Refinements of Dyck Paths with Flaws

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Abstract

The classical Chung-Feller theorem [2] tells us that the number of Dyck paths of length \( n \) with \( m \) flaws is the \( n \)-th Catalan number and independent on \( m \). In this paper, we consider the refinements of Dyck paths with flaws by four parameters, namely peak, valley, double descent and double ascent. Let \( p_{n,m,k} \) be the number of all the Dyck paths of semi-length \( n \) with \( m \) flaws and \( k \) peaks. First, we derive the reciprocity theorem for the polynomial \( P_{n,m}(x) = \sum_{k=1}^{n} p_{n,m,k}x^k \). Then we find the Chung-Feller properties for the sum of \( p_{n,m,k} \) and \( p_{n,m,n-k} \). Finally, we provide a Chung-Feller type theorem for Dyck paths of length \( n \) with \( k \) double ascents: the number of all the Dyck paths of semi-length \( n \) with \( m \) flaws and \( k \) double ascents is equal to the number of all the Dyck paths that have semi-length \( n \), \( k \) double ascents and never pass below the \( x \)-axis, which is counted by the Narayana number. Let \( v_{n,m,k} \) (resp. \( d_{n,m,k} \)) be the number of all the Dyck paths of semi-length \( n \) with \( m \) flaws and \( k \) valleys (resp. double descents). Some similar results are derived.

Keyword: Chung-Feller Theorem; Double ascent; Dyck path; Narayana number; Peak; Reciprocity

1 Introduction

Let \( Z \) denote the set of the integers. We consider \( n \)-Dyck paths in the plane \( Z \times Z \) using \textit{up} \((1,1)\) and \textit{down} \((1,-1)\) steps that go from the origin to the point \((2n,0)\). We say \( n \) the \textit{semilength} because

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there are $2n$ steps. Define $L_n$ as the set of all $n$-Dyck paths. Let $\mathcal{L} = \bigcup_{n \geq 0} L_n$. A $n$-flawed path is a $n$-Dyck path that contains some steps under the $x$-axis. The number of $n$-Dyck path that never pass below the $x$-axis is the $n$-th Catalan number $c_n = \frac{1}{n+1} \binom{2n}{n}$. Such paths are called the Catalan paths of length $n$. The generating function $C(z) := \sum_{n \geq 0} c_n z^n$ satisfies the functional equation $C(z) = 1 + zC(z)^2$ and $C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$ explicitly.

A Dyck path is called a $(n,m)$-flawed path if it contains $m$ up steps under the $x$-axis and its semilength is $n$. Clearly, $0 \leq m \leq n$. Let $\mathcal{L}_{n,m}$ denote the set of all the $(n,m)$-flawed paths and $l_{n,m} = |\mathcal{L}_{n,m}|$. The classical Chung-Feller theorem [2] says that $l_{n,m} = c_n$ for $0 \leq m \leq n$.

We can consider an $(n,m)$-flawed path $P$ as a word of $2n$ letters using only $U$ and $D$. In this word, let $P_i$ denote the $i$-th ($1 \leq i \leq 2n$) letter from the left. If a joint node in the Dyck path is formed by a up step followed by a down step, then this node is called a peak; if a joint node in the Dyck path is formed by a down step followed by a up step, then this node is called a valley; if a joint node in the Dyck path is formed by a up step followed by a up step, then this node is called a double ascent; if a joint node in the Dyck path is formed by a down step followed by a up step, then this node is called a double descent.

