E∞-RING STRUCTURES ON THE K-THEORY OF ASSEMBLERS AND POINT COUNTING

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ABSTRACT. We construct a monoidal structure on the category of assemblers. As an application of this, we construct a derived local zeta-function which takes a variety over a finite field to the set of points over the separable closure, and use the structure of this map to detect interesting elements in $K_1(\text{Var}_k)$.

INTRODUCTION

An additive invariant on varieties over a base field $k$ with values in an abelian group $A$ is a function $\mu: \{\text{Varieties}/k\} \to A$ such that for any closed immersion $Y \hookrightarrow X$, $[X] = [Y] + [X \setminus Y]$. Such an invariant must factor uniquely through a homomorphism from the Grothendieck ring of varieties, $K_0(\text{Var}_k)$, defined by $K_0(\text{Var}_k) \overset{\text{def}}{=} \text{free ab. gp. gen. by varieties over } k \sslash \text{closed } X \mapsto [X] = [Y] + [X \setminus Y]$.

In other words, $K_0(\text{Var}_k)$ is the universal additive invariant: if all structure on $K_0(\text{Var}_k)$ could be understood then all additive invariants would also be understood. Such an invariant is called multiplicative if it takes values in a ring and satisfies the additional condition that $\mu(X \times Y) = \mu(X)\mu(Y)$. Defining the ring structure on $K_0(\text{Var}_k)$ to be $[X][Y] = [X \times Y]$ implies that any multiplicative additive invariant must factor uniquely through a ring homomorphism from $K_0(\text{Var}_k)$.

The group $K_0(\text{Var}_k)$ can be modeled topologically as the connected components of a $K$-theory spectrum $K(\text{Var}_k)$, introduced in [Zak17A]. The higher homotopy groups of this spectrum encode further geometric information about piecewise-isomorphisms of varieties. The group $K_1(\text{Var}_k)$ can be thought of (by analogy with the $K$-theory of a ring) as a “determinant” for piecewise-automorphisms of varieties, in the following manner.

Consider the determinant of a matrix over a field $F$. The determinant is a collection of homomorphisms $\det_n: GL_n(F) \to F^{\times}$ for each positive integer $n$ satisfying the following conditions:

additivity: for positive integers $m$ and $n$, the following diagram commutes:

$$
\begin{array}{ccc}
GL_m(F) \oplus GL_n(F) & \longrightarrow & GL_{m+n}(F) \\
\det_m \oplus \det_n & \downarrow & \det_{m+n} \\
F^{\times} \oplus F^{\times} & \longrightarrow & F^{\times}
\end{array}
$$

initial conditions: the following computations hold:

$$
\det_2 \left( \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} \right) = -1 \quad \text{and} \quad \det |_{GL_1(F)} = \text{id}.
$$

In particular, the additivity of the determinant ensures stability: the homomorphisms $\det_n$ extend to a homomorphism $GL(F) \to F^{\times}$. If we generalize $F$ to a ring we have a choice about whether to enforce the initial conditions or not. If we enforce the initial conditions we can use the standard formula for the determinant and the additivity condition will hold; this determinant will take values
in $R^\times$. However, it also makes sense to say that the condition we truly care about is additivity; in this case, we can say that the determinant should take values in the largest abelian group possible (ensuring that additivity still holds)—the group $GL(R)^{ab} = K_1(R)$. The fact that $K_1(R)$ is not necessarily isomorphic to $R^\times$, as it is for a field, demonstrates that for some base rings the determinant contains more information than a single unit. Moreover, although the determinant on $GL_1(R)$ is no longer the identity, there is a natural inclusion $GL_1(R) \to K_1(R)$. For a more in-depth discussion of $K_1(R)$, see [Wei13, Chapter III].

In the case of varieties there is a similar description of the determinant for automorphisms of varieties. For a variety $X$, write $\text{Aut}(X)$ for the group of automorphisms of $X$. The additivity condition for the determinant can be described as the fact that for every variety $X$ there is a homomorphism

$$\det_X : \text{Aut}(X) \to K_1(\text{Var}_k)$$

satisfying the additivity condition that the diagram

$$\begin{array}{ccc}
\text{Aut}(X) & \oplus & \text{Aut}(Y) \\
\downarrow \det_X \oplus \det_Y & & \downarrow \det_{X \amalg Y} \\
K_1(\text{Var}_k) & \oplus & K_1(\text{Var}_1) \\
& \to & K_1(\text{Var}_k)
\end{array}$$

commutes for any varieties $X$ and $Y$ over $k$. The initial conditions are more complicated, although the condition on the swap has a natural interpretation. Finite sets can be considered 0-dimensional varieties, and this induces a map

$$\text{Aut}(\mathbb{F}) \equiv K(\text{Fin}) \to K(\text{Var}_k).$$

The non-identity element in the group $K(\text{Fin}) \cong \pi_1 \mathbb{S} \cong \mathbb{Z}/2$ thus has an image in $K(\text{Var}_k)$; this is the element which is the determinant of the “swap”. Note that this agrees with the classical definition: if finite sets are mapped to vector spaces of the appropriate dimension, then the two-point swap is mapped to the matrix

$$\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}$$

and the induced map on $K_1$ is exactly the homomorphism $\mathbb{Z}/2 \to F^\times$ taking $-1$ to itself.

Remark. It is natural to ask what the natural analog of the computation of the determinant of $GL_1(F)$ could be. One possible candidate is the following: the group $\text{Aut}(\mathbb{P}^1)$ contains as a subgroup $k^\times$, where $a \in k^\times$ represents the automorphism $[x : y] \mapsto [ax : y]$. There is therefore an induced map $k^\times \to K(\text{Var}_k)$. The question of whether this map is injective is still open.

In order to analyze elements in $K(\text{Var}_k)$ it is possible to “lift” invariants of $K_0(\text{Var}_k)$. More rigorously, suppose that we are given a homomorphism $K_0(\text{Var}_k) \to K_0(C)$ for some category $C$ which has an associated $K$-theory spectrum (such as finite sets, projective $R$-modules, etc.). It is often possible to lift this homomorphism to a map of spectra $K(\text{Var}_k) \to K(C)$; this map will induce a map $K_0(n,\text{Var}_k) \to K_0(n,\text{C})$ for all integers $n$. For example, when $k$ is finite, consider the usual local zeta function of a variety $X$, written in terms of the action of Frobenius on $\ell$-adic cohomology,

$$Z(X, t) = \prod_{i=0}^{2 \dim X} \det(1 - t \text{Frob} \mid H_i(\overline{\mathcal{X}}, \mathcal{Q}_\ell))^{(-1)^{i+1}}.$$

The local zeta function induces a homomorphism $K_0(\text{Var}_k) \to K_0(\text{GalRep}(\mathcal{Q}_\ell))$, where $\text{GalRep}(\mathcal{Q}_\ell)$ is the category of finitely-generated continuous Galois representations. In [CWZ19] the authors lift this homomorphism to a map of spectra and use it to show that when $|k| \equiv 3 \pmod{4}$ the group $K_1(\text{Var}_k)$ contains elements which are not in the image of $K_1(\text{Fin})$; in particular, they show that the element $[\mathbb{P}^1, 1/x] \in \text{Aut}(\mathbb{P}^1)$ is not in the image of $K_1(\text{Fin})$.

\footnote{This definition an discussion generalizes directly to piecewise-automorphisms of varieties; however, in the interest of readability we focus entirely on honest automorphisms here.}
However, the authors were not able to show that the induced map $K(\text{Var}_k) \longrightarrow K(\text{GalRep}(\mathbb{Q}_\ell))$ is a map of $E_\infty$-ring spectra; in less-technical language, the authors could lift the group homomorphism but not the multiplicative structure to the map of spectra. Although the spectrum $K(\text{Var}_k)$ was shown in [Cam15] to have an $E_\infty$-structure, the method for constructing the map of spectra could not ensure that the map is compatible with this structure.

In this paper, we develop alternate machinery for constructing an $E_\infty$-structure on $K(\text{Var}_k)$ and use it to construct an $E_\infty$-version of the local zeta function. In order to ensure that the map is $E_\infty$ we use a different, more combinatorial, model of the local zeta function:

$$Z(X,t) = \exp \sum_{n \geq 1} \frac{|X(\mathbb{F}_q^n)|}{n} t^n,$$

when $k = \mathbb{F}_q$. Noting that the set $X(\mathbb{F}_q^n)$ is uniquely determined by the set $X(\overline{\mathbb{F}}_q)$ together with the action of Frobenius, we consider the local zeta function to be a map $K_0(\text{Var}_k) \longrightarrow K_0(\text{AFSet}_{\hat{\mathbb{Z}}})$.

Here $\text{AFSet}_{\hat{\mathbb{Z}}}$ is the category of almost-finite sets; see Section 2.2 for a rigorous definition. This gives rise to the following theorem:

**Theorem A** (Theorems 7.13 and 7.15). The spectra $K(\text{Var}_k)$ and $K(\text{AFSet}_{\hat{\mathbb{Z}}})$ are $E_\infty$-ring spectra. The local zeta function induces a ring homomorphism $K_\ast(\text{Var}_k) \longrightarrow K_\ast(\text{AFSet}_{\hat{\mathbb{Z}}})$.

This ring structure allows us to do a more in-depth analysis of the structure of $K_1(\text{Var}_k)$. The $E_\infty$-structure on $K(\text{Var}_k)$ induces a multiplication

$$K_0(\text{Var}_k) \otimes K_1(\text{Fin}) \longrightarrow K_0(\text{Var}_k) \otimes K_1(\text{Var}_k) \longrightarrow K_1(\text{Var}_k).$$

The elements in this image of this map are those which can be represented by an automorphism which takes two copies of some variety $X$ and swaps them. The element in the image of $K_1(\text{Fin})$ is one such element, but there are others as well. We call elements in this image *permutative*, and those not in the image *non-permutative*. The model of the local zeta function constructed in this paper allows us to conclude the following:

**Theorem B** (Corollary 4.9). There exist non-permutative elements in $K_1(\text{Var}_k)$ for all finite fields $k$ with odd characteristic. In particular, let $b = \text{ord}_2(|k| - 1)$, and let $\alpha \in k$ be a primitive $2^b$-th root of unity. Then the element $[\mathbb{P}^1, x \mapsto \alpha x]$ is a non-permutative element of $K_1(\text{Var}_k)$.

**Remark.** Permutative elements exist in $K_n(\text{Var}_k)$ for all positive $n$, not just $n = 1$; the proof of [CWZ19, Theorem 6.6] uses such elements to show that when $k$ is a subfield of $\mathbb{C}$ there are infinitely many nontrivial groups $K_n(\text{Var}_k)$. (For finite fields this is clear, as the map $X \mapsto X(k)$ induces a map $K(\text{Var}_k) \longrightarrow K(\text{Fin})$ which splits the map $K(\text{Fin}) \longrightarrow K(\text{Var}_k)$.)

The proofs of these theorems use the machinery of assemblers, originally introduced in [Zak17A]. Assemblers contain the combinatorial data of how different objects of interest decompose but strip out the specific data of the context. Their combinatorial nature makes them amenable to analytic techniques, equipping them with dévissage and localization theorems analogous to Quillen’s theorems for abelian categories.

The assembler of varieties over $k$, generally denoted $\text{Var}_k$ to be consistent with the notation for the Grothendieck ring, has as objects the varieties over $k$ and as morphisms locally-closed immersions. It is also equipped with a pretopology generated by the coverage $\{Y \hookrightarrow X, X \setminus Y \hookrightarrow X\}$ (where $Y \hookrightarrow X$ is a closed immersion). In this paper we show that the assembler definition of the $K$-theory of varieties gives rise to an $E_\infty$-structure on the $K$-theory by showing that the $K$-theory functor on assemblers is “almost monoidal,” taking monoid objects in assemblers to $E_\infty$-ring spectra. In fact, this can be generalized a bit further, showing that not only monoid objects in the category of assemblers but “monoidal” assemblers (assemblers equipped with an operation which is associative, commutative, and unital up to a natural isomorphism) are taken to $E_\infty$-ring spectra.
**Theorem C.** The $K$-theory functor $K: Asm \rightarrow Sp$ is monoidal and takes symmetric monoidal assemblers to $E_\infty$-ring spectra and symmetric monoidal morphisms of assemblers to ring homomorphisms on $K$-groups.

For more rigorous definitions and theorem statements, see Section 7. Moreover, we show in Theorem 3.2 that the generators and relations given in [Zak17B, Theorem B] for $K_1(\text{Var}_k)$ interact in a natural manner with the multiplicative structure.

**Organization.** This paper is targeted towards those interested in the applications, including the derived $\zeta$-function. In the service of this, we front-load the applications, and leave the proofs of the structural theorems for later sections. Section 1 contains a quick review of assemblers and the results vital for an understanding of the applications. Section 2 gives several examples of interest to this paper. Section 3 analyzes non-permutative elements in $G$-sets, which is used in Section 4 to detect non-permutative elements in $K(\text{Var}_k)$; Section 4 has all of the main applications of the main theorem in this paper. The last four sections cover the technical underpinnings of Theorem C. Section 5 gives a run-down of the technical results necessary for the proofs. Section 6 proves that the claimed product on assemblers produces a symmetric monoidal structure. Section 7 proves Theorem C. Lastly, Section 8 proves that generators of $K_1$ interact with the monoidal structure in the expected manner.

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1. A Quick Run-down of Assemblers

**Definition 1.1** ([Zak17A, Definition 2.4]). In a Grothendieck site $\mathcal{C}$ with an initial object, a covering family is a family of morphisms which generates a covering sieve. A family $\mathcal{F}$ is disjoint if for any two morphisms $f: A \rightarrow C$ and $g: B \rightarrow C$ in $\mathcal{F}$, $A \times_C B$ exists and is equal to the initial object.

An assembler is a Grothendieck site $\mathcal{C}$, satisfying the following extra conditions:

(I) $\mathcal{C}$ has an initial object $\emptyset$, and $\emptyset$ has an empty covering family.

(R) For any two finite disjoint covering families of an object $A$, there is a common refinement which is itself finite and disjoint.

(M) All morphisms in $\mathcal{C}$ are monic.

An assembler is closed if the category $\mathcal{C}$ has all pullbacks. Note that in this case, axiom (R) holds automatically.

We generally assume that the initial object in $\mathcal{C}$ is unique, although this assumption does not affect any of the constructions in this paper.

For an assembler $\mathcal{C}$, we write $\mathcal{C}^o$ for the full subcategory of noninitial objects of $\mathcal{C}$.

A morphism of assemblers is a functor which preserves the initial objects and disjointness and which is continuous with respect to the topology. Write $Asm$ for the category of assemblers, and $cAsm$ for the subcategory of closed assemblers and pullback-preserving morphisms of assemblers.

