UNITARY BRAID REPRESENTATIONS WITH FINITE IMAGE

MICHAEL J. LARSEN AND ERIC C. ROWELL

Abstract. We characterize unitary representations of braid groups $B_n$ of degree linear in $n$ and finite images of such representations of degree exponential in $n$.

1. Introduction

In this paper, we prove two loosely connected results about unitary representations of the braid group $\phi: B_n \to U(d)$, when $n$ is sufficiently large and the degree $d$ is not too large compared to $n$. The original motivation goes back to the work of Jones on images of Braid groups in Hecke algebra representations $H(q, n)$. Jones showed [J1] that when $q = i$, the image of $B_n$ in every irreducible factor of the Hecke algebra is finite; more explicitly, each such image is an extension of a symmetric group by a 2-group. This is in sharp contrast to the usual behavior of irreducible factors of Hecke algebra representations, in which the closure of the image of $B_n$ contains all unimodular unitary matrices [FLW]. Birman and Wajnryb showed [BW] that when $q = e^{2\pi i/6}$, certain factors of $H(q, n)$ give rise to representations whose images are extensions of symplectic groups $Sp(2r, F_3)$ by 3-groups, where $n \approx 2r$ (see also [GJ]). It seems to be known by some experts, though so far as we know it has not appeared in print, that some other factors of $H(e^{2\pi i/6}, n)$ give rise to image groups which are extensions of $SU(r + 1, F_2)$ by 2-groups. Other (extensions of) symplectic groups appear as quotients of the braid group; Wajnryb [W] has found explicit relations exhibiting $Sp(2r, F_p)$ as a quotient of $B_{2r+1}$ for all $p$. We would like to explain in some sense or at least characterize the possibilities for finite images in such representations. Such a characterization is given in Theorem [15].

It appears to be typically the case that the image of $B_n$ in $U(d)$ can be regarded as a linear group, whose rank is comparable to $n$, over a
finite field. We would like to systematically study all representations of $B_n$ of dimension $O(n)$ over all fields. Such a study has been initiated for complex representations of degree $\leq n$ by Formanek and his coworkers in [F, FLSV, Sy]. In Theorem 3.3, we extend these results to higher multiples of $n$, but only for unitary representations. For general representations, we have only the very soft result Theorem 2.10 which is used to relate $n$ and $r$ in Theorem 4.5.

2. Braid Groups

In this section we establish some basic facts concerning the braid groups $B_n$ and their representations in the general sense of homomorphisms $\phi: B_n \rightarrow G$ where $G$ is any group. Propositions 2.2 and 2.9 can be found in [F], but we include full proofs for the reader’s convenience.

For each braid group $B_n$ we fix generators $x_1, \ldots, x_{n-1}$ such that
\begin{align}
  x_ix_jx_i &= x_jx_ix_j \text{ if } |i-j| = 1, \\
  x_ix_j &= x_jx_i \text{ if } |i-j| \neq 1.
\end{align}

**Definition 2.1.** We say a homomorphism $\phi: B_n \rightarrow G$ is constant if $\phi(x_1) = \phi(x_2) = \cdots = \phi(x_{n-1})$.

**Proposition 2.2.** If $\phi: B_n \rightarrow G$ is a homomorphism and $\phi(x_i)$ commutes with $\phi(x_{i+1})$ for some $i \leq n-2$, then $\phi$ is constant.

**Proof.** Applying (2.1) when $j = i + 1$, we get
\[ \phi(x_i)^2\phi(x_{i+1}) = \phi(x_{i+1})^2\phi(x_i), \]
which implies $\phi(x_i) = \phi(x_{i+1})$. As $x_i$ commutes with $x_{i+2}$, $\phi(x_{i+1}) = \phi(x_i)$ commutes with $\phi(x_{i+2})$. By induction on $i$,
\[ \phi(x_i) = \phi(x_{i+1}) = \cdots = \phi(x_{n-1}). \]
Likewise, $\phi(x_{i-1})$ and $\phi(x_i)$ commute, so $\phi(x_i) = \phi(x_{i-1})$, and by downward induction,
\[ \phi(x_i) = \phi(x_{i-1}) = \cdots = \phi(x_1). \]

\[ \Box \]

**Corollary 2.3.** If $\phi(x_i) \in Z(G)$ for some $i$, then $\phi$ is constant.

**Corollary 2.4.** If $\phi(x_i) = \phi(x_{i+1})$ for some $i$, then $\phi$ is constant.

If $j \geq i$, we use the notation $X_{[i,j]}$ for the product $x_ix_{i+1}\cdots x_j$; if $j < i$, we define $X_{[i,j]}$ to be the identity.

**Lemma 2.5.** For $k \geq 3$ and $1 \leq i \leq k-2$, we have
\[ X_{[1,k]}x_ix_{[1,k]}^{-1} = x_{i+1}. \]
Lemma 2.6. If \(1 \leq i, j, k, l \leq n - 1, \ |i - j| \geq 2, \ |k - l| \geq 2\), then there exists \(z = z_{i,j,k,l} \in B_n\) such that
\[
zz^{-1} = x_k, \ z^{-1}z = z_l.
\]

Proof. First we assume \(i < j\) and \(k < l\). By Lemma 2.5 without loss of generality we may assume \(j = l = n - 1\). As \(X_{[1,n-3]}\) commutes with \(x_{n-1}\), the ordered pair \((x_i, x_{n-1})\) can be conjugated to \((x_{i+1}, x_{n-1})\) as long as \(1 \leq i \leq n - 4\). By induction on \(i\), all the \((x_i, x_{n-1})\) with \(i \leq n - 3\) are conjugate.

To treat the case that \(i > j\) or \(k > l\), it suffices to prove that \((x_1, x_3)\) can be conjugated to \((x_3, x_1)\). Letting
\[
y = x_1x_2x_3x_1x_2x_1 = x_1x_2x_1x_3x_2x_1,
\]
we have
\[
yx_1 = x_1x_2x_3x_1x_2x_1x_1 = x_1x_2x_3x_2x_1x_2x_1 = x_1x_3x_2x_3x_1x_2x_1 = x_3y; \quad yx_3 = x_1x_2x_1x_3x_2x_3x_1 = x_1x_2x_1x_3x_2x_2x_1 = x_1x_1x_2x_1x_3x_2x_1 = x_1y.
\]

\[\square\]

Now let \(0 \to A \to G \to H \to 0\) be a central extension. We write \([h_1, h_2] = 1\) for the commutator \(g_1g_2g_1^{-1}g_2^{-1} \in G\), where \(g_i\) is any element mapping to \(h_i\). As the extension is central, this is well-defined.

