Fuzzy Soft Connected Sets in Fuzzy Soft Topological Spaces I

A. Kandil 1, O. A. E. Tantawy 2, S. A. El-Sheikh 3 and Sawsan S. S. El-Sayed 4

1 Mathematics Department, Faculty of Science, Helwan University, Helwan, Egypt
dr.ali_kandil@yahoo.com
2 Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt
drosamat@yahoo.com
3 Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt
elsheikh33@hotmail.com
4 Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt
sawsan_809@yahoo.com

ABSTRACT

In this paper we introduce some types of fuzzy soft separated sets and study some of their properties. Next, the notion of connectedness in fuzzy topological spaces due to Ming and Ming, Zheng etc., extended to fuzzy soft topological spaces. The relationship between these types of connectedness in fuzzy soft topological spaces is investigated with the help of number of counter examples.

Keywords: Fuzzy soft sets; fuzzy soft topological space; fuzzy soft separated sets; fuzzy soft Q-separated sets; fuzzy soft weakly separated sets; fuzzy soft strongly separated sets; fuzzy soft connected sets.

1 INTRODUCTION

After Zadeh [26] introduced the notion of a fuzzy set in 1965, Chang [4] used that concept to define fuzzy topology. In 1999, Molodtsov [15] introduced the concept of soft set theory which is a completely new approach for modeling uncertainty. In [15], Molodtsov established the fundamental results of this new theory and successfully applied the soft set theory into several directions, such as smoothness of functions, operations research, Riemann integration, game theory, theory of probability and so on. Maji et al. [13] defined and studied several basic notions of soft set theory in 2003. Shabir and Naz [21] introduced the concept of soft topological space.

Maji et al. [12] initiated the study involving both fuzzy soft sets and soft sets. In this paper, the notion of fuzzy soft sets was introduced as a fuzzy generalizations of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Maji et al. combined fuzzy sets and soft sets and introduced the concept of fuzzy soft sets. To continue the investigation on fuzzy soft sets, Ahmad and Kharal [1] presented some more properties of fuzzy soft sets and introduced the notion of a mapping on fuzzy soft sets. In 2011, Tanay et al. [22] gave the topological structure of fuzzy soft sets.

The notion of connectedness in fuzzy topological spaces has been studied by Ming and Ming [14], Lowen [10], Zheng Chong You [28], Fatteh and Bassan [7], Zhao [27], Saha [20], Ajmal and Kohli [2], and Chaudhuri and Das [6]. In soft setting, the notion of soft connectedness introduced by many authors such as Peyghan et al. [18], Yüksel et al. [25]. In 2013, Bayramov et al. [4] studied the soft path connectedness on soft topological spaces. In 2014, Munir et al. [16] studied some properties of soft connected spaces and soft locally connected spaces. In 2015, Hussain [8] wrote a paper entitled a note on soft connectedness. In fuzzy soft setting, connectedness has been introduced by Mahanta [11] and Karatas et al. [9].

In this paper, we extend the notion of connectedness of fuzzy topological space to fuzzy soft topological space. In Section 3, we introduce the different notions of fuzzy soft separated sets and study the relationship between them. Section 4 is devoted to introduce the different notions of connectedness in fuzzy soft topological space and study the implications that exist between them. Also, we study the characterization of connectedness in fuzzy soft setting.

2 PRELIMINARIES

Throughout this paper $X$ denotes initial universe, $E$ denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in $X$, and the set of all subsets of $X$ will be denoted by $P(X)$. In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel.

Definition 2.1. [5] A fuzzy set $A$ of a non-empty set $X$ is characterized by a membership function $\mu_A : X \rightarrow [0,1] = I$ whose value $\mu_A(x)$ represents the “degree of membership” of $x$ in $A$ for $x \in X$. Let $I^X$ denotes the family of all fuzzy sets on $X$.

Definition 2.2. [15] Let $A$ be a non-empty subset of $E$. A pair $(F, A)$ denoted by $F_A$ is called a soft set over $X$, where $F$ is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over $X$ is a parametrized family of subsets of
the universe $X$. For a particular $e \in A$, $F(e)$ may be considered the set of $e$·approximate elements of the soft set $(F, A)$ and if $e \notin A$, then $F(e) = \emptyset$ i.e., $F = \{F(e): e \in A \subseteq E, F: A \rightarrow P(X)\}$.

Aktas and Cagman [3] showed that every fuzzy set may be considered as a soft set. That is, fuzzy sets are a special class of soft sets.

**Definition 2.3.** [12] Let $A \subseteq E$. A pair $(f, A)$, denoted by $f_A$, is called fuzzy soft set over $X$, where $f$ is a mapping given by $f: A \rightarrow I^X$ defined by $f_A(e) = \mu_{f_A}^e$, where $\mu_{f_A}^e = \bar{0}$ if $e \notin A$, and $\mu_{f_A}^e \neq \bar{0}$ if $e \in A$, where $\bar{0}(x) = 0 \forall x \in X$. The family of all these fuzzy soft sets over $X$ is denoted by $FSS(X)_E$.

**Definition 2.4.** [12, 17, 19, 22, 23, 24] The complement of a fuzzy soft set $(f, A)$, denoted by $(f, A)^c$, and defined by $(f, A)^c = (f^c, A)$, $f^c_A: A \rightarrow I^X$ is a mapping given by $\mu^c_{f_A} = 1 - \mu^e_{f_A}$ $\forall e \in A$. Clearly, $(f_A)^c = f_A$.

**Definition 2.5.** [12, 17, 19, 22, 24] A fuzzy soft set $f_B$ over $X$ is said to be a null-fuzzy soft set, denoted by $\tilde{0}_B$, if for all $e \in E$, $f_B(e) = \tilde{1}_E$.

**Definition 2.6.** [12, 17, 19, 22, 24] A fuzzy soft set $f_B$ over $X$ is said to be an absolute fuzzy soft set, denoted by $\tilde{1}_B$, if $f_B(e) = \tilde{1}_E \forall e \in E$. Clearly, we have $(\tilde{0}_B)^c = \tilde{1}_B$ and $(\tilde{1}_B)^c = \tilde{0}_B$.

**Definition 2.7.** [12, 17, 19, 22, 23, 24] Let $f_A$ and $g_B \in FSS(X)_E$. Then $f_A$ is fuzzy soft subset of $g_B$, denoted by $f_A \subseteq g_B$, if $A \subseteq B$ and $\mu^e_{f_A} \leq \mu^e_{g_B}$ $\forall e \in A$. Also, $g_B$ is called fuzzy soft superset of $f_A$ denoted by $g_B \supseteq f_A$. If $f_A$ is not fuzzy soft subset of $g_B$, we written as $f_A \not\subseteq g_B$.

**Definition 2.8.** [12, 17, 19, 22, 23, 24] Two fuzzy soft sets $f_A$ and $g_B$ on $X$ are called equal if $f_A \subseteq g_B$ and $g_B \subseteq f_A$.

**Definition 2.9.** [12, 17, 19, 22, 24] The union of two fuzzy soft sets $f_A$ and $g_B$ over the common universe $X$, denoted by $f_A \cup g_B$, is also a fuzzy soft set $h_C$, where $C = A \cup B$ and for all $e \in C$, $h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \vee \mu^e_{g_B}$.

**Definition 2.10.** [12, 17, 19, 22, 24] The intersection of two fuzzy soft sets $f_A$ and $g_B$ over the common universe $X$, denoted by $f_A \cap g_B$, is also a fuzzy soft set $h_C$, where $C = A \cap B$ and for all $e \in C$, $h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \land \mu^e_{g_B}$.

**Definition 2.12.** [22] Let $FSS(X)_E$ be a collection of fuzzy soft sets over a universe $X$ with a fixed set of parameters $E$. Then $\tau \subseteq FSS(X)_E$ is called fuzzy soft topology on $X$ if:

1. $\tilde{0}_X, \tilde{1}_X \in \tau$, where $\tilde{0}_X(e) = \tilde{0}$ and $\tilde{1}_X(e) = \tilde{1}_E \forall e \in E$,
2. The union of any members of $\tau$ belongs to $\tau$.
3. The intersection of any two members of $\tau$ belongs to $\tau$.

The triplet $(X, \tau, E)$ is called fuzzy soft topological space over $X$. Also, each member of $\tau$ is called fuzzy soft open set in $(X, \tau, E)$.

**Examples 2.1.** The following are fuzzy soft topology on $X$:

1. $\tau_0 = \{\tilde{0}_X, \tilde{1}_X\}$ is called fuzzy soft indiscrete topology on $X$.
2. $\tau_0 = FSS(X)_E$ is called fuzzy soft discrete topology on $X$.

Note that, the intersection of any family of fuzzy soft topologies on $X$ is also a fuzzy soft topology on $X$.

**Definition 2.13.** [22] Let $(X, \tau, E)$ be a fuzzy soft topological space. A fuzzy soft set $f_A$ over $X$ is said to be fuzzy soft closed set in $X$, if its relative complement $f_A^c$ is fuzzy soft open set. The collection of all fuzzy soft closed sets will be denoted by $\tau^c$.
Definition 2.14. [17, 19] Let \((X, \tau, E)\) be a fuzzy soft topological space and \(f_A \in FSS(X)_E\). The fuzzy soft closure of \(f_A\), denoted by \(\text{Fcl}(f_A)\), is the intersection of all fuzzy soft closed supersets of \(f_A\), i.e., 
\[ \text{Fcl}(f_A) = \overline{\{h_C : h_C \in \tau', f_A \subseteq h_C\}}. \]
Clearly, \(\text{Fcl}(f_A)\) is the smallest fuzzy soft closed set over \(X\) which contains \(f_A\) and \(\text{Fcl}(f_A)\) is fuzzy soft closed set.

Definition 2.15. [19, 24] The fuzzy soft set \(f_A \in FSS(X)_E\) is called fuzzy soft point if there exist \(x \in X\) and \(e \in E\) such that \(\mu^e_{f_A}(x) = 0\), \(\mu^e_{f_A}(y) = 0\) \(\forall y \in X - \{x\}\) and this fuzzy soft point is denoted by \(x^e_{\alpha}\) or \(f_e\). The class of all fuzzy soft points of \(X\), denoted by \(\text{Fsp}(X)_E\).

