Generalised Lotka–Volterra model with hierarchical interactions

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In the analysis of complex ecosystems it is common to use random interaction coefficients, often assumed to be such that all species are statistically equivalent. In this work we relax this assumption by choosing interactions according to the cascade model, which we incorporate into a generalised Lotka–Volterra dynamical system. These interactions impose a hierarchy in the community. Species benefit more, on average, from interactions with species further below them in the hierarchy than from interactions with those above. Using dynamic mean-field theory, we demonstrate that a strong hierarchical structure is stabilising, but that it reduces the number of species in the surviving community, as well as their abundances. Additionally, we show that increased heterogeneity in the variances of the interaction coefficients across positions in the hierarchy is destabilising. We also comment on the structure of the surviving community and demonstrate that the abundance and probability of survival of a species is dependent on its position in the hierarchy.

I. INTRODUCTION

The study of complex ecosystems has been an active research area in theoretical ecology since the 1970s, when evidence emerged of a sharp transition from stability to instability in model ecosystems of increasing complexity [1, 2]. By suggesting fundamental limits on the size, connectedness, and interaction variability of a stable ecosystem, these results appeared to contradict the prevailing ecological view of the time. Many ecological networks are both large and highly interconnected [3, 4], leading to an a priori expectation that a more complex and well-connected ecosystem ought to be more stable than simpler, more sparsely connected counterpart [5, 6]. Such an apparent contradiction has led to an increasingly detailed and nuanced search for consistent definitions of ecological stability and complexity across theory and experiment [6–10].

One powerful theoretical tool for analysing which factors contribute to the stability of a large system of many interacting constituents, such as a complex ecosystem, is random matrix theory (RMT). This is the method that was employed by Robert May in his seminal work [2]. May started from the Jacobian of a hypothetical ecosystem, linearised about an ‘equilibrium’. He assumed that the entries of this Jacobian matrix were i.i.d. random variables. This allowed for the deduction of a stability criterion using Girko’s well-known circular law [11]. May’s initial and somewhat austere model has been extended in recent years, accounting for additional features of ecosystems such as food web structure [12, 13], spatial dispersal [14, 15], alternative interpretations of ‘interaction strength’ [16, 17] and varying self-regulation [18].

Despite providing a great deal of insight, RMT suffers from one crucial drawback: there is no guarantee that the random matrix under consideration corresponds to the Jacobian matrix of a feasible equilibrium [19, 20], that is, an equilibrium at which species abundances are non-negative. To remedy this, recent works [21–25] have instead studied the stability of complex ecosystems using the generalised Lotka–Volterra equations, which produce feasible equilibria by construction.

In this work, we extend a recent study [26], which used RMT to determine the impact of hierarchical interactions (known as the cascade model [4, 27]) on stability. Here, we incorporate such hierarchical interactions in the generalised Lotka–Volterra equations. We obtain analytical results for both stability and the composition of surviving communities using techniques from statistical physics and the theory of disordered systems, specifically dynamic mean-field theory [28, 41]. Our approach has several advantages. Foremost, as mentioned above, the equilibria we study are feasible by construction. Further, we are able to study the effects of hierarchical interactions not only on stability, but also on the emergent properties of the community, such as species abundances and survival probabilities. Finally, we are also able to identify not only when the ecosystem becomes unstable, but also the nature of the instability.

More specifically, we show that increasing the severity of hierarchical interaction in the community is a stabilising force, as is increasing the proportion of interactions which are of predator–prey type (in agreement with, for example [8, 23, 27, 29]). We also demonstrate that larger asymmetry in the variance of species’ interactions decreases stability. Further, we look at the properties of stable equilibria produced in such a hierarchical community. We find that the presence of hierarchical interaction leads to communities with non-Gaussian species abundance distributions, and that species lower in the hierarchy (i.e. species that benefit less from interactions) are both less likely to survive and less abundant.

This remainder of the paper is structured as follows: In Section II, we describe the generalised Lotka–Volterra model with hierarchical (cascade) interactions. In Section III, we outline how dynamic mean-field the-
ory is used to trade the set of coupled differential equations with random interaction coefficients constituting the generalised Lotka–Volterra model for a smaller number of statistically equivalent coupled stochastic differential equations. We then analyse the fixed-point solution found in Section IV, noting that the fixed-point equations were also obtained with the cavity method in Refs. [30] and [31] [Eqs. (13)], but no stability analysis was performed in these references. We then discuss the effect of hierarchical interactions on the distribution of species abundances and the fraction of surviving species in Section V. Finally, in Section VI, we study the stability of the fixed-point solution before concluding in Section VII.

II. MODEL DEFINITION

A. General Block-Structured Interactions

We consider a community of $N$ species, partitioned into $B$ distinct sub-communities, which we index with $a = 1, \ldots, B$. Sub-community $a$ contains $N^a$ species, so that $N = \sum_a N^a$. We first describe a more general model before introducing the specific case of the cascade model.

We denote the abundance of the $i$-th species in group $a$ by $x^a_i(t)$, so that $i = 1, \ldots, N^a$ when we are referring to species in group $a$. Species abundances evolve according to the generalised Lotka–Volterra (GLV) dynamics [21, 23, 30, 32–34]

$$
\dot{x}^a_i(t) = x^a_i(t) \left[ 1 - \frac{x^a_i(t)}{N} + \sum_{b=1}^{B} \sum_{j=1}^{N^b} A^a_{ij} x^b_j(t) \right],
$$

where the block-structured interaction matrix $(A^a_{ij})_{ab}$ dictates the influence of the $j$-th species in group $b$ on the $i$-th species in group $a$. The coefficients $A^a_{ij}$ are correlated random variables with statistics to be specified (which may vary between blocks). The diagonal elements $A^a_{ii}$ are set to zero. This general setup is illustrated in Fig. 1(a).

The statistics of the interactions between species depend only on their respective sub-communities. We write

$$
A^a_{ij} = \frac{\mu^{ab}}{N} + \frac{\sigma^{ab}}{\sqrt{N}} w^a_{ij},
$$

(2)

where the $w^a_{ij}$ are random variables with the following first and second moments,

$$
\bar{w}^a_{ij} = 0,
$$

$$
\langle (w^a_{ij})^2 \rangle = 1,
$$

$$
\bar{w}^a_{ij} w^a_{ji} = \gamma^{ab}.
$$

(3)

We have indicated an average over realisations of the matrix $w^a_{ij}$ with an overbar.

The model is thus fully specified by the parameters $\mu^{ab}, \sigma^{ab}$ and $\gamma^{ab}$. More specifically, $\mu^{ab}$ is the average influence of species in group $b$ on those in group $a$, $(\sigma^{ab})^2$ is the variance of those interactions and $\gamma^{ab} \in [-1, 1]$ is a correlation coefficient controlling the proportion of interactions between species in sub-communities $a$ and $b$ which are of a predator-prey type (i.e. $A^a_{ij} A^b_{ji} < 0$).

The factors of $1/N$ in Eq. (2) ensure a sensible thermodynamic limit, $N \to \infty$ (see e.g. [35]).

B. The Cascade Model

The cascade model is obtained from Eqs. (2) and (3) through a specific choice of model parameters $\mu^{ab}, \sigma^{ab}, \gamma^{ab}$. We imagine a ranking of the $B$ sub-communities in which, on average, species with higher rank gain more from lower ranked species than vice versa. Hence species higher up in the hierarchy are increasingly biased towards success.

Similar to Ref. [26], we achieve this with the choice

$$
\mu^{ab} = \begin{cases} 
\mu - \nu, & a < b \\
\mu, & a = b \\
\mu + \nu, & a > b
\end{cases},
$$

$$
\sigma^{ab} = \begin{cases} 
\sigma/\rho, & a < b \\
\sigma, & a = b \\
\sigma \rho, & a > b
\end{cases},
$$

(4)

where we mostly focus on the case $\nu > 0$. In this case species $a = 1$ is lowest in the hierarchy and species $a = B$ is highest. If we make the transformation $\nu \to -\nu$ and $\rho \to 1/\rho$, the ranking of species is reversed and we obtain identical results. We also require $\rho > 0$, ensuring the variance of all interactions remains positive. An example of the structure of the interaction matrix resulting from this choice is illustrated in Fig. 1 (b).

The model parameters $(\mu, \nu, \sigma, \rho, \gamma)$ can be separated into two sets: $\nu$ and $\rho$ describe the hierarchy of species, whereas $\mu, \sigma$, and $\gamma$ describe overall statistics. The parameter $\nu$ is a measure of the strength of the hierarchy. A larger value of $\nu$ increases the average benefit to species higher in the hierarchy, and correspondingly decreases the average benefit to species lower in the hierarchy. The parameter $\rho$ is a measure of the disparity between the variances of interactions $(\sigma^{ab})^2$ and $(\sigma^{ba})^2$ for $a \neq b$. The parameter $\mu$ characterises the mean interaction strength across all pairs of species in the community, $\sigma$ is a measure of the overall variability of interactions, and $\gamma$ can be related to the proportion of interactions that are of predator-prey type, $p$, via $\gamma = \cos(\pi p) + \mathcal{O}(1/\sqrt{N})$ [see Section S2 of the Supplemental Material (SM) for details]. We note that if $\nu = 0$ and $\rho = 1$, there is no distinction between any two positions in the hierarchy and all species are statistically equivalent, this is the case considered in [21, 23].

It is important to note that we distinguish between the notions of hierarchy, controlled by $\nu$, and the propor-
tion of predator-prey pairs in the model, controlled by $\gamma$. Hierarchy is present so long as $\nu \neq 0$. There is then a natural ordering to the subgroups of species. The choice $\nu > 0$ implies an average benefit of species higher in the hierarchy relative to those lower in the hierarchy. The proportion of predator-prey pairs, on the other hand, is not affected by hierarchy (i.e. a non-zero value of $\nu$) to leading order in $N$ (see Section S2 in the SM). This is due to the $1/N$ scaling of the average value of $A_{ij}$ in Eq. (2) compared to the $1/\sqrt{N}$ scaling of the random term.

III. DYNAMIC MEAN-FIELD THEORY

We use dynamic mean-field theory [28, 36–42] in order to analyse the stability and community properties of the GLV system in Eq. (1). Ultimately our analysis will focus on models with cascade interactions, determined by the parameters $\mu, \nu, \sigma, \rho$ and $\gamma$. However, for now, we return to the more general block-structured general case discussed in Section II A [Fig. 1(a)].

The analysis involves taking the limit $N \to \infty$ whilst keeping the ratios $n^a \equiv N^a/N$ constant, such that

$$\sum_b n^a = 1.$$  (5)

That is to say, each sub-community contains a large (formally infinite) number of species, but the proportions of species in each group remain fixed as $N$ is varied.

The calculation closely follows the lines of [21, 32, 36, 37], with modifications made to account for the block structure of the interactions in the community. Details are given in Section S3 of the SM.

The dynamic mean-field approach results in the reduction of the initial set of coupled ordinary equations with random coefficients $\xi$ given in Eq. (1), and with $N \to \infty$] to a set of $B$ stochastic integro-differential equations, one for each sub-community. These describe the ‘typical’ time evolution for the abundance of species in the different groups, $x^a(t)$. Carrying out the steps in Section S2 of the SM, we arrive at the following effective dynamics for species in sub-community $a$

$$\dot{x}^a(t) = x^a(t) \left(1 - x^a(t) + \sum_{b=1}^{B} n^b \mu^{ab} M^b(t) \right)$$

$$+ \sum_{b=1}^{B} n^b \gamma^{ab} \sigma^{ab} \int_0^t dt' C^b(t, t') x^b(t') + \eta^a(t).$$  (6)

The variables $\{\eta^a(t)\}$ are coloured Gaussian noise terms with the following statistics

$$\langle \eta^a(t) \rangle = 0,$$

$$\langle \eta^a(t) \eta^b(t') \rangle = \delta^{ab} \sum_{c=1}^{B} (\sigma^{ac})^2 \rho^c C(t, t').$$  (7)

where we have used $\langle \cdots \rangle$ to denote averages over realisations of the effective dynamics in Eq. (6), i.e. over realisations of the noise $\{\eta^a(t)\}$.

The macroscopic statistics of community $a$ are given by

$$M^a(t) \equiv \langle x^a(t) \rangle,$$

$$C^a(t, t') \equiv \langle x^a(t) x^a(t') \rangle,$$

$$G^a(t, t') \equiv \left\langle \frac{\delta x^a(t)}{\delta \eta^a(t')} \right\rangle.$$  (8)

The above quantities describe the average abundance of a species in sub-community $a$, the auto-correlations (in time) of a species abundance in the community, and the response to perturbations, respectively. The solution of Eqs. (6)-(8) determines the macroscopic statistics $M^a(t)$, $C^a(t, t')$ and $G^a(t, t')$ self-consistently.
depends on time differences with Eqs. (9) to (11) in mind, we find that the nontrivial
a
 expression was found for models without hierarchical structure in [21, 32, 37, 43]. We note that
depend on the value that the random variable \(x^a\) takes, some species will be extinct at the fixed point \((a^a = 0)\), whereas others will have positive abundance.

The average over realisations of the effective process, \(\langle \ldots \rangle\), is now an average over the static Gaussian random variables \(x^a\) at the fixed point. This allows us to obtain self-consistent conditions for the statistics of \(x^a\). We find (see Section S4.1 of the SM)
\[
\chi^a u^a = w_0(\Delta^a),
\]
\[
M^a u^a = w_1(\Delta^a) \sum_b (\sigma^{ab})^2 n^b q^b,
\]
\[
q^a (a^a)^2 = w_2(\Delta^a) \sum_b (\sigma^{ab})^2 n^b q^b,
\]
where we have abbreviated
\[
u^a = 1 - \sum_b \gamma^{ab} \sigma^{ab} \sigma^{ba} n^b x^b,
\]
\[
\Delta^a = \frac{1 + \sum_b \nu^{ab} n^b M^b}{\sqrt{\sum_b (\sigma^{ab})^2 n^b q^b}},
\]
and where we have defined the functions
\[
w_k(\Delta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Delta} dz e^{-z^2/2} e^{-z^2(z-\Delta)^k},
\]
for \(k \in \{0, 1, 2\}\). Eqs. (13) are equivalent to those in section IV.1 of the Supplementary Information of Ref. [30], derived with the cavity method. They also reduce to the fixed point equations for the model without hierarchy found in Refs. [21, 23] if one takes \(\mu^{ab} = \mu/B, \sigma^{ab} = \sigma/\sqrt{B}\) and \(\gamma^{ab} = \gamma\).
Eqs. (13) can be solved numerically for the quantities $\chi^a, M^a, q^a$ as functions of the model parameters $\mu^{ab}, \sigma^{ab}, \gamma^{ab}$ and $n^a$. We note that the fraction of surviving species in sub-community $a$, $\phi^a$, is given by

$$\phi^a = w_0(\Delta^a).$$

(16)

Hence the solution of Eqs. (13) also provides the fraction of surviving species in each sub-community.

