ON THE CANTOR–BENDIXSON RANK OF THE
GRIGORCHUK GROUP AND THE GUPTA–SIDKI 3
GROUP

RACHEL SKIPPER AND PHILLIP WESOLEK

ABSTRACT. We study the Cantor–Bendixson rank of the space of sub-
groups for members of a general class of finitely generated self-replicating
branch groups. In particular, we show for $G$ either the Grigorchuk group
or the Gupta–Sidki 3 group, the Cantor–Bendixson rank of $\text{Sub}(G)$ is $\omega$.
For each natural number $n$, we additionally characterize the subgroups
of rank $n$ and give a description of subgroups in the perfect kernel.

1. Introduction

Given a group $G$, the collection of subgroups, $\text{Sub}(G)$, admits a cano-
cical totally disconnected compact topology, called the Chabauty topology.
A natural invariant of compact topological spaces is the Cantor–Bendixson
rank, which is an ordinal. One naturally wishes to understand the con-
nections between this topological invariant and the subgroup structure of
the group under consideration. In this vein, we here answer the following
question, posed by R. Grigorchuk.

Question 1.1 (Grigorchuk). What is the Cantor–Bendixson rank of $\text{Sub}(\Gamma)$
for $\Gamma$ the Grigorchuk group?

Theorem 1.2 (see Corollary 6.3). For $G$ either the Grigorchuk group or
the Gupta–Sidki 3 group, the Cantor–Bendixson rank of $\text{Sub}(G)$ is $\omega$.

The above theorem follows from an analysis of a general class of finitely
generated self-replicating branch groups and subsequently showing the Grig-
orchuk group and the Gupta–Sidki 3 group are members of this class. This
class is given by two properties.

Definition 1.3. For $X^*$ a regular rooted tree, a group $G \leq \text{Aut}(X^*)$ is
said to have well-approximated subgroups if $\bigcap_{n \geq 0} H\text{St}_G(n) = H$ for
any finitely generated $H \leq G$, where $\text{St}_G(n)$ is the stabilizer of the $n$-th level
of the tree.

Having well-approximated subgroups is exactly the conjunction of two
widely studied properties.

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Lemma 1.4 (See Lemma 2.10). For $G \leq \text{Aut}(X^*)$, $G$ is LERF and has the CSP if and only if $G$ has well-approximated subgroups.

The next property which we call the Grigorchuk–Nagnibeda alternative seems, at the moment, rather exotic, and currently, only the Grigorchuk group and the Gupta–Sidki 3 group are known to enjoy it; see Theorem 6.2. Stating the property requires a weakening of the classical notion of a sub-direct product.

Definition 1.5. For $G$ a group and $Y$ a finite set, we say that $H \leq G^Y$ is an infra-direct product if $\psi_y(H)$ is of finite index in $G$ for all $y \in Y$, where $\psi_y$ is the projection onto the $y$-th coordinate.

Definition 1.6. A group $G \leq \text{Aut}(X^*)$ is said to obey the Grigorchuk–Nagnibeda alternative if for every finitely generated subgroup $H \leq G$ either

(a) there is $v \in X^*$ such that $\psi_v(H)$ is finite, or
(b) there is a spanning subtree $Y \subseteq X^*$ such that $\text{st}_H(Y)$ is an infra-direct product of $G^Y$, where $\text{st}_H(Y)$ is the pointwise stabilizer of $Y$ in $H$.

The Grigorchuk–Nagnibeda alternative is inspired by a characterization of finitely generated subgroups of the Grigorchuk group given by Grigorchuk and Nagnibeda in [1].

Theorem 1.7 (See Theorem 5.1). Suppose $G \leq \text{Aut}(X^*)$ is a finitely generated regular branch group that is just infinite, is strongly self-replicating, has well-approximated subgroups, and obeys the Grigorchuk–Nagnibeda alternative. For $H \leq G$, $H$ is in the perfect kernel of $\text{Sub}(G)$ if and only if either

(1) $\psi_v(H)$ is finite for some $v \in X^*$, or
(2) $H$ is not finitely generated.

The subgroups that do not lie in the perfect kernel have finite Cantor–Bendixson rank, and this topologically defined rank is equal to an algebraically defined rank.

Definition 1.8. For $G$ a group and $L \leq G$, the depth of $L$ in $G$ is the supremum of the natural numbers $n$ for which there is a series of subgroups $G = H_0 > H_1 > \cdots > H_n = L$ such that $|H_i/H_{i+1}| = \infty$ for all $i$. Depth zero subgroups are of finite index. We denote the depth by $\text{depth}(L)$.

Theorem 1.9 (See Corollary 5.3). Suppose $G \leq \text{Aut}(X^*)$ is a finitely generated regular branch group that is just infinite, is strongly self-replicating, has well-approximated subgroups, and obeys the Grigorchuk–Nagnibeda alternative. If $L \leq G$ is not a member of the perfect kernel of $\text{Sub}(G)$, then

(1) $L$ is finitely generated,
(2) $\text{rk}_{CB}(L) < \omega$, and
(3) $\text{rk}_{CB}(L) = \text{depth}(L)$.
Corollary 1.10 (See Corollary 5.4). Suppose \( G \leq \text{Aut}(X^*) \) is a finitely generated regular branch group that is just infinite, is strongly self-replicating, has well-approximated subgroups, and obeys the Grigorchuk–Nagnibeda alternative, then \( \text{Sub}(G) \) has Cantor–Bendixson rank \( \omega \).

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2. Preliminaries

2.1. Polish spaces and the Cantor–Bendixson derivative. In this section, we introduce and define the Cantor–Bendixson rank. For additional discussion, the reader is directed to [7].

Definition 2.1. A topological space \( X \) is called a Polish space if the topology of \( X \) is separable and can be given by a complete metric on \( X \). That is to say, \( X \) is the topological space left over when one takes a complete separable metric space and forgets the metric.

For any topological space \( X \), the derived set \( X' \) is the set of all limit points of \( X \). It is easy to see that \( X' = X \setminus Y \) where \( Y \) is the collection of isolated points. Thereby, \( X' \) is a closed subset of \( X \).

Derived sets allow us to define the \( \beta \)-th Cantor–Bendixson derivative, where \( \beta \) is an ordinal. The \( \beta \)-th Cantor–Bendixson derivative, denoted by \( X^\beta \), is defined by transfinite recursion as follows:

- \( X^0 := X \)
- \( X^{\alpha+1} := (X^\alpha)' \)
- for \( \lambda \) a limit ordinal, \( X^\lambda := \bigcap_{\alpha<\lambda} X^\alpha \).

Definition 2.2. For a Polish space \( X \), the Cantor–Bendixson rank is the least \( \alpha \) such that \( X^\alpha = X^{\alpha+1} \). We denote the Cantor–Bendixson rank by \( \text{rk}_{\text{CB}}(X) \).

Since Polish spaces are Lindelöf, it follows that for any Polish space \( X \), \( \text{rk}_{\text{CB}}(X) \) is a countable ordinal. The \( \text{rk}_{\text{CB}}(X) \)-th derivative \( X^{\text{rk}_{\text{CB}}(X)} \) is a perfect topological space; that is, every point is a limit point. The set \( X^{\text{rk}_{\text{CB}}(X)} \) is called the perfect kernel of \( X \).

This definition of rank easily extends to a rank on the points in a Polish space.

