Anisotropic enhancement of group velocity in a homogenized dielectric composite medium

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Abstract

Under certain circumstances, the group velocity in a homogenized composite medium (HCM) can exceed the group velocity in its component material phases. We explore this phenomenon for a uniaxial dielectric HCM comprising isotropic component material phases distributed as oriented spheroidal particles. The theoretical approach is based upon the Bruggeman homogenization formalism. Enhancement in group velocity in the HCM with respect to the component material phases is shown to be sensitively dependent upon the shape of the component spheroids and their alignment relative to the direction of propagation.

Keywords: Group–velocity enhancement, Bruggeman homogenization formalism, uniaxial dielectric

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1 Introduction

The process of homogenization involves the combination of two (or more) component material phases to produce a single, effectively homogeneous, composite medium [1, 2, 3]. Typically, the constitutive properties of the component material phases are relatively simple as compared with those of the homogenized composite medium (HCM). Through homogenization, novel and potentially useful material properties may be realized [4, 5]. Many examples of

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material properties being extended — or indeed entirely new material properties being realized — as a result of homogenization can be found within the regimes of linear and nonlinear electromagnetics [6].

An interesting result concerns the electromagnetic group velocity in HCMs. Under certain circumstances, the group velocity in an HCM can exceed the group velocities in its component material phases. This issue has been investigated for isotropic dielectric composite mediums using the Maxwell Garnett [7, 8] and the Bruggeman [9] homogenization formalisms. In these studies, an enhancement in group velocity is demonstrated through homogenizing two component material phases, one of which is characterized by a relatively large permittivity and relatively small frequency–dispersive term as compared with the other component material phase.

Enhancement of group velocity in a laminate composite medium has been considered by using a volume–weighted sum to estimate the HCM permittivity [7, 8]. The directional properties of group–velocity enhancement are further explored in this communication. Specifically, we consider a uniaxial dielectric HCM which develops from the homogenization of a random assembly of oriented spheroidal particles. The component material phases are themselves electromagnetically isotropic. Our theoretical analysis is founded upon the Bruggeman homogenization formalism [10].

2 Homogenization

Let us consider the homogenization of a composite medium containing two component material phases, labelled as $a$ and $b$. Both component material phases are taken to be isotropic dielectric mediums: $\epsilon^a$ and $\epsilon^b$ denote the permittivity scalars of phases $a$ and $b$, respectively. In order to focus in particular upon the phenomenon of enhancement of group velocity, without being distracted by the complications arising from dielectric loss, the component material phases are assumed to be nondissipative; i.e., $\epsilon^{a,b} \in \mathbb{R}$. The component material phases are envisioned as random distributions of identically oriented, spheroidal particles. The spheroidal shape — which is taken to be the same for all particles of phases $a$ and $b$ — is parameterized via the shape dyadic

$$\underline{U} = U_\perp \underline{I} + (U_\parallel - U_\perp) \hat{c} \hat{c},$$

where $\underline{I}$ is the identity $3 \times 3$ dyadic and the unit vector $\hat{c}$ is parallel to the spheroid’s axis of rotational symmetry. The spheroid’s surface is described by the vector

$$\underline{r}_s(\theta, \phi) = \eta \underline{U} \cdot \hat{r}(\theta, \phi),$$

with $\hat{r}$ being the radial unit vector from the spheroid’s centroid and specified by the spherical polar coordinates $\theta$ and $\phi$. The linear dimensions of the spheroid, as determined by the parameter $\eta$, are assumed to be small relative to the electromagnetic wavelength(s).

The permittivity dyadic of the resulting HCM,

$$\underline{\epsilon}^{Br} = \epsilon_\perp^{Br} \underline{I} + (\epsilon_\parallel^{Br} - \epsilon_\perp^{Br}) \hat{c} \hat{c},$$

