Quantum Correlations over Long-distances Using Noisy Quantum Repeaters

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Quantum correlations as the resource for quantum communication can be distributed over long distances by quantum repeaters. In this Letter, we introduce the notion of a noisy quantum repeater, and examine its role in quantum communication. Quantum correlations shared through noisy quantum repeaters are then characterized and their secrecy properties are studied. Remarkably, noisy quantum repeaters naturally introduce private states in the key distillation scenario, and consequently key distillation protocols are demonstrated to be more tolerant.

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Quantum Key Distribution (QKD) protocols such as the Bennett-Brassard 1984 (BB84) [1] have been implemented in laboratories [2, 3], and become one of the most important and promising applications of Quantum Information Theory. QKD is now no longer an experiment but an emerging market [4]. Further investigations on QKD protocols will improve their practical performance under realistic constraints [5]. To date, QKD reaches about 100 km in distance with photon sources through optical fibers, which however does not yet meet the distance standard of present-day communication.

The communication distance is somehow limited as any physical resource carrying quantum states suffer unwanted interactions with environment such as decoherence and losses during transmission. In this sense, it is natural to build a bridge for quantum correlations, for instance quantum relay or quantum repeater, to overcome the distance limit. Quantum repeaters are in fact known to efficiently extend the communication distance [2, 7], but unfortunately not feasible within current technology since the so-called quantum memory, that stores quantum states for a while, is experimentally challenging. Nevertheless, there have been remarkable experimental results that envisage a feasible quantum memory in the near future [5].

This work is therefore motivated by two perspectives. First, a quantum memory in the near future, as being in an earlier phase of development, would have a storage-time long enough to distribute quantum correlations over distances, but not sufficiently long to apply entanglement distillation. The next arises from the fact that a practical quantum repeater, being contacts to the quantum channels, would be susceptible to its surroundings. To be specific, as quantum repeaters are connected to one another by possibly noisy quantum channels, errors caused by the noisy channels will be ported to the quantum state of a quantum memory of the repeater, i.e. quantum repeaters become noisy. The question we address then is in what way do noisy quantum repeaters feature in QKD scenarios. It is actually not straightforward to conclude that the secret key rate decreases, since noise effects do not always degrade protocols [9]. Indeed, we will show that noisy repeaters degrade the power of eavesdropper, called Eve. In this Letter, we characterize quantum correlations distributed through noisy quantum repeaters, and then study distillation of secret key and entanglement. The distribution scenario is described in the entanglement-based scheme.

We first briefly review entanglement distribution through a single quantum repeater, denoted by $R$, assuming that all quantum channels are perfect but only limited in distance. The distribution scenario follows the standard scheme in Ref. [7], as follows. Alice first generates the maximally entangled state $|\phi_1\rangle$ where $|\phi_1\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, keeps the first qubit, and send the other one to the repeater. Bob does the same, and the repeater then has two qubits in store. The entanglement swapping (ES) protocol, denoted by $\Lambda_R$, is applied to the two qubits in store, and afterwards two honest parties share the state $|\phi_1\rangle_{AB}$. Here, the protocol $\Lambda_R$ is composed of Bell-basis measurement on the two qubits followed by the public announcement of the measurement outcome, in order that two honest parties apply local operations to rotate the shared state into $|\phi_1\rangle$. A quantum repeater being a quantum device whose physical state is described by a density operator, say $\eta_R$, as follows:

$$\Lambda_R(|\phi_1\rangle_A R \otimes |\phi_1\rangle_R B) = |\phi_1\rangle_{AB}(\phi_1 \otimes \eta_R).$$

It has been presumably of no interest to find out in which

![Diagram of quantum repeater](image)
quantum state a quantum repeater remains. This is because of mainly two reasons: as it is seen in (1), a repeater is factorized out from two honest parties by the Bell-basis measurement of the ES protocol, meaning that no secret correlations between the repeater and two honest parties would be exploited [10]. In addition, it is often supposed that a repeater stays in a constant state \( \eta_R \) all the time not being affected by any change of its surroundings.

We now turn to the realistic constraints to quantum channels and a quantum repeater. First, quantum channels are noisy in general, and therefore each of Alice and Bob shares mixed states with the repeater. Here we restrict to cases where the mixed state is Bell-diagonal in a single-copy level. If it is not the case, two honest parties can apply local filtering operations such that Bell-diagonal states are shared [11]. The filtering operation in fact increases the amount of entanglement, in terms of entanglement of formation [12], of shared states [13].