Definition 2.3. For a Polish space \( X \) and \( x \in X \), the Cantor–Bendixson rank of \( x \) is the ordinal \( \alpha \) such that \( x \in X^\alpha \setminus X^{\alpha+1} \). If no such \( \alpha \) exists, we say \( x \) has infinite rank. We denote the Cantor–Bendixson rank of \( x \in X \) by \( \text{rk}_{\text{CB}}(x) \).
2.2. The Chabauty space. Given a countable set $N$, the power set $\mathcal{P}(N)$ admits a compact Polish topology by identifying each element $X \in \mathcal{P}(N)$ with the indicator function for $X$ in $\{0,1\}^N$. Given a countable group $G$, the powerset $\mathcal{P}(G)$ is then a compact Polish space. It is an easy exercise to see that

$$\text{Sub}(G) := \{H \in \mathcal{P}(G) \mid H \leq G\}$$

is a closed subset of $\mathcal{P}(G)$, hence $\text{Sub}(G)$ is a compact Polish space. The space $\text{Sub}(G)$ with this topology is called the Chabauty space of $G$.

The topology of the Chabauty space $\text{Sub}(G)$ has a clopen basis consisting of sets of the form

$$O_{A,C} := \{H \in \text{Sub}(G) \mid \forall a \in A \ a \notin H \text{ and } \forall c \in C \ c \in H\}$$

where $A$ and $C$ range over finite subsets of $G$.

For $H \leq G$, the Cantor–Bendixson rank of the subgroup $H$ is defined to be $\text{rk}_{\text{CB}}(H)$ where $H$ is considered as an element of $\text{Sub}(G)$.

Let us make a few easy observations, which follow from the existence of the aforementioned basis. These observations will later be used implicitly.

**Observation 2.4.** For $G$ a group and a finitely generated subgroup $H \leq G$ with generating set $S$, $O_{\emptyset,S}$ is a neighborhood of $H$, so if $(H_i)_{i \in \mathbb{N}}$ is a sequence converging to $H$, then $H \leq H_i$ for all $i$ sufficiently large.

**Observation 2.5.** If $G$ is a finitely generated group, then every finite index subgroup has Cantor–Bendixson rank zero.

**Lemma 2.6.** Let $G$ be a countable group and $(H_i)_{i \in \mathbb{N}}$ with $H_i \in \text{Sub}(G)$ a sequence converging to a subgroup $H$. If $L \in \text{Sub}(G)$, then $H_i \cap L$ converges to $H \cap L$.

**Proof.** Fix $A$ and $C$ finite subsets of $G$ such that $O_{A,C}$ is a neighborhood of $H \cap L$. It suffices to consider the case that $A = \{a\}$ and $C = \{c\}$.

For all sufficiently large $i$, we infer that $c \in H_i \cap L$, since $H_i$ converges to $H$. If $a \notin H$, then that $H_i \rightarrow H$ ensures that $a \notin H_i \cap L$ for sufficiently large $i$. If $a \in H \setminus (L \cap H)$, then $a$ cannot be an element of $H_i \cap L$, since else $a \in L \cap H$. Hence, $H_i \cap L \in O_{A,C}$ for all sufficiently large $i$. We thus conclude that $H_i \cap L \rightarrow H \cap L$. \hfill $\Box$

2.3. Self-similar and branch groups. Letting $X$ be a finite set, the free monoid generated by $X$ is denoted by $X^*$. We write $X^n$ to indicate the words of length $n$ in the monoid. One may identify $X^*$ with its Cayley graph, so that $X^*$ is the rooted tree where each vertex has $|X|$ many children. It is, however, often more convenient to simply think of $X^*$ as the free monoid.

For $x \in X^*$, the level of $x$, denoted by $|x|$, is the length of the word $x$ in the alphabet $X$. If a word $x$ is a prefix of a word $y$, we write $x \subseteq y$. A finite subset $Y \subseteq X^*$ is a leaf set if $Y$ is finite and $x \nsubseteq y$ for all distinct $x$ and $y$ in $Y$. A collection of leaf sets $(Y_i)_{i \in I}$ is independent if $\bigcup_{i \in I} Y_i$ is a leaf set for all finite sets $F \subseteq I$. We say that $Y \subseteq X^*$ is a spanning leaf
set if it is a leaf set and there is \( N \) such that for every \( x \in X^* \) with \( |x| \geq N \) there is \( y \in Y \) with \( y \subseteq x \). The least such \( N \) is called the depth of \( Y \). One checks that every leaf set can be extended to a spanning leaf set.

The automorphism group of \( X^* \), \( \text{Aut}(X^*) \), is the set of bijections from \( X^* \) to \( X^* \) that preserve the prefix relation. That is, \( u \) is a prefix of \( v \) if and only if \( g(u) \) is a prefix of \( g(v) \) for any \( g \in \text{Aut}(X^*) \). Consequently, for any word \( uw \) in \( X^* \), \( g(uw) = g(u)g_w(w) \) for some other automorphism \( g_w \in \text{Aut}(X^*) \) which depends on \( u \). We call \( g_w \) the section of \( g \) at \( u \); note that some authors use “state” instead of “section.”

For any leaf set \( Y \subseteq X^* \) and \( G \leq \text{Aut}(X^*) \), we use \( G^Y \) to denote the group which acts as copy of \( G \) on each subtree rooted at a vertex in \( Y \). Note that \( G^Y \) is canonically isomorphic to \( \prod_{|Y|} G \).

For any \( G \leq \text{Aut}(X^*) \), there are several subgroups of particular importance. For any \( x \in X^* \), the stabilizer of the vertex \( x \), denoted by \( \text{st}_G(x) \), is the set of elements in \( G \) which fix the vertex \( x \). The rigid stabilizer of the vertex \( x \), denoted by \( \text{rist}_G(x) \), is the set of elements which fix every vertex outside of the subtree rooted at \( x \). For \( Y \subseteq X^* \) a leaf set, the rigid stabilizer of \( Y \) is

\[
\text{rist}_G(Y) := \langle \text{rist}_G(y) \mid y \in Y \rangle \cong \prod_{x \in Y} \text{rist}_G(x);
\]

Hence, \( \text{rist}_G(Y) \) is the internal direct product of the rigid stabilizers of the vertices in \( Y \). We also define \( \text{st}_G(Y) := \cap_{x \in Y} \text{st}_G(x) \). When \( Y = X^n \), we call \( \text{rist}_G(X^n) \) the rigid stabilizer of the level \( n \) and denote it by \( \text{rist}_G(n) \). Similarly, we denote \( \text{st}_G(X^n) \) by \( \text{st}_G(n) \) and call it the stabilizer of the level \( n \). Note that for any \( Y \), \( \text{rist}_G(Y) \leq \text{st}_G(Y) \).

The rigid stabilizer allows one to isolate an important class of groups.

**Definition 2.7.** A subgroup \( G \leq \text{Aut}(X^*) \) is a branch group if it acts transitively on every level and \( \text{rist}_G(n) \) has finite index in \( G \) for all \( n \in \mathbb{N} \).

For \( x \in X^* \), we define the section map \( \psi_x : \text{Aut}(X^*) \to \text{Aut}(X^*) \) by \( g \mapsto g_x \), and for a leaf set \( Y \), \( \psi_Y := \prod_{x \in Y} \psi_x \). Unless the domain of \( \psi_Y \) is restricted to a subgroup of \( \text{st}_{\text{Aut}(X^*)}(Y) \), \( \psi_Y \) is not a homomorphism. When the domain is restricted to a subgroup of \( \text{st}_{\text{Aut}(X^*)}(Y) \), \( \psi_Y \) can be thought of as a projection map onto the coordinates in \( Y \).

A group \( G \leq \text{Aut}(X^*) \) is called self similar if \( g_x \in G \) for all \( g \in G \) and \( x \in X^* \). A self-similar group is called self-replicating if \( \psi_x(\text{st}_G(x)) = G \) for all \( x \in X^* \). We say that \( G \) is strongly self-replicating if \( \psi_x(\text{st}_G(n)) = G \) for all \( x \in X^n \) and \( n \geq 1 \). For strongly self-replicating groups, \( \text{st}_G(n) \) is a subdirect product of \( G^{X^n} \).

