On the volume growth of the hyperbolic regular $n$-simplex

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Abstract. In this paper we give lower and upper bounds for the volume growth of a regular hyperbolic simplex, namely for the ratio of the $n$-dimensional volume of a regular simplex and the $(n-1)$-dimensional volume of its facets. In addition to the methods of U. Haagerup and M. Munkholm we use a third volume form based on the hyperbolic orthogonal coordinates of a body. In the case of the ideal, regular simplex our upper bound gives the best known upper bound. On the other hand, also in the ideal case our general lower bound, improved the best known one for $n = 3$.

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1. Introduction

1.1. The problem

The volume growth of the Euclidean regular $n$-simplex inscribed in the unit sphere can be calculated by the volume form

$$V_n(S) = \frac{m}{n} V_{n-1}(F),$$

where $V_n(S), V_{n-1}(F)$ and $m$ mean the $n$-dimensional volume of the simplex, the $(n-1)$-dimensional volume of its facet $F$ and the height of the simplex, respectively. Thus we have $V_n(S)/V_{n-1}(F) = (n+1)/n^2$. If $r$ is the radius of the circumscribed sphere of the regular simplex $rS$ then by similarity the ratio $V_n(rS)/V_{n-1}(rF)$ is equal to $r(n+1)/n^2$. In summary, the Euclidean “volume growth” is not an interesting quantity, it can be determined exactly and easily.

In contrast to the Euclidean case in hyperbolic $n$-space there is at least one kind of regular simplices whose volume growth is very important. This is the regular, ideal simplex. In hyperbolic $n$-space a simplex is called regular if
any permutation of its vertices can be induced by an isometry of the space. A simplex is ideal if all the vertices are on the sphere at infinity. There is only one ideal regular simplex in hyperbolic \( n \)-space up to isometry. It is known that the volume of a hyperbolic simplex is finite even if some of the vertices are on the sphere at infinity. Contrary to the case of the plane, there are non-congruent ideal simplices raising an interesting question: Which one has maximal volume? The answer that \textit{a simplex is of maximal volume if and only if it is ideal and regular} was conjectured (for any \( n \)) by Thurston [16]. In three space it was proved by Milnor (see in [12] or [13]). In higher dimensional spaces this result was published by Haagerup and Munkholm [9]. Their proof is based on the “volume growth” of an ideal, regular simplex, Proposition 2 in [9] says that \( (n-2)/(n-1)^2 \leq V_n(S)/V_{n-1}(F) \leq 1/(n-1) \). These bounds rely on an interplay between the volume forms of Poincare’s half-space model and the volume form of Cayley–Klein’s projective model using also Gauss’ divergence formula. Unfortunately, in hyperbolic \( n \)-space there is no formula analogous to (1), hence to compare the \( n \)-volume of a regular simplex and the \((n-1)\)-volume of its facets is a non-trivial exercise.

In recent decades the analytic investigations in hyperbolic space and even more the examination of computational methods of the \( n \)-dimensional hyperbolic volume proliferated. Without giving an exhaustive list I suggest studying the references [2–8,10,11,15]. Investigations of volume and area in a more general context can be found in [1]. Especially, the so-called Funk metrics come from the generalization of the Cayley–Klein model of the hyperbolic geometry, see [17] and [18].

In this paper we give estimations for the volume growth of a regular hyperbolic simplex in the above detailed sense. In addition to the methods of Haagerup and Munkholm we use a third volume form based on hyperbolic orthogonal coordinates. In the case of the ideal, regular simplex our upper bound gives the best known upper bound proved by Haagerup and Munkholm in 1981. On the other hand, also in the ideal case our general lower bound, improved the best known one for \( n=3 \).

1.2. Notation

We use the following notation in this paper:

- \( \mathbb{R}, E^n \) and \( H^n \): the set of real numbers, the Euclidean \( n \)-space and the hyperbolic \( n \)-spaces, respectively,
- \( | \cdot |, \rho(\cdot, \cdot) \): the Euclidean length and the hyperbolic distance function, respectively,
- \( V(\cdot) \): the volume function of the hyperbolic space,
$P^n$, $h : H^n \rightarrow P^n$: the half-space model of $H^n$ (Poincare’s second model) and the standard mapping sending the hyperbolic space onto the model, respectively,
- $H$: the boundary hyperplane of $P^n$,
- $CK^n$ and $p : H^n \rightarrow CK^n$: the projective model of $H^n$ (Cayley–Klein’s model) and the standard mapping on the hyperbolic space to the model
- $x_1, \ldots, x_n$: the Euclidean coordinates of the embedding Euclidean space $E^n$ with respect to an orthonormed basis in $E^n$,
- $S(n) = \text{conv}\{E_1, \ldots, E_{n+1}\}$: the Euclidean $n$-dimensional regular simplex inscribed in the unit sphere,
- $\tau[n, t] = \text{conv}\{p^{-1}(E_1), \ldots, p^{-1}(E_{n+1})\}$: the hyperbolic regular $n$-simplex with hyperbolic circumradius $r(t) = \tanh^{-1}(\sin t)$,
- $\tau_i[n, t] = \text{conv}\{p^{-1}(E_1), \ldots, p^{-1}(E_{i-1}), p^{-1}(E_{i+1}), \ldots, p^{-1}(E_{n+1})\}$: a facet of $\tau[n, t]$, it is a hyperbolic, regular simplex of dimension $n-1$.
- $r_k$: the hyperbolic radius of the circumscribed sphere of the $k$-dimensional faces of $\tau[n, t]$,
- $d_k$: the hyperbolic distance of the circumcenter of $\tau[n, t]$ and a $(k-1)$-dimensional face of $\tau[n, t]$.

