Topical Review

Multilayer coating for higher accelerating fields in superconducting radio-frequency cavities: a review of theoretical aspects

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Abstract

The theory of the superconductor–insulator–superconductor (SIS) multilayer structure for application in superconducting accelerating cavities is reviewed. The theoretical field limit, optimum layer thicknesses and material combination, and surface resistance are discussed for the SIS structure and are also reviewed for the superconductor–superconductor bilayer structure.

Keywords: superconducting accelerating cavity, multilayer, SRF, field limit

(Some figures may appear in colour only in the online journal)

1. Introduction

The science and technology of superconducting radio-frequency (SRF) cavities made of niobium (Nb) has been widely studied over the last decades [1]. Improvements in fabrication and processing technologies combined with progress in understanding SRF physics [2] have advanced the frontier of the accelerating electric field. At present, a peak surface magnetic field of around $B_0 \approx 150$ mT has been commonly achieved by using the set of modern surface-preparation techniques: electropolishing followed by a heat treatment for hydrogen degassing [3, 4], high-pressure rinsing [5–7], clean assembly [8], low-temperature baking [9–12] and local grinding combined with optical inspection [13–18]. Some laboratories have achieved $B_0 \approx 200$ mT $\sim B^{(Nb)}_c \sim B^{(Nb)}_x$ [19, 20], where $B^{(Nb)}_c$ and $B^{(Nb)}_x$ are the lower critical field and the thermodynamic critical field, respectively. Even higher fields, however, would not be expected because the present record field is thought to be close to the theoretical field limit, namely, the superheating field $B_s^{(Nb)} (\sim B^{(Nb)}_s)$.

The superheating field $B_s$ is the field at which the Meissner state becomes absolutely unstable. When $B_0 < B_s$, the Meissner state of the type II superconductor corresponds to the global minimum of the free energy. For $B_0 > B_s$, the vortex state, instead of the Meissner state, becomes the global minimum. However, transition from the Meissner state to the vortex state does not necessarily take place. This is because these two states are connected by a finite change of the order parameter and all the intermediate states have higher free energies than the Meissner states, which act as an energy barrier preventing the transition [21, 22]. The Meissner state may continue as a metastable state, even at $B_0 > B_s$. At $B_0 = B_c (> B_s)$ the free energy of all possible intermediates states achieved by perturbations to the Meissner state becomes smaller than that of the Meissner state: the Meissner state is unstable with respect to any small perturbation. Bean and Livingston [23] examined a specific and crucial intermediate state within the London theory: a vortex near the surface. They showed there exists an energy barrier for penetration of the vortex that originates in the attractive force between the surface and a single vortex (the Bean–Livingston barrier), and obtained a rough estimate of $B_s$ by finding the field at which the Bean–Livingston barrier disappears—which we call the vortex penetration field to distinguish the rough estimate from the true value of $B_s$. Rigorous calculations of $B_s$ have also been carried out within the Ginzburg–Landau (GL) theory [22, 24–26] and the quasiclassical theory [27, 28].
which are valid at the vicinity of the critical temperature $T_c$ and at an arbitrary temperature $0 < T < T_c$, respectively. Above $B_0$, only the highly dissipative vortex state, which yields much stronger dissipation than is acceptable in SRF applications, can exist. The superheating field $B_s$ at GHz frequencies defines the theoretical field limit of the SRF cavity. Then we may consider use of an alternative material that has a higher $B_s$ ($\sim B_i$) which may push up the ultimate limit (see the light blue regions of figure 1). Such a material, however, tends to have a small lower critical field $B_{c1}$ (see the deep blue regions of figure 1), above which the Meissner state ceases to be stable and can transition to the vortex state. The energy barrier may protect the material against penetration of vortices, as mentioned above, but it would not provides adequate protection: the actual cavity surface involves a tremendous number of material and topographic defects which reduce the energy barrier, causing local penetration of vortices at $B_0 \sim B_{c1}$. In particular, at a temperature as low as that required for SRF operations, vortices that locally penetrate at $B_0 \sim B_{c1}$. Thus it should be thick enough to protect the Nb substrate.

The applied magnetic field $B_0$ and the stability of the Meissner state at $T \approx 0$. The deep blue regions correspond to $B_0 < B_{c1}$ and represent the stable Meissner state. The light blue regions correspond to $B_{c1} < B_0 < B_c$, where the Meissner state is not stable but metastable and can transition to the more stable vortex state. Here we assumed the following material parameters [2]: $B^{(Nb)}_c = 170$ mT, $B^{(Nb)}_{c1} = 200$ mT and $B^{(Nb)}_s = 1.2B^{(Nb)}_c = 240$ mT for Nb; $B^{(NbSn)}_c = 20$ mT, $B^{(NbSn)}_{c1} = 230$ mT and $B^{(NbSn)}_s = 0.84B^{(NbSn)}_c = 190$ mT for NbN; $B^{(NbSnO)}_{c1} = 40$ mT, $B^{(NbSnO)}_s = 540$ mT and $B^{(NbSnO)}_c = 0.84B^{(NbSnO)}_c = 450$ mT for Nb$_3$Sn.

Figure 1. The applied magnetic field $B_0$ and the stability of the Meissner state at $T \approx 0$. The deep blue regions correspond to $B_0 < B_{c1}$ and represent the stable Meissner state. The light blue regions correspond to $B_{c1} < B_0 < B_c$, where the Meissner state is not stable but metastable and can transition to the more stable vortex state. Here we assumed the following material parameters [2]: $B^{(Nb)}_c = 170$ mT, $B^{(Nb)}_{c1} = 200$ mT and $B^{(Nb)}_s = 1.2B^{(Nb)}_c = 240$ mT for Nb; $B^{(NbSn)}_c = 20$ mT, $B^{(NbSn)}_{c1} = 230$ mT and $B^{(NbSn)}_s = 0.84B^{(NbSn)}_c = 190$ mT for NbN; $B^{(NbSnO)}_{c1} = 40$ mT, $B^{(NbSnO)}_s = 540$ mT and $B^{(NbSnO)}_c = 0.84B^{(NbSnO)}_c = 450$ mT for Nb$_3$Sn.

Figure 2. The simplest multilayer superconductor (SIS structure). The blue, green and gray regions correspond to a superconductor layer, an insulator layer and a superconductor substrate, respectively.

Figure 3. A multilayer superconductor without an insulator layer. When the superconductor layer (blue region) and the superconductor substrate (gray region) are dirty Nb and clean Nb, respectively, this can be regarded as a model of the Nb surface after low-temperature baking.

As mentioned above, the I layer is the essential constituent in the multilayer approach. However, the multilayer structure without I layers (as shown in figure 3) is also worth studying for the following two reasons. First, it can be regarded as a model of the surface of a superconductor that consists of superconductors with different penetration depths. As briefly mentioned in the discussion section of [40], after

bulk material and then lead to a thermal quench in the same manner as mentioned in the last paragraph. On the other hand, the S layer partly screens the surface magnetic field down to a level that the bulk Nb can withstand (i.e. $\sim B^{(Nb)}_{c1} \sim B^{(Nb)}_c$). Thus it should be thick enough to protect the Nb substrate.
low-temperature baking [9–12], the Nb surface, which has a depth-dependent mean-free path [41, 42] and then a depth-dependent penetration depth, can be described by a superconductor–superconductor (SS) bilayer [43] with a thin dirty Nb and a clean Nb substrate as the simplest model. The same would be true for modified baking [44]. Note here that the present approach cannot incorporate the impurity-concentration dependence of the density of states in the current-carrying state [2, 28]. Recent work on the Nb surface after low-temperature baking [45] also used a similar approach to the above. Second, some researchers have made SS bilayer structures such as MgB2–Nb and Nb3Sn–Nb, and have carried out sample testing [46, 47], which also needs to be understood theoretically. In the last part of the present article, some features of the SS bilayer structure are reviewed which are already known from studies of the SIS structure [36–39].

The main purpose of this article is to summarize the important formulae necessary for planning proof-of-concept experiments of the multilayer approach and to introduce some formulae for the SS bilayer structure obtained as by-products of studies of the SIS structure. The article is organized as follows. In section 2, the vortex penetration field and the superheating field are briefly reviewed, which are a necessary for calculating the field limit of a multilayer superconductor. In section 3, we review how to optimize layer thickness and the combination of materials in the SIS structure. First, an SIS structure with the ideal surface and negligibly thin I layer is studied. The results are expressed using the vortex penetration field from the London theory and the superheating fields of the GL and quasiclassical theories in a step by step procedure. The latter is valid at an arbitrary temperature (0 < T < Tc). Then the theory concerning the effects of a finite I-layer thickness is also investigated. Finally, the effects of surface defects are taken into account. The surface resistance of the SIS structure is also evaluated. In section 4, some known results of the SS bilayer structure are reviewed, using similar techniques to those in section 3. First the procedure for optimizing the layer thickness and material combination to maximize the theoretical field limit is reviewed. Then a barrier structure in the surface layer is examined: we see that the SS boundary has a role as a barrier to prevent vortex penetration. The surface resistance of the SS bilayer structure is also derived in much the same way as for the SIS structure. All the calculations are explained in detail for readers who want to follow the processes used to derive the formulae.

2. Brief review of the superheating field

Let us begin with a brief review of the basics of the superheating field. We treat a semi-infinite superconductor (as shown in figure 4) throughout this section. The surfaces of materials are assumed to be flat and parallel to the y–z plane. The applied magnetic field is parallel to the z-axis and is given by B0 = (0, 0, B0).

Figure 4. Model of a semi-infinite superconductor. The surface is parallel to the y–z plane and then perpendicular to the x-axis. The applied magnetic field is given by B0 = (0, 0, B0).

2.1. Vortex penetration field from the London theory

As mentioned in the section 1, the transition from the Meissner state to the vortex state is prevented by the existence of intermediate states with higher free energies than the Meissner state even when B0 > Bh. The purpose of this subsection is to estimate the superheating field by examining a specific intermediate state in the framework of the London theory as was done by Bean and Livingston [23]. We use the term ‘vortex penetration field’ instead of ‘superheating field’ in order to distinguish the rough estimate of the superheating field from the true one.

We assume the superconductor is made of an extreme type II material with a penetration depth λ and a coherence length ξ (ξ ≪ λ). Let us put a vortex with the flux quantum φ0 = 2.07 × 10−15 Wb parallel to the xy plane at x = (x0, 0). Then the vortex feels two distinct forces fB(x0) and fM(x0). The former is the force from a Meissner screening current J = Jy and is given by fM(x0) = J(x0) × φ0 ẑ = φ0 ẑ . The vortex is at the surface (x0 = ξ), we have fM(x0)|x0=ξ = φ0 ẑ , (1)

which pushes the vortex into the inside. The latter force, fB, is a force due to an interaction between the vortex and the boundary. The simplest way to calculate fB is use of the method of images: remove the boundary, regard all the space as the superconductor, put an image vortex to satisfy the boundary condition and evaluate the force due to the image. In this problem, the appropriate image is an antivortex for x0 > λ (see appendix A). When the vortex is at the surface (x0 = ξ), we have fB(x0)|x0=ξ = φ0 ẑ /4πμ0 ξ , (2)
which attracts the vortex to the surface. Instead of the method of images, $f_{d}$ can be evaluated by a brute-force approach: solve the London equation $-\lambda^2 \nabla^2 \mathbf{B} + \mathbf{B} = \phi_0 \delta^{(2)}(\mathbf{r} - \mathbf{r}_0)$ at the domain $x \geq 0$ with the boundary condition given by the zero current normal to the surface, evaluate the energy of the vortex interacting with the boundary and differentiate the energy over the position of the vortex (see appendix B).

