The exponential map of GL(N)

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April 9, 1996

Abstract

A finite expansion of the exponential map for a $N \times N$ matrix is presented. The method uses the Cayley-Hamilton theorem for writing the higher matrix powers in terms of the first N-1 ones. The resulting sums over the corresponding coefficients are rational functions of the eigenvalues of the matrix.

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1 Introduction

In the Lie theory of groups and their corresponding algebras the exponential map is a crucial tool because it gives the connection between a Lie algebra element \( H \in \mathfrak{g} \) and the corresponding Lie group element \( g \in G \)

\[
\exp : \quad \mathfrak{g} \longrightarrow G \\
H \longrightarrow g
\]

(For details see \([5]\) and \([1, 2]\) and ref. therein). In some low dimensional cases, like \( SU(2) \) and \( SO(3) \) the explicit expansion of the exponential map is known. Some years ago the exponential map for the Lorentz group was given by W. Rodrigues and J. Zeni \([7, 8]\). For the higher dimensional groups \( SU(2, 2) \) and \( O(2, 4) \) a method for the expansion was developed by A.O. Barut, J. Zeni, and A.L. \([1, 2]\). The subject of the present paper is a generalization of the method developed in \([1, 2]\) to the general linear groups \( GL(N) \). The result will be a method to calculate the exponential of a quadratic matrix \( H \), where only rational functions of the eigenvalues of \( H \) and the first \( N-1 \) powers of \( H \) are involved. The key points are the Cayley-Hamilton theorem and the introduction of a multiplier \( m \).

The organization of this paper is as follows. First the method is shown in the low dimensional case \( SU(3) \) which is a group occurring quite often in physics. Then the general case of the expansion of the exponential map for a \( N \times N \) matrix is presented. The method uses eigenvalues

The only real problem that remains is the determination of the eigenvalues of the matrix \( H \). Throughout the paper we assume that the groups \( GL(N) \) are represented as \( N \)-dimensional matrices and that the eigenvalues of \( H \) are all different (cf. remark in appendix B.3).

Some possible applications of these results will be presented in a future paper.

2 The exponential map for the group \( SU(3) \)

The group \( SU(3) \) is used in several branches of physics. The best known application is the model of the strong interaction (see e.g. \([3]\)). For this reason and because its a good exercise to follow the steps of the general method we will show the exponential mapping of \( SU(3) \) in great detail.

The calculations depend in some points on the fact that we consider a special group, i.e., the sum over the eigenvalues of the generator vanishes. There is no conceptional problem to extend the method to \( U(3) \). Like in the other cases (cf. \([1, 2]\)) a typical element \( U \in SU(3) \) can be written as exponential of the generator

\[
U = e^H = \sum_{n=0}^{\infty} \frac{1}{n!} H^n \quad \text{with} \quad H \in \mathfrak{su}(3) . \tag{1}
\]

The Cayley-Hamilton theorem and the iterated form in this case read

\[
H^3 = b_0 H + c_0 \quad \text{and} \quad H^{3+i} = a_i H^2 + b_i H + c_i
\]

where the coefficients \( b_0 \) and \( c_0 \) are functions of the eigenvalues of the eigenvalues \( x, y, z \) of \( H \). They satisfy the recurrence relations

\[
a_{i+1} = b_i , \quad b_{i+1} = a_i b_0 + c_i , \quad c_{i+1} = a_i c_0 . \tag{2}
\]
Hence the coefficients $a_i$ satisfy

$$a_{i+1} = a_{i-1} b_0 + a_{i-2} c_0$$

with the first few values

$$a_0 = 0, \quad a_1 = b_0, \quad a_2 = c_0, \quad a_3 = b_0^2, \quad a_4 = b_0 c_0.$$  \hfill (3)

The explicit form of $b_0$ and $c_0$ can easily be derived from the secular equation

$$0 = (\lambda - x) (\lambda - y) (\lambda - z) = \lambda^3 - \left(\sum_{a_0=0}^{\infty} \frac{1}{n+3}! \right) H^{n+3}.$$

