Bahadur efficiency for certain goodness-of-fit tests based on the empirical characteristic function

Simos Meintanis\textsuperscript{1,2} \cdot Bojana Milošević\textsuperscript{3} \cdot Marko Obradović\textsuperscript{3}

Received: 21 January 2022 / Accepted: 6 December 2022 / Published online: 26 December 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract

We study the Bahadur efficiency of several weighted L2-type goodness-of-fit tests based on the empirical characteristic function. The methods considered are for normality and exponentiality testing, and for testing goodness-of-fit to the logistic distribution. Our results are helpful in deciding which specific test a potential practitioner should apply. For the celebrated BHEP and energy tests for normality we obtain novel efficiency results, with some of them in the multivariate case, while in the case of the logistic distribution this is the first time that efficiencies are computed for any composite goodness-of-fit test.

Keywords Goodness-of-fit test \cdot Bahadur efficiency \cdot Empirical characteristic function \cdot Normality test \cdot Exponentiality test

1 Introduction

Let $X_1, ..., X_n$, denote independent copies of an arbitrary random variable $X \in \mathbb{R}^p$, and consider the problem of testing the composite goodness-of-fit (GOF) null hypothesis,

\[ H_0 : \text{The law of } X \in \mathcal{F}, \]

where \( \mathcal{F} := \{ F_\vartheta : \vartheta \in \Theta \} \) denotes a class of distributions indexed by the parameter $\vartheta \in \Theta \subseteq \mathbb{R}^q$, $q \geq 1$.

The work of B. Milošević is supported by the Ministry of Education, Science and Technological Development of Republic of Serbia.

\textsuperscript{1} Simos Meintanis
simosmei@econ.uoa.gr

\textsuperscript{2} Department of Economics, National and Kapodistrian University of Athens, Athens, Greece

\textsuperscript{3} Pure and Applied Analytics, North–West University, Potchefstroom, South Africa

\textsuperscript{3} Faculty of Mathematics, University of Belgrade, Belgrade, Serbia
For certain popular distributions, such as the normal and the exponential distribution, there exist many GOF tests, while for others such as the logistic distribution the range of methods available is not so extended. In either case though a potential practitioner would like to have some quality measure on the basis of which one could choose amongst the existing GOF methods. In this connection, one of the most popular methods for test comparison is the so-called Bahadur efficiency that allows to compare the efficiency of any given GOF test vis-à-vis its optimal counterpart, which is the likelihood ratio (LR) test for the distribution under test against a specified alternative.

In this work we consider GOF tests that are based on the weighted L2-type test statistic

$$T_{n,w} = n \int_{\mathbb{R}^p} \left| \varphi_n(t) - \varphi_{\hat{\theta}_n}(t) \right|^2 w(t) dt,$$

(1.2)

where $\varphi_{\theta}(\cdot)$ denotes the characteristic function corresponding to $F_{\theta}$,

$$\varphi_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{it^\top X_j},$$

(1.3)

is the empirical characteristic function (ECF) and $\hat{\theta}_n := \hat{\theta}_n(X_1, \ldots, X_n)$ is an estimator of $\theta$ obtained on the basis of $(X_j, \ j = 1, \ldots, n)$.

The test statistic $T_{n,w}$, besides the family $\mathcal{F}$ being tested and the corresponding estimator $\hat{\theta}_n$ employed, also depends on the the weight function $w(\cdot)$ figuring in (1.2). For certain choices of $w(\cdot)$, we obtain here efficiency results for the BHEP GOF test for normality of Epps and Pulley (1983) and Henze and Wagner (1997) as well as for the generalized energy (GE) test for normality first put forward by Székely and Rizzo (2005). In doing so we provide extension of the efficiency results of Ebner and Henze (2021) and Tenreiro (2009) for the BHEP test as well as extension of the results of Móri et al. (2021) from the original energy statistic to its generalized counterpart suggested by Székely and Rizzo (2013). Furthermore we obtain for the first time analogous results for the GOF test for the logistic distribution of Meintanis (2004) and for the exponentiality test suggested by Henze and Meintanis (2005), including efficiency comparisons with alternative tests. As already mentioned these efficiency results will facilitate the choice of the specific weight function $w(\cdot)$ that figures in the test statistics defined by (1.2), and thus provide some guidance on which method to apply amongst the many tests available for the normal, the exponential distribution and the logistic distribution.

The rest of this work unfolds as follows. In Sect. 2 we review the basic theory of Bahadur efficiency and the theory of the limit null distribution of the test statistics, while in Sect. 3 the necessary efficiency computations are discussed. In Sect. 4 we provide efficiency results and discussion for the BHEP and energy statistics, as well as for the two aforementioned ECF-based tests, one for the logistic and the other for the exponential distribution. Analogous efficiency results for the normality tests in the bivariate case are reported in Sect. 5, and we conclude in Sect. 6 with discussion. An Appendix contains technical material.
2 Limit null distribution and Bahadur slopes

The limit in distribution of the L2-type test statistics figuring in (1.2) is given by

\[ T_{n,w} \overset{D}{\to} \sum_{j=1}^{\infty} \lambda_j N_j^2, \]  

(2.1)

where \( N_j, j \geq 1, \) are independent copies of a standard normally distributed random variable, and \( \lambda_1 \geq \lambda_2 \geq \ldots, \) are the eigenvalues of the integral equation

\[ \lambda f(s) = \int K(s, t) f(t) w(t) \, dt; \]  

(2.2)

where \( K(s, t) \) is a covariance kernel associated, besides the weight function \( w(\cdot), \) with the given test statistic and the estimator employed in estimating the distributional parameter \( \vartheta; \) see for instance Meintanis and Swanepoel (2007).

In this section, we briefly review the essence of the Bahadur theory, and associate that theory with the limit law of \( T_{n,w} \) figuring in (2.1)–(2.2); for more details we refer to Bahadur (1971), Gregory (1980) and Nikitin (1995), and to the more recent article by Grané and Tchirina (2013). To this end let \( \mathcal{G} = \{ G_\vartheta, \vartheta > 0 \} \) be a family of alternative distribution functions, such that \( G_0 \) is the null family of distributions for some typical value \( \vartheta \in \Theta. \) Assume that regularity conditions B, given in Appendix, hold.

Also recall that LR tests are optimal tests in the Bahadur sense (see Bahadur 1967; Rublík 1989). Hence, for close alternatives from \( \mathcal{G}, \) the absolute local approximate Bahadur efficiency for any sequence of test statistics, say \( \{ T_n \}, \) is defined as the ratio of the Bahadur approximate slope of the considered test statistic to the corresponding slope of the LR test, i.e.

\[ \text{eff}(T_n) = \lim_{\vartheta \to 0} \frac{c_T(\vartheta)}{2K(\vartheta)}, \]  

(2.3)

where

\[ c_T(\vartheta) = \frac{b_T(\vartheta)}{\lambda_1} = \frac{b''_T(0)}{2\lambda_1} \vartheta^2 + o(\vartheta^2), \]  

(2.4)

(the last equation obtained by a Taylor expansion), with \( \lambda_1 \) being the largest eigenvalue figuring in (2.1) and

\[ \frac{T_n}{n} \overset{p}{\to} b_T(\vartheta), \]  

(2.5)

while \( K(\vartheta) \) is equal to minimal Kullback–Leibler (KL) distance from the given alternative to the class of distributions within the null hypothesis.
Before closing this section we wish to point out that the BHEP and GE tests are well known to be location/scale invariant, i.e. for both tests it holds,

\[ T_{n,w}(a + bX_1, \ldots, a + bX_n) = T_{n,w}(X_1, \ldots, X_n), \quad \text{for each } a \in \mathbb{R}, b > 0, \quad (2.6) \]

and therefore their respective limit null distributions are independent of the true values of the mean and variance of the underlying Gaussian law. Note that this property extends to the multivariate version of these tests. In this connection, the other tests studied herein are also invariant, whereby invariance is understood within the context of the specific family \( \mathcal{F} \) being tested. For instance if this family is confined to the positive real axis (such is the exponential family of distributions), invariance is understood only with respect to scale, i.e. (2.6) holds for \( a = 0, \) and \( b > 0. \)

### 3 KL distance and eigenvalue approximation

It can be seen from (2.3)–(2.5) that the basic components of an approximate Bahadur slope are the computation of the KL distance and of the largest eigenvalue. In this section we consider these components with emphasis on location-scale families of distributions.

#### 3.1 KL distance for location-scale families

Assume that the family \( \mathcal{F} \) is a location-scale family, i.e. a family generated by the density function \( f_\vartheta \) corresponding to \( F_\vartheta \), where \( \vartheta \) contains the location and scale parameters, \( \mu \) and \( \sigma \), respectively. We set \( f_0(\cdot) \) and \( F_0(\cdot) \) for the density and distribution function, respectively, of the standardized variable \( Z := \sigma^{-1}(X - \mu) \). Consider the KL distance \( K(\theta; \mu, \sigma) \) between \( g_\theta \in \mathcal{G} \) and arbitrary \( f_\vartheta \in \mathcal{F} \), i.e.

\[
K(\theta; \mu, \sigma) = \int \log(g_\theta(x))g_\theta(x)dx - \int \log(f_\vartheta(x))g_\theta(x)dx. \quad (3.1)
\]

Recall also that for the family \( \mathcal{G} \) with density \( g_\theta(\cdot), \ \theta > 0, \) we have that \( g_0 \equiv f_{\vartheta_0} \), where \( \vartheta_0 = (\mu_0, \sigma_0) \), and assume that regularity conditions \( A \) are satisfied (refer to the Appendix for details). Then the next theorem gives the local behavior of the minimal distance (3.1) over all \( f_\vartheta \in \mathcal{F} \). The proof of Theorem 3.1 is postponed to the Appendix.

