ON SPECIAL REGULARITY PROPERTIES OF SOLUTIONS OF THE ZAKHAROV-KUZNETSOV EQUATION

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

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Abstract. We study special regularity properties of solutions to the initial value problem associated to the Zakharov-Kuznetsov equation in three dimensions. We show that the initial regularity of the data in a family of half-spaces propagates with infinite speed. By dealing with the finite envelope of a class of these half-spaces we extend the result to the complement of a family of cones in $\mathbb{R}^3$.

1. Introduction. In this paper we consider solutions of the initial value problem (IVP) associated to the three dimensional (3D) Zakharov-Kuznetsov (ZK) equation

$$\begin{cases}
\partial_t u + \partial_x \Delta u + u \partial_x u = 0, & (x, y, z) \in \mathbb{R}^3, t \in \mathbb{R}, \\
u(x, y, z, 0) = u_0(x, y, z)
\end{cases}$$

(1.1)

where $u$ is a real function and $\Delta$ denotes the Laplace operator in space variables.

The equation above arises in the context of plasma physics, it was formally derived by Zakharov and Kuznetsov [30] as a long wave small-amplitude limit of the Euler-Poisson system in the “cold plasma” approximation. This formal long-wave limit was rigorously justified by Lannes, Linares and Saut in [21]. Recently, Han-Kwan [12] derived the ZK equation from the Vlasov-Poisson system in a combined cold ions and long wave limit.

The Zakharov-Kuznetsov equation can be seen as a natural multi-dimensional extension of the Korteweg-de Vries (KdV) equation, quite different from the well-known Kadomtsev-Petviashvili (KP) equation which is obtained as an asymptotic model of various nonlinear dispersive systems under a different scaling.

Contrary to the Korteweg-de Vries or the Kadomtsev-Petviashvili equations, the Zakharov-Kuznetsov equation is not completely integrable but it has a Hamiltonian structure and possesses two invariants, namely (for $u_0 = u(\cdot, 0)$)

$$M(t) = \int_{\mathbb{R}^3} u^2(\vec{x}, t) \, d\vec{x} = \int_{\mathbb{R}^3} u_0^2(\vec{x}) \, d\vec{x} = M(0),$$

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and
\[ E(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 - \frac{u^3}{3} \right) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 - \frac{u_0^3}{3} \right) = E(0). \]

Regarding well-posedness for the IVP (1.1) the best local result available was obtained by Ribaud and Vento [28] for initial data in \( H^s(\mathbb{R}^3) \), \( s > 1 \). In [26], Molinet and Pilod proved that this local result can be extended globally in time. Previous local well-posedness results for (1.1) were established by Linares and Saut in [25].

The equation in (1.1) has a two dimensional (2D) analogous
\[ \partial_t u + \partial_x \Delta u + u \partial_x u = 0, \quad (x, y) \in \mathbb{R}^2, \quad \Delta = \partial_x^2 + \partial_y^2, \tag{1.2} \]
which was also derived in [30]. As in the 3D case there have been a lot of interest in the study of well-posedness in low regularity spaces for the associated IVP. The best local result available in the literature was obtained for initial data in \( H^s(\mathbb{R}^2) \), \( s > 1/2 \), independently by Molinet and Pilod [26] and Grünrock and Herr [11]. Global well-posedness was proved for data in \( H^j(\mathbb{R}^2) \), \( j \in \mathbb{Z}^+ \), \( j \geq 1 \) by Faminskii [7] and for initial data in \( H^s(\mathbb{R}^2) \), \( s \geq 1 \), by Linares and Pastor [22].

The scaling argument suggests local well-posedness in the 3D case for data in \( H^s(\mathbb{R}^3) \) with \( s \geq -1/2 \) and in 2D for data in \( H^s(\mathbb{R}^2) \) with \( s \geq -1 \), therefore the local well-posedness results commented above are far from these values. It is an interesting open problem to establish the optimal Sobolev space where local well-posedness is attained in either case.

In the case of the so called generalized ZK equation, i.e.
\[ \partial_t u + \partial_x \Delta u + u^k \partial_x u = 0, \quad k \in \mathbb{Z}^+, \tag{1.3} \]
we shall briefly described what is known for the well-posedness of the associated IVP. In the 2D case, the scale argument suggests local well-posedness results for data in \( H^s(\mathbb{R}^2) \), \( s > s_k = 1 - 2/k \). Sharp local results were obtained by Ribaud and Vento [29], for \( k \geq 4 \). In [10], Grünrock proved the local well-posedness in \( H^s_n(\mathbb{R}^n) \) for \( n = 2, 3 \) and \( s_k = n/2 - 2/k \), \( k \geq 3 \). For the nonlinearity \( k = 2 \), for which \( L^2(\mathbb{R}^2) \) is the space suggested by the scaling argument, local result was shown for data in \( H^s(\mathbb{R}^2) \), \( s \geq 1/4 \) in [29]. Also for the nonlinearity \( k = 2 \) global well-posedness in \( H^s(\mathbb{R}^2) \), \( s > 53/63 \), for initial data with suitable \( L^2 \) norm, was established by Linares and Pastor in [23]. In the 3D case we refer to Grünrock [9] for low regularity well-posedness results regarding the equation with nonlinearity \( k = 2 \).

