Cohomology of $\mathfrak{sl}(2)$ acting on the space of $n$-ary differential operators on $\mathbb{R}$

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Abstract

We compute the cohomological space $H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda,\mu})$ where $\mu \in \mathbb{R}$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $D_{\lambda,\mu}$ is the space of multilinear differential operators from $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ to $\mathcal{F}_\mu$.

The structure of these spaces was conjectured in [M. Ben Ammar et al. in International Journal of Geometric Methods in Modern Physics Vol. 9, No. 4 (2012) 1250033 (15 pages).]

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1 Introduction

The space of weighted densities of weight $\mu$ on $\mathbb{R}$ (or $\mu$-densities for short), denoted by:

$$\mathcal{F}_\mu = \{ f dx^\mu, \ f \in C^\infty(\mathbb{R}) \}, \ \mu \in \mathbb{R},$$

is the space of sections of the line bundle $(T^*\mathbb{R})^\otimes^\mu$. The Lie algebra Vect($\mathbb{R}$) of vector fields $X_h = h \frac{d}{dx}$, where $h \in C^\infty(\mathbb{R})$, acts on $\mathcal{F}_\mu$ by the Lie derivative $L^\mu$. Alternatively, this action can be written as follows:

$$X_h \cdot (f dx^\mu) = L^\mu_{X_h}(f dx^\mu) := (hf' + \mu h' f) dx^\mu, \ (1.1)$$

where $f'$, $h'$ are $\frac{df}{dx}$, $\frac{dh}{dx}$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ we denote by $D_{\lambda,\mu}$ the space of multilinear differential operators $A$ from $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ to $\mathcal{F}_\mu$. The Lie algebra Vect($\mathbb{R}$) acts on the space $D_{\lambda,\mu}$ of these differential operators by:

$$X_h \cdot A = L^\mu_{X_h} \circ A - A \circ L^\lambda_{X_h} \ (1.2)$$

where $L^\lambda_{X_h}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ defined by the Leibnitz rule. If we restrict ourselves to the Lie algebra $\mathfrak{sl}(2)$ which is isomorphic to the Lie subalgebra of Vect($\mathbb{R}$) spanned by $\{X_1, X_x, X_{x^2}\}$,

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we have a family of infinite dimensional \( \mathfrak{sl}(2) \)-modules still denoted by \( D_{\lambda, \mu} \).

According to Nijenhuis-Richardson [3], the space \( H^1(\mathfrak{g}; \text{End}(V)) \) classifies the infinitesimal deformations of a \( \mathfrak{g} \)-module \( V \) and the obstructions to integrability of a given infinitesimal deformation of \( V \) are elements of \( H^2(\mathfrak{g}; \text{End}(V)) \). While the spaces \( H^1(\mathfrak{g}; L(\otimes^n V, V)) \) appear naturally in the problem of normalization of nonrepresentations of \( \mathfrak{g} \) in \( V \). To be more precise, let

\[
T : \mathfrak{g} \to \sum_{n \geq 0} L(\otimes^n V, V), \quad X \mapsto T_X = \sum T^n_X,
\]

be a nonlinear representation of \( \mathfrak{g} \) in \( V \), that is, \( T_{[X, Y]} = [T_X, T_Y] \). In [1], it is proved that the representation \( T \) is normalized if \( T^1_X \) is in a supplementary of \( B^1(\mathfrak{g}; L(\otimes^n V, V)) \) in \( Z^1(\mathfrak{g}; L(\otimes^n V, V)) \).