The main examples of assemblers appearing in this paper are the following:

**Example 1.2.** Let $G$ be a discrete group. The assembler $S_G^{Asm}$ has two objects, $\emptyset$ and $*$, with one morphism $\emptyset \rightarrow *$. In addition, $*$ has automorphism group $G$. The topology is generated by the trivial covering families on $\emptyset$ and $*$, together with the empty covering family of $\emptyset$. When $G$ is trivial we omit it from the notation.
Example 1.3. Let $G$ be a group. The assembler $\text{Fin}_G$ has as objects the finite $G$-sets, with morphism $G$-equivariant injections. A family is a covering family if it is mutually surjective. In other words, a family $\{f_i: A_i \to A\}_{i \in I}$ is a covering family if $\bigcup_{i \in I} f_i(A_i) = A$. (As before, when $G$ is trivial we omit it from the notation.) When $G$ is profinite, the assembler $\text{AFSet}_G$ is the assembler of almost-finite $G$-sets: those $G$-sets $S$ such that $S^H$ is finite for any closed subgroup $H \leq G$ and such that for all $x \in S$, the orbit $G \cdot x$ is finite. Again, the morphisms are $G$-equivariant inclusions and covering families are mutually surjective.

Example 1.4. Let $k$ be a field. The assembler $\text{Var}_k$ has as objects $k$-varieties (i.e. reduced separated schemes of finite type over $k$) and as morphisms locally closed immersions. The topology is generated by the coverage consisting of families $\{f: Y \hookrightarrow X, X \setminus Y \to X\}$, where $f$ is a closed immersion. More generally, for a Noetherian scheme $S$ the assembler $\text{Var}_S$ with objects varieties over $S$ is defined analogously to $\text{Var}_k$.

Assemblers have a $K$-theory which classifies “scissors congruence” of the objects of the assembler:

**Theorem 1.5** ([Zak17A, Theorem A]). There exists a functor $K: \text{Asm} \to \text{Sp}$ from the category of assemblers to the category of spectra such that for any assembler $C$, $\pi_0 K(C)$ is the free abelian group generated by objects of $C$ modulo the relations $[A] = \sum_{i \in I} [A_i]$ for any finite disjoint covering family $\{A_i \to A\}_{i \in I}$.

**Definition 1.6.** For an assembler $C$, write $K_n(C) \overset{\text{def}}{=} \pi_n K(C)$.

Although generators and relations for $K_n(C)$ for $n > 0$ akin to those given in the above theorem are difficult to come by, there is a simple description of those elements in $K_1(C)$ which are of interest in the current context:

**Theorem 1.7** (Corollary 8.2). Let $C$ be an assembler. Let $A$ be an object in $C$, an let $\sigma \in \text{Aut}(A)$. Then the pair $[A, \sigma]$ represents an element of $K_1(C)$. These satisfy the following relations:

- For any finite disjoint covering family $\{f_i: A_i \to A\}_{i \in I}$ such that for each $i \in I$, there is a $\sigma_i \in \text{Aut}(A_i)$ making the square

$$
\begin{array}{ccc}
A & \overset{\sigma}{\longrightarrow} & A \\
\downarrow f_i & & \downarrow f_i \\
A_i & \overset{\sigma_i}{\longrightarrow} & A_i
\end{array}
$$

commute,

$$
[A, \sigma] = \sum_{i \in I} [A_i, \sigma_i].
$$

- If $\sigma, \sigma' \in \text{Aut}(A)$ then

$$
[A, \sigma] + [A, \sigma'] = [A, \sigma \circ \sigma'].
$$

This theorem does not claim that these are the only relations satisfied by these elements, or that $K_1(C)$ is generated by these elements. However, as these are the only elements of interest in this paper we restrict our attention to this simpler statement. For a more comprehensive analysis, see [Zak17B, Theorem B].

There is a monoidal structure on the category of closed assemblers.\footnote{Although it should be possible to extend this construction to all assemblers, this would require more technical work which would distract from the main idea. As all examples of interest to us are closed we focus on this subclass of examples.} The intuition behind it comes from the following example:
Example 1.8. Let $\text{Seg}$ be the assembler whose objects are closed intervals in $\mathbb{R}$ and whose morphisms are isometric injections. Let $\text{Rec}$ be the assembler whose objects are sets of the form $[a, a'] \times [b, b']$ in $\mathbb{R}^2$, with morphisms isometric embeddings. (Again, the topologies consist of the mutually surjective families.) An object in $\text{Rec}$ is a “pair of objects” in $\text{Seg}$, and any decomposition of the elements of the pair produces a decomposition of the whole object:

The idea of the monoidal structure on $\text{Asm}$ is to produce covering families which are similarly “gridded” in the general setting.

Definition 1.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two closed assemblers. The assembler $\mathcal{C} \wedge \mathcal{D}$ has as underlying category the full subcategory of $\mathcal{C} \times \mathcal{D}$ consisting of those pairs $(C, D)$ where $C = \emptyset$ if and only if $D = \emptyset$. The topology on this assembler is generated by the coverage consisting of those families

$$\{(A_i, B_j) \rightarrow (A, B)\}_{(i,j) \in I \times J}$$

where both $\{A_i \rightarrow A\}_{i \in I}$ and $\{B_j \rightarrow B\}_{j \in J}$ are covering families in $\mathcal{C}$ and $\mathcal{D}$, respectively.

This is not the usual topology on the product of sites. The usual topology on the product would have as covering families those families $\{(A_i, B_i) \rightarrow (A, B)\}_{i \in I}$ where $\{A_i \rightarrow A\}_{i \in I}$ and $\{B_i \rightarrow B\}_{i \in I}$ are covering families in $\mathcal{C}$ and $\mathcal{D}$, respectively. In particular, the family of shaded rectangles gives a covering family under the standard topology, but not under the topology in $\text{Seg} \wedge \text{Seg}$. In $\text{Seg} \wedge \text{Seg}$ all 16 rectangles in the picture are necessary for a covering family.

Remark 1.10. This definition of the topology is interesting in that it allows us to produce a natural example of a Waldhausen category which does not satisfy the Saturation Axiom. See Example 6.7.

The main technical result of this paper is the following:

Theorem 1.11 (Lemma 6.6, Section 7). The structure $(\text{Asm}, \wedge, S)$ is a symmetric monoidal structure on the category of closed assemblers. The $K$-theory functor is monoidal and thus takes monoid objects to ring spectra.

Given a symmetric monoidal assembler $\mathcal{C}$ (see Definition 7.10), $K(\mathcal{C})$ is an $E_{\infty}$-ring spectrum. A monoidal morphism of symmetric monoidal assemblers $\mathcal{C} \rightarrow \mathcal{D}$ induces a ring homomorphism $K_{*}(\mathcal{C}) \rightarrow K_{*}(\mathcal{D})$.

Here, a “symmetric monoidal assembler” is a weakened form of a monoid object in assemblers; it is directly analogous to the definition of a symmetric monoidal category with the cartesian product of categories replaced by the $\wedge$-product of assemblers.

In particular, this theorem implies that if $\mathcal{C}$ is a symmetric monoidal assembler then $K_{*}(\mathcal{C})$ is a graded-commutative ring.
Proposition 1.12. Let $\mathcal{C}$ be a symmetric monoidal assembler with multiplication map $\mu$, and let $[A],[A'] \in K_0(\mathcal{C})$ and $[B,\sigma] \in K_1(\mathcal{C})$. Then

$$[A][A'] = [\mu(A,A')] \quad \text{and} \quad [A][B,\sigma] = [\mu(A,B),\mu(1_A,\sigma)].$$

2. Examples

2.1. Finite $G$-sets.

Notation 2.1. Let $G$ be a group. Denote by $C_G$ a set of representatives for conjugacy classes of subgroups of $G$.

Let $G$ be a finite group, and let $\text{Fin}_G$ be the assembler whose objects are finite $G$-sets, and whose morphisms are $G$-equivariant injections. The topology on the assembler is generated by mutually surjective families.

Every finite $G$-set is a disjoint union of its $G$-orbits. Since all morphisms in $\text{Fin}_G$ are injective, the image of any $G$-orbit is an isomorphic $G$-orbit. By picking a point in an orbit, we see that any $G$-orbit is isomorphic to $G/H$ for some subgroup $H$, and $G/H$ is isomorphic to $G/H'$ if and only if $H$ and $H'$ are conjugate in $G$. Let $\text{Fin}^H_G$ be the full subassembler of $\text{Fin}_G$ whose objects are disjoint unions of $G$-orbits, each of which is isomorphic to $G/H$. Then

$$\text{Fin}_G \cong \prod_{H \in C_G} \text{Fin}^H_G.$$  

Since $K$-theory of assemblers commutes with finite products, in order to analyze the $K$-theory of $\text{Fin}_G$ it is sufficient to understand the $K$-theory of each $\text{Fin}^H_G$.

Fix $H \in C_G$. Let $S \in \text{Fin}^H_G$ have exactly one $G$-orbit. Then every other object in $\text{Fin}^H_G$ is isomorphic to a disjoint union of copies of $S$, and the automorphism group of $S$ is $W_G H$, the Weyl group of $H$ in $G$. Thus by dévissage for assemblers [Zak17A, Theorem B] $K(\text{Fin}^H_G) \cong K(\mathbb{S}_G) \cong \sum_{i=0}^{\infty} B(W_G H)$. Putting this together, we have

$$K(\text{Fin}_G) \cong \prod_{H \in C_G} \sum_{i=0}^{\infty} B(W_G H).$$ (2.2)

The Cartesian product of finite $G$-sets produces a functor $\mu: \text{Fin}_G \wedge \text{Fin}_G \rightarrow \text{Fin}_G$. This preserves the initial object and disjointness by definition, so in order to check that it is a morphism of assemblers it suffices to show that it takes covering families to covering families; in particular, it suffices to check that for covering families $\{f_i: A_i \rightarrow A\}_{i \in I}$ and $\{g_j: B_j \rightarrow B\}_{j \in J}$, the family

$$\{(f_i,g_j): A_i \times B_j \rightarrow A \times B\}_{(i,j) \in I \times J}$$

is a covering family. This is true because for any $(a,b) \in A \times B$, if we choose $i \in I$ such that $a \in \text{im } f_i$ and $j \in J$ such that $b \in \text{im } g_j$, $(a,b)$ will be in the image of $(f_i,g_j)$, as desired.

By definition,

$$K_0(\text{Fin}_G) \cong A(G),$$

the Burnside ring of $G$; the monoidal structure above extends the ring structure of the Burnside ring to the higher $K$-groups.

To finish up this section we want to make a couple of observations which will be useful when we discuss almost-finite sets in Section 2.2. Suppose that $G = \mathbb{Z}/p^n$. We define $c_i: \text{Fin}_{\mathbb{Z}/p^n} \rightarrow \text{Fin}$ by

$$c_i(S) \overset{\text{def}}{=} \{s \in S \mid |G \cdot s| \leq p^i\}.$$  

The map $\prod_{i=0}^{n} c_i: \text{Fin}_G \rightarrow \prod_{i=0}^{n} \text{Fin}$ is the “ghost coordinate” map: on $K_0$ it takes a $G$-set to coordinates that add/multiply coordinatewise, and all information about the relative sizes of the $G$-orbits can be recovered from them.
This can also be coordinatized in an alternate manner, using “Burnside coordinates.” Define an “orbit counting map” \( b_i: \text{Fin}_{\mathbb{Z}/p^n} \to \text{Fin} \) by
\[
b_i(S) \stackrel{\text{def}}{=} \{ s \in S \mid |G \cdot s| = p^i \}_G.
\]
Note that the relationship between \( |b_i(S)| \) and \( |c_i(S)| \) looks closely related to the relationship between the Witt coordinates and the ghost coordinates:
\[
|c_i(S)| = \sum_{j=0}^{i} p^j |b_j(S)|.
\]
We can then define \( b = \prod_{i=0}^{n} b_i: \text{Fin}_{\mathbb{Z}/p^n} \to \text{Fin}^n \); note that this is an (additive) isomorphism on \( K_0 \).

We get the following commutative diagram:

\[
\begin{array}{ccc}
\text{Fin}^n \wedge \text{Fin}^n & \xrightarrow{b \wedge b} & \text{Fin}_{\mathbb{Z}/p^n} \wedge \text{Fin}_{\mathbb{Z}/p^n} \\
& \xrightarrow{c \wedge c} & \text{Fin}^n \wedge \text{Fin}^n \\
\text{Fin}^n & \xrightarrow{b} & \text{Fin}_{\mathbb{Z}/p^n} \\
& \xrightarrow{c} & \text{Fin}^n
\end{array}
\]

What must the dotted map be in order to make the diagram commute? On each pair of coordinates, the dotted map counts the number of each type of orbit which appears in the Cartesian product of orbits. Since \( b \) forgets all \( G \)-action information, this is functorial and it does not matter which representatives are taken when the dotted map is defined. Thus the dotted map exists. All of the horizontal maps in this diagram are isomorphisms on \( K_0 \), and the conversion between “ghost coordinates” and “Burnside coordinates” is the conversion between the right-hand side of the diagram and the left-hand side of the diagram.

2.2. Almost-finite \( G \)-sets. Now let \( G \) be a profinite group, and let \( S \) be a \( G \)-set. \( S \) is almost-finite if for all open subgroups \( U \) of \( G \), \( S^U \) is finite, and if for all \( x \in S \), the orbit \( G \cdot x \) is finite. For an in-depth discussion of almost-finite sets (and spaces), see [DS88]. We define \( \text{AFSet}_G \) to be the assembler whose objects are almost-finite \( G \)-sets and whose morphisms are \( G \)-equivariant inclusions. The topology on \( \text{AFSet}_G \) is given by the mutually surjective covering families.

Although we would like to use d\'evissage for assemblers to compute the \( K \)-theory of \( \text{AFSet}_G \), this is not directly possible, since an almost-finite set can be the union of infinitely many different \( G \)-orbits. Thus we need to be a little bit more clever. For an open subgroup \( U \) of \( G \), let \( C_{G/U} \) be the full subcategory of \( \text{SC}(\text{AFSet}_G) \) containing only those \( G \)-sets which are unions of orbits isomorphic to \( G/U \). Each such set is a finite disjoint union of copies of \( G/U \), and thus \( C_{G/U} \cong \text{SC}(\mathbb{S}_{W G U}) \). Thus
\[
\text{SC}(\text{AFSet}_G) \cong \prod_{\text{conj. class } U \leq G} C_{G/U} \cong \prod_{\text{conj. class } U \leq G} \text{SC}(\mathbb{S}_{W G U}).
\]
For any conjugacy class \( U \), the functor projecting to the \( U \)-coordinate is a morphism of assemblers, so induces a map on \( K \)-theory; in particular after applying \( K \)-theory there exists a map
\[
(2.3) \quad \psi: K(\text{AFSet}_G) \to \prod_{U \in C_G} \Sigma^\infty_+ B(W_G U).
\]
Write \( \psi_U: K(\text{AFSet}_G) \to \Sigma^\infty_+ BW_G U \) for the projection onto the \( U \)-th coordinate.
The ring $K_0(\text{AFSet}_G)$ is exactly the Burnside ring of $G$, so $\psi$ induces an isomorphism on $K_0$. By [DS89, Corollary 1 and Theorem 2.12.7], the Burnside ring is isomorphic to the big Witt ring, where the coordinates can be considered to be the “orbit counting” maps (analogous to $b_i$ in the previous section). Addition is represented by the usual disjoint union of sets, while multiplication is given by Cartesian product of almost-finite sets, with the unit the singleton set with the trivial $G$-action. Note that this multiplication, while induced by a morphism of assemblers $\text{AFSet}_G \to \text{AFSet}_G$ is not induced by multiplications on components $S_G/U \otimes S_G/U \to S_G/U$, illustrating that the decomposition above is not compatible with the multiplicative structure. (More concretely: the product of two $G/U$-orbits decomposes into separate orbits; it cannot be modeled by a product of singleton sets.) This is where the interesting multiplicative properties of Witt vectors come from.