For further analysis, it is useful to introduce the following average over communities

$$[X]_B = \sum_{a=1}^B n^a X^a$$

(17)

for a quantity $X^a$ defined in each community. The overall fraction of surviving species in the system is then $[\phi]_B$, and the average abundance per species is $[M]_B$.

### B. Cascade Model

We now analyse the fixed-point solution for a community with cascade-model interactions, for which the parameters $\mu^{ab}, \sigma^{ab}, \gamma^{ab}$ are as in Eq. (4). It is convenient to introduce $\alpha = a/B$, and to work in the limit $B \to \infty$, the resulting interaction matrix is illustrated in Fig. 1(c). The index $\alpha \in [0, 1]$ is now continuous, and the constraint $\sum_{b=1}^B n^b = 1$ becomes $\int_0^1 da \, n(\alpha) = 1$ (see also Section S4.1 of the SM). Quantities such as $M^a, \phi^a, \Delta^a$ etc. are now functions of $\alpha$, and averages over the index $a$ are integrals, which we write as follows

$$[X] \equiv \lim_{B \to \infty} [X]_B = \int_0^1 da \, n(\alpha)X(\alpha),$$

(18)

dropping the subscript $B$ in the limit. For example, the first relation in Eqs. (14) is now

$$u(\alpha) = 1 - \gamma \sigma^2 [\chi],$$

(19)

informing us that $u$ has no dependence on $\alpha$ in the cascade model when $B \to \infty$.

Recalling the definition of $\Delta^a$ in the second relation in Eq. (14) we also introduce the following quantities

$$\Delta_0 \equiv \Delta(\alpha = 0), \quad \Delta_1 \equiv \Delta(\alpha = 1).$$

(20)

To proceed, we first find expressions for the quantities $u, \Delta_0, \Delta_1$, in terms of $\nu, \sigma, \rho, \gamma$ using Eqs. (13). Then, we will find expressions for $[M]$ and $[\phi]$ [we also discuss expressions for the full functions $M(\alpha), \phi(\alpha)$ in Section V] in terms of $u, \Delta_0, \Delta_1$. In turn, this will enable us to calculate the macroscopic statistics of the system for any $\mu, \nu, \sigma, \rho, \gamma$. Details of all remaining calculations in this section can be found in Section S4 of the SM.

Manipulation of Eqs. (13) reveals that $u$ satisfies the following [see Eqs. (S49) and (S57) in Section S4 of the SM]

$$u^2 = \ell \sigma^2 \int_0^1 da \, n(\alpha)w_2[\Delta(\alpha)],$$

$$u(1-u) = \gamma \sigma^2 \int_0^1 da \, n(\alpha)w_0[\Delta(\alpha)],$$

(21)

where $\ell$ is the logarithmic mean [44] of $\rho^2$ and $1/\rho^2$

$$\ell \equiv \frac{\rho^2 - 1/\rho^2}{\ln \rho^2 - \ln 1/\rho^2},$$

(22)

which satisfies $\ell \geq 1$, with equality only if $\rho = 1$. We now show that the integrals in Eqs. (21) can be written explicitly in terms of only $u, \Delta_0, \Delta_1$. One achieves this by finding the following separable differential equation for $\Delta(\alpha)$ [Eq. (S67) in Section S4 of the SM]

$$\frac{d\Delta}{d\alpha} = n(\alpha),$$

(23)

with initial condition $\Delta(0) = \Delta_0$ and with

$$\theta(\Delta, u) = \frac{u^2}{2 u_0 w_2(\Delta) - \ell \sigma^2 \ln \rho^2 w_2(\Delta)}$$

(24)

Self-consistently, the solution to Eq. (23) will further have to satisfy the constraint $\Delta(1) = \Delta_1$.

One then uses Eq. (23) to change variables in the integrals in Eqs. (21), finding

$$\int_0^1 da \, n(\alpha)w_2[\Delta(\alpha)] = \int_{\Delta_0}^{\Delta_1} d\Delta \, \theta(\Delta, u)w_2(\Delta),$$

(25)

and similarly for the integral over $w_0[\Delta(\alpha)]$. If we also perform the same change of variables on the condition $\int_0^1 da \, n(\alpha) = 1$ then we arrive at the following simultaneous equations for $u, \Delta_0$ and $\Delta_1$

$$u^2 = \ell \sigma^2 \int_{\Delta_0}^{\Delta_1} d\Delta \, \theta(\Delta, u)w_2(\Delta),$$

$$u(1-u) = \gamma \sigma^2 \int_{\Delta_0}^{\Delta_1} d\Delta \, \theta(\Delta, u)w_0(\Delta),$$

$$1 = \int_{\Delta_0}^{\Delta_1} d\Delta \, \theta(\Delta, u).$$

(26)

Given the parameters $\nu, \sigma, \rho, \gamma$, Eqs. (26) can be solved numerically to obtain $\Delta_0, \Delta_1$ and $u$. As the parameter $\mu$ and the function $n(\alpha)$ do not appear in Eqs. (26), we conclude that $\Delta_0, \Delta_1$ and $u$ are independent of them. That is, $\Delta_0, \Delta_1$ and $u$ are functions of $\nu, \sigma, \rho$ and $\gamma$ only.

Once $\Delta_0, \Delta_1, u$ are determined, we calculate the average abundance and fraction of surviving species in the community with

$$\frac{1}{[M]} = \frac{\Delta_1 \rho^2 + \Delta_0 \nu - \mu}{\Delta_1 \rho^2 - \Delta_0 \nu},$$

(27a)

$$[\phi] = \frac{u(1-u)}{\gamma \sigma^2}.$$  

(27b)
See also Eqs. (S77) and (S78) in Section S4 of the SM. Eq. (27a) ceases to apply when \( \nu = 0 \), and a limit must be taken. Similar care has to be taken when finding \([\phi]\) in the limit \( \gamma \to 0 \). Both limits are found in Section S4.5.3 of the SM.

Inspection of Eqs. (27a) and (27b) reveals that, surprisingly, neither \([\phi]\) nor \([M]\) depend on \( n(\alpha) \), the relative sizes of the different sub-communities \( \alpha \). In fact, we will see in Sections V and VI that \( n(\alpha) \) does not affect any of the properties of the community that we are interested in (see also Section S8 in the SM). One also sees that \([\phi]\) is independent of \( \mu \).

Our theoretical predictions for the community average species abundance \([M]\) and the community average survival fraction \([\phi]\) are verified using the results of computer simulation of Eqs. (1) in Fig. 2.

V. FIXED POINT DISTRIBUTIONS

A. Abundance Distributions

At a stable equilibrium, we can calculate species abundance distributions (SADs) and rank abundance distributions (RADs). An SAD is obtained in simulations by binning species according to their abundances and producing a histogram of the number of species in each bin [panel (a) of Fig. 3]. An RAD is a plot of abundance against a ranking of species from 0 to 1, with the highest abundance species having a rank of 1 [panel (b) of Fig. 3]. For reviews of both SADs and RADs see Refs. [45, 46], or see Ref. [21] for similar calculations without hierarchical interactions, and [47] for RADs derived from random replicator equations rather than Lotka–Volterra.

We also introduce hierarchical abundance distributions (HADs). Similar to an RAD we rank species from 0 to 1, but this time such that a species with rank \( r \) is higher in the hierarchy of the cascade model than \( rN \) species, and lower than \((1-r)N\) species. An HAD is then a plot of abundance as a function of this hierarchical rank.

To calculate SADs and RADs we use \( P(x|\alpha) \), the probability that a species has abundance \( x \), given that it sits at position \( \alpha \) in the hierarchy. This is the probability density for \( x^2_n \) in Eq. (12). One arrives at (see also Section S5 in the SM) the following clipped Gaussian distribution

\[
P(x|\alpha) = [1 - \phi(\alpha)] \delta(x) + H(x) \varphi(x|\alpha),
\]

where \( H(\cdot) \) is the Heaviside step function \( H(x) = 1 \) for \( x \geq 0 \), and \( H(x) = 0 \) otherwise, and where the delta function \( \delta(x) \) represents species that have gone extinct and hence have an abundance of \( x = 0 \). The function \( \varphi(x|\alpha) \) is a (normalised) Gaussian in \( x \) such that \( \int_0^\infty dx \varphi(x|\alpha) = \phi(\alpha) \).

\[
\varphi(x|\alpha) = \frac{w_1[\Delta(\alpha)]}{M(\alpha)\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{w_1[\Delta(\alpha)]}{M(\alpha)} x - \Delta(\alpha) \right)^2 \right\}.
\]

The SADs in Fig. 3 are given by

\[
[P(x)] \equiv \int_0^1 d\alpha n(\alpha) P(x|\alpha).
\]

To calculate \([P(x)]\) explicitly, first Eqs. (26) are solved to obtain \( \Delta_0, \Delta_1 \) and \( u \). We then re-express Eq. (28) as a function of \( \Delta(\alpha) \) and use the same change of variables as in Eq. (25) to calculate the integral Eq. (30), obtaining

\[
[P(x)] = (1 - [\phi]) \delta(x) + H(x) \int_{\Delta_0}^{\Delta_1} d\Delta \theta(\Delta, u) \varphi(x|\Delta),
\]

details are found in Section S5 of the SM. Similarly to \([M]\) and \([\phi]\), the distribution \([P(x)]\) is independent of \( n(\alpha) \) (see Section S8 of the SM).
In the case where \((\nu = 0, \rho = 1)\) (i.e., no hierarchy), the underlying unclipped distribution \(\mathbb{E}\varphi(x) = \int_0^1 dx n(\alpha)\varphi(x|\alpha)\) is itself a Gaussian distribution. However, as demonstrated in Fig. 3(a), this simple form is lost in a hierarchical community, \(\mathbb{E}\varphi(x)\) is no longer Gaussian, and is not even symmetric around its maximum. Broadly speaking, raising \(\nu\) or \(\rho\) lowers the modal abundance and increases spread in abundances respectively.

RADs are also calculated from the distribution \([P(x)]\). We observe that if species are ranked on a scale of 0 to 1 by descending abundance, then the rank of a species with abundance \(x\) is \(\int_0^x [P(x')] dx'\). The plot in Fig. 3(b) shows abundance on the vertical axis, and rank on the horizontal axis. RADs are often the preferred representation of species abundances as they do not suffer from loss of information due to species binning, as SADs do [48].

To compute a hierarchical abundance distribution we rank species on a scale \(r \in [0, 1]\), a species with index \(\alpha\) has rank

\[
r(\alpha) = \int_0^\alpha d\alpha n(\alpha),
\]

ensuring that \(rN\) species are lower in the hierarchy and \((1 - r)N\) species are higher in the hierarchy. One then produces a HAD, such as in panel (c) of Fig. 3, parametrically, with the horizontal axis equal to the rank and the vertical axis equal to the abundance \(M(\alpha)\), which is derived from Eqs. (13) in Section S5 of the SM and given by

\[
M(\alpha) = A[M]w_1[\Delta(\alpha)]
\exp\left(\frac{\ell\sigma^2 \ln \rho}{u^2} \int_{\Delta_0}^{\Delta(\alpha)} d\Delta' \theta(\Delta', u)w_2(\Delta')\right),
\]

where the constant \(A\) ensures that \(\int_0^1 d\alpha n(\alpha)M(\alpha) = [M]\), for \([M]\) given by Eq. (27a). Surprisingly, just like \([M]\), \([\phi]\) and \([P(x)]\), HADs are also independent of \(n(\alpha)\), this is shown in Section S8 of the SM.

**B. Fraction of Surviving Species**

A given species’ survival probability as a function of the ranking \(r(\alpha)\) can be produced in a similar way to the hierarchical abundance distributions discussed in Section V A. We simply replace \(M(\alpha)\) by \(\phi(\alpha) = w_0[\Delta(\alpha)]\). By the same reasoning, we deduce that a plot of survival probability against rank \(r(\alpha)\) is independent of both \(\mu\) and \(n(\alpha)\) (see Section S8 of the SM). The distribution of survival probabilities is flat if \(\nu = 0, \rho = 1\) and is an increasing function of \(r\) in the presence of hierarchy (see Fig. 4). This is an indication that the composition of a hierarchical community before the dynamics are run is vastly different to the resulting stable community, species lower in the hierarchy are both less abundant and less likely to survive.

On further noting that the area under an HAD is \([M]\) and the area under the survival curve is \([\phi]\), we see that introducing hierarchy produces smaller communities dominated by high ranked species. Further, from Figs. 3 and 4, the mean abundance and the fraction of surviving species in communities with hierarchy are mostly lower than those in models without hierarchy (flat curve in both), demonstrating that few species benefit at all from hierarchical interactions.

**VI. STABILITY**

**A. Stability Conditions**

In Section IV, we found the statistics of the surviving species abundances by presuming a static solution to Eq. (6). In this section, we discuss when this fixed-point solution is valid and thus under what conditions we have a stable and feasible equilibrium. As in Refs. [21, 23, 30, 32], we find that instability can occur either through linear instability against small perturbations to the abundances, or through a divergence in species abundances.

