The relation between the turbulent Mach number and observed fractal dimensions of turbulent clouds

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ABSTRACT
Supersonic turbulence is a key player in controlling the structure and star formation potential of molecular clouds (MCs). The three-dimensional (3D) turbulent Mach number, $M$, allows us to predict the rate of star formation. However, determining Mach numbers in observations is challenging because it requires accurate measurements of the velocity dispersion. Moreover, observations are limited to two-dimensional (2D) projections of the MCs and velocity information can usually only be obtained for the line-of-sight component. Here we present a new method that allows us to estimate $M$ from the 2D column density, $\Sigma$, by analysing the fractal dimension, $D$. We do this by computing $D$ for six simulations, ranging between 1 and 100 in $M$. From this data we are able to construct an empirical relation, $\log M (D) = \xi_1 \left( \text{erfc}^{-1} \left( \frac{D - D_{\text{min}}}{\Omega} \right) + \xi_2 \right)$, where $\text{erfc}^{-1}$ is the inverse complimentary error function, $D_{\text{min}} = 1.55 \pm 0.13$ is the minimum fractal dimension of $\Sigma$, $\Omega = 0.22 \pm 0.07$, $\xi_1 = 0.9 \pm 0.1$ and $\xi_2 = 0.2 \pm 0.2$. We test the accuracy of this new relation on column density maps from Herschel observations of two quiescent subregions in the Polaris Flare MC, ‘saxophone’ and ‘quiet’. We measure $M \sim 10$ and $M \sim 2$ for the subregions, respectively, which is similar to previous estimates based on measuring the velocity dispersion from molecular line data. These results show that this new empirical relation can provide useful estimates of the cloud kinematics, solely based upon the geometry from the column density of the cloud.

Key words: hydrodynamics – turbulence – ISM: clouds – ISM: kinematics and dynamics – ISM: structure – methods: observational

1 INTRODUCTION
The dynamical evolution of molecular clouds (MCs) in the interstellar medium (ISM) is determined by supersonic, compressible turbulent flows (Larson 1981; Solomon et al. 1987; Klessen et al. 2000; Heitsch et al. 2001; Ossenkopf & Mac Low 2002; Elmegreen & Scalo 2004; Heyer & Brunt 2004; Mac Low & Klessen 2004; Scalo & Elmegreen 2004; Krumholz & McKee 2005; Ballesteros-Paredes et al. 2007; McKee & Ostriker 2007; Roman-Duval et al. 2011; Padoan et al. 2014; Federrath & Banerjee 2015). The turbulent dynamics of the clouds plays a diverse and vital role in the star formation process by providing support against collapse, giving rise to distinct statistical properties which are used in star formation models, and by providing high density, filamentary structures where star-forming cores are preferentially located (Scalo 1998; Ferrière 2001; Mac Low & Klessen 2004; Kainulainen et al. 2009; Arzoumanian et al. 2011; Federrath & Klessen 2012; Schneider et al. 2012; André et al. 2014; Konstandin et al. 2016; Könyves et al. 2015; Federrath 2016; Hacar et al. 2018; Mocz & Burkhardt 2018; Arzoumanian et al. 2019). Understanding the structure, kinematics and the statistics (density and velocity dispersions, for example) of the MCs has therefore been of interest. The aim of this study is to extend upon our recent effort in Beattie et al. (2019), herein called BFK19, to tie the fractal dimension, $D$, to the physical properties of the MCs, e.g. to cloud length scales, and to the relations on observational data, e.g. between 2D cloud projections and 3D cloud data.

In this study we present a new method for calculating the turbulent Mach number of the clouds, based purely upon two-dimensional (2D) projected cloud position-position (PP) data, i.e., the column density, $\Sigma$, which can

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be obtained by molecular lines, dust emission, or dust extinction observations. We also provide the first tests of the fractal dimension methods introduced by BFK19 using dust column density maps obtained from Herschel flux maps of the Polaris Flare. First, we will discuss the importance of the turbulent Mach number, and the diverse role it plays in star formation.

1.1 The Cloud Density and Turbulent Mach Number

The turbulent Mach number, $\mathcal{M}$, is a key ingredient for numerous star formation models (Krumholz & McKee 2005; Federrath et al. 2010; Hennebelle et al. 2011; Federrath & Klessen 2012; Federrath 2013; Konstandin et al. 2016). We make the distinction between the scale-dependent turbulent Mach number,

$$\mathcal{M}(\ell) = \sigma_v(\ell)/c_s, \quad (1)$$

where $\sigma_v(\ell)$ is the velocity dispersion of the cloud on length scale $\ell$, and $c_s$ is the sound speed, and the root-mean-squared (rms) Mach number,