A self-similar subgroup \( G \leq \text{Aut}(X^*) \) is said to be regular branch if it acts transitively on every level and there is a normal subgroup \( K \) with finite index in \( G \) such that \( K^{\{x\}} \leq K \) for all \( x \in X^* \) and such that \( K^{X^n} \) has finite index in \( G \) for all \( n \). In this case, \( K \) is called a branching subgroup for
If a group $G$ is a regular branch group, then it is also a branch group as $K^{(x)} \leq \text{rist}_G(x)$, and therefore, $K^n \leq \text{rist}_G(n)$.

For any subgroup $G \leq \text{Aut}(X^*)$, we say $G$ has the congruence subgroup property, or the CSP, if every subgroup of finite index contains a level stabilizer. Since in a branch group $\text{rist}_G(n) \leq \text{st}_G(n)$ and $\text{rist}_G(n)$ has finite index in $G$ for all $n$, a branch group has the CSP if and only if every subgroup of finite index contains a rigid stabilizer and every rigid stabilizer contains a level stabilizer. Many of the most studied branch groups have the congruence subgroup property including the Grigorchuk group [3] and the Gupta–Sidki groups [2, 4]; these will be discussed in more detail later.

2.4. Generalities on groups. A subgroup $L$ of a group $G$ is separable if it is the intersection of finite index subgroups. We say $L$ is separable in $G$ when we wish to emphasize the ambient group. We say that $G$ is LERF if every finitely generated subgroup is separable.

And element $g \in G$ is said to commensurate $L$, if $|L : L \cap gLg^{-1}|$ and $|gLg^{-1} : L \cap gLg^{-1}|$ are finite. The collection of $g \in G$ that commensurate $L$ is denoted by $\text{Comm}_G(L)$. We say that $L$ is commensurated in $G$ if $\text{Comm}_G(L) = G$.

**Theorem 2.8** (Caprace–Kropholler–Reid–Wesolek, [3, Main Theorem]). Let $G$ be a group and $L \leq G$ be a separable subgroup. If $G$ is generated by finitely many cosets of $L$ and $L$ is commensurated in $G$, then $L$ contains a finite index subgroup which is normal and separable in $G$.

We will also need a new notion for groups acting on trees.

**Definition 2.9.** A group $G \leq \text{Aut}(X^*)$ is said to have well-approximated subgroups if $\bigcap_{n \geq 0} \text{Hst}_G(n) = H$ for any finitely generated $H \leq G$.

**Lemma 2.10.** For $G \leq \text{Aut}(X^*)$, $G$ has well-approximated subgroups if and only if $G$ is LERF and has the CSP.

**Proof.** Suppose that $G$ is LERF and $G$ has the CSP. Fixing a finitely generated $H \leq G$, we may find an $\leq$-decreasing sequence of finite index subgroups $O_i$ such that $\bigcap_{i \in \mathbb{N}} O_i = H$ since $G$ is LERF. On the other hand, $G$ has the CSP, so there is $n_i$ such that $\text{st}_G(n_i) \leq O_i$ for each $i$. We now see that

$$H \leq \bigcap_{i \in \mathbb{N}} \text{Hst}_G(n_i) \leq \bigcap_{i \in \mathbb{N}} O_i = H.$$  

It follows that $H = \bigcap_{k \in \mathbb{N}} \text{Hst}_G(k)$.

Conversely, assume for all finitely generated $H \leq G$, $H = \bigcap_{n \geq 0} \text{Hst}_G(n)$.

Since $\text{Hst}_G(n)$ is of finite index in $G$ for any $n$, $G$ is LERF. For $O \leq G$ with finite index, $O = \bigcap_{n \geq 0} \text{Ost}_G(n)$. Since $O$ is finite index the sequence $(\text{Ost}_G(i))_{i \in \mathbb{N}}$ is eventually constant. We conclude that $O = \text{Ost}_G(n)$ for some $n$. Hence, $\text{st}_G(n) \leq O$, and it follows that $G$ has the CSP.

Finally, for a group $G$, the FC-center of $G$ is the set of all elements in $G$ whose conjugacy class is finite. Since the size of the conjugacy class of $gh$
is bounded by the product of the sizes of the conjugacy classes of $g$ and $h$, the FC-center forms a group. Since an element and its conjugate lie in the same conjugacy class, the FC-center is also normal.

A group is **just infinite** if it is an infinite group with every proper quotient finite.

**Lemma 2.11.** Let $K$ be a finitely generated just infinite group. If $K$ is not virtually abelian, then $K$ has a trivial FC-center.

**Proof.** Let $H$ be the FC-center for $K$. Since $K$ is just infinite, $H$ is either trivial or has finite index in $K$.

Assume to the contrary that $H$ is of finite index in $K$. Since $K$ is finitely generated, so is $H$. Moreover, since every element $g$ in $H$ has finitely many conjugates in $K$, every element has finitely many conjugates in $H$, so the centralizer of any element of $H$ in $H$, $C_H(g)$, has finite index in $H$. Choosing $S$ a finite generating set for $H$, the center of $H$, $Z(H)$, is

$$Z(H) = \bigcap_{g \in S} C_H(g).$$

The center $Z(H)$ is thus a finite intersection of subgroups of finite index in $H$ and therefore has finite index in $H$. We conclude that $Z(H)$ is of finite index in $K$, hence $K$ is virtually abelian, which is a contradiction. $\square$

### 3. Infra-direct products in branch groups

**Definition 3.1.** For $G$ a group and $Y$ a finite set, we say that $H \leq G^Y$ is an **infra-direct product** if $\psi_y(G)$ is of finite index in $G$ for all $y \in Y$.

We begin with an easy lemma.

**Lemma 3.2.** Let $G$ be a non-virtually abelian just infinite group, $H \leq G^Y$ be an infra-direct product, and $U \subseteq Y$ be least such that $(\psi_U) \upharpoonright_H: H \to G^U$ is injective. For any finite index subgroup $L \leq H$, $U$ is least such that $(\psi_U) \upharpoonright_L: L \to G^U$ is injective.

**Proof.** Suppose toward a contradiction there is $U' \subsetneq U$ such that $\psi_{U'} \upharpoonright_L$ is injective, and by passing to the normal core of $L$, we may assume that $L \leq H$. Since $\psi_{U'} \upharpoonright_H$ is not injective, there is $h \in H$ such that $\psi_{U'}(h) = 1$. For each $l \in L$, $hlh^{-1} \in L$, and $\psi_{U'}(hlh^{-1}) = \psi_{U'}(l)$. As $\psi_{U'} \upharpoonright_L$ is injective, we conclude that $h$ commutes with $L$. Finding $y \in Y$ such that $\psi_y(h)$ is non-trivial, the images $\psi_y(h)$ and $\psi_y(L)$ commute. This implies that $G$ has a non-trivial FC-center, which contradicts Lemma 2.11. $\square$

#### 3.1. Lower leaf sets

Let $G \leq \text{Aut}(X^*)$ be a strongly self-replicating group and suppose additionally that $G$ has the CSP and is just infinite. Let $Y \subseteq X^*$ be a spanning leaf set and $H \leq G$ be such that $\text{st}_H(Y)$ is an infra-direct product of $G^Y$. 
that claim (2) uses that the proofs are exercises in the definitions and so left to the reader. We note insight into the structure of subgroups $L$ is injective, where $\Omega$ the following hold:

For Observation 3.3. For $(Y_j)_{j=0}^n$ a system of lower leaf sets for $H$ and $Y = \{y_i\}_{i=1}^n$ as above, the following hold:

1. $y_j Z_j \subseteq Y_i$ for all $j \leq i$.
2. For each $z \in Z_i$, $\psi_{y_i}(\text{st}_H(Y_i)) = G$.
3. If $L$ contains $H$, then $(Y_j)_{j=0}^n$ is a system of lower leaf sets for $L$ and $Y$.

