From Andreev to Majorana bound states in hybrid superconductor-semiconductor nanowires

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(Dated: November 13, 2019)

Electronic excitations above the ground state must overcome an energy gap in superconductors with spatially-homogeneous pairing. In contrast, inhomogeneous superconductors such as those with magnetic impurities, weak links or heterojunctions containing normal metals can host subgap electronic excitations that are generically known as Andreev bound states (ABSs). With the advent of topological superconductivity, a new kind of ABS with exotic qualities, known as Majorana bound state (MBS), has been discovered. We review the main properties of all such subgap states and the state-of-the-art techniques for their detection. We focus on hybrid superconductor-semiconductor nanowires, possibly coupled to quantum dots, as one of the most flexible and promising experimental platforms. We discuss how the combined effect of spin-orbit coupling and Zeeman energy in these wires triggers the transition from ABSs into MBSs and show theoretical progress beyond minimal models in understanding experiments, including the possibility of a new type of robust Majorana zero mode without the need of a band topological transition. We examine the role of spatial non-locality, a special property of MBS wavefunctions that, together with non-Abelian braiding, is the key ingredient for realizing topological quantum computing.

I. INTRODUCTION

Ever since Kamerlingh Onnes discovered the “zero resistance state” of metals at very low temperatures in 1911 \cite{1}, the superconducting state of matter \cite{2, 3} has fascinated physicists. In the last century, the understanding of superconductivity has evolved extraordinarily and has garnered eight Nobel prizes, turning it into one of the most iconic topics in condensed matter physics \cite{4}. As described by the seminal Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity \cite{5}, the characteristic feature of superconductors (SCs) is the macroscopic occupation of bound pairs of electrons, known as Cooper pairs \cite{6}, in the same quantum-coherent ground state. The condensation of Cooper pairs into such ground state is associated with a superconducting complex order parameter $\Delta = \Delta e^{i\varphi}$ \cite{7, 8}, where $\varphi$ is the conjugate of the number of Cooper pairs. In a homogeneous $s$-wave BCS SC the single-particle spectrum develops an energy gap $\Delta$ for the creation of quasiparticle excitations above the ground state. These are propagating superpositions of electrons and holes with different energy-dependent weights. However, if the order parameter—also called the pair potential \cite{2}—varies in space, $\Delta(\mathbf{r})$, lower energy (“subgap”) excitations may develop. Such is the case of states trapped in magnetic flux vortices (so-called Caroli-Matricon-De Gennes states \cite{9}), at magnetic domains or impurities (Yu-Shiba-Rusinov states \cite{10, 11}), at weak links between SCs or at normal metal-superconductor (NS) contacts \cite{12}, to name a few. Collectively, these subgap states are dubbed Andreev bound states (ABSs), and are the focus of numerous theoretical and experimental efforts, as well as the basis of promising emerging quantum technologies, see Fig. 1.

The core physical mechanism behind the formation of subgap states in inhomogeneous systems with $\Delta(\mathbf{r})$ is a remarkable scattering process, predicted by Andreev \cite{14, 15}, in which an incoming particle-like excitation can convert into an outgoing hole-like one and vice versa, see central row of Fig. 1. Many of such Andreev scattering events coherently concatenated lead to the formation of subgap ABSs \cite{16, 17} that are localized near the region where the pair potential has strong spatial variations (for a recent review see \cite{18}).

In the last decade, a new twist in the possibilities afforded by the superconducting pairing of electrons has been possible with the advent of topological materials \cite{19, 20}. Inspired by notions of topology such as topological invariants \cite{21}, several authors have predicted the existence of a new phase of matter known as a topological superconducting state \cite{22, 23}. It arises in so-called $p$-wave SCs, which possess a rare triplet-like pair potential (an exotic form of superconductivity in which only a single spin band is involved \cite{25, 26}). This topological phase is characterized by the emergence of a rather special type of bound state occurring at topological defects such as vortices, boundaries or domain walls in these topologi-
cal SCs. Importantly, such bound states occur precisely at zero energy, and exhibit an electron and hole character with exactly equal probability. The second quantization operators describing these states are self-conjugate, \( \gamma = \gamma^\dagger \). They are in this sense a condensed matter realisation of the celebrated ‘particle-equals-antiparticle’ states known as Majorana fermions [33], and also of so-called Jackiw-Rossi states at field vortices in the Dirac equation [34, 35].

As opposed to standard ABSs, which can be pushed out of the gap by continuous deformations of the Hamiltonian, Majorana bound states (MBSs) cannot be removed from zero energy by any local perturbation or local noise that does not close the gap. This is possible owing to the bulk-boundary correspondence principle of band topology [36], which predicts that at the boundaries between topological materials with non-topological ones, edge states must appear that are protected against perturbations by the topology of the bulk. Quite remarkably, MBSs do not follow fermion statistics, unlike the original particles predicted by Majorana [33], but rather possess non-Abelian exchange statistics. Upon exchange of two MBSs (braiding), a non-trivial unitary operation will be performed on them. This property, together with their topological protection against local noise, holds promise for applications in fault-tolerant quantum computing [37, 38].

The interesting connection between Dirac physics, superconductivity and Majorana zero modes was fully exploited by Fu and Kane in 2008 [39], who put forward the conceptual breakthrough of effectively realizing the Dirac equation with exactly equal probability. The second quantization operators describing these states are self-conjugate, \( \gamma = \gamma^\dagger \). They are in this sense a condensed matter realisation of the celebrated ‘particle-equals-antiparticle’ states known as Majorana fermions [33], and also of so-called Jackiw-Rossi states at field vortices...