The leading coefficient $a_0$ vanishes since the generator $H$ is traceless, i.e., \( x + y + z = 0 \). The second coefficient can also be written as $b_0 = \frac{1}{2} (x^2 + y^2 + z^2)$. There are also some nice relations

$$b_0 x + c_0 = x^3, \quad b_0 y + c_0 = y^3, \quad b_0 z + c_0 = z^3.$$

The idea now is to use the Cayley-Hamilton theorem for writing the sum (1) like

$$U = I_4 + H + \frac{1}{2} H^2 + \sum_{n=0}^{\infty} \frac{1}{(n+3)!} H^{n+3}$$

$$= I_4 + H + \frac{1}{2} H^2 + \sum_{n=0}^{\infty} \frac{1}{(n+3)!} (a_n H^2 + b_n H + c_n)$$

This form contains only sum over rational functions, there are no higher powers of the generator present anymore. The next step is now to find an analytic expression for the sums over the coefficients.

A convenient form of the functions $a_n$, $b_n$, and $c_n$ can be obtained if we introduce the multiplier

$$m = (x - y) (x - z) (y - z) = (x^2 (y - z) + y^2 (z - x) + z^2 (x - y)).$$

Then we get for the group element

$$m U = m \left( I_4 + H + \frac{1}{2} H^2 \right) + \left[ \sum_{n=0}^{\infty} \frac{m a_n}{(n+3)!} \right] H^2 + \left[ \sum_{n=0}^{\infty} \frac{m b_n}{(n+3)!} \right] H + \left[ \sum_{n=0}^{\infty} \frac{m c_n}{(n+3)!} \right].$$

It can easily be shown that the following form for the coefficients satisfy the recurrence relations (2) and (3)

$$m a_n = (y - z) x^{n+3} + (z - x) y^{n+3} + (x - y) z^{n+3}$$

$$m b_n = (y - z) x^{n+4} + (z - x) y^{n+4} + (x - y) z^{n+4}$$

$$m c_n = y z (y - z) x^{n+3} + x z (z - x) y^{n+3} + y z (x - y) z^{n+3}$$

The three sums are now

$$\left[ \sum_{n=0}^{\infty} \frac{m a_n}{(n+3)!} \right] = (y - z) e^x + (z - x) e^y + (x - y) e^z - \frac{1}{2} m$$

$$\left[ \sum_{n=0}^{\infty} \frac{m b_n}{(n+3)!} \right] = x (y - z) e^x + y (z - x) e^y + z (x - y) e^z - m$$

$$\left[ \sum_{n=0}^{\infty} \frac{m c_n}{(n+3)!} \right] = y z (y - z) e^x + x z (z - x) e^y + y z (x - y) e^z.$$
Finally, we get the expansion of a SU(3) group element

\[ mU = [yz(y - z)\ e^x + xz(z - x)\ e^y + xy(x - y)\ e^z]\ I_3 \]
\[ + [x(y - z)\ e^x + y(z - x)\ e^y + z(x - y)\ e^z]\ H \]
\[ + [(y - z)\ e^x + (z - x)\ e^y + (x - y)\ e^z]\ H^2 \]  

\[ (4) \]

3 The exponential map of GL(N)

As we have seen in the cases of the groups SU(3) and SU(2,2) the exponential map can be written as sum over the first (N-1) powers of the generator \( H \in \mathfrak{su}(N) \). Where the coefficients are functions of the eigenvalues of \( H \). In this section we generalize the results we have gotten on the low dimensional examples. It seems that there is a relatively easy concept of generalization. The desired result is an expansion of the exponential map of the form

\[ g = e^H = \sum_{n=0}^{\infty} \frac{H^n}{n!} = \sum_{k=0}^{N-1} A_k H^k \]  

\[ (5) \]

where the coefficients \( A_k \) are rational functions of the eigenvalues \( \{\lambda_i ; i = 1,2,\ldots,N\} \) of the generator \( H \).

3.1 The secular equation

The first step will be to take a look at the eigenvalues, some auxiliary functions, and their interrelations.

