LIE GROUPOIDS AND THE FRÖLICHER-NIJENHUIS BRACKET

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Abstract. The space of vector-valued forms on any manifold is a graded Lie algebra with respect to the Frölicher-Nijenhuis bracket. In this paper we consider multiplicative vector-valued forms on Lie groupoids and show that they naturally form a graded Lie subalgebra. Along the way, we discuss various examples and different characterizations of multiplicative vector-valued forms.

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1. Introduction

Lie groupoids are ubiquitous in several areas of mathematics; they arise as models for singular spaces, in the study of foliations and group actions, noncommutative geometry, Poisson geometry, etc. (see e.g. [8, 9, 21, 23, 24] and references therein). In these settings, one is often led to consider Lie groupoids endowed with additional geometric structures compatible with the groupoid operation, referred to as multiplicative. Examples of interest include multiplicative symplectic and Poisson structures [20, 22, 25] (see also [1, 3, 5, 6, 14]), complex structures [18], and distributions [10, 12, 16]. The present paper fits into the broader project of studying multiplicative structures on Lie groupoids and should be seen as a companion to [7]. Here we focus on multiplicative vector-valued forms and study their compatibility with the Frölicher-Nijenhuis bracket [11].

There are several algebraic objects naturally associated with a smooth manifold $M$, such as the de Rham complex $(\Omega^\bullet(M), d)$, the Gerstenhaber algebra of multivector fields $(\Gamma(\bigwedge^\bullet TM), [\cdot, \cdot]_{SN})$, where $[\cdot, \cdot]_{SN}$ denotes the Schouten-Nijenhuis bracket (see e.g. [21 Sec. 7.5]), and the graded Lie algebra of vector-valued forms

$$(\Gamma(\bigwedge^\bullet T^* M \otimes TM), [\cdot, \cdot]_{FN}),$$

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where $\cdot,\cdot_{FN}$ is the Frölicher-Nijenhuis bracket \[11\] (see e.g. \[17\] Sec. 8)). These objects play a key role in measuring the integrability of geometric structures on $M$: for example, a differential form on $M$ is closed if it is a cocycle in the de Rham complex, a bivector field $\Lambda \in \Gamma(\wedge^2TM)$ is a Poisson structure if it satisfies $[\Lambda,\Lambda]_{SN} = 0$, and an almost complex structure $J \in \Gamma(T^*M \otimes TM)$ is a complex structure if $[J,J]_{FN} = 0$.

When $M$ is replaced by a Lie groupoid $G$, the relevant issue is whether these natural algebraic operations are compatible with multiplicative geometric structures. It is a simple verification that multiplicative forms define a subcomplex of $(\Omega^\bullet(G),d)$; it is also known that the Schouten-Nijenhuis bracket restricts to multiplicative multivector fields, making them into a Gerstenhaber subalgebra of $(\Gamma(\wedge^\bullet T^*G),[\cdot,\cdot]_{SN})$ \[14, Sec. 2.1\]. We verify in this paper that an analogous result holds for multiplicative vector-valued forms on $G$, i.e., we show that the space of multiplicative vector-valued forms is closed under the Frölicher-Nijenhuis bracket. Some of the applications of this result will be discussed in \[7\].

The paper is organized as follows. In Section 2 we review the key examples of Lie groupoids that are relevant to the paper. In Section 3 we consider multiplicative vector-valued forms on Lie groupoids and discuss examples, including relations with connections and curvature on principal bundles. Section 4 contains the main results: we give a direct proof of the compatibility of multiplicative vector-valued forms and the Frölicher-Nijenhuis bracket in Thm.$4.3$ and then see how this result follows from a broader, more conceptual, perspective, in which multiplicative vector-valued forms are characterized in terms of the Bott-Shulman-Stasheff complex of a Lie groupoid.

It is a pleasure to dedicate this paper to IMPA’s 60th anniversary.

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2. Lie groupoids and examples

This section recalls some examples of Lie groupoids relevant to the paper; further details can be found e.g. in \[8, 21, 24\].

Let $G$ be a Lie groupoid over a manifold $M$, denoted by $G \rightrightarrows M$. As usual, we refer to $G$ as the space of arrows and $M$ as the space of objects. We denote the source and target maps by $s, t : G \to M$, the multiplication map by $m : G^2 := \{(g,h) \in G \times G, |s(g) = t(h)\} \to G,$

the unit map by $\epsilon : M \to G$, and inversion by $\iota : G \to G, \iota(g) = g^{-1}$. We often identify $M$ with its image under the embedding $\epsilon$ and use the notation $\epsilon(x) = 1_x$. We also write $m(g,h) = gh$ to simplify notation. If there is any risk of confusion, we use the groupoid itself to label its structure maps: $s_G, t_G, m_G, \epsilon_G, \iota_G.$

A morphism from $G \rightrightarrows M$ to $H \rightrightarrows N$ is a pair of smooth maps $F : G \to H, f : M \to N$ that commute with source and target maps, and preserve multiplication (this implies that unit and inversion maps are also preserved).

A central observation to this paper is that, given a Lie groupoid $G \rightrightarrows M$, its tangent bundle $TG$ is naturally a Lie groupoid over $TM$: its source and target maps
are given by $T\mathcal{G}, Tt_\mathcal{G} : T\mathcal{G} \to TM$; for the multiplication, we notice that

$$(TG)^{(2)} = \{(X,Y) \in T\mathcal{G} \times T\mathcal{G} \mid T\mathcal{G}(X) = Tt_\mathcal{G}(Y)\} = T(\mathcal{G}^{(2)}),$$

so we set $m_{TG} = Tm_{\mathcal{G}}$. Similarly, the unit and inverse maps are $T\epsilon_\mathcal{G} : TM \to T\mathcal{G}$ and $T\tau_\mathcal{G} : T\mathcal{G} \to T\mathcal{G}$.

Another important remark is that the Whitney sum $\oplus^k T\mathcal{G}$ (of vector bundles over $\mathcal{G}$) is naturally a Lie groupoid over $\oplus^k TM$,

$$\oplus^k T\mathcal{G} \rightrightarrows \oplus^k TM,$$

with structure maps defined componentwise.

We list some basic examples of Lie groupoids and their tangent bundles.

**Example 2.1.** A Lie groupoid over a point is a Lie group $G$, in which case its tangent bundle $TG$ is also a Lie group. For $g,h \in G$ and $X \in T_gG, Y \in T_hG$, the multiplication on $TG$ is given by

$$Tm_G(X,Y) = Tr_h(X) + Tl_g(Y) \in T_{gh}G,$$

where $r_g, l_h : G \to G$ denote right, left translations. Using the trivialization $TG \simeq G \times \mathfrak{g}$ by right-translations, one sees that

$$Tm_G((g,u),(h,v)) = (gh, u + Ad_g(v)).$$

This identifies $TG$ with the Lie group $G \times \mathfrak{g}$ obtained by semi-direct product with respect to the adjoint action.