II. ABSs IN HIGH-DENSITY NANOWIRES AND QDs

A. Formation of ABSs

ABSs arise in superconducting systems as the result of an unusual form of quantum confinement caused by so-called Andreev reflection [14, 15]. In a metallic system in its normal phase, electrons become specularly reflected at planar interfaces with vacuum or insulating materials. This is known as normal reflection. However, at an NS boundary [13, 107], an incoming electron from the N side...
Andreev bound states (ABSs) in semiconducting nanowires. ABSs are at the heart of several physical mechanisms (central row), experimental techniques (lower row) and applications (upper row) in condensed matter physics. **Central row:** Andreev reflection at an NS junction, see central panel, is the retro-reflection of an electron into a hole (or vice versa) of opposite spin and wavevector, with the addition (or removal) of a Cooper pair to the SC condensate. In contrast, normal (specular) reflection leaves the particle and spin quantum numbers unchanged. The probability of each ($R_A$ vs. $1 - R_A$ below the gap) depends on normal-state transparency $T_N$ and energy of the incident electron $E$. The central-right panel shows the Andreev reflection probability versus $E$ at an NS junction of $T_N$ in the high density limit (chemical potential $\mu$ much larger than superconducting gap $\Delta$). Multiple coherent Andreev reflections in a short SNS Josephson junction produce an ABS confined to the N region with energy $E(\varphi)$ below the gap. The energy depends on $T_N$ and the phase difference $\varphi$ between the two SCs, see central-left panel. **Lower row:** Several experimental techniques have been developed to probe the ABS spectrum, amongst which we highlight Josephson spectroscopy using the AC Josephson effect of a capacitively coupled tunnel junction, microwave spectroscopy through the dispersive shift of a planar resonator, and tunneling spectroscopy using the differential conductance into the nanowire through an opaque barrier. In the bottom-left panel, a nanowire Josephson junction (denoted by yellow arrow) is embedded in a SQUID loop, which sets a phase bias of $\varphi = 2\pi \Phi / \Phi_0$, where $\Phi$ is the applied flux and $\Phi_0 = h/2e$ the superconducting flux quantum. Below is the schematics of the device, where $V_s = hf/2e$ is the spectrometer bias voltage and $V_g$ is the applied gate voltage to the junction. Nearby we show the measured excitation spectrum of a similar device in the single channel regime, where the phase-dependent Andreev level (upper line) and the Josephson plasma oscillations (lower line) contribute to the signal. Experimental data are reproduced from Ref. [16]. The bottom-central panel showcases experiments using the dispersive shift $\Delta f$ of an inductively coupled planar superconducting microwave resonator. The data and device image are reproduced from Ref. [17]. The bottom-right panel shows the setup and an experimental dataset for voltage bias spectroscopy, where the differential conductance $dI/dV$ is measured as a function of the voltage bias $V$. The tunnel barrier is created by depleting a section of the nanowire by the local gate voltage $V_{\text{tunnel}}$. The data and device images are reproduced from Ref. [18]. **Upper row:** Potential application domains of ABSs in quantum technologies include single spin readout [103], Andreev quantum bits [104], topological quantum electronics [105] and hybrid quantum simulators [106].
may transform into an outgoing hole with inverted spin and wave vector. This hole is said to be retro-reflected since both the parallel and normal wavevector components to the interface change sign, whereas in a normal reflection the parallel component remains the same. This process is known as Andreev reflection, and is accompanied by the injection of a Cooper pair into the SC. If the interface is highly transparent, below the gap such Andreev process dominates with high probability $R_A \approx 1$, whereas in the opposite limit the electron becomes normal-reflected ($1 - R_A \approx 1$), see central panel of Fig. 1.

Consider now an electron in a normal metal between two or more insulating interfaces. When the metallic region is small, multiple coherent normal reflections on the boundaries leads to the formation of electronic states of quantized energy. A similar process takes place when some or all of the confining insulators are replaced by SC boundaries [113]. This leads to the formation of ABSs, which are the superconducting analogue of the above particle-in-a-box states in quantum mechanics.

The formation of ABSs becomes particularly simple in the common case of high density SCs with negligible SO coupling and zero magnetic field. In such systems the superconducting gap is much smaller than the chemical potential, $\Delta \ll \mu$, a condition known as the Andreev limit [108]. The NS Andreev reflection probability $R_A$ then exhibits a simple dependence with energy $E$ (relative to the SC chemical potential) and normal-state junction transparency $T_N$, depicted in Fig. 1 central-right panel. It reaches $R_A = 1$ at $E = \Delta$ from $R_A \approx T_N^2/(2 - T_N)^2$ at $E = 0$ [109]. This result assumes a step-like pair potential at the interface, a usual approximation known as the rigid boundary-condition [110].

We now combine two such NS interfaces into a 1D SNS junction with normal length $L_N$. A computation in the Andreev limit of the energy $E(\phi)$ of ABSs below $\Delta$ for transparent interfaces, and as a function of SC phase difference $\phi$ across the junction, yields the following quantization condition [17]

$$\varphi - 2 \arccos \frac{E(\varphi)}{\Delta} - \frac{E(\varphi) L_N}{\Delta} = \frac{2\pi n}{\xi},$$

where $n$ is an integer and $\xi$ is the superconducting coherence length. A generalization to a multimode junction of finite transparency yields, in the short junction $L_N \ll \xi$ limit [17 108 111 112], an explicit ABS solution for mode $i$

$$E_i(\varphi) = \pm \Delta \sqrt{1 - T_N^2 \sin^2 \frac{\varphi}{2}},$$

where $T_N$ is the normal transmission for each independent scattering-matrix eigenmode $i$ in the normal phase [113]. The presence of such bound states has important consequences for transport, since, as argued by Kulik [17], it implies that a normal metal can carry a dissipationless supercurrent between two SCs over arbitrarily long lengths, provided that transport is coherent. This is the celebrated dc Josephson effect [114 115]. At zero temperature and neglecting the contribution to the supercurrent coming from the continuum of states above $\Delta$, $I_\varphi(\varphi) = -(2e/h) \sum_i \partial E_i(\varphi)/\partial \varphi$ (where the factor 2 accounts for spin degeneracy).

Figure 1 central-left panel illustrates the above $E(\varphi)$ for different $T_N$ in a single-channel junction. Near $\varphi = 0, \pi$ the ABSs touch the continuum of single quasiparticle states above $\Delta$ while they reach their minimum value at $\varphi = \pi$, with a minigap that decreases with increasing transparency until reaching an accidental zero-energy crossing as $T_N \rightarrow 1$. It is important to realize that, since $|E(\varphi)| < \Delta$, the ABS wavefunctions are confined to the junction, and exponentially decay into the bulk of the SC leads on a length scale $\xi_{ABS}^{-1} = \xi^{-1} \sqrt{T_N} \sin(\varphi/2)$.

Deviations from the Andreev limit, relevant in low-density nanowires, introduce important corrections to the Andreev reflection $R_A$ and ABS energies $E_i(\varphi)$, and will be discussed in Sec. [111].

B. ABS spectroscopy

Several measurement techniques have been developed to obtain information about ABSs in nanowire Josephson junctions. Here we focus on three broad classes: Josephson spectroscopy, microwave spectroscopy and tunneling spectroscopy, see bottom row of Fig. 1.

In a Josephson junction, parity-conserving transitions between the ground and excited states with an addition energy of $2E(\varphi)$ [see Eq. (2)] can be created by an incident photon with a frequency of $f = 2E(\varphi)/h$. Note that the SC gap $\Delta \approx 150 \mu eV$ of Al corresponds to a frequency range of $2\Delta/h \approx 90 \text{GHz}$. A precise treatment of the pair transition leads to an effective microwave impedance $Z(f)$ associated with the transition [116]. It can be detected via the inelastic Cooper-pair tunneling [117] in a capacitively coupled auxiliary Josephson junction [118], which is sensitive to the environmental impedance seen by this spectrometer junction. The probing frequency $f$ can be set by applying a voltage bias of $V_s = hf/2e$ (Fig. 1 lower-left panel). Measurements of this type confirmed the applicability of the short junction formula Eq. (2) in a wide range of excitation energies in InAs semiconductor channels with epitaxial Al leads and demonstrated that few-channel configurations of high channel transparency can be attained [116].

