CAPITULATION, AMBIGUOUS CLASSES AND THE COHOMOLOGY OF THE UNITS.

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Abstract. This paper presents results on both the kernel and cokernel of the $S$-capitulation map $\mathcal{C}_F \to \mathcal{C}_K^G$ for arbitrary finite Galois extensions $K/F$ of global fields (with Galois group $G$) and arbitrary finite sets of primes $S$ of $F$ (assumed to contain the archimedean primes in the number field case).

0. Introduction.

Let $K/F$ be a finite Galois extension of number fields with Galois group $G$, and let $\mathcal{C}_F$ and $\mathcal{C}_K$ denote, respectively, the ideal class groups of $F$ and $K$. The extension of ideals from $F$ to $K$ induces a natural capitulation map $j_{K/F}: \mathcal{C}_F \to \mathcal{C}_K^G$. An important problem in Number Theory is to determine the kernel of $j_{K/F}$, which is usually called the “capitulation kernel”. The classical Principal Ideal Theorem of global class field theory (see [11, Theorem II.5.8.3, p.168] for a generalized version of this theorem) asserts that $\text{Ker} j_{K/F}$ is all of $\mathcal{C}_F$ if $K$ is the Hilbert class field of $F$.

This fact motivated, quite early on, a great deal of interest in the study of $\text{Ker} j_{K/F}$ for subfields $K$ of the Hilbert class field of $F$. As a result, most of the existing literature on the Capitulation Problem is concerned with the study of $\text{Ker} j_{K/F}$ for unramified abelian extensions $K$ of $F$ (see, e.g., [4, 12, 17, 25]) or, more generally, with the kernel (and cokernel, in the case of [20]) of the $S$-$T$-capitulation map $j_{K/F,S}^T: \mathcal{C}_F^T \to (\mathcal{C}_K^T)^G$ for $T$-tamely ramified and $S$-split abelian extensions $K/F$ (see [19, 20] and [11, Corollary II.5.8.6, p.170]). Here $S$ denotes a finite set of primes of $F$, which we always assume to contain all archimedean primes in the number field case, and $T$ is a finite set of non-archimedean primes of $F$ which is disjoint from $S$. One of the few exceptions to this “general rule” is the work of B.Schmithals [23], who studied the kernel of $j_{K/F}$ for certain types of possibly ramified cyclic extensions $K$ of a quadratic number field $F$. However, the general problem of studying both the kernel and cokernel of $j_{K/F,S}^T$ for arbitrary finite Galois extensions $K$ of $F$ (i.e., not necessarily $S$-split or with ramification locus equal to $T$) has yet to be addressed. In particular, it is an unfortunate fact that, despite the long history of research on the Capitulation Problem, very little attention has been accorded the cokernel of $j_{K/F,S}^T$. In effect, apart from the contribution [20] already mentioned, the only other works known to us which study the capitulation cokernel are [9, Appendix] and, in the context of Iwasawa Theory, [13, 14, 15, 16, 18].

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In this paper we study both the kernel and cokernel of the $S$-capitulation map $j_{K/F,S} : C_{F,S} \to C_{G,K,S}$ for arbitrary finite Galois extensions $K/F$ of global fields and arbitrary sets $S$ as above. We show, for example, that $\text{Ker } j_{K/F,S}$ may be identified with the subgroup of $H^1(G,U_{K,S})$ of all cohomology classes which are locally trivial at all places outside $S$ (See Corollary 2.5). We also obtain a new generalization of Hilbert’s Theorem 94 (see Theorem 2.7), which applies to possibly ramified cyclic extensions of global fields (it is an open question to determine whether Theorem 2.7 below holds true for arbitrary abelian extensions, i.e., whether there exists a “ramified version” of Suzuki’s Theorem [25]). In Section 3 we obtain certain general results on $\text{Coker } j_{K/F,S}$ which have some rather interesting consequences. For example, Theorem 3.7 states that in the “semisimple case” (i.e., when the degree of $K/F$ is prime to the class number $h_{K,S}$) it is possible to determine the structure of $H^2(G,U_{K,S})$ completely. Sections 4-8 give applications of the main results of the paper. Theorem 4.1 states (roughly) that the structure of $H^1(G,U_K)$ is determined by that of $C_F$ and by the ramification indices of $K/F$ if $F$ belongs to a certain class of number fields and $K$ is equal to its own genus field relative to $F$ (the proof of this result uses well-known theorems of Tannaka-Terada, H.Furuya and C.Thiébaud). Section 5 deals with cyclic extensions and gives, under a certain hypothesis, a lower bound for the number of ambiguous $S$-classes of $K$ which do not come from $F$. See Theorem 5.2. This result may be regarded as a “dual” of Hilbert’s Theorem 94. The very brief Section 6 computes the kernel and cokernel of the capitulation map for certain types of imaginary extensions of function fields. An application to imaginary Artin-Schreir extensions is given. Section 7 discusses the case where $S$ is large relative to $K/F$, i.e., when $S$ contains all archimedean primes and all ramified primes of $K/F$ (we note that a significant portion of earlier work on the Capitulation Problem has taken place in this setting). We show that in this case the kernel of the capitulation map is naturally isomorphic to $H^1(G,U_{K,S})$, and that its cokernel is a certain group which measures the failure of the Hasse principle for $H^2(G,U_{K,S})$. In Section 8, which concludes the paper, we use results from [22] to show that the main theorems of Sections 2 and 3 have natural analogs in the context of divisor class groups. This Section also contains some results which supplement those of [22]. Finally, the paper contains an Appendix which relates certain integers that appear in the main text to the ramification indices of $K/F$.

Allow us to make here the following additional comments which may help clarify the approach adopted in this paper. Let the global field $F$ be given and let $S$ and $T$ be given finite sets of primes of $F$ satisfying the conditions stated above. Then a natural question is to study the $S$-$T$ capitulation map $j_{K/F,S}^T$ for varying choices of $K$. If $K$ is chosen so that all primes of $S$ split completely in $K$ and $K/F$ is (tamely) ramified exactly over $T$, then we are in the setting of [20]. But other choices of $K$ are possible. For example, $K$ could be chosen so that the set of primes of $F$ which ramify in $K$ is in fact disjoint from $T$ (which is certainly the case in this paper since here we consider $T = \emptyset$), but no conditions are imposed regarding the behavior in $K$ of the primes in $S$. By adopting this approach, we have been able to obtain results on the cohomology of the units whenever information on Capitulation is available (see, e.g., Theorem 4.1), and results on Capitulation whenever information on the units is available (see, e.g., Theorem 5.2 and Example 5.3).

One final comment is in order. Several of the results of this paper immediately

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1 This result may be well-known to all specialists in this area. See Remark 2.6.
yield divisibility relations which involve the class numbers $h_{F,S}$, $h_{K,S}$ and various other invariants (see, e.g., Theorem 5.2). Since explicitly stating all such divisibility relations would soon become quite repetitive, we have decided not to state them at all. We are certain that the interested readers will retrieve them without difficulty from the corresponding statements (for an illustration, see Example 5.3).

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1. Settings and notations.

Let $F$ be a global field, i.e. $F$ is a finite extension of $\mathbb{Q}$ (the “number field case”) or is finitely generated and of transcendence degree 1 over a finite field of constants $k$ of characteristic $p$ and cardinality $q$ (the “function field case”). Let $K/F$ be a finite Galois extension of global fields with Galois group $G$. In the function field case, we will write $k'$ for the field of constants of $K$ and $q'$ for its cardinality. The infinite primes of a function field $F$ are the primes which lie above the prime of $k(t)$ corresponding to the pole of $t$. A function field having only one infinite prime is called imaginary. Now let $S$ be any nonempty finite set of primes of the global field $F$, containing the archimedean primes in the number field case. Where confusion is unlikely, we will denote by $S$ (also) the set of primes of $K$ which lie above the primes in $S$. In the remaining instances, this set will be denoted by $S_K$. The set of non-archimedean primes of $F$ which ramify in $K$ will be denoted by $R$. Now, for each prime $v \in R \cup S$, we fix once and for all a prime $w$ of $K$ lying above $v$. We will write $\mathcal{I}_{F,S}$ for the group of fractional ideals (or divisors) of $F$ with support outside $S$. A similar notation will apply to $K$. Finally, the set of archimedean primes of a number field $F$ will be denoted by $S_\infty$.

