A Comment on the Geometric Entropy and Conical Space

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Abstract

It has been recently pointed out that a definition of the geometric entropy using the partition function in a conical space does not in general lead to a positive definite quantity. For a scalar field model with a non-minimal coupling we clarify the origin of the anomalous behavior from the viewpoint of the canonical formulation.
1 Introduction

The concept of the geometric entropy has attracted much attention for the past several years in elucidating the black hole entropy [2]-[8]. In this investigation a possibility was pointed out that the ultraviolet divergence of the entropy may exactly be identified with the divergence of the gravitational constant in the perturbative treatment of quantum gravity [9]. However the hypothesis on the equivalence of the two divergences has a crucial weak point in its discussion [9]-[11]. For a scalar field coupled non-minimally to the curvature like $L_\xi = -\xi R\phi^2/2$ with $\xi > 1/6$ and a gauge vector field in lower than eight dimensions, divergent corrections of the gravitational coupling take negative values. On the other hand, the divergence of the geometric entropy is expected to be positive. Thus the two quantities does not seem to be identical.

Even though such a severe problem is exposed, Larsen and Wilczek have recently argued [14] that the two divergences of the geometric entropy and of the gravitational constant, nevertheless, coincide for the case of fields of spin 0, 1/2 and 1 if one specifies the definition of the entropy by use of a partition function in a conical space. For example, their definition works well for a minimally coupled scalar and a spinor field such that it gives positive values precisely equal to the correction to the gravitational constant.

Their idea is quite appealing, but we think that there still remains an ambiguous point in their argument. This is because their entropy still yields negative values in the previously mentioned cases of the non-minimally coupled field and the gauge vector field. In general a definition of the entropy requires that the entropy is ensured to be positive and possesses a certain statistical origin. However they, instead, cite a work [6] in which it is only proved that the entropy of a minimally coupled field can be computed rigorously using the partition function on a cone.

In the present work we analyze carefully their definition for the non-minimal coupling case and argue that it cannot be interpreted directly as the geometric entropy.

The contents in subsequent sections are as follows. In Section 2 we briefly review definition of the geometric entropy using the partition function in a conical space taking a minimally coupled scalar field. In Section 3 we investi-
gate a proposal given by Larsen and Wilczek to define the geometric entropy for a non-minimally coupled field. In Section 4 we show that besides the standard entropy term: \(-Tr\rho \ln \rho\) their definition includes a correction term which has no statistical origin, thus this is unacceptable. This conclusion is similar to the result for a gauge vector field obtained by Kabat [12].

2 Geometric Entropy, Replica Trick and Partition Function on the Cone

In this section we review briefly a recent advance [3], that is, how the geometric entropy can be defined using the replica method and the Euclidean path integral taking an example of the minimally coupled scalar field in D dimensions.

To investigate the divergent part of the entropy, it is sufficient to discuss only the minimally coupled massless case. The action takes the form of

\[
S_{\text{min}} = -\frac{1}{2} \int d^Dx \sqrt{-g_{LL} \gamma_{\mu \nu}^L \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu}}
\]

where the form of the flat metric \(g_{LL} \gamma_{\mu \nu}^L\) can be taken in a variety of ways. In this discussion we use only two forms of the metric:

\[
ds^2 = -(dx^0)^2 + (dx^1)^2 + \sum_{i=2}^{D-1} (dx^i)^2
\]

and

\[
ds^2 = -r^2 dt^2 + dr^2 + \sum_{i=2}^{D-1} (dx^i)^2.
\]

The latter form, eqn(2), is usually called Rindler metric. The Rindler metric has a timelike Killing vector \(\partial_t\). Thus a conserved Hamiltonian \(H_R\) exists due to the symmetry:

\[
H_R = \int_0^\infty dr r \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial r} \right)^2 + \frac{1}{2} \sum_{i=2}^{D-1} \left( \frac{\partial \phi}{\partial x^i} \right)^2 \right]
\]
where $\Pi$ is a conjugate momentum of $\phi$.