Definition 2.16. [11] The fuzzy soft point \(x^e_{\alpha}\) is said to be belonging to the fuzzy soft set \(f_A\), denoted by \(x^e_{\alpha} \in f_A\), if for the element \(e \in A\), \(0 < \alpha \leq \mu^e_{f_A}(x)\). If \(x^e_{\alpha}\) is not belong to \(f_A\), we write \(x^e_{\alpha} \notin f_A\) and implies that \(\alpha > \mu^e_{f_A}(x)\).

Definition 2.17. [19, 24] A fuzzy soft point \(x^e_{\alpha}\) is said to be a quasi-coincident with a fuzzy soft set \(f_A\), denoted by \(x^e_{\alpha} \preceq f_A\), if \(\alpha + \mu^e_{f_A}(x) > 1\). Otherwise, \(x^e_{\alpha}\) is non-quasi-coincident with \(f_A\) and denoted by \(x^e_{\alpha} \npreceq f_A\).

Definition 2.18. [19, 24] A fuzzy soft set \(f_A\) is said to be quasi-coincident with \(g_B\), denoted by \(f_A \npreceq g_B\), if there exists \(x \in X\) such that \(\mu^e_{f_A}(x) + \mu^e_{g_B}(x) > 1\) for some \(e \in A \cap B\). If this is true we can say that \(f_A\) and \(g_B\) are quasi-coincident at \(x\).

Proposition 2.1. [19, 24] Let \(f_A\) and \(g_B\) be two fuzzy soft sets, \(f_A \subseteq g_B\) if and only if \(f_A \npreceq g_B\). In particular, \(x^e_{\alpha} \in f_A\) if and only if \(x^e_{\alpha} \npreceq f_A\).

Definition 2.19. [11] Let \((X, \tau, E)\) be a fuzzy soft topological space and \(g_B\) be a fuzzy soft subset of \(X\). Then \(\tau_{gB} = \{g_B \subset f_A : f_A \in \tau\}\) is called fuzzy soft relative topology and \((g_B, \tau_{gB}, B)\) is called fuzzy soft subspace. If \(g_B \in \tau\), then \((g_B, \tau_{gB}, B)\) is called fuzzy soft open subspace. If \(g_B \in \tau\), then \((g_B, \tau_{gB}, B)\) is called fuzzy soft closed subspace.

Lemma 2.1. [11] Let \((X, \tau, E)\) be a fuzzy soft topological space on \(X\) and \(g_B \subseteq f_A \in FSS(X)_E\). Then, 
\[ \text{Fcl}(g_B) = \text{Fcl}(g_B) \cap f_A. \]

Definition 2.19. [17] Let \(FSS(X)_E\) and \(FSS(Y)_K\) be families of fuzzy soft sets over \(X\) and \(Y\), respectively. Let \(u : X \rightarrow Y\) and \(p : E \rightarrow K\) be mappings. Then the map \(f_{pu}\) is called fuzzy soft mapping from \(FSS(X)_E\) to \(FSS(Y)_K\), denoted by \(f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K\), such that:

1. If \(g_B \in FSS(X)_E\), then the image of \(g_B\) under the fuzzy soft mapping \(f_{pu}\) is a fuzzy soft set over \(Y\) defined by \(f_{pu}(g_B)\) where \(\forall k \in p(E), \forall y \in Y, \)
\[ f_{pu}(g_B)(k)(y) = \begin{cases} \lor_{u(x) = y} \lor_{p(e) = k} (g_B(e))(x) & \text{if } x \in u^{-1}(y) \\ 0 & \text{otherwise} \end{cases} \]

2. If \(h_C \in FSS(Y)_K\), then the pre-image of \(h_C\) under the fuzzy soft mapping \(f_{pu}\) is a fuzzy soft set over \(X\) defined by \(\forall e \in p^{-1}(K), \forall x \in X, \)
\[ f_{pu}^{-1}(h_C)(e)(x) = \begin{cases} h_C(p(e))(u(x)) & \text{for } p(e) \in C \\ 0 & \text{otherwise} \end{cases} \]

Definition 2.20. [17] The fuzzy soft mapping \(f_{pu}\) is called surjective (respectively, injective) if \(p\) and \(u\) are surjective (respectively, injective), also \(f_{pu}\) is said to be constant if \(p\) and \(u\) are constant.

Definition 2.21. [17] Let \((X, \tau_1, E)\) and \((Y, \tau_2, K)\) be two fuzzy soft topological spaces and \(f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K\) be a fuzzy soft mapping. Then \(f_{pu}\) is called:
(1) Fuzzy soft continuous if \( f_{pu}^{-1}(h_c) \in \tau_1 \forall h_c \in \tau_2. 

(2) Fuzzy soft open if \( f_{pu}(g_f) \in \tau_2 \forall g_f \in \tau_1. 

**Definition 2.22.** [9] Two non-null fuzzy soft sets \( f_E \) and \( g_E \) are said to be fuzzy soft \( Q \)-separated in a fuzzy soft topological space \((X, \tau, E)\) if \( Fcl(f_E) \cap g_E = f_E \cap Fcl(g_E) = 0_E. \)

**Theorem 2.1.** [9] Let \((X, \tau, E)\) be a fuzzy soft topological space, \( f_E \) and \( g_E \) be two fuzzy soft closed sets in \( X \). If \( f_E \cap g_E \neq 0_E \), then \( f_E \) and \( g_E \) are fuzzy soft \( Q \)-separated sets.

**Theorem 2.2.** [9] Let \((X, \tau, E)\) be a fuzzy soft topological space and \( f_E \in FSS(X)_E \). \( f_E \) is called \( FSC_M \)-connected if and only if it cannot be written as a union of fuzzy soft \( Q \)-separated sets.

**Theorem 2.3.** [9] A fuzzy soft topological space \((X, \tau, E)\) is \( FSC_M \)-connected if and only if \( \tilde{1}_E \) cannot be written as a union of fuzzy soft \( Q \)-separated sets.

**Theorem 2.4.** [9] Let \((X, \tau, E)\) be a fuzzy soft topological space and \( f_E \in FSS(X)_E \) be an open \( FSC_M \)-connected set in \( X \). If \( f_E \subseteq g_E \subseteq Fcl(f_E) \), then \( g_E \) is a \( FSC_M \)-connected set.

**Remark 2.1.** [9] Let \((X, \tau, E)\) be a fuzzy soft topological space and \( f_E \in FSS(X)_E \) be an open \( FSC_M \)-connected set in \( X \). Then \( Fcl(f_E) \) is a \( FSC_M \)-connected set.

**Definition 2.23.** [9] Let \((X, \tau, E)\) be a fuzzy soft topological space and \( f_E \in FSS(X)_E \). Then, \( f_E \) is called:

1. \( FSC_1 \)-connected: if does not exist two non-null fuzzy soft open sets \( h_E \) and \( s_E \) such that \( f_E \subseteq h_E \cap s_E, h_E \cap s_E \subseteq f_E, \widetilde{h_E} \cap s_E = 0_E, f_E \cap h_E \neq 0_E, \text{and } f_E \cap s_E \neq 0_E. \)

2. \( FSC_2 \)-connected: if does not exist two non-null fuzzy soft open sets \( h_E \) and \( s_E \) such that \( f_E \subseteq h_E \cap s_E, h_E \cap s_E = 0_E, f_E \cap h_E \neq 0_E, \text{and } f_E \cap s_E \neq 0_E. \)

3. \( FSC_3 \)-connected: if does not exist two non-null fuzzy soft open sets \( h_E \) and \( s_E \) such that \( f_E \subseteq h_E \cap s_E, h_E \cap s_E \subseteq f_E, h_E \not\subset f_E, \text{and } s_E \not\subset f_E. \)

4. \( FSC_4 \)-connected: if does not exist two non-null fuzzy soft open sets \( h_E \) and \( s_E \) such that \( f_E \subseteq h_E \cap s_E, h_E \cap s_E = 0_E, h_E \not\subset f_E, \text{and } s_E \not\subset f_E. \)

Otherwise, \( f_E \) is called \( FSC_i \)-disconnected set for \( i = 1,2,3,4 \).

In the above definition, if we take \( \tilde{1}_E \) instead of \( f_E \), then the fuzzy soft topological space \((X, \tau, E)\) is called \( FSC_i \)-connected space \( (i = 1,2,3,4) \).

**Remark 2.2.** [9] The relationship between \( FSC_i \)-connectedness \((i = 1,2,3,4)\) can be described by the following diagram:

\[
\begin{array}{c}
FSC_1 \\
\downarrow \\
FSC_2 \\
\downarrow \\
FSC_3 \\
\downarrow \\
FSC_4
\end{array}
\]

**Remark 2.3.** [9] The reverse implications is not true in general (see Examples 3.14, 3.15, 3.16, 3.17 in [9]). But example 3.17 in [9] is incorrect, we must take \( \mu_{g(b)}(x) = \frac{2}{3} < \frac{2}{3} < x \leq 1. \)

**Theorem 2.5.** [9] Let \( f_{[b]}: (X, \tau, E) \rightarrow (Y, \sigma, K) \) be a fuzzy soft surjective continuous mapping and \( f_E \in FSS(X)_E \). If \( f_E \) is a \( FSC_i \)-connected set in \( X \), then \( f_{pu}(f_E) \) is a \( FSC_i \)-connected set in \( Y \) for \( i = 1,2,3,4 \).

### 3 FUZZY SOFT SEPARATED SETS IN FUZZY SOFT TOPOLOGICAL SPACES

In this section, we will introduce different notions of fuzzy soft separated sets and study the relation between these
notions. Also, we will investigate the characterizations of the fuzzy soft separated sets.

**Definition 3.1.** Two non-null fuzzy soft sets $f_E, g_E$ are said to be fuzzy soft weakly separated in a fuzzy soft topological space $(X, \tau, E)$ if $Fcl(f_E) \cap g_E$ and $Fcl(g_E) \cap f_E \neq \emptyset$.

**Theorem 3.1.** Let $(X, \tau, E)$ be a fuzzy soft topological space and $f_E, g_E \in FSS(X)_E$. Then, $f_E$ and $g_E$ are fuzzy soft weakly separated sets iff there exist fuzzy soft open sets $h_E$ and $s_E$ such that $f_E \subseteq h_E$, $g_E \subseteq s_E$, $f_E \cap s_E \neq \emptyset$, and $g_E \cap h_E \neq \emptyset$. The converse is obvious.

