Effects of the external string cloud on the Van der Waals like behaviour and efficiency of AdS-Schwarzschild black holes in massive gravity

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Abstract

In this paper we study AdS-Schwarzschild black holes in four and five dimensions in dRGT minimally coupled to a cloud of strings and investigate the effects of this string cloud on the thermodynamics of the black holes. It is observed that the entropy of the string cloud and massive terms does not affect the black hole entropy. The observations about four dimensions indicate that the massive term in the presence of external string cloud cannot exhibit Van der Waals like behaviour for AdS-Schwarzschild black holes and therefore there is only the Hawking-Page phase transition. In contrast, in five dimensions, the graviton mass modifies this behaviour through the third massive term, so that a critical behaviour and second order phase transition is deduced. The black hole stability conditions are also studied in four and five dimensions and a critical value for the string cloud parameter is presented. In five dimensions a degeneracy between states for extremal black holes is investigated. After studying black holes as thermodynamic systems, we consider such systems as heat engines, and finally the efficiency of them is calculated.

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1 Introduction

In general theory of relativity introduced by Albert Einstein, graviton is massless. In recent years, the idea of giving mass to the graviton increasingly being considered among cosmologists because they want to reveal the hidden angles of controversial and sophisticated phenomena such as dark energy and dark matter by modifying General Relativity (GR) in order to explain the acceleration of the universe and other unresolved problems in cosmology. The first attempt for constructing a massive theory of gravity is attributed to Fierz and Pauli [1] which was done in the context of a linear theory of gravity. Due to some issues of propagator in the massive gravity in the limit \( m = 0 \), it does not reduce to general relativity. Afterwards, people tend to nonlinear massive gravity theories. One of the first person that employed a nonlinear model for massive gravity was Vainshtein [2]. Accordingly, at some distance below the so-called Vainshtein radius, the linear regime breaks down and the model enters into a nonlinear framework. On the basis of Vainshtein mechanism, the recovery of GR has been established around massive bodies, more details are provided in [3]-[9]. But, regrettably, problem of generalization to nonlinear model by Boulware and Deser (BD) suffered from ghosts [10], [11]. Nevertheless, recently the BD ghost problem was resolved in [12], [13], [14] by a nonlinear massive gravity. In these theories, a fixed reference metric can be considered on which the massive gravity propagates. Dynamics of this reference metric is described in the background of theories at present called bigravities [15], [16], [17]. Also, higher dimensional nonlinear massive gravity and higher dimensional massive bigravity are investigated in Ref [18] and Ref [19] respectively. The phenomenology of massive gravity has also been interesting to scientists. To review empirical observations in the context of massive gravity see [20], [21], [22] and the recent LIGO results are given in [23], [24].

Black holes are one of the interesting predictions of Einstein theory (GR). The manifestation of black holes as thermodynamical systems have uncovered many aspects of them. As a forerunner of this category, it can be named the worthwhile work of Hawking and Page who discovered a phase transition between AdS black holes and a global AdS space [25]. Then Chamblin et al found a Van der Waals like phase transition in Reissner-Nordstrom AdS black hole [26], [27], [28]. As well as Kubiznak and Mann illustrated an interesting analogy between Reissner-Nordstrom AdS black holes and Van der Waals fluid-gas systems in the extended phase space of thermodynamics [29]. If the first law of black hole is corrected by a VdP term and the cosmological constant is treated as thermodynamical pressure of the black hole and its conjugate variable is regarded as a volume covered by the event horizon of the black hole, then the extended phase space is deduced [30], [31]. In this illustration, the gravitational mass is regarded as enthalpy. Another amazing application associated with the introduction of a mechanical work term on the P-V plane is the possibility of considering the black holes as heat engines. In an identified thermodynamic condition, which is determined by the equation of state, it is possible that the black hole burns some substance as fuel and produces mechanical work similar to heat engines. This idea was first introduced by C. Johnson [31]. Effects of a string cloud on the criticality and efficiency of AdS black holes as heat engines in the context of GR and f(R) gravity is investigated in [32]. Several au-
The authors also have studied black holes as heat engines in some modified theories of gravity \[33\]-\[38\]. On the other side, gravity is the low-energy limit of string theory. String theory is a promising theory for the unification of all known forces of nature that interprets particles as vibration modes of one dimensional string objects \[39\]. Letelier proposed a model for a cloud of strings, an aggregation of one dimensional objects in a defined geometrical frame that study the gravitational effects of matter such as black holes \[40\]. A cloud of strings is analogous to a pressureless perfect fluid. We are interested to study the effects of the cloud of strings on massive gravitational theory. Many authors have studied various gravitational models with different sources encompassed by a cloud of strings \[41\]-\[45\]. The impact of the cloud of strings on Schwarzschild AdS black hole was investigated and its thermodynamical properties in a non-extended phase space was introduced in \[46\]. A new extended phase space of Schwarzschild AdS black hole with an energy-momentum tensor coming from a cloud of strings related to the topological charge was illustrated by two formal approaches in \[47\] which leads to the same result in \[46\]. It is seen that the effect of string cloud can bring Van der Waals-like behaviour and second order phase transition in an extended phase space, while we know Schwarchild black hole cannot display a phase transition.