In fact if \( A \) is a differential operator on the line, \( A \) can be viewed as an homomorphism from \( \mathcal{F}_\tau \) to \( \mathcal{F}_\mu \). If \( A \) is with order \( n \), we can define its symbol as an element in \( S^n_\beta = \bigoplus_{j=0}^n \mathcal{F}_{\beta-j} \) for \( \beta = \tau - \lambda \). If \( n \) goes to \( +\infty \), the space \( S_{\beta} = \bigoplus_{j \geq 0} \mathcal{F}_{\beta-j} \) appears as the space symbols for all differential operators. The space \( H^1(\mathfrak{sl}(2); L(\otimes^n V, V)) \) can be decomposed as a sum of some spaces \( H^1(\mathfrak{sl}(2), D_{\lambda, \mu}) \) with \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \). Thus, for \( \lambda \in \mathbb{R}^n \), the knowledge of the spaces \( H^1(\mathfrak{sl}(2), D_{\lambda, \mu}) \) is useful to compute the terms \( T^\mu \) of a normalized nonlinear representation \( T \) of \( \mathfrak{sl}(2) \) in \( S_{\beta} \).

For \( \lambda \in \mathbb{R} \) the spaces \( H^1(\mathfrak{sl}(2), D_{\lambda, \mu}) \) are computed by Gargoubi [6] and Lecomte [7], and for \( \lambda \in \mathbb{R}^2 \) the spaces \( H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \mu}) \) are computed by Bouarroudj [4], while we are interested in this paper to generalize this study to \( \lambda \in \mathbb{R}^n \) solving a conjecture given by Ben Ammar et al in [3].

\section{The space \( H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \mu}) \)}

\subsection{Cohomology}

We will compute the first cohomology space of \( \mathfrak{sl}(2) \) with coefficients in \( D_{\lambda, \mu} \) where \( \lambda \in \mathbb{R}^n \) and \( \mu \in \mathbb{R} \). Let us first recall some fundamental concepts from cohomology theory (see, e.g., [5]). Let \( \mathfrak{g} \) be a Lie algebra acting on a space \( V \) and let \( \mathfrak{h} \) be a subalgebra of \( \mathfrak{g} \). (If \( \mathfrak{h} \) is omitted it assumed to be \( \{0\} \).) The space of \( \mathfrak{h} \)-relative \( n \)-cochains of \( \mathfrak{g} \) with values in \( V \) is the \( \mathfrak{g} \)-module

\[
C^n(\mathfrak{g}, \mathfrak{h}; V) := \text{Hom}_{\mathfrak{h}}(A^n(\mathfrak{g}/\mathfrak{h}); V).
\]

The \emph{coboundary operator} \( \partial^n : C^n(\mathfrak{g}, \mathfrak{h}; V) \to C^{n+1}(\mathfrak{g}, \mathfrak{h}; V) \) is a \( \mathfrak{g} \)-map satisfying \( \partial^n \circ \partial^{n-1} = 0 \). The kernel of \( \partial^n \), denoted \( Z^n(\mathfrak{g}, \mathfrak{h}; V) \), is the space of \( \mathfrak{h} \)-relative \( n \)-cocycles, among them, the elements in the range of \( \partial^{n-1} \) are called \( \mathfrak{h} \)-relative \( n \)-coboundaries. We denote \( B^n(\mathfrak{g}, \mathfrak{h}; V) \) the space of \( n \)-coboundaries.

By definition, the \( n \)th \( \mathfrak{h} \)-relative cohomology space is the quotient space

\[
H^n(\mathfrak{g}, \mathfrak{h}; V) = Z^n(\mathfrak{g}, \mathfrak{h}; V)/B^n(\mathfrak{g}, \mathfrak{h}; V).
\]

We will only need the formula of \( \partial^n \) (which will be simply denoted \( \partial \)) in degrees 0 and 1. For \( v \in C^0(\mathfrak{g}, \mathfrak{h}; V) = V^\mathfrak{h} \),

\[
\partial v(g) := g \cdot v,
\]

where \( V^\mathfrak{h} \) is the subspace of \( \mathfrak{h} \)-invariant elements of \( V \). For \( \Upsilon \in C^1(\mathfrak{g}, \mathfrak{h}; V) \) and \( g, h \in \mathfrak{g} \),

