Input and output in damped quantum systems III: 
Formulation of damped systems driven by Fermion fields

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Abstract.
A comprehensive input-output theory is developed for Fermionic input fields. Quantum stochastic differential equations are developed in both the Ito and Stratonovich forms. The major technical issue is the development of a formalism which takes account of anticommutation relations between the Fermionic driving field and those system operators which can change the number of Fermions within the system.

1. Introduction

The introduction of Input-Output formulations in the 1980s [1] was a response to the necessity for a theory of quantum damping which could deal with travelling wave situations. A formulation of a photodetector theory was suggested using input-output methods [2], in which the detection of a photon is envisaged as the conversion of an input light field into an output electron field. However, to do this requires a theory of inputs and outputs with Fermion fields, and an elementary theory was developed. Although basically satisfactory, this formulation was not complete, and in particular, could not be used to derive a master equation. The problem that arises is quite simple: the equations of motion for system generators are different, depending on whether a system operator is viewed as commuting or anticommuting with the fermionic heat bath operators.

In this paper this technical problem is overcome. It is shown how we may define “restricted” system operators so as to commute with all bath operators, and that these internal system operators all obey a quantum Langevin equation of the same form.

However, the equations of motion in the original operators are, for the two level atom, linear and thus exactly soluble. Thus, we have a description of a two level system interacting with a Fermi Bath which is essentially the same as that of a harmonic oscillator interacting with a Bosonic heat bath. The two level atom behaves like a Fermion coming to equilibrium with all the other Fermions.

It is possible to develop Fermionic quantum stochastic integration, and the corresponding Ito and Stratonovich formulations of quantum stochastic differential equations, and finally, from these, to derive the Master equation in the expected form.

To the best of the author’s knowledge there have only been two other papers dealing with input-output theory of Fermions, that of Sun and Milburn [3] and that of Search et al [4]. Neither of these attempts a comprehensive input-output formalism; the first concentrates essentially on counting statistics, while the focus of the second paper is on the theory of Fermions inside and outside of a linear cavity, and does not address the quantum stochastic issues which are the principal topic of this paper.
2. Beams of Fermions

2.1. Input and Output Fields

We want to consider here non-relativistic Fermi fields, in which we shall for simplicity and clarity make no mention of spin (though this is always present in Fermion, it plays no essential rôle in their non-relativistic description). For simplicity, we shall also consider propagation in only one dimension. A Fermi field can then be written:

\[
D(t, x) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} dk \, e^{i(kx - \omega t)} d(k)
\]  

where \(d(k)\) is the destruction operator, with anticommutation relations

\[
[d(k), d^\dagger(k')]_+ = \delta(k - k')
\]

\[
[d(k), d(k')]_+ = [d^\dagger(k), d^\dagger(k')]_+ = 0
\]  

leading to the equal time anticommutation relation,

\[
[D(t, x), D^\dagger(t, x')]_+ = \delta(x - x').
\]  

The dispersion relation

\[
\omega = \hbar k^2 / 2m
\]  

follows from non-relativistic mechanics, and ensures \(D(t, x)\) obeys the time dependent Schrödinger equation

\[
i\hbar \frac{\partial D(t, x)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 D(t, x)}{\partial x^2}
\]  

In the situation we wish to consider an input field radiating into a system, which itself radiates an output field in the opposite direction. In that case, it is more appropriate to consider the field as being defined on the half-line \(0 \leq x < \infty\), for which an appropriate expansion is

\[
D(t, x) = \sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} dk \cos(kx) e^{i\omega t} d(k)
\]  

which can also be written

\[
D(t, x) = \sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} dk \{ e^{i(kx - \omega t)} d_{\text{in}}(k) + e^{-i(kx + \omega t)} d_{\text{out}}(k) \}
\]  

where

\[
d_{\text{in}}(k) = d_{\text{out}}(k) = d(k).
\]  

We can also define

\[
D_{\text{in}}(t, x) = \sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} dk \, e^{-i(kx - \omega t)} d_{\text{in}}(k)
\]

\[
D_{\text{out}}(t, x) = \sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} dk \, e^{-i(kx + \omega t)} d_{\text{out}}(k)
\]  

and the boundary condition

\[
D_{\text{in}}(t, 0) - D_{\text{out}}(t, 0) = 0.
\]  

follows from (8).
2.2. Dispersion

Matter waves are dispersive; in fact the group and phase velocities differ by a factor of 2 for all frequencies. This means that the simple propagation of statistics as in light beams is not valid here—there is no solution analogous to the solution of the one dimensional wave equation which can be written in the form

\[ A(t, x) = A_{in}(t + x/v). \]  

We will therefore limit our considerations to very narrow bandwidth situations. In order to judge what sort of bandwidth can be considered “narrow”, let us consider the correlation functions.

2.3. Correlation Functions of Propagating Fermion Beams

We want to consider correlation functions

\[ G^{(1)}(x_1, t_1; x_2, t_2) = \langle D^\dagger(t_1, x_1)D(t_2, x_2) \rangle \]  

and

\[ G^{(2)}(x_1, t_1; x_2, t_2; x_3, t_3; x_4, t_4) = \langle D^\dagger(x_1, t_1)D^\dagger(x_2, t_2)D(x_3, t_3)D(x_4, t_4) \rangle \]

These are analogous to the similarly defined correlation functions for optical fields.

Let us now consider a stationary narrow bandwidth field, such that

\[ \langle a^\dagger(k)a(k') \rangle = \bar{N}(k)\delta(k - k') \]  

where

\[ \bar{N}(k) = 0 \quad \text{unless} \quad k \approx k_0. \]  

Now defining

\[ N(\omega)d\omega = \bar{N}(k)dk \]

then

\[ \langle D^\dagger(t, x)D(t', x') \rangle = \frac{1}{2\pi} \int d\omega N(\omega)e^{i\omega(t-t')-k(x-x')/\omega} \]

If \( \omega_0, k_0 \) are the central frequency and wavenumber of the range, and the central velocity of propagation is

\[ v = \omega_0/k_0 \]

then, provided that the range of frequencies, \( \delta\omega \), satisfies

\[ (x-x')\delta\omega/v \ll 1 \]

we can first write

\[ g(\tau) = e^{-i\omega_0\tau}\langle D^\dagger(\tau, x)D(0, x) \rangle \]

and then we can derive

\[ G^{(1)}(x, t; x', t') = e^{i\omega_0[t-t'-(x-x')/v]} \left( 1 + \frac{x-x'}{2v} \frac{\partial}{\partial t} \right) g(t - t' - \frac{x-x'}{v}) \]

This equation gives the correction due to dispersion of the “propagation approximation” to evaluating the correlation function of a fermion beam at two separated points. The condition of validity for such an approximation is (19), which can be interpreted to mean that the size of individual wavepackets, which is order of magnitude \( v/\delta\omega \), must be very much larger than the distance \( |x - x'| \) between the points considered.

The correction in (21) is relevant to the measurement of time correlation functions by delayed coincidence measurements, when the delay is induced by allowing one beam to propagate further than the other.
2.4. Thermal correlation functions for Fermionic beams

In the optical case a thermal light beam is considered to be Gaussian, and the factorizable property of Gaussian moments leads to a relationship between 1st and 2nd order correlation functions which is the characteristic of the “bunched” nature of thermal light. It is difficult to define what might be considered to be a “Gaussian” Fermion state, but a thermal Fermion state can be defined.

In this case of a thermal state we have

\[\langle d^\dagger(k) d(k') \rangle = \delta(k - k') \bar{N}(k) \]

\[\langle d^\dagger(k) d^\dagger(k'') d(k''') \rangle = [\delta(k - k'') \delta(k' - k''') - \delta(k - k''') \delta(k' - k'')] \bar{N}(k) \bar{N}(k') \bar{N}(k'') \bar{N}(k''').\]

Here, if

\[\bar{N}(\omega) d\omega = \bar{N}(k) dk,\]

then

\[\bar{N}(\omega) = 1/ [\exp(\hbar \omega/kT) + 1].\]

The antisymmetric requirement is in fact the natural analogy of the Gaussian factorization property of photon beams (there would be a + sign on the RHS of (23) in the case of photons instead of a sign.) It leads to the relationship

\[G^{(2)}(x_1, t_1; x_2, t_2; x_3, t_3; x_4, t_4) = G^{(1)}(x_1, t_1; x_3, t_3) G^{(1)}(x_2, t_2; x_4, t_4) - G^{(1)}(x_1, t_1; x_4, t_4) G^{(1)}(x_2, t_2; x_3, t_3).\]

The corresponding formula for a Gaussian Boson beam differs only by having a positive sign rather than a negative sign on the right hand side.

