ridiculous. By *Webster Dictionary* “several” means “more than two but fewer than many.”

– Page 1, col. 1, l. 22:

“[…] of Kochen-Specker contextual sets only four […] were explicitly provided for particular sets”

It is unclear what the authors mean by “were explicitly provided for particular sets?”

Is the referee serious? The full sentence reads: “of Kochen-Specker contextual sets only four in the 3-dim space [8,10] and several in 5- to 11-dim spaces [11–13] were explicitly provided for particular sets,” meaning that in those references sets with explicitly given vectors were provided. I gave the sentence to a group of students and no one had any problem with understanding the meaning of “explicitly,” so I presume the reader would not have it either.

– Page 1, col. 1, l. 38:

“An MMPH is a connected hypergraph k-l with k vertices and l hyperedges in which […] every hyperedge contains at least 2 and *at most n* vertices.” The number \(n\) here appears out of nowhere. Only “k” and “l” are bound by the definition of ”hypergraph k-l”, but ”n” means nothing in this context and therefore the requirement that every hyperedge contain at most n vertices seems to be an empty one.

The referee again shows that she/he is not at home with the theory of hypergraphs. The \(n\) does not appear out of nowhere, but out of the one but preceding sentence: “In this paper, we offer several methods for automated generation of arbitrarily many KS sets as well as contextual non-KS ones in \(n = 3, 5, 7, 9\) dimensiona spaces.”

**DETAILED COMMENTS**

An MMPH is a connected \(n\)-dim hypergraph \(k-l\) with \(k\) vertices and \(l\) hyperedges . . .

– Page 1, col.2, l. 1:

“(v) graphically, vertices are represented as dots and hyperedges as (curved) lines passing through them.”
This is given as one of the conditions in the definition of MMPH graphs, but in reality it is not a condition on hypergraphs. It is just an explanation of the convention used to draw hypergraphs to ease the readability of Figure 2, for example. It should be separated from the definition of MMPH hypergraph.

Nowhere do we speak about a "condition." It is just the point within a definition in the same way in which similar points are used in many hypergraph textbooks. We already clarified that point (v) above when we compare it with a similar point used by Berge, Ch. 1, Sec. 1, p. 1, 4th paragraph.

"We encode MMPHs by means of the following 90 ASCII characters: [list of ASCII characters] [...]" This is the kind of implementation detail that is completely irrelevant to the potential readers of this paper. Its inclusion is a waste of space that could be used to convey much more interesting information about the methods implemented. This does not concern any substantial/significant algorithmic choice, but simply a matter of convention chosen for the internal representation. It’s the kind of detail that is typically hidden from the user even in the documentation of a well-written library, and it’s even less relevant in the context of a journal article, which ought to focus on explaining the methods instead.

Once again, the referee clearly shows that she/he is not at home with the theory of hypergraphs at all. “ASCII completely irrelevant,” “waste of space,” “matter of convention,” “a journal article ought to focus on explaining the methods instead”. . . “There are no methods which could generate the contextual sets in odd dimensional spaces outside the MMPH language based on ASCII codification either in our minds or anywhere in the literature. MMPH language IS the method. A “method instead” does not exist. As the title of the paper stresses, the method consists in automated computer-program-enabled generation of such contextual sets based on the MMPH language.

"An MMPH is a special kind of a general hypergraph in the sense that none of the aforementioned points (i-v) holds for it.” The meaning of this sentence is unclear. What is the referent of the pronoun "it" at the end of the sentence? "it" = a general hypergraph. It is clear because points (i-v) obviously hold for MMPH.

"Orthogonality between vertices in an MMPH space just means that the vertices are contained in their hyperedges”. This is yet another sentence whose meaning isn’t clear. What is meant by "the vertices are contained in *their* hyperedges"?

For anyone who reads the paper so as to find out how MMPHs and the orthogonality within them are structured it is clear what is meant by *their*. Explanation is crystal clear: Let ABC be a hyperedge; we can say, for vertices A,B,C, the hyperedge is *their* hyperedge; so, the vertices A,B,C are mutually orthogonal just by being contained in *their* hyperedge. I’m convinced the reader will have much greater imagination and insight than the present referee.

Confusion is seemingly with the referee. Our paragraph clearly states that MMPHs are defined without any specification what their vertices might be. When vectors, states, projectors, or operators are assigned to vertices, then we speak about MMPHs with coordinatization. The former MMPHs with bare unspecified vertices are consequently MMPHs without coordinatization.

The acronym "MMPH" is long and awkward enough as it is. The strings "NBMMPH" and "(N)BMMPH", and later even "KS NBMMPH", are beyond what is reasonable to expect a reader to parse. It would be much better to keep the full phrase "non-binary".

The term “non-binary MMPH” introduced and abbreviated to NBMMPH on p. 2, is specific for this short article and the reader will want to go back to its definition a couple of times while reading it. So, “non-binary MMPH” vs. NBMMPH would not make a difference in this respect; the full phrase would only make the article longer.

I would even go further to suggest finding an alternative to MMPH. Also because two of the initials in this acronym refer to the authors of the present paper...

Is the referee serious? The acronym appears in more than 20 papers since 2005 and no one of the former referees has ever brought it into question. Besides, one of the authors (N.M.) is unfortunately not alive any more.
"we consider* them to be (non)contextual". Is this supposed to be a definition?

What is that about? We simply state that we *consider* them in the same way as they are *considered* to be of such a kind in the literature over and over. For instance, “In noncontextual hidden-variable theories the predetermined results of an observable are independent of any other observables that are measured jointly with it” (C. Brukner and M. Zukowski, Bell’s Inequalities: Foundations and Quantum Communication, in Handbook of Natural Computing, Eds: G. Rozenberg, T.H.W. Baek, and J.N. Kok, Springer (2010), pp. 1413-1450). Contextual sets as ones that do not admit predetermined values and noncontextual as ones that do go back to the original Kochen and Specker paper [20].

When conditional or unconditional contextual operators [...] The terms conditionally contextual, unconditionally contextual, and conditionally noncontextual are never defined or even explained informally.

Yes, the terms might have required some comments.

Instead, in the present version of the paper I substituted the following sentence for the sentences containing the disputed terms (3rd paragraph of the left column, on p. 2):

When either state-dependent or state-independent tests of operators defined on vertices of an NBMMPH with \(\kappa(i) \leq n\) confirm the contextuality, e.g. [13, 18], then the NBMMPH turns out to be contextual in all considered cases so far.

State-independent contextuality (SIC) is a well-known term which does not require any further explanation.

To generate (N)BMMPHs in the odd-dim spaces we make use of three methods [...] These short paragraphs are what is included as a description of the methods used. The descriptions of each of M1 to M3 provided are extremely brief and lacking in detail.

