On the Left Connected Subalgebra of the Descent Algebra of a Coxeter Group of Classical Type

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Abstract. A Coxeter group of classical type $A_n$, $B_n$ or $D_n$ contains a chain of subgroups of the same type. We show that intersections of conjugates of these subgroups are again of the same type, and make precise in which sense and to what extent this property is exclusive to the classical types of Coxeter groups. As the main tool for the proof, we use Solomon’s descent algebra. Using Stirling numbers, we express certain basis elements of the descent algebra as polynomials and derive explicit multiplication formulas for a commutative subalgebra of the descent algebra.

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1. Introduction

For subgroups $J$ and $K$ of a finite group $W$, Mackey’s Theorem describes the restriction to $K$ of a character induced from $J$ to $W$ in terms of intersections of $W$-conjugates of both $J$ and $K$, which are parametrized by the $(J, K)$-double cosets of $W$. In such a context, it is natural to ask: which subgroups of $W$ do occur as intersections of conjugates of $J$ and $K$?

In [4], for example, this question arises for a Coxeter group $W$ of type $B_n$, with Coxeter groups $J, K \leq W$ of types $B_j, B_k$, for some $j, k \leq n$. In this case, the intersections turn out to be Coxeter groups of type $B_l$, $1 \leq l \leq n$. Representatives of the conjugacy classes of these subgroups can be generated by subsets of the Coxeter generators of $W$, and lie in a maximal chain of such subgroups of $W$.

In this paper, we classify all such chains of subgroups of an irreducible finite Coxeter group $W$, generated by a set $S$ of simple reflections. The natural place to study such questions is the descent algebra of $W$. 
The descent algebra \( \Sigma(W) \) is a remarkable subalgebra of the group algebra \( \mathbb{Q}[W] \), introduced by Solomon [6] in 1976. It has a linear basis consisting of \( 2^{|S|} \) elements \( x_J, J \subseteq S \), and its radical coincides with the kernel of the homomorphism from \( \Sigma(W) \) into the character ring of \( W \) which maps \( x_J \) to the permutation character \( \pi_J \) of the action of \( W \) on the cosets of the subgroup \( W_J \) generated by \( J \subseteq S \).

In this setup, the search for maximal chains of (conjugacy classes of) subgroups which are closed under intersection leads us to consider subalgebras \( \mathbb{Q}[x_M] \) of \( \Sigma(W) \), where \( M \) is a maximal subset of \( S \). We have \( \dim \mathbb{Q}[x_M] = \# \{ \pi_M(w) : w \in W \} \), since the minimal polynomial of \( x_J, J \subseteq S \), is \( \prod_{a \in \{ \pi_J(w) : w \in W \}} (x - a) \in \mathbb{Q}[x] \), by a theorem of Bonnafé–Pfeiffer [2]. We classify all cases where \( \mathbb{Q}[x_M] \) is spanned as \( \mathbb{Q} \)-vector space by a “native” basis, i.e., a subset of the basis \( \{ x_J : J \subseteq S \} \). It turns out that this is almost exclusively the case when \( W \) is of classical type \( A_n, B_n \) or \( D_n \) and, as in the above example of the Coxeter group of type \( B_n \), the subset \( M \) is “left connected” in \( S \), in a sense that we will make precise shortly. We also show that, if \( M \) is left connected in \( S \), the native basis elements \( x_J \) of \( \mathbb{Q}[x_M] \) are integer polynomials in \( x_M \). For each of the three classical types of Coxeter groups, we provide explicit formulas for the structure constants of the algebra \( \mathbb{Q}[x_M] \) relative to its native basis.

We proceed as follows. In Sect. 2, we set up notation for a finite Coxeter group \( W \) and its descent algebra \( \Sigma(W) \), and we review some useful properties of Coxeter groups of classical type. In Sect. 3, we define \( W_M \) in \( W \) as maximally left connected, if \( W_M \) is of type \( X_{n-1} \) in \( W \) of type \( X_n \), where \( X \) is any of the classical types \( A, B \) or \( D \), subject to some natural constraints on \( n \). The choice of name is derived from the position of \( M \) in the Coxeter diagram of \( (W,S) \). We show that in these maximally left connected cases, the algebra \( \mathbb{Q}[x_M] \) has a native basis. In Sect. 4, we consider the converse statement and argue that in almost all cases where \( M \) is maximal, but not left connected in \( W \), the algebra \( \mathbb{Q}[x_M] \) does not have a native basis. The only exceptions are dihedral types and that of \( B_1 \times A_1 \) in \( B_3 \). We can now state our main result.

**Theorem 1.1.** Let \( W \) be an irreducible finite Coxeter group with Coxeter generators \( S \). Let \( M \subseteq S \) be a maximal subset. Then, the subalgebra \( \mathbb{Q}[x_M] \) of the descent algebra admits a native basis if and only if

(i) \( W \) is of classical type \( A_n, B_n \) or \( D_n \), and \( M \) is maximally left connected in \( S \);

(ii) \( W \) is of dihedral type \( I_2(m) \) with \( m \geq 5 \); or

(iii) \( W \) is of type \( B_3 \) and \( W_M \) is of type \( B_1 \times A_1 \).

