ON THE RIEZ MEANS OF $\delta_k(n)$

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Abstract. Let $k \geq 1$ be an integer. Let $\delta_k(n)$ denote the maximum divisor of $n$ which is co-prime to $k$. We study the error term of the general $m$-th Riesz mean of the arithmetical function $\delta_k(n)$ for any positive integer $m \geq 1$, namely the error term $E_m(x)$ where

$$\frac{1}{m!} \sum_{n \leq x} \delta_k(n) \left(1 - \frac{n}{x}\right)^m = M_{m,k}(x) + E_{m,k}(x).$$

We establish a non-trivial upper bound for $|E_{m,k}(x)|$, for any integer $m \geq 1$.

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1. Introduction

For any fixed positive integer $k$, we define

$$(1) \quad \delta_k(n) = \max\{d : d \mid n, \ (d,k) = 1\}.$$

Joshi and Vaidya [6] proved that

$$(2) \quad \sum_{n \leq x} \delta_k(n) = \frac{k}{2\sigma(k)} x^2 + E_k(x),$$

with $E_k(x) = O(x)$ and $\sigma(k) = \sum_{d \mid k} d$, when $k$ is a square free positive integer. They also proved that when $k = p$, a prime,

$$\lim_{n \to \infty} \frac{E_p(x)}{x} = - \frac{p}{p+1}, \quad \text{and} \quad \lim_{n \to \infty} \frac{E_p(x)}{x} = \frac{p}{p+1}.$$  

It was proved by Maxsein and Herzog [7] that for any square free positive integer $k$,

$$\lim_{n \to \infty} \frac{E_k(x)}{x} \leq - \frac{k}{\sigma(k)}, \quad \text{and} \quad \lim_{n \to \infty} \frac{E_k(x)}{x} \geq \frac{k}{\sigma(k)}.$$  

Around the same time, Adhikari, Balasubramanian and Sankaranarayanan [1] proved the above results by a different method. While a tauberian theorem of Hardy-Littlewood and Karamata was used in [7] to get the asymptotic formula for $\sum_{n \leq x} \gamma_k(n)$, where $\gamma_k(n)$ is defined by the relation $\delta_k(n) = \gamma_k \ast I(n)$ where $\ast$ is the Dirichlet convolution and $I$ is the identity function, the method of [1] consists of averaging over arithmetical progressions.

For $k \geq 1$ and square free, Harzog and Maxsein [7] had also observed that

$$\limsup_{x \to \infty} \frac{E_k(x)}{x} \leq \frac{1}{2} d(k),$$

where $d(k)$ denotes the number of divisors of $k$. Later Adhikari and Balasubramanian [2] improved this result of Maxsein and Herzog by showing that

$$\lim_{n \to \infty} \frac{|E_k(x)|}{x} \leq \frac{1}{2} \left(1 - \frac{1}{p+1}\right) d(k),$$

where $p$ denotes the smallest prime dividing $k$.

Writing

$$H_k(x) = \sum_{n \leq x} \frac{\delta_k(n)}{n} - \frac{kx}{\sigma(k)},$$
one observes (see [1]) that

\[ E_k(x) = H_k(x) + O(1). \]

In [3], more precise upper and lower bounds for the quantities \( \lim H_k(x) \) and \( \lim H_k(x) \) were established. The aim of this article is to study the error term of the general \( m \)-th Riesz mean related to the arithmetic function \( \delta_k(n) \) for any positive integer \( m \geq 1 \) and \( k \geq 1 \) (need not be a square free integer). More precisely, we write

\[ \frac{1}{m!} \sum_{n \leq x} \delta_k(n) \left( 1 - \frac{n}{x} \right)^m = M_{m,k}(x) + E_{m,k}(x) \]

where \( M_{m,k}(x) \) is the main term (exists) and \( E_{m,k}(x) \) is the error term of the sum under investigation. We prove the following.

**Theorem 1.1.** Let \( x \geq x_0 \) where \( x_0 \) is a sufficiently large positive number and let \( c(\eta) = \frac{2}{1 - 2^{-\eta}} \) for any \( \eta > 0 \). For any integer \( m \geq 1 \) and for any integer \( k \geq 1 \), we have

\[ \frac{1}{m!} \sum_{n \leq x} \delta_k(n) \left( 1 - \frac{n}{x} \right)^m = \frac{x^2}{(m + 2)!} \prod_{p \mid k} \frac{p}{p + 1} + E_{m,k}(x), \]

where

\[ E_{1,k}(x) \ll kc(1/2)^{\omega (k)} x^{1/2} \log x, \]

and for \( m \geq 2 \), we have

\[ E_{m,k}(x) \ll kc(\eta)^{\omega (k)} x^\eta \]

for any small fixed positive constant \( \eta \) and the implied constant is independent of \( m \).