$$\mathcal{M} \approx \sigma_v(L)/c_s = \mathcal{M}(L), \quad (2)$$

where $L$ is the cloud diameter, which corresponds to the outer scale of turbulence in our study (Federrath 2013). For an isothermal cloud with purely turbulent dynamics, $\mathcal{M}$ sets the width of the log-normal cloud density distribution,

$$\sigma_s^2 = \ln (1 + b^2 \mathcal{M}^2), \quad (3)$$

where the $s$ subscript denotes the variance of the normalised cloud density, $s = \ln(p/p_0)$, where $p_0$ is the mean density of the cloud and $b$ is the turbulent forcing parameter (Padoan et al. 1997; Passot & Vázquez-Semadeni 1998; Kritsuk et al. 2007; Federrath et al. 2008; Federrath et al. 2010; Konstandin et al. 2012b). The dispersion has been studied extensively and there have been many modifications to account for 2D projections of the 3D cloud (Burkhart & Lazarian 2012), thermal and magnetic pressures (Padoan et al. 1997; Passot & Vázquez-Semadeni 1998; Federrath et al. 2008; Price et al. 2011; Molina et al. 2012; Gazol & Kim 2013), and non-isothermal (Nolan et al. 2015) and polytropic gases (Passot & Vázquez-Semadeni 1998; Li et al. 2002; Federrath & Banerjee 2015). Calculating the density dispersion is important for star formation models that predict the star formation rate (SFR) directly from integrating the density and free-fall time weighted cloud density distribution to determine the mass fraction of the cloud that could collapse into new stars (Krumholz & McKee 2005; Padoan & Nordlund 2011; Hennebelle et al. 2011; Federrath & Klessen 2012; Kainulainen et al. 2014). Beyond the density distribution, the turbulent Mach number may also play an important role in the distribution of cloud filament widths.

1.2 Filaments and the Turbulent Mach Number

Filament structures have been observed in star-forming and quiescent clouds, and may play an important role in star formation, since star clusters and star-forming cores have been found to be preferentially located in them (André et al. 2010; Men'shchikov et al. 2010; Schneider et al. 2012; André et al. 2014; Padoan et al. 2014; Arzoumanian 2015; Federrath 2016). Interstellar filament widths are distributed with a peak at $\sim 0.1$ pc, which seems to be a universal feature of filaments and has been found in both observations and simulations of star-forming clouds (Arzoumanian et al. 2011; Juvela et al. 2012; Palmeirim et al. 2013; André et al. 2014; Smith et al. 2014; Benedettini et al. 2015; Kirk et al. 2015; Federrath 2016; Federrath et al. 2016; Smith et al. 2016; Arzoumanian et al. 2019). The standard deviation of the filament width distribution is thought to be associated with the sonic scale, $\ell_s$, in the clouds (Federrath et al. 2018). The sonic scale marks the transition between supersonic and subsonic velocity dispersions, and is theorised to be at length scale,

$$\ell_s = L \left[ \frac{1}{\mathcal{M}} (1 + \beta)^{1/2} \right]^{1/p}, \quad (4)$$

in the cloud, where $p \approx 1/2$ has been measured using both Galactic cloud observations and simulations, and $\beta = p_{\text{thermal}}/p_{\text{magnetic}}$ is the ratio between the thermal and magnetic pressures at the cloud diameter scale $L$ (Larson 1981; Solomon et al. 1987; Ossenkopf & Mac Low 2002; Heyer & Brunt 2004; Kritsuk et al. 2007; Schmidt et al. 2009; Federrath et al. 2010; Roman-Duval et al. 2011; Federrath & Klessen 2012; Federrath 2016; Federrath et al. 2018). Being able to measure the turbulent Mach number is thus essential for testing theories about the filament width distribution. The turbulent driving that the clouds undergo may lead to the formation of filaments through interacting planar shocks (Federrath 2016; Tokuda et al. 2018). However these are not the only structures that are formed and the densities of turbulent clouds have been shown to respect a fractal geometry.