In this paper we study the thermodynamics of Massive-AdS Black holes minimally coupled to a cloud of strings in an extended phase space in four and five dimensions and we investigate criticality of these black holes. Our motivation is to simultaneously investigate the effect of string cloud and massive terms on the criticality. Then we consider black holes as heat engines and we calculate the efficiency of them. Finally we present our work results.

2 Gravity setup and thermodynamics in d=4 dimensions

The action of GR-A-massive gravity coupled to a cloud of strings is

\[
I = \frac{1}{16\pi} \int d^d x \sqrt{-g} [R - 2\Lambda + m^2 \sum_{i=1}^{4} c_{i} U_{i}(g, f)] + \int_{\Sigma} m' \sqrt{-\gamma} d\tau d\sigma, \tag{1}
\]

where \(R\) is the scalar curvature of the metric \(g_{\mu\nu}\), \(\Lambda = -\frac{n(n-1)}{2l^2}\) is the negative cosmological constant \((n = d - 1)\), \(m\) is massive parameter and \(f_{\mu\nu}\) is fixed symmetric tensor. The \(c_i\)'s are constant and the \(U_i\)'s are symmetric polynomials of the eigenvalues of the \(d \times d\) matrix \(K_{\mu\nu}^\alpha = \sqrt{g^{\mu\alpha}} f_{\alpha\nu}\), where

\[
U_1 = [K], \\
U_2 = [K]^2 - [K]^2, \\
U_3 = [K]^3 - 3[K][K^2] + 2[K^3], \\
U_4 = [K]^4 - 6[K^2][K^2] + 8[K^3][K] + 3[K^2]^2 - 6[K^4]. \tag{2}
\]
The square root in $\mathcal{K}$ means $(\sqrt{A})^\nu_\lambda (\sqrt{A})^\nu_\lambda = A^\nu_\lambda$ and the rectangular brackets denote traces. The second integral called a Nambu-Goto action. $(\tau, \sigma)$ is a parametrization of the world sheet and $m'$ is positive and is related to the tension of the string. $\gamma$ is the determinant of the induced metric

$$\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b},$$

(3)

The action can also be described by a spacetime bi-vector $\Sigma^{\mu\nu}$, given by

$$\Sigma^{\mu\nu} = \varepsilon^{ab} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b},$$

(4)

such that Nambu-Goto action can be written as

$$I_{GN} = m' \int_\Sigma \frac{1}{2} \Sigma_{\mu\nu} \Sigma^{\mu\nu} d\lambda^0 d\lambda^1.$$  

(5)

The energy-momentum tensor for the string can be calculated from the relation $T_{\mu\nu} = -\frac{\partial L}{\partial g^{\mu\nu}}$, where $L = m' \sqrt{-\frac{1}{2} \Sigma_{\mu\nu} \Sigma^{\mu\nu}}$. The energy-momentum tensor of the string cloud is then given by

$$T^{\mu\nu} = \rho m' \frac{\Sigma^{\mu\sigma} \Sigma^{\nu\sigma}}{\sqrt{-\gamma}},$$

(6)

where $\rho$ is the density of the string cloud.