We observe that in 2D case there exists a nonsingular linear transformation that symmetrizes the equation (1.2) (see [7, 11] and references therein). More precisely, the change of variables
\[ x' = \alpha x + \mu y, \]
\[ y' = \alpha x - \mu y, \]
with \( \alpha \mu \neq 0 \), suitable chosen takes the equation in (1.2) into the equation
\[ \partial_t u + \partial_{x'}^3 u + \partial_{y'}^3 u + u(\partial_{x'} + \partial_{y'}) u = 0. \tag{1.4} \]

This in particular will allow to consider the IVP associated to (1.4) instead of the IVP (1.2) without changing the well-posedness theory.

As is well-known, energy estimates obtained by using the commutator estimates in [17] permit to establish local well-posedness results for the IVP (1.2) in Sobolev spaces \( H^s(\mathbb{R}^2) \), \( s > 2 \) (see [5] for instance). One can loose (or strength) a little
Theorem 1.1. positive times. More precisely, established the propagation of regularity in the right hand side of the data for the IVP associated to the generalized KdV equation in [15] Isaza, Linares and Ponce, considering suitable solutions of [24].

Another interesting issue is that concerning the traveling wave solutions or solitary waves for the ZK equation and for the generalized ZK equation. In [5] de Bouard studied the existence and the orbital stability of such solutions. Recently, Côte, Muñoz, Pilod, and Simpson [6] among other results, showed the asymptotic stability of the solitons in the 2D case.

For further results affiliated to the ZK equations we refer to [8, 27, 3, 4, 1], and [19].

Our purpose here is the study of the propagation of regularity for solutions of the IVP (1.1). In [15] Isaza, Linares and Ponce, considering suitable solutions of the IVP associated to the generalized KdV equation

\[
\begin{align*}
\partial_t u + \partial_x^3 u + u^k \partial_x u &= 0, \quad x, t \in \mathbb{R}, \; k \in \mathbb{Z}^+, \\
u(x, 0) &= u_0(x),
\end{align*}
\]  

established the propagation of regularity in the right hand side of the data for positive times. More precisely,

**Theorem 1.1.** ([15]). If \( u_0 \in H^{3/4+} (\mathbb{R}) \) and for some \( l \in \mathbb{Z}^+ \), \( l \geq 1 \) and \( x_0 \in \mathbb{R} \)

\[
\| \partial_x^l u_0 \|_{L^2((x_0, \infty))}^2 = \int_{x_0}^{\infty} |\partial_x^j u_0(x)|^2 \, dx < \infty,
\]

then the solution of the IVP (1.5) provided by the local theory in [19] satisfies that for any \( v > 0 \) and \( \epsilon > 0 \)

\[
\sup_{0 \leq t \leq T} \int_{x_0 + \epsilon - vt}^{x_0 + \epsilon + vt} (\partial_x^j u)^2(x, t) \, dx < c,
\]

for \( j = 0, 1, \ldots, l \) with \( c = c(l; \| u_0 \|^{3/4+}_{s, 2}, \| \partial_x^l u_0 \|_{L^2((x_0, \infty))}; v; \epsilon; T) \).

In particular, for any \( t \in (0, T] \), the restriction of \( u(\cdot, t) \) to any interval \( (x_0, \infty) \) belongs to \( H^l((x_0, \infty)) \).

Moreover, for any \( v \geq 0 \), \( \epsilon > 0 \) and \( R > 0 \)

\[
\int_0^T \int_{x_0 + R - vt}^{x_0 + \epsilon - vt} (\partial_x^{l+1} u)^2(x, t) \, dx \, dt < c,
\]

with \( c = c(l; \| u_0 \|^{3/4+}_{s, 2}, \| \partial_x^l u_0 \|_{L^2((x_0, \infty))}; v; \epsilon; R; T) \).

The property described in the Theorem 1.1 is intrinsic to suitable solutions of some nonlinear dispersive models, see for instance [13, 14], where analogous results for the Kadomtsev-Petviashvili II and Benjamin-Ono equations were proved.

To state our results we need to describe the class of solutions in which it applies. Thus, we first recall a result which is direct consequence of the energy estimates obtained by combining the commutator estimates in [17], the Sobolev embedding theorem and the argument in [2].

**Theorem 1.2.** Given \( u_0 \in H^s(\mathbb{R}^3) \) with \( s > 5/2 \), there exist \( T = T(\| u_0 \|_{s, 2}) > 0 \) and a unique solution \( u = u(\bar{x}, t) \) of the IVP (1.1) such that

\[
u \in C([0, T) : H^s(\mathbb{R}^3)).
\]

Moreover, the map data-solution \( u_0 \mapsto u(\bar{x}, t) \) from \( H^s(\mathbb{R}^3) \) into \( C([0, T) : H^s(\mathbb{R}^3)) \) is (locally) continuous.
We observe that
\[ \partial_x u, \partial_y u, \partial_z u \in C([0, T] : H^{s-1}([\mathbb{R}^3])) \subset L^1([0, T] : L^\infty([\mathbb{R}^3])). \]
(1.7)

Also, we shall use the following result which is a consequence of the arguments given in [25].