Let us consider the secular equation of the matrix \( H \)

\[ 0 = \prod_{i=1}^{N}(\lambda - \lambda_i) = - \left( \sum_{k=0}^{N} C_k \lambda^{N-k} \right), \]  

\[ (6) \]

where the coefficients \( C_k \) are functions of the eigenvalues of \( H \). In section 3.4 some coefficients are listed in their explicit form. For later convenience we introduce also the “truncated” version \( C_{(i)k} \) of the coefficients \( C_k \), defined by

\[ \prod_{j \neq i}(\lambda - \lambda_j) =: - \sum_{k=0}^{N-1} C_{(i)k} \lambda^{N-1-k} \]  

\[ (7) \]

Essentially the \( C_{(i)k} \) contain all terms of \( C_k \) without \( \lambda_i \). The connection between these coefficients can be seen easily via
\[
\prod_{i=1}^{N} (\lambda - \lambda_i) = (\lambda - \lambda_1) \prod_{i=2}^{N} (\lambda - \lambda_i)
\]

\[
= (\lambda - \lambda_1) \left(\lambda^{N-1} - \sum_{k=1}^{N-1} C(1)_k \lambda^{N-1-k}\right)
\]

\[
= \lambda^N + \lambda_1 C(1)_{N-1} - \sum_{k=0}^{N-2} \left[C(1)_{k+1} - \lambda_1 C(1)_k\right] \lambda^{N-1-k}
\]

\[
= \lambda^N - \sum_{k=0}^{N-1} C_{k+1} \lambda^{N-1-k}.
\]

Since the calculations above can be generalized to all eigenvalues we get the relations

\[
C_k = C(i)_k - \lambda_i C(i)_{k-1} \quad \text{for} \quad k = 1, 2, \ldots, N - 1 \tag{8}
\]

\[
C_N = -\lambda_i C(i)_{N-1}.
\]

Let us define the multiplier \( m \); i.e., the discriminant of the secular equation

\[
m := \prod_{i<j} (\lambda_i - \lambda_j) \tag{9}
\]

and the functions (see also \( m \))

\[
m_i := m (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_N) = \prod_{j<k \neq i} (\lambda_j - \lambda_k). \tag{10}
\]

In what follows we mainly use the following form of \( m \) which can be obtained by expanding the Slater determinant (see (26) and (29))

\[
m = \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1}. \tag{11}
\]

This formula can be generalized to

\[
m \delta_{kl} = \sum_{i=1}^{N} (-1)^i m_i C(i)_{k-1} \lambda_i^{N-l} \quad \text{for} \quad k, l = 1, 2, \ldots, N. \tag{12}
\]

For the proof see section 3.4.

### 3.2 Recurrence relations

The described method relies on the Cayley-Hamilton theorem which gives us the ability to write all powers \( H^{N+n} \) for \( n \in \mathbb{N} \) in terms of the first \( N-1 \) powers of \( H \). The Cayley-Hamilton theorem for \( H \in \mathfrak{gl}(N) \) reads

\[
H^N = \sum_{k=1}^{N} C_k H^{N-k} \tag{13}
\]
The coefficients \( C_k \) are the same as those in the secular equation and satisfy the recurrence relations \((15)\) derived below. For the special groups, i.e., \( \det g = 1 \) for \( g \in \text{SL}(N) \) the first coefficient vanishes since the sum over the eigenvalues is zero.

Multiplication of \((13)\) with \( H \) and using \((13)\) again gives the iterated form

\[
H^{N+n} = \sum_{k=1}^{N} C_k^n H^{N-k} \tag{14}
\]

