Generalized Symmetric Metric Connection for Kenmotsu Manifolds

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Mathematics Subject Classification: 53C15, 53C25, 53C40.
Keywords and phrases: Kenmotsu manifold, symmetric connection, generalized symmetric metric connection.

Abstract. We get new metric connection for Kenmotsu manifolds which is called generalized symmetric metric connection of type \((\alpha, \beta)\). Quarter-symmetric and semi-symmetric connections are two samples of this connection such that \((\alpha, \beta) = (0, 1)\) and \((\alpha, \beta) = (1, 0)\), respectively. The object of the present paper is to study Kenmotsu manifolds with generalized symmetric metric connection of type \((\alpha, \beta)\).

1 Introduction

A linear connection \(\nabla\) is said to be generalized symmetric connection if its torsion tensor \(T\) is of the form

\[
T(X, Y) = \alpha \{u(Y)X - u(X)Y\} + \beta \{u(Y)\varphi X - u(X)\varphi Y\},
\]

for any vector fields \(X, Y\) on a manifold, where \(\alpha\) and \(\beta\) are smooth functions. \(\varphi\) is a tensor of type \((1, 1)\) and \(u\) is a \(1\)-form associated with a non-vanishing smooth non-null unit vector field \(\xi\). Moreover, the connection \(\nabla\) is said to be a generalized symmetric metric connection if there is a Riemannian metric \(g\) in \(M\) such that \(\nabla g = 0\), otherwise it is non-metric.

In the equation (1), if \(\alpha = 0\) (\(\beta = 0\)), then the generalized symmetric connection is called \(\beta\) quarter-symmetric connection (\(\alpha\) semi-symmetric connection), respectively. Moreover, if we choose \((\alpha, \beta) = (1, 0)\) and \((\alpha, \beta) = (0, 1)\), then the generalized symmetric connection is reduced to a semi-symmetric connection and quarter-symmetric connection, respectively. Hence a generalized symmetric connections can be viewed as a generalization of semi-symmetric connection and quarter-symmetric connection. This two connection are important for both the geometry study and applications to physics. In [1], H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors [2]-[7]. The idea of quarter-symmetric linear connections in a differential manifold was introduced by S.Golab [3]. In [14], Sharfuddin and Hussain defined a semi-symmetric metric connection in an almost contact manifold, by setting

\[
T(X, Y) = \eta(Y)X - \eta(X)Y.
\]
In [15] and [16] the authors studied the semi-symmetric metric connection and semi-symmetric non-metric connection in a Kenmotsu manifold, respectively.

In the present paper, we defined new connection for Kenmotsu manifold, generalized symmetric metric connection. This connection is the generalized form of semi-symmetric metric connection and quarter-symmetric metric connection. Section 2 is devoted to preliminaries. In section 3, we get generalized symmetric metric connection for a Kenmotsu manifold. In section 4, we calculate curvature tensor and ricci tensor and scalar curvature of a Kenmotsu manifold with respect to generalized metric connection. Moreover we get that first Bianchi identity is provided and Ricci tensor is symmetric with respect to generalized metric connection of type $(-1, \beta)$ and $(\alpha, 0)$. In section 5, it is shown that if a Kenmotsu manifold is $\phi-$ projectively flat with respect to generalized metric connection, then the manifold is a generalized $\eta-$ Einstein manifold with respect to generalized metric connection. In section 6, we find an expression for concircular curvature tensor with respect to generalized metric connection. In section 7, we give an example which is verifying some results of Section 4 and Section 5.

