Connecting Core Percolation and Controllability of Complex Networks

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Core percolation is a fundamental structural transition in complex networks related to a wide range of important problems. Recent advances have provided us an analytical framework of core percolation in uncorrelated random networks with arbitrary degree distributions. Here we apply the tools in analysis of network controllability. We confirm analytically that the emergence of the bifurcation in control coincides with the formation of the core and the structure of the core determines the control mode of the network. We also derive the analytical expression related to the controllability robustness by extending the deduction in core percolation. These findings help us better understand the interesting interplay between the structural and dynamical properties of complex networks.

Results

Analytical Framework of Core Percolation. In core percolation leaf nodes and their neighbors are taken off iteratively from the network according to GLR procedure21,22. Specifically, a node with degree one is randomly chosen. This node and its neighbor are removed with all their links. Nodes that becomes isolated are also removed. This procedure is repeated until no node with degree one is left and a core emerges as a compact cluster of nodes left. To systematically study the core percolation, two categories of removable nodes are introduced27: \( x \) removable...
Figure 1 | A bipartite graph with a core (highlighted). Node 1+, 2+, 1− and 2− are the core nodes. Node 3+ and 4− are z removable because they can become isolated without directly removing themselves (e.g. node 3− is chosen in the GLR procedure, node 3− and 3− with all their links are removed and node 4− becomes isolated). Node 3+ and 5+ and 5− are β removable as they can become a neighbor of a leaf.

Controllability of Complex Networks. The controllability of complex systems is a fundamental challenge of contemporary science that draws considerable interests in multidisciplinary fields.28,29,31–50. According to control theory51–52, the dynamic process of controlling a linear time-invariant system can be described by the equation \(\dot{x}(t) = Ax(t) + Bu(t)\), where the state vector \(x(t) = (x_1(t), \ldots, x_N(t))^T\) captures the state of a system of N components at time t. The \(N \times N\) state matrix A corresponds to the internal interactions of the system. The input matrix B is an \(N \times M\) matrix indicating how the M external signals \(u(t) = (u_1(t), \ldots, u_M(t))^T\) are exerted to the system to drive it from any initial state to any desired final state within finite time. Recently an efficient methodology has been introduced to identify the minimum driver node set (MDS), the smallest set of nodes whose time-dependent control yields the control over the whole system.29,30. The procedure is to convert a directed network into a bipartite graph (Fig. 2a, b) and find the maximum matching of bipartite graph.53. The minimum driver nodes are the unmatched nodes in the − set. If a perfect matching exists and all nodes in the − set are matched, one input signal would be sufficient to drive the system. In this case the number of driver node is one.

The methodology proposed indicates the existence of multiple MDSs, hence a node does not necessarily participate in all MDSs (Fig. 2c).30. Accordingly a node can be categorized by its participation in control: critical if it participates in all MDSs, redundant if it is not included in any MDSs and intermittent if it is in some MDSs but not all. The fraction of critical nodes \(n_c\) is purely determined by the in-degree distribution as \(n_c = P_{in}(0)\). However, the fraction of redundant nodes \(n_r\) displays a bifurcation after some critical average degree \(\langle k_c \rangle\); networks with identical degree distribution and average degree can have very high or low value of \(n_r\) (Fig. 3, 4). Such a bimodality feature leads to the two distinct control modes, with significant difference in the total number of MDS choices. While the two control modes coexist with equal probability in dense networks with identical in- and out-degree distributions, networks with different in- and out-degree distribution may have one mode dominate or follow only control mode.29.
Before the formation of the core, a node is either an $a$ or $\beta$ node, hence the fraction of redundant nodes $n_r$ can be found by Eqs.(3) and (4).

When core emerges, however, $a$ and $\beta$ nodes are not the only nodes in the network. The number of core nodes in the $1^{+}$ set is $N_{1}^+$, and the number of core nodes in the $1^{-}$ set is $N_{1}^-$. When $N_{1}^+ < N_{1}^-$, all the core nodes in the $1^{+}$ set are not always matched. This corresponds to the lower branch of the bifurcation with $n_r = n_{1}^+ - n_{1}^-$. When $N_{1}^+ > N_{1}^-$, all the core nodes in the $1^{+}$ set are always matched, giving rise to a large number of always matched nodes. In this case, the network is on the upper branch of the bifurcation curve and $n_r = 1 - n_{1}^-$ as only $a$ nodes in the $1^{-}$ set are not redundant (Fig. 4a, c). In summary, depending on the core structure, $n_r$ can be high or low as

$$n_r = \begin{cases} 1 - n_{1}^-, & \text{if } N_{1}^+ \geq N_{1}^- \\ n_{1}^+, & \text{if } N_{1}^+ < N_{1}^- \end{cases} \tag{6}$$