Define $\mathcal{P}_{n,m,k}$ (resp. $\mathcal{V}_{n,m,k}$) as the set of all the $(n,m)$-flawed paths with $k$ peaks (resp. valleys). Let $p_{n,m,k} = |\mathcal{P}_{n,m,k}|$ and $v_{n,m,k} = |\mathcal{V}_{n,m,k}|$. We also define $\mathcal{A}_{n,m,k}$ (resp. $\mathcal{D}_{n,m,k}$) as the set of $(n,m)$-flawed path with $k$ double ascents (resp. $k$ double descents). Let $a_{n,m,k} = |\mathcal{A}_{n,m,k}|$ and $d_{n,m,k} = |\mathcal{D}_{n,m,k}|$. Let $\varepsilon$ be a mapping from the set $\{U, D\}$ to itself such that $\varepsilon(U) = D$ and $\varepsilon(D) = U$. Furthermore, for any path $P = P_1 P_2 \ldots P_{2n} \in \mathcal{P}_{n,m,k}$, let $\phi(P) = \varepsilon(P_1)\varepsilon(P_2)\ldots\varepsilon(P_{2n})$. It is easy to see that $\phi$ is a bijection between the sets $\mathcal{P}_{n,m,k}$ and $\mathcal{V}_{n,n-m,k}$. For any $P = P_1 P_2 \ldots P_{2n} \in \mathcal{A}_{n,m,k}$, let $\psi(P) = \varepsilon(P_{2n})\varepsilon(P_{2n-1})\ldots\varepsilon(P_1)$. Clearly, $\psi$ is a bijection from the set $\mathcal{A}_{n,m,k}$ to the set $\mathcal{D}_{n,m,k}$. Hence, in this paper, we focus on the polynomials $P_{n,m}(x) = \sum_{k=1}^{n} p_{n,m,k} x^k$ and
$A_{n,m}(x) = \sum_{k=0}^{n-1} a_{n,m,k} x^k$. Table 1 shows the polynomials $P_{n,m}(x)$ for small values of $n$ and $m.$

| $(n, m)$ | $P_{n,m}(x)$ | $(n, m)$ | $P_{n,m}(x)$ |
|----------|-------------|----------|--------------|
| $(1, 0)$ | $x$         | $(5, 0)$ | $x^5 + 10x^4 + 20x^3 + 10x^2 + x$ |
| $(1, 1)$ | $1$         | $(5, 1)$ | $5x^4 + 20x^3 + 15x^2 + 2x$ |
| $(2, 0)$ | $x^2 + x$   | $(5, 2)$ | $4x^4 + 18x^3 + 17x^2 + 3x$ |
| $(2, 1)$ | $2x$        | $(5, 3)$ | $3x^4 + 17x^3 + 18x^2 + 4x$ |
| $(2, 2)$ | $x + 1$     | $(5, 4)$ | $2x^4 + 15x^3 + 20x^2 + 5x$ |
| $(3, 0)$ | $x^3 + 3x^2 + x$ | $(5, 5)$ | $x^4 + 10x^3 + 20x^2 + 10x + 1$ |
| $(3, 1)$ | $3x^2 + 2x$ | $(6, 0)$ | $x^6 + 15x^5 + 50x^4 + 50x^3 + 15x^2 + x$ |
| $(3, 2)$ | $2x^2 + 3x$ | $(6, 1)$ | $6x^5 + 40x^4 + 60x^3 + 24x^2 + 2x$ |
| $(3, 3)$ | $x^2 + 3x + 1$ | $(6, 2)$ | $5x^5 + 35x^4 + 60x^3 + 29x^2 + 3x$ |
| $(4, 0)$ | $x^4 + 6x^3 + 6x^2 + x$ | $(6, 3)$ | $4x^5 + 32x^4 + 60x^3 + 32x^2 + 4x$ |
| $(4, 1)$ | $4x^3 + 8x^2 + 2x$ | $(6, 4)$ | $3x^5 + 29x^4 + 60x^3 + 35x^2 + 5x$ |
| $(4, 2)$ | $3x^3 + 8x^2 + 3x$ | $(6, 5)$ | $2x^5 + 24x^4 + 60x^3 + 40x^2 + 6x$ |
| $(4, 3)$ | $2x^3 + 8x^2 + 4x$ | $(6, 6)$ | $x^5 + 15x^4 + 50x^3 + 50x^2 + 15x + 1$ |
| $(4, 4)$ | $x^3 + 6x^2 + 6x + 1$ |                |              |