They are brief but not lacking in detail. M1: dropping of \(m = 1\) vertices means automated dropping of \(m = 1\); there is nothing more to it. M2: automated random addition of hyperedges means automated random addition of hyperedges; there is nothing more to it; M3: combining simple vector components so as to exhaust all possible collections of \(n\) mutually orthogonal \(n\)-dim vectors is just that; e.g., \((0,1)\) components in a 3-dim space yield vectors: \(A=(0,0,1), B=(0,1,0), C=(1,0,0)\) which form the hyperedge ABC; there is nothing more to it. The referee apparently cannot believe that everything turns out to be so simple.

A more detailed exposition of the methods and some examples illustrating them, especially for M3, would be useful and make the paper more interesting to potential readers. Also, it would be good to see more details on how the code deals with / finds / manipulates coordinatizations (i.e., realizations with quantum states and measurements) for the graphs in question.

On the contrary. As I just stressed, what is useful and interesting to the reader is to see how simple and straightforward the algorithms are. Detailed examples of the three methods in all considered dimensions are given in the body of the article.

As for the realizations/implementations, in the present version of the paper I add the following sentences at the end of the 2nd paragraph of the left column, on p. 2:

The most direct implementation of an MMPH can be carried out by measuring its vectors/states coming out of the gates determined by its hyperedges. Operator based implementation, e.g. the one in which projectors determined by states/vertices are projected on many other chosen states, is more demanding.

Again and again. The MMPH language and its algorithms and programs is a sine qua non for generating the MMPHs. How? Randomly? On p. 2 the referee can read: “M2 consists in an automated random addition of hyperedges...” How are they stripped to critical sets? Or how do we know that? Because by removing any of the remaining hyperedges it stops being an NBMMPH (contextual) and turns into a BMMPH (noncontextual).
> what are the shaded black stripes. The same applies to Fig 3 and Fig 4 later on.

The referee is bold again. Abscissa = \( l \) (number of hyperedges); ordinate = \( k \) (number of vertices). Dots represent \((k, l)\). Consecutive dots (same \( l \)) are shown as strips. Analogous figures have been presented in more than 20 papers of ours and all referees or those who cited or commented on our papers or to whom the papers were presented to conferences have obviously immediately grasped what the axes represent since no one has ever posed such a question.

> – Page 2, col. 2, l. 13-14:
> "When applying M3 we obtain that Bub's is the only 49-36 NBIMPH and that there are no smaller KS ones.
> > for the considered vector components." Again, this statement is unnecessarily vague. What is meant by "for the
> > considered vector components"? Which vector components are being considered which allow the authors to make
> > this statement?

Another failed weaponized pedantry. Vector components are shown in Fig. 1 \((0, \pm 1, \ldots, 2\omega^2)\).

> This goes back, I believe, to the overly vague explanation given of M3.

Such comments, I'm sure, do not go anywhere.

> – Page 5, col.1, l. 26-30
> "The methods are especially needed in odd dimensional Hilbert spaces since, in contrast to even dimensional
> ones, we cannot make use of polytopes, Pauli operators, qubit states, parities, and other approaches specific
> to qubit spaces." Are analogous methods not available for qudits? Is there a reason why this is the case?

It is so good that this is the last referee's comment. We cannot make use of them because they are "specific to qubit spaces" for whatever reason. We do not want to speculate about reasons. It is outside the scope of the article. It suffices that there no such methods in the literature and this is transparent from the fact that so far there were only a dozen of MMPHs found in the odd-dimensional spaces.

APPENDIX 2: PHYSICAL REVIEW LETTER EDITOR’S 2ND DECISION, 2ND REFEREE REPORT, AND MY RESPONSE TO IT

Mladen Pavičić

Editor’s 2nd decision

Journal: Physical Review Letters; Paper: LP17301; Received: 17Feb2022
16Sep22 Editorial decision
29Jul22 01Sep22 Review request to referee; report received
Date: 14 Sep 2022
Re: LP17301 Automated generation of arbitrarily many Kochen-Specker and other contextual sets in odd dimensional Hilbert spaces by Mladen Pavičić and Norman D. Megill

Dear Prof. Pavicic,

This manuscript has been reviewed by our [2nd] referee. A critique from the report appears below. Based on this we judge that the work probably warrants publication in some form, but does not meet the Physical Review Letters criteria of impact, innovation, and interest. In accordance with our standard practice, this concludes our review of your manuscript.

The paper, with revision as appropriate, might be suitable for publication in one of the topical Physical Review journals or Physical Review Research. The editors of that journal will make the decision on publication, and may seek further review.

Yours sincerely,
Robert Garisto (he/him/his) Editor Physical Review Letters

Report of the 2nd Referee – LP17301/Pavicic

The work “Automated generation of arbitrarily many Kochen-Specker and other contextual sets in odd-dimensional Hilbert spaces” by Pavicic and Megill introduces numerical methods to obtain Kochen-Specker (KS) sets (or in general contextual sets) in odd dimensions. Their methods yield a huge number of KS sets in dimensions 3, 5, 7, and 9 that are not known before. KS set is a set of projectors that cannot be assigned pre-determined values independent of the context of the observables in which the projector belongs. KS sets are genuine signatures of nonclassicality and have found applications in quantum computation and communication. There are, in general, two ways that contextual sets are associated with quantum information. Contextuality is shown to be necessary for quantum advantages over classical systems in certain aspects, that is, an advantage by quantum systems in certain aspects of
In contrast to the 1st referee, the present referee is at home with both the hypergraph theory and contextual sets. Her/his understanding and evaluation of the results achieved in the paper are accurate—up to the last point below—which is not.

> Before I present my opinion regarding the suitability of this work for publishing in PRL, I would like to pose a few queries and suggestions.

1. KS MMPH is defined as the NBMMPH with \( k(i) = n \) for all \( i \). If an NBMMPH does not have coordinatization then the graph has no quantum realization. However, one may get confused with the terminology of ‘KS MMPH’ since the “KS set” is usually referred to a set that already has a quantum realization. I would recommend adding one line in the text to clarify this. Are there examples of non-KS NBMMPH without known coordinatization?

2. Do non-KS NBMMPHs always give rise to proof of contextuality? In other words, given any non-KS NBMMPH, is it always possible to propose a noncontextuality inequality that quantum theory violates?

3. I would request the authors to mention some details about the software or/and tools used to execute the numerics so that it would be convenient to study this technique in the future.

4. A huge number of critical KS sets are found, and it would be nice if some sort of comparison is made among those critical sets. For instance, one can consider any measure of robustness for contextuality and study which set is more robust, or, one can see which set has the highest ratio between the chromatic number of the graph and \( n \) (the dim of the Hilbert space). This is just a suggestion.