2. **Notation**

1. Throughout the paper, \( s = \sigma + it \); the parameters \( T \) and \( x \) are sufficiently large real numbers and \( m \) is an integer \( \geq 1 \).
2. \( \eta, \epsilon \) always denote sufficiently small positive constants.
3. As usual \( \zeta(s) \) denotes the Riemann zeta-function.
4. \( k \) is any square free positive integer.

3. **Some Lemmas**

Generating function for \( \delta_k(n) \) is given by:

**Lemma 3.1.** We have

\[ \sum_{n=1}^{\infty} \frac{\delta_k(n)}{n^s} = \zeta(s - 1)G(s), \]

where

\[ G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_{p \mid k} \left( \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^s}} \right) \ll k c(\eta)^{\omega (k)}, \]

for \( \sigma \geq \eta \) and

\[ c(\eta) = \frac{2}{1 - 2^{-\eta}}. \]
Proof. We have (see [1, equation 2.2]),
\[
\sum_{n=2}^{\infty} \frac{\delta_k(n)}{n^s} = \prod_p \left( 1 + \frac{\delta_k(p)}{p^s} + \frac{\delta_k(p^2)}{p^{2s}} + \cdots \right)
= \prod_{p|k} \left( 1 + \frac{1}{p^s} \right) \prod_{p|k} \left( 1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots \right)
= \zeta(s-1) \prod_{p|k} \frac{1 - \frac{p^s}{1 - \frac{1}{p^s}}} {1 - \frac{1}{p^s}} := \zeta(s-1) G(s),
\]
since
\[
\delta_k(p^m) = \begin{cases} 1 & \text{if } p | k \\ p^m & \text{if } p \nmid k. \end{cases}
\]

And for \( \sigma \geq \eta (>0) \), we observe that
\[
|G(s)| = \prod_{p|k} \left| \frac{1 - \frac{p^s}{1 - \frac{1}{p^s}}} {1 - \frac{1}{p^s}} \right| \leq \prod_{p|k} \frac{1 + p^{1-\eta}} {1 - \frac{1}{p^s}} \leq \prod_{p|k} \frac{2p} {1 - \frac{2p}{p^s}} \leq kc(\eta)^{\omega(k)}.
\]

\[\blacksquare\]

**Lemma 3.2.** Let \( m \) be an integer \( \geq 1 \). Let \( c \) and \( y \) be any positive real numbers and \( T \geq T_0 \) where \( T_0 \) is sufficiently large. Then we have,
\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s} {s(s+1) \cdots (s+m)} ds = \begin{cases} \left( \frac{1}{m!} \right) \left( \frac{1 - \frac{1}{y}} {1 - \frac{1}{p^s}} \right)^m + O \left( \frac{4^m y^\sigma} {p^s} \right) & \text{if } y \geq 1, \\ O \left( \frac{1} {p^s} \right) & \text{if } 0 < y \leq 1. \end{cases}
\]

Proof. See [8, Lemma 3.2] and also [4, p.31 Theorem B]). \( \blacksquare \)

**Lemma 3.3.** The Riemann zeta-function \( \zeta(s) \) is extended as a meromorphic function in the whole complex plane \( \mathbb{C} \) with a simple pole at \( s = 1 \) and it satisfies a functional equation \( \zeta(s) = \chi(s)\zeta(1-s) \) where
\[
\chi(s) = \pi^{-(1-s)/2} \Gamma \left( \frac{1-s} {2} \right) \frac{1} {\pi^{s/2} \Gamma \left( \frac{s} {2} \right)}.
\]

Also, in any bounded vertical strip, using Stirling’s formula, we have
\[
\chi(s) = \left( \frac{2\pi} {t} \right)^{\sigma+it-1/2} e^{i(t+\frac{\pi}{4})} (1 + O(t^{-1}))
\]
as \( |t| \to \infty \). Thus, in any bounded vertical strip,
\[
|\chi(s)| \asymp t^{1/2-\sigma} (1 + O(t^{-1}))
\]
as \( |t| \to \infty \).