Table 1. The polynomials $D_{n,m}(x)$ for small values of $n$ and $m.$

From the classical Chung-Feller theorem, we have $P_{n,m}(1) = A_{n,m}(1) = c_n$ for $0 \leq m \leq n$. The classical Chung-Feller theorem was proved by using analytic method in [2]. T.V.Narayana [6] showed the theorem by combinatorial methods. S.P.Eu et al. [4] studied the theorem by using the Taylor expansions of generating functions and proved a refinement of this theorem. Y.M. Chen [1] revisited the theorem by establishing a bijection. Recently, Shu-Chung Liu et al. [5] use an unify algebra approach to prove chung-Feller theorems for Dyck path and Motzkin path and develop a new method to find some combinatorial structures which have the Chung-Feller property. However, the macroscopical structures should be supported by some microcosmic structures. We want to find the Chung-Feller phenomenons in the more exquisite structures.
Richard Stanley’s book [7], in the context of rational generating functions, devotes an entire section to exploring the relationships (called reciprocity relationships) between positively- and nonpositively-indexed terms of a sequence. First, we give the reciprocity theorem for the polynomial $P_{n,m}(x)$. Particularly, we prove that the number of Dyck paths of semi-length $n$ with $m$ flaws and $k$ peaks is equal to the number of Dyck paths of semi-length $n$ with $n-m$ flaws and $n-k$ peaks.

One observes that the sum of $p_{n,m,k}$ and $p_{n,m,n-k}$ are independent on $m$ for any $1 \leq m \leq n-1$ and $1 \leq k \leq \lfloor n/2 \rfloor$ in Table 1. This is proved in Theorem 3.2 by using the algebra methods. Given $n$ and $k$, we also show that the polynomials $A_{n,m}(x)$ have the Chung-Feller property on $m$. Particularly, we conclude that the number of all the Dyck paths of semi-length $n$ with $m$ flaws and $k$ double ascents is equal to the number of all the Dyck paths that have semi-length $n$, $k$ double ascents and never pass below the $x$-axis, which is counted by the Narayana number. So, the Classical Chung-Feller theorem can be viewed as the direct corollary of this result.

This paper is organized as follows. In Section 2, we will prove the reciprocity theorem for the polynomial $P_{n,m}(x)$. In Section 3, we will show that $p_{n,m,k} + p_{n,m,n-k}$ have the Chung-Feller property on $m$ for any $1 \leq m \leq n-1$ and $1 \leq k \leq \lfloor n/2 \rfloor$. In Section 4, we will prove that the polynomials $A_{n,m}(x)$ have the Chung-Feller property on $m$.

2 The reciprocity theorem for the polynomial $P_{n,m}(x)$

In this section, first, define the generating functions $P_m(x,z) = \sum_{n \geq m} P_{n,m}(x)z^n$. When $m = 0$, $p_{n,0,k} = \frac{1}{k}(\frac{n-1}{k-1})$ is the Narayana numbers. It is well known that

$$P_0(x,z) = 1 + P_0(x,z)[x + P_0(x,z) - 1],$$

equivalently,

$$P_0(x,z) = \frac{1 + (1-x)z - \sqrt{1 - 2(1+x)z + (1-x)^2z^2}}{2z}.$$ 

Similarly, let $V_m(x,z) = \sum_{n,k \geq 0} v_{n,m,k}x^kz^n$. It is easy to obtain

$$V_0(x,z) = 1 + z + z[V_0(x,z) - 1](1 + xV_0(x,z)), $$

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equivalently,
\[ V_0(x, z) = \frac{1 - (1 - x)z - \sqrt{1 - 2(1 + x)z + (1 - x)^2z^2}}{2zx}. \]

In fact, we have \( v_{n,0,k} = p_{n,0,k+1} \) since the number of the valleys is equal to the number of the peaks minus one for each Catalan path. So, \( P_n(x, z) = V_0(x, z). \)

Now, let \( P(x, y, z) = \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{k=1}^{n} p_{n,m,k} x^k y^m z^n \). Let \( P \in \mathcal{L} \) contain some step over \( x \)-axis. We decompose \( P \) into \( P_1 UP_2 DP_3 \), where \( U \) and \( D \) are the first up and down steps leaving and returning to \( x \)-axis and on \( x \)-axis respectively. Note that all the steps of \( P_1 \) are below \( x \)-axis, \( P_2 \) is a Catalan path and \( P_3 \in D \). If \( P_2 = \emptyset \), then we get a peak \( UD \). So, we obtain the following lemma.