Let us first start from the standard definition of the entropy. Consider the wavefunctional of the Minkowskian vacuum state in the cartesian coordinates, eqn(1):

$$\langle \phi | 0 \rangle = \Psi_0[\phi(x^1, \ldots, x^{D-1})] = \Psi_o[\phi_+, \phi_-],$$

where $\phi_+ = \phi(x^1 \geq 0)$ and $\phi_- = \phi(x^1 < 0)$. Tracing over fields in the $x^1 < 0$ half-space reduces the pure wavefunctional $\Psi_o$ to a mixed density matrix:

$$\langle \phi_1^+ | \rho | \phi_2^+ \rangle = \rho[\phi_1^+, \phi_2^+] = \int D\phi_- \Psi_o[\phi_1^+, \phi_-] \Psi_o^*[\phi_2^+, \phi_-].$$

(3)

The definition of geometric entropy is given as follows.

$$S_{geo} \equiv -Tr \rho \ln \rho.$$  

(4)

It is worthwhile to recall the well known fact [1, 4] that the density matrix (3) takes a thermal form such as

$$\rho = \frac{e^{-2\pi H_R}}{Tr(e^{-2\pi H_R})}. $$

(5)

This relation will be used in the following analysis.

The definition (4) can be also rewritten by use of the replica trick:

$$S_{geo} = \left(1 - n \frac{\partial}{\partial n}\right) \ln Tr \rho^n |_{n=1}. $$

(6)

This expression proposes a nice way of defining the geometric entropy invoking the thermalization theorem, eqn(5), and path-integral formulation. Substituting eqn(3) into eqn(6), we obtain

$$S_{geo} = \left(1 - n \frac{\partial}{\partial n}\right) \ln \left[Tr e^{-2\pi H_R} \right] |_{n=1}, $$

(7)

Here notice that a factor $(Tr e^{-2\pi H_R})^{-n}$ coming from normalization of the density matrix does not contribute to the entropy itself. Using the standard
path integration, it is proved that the kernel part can be regarded as a
partition function of the field in a periodic manifold in time.

\[
Tr \left[ \left( e^{-2\pi H_R} \right)^n \right] = Tr e^{-2\pi n H_R}
\]

\[
= \int_{\text{periodic}} D\Pi D\phi \exp \left[ i \int_0^{2\pi n} d\tau \int_0^{\infty} dr \int d^{D-2} x \Pi \frac{\partial \phi}{\partial \tau} 
- \int_0^{2\pi n} d\tau \int_0^{\infty} drr \int d^{D-2} x \left\{ \frac{1}{2} \Pi^2 + \frac{1}{2} \frac{\partial \phi}{\partial r}^2 + \frac{1}{2} \sum_{i=2}^{D-1} \left( \frac{\partial \phi}{\partial x^i} \right)^2 \right\} \right]
\]

\[
= \int_{\text{periodic}} D\phi \exp \left[ - \int_0^{2\pi n} d\tau \int_0^{\infty} dr \int d^{D-2} x \left\{ \frac{1}{2r} \left( \frac{\partial \phi}{\partial \tau} \right)^2 + \frac{r}{2} \left( \frac{\partial \phi}{\partial r} \right)^2 + \frac{r}{2} \sum_{i=2}^{D-1} \left( \frac{\partial \phi}{\partial x^i} \right)^2 \right\} \right]
\]

where a periodic condition, \( \phi(\tau+2\pi n) = \phi(\tau) \), is imposed in the path integral. By virtue of a term:

\[
\frac{1}{2r} \left( \frac{\partial \phi}{\partial \tau} \right)^2
\]

in the path-integral action, configurations with \( \partial_\tau \phi(r = 0) \neq 0 \) are highly suppressed and thus an equation:

\[
\phi(\tau, r = 0) = \phi(0, 0)
\]

can be used in the above path integral. Therefore it is justified that the surface region defined by \( r = 0 \) is exactly a point, that is, the space has a conical structure around \( r = 0 \). Consequently we obtain an amazing expression of the geometric entropy \([8]\):

\[
S_{geo} = \left( 1 - n \frac{\partial}{\partial n} \right) \ln Z(n) \bigg|_{n=1}, \quad (8)
\]

where \( Z(n) \) is a partition function of the scalar field in a conical space of the deficit angle \( \delta = 2\pi (1 - n) \):

\[
Z(n) = \int_{\text{cone}} D\phi \exp \left[ - \frac{1}{2} \int_0^{2\pi n} d\tau \int_0^{\infty} dr \int_{R^{D-2}} d^{D-2} x \sqrt{g} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right]
\]

and

\[
g_{\mu \nu} dx^\mu dx^\nu = r^2 d\tau^2 + dr^2 + \sum_{a=2}^{D-1} (dx^a)^2.
\]