**Proof.** Let $f_E, g_E$ be fuzzy soft weakly separated sets in $(X, \tau, E)$. Then $Fcl(f_E) \cap g_E$ and $Fcl(g_E) \cap f_E$. Therefore, $g_E \subseteq Fcl(f_E)$ and $f_E \subseteq Fcl(g_E)$. Taking $h_E = [Fcl(g_E)]^c$ and $s_E = [Fcl(f_E)]^c$. Then $h_E \cap s_E \neq \emptyset$. The converse is obvious.

**Remark 3.1.** If $f_E$ and $g_E$ are fuzzy soft $Q$-separated, then $f_E$ and $g_E$ are fuzzy soft weakly separated.

**Proof.** The result follows from Definitions 3.1, 2.22.

**Remark 3.2.** If $f_E$ and $g_E$ are fuzzy soft weakly separated, then they may not be fuzzy soft $Q$-separated as shown by the following example.

**Example 3.1.** Let $X = \{a, b\} = E = \{e_1, e_2\}$ and $A = \{e_1, e_2\} \subseteq E$ and $\tau = \{\emptyset, \{e_1, a_{0.5}, b_{0.3}\}, \{e_2, a_{0.5}, b_{0.3}\}\}$ be a fuzzy soft topology on $X$. If $f_A = \{e_1, a_{0.1}\}$ and $g_A = \{e_1, a_{0.1}, b_{0.1}\}$, then $f_E$ and $g_A$ are fuzzy soft weakly separated sets. But $f_A$ and $g_A$ are not fuzzy soft $Q$-separated.

**Definition 3.2.** Two non-null fuzzy soft sets $f_E$ and $g_E$ are said to be fuzzy soft separated in a fuzzy soft topological space $(X, \tau, E)$ if there exist non-null fuzzy soft open sets $h_E$ and $s_E$ such that $f_E \subseteq h_E$, $g_E \subseteq s_E$ and $f_E \cap s_E \neq \emptyset$, and $g_E \cap h_E \neq \emptyset$. The support of $f_E$, denoted by $S(f_E)$, is the set, $S(f_E) = \{x \in X; f_E(x) > 0\}$.

**Example 3.2.** Let $X = \{a, b\} = E = \{e\}$ and $\tau = \{\emptyset, \{e\}\}$, $h_E = \{(e, a_{0.5})\}$, $s_E = \{(e, b_{0.5})\}$, $h_E \subseteq s_E$ be a fuzzy soft topology on $X$. If $f_E = \{e, a_{0.5}\}$ and $g_E = \{e, b_{0.5}\}$, then $f_E$ and $g_E$ are fuzzy soft $Q$-separated sets. But $f_E \cap s_E \neq \emptyset$ and $g_E \cap h_E \neq \emptyset$. So, $f_E$ and $g_E$ are not fuzzy soft separated sets.

**Definition 3.3.** Let $f_E \in FSS(X)_E$. The support of $f_E$, denoted by $S(f_E)$, is the set, $S(f_E) = \{x \in X; f_E(x) > 0\}$.

**Example 3.3.** Let $X = \{a, b\} = E = \{e\}$ and $\tau = \{\emptyset, \{e\}\}$, $h_E = \{(e, a_{0.5})\}$, $s_E = \{(e, b_{0.5})\}$, $h_E \subseteq s_E$ be a fuzzy soft topology on $X$. If $f_E = \{e, a_{0.5}\}$ and $g_E = \{e, b_{0.5}\}$, then $f_E$ and $g_E$ are fuzzy soft $Q$-separated sets. But $f_E \cap s_E \neq \emptyset$ and $g_E \cap h_E \neq \emptyset$. So, $f_E$ and $g_E$ are not fuzzy soft separated sets.

**Definition 3.4.** Two fuzzy soft sets $f_E$ and $g_E$ are said to be quasi-coincident with respect to $f_E$ if $\mu_{f_E}(x) + \mu_{g_E}(x) > 1$ for every $x \in S(f_E)$. 
Definition 3.5. Two non-null fuzzy soft sets \( f_E \) and \( g_E \) are said to be fuzzy soft strongly separated in a fuzzy soft topological space \( (X, \tau, E) \) if there exist \( h_E \) and \( s_E \in \tau \) such that \( f_E \subseteq h_E \), \( g_E \subseteq s_E \), \( f_E \cap s_E = g_E \cap h_E = \emptyset_E \), and \( f_E \) and \( h_E \) are fuzzy soft quasi-coincident with respect to \( f_E \), and \( g_E \) and \( s_E \) are fuzzy soft quasi-coincident with respect to \( g_E \).

Remark 3.6. If \( f_E \) and \( g_E \) are fuzzy soft strongly separated, then \( f_E \) and \( g_E \) are fuzzy soft separated and fuzzy soft weakly separated.

**Proof.** The result follows from Definitions 3.5, 3.2 and Remark 3.3.

Remark 3.7. If \( f_E \) and \( g_E \) are fuzzy soft separated, then they may not be fuzzy soft strongly separated as shown by the following example.

Example 3.4. Let \( X = \{a, b\} \), \( E = \{e_1, e_2\} \), \( A = \{e_1\} \), \( B = \{e_2\} \subseteq E \) and \( \tau = \{1_E, \emptyset_E\} \). \( \{(e_1, \{a_{0.3}, b_{0.2}\})\}, \{(e_2, \{a_{0.3}, b_{0.2}\})\} \) is a fuzzy soft topology on \( X \). Let \( f_A = \{(e_1, \{a_{0.3}\})\} \) and \( g_B = \{(e_2, \{b_{0.2}\})\} \). Then, \( f_A \) and \( g_B \) are fuzzy soft separated sets, but \( f_A \) and \( g_B \) are not fuzzy soft strongly separated.

Remark 3.8. The notions of fuzzy soft \( Q \)-separated and fuzzy soft strongly separated are independent to each others as shown by the following examples:

Example 3.5. Let \( X = \{a, b\} \), \( E = \{e_1, e_2\} \), \( A = \{e_1\} \), \( B = \{e_2\} \subseteq E \) and \( \tau = \{1_E, \emptyset_E\} \). \( \{(e_1, \{a_{0.7}, b_{0.2}\})\}, \{(e_2, \{a_{0.7}, b_{0.2}\})\} \) is a fuzzy soft topology on \( X \). Let \( f_A = \{(e_1, \{a_{0.5}\})\} \) and \( g_B = \{(e_2, \{b_{0.4}\})\} \). Then, \( f_A \) and \( g_B \) are fuzzy soft strongly separated sets, but \( f_A \) and \( g_B \) are not fuzzy soft \( Q \)-separated.

Example 3.6. Let \( X = \{a, b\} \), \( E = \{e_1, e_2\} \), \( A = \{e_1, e_2\} \subseteq E \) and \( \tau = \{1_E, \emptyset_E\} \). \( \{(e_1, \{a_{0.3}, b_{0.2}\})\}, \{(e_2, \{a_{0.1}, b_{0.4}\})\}, \{(e_1, \{a_{0.3}, b_{0.2}\})\}, \{(e_2, \{a_{0.1}, b_{0.4}\})\} \) is a fuzzy soft topology on \( X \). Let \( f_A = \{(e_1, \{a_{0.2}\})\} \) and \( g_B = \{(e_2, \{b_{0.3}\})\} \). Then, \( f_A \) and \( g_B \) are fuzzy soft \( Q \)-separated sets, but \( f_A \) and \( g_B \) are not fuzzy soft strongly separated.

Remark 3.9. In fuzzy soft topological space \( (X, \tau, E) \) the relationship between different notions of fuzzy soft separated sets can be described by the following diagram.

\[
\begin{align*}
\text{fuzzy soft strongly separated} & \downarrow \quad \text{fuzzy soft separated} \\
\text{fuzzy soft } Q \text{- separated} & \downarrow
\end{align*}
\]

\[\text{fuzzy soft } Q \text{- separated } \Rightarrow \text{fuzzy soft weakly separated}\]

**Theorem 3.2.** Let \( f_E \) and \( g_E \) be fuzzy soft \( Q \)-separated (respectively, separated, strongly separated, weakly separated) sets in \( X \) and \( h_E \subseteq f_E, s_E \subseteq g_E \). Then, \( h_E \) and \( s_E \) are fuzzy soft \( Q \)-separated (respectively, separated, strongly separated, weakly separated) sets in \( X \).

**Proof.** As a sample, we will prove the case fuzzy soft \( Q \)-separated. Let \( f_E \) and \( g_E \) be fuzzy soft \( Q \)-separated sets in \( X \).

Then, \( \text{Fcl}(f_E) \cap g_E = f_E \cap \text{Fcl}(g_E) = \emptyset_E \). Since \( h_E \subseteq f_E \) and \( s_E \subseteq g_E \), then \( \text{Fcl}(h_E) \cap s_E = h_E \cap \text{Fcl}(s_E) = \emptyset_E \). Therefore, \( h_E \) and \( s_E \) are fuzzy soft \( Q \)-separated sets in \( X \).

**Theorem 3.3.** Let \( (X, \tau, E) \) be a fuzzy soft topological space and \( f_E, g_E \in \text{FSS}(X)_E \). Then, \( f_E \) and \( g_E \) are fuzzy soft \( Q \)-separated in \( X \) if there are fuzzy soft closed sets \( h_E \) and \( s_E \) such that \( f_E \subseteq h_E, g_E \subseteq s_E \) and \( f_E \cap s_E = g_E \cap h_E = \emptyset_E \).