Furthermore, the native basis consists of integer polynomials in \( x_M \) exclusively for (i).
Let \((W, S)\) be a finite Coxeter system, and let \(\ell(w)\) denote the minimal length of \(w \in W\) as a word in \(S\). For each subset \(J \subseteq S\), let \(W_J\) be the subgroup of \(W\) generated by \(J\). The set
\[
X_J = \{w \in W : \ell(ws) > \ell(w) \text{ for all } s \in J\}
\]
is a distinguished transversal of the left cosets \(wW_J\) in \(W\) consisting of representatives of minimal length in their coset. For \(K \subseteq S\), the set \(X_K^{-1} = \{w^{-1} : w \in X_K\}\) is a distinguished transversal of the right cosets \(W_Kw\) in \(W\), and the set \(X_{JK} = X_J^{-1} \cap X_K\) is a distinguished transversal of the double cosets \(W_JwW_K\) in \(W\).

For \(J \subseteq S\), define \(x_J = \sum_{w \in X_J} w\). The subspace \(\Sigma(W) = \sum_{J \subseteq S} \Q X_J\) of the group algebra \(\Q W\) is a ring with identity \(x_S = 1\), and thus a subalgebra of \(\Q W\), called the descent algebra of \(W\). The elements \(x_J, J \subseteq S\), are linearly independent, and thanks to Solomon [6]
\[
x_J x_K = \sum_{d \in X_{JK}} x_{d \cap K} \sum_{L \subseteq S} a_{JKL} x_L,
\]
where \(a_{JKL} = |\{d \in X_{JK} \mid J \cap K = L\}|\), for \(J, K, L \subseteq S\).

For \(J \subseteq K \subseteq S\), we denote \(X_J(\bar{K}) = X_J \cap W_K\). Then, \(X_J(\bar{K})\) is a distinguished left transversal of \(W_J\) in \(W_K\), and \(X_J = X_K X_J(\bar{K}) = \{vw : v \in X_K, w \in X_J(\bar{K})\}\), by [3, 2.1.5]. Accordingly, we have
\[
x_J = x_K x_J(\bar{K}),
\]
where \(x_J(\bar{K}) = \sum_{w \in X_J(\bar{K})} w\).

### 2.1. Left Connected Subgroups

In the following section, we are mostly concerned with Coxeter groups of type \(A_n, B_n\) and \(D_n\), whose Coxeter graphs of the respective types are given in Fig. 1.

The types \(A_n, B_n\) and \(D_n\) are in most parts well understood. For example, there exists a well-known faithful permutation representation for these types, which in detail is presented in [1, 1.5; 8.1; 8.2]:

![Coxeter graphs](image-url)
Remark 2.1. (i) For type $A_n$, there exists an isomorphism between $W$ and the symmetric group $S_{n+1}$ of permutations of the set $\{1, 2, \ldots, n+1\}$ which sends the basis element $s_1$ to the involution $(i, i+1)$ interchanging $i$ and $i+1$. This gives rise to an action of $W$ on the set $\{1, 2, \ldots, n+1\}$.

(ii) Denote by $S^B_n$ the group of signed permutations on the set $\{-n, -n + 1, \ldots, n - 1, n\}$, which consists of the permutations $\pi$ satisfying $\pi(-i) = -\pi(i)$ for $0 \leq i \leq n$. For type $B_n$, there exists an isomorphism between $W$ and $S^B_n$ sending $s_i$ to the involution $(-i, -i + 1)(i - 1, i)$ for $2 \leq i \leq n$, and $s_1$ to $(-1, 1)$. This gives rise to an action of $W$ on the set $\{-n, -n + 1, \ldots, n-1, n\}$.

(iii) Denote by $S^D_n$ the subgroup of $S^B_n$ consisting of all signed permutations $\pi$ which map an even number of positive integers onto negative integers. For type $D_n$, there exists an isomorphism between $W$ and $S^D_n$ which sends $s_i$ to the involution $(-i, -i + 1)(i - 1, i)$ for $2 \leq i \leq n$, and $s_1$ to $(-2, 1)(-1, 2)$.

Let $W$ be of type $A_n$ ($n \geq 1$), $B_n$ ($n \geq 2$) or $D_n$ ($n \geq 3$), with $S = \{s_1, s_2, \ldots, s_n\}$ as in the diagrams in Fig. 1. For $J = \{s_1, s_2, \ldots, s_j\}$ and $K = \{s_1, s_2, \ldots, s_k\}$, $0 \leq j, k \leq n$, we set $W_J = W_{\emptyset}$, $X_j = X_{\emptyset}$ and $X_{jk} = X_{\emptyset}$. Moreover, in $\mathbb{Q}W$, we set $x_j = x_{\emptyset}$ and $x_{j(k)}^{(K)} = x_j^{(K)}$. Thus, (2.2) becomes $x_j = x_{j(k)}^{(K)} x_k$, for $0 \leq j \leq k \leq n$. For type $D_n$, we exclude the occurrence of $J = \{s_1\}$ or $K = \{s_1\}$ by instead putting $W_1 := W_\emptyset$, $X_1 := X_\emptyset$ and so forth. In Sect. 3.4, we furthermore put $x_0 := 2x_1$ for type $D_n$, for a simplified notation.