Eq.(6) confirms the emergence of bifurcation at the formation of the core (Fig. 3). More importantly, it explains the condition for the coexistence of the two control modes. When the in- and out-degree distribution are the same, the expected $n_{1}^+$ and $n_{1}^-$ are identical, giving the same probability of $N_{1}^+ \geq N_{1}^-$ and $N_{1}^+ < N_{1}^-$. When the in- and out-degrees are asymmetric, however, Eq.(5) gives different $n_{1}^+$ and $n_{1}^-$ values\(^2\), implying that in sufficiently large networks where the mean-field equation Eq.(5) applies, only one type of the core structure is allowed. Consequently any tiny differences between the in- and out-degree distribution eliminates the
bifurcation of \( n_r \) and forces the large network to follow one or the other control mode. Yet, in small systems the mean-field equation may not hold and the core structure different from the expectation can emerge due to random fluctuations. As an example, networks with \( P_{\text{out,in}}^{\text{out,in}} = k^{-\gamma_{\text{out,in}}}, \gamma_{\text{out}} = 3, \gamma_{\text{in}} = 2.85 \) have very small difference between \( n_{\text{core}}^{+} \) and \( n_{\text{core}}^{-} \) almost overlap. The above results reveal the importance of core as a fundamental structure that controls the two control modes. This feature allows us to switch the control modes by changing the core balance. The most intuitive way to induce such a switch is to reverse the direction of all links in the network. A network originally with \( N_{\text{core}}^{+} > N_{\text{core}}^{-} \) on the upper branch of the bifurcation will have \( N_{\text{core}}^{+} < N_{\text{core}}^{-} \) in the transpose network staying on the lower branch and vice versa. While

![Figure 3](http://www.nature.com/scientificreports/45379)

(a–c) Dependency between the core and the control modes for networks with \( P_{\text{out,in}}^{\text{out,in}} = k^{-\gamma_{\text{out,in}}}, \gamma_{\text{out}} = 3, \gamma_{\text{in}} = 2.85 \) and different sizes. The bifurcation emerges at the formation of the core. The solid lines in (a) and (b) correspond to Eq.(6) with the expected core structure \( N_{\text{core}}^{+} \geq N_{\text{core}}^{-} \). The dashed lines are based on Eq.(6) when \( N_{\text{core}}^{+} < N_{\text{core}}^{-} \), as the minority of network realizations different from expectation. (c) The expected fraction of core nodes on the + and − set (\( n_{\text{out}}^{+} \) and \( n_{\text{core}}^{-} \)) based on Eq.(5). The difference between \( n_{\text{core}}^{+} \) and \( n_{\text{core}}^{-} \) is very small (\( \sim 0.005 \)) and the two curves of \( n_{\text{core}}^{+} \) and \( n_{\text{core}}^{-} \) almost overlap. (d,e) The distribution of \( n_r \) of an ensemble of network realizations with \( P_{\text{out,in}}^{\text{out,in}} = k^{-\gamma_{\text{out,in}}}, \gamma_{\text{out}} = 3, \gamma_{\text{in}} = 2.85, \langle k \rangle = 10 \) and different sizes. As the networks size increases, the core structure is closer to the expectation, eliminating the lower branch of the bifurcation curve.
the change of all links’ directions is too drastic, sometimes the switch of the control mode can be induced by only local changes, occasionally as little as flipping the direction of a single well chosen link. Our chance of finding such a link depends on the core structure: if $N_{\text{core}}^+$ and $N_{\text{core}}^-$ is close, the network is sensitive to link changes. Indeed in a sample of 1000 realizations of Erdős-Rényi networks, all networks that are able to switch the control modes via a link’s reversal have very close core size (Fig. 5a, b). As the core size depends on the network size and degree distribution, the control mode is more stable in large networks and networks with asymmetric in- and out-degrees, in which more structural changes are required to change the control mode. Small networks with identical in- and out-degree distribution, however, are more likely to have close $N_{\text{core}}^+$ and $N_{\text{core}}^-$ therefore sensitive to structural perturbations (Fig. 5c, d).