2
is estimated using the Bruggeman homogenization formalism as the solution of the equation

\[ f_a a^a + f_b a^b = 0, \quad (4) \]

where \( f_a \) and \( f_b = 1 - f_a \) denote the respective volume fractions of the material component phases \( a \) and \( b \) [10]. The polarizability dyadics in (4) are defined as

\[ a^\ell = (\epsilon^\ell I - \epsilon^{Br}) \cdot \left[ I + D \cdot (\epsilon^\ell I - \epsilon^{Br}) \right]^{-1}, \quad (\ell = a, b), \quad (5) \]

wherein the depolarization dyadic is given by the surface integral [11, 12, 13]

\[ D = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \frac{1}{\hat{r} \cdot U^{-1} \cdot \epsilon^{Br} \cdot U^{-1} \cdot \hat{r}} \right) U^{-1} \cdot \hat{r} \cdot \hat{c} \cdot U^{-1}. \quad (6) \]

The depolarization dyadic may be expressed as

\[ \mathbf{D} = D_{\perp} \mathbf{I} + (D_{\parallel} - D_{\perp}) \hat{c} \hat{c}, \quad (7) \]

where

\[ D_{\parallel} = \frac{\gamma}{\epsilon_{\parallel}} \Gamma_{\parallel}(\gamma), \quad \Gamma_{\parallel}(\gamma) = \frac{U^2 \epsilon^{Br}}{U_{\parallel}^2 \epsilon_{\parallel}}, \quad (8) \]

\[ D_{\perp} = \frac{1}{\epsilon_{\perp}} \Gamma_{\perp}(\gamma), \quad \Gamma_{\perp}(\gamma) = \frac{U_{\perp}^2 \epsilon^{Br}}{U_{\parallel}^2 \epsilon_{\parallel}}, \quad (9) \]

The terms \( \Gamma_{\parallel} \) and \( \Gamma_{\perp} \) herein are functions of the real–valued parameter

\[ \gamma = \frac{U_{\parallel}^2 \epsilon^{Br}}{U_{\parallel}^2 \epsilon_{\parallel}}; \quad (10) \]

they have the representations

\[ \Gamma_{\parallel}(\gamma) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{\cos^2 \phi \sin^3 \theta}{\cos^2 \theta + \sin^2 \theta (\gamma \cos^2 \phi + \sin^2 \phi)}, \quad (11) \]

\[ \Gamma_{\perp}(\gamma) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{\sin^2 \phi \sin^3 \theta}{\cos^2 \theta + \sin^2 \theta (\gamma \cos^2 \phi + \sin^2 \phi)}. \quad (12) \]
The surface integrals (11) and (12) may be evaluated as

\[
\Gamma_{\parallel}(\gamma) = \begin{cases} \sinh^{-1}\sqrt{\frac{1-\gamma}{1-\gamma}} - \frac{1}{1-\gamma} & \text{for } 0 < \gamma < 1 \\ \frac{1}{\gamma - 1} - \frac{\sec^{-1}\sqrt{\gamma}}{(\gamma - 1)^{\frac{3}{2}}} & \text{for } \gamma > 1 \end{cases}, \\
\Gamma_{\perp}(\gamma) = \begin{cases} \frac{1}{2} \left( \frac{1}{1-\gamma} - \frac{\gamma \sinh^{-1}\sqrt{\frac{1-\gamma}{1-\gamma}}}{(1-\gamma)^{\frac{3}{2}}} \right) & \text{for } 0 < \gamma < 1 \\ \frac{1}{2} \left( \frac{\gamma \sec^{-1}\sqrt{\gamma}}{(\gamma - 1)^{\frac{3}{2}}} - \frac{1}{\gamma - 1} \right) & \text{for } \gamma > 1 \end{cases},
\]

(13) (14)

We exclude the cases of

- the isotropic HCM with \( \gamma = 1 \), and
- the anomalous hyperbolic HCM with \( \gamma < 0 \) [14]

from consideration.

The dyadic Bruggeman equation (4) provides the two nonlinear scalar equations

\[
\frac{\epsilon^a - \epsilon_{\parallel}^{Br}}{1 + D_{\parallel}(\epsilon^a - \epsilon_{\parallel}^{Br})} f_a + \frac{\epsilon^b - \epsilon_{\parallel}^{Br}}{1 + D_{\parallel}(\epsilon^b - \epsilon_{\parallel}^{Br})} f_b = 0,
\]

(15)

\[
\frac{\epsilon^a - \epsilon_{\perp}^{Br}}{1 + D_{\perp}(\epsilon^a - \epsilon_{\perp}^{Br})} f_a + \frac{\epsilon^b - \epsilon_{\perp}^{Br}}{1 + D_{\perp}(\epsilon^b - \epsilon_{\perp}^{Br})} f_b = 0,
\]

(16)

coupled via \( D_{\perp,\parallel} \), which can be solved straightforwardly for \( \epsilon_{\parallel}^{Br} \) and \( \epsilon_{\perp}^{Br} \) using standard numerical techniques.