**Theorem 3.1** The KL distance from the alternative \( g_\theta(\cdot) \) in \( \mathcal{G} \) to the closest null distribution

\[
K(\theta) = \inf_{\mu, \sigma} K(\theta; \mu, \sigma)
\]
admits the representation

\[ 2K(\theta) = \left( \sigma_0^2 \int h^2(\mu_0 + x\sigma_0) dx + \frac{1}{\sigma_0} \int \frac{(f_0'(x))^2}{f_0(x)} (\mu'(0) + x\sigma'(0))^2 dx \right. \]

\[- \left. \frac{(\sigma'(0))^2}{\sigma_0^2} + 2 \int \frac{f_0'(x)}{f_0(x)} (\mu'(0) + x\sigma'(0)) h(\mu_0 + x\sigma_0) dx \right) \theta^2 + o(\theta^2), \]

\[ \theta \to 0, \]

\[ (3.2) \]

where \((\mu(\theta), \sigma(\theta)) = \arg\inf K(\theta; \mu, \sigma)\) and \(h(x) = \frac{\partial}{\partial \theta} g_{\theta}(x)|_{\theta=0}\).

In the case of some specific distribution the expression above can be simplified. Since the maximum likelihood (ML) estimators minimize the KL distance, the functions \(\mu(\theta)\) and \(\sigma(\theta)\) are equal to population counterparts, i.e. probability limits of the ML estimators of the location and scale parameter, respectively.

For example, in the case that \(F\) is the normal location-scale family, we have

\[ \mu(\theta) = \int_{-\infty}^{\infty} x g_{\theta}(x) dx; \quad \sigma(\theta) = \left( \int_{-\infty}^{\infty} x^2 g_{\theta}(x) dx - \mu^2(\theta) \right)^{1/2} \]

and thus the KL distance reduces to (see also Milošević et al. 2021)

\[ 2K(\theta) = \left( \int_{-\infty}^{\infty} \sqrt{2\pi} \sigma_0 h^2(x) e^{\frac{(x-\mu)^2}{2\sigma_0^2}} dx - \frac{1}{\sigma_0^2} \left( \int_{-\infty}^{\infty} x h(x) dx \right)^2 \right) \theta^2 + o(\theta^2). \]

For the exponential scale family, we have

\[ \mu(\theta) = 0; \quad \sigma(\theta) = \int_0^{\infty} x g_{\theta}(x) dx, \]

and thus the KL distance reduces to (see also Nikitin and Tchirina (1996))

\[ 2K(\theta) = \left( \int_0^{\infty} \sigma_0 h^2(x) e^{\frac{x}{\sigma_0}} dx - \frac{1}{\sigma_0^2} \left( \int_0^{\infty} x h(x) dx \right)^2 \right) \theta^2 + o(\theta^2). \]

However, in the case that \(F\) is the logistic location-scale, the parameters \((\mu(\theta), \sigma(\theta))\) that minimize the KL distance figuring in (3.1) admit no closed-form expression. Nevertheless, by applying the implicit function theorem we obtain

\[ \mu'(0) = \frac{6}{\sigma_0^2} \int_{-\infty}^{\infty} \frac{h(x)}{1 + e^{\frac{x-\mu_0}{\sigma_0}}} dx; \]
\[ \sigma'(0) = \frac{9}{\pi^2 + 3} \left( \int_{-\infty}^{\infty} \frac{1 - e^{\frac{-x - \mu_0}{\sigma_0}} x h(x)}{1 + e^{\frac{-x - \mu_0}{\sigma_0}}} \, dx \right), \]

and thus by plugging these expressions in (3.2) we can obtain the corresponding KL distance.

### 3.2 Eigenvalue approximation

In this section we briefly review a method for eigenvalue approximation proposed in Božin et al. (2020). Recall that these eigenvalues are solutions of an integral equation involving a specific operator. At first we replace the original operator by a symmetric operator that has the same eigenvalues, and then (i) consider a truncated version of the symmetrized operator, and (ii) consider a discretized version of the truncated operator. In these two steps the truncated operator is chosen so that as the amount of truncation diminishes this operator converges to the symmetrized operator, and likewise, the discretized operator approaches the truncated operator as the grid of discretization becomes more fine and extended.

Specifically we replace the original operator

\[ A f(s) = \int_{-\infty}^{\infty} K(s, t) f(t) w(t) \, dt \]

with the operator

\[ \overline{A} f(s) = \int_{-\infty}^{\infty} K(s, t) f(t) \sqrt{w(t)w(s)} \, dt, \quad (3.3) \]

that has the same spectrum as \( A \), but is symmetric. Then in the first step we define the truncated operator \( \overline{A}_B \) acting on the set of real functions with support \([-B, B]\), \( B > 0 \), defined by

\[ \overline{A}_B f(s) = \int_{-\infty}^{\infty} K(s, t) f(t) \sqrt{w(t)w(s)} 1(|t| \leq B) \, dt, \]

which clearly and for sufficiently large \( B \), is close to \( \overline{A} \).

In the second step of approximation we employ a sequence of symmetric linear operators \( M^{(m)} \), which converges in norm to \( \overline{A}_B \), as \( m \to \infty \). This discretized sequence can be defined by \((2m+1) \times (2m+1)\) matrices \( M^{(m)} = \| m^{(m)}_{i,j} \|, \) \(-m \leq i \leq m, -m \leq j \leq m\), with elements

\[ m^{(m)}_{i,j} = \frac{2B}{(2m+1)} K \left( \frac{iB}{m}, \frac{jB}{m} \right) \sqrt{w\left(\frac{iB}{m}\right) w\left(\frac{jB}{m}\right)}. \quad (3.4) \]
Using the perturbation theory—see (Kato 2013, Theorem 4.10, page 291)—we have that the spectra of these two operators are at a distance that tends to zero. Hence within the degree of approximation, the sequence $\lambda_1^{(m)}$ of the largest eigenvalues of $M^{(m)}$ will converge to the largest eigenvalue $\lambda_1(B)$ of $\overline{A_B}$, which in turn approaches $\lambda_1$. Consequently, the eigenvalues $\lambda_1^{(m)}$ and $\lambda_1$ will coincide up to any desired accuracy, provided that the pair of approximation parameters $(m, B)$ is large enough.

4 Tests for the normal and the logistic distribution

When testing for GOF to a symmetric distribution such as the normal or the logistic, it is customary to use general purpose nonparametric alternatives parametrized by $\theta$, a parameter that usually controls skewness. A discussion on the construction and applications of these alternatives is available in Jones (2015) and Ley (2015). Specifically we consider the following alternatives:

- Lehmann alternatives
  $$g_\theta^{(1)}(x) = (1 + \theta) F_{\vartheta_0}(x) f_{\vartheta_0}(x)$$

- first Ley-Paindaveine alternatives
  $$g_\theta^{(2)}(x) = f_{\vartheta_0}(x) e^{-\theta(1-F_{\vartheta_0}(x))} (1 + \theta F_{\vartheta_0}(x))$$

- second Ley-Paindaveine alternatives
  $$g_\theta^{(3)}(x) = f_{\vartheta_0}(x) (1 - \theta \pi \cos(\pi F_{\vartheta_0}(x)))$$

- contamination alternatives
  $$g_\theta^{(4)}(x; \mu, \sigma^2) = (1 - \theta) f_{\vartheta_0}(x) + \theta \frac{1}{\sigma} f_{\vartheta_0} \left( \frac{x - \mu}{\sigma} \right)$$

Note that in the above alternatives, $f_{\vartheta_0}(\cdot)$ and $F_{\vartheta_0}(\cdot)$ are the null density and distribution function, respectively, corresponding to an arbitrary member of the null location-scale family (in our case the normal or logistic) with $\vartheta_0 = (\mu_0, \sigma_0)$, so that for $\theta = 0$ each of these alternatives reduces to the null distribution under test, e.g. the standard normal distribution with the density $(2\pi)^{-1/2} e^{-x^2/2}$ or the standard logistic distribution with density $e^x / (1 + e^x)^2$.

In all considered alternatives the function $h(x) = \frac{\partial}{\partial \theta} g_\theta(x)|_{\theta=0}$ can be expressed as $h(x) = \frac{1}{\sigma_0} h_0(\frac{x - \mu_0}{\sigma_0})$, where $h_0(x)$ is the partial derivative at zero of $g_\theta(x)$ in the standard case $g_0(x) = f_0(x)$. In such case it can be shown that the local KL distance does not depend on $\vartheta_0 = (\mu_0, \sigma_0)$. The same holds for $b''(0)$ from (2.4), hence the local approximate Bahadur relative efficiencies is invariant with respect to $\vartheta_0$, and, without loss of generality we can consider the alternatives arising from the standard member of the location-scale family.
4.1 Tests for normality

4.1.1 The GE test

The GE test may be formulated as

\[ T_{n,w} = n \int_{-\infty}^{\infty} \left| \phi_n(t) - e^{-\frac{t^2}{2}} \right|^2 w(t) \, dt, \]  

(4.1)

with the ECF \( \phi_n(t) \) obtained as in (1.3) with \( X_j \) replaced by

\[ Z_j = \frac{X_j - \bar{X}_n}{S_n}, \quad j = 1, ..., n, \]  

(4.2)

where \( \bar{X}_n \) denotes the sample mean and \( S^2_n \) the sample variance of \( (X_j, j = 1, ..., n) \).

As weight function Móri et al. (2021) use \( w(t) = |t|^{-2} \). Here we consider the GE test of Székely and Rizzo (2013) whereby \( w(t) = |t|^{-1 - \gamma}, \quad 0 < \gamma < 2 \), is adopted as weight function, thus rendering the convenient form

\[ T_{n,w} = 2 \sum_{j=1}^{n} \mathbb{E}|Z_j - X_1|^{\gamma} - n \mathbb{E}|X_1 - X_2|^{\gamma} - \frac{1}{n} \sum_{j,k=1}^{n} |Z_j - Z_k|^{\gamma}. \]  

(4.3)

1 Note that the expectations in (4.3) are taken with respect to the standard Gaussian distribution, and consequently the GE test statistic may be explicitly expressed by using the equations

\[ \mathbb{E}|X_1 - X_2|^{\gamma} = \frac{2^{\gamma}}{\sqrt{\pi}} \Gamma \left( \frac{1 + \gamma}{2} \right), \]

and

\[ \mathbb{E}|x - X_1|^{\gamma} = \frac{2^{\gamma}}{\sqrt{\pi}} e^{-\frac{x^2}{2}} \Gamma \left( \frac{1 + \gamma}{2} \right) _1 F_1 \left( \frac{1}{2} + \frac{\gamma}{2}, \frac{\gamma}{2}, \frac{x^2}{2} \right) = \frac{2^{\frac{\gamma}{2}}}{\sqrt{\pi}} \Gamma \left( \frac{1 + \gamma}{2} \right) _1 F_1 \left( -\frac{\gamma}{2}, \frac{1}{2}, -\frac{x^2}{2} \right), \]

where \(_1 F_1(a, b, c)\) stands for the Kummer confluent hypergeometric function; see Gradshteyn and Ryzhik (1994). For the covariance kernel \( K(s, t) \) corresponding to the GE statistic \( T_{n,w} \) we refer to Móri et al. (2021), while calculation of the eigenvalues \( \lambda_j, j \geq 1 \), figuring in (2.1)–(2.2) is carried out using method described in Sect. 3.2.