Core Percolation and the Robustness of Control. The control mode can be sensitive to structural changes, but the network controllability is relatively robust. Removing a node or link can only change the number of driver nodes $N_D$ by 1 (4.23). To measure the importance of nodes in sustaining the controllability over the network, a different set of node category has been introduced (28). A node is structurally critical if the number of driver nodes has to be increased to maintain full control in its absence ($N_D$ increases by 1), a node is structurally redundant if it can be removed without affecting the current set of driver nodes ($N_D$ does not change) and a node is structurally ordinary if it is neither structurally critical or structurally redundant ($N_D$ decreases by 1) (Fig. 2d). While the fraction of nodes in each category can be numerically studied, the analytical approach is missing.

Here we use the tools in core percolation to derive the analytical expression of the fraction of nodes in each category. In the controllability problem, a node has dual roles. On one hand, a node’s dynamics is controlled via internal or external channels pointed to it. On the other hand, a node serves as means to control other neighboring nodes it points to. Such dual roles can be best seen in a network’s bipartite representation. The nodes in the $+$ set can be considered as “superiors” that influence others internally and the nodes in the $-$ set are “subordinates” that need to be controlled. Accordingly the consequence of a node’s removal relies on the node’s role in both $+$ and $-$ set. If a node in the $-$ set is always matched, its removal will not change the number of unmatched nodes in the $-$ set. Otherwise, the number of unmatched nodes in the $-$ set will decrease by 1. Similarly if a node in the $+$ set is always matched, it matches a node in the $-$ set in all matchings. Removing this node increases the number of unmatched nodes in the $-$ set by 1. Otherwise, the number of unmatched nodes in the $-$ set will not change as there exits alternative configurations matching the same number of nodes. The impact of node $i$’s removal (node $i^+$ and node $i^-$ in the bipartite graph) is summarized in Table 1. As the number of driver nodes equals the number of unmatched nodes in the $-$ set, we readily have the relationship between a node’s structural role in control and its matching status. A node $i$ in a directed network is structurally critical if in its bipartite representation both nodes $i^+$ and $i^-$ are always matched. A node $i$ is structurally ordinary if neither node $i^+$ nor $i^-$ is always matched. Otherwise node $i$ is structurally redundant.

The expression of always matched nodes in the $-$ set ($n_i$) is obtained in Eq.(6). With the symmetry between the $+$ and $-$ sets, we can find the expression of structurally critical, ordinary and redundant nodes as

$$n_{ic} = \begin{cases} n_i^+ (1 - n_i^-), & \text{if } N_{\text{core}}^+ \geq N_{\text{core}}^- \\ n_i^- (1 - n_i^+), & \text{if } N_{\text{core}}^+ < N_{\text{core}}^- \end{cases}$$

$$n_{io} = \begin{cases} (1 - n_i^+) n_i^-, & \text{if } N_{\text{core}}^+ \geq N_{\text{core}}^- \\ (1 - n_i^-) n_i^+, & \text{if } N_{\text{core}}^+ < N_{\text{core}}^- \end{cases}$$

and

$$n_{ir} = 1 - n_{ic} - n_{io},$$

where $n_{ic}$, $n_{io}$ and $n_{ir}$ are fractions of structurally critical, structurally ordinary and structurally redundant nodes.

The results of Eqs.(7–9) are numerically tested (Fig. 6a). There is a sudden change of $n_{ic}$, $n_{io}$ and $n_{ir}$ on the formation of the core ($k_c = 2e$ in the Erdős–Rényi network) accompanied by the change of fraction of $\alpha$ and $\beta$ nodes. Note that whether a node is a driver node depends on the matching in the $-$ set only. The always matched nodes in the $-$ set can be high or low depending on the two structures of the core. This generates the bifurcation feature where nodes’ participations in control differ dramatically. When a node’s role in controllability robustness is concerned, however, it depends on matching in both the $+$ and $-$ sets. Therefore $n_{ic}$, $n_{io}$ or $n_{ir}$ do not show a bifurcation feature. Indeed, when the in- and out-degree distribution are the same, $N_{\text{core}}^+ \geq N_{\text{core}}^-$ and $N_{\text{core}}^+ < N_{\text{core}}^-$ are equally likely, but $\beta^- = \beta^+$ and $\alpha^+ = \alpha^-$, giving rise to a single value of $n_{ic}$, $n_{io}$ and $n_{ir}$ regardless of the shape of the core. When the in- and

![Figure 4](https://example.com/figure4.png)

Figure 4 | (a), (b) Two networks on different control modes, with high (a) and low (b) fraction of redundant nodes. (c), (d) The core of the directed network in (a) and (b), respectively. Network with a high fraction of redundant nodes (a) yields a core with $N_{\text{core}}^+ > N_{\text{core}}^-$ (c), and vice versa.