**Example 2.2.** Any vector bundle $\pi : E \to M$ can be naturally seen as a Lie groupoid: source and target maps coincide with the projection $\pi$, and the multiplication is given by addition on the fibers. In this case, the tangent groupoid $TE$ over $TM$ is defined by the vector bundle $T\pi : TE \to TM$, known as the tangent prolongation of $E$.

**Example 2.3.** Let $G$ be a Lie group, and let $\pi : P \to M$ be a (right) principal $G$-bundle. We denote the $G$-action on $P$ by $\psi : P \times G \to P$,

$$p \mapsto \psi_g(p), \quad p \in P.$$

The corresponding gauge groupoid $\mathcal{G}(P) \rightrightarrows M$ is defined as the orbit space of the diagonal action of $G$ on $P \times P$; we write $[p,q]$ for the image of $(p,q) \in P \times P$ in $\mathcal{G}(P)$. Source and target maps on $\mathcal{G}(P)$ are given by the composition of the natural projections $P \times P \to P$ with $\pi$, and multiplication is given by

$$[p,q] \cdot [p',q'] = [p,q'],$$

where we assume in this composition that $q = p'$ (given any representatives $(p,q)$ and $(p',q')$, we have that $\pi(q) = \pi(p')$, so for a fixed $(p,q)$ one may always replace $(p',q')$ by a unique point in its $G$-orbit satisfying the desired property). The unit map is

$$\epsilon : M \to \mathcal{G}(P), \quad x \mapsto [p,p],$$

where $p \in P$ is any point such that $\pi(p) = x$, whereas the inversion is given by

$$[p,q] \mapsto [q,p].$$

The $G$-action on $P$ naturally induces a $TG$-action on $TP$ by

$$\Psi_{(g,u)}(X_q) = T\psi_g(X_q) + u_P(\psi_g(q)), \quad (2.3)$$
for \( X_q \in T_q P \) and \((g, u) \in TG \cong G \ltimes g\); here \( u_p \in \mathfrak{X}(P) \) is the infinitesimal generator of the \( G \)-action on \( P \). This action makes \( T\pi : TP \to TM \) into a principal \( TG \)-bundle, so we have a corresponding gauge groupoid \( G(TP) \). One may verify that there is a natural identification between \( G(TP) \) and the tangent groupoid \( TG(P) \Rightarrow TM \):

\[
TG(P) = G(TP).
\]

We denote the image of an element \((X, Y) \in TP \times TP \) in \( G(TP) \) by \((\overline{X}, \overline{Y})\). The induced vector bundle structure \( G(TP) \to G(P) \) is given by

\[
\overline{(X_1, Y_1)} + \overline{(X_2, Y_2)} = \overline{(X_1 + X_2, Y_1 + Y_2)}, \quad \lambda(X, Y) = \overline{(\lambda X, \lambda Y)},
\]

where, for the addition, the representatives are chosen over the same fiber of \( TP \times TP \to P \times P \).

For a vector bundle \( E \to M \) (of rank \( n \)), let \( GL(E) \) be the gauge groupoid of the frame \( GL(n) \)-bundle \( Fr(E) \to M \). More concretely, \( GL(E) \Rightarrow M \) is the Lie groupoid whose arrows between \( x, y \in M \) are linear isomorphisms from \( E_x \) to \( E_y \). A representation of \( G \Rightarrow M \) on a vector bundle \( E \to M \) is a groupoid homomorphism from \( G \) into \( GL(E) \).

**Example 2.4.** Given a representation of a Lie groupoid \( G \Rightarrow M \) on a vector bundle \( \pi : E \to M \), there is an associated semi-direct product Lie groupoid \( G \ltimes E \Rightarrow M \): its space of arrows is

\[
t^*E = G \ltimes t_{\pi} E = \{(g, e) \mid t(g) = \pi(e)\},
\]

with source and target maps given by \( (g, e) \mapsto s_G(g) \) and \( (g, e) \mapsto t_G(g) \), respectively, and multiplication given by

\[
((g_1, e_1), (g_2, e_2)) \mapsto (g_1g_2, e_1 + g_1 \cdot e_2),
\]

where we write \( g \cdot e \) for the action \( G \ltimes \pi E \to E \) induced by the representation.

There is an induced representation of \( TG \Rightarrow TM \) on \( TE \to TM \), and the tangent groupoid to \( G \ltimes E \) is the corresponding semi-direct product.

### 3. Multiplicative vector-valued forms

**3.1. Definition and first examples.** A vector-valued \( k \)-form on a manifold \( N \) is an element in \( \Omega^k(N, TN) := \Gamma(\wedge^k T^*N \otimes TN) \). It will be convenient to think of vector-valued \( k \)-forms as maps

\[
\oplus^k TN \to TN.
\]

In particular, vector-valued 1-forms \( K \in \Omega^1(N, TN) \) are naturally identified with endomorphisms \( TN \to TN \) (covering the identity).

Given a Lie groupoid \( G \Rightarrow M \), we will be concerned with vector-valued forms on \( G \) which are compatible with the groupoid structure in the following sense \([7, 18]\).

**Definition 3.1.** A vector-valued form \( K \in \Omega^k(G, TG) \) is multiplicative if there exists \( K_M \in \Omega^k(M, TM) \) such that

\[
\begin{array}{ccc}
\oplus^k TG & \xrightarrow{K} & TG \\
\downarrow & & \downarrow \\
\oplus^k TM & \xrightarrow{K_M} & TM
\end{array}
\]
is a groupoid morphism.

In this case, we say that $K$ covers $K_M$.

**Example 3.2.** Let $G$ be a Lie group. An endomorphism $J : TG \rightarrow TG$, viewed as a vector-valued 1-form $J \in \Omega^1(G,TG)$, is multiplicative if and only if

$$J \circ Tm = Tm \circ (J \times J).$$

In particular, if $J$ is an integrable almost complex structure on $J$, then it is multiplicative if and only if $m : G \times G \rightarrow G$ is a holomorphic map, i.e., $J$ makes $G$ into a complex Lie group (the fact that the inversion map is holomorphic automatically follows).

In general, a multiplicative vector-valued $k$-form on a Lie group $G$ may be equivalently viewed as a multiplicative $k$-form on $G$ with values on the adjoint representation $[1]$. To verify this fact, we use the identification $TG = G \times \mathfrak{g} = \mathfrak{t}^* \mathfrak{g}$, recalling that the target map is the trivial map $t : G \rightarrow \{*\}$, and notice that, for $K \in \Omega^k(G,TG)$, $(3.2)$ implies that $(3.1)$ is a Lie groupoid morphism if and only if

$$(m^* K)_{(g,h)} = pr_1^* K + Ad_g(pr_2^* K), \quad \text{for} \ g, h \in G,$$

where $pr_1, pr_2 : G \times G \rightarrow G$ are the natural projections.