To illustrate this last observation, consider the case $G = \mathbb{Z}_p$ and the maps $b$ and $c$ defined in the previous section. There is an analogous commutative diagram

Again, the multiplication induced on the $b$-coordinates is not a simple coordinate-wise one, as the product of two orbits of size $p^i$ (for example) is not a single orbit of size $p^i$. The formula for the relationship between the $c$-coordinates and the $b$-coordinates is exactly that between the ghost coordinates and the Burnside ring. If we compose with the canonical isomorphism between the Burnside ring and the Witt ring, this becomes the standard coordinate transformation between the Witt ring and the ghost coordinates. For more details, see [DS88].

Moreover, these two separate perspectives illustrate where many of the strange multiplicative properties of the Witt vectors come from. The Witt coordinates are closely related to the Burnside coordinates; however, under multiplication, the Burnside coordinates do not simply multiply. Consider that the product of a single orbit of size $p^i$ and an orbit of size $p^j$ (with, WLOG, $j \geq i$) is $p^i$ orbits of size $p^j$; thus the product of a vector with a single 1 in position $i$ with a vector with a single 1 in position $j$ is a vector with a $p^i$ in position $j$.

**Remark 2.4.** In the discussion above it is tempting to use a result showing that $K$-theory commutes with infinite products, such as [Car95] or [KW20], to conclude that $\psi$ is an isomorphism. Unfortunately, the $K$-theory of assemblers cannot directly use either of these results—the former assumes Waldhausen categories with cylinder functors (which $\text{SC}(C)$ does not have) and the latter assumes exact categories—and thus these results are not accessible to us at this point.

As we expect that $K$-theory of assemblers does commute with infinite products, we conjecture that the map $\psi$ is actually a weak equivalence.

2.3. **Varieties.** Let $\text{Var}_k$ be the assembler of varieties over a base field $k$. The morphisms in the assembler are locally closed embeddings; the topology is generated by the coverage $\{Y \hookrightarrow X, X \smallsetminus Y \twoheadrightarrow X\}$, where $Y \hookrightarrow X$ is a closed embedding. This example is discussed in detail in [Zak17A, Section 5.1]. The fiber product (over Spec $k$) of varieties induces a product structure on $\text{Var}_k$, with Spec $k$ as the unit. This produces an $E_\infty$-ring structure on $K(\text{Var}_k)$ which on $K_0$ gives the Grothendieck ring of varieties.

If we would like to work over a base Noetherian scheme $S$, instead of a base field, an analogous construction works with the ring structure given by taking the fiber product over $S$ (in which case $S$ is the unit). (Again, we assume that varieties over $S$ are reduced separated schemes of finite type over $S$.) It is important to note that the technical condition in the following lemma requires
equalities and not canonical isomorphisms; the desired formulas always hold up to canonical isomorphism.

**Lemma 2.5.** Let \( f: T \to S \) be a morphism of schemes. Base change along \( f \) induces a symmetric monoidal morphism of assemblers \( \text{Var}_S \to \text{Var}_T \). In particular, \( K_*(\text{Var}_S) \to K_*(\text{Var}_T) \) is ring homomorphism.

**Proof.** Base change is a morphism of assemblers because it preserves disjointness and covering families in the generating coverage. To check that it is a symmetric monoidal map it suffices to check that the following diagrams commute up to natural isomorphism:

\[
\begin{array}{ccc}
\text{Var}_S & \xrightarrow{\mu} & \text{Var}_S \\
\downarrow & & \downarrow \\
\text{Var}_S & \xrightarrow{\mu} & \text{Var}_S \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\text{Var}_S & \xrightarrow{f' \land f'} & \text{Var}_T \\
\downarrow & & \downarrow \\
\text{Var}_S & \xrightarrow{f'} & \text{Var}_T \\
\end{array}
\]

Since \( f' \) maps \( S \) to \( T \), the right-hand diagram commutes. Since both base change and the multiplication are via fiber products, the left-hand diagram commutes up to natural isomorphism. \( \Box \)

**Remark 2.6.** It may seem that this lemma is overly-complicated: the given formulas always hold up to unique isomorphism, and for most purposes this is sufficient. However, this is not sufficient in the current case: in order to define a monoid object in a category, the given diagrams must commute exactly, not simply up to unique isomorphism. This is, in fact, precisely the problem that \( \infty \)-categories are designed to address: situations where coherence issues occur because of imprecise commutativity. In the current situation it ought to be the case that the weaker commutativity should be sufficient, and that the map \( K(\text{Var}_S) \to K(\text{Var}_T) \) should be \( E_\infty \) regardless of which model is taking. However, proving this would require keeping track of the \( E_\infty \)-operad structure through the \( K \)-theoretic machinery, which can be quite complicated. (See for example [EM06] for a description of this; note that their example is not sufficient for the current application, and would need to be weakened further, as the current situation would not be bipermutative.) We therefore take the alternate track of focusing on the special case of interest to us.

**Example 2.7.** Let \( A \) and \( B \) be rings, with a map \( f: \text{Spec} B \to \text{Spec} A \). Write \( \text{Var}_A \) for the full subassembler of \( \text{Var}_{\text{Spec} A} \) of reduced separated affine schemes of finite type over \( \text{Spec} A \). Since all varieties have a finite disjoint covering family by affines, the inclusion \( \text{Var}_A \to \text{Var}_{\text{Spec} A} \) induces an equivalence on \( K \)-theory (by dévissage, [Zak17A, Theorem B]). Define the assembler \( \text{Var}_B \) analogously.

The multiplication and the base change to working over \( B \) on \( \text{Var}_A \) can be modeled by the tensor product of \( A \)-algebras, and it suffices to define a tensor product for which the formulas \( A \otimes_A B = B \) and

\[
(R \otimes_A R') \otimes_A B = (R \otimes_A B) \otimes_B (R' \otimes_A B)
\]

hold. (Note, again, that these are equalities, and not isomorphisms; these will of course always be canonically isomorphic.)

For example, here is a method for building such a model. Pick a tensor product functor on \( \text{Var}_A \). For every \( A \)-algebra \( R \) there is a \( B \)-algebra \( R \otimes_A B \) which is generated by pairs \( (r,b) \).

Define a tensor product on \( \text{Var}_B \) by defining \( (R \otimes_A B) \otimes_B (R' \otimes_A B) \) to be generated by classes of triples \( (r,r',b) \) modulo the necessary relations. (This differs from the usual definition: usually we would take pairs of pairs \( ((r,b),(r',b')) \) and define the appropriate relations on those.) Thus, on the subcategory of those modules which are in the image of base change from \( \text{Var}_A \), we impose the formula by definition.
In a more combinatorial example, we can construct a derived \( \zeta \)-function. Let \( k \) be a finite field, \( \text{Var}_k \) be the assembler of varieties over \( k \), and \( \text{AFSet}_{\mathbb{Z}} \) be the assembler of almost-finite \( \mathbb{Z} \)-sets. There is a morphism of assemblers
\[
(2.8) \quad \zeta : \text{Var}_k \longrightarrow \text{AFSet}_{\mathbb{Z}} \quad \text{given by } X \longmapsto X(\kappa).
\]

**Lemma 2.9.** The morphism of assemblers \( \zeta \) is a symmetric monoidal morphism, in the sense that the following diagrams commute up to natural isomorphism:
\[
\begin{align*}
\text{Var}_k \wedge \text{Var}_k & \xrightarrow{\mu} \text{AFSet}_{\mathbb{Z}} \wedge \text{AFSet}_{\mathbb{Z}} \\
\text{Var}_k & \xrightarrow{\zeta} \text{AFSet}_{\mathbb{Z}}
\end{align*}
\]
and
\[
\begin{align*}
\text{S}^{\text{Sm}} & \longmapsto \text{Var}_k \\
\eta & \longmapsto \zeta
\end{align*}
\]
\[
\begin{align*}
\text{Var}_k & \hfill \text{AFSet}_{\mathbb{Z}}
\end{align*}
\]

In particular, \( K_*(\zeta) \) is a ring homomorphism.

3. Detecting non-permutative elements in \( G \)-sets

In this section we discuss how to use the theory developed above to detect nontrivial elements in \( K_1 \) of an assembler. The particular elements we are concerned with are the “non-permutative” elements:

**Definition 3.1.** Let \( E \) be a connective \( E_\infty \)-ring spectrum with unit map \( S \longrightarrow E \). An element in \( \pi_nE \) is 0-dimensional if it is in the image of \( \pi_nS \longrightarrow \pi_nE \). An element in \( \pi_nE \) is permutative if it is in the image of the map
\[
\pi_0E \otimes \pi_nS \longrightarrow \pi_0E \otimes \pi_nE \longrightarrow \pi_nE.
\]

If an element is not permutative then it is non-permutative.

In \cite{CWZ19} the authors showed that there exist elements in \( K_n(\text{Var}_k) \) which are not 0-dimensional; however, the question of whether there exist non-permutative elements was left open. The goal of the rest of this paper is to show that such elements exist in \( K(\text{Var}_k) \). The existence of non-permutative elements in the higher homotopy of \( E \) demonstrates that \( E \) is not uniquely determined by \( \pi_0E \) and the higher homotopy groups of \( S \). In particular, we will use it to demonstrate that the higher \( K \)-groups of \( \text{Var}_k \) contain nontrivial information about the geometry of varieties.

The important theorem for determining that certain elements are non-permutative is the following theorem, whose proof is put off until later:

**Theorem 3.2.** For a closed symmetric monoidal assembler \( C \), \( [X] \in K_0(C) \), and \( [A, \tau] \in K_1(C) \),
\[
[X][A, \tau] = [X \times A, 1 \times \tau] \in K_1(C).
\]

This is a special case of Theorem 8.3.

3.1. Finite \( G \)-sets. First, consider the simple case of finite \( G \)-sets. From (2.2) it follows that
\[
K_1(\text{Fin}_G) \cong \bigoplus_{H \in C_G} \mathbb{Z}/2 \oplus (W_G/H)^{ab}.
\]

By Theorem 3.2, the product of an element in \( K_0(\text{Fin}_G) \), represented as \( [A] \) for some finite \( G \)-set \( A \), and the nontrivial element in \( K_1(\text{Fin}) \), represented by \([\{1, 2\}, \tau] \) is represented by the element \([A \amalg A, \tau] \), where \( \tau \) swaps the two copies of \( A \). In particular, neither of the morphisms in this pair has a nontrivial action on any \( G \)-orbit of \( A \); thus this lands in the subgroup \( \bigoplus_{H \in C_G} \mathbb{Z}/2 \oplus 0 \). Conversely, any term in this subgroup is represented, since \([G/H \amalg G/H, \tau] \) represents the element with a 1 in the \( H \)-coordinate and 0's everywhere else. Thus the non-permutative elements in the group are exactly those that are outside this subgroup.
3.2. Almost-finite $G$-sets. The analysis of the previous subsection can be extended to almost-finite $G$-sets, although there is again the additional difficulty that we do not know whether $K$-theory commutes with infinite products. However, we can show the following:

**Proposition 3.3.** Recall the map $\psi$ from (2.3). All permutative elements $\alpha \in K_1(\text{AFSet}_G)$ have

$$\psi_\ast \alpha \in \prod_{H \in C_G} Z/2 \oplus 0 \subset \prod_{H \in C_G} Z/2 \oplus (W_G H)^{ab}.$$  

Moreover, the subgroup of permutative elements surjects onto this subgroup.

**Proof.** The proof that the image of any permutative element lies in this group is identical to the previous subsection. That every element in this subgroup is represented can be seen by taking an element in the subgroup, which is of the form $\{\epsilon_H\}_{H \in C_G}$ with $\epsilon_H = \pm 1$. Consider the almost-finite $G$-set $A \overset{\text{def}}{=} \prod_{\epsilon_H = 1} G/H$. Then $[A \amalg A, \tau]$ exactly represents $\{\epsilon_H\}_{H \in C_G}$. \hfill \square

4. A combinatorial derived zeta function

Let $k$ be a finite field, and let $L_n$ be the unique extension of $k$ of degree $n$. The local zeta function of a variety $X$ over $k$ is defined by

$$Z(X, t) \overset{\text{def}}{=} \exp \sum_{n \geq 1} |X(L_n)| \frac{t^n}{n}.$$  

There are two other classical ways of writing this function. We can think of the function $Z(\cdot, t)$ as taking a variety to a power series in $t$ with integer coefficients. (We know that it will have integer coefficients because we can rewrite the expression above as $\sum_{n \geq 0} (\text{Sym}^n X)(k)[t^n]$.) We can see by analyzing the expression for $Z(X, t)$ that, for a closed embedding $Y \hookrightarrow X$,

$$Z(X, t) = Z(Y, t)Z(X \setminus Y, t).$$

Thus $Z(-, t)$ is actually a homomorphism $K_0(\text{Var}_k) \rightarrow (1 + t \mathbb{Z}[t], \times)$. If we consider the codomain as the big Witt ring with the multiplication of Witt vectors, $Z(-, t)$ is a ring homomorphism.

By the discussion in Section 2.1, the big Witt ring is $K_0(\text{AFSet}_{\mathbb{Z}})$, and, indeed, we can see that all of the data necessary to construct $Z(X, t)$ is contained in the $\mathbb{Z}$-structure of $X(\bar{k})$. Thus the morphism of assemblers $\zeta$ in (2.8) gives the desired derived map

$$K(\zeta): K(\text{Var}_k) \longrightarrow K(\text{AFSet}_{\mathbb{Z}}).$$

Since $\zeta$ is a morphism of monoidal objects, this map induces a map of rings. Moreover, because this map is compatible with the unit maps it takes permutative elements to permutative elements. In order to demonstrate that an element in $K_1(\text{Var}_k)$ is non-permutative it therefore suffices to show that it maps to a non-permutative element in $K_1(\text{AFSet}_{\mathbb{Z}})$.

The map $\psi$ defined in Section 2.2 induces a map

$$K_1(\text{AFSet}_{\mathbb{Z}}) \longrightarrow \prod_{n \geq 1} K_1(\mathbb{S}^{\text{Asm}}_{\mathbb{Z}/n}) \cong \prod_{n \geq 1} Z/2 \oplus Z/n.$$  

We write an element in the codomain as $\prod_{n \geq 1}(\pm 1, g)$. In each pair we call the first coordinate the *external* coordinate and the second the *internal* coordinate; these correspond to the sign of the permutation of $\ast$'s in the objects of $\text{SC}(\mathbb{S}^{\text{Asm}}_{\mathbb{Z}/n})$ and the action of $\mathbb{Z}/n$, respectively.