\((0 \leq \tau \leq 2\pi n, \ 0 \leq r \leq \infty.)\)
Due to the ultraviolet behavior of the partition function $Z(n)$ the geometric entropy calculated from eqn(8) is also divergent. We thus need some regularization scheme and adopt the heat kernel regularization:

$$\ln Z(n) = -\frac{1}{2} \ln \text{Det}(-\nabla^2) \rightarrow \frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{dT}{T} \text{Tr} e^{-T(-\nabla^2)},$$  \(9\)

where $\epsilon$ is a covariant short distance cutoff. Then the most singular term of the geometric entropy is evaluated straightforwardly as follows [6, 12, 14]:

$$S_{\text{geo}} \sim A_{\perp}^{(D-2)} \frac{2}{3(D-2)} \left[ \frac{1}{4\pi \epsilon^2} \right]^{D/2-1},$$  \(10\)

where $A_{\perp}^{(D-2)} = \int \prod_{i=2}^{D-2} dx^i$.

The original definition of the entropy: $-Tr \rho \ln \rho$ is known to take a non-negative value. Consistent with this result, the positive term appears in eqn(10).

Furthermore it has been established [12, 14] that this divergence is automatically renormalized by the gravitational constant $G$ if the divergent term is added to the bare Bekenstein-Hawking entropy, $A_{\perp}^{(D-2)}/(4G_o)$:

$$\frac{1}{G} = \frac{1}{G_o} + \frac{2}{3(D-2)} \left[ \frac{1}{4\pi \epsilon^2} \right]^{D/2-1}.$$

This equivalence between the entropy and the gravitational constant renormalizations is proved explicitly not only for the minimally coupled scalar field but also for a spinor field [8, 12, 14] in any spacetime dimension by use of a similar definition to eqn(8).

Thus it can be summarized that the definition (8) works effectively for the minimally coupled field and the spinor field.

3 Non-Minimal Coupling Case

Larsen and Wilczek [14] argue that a proper definition of the geometric entropy is given by eqn(8) not only for the minimally coupled scalar field and the spinor field but also for other fields such as a non-minimally coupled
scalar field and a gauge vector field by substituting each partition function of the field on the cone into eqn(8).

In this and the next section we examine the non-minimal coupling case and check whether this definition is truly suitable for the geometric entropy or not.

Let us start from considering a non-minimally coupled massless field in a general static space. The Lorentzian action reads as

$$S_{nm} = \int d^Dx \sqrt{-g_L} \left( -\frac{1}{2} g_L^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\xi}{2} R_L \phi^2 \right),$$

where $g_{L\mu\nu}$ is given by

$$g_{L\mu\nu} dx^\mu dx^\nu = -N(\vec{x}) (dx^0)^2 + h_{ab}(\vec{x}) dx^a dx^b,$$  \hspace{1cm} (11)

with $N$ and $h_{ab}$ independent of $x^0$.

Now let us define an entropy of the field in a similar way to that \[14\] :

$$S_\xi = \left( 1 - n \frac{\partial}{\partial n} \right) \ln Z(n, \xi) \mid_{n=1},$$  \hspace{1cm} (12)

where

$$Z(n, \xi) = \int_{\text{periodic}} \mathcal{D}\phi \exp \left[ -\frac{1}{2} \int_0^{2\pi n} d\tau \int d^{D-1}x \sqrt{g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi R_L \phi^2 \right) \right],$$

$$g_{\mu\nu} dx^\mu dx^\nu = N(\vec{x}) d\tau^2 + h_{ab}(\vec{x}) dx^a dx^b, \hspace{1cm} (0 \leq \tau \leq 2\pi n),$$  \hspace{1cm} (13)

and the periodic condition, $\phi(\tau + 2\pi n) = \phi(\tau)$, is imposed. Then we can prove formally the positivity of the entropy $S_\xi$. To show this, rewrite the partition function $Z(n, \xi)$ as follows.