2. The theorem

In this paper we prove the following theorem:

**Theorem 1.** We have the following two inequalities

$$\left(\frac{n+1}{2}\right)^{n-1} \frac{\sqrt{1 - \sin^2 t \frac{n-1}{2n}}}{\sqrt{1 - \frac{n-1}{2n} \sqrt{n^2 - \sin^2 t}}} (\cos t) \tanh^{-1}\left(\frac{\sin t \sqrt{1 - \frac{n-1}{2n}}}{\sqrt{1 - \sin^2 t \frac{n-1}{2n}}}\right) \leq \frac{V(\tau[n, t])}{V(\tau_i[n, t])}$$

and

$$\frac{V(\tau[n, t])}{V(\tau_i[n, t])} \leq \frac{1}{n-1} \left(1 - \left(\frac{n^2 (1 - \sin t)^2 (1 + \sin t)}{(n + \sin t)^2 (1 + \sin t) - (n^2 - 1) \sin^2 t (1 - \sin t)^2}\right)^{n-1}\right).$$

**Note 1.** For $n = 3$ and $t = \pi/2$, (2) is stronger than the left hand side of (3.1) in [9]. In fact,

$$\lim_{t \to \pi/2} (\cos t) \tanh^{-1}\left(\frac{\sin t \sqrt{1 - \frac{n-1}{2n}}}{\sqrt{1 - \sin^2 t \frac{n-1}{2n}}}\right) = \lim_{t \to \pi/2} (\cos t) \tanh^{-1}(\sin t) = 1,$$

hence we have for $n = 3$ that

$$\frac{n - 2}{(n - 1)^2} = \frac{1}{4} \leq \frac{4}{4\sqrt{8}} = \frac{n + 1}{2^{n-1} \sqrt{n^2 - 1}} \leq \frac{1}{n - 1} = \frac{1}{2}.$$
Consequently, in dimension three our result is a generalization of the inequalities of [9].

**Note 2.** In the case of $t = \pi/2$, (3) gives the same bound as we saw in (3.1) in [9]. Really,

$$
\frac{n}{n-1} \left( 1 - \left( \frac{n^2(1 - \sin t)^2(1 + \sin t)}{(n + \sin t)^2(1 + \sin t) - (n^2 - 1) \sin^2 t(1 - \sin t)^2} \right)^{n-1} \right),
$$

if $t = \pi/2$, is equal to $\frac{n}{n-1}$ (in all dimensions).

### 3. The proof of the theorem

The proof can be divided into three steps.

The *first step* uses calculations which can be got from the Cayley–Klein (or projective) model. We determine certain metric properties of the regular hyperbolic simplex of circumradius $r_n$.

In the *second step* using hyperbolic orthogonal coordinates we prove the general lower bound.

The *third step* contains the proof of the upper bound. In this section we use Poincare’s half-space model.

For good readability these steps can be found in three subsection and the result is the union of the statements of Lemmas 1 and 2.
3.1. Calculations in the projective model

We consider the map \( p : H^n \to CK^n \) sending the regular ideal simplex \( \tau[n, \pi/2] \) into the Euclidean regular simplex \( p(\tau[n, \pi/2]) \) inscribed in the unit sphere. (Clearly, the map \( p \) can be given concretely but it is not important for our purpose.) For every \( t \in [0, \pi/2] \), \( p(\tau[n, t]) \) denotes the regular simplex concentric with \( p(\tau[n, \pi/2]) \) and with circumradius \( r = \sin t \). For simplicity \( S(n) \) denotes the regular simplex \( p(\tau[n, \pi/2]) \). Observe that the volume of \( \tau[n, t] \) can be calculated from the volume form of \( CK^n \) (see in [5])

\[
dV = (1 - r^2)^{-(n+1)/2} \, dx_1 \ldots dx_n, \quad r^2 := \sum_{i=1}^{n} x^2_i,
\]

and we get that

\[
V(\tau[n, t]) = \int_{(\sin t)S(n)} (1 - r^2)^{-\frac{n+1}{2}} \, dr = \sin^n t \int_{S(n)} (1 - (\sin^2 t) r^2)^{-\frac{n+1}{2}} \, dr.
\]

By spherical symmetry we can choose a coordinate system in such a way, that \( E_{n+1} = (0, \ldots, 0, \sin t)^T \). Denote by \( O \) and \( K_{n-1} \) the center of the simplex \( S(n) \) and the center of its facet \( \text{conv}\{E_1, \ldots, E_n\} \), respectively. Then \( |OK_{n-1}| = \sin t/n \) and the hyperbolic distance of the corresponding points is

\[
\rho(p^{-1}(O), p^{-1}(K_{n-1})) = \frac{1}{2} \ln \frac{n + \sin t}{n - \sin t} = \tanh^{-1}(\sin t).
\]

Immediate calculation with cross-ratio gives the circumradius of the facets \( \tau_i[n, t] \) \((i = 1, \ldots, n + 1)\) of \( \tau[n, t] \):

\[
\rho(p^{-1}(E_1), p^{-1}(K_{n-1})) = \frac{\ln \sqrt{n^2 - \sin^2 t} + \sin t \sqrt{n^2 - 1}}{n \cos t}.
\]