**Lemma 4.2.** Let $[X] \in K_0(\text{Var}_k)$, and let $\eta \in K_1(\text{Fin})$ be the nonzero element, and write $L_j$ for the extension of $k$ of degree $j$. Let

$$X_n = \{ x \in X(L_n) \mid x \notin X(L_m), \ m < n \}.$$
Then

\[ \psi_* \circ \zeta_*([X]\eta) = \prod_{n \geq 1}((-1)^{|X_n|/n}, 0). \]

Proof. Write \( \psi_n \) for the composition of \( \psi_* \) and the projection onto the coordinate indexed by \( n \). Write \( \text{Fin}^\mathbb{Z}/n \) for the assembler of finite sets with free \( \mathbb{Z}/n \)-action; then \( K(\text{Fin}^\mathbb{Z}/n) \simeq \Sigma_\infty^B \mathbb{Z}/n \), and the map \( \text{Var}_k \to \text{Fin}^\mathbb{Z}/n \) mapping \( X \) to \( X_n \) gives the \( n \)-th coordinate of the map \( \psi_* \zeta_* \). There is a map of assemblers \( \text{Fin} \to \text{Fin}^\mathbb{Z}/n \) defined by \( S \mapsto S \times \mathbb{Z}/n \), with \( \mathbb{Z}/n \) acting on the second coordinate.

It suffices to show that for all \( n \) there exists a morphism \( \sigma_n \) which makes the following square commute:

\[
\begin{array}{ccc}
K_1(\text{Var}_k) & \xrightarrow{\psi_n} & K_1(\text{Fin}^\mathbb{Z}/n) \\
\mu \downarrow & & \downarrow \\
K_0(\text{Var}_k) \otimes K_1(\text{Asm}) & \xrightarrow{\sigma_n} & K_1(\text{Fin}).
\end{array}
\]

Pick a generator of \( K_0(\text{Var}_k) \), represented by a variety \([X]\) and the generator \([* \Pi *, \text{swap}]\) of \( K_1(\text{Asm}) \). The map \( \mu \) then maps this generator to the generator \([X \Pi X, \text{swap}]\); the map \( \psi_n \) maps this to the generator

\[ [X_n \Pi X_n, \text{swap}]. \]

Note that \( X_n \cong (X_n)_\text{Frob} \times \mathbb{Z}/n \). Thus if we define

\[ \sigma_n([X] \otimes \eta) \overset{\text{def}}{=} [(X_n)_\text{Frob} \Pi (X_n)_\text{Frob}, \text{swap}] \]

the diagram commutes, as desired. Moreover, the sign of the induced permutation is exactly the sign of swapping \( |X_n|/n \) orbits, giving the desired formula. \( \square \)

Since \( K_* (\zeta) \) is a ring homomorphism, the image of a permutative element in \( K_* (\text{Var}_k) \) is a permutative element in \( K_* (\text{Asm}) \). In particular, permutative elements have all internal coordinates equal to 0. In order to find a nonpermutative element it suffices to find an element which has a nonzero internal coordinate.

By [Zak17B, Theorem B], any automorphism of a variety represents an element of \( K_1(\text{Var}_k) \). The functor \( \zeta \) takes this data to a \( \mathbb{Z} \)-set \( X(\mathbb{Z}) \) together with a \( \mathbb{Z} \)-equivariant permutation; projecting onto the \( \mathbb{Z}/2 \)-coordinate in (4.1) induces a map

\[ \psi_2 : K_1(\text{Var}_k) \to K_1(\text{Asm}_{\mathbb{Z}/2}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2. \]

Here, the first coordinate is the external coordinate, and the second is the internal coordinate. As before, we write the external coordinate multiplicatively and the internal coordinate additively.

Definition 4.3. Let \( S \) be a finite set equipped with an action of \( \mathbb{Z}/m \oplus \mathbb{Z}/n \). Suppose that this action is free when restricted to both \( \mathbb{Z}/m \oplus 1 \) and \( 1 \oplus \mathbb{Z}/n \). For a point \( x \in S \) write \([x]\) for the orbit of \( x \) under the \( \mathbb{Z}/m \oplus \mathbb{Z}/n \) action. This orbit has type \((d,a)\) if \((d,0) \cdot x = (0,a) \cdot x\) for \( 0 \leq a < n \) and \( d \) is the minimal positive integer for which such an integer \( a \) exists.

Define \#S_{(d,a)} \label{S} for the number of orbits of type \((d,a)\).

Let \( X \) be a variety over \( \mathbb{F}_q \) equipped with an automorphism \( \varphi \) which acts on \( X_n \) freely with order \( m \). We consider \( X_n \) to be equipped with action of \( \mathbb{Z}/m \oplus \mathbb{Z}/n \) by having \((1,0)\) act by \( \varphi \) and \((0,1)\) act by Frobenius. Define \#X_{(d,a)}^{m,n} \label{X} to be the number of orbits of type \((d,a)\) in \( X_n \); when \( n \) and \( \varphi \) are clear from context we omit them.
If there exists an orbit of type \((d,a)\) in \(S\) then it is necessarily the case that \(d|m\) and that \(\frac{m}{d} = \frac{n}{(n,a)}\). In particular, \(d = m\) if and only if \(a = 0\).

We can use this to describe the image of an element \([X,\varphi] \in K_1(\text{Var}_k)\) under \(\psi_n\) explicitly in terms of the actions on orbits.

**Proposition 4.4.** Let \(X\) be a variety over \(k\) and let \(\varphi\) be an automorphism of \(X\); suppose that \(\varphi\) acts freely on \(X_n\) with order \(m\). For each pair \((d,a)\) write \(#X_{(d,a)}\) be the number of orbits of \(X_n\) of type \((d,a)\). Then

\[
\psi_n([X,\varphi]) = \sum_{(d,a)} (#X_{(d,a)})((-1)^{d+1},a).
\]

**Proof.** The element in \(K_1(\text{Fin}^{\mathbb{Z}/n})\) is the sum of its actions on orbits, so it suffices to consider a single orbit \([x]\) of type \((d,a)\). This orbit consists of \(d\) disjoint Frobenius orbits. If we write \(x_i = \varphi^i \cdot x\) for \(i = 0, \ldots, d - 1\) we can think of the \(d\) Frobenius orbits as the sets \(\{x, \text{Frob}_n \cdot x, \ldots, \text{Frob}_1^{-1} \cdot x\}\) then the action of \(\varphi\) takes the \(i\)-th orbit to the \(i+1\)-st orbit with no \(\mathbb{Z}/n\)-action for all \(0 \leq i < d - 1\); however, when \(i = d - 1\) it takes the \(i\)-th orbit to the 0-th orbit with an additional \(\mathbb{Z}/n\) action by \(a\). We can thus write it as a cyclic permutation of \(d\) orbits, followed by a twist by \(a\) on a single orbit. Thus the representative in \(K_1(\text{Fin}^{\mathbb{Z}/n})\) is

\[
((-1)^{d+1}, 0) + (1, a) = ((-1)^{d+1}, a),
\]
as desired.

Summing over all orbits gives the desired formula. \(\square\)

Proposition 4.4 and Lemma 4.2 can be used to find non-permutative elements in \(K_1(\text{Var}_{\mathbb{Z}/n})\).

Our first result is a special case of the above result.

**Proposition 4.5.** Fix an integer \(n > 1\). Let \(\lambda \in k^\times\) have order \(m\), and define \(P_1 = \#(\mathbb{P}^1)_{(m,0)}\) and for any other \(d|n\) let \(P_d = \#(\mathbb{P}^1)_{(m/d,n/d)}\); for all other \(d\) we define \(P_d = 0\). Let \(\phi\) be the Euler \(\phi\)-function. Then

\[
\psi_n([\mathbb{P}^1, \lambda x]) = \left((-1)^{P_1(m+1)+P_2(m/2+1)} \sum_{d|n,m} P_d \phi(d) \cdot \frac{n}{d}\right),
\]

where \(\phi\) is the Euler \(\phi\)-function. In particular, if \((n,m) = 1\) then the second coordinate is 0.

**Proof.** From the formula in Proposition 4.4, and writing \(X = \mathbb{P}^1\) for conciseness,

\[
\psi_n([\mathbb{P}^1, \lambda x]) = \sum_{(d,a)} (#X_{(d,a)})((-1)^{d+1},a).
\]

Suppose that \(d, a, a'\) are such that both \(X_{(d,a)}\) and \(X_{(d,a')}\) are nonempty. The necessary conditions on \(a\) and \(a'\) ensure that there exists a constant \(c\) and two primitive roots \(g, g'\) such that

\[
\lambda^a = g^c \quad \text{and} \quad \lambda^{a'} = (g')^c.
\]

With these primitive roots we can construct a bijection between \(X_{(d,a)}\) and \(X_{(d,a')}\) in the following manner. Given an orbit \([x] \in X_{(d,a)}\), by definition

\[
x^{q^d} = \lambda^a x \iff x^{q^d - 1} = \lambda^a.
\]

Writing \(x = g^y\), such a point \(x\) is a solution to \(y\) to the equation

\[
y(q^d - 1) \equiv ca \pmod{q^n - 1}.
\]

In particular, this point \(x\) corresponds to a point \(x' \overset{\text{def}}{=} (g')^y\). This gives a function \(X_{(d,a)} \to X_{(d,a')}\); the inverse is given by the reverse choice of primitive roots. This bijection shows that for any two \(a\) and \(a'\) satisfying \(\frac{m}{d} = \frac{n}{(n,a)}\) it is the case that \(\#X_{(d,a)} = \#X_{(d,a')}\). Thus in the sum above we can
choose \( a = (n, a) = \frac{n}{m/d} \) for every \( d \) and multiply by the number of choices of \( a \), which is exactly \( \phi(m/d) \). Moreover, such a choice is possible only if \( (m/d) \mid n \). Thus the sum can be rewritten as

\[
\psi_n([\mathbb{P}^1, \lambda x]) = \sum_{d|m \atop m/d|n} \phi\left(\frac{m}{d}\right) \#X(d, \frac{m}{m/d}) \left((-1)^{d+1}, \frac{m}{d}\right) = \sum_{d|(n, m)} \phi(d)\#X(m/d, n/d)\left((-1)^{m/d+1}, d\right).
\]

Since \( \phi(d) \) is even unless \( d = 1 \) or 2, the first coordinate of each summand will almost always be 1. Taking this into account gives the desired formula.

Using this we can do a a complete analysis of the image under \( \psi_n \) of \([\mathbb{P}^1, 1/x] = [\mathbb{P}^1, -x] \), in order to illustrate both the benefits and the drawbacks of the approach.

**Corollary 4.6.** Let \( k = \mathbb{F}_q \). Writing \( n = 2^m n' \) with \( n' \) odd,

\[
\psi_n([\mathbb{P}^1, -x]) = \begin{cases} 
\left((-1)^{\frac{n-1}{2}}, 0\right) & \text{if } n = 1, \\
\left((-1)^{\frac{n-1}{2}}, \frac{q-1}{2}\right) & \text{if } n = 2, \\
(1, 0) & \text{otherwise.}
\end{cases}
\]  

In particular, if \( q \equiv 3 \pmod{4} \) then \([\mathbb{P}^1, -x] = [\mathbb{P}^1, 1/x] \) is non-permutative.

**Proof.** When \( n = 1 \) Galois orbits are trivial, and thus the internal coordinate is 0. Thus the question becomes to compute the sign of the permutation that \( 1/x \) induces on \( \mathbb{F}_q \setminus \{0\} \). There are two fixed points \((\pm 1)\) and the rest are paired up into transpositions, so the sign is \((-1)^{\frac{n-1}{2}}\), as desired.

For \( n > 1 \) we use the formula in Proposition 4.5. In this the formula simplifies to:

\[
\psi_n([\mathbb{P}^1, -x]) = \begin{cases} 
\left((-1)^{P_1}, P_2 \frac{q}{2}\right) & \text{if } n \text{ is even} \\
\left((-1)^{P_1}, 0\right) & \text{otherwise.}
\end{cases}
\]

In particular, the result only depends on the parities of \( P_1 = \#X(2,0) \) and \( P_2 = \#X(1, \frac{q}{2}) \).

Before we begin the more complicated cases, some notation for the rest of the proof. For integers \( a \) and \( b \), \( M_a(b) \) is the number of aperiodic necklaces of length \( b \) with beads of \( a \) colors. The function \( \mu(m) \) is the Mobius function, which is 0 if \( m \) is not squarefree, and otherwise is \(-1\) to the power of the number of distinct prime factors of \( m \). The symbol \( \delta_{ij} \) is the Kronecker delta function. The important facts about these to know are that

\[
M_a(b) = \frac{1}{b} \sum_{d|b} \mu\left(\frac{b}{d}\right) a^d \quad \text{and} \quad \delta_{1a} = \sum_{d|a} \mu(d).
\]

Suppose \( n \) is odd, so \( X(1, \frac{q}{2}) \) is empty. We have (by Mobius inversion)

\[
P_1 = \frac{1}{2n} (q^n - \#\{\text{points in smaller extensions}\}) = \frac{1}{2n} \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d.
\]

Since we only care about the parity of this number it suffices to consider the sum \( \sum_{d|n} \mu(n/d)q^d \) modulo 4. Since \( n \) is odd, \( d \) must also always be odd; in particular, \( q^d \equiv q \pmod{4} \) for all \( d \). Thus, since \( n > 1 \),

\[
(2n)P_1 \equiv q \sum_{d|n} \mu\left(\frac{n}{d}\right) = 0 \pmod{4}.
\]

This completes the odd case.

When \( n \) is even, write \( n = 2^m n' \); there are two types of orbits \((2,0) \) and \((1, \frac{q}{2}) \). Consider first orbits of type \((1, \frac{q}{2}) \). A point in an orbit of type \((1, \frac{q}{2}) \) satisfies \(-x = \text{Frob}^{n/2} x \). There are exactly \( q^{n/2} - 1 \) of these; if any solution lies in a subextension of even index then it must lie in \( \mathbb{F}_q \), which
contains exactly 2 solutions. Call a point $x$ good if $x^{q^{n/2}-1} = 1$. Let $L_1, \ldots, L_b$ be the maximal proper subfields of $\mathbb{F}_q$ of odd index. Then, using the principle of inclusion/exclusion (or Mobius inversion),

$$nP_2 = (q^{n/2} - 1) - \sum_{i=1}^{b} \#\{\text{good points in } L_i\} + \sum_{i,j} \#\{\text{good points in } L_i \cap L_j\} - \cdots$$

$$= \sum_{d|n, \frac{n}{d} \text{ odd}} \mu\left(\frac{n}{d}\right) (q^{d/2} - 1) = \sum_{d|n'} \mu\left(\frac{n'}{d}\right) (q^{2m-1}d - \delta_{1n'}).$$

Only the parity of $P_2$ matters, so it suffices to consider the right-hand side modulo $2^{m+1}$. If $m > 1$, since $q$ is odd then $q^{2m-1} \equiv 1 \pmod{2^{m+1}}$, so

$$nP_2 = \sum_{d|n'} \mu\left(\frac{n'}{d}\right) - \delta_{1n'} \equiv 0 \pmod{2^{m+1}}.$$

On the other hand, if $m = 1$ we have $q^d \equiv q \pmod{4}$ and thus

$$nP_2 \equiv (q - 1)\delta_{1n'} \pmod{4},$$

thus giving the desired formula.