Lemma 2.7. If \(0 \to A \to G \to H \to 0\) is a central extension and \(\phi: B_n \to G\) is a homomorphism such that \(\pi \circ \phi\) is constant, then \(\phi\) is constant.

Proof. Any two elements of \(G\) which map to the same element of \(H\) must commute. The lemma therefore follows from Proposition 2.2.

\[\square\]

Proposition 2.8. If \(0 \to A \to G \to H \to 0\) is a central extension and \(\phi: B_n \to H\) is a homomorphism such that \([\phi(x_i), \phi(x_j)] = 1\) for some \(i, j\) with \(|i - j| \geq 2\), then \(\phi\) lifts to a homomorphism \(\phi: B_n \to G\).
Proof. As $[\cdot]^\sim$ respects conjugation, Lemma 2.6 implies
\[ [\phi(x_i), \phi(x_j)]^\sim = 1 \]
for all $i, j$ with $|i - j| \geq 2$. Fix an element $\tilde{x}_1 \in G$ with $\pi(\tilde{x}_1) = \phi(x_1)$ and an element $\tilde{y} \in G$ with $\pi(\tilde{y}) = \phi(X_{[1,n-1]})$. By Lemma 2.6,
\[ \pi(\tilde{y}^k \tilde{x}_1 \tilde{y}^{-k}) = \phi(x_{k+1}), \quad k = 0, 1, \ldots, n - 2. \]
Let
\[ g_i = \tilde{y}^{i-1} \tilde{x}_1 \tilde{y}^{1-i}. \]
Thus $g_i$ and $g_j$ commute when $|i - j| \neq 1$, and the elements
\[ a_i := g_i g_{i+1} g_i^{-1} g_{i+1}^{-1} \]
are all conjugate in $G$ and lie in $A$. Thus, they all coincide; denoting this common element $a$, and setting $\tilde{x}_i = a^i g_i$, we have $\pi(\tilde{x}_i) = \phi(x_i)$, and the $\tilde{x}_i$ satisfy the relations (2.1) and (2.2). Defining a homomorphism $\tilde{\phi}$ by the equations $\tilde{\phi}(x_i) = \tilde{x}_i$, we see that $\tilde{\phi}$ is a lift of $\phi$. \qed

**Proposition 2.9.** If $n \geq 6$, then every homomorphism from $B_n$ to a solvable group $G$ is constant.

**Proof.** We use induction on the length of the derived series. The proposition follows immediately from Corollary 2.3 when $G$ is abelian, so without loss of generality we may assume that the last non-trivial term $A$ in the derived series of $G$ is a proper subgroup of $G$. By the induction hypothesis, any homomorphism $B_n \to G/A$ is constant. We therefore choose an element $g \in G$ and a sequence $a_1, \ldots, a_{n-1} \in Z$ such that $\phi(x_i) = a_i g$ for $i = 1, \ldots, n - 1$. Writing $a^g$ for $g a g^{-1}$, we have
\[ a_i a_j^g g^2 = \phi(x_i x_j) = \phi(x_j x_i) = a_j a_i^g g^2 \]
and therefore
\[ a_i^{-1} a_i^g = a_j^{-1} a_j^g \]
whenever $|i - j| \geq 2$. The graph on the vertex set $\{1, 2, \ldots, n-1\}$ defined by the relation $|i - j| \geq 2$ is connected for $n \geq 6$. Thus,
\[ a_1^{-1} a_1^g = \cdots = a_{n-1}^{-1} a_{n-1}^g = a \]
for some $a \in A$. The braid relation (2.1) for $j = i + 1$ implies
\[ a^3 a_i^2 a_{i+1} = a_i a_{i+1} a_i^2 = a_{i+1} a_i^2 a_{i+1}^2 = a^3 a_i a_{i+1}^2, \]
so $a_1 = \cdots = a_{n-1}$, and $\phi$ is constant as claimed. \qed

**Theorem 2.10.** If $G$ is a linear algebraic group over a field $K$ with solvable component group, and $n \geq 6 \sqrt{\dim G} + 3$, then every homomorphism $B_n \to G(K)$ is constant.
Proof. We assume without loss of generality that \( K \) is algebraically closed. We prove by induction on \( r \) that the theorem is true whenever \( 2\sqrt{\dim G} \leq r \) and \( n \geq 3r \) the base case \( r = 1 \) being trivial. If \( r \geq 2 \), then \( n \geq 6 \), so the composition of \( \phi: B_n \rightarrow G(K) \) with the quotient map \( G(K) \rightarrow G(K)/G^o(K) \) is constant. We may therefore assume that \( G/G^o \) is cyclic. We may also assume that the theorem is true in dimension \(< \dim G \). If \( U \) denotes the unipotent radical of \( G^o \), then \( U \) is a normal algebraic subgroup of \( G \). If the composition homomorphism \( B_n \rightarrow (G/U)(K) \) is constant, then \( B_n \) maps to a solvable subgroup of \( G(K) \), namely, an extension of the (cyclic) image of this homomorphism by \( U(K) \). By Proposition 2.9, this implies that \( \phi \) is constant. Without loss of generality, therefore, we may assume that \( G^o \) is adjoint semisimple.

If there exist positive dimensional normal subgroups \( N_1, \ldots, N_t \) of \( G \) such that \( N_1(K) \cap \cdots \cap N_t(K) = \{1\} \), then the compositions of \( \phi \) with the projections \( G(K) \rightarrow (G/N_i)(K) \) are all constant, and therefore \( \phi \) is constant. If \( G^o \) has at least two non-isomorphic simple factors, then the product of all factors of any one type is a proper normal subgroup of \( G \). We may therefore assume that \( G^o \cong H^k \) for some positive integer \( k \) and some (adjoint) simple algebraic group \( H \). Moreover, conjugation by a generator of \( G/H^k \) induces a well-defined outer automorphism of \( H^k \) and therefore a permutation \( \sigma \) of the factors, which are the minimal non-trivial normal subgroups of \( H^k \). Without loss of generality we may assume that this permutation is a \( k \)-cycle, since otherwise, each orbit of \( \sigma \) determines a product of factors \( H \) which is a normal subgroup of \( G \).