Denote by \( K_i \) the center of the face \( \text{conv}\{E_1, E_2, \ldots, E_{i+1}\} \) for \( i = 1, \ldots, n - 1 \). Then the fundamental orthoseme \( O \) of the simplex \( \tau[n, t] \) is the convex hull of the points \( \{E_1, K_1, K_2, \ldots, K_{n-1}, O\} \). Clearly \( \tau[n, t] \) is the disjoint union of \((n+1)!\) congruent copies of \( O \) hence \( V(\tau[n, t]) = (n+1)!V(O) \). The successive edge lengths of \( O \) can be determined by induction. As we saw

\[
r_n := \rho(p^{-1}(O), p^{-1}(E_1)) = \tanh^{-1}(\sin t) \quad \text{and} \quad d_n := \rho(p^{-1}(O), p^{-1}(K_{n-1})) = \tanh^{-1}\left(\frac{\sin t}{n}\right)
\]
\[ r_{n-1} := \rho(p^{-1}(K_{n-1}), p^{-1}(E_1)) = \ln \frac{\sqrt{n^2 - \sin^2 t + \sin t\sqrt{n^2 - 1}}}{n \cos t} \]

\[ = \frac{1}{2} \ln \frac{1 - \sin^2 t \left(\frac{1}{n^2}\right) + \sin t\sqrt{1 - \frac{1}{n^2}}}{\sqrt{1 - \sin^2 t \left(\frac{1}{n^2}\right)} - \sin t\sqrt{1 - \frac{1}{n^2}}} = \tanh^{-1} \left(\frac{\sin t\sqrt{1 - \frac{1}{n^2}}}{\sqrt{1 - \sin^2 t \left(\frac{1}{n^2}\right)}}\right) \]

and

\[ d_{n-1} := \rho(p^{-1}(K_{n-1}), p^{-1}(K_{n-2})) = \frac{1}{2} \ln \frac{1 - \sin^2 t \left(\frac{1}{n^2}\right) + \frac{1}{(n-1)} \sin t\sqrt{1 - \frac{1}{n^2}}}{\sqrt{1 - \sin^2 t \left(\frac{1}{n^2}\right)} - \frac{1}{(n-1)} \sin t\sqrt{1 - \frac{1}{n^2}}} \]

\[ = \tanh^{-1} \left(\frac{\sin t\sqrt{1 - \frac{1}{n^2}}}{(n-1)\sqrt{1 - \sin^2 t \left(\frac{1}{n^2}\right)}}\right). \]

It can be proved by induction that

\[ r_{n-k} := \rho(p^{-1}(K_{n-k}), p^{-1}(E_1)) = \tanh^{-1} \left(\frac{\sin t\sqrt{1 - \frac{k}{n(n-k+1)}}}{\sqrt{1 - \sin^2 t \left(\frac{k}{n(n-k+1)}\right)}}\right) \] (5)

holds for \( k = 1, \ldots, (n-1) \) and

\[ d_{n-k} := \rho(p^{-1}(K_{n-k}), p^{-1}(K_{n-k-1})) = \tanh^{-1} \left(\frac{\sin t\sqrt{1 - \frac{k}{n(n-k+1)}}}{(n-k)\sqrt{1 - \sin^2 t \left(\frac{k}{n(n-k+1)}\right)}}\right) \] (6)

holds also for \( k = 1, \ldots, (n-1) \) if \( K_0 \) means \( E_1 \).

With respect to an \((n-1)\)-dimensional projective model we can determine the volume of a facet of the regular \( n \)-simplex with circumradius \( r_n \). In fact, the circumradius of its facets is \( r_{n-1} \) hence using the formula (4) we get

\[ V(\tau_1[n,t]) = \int_{(\tanh r_{n-1})S(n-1)} (1-r^2)^{-\frac{n}{2}} \, dr = \int_{\frac{\sin t\sqrt{1 - \frac{1}{n^2}}}{\sqrt{1 - \sin^2 t \left(\frac{1}{n^2}\right)}}S(n-1)} \left(\frac{1}{\sqrt{1-r^2}}\right)^n \, dr. \] (7)
3.2. Hyperbolic orthogonal coordinates and the lower bound

Following the method of [5] (see paragraph 3.3.3) we can express the volume in terms of hyperbolic orthogonal coordinates as follows:

\[
V(O) = \int_0^{\alpha_1} \int_0^{\alpha_2} \cdots \int_0^{\alpha_{n-1}} (\cosh^{n-1} x_{n-1}) (\cosh^{n-2} x_{n-2}) \cdots (\cosh x_1) dx_{n-1} \cdots dx_1 dx_n,
\]

where

\[
\begin{align*}
\alpha_{n-1} &= \tanh^{-1}\left(\frac{\tanh d_n}{\sinh d_{n-1}} \sinh x_{n-2}\right), \\
\alpha_{n-2} &= \tanh^{-1}\left(\frac{\tanh d_{n-1}}{\sinh d_{n-2}} \sinh x_{n-3}\right), \\
&\vdots \\
\alpha_2 &= \tanh^{-1}\left(\frac{\tanh d_3}{\sinh d_2} \sinh x_1\right), \\
\alpha_1 &= \tanh^{-1}\left(\frac{\tanh d_2}{\sinh d_1} \sinh x_n\right), \\
\alpha_n &= d_1.
\end{align*}
\]