The equation of motion is as follows,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + m^2 \chi_{\mu\nu} = T_{\mu\nu},$$

(7)

in which $G_{\mu\nu}$ is the Einstein tensor and $\chi_{\mu\nu}$ is

$$\chi_{\mu\nu} = \frac{c_1}{2} (K_{\mu\nu} - U_1 g_{\mu\nu}) - \frac{c_2}{2} (U_2 g_{\mu\nu} - 2U_1 K_{\mu\nu} + 2K^2_{\mu\nu})$$

$$- \frac{c_3}{2} (U_3 g_{\mu\nu} - 3U_2 K_{\mu\nu} + 6U_1 K^2_{\mu\nu} - 6K^3_{\mu\nu}) - \frac{c_4}{2} \times$$

$$(U_4 g_{\mu\nu} - 4U_3 K_{\mu\nu} + 12U_2 K^2_{\mu\nu} - 24U_1 K^3_{\mu\nu} + 24K^4_{\mu\nu}),$$

and conservation of the energy-momentum tensor $\nabla_\nu T^{\mu\nu} = 0$ leads

$$\partial_\mu (\sqrt{-g} \rho \Sigma^{\mu\sigma}) = 0.$$  

(8)

$f_{\mu\nu}$ was proposed in [48] with the form $f_{\mu\nu} = \frac{c^2}{r^2} \text{diag}(0, 0, 1, 1)$. Then considering this ansatz for the metric,

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 h_{ij} dx_i dx_j,$$

(9)

where

$$h_{ij} = \delta_{ij},$$

(10)

yielding [49]

$$U_1 = \frac{3c_0}{r}, \quad U_2 = \frac{6c_0^2}{r^2}, \quad U_3 = 0, \quad U_4 = 0.$$
Finally, the field equations yield

\[ f(r) = k - \frac{b}{r} - a - \frac{\Lambda}{3} r^2 + m^2 \left( \frac{c_0 c_1}{2} r + c_0^2 c_2 \right), \]  

(11)

where \( b \) is an integration constant which is calculated from solving \( f(r_0) = 0 \) that \( r_0 \) is location of the event horizon

\[ b = r_0 \left[ k - \frac{\Lambda}{3} r_0^2 - a + \Delta \right], \]  

(12)

\[ \Delta \equiv m^2 \left( \frac{c_0 c_1}{2} r_0 + c_0^2 c_2 \right), \]  

(13)

\[ a \equiv \sigma^2. \]  

(14)

To investigate the effect of the string cloud on our solution we plot \( f(r) - r \) diagrams (see Figure 1). We see that based on the value of the string cloud parameter, for \( a < a_c \) maximum three roots appear, while for \( a > a_c \) maximum one root appears, where the value of \( a_c \) is given in the following. We note that the metric function has a maximum of three roots regardless of curvature geometry which means, it has a maximum of three event horizon apart from the topology of the horizon: spherical \((k = 1)\), flat \((k = 0)\) or hyperbolic \((k = -1)\) in 4 dimensions.

In an extended phase space we consider the cosmological constant as pressure of the black hole with \( P = -\frac{\Lambda}{8\pi} \), then its conjugated variable plays role of the black hole’s volume. In this conjecture the black hole mass is interpreted as enthalpy

\[ M = \frac{V_2 b}{8\pi}, \]  

(15)
where $V_2$, the area of a unit volume of constant $(t,r)$ space is equal to $4\pi$.

Entropy $S$ and temperature $T$ of the black hole can be derived respectively as follow

$$S = \int_{r_0}^{r_f} \frac{1}{T} \left( \frac{\partial M}{\partial r} \right) dr_0 = \frac{V_2 r_f^2}{4}, \quad (16)$$

$$T = \frac{1}{4\pi} \partial_r f(r) \mid_{r=r_0} = \frac{k}{4\pi r_0} - \frac{r_0 \Lambda}{4\pi} - \frac{a}{4\pi r_0} + \frac{m^2}{4\pi r_0} (c_0 c_1 r_0 + c_0^2 c_2)]. \quad (17)$$