Next, a practical quantum repeater, as a quantum device, would react susceptibly to a change of its surroundings. In particular, a quantum memory in the repeater interacts with two qubits sent by two honest parties, and thus two qubits stored become the effective environment of the repeater. Then, suppose that the two-qubit state encoded by two honest parties are sent through and perturbed in noisy channels. The isomorphism between quantum channels and quantum states tells us that all the properties of the noisy channels can be found in the quantum state that have arrived at the repeater [13]. This means that, an error caused by noisy channels corresponds to a change in the repeater’s environment. In terms of noise parameters, when \( |\phi_i\rangle_{AR} \) and \( |\phi_j\rangle_{RB} \) are shared for some \( i \) and \( j \) depending on channel properties, a repeater would be perturbed according to the noise \( i \) and \( j \). This is what we mean that a practical quantum repeater is noisy.

We need to clarify here that it is two honest parties who prepare and put a repeater in the middle. This means that they already know the properties of the repeater, how it reacts to each of phase-, bit-, and both errors, and is perturbed. As well, the ES protocol in a repeater cannot be designed to work by recognizing errors instance per instance i.e. which pair of Bell-states is kept, but always assumes the ideal case that \( |\phi_1\rangle_{AR} \otimes |\phi_1\rangle_{RB} \) is shared. In addition, we only suppose the minimal responsibilities to a repeater, performing the ES protocol, and do not consider cases where repeaters collaborate with two honest parties or an eavesdropper.

After the ES protocol on \( |\phi_i\rangle_{AR} \otimes |\phi_j\rangle_{RB} \) in the repeater, two honest parties share one of Bell states, denoted by \( |\phi_k\rangle \) where \( k \) is completely determined by the error kinds \( i \) and \( j \). The noisy quantum repeater can also be captured by the single index, \( k \), as follows,

\[
\Lambda_R(|\phi_i\rangle_{AR} \otimes |\phi_j\rangle_{RB}) = |\phi_k\rangle_{AB} \langle \phi_k| \otimes \eta_R(k). \tag{2}
\]

The repeater’s quantum state \( \eta_R(k) \) are known to two honest parties since they have prepared a repeater. Likewise, in a general case when \( N \) noisy quantum repeaters are located, Bell-diagonal states will also end up between two honest parties, and the shared state would be finally of the following form,

\[
\rho_{ABR} = \bigotimes_{n=1}^{N} \Lambda(|\phi_1\rangle_{R_{n-1}R_n} \otimes |\phi_1\rangle_{R_nR_{n+1}})
= \beta_1|\phi_1\rangle_{AB} \langle \phi_1| \otimes \eta_1 + \beta_2|\phi_2\rangle_{AB} \langle \phi_2| \otimes \eta_2
+ \beta_3|\phi_3\rangle_{AB} \langle \phi_3| \otimes \eta_3 + \beta_4|\phi_4\rangle_{AB} \langle \phi_4| \otimes \eta_4 \tag{3}
\]

where \( \eta_j \) is the normalized quantum state of \( N \) repeaters when two honest parties share \( |\phi_j\rangle \). Interestingly, this is exactly the state that has been considered in the context of distillation of private states in Ref. [14], except only that the states \( \eta_j \) do not belong to Alice and Bob but an independent party, repeaters.

We have characterized quantum states shared through noisy repeaters and noisy channels, exploiting the independent party, repeaters, in the shared state. Before starting secrecy analysis, we should first introduce the important assumption on shared states, that quantum states shared by two honest parties are made invariant under any permutation of pairs i.e. symmetric states. This can be done by random permutations. The quantum de Finetti theorem then states that the most general \( N \)-symmetric state, i.e. invariant under any permutations, \( \rho^{(N)} \) converges efficiently to the identically and independently distributed (i.i.d.) one \( \rho_A^{N}, \) as \( N \) becomes a very large number [15]. This implies that in the context of QKD, one does not have to go through the most general case of \( N \)-symmetric states \( \rho_A^{N} \) but it suffices to consider the i.i.d. one, the so-called collective attacks, \( \rho_A^{N} \) [15]. The bound for the general security can be obtained by analyzing collective attacks. In what follows, we assume that the number of copies, \( N \), becomes a very large number to ensure applicability of the quantum de Finetti theorem.

We now translate the scenario introduced in Ref. [14] of distilling private states to that of distilling secret key with noisy quantum repeaters. Since repeaters are supposed to comprise an independent party not belonging to two honest parties nor the eavesdropper, errors caused by those repeaters have nothing to do with eavesdropping strategies and therefore do not have to be necessarily corrected to distill secret key. This also means that the quantum state representing the general security does not necessarily correspond to \( |\phi_1\rangle_{AB} \). As it was pointed out in [14], by introducing a party independent to two honest parties and Eve, Eve’s purification power can be
degraded. All this can be encapsulated by the so-called private states, corresponding to the general security, as follows,

$$\gamma_{ABR} = U|\phi_1\rangle_{AB}|\phi_1\rangle \otimes \rho_R U^\dagger,$$  

(4)

where $\rho_R$ is a repeater state and $U = \sum_{ij} |ij\rangle_{AB} \otimes W_R^{ij}$ is a unitary operation called twisting. Note that a twisting $U$ provides the equivalence class of private states. The only difference is that the local assistance by $A'$ and $B'$ in Ref. [14] is replaced with noisy quantum repeaters.