**1. Linear Instability**

Following along the lines of [21, 37] (see Section S6 of the SM), we use a linear stability analysis to find that the system is unstable to perturbations in species abundances
FIG. 5. Phase diagrams for the cascade model (horizontal axis shows the proportion of predator-prey pairs). The onset of instability [linear instability in panel (a), and diverging abundances in panel (b)] are shown as black lines for $(ν, ρ) = (0, 1)$, as well as for $(ν, ρ) = (3, 1)$ and $(ν, ρ) = (0, 3)$ as indicated. Stability is coloured as labelled for the case $(ν, ρ) = (0, 1)$. Length of arrows indicate the effect of increasing the values of $ν$ or $ρ$ by one $[ν → ν + 1, or ρ → ρ + 1 respectively].$

when

$$σ^2 ≥ \frac{ℓ}{(ℓ + γ)^2} \frac{1}{[φ]_0},$$

(34)

where the average survival probability $[φ]$ is to be determined from Eqs. (26) and Eq. (27b).

Interestingly, the same criterion as in Eq. (34) can be obtained with the machinery of random matrix theory, noting that $N^* = [φ] N$ species survive the dynamics asymptotically and reach a fixed point. The bulk of the eigenvalue spectrum of an $N^* × N^*$ random matrix with cascade statistics as in Eq. (4) (c.f. Fig. 1) crosses the imaginary axis precisely when Eq. (34) is satisfied [26]. Similar observations were made in the case without hierarchy $(ν = 0$ and $ρ = 1$) in Ref. [22]. A crucial difference between the dynamical and random matrix approaches is that the fraction of survivors $[φ] = N^* / N$ is determined from Eq. (27b) in the dynamical theory, but $[φ]$ is an independent parameter of the model in the random matrix approach. In particular, Eq. (34) would lead one to conclude that $ν$ has no effect on linear stability if a random matrix approach were used. However, as $[φ]$ is itself a function of the parameters $ν, σ, ρ, γ$, no such conclusion is drawn here.

2. Diverging Abundances

To find the point at which species abundances diverge, we solve Eqs. (26), together with Eq. (27a), for the point at which $1/∥M∥ = 0$. We find that abundances diverge if

$$μ ≥ \frac{Δνρ^2 + Δν}{Δνρ^2 - Δ0},$$

(35)

On eliminating $Δ0$ and $Δ1$ using Eqs. (26), Eq. (35) can be solved to yield the critical value of one of the parameters $(μ, σ, ν, ρ, γ)$ given the others. The resulting predictions for the point at which the mean abundance diverges is exact when the system is stable with respect to small perturbations, but is only approximate when the system becomes linearly unstable before the point of divergence is reached (see Section S7 of the SM for numerical justification). We include both cases in Fig. 6. The stability conditions Eqs. (34) and (35) provide a comprehensive analytical picture of stability in the cascade system.

B. Phase Diagrams

The phase diagrams in Figs. 5 and 6 illustrate the effects of hierarchical interaction $(ν, ρ)$ on stability.

When $ν$ is sufficiently large, increasing $ν$ typically decreases average abundances and the fraction of surviving species (see Fig. 2), thereby pushing the system away from diverging abundances and from linear instability. Larger deviations of $ρ$ from unity push the system both towards linear instability and generally towards infinite abundances. These effects are demonstrated in panels (b) and (c) of Fig. 6, which indicate that a large enough value of $ρ$ will always result in an unstable system and a large enough value of $ν$ will always result in a stable one. We also demonstrate the (separate) effects of changing $ν$ and $ρ$ in Fig. 5. If our parameters are restricted such that either $ν = 0$ or $ρ = 1$ [shown in panel (a) of Fig. 5], then we can analytically demonstrate the effects of varying only one parameter (either $ν$ or $ρ$) on linear instability. Details can be found in Section S6.3 of the SM.

Panels (b) and (c) in Fig. 6 also reveal how the precise combination of $ν$ and $ρ$ can affect stability and community composition. Specifically, communities with $ν > 0$ and $ρ > 1$ tend to have lower abundances than those
with $\nu > 0$ and $\rho < 1$ and are further from the diverging abundances transition.

From Fig. 5, we find that the influence of $\nu$ on linear stability is relatively small compared with its influence on the point at which abundances diverge. This is reminiscent of the fact that the overall average interaction strength $\mu$ has no effect on linear stability but affects the average abundance. Fig. 5 also reveals that $\rho$ has a more dramatic effect on stability in communities with a large proportion of predator-prey pairs. In particular, any value of $\rho \neq 1$ removes the special behaviour of the model without hierarchy and all interactions of the predator-prey type ($\rho = 1, \nu = 0, \gamma = -1$). In this special case, there is no linear instability [21, 23, 43], as indicated in panel (a) of Fig. 5. Any value $\rho \neq 1$ will however lead to the possibility of instability in the system with only predator prey interactions. The stabilising influence of $\nu$, on the contrary, is only mildly affected by the proportion of predator-prey pairs.

The influence of the remaining parameters $\mu, \sigma$ and $\gamma$ on stability is relatively straightforward, and similar to the model without block structure ($\nu = 0$ and $\rho = 1$) previously studied in [21, 23, 32, 43], and in studies of complex ecosystems based on the spectra of random matrices [2, 29]. The parameter $\mu$ has no effect on linear stability, as Eq. (34) has no $\mu$ dependence. Instead, from Eq. (27a) we see that $\mu$ increases average abundances (noting that $\Delta_0$ and $\Delta_1$ are independent of $\mu$), thereby moving the system towards instability via diverging abundances. This can be seen in panel (a) of Fig. 6 and panel (b) of Fig. 5: a large enough value of $\mu$ will always lead to diverging abundances in the community. Increasing $\sigma$, on the other hand, pushes the system towards both linear instability and diverging abundances, as demonstrated in Fig. 6(a) and in Fig. 5(a). Increasing the correlation parameter $\gamma$ (i.e., decreasing the proportion of predator-prey interaction pairs), also pushes the system towards linear instability and diverging abundances, as demonstrated in Fig. 5.

VII. DISCUSSION

Our analysis of the generalised Lotka–Volterra model with cascade interactions has focused on the effect of hierarchical interactions on both stability and structure in complex ecological communities. We have extended previous work on the stabilising impact of predator-prey like relationships [13, 21, 23, 29] by considering both the average severity of the hierarchy ($\nu$) and the proportion of interactions of predator-prey type ($\gamma$) in a single dynamical model. We find that increases to both factors are stabilising. We also find that increased heterogeneity in interaction variances ($\rho$) is a destabilising force.

The dynamic mean-field theory approach, unlike an approach based on the spectra of random matrices, guarantees a feasible equilibrium, and gives access to properties of the ecosystem other than stability. We find that communities with a strong hierarchy are dominated by species at the top, which are both more abundant and more likely to survive asymptotically. Further, hierarchy leads to more complex, non-Gaussian abundance distributions.

In order to find fixed point equations for the cascade model with an infinite number of trophic levels, we first considered a related and more general community, divided into a finite number of sub-communities. Fixed-point equations were then obtained, resulting in an effective abundance for a representative species $x^a(t)$ in each community. This is in contrast to most similar studies employing dynamic mean-field theory [21, 37, 43], which do not have a structured population, and accord-

![Diagram](image-url)
ingly only a single effective species. Our more general approach could allow for progress to be made investigating more heterogeneous interaction structures in and beyond ecology, such as other block-structured interaction matrices [49], meta-population models on complex networks [15, 50, 51], or trophic levels [14, 52, 53].

CODE AVAILABILITY

Codes can be found at https://github.com/LylePoley/Cascade-Model.git. A method for simulating the community dynamics with Eq. (1), numerical solution procedures for solving the cascade model fixed point equations Eqs. (26) and the data/code for producing all figures.

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/ — Supplemental Material —

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References
S1. OVERVIEW

This supplement contains further technical detail on the results in the main paper. In particular we report the generating functional analysis that was used to derive the effective dynamics and the subsequent stability analysis that was used deduce the phase diagrams in the main text.

The document is structured as follows:

First, in Section S2 we derive the formula for the relation between the proportion of predator prey interaction pairs \( p \) in the community to the correlation parameter \( \gamma, \gamma = \cos(\pi p) \).

Then, in Section S3 we outline the derivation of the effective single-species dynamics in Eq. (6) of the main text. We then derive Eqs. (13) in the main text, a set of self-consistent equations which provide the statistics of the species abundances, assuming a unique stable fixed-point.

In Section S4 we solve these fixed-point equations for the specific case of the Cascade Model, arriving at Eqs. (26) of the main text.

In Section S5 we then use these results to derive expressions for hierarchy [Eq. (34)], rank, and species abundance distributions [Eq. (32)].

In Section S6, we then examine the stability of our system, detailing conditions for both linear instability and the point at which abundances diverge. We then compare our results against simulation, finding excellent agreement in Section S7. We also demonstrate that the parameters \( \nu \) and \( \rho \) are stabilising and destabilising respectively by examining some informative special cases.

Finally, in Section S8 we demonstrate that all fixed point properties of our system that are of interest to us are independent of \( n(\alpha) \).

S2. RELATION BETWEEN FRACTION OF PREDATOR-PREY PAIRS AND \( \gamma \), ASSUMING GAUSSIAN DISTRIBUTED INTERACTIONS

A pair of interaction coefficients \( A_{ij}^{ab}, A_{ji}^{ba} \) are a predator-prey pair if their product is negative. We will use \( X_1 = A_{ij}^{ab} \) and \( X_2 = A_{ji}^{ba} \) for the sake of this section and call \( \mu_1 = \mu^{ab}, \mu_2 = \mu^{ba}, \sigma_1 = \sigma^{ab}, \sigma_2 = \sigma^{ba}, \gamma^{ab} = \gamma \) to avoid cluttering the argument with superfluous indices.

The proportion of interaction pairs \( X_1, X_2 \) which are of predator prey type is then the sum of two probabilities

\[
P(X_1X_2 < 0) = P(X_1 < 0, X_2 > 0) + P(X_1 > 0, X_2 < 0). \tag{S1}
\]

This can be evaluated using the Cholesky decomposition of \( X_1, X_2 \)

\[
X_1 = \frac{\mu_1}{\sqrt{N}} + \frac{\sigma_1}{\sqrt{N}} z_1, \\
X_2 = \frac{\mu_2}{\sqrt{N}} + \frac{\sigma_2}{\sqrt{N}} \left( \gamma z_1 + \sqrt{1 - \gamma^2} z_2 \right), \tag{S2}
\]

where \( z_1, z_2 \) are uncorrelated, mean zero, unit variance random variables. We then have

\[
P(X_1 > 0, X_2 < 0) = P \left( z_1 < -\frac{\mu_1}{\sigma_1 \sqrt{N}}, z_1 \gamma + z_2 \sqrt{1 - \gamma^2} > -\frac{\mu_2}{\sigma_2 \sqrt{N}} \right) \\
= P \left( z_1 < 0, z_1 \gamma + z_2 \sqrt{1 - \gamma^2} > 0 \right) + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right), \tag{S3}
\]
where we have assumed that $\mu_{1,2}$ and $\sigma_{1,2}$ are $O(1)$. Therefore, to leading order in $N$, the probability we want is a region bounded by the lines $z_1 = 0$ and $\gamma z_1 = -\sqrt{1-\gamma^2} z_2$ in the $(z_1, z_2)$-plane. The joint distribution of $z_1, z_2$ is circularly symmetric as $z_1$ and $z_2$ are uncorrelated, hence the probability we want is proportional to the angle between these two lines, or

$$P \left( z_1 < 0, \ z_1 \gamma + z_2 \sqrt{1-\gamma^2} > 0 \right) = \frac{\arccos(\gamma)}{2\pi}. \quad (S4)$$

A similar line of reasoning shows that $P (X_1 > 0, X_2 < 0)$ has the same value, and so the proportion of predator-prey pairs $p$ is

$$p = \frac{\arccos(\gamma)}{\pi} + O \left( \frac{1}{\sqrt{N}} \right). \quad (S5)$$

If $\gamma$ is not the same for all sub-communities, we find the proportion of predator-prey pairs by summing over all blocks

$$p = \frac{1}{\pi} \sum_{ab} n^a n^b \arccos(\gamma^{ab}). \quad (S6)$$

### S3. DERIVATION OF THE EFFECTIVE DYNAMICS

To find the effective process in Eq. (6) in the main text, we follow [S36, S54] and consider the MSRJD [S28, S41, S55, S56] functional integral of the GLV dynamics in Eq. (1)

$$Z[h, \psi, A] = \int D[x, \hat{x}] \exp \left( i \sum_{(a,i)} \int ds \left[ \hat{x}^a_i(s) \left( \frac{\dot{x}^a_i(s)}{x^a_i(s)} - \left[ 1 + \sum_{(b,j)} A_{ij}^a x^b_j(s) + h^a_i(s) \right] \right) \right] \right) \times \exp \left( i \sum_{(a,i)} \int ds x^a_i(s) \psi^a_i(s) \right), \quad (S7)$$

where

$$D[x, \hat{x}] := \prod_{(a,i)} \prod_{s=0}^t \frac{dx^a_i(s) d\hat{x}^a_i(s)}{\sqrt{2\pi}}, \quad (S8)$$

and where $(a, i)$ indicates a sum or product over both $a$ and $i$, so that, for example

$$\sum_{(a,i)} x^a_i(t) \equiv \sum_{a=1}^B \sum_{i=1}^{N^a} x^a_i(t). \quad (S9)$$

The functions $h^a_i(s)$ are perturbation fields, and the $\psi^a_i(s)$ are source terms. Taking functional derivatives of $Z[h, \psi, A]$ with respect to these functions generates moments of species abundances $x^a_i(s)$ as well as of the conjugate variables $\hat{x}^a_i(s)$. For example we have

$$\frac{\delta Z}{\delta h^a_i(s)} = -\langle i \hat{x}^a_i(s) \rangle_Z, \quad \frac{\delta Z}{\delta \psi^a_i(s)} = \langle x^a_i(s) \rangle_Z, \quad \frac{\delta^2 Z}{\delta \psi^a_i(s) \delta h^a_i(s')} = -i \langle x^a_i(s) \hat{x}^a_i(s') \rangle_Z. \quad (S10)$$
The notation \( \langle \ldots \rangle_Z \) denotes an average over the integral in Eq. (S7)

\[
\langle \ldots \rangle_Z \equiv \int D[x, \hat{x}] \langle \ldots \rangle \exp \left( i \sum_{(a,i)} \int ds \left[ \hat{x}_a^i(s) \left( \frac{\dot{x}_a^i(s)}{x_a^i(s)} - \left[ 1 + \sum_{(b,j)} A^{ab}_{ij} x_j^b(s) + h_i^a(s) \right] \right) \right] \right) 
\times \exp \left( i \sum_{(a,i)} \int ds x_a^i(s) \psi_i^a(s) \right),
\]

(S11)

If \( \psi_i^a(s) = 0 \) and \( h_i^a(s) = 0 \) in Eq. (S7), then the average \( \langle \ldots \rangle_Z \) constrains the system to follow the dynamics in Eq. (1) in the main text. Further, if \( \psi_i^a(s) = 0 \) in Eq. (S7), then the average \( \langle \ldots \rangle_Z \) is normalised, so that \( Z[h, \psi = 0, A] = 1 \) [S28].