Now consider $P_1$. Using the fact that an orbit of type $(2,0)$ has $2n$ points and an orbit of type $(1,\frac{n}{2})$ has $n$ points, in terms of point counts over $\mathbb{F}_q^n$,

$$P_1 = \frac{1}{2n}(q^n - \#\{\text{points in smaller extensions}\} - \#\{\text{points in orbits of type } (1,\frac{n}{2})\})$$

$$= \frac{1}{2n}(q^n - \#\{\text{points in smaller extensions}\} - nP_2).$$

By Mobius inversion,

$$q^n - \#\{\text{points in smaller extensions}\} = \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d.$$

Thus the parity of $P_1$ can be determined by considering $2n\#P_1$ modulo $2^{m+2}$.

$$2nP_1 \equiv \left(\sum_{d|n} \mu\left(\frac{n}{d}\right) q^d\right) - \delta_{1n'}(q^{2m-1} - 1) \pmod{2^{m+2}}.$$

Consider the first sum. If $\text{ord}_d d < m - 1$ then $\mu(n/d) = 0$. Thus the only summands which are nonzero must have $\text{ord}_d(d) = m$ or $m - 1$. Thus

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) q^d = \sum_{d|n'} \mu\left(\frac{n'}{d}\right) q^{2m-1} - \sum_{d|n'} \mu\left(\frac{n'}{d}\right) q^{2m-1} - \sum_{d|n'} \mu\left(\frac{n'}{d}\right) (q^{2m}d - q^{2m-1}d).$$

Modulo $2^{m+2}$, $q^{2m}d \equiv 1$. As $d$ is odd, $q^{2m-1}d \equiv q^{2m-1} \pmod{2^{m+2}}$. We can therefore conclude that

$$2n\#X_{(2,0)} \equiv -2\delta_{1n'}(q^{2m-1} - 1) \pmod{2^{m+2}}.$$  

In particular, if $n' > 1$ this is 0. If $m > 1$ then $q^{2m-1} \equiv 1 \pmod{2^{m+1}}$, this must be 0. Lastly, if $n = 2$ this is $-2(q - 1) \pmod{8}$; in other words, $\#X_{(2,0)}$ is even if $q$ is $q \equiv 1 \pmod{4}$ and odd if $q \equiv 3 \pmod{4}$.

**Remark 4.8.** The question of whether $[\mathbb{P}^1,1/x]$ is non-permutative (or even non-0-dimensional!) when $q \equiv 1 \pmod{4}$ remains open.

The previous result implies that multiplying by $-1$ is non-permutative if $-1$ is not a square. We can use the intuition behind this to show that non-permutative elements always exist:
Corollary 4.9. Let \( \mathbb{F}_q \) be a finite field such that \( \text{ord}_2(q-1) = \ell \). Let \( \alpha \) be a primitive \( 2^{\ell} \)-th root of unity. Then the element \([\mathbb{P}^1, \alpha x]\) is non-permutative.

Proof. When \( \ell = 1 \) this is simply Corollary 4.6, so we focus on the case \( \ell > 1 \). By Proposition 4.5, keeping in mind that \( m = 2^\ell \),

\[
\psi_{2^\ell}[\mathbb{P}^1, \alpha x] = \left((-1)^{P_1+P_2}, 2^{\ell-1} \sum_{j=1}^{\ell} P_{2j}\right).
\]

To show that \([\mathbb{P}^1, \alpha x]\) is non-permutative it therefore suffices to check that \( \sum_{j=1}^{\ell} P_{2j} \) is odd. In fact, we claim that \( P_{2\ell} \) is odd and \( P_{2j} \) is even for \( 1 \leq j < \ell \).

First, consider \( P_{2\ell} \), which counts orbits of type \((1,1)\). Solutions to \( x^{2\ell} = \alpha x \) correspond to solutions to

\[
a(q-1) \equiv \frac{q^{2\ell}-1}{2^\ell} \pmod{q^{2\ell}-1}.
\]

Since \( (q-1)/2^\ell \mid q-1, \frac{q^{2\ell}-1}{2^\ell}, q^{2\ell}-1 \), solutions to this equation exist exactly when solutions to

\[
2^{\ell} a \equiv q^{2\ell-1} + \cdots + q + 1 \pmod{2^{\ell}(q^{2\ell-1} + \cdots + 1)}
\]

exist. As there are \( 2^\ell \) terms in the right-hand side of the equivalence, solutions exist—and thus there are exactly \( q-1 \) different solutions to the original equation. Note, in addition, that none of these are in extensions of lower degree. If we instead consider solutions to \( x^{d} = \alpha x \) in \( \mathbb{F}_{2d} \) with \( d < \ell \) then they correspond to solutions to

\[
2^{\ell} a \equiv q^{2d-1} + \cdots + q + 1 \pmod{2^{\ell}(q^{2d-1} + \cdots + 1)}.
\]

The left-hand side of the equivalence is \( 0 \mod 2^\ell \), while the right-hand side is \( 2^d \neq 0 \pmod{2^\ell} \); thus there are no solutions in lower extensions. Thus

\[
P_{2\ell} = \frac{q-1}{2^\ell} \equiv 1 \pmod{2}.
\]

Now consider \( P_{2\ell-1} \), which counts orbits of type \((2,2)\). Solutions to \( x^{2^{\ell-1}} = \alpha^2 x \) correspond to solutions to

\[
a(q^2-1) \equiv \frac{q^{2^{\ell-1}}-1}{2^{\ell-1}} \pmod{q^{2^{\ell-1}}-1}.
\]

As above, there are exactly \( q^2 - 1 \) solutions to this equation. Now consider \( x^{q^2} = \alpha^2 x \) over \( \mathbb{F}_{q^{2d}} \) for \( d < \ell \). If \( d = 0, 1 \) there are no solutions, since \( x^{q^2} = x \). If \( d \geq 2 \) then solutions to this equation correspond to solutions to

\[
a(q^2-1) \equiv \frac{q^{2d}-1}{2^{\ell-1}} \pmod{q^{2d}-1}.
\]

Dividing all three terms by \( \frac{q^2-1}{2^{\ell-1}} \) gives

\[
2^{\ell-1} a \equiv q^{2^{\ell-2}} + q^{2^{\ell-4}} + \cdots + q^2 + 1 \pmod{2^{\ell-1}(q^{2\ell-2} + q^{2\ell-4} + \cdots + q^2 + 1)}.
\]

The left-hand side of the equivalence is \( 0 \mod 2^{\ell-1} \), but the right-hand side is equivalent to \( 2^{d-1} \neq 0 \pmod{2^{\ell-1}} \), since \( d < \ell \). Thus there are no solutions, and thus no points over lower-degree extensions. However, every orbit of type \((1,1)\) also gives solutions to this equation, as do orbits of type \((1,2^{\ell-1}+1)\) (which are in bijection with orbits of type \((1,1)\)). Thus

\[
P_{2^{\ell-1}} = \frac{1}{2^{\ell+1}} \left((q^2-1) - 2^{\ell+1} P_1\right) = \frac{1}{2^{\ell+1}} \left((q^2-1) - 2(q-1)\right) = \frac{(q-1)^2}{2^{\ell+1}}.
\]

Thus \( \text{ord}_2 P_{2^{\ell-1}} = 2\ell - (\ell + 1) = \ell - 1 \); since \( \ell > 1 \) this is positive, and thus \( P_{2^{\ell-1}} \) is even.
We now claim that for $1 \leq r \leq \ell$, 
\[ P_{2^l-r} = \frac{(q^{2r-1} - 1)^2}{2^{l+r}}, \]
so that 
\[ \text{ord}_q P_{2^l-r} = \ell + r - 2 \geq 1; \]
in particular $P_{2^l-r}$ is always even, from which the result follows. We prove this by induction on $r$. The base cases $r = 0, 1$ were done above, so we proceed to the inductive step. This is analogous to the base case $r = 1$, although with slightly more bookkeeping. To compute $P_{2^l-r}$ we first count solutions to $x^{2r} = a^{2r}x$. In $\mathbb{F}_{q^2}$ solutions to this correspond to solutions to 
\[ a(q^{2r} - 1) \equiv \frac{q^{2r} - 1}{2^{l-r}} \pmod{q^{2r} - 1}. \]
Since all three terms are divisible by $q^{2r} - 1$ solutions to the equation exist, and thus there are $q^{2r} - 1$ solutions. For $d < \ell$, solutions to the equation in $\mathbb{F}_{q^d}$ correspond to solutions to 
\[ a(q^{2r} - 1) \equiv \frac{q^{2d} - 1}{2^{l-r}} \pmod{q^{2d} - 1}. \]
Dividing both sides by $\frac{q^{2r} - 1}{2^{l-r}}$ shows that solutions to this equation exist exactly when there exist solutions to 
\[ 2^{l-r}a \equiv q^{2d-2r} + \cdots + q^{2d} + 1 \pmod{2^{l-r}(q^{2d-2r} + \cdots + q^{2d} + 1)}. \]
Modulo $2^{l-r}$ the left-hand side is 0 but the right-hand side is $2^{d-r}$, which is not 0; thus there are no solutions over lower-degree extensions. However, some of these extensions come from orbits of type $P_{2^l-r}$ for $r' < r$ and we have 
\[ P_{2^{l-r}} = \frac{1}{2^{l+r}} \left( (q^{2r} - 1) - \sum_{j=0}^{r-1} 2^{r-j} \cdot 2^{\ell+j} P_{2^{l-j}} \right) = \frac{1}{2^{l+r}} \left( q^{2r-1} - 1 \right)^2, \]
by applying the induction hypothesis and the two base cases inside the summation. \qed 

As an alternate approach, one can consider elliptic curves with complex multiplication by $i$.

Example 4.10. Let $k = \mathbb{F}_q$ with $q \equiv 1 \pmod{4}$. Then the elliptic curve $E$ given by $y^2 = x^3 + x$ has an automorphism $\varphi: (x, y) \mapsto (-x, iy)$. This has order 4 on all finite points except for those where $y = 0$, which are all over $\mathbb{F}_q$. Consider $\psi_2[E, \varphi]$. There are two types of orbits: $(4, 0)$ and $(2, 1)$. A point $(x, y)$ in an orbit of type $(2, 1)$ has $(x, y) = (x, -y)$; in other words, if we write $\mathbb{F}_q^2 = \mathbb{F}_q[\sqrt{\alpha}]$ for some $\alpha \in \mathbb{F}_q$ then $x \in \mathbb{F}_q$ and $y = y'\sqrt{\alpha}$. Thus the point $(x, y')$ is an $\mathbb{F}_q$-point of the curve $E'$ given by $\alpha y^2 = x^3 + x$, the quadratic twist of $E$. Conversely, any finite point on $E'$ where $y \neq 0$ corresponds to a point in an orbit of type $(2, 1)$; thus the number of orbits of type $(2, 1)$ is a quarter of the number of finite points with nonzero $y$-coordinate: 
\[ \#X_{(2,1)} = \frac{1}{4} \left( E'(\mathbb{F}_q) - 1 \right). \]
Therefore 
\[ \#X_{(4,0)} = \frac{1}{8} \left( E(\mathbb{F}_q^2) - E(\mathbb{F}_q) - 4 \#X_{(2,1)} \right) = \frac{1}{8} E(\mathbb{F}_q^2) - \frac{q-1}{4}. \]
Using Proposition 4.4 we conclude that 
\[ \psi_2[E, \varphi] = \left( -1 \right)^{E(\mathbb{F}_q^2)/8 - (q-1)/4}, \frac{1}{4} \#E'(\mathbb{F}_q) - 1 \right) \in \mathbb{Z}/2 \times \mathbb{Z}/2. \]
In particular, $[E, \varphi]$ is non-permutative if $\#E'(\mathbb{F}_q)$ is a multiple of 8.
4.6 Its image under base change is $C$. 

4.9 $\{J, CWZ19\}$. 

Remark 4.12. In [CWZ19], the author and collaborators give an alternate construction of such a “derived zeta function.” By using the observation that $|X(L_n)|$ is exactly the number of fixed points of Frobenius and using the Grothendieck–Lefschetz fixed point theorem, the collaborators construct a map of spectra $K(\text{Var}_k) \to K(\text{End}(\mathbb{Q}_\ell))$ which on $K_0$ is exactly the zeta function (using a result of Almkvist [Gra78] to show that the composition of that map with $K_0(\text{End}(\mathbb{Q}_\ell)) \to (1 + t\mathbb{Z}[t], \times)$ is $Z(-, t)$). As $K_0(\text{End}(\mathbb{Q}_\ell))$ is the rational Witt vectors, this produces a derived zeta function which recalls that zeta functions should be rational. Our current construction cannot do that, as the data about all orbits of different sizes are independent. However, the construction of [CWZ19] has the weakness that it was not a map of $E_\infty$-ring spectra, as it was constructed using two formal inverses to weak equivalences. The construction was also significantly more complicated than the construction in this paper, leading to much more difficulty with analysis. In future work, the author hopes to find a construction that unifies the strength of these two approaches, so that it can retain the rationality data as well as the ring structure.

5. Technical Preliminaries

In this section we review some of the technical preliminaries necessary for the proofs. Most of the results in this section can be found in [Zak17A, Section 2]; we revisit them here in the interest of readability.

Definition 5.1 ([Zak12, Definition 2.1]). Let $C$ be an assembler. The category $\text{Tw}(C)$ is defined to have

- **objects:** tuples $\{A_i\}_{i \in I}$, where $I$ is a finite set and each $A_i$ is a noninitial object in $C$.
- **morphisms:** A morphism $f: \{A_i\}_{i \in I} \to \{B_j\}_{j \in J}$ is a map of finite sets $f: I \to J$ (called the set map), together with morphisms $f_i: A_i \to B_{f(i)}$ in $C$, for each $i \in I$ (called the component maps). The component maps must satisfy the condition that for $i \neq i'$, if $f(i) = f(i')$ then $f_i$ and $f_{i'}$ are disjoint.
- **composition:** The composition of $f: \{A_i\}_{i \in I} \to \{B_j\}_{j \in J}$ and $g: \{B_j\}_{j \in J} \to \{C_k\}_{k \in K}$ is given by the set map $g \circ f$, together with the component maps $g_{f(i)} \circ f_i: A_i \to C_{g(f(i))}$

The category $\text{W}(C)$ is the subcategory of $\text{Tw}(C)$ containing all morphisms $\{A_i\}_{i \in I} \to \{B_j\}_{j \in J}$ such that for all $j \in J$, the family $\{f_i: A_i \to B_j\}_{i \in f^{-1}(j)}$ is a finite disjoint covering family.

In this paper, we use two distinct constructions of the $K$-theory of an assembler: the $\Gamma$-space definition from [Zak17A] and the Waldhausen category definition from [Zak17B]. In [Zak17B,
Theorem 2.1] it is shown that these two constructions produce equivalent $K$-theories for closed assemblers; in this paper, we will also show that this equivalence respects the monoidal structure.

We give a short review of these definitions here. Write $\mathbf{ΓSp}$ for the category of $\mathbf{Γ}$-spaces, $\mathbf{WaldCat}$ for the category of Waldhausen categories, and $\mathbf{Sp}$ for the category of symmetric spectra.

We begin by recalling the two definitions.

**Definition 5.2** ([Zak17A, Definition 2.12]). Let $X$ be a pointed set; write $X^* \overset{\text{def}}{=} X \setminus \{\ast\}$. For an assembler $\mathcal{C}$, we write $X \land \mathcal{C}$ for the assembler $\bigvee_{x \in X^*} \mathcal{C}$; here, the wedge product of assemblers is obtained by taking their unions and associating the initial objects. This gives a functor $\mathbf{FinSet}_* \times \mathbf{Asm} \to \mathbf{Asm}$. In addition, there is a natural transformation $\cdot \land \mathcal{C} \to \mathcal{C}$ given for each $X$ by the composition

$$X \land \mathcal{C} \cong \bigvee_{X^*} \mathcal{C} \longrightarrow N \left( \bigoplus_{X^*} \mathcal{C} \right) \cong N(\mathcal{C} \land X^*).$$

For an assembler $\mathcal{C}$, we define

$$K^\Gamma(\mathcal{C}) \overset{\text{def}}{=} \mathbf{B}(X \to \mathcal{C}).$$

Here, $\mathbf{B}$ is the classifying spectrum functor which takes a $\Gamma$-space to a spectrum. When not comparing this construction to $K^W$ (defined below), we write $K$ instead of $K^\Gamma$.