Observe that the volume of the facet \(O_{n-1} := \text{conv}\{p^{-1}(E_1), p^{-1}(K_1), p^{-1}(K_2) \cdots p^{-1}(K_{n-1})\}\) is

\[
V(O_{n-1}) = \int_0^{\alpha_1} \int_0^{\alpha_2} \cdots \int_0^{\alpha_{n-2}} (\cosh^{n-2} x_{n-2}) (\cosh x_1) dx_{n-2} \cdots dx_1 dx_n,
\]

with the same \(\alpha_i\)'s. Consequently these formulas lead to a connection between \(V(O)\) and \(V(O_{n-1})\) and between \(V(\tau[n, t]) = (n+1)!V(O)\) and \(V(\tau_1[n, t]) = n!V(O_{n-1})\). In the proof we use Chebyshev’s integral inequality (see in [14]), saying that if \(f, g : [a, b] \to \mathbb{R}\) have the same monotony then

\[
\int_a^b f(x)g(x)dx \geq \frac{1}{b-a} \int_a^b f(x)dx \int_a^b g(x)dx,
\]

and if \(f; g\) have opposite monotony, then the inequality should be reversed.
Lemma 1. For \( n \geq 3 \) we have

\[
V(\tau[n,t]) \geq (n+1) \left( \frac{1}{2} \right)^{n-1} \sqrt{1 - \frac{2}{n-1} \frac{\sin^2 t}{1 - \frac{1}{2n} \sqrt{n^2 - \sin^2 t}}} \frac{1}{(n-1)^{1/2}} \frac{1}{(1 - \frac{n-1}{2n} \frac{\sin^2 t}{n-1})} V(\tau_i[n,t]).
\]

(8)

Proof. We compare the following two integrals

\[
V(O) = \int_0^{\alpha_1} \cdots \int_0^{\alpha_{n-2}} \left( \int_0^{\alpha_{n-1}} \cosh^{n-1} x_{n-1} dx_{n-1} \right) \cosh^{-2} x_{n-2} \cdots \cosh x_1 dx_{n-2} \cdots dx_1 dx_n,
\]

and

\[
V(O_{n-1}) = \int_0^{\alpha_1} \cdots \int_0^{\alpha_{n-2}} \cosh^{n-2} x_{n-2} \cdots \cosh x_1 dx_{n-2} \cdots dx_1 dx_n.
\]

Since

\[
\cos^{2k}(x) = \frac{1}{4^k} \binom{2k}{k} + \frac{1}{2^{2k-1}} \sum_{l=0}^{k-1} \binom{2k}{l} \cos(2(k-l)x)
\]

\[
\cos^{2k+1} x = \frac{1}{4^k} \sum_{l=0}^{k} \binom{2k+1}{l} \cos((2(k-l)+1)x),
\]

using the identity \( \cosh x = \cos(ix) \) we have

\[
\cosh^{2k} x = \frac{1}{4^k} \binom{2k}{k} + \frac{1}{2^{2k-1}} \sum_{l=0}^{k-1} \binom{2k}{l} \cosh(2(k-l)x)
\]

\[
\cosh^{2k+1} x = \frac{1}{4^k} \sum_{l=0}^{k} \binom{2k+1}{l} \cosh((2(k-l)+1)x).
\]

Let \( F_{n-1}(x) \) denote the antiderivative of the functions \( \cosh^{n-1} x \). Then we have

\[
F_{n-1}(x) = \begin{cases} 
\frac{1}{4^k} \binom{2k}{k} x + \frac{1}{2^{2k-1}} \sum_{l=0}^{k-1} \binom{2k}{l} \frac{1}{2(k-l)} \sinh(2(k-l)x) & \text{if } n = 2k \\
\frac{1}{4^k} \sum_{l=0}^{k} \binom{2k+1}{l} \frac{1}{(2(k-l)+1)} \sinh((2(k-l)+1)x) & \text{if } n = 2k + 1.
\end{cases}
\]

(9)
Since $F_{n-1}(0) = 0$, we get

$$f_{n-1}(x_{n-2}) := \int_0^{\alpha_{n-1}} \cosh^{n-1}(x_{n-1}) dx_{n-1}$$

$$= \begin{cases} \frac{1}{4^k} \sum_{l=0}^{k-1} \left( \frac{2k}{l} \right) \alpha_{n-1}(x_{n-2}) + \frac{1}{2^{k-l}} \sum_{l=0}^{k-1} \left( \frac{2k}{l} \right) \frac{1}{2^{k-l}} \sinh(2(k-l)\alpha_{n-1}(x_{n-2})) & \text{if } n = 2k \\ \frac{1}{4^k} \sum_{l=0}^{k} \left( \frac{2k+1}{l} \right) \frac{1}{(2k-l+1)} \sinh((2(k-l)+1)\alpha_{n-1}(x_{n-2})) & \text{if } n = 2k + 1 \end{cases}$$

where

$$\alpha_{n-1}(x_{n-2}) = \tanh^{-1} \left( \frac{\tanh d_n}{\sinh d_{n-1}} \sinh x_{n-2} \right) = \tanh^{-1} \left( \frac{\sqrt{n-1}}{n+1} \sqrt{1 - \frac{2 \sin^2 t}{n(n-1)} \sinh x_{n-2}} \right).$$

Without loss of generality we can assume that $n = 2k + 1$, the other case when $n = 2k$ can be proved analogously leading to the same result. Since $f_{n-1}(x_{n-2})$ is strictly increasing and $\sinh x, \tanh^{-1}(x) \geq x$ for all positive $x$, we have

$$f_{n-1}(x_{n-2}) \geq \frac{1}{4^k} \sum_{l=0}^{k} \left( \frac{2k+1}{l} \right) \frac{1}{(2k-l+1)} (2(k-l) + 1) \alpha_{n-1}(x_{n-2})$$

$$= \alpha_{n-1}(x_{n-2})$$

$$= \tanh^{-1} \left( \frac{\tanh d_n}{\sinh d_{n-1}} \sinh x_{n-2} \right) \geq \frac{\tanh d_n}{\sinh d_{n-1}} \sinh x_{n-2},$$

implying that

$$V(O) = \int_0^{\alpha_n} \int_0^{\alpha_1} \cdots \int_0^{\alpha_2} \left( \int_0^{\alpha_{n-1}} \cosh^{n-1} x_{n-1} dx_{n-1} \right) \cosh^{n-2} x_{n-2} \cdots$$