Letting \( (\mu(\theta), \sigma^2(\theta)) \) be the probability limit of the estimator \((\bar{X}_n, S^2_n)\), i.e.,

\[ \mu(\theta) = \int_{-\infty}^{\infty} x g_\theta(x) \, dx; \quad \sigma^2(\theta) = \int_{-\infty}^{\infty} (x - \mu(\theta))^2 g_\theta(x) \, dx, \]

\[ \text{and} \]

\[ 1 \text{ The original weight function also includes the constant } 2 \sqrt{\pi} \Gamma(1 - (\gamma/2))/\Gamma((1 + \gamma)/2). \]
we obtain the probability limit figuring in (2.5) for the GE test statistic as

\[
\frac{T_{n,w}}{n} \xrightarrow{p} b(\theta) = \frac{2^{1+\frac{\gamma}{2}}}{\sqrt{\pi}} \frac{1 + \gamma}{2} \Gamma \left( \frac{1 + \gamma}{2} \right) \mathbb{E}_\theta \left[ 1 F_1 \left( -\frac{\gamma}{2}, 1, \frac{1}{2}, -\frac{(X_1 - \mu(\theta))^2}{2\sigma^2(\theta)} \right) \right] - \frac{2\gamma}{\sqrt{\pi}} \Gamma \left( \frac{1 + \gamma}{2} \right) - \frac{1}{\sigma(\theta)} \sqrt{\pi} \Gamma \left( \frac{1 + \gamma}{2} \right) \mathbb{E}_\theta \left[ |X_1 - X_2| \right].
\]

(4.4)

In order to calculate the quantity \( b''(0) \) figuring in (2.4) we use numerical integration in Wolfram Mathematica facilitated by differentiation under the integral sign, and thereby obtain the expressions

\[
\mu'(0) = \int_{-\infty}^{\infty} x h(x) dx; \quad \mu''(0) = \int_{-\infty}^{\infty} x u(x) dx
\]

\[
\sigma'(0) = \frac{1}{2} \int_{-\infty}^{\infty} x^2 h(x) dx
\]

\[
\sigma''(0) = \frac{1}{2} \int_{-\infty}^{\infty} x^2 u(x) dx - \left( \int_{-\infty}^{\infty} x h(x) dx \right)^2 - \frac{1}{4} \left( \int_{-\infty}^{\infty} x^2 h(x) dx \right)^2,
\]

(4.5)

The asymptotic null distribution of the BHEP test along with the expression for the covariance kernel \( K(s, t) \) may be found in Henze and Wagner (1997), and corresponding eigenvalues have been recently computed by Ebner and Henze (2022).

The probability limit of the BHEP test is obtained analogously to (4.4) as

\[
b(\theta) = \sqrt{\frac{\pi}{1 + \gamma}} + \sqrt{\frac{\pi}{\gamma}} \mathbb{E}_\theta \left[ \exp \left( -\frac{(X_1 - X_2)^2}{4\gamma\sigma^2(\theta)} \right) \right] - 2 \sqrt{\frac{2\pi}{1 + \gamma}} \mathbb{E}_\theta \left[ \exp \left( -\frac{(X_1 - \mu(\theta))^2}{(2 + 4\gamma)\sigma^2(\theta)} \right) \right].
\]

while the calculation of efficiencies is also carried out in the same way as in the previous subsection by means of the approximation outlined in Sect. 3.2. These efficiency results (LABEs) for the BHEP test are reported in Table 2 and in Fig. 2.
### Table 1  LABE of the energy test for normality

| $\gamma$ | $g^{(1)}$ | $g^{(2)}$ | $g^{(3)}$ | $g^{(4)}(1,1)$ | $g^{(4)}(0.5,1)$ | $g^{(4)}(0,0.5)$ |
|----------|----------|----------|----------|----------------|----------------|----------------|
| 0.1      | 0.501    | 0.714    | 0.843    | 0.323          | 0.431          | 0.630          |
| 0.2      | 0.520    | 0.734    | 0.861    | 0.336          | 0.447          | 0.636          |
| 0.3      | 0.538    | 0.754    | 0.877    | 0.349          | 0.464          | 0.640          |
| 0.4      | 0.556    | 0.772    | 0.892    | 0.362          | 0.480          | 0.643          |
| 0.5      | 0.573    | 0.790    | 0.906    | 0.374          | 0.496          | 0.645          |
| 0.6      | 0.590    | 0.806    | 0.918    | 0.387          | 0.512          | 0.645          |
| 0.7      | 0.608    | 0.821    | 0.930    | 0.399          | 0.527          | 0.645          |
| 0.8      | 0.623    | 0.837    | 0.940    | 0.411          | 0.542          | 0.643          |
| 0.9      | 0.639    | 0.851    | 0.950    | 0.423          | 0.558          | 0.640          |
| 1.0      | 0.655    | 0.865    | 0.958    | 0.434          | 0.572          | 0.636          |
| 1.1      | 0.670    | 0.877    | 0.966    | 0.446          | 0.586          | 0.632          |
| 1.2      | 0.685    | 0.889    | 0.973    | 0.457          | 0.600          | 0.626          |
| 1.3      | 0.699    | 0.900    | 0.978    | 0.468          | 0.614          | 0.620          |
| 1.4      | 0.713    | 0.911    | 0.984    | 0.479          | 0.628          | 0.613          |
| 1.5      | 0.727    | 0.921    | 0.988    | 0.490          | 0.641          | 0.605          |
| 1.6      | 0.740    | 0.930    | 0.991    | 0.501          | 0.654          | 0.596          |
| 1.7      | 0.753    | 0.939    | 0.994    | 0.511          | 0.666          | 0.587          |
| 1.8      | 0.765    | 0.947    | 0.997    | 0.521          | 0.679          | 0.577          |
| 1.9      | 0.777    | 0.954    | 0.998    | 0.531          | 0.691          | 0.567          |

**Fig. 1** LABE for the energy test for the normal distribution
Table 2 LABE of the BHEP test for normality

| γ   | $g^{(1)}$ | $g^{(2)}$ | $g^{(3)}$ | $g^{(4)}(1, 1)$ | $g^{(4)}(0.5, 1)$ | $g^{(4)}(0, 0.5)$ |
|-----|-----------|-----------|-----------|-----------------|-----------------|-----------------|
| 0.1 | 0.477     | 0.701     | 0.840     | 0.302           | 0.406           | 0.654           |
| 0.2 | 0.582     | 0.814     | 0.929     | 0.376           | 0.501           | 0.676           |
| 0.3 | 0.655     | 0.879     | 0.968     | 0.429           | 0.568           | 0.658           |
| 0.4 | 0.710     | 0.921     | 0.986     | 0.471           | 0.620           | 0.628           |
| 0.5 | 0.752     | 0.948     | 0.992     | 0.505           | 0.661           | 0.593           |
| 0.6 | 0.785     | 0.967     | 0.993     | 0.532           | 0.695           | 0.559           |
| 0.7 | 0.812     | 0.979     | 0.990     | 0.555           | 0.722           | 0.527           |
| 0.8 | 0.834     | 0.987     | 0.986     | 0.574           | 0.745           | 0.497           |
| 0.9 | 0.853     | 0.993     | 0.980     | 0.591           | 0.764           | 0.469           |
| 1   | 0.868     | 0.997     | 0.974     | 0.605           | 0.780           | 0.443           |
| 2   | 0.941     | 0.992     | 0.917     | 0.681           | 0.865           | 0.281           |
| 3   | 0.963     | 0.975     | 0.879     | 0.711           | 0.896           | 0.202           |
| 4   | 0.972     | 0.961     | 0.855     | 0.726           | 0.910           | 0.158           |
| 5   | 0.977     | 0.951     | 0.839     | 0.735           | 0.918           | 0.129           |
| 6   | 0.979     | 0.942     | 0.826     | 0.741           | 0.923           | 0.109           |
| 7   | 0.980     | 0.936     | 0.817     | 0.745           | 0.926           | 0.094           |
| 8   | 0.981     | 0.931     | 0.810     | 0.746           | 0.929           | 0.083           |
| 9   | 0.981     | 0.926     | 0.804     | 0.750           | 0.930           | 0.074           |
| 10  | 0.981     | 0.923     | 0.799     | 0.751           | 0.932           | 0.067           |

Fig. 2 LABE for the BHEP test for the normal distribution
4.1.3 Discussion

From Tables 1 and 2 and Figs. 1 and 2 we can see that there is a significant influence of the tuning parameter on the efficiency of both tests. Specifically, in the case of the GE test, the efficiencies generally grow with \( \gamma \), hence a high value of the tuning parameter (close to the boundary value \( \gamma = 2 \)) can be recommended for this test. On the other hand, the corresponding LABEs of the BHEP test exhibit no such consistent pattern, with the impact on the tuning parameter depending very much on the specific alternative. Specifically for Lehmann and location/scale contamination alternatives higher values of the tuning parameter should be used, while for all other alternatives an “in-between” value in the neighborhood of \( \gamma = 1 \) yields better efficiency for the corresponding BHEP test. Moreover the aforementioned value seems to be a good compromise, and if one must choose a single test between the GE and BHEP tests, then the latter test with \( \gamma = 1 \) appears to yield a good overall efficiency. We can also compare the efficiency of the ECF-based tests to the efficiencies of corresponding tests based on the empirical distribution function (EDF) provided by Milošević et al. (2021). In this connection, a close inspection of the corresponding efficiency figures shows a significant advantage of the ECF-based tests over their EDF counterparts.