$$Z(n, \xi)$$

$$= \int_{\text{periodic}} \mathcal{D}\phi \exp \left[ -\int_0^{2\pi n} d\tau \int d^{D-1}x N \sqrt{h} \left[ \frac{1}{2N^2} \phi^2 + \frac{1}{2} h^{ab} \partial_a \phi \partial_b \phi + \frac{\xi}{2} R_L \phi^2 \right] \right]$$

$$= \int_{\text{periodic}} \mathcal{D}\Pi \mathcal{D}\phi \exp \left[ i \int_0^{2\pi n} d\tau \int d^{D-1}x \Pi \dot{\phi} \right]$$

$$- \int_0^{2\pi n} d\tau \int d^{D-1}x N \sqrt{h} \left[ \frac{1}{2} \left( \frac{\Pi}{\sqrt{h}} \right)^2 + \frac{1}{2} h^{ab} \partial_a \phi \partial_b \phi + \frac{\xi}{2} R_L \phi^2 \right]$$

$$= Tr e^{-2\pi n H_{can}}$$  \hspace{1cm} (14)
where
\[ \hat{H}_{\text{can}} = \int d^{D-1}x \sqrt{h} \left[ \frac{1}{2} \left( \frac{\hat{\Pi}}{\sqrt{h}} \right)^2 + \frac{1}{2} h^{ab} \partial_a \hat{\phi} \partial_b \hat{\phi} + \frac{\xi}{2} R \hat{\phi}^2 \right] \]

and \( \hat{\Pi} \) is a canonical momentum operator conjugate to \( \hat{\phi} \). Substituting \( Z(n, \xi) = \text{Tr} e^{-2\pi n \hat{H}_{\text{can}}} \) into eqn(12) and using a "fact" that \( \hat{H}_{\text{can}} \) is independent of \( n \), we get
\[ S_{\xi} = \ln \text{Tr} e^{-2\pi \hat{H}_{\text{can}}} + 2\pi \frac{\text{Tr}(\hat{H}_{\text{can}} e^{-2\pi \hat{H}_{\text{can}}})}{\text{Tr} e^{-2\pi \hat{H}_{\text{can}}}} \]
\[ = -\text{Tr} \rho \ln \rho, \]

where
\[ \rho = \frac{e^{-2\pi \hat{H}_{\text{can}}}}{\text{Tr} e^{-2\pi \hat{H}_{\text{can}}}}. \]

Therefore if the analysis is entirely correct, the entropy \( S_{\xi} \) really has an explicit statistical origin and is guaranteed to yield a non-negative value.

It should be stressed here that the metric of the cone, which is of our concern, is also written in the above form (13) with \( N = x^1 = r \) and \( h_{ab} = \delta_{ab} \). Moreover, from the replica trick point of view as seen in Section 2, the entropy \( S_{\xi} \) for the conical space could be naively identified with the geometric entropy \( S_{\text{geo}} \) itself. Larsen and Wilczek \[14\] insist that eqn(12) with the conical metric defines naturally the geometric entropy even for the non-minimally coupled scalar field and calculate it explicitly. Then they give the following result:
\[ S_{\text{geo}} = S_{\xi} \sim A_\perp^{(D-2)} \frac{4}{(D-2)} \left( \frac{1}{6} - \xi \right) \left[ \frac{1}{4\pi \epsilon^2} \right]^{D/2-1}, \quad (15) \]

adopting the heat kernel regularization just like in Section 2.

Despite the apparent naturalness of the definition, it is noticed that \( S_{\xi} \) fails to take a non-negative value for \( \xi > 1/6 \). Though this negativeness of the entropy seems very queer, they still argues that the above result is natural and correct \[14\].

Our opinion for the negative value is quite different from that of Larsen and Wilczek and rather similar to that of Kabat \[12\]. It should be emphasized, we believe, that any definition of the geometric entropy must both
intrinsically be non-negative and possess a manifest statistical meaning. The negative value just makes us doubt deeply of the validity of their definition. In fact, as to spaces with conical structure, the previous argument on positivity of $S_\xi$ is too formal and clearly incorrect. It completely misses out an effect of the delta functional curvature at $r = 0$. We shall argue in the next section, treating the conical singularity carefully, that the quantity $S_\xi$ in the conical space cannot be identified exactly with the geometric entropy $S_{geo}$.

4 Validity of a Definition of the Geometric Entropy using Partition Function on the Cone

The definition of the geometric entropy proposed by Larsen and Wilczek [14] is certainly interesting. However in this section we argue that their definition needs some modification for that to be regarded as the entropy of the non-minimally coupled scalar field.