Overall I find this work interesting. But, there is a substantial limitation of this work. The proposed numerical methods are computationally demanding for higher dimensions, and the KS sets are found only in a few small dimensions. Thus the proposed methods for automated generation of contextual sets are applicable to a certain extent. This is the reason I think the paper is suitable for specialized audiences, and I am unable to give a reason why the manuscript deserves publication in PRL. Given the queries mentioned above are appropriately addressed, this work can be published in a specialized journal.
then the graph has no quantum realization. However, one may get confused with the terminology of ‘KS MMPH’
since the “KS set” is usually referred to a set that already has a quantum realization. I would recommend adding
one line in the text to clarify this. Are there examples of non-KS NBMMPH without known coordinatization?

**** In the 2nd paragraph of the left column on p. 2, the following sentences are added: ****

************** The assignments of 0s and 1s do not require a coordinatization ... **************

************** An example of a non-KS NBMMPH without known coordinatization is the 33-27 in Fig. 1(d) in
the Supplemental Material (SM). **************

> 2. Do non-KS NBMMPHs always give rise to proof of contextuality? In other words, given any non-KS
> NBMMPH, is it always possible to propose a noncontextuality inequality that quantum theory violates?

**** Yes. See Table IX in [14]. ****

> 3. I would request the authors to mention some details about the software or/and tools used to execute the
> numerics so that it would be convenient to study this technique in the future.

**** What is of the primary interest here are the algorithms. They are universal and they hold in any dimension.
I added the following paragraph to the left column on p. 1: ****

************** In this paper, we offer universal and general algorithms for automated generation of arbitrarily
many contextual sets in any dimension. In contrast to them, the programs we wrote to implement them are computa-
tionally demanding and therefore, here, we use them to generate sets that have not been generated so far: billions
of KS and contextual non-KS sets in $n = 3, 5, 7, 9$-dim spaces. The programs themselves are freely available from our
repository and technical details of their previous versions are given in [5-9]. **************

> 4. A huge number of critical KS sets are found, and it would be nice if some sort of comparison is made among
> those critical sets. For instance, one can consider any measure of robustness for contextuality and study which
> set is more robust, or, one can see which set has the highest ratio between the chromatic number of the graph
> and $n$ (the dim of the Hilbert space). This is just a suggestion.

**** I have done this in [14]. ****

> Overall I find this work interesting. But, there is a substantial limitation of this work. The proposed numerical
> methods are computationally demanding for higher dimensions, and the KS sets are found only in a few small
> dimensions. Thus the proposed methods for automated generation of contextual sets are applicable to a certain
> extent. This is the reason I think the paper is suitable for specialized audiences, and I am unable to give a reason
> why the manuscript deserves publication in PRL. Given the queries mentioned above are appropriately addressed,
> this work can be published in a specialized journal.

**** The referee misunderstood the main accomplishment of the paper. Our algorithm and method are applicable
to any dimension and are therefore universal and general—they are not “computationally demanding”—our programs
based on them are. The present computational barrier which prevents us from providing outputs in higher dimensions
is not their “limitation” in any way that might diminish their importance for the PRL readership. Following our
algorithms one can obtain outputs in higher dimensions by making use of more powerful supercomputers or new
linear programs one might invent in the future. Also, a quantum computer would allow for outputs in any dimension.
Is the discovery of the DNA structure unimportant just because we still cannot calculate how to manipulate all of its
sequences? ****

**** The algorithm we propose is the only universal method for generating arbitrary contextual sets known in the
literature and that, in my opinion, warrants its publication in PRL. I hope the referee will concur.****
APPENDIX 3: PHYSICAL REVIEW LETTER EDITOR’S 3RD DECISION

Mladen Pavičić

Acknowledgement LP17301 Pavicic

Date: Mon, 19 Sep 2022 11:49:44 +0000
From: prltex@aps.org
Subject: Acknowledgement LP17301 Pavicic
Re: LP17301 Automated generation of arbitrarily many Kochen-Specker and other contextual sets in odd dimensional Hilbert spaces by Mladen Pavičić and Norman D. Megill
Dear Prof. Pavičić,
We have successfully generated output of the above manuscript that you sent for resubmission. The manuscript has been forwarded for further processing.
Yours sincerely,
Editorial Systems Physical Review Letters

Editor’s 3rd decision

Date: Fri, 30 Sep 2022 14:53:08 +0000
Subject: Your manuscript LP17301 Pavicic
Re: LP17301
Dear Prof. Pavicic,
This is in response to your resubmission.
Reopening review at this stage would be an extraordinary step, and we see insufficient cause for that.
Thus, we turn the matter back to you. We suggest that you submit to a more specialized journal, perhaps Physical Review A.
Yours sincerely,
Robert Garisto (he/him/his) Editor Physical Review Letters

APPENDIX 4: COMMENT ON PHYSICAL REVIEW LETTER EDITOR’S 3RD DECISION

Mladen Pavičić

Date: Sat, 1 Oct 2022 23:25:42 +0200
Subject: To the exclusive attention of Dr. Garisto - LP17301 Pavicic

Dear Dr. Garisto,
On Fri, Sep 30, 2022 you wrote:
> Reopening review at this stage would be an extraordinary step, and we see insufficient cause for that.

What is extraordinary here is that I have received no comments on my responses and changes from either of the referees.

With the best regards,
Mladen Pavicic.

APPENDIX 5: QUANTUM JOURNAL EDITOR’S DECISION AND MY RESPONSE TO IT

Publication Decision from Quantum

Automated Generation of Arbitrarily Many Kochen-Specker and Other Contextual Sets in Odd Dimensional Hilbert Spaces

Decision made on October 20th, 2022 Editorial board’s determination: Reject
Dear Mladen Pavicic,

Thank you for submitting to Quantum. Unfortunately, we have to inform you that the editorial board of Quantum have decided not to publish your paper “Automated Generation of Arbitrarily Many Kochen-Specker and Other Contextual Sets in Odd Dimensional Hilbert Spaces”. This decision is due to the following reason:

We have noticed that you have made public, in [arXiv:2202.08197](https://arxiv.org/abs/2202.08197), the referee reports which you received for this manuscript. The reports were anonymous and formed a confidential communication between author and editor. By making them public, including your denigrating comments on the scientific expertise of one of the referees, you have breached this confidentiality. Moreover, by this conduct you have put an undue pressure on the refereeing process at Quantum, since potential referees can expect a similar action in response to their own reports, if these happen to be critical as well. (This case of “pressure” by the authors on editors or referees is explicitly mentioned in our editorial policies, as a ground for desk rejection.)

For this reason we are not able to consider your paper for publication in Quantum.

We hope that this quick response will allow you to submit to another journal, and we look forward to receiving your future work. Thank you for supporting our efforts towards open, community-led and fair science publishing.