Then we find that these thermodynamic quantities satisfy the first law of the black hole thermodynamics in the extended phase space with the following form

$$dM = TdS + VdP + Ada + C_1 dc_1 + C_2 dc_2, \quad (18)$$

with

$$T = \left( \frac{\partial M}{\partial S} \right)_{a,P,c_i}, \quad (19)$$

$$V = \left( \frac{\partial M}{\partial P} \right)_{S,a,c_i} = \frac{1}{3} V_2 r_0^3, \quad (20)$$

$$A = \left( \frac{\partial M}{\partial a} \right)_{S,P,c_i} = -\frac{V_2}{8\pi} r_0, \quad (21)$$

$$C_1 = \left( \frac{\partial M}{\partial c_1} \right)_{S,a,P,c_2,c_3} = \frac{V_2 m^2 c_0 r_0^2}{16\pi}, \quad (22)$$

$$C_2 = \left( \frac{\partial M}{\partial c_2} \right)_{S,a,P,c_1,c_3} = \frac{V_2 m^2 c_0 r_0^2}{8\pi}. \quad (23)$$

where $V$ is the thermodynamic volume, $A$ and $C_i$’s stand for the physical quantities conjugated to the parameters $a$ and $c_i$’s respectively. Besides the corresponding Smarr relation can be extracted by a scaling argument as

$$M = 2TS - 2VP - C_1 c_1. \quad (24)$$

Since the $c_2$-term and $a$-term in the metric function are constant terms in four dimensions do not appear in first law of black hole thermodynamics and we set $dc_2 = 0, da = 0$. Also, if we define an equipotential surface $f(r) = cte$ and varying it with respect to variables $k$, $r$, $m_0 \equiv b/2$, $a$ and $c_i$, then a term $\omega d\varepsilon$ is added to equation (18) where $\varepsilon = V_2 k$ is named topological charge and $\omega = \frac{k^2 r}{8\pi}$ is its conjugated potential. Anyway since the dimention of $\varepsilon$ is proportional to $[L]^0$ this term has no role in coresponding Smarr relation.

By inserting relationships (15),(12),(16),(20),(22) in (24) we will achieve the equation of state

$$P = \frac{1}{2r_0} \left( T - \frac{m^2 c_0 c_1}{4\pi} \right) + \frac{a - k - m^2 c_0^2 c_2}{8\pi r_0^2}, \quad (25)$$

where the value of $(-\frac{m^2 c_0 c_1}{4\pi})$ can be considered as a correction to the Hawking temperature.

The ideal gas law for one mole of gas composed of non-interacting point particles satisfy $P \propto \frac{T}{V}$. Van der Waals proposed that all particles are hard spheres of the same finite radius $r$ (the van der Waals radius). The available space in which the particles
are free to move is limited by the amount of space occupied by themselves, so van der Waals corrected the gas state equation by replacing \( V - b \) instead \( V \), where \( b \) is called the co-volume. Then we will have \( P \propto \frac{T}{\sqrt{r_0}} = T \left( \frac{1}{T} + \frac{b}{V} + \frac{b^2}{2V^2} + \ldots \right) \). Consequently in order to the equation of state display van der Waals like behaviour, as a necessary condition, it should include at least the \( \frac{1}{T} \) term and the higher powers in the denominator. Thus, in our case study, the equation of state (25), does not show the van der Waals like behavior.

Also we can explore critical phenomena and van der Waals like behaviour by computing the inflection point \( \frac{\partial^2 P}{\partial T^2} \bigg|_T = \frac{\partial^2 P}{\partial c^2} \bigg|_T = 0 \), we see that no critical points are found. Therefore we observe that adding the graviton mass to the AdS-Schwarzschild black holes coupled to a cloud of strings in 4 dimensions have no critical behaviour. This fact can also be seen from the \( P - r \) diagrams (see Figure 2). Here, as in the case of AdS-Schwarzschild black hole, there is only the Hawking-Page phase transition. This is evident from the \( T - r \) diagrams (see Figure 2). These diagrams are plotted for two conditions: \( a > a_c \) and \( a < a_c \) where \( a_c = k + m^2 c_0^2 c_2 \). For \( a < a_c \), we see that there is a minimum temperature in which a phase transition between a small black hole and a large black hole takes place. It should be noted that these diagrams are plotted for two modes, \( c_1 > 0 \) and \( c_1 < 0 \).