4.2 Tests for the logistic distribution

In complete analogy to the GE and BHEP tests formulated as in (1.2) and (4.1), Meintanis (2004) defines a GOF test statistic for the logistic distribution as

\[
T_{n,w} = n \int_{-\infty}^{\infty} \left| \phi_n(t) - \frac{\pi t}{\sinh(\pi t)} \right|^2 w(t) dt,
\]

(4.6)

with the ECF \( \phi_n(t) \) obtained as in (1.3) with \( X_j \) replaced by \( Z_j = (X_j - \hat{\mu}_n)/\hat{\sigma}_n \), where \((\hat{\mu}_n, \hat{\sigma}_n)\) denote the moment estimators or the ML estimators of the parameters \((\mu, \sigma)\) of the logistic distribution. An explicit test statistic formula corresponding to the weight function \( w(t) = e^{-\gamma |t|}, \gamma > 0 \), is given by

\[
T_{n,w} = \frac{2\gamma}{n} \sum_{j,k=1}^{n} \frac{1}{\gamma^2 + (Z_j - Z_k)^2} - \frac{2}{\pi} \sum_{j=1}^{n} \left[ S^{(1)}_\gamma(Z_j) - \frac{Z_j^2}{2\pi^2} S^{(2)}_\gamma(Z_j) \right] + \frac{n}{\pi} \left[ 2\xi_2 \left( 1 + \frac{\gamma}{2\pi} \right) - \frac{\gamma}{\pi} \xi_3 \left( 1 + \frac{\gamma}{2\pi} \right) \right],
\]

(4.7)

where \( \xi_\gamma(x) = \sum_{k=0}^{\infty} (k + x)^{-\gamma} \), and

\[
S^{(m)}_\gamma(x) = \sum_{k=0}^{\infty} \left[ \left( \frac{x}{2\pi} \right)^2 + \frac{\gamma + \pi}{2\pi} + k \right]^{-m}.
\]

The expression for the covariance kernel \( K(s, t) \) may also be found in Meintanis (2004), along with a Monte Carlo study of the power of the test based on \( T_{n,w} \) against
the classical tests based on the EDF. These results, nicely complemented by results from Gulati and Shapiro (2009), suggest that the test based on (4.7) is an overall competitive test.

Turning to efficiency calculations we obtain the corresponding probability limit (recall (2.5)) as

\[ b(\theta) = \mathbb{E}_\theta \left[ \frac{2\gamma}{\gamma^2 + \left( \frac{X_1 - X_2}{\sigma(\theta)} \right)^2} \right] - \frac{2}{\pi} \mathbb{E}_\theta \left[ S^{(1)}_\gamma \left( \frac{X - \mu(\theta)}{\sigma(\theta)} \right) \right] - \frac{1}{\pi^3} \mathbb{E}_\theta \left[ \frac{X - \mu(\theta)}{\sigma(\theta)} \right]^2 S^{(2)}_\gamma \left( \frac{X - \mu(\theta)}{\sigma(\theta)} \right) \]

\[ + \frac{1}{\pi} \left( 2\xi_2 \left( 1 + \frac{\gamma}{2\pi} \right) - \frac{\gamma}{\pi} \xi_3 \left( 1 + \frac{\gamma}{2\pi} \right) \right), \]

where \( \mu(\theta) \) and \( \sigma(\theta) \) denote the probability limits of \( \hat{\mu}_n \) and \( \hat{\sigma}_n \), respectively.

In order to calculate \( b''(0) \), we use as before numerical integration in Wolfram Mathematica facilitated by the formulae for the first and second derivatives at zero of \( \mu(\theta) \) and \( \sigma(\theta) \). In this connection and since for the ML estimators there exist no closed expressions for \( \mu(\theta) \) and \( \sigma(\theta) \), the necessary derivatives are obtained via the implicit function theorem as

\[ \mu'(0) = 6 \int_{-\infty}^{\infty} \frac{h(x)}{1 + e^{-x}} \mathrm{d}x; \quad \mu''(0) = 6 \left( \int \frac{u(x)}{1 + e^{-x}} \mathrm{d}x + \frac{1}{6} \mu'(0) \sigma'(0) - 2 \int \frac{(\mu'(0) + x\sigma'(0))e^{-x}h(x)}{(1 + e^{-x})^2} \mathrm{d}x \right), \]

\[ \sigma''(0) = \frac{3}{2\pi^2} \left( \int xu(x) \mathrm{d}x + \frac{1}{2}\mu'(0)^2 - 4\mu'(0)\sigma'(0) + \frac{1}{18}(\pi^2 - 6)\sigma'(0)^2 \right) 
- 4 \int \frac{(e^x(x - 1) - 1)\mu'(0) + e^x x^2 \sigma'(0)}{(1 + e^x)^2} h(x) \mathrm{d}x 
- 2 \int \frac{xe^{-x}u(x)}{(1 + e^{-x})} \mathrm{d}x \right). \]

On the other hand, when \( \hat{\mu}_n \) and \( \hat{\sigma}_n \) are obtained by the method of moments, and by using the corresponding probability limits

\[ \mu(\theta) = \int_{-\infty}^{\infty} x g_\theta(x) \mathrm{d}x, \]

\[ \sigma(\theta) = \frac{\sqrt{3}}{\pi} \left( \int_{-\infty}^{\infty} x^2 g_\theta(x) \mathrm{d}x - \mu^2(\theta) \right)^{\frac{1}{2}}, \]

we obtain the necessary derivatives as

\[ \mu'(0) = \int_{-\infty}^{\infty} x h(x) \mathrm{d}x; \quad \mu''(0) = \int_{-\infty}^{\infty} x u(x) \mathrm{d}x; \]

\[ \sigma'(0) = \frac{3}{2\pi^2} \int_{-\infty}^{\infty} x^2 h(x) \mathrm{d}x; \]
Table 3  LABE of the test for the logistic distribution (ML estimation)

| γ  | \(g^{(1)}\) | \(g^{(2)}\) | \(g^{(3)}\) | \(g^{(4)}(1, 1)\) | \(g^{(4)}(0.5, 1)\) | \(g^{(4)}(0, 0.5)\) |
|----|-------------|-------------|-------------|----------------|----------------|----------------|
| 0.1| 0.274       | 0.456       | 0.641       | 0.468          | 0.463          | 0.702          |
| 0.2| 0.314       | 0.508       | 0.710       | 0.525          | 0.516          | 0.759          |
| 0.3| 0.347       | 0.548       | 0.762       | 0.570          | 0.558          | 0.795          |
| 0.4| 0.378       | 0.581       | 0.804       | 0.607          | 0.593          | 0.820          |
| 0.5| 0.406       | 0.608       | 0.839       | 0.641          | 0.622          | 0.837          |
| 0.6| 0.432       | 0.632       | 0.868       | 0.670          | 0.648          | 0.847          |
| 0.7| 0.457       | 0.652       | 0.893       | 0.695          | 0.669          | 0.852          |
| 0.8| 0.481       | 0.668       | 0.913       | 0.718          | 0.688          | 0.854          |
| 0.9| 0.503       | 0.683       | 0.931       | 0.738          | 0.704          | 0.853          |
| 1  | 0.524       | 0.695       | 0.945       | 0.756          | 0.718          | 0.850          |
| 2  | 0.693       | 0.728       | 0.989       | 0.844          | 0.768          | 0.745          |
| 3  | 0.802       | 0.673       | 0.945       | 0.837          | 0.728          | 0.605          |
| 4  | 0.870       | 0.581       | 0.876       | 0.787          | 0.650          | 0.473          |
| 5  | 0.908       | 0.479       | 0.812       | 0.722          | 0.559          | 0.362          |
| 6  | 0.928       | 0.381       | 0.767       | 0.663          | 0.473          | 0.274          |
| 7  | 0.938       | 0.299       | 0.745       | 0.616          | 0.402          | 0.210          |
| 8  | 0.943       | 0.237       | 0.739       | 0.584          | 0.351          | 0.166          |
| 9  | 0.946       | 0.194       | 0.741       | 0.565          | 0.315          | 0.136          |
| 10 | 0.948       | 0.163       | 0.744       | 0.552          | 0.230          | 0.116          |

\[
\sigma''(0) = \frac{3}{2\pi^2} \int_{-\infty}^{\infty} x^2 u(x) dx - \frac{3}{\pi^3} \left( \int_{-\infty}^{\infty} x h(x) dx \right)^2 - \frac{3}{4\pi^2} \int_{-\infty}^{\infty} x^2 h(x) dx,
\]

(recall \(h(x) = \frac{\partial}{\partial \theta} g_\theta(x)|_{\theta=0}\) and \(u(x) = \frac{\partial^2}{\partial \theta^2} g_\theta(x)|_{\theta=0}\)), and thereby calculate the efficiencies of the test for the logistic distribution based on (4.7). These efficiencies (LABEs) are reported in Table 3 (ML estimation) and Table 4 (moment estimation). Corresponding graphs of LABEs are given in Fig. 3.

For comparison purposes we also provide in Table 5 corresponding efficiencies for three EDF-based tests, namely the Kolmogorov-Smirnov (KS), the Cramér-von Mises (CM), and the Anderson-Darling (AD) test. The calculations were carried out using the same method used by Milošević et al. (2021) for normality tests, while the covariance function of the corresponding empirical process is available from Stephens (1979).

4.2.1 Discussion

We note that this is the first time that efficiency results are obtained for any GOF test to the logistic distribution in the context of the “fully” composite null hypothesis, i.e. when both the location and scale parameters are unknown.