\[
\partial(\Upsilon)(g, h) := g \cdot \Upsilon(h) - h \cdot \Upsilon(g) - \Upsilon([g, h]).
\]
Here we consider \( \mathfrak{g} = \mathfrak{sl}(2), \mathfrak{h} = \text{aff}(1) \) the subalgebra of \( \mathfrak{sl}(2) \) spanned by \( \frac{d}{dx} \) and \( x \frac{d}{dx} \) and we consider \( V = D_{\lambda, \mu} \). We compute \( H^1(\mathfrak{sl}(2); D_{\lambda, \mu}) \) and \( H^1(\mathfrak{sl}(2), \text{aff}(1); D_{\lambda, \mu}) \).

### 2.2 The spaces \( H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \mu}) \) and \( H^1(\mathfrak{sl}(2), \text{aff}(1); D_{\lambda, \mu}) \)

Let \( \mu \in \mathbb{R}, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), we consider \( \delta = \mu - \sum_{i=1}^n \lambda_i \) and \( |\alpha| = \sum \alpha_i \). For \( F = f_1 \otimes \cdots \otimes f_n \in \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \), we define \( F^{(\alpha)} := f_1^{(\alpha_1)} \cdots f_n^{(\alpha_n)} \).

We also define

\[
\alpha^i = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_n) \quad \text{and} \quad \alpha^{-i} = (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n).
\]

Recall that the space \( \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \) is an \( \mathfrak{sl}(2) \)-module:

\[
X_h \cdot F := L^\lambda_{X_h}(F) = \sum_{i=1}^n f_1 \otimes \cdots \otimes L^\lambda_{X_h}(f_i) \otimes \cdots \otimes f_n.
\]

The following lemma gives the general form of any 1-cocycle.

**Lemma 2.1.** Up to a coboundary, any 1-cocycle \( c \in Z^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \mu}) \) can be expressed as follows. For all \( F = f_1 \otimes \cdots \otimes f_n \in \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \) and for all \( X_h \in \mathfrak{sl}(2) \):

\[
c(X_h, F) = \sum_{\alpha} B_{\alpha} h' F^{(\alpha)} + \sum_{\alpha} C_{\alpha} h'' F^{(\alpha)},
\]

where \( B_{\alpha} \) and \( C_{\alpha} \) are constants satisfying:

\[
2(\delta - |\alpha| - 1)C_{\alpha} + \sum_i (\alpha_i + 1)(\alpha_i + 2\lambda_i)B_{\alpha^i} = 0.
\]

**Proof.** Any 1-cocycle on \( \mathfrak{sl}(2) \) should retains the following general form:

\[
c(X_h, F) = \sum_{\alpha} U_{\alpha} h F^{(\alpha)} + \sum_{\alpha} V_{\alpha} h' F^{(\alpha)} + \sum_{\alpha} W_{\alpha} h'' F^{(\alpha)},
\]

where \( U_{\alpha}, V_{\alpha} \) and \( W_{\alpha} \) are, a priori, functions. First, we prove that the terms in \( h \) can be annihilated by adding a coboundary. Let \( b : \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \rightarrow \mathcal{F}_{\mu} \) be a multilinear differential operator defined by

\[
b(F) = \sum_{\alpha} D_{\alpha} F^{(\alpha)},
\]

We have

\[
\partial b(X_h, F) = h(b(F))' + \mu h'b(F) - b(X_h \cdot F)
\]

\[
= \sum_{\alpha} D_{\alpha}' h F^{(\alpha)} + \sum_{\alpha} (\delta - |\alpha|) D_{\alpha} h' F^{(\alpha)}
\]

\[
- \frac{1}{2} \sum_{\alpha} \sum_{i=1}^n \alpha_i(\alpha_i + 2\lambda_i - 1)D_{\alpha} h'' F^{(\alpha^{-i})}.
\]

Note that the terms in \( F^{(\alpha^{-i})} \) do not appear for \( \alpha_i = 0 \).