**Proof.** Let \( f_E \) and \( g_E \) be fuzzy soft \( Q \)-separated in \( X \). Then \( \text{Fcl}(f_E) \cap g_E = f_E \cap \text{Fcl}(g_E) = \emptyset_E \). Taking \( h_E = \text{Fcl}(f_E) \) and \( s_E = \text{Fcl}(g_E) \). Therefore, \( h_E \) and \( s_E \) are fuzzy soft closed sets in \( X \) such that \( f_E \subseteq h_E, g_E \subseteq s_E \) and \( f_E \cap s_E = g_E \cap h_E = \emptyset_E \).
Theorem 3.4. Let \((X, \tau, E)\) be a fuzzy soft topological space and \(g_E \subseteq f_E \in FSS(X)_E\). Two fuzzy soft sets \(h_E\) and \(s_E\) are fuzzy soft separated (respectively, \(Q\)-separated, strongly separated) in \((g_E, \tau_{g_E}, E)\) if and only if \(h_E \cap s_E = \emptyset\). If \(g_E \subseteq f_E\), then \(Fcl_{g_E}(h_E) \cap s_E = \emptyset\). Since \(g_E \subseteq f_E\), then \(Fcl_{g_E}(h_E) = Fcl_{f_E}(h_E) \cap g_E\) and \(Fcl_{g_E}(s_E) = Fcl_{f_E}(s_E)\). Therefore, \(Fcl_{f_E}(h_E) \cap s_E = h_E \cap s_E = g_E \cap s_E = \emptyset\) and \(Fcl_{f_E}(s_E) = s_E\). Hence, \(h_E\) and \(s_E\) are fuzzy soft \(Q\)-separated in \((f_E, \tau_{f_E}, E)\).

Proof. As a sample, we will prove the case fuzzy soft \(Q\)-separated. Let \(h_E\) and \(s_E\) be fuzzy soft \(Q\)-separated in \((g_E, \tau_{g_E}, E)\). Then \(Fcl_{g_E}(h_E) \cap s_E = h_E \cap s_E = g_E \cap s_E = \emptyset\). Since \(g_E \subseteq f_E\), then \(Fcl_{f_E}(h_E) = Fcl_{f_E}(h_E) \cap g_E\) and \(Fcl_{g_E}(s_E) = Fcl_{f_E}(s_E)\). Therefore, \(Fcl_{f_E}(h_E) \cap s_E = h_E \cap s_E = g_E \cap s_E = \emptyset\). Hence, \(h_E\) and \(s_E\) are fuzzy soft \(Q\)-separated in \((f_E, \tau_{f_E}, E)\).

Conversely, let \(h_E\) and \(s_E\) be fuzzy soft \(Q\)-separated in \((f_E, \tau_{f_E}, E)\). Then \(Fcl_{f_E}(h_E) \cap s_E = h_E \cap s_E = g_E \cap s_E = \emptyset\). Therefore, \((Fcl_{f_E}(h_E) \cap g_E) \cap s_E = h_E \cap s_E = g_E \cap s_E = \emptyset\). And so, \(Fcl_{g_E}(h_E) \cap s_E = h_E \cap s_E = g_E \cap s_E = \emptyset\). Hence, \(h_E\) and \(s_E\) are fuzzy soft \(Q\)-separated in \((g_E, \tau_{g_E}, E)\).

Theorem 3.5. Let \((X, \tau, E)\) be a fuzzy soft topological space and \(g_E \subseteq f_E \in FSS(X)_E\). If \(h_E\) and \(s_E\) are fuzzy soft weakly separated sets in \((f_E, \tau_{f_E}, E)\), then \(h_E\) and \(s_E\) are fuzzy soft weakly separated in \((g_E, \tau_{g_E}, E)\).

Proof. Let \(h_E\) and \(s_E\) be fuzzy soft weakly separated sets in \((f_E, \tau_{f_E}, E)\). Then \(Fcl_{f_E}(h_E) = Fcl_{f_E}(h_E) \cap g_E \subseteq Fcl_{f_E}(s_E)\). Therefore, \(Fcl_{f_E}(h_E) \cap s_E \subseteq Fcl_{f_E}(s_E)\). Thus, \(h_E\) and \(s_E\) are fuzzy soft weakly separated sets in \((g_E, \tau_{g_E}, E)\).

Remark 3.10. The converse of Theorem 3.5 is not true in general as shown by the following example:

Example 3.7. Let \(X = \{a, b\}, E = \{e_1, e_2\}, A = \{e_1\} \subseteq E\) and \(\tau_0 = \{\emptyset, \{b\}\}\) be the fuzzy soft indiscrete topology on \(X\). If \(h_A = \{e_1, (a_{0.1}, b_{0.2})\}\) and \(s_A = \{e_1, (a_{0.1}, b_{0.3})\}\), then \(h_A\) and \(s_A\) are fuzzy soft weakly separated sets in \((f_E, \tau_{f_E}, E)\) but \(h_A\) and \(s_A\) are not fuzzy soft weakly separated sets in \((X, \tau, E)\).

4 FUZZY SOFT CONNECTED SETS IN FUZZY SOFT TOPOLOGICAL SPACES

In this section, we introduce different notions of connectedness of fuzzy soft sets and study the relation between these notions. Also, we will investigate the characterizations of the fuzzy soft connected sets.

Definition 4.1. A fuzzy soft set \(f_E\) in a fuzzy soft topological space \((X, \tau, E)\) is called \(FSC_M\)-connected set if there exist two non-null fuzzy soft \(Q\)-separated sets \(h_E\) and \(s_E\) in \(X\) such that \(f_E = h_E \cup s_E\). Otherwise, \(f_E\) is called \(FSC_M\)-connected set.

Definition 4.2. A fuzzy soft set \(f_E\) in a fuzzy soft topological space \((X, \tau, E)\) is called \(FSC_L\)-connected set if there exist two non-null fuzzy soft weakly-separated sets \(h_E\) and \(s_E\) in \(X\) such that \(f_E = h_E \cup s_E\). Otherwise, \(f_E\) is called \(FSC_L\)-connected set.

Definition 4.3. A fuzzy soft set \(f_E\) in a fuzzy soft topological space \((X, \tau, E)\) is called \(FSO\_q\)-connected (respectively, \(FSO_q\)-disconnected) set if there exist two non-null fuzzy soft separated (respectively, strongly separated) sets \(h_E\) and \(s_E\) in \(X\) such that \(f_E = h_E \cup s_E\). Otherwise, \(f_E\) is called \(FSO\)-connected (respectively, \(FSO_q\)-disconnected) set.

Definition 4.4. A fuzzy soft set \(f_E\) in a fuzzy soft topological space \((X, \tau, E)\) is called \(FSC_{\_L}\)-connected set if there does not exist any non-null proper fuzzy soft clopen set in \((f_E, \tau_{f_E}, E)\). Note that, this kind of fuzzy soft connectedness was studied by Mahanta [11].

In the above definitions, if we take \(\bar{1}_E\) instead of \(f_E\), then the fuzzy soft topological space \((X, \tau, E)\) is called \(FSC_M\)-connected set.
connected (respectively, $FSC_S$-connected, $FSO$-connected, $FSO_q$-connected, $FSC_S$-connected) space.

**Theorem 4.1.** Let $(X, \tau, E)$ be a fuzzy soft topological space, $f_E \in FSS(X)_E$. If $f_E$ is a $FSC_S$-connected set in $X$, then $f_E$ is a $FSC_M$-connected.

**Proof.** Let $f_E$ be a $FSC_S$-connected set in $X$. Suppose $f_E$ is a $FSC_M$-disconnected. Then, there exist two non-null fuzzy soft $Q$-separated sets $h_E$ and $s_E$ in $X$ such that $f_E = h_E \cup s_E$. By Remark 3.1, $h_E$ and $s_E$ are non-null fuzzy soft weakly-separated sets in $X$ such that $f_E = h_E \cup s_E$. Therefore, $f_E$ is a $FSC_S$-disconnected set in $X$. This is a contradiction. Hence, $f_E$ is a $FSC_M$-connected.

**Remark 4.1.** If $f_E$ is a $FSC_M$-connected, then it may not be a $FSC_S$-connected as shown by the following example.

**Example 4.1.** Let $X = \{ a, b \}$, $E = \{ e_1, e_2 \}$, $A = \{ e_1 \} \subseteq E$, $\tau = \{ \emptyset, \bar{E}, \{ e_1 \}, \{ e_2 \}, \{ e_1, e_2 \} \}$, $f_A = \{ (e_1, \{ a_0, b_0, b_0 \} \}$ and $s_A = \{ (e_1, \{ b_0 \}) \}$ such that $Cl(h_A) \subseteq s_A$. Then, there exist $h_A = \{ (e_1, \{ a_0, b_0 \}) \}$ and $s_A = \{ (e_1, \{ a_0, b_0 \}) \}$ such that $Cl(h_A) \subseteq s_A$. So, $f_A$ is not a $FSC_S$-connected. But $f_A$ is a $FSC_M$-connected.

**Theorem 4.2.** Let $(X, \tau, E)$ be a fuzzy soft topological space, $f_E \in FSS(X)_E$. If $f_E$ is a $FSC_1$-connected set in $X$, then $f_E$ is a $FSC_S$-connected.

**Proof.** Let $f_E$ be a $FSC_1$-connected set in $X$. Suppose $f_E$ is a $FSC_S$-disconnected. Then, there exist two non-null fuzzy soft weakly-separated sets $g_E$ and $u_E$ in $X$ such that $f_E = g_E \cup u_E$. By Theorem 3.1, there exist two fuzzy soft open sets $g_E$ and $u_E$ such that $h_E \subseteq g_E \cup u_E$. Then, $f_E \subseteq g_E \cup u_E$. Also, $f_E \cap g_E \neq \emptyset$. For, if $f_E \cap g_E = \emptyset$, then $f_E \cap h_E = \emptyset$ so that $h_E = \emptyset$ (since $f_E = h_E \cup s_E$ implies $h_E \subseteq f_E$), which contradicts that $h_E$ is a non-empty. Similarly, $f_E \cap u_E \neq \emptyset$.

Also, $g_E \cap u_E \subseteq (f_E)^c$. For, if $g_E \cap u_E \subseteq (f_E)^c$, then there exist $x \in X$, $e \in E$ such that $\mu_{g_E}^{e}(x) > 1 - \mu_{f_E}^{e}(x)$. This means $\mu_{g_E}^{e}(x) > 1$ and $\mu_{u_E}^{e}(x) > 1$. Since $f_E = h_E \cup s_E$, then $\mu_{u_E}^{e}(x) > 1$. Hence, $h_E q u_E$ or $s_E q u_E$ and $(s_E q g_E)$ or $(s_E q h_E)$. This is a contradiction. So, $f_E$ is a $FSC_S$-connected.

**Remark 4.2.** If $f_A$ is a $FSC_S$-connected, then it may not be a $FSC_1$-connected as shown by the following example.