When the screening current density $J(0)$ is so small that $|f_{d,\mathbf{M}}| < |f_{d}|$, the total force directs the negative direction of the $x$-axis, which acts as a barrier preventing penetration of vortices. This barrier is called the Bean–Livingston surface barrier [23]. When $|f_{d,\mathbf{M}}(\xi)| > |f_{d}(\xi)|$, the barrier disappears and the vortex is drawn into the material. Then the maximum current that the material can withstand against vortex penetration is derived from the condition $|f_{d,\mathbf{M}}(\xi)| = |f_{d}(\xi)|$ and is given by

$$J_{\text{max,L}} = \frac{\phi_0}{4\pi \mu_0 \lambda^2 \xi}$$

(3)

where the subscript L represents the London theory. By using the London equation $J(0) = -\lambda \mathbf{A}(0)/\mu_0 \lambda^2$, equation (3) can be expressed as $A_{\text{max,L}} = | - \phi_0 \lambda^2 J_{\text{max,L}} |$ or

$$A_{\text{max,L}} = \frac{\phi_0}{4\pi \lambda^2 \xi}.$$  

(4)

The applied field corresponding to equations (3) or (4) is the vortex penetration field, $B_c$. In order to obtain $B_c$, we need to know the relation between $B_0$ and $J$ (or $A$). Then the next task is to solve the London equation,

$$A'' = \frac{1}{\lambda^2} A = 0,$$

(5)

where the prime denotes the derivative over $x$ and then the double prime means the second derivative. The solution of equation (5) under the boundary condition $B_0 = A'(0)$ is given by $A(x) = -\lambda B_0 e^{-x/\lambda}$ or $J(x) = B_0/\mu_0 \lambda e^{-x/\lambda}$. Since $B_0 = \mu_0 \lambda J(0)$, $B_c$ is given by $B_c = \mu_0 \lambda J_{\text{max,L}}$ or [23]

$$B_c = \frac{\phi_0}{4\pi \lambda^2 \xi} = \frac{1}{\sqrt{2}} B_0 \simeq 0.71 B_0.$$  

(6)

It should be noted that the balance of forces at the surface means that the free energy at the surface is flat (i.e. the energy barrier disappears). The force approach is equivalent to the free energy approach [23] in the evaluation of the vortex penetration field in the London theory [48, 49].

Clearly, the definition of the vortex penetration field is unsatisfactory. The London theory ignores the pair-breaking effect due to the current density, and the vortex core is replaced by the normal conducting filament with radius $\sim \xi$. In the above, we put a vortex at $x_0 = \xi$ by hand and examine how large a field is necessary to make it penetrate into the inside, where we necessarily introduce an ambiguity resulting from the short distance cutoff $\sim \xi$. The vortex penetration field only gives the order of magnitude of the true superheating field. For a rigorous discussion, at least the GL theory is necessary.

2.2. Superheating field at $T \approx T_c$

Let us examine the superheating field within the GL theory, which is valid only at $T \approx T_c$ [22, 24–26]. We use the same unit as [22]: $\vec{\nabla} \equiv \lambda \nabla$, $\mathbf{A} \equiv \mathbf{A}/\sqrt{2} B_0$, $\mathbf{B} \equiv \vec{\nabla} \times \mathbf{A} = \mathbf{B}/(\sqrt{2} B_0)$. In the following, we omit all the tildes for brevity. Then the free energy of a semi-infinite superconductor is given by

$$\Omega = \int d^3r \left[ \frac{1}{\kappa^2} (\nabla f)^2 + \frac{1}{2} (1 - f^2)^2 \right] + f^2 \mathbf{A}^2 + (\mathbf{B}_0 - \nabla \times \mathbf{A})^2,$$

(7)

where $\kappa = \lambda/\xi$ is the GL parameter, $f$ represents the real and dimensionless order parameter, and $B_0$ is the applied magnetic field. In the absence of vortices, it is possible to choose the gauge in which $f$ is real, and the superfluid velocity is simply proportional to $A$. The GL equations are given by

$$\nabla \times \nabla \times \mathbf{A} = -f^2 \mathbf{A}.$$  

(8)

The stability of the Meissner state can be discussed by considering the second variation of the free energy under small perturbations $f + \delta f$ and $A + \delta A$, namely,

$$\delta^2 \Omega = \int d^3r \left[ \frac{1}{\kappa^2} (\nabla \delta f)^2 + (3f^2 + \mathbf{A}^2 - 1) \delta f^2 + 4f \mathbf{A} \cdot \delta \mathbf{A} \delta f + f^2 \delta \mathbf{A}^2 + (\nabla \times \delta \mathbf{A})^2 \right].$$  

(9)

As long as $\delta^2 \Omega$ is positive definite, the Meissner state corresponds to the global minimum or a metastable local minimum [22]. The perturbations are generally given by $\delta f = \bar{\delta f}(x, y)$ and $\delta A = (\delta A_x(x, y), \delta A_y(x, y), 0)$ and can be expanded as $\bar{\delta f}(x, y) = \hat{\delta f}(x) \cos ky, \delta A_x(x, y) = \hat{\delta A}_x(x) \sin ky$ and $\delta A_y(x, y) = \hat{\delta A}_y(x) \cos ky$.

Let us consider the case $\kappa \to \infty$ for simplicity. Then, after some calculations, we find $\delta^2 \Omega$ is positive definite as long as $A^2 \leq 1/3$, and the Meissner state becomes absolutely unstable when $|A| = 1/\sqrt{3} \equiv A_{\text{max,GL}}$. The subscript shows that the result is obtained by the GL theory. Restoring the dimensional units, we obtain

$$A_{\text{max,GL}} = \sqrt{2} B_0 \lambda \frac{1}{\sqrt{3}} = \frac{\phi_0}{2\sqrt{3} \pi \xi}.$$  

(10)

The applied field corresponding to equation (10) is the superheating field. The applied field is related to $A$ through the relation $B_0 = (\text{rot} \mathbf{A}(0))_x = \lambda(0)$, where $A^2$ can be obtained by solving the GL equation $A''$.

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$$B_{c,\text{GL}} = \sqrt{2} B_c \frac{\sqrt{5}}{18} = \frac{\sqrt{5}}{3} B_c \simeq 0.745 B_c.$$  

(11)
which is the superheating field of the superconductor with \( \kappa \to \infty \) at \( T \approx T_c \). Note that equation (11) is modified for a finite \( \kappa \) [24, 26]. For example, the superheating field of Nb \( (\kappa \simeq 1) \) is given by \( B_c^{(\text{NB})} \approx 1.2B_c^{(\text{Nb})} \) at \( T \approx T_c \). (See [26] for \( B_{s,\text{GL}} \) for an arbitrary \( \kappa \).)

2.3. Superheating field at \( T = 0 \)

The superheating field evaluated in the GL theory, which is valid only at \( T \approx T_c \), is not applicable to the SRF cavity operated at \( T \ll T_c \) in accelerator applications. The quasiclassical formalism [50], which is applicable to an arbitrary temperature, is available for calculations of the superheating field at \( T \ll T_c \). The superheating field for a clean superconductor with \( \kappa \to \infty \) at \( T \to 0 \) is given by [21, 27, 28]

\[
B_c(0) = B_c(0) \sqrt{1 - (25/3 - 3) \exp(24/3 - 2)} \approx 0.84B_c(0).
\] (12)

(see appendix C for the derivation of equation (12)). Extended results for \( T = 0 \) are seen in [27] and those for a superconductor with impurities are in [28]. Equation (12) is approximately applicable to a superconductor with \( \kappa \to \infty \) containing non-magnetic impurities [28]. Note here that quasiclassical theory is valid in the temperature range \( 0 < T < T_c \), and equation (11) can also be derived using the quasiclassical formalism by considering the case that \( T \approx T_c \).

3. A multilayer superconductor

Now we examine a SIS multilayer superconductor. The theoretical field limit of the SIS structure \( B_{\text{max}} \) and the optimum layer thicknesses and material combination to maximize \( B_{\text{max}} \) are discussed. We start from an investigation of a model with an ideally flat surface and a negligibly thin insulator in the London theory. Then we develop it step by step towards a more quantitative model. At the end of this section, we arrive at a realistic model with an imperfect surface and a finite insulator thickness; its field limit and the optimum parameters are expressed using the superheating field of the quasiclassical theory, which is valid at an arbitrary temperature \( 0 < T < T_c \). The surface resistance of the SIS structure is also derived. This step-by-step approach seems to be redundant, but would be beneficial for readers who want to follow all the calculations. Throughout this section, we consider the model shown in figure 5.

3.1. SIS structure with a thin \( I \) layer in the London theory

While the London theory provides only a rough estimate of the field limit of the SIS structure, the analysis based on the London theory contains the essence of the optimization procedure of layer thicknesses and a material combination [36].

As mentioned in the section 2.1, the vortex penetration field is defined by the balance of the two forces acting on a vortex at the surface: the force from the screening current, \( f_\text{M} \), and that from the boundary, \( f_B \). As seen in the section 2.1, the former is given by

\[
\begin{align*}
f_\text{M}(x_0) |_{x_0=\xi} &= J(\xi) \times \phi_0 \hat{\mathbf{k}} \\
&= \phi_0 J(\xi)\hat{\mathbf{x}} \approx \phi_0 J(0)\hat{\mathbf{x}},
\end{align*}
\] (13)

where \( J = f\hat{\mathbf{y}} \) is the screening current density. The latter, \( f_B \), can be evaluated by the method of images: remove both the boundaries at \( x = 0 \) and \( x = d_S \), extend the S-layer material to the whole space, put appropriate images to satisfy the boundary conditions (zero current normal to the boundaries at \( x = 0 \) and \( x = d_S \)), and evaluate the force due to all the images. This time, unlike in the section 2.1, an infinite number of images are necessary to satisfy the boundary conditions. Suppose a vortex is placed at an arbitrary position \( x_0 \) in the S layer. Then we need to introduce: (i) an antivortex at \( x = -x_0 \) to satisfy the condition at \( x = 0 \), (ii) an antivortex at \( x = 2d_S - x_0 \) and a vortex at \( x = 2d_S + x_0 \) to satisfy the condition at \( x = d_S \), which violates the condition at \( x = 0 \), (iii) a vortex at \( x = -2d_S + x_0 \) and an antivortex at \( x = -2d_S - x_0 \) to satisfy the condition at \( x = 0 \) again, which violates the condition at \( x = d_S \), (iv) an antivortex at \( x = 4d_S - x_0 \) and a vortex at \( x = 4d_S + x_0 \) to satisfy the condition at \( x = d_S \), and so on. Finally, an infinite number of image vortices are introduced, as shown in figure 6. All the images act on the vortex at \( x = x_0 \). When \( d_S \lesssim \lambda_1 \) the total force can be calculated as (see appendix D)

\[
\begin{align*}
\begin{aligned}
f_B(x_0) &= \frac{\phi_0^2}{4\pi\mu_0\lambda_1^2} \left[ -\frac{1}{x_0} + \sum_{n=1}^{\infty} \left( \frac{1}{nd_S - x_0} - \frac{1}{nd_S + x_0} \right) \right] \hat{\mathbf{x}} \\
&= \frac{\phi_0^2}{4\pi\mu_0\lambda_1^2d_S} \pi \cot \pi x_0 \frac{d_S}{d_S} \hat{\mathbf{x}}.
\end{aligned}
\end{align*}
\] (14)

When the vortex is placed at the surface \( (x_0 = \xi) \) and
\( \xi_i \ll d_s \), equation (14) is reduced to
\[
f_b(x_0)|_{x_0=\xi_i} = -\frac{\phi_0^2}{4\pi \mu_0 \lambda_i^2} \hat{S}, \quad (15)
\]
which corresponds to equation (2) obtained for a semi-infinite superconductor in the section 2.1. Equations (14) and (15) can be derived by directly solving the London equation (see [52] and appendix E).

When \( J(0) \) is so large that \( |f_0(\xi_i)| > |f_1(\xi_i)| \), the barrier disappears and the vortex is drawn into the material. The maximum current can be obtained by balancing \( f_0 \) and \( f_1 \) and is given by
\[
J_{\text{max},L}^{(S)} = \frac{\phi_0}{4\pi \mu_0 \lambda_i^2}. \quad (16)
\]
By using the London equation, this can be written as
\[
A_{\text{max},L}^{(S)} = -\mu_0 \lambda_i^2 J_{\text{max},L}^{(S)} \text{ or } A_{\text{max},L}^{(S)} = \frac{\phi_0}{4\pi \xi_i}. \quad (17)
\]
Equations (16) and (17) also correspond to those obtained for the semi-infinite superconductor in the section 2.1.