2. The capitulation kernel.

We begin by considering the exact sequence of $G$-modules

$$1 \to U_{K,S} \to K^* \to \mathcal{I}_{K,S} \to C_{K,S} \to 0,$$

which we split into two short exact sequences of $G$-modules as follows:

$$1 \to U_{K,S} \to K^* \to K^*/U_{K,S} \to 1$$

(1)

and

$$1 \to K^*/U_{K,S} \to \mathcal{I}_{K,S} \to C_{K,S} \to 0.$$  

(2)

Lemma 2.1. We have

$$H^1(G, \mathcal{I}_{K,S}) = 0.$$  

Proof. This is well-known. See, for example, [28, Lemma 2.1].
Set $\text{Br}(K/F) = H^2(G, K^*) = \text{Ker} [\text{Br}F \to (\text{Br}K)^G]$. By the preceding lemma and Hilbert's Theorem 90, (1) and (2) yield the following exact sequences:

$$1 \to F^*/U_{F,S} \to (K^*/U_{K,S})^G \to H^1(G, U_{K,S}) \to 0,$$  \hspace{1cm} (3)

$$0 \to H^1(G, K^*/U_{K,S}) \to H^2(G, U_{K,S}) \to \text{Br}(K/F) \to H^2(G, K^*/U_{K,S}) \to H^3(G, U_{K,S}),$$ \hspace{1cm} (4)

$$1 \to (K^*/U_{K,S})^G \to I_{K,S}^G \to (C_{K,S})^G_{\text{trans}} \to 0,$$ \hspace{1cm} (5)

$$0 \to (C_{K,S})^G_{\text{trans}} \to C_{K,S}^G \to H^1(G, K^*/U_{K,S}) \to 0,$$ \hspace{1cm} (6)

and

$$0 \to H^1(G, C_{K,S}) \to H^2(G, K^*/U_{K,S}) \to H^2(G, I_{K,S}),$$ \hspace{1cm} (7)

where, by definition\(^2\),

$$(C_{K,S})^G_{\text{trans}} = \text{Ker} [C_{K,S}^G \to H^1(G, K^*/U_{K,S})].$$

Note that, by (5), $(C_{K,S})^G_{\text{trans}}$ is trivial exactly when every ambiguous $S$-ideal of $K$ is principal.

For subsequent use, we note that the connecting homomorphism $(K^*/U_{K,S})^G \to H^1(G, U_{K,S})$ appearing in (3) maps a class $\beta U_{K,S} \in (K^*/U_{K,S})^G$ to the cohomology class $\{\xi_\beta\} \in H^1(G, U_{K,S})$ which is represented by the 1-cocycle $\xi_\beta: G \to U_{K,S}$ defined by $\xi_\beta(\sigma) = \beta^\sigma/\beta \ (\sigma \in G)$.

Now let $j_{K/F,S}: C_{F,S} \to C_{K,S}^G$ be the natural capitulation map. It is not difficult to see, using the general description of the connecting homomorphism $C_{K,S}^G \to H^1(G, K^*/U_{K,S})$ appearing in (6) (see, e.g., [2, p.97]), that the image of $j_{K/F,S}$ is contained in $(C_{K,S})^G_{\text{trans}}$. We will write $j_{K/F,S}': C_{F,S} \to (C_{K,S})^G_{\text{trans}}$ for the map induced by $j_{K/F,S}$. Clearly, $\text{Ker} j_{K/F,S}' = \text{Ker} j_{K/F,S}$ and (6) immediately yields the following proposition.

**Proposition 2.2.** There exists a natural exact sequence

$$0 \to \text{Coker } j_{K/F,S}' \to \text{Coker } j_{K/F,S} \to H^1(G, K^*/U_{K,S}) \to 0. \quad \Box$$

Now, we have a natural exact commutative diagram

$$
\begin{array}{cccccccc}
1 & \to & F^*/U_{F,S} & \to & I_{F,S} & \to & C_{F,S} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow j_{K/F,S}' & \\
1 & \to & (K^*/U_{K,S})^G & \to & I_{K,S}^G & \to & (C_{K,S})^G_{\text{trans}} & \to & 0
\end{array}
$$  \hspace{1cm} (8)

where the top row is the exact sequence (2) over $F$, the bottom row is (5), and the left vertical map comes from (3). The middle vertical map is injective and its cokernel has the following well-known description.

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\(^2\)The composite $C_{K,S}^G \to H^1(G, K^*/U_{K,S}) \to H^2(G, U_{K,S})$ is known as the *transgression map*, and $(C_{K,S})^G_{\text{trans}}$ might well be called the group of *transgressive ambiguous classes*. However, $(C_{K,S})^G_{\text{trans}}$ is better known as the group of *strongly ambiguous classes*.}


Lemma 2.3. There exists a canonical isomorphism
\[ \text{Coker} \left[ I_{F,S} \to (I_{K,S})^G \right] \cong \bigoplus_{v \in R \setminus S} H^1(G_w, U_w). \]

Proof. We give a proof of this well-known result because we will need some of the maps defined below. Let \( D = \sum_{v \notin S} \sum_{w \mid v} n_{w}(D) w \in I_{K,S}. \) Since \( G \) permutes transitively the primes of \( K \) lying above the same prime of \( F \), we have \( D \in (I_{K,S})^G \) if and only if, for every \( v \notin S \), the coefficients \( n_{w}(D) \) coincide for all \( w \mid v \). Write \( n_{v}(D) \) for this common value. Then \( D \) belongs to the image of the map \( I_{F,S} \to I_{K,S}, \sum m_{v} v \mapsto \sum m_{v} e_{v} \sum_{w \mid v} w \), if and only if \( n_{v}(D) \) is divisible by \( e_{v} \) for each \( v \notin S \). Thus there exists a canonical isomorphism
\[ \text{Coker} \left[ I_{F,S} \to (I_{K,S})^G \right] \cong \bigoplus_{v \in R \setminus S} \mathbb{Z}/e_{v} \mathbb{Z} \]
which maps the class of \( D \in (I_{K,S})^G \) to \((n_{v}(D) \mod e_{v})_{v \in R \setminus S} \in \bigoplus_{v \in R \setminus S} \mathbb{Z}/e_{v} \mathbb{Z}.

On the other hand, for each \( w \notin S_{K} \), there exists a natural exact sequence
\[ 1 \to U_{w} \to K^{*}_{w} \xrightarrow{\text{ord}_{w}} \mathbb{Z} \to 0, \]
where the map \( \text{ord}_{w} \) assigns the value 1 to a fixed uniformizing parameter of \( K_{w} \). Let \( v \) be the prime of \( F \) lying below \( w \) and write \( G_{w} = \text{Gal}(K_{w}/F_{v}). \) We have a natural exact commutative diagram
\[ \begin{array}{cccccc}
1 & \to & U_{v} & \to & F^{*}_{v} & \to & e_{v} \mathbb{Z} & \to & 0 \\
& & \| & & \| & & \| & & \| \\
1 & \to & U_{w}^{G_{w}} & \to & (K^{*}_{w})^{G_{w}} & \to & \mathbb{Z} & \to & H^1(G_{w}, U_{w}) & \to & 0.
\end{array} \]
Thus we have a canonical isomorphism
\[ \mathbb{Z}/e_{v} \mathbb{Z} \cong H^1(G_{w}, U_{w}) \]
which maps a class \( m \mod e_{v} \in \mathbb{Z}/e_{v} \mathbb{Z} \) to the cohomology class \( \{\xi_{m}\} \in H^1(G_{w}, U_{w}) \) represented by the 1-cocycle \( \xi_{m}: G_{w} \to U_{w} \) given by \( \xi_{m}(\sigma) = \beta^{\sigma}/\beta \) \( (\sigma \in G_{w}), \)
where \( \beta \in K^{*}_{w} \) satisfies \( \text{ord}_{w}(\beta) = m. \)