Thus, if \( D_{\alpha}' = U_{\alpha} \) then \( c - \partial b \) does not contain terms in \( h \). So, we can replace \( c \) by \( c - \partial b \). That is, up to a coboundary, any 1-cocycle on \( \mathfrak{sl}(2) \) can be expressed as follows:

\[
c(X_h, F) = \sum_{\alpha} B_{\alpha} h' F^{(\alpha)} + \sum_{\alpha} C_{\alpha} h'' F^{(\alpha)},
\]

\[
X_{\alpha}\cdot F := L_{X_{\alpha}}(F) = \sum_{i=1}^n f_1 \otimes \cdots \otimes L_{X_{\alpha}}(f_i) \otimes \cdots \otimes f_n.
\]
Now, consider the 1-cocycle condition:

\[ c([X_{h_1}, X_{h_2}], F) - X_{h_1} \cdot c(X_{h_2}, F) + X_{h_2} \cdot c(X_{h_1}, F) = 0, \]

where \( X_{h_1}, X_{h_2} \in \mathfrak{sl}(2) \). That is,

\[
\sum_{\alpha} B'_{\alpha} (h_1 h_2' - h_1' h_2) F^{(\alpha)} + \sum_{\alpha} C'_{\alpha} (h_1 h_2'' - h_1'' h_2') F^{(\alpha)} + \frac{1}{2} \sum_{\alpha} \left( 2(\delta - |\alpha| - 1) C_{\alpha} + \sum_{i} (\alpha_i + 1)(\alpha_i + 2\lambda_i) B_{\alpha_i} \right) (h_1 h_2'' - h_1'' h_2') F^{(\alpha)} = 0.
\]

So, for all \( \alpha \), we have \( B'_{\alpha} = C'_{\alpha} = 0 \) and

\[
2(\delta - |\alpha| - 1) C_{\alpha} + \sum_{i} (\alpha_i + 1)(\alpha_i + 2\lambda_i) B_{\alpha_i} = 0.
\]

\[ \square \]

**Theorem 2.2.**

1) If \( \delta \notin \mathbb{N} \) then \( H_1^{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \mu}) = 0 \).

2) If \( \delta = k \in \mathbb{N} \) then, up to a coboundary, any 1-cocycle \( c \in Z_1^{\text{diff}}(\mathfrak{sl}(2), D_{\lambda, \mu}) \) can be expressed as follows. For all \( F = f_1 \otimes \cdots \otimes f_n \in \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \) and for all \( X_h \in \mathfrak{sl}(2) \):

\[
c(X_h, F) = \sum_{|\alpha| = k} B_{\alpha} h' F^{(\alpha)} + \sum_{|\beta| = k-1} C_{\beta} h'' F^{(\alpha)}, \tag{2.4}
\]

where, for any \( \alpha \) with \( |\alpha| = k - 1 \), the \( B_{\alpha_i} \) are constants satisfying:

\[
\sum_{i} (\alpha_i + 1)(\alpha_i + 2\lambda_i) B_{\alpha_i} = 0.
\]

3) If the \( B_{\alpha} \) are not all zero then the cocycles (2.4) are nontrivial.

Proof. 1) Indeed, according to Lemma 2.1, we can easily show the 1-cocycle \( c \) defined by (2.1) is nothing but the operator \( \partial b \) where

\[
b(F) = \sum_{\alpha} \frac{B_{\alpha}}{\delta - |\alpha|} F^{(\alpha)},
\]

2) Consider the 1-cocycle \( c \) defined by (2.1) and consider the operator \( \partial b \) where

\[
b(F) = \sum_{|\alpha| \neq k} \frac{1}{k - |\alpha|} B_{\alpha} F^{(\alpha)}.
\]

We have

\[
\partial b(X_h, F) = \sum_{|\alpha| \neq k} B_{\alpha} h' F^{(\alpha)} - \frac{1}{2} \sum_{|\alpha| \neq k} \sum_{i=1}^{n} \frac{\alpha_i (\alpha_i + 2\lambda_i - 1)}{k - |\alpha|} B_{\alpha} h'' F^{(\alpha^{-i})}.
\]

