EXOTIC SPACES IN QUANTUM GRAVITY I: EUCLIDEAN QUANTUM GRAVITY IN SEVEN DIMENSIONS

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Abstract. It is well known that in four or more dimensions, there exist exotic manifolds; manifolds that are homeomorphic but not diffeomorphic to each other. More precisely, exotic manifolds are the same topological manifold but have inequivalent differentiable structures. This situation is in contrast to the uniqueness of the differentiable structure on topological manifolds in one, two and three dimensions. As exotic manifolds are not diffeomorphic, one can argue that quantum amplitudes for gravity formulated as functional integrals should include a sum over not only physically distinct geometries and topologies but also inequivalent differentiable structures. But can the inclusion of exotic manifolds in such sums make a significant contribution to these quantum amplitudes? This paper will demonstrate that it will. Simply connected exotic Einstein manifolds with positive curvature exist in seven dimensions. Their metrics are found numerically; they are shown to have volumes of the same order of magnitude. Their contribution to the semiclassical evaluation of the partition function for Euclidean quantum gravity in seven dimensions is evaluated and found to be nontrivial. Consequently, inequivalent differentiable structures should be included in the formulation of sums over histories for quantum gravity.

1 Introduction

The sum over histories formulation of quantum amplitudes is a method suited to the study of many issues in field theory, especially those involving nonperturbative effects. This approach is particularly useful in the study of quantum gravity, as it provides a natural connection between the quantum mechanics of gravity and its classical limit. Many interesting results have been formulated and calculated using such sums over histories, for example certain proposals for initial conditions for the wavefunction of the universe [1,2] and tunnelling amplitudes for pair creation of black holes [3,4,5,6].

In particular, in Euclidean quantum gravity, a history formally consists of a manifold $M^n$ and a riemannian metric $g$ on that manifold. The partition function, for example, is then given by

$$ Z = \sum_{(M^n,g)} \exp(-I[g]) \quad (1.1) $$

where the sum is over physically distinct riemannian histories weighted by the euclidean action,

$$ I[g] = -\frac{1}{16\pi G} \int (R - 2\Lambda) d\mu(g). \quad (1.2) $$

Other quantities such as the wavefunction of the universe can be similarly constructed in terms of a sum over a suitable set of physically distinct riemannian histories. Clearly these formulations are

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somewhat heuristic as the many issues involved with the concrete specification of these sums are not addressed. Nonetheless, these formulations directly include the contribution from histories with both different topologies and different geometries. Thus they provide a natural starting point for consideration of many issues in quantum gravity, especially those involving topology.

Indeed the semiclassical evaluation of such quantum amplitudes has been used as the starting point for studying the qualitative effects of quantum gravity and topology change [see for example Refs. 7,8 as well as the preceding references]. This work has produced interesting insights into possible large scale effects of quantum gravity even though it is clear that the results rely on known solutions of the theory. Given this success, one is motivated to return to the formal expression for the sums over histories in (1.1) and attempt to formulate it more carefully. One of the first questions that naturally arises is, what properties characterize a physically distinct history?

This question, as discussed at length in section 2 of [9], is a difficult one; a complete specification of the set of histories and the measure used in expressions such as (1.1) is unknown (and perhaps nonexistent for Einstein gravity as such a specification is equivalent to a quantization of the theory). Nonetheless, it is useful and possible to address this question in the context of the histories relevant in semiclassical approximation as such expressions should be related to the low energy limit of a correct quantum theory of gravity. It is clear that riemannian metrics $g$ not related by coordinate transformations are physically distinct. It is also clear that manifolds that have different topology are also physically distinct. In one, two and three dimensions, these two properties suffice to characterize physically distinct histories. However, in four or more dimensions there is a third property that characterizes physically distinct spaces; the specification of a differentiable structure on the manifold. Manifolds with the same topology can have inequivalent differentiable structures in four or more dimensions. Such manifolds are termed exotic manifolds. The inequivalent differentiable structures will lead to different properties for physics on these manifolds. Therefore as argued in [9], not only different topologies, but also all inequivalent differentiable structures on each topology should be considered in the specification of physically distinct histories.

The appearance of exotic manifolds as riemannian histories for quantum gravity in four or more dimensions should not be too surprising. Similar arguments for the inclusion of inequivalent differentiable structures were proposed in the context of Kaluza - Klein theories by Freund [11,12]. The issue of exotic manifolds has also been raised by Brans in the context of classical gravity [13]. However, a key issue not addressed in these previous arguments for the inclusion of inequivalent differentiable structures is their significance: will the possibility of having inequivalent differentiable structures on a given manifold make a difference to the results of a calculation?

In this paper we will demonstrate that it can; the inclusion of exotic manifolds is significant in the semiclassical limit of functional integrals for quantum gravity. We do so by analyzing a particular set of exotic manifolds; the Wallach spaces $M_{-42652,61213}$ and $M_{-56788,5227}$. These manifolds are simply connected homogeneous spaces in seven dimensions [14]. They admit Einstein metrics with positive curvature [15]. Furthermore, Kreck and Stolz [16] showed that in this family, one can find homeomorphic manifolds that are not diffeomorphic; in particular the Wallach spaces $M_{-42652,61213}$ and $M_{-56788,5227}$ are a pair of such exotic manifolds.

We will numerically solve the solutions for the Einstein metrics with positive curvature in seven dimensions. We find that there are actually two such Einstein metrics on each space. The Euclidean actions for these Einstein metrics are the same order of magnitude. These actions on both exotic Wallach spaces are also of the same order of magnitude. Finally they are also comparable to those of Wallach spaces with the same homotopy type. As these Einstein metrics are stationary points of (1.2), they contribute in semiclassical evaluations of the partition function for gravity. Therefore, given that these exotic manifolds contribute in the semiclassical limit, a sum over histories formulation of quantum amplitudes for gravity must include a sum over inequivalent differentiable structures in addition to those over distinct geometries and topologies. Furthermore, as the Wallach spaces are simply connected, restricting the topologies summed over in (1.1) to simply connected spaces as suggested by some authors [see for example Ref. 17
and references therein], cannot remove their contribution. Thus the importance of inequivalent differentiable structures must not be discounted in the computation of quantum amplitudes.

Section 2 provides a summary and discussion of topology and differentiable structure. The topology, geometry and differentiable structure of Wallach spaces are presented in Section 3. Section 4 gives the proof of existence of two Einstein metrics with positive curvature on each Wallach space. It then presents the numerical solutions for these metrics. The contribution of these exotic manifolds to the partition function in the semiclassical limit is analyzed in section 5. Finally, a table of numerical solutions for the Einstein metrics for the Wallach spaces of of the same homotopy type as these exotic manifolds is given in the Appendix.

2 Basics

In order to discuss the consequences of exotic manifolds as riemannian histories in sums over histories for quantum gravity, it is necessary to have precise definitions. We will begin with summary of these definitions and then discuss their meaning and consequences to this paper.

The topology of a riemannian history is that of a metrizable space corresponding to a smooth manifold. A metrizable space is one for which open sets can be defined in terms of a distance function. A distance function is a real valued symmetric function for which given any two points $x, y$ in the space, $d(x, y) = d(y, x) \geq 0$ with the equality holding if and only if $x = y$ and the triangle inequality holds. Then

**Definition 1.** A metrizable space $M^n$ is a smooth manifold if

1. every point has a neighborhood $U_\alpha$ which is homeomorphic to a subset of $\mathbb{R}^n$ via a mapping $\phi_\alpha : U_\alpha \to \mathbb{R}^n$.
2. Given any two neighborhoods with nonempty intersection, the mapping
   \[
   \phi_\beta \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)
   \]
   is a smooth mapping between subsets of $\mathbb{R}^n$.

The condition that the space be metrizable is exactly equivalent to restricting the space to be paracompact and Hausdorff as used in other definitions. A manifold that satisfies condition (1) only is a topological manifold. A manifold such that the mapping in (2) is $C^k$ differentiable\(^1\) with $k \geq 1$ instead of smooth is called a $C^k$ differentiable manifold. One such that the mapping in (2) is PL homeomorphism called a PL manifold.\(^2\) A smooth manifold with boundary is one for which condition (1) is replaced by the requirement that every point has a neighborhood $U_\alpha$ which is homeomorphic to a subset of the upper half plane $\mathbb{R}^n_+$. The set $\{(U_\alpha, \phi_\alpha)\}$ is called the atlas of the manifold. Each element $(U_\alpha, \phi_\alpha)$ in the atlas is called a chart.