We now average the generating functional \( Z \) over realisations of the interaction matrix \( A^{ab}_{ij} \), and note that such a procedure can also be performed using the cavity method [S23, S30, S57].

### A. Disorder average

Denoting averages over realisations of \( A \) by an overbar, we find

\[
\exp \left( -i \sum_{(a,i),(b,j)} \int ds A^{ab}_{ij} \hat{x}_a^i(s)x_j^b(s) \right) = 1 - i \sum_{(a,i),(b,j)} \int ds A^{ab}_{ij} \hat{x}_a^i(s)x_j^b(s) 
- \sum_{(a,i),(b,j)} \sum_{(c,k),(d,l)} \int dsds' A^{ac}_{ij}A^{bd}_{kl} \hat{x}_a^c(s)x_j^d(s)\hat{x}_k^c(s')x_k^d(s') + \cdots.
\]

(S12)

To leading order in \( N \), the averaged functional \( \overline{Z[h, \psi, A]} \) only depends on the first two moments of \( A \), provided higher moments drop sufficiently fast with growing \( N \). Upon averaging the above and re-exponentiating we find

\[
\exp \left( -i \sum_{(a,i),(b,j)} \int ds A^{ab}_{ij} \hat{x}_a^i(s)x_j^b(s) \right) = \exp \left( -N \sum_{a,b} n^a n^b \int ds \mu^{ab} P^a(s)M^b(s) \right) 
\exp \left( -N \sum_{a,b} n^a n^b \int dsds' \left[ \frac{1}{2} \langle \sigma^{ab} \rangle^2 L^a(s, s')C^b(s, s') + \gamma^{ab} \sigma^{ab} \sigma^{ba} K^a(s, s')K^b(s', s') \right] + \mathcal{O}(N^0) \right),
\]

(S13)
where \( n^a \equiv N^a/N \), and where we have introduced the following macroscopic parameters as the level of a single sub-community

\[
M^a(t) := \frac{1}{N^a} \sum_{i=1}^{N^a} x^a_i(t),
\]

\[
P^a(t) := \frac{i}{N^a} \sum_{i=1}^{N^a} \hat{x}^a_i(t),
\]

\[
C^a(t, t') := \frac{1}{N^a} \sum_{i=1}^{N^a} x^a_i(t)x^a_i(t'),
\]

\[
L^a(t, t') := \frac{1}{N^a} \sum_{i=1}^{N^a} \hat{x}^a_i(t)\hat{x}^a_i(t'),
\]

\[
K^a(t, t') := \frac{1}{N^a} \sum_{i=1}^{N^a} \hat{x}^a_i(t)x^a_i(t').
\]  

We impose these definitions in the generating functional \( Z \) using Dirac delta functions in their complex exponential representation, e.g.

\[
1 = \int \prod_{a=1}^{B} \prod_s dM^a(s) \ \delta \left( M^a(s) - \frac{1}{N^a} \sum_i x^a_i(s) \right)
\equiv \int D\{M, \hat{M}\} \exp \left( i \sum_a N^a \int ds \ M^a(s) \left[ M^a(s) - \frac{1}{N^a} \sum_i x^a_i(s) \right] \right). \tag{S15}
\]

We absorb prefactors of \( 2\pi \) and \( N \) that result from this procedure into the measure \( D\{M, \hat{M}\} \). In the saddle-point approximation, this prefactor has no bearing on our results.

After averaging over \( A \) and insertion of the order parameters into \( Z[h, \psi, A] \) we have

\[
Z[h, \psi, A] = \int D\{M, \hat{M}\} \cdots D\{K, \hat{K}\} \exp \left( N \left[ \Psi[M, \hat{M}, \cdots, K, \hat{K}] + \Phi[M, \cdots, K] + \Omega[\hat{M}, \cdots, \hat{K}] \right] \right), \tag{S16}
\]
where

\[ \Psi = i \sum_a n^a \int ds \left( M^a(s) \hat{M}^a(s) + P^a(s) \hat{P}^a(s) \right) \]

\[ + i \sum_a n^a \int ds ds' \left( L^a(s, s') \hat{L}^a(s, s') + C^a(s, s') \hat{C}^a(s, s') + K^a(s, s') \hat{K}^a(s, s') \right), \]

\[ \Phi = - \sum_{a,b} n^a n^b \int ds \left( \mu^{ab} P^a(s) M^b(s) \right) \]

\[ - \sum_{a,b} n^a n^b \int ds ds' \left( \frac{1}{2} (\sigma^{ab})^2 L^a(s, s') C^b(s, s') + \gamma^{ab} \sigma^{ab} \sigma^{ba} K^a(s, s') K^b(s, s') \right), \]

\[ \Omega = \ln \int D[x, \hat{x}] \exp \left( i \sum_a \int ds \left[ \hat{\omega}^a(s) \left( \frac{\hat{x}^a(s)}{x^a(s)} - (1 - x^a(s) + h^a(s)) \right) + \psi^a(s) x^a(s) \right] \right) \]

\[ \times \exp \left( -i \sum_a \int ds \left[ \hat{\omega}^a(s) x^a(s) + \hat{P}^a(s) i \hat{x}^a(s) \right] \right) \]

\[ \times \exp \left( -i \sum_a \int ds ds' \left[ \hat{L}^a(s, s') \hat{x}^a(s') + \hat{C}^a(s, s') x^a(s') - K^a(s, s') \hat{x}^a(s') \right] \right). \]

The common factor of \( N \) in the exponent in Eq. (S16) makes the integral amenable to a saddle-point approximation for large \( N \). At the saddle point, we find that the macroscopic hatted and un-hatted parameters satisfy

\[ i \hat{M}^a(t) = \sum_{b=1}^{B} n^b \delta^a_b(t) \mu^b, \]

\[ i \hat{P}^a(t) = \sum_{b=1}^{B} \mu^{ab} n^b M^b(t), \]

\[ i \hat{C}^a(t, t') = \frac{1}{2} \sum_{b=1}^{B} n^b L^b(t, t') (\sigma^{ba})^2, \]

\[ C^a(t, t') = \frac{1}{N^a} \sum_{i=1}^{N^a} \langle x_i^a(t) x_i^a(t') \rangle_Z, \]

\[ i \hat{L}^a(t, t') = \frac{1}{2} \sum_{b=1}^{B} (\sigma^{ab})^2 n^b C^b(t, t'), \]

\[ L^a(t, t') = \frac{1}{N^a} \sum_{i=1}^{N^a} \langle \hat{x}_i^a(t) \hat{x}_i^a(t') \rangle_Z, \]

\[ i \hat{K}^a(t, t') = \frac{1}{2} \sum_{b=1}^{B} \gamma^{ab} \sigma^{ab} \sigma^{ba} K^b(t', t). \]

In particular, we see that \( P^a(s) \) and \( L^a(s, s') \) vanish at the saddle point as

\[ \langle \hat{x}_i^a(t) \rangle_Z = \frac{\delta Z [h, \psi = 0, A]}{\delta h_i^a(s)} = \frac{\delta (1)}{\delta h_i^a(s)} = 0, \]

and similarly for \( \langle \hat{x}_i^a(t) \hat{x}_i^a(t') \rangle_Z \) (see e.g. [S21, S36] for similar calculations).
For the rest of the calculation we set $\psi(s) = h(s) = 0$, they are fields introduced with the sole purpose of producing correlation functions such as that in Eq. (S21) and are no longer needed. Substituting the saddle-point conditions in Eq. (S20) into the disorder-averaged generating functional in Eq. (S16), we obtain a generating functional that factorises as follows

$$Z[h = 0, \psi = 0, A] = \prod_{a=1}^{B} (Z_{\text{eff}}^a)^{N_a},$$

where $Z_{\text{eff}}^a$ is an effective generating functional for the $a^{th}$ block

$$Z_{\text{eff}}^a = \int \prod_{s} dx^{a}(s)d\hat{x}^{a}(s) \sqrt{2\pi} \times \exp \left( i \int ds \hat{x}^{a}(s) \left[ \frac{\dot{x}^{a}(s)}{x^{a}(s)} - (1 - x^{a}(s)) \right] \right),$$

$$\times \exp \left( -i \int ds \hat{x}^{a}(s) \left[ \sum_{b} n^{b} \left[ \mu^{ab} M^{b}(s) + \gamma^{ab} \sigma^{ab} \sigma^{ba} \int ds' x^{a}(s') G^{b}(s, s') \right] \right) \right),$$

$$\times \exp \left( -i \int ds ds' \left[ \frac{1}{2} \sum_{b} n^{b} (\sigma^{ab})^{2} C^{b}(s, s') \hat{x}^{a}(s) \hat{x}^{a}(s') \right] \right),$$

where we define $G^{a}(t, t') = -iK^{a}(t', t)$.

The object $Z_{\text{eff}}^a$ is recognised as the generating functional of the process in Eq. (6) of the main text

$$\dot{x}^{a}(t) = x^{a}(t) \left[ 1 - x^{a}(t) + \sum_{b=1}^{B} n^{b} \left[ \mu^{ab} M^{b}(t) + \gamma^{ab} \sigma^{ab} \sigma^{ba} \int_{0}^{t} dt' x^{a}(t') G^{b}(t, t') \right] + \eta^{a}(t) \right],$$

with $M^{a}(t), C^{a}(t, t'), G^{a}(t, t')$ the average abundance, correlation, and response functions for abundances in block $a$,

$$M^{a}(t) := \langle x^{a}(t) \rangle, \quad C^{a}(t, t') := \langle x^{a}(t)x^{a}(t') \rangle, \quad G^{a}(t, t') := \frac{\delta \langle x^{a}(t) \rangle}{\delta \eta^{a}(t')}$$

and $\eta^{a}(t)$ colored Gaussian noise

$$\langle \eta^{a}(t) \rangle = 0, \quad \langle \eta^{a}(t)\eta^{a}(t') \rangle = \sum_{b=1}^{B} n^{b} (\sigma^{ab})^{2} C^{a}(t, t'),$$

where $\langle \ldots \rangle$ is an average over the effective process in Eq. (S24).

C. Fixed Point Equations

We now suppose that the system Eq. (S24) reaches a fixed point such that $\dot{x}^{a}(t) \rightarrow 0, x^{a}(t) \rightarrow x^{a}_s$ as $t \rightarrow \infty$. Under this assumption, we replace $M^{a}(t) \rightarrow M^{a}, C^{a}(t, t') \rightarrow q^{a}, \eta^{a}(t) \rightarrow \sqrt{\sum b (\sigma^{ab})^{2} n^{b} q^{b}} \cdot z^{a},$
where the $z^a$ are independent mean-zero static Gaussian random variables with unit variance. Further assuming that the response function $G^a(t,t')$ is a function only of time differences $\tau = t - t'$, we deduce that

$$0 = x^a_*(1 - x^a_* + \sum_b n^b \left[ \mu^{ab} M^b + \gamma^{ab} \sigma^{ab} \sigma^{ba} x^a_* \chi^a \right] + \sqrt{\sum_b (\sigma^{ab})^2 n^b q^b z^a}) ,$$

(S27)

where we have defined the integrated response function $\chi^a = \int_0^\infty d\tau G^a(\tau)$.

Similarly to cases with no block structure [S21, S23], we assume that $1 - \sum_b \gamma^{ab} \sigma^{ab} \sigma^{ba} n^b \chi^b > 0$ and find

$$x^a_* = \max \left( 0, \frac{1 + \sum_b \mu^{ab} n^b M^b + \sqrt{\sum_b (\sigma^{ab})^2 n^b q^b z^a}}{1 - \sum_b \gamma^{ab} \sigma^{ab} \sigma^{ba} n^b \chi^b} \right) ,$$

(S28)

so that $x^a_* \geq 0$, and has a non-zero value when the non-zero fixed point is positive.

Following [S21, S37], we can self-consistently determine $\chi^a, M^a, q^a$ from Eq. (S28) together with Eqs. (S24) to (S26). One obtains

$$\chi^a = \frac{1}{1 - \sum_b \gamma^{ab} \sigma^{ab} \sigma^{ba} n^b \chi^b w_0(\Delta^a)} ,$$

$$M^a = \frac{\sqrt{\sum_b (\sigma^{ab})^2 n^b q^b}}{1 - \sum_b \gamma^{ab} \sigma^{ab} \sigma^{ba} n^b \chi^b w_1(\Delta^a)} ,$$

$$q^a = \frac{\sum_b (\sigma^{ab})^2 n^b q^b}{(1 - \sum_b \gamma^{ab} \sigma^{ab} \sigma^{ba} n^b \chi^b)^2 w_2(\Delta^a)} ,$$

(S29)

where

$$\Delta^a = \frac{1 + \sum_b \mu^{ab} n^b M^b}{\sqrt{\sum_b (\sigma^{ab})^2 n^b q^b}} ,$$

$$w_k(\Delta^a) = \int_{-\infty}^{\Delta^a} \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} (\Delta^a - z)^k .$$

(S30)

(S31)

Hence we arrive at Eqs. (13) in the main text.