Let \( x = \phi(x_{n-2})^{-1}\phi(x_{n-1}) \), and let \( B_{n-3} \) denote the subgroup of \( B_n \) generated by \( x_1, \ldots, x_{n-4} \). Thus, \( x \) lies in \( H(K)^k \) and \( \phi(B_{n-3}) \) lies in the centralizer of \( x \) in \( G(K) \). If \( x \) is the identity, then \( \phi \) is constant by Corollary 2.4. Therefore, the image of \( x \) in one of the factors of \( H(K)^k \) is non-trivial, so without loss of generality we may assume that the centralizer \( Z_x \subset G \) satisfies

\[
Z_x \cap H^k \subset H_1 \times H^{k-1},
\]

where \( H_1 \) is the centralizer of a non-trivial element \( x_1 \) of \( H \). Since every element \( x_1 \in H(K) \) has a Jordan decomposition, and the centralizer of \( x_1 \) is contained both in the centralizer of its unipotent part and in that of its semisimple part, without loss of generality we may assume that \( H_1 \) is the centralizer of either a semisimple or a unipotent element. We have seen that \( \phi(B_{n-3}) \) lies in \( Z_x(K) \). Suppose \( Z_x \) fails to meet every component of \( G \). Then the factors of \( H^k \) form \( t \geq 2 \) orbits of
equal cardinality under conjugation by $\mathbb{Z}_x$. Let $\mathcal{N}_i$ denote the algebraic subgroup of elements in $\mathbb{Z}_x \cap \mathcal{H}^k$ which are trivial on all factors of $\mathcal{H}$ belonging to the $i$th orbit of this action, it suffices to prove that the composition $\phi_i$ of $\phi|_{B_{n-3}}$ with the projection $\mathbb{Z}_x(K) \to (\mathbb{Z}_x/\mathcal{N}_i)(K)$ is constant for each $i$. However,

$$\dim \mathbb{Z}_x/\mathcal{N}_i \leq \frac{\dim \mathcal{H}^k}{t} = \frac{\dim \mathcal{G}}{t} \leq \frac{r^2}{8} \leq \frac{(r - 1)^2}{4}$$

since $\mathcal{G}$ non-trivial and semisimple implies $r \geq 4$. Thus, the induction hypothesis implies that each $\phi_i$ is constant and therefore that $\phi$ is constant.

This leaves the case that $t = 1$. In this case, $\mathbb{Z}_x \cap \mathcal{H}^k$ is contained in a product $\mathcal{H}_1 \times \cdots \times \mathcal{H}_k$ where each $\mathcal{H}_i$ is isomorphic to $\mathcal{H}_1$. If $\mathcal{R}_i$ denotes the radical of $\mathcal{H}_1$, then $\mathcal{R}_1 \times \cdots \times \mathcal{R}_k$ is a normal subgroup of $\mathbb{Z}_x$, and it suffices to prove that every homomorphism $B_n \to (\mathbb{Z}_x/(\mathcal{R}_1 \times \cdots \times \mathcal{R}_k))(K)$ is constant. The component group of $\mathbb{Z}_x$ is isomorphic to an extension of a cyclic group by $(\mathcal{H}_1/\mathcal{H}_1^\circ)^k$. If $\mathcal{H}_1$ is the centralizer of a semisimple element, the group of components is commutative. This is an immediate consequence of the theorem of Springer-Steinberg asserting that the centralizer of a semisimple element in a simply-connected semisimple group is connected. If $\mathcal{H}_1$ is the centralizer of a unipotent element, the component group $\mathcal{H}_1/\mathcal{H}_1^\circ$ is either solvable or isomorphic to $S_5$, by a theorem of Alexeevski [A] and Mizuno [M]. Except in the last case, which can only occur if $\mathcal{H} \cong E_8$, the component group of $\mathbb{Z}_x/(\mathcal{R}_1 \times \cdots \times \mathcal{R}_k)$ is solvable, so the induction hypothesis applies. Every automorphism of $S_5 \cong \text{PGL}_2(\mathbb{F}_3)$ is inner and therefore extends to an automorphism of the algebraic group PGL$_2$ in characteristic 5. In the case of $S_5$, $n \geq 6\sqrt{248k} + 3$, so $n - 3 \geq 6\sqrt{3k} + 3$. Thus, the induction hypothesis implies that any homomorphism from $B_{n-3}$ to an extension of $\mathbb{Z}/k\mathbb{Z}$ by $S_5^k$ is constant, and replacing $\mathbb{Z}_x$ with a suitable open subgroup, we may again assume that $\mathbb{Z}_x/\mathbb{Z}_x^\circ$ is solvable.

Finally, applying the classification of maximal subgroups of simple algebraic groups [Se1, Se2], we see that

$$2\sqrt{\dim(\mathcal{H}_1/\mathcal{R}_1)} < 2\sqrt{\dim \mathcal{H}} - 1.$$ 

The theorem now follows by induction.

A variant of this argument gives the following:
Theorem 2.11. If $H$ is a finite simple group, $G := H^k \rtimes C$, where $C$ is solvable, and $n \geq 3 \log_2 k |H|$, then every homomorphism $B_n \to G$ is constant.