A given topological manifold can have many different atlases. For example, one can produce a different atlas by choosing different mappings $\phi'_\alpha$ for the same neighborhoods. One can also subdivide a neighborhood $U_\alpha$ into new neighborhoods with nonempty intersection and use the restriction of $\phi_\alpha$ to each new neighborhood to define the new charts. Clearly the manifolds defined with these atlases are equivalent. One can more precisely formulate this equivalence in terms of homeomorphisms and diffeomorphisms:

**Definition 2.** A homeomorphism is a continuous invertible map $h : M^n \to N^n$ such that its inverse $h^{-1} : N^n \to M^n$ is also continuous.

**Definition 3.** A diffeomorphism is a homeomorphism $h : M^n \to N^n$ such that $h$ is a differentiable invertible map whose inverse $h^{-1} : N^n \to M^n$ is also differentiable.

A homeomorphism characterizes the topological equivalence of the two manifolds. A diffeomorphism also characterizes the topological equivalence, but in addition preserves the additional

\(^1\)A $C^k$ differentiable map is a continuous map with $k$ continuous derivatives.

\(^2\)A map $f : M^n \to N^n$ between two manifolds $M^n$ and $N^n$ is PL if each point $p$ in $M^n$ has a cone neighborhood $N = pS^{n-1}$ such that $f(\lambda p + \mu x) = \lambda f(p) + \mu f(x)$ where $x$ is in $S^{n-1}$ and $\lambda, \mu \geq 0, \lambda + \mu = 1$. A PL homeomorphism is a continuous invertible PL map with continuous inverse.
structure carried in the atlases of the manifolds. This additional structure is physical; inequivalent differentiable structures will result in different values for physical quantities. Thus the appropriate notion of physically equivalent manifolds for physics is that of a diffeomorphism.

With Definition 3, one can show that a $C^1$ atlas is $C^1$ diffeomorphic to a smooth atlas [18].

Therefore, questions regarding the properties of atlases can be restricted to a consideration of smooth atlases without loss of generality.

The atlas on a manifold provides the information for determining the smoothness and differentiability of functions and other quantities. The differentiability of a function is given by the differentiability of its pullback by $\phi_{\alpha}$ to each neighborhood $U_{\alpha}$ in $\mathbb{R}^n$. The determination of the differentiability of the pullback of the function then proceeds as in ordinary multivariable calculus. Consequently, the differentiability of a function will depend not only on its form but also on the atlas.

A simple example of how this works is provided by choosing different atlases on the real line. Denote $M = \mathbb{R}$ with atlas consisting of one chart $(\mathbb{R}, \phi)$ where $\phi(x) = x$. This first atlas is just the identity map to $\mathbb{R}$.

Let $N = \mathbb{R}$ with atlas consisting of one chart $\{\mathbb{R}, \psi\}$ where $\psi(x) = x^{\frac{3}{2}}$.

Observe that both of these atlases are smooth. There is only one chart for each; thus condition (2) in Def. 1 is satisfied trivially.

Next consider the function $f : M \to \mathbb{R}$ given by $f(x) = x^{\frac{3}{2}}$. This $f$ is not smooth on $M$.

Recall that the analysis of smoothness must be carried out in the chart. Now the pullback of $f$ to the chart of $M$, $f \cdot \phi^{-1} : \mathbb{R} \to \mathbb{R}$, is $f \cdot \phi^{-1}(y) = f(y) = y^{\frac{3}{2}}$. This function is continuous, but not everywhere differentiable. Thus $f$ is not smooth on $M$.

By the same analysis observe that the function $g : N \to \mathbb{R}$ given by $g(x) = x^{\frac{3}{2}}$ is smooth. The pullback of the function to the chart of $N$ is $g \cdot \psi^{-1}(y) = g(y^3) = y^2$. It is clear that this pullback is smooth. Thus although $f$ is not a smooth function on $M$, $g$ is a smooth function on $N$.

Now although the sets of smooth functions on $\mathbb{R}$ depend on the atlas, the differentiable structure of the real line for both atlases is equivalent; $M$ is diffeomorphic to $N$, $h : M \to N$ with $h(x) = x^{\frac{3}{2}}$. This implies that a function that is smooth in $N$ will be diffeomorphic to a smooth function in $M$.

In the above case, $g$ will correspond to the smooth function $g \cdot h^{-1} : M \to \mathbb{R}$ given by $g \cdot h^{-1}(x) = g(x^3) = x^2$. Pulling $g \cdot h^{-1}$ back to the coordinate chart yields $g \cdot h^{-1} \phi^{-1}(y) = y^2$; clearly a smooth function. Thus although a smooth function with respect to one atlas will be diffeomorphic to a smooth function in another atlas, its form will change. This is no surprise; a change of coordinates generically has this effect.

Now it is intuitively obvious that different atlases on the same topological manifold will result in the same physics if they are related by a diffeomorphism. However, will all possible choices of atlases on a given topological manifold be diffeomorphic to each other? If $M^n$ were homeomorphic to $N^n$ but not diffeomorphic, then $M^n$ and $N^n$ would be inequivalent. This inequivalence would not be characterized by their topology but rather carried in their atlases. Even so, it would be reflected in the physics observed on each space. For example, the set of smooth functions on $M^n$ would not correspond to the set of smooth functions on $N^n$; any homeomorphism $h : M^n \to N^n$ will map smooth functions on $M^n$ to continuous ones on $N^n$. Thus a smooth solution to Laplace’s equation on $M^n$ would only be continuous in $N^n$ and consequently not be a smooth solution on that space. Thus the inequivalence of the differentiable structures of $M^n$ and $N^n$ would be reflected in the basic properties of the space such as its geometry and the spectra of differential operators such as the Laplacian.

Clearly, the first question is whether this is an issue at all; are all atlases on a given topological manifold diffeomorphic? The answer depends on dimension. In one, two and three dimensions one can show that all manifolds admit a differentiable structure. Furthermore any two manifolds that are homeomorphic are also diffeomorphic [19]. Thus any two atlases on a given topological

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3This example is very degenerate. Note that as all spaces are $\mathbb{R}$, the mappings are functions. Thus we can use the standard notation for functions on $\mathbb{R}$ to describe them. Furthermore cartesian coordinates can be used to explicitly characterize points on all copies of $\mathbb{R}$. We will use the convention that $x$ corresponds to points in the manifold and $y$ to the coordinates in the chart.
manifold are equivalent in these dimensions. In four or more dimensions, this is no longer the case; manifolds that are homeomorphic are not necessarily diffeomorphic:

**Definition 4.** If $M^n$ is homeomorphic to $N^n$ but not diffeomorphic, then $M^n$ and $N^n$ have inequivalent differentiable structures.

Manifolds with inequivalent differentiable structures are termed exotic manifolds.

The result that manifolds admit inequivalent differentiable structures was first shown by Milnor [20]. He explicitly constructed exotic 7-spheres which are not diffeomorphic to each other. These exotic 7-spheres $M^7_k$ are constructed from 3-sphere bundles over $S^4$. The atlas on these spaces consists of two copies of $\mathbb{R}^4 \times S^3$. The first copy is the complement of the north pole of four-sphere, the second is the complement of south pole. Choose as coordinates the quaternion pairs $(u, v)$ on each copy with $u$ the quaternionic coordinates on $\mathbb{R}^4$. Now identify the subsets $(\mathbb{R}^4 - 0) \times S^3$ of each chart under the diffeomorphism

$$(u, v) \to (u', v') = (u/||u||^2, u^h v u^j/||u||)$$

where $h$ and $j$ are integers. Intuitively the integers $h$ and $j$ describe how one bundle is twisted before it is glued to the other. By calculation of a differentiable characteristic class $\lambda(M^7)$, Milnor showed that for $(h - j)^2 \neq 1 \mod 7$, the manifold has no orientation reversing maps. Thus, as $S^7$ with its usual differentiable structure has orientation reversing maps, $M^7_k$ is not diffeomorphic to $S^7$.