We say that a subgroup $U$ of $W$ is left connected in $W$, if $U = W_j$ for some $0 \leq j \leq n$, where for type $D_n$, we require $j \geq 1$. This definition generalizes the notion of left connected subsets of $S$ for $W$ of type $B_n$, introduced by Gerber, Hiß, Jacon [4], to all classical types of finite Coxeter groups. We will show in Corollary 3.5, 3.8 and 3.12 for each of the types $A_n$, $B_n$ and $D_n$ that

$$x_j x_k = \sum_{l=0}^{n} a_{jkl} x_l$$

for nonnegative integer coefficients $a_{jkl}$, which generalizes [4, 2.2 Lemma] for type $A_n$ and $D_n$.

For completeness, we say that a subgroup $U$ of $W$ is not left connected in $W$, if the Coxeter group $W$ is not of type $A_n$, $B_n$ or $D_n$.

3. Native Bases in Classical Types

In this section, we define a particular class of subalgebras of $\Sigma(W)$ for any finite Coxeter system. By applying some general properties of the descent algebra to study this algebra for the maximal left connected subgroup in type $A_n$, $B_n$ and $D_n$, we will see that for this particular case, the algebra admits a basis which is a subset of the standard basis of $\Sigma(W)$ and corresponds to a set of falling factorials.
Definition 3.1. Let \((W, S)\) be a finite Coxeter system and \(E \subseteq \Sigma(W)\) be a subalgebra of the descent algebra of \(W\). We call a basis of \(E\) native if it is a subset of the standard basis \(\{x_J | J \subseteq S\}\) of \(\Sigma(W)\).

A trivial example for native bases is the standard basis \(\{x_J | J \subseteq S\}\) of the descent algebra itself, which is a native basis of \(\Sigma(W)\). In the following, we will explore nontrivial examples of subalgebras which admit native bases.

Definition 3.2. Let \((W, S)\) be a finite Coxeter system and let \(J \subseteq S\). Define \(E(W, x_J) := Q[x_J]\), the smallest subalgebra of the descent algebra \(\Sigma(W)\), and the group ring \(QW\), containing the element \(x_J\). For type \(A_n, B_n\) and \(D_n\), put \(E(W) := E(W, x_{n-1})\).

It is clear by its definition that \(E(W)\) is a commutative algebra isomorphic to a quotient of \(Q[x]\) for all three types via the insertion homomorphism. We now explore the algebras \(E(W)\) separately for those types. We will see that \(E(W)\) admits a native basis and determine its structural constants with respect to this basis. But first, we review some properties of falling powers.

3.1. Falling Factorial Powers

Following [5], for \(k \geq 0\), the \(k\)th falling power of \(x\) is

\[
x^k := \prod_{i=0}^{k-1} (x - i) = \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} x^m,
\]

where \(\binom{k}{m}\) is a Stirling number of the first kind, the number of ways to arrange \(k\) elements in \(m\) cycles. Conversely, for \(k \geq 0\)

\[
x^k = \sum_{m=0}^{k} \left\{ \binom{k}{m} \right\} x^m,
\]

where \(\left\{ \binom{k}{m} \right\}\) is a Stirling number of the second kind, the number of ways to partition an \(k\)-element set into \(m\) nonempty subsets. An application of the Principle of Inclusion and Exclusion yields

\[
\sum_{a+b=c} \binom{j}{a} \binom{k}{b} = \sum_{l} (-1)^l \binom{j}{l} \binom{k}{l} l! \binom{j+k-l}{c},
\]

where the coefficient \(\binom{j}{l}l!\) counts the number of partial bijections between \(l\) elements of a set of size \(j\) and a set of size \(k\). From this, it is easy to see that

\[
x^j x^k = \sum_{l=0}^{\min(j,k)} \binom{j}{l} \binom{k}{l} l! x^{l+k-1},
\]

by expanding the falling powers \(x^j = \sum_a (-1)^{j-a} \binom{j}{a} x^a\) and \(x^k = \sum_b (-1)^{k-b} \binom{k}{b} x^b\), and comparing the coefficients of \(x^c\) for \(c = a + b\).
3.2. Type A
In the following, consider \((W, S)\) a Coxeter system of type \(A_n\).

**Lemma 3.3.** We have
\[
x_{n-1} x_{n-k} = k x_{n-k} + x_{n-k-1}, \quad \text{for } 0 \leq k \leq n,
\]  
(3.5)
where we set \(x_{-1} := x_0\). Therefore, in particular, \(x_{n-1} x_0 = n x_0 + x_{-1} = (n + 1) x_0\).