Proof. If $|G| \leq 5$, then $|G|$ is abelian, so without loss of generality we may assume $n \geq 6$. Following the proof of Theorem 2.10, we may therefore assume that $C \cong \mathbb{Z}/k\mathbb{Z}$ cyclically permutes the factors of $H^k$. Given $\phi: B_n \to G$, we let $x = \phi(x_{n-2})^{-1} \phi(x_{n-1})$, and let $Z_x$ denote the centralizer of $x$ in $G$. As $x \in H^k$, we fix a factor on which the projection of $x$ is non-trivial, and let $H_1$ denote the centralizer of this projection of $x$ in this factor. If $Z_x$ does not map onto $C$, then there exist normal subgroups $N_1, \ldots, N_t$ of $Z_x$ with $N_1 \cap \cdots \cap N_t = \{1\}$ such that each $Z_x/N_i$ is contained in a group of the form $H^d \rtimes \mathbb{Z}/d\mathbb{Z}$, where $d$ is a proper divisor of $k$, so $\log_2 d \leq \log_2 k - 1$. By the induction hypothesis every homomorphism from $B_{n-3}$ to $Z_x/N_i$ is constant, so $\phi|_{B_{n-3}}$ is constant, and as $n \geq 6$, this means that $\phi(x_1) = \phi(x_2)$ and therefore that $\phi$ is constant.

If $Z_x$ does map onto $C$, then $Z_x$ is a subgroup of a group isomorphic to $H_1^k \rtimes C$, where $H_1$ is the centralizer of a non-trivial element of $H$. As $\log_2 |H_1| \leq \log_2 |H|$, the induction hypothesis implies that $\phi|_{B_{n-3}}$ is constant and therefore that $\phi$ is constant.

\[\square\]

3. Representations of Linearly Bounded Degree

In this section, we examine the possible degrees of low-dimensional unitary representations of a braid group $B_n$. The complex irreducible representations of degree $\leq n$ of $B_n$ have been completely described \cite{FLSV, Sy}. The constant representations have degree 1, and the non-constant representations in this range have degree $n-2$, $n-1$, or $n$. Sysoeva \cite{Sy} has announced that there are no irreducible representations of degree $n+1$ for $n$ sufficiently large, and has conjectured that such a statement holds for degree $n+k$ as well.

In this section, we consider the irreducible unitary representations of $B_n$ of degree $\leq ln$ where $l$ is a fixed integer and $n$ is sufficiently large in terms of $l$.

We say that a sequence $d_0, d_1, d_2, \ldots$ is weakly convex if the sequence of differences $d_1 - d_2, d_2 - d_3, \ldots$ is non-increasing.

Lemma 3.1. If $d_0, d_1, \ldots$ is a weakly convex sequence and $i < j < k$, then there exists an integer $s$ such that

$$\frac{d_j - d_i}{j - i} \leq s \leq \frac{d_k - d_j}{k - j}$$


Proof. Setting $s = d_{j+1} - d_j$, the lemma follows immediately. \hfill \Box

\textbf{Lemma 3.2.} Let $V$ be a finite-dimensional vector space, $W \subset V$ a subspace, and $T: V \to V$ an invertible linear transformation. The sequence $d_0, d_1, d_2, \ldots$ defined by $d_0 := \dim V$ and

$$d_k := \dim W \cap T(W) \cap T^2(W) \cap \cdots \cap T^{k-1}(W), \quad k \geq 1$$

is weakly convex.

\textit{Proof.} Define $W_0 = V$, and

$$W_k := W \cap T(W) \cap T^2(W) \cap \cdots \cap T^{k-1}(W), \quad k \geq 1.$$

Then

$$d_k - d_{k+1} = \dim W_k - \dim W_{k+1} = \dim W_k/W_{k+1}$$

As $T^{-1}$ maps to $W_{k+1}$ to $W_k$ and $W_{k+2}$ to $W_{k+1}$, it induces a map

$$W_{k+1}/W_{k+2} \to W_k/W_{k+1}.$$

As

$$W_{k+1} \cap T(W_{k+1}) = W_{k+2},$$

this linear transformation is injective, so

$$d_k - d_{k+1} \geq d_{k+1} - d_{k+2}.$$

\hfill \Box

We apply this lemma in the following way. Let $V$ be a finite-dimensional complex vector space endowed with a Hermitian inner product, and $\phi: B_n \to U(V)$ an irreducible unitary representation. For each $\lambda \in \mathbb{C}$, we define $W = W^\lambda$ to be the $\lambda$-eigenspace of $\phi(x_1)$. By Lemma 2.5, there exists $y \in B_n$ such that $yx_iy^{-1} = x_{i+1}$ for $1 \leq i \leq n - 2$. We set $T = \phi(y)$. Now, $w \in W^\lambda$, if and only if

$$(\phi(x_1) - \lambda)(w) = 0.$$

For any $k$, this is equivalent to

$$(\phi(y^{1-k}x_ky^{k-1}) - \lambda)(w) = 0,$$

or to

$$(\phi(x_k) - \lambda)(\phi(y^{k-1})(w)) = 0.$$

Thus, the $\lambda$-eigenspace of $\phi(x_k)$ is $T^{k-1}(W^\lambda)$.

We say that an irreducible representation $\phi: B_n \to U(m)$ is of \textit{level} $k$ if one of the following is true:

1. $k = 0$ and $m = 1$.
2. $k \geq 1$ and $kn - (k^2 + 3k - 2) \leq m \leq kn$. 


Theorem 3.3. For every integer \( l \geq 1 \) and every integer \( n \) sufficiently large in terms of \( l \), every irreducible unitary representation of the braid group \( B_n \) of degree \( \leq ln \) is of some (unique) level \( k \leq l \).

Proof. As

\[
(k - 1)n < kn - (k^2 + 3k - 2)
\]

when \( n \) is sufficiently large, uniqueness is clear. For existence, we use induction on \( l \), the \( l = 1 \) case being known \([FLSV]\). For given \( l \geq 2 \), let \( \phi : B_n \to \text{Aut}(V) \) be an irreducible unitary representation of degree \( \leq ln \). We may therefore assume that

\[
(l - 1)n + 1 \leq \dim V \leq ln - (l^2 + 3l - 1)
\]

We write \( B_{n-1} \) and \( B_{n-2} \) for the subgroups of \( B_n \) generated by \( x_i \) with \( 1 \leq i \leq n - 2 \) and \( 1 \leq i \leq n - 3 \) respectively.

For each eigenvalue \( \mu \) of \( \phi(x_{n-1}) \), let \( X^\mu \) denote the \( \mu \)-eigenspace. As \( B_{n-2} \) commutes with \( x_{n-1} \), \( \phi(B_{n-2}) \) acts on \( X^\mu \). We say that \( X^\mu \) splits if it is a direct sum of constant representations of \( B_{n-2} \). A sufficient condition that \( X^\mu \) splits is \( \dim X^\mu \leq n - 5 \), as the minimum degree of a non-constant representation of \( B_{n-2} \) is \( n - 4 \).