Considerable work has been carried out regarding differentiable structures since Milnor’s beautiful result and much more is now known. In particular, there are topological manifolds that do not admit any differentiable structure in four or more dimensions [21]. However, such manifolds do not appear relevant to physics as a differentiable structure is required to define derivatives of fields and other physical quantities. In five or more dimensions, the number of inequivalent differentiable structures is determined by an invariant $ks(M^n)$ that characterizes whether the structure group of the topological manifold $Top(n)$ can be replaced by $PL(n)$:

**Theorem 1 [Kirby and Siebenmann [22]].** Let $M^n$ be a topological manifold with $n \geq 5$. Then $M^n$ has a PL structure if and only if the invariant $ks(M^n) \in H^4(M^n, \partial M^n; \mathbb{Z}_2)$ satisfies $ks(M^n) = 0$. Furthermore, given a continuous homeomorphism $h : M^n \to N^n$ between PL manifolds, it is equivalent to a PL homeomorphism if and only if the invariant $ks(h) \in H^4(M^n, \partial M; \mathbb{Z}_2)$ satisfies $ks(h) = 0$.

Using these results, it can be proven that there are a finite number of inequivalent PL structures on all $n$-manifolds in five or more dimensions if the cohomology is finitely generated. In particular, it implies that as all compact $n$-manifolds have finitely generated cohomology, they admit a finite number of inequivalent differentiable structures.

The above theorem classifies PL structures on the manifold. One can show that in less than eight dimensions, every PL manifold has a smoothing; that is one can smooth the PL structure to obtain a differentiable structure. In less than seven dimensions, the smoothings are unique. Thus the above theorem classifies inequivalent differentiable structures in five and six dimensions. Furthermore, it can be extended to classify the number of inequivalent differentiable structures in seven dimensions using a calculation of characteristic classes that determines the number of smoothings of a given PL manifold [22].

Theorem 1 breaks down in four dimensions. The vanishing of $ks(M^n)$ is a necessary condition, but not sufficient in this case. This is why many issues concerning differentiable structures remain open in four dimensions. However, several important facts are known. First, there are well known examples of 4-manifolds that admit more than one inequivalent differentiable structure. In fact there can be a countably infinite number of inequivalent differentiable structures on compact 4-manifolds. For example, the connected sum of complex projective space with nine copies of itself

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4See, for example, Appendix B of [10] for a summary of how the results of Freedman and Donaldson show that the topological manifold $||E8||$ has only a continuous atlas.
with opposite orientation, $\mathbb{CP}^2\#9(-\mathbb{CP}^2)$, admits a countably infinite number of inequivalent differentiable structures [22]. More surprisingly, certain open 4-manifolds, notably $\mathbb{R}^4$, admit an uncountably infinite number of inequivalent differentiable structures [see for example Ref. 21 and references therein]. This result is shocking as all other $\mathbb{R}^n$ for $n > 4$ admit a unique differentiable structure.

From the above, it is clear that exotic manifolds exist and are numerous in four dimensions. However, it is difficult to directly investigate their consequences to physics in this dimension. This is because simple constructions of exotic 4-manifolds in a form amenable to further global geometric analysis are unknown. Therefore it is worth studying the physical contributions of higher dimensional exotic manifolds as a guide to the possible physical significance of exotic manifolds in quantum gravity.

3 Wallach Spaces

A natural place for the study of physical consequences of exotic manifolds is in quantum gravity. Do these spaces contribute to semiclassical evaluations of functional integrals such as (1.1)? Do to so, they must have an Einstein metric with positive curvature. One might be tempted to begin searching for such metrics by considering exotic 7-spheres. The manifold $S^7$ with its usual differentiable structure clearly admits an Einstein metric with positive curvature. Some exotic 7-spheres are known not to admit an Einstein metric with positive curvature. However, it remains an open problem as to whether or not any of the exotic 7-spheres also admit such an Einstein metric [23; see also 11]. Fortunately, there are known examples of other 7-manifolds that admit both inequivalent differentiable structures and Einstein metrics with positive curvature. In particular, a set of homogeneous exotic 7-manifolds admitting Einstein metrics is provided by the Wallach spaces [24].

Explicitly, the Wallach spaces $M_{k,l}$ are coset spaces of the form $SU(3)/i_{k,l}(S^1)$ where $i_{k,l}$ is the embedding of $S^1$ in $SU(3)$. The generator of $i_{k,l}$ can be written in terms of the generators of the standard maximal torus as

$$b_{k,l} = \frac{i}{\sqrt{2\Gamma_{k,l}}} \begin{pmatrix} k & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & -k-l \end{pmatrix}$$

(3.1)

where $\Gamma_{k,l} = k^2 + kl + l^2$. The remaining generator is then

$$b_{k,l}^\perp = \frac{i}{\sqrt{6\Gamma_{k,l}}} \begin{pmatrix} k+2l & 0 & 0 \\ 0 & -2k-l & 0 \\ 0 & 0 & k-l \end{pmatrix}$$

(3.2)

that is $t = b_{k,l}^\perp \oplus b_{k,l}$. Insight into the construction of these spaces can be gained by considering the action of the exponential of the generator (3.1) on the subspace of $SU(3)$ corresponding to the maximal torus. This generator identifies points along a circle winding around this maximal torus; $e^{2\pi i \theta} \rightarrow \text{diag}(e^{2\pi ik\theta}, e^{2\pi il\theta}, e^{-2\pi i(k+l)\theta})$. Different choices of the integers $k, l$ result in different points being identified. In particular, the circle will wind around the maximal torus a different number of times before closing. Thus the homotopy of this circle corresponding to restriction of the action of the generator of $i_{k,l}$ to the maximal torus depends on $k, l$. Of course, this generator also acts nontrivially on the other elements of lie group. Therefore the identification of points on maximal torus really corresponds to a nontrivial identification of a whole set of coordinates on the manifold. This identification produces a particular $S^1$ bundle over the space $M_{k,l}$; that which

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5Recall that each element $g$ in a Lie group corresponds to a point in a manifold. The coset space is thus constructed by performing the identification of points in this manifold as given by the coset of the group. That is two points $g$ and $g'$ are identified if $g = hg'$ for some element $h$ in the subgroup.

6The maximal torus is the subgroup generated by the maximal number of commuting lie algebra generators of the group.
is the manifold $SU(3)$. Now as different choices of the generator $i_{k,l}$ produce different windings of the circle, it is intuitively plausible that they will produce $S^1$ bundles over different $M_{k,l}$, all of which are isomorphic to $SU(3)$. Indeed this is the case.

The tangent space at a point, $T_0 M_{k,l}$, is given by the vector fields associated with the lie algebra of the resulting coset space. One can denote these vector fields by their corresponding lie algebra elements;

$$X^0 = 2\pi \sqrt{6} h_{k,l}$$

$$X^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X^2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X^4 = \begin{pmatrix} 0 & 0 & i \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X^5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$X^6 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

The metric for this space can be written in terms of the left invariant covectors $\sigma_i$ associated with each $X^i$,

$$ds^2 = \kappa \sigma_0^2 + a(\sigma_1^2 + \sigma_2^2) + b(\sigma_3^2 + \sigma_4^2) + c(\sigma_5^2 + \sigma_6^2). \quad (3.3)$$

This space is a homogeneous space for all choices of the parameters $\kappa, a, b, c$. It is thus a seven dimensional analog of the Bianchi spaces. The bi-invariant metric on $M_{k,l}$ is that naturally induced by the group structure on $SU(3)$ by $-\text{Re}(\text{tr}(XY'))$. This metric has coefficients $\kappa = 24\pi^2 \Gamma_{k,l}, a = b = c = 2$.

From the previous discussion, it is clear that the topology of $M_{k,l}$ is determined by exactly how the circle is taken out of $SU(3)$. It turns out that several conditions on the integers $k, l$ arise when precisely formulating this removal. For the case of homogeneous manifolds admitting strictly positive sectional curvature the action of the subgroup $i_{k,l}$ must be nontrivial. Thus no diagonal term in the generator (3.1) may vanish. This restriction is enforced by the requirement $kl(k+l) \neq 0$. For $k, l$ relatively prime, this identification is on all points for $\theta$ in the range $0 \leq \theta \leq 1$. Multiplication of both integers by another integer $r$ changes the range of theta to $0 \leq \theta \leq 1/r$ but not the identification. Therefore one can restrict $k, l$ to relatively prime integers.

Additionally as $SU(3)$ contains a $Z_3$ action, integer pairs $k, l$ that are equivalent mod 3 result in a group action that contains a cyclic subgroup. For example, the exponential of (3.1) with the substitution $k = l + 3n$,

$$g_{t+3n,l}(t) = \exp\left(\frac{i t}{\sqrt{24} l} \begin{pmatrix} \exp(\frac{3 i t}{24}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(\frac{-3 i (n+b) t}{24}) \end{pmatrix} \right)$$

where $\Gamma = 3l^2 + 12nl + 9n^2$ is manifestly the cube of another group element. Thus the subgroup $i_{t+3n,t}$ contains $Z_3$. Therefore $SU(3)$ will not act effectively on a coset space $SU(3)/i_{t+3n,t}$ as its action will have fixed points.