**Theorem 1.3.** Given \( u_0 \in H^s([\mathbb{R}^3] \) with \( s > 3/2 \), there exist \( T = T(\|u_0\|_{s,2}) > 0 \) and a unique solution \( u = u(\bar{x}, t) \) of the IVP (1.1) such that
\[ u \in C([0, T] : H^s([\mathbb{R}^3])), \]
with
\[ \partial_x u \in L^1([0, T] : L^\infty([\mathbb{R}^3])). \]
Moreover, the map data-solution \( u_0 \mapsto u(\bar{x}, t) \) from \( H^s([\mathbb{R}^3]) \) into \( C([0, T] : H^s([\mathbb{R}^3])) \) is (locally) continuous.

To delineate our results we introduce some notations: for \( a, b, c, d, f \in \mathbb{R} \) we define the half-space
\[ P_{\{a,b,c,d\}} = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz \geq d\}, \]
and the strip
\[ H_{\{a,b,c,d,f\}} = \{(x, y, z) \in \mathbb{R}^3 : d \leq ax + by + cz \leq f\}. \]

Our main result in this paper is the following:

**Theorem 1.4.** Let \( u_0 \in H^s([\mathbb{R}^3] \) with \( s > 5/2 \). If for some \( (a, b, c) \in \mathbb{R}^3 \) with
\[ a > 0, \quad b, c \geq 0 \quad \text{and} \quad \sqrt{3}a > \sqrt{b^2 + c^2}, \]
and for some \( j \in \mathbb{Z}^+ \), \( j \geq 3 \)
\[ N_j = \sum_{|a|=j} \int_{P_{\{a,b,c,d\}}} (\partial^a u_0(\bar{x}))^2 d\bar{x} < \infty, \]
(1.9)
then the corresponding solution \( u = u(\bar{x}, t) \) of the IVP for the ZK equation (1.1) provided by Theorem 1.2 satisfies that for any \( v \geq 0, \epsilon > 0 \) and \( R > 4\epsilon \)
\[ \sup_{0 \leq t \leq T} \sum_{|a|\leq j} \int_{P_{\{a,b,c,d-vt+\epsilon\}}} (\partial^a u(\bar{x}, t))^2 d\bar{x} \]
\[ + \sum_{|a|=j+1} \int_{0}^{T} \int_{H_{\{a,b,c,d-vt+\epsilon,c-d-vt+R\}}} (\partial^a u(\bar{x}, t))^2 d\bar{x} dt \leq c = c(\|u_0\|_{s,2}; \{N_l : 1 \leq l \leq j\}; j; a; b; c; v; T; c; R). \]
(1.10)

**Remark 1.5.** It is clear that if the hypothesis (1.9) is satisfied by a finite family of half-spaces \( P_{\{a_r, b_r, c_r, d_r\}}, r = 1, \ldots, N \) with the \( (a_r, b_r, c_r)'s \) as in (1.8), then the result in (1.10) holds for the union of the half-spaces \( P_{\{a_r, b_r, c_r, d_r-vt+\epsilon\}} \) and strips \( H_{\{a_r, b_r, c_r, d_r-vt+\epsilon,c-d-vt+R\}} \).

In particular, this tells us that the result extends to sets which are complement of a class of cones. More precisely, define the solid cone with vertex \( \bar{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3 \), axis \( \hat{w} \in \mathbb{S}^2 \) and opening \( \theta \in (0, \pi) \) as
\[ E_{\bar{x}_0, \hat{w}, \theta} = \{\bar{x} = (x, y, z) \in \mathbb{R}^3 : (\bar{x} - \bar{x}_0, \hat{w}) \leq \|\bar{x} - \bar{x}_0\| \cos(\theta)\}. \]
(1.11)
Thus, if for some \( j \in \mathbb{Z}^+ \), \( j \geq 2 \)
\[
N_j = \sum_{|\alpha| = j} \int_{(e_{\xi_0,-1,\pi/6})^c} (\partial^\alpha u_0(\vec{x}))^2 d\vec{x} < \infty,
\]
(1.12)
then for any \( v \geq 0 \) and \( \epsilon > 0 \)
\[
\sup_{0 \leq t \leq T} \sum_{|\alpha| \leq j} \int_{(e_{\xi_0,vt+\epsilon,\pi/6})^c} (\partial^\alpha u(\vec{x},t))^2 d\vec{x} < c
\]
(1.13)
with \( c = c(\|u_0\|_{s,2}; \{N_l : 1 \leq l \leq j\}; j; a; b; c; \epsilon) \).

A similar result to that described in (1.11)-(1.13) holds in \((e_{\xi_0,\vec{w},\theta})^c\) (see (1.11)) assuming that
\[
e_{\xi_0,-1,\pi/6} \subseteq e_{\xi_0,\vec{w},\theta}.
\]
(1.14)
One notices that the fact \( \epsilon > 0 \) allows us to write \((e_{\xi_0,-vt+\epsilon,\vec{w},\theta})^c\) as the union of finitely many half-spaces \( P_{a,b,c,d,-vt+\epsilon} \)'s, and that the condition (1.14) guarantees that the \( a, b, c, \epsilon \)'s satisfy (1.8).

