New results on the AM-GM inequality

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Abstract. There is a large number of discrete inequalities between geometric and arithmetic means. Our intention is to investigate possible generalization and integral analogues of the inequalities obtained by Redhefer, Bullen and Godunova. At the end, we will consider refinements of these inequalities by different separations of their domains.

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1. Preliminaries

Consider the following sequences of positive real numbers:

\((a) = (a_1, \ldots, a_n); \ (b) = (b_1, \ldots, b_n); \ (p) = (p_1, \ldots, p_n); \ (q) = (q_1, \ldots, q_n)\).

We use the notations:

\[ P_n = \sum_{k=1}^{n} p_k, \quad Q_n = \sum_{k=1}^{n} q_k, \]

\[ A_n(a, q) = \frac{1}{Q_n} \sum_{k=1}^{n} q_k a_k, \quad G_n(a, q) = \prod_{k=1}^{n} a_k^{\frac{q_k}{Q_n}}, \]

\[ \Gamma_n(a, q) = \frac{1}{Q_n} \sum_{k=1}^{n} q_k G_k(a, q). \]  

A function \( F : [a, b] \to \mathbb{R} \) is said to be convex if for all \( x, y \in [a, b] \) and all \( \alpha \in [0, 1] \),

\[ F(\alpha x + (1-\alpha)y) \leq \alpha F(x) + (1-\alpha)F(y). \]  

If the inequality (1.2) is reversed, then \( F \) is said to be concave.

Investigating recurrent inequalities in [5], Redhefer has obtained the following result involving the arithmetic and geometric means. The statement is given in terms of (1.1), where \( G_n(a, 1) = G_n(a, q), \ q_k = 1, k = 1, \ldots, n \).
Theorem 1 (Redheffer). If \( t > 0 \), then
\[
G_n(a, 1) < A_n(a, 1)e^{-t} + t \Gamma_n(a, 1).
\]

Bullen [2] proved an inequality of Rado type generalizing Redheffer’s inequality.

Theorem 2 (Bullen). If \( 0 \leq tq_n \leq Q_n \), then
\[
Q_n(A_n(a, q)e^{-t} + t \Gamma_n(a, q) - G_n(a, q)) \\
\geq Q_{n-1}(A_{n-1}(a, q)e^{-t} + t \Gamma_{n-1}(a, q) - G_{n-1}(a, q)).
\]

A more general result (given below) is obtained in [3].

Theorem 3 (Godunova). Suppose that \( G \) and \( F \) are twice continuously differentiable functions such that \( G \) is concave and \( GF^\prime \) is convex.

For \( t \in \mathbb{R} \) the next inequality is true:
\[
Q_n \left\{ G(A_n(F(t + a_k), q)) - t A_n \left( (GF)^\prime \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right) \right) \right\} \geq \frac{q_n P_n}{p_n Q_n} A_n(a, p) \tag{1.3}
\]
\[
- G(F \left( \frac{q_n P_n}{p_n Q_n} A_n(a, p) \right)) \tag{1.4}
\]
\[
\geq Q_{n-1} \left\{ G(A_{n-1}(F(t + a_k), q)) - t A_{n-1} \left( (GF)^\prime \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right) \right) \right\} \geq \frac{q_n P_n}{p_n Q_n} A_{n-1}(a, p) \tag{1.5}
\]
\[
- G(F \left( \frac{q_n P_n}{p_n Q_n} A_{n-1}(a, p) \right)) \tag{1.6}
\]

In this article we will present some generalizations and refinements of these results.

2. Main result

Our results are based on the well-known properties of convex functions. The conditions are simpler than the ones of Theorem 3. It is obvious that Theorem 3 follows from this result.