### 3 Group velocity

Let us consider a wavepacket which is a superposition of planewaves with phasors

\[
E(r) = E_0 \exp(ik \cdot r) \\
H(r) = H_0 \exp(ik \cdot r)
\]

(17)

The group velocity \( \nu_g \) of the wavepacket is conventionally defined in terms of the gradient of the angular frequency \( \omega \) with respect to \( k \) [15]; i.e.,

\[
\nu_g = \nabla_k \omega \bigg|_{\omega = \omega(k_{avg})},
\]

(18)
where \( k_{\text{avg}} \) denotes the average wavenumber of the wavepacket. Herein we adopt the compact notation

\[
\nabla_k \equiv \left( \frac{\partial}{\partial k_x}, \frac{\partial}{\partial k_y}, \frac{\partial}{\partial k_z} \right)
\]

(19)

for the gradient operator with respect to \( \mathbf{k} \), where \((k_x, k_y, k_z)\) is the representation of \( \mathbf{k} \) in terms of its Cartesian components.

In order to calculate the group velocity in the uniaxial dielectric HCM (3), denoted as \( v_{Br} \), we exploit the corresponding planewave dispersion relation as follows. The combination of (3) with the source–free Maxwell curl postulates

\[
\begin{align*}
\nabla \times \mathbf{E}(\mathbf{r}) &= i\omega \mathbf{B}(\mathbf{r}) \\
\nabla \times \mathbf{H}(\mathbf{r}) &= -i\omega \mathbf{D}(\mathbf{r})
\end{align*}
\]

(20)

delivers the vector Helmholtz equation

\[
\left[ (\nabla \times \mathbf{I}) \cdot (\nabla \times \mathbf{I}) - \mu_0 \omega^2 \mathbf{E}_{Br}^2 \right] \cdot \mathbf{E}_0 = 0,
\]

(21)

with \( \mu_0 \) being the permeability of free space. The requirement that (21) provide nonzero solutions for the planewave phasors (17) yields the dispersion relation

\[
W(\mathbf{k}, \omega) = 0,
\]

(22)

wherein the scalar function \( W \) is defined as

\[
W(\mathbf{k}, \omega) = (\mathbf{k} \cdot \mathbf{k} - \epsilon_{\perp}^{Br} \mu_0 \omega^2) \left( \mathbf{k} \cdot \epsilon_{\parallel}^{Br} \mathbf{k} - \epsilon_{\parallel}^{Br} \epsilon_{\parallel}^{Br} \mu_0 \omega^2 \right).
\]

(23)

The dispersion relation (22) admits two wavevector solutions: the ordinary wavevector \( \mathbf{k}_o \) and the extraordinary wavevector \( \mathbf{k}_{ex} \), satisfying

\[
\begin{align*}
\mathbf{k}_o \cdot \mathbf{k}_o - \epsilon_{\perp}^{Br} \mu_0 \omega^2 &= 0 \\
\mathbf{k}_{ex} \cdot \epsilon_{\parallel}^{Br} \mathbf{k}_{ex} - \epsilon_{\parallel}^{Br} \epsilon_{\parallel}^{Br} \mu_0 \omega^2 &= 0
\end{align*}
\]

(24)

We note that the magnitude of the ordinary wavevector is direction–independent, and the ordinary and extraordinary wavevectors coincide when \( \mathbf{k}_{ex} \) is directed along \( \hat{c} \).