2 Preliminaries

A differentiable manifold of dimension $n = 2m + 1$ is called almost contact metric manifold, if it admit a $(1, 1)$ tensor field $\phi$, a contravaryant vector field $\xi$, a $1-$ form $\eta$ and Riemannian metric $g$ which satify

\[
\phi \xi = 0, \tag{2}
\]
\[
\eta(\phi X) = 0 \tag{3}
\]
\[
\eta(\xi) = 1, \tag{4}
\]
\[
\phi^2(X) = -X + \eta(X)\xi, \tag{5}
\]
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{6}
\]
\[
g(X, \xi) = \eta(X), \tag{7}
\]

for all vector fields $X$, $Y$ on $M$. If we write $g(X, \phi Y) = \Phi(X, Y)$, then the tensor field $\phi$ is a anti-symmetric $(0, 2)$ tensor field [11]. If an almost contact metric manifold satisfies

\[
(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{8}
\]
\[
\nabla_X \xi = X - \eta(X)\xi, \tag{9}
\]

then $M$ is called a Kenmotsu manifold, where $\nabla$ is the Levi-Civita connection of $g$ [10].
In Kenmotsu manifolds the following relations hold [10]:

\[
(\nabla_X \eta)Y = g(\phi X, \phi Y) \tag{10}
\]

\[
g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{11}
\]

\[
R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{12}
\]

\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{13}
\]

\[
R(\xi, X)\xi = X - \eta(X)\xi, \tag{14}
\]

\[
S(X, \xi) = -(n - 1)\eta(X), \tag{15}
\]

\[
S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y) \tag{16}
\]

for any vector fields \(X, Y\) and \(Z\), where \(R\) and \(S\) are the curvature and Ricci tensors of \(M\), respectively.

A Kenmotsu manifold \(M\) is said to be generalized \(\eta\) Einstein if its Ricci tensor \(S\) is of the form

\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + cg(\phi X, Y), \tag{17}
\]

for any \(X, Y \in \Gamma(TM)\), where \(a\), \(b\) and \(c\) are scalar functions such that \(b \neq 0\) and \(c \neq 0\). If \(c = 0\) then \(M\) is called \(\eta\) Einstein manifold.

### 3 Generalized Symmetric Metric Connection in a Kenmotsu manifold

Let \(\nabla\) be a linear connection and \(\nabla\) be a Levi-Civita connection of an almost contact metric manifold \(M\) such that

\[
\nabla_X Y = \nabla_X Y + H(X, Y), \tag{18}
\]

for any vector field \(X\) and \(Y\). Where \(H\) is a tensor of type (1, 2). For \(\nabla\) to be a generalized symmetric metric connection of \(\nabla\), we have

\[
H(X, Y) = \frac{1}{2}[T(X, Y) + T'(X, Y) + T'(Y, X)], \tag{19}
\]

where \(T\) is the torsion tensor of \(\nabla\) and

\[
g(T'(X, Y), Z) = g(T(Z, X), Y). \tag{20}
\]

From (11) and (20) we get

\[
T'(X, Y) = \alpha\{\eta(X)Y - g(X, Y)\xi\} + \beta\{-\eta(X)\phi Y - g(\phi X, Y)\xi\}. \tag{21}
\]

Using (11), (19) and (21) we obtain

\[
H(X, Y) = \alpha\{\eta(Y)X - g(X, Y)\xi\} + \beta\{-\eta(X)\phi Y\}. \tag{22}
\]
Corollary 1 For a Kenmotsu manifold, generalized symmetric metric connection $\nabla$ is given by

$$\nabla_X Y = \nabla_X Y + \alpha \{\eta(Y)X - g(X, Y)\xi\} - \beta \eta(X)\phi Y. \tag{23}$$

If we choose $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, generalized metric connection is reduced a semi-symmetric metric connection and quarter-symmetric metric connection as follows:

$$\nabla_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \tag{24}$$

$$\nabla_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{25}$$

From (23) we have the following proposition

Proposition 2 Let $M$ be a Kenmotsu manifold with generalized metric connection. We have the following relations:

$$\nabla_X \phi Y = (\alpha + 1)\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{26}$$

$$\nabla_X \xi = (\alpha + 1)\{X - \eta(X)\xi\}, \tag{27}$$

$$\nabla_X \eta Y = (\alpha + 1)\{g(X, Y) - \eta(Y)\eta(X)\}, \tag{28}$$

for any $X, Y, Z \in \Gamma(TM)$.