**Example 4.2.** Let $X = \{ a, b \}$, $E = \{ e_1, e_2 \}$, $A = \{ e_1 \} \subseteq E$, $\tau = \{ \emptyset, \bar{E}, \{ e_1 \}, \{ e_2 \}, \{ e_1, e_2 \} \}$, $f_A = \{ (e_1, \{ a_0, b_0, b_0 \} \}$ and $s_A = \{ (e_1, \{ a_0, b_0 \}) \}$. Then, there exist two fuzzy soft open sets $h_A = \{ (e_1, \{ a_0, b_0 \}) \}$ and $s_A = \{ (e_1, \{ a_0, b_0 \}) \}$ such that $f_A \subseteq h_A \cup s_A$. Then, $f_A \subseteq h_A \cup s_A$. Therefore, $g_A$ and $u_A$ are not fuzzy soft weakly-separated sets. Hence, $f_A$ is a $FSC_S$-connected.

**Theorem 4.3.** Let $(X, \tau, E)$ be a fuzzy soft topological space, $f_E \in FSS(X)_E$. If $f_E$ is a $FSC_S$-connected set in $X$, then $f_E$ is a $FSC_2$-connected.

**Proof.** Let $f_E$ be a $FSC_S$-connected set in $X$. Suppose $f_E$ is a $FSC_2$-disconnected. Then, there exist $h_E$ and $s_E$ such that $f_E = h_E \cup s_E$. Then, $f_E = g_E \cup u_E$ where $g_E = f_E \cap h_E \subseteq h_E$ and $u_E = f_E \cap s_E \subseteq s_E$.

Since $f_E \cap h_E \cup s_E = \emptyset$ and $g_E \subseteq h_E$, then $f_E \cap g_E \subseteq h_E$. Also, since $g_E \subseteq f_E$, then $g_E \subseteq g_E \subseteq h_E$. Therefore, $g_E \subseteq h_E$. Similarly, $u_E \subseteq h_E$. Hence, $f_E$ is not a $FSC_S$-connected. This complete the
Theorem 4.4. Let \((X, \tau, E)\) be a fuzzy soft topological space. \(f_E \in FSS(X)_E\). If \(f_E\) is a FSC\(_3\)-connected set in \(X\), then \(f_E\) is a FSC\(_3\)-connected.

Proof. Let \(f_E\) be a FSC\(_3\)-connected set in \(X\). Suppose \(f_E\) is a FSC\(_3\)-disconnected. Then, there exist two fuzzy soft open sets \(h_E\) and \(s_E\) such that \(f_E \subseteq h_E \cap s_E\), \(h_E \cap s_E \subseteq (f_E)^c\), \(h_E \not\subset (f_E)^c\) and \(s_E \not\subset (f_E)^c\). Then \(f_E = g_E \cup u_E\) where \(g_E = f_E \cap h_E \subseteq h_E\) and \(u_E = f_E \cap s_E \subseteq s_E\). Let \(v_E\) and \(j_E\) be FSS\((X)_E\) defined by:

\[
\mu^{e}_v(x) = \begin{cases} 
\mu^{e}_{s_E}(x) & \text{if } \mu^{e}_{h_E}(x) \geq \mu^{e}_{s_E}(x), \\
0 & \text{otherwise}
\end{cases}
\]

\[
\mu^{e}_j(x) = \begin{cases} 
\mu^{e}_{u_E}(x) & \text{if } \mu^{e}_{s_E}(x) > \mu^{e}_{u_E}(x), \\
0 & \text{otherwise}
\end{cases}
\]

Then \(f_E = v_E \cup j_E\).

Now, \(\mu^{e}_j(x) \neq 0\). For, \(\mu^{e}_j(x) = 0\). Since \(h_E \not\subset (f_E)^c\), then there exist \(x \in X, e \in E\) such that \(\mu^{e}_{h_E}(x) > 1\). Then \(\mu^{e}_{h_E}(x) > \mu^{e}_{s_E}(x)\). For, \(\mu^{e}_{h_E}(x) \leq \mu^{e}_{s_E}(x)\) implies \(\mu^{e}_{h_E}(x) + \mu^{e}_{j_E}(x) > 1\) and hence \(\mu^{e}_{h_E \cap s_E}(x) > 1 - \mu^{e}_{j_E}(x)\). This is a contradiction with \(h_E \cap s_E \subseteq (f_E)^c\). So, \(\mu^{e}_j(x) \neq 0\).

Similarly, \(\mu^{e}_v(x) \neq 0\).

Also, \(v_E \subseteq g_E \subseteq h_E\) and \(j_E \subseteq u_E \subseteq s_E\). Now, \(v_E \cap s_E\). For, if \(v_E \cap s_E\), then there exist \(x \in X, e \in E\) such that \(\mu^{e}_{v_E}(x) + \mu^{e}_{j_E}(x) > 1\) and hence \(\mu^{e}_{v_E}(x) > 0\). This means \(\mu^{e}_{h_E}(x) \geq \mu^{e}_{v_E}(x)\) and so \(\mu^{e}_{j_E}(x) = \mu^{e}_{v_E}(x)\), implying \(\mu^{e}_{h_E \cap s_E}(x) > 1 - \mu^{e}_{j_E}(x)\) which is a contradiction with \(h_E \cap s_E \subseteq (f_E)^c\). Similarly, \(j_E \cap h_E\). Thus, \(v_E\) and \(j_E\) are fuzzy soft weakly separated and \(f_E = v_E \cup j_E\). So, \(f_E\) is not a FSC\(_3\)-connected. This a contradiction. Then \(f_E\) is a FSC\(_3\)-connected. \(\blacksquare\)

Remark 4.3. If \(f_E\) is a FSC\(_3\)-connected (respectively, C\(_2\)-connected) set, then it may not be a FSC\(_3\)-connected as shown by the following example.

Example 4.3. Let \(X = \{a, b\}, E = \{e_1, e_2\}\), \(A = \{e_1\} \subseteq E\), \(\tau = \{\emptyset, \emptyset, \{e_1\}, \{a_1, a_2, b_2\}\}\) be a fuzzy soft topology on \(X\) and \(f_A = \{(e_1, \{a_1, a_2, b_2\})\}\). Then \(f_A\) is a FSC\(_3\)-connected (respectively, FSC\(_3\)-connected) set. But \(f_A\) is not a FSC\(_3\)-connected as there exist \(h_A = \{(e_1, \{b_1\})\}\) and \(s_A = \{(e_1, \{a_2\})\}\) fuzzy soft weakly separated sets such that \(f_A = h_A \cup s_A\).

Theorem 4.5. Let \((X, \tau, E)\) be a fuzzy soft topological space. \(f_E \in FSS(X)_E\). If \(f_E\) is a FSC\(_3\)-connected set in \(X\), then \(f_E\) is a FSC\(_3\)-connected.

Proof. Let \(f_E\) be a FSC\(_3\)-connected set in \(X\). Suppose \(f_E\) is a FSC\(_3\)-disconnected. Then, there exist non-null fuzzy soft \(Q\)-separated sets \(h_E\) and \(s_E\) in \(X\) such that \(f_E = h_E \cup s_E\). Let \(g_E = [Fcl(h_E)]^c\) and \(u_E = [Fcl(s_E)]^c\). Then \(g_E\) and \(u_E\) are non-null fuzzy soft open sets.

Now, \(g_E \cap u_E = [Fcl(h_E)]^c \cap [Fcl(s_E)]^c = [Fcl(h_E)]^c \cup [Fcl(s_E)]^c = [Fcl(h_E) \cup Fcl(s_E)]^c \subseteq (f_E)^c\).

Also, \(g_E \not\subset (f_E)^c\). For, \(g_E \not\subset (f_E)^c\) then \(f_E \not\subseteq g_E^c = Fcl(h_E)\) which should imply \(s_E \subseteq \emptyset\) (since \(Fcl(h_E) \cap s_E \subseteq \emptyset\)). This is a contradiction. Similarly, \(u_E \not\subset (f_E)^c\).

Therefore, \(f_E\) is a FSC\(_3\)-disconnected. So, \(f_E\) is a FSC\(_3\)-connected.
Remark 4.4. If $f_E$ is a FSC$_M$-connected, then it may not be a FSC$_3$-connected as shown by the following example.

Example 4.4. Let $X = \{a, b\}, E = \{e_1, e_2\}, A = \{e_1\} \subseteq E$ and $\tau = \{1_E, \overline{0}_E\} = \{1_E, \overline{0}_E\}$. Then there exist non-null fuzzy soft open sets $h_A = \{(e_1, \{a, 0.6, b, 0.2\})\}$ and $s_A = \{(e_1, \{a, 0.2, b, 0.7\})\}$ such that $f_A \subseteq h_A \cap s_A$, $h_A \cap s_A \subset (f_A)^c$ and $s_A \subset (f_A)^c$. So, $f_A$ is not a FSC$_3$-connected. However, $f_A$ is a FSC$_M$-connected.

Theorem 4.6. Let $(X, \tau, E)$ be a fuzzy soft topological space. A fuzzy soft set $f_E$ in $X$ is a FSC$_2$-connected if and only if $f_E$ is a FSO$_q$-connected.

Proof. Let $f_E$ be a FSC$_2$-connected set in $X$. Suppose $f_E$ is not a FSO$_q$-connected. Then there exist non-null fuzzy soft separated sets $h_E$ and $s_E$ in $X$ such that $f_E = h_E \cap s_E$. By Theorem 3.1 and Remark 3.3, there exist two non-null fuzzy soft open sets $g_E$ and $u_E$ such that $h_E \subseteq g_E$, $s_E \subseteq u_E$, and $g_E \cap s_E = u_E \cap h_E = \overline{0}_E$. Then, $f_E \subseteq g_E \cap u_E$.

Now, for $h_E \cap g_E \cap u_E = (h_E \cap g_E) \cap u_E = (h_E \cap g_E) \cap u_E = (h_E \cap g_E) \cap u_E = \overline{0}_E$. Similarly, $f_E \not\subseteq g_E \cap u_E$.

Conversely, let $f_E$ be a FSO$_q$-connected. Suppose $f_E$ is not a FSC$_2$-connected. There exist two non-null fuzzy soft open sets $g_E$ and $u_E$ such that $f_E \subseteq g_E \cup u_E$, $f_E \cap g_E \cap u_E = \overline{0}_E$, $f_E \cap u_E \not= \overline{0}_E$, and $f_E \cap g_E \not= \overline{0}_E$.