In the case of stationary statistics, evaluated with time difference \(\tau\) at \(x = 0\), we have for

\[g^{(2)}(\tau) \equiv G^{(2)}(0, t; 0, t + \tau; 0, t + \tau; 0, 0)\]

\[g^{(1)}(\tau) = G^{(1)}(0, t; 0, t + \tau)\]

that

\[g^{(2)}(\tau) = g^{(1)}(\tau) g^{(1)}(-\tau) - [g^{(1)}(0)]^2\]

\[= |g^{(1)}(\tau)|^2 - |g^{(1)}(0)|^2\]

Clearly \(g^{(2)}(0) = 0\), corresponding to perfect antibunching, as expected from a Fermion beam.

3. Interaction of a System with a Fermionic heat bath

In order to fix our ideas, let us consider the ionization of an atom under the influence of an impinging electron beam. The Hamiltonian is

\[H = H_{\text{sys}} + H_{\text{Int}} + H_B.\]

Here, \(H_{\text{sys}}\) is the free atom Hamiltonian, whose precise form will be left open. The bath Hamiltonian, corresponding to a field on a half line, \(0 < x < \infty\), can be written

\[H_B = \int_0^\infty dk \hbar \omega(k) d^\dagger(k) d(k)\]

where in this case, for an electron of mass \(m\),

\[\omega(k) = \hbar k^2/2m.\]
The Fermion operators \( d(k) \) are as in the previous section. Finally, the interaction is conceived as representing the absorption or emission of an electron, and is written

\[
H_{\text{Int}} = i\hbar \int_0^\infty dk \kappa(k) \{ d^\dagger(k) \tilde{c} - \tilde{c} d(k) \}
\]  

Here \( \tilde{c}, \tilde{c}^\dagger \) are system operators. The action of \( \tilde{c}^\dagger \) on a system state increases the number of constituent electrons by 1.

### 3.1. Fermionic and Bosonic System Operators

The commutation relations for the system operators depend on the systems being studied. In the usual case of a Boson bath [1] the system and bath operators commute at equal times. However, when dealing with a Fermionic bath the situation is different, depending on whether a system operator can be considered as changing the number of Fermions which make up the system or not. The separation into “system” and “bath” tends to obscure the fact that the system does have an internal structure, and that (for example) an ion and a neutral atom must have different numbers of constituent electrons. An operator such as \( \tilde{c} \), which can be regarded as removing an electron from the neutral atom, must anticommute with all bath operators, since it must be composed of an odd number of creation and destruction operators. On the other hand, there are operators (such as \( H_{\text{sys}} \)) which do not change the number of constituent electrons, and hence commute with the bath operators at equal times. We thus conclude that these are two kinds of system operators.

- **a)** Bosonic—these commute with all bath operators, \( d(k), d^\dagger(k) \) (at equal times)
- **b)** Fermionic—these anticommute with all bath operators \( d(k), d^\dagger(k) \) (at equal times)

We will use the notation \( \hat{a}, \hat{b}, \tilde{c}, \) etc. for the system operators to emphasize that such operators may anticommute with the bath operators, and hence are not necessarily independent of them. We will shortly introduce “restricted” system operators, which are independent of and hence commute with the bath operators.

### 3.2. Derivation of Quantum Langevin Equations

Because all Fermion fields of interest are massive, and therefore the wave propagation is dispersive it is not quite as easy to derive quantum Langevin equations as in the optical case. Added to this is the complication that some of the system operators are Fermionic, and others Bosonic.

The operator \( \tilde{c} \) which occurs in the interaction Hamiltonian must be Fermionic, since it changes the number of constituent electrons in the system by 1. Using this, we derive the equation of motion for \( d(k) \).

\[
\dot{d}(k) = -i\omega(k) d(k) - \kappa \tilde{c}
\]  

which we can integrate to get

\[
d(k, t) = e^{-i\omega(k) (t-t_0)} d(k, t_0) - \kappa(k) \int_{t_0}^t e^{-i\omega(k)(t-t')} \tilde{c}(t') dt'
\]  

We now define

\[
d(t) \equiv \frac{1}{2} D(t, 0) = \sqrt{\frac{1}{2\pi}} \int_0^\infty dk \, d(k, t)
\]  

\[
d_{\text{in}}(t) = \sqrt{\frac{1}{2\pi}} \int_0^\infty dk \, e^{-i\omega(k)(t-t_0)} d(k, t_0)
\]
The quantity \(d(t)\) is a genuine Heisenberg operator for the time \(t\), whereas \(d_{in}(t)\) depends only on the initial values \(d(k, \omega_0)\), of the destruction operators. We now write the equation of motion for an arbitrary system operator \(\hat{a}\),

\[
\dot{\hat{a}} = -\frac{i}{\hbar} [\hat{a}, H_{sys}] + \int_0^{\infty} dk \kappa(k) \left[ \hat{a}, d^\dagger(k) \hat{c} - \hat{c}^\dagger(k) d(k) \right] \tag{39}
\]

\[
= -\frac{i}{\hbar} [\hat{a}, H_{sys}] + \int_0^{\infty} dk \kappa(k) \left\{ \mp d^\dagger(k) [\hat{a}, \hat{c}] \pm - [\hat{a}, \hat{c}^\dagger] \pm d(k) \right\} \tag{40}
\]

where the top signs apply if \(\hat{a}\) is Fermionic, and the bottom signs apply if \(\hat{a}\) is Bosonic.

3.2.1. The white noise approximation To obtain Langevin equations we must make approximations. There are two principal approximations.

a) The interaction is weak, and the free motion of \(\hat{c}(t)\) is proportional to \(e^{-i\omega_0 t}\).

b) The frequency \(\omega_0\) is rather large. Since the equation (40) is homogeneous in \(\hat{a}\), this means that the main contribution from the integrals will occur where \(\omega(k) \approx \omega_0\).

We can thus write an approximate expression for the \(k\) integrals in (40) by evaluating the c-number \(\kappa(k)\) at \(k_0\), i.e.,

\[
\int_0^{\infty} dk \kappa(k) d(k) \approx \kappa(k_0) \int_0^{\infty} dk d(k) \approx \sqrt{2\pi} \kappa(k_0) d(t). \tag{41}
\]

provided \(\kappa(k)\) is a smooth function of \(k\) around \(\omega(k) = \omega_0\). From (36)

\[
d(t) = d_{in}(t) + \sqrt{\frac{1}{2\pi}} \int_0^{\infty} d(k') \kappa(k') e^{-i\omega(k)(t-t')} \hat{c}(t') dt' \tag{42}
\]

\[
= d_{in}(t) + \sqrt{\frac{1}{2\pi}} \int_{t_0}^{t} dt' \int d\omega \frac{dk(\omega)}{d\omega} \kappa(k(\omega)) e^{-i\omega(t-t')} \hat{c}(t') \tag{43}
\]

and provided \(\kappa(k(\omega))\) and \(dk(\omega)/d\omega\) are smooth around \(\omega = \omega_0\), we can again approximate:

\[
d(t) = d_{in} + \sqrt{\frac{2\pi}{2}} \int_{t_0}^{t} \delta(t - t') \hat{c}(t') \left[ \kappa(k(\omega)) \frac{dk(\omega)}{d\omega} \right]_{\omega = \omega_0} \tag{44}
\]

that is

\[
d(t) = d_{in}(t) + \sqrt{\frac{\pi}{2}} \kappa(k(\omega_0)) \frac{dk(\omega_0)}{d\omega_0} \hat{c}(t) \tag{45}
\]

Before finally substituting to derive the quantum Langevin equations, notice that

\[
\left[ d_{in}(t), d_{in}^\dagger(t') \right]_+ = \frac{1}{2\pi} \int_0^{\infty} dk e^{-i\omega(k)(t-t')} \tag{46}
\]

\[
= \frac{1}{2\pi} \int_0^{\infty} d\omega \frac{dk(\omega)}{d\omega} e^{-i\omega(t-t')} \tag{47}
\]

and if this is being used mostly at frequency \(\omega_0 \gg 0\), then we can approximate to get

\[
\left[ d_{in}(t), d_{in}^\dagger(t') \right]_+ = \frac{dk(\omega_0)}{dk} \delta(t - t') \tag{48}
\]
We want a noise input with anticommutator normalized to $\delta(t - t')$; we therefore define
\[ f_{in}(t) = \left[ \frac{dk(\omega_0)}{d\omega_0} \right]^{-\frac{1}{2}} d_{in}(t), \]
so that (51) becomes
\[ \int_{0}^{\infty} dk \kappa(k) d(k) \approx \frac{\gamma}{2} \hat{c} + \sqrt{\gamma} f_{in}(t), \]
where
\[ \gamma = 2\pi\kappa(\omega_0)^2 \left( \frac{dk(\omega_0)}{d\omega_0} \right). \]
Using this approximation, the interaction Hamiltonian can also be approximated by
\[ H_{int} \approx i\hbar \left\{ \left( \frac{\gamma}{2} \hat{c} + \sqrt{\gamma} f_{in}^\dagger(t) \right) \hat{c} - \hat{c}^\dagger \left( \frac{\gamma}{2} \hat{c}^\dagger + \sqrt{\gamma} f_{in}(t) \right) \right\}, \]
and this is a form which will be useful in the remainder of this paper.