To further explore thermodynamical properties of the black holes, we calculate the specific heat in canonical ensemble to check the stability of the black holes. The specific heat can be derived as

\[
C = \frac{\partial M}{\partial T} = \frac{2\pi r_0^2 (-\Lambda r_0^2 + k - a + m^2 (c_0 c_1 r_0 + c_0^2 c_2))}{-\Lambda r_0^2 - (m^2 c_0^2 c_2 + k - a)}.
\] (26)

The \( C - r \) diagrams in the range \( a < a_c \) of string cloud parameter and for two possible modes \( c_1 > 0 \) and \( c_1 < 0 \) are plotted in Figure 3. We see that for \( c_1 < 0 \) the black holes with radius from \( r_0 = \frac{1}{2\pi \Lambda} \left[ -m^2 c_0 c_1 - \sqrt{m^4 c_0^4 c_1^2 - 4 \Lambda (m^2 c_0^2 c_2 + k - a)} \right] \) to \( r_0 = \frac{\sqrt{m^2 c_0^2 c_2 + k - a}}{\Lambda} \) and \( r_0 > \frac{1}{2\pi \Lambda} \left[ -m^2 c_0 c_1 + \sqrt{m^4 c_0^4 c_1^2 - 4 \Lambda (m^2 c_0^2 c_2 + k - a)} \right] \) are stable and for the rest of them are unstable. Also for \( c_1 > 0 \) the black holes are unstable. The conditions for \( a > a_c \) are obvious in Figure 4.

It should be noted that, as can be deduced from reference [50], the physical condition of unitarity is the first massive term, causality requires that \( c_1 < 0 \). Therefore, only those solutions that satisfy this condition are physical.

The Gibbs free energy for \( d = 4 \) is given by

\[
G = M - TS = -\frac{1}{4} (a - k - m^2 c_0^2 c_2) r_0 - \frac{2\pi}{3} P r_0^3.
\] (27)

The G-T diagrams for \( d = 4 \) are plotted in Figure 5. It is known that in the second order phase transition, due to a thermodynamic potential such as Gibbs free energy becomes non-analytic, a swallow tail-like behaviour and discontinuity appears in \( G - T \) diagrams, which is not observed here.
Figure 2: (a) $P - r$ for $d = 4$, $T = 5$, $c_0 = 1$, $c_1 = -2$, $c_2 = 3.18$, $m = 2.1$ and $a = 1$, (b): $T - r$ for $d = 4$, $\Lambda = -1$, $c_0 = 1$, $c_1 = -2$, $c_2 = 3.18$, $m = 2.1$, $a = 1 < a_c$, (c) : $a = 1 < a_c$, $c_1 = 2$, (d) : $a = 60 > a_c$, $c_1 = 2$ and (e) : $a = 60 > a_c$, $c_1 = -2$.

3 Computations and remarks for $d = 5$ dimensions

$U_i$'s are the following [51, 52]

$$U_1 = \frac{3c_0}{r}, \quad U_2 = \frac{6c_0^2}{r^2}, \quad U_3 = \frac{6c_0^3}{r^3}, \quad U_4 = 0,$$

where the metric function for $d = 5$ dimensions is given by

$$f(r) = k - \frac{b}{r^2} - \frac{2a}{3r} - \frac{\Lambda}{6} r^2 + m^2 \left( \frac{c_0 c_1}{2} r + c_0^2 c_2 + \frac{c_0^3 c_3}{r} \right), \quad (28)$$

with

$$b = r_0^2 \left( k - \frac{\Lambda}{6} r_0^2 - \frac{2a}{3r_0} + m^2 \left( \frac{c_0 c_1}{2} r_0 + c_0^2 c_2 + \frac{c_0^3 c_3}{r_0} \right) \right). \quad (29)$$

$f(r)$-r diagrams show that in five dimensions we have at most three horizons, so the effect of string cloud on the number of event horizons is similar to the four dimensions.
Figure 3: $C - r$ for $d = 4$, $\Lambda = -1$, $c_0 = 1$, $c_2 = 3.18$, and $m = 2.1$, $a = 1$, (a): $c_1 = -2$ and (b): $c_1 = 2$.