$$\cosh x_1 dx_{n-2} \cdots dx_1 dx_n$$

$$\geq \frac{\tanh d_n}{\sinh d_{n-1}} \int_0^{\alpha_n} \int_0^{\alpha_1} \cdots \int_0^{\alpha_2} \sinh x_{n-2} \cosh^{n-2} x_{n-2} dx_{n-2}$$

$$\cosh^{n-3} x_{n-3} \cdots \cosh x_1 dx_{n-3} \cdots dx_1 dx_n.$$

Using Chebysev’s inequality we get that

$$\int_0^{\alpha_{n-2}} \sinh x_{n-2} \cosh^{n-2} x_{n-2} dx_{n-2} \geq \frac{\cos \alpha_{n-2} - 1}{\alpha_{n-2}} \int_0^{\alpha_{n-2}} \cosh^{n-2} x_{n-2} dx_{n-2}$$

$$\geq \frac{\alpha_{n-2}}{2} \int_0^{\alpha_{n-2}} \cosh^{n-2} x_{n-2} dx_{n-2} \geq \frac{1}{2} \frac{\tanh d_{n-1}}{\sinh d_{n-2}} \sinh x_{n-3} \int_0^{\alpha_{n-2}} \cosh^{n-2} x_{n-2} dx_{n-2}.$$
So

\[ V(O) \geq \frac{1}{2} \frac{\tanh d_n \tanh d_{n-1}}{\sinh d_{n-1} \sinh d_{n-2}} \int_0^{\alpha_1} \int_0^{\alpha_1} \cdots \int_0^{\alpha_{n-3}} \left( \int_0^{\alpha_{n-2}} \cosh^{-2} x dx \right)^{n-2} x^{-2} dx \left( \int_0^{\alpha_{n-2}} \sinh^{-2} x dx \right)^{n-1} x^{-1} dx \cdot \sinh^{-2} x_{n-3} dx_{n-3} \cosh^{n-4} x_{n-4} \cosh x_1 dx_{n-4} \cdots dx_1 dx_n. \]

From (9) it can be easily seen that the function

\[ \left( \int_0^{\alpha_{n-2}} \cosh^{-2} x dx \right)^{n-2} x^{-2} dx \cosh^{-3} x_{n-3} \]

satisfies the assumption of Chebysev’s inequality and hence we can continue the process.

\[ V(O) \geq \left( \frac{1}{2} \right)^2 \frac{\tanh d_n \tanh d_{n-1}}{\sinh d_{n-1} \sinh d_{n-2}} \int_0^{\alpha_1} \int_0^{\alpha_1} \cdots \int_0^{\alpha_{n-4}} \left( \int_0^{\alpha_{n-3}} \cosh^{-2} x dx \right)^{n-4} x^{-4} dx \left( \int_0^{\alpha_{n-3}} \cosh^{-2} x dx \right)^{n-3} x^{-3} dx_{n-3} \cosh^{n-4} x_{n-4} dx_{n-4} \cdot \sinh^{-5} x_{n-5} \cosh x_1 dx_{n-5} \cdots dx_1 dx_n \]

and so on... By induction we get the inequality

\[ \frac{1}{(n+1)!} V(\tau[n, t]) = V(O) \geq \left( \frac{1}{2} \right)^{n-1} \frac{(\tanh d_n \cdots \tanh d_2) d_1}{\sinh d_{n-1} \cdots \sinh d_1} V(O_{n-1}) \]

\[ = \left( \frac{1}{2} \right)^{n-1} \frac{(\sinh d_n) d_1}{\cosh d_n \cdots \cosh d_2 \sinh d_1} V(O_{n-1}) \]

\[ = \left( \frac{1}{2} \right)^{n-1} \frac{(\sinh d_n \cosh d_1) d_1}{\cosh r_n \sinh d_1} \frac{V(\tau_i[n, t])}{n!}. \]

But

\[ \sinh d_n = \frac{\sin t}{\sqrt{n^2 - \sin^2 t}}, \quad \cosh r_n = \frac{1}{\cos t}, \]

\[ d_1 = \tanh^{-1} \left( \frac{\sin t \sqrt{1 - \frac{n-1}{2n}}}{1 - \sin^2 t \frac{n-1}{2n}} \right), \quad \sinh d_1 = \frac{\sin t \sqrt{1 - \frac{n-1}{2n}}}{\cos t} \]

\[ \cosh d_1 = \frac{\sqrt{1 - \sin^2 t \frac{n-1}{2n}}}{\cos t}, \]
we get the inequality

\[ V(\tau[n,t]) \geq (n+1) \left( \frac{1}{2} \right)^{n-1} \sqrt{1 - \frac{\sin^2 t}{\frac{n-1}{2n}}} (\cos t) \]

\[ \tanh^{-1} \left( \frac{\sin t \sqrt{1 - \frac{n-1}{2n}}}{1 - \sin^2 t \frac{n-1}{2n}} \right) V(\tau_i[n,t]), \]

as we stated. \qed

3.3. Half-space model and the upper bound

Consider a regular \(n\)-simplex in the upper half-space model in the following special position conv\{\(E_1, \ldots, E_n\)\} form a regular (Euclidean) simplex lying on the horizontal hyperplane \(H\), \(E_{n+1}\) and the center of the hyperbolic ball circumscribed around the simplex is on the vertical coordinate axis, see Fig. 2 for the notations. If \(K_{n-1}\) is of vertical (Euclidean) coordinate 1 then, by formulas in Subsection 3.1, we have \(|OP| = \sqrt{\frac{n+\sin t}{n-\sin t}}\) and \(|OP E_{n+1}| = \sqrt{\frac{(n+\sin t)(1+\sin t)}{(n-\sin t)(1-\sin t)}}\). If \(\mu\) is the Euclidean angle \(E_i O P O \angle (1 \leq i \leq n-1)\) then we have:

\[ \rho(E_1, K_{n-1}) = \ln \frac{1 + \sin \mu}{\cos \mu} \]
and the equation between $\mu$ and $t$ is

$$\frac{\sqrt{n^2 - \sin^2 t} + \sin t \sqrt{n^2 - 1}}{n \cos t} = 1 + \sin \mu$$

implying that

$$\sin t = \frac{n \sin \mu}{\sqrt{n^2 - \cos^2 \mu}} \quad \text{or} \quad \sin \mu = \frac{\sqrt{n^2 - 1} \sin t}{\sqrt{n^2 - \sin^2 t}} = \frac{\sin t \sqrt{1 - \frac{1}{n^2}}}{\sqrt{1 - \frac{\sin^2 t}{n^2}}}.$$  

The Euclidean coordinates of the points $E_i = (x^i_j)$, $j = 1, \ldots, n$, $i = 1, \ldots, n+1$ satisfy the equalities:

$$\sum_{j=1}^{n-1}(x^i_j)^2 = \sin^2 \alpha = \frac{(n^2 - 1) \sin^2 t}{n^2 - \sin^2 t} \quad \text{for all} \quad i = 1 \ldots n,$$

$$(x^i_n)^2 = \cos^2 \alpha = \frac{n^2 \cos^2 t}{n^2 - \sin^2 t} \quad \text{for all} \quad i = 1 \ldots n,$$

$$\sum_{j=1}^{n-1}(x^{i+1}_j)^2 = 0, \quad (x^{n+1}_n)^2 = \frac{(n + \sin t)(1 + \sin t)}{(n - \sin t)(1 - \sin t)}.$$  

(10)

Let us denote by $C_i = (y^i_1, \ldots, y^i_{n-1}, 0)^T$ and $\gamma_i$ the center and the radius of the sphere through the points $\{E_1, \ldots, E_{i-1}, E_{i+1}, \ldots, E_{n+1}\}$. This sphere contains a facet of $h(\tau[n, t])$. Hence for all $i = 1, \ldots, n$ we have $(n-1)$ equalities holding for the coordinates above. These are

$$\sum_{j=1}^{n-1} x^k_j y^j_i = \frac{-(n + 1) \sin t}{(n - \sin t)(1 - \sin t)} = c \quad \text{where} \quad k \neq i, \quad k = 1, \ldots, n. \quad (11)$$

Introduce the notation $X^i =: [x^k_j]_{j=1}^{n-1} \in \mathbb{R}^{(n-1) \times (n-1)}$ and $y^i = [y^i_1, \ldots, y^i_{n-1}]^T$, respectively. Then we get

$$(y^i)^T X^i = [c, \ldots, c] \quad \text{or equivalently} \quad (X^i)^T y^i = [c, \ldots, c]^T = c[1, \ldots, 1]^T.$$  

Since $\text{conv}\{E_1, \ldots, E_n\}$ is a regular simplex we get

$$R(n - 1) := (X^i)^T (X^i)$$

$$= \frac{(n^2 - 1) \sin^2 t}{n^2 - \sin^2 t} \begin{pmatrix}
1 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\
-\frac{1}{n-1} & 1 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\frac{1}{n-1} & \cdots & -\frac{1}{n-1} & 1 & -\frac{1}{n-1} \\
-\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} & 1 
\end{pmatrix}.$$  

It follows that

$$(X^i)^T = R(n - 1)(X^i)^{-1} \quad \text{and so} \quad ((X^i)^T)^{-1} = (X^i)(R(n - 1))^{-1}.$$
Since the inverse of $R(n - 1)$ is equal to

$$\begin{pmatrix}
2 & 1 & 1 & \ldots & 1 \\
1 & 2 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \ldots & 1 & 2 & 1 \\
1 & 1 & \ldots & 1 & 2
\end{pmatrix}$$

we get

$$y^i = X^i (R(n - 1))^{-1} [c, \ldots, c]^T$$

$$= - \frac{(n + \sin t)}{n \sin t(1 - \sin t)} X^i \begin{pmatrix}
1 \\
1 \\
\vdots \\
1 \\
1
\end{pmatrix} = - \frac{(n + \sin t)}{\sin t(1 - \sin t)} \begin{pmatrix}
\sum_{k=1 \atop k \neq i}^{n} x_k^1 \\
\sum_{k=1 \atop k \neq i}^{n} x_k^2 \\
\vdots \\
\sum_{k=1 \atop k \neq i}^{n} x_k^{n-2} \\
\sum_{k=1 \atop k \neq i}^{n} x_k^{n-1}
\end{pmatrix}$$

$$= - \frac{(n + \sin t)}{\sin t(1 - \sin t)} \sum_{k=1 \atop k \neq i}^{n} v_k,$$

where $v_k := [x_1^k, \ldots, x_{n-1}^k]^T$. Since $\sum_{k=1}^{n} v_k = 0$, we get that

$$y^i = \frac{(n + \sin t)}{\sin t(1 - \sin t)} v_i.$$

Hence

$$|y^i|^2 = \frac{(n + \sin t)^2}{\sin^2 t(1 - \sin t)^2} |v_i|^2 = \frac{(n^2 - 1)(n + \sin t)}{(n - \sin t)(1 - \sin t)^2},$$

implying that

$$\gamma_i^2 = \left( \frac{n + \sin t}{1 - \sin t} \right)^2.$$
Let us denote by $\epsilon[n-1,t] = \text{conv}\{F_1,\ldots,F_n\}$ where $F_i$ is the orthogonal projection of $E_i$ to $H$. $\epsilon[n-1,t]$ can be dissected into $n$ congruent simplices with a common vertex $O_P$. Denote by $\epsilon_i[n-1,t]$ these simplices, concretely, we set $\epsilon_i[n-1,t] := \{\sum_{j \neq i} \alpha_j v_j \mid \alpha_j \geq 0, \sum_{j \neq i} \alpha_j \leq 1\}$. If a point $z \in P^n$ is in $h(\tau[n,t])$, then it is of the form $z = v + z_n(v)e_n$, where $v \in \epsilon([n-1,t])$. If $z \in h(\tau[n,t])$ with $v \in \epsilon[n-1,t]$ then $\|z - C_n\|^2 \leq \gamma_n^2$ hence we have that

$$\sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} \alpha_j x_i^j - y_i^\perp \right)^2 + z_n^2 \leq (\gamma_n)^2.$$ 