The existence of lower leaf sets provides us with a key lemma, which gives insight into the structure of subgroups $L$ such that $\text{st}_L(Y)$ is an infra-direct product of $G^Y$ for $Y$ some spanning leaf set.

Lemma 3.4. For $G$, $H$, and $(Y_j)_{j=0}^n$ as above, there is a non-empty set $W \subseteq Y_n$ such that $\psi_W : \text{st}_H(Y_n) \rightarrow G^W$ is injective with a finite index image.

Proof. Let $U \subseteq Y_0$ be least such that $\psi_U : \text{st}_H(Y_0) \rightarrow G^U$ is injective and say $U = \{y_{i_1}, \ldots, y_{i_m}\}$; note that for any finite index subgroup of $\text{st}_H(Y_0)$, the subset $U$ is also minimal such that the map $\psi_U$ is injective by Lemma 3.2. We inductively build a sequence of pairs $(W_j, N_j)$ with $1 \leq j \leq m$ such that the following hold:

(i) $W_j$ is a non-empty subset of $y_{i_j} Z_{i_j}$, where $y_{i_j} Z_{i_j}$ is as in the construction of the lower leaf sets,
(ii) $N_j \leq \text{st}_H(Y_{i_j})$,
(iii) $\psi_{W_j}(N_j)$ is of finite index in $G^{W_j}$, $\psi_{W_i}(N_j) = \{1\}$ for any $i < j$, and $\psi_{y_{i_j}}(N_j) = \{1\}$ for $l > j$, and
(iv) Setting $\Omega_j := \bigcup_{i \leq j} W_i \cup \{y_{i_l}\}_{l>j}$, the map $\psi_{\Omega_j} : \text{st}_H(Y_{i_j}) \rightarrow G^{\Omega_j}$ is injective, and $\Omega_j \subseteq Y_{i_j}$ is minimal such that $\psi_{\Omega_j}$ is injective.

For the base case, let $W_1 \subseteq y_{i_1} Z_{i_1}$ be least such that $\psi_{\Omega_1} : \text{st}_H(Y_{i_1}) \rightarrow G^{\Omega_1}$ is injective, where $\Omega_1 = W_1 \cup \{y_{i_l}\}_{l>1}$. The subgroup $\text{st}_H(Y_{i_1})$ is of finite index in $\text{st}_H(Y_0)$, so $U$ must be a minimal subset of $Y_0$ such that $\psi_U$ restricted to $\text{st}_H(Y_{i_1})$ is injective, by Lemma 3.2. Hence, $W_1$ is non-empty. Condition (i) is clearly satisfied, and condition (iv) follows from our choice of $U$ and Lemma 3.2.

For each $w \in W_1$, the map $\psi_{\Omega_1 \setminus \{w\}} : \text{st}_H(Y_{i_1}) \rightarrow G^{\Omega_1 \setminus \{w\}}$
fails to be injective. Let $N_w$ be the kernel. In view of Observation 3.3, $\psi_w(st_H(Y_i)) = G$. Therefore, $\psi_w(N_w)$ is of finite index in $G$, as $G$ is just infinite. Setting $N_1 := \langle N_w \mid w \in W_1 \rangle$, it now follows that $\psi_W(N_1)$ is of finite index in $G^{W_1}$. Furthermore, $\psi_{y_l}(N_1) = \{1\}$ for $l > 1$. Hence, (ii) and (iii) hold.

Suppose we have built our sequence up to $(W_j, N_j)$. By recursion, $\psi_{\Omega_j} : st_H(Y_{ij}) \to G^{\Omega_j}$ is injective and $\Omega_j$ is minimal for which the projection is injective, where

$$\Omega_j := \bigcup_{i \leq j} W_i \cup \{y_i\}_{l > j}.$$  

Lemma 3.2 ensures $\Omega_j$ must be a minimal subset of $Y_{ij}$ such that $\psi_{\Omega_j}$ restricted to $st_H(Y_{ij+1})$ is injective. Let $W_{j+1} \subseteq y_{ij+1}Z_{ij+1}$ be least such that $\psi_{\Omega_{j+1}} : st_H(Y_{ij+1}) \to G^{\Omega_{j+1}}$ is injective, where

$$\Omega_{j+1} := \bigcup_{i \leq j+1} W_i \cup \{y_i\}_{l > j+1}.$$  

As in the base case, Lemma 3.2 ensures that $W_{j+1}$ is non-empty and $\Omega_{j+1}$ is minimal such that $\psi_{\Omega_{j+1}}$ is injective. Conditions (i) and (iv) are thus satisfied.

For each $w \in W_{j+1}$, the map

$$\psi_{\Omega_{j+1}\{w\}} : st_H(Y_{ij+1}) \to G^{\Omega_{j+1}\{w\}}$$

fails to be injective. Let $N_w$ be the kernel and setting $N_{j+1} := \langle N_w \mid w \in W_{j+1} \rangle$, it follows as in the base case that $\psi_{W_{j+1}}(N_{j+1})$ is of finite index in $G^{W_{j+1}}$, $\psi_{W_l}(N_{j+1}) = \{1\}$ for any $i < j + 1$, and $\psi_{y_l}(N_{j+1}) = \{1\}$ for $l > j + 1$. Hence, (ii) and (iii) hold, and our construction is complete.

Set $W := \bigcup_{j=1}^m W_j$. We see that $W = \Omega_m$, so $\psi_W : st_H(Y_n) \to G^W$ is injective. Lemma 3.2 implies $\psi_W : st_H(Y_n) \to G^W$ is also injective. Set $M_j := N_j \cap st_H(Y_n)$. For each $j$, $\psi_W(M_j) = \{1\}$ for $l \neq j$. Indeed, our recursive construction ensures that $\psi_{W_l}(M_j) = \{1\}$ for $l < j$. For $l > j$, $\psi_{W_l}(N_j) = \{1\}$. Since $\psi_{W_l}(M_j) = \psi_Z \circ \psi_{y_l}(M_j)$ for some $Z$ such that $y_lZ = W_l$, we conclude that $\psi_{W_l}(M_j) = \{1\}$. On the other hand, $M_j$ is a finite index subgroup of $N_j$, so $\psi_{W_j}(M_j)$ is of finite index in $G^{W_j}$. It now follows that $\psi_W(\langle M_j \mid 1 \leq j \leq m \rangle)$ is of finite index in $G^W$. Hence, $\psi_W(st_H(Y_n))$ is of finite index in $G^W$. \[\Box\]

3.2. Structure results. We now deduce several consequences of Lemma 3.3 which will later be used to analyse the Chabauty space.

**Lemma 3.5.** Suppose that $G \leq Aut(X^*)$ is finitely generated and just infinite, is strongly self-replicating, and has the CSP. For $Y$ a spanning leaf set and $L \leq G$, if $st_L(Y) \leq G^Y$ is an infra-direct product, then $L$ is finitely generated.
Proof. Letting \((Y_j)_{j=0}^n\) be a system of lower leaf sets for \(L\) and \(Y\), Lemma 3.4 supplies a non-empty \(W \subseteq Y_n\) such that \(\psi_W : st_L(Y_n) \to G^W\) is injective with a finite index image. Hence, \(st_L(Y_n)\) is finitely generated, and as \(st_L(Y_n)\) is of finite index in \(L\), \(L\) is finitely generated.