The microwave impedance of the Andreev level transitions can also be detected by the shift in the resonance frequency of a coupled microwave resonator (Fig. 1 lower-central panel). The well-established techniques of circuit quantum electrodynamics [119] enabled a real-time tracking of the junction charge parity [114] yielding characteristic parity lifetimes in excess of 100 $\mu s$ in InAs nanowire Josephson junctions. Furthermore, typical relaxation times ranging up to $\sim 10 \mu s$ allowed for the coherent manipulation of the nanowire-based Andreev level
quantum bit.

Direct quasiparticle tunneling into the ABSs can also probe the ABS spectrum (Fig. 1 lower-right panel). These experiments utilize a gate-defined depleted section of the nanowire [45] or an in-situ grown axial tunnel barrier [53] [120] as the opaque probe junction. This measurement geometry allows for the characterization of energy spectra in proximitized semiconductor segments [121], quantum dots [49] [51] and makes non-local correlation experiments possible [122], however mesoscopic interference effects in the leads may yield additional features in the differential conductance [52].

It is worth noting that the ABS spectrum can indirectly be characterized via the measurement of the phase-dependent supercurrent \( I_S(\varphi) \sim dE/d\varphi \), which was performed by an inductively coupled SQUID loop [123] [124]. These experiments yielded strongly skewed current-phase relations, the signature of highly transparent channels in an InAs nanowire with Al superconducting leads. Similarly, the Josephson inductance, \( L_J = dI_S(\varphi)/d\varphi \) could serve as another probe of the anharmonicity in the current phase relationship [124]. Finally, external tunnel barriers, typically \( \text{AlO}_x \) of a few atomic layers, attached to a metallic probe also became an established technique to detect ABSs in other systems, such as carbon nanotubes [126] and graphene flakes [127].

C. ABSs in QDs

For the junctions above, it was assumed that the channel connecting the SC leads allowed for coherent transport through a segment of ballistic nanowire. By contrast, in QDs, charges localize in the channel and the effect of a finite electrostatic charging energy \( U \) must be taken into account. QDs can be formed in a nanowire by e.g. inducing barriers with electrostatic gates (Fig. 2(a)). At low temperatures and for low bias voltages, transport is blocked by the large \( U \) and the system is in the so-called Coulomb blockade regime with a well defined number of electrons \( n \). Current flow is only possible at discrete degeneracy points where the energies of the \( n \) and \( n + 1 \) charge states become degenerate. Given the strong confinement in nanoscale QDs, \( U \) can easily exceed \( \Delta \) in the electrodes, resulting in an interesting interplay between single-electron charge transport, localized spins and superconductivity [131].

The formation of ABSs can be understood by considering a single QD level coupled to a superconducting electrode. If the level is singly occupied, it holds an unpaired spin, i.e. a spin-doublet ground state (Fig. 2(b)). Conceptually, this scenario is identical to having an isolated magnetic impurity in a superconducting host. As shown by Yu, Shiba and Rusinov (YSR) in the 1960s [10] [12], the magnetic impurity induces localized bound states within the SC gap. At a critical exchange coupling the system undergoes a quantum phase transition to a magnetically screened, spin-singlet ground state. Conversely, at weaker coupling, the system maintains its original doublet state. While the above YSR picture applies for classical magnetic impurities, a full quantum treatment naturally leads to the physics of the Kondo effect [132], where, despite the absence of screening electrons within \( \Delta \) of the electrodes, the localized spin can still be screened by the above-gap quasiparticles in the SC. As in normal metals, Kondo physics sets in below a characteristic temperature \( T_K \), which results in singlet-doublet transitions occurring at \( k_B T_K/\Delta \sim 0.3 \). Early work on hybrid dots indicated the importance of Kondo-like correlations [133] [134], while more recent experimental work has provided precise boundaries for the transition [50]. Figure 2(b) shows the generic phase diagram of a hybrid QD as a function of dot parameters [49] [50] [130].

ABSs in QDs can be detected by transport spectroscopy [49] [126] [128] [129] [133] [137], whereby \( dI/dV \) is measured as a function of bias voltage \( V \). The sub-gap transport reflects resonant Andreev reflection processes (involving a parity change of the system) at voltages matching the energy difference \( E_{BS} \) between the ground and the excited state of the QD (Figs. 2(c) and (d)). This results in \( dI/dV \) peaks located symmetrically around \( V = 0 \), corresponding to ABS resonances at energies \( \pm E_{BS} \). Figure 2(c) shows a typical transport spectrum, where ABSs are visible as ridges below the gap. As the charge state, and thereby the parity, of the dot is tuned, the ground state switches between the singlet and doublet states, as reflected by the ABS crossings at zero bias. Remarkably, the ground state remains a singlet in some odd-occupancy regions due to the strong screening discussed above, which leads to avoided ABS crossings in the spectra. The experimental phase diagram of the QD-S system has been explored [50] [128] with excellent quantitative agreement with theory [130] [138]. In some cases, however, one needs to go beyond the bulk treatment of the SC above (to include soft gap effects, finite-length effects, etc) in order to understand the complex ABS spectra of finite-length proximitized nanowires [52] [53]. Transport spectroscopy of ABSs can also be performed by replacing the N probe by a weakly coupled superconductor. Here, all spectroscopical features are shifted by \( \Delta \) [48]. ABSs exist also in coupled hybrid dot systems [51] [139] where one can observe YSR screening of higher spin states and a more intricate phase diagram than Fig. 2(b) [51] [140].

In an external magnetic field, the Zeeman effect lifts the spin degeneracy of the doublet state. This strongly impacts the transport spectra of the ABSs (Fig. 2(d)). In case of a singlet ground state, two (parity-changing) transitions are allowed owing to the splitting of the excited doublet state. In contrast, when the ground state is a doublet, only one transition remains accessible independent of \( B \). As a result, the ABSs shift to higher energies but do not split. Figure 2(f) depicts these two distinct behaviors of the ABSs at finite \( B \) [49]. Interestingly, for high enough fields, the lowest-energy, spin-split ABSs can cross the Fermi level, denoting a quantum phase transi-
FIG. 2. ABSs in hybrid quantum dots (QDs). (a) Schematic of a semiconductor nanowire contacted by a normal metal (N) and a superconductor (SC). Local gates can be used to confine a QD, and to tune the dot-electrode tunnel couplings and the dot occupation/parity [51,128]. QDs can also form unintentionally in a nanowire, e.g., by barriers at interfaces [49,50,99,121,129]. (b) Top panel: charge stability diagram of a normal QD as a function of the bias voltage, \( V \), and the gate voltage, \( V_g \). The dot occupation (0, 1 or 2) is well-defined inside the Coulomb diamonds. Bottom panel: phase diagram of a hybrid QD as a function of \( V_g \) and the QD-S coupling, \( \Gamma_S \), normalized by the charging energy, \( U = e^2/2C \) (\( e \) being the electron charge and \( C \) the QD capacitance). In the weak coupling limit, \( \Gamma_S/U \ll 1 \), the ground state is a spin-doublet when the dot is occupied by an odd number of electrons. Conversely, for \( \Gamma_S/U \gg 1 \), the ground state is a spin-singlet irrespective of the dot occupancy. The precise boundary between both states can be obtained by experimentally tuning the ratio \( \Gamma_S/U \) [50] in very good agreement with theoretical results obtained by a superconducting analog of the Anderson model [130]. (c) ABSs formed within the SC gap by the Yu-Shiba-Rusinov mechanism where the confined spin (impurity) is screened by itinerant quasiparticles. The bound state energy, \( E_{BS} \), represents the excitation energy from the ground state of the QD-S system to an excited state. (d) Schematics of the possible transitions between ground and excited states of a hybrid QD. An external magnetic field, \( B \), splits the doublet state by the Zeeman energy, \( 2V_Z \). Top panel: when the ground state is the doublet, the bound state energy increases with \( B \) (green arrow). The transition between the two spin-polarized states is not visible by tunneling spectroscopy (red arrow). Bottom panel: when the ground state is the singlet, both transitions to the spin-polarized excited states are visible, \( E_{BS}^+ \) and \( E_{BS}^- \). (e) Subgap spectrum of a QD with a single SC electrode as a function of \( V_g \) at \( B = 0 \). Crossings of the bound states at \( V = 0 \) denote transitions between singlet and doublet ground states. Data reproduced from Ref. [128]. (f) The bound states only split in the presence of an external \( B \) when the ground state is the singlet. (g) Zeeman splitting of the bound states as a function of \( B \). The dashed vertical line underscores a quantum phase transition (QPT) whereby the ground state of the system turns from the singlet to a spin-polarized state. Data reproduced from Ref. [49].
III. LOW-DENSITY NANOWIRES AND MBs