1.3 Fractal Cloud Structures

Molecular clouds have a highly complex structure which includes sheets, filaments and dense cores. Observations of MCs through CO lines, dust emission as well as dust extinction show that they are organised into self-similar fractal structures, with substructures of clouds being continuously resolved, even at the highest spatial resolution achievable (Falgarone & Phillips 1996; Stutzki et al. 1998; Chappell & Scalo 2001; Kauffmann et al. 2010; Schneider et al. 2013; Kainulainen et al. 2014; Rathborne et al. 2015). There is a strong agreement between simulations and observations that the three-dimensional (3D) fractal dimension, i.e. the fractal dimension of the position-position-position (PPP) data of turbulent MCs falls between $\approx 2$ and $3$ (Scalo 1990; Elmegreen & Falgarone 1996; Sanchez et al. 2005; Kowal & Lazarian 2007; Federrath et al. 2009; Roman-Duval et al. 2010; Donovan Meyer et al. 2013; Konstandin et al. 2016; BFK19). However, where it falls between 2 and 3 depends upon the type of turbulent driving (Federrath et al. 2009), the rms $\mathcal{M}$ (Konstandin et al. 2016; BFK19), the length scales in the clouds, and the amount of shocks and filamentary structures in the cloud (BFK19). BFK19 also found that the fractal dimension is significantly higher in the column density map ($\approx 1.6$ in the high $\mathcal{M}$ limit) compared to 2D density slices ($\approx 1$ in the high $\mathcal{M}$ limit). In this study we show how by expanding upon the methods outlined in BFK19 one can utilise the fractal structure of the column density from the cloud, specifically the
For each of the simulations we solve the compressible Euler equations, \( \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \)
\( \nabla \cdot (\mathbf{v} \mathbf{v}) = -\frac{1}{\rho} \nabla P + \mathbf{F}, \)
\[ \text{where } \rho \text{ is the density, } \mathbf{v} \text{ the velocity, } P \text{ the pressure, following an isothermal equation of state, } P = c_s^2 \rho, \text{ where } c_s \text{ is the speed of sound, and } \mathbf{F} \text{ is an Ornstein-Uhlenbeck (OU) forcing function that drives the turbulence through a mixture of solenoidal } (\nabla \cdot \mathbf{F} = 0) \text{ and compressive } (\nabla \times \mathbf{F} = 0) \text{ modes.} \]

1. TURBULENT MOLECULAR CLOUD MODELS

In this study we use six purely hydrodynamical simulations of quiescent (non-star-forming) molecular clouds, with no self-gravitating, to construct our \( M - D \) relation. The parameter set of the molecular cloud models is listed in Table 1. For each of the simulations we solve the compressible Euler equations, for example, Federrath et al. (2009, 2010). This lets us perform the Euler equations in a cube frame of reference, which is discussed in detail in Beattie et al. (2019). We solve the Euler equations in a cube frame of reference, which is discussed in detail in Beattie et al. (2019).

Table 1. Simulation parameters.

| \( M \) (±σ) | Native Simulation Grid Resolution | Number of Time Slices | Time Interval |
|---|---|---|---|
| 1.01 ± 0.05 | 1024 \(^3\) | 71 | \( 2 \leq t/T \leq 9 \) |
| 4.1 ± 0.2 | 1006 \(^3\) | 71 | \( 2 \leq t/T \leq 9 \) |
| 10.2 ± 0.5 | 1024 \(^3\) | 71 | \( 2 \leq t/T \leq 9 \) |
| 20 ± 1 | 1024 \(^3\) | 71 | \( 2 \leq t/T \leq 9 \) |
| 40 ± 2 | 1024 \(^3\) | 71 | \( 2 \leq t/T \leq 9 \) |
| 100 ± 5 | 1024 \(^3\) | 71 | \( 2 \leq t/T \leq 9 \) |

Notes: Column (1): the rms turbulent Mach number of the simulation ± the 1σ temporal fluctuations. Column (2): the native 3D grid resolution of each of the simulations. Column (3): the number of time slices that we use for temporal averaging. Column (4): the time interval in units of large-scale turnover times.

From 2D projections to 3D turbulent Mach number

| \( D \) min ±1σ | \( D \) max | \( \beta_0 \) ±1σ | \( \beta_1 \) ±1σ | \( M \) ±1σ |
|---|---|---|---|---|
| 1.55 ± 0.13 | 2 | 0.46 ± 0.08 | 0.56 ± 0.08 | 4.1 ± 0.2 |

Notes: \( \beta_0, \beta_1 \) and \( D \) min are calculated in Beattie et al. (2019) using weighted non-linear regression. Column (1): \( D \) min is the minimum fractal dimension of the column density. Column (2): \( D \) max is assumed to be 2 for the maximum fractal dimension of the column density. This corresponds to completely space-filling flows on the 2D plane. Column (3): \( \beta_0 \) is a fitting parameter that corresponds to the translation of the complimentary error function over the \( t/L \) axis. Column (4): \( \beta_1 \) is a fitting parameter that corresponds to the rate in which the complimentary error function changes between the high D and low D states. Column (5): The rms Mach number, \( M \) is measured by averaging over all rms Mach numbers from \( 2 \leq t/T \leq 9 \) in the \( M = 4 \) simulation. The 1σ uncertainty is the standard deviation of the averaging process. We use \( M = 4 \) because the curves are in the Mach 4 simulation frame of reference, which is discussed in detail in Beattie et al. (2019).

In this study we use the column density data, \( \Sigma \), integrated along the z-axis. Figure 1 shows a single snapshot in time, \( t \approx 2T \) of the column density in each simulation. In the absence of magnetic fields our turbulence simulations are isotropic in a statistical sense, i.e. when averaged over time, e.g., Federrath et al. (2009, 2010). This lets us perform our study only on the xy projections (column densities), whilst still being representative 2D projections through any viewing angle.