Finally, any two coset spaces related by group conjugation are equivalent as group conjugation corresponds to a diffeomorphism on the manifold.\(^7\) This property allows one to show that $M_{k,l}$ is diffeomorphic to $M_{-k,-l}$. For example, conjugation with the element

$$g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

of elements in $i_{k,l}$ results in the subgroup $i_{l,k}$. This conjugation of the coset space $SU(3)/i_{k,l}$ will result in $SU(3)/i_{l,k}$. Additionally combining this group conjugation with the outer automorphism

\(^7\)Group conjugation by an element $g$ is given by the map $g \to ghg^{-1}$ applied to all $h \in SU(3)/i_{k,l}$. If $SU(3)$ acts effectively, this map is invertible, 1-1, onto and smooth - in other words a diffeomorphism.
given by complex conjugation will result in $SU(3)/i_{-l,-k}$. This space is also diffeomorphic to $SU(3)/i_{k,l}$. Given these equivalences it is clear that one can restrict the integers $k, l$ to the range $k < 0, l > 0$ without loss of generality.

The Wallach space $M_{k,l}$ is a simply connected manifold for all integers $k$ and $l$ satisfying the above restrictions. This can be readily proven using the fiber bundle exact sequence:

$$\ldots \pi_1(i_{k,l}) \rightarrow \pi_1(SU(3)) \rightarrow \pi_1(M_{k,l}) \rightarrow \pi_0(i_{k,l}) \rightarrow \ldots$$

Now $\pi_1(SU(3)) = 1$ as $SU(3)$ is simply connected; also $\pi_0(i_{k,l}) = 1$ as it is connected. Therefore the image of the map to $\pi_1(M_{k,l})$ is the kernel of the map to $\pi_0(i_{k,l})$; thus $\pi_1(M_{k,l}) = 1$.

Its fourth cohomology is determined from $k,l$:

**Theorem 2 (Aloff and Wallach[14]).** Suppose that $k, l$ are relatively prime. Then

$$H^4(M_{k,l}; \mathbb{Z}) = \mathbb{Z}/\Gamma_{k,l}\mathbb{Z}$$

where $\Gamma_{k,l} = k^2 + kl + l^2$.

Therefore the family of manifolds $M_{k,l}$ admits infinitely many homotopy types. A characterization of the topology of this family is given by Kreck and Stolz. They derive this characterization in the oriented category; that is the manifolds are oriented manifolds and the diffeomorphisms (and homeomorphisms) are orientation preserving.

**Theorem 3 (Kreck and Stolz 1991 [16]).** Assume that both $k, l$ and $\bar{k}, \bar{l}$ are relatively prime. Then

1. $M_{k,l}$ is homeomorphic to $M_{\bar{k},\bar{l}}$ if and only if $\Gamma_{k,l} = \Gamma_{\bar{k},\bar{l}}$ and $\lambda_{k,l} \equiv \lambda_{\bar{k},\bar{l}} \mod 2^3 \cdot 3 \cdot \Gamma_{k,l}$ where $\Gamma_{k,l} = k^2 + kl + l^2$ and $\lambda_{k,l} = kl(k + l)$.
2. $M_{k,l}$ is diffeomorphic to $M_{\bar{k},\bar{l}}$ if and only if $\Gamma_{k,l} = \Gamma_{\bar{k},\bar{l}}$ and $\lambda_{k,l} \equiv \lambda_{\bar{k},\bar{l}} \mod 2^5 \cdot 7^2 \mu(\Gamma) \cdot 3 \cdot \Gamma_{k,l}$ where $\Gamma_{k,l} = k^2 + kl + l^2$, $\lambda_{k,l} = kl(k + l)$ and

$$\mu(\Gamma) = \begin{cases} 0 & \text{if } \Gamma \equiv 0 \mod 7 \\ 1 & \text{otherwise.} \end{cases}$$

The diffeomorphism and homeomorphism classifications immediately imply that certain Wallach spaces with different values of $k, l$ are diffeomorphic. This follows from the observation that $\lambda_{k,l}$ is left unchanged by certain permutations of the integers $k, l, (k + l)$. For example

$$k, l \rightarrow -(k + l), l \quad \text{when } k + l > 0$$

$$k, l \rightarrow k, -(k + l) \quad \text{when } k + l < 0. \quad (3.4)$$

These correspond to orientation preserving diffeomorphisms. Other permutations of $k, l, (k + l)$ change the sign of $\lambda_{k,l}$, for example

$$k, l \rightarrow -l, (k + l) \quad \text{when } k + l > 0$$

$$k, l \rightarrow (k + l), -k \quad \text{when } k + l < 0. \quad (3.5)$$

These permutations correspond to orientation reversing diffeomorphisms. One can explicitly see this by finding the group element that produces this identification under group conjugation. These Wallach spaces will be distinct in the oriented category as such diffeomorphisms are excluded. However, for the purposes of this paper, one does not wish to consider these spaces as physically distinct manifolds. One can easily augment the classification scheme to handle these cases. As orientation reversing diffeomorphisms change the sign of $\lambda_{k,l}$, manifolds related by an orientation reversing diffeomorphism can be classified using $-\lambda_{k,l}$ in place of $\lambda_{k,l}$ in the equivalences in Theorem 3.

This result of Kreck and Stolz allows one to answer an interesting question; are there Wallach spaces that are homeomorphic but not diffeomorphic? The answer is yes; a difficult calculation involving recasting the computation of the modular equivalence in Theorem 3 in terms of number theory results in
Corollary (Kreck and Stolz 1991 [16]). If \( M_{k,l} \) and \( M_{k,l} \) are homeomorphic Wallach spaces with \( \Gamma_{k,l} < 2955367597 \), then \( M_{k,l} \) is diffeomorphic to \( M_{k,l} \). On the other hand, the Wallach spaces \( M_{56788,5227} \) and \( M_{-42652,61213} \) are homeomorphic but not diffeomorphic. The order of their fourth cohomology is 2955367597.

Thus the Wallach spaces with fourth cohomology of order 2955367597 are a natural set to study for understanding the contribution of exotic manifolds to quantum gravity.

4 Einstein Metrics

An Einstein metric with positive scalar curvature satisfies \( R_{ab} = E g_{ab} \) for some positive constant \( E \). All Wallach spaces with \( k,l \) such that \( k \neq l \mod 3 \) admit an Einstein metric [15]. However, this method of proof cannot be applied to compute the metric numerically for a given \( k,l \). Thus it is useful to have an alternate proof of this result; it will show that actually two Einstein metrics exist for a given \( k,l \). Furthermore, this derivation can be implemented numerically to solve for the metrics. These will be computed at the end of this section.

The derivation of the Ricci components of the metric (3.3) can be carried out using standard techniques for coset spaces [24]. Alternately one can find these components from the metric using Cartan calculus to compute the curvature of the appropriate form of the metric of \( SU(3) \) and the Gauss-Codazzi equations to compute the curvature of the space orthogonal to the generator of \( i_{k,l} \). Fixing the overall scale of the metric by the condition \( \kappa = 24 \pi^2 \Gamma \) where \( \Gamma = \Gamma_{k,l} \), one finds

\[
E = 3 \Gamma \left( \frac{(k+l)^2}{a^2} + \frac{l^2}{b^2} + \frac{k^2}{c^2} \right) \quad (4.1)
\]

\[
E = -\frac{3(k+l)^2}{\Gamma a^2} + \frac{a^2 - c^2 - b^2 + 2bc}{abc} \quad (4.2)
\]

Eliminating \( E \) by substitution of (4.1) in (4.2) results in

\[
\frac{3}{\Gamma} \left( \frac{(k+l)^2}{a^2} + \frac{3l^2}{b^2} + \frac{2k^2}{c^2} \right) = \frac{6}{a} + \frac{6}{b} - \frac{2c}{ab} \quad (4.3)
\]

\[
\frac{3}{\Gamma} \left( \frac{2(k+l)^2}{a^2} + \frac{3l^2}{b^2} + \frac{3k^2}{c^2} \right) = \frac{6}{b} + \frac{6}{c} - \frac{2a}{bc} \quad (4.4)
\]

\[
\frac{3}{\Gamma} \left( \frac{3(k+l)^2}{a^2} + \frac{2l^2}{b^2} + \frac{3k^2}{c^2} \right) = \frac{6}{a} + \frac{6}{c} - \frac{2b}{ac} \quad (4.5)
\]