We now apply the snake lemma to diagram (8) and obtain an exact sequence
\[ 0 \to \text{Ker} j_{K/F,S} \to \text{Coker} \left[ F^{*}/U_{F,S} \to (K^{*}/U_{K,S})^{G} \right] \xrightarrow{i} \text{Coker} \left[ I_{F,S} \to (I_{K,S})^G \right] \]
\[ \xrightarrow{\lambda} \text{Coker} j'_{K/F,S} \to 0, \]
where the map \( i \) is induced by the natural map \( K^{*}/U_{K,S} \to I_{K,S}. \) This map fits into a commutative diagram
\[ \begin{array}{ccc}
\text{Coker} \left[ F^{*}/U_{F,S} \to (K^{*}/U_{K,S})^{G} \right] & \xrightarrow{i} & \text{Coker} \left[ I_{F,S} \to (I_{K,S})^G \right] \\
\| & & \| \\
H^1(G, U_{K,S}) & \xrightarrow{\lambda} & \bigoplus_{v \in R \setminus S} H^1(G_{w}, U_{w}),
\end{array} \]
where the right vertical map is the isomorphism of Lemma 2.3, the left vertical map is induced by the connecting homomorphism \((K^*/U_{K,S})^G \to H^1(G, U_{K,S})\) described earlier, and the bottom ("localization") map \(\lambda\) may be described as follows: let \(c \in H^1(G, U_{K,S})\) be represented by the 1-cocycle \(\xi: G \to U_{K,S}, \sigma \mapsto \beta^\sigma/\beta\), where \(\beta U_{K,S} \in (K^*/U_{K,S})^G\). Then the \(v\)-component of \(\lambda(c)\) \((v \in R \setminus S)\) is the cohomology class in \(H^1(G_w, U_w)\) represented by the 1-cocycle \(\xi_v: G_w \to U_w\) given by \(\xi_v(\sigma) = \beta^\sigma/\beta\) \((\sigma \in G_w)\).

The above argument yields the following result.

**Theorem 2.4.** There exists an exact sequence

\[
0 \to \text{Ker} j_{K/F,S} \to H^1(G, U_{K,S}) \xrightarrow{\lambda} \bigoplus_{v \in R \setminus S} H^1(G_w, U_w) \to \text{Coker} j'_{K/F,S} \to 0,
\]

where \(j_{K/F,S}\) is the capitulation map, \(j'_{K/F,S}: C_F \to (C_{K,S})_{\text{trans}}^G\) is induced by \(j_{K/F,S}\) and \(\lambda\) is the localization map described above.

By the description of the map \(\lambda\) given above and the proof of Lemma 2.3, the following is an immediate consequence of the theorem.

**Corollary 2.5.** The capitulation kernel \(\text{Ker} j_{K/F,S}\) is canonically isomorphic to the subgroup of \(H^1(G, U_{K,S})\) of all cohomology classes which are represented by a 1-cocycle \(\xi: G \to U_{K,S}\) of the form \(\xi(\sigma) = \beta^\sigma/\beta\) \((\sigma \in G)\), where \(\beta U_{K,S} \in (K^*/U_{K,S})^G\) satisfies \(\text{ord}_w(\beta) \equiv 0 \mod e_v\) for all \(v \in R \setminus S\). \(\square\)

**Remark 2.6.** An equivalent form of the exact sequence of Theorem 2.4 was previously obtained by H.van der Wall [28, proof of Theorem 1.3, bottom of p.7] in the case that \(F\) is a number field and \(S\) is the set of all archimedean primes of \(F\). See also [23, Theorem 2, p.46]. Further, we invite the reader to compare Corollary 2.5 (for number fields and \(S = S_\infty\)) with [23, Corollary, p.46].

Now set

\[S' = S \cup (R \setminus S) = S \cup R.\]

We define integers \(d_v\), for \(v \in S'\), when \(K/F\) is a cyclic extension of degree \(n\) as follows:

\[
d_v = \begin{cases} [K_w: F_v] & \text{if } v \in S, \\ e_v & \text{if } v \in R \setminus S. \end{cases}
\]

Clearly, each \(d_v\) is a divisor of \(n\). Set

\[D = \text{l.c.m.}\{d_v : v \in S'\}\]

and

\[n_0 = n/D.\]

Then we have the following generalization of Hilbert’s Theorem 94.
Theorem 2.7. Let $F$ be a global field and let $K/F$ be a cyclic extension of degree $n$. Let $d_v$, $D$ and $n_0$ be the integers (9), (10) and (11), respectively. Then at least $n_0/(n_0,\prod_{v\in S}d_v/D)$ $S$-ideal classes of $F$ capitulate in $K$.

Proof. The proof of Lemma 2.3 and Theorem 2.4 immediately yield the order relation

$$[\operatorname{Ker} j_{K/F,S}] \prod_{v\in R\setminus S} e_v = [H^1(G,U_{K,S})] [\operatorname{Coker} j'_{K/F,S}].$$

On the other hand, the well-known formula for the Herbrand quotient of the $G$-module $U_{K,S}$ (see, e.g., [26, Proof of Theorem 8.3, p.178]) yields the identity

$$n [\widehat{H}^0(G,U_{K,S})] = [H^1(G,U_{K,S})] \prod_{v\in S} [K_w:F_v].$$

Combining the preceding formulas, we obtain the equality

$$\frac{n}{n,\prod_{v\in S}d_v} [\widehat{H}^0(G,U_{K,S})] [\operatorname{Coker} j'_{K/F,S}] = \frac{\prod_{v\in S}d_v}{n,\prod_{v\in S}d_v} [\operatorname{Ker} j_{K/F,S}].$$

It follows that $n/(n,\prod_{v\in S}d_v) = n_0/(n_0,\prod_{v\in S}d_v/D)$ divides $[\operatorname{Ker} j_{K/F,S}]$, as asserted. $\square$

The group $\operatorname{Coker} j'_{K/F,S}$ appears to be as difficult to compute as $\operatorname{Ker} j_{K/F,S}$. We close this section by giving an alternative description of $\operatorname{Coker} j'_{K/F,S}$ (see Proposition 2.9 below) which may prove useful in future research on this group. We need the following approximation lemma.

Lemma 2.8. The natural map

$$F^*/U_{F,S} \rightarrow \bigoplus_{v\in R\setminus S} F^*/U_v$$

is surjective.

Proof. For each $v \in R \setminus S$, let $x_vU_v \in F^*/U_v$ and set $m = \max\{\operatorname{ord}_v(x_v): v \in R \setminus S\}$. By the strong approximation theorem [2, p.67], there exists a $\beta \in F^*$ such that $\operatorname{ord}_v(\beta - x_v) > m$ for all $v \in R \setminus S$. It follows that $\operatorname{ord}_v(\beta) = \operatorname{ord}_v(x_v)$ for every $v \in R \setminus S$, i.e., $\beta x_v^{-1} \in U_v$ for every $v \in R \setminus S$. Consequently $\beta U_{F,S} \in F^*/U_{F,S}$ maps to $(x_vU_v)_{v\in R\setminus S} \in \bigoplus_{v\in R\setminus S} F^*/U_v$. $\square$

We now consider the exact commutative diagram

$$\begin{array}{cccccc}
1 & \rightarrow & F^*/U_{F,S} & \rightarrow & (K^*/U_{K,S})^G & \rightarrow & H^1(G,U_{K,S}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \lambda & & \\
1 & \rightarrow & \bigoplus_{v\in R\setminus S} F^*/U_v & \rightarrow & \bigoplus_{v\in R\setminus S} (K^*_w/U_w)^G & \rightarrow & \bigoplus_{v\in R\setminus S} H^1(G_w,U_w) & \rightarrow & 0,
\end{array}$$

where the top row is the exact sequence (3), the bottom row is the direct sum over $v \in R \setminus S$ of exact sequences analogous to (3) for the extension $K_w/F_v$, and the

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$^3$See [28] for the computation of $[\operatorname{Coker} j'_{K/F,S}]$ when $S = S_{\infty}$ in some particular cases, notably when $K/F$ is an extension of prime degree of a quadratic number field $F$. 