Let \( \beta = \alpha^{-i} \), so, we have

\[
\alpha = \beta^i, \quad |\beta| \neq k - 1, \quad \beta_j = \alpha_j \quad \text{for} \quad j \neq i \quad \text{and} \quad \beta_i = \alpha_i - 1.
\]
Therefore
\[
\partial b(X_h, F) = \sum_{|\alpha|\neq k} B_\alpha h' F^{(\alpha)} - \frac{1}{2} \sum_{|\beta|\neq k-1} \sum_{i=1}^n \frac{\beta_i + 1)(\beta_i + 2\lambda_i)}{k - |\beta| - 1} B_\beta h'' F^{(\beta)}.
\]

According to (2.2) we have
\[
-\frac{1}{2} \sum_{i=1}^n (\beta_i + 1)(\beta_i + 2\lambda_i) \frac{1}{k - |\beta| - 1} B_\beta = C_\beta
\]

Thus,
\[
\partial b(X_h, F) = \sum_{|\alpha|\neq k} B_\alpha h' F^{(\alpha)} + \sum_{|\alpha|\neq k-1} C_\alpha h'' F^{(\alpha)}
\]

and then
\[
(c - \partial b)(X_h, F) = \sum_{|\alpha|=k} B_\alpha h' F^{(\alpha)} + \sum_{|\alpha|=k-1} C_\alpha h'' F^{(\alpha)}.
\]

In this case (2.2) becomes:
\[
\sum_i (\alpha_i + 1)(\alpha_i + 2\lambda_i) B_\alpha = 0.
\]

3) In fact, for \( \delta = \mu - |\lambda| = |\alpha| = k \) there are no terms in \( h' F^{(\alpha)} \) in the expression of \( \partial b(X_h, F) \) for any \( b \in D_{\lambda,\mu} \) (see (2.3)).

Now, we prove that, generically, we can annihilate the term \( h'' \) in the expression of the 1-cocycle (2.4) by adding a coboundary.

**Theorem 2.3.** If \( \delta = k \in \mathbb{N}^* \) and \( -2\lambda \notin \{0, \ldots, k-1\} \) then, any 1-cocycle \( c \in \mathbb{Z}_1^{\text{odd}}(\mathfrak{sl}(2), D_{\lambda,\mu}) \) can be expressed as follows. For all \( F \in \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \) and for all \( X_h \in \mathfrak{sl}(2) \):
\[
c(X_h, F) = \sum_{|\alpha|=k} B_\alpha h' F^{(\alpha)},
\]
where, for \( |\alpha| = k-1 \), the \( B_\alpha \) are constants satisfying:
\[
\sum_i (\alpha_i + 1)(\alpha_i + 2\lambda_i) B_\alpha = 0.
\]

Proof. By Theorem 2.2, for \( \delta = k \in \mathbb{N} \), any 1-cocycle \( c \) can be expressed as follows:
\[
c(X_h, F) = \sum_{|\alpha|=k} B_\alpha h' F^{(\alpha)} + \sum_{|\beta|=k-1} C_\beta h'' F^{(\beta)}.
\]

Let \( b : \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \rightarrow \mathcal{F}_\mu \) be a multilinear differential operator defined by
\[
b(F) = \sum_{|\alpha|=k} D_\alpha F^{(\alpha)}.
\]

We have
\[
\partial b(X_h, F) = -\frac{1}{2} \sum_{|\alpha|=k} \sum_{i=1}^n \alpha_i (\alpha_i + 2\lambda_i - 1) D_\alpha h'' F^{(\alpha^{-1})}.
\]