Let \( X \) denote the direct sum of all irreducible 1-dimension factors of \( B_{n-2} \) in \( V \), so \( X \) contains the sum of all split \( X^\mu \). Let \( \lambda_1, \ldots, \lambda_r \) be the constants appearing in \( X \) regarded as a \( B_{n-2} \)-representation, and let \( W_{\lambda_i} \) denote the \( \lambda_i \)-eigenspace of \( \phi(x_1) \) on \( V \), which of course contains the \( \lambda_i \)-eigenspace of \( \phi(x_1) \) on \( X \). Thus \( W_{\lambda_i} \) is the intersection of the \( \lambda_i \)-eigenspaces of \( \phi(x_1), \ldots, \phi(x_j) \). As \( W_{\lambda_i} = \{0\} \), Lemma 3.2 implies

\[
\dim W_{\lambda_i} \geq \frac{n - 1 - j}{2} \dim W_{\lambda_i}^{n-3},
\]

for \( 1 \leq j \leq n - 3 \). If \( \dim X \geq 2l + 1 \),

\[
ln \geq \sum_i \dim W_{\lambda_i}^{n-3} \geq \frac{n - 2}{2} \dim X \geq ln + (n/2 - 2l - 1),
\]

Assuming \( n > 4l + 2 \), we may therefore conclude that \( \dim X \leq 2l \).

We consider first the case that there are at least two different eigenvalues \( \mu_1 \) such that \( X^{\mu_1} \) does not split. For each \( \mu \in \{\mu_1, \ldots, \mu_r\} \), let \( X_{ns}^{\mu} \) denote the orthogonal complement in \( X^\mu \) of the direct sum of all constant representations of \( B_{n-2} \). Then

\[
\dim V - \dim X_{ns}^{\mu} \leq ln - (l^2 + 3l - 1) - (n - 4)
\]

\[
= (l - 1)(n - 2) - (l^2 + l - 3)
\]

\[
< (l - 1)(n - 2),
\]
so $\bigoplus_{\mu \neq \mu} X_{\mu}^{\mu}$ satisfies the induction hypothesis for representations of $B_{n-2}$, and the same is true of each irreducible factor of each $X_{\mu}^{\mu}$. Each irreducible factor of $X_{\mu}^{\mu}$ therefore has a level. Letting $k_1, k_2, \ldots, k_s \geq 1$ denote the sequence of levels, we have

$$\dim V = \dim X + \sum_{i=1}^{s} \dim X_{\mu_i}^{\mu_i},$$

so

$$(k_1 + \cdots + k_s)(n-2) - \sum_{i=1}^{s} (k_i^2 + 3k_i - 2) \leq \dim V \leq 2l + (k_1 + \cdots + k_s)(n-2)$$

For $n$ sufficiently large in terms of $l$, this, together with (3.1) implies $k_1 + \cdots + k_s = l$. As $x^2 + 3x - 2$ is convex, for any fixed values of $s \geq 2$ and $l$, the sum of $k_i^2 + 3k_i - 2$ is minimized, subject to the constraints $k_i \geq 1$ and $k_1 + \cdots + k_s = l$, when all but one value of $k_i$ is 1. As the difference between values of $x^2 + 3x - 2$ for consecutive positive integers exceeds the value at $x = 1$, if $s$ is constrained to be greater than 1 but otherwise can be chosen freely, the sum of $k_i^2 + 3k_i - 2$ is maximized when $s = 2$. Thus,

$$\dim V \geq (k_1 + \cdots + k_s)(n-2) - \sum_{i=1}^{s} (k_i^2 + 3k_i - 2)$$

$$\geq ln - 2l - (l - 1)^2 - 3(l - 1) + 2 - 2$$

$$= ln - (l^2 + 3l - 2).$$

This leaves the case that there exists a unique $\mu$ such that $X_{\mu}^{\mu}$ is not zero. Let $X_{\mu}^{\mu}$ denote the intersection of the $\mu$-eigenspaces of $x_{n-1}, x_{n-2}, \ldots, x_{n-i}$. By Lemma 3.2, applying Lemma 3.1 for $0 < i < j$,

$$\dim X_{\mu}^{\mu} - \dim X_{\mu}^{\mu} \leq (j - i) \left[ \frac{\dim V - \dim X_{\mu}^{\mu}}{i} \right].$$

If

$$\dim V - \dim X_{\mu}^{\mu} < l^2,$$

then setting $j = n - 1$ and $i = l$, we have

$$\dim V \leq l^2 - 1 + \dim X_{l}^{\mu} \leq l^2 - 1 + \dim X_{l}^{\mu} - \dim X_{n-1}^{\mu}$$

$$\leq l^2 - 1 + (n - l - 1) \left[ \frac{l^2 - 1}{l} \right]$$

$$= (l - 1)n,$$
which for \( n \) sufficiently large is inconsistent with (3.1). On the other hand,

\[
\dim V - \dim X^\mu \leq 2l,
\]

so

\[
\dim V - \dim X^\mu_i \leq 2l^2.
\]

Assuming that \( 2l^2 \leq n - l - 6 \), this implies that the orthogonal complement of \( X^\mu_i \) is a split representation of \( B_{n-l-1} \), the subgroup of \( B_n \) generated by \( x_1, \ldots, x_{n-l-2} \).

Let \( \lambda_i \) denote the eigenvalues of this representation. We have

\[
\sum_i \dim W_{n-l-2}^\lambda_i \geq l^2.
\]

On the other hand, \( \dim W_{n-1}^\lambda = 0 \). By Lemmas 3.1 and 3.2,

\[
\dim W_1^\lambda - \dim W_{n-l-2}^\lambda \geq (n - l - 3) \left[ \frac{\dim W_{n-l-2}^\lambda}{l+1} \right].
\]

As \( \lceil x/(l+1) \rceil \) is superadditive in \( x \) and \( \lceil l^2/(l+1) \rceil = l \),

\[
\sum_i \dim W_1^\lambda_i \geq \sum_i \dim W_{n-l-2}^\lambda_i + (n - l - 3) \left[ \frac{\dim W_{n-l-2}^\lambda}{l+1} \right] \\
\geq l^2 + (n - l - 3)l = nl - 3l,
\]

contrary to (3.1). \( \square \)

In particular by the proof of Theorem 3.3 we see that \( B_n \) has no irreducible \( (n+1) \)-dimensional unitary representations for \( n \geq 16 \). The actual lower bound is at least 8 as \( B_7 \) has irreducible 8-dimensional unitary representations (factoring over the Hecke algebra \( H(i,7) \), see [J1]).