3 FRACTAL DIMENSION CURVES

We follow the mass-length fractal dimension method outlined and discussed with detail in BFK19. This method allows us to calculate a mass-length \( D \) on each length scale in the cloud. It is important that we are able to access \( D \) on each \( \ell \), since our aim is to relate \( D \) with \( M \) through \( \ell \). We provide a summary of the method, and the key results below. Please note in BFK19 we use \( D_p \) to indicate the 2D projected (column density) fractal dimension but in this study we use \( D \) for simplicity.

3.1 Method Summary

There exists a power-law scaling between the mass and length scales (size) in real MCs (Larson 1981; Myers 1983; Falgarone & Phillips 1996; Roman-Duval et al. 2010; Donovan Meyer et al. 2013). To calculate the mass-length dimen-
Figure 1. The column densities, $\Sigma$, for the six simulations that we use to study the dependence of the turbulent Mach number, $M$, on the fractal dimension, $D$. Indicated in white is the root-mean-squared (rms) $M$ of the simulation, ranging from the transonic $M = 1$ clouds to the highly supersonic $M = 100$ clouds. All column densities are shown at $t = 2T$, in the regime of fully-developed turbulence. The densities are shown in units of log average column density, $\Sigma_0$. An animation of the time evolution for the 2D projections (and 2D slices) is available in the online version of Beattie et al. (2019).
Of proportionality is \( D = 2 \) in the 2D projection) and clouds saturated with planar shocks \((\mathcal{D} = 1.55 \pm 0.13, \text{which is a key result from BFK19})\). A simple power-law relation is fit for high fractal dimensions and clouds driven at high \( \mathcal{M} \) limits, corresponding to a transition between space-filling and shock-dominated geometries in the cloud.

We apply the four steps above on each of the 71 time slices in the interval \( 2 \leq \tau / T \leq 9 \), the statistically fully-developed turbulence regime, averaging over them to construct a single curve for each simulation with 1\(\sigma\) uncertainties.

### 3.2 Key Results from the \( \mathcal{D}(\ell) \) Curves

In BFK19 we show that the fractal dimension curves from each simulation (shown in Figure 2) can be combined together into the same common reference frame to create a composite fractal dimension curve. This lets us map the fractal dimension of \( \Sigma \) over seven orders of magnitude in spatial scales, encompassing clouds undergoing subsonic to highly supersonic turbulent dynamics. After combining the curves we find that a complimentary error function is a good fit for \( \mathcal{D}(\ell/L) \), which models a smooth transition between space-filling clouds \((\mathcal{D} = 2 \text{ in the 2D projection})\) and clouds saturated with planar shocks \((\mathcal{D} = 1.55 \pm 0.13, \text{which is a key result from BFK19})\). A simple power-law relation is not sufficient, because at both low and high rms Mach flows
Figure 3. Large panel: The turbulent Mach number as a function of fractal dimension, fit on two decades of Mach number data ($M = 1 - 100$). The different colours correspond to each of the six simulations, with rms Mach number indicated in the legend. The fit (Equation 16) is shown in black, with 1σ uncertainties for the fit shown as dashed lines. Small (inset) panel: The fractal dimension as a function of length scale for the column density. This is a key result from Beattie et al. (2019), where it is discussed in detail. The black line is the fit shown in Equation 12. We use the same data, but transform the length scales into Mach numbers using a power-law scaling based on supersonic turbulence theory, $\ell \sim M^2$, to create the $M - D$ relation (Burgers 1948; Konstandin et al. 2012a; Federrath 2013).

The fractal dimension curves begin to flatten out as they approach the high and low limits. The empirical fit is,

$$D(\ell/L) = \frac{D_{\text{max}} - D_{\text{min}}}{2} \text{erfc} \left( \beta_1 \log (\ell/L) + \beta_0 \right) + D_{\text{min}},$$  \hspace{1cm} (12)

where erfc is the complimentary error function, $D_{\text{min}}$ and $D_{\text{max}}$ are the minimum and maximum fractal dimension, respectively, and $\beta_1$ and $\beta_0$ are fitting parameters, determined using nonlinear least squares and tabulated in Table 2. This fit encodes the limits $\lim_{(\ell/L)\to1} D(\ell/L) = D_{\text{min}}$ and $\lim_{(\ell/L)\to0} D(\ell/L) = D_{\text{max}}$ that we find in the data, and encompasses the smooth transition that we find between them. The composite curve data, along with the complimentary error function fit for $D(\ell/L)$ are shown in the sub-panel of Figure 3. The plot shows that the $D$ of the column density is bounded between $D_{\text{max}} = 2$ and $D_{\text{min}} = 1.55 \pm 0.13$, where the former is assumed, and the latter is measured through the fitting process. Next, we use this curve to construct the $M - D$ relation.

