Pairwise Relative Primality of Positive Integers

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A classic result in number theory is that the probability that two positive integers are relatively prime is \( \frac{6}{\pi^2} \). More generally the probability that \( k \) positive integers chosen arbitrarily and independently are relatively prime is \( 1/\zeta(k) \), where \( \zeta(k) \) is Riemann’s zeta function. A short accessible proof of this result was given by J. E. Nymann [3]. In a recent paper L. Tóth [5] solved the problem of finding the probability that \( k \) positive integers are pairwise relatively prime by the recursion method, he proved that, for \( k \geq 2 \), the probability that \( k \) positive integers are pairwise relatively prime is

\[
A_k = \prod_p \left( 1 - \frac{1}{p} \right)^{k-1} \left( 1 + \frac{k-1}{p} \right).
\]

Given a graph \( G = (V, E) \) with \( V = \{1, 2, ..., k\} \), the \( k \) positive integers \( a_1, a_2, ..., a_k \) are \( G \)-wise relatively prime if \( (a_i, a_j) = 1 \) for \( \{i, j\} \in E \). In this note we consider the problem of finding the probability \( A_G \) that \( k \) positive integers are \( G \)-wise relatively prime.

For a \( k \)-tuple positive integers \( u = (u_1, u_2, ..., u_k) \), let \( Q_G(u)(n) \) denote the number of \( k \)-tuples of positive integers \( a_1, a_2, ..., a_k \) with \( 1 \leq a_1, a_2, ..., a_k \leq n \) such that \( (a_i, u_i) = 1 \) for \( i = 1, ..., k \) and they are \( G \)-wise relatively prime.

The next theorem gives an asymptotic formula for \( Q_G(u)(n) \) and the exact values of \( A_G \). Before we state the theorem, let us introduce some notations. Given a graph \( G = (V, E) \) with \( V = \{1, 2, ..., k\} \), a subset \( S \subset V \) is called independent if no two vertices of \( S \) are adjacent in \( G \). We denote by \( i_m(G) \) the number of independent sets of cardinality \( m \) in \( G \), and for a subset \( S \subset V \), we denote by \( i_m,S(G) \) the number of independent sets of cardinality \( m \) in \( G \) which contains at least one vertex in \( S \). For a \( k \)-tuple positive integers \( u = (u_1, u_2, ..., u_k) \), and an integer \( d \), the set of positive integers \( i \) with \( 1 \leq i \leq k \) such that \( d \) divides \( u_i \) is denoted by \( S(u, d) \).

**Theorem 1** For a graph \( G = (V, E) \) with \( V = \{1, 2, ..., k\} \), we have uniformly for \( n, u_i \geq 1 \),

\[
Q_G^{(u)}(n) = A_G f_G(u)n^k + O(\theta(u)n^{k-1} \log^{k-1} n),
\]

where

\[
A_G = \prod_p \left( \sum_{m=0}^{k} i_m(G) \left( 1 - \frac{1}{p} \right)^{k-m} \frac{1}{p^m} \right).
\]
\[ f_G(u) = \prod_{p|u_1 u_2 \cdots u_k} \left( 1 - \frac{\sum_{m=0}^{k} i_m S(u,p)(G)(p-1)^{k-m}}{\sum_{m=0}^{k} i_m (G)(p-1)^{k-m}} \right), \]

and if \( \theta(u_i) \) denotes the number of square free divisors of \( u_i \), then \( \theta(u) = \max\{\theta(u_i), i = 1, 2, \ldots, k\} \).

**Corollary 2** The probability that \( k \) positive integers \( a_1, a_2, \ldots, a_k \) are \( G \)-wise relatively prime and \( (a_i, u_i) = 1 \) for \( i = 1, \ldots, k \) is

\[
\lim_{n \to \infty} \frac{Q(n)}{n^k} = A_G f_G(u).
\]

For \( u_i = 1 \), the probability that \( k \) positive integers are \( G \)-wise relatively prime is

\[
A_G = \prod_p \left( \sum_{m=0}^{k} i_m (G) \left( 1 - \frac{1}{p} \right)^{k-m} \frac{1}{p^m} \right).
\]

In [2], P. Moree proposed the problems of finding probabilities that \( k \) positive integers have exact (or at least) \( r \) relatively prime pairs. Using Theorem 1 and an Inclusion-Exclusion formula in combinatorics (see exercise 1 of chapter 2 in [4]), we can give a solution to his problems.