By taking the gradient of the dispersion relation (22) with respect to \( \mathbf{k} \), we find

\[
\nabla_k W + \frac{\partial W}{\partial \omega} \nabla_k \omega = 0.
\]

(25)

Hence, the HCM group velocity (18) may be expressed as

\[
v_{Br}^{B} = -\frac{1}{\partial W/\partial \omega} \left. \nabla_k W \right|_{\omega = \omega(k_{\text{avg}})}.
\]

(26)
The partial derivative terms involving $W$ are found to be

$$\nabla_k W = 2 \left[ (k \cdot \epsilon^{Br} \cdot k - \omega^2 \mu_0 \epsilon^{Br} \epsilon^{Br}) k + (k \cdot k - \omega^2 \mu_0 \epsilon^{Br}) \epsilon^{Br} \cdot k \right],$$

(27)

$$\frac{\partial W}{\partial \omega} = (k \cdot k - \omega^2 \mu_0 \epsilon^{Br}) \left\{ \frac{d \epsilon^{Br}}{d \omega} \cdot k - \mu_0 \omega \left[ 2 \epsilon^{Br} \epsilon^{Br} + \omega \left( \frac{d \epsilon^{Br}}{d \omega} \epsilon^{Br} + \epsilon^{Br} \frac{d \epsilon^{Br}}{d \omega} \right) \right] \right\}$$

$$- \mu_0 \omega \left( 2 \epsilon^{Br} + \omega \frac{d \epsilon^{Br}}{d \omega} \right) (k \cdot \epsilon^{Br} \cdot k - \epsilon^{Br} \epsilon^{Br} \mu_0 \omega),$$

(28)

with

$$\frac{d \epsilon^{Br}}{d \omega} = \frac{d \epsilon^{Br}}{d \omega} I + \left( \frac{d \epsilon^{Br}}{d \omega} - \frac{d \epsilon^{Br}}{d \omega} \right) \hat{c} \hat{c}.$$  

(29)

By virtue of (24), we see that the ordinary and the extraordinary group velocities are given by

$$v^g_{Br} \bigg|_{k=k_{or}} = \frac{2}{\omega \mu_0 \left( 2 \epsilon^{Br}_\perp + \omega \frac{d \epsilon^{Br}_\perp}{d \omega} \right)} k_{or},$$

(30)

and

$$v^g_{Br} \bigg|_{k=k_{ex}} = \frac{2}{\omega \mu_0 \left[ 2 \epsilon^{Br}_\perp \epsilon^{Br} + \omega \left( \frac{d \epsilon^{Br}_\perp}{d \omega} \epsilon^{Br} + \epsilon^{Br}_\perp \frac{d \epsilon^{Br}_\perp}{d \omega} \right) \right] - k_{ex} \cdot \frac{d \epsilon^{Br}_\perp}{d \omega} \cdot k_{ex},$$

(31)

respectively.

In order to find the derivatives of $\epsilon^{Br}_\perp$ and $\epsilon^{Br}_\parallel$ needed to evaluate the group velocities (30) and (31), we have to exploit the Bruggeman equations (15) and (16). As a precursor, let us first note the derivatives of the depolarization dyadic components

$$\frac{d D_\parallel}{d \omega} = \alpha_{11} \frac{d \epsilon^{Br}_\parallel}{d \omega} + \alpha_{12} \frac{d \epsilon^{Br}_\perp}{d \omega},$$

(32)

$$\frac{d D_\perp}{d \omega} = \alpha_{21} \frac{d \epsilon^{Br}_\parallel}{d \omega} + \alpha_{22} \frac{d \epsilon^{Br}_\perp}{d \omega},$$

(33)

with

$$\alpha_{11} = \frac{U^2_\parallel}{U^2_\parallel \epsilon^{Br}_\parallel \epsilon^{Br}_\parallel} \left( \Gamma_\parallel + \gamma \frac{d \Gamma_\parallel}{d \gamma} \right) - \frac{\gamma \Gamma_\parallel}{(\epsilon^{Br}_\parallel)^2},$$

(34)

$$\alpha_{12} = - \frac{U^2_\perp}{U^2_\parallel \epsilon^{Br}_\parallel \epsilon^{Br}_\parallel} \left( \Gamma_\parallel + \gamma \frac{d \Gamma_\parallel}{d \gamma} \right),$$

(35)

$$\alpha_{21} = \left( \frac{U^2_\parallel}{U^2_\parallel \epsilon^{Br}_\parallel \epsilon^{Br}_\parallel} \right) \frac{d \Gamma_\perp}{d \gamma},$$