A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called holomorphic if it is equipped with a complex structure $J \in \Omega^1(\mathcal{G},\mathcal{T}\mathcal{G})$ that is multiplicative. Besides complex Lie groups, holomorphic vector bundles provide natural examples:

**Example 3.3.** Let $(M,J_M)$ be a complex manifold and consider a (real) vector bundle $\pi : E \rightarrow M$, viewed as a Lie groupoid as in Example 2.2. A vector-valued $k$-form $K \in \Omega^k(E,TE)$ is multiplicative in this case if and only if the associated map $\oplus^kTE \rightarrow TE$ is a vector-bundle morphism with respect to the vector-bundle structures $\oplus^kTE \rightarrow \oplus^kTM$ and $TE \rightarrow TM$. It is observed in [18] that an integrable almost complex structure $J \in \Omega^1(E,TE)$ which is multiplicative and covers $J_M \in \Omega^1(M,TM)$ is equivalent to equipping $E$ with the structure of a holomorphic vector bundle over $M$.

Other examples of multiplicative vector-valued forms arise in the context of connections on principal bundles, as we now discuss.

**3.2. Principal connections and curvature.** Let $G$ be a Lie group and $\pi : P \rightarrow M$ be a principal (right) $G$-bundle. We will follow the notation of Example 2.3.

Let $V \subseteq TP$ be the vertical bundle over $P$, i.e., the fiber of $V \rightarrow P$ over $p \in P$ is

$$V_p = \{ u_P(p) \mid u \in \mathfrak{g} \},$$

where $u_P \in \mathfrak{X}(P)$ is the infinitesimal generator of the $G$-action on $P$. The vertical bundle $V \rightarrow P$ induces a distribution $[2]$

$$\Delta_V \subseteq T\mathcal{G}(P)$$

on the gauge groupoid $\mathcal{G}(P)$ given by the image of $V \times V \subseteq TP \times TP$ under the quotient map $TP \times TP \rightarrow \mathcal{G}(TP) = T\mathcal{G}(P)$.

---

1. Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ along with a representation on $E \rightarrow M$, recall from [10, Sec. 2.1] that a form $\omega \in \Omega^k(\mathcal{G},\mathfrak{t}^*E)$ is multiplicative if it satisfies $m^* \omega |_{(g,h)} = pr_1^* \omega + g \cdot pr_2^* \omega$, where $(g,h) \in \mathcal{G}^{(2)}$ and $pr_1, pr_2 : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ are the natural projections.

2. We always assume distributions to be of constant rank, i.e., subbundles of the tangent bundle.
3.2.1. Principal connections.

Let $\theta \in \Omega^1(\mathcal{P}, \mathfrak{g})$ be a principal connection on $P$. By using the identification (of $G$-equivariant vector bundles over $P$)

$$P \times \mathfrak{g} \to V, \quad (p, u) \mapsto u_P(p),$$

we may equivalently describe it as a $G$-equivariant 1-form $\Theta \in \Omega^1(P, V)$ such that

$$\text{Im}(\Theta) = \Delta = \{} D \cap V \setminus \theta \in G \text{-orbit} \}$$

so that $\Theta(X) = (\theta(X))_p$. We denote the horizontal bundle defined by the connection by $H_\text{horizontal} = \ker(\theta) = \ker(\Theta) \subseteq TP$.

We observe that principal connections on $P$ are naturally associated with certain multiplicative vector-valued 1-forms on $\mathcal{G}(P)$:

**Proposition 3.4.** There is a one-to-one correspondence between principal connections $\theta \in \Omega^1(\mathcal{P}, \mathfrak{g})$ on $P$ and multiplicative $K \in \Omega^1(\mathcal{G}(P), T\mathcal{G}(P))$ satisfying $K^2 = K$ and $\text{Im}(K) = \Delta$. For the proof, we need some general observations.

**Lemma 3.5.** Let $D \subset TP$ be a $G$-invariant distribution on $P$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an $\text{Ad}$-invariant subspace, and suppose that

$$D \cap V|_p = \{ u_P(p) \mid u \in \mathfrak{h} \},$$

at each $p \in P$. Then the image $\Delta_D$ of $D \times D \subset TP \times TP$ under the quotient map $TP \times TP \to T\mathcal{G}(P)$ is distribution which is a Lie subgroupoid of $T\mathcal{G}(P)$:

$$\text{Im}(\Theta) = \Delta = \{} D \cap V \setminus \theta \in G \text{-orbit} \}$$

for $D_M = T\pi(D) \subset TM$.

**Proof.** One may directly check from [23] that $D \subset TP$ is $G \ltimes \mathfrak{h}$-invariant, where we view the semi-direct product Lie group $G \ltimes \mathfrak{h} \subseteq TG$ as a subgroup of $TG$.

For $X, Y \in D$, $T\pi(X) = T\pi(Y)$ if and only if

$$X = \Psi_{(g, u)}Y = T \psi_g(Y) + u_P,$$

but since $D$ is $G$-invariant, it follows that $u_P \in D$, hence $u \in \mathfrak{h}$. It follows that $T\pi(X) = T\pi(Y)$ if and only if $X$ and $Y$ are on the same $G \ltimes \mathfrak{h}$-orbit.

Since $V \cap D$ has constant rank, $\Delta_D$ is a subbundle of $T\mathcal{G}(P)$, and $D_M = T\pi(D)$ is a subbundle of $TM$. To verify that $\Delta_D \Rightarrow D_M$ is Lie subgroupoid of $T\mathcal{G}(P)$, let

$$(X_i, Y_i) \in \Delta_D^{(p_i, q_i)},$$

for $i = 1, 2$, be composable, i.e., $T\pi(Y_1) = T\pi(X_2)$. For $Y_1 \in D|_{q_1}$ and $X_2 \in D|_{p_2}$, we saw that this implies the existence of $(g, u) \in G \ltimes \mathfrak{h}$ such that $Y_1 = \psi_{(g, u)}(X_2)$, where $q_1 = \psi_g(p_2)$. Hence

$$Tm((X_1, Y_1), (X_2, Y_2)) = Tm((X_1, Y_1), (\psi_{(g, u)}(X_2), \psi_{(g, u)}(Y_2)))$$

$$= (X_1, \psi_{(g, u)}(Y_2)),$$

which belongs to $\Delta_D$ since $D$ is $G \ltimes \mathfrak{h}$-invariant. □

The following are particular instances of Lemma 3.5:
• The vertical bundle $V$ satisfies the conditions of Lemma 3.5 with $\mathfrak{h} = \mathfrak{g}$. In this case $V_M = T\pi(V) = M$, and we have a corresponding subgroupoid

$$\Delta_V \rightrightarrows M$$

of $T\mathcal{G}(P) \rightrightarrows TM$.

• For $D = H$ the horizontal bundle of a principal connection, the conditions in Lemma 3.5 hold for $\mathfrak{h} = \{0\}$; then $H_M = T\pi(H) = TM$, and we have a subgroupoid

$$\Delta_H \rightrightarrows TM.$$

For an arbitrary Lie groupoid $\mathcal{G} \rightrightarrows M$, recall that a distribution $\Delta \subset T\mathcal{G}$ is called multiplicative if it is a Lie subgroupoid of $T\mathcal{G} \rightrightarrows TM$. In this case, the space of objects of $\Delta$ is a subbundle $\Delta_M \subset TM$ (see e.g. [15]).