**Definition 5.3** ([Zak17B, Definition 1.7]). Suppose that $\mathcal{C}$ is a closed assembler. We define the category $\text{SC}(\mathcal{C})$ to have

- **objects**: $\text{ob} \text{Tw}(\mathcal{C})$
- **morphisms**: A morphism $f: \{A_i\}_{i \in I} \to \{B_j\}_{j \in J}$ is represented by a span
  $$\{A_i\}_{i \in I} \leftarrow \{C_k\}_{k \in K} \longrightarrow \{B_j\}_{j \in J}.$$  
  Here $p$ is a morphism in $\text{Tw}(\mathcal{C})$, and $\sigma$ is represented by a set map $\sigma: K \to J$, together with component maps $\sigma_k: C_k \to B_{\sigma(k)}$ which are isomorphisms in $\mathcal{C}$.
- **composition**: The composition of two morphisms $f: \{A_i\}_{i \in I} \to \{B_j\}_{j \in J}$ and $g: \{B_j\}_{j \in J} \to \{C_k\}_{k \in K}$ represented by a diagram
  $$\{A_i\}_{i \in I} \overset{p}{\longrightarrow} \{A_i'\}_{i' \in I'} \overset{\sigma}{\longrightarrow} \{B_j\}_{j \in J} \overset{q}{\longrightarrow} \{B_j'\}_{j' \in J'} \overset{\tau}{\longrightarrow} \{C_k\}_{k \in K}$$
  is defined by pulling back $q$ along $\sigma$ and composing down the two sides. It is necessary to check that such a pullback produces a well-defined composition; see [Zak12, Lemma 6.4].

We give $\text{SC}(\mathcal{C})$ the structure of a Waldhausen category by defining

- **cofibrations**: to be those morphisms where $p$ is in $\mathcal{W}(\mathcal{C})$ and $\sigma$ has an injective set map, and
- **weak equivalences**: to be those cofibrations where $\sigma$ has a bijective set map.

For a closed assembler $\mathcal{C}$, we define

$$K^W(\mathcal{C}) \overset{\text{def}}{=} K(\text{SC}(\mathcal{C})).$$

In assemblers it is often possible to compute the quotient of the $K$-theory of an assembler by the $K$-theory of a subassembler by simply “removing” the objects of the subassembler.

**Definition 5.4** ([Zak17A, Definition 2.9]). Let $\mathcal{C}$ be an assembler and $\mathcal{D}$ a sieve in $\mathcal{C}$. The assembler $\mathcal{C} \setminus \mathcal{D}$ is defined to have as its underlying category the full subcategory of $\mathcal{C}$ containing all objects not in $\mathcal{D}^\circ$. A family $\{f_i: A_i \to A\}_{i \in I}$ is a covering family in $\mathcal{C} \setminus \mathcal{D}$ if there exists a family $\{f_j: A_j \to A\}_{j \in J}$ with each $A_j \in \mathcal{D}$ such that $\{f_i: A_i \to A\}_{i \in I \cup J}$ is a covering family in $\mathcal{C}$.
Often it is the case that $K(C)/K(D) \simeq K(C \smallsetminus D)$; see [Zak17A, Theorem D] for more detail. Here, we consider a situation where this does not hold, as it will be important intuition for the construction of the monoidal structure on $c\textbf{Asm}$.

Consider an object $A$ which has an empty covering family. The morphism $\{\emptyset \to \{A\}_*\}$ is a weak equivalence in $\text{SC}(C)$. Thus, morally speaking, $A$ should not contribute to the $K$-theory of $C$. However, this can be deceiving, as the underlying categorical structure of $C$ can contribute to the $K$-theory of $C$ despite this. To help illustrate this, we present an example, which will be helpful in understanding the difference between $C \boxtimes D$ and $C \land D$ in Section 6:

**Example 5.5.** Let $C$ be the assembler with the following underlying category:

```
  B
  ▽
  A  \quad D.
  ▽
  C
```

The topology on $C$ is generated by the covering families $\{B \to D, C \to D\}$ and the empty covering families of $A$ and $\emptyset$. Note that $D$ has no finite disjoint covering families. Thus, despite the fact that $D$ is covered by $B$ and $C$,

$$K(C) \simeq S \lor S \lor S,$$

with one copy of $S$ for each of $B$, $C$, and $D$.

Now consider $C \smallsetminus \{\emptyset \to A\}$; this assembler is given by the diagram

```
  B
  ▽
  \emptyset \quad D.
  ▽
  C
```

The topology on this assembler is generated by the covering family $\{B \to D, C \to D\}$ and the empty covering family on $\emptyset$. Here, $D$ does have a finite disjoint covering family; thus

$$K(C \smallsetminus \{\emptyset \to A\}) \simeq S \lor S,$$

with one copy of $S$ for each of $B$ and $C$.

### 6. A Monoidal Structure on the Category of Assemblers

The goal of this section is to construct a symmetric monoidal structure on $c\textbf{Asm}$ in such a way that the $K$-theory functor is symmetric monoidal and $S^\textbf{Asm}$ is the unit.

We begin with a helper definition.

**Definition 6.1.** Let $C, D$ be two closed assemblers. The assembler $C \boxtimes D$ has as its underlying category the category $C \times D$, and its topology is generated by the coverage in which the covering families are families $\{(A_i, B_j) \to (A, B)\}_{(i,j) \in I \times J}$, where $\{A_i \to A\}_{i \in I}$ is a covering family in $C$ and $\{B_j \to B\}_{j \in J}$ is a covering family in $D$.

**Lemma 6.2.** $C \boxtimes D$ is a closed assembler.
Proof. First we need to check that the Grothendieck topology is well-defined. To check this we simply need to check that the pullback of a family in the coverage is still in the coverage; this follows because covering families in $\mathcal{C}$ and $\mathcal{D}$ are closed under pullbacks.

Axioms (I) and (M) follow directly from the definition. Axiom (R) holds because $\mathcal{C} \boxtimes \mathcal{D}$ has pullbacks. □

Remark 6.3. This construction is analogous to the construction of the product topology. In the product topology, the generating open sets are the products of the opens in the two categories. More concretely, suppose $U \subseteq X$ is covered by $\{A_1, \ldots, A_n\}$ and $V \subseteq Y$ is covered by $\{B_1, \ldots, B_m\}$. Then to cover $U \times V$ we need to take $\{A_i \times B_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$.

Contrast this with the usual disjoint union topology on $\mathcal{C} \times \mathcal{D}$ (where $\mathcal{C}$ and $\mathcal{D}$ are sites) where $\{(A_i, B_j) \rightarrow (A, B)\}_{i \in I}$ is a covering family if $\{A_i \rightarrow A\}_{i \in I}$ and $\{B_j \rightarrow B\}_{j \in I}$ are both covering families.

Let $\alpha_{\mathcal{C},\mathcal{D},\mathcal{E}}:(\mathcal{C} \boxtimes \mathcal{D}) \boxtimes \mathcal{E} \rightarrow \mathcal{C} \boxtimes (\mathcal{D} \boxtimes \mathcal{E})$ be the functor taking $((A, B), C)$ to $(A, (B, C))$. Let $\gamma_{\mathcal{C},\mathcal{D}}:\mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{D} \boxtimes \mathcal{C}$ be the functor taking $(A, B)$ to $(B, A)$. Let $\lambda_{\mathcal{C}}: S^{\text{Asm}} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ be the projection onto the second coordinate and let $\rho_{\mathcal{C}}: \mathcal{C} \boxtimes S^{\text{Asm}} \rightarrow \mathcal{C}$ be the projection onto the first coordinate.

Lemma 6.4. The natural transformations $\alpha$, $\gamma$, $\lambda$ and $\rho$ satisfy all of the axioms of a symmetric monoidal structure except the condition that $\lambda$ and $\rho$ be natural isomorphisms.

Proof. The only part that is not direct from the definitions is checking that $\lambda_{\mathcal{C}}$ and $\rho_{\mathcal{C}}$ are well-defined morphisms of assemblers. We focus on $\lambda_{\mathcal{C}}$; the result for $\rho_{\mathcal{C}}$ will follow analogously. Since the topology on $S^{\text{Asm}} \boxtimes \mathcal{C}$ is generated by a pretopology and since $\lambda_{\mathcal{C}}$ commutes with pullbacks (since pullbacks in $S^{\text{Asm}} \boxtimes \mathcal{C}$ are done coordinatewise) it suffices to check that for any covering family $\mathcal{F}$, $\lambda_{\mathcal{C}} \mathcal{F}$ is a covering family. A covering family in $S^{\text{Asm}} \boxtimes \mathcal{C}$ is a finite refinement of families in the coverage. The projection of a family in the coverage is a covering family. The refinement of a covering family can either refine the projection, or it can add some morphisms to the covering family (which still keeps it a covering family). Either way, the projection of a covering family is a covering family, as desired. □

Thus $\boxtimes$ does not produce a symmetric monoidal structure on $\text{Asm}$. To make this into a monoidal structure it is necessary to rectify this problem.

Definition 6.5. We consider $\mathcal{C} \vee \mathcal{D}$ to be the full subassembler of $\mathcal{C} \boxtimes \mathcal{D}$ containing those objects where one coordinate or the other is the initial object. The subassembler $\mathcal{C} \vee \mathcal{D}$ is a sieve in $\mathcal{C} \boxtimes \mathcal{D}$.

Define

$$\mathcal{C} \land \mathcal{D} \overset{\text{def}}{=} (\mathcal{C} \boxtimes \mathcal{D}) \setminus (\mathcal{C} \vee \mathcal{D}).$$

The relationship of $\mathcal{C} \land \mathcal{D}$ to $\mathcal{C} \boxtimes \mathcal{D}$ has the exact flavor of Example 6.7, with objects in $\mathcal{C} \vee \mathcal{D}$ being obstructions to objects in $\mathcal{C} \boxtimes \mathcal{D}$ being disjoint. Once these objects are removed, moreover, the natural transformations $\alpha$, $\gamma$, $\lambda$, $\rho$ are all well-defined with $\boxtimes$ replaced by $\land$, since as categories $\mathcal{C} \land \mathcal{D}$ can be thought of as a full subcategory of $\mathcal{C} \boxtimes \mathcal{D}$.

Lemma 6.6. $(\text{cAsm}, S^{\text{Asm}}, \land)$ is a symmetric monoidal category.

Proof. By Lemma 6.4 all that remains to show is that $\lambda$ and $\rho$ are natural isomorphisms.

We check that $\lambda_{\mathcal{C}}$ is an isomorphism $S^{\text{Asm}} \land \mathcal{C} \rightarrow \mathcal{C}$. The objects of $S^{\text{Asm}} \land \mathcal{C}$ are the initial object and pairs $(\ast, A)$ with $A \in \mathcal{C}$. $\lambda_{\mathcal{C}}$ takes the first to the initial object and the second to itself. The structure on morphisms is analogous. Since the functor is a bijection on both objects and morphisms, it is an isomorphism, as desired.

We must also check that $\lambda_{S^{\text{Asm}}} = \rho_{S^{\text{Asm}}}$. This is the case because $S^{\text{Asm}} \land S^{\text{Asm}}$ has a unique nontrivial map to $S^{\text{Asm}}$; since both $\lambda_{S^{\text{Asm}}}$ and $\rho_{S^{\text{Asm}}}$ are isomorphisms, they must be equal. □
Each object of $\mathcal{C} \vee \mathcal{D}$ sitting inside $\mathcal{C} \otimes \mathcal{D}$ has an empty covering family. It is thus tempting to conclude that $K(\mathcal{C} \otimes \mathcal{D})$ may already have the correct symmetric monoidal structure, at least up to homotopy. In general this is not the case, as structures similar to those in Example 5.5 arise. In fact, most objects in $\mathcal{C}^\circ \times \mathcal{D}^\circ$ have no nontrivial finite disjoint covering families. Indeed, suppose that $(A, B)$ is an object for which the empty family is not a covering family. Then any nontrivial covering family will have two elements that share a coordinate, and will therefore not be disjoint. On the other hand, in $\mathcal{C} \vee \mathcal{D}$, any pair of finite disjoint covering families produces a finite disjoint covering family, since only one of the coordinates being disjoint is sufficient for disjointness.

For a more concrete example, consider $K(S \Asm \vee S \Asm)$. $S \Asm \vee S \Asm$ has three noninitial objects and no nontrivial finite disjoint covering families. Thus $K(S \Asm \vee S \Asm) \simeq S \vee S \vee S$. If the functor were correctly monoidal it would instead be $S$, so we see that $\otimes$ is not the desired structure, even up to homotopy.

The monoidal structure of assemblers gives rise to an interesting phenomenon: in general, the category $S\mathcal{C}(\mathcal{C} \otimes \mathcal{D})$ will not be saturated. We give an explicit example to illustrate how this can arise, and we stress that such examples are the norm and not the exception:

**Example 6.7.** Let $\mathcal{C}$ be the assembler $\text{Seg}$, discussed in Example 1.8. In $S\mathcal{C}(\text{Seg} \otimes \text{Seg})$ there is the following diagram of morphisms:

```
  \begin{tikzpicture}
    \node (A) at (0,0) {\cell{0}{0}};
    \node (B) at (2,0) {\cell{1}{1}};
    \node (C) at (4,0) {\cell{2}{2}};
    \node (D) at (2,2) {\cell{1}{1}};
    \node (E) at (4,2) {\cell{2}{2}};
    \node (F) at (6,2) {\cell{3}{3}};
    \node (G) at (8,2) {\cell{4}{4}};
    \node (H) at (6,0) {\cell{3}{3}};
    \node (I) at (8,0) {\cell{4}{4}};
    \draw[->] (A) -- (B) node[midway, above] {$\sim$};
    \draw[->] (B) -- (C) node[midway, above] {$\sim$};
    \draw[->] (C) -- (D) node[midway, above] {$\sim$};
    \draw[->] (D) -- (E) node[midway, above] {$\sim$};
    \draw[->] (E) -- (F) node[midway, above] {$\sim$};
    \draw[->] (F) -- (G) node[midway, above] {$\sim$};
    \draw[->] (G) -- (H) node[midway, above] {$\sim$};
    \draw[->] (H) -- (I) node[midway, above] {$\sim$};
  \end{tikzpicture}
```

A weak equivalence is a morphism that can be written as a finite composition of decompositions into “grids” on each rectangle; the two marked morphisms are therefore weak equivalences, but the unmarked one is not. Thus $S\mathcal{C}(\text{Seg} \otimes \text{Seg})$ does not satisfy the saturation axioms.

**7. The interaction of K-theory and the monoidal structure**

In an ideal world, the $K$-theory functor would be monoidal and we could construct ring spectra simply by finding monoid objects inside $\Asm$. However, that is not the case: even the category of pointed finite sets does not produce an honest ring spectrum, but rather an $E_\infty$-spectrum, as it is not possible to make a completely rigid model of both of the monoidal structures (disjoint union and product) on finite sets.