Best regards,

Lukas Schalleck (Admin) on behalf of the Editorial Board of Quantum

This rejection letter is final and this discussion is now closed. Under certain conditions authors can appeal against editorial decisions.

My response to the Decision from Quantum

Mladen Pavicic Oct 25, 2022 - 12:12 pm CEST

Dear Mr. Schalleck,

Lukas Schalleck (Admin) replied to a discussion on the manuscript:

Automated Generation of Arbitrarily Many Kochen-Specker and Other Contextual Sets in Odd Dimensional Hilbert Spaces. (Quantum)

> Further to this decision and as previously indicated, we would like to inform you that desk-rejections are not eligible for appeals.

OK. But you write: “This case of “pressure” by the authors on editors or referees is explicitly mentioned in our editorial policies, as a ground for desk rejection.”

Where exactly? I was unable to spot it in your documents. Please, help me with that and direct me, as an author, to the relevant link and lines in it.

With the best regards,

M. Pavicic.

Admin’s response

Lukas Schalleck (admin) Oct 25, 2022 - 12:24 pm CEST

Dear Mladen Pavicic,

Thanks for reaching out again. Happy to help, of course: this may be found under “Acceptance criteria” → “Editorial pre-selection” on the following page: [https://quantum-journal.org/editorial-policies/](https://quantum-journal.org/editorial-policies/)

The pertinent lines state: “External pressure by the authors on editors or referees outside the due editorial process may also result in editorial rejection.”

Best regards, Lukas Schalleck

My response to Admin’s response

Mladen Pavicic, Oct 25, 2022 - 1:26 pm CEST

Dear Mr. Schalleck,
I fail to grasp how my publication of anonymous previous reports can be possibly rendered as: “External pressure by the authors on editors or referees” when there is a standard procedure to send previous reports to next referees of a paper in most journals and to send previous reports to other journals of the same publisher.

By these standards, insight into previous reports can only be helpful and by no means a “pressure.”

With the best regards,
M. Pavicic.

APPENDIX 6: PHYSICAL REVIEW A EDITOR’S LETTER, 3 REFEREE REPORTS, MY RESPONSES TO THEM, AND EDITOR’S DECISION

Physical Review A has taken the manuscript and the reports over from the Physical Review Letters

Editor’s letter

Date: Tue, 13 Dec 2022
Re: LP17301AR

Automated generation of arbitrarily many Kochen-Specker and other contextual sets in odd-dimensional Hilbert spaces by Mladen Pavičić and Norman D. Megill

Dear Prof. Pavicic,

The above manuscript has been reviewed by three of our referees. Comments from the reports appear below for your consideration.

When you resubmit your manuscript, please include a summary of the changes made and a brief response to all recommendations and criticisms.

Yours sincerely,
Dr. Erin Knutson Associate Editor Physical Review A

P.S. We are taking referee A’s comments seriously. It will be easy for the review process to become (more) argumentative at this point, and we want to avoid that outcome. Please be sure to thoroughly and politely address their comments, which are designed to improve the impact and readability of your manuscript.

Three reports

Second Report of Referee A – LP17301AR/Pavicic

Overview

This is a resubmission of a manuscript originally submitted to PRL. In my original referee report I raised some concerns about the paper and made a number of suggestions for improvement, encouraging the author to submit a carefully revised version to e.g. PRA. When I received a revised submission I was convinced this would be an easy case of recommending acceptance to PRA, with at most only some minor remarks.

However, I am sad to say that I maintain most of my reservations about the manuscript. Unfortunately the author seems to have followed only the last part of the recommendation and resubmitted an almost identical version to PRA. I was disappointed to see that they made very little effort to address or engage with almost all of the points raised in the report.

Even regarding fairly minimal comments (such as a request to clarify the caption of a figure, see point 11 below), the author simply did not bother to make the very small, harmless change suggested to improve readability (instead suggesting in their reply that the referee should have “immediately grasped what the axes represent”). This was of course a very minor and unimportant point in the scheme of the paper, but I think this reply is indicative of the author’s failure to take the peer-review process seriously enough and to make an effort to engage with criticism and suggestions.

More worryingly, their reply to some of the more substantial points seems to betray some basic misunderstandings. The most shocking is the insistence on presenting unimportant implementation details, such as keeping the list of ASCII characters used to encode hypergraphs in their programs as “an integral part of the MMPH language,” and even doubling down on this. This seems to show a failure to grasp the distinction between (high-level) algorithms (which
are relevant to the reader) and (low-level) implementation choices (which are incidental and completely irrelevant).

I had commented that choosing to present such implementation details obfuscates the actual contributions of the paper—however, this reply made me think that the distinction is not even clear to them.

As another example, similar remarks apply to the suggestion regarding the definition of MMPH hypergraphs. The author’s reply mentions a standard textbook to counter the suggestion raised in the report, claiming that in that book the definition appears in a similar form as in their manuscript. Looking at the book, one actually finds out that this is not the case: in the book the presentation follows the suggestion I had given in my report. Once again, no change was made to the manuscript regarding this point.

In the remainder of this review, I go into more detail into the issues mentioned above and consider the author’s reply point by point. But to summarize: in addition to the reservations I had expressed about the adequacy of the way their methods are presented (which were not addressed), the replies quoted above and the underlying misconceptions made me have bigger doubts about the quality of this paper.

For the reasons just outlined, I believe that the manuscript should not be accepted in its current form. As I still believe that there is worthwhile work behind this paper, I recommend giving the author another opportunity to revise the manuscript and resubmitting it as a PRA Letter, in the hope that they’ll engage with criticism and suggestions in a more constructive fashion to improve the presentation.

Detailed comments

Quotes in pink are from the previous referee report and in blue from the author’s response.

1. < ‘[. . . ] It claims that “development of quantum computation and communication, recently shown to be supported by contextuality, arguably asks for an abundant supply of contextual sets,” but the authors do not bother to explain why such sets might be necessary [. . . ]’

> ‘On the contrary, the motivation is expressed in the second paragraph of the paper: “none of the methods previously employed in the even-dimensional spaces is available in the odd-dimensional spaces. In this paper, we offer several methods for automated generation of arbitrarily many KS sets as well as contextual non-KS ones in \( n = 3, 5, 7, 9 \) dimensional spaces.” The sentence: “quantum computation and communication . . . arguably asks for an abundant supply of contextual sets” is taken from the abstract which is limited in size. In the very body of the article, it is substantiated by the motivation sentences cited above.’