**Lemma 3.6.** Let $\Delta^1, \Delta^2$ be distributions on $\mathcal{G}$ satisfying $T\mathcal{G} = \Delta^1 \oplus \Delta^2$. If $K \in \Omega^1(\mathcal{G}, T\mathcal{G})$ is a projection so that $\Delta^1 = \text{Im}(K)$ and $\Delta^2 = \text{ker}(K)$, then $K$ is multiplicative if and only if both $\Delta^1$ and $\Delta^2$ are multiplicative distributions.

**Proof.** Suppose that $K$ is multiplicative, i.e., a groupoid morphism $T\mathcal{G} \to T\mathcal{G}$. The fact that $\Delta^2 = \text{ker}(K)$ is multiplicative follows from the more general fact that the kernel of morphisms of VB-groupoids (see e.g. [21, Ch. 11]) is a VB-subgroupoid whenever it has constant rank, see e.g. [4, 19]. The analogous result for $\Delta^1$ follows since $\text{Id} - K$ is also a multiplicative projection and $\Delta^1$ is its kernel.

To prove the converse, note that the spaces of units of the groupoids $\Delta^1$ and $\Delta^2$, denoted by $\Delta^1_M$ and $\Delta^2_M$, are subbundles of $TM$ satisfying $TM = \Delta^1_M \oplus \Delta^2_M$. Let $K_M : TM \to TM$ be the projection on $\Delta^1_M$ along $\Delta^2_M$. It is clear that $K$ and $K_M$ intertwine the source and target maps for $T\mathcal{G} \rightrightarrows TM$. For

$$X = X_1 + X_2 \in \Delta^1 \oplus \Delta^2 \quad \text{and} \quad Y = Y_1 + Y_2 \in \Delta^1 \oplus \Delta^2$$

satisfying $Ts(X) = Tt(Y)$, we see that $Ts(X_1) = Tt(Y_1)$, $Ts(X_2) = Tt(Y_2)$ and

$$K(Tm(X,Y)) = K(Tm(X_1,Y_1) + Tm(X_2,Y_2)) = K(Tm(X_1,Y_1)) = Tm(X_1,Y_1) = Tm(K(X),K(Y)).$$

So $K$ preserves groupoid multiplication. □

We can now prove Prop. 3.4.

**Proof.** Consider a connection on $P$ given by $\Theta \in \Omega^1(P,V)$. Let $K : T\mathcal{G}(P) \to T\mathcal{G}(P)$ be defined by

$$K((X,Y)) = (\Theta(X), \Theta(Y)).$$

(3.4)

The properties of $\Theta$ (see (3.3)) imply that $K$ is well defined (by the $G$-equivariance of $\Theta$), satisfies $K^2 = K$, and that $\text{Im}(K) = \Delta_V$ and $\text{Ker}(K) = \Delta_H$. By Lemmas 3.5 and 3.6 $K$ is multiplicative.

Conversely, let $K$ be a multiplicative vector-valued 1-form satisfying $K^2 = K$ and $\text{Im}(K) = \Delta_V$. Let us consider the vector bundle $TP/G \to M$, and its subbundle $V/G \to M$. We note that $K$ naturally induces a projection map

$$\tilde{\Theta} : TP/G \to V/G$$

(3.5)
as follows. First recall that there is a natural identification of $TP/G$ with $\ker(Ts_G(P))|_M$ as vector bundles over $M$: indeed, noticing that

$$\ker(Ts_G(P))|_x = \{(X_p, Y_p), \, T\pi(Y) = 0\},$$

where $\pi(p) = x \in M$, the identification $TP/G \to \ker(Ts_G(P))|_M$ is given by

$$\overline{X}|_{\pi(p)} \mapsto (X_p, 0_p),$$

where $\overline{X}$ denotes the class of $X \in TP$ in $TP/G$. The inverse map is $(X_p, Y_p) \mapsto \overline{X_p - Y_p} \in (TP/G)|_{\pi(p)}$. Under this identification, the subbundle $V/G \subset TP/G$ corresponds to $\Delta_V|_M \subset \ker(Ts_G(P))|_M$. The projection map (3.5) is defined by the diagram

\[
\begin{array}{ccc}
TP/G & \sim & \ker(Ts_G(P))|_M \\
\downarrow \Theta & & \downarrow K \\
V/G & \sim & \Delta_V|_M.
\end{array}
\]

The map $\Theta$ is equivalent to a connection $\Theta \in \Omega^1(P, V)$ through $\Theta(\overline{X}) = \overline{\Theta(X)}$. This is the connection defined by $K$.

More explicitly, the relation between $\Theta$ and $K$ in diagram (3.6) is

\[
K((X_p, 0_p)) = (\Theta(X_p), 0_p),
\]

and, as we now see, this condition completely determines $K$: Using the groupoid structure on $T\mathcal{G}(P)$, we can write an arbitrary $(\overline{X_p, Y_q})$ as

\[
\overline{X_p, Y_q} = (X_p, 0_p) \cdot (0_p, 0_q) \cdot (0_q, Y_q) = (X_p, 0_p) \cdot (0_p, 0_q) \cdot (Y_q, 0_q)^{-1},
\]

and, since $K$ is multiplicative, (3.7) implies that

\[
K((X_p, Y_q)) = (\Theta(X_p), 0_p) \cdot (0_p, 0_q) \cdot (\Theta(Y_q), 0_q)^{-1} = (\Theta(X_p), \Theta(Y_q)).
\]

It follows (see (3.3)) that the construction relating $K$ and $\Theta$ just described are inverses of one another. \hfill \Box

3.2.2. Curvature.

For a manifold $N$, the \textit{curvature} of a projection $K : TN \to TN$ is the vector-valued 2-form $R_K \in \Omega^2(N, TN)$ given by

\[
R_K(X, Y) = K([\text{Id} - K](X), (\text{Id} - K)(X)), \quad X, Y \in \mathcal{X}(N),
\]

where $[\cdot, \cdot]$ is the Lie bracket of vector fields (see e.g. [17]). So $R_K$ measures the integrability of the distribution $\text{Ker}(K) \subseteq TN$. The \textit{co-curvature} of $K$ is the curvature of $\text{Id} - K$.

A direct consequence of the results in Section 4 (see Theorem 4.3) is that, on a Lie groupoid, the curvature of any multiplicative projection is a multiplicative vector-valued 2-form. We will now verify this fact in the case of projections on gauge groupoids $\mathcal{G}(P)$ arising from principal connections, as in Prop. 3.11. In this particular context, the result follows from the explicit relation between $R_K \in \Omega^2(\mathcal{G}(P), T\mathcal{G}(P))$ and the curvature of the connection corresponding to $K$, as explained in Prop. 3.10 below.
Let $\text{Ad}(P) \to M$ be the vector bundle associated with the adjoint action on $\mathfrak{g}$, i.e., $\text{Ad}(P) = (P \times \mathfrak{g})/G$. We denote elements in $\text{Ad}(P)$ by $(p, v)$, for $p \in P$ and $v \in \mathfrak{g}$. There is a natural representation of the gauge groupoid $G(P)$ on $\text{Ad}(P)$ by

$$(q, p) \cdot (p, v) = (q, v).$$

As in Example 2.3, we consider the semi-direct product groupoid $G(P) \ltimes \text{Ad}(P)$, that we denote by $t^*\text{Ad}(P) \to M$.