In order to evaluate the maximum field that the S layer can withstand, we need to know the relation between \( B_0 \) and \( J \) (or \( A \)). Here, for simplicity, we consider the case that \( d_t \) is negligibly small and solve the London equation
\[
A'' = \frac{1}{N_i} A, \quad (18)
\]
where \( \lambda = \lambda_1 \) at \( 0 \leq x \leq d_s \) and \( \lambda = \lambda_2 \) at \( x > d_s + d_t \). The general solution is written as
\[
A = C_1 e^{-\frac{x}{\lambda_1}} + C_2 e^{\frac{x}{\lambda_1}} \text{ at } 0 \leq x \leq d_s \text{ and } A = C_3 e^{-\frac{x}{\lambda_2}} \text{ at } x > d_s, \quad (19)
\]
where \( C_i \) \((i = 1, 2, 3)\) are constants determined by boundary conditions. The boundary conditions are given by
\[
B_0 = (\text{rot } A(0))_i = A'(0) = -C_i/\lambda_i + C_i/\lambda_1 \text{ and the continuity conditions of } B \text{ and } A \text{ at } x = d_s, \quad \text{namely, } C_1 e^{\frac{d_t}{\lambda_1}} + C_2 e^{\frac{d_t}{\lambda_1}} = C_3 \text{ and } -C_1 e^{-\frac{d_t}{\lambda_1}} + C_2 e^{-\frac{d_t}{\lambda_1}} = -(\lambda_1/\lambda_2) C_3. \quad (20)
\]
The solution is given by
\[
A = -\lambda_1 B_0 \left( \frac{\sinh \frac{d_t}{\lambda_1} + \lambda_1 \sinh \frac{d_t}{\lambda_1}}{\cosh \frac{d_t}{\lambda_1} + \lambda_1 \sinh \frac{d_t}{\lambda_1}} \right), \quad (0 \leq x \leq d_s), \quad (19)
\]
and
\[
A = -\lambda_2 B_0 \left( \frac{e^{-\frac{x}{\lambda_2}}}{\cosh \frac{d_s}{\lambda_2} + \lambda_2 \sinh \frac{d_s}{\lambda_2}} \right), \quad (d_s < x < \infty). \quad (20)
\]
The magnetic field distribution [36] is given by
\[
B(x) = A'(x), \quad \text{or} \quad B = B_0 \left( \frac{\cosh \frac{d_t}{\lambda_1} + \lambda_1 \sinh \frac{d_t}{\lambda_1}}{\gamma_1/\mu_0 \lambda_1} \right) \left( \frac{\cosh \frac{d_t}{\lambda_2} + \lambda_2 \sinh \frac{d_t}{\lambda_2}}{\gamma_2/\mu_0 \lambda_2} \right), \quad (0 \leq x \leq d_s) \quad (21)
\]
and
\[
B = B_0 \left( \frac{e^{-\frac{x}{\lambda_2}}}{\cosh \frac{d_s}{\lambda_2} + \lambda_2 \sinh \frac{d_s}{\lambda_2}} \right), \quad (d_s < x < \infty). \quad (22)
\]
The current density distribution [36] is given by
\[
J = \frac{B_0}{\mu_0 \lambda_1} \frac{\sinh \frac{d_t}{\lambda_1} + \lambda_1 \cosh \frac{d_t}{\lambda_1}}{\gamma_1/\mu_0 \lambda_1} \left( \frac{\cosh \frac{d_t}{\lambda_2} + \lambda_2 \sinh \frac{d_t}{\lambda_2}}{\gamma_2/\mu_0 \lambda_2} \right), \quad (0 \leq x \leq d_s), \quad (23)
\]
and
\[
J = \frac{B_0}{\mu_0 \lambda_2} \left( \frac{e^{-\frac{x}{\lambda_2}}}{\cosh \frac{d_s}{\lambda_2} + \lambda_2 \sinh \frac{d_s}{\lambda_2}} \right), \quad (d_s < x < \infty). \quad (24)
\]
Examples of the magnetic field and current density distributions are shown in figure 7. Then, at the surface, we have [36]
\[
J(0) = \gamma_1 B_0/\mu_0 \lambda_1, \quad (25)
\]
where the factor \( \gamma_i \), defined by
\[
\gamma_i = \frac{\sinh \frac{d_s}{\lambda_i} + \lambda_i \cosh \frac{d_s}{\lambda_i}}{\cosh \frac{d_s}{\lambda_i} + \lambda_i \sinh \frac{d_s}{\lambda_i}}, \quad (26)
\]
represents the difference in the surface current density between the SIS and a simple semi-infinite superconductor. This factor comes from the counterflow induced by the substrate [36]. An intuitive explanation is as follows. Let us consider the magnetic field at the interface of the S layer and the substrate, \( B_i \). The magnetic field generated by the S layer current is parallel to \( \hat{z} \) at the interface and negatively contributes to \( B_i \); on the other hand, the one due to the substrate current is parallel to \( \hat{z} \) at the interface and positively contributes to \( B_i \); these two contributions determine \( B_i \). When the substrate is made of the same material as the S layer \( (\lambda_2 = \lambda_1) \), the magnetic field distribution becomes the well-known exponential decay for a simple semi-infinite superconductor: \( B_i = B_0 e^{-d_t/\lambda_1} \). If we replace the substrate material with a material that has a smaller penetration depth \( \lambda_2 < \lambda_1 \), the magnetic field generated by the substrate increases, and the magnetic field at the interface does too. Thus we have \( B_i > B_0 e^{-d_t/\lambda_1} \); magnetic field attenuation in the S layer is prevented by the counterflow induced by the substrate with a smaller penetration depth (see figure 7(a)). Since the current density is given by the slope of the magnetic field attenuation,
and magnetic field attenuation in the S layer is promoted. This means that the surface current is enhanced and \( \gamma_1 \times 1 > 1 \).

By using equation (25) or \( B_0 = \gamma_1^{-1} \mu_0 \gamma_1 J(0) \), the applied magnetic field corresponding to \( J_{\text{max},L}^{(S)} \) or \( A_{\text{max},L}^{(S)} \) is given by [36]

\[
B_{\text{max},L}^{(S)} = \gamma_1^{-1} \mu_0 \gamma_1 J_{\text{max},L}^{(S)} = \gamma_1^{-1} \frac{\phi_0}{4 \pi \lambda_1 \xi_1} = \gamma_1^{-1} \frac{B_{\text{c}}^{(S)}}{\sqrt{2}}
\]

(27)

where \( B_{\text{c}}^{(S)} \) and \( B_{\text{c}}^{(S)} \) are the thermodynamic critical field and the vortex penetration field of the S-layer material, respectively. \( g_{\text{max},L}^{(S)} \) is the maximum field that the S layer can withstand. As mentioned above, \( \gamma_1 < 1 \) or \( \gamma_1^{-1} > 1 \) when the condition \( \gamma_1 \times 1 > 1 \) is satisfied. Then \( B_{\text{max},L}^{(S)} \) can exceed the vortex penetration field of the S-layer material \( B_{\text{c}}^{(S)} \) by the factor \( \gamma_1^{-1} \). This enhancement comes from the suppression of surface current by \( \gamma_1^{-1} \). Conversely, when \( \lambda_1 < \lambda_2 \), the surface current is enhanced by \( \gamma_1 \times 1 > 1 \) and \( B_{\text{max},L}^{(S)} \) is suppressed by \( \gamma_1^{-1} \times 1 < 1 \).

Figure 8 shows \( \gamma_1^{-1} \) as a function of the S-layer thickness. When \( \lambda_1 > \lambda_2 \), the factor \( \gamma_1^{-1} \) increases as \( d_S \) decreases: the thinner the S layer the larger the \( B_{\text{max},L}^{(S)} \) (see the solid blue curve).

In the following discussion, equation (28) is assumed to be satisfied. While a thin S layer pushes up \( B_{\text{max},L}^{(S)} \), an extremely small \( d_S \) cannot protect the substrate. When the magnetic field at the interface of the substrate [36],

\[
B_1 \equiv B(d_S) = \gamma_2 B_0,
\]

\[
\gamma_2 = \frac{1}{\cosh \frac{d_S}{\lambda_1} + \frac{\lambda_2}{\lambda_1} \sinh \frac{d_S}{\lambda_1}},
\]

(29)

exceeds the field limit of the substrate \( B_{\text{max}}^{(\text{sub})} \), it causes a breakdown, where \( B_{\text{max}}^{(\text{sub})} \) is an empirical field limit of the substrate material (e.g. \( B_{\text{max}}^{(\text{Nb})} \sim B_{\text{c}}^{(\text{Nb})} \sim B_{\text{c}}^{(\text{Nb})} \) for a bulk Nb). Thus, in order to improve the field limit of the whole SIS structure \( B_{\text{max},L}^{(S)} \), we need to optimize \( d_S \) so as to simultaneously increase \( B_{\text{max},L}^{(S)} \) and suppress \( B_1 \). For a given \( d_S \), \( B_{\text{max},L}^{(S)} \) is given by \( B_0 \) that satisfies \( B_0 < B_{\text{max},L}^{(S)} \) and \( B_0 < B_{\text{max}}^{(\text{sub})} \) simultaneously [36]:

\[
B_{\text{max},L}^{(S)} = \min \{ \gamma_1^{-1} B_{\text{c}}^{(S)}, \gamma_2^{-1} B_{\text{max}}^{(\text{sub})} \}.
\]

(30)

To find the maximum value of \( B_{\text{max},L}^{(S)} \), let us see the solid curves in figure 8, corresponding to \( \lambda_1 > \lambda_2 \). While \( \gamma_1^{-1} \) increases as \( d_S \) decreases, \( \gamma_2^{-1} \) increases as \( d_S \) increases. \( B_{\text{max}}^{(S)} \) is maximized when the condition \( \gamma_1^{-1} B_{\text{c}}^{(S)} = \gamma_2^{-1} B_{\text{max}}^{(\text{sub})} \) is satisfied. By substituting the definitions of \( \gamma_1 \) and \( \gamma_2 \), this condition becomes the quadratic equation

\[
(1 + \lambda_2/\lambda_1)u^2 - 2ru - (1 - \lambda_2/\lambda_1) = 0,
\]

where \( u \equiv e^{d_S/\lambda_1} \) and \( r \equiv B_{\text{c}}^{(S)}/B_{\text{max}}^{(\text{sub})} \).

The solution is given by

\[
u = (r + \sqrt{r^2 + 1 - \lambda_2^2/\lambda_1^2})/(1 + \lambda_2/\lambda_1) \equiv u_0 \text{ or } [37] \]
\[ d_s^{\text{opt}} = \lambda_1 \log u_0 = \lambda_1 \log \left[ \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{B_{(s)}^{(S)}}{B_{(s)}^{(sub)}} \right] ^2 + \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^2 \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2 \] (31)

Substituting \( u = u_0 \) into \( \gamma_2^{-1} = (1/2)(u + u^{-1}) + (1/2) \) \((\lambda_2/\lambda_1)(u - u^{-1})\), we find \( \gamma_2^{-1} = \sqrt{r^2 + 1 - (\lambda_2/\lambda_1)^2} \). Then the optimized \( B_{\text{opt},L} \) is given by [37]

\[ B_{\text{opt},L} = \gamma_2^{-1} B_{(s)}^{(S)} = \gamma_2^{-1} B_{(s)}^{(sub)} \]

\[ = \frac{1}{2} \left( B_{(s)}^{(S)} \right)^2 + \frac{1 - \lambda_2^2}{\lambda_1^2} \left( B_{(s)}^{(sub)} \right)^2 \] (32)

which is the main result in this subsection together with the optimum conditions given by equations (28) and (31).

So far, we have examined the SIS structure in the framework of the London theory, where the main results explicitly depend on the vortex penetration field of the S-layer material, \( B_{(s)}^{(S)} \). As mentioned in the section 2.1, however, the vortex penetration field defined in the London theory is unsatisfactory. The main results should be expressed by the superheating field of the GL or quasiclassical theories.

### 3.2. SIS structure with a thin I layer at \( T \approx T_c \)

Next we investigate the same system as above, an SIS structure with a negligibly thin I layer, in the framework of the GL theory [37, 38] and rewrite the main results using the GL superheating field. We follow the discussion in [37].

For simplicity, we assume that the S layer and the substrate are made of materials with \( \kappa \gg 1 \). Then the GL equation is given by \( A'' = A - A^3 \) in the usual dimensionless expression (see also the discussion below equation (10)).