The same argument given below for the proof of Theorem 1.2 shows that :

**Theorem 1.6.** Let \( u_0 \in H^s(\mathbb{R}^3) \) with \( s > 3/2 \). If for some \((a, b, c) \in \mathbb{R}^3\) with
\[
a > 0, \quad b, c \geq 0 \quad \text{and} \quad \sqrt{3a} > \sqrt{b^2 + c^2},
\]
and for some \( j \in \mathbb{Z}^+ \), \( j \geq 2 \)
\[
\tilde{N}_j = \int_{P_{a,b,c,d}} (\partial^j u_0(\vec{x}))^2 d\vec{x} < \infty,
\]
then the corresponding solution \( u = u(\vec{x},t) \) of the IVP for the ZK equation (1.1) provided by Theorem 1.3 satisfies that for any \( v \geq 0 \), \( \epsilon > 0 \) and \( R > 4\epsilon \)
\[
\sup_{0 \leq t \leq T} \int_{P_{a,b,c,d-\epsilon}} (\partial^j u(\vec{x},t))^2 d\vec{x}
\]
\[
+ \int_0^T \int_{H_{a,b,c,d-\epsilon}} (\partial^{j+1} u(\vec{x},t))^2 d\vec{x} dt
\]
\[
\leq c = c(\|u_0\|_{s,2}; \{N_l : 1 \leq l \leq j\}; j; a; b; c; v; T; \epsilon; R).
\]

**Remark 1.7.** The same comments stated in Remark 1.5 apply to the results in Theorem 1.6 with \( \partial_x \) instead of \( \partial \) in (1.12) and (1.13).

**Remark 1.8.** Concerning the sectorial regularity of solutions of the IVP (1.1) we consider the so called Strichartz estimates for the linear IVP associated to (1.1)
\[
\left\{ \begin{array}{l}
\partial_t v + \partial_x \Delta v = 0, \quad (x, y, z) \in \mathbb{R}^3, t \in \mathbb{R}, \\
v(x, y, z, 0) = v_0(x, y, z).
\end{array} \right.
\]
(1.15)

The solution \( v(x, y, z, t) \) of (1.15) is described by the unitary group \( \{U(t) : t \in \mathbb{R}\} \) with
\[
U(t) v_0(\vec{\xi}) = e^{it\varphi(\vec{\xi})} v_0(\vec{\xi}), \quad \varphi(\vec{\xi}) = \xi_1^3 + \xi_1(\xi_2^2 + \xi_3^2)
\]
with \( \vec{\xi} = (\xi_1, \xi_2, \xi_3) \).

In [25] (Proposition 1) Linares-Saut proved : Let \( \epsilon \in (0,1) \) and \( \theta \in (0, (1 + \epsilon/3)^{-1}\), Then
\[
\|D^{2/3}_{x} U(t) v_0\|_{L^q_t L^r_{xyz}} \leq c\|v_0\|_{L^q_{xyz}},
\]
where \( 1/q + 1/q' = 1/p + 1/p' = 1 \), \( p = 1/(1 - \theta) \) and \( 2/q = \theta(1 + \epsilon/3) \).
Thus, for $\epsilon \sim 1$ one gets a gain of $3/8^-$ derivatives (in the $x$-variable).

Notice that (roughly speaking) this gain of derivative in the $x$-variable extends
to all variables $(x,y,z)$ if $\hat{v}_0$ is supported inside the cone $\sqrt{3}\xi_1 > \sqrt{\xi_2^2 + \xi_3^2}$ which
is similar to that described in (1.8) for the physical space.

In [20] (Theorem 3.1) Kenig-Ponce-Vega showed : for $\mu \geq 0$ define
\[
U_\mu(t)v_0(\vec{x}) = \int e^{i(t\varphi(\vec{\xi}) + \vec{\xi} \cdot \vec{x})} |H\varphi|^{\mu/2} \hat{v}_0(\vec{\xi}) \phi_\varphi(\vec{\xi}) d\vec{\xi},
\]
with
\[
H\varphi(\vec{\xi}) = \det(\partial_{jk}^2 \varphi) = 8\xi_1 (3\xi_1^2 - (\xi_2^2 + \xi_3^2)),
\]
(hence
\[
H\varphi(\vec{\xi}) \geq \delta \|\vec{\xi}\|^3 \quad \text{if} \quad \sqrt{3}\xi_1 > \sqrt{\xi_2^2 + \xi_3^2},
\]
for some $\delta > 0$ and $\phi_\varphi \in C^\infty(\mathbb{R}^3)$ supported in the cone
\[
\sqrt{3}\xi_1 > \sqrt{\xi_2^2 + \xi_3^2}
\]
with $\phi_\varphi(\vec{\xi}) = 1$ for $\vec{\xi}$ sufficiently large inside the cone
\[
\sqrt{3}\xi_1 > (1 + \epsilon)\sqrt{\xi_2^2 + \xi_3^2}, \quad \epsilon > 0,
\]
then
\[
\|U_{\mu/2}(t)v_0\|_{L_t^q L_x^p} \leq c\|\hat{v}_0 \phi_\varphi\|_{L_{xyz}^p}, \quad (1.16)
\]
where
\[
\mu \in [0, 2/3), \quad 1/q + 1/q' = 1/p + 1/p' = 1, \quad \text{and} \quad (q,p) = (4/3\theta, 2/(1 - \theta)).
\]

Thus, for $\mu \sim 2/3^-$ (1.16) represents a gain of $1/2^-$ derivatives.