A. ABSs in trivial SNS junctions with SO coupling and Zeeman field

As the Fermi energy $\mu$ of the nanowire is reduced (low density junction), it may become comparable to other energy scales in the problem, such as the SO energy $E_{SO}$ or the Zeeman energy $V_Z$ at the junction, or the gap $\Delta$ of the SCs at either side, see Box Figure 7. The Andreev reflection at a low-density NS interface, where $\mu \lesssim \Delta$, deviates considerably from the standard picture described in Sec. II A. Figure 3 (a-c) shows the typical dependence of $R_A$ with energy for a single channel contact when both N and SC sides have a common Fermi energy, SO coupling and Zeeman. Similarly, the Andreev spectrum of the corresponding low-density SNS nanowire junction is no longer well described by the conventional Eq. (2), even in the short junction limit, see Fig. 3 (e-g). Note in particular that the parity crossing present at $\phi = \pi$ in high-density transparent junctions becomes an anticrossing in the low-density regime even at $T_N = 1$.

In realistic low-density SNS nanowire junctions the situation is further complicated by the fact that $\alpha$ and $V_Z$ are largely confined to the normal nanowire, whose Fermi energy $\mu_N$ also differs strongly from that of the SC contacts $\mu_S$. The corresponding bandstructures will thus exhibit a strong Fermi momentum mismatch, which reduces Andreev reflection and affects the resulting ABS spectrum. In a nominally perfect, single mode SNS junction of nanowire length $L_N$ with $V_Z = 0$, Eq. (2) can be generalized to

$$E(\varphi) = \Delta \sqrt{1 - \frac{\sin^2(\varphi/2)}{1 + \kappa \sin^2(k_0 L_N)}},$$

where $\kappa = [(k_F^S)^2 - k_0^2]/(2k_F^S k_0)$ captures the effect of momentum mismatch acting as an effective barrier at each interface, with a transmission $T_N = \frac{1}{1 + \kappa \sin^2(k_0 L_N)}$ that is smaller than 1, except at resonant values of the nanowire length $k_0 L_N = n \pi$, $n \in \mathbb{Z}$. Here the SC and N Fermi wavevectors are $k_F^{S,N} = \sqrt{2m^* \mu_{S,N}/\hbar}$, and $k_0 = \sqrt{(k_F^N)^2 + 4k_{SO}^2}$. This $k_0$ depends also on the SO momentum $k_{SO} = m^* \alpha/\hbar^2$, that captures the momentum band shift of the two spin sectors in the nanowire (see Box 1). $E(\varphi)$ of Eq. (3) remains doubly degenerate for all $\varphi$ despite the shift $k_{SO}$ of the two spin sectors; electron-hole pairs can still form in a similar manner as for a spin-degenerate single parabolic dispersion, see Fig. 7.

In the absence of a Zeeman field, spin splitting of the ABS spectrum can be achieved in a two-subband model with intersubband coupling. Specifically, mixing between the two lowest transverse subbands produces a spin-dependent Fermi velocity $v_F^\uparrow \neq v_F^\downarrow$, and hence coherence lengths $\xi^i = \frac{\hbar k_F^i}{\Delta}$, which leads to spin-dependent quantization conditions according to Eq. (1). In this situation, the ABSs can be written as

$$E_i(\varphi) = \pm \frac{\Delta}{2} \frac{\cos(\varphi/2)}{1 + \lambda_i \sin(\varphi/2)},$$

with $\lambda_i = L_N/\xi_i$. The spin splitting between ABSs reads

$$E_{\uparrow}(\varphi) - E_{\downarrow}(\varphi) = \frac{\Delta(\lambda_\uparrow - \lambda_\downarrow) \sin(\varphi/2)}{2[1 + \lambda_\uparrow \sin(\varphi/2)][1 + \lambda_\downarrow \sin(\varphi/2)]}.$$
est energy state clearly traces the topological phase diagram for large $L$. It is interesting to note the role of finite $L$ in the topological Josephson effect (panels $g,h$). Due to the overlap of the MBSs in the junction and the ‘outer’ MBSs at the opposite ends of the nanowires, the $4\pi$ Josephson periodicity is destroyed under an adiabatic $\phi(t)$, and a non-topological $2\pi$-periodic Josephson effect is restored \[151,153\]. A similar effect is expected from quasiparticle poisoning (exchange of quasiparticles with the junction’s environment which breaks parity conservation) and by higher-energy quasiparticle excitation \[153\].

The role of SO coupling is crucial for the physics of MBSs. For $\alpha = 0$ and $V_Z$ larger than $\Delta$ the spectrum is gapless (the magnetic field just kills superconductivity), so that no localized MBSs emerge, while for $V_Z < \Delta$ the system has a gap. The addition of SO coupling radically changes this picture, and enables a topological minigap to emerge at $V_Z > V_Z^c$. The minigap can be shown to be effectively $p$-wave, and hence topologically non-trivial. The Majorana zero modes at the ends of a $V_Z > V_Z^c$ nanowire are in fact a manifestation of the bulk-boundary correspondence of this topological gap. They are thus topologically protected states. The extension of the Majorana wavefunction is the coherence length corresponding to the minigap (also known as the Majorana length $\xi_M$) and is hence smaller for stronger SO coupling. The Majorana oscillatory hybridization is thus exponentially suppressed by both a strong SO (minigap) and nanowire length. In both limits, an exact Majorana topological protected zero mode is recovered at each end of the nanowire.