**Lemma 3.7.** The following holds:

(a) There is a natural groupoid isomorphism

$$\varphi : (\Delta_V \rightrightarrows M) \to (t^*\text{Ad}(P) \rightrightarrows M),$$

which is also a isomorphism of vector bundles over $G(P)$. (I.e., this is a isomorphism of VB-groupoids [21, Ch. 11].)

(b) Assume that a vector-valued $k$-form $R \in \Omega^k(G(P), T\mathcal{G}(P))$ takes values in $\Delta_V \subseteq T\mathcal{G}(P)$. Then $R$ is multiplicative if and only if $R' := \varphi \circ R \in \Omega^k(G(P), t^*\text{Ad}(P))$ satisfies

$$m^*R'_{(g,h)} = pr_1^*R' + g \cdot pr_2^*R',$$

for $(g, h) \in G(P)^{(2)}$. (I.e., $R'$ is multiplicative as a $k$-form with values on the representation $\text{Ad}(P)$, as in [10].)

**Proof.** We define the map $\varphi : \Delta_V \to t^*\text{Ad}(P)$ by $\varphi((uP(p), vP(q))) = (p, u - v)$. One can directly verify that this map is well-defined, and that it is a morphisms of vector bundles over $G(P)$; the inverse map $t^*\text{Ad}(P) \to \Delta_V$ is defined, on each fiber over $(p, q) \in G(P)$, by $(p, v) \mapsto (vP(p), 0P(q))$. To verify that $\varphi$ is a groupoid morphism, fix $g = (p_1, q_1), h = (p_2, q_2) \in G(P)$, and

$$X = (uP^1(p_1), vP^1(q_1)) \in \Delta_V|_{(p_1, q_1)}, \quad Y = (uP^2(p_2), vP^2(q_2)) \in \Delta_V|_{(p_2, q_2)}.$$

Since $Tt(Y) = Ts(X)$, we can assume that $q_1 = p_2$ and $v^1 = u^2$. So

$$\varphi(Tm(X, Y)) = \varphi((uP^1(p_1), vP^2(q_2))) = (p_1, u^1 - v^2).$$

On the other hand,

$$\varphi(X) + g : \varphi(Y) = (p_1, u^1 - v^1) + (p_1, q_1) \cdot (p_2, u^2 - v^2)$$

$$= (p_1, u^1 - v^1) + (p_1, u^1 - v^2) = (p_1, u^1 - v^2),$$

hence multiplication is preserved (c.f. Example 2.4).

The claim in part (b) follows directly from (a) (and (2.5)). □

**Remark 3.8.** The observation in Lemma 3.7 part (a), is an instance of a more general fact: on any regular Lie groupoid $G \rightrightarrows M$, there is a natural representation of $G$ on the vector subbundle $\ker(\rho) \subset A$, where $A$ is the Lie algebroid of $G$ and $\rho$ is its anchor; in this case the distribution $\ker(Ts) \cap \ker(Tt) \subseteq TG$ is multiplicative, and naturally isomorphic to the semi-direct product groupoid $G \ltimes \ker(\rho)$ (as a groupoid and as a vector bundle over $G$). When $G$ is a gauge groupoid $G(P)$, $\ker(\rho) = \text{Ad}(P)$, and $\ker(Ts) \cap \ker(Tt) = \Delta_V$.
For a connection $\theta \in \Omega^1(P,g)$, let $H \subset TP$ be its horizontal bundle. For a vector field $X \in \mathfrak{X}(P)$, let $X^H$ be its projection on $H$: $X^H = (\text{Id} - \Theta)(X)$. Let $F_\theta \in \Omega^2(P,g)$ be the curvature of $\theta$,

$$F_\theta(X,Y) = -\theta([X^H, Y^H]).$$

Since $F_\theta$ is invariant and $i_X F_\theta = 0$ for $X \in \mathfrak{X}$, it may be alternatively viewed as an element in $\Omega^2(M, \text{Ad}(P))$. Let $K \in \Omega^1(\mathcal{G}(P), T\mathcal{G}(P))$ be the projection corresponding to $\theta$, and let $R_K \in \Omega^2(\mathcal{G}(P), T\mathcal{G}(P))$ be its curvature \[10\]. Using \[3.10\], we consider

$$R'_K = \varphi \circ R_K \in \Omega^2(\mathcal{G}(P), t^*\text{Ad}(P)).$$

**Lemma 3.9.** $R'_K$ satisfies

\[3.12\]

$$R'_K|_g = g \cdot (s^* F_\theta) - t^* F_\theta,$$

where $F_\theta \in \Omega^2(M, \text{ad}(P))$ is the curvature of $\theta$.

**Proof.** Let us fix $g = (p,q) \in \mathcal{G}(P)$, $\mathcal{X} = (X_1,Y_1)$, $\mathcal{Y} = (X_2,Y_2) \in T\mathcal{G}(P)|_g$, for $X_1, X_2 \in T_P$ and $Y_1, Y_2 \in T_P$. By definition (see \[3.8\]),

$$R_K(\mathcal{X}, \mathcal{Y}) = K \left( ([X_1^H, X_2^H]_p, [Y_1^H, Y_2^H]_q) \right) = (\Theta([X_1^H, X_2^H](p)), \Theta([Y_1^H, Y_2^H](q))),$$

where $X_i^H, Y_i^H \in \mathfrak{X}(P)$ are horizontal vector fields extending $(\text{Id} - \Theta)(X_i)$ and $(\text{Id} - \Theta)(Y_i)$, respectively, for $i = 1, 2$. Hence,

$$\varphi(R_K(\mathcal{X}, \mathcal{Y})) = (p, \theta([X_1^H, X_2^H](p)) - \theta([Y_1^H, Y_2^H](q))).$$

On the other hand,

$$F_\theta(Tt(\mathcal{X}), Tt(\mathcal{Y})) = F_\theta(T\pi(X_1), T\pi(X_2)) = -(p, \theta([X_1^H, X_2^H](p))),$$

and

$$(p,q) \cdot F_\theta(T\pi(Y_1), T\pi(Y_2)) = -(p,q) \cdot (q, \theta([Y_1^H, Y_2^H])) = -(p, \theta([Y_1^H, Y_2^H])).$$

\[\square\]

**Proposition 3.10.** If $K \in \Omega^1(\mathcal{G}(P), T\mathcal{G}(P))$ is a projection corresponding to a connection on $\mathcal{G}(P)$, then $R_K \in \Omega^2(\mathcal{G}(P), T\mathcal{G}(P))$ is multiplicative.