These sentences motivate why one would look at odd-dimensional subspaces given that prior work has focused on even-dimensional subspaces. These sentences do not address the following point I was referring to: why does the ‘development of quantum computation and communication’ ask for a requisite/abundant supply of contextual sets? I understand that quantum communication and computation are ‘supported by contextuality’, but what is the actual use of having a large supply of contextual sets? Again, I am not claiming that finding such contextual sets is not necessary. I am simply stating that the article does not argue for this need, beyond a this flimsy reference to contextuality supporting communication and communication. More motivation on this point would make the manuscript stronger and more broadly appealing. Having said that, I think this is a less important point than previously for PRL.

2. < ‘The presentation falls short of the standard expected in a research article, especially in a journal as good as PRL, in terms of clarity and rigor. Various passages are written in a confusing fashion.’

> ‘We published in Phys. Rev. Lett., Scientific Reports, Phys. Rev. A & D, Optics Express, J. Math. Phys., J. Phys. A, J. Opt. Soc. Am. B, Entropy, Phys. Lett. A, Found. Phys., Ann. H. Poincare . . . over almost half a century, and we do not think we wrote this paper in a “confusing fashion.”’

The authors have produced great research and published excellent papers over their careers. But as far as I am concerned their prior publishing record is irrelevant in this context where we’re only being asked to judge the one manuscript at hand. I fail to understand the relevance of bringing this up. As for the substance of the criticism, the overview paragraph (as well as the detailed comments in my first report) highlight various passages where the presentation was, in my opinion, confusing. The author would do better to respond to these concerns instead of bragging about their publication list.

3. < ‘For example, it is often unclear from the text that a passage is to be taken as the definition of a new concept (e.g. when the notion of coordinatization is introduced).’

> ‘Wiktionary has a single item for: coordinatization (plural coordinatizations): (mathematics) “The application of coordinates to a system,” and in the 3rd paragraph of the right column on p. 1 we additionally clarify: “The MMPHs above are defined without a coordinatization, i.e., neither their vertices nor their hyperedges are related
to either vectors or operators.” So, no “new concept” is introduced. The “coordinatization” is a word from a dictionary the referee should have consulted.

The dictionary definition is characteristically vague, as suits a dictionary of a natural language. It does not explain what the word means in specific contexts where it is used with a precise, formal, mathematical meaning. There are many ways one could instantiate this broad natural-language definition (‘applying coordinates to a system’) in the concrete setting of hypergraphs. ‘Coordinatization of a MMPH’ is a concept that is first (i.e. newly) introduced in this text in the last paragraph of page 1. There, its meaning is not precisely set out. One infers from the way the sentence is written (and from one’s knowledge of the English language) that a coordinatization of an MMPH is a map from vertices or from hyperedges (which?) to vectors or operators (which?!). This is sloppy and imprecise. Which is it? Do you mean that a coordinatization associates to each vertex a vector or an operator? Or are such vectors (or operators) associated to hyperedges instead of vertices? Are such associations supposed to satisfy some constraints, such as the vectors needing to be orthogonal when they belong to the same hyperedge (as one seems to infer in between the lines from reading a few sentences ahead)?

Concepts should be defined precisely and fully in scientific (especially mathematical) text. The reader is not supposed to infer a definition by trying to reconstruct what was in the authors’ head from pieces of sentences here and there.

4. ‘On the other hand, e.g. in the definition of MMPH hypergraph, some extra commentary is written as a part of the definition when it does not belong there, namely item (v).’

‘In the theory of hypergraphs, e.g., Berge, Hypergraphs, North Holland (1989), Ch. 1, Sec. 1, p. 1, 4th paragraph it is assumed that “A hypergraph may be drawn as a set of points representing the vertices and a hyperedge is represented ... by a single closed curve enclosing the vertices is a part of the definition.” There is no particular kind of a “definition” of ours which would differ from Berge’s or any other standardly accepted definition except that we introduce, in point (v), a curve passing through vertices instead of Berge’s curve which encloses them.’

‘(v) graphically, vertices are represented as dots and hyperedges as (curved) lines passing through them.” This is given as one of the conditions in the definition of MMPH graphs, but in reality it is not a condition on hypergraphs. It is just an explanation of the convention used to draw hypergraphs to ease the readability of Figure 2, for example. It should be separated from the definition of MMPH hypergraph.’

‘Nowhere do we speak about a condition. It is just the point within a definition in the same way in which similar points are used in many hypergraph textbooks. We already clarified that point (v) above when we compared it with a similar point used by Berge, Ch. 1, Sec. 1, p. 1, 4th paragraph.’

I located a copy of the book mentioned. It is surprising that the author chose to point to it. The beginning of Chapter 1, which introduces the notion of hypergraph, is indeed clearly written. But it does not do what I criticised in this manuscript; instead, the defining conditions and additional commentary on notational or pictorial conventions are clearly delineated from each other. I suggest that the author actually take this passage of the book as a model.

Berge gives the definition of a hypergraph (‘A hypergraph on X is a family ... such that (1) ... (2) ... A simple hypergraph ... is a hypergraph ... such that (3) ... ’). Then in the following paragraph Berge goes on to explaining that elements of X are called vertices etc. as well as setting out some other conventions. Then the paragraph after that contains the sentence quoted, explaining the book’s convention for depicting/drawing hypergraphs. This is most definitely not part of the definition, which Berge clearly demarcates.

Quite differently, the manuscript reads: ‘An MMPH is a connected n-dim hypergraph k-l with k vertices and l hyperedges in which (i) ...; (ii) ...; (iii) ...; (iv) ...; (v) graphically, vertices are represented ...’. Here, these are presented as five conditions (on hypergraphs) defining what it means to be an MMPH hypergraph. Item (v) is not (as in Berge’s book) presented as what it is: a convention on how to depict MMPH hypergraphs, rather than an item that is part of the definition, i.e. of what it means to be an MMPH hypergraph. It should, as in Berge’s text, be clearly separated.

The author claims that this is never spoken about as a ‘condition’. But in fact the paragraph defining MMPH hypergraphs is phrased in a way that places (v) on par with the other conditions defining what it means to be a hypergraph. And moreover, later in the text one reads: ‘An MMPH is a special kind of a general hypergraph in the sense that none of the aforementioned points (i-v) holds for it’, which implicitly says that (i-v) are to be taken as conditions on hypergraphs that define what it means to be an MMPH hypergraph.
The choice of material included in the text is odd, and it is inadequate for a journal article. At points, the authors choose to give too much information on irrelevant implementation details (e.g., the internal ASCII encoding of the graphs being manipulated, or the name of functions in their computer code).

