q-EULER AND GENOCCHI NUMBERS

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Abstract. Carlitz has introduced an interesting q-analogue of Frobenius-Euler numbers in [4]. He has indicated a corresponding Stadudt-Clausen theorem and also some interesting congruence properties of the q-Euler numbers. In this paper we give another construction of q-Euler numbers, which are different than his q-Euler numbers. By using our q-Euler numbers, we define the q-analogue of Genocchi numbers and investigate the relations between q-Euler numbers and q-analogs of Genocchi numbers.

1. Introduction

Throughout this paper, we consider a complex number $q \in \mathbb{C}$ with $|q| < 1$ as an indeterminate. The q-analogue of $n$ is defined by $[n]_q = \frac{1 - q^n}{1 - q}$. The ordinary Euler numbers are defined by the generating function as follows:

\begin{equation}
F(t) = \frac{2}{e^t + 1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \pi,
\end{equation}

where we use the technique method notation by replacing $E^m$ by $E_m$ $(m \geq 0)$, symbolically, cf.\,[2, 6].

From Eq.\,(1), we can derive the Genocchi numbers as follows:

\begin{equation}
G(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi.
\end{equation}

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It satisfies $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given $G_m = 2(1 - 2^{2m})B_{2m} = 2mE_{2m-1}$, where $B_m$ are the $m$-th ordinary Bernoulli numbers, cf.[6]. It follows from (2) and Staduclt-Clasusen theorem that Genocchi numbers are integers. For $x \in \mathbb{R}$ (=the field of real numbers) the Euler polynomials are defined by

$$F(x, t) = F(t)e^{xt} = \frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad (|t| < \pi).$$

From (3), we can also derive the definition of Genocchi polynomials as follows:

$$\frac{2t}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad (|t| < \pi).$$

The following formulae ((5)-(6)) are well known in [6].

$$E_m(x) = \sum_{k=0}^{m} \binom{m}{k} G_{k+1}x^{m-k}.$$  \hfill (5)

For $n, m \geq 1$, and $n$ odd, we have

$$E_m(x) = \sum_{k=1}^{m-1} \binom{m}{k} n^k G_k Z_{m-k}(n-1),$$  \hfill (6)

where $Z_m(n) = 1^m - 2^m + 3^m - \cdots + (-1)^{n+1}n^m$. In this paper we give the $q$-analogs of the above Eq.(5) and Eq.(6). The purpose of this paper is to give another construction of $q$-Euler numbers, which are different than a $q$-Eulerian numbers of Carlitz. From the definition of our $q$-Euler numbers, we derive the $q$-analogs of Genocchi numbers and investigate the properties of $q$-Genocchi numbers which are related to $q$-Euler numbers.

### 2. $q$-Euler Numbers and Polynomials

Let $q$ be a complex number with $q < 1$. In [3, 4] Carlitz constructed $q$-analogue of Eulerian numbers. We now consider another construction of a $q$-Eulerian numbers, which are different than his $q$-Eulerian numbers. First we consider the following generating functions:

$$F_q(t) = [2]_q e^{\frac{1}{1-q}} \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!} \frac{t^j}{1 + q^{j+1}} = e^{E_q t} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!},$$ \hfill (7)
and
\[ F_q(x, t) = [2]_q e^{\frac{1}{t-q}} \sum_{j=0}^{\infty} \frac{(-1)^j q^j x}{1 + q^{j+1}} \left( \frac{1}{1 - q} \right)^j \frac{t^j}{j!} = e^{E_q(x)t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}, \]
where we use the technique method notation by replacing \( E_n \) by \( E_{n,q} \), symbolically. Thus we have
\[ E_{n,q} = \frac{[2]_q}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1 + q^{l+1}}, \]
and
\[ E_{n,q}(x) = \frac{[2]_q}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{lx}}{1 + q^{l+1} q^x}, \]
where \( \binom{n}{l} \) is binomial coefficient.

By (8-1), we easily see that \( \lim_{q \to 1} E_{n,q} = E_n \) and \( \lim_{q \to 1} E_{n,q}(x) = E_n(x) \). From Eq.(8), we can derive the below Eq.(9):
\[ F_q(x, t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \]
By (9), we easily see that
\[ E_{n,q} = \frac{[2]_q}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{m-1} q^{lx}}{1 + q^{l+1} q^x}, \]
where \( m \) is odd.