**Proof.** Using Lemmas \[3.7\] and \[3.9\] the result follows once we check that condition \[3.12\] implies that \[3.11\] holds. Considering the maps $m, pr_1, pr_2 : \mathcal{G}(P)^{(2)} \to \mathcal{G}(P)$, this can be directly verified using the identities $t \circ m = t \circ pr_1$, $s \circ m = s \circ pr_2$, and $t \circ pr_2 = s \circ pr_1$.

\[\square\]

**Remark 3.11.** In the context of multiplicative forms on Lie groupoids with coefficients on representations \[10\], Lemma \[3.4\] may be interpreted as the fact that $R'_K$ is “exact”, or “cohomologically trivial”, while the weaker condition \[3.11\], that guarantees multiplicity, corresponds to “closedness” (see \[10\] Sec. 2.1 and Sec. 3.4).
4. The Frölicher-Nijenhuis Bracket

Let $N$ be a manifold and $\Omega^*(N)$ be its graded algebra of differential forms. A degree $l$ derivation of $\Omega^*(N)$ is a linear map $D : \Omega^*(N) \to \Omega^{*+l}(N)$ such that $D(\alpha \wedge \beta) = D(\alpha) \wedge \beta + (-1)^p \alpha \wedge D(\beta)$, for $\alpha \in \Omega^p(N)$. Any vector-valued form $K \in \Omega^k(N,TN)$ gives rise to a degree $(k-1)$ derivation of $\Omega^*(N)$ by

$$i_K \omega(X_1, \ldots, X_{k+p-1}) = \frac{1}{k!(p-1)!} \sum_{\sigma \in S_{k+p-1}} \text{sgn}(\sigma) \omega(K(X_{\sigma(1)}, \ldots, X_{\sigma(k)}), X_{\sigma(k+1)}, \ldots, X_{\sigma(k+p-1)}),$$

for $\omega \in \Omega^p(N)$, $X_1, \ldots, X_{k+p-1} \in TN$. It also gives rise to a degree $k$ derivation of $\Omega^*(N)$ via

$$L_K = [d, i_K] = di_K - (-1)^{k-1}i_Kd,$$

where $d$ is the exterior differential on $N$.

Given $K \in \Omega^k(N,TN)$ and $L \in \Omega^l(N,TN)$, their Frölicher-Nijenhuis bracket is the vector-valued form $[K, L] \in \Omega^{k+l}(N,TN)$ uniquely defined by the condition

$$[K, L] = [L_K, L] = [L, L_K] - (-1)^{k+l}L_L L_K.$$

When $K$ and $L$ have degree zero (i.e., they are vector fields on $N$), (1.1) agrees with the usual Lie bracket of vector fields. The Frölicher-Nijenhuis bracket makes $\Omega^*(N,TN)$ into a graded Lie algebra, and it satisfies the following additional properties (see e.g. [7, Ch. 2]):

(a) For $K \in \Omega^1(N,TN)$,

$$\frac{1}{2} [K, K] = N_K,$$

where $N_K$ is the Nijenhuis tensor of $K$,

$$N_K(X,Y) = [K(X), K(Y)] - K([KX, Y] + [KY, X]) + K^2[X,Y],$$

for $X,Y \in TN$.

(b) When $K \in \Omega^1(N,TN)$ is a projection, then

$$\frac{1}{2} [K, K] = R_K + \overline{R}_K,$$

where $R_K$ is its curvature and $\overline{R}_K$ is its co-curvature.

(c) Let $f : N_1 \to N_2$ be a smooth map, and $K_i \in \Omega^k(N_i,TN_i)$, $L_i \in \Omega^l(N_i,TN_i)$, $i = 1, 2$, be such that $K_1$ is $f$-related to $K_2$ and $L_1$ is $f$-related to $L_2$. Then $[K_1,L_1]$ is $f$-related to $[K_2,L_2]$.

Regarding property (c), recall that $K_1 \in \Omega^k(N_1,TN_1)$ is $f$-related to $K_2 \in \Omega^k(N_2,TN_2)$ if

$$K_2(Tf(X_1), \ldots, Tf(X_k)) = Tf(K_1(X_1, \ldots, X_k)),$$

for all $X_1, \ldots, X_k \in T_xN$, and $x \in N$. Alternatively, $K_1$ and $K_2$ are $f$-related if and only if

$$L_{K_1} \circ f^* = f^* \circ L_{K_2}$$

where $f^* : \Omega(N_2) \to \Omega(N_1)$ is the pull-back of differential forms. We refer to the property in (c) above as the naturality of the Frölicher-Nijenhuis bracket.
4.1. The bracket on Lie groupoids. We now verify that the Frölicher-Nijenhuis bracket on a Lie groupoid \( G \rightrightarrows M \) preserves multiplicative vector-valued forms.

We start by giving an alternative characterization of multiplicative vector-valued forms. We say that \( K \in \Omega^k(G, TG) \) is \((s, t)\)-projectable if there exists \( K_M \in \Omega^k(M, TM) \) such that \( K \) is both \( s \) and \( t \)-related to \( K_M \).

Any \( K \in \Omega^k(G, TG) \) gives rise to a vector valued \( k \)-form \( K \times K \) on \( G \times G \) given by

\[
K \times K((X_1, Y_1), \ldots, (X_k, Y_k)) = (K(X_1, \ldots, X_k), K(Y_1, \ldots, Y_k)),
\]

for \( X_1, \ldots, X_k \in T_gG \) and \( Y_1, \ldots, Y_k \in T_hG \); this form is uniquely characterized by the fact that it is both \( pr_1 \) and \( pr_2 \)-related to \( K \), where \( pr_1, pr_2 : G \times G \rightarrow G \) are the natural projections.

**Lemma 4.1.** If \( K \) is \((s, t)\)-projectable, then \( K \times K \) restricts to a vector valued \( k \)-form \( K^{(2)} \) on the space of composable arrows \( G^{(2)} \). Moreover, \( K \) is multiplicative if and only if \( K \) is \((s, t)\)-projectable and \( K^{(2)} \) is \( m \)-related to \( K \).

**Proof.** A direct computation shows that \( K \times K \) restricts to \( G^{(2)} \) when \( K \) is \((s, t)\)-projectable.

Recall that \( K \) is multiplicative if and only if there exists \( K_M \in \Omega^k(M, TM) \) such that \([K, L]_1\) is a groupoid morphism. The existence of \( K_M \) is equivalent to \( K \) being \((s, t)\)-projectable, whereas the identity

\[
K(Tm(X_1, Y_1), \ldots, Tm(X_k, Y_k)) = Tm(K(X_1, \ldots, X_k), K(Y_1, \ldots, Y_k)) = Tm(K^{(2)}((X_1, Y_1), \ldots, (X_k, Y_k))),
\]

for \( (X_1, Y_1), \ldots, (X_k, Y_k) \in T_{(g,h)}G^{(2)} \), shows that \( K \) intertwines the multiplication if and only if \( K \) and \( K^{(2)} \) are \( m \)-related. \( \square \)

We also need the following observation:

**Lemma 4.2.** If \( K \) and \( L \) are \((s, t)\)-projectable, then

\[
[K, L]^{(2)} = [K^{(2)}, L^{(2)}].
\]

**Proof.** Since \( K^{(2)} \) (resp. \( L^{(2)} \)) is \( pr_1 \) and \( pr_2 \)-related to \( K \) (resp. \( L \)), it follows from the naturality of the Frölicher-Nijenhuis bracket that \([K^{(2)}, L^{(2)}] \) is both \( pr_1 \) and \( pr_2 \)-related to \([K, L] \). Since \([K, L]^{(2)} \) is the unique vector-valued form satisfying this property, it follows that \([K, L]^{(2)} = [K^{(2)}, L^{(2)}] \). \( \square \)

**Theorem 4.3.** Let \( K \in \Omega^k(G, TG) \) and \( L \in \Omega^l(G, TG) \) be multiplicative. Then \([K, L] \) is multiplicative.