However, we can produce the next best thing: a bipermutative category.

**Definition 7.1.** A category $\mathcal{C}$ is permutative if it is equipped with a functor $\oplus: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $0 \in \mathcal{C}$, and a natural isomorphism $\gamma: a \oplus b \cong b \oplus a$ satisfying the extra conditions that

1. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
2. $a \oplus 0 = a = 0 \oplus a$, and
3. $\gamma_{a,0} = 1_a$ and the following diagrams commute:

\[
\begin{array}{ccc}
  a \oplus b & \gamma & b \oplus a \\
  \downarrow 1_{a \oplus b} & & \downarrow \gamma \\
  a & \gamma & b \oplus a \\
\end{array}
\]

\[
\begin{array}{ccc}
  a \oplus b & \gamma & c \oplus a \oplus b \\
  \downarrow 1 \gamma & & \downarrow \gamma \\
  a \oplus c \oplus b & \gamma & c \oplus a \oplus b \\
\end{array}
\]

In $\mathcal{C}^\circ \times \mathcal{D}^\circ$, any pair of finite disjoint covering families produces a finite disjoint covering family, since only one of the coordinates being disjoint is sufficient for disjointness.
In other words, a permutative category is a symmetric monoidal category with strict associativity and unit. This is referred to as the additive structure on the permutative category.

A permutative category \( \mathcal{C} \) is bipermutative if it is equipped with a second permutative structure \((\mathcal{C}, \otimes, 1)\) (which is referred to as the multiplicative structure) and natural distributivity maps
\[
d_l: (a \otimes b) \oplus (a' \otimes b) \longrightarrow (a \oplus a') \otimes b
\]
and
\[
d_r: (a \otimes b) \oplus (a \otimes b') \longrightarrow a \otimes (b \oplus b')
\]
satisfying certain compatibility requirements, described in [EM06, Definition 3.3, 3.6].

It is not immediately obvious why distributivity maps cannot be “rigidified” away, when it is known why monoidal structures can generally be replaced with rigid versions. To help with this, and as it will be used in Proposition 7.6, we give an explicit description of a bipermutative structure on the category of finite sets.

**Example 7.2.** Let \( \text{FinSet} \) be the category with
- **objects:** the sets \( \emptyset \) and \( \{1, \ldots, n\} \) for all natural numbers \( n \) and
- **morphisms:** functions between finite sets.

The additive permutative structure on finite sets is given by disjoint union, where we think of “concatenating the two sets in order”. Thus in \( \{1, \ldots, k\} \oplus \{1, \ldots, \ell\} = \{1, \ldots, k + \ell\} \) we think of the first \( k \) elements as coming from \( \{1, \ldots, k\} \) and the rest as coming from \( \{1, \ldots, \ell\} \), in the correct order. (Although morphisms are not required to preserve order, we keep track of it here so as to analyse the symmetry more precisely.) The map \( \gamma \) is the \( k, \ell \)-shuffle which preserves the order of the two sets and moves the elements past one another.

The multiplicative permutative structure is via the cartesian product of sets, in which the isomorphism \( \{1, \ldots, k\} \times \{1, \ldots, \ell\} \cong \{1, \ldots, k\ell\} \) is given via the lexicographic ordering of pairs. With this second structure, the natural transformation \( d_r \) is the identity map, but the natural transformation \( d_l \) is not, as illustrated below:

\[
\begin{align*}
(a \otimes (b \oplus b')) & \quad (a \otimes b) \oplus (a \otimes b')
\end{align*}
\]

In this picture, the rectangle represents the set of pairs, with the arrows showing the induced ordering, with greener arrows before bluer arrows. Note that in the two pictures the orderings are not the same.

We now investigate the structures on assemblers that produce morphisms of bipermutative categories. We begin with a simple observation.

**Lemma 7.3.** Let \( \mathcal{C} \) be an assembler. Then \( \mathcal{W}(\mathcal{C}) \) is a permutative category with the permutative structure induced by the permutative structure on \( \text{FinSet} \).

**Proof.** The category \( \mathcal{W}(\mathcal{C}) \) has as objects tuples \( \{A_i\}_{i \in I} \) with \( I \in \text{FinSet} \) and \( A_i \in \mathcal{C}^\circ \) for all \( i \). We define
\[
\{A_i\}_{i=1}^n \oplus \{B_j\}_{j=1}^m = \{C_k\}_{k=1}^{n+m},
\]
where \( C_k = A_k \) if \( k \leq n \) and \( C_k = B_{k-n} \) for \( k > n \). The 0 object is the empty tuple \( \{\} \). As \( \text{FinSet} \) has a permutative structure, this inherits the same structure. \( \square \)
Given a sufficiently strict monoidal product on \( C \), this permutative structure can be extended to a bipermutative structure.

**Notation 7.4.** Given \( I,J \in \text{FinSet} \), write \( I \otimes J \) for the set \( I \times J \) with the lexicographic ordering, associated via this ordering with the set \( \{1,\ldots,|I| \cdot |J|\} \). We will write \( (i,j) \in I \otimes J \) for the image of \( (i,j) \) under this ordering.

**Definition 7.5.** Let \( C \) be an assembler with a multiplication \( \mu: C \wedge C \rightarrow C \). A symmetry for \( \mu \) is a natural isomorphism \( \gamma: \mu \rightarrow \mu \circ \tau \), where \( \tau \) swaps the two factors of \( C \), satisfying the extra condition that \( \gamma_{A,B} \circ \gamma_{B,A} = 1_{\mu(A,B)} \) for all \( A,B \).

**Proposition 7.6.** Let \( C \) be a monoid object in \( \text{Asm} \) with product \( \mu: C \wedge C \rightarrow C \). Then the permutative structure on \( \mathcal{W}(C) \) extends to a ring structure, with

\[
\{A_i\}_{i \in I} \otimes \{B_j\}_{j \in J} \cong \{\mu(A_i,B_j)\}_{(i,j) \in I \otimes J}.
\]

If in addition \( C \) is equipped with a symmetry for \( \mu \) then this ring structure is a bipermutative structure.

**Proof.** First we must check that the given tensor product is a well-defined functor \( \otimes: \mathcal{W}(C) \times \mathcal{W}(C) \rightarrow \mathcal{W}(C) \). We factor this as

\[
\mathcal{W}(C) \times \mathcal{W}(C) \xrightarrow{\nu} \mathcal{W}(C \wedge C) \xrightarrow{\mathcal{W}(\mu)} \mathcal{W}(C).
\]

Here, we define \( \nu \) by

\[
\nu(\{A_i\}_{i \in I}, \{A'_i\}_{i' \in I'}) = \{(A_i, A'_i)\}_{(i,i') \in I \times I'}
\]

on objects, and define it on morphisms by taking the pair \( f: \{A_i\}_{i \in I} \rightarrow \{B_j\}_{j \in J} \) and \( f': \{A'_i\}_{i' \in I'} \rightarrow \{B'_j\}_{j' \in J'} \) to the morphism defined by the set map \( f \times f': I \times I' \rightarrow J \times J' \) and the component maps \( (f_i,f_i'):(A_i,A'_i) \rightarrow (B_j,B'_j) \). It remains to check that this is well-defined: that if both \( f \) and \( f' \) were morphisms in \( \mathcal{W}(C) \) then \( \nu(f,f') \) is a morphism in \( \mathcal{W}(C \wedge C) \). In particular it is necessary to check:

**disjointness:** Given distinct \((i_0,i'_0)\), \((i_1,i'_1)\) \( \in I \times I' \) with \( f(i_0) = f(i_1) \) and \( f'(i'_0) = f'(i'_1) \) we must check that \((f_0,f'_0)\) and \((f_1,f'_1)\) are disjoint. it suffices to check that this is equal to \( \emptyset \). However, since the pairs were distinct, one of the two coordinates must be different; suppose WLOG that it is the first one, so that \( i_0 \neq i_1 \). Then the maps \( f_0 \) and \( f_1 \) are disjoint, so \( A_{i_0} \times B_{f(i_0)} \), \( A_{i_1} \neq \emptyset \). But then the pullback has a single initial coordinate when computed inside \( C \times C \), and is therefore equal to \( \emptyset \) in \( C \wedge C \), as desired.

**covering:** A morphism \( f: \{A_i\}_{i \in I} \rightarrow \{B_j\}_{j \in J} \) in \( \mathcal{W}(C) \) is a collection of finite disjoint covering families \( \{f_i: A_i \rightarrow B_j\}_{i \in f^{-1}(j)} \). We have already checked that \( \nu(f,f') \) is a collection of disjoint morphisms; it remains to check that they give covering families. In particular, it must be the case that for all \((j,j') \in J \times J' \), the family

\[
\{(f_i,f'_i):(A_i,A'_i) \rightarrow (B_j,B'_j)\}_{(i,i') \in (f,f')^{-1}(j,j')}
\]

is a covering family. It is an element of the coverage that generates the topology on \( C \wedge C \), so it is a covering family, as desired.

The unit map for \( C \) is a morphism of assemblers \( S \rightarrow C \); denote by \( \nu \in C \) the image of \( * \). We claim that \( \otimes \) is strictly associative and \( \{\nu\}_1 \) is a strict unit for \( \otimes \).

We have that

\[
(\{A_i\}_{i \in I} \otimes \{B_j\}_{j \in J}) \otimes \{C_k\}_{k \in K} = \{\mu(\mu(A_i,B_j),C_k)\}_{((i,j),k) \in (I \otimes J) \otimes K}
\]

and

\[
\{A_i\}_{i \in I} \otimes (\{B_j\}_{j \in J} \otimes \{C_k\}_{k \in K}) = \{\mu(A_i,\mu(B_j,C_k))\}_{((i,j),k) \in I \otimes (J \otimes K)}.
\]
Since \( \text{FinSet} \) is bipermutative, the two indexing sets are equal. Since \( \mathcal{C} \) is a monoid object, \( \mu \) is strictly associative, so the two objects are equal, as well. Thus \( \otimes \) is strictly associative.

Similarly,
\[
\{\nu\}_{1} \otimes \{A_{i}\}_{i \in I} = \{\mu(\nu, A_{i})\}_{(i, j) \in I} \otimes \{\nu\}_{1};
\]

since \( \mu \) is strictly unital, this is equal to \( \{A_{i}\}_{i \in I} \), as desired. The analogous proof works for \( \{A_{i}\}_{i \in I} \otimes \{\nu\}_{1} \).

If we have the structure map \( \gamma \) then symmetry also holds. Indeed, the map
\[
\{A_{i}\}_{i \in I} \otimes \{B_{j}\}_{j \in J} = \{\mu(A_{i}, B_{j})\}_{(i, j) \in I \otimes J} \longrightarrow \{\mu(B_{j}, A_{i})\}_{(j, i) \in J \otimes I} = \{B_{j}\}_{j \in J} \otimes \{A_{i}\}_{i \in I}
\]
is given by sending \((i, j) \in I \otimes J \) to \((j, i) \in J \otimes I \) using the symmetry map in \( \text{FinSet} \), and mapping \( \mu(A_{i}, B_{j}) \longrightarrow \mu(B_{j}, A_{i}) \) via \( \gamma_{A_{i}, B_{j}} \). This satisfies the relation for the symmetry map in a permutative category because the set map does (as \( \text{FinSet} \) is a bipermutative category) and the component maps satisfy it by the condition on \( \gamma \).

We now turn to checking the relations for the ring structure. We define distributivity maps
\[
d_{l} : \{\{A_{i}\}_{i \in I} \otimes \{B_{j}\}_{j \in J}\} \otimes \{\{A'_{i'}\}_{i' \in I'} \otimes \{B_{j}\}_{j \in J}\} \longrightarrow \{\{A_{i}\}_{i \in I} \otimes \{A'_{i'}\}_{i' \in I'}\} \otimes \{B_{j}\}_{j \in J}
\]
\[
d_{r} : \{\{A_{i}\}_{i \in I} \otimes \{B_{j}\}_{j \in J}\} \otimes \{\{A_{i}\}_{i \in I} \otimes \{B'_{j'}\}_{j' \in J'}\} \longrightarrow \{\{A_{i}\}_{i \in I} \otimes \{B'_{j'}\}_{j' \in J'}\} \otimes \{B_{j}\}_{j \in J}
\]
to be induced from the distributivity maps on \( \text{FinSet} \), with identity maps as the components.

We follow the naming from [EM06, Definition 3.3, Definition 3.6]. Axiom (a) holds because the product of any set with the empty set is empty. Axioms (b), (c), (d), (e) and (f) hold because they hold in \( \text{FinSet} \) and all component maps in the given diagrams are identities.

Given the extra structure of the symmetry \( \gamma \), Axiom (e') holds in \( \text{FinSet} \), and over each index this reduces to the diagram
\[
\begin{array}{c}
\mu(A, B) \quad \mu(A, B) \\
\frac{\gamma_{A, B}}{\gamma_{A, B}} \\
\mu(B, A) \quad \mu(B, A)
\end{array}
\]
which commutes. \( \square \)

**Corollary 7.7.** The K-theory of any monoid object in \( \mathcal{C} \) is an \( A_{\infty} \)-ring spectrum. The K-theory of a monoid object equipped with a symmetry for the multiplication is an \( E_{\infty} \)-ring spectrum.

Using this we can construct some examples of \( E_{\infty} \)-ring spectra.

**Example 7.8.** For a group \( G \), let \( \text{Fin}_{G} \) be the assembler with

**objects:** pairs of an integer \( m \geq 0 \) together with a finite set of tuples of integers of length \( m \) equipped with a \( G \)-action, and

**morphisms:** \( G \)-equivariant inclusions of sets.

This has a multiplication \( \mu : \text{Fin}_{G} \wedge \text{Fin}_{G} \longrightarrow \text{Fin}_{G} \) taking a pair \((m, S) \wedge (n, T)\) to \((m + n, S \times T)\), where an element in \( S \times T \) is modeled as a tuple of length \( m + n \), except when \( m = 1 \) and \(|S| = 1\), in which case \((m, S) \wedge (n, T) = (n, T)\) (and similarly for the case when \( n = 1 \) and \(|T| = 1\)). The \( G \)-action on \( S \times T \) is diagonal.

The multiplication is strictly associative and strictly unital, with unit \((1, \{1\})\). It is also equipped with a symmetry for the multiplication, taking \( S \times T \) to \( T \times S \) via a shuffle swapping the first \( m \) and last \( n \) coordinates applied to each tuple.

Thus, by the corollary, \( K(\text{Fin}_{G}) \) is an \( E_{\infty} \)-ring spectrum.

Moreover, [EM06, Theorem 9.3.8] immediately implies the following:
Proposition 7.9. A symmetric monoid map of symmetric monoid objects in assemblers induces an $E_{\infty}$-map on the $K$-theories.

Unfortunately, much of the time we do not immediately have a strict model for the multiplication. Consider the example we are most interested in: varieties. Given varieties $X$ and $Y$ over $k$ the product should be the fiber product $X \times_k Y$. However, to make $\operatorname{Var}_k$ a monoid object, we must be able to model this fiber product rigidly. Although this is possible we prefer to apply a more general technique and work with strong morphisms of monoidal assemblers.