4 Curvature Tensor

Let $M$ be an $n-$ dimensional Kenmotsu manifold. The curvature tensor $\overline{R}$ of the generalized metric connection $\nabla$ on $M$ is defined by

$$\overline{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \tag{29}$$

Using proposition 68 from (23) and (29) we have

$$\overline{R}(X, Y)Z = R(X, Y)Z + \{(-\alpha^2 - 2\alpha)g(Y, Z) + (\alpha^2 + \alpha)\eta(Y)\eta(Z)\}X$$

$$+\{(\alpha^2 + 2\alpha)g(X, Z) + (-\alpha^2 - \alpha)\eta(X)\eta(Z)\}Y$$

$$+\{(\alpha^2 + \alpha)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + (\beta + \alpha\beta)[g(X, \phi Z)\eta(Y) - g(Y, \phi Z)\eta(X)]\}\xi$$

$$+(\beta + \alpha\beta)\eta(\eta(Y)\eta(Z)\phi X - (\beta + \alpha\beta)\eta(X)\eta(Z)\phi Y \tag{30}$$

is the curvature tensor with respect to the Levi-Civita connection $\nabla$. Using (30) and the first Bianchi identity we have

$$\overline{R}(X, Y)Z + \overline{R}(Y, Z)X + \overline{R}(Z, X)Y = 2(\beta + \alpha\beta)\{\eta(X)g(\phi Y, Z) + \eta(Y)g(X, \phi Z) + \eta(Z)g(Y, \phi X)\}. \tag{32}$$

Hence we have the following proposition
Proposition 3 Let $M$ be an $n$–dimensional Kenmotsu manifold with generalized symmetric metric connection of type $(\alpha, \beta)$. If $(\alpha, \beta) = (−1, \beta)$ or $(\alpha, \beta) = (\alpha, 0)$ then the first Bianchi identity of the generalized symmetric metric connection $\nabla$ on $M$ is provided.

Using (11), (12), (13), (14) and (30) we give the following proposition:

Proposition 4 Let $M$ be an $n$–dimensional Kenmotsu manifold with generalized symmetric metric connection of type $(\alpha, \beta)$. Then we have the following equations:

$$R(\xi, Y)\xi = (\alpha + 1)\{\eta(Y)\xi - \eta(\xi)Y + \beta[\eta(Y)x - \eta(X)\phi Y]\},$$  \hspace{1cm} (33)$$

$$R(\xi, X)Y = (\alpha + 1)\{\eta(Y)\xi - g(X,Y)\xi + \beta[\eta(Y)\phi X - g(X,\phi Y)x]\},$$  \hspace{1cm} (34)$$

$$R(\xi, Y)\xi = (\alpha + 1)\{Y - \eta(\xi)\xi - \beta\phi Y\},$$  \hspace{1cm} (35)$$

$$\eta(\nabla(X, Y)Z) = (\alpha + 1)\{\eta(Y)g(X, Z) - \eta(X)g(Y, Z) + \beta[\eta(Y)g(X, \phi Z) - \eta(Y)g(Y, \phi Z)]\}$$  \hspace{1cm} (36)$$

for any $X, Y, Z \in \Gamma(TM)$.

The Ricci tensor $\overline{\nabla}$ and the scalar curvature $\overline{\tau}$ of a Kenmotsu manifold with generalized symmetric metric connection $\nabla$ is given by

$$\overline{\tau}(X, Y) = \sum_{i=1}^{n} g(\overline{\nabla}(e_i, X)Y, e_i),$$

$$\overline{\tau} = \sum_{i=1}^{n} \overline{\tau}(e_i, e_i),$$

where $X, Y \in \Gamma(TM)$, $\{e_1, e_2, ..., e_n\}$ is an orthonormal frame. Then by using (11) and (30) we get

$$\overline{\tau}(Y, Z) = S(Y, Z) + (2n\alpha^2 + (3 - 2n)\alpha)g(Y, Z) + (n - 2)(\alpha^2 + \alpha)\eta(Y)\eta(Z)$$

$$-(\beta + \alpha\beta)g(Y, \phi Z),$$  \hspace{1cm} (37)$$

where $S$ is the Ricci tensor with respect to Levi-Civita connection.