When ECF based tests are compared among themselves, from Tables 3 and 4, and Fig. 3, it may be inferred that the efficiencies vary considerably between estimation
Table 4  LABE of the test for the logistic distribution (moment estimation)

| γ  | $g^{(1)}$ | $g^{(2)}$ | $g^{(3)}$ | $g^{(4)}(1, 1)$ | $g^{(4)}(0.5, 1)$ | $g^{(4)}(0, 0.5)$ |
|----|---------|---------|---------|----------------|----------------|----------------|
| 0.1| 0.485   | 0.695   | 0.811   | 0.680          | 0.693          | 0.878          |
| 0.2| 0.539   | 0.754   | 0.874   | 0.741          | 0.752          | 0.924          |
| 0.3| 0.582   | 0.794   | 0.915   | 0.786          | 0.794          | 0.947          |
| 0.4| 0.619   | 0.824   | 0.944   | 0.820          | 0.825          | 0.957          |
| 0.5| 0.651   | 0.846   | 0.965   | 0.846          | 0.849          | 0.958          |
| 0.6| 0.670   | 0.863   | 0.980   | 0.868          | 0.867          | 0.954          |
| 0.7| 0.706   | 0.875   | 0.989   | 0.884          | 0.880          | 0.945          |
| 0.8| 0.729   | 0.883   | 0.995   | 0.890          | 0.890          | 0.933          |
| 0.9| 0.751   | 0.889   | 0.998   | 0.908          | 0.896          | 0.919          |
| 1  | 0.770   | 0.891   | 0.998   | 0.916          | 0.901          | 0.904          |
| 2  | 0.898   | 0.841   | 0.931   | 0.910          | 0.863          | 0.724          |
| 3  | 0.941   | 0.729   | 0.816   | 0.828          | 0.760          | 0.559          |
| 4  | 0.773   | 0.504   | 0.580   | 0.601          | 0.532          | 0.355          |
| 5  | 0.581   | 0.323   | 0.387   | 0.405          | 0.348          | 0.213          |
| 6  | 0.422   | 0.202   | 0.254   | 0.266          | 0.222          | 0.126          |
| 7  | 0.304   | 0.127   | 0.169   | 0.176          | 0.142          | 0.076          |
| 8  | 0.221   | 0.081   | 0.114   | 0.118          | 0.092          | 0.047          |
| 9  | 0.161   | 0.052   | 0.079   | 0.080          | 0.061          | 0.028          |
| 10 | 0.120   | 0.034   | 0.055   | 0.056          | 0.041          | 0.019          |

Fig. 3  LABE for the ECF tests for the logistic distribution: MLE—solid line, MoM—dashed line
methods, and that the variant of the test which uses the method of moments is somewhat more efficient than its counterpart based on ML estimation, a finding that is in line with the Monte Carlo results of Meintanis (2004). This phenomenon of (generally) less efficient estimators, such as the moment estimators, producing more powerful tests has already been pointed out by Gürtler and Henze (2000) under similar circumstances, and dates back at least to Drost et al. (1990). Turning to the tuning parameter we observe that its impact on test performance is noticeable, and that, by taking into account all considered alternatives, we can recommend $\gamma = 3$ in the case of ML estimation and $\gamma = 1$ in the case of moment estimation as overall good choices.

As far as classical tests are concern, we notice from Table 5 that the AD test is more efficient than the other EDF-based tests and competitive with the ECF-based tests. Nevertheless the moment-based ECF test appears to be preferable overall against the AD in the neighborhood of the suggested value $\gamma = 1$. The other EDF-based tests are also less efficient than the ECF tests for the recommended value of the tuning parameter $\gamma$.

### 4.3 Tests for the exponential distribution

In extensive Monte Carlo studies (see for instance Henze and Meintanis 2005; Grané and Fortiana 2011), certain tests based on the ECF are found to be competitive for testing GOF to the exponential distribution. One such test is also a weighted L2-type test, but the corresponding test statistic admits a slightly different formulation from that in (1.2). Specifically we have

$$T_{n,w} = n \int_{-\infty}^{\infty} \left( |\phi_n(t)|^2 - C_n(t) \right)^2 w(t) dt, \quad (4.8)$$

with $|\phi_n(t)|$ denoting the modulus and $C_n(t)$ the real part of the ECF obtained from (1.3) by replacing $X_j$ by

$$Z_j = X_j / \bar{X}_n, \quad j = 1, \ldots, n. \quad (4.9)$$

As weight functions Henze and Meintanis (2005) suggest $w_\beta(t) = e^{-\gamma |t|^\beta}, \gamma > 0, \beta = 1, 2$, and provide explicit expressions for the resulting test statistics. The
corresponding kernel \( K(s, t) \) figuring in (2.2) may be computed as

\[
K(s, t) = \frac{s^2 t^2 (1 + s^2 + t^2)}{(1 + s^2)(1 + t^2)(1 + (s - t)^2)(1 + (s + t)^2)}.
\]

For the weight function \( w_1(t) = e^{-|t|} \), the probability limit figuring in (2.5) is given by

\[
b_1(\theta) = \gamma E_\theta \left[ \frac{1}{\gamma^2 + (\frac{X_1-X_2}{\sigma(\theta)})^2} + \frac{1}{\gamma^2 + (\frac{X_1+X_2}{\sigma(\theta)})^2} \right]
\]

\[
- 2\gamma E_\theta \left[ \frac{1}{\gamma^2 + (\frac{X_1-X_2-X_3}{\sigma(\theta)})^2} + \frac{1}{\gamma^2 + (\frac{X_1-X_2+X_3}{\sigma(\theta)})^2} \right]
\]

\[
+ \gamma E_\theta \left[ \frac{1}{\gamma^2 + (\frac{X_1-X_2-X_3+X_4}{\sigma(\theta)})^2} + \frac{1}{\gamma^2 + (\frac{X_1-X_2+X_3-X_4}{\sigma(\theta)})^2} \right],
\]

while for \( w_2(t) = e^{-\gamma t^2} \), the same probability limit is given by

\[
b_2(\theta) = \frac{1}{2} \sqrt{\frac{\pi}{\gamma}} E_\theta \left[ \exp \left( \frac{(X_1 - X_2)^2}{4\gamma \sigma^2(\theta)} \right) + \exp \left( \frac{(X_1 + X_2)^2}{4\gamma \sigma^2(\theta)} \right) \right]
\]

\[
- \sqrt{\frac{\pi}{\gamma}} E_\theta \left[ \exp \left( \frac{(X_1 - X_2 - X_3)^2}{4\gamma \sigma^2(\theta)} \right) + \exp \left( \frac{(X_1 - X_2 + X_3)^2}{4\gamma \sigma^2(\theta)} \right) \right]
\]

\[
+ \frac{1}{2} \sqrt{\frac{\pi}{\gamma}} E_\theta \left[ \exp \left( \frac{(X_1 - X_2 - X_3 + X_4)^2}{4\gamma \sigma^2(\theta)} \right) \right]
\]

\[
+ \exp \left( \frac{(X_1 - X_2 + X_3 - X_4)^2}{4\gamma \sigma^2(\theta)} \right),
\]

where, recall, \( \sigma(\theta) = \int_{-\infty}^{\infty} x g_\theta(x) dx \).

As far as alternatives are concerned, the exponential distribution has some common close alternatives. Specifically we consider the following alternatives:

– the Weibull distribution with density

\[
g_\theta(x) = e^{-x^{1+\theta}} (1 + \theta)x^\theta, \theta > 0, x \geq 0;
\]

– the gamma distribution with density

\[
g_\theta(x) = \frac{x^\theta e^{-x}}{\Gamma(\theta + 1)}, \theta > 0, x \geq 0;
\]

– the Makeham distribution with density

\[
g_\theta(x) = e^{-x-\theta e^x} (1 + \theta e^x), \theta > 0, x \geq 0;
\]
Table 6 Labe of the test for exponentiality with weight function $w_1$

| $\gamma$ | Weibull | Gamma | Makeham | LFR | ME(3) | ME(6) |
|----------|----------|--------|----------|-----|-------|-------|
| 0.1      | 0.502    | 0.619  | 0.404    | 0.125 | 0.585 | 0.831 |
| 0.2      | 0.552    | 0.625  | 0.470    | 0.155 | 0.669 | 0.841 |
| 0.3      | 0.584    | 0.625  | 0.523    | 0.183 | 0.724 | 0.834 |
| 0.4      | 0.610    | 0.632  | 0.580    | 0.203 | 0.763 | 0.819 |
| 0.5      | 0.625    | 0.627  | 0.621    | 0.231 | 0.792 | 0.802 |
| 0.6      | 0.637    | 0.626  | 0.657    | 0.252 | 0.813 | 0.784 |
| 0.7      | 0.649    | 0.626  | 0.688    | 0.275 | 0.829 | 0.767 |
| 0.8      | 0.659    | 0.624  | 0.713    | 0.295 | 0.842 | 0.750 |
| 0.9      | 0.671    | 0.622  | 0.737    | 0.315 | 0.852 | 0.735 |
| 1        | 0.676    | 0.620  | 0.757    | 0.334 | 0.859 | 0.720 |
| 2        | 0.722    | 0.596  | 0.881    | 0.494 | 0.874 | 0.616 |
| 3        | 0.738    | 0.573  | 0.925    | 0.611 | 0.850 | 0.553 |
| 4        | 0.741    | 0.553  | 0.938    | 0.699 | 0.821 | 0.510 |
| 5        | 0.737    | 0.534  | 0.936    | 0.765 | 0.792 | 0.478 |
| 6        | 0.730    | 0.518  | 0.928    | 0.815 | 0.766 | 0.453 |
| 7        | 0.721    | 0.504  | 0.917    | 0.853 | 0.743 | 0.434 |
| 8        | 0.713    | 0.487  | 0.905    | 0.882 | 0.723 | 0.418 |
| 9        | 0.704    | 0.478  | 0.893    | 0.905 | 0.705 | 0.404 |
| 10       | 0.696    | 0.461  | 0.881    | 0.923 | 0.690 | 0.393 |

- the linear failure rate (LFR) distribution with density

$$g_\theta(x) = e^{-x-\theta x^2} (1 + \theta x), \theta > 0, x \geq 0;$$

- the mixture of exponential distributions with negative weights (ME(\(\beta\))) with density

$$g_\theta(x; \beta) = (1 + \theta) e^{-x} - \theta \beta e^{-\beta x}, \theta \in \left(0, \frac{1}{\beta - 1}\right], x \geq 0;$$