**S4. DERIVATION OF FIXED POINT EQUATIONS (26)**

There are a number of steps to the following derivation, the results of which allow us to calculate the average abundance $\langle M \rangle$ and survival rate $\langle \phi \rangle$ in the $B \to \infty$ limit of the cascade model.

First, we take the $B \to \infty$ limit of Eqs. (S29) to get Eqs. (S35). Then we impose cascade model interactions, deriving Eqs. (S42) through (S46).

Once we have found the set of equations to be solved we set about solving them. We first set up Eqs. (21) in Section S4B as self consistent integrals over the species index $\alpha$. We then change variables from $\alpha$ to $\Delta$ in Section S4D to make these integrals explicit. The result of this change in variables is Eqs. (26), equations for determining $u, \Delta_0, \Delta_1$ as functions of $\nu, \sigma, \rho, \gamma$.

Finally, we derive expressions for $\langle M \rangle$ and $\langle \phi \rangle$ as explicit functions of $\mu, \nu, \sigma, \rho, \gamma$ and $u, \Delta_0, \Delta_1$. 

A. Taking the $B \to \infty$ Limit of Eqs. (13) in the main text

We will take for example the term $\sum_{b=1}^{B} \mu_{b}^{a} n^{b} M^{b}$, which appears in the second of Eqs. (14). The same steps apply analogously to all similar terms in Eqs. (13) and (14). Writing $\alpha = a/B$, $\beta = b/B$ as well as $n(\beta) := B n^{b}$, we have

$$\lim_{B \to \infty} \sum_{b=1}^{B} \mu_{b}^{a} n^{b} M^{b} = \int_{0}^{1} d\beta \mu(\alpha, \beta) n(\beta) M(\beta),$$  \hspace{1em} (S32)$$

where we have introduced

$$\mu(\alpha, \beta) \equiv \mu_{ab}, \quad M(\beta) \equiv M^{b}. \quad (S33)$$

The condition $\sum_{b} n^{b} = 1$ becomes an integral as well

$$\lim_{B \to \infty} \sum_{b} n^{b} = \int_{0}^{1} d\beta n(\beta) = 1. \quad (S34)$$

In the $B \to \infty$ limit our general fixed point equations [see Eqs. (13)] are therefore

$$u(\alpha) = 1 - \int_{0}^{1} d\beta \gamma(\alpha, \beta) \sigma(\alpha, \beta) \sigma(\beta, \alpha) n(\beta) \chi(\beta),$$

$$\chi(\alpha) u(\alpha) = w_{0}(\Delta(\alpha)),$$

$$M(\alpha) u(\alpha) = w_{1}(\Delta(\alpha)) \sqrt{\int_{0}^{1} d\beta \sigma(\alpha, \beta)^{2} n(\beta) q(\beta)},$$

$$q(\alpha) u(\alpha)^{2} = w_{2}(\Delta(\alpha)) \int_{0}^{1} d\beta \sigma(\alpha, \beta)^{2} n(\beta) q(\beta),$$

$$\Delta(\alpha) = \frac{1 + \int_{0}^{1} d\beta \mu(\alpha, \beta) n(\beta) M(\beta)}{\sqrt{\int_{0}^{1} d\beta \sigma(\alpha, \beta)^{2} n(\beta) q(\beta)}} \quad (S35)$$

with

$$\mu \left( \frac{a}{B}, \frac{b}{B} \right) = \mu_{ab}, \quad \sigma \left( \frac{a}{B}, \frac{b}{B} \right) = \sigma_{ab}, \quad \gamma \left( \frac{a}{B}, \frac{b}{B} \right) = \gamma_{ab}. \quad (S36)$$

In the cascade model we choose the matrices $\mu_{ab}, \sigma_{ab}, \gamma_{ab}$ as in Eq. (4) in the main paper. For the matrix $\mu_{ab}$, this leads to

$$\mu(\alpha, \beta) = \begin{cases} 
\mu - \nu, & \alpha < \beta \\
\mu(\alpha), & \alpha = \beta \\
\mu + \nu, & \alpha > \beta 
\end{cases} \quad (S37)$$

The $\alpha = \beta$ contribution is negligible in the limit $B \to \infty$. We therefore conclude

$$\lim_{B \to \infty} \sum_{b} \mu_{b}^{a} n^{b} M^{b} = \mu [M] + \nu \left( \int_{0}^{\alpha} d\beta n(\beta) M(\beta) - \int_{\alpha}^{1} d\beta n(\beta) M(\beta) \right). \quad (S38)$$
The notation \([\ldots]\) indicates an average over the index \(\alpha\), that is
\[
[\ldots] = \int_0^1 d\alpha \ n(\alpha)(\ldots). \tag{S39}
\]

Similarly, one finds
\[
\lim_{B \to \infty} \sum_b (\sigma^{ab})^2 n^b \rho^b = \sigma^2 \left( \rho^2 \int_0^\alpha d\beta \ n(\beta)q(\beta) + \frac{1}{\rho^2} \int_\alpha^1 d\beta \ n(\beta)q(\beta) \right). \tag{S40}
\]

Inspired by the similarity in form of the above, we define two operators with action on a function \(f(\alpha)\),
\[
f_\nu(\alpha) \equiv \nu \int_0^\alpha d\beta \ n(\beta)f(\beta) - \nu \int_\alpha^1 d\beta \ n(\beta)f(\beta),
\]
\[
f_\rho(\alpha) \equiv \rho^2 \int_0^\alpha d\beta \ n(\beta)f(\beta) + \frac{1}{\rho^2} \int_\alpha^1 d\beta \ n(\beta)f(\beta). \tag{S41}
\]

Eqs. (13) may now be written as
\[
\begin{align*}
u &= 1 - \gamma \sigma^2 \chi, \tag{S42} \\
\chi(\alpha) &= w_0[\Delta(\alpha)], \tag{S43} \\
M(\alpha) &= w_1[\Delta(\alpha)]\sigma \sqrt{q_\rho(\alpha)}, \tag{S44} \\
q(\alpha) &= w_2[\Delta(\alpha)]\sigma^2 q_\rho(\alpha), \tag{S45} \\
\Delta(\alpha) &= \frac{1 + \mu [M] + M_\nu(\alpha)}{\sigma \sqrt{q_\rho(\alpha)}}. \tag{S46}
\end{align*}
\]

A specific use of the average notation \([\ldots]\) we will use frequently is the average of a composite function \(f[\Delta(\alpha)]\), where \(f\) is again any function. We write
\[
[f(\Delta)] \equiv \int_0^1 d\alpha \ n(\alpha)f(\Delta(\alpha)). \tag{S47}
\]

B. Derivation of Eqs. (21)

To derive the first of Eqs. (21), we first average Eq. (S43) to obtain
\[
[\chi] u = [w_0(\Delta)], \tag{S48}
\]
Now, substituting this result into Eq. (S42) gives
\[
u^2 = u - \gamma \sigma^2 [w_0(\Delta)]. \tag{S49}
\]
Eq. (S49) is the first equation in Eqs. (26).

To derive the second of Eqs. (21), we differentiate \(q_\rho(\alpha)\) with respect to \(\alpha\), finding
\[
\frac{d}{d\alpha} q_\rho(\alpha) = \left( \rho^2 - \rho^{-2} \right) n(\alpha)q(\alpha),
\]
\[
= \frac{\ell \sigma^2 \ln \rho^2}{u^2} n(\alpha)w_2(\Delta(\alpha))q_\rho(\alpha), \tag{S50}
\]

To derive the second of Eqs. (21), we differentiate \(q_\rho(\alpha)\) with respect to \(\alpha\), finding
\[
\frac{d}{d\alpha} q_\rho(\alpha) = \left( \rho^2 - \rho^{-2} \right) n(\alpha)q(\alpha),
\]
\[
= \frac{\ell \sigma^2 \ln \rho^2}{u^2} n(\alpha)w_2(\Delta(\alpha))q_\rho(\alpha), \tag{S50}
\]
where the second line follows from use of Eq. (S45), and where $\ell$ is the logarithmic mean of $\rho^2$ and $\rho^{-2}$, that is

$$
\ell \equiv \text{Lm} \left( \rho^2, \frac{1}{\rho^2} \right),
$$

(S51)

with

$$
\text{Lm} (x, y) \equiv \frac{x - y}{\ln x - \ln y}.
$$

(S52)

Eq. (S50) is a linear differential equation for $q_\rho(\alpha)$, with boundaries [from Eqs. (S41)]

$$
q_\rho(0) = \lbrack q \rbrack \rho^{-2},
$$

(S53)

$$
q_\rho(1) = \lbrack q \rbrack \rho^2,
$$

(S54)

we note that either boundary can be used as an initial or final condition to solve Eq. (S50) and the other will automatically be satisfied. The solution is

$$
q_\rho(\alpha) = A \exp \left( \frac{\ell \sigma^2 \ln \rho^2}{u^2} \int_0^\alpha d\beta \, n(\beta) w_2(\beta) \right),
$$

(S55)

for some constant $A$. Substituting in the boundary conditions gives

$$
A = \rho^{-2} \lbrack q \rbrack,
$$

(S56)

$$
\sigma^2 = \ell \sigma^2 \lbrack w_2(\Delta) \rbrack.
$$

(S57)

C. An implicit expression for $\Delta(\alpha)$

So far, we have the following three relations [see Eqs. (S34), (S49) and (S57)]

$$
u(1 - u) = \gamma \sigma^2 \lbrack w_0(\Delta) \rbrack, \quad u^2 = \ell \sigma^2 \lbrack w_2(\Delta) \rbrack, \quad 1 = \int_0^1 d\alpha \, n(\alpha).
$$

(S58)

One obtains Eqs. (26), the equations necessary for obtaining $u, \Delta_0, \Delta_1$ from our model parameters, from the above by changing variables from $\alpha$ to $\Delta$ in the three integrals, $\lbrack w_0(\Delta) \rbrack, \lbrack w_2(\Delta) \rbrack$ and $\int_0^1 d\alpha \, n(\alpha)$. In order to find this change of variables, we will first derive Eq. (23) in the main text, that is, we find a differential equation linking $\Delta(\alpha)$ to $\alpha$. To do this, we find an expression for $M_e(\alpha)$, which will give us an implicit expression for $\Delta(\alpha)$ by Eq. (S46), which in turn yields Eq. (23) when differentiated.

First, we substitute Eq. (S44) into Eq. (S46) to get

$$
M(\alpha) u = \frac{w_1(\Delta(\alpha))}{\Delta(\alpha)} \left( 1 + \mu \lbrack M \rbrack + M_e(\alpha) \right).
$$

(S59)
We can find $1 + \mu [M] + M_\nu(\alpha)$ in a similar manner to our derivation of $q_\nu(\alpha)$. We have
\[
\frac{d}{d\alpha} (1 + \mu [M] + M_\nu(\alpha)) = 2\nu n(\alpha) M(\alpha),
\]
\[
= \frac{2\nu}{u} n(\alpha) \frac{w_1(\Delta(\alpha))}{\Delta(\alpha)} (1 + \mu [M] + M_\nu(\alpha)).
\]
(S60)

This is a differential equation for $1 + \mu [M] + M_\nu(\alpha)$, with boundaries [see Eqs. (S41)]
\[
1 + \mu [M] + M_\nu(0) = 1 + (\mu - \nu) [M], \quad \text{(S61)}
\]
\[
1 + \mu [M] + M_\nu(1) = 1 + (\mu + \nu) [M]. \quad \text{(S62)}
\]

The solution is therefore
\[
1 + \mu [M] + M_\nu(\alpha) = A \exp \left( \frac{2\nu}{u} \int_0^\alpha d\beta n(\beta) \frac{w_1(\Delta(\beta))}{\Delta(\beta)} \right),
\]
(S63)

where the boundary conditions give
\[
A = 1 + (\mu - \nu) [M], \quad \text{(S64)}
\]
\[
u = \text{Lm} \left( \frac{1}{[M]} + \mu + \nu, \frac{1}{[M]} + \mu - \nu \right) \frac{[w_1(\Delta)]}{\Delta}, \quad \text{(S65)}
\]

where we write $[w_1(\Delta)/\Delta] \equiv \int_0^1 d\alpha n(\alpha) w_1(\Delta(\alpha))/\Delta(\alpha)$.