Ricci tensor of the Levi-Civita connection is symmetric, thus from (37) we get

$$\overline{\tau}(Y, Z) - \overline{\tau}(Z, Y) = -2(\beta + \alpha\beta)g(Y, \phi Z).$$  \hspace{1cm} (38)$$

Thus we have the following theorem

Theorem 5 Let $M$ be an $n$–dimensional Kenmotsu manifold. Then the Ricci tensor $\overline{\nabla}$ of generalized symmetric metric connection $\nabla$ is symmetric if and only if $(\alpha, \beta) = (−1, \beta)$ or $(\alpha, \beta) = (\alpha, 0)$.

Using (37) we get the following theorem
**Theorem 6** Let $M$ be an $n$-dimensional Kenmotsu manifold with generalized symmetric metric connection of type $(\alpha, \beta)$. Then the scalar curvature with respect to generalized symmetric metric connection as follows:

$$r = r + (n - 2)(1 - n)\alpha^2 - 2(n - 1)^2 \alpha,$$

where $r$ is scalar curvature of Levi-Civita connection.

**Lemma 7** Let $M$ be an $n$-dimensional Kenmotsu manifold with generalized symmetric metric connection of type $(\alpha, \beta)$. Then we have:

$$S(Y, \xi) = (1 - n)(\alpha + 1)\eta(Y),$$

$$S(\phi Y, \phi Z) = S(Y, Z) + (n - 1)(1 + \alpha),$$

for any $X, Y \in \Gamma(\mathcal{TM})$.

**Proof.** From (4), (15) and (37) we have (40). By using (6), (16) and (37) we obtain (41).

**Theorem 8** Let $M$ be an $n$-dimensional Kenmotsu manifold with generalized symmetric metric connection of type $(\alpha, \beta)$. If $M$ is Ricci semi-symmetric with respect to generalized symmetric metric connection, then $M$ is an $\eta$ Einstein manifold with respect to generalized symmetric metric connection ($\beta \neq 1, \beta \neq -1$).

**Proof.** Let $\mathcal{R}(X, Y)\mathcal{S} = 0$ be on $M$ for any $X, Y, Z, U \in \Gamma(\mathcal{TM})$, then we have

$$S(\mathcal{R}(X, Y)Z, U) + S(Z, \mathcal{R}(X, Y)U) = 0.$$ 

If we choose $Z = \xi$ and $X = \xi$ in (42), we get

$$S(\mathcal{R}(\xi, Y)\xi, U) + S(\xi, \mathcal{R}(\xi, Y)U) = 0.$$ 

Using (34), (35) and (40) in (43), we obtain

$$S(Y, U) - \beta S(\phi Y, U) = (1 - n)\{g(Y, U) + \beta g(Y, \phi U)\}. $$

If one substitutes $Y = \phi Y$ in the equation (44) and using (40), we get

$$S(\phi Y, U) + \beta S(Y, U) = (1 - n)\{g(\phi Y, U) + \beta g(Y, U) + (\alpha - \beta + 1)\eta(Y)\eta(U)\}. $$

From the (44) and (45), we obtain

$$S(Y, U) = \frac{1 - n}{1 - \beta^2}\{(1 + \beta^2)g(Y, U) + (\alpha - \beta + 1)\eta(Y)\eta(U)\}. $$

This equation tell us $M$ is an $\eta$ Einstein manifold with respect to generalized metric connection.

**Corollary 9** Let $M$ be an $n$-dimensional Kenmotsu manifold with generalized symmetric metric connection of type $(\alpha, \beta)$. If $M$ is Ricci semi-symmetric with respect to generalized symmetric metric connection, then we have the following statements:

(i) $M$ is reduces to an Einstein manifold with respect to generalized symmetric metric connection of type $(\beta - 1, \beta)$.