**Proof.** We show that Eq. (3.5) holds for \(k < n\), by induction on \(k\). For \(k = 0\), we have \(x_{n-k} = x_n = 1\), and thus, \(x_{n-1} x_{n-0} = x_{n-1} + 0 x_{n-0}\). For \(k = 1\), we have only two double cosets of \(W_{n-1}\) in \(W\) with distinguished coset representatives 1 and \(s_n\). Thus
\[
x_{n-1}^2 = x_{n-2} + x_{n-1}
\]  
(3.6)
by Eq. (2.1), with an exception for \(n = 1\) where \(x_1^2 = 2x_0\) holds. Thus, (3.5) holds for \(k \leq 1\), which also proves the case \(n = 2\). Suppose \(n > 2\). By Eq. (2.2), we have
\[
x^{(n)}_{j} = x^{(n)}_{k} x^{(k)}_{j},
\]
for all \(0 \leq j \leq k \leq n\). The cases \(k \in \{0, 1\}\) have already been dealt with, and thus, let \(k \geq 2\) and (3.5) be true for \(k - 1\). Then
\[
x^{(n)}_{n-1} x^{(n)}_{n-k} (2.2) = x^{(n)}_{n-1} x^{(n)}_{n-k+1} x^{(n-k+1)}_{n-k}
\]
\[
\text{Ind.} \quad ((k - 1)x^{(n)}_{n-k+1} + x^{(n)}_{n-k}) x^{(n-k+1)}_{n-k}
\]
\[
\begin{align*}
(2.2) & (k - 1)x^{(n)}_{n-k} + x^{(n)}_{n-k+1} x^{(n-k+1)}_{n-k} x^{(n-k+1)}_{n-k} \\
(3.6) & (k - 1)x^{(n)}_{n-k} + x^{(n)}_{n-k+1} (x^{(n-k+1)}_{n-k} + x^{(n-k-1)}_{n-k-1})
\end{align*}
\]
(3.7)
Thus, by induction, (3.5) holds for all \(0 \leq k \leq n\), for \(n \in \mathbb{N}_+\). Multiplying \(x_{n-1}\) with \(x_0\) yields
\[
x^{(n)}_{n-1} x^{(n)}_0 = (n + 1)x_0^{(n)},
\]  
(3.8)
since \(x_0\) is the sum over all elements in \(W\) and \(x_{n-1}\) consists of \(n + 1\) summands, as its number of summands is equal to the index of \(W_{n-1}\) in \(W\).

As an immediate consequence of Lemma 3.3
\[
x_{n-k} = \prod_{i=0}^{k-1} (x_{n-1} - i) = x^k_{n-1}, \quad \text{for } 0 \leq k \leq n,
\]  
(3.9)
and
\[
(x_{n-1} - n - 1) x_0 = x_0 (x_{n-1} - n) - x_0 = 0.
\]
It follows that \(x^{n+1}_{n-1} = x^n_{n-1}\) and that \(x^{n+t}_{n-1} = 0\), for \(t > 1\). This shows the following.
Proposition 3.4. As \( \mathbb{Q}\)-algebras, \( \mathfrak{E}(W) = \mathbb{Q}[x_{n-1}] \) is isomorphic to \( \mathbb{Q}[x]/(x^{n+1} - x^n) \). Its \( \mathbb{Q}\)-bases \( \{x_0, x_1, \ldots, x_n\} \) and \( \{x^0_{n-1}, \ldots, x^n_{n-1}\} \) are related by the base change formulas

\[
x_{n-k} = \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} x_{n-m}^m, \quad \text{and} \quad x_{n-1}^k = \sum_{m=0}^{k} \binom{k}{m} x_{n-m}^m, \quad (3.10)
\]

for \( 0 \leq k \leq n \).

This confirms that the basis elements \( x_j \) of \( \mathfrak{E}(W) \) are integral polynomials in \( x_{n-1}, 0 \leq j \leq n \). Moreover, the identification with falling powers yields a multiplication formula in terms of explicit structure constants of \( \mathfrak{E}(W) \) with respect to the basis \( \{x_0, x_1, \ldots, x_n\} \).

Corollary 3.5. We have

\[
x_j x_k = \sum_{l=-1}^{\min(j,k)} \binom{n-j}{k-l} \binom{n-k}{j-l} (n-j-k+l)! x_l, \quad (3.11)
\]

for \( 0 \leq j, k \leq n \).