Theorem 4. If \( F \) is a convex differentiable function, then for all \( t \in \mathbb{R} \)
\[
Q_n \left[ A_n(F(t + a_k), q) - t \cdot A_n \left( F^\prime \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right) \right) \right] \geq \frac{q_n P_n}{p_n Q_n} A_n(a, p) \tag{2.1}
\]
\[
- F \left( \frac{q_n P_n}{p_n Q_n} A_n(a, p) \right) \]
\[
\geq Q_{n-1} \left[ A_{n-1}(F(t + a_k), q) - t \cdot A_{n-1} \left( F^\prime \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right) \right) \right] \geq \frac{q_n P_n}{p_n Q_n} A_{n-1}(a, p) \tag{1.5}
\]
\[
- F \left( \frac{q_n P_n}{p_n Q_n} A_{n-1}(a, p) \right) \]
Proof. Obviously, we have
\[
\sum_{k=1}^{n} q_k F(t + a_k) - \sum_{k=1}^{n-1} q_k F(t + a_k) = q_n F(t + a_n)
\]
and
\[
Q_n \cdot A_n \left( F' \left( \frac{q_k P_k}{p_k Q_k} \cdot A_k(a, p) \right), q \right)
\]
\[
- Q_{n-1} \cdot \text{dot} \cdot A_{n-1} \left( F' \left( \frac{q_k P_k}{p_k Q_k} \cdot A_k(a, p) \right), q \right)
\]
\[
= q_n \cdot F' \left( \frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right).
\]

Statement (2.1) is equivalent to
\[
q_n F(t + a_n) + Q_{n-1} \cdot F \left( \frac{q_n P_{n-1}}{p_n Q_{n-1}} \cdot A_{n-1}(a, p) \right) \geq Q_n \cdot \left( \frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right),
\]
so it remains to show (2.2). Because of \(\frac{q_n}{Q_n} + \frac{Q_{n-1}}{Q_n} = 1\), the definition of convex functions implies
\[
q_n \cdot F(t + a_n) + Q_{n-1} \cdot F \left( \frac{q_n P_{n-1}}{p_n Q_{n-1}} \cdot A_{n-1}(a, p) \right)
\]
\[
\geq Q_n \cdot F \left( \frac{t q_n}{Q_n} + \frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right).
\]

Presuming that the convex function \(F\) is differentiable, we have (see [4], p.5, Theorem 1.6.):
\[
F(u + \alpha) \geq F(u) + \alpha \cdot F'(u).
\]

Upon application of these results, the following inequality is obtained
\[
Q_n \cdot F \left( \frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) + \frac{t q_n}{Q_n} \right)
\]
\[
\geq Q_n \cdot F \left( \frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right) + t \cdot q_n \cdot F' \left( \frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right).
\]

Now (2.3) and (2.5) imply (2.2). \(\square\)
Corollary 1. If $F$ is a strictly monotone convex function, then
\[ Q_n \left[ A_n(b,q) - F \left( \frac{q_n P_n}{p_n Q_n} A_n(F^{-1}(b),p) \right) \right] \geq Q_{n-1} \left[ A_{n-1}(b,q) - F \left( \frac{q_n P_{n-1}}{p_n Q_{n-1}} A_{n-1}(F^{-1}(b),p) \right) \right]. \]

Proof. It is enough to substitute $t = 0$ and $a_k = F^{-1}(b_k)$, $k = 1,2,\ldots,n$ in (2.1).

Corollary 2. If $F$ is a convex, differentiable function, then for every $t \in \mathbb{R}$
\[ F(A_n(a,p)) \leq A_n(F(t + a_k), p) - t A_n(F'(a_k,p),p). \] (2.6)

Proof. Substitute $(p) = (q)$ in (2.1) for $n = 2,3,\ldots,n-1,n$ in order to get a sequence of inequalities. Then sum these inequalities.

Inequality (2.6) turns into the inequality of Redhefer upon taking $F(u) = e^{-u}$, $a_i = \ln \frac{1}{b_i}$ and $(p) = (1,\ldots,1)$. This inequality is proved in [5] for $t \geq 0$. The same inequality is proved in [2], but only for $0 \leq t \leq 2$.

Theorem 4 ensures the statement of Theorem 2 proved in [2] only for $0 \leq t q_n \leq Q_n$.

Corollary 3. For every $t \in \mathbb{R}$
\[ Q_n \left( A_n(b,q) e^{-t} + t I_n(b,q) - G_n(b,q) \right) \geq Q_{n-1} \left( A_{n-1}(b,q) e^{-t} + t I_{n-1}(b,q) - G_{n-1}(b,q) \right). \] (2.7)

Proof. To obtain (2.7), substitute $(p) = (q)$ and $F(u) = e^{-u}$ in (2.1). Then $F(a_k) = b_k = e^{-a_k}$ and $a_k = -\ln b_k = \ln \frac{1}{b_k}$.