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unlabeled vertical maps are the natural ones. By the preceding lemma, the above diagram yields an isomorphism

$$\text{Coker } \lambda \simeq \text{Coker } \left[ (K^*/U_{K,S})^G \to \bigoplus_{v \in R\backslash S} (K^*_w/U_w)^{G_w} \right].$$  \hfill (12)

Let

$$\varphi: (K^*/U_{K,S})^G \to \bigoplus_{v \in R\backslash S} \mathbb{Z}$$

be given by

$$\varphi(xU_{K,S}) = (\text{ord}_w(x))_{v \in R\backslash S} \quad (x \in K^*),$$  \hfill (13)

i.e., $\varphi$ is the composite of the natural map $(K^*/U_{K,S})^G \to \bigoplus_{v \in R\backslash S} (K^*_w/U_w)^{G_w}$ and the isomorphism $\bigoplus_{v \in R\backslash S} (K^*_w/U_w)^{G_w} \cong \bigoplus_{v \in R\backslash S} \mathbb{Z}$. Then (12) induces an isomorphism

$$\text{Coker } \lambda \simeq \text{Coker } \varphi.$$  

On the other hand, Theorem 2.4 yields an isomorphism $\text{Coker } j'_{K/F,S} \simeq \text{Coker } \lambda$. Thus the following holds.

**Proposition 2.9.** There exists an isomorphism $\text{Coker } j'_{K/F,S} \simeq \text{Coker } \varphi$, where $\varphi$ is the map (13). \hfill □

### 3. The capitulation cokernel.

We now recall the exact sequence (4):

$$0 \to H^1(G, K^*/U_{K,S}) \to H^2(G, U_{K,S}) \to \text{Br}(K/F) \to H^2(G, K^*/U_{K,S}) \to H^3(G, U_{K,S}).$$

We note that the map $H^2(G, U_{K,S}) \to \text{Br}(K/F) = H^2(G, K^*)$ appearing above is induced by the inclusion $U_{K,S} \subset K^*$. Now let

$$B(\mathcal{O}_{F,S}, \mathcal{O}_{K,S}) = \text{Ker } \left[ \text{Br}(K/F) \to H^2(G, \mathcal{I}_{K,S}) \right],$$  \hfill (14)

where the map involved is induced by the natural map $K^* \to \mathcal{I}_{K,S}$, $x \to (x)$. Then, by (7), the above exact sequence induces an exact sequence

$$0 \to H^1(G, K^*/U_{K,S}) \to H^2(G, U_{K,S}) \to B(\mathcal{O}_{F,S}, \mathcal{O}_{K,S}) \to H^1(G, C_{K,S}) \to H^3(G, U_{K,S}).$$  \hfill (15)

See [16, §1] for the details. By combining the preceding exact sequence with Proposition 2.2, we obtain the following result.

**Proposition 3.1.** There exists a natural six-term exact sequence

$$0 \to \text{Coker } j'_{K/F,S} \to \text{Coker } j_{K/F,S} \to H^2(G, U_{K,S}) \to B(\mathcal{O}_{F,S}, \mathcal{O}_{K,S}) \to H^1(G, C_{K,S}) \to H^3(G, U_{K,S}),$$

where $B(\mathcal{O}_{F,S}, \mathcal{O}_{K,S})$ is the group (14). \hfill □

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4This exact sequence was discovered independently by M.Auslander and A.Brumer in [3] and by S.Chase, D.Harrison and A.Rosenberg in [6]. It was used in [5] and in [16]. More recently, it resurfaced in [9, Appendix], where a particular case of (15) was derived from a Hochschild-Serre spectral sequence in étale cohomology.
We will now use class field theory to compute $B(\mathcal{O}_F, \mathcal{O}_K, S)$. This computation generalizes [9, Lemma A.1].

By global class field theory, $\text{Br} F$ is naturally isomorphic to the kernel of the map
\[ \sum \text{inv}_v : \bigoplus_{v \notin S} \text{Br}_{F_v} \to \mathbb{Q}/\mathbb{Z}. \]
Under this isomorphism, the subgroup $\text{Br}(K/F)$ of $\text{Br} F$ may be identified with the kernel of the induced map
\[ \sum \text{inv}_v : \bigoplus_{v \in S} H^2(G_w, \mathcal{K}_w) \to \mathbb{Q}/\mathbb{Z}, \]
where, for each $v$, we regard $H^2(G_w, \mathcal{K}_w)$ as a subgroup of $\text{Br}_{F_v}$ via the inflation map $\text{Inf}_{w/v} : H^2(G_w, \mathcal{K}_w) \to \text{Br}(F_v)$ (which is injective). On the other hand, there exist well-known canonical isomorphisms
\[ H^2(G, \mathcal{I}_{K,S}) \simeq \bigoplus_{v \notin S} H^2(G_w, (\mathfrak{p}_v)) \simeq \bigoplus_{v \notin S} H^2(G_w, \mathbb{Z}) \]
\[ \simeq \bigoplus_{v \not\in S} H^2(G_w, \mathcal{K}_w/\mathcal{U}_w). \]
Let $\psi : H^2(G, \mathcal{I}_{K,S}) \to \bigoplus_{v \notin S} H^2(G_w, \mathcal{K}_w/\mathcal{U}_w)$ be the composite of the above isomorphisms. Then the diagram
\[ \begin{array}{ccc}
\text{Br}(K/F) & \rightarrow & \left( \bigoplus_{v \in S} H^2(G_w, \mathcal{K}_w) \right) \oplus \left( \bigoplus_{v \not\in S} H^2(G_w, \mathcal{K}_w) \right) \\
& & \downarrow \\
H^2(G, \mathcal{I}_{K,S}) & \rightarrow \psi & \bigoplus_{v \not\in S} H^2(G_w, \mathcal{K}_w/\mathcal{U}_w),
\end{array} \]
where $f_1$ is the zero map and $f_{2,v} : H^2(G_w, \mathcal{K}_w) \to H^2(G_w, \mathcal{K}_w/\mathcal{U}_w)$, for each $v \not\in S$, is induced by the natural map $\mathcal{K}_w \to \mathcal{K}_w/\mathcal{U}_w$, commutes. It follows that the identification
\[ \text{Br}(K/F) = \text{Ker} \left[ \bigoplus_{v \in S} H^2(G_w, \mathcal{K}_w) \sum \text{inv}_v \rightarrow \mathbb{Q}/\mathbb{Z} \right] \]
induces an identification
\[ B(\mathcal{O}_F, \mathcal{O}_K, S) = \text{Ker} \left[ \bigoplus_{v \in S} H^2(G_w, \mathcal{K}_w) \oplus \text{Ker} f_{2,v} \sum \text{inv}_v \rightarrow \mathbb{Q}/\mathbb{Z} \right]. \quad (16) \]
Now, since $H^1(G_w, \mathcal{K}_w/\mathcal{U}_w) \simeq H^1(G_w, \mathbb{Z}) = 0$ for each $v \not\in S$, there exist canonical isomorphisms
\[ \text{Ker} f_{2,v} \simeq H^2(G_w, \mathcal{U}_w) \quad (v \not\in S). \quad (17) \]
Note that the latter group is zero if $v \not\in R$. Now recall the set $S' = S \cup (R \setminus S) = S \cup R$. We define integers $d_v$, for $v \in S'$, as follows:\footnote{If $K/F$ is cyclic, then the integers (18)-(19) agree with the integers (9)-(10). See, for example, [1].}
\[ d_v = \begin{cases} 
[K_w : F_v] & \text{if } v \in S \\
H^2(G_w, \mathcal{U}_w) & \text{if } v \in R \setminus S.
\end{cases} \quad (18) \]
Further, set
\[ D = \text{l.c.m.}\{d_v : v \in S'\}. \]  
(19)

Then the invariant map \( \text{inv}_v \) induces isomorphisms
\[ H^2(G_w, K^*_w) \simeq d_v^{-1}\mathbb{Z}/\mathbb{Z} \quad (v \in S) \]  
(20)
\[ H^2(G_w, U_w) \simeq d_v^{-1}\mathbb{Z}/\mathbb{Z} \quad (v \in R \setminus S). \]  
(21)

It follows from (16), (17), (20) and (21) that there exists a natural isomorphism
\[ B(\mathcal{O}_F, \mathcal{O}_K, S) \simeq \text{Ker} \left[ \bigoplus_{v \in S'} d_v^{-1}\mathbb{Z}/\mathbb{Z} \xrightarrow{\Sigma} D^{-1}\mathbb{Z}/\mathbb{Z} \right], \]
where \( \Sigma \) is the summation map \( (x_v) \rightarrow \sum x_v \). The latter map is surjective (see [10, Lemma 1.2]), whence the following holds.