In order to properly treat the conical singularity, let us first express the conical space as a limit of a non-singular static curved space. Consider the following metric:

$$ds^2 = \sum_{n,m=0}^{1} \tilde{g}_{nm} dx^n dx^m + \sum_{i=2}^{D-1} (dx^i)^2 = F(r)^2 d\tau^2 + dr^2 + \sum_{i=2}^{D-1} (dx^i)^2$$

with $0 \leq \tau \leq 2\pi n$, $F(r = 0) = 0$, $F(r \sim \infty) = r$ and $\partial_r F(r = 0) = c$. Here the point $(\tau = 2\pi n, r, x^i)$ is identified with $(0, r, x^i)$.

The scalar curvature $R$ is obtained by a simple calculation:

$$R = 4\pi (1 - cn) \frac{1}{\sqrt{g}} \delta(x^0)\delta(x^1) - \frac{2}{F} \frac{\partial^2 F}{\partial r^2}.$$

To remove the delta function in the curvature, we set

$$c = \frac{1}{n}.$$
Note that this choice of $c$ demands a compensation that the metric itself has $n$ dependence. This $n$ dependence is crucial to understand the anomalous negative value of $S_\xi$, as shown subsequently. To make our analysis more concrete, let us give an explicit example form of $F$ such as

$$F(r, n) = r \left[ \frac{1}{n} + \left( 1 - \frac{1}{n} \right) \frac{\lambda^2 r^2}{1 + \lambda^2 r^2} \right].$$  \hspace{1cm} (16)$$

If we take $\lambda \to \infty$ limit, the conically flat space appears again:

$$\lim_{\lambda \to \infty} F(r, n) = r.$$  

Eqn(16) yields a scalar curvature without the delta function:

$$R(r, n) = \frac{4\lambda^2}{\left(1 + \lambda^2 r^2\right)^2} \frac{3 - \lambda^2 r^2}{1 + \lambda^2 r^2}.$$  \hspace{1cm} (17)

For this metric it is easy to repeat the same argument in Section 3 and we obtain

$$Z(n, \xi) = Tr e^{-2\pi n \hat{H}_{\text{can}}(n)}$$

$$= Tr \exp \left[ -2\pi n \int_0^\infty dr F(r, n) \int d^{D-2}x \left[ \frac{1}{2} \hat{\Pi}^2 + \frac{1}{2} \hat{g}^{ab} \partial_a \hat{\phi} \partial_b \hat{\phi} + \frac{\xi}{2} R(r, n) \hat{\phi}^2 \right] \right].$$

It is worth noting that the $n$ dependence still remains in $\hat{H}_{\text{can}}(n)$. Thus when the derivative with respect to $n$ in eqn(12) is taken, we cannot neglect naively a term proportional to $\partial_n \hat{H}_{\text{can}}$:

$$S_\xi = \left( 1 - n \frac{\partial}{\partial n} \right) \ln Tr e^{-2\pi n \hat{H}_{\text{can}}|_{n=1}}$$

$$= -Tr \varrho \ln \varrho + 2\pi Tr \left[ \varrho \frac{\partial \hat{H}_{\text{can}}}{\partial n} |_{n=1} \right],$$  \hspace{1cm} (18)

where

$$\varrho = \frac{e^{-2\pi \hat{H}_{\text{can}}(1)}}{Tr e^{-2\pi \hat{H}_{\text{can}}(1)}}.$$  

The first term $-Tr \varrho \ln \varrho$ clearly takes a non-negative value and should be identified with the geometric entropy $S_{\text{geo}}$ after taking $\lambda \to \infty$. Now a crucial question is whether the limit of the second correction term:

$$\lim_{\lambda \to \infty} 2\pi Tr \left[ \frac{\partial \hat{H}_{\text{can}}}{\partial n} |_{n=1} \right]$$
vanishes or not. It is easily shown from eqns (16) and (17) that the following relations are satisfied:

\[
\lim_{\lambda \to \infty} \frac{\partial F}{\partial n} \bigg|_{n=1} = 0, \\
\lim_{\lambda \to \infty} \frac{\partial R}{\partial n} \bigg|_{n=1} = -\frac{1}{r} \delta(r).
\]