Note that these three equations are related under simultaneous cyclic permutations of \( a, b, c \) and \((k+l), l, k\). This is a reflection of the group structure of the Wallach space. One can conveniently reexpress this set of equations; defining

\[
x = \frac{c}{a} \quad y = \frac{c}{b} \quad (4.6)
\]

one finds

\[
c = \frac{3}{\Gamma} \left( \frac{3(k+l)^2 x^2 + 3l^2 y^2 + 2k^2}{6x + 6y - 2xy} \right) \quad (4.7)
\]

where \( x \) and \( y \) are solutions of

\[
(6x + 6y - 2xy)(2(k+l)^2 x^2 + 3l^2 y^2 + 3k^2) - (3(k+l)^2 x^2 + 3l^2 y^2 + 2k^2)(6y + 6 - \frac{y}{x}) = 0 \quad (4.8)
\]

\[
(6x + 6y - 2xy)(3(k+l)^2 x^2 + 2l^2 y^2 + 3k^2) - (3(k+l)^2 x^2 + 3l^2 y^2 + 2k^2)(6x + 6 - \frac{x}{y}) = 0. \quad (4.9)
\]
Positive solutions to the above set of equations yield Einstein metrics.

These equations inherit a useful symmetry from the Wallach spaces; equation (4.6) is equivalent to (4.7) under the simultaneous exchange of \( \{x, y\} \rightarrow \{y, x\} \) and \( \{(k + l)^2, l^2\} \rightarrow \{l^2, (k + l)^2\} \). Therefore knowledge of the behavior of one of these equations for generic coefficients is applicable to the other. This allows a simple method to show the existence of a solution. Note that (4.6) and (4.7) are polynomials in both \( x \) and \( y \) of degree less than five. Thus one can solve for the roots of each equation for either variable. Roots obtained by treating \( y \) as the dependent variable will be parameterized in terms of \( x \) and conversely. Thus these roots generate curves in the \( xy \) plane. The simultaneous solutions of these two equations are the points of intersection of these curves. Einstein metrics (if there are any) are at points of intersection with both \( x \) and \( y \) positive. By analyzing these roots, one can prove that there are two positive intersections of these curves. Therefore there are two Einstein metrics on each Wallach space.

We now proceed to do so; observe that (4.6) can be rewritten explicitly as a cubic polynomial in \( y \) with \( x \) dependent coefficients;

\[
\alpha(x)y^3 + \beta(x)y^2 + \gamma(x)y + \delta(x) = 0 \tag{4.8}
\]

where

\[
\begin{align*}
\alpha(x) &= 6l^2(x^2 - 1) \\
\beta(x) &= 18l^2x(1-x) \\
\gamma(x) &= 4(k + l)^2x^4 + 6(k + l)^2x^3 + 6(k^2 - (k + l)^2)x^2 - 6k^2x - 4k^2 \\
\delta(x) &= -12(k + l)^2x^4 + 18(k + l)^2x^3 - 18k^2x^2 + 12k^2x
\end{align*}
\]

Cubic polynomials always have at least one real root. Furthermore as the coefficients of (4.8) are polynomials in \( x \), the roots are piecewise continuous functions of \( x \). The behavior of these roots as curves in the \( xy \) plane are controlled by the following:

(1) The roots of its polynomial coefficients in \( x \) will provide an upper bound on the number of positive roots of (4.6) for all values of \( x \). Those of \( \alpha(x) \) and \( \beta(x) \) are obvious. The coefficient \( \gamma(x) \) has one nonnegative zero at \( x = x_\gamma \) with \( \gamma(x) < 0 \) for \( x < x_\gamma \) and \( \gamma(x) > 0 \) otherwise; \( \delta(x) \) has two nonnegative zeros at \( x = 0 \) and \( x = x_\delta \) with \( \delta(x) > 0 \) \( 0 < x < x_\delta \) and \( \delta(x) < 0 \) otherwise. Both \( x_\gamma \) and \( x_\delta \) depend only on the integers \( k, l \). Furthermore, one has that \( \gamma(1) = \frac{4}{3}\delta(1) \). From this one can deduce that the nonnegative roots of (4.8) fall into either of two cases: (1) \( x_\gamma < 1, x_\delta > 1 \) corresponding to \( l > -2k \) or (2) \( x_\gamma > 1, x_\delta < 1 \) corresponding to \( l < -2k \). Now Descartes’ rule of signs for the number of solutions of (4.8) as a function of \( y \) with coefficients parameterized by \( x \) yields

(i) \( 0 < x < x_\gamma \) \( 3 \) positive roots and \( 0 \) negative roots, \( x_\gamma < x < 1 \) \( 1 \) positive root and \( 2 \) negative roots, \( 1 < x < x_\delta \) \( 2 \) positive roots and \( 1 \) negative root, \( x_\delta < x \) \( 3 \) positive roots and \( 0 \) negative roots.

(ii) \( 0 < x < x_\delta \) \( 3 \) positive roots and \( 0 \) negative roots, \( x_\delta < x < 1 \) \( 2 \) positive roots and \( 1 \) negative root, \( 1 < x < x_\gamma \) \( 1 \) positive root and \( 2 \) negative roots, \( x_\gamma < x \) \( 3 \) positive roots and \( 0 \) negative roots.

These results set the behavior of the roots as parameterized by \( x \). In particular, there is at least one positive root for \( 0 < x < x_\gamma \) and \( x > x_\delta \) \( \left( 0 < x < x_\delta \text{ and } x > x_\gamma \right) \).

(2) One can easily solve for \( y \) at certain points, namely \( y = 0, \pm i \sqrt{\frac{2k^2}{3l^2}} \) at \( x = 0 \) and \( y = 3, \pm i \sqrt{\frac{2m^2}{3l^2}} \) as \( x \to \infty \). Additionally, one root of (4.8) at \( x = x_\delta \) is \( y = 0 \). Note that \( y = 0 \) only at \( x_\delta \) and 0. Additionally at \( x = 1 \), one root is \( y = -\frac{3}{2} \). As one approaches this
point, there are two behaviors for any real solution of (4.8); either \( y \) approaches \( y = -\frac{3}{2} \) or \( y \to \infty \).

(3) One can show that \( \frac{dy}{dx} = 3 \) at \( x = 0, y = 0 \) by taking the derivative of the equation (4.8).

Therefore \( y \) is an increasing function of \( x \) there implying that there is one positive root in a neighborhood of this point.

Fourth, the discriminant of the cubic equation is negative except possibly between \( x_3 < x < x_\gamma \) for case (1) and \( x_\gamma < x < x_\delta \) for case (2). Therefore there is only one real root outside these ranges of \( x \). One can deduce from the results of Descartes’ rule of signs that this root is positive outside these ranges. Furthermore, there are no other positive roots for any value of \( x \).

Putting these facts together one sees that there are two behaviors for the roots of (4.8) as a function of \( x \) depending on the values of \( k, l \). For case (1), there is a nonnegative root which is an increasing function of \( x \) with initial slope 3 at the point \( x = 0, y = 0 \) that increases to infinity as \( x \to 1 \). For \( x > 1 \) there is a root that crosses the \( y \) axis at \( x = x_3 \) and increases to \( y = 3 \) as \( x \to \infty \). (This root continues to negative values; at \( x = 1 \) it has value \( y = -\frac{3}{2} \).) This case is illustrated in Figure 1. For case (2), there is a nonnegative root which is an increasing function of \( x \) with slope 3 at the point \( x = 0, y = 0 \) that turns around and crosses the \( x \) axis at \( x_\delta \). (It continues to negative values to \( x = 1, y = -\frac{3}{2} \).) For \( x > 1 \) there is a root that is asymptotically decreasing from infinity to a minimum value, then increases to \( y = 3 \) as \( x \to \infty \). This case is illustrated in Figure 2.

Now, having characterized the behavior of the nonnegative roots of (4.6), one has also done so for the nonnegative roots of (4.7) as these equations are related simply by an exchange of \( \{x, y\} \to \{y, x\} \) and \( \{(k + l)^2, l^2\} \to \{l^2, (k + l)^2\} \). Note that under the exchange of parameters, case (1) behavior may turn into case (2) behavior. Therefore there are four possible combinations to analyze when searching for intersections of the curves of zeros. However it is easy to see that each combination has exactly two intersections.