It is possible to consider a scaled version of alternatives \(\tilde{g}_\theta(x) = \frac{1}{\sigma_0} g_\theta(\frac{x}{\sigma_0})\) to make them close to an arbitrary member of the null exponential family. However, as discussed previously, the Bahadur efficiency does not depend on the scale parameter, so, without loss of generality, we can consider the alternatives close to the unit exponential distribution. These alternatives are also used by e.g. Milošević and Obadović (2016), Nikitin and Volkova (2016), Milošević (2016) and Cuparić et al. (2019). The resulting efficiencies are calculated by following analogous steps as in the previous sections and are reported in Tables 6 and 7, for the test based on \(w_1(t) = e^{-\gamma |t|}\) and \(w_2(t) = e^{-\gamma |t|^2}\), respectively. Corresponding graphs of LABEs are given in Fig. 4.
Table 7  LABE of the test for exponentiality with weight function $w_2$

| $\gamma$ | Weibull | Gamma | Makeham | LFR  | ME(3) | ME(6) |
|----------|---------|-------|---------|------|-------|-------|
| 0.1      | 0.581   | 0.585 | 0.580   | 0.198| 0.756 | 0.749 |
| 0.2      | 0.619   | 0.584 | 0.676   | 0.261| 0.807 | 0.698 |
| 0.3      | 0.639   | 0.581 | 0.735   | 0.312| 0.828 | 0.665 |
| 0.4      | 0.657   | 0.579 | 0.773   | 0.350| 0.838 | 0.640 |
| 0.5      | 0.670   | 0.576 | 0.802   | 0.384| 0.843 | 0.621 |
| 0.6      | 0.679   | 0.573 | 0.824   | 0.413| 0.845 | 0.605 |
| 0.7      | 0.687   | 0.571 | 0.842   | 0.439| 0.845 | 0.592 |
| 0.8      | 0.693   | 0.568 | 0.856   | 0.463| 0.844 | 0.580 |
| 0.9      | 0.698   | 0.566 | 0.867   | 0.485| 0.842 | 0.570 |
| 1        | 0.703   | 0.564 | 0.877   | 0.504| 0.840 | 0.562 |
| 2        | 0.722   | 0.544 | 0.918   | 0.643| 0.811 | 0.505 |
| 3        | 0.725   | 0.529 | 0.925   | 0.724| 0.785 | 0.474 |
| 4        | 0.723   | 0.516 | 0.921   | 0.778| 0.763 | 0.452 |
| 5        | 0.719   | 0.505 | 0.915   | 0.816| 0.746 | 0.436 |
| 6        | 0.714   | 0.497 | 0.908   | 0.844| 0.731 | 0.424 |
| 7        | 0.710   | 0.489 | 0.900   | 0.866| 0.718 | 0.414 |
| 8        | 0.705   | 0.481 | 0.892   | 0.892| 0.708 | 0.406 |
| 9        | 0.700   | 0.476 | 0.885   | 0.885| 0.698 | 0.399 |
| 10       | 0.697   | 0.467 | 0.879   | 0.910| 0.690 | 0.393 |

Fig. 4 LABE for the ECF tests for exponentiality: $w_1$—solid line, $w_2$—dashed line
4.3.1 Discussion

From Tables 6 and 7 and Fig. 4 one can notice that the type of monotonicity of the efficiencies with respect to the tuning parameter $\gamma$ varies amongst alternatives. However, for any fixed alternative, both tests exhibit the same behaviour, and if we compare maximal reached efficiencies, the test with weight function $w_1(\cdot)$ appears to have a slight edge. Next we compare the efficiencies obtained here with the efficiencies reported by Cuparić et al. (2022) for a wide variety of exponentiality tests and for four of the alternatives considered herein (e.g., Weibull, Gamma, LFR and ME(3)). Specifically the tests based on $T_{n,w}$ seem to be less efficient than some of the best tests considered by Cuparić et al. (2022). When restricting comparison to the classical tests based on the EDF, we see that the new tests are more efficient than the KS test, but overall less efficient than the CM and AD tests, with the exception of the LFR alternative where, with proper choice of $\gamma$, our tests compete well with these two tests.

5 Tests for the bivariate normal distribution

We consider the problem of testing the null hypothesis $H_0$ that the sample comes from a bivariate normal distribution, using the BHEP and GE tests presented in Sect. 4.1. Efficiencies are calculated for two cases: The simple hypothesis of known mean/covariance, that the law of $X$ is $N_2(0, I)$, and the hybrid case of unknown mean/known covariance, i.e. that the law of $X$ is $N_2(\mu, I)$, where $N_p(\mu, \Sigma)$ denotes the multivariate normal distribution with mean vector $\mu$ and covariance matrix equal to $\Sigma$, and $I$ stands for the identity matrix in the corresponding dimension. The covariance kernels of the test statistics are available from Móri et al. (2021) (GE test) and Henze and Wagner (1997) (BHEP test), while for the simple hypothesis case with $\gamma = 1/2$, eigenvalues of the BHEP test in arbitrary dimension may be obtained from Baringhaus (1996).\footnote{As a numerical confirmation, we have compared the efficiencies obtained by using Baringhaus’ eigenvalues with those using the Monte Carlo method and found them to be quite close in all cases.}

The method of calculation of efficiencies is the same as for univariate tests. Specifically, consider an alternative with density $g(x, y; \theta)$, such that $g(x, y; 0)$ is the density of $N_2(0, I)$ in the simple null hypothesis case, and of $N_2(\mu_0, I)$, for an arbitrary mean vector $\mu_0 = (\mu_{10}, \mu_{20})^\top$, in the unknown mean case. For small $\theta$, compute the double KL distance of this alternative to the null hypothesis, as

- simple hypothesis case:

$$2K(\theta) = \left( \int \int_{\mathbb{R}^2} \frac{h^2(x, y)}{f_0(x, y)} dx dy \right) \theta^2 + o(\theta^2),$$

where $f_0(\cdot, \cdot)$ denotes the density of the standard normal distribution.
Bahadur efficiency for certain goodness-of-fit tests… 743

estimated mean case:

\[ 2K(\theta) = \left( \int \int_{\mathbb{R}^2} \frac{(h(x, y) - f_{\theta_0}(x, y)(x - \mu_{10})\mu_X'(0) + (y - \mu_{20})\mu_Y'(0))^2}{f_{\theta_0}(x, y)} \right) dxdy \]

\[ \theta^2 + o(\theta^2), \]

where \( f_{\theta_0}(x, y) \) stands for the density of the bivariate normal distribution with the vector of parameters \( \theta_0 = (\mu_{10}, \mu_{20}, 1, 1, 0) \), and

\[ \mu_X(\theta) = \int \int_{\mathbb{R}^2} xg(x, y; \theta) dxdy; \quad \mu_Y(\theta) = \int \int_{\mathbb{R}^2} yg(x, y; \theta) dxdy, \]

The proof is analogous to that of Theorem 3.1, so we omit it here.

The multidimensional version of the eigenvalue approximation procedure described in Sect. 3 is computationally very complex, hence here we obtained the largest eigenvalues using the Monte Carlo procedure described in Móri et al. (2021). In this connection we have noticed that the variance of the Monte Carlo estimator of the eigenvalue is small in the case of BHEP test, which makes the obtained values reliable. On the other hand, for the GE test, the variance is significant, making the approximation not so accurate. In addition, this problem drastically increases with the increase of the tuning parameter \( \gamma \), therefore we excluded the cases when \( \gamma > 1 \) from the study.

In order to introduce our alternatives write \( f(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) \) for the bivariate normal density with mean vector \( \mu = (\mu_1, \mu_2)^\top \) and covariance matrix \( \Sigma \) with elements \( \sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2 \), and \( \sigma_{12} = \sigma_{21} = \rho \).

With this notation we consider the following alternatives for the case of the simple hypothesis tests:

- location alternative \( g_l(\theta) = f(\theta, 0, 1, 1, 0) \)
- correlation alternative \( g_c(\theta) = f(0, 0, 1, 1, \theta) \)
- single scale alternative \( g_{s1}(\theta) = f(0, 0, 1 - \theta, 1, 0) \)
- double scale alternative \( g_{s2}(\theta) = f(0, 0, 1 - \theta, 1 - \theta, 0) \)
- contamination alternative

\[ g_{cn}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho, \theta) = (1 - \theta) f(0, 0, 1, 1, 0) + \theta f(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho), \]

for the specific choices of parameters given in Table 8.

5.1 Discussion

Table 9 (simple hypothesis case) and Table 10 (estimated mean case) report efficiency results for the GE and BHEP tests for bivariate normality. To facilitate comparison,
Table 8  Parameters of the contaminating distribution

| g cn | μ_1 | μ_2 | σ_1 | σ_2 | ρ |
|------|-----|-----|-----|-----|---|
| (1)  | 0.1 | 0   | 1   | 1   | 0 |
| (2)  | 0.5 | 0   | 1   | 1   | 0 |
| (3)  | 0.9 | 0   | 1   | 1   | 0 |
| (4)  | 1.5 | 0   | 1   | 1   | 0 |
| (5)  | 0   | 0   | 1   | 1   | 0.1|
| (6)  | 0   | 0   | 1   | 1   | 0.5|
| (7)  | 0   | 0   | 1   | 1   | 0  |
| (8)  | 0   | 0   | 0.5 | 1   | 0  |
| (9)  | 0   | 0   | 0.7 | 1   | 0  |
| (10) | 0   | 0   | 0.9 | 1   | 0  |
| (11) | 0   | 0   | 1.1 | 1   | 0  |