**Lemma 2.1**

\[ P(x, y, z) = V_0(x, yz) \{1 + z[x + P_0(x, z) - 1]P(x, y, z)\}. \]

Equivalently,
\[ P(x, y, z) = \frac{2}{\sqrt{f(x, z)} + \sqrt{f(x, yz)} + (1 - x)(1 - y)z} \]

where \( f(x, y) = 1 - 2(1 + x)y + (1 - x)^2y^2 \).

We state the reciprocity relationships for the polynomials \( P_{n,m}(x) \) as the following theorem.

**Theorem 2.2** Let \( n \geq 1 \). \( P_{n,m}(x) = x^n P_{n,n-m} \left( \frac{1}{x} \right) \) for all \( 0 \leq m \leq n \). Equivalently, \( p_{n,m,k} = p_{n,n-m,n-k} \)

**Proof.** Let \( f(x, y) = 1 - 2(1 + x)y + (1 - x)^2y^2 \). Note that (1) \( f(x^{-1}, xyz) = f(x, yz) \); (2) \( f(x^{-1}, xz) = f(x, z) \); and (3) \( (1 - x^{-1})(1 - y^{-1})xyz = (1 - x)(1 - y)z \).

By Lemma 2.1, we have
\[ P(x, y, z) = P(x^{-1}, y^{-1}, xyz). \]
Since $P(x, y, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} P_{n,m}(x)y^{m}z^{n}$, we have

$$P(x^{-1}, y^{-1}, xyz) = 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} P_{n,m}(\frac{1}{x})y^{-m}(xyz)^{n}$$

$$= 1 + \sum_{n=1}^{\infty} x^{n} P_{n,m}(\frac{1}{x})y^{n-m}z^{n}$$

$$= 1 + \sum_{n=1}^{\infty} x^{n} P_{n,n-m}(\frac{1}{x})y^{m}z^{n}.$$ 

This implies $P_{n,m}(x) = x^{n}P_{n,n-m}(\frac{1}{x})$ for all $0 \leq m \leq n$. Comparing the coefficients on the sides of the identity, we derive $p_{n,m,k} = p_{n-n,m,k}$.

Recall that $v_{n,m,k}$ is the number of Dyck paths of semi-length $n$ with $m$ flaws and $k$ valleys and $v_{n,m,k} = p_{n,n-m,k}$.

**Corollary 2.3** Let $n \geq 1$. Then $v_{n,m,k} = v_{n,n-m,n-k}$.

### 3 The refinement of $(n, m)$-flawed paths obtained by peak

In this section, we will consider the refinement of $(n, m)$-flawed paths obtained by peak and prove that the values of $p_{n,m,k} + p_{n,n-m-k}$ have the Chung-Feller property on $m$ for any $1 \leq m \leq n-1$ and $1 \leq k \leq \lceil \frac{n}{2} \rceil$.

**Lemma 3.1**

$$P_{1}(x, z) = (1 + z - xz)P_{0}(x, z) - 1.$$ 

Furthermore, we have

$$p_{n,1,k} = \frac{2(n-k)}{n(n-1)} \binom{n}{k-1} \binom{n}{k}$$

for any $n \geq 2$.

**Proof.** Let $P$ be a Dyck path containing exact one up step under the $x$-axis. Then we can decompose the path $P$ into $P_{1}DP_{2}$, where $P_{1}$ and $P_{2}$ are both Catalan paths. So, $P_{1}(x, z) = z[P_{0}(x, z)]^{2}$. Hence, we have $P_{1}(x, z) = (1 + z - xz)P_{0}(x, z) - 1$ since $P_{0}(x, z) = 1 + P_{0}(x, z)z[x + P_{0}(x, z) - 1]$. 