Proof. From (3.4), we have

\[
x_{n-j} x_{n-k} = \sum_{s=0}^{\min(n-j,n-k)} \binom{n-j}{n-j-s} \binom{n-k}{n-k-s} s! x_{n-j+n-k-s}^{n-j+n-k-s}
\]

And, as \( x_{n-j}^{-1} = 0 \) unless \( n-j \leq n+1 \), it follows that:

\[
x_j x_k = x_{n-j}^{n-j} x_{n-k}^{n-k} = \sum_{l=-1}^{\min(j,k)} \binom{n-j}{k-l} \binom{n-k}{j-l} (n+l-j-k)! x_l,
\]

as desired. \( \Box \)

Alternatively, we can prove Corollary 3.5 using formula (2.1) by counting for each \( l \in \{1, 2, \ldots, n\} \) the coset representatives \( x \in X_{jk} \) satisfying

\[
\{1, 2, \ldots, j\}^x \cap \{1, 2, \ldots, k\} = \{1, 2, \ldots, l\}. \quad (3.12)
\]

By [1, 1.5.3], we have \( \ell(xs_t) > \ell(x) \), if and only if \( x(i) < x(i+1) \), which implies

\[
X_j = \{x \in W | x(i) < x(i+1), 1 \leq i \leq j\}.
\]

Therefore

(i) \[ x(1) < x(2) < \cdots < x(k+1), \]

(ii) \[ x^{-1}(1) < x^{-1}(2) < \cdots < x^{-1}(j+1), \]

respectively.
for each \( x \in X_{jk} \). Thus, for a fixed \( l \), any coset representative satisfying Eq. (3.12) stabilizes the elements \( 1, 2, \ldots, l + 1 \) point-wise while not stabilizing \( l + 2 \). The coefficient

\[
\frac{n - j}{n - j - k + 1} \frac{n - k}{n - j - k + l} (n - j - k + l)!
\]

can then be read as the number of all bijections between the \( n - j - k + l \) elements in \( \{k + 2, \ldots, n + 1\} \) whose image is not determined by (ii), and the \( n - j - k + l \) elements in \( \{j + 2, \ldots, n + 1\} \) in the image whose pre-image is not determined by (i).

### 3.3. Type B

In the following consider \((W, S)\) a Coxeter system of type \( B_n \). The set of distinguished double coset representatives of \( W_{n-1} \) in \( W \) is

\[
X_{n-1,n-1} = \{1, s_n, s_n s_{n-1} \cdots s_2 s_1 s \cdots s_n\};
\]

see, e.g., [3, Example 2.2.5]. Thus, \( x_{n-1}^2 = 2x_{n-1} + x_{n-2} \). Analogously to Lemma 3.3, induction over \( k \) yields:

**Lemma 3.6.** We have

\[
x_{n-1} x_{n-k} = 2k x_{n-k} + x_{n-k-1},
\]

for \( 0 \leq k \leq n \), where we put \( x_{-1} := 0 \). Therefore, in particular, \( x_{n-1} x_0 = 2nx_0 + x_{-1} = 2nx_0 \). \( \square \)

As an immediate consequence of Lemma 3.6

\[
x_{n-k} = \prod_{i=0}^{k-1} (x_{n-1} - 2i) = 2^k \left( \frac{x_{n-1}}{2} \right)^k, \text{ for } 0 \leq k \leq n,
\]

and

\[
(x_{n-1} - 2n)x_0 = 0.
\]

This shows the following.

**Proposition 3.7.** As \( \mathbb{Q} \)-algebras \( \mathcal{E}(W) = \mathbb{Q}[x_{n-1}] / (x_{n+1}) \), where \( x_{n-1} \) is sent to the coset of \( 2x_1 \). Its \( \mathbb{Q} \)-bases \( \{x_0, x_1, \ldots, x_n\} \) and \( \{x_0^0, \ldots, x_n^0\} \) are related by the base change formulas

\[
x_{n-k} = \sum_{m=0}^{k} (-2)^{k-m} \binom{k}{m} x_{n-1}^m, \quad \text{and} \quad x_{n-1}^k = \sum_{m=0}^{k} \binom{k}{m} 2^{k-m} x_{n-m},
\]

for \( 0 \leq k \leq n \).
Proof. The minimal polynomial of $x_{n-1}$ arises from Eqs. (3.14) and (3.15) as
\[ \mu = \prod_{i=0}^{n}(x - 2i). \]
Furthermore, the natural isomorphism $\mathcal{E}(W) \cong \mathbb{Q}[x]/(\mu)$
 sends $x_{n-k}$ to the coset of $\prod_{i=0}^{k-1}(x - 2i)$. Sending a polynomial $f(x) \in \mathbb{Q}[x]$ to the coset $f(2x) + (x^{n+1})$ yields an algebra epimorphism from $\mathbb{Q}[x]$ to $\mathbb{Q}[x]/(x^{n+1})$. Since $\mu(2x) = 2^{n+1}x^{n+1}$ and $x^{n+1}$ generate the same ideal of $\mathbb{Q}[x]$ the kernel of this epimorphism is exactly $(\mu)$, thus it induces an isomorphism between $\mathcal{E}(W)$ and $\mathbb{Q}[x]/(x^{n+1})$. Under this isomorphism, $x_{n-k}$ is sent to the coset of $2^k x^k$. The change of basis formulas of the native basis to the
standard basis arise from Eqs. (3.1) and (3.2) under this isomorphism.