Lemma 3.6. Suppose that \(G \leq \text{Aut}(X^*)\) is finitely generated and just infinite, is strongly self-replicating, and has the CSP. For \(Y\) some spanning leaf set and \(L \leq G\), if \(st_L(Y) \leq G^Y\) is an infra-direct product and separable in \(G\), then \(|\text{Comm}_G(L) : L| < \infty\).

Proof. Applying Lemma 3.4 we may find a spanning leaf set \(Y'\) and a non-empty \(W \subseteq Y'\) such that \(\psi_W : st_L(Y') \to G^W\) has a finite index image. Note that \(\text{Comm}_G(st_L(Y')) = \text{Comm}_G(L)\) since \(st_L(Y')\) is of finite index in \(L\). Replacing \(L\) with \(st_L(Y')\) and \(Y\) with \(Y'\), we may assume that there is \(W \subseteq Y\) such that \(\psi_W : L \to G^W\) is an injective homomorphism with a finite index image.

Put \(J := \text{Comm}_G(L)\). The stabilizer \(st_J(Y)\) is an infra-direct product of \(G^Y\), so by Lemma 3.5, \(st_J(Y)\) is finitely generated. Theorem 2.8 now supplies \(\tilde{L} \leq st_J(Y)\) such that \(\tilde{L}\) is a finite index subgroup of \(L\). Suppose toward a contradiction that \(L\) is of infinite index in \(st_J(Y)\). The map \((\psi_W)_{st_J(Y)} : st_J(Y) \to G^W\) must be non-injective, so \((\psi_W)_{st_J(Y)}\) has a non-trivial kernel \(I\). The subgroup \(I\) is normal in \(st_J(Y)\) and intersects \(L\) trivially, hence \(I\) and \(\tilde{L}\) commute. Fix some coordinate \(y \in Y\) such that \(\psi_y(I)\) is non-trivial. The projection \(\psi_y(\tilde{L})\) is then a finite index subgroup of \(G\) that centralizes \(\psi_y(I)\). The non-trivial group \(\psi_y(I)\) is thus contained in the FC-center of \(G\) which is impossible in view of Lemma 2.11. We conclude that \(L\) is of finite index in \(st_J(Y)\). Therefore, \(|J : L| < \infty\), since \(|J : st_J(Y)| < \infty\).

Corollary 3.7. Suppose that \(G \leq \text{Aut}(X^*)\) is finitely generated and just infinite, is strongly self-replicating, and has the CSP. For \(Y\) some spanning leaf set and \(L \leq G\), if \(st_L(Y) \leq G^Y\) is an infra-direct product and separable in \(G\), then there are only finitely many subgroups \(H \leq G\) such that \(L \leq H \leq G\) and \(|H : L| < \infty\).

Proof. Every such subgroup \(H\) is such that \(L \leq H \leq \text{Comm}_G(L)\). Lemma 3.6 ensures that \(|\text{Comm}_G(L) : L| < \infty\), so there are finitely many such \(H\).

Definition 3.8. For \(L \leq G\), the depth of \(L\) in \(G\) is the supremum of the natural numbers \(n\) for which there is a series of subgroups \(G = H_0 > H_1 > \cdots > H_n = L\) such that \(|H_i / H_{i+1}| = \infty\) for all \(i\). Depth zero subgroups are of finite index. We denote the depth by \(\text{depth}_G(L)\). When the ambient group \(G\) is clear from context, we write \(\text{depth}(L)\).
Lemma 3.9. Suppose that $G \leq \text{Aut}(X^*)$ is finitely generated and just infinite, is strongly self-replicating, and has the CSP. For $Y$ some spanning leaf set and $L \leq G$, if $\text{st}_L(Y) \leq G^Y$ is an infra-direct product, then $\text{depth}(L) < \infty$.

Proof. Let $(Y_j^n)_{j=0}^n$ be a system of lower leaf sets for $L$ and $Y$ and put $m = 2^{|Y_n|}$, which is the size of the power set of $Y_n$. Let $G = H_0 > H_1 > \cdots > H_m = L$ be a sequence of subgroups. By Observation 3.3, $(Y_j^n)_{j=0}^n$ is a system of lower leaf sets for each $H_i$. For each $i$, Lemma 3.4 supplies a non-empty $W_i \subseteq Y_n$ such that $\psi_{W_i} : \text{st}_{H_i}(Y_n) \to G^{W_i}$ is injective with a finite index image. Since $m + 1$ is larger than the size of the power set of $Y_n$, there are $H_i$ and $H_j$ with $i < j$ such that $W_i = W_j$. Hence, $|H_j : H_i| < \infty$. We conclude that $\text{depth}(L) \leq m$.

4. ON THE GRIGORCHUK–NAGNIBEDA ALTERNATIVE

Our main theorem will consider groups which satisfy the following alternative.

Definition 4.1. A group $G \leq \text{Aut}(X^*)$ is said to obey the Grigorchuk–Nagnibeda alternative if for every finitely generated subgroup $H \leq G$, either
(a) there is $v \in X^*$ such that $\psi_v(H)$ is finite, or
(b) there is a spanning subtree $Y \leq X^*$ such that $\text{st}_H(Y)$ is an infra-direct product of $G^Y$.

Towards establishing our main theorem, we here examine the groups described in the cases of the alternative.

4.1. Subgroups with a finite section group: Case (a). For a leaf set $T \subseteq X^*$ and $n$ greater that or equal to the depth of $T$, the shadow of $T$ on level $n$ is
$$S(T, n) := \{ v \in X^n \mid \exists t \in T \ t \sqsubseteq v \}.$$ By the choice of $n$, the shadow is always a non-empty leaf set.

Lemma 4.2. Let $G \leq \text{Aut}(X^*)$ and $H \leq G$. If $T := \{ v \in X^k \mid \psi_v(H) \text{ is finite} \}$ is non-empty, then $S(T, n)$ is setwise invariant under the action of $H$ on $X^n$ for all $n \geq k$.

Proof. The claim is immediate once we establish that $T$ is setwise invariant under the action of $H$. For $v \in T$ and $h \in H$, it suffices to show that $\psi_{h(v)}(\text{st}_H(h(v)))$ is finite, since $\text{st}_H(h(v))$ is of finite index in $H$. We see that
$$\psi_{h(v)}(\text{st}_H(h(v))) = (h_v)\psi_v(\text{st}_H(v))(h_v)^{-1},$$ and in particular, $|\psi_{h(v)}(\text{st}_H(h(v)))| = |(h_v)\psi_v(\text{st}_H(v))(h_v)^{-1}| < \infty$. Thus, $h(v) \in T$, and $T$ is setwise invariant under the action of $H$. \qed
Lemma 4.3. Let $G \leq \text{Aut}(X^*)$ and $H \leq G$. If there is $v \in X^*$ such that $\psi_v(H)$ is finite, then there is a countable independent family $(Y_i)_{i \in \mathbb{N}}$ of $H$-invariant leaf sets such that $\psi_v(H)$ is finite for all $v \in Y_i$ and $i \in \mathbb{N}$. Furthermore, for any $M \in \mathbb{N}$, we may take $|x| \geq M$ for every $x \in Y_i$ and $i \in \mathbb{N}$.

Proof. For $N \geq 1$, let $T := \{v \in X^N \mid \psi_v(H) \text{ is finite}\}$. The set $T$ is non-empty for any suitably large $N$; fix such an $N$. Let $X_T$ be the subset of $X^*$ consisting of all vertices that contain a vertex of $T$ as a prefix. In other words, $X_T$ is the union of $S(T, n)$ for all $n \geq N$. Note in particular, that $X_T$ is $H$-invariant by Lemma 4.2.