**Lemma 3.2.** There exists an exact sequence
\[ 0 \rightarrow B(\mathcal{O}_F, \mathcal{O}_K, S) \rightarrow \bigoplus_{v \in S'} \mathbb{Z}/d_v\mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z}/D\mathbb{Z} \rightarrow 0, \]
where \( d_v \) and \( D \) are the integers (18) and (19), respectively. In particular,
\[ [B(\mathcal{O}_F, \mathcal{O}_K, S)] = \left( \prod_{v \in S'} d_v \right)/D. \]

Now, combining Proposition 3.1 and Lemma 3.2, we obtain the following result.

**Theorem 3.3.** There exists an exact sequence
\[ 0 \rightarrow \text{Coker} j'_{K/F,S} \rightarrow \text{Coker} j_{K/F,S} \rightarrow H^2(G, U_{K,S}) \rightarrow B(\mathcal{O}_F, \mathcal{O}_K, S) \]
\[ \rightarrow H^1(G, C_{K,S}) \rightarrow H^3(G, U_{K,S}), \]
where \( B(\mathcal{O}_F, \mathcal{O}_K, S) \) is a group of order \( \left( \prod_{v \in S'} d_v \right)/D. \)

**Remark 3.4.** We note that, via the isomorphisms (16) and (17), the map
\[ H^2(G, U_{K,S}) \rightarrow B(\mathcal{O}_F, \mathcal{O}_K, S) \]
appearing in the exact sequence of the theorem is induced by the inclusions \( U_{K,S} \subset U_w (v \notin S) \) and \( U_{K,S} \subset K^*_w (v \in S) \). Thus, by (15),
\[ H^1(G, K^*/U_{K,S}) \simeq \text{Ker} \left[ H^2(G, U_{K,S}) \rightarrow B(\mathcal{O}_F, \mathcal{O}_K, S) \right] \]
\[ \simeq \text{Ker} \left[ H^2(G, U_{K,S}) \rightarrow \bigoplus_{v \in R \setminus S} H^2(G_w, U_w) \oplus \bigoplus_{v \in S} H^2(G_w, K^*_w) \right], \]
where the last map is induced by the inclusions \( U_{K,S} \subset U_w (v \notin S) \) and \( U_{K,S} \subset K^*_w (v \in S) \).

We now derive some consequences of Theorem 3.3. The first one generalizes [22, Theorem 2(C), p.161] (see Proposition 2.2).

**Corollary 3.5.** Assume that the integers \( d_v \), where \( v \in S' \), are pairwise coprime. Then there exists a natural exact sequence
\[ 0 \rightarrow \text{Coker} j'_{K/F,S} \rightarrow \text{Coker} j_{K/F,S} \rightarrow H^2(G, U_{K,S}) \rightarrow 0. \]

**Corollary 3.6.** Assume that \( H^2(G, U_{K,S}) = 0 \). Then \( H^1(G, C_{K,S}) \) contains at least \( \left( \prod_{v \in S'} d_v \right)/D \) elements.
The next theorem refers to the “limit case” where $[K:F]$ is prime to $h_{K,S}$.

**Theorem 3.7.** Let $K/F$ be a Galois extension of global fields, of degree $n$. Assume that $n$ is prime to $h_{K,S}$. Then there exists a natural isomorphism

$$H^2(G, U_{K,S}) \simeq \text{Ker} \left[ \bigoplus_{v \in S'} \mathbb{Z}/d_v \mathbb{Z} \to \mathbb{Z}/D \mathbb{Z} \right].$$

In particular,

$$[H^2(G, U_{K,S})] = (\prod_{v \in S'} d_v) / D.$$

**Proof.** Let $N: C_{K,S} \to C_{K,S}^G$ be the map $c \mapsto \prod_{\sigma \in G} c^\sigma$. Then $N = j_{K/F,S} \circ N_{K/F}$, where $N_{K/F}: C_{K,S} \to C_{F,S}$ is induced by the norm of ideals. Consequently, Coker $j_{K/F,S}$ is a quotient of $C_{K,S}^G / NC_{K,S} = \hat{H}^0(G, C_{K,S})$. Since the latter group is annihilated by multiplication by $(n, h_{K,S}) = 1$, we conclude that Coker $j_{K/F,S} = 0$. The theorem now follows from Theorem 3.3 since $H^1(G, C_{K,S})$ is also annihilated by multiplication by $(n, h_{K,S}) = 1$. □

**Remark 3.8.** If $K/F$ is cyclic of degree $n$, and $n$ is prime to $h_{K,S}$, then the order of Ker $j_{K/F,S}$ may be computed explicitly. Indeed, by the theorem, $[\hat{H}^0(G, U_{K,S})] = [H^2(G, U_{K,S})] = (\prod_{v \in S'} d_v) / D$. On the other hand, the proof of the above theorem and Proposition 2.2 show that Coker $j_{K/F,S} = 0$. Finally the proof of Theorem 2.7 yields

$$[\text{Ker } j_{K/F,S}] = n_0,$$

where $n_0 = n / D$.

4. Genus fields.

In this section we consider abelian extensions of certain base fields $F$ which are equal to their own genus field relative to $F$. For such extensions (and a minimal set $S$), the group $(C_{K,S})_{\text{trans}}^G$ is zero and the following holds.

**Theorem 4.1.** Let $K/F$ be a finite abelian extension of number fields. Assume that $K$ is its own genus field relative to $F$. Assume, furthermore, that one of the following conditions holds:

(a) $K/F$ is cyclic, or

(b) $F$ is either the rational field $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$ of discriminant $< -4$ and conductor prime to 2.

Then there exists an exact sequence

$$0 \to C_F \to H^1(G, U_K) \to \bigoplus_{v \nonarch} \mathbb{Z}/e_v \mathbb{Z} \to 0.$$

Furthermore, there exists a canonical isomorphism

$$C^G_K \simeq H^1(G, K^*/U_{K,S}).$$

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6This condition can always be satisfied by enlarging $S$ appropriately. We also note that this case has been previously studied by H.Yokoi in [29, §4].
In particular, if $h_F = 1$, there exists an isomorphism\(^7\)
\[
H^1(G, U_K) \simeq \bigoplus_{v \text{ non-arch.}} \mathbb{Z}/e_v \mathbb{Z}.
\]

**Proof.** We apply results from the preceding sections with $S$ equal to the set of all archimedean primes of $F$. By theorems of Tannaka-Terada, H.Furuya [8] and C.Thiebaud [27], every ambiguous ideal of $K$ is principal. Therefore (\(G\times))^\text{trans}_K = 0 \text{ (see (5))}, \text{ whence Ker } j_{K/F} = C_F, \text{ Coker } j'_{K/F} = 0 \text{ and Coker } j_{K/F} = C_K^2. \text{ The theorem is now immediate from Theorem 2.4 and Proposition 2.2.} \Box

We note that the preceding theorem applies, in particular, to ray class fields over $F$ since such fields are equal to their own genus field relative to $F$.\(^8\)

**Remark 4.2.** There exists a function field analog of the preceding theorem. Indeed, let $F$ be a function field, let $\infty$ be a fixed place of $F$ and let $S = \{\infty\}$. Further, let $K$ be a finite abelian extension of $F$ where $\infty$ is tamely ramified and the decomposition and inertia groups of $\infty$ agree. These extensions were called “of type (\(\ast\))” in [7]. Assume further that $K$ is its own $S$-genus field. Then the main theorem of [7], combined with (5), shows that (\(G\times)^\text{trans}_K = 0. \text{ Consequently, Theorem 2.4 yields an exact sequence}
\[
0 \rightarrow C_{F,S} \rightarrow H^1(G, U_{K,S}) \rightarrow \bigoplus_{v \neq \infty} \mathbb{Z}/e_v \mathbb{Z} \rightarrow 0.
\]
In particular, if $h_{F,S} = 1$, there exists an isomorphism
\[
H^1(G, U_{K,S}) \simeq \bigoplus_{v \neq \infty} \mathbb{Z}/e_v \mathbb{Z}.
\]
The above generalizes [7, Corollary 4.3].