Restoring the dimensional units, we have

\[ A'' = \frac{1}{\lambda^2} A - \frac{4\pi^2\xi^2}{\phi_0^2} A^3, \] (33)

where \( \lambda = \lambda_1 \) and \( \xi = \xi_1 \) if \( 0 \leq x \leq d_S \) and \( \lambda = \lambda_2 \) and \( \xi = \xi_2 \) at \( x > d_S \). Multiplying \( A' \) and integrating \( \lambda^2 A'' = A' - (4\pi^2\xi^2/\phi_0^2) A^3 \), we obtain

\[ \lambda^2 A' - A^2 + \frac{2\pi^2\xi^2}{\phi_0^2} A^4 = C \quad (0 < x < d_S), \] (34)

\[ \lambda^2 A' - A^2 + \frac{2\pi^2\xi^2}{\phi_0^2} A^4 = 0 \quad (d_S < x), \] (35)

where \( C \) is a constant. The S layer and the substrate of the optimized SIS structure can achieve \( A(0) = \phi_0/2\sqrt{3} \pi \xi_1 \) and \( A(d_S) = \phi_0/2\sqrt{3} \pi \xi_2 \), respectively (see equation (10)), when the applied field is \( B_0 = A'(0) = B_{(s)}^{\text{opt},L} \). Substituting \( x = 0 \) into equation (34), \( x = d_S \) into equation (34), and \( x = d_S \) into equation (35), we have

\[ \lambda^2 (B_{(s)}^{\text{opt},L})^2 - \frac{5\phi_0^2}{\sqrt{2}\pi\xi_2} = C, \] (36)

\[ \lambda^2 A'(d_S)^2 - \frac{4\phi_0^2}{72\pi^2\xi_0^2} \left( 6 - \frac{\xi_1^2}{\xi_0^2} \right) = C, \] (37)

\[ \lambda^2 A'(d_S)^2 - \frac{5\phi_0^2}{72\pi^2\xi_0^2} = 0. \] (38)

Solving these three equations, we find [37]

\[ B_{(s)}^{\text{opt},L,\text{GL}} = \sqrt{\left( 1 - \frac{\xi_1^2}{\xi_0^2} \right) + \frac{\xi_1^2}{\xi_0^2} \left( B_{(s)}^{(S)} \right)^2 + \left( 1 - \frac{\lambda_2^2}{\lambda_1^2} \right) \left( B_{(s)}^{(sub)} \right)^2}, \] (39)

where \( B_{(s)}^{(S)} = (\sqrt{5}/3)(\phi_0/2\sqrt{2}\pi \xi_1) \) and \( B_{(s)}^{(sub)} = (\sqrt{5}/3) \). In the above calculation, we have assumed the substrate can withstand fields up to its superheating field \( B_{(s)}^{(sub)} \), but this can be replaced by an empirical field limit \( B_{(s)}^{(sub)} \). Furthermore, when \( \xi_1 \ll \xi_2 \), the second and third terms in the first parenthesis are negligible. Then equation (39) is reduced to [37]

\[ B_{(s)}^{\text{opt},L,\text{GL}} = \sqrt{\left( B_{(s)}^{(S)} \right)^2 + \left( 1 - \frac{\lambda_2^2}{\lambda_1^2} \right) \left( B_{(s)}^{(sub)} \right)^2}, \] (40)

which has the same form as equation (32), except that \( B_{(s)}^{(S)} \) is replaced by \( B_{(s)}^{(S)} \).

Equation (40) can be obtained in an easier way as follows. We disregard the nonlinear term in equation (33) and obtain the London equation, equation (18)—we assume that the magnetic field attenuation is well described by the London equation. Its solution is given by equations (19) and (20). Then the surface current density is given by equation (25), and the magnetic field at the interface by equation (29). The surface current density must be smaller than the deparing limit \( B_{(s)}^{(S)}(\mu_0 \lambda_1) \), namely \( \gamma_1 B_0/\mu_0 \lambda_1 < B_{(s)}^{(S)}/\mu_0 \lambda_1 \) or \( B_0 < \gamma_1 B_{(s)}^{(S)} \). Furthermore, the magnetic field at the interface must be smaller than the empirical field limit of the substrate: \( B_l = \gamma_2 B_0 < B_{(s)}^{(sub)} \). Then the maximum \( B_0 \) is given by \( B_{(s)}^{(S)} = \min \{ \gamma_1 B_{(s)}^{(S)}, \gamma_1 B_{(s)}^{(sub)} \} \), which corresponds to equation (30) except that \( B_{(s)}^{(S)} \) is replaced by \( B_{(s)}^{(S)} \). \( B_{(s)}^{(S)} \) is maximized when \( \gamma_1 B_{(s)}^{(S)} = \gamma_2 B_{(s)}^{(sub)} \), and finally we obtain equation (40).

By using the same scheme as the above, the optimum conditions and the optimized field limit can be expressed by using the superheating field of the quasiclassical theory as shown below, which is valid at an arbitrary temperature \( 0 < T < T_c \).

### 3.3. SIS structure with a thin I layer at \( 0 < T < T_c \)

We repeat the same scheme as above. The only difference is the deparing limit: \( B_{(s)}^{(S)}(\mu_0 \lambda_1) \) is replaced by \( B_{(s)}^{(S)}/\mu_0 \lambda_1 \) which is obtained in the framework of the quasiclassical theory. Let us summarize the results. The field limit for a
given $d_s$ is given by [36]

$$B_{\text{max}} = \min \left( \gamma_1^{-1} B^{(S)}_s, \gamma_2^{-1} B^{(\text{sub})}_{\text{max}} \right),$$

where $\gamma_1$ and $\gamma_2$ are given by equations (26) and (29), respectively. When the conditions [36]

$$\lambda_1 > \lambda_2,$$  

and [37]

$$d_s = \lambda_1 \log \left[ \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{B^{(S)}_s}{B^{(\text{sub})}_{\text{max}}} + \frac{1}{\lambda_1 + \lambda_2} \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2 \right]$$

are satisfied, $B_{\text{max}}$ is maximized and is given by [37]

$$B_{\text{max}}^{\text{opt}} = \sqrt{(B^{(S)}_s)^2 + \left(1 - \frac{\lambda_2^2}{\lambda_1^2}\right)(B^{(\text{sub})}_{\text{max}})^2}.$$  

Note that all $B^{(S)}_s$ or $B^{(S)}_{\text{sub}}$ have been replaced by those obtained in the quasiclassical theory, $B^{(S)}_s$; these formulae are valid at an arbitrary temperature $0 < T < T_c$. When the S-layer material is a superconductor with $\kappa \gg 1$ and an accelerator is operated at $T \ll T_c$, $B^{(S)}_s$ is approximately given by $B^{(S)}_s = 0.84 B^{(S)}_c$, which is derived in section 2.3 for a superconductor with $\kappa \to \infty$ at $T \to 0$. The same $B^{(S)}_s$ is available as an approximate value when non-magnetic impurities are included [28].

3.4. SIS structure with a finite $d_l$, at $0 < T < T_c$

So far we have neglected the thickness of the I layer. Now we incorporate effects of a finite $d_l$. When the frequency of the electromagnetic field is of the order of GHz and $d_l \ll 1$ cm, the magnetic field distribution in the SIS structure is given by (see appendix F and [36, 53])

$$B = B_0 \cosh \frac{d_s - x}{\lambda_l} \frac{\lambda_2 + d_l}{\lambda_1} \sinh \frac{d_s - x}{\lambda_l} \frac{\lambda_2 + d_l}{\lambda_1}$$

$$0 \leq x \leq d_s,$$  

$$B = B_0 \frac{1}{\cosh \frac{d_s}{\lambda_l} + \frac{\lambda_2 + d_l}{\lambda_1} \sinh \frac{d_s}{\lambda_l}}$$

$$= \gamma_2 B_0 \left( d_5 < x \leq d_s + d_l \right),$$

$$B = B_0 \frac{e^{-\frac{x - d_s}{\lambda_l}}}{\cosh \frac{d_s}{\lambda_l} + \frac{\lambda_2 + d_l}{\lambda_1} \sinh \frac{d_s}{\lambda_l}}$$

$$= \gamma_2 B_0 e^{-\frac{x - d_s}{\lambda_l}} \left( x \geq d_s + d_l \right),$$

where

$$\gamma_2 = \frac{1}{\cosh \frac{d_s}{\lambda_l} + \frac{\lambda_2 + d_l}{\lambda_1} \sinh \frac{d_s}{\lambda_l}}.$$  

Then the surface current density is given by $J(0) = -B(0)/\mu_0$ or [36]
The above considerations, a small $d_1$ is desirable taking into account the dielectric loss and the low thermal conductivity of the I layer. The dielectric loss is discussed in section 3.6.

The optimum conditions for maximizing $B_{\text{max}}$ are derived in much the same way as before, and are given by \cite{37, 39}

$$\lambda_1 > \lambda_2 + d_1, \quad d_1 \lesssim \mathcal{O}(10) \text{ nm},$$

$$d_s = \lambda_1 \log \left[ \frac{\lambda_1}{\lambda_1 + \lambda_2 + d_1} \frac{B_{\text{max}}^{(S)}}{B_{\text{max}}^{(\text{sub})}} \right] + \sqrt{\left( \frac{\lambda_1}{\lambda_1 + \lambda_2 + d_1} B_{\text{max}}^{(S)} \right)^2 + \left( \frac{\lambda_1 - \lambda_2 - d_1}{\lambda_1 + \lambda_2 + d_1} \right)^2}$$

The optimized $B_{\text{max}}$ is given by \cite{37, 39}

$$B_{\text{max}}^{\text{sub}} = \sqrt{(B_{\text{max}}^{(S)})^2 + \left[ 1 - \left( \frac{\lambda_2 + d_1}{\lambda_1} \right)^2 \right] (B_{\text{max}}^{(\text{sub})})^2}.$$ \hspace{1cm} (54)

When $d_1 \ll \lambda_2$, these formulae are reduced to equations (42)-(44).

3.5. Incorporating the effect of defects

According to studies on surface topographies \cite{54, 55}, material surfaces are covered by multi-scale structures characterized by a fractal nature \cite{56, 57}. Nano-scale defects almost continuously exist on the surface, and $B_s$ is reduced at each defect (see \cite{40} for example). Furthermore, precipitates or variations of chemical composition also reduce $B_s$. Then $B_s$ of the real surface is effectively reduced to $\eta B_s$, where $\eta$ is a suppression factor that contains the effects of surface defects.

In the context of a multilayer superconductor, the superheating field of the S layer would be reduced to $\eta B_{\text{max}}^{(S)}$. This does not affect the field and current distributions: the field distribution is given by equations (45)-(48) and the surface current by equation (49). Then the field limit can be derived by replacing $B_{\text{max}}^{(S)}$ by $\eta B_{\text{max}}^{(S)}$:

$$B_{\text{max}} = \min \{ \tilde{\gamma}_1 - \eta B_{\text{max}}^{(S)}, \tilde{\gamma}_2 - \eta B_{\text{max}}^{(\text{sub})} \}.$$ \hspace{1cm} (55)

The optimum conditions and the optimized $B_{\text{max}}$ are given by \cite{37, 39}

$$\lambda_1 > \lambda_2 + d_1, \quad d_1 \lesssim \mathcal{O}(10) \text{ nm},$$

$$d_s = \lambda_1 \log \left[ \frac{\lambda_1}{\lambda_1 + \lambda_2 + d_1} \frac{\eta B_{\text{max}}^{(S)}}{B_{\text{max}}^{(\text{sub})}} \right] + \sqrt{\left( \frac{\lambda_1}{\lambda_1 + \lambda_2 + d_1} \frac{\eta B_{\text{max}}^{(S)}}{B_{\text{max}}^{(\text{sub})}} \right)^2 + \left( \frac{\lambda_1 - \lambda_2 - d_1}{\lambda_1 + \lambda_2 + d_1} \right)^2}$$

$$B_{\text{max}}^{\text{opt}} = \sqrt{(\eta B_{\text{max}}^{(S)})^2 + \left[ 1 - \left( \frac{\lambda_2 + d_1}{\lambda_1} \right)^2 \right] (B_{\text{max}}^{(\text{sub})})^2}.$$ \hspace{1cm} (58)

Assuming some concrete values of $\eta$, we can make similar contour plots to those in section 3.4. Figures 12, 13, and 14 show contour plots of $B_{\text{max}}$ for the cases of $\eta = 0.9$ and $\eta = 0.5$. Note here the optimum S-layer thickness decreases.

---

**Figure 10.** $B_{\text{max}}$ of a Nb₃Sn–I–Nb system in units of mT. Assumed parameters are $B_{\text{max}}^{(\text{NbN})} = 230 \text{ mT}$ and $\lambda_1 = 200 \text{ nm}$ for the S-layer material, $B_{\text{max}}^{(\text{sub})} = 170 \text{ mT}$ and $\lambda_2 = 40 \text{ nm}$ for the substrate. See also \cite{39}.

**Figure 11.** $B_{\text{max}}$ of a Nb₃Sn–I–Nb system in units of mT. Assumed parameters are $B_{\text{max}}^{(\text{NbSn})} = 540 \text{ mT}$ and $\lambda_1 = 120 \text{ nm}$ for the S-layer material, $B_{\text{max}}^{(\text{sub})} = 170 \text{ mT}$ and $\lambda_2 = 40 \text{ nm}$ for the substrate. See also \cite{39}.
as \( \eta \) decreases (see figures 9 and 12, figures 10 and 13, and figures 11 and 14). This can be understood as follows. As \( \eta \) decreases, the field limit of the S-layer decreases. The decreased field limit can be compensated by suppressing the surface current, which is possible by reducing \( d_S \). However, a complete compensation leads to a \( d_S \) that is too thin to protect the substrate. As a result, the optimum \( d_S \) takes into a moderately reduced value that can partially compensate the decreased field limit.