Proof. See [9] p.116] or [5] p.8-12]. \( \blacksquare \)

**Lemma 3.4.** We have for \( t \geq t_0 \) (sufficiently large),
\[
\zeta \left( \frac{1} {2} + it \right) \ll t^{1/6} (\log t)^{3/2}
\]
and
\[
\zeta(1+it) \ll \log t.
\]

Proof. See [9] page 99, Theorem 5.5 and [9] page 49, Theorem 3.5] \( \blacksquare \)
4. PROOF OF THEOREM 1.1

From Lemma 3.2 with \( c = 2 + \frac{1}{\log x} \) and writing \( F(s) := \zeta(s-1)G(s) \), we have

\[
S := \sum_{n \leq x} \delta_k(n) \left( 1 - \frac{n}{x} \right) \left( 1 - \frac{n}{x} \right) \cdot \ldots \cdot \left( 1 - \frac{n}{x} \right) m = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s(s+1) \cdots (s+m)} ds + O \left( \frac{4^m x^c \log x}{T^m} \right).
\]

Note that the tail portion error term in the above expression is actually

\[
\ll \frac{4^m x^c \sum_{n \leq x} \delta_k(n)}{n^c} \ll \frac{4^m x^c \log x}{T^m},
\]

since \( \delta_k(n) \leq n \).

**Case 1:** Let \( m = 1 \). We move the line of integration in the above integral to \( \Re s = \frac{1}{2} \). In the rectangular contour formed by the line segments joining the points \( c - iT, \ c + iT, \frac{1}{2} + iT \) and \( c - iT \) in the anticlockwise order, we observe that \( s = 2 \) is a simple pole of the integrand. Thus we get the main term \( \frac{1}{(m+2)!} \prod_{p | k} \frac{p}{p+1} \) from the residue coming from the pole \( s = 2 \).

We note that

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)} ds = \frac{1}{2\pi i} \left\{ \int_{\frac{1}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{\frac{1}{2}-iT} \right\} + \text{sum of the residues}.
\]

The left vertical line segment contributes the quantity:

\[
Q_1 := \frac{1}{2\pi i} \int_{-T}^{T} F(1/2 + it) \frac{x^{1/2 + it} dt}{(-1/2 + it)(1/2 + it)} dt
\]

\[
= \frac{1}{2\pi} \left( \int_{|t| \leq t_0} + \int_{t_0 < |t| \leq T} \right) \frac{x^{1/2 + it} \zeta(-\frac{1}{2} + it) G\left(\frac{1}{2} + it\right)}{\left(\frac{1}{2} + it\right) \left(\frac{1}{2} + it\right)} dt
\]

\[
\ll k \ c(1/2)^{\omega(k)} x^{1/2} + k \ c(1/2)^{\omega(k)} x^{1/2} \int_{t_0 < |t| \leq T} \left| \zeta(3/2 + it) G\left(\frac{1}{2} + it\right) \right| \frac{dt}{t^2}
\]

\[
\ll k \ c(1/2)^{\omega(k)} x^{1/2} + k \ c(1/2)^{\omega(k)} x^{1/2} \int_{t_0 < |t| \leq T} dt.
\]

Now we will estimate the contributions coming from the upper horizontal line (lower horizontal line is similar).
The horizontal lines in total contribute a quantity which is in absolute value
\[
\ll \int_{1/2}^c | \zeta(\sigma - 1 + iT)G(\sigma + iT) \frac{x^{\sigma + iT}}{(\sigma + iT)(\sigma + 1 + iT)} | d\sigma
\]
\[
\ll \left( \int_{1/2}^1 + \int_{1}^{3/2} + \int_{3/2}^c \right) | \zeta(\sigma - 1 + iT)G(\sigma + iT) \frac{x^{\sigma}}{T^2} | d\sigma
\]
\[
\ll k \ c(1/2) \omega(k) \left\{ \left( \int_{1/2}^1 + \int_{1}^{3/2} \right) T^{1/2 - \sigma + 1} | \zeta(2 - \sigma + iT) | \frac{x^{\sigma}}{T^2} d\sigma
\]
\[
+ \int_{3/2}^c | \zeta(\sigma - 1 + iT) | \frac{x^{\sigma}}{T^2} d\sigma \} \text{ (by Lemma 3.3)}
\]
\[
\ll k \ c(1/2) \omega(k) \left( \frac{x \log T}{T} + \frac{x^{3/2} \log T}{T^{3/2}} + \frac{x^2 \log T}{T^{11/6}} \right) \text{ (by Lemma 3.4)}.
\]

Collecting all the estimates, and taking \( T = x^{10} \) we get:
\[
E_{1,k}(x) \ll k \ c(1/2) \omega(k) \left( x^{1/2} \log T + \frac{x^2}{T} + \frac{x \log T}{T} + \frac{x^{3/2} \log T}{T^{3/2}} + \frac{x^2 \log T}{T^{11/6}} \right)
\]
\[
(7) \ll k \ c(1/2) \omega(k) x^{1/2} \log x.
\]