The kernel $\ker(H \curvearrowright X_T)$ equals the collection of $h \in H$ such that $h \in \text{st}_H(T)$ and $\psi_T(h) = 1$. The image $\psi_T(\text{st}_H(T))$ is finite, since $\psi_v(H)$ is finite for each $v \in T$. Thus $(\psi_T)^{-1}(1)$ is of finite index in $\text{st}_H(T)$. We deduce that $|H : \ker(H \curvearrowright X_T)| < \infty$.

We now build a family of $H$-invariant leaf sets $(W_i)_{i \in \mathbb{N}}$ along with natural numbers $k_i \geq 1$ such that

(i) $W_i = Y_i \sqcup Z_i$ where $Y_i$ and $Z_i$ are non-empty and $H$-invariant,
(ii) $W_i \subseteq S(Z_{i-1}, k_i)$.

For the base case, the size of any orbit of $H$ on $S(T, k)$ is bounded by $|H : \ker(H \curvearrowright X_T)|$. Fixing $k_0 \geq \max\{N, |H : \ker(H \curvearrowright X_T)| + 1\}$, the action of $H$ on $S(T, k_0)$ has at least two orbits. Set $W_0 = S(T, k_0)$, observe that $W_0$ is a leaf set, and fix a partition $W_0 = Y_0 \sqcup Z_0$ where $Y_0$ and $Z_0$ are non-empty $H$-invariant subsets. Condition (i) is satisfied, and (ii) is vacuous.

Suppose we have built the sequence up to $n$. The shadow $S(Z_n, l)$ is $H$-invariant for all $l > k_n$. As in the base case, we may find $k_{n+1}$ large enough such that $H$ has at least two orbits on $S(Z_n, k_{n+1})$. Set $W_{n+1} = S(Z_n, k_{n+1})$, observe that $W_{n+1}$ is a leaf set, and fix a partition $W_{n+1} = Y_{n+1} \sqcup Z_{n+1}$ where $Y_{n+1}$ and $Z_{n+1}$ are non-empty $H$-invariant subsets. Conditions (i) and (ii) are clearly satisfied.

A straightforward induction argument shows the collection $(Y_i)_{i \in \mathbb{N}}$ is the desired independent family of leaf sets. That $\psi_v(H)$ is finite for each $v \in Y_i$ follows from the fact that such a $v$ contains an element of $T$ as a prefix. Taking $k_0 > M$ at stage zero of our construction ensures that $|x| \geq M$ for all $x \in Y_i$ and $i \in \mathbb{N}$. \qed

Lemma 4.4. Let $G \leq \text{Aut}(X^*)$ be a self-similar regular branch group with well-approximated subgroups. If $H \leq G$ is finitely generated and there is $v \in X^*$ such that $\psi_v(H)$ is finite, then $H$ is in the perfect kernel of Sub($G$).

Proof. Suppose that $K$ is a branching subgroup of $G$ and recall that $K$ is normal in $G$.

It suffices to show that every neighborhood of $H$ in Sub($G$) has continuum many elements. Since $H$ is finitely generated, a neighborhood base at $H$ has
the form

\[ V_A := \{ I \in \text{Sub}(G) \mid H \leq I \text{ and } \forall a \in A \ a \notin I \} \]

where \( A \) ranges over finite subsets of \( G \setminus H \). It therefore suffices to show that each \( V_A \) has size continuum.

Fix a finite set \( A \subseteq G \setminus H \). As \( G \) has well-approximated subgroups, there is a level \( M \) such that \( \text{Hrist}_G(M) \cap A = \emptyset \). Applying Lemma 4.3, we obtain a countable independent family \( (Y_i)_{i \in \mathbb{N}} \) of \( H \)-invariant leaf sets such that \( |v| \geq M \) for \( v \in Y_i \) and \( \psi_v(H) \) is finite for all \( v \in Y_i \).

For each \( \alpha \in \{0,1\}^\mathbb{N} \), define

\[ J_\alpha = \langle K^{\{x\}} \mid x \in Y_i \text{ and } \alpha(i) = 1 \rangle \]

where \( K^{\{x\}} \) is the copy of \( K \) which acts only on the tree below \( x \). The sequence \( (Y_i)_{i \in \mathbb{N}} \) is independent, so letting \( Z := \{ i \in \mathbb{N} \mid \alpha(i) = 1 \} \),

\[ J_\alpha = \bigoplus_{i \in Z} K^{Y_i} \times \bigoplus_{j \in \mathbb{N} \setminus Z} \{1\}^{Y_j}. \]

For \( y \in Y_i \) and \( g \in H \), \( gK^{\{y\}}g^{-1} \) acts only on the tree below \( y' = g(y) \in Y_i \), since \( Y_i \) is \( H \)-invariant. The image \( \psi_{y'}(gK^{\{y\}}g^{-1}) \) equals \( g_yK(g_y)^{-1} \). Since \( K \) is normal in \( G \) and \( G \) is self-similar, it follows that \( g_yK(g_y)^{-1} = K \). Hence, \( gK^{\{y\}}g^{-1} = K^{\{g(y)\}} \), and \( H \) normalizes \( J_\alpha \). Furthermore, \( HJ_\alpha \leq \text{Hrist}_G(M) \), so \( HJ_\alpha \subseteq V_A \).

Suppose that \( \alpha \) and \( \beta \) are distinct elements of \( \{0,1\}^\mathbb{N} \) and find \( i \) such that \( \alpha(i) \neq \beta(i) \). Without loss of generality, we assume that \( \alpha(i) = 1 \) while \( \beta(i) = 0 \). Fixing \( v \in Y_i \), \( (hy)_v = h_vy_v \) for any \( hy \in (HJ_{\gamma})_v \) and \( \gamma \in \{0,1\}^\mathbb{N} \), since \( y \) must fix \( v \). It now follows that \( \psi_v(HJ_\alpha) \) is infinite while \( \psi_v(HJ_\beta) = \psi_v(H) \) is finite. Hence, \( HJ_\alpha \neq HJ_\beta \). We conclude that \( V_A \) contains uncountably many subgroups of \( G \), and lemma follows. \( \square \)

4.2. Infra-direct product subgroups: Case \( (b) \). We now turn our attention to the second type of subgroup described by the alternative. We begin with a proposition.

**Proposition 4.5.** For \( G \leq \text{Aut}(X^*) \) a finitely generated group that is LERF, the finite index subgroups of \( G \) are exactly the subgroups with Cantor–Bendixson rank 0.

**Proof.** Any finite index subgroup has rank zero since \( G \) is finitely generated. Conversely, suppose that \( H \in \text{Sub}(G) \) has rank 0. The subgroup \( H \) is isolated in \( \text{Sub}(G) \) and must be finitely generated as otherwise \( H \) can be approximated by its finitely generated subgroups. As \( G \) is LERF, \( H \) is the intersection of finite index subgroups. Since \( H \) is isolated in \( \text{Sub}(G) \), it must itself be a finite index subgroup. \( \square \)

**Theorem 4.6.** Suppose that \( G \leq \text{Aut}(X^*) \) is finitely generated, just infinite, and strongly self-replicating, and has well-approximated subgroups. For \( Y \) a spanning leaf set, if \( H \leq G \) is such that \( \text{st}_H(Y) \) is an infra-direct product of \( G^V \), then the \( \text{rk}_{\text{CB}}(H) = \text{depth}(H) \). In particular, \( \text{rk}_{\text{CB}}(H) < \omega \).
Proof. The hypothesis of Lemma 3.9 are satisfied, so depth$(H) < \infty$.