### 3.2.2. Fermionic quantum Langevin equations

Now substitute into (40) to get
\[
\dot{\hat{a}} = -\frac{i}{\hbar} [\hat{a}, H_{sys}] - \frac{1}{\hbar} \left\{ \left( \frac{\gamma}{2} \hat{c} + \sqrt{\gamma} f_{in}^\dagger(t) \right) \hat{c} - \hat{c}^\dagger \left( \frac{\gamma}{2} \hat{c}^\dagger + \sqrt{\gamma} f_{in}(t) \right) \right\} |\hat{a}, \hat{c}|\pm \tag{53}
\]
Equations (53) are the Fermionic quantum Langevin equations for the full system operators. At this stage all we know is that the $f_{in}(t)$ are determined by the $d(k, t_0)$, whose statistics are determined from the initial state of the Fermionic heat bath, and that
\[ \left[ f_{in}(t), f_{in}^\dagger(t') \right]_+ = \delta(t - t'). \tag{54} \]
The validity of (53) and (54) is restricted to situations in which the interaction is rather weak, and in which the time dependence of $\hat{c}(t)$ in the case of no interaction is $e^{-i\omega_0 t}$, for some rather large $\omega_0$.

### 3.2.3. “Out” operators

As in the case of Bosonic quantum white noise, we can define “out” operators, by considering solutions of (55) in terms of a final condition at a time $t_1 > t$. Thus, (55) becomes
\[ d(k, t) = e^{-i\omega(k)(t-t_1)} d(k, t_1) + \int_{t}^{t_1} \kappa(k)e^{-i\omega(k)(t-t')} \hat{c}(t')dt'. \tag{55} \]
and we define
\[ d_{out}(t) = \sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} dk e^{-i\omega(k)(t-t_1)} d(k, t_1). \tag{56} \]
and from (56) and (53)
\[ d_{out}(t) - d_{in}(t) = \sqrt{\frac{1}{2\pi}} \int_{0}^{t_1} dt' \int_{0}^{\infty} dk \kappa(k) e^{i\omega(k)(t-t')} \hat{c}(t')dt'. \tag{57} \]
and using the same methods as in (44) to (48)
\[ d_{out}(t) - d_{in}(t) = \sqrt{2\pi\kappa(\omega_0)} \frac{dk(\omega_0)}{d\omega_0} \hat{c}(t). \tag{58} \]
or, in terms of $f_{in}(t), f_{out}(t)$,
\[ f_{out}(t) - f_{in}(t) = \sqrt{\gamma} \hat{c}(t). \tag{59} \]
We can derive “time reversed” quantum Langevin equations in terms of these “out” noises.
3.2.4. Commutation Relations between System and Input Operators

The operators $d(t), d^\dagger(t)$, defined by (37), commute (anticommute) with all Bosonic (Fermionic) system operators at the same time, since these describe independent degrees of freedom. Thus, using (45) and (49), we can say that if $\tilde{a}(t)$ is an arbitrary system operator

$$[\tilde{a}(t), d(t)]_\pm = 0 \implies [\tilde{a}(t), f_{\text{in}}(t)]_\pm = -\frac{1}{2} \sqrt{\gamma} [\tilde{a}(t), \tilde{c}(t)]_\pm$$

(60)

If we rewrite (59) as

$$f_{\text{in}}(t) + \frac{1}{2} \sqrt{\gamma} \tilde{c}(t) = f_{\text{out}}(t) - \frac{1}{2} \sqrt{\gamma} \tilde{c}(t)$$

(61)

it is easy to rewrite the quantum Langevin equation (53) in the “out” form

$$\dot{\tilde{a}} = -\frac{i}{\hbar} [\tilde{a}, H_{\text{sys}}] - [\tilde{a}, c^\dagger]_\pm \{ -\frac{\gamma}{2} \tilde{c} + \sqrt{\gamma} f_{\text{out}}(t) \}$$

$$\mp \{ -\frac{\gamma}{2} c^\dagger + \sqrt{\gamma} f^\dagger_{\text{out}}(t) \} [\tilde{a}, \tilde{c}]_\pm.$$ 

(62)

3.3. Restricted System Operators

Because the bath operators do not commute with those system operators $\tilde{a}$ which are Fermionic, we have different forms for the quantum Langevin equation depending on whether or not the operator under consideration is Fermionic.

We shall introduce a different set of system operators, called restricted system operators, which do not have this problem, that is, the equations of motion take the same form for all restricted system operators.

To do this we introduce the operator $I$, in the bath space, which anticommutes with all bath operators.

$$[I, d(k)]_+ = [I, d^\dagger(k)]_+ = 0$$

(63)

This operator is easy to construct explicitly. If $|A\rangle$ is any bath state with a definite number $n_A$ of bath Fermions, then we define

$$I|A\rangle = (-1)^{n_A}|A\rangle.$$ 

(64)

Clearly $I$ is Hermitian, $I^2 = 1$, and $I$ commutes with all system operators.

We now define restricted system operators $x$ by

$$x = \begin{cases} 
I \hat{x} & \text{if } \hat{x} \text{ is Fermionic} \\
\hat{x} & \text{if } \hat{x} \text{ is Bosonic} 
\end{cases}$$

(65)

Independently of whether $\hat{x}$ is Fermionic of Bosonic, $x$ commutes with all bath operators. These restricted system operators, $x$, are to be contrasted with the full system operators $\tilde{x}$. The $\hat{x}$ are the more physical operators, since they are the ones that turn up in the Hamiltonian.

The restricted system operators are necessary when we wish to consider operations on the reduced density operator $\rho_{\text{sys}} \equiv \text{Tr}_B \{ \rho \}$.

In order to rewrite the Hamiltonian in terms of the restricted operators, it is necessary only to rewrite $H_{\text{Int}}$ as

$$H_{\text{Int}} = i\hbar \int_0^\infty dk \kappa(k) \{ d^\dagger(k) Ic - c^\dagger Id(k) \}$$

(66)
3.3.1. Commutation Relations for \(I(t)\) Because \(I\) does not commute with \(H_{\text{int}}\), it is time dependent, and we re-write is as \(I(t)\). Because \(I(t)\) anticommutes with \(d(k,t), d^\dagger(k,t)\), we deduce from (45) and (47) that

\[
[I(t), f_{\text{in}}(t)]_+ = -\sqrt{\gamma} c(t) = -\sqrt{\gamma} I \bar{c}(t) \tag{67}
\]

\[
[I(t), f^\dagger_{\text{in}}(t)]_+ = -\sqrt{\gamma} c^\dagger(t) = -\sqrt{\gamma} I \bar{c}^\dagger(t) \tag{68}
\]

The commutators of \(I(t)\) with either restricted system operators \(x(t)\) or the full system operators \(\bar{x}(t)\) are zero. The fact that \(I(t)\) does not commute with \(f_{\text{in}}(t)\) or \(f^\dagger_{\text{in}}(t)\), and is time dependent, makes the quantum Langevin equations for the restricted operators, \(\ref{eq:quantum_langevin_72}\), in general quite different from those for the full operators, \(\ref{eq:quantum_langevin_15}\).

However, because \(I(t)^2 = 1\), and because \(I(t)\) commutes with system operators of either kind, all equal times algebraic relations between different system operators are the same for both full and restricted system operators.