This confirms that the basis elements $x_j$ of $\mathcal{E}(W)$ are integral polynomials
in $x_{n-1}$, $0 \leq j \leq n$. Moreover, the identification with multiples of falling powers
yields a multiplication formula in terms of explicit structure constants of $\mathcal{E}(W)$
with respect to the basis \{ $x_0, x_1, \ldots, x_n$ \}. The proof proceeds analogously to
Corollary 3.5 with the difference that the coset $x_{n+1}$ already equals 0, thus
eliminating the second term for $x_0$. This yields:

**Corollary 3.8.** We have

\[ x_j x_k = \sum_{l=0}^{\min(j,k)} \binom{n-j}{k-l} \binom{n-k}{j-l} (n-j-k+l)! 2^{n-j-k+l} x_l, \quad (3.17) \]

for $0 \leq j, k \leq n$.

A similar interpretation of the coefficient of $x_i$ as for type $A_n$ exists.

Here, the action of $W$ on \{-$n$, $-n+1$, $\ldots$, $n$\} yields $\ell(ws_i) > \ell(w)$, if and only
if $w(i-1) < w(i)$, for all $w \in W$ and $1 \leq i \leq n$, by [1, 8.1.2]. The factor
$2^{n-j-k+i}$ arises, since for the bijection between the two sets with $n - j - k + i$
elements, we can also choose the sign of each element freely.

### 3.4. Type $D$

In the following, consider $(W,S)$ a Coxeter system of type $D_n$. The set of
distinguished double coset representatives of $W_{n-1}$ in $W$ is

\[ X_{n-1,n-1} = \{1, s_n, s_n s_{n-1} \cdots s_3 s_1 s_2 \cdots s_n \}; \]

see, e.g., [3, Example 2.2.6]. Thus, $x_{n-1}^2 = 2x_{n-1} + x_{n-2}$. Analogously to
Lemma 3.3, induction over $k$ yields:

**Lemma 3.9.** We have

\[ x_{n-1} x_{n-k} = 2k x_{n-k} + x_{n-k-1} \quad \text{for} \quad k \in \{0, 1, \ldots, n-1\}, \quad (3.18) \]

where we set $x_0 := 2x_1$. Therefore, in particular, $x_{n-1} x_1 = 2nx_1$.

As an immediate consequence of Lemma 3.9

\[ x_{n-k} = \prod_{i=0}^{k-1}(x_{n-1} - 2i) = 2^k \left( \frac{x_{n-1}}{2} \right)^k, \quad \text{for} \quad 0 \leq k \leq n-1, \quad (3.19) \]

and

\[ (x_{n-1} - 2n)x_1 = 0. \quad (3.20) \]

This shows the following.
Proposition 3.10. As \( \mathbb{Q}\)-algebras \( \mathcal{E}(W) = \mathbb{Q}[x_{n-1}] \) is isomorphic to \( \mathbb{Q}[x]/(x^n - x^{n-1}) \), where \( x_{n-1} \) is sent to the coset of \( 2x^1 \). Its \( \mathbb{Q}\)-bases \( \{x_1, \ldots, x_n\} \) and \( \{x_{n-1}^0, \ldots, x_{n-1}^{n-1}\} \) are related by the base change formulas

\[
\sum_{m} (-2)^{k-m} \binom{k}{m} x_{n-1}^m, \quad \text{and} \quad x_{n-1}^k = \sum_{m} \binom{k}{m} 2^{k-m} x_{n-m},
\]

for \( 0 \leq k \leq n - 1 \).

Proof. The minimal polynomial of \( x_{n-1} \) arises from Eqs. (3.19) and (3.20) as

\[
\mu = (x - 2n) \prod_{i=0}^{n-2} (x - 2i).
\]

Furthermore, the natural isomorphism \( \mathcal{E}(W) \cong \mathbb{Q}[x]/(\mu) \) sends \( x_{n-1} \) to the coset of \( \prod_{i=0}^{n-1} (x - 2i) \). Sending a polynomial \( f(x) \in \mathbb{Q}[x] \) to the coset \( f(2x) + (x^n - x^{n-1}) \) yields an algebra epimorphism from \( \mathbb{Q}[x] \) to \( \mathbb{Q}[x]/(x^n - x^{n-1}) \). Since \( \mu(2x) = 2^n (x^n - x^{n-1}) \) and \( x^n - x^{n-1} \) generate the same ideal in \( \mathbb{Q}[x] \), the kernel of this epimorphism is exactly \( (\mu) \); thus, it induces an isomorphism between \( \mathcal{E}(W) \) and \( \mathbb{Q}[x]/(x^n - x^{n-1}) \). Under this isomorphism \( x_{n-1} \) is sent to the coset of \( 2^n x^k \). The change of basis formulas of the native basis to the standard basis arise from Eqs. (3.1) and (3.2) under this isomorphism. \( \square \)

This confirms that the basis elements \( x_j \) of \( \mathcal{E}(W) \) are integral polynomials in \( x_{n-1} \), \( 1 \leq j \leq n \). Comparing Proposition 3.10 with Proposition 3.4, we see that the ideals factoring \( \mathbb{Q}[x] \) for type \( \mathbb{D}_n \) and type \( \mathbb{A}_{n-1} \) are identical. This gives us a surprising connection between \( \mathcal{E}(W) \) for type \( \mathbb{D}_n \) and \( \mathcal{E}(W) \) for type \( \mathbb{A}_{n-1} \).