Note that $P_0(x, z) = 1 + \sum_{n \geq 1} \sum_{k=1}^{n} p_{n,0,k} x^k z^n$, where $p_{n,0,k} = \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k}$. Therefore,

$$p_{n,1,k} = \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k} + \frac{1}{k} \binom{n-2}{k-1} \binom{n-1}{k-1} - \frac{1}{k-1} \binom{n-2}{k-2} \binom{n-1}{k-2}$$

$$= \frac{2(n-k)}{n(n-1)} \binom{n}{k} \binom{n}{k+1}.$$

\[ \square \]

**Theorem 3.2** Let $n$ be an integer with $n \geq 1$ and $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. Then

$$p_{n,m,k} + p_{n,m,n-k} = p_{n,m,k} + p_{n,n-m,k} = \frac{2(n+2)}{n(n-1)} \binom{n}{k+1}$$

for any $1 \leq m \leq n-1$.

**Proof.** Theorem 2.2 implies that $p_{n,m,k} + p_{n,m,n-k} = p_{n,m,k} + p_{n,n-m,k}$. We consider the generating function $R(x,y,z) = \sum_{n \geq 1} \sum_{m=1}^{n-1} \sum_{k=1}^{n-1} (p_{n,m,k} + p_{n,n-m,k}) x^k y^m z^n$. It is easy to see

$$R(x,y,z) = P(x,y,z) + P(x,y^{-1},yz) + 2$$

$$-[V_0(x,z) + V_0(x,yz)] - [P_0(x,z) + P_0(x,yz)]$$

Let $\alpha(x,z) = \frac{1 + x - (1-x)z}{x} P_0(x,z) - \frac{P_0(x,z)}{V_0(x,z)} - \frac{1}{x}$. Then

$$R(x,y,z) = \frac{y \alpha(x,z) - \alpha(x,yz)}{1-y}.$$ 

Suppose $\alpha(x,z) = \sum_{n \geq 1} \sum_{k=1}^{n-1} a_{k,n} x^k z^n$. Then

$$R(x,y,z) = \sum_{n \geq 1} \sum_{k=1}^{n-1} a_{k,n} x^k y^m z^n \frac{1 - y^{n-1}}{1-y}$$

$$= \sum_{n \geq 1} \sum_{m=1}^{n-1} \sum_{k=1}^{n-1} a_{k,n} x^k y^m z^n.$$
Hence, given \( n \geq 1 \) and \( 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \), we have \( p_{n,m,k} + p_{n,n-m,k} = p_{n,n-k} + p_{n,m,n-k} = a_{k,n} \) for all \( 1 \leq m \leq n-1 \). By Lemma 3.1, we have

\[
\begin{align*}
p_{n,m,k} + p_{n,n-m,k} &= p_{n,1,k} + p_{n,1,n-k} \\
&= \frac{2(n+2)}{n(n-1)} \binom{n}{k-1} \binom{n}{k+1}.
\end{align*}
\]

**Corollary 3.3** Let \( n \) be an integer with \( n \geq 1 \). Then

\[
p_{2n,m,n} = \frac{1}{2n-1} \binom{2n}{n} \binom{2n}{n}
\]

for any \( 1 \leq m \leq 2n-1 \).

Note that \( v_{n,m,k} = p_{n,n-m,k} \). We obtain the following corollaries.

**Corollary 3.4** Let \( n \) be an integer with \( n \geq 1 \) and \( 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \). Then

\[
v_{n,m,k} + v_{n,m,n-k} = v_{n,m,k} + v_{n,n-m,k} = \frac{2(n+2)}{n(n-1)} \binom{n}{k-1} \binom{n}{k+1}
\]

for any \( 1 \leq m \leq n-1 \).