On substituting Eqs. (S55) and (S63) into Eq. (S46) we now have an expression for $\Delta(\alpha)$
\[
\Delta(\alpha) = \Delta_0 \exp \left( \frac{2\nu}{u} \int_0^\alpha d\beta n(\beta) \frac{w_1(\Delta(\beta))}{\Delta(\beta)} - \frac{\sigma^2 \ln \rho}{u^2} \int_0^\alpha d\beta n(\beta) w_2(\Delta(\beta)) \right), \quad \text{(S66)}
\]

**D. Derivation of Eq. (23) and Eqs. (26)**

We may now differentiate both sides of Eq. (S66) with respect to $\alpha$ to get Eq. (23) in the main text
\[
\frac{d\Delta}{d\alpha} = \frac{n(\alpha)}{\theta(\Delta, u)}, \quad \text{(S67)}
\]

where
\[
\theta(\Delta, u) \equiv \frac{u^2}{2 \nu \Delta} - \frac{1}{2 \nu \Delta^2 \ln \rho \Delta^2 w_2(\Delta)}. \quad \text{(S68)}
\]

We are now in a position to change variables from $\alpha$ to $\Delta$ in Eqs. (S58). Using Eq. (S67) we can write
\[
[w_0(\Delta)] = \int_0^1 d\alpha n(\alpha) w_0(\Delta(\alpha)) = \int_{\Delta_0}^{\Delta_1} d\Delta \theta(\Delta, u) w_0(\Delta), \quad \text{(S69)}
\]
and similarly for \(\omega_2(\Delta)\) and \(\omega_1(\Delta)/\Delta\). The result is Eqs. (26) in the main text

\[
\begin{align*}
u(1 - \nu) &= \gamma^2 \int_{\Delta} \Delta \theta(\Delta, \nu) w_0(\Delta), \\ u^2 &= \ell^2 \int_{\Delta} \Delta \theta(\Delta, \nu) w_2(\Delta), \\ 1 &= \int_{\Delta} \Delta \theta(\Delta, \nu).
\end{align*}
\] (S70)

Hence, given the parameters \(\nu, \sigma, \rho, \gamma\), Eqs. (S70) can be solved to find \(u, \Delta_0, \) and \(\Delta_1\). Therefore we can readily obtain \(u, \Delta_0, \) and \(\Delta_1\) as known functions of our system parameters, i.e.

\[
\begin{align*}
u &\equiv \nu(\nu, \sigma, \rho, \gamma), \\ \Delta_0 &\equiv \Delta_0(\nu, \sigma, \rho, \gamma), \\ \Delta_1 &\equiv \Delta_1(\nu, \sigma, \rho, \gamma).
\end{align*}
\] (S71, S72, S73)

E. Expressions for \([M], [\phi]\) [derivation of Eqs. (27) and (28)]

1. Expression for \([M]\)

To find an expression for \([M]\), we evaluate Eq. (S46) at \(\alpha = 0\) and \(\alpha = 1\), giving

\[
\begin{align*}
\Delta_0 &= 1 + (\mu - \nu) \frac{[M]}{(\sigma/\rho) \sqrt{[q]}}, \\
\Delta_1 &= 1 + (\mu + \nu) \frac{[M]}{\sigma \rho \sqrt{[q]}}. 
\end{align*}
\] (S74, S75)

Alternatively, one can evaluate Eq. (S66) at \(\alpha = 0\) and \(\alpha = 1\) and use Eqs. (S57) and (S65) to obtain the same expressions. Taking the ratio of the above expressions, we obtain

\[
\frac{\Delta_1 \rho^2}{\Delta_0} = 1 + (\mu + \nu) \frac{[M]}{1 + (\mu - \nu) [M]}.
\] (S76)

Re-arranging for \([M]\) then gives Eq. (27) in the main text:

\[
\frac{1}{[M]} + \mu = \frac{\Delta_1 \rho^2 + \Delta_0}{\Delta_1 \rho^2 - \Delta_0} \nu.
\] (S77)

2. Expression for \([\phi]\)

Eq. (28) follows from recognising that \([\phi] = [w_0(\Delta)]\) [since \(\phi(\alpha) = w_0(\Delta(\alpha))\)]. Therefore, by the first equation in Eqs. (S70) [or the first equation in Eqs. (S58), or Eq. (S49)] we find Eq. (28) of the main text

\[
[\phi] = \frac{u(1 - u)}{\gamma \sigma^2}.
\] (S78)
As mentioned in the main text, we have to be careful with Eqs. (27) and (28) [equivalently, Eqs. (S77) and (S78)] if either of \( \nu \) or \( \gamma \) is set to zero. Similar care must be taken when \( \sigma \) is set to zero. However, this would imply that there is no disorder in our system, and we are only interested in disordered interactions in this work, therefore we assume \( \sigma > 0 \).

To find the \( \nu \to 0 \) limit of Eq. (S77) we first note that if \( \nu = 0 \), then by Eq. (S76) \( \Delta_0 \rho^2 = \Delta_0 \), so that the right hand side of Eq. (S77) is in an indeterminate form. To find the appropriate limit we use Eq. (S65), which we repeat here

\[
\begin{align*}
\nu = Lm \left( \frac{1}{[M]} + \mu - \nu, \frac{1}{[M]} + \mu + \nu \right) \left[ \frac{w_1(\Delta)}{\Delta} \right].
\end{align*}
\]

In principle, the above applies for all \( \nu, \sigma, \rho, \gamma \) and therefore has greater applicability than Eq. (S77). However, when it does apply, Eq. (S77) is simpler, less computationally expensive, and makes clearer the relationship between \([M]\) and \( \mu \). It is easy to check that \( Lm(x, x) = x \) for any \( x \), and so if \( \nu = 0 \) the above becomes

\[
\begin{align*}
\nu = \left( \frac{1}{[M]} + \mu \right) \left[ \frac{w_1(\Delta)}{\Delta} \right].
\end{align*}
\]

from which \([M]\) is readily obtained.

To find the \( \gamma \to 0 \) limit of Eq. (S78) we must look back at Eqs. (S70), if we rearrange the first two of these equations for \( u \) we find

\[
\begin{align*}
\nu = \frac{[w_2(\Delta)]}{[w_2(\Delta)] + \gamma/\ell [w_0(\Delta)]}.
\end{align*}
\]

Hence, if \( \gamma = 0 \) then \( u = 1 \). Now we can simply solve the first and last of Eqs. (S70) for \( \Delta_0, \Delta_1 \)

\[
\begin{align*}
1 &= \ell \sigma^2 \int_{\Delta_0}^{\Delta_1} d\Delta \, \theta(\Delta, u = 1) w_2(\Delta),
\end{align*}
\]

\[
\begin{align*}
1 &= \int_{\Delta_0}^{\Delta_1} d\Delta \, \theta(\Delta, u = 1).
\end{align*}
\]

With the values of \( \Delta_0 \) and \( \Delta_1 \) known, one then has two methods for computing \([\phi] = [w_0(\Delta)]\). We can directly compute it as an integral over \( \Delta \)

\[
\begin{align*}
[\phi] &= \int_{\Delta_0}^{\Delta_1} d\Delta \, \theta(\Delta, u = 1) w_0(\Delta),
\end{align*}
\]

or we can solve the differential equation Eq. (S67) for the whole function \( \Delta(\alpha) \), and compute

\[
\begin{align*}
[\phi] &= \int_0^1 d\alpha \, n(\alpha) w_0(\Delta(\alpha)).
\end{align*}
\]
S5. ABUNDANCE DISTRIBUTIONS

A. Species abundance distributions (SADs), derivation of Eq. (32)

We wish to find the probability that the abundance in sub-community $a$ is $x$, $P(x|a)$. Setting $x_a^a(\eta_a^a) = x$ in Eq. (12) of the main text gives

$$x = \max \left(0, \frac{1 + \sum_b \mu^{ab} q^b M^b + \sqrt{\sum_b (\sigma^{ab})^2 n^b q^b z^a}}{1 - \sum_b \gamma^{ab} \sigma^{ab} \sigma^{ba} \chi^b} \right),$$  (S86)

where we have written $\eta_a^a = \sqrt{\sum_b (\sigma^{ab})^2 n^b q^b z^a}$ for a mean-zero, unit variance Gaussian random variable $z^a$.

Eq. (S86) implies that the distribution of $x_a^a$ has two contributions. First, a delta function at $x = 0$ with a weight equal to the probability of that species going extinct (having zero abundance). Second, a Gaussian distribution with mean $\frac{1 + \sum_b \mu^{ab} q^b M^b}{1 - \sum_b \gamma^{ab} \sigma^{ab} \sigma^{ba} \chi^b}$ and variance $\frac{\sum_b (\sigma^{ab})^2 n^b q^b}{(1 - \sum_b \gamma^{ab} \sigma^{ab} \sigma^{ba} \chi^b)^2}$, truncated to impose $x > 0$. With some straightforward substitutions from Eqs. (13) we arrive at

$$P(x|a) = \delta(x) (1 - \phi^a) + H(x) \frac{w_1(\Delta^a)}{M^a \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{xw_1(\Delta^a)}{M^a} - \Delta^a \right)^2 \right),$$  (S87)

where $H(x)$ is the Heaviside step function.

When we take the $B \to \infty$ limit and use the parameters of the cascade model, we arrive at Eq. (29) of the main text

$$P(x|\alpha) = \delta(x) (1 - \phi(\alpha)) + \frac{H(x)}{\sqrt{2\pi}} \frac{w_1[\Delta(\alpha)]}{M(\alpha)} \exp \left( -\frac{1}{2} \left( \frac{xw_1[\Delta(\alpha)]}{M(\alpha)} - \Delta(\alpha) \right)^2 \right).$$  (S88)

To obtain a species abundance distribution, we find the probability that any species in the community has abundance $x$. We must therefore integrate the above over $\alpha$

$$P(x) = \int_0^1 d\alpha \ n(\alpha) P(x|\alpha).$$  (S89)

The species abundance distribution is therefore

$$P(x) = \delta(x) (1 - \left[ \phi \right]) + \frac{H(x)}{\sqrt{2\pi}} \left[ \frac{w_1(\Delta)}{M} \exp \left( -\frac{1}{2} \left( \frac{xw_1(\Delta)}{M} - \Delta \right)^2 \right) \right],$$  (S90)

where we recall the meaning of the average $[ \ldots ]$ in Eq. (S47). To compute the expression on the right-hand side of Eq. (S90), we first solve Eqs. (26) for $\Delta_0, \Delta_1$ and $u$, then compute $M(\alpha)$ using either Eq. (34) or Eq. (S98). Finally, we then use Eq. (28) [equivalently Eq. (S78)] to obtain $[\phi]$. The integral

$$\left[ \frac{w_1(\Delta)}{M} \exp \left( -\frac{1}{2} \left( \frac{xw_1(\Delta)}{M} - \Delta \right)^2 \right) \right] = \int_0^1 d\alpha \ n(\alpha) \frac{w_1(\Delta(\alpha))}{M(\alpha)} \exp \left( -\frac{1}{2} \left( \frac{xw_1(\Delta(\alpha))}{M(\alpha)} - \Delta(\alpha) \right)^2 \right),$$  (S91)

can be computed either by first finding $\Delta(\alpha)$ using Eq. (23) [equivalently Eq. (S67)], or by changing variables to $\Delta$, again using Eq. (23) [equivalently Eq. (S67)].
1. \( P(x) \) in Eq. (32) is a probability distribution

Here we demonstrate that \( P(x) \), as given in Eq. (S90) [equivalently Eq. (32)], is a probability distribution, that is, we show that \( \int_{-\infty}^{\infty} dx \ P(x) = 1 \). We start by demonstrating

\[
\int_{-\infty}^{\infty} dx \ \frac{H(x)}{\sqrt{2\pi}} \left[ \frac{w_1(\Delta)}{M} \exp \left( -\frac{1}{2} \left( \frac{wx_1(\Delta)}{M} - \Delta \right)^2 \right) \right] = [\phi]. \quad (S92)
\]

This is done in the following steps

\[
\left[ \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dx \ \frac{w_1(\Delta)}{M} \exp \left( -\frac{1}{2} \left( \frac{wx_1(\Delta)}{M} - \Delta \right)^2 \right) \right] = \left[ \frac{1}{\sqrt{2\pi}} \int_\Delta^{\infty} dy \ \exp \left( -\frac{1}{2} y^2 \right) \right] = [w_0(\Delta)] = [\phi], \quad (S93)
\]

where the substitution \( y = \frac{xw_1(\Delta)}{M} - \Delta \) gives the first equality. The second equality is the definition of \( w_0(\Delta) \). We therefore find that

\[
\int_{-\infty}^{\infty} dx \ P(x) = (1 - [\phi]) + [\phi] = 1, \quad (S94)
\]
as claimed.

B. Rank abundance distributions (RADs)

As mentioned in the main text, a rank abundance distribution is obtained from ranking species by abundance, with the largest abundance species being given rank 0 and lowest abundance species being given rank 1.

The function \( P(x) \) in Eq. (S90) can be used to construct such a ranking: there are \( \int_0^x dx' \ P(x') \) species with an abundance smaller than \( x \). Hence a plot of abundance \( x \) against \( 1 - \int_0^x dx' \ P(x') \) provides the rank abundance distribution.

C. Derivation of Eq. (34) and of survival distributions

For a species with index \( \alpha \) (recall that \( \alpha = a/B \) in the \( B \to \infty \) limit, see Section IV.2 in the main text), there are \( N \int_0^\alpha d\beta \ n(\beta) \) species lower in the hierarchy. Hence we use

\[
r(\alpha) \equiv \int_0^\alpha d\beta \ n(\beta), \quad (S95)
\]
as an alternative measure of a species position in the hierarchy. To find \( M(\alpha) \), we use Eq. (S44), which we repeat here

\[
M(\alpha) = w_1[\Delta(\alpha)]\sqrt{q_\rho(\alpha)}. \quad (S96)
\]

Dividing the above by its average gives

\[
\frac{M(\alpha)}{[M]} = \frac{w_1[\Delta(\alpha)]\sqrt{q_\rho(\alpha)}}{[w_1(\Delta)]\sqrt{q_\rho}}. \quad (S97)
\]
Substituting for \( q(\alpha) \) using Eq. (S54) gives Eq. (34) from the main text. Explicitly

\[
\frac{M(\alpha)}{M} = A w_1[\Delta(\alpha)] \exp \left( \frac{\ell \sigma^2 \ln \rho}{u^2} \int_0^\alpha d\beta \ n(\beta) w_2(\Delta(\beta)) \right),
\]

(S98)

where

\[
A = \int_0^1 d\alpha \ n(\alpha) w_1(\Delta(\alpha)) \exp \left( \frac{\ell \sigma^2 \ln \rho}{u^2} \int_0^\alpha d\beta \ n(\beta) w_2(\Delta(\beta)) \right).
\]

(S99)

To explicitly compute \( A \), we first solve Eqs. (S70) for \( \Delta_0, \Delta_1, u \), which we then use to find \( \Delta(\alpha) \) with Eq. (S67) [the value of \( u \) is substituted into \( \theta(\Delta, u) \) and \( \Delta_0, \Delta_1 \) are the boundary conditions on \( \Delta(\alpha) \)]. Once \( \Delta(\alpha) \) is known, we can explicitly compute the above integrals numerically. A parametric plot of \( M(\alpha) \) against \( r(\alpha) \) constitutes an HAD, as in panel (c) of Fig. 3 in the main text.

To produce a survival distribution we first find \( \Delta(\alpha) \) for given parameters \( \nu, \sigma, \rho, \gamma \) and then produce a parametric plot with \( w_0[\Delta(\alpha)] \) on the y axis and \( r(\alpha) \) on the x axis.