Table 9  LABE for the GE and BHEP tests—simple hypothesis

| Test | GE | BHEP |
|------|----|------|
| γ    | 0.5| 0.7  | 1   |
|      |    |      | 0.25| 0.5| 1   | 2   |

| g_l  | 0.285| 0.564| **0.961**| 0.631| 0.762| 0.860| 0.927|
| g_c  | 0.054| 0.092| **0.119**| 0.252| **0.254**| 0.215| 0.154|
| g_s1 | 0.080| 0.138| **0.180**| 0.379| **0.381**| 0.323| 0.232|
| g_s2 | 0.106| 0.184| **0.240**| 0.505| **0.508**| 0.430| 0.309|
| g cn (1)| 0.284| 0.562| **0.956**| 0.627| 0.735| 0.853| **0.918**|
| g cn (2)| 0.248| 0.492| **0.840**| 0.542| 0.637| 0.743| **0.808**|
| g cn (3)| 0.178| 0.355| **0.609**| 0.378| 0.449| 0.531| **0.582**|
| g cn (4)| 0.068| 0.137| **0.238**| 0.134| 0.163| 0.200| **0.225**|
| g cn (5)| 0.053| 0.092| **0.119**| 0.253| 0.245| 0.213| **0.150**|
| g cn (6)| 0.046| 0.078| **0.101**| 0.237| 0.215| 0.172| **0.117**|
| g cn (7)| 0.019| 0.031| **0.038**| 0.110| 0.082| 0.055| 0.033|
| g cn (8)| 0.098| 0.159| **0.193**| 0.588| 0.454| 0.310| **0.180**|
| g cn (9)| 0.102| 0.170| **0.213**| 0.563| 0.486| 0.369| **0.233**|
| g cn (10)| 0.090| 0.156| **0.199**| 0.451| 0.425| 0.355| **0.245**|
| g cn (11)| 0.067| 0.116| **0.154**| 0.303| 0.303| 0.275| **0.206**|

for each alternative we report in italics the best efficiency of the test under discussion against γ, while a bold entry indicates the best efficiency value across both tests.

For the GE test the monotonicity is clear, since better efficiencies are obtained as γ increases, in both the simple and the hybrid case test, a behaviour that mimics the univariate tests. On the other hand, also analogously to the univariate case, no such pattern is visible for the BHEP test in the simple hypothesis case, as efficiencies are
better at the one or the other end of the tuning parameter interval, or even in-between values of $\gamma$, depending on the type of alternative. Things are somewhat more clear though for the BHEP test with estimated mean, and specifically in this case better efficiencies are observed for larger values of $\gamma$, nearly uniformly over alternatives.

When comparing the GE and BHEP tests with each other, one cannot suggest a single test as being uniformly more efficient over all alternatives. Nevertheless in the simple hypothesis case, the GE test with $\gamma = 1$ appears to perform better against location contamination alternatives, while the BHEP test with $\gamma = 0.25$, 0.50 shows higher efficiency in nearly all other cases of alternatives. With a few exceptions, the BHEP test with $\gamma = 2$ seems to be also preferable over the GE test in the case of testing for bivariate normality with unknown mean.

### 6 Conclusion

We consider test optimality in the Bahadur sense for certain ECF-based GOF tests for the normal, the logistic and the exponential distribution. In the case of testing for normality we compare the efficiencies of the celebrated generalized energy and BHEP tests, and we offer suggestions as to which tests should be used on the basis of our efficiency comparisons, in the univariate as well as the bivariate case, with or without estimated parameters. In the case of testing for the logistic distribution efficiency comparisons of an ECF-based test against classical tests based on the EDF are reported for moment as well as maximum likelihood estimation of parameters, whereas for testing exponentiality we compare our efficiency findings with those of Cuparić et al. (2022) for a wide range of alternative tests, including classical ones. Overall ECF tests appear to compare well and often outperform competitors. Moreover as ECF-based tests involve a tuning parameter, the results reported herein may be
used in determining which value of this parameter should be employed by the user, a problem that in various forms occupies researchers to this date; see for instance the contributions of Ebner and Henze (2021, 2022), Tenreiro (2019), and Allison and Santana (2015).

**Funding** Funding is provided by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia.

**Appendix**

**Regularity conditions A**

There exists $\delta > 0$ such that for $|\theta| \leq \delta$ it holds:

A1 Let MLEs $\hat{\mu}$ and $\hat{\sigma}$ of $\mu$ and $\sigma$ exist (under the null model) such that $\hat{\mu} \xrightarrow{P_\theta} \mu(\theta)$ and $\hat{\sigma} \xrightarrow{P_\theta} \sigma(\theta)$;

A2 Functions $\mu(\theta)$ and $\sigma(\theta)$ are three times continuously differentiable;

A3 Functions $L_1(x; \theta) = \log g_\theta(x)$ and $L_2(x; \theta) = \log \left( \frac{1}{\sigma(\theta)} f_0\left( \frac{x-\mu(\theta)}{\sigma(\theta)} \right) \right)$ are three times differentiable;

A4 $\left| \frac{\partial L_k(x; \theta)}{\partial \theta^i} \frac{\partial g_\theta(x)}{\partial \theta^j} \right| < M_{i,j}(x)$ for $i, j = 0, 1, 2, i + j \leq 3, k = 1, 2$, where $M_{i,j}(x)$ are integrable functions.

**Proof of Theorem 3.1** Let $\mu = \mu(\theta)$ and $\sigma = \sigma(\theta)$ be the values that minimize (3.1), their existence following from Condition A1, by considering the sample Kullback–Leibler distance $\frac{1}{n} \sum_{i=1}^n \log \left( \frac{g_\theta(x)}{\frac{1}{\sigma(\theta)} f_0\left( \frac{x-\mu(\theta)}{\sigma(\theta)} \right)} \right)$ as in Villa (2016).

Conditions A2–A4 enable differentiation under the integral sign. Hence, differentiating (3.1) with respect to $\theta$ we get

$$K'(\theta) = \int \log g_\theta(x) g'_\theta(x) dx - \int \frac{1}{f_0\left( \frac{x-\mu(\theta)}{\sigma(\theta)} \right)} f'_0\left( \frac{x-\mu(\theta)}{\sigma(\theta)} \right) \left( \frac{x-\mu(\theta)}{\sigma(\theta)} \right)' g_\theta(x) dx + \frac{\sigma'(\theta)}{\sigma(\theta)} - \int \log f_0\left( \frac{x-\mu(\theta)}{\sigma(\theta)} \right) g'_\theta(x) dx.$$  

It is easy to show that $K'(0) = 0$. Differentiating (3.1) once more we obtain at $\theta = 0$,

$$K''(0) = \int \frac{h^2(x)}{\sigma_0 f_0\left( \frac{x-\mu_0}{\sigma_0} \right)} dx + \int \log \frac{1}{\sigma_0} f_0\left( \frac{x-\mu_0}{\sigma_0} \right) u(x) dx + \frac{\sigma''(0)\sigma_0 - (\sigma'(0))^2}{\sigma_0^2}$$

$$+ \int \left( \frac{f'_0\left( \frac{x-\mu_0}{\sigma_0} \right)}{\sigma_0 f_0\left( \frac{x-\mu_0}{\sigma_0} \right)} \right)^2 \left( \frac{\mu'(0)\sigma_0 + (x-\mu_0)\sigma'(0)}{\sigma_0^2} \right)^2 dx$$

$$- \int \frac{1}{\sigma_0} f''_0\left( \frac{x-\mu_0}{\sigma_0} \right) \left( \frac{\mu'(0)\sigma_0 + (x-\mu_0)\sigma'(0)}{\sigma_0^2} \right)^2 dx.$$
\[-\int \frac{1}{\sigma_0} f_0' \left( x - \mu_0 \right) \left( -\mu''(0)\sigma_0 + 2\mu'(0)\sigma'(0) + (x - \mu_0) \left( \frac{\sigma'(0)^2}{\sigma_0} - \sigma''(0) \right) \right) dx\]

\[+ \int \frac{f_0'}{f_0} \left( x - \mu_0 \right) \left( \mu'(0)\sigma_0 + (x - \mu_0)\sigma'(0) \right) h(x) dx\]

\[+ \int \frac{f_0'}{f_0} \left( x - \mu_0 \right) \left( \mu'(0)\sigma_0 + (x - \mu_0)\sigma'(0) \right) h(x) dx - \int \log f_0 \left( \frac{x - \mu_0}{\sigma_0} \right) u(x) dx,\]

where \( h(x) = \frac{\partial}{\partial \theta} g_\theta(x)|_{\theta = 0} \) and \( u(x) = \frac{\partial^2}{\partial \theta^2} g_\theta(x)|_{\theta = 0} \). Using the change of variable, rearranging terms in the above expression, and expanding \( K(\theta) \) into a Maclaurin expansion completes the proof. \( \square \)

**Regularity conditions B**

Let the test statistic be a V-statistic of the form

\[T_n = \frac{1}{n^2} \sum_{i,j} \Phi \left( \frac{X_i - \hat{\mu}}{\hat{\sigma}}, \frac{X_j - \hat{\mu}}{\hat{\sigma}} \right)\]

for some degenerate kernel \( \Phi \). Then \( b(\theta) \) has representation

\[b(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi \left( \frac{x_1 - \mu(\theta)}{\sigma(\theta)}, \frac{x_2 - \mu(\theta)}{\sigma(\theta)} \right) g_\theta(x_1) g_\theta(x_2) dx_1 dx_2 \]

\[= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(z_1, z_2) \widehat{g}(z_1; \theta) \widehat{g}(z_2; \theta) dz_1 dz_2,\]

where

\[\widehat{g}(z; \theta) = \sigma(\theta) g_\theta(z \sigma(\theta) + \mu(\theta)).\]

To allow a Taylor expansion of \( b(\theta) \), suppose the following regularity conditions similar to A1–A4 are satisfied.

**B1** Let \( \hat{\mu} \) and \( \hat{\sigma} \) be consistent estimators of \( \mu(\theta) \) and \( \sigma(\theta) \), i.e. \( \hat{\mu} \xrightarrow{P} \mu(\theta) \) and \( \hat{\sigma} \xrightarrow{P} \sigma(\theta) \);

**B2** Functions \( \mu(\theta) \) and \( \sigma(\theta) \) are three times continuously differentiable;

**B3** Function \( \widehat{g}(x; \theta) \) is three times differentiable;

**B4** \[ \left| \frac{\partial^2 \widehat{g}(x; \theta)}{\partial \theta^i \partial \theta^j} \right| \leq \tilde{M}_{i,j}(x, y) \] for \( i, j = 0, 1, 2, i + j \leq 3, k = 1, 2 \), where \( \tilde{M}_{i,j}(x, y) \) are integrable functions.