Hence, $f_E = h_E \cap g_E \cap u_E$ where $h_E = f_E \cap g_E \cap u_E = \overline{0}_E$, $f_E \cap u_E \not= \overline{0}_E$, and $f_E \cap g_E \not= \overline{0}_E$. Also, $g_E \cap u_E = g_E \cap u_E = \overline{0}_E$. Similarly, $g_E \cap u_E = \overline{0}_E$. So, $f_E$ is not a FSO$_q$-connected and hence the proof.

Theorem 4.7. Let $(X, \tau, E)$ be a fuzzy soft topological space. If $f_E \in FSS(X)_E$. If $f_E$ is a FSC$_4$-connected set in $X$, then $f_E$ is a FSO$_q$-connected.

Proof. Let $f_E$ be a FSC$_4$-connected set in $X$. Suppose $f_E$ is a FSO$_q$-disconnected. Then there exist non-null fuzzy soft strongly separated sets $h_E, s_E$ in $X$ such that $f_E = h_E \cap s_E$.

So, there exist two non-null fuzzy soft open sets $g_E$ and $u_E$ such that $h_E \subseteq g_E$, $s_E \subseteq u_E$, $g_E \cap s_E = u_E \cap h_E = \overline{0}_E$, $h_E \cap g_E$ and $s_E \cap g_E$ quasi-coincident with respect to $h_E$ and $s_E$ with respect to $g_E$. Then, for every $x \in S(h_E(e))$, we have $\mu_{h_E}^g(x) + \mu_{h_E}^u(x) > 1$ and for every $x \in S(s_E(e))$, we have $\mu_{s_E}^g(x) + \mu_{s_E}^u(x) > 1$. Then, $f_E \subseteq g_E \cup u_E$. Also, $f_E \cap g_E \cap u_E = \overline{0}_E$.

Again, $\mu_{g_E}^x(x) + \mu_{u_E}^x(x) \geq \mu_{h_E}^g(x) + \mu_{h_E}^u(x) > 1$ for every $x \in S(h_E(e))$. Therefore, $g_E \not\subseteq f_E$. Similarly, $u_E \not\subseteq f_E$. Thus, $f_E$ is not a FSC$_4$-connected. This is a contradiction. So, $f_E$ is a FSO$_q$-connected.

Remark 4.5. If $f_E$ is a FSO$_q$-connected, then it may not be a FSC$_4$-connected as shown by the following example.

Example 4.5. Let $X = \{a, b, c\}, E = \{e_1, e_2\}, A = \{e_2\} \subseteq E$ and $\tau = \{1_E, \overline{0}_E\} = \{1_E, \overline{0}_E\}$. Then, $A = \{(e_1, \{a, c\}, b, 0.7), \{a, 0.7, b, 0.7\})\}$ and $g_A = \{(e_1, \{a, 0.7, b, 0.7\})\}$, $u_A = \{(e_1, \{a, 0.7, b, 0.7\})\} \in \tau$. Then, $f_A \subseteq g_A \cup u_A$, $f_A \cap g_A \cap u_A = \overline{0}_E$, $g_A \cap f_E$ and $u_E \cap f_E$ connect. So, $f_A$ is not a FSC$_4$-connected. However, $f_A$ is a FSO$_q$-connected.

Remark 4.6. If $f_A$ is a FSC$_M$-connected, then it may not be a FSO$_q$-connected as shown by the following example.

Example 4.6. Let $X = \{a, b, c\}, E = \{e_1, e_2\}, A = \{e_1\} \subseteq E$ and $\tau = \{1_E, \overline{0}_E\} = \{1_E, \overline{0}_E\}$. Then there
exist two non-null fuzzy soft strongly separated $h_A = \{(e_1, \{a_{0.6}\})\}$ and $s_A = \{(e_1, \{b_{0.7}, c_{0.8}\})\}$ such that $f_A = h_A \cup s_A$. So, $f_A$ is not a $FSC_2$-connected. However, $f_A$ is a $FSC_1$-connected as $Fcl(s_A) \cap h_A \neq \emptyset$. Also, $Fcl(h_A) \cap s_A \neq \emptyset$.

**Remark 4.7.** If $f_A$ is a $FSC_2$-connected, then it may not be a $FSC_1$-connected as shown by the following example.

**Example 4.7.** Let $X = \{a, b\}$, $E = \{e_1\}$ and $\tau = \{\emptyset_E, \{e_1\}, \{a, b\}, \{a, e_1\}, \{b, e_1\}, \{a, b, e_1\}\}$ be a fuzzy soft topology on $X$. Let $f_A = \{\{a_1, b_2\}\}$. Then $f_A$ can be expressed as a union of two non-null fuzzy soft $Q$-separated sets $h_A = \{(e_1, \{a_2\})\}$ and $s_A = \{(e_1, \{b_2\})\}$. So, $f_A$ is not a $FSC_1$-connected.

However, $f_A$ is a $FSC_2$-connected as if we take $g_E = \{(e_1, \{a_1, b_1\})\}$ and $u_E = \{(e_1, \{a_1, b_2\})\}$, then $f_A \subseteq g_E \cup u_E$ but $f_A \cap \emptyset_E \neq \emptyset$.

**Remark 4.8.** If $f_A$ is a $FSC_1$-connected, then it may not be a $FSC_2$-connected as shown by the following example.

**Example 4.8.** Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$ and $\tau = \{\emptyset_E, \{e_1\}, \{a_1, e_1\}, \{b_1, e_1\}, \{a_1, b_1\}\}$ be a fuzzy soft topology on $X$. Let $f_A = \{(e_1, \{a_1, b_2\})\}$, then there exist two non-null fuzzy soft $Q$-separated sets $h_A = \{(e_1, \{a_0.4\})\}$ and $s_A = \{(e_1, \{b_0.4\})\}$ such that $f_A = h_A \cup s_A$. So, $f_A$ is not a $FSC_2$-connected. However, $f_A$ is a $FSC_1$-connected as $h_A$ and $s_A$ are not fuzzy soft strongly separated.

**Remark 4.9.** If $f_A$ is a $FSC_1$-connected set, then it may not be a $FSC_2$-connected (respectively, $FSC_q$-connected, $FSC_i$-connected for $i = 1, 2, 3, 4, S, M$) set. In fact, $f_A$ defined in Example 4.6 (or 4.7) is a $FSC_2$-connected, but it is not a $FSC_q$-connected set and not a $FSC_1$-connected set. Therefore, it is not a $FSC_2$-connected set and not a $FSC_i$-connected set for $i = 1, 2, 3, 4, S$.

**Remark 4.10.** If $f_A$ is a $FSC$-connected (respectively, $FSC_q$-connected, $FSC_i$-connected for $i = 1, 2, 3, 4, S, M$) set, it may not be a a $FSC_2$-connected as shown by the following example.

**Example 4.9.** Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$ and $\tau = \{\emptyset_E, \{e_1\}, \{a_0.3\}\}$ be a fuzzy soft topology on $X$. Let $f_A = \{(e_1, \{b_0.7\})\}$. Then $f_A$ is a $FSC_1$-connected for $i = 1, 2, 3, 4, S, M$. But since $\{e_1, \{b_0.5\}\}$ is a non-null proper clopen fuzzy soft set in $f_A$. So, $f_A$ is not a $FSC_2$-connected.

**Remark 4.11.** In a fuzzy soft topological space $(X, \tau, E)$, the classes of $FSC$-connected, $FSC_q$-connected, and $FSC_i$-connected sets for $i = 1, 2, 3, 4, S, M$ can be described by the following diagram.

![Diagram](attachment:image.png)

We observe that, if a fuzzy soft point $x_{0.5}^2$ is $FSC_i$-connected set ($i = 2, 3$) hence $FSC_4$-connected, but not necessarily $FSC_1$-connected which is a departure from general topology where points are connected sets.
Example 4.10. Let $X = \{a, b\}, E = \{e_1, e_2\}, \tau = \{I_E, \emptyset_E\}, \{\{e_1, \{a_1, b_2\}\}, \{e_1, \{a_2, b_1\}\}, \{e_1, \{a_2, b_2\}\}\}$. Here, the fuzzy soft point $a_1^s = \{\{a_1\}\}$ is not a $FSC_4$-connected

Moreover, we observe that the null-fuzzy soft set $\emptyset_E$ is $FSC_4$-connected and hence $FSC_4$-connected ($i = 2, 3, 4$).

Theorem 4.8. Let $(X, \tau, E)$ and $(Y, \sigma, K)$ be a fuzzy soft topological spaces and $f_{pu}: (X, \tau, E) \rightarrow (Y, \sigma, K)$ be fuzzy soft bijective continuos mapping. If $g_B$ is a $FSC_i$-connected (respectively, $FSO$-connected, $FSO_q$-connected) set in $X$ for $i = 5, S, M$, then $f_{pu}(g_B)$ is a $FSC_i$-connected (respectively, $FSO$-connected, $FSO_q$-connected) set in $Y$ for $i = 5, S, M$.

**Proof.** As a sample, we will prove the case $i = 5$. Let $g_B$ be a $FSC_5$-connected set in $X$. Suppose, $f_{pu}(g_B)$ is not a $FSC_5$-connected set in $Y$. Then, $f_{pu}(g_B)$ has a non-null proper clopen fuzzy soft subset $h_C$.

So, there exist $s_D \in \sigma$ and $u_N \in \sigma^c$ such that

$$h_C = f_{pu}(g_B) \cap s_D \neq \emptyset$$

Since $f_{pu}$ is a bijective function, then

$$f_{pu}^{-1}(h_C) = g_B \cap f_{pu}^{-1}(s_D) = g_B \cap f_{pu}^{-1}(u_N).$$

Also, since $s_D \in \sigma$ and $u_N \in \sigma^c$ and $f_{pu}$ is a fuzzy soft continuos function, then $f_{pu}^{-1}(s_D) \in \tau$ and $f_{pu}^{-1}(u_N) \in \tau^c$.

Hence, $f_{pu}^{-1}(h_C)$ is a non-null proper clopen fuzzy soft subset of $g_B$ which is a contradiction. Therefore, $f_{pu}(g_B)$ is a $FSC_5$-connected set in $Y$.