(ii) There is no Einstein manifold with generalized symmetric metric connection of type $(\alpha, 1)$ and $(\alpha, -1)$.
5 Projective Curvature Tensor

Let $M$ be an $n-$ dimensional Kenmotsu manifold with generalized symmetric metric connection of type $(\alpha, \beta)$. The projective curvature tensor $\mathcal{P}$ of type $(1, 3)$ of $M$ with respect to generalized symmetric metric connection $\nabla$ is defined by

$$\mathcal{P}(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-1}\{\mathcal{S}(Y, Z)X - \mathcal{S}(X, Z)Y\}. \quad (47)$$

**Definition 10** Let $M$ be an $n-$ dimensional Kenmotsu manifold. Then $M$ is said to be $\xi-$ projectively flat with respect to generalized symmetric metric connection $\nabla$ if $\mathcal{P}(X, Y)\xi = 0$ on $M$.

Using (33), (40) and (47) we have

$$\mathcal{P}(X, Y)\xi = (\alpha + 1)\beta\{\eta(Y)\phi X - \eta(X)\phi Y\}. \quad (48)$$

Then we have the following theorem

**Theorem 11** Let $M$ be an $n-$ dimensional Kenmotsu manifold. Then $M$ is $\xi-$ projectively flat with respect to generalized symmetric metric connection if and only if $(\alpha, \beta) = (-1, \beta)$ or $(\alpha, \beta) = (\alpha, 0)$.

**Corollary 12** Let $M$ be an $n-$ dimensional Kenmotsu manifold. Then we have the following expressions:

(i) The manifold is $\xi-$ projectively flat with respect to $\beta-$ quarter symmetric metric connection.

(ii) The manifold is $\xi-$ projectively flat with respect to semi-symmetric metric connection.

**Definition 13** Let $M$ be an $n-$ dimensional Kenmotsu manifold. Then $M$ is said to be $\phi-$ projectively flat with respect to generalized symmetric metric connection if $g(\mathcal{P}(\phi X, \phi Y)\phi Z, \phi U) = 0$ on $M$.

From (47)

$$\mathcal{P}(\phi X, \phi Y)\phi Z = \mathcal{R}(\phi X, \phi Y)\phi Z - \frac{1}{n-1}\{\mathcal{S}(\phi Y, \phi Z)X - \mathcal{S}(\phi X, \phi Z)Y\}. \quad (48)$$

Using (41) and (48), $\phi-$ projectively flatness implies

$$\mathcal{K}(\phi X, \phi Y, \phi Z, \phi U) = \frac{1}{n-1}\{[\mathcal{S}(Y, Z) + (n-1)(1+\alpha)]g(\phi X, \phi U)$$

$$+ [\mathcal{S}(X, Z) + (n-1)(1+\alpha)]g(\phi Y, \phi U). \quad (49)$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthogonal basis of the vector fields in $M$ and using the fact that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthogonal basis, putting $X = U = e_i$ and summing up with respect to $i = 1, 2, ..., n-1$ we obtain

$$\sum_{i=1}^{n-1} \mathcal{K}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \frac{1}{n-1}\{[\mathcal{S}(Y, Z) + (n-1)(1+\alpha)]g(\phi e_i, \phi e_i)$$

$$+ [\mathcal{S}(e_i, Z) + (n-1)(1+\alpha)]g(\phi Y, \phi e_i). \quad (50)$$
We know that
\[
\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \quad (51)
\]
and from (41)
\[
\sum_{i=1}^{n-1} S(e_i, Z)g(\phi Y, \phi e_i) = S(Y, Z) + (n - 1)(1 + \alpha)(1 - \sum_{i=1}^{n-1} g(\phi Y, \phi e_i)). \quad (52)
\]
Thus from (51) and (52), the equation (50) becomes
\[
\sum_{i=1}^{n-1} K(\phi e_i, \phi Y, \phi Z, \phi e_i) = n - 2n - 1 S(Y, Z) + (n - 1)(1 + \alpha). \quad (53)
\]
Moreover we have
\[
\sum_{i=1}^{n-1} K(\phi e_i, \phi Y, \phi Z, \phi e_i) = S(\phi Y, \phi Z) - g(R(\xi, \phi Y)\phi Z, \xi). \quad (54)
\]
Using (34), (41) and (54) we get
\[
\sum_{i=1}^{n-1} K(\phi e_i, \phi Y, \phi Z, \phi e_i) = S(Y, Z) + (\alpha + 1)\{n - 1 + g(\phi Y, \phi Z) + \beta g(\phi Y, Z)\} \quad (55)
\]
Therefore by using (53) and (55) we obtain
\[
S(Y, Z) = (1 - n)(\alpha + 1)\{g(Y, Z) - \eta(Y)\eta(Z) + \beta g(\phi Y, Z)\} \quad (56)
\]
Hence we have the following theorem