Again, Berge, Ch. 1, Sec. 1, p. 1, 5th paragraph: “One may also define a hypergraph by its incidence matrix, with columns representing the [hyper]edges and rows representing the vertices (cf. Figure 1, p. 2).++ We, instead of an incidence matrix, make use of MMPH ASCII strings. Some kind of codification is unavoidable and our MMPH string codification is an integral part of the MMPH language and these strings are generated by our algorithms and programs and they then serve us as further direct inputs to our algorithms and programs whose outputs are all MMPH properties and features including graphical representations of the obtained MMPHs. The difference between incidence matrices and MMPHs is obvious; for instance, the MMPH string of Fig. 2(c) can be written in a single line, while the incidence matrix takes 37 columns and 51 lines. Also an MMPH string can be put in any of our programs while no such programs for arbitrary incidence matrices are known to us.’

This is the kind of implementation detail that is completely irrelevant to the potential readers of this paper. Its inclusion is a waste of space that could be used to convey much more interesting information about the methods implemented. This does not concern any substantial/significant algorithmic choice, but simply a matter of convention chosen for the internal representation. Its the kind of detail that is typically hidden from the user even in the documentation of a well-written library, and its even less relevant in the context of a journal article, which ought to focus on explaining the methods instead.’

There are no methods which could generate the contextual sets in odd dimensional spaces outside the MMPH language based on ASCII codification either in our minds or anywhere in the literature. The MMPH language is the method. A “method instead” does not exist.

This is problematic for two reasons. First, it detracts from the interest that the manuscript may have to a journals audience: most readers would read the paper to learn the main ideas underlying the methods so they could replicate and adapt them (anyone keen on learning implementation details would refer to the codes documentation and/or to the code itself). . . .

The 2nd sentence seems to contradict itself: “most readers would read the paper to learn the main ideas underlying the methods so they could replicate and adapt them,” but they would not like “to learn the code itself.” Note that without the MMPH language the results obtained in the article are inaccessible as they were in the last half of century.

These replies contributed to seriously aggravate my misgivings, in particular the claims that the string codification chosen is ‘an integral part of the MMPH language’, or that ‘there are no methods . . . outside the MMPH language based on ASCII codification’ and that this ‘MMPH language is the method,’ or the outrageous suggestion that ‘without the MMPH language the results obtained in the article are inaccessible’. These seems to reveal some very basic misunderstanding on the author’s part of the distinction between the level of ideas, methods, algorithms and that of concrete implementation details. The higher level abstracts from such irrelevant details and it should be conveyed in a way that is oblivious to it.

For illustration, let me first remark the obvious point that the use of ASCII characters is totally immaterial. If instead the author had used Unicode characters or 16-bit integers to encode hypergraphs, nothing of relevance would change whatsoever (except that it would take longer for the available ‘characters’ to be exhausted).

Similarly, the point about a string being written ‘in a single line while the incidence matrix takes 37 columns and 51 lines’ is misguided. For example, many graph algorithms are typically presented in terms of incidence matrices (or even more abstractly), but of course ultimately everything is represented in the computer as a sequence of bits, which can in theory be displayed in a single line. One doesn’t usually do this for a reason: such encodings are cumbersome and distract from the essence of the algorithms. The great power of abstraction is precisely that it allows one to focus on the relevant structure at each level, without concerning oneself with irrelevant, low-level details.

Finally, regarding comment that ‘an MMPH string can be put in any of our programs while no such programs for arbitrary incidence matrices are known to us’ let me just reiterate that a research article is not a manual documenting the usage of a computer program. In such a context, it would be natural to describe the format in which the program expects to receive its input, so that a potential user could interface with the program. But that’s completely beside the point for a journal article.

I would therefore encourage the authors to submit a carefully *revised* version, with a more detailed presentation of the main novel technical contributions, to a journal such as PRA.

Our algorithms do require sophisticated programs but that does not make the contributions “technical,” because there is no other way to obtain them.”
The last sentence makes very little sense. If the authors don’t recognise any new technical contributions, why are they submitting a research article?! What does being a technical contribution have to do with there not being any other way to obtain them? Anyway, I did not use the word “technical” as a criticism in any way. My point was that it would be good to see more about the actual new contributions enabling those “sophisticated programs”.

7. < ‘Development of quantum computation and communication, recently shown to be supported by contextuality, *arguably asks for an abundant supply of contextual sets*”. Contextuality has been shown to underlie quantum advantage in various computation and communication tasks and protocols. However, the second part of the sentence does not follow directly from this. That is, why does the development of quantum computation and communication ask for an *abundant supply of contextual sets*? This may well be the case, but (1) it is not an obvious consequence of the results mentioned from the literature, and (2) the authors fail to argue for this. Given that this is presented as the main motivation for this work and for its broad appeal, I would expect to see a better case made for this.’

> ‘Since “what gives quantum computers that extra oomph over their classical digital counterparts . . . is an intrinsic, measurable aspect of quantum mechanics called contextuality” [1,2], we should have more than a few sets per dimension. Those recent results are well-known and we considered that it is outside the scope of the present paper to elaborate on it.’

I agree that those recent results about contextuality and quantum advantage are well known and there is no need to elaborate on them. But it is not clear from any of these results that finding contextual sets is required or useful.

8. < ‘and *several* in 5- to 11-dim spaces” This is a rather vague statement: what is meant by “several”?’

> ‘By Webster Dictionary “several” means “more than two but fewer than many.”’

This is yet another flippant reply, not to say rude. The dictionary meaning of “several” is familiar—that was of course not the point of my comment. Note that this dictionary definition is again characteristically and unavoidably vague. The boundary between “several” and “many” is context dependent. Therefore saying that “several” KS sets were found in 5- to 11-dim spaces conveys very little information: what order of magnitude are the authors talking about? Is a dozen “several” or “many”? A hundred? A thousand?

9. < ‘An MMPH is a special kind of a general hypergraph in the sense that none of the aforementioned points (i-v) holds for it.” The meaning of this sentence is unclear. What is the referent of the pronoun “it” at the end of the sentence?’

> ‘it” = a general hypergraph. It is clear because points (i-v) obviously hold for MMPH.’

Well, yes, one can infer what the author meant to write. But grammatically this sentence is far from being clear. It really is just poor English, but it is astonishing that the author hasn’t even attempted to rephrase the sentence after this was pointed out.

10. < ‘Orthogonality between vertices in an MMPH space just means that the vertices are contained in their hyperedges” This is yet another sentence whose meaning isn’t clear. What is meant by “the vertices are contained in *their* hyperedges”?’

> ‘Meaning is clear. Let me put it this way: Let ABC be a hyperedge; for vertices A,B,C, the hyperedge ABC is *their* hyperedge; so, the vertices A,B,C are mutually orthogonal just by being contained in *their* hyperedge. I’m convinced the reader will have no problem with the sentence.’

The same vertex might belong to more than one hyperedge, so it makes no sense to speak of ‘its’ hyperedge. A vertex is always contained, by definition, in any of its hyperedges, so the sentence as it stands is trivial. What the author wanted to say is that two vertices being orthogonal means that they are both contained in a common hyperedge. But this is not what the text says.