**Corollary 3.5** Let \( n \) be an integer with \( n \geq 1 \). Then

\[
v_{2n,m,n} = \frac{1}{2n-1} \binom{2n}{n} \binom{2n}{n}
\]

for any \( 1 \leq m \leq 2n-1 \).

In the following theorem, we derive a recurrence relation for the polynomial \( P_{n,m}(x) \).

**Theorem 3.6** For any \( m, r \geq 0 \), we have

\[
P_{m+r,m}(x) = \begin{cases} 
1 & \text{if } (m,r) = (0,0) \\
\sum_{k=1}^{m} \binom{m-1}{k-1} \binom{m}{k} x^{k-1} & \text{if } r = 0 \text{ and } m \geq 1 \\
x \sum_{i=0}^{m} \sum_{j=0}^{r-1} P_{m-i,m-i}(x) P_{r-j-1,r-j-1}(x) P_{j+i,i}(x) & \text{if } r \geq 1 
\end{cases}
\]
Proof. It is trivial for the case with \( r = 0 \). We only consider the case with \( r \geq 1 \). Note that
\[ x + P_0(x, z) - 1 = xV_0(x, z). \]
Lemma 2.1 tells us that
\[ P(x, y, z) = V_0(x, yz) + xzV_0(x, z)V_0(x, yz)P(x, y, z). \]

It is well known that
\[ V_0(x, z) = \sum_{n \geq 0} b_n(x)z^n, \]
where \( b_0(x) = 1 \) and
\[ b_n(x) = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k} x^{k-1} \]
for all \( n \geq 1 \). Comparing the coefficients of \( y^m \) on both sides of Identity (1), we get
\[ P_m(x, z) = b_m(x)z^m + xzV_0(x, z) \sum_{i=0}^{m} P_i(x, z)b_{m-i}(x)z^{m-i}. \]

Finally, since \( P_m(x, z) = \sum_{n \geq m} P_{n,m}(x)z^n \), comparing the coefficients of \( z^n \) on both sides of Identity (2), we obtain
\[ P_{m,m}(x) = b_m(x), \]
and
\[ P_{n,m}(x) = x \sum_{i=0}^{m} \sum_{j=i}^{n-m+i-1} b_{m-i}(x)b_{n-m+i-j-1}(x)P_{j,i}(x). \]

This completes the proof.

4 The refinement of \((n, m)\)-flawed paths obtained by double ascent

In this section, we will consider the refinement of \((n, m)\)-flawed paths obtained by double ascent and prove the value of \( a_{n,m,k} \) have the Chung-Feller property on \( m \). Define the generating functions
\[ A_m(x, z) = \sum_{n \geq m} A_{n,m}(x)z^n. \]
When \( m = 0 \), \( a_{n,0,k} = \frac{1}{k+1} \binom{n-1}{k} \binom{n}{k} \). It is well known that
\[ A_0(x, z) = 1 + \frac{zA_0(x, z)}{1 - xzA_0(x, z)}, \]
equivalently,
\[ A_0(x, z) = \frac{1 + (x - 1)z - \sqrt{(1 + xz - z)^2 - 4xz}}{2xz}. \]

Define a generating function \( A(x, y, z) = \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{k=1}^{n} a_{n,m,k}x^ky^n z^n. \)

Lemma 4.1
\[ A(x, y, z) = \frac{A_0(x, z)A_0(x, yz)}{1 - x[A_0(x, z) - 1][A_0(x, yz) - 1]}. \]
Proof. Let the mapping $\phi$ be defined as that in Introduction. An alternating Catalan path is a Dyck path which can be decomposed into $RT$, where $R \neq \emptyset$ and $T \neq \emptyset$, such that $\phi(R)$ and $T$ are both Catalan paths.