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Let \( \beta = \alpha^{-i} \), so, we have
\[
\alpha = \beta^i, \quad |\beta| = k - 1, \quad \beta_j = \alpha_j \quad \text{for} \quad j \neq i \quad \text{and} \quad \beta_i = \alpha_i - 1.
\]
Therefore, we have
\[
\partial b(X_h, F) = -\frac{1}{2} \sum_{|\beta|=k-1} \sum_{i=1}^{n} (\beta_i + 1)(\beta_i + 2\lambda_i)D_{\beta}h^n F(\beta).
\]

Consider the linear system
\[
\frac{1}{2} \sum_{i=1}^{n} (\beta_i + 1)(\beta_i + 2\lambda_i)D_{\beta} = C_{\beta}, \quad |\beta| = k - 1.
\]

(2.6)

The number \( N_k \) of unknowns \( D_{\beta} \) is the cardinal of the set \( \{ \alpha \in \mathbb{N}^n, |\alpha| = k \} \) and the number of equations is the cardinal of the set \( \{ \alpha \in \mathbb{N}^n, |\alpha| = k - 1 \} \). We prove recurrently that
\[
N_k = \binom{n + k - 1}{k}.
\]

We consider the lexicographic order and we denote by \( a_1 > \cdots > a_{N_{k-1}} \) the elements of the set \( \{ \alpha \in \mathbb{N}^n, |\alpha| = k - 1 \} \). So, we choose an order on the unknowns \( D_{\beta_i} \) such that the first are those indexed by \( a_1^1, \ldots, a_{N_{k-1}}^1 \).

Thus, the matrix of the system (2.6) has the following form:
\[
\Lambda = \left( \begin{array}{cccccc}
\Lambda_1^{k-1} & \Lambda_2^0 & \Lambda_3^0 & \cdots & \Lambda_{n}^0 & 0 & \cdots & \cdots \\
0 & \Lambda_1^{k-2} & 0 & \cdots & 0 & \Lambda_2^0 & \cdots & \cdots \\
0 & 0 & \Lambda_1^{k-2} & 0 & \cdots & 0 & \Lambda_2^0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \Lambda_1^j & 0 & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \Lambda_1^j & 0 & \cdots & \cdots & \cdots \\
\end{array} \right)
\]
(2.7)

where \( \Lambda_j^i = (j + 1)(j + 2\lambda_i) \), \( j = 0, \ldots, k - 1 \) and \( i = 1, \ldots, n \).

Without loss of generality, assume that \(-2\lambda_1 \notin \{0, \ldots, k - 1\}\) then \( \Lambda_1^i \) does never vanish, therefore obviously the system (2.6) is of rank \( N_{k-1} \). Thus, we can choose the operator \( b \) such that \( (c + \partial b)(X_h, F) \) does not contain terms in \( h^n \).

For \( n = 2 \), we have:
\[
\Lambda = \left( \begin{array}{cccc}
\Lambda_1^{k-1} & \Lambda_2^0 & 0 & \cdots & 0 \\
0 & \Lambda_1^{k-2} & \Lambda_2^1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \cdots & \ddots \\
0 & \cdots & 0 & \Lambda_1^0 & \Lambda_2^{k-1} \\
\end{array} \right)
\]
(2.8)

\[ \square \]

**Theorem 2.4.** If \( \delta = k \geq 1 \) and \(-2\lambda \notin \{0, \ldots, k - 1\}^n \) then, we have
\[
\dim \Pi_1(sl(2), D_{\lambda, \mu}) = \binom{n + k - 2}{k}.
\]
These spaces are spanned by the cocycles:

$$c(X_h, F) = \sum_{|\alpha| = k} B_{\alpha} h' F(\alpha),$$

where, for $|\alpha| = k - 1$, the $B_{\alpha}$ are constants satisfying:

$$\sum_i (\alpha_i + 1)(\alpha_i + 2\lambda_i)B_{\alpha} = 0.$$  \hspace{1cm} (2.9)