C. MBS spectroscopy

The Majorana zero mode can be experimentally probed by using similar techniques as for ABSs. In particular, tunneling spectroscopy has been performed extensively in nanowires. A typical nanowire device used in experiments is presented in Fig. 3 (a,b). It consists of a semiconductor nanowire (gray) which is partially covered by a SC (green). This section of the nanowire acquires superconducting correlations due to the proximity effect. The superconducting wire is connected to a normal metal lead (yellow) via an NS-junction. The transparency of the junction is controlled by a local tunnel gate (red), while the chemical potential of the superconducting wire is tuned by the depletion gate (purple). In the tunneling
limit, the conductance through the junction is a direct measure of the local density of states at the end of the hybrid nanowire, which should therefore exhibit a prominent peak at zero bias in the presence of a MBS.

Since initial reports of robust zero energy states measured in hybrid SC-semiconductor nanowire systems, several groups have employed this type of set-up to find robust zero energy states as the magnetic field is increased beyond a critical value \[45, 99, 102, 141, 154–156\], see Fig. 5 (c). Additionally, recent works have also demonstrated that the peak conductance value can be quantized at the theoretical value of 2/\(e^2\) (Fig. 5 (d)), further solidifying this feature as a hallmark of Majorana physics.

Apart from tunneling spectroscopy measurements, in order to detect MBSs one can also explore dynamical detection techniques in SNS junctions. In Sec. II A we discussed the ABS spectrum and concluded that in a finite length system of length \(L\), the overlap between the ’inner’ and ’outer’ Majorana wavefunctions restores the avoided crossing with an energy scale \(\sim \exp(-L/\xi)\) for a Majorana coherence length of \(\xi_M [151–153]\). In addition, the tunneling of unpaired non-equilibrium quasiparticles enables relaxation to the parity ground state, resulting in a trivial, 2\(\pi\)-periodic behavior on timescales much longer than the parity poisoning time of the system [40, 157].

Therefore, the experimental detection efforts of the 4\(\pi\)-periodic Andreev levels focused on dynamical detection techniques based on the ac Josephson effect [114], which links the frequency \(f\) of the oscillating supercurrent \(I_s(t) = Ic \sin(2\pi ft)\) to the voltage bias \(V\) over the junction with \(f/V = 2e/h \approx 486\text{ MHz}/\mu\text{V}\) for conventional Josephson junctions [158]. Note that the superconducting flux quantum is \(\Phi_0 = h/2e\), consequently for the 4\(\pi\)-periodic Josephson effect \(f/V = e/h\) holds, resulting in a frequency halving with respect to the conventional case.

This transition can manifest in Shapiro step measurements [159] when the junction is irradiated at a frequency \(f\) in the microwave domain, and the dc \(I(V)\) characteristics develops discrete voltage steps with a spacing of \(V_{2\pi} = hf/2e\) and \(V_{4\pi} = hf/e\) for the trivial and topological state, respectively [157] [190] [101]. While the dis-
appearance of the first voltage step was repeatedly observed, higher odd steps typically persist in experiments [162, 163]. However, the interpretation of the measurements has to include the deviations from the tunnel junction behavior, such as non-sinusoidal supercurrent [160], overheating effects [163, 165], Landau-Zener tunneling between the Andreev bands and to the quasiparticle continuum [152, 161]. Furthermore, the addition of several non-topological ABSs has a non-trivial effect on the observed Shapiro steps [160, 166].

Another class of experiments rely on the direct spectroscopic detection of the Josephson radiation of voltage-biased junctions, which is expected to be centered at $f_{2\pi} = 2eV/h$ or at $f_{4\pi} = eV/h$ [153]. This transition has been observed in InAs/Al nanowire Josephson junctions integrated with an on-chip SIS microwave detector [167], and by using a conventional microwave amplifier chain [163].

It is to be noted that additional measurement schemes were proposed to observe the $4\pi$-periodic Josephson effects as a probe for topological superconductivity. These utilize Shapiro steps in the low-frequency regime [161], Andreev level pair excitations in long junctions [158], critical current measurements [169, 171], and the shape of switching current histograms [172], respectively.

IV. MBSs BEYOND THE MINIMAL MODEL

A. Extensions of the minimal model

The minimal Oreg-Lutchyn model has proven to be a first useful guide to investigate the physics of Majorana nanowires. However, discrepancies between its predictions and experimental observations have motivated extensions that provide a more complete understanding of the experimental system. A natural extension of the 1D single band model is to allow for multiple subbands in the nanowire [54–57]. This leads to a more complicated phase diagram, depending on the number of occupied bands and their relative energies. Additionally, the orbital effects of the magnetic field may become relevant, especially when the number of occupied subbands is increased [173]. The orbital effect has been shown to dramatically alter the topological phase diagram [68] (see Fig. 6 (b)) and the dispersion of states in the nanowire, leading to large effective $g$-factors and suppressed topological gaps. Although numerical simulations of multi-band can shed additional light on the experimental results, they tend to depend more strongly on details such as the geometry and effective parameter values which are not always experimentally accessible.
While initial experiments generally suffered from unwanted quasiparticle states inside the superconducting gap (referred to as “soft gap” [45, 68, 174], see Fig. 6(c)), clean superconducting gaps comparable to the bulk gap of the parent SC have since been achieved [101, 121] by engineering epitaxial interfaces between the two material systems [175, 176]. Both the “soft gap” issue [71] and the large gaps measured in later experiments ignited interest in a more complete description of the superconducting proximity effect in these systems. This includes pair breaking effects that suppress superconductivity beyond a critical value of the magnetic field, or a more accurate model of the induced pairing in the form of an energy-dependent anomalous self-energy. The latter extends the regime of weak coupling between the semiconductor and the SC, wherein the induced superconducting gap is simply proportional to the coupling strength between the two systems. It was found that in the opposite, strong coupling regime, the band structure of the nanowire is significantly altered, resulting in a strong renormalization of model parameters [58]. It has also been demonstrated that the proximity effect can strongly depend on the thickness of the SC film [59, 63]. The SC-semiconductor coupling has furthermore been found to depend on the details of the electrostatic environment [60, 66], resulting in gate voltage dependent effective parameters such as the $g$-factor [61] and the induced gap [67].

Another notable disagreement between most experiments and the minimal model revolves around Majorana oscillations. The oscillatory energy splittings are predicted to be regular and grow with Zeeman field [68, 177, 178], while in most experiments robust zero-bias peaks appear without oscillations [99, 101]. Several model extensions have been explored that predict a reduction or suppression of oscillation, such as interactions with a dielectric environment or among carriers [65, 177, 179], orbital effects [80], dissipation [79, 181, 182] or non-uniform pairing [72]. A further common disagreement is a lack of visible bandgap-closing and reopening in some experiments [45, 61, 99], which is a key feature of the model’s topological transition. This has been explained as the result of poor visibility resulting from tunnel probe smoothness [69, 183] and even by a lack of bulk transition altogether [184], as will be discussed in Sec. 11.