Multiplying once more with \( H \) gives

\[
H^{N+n+1} = (C_2^n + C_1^n C_1) H^{N-1} + (C_3^n + C_1^n C_2) H^{N-2} + \ldots + (C_{n+1}^n + C_1^n C_n) H^{N-n} + \ldots + (C_N^n + C_1^n C_{N-1}) H + C_1^n C_N
\]

\[
= \sum_{k=1}^{N} C_k^{n+1} H^{N-k}
\]

and hence we get the recurrence relations

\[
C_1^{n+1} = C_2^n + C_1^n C_1 \quad \quad C_2^{n+1} = C_3^n + C_1^n C_2 \quad \quad \ldots \quad \quad C_k^{n+1} = C_{k+1}^n + C_1^n C_k \quad \quad \ldots \quad \quad C_{N-1}^{n+1} = C_N^n + C_1^n C_{N-1}
\]

\[
C_N^{n+1} = C_1^n C_N \tag{15}
\]

If we successively plug in the \( C_j^k \) in the recurrence relation of \( C_1^n \) we get a formula which contains only terms with \( C_1^n \) and the coefficients of the original Cayley-Hamilton equation \((13)\)

\[
C_1^{n+1} = \begin{cases} 
\sum_{j=1}^{N} C_1^{n+1-j} C_j & \text{for } n \geq N-1 \\
\sum_{j=0}^{n} C_1^{n-j} C_{1+j} + m C_{n+2} & \text{for } n < N-1 
\end{cases} \tag{16}
\]

For the other coefficients \( C_k^{n+1} \) \( (k = 1, 2, \ldots, N) \) we get analogous formulae

\[
C_k^{n+1} = \begin{cases} 
\sum_{j=0}^{N-k} C_1^{n-j} C_{k+j} & \text{for } n \geq N-k \\
\sum_{j=0}^{n} C_1^{n-j} C_{k+j} + m C_{k+n+1} & \text{for } n < N-k 
\end{cases} \tag{17}
\]

The coefficients of the secular equation have the explicit form

\[
m C_k = (11) \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1} C_k \quad \quad \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1} \left( C_{(i)k} - \lambda_i C_{(i)k-1} \right) = \quad \quad \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1} C_{(i)k}
\]

\[
= - \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N} C_{(i)k-1} + \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1} C_{(i)k} = 0 \quad \quad \text{for } k \neq 0
\]

\[
= 0 \quad \quad \text{for } k = 0
\]
where we applied Eq.\((34)\) to the second term in the last equation. We will need this form as first values in the proof of Eq.\((21)\).

\[
m C_k = \sum_{i=1}^{N} (-1)^i m_i \lambda_i^k \quad \text{for } k = 1, 2, \ldots, N
\]

(18)

From the SU(3) and SU(2,2) cases one may assume that the recurrence relation \((16)\) has the solution

\[
m C^m = \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n}.
\]

(19)

Proof:
The proof of Eq.\((19)\) is done by induction over \(n\).
The first coefficient \((n = 0)\) is given by

\[
m C_1 = \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^N
\]

(20)

which is easy to prove if one writes \(C_1\) in the form

\[
C_1 = \sum_{j=1}^{N} \lambda_j = \lambda_i + \sum_{j \neq i} \lambda_j = \lambda_i + C_{(i)1}.
\]

For the product \(m C_1\) we take \(m\) in the form of Eq.\((29)\) we get

\[
m C_1 = \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1} \left( \lambda_i + C_{(i)1} \right)
\]

\[
= \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^N + \sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)1} \lambda_i^{N-1}
\]

\[
= 0
\]

The last equation holds since the exponent of \(\lambda_i\) should be \(N - 2\) in order to yield a non-vanishing sum (see Eq.\((34)\)).

First we treat the case of \(n \geq N\). The next step is to assume the validity of \((19)\) for \(n\) and to show that then it follows also for \(n + 1\)

\[
m C^{m+1}_1 = \sum_{j=1}^{N} m C^{n+1-j}_1 C_j \overset{\text{(16)}}{=} \sum_{j=1}^{N} \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n+1-j} C_j
\]

\[
\overset{\text{(18)}}{=} \sum_{j=1}^{N} \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n+1-j} \left( C_{(i)j} - \lambda_i C_{(i)j-1} \right)
\]
\[
\sum_{i=1}^{N} (-1)^{i+1} m_i \left( \sum_{j=1}^{N} C_{(i)j} \lambda_i^{N+n+1-j} - \sum_{j=1}^{N} C_{(i)j-1} \lambda_i^{N+n+2-j} \right)
\]