**Corollary 3** The probability that \( k \) positive integers have exactly \( r \) relatively prime pairs is

\[
A_{k,=r} = \sum_{i=r}^{k(k-1)/2} (-1)^{i-r} \binom{i}{r} B_{k,i},
\]

and the probability that \( k \) positive integers have at least \( r \) relatively prime pairs is

\[
A_{k,\geq r} = \sum_{i=r}^{k(k-1)/2} (-1)^{i-r} \binom{i-1}{r-1} B_{k,i},
\]

where

\[
B_{k,i} = \sum_{|E|=i} A_G.
\]

In particular, the probability that \( k \) positive integers are pairwise not relatively prime is

\[
A_{k,=0} = \sum_{i=0}^{k(k-1)/2} (-1)^i B_{k,i}.
\]

In [1], J. L. Fernández and P. Fernández proved that the number of relatively prime pairs is asymptotically normal as \( k \) tends to \( \infty \).

To prove Theorem 1 we need the following lemmas.
Lemma 4  For \( k, n \geq 1 \), a graph \( G = (V, E) \) with \( V = \{1, 2, ..., k + 1\} \), and \( u = (u_1, u_2, ..., u_{k+1}) \) with \( u_i \geq 1 \),

\[
Q_G^{(u)}(n) = \sum_{j=1}^{n} Q_{G-v}^{(j\ast u)}(n),
\]

where \( G - v \) is the graph obtained from \( G \) by deleting the vertex \( v=k+1 \) together with all the edges incident to \( v \), and if \( (j \ast u)_i \) denotes the \( i \)th component of \( j \ast u \), then

\[
(j \ast u)_i = \begin{cases} j u_i & \text{if } i \text{ is adjacent to } v \text{ in } G, \\ u_i & \text{otherwise}, \end{cases}
\]

for \( i = 1, 2, ..., k \).

Proof.  The \( k + 1 \) positive integers \( a_1, a_2, ..., a_{k+1} \) are \( G \)-wise relatively prime and \( (a_i, u_i) = 1 \) for \( i = 1, 2, ..., k+1 \) if and only if the first \( k \) positive integers \( a_1, a_2, ..., a_k \) are \( (G - v) \)-wise relatively prime and \( (a_i, u_i) = 1 \) for \( i = 1, 2, ..., k \), and \( (a_k+1, u_{k+1}) = 1 \) when the vertex \( i \) is adjacent to the vertex \( v = k + 1 \), and \( (a_{k+1}, u_{k+1}) = 1 \). We have

\[
Q_G^{(u)}(n) = \sum_{a_{k+1} = 1}^{n} Q_{G-v}^{(a_{k+1} \ast u)}(n) = \sum_{j=1}^{n} Q_{G-v}^{(j \ast u)}(n).
\]

Lemma 5  For \( k, u_i \geq 1 \), a graph \( G = (V, E) \) with \( V = \{1, 2, ..., k\} \), and \( S \) a subset of vertices in \( V \),

\[
\frac{f_G(j \ast u)}{f_G(u)} = \mu(d)\frac{\alpha_{G,S}(u, d)}{\alpha_G(u, d)}
\]

if \( d \) is square free, then

\[
\frac{\alpha_{G,S}(u, d)}{\alpha_G(u, d)} \leq \frac{k^{\omega(d)}}{d},
\]

where

\[
\alpha_G(u, d) = \prod_{p | d} \left( \sum_{m=0}^{k} i_m (G - S(u, p))(p - 1)^{k-m} \right),
\]

\[
\alpha_{G,S}(u, d) = \prod_{p | d} \left( \sum_{m=S}^{k} i_m (G - S(u, p))(p - 1)^{k-m} \right),
\]

and if \( (j \ast u)_i \) denotes the \( i \)th component of \( j \ast u \), then

\[
(j \ast u)_i = \begin{cases} j u_i & \text{if } i \text{ is in } S, \\ u_i & \text{otherwise}, \end{cases}
\]

for \( i = 1, 2, ..., k \), and \( \omega(d) \) denote the number of distinct prime factors of \( d \).
Proof. It suffices to verify the equality for \( j = p^a \) a prime power:

\[
\sum_{d | p^a} \frac{\mu(d) \alpha_{G,S}(u,d)}{\alpha_G(u,d)} = 1 - \frac{\sum_{m=0}^{k} i_m(G-S(u,p))(p-1)^{k-m}}{\sum_{m=0}^{k} i_m(G-S(u,p))(p-1)^{k-m}}
\]

\[
= \frac{\sum_{m=0}^{k} i_m(G-S(u,p))(p-1)^{k-m} - \sum_{m=0}^{k} i_m(G-S(u,p))(p-1)^{k-m}}{\sum_{m=0}^{k} i_m(G-S(u,p))(p-1)^{k-m}}
\]

\[
= \frac{\sum_{m=0}^{k} i_m(G)(p-1)^{k-m} - \sum_{m=0}^{k} i_m(G)(p-1)^{k-m}}{\sum_{m=0}^{k} i_m(S(u,p))(p-1)^{k-m}}
\]

\[
= \frac{f_G(p^a * u)}{f_G(u)}.
\]