We now argue by induction on depth$(H)$ for the claim. If depth$(H) = 0$, then $H$ has finite index in $G$, and Proposition 4.5 ensures that $\text{rk}_{CB}(H) = 0$. For the successor case, suppose that depth$(H) = n + 1$ and say that $A_i$ is a sequence of distinct subgroups that converges to $H$ in Sub$(G)$. We argue that all but finitely many terms of the sequence are such that $\text{rk}_{CB}(A_i) \leq n$. By Lemma 3.3, $H$ is finitely generated, so we may assume, by possibly deleting finitely many terms from the sequence, that $|A_i : H| < \infty$. Possibility deleting finitely many terms, we may assume that $|A_i : H| = \infty$ for all $i$.

For each $i$, $|A_i : H| = \infty$, so depth$(A_i) < \text{depth}(H) = n + 1$. Applying the inductive hypothesis, $\text{rk}_{CB}(A_i) \leq n$. It now follows that there is a neighborhood of $H$ in Sub$(G)$ consisting of subgroups of rank at most $n$. Hence, $\text{rk}_{CB}(H) \leq n + 1$.

Conversely, find a sequence $G = L_0 > \cdots > L_{n+1} = H$ such that $|L_i : L_{i+1}| = \infty$ for each $i$. The stabilizer $\text{st}_{L_n}(Y)$ is an infra-direct product of $G^Y$, so by Lemma 3.9, $\text{st}_{L_n}(Y)$ is finitely generated. The group $\text{st}_{L_n}(Y)$ is of finite index in $L_n$, so $L_n$ is finitely generated. The group $L_n$ is thus finitely generated with depth $n$, so by the inductive hypothesis, $\text{rk}_{CB}(L_n) = n$.

Let $O_i$ be an $\subseteq$-decreasing sequence of finite index subgroups of $G$ such that $\bigcap_{i \in \mathbb{N}} O_i = H$. The sequence $O_i \cap L_n$ converges to $H$ in Sub$(H)$. The terms $O_i \cap L_n$ have depth at most $n$, since else we contradict the depth of $H$, and on the other hand, they have depth at least $n$, witnessed by the sequence $L_0 > \cdots > L_{n-1} > O_i \cap L_n$. Applying the inductive hypothesis, $\text{rk}_{CB}(O_i \cap L_n) = n$. The sequence $(O_i \cap L_n)_{i \in \mathbb{N}}$ is thus a sequence of rank $n$ groups converging to $H$, so $\text{rk}_{CB}(H) \geq n + 1$. Hence, $\text{rk}_{CB}(H) = n + 1$, and the induction is complete. \hfill \Box

5. The Structure of Sub$(G)$

We are now prepared to give a detailed picture of Sub$(G)$ for $G$ from a certain class of well-behaved branch groups.

Theorem 5.1. Suppose $G \leq \text{Aut}(X^*)$ is a finitely generated regular branch group that is just infinite, is strongly self-replicating, has well-approximated subgroups, and obeys the Grigorchuk–Nagnibeda alternative. For $H \leq G$, $H$ is in the perfect kernel of Sub$(G)$ if and only if either

1. $\psi_v(H)$ is finite for some $v \in X^*$, or
2. $H$ is non-finitely generated.

Proof. Let $K$ be a branching subgroup for $G$.

Let’s first see that the perfect kernel contains all subgroups of the two forms stated. If $H \leq \Gamma$ has form (1), then Lemma 4.4 ensures that $H$ is in the perfect kernel.

Suppose that $H$ is non-finitely generated and let $B \leq H$ be a finitely generated subgroup. If there is a spanning leaf set $Y \subseteq X^*$ such that $\text{st}_B(Y)$
is an infra-direct product in $G^Y$, then $\text{st}_H(Y)$ is an infra-direct product of $G^Y$. In view of Lemma 3.5, $\text{st}_H(Y)$ is finitely generated, and it follows that $H$ is finitely generated, which contradicts our assumption on $H$. It is thus the case that no finitely generated $B \leq H$ admits a spanning leaf set $Y$ such that $\text{st}_B(Y)$ is an infra-direct product of $G^Y$. Since $G$ obeys the Grigorchuk–Nagnibeda alternative, each finitely generated subgroup admits $v$ such that the group of sections at $v$ is finite. Each finitely generated subgroup of $H$ is thereby an element of the perfect kernel. Noting that $H$ is the limit of its finitely generated subgroups in $\text{Sub}(G)$ and that the perfect kernel is closed, $H$ is in the perfect kernel.

Conversely, suppose that $H$ is an element of the perfect kernel. If $H$ is non-finitely generated, (2) holds, and we are done. Let us then suppose that $H$ is finitely generated. As $G$ satisfies the Grigorchuk–Nagnibeda alternative, either $\psi_v(H)$ is finite for some $v$, or there is a spanning leaf set $Y$ such that $\text{st}_H(Y)$ is an infra-direct product of $G^Y$. In the latter case, Theorem 4.6 implies that $H$ has finite rank, which is absurd. We conclude that $\psi_v(H)$ is finite for some $v$, so (1) holds. □

**Theorem 5.2.** Suppose $G \leq \text{Aut}(X^*)$ is a finitely generated regular branch group that is just infinite, is strongly self-replicating, has well-approximated subgroups, and obeys the Grigorchuk–Nagnibeda alternative. For $L \leq G$, the following are equivalent:

1. $L$ is not a member of the perfect kernel of $\text{Sub}(G)$.
2. There is a spanning leaf set $Y$ such that $\text{st}_L(Y)$ is an infra-direct product of $G^Y$.
3. $\text{rk}_{CB}(L) < \omega$.
4. $\text{depth}(L) < \infty$ and $L$ is finitely generated.

**Proof.** (1)$\Rightarrow$(2). If (1) holds, then $L$ is finitely generated and there is no $v \in X^*$ such that $\psi_v(L)$ is finite. The Grigorchuk–Nagnibeda alternative implies that (2) holds.

(2)$\Rightarrow$(3) and (2)$\Rightarrow$(4). Suppose (2) holds. From Theorem 4.6, (3) holds, and $\text{depth}(L) < \infty$. In view of Lemma 3.5, $L$ is also finitely generated, so (4) holds.

That (3)$\Rightarrow$(1) holds is immediate, and it then follows from the previous paragraph that (3)$\Rightarrow$(4).

(4)$\Rightarrow$(1). We will prove the contrapositive of this implication. Suppose (1) fails. In view of Theorem 5.1, either $L$ is not finitely generated or there is $v \in X^*$ such that $\psi_v(L)$ is finite. In the former case, we are done. Suppose that there is $v$ with $\psi_v(L)$ is finite. Appealing to Lemma 4.3, we may find an independent family $(Y_i)_{i \in \mathbb{N}}$ of $L$-invariant leaf sets such that $\psi_x(L) = \{1\}$ for all $x \in Y_i$. 


Let $K$ be the branching subgroup of $G$ and for each $m \in \mathbb{N}$, define
\[ J_m := \langle K^x \mid x \in \bigcup_{i=0}^{m} Y_i \rangle = \bigoplus_{i \leq m} K^Y_i \times \bigoplus_{j > m} \{1\}^Y_j. \]

The group $L$ normalizes $J_m$. Furthermore, $LJ_m < LJ_{m+1}$ and $|LJ_m| = \infty$. The sequence $(LJ_m)_{m \in \mathbb{N}}$ thus demonstrates that $\text{depth}(H) = \infty$. Hence, (4) fails. \hfill \Box

Theorems 5.2 and 4.6 now give a clean description of the subgroups not in the perfect kernel.