(36)

$$\alpha_{22} = - \left( \frac{U^2_\parallel \epsilon^{Br}_\parallel}{U^2_\parallel \epsilon^{Br}_\parallel \epsilon^{Br}_\parallel} \right) \frac{d \Gamma_\perp}{d \gamma} - \frac{\Gamma_\perp}{(\epsilon^{Br}_\perp)^2},$$

(37)
and

\[
\frac{d\Gamma_\parallel}{d\gamma} = \begin{cases} 
\frac{1}{2} \left( \frac{3 \sinh^{-1} \sqrt{\frac{1-\gamma}{\gamma}} - 1 + 2\gamma}{(1-\gamma)^{\frac{3}{2}}} \right) & \text{for } 0 < \gamma < 1 \\
\frac{1}{2} \left( -\frac{1 + 2\gamma}{(\gamma - 1)^2 \gamma} + \frac{3 \sec^{-1} \sqrt{\gamma}}{(\gamma - 1)^{\frac{3}{2}}} \right) & \text{for } \gamma > 1
\end{cases},
\]

\[
\frac{d\Gamma_\perp}{d\gamma} = \begin{cases} 
\frac{1}{4} \left( \frac{3}{(1-\gamma)^2} - \frac{(2 + \gamma) \sinh^{-1} \sqrt{\frac{1-\gamma}{\gamma}}}{(1-\gamma)^{\frac{3}{2}}} \right) & \text{for } 0 < \gamma < 1 \\
\frac{1}{4} \left( -\frac{(2 + \gamma) \sec^{-1} \sqrt{\gamma}}{(\gamma - 1)^{\frac{3}{2}}} + \frac{3}{(\gamma - 1)^2} \right) & \text{for } \gamma > 1
\end{cases}.
\]

Now we turn to the Bruggeman equations (15) and (16). Their derivatives with respect to \(\omega\) may be written as

\[
\beta_{11} \frac{d\epsilon_{\parallel}^{Br}}{d\omega} + \beta_{12} \frac{d\epsilon_{\perp}^{Br}}{d\omega} + \beta_{13} = 0,
\]

\[
\beta_{21} \frac{d\epsilon_{\parallel}^{Br}}{d\omega} + \beta_{22} \frac{d\epsilon_{\perp}^{Br}}{d\omega} + \beta_{23} = 0,
\]

with

\[
\beta_{11} = \alpha_{11} (\epsilon^a - \epsilon_\parallel^{Br}) (\epsilon^b - \epsilon_\parallel^{Br}) + D_\parallel (2\epsilon_\parallel^{Br} - \epsilon^a - \epsilon^b) - 1,
\]

\[
\beta_{12} = \alpha_{12} (\epsilon^a - \epsilon_\parallel^{Br}) (\epsilon^b - \epsilon_\parallel^{Br}),
\]

\[
\beta_{13} = \left[ f_a + D_\parallel (\epsilon^b - \epsilon_\parallel^{Br}) \right] \frac{d\epsilon^a}{d\omega} + \left[ f_b + D_\parallel (\epsilon^a - \epsilon_\parallel^{Br}) \right] \frac{d\epsilon^b}{d\omega},
\]

\[
\beta_{21} = \alpha_{21} (\epsilon^a - \epsilon_\perp^{Br}) (\epsilon^b - \epsilon_\perp^{Br}),
\]

\[
\beta_{22} = \alpha_{22} (\epsilon^a - \epsilon_\perp^{Br}) (\epsilon^b - \epsilon_\perp^{Br}) + D_\perp (2\epsilon_\perp^{Br} - \epsilon^a - \epsilon^b) - 1,
\]

\[
\beta_{23} = \left[ f_a + D_\perp (\epsilon^b - \epsilon_\perp^{Br}) \right] \frac{d\epsilon^a}{d\omega} + \left[ f_b + D_\perp (\epsilon^a - \epsilon_\perp^{Br}) \right] \frac{d\epsilon^b}{d\omega}.
\]