For example, take the case of the Wallach space \( M_{-40388.61811} \) illustrated in Figure 3. The exchange of \( l \) and \( (k + l) \) takes a case (1) curve into a case (2) curve. In the region of the \( xy \) plane for which \( 0 < x < 1 \), the root of (4.6) has initial slope 3, that of (4.7) has initial slope \( \frac{1}{3} \). Therefore the second root starts under the first. The first root goes to infinity as \( x \to 1 \); the second crosses the \( y \) axis at finite \( y \). It follows that these curves must intersect somewhere in this region of the \( xy \) plane. Therefore there is an Einstein metric with \( 0 < x < 1 \). Next consider the region in the \( xy \) plane with \( x > 1 \). The root of (4.6) is monotonically increasing from \( y = 0 \) at \( x = x_3 \) to \( y = 3 \) as \( x \to \infty \). The root of (4.7) is always positive. It approaches \( y = 1 \) as \( x \to \infty \). Also \( y \to \infty \) as \( x \to 3 \). As (4.6) begins from zero and approaches an asymptote that is greater than that of (4.7) as \( x \to \infty \), these curves must intersect. Therefore there is a second Einstein metric with \( x > 1 \). Thus there are two Einstein metrics on each Wallach space.

One can similarly analyze each of the remaining three combinations of roots and show that there are two intersections in each of these. For example, the Wallach space \( M_{-56788.5227} \) is illustrated in Figure 4. The behavior of the roots is slightly different, but it is straightforward to see that again there will be two intersections. Therefore, each Wallach space has two Einstein metrics.

Finally, one can show with more work that there can be no more than two Einstein metrics on each Wallach space. Additional positive roots can only occur in the regions \( 0 < x < x_\gamma \) and \( x_\delta < x \) for case (1) or \( 0 < x < x_\delta \) and \( x_\gamma < x \) for case (2). For such a root to occur, one must have a double positive root in this region. Comparison of the form of (4.8) to the desired form \( (x - x_1)(x - B)^2 = 0 \) where \( x_i \) is the known positive root leads to an overdetermined set of
equations for $B$;

\[ 3B^2 + 2\frac{\beta}{\alpha}B + \frac{\gamma}{\alpha} = 0 \]
\[ 2B^3 + \frac{\beta}{\alpha}B^2 - \frac{\delta}{\alpha} = 0. \]

A simultaneous solution to both is needed for the double positive root. Necessary conditions for this to occur are given by the discriminants of the above equations. One finds that

\[ \left(\frac{\beta}{\alpha}\right)^2 > 3\frac{\gamma}{\alpha} \]
\[ \left(\frac{\beta}{\alpha}\right)^3 < 27\frac{\delta}{\alpha} \]

One can bound the values of both sides of these two equations using the properties of the coefficients in the regions $0 < x < 1$ and $1 < x$ of the $xy$ plane. From these bounds one can constrain the possible values of $k, l$ for which a simultaneous solution might occur. For example, one finds that for the region $0 < x < 1$, bounding the coefficients $\beta, \gamma, \delta$ by their minimum value and $\alpha$ by its maximum value leads to the relation $-\frac{16}{17}k < l < -\frac{16}{7}k$. Thus if one has a Wallach space such that both $l < -k$ and $k + l < -k$, this relation will never be satisfied and no double root exists. However, the diffeomorphism equivalence of different permutations of $k, l, (k + l)$ can be used to choose a set of integers $k, l$ for each diffeomorphism equivalent Wallach space that satisfies this constraint. Therefore, there are no additional positive roots to (4.8). Thus the two solutions found before are the only Einstein metrics on each Wallach space.

The above proof justifies the derivation of the Einstein metrics for the Wallach spaces by numerical methods. The actual values $x, y$ at which the curves intersect can be solved for using the Newton-Raphson method [see for example Ref. 25]. The corresponding values of $a, b, c$ and $E$ can then be calculated using (4.1), (4.4) and (4.5). This procedure gives metrics with different values of $E$ as the derivation of the equations (4.6) and (4.7) did not hold $E$ fixed. However, one can find the corresponding Einstein metrics with fixed curvature by rescaling the metric by $\lambda$, that is

\[ \kappa, a, b, c \to \lambda\kappa, \lambda a, \lambda b, \lambda c \]  

(4.10)

Observe that under this rescaling,

\[ R_{ab} \to \frac{1}{\lambda} R_{ab} \]
\[ E \to \frac{E}{\lambda^2}. \]  

(4.11)

Therefore, one can rescale a metric with a curvature constant of $E$ to a curvature constant of 1 by choosing $\lambda^2 = E$. Thus one can find the Einstein metrics of constant curvature 1 by first numerically solving for $x, y$, computing the corresponding values of $a, b, c$ and $E$, then rescaling the metric using $\lambda = \sqrt{E}$ in (4.10).

Using the above, the metrics and volume for the homeomorphic exotic Wallach spaces $M_{-56788,5227}$ and $M_{-42652,61213}$ are summarized in Table 1. The volume is given in terms of $\beta$ where

\[ V = \beta V_0 \]  

(4.12)

and $V_0$ is the volume of the metric on the Wallach space induced by the biinvariant metric on $SU(3)$.

Notice that the volumes for the distinct solutions are the same order of magnitude; the smallest volume is about 40% of the largest. Furthermore, this difference occurs between two Einstein
metrics on the same coset space, $M_{-42652,61213}$. The Einstein metrics on $M_{-56788,5227}$ have volumes closer to that of the lower action solution.

It is also of interest to compare the properties and magnitudes of the volumes of topologically inequivalent manifolds in the same cohomology class as the exotic Wallach spaces. Again these can be calculated numerically using the Newton-Raphson method. The results of this tabulation are given in the Appendix. This table is ordered by the homeomorphism classification number $\lambda_h \mod 2^3 \cdot 3 \cdot \Gamma$. Some interesting properties appear. The Wallach space $M_{-62773,31212}$ has both extremal volume metrics in this set. Both of these metrics are nearly identical in two of the three parameters. That with largest volume has $a \sim b \sim c/2$; that with smallest volume has $a \sim b \sim 2c$. The manifold $M_{-54193,54532}$ has metrics with nearly equal volumes. For each metric the metric coefficients $a,b,c$ differ significantly from each other. However the two sets of metric coefficients are, not surprisingly, nearly the same.

A different presentation of the data, not given explicitly in this paper, reveals an order behind these geometric properties. This presentation is created by relabeling the $M_{k,l}$ using the relations for orientable and nonorientable diffeomorphisms, (3.4) and (3.5), as needed such that each diffeomorphism inequivalent Wallach space is labeled by the maximal negative value of $k$.

Now a sort by increasing $k$ reveals a pattern of monotonic decrease of the larger volume metric and a monotonic increase of the smaller volume metric until comparable metrics are reached. The coefficients $a,b,c$ reveal a similar pattern of monotonic behavior. A careful examination of Table 2 indicates that the volumes of the two Einstein metrics on the exotic Wallach spaces $M_{-42652,61213}$ and $M_{-56788,5227}$ are in the middle of the pack. Therefore, there is nothing particularly distinguishing about the Einstein metrics of these exotic Wallach spaces, either from each other or Wallach spaces of the same homotopy type.

5 Semiclassical Evaluation of the Euclidean Functional Integral

Given these numerical solutions for the Einstein metrics on the exotic Wallach spaces, one can evaluate their contribution to semiclassical evaluations of Euclidean functional integrals. Clearly, as the volume of each Einstein metric differs, their action will too. However, it is useful to analyze these effects more quantitatively in the context of a simple example. For this purpose, this section will consider the logarithmic derivative of the partition function (1.1);

$$\frac{\partial \ln Z[\Lambda]}{\partial \Lambda} = \frac{1}{8\pi G Z[\Lambda]} \sum_{(M^n,g)} \exp(-I[g]) \int d\mu(g)$$  \hspace{1cm} (5.1)

This quantity is formally the expectation value of the volume over all 7-manifolds for fixed cosmological constant $\Lambda$. In semiclassical approximation, the partition function becomes a sum over stationary points of the Einstein action and (5.1) becomes

$$\frac{\partial \ln Z_{sc}[\Lambda]}{\partial \Lambda} = -\frac{1}{Z_{sc}[\Lambda]} \sum_i \frac{\partial I[g_i]}{\partial \Lambda} \exp(-I[g_i])$$  \hspace{1cm} (5.2)
where \(i\) indexes the classical Einstein actions corresponding to the riemannian histories admitting Einstein metrics. Although (5.2) is formally a sum over all physically distinct riemannian histories, one can also consider quantities defined in terms of sums over well defined subsets of these histories. In particular, one can consider the subset to be Wallach spaces of a given topology or homotopy type. In this case, \(i\) in (5.2) will index the Einstein actions on the subset of histories of interest.