3.6. Surface resistance of a multilayer superconductor

The surface resistance of an SIS structure can be obtained by calculating the total Joule dissipation [37], which is given by

\[
\begin{align*}
\text{Figure 12.} & \text{ } B_{\text{max}} \text{ of a dirty Nb–I–Nb system in units of mT. Assumed parameters are } B_{\text{Nb}}^{(\text{I})} = 200 \text{ mT, } \lambda_1 = 180 \text{ nm and (a) } \eta = 0.9 \text{ and (b) } \eta = 0.5 \text{ for the S-layer material; } B_{\text{max}}^{(\text{sub})} = 170 \text{ mT and } \lambda_2 = 40 \text{ nm for the substrate. See also [39].}
\end{align*}
\]

\[
\begin{align*}
\text{Figure 13.} & \text{ } B_{\text{max}} \text{ of a NbN–I–Nb system in units of mT. Assumed parameters are } B_{\text{NbN}}^{(\text{I})} = 230 \text{ mT, } \lambda_1 = 200 \text{ nm, and (a) } \eta = 0.9 \text{ and (b) } \eta = 0.5 \text{ for the S layer material; } B_{\text{max}}^{(\text{sub})} = 170 \text{ mT and } \lambda_2 = 40 \text{ nm for the substrate. See also [39].}
\end{align*}
\]
(see appendix F)

\[ R_s = 2 \frac{\mu_0^2}{B_0^2} R_s^{(S)} \int_0^{d_s} J^2 dx + \frac{2}{B_0^2} \frac{\mu_0^2}{R_s^{(sub)}} \int_{d_s + d_l}^\infty J^2 dx + \frac{2}{B_0^2} R_1, \]

(59)

where \( J \) is the screening current distribution derived from the London equation, \( R_s^{(S)} \) is the surface resistance of the semi-infinite superconductor made of the S-layer material, \( R_s^{(sub)} \) is the surface resistance of the semi-infinite superconductor made of the substrate material, and \( R_1 \) is the dielectric loss. The evaluation of equation (59) is straightforward [37]:

\[
R_s = \left[ \frac{1 + \frac{\gamma_2}{2} \sinh \frac{2d_s}{\lambda_s}}{2} + r_s \left( \cosh \frac{2d_s}{\lambda_s} - 1 \right) \right] R_s^{(S)}
- (1 - \frac{\gamma_1}{\lambda_s}) \frac{d_s}{\lambda_s} \gamma_2 R_s^{(S)}
+ \gamma_2^2 R_s^{(sub)} + \gamma_2^2 \mu_0^2 \nu^2 \lambda_2^2 d_1,
\]

(60)

where \( r_s = (\lambda_2 + d_1)/\lambda_1 \) and we used the fact that the electric field in the I layer is given by \(- \nu \omega \lambda_2 \gamma_2 B_0\). The first, second and third terms correspond to a contribution from the S layer, substrate and I layer, respectively.

Let us roughly evaluate the third term, the dielectric loss contribution. Substituting \( \gamma_2 \approx 1, \omega \approx 10^{10} \text{ s}^{-1}, \epsilon'' < \epsilon_0, \lambda_2 \approx 10^{-7} \text{ m}, \) we find that it is smaller than \((d_I/\text{nm}) \times 10^{-7} \text{ mT}\). For example, when \( d_I = 100 \text{ nm} \), the dielectric loss contribution is smaller than \(10^{-5} \text{ mT}\) and is negligible. This smallness can be understood by recalling that the electric field in the I layer is given by \( |E| = \omega \lambda_2 \gamma_2 B_0 \approx 10^{-5} \text{ V m}^{-1} \) for \( B_0 = 10 \text{ mT} \), which is much smaller than that of the plane wave in the vacuum \( |E| \sim c B_0 \sim 1 \text{ MV m}^{-1} \) for the same \( B_0 \).

See also references [58, 59] for the multilayer normal conductor (NIN structure), where a reduction of power loss of a normal conducting RF cavity by using the NIN structure is proven theoretically and experimentally.

### 3.7. Summary of section 3

The main results of this section are as follows:

(i) We started with an investigation of the SIS structure with an ideal surface and a negligibly thin I layer in the framework of the London theory. Typical field and current distributions in the SIS structure are given in figure 7. The field limit is given by equation (30). The optimum conditions for maximizing the field limit and the optimized field limit are given by equations (28), (31) and (32).

(ii) The same system was examined in the GL theory, which is valid only at \( T \approx T_c \). The optimized field limit is given by equation (40) when the coherence length of the S layer is smaller than that of the substrate.

(iii) At \( 0 < T < T_c \), the field limit is given by equation (41), and the optimum conditions and the optimized field limit are given by equations (42)–(44), which are expressed using the superheating field derived in the quasiclassical theory.

(iv) In much the same way, a generalized model with a finite \( d_I \) was studied. The field limit is given by equation (51), and the optimum conditions and the optimized field limit are given by equations (52)–(54), which depend on \( d_I \) (see also figures 9–11).
(v) Furthermore, effects of material and topographic defects were incorporated. The field limit is given by equation (55), and the optimum conditions and the optimized field limit are given by equations (56)–(58), where the superheating field of the S-layer material is reduced by a factor $\eta$ (see also figures 12–14). These are the most general formulae, which can be applied to an SIS structure with surface defects and a finite $d_1$ under an arbitrary temperature $0 < T < T_c$.

(vi) Finally, the surface resistance formula was derived. See equation (60).

4. A multilayer superconductor without an insulator layer

As mentioned in the section 1, the role of the I layer is to intercept propagating vortex loops and to localize vortex dissipation in the S layer. The I layer is essential in the multilayer approach. Nonetheless, a multilayer superconductor without an I layer is also an interesting system and worth studying. Here we summarize the two reasons for this that were mentioned in section 1. (i) It can be regarded as a model of the surface of baked Nb, in which the penetration depth decreases in the first several tens of nanometers from the surface due to a depth-dependent mean-free path [41, 42]. The simplest model of baked Nb is the SS bilayer structure (see also the discussion section of [40]). Studying this system may help our understanding of the way that the low-temperature baking works. (ii) Some SRF researchers have made SS bilayer structures such as MgB$_2$–Nb or Nb$_3$Sn–Nb. The results of the sample tests [46, 47] need to be understood theoretically [43]. In this section, we review some features of the SS bilayer structure that have already been revealed through studies of the SIS structure.

4.1. Theoretical field limit

We consider the model shown in figure 15. Materials of the surface layer and the substrate are assumed to be superconductors with $\lambda_1$ and $\lambda_2$, respectively. The theoretical field limit of the SS bilayer structure [37, 43] can be derived by exactly same procedure as the SIS structure. To obtain the current and field distribution, we solve the London equation, $\lambda_1^2 A - A = 0$, where $\lambda = \frac{\lambda_1}{x}$ at $0 \leq x \leq d$ and $\lambda = \frac{\lambda_2}{x}$ at $x > d$. Its solution is given by equations (19) and (20). Then the current densities at the surface and the SS boundary can be obtained by using $J = -A/\mu_0\lambda^2$ and are given by

$$J(0) = \frac{B_0}{\mu_0\lambda_1},$$

$$\gamma_1 = \frac{\sinh \frac{d}{\lambda_1} + \frac{\lambda_2}{\lambda_1}\cosh \frac{d}{\lambda_1}}{\cosh \frac{d}{\lambda_1} + \frac{\lambda_2}{\lambda_1}\sinh \frac{d}{\lambda_1}}.$$

Figure 15. Model of the SS bilayer structure. The surface and boundary are parallel to the y–z plane and then perpendicular to the x-axis. The thickness of the surface layer is given by $d$. The applied magnetic field is given by $B_0 = (0, 0, B_0)$.

These current densities must be smaller than the depairing limit of the surface layer $B_{s} \langle S \rangle / \mu_0 \lambda_1$ and that of the substrate $B_{s} \langle s \rangle / \mu_0 \lambda_2$, respectively, where $B_{s} \langle S \rangle$ and $B_{s} \langle s \rangle$ are the superheating fields of the surface and substrate material for an arbitrary temperature derived using the quasiclassical theory. Then we have [36]

$$B_{\text{max}} = \min \{ \gamma_1^{-1} B_{s} \langle S \rangle, \gamma_2^{-1} B_{s} \langle s \rangle \}. \quad (63)$$

Note that $\gamma_i$ ($i = 1, 2$) are functions of $d$ as shown in figure 7, and then $B_{\text{max}}$ is also a function of $d$. The optimization of $d$ can also be carried out in much the same way as the SIS structure. $B_{\text{max}}$ is maximized when $\gamma_1^{-1} B_{s} \langle S \rangle = \gamma_2^{-1} B_{s} \langle s \rangle$, and finally we obtain the optimum conditions to maximize the field limit [36, 37],

$$d = \lambda_1 \log \left[ \frac{\lambda_1}{\lambda_2} \frac{B_{s} \langle S \rangle}{B_{s} \langle s \rangle} + \sqrt\left\{ \left( \frac{\lambda_1}{\lambda_2} \frac{B_{s} \langle S \rangle}{B_{s} \langle s \rangle} \right)^2 + \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right\} \right]. \quad (65)$$

The optimized field limit is given by [37]

$$B_{\text{max}} = \sqrt\left\{ (B_{s} \langle S \rangle)^2 + \left( 1 - \frac{\lambda_2^2}{\lambda_1^2} \right) (B_{s} \langle s \rangle)^2 \right\}, \quad (66)$$

which is the same as for the SIS structure with negligibly thin $d_1$ (see equation (44)). Figures 16 and 17 show examples of $B_{\text{max}}$ as functions of $d$. The peak values correspond to $B_{\text{max}}^{\text{opt}}$. 
It should be noted that even if the theoretical field limit is high such a high field cannot be necessarily be achieved in reality. As mentioned in section 1, the Meissner state ceases to be stable at \( B_0 > B_{c1} \) (see figure 1). While the surface barrier still protects the material against penetration of vortices, taking into account that the surface barrier is reduced at material and topographic defects that cover the cavity surface, achieving a field much higher than \( B_{c1} \) would not be easy without an additional mechanism to stabilize the Meissner state. In the SIS structure, the stability of the Meissner state at \( B_0 > B_{c1} \) is assured by the existence of the I layer, which stops penetration of vortices and suppresses vortex dissipation. In the SS bilayer structure, however, the I layer is absent: we have only the SS boundary. Is there any mechanism to stabilize the Meissner state in the SS bilayer structure? Our next task is to examine a role of the SS boundary.

4.2. Interaction between a vortex and the SS boundary

4.2.1. Infinite superconductor with two regions. As an instructive exercise we first consider an infinite superconductor that consists of two regions, \( x < 0 \) with \( \lambda = \lambda_1 \) and \( x \geq 0 \) with \( \lambda = \lambda_2 \). We examine the interaction between a vortex and the boundary. Suppose there exists a vortex parallel to \( \mathbf{k} \) at \( x = x_0 = -|x_0| \), where \( |x_0| \) is assumed to be smaller than \( \lambda_1 \) and \( \lambda_2 \) for simplicity. The force acting on the vortex can be evaluated by the method of images as usual. By using an analogy with a line charge embedded in an infinite dielectric with two regions, we find that the current distribution for \( x < 0 \) can be expressed by the superposition of the current circulating the vortex at \( x = -|x_0| \) and an image vortex with flux \( \phi_1' = \tau \phi_0 \) at \( x = +|x_0| \), and the current distribution for \( x > 0 \) can be expressed by an image vortex with \( \phi_1' = \tau' \phi_0 \) at \( x = -|x_0| \). Imposing the continuity conditions of \( j_x \) and \( A_y \) at the boundary, we find [43]

\[
\tau = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2}, \quad \tau' = 1 - \tau. \tag{67}
\]

Then the force acting on the vortex \( \mathbf{f}_B \) is given by [43]

\[
\mathbf{f}_B = \mathbf{j}_{\text{img}} \times \phi_0 \hat{\mathbf{z}} = -\frac{\phi_0 \phi_1}{4\pi\mu_0 \lambda^2 |x_0|} \hat{\mathbf{x}}, \tag{68}
\]

where \( \mathbf{j}_{\text{img}} \) is the current circulating the image vortex with flux \( \phi_1 \) at \( x = |x_0| \). Thus the SS boundary pushes the vortex to the direction of the material with a larger penetration depth. Note that instead of using the method of images we can directly solve the London equation and obtain the same result as the above (see appendix G).