Figure 4: $C - r$ for $d = 4$, $\Lambda = -1$, $c_0 = 1$, $c_2 = 3.18$, and $m = 2.1$, $a = 60$, (a): $c_1 = -2$ and (b): $c_1 = 2$.

(see Figure 6). The ADM mass $M$, Wald entropy $S$ and Hawking temperature $T$ of the black hole can be calculated respectively as follow

$$M = \frac{3V_3}{16|\mathbf{r}_0^3|[k - \frac{2a}{3r_0} + \frac{4\pi}{3}Pr_0^2 + m^2(\frac{c_0c_1}{2}r_0 + c_0^2c_2 + \frac{c_0^3c_3}{r_0})]}$$

$$S = \frac{V_3r_0^3}{4}$$

$$T = \frac{k}{2\pi r_0} - \frac{r_0\Lambda}{6\pi} - \frac{a}{6\pi r_0} + \frac{m^2}{4\pi r_0}(\frac{3}{2}c_0^2c_1r_0 + 2c_0^2c_2 + \frac{c_0^3c_3}{r_0})$$

where $V_3$, volume of the three dimensional unit sphere as plane or hyperbola, is equal to $\frac{4\pi}{3}$. These relations satisfy the first law of black hole thermodynamics in the extended
phase space. The corresponding Smarr relation can be derived as

\[ 2M = 3TS - 2VP + Aa - C_1c_1 + C_3c_3, \]  

(33)

where

\[ V = (\frac{\partial M}{\partial P})_{S,a,c_1} = \frac{1}{4}V_3r_0^4, \]  

(34)

\[ A = (\frac{\partial M}{\partial a})_{S,P,c_1,c_2} = -\frac{r_0V_3}{8\pi}, \]  

(35)

\[ C_1 = (\frac{\partial M}{\partial c_1})_{S,a,P,c_2,c_3} = \frac{3V_3m^2c_0r_0^3}{32\pi}, \]  

(36)

\[ C_3 = (\frac{\partial M}{\partial c_3})_{S,a,P,c_1,c_2} = \frac{3V_3m^2c_0^3r_0}{16\pi}. \]  

(37)

By inserting these relations in Eq.(33), and using (30) and (31), the pressure is obtained

\[ P = \frac{3}{4r_0}(T - \frac{3m^2c_0c_1}{8\pi}) - \frac{3k + 3m^2c_0^2c_2}{8\pi r_0^2} + \frac{2a - 3m^2c_0^3c_3}{16\pi r_0^3}. \]  

(38)

With setting \( (\frac{\partial P}{\partial r_0})_T = (\frac{\partial^2 P}{\partial r_0^2})_T = 0 \), the critical radius, temperature and pressure as follow

\[ r_{0c} = \frac{2a - 3m^2c_0^3c_3}{2k + 2m^2c_0^2c_2}, \]  

(39)

\[ T_c = \frac{(k + m^2c_0^2c_2)^2}{\pi(2a - 3m^2c_0^3c_3)} + \frac{3m^2c_0c_1}{8\pi}, \]  

(40)

\[ P_c = \frac{(k + m^2c_0^2c_2)^3}{2\pi(2a - 3m^2c_0^3c_3)^2}. \]  

(41)

The appearance of these values indicates critical behavior in five dimensions (see

Figure 5: \( G - T \) for \( d = 4, \ P = 1, \ c_0 = 1, \ c_1 = -2, \ c_2 = 3.18, \) and \( m = 2.1, \) (a): \( a=1 \) and (b): \( a=60. \)
Figure 6: $f(r) - r$ for $d = 5$, $\Lambda = -1$, $c_0 = 1$, $c_1 = -2$, $c_2 = 3.18$, $c_3 = 4$, $b = 20$ and $m = 2.1$, (a): $a=1$ and (b): $a=60$.

Figure 7: $P - r$ for $d = 5$, $c_0 = 1$, $c_1 = -2$, $c_2 = 4$, $c_3 = -4$ and $m = 2.1$, (a): $a=1$, $T = 0.3125T_c$ and (b): $a=0.1$, $T = 0.2912T_c$.