3.3.2. Langevin equation for \(I(t)\) Using the commutation relations \(\ref{eq:commutation_67,68}\), the equation of motion for \(I(t)\) can be deduced by much the same reasoning as above, to be

\[
\dot{I} = -2\sqrt{\gamma} \left\{ f^\dagger_{\text{in}}(t) c + c^\dagger f_{\text{in}}(t) + \sqrt{\gamma} I \bar{c} c \right\}, \tag{69}
\]

\[
\dot{I} = -2\sqrt{\gamma} \left\{ f^\dagger_{\text{in}}(t) \bar{c} + \bar{c}^\dagger f_{\text{in}}(t) + \sqrt{\gamma} I \bar{c}^\dagger \bar{c} \right\}. \tag{70}
\]

Finally, notice that \(I(t)^2 = 1\) implies that \(\dot{I} + I I = 0\). This can be explicitly shown from \(\ref{eq:commutation_69}\) by using \(\ref{eq:commutation_67,68}\).

The initial condition, that is the operator \(I(t_0)\), is determined by the formulae \(\ref{eq:initial_condition_65,66,67}\), using \(d(k,t_0), d^\dagger(k,t_0)\), and by definition \(f_{\text{in}}(t), f^\dagger_{\text{in}}(t)\), are linear functions of \(d(k,t_0), d^\dagger(k,t_0)\). Thus, we can say that

\[
[I(t_0), f_{\text{in}}(t)]_+ = [I(t_0), f^\dagger_{\text{in}}(t)]_+ = 0. \tag{71}
\]

3.3.3. Langevin equations for the restricted system operators Similar reasoning can be used to deduce that for all restricted system operators \(a\), the quantum Langevin equations take the form

\[
\dot{a} = \frac{\gamma}{\hbar} [a, H_{\text{sys}}] - [a, c^\dagger] \left\{ \frac{\gamma}{2} c + \sqrt{\gamma} f_{\text{in}}(t) \right\} + \left\{ \frac{\gamma}{2} c^\dagger + \sqrt{\gamma} f^\dagger_{\text{in}}(t) I \right\} [a, c]. \tag{72}
\]

This equation can also be deduced by substituting \(\dot{a} = a, or Ia\) as the case may be, into \(\ref{eq:quantum_langevin_53}\), and using \(\ref{eq:commutation_57}\), together with \(\ref{eq:commutation_67,68}\).

The major advantage of the use of restricted system operators is that they are truly independent of the bath operators evaluated at the same time. The whole burden of antisymmetrization required between bath and system is borne by the operator \(I(t)\), which is not purely a function of \(f_{\text{in}}(t), f^\dagger_{\text{in}}(t)\), but is a dynamical variable whose equation of motion must be considered alongside that of the system operators \(a\)—namely the restricted operator quantum Langevin equation \(\ref{eq:quantum_langevin_72}\).

3.3.4. Fermion conservation superselection rule Fermions can only be created and destroyed in pairs. Hence if \(N_B\) is the number of Fermions in the bath, and \(N_{\text{sys}}\) is the number contained in the system, the quantity

\[
K \equiv (-1)^{N_B + N_{\text{sys}}} \tag{73}
\]
where we have defined the notation which amounts in practice to a restricted operator expression of the full system operator. Only if there are an odd number of Fermionic system operators in the correlation function conditions at time $t$ of the left hand sides of (83,84), and then use the relations (83,84) for $t' = s$ to show that the results vanish. Hence that the only solution of the resultant differential equations corresponds to the truth of (83,84). In a similar way, one can show that 

$$[a(t'), f_{\text{out}}(t)] = [a(t'), f_{\text{out}}^I(t)] = 0 \quad \text{for } t \leq t' \leq t_1;$$

$$[I(t'), f_{\text{out}}(t)]_+ = [I(t'), f_{\text{out}}^I(t)]_+ = 0 \quad \text{for } t \leq t' \leq t_1.$$
Noting now (59), we can derive
\[ [a(t'), f_{in}(t)] = -u(t' - t)\sqrt{\gamma}[a(t'), \bar{c}(t)], \]
\[ [a(t'), f_{in}^\dagger(t)] = -u(t' - t)\sqrt{\gamma}[a(t'), \bar{c}^\dagger(t)], \]
\[ [I(t'), f_{in}(t)]_+ = -u(t' - t)\sqrt{\gamma}[I(t'), \bar{c}(t)], \]
\[ [I(t'), f_{in}^\dagger(t)]_+ = -u(t' - t)\sqrt{\gamma}[I(t'), \bar{c}^\dagger(t)]. \]
From these it also follows that for any Fermionic full system operator \( \tilde{g} \)
\[ [\tilde{g}(t'), f_{in}(t)]_+ = -u(t' - t)\sqrt{\gamma}[\tilde{g}(t'), \bar{c}(t)]_+, \]
\[ [\tilde{g}(t'), f_{in}^\dagger(t)]_+ = -u(t' - t)\sqrt{\gamma}[\tilde{g}(t'), \bar{c}^\dagger(t)]_+. \]
These results are derived by writing \( \tilde{g}(t) = g(t)f(t) \), and expanding the commutator using (87, 90). The corresponding results for a Bosonic full system operator, which is identical with its corresponding restricted form, have the same form as (87, 88).

4. Fermionic Quantum White Noise and Quantum Stochastic Differential Equations

The operators \( f_{in}(t), f_{in}^\dagger(t) \) have the idealized anticommutation relations (54), which leads naturally to a formulation of Fermionic quantum white noise. We define a Fermionic Quantum Wiener Process by
\[ F(t, t_0) = \int_{t_0}^t f_{in}(t')dt', \]
and we assume the averages,
\[ \langle F^\dagger(t, t_0)F(t, t_0) \rangle = \bar{N}(t - t_0) \]
\[ \langle F(t, t_0)F^\dagger(t, t_0) \rangle = (1 - \bar{N})(t - t_0) \]
and the anticommutator, from (54)
\[ \langle [F(t, t_0), F^\dagger(t, t_0)]_+ \rangle = (t - t_0) \]
We of course also assume the independence of the \( F \) operators defined on non-overlapping time intervals;
\[ \langle F^\dagger(t, t_0)F(s, s_0) \rangle = \langle F(s, s_0)F^\dagger(t, t_0) \rangle \\
= [F(t, t_0), F(s, s_0)]_+ = 0 \]
if \( (s, s_0) \) and \( (t, t_0) \) are disjoint. In frequency space, this can be obtained by writing
\[ f_{in}(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f_0(\omega)e^{-i\omega t}d\omega \]
with
\[ [f_0(\omega), f_0^\dagger(\omega')]_+ = \delta(\omega - \omega') \]
\[ [f_0^\dagger(\omega)f_0(\omega')] = \bar{N}\delta(\omega - \omega') \]
\[ f_0(\omega)f_0^\dagger(\omega') = (1 - \bar{N})\delta(\omega - \omega') \]
The \( f_0(\omega) \) are therefore like idealized Fermion destruction operators, defined on a frequency range \((-\infty, \infty)\). In practice, the \( d_0(k) \) are the true destruction operators, and the correspondence between \( f_0(\omega) \) and \( d_0(k) \) is made via the relationship (50), and is only valid for a narrow bandwidth around the frequency \( \omega_0 \). The components of \( f_{in}(t) \) and \( d_{in}(t) \) outside this narrow bandwidth have little effect on the solutions of the quantum Langevin equations.
Corresponding to the formula for Bosonic white noise, the density operator which gives the averages \( \rho_W(t, t_0) \) has the form
\[
\rho_W(t, t_0) = (1 + e^{-\mu}) \exp \left[ -\frac{\mu F(t, t_0) F(t, t_0)}{t - t_0} \right]
\]
in which
\[
\bar{N} = \frac{1}{e^{\mu} + 1}
\]

The formulation of \( F(t, t_0) \) as above enables us to develop a formal theory of quantum stochastic integration and quantum stochastic differential equations which is quite simple and easy to use, and whose use gives essentially the same results as any exact formulation. We will also be able to show that the quantum stochastic differential equations so developed are exactly equivalent to the master equation for systems interacting with a Fermionic heat bath.

To do this, we partition the time interval \((t_0, t_f)\), inside which we are interested in treating the motion, into subintervals at bounded by times \(t_0, t_1, t_2, \ldots, t_n \equiv t_f\) corresponding to the increments
\[
\Delta t_i \equiv t_{i+1} - t_i
\]
\[
\Delta F_i \equiv F(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} f_{in}(t')dt'.
\]

Depending on whether \( \tilde{g}(t) \) is a Bosonic or a Fermionic operator, \( \Delta F_i \) will commute or anticommute with \( \tilde{g}(t_i) \).