The topological phase transitions in these extended models are generally calculated using the chemical potential $\mu$ and the Zeeman energy $V_Z$. However, the control parameters used in experiments are gate voltages and magnetic fields. Calculating the phase diagram in terms of gate voltages requires a self-consistent treatment of the electrostatics [64]. While some progress has been made in self-consistent Schrödinger-Poisson calculation for 3D device geometries [68, 179, 185] (see Fig. 6(a)), this remains a difficult problem to solve reliably. In addition to electrostatic modifications of the phase diagram, interaction effects have been demonstrated to play a role in the low energy spectrum of Majorana nanowires [65, 179].

An immediate effect of a self-consistent description of nanowire junctions, both for electrostatics and the proximity effect, is a smoothening of the pairing and Fermi energy profiles [69], which can no longer be assumed piecewise-constant as in the minimal model. Smooth $\Delta(r)$, $\mu(r)$ at a junction have been shown to give rise to near-zero modes without the need of a topological bulk. In the next subsections we discuss the emergence of these and other types of zero modes unrelated to topology, their connection to topological MBSs and their possible role in experiments.

### B. Majorana zero modes with a topologically trivial bulk

The combination of multiband wires with disorder has been shown [175, 186, 187] to produce topologically trivial zero energy states. Since the advent of cleaner experiments, it has become possible to distinguish disorder-based mechanisms from zero bias peaks of topological origin, as the former are associated to specific observable features (e.g. soft gap, low transport peak heights) that have been optimized away.

Near-zero bound states can also be generically present in a tunneling spectroscopy nanowire setup if there is a non-superconducting section between the tunnel barrier and the superconducting wire [63, 69, 73, 75, 78, 80, 188] [Fig. 6(d)]. Such a N region can host ABSs that become spin-polarized under a Zeeman field and may thus be tuned to zero energy, much like the Shiba states, possibly with a strongly renormalized $g$-factor due to SO coupling [92]. In the simplest situation, these are readily distinguished because their zero energy is a matter of fine tuning parameters such as $B$ to specific values, unlike the case of topological MBSs. Under some circumstances, however, these modes can become pinned to zero or near-zero energy for an extended range in magnetic field and other control parameters, resembling the behavior expected from MBSs, but with the SC in the topologically trivial phase [69, 71, 73, 80]. This type of zero mode has been dubbed a quasi-MBS, pseudo-MBS, partially separated MBS, etc. In what follows we call them non-topological MBSs to emphasize the fact that they satisfy the Majorana self-conjugate property. This is guaranteed by virtue of their zero energy. This implies the degeneracy of the ABS and its conjugate, which allows them to be rotated into the self-conjugate Majorana basis while remaining eigenstates, see Box 2. These particular MBSs do not arise as a result of a topological bulk in the band-topology sense. They have been shown, however, to mimic much of the phenomenology of topological MBSs [77, 80].

In the case of isolated NS nanowire junctions, we can distinguish two main mechanisms for zero-energy pinning of a non-topological MBS: smooth confinement [71, 77, 79, 189] and SO-induced pinning [75]. Both effects ultimately cause an enhanced Andreev reflection of a normal
electron on the trivial SC. In the case of a junction with spatially smooth parameters, the momentum transfer required for normal reflection at the junction is suppressed, and hence Andreev reflection dominates. Under these conditions, states at the junctions decouple into two sectors around different Fermi wavevector and spin (due to the SO coupling and the Zeeman field) [70, 74, 78], each of which behaves as an independent topological p-wave SC that gives rise to a zero-energy MBS decoupled from its partner. The Majorana wavefunction corresponding to the two wavevectors are centered at different positions in space and exhibit different spatial profiles (oscillatory exponential and smooth gaussian, respectively [74, 80]), see Fig. 6 (e).

A similar pinning effect can be caused by SO coupling. For large SO, the effective $g$-factor is strongly renormalized and ABSs can become largely insensitive to magnetic fields [62]. When the length of the N section is further tuned to an approximately odd-integer multiple of the SO length, an ABS will appear pinned near zero energy [75]. This SO-induced pinning does not require junction smoothness, but the above Fabry-Perot resonance condition on length must be satisfied.

A third route towards stabilising zero modes belonging to a nominally trivial bulk has been proposed in topologically trivial nanowires open to fermion reservoirs (which is a standard geometry in NS junctions used to perform transport spectroscopy). When such a nanowire becomes coupled to the reservoir, it can develop an ‘exceptional point’ bifurcation in its complex (non-Hermitian) spectrum, see Fig. 6 (f), where the real part of the lowest quasibound Bogoliubov mode becomes robustly pinned to zero energy as the imaginary part bifurcates. Such non-Hermitian topological transition stabilizes a couple of quasibound states at the contact with different decay rates. One of the two becomes essentially non-decaying after the exceptional point bifurcation, thus becoming a stable Majorana zero mode without the need of a bulk topological transition. While the microscopic mechanism leading to coupling asymmetry, and hence to an exceptional point bifurcation, is not universal (sources of asymmetry include finite length [79, 190], smooth potentials [79], spin-polarized leads [83], etc), the physical consequences and underlying mathematical structure are [79]. Research into Majorana states in open systems for quantum computation purposes is still in its early stages. The field is advancing rapidly, however, with e.g. new non-Hermitian topological classification theories being developed recently [191, 194] that extend band-topological concepts to open systems where these do not strictly apply.

Smooth junctions and coupling to reservoirs are a common occurrence in experiments. While ideal tunnel barriers are often assumed in calculations, actual tunneling spectroscopy experiments involve screened electrostatic potentials, smooth on the scale of the Fermi wavelength, that are used to create barriers for electronic transport (Fig. 5 (b)). A considerable number of experiments exhibiting signatures of zero energy states are consistent with this picture of Majorana modes with a trivial bulk. One characteristic feature of the smoothly-confined MBS’s evolution with Zeeman field is that the zero mode does not emerge from a band inversion at a critical $V_Z$ [69, 195] (there is no bulk topological transition involved), but appears instead to evolve smoothly from a lone ABS that detaches from the continuum as $V_Z$ increases, and is gradually pinned to zero, see Fig. 6 (d). This telltale feature is often seen in experiments, see e.g. Fig. 5 and should be taken as a strong hint that the zero mode might be a Majorana without an underlying bulk topological transition.

C. Protection against errors and MBS overlaps

Topological quantum computation was proposed as a way to achieve scalability through the hardware-level resilience of Majorana-based qubits. The Majorana qubit is defined in terms of the occupation of non-local fermion states such as $c = \gamma_1 + i\gamma_2$ [31], see Box 2. The original promise of these qubits stems from a basic idea: the wavefunction of the MBS $\gamma_1$ has exponentially small overlap with its partner $\gamma_2$ at the other edge of the wire for increasing nanowire length. In principle, such non-locality leads to exponentially suppressed susceptibility of the Majorana qubit to arbitrary local electrostatic noise.