Using these relations we finally obtain the following result with the non-vanishing correction term:

\[
\lim_{\lambda \to \infty} S_\xi = -Tr \rho \ln \rho - 2\pi \xi A^{(D-2)}_0 \frac{\langle 0_M | \phi(0)^2 | 0_M \rangle}{\langle 0_M |0_M \rangle},
\]

where

\[\rho = \frac{e^{-2\pi H_R}}{Tr(e^{-2\pi H_R})}\]

\[|0_M\rangle\] is the Minkowskian vacuum state and we have used the thermalization theorem (5):

\[Tr_{x^1<0} [\langle 0_M |0_M \rangle] = \rho.\]

We also comment that even when the \(\lambda \to \infty\) limit is performed before taking \(n \to 1\) the same result (13) appears from relations for the conical space:

\[R = 4\pi (1 - n) \frac{1}{\sqrt{g}} \delta(x^0)\delta(x^1),\]

and

\[\lim_{\lambda \to \infty} Z(n, \xi) = Tr \exp \left[ -2\pi \xi \int d^{D-2}x \phi(r = 0)^2 - 2\pi n H_R(\xi) \right],\]

where

\[H_R(\xi) = H_R - \xi \int d^{D-2}x \phi(r = 0)^2.\]

Consequently the entropy proposed by Larsen and Wilczek evidently differs from the geometric entropy \(S_{geo}\) due to the existence of the non-statistical correction term proportional to \(\langle \phi(0)^2 \rangle\) when \(\xi \neq 0\). Because \(\langle \phi(0)^2 \rangle\) is positive, irrespective of the detail of the regularization, the value of \(S_\xi\) reverses its
sign for positive $\xi$ large enough. The geometric entropy for the non-minimal coupling case is more naturally defined by $S_\xi$ in eqn[12] with the conical metric and $\xi = 0$, just like the minimal coupling case.

It has been already pointed out for the gauge vector field case by Kabat [12] that the similar entropy correction term like in eqn[19] follows and that the entropy defined by use of the partition function cannot be equated with the geometric entropy.

We can also prove explicitly that the $<\phi(0)^2>$ term in eqn[19] is precisely equal to the deviation of $S_\xi$ in eqn (15) from the geometric entropy, using the heat kernel regulator. The proof is as follows.

$$
\Delta S = -2\pi\xi A^{(D-2)}_{\perp} \frac{<0M|\phi^2(0,0)|0M>}{<0M|0M>}
$$

$$
= -2\pi\xi \frac{A^{(D-2)}_{\perp}}{V_D} \int d^Dx G_E(x, x)
$$

$$
= -2\pi\xi \frac{A^{(D-2)}_{\perp}}{V_D} Tr \left[ \frac{1}{-\partial^2} \right]
$$

$$
\rightarrow -2\pi\xi \frac{A^{(D-2)}_{\perp}}{V_D} \int e^2 dT Tr e^{T\partial^2}
$$

$$
= -2\pi \int e^2 dT \frac{A^{(D-2)}_{\perp}}{4\pi^2} \frac{4}{D-2} \left[ \frac{1}{4\pi e^2} \right]^{D/2-1}
$$

where $V_D = \int d^Dx$ and the following relations have been used:

$$
G_E(x, y) = <\phi(x)\phi(y)>
$$

$$
= \frac{1}{Z_o} \int D\phi \phi(x)\phi(y) \exp \left[ -\int d^4x \frac{1}{2}(\partial\phi)^2 \right]
$$

$$
= \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} \frac{1}{Z_o} \int D\phi \exp \left[ -\int d^4x \left[ \frac{1}{2}(\partial\phi)^2 + J\phi \right] \right] \bigg|_{J=0}
$$

$$
= -\frac{1}{\partial^2} \delta^4(x-y).
$$

Thus the correction term $\Delta S$ precisely reproduces the term linearly depending on $\xi$ in eqn[13], which demonstrates that the quantity (12) is negative.
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Note added.

After submitting this paper we received a few comments.
We were informed that the negative entropy for the non-minimal coupling case has independently been derived by S.N.Solodukhin\cite{13} using the heat kernel method.
Another comment is that the relation in eqn(19) has been also derived in the different context, namely, in a special class of the induced gravity theory by V.P.Frolov, D.V.Fursaev and A.I.Zelnikov\cite{15}.

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