5. CYCLIC EXTENSIONS.

In this Section we assume that $K/F$ is a cyclic extension of degree $n$.

Set
\[
W_{F,S} = U_{F,S} \cap N_{K/F} K^*.
\] (22)

Then, by (4) and the periodicity of the cohomology of cyclic groups, we have
\[
H^1(G, K^*/U_{K,S}) \simeq \text{Ker } [H^2(G, U_{K,S}) \rightarrow H^2(G, K^*)]
\simeq \text{Ker } [U_{F,S}/N_{K/F} U_{K,S} \rightarrow F^*/N_{K/F} K^*]
= W_{F,S}/N_{K/F} U_{K,S}
\]
(cf. [22, Theorem 2(B), p.161]). Thus, by Proposition 2.2, the following holds.

\(^7\)More generally, if $C_F = C_F^e_v$ for every non-archimedean prime $v$, then $H^1(G, U_K) \simeq C_F \oplus (\bigoplus_{v \text{ non-arch.}} \mathbb{Z}/e_v \mathbb{Z})$.

\(^8\)See [27, Lemma 2.1] for a general description of the abelian extensions of $F$ which are equal to their own genus field relative to $F$. 

Lemma 5.1. Assume that $K/F$ is a cyclic extension of global fields. Then there exists a natural exact sequence
\[ 0 \to \text{Coker} \ j'_{K/F,S} \to \text{Coker} \ j_{K/F,S} \to W_{F,S}/N_{K/F}U_{K,S} \to 0, \]
where $W_{F,S}$ is the group (22). □

The next result may be regarded as a “dual” of Theorem 2.7.

Theorem 5.2. Let $K/F$ be a cyclic extension of global fields. Assume that $U_{F,S} \subset N_{K/F}K^*$. Then $\text{Coker} \ j_{K/F,S}$ contains at least
\[ \prod_{v \in S'} d_v / D \]
\[ (n_0, \prod_{v \in S'} d_v / D) \]
classes, where $d_v$, $D$ and $n_0$ are the integers (9), (10) and (11), respectively.

Proof. The lemma and the proof of Theorem 2.7 immediately yield the formula
\[ \frac{n}{(n, \prod_{v \in S'} d_v)} [U_{F,S} : W_{F,S}] [\text{Coker} \ j_{K/F,S}] = \frac{\prod_{v \in S'} d_v}{(n, \prod_{v \in S'} d_v)} [\text{Ker} \ j_{K/F,S}]. \]

Therefore
\[ \frac{\prod_{v \in S'} d_v}{(n, \prod_{v \in S'} d_v)} = \frac{\prod_{v \in S'} d_v / D}{(n_0, \prod_{v \in S'} d_v / D)} \]
divides $[U_{F,S} : W_{F,S}] [\text{Coker} \ j_{K/F,S}]$. The result is now clear since $W_{F,S} = U_{F,S}$ by hypothesis. □

Example 5.3. Let $F$ be either $\mathbb{Q}$, an imaginary quadratic number field (with $S = S_\infty$ in both cases) or a function field with $\#S = 1$. Let $K$ be a cyclic extension of $F$ of degree $l^m$, where $m$ is a positive integer and $l$ is a rational prime which is either $\geq 5$ in the number field case or prime to $q - 1$ in the function field case. Then $l^m$ is prime to the order of the finite group $U_{F,S}$, whence $U_{F,S} = N_{K/F}U_{K,S} \subset N_{K/F}K^*$. Thus the hypothesis of Theorem 5.2 is satisfied. Assume now, for simplicity, that in the function field case the prime in $S$ splits completely in $K$. Let $l_{i_1}, l_{i_2}, \ldots, l_{i_r}$ be the ramification indices of the ramified primes of $K/F$, and assume that $m < \sum_{i=1}^r t_i$. Then the theorem asserts that $l^t \mid [\text{Coker} \ j_{K/F,S}]$, where
\[ t = \sum_{i=1}^r t_i - m. \]

In particular, if $K/k(t)$ is a real Artin-Schreier extension, i.e., $K = F(\alpha)$, where $\alpha$ is a root of $x^p - x + Q(t) = 0$ and $Q(t)$ is such that the infinite prime of $k(t)$ splits in $K$, then
\[ \text{rank}_p(C_K^G) \geq r - 1, \]
where $r$ is the number of finite primes of $k(t)$ which ramify in $K$. \footnote{See Corollary 6.2 for the case of imaginary Artin-Schreier extensions.}
We conclude this Section by noting that (15) and the identifications
\[ H^1(G, K^*/U_{K,S}) \simeq W_{F,S} / N_{K/F} U_{K,S} \]
and
\[ H^2(G, U_{K,S}) \simeq U_{F,S} / N_{K/F} U_{K,S} \]
show that \( U_{F,S} / W_{F,S} \) is isomorphic to a subgroup of \( B(O_{F,S}, O_{K,S}) \). Consequently, Lemma 3.2 yields the following generalization of [29, Theorem 1(iv)] (see also [op.cit., Lemma 5]).

**Theorem 5.4.** The index \([U_{F,S} : W_{F,S}]\) divides \( \prod_{v \in S'} d_v / D \). Consequently, if the integers \( d_v \), for \( v \in S' \), are pairwise coprime, then every \( S \)-unit of \( F \) is a norm from \( K \).

**Remark 5.5.** Note that in the “limit case” \([U_{F,S} : W_{F,S}] = \prod_{v \in S'} d_v / D \) (cf. Theorem 3.7), the proof of Theorem 5.2 yields the identity
\[ \ker j_{K/F,S} = n_0 \coker j_{K/F,S} , \]
where \( n_0 = n / D \). In particular, \( \ker j_{K/F,S} \) contains at least \( n_0 \) elements. Compare Remark 3.8.

6. Imaginary extensions of function fields.

The main result of this section is the following

**Theorem 6.1.** Let \( F \) be a function field, let \( K \) be a Galois extension of \( F \) of degree \( n \) and let \( k' \) be the field of constants of \( K \). Assume that \# \( S_K = 1 \) and that \( n \) is prime to \( q' - 1 \), where \( q' \) is the cardinality of \( k' \). Then there exist an exact sequence
\[ 0 \to C_{F,S} \to C_{K,S} \to \bigoplus_{R \setminus S} \mathbb{Z}/e_v \mathbb{Z} \to 0 \]
and an isomorphism
\[ H^1(G, C_{K,S}) \simeq \ker \left[ \bigoplus_{v \in S'} \mathbb{Z}/d_v \mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z}/D \mathbb{Z} \right] . \]

**Proof.** The hypothesis \# \( S_K = 1 \) (i.e., \( S \) consists of exactly one prime of \( F \) and there is only one prime of \( K \) lying above it) implies, by Dirichlet’s Unit Theorem, that \( U_{K,S} = (k')^* \). Hence \( H^1(G, U_{K,S}) = H^1(G, (k')^*) \) is annihilated by multiplication by \( (n, q' - 1) \) for all \( i \geq 1 \). As \( (n, q' - 1) = 1 \) by hypothesis, we conclude that \( H^1(G, U_{K,S}) = 0 \) for all \( i \geq 1 \). The theorem now follows by combining Theorems 2.4 and 3.3. \( \square \)

The following result is one possible application of Theorem 6.1.

**Corollary 6.2.** Let \( F = k(t) \) with \( k \) a finite field of characteristic \( p \), let \( Q(t) \in F^* \) and let \( K = F(\alpha) \), where \( \alpha \) is a root of \( x^p - x + Q(t) = 0 \). Assume that \( K \) is imaginary, i.e., that there exists only one prime of \( K \) lying above the infinite prime of \( F \). Then there exists a canonical isomorphism
\[ C_{K}^{\infty} \simeq (\mathbb{Z}/p\mathbb{Z})^r , \]
where \( r \) is the number of finite primes of \( F \) which ramify in \( K \).

**Proof.** Note that \( K \) is a Galois extension of \( F \) of degree \( p \), which is prime to \( q' - 1 \). Further, \( h_F = 1 \). The corollary is immediate from the theorem. \( \square \)
7. LARGE \( S \).