For $\delta = 0$ the space $H^1(\mathfrak{sl}(2), D_{\lambda, \mu})$ is one dimensional spanned by the 1-cocycle $C_0$ defined by

$$C_0(X_h, f_1 \otimes \cdots \otimes f_n) = h' f_1 \cdots f_n.$$

**Remark 2.1.** For $n = 1$ and for $n = 2$ we refined the results of Lecomte and Bouarroudj (see [7], [4]).

**Proof.** According to Theorem 2.3 the space $H^1(\mathfrak{sl}(2), D_{\lambda, \mu})$ is isomorphic to the space of solutions of the system of linear equations (2.5). The linear system (2.5) has \(\binom{n + k - 2}{k - 1}\) equations with \(\binom{n + k - 1}{k}\) unknowns $B_{\alpha}$. As in Theorem 2.3 we see that this system is with maximal rank, so, we have

$$\dim H^1(\mathfrak{sl}(2), D_{\lambda, \mu}) = \binom{n + k - 1}{k} - \binom{n + k - 2}{k - 1} = \binom{n + k - 2}{k}.$$

\hspace{1cm} \(\square\)

**Corollary 2.2.** If $\delta = k \geq 0$ and $-2\lambda \notin \{0, \ldots, k - 1\}^n$ then, we have

$$H^1(\mathfrak{sl}(2), \text{aff}(1); D_{\lambda, \mu}) = 0.$$

In fact, in this case, there are no nontrivial 1-cocycles vanishing on \(\text{aff}(1)\).

**Theorem 2.5.** If $\delta = k$ and $-2\lambda \in \{0, \ldots, k - 1\}^n$ then

$$\binom{n + k - 2}{k} \leq \dim H^1(\mathfrak{sl}(2), D_{\lambda, \mu}) \leq \binom{n + k - 2}{k} + 2 \binom{n + k - r - 3}{k - r - 1}$$

where $r = \max(-2\lambda_i)$.

**Proof.** Without loss of generality, assume that $r = -2\lambda_1$, then $\Lambda^r_1 = 0$ and $\Lambda^r_1$ appears \(\binom{n + k - r - 3}{k - r - 1}\) times in the matrix $\Lambda$ defined by (2.7). So,

$$N_{k-1} - \binom{n + k - r - 3}{k - r - 1} \leq \text{rank}(\Lambda) \leq N_{k-1}.$$  \hspace{1cm} (2.10)

Obviously the number \(\binom{n + k - r - 3}{k - r - 1}\) is minimal if $r$ is maximal.
Now, any nontrivial cocycle
\[ c(X_h, F) = \sum_{|\alpha|=k} B_\alpha h' F^{(\alpha)} + \sum_{|\beta|=k-1} C_\beta h'' F^{(\beta)} \]
can be decomposed into two cocycles
\[ c_1(X_h, F) = \sum_{|\alpha|=k} B_\alpha h' F^{(\alpha)} \quad (2.11) \]
and
\[ c_2(X_h, F) = \sum_{|\beta|=k-1} C_\beta h'' F^{(\beta)} \quad (2.12) \]
indeed, the coefficients \( B_\alpha \) are independent of the coefficients \( C_\beta \) (see (2.2)). The space of nontrivial cocycles (2.11) is generated by the system of linear equations (2.5), so, it is with dimension \( N_k - \text{rank}(\Lambda) \). The space of nontrivial cocycles (2.12) is managed by the system of linear equations (2.6), so, it is with dimension \( N_{k-1} - \text{rank}(\Lambda) \).

Thus, \( \dim H^1(\mathfrak{sl}(2), D_{\lambda,\mu}) = N_k + N_{k-1} - 2\text{rank}(\Lambda) \).

We conclude by using (2.10). \( \square \)

Obviously, the nontrivial cocycles (2.12) managed by the system of linear equations (2.6) are \text{aff}(1)-invariant vanishing on \text{aff}(1), so, we have:

**Theorem 2.6.**
\[ \dim H^1(\mathfrak{sl}(2), \text{aff}(1); D_{\lambda,\mu}) = N_{k-1} - \text{rank}(\Lambda) \]

These spaces are generated by the cocycles (2.12).

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