The previous observation can also be applied to solutions of the linear problem
associated to the ZK equation in 2D. More precisely, it was proved in [22] that solu-
tions of the linear problem, $U(t)\hat{f} = e^{it((\xi^2 + \eta^2))} \hat{f}$ satisfy the Strichartz
estimates.

**Lemma 1.9.** Let $0 \leq \epsilon < 1/2$ and $0 \leq \theta \leq 1$. Then,
\[
\|D_x^{\epsilon/2} U(t)f\|_{L_t^q L_x^p} \lesssim \|f\|_{L_{xyz}^p},
\]
where $p = \frac{2}{1+\epsilon}$ and $\frac{2}{q} = \frac{6(2+\epsilon)}{3}$. 

**Remark 1.10.** Our second result describes the persistence properties and regularity
effects, for positive times, in solutions associated with data having polynomial
decay in an appropriate half-space.

**Theorem 1.11.** If $u_0 \in H^s(\mathbb{R}^3)$, $s > 5/2$ and for some $j \in \mathbb{Z}^+$, $j \geq 2$ and some
$a, b, c$ satisfying (1.8)
\[
M_j = \| (ax + by + cz - d)^{j/2} u_0 \|_{L_2(P_{(a,b,c,d))}}^2
\]
\[
= \int_{P_{(a,b,c,d)}} (ax + by + cz - d)^j |u_0(\vec{x})|^2 d\vec{x} < \infty,
\]
then the solution $u = u(\vec{x}, t)$ of the IVP (1.1) provided by Theorem 1.2 satisfies that
for any $v \geq 0$ and $\epsilon > 0$
\[
\sup_{0 \leq t \leq T} \int_{P_{(a,b,c,d+t-\epsilon)}} |ax + by + cz - d + vt - \epsilon|^j |u(\vec{x}, t)|^2 d\vec{x} \leq c,
\]
with $c = c(\|u_0\|_{s,2}; \{\mathcal{M}_l : 1 \leq l \leq j\}; j; a; b; c; v; c; T)$. Moreover, for any $\epsilon, \delta, R > 0, v \geq 0$

$$
\sup_{\delta \leq t \leq T} \sum_{|\alpha| \leq j} \int_{P_{(u,b,c,d,-v+\epsilon)}} (\partial^\alpha u(\vec{x}, t))^2 d\vec{x} + \sum_{|\alpha| \leq j+1} \int_{\delta}^T \int_{H_{(u,b,c,d,-v+\epsilon,+R)}} (\partial^\alpha u)^2(\vec{x}, t) d\vec{x} dt \leq c,
$$

with $c = c(\|u_0\|_{s,2}; \{\mathcal{M}_l : 1 \leq l \leq j\}; j; a; b; c; v; c; R; T)$.

The proof of Theorem 1.11 follows an argument similar to that given below in the proof of Theorem 1.4 so it will be omitted (for details see [15]).

**Remark 1.12.** Our argument of proof is mainly based on weighted energy estimates for which the assumption (1.7) is essential.

**Remark 1.13.** The results in Theorem 1.2 extend (with the pertinent modifications) to the 2D ZK equation and to the IVP for the generalized ZK equation (1.3) in 2D and 3D for solutions satisfying the appropriate version of the hypothesis (1.7).

### 2. Proof of Theorem 1.4

Our starting point will be a weighted energy identity. This was motivated by the original proof of the so called Kato smoothing effect deduced in [16]. To obtain it we apply the operator $\partial^\alpha = \partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3}$ to the equation in (1.1), multiply the result by $2\partial^\alpha u_\chi$ with $\chi = \chi(ax+by+cz-d+vt)$, $a, b, c, d, v \in \mathbb{R}$ and $\chi, \chi' \geq 0$ to be chosen, and integrate the result in $\mathbb{R}^3$ to get after some integrations by parts that