\[
= \sum_{i=1}^{N} (-1)^{i+1} m_i \left( \sum_{j=2}^{N+1} C_{(i)j-1} \lambda_i^{N+n+2-j} - \sum_{j=1}^{N} C_{(i)j-1} \lambda_i^{N+n+2-j} \right)
\]

\[
= \sum_{i=1}^{N} (-1)^{i+1} m_i \left( C_{(i)N} \lambda_i^{n+1} - C_{(i)0} \lambda_i^{N+n+1} \right)
\]

\[
= \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n+1} = m C_1^{n+1}.
\]

In the case of \( n < N - 1 \) there is an additional term

\[
m C_1^{n+1} = \sum_{j=0}^{n} m C_1^{n-j} C_1^{j+1} + m C_{n+2} = \ldots =
\]

\[
= \sum_{i=1}^{N} (-1)^{i+1} m_i \left( C_{(i)N} \lambda_i^{n+1} - C_{(i)0} \lambda_i^{N+n+1} \right) + m C_{n+2}
\]

\[
= \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n+1} + \sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)N} \lambda_i^{n+1} + m C_{n+2}
\]

\[
= \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n+1}.
\]

\[\Box\]

The coefficients \( C_k^n \) can be written in the form

\[
m C_k^n = \sum_{i=1}^{N} (-1)^i m_i C_{(i)k-1} \lambda_i^{N+n} \quad \text{for } k = 1, \ldots, N
\]

\[\text{(21)}\]

**Proof**

The proof is analogous to the one of \( m C_1^n \) but uses the explicit form (13) of these coefficients.

\[
m C_k^{n+1} = \sum_{j=0}^{N-k} m C_1^{n-j} C_{k+j} \quad \text{(19)} = \sum_{j=0}^{N-k} \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n-j} C_{k+j}
\]

\[
= \sum_{j=0}^{N-k} \sum_{i=1}^{N} (-1)^{i+1} m_i \left( C_{(i)k+j} - \lambda_i C_{(i)k+j-1} \right) \lambda_i^{N+n-j}
\]

\[\Box\]
\[ N - k \sum_{j=0}^{N} (-1)^{i+1} m_i C_{(i)k+j} \lambda_i^{N+n-j} \]

\[ - \sum_{j=1}^{N-k} N \sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)k+j-1} \lambda_i^{N+1+n-j} \]

\[ - \sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)k-1} \lambda_i^{N+n+1}. \]

If we now shift the summation index \( j \) in the second sum most of the terms cancel with those of the first sum. The remaining term \( j = N - k \) in the first sum contains \( C_{(i)N} \), hence, vanishes also. Therefore, only the third sum remains what proofs the assumed form (21) of \( C_k^n \).

Again there are the cases \( n < N - k \) which need to be treated separately

\[ m C_k^{n+1} = \sum_{j=0}^{N-k} m C_1^{n-j} C_{k+j} + m C_{k+n+1} = \ldots = \]

\[ = \sum_{i=1}^{N} (-1)^i m_i C_{(i)k-1} \lambda_i^{N+n+1} + \sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)k+n} \lambda_i^N + m C_{k+n+1} \]

\[ = \sum_{i=1}^{N} (-1)^i m_i C_{(i)k-1} \lambda_i^{N+n+1}. \]

\[ \square \]

### 3.3 The exponential map

The expansion of a group element \( g \in G \) with generator \( H \in \mathfrak{gl}(N) \) can now be written like

\[ g = e^H = \sum_{n=0}^{\infty} \frac{H^n}{n!} = \sum_{n=0}^{N-1} \frac{H^n}{n!} + \sum_{n=0}^{\infty} \frac{H^{N+n}}{(N+n)!} \]

\[ = \sum_{n=0}^{N-1} \frac{H^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{(N+n)!} \left( \sum_{k=1}^{N} C_k^n H^{N-k} \right). \quad (22) \]