**Theorem 14** If a Kenmotsu manifold is \(\phi\)— projectively flat with respect to generalized symmetric metric connection, then the manifold is generalized \(\eta\) Einstein manifold with respect to generalized symmetric metric connection.

**Corollary 15** Let \(M\) be an \(n\)— dimensional Kenmotsu manifold. If the manifold is \(\phi\)— projectively with respect to generalized symmetric metric connection, then we have the following expressions:

(i) The manifold is \(\eta\)— Einstein manifold with respect to \(\beta\)— quarter symmetric metric connection.

(ii) The manifold is Ricci flat with respect to generalized symmetric metric connection of type \((-1, \beta)\).

(iii) The manifold is Ricci flat with respect to semi-symmetric metric connection.
6 Concircular Curvature Tensor

Let $M$ be an $n-$ dimensional Kenmotsu manifold. The concircular curvature tensor of $M$ with respect to generalized symmetric metric connection $\nabla$ is defined by

$$C^\ast(X, Y)Z = R(X, Y)Z - \frac{\tau}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\}$$

(57)

By using (30), (39) and (57) we get

$$g(C^\ast(X, Y)Z, \xi) = g(C^\ast(X, Y)Z, \xi) + \{(\beta + \alpha)g(Y, Z) - \eta(Y)\eta(Z)\}$$

(58)

where $C^\ast$ is concircular curvature tensor of $M$ with respect to Levi-Civita connection. If we consider $C^\ast = C^\ast$, then putting $X = \xi$ of the equation (58) we have

$$\{(\beta + \alpha)g(Y, Z) = (\beta + \alpha)g(Y, \phi Z).$$

(59)

From (37), (58) and (59) we have the following theorem

**Theorem 16** In a Kenmotsu manifold, if concircular curvature tensor is invariant under generalized metric connection, then we have

$$S(Y, Z) = S(Y, Z) - \{(4+2n)^2 + (6+4n)\alpha\}g(Y, Z) + \{(2n-4)\alpha^2 + (3n-5)\alpha\}g(Y, Z)$$

for any $X, Y \in \Gamma(TM)$.

**Corollary 17** Let $M$ be an $n-$ dimensional Kenmotsu manifold. If concircular curvature tensor is invariant under generalized symmetric metric connection, then we have the following expressions:

(i) If the manifold is Ricci flat, then manifold is $\eta-$ Einstein manifold with respect to generalized symmetric metric connection.

(ii) Ricci tensor is invariant with respect to $\beta-$ quarter symmetric metric connection.

7 Example

We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Let $E_1, E_2, E_3$ be a linearly independent global frame on $M$ given by

$$E_1 = x \frac{\partial}{\partial z}, \quad E_2 = z \frac{\partial}{\partial y}, \quad E_3 = -x \frac{\partial}{\partial x}.$$ 

(60)
Let $g$ be the Riemannian metric defined by
\[ g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0, g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1, \]
Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_1)$, for any $U \in TM$. Let $\phi$ be the $(1,1)$ tensor field defined by $\phi E_1 = E_2, \phi E_2 = -E_1$ and $\phi E_3 = 0$. Then, using the linearity of $\phi$ and $g$ we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any $U, W \in TM$. Thus for $E_3 = \xi$, $(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.