4 THE $M - D$ RELATION

After establishing the length scale dependence of the fractal dimension we may immediately ask how then does the fractal dimension change with the velocity dispersion of the cloud, since the velocity dispersion also depends upon length scale, as indicated in Equation 1. This link is made available to us
by approximating \( M(\ell) \) using scaling relations from models of supersonic turbulence.

### 4.1 Constructing the Relation

Using the second-order structure function, 
\[
\text{SF}_2(\ell/L) = \langle |v(\mathbf{r}) - v(\mathbf{r} + \ell/L)|^2 \rangle, \tag{13}
\]
where \( v(\mathbf{r}) \) is the velocity of the cloud at position \( \mathbf{r} \), and the operator \( \langle \ldots \rangle \) is an average over a large ensemble of spatial positions, one can construct the turbulent Mach number at length scale \( \ell/L \) (using the definition of the second-order structure function and Equation 4.3 from Konstandin et al. 2012a). This construction of the Mach number follows a power-law of the form,
\[
\mathcal{M}(\ell/L) = \sqrt{\text{SF}_2(\ell/L)/(2c_s^2)} \sim (\ell/L)^p, \tag{14}
\]
where \( p \approx 1/2 \) for supersonic turbulence and \( p \approx 1/3 \) for subsonic turbulence (Kolmogorov 1941; Burgers 1948; Kritsuk et al. 2007; Schmidt et al. 2009; Federrath et al. 2018). We use the \( p = 1/2 \) case, \( \ell/L \sim \mathcal{M}(\ell/L)^p \), to transform all length scales into Mach numbers. We set \( \ell = L \) to find the constant of proportionality, \( \mathcal{M}^{-2} \), i.e. on large scales in the cloud \( \mathcal{M} \) is \( \approx M \) (Federrath 2013). Hence the transformation is
\[
\ell/L = [\mathcal{M}(\ell/L)/M]^2. \tag{15}
\]

We apply this transformation to our \( \ell/L \) composite data shown in the sub-panel of Figure 3. This provides us with an estimate for \( D(M) \),
\[
D(M) = \frac{D_{\text{max}} - D_{\text{min}}}{2} \text{erfc} \left[ \frac{2\beta_1 \log \left( \frac{M}{M} \right) + \beta_0}{\Omega} \right] + D_{\text{min}}. \tag{16}
\]

We invert the equation to obtain \( M(D) \), which means that from measurements of the fractal dimension one can infer the scale-dependent turbulent Mach number,
\[
\log M(D) = \xi_1 \left( \text{erfc}^{-1} \left[ \frac{D - D_{\text{min}}}{\Omega} \right] + \xi_2 \right), \tag{17}
\]
where
\[
\Omega = \frac{D_{\text{max}} - D_{\text{min}}}{2}, \tag{18}
\]
and
\[
\xi_1 = (2\beta_1)^{-1} \quad \text{and} \quad \xi_2 = \xi_1^{-1} \log M - \beta_0. \tag{19}
\]

This corresponds to the turbulent Mach number on the length scale that \( D \) was measured, since \( M(D(\ell)) \). The values of the estimated and derived parameters for this fit are shown in Table 3.
Table 3. Derived parameter values for $\mathcal{M}(\mathcal{D})$, shown in Equation 16.

| $\mathcal{D}_{\text{min}} \pm 1 \sigma$ | $\Omega \pm 1 \sigma$ | $\xi_1 \pm 1 \sigma$ | $\xi_2 \pm 1 \sigma$ |
|---------------------------------------|-----------------------|-----------------------|-----------------------|
| $1.55 \pm 0.13$                      | $0.22 \pm 0.07$       | $0.9 \pm 0.1$         | $0.2 \pm 0.2$         |

Notes: The values for $\mathcal{D}_{\text{max}}, \beta_0, \beta_1$, and $\mathcal{M}$ can be found in Table 2, which are used to derive the parameters in this table. Column (1): the same as column one from Table 2, but included for completeness. Column (2): $\Omega = (\mathcal{D}_{\text{max}} - \mathcal{D}_{\text{min}})/2$. Column (3): $\xi_1 = (2\beta_1)^{-1}$. Column (4): $\xi_2 = \xi_1^{-1} \log \mathcal{M} - \beta_0$.