In particular, we take a unitary operation, the untwisting one $U^\dagger$ in Ref. [16] that works for a state of the form in (3) as follows,

$$\rho_{ABR} = U^\dagger \sigma_{ABR} U$$  

(5)

where $\sigma_{AB} = \text{tr}_R[\sigma_{ABR}] = \sum_j \lambda_j |\phi_j\rangle \langle \phi_j|$ is again Bell-diagonal, with

$$\lambda_{1,2} = \frac{1}{2}(\|\beta_1 \eta_1 + \beta_2 \eta_2\| \pm \|\beta_1 \eta_1 - \beta_2 \eta_2\|)$$

$$\lambda_{3,4} = \frac{1}{2}(\|\beta_3 \eta_3 + \beta_4 \eta_4\| \pm \|\beta_3 \eta_3 - \beta_4 \eta_4\|),$$  

(6)

where the trace-norm of an operator $A$ has been denoted by $\|A\|$. This untwisting operation shows the phase errors that are necessary to be corrected for the $\rho_{ABR}$ to be a private state. Note that repeaters are classically correlated with two honest parties in the state $\sigma_{ABR}$, which means that measurement outcomes of two honest parties are independent to quantum states of repeaters.

The relations above (5) and (6) reveal not only the reason why not all errors existing in $\rho_{ABR}$ have to be corrected but also how much errors are necessarily to be corrected. First, the state that we are aiming to distill is a private state. Note that repeaters are classically correlated with two honest parties in the state $\sigma_{ABR}$, which means that measurement outcomes of two honest parties are independent to quantum states of repeaters. Next, $\sigma_{AB}$ has less errors than $\rho_{AB}$, which can be seen by comparing $\beta_j$ of $\rho_{AB}$ in (3) to $\lambda_j$ of $\sigma_{AB}$ in (6) for $j = 1, 2, 3, 4$. To be precise, it holds that $\lambda_j \leq \beta_j$ for $j = 2, 3, 4$ [17, 18]. Therefore, correcting less errors that exist in $\sigma_{AB}$, two honest parties will share secret key related to a private state. The question followed is whether key distillation techniques such as advantage distillation and the standard one-way distillation protocol commute with a twisting operation $U$. In fact, the distillation techniques commute with twisting operations [17, 18] and therefore, no more additional step is required in the key distillation scenario. The only difference in the classical step is that less errors are estimated to be corrected.

We now analyze secrecy properties of quantum states in (3) shared through noisy repeaters. Measuring $\rho_{ABR}^{\otimes N}$ in the computational basis, two honest parties are with measurement outcomes of probability distribution,

$$p_{AB}(i,j) = \langle i_{AB}|tr_R \rho_{ABR}^{\otimes N}|i_{AB}\rangle,$$  

(7)

to which key distillation techniques are then applied. Note here the precondition for key distillation, that quantum state from which secrecy is to be distilled must be entangled [10]. In this case, $\sigma_{AB}$ in (3) is the state where secrecy is to be extracted, and once the state $\rho_{ABR}$ is identified through state tomography, from the relation (3) it can be easily checked if the state $\sigma_{AB}$ is entangled. The state $\sigma_{AB}$ is entangled if and only if the follows holds

$$\|\beta_1 \eta_1 - \beta_2 \eta_2\| > \|\beta_3 \eta_3 + \beta_4 \eta_4\|.$$  

(8)

It is also worth mentioning a sufficient condition: if the state $\rho_{AB}$ is entangled, i.e. $\beta_1 - \beta_2 > \beta_3 + \beta_4$, so is $\sigma_{AB}$. This is clear from the inequality, $\beta_1 - \beta_2 \leq \|\beta_1 \eta_1 - \beta_2 \eta_2\|$, that holds true for all $\beta_{1,2}$ and $\eta_{1,2}$.