The corresponding joint density operator for the Fermi noises partitioned this way is then the direct product
\[
\rho_F = \rho_W(t_n, t_{n-1}) \otimes \rho_W(t_{n-1}, t_{n-2}) \otimes \ldots \otimes \rho_W(t_2, t_1) \otimes \rho_W(t_1, t_0).
\]

This form means that we can consider a trace operation over the Fermion bath to be taken over each time interval in a discretization; thus if \( Q \) is some operator which acts on the fermion bath, we can write
\[
\text{Tr}_B \{ Q \rho_F \} = \text{Tr}_{(t_n, t_{n-1})} \{ \text{Tr}_{(t_{n-1}, t_{n-2})} \{ \ldots \text{Tr}_{(t_2, t_1)} \{ \text{Tr}_{(t_1, t_0)} \{ Q \rho_F \} \} \} \}.
\]

This form is particularly useful in deriving correlation function identities.

4.1. Quantum Stochastic Integration

As in the case of classical stochastic integration with respect to white noise, there are two natural definitions of integration, the Ito and Stratonovich methods. The definition of these is relatively straightforward.

4.1.1. Fermionic quantum Ito integral If \( \tilde{g}(t) \) is a full system operator (not a restricted system operator), the Fermionic quantum Ito integral is defined by
\[
(I) \int_{t_0}^{t} \tilde{g}(t') dF(t') = \lim_{n \to \infty} \sum_i \tilde{g}(t_i) \Delta F_i
\]
where \( t_0 < t_1 < t_2 < \ldots < t_n = t \), and the limit is a mean-square limit. A similar definition is used for \( \int_{t_0}^{t} \tilde{g}(t') dF^\dagger(t') \). The advantage of the Ito definition \( (108) \) of the integral is that the increment, \( \Delta F_i \) is seen in the explicit definition on the right hand side, to be in the future of \( t_i \).
In particular, as is the case for classical and Bosonic stochastic integration,
\[
\langle (I) \int_{t_0}^{t} \tilde{g}(t') \, dF(t') \rangle = 0, \quad (109)
\]
\[
\langle (I) \int_{t_0}^{t} \tilde{g}(t') \, dF^\dagger(t') \rangle = 0. \quad (110)
\]
Further, depending on whether \( \tilde{g}(t) \) is Bosonic or Fermionic
\[
(I) \int_{t_0}^{t} [\tilde{g}(t'), dF(t')]_{\pm} = 0, \quad (111)
\]
\[
(I) \int_{t_0}^{t} [\tilde{g}(t'), dF^\dagger(t')]_{\pm} = 0. \quad (112)
\]

### 4.1.2. Fermionic quantum Stratonovich integral

The Stratonovich integral can be defined in the same way as it is for Bosonic noise
\[
(S) \int_{t_0}^{t} \tilde{g}(t') \, dF(t') = \lim_{n \to \infty} \sum_{i} \frac{\tilde{g}(t_i) + \tilde{g}(t_{i+1})}{2} \Delta F_i. \quad (113)
\]
As in the case of classical and Bosonic noise, we cannot make any connection between these two forms of integral without knowing what kind of stochastic differential equation is obeyed by the system operators.

Simple relations of the kind (109–112) do not hold. For the first two we need to establish the relationship between the two kinds of integral first, and this will be done in Sect. 4.3.

### 4.2. Ito quantum stochastic differential equation

We will define the Ito quantum stochastic differential equation obeyed by a restricted system operator \( \hat{a} \) as
\[
(I) \, d\hat{a} = -\frac{i}{\hbar} [\hat{a}, H_{\text{sys}}] \, dt + \frac{\gamma}{2} (1 - \bar{N}) \left\{ 2c^\dagger c^\dagger a - ac^\dagger c - c^\dagger a^\dagger \right\} \, dt \\
+ \frac{\gamma}{2} \bar{N} \left\{ 2ca^\dagger c - acc^\dagger - cc^\dagger a \right\} \, dt \\
- \sqrt{\gamma} [a, c^\dagger] \, dF(t) + \sqrt{\gamma} dF^\dagger(t) \, I [a, c]. \quad (114)
\]
This can be written in the alternative form
\[
(I) \, d\hat{a} = -\frac{i}{\hbar} [\hat{a}, H_{\text{sys}}] \, dt + \left\{ \frac{\gamma}{2} c^\dagger [a, c] - \frac{\gamma}{2} [a, c^\dagger] c \right\} \, dt \\
+ \frac{\gamma}{2} \bar{N} \left\{ [c, [a, c^\dagger]]_+ - [c^\dagger, [a, c]]_+ \right\} \, dt \\
- \sqrt{\gamma} [a, c^\dagger] \, dF(t) + \sqrt{\gamma} dF^\dagger(t) \, I [a, c]. \quad (115)
\]
These definitions are made with foresight; using them we will show how to connect the Ito and Stratonovich definitions of the quantum stochastic differential equation, ultimately showing that the Fermionic Langevin equations as derived above are then to be interpreted as Stratonovich quantum stochastic differential equations.
4.3. Connection between Ito and Stratonovich integrals

We write the term \( \tilde{g}(t_{i+1}) \) in the definition of the Stratonovich integral as

\[
\tilde{g}(t_{i+1}) = \tilde{g}(t_i) + \Delta \tilde{g}(t_i),
\]

where \( \Delta \tilde{g}(t_i) \) is to be calculated using the Ito quantum stochastic differential equation \([14]\).

Since \( \Delta \tilde{g}(t_i) \) is to be used in \([13]\), only the terms involving the noise operators will be significant, since in the limit terms involving \( \Delta t_i \) as well as a noise operator vanish.

The conversion formula will differ, depending on whether \( \tilde{g} \) is Fermionic or Bosonic; we will consider first the Fermionic case. The quantum stochastic differential equation \([14]\) is written for the restricted system operators, but we can write

\[
\tilde{g}(t_{i+1}) = I(t_{i+1})g(t_{i+1}) = K J(t_{i+1})g(t_{i+1}) = K \tilde{g}(t_{i+1}).
\]

Since \( \tilde{g} \) is a restricted system operator, it obeys the quantum stochastic differential equation \([14]\), which means that we can write for the stochastic part of \( \Delta \tilde{g}(t_i) \)

\[
\Delta \tilde{g}(t_i)_{\text{stochastic}} = K \left\{ -\sqrt{\gamma} [\tilde{g}, c^\dagger] I \Delta F_i + \sqrt{\gamma} \Delta F_i^\dagger I[\tilde{g}, c] \right\}.
\]

From \([24, 95]\) we can write in stochastic integrals

\[
\Delta F_i^\dagger \Delta F_i = \bar{N} \Delta t_i,
\]

\[
\Delta F_i \Delta F_i^\dagger = (1 - \bar{N}) \Delta t_i,
\]

and the anticommutation relations also give

\[
\Delta F_i \Delta F_i^\dagger = \Delta F_i^\dagger \Delta F_i = 0.
\]

Carrying out similar reasoning for other integrals, we find that we can write

**Fermionic integrand \( \tilde{g} \):**

\[
\begin{align*}
(\text{S}) & \int_{t_0}^{t} \tilde{g}(t') dF(t') = (\text{I}) \int_{t_0}^{t} \tilde{g}(t') dF(t') - \frac{\sqrt{\gamma} \bar{N}}{2} \int_{t_0}^{t} [g(t'), c(t')]_+ dt', \\
(\text{S}) & \int_{t_0}^{t} dF(t') \tilde{g}(t') = (\text{I}) \int_{t_0}^{t} dF(t') \tilde{g}(t') - \frac{\sqrt{\gamma} (1 - \bar{N})}{2} \int_{t_0}^{t} [g(t'), c(t')]_+ dt', \\
(\text{S}) & \int_{t_0}^{t} \tilde{g}(t') dF^\dagger(t') = (\text{I}) \int_{t_0}^{t} \tilde{g}(t') dF^\dagger(t') - \frac{\sqrt{\gamma} (1 - \bar{N})}{2} \int_{t_0}^{t} [g(t'), c^\dagger(t')]_+ dt', \\
(\text{S}) & \int_{t_0}^{t} dF^\dagger(t') \tilde{g}(t') = (\text{I}) \int_{t_0}^{t} dF^\dagger(t') \tilde{g}(t') - \frac{\sqrt{\gamma} \bar{N}}{2} \int_{t_0}^{t} [g(t'), c^\dagger(t')]_+ dt'.
\end{align*}
\]