Remark. Theorem 3, (the first theorem of [3]), follows from Theorem 4. Substitute the composition $GF$ instead of the convex function $F$ in (2.2) and use the Jensen inequality for the concave function $G$:
\[ Q_n G(A_n(F(t + a_k),q)) \geq Q_{n-1} G(A_{n-1}(F(t + a_k),q)) + q_n G(F(t + a_n)). \]

The next inequality from [1] is an immediate consequence of Theorem 3.

Corollary 4. Suppose that $F$ is a strictly monotone function, $G$ is a concave function and $GF$ is a convex function. Then
\[ Q_n \left\{ G(A_n(b,q)) - (GF) \left( \frac{q_n P_n}{p_n Q_n} A_n(F^{-1}(b),p) \right) \right\} \geq Q_{n-1} \left\{ G(A_{n-1}(b,q)) - (GF) \left( \frac{q_n P_{n-1}}{p_n Q_{n-1}} A_{n-1}(F^{-1}(b),p) \right) \right\}. \]
Taking \((p) = (q)\), the next corollary is obtained from (2.1).

**Corollary 5.** If \(F\) is a differentiable convex function, then

\[
\sum_{k=1}^{n} p_k F(t + a_k) - t \cdot \sum_{k=1}^{n} p_k F' \left( \frac{1}{P_n} \sum_{j=1}^{k} p_j a_j \right) \\
- \sum_{k=1}^{n} p_k \cdot F \left( \frac{1}{P_n} \sum_{k=1}^{n} p_k a_k \right) \\
\geq \sum_{k=1}^{n-1} p_k F(t + a_k) - t \cdot \sum_{k=1}^{n-1} p_k F' \left( \frac{1}{P_{n-1}} \sum_{k=1}^{n-1} p_k a_k \right) \\
- \sum_{k=1}^{n-1} p_k \cdot F \left( \frac{1}{P_{n-1}} \sum_{k=1}^{n-1} p_k a_k \right),
\]

whereby \(t \in \mathbb{R}\).

### 3. An Application on Integral analogue

Integral analogue is given in [3] as Theorem 2.

**Theorem 5** (Godunova). Let \(\psi\) be a convex, differentiable function and let \(p(\tau)\) and \(a(\tau)\) be positive functions for \(\tau > 0\). Then the function

\[
\Phi(x) = \int_{0}^{x} \psi(t + a(\tau)) p(\tau) d\tau - t \int_{0}^{x} \psi' \left( \frac{\int_{0}^{t} a(\xi) p(\xi) d\xi}{\int_{0}^{x} p(\xi) d\xi} \right) p(\tau) d\tau \\
- \psi \left( \frac{\int_{0}^{x} a(\tau) p(\tau) d\tau}{\int_{0}^{x} p(\tau) d\tau} \right) \cdot \int_{0}^{x} p(\tau) d\tau \quad (3.1)
\]

is an increasing function on \((0, \infty)\).

The next corollary estimates the difference between the values, so that composition \(GF\) is calculated at the mean values on the intervals \([0, x] \subset [0, y]\).

**Corollary 6.** Let \(GF\) be a convex function, let \(G\) be a concave function and suppose that \(a, p\) are positive on \((0, \infty)\). Then

\[
GF \left( \frac{\int_{0}^{y} a(\tau) p(\tau) d\tau}{\int_{0}^{y} p(\tau) d\tau} \right) \cdot \int_{0}^{y} p(\tau) d\tau \\
- GF \left( \frac{\int_{0}^{x} a(\tau) p(\tau) d\tau}{\int_{0}^{x} p(\tau) d\tau} \right) \cdot \int_{0}^{x} p(\tau) d\tau
\]
\[ \leq \int_x^y p(\tau) d\tau \cdot G \left( \frac{\int_x^y F(t + a(\tau)) p(\tau) d\tau}{\int_x^y p(\tau) d\tau} \right) - t \int_x^y (GF)' \left( \frac{\int_0^\tau a(\xi) p(\xi) d\xi}{\int_0^\tau p(\xi) d\xi} \right) p(\tau) d\tau. \]

**Proof.** Substitute \( \psi = GF \) in (5). Notice that if 
\[ t \int_x^y \psi d\tau \mid DGF \mid \leq P(y) \]
with \( P(y) = \int_0^y p(\tau) d\tau \), makes it possible to define the function 
\[ F(y) = P(y) \cdot G \left( \frac{1}{P(y)} \int_0^y F(t + a(\tau)) p(\tau) d\tau \right) - \int_0^y GF(t + a(\tau)) p(\tau) d\tau. \]