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**Reference:**

1. Erdős, P. & Rényi, A. Publication. 2, 125 (1959).

2. Ref. 1, Eq. 3

3. Ref. 1, Eq. 4

4. Ref. 1, Eq. 5

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**Note:** The above text is a natural language representation of the content of a page from a scientific document.
out-degree distribution are different, only one condition (\(N_{\text{core}}^+ \geq N_{\text{core}}^-\) or \(N_{\text{core}}^- > N_{\text{core}}^+\)) is allowed in infinite networks, generating only one value of \(n_{\text{in}}, n_{\text{so}},\) and \(n_{\text{out}}\). Small networks with different in- and out-degree distribution do have cores with \(N_{\text{core}}^+ \geq N_{\text{core}}^-\) or \(N_{\text{core}}^- > N_{\text{core}}^+\) (Fig. 3). But the difference in the expected value of \(n_{\text{in}}, n_{\text{so}},\) and \(n_{\text{out}}\) in the two cases are typically small (Fig. 6b). As the result no obvious bifurcation curve can be observed with the random noise due to the small system size.

**Discussion**

In summary, we apply the recent advances in core percolation to the controllability of complex networks. Historically, core percolation has been discovered to be related to a wide range of important problems in complex networks. Here we add a new connection to the network controllability. In particular, we reveal the importance of core as a fundamental structure that generates the two control modes. The core structure determines the control mode that is related to the participation of nodes in control under the minimum driver nodes. The stability of the control mode under structural perturbations also relies on the balance of the core. Moreover, we derive the analytical expression of the fraction of nodes with different controllability robustness. The expression obtained demonstrates dependency on the structure of the core.

The results presented raise several intriguing questions awaiting answers. For example, it is found that switching the balance of the core is crucial in changing the control modes. However, an efficient algorithm to identify a series of structure variations needed to change the core size is still missing. We lack the method to change the control mode in arbitrary networks. The calculation in controllability robustness is based on uncorrelated in- and out-degree distribution. The effects of higher order correlations require further investigations. Finally, the analytical framework of core percolation is limited to model networks. In many real systems, core is quite different from the analytical expectations. For example, many real networks have multiple pieces of cores which is not observed in model networks. In many real systems, core is quite different from the analytical expectations. For example, many real networks have multiple pieces of cores which is not observed in model networks. In many real systems, core is quite different from the analytical expectations. For example, many real networks have multiple pieces of cores which is not observed in model networks.
is a sudden change at correlations. For example, node 1 in both the distribution. The scale free network directly generated by static model has degree structures different from the expectation ($P_{\text{in}} \sim k^{-\gamma_{\text{in}}}$, $P_{\text{out}} \sim 3^{\gamma_{\text{out}}} = 2.85$ and different $k$). The solid lines correspond to the expected core structures ($N_{\text{core}} \geq N_{\text{con}}$) and the dashed lines are the results for core structures different from the expectation ($N_{\text{core}} < N_{\text{con}}$). The difference of the results between the two core structures are small. Therefore even though in small systems both core structures are possible, there is no bifurcation observed.

**Methods**

Generating a scale free network. The scale-free networks analyzed are generated via the static model. We start from $N$ disconnected nodes indexed by integer number ($i = 1, \ldots, N$). The weight $w_i^j = 1^{-k}$ is assigned to each node in the out and the in set, with $x_i^j$ a real number in the range $[0, 1]$. Randomly selected two nodes $i$ and $j$ respectively from the out set and the in set, with probability proportional to $w_i^j$ and $w_j^i$. Connect node $i$ and $j$ if there is no connection between them, corresponding to a directed link from node $i$ to node $j$ in the digraph. Otherwise randomly choose another pair. Repeated the procedure until ($k_{\text{in}} = k_{\text{out}} = \gamma_{\text{in}} = 2.85$ and different $k$) links are created. The degree distribution under this construction is $p(k) \sim (k+1)^{-(1+\gamma_{\text{in}})} = k^{-\gamma_{\text{in}}}$.

Eliminating correlations between in- and out-degree distribution. The calculation of controllability robustness is based on the assumption that a node’s role in the + set is independent of its role in the − set, which requires independent in- and out-degree distribution. The scale free network directly generated by static model has degree correlations. For example, node 1 in both the + and − set have the largest expected degree. To eliminate the degree correlation, we randomize the weight $w_i^j$.

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**Author contributions**

T.J. and M.P. designed the research. T.J. and M.P. performed numerical simulation and analytical deduction. T.J. prepared the initial draft of the manuscript.

**Additional information**

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