Notice that stochastic integrals use the full operator \( \tilde{g} \), while the correction terms all use the restricted system operators \( g, c \).
Bosonic integrand $g$: In this case the restricted operator $g$ is identical with the full system operator $\tilde{g}$, so we can write everything in terms of $g$.

\begin{align}
(S) \int_{t_0}^{t} g(t') \, dF(t') &= (I) \int_{t_0}^{t} g(t') \, dF(t') + \sqrt{\tilde{N}} \int_{t_0}^{t} [g(t'), c(t')] \, dt', \\
(S) \int_{t_0}^{t} dF(t') \, g(t') &= (I) \int_{t_0}^{t} dF(t') \, g(t') - \sqrt{\tilde{N}} (1 - \tilde{N}) \int_{t_0}^{t} [g(t'), c(t')] \, dt', \\
(S) \int_{t_0}^{t} g(t') \, dF^\dagger(t') &= (I) \int_{t_0}^{t} g(t') \, dF^\dagger(t') + \sqrt{\tilde{N}} (1 - \tilde{N}) \int_{t_0}^{t} [g(t'), c^\dagger(t')] \, dt', \\
(S) \int_{t_0}^{t} dF^\dagger(t') \, g(t') &= (I) \int_{t_0}^{t} dF^\dagger(t') \, g(t') - \sqrt{\tilde{N}} \int_{t_0}^{t} [g(t'), c^\dagger(t')] \, dt'.
\end{align}

4.4. Stratonovich quantum stochastic differential equation

We want to write the Stratonovich equivalent of the quantum stochastic differential equations (114,115), from which we see that the Ito integrals to be converted into Stratonovich integrals correspond to making the choices

\begin{align}
\tilde{g}(t) \, dF(t) &\rightarrow - \sqrt{\bar{N}} [a(t), c^\dagger(t)] \, I(t) \, dF(t), \\
dF^\dagger(t) \, \tilde{g}(t) &\rightarrow \sqrt{\bar{N}} dF^\dagger(t) \, I(t) \, [a(t), c(t)].
\end{align}

The combinations $[a(t), c^\dagger(t)] \, I(t)$ and $I(t) \, [a(t), c(t)]$ thus form the appropriate Fermionic full system operators. Using therefore (129,132), we find the total correction term is

\begin{equation}
- \frac{\bar{N}}{2} \left\{ [c, [a, c^\dagger]]_+ - [c^\dagger, [a, c]]_+ \right\},
\end{equation}

and this is precisely the negative of the third line of (115), which leads to the Stratonovich form

\begin{equation}
(S) \, da = - \frac{i}{\hbar} [a, H_{\text{sys}}] \, dt + \left\{ \frac{\gamma}{2} c^\dagger [a, c] - \frac{\gamma}{2} [a, c^\dagger]c \right\} \, dt \\
- \sqrt{\bar{N}} [a, c^\dagger] \, I \, dF(t) + \sqrt{\bar{N}} \, dF^\dagger(t) \, I \, [a, c].
\end{equation}

This corresponds exactly to the Langevin equation for the restricted system operator as given in (12), and thus justifies the form (114,115) chosen for the Ito quantum stochastic differential equation.

5. The master equation for the system density operator

In the Heisenberg picture, the density operator is time independent, and takes the form

\begin{equation}
\rho = \rho_{\text{sys}}(t_0) \otimes \rho_F.
\end{equation}

This corresponds to an assumption that the initial density operator in the Schrödinger picture can be factorized into a bath and a system term.

Using this density operator, we would say that

\begin{align}
\langle a(t) \rangle &= \text{Tr}_{\text{sys}} \{ a \rho_{\text{sys}}(t) \}, \\
d_{dt} \langle a(t) \rangle &= \text{Tr}_{\text{sys}} \left\{ a \frac{d \rho_{\text{sys}}(t)}{dt} \right\}.
\end{align}
On the other hand, the quantum stochastic differential equation (144) can be used in conjunction with the fact that the means of the Ito integrals are zero to show that
\[
\frac{d}{dt} \langle a(t) \rangle = \left\langle -\frac{i}{\hbar}[a, H_{\text{sys}}] + \frac{\gamma}{2}(1 - \bar{N}) \left\{ 2c^{\dagger}ac - acc^{\dagger} - c^{\dagger}ca \right\} \right. \\
\left. + \frac{\gamma}{2}\bar{N} \left\{ 2cac^{\dagger} - acc^{\dagger} - cc^{\dagger}a \right\} \right\rangle 
\]
from which we deduce, since this equation is true for any restricted system operator \(a\)
\[
\frac{d \rho_{\text{sys}}}{dt} = -\frac{i}{\hbar}[\rho_{\text{sys}}, H_{\text{sys}}] + \frac{\gamma}{2}(1 - \bar{N})\left\{ 2c\rho_{\text{sys}}c^{\dagger} - \rho_{\text{sys}}c^{\dagger}c - c^{\dagger}c\rho_{\text{sys}} \right\} \\
+ \frac{\gamma\bar{N}}{2}\left\{ 2c\rho_{\text{sys}}c - cc^{\dagger}\rho_{\text{sys}} - c^{\dagger}\rho_{\text{sys}}cc^{\dagger} \right\}. 
\]  

6. Quantum stochastic differential equations in the interaction picture

For a discussion of many aspects of input-output theory it is advantageous to formulate an appropriate quantum stochastic differential equation theory in the interaction picture, as explained in Chap. 11 of [5] in the case of Bosonic noise. In such a situation, the time-dependent state vectors can be written in terms of the evolution operator \(U(t, t_0)\) as
\[
|\psi, t\rangle = U(t, t_0)|\psi, t_0\rangle. 
\]  
In much the same way as for Bosonic noise, we can write a Stratonovich quantum stochastic differential equation for the evolution operator corresponding to the Hamiltonian (31) in the form
\[
(S) dU(t, t_0) = -\frac{i}{\hbar}H_{\text{sys}}U(t, t_0)dt + \left\{ \sqrt{\gamma}dF^{\dagger}(t)c + \sqrt{\gamma}dF(t)c^{\dagger} \right\} U(t, t_0). 
\]  
Our aim now is to transform this to a corresponding Ito form. We assume there exists an Ito equation in the form
\[
(I) dU(t, t_0) = \left\{ \alpha(t) dt + \beta(t) dF^{\dagger}(t) + \beta^{\dagger}(t) dF(t) \right\} U(t, t_0), 
\]
and from this derive the relationship between the Stratonovich and Ito integrals.

A Stratonovich integral of the evolution operator is defined by
\[
(S) \int dF(s)U(s) = \lim_{n \to \infty} \sum_{i} \Delta F_{i} \frac{U(t_{i+1}) + U(t_{i})}{2}, 
\]  
We now express \(U_{i+1}\) in terms of \(U_{i}\) using the Ito quantum stochastic differential equation (148), and neglecting terms of order of magnitude \(\Delta t^{3/2}\) and higher, we get
\[
(S) \int dF(s)U(s) = \lim_{n \to \infty} \sum_{i} \Delta F_{i} \left\{ 1 + \frac{1}{2}\beta_{i} \Delta F_{i}^{\dagger} \right\} U(t_{i}) \\
= \lim_{n \to \infty} \sum_{i} \left\{ \Delta F_{i} - \frac{1}{2}\beta_{i}(1 - \bar{N})\Delta t_{i} \right\} U(t_{i}) \\
= (I) \int dF(s)U(s) - \frac{1}{2}(1 - \bar{N}) \int \beta(s)U(s) ds. 
\]
Similarly
\[
(S) \int dF^{\dagger}(s)U(s) = (I) \int dF^{\dagger}(s)U(s) - \frac{1}{2}\bar{N} \int \beta^{\dagger}(s)U(s) ds. 
\]
Using these relations, we can then convert the Stratonovich quantum stochastic differential equation to the equivalent Ito form

$$dU(t, t_0) = -\left\{ \frac{i}{\hbar} H_{\text{sys}} + \frac{\gamma N}{2} \hat{c}\hat{c} + \frac{\gamma (1 - N)}{2} \hat{c}\hat{c} + \frac{\gamma i}{2} \hat{c}\hat{c} \right\} U(t, t_0) dt$$

$$+ \left\{ \sqrt{\gamma} dF^\dagger(t)\hat{c} + \sqrt{\gamma} dF(t)\hat{c} \right\} U(t, t_0), \quad (154)$$