$$
\frac{d}{dt} \int (\partial^\alpha u)^2 \chi d\vec{x} - v \int (\partial^\alpha u)^2 \chi' d\vec{x} + 3a \int (\partial_x \partial^\alpha u)^2 \chi' d\vec{x} - a^3 \int (\partial^\alpha u)^2 \chi''' d\vec{x} + a \int (\partial_y \partial^\alpha u)^2 \chi' d\vec{x} + 2b \int (\partial_x \partial^\alpha u)(\partial_y \partial^\alpha u)\chi' d\vec{x} - ab^2 \int (\partial^\alpha u)^2 \chi''' d\vec{x} + a \int (\partial_z \partial^\alpha u)^2 \chi' d\vec{x} + 2c \int (\partial_x \partial^\alpha u)(\partial_z \partial^\alpha u)\chi' d\vec{x} - ac^2 \int (\partial^\alpha u)^2 \chi''' d\vec{x} + 2 \int \partial^\alpha(u\partial_x u)\partial^\alpha u \chi d\vec{x} = \frac{d}{dt} \int (\partial^\alpha u)^2 \chi d\vec{x} - vE_1 + 3aE_2 - a^3E_3 + aE_4 + 2bE_5 - ab^2E_6 + aE_7 + 2cE_8 - ac^2E_9 + 2E_{10} = 0.
$$

We claim : if (1.8) holds, i.e.

$$
a > 0, \quad b, c \geq 0 \quad \text{and} \quad \sqrt{3}a > \sqrt{b^2 + c^2},
$$

then there exists $\epsilon > 0$ such that

$$
3aE_2 + aE_4 + aE_7 + 2bE_5 + 2cE_8 \geq \epsilon(E_2 + E_4 + E_7).
$$
To prove the claim we shall use the notations
\[
\mu^2 = E_2 = \int (\partial_x \partial^a u)^2 \chi' \, d\vec{x},
\beta^2 = E_4 = \int (\partial_y \partial^a u)^2 \chi' \, d\vec{x},
\gamma^2 = E_7 = \int (\partial_z \partial^a u)^2 \chi' \, d\vec{x},
\]
so that
\[
|E_5| \leq \mu \beta \quad \text{and} \quad |E_8| \leq \mu \gamma.
\]  
(2.4)

Hence, since
\[
3a\mu^2 + a\beta^2 + a\gamma^2 \geq 2\sqrt{3}a(\sqrt{\mu} \beta + \sqrt{\gamma}) \quad \text{for} \quad \theta \in [0, 1],
\]  
(2.5)

(2.3) follows by combining (2.2), (2.4) and (2.5).

Without loss of generality we shall assume from now on that \(d = 0\).

First, we shall consider the case:
\[
j = |\alpha| = 2.
\]

In this case we use the identity (2.1) for all \(\alpha\) such that \(|\alpha| = 2\) and \(v = 0\), so
\[
E_1 = 0.
\]
Since \(u \in C([0, T] : H^{5/2}((\mathbb{R}^3)))\) the absolute value of the terms \(E_3, E_6, E_9\) are uniformly bounded for \(t \in [0, T]\), and the nonlinear term contribution \(E_{10}\) can be estimated, using integration by parts, as
\[
|E_{10}| \leq c\|\partial u(t)\| \int (\partial^a u)(\vec{x}, t)^2 \chi d\vec{x} + c\|u(t)\| \int (\partial^a u)(\vec{x}, t)^2 \chi' d\vec{x}.
\]
Combining this information and (2.3) one concludes by Gronwall’s lemma that for any \(\alpha'\) with \(|\alpha'| = 3\)
\[
\int_0^T \int (\partial^a u(\vec{x}, t))^2 \chi'(ax + by + cz) d\vec{x} dt \leq c = c(\|u_0\|_{s, 2}; a; b; c; 0; T; \chi),
\]  
(2.6)

for any \(\chi \in C^3(\mathbb{R})\) with \(\chi, \chi' \geq 0\) and \(\chi'\) having compact support.

Case: \(j = |\alpha| = 3\).

By hypothesis on the data \(u_0\) for some \(a, b, c\) for which (2.2) holds one has
\[
N_3 \equiv \sum_{|\alpha| = 3} \int_{P_{\{a, b, c, 0\}}} (\partial^a u_0(\vec{x}))^2 \, d\vec{x} < \infty,
\]
where
\[
P_{\{a, b, c, d\}} = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz \geq d\}.
\]

For any \(\epsilon > 0\) and \(R > 4\epsilon\) let
\[
\chi_{\epsilon, R} = \chi \in C^3(\mathbb{R}), \quad \chi, \chi' \geq 0 \quad \text{supp} \chi' \subset (\epsilon, R),
\]
\[
\chi'(l) \geq 1/10\epsilon \quad \text{for} \quad l \in (2\epsilon, R - \epsilon).
\]  
(2.7)

Let \(v > 0\). Consider the identity (2.1) for all \(\alpha\) with \(|\alpha| = 3\). Adding in \(\alpha\) the previous argument combined with (2.6) provides, after integration in time, the bounds for the terms \(E_1, E_3, E_6, E_9\). Also we have (omitting the summation in \(|\alpha| = 3\)) by integration by parts that
\[ |E_{10}| \leq c\|\partial u(t)\|_\infty \int (\partial^\alpha u(\vec{x}, t))^2 \chi d\vec{x} \]
\[ + c\|u(t)\|_\infty \int (\partial^\alpha u(\vec{x}, t))^2 \chi d\vec{x} + c E_{10,3}, \quad (2.8) \]

with
\[ E_{10,3} = \sum_{|\beta|, |\beta'| = 2} |\int \partial^\beta u \partial^\beta' u \partial^\alpha u \chi d\vec{x}|. \]

Thus, the first term in the r.h.s of (2.8) will be estimated by Gronwall’s inequality and (1.7) and the second one, after integration in time, by (1.6) and (2.6). To bound \( E_{10,3} \) we see that
\[ E_{10,3} \leq \|\partial^\alpha u\chi^{1/4}\|_4 \|\partial^\beta u\chi^{1/4}\|_4 \|\partial^\alpha u\chi^{1/2}\|_2, \quad (2.9) \]

where we have omitted the summation sign in \( \beta, \beta', \alpha \) with \( |\beta| = |\beta'| = 2, |\alpha| = 3 \).