The derivatives of \(\epsilon_\perp^{Br}\) and \(\epsilon_\parallel^{Br}\) therefore finally emerge as

\[
\frac{d\epsilon_\perp^{Br}}{d\omega} = \frac{\beta_{12} \beta_{23} - \beta_{22} \beta_{13}}{\beta_{11} \beta_{22} - \beta_{12} \beta_{21}},
\]

\[
\frac{d\epsilon_\parallel^{Br}}{d\omega} = \frac{\beta_{21} \beta_{13} - \beta_{11} \beta_{23}}{\beta_{11} \beta_{22} - \beta_{12} \beta_{21}}.
\]

To summarize, given a uniaxial dielectric HCM with permittivity dyadic \(\epsilon^{Br}\) estimated using the Bruggeman homogenization formalism, the group velocity (18) may be computed using the expression (26), with (27) and (28), wherein the derivatives of \(\epsilon_\perp^{Br}\) and \(\epsilon_\parallel^{Br}\) are provided by (48) and (49).
4 Numerical studies

Without loss of generality, let us choose the axis of rotational symmetry of the component spheroids to lie along the $x$ axis, i.e., $\hat{c} = \hat{x}$. We consider wavevectors lying in the $xy$ plane, oriented at an angle $\theta$ to the $x$ axis. That is, we take

$$ \vec{k} = k \{\cos \theta, \sin \theta, 0\} . $$

(50)

Thus, the magnitudes $k = k_{or} \equiv |\vec{k}_{or}|$ and $k = k_{ex} \equiv |\vec{k}_{ex}|$ of the ordinary and extraordinary wavevectors arise from (24) as [15]

$$ k_{or} = \omega \sqrt{\mu_0 \epsilon_{\perp_{Br}}}, $$

(51)

$$ k_{ex} = \omega \sqrt{\frac{\mu_0 \epsilon_{||_{Br}} \epsilon_{\perp_{Br}}}{\epsilon_{||_{Br}} \cos^2 \theta + \epsilon_{\perp_{Br}}^2 \sin^2 \theta}} . $$

(52)

Let us explore numerically the enhancement in group velocity that can arise through homogenization, paying special attention to directional effects induced by the shape of the component spheroidal particles. In particular, we choose the component material phase $a$ to have a relatively high permittivity $\epsilon^a$ and a relatively small frequency–dispersion term $d\epsilon^a/d\omega$, compared with the component material phase $b$. As representative constitutive parameter values, we set: $\epsilon^a = 30\epsilon_o$, $(d\epsilon^a/d\omega)|_{\omega = \omega_o} = 6\epsilon_o/\omega_o$, $\epsilon^b = 1.2\epsilon_o$ and $(d\epsilon^b/d\omega)|_{\omega = \omega_o} = 12\epsilon_o/\omega_o$, where $\epsilon_o$ is the permittivity of free space.

In Figure 1, the Bruggeman estimates of the HCM permittivity parameters $\epsilon_{||_{Br}}$ and $\epsilon_{\perp_{Br}}$ are plotted as functions of volume fraction $f_a$, for the range of values of $\rho = U_{||}/U_{\perp}$ shown in Table 1. Clearly, $\epsilon_{\perp_{Br}} \to \epsilon^b$ as $f_a \to 0$ and $\epsilon_{\perp_{Br}} \to \epsilon^a$ as $f_a \to 1$. We see that $\epsilon_{||_{Br}}$ becomes an increasingly nonlinear function of $f_a$ as $\rho$ decreases, whereas $\epsilon_{\perp_{Br}}$ becomes an increasingly nonlinear function of $f_a$ as $\rho$ increases.

In Figure 2, the magnitude of the group velocity $v_{g_{Br}} = |\vec{v}_{g_{Br}}|$ of a wavepacket in the chosen HCM is plotted against volume fraction. The group velocities are calculated with $\vec{k} = \vec{k}_{ex}$ for $\theta = 0^\circ, 30^\circ, 60^\circ$ and $90^\circ$. The corresponding graphs for $180^\circ - \theta$ are the same as those for $\theta$.

Since the ordinary wavevector $\vec{k}_{or} = \vec{k}_{ex}$ at $\theta = 0^\circ$, the ordinary group velocities for any $\theta$ are identical to those provided in Figure 2(a) wherein the results for $\theta = 0^\circ$ are presented. The group velocity magnitudes for the component material phases $a$ and $b$ are $v_{g}^a = 0.166c$ and $v_{g}^b = 0.152c$, respectively (as is confirmed in Figure 2 by the group velocity values at $f_a = 1$ and $f_a = 0$, respectively), where $c = 1/\sqrt{\epsilon_o \mu_o}$. Hence, for this particular homogenization example, group–velocity enhancement arises when $v_{g}^{Br} > \max \{v_{g}^a, v_{g}^b\} = 0.166c$. The group–velocity–enhancement region is identified by shading in Figure 2.