The full density operator at time $t$ can be written

$$\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0) \quad (155)$$

and so obeys the quantum stochastic differential equation

$$d\rho(t) = \left\{ -\frac{i}{\hbar} [H_{\text{sys}}, \rho(t)] - \frac{\gamma N}{2} [\rho(t), \hat{c}\hat{c}^\dagger]_+ - \frac{\gamma (1 - N)}{2} [\rho(t), \hat{c}\hat{c}^\dagger]_+ \right\} dt$$

$$- \gamma \hat{c} dF^\dagger(t)\rho(t)dF(t)\hat{c}^\dagger - \gamma \hat{c}\hat{c}^\dagger dF(t)\rho(t)dF^\dagger(t)\hat{c}$$

$$+ \sqrt{\gamma} [dF^\dagger(t)\hat{c} + dF(t)\hat{c}^\dagger, \rho(t)] \quad (156)$$

### 6.1. Alternative derivation of the master equation

To derive the master equation from the evolution equation (156), still requires us to use the restricted system operators as follows. Firstly, note that the for the product operators we have

$$\hat{c}\hat{c} = c^\dagger c, \quad \hat{c}\hat{c}^\dagger = c^\dagger c\hat{c}, \quad (157)$$

and of course the restricted operator form of $H_{\text{sys}}$ is $H_{\text{sys}}$ itself. These operators are therefore proportional to the identity in the bath space, and we can write, for example

$$\text{Tr}_B \left\{ [\rho(t), \hat{c}\hat{c}^\dagger]_+ \right\} = \text{Tr}_B \left\{ [\rho(t), c^\dagger c^\dagger]_+ \right\} = \text{Tr}_B \left\{ \rho(t) \right\} c^\dagger c^\dagger = [\rho_{\text{sys}}(t), c^\dagger c^\dagger]_+ \quad (158)$$

However, for the terms on the second line of (156), this is not immediately possible, since $\hat{c}$ and $\hat{c}^\dagger$ do act in the bath space, since they do not commute with the noises. We therefore have to convert to restricted system operators. In the interaction picture the operator $I$ introduced in (64) is time independent, so we can write, for example

$$\text{Tr}_B \left\{ \hat{c} dF^\dagger(t)\rho(t)dF(t)\hat{c}^\dagger \right\} = \text{Tr}_B \left\{ cI dF^\dagger(t)\rho(t)dF(t) I \right\}$$

$$= c \text{Tr}_B \left\{ I dF^\dagger(t)\rho(t)dF(t) I \right\}$$

$$= c \text{Tr}_B \left\{ dF(t) I^2 dF^\dagger(t)\rho(t) \right\}$$

$$= (1 - \hat{N})c\rho_{\text{sys}} c\hat{c} \quad (159)$$

The trace over the last line of (156) gives zero, so we derive again the master equation in the form (145).

### 6.2. Correlation functions

From the density operator one can in principle compute all correlation functions for the restricted system operators. Using the relationships between the operator forms $\hat{a}$, $\hat{a}^\dagger$ and $a$, as applied in Sect.3.3.4, it is then possible compute the correlation functions for the full system operators.

The importance of the full system operators comes from the need to compute the correlation functions of the output operators which arises because of the relationship (59) between outputs, inputs and the full system operators.
6.2.2. White noise input  

We can compute

\[ \langle f_{\text{out}}^d(t')f_{\text{out}}(t) \rangle = \left\langle \left( f_{\text{in}}^d(t') - \sqrt{\gamma} \tilde{c}^\dagger(t') \right) \left( f_{\text{in}}(t) - \sqrt{\gamma} \tilde{c}(t) \right) \right\rangle. \] (160)

The simplest case is if the input field corresponds to the vacuum, in which case we get

\[ \langle f_{\text{out}}^d(t')f_{\text{out}}(t) \rangle = \gamma \langle \tilde{c}^\dagger(t')\tilde{c}(t) \rangle \] (161)

\[ = \gamma \langle \tilde{c}^\dagger(t')\tilde{c}(t) \rangle \] (162)

If we want the number counting correlation function, then we find in much the same way as the Bosonic case that for the \textit{time ordered} correlation function, in which \( t' > t \)

\[ \langle f_{\text{out}}^d(t)f_{\text{out}}^d(t')(f_{\text{out}}(t')f_{\text{out}}(t)) \rangle = \gamma^2 \langle \tilde{c}^\dagger(t)\tilde{c}(t')\tilde{c}(t')\tilde{c}(t) \rangle \] (163)

\[ = \gamma^2 \langle \tilde{c}^\dagger(t)\tilde{c}(t')\tilde{c}(t')\tilde{c}(t) \rangle. \] (164)

6.2.2. White noise input  

Let us consider again the correlation function \( \langle f_{\text{out}}^d(t')f_{\text{out}}(t) \rangle \) in the case that the input is non-vacuum, and for \( t < t' \). In this case (160) becomes

\[ \langle f_{\text{out}}^d(t')f_{\text{out}}(t) \rangle = \left\langle \left( f_{\text{in}}^d(t') - \sqrt{\gamma} \tilde{c}^\dagger(t') \right) \left( f_{\text{in}}(t) - \sqrt{\gamma} \tilde{c}(t) \right) \right\rangle \] (165)

\[ \equiv - \sqrt{\gamma} \tilde{c}(t')(f_{\text{in}}(t) - \sqrt{\gamma} \tilde{c}(t)) \] (166)

\[ = - \sqrt{\gamma} \langle \tilde{c}(t') \{ dF(t) - \sqrt{\gamma} \tilde{c}(t) dt \} \rangle / dt. \] (167)

Here we will consider \( dF(t) \) to be an Ito increment, but there will be no difference between an Ito and a Stratonovich version of the increment in this formula except when the two times \( t \) and \( t' \) are equal.

Thus it is necessary to compute \( \langle \tilde{c}^\dagger(t') dF(t) \rangle \) where \( t < t' \). For a general full system operator \( \hat{a}(t') \) we can write (where \( t_1 \equiv t + \Delta t \), for brevity)

\[ \langle \hat{a}(t') \Delta F(t) \rangle = \text{Tr}_{\text{sys}} \left\{ \text{Tr}_{(t, t')\lbrace \hat{a}U(t', t)U(t, t)\Delta F(t)\rho(t)U^{-1}(t, t)U^{-1}(t', t_1) \rbrace} \right\} \] (168)

The evolution operator \( U(t', t_1) \) contains no dependence on the noise in the interval \( (t, t_1) \), so we can than write

\[ \langle \hat{a}(t') \Delta F(t) \rangle = \text{Tr}_{\text{sys}} \left\{ \hat{a} \text{Tr}_{(t, t')\lbrace U(t', t_1)U(t, t)\Delta F(t)\rho(t)U^{-1}(t, t)U^{-1}(t', t_1) \rbrace} \right\} \] (169)

We now write the infinitesimal form

\[ U(t_1, t) \equiv U(t + \Delta t, t) = - \left\{ \frac{1}{\hbar} H_{\text{sys}} + \frac{\gamma}{2} \tilde{c}^\dagger \tilde{c} + \frac{\gamma(1 - \bar{N})}{2} \tilde{c}^\dagger \tilde{c} \right\} \Delta t \]

\[ + \left\{ \sqrt{\gamma} \Delta F^\dagger(t) \tilde{c} + \sqrt{\gamma} \Delta F(t) \tilde{c}^\dagger \right\} \] (170)

Now computing the trace over the interval \( (t, t_1) \equiv (t, t + \Delta t) \), and keep only terms of order \( \Delta t \), we get

\[ \text{Tr}_{(t, t_1)} \left\{ U(t_1, t)\Delta F(t)\rho(t)U^{-1}(t, t) \right\} = \sqrt{\gamma} \tilde{N} \left[ \tilde{c}, \rho(t) \right] \Delta t, \] (171)

so that

\[ \langle \hat{a}(t') \Delta F(t) \rangle = \sqrt{\gamma} \tilde{N} \text{Tr}_{\text{sys}} \left\{ \hat{a} \text{Tr}_{(t, t')\left\lbrace U(t', t_1)\tilde{c}, \rho(t)U^{-1}(t', t_1) \right\rbrace} \right\} \Delta t \] (172)

Taking account of the fact that for \( t' < t \) the average obviously vanishes, we get

\[ \langle \hat{a}(t') dF(t) \rangle = \sqrt{\gamma} \tilde{N} u(t' - t) \langle [\tilde{a}(t'), \tilde{c}(t)] \rangle \Delta t. \] (173)
Using this result we get
\[
\langle f_{\text{out}}^\dagger(t') f_{\text{out}}(t) \rangle = \gamma(1 - \bar{N})\langle \hat{c}(t')\hat{c}(t) \rangle - \gamma\bar{N}\langle \hat{c}(t)\hat{c}(t') \rangle + \bar{N}\delta(t - t').
\] (174)

The result is valid for all \(t, t'\); the case for \(t > t'\) is given by the complex conjugate of that for \(t < t'\).