As efforts develop towards realising this promise, different error-inducing mechanisms have been identified and studied for topological MBS qubits, such as those created by a coupling to ungapped [86] or gapped [85, 87] fermionic baths, as well as to fluctuating bosonic fields [92] (e.g., phonons [95], photons [88, 90, 93], thermal fluctuations of a gate potential [89, 96, 97], or electromagnetic environments [95]). While these are often controlled by wavefunction overlap, one needs also to consider the errors induced by qubit manipulation, such as unwanted excitations created by nonadiabatic manipulation [91, 94], which are largely controlled by the superconducting minigap.

Given the likely ubiquity of non-topological MBSs in realistic devices, particularly when including QDs as basic elements of many proposed schemes for topological quantum computing [103, 196, 198], it has become important to understand whether the protection of topological MBSs applies in some form also to non-topological MBSs. In contrast to topological MBSs, pairs of non-topological MBSs typically occupy the same neighbourhood of the N region or the junction, so that they usually have partially overlapping wavefunctions. Due to their specific profiles, however, their overlap can be quite small in practice, and even be comparable or smaller than that of topological MBSs in nanowires of realistic length [74]. In such finite nanowires, non-topological and topological MBSs are in fact continuously connected. Given the same degree of wavefunction spatial overlap between two non-topological and two topological MBSs in a finite
nanowire (see Ref. 74 for precise overlap definitions), all relevant observables, including conductance quantisation and qubit decoherence rates under generic noise will be the same. In the context of Majorana qubits, therefore, the susceptibility to noise is controlled by the degree of Majorana wavefunction overlap, as well as the effective gap between these zero modes and other possible subgap ABSs, rather than the topological or trivial origin of the corresponding MBS (a nominally topological nanowire may have large Majorana overlaps for $L \sim \xi$). However, we note that for finite overlaps, the resilience of different types of MBSs to perturbations can differ strongly. In this sense, only complete non-locality and exponentially vanishing overlaps can be considered as true protection against perturbations 199.

Traditional experimental schemes to measure the sub-gap spectrum of nanowires, such as tunneling spectroscopy, rely on local probes, so that they do not directly access the degree of non-locality of a given zero bias anomaly. An alternative, though still local scheme has been proposed to extract a quantitative estimate of the degree of MBS overlap 74.82.200.201. It consists of measuring tunneling spectroscopy into the end of the nanowire through a QD in series, which reveals the asymmetric coupling of spin-polarized states in the QD with the two spatially separated Majorana components of the zero mode. Such a scheme was implemented in a recent experiment 83 that demonstrated a varying degree of wavefunction overlap in otherwise similar zero modes. Other, truly non-local probes have been realized very recently that could detect the presence of non-local Majoranas using multiple tunnel probes 122,156,202.

V. SUMMARY AND OUTLOOK

We have reviewed the remarkable recent advances towards probing and characterizing the detailed structure of ABSs in hybrid nanowires and related systems, particularly in regards to their internal spin texture as a result of magnetic and spin-orbit couplings. Researchers have begun to understand how these delicate spin effects con-
nect to the spatial separation of ABS wavefunction components, ultimately resulting in robust Majorana zero energy modes. Theoretically, we have also begun to clarify the different routes towards stabilizing MBSs, from the well established bulk topological transition to using smooth and/or spin-dependent confinement, to the non-Hermitian approach based on exceptional point bifurcations. Having understood the importance of wavefunction non-locality for the protection of MBSs, their resilience against noise and the possibility of carrying out braiding operations, it has become a major focus point in current experiments to detect and quantify the degree of Majorana overlap. We have reviewed some promising first results using purely local probes. These have intrinsic limitations, unfortunately, and can only suggest, not demonstrate, Majorana non-locality. Experiments are underway, however, to exploit truly non-local measurements in more complex nanowire setups without such limitations. The ultimate demonstration, non-Abelian geometric braiding, remains an open challenge. Braiding and non-locality are the cornerstones behind the original promise of Majorana applications, i.e. to harness the hardware-level resilience of Majorana qubits to solve the scalability problem of quantum computers. Regardless of the outcome of such long-term endeavour, the leading efforts have already revealed a remarkably fertile field for condensed matter research.

ACKNOWLEDGMENTS

Research supported by the Spanish Ministry of Science, Innovation and Universities through grants FIS2015-65706-P, FIS2015-64654-P, FIS2016-80434-P, FIS2017-84860-R and PGC2018-097018-B-I00 (AEI/FEDER, EU), the Ramón y Cajal programme grant RYC-2011-09345 and RYC-2015-17973, the Maria de Maeztu Programme for Units of Excellence in R&D (MDM-2014-0377), the European Union’s Horizon 2020 research and innovation programme under grant agreements Nos 828948 (FETOPEN AndQC), 127900 (Quantera SuperTOP), 788715 (LEGOTOP) and European Research Council (ERC) grant agreements 804988 (SiMS) and 757725 (Starting Grant), the Netherlands Organization for Scientific Research (NWO), Microsoft, the Danish National Research Foundation, the Carlsberg Foundation, the Swiss National Science Foundation and NCCR QSIT. We also acknowledge support from CSIC Research Platform on Quantum Technologies PTI-001.

Appendix A: Box 1 – Proximitized nanowire model

The starting point of the proximitized nanowire model (R. M. Lutchyn et al and Y. Oreg et al) is a Hamiltonian describing a 1D semiconducting nanowire with Rashba spin-orbit (SO) interaction and in the presence of an external magnetic field $B$ perpendicular to the Rashba field (here, we assume that $B$ is applied parallel to the nanowire axis $x$):

$$H = \int dx \Psi^\dagger(x) \mathcal{H}(x) \Psi(x),$$

with

$$\mathcal{H}(x) = \left(-\frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial x^2} - \mu - i \alpha \partial_z \sigma_y\right) \tau_z + V_Z \sigma_x. \quad (A1)$$

where $m^*$ is the effective mass of the semiconductor, $\mu$ its chemical potential, $\alpha$ the SO coupling and $V_Z = \frac{1}{2} g_H B$ the Zeeman energy produced by $B$, given in terms of the nanowire’s $g$-factor (with $g_B$ being the Bohr’s magneton). $\Psi(x) = (\psi_1, \psi_2, -\psi_1, -\psi_2)$ are Nambu spinors and $\sigma$ and $\tau$ Pauli matrices in spin and particle-hole space, respectively.