4.2 $\mathcal{M} - \mathcal{D}$ Relation Results

In the main plot of Figure 3 we show $\mathcal{M}$ as a function of $\mathcal{D}$, derived from the column density data. The black line shows Equation 16, which is the equation we will use to convert fractal dimensions into Mach numbers. There are three main limitations to this method. The first is that the inverse complimentary error function has steep tails. This means that for low and high $\mathcal{D}$, the relation is extremely sensitive to small changes in $\mathcal{D}$. The second is that for high $\mathcal{M}$ the temporal fluctuations of $\mathcal{D}$ become significant, spanning over $>0.2 \mathcal{D}$. This means that our relation will perform best at measuring regions of MCs with $10 \gtrsim \mathcal{M} \gtrsim 0.1$. Finally, since in our construction of the relation we use clouds driven by compressible, supersonic, isothermal and isotropic turbulence, with a natural mixture between solenoidal and compressive modes ($b \sim 0.4$), the fit shown in Figure 3 may only work well, without modification, on quiescent clouds, without significant deviation from natural mixing, since $\mathcal{D}$ is sensitive to changes in driving Federrath et al. (2009). Acknowledging these limitations, we now test the performance of the relation on Herschel observations of subregions from the Polaris Flare cloud.

5 APPLICATION ON QUIESCENT SUBREGIONS IN THE POLARIS FLARE CLOUD

The Polaris Flare is a high Galactic latitude cloud which is located at a distance $\lesssim 150$pc (Falgarone et al. 1998; Miville-Deschênes et al. 2010; Schlafly et al. 2014). It has weak, but significant CO emissions in regions of higher hydrogen column density (Falgarone et al. 1998; Meyer, Heithausen 1996; Miville-Deschênes et al. 2010). There is no active star-formation in Polaris, only 5 starless cores were detected (André et al. 2010; Ward-Thompson et al. 2010) that are most likely not gravitationally bound. Polaris is thus a perfect candidate for testing our new relation, which was calibrated upon quiescent cloud simulations.

5.1 Observational Data

The Polaris region was observed as part of the Herschel Gould Belt survey (HGBS, André et al. 2010) using the PACS and SPIRE instruments on-board Herschel. For all observational details, we refer to (André et al. 2010; Men'shchikov et al. 2010; Ward-Thompson et al. 2010; Miville-Deschênes et al. 2010). We employ publicly available level 3 data products produced with HIPE13 (Herschel Interactive Processing Environment) from the Herschel archive. The angular resolution of the maps is 11.7", 18.2", 24.9", and 36.3" for 160 $\mu$m (PACS) and 250, 350, and 500 $\mu$m (SPIRE), respectively. For an absolute calibration of the maps (included in the SPIRE level 3 data), the Planck High Frequency Instrument (HFI) observations were used for the HIPE-internal zero-point correction task that calculates the absolute offsets, based on cross-calibration with HFI-545 and HFI-857 maps, including colour-correcting HFI to SPIRE wavebands, assuming a grey-body function with fixed beta. For the PACS 160 $\mu$m map, we obtained the zero-point correction following the procedure outlined in Bernard et al. (2010). Column density and temperature maps were then produced at an angular resolution of 18", following the procedure outlined in Palmeirim et al. (2013) that employs a multi-scale decomposition of the imaging data and assumes a constant line-of-sight temperature. We performed a pixel-by-pixel SED (Spectral Energy Distribution) fit from 160 to 250$\mu$m, using a dust opacity law $\kappa_{\beta} = 0.1 \times (\nu / 1000 \text{GHz})^\beta$ cm$^2$ g$^{-1}$ with $\beta = 2$ and assuming a gas-to-dust ratio of 100. This dust opacity law is commonly adopted in other HGBS publications and we refer to André et al. (2010) and Könyves et al. (2015) for further details. We estimate that the final uncertainties of the column density map are around 20–30%.

The resulting large, high-angular resolution (18") hydrogen column density map traces structures between 0.01 to 8 pc (Miville-Deschênes et al. 2010; Schneider et al. 2013). For our study, we cut out the two subregions that have previously been used to investigate the link between the probability distribution function of column density (N-PDF) and $\mathcal{M}$ in Polaris (Schneider et al. 2013), the ‘saxophone’ and ‘quiet’ subregions, shown in Figure 4. Schneider et al. (2013) made estimates of the turbulent Mach number for the two regions based on the CO$^1$ velocity dispersion. The authors calculated the $\mathcal{M}$ for each subregion using,

$$\mathcal{M} = \left(\sqrt{3}\text{FWHM}\right) / \left(c_s \sqrt{8 \ln 2}\right) \quad (17)$$

where FWHM [km s$^{-1}$] is the full width at half maximum of the CO molecular line data, and under the LTE assumption the sound speed is $c_s \approx 0.188 \sqrt{T_{\text{ex}}/10K}$, where $T_{\text{ex}}$ is the excitation temperature, $T_{\text{ex}} = 5.53 [\text{ln}(5.33/T_{\text{CMB}}) + 1]^{-1}$ and $T_{\text{CMB}} = 2.728K$ is the temperature of the cosmic microwave background. The values for these estimates of $\mathcal{M}$ are shown in column (2) of Table 4.