Corollary 3.11. Let \( (W', S') \) be a Coxeter system of type \( \mathbb{A}_{n-1} \) for \( n \geq 3 \). To distinguish between elements in \( \mathcal{E}(W) \) and \( \mathcal{E}(W') \) denote by \( x_j^D \) the element \( x_i \in \mathcal{E}(W) \) and by \( x_i^A \) the element \( x_i \in \mathcal{E}(W') \). There exists an \( \mathbb{Q}\)-algebra isomorphism

\[
\varphi : \mathcal{E}(W) \rightarrow \mathcal{E}(W'),
\]

with

\[
\varphi(x_j^D) = 2^{n-j} x_j^A,
\]

for \( 1 \leq j \leq n \). \( \square \)

We can solve multiplication in \( \mathcal{E}(W) \) by either using the multiplication formula (3.4) for falling factorials and the isomorphism onto \( \mathbb{Q}[x]/(x^n - x^{n-1}) \) from Proposition 3.10, or using the isomorphism between \( \mathcal{E}(W) \) and \( \mathcal{E}(W') \) in Corollary 3.11 and the multiplication formula for type \( \mathbb{A}_{n-1} \) in Corollary 3.5. Both methods yield the following.

Corollary 3.12. We have

\[
x_j x_k = \sum_{l=0}^{\min(j,k)} \binom{n-j}{k-l} \binom{n-k}{j-l} (n-j-k+l)! 2^{n-j-k+l} x_l,
\]

for \( 1 \leq j, k \leq n \), where as a reminder \( x_0 = 2x_1 \). Therefore, in particular, \( x_j x_1 = \frac{n!}{j!} 2^{n-j} x_1 \). \( \square \)
To conclude our study of $\mathcal{E}(W)$, we summarize the general results of Propositions 3.4, 3.7 and 3.10.

**Theorem 3.13.** Let $(W,S)$ be a Coxeter system of type $A_n$, $B_n$ or $D_n$. Then, $\mathcal{E}(W)$ admits a native basis, consisting of all basis elements of $\Sigma(W)$ corresponding to left connected subgroups of $W$. Moreover, all elements of the native basis are integral polynomials in $x_{n-1}$.

We will see in the next section that native bases for $\mathcal{E}(W,x_M)$ are not exclusive for the left connected case. However, demanding that the elements of the native basis are integral polynomials in $x_M$ only holds for $\mathcal{E}(W)$.

### 4. No Native Bases

The following criterion explains why, in most cases of $M$ maximal in $S$, $\mathbb{Q}[x_M]$ does not have a native basis.

**Lemma 4.1.** Let $s \in S$ and set $M := S \setminus \{s\}$ and $K := M^s \cap M$.

(i) If $x_Mx_K \neq x_Kx_M$, then $\mathbb{Q}[x_M]$ does not have a native basis.

(ii) If $t^v \in K$ for some $t \in M \setminus K$ and $v \in X_{MK}$, then $x_Mx_K \neq x_Kx_M$.

**Proof.** (i) Assume, for a contradiction, that $\mathbb{Q}[x_M]$ has a native basis. This basis contains $x_M$ and $x_K$, as both $M,K \in \text{supp}(x_M^2)$ (since by (2.1), $x_M^2 = \sum_{x \in X_{MM}} x_M x \cap M$, and clearly, $1,s \in X_{MM}$). However, $x_Mx_K \neq x_Kx_M$ is absurd, since $\mathbb{Q}[x_M]$ is commutative. Therefore, $\mathbb{Q}[x_M]$ does not have a native basis.

(ii) Set $u = v^{-1} \in X_{KM}$. Then, $t \in K^u \cap M$, but $t \notin K$. Hence, $K^u \cap M \notin \text{supp}(x_Mx_K)$ as clearly $L \subseteq K$ for all $L \in \text{supp}(x_Mx_K)$. However, $K^u \cap M \in \text{supp}(x_Kx_M)$ as $u \in X_{KM}$. Therefore, $\text{supp}(x_Mx_K) \neq \text{supp}(x_Kx_M)$, whence $x_Mx_K \neq x_Kx_M$. \hfill $\square$

We list some explicit cases in Table 1. Here, for $J \subseteq K \subseteq S$, we write $d_J^K$ for the longest right coset representative of $W_J$ in $W_K$. With this notation, in each case, it is straightforward to verify that the listed element $v$ satisfies the conditions $v \in X_{MK}$ and $t^v \in K$. Here, and in the remainder of this section, the labelling of the elements of $S$ follows the conventions used, e.g., in [3, 1.3].

In fact, Table 1 almost suffices to classify all native bases, as follows.