S6. LOCAL STABILITY ANALYSIS

A. Derivation for block structured matrices

We first find the local stability of the fixed point of the system with general block-structured interaction and then move on to the specific case of the cascade model. We follow along the lines of the stability analyses in [S21, S37].

The local stability of possible fixed points can be probed by addition of an infinitesimal independent and identically distributed Gaussian perturbation \( \epsilon \xi^a(t) \) to each block in the effective dynamics Eq. (6). In a stable regime we expect the system to return to the fixed point when perturbed.

Applying these perturbations, we have

\[
\dot{x}^a(t) = x^a(t) \left( 1 - x^a(t) + \sum_b n^b \left[ \mu^{ab} M^b(t) + \gamma^{ab} \sigma^{ab} \sigma^{ba} \int_0^t dt' G^{ab}(t, t') x^a(t') \right] \right) + \eta^a(t) + \epsilon \xi^a(t).
\]

(S100)

We quantify the linear perturbations of \( x^a(t) \) and \( \eta^a(t) \) about the fixed point (which we assume are of the order \( \epsilon \)) by \( y^a(t), \kappa^a(t) \) respectively, such that

\[
x^a(t) = x_*^a + \epsilon y^a(t)
\]

\[
\eta^a(t) = \eta_*^a + \epsilon \kappa^a(t).
\]

(S101)

We obtain the following self-consistency conditions [see Eq. (S26)]

\[
\langle \kappa^a(t) \kappa^a(t') \rangle = \sum_b n^b (\sigma^{ab})^2 \langle y^a(t) y^a(t') \rangle.
\]

(S102)

Assuming time translation invariance in the long-time limit, linearising Eq. (S100) around the non-zero fixed point gives

\[
\dot{y}^a(t) = y^a(t) - y^a(t) + \sum_b n^b \gamma^{ab} \sigma^{ab} \sigma^{ba} \int_0^t dt' G^{ab}(t - t') y^a(t') + \kappa^a(t) + \xi^a(t).
\]

(S103)
We now follow [S37] by going to Fourier space,

\[ i\omega \hat{y}^a(\omega) = x^a_s \left( -\hat{y}^a(\omega) + \sum_b n^b \gamma^{ab} \sigma^{ab} \sigma^{ba} \hat{G}^b(\omega) \hat{y}^a(\omega) + \hat{\kappa}^a(\omega) + \hat{\xi}^a(\omega) \right). \]  

(S104)

Squaring and averaging over \(\kappa^a\) and \(\xi^a\) we find

\[ \frac{|\omega|^2}{x_s^2} \left( 1 - \sum_b n^b \gamma^{ab} \sigma^{ab} \sigma^{ba} \hat{G}^b(\omega) \right)^2 \langle |\hat{y}^a(\omega)|^2 \rangle = \phi^a \left( \sum_b n^b (\sigma^{ab})^2 \langle |\hat{y}^b(\omega)|^2 \rangle + 1 \right), \]  

(S105)

where the factor of \(\phi^a\) is due to the fact that Eq. (S103) only applies to non-zero fixed points, fluctuations around the zero point decay and hence do not contribute to \(\langle |\hat{y}^a(\omega)|^2 \rangle\). Noting that \(\hat{G}^a(0) = \chi^a\), we now set \(\omega = 0\) (see [S37]) and find

\[ \left( 1 - \sum_b n^b \gamma^{ab} \sigma^{ab} \sigma^{ba} \chi^b \right)^2 Y^a = \phi^a \left( \sum_b n^b (\sigma^{ab})^2 Y^b + 1 \right), \]  

(S106)

where \(Y^a \equiv \langle |\hat{y}^a(0)|^2 \rangle\). Assuming a stationary state in which \(\langle y(t)y(t+\tau)\rangle\) depends on \(\tau\) only, then \(Y^a = \int d\tau \langle y(t)y(t+\tau)\rangle\), and we may conclude that if \(Y^a\) diverges, then perturbations do not decay to zero. Hence, a non-zero fixed point is unstable if the only solution to Eq. (S106) is one in which \(Y^a\) diverges for some \(a\).

### B. Specific case of the cascade model

We now obtain the stability criterion in the specific case of the Cascade model. We follow very similar lines to Section S4 A to find the \(B \to \infty\) limit of Eq. (S106). The result is

\[ u(\alpha)^2 Y(\alpha) = w_0[\Delta(\alpha)] \left( 1 + \int_0^1 d\beta n(\beta)\sigma(\alpha,\beta)^2 Y(\beta) \right), \]  

(S107)

which, again following very similar lines to Section S4 A, takes the following form in the cascade model

\[ u^2 Y(\alpha) = w_0[\Delta(\alpha)] \left( \sigma^2 \rho^2 \int_0^\alpha d\beta n(\beta)Y(\beta) + \frac{\sigma^2}{\rho^2} \int_0^1 d\beta n(\beta)Y(\beta) + 1 \right). \]  

(S108)

This can be written as

\[ u^2 Y(\alpha) = w_0[\Delta(\alpha)] \left( 1 + \sigma^2 Y_\rho(\alpha) \right), \]  

(S109)

where we recall the definition in Eqs. (S41). From the general case considered in Section S6 we know that our system is linearly unstable if the solution \(Y(\alpha)\) to the condition in Eq. (S109) is unbounded. For the purposes of our analysis we will assume the weaker condition that the integral \([Y]\) is unbounded, the justification for this assumption is the numerical agreement we find in Section S7.

Following a very similar procedure to that in Section S4B, we find

\[ u^2 = \text{Lm} \left( \frac{1}{[Y]} + \frac{\sigma^2}{\rho^2}, \frac{1}{[Y]} + \sigma^2 \rho^2 \right) [w_0(\Delta)]. \]  

(S110)
The following property of the logarithmic mean 

\[ \text{Ln} (|a| + x, |a| + y) > \text{Ln} (x, y), \]  

(S111)

tells us that, when \([Y]\) is finite, the following must hold

\[ u^2 < \text{Ln} \left( \frac{\sigma^2}{\rho^2}, \sigma^2 \rho^2 \right) \left[ w_0 \right] = \ell \sigma^2 \left[ w_0(\Delta) \right], \]  

(S112)

When \([Y]\) diverges, that is, on the edge of linear instability, the inequality becomes an equality, giving a sufficient condition for linear instability. When combined with Eq. (S57), we see that the system is unstable to linear perturbation when

\[ \left[ w_0(\Delta) \right] \geq \left[ w_2(\Delta) \right]. \]  

(S113)

Plugging the above condition into Eqs. (26) [equivalently Eqs. (S70)] gives Eq. (35) from the main text

\[ \sigma^2 \geq \frac{\ell}{(\ell + \gamma)^2} \left[ \phi \right]. \]  

(S114)

Note that nothing in this derivation depended explicitly on \(\mu(\alpha, \beta)\) (equivalently \(\mu^{ab}\)). Hence the choice of \(\mu(\alpha, \beta)\) only enters into the stability analysis through its effect on average species survival rates \(\left[ \phi \right]\).

C. Local Stability in the cases when \(\nu = 0\) or \(\rho = 1\) or both

In this section we will look at Eq. (35) [equivalently Eq. (S114)] in three special cases. Firstly, when \(\nu = 0\) and \(\rho = 1\) the model reduces to that in [S21]. In those references the equivalent of Eq. (35) is derived [see Eq. (13) in Ref. [S21]]

\[ \sigma^2_{\nu=0, \rho=1} \geq \frac{2}{(1 + \gamma)^2}. \]  

(S115)

By Eq. (22) [equivalently Eq. (S51)], when \(\rho = 1\) we have \(\ell = 1\). Hence, by inspection of both Eq. (S115) and Eq. (S114), we see that the average survival rate \(\left[ \phi \right]\) at the onset of linear instability in the case where \(\nu = 0\) and \(\rho = 1\) is equal to 1/2.

We will now derive similar conditions in the cases when either \(\nu \neq 0\) or \(\rho \neq 1\). This will demonstrate that the critical value of \(\sigma\) at which the system becomes unstable is larger with larger values of \(|\nu|\), and smaller with smaller values of \(|\rho - 1|\). That is, a larger value of \(|\rho - 1|\) destabilises and larger \(|\nu|\) stabilises.

1. The case \(\nu = 0\)

When \(\nu = 0\) we find that Eq. (24) [equivalently Eq. (S68)] becomes

\[ \theta(\Delta) = -\frac{u^2}{\ell \sigma^2 \ln \rho^2 \Delta w_2(\Delta)}, \]  

(S116)
so that Eqs. (26) [equivalently Eqs. (S70)] are

$$\Delta_1 \rho^2 = \Delta_0,$$  \hspace{1cm} (S117)

$$\frac{1}{u} - 1 = \frac{\gamma}{\ell \ln \rho^2} \int_{\Delta_1}^{\Delta_0} \frac{w_0(x)}{x w_2(x)} \, dx,$$  \hspace{1cm} (S118)

$$1 = \frac{u^2}{\sigma^2 \ell \ln \rho^2} \int_{\Delta_1}^{\Delta_0} \frac{dx}{x w_2(x)}.$$  \hspace{1cm} (S119)

We want to solve these equations when the system is on the edge of linear instability, which, by Eq. (S113), occurs when $$[w_0(\Delta)] = [w_2(\Delta)]$$. Using the identity $$w_2(\Delta) = w_0(\Delta) + \Delta w_1(\Delta)$$ [see Eq. (S31) for the definition of $$w_0, w_1, w_2$$ we may more conveniently write this condition as $$[\Delta w_1(\Delta)] = 0$$. We then conclude that the system is on the edge of instability when Eqs. (S117) to (S119) are satisfied, and when simultaneously

$$\Delta w_1(\Delta) \int_{\Delta_0}^{\Delta_1} dx \theta(x) x w_1(x) = 0.$$  \hspace{1cm} (S120)

As $$w_1'(x) = 2w_1(x)$$, we can explicitly write the above as

$$\ln \frac{w_2(\Delta_0)}{w_2(\Delta_1)} = 0,$$  \hspace{1cm} (S121)

implying that $$\Delta_0 = \Delta_1$$. Combining this with Eq. (S117) we see that both of $$\Delta_0 = \Delta_1 \rho^2$$ and $$\Delta_0 = \Delta_1$$ must hold, and therefore $$\Delta_0 = \Delta_1 = 0$$ as we have assumed that $$\rho \neq 1$$.

For the remainder of this derivation we write $$\Delta_1 = \Delta$$ so that $$\Delta_0 = \Delta \rho^2$$. By the above considerations, we are interested in Eqs. (S118) and (S119) in the limit $$\Delta \to 0$$. Examining Eq. (S118) we have

$$\frac{1}{u} - 1 = \frac{\gamma}{\ell \ln \rho^2} \lim_{\Delta \to 0} \int_{\Delta}^{\Delta_0} \frac{w_0(x)}{x w_2(x)} \, dx = \frac{\gamma}{\ell \ln \rho^2} \lim_{\Delta \to 0} \int_{\Delta}^{\Delta_0} \frac{1}{x} + O(1) \, dx = \frac{\gamma}{\ell},$$  \hspace{1cm} (S122)

so that

$$u = \frac{1}{1 + \gamma/\ell}.$$  \hspace{1cm} (S123)

Similarly, from Eq. (S119) we find

$$u^2 = \frac{1}{2} \ell \sigma^2,$$  \hspace{1cm} (S124)

so that the equivalent of Eq. (35) when $$\nu = 0$$ is

$$\sigma^2_{\nu=0} \geq \frac{2\ell}{(\ell + \gamma)^2}.$$  \hspace{1cm} (S125)

In particular, we see that at the edge of instability, $$[\phi] = 1/2$$.

Eq. (S125) tells us that, when $$\nu = 0$$, increasing $$|\rho - 1|$$ is always destabilising. To see this, we look at the sign of the derivative of $$\sigma^2$$ with respect to $$\rho$$. Taking the logarithmic derivative of both sides of Eq. (S125) gives

$$\frac{1}{\sigma^2} \frac{(\sigma^2)'}{\ell'} = \frac{1}{\ell} - 2 \frac{1}{\ell + \gamma} \leq 0.$$  \hspace{1cm} (S126)
The last inequality follows from
\[ \frac{1}{1 + \frac{\gamma}{\ell}} \geq \frac{1}{2} \]  
(S127)
which itself follows from the fact that \( \gamma \in [-1, 1] \) and \( \ell \in [1, \infty) \), so that \( \gamma/\ell \in [-1, 1] \). Therefore, we find
\[ \text{sign}[(\sigma^2')] = -\text{sign}(\ell'), \]  
(S128)
it is easily verified that \( \ell \) increases if \( |\rho - 1| \) is increased, and so therefore the critical value of \( \sigma^2 \) decreases if \( |\rho - 1| \) is increased. Hence, in the case \( \nu = 0 \) increasing \( |\rho - 1| \) always moves the system towards linear instability.