**Corollary 5.3.** Suppose $G \leq \text{Aut}(X^*)$ is a finitely generated regular branch group that is just infinite, is strongly self-replicating, has well-approximated subgroups, and obeys the Grigorchuk–Nagnibeda alternative. If $L \leq G$ is not a member of the perfect kernel of $\text{Sub}(G)$, then

1. $L$ is finitely generated,
2. $\text{rk}_{\text{CB}}(L) < \omega$, and
3. $\text{rk}_{\text{CB}}(L) = \text{depth}(L)$.

We can also compute exactly the Cantor–Bendixson rank of $\text{Sub}(G)$.

**Corollary 5.4.** If $G \leq \text{Aut}(X^*)$ is a finitely generated branch group that is just infinite, is strongly self-replicating, has well-approximated subgroups, and obeys the Grigorchuk–Nagnibeda alternative, then $\text{Sub}(G)$ has Cantor–Bendixson rank $\omega$.

**Proof.** Theorem 5.2 ensures the Cantor–Bendixson rank is at most $\omega$. Conversely, given a $n \geq 1$, let $Y \subseteq X^*$ be spanning leaf set with $|Y| > n$. Let $K$ be the branching subgroup of $G$ and for each $k \in K$ let $f_k \in K^Y$ be such that $\psi_y(f_k) = k$ for all $y \in Y$. Setting $H := \langle f_k \mid k \in K \rangle$, we see that $H$ is an infra-direct product of $G^Y$. The depth of $H$ in $K^Y$ is at least $|Y| - 1$, so $\text{depth}_G(H) \geq n$. Theorem 4.6 ensures that $\text{rk}_{\text{CB}}(H) = \text{depth}(H)$, so $\text{rk}_{\text{CB}}(H) \geq n$. The Cantor–Bendixson rank of $G$ is thus $\omega$. \hfill \Box

Finally, we make an observation concerning the commensurators of subgroups outside the perfect kernel.

**Corollary 5.5.** Suppose $G \leq \text{Aut}(X^*)$ is a finitely generated regular branch group that is just infinite, is strongly self-replicating, has well-approximated subgroups, and obeys the Grigorchuk–Nagnibeda alternative. If $L \leq G$ is not a member of the perfect kernel of $\text{Sub}(G)$, then $|\text{Comm}_G(L) : L|$ is finite.

**Proof.** By Theorem 5.2 there is a spanning leaf set $Y$ such that $\text{st}_L(Y)$ is an infra-direct product of $G^Y$. Lemma 3.6 now implies that $|\text{Comm}_G(L) : L| < \infty$. \hfill \Box
6. Applications

We here show that both the Grigorchuk group and the Gupta–Sidki 3-group obey the Grigorchuk–Nagnibeda alternative. Our proof will rely on the following “subgroup induction” theorem due to Grigorchuk–Wilson, for the Grigorchuk group, and Garrido, for the Gupta–Sidki 3-group.

**Theorem 6.1** ([6, Theorem 3], [4, Theorem 6]). Let $G$ be either the Grigorchuk group or the Gupta–Sidki 3-group. Let $X$ be a family of subgroups of $G$ such that the following hold:

(i) $\{1\} \in X$, and $G \in X$.

(ii) If $H \in X$, then $L \in X$ for any $L \leq G$ for which $H$ is a finite index subgroup of $L$.

(iii) If $H$ is a finitely generated subgroup of $\text{st}_G(1)$ and all first level sections of $H$ are in $X$, then $H \in X$.

Then, all finitely generated subgroups of $G$ are elements of $X$.

The case of the Grigorchuk group in the next theorem already follows from [1].

**Theorem 6.2.** Both the Grigorchuk group and the Gupta–Sidki 3-group obey the Grigorchuk–Nagnibeda alternative.

**Proof.** Let $G$ be the Gupta–Sidki 3-group; the proof for the Grigorchuk group is similar. Let $X$ be the collection of subgroups that satisfy the Grigorchuk–Nagnibeda alternative. That is to say, $X$ is the collection of all subgroups $H$ such that either $|\psi_v(H)| < \infty$ for some $v \in X^*$, or there is a spanning leaf set $Y$ such that $\text{st}_H(Y)$ is an infra-direct product of $G^Y$. It suffices to show $X$ satisfies the conditions of Theorem 6.1.

That condition (i) holds is immediate. For (ii), suppose that $H \in X$ and $L \leq G$ is such that $H \leq L$ with $|L : H| < \infty$. First, if there is a spanning leaf set $Y$ such that $\text{st}_H(Y)$ is an infra-direct product of $G^Y$, then clearly $\text{st}_L(Y)$ is an infra-direct product of $G^Y$. Hence, $L \in X$. Suppose next that $H$ is such that $\psi_v(H)$ is finite for some $v \in X^*$. The subgroup $\psi_v(\text{st}_L(v))$ is a finite extension of $\psi_v(\text{st}_H(v))$ and is thus also finite. It follows that $\psi_v(L)$ is finite. We conclude that $L \in X$.

Finally, let us argue for (iii). Suppose that $H \leq \text{st}_G(1)$ and $H_i := \psi_i(H) \in X$ for all $i \in X = \{0, 1, 2\}$. If $\psi_v(H_i)$ is finite for some $i$, then $\psi_v(H_i)$ is finite, so $H \in X$. Otherwise, say that $Y_i$ is a spanning leaf set such that $\text{st}_{H_i}(Y_i)$ is an infra-direct product of $G^{Y_i}$. Put $Y := 0Y_0 \cup Y_1 \cup 2Y_2$ and consider $\text{st}_H(Y)$. The group $\text{st}_H(Y)$ is of finite index in $H$, and $\psi_i(\text{st}_H(Y))$ is contained in $\text{st}_{H_i}(Y_i)$. Hence, $\psi_i(\text{st}_H(Y))$ is of finite index in $\text{st}_{H_i}(Y_i)$. For each $y \in Y_i$, we conclude that $\psi_{iy}(\text{st}_H(Y)) = \psi_y \circ \psi_i(\text{st}_H(Y))$ is of finite index in $G$. It now follows that $\text{st}_H(Y)$ is an infra-direct product of $G^Y$. □

It is well-known that both the Grigorchuk group and the Gupta–Sidki 3-group are finitely generated regular branch groups that are just infinite and have well-approximated subgroups (i.e. have CSP and are LERF); see...
They are both strongly self-replicating by [4, Page 673] and [4, Proposition 2.1].

**Corollary 6.3.** For $G$ either the Grigorchuk group or the Gupta–Sidki 3-group, the Cantor–Bendixson rank of $\text{Sub}(G)$ is $\omega$.

**References**

1. Profinite and asymptotic group theory, Oberwolfach Rep. 5 (2008), no. 2, 1537–1587. Abstracts from the workshop held June 22-28, 2008, Organized by Fritz Grunewald and Dan Segal, Oberwolfach Reports. Vol. 5, no. 2. MR 2508631
2. Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunić, Branch groups, Handbook of Algebra 3 (2003), 989–1112.
3. P.E. Caprace, P.H Kropholler, C.D. Reid, and P. Wesolek, On the profinite closure of commensurated subgroups, preprint, arxiv:1706.06853, 2017.
4. Alejandra Garrido, Abstract commensurability and the Gupta-Sidki group, Groups Geom. Dyn. 10 (2016), no. 2, 523–543. MR 3513107
5. R. I. Grigorchuk, Just infinite branch groups, New horizons in pro-$p$ groups, Progr. Math., vol. 184, Birkhäuser Boston, Boston, MA, 2000, pp. 121–179. MR 1765119 (2002f:20044)
6. R. I. Grigorchuk and J. S. Wilson, A structural property concerning abstract commensurability of subgroups. J. London Math. Soc. (2) 68 (2003), no. 3, 671–682. MR 2009443
7. Alexander S. Kechris, Classical descriptive set theory, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.