Let $\nabla$ be the Levi-Civita connection with respect to the Riemannian metric $g$. Then we have
\[ [E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2, \quad (61) \]
Using Koszul formula for the Riemannian metric $g$, we can easily calculate
\[
\begin{align*}
\nabla_{E_1} E_1 &= -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = E_1, \\
\nabla_{E_2} E_1 &= 0, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_2} E_3 = 0, \\
\nabla_{E_3} E_1 &= 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.
\end{align*}
\]
From the above relations, it can be easily seen that
\[
(\nabla_X \phi) Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad \nabla_X \xi = X - \eta(X)\xi, \text{ for all } E_3 = \xi.
\]
Thus the manifold $M$ is a Kenmotsu manifold with the structure $(\phi, \xi, \eta, g)$. Using the above relations, we can calculate the non-vanishing components of the curvature tensor as follows:
\[
\begin{align*}
R(E_1, E_2)E_1 &= E_2, \quad R(E_1, E_2)E_2 = -E_1, \quad R(E_1, E_3)E_1 = E_3, \\
R(E_1, E_3)E_3 &= -E_1, \quad R(E_2, E_3)E_2 = E_3, \quad R(E_2, E_3)E_3 = -E_2
\end{align*}
\]
From the equations $\quad (63)$ we can easily calculate the non-vanishing components of the Ricci tensor as follows:
\[ S(E_1, E_1) = -2, \quad S(E_2, E_2) = -2, \quad S(E_3, E_3) = -2 \quad (64) \]
Now, we can make similar calculations for generalized metric connection. Using $\quad (23)$ in the above equations, we get
\[
\begin{align*}
\nabla_{E_1} E_1 &= -(1 + \alpha)E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = (1 + \alpha)E_1, \\
\nabla_{E_2} E_1 &= 0, \quad \nabla_{E_2} E_2 = -(1 + \alpha)E_3, \quad \nabla_{E_2} E_3 = \alpha E_2, \\
\nabla_{E_3} E_1 &= -\beta E_2, \quad \nabla_{E_3} E_2 = \beta E_1, \quad \nabla_{E_3} E_3 = 0.
\end{align*}
\]
From the equations $\quad (65)$ we can calculate the non-vanishing components of curvature tensor with respect to generalized metric connection as follows:
\[
\begin{align*}
\overline{R}(E_1, E_2)E_1 &= (1 + \alpha)^2 E_2, \quad \overline{R}(E_1, E_2)E_2 = -(1 + \alpha)^2 E_1, \\
\overline{R}(E_1, E_3)E_1 &= (1 + \alpha)E_3, \quad \overline{R}(E_1, E_3)E_3 = (1 + \alpha)(\beta E_2 - E_1), \\
\overline{R}(E_2, E_3)E_2 &= (1 + \alpha)E_3, \quad \overline{R}(E_2, E_3)E_3 = -(1 + \alpha)(-\beta E_1 + E_2), \\
\overline{R}(E_3, E_2)E_1 &= -(1 + \alpha)\beta E_3, \quad \overline{R}(E_3, E_3)E_2 = (1 + \alpha)\beta E_3.
\end{align*}
\]
From (66), the non-vanishing components of the Ricci tensor as follows:

\[ \mathcal{S}(E_1, E_1) = -(1 + \alpha)(2 + \alpha), \quad \mathcal{S}(E_2, E_2) = -(1 + \alpha)(2 + \alpha), \]
\[ \mathcal{S}(E_3, E_3) = -2(1 + \alpha), \quad \mathcal{S}(E_1, E_2) = -(1 + \alpha)\beta. \]  

(67)

The equations (66) and (67) verify (30) and (37), respectively. Moreover, the scalar curvature with respect to the Levi-Civita connection and quarter symmetric non-metric connection are \( r = -6 \) and \( \mathcal{P} = -2(1 + \alpha)(3 + \alpha) \).

From (67), we get the following expressions:

\[ \mathcal{P}(E_1, E_3)E_3 = (1 + \alpha)\beta E_2, \quad \mathcal{P}(E_2, E_3)E_3 = -(1 + \alpha)\beta E_1, \quad \mathcal{P}(E_1, E_2)E_3 = 0. \]  

(68)

This expressions are verifying the theorem 5, theorem 6 and theorem 11.

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http://dx.doi.org/10.4134/BKMS.b150590 pISSN: 1015-8634 / eISSN: 2234-3016

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