Using the multiplier \( m \) we get

\[ m g = m \sum_{n=0}^{N-1} \frac{H^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{(N+n)!} \left( \sum_{k=1}^{N} C_k^n m C_k^m \right) H^{N-k}. \quad (23) \]

We can now treat the sums for different \( k \) separately

\[ \sum_{n=0}^{\infty} \frac{1}{(N+n)!} m C_k^n = \sum_{n=0}^{\infty} \frac{1}{(N+n)!} \sum_{i=1}^{N} (-1)^i m_i C_{(i)k-1} \lambda_i^{N+n} \]
\[
\sum_{i=1}^{N} (-1)^i C_{(i)k-1} m_i \sum_{n=0}^{\infty} \frac{1}{(N+n)!} \lambda_i^{N+n} \\
= \sum_{i=1}^{N} (-1)^i C_{(i)k-1} m_i \left( e^{\lambda_i} - \sum_{n=0}^{N-1} \frac{\lambda_i^n}{n!} \right) \\
= \sum_{i=1}^{N} (-1)^i C_{(i)k-1} m_i e^{\lambda_i} + \sum_{n=0}^{N-1} \frac{1}{n!} \sum_{i=1}^{N} (-1)^{i+1} C_{(i)k-1} m_i \lambda_i^n \\
= \sum_{i=1}^{N} (-1)^i C_{(i)k-1} m_i e^{\lambda_i} - \frac{m}{(N-k)!}
\]

The last equation relies on Eq. (33) and (34). The terms \(-\frac{m}{(N-k)!}\) cancel the first sum in Eq. (23).

The final result turns out to be

\[
m e^H = (-1)^N \det H \left( \sum_{i=1}^{N} (-1)^i m_i e^{\lambda_i} \right) I_N + \left( \sum_{i=1}^{N} (-1)^i C_{(i)N-2} m_i e^{\lambda_i} \right) H \\
+ \ldots + \left( \sum_{i=1}^{N} (-1)^i C_{(i)k} m_i e^{\lambda_i} \right) H^{N-1-k} + \ldots \\
+ \left( \sum_{i=1}^{N} (-1)^i m_i \lambda_i e^{\lambda_i} \right) H^{N-2} + \left( \sum_{i=1}^{N} (-1)^{i+1} m_i e^{\lambda_i} \right) H^{N-1}
\]

or in closed form

\[
m e^H = \sum_{n=1}^{N} \left( \sum_{i=1}^{N} (-1)^i C_{(i)n-1} m_i e^{\lambda_i} \right) H^{N-n} 
\]

which reads in terms of the adjoints of the Slater determinant

\[
m e^H = \sum_{n=0}^{N-1} \left( \sum_{i=1}^{N} A_{(i)n} e^{\lambda_i} \right) H^n 
\]

3.4 The Slater determinant

One crucial ingredient of the method is the usage of a multiplier \(m\), defined in Eq. (9). From low dimensional examples one may assume the forms (11) and (12) of \(m\). The general proofs can be done by writing \(m\) as Slater determinant. The Slater determinant is defined as (cf. [4, 6])

\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
\lambda_N & \lambda_{N-1} & \ldots & \lambda_1 \\
\lambda_N^2 & \lambda_{N-1}^2 & \ldots & \lambda_1^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_N^{N-1} & \lambda_{N-1}^{N-1} & \ldots & \lambda_1^{N-1}
\end{vmatrix} = \prod_{i<j} (\lambda_i - \lambda_j) = m.
\]
We can now use the Laplacian method of expanding the Slater determinant
\[ \det A \quad = \quad \sum_{j=1}^{n} a_{ij} A_{ij} \quad = \quad \sum_{i=1}^{n} a_{ij} A_{ij} \]  
(27)
where the so-called adjoints \( A_{ij} \) are the subdeterminants of \( a_{ij} \), multiplied by the sign factor \( (-1)^{i+j} \).