Theorem 4.9. Let $(X, \tau, E)$ and $(Y, \sigma, K)$ be a fuzzy soft topological spaces and $f_{pu}: (X, \tau, E) \rightarrow (Y, \sigma, K)$ be fuzzy soft open function such that $p: E \rightarrow K, u: X \rightarrow Y$ are bijective mapping. If $g_B$ is a $FSC_i$-connected (respectively, $FSO$-connected, $FSO_q$-connected) set in $Y$ for $i = 1, 2, 3, 4, 5, S, M$, then $f_{pu}^{-1}(g_B)$ is a $FSC_i$-connected (respectively, $FSO$-connected, $FSO_q$-connected) set in $X$ for $i = 1, 2, 3, 4, 5, S, M$.

**Proof.** As a sample, we will prove the case of $FSO$-connected. Let $g_B$ be a $FSO$-connected set in $Y$. Suppose $f_{pu}^{-1}(g_B)$ is not a $FSO$-connected set in $X$. Then there exist two non-null fuzzy soft separated sets $h_C$ and $s_D$ in $X$ such that

$$f_{pu}^{-1}(g_B) = h_C \cup s_D.$$ 

Therefore, there exist two non-null fuzzy soft open sets $u_N$ and $j_L$ in $X$ such that

$$h_C \subseteq u_N, s_D \subseteq j_L$$

and $h_C \cap j_L = s_D \cap u_N = \emptyset_E$. Since $f_{pu}$ is a surjective fuzzy soft function, then

$$f_{pu}[f_{pu}^{-1}(g_B)] = g_B$$

and so $g_B = f_{pu}[h_C \cup s_D] = f_{pu}(h_C) \cup f_{pu}(s_D)$. Since $f_{pu}$ is a fuzzy soft open function, then $f_{pu}(u_N)$ and $f_{pu}(j_L)$ are non-null fuzzy soft open sets in $Y$ such that $f_{pu}(h_C) \subseteq f_{pu}(u_N)$ and $f_{pu}(s_D) \subseteq f_{pu}(j_L)$. Also, since $f_{pu}$ is a fuzzy soft injective function, then $f_{pu}(h_C) \cap f_{pu}(j_L) = f_{pu}(h_C \cap j_L) = \emptyset_K$ and $f_{pu}(s_D) \cap f_{pu}(s_D) = \emptyset_K$. It follows that $g_B$ is a $FSC$-disconnected set, which a contradiction.

Definition 4.6. Two non-null fuzzy soft sets $f_A$ and $g_B$ are said to be intersecting if there exist $x \in X, e \in E$ such that $\min\{f_A(x)(e), g_B(x)(e)\} \neq 0$. If $f_A$ and $g_B$ are non-intersecting, then $f_A$ and $g_B$ are said to be disjoint.

Theorem 4.10. If $f_A$ and $g_B$ are intersecting $FSC_i$-connected (respectively, $FSC_4$-connected, $FSC_i$-connected, $FSC_5$-connected, $FSC_M$-connected, $FSO_q$-connected) sets in $X$, then $f_A \cup g_B$ is a $FSC_i$-connected (respectively, $FSC_4$-connected, $FSC_M$-connected, $FSC_5$-connected, $FSC_M$-connected, $FSO_q$-connected) set in $X$.

**Proof.** The cases of $FSC_4$-connected and $FSC_4$-connected sets previously proved (see Theorem 3.22, 3.23 and 3.24 in [9]). As a sample we will prove the case of $FSC_i$-connected sets. Let $f_A$ and $g_B$ be intersecting $FSC_M$-connected sets in $X$. Suppose $f_A \cup g_B$ is a $FSC_M$-disconnected set. Then, there exist two non-null fuzzy soft $Q$-separated sets $h_C$ and $s_D$ in $X$ such that $f_A \cap g_B = h_C \cap s_D$. Therefore, $f_A \cap h_C, f_A \cap s_D, g_B \cap h_c$ and $g_B \cap s_D$ are non-null fuzzy soft $Q$-separated sets in $X$ as subsets of $h_C$ and $s_D$. Since

$$f_A = (f_A \cap h_C) \cup (f_A \cap s_D)$$

and

$$g_B = (g_B \cap h_C) \cup (g_B \cap s_D),$$

then $f_A$ and $g_B$ are $FSC_M$-disconnected which is a contradiction.
Theorem 4.11. Let \( \{(f_A)_i; i \in J \} \) be a family of a \( FSC_T \)-connected (respectively, \( FSC_2 \)-connected, \( FSO \)-connected, \( FSC_S \)-connected, \( FSC_M \)-connected, \( FSO_q \)-connected) sets in \( X \) such that for \( i, j \in J; i \neq j \) the fuzzy soft sets \((f_A)_i\) and \((f_A)_j\) are intersecting. Then, \( f_A = \bigcup_{i \in J} (f_A)_i \) is a \( FSC_T \)-connected (respectively, \( FSC_2 \)-connected, \( FSO \)-connected, \( FSC_S \)-connected, \( FSC_M \)-connected, \( FSO_q \)-connected) set in \( X \).

Proof. Let \( \{(f_A)_i; i \in J \} \) be a family of a \( FSC_T \)-connected sets in \( X \). Suppose that \( f_A \) is not a \( FSC_T \)-connected set in \( X \). Then, there exist two fuzzy soft open sets \( h_C \) and \( S_D \) in \( X \) such that \( f_A \subseteq h_C \circ S_D \) and \( h_C \cap S_D \subseteq f_A^c \).

Now, let \( \{(f_A)_i; i \in J \} \) be any fuzzy soft set of the given family. Then, \( (f_A)_i \subseteq h_C \circ S_D \) and \( h_C \cap S_D \subseteq (f_A)_i^c \). But, \( (f_A)_i \) is a \( FSC_T \)-connected set. Hence, \( (f_A)_i \cap h_C = \tilde{0}_E \) or \( (f_A)_i \cap S_D = \tilde{0}_E \). Now if \( (f_A)_i \cap h_C = \tilde{0}_E \), we can prove that \( (f_A)_i \cap h_C = \tilde{0}_E \) for each \( i \in J - \{i_0\} \) and so \( f_A \cap h_C = \tilde{0}_E \).

This complete the proof.

Corollary 4.1. If \( \{(f_A)_i; i \in J \} \) is a family of a \( FSC_T \)-connected (respectively, \( FSC_2 \)-connected, \( FSO \)-connected, \( FSC_S \)-connected, \( FSC_M \)-connected, \( FSO_q \)-connected) sets in \( X \) and \( \bigcap_{i \in J} (f_A)_i \neq \tilde{0}_E \), then \( \bigcap_{i \in J} (f_A)_i \) is a \( FSC_T \)-connected (respectively, \( FSC_2 \)-connected, \( FSO \)-connected, \( FSC_S \)-connected, \( FSC_M \)-connected, \( FSO_q \)-connected) set in \( X \).

Proof. Straightforward in view of Theorem 4.10.

The following example shows that Theorem 4.10 fails for \( FSC_3 \)-connected (respectively, \( FSC_4 \)-connected) spaces.

Example 4.11. Let \( X = \{a, b\} \), \( E = \{e_1, e_2\} \), \( A = \{e_1\} \subseteq E \) and \( \tau = \{\tilde{1}_E, \tilde{0}_E, ((e_1, \{a, b\}))\} \). Then, \( (e_1, \{a, b\}) \) be a fuzzy soft topology on \( X \). Let \( f_A = ((e_1, \{a, b\})) \) and \( g_A = \{(e_1, \{a, b\})\} \). Here, \( f_A \cap g_A \neq \tilde{0}_E \) and \( f_A \cup g_A \) are \( FSC_T \)-connected sets in \( X \), but \( f_A \cap g_A \) is not \( FSC_3 \)-connected set in \( X \).

Example 4.12. Let \( X = \{a, b\} \), \( E = \{e_1, e_2\} \) and \( \tau = \{\tilde{1}_E, \tilde{0}_E, ((e_1, \{a, b\}))\} \). Then, \( (e_1, \{a, b\}) \) be a fuzzy soft topology on \( X \). Let \( f_E = ((e_1, \{a, b\})) \) and \( g_E = ((e_1, \{a, b\}) \cup (e_2, \{a, b\})) \). Here, \( f_E \cap g_E \neq \tilde{0}_E \) and \( f_E \cup g_E \) are \( FSC_4 \)-connected sets in \( X \), but \( f_E \cap g_E \) is not \( FSC_4 \)-connected set in \( X \).

Theorem 4.12. If \( f_A \) and \( g_B \) are quasi-coincident \( FSC_T \)-connected (respectively, \( FSC_2 \)-connected) sets in \( X \), then \( f_A \cap g_B \) is a \( FSC_T \)-connected (respectively, \( FSC_2 \)-connected) set in \( X \).

Proof. As a sample, we will prove the case \( FSC_3 \)-connected. Let \( f_A \) and \( g_B \) be quasi-coincident \( FSC_3 \)-connected sets in \( X \). Suppose there exist two non-null fuzzy soft open sets \( h_C \) and \( S_D \) in \( X \) such that

\[
f_A \cap g_B \subseteq h_C \circ S_D \quad \text{and} \quad h_C \cap S_D \subseteq (f_A \cap g_B)^c
\]

Therefore, \( f_A \subseteq h_C \circ S_D \), \( h_C \cap S_D \subseteq f_A^c \cdot g_B \subseteq h_C \circ S_D \) and \( h_C \cap S_D \subseteq g_B^c \). Since \( f_A \) and \( g_B \) are \( FSC_3 \)-connected, then \( (h_C \subseteq f_A^c \circ S_D \subseteq f_A^c) \) and \( (h_C \subseteq g_B \circ S_D \subseteq g_B^c) \).