This is equivalent to
\[ [2]_q \sum_{n=0}^{m-1} (-1)^n q^n E_{n,q}(\frac{a + x}{m}) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} E_{k,q}, \]
for \( m \) odd.

If we put \( x = 0 \) in Eq.(11), then we have
\[ [m]_{-q} E_{n,q} - [m]_{q} [m(n + 1)]_{-q} E_{n,q} = \sum_{l=0}^{n-1} \binom{n}{l} [m]_{l} q \sum_{a=1}^{m-1} (-1)^a q^{a(l+1)} [a]_{q}^{n-l}, \]
where \( [m]_{-q} = \frac{1 + q^m}{1 + q} \) for \( m \) odd.

Define the operation \(*\) on \( f_n(q) \) as follows:
\[ (1 - [m]_{q}) * f_n(q) = [m]_{-q} f_n(q) - [m]_{q} \frac{[m(n + 1)]_{-q} - [m]_{[n + 1]_{-q}}}{[n + 1]_{-q}} f_n(q^m). \]
By (12) and (13), we obtain the following:
Proposition 1. For $m, n \in \mathbb{N}$ and $m$ odd, we have

$$(1 - [m]_q^n) \ast E_{n,q} = \sum_{l=0}^{n-1} \binom{n}{l} [m]_q^l E_{l,q} \sum_{a=1}^{m-1} (-1)^a q^{a(l+1)} [a]_q^{n-l}.$$  

For any positive integer $n$, it is easy to see that

$$(14) \quad -[2]_q \sum_{l=0}^{\infty} (-1)^l q^{l+n} e^{[l+n]_q t} + [2]_q \sum_{l=0}^{\infty} (-1)^l q^l e^{[l]_q t} = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l e^{[l]_q t}.$$  

From (9) and (14) we can derive the below:

$$\sum_{l=0}^{n-1} (-1)^l q^l [l]_q^m = \frac{1}{[2]_q} \left( (-1)^{n+1} q^n E_{m,q}(n) - E_{m,q} \right).$$

Therefore we obtain the following:

Proposition 2. For $n, m \in \mathbb{N}$, we have

$$\sum_{l=0}^{n-1} (-1)^l q^l [l]_q^m = \frac{1}{[2]_q} \left( (-1)^{n+1} q^n E_{m,q}(n) - E_{m,q} \right).$$

In the recent many authors have studies the sums of powers of consecutive integers, cf.[1, 5, 7, 10, 11]. The above Proposition 2 is the another $q$-analogue of the sums of powers of consecutive integers. The Genocchi numbers $G_n$ are defined by the generating function:

$$G(t) = \frac{2t}{e^t + 1} = e^{Gt} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi),$$

where we use the technique method notation by replacing $G^m$ by $G_m$ ($m \geq 0$), symbolically. It satisfies $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = 0$ and even coefficients are given $G_m = 2(1 - 2^m) B_{2m} = 2m E_{2m-1}$, cf.[6]. We now derive the $q$-extension of the above Genocchi numbers from the definition of our $q$-Euler numbers.
3. \(q\)-Genocchi Numbers and polynomials

By the meaning of (1) and (2), let us define the \(q\)-extension of Genocchi numbers as follows:

\[
G_q(t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \quad (|t| < \pi).
\]

Note that \(\lim_{q \to 1} G_q(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}\). Hence, \(\lim_{q \to 1} G_{n,q} = G_n\). In [8], the \(q\)-Bernoulli numbers are defined by

\[
(15) -\sum_{n=0}^{\infty} q^n e^{[n]_q t} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}, \quad (|t| < 2\pi).
\]

It was known that \(\lim_{q \to 1} B_{n,q} = B_n\), cf.[8, 9]. By (15), we easily see that

\[
(16) -[2]_q t \sum_{n=0}^{\infty} q^n e^{[n]_q t} + 2[2]_q t \sum_{n=0}^{\infty} q^{2n} e^{[2n]_q t} = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t}.
\]

From (15) and (16), we can derive the below Eq.(17):

\[
(17) G_{n,q} = [2]_q B_{n,q} - 2[2]_q^n B_{n,q^2}.
\]