From this inequality we get

$$z_n^2 \leq \frac{(n + \sin t)^2}{(1 - \sin t)^2} - \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n} \alpha_j x_i^j - \frac{(n + \sin t)}{\sin t(1 - \sin t)} x_i^n \right)^2$$

$$= \frac{(n + \sin t)^2}{(1 - \sin t)^2} - \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n} \left( \alpha_j + \frac{(n + \sin t)}{\sin t(1 - \sin t)} \right) x_i^j \right)^2$$

$$= \frac{(n + \sin t)^2}{(1 - \sin t)^2} - \sum_{j,k=1}^{n-1} \left( \alpha_j + \frac{(n + \sin t)}{\sin t(1 - \sin t)} \right) \left( \alpha_k + \frac{(n + \sin t)}{\sin t(1 - \sin t)} \right) \sum_{i=1}^{n-1} x_i^j x_i^k.$$ 

Using the Eqs. (10) and (11) we get that it is equal to:

$$\frac{(n + \sin t)^2}{(1 - \sin t)^2} - \frac{(n^2 - 1) \sin^2 t}{n^2 - \sin^2 t} \left( \sum_{j=1}^{n-1} \left( \alpha_j + \frac{(n + \sin t)}{\sin t(1 - \sin t)} \right)^2 \right)$$

$$- \frac{1}{n-1} \sum_{j,k=1}^{n-1} \left( \alpha_j + \frac{(n + \sin t)}{\sin t(1 - \sin t)} \right) \left( \alpha_k + \frac{(n + \sin t)}{\sin t(1 - \sin t)} \right) \left( \alpha_j + \alpha_k \right) \frac{(n + \sin t)^2}{\sin^2 t(1 - \sin t)^2}$$

$$= \frac{(n + \sin t)^2}{(1 - \sin t)^2} - \frac{(n^2 - 1) \sin^2 t}{n^2 - \sin^2 t} \left( \sum_{j=1}^{n-1} \left( \alpha_j^2 + 2\alpha_j \frac{(n + \sin t)}{\sin t(1 - \sin t)} \right) \frac{(n + \sin t)}{\sin t(1 - \sin t)} \right)$$

$$+ \frac{(n + \sin t)^2}{\sin^2 t(1 - \sin t)^2} - \frac{1}{n-1} \sum_{j,k=1}^{n-1} \left( \alpha_j + \alpha_k \right) \frac{(n + \sin t)^2}{\sin^2 t(1 - \sin t)^2} \left( \frac{n + \sin t}{\sin t(1 - \sin t)} + \frac{(n + \sin t)^2}{\sin^2 t(1 - \sin t)^2} \right)$$

$$= \frac{(n + \sin t)^2}{(1 - \sin t)^2} - \frac{(n^2 - 1) \sin^2 t}{n^2 - \sin^2 t} \left( (n - 1) - \frac{1}{n-1} (n - 1)(n - 2) \right) \frac{(n + \sin t)^2}{\sin^2 t(1 - \sin t)^2}$$

$$- \frac{(n^2 - 1) \sin^2 t}{n^2 - \sin^2 t} \left( 2 \sum_{j=1}^{n-1} \alpha_j - \frac{1}{n-1} \sum_{j \neq k=1}^{n-1} \alpha_j + \alpha_k \right) \frac{(n + \sin t)}{\sin t(1 - \sin t)}.$$
\[ - \frac{(n^2 - 1) \sin^2 t}{n^2 - \sin^2 t} \left( \sum_{j=1}^{n-1} \alpha_j^2 - \frac{1}{n-1} \sum_{j \neq k=1}^{n-1} \alpha_j \alpha_k \right) = \frac{(n + \sin t)(1 + \sin t)}{(n - \sin t)(1 - \sin t)} - \frac{2(n + 1) \sin t}{(n - \sin t)(1 - \sin t)} \sum_{j=1}^{n-1} \alpha_j - |v|^2, \]

Since \(\|z\|^2 \geq 1\),

\[ 1 - |v|^2 \leq z_n^2 \leq \frac{(n + \sin t)(1 + \sin t)}{(n - \sin t)(1 - \sin t)} - \frac{2(n + 1) \sin t}{(n - \sin t)(1 - \sin t)} \sum_{j=1}^{n-1} \alpha_j - |v|^2, \quad (12) \]

where \(|v|^2 = \frac{(n^2 - 1) \sin^2 t}{n^2 - \sin^2 t} \left( \sum_{j=1}^{n-1} \alpha_j^2 - \frac{1}{n-1} \sum_{j \neq k=1}^{n-1} \alpha_j \alpha_k \right) \) is the square of the Euclidean norm of the vector \(v\). Denote the quantity \(\sum_{j=1}^{n-1} \alpha_j \) by \(\alpha(v)\). Then the volume is:

\[ V(\tau[n, t]) = \sum_{i=1}^{n} \int_{v \in \epsilon_i[n-1, t]} \int_{z_n = \sqrt{1 - |v|^2}} z_n^{-n} dz_n d\rho. \]