Now, Let $P \in \mathcal{D}$. We can uniquely decompose $P$ into $PQ_1 \ldots Q_mR$ such that $P$ and $\phi(R)$ are Catalan paths and $Q_i$ is the alternating Catalan path for all $i$. Hence,

$$A(x, y, z) = A_0(x, z) \left( \sum_{m \geq 0} x[A_0(x, z) - 1][A_0(x, yz) - 1] \right) A_0(x) yz$$

$$= \frac{A_0(x, z)A_0(x, yz)}{1 - x[A_0(x, z) - 1][A_0(x, yz) - 1]}.$$

Theorem 4.2 Let $n$ be an integer with $n \geq 0$ and $0 \leq k \leq n - 1$. Then

$$a_{n, m, k} = \frac{1}{k+1} \binom{n-1}{k} \binom{n}{k}$$

for any $0 \leq m \leq n$.

Proof. First, we give an algebra proof of this theorem. Since $xz[A_0(x, z)]^2 = A_0(x, z)[1+xz-z]-1$, simple calculations tell us

$$z \{1 - x[A_0(x, z) - 1][A_0(x, yz) - 1]\} [yA_0(x, yz) - A_0(x, z)]$$

$$= z(y - 1)A_0(x, z)A_0(x, yz).$$

By Lemma 4.1, we have

$$A(x, y, z) = \frac{A_0(x, z)A_0(x, yz)}{1 - x[A_0(x, z) - 1][A_0(x, yz) - 1]}$$

$$= \frac{yA_0(x, yz) - A_0(x, z)}{y - 1}$$

$$= \sum_{n \geq 0} \sum_{m=0}^n A_{n,0}(x)y^m z^n.$$ 

This implies $A_{n,m}(x) = A_{n,0}(x)$ for any $0 \leq m \leq n$. Therefore, $a_{n,m,k} = a_{n,0,k} = \frac{1}{k+1} \binom{n-1}{k} \binom{n}{k}$. 

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Now, we give a bijection proof of this theorem. Let $P$ be a path of semi-length $n$ with $m$ flaws and $k$ double ascent, where $0 \leq m \leq n - 1$. We say that a Catalan path is prime if the path touches $x$-axis exact twice. We can decompose $P$ into $SRUQDT$ such that

1. $UQD$ is the right-most prime Catalan path in $P$
2. $\phi(R)$ is a Catalan path, where $\phi$ is defined as that in Introduction, and
3. the final step of $S$ is $D$ on $x$-axis or $S = \emptyset$.

It is easy to see $\phi(T)$ is a Catalan path. We define a path $\varphi(P)$ as

$$\varphi(P) = STDRUQ.$$  

Clearly, the number of double ascents in $\varphi(P)$ is equal to the number of double ascents in $P$ and the number of flaws in $\varphi(P)$ is $m + 1$.

To prove the mapping $\varphi$ is a bijection, we describe the inverse $\varphi^{-1}$ of the mapping $\varphi$ as follows: Let $P'$ be a path of semi-length $n$ with $m + 1$ flaws and $k$ double ascent, where $0 \leq m \leq n - 1$. We can decompose $P'$ into $STDRUQ$ such that

1. $D$ and $U$ are the right-most steps leaving and returning to the $x$-axis steps and under the $x$-axis in $P'$;
2. $\phi(T)$ is a Catalan path, where $\phi$ is defined as that in Introduction, and
3. the final step of $S$ is $D$ on $x$-axis or $S = \emptyset$.

Clearly, $Q$ and $\phi(DRU)$ are both Catalan paths. We define a path $\varphi^{-1}(P')$ as $\varphi^{-1}(P') = SRUQDT$. 

$\blacksquare$
Corollary 4.3 (Chung-Feller.) The number of n-Dyck path with m-flaws is the Catalan number $c_n$ for any $0 \leq m \leq n$.

Recall that $d_{n,m,k}$ is the number of Dyck paths of semi-length $n$ with $m$ flaws and $k$ double descents and $d_{n,m,k} = a_{n,m,k}$.

Corollary 4.4 Let $n$ be an integer with $n \geq 0$ and $0 \leq k \leq n - 1$. Then

$$d_{n,m,k} = \frac{1}{k+1} \binom{n-1}{k} \binom{n}{k}$$

for any $0 \leq m \leq n$.

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