Figure 7). We note that if we exclude the massive term ($m \to 0$), the above equations are converted to the results in Ref [47] which is valid for $k = 1$, therefore in the absence of the massive term, we have critical behaviour only for spherical topology. In other words Einstein gravity modification with massive graviton in the presence of external string cloud can display second order phase transition and van der Waals like behaviour for topological black holes ($k = 0, -1$). We can consider the value of $(-\frac{4m^2c_0c_1}{\pi^2})$ for the correction to the Hawking temperature. If the graviton mass corrects Hawking’s temperature in this way, we can estimate the compression factor or the gas deviation factor as $P = \frac{v}{T} = \frac{1}{3} \simeq 0.66$, which is different from $P = \frac{v}{T} = \frac{3}{8}$ for the Van der Waals fluid, where $v = \frac{4}{3}r_0$ is defined as specific volume.

In critical behaviour the conditions of positive pressure and radius are required

$$k + m^2c_0c_2 > 0,$$

(42)
Figure 8: $T - r$ for $d = 5$, $\Lambda = -1$, $c_0 = 1$, $c_1 = -2$, $c_2 = 4$, $c_3 = -4$ and $m = 2.1$, (a) $a = 0.1 < a_c = 54.01$, (b) $a = 60 > a_c = 54.01$ and (c) $a = 60 > a_c = 54.01$, $c_3 = 4$.

\begin{equation}
2a - 3m^2c_3^3c_3 > 0.
\end{equation}

In Figure 7, the $P - r$ diagrams are plotted for $c_3 < 0$. It should be noted that if $c_3 > 0$, then it is necessary that $a > \frac{3}{2}m^2c_0^2c_3$ to maintain the condition (43). The condition of positive pressure results

\begin{equation}
T > \frac{3m^2c_0c_1}{8\pi} + \frac{2(k + m^2c_0^2c_2)^2}{3\pi(2a - 3m^2c_0^2c_3)},
\end{equation}

which equation (40) applies in this constraint.

To check the behaviour of temperature with respect to the radius one can plot $T - r$ diagrams (see Figure 8) for string cloud parameter $a$ less or greater than critical value $a_c$ where

\begin{equation}
a_c = \frac{1}{2} \sqrt{-\frac{4(k + m^2c_0^2c_2)^3}{\Lambda}} + \frac{3}{2}m^2c_0^3c_3.
\end{equation}

For $a < a_c$, there is a black hole from zero to one critical temperature and as the black hole grows its temperature rises. From this critical temperature to a definite amount
Figure 9: $C - r$ for $d = 5$, $\Lambda = -1$, $c_0 = 1$, $c_1 = -2$, $c_2 = 4$, $c_3 = -4$, and $m = 2.1$, (a) : $a=0.1$, (b): $a=60$, (c): $a=0.1$, $c_3 = 4$ and (d): $a=60$, $c_3 = 4$.

of temperature, we have three black holes, which, depending on the size of these three black holes, we call them small, medium and large. Then, as the temperature rises, two smaller black holes disappear and only the large black hole will be at high temperatures. For $a > a_c$ and $c_3 < 0$ at any given temperature, we have only one black hole. 

The specific heat in canonical ensemble for $d = 5$ is deduced

$$C = \frac{\pi [3kr_0^4 - ar_0^3 - \Lambda r_0^6 + m^2 (\frac{3c_0c_1}{8}r_0^5 + \frac{c_1^2c_2}{2}r_0^4 + \frac{c_1^2c_3}{4}r_0^3)]}{-3kr_0 + 2a - \Lambda r_0^3 - 3m^2c_0^2c_2r_0 - 3m^2c_0^2c_3}.$$  \hspace{1cm} (46)

The conditions of the stability of black holes are obvious from $C - r$ diagrams (see Figure 9).

The Gibbs free energy for $d = 5$ is as follows

$$G = -\frac{1}{18} (2a - 3m^2c_0^3c_3)r_0 + \frac{1}{12} (k + m^2c_0^2c_2)r_0^2 - \frac{\pi}{9} Pr_0^4,$$  \hspace{1cm} (47)

which has the following critical value

$$G_c = \frac{-(2a - 3m^2c_0^3c_3)^2}{96(k + m^2c_0^2c_2)}.$$  \hspace{1cm} (48)
The second order phase transition is clearly evident from the G-T diagrams (see Figure 10).