The function \( F \) is monotone on \((0, \infty)\).
Proposition 1. If $G$ is a concave function, then $0 \leq x \leq y$ ensures

$$\mathcal{F}(x) \leq \mathcal{F}(y).$$  \hfill (3.2)

Proof. This statement follows from the concavity of $G$. There is a chain of inequalities:

$$P(x)G \left( \frac{1}{P(x)} \int_0^x F(t + a(\tau)) p(\tau) d\tau \right) - \int_0^x GF(t + a(\tau)) p(\tau) d\tau + \int_0^y GF(t + a(\tau)) p(\tau) d\tau

= P(x)G \left( \frac{1}{P(x)} \int_0^x F(t + a(\tau)) p(\tau) d\tau \right)

+ \int_0^y p(\tau) d\tau \cdot G \left( \frac{\int_0^y F(t + a(\tau)) p(\tau) d\tau}{\int_0^y p(\tau) d\tau} \right)

\leq \frac{P(y)}{P(y)} \left[ P(x) \cdot G \left( \frac{1}{P(x)} \int_0^x F(t + a(\tau)) p(\tau) d\tau \right)

+ \int_0^y p(\tau) d\tau \cdot G \left( \frac{\int_0^y F(t + a(\tau)) p(\tau) d\tau}{\int_0^y p(\tau) d\tau} \right) \right]

= P(y) \left[ \frac{P(x)}{P(y)} \cdot G \left( \frac{1}{P(x)} \int_0^x F(t + a(\tau)) p(\tau) d\tau \right)

+ \int_0^y p(\tau) d\tau \cdot G \left( \frac{\int_0^y F(t + a(\tau)) p(\tau) d\tau}{\int_0^y p(\tau) d\tau} \right) \right]

\leq P(y) \cdot G \left( \frac{1}{P(y)} \int_0^x F(t + a(\tau)) p(\tau) d\tau \right) + \frac{1}{P(y)} \int_0^y F(t + a(\tau)) p(\tau) d\tau

= P(y) \cdot G \left( \frac{1}{P(y)} \int_0^x F(t + a(\tau)) p(\tau) d\tau \right).$$

At this point observe certain refinements when $G$ is concave and $GF$ is a convex function.

Remark 1. Applying (3.2) in

$$D(x) - D(y) \leq \int_0^y GF(t + a(\tau)) p(\tau) d\tau - \int_0^x GF(t + a(\tau)) p(\tau) d\tau,$$

the following is obtained:

$$P(y) \cdot GF \left( \frac{\int_0^y a(\tau) p(\tau) d\tau}{\int_0^y p(\tau) d\tau} \right) - P(x) \cdot GF \left( \frac{\int_0^x a(\tau) p(\tau) d\tau}{\int_0^x p(\tau) d\tau} \right).$$
\begin{align*}
\leq & \, P(y) \cdot G \left( \frac{\int_0^y F(t + a(\tau))p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) - P(x) \cdot G \left( \frac{\int_0^x F(t + a(\tau))p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right) \\
& - t \int_x^y (GF)' \left( \frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) p(\tau)d\tau.
\end{align*}

To sum up, if GF is a non-decreasing function, we can make the following observation.

Remark 2. If GF is a nondecreasing differentiable function, then \( GF' \geq 0 \) and for \( t \geq 0 \) one can obtain

\[ P(y) \cdot GF \left( \frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) - P(x) \cdot GF \left( \frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right) \]

\[ \leq P(y) \cdot G \left( \frac{\int_0^y F(t + a(\tau))p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) - P(x) \cdot G \left( \frac{\int_0^x F(t + a(\tau))p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right). \]

For \( t = 0 \) we get ordinary monotonicity. If \([0,x] \subseteq [0,y]\), then

\[ P(x) \cdot G \left( \frac{\int_0^x F(a(\tau))p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right) - P(x) \cdot GF \left( \frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) \]

\[ \leq P(y) \cdot G \left( \frac{\int_0^y F(a(\tau))p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) - P(y) \cdot GF \left( \frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right). \]

4. DECREASING PROPERTY UPON TAKING SUBSETS

In the next theorem an integral version of the statement obtained in (1.6) is given whereby \((p) = (q)\) and \( t = 0 \). The statement appeared as particular case of Bullen’s inequality (see [4]). In [3] the author omitted the proof. We are providing it here to refresh certain basic integral inequalities.