2. The case \( \rho = 1 \)

When \( \rho = 1 \), Eq. (24) becomes
\[ \theta(\Delta) = \frac{u}{2\nu w_1(\Delta)}, \]  
(S129)
so that Eqs. (26) are
\[ 1 = \frac{\sigma^2}{2\nu u} \left( \ln\left(\frac{w_1(\Delta_1)}{w_1(\Delta_0)}\right) + \frac{1}{2} (\Delta_1^2 - \Delta_0^2) \right), \]  
(S130)
\[ 1 - u = \frac{\gamma \sigma^2}{2\nu} \ln\left(\frac{w_1(\Delta_1)}{w_1(\Delta_0)}\right), \]  
(S131)
\[ 1 = \frac{u}{2\nu} \int_{\Delta_0}^{\Delta_1} \frac{dx}{w_1(x)}. \]  
(S132)
As with the case of \( \nu = 0 \), we know that the system is on the edge of linear instability when \( [\Delta w_1(\Delta)] = 0 \), therefore
\[ \int_{\Delta_0}^{\Delta_1} \frac{w_1(x) dx}{w_1(x)} = 0, \]  
(S133)
as this integral simply evaluates to \( 1/2(\Delta_1^2 - \Delta_0^2) \), we conclude that \( \Delta_0 = \pm \Delta_1 \). Suppose \( \Delta_0 = \Delta_1 \), then by Eq. (S132), \( u \) must diverge, which is inconsistent with Eq. (S131). Hence, the only consistent choice is \( \Delta_0 = -\Delta_1 \), and so we call \( \Delta_1 = \Delta \) and \( \Delta_0 = -\Delta \). Eliminating \( u \) from Eqs. (S130) to (S132) then gives
\[ \sigma^2_{\rho=1} = \frac{1}{(1 + \gamma)^2} \left( \int_{-\Delta}^{\Delta} \frac{dx}{w_1(x)} \right)^2 \int_{-\Delta}^{\Delta} \frac{w_0(x)}{w_1(x)} dx, \]  
(S134)
\[ \nu = \frac{1}{2} \frac{1}{1 + \frac{\gamma}{\Delta}} \int_{-\Delta}^{\Delta} \frac{dx}{w_1(x)}. \]  
(S135)
The above are unchanged under the transformation \( \nu \to -\nu, \Delta \to -\Delta \), so for the remainder of this section we will take \( \nu > 0, \Delta > 0 \) without loss of generality, one can make the same conclusions
for $\nu < 0$ by reversing the sign of $\Delta$. Taking $\Delta > 0$ allows us to use some useful properties of the functions $w_0(\Delta)$ and $w_1(\Delta)$ in the following.

We will look at the logarithmic derivative of $\sigma^2$ with respect to $\nu$ and show that this quantity is positive. Firstly, we note that $w_1(x) > 0$ for all $x$, implying that $\nu$ is an increasing function of $\Delta$ by Eq. (S135). In turn this implies that the derivative of $\sigma^2$ with respect to $\nu$ has the same sign as the derivative of $\sigma^2$ with respect to $\Delta$ by the chain rule. Hence, it is enough to show that the following is positive

$$\frac{d \ln \sigma^2}{d\Delta} = \frac{1}{w_1(\Delta)} \frac{d}{d\Delta} \int_{-\Delta}^\Delta \frac{dx}{w_1(x)} - \frac{w_0(\Delta)/w_1(\Delta) + w_0(-\Delta)/w_1(-\Delta)}{\int_{-\Delta}^\Delta \frac{dx}{w_0(x)}}. \quad (S136)$$

First, we show that for $\Delta \geq 0$

$$\int_{-\Delta}^\Delta \frac{dx}{w_1(x)} \leq \frac{1}{2} \int_{-\Delta}^\Delta \frac{dx}{w_1(x)}, \quad (S137)$$

which follows from the following observations. Firstly, $w_0(\Delta) - 1/2$ is an odd increasing function of $\Delta$ and is positive for $\Delta \geq 0$. Secondly, $1/w_1(\Delta)$ is a decreasing function of $\Delta$. Hence we may write

$$\int_0^\Delta dx \frac{w_0(x) - 1/2}{w_1(x)} \leq \int_0^\Delta dx \frac{w_0(x) - 1/2}{w_1(-x)}, \quad (S138)$$

or, using that $w_0(\Delta) - 1/2$ is odd

$$2 \int_{-\Delta}^\Delta dx \frac{w_0(x) - 1/2}{w_1(x)} \leq 0, \quad (S139)$$

from which our claim follows.

We can re-write the claim in Eq. (S136) as

$$\left(\frac{1}{w_1(\Delta)} + \frac{1}{w_1(-\Delta)}\right) \int_{-\Delta}^\Delta dx \frac{w_0(x)}{w_1(x)} - \left(\frac{w_0(\Delta)}{w_1(\Delta)} + \frac{w_0(-\Delta)}{w_1(-\Delta)}\right) \int_{-\Delta}^\Delta dx \frac{1}{w_1(x)} \geq 0. \quad (S140)$$

We now use Eq. (S137) to write

$$\left(\frac{1}{w_1(\Delta)} + \frac{1}{w_1(-\Delta)}\right) \int_{-\Delta}^\Delta dx \frac{w_0(x)}{w_1(x)} - \left(\frac{w_0(\Delta)}{w_1(\Delta)} + \frac{w_0(-\Delta)}{w_1(-\Delta)}\right) \int_{-\Delta}^\Delta dx \frac{1}{w_1(x)} \geq \left\{\left(\frac{1}{w_1(\Delta)} + \frac{1}{w_1(-\Delta)}\right) - 2 \left(\frac{w_0(\Delta)}{w_1(\Delta)} + \frac{w_0(-\Delta)}{w_1(-\Delta)}\right)\right\} \int_{-\Delta}^\Delta dx \frac{1}{w_1(x)}. \quad (S141)$$

We now show that the quantity in the curly brackets is positive with the following manipulations

$$\left(\frac{1}{w_1(\Delta)} + \frac{1}{w_1(-\Delta)}\right) - 2 \left(\frac{w_0(\Delta)}{w_1(\Delta)} + \frac{w_0(-\Delta)}{w_1(-\Delta)}\right) = \frac{1 - 2w_0(\Delta)}{w_1(\Delta)} - \frac{1 - 2w_0(-\Delta)}{w_1(-\Delta)} = (1 - 2w_0(\Delta)) \left(\frac{1}{w_1(-\Delta)} - \frac{1}{w_1(\Delta)}\right), \quad (S142)$$

which is positive for positive $\Delta$, as $w_0(\Delta) \geq 1/2$ and $1/w_1(\Delta)$ is a decreasing function. Hence, $\sigma^2_{\rho=1}$ is an increasing function of $\nu$, in other words, increasing $\nu$ stabilises the system.
FIG. S1. Simulation data for central and right plots of Fig. 6 in the main text. Parameters are $\mu = -0.5, \sigma = 0.7, \gamma = -0.8$ for the left plot and $\mu = 0.5, \sigma = 0.8, \gamma = -0.8$ for the right plot. Each data point is stability data for a community of $N = 200$ species, averaged over 20 runs, $(\nu, \rho)$ pairs are randomly sampled. Points are colored according to their measured stability, purple (darkest) indicates that abundances diverge, blue (medium darkness) indicates linear instability and yellow (lightest) indicates stability. There is a colour gradient between stable and linearly unstable as the instability boundary is only perfectly sharp in the $N \to \infty$ limit.

S7. VERIFICATION OF THE CRITERIA FOR STABILITY USING COMPUTER SIMULATION

In order to test the stability criteria in Eqs. (35) and (36), we randomly sample a large number of pairs $(\nu, \rho)$. $\nu$ is sampled uniformly from the range $(-5, 5)$ and $\rho$ is obtained as $\rho = \exp(X)$, where $X$ is uniformly sampled in the range $(-\ln 10, \ln 10)$. This ensures that the points shown in Fig. S1 are uniformly distributed on the logarithmic scale. The values of $\mu, \sigma, \gamma$ are fixed. For each pair $(\nu, \rho)$ we simulate the system 20 times with input parameters $\mu, \nu, \sigma, \rho, \gamma$, and determine an approximate probability of the systems stability, or lack thereof. Each point is then plotted in the $\nu - \rho$ plane, and given a colour representative of this probability, for example, if 10 of the twenty runs are deemed stable and 10 linearly unstable, then that particular point will have a colour half way between the stable and unstable colours on the colour map we have chosen. The result, as well as comparison to theory lines predicted by Eqs. (35) and (36), are shown in two particular instances in Fig. S1.

S8. INDEPENDENCE OF RESULTS FROM $n(\alpha)$

Here we prove the following statement about our system: Let $f(\alpha)$ be some function which does not explicitly depend on the relative number of species with index $\alpha$ $[n(\alpha)]$. We also imagine that $f(\alpha) = g[\Delta(\alpha)]$ for some function $g$. When both of these conditions are satisfied, then $[f]$ (equivalently $[g(\Delta)]$) is independent of $n(\alpha)$.
To see this, we use Eq. (23) [equivalently Eq. (S67)] to change variables $\alpha \rightarrow \Delta$ in the following

$$\int_0^1 d\alpha \ n(\alpha) f(\alpha) = \int_{\Delta_0}^{\Delta_1} d\Delta \ \theta(\Delta, u) g(\Delta). \quad (S143)$$

We recall Eqs. (26) [equivalently Eqs. (S70)], which allow us to compute $\Delta_0, \Delta_1$ and $u$ as functions of $\nu, \sigma, \rho, \gamma$ only. Therefore, the integral over delta in the above depends only on $\nu, \sigma, \rho, \gamma$, as well as any parameters which $g$ depends on, which we have assumed excludes $n(\alpha)$.

A. $[M]$ and $[\phi]$ [Eqs. (27) and (28)]

In light of the preceding subsection, $[M]$ and $[\phi]$ must be independent of $n(\alpha)$ by Eqs. (27) and (28) [equivalently Eqs. (S77) and (S78)]. Here, we give an alternative proof by demonstrating that $M(\alpha) = M[\Delta(\alpha)]$ and $\phi(\alpha) = \phi[\Delta(\alpha)]$. We recall Eq. (S98),

$$\frac{M(\alpha)}{\int_0^\alpha d\beta \ n(\beta) w_2(\Delta(\beta))} = Aw_1(\Delta(\alpha)) \exp\left(\frac{\ell \sigma^2 \ln \rho}{u^2} \int_0^\alpha d\beta \ n(\beta) w_2(\Delta(\beta))\right), \quad (S144)$$

changing variables with Eq. (S67) in the exponentiated integral gives

$$\int_0^\alpha d\beta \ n(\beta) w_2(\Delta(\beta)) = \int_{\Delta_0}^{\Delta(\alpha)} d\Delta \ \theta(\Delta, u) w_2(\Delta), \quad (S145)$$

demonstrating that $M(\alpha) = M(\Delta(\alpha))$, so that $[M]$ is independent of $n(\alpha)$. Noting that $\phi(\alpha) = w_0(\Delta(\alpha))$ gives the corresponding claim for $[\phi]$.

B. SADs [Eq. (32)]

This follows from changing variables $\alpha \rightarrow \Delta$ in the integral in the second term in Eq. (S90). We find

$$\left[\frac{w_1(\Delta)}{M} \exp\left(-\frac{1}{2} \left(\frac{xw_1(\Delta)}{M} - \Delta\right)^2\right)\right] = \int_{\Delta_0}^{\Delta_1} d\Delta \ \theta(\Delta, u) w_1(\Delta) \exp\left(-\frac{1}{2} \left(\frac{xw_1(\Delta)}{M(\Delta)} - \Delta\right)^2\right), \quad (S146)$$

where, by Eqs. (S67) and (S98), we have

$$\frac{M(\Delta)}{\int_{\Delta_0}^{\Delta_1} d\Delta' \ w_1(\Delta')} = \frac{w_1(\Delta) \exp\left(\frac{\ell \sigma^2 \ln \rho}{u^2} \int_{\Delta_0}^{\Delta'} d\Delta' \ \theta(\Delta', u) w_2(\Delta')\right)}{\int_{\Delta_0}^{\Delta_1} d\Delta' \ w_1(\Delta') \exp\left(\frac{\ell \sigma^2 \ln \rho}{u^2} \int_{\Delta_0}^{\Delta'} d\Delta' \ \theta(\Delta'', u) w_2(\Delta'')\right)}, \quad (S147)$$

which has no $n(\alpha)$ dependence, hence $P(x)$ in Eq. (32) [equivalently Eq. (S90)] also has no dependence on $n(\alpha)$.

C. RADs

This follows from the independence of $P(x)$ from $n(\alpha)$, as this function is used to define an RAD (see Section S5 B).
D. HADs [Eq. (34)]

To demonstrate that HADs and survival distributions are independent of \( n(\alpha) \), we will derive explicit expressions for \( M \) and \( \phi \) as functions of \( r \), where \( r \) is given by Eq. (S95). First we note that Eq. (S67) can be written as

\[
\frac{d\Delta}{dr} = \frac{1}{\theta(\Delta, u)},
\]

and therefore \( \Delta(r) \) is independent of \( n(\alpha) \). Further, we can now write

\[
\frac{M(r)}{[M]} = Aw_1[\Delta(r)] \exp \left( \frac{\ell \sigma^2 \ln \rho}{u^2} \int_0^r dr' w_2[\Delta(r')] \right),
\]

where

\[
A = \int_0^1 dr \, w_1[\Delta(r)] \exp \left( \frac{\ell \sigma^2 \ln \rho}{u^2} \int_0^r dr' w_2[\Delta(r')] \right),
\]

and hence \( M(r) \) has no \( n(\alpha) \) dependence.

E. Survival Distributions

Similarly to \( M(r) \), we consider \( \phi(r) = w_0[\Delta(r)] \), which also has no \( n(\alpha) \) dependence, as \( \Delta(r) \) has none.

F. Linear Instability [Eq. (35)]

We recall Eq. (35) [equivalently Eq. (S114)]

\[
\sigma^2 \geq \frac{\ell}{(\ell + \gamma)^2 \lbrack \phi \rbrack},
\]

which is independent of \( n(\alpha) \) as \( \ell \) depends only on \( \rho \) [Eq. (22)] and \( \lbrack \phi \rbrack \) is independent of \( n(\alpha) \).

G. Diverging abundances instability [Eq. (36)]

This follows from the independence of \( [M] \) on \( n(\alpha) \), hence the point at which \( [M] \rightarrow \infty \) is also independent of the choice of the function \( n(\alpha) \). Alternatively, recall Eq. (36) from the main text, abundances will diverge provided

\[
\mu \geq \frac{\Delta_1 \rho^2 + \Delta_0}{\Delta_1 \rho^2 - \Delta_0} \nu,
\]

and, by Eqs. (26) [alternatively Eqs. (S70)], both of \( \Delta_0 \) and \( \Delta_1 \) are independent of \( n(\alpha) \).

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[1] M. R. Gardner and W. R. Asgby. Connectance of large dynamic (cybernetic) systems: Critical values for stability. Nature, 228(5273):784–784, Nov 1970. ISSN 1476-4687. doi:10.1038/228784a0.