Thereafter, the presence of the massive coefficient $c_1$ in the pressure and temperature and its absence in Gibbs free energy allows us to examine the degeneracy of states in extremal black holes. An extremal black hole is the smallest possible black hole that can exist while rotating at a given fixed constant speed. It has been suggested by Sean Carroll [53] that the entropy of an extremal black hole is equal to zero. If we set $S = 0$ in Gibbs free energy formula then we obtain $G = E$. Therefore, by changing the coefficient $c_1$, the pressure and temperature change in each state, but the energies of the states are the same and we get the degeneracy between states.

4 black hole as heat engine

After considering black hole as thermodynamic system, one can consider such system as a heat engine. That is, black hole generates mechanical work by burning some substance. The substance used by such engine as fuel follow the equation of state.
An interesting advantage of examining the thermodynamics of the black holes in the extended phase space is that the mechanical term \( PdV \) in the first law provides the possibility of calculating the efficiency of these heat engines. Carnot showed that the heat engine that operates a reversible cycle composed of two isothermals and two adiabatics, has the highest efficiency compared with the other heat engines. The efficiency of a heat engine that working between two reservoirs of temperature is given by

\[
\eta = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H},
\]

where \( Q_H, Q_C \) and \( W \) stand for input heat to the system, output heat from the system and mechanical work done by the system, respectively. Apart from the Carnot cycle, which is the simplest cycle that can be considered, it is difficult to find an analytical formula for the efficiency of other cycles. The first time Johnson \[31\] proposed a simple cycle in \( P - V \) plane, known as squared cycle constructed by two adiabatics/isochorics and two isobarics as shown in Figure 11. Along the isobar \( 1 \to 2 \) and \( 3 \to 4 \), system produces work so that the total work done by this heat engine is equal to the area of the rectangle \( 1 \to 2 \to 3 \to 4 \). Also the input heat to the system along the isobar \( 1 \to 2 \) and the output heat from the system along the isobar \( 3 \to 4 \) is obtained from the first law of thermodynamics

\[
dH = \delta Q + VdP.
\]

Since the pressure is constant during the isobaric process it results that the heat absorbed/emitted is equal to the difference between the enthalpy of the two points at the beginning and the end of the process so the efficiency is as follows

\[
\eta = 1 - \frac{M_3 - M_4}{M_2 - M_1}.
\]

For the case of our black hole we plot the diagrams for efficiency according to the string cloud parameter \( a \) in four and five dimensions (see Figure 12). We see that the

Figure 12: \( \eta - a \) for \( S_1 = S_4 = \pi, S_2 = S_3 = 3\pi, P_1 = P_2 = 4, P_3 = P_4 = 1, c_0 = 1, c_1 = -2, c_2 = 4, c_3 = 4 \), and \( m = 2.1 \), (a): \( d = 4 \) and (b): \( d = 5 \).
efficiency increases by rising the string cloud parameter. On the other hand, given that the maximum value of efficiency is equal to one, an upper limit on the string cloud parameter can be applied. Also, the gradient of the graph in four dimensions is more than five dimensions.

5 Conclusion

In this paper we observed that the massive gravity minimally coupled to a cloud of strings cannot exhibit Van der Waals like behaviour for AdS–Schwarzschild black hole in four dimensions, but in five dimensions this modification takes place. This behaviour can be attributed to the absence of one term from the mass episode namely the third massive term in the equation of state in four dimensions. But in 5 dimensions, the massive parameter and the string cloud parameter, the latter only in flat topology, play a decisive role in the van der Waals like behaviour. It is interesting to note that the string cloud term in each dimensions treats like the final massive term. Also the string cloud parameter affects the number of event horizons and the quality of the black holes stability, so we found a critical value for this parameter. Finally we saw that the string cloud parameter affects the efficiency of the black hole heat engine and we showed for squared cycle, the efficiency grows up as the string cloud parameter increases and approaches to one. The upper limit that can be considered for a string cloud parameter becomes larger by increasing the dimensions.

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