**Theorem 6.** Let \( \mathcal{D} \) be a measurable domain and let \( \mathcal{D}_1 \subset \mathcal{D} \) be its regular measurable sub-domain. Further, suppose \( a, q : \mathcal{D} \to [0, +\infty) \) is such that \( Q = \int_{\mathcal{D}_1} q(u)dV_u \neq 0 \) and \( Q_1 = \int_{\mathcal{D}_1} q(u)dV_u \neq 0 \). If \( G \) is concave, \( F \) is monotone and \( GF \) is a convex function, then

\[ Q_1 \left[ G \left( \frac{1}{Q_1} \int_{\mathcal{D}_1} F(a(u))q(u)dV_u \right) - GF \left( \frac{1}{Q_1} \int_{\mathcal{D}_1} a(u)q(u)dV_u \right) \right] \]

\[ \leq Q \left[ G \left( \frac{1}{Q} \int_{\mathcal{D}} F(a(u))q(u)dV_u \right) - GF \left( \frac{1}{Q} \int_{\mathcal{D}} a(u)q(u)dV_u \right) \right]. \tag{4.1} \]
Proof. Concavity of $G$ ensures
\[
Q \cdot G \left( \frac{1}{Q} \int_{\mathcal{D}} F(a(u))q(u)dV_u \right)
\geq Q_1 \cdot G \left( \frac{1}{Q_1} \int_{\mathcal{D}_1} F(a(u))q(u)dV_u \right)
+ Q_2 \cdot G \left( \frac{1}{Q_2} \int_{\mathcal{D}_2} F(a(u))q(u)dV_u \right),
\]
wherby $\mathcal{D}_2 = \mathcal{D} \setminus \mathcal{D}_1$ with $Q_2 = \int_{\mathcal{D}_2} q(u)dV_u \neq 0$. Besides, the convexity of $GF$ implies
\[
Q \cdot GF \left( \frac{1}{Q} \int_{\mathcal{D}} a(u)q(u)dV_u \right)
\leq Q_1 \cdot GF \left( \frac{1}{Q_1} \int_{\mathcal{D}_1} a(u)q(u)dV_u \right) + Q_2 \cdot GF \left( \frac{1}{Q_2} \int_{\mathcal{D}_2} a(u)q(u)dV_u \right).
\]
Integral version of Jensen’s inequality (see [4], p. 45) for concave function $G$ provides
\[
Q_2 \cdot G \left( \frac{1}{Q_2} \int_{\mathcal{D}_2} F(a(u))q(u)dV_u \right) \geq \int_{\mathcal{D}_2} GF(a(u))q(u)dV_u.
\]
Furthermore, the convexity of $GF$ provides the statement
\[
\int_{\mathcal{D}_2} GF(a(u))q(u)dV_u \geq Q_2 \cdot GF \left( \frac{1}{Q_2} \int_{\mathcal{D}_2} a(u)q(u)dV_u \right).
\]
Now there is a chain
\[
Q \cdot G \left( \frac{1}{Q} \int_{\mathcal{D}} F(a(u))q(u)dV_u \right) - Q_1 \cdot G \left( \frac{1}{Q_1} \int_{\mathcal{D}_1} F(a(u))q(u)dV_u \right)
= Q_2 \cdot G \left( \frac{1}{Q_2} \int_{\mathcal{D}_2} F(a(u))q(u)dV_u \right) \geq Q_2 \cdot GF \left( \frac{1}{Q_2} \int_{\mathcal{D}_2} a(u)q(u)dV_u \right)
\geq Q \cdot GF \left( \frac{1}{Q} \int_{\mathcal{D}} a(u)q(u)dV_u \right) - Q_1 \cdot GF \left( \frac{1}{Q_1} \int_{\mathcal{D}_1} a(u)q(u)dV_u \right).
\]
The first line and the last line from this give (4.1).

Finally, we generalize (4.1).

