Comment on “General Non-Markovian Dynamics of Open Quantum System”

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(Dated: January 23, 2014)

The existence of a “non-Markovian dissipationless” regime, characterized by long lived oscillations, was reported in Ref. [1] for a class of quantum open systems. It is claimed this could happen in the strong coupling regime, a surprising result which has attracted some attention. We show that this regime exists if and only if the total Hamiltonian is unbounded from below, casting serious doubts on the usefulness of this result.

We focus on the simplest bosonic model discussed in [1]: An oscillator $H_s = \Omega a^\dagger a$ couples to an environment $H_E = \sum_k \omega^k b_k^\dagger b_k$ through the interaction $H_I = \sum_k \lambda_k (a^k b_k^\dagger + a^k b_k)$. The total Hamiltonian $H_T = H_s + H_E + H_I$ commutes with the excitation number $N = a^\dagger a + \sum_k b_k^\dagger b_k$. Thus, an eigenstate of $H_T$ in the single excitation sector can be written $| \Phi \rangle = C^1 | 0, 0 \rangle$, with $C^1 = c_s a^\dagger + \sum_k c_k b_k^\dagger$ a generalized creation operator and $| 0, 0 \rangle$ the vacuum. Requiring $H_T | \Phi \rangle = E | \Phi \rangle$ implies $E c_s = \Omega c_s + \sum_k \lambda_k c_k$ and $E c_k = \omega_k c_k + \lambda_k c_s$ [2]. The energy $E$ therefore satisfies $E = \Omega + \sum k \lambda_k^2 / (\omega - \omega_k)$, which for negative solutions $E = -|\omega| = \sum k \lambda_k^2 / (|\omega| + \omega_k)$. This equation has solutions if and only if $\Omega + \delta \Omega < 0$ with $\delta \Omega = -\sum k \lambda_k^2 / \omega_k$. Thus, in this regime the total Hamiltonian acquires a negative eigenvalue. Moreover, noticing that $[H_T, C^1] = EC^1$, we see that there also exist eigenstates $| \Phi_\omega \rangle = (C^1)^n | 0, 0 \rangle$ with eigenvalues $nE = -n|\omega|$, extending to negative infinity: the total Hamiltonian is unbounded from below in this regime. Unbounded Hamiltonians, such as $H = -\omega a^\dagger a$ have no ground state, no thermal state (divergent partition functions), and which would act as infinite sources of energy when weakly coupled with any other system (they are thermodynamically unstable).

We now follow the argument presented in [1] to show that the dissipationless regime at strong coupling is precisely the regime when $\Omega + \delta \Omega < 0$, and the total Hamiltonian is unbounded. The density matrix of the system satisfies [1, 3], $\rho = -i[\hat{\Omega}(t) a^\dagger a, \rho] + (1 + \hat{n}(t))(2\rho a^\dagger a + a^\dagger a \rho - \rho a^\dagger a + \gamma(t) \hat{n}(t)(2\rho a^\dagger a - a^\dagger a \rho - \rho a^\dagger a))$. Coefficients depend on the Green’s function which satisfies $\hat{u}(t) + i\Omega u(t) + \int_0^t ds \eta(t-s) u(s) = 0$, where the dissipation kernel is $\eta(s) = \int_0^\infty d\omega J(\omega) \exp[-i\omega s]$, and spectral density $|\Omega(\omega)| = \sum k \lambda_k^2 \delta(\omega - \omega_k)$. The frequency and damping rate satisfy $i\Omega(\omega) + \gamma(\omega) = -i\Omega(\omega) + \gamma(t)(\omega) \hat{n}(t)$, while $\gamma(\omega) = \int_0^\infty d\tau \sum_k \lambda_k \delta(\omega - \omega_k)$. Thus, $\gamma(\omega) = \sum k \lambda_k \omega_k \delta(\omega - \omega_k)$, and $\hat{u}(s) = -\int_0^\infty d\omega J(\omega) \exp[i\omega s] / (\exp(\omega/k_B T) - 1)$ (T is the temperature of the environment). This equation is valid for all spectral densities and temperatures, and is the tool used to study non-perturbative and non-

FIG. 1: Damping rate and renormalized frequency for the dissipationless sub-Ohmic model shown in Fig. (2) of Ref. [1].

Markovian effects.

A dissipationless regime exists when $u(t) \rightarrow r \exp[-i\omega_0 t]$ at long times. In this case, $\Omega(t) = \omega_0$, $\gamma(t) = \gamma(t)|\hat{n}(t)| = 0$: as $t \rightarrow \infty$ the system evolves unitarily with a Hamiltonian $H_s = \omega_0 a^\dagger a$. For $u(t)$ to behave in this way, its Laplace transform must have a purely imaginary pole, i.e. $\omega_0 - \Omega + i\eta(-i\omega_0) = 0$, where the Laplace transform of $\eta(t)$ is $\eta(s)$. The imaginary part of this equation is $J(\omega_0) = 0$. For spectral densities of any type (Ohmic, sub-Ohmic etc.) satisfying $J(\omega) > 0$ for all $\omega > 0$, this condition can be satisfied for $\omega_0 < 0$ [1]. With $\omega_0 = -|\omega_0|$ the real part gives $\Omega - |\omega| = \int_0^\infty d\omega J(\omega)/(|\omega| + |\omega_0|)$, which has solutions if and only if $\Omega + \delta \Omega < 0$, with $\delta \Omega = -\int_0^\infty d\omega J(\omega)/|\omega|$. This is precisely the condition under which the total Hamiltonian becomes unbounded. This also manifests in the master equation. In Fig. (1) we plot $\gamma(t)$ and $\Omega(t)$ for the same parameters as in Fig. (2) of Ref. [1] (where $\Omega(t)$ was not analyzed). This exact calculation shows that $\hat{n}(t)$ approaches a negative value while the damping rate vanishes, making the renormalized Hamiltonian also unbounded.

Thus, we have proved that the dissipationless regime of Ref. [1] for strong coupling exists if and only if the total Hamiltonian is unbounded from below, and therefore thermodynamically unstable. An analogous instability is well known for the famous model where the system Hamiltonian is $H_s = p^2/2m + \kappa \omega^2/2$, the environment is $H_E = \sum_k (p_k^2/2m_k + m_k \omega_k^2 q_k^2/2)$ and the interaction is $H_I = \sum_k c_k q_k$. A simple calculation shows that the total Hamiltonian is $H_T = H_R + \sum_k (p_k^2/2m_k + (m_k \omega_k^2 q_k + c_k x)^2/2m_k \omega_k^2)$]. Here, $H_R = p^2/2m + \kappa \omega^2/2$ with $\kappa = 0$ and $\delta \kappa = -\sum \lambda_k^2 / m_k \omega_k^2$. Thus, when $\kappa < 0$, the total Hamiltonian $H_T$ is unbounded (and, as opposed to the previous case, is both thermodynamically and dynamically unstable).

In Ref. [1] the authors identify long lived oscillations when $J(\omega)$ has band gaps. We do not question this result, which is indeed valid but far from surprising.
There are various ways to demonstrate the unbounded nature of the Hamiltonian. Here we extend a simple method presented in A. Rancon and J. Bonart, Euro. Phys. Lett., 104, 50010 (2014)

[1] W.-M. Zhang and P.-Y. Lo and H.-N. Xiong and M. W.-Y. Tu and F. Nori, Phys. Rev. Lett., 109, 170402 (2012)

[2] J.P. Paz and A.J. Roncaglia, Quantum Inf. Processing 8, 535 (2009)

[3] E.A. Martinez and J.P. Paz, Phys. Rev. Lett., 110, 130406 (2013)