5.2 Application and considerations

Using the method summarised in §3.1 we construct the fractal dimension curves for each of the two regions and then use Equation 16 to calculate the Mach number$^2$. The application differs in two ways compared to BFK19. (1) Here we use terminating boundary conditions, while on the simulation data

1 The CO data stem from $^{12}$CO 2–1 and $^{13}$CO 1–0 observations from the KOSMA and FCRAO telescopes published in Bensch et al. (2003).

2 Our fractal dimension implementation on the Polaris Flare cloud is available here: https://github.com/AstroJames/FractalGeometryofPolaris
we use periodic boundaries. Since the observational data is not periodic we terminate the \( l \times l \) expansion on the boundaries. (2) We calculate the monofractal mass-length dimension instead of our length-dependent curves. Our method reduces exactly to the monofractal \( D \) when \( D(l = L) \), where \( L \) is the largest scale of the \( l \times l \) expansion. This means that we use all nested scales \( l \leq L \) to calculate \( D \), as described in §3.1. We do this because it allows us to calculate \( M \), the turbulent Mach number on the cloud scale, which is useful for constraining the 3D density PDF, calculating the star formation rate and estimating the sonic scale in the cloud (as discussed in §1.1 and 1.2, respectively). It also allows for comparison with previous estimates made for \( M \) in the ‘quiet’ and ‘saxophone’ subregions of the Polaris Flare molecular cloud using Equation 17.

5.3 \( M(D) \) Results

In Figure 5 we show our \( M \) estimates for the two subregions of Polaris, indicated in red, and compare them with the \( M \) estimated by Schneider et al. (2013), shown as blue-dashed regions. The \( M \) calculated by Schneider et al. (2013) and the \( M \) calculated in this study are shown in Table 4. We find \( M = 2 \pm 1 \) and \( M = 10^{+6}_{-4} \); whereas Schneider et al. (2013) finds \( M = 3 \pm 1 \) and \( M = 7 \pm 3 \), for the ‘quiet’ and ‘saxophone’ subregions of Polaris, respectively. These estimates are consistent to within 1σ. However the Mach number measurements are very sensitive to small changes in \( D \), especially for \( M \gtrsim 10 \), where the relation becomes extremely steep. This translates into large, and not necessarily symmetric uncertainties, as shown for the ‘saxophone’ subregion, in Table 4. To understand why there may be differences between the \( M(D) \) and previous estimates based on the CO velocity dispersion we turn to the calculation of \( D \).

To estimate \( M \) we first calculate \( D \), which is shown in column (3) of Table 4. We find ‘quiet’, the lower \( M \) region, has a \( D = 1.76 \pm 0.05 \), and ‘saxophone’, the higher \( M \) region, \( D = 1.60 \pm 0.04 \). This is consistent with BFK19, who argues that for higher \( M \) flows we should expect lower \( D \), corresponding to the introduction of compressive shocks into a diffuse cloud with increasing \( M \), and previous studies have calculated \( D = 1.4 - 1.8 \) for column densities (Elmegreen & Falgarone 1996; Elmegreen & Scalo 2004; Sanchez et al. 2005; Rathborne et al. 2015). This suggests that the \( D \) values we calculate are reasonable, however, the ‘saxophone’ region has a clear filamentary structure (see the high-density filament feature on the left column density map in Figure 4), which has a density-length scaling relation \( \rho \sim \ell^{-2} \), or \( \Sigma \sim \ell^{-1} \), and will act to reduce \( D \) in the vicinity of \( \Sigma_{\text{max}} \) (Schneider et al. 2013; Federrath 2016; Andrè 2017). This may account for the slightly higher \( M \) that we estimate for ‘saxophone’, compared to the Mach estimate in Schneider et al. (2013). For the ‘quiet’ subregion we slightly underestimate \( M \). This is because the column density (see the right column density map in Figure 4) is diffuse, and lacks the shock structures that we see introduced between the \( \Sigma = 1 \) and \( \Sigma = 4 \) simulations in Figure 1. The different column density geometry in the ‘quiet’ cloud may be due to a deviation away from the natural mixing of driving modes, towards stronger, compressive driving which we do not currently include in our relation, and which can change the

| Subregion       | \( M \) (based on Eq. 17) | \( D \pm 1\sigma \) | \( M(D) \pm 1\sigma \) |
|-----------------|---------------------------|---------------------|-----------------------|
| Saxophone       | 7 \pm 3                   | 1.60 \pm 0.04       | 10^{+55}_{-4}         |
| Quiet           | 3 \pm 1                   | 1.76 \pm 0.05       | 2 \pm 1               |

Notes: Column (1): The subregion in the Polaris Flare cloud. Column (2): the estimated \( M \) from Schneider et al. (2013), which is stated to have an error \( \sim 30-40\% \). Column (3): \( D \) calculated from our mass-length method. We take the \( D \) on the largest scale calculated, which reduces to the regular monofractal mass-length dimension. Column (4): The Mach number calculated from Equation 16, with 1σ uncertainties propagated from column (3).