Theorem 7. Let $(\Omega, \Sigma, \mu)$ be a space with a positive finite measure and let $a : \Omega \to \mathbb{R}$ be a $\mu$-measurable function. Suppose that $G$ and $F$ are differentiable functions such that $G$ is concave and $GF$ is convex. For every $t \in \mathbb{R}$, if $\Omega_1 \in \Sigma$, and $\Omega_1 \neq \Omega$, then
\[
\mu(\Omega) \cdot G \left( \frac{1}{\mu(\Omega)} \int_{\Omega} F(t+a)d\mu \right) \tag{4.2}
\]
\[
-\mu(\Omega_1) \cdot G \left( \frac{1}{\mu(\Omega_1)} \int_{\Omega_1} F(t+a) \, d\mu \right) \\
\geq (\mu(\Omega) - \mu(\Omega_1)) \left[ G \left( \frac{\int_{\Omega} a \, d\mu - \int_{\Omega_1} a \, d\mu}{\mu(\Omega) - \mu(\Omega_1)} \right) \\
+ t \cdot (GF)' \left( \frac{\int_{\Omega} a \, d\mu - \int_{\Omega_1} a \, d\mu}{\mu(\Omega) - \mu(\Omega_1)} \right) \right].
\]

Proof. Take \( \Omega \setminus \Omega_1 = \Omega_2 \). Then concavity of \( G \) ensures
\[
\mu(\Omega) \cdot G \left( \frac{1}{\mu(\Omega)} \int_{\Omega} F(t+a) \, d\mu \right) - \mu(\Omega_1) \cdot G \left( \frac{1}{\mu(\Omega_1)} \int_{\Omega_1} F(t+a) \, d\mu \right) \\
\geq \mu(\Omega_2) \cdot G \left( \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} F(t+a) \, d\mu \right). \tag{4.3}
\]
Concavity of \( G \) implies (see [4], p.51) that
\[
\mu(\Omega_2) \cdot G \left( \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} F(t+a) \, d\mu \right) \geq \int_{\Omega_2} GF(t+a) \, d\mu.
\]
Now, convexity of \( GF \) implies
\[
\int_{\Omega_2} GF(t+a) \, d\mu \geq \mu(\Omega_2) \cdot GF \left( \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} (t+a) \, d\mu \right).
\]
Using (2.4) and the equality \( \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} (t+a) \, d\mu = t + \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} a \, d\mu \), we claim that
\[
GF \left( t + \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} a \, d\mu \right) \geq GF \left( \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} a \, d\mu \right) \\
+ t (GF)' \left( \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} a \, d\mu \right). \tag{4.4}
\]
Finally, (4.3) and (4.4), together with the well-known properties
\( \mu(\Omega \setminus \Omega_1) = \mu(\Omega) - \mu(\Omega_1) \) and \( \int_{\Omega \setminus \Omega_1} a \, d\mu = \int_{\Omega} a \, d\mu - \int_{\Omega_1} a \, d\mu \) give (4.2). \( \square \)

Theorem 7 contains (4.1) as a particular case.

Remark 3. Notice that convexity of \( GF \), assumed in (4.4), implies
\[
\mu(\Omega) \cdot G \left( \frac{1}{\mu(\Omega)} \int_{\Omega} F(t+a) \, d\mu \right) - \mu(\Omega_1) \cdot G \left( \frac{1}{\mu(\Omega_1)} \int_{\Omega_1} F(t+a) \, d\mu \right) \\
\geq \mu(\Omega) \cdot GF \left( \frac{1}{\mu(\Omega)} \int_{\Omega} a \, d\mu \right) - \mu(\Omega_1) \cdot GF \left( \frac{1}{\mu(\Omega_1)} \int_{\Omega_1} a \, d\mu \right)
\]
\[ + t \cdot (GF)' \left( \frac{\int_{\Omega} a \, d\mu - \int_{\Omega_1} a \, d\mu}{\mu(\Omega) - \mu(\Omega_1)} \right) \]  (4.5)

and (4.1) can be obtained from (4.5) for \( t = 0 \).

In the similar manner as in [6], where the authors have refined Jensen’s inequality, here we give an estimate after dividing the \( n \)-tuples \((a_1, \ldots, a_n), (p_1, \ldots, p_n)\) and \((q_1, \ldots, q_n)\) into groups. Firstly, we obtain the next Theorem.