11. < ‘Page 2, col. 2, Fig. 1: This figure is hard to interpret. The caption does not adequately explain, e.g., what do the axes represent, or what are the shaded black stripes. The same applies to Fig 3 and Fig 4 later on.’

> ‘Abscissa = l (number of hyperedges); ordinate = k (number of vertices). Dots represent (k,l). Consecutive dots (same l) are shown as strips. Analogous figures have been presented in more than 20 papers of ours and all the referees and all from many conferences have immediately grasped what the axes represent.’

I was disappointed to see no change made to the caption to clarify this point, which could have been very easily addressed.
Second Report of Referee B – LP17301AR/Pavicic

The authors have addressed my comments in the revised version of the manuscript. A couple of comments are addressed in a later work, Ref.[14]. I think the current version is suitable for publication in PRA as a Letter. The phrase ‘do not require’ is repeated in the added sentence (2nd paragraph on page 2).

Report of the Third Referee – LP17301AR/Pavicic

Having read the manuscript and the previous referees’ report, I feel the authors have successfully attended all the concerns of the referees and this manuscript, containing much new and important material, should be published in PRA.

The authors state that these results may have future applications in quantum computing or communication. These hopes are often mentioned in quantum mechanics manuscript, but no concrete idea is presented. Some further discussion would be useful.

On page 2 second paragraph line 14 the phrase “do not require” is repeated.

My responses to the Editor and to the three reports

Date: Tue, 13 Dec 2022
Response to Editor’s comments on LP17301AR
Dear Dr. Knutson,

... In my response to referee A’s report I’m now as polite and cooperative as humanly possible.

... With the best regards,
Mladen Pavicic

Response to the Second Report of Referee A – LP17301AR/Pavicic

I thank the referee for her/his benevolent attempts to convince me to “engage with [her/his] criticism and suggestions in a more constructive fashion to improve the presentation.” The referee made use of the first page of her/his reports to elaborate on her/his conviction and since real changes are at stake here, I’m dwelling on them straight away. Here the opinion of the third referee “Having read the manuscript and the previous referees’ report, I feel the authors have successfully attended all the concerns of the referees and this manuscript, containing much new and important material, should be published in PRA” suggests that I would hopefully be able to successfully meet all suggestions put forward by the present referee. Note that the changes suggested on p. 1 of the reports are specified in its Detailed comments to which I respond below.

Pink quotes from the previous referee report are prefixed by >>>, from my previous blue answers by >>, and from the present report by > (I’m writing my response in the plain text mode.)

> 1. why does the ‘development of quantum computation and communication’ ask for a requisite/abundant supply > of contextual sets? I understand that quantum communication and computation are ‘supported by contextuality’, > but what is the actual use of having a large supply of contextual sets? Again, I am not claiming that finding such > contextual sets is not necessary. I am simply stating that the article does not argue for this need, beyond this > flimsy reference to contextuality supporting communication and communication. More motivation on this point would > make the manuscript stronger and more broadly appealing. Having said that, I think this is a less important > point than previously for PRL.

The referee is right. I added the following paragraph to the left column on p. 1:

“Since the quantum communication and computation are supported by contextuality [1,2], the actual potential use of a large supply of contextual sets is twofold. First, quantum computation algorithms which would rely on contextual
sets would arguably rely on a variety of such sets and on their automated generation. Second, structural properties of contextual sets differ according to their coordinatization, parities, dimensions, sizes, etc., and that can lead us to better understanding and applications of the sets."

> 2. The author would do better to respond to these concerns instead of bragging about their publication list.

I’ll do my best.

> 3. ‘Coordinatization of a MMPH’ is a concept that is first (i.e., newly) introduced in this text in the last paragraph of page 1. There, its meaning is not precisely set out. One infers from the way the sentence is written that a coordinatization of an MMPH is a map from vertices or from hyperedges (which?) to vectors or operators (which?!). This is sloppy and imprecise. Which is it? Do you mean that a coordinatization associates to each vertex a vector or an operator?

Yes.

> are such vectors (or operators) associated to hyperedges instead of vertices? Are such associations supposed to satisfy some constraints, such as the vectors needing to be orthogonal when they belong to the same hyperedge?

Yes. I added the following sentence “The meaning of coordinatization is that a vector is assigned to each vertex and that all vectors assigned to vertices belonging to a common hyperedge are orthogonal to each other.” as the 2nd sentence of that paragraph which is now the 2nd paragraph on p. 2.

> 4. the manuscript reads: ‘An MMPH is a connected n-dim hypergraph k-l with k vertices and l hyperedges in which (i) . . . ; (ii) . . . ; (iii) . . . ; (iv) . . . ; (v) graphically, vertices are represented . . . ’. Here, these are presented as five conditions (on hypergraphs) defining what it means to be an MMPH hypergraph. Item (v) is not presented as what it is: a convention on how to depict MMPH hypergraphs, rather than an item that is part of the definition, i.e., of what it means to be an MMPH hypergraph. It should be clearly separated.

The referee is right. (v) is not a “condition” and now I clearly separate “graphically, vertices are represented . . .” from the conditions “(i)-(iv)” and I don’t denote it as “(v).” I changed the previous last sentence of the 2nd paragraph of the right column on p. 1 “. . .; (v) graphically, . . .” so as to read: . . . Graphically, . . .”

> later in the text one reads: ‘An MMPH is a special kind of a general hypergraph in the sense that none of the aforementioned points (i-v) holds for it’, which implicitly says that (i-v) are to be taken as conditions on hypergraphs that define what it means to be an MMPH hypergraph.

The referee is right. I changed ”(i-v)” so as to read ”(i-iv).”

> 5. . . . seriously aggravate my misgivings, in particular the claims that the string codification chosen is ‘an integral part of the MMPH language’, or that ‘there are no methods . . . outside the MMPH language based on ASCII codification’ and that this ‘MMPH language is the method,’ or the outrageous suggestion that ‘without the MMPH language the results obtained in the article are inaccessible’.

The referee is right. The sentences are too strong. But the sentences “an integral part of the MMPH language” or “there are no methods . . . outside the MMPH language based on ASCII codification” or “without the MMPH language the results obtained in the article are inaccessible” do not appear in the paper and I completely withdraw such opinions expressed in my previous response to the previous referee’s report.

> These seems to reveal some very basic misunderstanding on the author’s part of the distinction between the level of ideas, methods, algorithms and that of concrete implementation details.