We now consider the standard one-way communication of error correction and privacy amplification to the measurement outcomes of the probability distribution (7). A lower bound to the one-way secret key rate has been derived in Refs. [19], $K_{\rightarrow} \geq I_{AB} - I_{AE}$, where the mutual information is denoted by $I$. It is then straightforward to compute the lower bound to the key rate for measurement outcomes in (7), as follows

$$K_{\rightarrow} \geq 1 - h(x) - \sum_{i=1,3} \|\beta_i \eta_i + \beta_{i+1} \eta_{i+1}\| h(y_i)$$  

(9)

where $h(\cdot)$ is the binary entropy, $x = \|\beta_1 \eta_1 + \beta_2 \eta_2\|$ and $y_i = (1 + \|\beta_i \eta_i - \beta_{i+1} \eta_{i+1}\|/\|\beta_i \eta_i + \beta_{i+1} \eta_{i+1}\|)/2$. One can see from the security condition (4) how the quantum states of quantum repeaters are relevant. In the following example, we consider the BB84 protocol with noisy quantum repeaters, and show that noisy quantum repeaters make the protocol more tolerant.

**Example.** (BB84) In the entanglement-based scheme, the shared quantum state in the BB84 protocol is identified in a single-copy level as

$$\rho_{AB} = (1 - Q)^2|\phi_1\rangle \langle \phi_1| + Q(1 - Q)|\phi_2\rangle \langle \phi_2|$$

$$+ Q(1 - Q)|\phi_3\rangle \langle \phi_3| + Q^2|\phi_4\rangle \langle \phi_4|$$  

(10)

where $Q$ is the quantum-bit-error-rate (QBER) [20]. As a toy model of noisy quantum repeaters, we here take the shield states considered in Ref. [14] as their quantum states, $\eta_{1,3,4} = (\rho_s + \rho_s)^{Gy}/2^l$ and $\sigma_2 = \rho_0^{Gy}$, where $\rho_s$ is the normalized $d$-dimensional projection operator onto asymmetric(symmetric) space. These noisy quantum repeaters, where only $\eta_2$ is different from the others, can be interpreted as being sensitive to phase errors. Then, the shared state in a single-copy level can be written as follows,

$$\rho_{ABR} = (1 - Q)^2|\phi_1\rangle \langle \phi_1| \otimes \eta_1 + (1 - Q)Q|\phi_2\rangle \langle \phi_2| \otimes \eta_2$$

$$+ (1 - Q)Q|\phi_3\rangle \langle \phi_3| \otimes \eta_3 + Q^2|\phi_4\rangle \langle \phi_4| \otimes \eta_4.$$  

(11)
As we have shown, it suffices to consider errors existing in the untwisted state \( \sigma_{AB} \) in [13] so that private states in [16] are to be distilled. The lower bound to the secret key rate can be computed from [9], and is depicted in Fig. 2. When \( l \) becomes a very large number, the QBER converges to \( Q = 24.5\% \), much higher than the known bound 11.0\% in Ref. [21].

The key distillation technique applying the two-way (AD) followed by the one-way key distillation tolerates higher values of QBER. This has been completely analyzed in Ref. [17], which shows that if shared states satisfy

\[
\|\beta_1 \eta_1 - \beta_2 \eta_2\|^2 > \|\beta_1 \eta_1 + \beta_2 \eta_2\| \|\beta_3 \eta_3 + \beta_4 \eta_4\|,
\]

(12) then secret key can be distilled. By this, quantum states in a wider range are shown to be distilled to secret key. In particular, if \( \eta_1 \) is orthogonal to \( \eta_2 \), i.e. \( \text{tr}[\eta_1 \eta_2] = 0 \), it holds that \( \|\beta_1 \eta_1 + \beta_2 \eta_2\| = \|\beta_1 \eta_1 - \beta_2 \eta_2\| \), remarkably meaning that the security condition in (12) coincides to the precondition for key distillation [8]. Therefore, for this particular case, all secret correlations derived from entangled states \( \sigma_{AB} \) can be converted to secrecy. To our knowledge, this is the first case that entanglement itself implies secrecy, although its general connection remains open.

Finally, we would like to mention that noisy quantum repeaters do not really play a role in distilling entanglement. This is because, differently to key distillation where secret key rate matters, entanglement distillation is concerned with the singlet fidelity \( F = \langle \phi_1 | \text{tr} [\rho_{AB}] | \phi_1 \rangle \), where quantum repeaters are traced out and therefore not considered. However, it would be interesting still to characterize errors that pertain to the distillation rate of entanglement.

To conclude, we have characterized quantum correlations distributed through noisy quantum repeaters, and shown that the scenario corresponds to the distillation of private states. This helps to close the gap between two extreme sides of QKD, one theoretical and the other practical. Remarkingly, QKD protocols are shown to tolerate higher values of QBER by noisy quantum repeaters. Here the lesson to practical QKD over long distances, usually having higher values of QBER, is that noise effects independent to Eve would make more tolerant protocols, for instance, where noisy ES protocols are included. In addition, although the scheme of quantum repeater introduced in Ref. [2] is considered, our results can be easily generalized to other variant schemes that are basically equivalent to that in Ref. [2].

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