Solving the above Hamiltonian in reciprocal space, one gets a dispersion relation of the form $E_{k,\pm} = \frac{\hbar^2 k^2}{2m^*} + \mu \pm \sqrt{V_Z^2 + \alpha^2 k^2}$. In the absence of Zeeman field, $V_Z = 0$, the Rashba term removes the spin degeneracy of the 1D parabolic band and gives rise to two parabolas shifted relative to each other along the momentum axis (each by an amount $k_{SO} = m^* \alpha/\hbar^2$) and displaced down in energy by an amount $E_{SO} = m^* \alpha^2/2\hbar^2$, where $\alpha$ the Planck’s constant, see Fig. 7(a). These parabolas correspond to spin up and spin down projections along the spin quantization axis fixed by the Rashba coupling (here $\sigma_y$). On the other hand, a finite Zeeman $V_Z \neq 0$ mixes both spins and hence removes the spin degeneracy at $k = 0$ by opening up a gap of size $2V_Z$, Fig. 7(b). These split bands are helical, with a degree of spin canting which depends on $k$ (spins...
at low momenta are almost aligned with $B$ while canting towards the Rashba axis occurs for larger $k$. This spin canting is crucial for obtaining $p$-wave superconductivity: it can be shown that by projecting a standard $s$-wave pairing term $H_s = \sum_k \Delta \{ \psi_{i,k} \psi_{i,-k} + H.c. \}$ onto the helical basis one obtains intraband (spinless) pairing terms of the form $H_p = \sum_k \sum_{i=\pm} \Delta_k \{ \psi_{i,k} \psi_{i,-k} + H.c. \}$, with $\Delta_k = \frac{\pi |c| k}{2\sqrt{V_Z^2 + \sigma^2 k^2}}$ having so-called $p$-wave symmetry $\Delta_k = -\Delta_{-k}$. This minimal Hamiltonian is a realistic implementation of Kitaev’s model for 1D $p$-wave superconductivity [21]. When the applied Zeeman field is larger than the critical value $V_Z^c = \sqrt{\Delta^2 + \mu^2}$, the 1D SC becomes topological and hosts MBSs at its ends.

To implement this proposal in experimentally realizable systems, one needs semiconductors with large $g$-factors in order to achieve a large $V_Z$ under moderate external magnetic fields $B$ below the critical field of the SC. A good proximity effect with conventional SCs and a large Rashba energy are also necessary. Last but not least, one needs to be able to keep the chemical potential $\mu$ of the nanowire close to zero (in order to reach the helical regime with spin-momentum locking for moderate $B$), despite the proximity to the SC.

Appendix B: Box 2 – Majorana basis

A Bogoliubov-de Gennes eigenstate in a superconducting system is an excitation $|\psi_n\rangle = \psi_n^\dagger |\text{BCS}\rangle$ of energy $\epsilon_n$ over its ground state $|\text{BCS}\rangle$ that consists of a superposition of one electron and one hole quasiparticles,

$$\psi_n = \int dx \sum_\sigma \left[ u_{n\sigma}(x) \psi_{\sigma}(x) + v_{n\sigma}(x) \psi_{\sigma}^\dagger(x) \right].$$

Here $\psi_{\sigma}^\dagger(x)$ and $\psi_{\sigma}(x)$ create and destroy a quasiparticle of spin $\sigma$ perfectly localized at point $x$, respectively, and $u(x), v(x)$ are electron/hole wavefunctions.

If the energy of a given eigenstate $|\psi_0\rangle = \psi_0^\dagger |\text{BCS}\rangle$ becomes negligibly small $\epsilon_0 \approx 0$ as in the case of a topological Majorana nanowire, the eigenstates $|\text{BCS}\rangle$ and $|\psi_0\rangle$ are both degenerate ground states, of even and odd fermionic parity, respectively. If we denote $|\psi_{\text{even}}\rangle \equiv |\text{BCS}\rangle$ and $|\psi_{\text{odd}}\rangle \equiv |\psi_0\rangle$, we find that $|\psi_0$ and $|\psi_0^\dagger$ switch between the two

$$\left( \begin{array}{c} |\psi_{\text{even}}\rangle \\ |\psi_{\text{odd}}\rangle \end{array} \right) = \left( \begin{array}{cc} 0 & \psi_0^\dagger \\ \psi_0 & 0 \end{array} \right) \left( \begin{array}{c} |\psi_{\text{even}}\rangle \\ |\psi_{\text{odd}}\rangle \end{array} \right).$$

The matrix elements of $\psi_0, \psi_0^\dagger$ in this subspace are therefore $\langle \psi_{\text{even,odd}} | \psi_{\text{even,odd}} \rangle = (\sigma_1 + \sigma_2)/2$ and $\langle \psi_{\text{even,odd}} | \psi_{\text{even,odd}} \rangle = (\sigma_1 - i\sigma_2)/2$, where $\sigma_i$ are Pauli matrices.

By performing a unitary rotation to the so-called Majorana basis, the eigenstate operators $\psi_0, \psi_0^\dagger$ can be decomposed into two Majorana operators that satisfy self-conjugation, $\gamma_1 = \gamma_1^\dagger$ and $\gamma_2 = \gamma_2^\dagger$, so that

$$\psi_0 = (\gamma_1 + i\gamma_2)/2, \quad \psi_0^\dagger = (\gamma_1 - i\gamma_2)/2; \quad \gamma_1 = \psi_0^\dagger + \psi_0, \quad \gamma_2 = i(\psi_0^\dagger - \psi_0). \quad (B1)$$

Each Majorana operator corresponds to a fermionic eigenstate, in the sense that $\{ \gamma_i^\dagger, \gamma_j \} = 2\delta_{ij}$, but the Majorana reality property also implies that, unlike a conventional fermion, $\gamma_i^2 = 1$. The matrix elements of $\gamma_i$ in the ground state subspace are $\langle \psi_{\text{even,odd}} | \gamma_i |\psi_{\text{even,odd}}\rangle = \sigma_i$.

The Majorana states created by $\gamma_1$ and $\gamma_2$ are sometimes intuitively described as half-fermions, as they always come in pairs and any two in a system can be combined to create a conventional fermion as above. In a topological Majorana nanowire, the wavefunction $u_{1\sigma}^M(x)$ of

$$\gamma_i = \int dx \sum_\sigma \left[ u_{1\sigma}^M(x) \psi_{\sigma}(x) + u_{1\sigma}^{M*}(x) \psi_{\sigma}^\dagger(x) \right]$$

is localized at either end of the nanowire, unlike $u_{0\sigma}(x)$, $v_{0\sigma}(x)$ of $|\psi_0\rangle$, that occupies both ends, see Fig. [8]. The latter is hence called a non-local fermion.

The above transformations, Eqs. (B1), can be applied to an arbitrary ABS $\psi_0$ of finite energy. In such case, the resulting Majorana states are not eigenstates. However, the decomposition still allows to examine the degree of Majorana non-locality of $\psi_n$ by computing the overlap between the corresponding $u_{1\sigma}^{M*}(x)$ and $u_{2\sigma}^M(x)$.

FIG. 8. Wavefunction of the lowest energy state $\psi_0$ in a uniform $L = 1 \mu m$ Majorana nanowire for a trivial $V_Z = 0.5 V_Z^c$ (a) and a non-trivial $V_Z = 1.4 V_Z^c$ (b) cases. (c,d) Wavefunctions of the corresponding Majorana components $\psi_{1n}^M(x)$ in the Majorana basis. In the trivial regime the left (red) and right (blue) Majorana components strongly overlap (c), whereas in the topological regime they move apart and concentrate at the ends of the nanowire (d).
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