Theorem 3.3 can be extended to projective unitary representations. In fact, we have the following proposition:

**Proposition 3.4.** Every irreducible projective unitary representation of \( B_n \) of degree \( d \leq 2n/6 \) lifts to a linear representation of \( B_n \).

**Proof.** The proposition is trivial for \( n \leq 5 \). We may therefore assume \( n \geq 6 \). Thus there exists a sequence \( a_1 < \cdots < a_{2m} \) of positive odd integers less than \( n \), with \( m \geq n/6 \). Let \( y_i = x_{a_i} \). The generators \( y_i \) commute with one another. The central extension

\[
0 \to U(1) \to U(d) \to PSU(d) \to 0
\]

defines a commutator map \( [ \ ] \). By Lemma 2.6, \( [\phi(x_i), \phi(x_j)] \) is independent of the pair \( (i,j) \) provided \( |i - j| \geq 2 \). It is therefore symmetric as well as antisymmetric and consequently takes values \( \pm 1 \). If
\[
\begin{align*}
\phi(x_i), \phi(x_j) &\sim 1 \text{ for some (and therefore all) } (i, j) \text{ with } |i - j| \geq 2, \\
\text{then by Proposition 2.7, } \phi \text{ lifts to a homomorphism to } U(m). \\
\text{We therefore assume that } [\phi(y_i), \phi(y_j)] &\sim -1 \text{ for all } i \neq j. \\
\text{Let } a_i = y_1y_2 \cdots y_{2i-1}, \quad b_i = y_1y_2 \cdots y_{2i-2}y_{2i}. \\
\text{Then } [a_i, a_j] &\sim [b_i, b_j] = 1, \quad [a_i, b_j] \sim (-1)^{\delta_{ij}}. \\
\end{align*}
\]

Let 
\[
G_i := \pi^{-1}(\phi(\langle a_i, b_i \rangle)).
\]

Clearly, the restriction of the standard representation of \(U(m)\) to \(G_i\) has no 1-dimensional components. The subgroups \(G_1, \ldots, G_m \subset U(d)\) commute in pairs and give rise to a homomorphism \(G_1 \times \cdots \times G_m \to U(d)\). The restriction of the standard representation of \(U(m)\) to this product decomposes as a sum of irreducible representations of \(G_1 \times \cdots \times G_m\), each of which is an external tensor product of representations of the \(G_i\), each of degree \(> 1\). Therefore, \(d \geq 2^m\).

4. Representations of Exponentially Bounded Degree

In this section we fix a constant \(c\) and consider non-constant unitary representations of \(B_n, n \geq 6\), of degree \(d \leq c^n\) with finite image. We are interested in the behavior of \(G := \rho(B_n)\). By Proposition 2.9, \(G\) cannot be solvable.

**Definition 4.1.** A finite group \(G\) is almost characteristically simple if there exists a (non-abelian) finite simple group \(H\) and a positive integer \(k\) such that \(H^k < G < \text{Aut}(H^k)\). We say \(G\) is of permutation type if \(H\) is isomorphic to the alternating group \(A_n\) for some \(n \geq 5\).

**Proposition 4.2.** If \(G\) is any finite group which is not solvable and \(K\) is maximal among normal subgroups of \(G\) such that \(G/K\) is not solvable, then \(G/K\) is almost characteristically simple.

**Proof.** Replacing \(G\) by \(G/K\), we may assume that \(G\) is not solvable but every non-trivial quotient group of \(G\) is. In particular, \(G\) has no non-trivial normal abelian subgroup. Let \(H^k\) denote a characteristically simple normal subgroup of \(G\). Thus, \(H\) is non-abelian. Let \(K\) denote the centralizer of \(H^k\) in \(G\). Then \(K\) is a normal subgroup. The quotient \(G/K\) is not solvable because it contains the homomorphic image of \(H^k\), which is isomorphic to \(H^k\) itself since the center of \(H\) is trivial. It follows that \(G/K\) is trivial and therefore that the action of \(G\) on \(H^k\) by conjugation is faithful, i.e., \(H^k \subset G \subset \text{Aut}(H^k)\).
**Definition 4.3.** If $G$ is a finite group which is not solvable, a minimal quotient is any group of the form $G/K$ where $K$ is maximal among normal subgroups of $G$ such that $G/K$ is not solvable.

**Definition 4.4.** A finite group is of classical type of rank $r$ if it is a finite simple group of the form $A_r(q), 2A_r(q), B_r(q), C_r(q), D_r(q), \text{ or } 2D_r(q)$.

Roughly speaking, a finite simple group is of classical type if it is a linear, unitary, orthogonal, or symplectic group over a finite field.

**Theorem 4.5.** For every constant $c$ there exist positive constants $A$, $B$, $K$, $N$, and $Q$ such that for all $n > N$ and all $\rho: B_n \to U(d)$ with $d \leq c^n$ and finite image $G$, every minimal quotient of $G$ is either of permutation type or of the form $H^k \rtimes \mathbb{Z}/m\mathbb{Z}$, where $H$ is a finite simple group of classical type of rank $r$. In the latter case, $1 \leq k \leq K$, $2 \leq q \leq Q$, and $An \leq r \leq Bn$.