6 SUMMARY AND KEY FINDINGS

In this study we construct a new empirical relation for the scale-dependent three-dimensional (3D) Mach number, \( \mathcal{M} \), and the fractal dimension, \( D \), of the column density for turbulent clouds. We use the mass-length fractal dimension method introduced in Beattie et al. (2019) (BFK19) on six hydrodynamical cloud simulations, with root-mean-squared (rms) Mach number, \( M \), varying from \( M = 1 \) to 100. We apply the method on the column densities, with an example of the densities shown Figure 1, to construct \( D \) as a function of length scale, \( \ell \). We then transform the cloud length scales to \( \mathcal{M} \) using the scaling relation \( \ell \sim \mathcal{M}^2 \), for supersonic turbulence (Burgers 1948; Federrath 2013). Using this data we are able to construct \( D(\mathcal{M}) \), and finally \( \mathcal{M}(D) \). Using \( \mathcal{M}(D) \) we construct \( M \), where \( M \sim \mathcal{M}(L) \), and \( L \) is the cloud scale, for the dust column density maps of two quiescent subregions from the Polaris Flare, that are termed
structures, that locally scale the cloud by $\Sigma \sim \text{cloud geometries}$ or how the presence of large filamentary for how different types of turbulent driving influence the is acceptable, especially considering we do not account the number estimate based on the CO velocity dispersion and velocity dispersion. The agreement between the Mach for the two respective subregions, but based on the CO made in Schneider et al. (2013), $\xi$ column density, $\text{dependent Mach number}$ and the fractal dimension of the $(2013)$. We summarise our key findings below:

'quiet' and 'saxophone', studied earlier in Schneider et al. (2013). We thank the anonymous reviewer for the detailed and crit- pr48pi and GCS Large-scale project 10391), the Partner- and the Gauss Centre for Supercomputing (grants pr32lo, pr89mu), the Australian National Computational Infra-structure (grant ek9), and the Pawsey Supercomputing Cen-tre with funding from the Australian Government and the Government of Western Australia, in the framework of the National Computational Merit Allocation Scheme and the ANU Allocation Scheme. The simulation software FLASH was in part developed by the DOE-supported Flash Centre for Computational Science at the University of Chicago. N. S. acknowledges support by the French ANR and the German DFG through the project "GENESIS" (ANR-16-CE92-0035-01/DFG1591/2-1).

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\begin{align*}
\log \mathcal{M}(D) &= \xi_1 \left( \text{erfc}^{-1} \left( \frac{D - D_{\text{min}}}{\Omega} \right) + \xi_2 \right),
\end{align*}

as defined in Equation 16 and plotted in Figure 3, where $\xi_1 = 0.9 \pm 0.1$, $\xi_2 = 0.2 \pm 0.2$, $\Omega = 0.22 \pm 0.07$ and $D_{\text{min}}$. the fractal dimension of the column density in the high $M$ limit is $1.55 \pm 0.13$. This relation allows for the calculation of $M$ for clouds in the range $10 \geq M \geq 0.1$. Very large and very low $M$ are inappropriate for the model due to the steep tails in the inverse complimentary error function.

We use the mass-length fractal dimension method in BFK19 to calculate the $D$ of the ‘saxophone’ and ‘quiet’ subregions of the Polaris Flare, shown in Figure 4. We find $D = 1.60 \pm 0.04$ and $D = 1.76 \pm 0.05$ for ‘saxophone’ and ‘quiet’, respectively, consistent with the thesis that higher Mach number flows reduce $D$, by turning diffuse, space-filling structures into compressive shocks and filaments.

Using $D$ we estimate $M$ for each of the subregions. We find ‘quiet’ has a $M \sim 2$ and ‘saxophone’ has a $M \sim 10$, shown in Figure 5. This is comparable to the estimates made in Schneider et al. (2013), $M \sim 3$ and $M \sim 7$ for the two respective subregions, but based on the CO velocity dispersion. The agreement between the Mach number estimate based on the CO velocity dispersion and based on our new fractal dimension relation, Equation 16, is acceptable, especially considering we do not account for how different types of turbulent driving influence the cloud geometries or how the presence of large filamentary structures, that locally scale the cloud by $\Sigma \sim \ell^{-1}$, act to reduce $D$.

Our results suggest that the new empirical relation between the fractal dimension of column densities and 3D turbulent Mach number is a useful tool for extracting the Mach number purely from the structure and geometry of column density data from the cloud.