The classical action for the Wallach spaces can be computed using the results of section 4 by relating \(E\) to \(\Lambda\). The Euclidean Einstein equation for positive cosmological constant in seven dimensions implies that \(R = \frac{7}{5} \Lambda\). Therefore \(E = \frac{1}{5} \Lambda\). The action (1.2) evaluated for the Einstein metric is then

\[
I_{[g_i]} = \frac{3V_i \Lambda}{80 \pi G} \tag{5.3}
\]

where \(V_i\) is the volume of the Wallach space with Einstein metric \(g_i\) with curvature constant \(E = \frac{1}{5} \Lambda\). Using the scaling properties of (4.11) and (4.12), one can relate this volume to \(\Lambda\):

\[
I_{[g_i]} = \frac{3V_0}{16 \pi G} \left( \frac{5}{\Lambda} \right)^{\frac{7}{4}} \beta_i \tag{5.4}
\]

Note that \(V_0\) is the same for all Wallach spaces. The factor \(\frac{3V_0}{16 \pi G}\) is universal to all Wallach spaces; one can thus choose units such that its value is one. Now as

\[
\frac{\partial I_{[g_i]}}{\partial \Lambda} = -\frac{3}{20} \left( \frac{5}{\Lambda} \right)^{\frac{7}{4}} \beta_i
\]

the semiclassical expectation value of the volume is

\[
\frac{\partial \ln Z_{sc}[\Lambda]}{\partial \Lambda} = \frac{3}{20} \left( \frac{5}{\Lambda} \right)^{\frac{7}{4}} \beta_i
\]

< \beta > = \frac{1}{Z_{sc}[\Lambda]} \sum_i \beta_i \exp\left( -\left( \frac{5}{\Lambda} \right)^{\frac{7}{4}} \beta_i \right) \tag{5.5}

One can now evaluate (5.5) for sets of Wallach spaces. Clearly, the cosmological constant will determine the relative importance of differences in the volume of different Einstein metrics. First consider the computation of < \(\beta\) > for Einstein metrics on a single smooth Wallach space, in particular \(M_{-42652,61213}\) and \(M_{-56788,5227}\). As \(\Lambda \to 0\), the metric with least \(\beta\) will dominate; thus

\[
< \beta >_{-42652,61213} = 7.6311 \quad < \beta >_{-56788,5227} = 12.2181
\]

In this limit the \(M_{-42652,61213}\) has a smaller mean \(\beta\) as it has the smallest volume metric. As \(\Lambda \to \infty\) both metrics on the Wallach space will contribute equally; thus

\[
< \beta >_{\Lambda \to \infty} = 13.2961 \quad < \beta >_{\Lambda \to \infty} = 13.9293
\]

The two exotic manifolds have very similar values of \(\beta\) for this case. The above determines the behavior of (5.5) for the relevant case of \(M_{\text{hom}}\) the topological manifold common to both \(M_{-42652,61213}\) and \(M_{-56788,5227}\). At \(\Lambda \to 0\) the contribution from \(M_{-42652,61213}\) will dominate, but as \(\Lambda \to \infty\) all metrics contribute to the result. Thus

\[
< \beta >_{\Lambda \to 0} = 7.6311 \quad < \beta >_{\Lambda \to \infty} = 13.6126.
\]

Finally, one can compute \(\beta\) for all Wallach spaces of the same cohomology from the results tabulated in the appendix; one finds

\[
< \beta >_{H^4(\Gamma;\mathbb{Z})} = 4.8364 \quad < \beta >_{H^4(\Gamma;\mathbb{Z})} = 13.3520.
\]
The graphs of the four values of $\beta$ as a function of $\Lambda$ are given in Figure 5.

Much is apparent from this simple calculation. First, there is a small difference between $\beta_{\text{hom}}$ and $\beta_{-42652,61213}$ at $\Lambda \to \infty$. There is a similar difference comparing the answers on $\beta_{-56788,5227}$ and $\beta_{\text{hom}}$ at this $\Lambda$. However, for $\Lambda = 0$ the story is quite different; $\beta_{\text{hom}}(0) = \beta_{-42652,61213}$ but $\beta_{-56788,5227}$ is 60% greater. Therefore one cannot predict which differentiable structure on a topological manifold will provide the dominant Einstein metric without carrying out the analysis for the value of the cosmological constant of interest.

6 Conclusions

The examples constructed in this paper show several key features: First inequivalent differentiable structures will contribute on an equal footing with topology in functional integrals for gravity. Secondly, there is no way to predict a priori which differentiable structure if any will dominate the contribution to the functional integral. Furthermore, as the Wallach spaces are simply connected manifolds, they will contribute to semiclassical approximations of sums over histories restricted to simply connected manifolds. Therefore the contribution of exotic manifolds to functional integrals for gravity must be considered.

Although this paper examined only the Wallach spaces of order 2955367597, more examples of exotic Wallach spaces exist; 30 additional exotic Wallach manifolds are known and 14 are tabulated in [16]. Observe also that the Wallach spaces are not the only known examples of exotic manifolds that admit Einstein metrics. Kreck and Stoltz showed that the coset spaces $SU(3) \times SU(2) \times U(1)/(SU(2) \times U(1) \times U(1))$ in which the subgroup $(SU(2) \times U(1) \times U(1))$ is removed in a nontrivial fashion also contain exotic manifolds [26]. These spaces also appear in the physics literature as manifolds for compactification of supergravity theories via the Kaluza-Klein construction [27]. They are not simply connected. However, one can show that there are topological manifolds in this sequence that exhibit all 28 possible inequivalent differentiable structures. Furthermore, these exotic manifolds admit an infinite family of Einstein metrics.

A key feature in the construction of the Einstein metrics on these Wallach spaces is their formulation as a coset space. This feature yields the remarkably simple form of their metric (3.3). It is amusing to note that due to this, the differentiable structure for the Wallach spaces is encoded in a single analytic chart. Therefore, although exotic manifolds may seem quite mysterious, they may actually be rather simple once the appropriate description is found.

One might be curious as to the implications of such exotic manifolds for the wavefunction of the universe. One could divide these spaces by a six dimensional manifold and therefore have a stationary point for the Einstein action with boundary. A natural choice for the topology of the boundary manifold is $SU(3)/T$, that is the coset space obtained by removing the maximal torus. It is known that there are only two distinct homogeneous Einstein metrics on this space: that induced by the submersion of the negative of the Killing form on $SU(3)$ and a Kähler-Einstein metric. Both of these have two equal metric components and cannot come from $M_{k,l}$ by submersion except when $k = l$, or $k = 0$ or $l = 0$. As the exotic manifolds do not satisfy these conditions, they cannot have a boundary with a homogeneous Einstein metric. This would seem to imply that they do not produce wavefunctions of the universe peaked toward isotropy.

Clearly, an important question is the relevance of these results to four dimensional Euclidean functional integrals. Unfortunately, one cannot use techniques involving coset spaces similar to those in this paper to construct examples of exotic manifolds in four dimensions. However, the lack of an easy technique for constructing exotic Einstein manifolds by no means implies that these spaces do not exist! Furthermore, as exotic manifolds are in some sense most numerous in four dimensions, it could be argued that their contribution would be most important in this dimension.

It is also natural to ask what these results imply for the significance of differentiable structures in Lorentzian functional integrals for quantum gravity. This is an important question that deserves its own forum due to the nature of the issues involved in formulating such integrals. Indeed preparation of such a discussion is a current focus of the authors [28]. However, it is perhaps
useful to provide here a brief discussion of potential significance of inequivalent differentiable structures in Lorentzian functional integrals.

Lorentzian functional integrals for gravity involve sums over Lorentzian spacetimes; however it is not yet clear as to what kinds of Lorentzian spacetimes should be included in these sums. For example, one could formulate Lorentzian functional integrals as sums over globally hyperbolic spacetimes that interpolate between two spatial hypersurfaces. One could also formulate them as sums over more general classes of Lorentzian spacetimes, for example a class allowing regular Lorentzian spacetimes that contain closed timelike curves or one that allows Lorentzian spacetimes that contain certain "nice" singularities. The type of Lorentzian spacetime allowed will critically impact at what points inequivalent differentiable structures will contribute.