By integration by parts one has that
\[ \|\partial^\beta u\chi^{1/4}\|_4 \leq c\|\partial u\|_\infty \|\partial^\alpha u\chi^{1/2}\|_2 + c\|\partial u\|_\infty \int (\partial^\beta u)^3 \chi' d\vec{x}. \quad (2.10) \]

For the last term in (2.10) it follows from Gagliardo-Nirenberg and Young inequalities that
\[ |\int (\partial^\beta u)^3 \chi' d\vec{x}| \leq c\|(\partial^\beta u\tilde{\chi}')\|_2 \leq c\|\partial^\beta u\tilde{\chi}'\|_2 \|\partial (\partial^\beta u\tilde{\chi}')\|_2 \]
\[ \leq c\|\partial^\beta u\tilde{\chi}'\|_2^6 + c\|\partial (\partial^\beta u\tilde{\chi}')\|_2^2 \quad (2.11) \]

(with an appropriate \( \tilde{\chi} \) of which from (1.6) and (2.6) (with a suitable choice of \( \chi \)) is bounded after integration in time. Therefore, inserting (2.10) and (2.11) in (2.9) and using (2.6), (2.3) and (2.1) with \( |\alpha| = 3 \) it follows that for any \( v \geq 0, \epsilon > 0 \) and \( R > 4\epsilon \)

\[
\begin{align*}
&\sup_{0 \leq t \leq T} \sum_{|\alpha'|=3} \int_{P_{(u, a, b, c, -v, t+1)}} (\partial^\alpha u(\vec{x}, t))^2 d\vec{x} \\
&+ \sum_{|\alpha'|=4} \int_{0}^{T} \int_{H_{(u, a, b, c, -v, t+1, -v, t+R)}} (\partial^\alpha u(\vec{x}, t))^2 d\vec{x} dt \\
&\leq c = c(\|u_0\|_{s, 2}; N_3; a; b; c; v; T; c; R).
\end{align*}
\] (2.12)

Case: \( j = |\alpha| = 4 \).

Inequality (2.12) provides the estimate for \( E_1, E_4, E_6, E_9 \) and (2.2)-(2.3) that for the terms in (2.1) involving \( E_2, E_4, E_5, E_7, E_9 \). Hence, it remains to consider the term \( E_{10} \) in (2.1) which carries the contribution for the nonlinear part. Omitting the summation in \( |\alpha| = 4 \) integration by parts leads to
\[ |E_{10}| \leq c\|\partial u(t)\|_\infty \int (\partial^\alpha u(\vec{x}, t))^2 \chi d\vec{x} \]
\[ + c\|u(t)\|_\infty \int (\partial^\alpha u(\vec{x}, t))^2 \chi' d\vec{x} + c E_{10,4}, \quad (2.13) \]

with
\[ E_{10,4} = \sum_{|\beta|, |\beta'| = 3} |\int \partial^\beta u \partial^\beta' u \partial^\alpha u \chi d\vec{x}|. \]
As in the previous case, the first term in the r.h.s of (2.13) will be estimated by (1.7) and Gronwall’s inequality and the second one, after integration in time, by (1.6) and (2.12). To handle $E_{10.4}$ we write

$$E_{10.4} \leq \|\partial^\beta u\|_3 \|\partial^\beta u\|_6 \|\partial^\alpha u\|^2_2,$$

with $\chi$ in the class defined in (2.7) with $\chi(l) = 1$ for $l \in \text{supp}(\chi)$. This can be done by changing the values of $\epsilon$ and $R$. By the Gagliardo-Nirenberg inequality, as in (2.11), $\|\partial^\beta u\|_3$ is uniformly bounded in the time interval $[0, T]$. The Sobolev embedding yields

$$\|\partial^\beta u\|_6 \leq c\|\partial(\partial^\beta u\|_2) \leq c\|\partial^\beta u\|_2 + c\|\partial^\beta u(\chi)^{1/2}\|_2,$$

(2.14)

where by (2.6) the second term in the r.h.s of (2.14) after integration in time is bounded, and the first term in the r.h.s. of (2.14) is what we are estimating. Therefore, a familiar argument yields that any $v \geq 0$, $\epsilon > 0$ and $R > 4\epsilon$

$$\sup_{0 \leq t \leq T} \sum_{|\alpha| = k} \int_{P_{(a,b,c,-vt+)}^{(a,b,c,-vt+e)}} (\partial^\alpha u(\vec{x},t))^2 d\vec{x}$$

$$+ \sum_{|\alpha'| = 5} \int_0^T \int_{H_{(a,b,c,-vt+e,-vt+e)}} (\partial^\alpha u(\vec{x},t))^2 d\vec{x}dt$$

$$\leq c = c(\|\partial u\|_{s,2}; N_4; a; b; c; T; c; R).$$

Case: $j = |\alpha| \geq 5$.