**Proof.** By naturality, \([K, L] \) is \( s \) and \( t \)-related to \([K_M, L_M] \). So \([K, L] \) is \((s, t)\)-projectable. Similarly, since \( K^{(2)} \) (resp. \( L^{(2)} \)) and \( K \) (resp. \( L \)) are \( m \)-related, it follows that \([K^{(2)}, L^{(2)}] \) and \([K, L] \) are \( m \)-related. By Lemma 4.2 \([K, L]^{(2)} \) and \([K, L] \) are \( m \)-related. The result now follows from Lemma 4.1. \( \square \)

**Corollary 4.4.** The following holds:

(a) If \( K \in \Omega^1(G, TG) \) is multiplicative, then its Nijenhuis tensor \( N_K \in \Omega^2(G, TG) \) is multiplicative.

(b) If \( K \in \Omega^1(G, TG) \) is a multiplicative projection, then its curvature \( R_K \in \Omega^2(G, TG) \) is multiplicative.
we provide an alternative characterization of multiplicative vector-valued forms lead-

\[ 4.2. \]

\[ i.e., \text{groupoid morphisms (and hence so is their composition).} \]

\[ \square \]

Note that part (a) recovers [18, Prop. 3.3]; Part (b) generalizes Prop. 3.10.

4.2. Relation with the Bott-Shulman-Stasheff complex. In this final section, we provide an alternative characterization of multiplicative vector-valued forms leading to another viewpoint to Thm. 4.3.

For a smooth manifold \( N \), as previously mentioned, the Fr"{o}licher-Nijenhuis bracket makes \( \Omega^\bullet(N, TN) \) into a graded Lie algebra. This is a consequence of fact that the map

\[ (4.4) \]

\[ K \mapsto \mathcal{L}_K \]

identifies vector-valued forms on \( N \) with derivations of \( \Omega^\bullet(N) \) commuting (always in the graded sense) with the exterior differential \([11]\), which is itself a graded Lie algebra with respect to commutators. The Fr"{o}licher-Nijenhuis bracket is the unique bracket on \( \Omega^\bullet(N, TN) \) for which \([14]\) is an isomorphism of graded Lie algebras (see \([11]\)). We will show that this result extends to multiplicative vector-valued forms on Lie groupoids.

For a Lie groupoid \( \mathcal{G} \rightrightarrows M \), let us consider the associated simplicial manifold \( N(\mathcal{G}) \), known as its nerve, defined as follows: for each \( p \in \mathbb{N} \), its \( p \) component is \( \mathcal{G}^{(p)} \), the string of \( p \) composable arrows (i.e., \((g_1, \ldots, g_p) \in \mathcal{G}^p \) satisfying \( s(g_{i+1}) = t(g_i) \)); its face maps are \( \partial^{p-1}_i : \mathcal{G}^p \to \mathcal{G}^{(p-1)}, \) \( i = 0, \ldots, p \), given by

\[ \partial^{p-1}_i (g_1, \ldots, g_p) = \begin{cases} (g_2, \ldots, g_p), & \text{if } i = 0, \\ (g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_p), & \text{if } 1 \leq i \leq p - 1, \\ (g_1, \ldots, g_{p-1}), & \text{if } i = p, \end{cases} \]

for \( p \geq 1 \), and \( \partial^0_0 = s, \partial^0_1 = t, \) for \( p = 1 \); the degeneracy maps \( s^p_i : \mathcal{G}^{(p-1)} \to \mathcal{G}^p, \) \( i = 0, \ldots, p - 1 \), are defined by

\[ s^p_i (g_1, \ldots, g_{p-1}) = (g_1, \ldots, g_i, 1_{t(g_i+1)}, g_{i+1}, \ldots, g_{p-1}) \]

\[ = (g_1, \ldots, g_i, 1_{s(g_i)}, g_{i+1}, \ldots, g_{p-1}). \]

For convenience, we recall the identities relating the face and degeneracy maps:

\[ (4.5) \]

\[ \partial^{p-1}_j \circ s^p_i = \begin{cases} s^{p-1}_{i-1} \circ \partial^{p-2}_j, & \text{if } j < i, \\ \text{Id}_{\mathcal{G}^{(p-1)}}, & \text{if } j = i, i + 1, \\ s^{p-1}_i \circ \partial^{p-2}_{i+1}, & \text{if } j > i. \end{cases} \]

We consider the associated double complex \( \Omega^\bullet(\mathcal{G}^{\bullet}) \), referred to as the Bott-Shulman-Stasheff complex, with differentials given by the exterior derivative \( d : \Omega^\bullet(\mathcal{G}^{\bullet}) \to \Omega^{\bullet+1}(\mathcal{G}^{\bullet}) \) and by

\[ \delta : \Omega^\bullet(\mathcal{G}^{(p-1)}) \to \Omega^\bullet(\mathcal{G}^{(p)}), \quad \delta = \sum_{i=0}^{p} (-1)^i (\partial^p_i)^*, \]

where \( \partial^p_i \) are the face maps and \( (\partial^p_i)^* \) are the dual face maps.
and whose total cohomology (also known as the de Rham cohomology of $G$) agrees with the cohomology of the geometric realization of $N(G)$ \[^2\] (see also \[^1\] and references therein).

A degree $l$ derivation of $\Omega^\bullet(G^{\bullet})$ is a sequence $D = (D_0, D_1, \ldots)$, where each $D_p$ is a degree $l$-derivation of $\Omega^\bullet(G^{(p)})$ and

\begin{equation}
(s^p_i)^* \circ D_p = D_{p-1} \circ (s^p_i)^*, \tag{4.6}
\end{equation}

for all $p$ and $i = 0, \ldots, p - 1$. The componentwise commutator turns the space of derivations of $\Omega^\bullet(G^{\bullet})$ into a graded Lie algebra. The subspace of derivations of $\Omega^\bullet(G^{\bullet})$ commuting with its total differential is a graded Lie subalgebra.

**Proposition 4.5.** There is a (graded) linear isomorphism between the space of multiplicative vector-valued forms on $G$ and the space of derivations of $\Omega^\bullet(G^{\bullet})$ commuting with the total differential. Explicitly, the map taking a multiplicative $K \in \Omega^k(G, TG)$ to a degree $k$ derivation is given by

\begin{equation}
K \mapsto (\mathcal{L}_{K_0}, \mathcal{L}_{K_1}, \ldots, \mathcal{L}_{K^{(p)}}, \ldots), \tag{4.7}
\end{equation}

where $K^{(p)}$ is the restriction of $(K \times \cdots \times K) \in \Omega^k(G^{p}, TG^p)$ to $G^{(p)}$.