Let us consider the \(q\)-analogue of Genocchi polynomials as follows:

\[
(18) G_q(x, t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}.
\]

By (18), we easily see that

\[
(19) G_q(x, t) = [2]_q q^x t e^{\frac{t}{1-q}} \sum_{l=0}^{\infty} \frac{(-1)^l t^l}{1 + q^{l+1}} q^l x \left( \frac{1}{1-q} \right)^l \frac{t^l}{l!}.
\]

Thus, we have

\[
G_{n,q}(x) = n \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \frac{(-1)^l}{1 + q^{l+1}} q^{(l+1)x}.
\]
From (8) and (19), we can derive the below equality:

$$F_q(x, t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q} t^n = \frac{e^{[x]_q t}}{q^x t} [2]_q q^x t \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t}$$

(20)

$$= e^{[x]_q t} \sum_{n=0}^{\infty} q^n x q^{n+1} \frac{t^n}{n+1} n! = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) [x]_q x^{n-k} q^{n} \frac{G_{n+1,q}}{n+1} \right) \frac{t^n}{n!}.$$  

By (20), we easily see that

$$E_{n,q}(x) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) [x]_q x^{n-k} q^{n} \frac{G_{n+1,q}}{n+1}.  \tag{21}$$

Remark. The Eq.(21) is the $q$-analogue of Eq.(5).

Therefore we obtain the following theorem:

**Theorem 3.** For any positive integer $n$, we have

(a) $G_{n,q}(x) = n \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \left( \begin{array}{c} n-1 \\ l \end{array} \right) \frac{(-1)^l}{1+q^{l+1}} q^{(l+1)x}$,

(b) $E_{n,q}(x) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) [x]_q x^{n-k} q^{n} \frac{G_{n+1,q}}{n+1},$

(c) $G_{n,q} = [2]_q B_{n,q} - 2[2]_q B_{n,q^2},$

where $B_{n,q}$ are the $q$-Bernoulli numbers which are defined in [8].

By (18), we easily see that

$$\sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{[2]_q}{[2]_q m} \right)^{n-1} \sum_{a=0}^{m-1} (-1)^a q^{a+x} G_{n,q^m} \left( \frac{x+a}{m} \right) \frac{t^n}{n!}, \text{ for } m \text{ odd}.$$

(22)

Thus we obtain the following:

**Theorem 4.** Let $m \in \mathbb{N}$ and $m$ odd. Then we see that

$$G_{n,q}(x) = \frac{[2]_q}{[2]_q m} \sum_{a=0}^{m-1} (-1)^a q^{a+x} G_{n,q^m} \left( \frac{x+a}{m} \right) = \sum_{k=0}^{\infty} \left( \begin{array}{c} n \\ k \end{array} \right) q^{kx} G_{k,q} [x]_q^{n-k}.$$
This is equivalent to

\begin{equation}
G_{n,q}(mx) = \frac{[2]_q}{[2]_q m} [m]_q [m]_q \sum_{a=0}^{m-1} (-1)^a q^{a+mx} G_{n,q}(x + \frac{a}{m}).
\end{equation}

If we take \( x = 0 \) in Eq. (23), then we easily see that

\begin{equation}
[2]_q [m]_q G_{n,q} - [2]_q [m]_q^n G_{n,q} m \frac{[2]_q q^{m+1}}{[2]_q q^{n+1}} = [2]_q \sum_{k=0}^{n-1} \binom{n}{k} [m]_q [m]_q^n G_{k,q} m \sum_{a=0}^{m-1} (-1)^a q^{a(k+1)} [a]_q^{n-k}.
\end{equation}

From the definition of the operation \(*\) in the previous section, we note that

\begin{equation}
([m]_q - [m]_q^n) * f_n(q) = [2]_q [m]_q f_n(q) - [2]_q [m]_q^n \frac{[2]_q q^{m+1}}{[2]_q q^{n+1}} f_n(q^n).
\end{equation}

By (24) and (25), we easily see that

\begin{equation}
([m]_q - [m]_q^n) * G_{n,q} = [2]_q \sum_{k=0}^{n-1} \binom{n}{k} [m]_q [m]_q^n G_{n,q} m \sum_{a=0}^{m-1} (-1)^a q^{a(k+1)} [a]_q^{n-k}.
\end{equation}

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