Now we prove the inequality. Notice that

\[
i_0(G-S(u,p)) = 1, \quad i_{0,S}(G-S(u,p)) = 0, \quad i_{m,S}(G-S(u,p)) \leq i_m(G-S(u,p)).
\]

Then when \( d \) is square free,

\[
\frac{\alpha_{G,S}(u,d)}{\alpha_G(u,d)} = \prod_{p \mid d} \frac{\sum_{m=0}^{k} i_m(G-S(u,p))(p-1)^{k-m}}{\sum_{m=0}^{k} i_m(G-S(u,p))(p-1)^{k-m}}
\]

\[
\leq \prod_{p \mid d} \frac{\sum_{m=1}^{k} i_m(G-S(u,p))(p-1)^{k-m}}{(p-1)^k + \sum_{m=1}^{k} i_m(G-S(u,p))(p-1)^{k-m}}
\]

\[
\leq \prod_{p \mid d} \frac{\sum_{m=1}^{k} \binom{k}{m} (p-1)^{k-m}}{(p-1)^k + \sum_{m=1}^{k} \binom{k}{m} (p-1)^{k-m}}
\]

\[
= \prod_{p \mid d} \frac{(p^k - (p-1)^k)}{p^k}
\]

\[
\leq \prod_{p \mid d} \frac{k p^{k-1}}{p^k}
\]

\[
= \prod_{p \mid d} \frac{k}{d} = \frac{k}{d} \omega(d)
\]

For the proof of the theorem, we proceed by induction on \( k \). For \( k = 1 \), we have by the Inclusion-Exclusion Principle

\[
Q_{\{1\}}^{(u_1)}(n) = \sum_{j = 1}^{n} \sum_{d | u_1} \mu(d) \frac{\mu_j}{\varphi_j} = \sum_{d | u_1} \mu(d) \left( \frac{\mu_j}{\varphi_j} + O(1) \right)
\]
\[ = n \sum_{d \mid u_1} \frac{\mu(d)}{d} + O(\sum_{d \mid v} \mu^2(d)). \]

Hence,

\[ Q^{(u_1)}_\{1\}(n) = \sum_{j = 1}^{n} 1 = n \frac{\phi(u_1)}{u_1} + O(\theta(u_1)) \quad (2) \]

and (3) is true for \( k = 1 \) with \( A_{\{1\}} = 1, f_{\{1\}}(u_1) = \frac{\phi(u_1)}{u_1}, \phi \) denoting the Euler function.

Suppose that (3) is valid for \( k \), we prove it for \( k + 1 \). Let \( u = (u_1, u_2, ..., u_{k+1}) \) and \( u' = (u_1, u_2, ..., u_k) \), from Lemma 4 we have

\[ Q_G^{(u)}(n) = \sum_{j = 1}^{n} Q_G^{(j*u)}(n) \]

\[ = \sum_{j = 1}^{n} A_G - v[j*u]n^k + O(\theta(j*u)n^{k-1} \log^{k-1} n) \]

\[ = A_G - v f_G - v(u'n)^k \sum_{j = 1}^{n} \frac{f_G - v[j*u']}{f_G - v(u')} \]

\[ + O(\theta(u'n^{k-1} \log^{k-1} n) \sum_{j = 1}^{n} \theta(j)). \]

Here \( \sum_{j = 1}^{n} \theta(j) \leq \sum_{j = 1}^{n} \tau_2(j) = O(n \log n) \), where \( \tau_2 = \tau \) is the divisor function.

Furthermore, in Lemma 5 choosing the subset \( S \) to be the open neighbourhood \( N(v) \) of \( v \), which is the set of vertices adjacent to \( v \), we have

\[ \sum_{j = 1}^{n} \frac{f_G - v[j*u']}{f_G - v(u')} \]

\[ = \sum_{d \leq n} \frac{\mu(d)\alpha_G - v, N(v)(u', d)}{\alpha_G - v(u', d)} \]

\[ = \sum_{d \leq n} \frac{\mu(d)\alpha_G - v, N(v)(u', d)}{\alpha_G - v(u', d)} \sum_{e \leq \frac{n}{d}} 1 \]

Using (2), we have

\[ \sum_{j = 1}^{n} \frac{f_G - v[j*u']}{f_G - v(u')} \]
by Lemma 5.