For example, if one restricts the Lorentzian geometries to be those corresponding to the evolution of initial data, their topology is then that of a n-manifold cross the real interval. Furthermore, the initial value problem determines the differentiable structure in terms of that of the initial Cauchy surface. Thus the differentiable structure of the Lorentzian history is determined by that of the initial hypersurface. In the case of functional integrals in 4 dimensions, the spacelike hypersurfaces are 3-manifolds. As all 3-manifolds admit a unique differentiable structure, the differentiable structure on any evolution is unique. However, for functional integrals in 5 or more dimensions, the Cauchy surfaces themselves will potentially admit more than one differentiable structure. For these cases, exotic differentiable structures will contribute as part of the initial conditions on an equal footing with the topology.

However, one can argue that allowing only Lorentzian geometries of this type is needlessly restrictive. For example, no topology change is allowed for a Lorentzian spacetime that is an evolution. If one generalizes the Lorentzian manifolds to a larger class, for example those containing closed timelike curves, then inequivalent differentiable structures may indeed play an important physical role. For example, there are an uncountable number of differentiable structures on the manifold $S^3 \times \mathbb{R}$ [See ref. 21 and references therein]. Smooth Lorentzian metrics can be found on each of the inequivalent differentiable structures. Therefore, it clearly becomes an issue whether or not any of these is an appropriate Lorentzian solution to the Einstein equations. Therefore, the issue of differentiable structures may have physical significance in functional formulations allowing these classes of Lorentzian 4-manifolds.

Finally, if one allows the use of Euclidean instantons as a technique for evaluating Lorentzian functional integrals, it is clear that the results on the physical significance of inequivalent differentiable structures immediately apply in four or more dimensions.


**Appendix A**

The Einstein metrics and volumes for the 32 diffeomorphism inequivalent Wallach spaces with $H^4(M_{k,l}: Z) = \frac{Z}{\Gamma Z}$ for $\Gamma = 2955367597$ are summarized in the table below. The table is ordered by ascending value of the homeomorphism classification number $\lambda_h \mod 2^3 \cdot 3 \cdot \Gamma$.

| Wallach Space | Metric Parameters | $k$ | $l$ | $\lambda_h \mod 2^3 \cdot 3 \cdot \Gamma$ | $a$ | $b$ | $c$ | $\beta$ |
|---------------|-------------------|-----|-----|----------------------------------------|-----|-----|-----|--------|
| $-40388$      | $61811$           | $530781252$ | $2.51431$ | $3.85991$ | $1.57678$ | $4.96699$ | $2.05195$ | $5.25276$ | $19.4401$ |
| $-58137$      | $49573$           | $1563779580$ | $3.49151$ | $1.73778$ | $4.52134$ | $4.46916$ | $5.09034$ | $1.95248$ | $16.6112$ |
| $-57748$      | $50187$           | $3724397916$ | $3.55872$ | $1.75197$ | $4.56521$ | $4.42169$ | $5.06777$ | $1.94214$ | $16.3275$ |
| $-53223$      | $55436$           | $3938960388$ | $3.89576$ | $4.77725$ | $1.82538$ | $4.14816$ | $1.88159$ | $4.92452$ | $14.7037$ |
| $-45964$      | $60007$           | $5728436076$ | $3.10134$ | $4.25978$ | $1.66055$ | $4.70684$ | $2.00273$ | $5.189$ | $18.017$ |
| $-50277$      | $57689$           | $6552905052$ | $3.5686$ | $4.57161$ | $1.75407$ | $4.41453$ | $1.94057$ | $5.0643$ | $16.2848$ |
| $-49197$      | $58364$           | $6923017548$ | $3.45048$ | $4.49436$ | $1.72922$ | $4.49712$ | $1.95853$ | $5.10324$ | $16.7781$ |
| $-54193$      | $54532$           | $8827144572$ | $4.00552$ | $4.84275$ | $1.84978$ | $4.04418$ | $1.8584$ | $4.86531$ | $14.0975$ |
| $-62508$      | $26261$           | $9012685956$ | $1.57654$ | $2.10501$ | $3.60363$ | $5.24754$ | $5.0972$ | $2.07113$ | $20.0076$ |
| $-40273$      | $61837$           | $9036905268$ | $2.50249$ | $3.85203$ | $1.5758$ | $4.97125$ | $2.05266$ | $5.25315$ | $19.4611$ |
| $-39492$      | $62003$           | $9245608380$ | $2.42268$ | $3.79933$ | $1.57034$ | $4.99919$ | $2.05722$ | $5.255$ | $19.5955$ |
| $-61167$      | $18364$           | $10381999620$ | $1.60559$ | $2.76577$ | $4.03016$ | $5.23637$ | $4.86781$ | $2.0343$ | $18.9245$ |
| \( k \)  | \( l \)  | \( \lambda_h \mod 2^3 \cdot 3 \cdot \Gamma \) | \( a \)     | \( b \)     | \( c \)     | \( \beta \) |
|-------|-------|----------------|--------|--------|--------|-------|
| -20148 | 61561 | 13361497092    | 1.58721 | 3.93119 | 2.62036 | 7.12507 |
| -62773 | 31212 | 13384952580    | 1.72257 | 1.74183 | 3.4643  | 4.8364  |
| -48011 | 59028 | 13422449916    | 3.32168 | 4.40874 | 1.70292 | 10.1993 |
| -42652 | 61213 | 15728825412    | 2.74999 | 4.01931 | 1.60339 | 7.63107 |
| -56788 | 5227  | 15728825412    | 1.78458 | 3.71019 | 4.6623  | 12.2181 |
| -44489 | 60597 | 17116856028    | 2.94401 | 4.15224 | 1.63302 | 8.44413 |
| -38564 | 62177 | 17905442820    | 2.32915 | 3.73889 | 1.56682 | 6.10593 |
| -62628 | 27617 | 18277035876    | 1.59599 | 1.99402 | 3.54518 | 5.18889 |
| -37761 | 62308 | 18424313244    | 2.24968 | 3.68906 | 1.56687 | 5.85991 |
| -60689 | 16452 | 19983541908    | 1.62862 | 2.91726 | 4.13391 | 8.32831 |
| -53411 | 55268 | 20194538580    | 3.91694 | 4.79004 | 1.83008 | 13.368  |
| -62772 | 31033 | 22250745252    | 1.7132  | 1.75215 | 3.4649  | 4.83898 |
| -60636 | 16253 | 22886448612    | 1.63115 | 2.93276 | 4.14453 | 8.39525 |
| -52607 | 55964 | 24231179244    | 4.20956 | 1.89527 | 4.95837 | 15.0652 |
| -60183 | 14636 | 26167379412    | 1.65248 | 3.05674 | 4.22936 | 8.94485 |
| -59739 | 13172 | 27462020340    | 1.67259 | 3.166   | 4.30375 | 9.44933 |
| -37068 | 62407 | 29569264236    | 2.18249 | 3.64838 | 1.56959 | 5.66433 |
Wallach Space

| $k$   | $l$   | $\lambda_h \mod 2^3 \cdot 3 \cdot \Gamma$ | $a$     | $b$     | $c$     | $\beta$     |
|-------|-------|----------------------------------------|---------|---------|---------|-------------|
| −35748 | 62561 | 30559561404                           | 2.05927 | 3.5786  | 1.583   | 5.33967     |
|       |       |                                        | 5.10987 | 2.0726  | 5.24438 | 20.0512     |
| −62739 | 29572 | 30895359468                           | 1.64926 | 1.84639 | 3.48449 | 4.9228      |
|       |       |                                        | 5.22013 | 5.16602 | 2.07751 | 20.1979     |
| −55339 | 2007  | 34682602716                           | 1.8281  | 3.90803 | 4.78467 | 13.3176     |
|       |       |                                        | 4.91823 | 4.13694 | 1.87909 | 14.6378     |

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Figure 1. Curves of zeros for the Wallach space $M_{-42652,61213}$. 
Figure 2. Curves of zeros for the Wallach space $M_{-42652, 61211}$ under interchange of $l$ and $(k + l)$. 
Figure 3. The intersection of both curves of zeros for the Wallach space $M_{-42652,61213}$. Points of intersection correspond to Einstein metrics of nonegative curvature.
Figure 4. The intersection of both curves of zeros for the Wallach space $M_{-5678,5227}$. Note that again there are exactly two intersections.
Figure 5. The expectation value $<\beta>$ as a function of cosmological constant for different sets of Wallach spaces.