For the Lie groupoid $N \Rightarrow N$, with $s = t = id_N$, Proposition 4.5 boils down to the correspondence (c.f. \[^2\]) between vector-valued forms on $N$ and derivations of $\Omega^\bullet(N)$ commuting with $d$. Note also that Thm. 4.3 is a consequence of Prop. 4.5: the map \[^4\] induces a graded Lie bracket on multiplicative vector-valued forms on $G$ which is nothing but the restriction of the Frölicher-Nijenhuis bracket.

**Proof.** For a multiplicative $K \in \Omega^k(G, TG)$, one may directly verify that the restrictions $K^{(p)}$ are well defined (see Lemma 4.1), and that the right-hand side of (4.7) satisfies (4.6) and commutes with the total differential.

Let $D$ be a degree-$k$ derivation of $\Omega^\bullet(G^{\bullet})$. The fact that $D$ commutes with the total differential gives the conditions

\begin{align}
D_p \circ \delta &= \delta \circ D_{p-1} \tag{4.8} \\
D_p \circ d &= (-1)^k d \circ D_p \tag{4.9}
\end{align}

for each $p$. Condition (4.9) is simply that $[D_p, d] = 0$, so it implies that there exists $K_p \in \Omega^k(G^{(p)})$ such that $D_p = \mathcal{L}_{K_p}$ (see \[^1\] Sec. 8.5). So we are left with proving that $K = K_1$ is multiplicative, covers $K_M = K_0$, and $K_p = K^{(p)}$.

From (4.6) we see that $K_p$ and $K_{p-1}$ are $s^p_i$-related, for $i = 0, \ldots, p - 1$. In particular, $K$ and $K_M$ are $\epsilon$-related.

When $p = 1$, \[^4\] and the fact that $K_2$ and $K$ are $s^2_0$-related imply that

\begin{equation}
\mathcal{L}_{K} \circ (s^2_0)^* \circ \delta = (s^2_0)^* \circ \mathcal{L}_{K_2} \circ \delta = (-1)^k (s^2_0)^* \circ \delta \circ \mathcal{L}_{K}. \tag{4.10}
\end{equation}

Since $(s^2_0)^* \circ \delta = (\epsilon \circ t)^*$, from (4.5) one gets

\[ \mathcal{L}_{K} \circ t^* \circ \epsilon^* = t^* \circ \epsilon^* \circ \mathcal{L}_{K} = t^* \circ \mathcal{L}_{K_M} \circ \epsilon^*. \]

The fact that $\epsilon$ is an immersion implies that $\mathcal{L}_{K} \circ t^* = t^* \circ \mathcal{L}_{K_M}$, thus proving that $K$ and $K_M$ are $t$-related. To obtain the analogous result for the source map, it suffices to apply $(s^2_1)^*$ to (4.8). This proves that $K$ covers $K_M$. 

To verify the compatibility of $K$ with the multiplication on $\mathcal{G}$, one needs to work at the level $p = 2$. In this case, (4.8) reads
\begin{equation}
\mathcal{L}_{K_3} \circ \delta = \delta \circ \mathcal{L}_{K_2}.
\end{equation}

By applying $(s_0^3)^*$ to (4.11) and using that $K_3$ and $K_2$ are $s_0^3$-related, one gets
\[ \mathcal{L}_{K_2} \circ (s_0^3)^* \circ \delta = (s_0^3)^* \circ \delta \circ \mathcal{L}_{K_2}. \]

The identities (4.5) imply that $(s_0^3)^* \circ \delta = ((\partial_0^1)^* - \delta) \circ (s_0^3)^*$. Hence
\[ \mathcal{L}_{K_2} \circ (\partial_0^1)^* \circ (s_0^3)^* - \mathcal{L}_{K_2} \circ \delta \circ (s_0^3)^* = (\partial_0^1)^* \circ (s_0^3)^* \circ \mathcal{L}_{K_2} - \delta \circ (s_0^3)^* \circ \mathcal{L}_{K_2} \]
\[ = (\partial_0^1)^* \circ \mathcal{L}_{K} \circ (s_0^3)^* - \delta \circ \mathcal{L}_{K} \circ (s_0^3)^*, \]

which implies that $\mathcal{L}_{K_2(2)} \circ (\partial_0^1)^* \circ (s_0^3)^* = (\partial_0^1)^* \circ \mathcal{L}_{K_2} \circ (s_0^3)^*$. Since $s_0^3$ is an immersion, we conclude that $K_2$ and $K$ are $\partial_2^1$-related. Arguing similarly with $s_2^3$, one obtains that $K_2$ and $K$ are $\partial_2^1$-related. These two facts imply that $K_2 = K^{(2)} = (K \times K)|_{\mathcal{G}(2)}$. Moreover, as $m = \partial_1^3$, one has that
\[ m^* \circ \mathcal{L}_{K} = (\partial_0^1)^* + (\partial_2^1)^* - \delta) \circ \mathcal{L}_{K} = \mathcal{L}_{K^{(2)}} \circ (\partial_0^1)^* + (\partial_2^1)^* - \delta) = \mathcal{L}_{K^{(2)}} \circ m^*, \]

which proves that $K^{(2)}$ and $K$ are $m$-related. This concludes the proof that $K$ is multiplicative.

To prove that $K_p = K^{(p)}$, one proceeds by induction. Assume that $K_{p-1} = K^{(p-1)}$ and use the identity $(s_0^{p+1})^* \circ \delta = ((\partial_0^{p-1})^* - \delta) \circ (s_0^p)^*$ to prove as above that $K_p$ and $K^{(p-1)}$ are $\partial_0^{p-1}$-related. Arguing similarly with $s_0^{p+1}$ proves that $K_p$ and $K^{(p-1)}$ are $\partial_0^{p-1}$-related, which implies that $K_p = K^{(p)}$.

Remark 4.6. Along the lines of the proof of Proposition 4.5, one can prove that the relations
\begin{align*}
\{ & (s_0^i)^* \circ \mathcal{L}_{K_p} = \mathcal{L}_{K_{p-1}} \circ (s_0^i)^*, \quad i = 0, \ldots, p - 1, \\
& \mathcal{L}_{K_p} \circ \delta = \delta \circ \mathcal{L}_{K_{p-1}} \}
\end{align*}

are equivalent to each $K_p$ being both $s_0^i$- and $\partial_2^{p-1}$-related to $K_{p-1}$, for $i = 0, \ldots, p - 1$, and $j = 0, \ldots, p$. When $K$ is a vector field, this indicates that $(K_M, K, K^{(2)}, \ldots)$ may be thought of as a “vector field” on the simplicial manifold $N(\mathcal{G})$, in the sense that it defines a section of the natural projection $T(N \mathcal{G}) \to N(\mathcal{G})$ (where $T(N \mathcal{G})$ is the simplicial manifold obtained by taking the tangent functor on each component of $N(\mathcal{G})$). We refer to [13] for a related approach to vector fields on differentiable stacks. It would be interesting to extend this picture to higher degrees.

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