7. Examples

7.1. The two-level ion

We consider a two level system in which we have a lower energy level with an even number of electrons and an upper level with an odd number of electrons—hence the terminology “ion”. The system interacts with an electron field, and thus may be described by the choices
\[
c = \sigma^-,
\] (175)
\[
c^\dagger = \sigma^+,
\] (176)
\[
H_{\text{sys}} = \frac{\hbar \omega}{2} \sigma_z,
\] (177)

so that
\[
\tilde{c} = I\sigma^- \equiv \tilde{\sigma}^-,
\] (178)
\[
\tilde{c}^\dagger = I\sigma^+ \equiv \tilde{\sigma}^+,
\] (179)
\[
J = \sigma_z,
\] (180)
\[
\tilde{c} = J\sigma^- = -\sigma^-, \quad (\tilde{c})^\dagger = \sigma^+ J = -\sigma^+.
\] (181)

Quantum Langevin equations for the full system operators

Using the usual commutation relations in (53), we get the explicit equations of motion
\[
\dot{\tilde{\sigma}}^+ = \left( i\omega - \frac{\gamma}{2} \right) \tilde{\sigma}^+ - \sqrt{\gamma} f_{\text{in}}^\dagger(t),
\] (182)
\[
\dot{\tilde{\sigma}}^- = - \left( i\omega + \frac{\gamma}{2} \right) \tilde{\sigma}^- - \sqrt{\gamma} f_{\text{in}}(t),
\] (183)
\[
\dot{\tilde{\sigma}}_z = - \gamma (\tilde{\sigma}_z + 1) - 2\sqrt{\gamma} \tilde{\sigma}^+ f_{\text{in}}(t) - 2\sqrt{\gamma} \tilde{\sigma}^- f_{\text{in}}^\dagger(t).
\] (184)

The equations (182, 183) are linear in \(\tilde{\sigma}^\pm, f_{\text{in}}, f_{\text{in}}^\dagger\), and are thus exactly solvable. Although the equation for \(\tilde{\sigma}_z\) is not linear, its solution follows from the other equations by using identity \([\tilde{\sigma}^+, \tilde{\sigma}^-] = \tilde{\sigma}_z\).

This solvability arises because a two level ion is in fact the same thing as another Fermion degree of freedom, that is, in this case the operators \(\tilde{c}, \tilde{c}^\dagger\) are like Fermion creation and destruction operators since \([\tilde{c}, \tilde{c}^\dagger] = 1\) and they anticommute with the bath Fermion operators.

This is exactly the same situation as for a harmonic oscillator interacting with a Bose input field, which is exactly solvable in the same way.

Stratonovich quantum stochastic differential equations for the restricted system operators

These take the form, from (140)
\[
(S)\, d\tilde{\sigma}^+ = \left( i\omega - \frac{\gamma}{2} \right) \sigma^+ dt + \sqrt{\gamma} dF^\dagger(t) (I\sigma_z),
\] (185)
\[
(S)\, d\tilde{\sigma}^- = - \left( i\omega + \frac{\gamma}{2} \right) \sigma^- dt + \sqrt{\gamma} (I\sigma_z) dF(t),
\] (186)
\[
(S)\, d\tilde{\sigma}_z = - \gamma (\sigma_z + 1) dt + 2\sqrt{\gamma} \sigma^+ (I\sigma_z) dF(t) + 2\sqrt{\gamma} dF^\dagger(t) (I\sigma_z)\sigma^-.
\] (187)
Note that from Sect. 3.3.4 the quantity which multiplies all the noise terms, $I\sigma_z$ is in this case the conserved operator $K$. Thus the first two equations are again exactly solvable.

It is surprising to see the apparent difference between the form of the set of equations (182–184) and that of the set (185–187)—however, it must be borne in mind that the transformation from the full system operators to the restricted system operators involves the time dependent operator $I(t)$, and this accounts for the apparent difference. Moreover, once the conserved nature of $K = I\sigma_z$ is noted, one sees that the only difference between the two sets of equations is a sign change when the total number of Fermions is odd, which is of course of no consequence.

**Ito quantum stochastic differential equations for the restricted system operators**

These take the form, from (115),

(I) $d\sigma^+ = \left(i\omega - \frac{\gamma}{2}\right)\sigma^+ dt + \sqrt{\gamma} dF^\dagger(t) (I\sigma_z),$  
(188)

(I) $d\sigma^- = -\left(i\omega + \frac{\gamma}{2}\right)\sigma^- dt + \sqrt{\gamma} (I\sigma_z) dF(t),$  
(189)

(I) $d\sigma_z = -\gamma(\sigma_z + 1 - 2\bar{N}) dt + 2\sqrt{\gamma}\sigma^+ (I\sigma_z) dF(t) + 2\sqrt{\gamma} dF^\dagger(t) (I\sigma_z)\sigma^-.$  
(190)

### 7.2. The harmonic oscillator

The harmonic oscillator coupled to a fermion bath is a problem which cannot be solved exactly, in the same way as the two level atom coupled to a Bosonic bath is not exactly solvable. In this case we have harmonic oscillator creation and destruction operators $a, a^\dagger$,

$c = a,$  
(191)
$c^\dagger = a^\dagger,$  
(192)

$H_{sys} = \hbar\omega a^\dagger a,$  
(193)

so that

$c = Ia \equiv a,$  
(194)
$c^\dagger = Ia^\dagger \equiv a^\dagger,$  
(195)

$J = (-1)^a^\dagger a,$  
(196)

$\bar{a} = Ja, \quad (\bar{a})^\dagger = a^\dagger J.$  
(197)

**Quantum Langevin equations for the full system operators**

Using the usual commutation relations in (53), we get the explicit equation of motion

$\dot{a} = -i\omega a - (2\bar{a}^\dagger a + 1) \left\{ \frac{\gamma}{2} \bar{a} + \sqrt{\gamma} f_{in}(t) \right\} - \left\{ \frac{\gamma}{2} \bar{a}^\dagger + \sqrt{\gamma} f_{in}^\dagger(t) \right\} (2\bar{a}^2).$  
(198)

**Stratonovich quantum stochastic differential equations for the restricted system operators**

We get the form, from (140)

(S) $da = -\left(i\omega + \frac{\gamma}{2}\right) a dt - \sqrt{\gamma} I dF(t)$  
(199)

The difference between the equations for the full and the restricted operators is now quite dramatic. The simple form of (199) is deceptive, since it involves the time dependent operator $I$, whose equation of motion is not solvable.
Ito quantum stochastic differential equations for the restricted system operators

We get, from \( (115) \),

\[
\begin{align*}
(\mathbf{I}) \; da &= - \left( i\omega + \frac{\gamma(1-2\bar{N})}{2} \right) a \, dt - \sqrt{\gamma} \, I \, dF(t) \\
(\mathbf{I}) \; d \left( a^\dagger a \right) &= - \gamma \{ (1-2\bar{N})a^\dagger a + \bar{N} \} \, dt - \sqrt{\gamma} \, a^\dagger I \, dF(t) - \sqrt{\gamma} \, a \, dF^\dagger(t) 
\end{align*}
\]

Using the last equation, we can find the equation of motion for the mean number \( n(t) \) to be

\[
\dot{n}(t) = - \gamma \{ (1-2\bar{N})n(t) + \bar{N} \}
\]

with the stationary solution

\[
n_s = \frac{\bar{N}}{1 - 2\bar{N}}
\]

For Fermionic noise at a predominant frequency \( \omega \), we know that

\[
\bar{N} = \frac{1}{e^{\hbar \omega/kT} + 1}
\]

and this yields the correct result for the harmonic oscillator

\[
n_s = \frac{1}{e^{\hbar \omega/kT} - 1}
\]

8. Conclusions

This paper answers what is in some sense an academic exercise—how do we deal with the Fermion inputs and outputs that are so common in the real world, turning up in electronic and many other systems. The two previous treatments \([8, 7]\) gave partial answers, the first dealing only with counting statistics, the second dealing only with noninteracting particles in cavities. The treatment given here is compatible with both.

As coherent Fermion physics becomes more important, for example in highly degenerate trapped cold Fermi vapors \([6]\) this kind of formalism will undoubtedly become more relevant. I look forward to that time.

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