Hence, the main term of (4) is

\[
\frac{\phi(u_{k+1})}{u_{k+1}} n \sum_{d=1}^{\infty} \frac{\mu(d) \alpha_{G-v,N(v)}(u',d)}{\alpha_{G-v}(u',d)} \prod_{p | u_1 u_2 \cdots u_k} \left( 1 - \frac{\sum_{m=0}^{k} i_{m,N(v)}(G-v)(p-1)^{k-m}}{p (\sum_{m=0}^{k} i_{m}(G-v)(p-1)^{k-m})} \right) \prod_{p | u_{k+1}} \left( 1 - \frac{1}{p}\right)^{-1} \prod_{p | u_1 u_2 \cdots u_k} \left( 1 - \frac{\sum_{m=0}^{k} i_{m,N(v)}(G-v-S(u,p))(p-1)^{k-m}}{p (\sum_{m=0}^{k} i_{m}(G-v-S(u,p))(p-1)^{k-m})} \right),
\]

and its O-terms are

\[
O(n \sum_{d>n} \frac{k^{\omega(d)}}{d^2}) = O(n \sum_{d>n} \frac{\tau_k(d)}{d^2}) = O(\log^{k-1} n)
\]

by Lemma 3(b) in [5], which gives an asymptotic estimate of the sum

\[
\sum_{n>x} \frac{\tau_k(n)}{n^2} = O\left(\frac{\log^{k-1} x}{x}\right)
\]

and
from Lemma 3(a) in [5], which gives an asymptotic estimate of the sum

\[ \sum_{n \leq x} \tau_k(n) = O(\log^k x). \]

Substituting into (3), we get

\[
Q_G(u) G(n) = A_G f_G(u) n^{k+1} + O(\theta(u) n^k \log^k n)
\]

by a simple computation, which shows that the formula is true for \( k + 1 \) and we complete the proof.

T. Freiberg computed the probability that three positive integers are pairwise not relatively prime, see [2]. As an application of our method, we compute the probability that four positive integers are pairwise not relatively prime.

Let \( A_{3,i} \) denote the probability that three positive integers have \( i \) relatively prime pairs, for \( i = 1, 2, 3 \). By Theorem 1 and Corollary 3, we have

\[
A_{3,0} = 1, \quad A_{3,1} = \prod_p \left(1 - \frac{1}{p^2}\right), \quad A_{3,2} = \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \left(1 - \frac{1}{p}\right) \frac{1}{p}\right),
\]

\[
A_{3,3} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 - \frac{2}{p}\right),
\]

and

\[
A_{3,0} = 1 - 3A_{3,1} + 3A_{3,2} - A_{3,3}.
\]

This recovers T. Freiberg’s result.

When \( k = 4 \), the number of graphs with given number of edges and the number of independent sets are summarized in the following table.
Given graph $G$ with four vertices and $i$ edges, let $A_{4,i}$ denote the probability that integers are $G$-wise relatively prime for $i = 1, 2, ..., 6$, if there are more than one type of graphs with fixed number of edges $i$, for the graphs of type $j$, the probability is denoted by $A_{4,i,j}$. Again by Theorem 1 and Corollary 3, we have

$$A_{4,0} = 1, A_{4,1} = \prod_p \left(1 - \frac{1}{p^2}\right), \quad A_{4,2,1} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \left(1 - \frac{1}{p}\right) \frac{1}{p}\right),$$

$$A_{4,2,2} = \prod_p \left(1 - \frac{1}{p^2}\right)^2, \quad A_{4,3,1} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \left(1 - \frac{1}{p}\right)^2 \frac{1}{p}\right),$$

$$A_{4,3,2} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 - \frac{2}{p}\right), \quad A_{4,4} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p} - \frac{1}{p^2}\right),$$

$$A_{4,5} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + 2 \left(1 - \frac{1}{p}\right) \frac{1}{p}\right), \quad A_{4,6} = \prod_p \left(1 - \frac{1}{p}\right)^3 \left(1 + \frac{1}{p}\right),$$

and

$$A_{4,0} = 1 - 6A_{4,1} + 12A_{4,2,1} + 3A_{4,2,2} - 4A_{4,3,1} - 16A_{4,3,2} + 15A_{4,4} - 6A_{4,5} + A_{4,6}.$$

### References

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