**Theorem 8.** Suppose that \( G \) and \( F \) are differentiable functions such that \( G \) is concave and \( GF \) is convex. If \( m \leq n \), then

\[
Q_n \left[ G(A_n(F(t + a_k), q)) - tA_n \left( (GF)' \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right.
- GF \left( q_n P_n \frac{A_n(a, p)}{p_n Q_n} \right) \]
- \( Q_m \left[ G(A_m(F(t + a_k), q)) - tA_m \left( (GF)' \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right) \right), q \right) \)
- GF \left( q_m P_m \frac{A_m(a, p)}{p_m Q_m} \right) \]
\[ \geq \sum_{k=m}^{n-1} Q_k \left[ GF \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right) - \left( \frac{q_{k+1} P_k}{p_{k+1} Q_k} A_k(a, p) \right) \right]. \]

*Proof.* No generality is lost if we separate each of \( n \)-tuples \((a_1, \ldots, a_n), (p_1, \ldots, p_n)\) and \((q_1, \ldots, q_n)\) by the choosing the first \( m \) members in the first group. For example: \((a_1, \ldots, a_m, a_{m+1}, \ldots, a_n)\).

Then the first inequality is

\[
Q_n \left[ G(A_n(F(t + a_k), q)) - tA_n \left( (GF)' \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right.
- GF \left( q_n P_n \frac{A_n(a, p)}{p_n Q_n} \right) \]
\[ \geq Q_{n-1} \left[ G(A_{n-1}(F(t + a_k), q)) - tA_{n-1} \left( (GF)' \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right.
- GF \left( q_{n-1} P_{n-1} \frac{A_{n-1}(a, p)}{p_{n-1} Q_{n-1}} \right) \]
\[ + Q_{n-1} \left[ GF \left( \frac{q_{n-1} P_{n-1}}{p_{n-1} Q_{n-1}} A_{n-1}(a, p) \right) - GF \left( \frac{q_n P_n}{p_n Q_n} A_{n-1}(a, p) \right) \right]. \]

The second one is

\[
Q_{n-1} \left[ G(A_{n-1}(F(t + a_k), q)) - tA_{n-1} \left( (GF)' \left( \frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right.
- GF \left( q_{n-1} P_{n-1} \frac{A_{n-1}(a, p)}{p_{n-1} Q_{n-1}} \right) \]
\[ + Q_{n-1} \left[ GF \left( \frac{q_n P_n}{p_n Q_n} A_{n-1}(a, p) \right) - GF \left( \frac{q_{n-1} P_{n-1}}{p_{n-1} Q_{n-1}} A_{n-1}(a, p) \right) \right]. \]
When \( p/a \), then the positive weights \( n_l \) and \( m_k \) satisfy conditions given in Theorem 8. Therefore it could be positive, negative or zero. When \( p/q \), the sum is zero and we obtain the following Corollary.

Corollary 7. Suppose \( G \) and \( F \) fulfill the assumptions of Theorem 8. If \( n \geq m \), then

\[
Q_m \left[ \frac{q_m p_m}{p_m Q_m} A_m(a, p) \right] = \left( \frac{q_{m+1} p_{m+1}}{p_{m+1} Q_{m+1}} A_{m+1}(a, p) \right) - \left( \frac{q_k p_k}{p_k Q_k} A_k(a, p) \right)
\]

The sum \( \sum_{k=m}^{n-1} Q_k \left[ \frac{q_k p_k}{p_k Q_k} A_k(a, p) \right] - \left( \frac{q_{k+1} p_{k+1}}{p_{k+1} Q_{k+1}} A_{k+1}(a, p) \right) \) depends on the positive weights \( p/a \) and \( q \). Therefore it could be positive, negative or zero.

Finally, we state the following Corollary in the case of disjunct separation of the \( n \)-tuples as: \( (a_1, \ldots, a_n) = (a_1, \ldots, a_m, a_{m+1}, \ldots, a_n) \).

Corollary 8. Suppose that \( G(u) \) and \( F(u) \) satisfy conditions given in Theorem 8. If \( l + m = n \) and \( (b_1, \ldots, b_l, c_1, \ldots, c_m) = (a_1, \ldots, a_n) \), then

\[
Q_l \left[ \frac{q_l p_l}{p_l Q_l} A_l(F(t + b_k), p) \right] - t A_l \left( \frac{q_k p_k}{p_k Q_k} A_k(b, p) \right)
\]
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\[ -GF(A_1(b, p)) + Q_m \{ G(A_m(F(t + c_k), q)) - t A_m \left( (GF)'(A_k(c, p)), p \right) \} \]
\[ \leq 2 \cdot Q_n \{ G(A_n(F(t + a_k), p)) - t A_n \left( (GF)'(A_k(a, p)), p \right) \} - GF(A_n(a, p)) \} \].

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