It is also well known that the Laplace expansion with “wrong” adjoints gives zero
\[ 0 \quad = \quad \sum_{j=1}^{n} a_{ij} A_{lj} \quad \text{for} \quad l \neq i. \]  
(28)

The Laplacian method applied with respect to the last row gives then the expansion
\[ m \quad = \quad \sum_{i=1}^{N} \lambda_i^{N-1} A_{(N+1-i)N} = \sum_{i=1}^{N} (-1)^{N+(N+1-i)} m_i \lambda_i^{N-1} \]
\[ = \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1}, \]  
(29)
where \( m_i \) are the subdeterminants of \( m \)
\[ m_i = \prod_{k<i, k,j \neq i} \lambda_k - \lambda_j. \]  
(30)

We get
\[ m \quad = \quad m_i \prod_{k<i} (\lambda_k - \lambda_i) \prod_{j<i} (\lambda_i - \lambda_j) \quad \text{for} \quad i = 1, 2, \ldots, N \]
\[ = \quad (-1)^{i-1} m_i \prod_{j \neq i} (\lambda_i - \lambda_j) \]
\[ \equiv \quad (-1)^{i-1} m_i \sum_{n=0}^{N-1} \left( - C_{(i)n} \lambda_i^{N-1-n} \right) \]
\[ = \quad \sum_{n=0}^{N-1} (-1)^i m_i C_{(i)N-1-n} \lambda_i^n \]  
(31)
what proves Eq.(33). Also Eq.(34) is proven since if the exponent of \( \lambda_i \) is not \( n \) the sum vanishes because it is an expansion with the “wrong” adjoints \( A_{(i)n} \). Hence we get an explicit expression for the adjoints
\[ A_{(i)n} = (-1)^i m_i C_{(i)N-n-1}. \]  
(32)

We can also expand \( m \) with respect to the \((n+1)\)-th line and then use Eq.(32)
\[ m \quad = \quad \sum_{i=1}^{N} A_{(i)n} \lambda_i^n \quad \text{for} \quad n = 0, 1, \ldots, N - 1 \]
\[ = \quad \sum_{i=1}^{N} (-1)^i m_i C_{(i)N-1-n} \lambda_i^n. \]  
(33)
Writing the Laplacian expansion with “wrong” adjoints leads to

\[ 0 = \sum_{i=1}^{N} A_{(i)k} \lambda_i^n \quad \text{for} \quad k, n = 0, 1, \ldots, N - 1 \quad k \neq n \]

\[ = \sum_{i=1}^{N} (-1)^i m_i C_{(i)N-1-k} \lambda_i^n \]  \hspace{1cm} (34)

Ergo (cf. Eq.(12))

\[ m \delta_{kl} = \sum_{i=1}^{N} (-1)^i m_i C_{(i)N-k} \lambda_i^{N-l} \quad \text{for} \quad k, l = 1, 2, \ldots, N \]  \hspace{1cm} (35)

**Appendix**

**A Some details**

This section contains explicit forms of some coefficients and some proofs. Almost all of the equations hold in the general case, but those which hold only in the case of the special groups, i.e., vanishing sum of eigenvalues, are denoted by the sign \( \simeq \).

For the coefficients of the secular equation we get, e.g.,

\[ C_N = (-1)^{N+1} \prod_{i=1}^{N} \lambda_i = (-1)^{N+1} \det H \]

\[ C_{N-1} = (-1)^N \sum_{i=1}^{N} \prod_{j \neq i} \lambda_j = (-1)^N \sum_{i=1}^{N} \frac{\det H}{\lambda_i} \]  \hspace{1cm} (36)

\[ C_2 = (-1) \sum_{i<j} \lambda_i \lambda_j \simeq \frac{1}{2} \sum_{k=1}^{N} \lambda_k^2 \]

\[ C_1 = \sum_{i=1}^{N} \lambda_i \simeq 0, \quad C_0 = -1. \]