Following Nikitin and Peaucelle (2004) and arguments therein the regularity conditions can be expressed in terms of \( \tilde{g}(z; \theta) \), i.e. Assumptions WD:
Remark If the kernel function $\Phi(\cdot)$ is uniformly bounded, then conditions B3 and B4 can be simplified by replacing $\tilde{g}$ with $g$, while WD1 and WD2 follow from B3 and B4.

By way of example we consider the regularity conditions for the second Ley-Paindaveine alternative to the logistic null distribution. (If we consider the same alternative in the case of the normal distribution the regularity conditions may be shown by analogous arguments). Specifically for regularity conditions A, note that the density is:

$$g_\theta(x) = e^{-x}(1 + e^{-x})^2 \left(1 - \theta \pi \cos \left(\frac{\pi}{1 + e^{-x}}\right)\right).$$

(For simplicity and without loss of generality here we can set the null parameters to $\mu = 0$ and $\sigma = 1$).

Condition A1. Given the MLEs for $\mu$ and $\sigma$ in the logistic model, the consistency of estimators under the considered alternative can be justified using arguments from Theorem 2.6.1. (from Subba Rao (2017)). In particular, the functions $\log g_\theta(x)$ and $\frac{\partial^2}{\partial \theta^2} \log g_\theta(x)$ are Lipschitz-continuous for $|\theta| \leq \delta$, the parameter space is compact ($\{\theta : |\theta| \leq \delta\}$), and point-wise convergence of the log-likelihood and its second derivative to their expectations hold due to law of large numbers.

Conditions A2 follows from the implicit function theorem.

Condition A3 is obviously satisfied.

Condition A4. Let $|\theta| \leq \delta$. Then functions $\mu(\theta)$ and $\sigma(\theta)$ are bounded due to their continuity. The integrable dominants are available from the following bounds, where all constants $C$ and $C_j$ depend on $\delta$ (and are different in each equation).

From the inequality $|\log \frac{e^{-x}}{(1 + e^{-x})^2}| \leq |x| + \ln 2$ we get

$|L_1(x; \theta)| \leq |x| + C$;

$|L_2(x; \theta)| \leq C_1|x| + C_0$;

For the derivatives of $L_1$ we have

$$\left|\frac{\partial^i}{\partial \theta^i} L_1(x; \theta)\right| = \left|\frac{\pi^i \cos^i \left(\frac{\pi}{1 + e^{-x}}\right)}{(1 - \pi \theta \cos \left(\frac{\pi}{1 + e^{-x}}\right))^i}\right| \leq C;$$
The expressions for the derivatives of $L_2$ are too cumbersome to display. However, using boundness of functions $e^{-ax}/(1 + e^{-x})^b$ for $a \leq b$ and boundedness of derivatives of $\mu(\theta)$ and $\sigma(\theta)$, we get

$$
\left| \frac{\partial}{\partial \theta} L_2(x; \theta) \right| \leq C_0 + C_1 x;
$$

$$
\left| \frac{\partial^2}{\partial \theta^2} L_2(x; \theta) \right| \leq C_0 + C_1 x + C_2 x^2;
$$

$$
\left| \frac{\partial^3}{\partial \theta^3} L_2(x; \theta) \right| \leq C_0 + C_1 x + C_2 x^2 + C_3 x^3.
$$

The density $g_\theta(x)$ itself and its first derivative is simply bounded by the standard logistic density as

$$
|g_\theta(x)| = \frac{e^{-x}}{(1 + e^{-x})^2} \left| 1 - \pi \theta \cos \left( \frac{\pi}{e^{-x} + 1} \right) \right| \leq C \cdot \frac{e^{-x}}{(1 + e^{-x})^2};
$$

$$
\left| \frac{\partial}{\partial \theta} g_\theta(x) \right| = \frac{e^{-x}}{(1 + e^{-x})^2} \left| \pi \cos \left( \frac{\pi}{e^{-x} + 1} \right) \right| \leq C \cdot \frac{e^{-x}}{(1 + e^{-x})^2},
$$

and its higher derivatives are equal to zero.

The integrability of all dominants now follows form the finiteness of moments of the logistic distribution.

Turning to regularity conditions B, Condition B1 in the case of MLEs, coincides with Condition A1, while in the case of moment estimators, which we have in analytic form, it follows from the law of large numbers. Condition B2 in the case of MLEs, again coincides with Condition A2, while in the case on moment estimators it follows from the differentiability of the function $g(\cdot)$ and the finiteness of $\int_{-\infty}^{\infty} x^k |\frac{\partial^k g_\theta(x)}{\partial \theta^k}| dx$, $k = 0, 1, 2, 3$. Since the kernel $\Phi$ of the test for logistic distribution is bounded, from the Remark above, conditions B3 and B4 follow from A3 and A4.

References

Allison J, Santana L (2015) On a data-dependent choice of the tuning parameter appearing in certain goodness-of-fit tests. J Stat Comput Simul 85(16):3276–3288

Bahadur R (1967) An optimal property of the likelihood ratio statistic. In: Proceedings of the fifth Berkeley symposium on mathematical statistics and probability, vol 1, pp 13–26

Bahadur RR (1971) Some limit theorems in statistics. SIAM

Baringhaus L (1996) Fibonacci numbers, Lucas numbers and integrals of certain Gaussian processes. Proc Am Math Soc 124(12):3875–3884

Božin V, Milošević B, Nikitin YY, Obradović M (2020) New characterization-based symmetry tests. Bull Malays Math Sci Soc 43(1):297–320

Cuparić M, Milošević B, Obradović M (2019) New $L^2$-type exponentiality tests. SORT Stat Oper Res Trans 43(1):25–50

Cuparić M, Milošević B, Obradović M (2022) New consistent exponentiality tests based on $V$-empirical Laplace transforms with comparison of efficiencies. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A Matemáticas 116(42):1–26
Drost F, Kallenberg W, Oosterhoff J (1990) The power of EDF tests of fit under non-robust estimation of nuisance parameters. Stat Decis 8:167–182
Ebner B, Henze N (2021) Bahadur efficiencies of the Epps–Pulley test for normality. Zapiski Nauchnykh Seminarov POMI 501:302–314
Ebner B, Henze N (2022) On the eigenvalues associated with the limit null distribution of the Epps–Pulley test for normality. Stat Papers. https://doi.org/10.1007/s00362-022-01336-6
Epps T, Pulley L (1983) A test for normality based on the empirical characteristic function. Biometrika 70(3):723–726
Gradshteyn I, Ryzhik I (1994) Tables of integrals, series, and products. Academic Press, New York
Grané A, Fortiana J (2011) A directional test of exponentiality based on maximum correlations. Metrika 73(2):255–274
Grané A, Tchirina A (2013) Asymptotic properties of goodness-of-fit test based on maximum correlations. Statistics 47(1):202–205
Gregory GG (1980) On efficiency and optimality of quadratic tests. Ann Stat 8(1):116–131
Gulati S, Shapiro S (2009) A new goodness of fit test for the logistic distribution. J Stat Theory Pract 3(3):567–576
Gürtler N, Henze N (2000) Goodness-of-fit tests for the Cauchy distribution based on the empirical characteristic function. Ann Inst Math 52(2):267–286
Henze N, Wagner T (1997) A new approach to the BHEP tests for multivariate normality. J Multivar Anal 62(1):1–23
Henze N, Meintanis SG (2005) Recent and classical tests for exponentiality: a partial review with comparisons. Metrika 61(1):29–45
Jones MC (2015) Flexible modelling in statistics: past, present and future. Journal de la Société Française de Statistique 156(1):76–96
Meintanis SG (2004) Goodness-of-fit tests for the logistic distribution based on empirical transforms. Sankhya Indian J Stat 66(2):306–326
Meintanis SG, Swanepoel J (2007) Bootstrap goodness-of-fit tests with estimated parameters based on empirical transforms. Stat Probab Lett 77(10):1004–1013
Milošević B (2016) Asymptotic efficiency of new exponentiality tests based on a characterization. Metrika 79(2):221–236
Milošević B, Obradović M (2016) Some characterization based exponentiality tests and their Bahadur efficiencies. Publications de L’Institut Mathematique 100(114):107–117
Milošević B, Nikitin YY, Obradović M (2021) Bahadur efficiency of EDF based normality tests when parameters are estimated. Zapiski Nauchnykh Seminarov POMI 501:203–217
Móri TF, Székely GI, Rizzo ML (2021) On energy tests of normality. J Stat Plan Inference 213:1–15
Nikitin YY (1995) Asymptotic efficiency of nonparametric tests. Cambridge University Press, New York
Nikitin YY, Peaucelle I (2004) Efficiency and local optimality of nonparametric tests based on U- and V-statistics. Metron Int J Stat LXII(2):185–200
Nikitin YY, Volkova KY (2016) Efficiency of exponentiality tests based on a special property of exponential distribution. Math Methods Stat 25(1):54–66
Rublik F (1989) On optimality of the LR tests in the sense of exact slopes I. General case. Kybernetika 25(1):13–14
Stephens MA (1979) Tests of fit for the logistic distribution based on the empirical distribution function. Biometrika 66(3):591–595
Subba Rao S (2017) Lecture notes: advanced statistical inference. http://web.stat.tamu.edu/~suhasini/teaching613/teaching613_2017.html
Székely GI, Rizzo ML (2005) A new test for multivariate normality. J Multivar Anal 93(1):58–80
Székely GI, Rizzo ML (2013) Energy statistics: a class of statistics based on distances. J Stat Plan Inference 143(8):1249–1272
Tenreiro C (2009) On the choice of the smoothing parameter for the BHEP goodness-of-fit test. Comput Stat Data Anal 53(4):1038–1053
Tenreiro C (2019) On the automatic selection of the tuning parameter appearing in certain families of goodness-of-fit tests. J Stat Comput Simul 89(10):1780–1797

© Springer
Villa C (2016) A property of the Kullback–Leibler divergence for location-scale models. arXiv preprint arXiv:1604.01983

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.