**Proof.** A minimal quotient is of the form $H^k \rtimes C$, where $C$ is solvable and $H$ is simple. By hypothesis, $H$ is not an alternating group. By Theorem [2.11] if $n$ is sufficiently large, then $|H|$ can be taken to be as large as we wish; in particular, we exclude that case that $H$ is sporadic. By Theorem [2.10] if $n$ is sufficiently large and $H$ is of Lie type, the dimension of the underlying simple algebraic group must be $> \epsilon n^2$ for some absolute constant $\epsilon > 0$, so the rank $r$ of the group must be greater than $An$ for some absolute constant $A > 0$. Thus, we may assume that $H$ is a perfect group whose universal central extension is $\mathcal{H}(\mathbb{F})$, where $\mathcal{H}$ is a simply connected semisimple algebraic group over $\mathbb{F}$ which is absolutely simple modulo its center and of rank $r \geq 9$.

Let $G_0$ denote the inverse image of $H^k \subset H^k \rtimes C$ in $G$. We have a short exact sequence

$$0 \to J \to G_0 \to H^k \to 0,$$

which we pull back to a short exact sequence

$$(4.1) \quad 0 \to J \to \tilde{G}_0 \to \mathcal{H}(\mathbb{F})^k \to 0.$$ 

As $\tilde{G}_0$ is a central extension of $G_0$, the faithful representation $G_0 \to U(d)$ gives rise to an almost faithful $d$-dimensional representation of $\tilde{G}_0$. We claim that this implies that $d$ is greater than or equal to the degree of the minimal non-trivial representation of $\mathcal{H}(\mathbb{F})$. Let $X \subset \text{Hom}(Z(J), \mathbb{C}^\times)$ denote the set of characters obtained by restricting $\tilde{G}_0 \to U(d)$ to the abelian group $Z(J)$. Thus $\mathcal{H}(\mathbb{F})^k$ acts on $X$. If this action is non-trivial, then the permutation representation of $\mathcal{H}(\mathbb{F})^k$ acting on $X$ is non-trivial and therefore contains a non-trivial factor.
The minimal degree for a non-trivial representation of $\mathcal{H}(\mathbb{F})^k$ is the same as that for $\mathcal{H}(\mathbb{F})$. We may therefore assume that $\mathcal{H}(\mathbb{F})^k$ acts trivially on $X$. This implies that the action of $\mathcal{H}(\mathbb{F})^k$ on $Z(J)$ preserves both $Z(\tilde{G}_0) \subset Z(J)$ and $Z(\tilde{G}_0)/Z(J)$ pointwise. As $\mathcal{H}(\mathbb{F})^k$ is perfect, any action of this group on an abelian group which fixes a subgroup and quotient group pointwise is trivial. It follows that $Z(J)$ lies in the center of $\tilde{G}_0$. The non-abelian cohomology class which determines whether (4.1) splits lies in $H^2(\mathcal{H}(\mathbb{F})^k, J)$, which is a principal homogeneous space of $H^2(\mathcal{H}(K)^k, Z(J))$. The latter is trivial since $\mathcal{H}(K)^k$ is centrally closed. Therefore, $G_0$ contains a subgroup isomorphic to $\mathcal{H}(\mathbb{F})^k$, and restricting $V$ to this subgroup, we see that our claim holds.

The Seitz-Landazuri bound [LS] on the minimal degree projective representations of finite simple groups of Lie types now implies that $q^{rk/n}$ is bounded in terms of $c$. Given that $r/n > A$, this gives upper bounds $Q$ and $K$ for $q$ and $k$, and given that $q \geq 2$, $k \geq 1$, this gives an upper bound $B$ for $r/n$.

We remark that the theorem can be extended in two ways without essentially modifying the proof. On the one hand, we need not assume that the representation $V$ is unitary. On the other hand, if $V$ is unitary, we need not assume that $\rho(B_n)$ is finite; we can take the closure of the image, obtain a compact Lie group, and characterize the group of components of this Lie group without assuming that the identity component is trivial.

5. An Application

We would like to describe a general setting in which one obtains sequences of unitary representations of the braid group of exponentially bounded degree. Let $\mathcal{C}$ be any unitary premodular (=ribbon fusion) category (see [T, Chapter II.5]). In particular this means that $\mathcal{C}$ is semisimple with finitely many (isomorphism classes of) simple objects \{X_0, \cdots, X_r\} and the morphism spaces are finite dimensional \mathbb{C}-vector spaces. Moreover, such a category is equipped with a conjugation and a positive definite Hermitian form with respect to which each $\text{End}(X^{\otimes n})$ is a Hilbert space. The braiding isomorphisms $c_{X,Y} : X \otimes Y \cong Y \otimes X$ induce unitary representations $\rho_n^X : B_n \to U(\text{End}(X^{\otimes n}))$ via:

$$\rho_n^X(\sigma_i)f = \text{Id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \text{Id}_X^{\otimes n-i-1} \circ f$$

for any object $X$, where the $B_n$-invariance of the Hermitian form is included in the axioms. By semisimplicity of $\text{End}(X^{\otimes n})$ the spaces
Hom($X_j, X^{\otimes n}$) for simple $X_j$ are equivalent to (potentially reducible) unitary $B_n$ subrepresentations of $\text{End}(X^{\otimes n})$.

We will show that $\dim \text{Hom}(X_j, X^{\otimes n})$ is exponentially bounded. For simplicity of notation we assume that $X = X_i$ is a simple object and each object is isomorphic to its dual; the general case is essentially the same. For each simple object $X_i$ we define a (symmetric) matrix $N_i$ whose $(j, k)$-entry is $\dim \text{Hom}(X_k, X_i \otimes X_j)$. The matrices $N_i$, $0 \leq i \leq r$ pairwise commute, and are clearly non-negative. Let $d_i$ be the Perron-Frobenius eigenvalue of $N_i$, i.e. the largest eigenvalue. Setting $D = \max\{d_i\}$ we will show that $\dim \text{Hom}(X_j, X^{\otimes n}) \leq D^n$.

First observe that $d_i \geq 1$, since $|\lambda| \leq d_i$ for all other eigenvalues $\lambda$ and clearly $(N_i)^n \neq 0$ for all $n$. It follows from the Perron-Frobenius Theorem that the vector $d = (d_0, d_1, \ldots, d_r)^T$ is a strictly positive eigenvector with eigenvalue $d_i$ for each $N_i$, uniquely determined up to rescaling (one applies the Perron-Frobenius Theorem to the strictly positive matrix $M := \sum_i N_i$, see e.g. [ENO]). Now denoting by $e_i$ the $i$th standard basis vector for $\mathbb{R}^r$, we see that $\dim(X_j, X_i^{\otimes n})$ is the $j$th entry of $(N_i)^{n-1}e_i$ which is less than or equal to the $j$th entry of $(N_i)^{n-1}d = (d_i)^{n-1}d$ which in turn is bounded by $D^n$.