Yes, I admit that we “misunder[ood] . . . the distinction between the level of ideas, methods, algorithms and that of concrete implementation details.” And in particular the referee is right that “The higher level abstracts from such irrelevant details and it should be conveyed in a way that is oblivious to it.” Therefore I’m ready to, along these lines, adopt the suggestions made by the referee and implement the following “concrete implementation details:”

> For illustration, let me first remark the obvious point that the use of ASCII characters is totally immaterial. If instead the author had used Unicode characters or 16-bit integers to encode hypergraphs, nothing of relevance would change whatsoever.

The referee is quite right. We now added the following paragraph at the top of p. 2: “Of course, instead of ASCII characters we could have used Unicode characters or 16-bit integers but 20 years ago we decided to proceed with the ASCII characters to encode MMPH strings and design our algorithms and programs which in turn yield all properties and features of MMPHs as well as their figures within the MMPH language. All our papers since 2000 [15] make use of the ASCII characters for the purpose.”
Similarly, the point about a string being written "in a single line while the incidence matrix takes 37 columns and 51 lines is misguided."

I took out the paragraph completely.

Finally, regarding comment that "an MMPH string can be put in any of our programs while no such programs for arbitrary incidence matrices are known to us" let me just reiterate that a research article is not a manual documenting the usage of a computer program. In such a context, it would be natural to describe the format in which the program expects to receive its input, so that a potential user could interface with the program. But that's completely beside the point for a journal article.

The referee is quite right. The mentioned sentence is not any more in the paper and actually neither "incidence" nor "matrix" nor "matrices" do appear in the paper and the "incidence matrices" are not even mentioned in the present revised version of the paper.

"I would therefore encourage the authors to submit a carefully revised version, with a more detailed presentation of the main novel technical contributions, to a journal such as PRA."

"Our algorithms do require sophisticated programs but that does not make the contributions "technical," because there is no other way to obtain them."

The last sentence makes very little sense. If the authors don’t recognise any new technical contributions, why are they submitting a research article?! What does being a technical contribution have to do with there not being any other way to obtain them? Anyway, I did not use the word “technical” as a criticism in any way. My point was that it would be good to see more about the actual new contributions enabling those “sophisticated programs”.

The referee is quite right. I’m now submitting a “carefully revised version.” As for “there is no other way to obtain them,” I meant: for us—after more than 20 years of our usage of our algorithms and programs. Anyhow, the sentence does not appear in the paper. As for referee’s last sentence, we now point the role of algorithms at several places in the paper and we also added the following paragraph to p. 2, right column, 4th paragraph from the top: “We carry out methods M1-M3 and by means of our programs MMPStrip (for stripping and adding hyperedges), States01 (for verifying the contextuality), MMPShuffle (for reorganizing MMPHs), and ONE (for evaluating the structural properties of MMPHs [14]).”

I agree that those recent results about contextuality and quantum advantage are well known and there is no need to elaborate on them. But it is not clear from any of these results that finding contextual sets is required or useful.

The referee is right. I discussed that under point 1 above and added a relevant paragraph to the left column on p. 1 (also cited above under point 1).

"and *several* in 5- to 11-dim spaces" This is a rather vague statement: what is meant by “several”? Is a dozen “several” or “many”? A hundred? A thousand?

The referee is right. The sentence now reads: “. . . , four in the 5-dim space [11],[12, Supp. Material], five in the 7-dim space [11, 13],[12, Supp. Material], two in the 9-dim space [12, Supp. Material], and two in the 11-dim space [12, Supp. Material]."

"An MMPH is a special kind of a general hypergraph in the sense that none of the aforementioned points (i-v) holds for it.” The meaning of this sentence is unclear. What is the referent of the pronoun “it” at the end of the sentence? Well, yes, one can infer what the author meant to write. But grammatically this sentence is far from being clear.

The referee is right. The sentence now reads: “An MMPH is a special kind of a general hypergraph in the sense that none of the aforementioned points (i-iv) holds for such a hypergraph.”

"Orthogonality between vertices in an MMPH space just means that the vertices are contained in their hyperedges” This is yet another sentence whose meaning isn’t clear. What is meant by “the vertices are contained in *their* hyperedges?” What the author wanted to say is that two vertices being orthogonal means that they are both contained in a common hyperedge. But this is not what the text says.

The referee is right. The sentence now reads: “Orthogonality between vertices in an MMPH space just means that they are contained in common hyperedges.”

Page 2, col. 2, Fig. 1: This figure is hard to interpret. The caption does not adequately explain, e.g., what do the axes represent, or what are the shaded black stripes. The same applies to Fig 2 and Fig 3 later on.
I was disappointed to see no change made to the caption to clarify this point, which could have been very easily addressed.

The referee is right. I now added the following sentences to the caption of Fig. 1: “Abscissa is \( l \) (number of hyperedges); ordinate is \( k \) (number of vertices). Dots represent \((k, l)\). Consecutive dots (same \( l \)) are shown as strips. The same applies to Figs. 2 and 3.”

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Response to the Second Report of Referee B – LP17301AR/Pavicic

> The authors have addressed my comments in the revised version of the manuscript. A couple of comments are addressed in a later work, Ref. [14]. I think the current version is suitable for publication in PRA as a Letter. The phrase ‘do not require’ is repeated in the added sentence (2nd paragraph on page 2).

I thank the referee. The typo is corrected.

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Response to Report of the Third Referee – LP17301AR/Pavicic

> Having read the manuscript and the previous referees’ report, I feel the authors have successfully attended all the concerns of the referees and this manuscript, containing much new and important material, should be published in PRA.

I thank the referee.

> The authors state that these results may have future applications in quantum computing or communication. These hopes are often mentioned in quantum mechanics manuscript[s], but no concrete idea is presented. Some further discussion would be useful.

A discussion is now included in the bottom paragraph of the left column on p. 1. It reads: “Since the quantum communication and computation are supported by contextuality [1, 2], the actual potential use of a large supply of contextual sets is twofold. First, quantum computation algorithms which would rely on contextual sets would arguably rely on a variety of such sets and on their automated generation. Second, structural properties of contextual sets differ according to their coordinatization, parities, dimensions, sizes, etc., and that can lead us to better understanding and applications of the sets.”

> On page 2 second paragraph line 14 the phrase “do not require” is repeated.

The typo is corrected.

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Editor’s decision—on Dec 16; my responses to referee’s reports above were submitted on Dec 15

Acceptance LP17301AR Pavicic
From pra@aps.org
Date 2022-12-16 20:25

*Automated generation of arbitrarily many Kochen-Specker and other contextual sets in odd-dimensional Hilbert spaces* by Mladen Pavićić and Norman D. Megill

Dear Prof. Pavicić,

We are pleased to inform you that your manuscript has been accepted for publication as a Letter in Physical Review A.

Yours sincerely,

Davide Girolami, Associate Editor, Physical Review A

*Physical Review A* Twitter of the paper:
https://twitter.com/PhysRevA/status/1611369522254041091