Since the set of the vectors \(v\) belonging to more than one \(\epsilon_i[n-1, t]\) has measure zero, the above equality can be written of the form:

\[ V(\tau[n, t]) = \frac{1}{n-1} \int_{\epsilon[n-1, t]} \left( \frac{1}{\sqrt{1-\rho^2}} \right)^{n-1} d\rho - \frac{1}{n-1} \int_{\epsilon[n-1, t]} \left( \frac{1}{\sqrt{(n + \sin t)(1 + \sin t) - \frac{2(n + 1) \sin t}{(n - \sin t)(1 - \sin t)} \alpha(v) - |v|^2}} \right)^{n-1} d\rho, \quad (13) \]

where \(\rho^2 = |v|^2\) and \(\alpha(v)\) is uniquely determined by the vector \(v\).

By formula (13) we can get an upper bound for the volume of \(\tau[n, t]\).

**Lemma 2.**

\[ V(\tau[n, t]) \leq \frac{1}{n-1} \left( 1 - \frac{n^2(1 - \sin t)^2(1 + \sin t)}{(n + \sin t)(1 + \sin t) - (n^2 - 1) \sin^2 t(1 - \sin t)^2} \right)^{n-1} V(\tau_1[n, t]). \quad (14) \]

**Proof.** Clearly

\[ \frac{\sqrt{(n + \sin t)(1 + \sin t)}}{(n - \sin t)(1 - \sin t)} - \frac{2(n + 1) \sin t}{(n - \sin t)(1 - \sin t)} \alpha(v) - |v|^2 \leq \sqrt{\frac{(n + \sin t)(1 + \sin t)}{(n - \sin t)(1 - \sin t)} - |v|^2} \]
and using the fact that if $\zeta > 1$ then the function $f(x) = (\zeta - x)/(1 - x)$ is strictly increasing for positive $x$, we get

$$
\sqrt{\frac{(n + \sin t)(1 + \sin t)}{(n - \sin t)(1 - \sin t)}} - |v|^2 \leq \sqrt{\frac{(n + \sin t)(1 + \sin t)}{(n - \sin t)(1 - \sin t)}} - \frac{|v|^2_{max}}{1 - |v|^2_{max}} \left( \sqrt{1 - |v|^2} \right).
$$

Since $v \in \epsilon[n - 1, t]$, we have $|v|^2_{max} = h(r_{n - 1})^2 = \left( \frac{\sqrt{n^2 - 1}\sin t}{\sqrt{n^2 - \sin^2 t}} \right)^2$. Since the radius of the circumscribed sphere of $\epsilon[n - 1, t]$ is equal to $\sin \alpha = \frac{\sqrt{n^2 - 1}\sin t}{\sqrt{n^2 - \sin^2 t}}$, we get

$$
V(\tau[n, t]) \leq \frac{1}{n - 1} \left( 1 - \left( \frac{1 - \frac{|v|^2_{max}}{n^2(1 - \sin t)^2(1 + \sin t)}}{1 - \frac{|v|^2_{max}}{n^2(1 + \sin t)}} \right)^{n - 1} \right) \int_{\sqrt{n^2 - \sin^2 t}}^{\sqrt{n^2 - 1}\sin t} S(n - 1) \cdot \frac{1}{\sqrt{1 - |v|^2}} |v| \, d|v|,
$$

and using (7) we get

$$
V(\tau[n, t]) \leq \frac{1}{n - 1} \left( 1 - \left( \frac{n^2(1 - \sin t)^2(1 + \sin t)}{n^2(1 + \sin t)} \right)^{n - 1} \right) V(\tau_1[n, t]),
$$

as we stated.

\[\square\]

\textbf{Remark 1.} An analogue of (13) can be determined in such a case when the simplex $F$ is not regular. If we assume that the circumcenter $h^{-1}(O)$, the vertex $h^{-1}(E_{n+1})$ and the center $h^{-1}(K_{n-1})$ of the facet $F_{n+1}$ are collinear and $h^{-1}(K_{n-1})$ is an inner point of $F_{n+1}$ then the half-space model representation of $F$ is very similar to the regular case. We have to make a distinction only at those points in the calculation when (firstly) we determined and used the Gramm matrix $R(n - 1)$ of the regular vector system, and (secondly) when we determined the distances of the points $K_{n-1}, O$ and $E_{n+1}$, respectively. Let us denote $\rho(h^{-1}(K_{n-1}), h^{-1}(O))$ by $d$ and $\rho(h^{-1}(O), h^{-1}(E_{n+1}))$ by $r$, respectively. As in the regular case we can determine the coordinate $z_n$ of a point $z = v + z_n e_n$, $v \in \epsilon[n - 1, t]$. It is bounded above by the value $\left( (e^r + d)^2 - e^{r + d}(e^{r + d} + 2)\alpha(v) - |v|^2 \right)$ where $\alpha(v)$ is the same quantity as in
the regular case. We denote by $\varepsilon[n-1]$ the orthogonal projection of $h(\mathcal{F}_{n+1})$ to $H$. The general formula now is

$$V(\mathcal{F}) = \frac{1}{n-1} \int_{\varepsilon[n-1]} \left( \frac{1}{\sqrt{1 - \rho^2}} \right)^{n-1} \left( \frac{1}{\sqrt{(e^{r+d} + 1)^2 - e^{r+d}(e^{r+d} + 2)\alpha(v) - \rho^2}} \right)^{n-1} d\rho. \quad (15)$$

From this formula, an upper bound similar to (14) follows immediately for the volume.

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