There are two well-known constructions of unitary premodular categories. The first is $\text{Rep}(D^\omega G)$: the representation category of the twisted quantum double of a finite group $G$. $D^\omega G$ is a semisimple $|G|^2$-dimensional quasi-triangular quasi-Hopf algebra (see [BK]), and $\text{Rep}(D^\omega G)$ is a modular category. The braid group representations were studied in [ERW] and found to have finite images. In particular the image of $\rho_n^H$ where $H = D^\omega G$ is the left regular representation of $D^\omega G$ is found to be a subgroup of the full monomial group $S_n \rtimes \mathbb{Z}_s^n$ for some $s$ and hence of permutation type. Since any simple object appears as a subobject of $H$, it follows that all images are of permutation type. The second set of examples come from representations of quantum groups at roots of unity (see e.g. [R]) or, equivalently, from fixed level representations of affine Kac-Moody algebras. Quantum groups of type $A_k$ at 4th and 6th roots of unity yield modular categories supporting braid group representations with finite images. In fact, these representations factor over quotients of Hecke algebras $H(q, n)$ and are precisely those alluded to in the introduction. Quantum groups of type $C_2$ at 10th roots of unity also yield finite braid group images [3], with images $\text{Sp}(n - 1, \mathbb{F}_5)$. Here the object $X$ of interest has $d_X = \sqrt{5}$, and for $B_n$ with $n$ odd, $\dim \text{End}(X^{\otimes n}) = (\sqrt{5})^{n-1}$ and is the metaplectic representation of $\text{Sp}(n - 1, \mathbb{F}_5)$ with two irreducible subrepresentations of dimension $\frac{(\sqrt{5})^{n-1} \pm 1}{2}$. It appears that this can be generalized: there
is evidence that quantum groups of type $B_k$ at $(4k+2)$th roots of unity and $D_k$ at $4k$th roots of unity support braid group representations with finite symplectic groups as images.

**References**

[A] A. Alexeevski: Component groups of the centralizers of unipotent elements in semisimple algebraic groups. *Lie groups and invariant theory*, 15–31, Amer. Math. Soc. Transl. Ser. 2, 213, Amer. Math. Soc., Providence, RI, 2005.

[BK] B. Bakalov and A. Kirillov, Jr.: *Lectures on Tensor Categories and Modular Functors*. University Lecture Series, 21, Amer. Math. Soc., 2001.

[BW] J. Birman and B. Wajnryb: Markov classes in certain finite quotients of Artin’s braid group. *Israel J. Math.* 56 (1986), no. 2, 160–178.

[ENO] P. Etingof, D. Nikshych, and V. Ostrik: On fusion categories. *Ann. of Math. (2)* 162 (2005), no. 2, 581–642.

[ERW] P. Etingof, E. C. Rowell, and S. J. Witherspoon: Braid group representations from quantum doubles of finite groups. *Pacific J. Math.* 234, no. 1 (2008), 33–41.

[FLW] M. H. Freedman, M. J. Larsen, and Z. Wang: The two-eigenvalue problem and density of Jones representation of braid groups. *Comm. Math. Phys.* 228 (2002), no. 1, 177–199.

[F] E. Formanek: Braid group representations of low degree. *Proc. London Math. Soc. (3)* 73 (1996), no. 2, 279–322.

[FLSV] E. Formanek, W. Lee, I. Sysoeva, and M. Vazirani: The irreducible complex representations of the braid group on $n$ strings of degree $\leq n$. *J. Algebra Appl.* 2 (2003), no. 3, 317–333.

[GJ] D. M. Goldschmidt and V. F. R. Jones: Metaplectic link invariants. *Geom. Dedicata* 31 (1989), no. 2, 165–191.

[J1] V. F. R. Jones: Braid groups, Hecke algebras and type II1 factors. *Geometric methods in operator algebras (Kyoto, 1983)*, 242–273, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, 1986.

[J2] V. F. R. Jones: *Subfactors and Knots*. CBMS Regional Conference Series in Mathematics, 80. American Mathematical Society, Providence, RI, 1991.

[J3] V. F. R. Jones: On a certain value of the Kauffman polynomial. *Comm. Math. Phys.* 125 (1989), no. 3, 459–467.

[LS] V. Landazuri and G. M. Seitz: On the minimal degrees of projective representations of the finite Chevalley groups. *J. Algebra* 32 (1974), 418–443.

[LRW] M. J. Larsen, E. C. Rowell, and Z. Wang: The $N$-eigenvalue problem and two applications. *Int. Math. Res. Not.* 2005 (2005), no. 64, 3987–4018.

[M] K. Mizuno: The conjugate classes of unipotent elements of the Chevalley groups $E_7$ and $E_8$. *Tokyo J. Math.* 3 (1980), no. 2, 391–461.

[R] E. C. Rowell: From quantum groups to unitary modular tensor categories. in *Contemp. Math.* 413 (2006), 215–230.

[Se1] G. M. Seitz: The maximal subgroups of classical algebraic groups. *Mem. Amer. Math. Soc.* 67 (1987), no. 365.
G. M. Seitz: Maximal subgroups of exceptional algebraic groups. *Mem. Amer. Math. Soc.* 90 (1991), no. 441.

R. Steinberg: *Endomorphisms of linear algebraic groups.* Memoirs of the American Mathematical Society, No. 80.

I. Sysoeva: Dimension $n$ representations of the braid group on $n$ strings. *J. Algebra* 243 (2001), no. 2, 518–538.

V. G. Turaev: *Quantum Invariants of Knots and 3-Manifolds.* de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994.

B. Wajnryb: A braidlike presentation of $\text{Sp}(n,p)$. *Israel J. Math.* 76 (1991), no. 3, 265–288.

E-mail address: mjlarsen@indiana.edu

Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.

E-mail address: rowell@math.tamu.edu

Department of Mathematics, Texas A&M University, College Station, TX 77843, U.S.A.