4.2.2. Thin superconductor layer on a superconductor substrate. Now we go back to the system shown in figure 15. Suppose there exists a vortex parallel to \( \hat{\mathbf{z}} \) at \( x = x_0 \) inside the surface layer. The easiest way to evaluate the force acting on the vortex is to use the method of images. In order to satisfy the boundary conditions at \( x = 0 \) and \( x = d \), an infinite number of image vortices are necessary in common with the multilayer superconductor. We need: (i) an antivortex at \( x = -x_0 \) to satisfy the condition at \( x = 0 \), (ii) a vortex with flux \( \tau \phi_0 \) at \( x = 2d - x_0 \) and an antivortex with flux \( \tau \phi_0 \) at \( x = 2d + x_0 \) to satisfy the condition at \( x = d \), which violate the condition at \( x = 0 \), (iii) an antivortex with flux \( \tau \phi_0 \) at \( x = -2d + x_0 \) and a vortex with flux \( \tau \phi_0 \) at \( x = -2d - x_0 \) to satisfy the condition at \( x = 0 \) again, which violate the condition at \( x = d \), (iv) an antivortex with flux \( \tau^2 \phi_0 \) at \( x = 4d - x_0 \) and a vortex with flux \( \tau^2 \phi_0 \) at \( x = 4d + x_0 \) to satisfy the condition at \( x = d \), and so on. Finally an infinite number of image vortices are introduced.
the factor $\tau$ is given by equation (67).

(see figure 18). The total force is given by (see appendix H)

$$
\mathbf{f}_B = \frac{\phi_0^2}{4\pi \mu_0 \lambda_l} \left[ -\frac{1}{x_0} + \sum_{n=1}^{\infty} (-1)^n \tau^n \right] \times \left( \frac{1}{nd - x_0} - \frac{1}{nd + x_0} \right) + \frac{\tau}{1 - x_0/d} F \left( 1, 1 - \frac{x_0}{d}; 2 - \frac{x_0}{d}; -\tau \right),
$$

where $F(a, b; c; z) = [\Gamma(c)/\Gamma(b)\Gamma(c - b)] \int_0^1 dt (1 - tz)^{-a+b-1}(1 - t)^{-b-1}$ is the Gaussian hypergeometric function. Note that equation (69) is reduced to equation (2) as $x_0 \to 0$ and to equation (68) as $x_0 \to d$. The same result can be obtained by directly solving the London equation (see appendix I).

Figure 19 shows $f_B$ in units of $f_{B_{\text{GL}}}$ as functions of the vortex position $x_0/d$, where $f_{B_{\text{GL}}} = \phi_0^2/4\pi \mu_0 \lambda_l^2$. Note that the sign of $f_B/f_{B_{\text{GL}}}$ is positive when the force directs the surface and then acts as a barrier. When $\lambda_1 = \lambda_2$, the present system is reduced to a simple semi-infinite superconductor, and only the Bean–Livingston barrier exists, which attenuates as $x_0$ increases (black dashed curve). On the other hand, when $\lambda_1 \neq \lambda_2$, the vortex feels not only the Bean–Livingston barrier but also the force due to the SS boundary (blue solid curve and red dashed-dotted curve). In particular, when $\lambda_1 > \lambda_2$, the force due to the SS boundary acts as a barrier to prevent the penetration of vortices.

As seen in the above, the SS bilayer structure is protected by the double barriers: the Bean–Livingstone barrier and the barrier due to the SS boundary. Both barriers can be reduced by defects and have weak spots, but a vortex that penetrates from a weak spot in the Bean–Livingstone barrier may be stopped by the SS boundary: there is a second chance to stop the vortex. While the SS boundary is not as robust as the I layer in the SIS structure, it is also expected to contribute to preventing the penetration of vortices. Low-temperature baking [9–12] transforms the Nb surface from a simple semi-infinite clean Nb to a layered structure with $\lambda_1 > \lambda_2$ that consists of a dirty Nb layer and a clean Nb substrate [41, 42], where the boundary between dirty and clean Nb plays the role of a barrier and may be related to curing the high-field Q drop [40] together with other factors that would significantly affect SRF performance at a high field such as the difference in the density of states between the dirty and clean Nb [2]. The same would be true for the modified low-temperature baking [44]. The SS boundary in Nb$_3$Sn–Nb also satisfies $\lambda_1 > \lambda_2$ and plays the role of a barrier against the penetration of vortices.

It should be noted that the I layer in the SIS structure has a role not only in stopping penetration of vortices but also in suppressing vortex dissipation, because the dissipative vortex core disappears in the I layer. On the other hand, in the SS bilayer structure, the double barrier would contribute to stopping vortex penetration, but the dissipative vortex core is conserved in contrast to the SIS structure: the whole length of an oscillating vortex inside the surface layer contributes to dissipation.

### 4.3. Surface resistance of the SS bilayer structure

The surface resistance of the SS bilayer structure can be derived in much the same way as that of the SIS structure [37].
The main results of this section are as follows:

(i) The theoretical field limit of the SS bilayer structure was examined in much the same way as the SIS structure. The field limit is given by equation (63), which is maximized when equations (64) and (65) are satisfied. The optimized field limit is given by equation (66) (see figures 16 and 17). It should be noted that in order to achieve a theoretical field limit much higher than the lower critical field a mechanism to stabilize the Meissner state, such as the I layer of the SIS structure, is necessary.

(ii) The interaction among a vortex, the surface and the SS boundary was examined. The force acting on a vortex inside the surface layer is given by equation (69) (see also figure 19). The SS boundary provides an additional barrier to prevent penetration of vortices. It would not be as robust as the I layer of the SIS structure, but it also contributes to pushing up the onset of vortex penetration.

(iii) Finally the surface resistance of the SS bilayer structure was examined. The surface resistance formula is given by equation (70) (see also figures 20 and 21).

5. Summary

We have reviewed recent progress in theoretical understanding of the SIS structure and summarized important formulae that will be necessary for planning proof-of-concept experiments. Some results for the SS bilayer structure obtained in studies of the SIS structure have also been introduced, which would be useful for studying a system that can be modeled by the SS bilayer structure such as Nb3Sn–Nb, MgB2–Nb and the Nb surface after low-temperature baking. Important results are summarized in the end of each section (see sections 3.7 and 4.4 for the SIS and SS structures, respectively).
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Appendix A. Vortex in an infinite superconductor

The magnetic field distribution in an infinite superconductor can be derived by solving the London equation \(-\lambda^2 \nabla^2 B + B = \phi_0 \delta_3(r - r_0)\) or
\[
- \lambda^2 (\partial_x^2 + \partial_y^2) B(x, y) + B(x, y) = \phi_0 \delta(x - x_0) \delta(y),
\]
where \(B = B(x, y)\) and \(r_0 = (x_0, 0)\). While we can treat this equation in polar coordinates, here we use Cartesian coordinates as an instructive exercise for problems without rotational symmetry. Equation (A.1) can be written as
\[
B_k'' - p^2 B_k = -\frac{\phi_0}{\lambda^2} \delta(x - x_0),
\]
where
\[
p = \sqrt{k^2 + \frac{1}{\lambda^2}},
\]
\[
B_k(x) = \int_0^\infty dy B(x, y)e^{-iky}. By introducing the Fourier transformation \(B_k = \int_0^\infty dx B(x)e^{-ikx}\), equation (A.2) becomes an algebraic equation, the solution of which can be inverse Fourier transformed on the complex \(k'\)-plane with poles at \(x = \pm ip\). Then we find
\[
B_k(x) = \frac{\phi_0}{2\lambda^2 p}e^{-\pi(k - x_0)}.n
\]
The self-energy of the vortex is given by \(\epsilon_v = (\phi_0/2\mu_0)B(r_0)\) or
\[
\epsilon_v = \frac{\phi_0^2}{4\mu_0 \lambda^2} \int_0^\infty \frac{dk}{2\pi} e^{-\rho^2} = \frac{\phi_0^2}{4\mu_0 \lambda^2} K_0\left(\frac{\xi}{\lambda}\right),
\]
where the standard prescription \(r_0 = (x_0, 0) \rightarrow (x_0 + \xi, 0)\) is used, and \(K_0(z) = (1/2) \int_0^\infty dr \exp(-z \cosh r)\) is the modified Bessel function. By using \(K_0(\xi) \approx \log(1/\xi) + \log 2 - \gamma + O(\xi^2)\), where \(\gamma = 0.577\) is the Euler constant, equation (A.5) is reduced to
\[
\epsilon_v \approx \frac{\phi_0^2}{4\mu_0 \lambda^2} \log \frac{\lambda}{\xi},
\]
for \(\lambda/\xi \gg 1\). The current density can be derived by
\[
J = -(1/\mu_0)B' = -(1/\mu_0) \int_0^\infty (dk/2\pi) B_k'(x)e^{-iky}.
\]
When we are interested in a scale smaller than \(\lambda, p\) can be replaced by \(|k|\), and the current density at a distance \(r\) from the vortex core is given by
\[
J(r)\big|_{r = r_0 = r} = J(x_0 + r, 0) = \frac{\phi_0}{2\pi \mu_0 \lambda^2} \int_0^\infty e^{-kr} = \frac{\phi_0}{2\pi \mu_0 \lambda^2 r}. (A.7)
\]

Appendix B. Vortex in a semi-infinite superconductor

A system with a single vortex in a semi-infinite superconductor can be treated in much the same way as in appendix A. The governing equation is equation (A.2), and the general solution can be written as \(B_k(x) = (\phi_0/2\lambda^2)(1/p) e^{-\rho^2} + C e^{-\rho^2}\), where \(C\) is a constant. Since \(\partial_z B(x, y) = \int dk/2\pi B(x)ik e^{iky}\), the boundary condition, \(\partial_z = 0\) at the surface, can be written as \(B_k = 0\). Then we have
\[
C = -(\phi_0/2\lambda^2)(1/p) e^{-\rho^2}, \quad B_k(x) = \frac{\phi_0}{2\lambda^2} e^{-\rho^2} - e^{-p(x + x_0)}, (B.1)
\]
The self-energy of the vortex \(\epsilon_v = (\phi_0/2\mu_0)B(r_0)\) depends on its position due to the existence of the surface, in contrast to that of the free vortex treated in appendix A. This means that the vortex is attracted to a direction that yields a smaller \(\epsilon_v\) with a force given by \(f_B = -\partial_{\rho} \epsilon_v = -(\phi_0/2\mu_0) \int dk/2\pi \partial_\rho B_k(x_0)\) or
\[
f_B = -\frac{\phi_0^2}{2\mu_0 \lambda^2} \int_0^\infty dk e^{-2p\rho} = -\frac{\phi_0^2}{2\mu_0 \lambda^2} K_1(\frac{2\rho}{\lambda}). (B.2)
\]
where \(K_1(\xi) = \int_0^\infty dr e^{-z \cosh t} \cosh \xi t\) is used. When the vortex is placed at the vicinity of the surface, \(x_0/\lambda \ll 1\), equation (B.2) is reduced to
\[
f_B = -\frac{\phi_0^2}{4\mu_0 \lambda^2 x_0}, (B.3)
\]
where the asymptotic behavior \(\lim_{\rho \rightarrow 0} K_1(\xi) = (\nu - 1)! x^{2\nu - 1} e^{-x}\) is used. It should be noted that equation (B.3) can be derived in an easier way. Since we are interested only in a scale much smaller than \(\lambda\), we can replace \(p\) by \(|k|\). Then equation (B.2) becomes
\[
f_B = -\frac{\phi_0^2}{2\mu_0 \lambda^2} \int_0^\infty dk e^{-|k| x_0} = -\frac{\phi_0^2}{4\mu_0 \lambda^2 x_0}.
\]
which corresponds to equation (B.3).