Moreover, since \( f_A \) and \( g_B \) are quasi-coincident, there exist \( x \in X, e \in E \) such that

\[
\mu_f^x(x) > 1 - \mu_g^x(x)
\]

Now, consider the following cases:
Case I. Suppose \( h_c \subseteq f_A^c \). Then by (2) we have,
\[
\mu_{h_c}(x) < \mu_{g_B}(x) \tag{3}
\]
We claim that, \( s_D \subsetneq g_B^c \). For if not, then
\[
\mu_{s_D}(x) \leq 1 - \mu_{g_B}(x) < \mu_{f_A}(x) \tag{4}
\]
Now by (3) and (4), we have \( \mu_{h_c}(x) < \mu_{f_A \cup g_B}(x) \) which implies \( f_A \cup g_B \not\subset h_c \cup s_D \), this contradicts (1). Hence, \( h_c \subseteq g_B^c \). Therefore, \( h_c \subseteq f_A^c \cap g_B^c = (f_A \cup g_B)^c \).

Case II: Suppose, \( s_D \subseteq f_A^c \). Here, we can show as in Case I that \( h_c \not\subset h_B^c \). Therefore, \( s_D \not\subset g_B^c \). Hence, \( s_D \subseteq f_A^c \cap g_B^c = (f_A \cup g_B)^c \). This complete the proof.

Theorem 4.13. Let \( \{(f_A)_i; i \in J\} \) be a family of a FSC_{\alpha} (respectively, FSC_{\beta} )-connected sets in \( X \) such that for \( i, j \in J; i \neq j \) the fuzzy soft sets \( (f_A)_i \) and \( (f_A)_j \) are quasi-coincident. Then \( f_A = \bigcup_{i \in J} (f_A)_i \) is a FSC_{\alpha} (respectively, FSC_{\beta} )-connected set in \( X \).

**Proof.** Let \( \{(f_A)_i; i \in J\} \) be a family of a FSC_{\alpha} (respectively, FSC_{\beta} )-connected sets in \( X \). Suppose there exist two fuzzy soft open sets \( h_c \) and \( s_D \) in \( X \) such that \( f_A \subseteq h_c \cup s_D \) and \( h_c \cap s_D \subseteq f_A^c \). Let \( (f_A)_i \) be any fuzzy soft set of the given family. Then, \( f_A \subseteq (f_A)_i \) or \( s_D \subseteq (f_A)_i^c \). Now, the result follows in view of the facts that \( (f_A)_i \subseteq (h_c)_i \) if \( i \in J \) or \( (f_A)_i \subseteq (h_c)_i \) if \( i \notin J \) and \( (f_A)_i \) are quasi-coincident FSC_{\alpha} (respectively, FSC_{\beta} )-connected sets in \( X \). Hence, \( f_A = \bigcup_{i \in J} (f_A)_i \) is a FSC_{\alpha} (respectively, FSC_{\beta} )-connected set in \( X \).

**Corollary 4.2.** Let \( \{(f_A)_i; i \in J\} \) be a family of a FSC_{\alpha} (respectively, FSC_{\beta} )-connected sets in \( X \) and \( x_\alpha^e \) be a fuzzy soft point such that \( \alpha > \frac{1}{2} \) and \( x_\alpha^e \subseteq \bigcup_{i \in J} (f_A)_i \). Then \( \bigcup_{i \in J} (f_A)_i \) is a FSC_{\alpha} (respectively, FSC_{\beta} )-connected set in \( X \).

**Proof.** Since \( x_\alpha^e \subseteq \bigcup_{i \in J} (f_A)_i \), then \( x_\alpha^e \subseteq (f_A)_i \) for each \( i \in J \). Therefore, \( (f_A)_i \) and \( (f_A)_j \) are quasi-coincident fuzzy soft sets for each \( i, j \in J \). By Theorem 4.13, \( \bigcup_{i \in J} (f_A)_i \) is a FSC_{\alpha} (respectively, FSC_{\beta} )-connected set in \( X \).

**Theorem 4.15.** If \( f_A \) is a FSC_{\alpha} (respectively, FSC_{\beta} )-connected set in \( X \) and \( f_A \subseteq g_B \subseteq Fcl(f_A) \), then \( g_B \) is also a FSC_{\alpha} (respectively, FSC_{\beta} )-connected set in \( X \).

**Proof.** As a sample, we will prove the case of FSC_{\alpha} -connected set. Let \( h_c \) and \( s_D \) be fuzzy soft open sets in \( X \) such that \( g_B \subseteq h_c \cup s_D \) and \( h_c \cap s_D \subseteq g_B^c \). Then, \( f_A \subseteq h_c \cup s_D \) and \( h_c \cap s_D \subseteq f_A^c \). Since \( f_A \) is a FSC_{\alpha} -connected set, we have \( f_A \subseteq (h_c)^c \) or \( f_A \subseteq (s_D)^c \). But, if \( f_A \subseteq (h_c)^c \), then \( Fcl(f_A) \subseteq (h_c)^c \) and on the other hand, if \( f_A \subseteq (s_D)^c \), then \( Fcl(f_A) \subseteq (s_D)^c \). Therefore, \( g_B \subseteq Fcl(f_A) \subseteq (h_c)^c \) or \( g_B \subseteq Fcl(f_A) \subseteq (s_D)^c \). Hence, \( g_B \) is a FSC_{\alpha} -connected set in \( X \).
However, the above theorem fails in case of $FSC_1$-connectedness (respectively, $FSC_2$-connectedness, $FSC_5$-connectedness, $FSC_3$-connectedness, $FSO$-connectedness) which is a departure from general topology. The following example will illustrate that the closure of a $FSC_1$-connected (respectively, $FSC_2$-connected, $FSC_5$-connected, $FSC_3$-connected, $FSO$-connected) set need not be a $FSC_1$-connected (respectively, $FSC_2$-connected, $FSC_5$-connected, $FSC_3$-connected, $FSO$-connected). By the following examples we show that Theorem 4.9 and Remark 4.7 of [11] are incorrect.

**Example 4.13.** Let $X = \{a, b\}$, $E = \{e_1\}$ and $\tau = \{1_E - \bar{0}_E, \{(e_1, \{a_1\})\}, \{(e_1, \{b_2\})\}, \{(e_1, \{a_1, b_2\})\}\}$ be a fuzzy soft topology on $X$. Here, $f_B = \{(e_1, \{a_1\})\}$ is a $FSC_1$-connected (respectively, $FSC_2$-connected, $FSC_5$-connected, $FSC_3$-connected, $FSO$-connected) set, but $Fcl(f_B) = \{(e_1, \{a_1\}, b_2)\}$ is not a $FSC_1$-connected (respectively, $FSC_2$-connected, $FSC_5$-connected, $FSC_3$-connected, $FSO$-connected).

**REFERENCES**

1. Ahmad and A. Kharal, Mappings of Fuzzy Soft Classes, Adv. Fuzzy Syst. (2009).
2. N. Ajmal and J.K. Kohli, Connectedness in fuzzy topological spaces, Fuzzy Sets and Systems 31 (1989) 369-388.
3. H. Aktaş and N. Çağman, Soft Sets and Soft Groups, Inform. Sci. 177 (2007) 2726–2735.
4. S. Bayramov, C. Gunduz and A. Erdem, Soft Path Connectedness on Soft Topological Spaces, Proceedings of the 13th International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE 2013.
5. C. L. Chang, Fuzzy Topological Spaces, J. Math. Appl. 24 (1968) 182–193.
6. A.K. Chaudhuri and P. Das, Fuzzy connected sets in fuzzy topological spaces, Fuzzy Sets and Systems 49 (1992) 223-229.
7. U.V. Fatteh and D.S. Bassan, Fuzzy connectedness and its stronger forms, J. Math. Anal. Appl. III (1985) 449-464.
8. S. Hussain, A note on soft connectedness, Journal of Egyptian Mathematical Society, 23 (1) 2015, 6-11.
9. S. Karataş, B. Kihç and M. Telloğlu, On fuzzy soft connected topological spaces, Journal of Linear and Topological Algebra, 3(4) (2015), 229-240.
10. R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976) 621-633.
11. J. Mahanta and P. K. Das, Results on Fuzzy Soft Topological Spaces, arXiv:1203.0634v1, 2012.
12. P. K. Maji, R. Biswas, and A.R. Roy, Fuzzy Soft Sets, J. Fuzzy Math. 9 (3) (2001) 589–602.
13. P. K. Maji, R. Biswas, and A.R. Roy, Soft Set Theory, Computers Math. Appl. 45 (2003) 555–562.
14. P.P. Ming and L.Y. Ming, Fuzzy topology I, Neighbourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980) 571-599.
15. D. Molodtsov, Soft Set Theory-First Results, Computers Math. Appl. 37 (4-5) (1999) 19–31.
16. Munir Abdul Khalik Al-Khafaj and Majid Hamid Mahmood, Some Properties of Soft Connected Spaces and Soft Locally Connected Spaces, IOSR Journal of Mathematics, 10 (2014), 102-107.
17. B. Pazar Varol and H. Aygün, Fuzzy Sot Topology, Hacettepe Journal of Mathematics and Statistics, 41 (3) (2012) 407-419.
18. E. Peyghan, B. Samadi and A. Tayebi, On Soft Connectedness, arXiv:1202.1668v1, 2012.
19. S. Roy and T. K. Samanta, An Introduction to Open and Closed Sets on Fuzzy Topological Spaces, Annals of Fuzzy Mathematics and Informatics, 2012.
20. S. Saha, Local connectedness in fuzzy setting, Simon Stevin 61 (1987) 3-13.
21. M. Shabir and M. Naz, On Soft Topological Spaces, Computers and Mathematics with Applications 61 (2011) 1786–1799.
22. B. Tanay and M. B. Kandemir, Topological Structures of Fuzzy Soft Sets, Computers and Mathematics with Applications 61 (2011) 412–418.
23. Tridiv Jyotinog and Dusmanta Kumar Sut, Some New Operations of Fuzzy Soft Sets, J. Math. Comput. Sci. 2 (5)
24. Tugbahan Simsekler, Saziye Yuksel, Fuzzy Soft Topological spaces, Ann. Fuzzy Math. Inform, vol. x(2012) 1-x.

25. Ş. Yüksel, Z. G. Ergül and Z. Güven, Soft Connected Spaces, International J. of Pure & Eng. Mathematics (IJPEM), 2 (2014), 121-134.

26. L.A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338-353.

27. X.A. Zhao, Connectedness on fuzzy topological spaces, Fuzzy Sets and Systems 20 (1986) 223-240.

28. Zheng Chong You, On connectedness of fuzzy topological spaces, Fuzzy Mathematics 3 (1982) 59-66.