It may be discerned from Figure 2(a) that group–velocity enhancement occurs over an increasingly large range of $f_a$ values as $\rho$ decreases. Furthermore, the degree of enhancement at $\rho = 20$ is much smaller than it is at $\rho = 0.05$.

As $\theta$ increases, the range of $f_a$ values at which group–velocity enhancement occurs progressively decreases for small values of $\rho$. In fact, at $\theta = 60^\circ$ there is no longer any enhancement in group velocity for $\rho = 0.05$. At $\theta = 90^\circ$, the group–velocity enhancement
characteristics at low and high values of \( \rho \) are approximately the reverse of their respective characteristics at \( \theta = 0^\circ \). That is, group–velocity enhancement occurs over a wide range of \( f_a \) values for high values of \( \rho \) at \( \theta = 90^\circ \), but there is no enhancement in group velocity at low values of \( \rho \).

Clearly therefore, enhancement of group velocity is maximum in a direction parallel to the longest semi–axis of the spheroidal particles, which can be prolate \((\rho < 1)\) or oblate \((\rho > 1)\). For spherical particles \((\rho = 1)\), group–velocity enhancement is direction–independent, and we recover the results of the predecessor study \cite{9}.

\section{Concluding remarks}

The enhancement in group velocity brought about by homogenization is sensitively dependent upon directional properties. Both the shape of the component spheroidal particles, and their orientation relative to the direction of propagation, strongly influence the group–velocity enhancement.

The homogenization scenario presented here deals with the conceptualization of a uniaxial HCM as arising from identically oriented spheroidal particles of isotropic component material phases. The homogenization of two uniaxial dielectric component phases distributed as spherical particles is mathematically equivalent, provided that the distinguished axes of the component material phases have the same orientation \cite{16}.

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| Value of $\rho$ | Key |
|----------------|-----|
| 0.05           | —   |
| 0.1            | —   |
| 0.2            | —   |
| 0.5            | —   |
| 1              | —   |
| 2              | —   |
| 5              | —   |
| 10             | —   |
| 20             | —   |

Table 1. Key for the values of $\rho = U_{||}/U_{\perp}$ used in Figures 1 and 2.
Figure 1: The HCM relative permittivity parameters $\varepsilon_{\parallel Br}^{Br}/\varepsilon_0$ (above) and $\varepsilon_{\perp Br}^{Br}/\varepsilon_0$ (below) plotted against volume fraction $f_a$. Permittivities of component material phases: $\varepsilon_a = 30\varepsilon_0$ and $\varepsilon_b = 1.2\varepsilon_0$. A key for the $\rho = U_{\parallel}/U_{\perp}$ values is given in Table 1.
Figure 2: The magnitude of the HCM group velocity $v_g^{Br} = |\Omega_g^{Br}|$, as estimated using the Bruggeman formalism, plotted against volume fraction $f_a$. The group velocity is normalized with respect to $c = 1/\sqrt{\varepsilon_0\mu_0}$. Constitutive parameters of component material phases: $\varepsilon^a = 30\varepsilon_0$, $(d\varepsilon^a/d\omega)|_{\omega=\omega_o} = 6\varepsilon_0/\omega_o$, $\varepsilon^b = 1.2\varepsilon_0$ and $(d\varepsilon^b/d\omega)|_{\omega=\omega_o} = 12\varepsilon_0/\omega_o$. A key for the $\rho = U_{\parallel}/U_{\perp}$ values is given in Table 1. Shading indicates the region of group–velocity enhancement. (a) Extraordinary wavevector angle $\theta = 0^\circ$.

Figure 2: (b) Extraordinary wavevector angle $\theta = 30^\circ$. 
Figure 2: (c) Extraordinary wavevector angle $\theta = 60^\circ$.

Figure 2: (d) Extraordinary wavevector angle $\theta = 90^\circ$. 