Some “truncated” coefficients

\[ C_{(i)N-1} = (-1)^{N-2} \prod_{j \neq i} \lambda_j = (-1)^N \frac{\det H}{\lambda_i} \]

\[ C_{(i)N-2} = (-1)^{N-3} \sum_{k \neq i} \prod_{j \neq k, i} \lambda_j \]

\[ C_{(i)2} = (-1) \sum_{j < k \atop j, k \neq i} \lambda_j \lambda_k, \quad C_{(i)1} = \sum_{j \neq i} \lambda_j \simeq -\lambda_i \]

\[ C_{(i)0} = -1, \quad C_{(i)N} = 0. \]
B Additional checks

B.1 One-dimensional subgroups

One-dimensional subgroups (cf. [3]) of $GL(N)$ can be generated by
\[
\left\{ e^{tH} : H \in \mathfrak{gl}(N) , t \in \mathbb{R} \right\}.
\] (37)

From the known expansion of $e^H$ we can derive the expansion of $e^{tH}$ by multiplying the occurring expressions with an appropriate factor. Obviously, the eigenvalues of the $t$-dependent generator $tH$ are $t \lambda_i$ if the $\lambda_i$ are the eigenvalues of $H$. Therefore, we need to make the replacements
\[
\begin{align*}
\lambda_i & \rightarrow t \lambda_i \\
C_k & \rightarrow t^k C_k \\
C_{(i)k} & \rightarrow t^k C_{(i)k} \\
m & \rightarrow t^{N(N-1)/2} m \\
m_i & \rightarrow t^{(N-1)(N-2)/2} m_i
\end{align*}
\]

The expansion (25) reads now
\[
t^{N(N-1)/2} m e^{tH} = \sum_{n=0}^{N-1} \left( \sum_{i=1}^{N} (-1)^i t^{n-1} C_{(i)n-1} t^{(N-1)(N-2)/2} m_i e^{tH} \right) (tH)^{N-n}
\]
\[
= t^{N(N-1)/2} \sum_{n=0}^{N-1} \left( \sum_{i=1}^{N} (-1)^i C_{(i)n-1} m_i e^{tH} \right) H^{N-n}
\]
or
\[
m e^{tH} = \sum_{n=0}^{N-1} \left( \sum_{i=1}^{N} (-1)^i C_{(i)n-1} m_i e^{tH} \right) H^{N-n}.
\] (38)

Differentiation of the r.h.s of Eq. (38) and setting $t = 0$ gives the derivation of the unit element
\[
\sum_{n=0}^{N-1} \left( \sum_{i=1}^{N} (-1)^i C_{(i)n-1} m_i \lambda_i \right) H^{N-n} = m H .
\]

Since this result coincides with the one we get by differentiating the l.h.s it is an additional proof of the expansion (24).

B.2 Eigenvalues

It is easy to demonstrate that Eq. (25) gives also the right connection between the eigenvalues of the generator $H$ and the ones of the corresponding group element $g = e^H$. Let $x_i$ be the eigenvectors of $H$ with eigenvalues $\lambda_i$
\[
H x_i = \lambda_i x_i \quad \text{for} \quad i = 1, 2, \ldots, N.
\]
For the powers of $H$ we get
\[
H^n x_i = \lambda_i^n x_i \quad \text{for} \quad n \in \mathbb{N}.
\]
Plugging this in Eq.(24) yields

\[ m e^H x_j = \sum_{n=0}^{N-1} \left( \sum_{i=1}^N A_{(i)n} e^{\lambda_i} \right) H^n x_j = \sum_{n=0}^{N-1} \left( A_{(i)n} e^{\lambda_i} \lambda_i^n \right) x_j \]
\[ = \sum_{i=1}^N \left( \sum_{n=0}^{N-1} A_{(i)n} \lambda_i^n \right) x_j = m e^{\lambda_j} x_j. \]

Therefore, we get the desired result

\[ g x_j = e^{\lambda_j} x_j \]

which again confirms the expansion (24).

B.3 Remark

In the cases where some of the eigenvalues coincide, the multiplier \( m \) will be zero. But in these cases \( m \) can be chosen in a simpler fashion so that there occur only non-vanishing factors. Essentially all factors which will become zero can be canceled out in Eq.(24).

Acknowledgement

The author would like to thank Dr. K-P. Marzlin for the discussions of some points.

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