Appendix C. The superheating field of a clean superconductor at \(T \rightarrow 0\)

We use the same units as [27]: \(\nabla = \lambda e \nabla, \hat{\Lambda} = (2\pi \xi_e/\phi_0) A, \hat{B} = \nabla \times \hat{A} = (3/2\mu_0 N(0)(B/\Delta_0), \hat{\Delta} = \Delta/\Delta_0, \hat{T} = k_B T/\Delta_0, \hat{\omega}_e = \hbar \omega_e/\Delta_0, \lambda_e^2 = (4\mu_0/3)(2\pi \xi_e/\phi_0)^2 \Delta_0^2 N(0), \xi_e = \hbar v_F/2\Delta_0, \nu = \Delta_0 N(0),\) where \(N(0)\) is the
density of states per one spin at the Fermi surface, $\Delta_{00}$ is the zero-temperature and zero-field order parameter, $v_F$ is the Fermi velocity, $k_B$ is the Boltzmann constant and $\omega_n = (2n \pi k_B T / h)(n + 1/2)$ is the Matsubara frequency [51]. In the following, we omit all the tildes for brevity. Then the free energy in unit of $\Delta_{00}$ is given by

$$\Omega = \nu \int d^3r \left\{ \frac{1}{3} \left( \nabla \times A - B_0 \right)^2 + \Delta^2 \log \frac{T}{T_c} \right. $$

$$+ 2\pi T \sum_n \left\{ \frac{\Delta^2}{\omega_n^2} - 2\Delta (\langle f \rangle - 2\omega_n (g) - 1) \right. $$

$$+ 2i (\mathbf{g} \cdot \mathbf{A}) \right\}, \tag{C.1}$$

where $\mathbf{n}$ is the unit vector normal to the Fermi surface, the angular brackets indicate angular averaging over the Fermi surface and the quasi-classical Green functions are given by $f = \Delta / \sqrt{\Omega_n^2 + \Delta^2}$ and $g = \Omega_n / \sqrt{\Omega_n^2 + \Delta^2}$ with $\Omega_n \equiv \omega_n + i \mathbf{n} \cdot \mathbf{A}$, which satisfy the constraint $g^2 + f^2 = 1$ and the Eilenberger equation $\Omega_n f = \Delta g$ for $\kappa \equiv \lambda_0 / \xi_s \rightarrow \infty$. The self-consistency condition is given by

$$\Delta \log \frac{T}{T_c} + 2\pi T \sum_n \left\{ \frac{\Delta}{\omega_n^2} - \langle f \rangle \right\} = 0. \tag{C.2}$$

In this unit, the energy density of the magnetic field $B^2 / 2\mu_0$ is reduced to $\nu T^2 / 3\mu_0$, and the condensation energy is given by $-(\nu / 3)B_0(T)^2 \equiv \nu \left\{ \Delta^2 \log (T / T_c) + 2\pi T \sum_n (\Delta^2 / \omega_n^2 - 2\Delta \omega_n (g) - 1) \right\}$. When $T = 0$, we have $B_0(0) = \sqrt{3} / 2$. The dimensional units, the well-known result $B_0(0) = \sqrt{3} T_c N(0) / \mu_0$ is reproduced.

In the same way as in subsection 2.2, we consider the second variation of $\Omega$ under small perturbations $\Delta + \delta \Delta$ and $A + \delta A$, which is given by

$$\delta^2 \Omega = \nu \int d^3r \left\{ \frac{1}{3} (\nabla \times \delta A)^2 + 2\pi T \sum_n \left\{ \delta^2 \left( \frac{\Delta^2}{\omega_n^2} + \langle n \cdot \delta A \rangle^2 \right) \right. $$

$$\left. + 2i \left( \mathbf{g} \cdot \mathbf{A} \right) \right\}, \tag{C.3}$$

where equation (C.2) is used. Expanding the perturbations as $\delta \Delta(x, y) = \delta \Delta_A(x) \cos ky$, $\delta A(x, y) = \delta \Delta_A(x) \sin ky$ and $\delta \Delta_A(x) \cos ky$, we obtain $\delta^2 \Omega \propto \int d x [(1 / 3) (\delta \Delta_A')^2 + F_0 \delta \Delta_A^2 + F_1 \delta \Delta_A^2 + 2G \delta \Delta_A \delta \Delta_A'],$ where

\begin{equation}
F_0 \equiv 2\pi T \sum_n (\Delta^2 / (\Omega_n^2 + \Delta^2)^2), \quad F_1 \equiv 2\pi T \sum_n (\Delta \omega_n / (\Omega_n^2 + \Delta^2)^{3/2}), \quad G \equiv 2\pi T \sum_n (\Delta \omega_n n / (\Omega_n^2 + \Delta^2)^{3/2}).
\end{equation}

Minimizing $\delta^2 \Omega$ with respect to $\delta \Delta$ and $\delta \Delta_A$, we find $\delta \Delta = - (G / F_0) \delta \Delta_A$ and $\delta \Delta_A = k \Delta_A / (3F_0 + k^2 \Delta_A')$. Substituting these into $\delta^2 \Omega$, we find the $\delta^2 \Omega$ is positive definite as long as $F_0 F_2 = G^2$. At the limit $T \rightarrow 0$, $F_0$, $F_2$, $G$ and $\mathbf{g}$ are analytically calculable. Using the notation $b \equiv \Delta / \Delta_A$, we obtain $(1 - \sqrt{1 - b^2}) (1 / 3) [1 - (1 + 2b^2) \sqrt{1 - b^2}] = (b \sqrt{1 - b^2})^2 [27]$ or

$$b = \sqrt{1 - (2/3 - 1)^2} \equiv b_0. \tag{C.4}$$

Then we have $\sqrt{1 - b_0^2} = 2/3 - 1$.

When $A \rightarrow 0$, equation (C.2) is reduced to the zero-field self-consistency condition: $\log (T / T_c) + 2\pi T \sum_n (\Delta^2 / \omega_n - 1 / \sqrt{\omega_n^2 + \Delta^2}) = 0$. Combining this with equation (C.2), we obtain $2\pi T \sum_n (\Delta / \sqrt{\omega_n^2 + \Delta^2} - \langle f \rangle) = 0$. At $T \rightarrow 0$, we find $\log [\Delta(1 + \sqrt{1 - b^2})] = (1 - \sqrt{1 - b^2}) / (1 + \sqrt{1 - b^2})$. Substituting $b = b_0$ into this, we obtain $A_{\max} = 2/3 b_0 / \exp(2/3 - 1)$. Restoring the dimension, we have

$$A_{\max} = \frac{\exp(2/3 - 1) \phi_0}{2/3} \frac{\phi_0}{2\pi \xi_s}. \tag{C.5}$$

The relation among $\Delta_A$, $\Delta$ and $A$ for a superconductor with $\kappa \gg 1$ is given by $B_s^2 = B_A^2 + (3 / \nu) \nu \left[ \frac{\Delta^2 \log (T / T_c) + 2\pi T \sum_n (\Delta^2 / \omega_n - 1 / \sqrt{\omega_n^2 + \Delta^2}) - 2\Delta \omega_n (g) - 1} {2\pi T \sum_n (\Delta^2 / \omega_n - 1 / \sqrt{\omega_n^2 + \Delta^2}) - 2\Delta \omega_n (g) - 2} \right] \right]$. By using equation (C.2), this becomes $B_s^2 / B_A^2 = 1 + B_s^{-2} \pi T \sum_n (2\omega_n - \Delta (f) - 2\omega_n (g) - 2i (\mathbf{g} \cdot \mathbf{A})) / \Delta (f)$. The angular averaging are given by $\langle f \rangle = - i b (z_2 - z_1)$, and $2i (\mathbf{g} \cdot \mathbf{A}) = \omega_n (- (g) + 4 / (2f)) - \Delta (f)$, respectively, where $z_1 \equiv (a - i) / b$, $z_2 \equiv (a + i) / b$, $a = \omega_n / A$, and $b = \Delta / A$. Then we have

$$B_s^2 / B_A^2 = 1 - B_s^{-2} \pi T \sum_n (2\omega_n - (2 / A) \text{Im}(\omega_n (\Omega_n^2 + \Delta^2))) / \Delta (f),$$

where $\Omega_n \equiv \omega_n + i a$. When $T \rightarrow 0$, substituting $b = b_0$ and $A = A_{\max}$ into $B_s^2 / B_A^2 = 1 - (2 / 3) A^2 \left( (1 / 2) - (3 / 2)(1 - b^2) + (1 / 2) \right)$, we find $[21, 27, 28]$

$$B_s(0) = B_c(0) \sqrt{1 - \frac{2(5/3 - 3) \exp(2/3 - 2)}{0.84 B_c(0)}. \tag{C.6}$$

### Appendix D. The summation in equation (14)

Let us evaluate the summation

$$S \approx \sum_{n=0}^{N} \left( \frac{1}{nds - x_0} - \frac{1}{nds + x_0} \right) = \frac{1}{\Delta_s} \sum_{n=-N}^{N} \left( \frac{1}{n - a} - \frac{1}{n + a} \right), \tag{D.1}$$

where $a \equiv x_0 / d_3$. By using the difference equation of the digamma function, $\psi(z + N) - \psi(z) = \sum_{n=1}^{N} \frac{1}{n + a}$
\((n + z - 1)^{-1}\), we have
\[
S = \lim_{N \to \infty} \left[ \psi(-a + 1 + N) - \psi(-a + 1) - \psi(a + 1 + N) + \psi(a + 1) \right] \\
= \frac{1}{ds} \left[ \lim_{N \to \infty} \log \frac{-a + 1 + N}{a + 1 + N} + \psi(a + 1) - \psi(1-a) \right] \\
= \frac{1}{ds} \left[ \psi(a + 1) - \psi(1-a) \right] \\
= \frac{1}{ds} \left[ \frac{1}{a} + \psi(a) - \psi(1-a) \right], \tag{D.2}
\]
where the relation \(\psi(a + 1) = \psi(a) + 1/a\) is used. Then, using the reflection formula, \(\psi(z) - \psi(1-z) = -\pi \cot \pi z\), we find
\[
S = \frac{1}{ds} \left[ \frac{1}{a} - \pi \cot \pi a \right] = \frac{1}{x_0} - \frac{1}{ds} \pi \cot \frac{\pi x_0}{d} ds. \tag{D.3}
\]

### Appendix E. Vortex in a thin film

Next we tackle a system with a single vortex in a thin film with thickness \(d \ll \lambda\) using the same technique as in appendix B (see [52] for more detailed discussions). The general solution of equation (A.2) can be written as
\[
B_k(x) = \left(\phi_0/2\lambda^2\right)(1/p)e^{-p/d} + C_1 e^{p/d} + C_2 e^{-p/d},
\]
where \(C_1\) and \(C_2\) are constants. The boundary conditions are given by \(i_0 = B_0 = 0\) and \(B_0 = B_0(d) = 0\). Then we find
\[
C_1 = -(\phi_0/2\lambda^2 p)[e^{-p(d-x_0)} - e^{-p(d+x_0)}]/(e^{p/d} - e^{-p/d}),
\]
\[
C_2 = (\phi_0/2\lambda^2 p)[e^{p(d-x_0)} + e^{-p(d+x_0)}]/(e^{p/d} - e^{-p/d}),
\]
and
\[
B_k(x) = \frac{\phi_0}{2\lambda^2 p \sinh pd} \left[ \cosh p(d - |x - x_0|) - \cosh p(x + x_0 - d) \right]. \tag{E.1}
\]
which corresponds with that given in [52] if we translate the coordinate as \(x \to x + d/2\) and \(x_0 \to x_0 + d/2\). In much the same way as the above, the self-energy of the vortex \(\varepsilon_v(r_0) = \langle \phi_0^2/2\mu_0 \rangle B(r_0)\) depends on its position due to the existence of the surfaces, and the vortex is exerted a force given by
\[
f_b = \frac{\phi_0^2}{2\pi\mu_0 \lambda^2} \lim_{\Delta \to 0} \int_{\Delta} \frac{dk}{2\pi} \sinh[pd(1 - 2a)] \sinh pd, \tag{E.2}
\]
where \(a \equiv x_0/d\). Since we are interested in a scale much smaller than \(\lambda\), we may replace \(F\) by \(|k|\). Substituting \(t = e^{-2kd}\), the integral becomes \((1/2d) \int_0^d dt [t^{a-1} - t^{1-a-1}]/(1 - t)\)
\[
= \left[ -\psi(a) + \psi(1-a) \right]/2d, \tag{E.3}
\]
and using the reflection formula, \(\psi(z) - \psi(1-z) = -\pi \cot \pi z\), equation (E.3) becomes
\[
f_b = -\frac{\phi_0^2}{4\pi\mu_0 \lambda^2} \pi \cot \frac{\pi x_0}{d}. \tag{E.3}
\]
When the vortex is at the edge of the film \(x_0/d \ll 1\), equation (E.3) is reduced to
\[
f_b = -\frac{\phi_0^2}{4\pi\mu_0 \lambda^2} \pi \cot \frac{\pi x_0}{d}. \tag{E.4}
\]
where \(\cot(\pi x_0/d) \approx d/\pi x_0\) is used. Note that equation (E.4) is equal to the force acting on the vortex at the edge of a semi-infinite superconductor.