**Proposition 4.2.** Let $W$ be an irreducible finite Coxeter group with simple reflections $S$. Let $s \in S$ and $M := S \setminus \{s\}$ be such that $M$ is not left connected in $S$. Then, the subalgebra $\mathbb{Q}[x_M]$ of $\Sigma(W)$ has a native basis if and only if

(i) $W$ is of type $I_2(m)$ with $m \geq 5$; or

(ii) $W$ is of type $B_3$ and $s = s_2$.

**Proof.** Note that if $L \subseteq S$ is such that $s \in L$ and the subgroup $W_L$ of $W$ together with the reflection $s$ occurs as one of the cases in Table 1, then, with the same notation $M = S \setminus \{s\}$ and $K = M^s \cap M$, it still follows that $x_Mx_K \neq x_Kx_M$. This is because one can choose the same elements $t \in M \setminus K$
and \( v \in X_{MK} \) as in the table, so that \( t^v \in K \). Hence, in these cases, \( \mathbb{Q}[x_M] \) does not have a native basis by Lemma 4.1.

A careful inspection of the classification of finite irreducible Coxeter groups shows that, apart from rank 2 and the left connected maximal subsets \( M \subseteq S \), this leaves only three cases to consider: \( W \) of type \( B_3 \) with \( s = s_2 \), \( W \) of type \( H_3 \) with \( s = s_2 \) and \( W \) of type \( F_4 \) with \( s = s_1 \). Case (i), where \( W \) is of rank 2, is described in detail in Example 4.3. Case (ii), where \( W \) is of type \( B_3 \) and \( s = s_2 \) is described in detail in Example 4.4.

In both remaining cases, it can be explicitly verified that \( x_M x_K \neq x_K x_M \) for some \( K \in \text{supp}(x_M^2) \):

- \( W = H_3 \), \( s = s_2 \): \( M = \{ s_1, s_3 \} \), \( u = d_{12}^1 d_{23}^3 \), \( M^u \cap M = K = \{ s_3 \} \). Then, \( K^v \cap M = \{ s_1 \} \not\subseteq K \) for \( v = s_2 s_1 s_3 s_2 s_1 s_2 \).
- \( W = F_4 \), \( s = s_1 \): \( M = \{ s_2, s_3, s_4 \} \), \( u = d_{23}^{12} \), \( M^u \cap M = K = \{ s_2, s_3 \} \). Then, \( K^v \cap M = \{ s_4 \} \not\subseteq K \) for \( v = s_1 s_2 s_3 s_4 s_3 s_2 s_1 \).

This concludes the proof of Proposition 4.2. \( \square \)

**Example 4.3.** If \( W \) is of type I_2(m), \( m = 2k + l \) (for \( k \geq 1 \), \( l = 1, 2 \)), with \( s = s_1 \) and \( M = \{ s_2 \} \), we have

\[
\begin{align*}
x_M^0 &= x_S \\
x_M^1 &= x_M \\
x_M^2 &= \lambda x_M + k x_0 \\
x_S^0 &= x_M^0 \\
x_M^1 &= x_M \\
x_0^2 &= -\frac{1}{k} x_M^2 + \frac{1}{k} x_M^2.
\end{align*}
\]

Thus, \( \mathbb{Q}[x_M] \) has a native basis, but, unless \( k = 1 \), not all of its elements are integer polynomials in \( x_M \). If \( k = 1 \), then \( W \) is of type \( A_2 \) (\( l = 1 \)) or \( B_2 \) (\( l = 2 \)) and \( M \) is left connected in \( S \).
Example 4.4. If $W$ is of type $B_3$, with $s = s_2$, $M = \{s_1, s_3\}$, and $K = \{s_1\}$, we have

\[
\begin{align*}
 x_M^0 &= x_S \\
 x_M^1 &= x_M \\
 x_M^2 &= 2x_M + x_K + 2x_\emptyset \\
 x_M^3 &= 4x_M + 6x_K + 32x_\emptyset
\end{align*}
\]

Thus, $Q[x_M]$ has a native basis, but not all of its elements are integer polynomials in $x_M$.

Our Main Theorem 1.1 now follows from Theorem 3.13 and Proposition 4.2.

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Declarations

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References

[1] Anders Björner and Francesco Brenti, Combinatorics of Coxeter groups, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR 2133266

[2] C. Bonnafé and G. Pfeiffer, Around Solomon’s descent algebras, Algebr. Represent. Theory 11 (2008), no. 6, 577–602. MR 2453230
[3] Meinolf Geck and Götz Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, London Mathematical Society Monographs. New Series, vol. 21, The Clarendon Press, Oxford University Press, New York, 2000. MR1778802

[4] Thomas Gerber, Gerhard Hiss, and Nicolas Jacon, *Harish-Chandra series in finite unitary groups and crystal graphs*, Int. Math. Res. Not. IMRN (2015), no. 22, 12206–12250. MR 3456719

[5] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete mathematics*, second ed., Addison-Wesley Publishing Company, Reading, MA, 1994, A foundation for computer science. MR1397498

[6] Louis Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra 41 (1976), no. 2, 255–264. MR444756

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