A (nonempty) set \( S \) of primes of a global field \( F \) is **large relative to** \( K/F \) if \( S \) contains all archimedean primes of \( F \) and all primes that ramify in \( K/F \). In this section we assume that our set \( S \) is large. Note then that \( S' = S \cup (R \setminus S) = S \).

**Theorem 7.1.** Let \( K/F \) be a finite Galois extension of global fields with Galois group \( G \), and let \( S \) be a set of primes of \( F \) which is large relative to \( K/F \) (as defined above). Then there exists an exact sequence

\[
0 \to H^1(G, U_{K,S}) \to C_{F,S} \to C_{K,S}^G \to \Pi^2(G, U_{K,S}) \to 0,
\]

where

\[
\Pi^2(G, U_{K,S}) = \text{Ker} \left[ H^2(G, U_{K,S}) \to \bigoplus_{v \in S} H^2(G_{w}, K_{v}^*) \right],
\]

the map involved being induced by the inclusions \( U_{K,S} \subset K_{v}^* \) \((v \in S)\).

**Proof.** Since \( S \supset R \), Theorem 2.4 shows that \( \text{Ker} j_{K/F,S} = H^1(G, U_{K,S}) \) and that \( \text{Coker} j_{K/F,S} = 0 \). Now Proposition 2.2 and Remark 3.4 yield \( \text{Coker} j_{K/F,S} = \Pi^2(G, U_{K,S}) \), where \( \Pi^2(G, U_{K,S}) \) is as in the statement. \( \square \)

The following corollary of the above theorem should be compared with [9, Proposition A.2].

**Corollary 7.2.** Let \( K/F \) be a finite Galois extension of global fields with Galois group \( G \). Assume that

(a) exactly one prime \( v_0 \) of \( F \) ramifies in \( K \),
(b) \( S \supset S_\infty \cup \{v_0\} \) (i.e., \( S \) is large relative to \( K/F \)), and
(c) every prime in \( S \setminus \{v_0\} \) splits completely in \( K \).

Then there exists an exact sequence

\[
0 \to H^1(G, U_{K,S}) \to C_{F,S} \to C_{K,S}^G \to H^2(G, U_{K,S}) \to 0.
\]

**Proof.** By Lemma 3.2, \( B(\mathcal{O}_{F,S}, \mathcal{O}_{K,S}) \) is a group of order

\[
\left( \prod_{v \in S} [K_w: F_v] \right) / \text{l.c.m.} \{[K_w: F_v] : v \in S\} = [K_{w_0}: F_{v_0}] / [K_{w_0}: F_{v_0}] = 1.
\]

Consequently

\[
\Pi^2(G, U_{K,S}) = \text{Ker} \left[ H^2(G, U_{K,S}) \to B(\mathcal{O}_{F,S}, \mathcal{O}_{K,S}) \right] = H^2(G, U_{K,S})
\]
(see Remark 3.4). The corollary is now immediate from the theorem. \( \square \)

8. **Divisor class groups.**

This Section may be regarded as the “\( S = \emptyset \)” version of Sections 2 and 3 in the function field case. We follow [22] closely.
Let \( K/F \) be a Galois extension of degree \( n \) of function fields with Galois group \( G \). Let \( X' \) denote the unique smooth complete curve over \( k' \) with function field \( K' \). Similarly, let \( X \) be the smooth complete curve over \( k \) with function field \( F \). Further, let \( H = \text{Gal}(K/Fk') \) and \( g = \text{Gal}(k'/k) \), so that \( G/H = g \). The group of \( k' \)-rational points of the Jacobian variety of \( X' \) will be denoted by \( J_{K'} \). Since \( X' \) has a \( k' \)-rational point, we have \( J_K = \text{Pic}^0(\tilde{X}') \). Similarly, let \( J_F = \text{Pic}^0(X) \). Then there exists a natural \textit{capitulation map} \( j_{K/F} : J_F \to J_K^{G} \), which is defined by pulling back line bundles. Its kernel was determined in [22, Theorem 5] under the assumption that \( k = k' \). In general, the following holds

**Theorem 8.1.** Let \( K/F \) be a finite Galois extension of function fields and let \( M \) be the maximal abelian unramified extension of \( k'F \) in \( K \). Then the kernel of the capitulation map \( j_{K/F} : J_F \to J_K^{G} \) is naturally isomorphic to \( \text{Hom}_g(\text{Gal}(M/Fk'),(k')^*) \).

**Proof.** The Hochschild-Serre spectral sequence

\[
H^p(g, H^q(H,(k')^*)) \implies H^{p+q}(G,(k')^*)
\]

yields an exact sequence

\[
0 \to H^1(g,(k')^*) \to H^1(G,(k')^*) \to H^1(H,(k')^*)^g \to \text{Br}(k'/k)
\]

(see, for example, [21, p.309]). Now \( H^1(g,(k')^*) = 0 \) by Hilbert’s Theorem 90 and \( \text{Br}(k'/k) = 0 \) since the Brauer group of a finite field is zero (see, e.g., [24]). Therefore

\[
H^1(G,(k')^*) \cong H^1(H,(k')^*)^g \cong \text{Hom}_g(H,(k')^*)
\]

\[
\cong \text{Hom}_g(\text{Gal}(M'/Fk'),(k')^*)
\]

where \( M' \) is the maximal abelian extension of \( Fk' \) in \( K \). Taking into account these facts, it is not difficult to adapt the proof of Theorem 5 of [22] to the case where \( k' \) is not necessarily equal to \( k \). We leave the details to the reader. \( \square \)

We will now study the cokernel of the capitulation map \( J_F \to J_K^{G} \). Let \( \mathcal{T}_K^0 = \text{Div}^0(X') \). Then there exists a natural exact sequence

\[
1 \to (k')^* \to K^* \to \mathcal{T}_K^0 \to J_K \to 0
\]

inducing an exact sequence

\[
1 \to (K^*/(k')^*)^G \to (\mathcal{T}_K^0)^G \to J_K^G \to H^1(G,K^*/(k')^*) \to H^1(G,\mathcal{T}_K^0) \quad .
\]

Define

\[
(J_K^G)_{\text{trans}} = \text{Ker} \left[ J_K^G \to H^1(G,K^*/(k')^*) \right].
\]

Then the image of \( j_{K/F} \) is contained in \( (J_K^G)_{\text{trans}} \). Let \( j_{K/F}' : J_F \to (J_K^G)_{\text{trans}} \) be the map induced by \( j_{K/F} \). Arguing as in §2 (see, especially, diagram (8)), we obtain the following result.
Proposition 8.2. There exists a natural exact sequence

\[ 0 \to \text{Ker} j_{K/F} \to H^1(G, (k')^*) \to (I_0^G) G/F \to \text{Coker} j'_{K/F} \to 0. \] \[ \square \]

Now define positive integers \( \delta \) and \( \delta' \) by

\[ \text{Im} \left[ \text{Div}(X')^G \to \mathbb{Z} \right] = \delta \mathbb{Z} \]
\[ \text{Im} \left[ \text{Pic}(X')^G \to \mathbb{Z} \right] = \delta' \mathbb{Z}. \]

The integer \( \delta \) is “fairly easy to compute” [22, p.164]. Indeed, it can be shown that, if \( P_1, P_2, \ldots, P_r \) are the primes of \( F \) that ramify in \( K \) and \( e_1, e_2, \ldots, e_r \) are their respective ramification indices, then

\[ \delta = (n, (n/e_1) \text{deg}_F P_1, \ldots, (n/e_r) \text{deg}_F P_r). \]

See [22, proof of Proposition 1, p.163] (by contrast, \( \delta' \) is a “more subtle invariant” [op.cit., p.164]). Clearly, \( \delta' | \delta | n \).

Lemma 8.3. There exists a natural exact sequence

\[ 0 \to (I_0^G) G/F \to \bigoplus_{i=1}^r \mathbb{Z}/e_i \mathbb{Z} \to \delta \mathbb{Z}/n \mathbb{Z} \to 0. \]

Proof. This follows at once from [22, Proposition 1 and Theorem 3(B)]. \[ \square \]

We also have the following exact sequence, which is the analog of Proposition 2.2 in this context.