### Appendix F. Electromagnetic field in a superconductor

We briefly summarize some results necessary for calculating the electromagnetic field distribution and the surface resistance in the SIS structure. Let us introduce the complex conductivity
\[
\sigma = \sigma' + i\sigma''. \tag{F.1}
\]
Then the current density can be written as \(j = \sigma E\). Then starting from the Maxwell equation \(\nabla \times E = -\partial_t B\), we obtain \(-\partial_t E = -\partial_i \nabla \times B = i\mu_0 \sigma \omega E\), where the displacement current term is always negligible. In much the same way as the above, starting from \(\nabla \times B = \mu_0 j\), we obtain \(-\Delta B = i\mu_0 \sigma \omega B\). Then the London equations for the electromagnetic field are given by [1]
\[
\Delta E = \frac{1}{\ell^2} E, \quad \Delta B = \frac{1}{\ell^2} B, \tag{F.2}
\]
Then the current density can be written as \(j = \sigma E\). Then starting from the Maxwell equation \(\nabla \times E = -\partial_t B\), we obtain \(-\partial_t E = -\partial_i \nabla \times B = i\mu_0 \sigma \omega E\), where the displacement current term is always negligible. In much the same way as the above, starting from \(\nabla \times B = \mu_0 j\), we obtain \(-\Delta B = i\mu_0 \sigma \omega B\). Then the London equations for the electromagnetic field are given by [1]
\[
\Delta E = \frac{1}{\ell^2} E, \quad \Delta B = \frac{1}{\ell^2} B. \tag{F.4}
\]
By using equations (F.4), we can calculate the electromagnetic field distribution in the SIS structure. The result and its derivation processes are shown in [36, 53].

On the other hand, for calculating the surface resistance, the second term of equation (F.3) is essential. For example, let \(\omega \to 0\), the second term approaches zero, and \(\ell \to \lambda\). In calculations of the electromagnetic field distribution, we can replace \(\ell\) by \(\lambda\). For example, using \(\omega \sim 10^{10} \text{ s}^{-1}\), \(\lambda \sim 10^{-7} \text{ m}\) and \(\sigma' \sim 10^7 \text{ S m}^{-1}\), we obtain \(\mu_0 \omega \sigma' \lambda^2 \approx 10^{-3}\), and the second term of equation (F.3) is negligible. Then equations (F.2) are reduced to
\[
\Delta E = \frac{1}{\lambda^2} E, \quad \Delta B = \frac{1}{\lambda^2} B. \tag{F.3}
\]
When we neglect terms with \(O(\sigma'^2/\sigma'')\), equation (F.5) is reduced to a simple form. Since \(1/\sigma' \approx i/\sigma'' + \sigma'/\sigma'\), we obtain \(R_1(1/\sigma') = (1/\sigma'') (\sigma'/\sigma'')\). Then we may consider
only the zeroth order for the contribution from the factor $|J|^2$ and we can regard $|J|^2$ as $J_{\nu=\lambda}^2$. Thus we have

$$R_s = \frac{\mu^2}{B_0} \int_0^\infty \frac{\sigma'}{\sigma_{\eta}} dJ_{\nu=\lambda}^2 \sigma' \sigma_{\eta}^2$$

$$= \frac{\mu^2}{B_0} \int_0^\infty \frac{\sigma'}{\sigma_{\eta}} dJ_{\nu=\lambda}^2 \sigma' \sigma_{\eta}^2 \lambda^4,$$  \hspace{1cm} (F.6)

where $\lambda^2 = \mu_0 \omega^2 |\eta|^2$ is used. Substituting $B_{\nu=\lambda} = B_0 e^{-\frac{r}{d}}$ or

$J_{\nu=\lambda} = -B'/\mu_0 = (B_0/\mu_0 \lambda) e^{-\frac{r}{d}}$ into equation (F.6), we obtain

$$R_s = \frac{1}{2} \sigma' \mu_0^2 \omega^2 \lambda^4.$$  \hspace{1cm} (F.7)

By using equation (F.5), the surface resistance of the SIS structure and the SS bilayer can also be evaluated.

**Appendix G. Vortex in an infinite superconductor with two regions**

Suppose the vortex is at $x = x_0 = -|x_0|$. Then the magnetic field distribution in this system can be obtained by solving the set of equations

$$B_{\nu=\lambda}^x - p_1^2 B_k = -\frac{\phi_0}{\lambda^2} \delta (x - x_0) \hspace{1cm} (x \leq 0),$$

$$B_{\nu=\lambda}^n - p_2^2 B_k = 0 \hspace{1cm} (x > 0),$$  \hspace{1cm} (G.1)

where $p_1 = \sqrt{k^2 + \lambda_1^2}$ and $p_2 = \sqrt{k^2 + \lambda_2^2}$. The general solution of equation (G.1) can be written as $B_k (x) = (\phi_0/2\lambda^2) \left( (1/p_1) e^{p_1|x-x_0|} + C e^{p_2 x} \right)$, and that of equation (G.2) is given by $B_k (x) = C e^{p_2 x}$, where $C_1$ and $C_2$ are constants.

The boundary conditions are given by $\xi_1 (0) = \xi_1 (0)$ and $\xi_1 (-x_0) = \xi_1 (-x_0)$, which reduce to $B_1 (0) = B_1 (-x_0)$ and $\lambda_1^2 B_1 (0) = \lambda_2^2 B_1 (-x_0)$, respectively. Then we find $A_1 = (\phi_0/2p_1 \lambda^2) [(p_1 \lambda_1^2 - p_2 \lambda_2^2)/(p_1 \lambda_1^2 + p_2 \lambda_2^2)] e^{p_1 x_0}$, $A_2 = \phi_0/(p_1 \lambda_1^2 + p_2 \lambda_2^2) e^{-p_2 x_0}$ and

$$B_k (x) = -\frac{\phi_0}{p_1 \lambda^2} \left( (1/p_1) e^{-p_1|x-x_0|} + \frac{p_1 \lambda_1^2 - p_2 \lambda_2^2}{p_1 \lambda_1^2 + p_2 \lambda_2^2} e^{p_1 x_0} \right) \hspace{1cm} (x \leq 0),$$

$$B_k (x) = \frac{\phi_0}{p_1 \lambda^2} \left( (1/p_1) e^{p_1 x} + \frac{p_1 \lambda_1^2 - p_2 \lambda_2^2}{p_1 \lambda_1^2 + p_2 \lambda_2^2} e^{-p_1 x_0} \right) \hspace{1cm} (x > 0).$$  \hspace{1cm} (G.3)

The force acting on the vortex is given by

$$f_B = -\frac{\phi_0}{2\mu_0} \int (dk/2\pi) \partial_{\nu x_0} B_k (x_0)$$

$$= -\frac{\phi_0}{2\mu_0 \lambda^2} \int_0^\infty dk p_1 \lambda_1^2 - p_2 \lambda_2^2 e^{p_2 x_0}.$$  \hspace{1cm} (G.5)

When we focus attention on a scale smaller than $\lambda_1$ and $\lambda_2$, we may replace $p_1$ and $p_2$ by $|k|$. Then the force is given by

$$f_B = -\frac{\tau_0 \phi_0}{2\mu_0 \lambda^2} \int_0^\infty dk e^{2kx_0} = -\frac{\phi_0 \phi_1}{4\mu_0 \lambda^2 |x_0|}$$

$$= \frac{\phi_1}{\tau_0 \phi_0}, \quad \tau \equiv \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2}.$$  \hspace{1cm} (G.6)

where

$$\phi_1 \equiv \tau \phi_0, \quad \tau \equiv \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2}.$$  \hspace{1cm} (G.7)

Equation (G.6) can be interpreted as a force due to the image with flux $\phi_1 = \tau \phi_0$ at $x = |x_0|$ (see also [43]).

**Appendix H. Summation in equation (69)**

Let us evaluate the summation in equation (69),

$$S' \equiv \frac{1}{x_0} - \sum_{n=1}^{\infty} \left( -1 \right)^n \left( \frac{1}{nd - x_0} - \frac{1}{nd + x_0} \right)$$

$$= \frac{1}{d} \left( \frac{1}{a} + \sum_{n=1}^{\infty} \left( -1 \right)^n \frac{1}{n + a} - \sum_{n=1}^{\infty} \left( -1 \right)^n \frac{1}{n - a} \right),$$

$$= \frac{1}{d} \left[ \Phi(-\tau, 1; a) + \tau \Phi(-\tau, 1, 1 - a) \right],$$

where $\Phi(z, s, a) = \sum_{n=0}^{\infty} z^n / (n + a)^s$ is the Lerch transcendent. Through its integral representation, $\Phi(z, s, a) = \Gamma(s)^{-1} \int_0^1 t^{s-1} e^{-at} / (1 - ze^{-t}) dt$, we arrive at

$$S' = \frac{1}{d} \left[ \frac{1}{1 + x_0} \int_0^1 F (1; 1; 1 - a) \right]$$

$$+ \frac{\tau}{1 - a} \int_0^1 F (1, 1; 1 - a; -\tau \gamma),$$

where $F$ is the Gaussian hypergeometric function.

**Appendix I. Vortex in a thin layer formed on a semi-infinite superconductor**

Suppose the vortex is at $x = x_0$ ($0 < x_0 < d$). Then the magnetic field distribution in the surface layer can be obtained by solving

$$B_{\nu=\lambda}^x - p_1^2 B_k = -\frac{\phi_0 \delta (x - x_0)}{\lambda^2} \hspace{1cm} (0 < x < d),$$

$$B_{\nu=\lambda}^n - p_2^2 B_k = 0 \hspace{1cm} (x > d)$$  \hspace{1cm} (I.1)

$$B_{\nu=\lambda}^n - p_2^2 B_k = 0 \hspace{1cm} (x > d).$$  \hspace{1cm} (I.2)

where $p_1 = \sqrt{k^2 + \lambda_1^2}$ and $p_2 = \sqrt{k^2 + \lambda_2^2}$. The general solution of equation (I.1) can be written as $B_k (x) = (\phi_0/2\lambda^2) \left( (1/p_1) e^{p_1|x-x_0|} + C e^{p_2 x} \right)$, and that of equation (I.2) is given by $B_k (x) = C e^{p_2 x}$, where $C_1$, $C_2$, and $C_3$ are constants.

The boundary conditions are given by $\xi_1 (0) = \xi_1 (0)$ and $\xi_1 (d - 0) = \xi_1 (d + 0)$, which reduce to $B_1 (0) = B_1 (d - 0)$ and $B_1 (d - 0) = \lambda_1^2 B_1 (d - 0)$, and $\lambda_2^2 B_1 (d - 0) = \lambda_2^2 B_1 (d + 0)$, respectively. Then we find $C_1 = (\phi_0/2p_1 \lambda^2) [p_1 \lambda_1^2 - p_2 \lambda_2^2] e^{p_1 x_0}$, $C_2 = -\phi_0/(2p_1 \lambda_2^2) e^{-p_2 x_0}$, and $C_3 = (\phi_0/2p_1 \lambda_1^2) e^{p_2 x_0}$.
\[ e^{-p_1d}(1 + \tau^{-1}e^{2\pi d}), \text{ where } \tau \equiv (p_1 \lambda_2^2 - p_2 \lambda_1^2)/(p_1 \lambda_1^2 + p_2 \lambda_2^2). \text{ Then } B_k \text{ at } 0 < x < d \text{ is given by} \]

\[
B_k(x) = \frac{\phi_0}{2p_1 \lambda_1^2} \frac{e^{-p_1|x-x_0|} - e^{-p_1(x+x_0)} + \tau e^{-2\pi d}(e^{p_1(x+x_0)} - e^{p_1|x-x_0|})}{1 + \tau e^{-2\pi d}}.
\]

The force acting on the vortex is given by

\[ f = -\frac{\phi_0}{2\mu_0} \int d\gamma [2\pi \partial_x B_k(x_0)] \]

\[ f_B = -\frac{\phi_0^2}{2\pi\mu_0 \lambda_1^2} \int_0^\infty \frac{d\epsilon}{1 + \tau e^{-2\pi d}} [e^{-2\pi \epsilon^2} + \tau e^{-2\pi d}\epsilon^2]. \]

Since we are focusing on a scale smaller than \( \lambda_1 \) and \( \lambda_2 \), \( p_1 \) and \( p_2 \) can be replaced by \( |k| \), and \( \tau \) by \( \tau \equiv (\lambda_2^2 - \lambda_1^2)/(\lambda_1^2 + \lambda_2^2) \). Substituting \( t \equiv e^{-2\pi d} \) and \( a \equiv x_0/d \), we have

\[ f_B = -\frac{\phi_0^2}{2\pi\mu_0 \lambda_1^2} \int_0^\infty \frac{d\epsilon}{1 + \tau e^{-2\pi d}} F(1, 1; 1+a; -\tau)
+ \frac{\tau}{1-a} F(1, 1-a; 2-a; -\tau), \]

where \( F \) is the Gaussian hypergeometric function. When the vortex is at the surface \( a = x_0/d \ll 1 \), the contribution from the first term becomes dominant, and we have

\[ f_B = -\frac{\phi_0^2}{4\pi\mu_0 \lambda_1^2} \int \frac{d\epsilon}{1 + \tau e^{-2\pi d}} F(1, 1; 1; -\tau)
+ \frac{\tau}{1-a} F(1, 1-a; 2-a; -\tau).
\]

Equation (I.6) corresponds with the force acting on the vortex near the boundary of two infinite superconductors given by equation (G.6).

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