**Case 2:** Let \( m \geq 2 \). We move the line of integration to \( \Re s = \eta \ (> 0) \).

We note that
\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1) \cdots (s+m)} ds
\]
\[
= \frac{1}{2\pi i} \left\{ \int_{c+iT}^{c+iT} \cdots + \int_{\delta+iT}^{\delta+iT} \cdots + \int_{c-iT}^{c-iT} \cdots \right\} \text{ sum of the residue.}
\]

The left vertical line segment contributes the quantity:
\[
Q_m := \frac{1}{2\pi} \int_{-T}^T F(\eta + it) \frac{x^{\eta + it}}{(\eta + it)(\eta + 1 + it) \cdots (\eta + m + it)} dt
\]
\[
= \frac{1}{2\pi} \left( \int_{|t| \leq t_0} + \int_{t_0 < |t| \leq T} \right) \frac{x^{\eta + it} \zeta(\eta - 1 + it) G(\eta + it) dt}{(\eta + it)(\eta + 1 + it) \cdots (\eta + m + it)}
\]
\[
\ll k \ c(\eta) \omega(k) x^\eta + k \ c(\eta) \omega(k) x^\eta \int_{t_0 < |t| \leq T} \frac{t^{1/2 - (\eta - 1) \zeta(3/2 - \eta + it) G(\eta + it)}}{t^{m+1}} dt
\]
\[
\ll k \ c(\eta) \omega(k) x^\eta + k \ c(\eta) \omega(k) x^\eta \int_{t_0 < |t| \leq T} \frac{t^{3/2 - \eta}}{t^3} dt.
\]
\[
(9) \ll k \ c(\eta) \omega(k) x^\eta.
\]

Now we will estimate the contributions coming from the upper horizontal line (lower horizontal line is similar).
The horizontal lines in total contribute a quantity which is in absolute value
\[ \ll \int \frac{\zeta'(\sigma - 1 + iT)G(\sigma + iT)}{(\sigma + iT)(\sigma + 1 + iT)\cdots(\sigma + m + iT)} d\sigma \]
\[ \ll c(\eta)^{(k)} \left( \int_{\frac{1}{2}}^{1} + \int_{\frac{1}{2}}^{c} \right) \frac{x^\sigma}{T^{k+1}} \]
\[ \ll k c(\eta)^{(k)} \left( \left( \int_{\frac{1}{2}}^{1/2} + \int_{\frac{1}{2}}^{3/2} \right) T^{1/2-\sigma+1} |\zeta(2-\sigma+iT)| \frac{x^\sigma}{T^{m+1}} d\sigma \right) \]
\[ \ll k c(\eta)^{(k)} \left( \frac{x^{1/2}}{T^{m-1/2+\eta}} + \frac{x \log T}{T^m} + \frac{x^{3/2} (\log T)^{3/2}}{T^{m+5/6}} + \frac{x^2 (\log T)^{3/2}}{T^{m+5/6}} \right) \]
Collecting all the estimates, and taking \( T = x^{10} \), for \( m \geq 2 \) we get:
\[ (10) \]
\[ E_{m,k}(x) \ll k c(\eta)^{(k)} x^\eta. \]
This proves Theorem 1.1.

**Remark 4.1.** For \( m \geq 2 \) we may try to move the line of integration slightly left of vertical line 0. On the line \( \Re s = 0 \), the function \( G(s) \) has simple poles at the points \( s(\ell,p) = \frac{2\pi i \ell}{\log p} \) \( \forall \ell \in \mathbb{Z} \) and for each prime \( p \mid k \), let \( p_1, p_2, \ldots, p_k \) be the primes dividing \( k \). The total contribution from the simple poles at the points \( s(\ell,p) = \frac{2\pi i \ell}{\log p} \) for \( 1 \leq j \leq r_k \) is given by:
\[ M = \sum_{j=1}^{r_k} \sum_{\ell \mid \eta} \zeta(\frac{2\pi i \ell}{\log p_j} - 1) \prod_{p_i \neq p_j} \left( 1 - \frac{p_i}{p_j} \right)^{\frac{2\pi i \ell}{\log p_j}} \frac{x^{\frac{2\pi i \ell}{\log p_j} + m}}{x^{\log_p j}} \]
If one establishes that \( M = o(x^{\eta}) \), then this will improve the error term. This seems to be really difficult.

**Remark 4.2.** From the Theorem 1.1 observe that
\[ E_{1,k}(x) \ll x^{1/2 + 10\epsilon} \]
uniformly for \( 3 \leq k \ll x^\epsilon \) since \( \omega(k) \ll \frac{\log \log k}{\log k} \) for \( k \geq 3 \). Also \( E_{m,k}(x) \ll x^{c_1 \eta} \) uniformly for \( 3 \leq k \ll x^c \), where \( c_1 \) is effective positive constant.

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