By hypothesis we have :

$$N_j \equiv \sum_{|\alpha| = j} \int_{P_{(a,b,c,0)}} (\partial^\alpha u_0(\vec{x}))^2 d\vec{x} < \infty.$$

and for any $v > 0$, $\epsilon > 0$ and $R > 4\epsilon$

$$\sup_{0 \leq t \leq T} \sum_{|\alpha| = j-1} \int_{P_{(a,b,c,-vt+)}^{(a,b,c,-vt+e)}} (\partial^\alpha u(\vec{x},t))^2 d\vec{x}$$

$$+ \sum_{|\alpha'| = j} \int_0^T \int_{H_{(a,b,c,-vt+e,-vt+e)}} (\partial^\alpha u(\vec{x},t))^2 d\vec{x}dt$$

$$\leq c = c(\|\partial u\|_{s,2}; \{N_l : l = 1, 2, ..., j - 1; a; b; c; v; T; c; R\}).$$

(2.15)

We consider the identity (2.1) with $\alpha$ such that $|\alpha| = j$. The property (2.15) gives the estimate for $E_1, E_3, E_6, E_9$ and (2.2)-(2.3) that for the terms in (2.1) involving $E_2, E_4, E_5, E_7, E_8$. Hence, one just needs to handle the contribution of the nonlinear term $E_{10}$ in (2.1)

$$E_{10} = \int \partial^\alpha (u \partial_x u) \partial^\gamma u \partial^\gamma u \partial^\alpha u \partial^\gamma u d\vec{x}$$

$$+ c_{\alpha} \int \partial^\alpha u \partial^\gamma u \partial^\gamma u \partial^\alpha u \partial^\gamma u d\vec{x} + c_\beta \int \partial^\alpha u \partial^\gamma u \partial^\gamma u \partial^\alpha u \partial^\gamma u d\vec{x}$$

$$+ \sum_{\beta + \beta' = \alpha_0} \sum_{|\beta|, |\beta'| \leq j - 2} c_{\beta, \beta'} \int \partial^\beta u \partial^\gamma u \partial^\gamma u \partial^\alpha u \partial^\gamma u d\vec{x}$$

(2.16)
omitting the summation signs in \( \alpha, \alpha' \) with \( |\alpha| = |\alpha'| = j \) and with \( \mu \) and \( \alpha'' \) such that \( |\mu| = 2, \mu + \alpha'' = \alpha + (1,0,0) \) and \( \alpha_0 = \alpha + (1,0,0) \).

By integration by parts it is clear that
\[
\int u \partial \alpha u \partial \alpha' u \chi d\vec{x} + c_{\alpha, \alpha'} \int u \partial^\alpha u \partial^\alpha u \chi d\vec{x}
\leq c \|\partial u(t)\|_{\infty} \int (\partial^\alpha u(\vec{x}, t))^2 \chi d\vec{x} + c \|u(t)\|_{\infty} \int (\partial^\alpha u(\vec{x}, t))^2 \chi' d\vec{x}.
\]

Since \( |\mu| = 2 \) it follows that
\[
\int \partial^\mu u \partial^\nu u \partial^\alpha u \chi d\vec{x}
\leq c \|\partial^\mu u(t)\|_{\infty} \int (\partial^\nu u(\vec{x}, t))^2 \chi d\vec{x}^{1/2} \int (\partial^\alpha u(\vec{x}, t))^2 \chi d\vec{x}^{1/2}
\]
for an appropriate \( \chi \) in the class in (2.7) with \( \chi(l) = 1 \) for \( l \in \text{supp}(\chi) \). Using (2.15) we have that \( \|\partial^\mu u(t)\|_{\infty} \) is uniformly bounded in the time interval \([0,T]\). Hence, it remains to bound the last term in the r.h.s. of (2.16). Since \( \beta + \beta' = \alpha_0 = \alpha + (1,0,0) \) and \( |\beta|, |\beta'| \leq j - 2 \) with \( j \geq 5 \) one has
\[
\int \partial^\beta u \partial^\beta' u \partial^\alpha u \chi d\vec{x} \leq c \|\partial^\beta u(t)\|_{4} \|\partial^\beta u(t)\|_{4} \|\partial^\alpha u(\vec{x}, t)\|_{4} \int (\partial^\alpha u(\vec{x}, t))^2 \chi d\vec{x}^{1/2},
\]
where by the Gagliardi-Nirenberg inequality \( \|\partial^\beta u(t)\|_{4} \) can be bounded, uniformly in the time interval \([0,T]\), in terms of the known previous estimates in (2.15).

Gathering this information a familiar argument leads to the desired result and completes the proof of Theorem 1.4.

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