Proposition 8.4. There exists an exact sequence

\[ 0 \to \text{Coker} j'_{K/F} \to \text{Coker} j_{K/F} \to H^1(G, K^*/k'^*) \to \Gamma \to 0, \]

where \( \Gamma \) is a cyclic group of order \( \delta/\delta' \).

Proof. This is immediate from (24) and [22, Theorem 3(A), p.163], which shows that the image, \( \Gamma \), of \( H^1(G, K^*/(k')^*) \) in \( H^1(G, I_0^G) \) under the map appearing in (24) is a cyclic group of the indicated order. \[ \square \]

Combining statements 8.2, 8.3 and 8.4, we obtain the following analogue of C.Chevalley’s classical “ambiguous class number formula” (cf. [22, Theorem 8, p.166]):

Theorem 8.5. We have

\[ \left[ \frac{J_K^G}{J_F} \right] = \left[ \frac{H^1(G, K^*/(k')^*)}{(n/\delta') \left[ H^1(G, (k')^*) \right]} \right] \prod_{i=1}^r e_i. \] \[ \square \]

\[ ^{16} \text{One might call these integers the “index” and the “period” of } K/F, \text{ respectively.} \]
Now we note that (23) induces an exact sequence
\[ 0 \to H^1(G, K^*/(k')^*) \to H^2(G, (k')^*) \to Br(K/F) \to H^2(G, K^*/(k')^*) \]
(cf. (4)). On the other hand, there exists a natural map \( Br(K/F) \to H^2(G, I_K) \) (where \( I_K = \text{Div}(X') \)) which factors as
\[
Br(K/F) \to H^2(G, K^*/(k')^*) \to H^2(G, I_K),
\]
where the second map is induced by the canonical map \( K^*/(k')^* \to I_K \). We conclude that there exists an exact sequence
\[ 0 \to H^1(G, K^*/(k')^*) \to H^2(G, (k')^*) \to B, \quad (25) \]
where
\[ B = \text{Ker} \left[ Br(K/F) \to H^2(G, I_K) \right]. \]
Now essentially the same argument which proves Lemma 3.2 yields an isomorphism
\[ B \simeq \text{Ker} \left[ \bigoplus_{i=1}^r \mathbb{Z}/d_i \mathbb{Z} \to \mathbb{Z}/[d_1, d_2, \ldots, d_r] \mathbb{Z} \right], \quad (26) \]
where \( d_i = [H^2(G_{P_i}, U_{P_i})] \) for each \( i = 1, 2, \ldots, r \) and \([d_1, d_2, \ldots, d_r] \) denotes the least common multiple of \( d_1, d_2, \ldots, d_r \). In particular, \( B \) is a group of order \((\prod_{i=1}^r d_i)/[d_1, d_2, \ldots, d_r] \). Consequently, if \( K/F \) is cyclic, we have
\[ [B] = \prod_{i=1}^r e_i \mid \prod_{i=1}^r e_i. \]
Now define
\[ m(P_i) = \left( \frac{q^{\deg P_i} - 1}{(q^{\deg P_i} - 1, e_i)}, q - 1 \right) \quad (1 \leq i \leq r) \]
and set
\[ m = (m(P_1), m(P_2), \ldots, m(P_r)). \quad (27) \]
The following proposition shows that the degrees and the ramification indices of the primes that ramify in a cyclic extension \( K/F \) are subject to certain non-obvious constraints.

**Proposition 8.6.** Assume that \( K/F \) is a cyclic extension. Then
\[ \prod_{i=1}^r e_i \equiv 0 \pmod{q-1}, \]
where \( m \) is the integer (27). Consequently, if the ramification indices \( e_i \) are pairwise coprime, then
\[ \frac{q^{\deg P_i} - 1}{(q^{\deg P_i} - 1, e_i)} \equiv 0 \pmod{q - 1} \]
for every \( i = 1, 2, \ldots, r \).

**Proof.** The periodicity of the cohomology of cyclic groups and [22, Theorem 2(B), p.161, and Proposition 2, p.166] show that
\[ \frac{[H^2(G, (k')^*)]}{[H^1(G, K^*/(k')^*)]} = q - 1 \]
\[ \frac{m}{m}. \]
Therefore, \((q - 1)/m \) divides \([B] = (\prod_{i=1}^r e_i)/[e_1, e_2, \ldots, e_r] \), as claimed. Now, if the ramification indices \( e_i \) are pairwise coprime, then necessarily \( m = q - 1 \). The definition of \( m \) now yields the second assertion of the proposition. \( \Box \)
We conclude this paper with the following strengthening of [22, Theorem 14].

**Theorem 8.7.** Assume that $K/F$ is a Galois extension with Galois group $G \cong \bigoplus_{i=1}^{s} Z/\ell Z$, where $s$ is a positive integer and $\ell$ is a prime which divides $q - 1$. Assume, furthermore, that the field of constants of $F$ is algebraically closed in $K$. Then

$$\text{rank}_\ell(J^G_K) \geq (s+1)/2 - r.$$  

**Proof.** By (26), $\text{rank}_\ell(B) \leq r - 1$. Consequently, (25) shows that

$$\text{rank}_\ell(H^2(G, (k^*)^r)) = \text{rank}_\ell(H^2(G, K^*)^r) \leq \text{rank}_\ell(H^1(G, K^*/k^*)) + r - 1.$$  

The rest of the proof is similar to that of Theorem 14 of [22]. \hfill \Box

**APPENDIX.**

The integers $[H^2(G_w, U_w)]$, where $w$ lies above a ramified prime of $K/F$, intervene at various places in the paper. The following result (which may be well-known) relates these integers to the ramification indices $e_v$ of $K/F$.

**Proposition A.1.** Let $v$ be a non-archimedean prime of $F$ and let $w$ be a fixed prime of $K$ lying above $v$. Then there exists an exact sequence

$$0 \to Z/(e_v, f_v)Z \to H^2(G_w, U_w) \to Z/e_vZ,$$

where $f_v$ is the residue class degree. In particular, $[H^2(G_w, U_w)]$ divides $e_v^2$.

**Proof.** Let $F'_v$ be the maximal unramified extension of $F_v$ contained in $K_w$. Set $I_w = \text{Gal}(K_w/F'_v)$. Then $I_w$ is a subgroup of $G_w$ of order $e_v$ and $G_w/I_w = \text{Gal}(F'_v/F_v)$ is a cyclic group of order $f_v$. The exact sequence of terms of low degree belonging to the Hochschild-Serre spectral sequence $H^p(G_w/I_w, H^q(I_w, U_w)) \Rightarrow H^{p+q}(G_w, U_w)$ yields, since $H^1(G_w/I_w, U_w) = 0$ for all $i \geq 1$, an exact sequence

$$0 \to H^1(G_w/I_w, H^1(I_w, U_w)) \to H^2(G_w, U_w) \to H^2(I_w, U_w)$$

(see [21, p.309]). On the other hand, since $K_w/F'_v$ is totally ramified, we have $H^2(I_w, U_w) \simeq H^2(I_w, K^*_w) \simeq Z/e_vZ$, by [24, Ch.XII, Exer.2(b), p.182] and local class field theory. Further, $H^1(I_w, U_w) = Z/e_vZ$ with trivial $G_w/I_w$-action [op.cit.], whence

$$H^1(G_w/I_w, H^1(I_w, U_w)) \simeq \text{Hom}(Z/f_vZ, Z/e_vZ) \simeq Z/(e_v, f_v)Z.$$

We conclude that there exists an exact sequence

$$0 \to Z/(e_v, f_v)Z \to H^2(G_w, U_w) \to Z/e_vZ,$$

as asserted. \hfill \Box

Now define integers $d'_v$ by

$$d'_v = \begin{cases} \lbrack K_w : F_v \rbrack & \text{if } v \in S \\ e_v & \text{if } v \in R \setminus S. \end{cases}$$

The following corollary shows that the hypothesis of Corollary 3.5 is satisfied if the integers $d'_v$ defined above (which are relatively easy to compute) are pairwise coprime.

**Corollary A.2.** Assume that the integers $d'_v$, where $v \in S'$, are pairwise coprime. Then so also are the integers $d_v$ defined by (18). \hfill \Box
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