Definition 7.10. A (symmetric) monoidal assembler is an assembler $C$ equipped with a map $\mu:C \wedge C \to C$ and a map $\eta:S \to C$ satisfying the axioms of a (symmetric) monoidal category. More concretely, a (symmetric) monoidal assembler is equipped with a natural isomorphism $\alpha_{A,B,C}:\mu(A,\mu(B,C)) \to \mu(\mu(A,B),C)$ and natural isomorphisms $\lambda_A:\mu(A,\eta(\ast)) \to A$ and $\rho_A:\mu(\eta(\ast),A) \to A$ (as well as a natural isomorphism $\gamma_{A,B}:\mu(A,B) \to \mu(B,A)$, in the symmetric case) satisfying the usual relations of a symmetric monoidal category.

A symmetric monoidal morphism of symmetric monoidal assemblers $F:C \to D$ is a morphism of assemblers together with a natural transformation $\nu_{A,B}:\mu_D(F(A),F(B)) \to F(\mu_C(A,B))$ and a morphism $\epsilon:\eta_D(\ast) \to F(\eta_C(\ast))$ satisfying the relations of a symmetric monoidal functor.

We begin by showing that the $K$-theory of a symmetric monoidal assembler is an $E_{\infty}$-ring spectrum.

Definition 7.11. For any sequence of objects $[A_1, \ldots, A_n]$ in $C$, define

$$M[A_1, \ldots, A_n] \overset{\text{def}}{=} \mu(A_1, \mu(A_2, \ldots, \mu(A_{n-1}, A_n))).$$

Define $M[\ast] = \eta(\ast)$. Let $\widehat{C}$ be the assembler with

- **noninitial objects**: finite sequences $[A_1, \ldots, A_n]$ of noninitial objects in $C$, and

- **morphisms**: given by

$$\operatorname{Hom}_C([A_1, \ldots, A_n], [B_1, \ldots, B_m]) = \operatorname{Hom}_C(M([A_1, \ldots, A_n]), M([B_1, \ldots, B_m])).$$

- **topology**: covering families are those which are mapped to covering families by $M$.

Thus $M$ extends to a functor $M: \widehat{C} \to C$ which is essentially surjective, full, and faithful. We can then define the topology by defining the covering families to be the preimages of covering families under $M$. Then $M$ is a continuous equivalence of families; the inverse equivalence is given by the functor mapping $A$ to the sequence $[A]$—except for objects isomorphic to $\eta(\ast)$, which are mapped to $[\ast]$.

We define a symmetric monoidal structure on $\widehat{C}$ on objects by defining

$$\mu([A_1, \ldots, A_n], [B_1, \ldots, B_m]) = [A_1, \ldots, A_n, B_1, \ldots, B_m].$$

On morphisms, we define $\mu(f: A \to A', g: B \to B')$ to be the morphism given by

$$\mu(M(A), M(B)) \overset{\cong}{\longrightarrow} M(\mu(A, B)) \overset{\mu(f,g)}{\longrightarrow} M(\mu(A', B')) \overset{\cong}{\longrightarrow} \mu(M(A'), M(B')).$$

The two marked isomorphisms are uniquely defined using the associator, so this is a well-defined morphism. The associator and the two projections are identities. The symmetry map

$$\gamma_{A,B}:\mu([A_1, \ldots, A_n], [B_1, \ldots, B_m]) \longrightarrow \mu([B_1, \ldots, B_m], [A_1, \ldots, A_n])$$

is induced by the unique shuffle map

$$\mu(M(A), M(B)) \overset{\cong}{\longrightarrow} \mu(M(B), M(A)).$$

Lemma 7.12. For a (symmetric) monoidal assembler $C$, the functor $M$ induces a homotopy equivalence $K(\widehat{C}) \to K(C)$. In addition, $\widehat{C}$ is a monoid object in $\text{Asm}$. 
Proof. By definition, \( \mathcal{C} \) is the full subassembler of \( \mathcal{C} \) containing all sequences of length at most 1. As this is an equivalence of categories it induces an equivalence \( \mathcal{W}(\mathcal{C}) \to \mathcal{W}(\mathcal{C}) \), and thus induces an equivalence on \( K \)-theories, as desired.

We now check that \( \mathcal{C} \) is a monoid object. First we show that \( \mu \) and \( \eta \) are well-defined morphisms of assemblers. For \( \eta \) this is clear: since \( S^{\text{Asm}} \) has no nontrivial covering families and no disjoint objects, this simply states that the map \( S^{\text{Asm}} \to \mathcal{C} \) taking \( * \) to \( [ ] \) is a functor.

Now consider \( \mu \). It preserves the initial object by definition. To check that it preserves covering families it suffices to check that it preserves all families in the coverage. Thus consider a covering family \( \{(A_i, B_j) \to (A, B)\}_{(i,j) \in I \times J} \) in the coverage generating the topology of \( \mathcal{C} \). By definition, this implies that \( \{M(A_i) \to M(A)\}_{i \in I} \) and \( \{M(B_j) \to M(B)\}_{j \in J} \) are covering families in \( \mathcal{C} \), and thus \( \{\mu(M(A_i), M(B_j)) \to \mu(M(A), M(B))\}_{(i,j) \in I \times J} \) is a covering family in \( \mathcal{C} \). By definition, the associator induces an isomorphism

\[
\mu(M(X), M(Y)) \to M(\mu(X, Y))
\]

for all \( X \) and \( Y \). Thus the covering family \( \{\mu(M(A_i), M(B_j)) \to \mu(M(A), M(B))\}_{(i,j) \in I \times J} \) implies that \( \{\mu(\mu(A_i, B_j)) \to M(\mu(A, B))\}_{(i,j) \in I \times J} \) is a covering family as well. Thus \( \mu \) preserves covering families. To check that it preserves disjointness note that

Since \( \mathcal{C} \) is strictly associative and strictly unital the relevant commutative diagrams in \( \text{Asm} \) commute on-the-nose, and \( \mathcal{C} \) is a monoid object. \( \Box \)

Together with [EM06, Corollary 3.9], the above results prove the following theorem.

**Theorem 7.13.** Let \( \mathcal{C} \) be a symmetric monoidal assembler. Then \( K(\mathcal{C}) \) is equivalent to a strictly commutative ring symmetric spectrum.

**Example 7.14.** Let \( \text{Var}_S \) be the assembler of varieties over \( S \), and define the monoidal structure by \( \mu(X, Y) \overset{\text{def}}{=} X \times_S Y \). Then \( K(\text{Var}_S) \) is an \( E_\infty \)-ring spectrum.

We now turn our attention to symmetric monoidal morphisms.

**Theorem 7.15.** A symmetric monoidal morphism of symmetric monoidal assemblers induces a ring homomorphism on \( K \)-groups.

**Proof.** The main issue is the question of how strict the symmetric monoidal morphism is. Let \( F: \mathcal{C} \to \mathcal{D} \) be a symmetric monoidal morphism of symmetric monoidal assemblers. Using strictification, there is a diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\mathcal{C} & \xrightarrow{\mathcal{C}} & \mathcal{D} \\
\end{array}
\]

commuting up to natural isomorphism; thus the map \( K_*(\mathcal{C}) \to K_*(\mathcal{D}) \) is the same as the composition

\[
K_*(\mathcal{C}) \to K_*(\mathcal{C}) \to K_*(\mathcal{D}) \to K_*(\mathcal{D}).
\]

The functor \( \mathcal{W}(\mathcal{F}) \) is strict on the additive and multiplicative structure, since \( \mathcal{F} \) is strict on the multiplicative structure by definition, and the rest of the structure arises from the structure on \( \text{FinSet} \), which is the same in both domain and codomain. Thus by [EM06, Theorem 9.3.7] it is equivalent to a map of strictly commutative ring spectra, and thus induces a ring homomorphism on homotopy groups. Since the diagram commutes up to natural isomorphism, it commutes up to homotopy after applying \( K \)-theory; in particular, the homomorphisms induced on \( \pi_* \) must

[EM06] E. M. , Corollary 3.9.

[FinSet] F. S., Theorem 9.3.7.
be the same. Thus, since the vertical maps are ring isomorphisms and the top map is a ring homomorphism, the induced map \( K_*(C) \to K_*(D) \) must also be a ring homomorphism. □

With this theorem we now have the two desired examples:

**Example 7.16.** Let \( \text{Var}_k \) be the assembler of \( k \)-varieties for \( k \) a finite field, and let \( G = \text{Gal}(\overline{k}/k) \). Then the map \( \zeta: \text{Var}_k \to \text{AFSet}_G \) given by \( X \mapsto X(\overline{k}) \) is a strong monoidal map of assemblers, and thus induces an \( E_\infty \)-map on the \( K \)-theory.

**Example 7.17.** Let \( f: T \to S \) be a map of finite type. Then base change induces an \( E_\infty \)-map of \( K \)-theories \( K(\text{Var}_S) \to K(\text{Var}_T) \). In particular, for \( k \) a number field with \( k \) a residue field of its ring of integers, the map \( K(\text{Var}_{\text{Spec}O_k}) \to K(\text{Var}_k) \) is an \( E_\infty \)-map of spectra.

8. Interaction with \( K_1 \)

A model for the \( K_1 \) of an assembler was given in [Zak17B], which contains the following theorem:

**Theorem 8.1 ([Zak17B, Theorem B]).** For any assembler \( C \), every element of \( K_1(C) \) can be represented by a pair of morphisms

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \quad \downarrow \\
B \xrightarrow{g} A
\end{array}
\]

in \( W(C) \). These satisfy the relations

\[
[A \xrightarrow{f} B] = 0, \quad [B \xrightarrow{g_1} A] + [A \xrightarrow{f_1} B] = [A \xrightarrow{g_1 f_1} B]
\]

and

\[
[A \xrightarrow{f_2} B] + [C \xrightarrow{g_2} D] = [A \xrightarrow{g_2 f_2} B] = [A \xrightarrow{f_1 \coprod g_1} B] \coprod [B \xrightarrow{f_2 \coprod g_2} D]
\]

The particular case of interest in this paper is when \( A = B \) is an object of \( C \) (which can be considered objects of \( W(C) \) by indexing over a singleton) and \( g \) is the identity morphism. Then \( f \) must be an automorphism of \( A \). We write the generator

\[
[A, f] \overset{\text{def}}{=} [A \xrightarrow{f} A].
\]

When restricted to such generators, the above theorem implies the following:

**Corollary 8.2.** Let \( C \) be an assembler which contains disjoint unions, in the sense that for any two objects \( A \) and \( B \), the pushout of the diagram

\[
\begin{array}{c}
A \\
\downarrow \varnothing \downarrow \quad \quad \downarrow B
\end{array}
\]

exists and has a covering family by the induced inclusions from \( A \) and \( B \); we denote this by \( \Pi \).

Every element \([A, f] \) with \( A \in \text{ob} C \) and \( f \in \text{Aut}(A) \) represents an element in \( K_1(C) \). These elements satisfy the relations

\[
[A, 1_A] = 0, \quad [A, g] + [A, f] = [A, g f] \quad [A, f] + [B, g] = [A \coprod B, f \coprod g].
\]

**Proof.** The first two relations follow directly from Theorem 8.1 so we focus on proving the last case. The \( \Pi \) in the statement of the theorem is the disjoint union in \( W(C) \), which simply takes disjoint unions on indexing sets, so it implies that

\[
[A, f] + [B, g] = [\{A, B\} \xrightarrow{\text{coprod}} \{A, B\}].
\]
It remains to check that the representative on the right is \([A \amalg B, f \amalg g]\).

Let \(f \amalg g: A \amalg B \to A \amalg B\) be the morphism on the disjoint union in \(C\). Let \((f, g): \{A, B\} \to \{A, B\}\) be the morphism induced by \(f\) and \(g\) inside \(\mathcal{W}(C)\). Let \(\delta: \{A, B\} \to \{A \amalg B\}\) be the morphism given by the covering family \(\{A \to A \amalg B, B \to A \amalg B\}\) induced from the coproduct in \(C\). Inside \(\mathcal{W}(C)\), the square

\[
\begin{array}{ccc}
\{A, B\} & \xrightarrow{(f, g)} & \{A, B\} \\
\delta & \downarrow & \delta \\
\{A \amalg B\} & \xrightarrow{f \amalg g} & \{A \amalg B\}
\end{array}
\]

commutes. Therefore in \(K_1(C)\),

\[
[\{A, B\}] \xrightarrow{f \amalg g} [\{A, B\}] = [\{A, B\}] + [\{A, B\}] = [\{A, B\}] + [A \amalg B, f \amalg g],
\]
as desired. \(\square\)

We now turn to examining how these interact with the product structure. From the previous section we know that for any pair of closed assemblers \(C\) and \(D\), there is a map

\[
\phi: K(C) \wedge K(D) \to K(C \wedge D).
\]

Applying \(\pi_1\) induces a map

\[
(K_0(C) \otimes K_1(D)) \oplus (K_1(C) \otimes K_0(D)) \xrightarrow{\pi_1} K_1(C \wedge D).
\]

Since \(K_0(C)\) is generated by symbols \([C]\), where \(C \in \text{ob SC}(C)\), and \(K_1(D)\) is generated by symbols \([ A \xleftarrow{f} \xrightarrow{g} B ]\), it makes sense to ask whether there is a good representative for

\[
\phi\left([C] \otimes [ A \xleftarrow{f} \xrightarrow{g} B ]\right).
\]

**Theorem 8.3.** Let \(C\) and \(D\) be closed assemblers, and let \(\phi: K^W(C) \wedge K^W(D) \to K^W(C \wedge D)\) be given by the monoidal structure of \(K^W\). For any \([C] \in K_0(C)\) and \([ A \xleftarrow{f} \xrightarrow{g} B ]\) in \(K_1(D)\),

\[
\phi_*\left([C] \otimes [ A \xleftarrow{f} \xrightarrow{g} B ]\right) = \left((C, A) \xleftarrow{(1_C, f)} \xrightarrow{(1_C, g)} (C, B)\right) \in K_1(C \wedge D).
\]

Analogously, for any \([ C \xleftarrow{f} \xrightarrow{g} D ]\) in \(K_1(C)\) and \([A] \in K_0(D)\),

\[
\phi_*\left([ C \xleftarrow{f} \xrightarrow{g} D ] \otimes [A]\right) = \left((C, A) \xleftarrow{(f, 1_A)} \xrightarrow{(g, 1_A)} (D, A)\right) \in K_1(C \wedge D).
\]
Proof. In [MT07, Theorem 2.5] Muro and Tonks give a presentation for the 1-type of a Waldhausen category in a way which is compatible with its multiplicative structure. More concretely, they show that for any Waldhausen category $\mathcal{E}$ there exists a stable quadratic module (see [MT07, Definition 1.4]) generated by the objects and weak equivalences in $\mathcal{E}$, whose homology encodes $K_0(\mathcal{E})$ and $K_1(\mathcal{E})$. This is compatible with the multiplicative structures on Waldhausen categories, in the sense that for a biexact functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ of Waldhausen categories, the generators map in a predictable manner, with (for example) $F_\ast([A], [C \sim D]) = [F(A, C) \sim F(A, D)]$. Muro and Tonks then give generators and relations for $K_1$ of a Waldhausen category based on the structure of the stable quadratic module.

In [Zak17B, Definition 3.2, Proposition 3.4] a special case of the structure is worked out, in the case when the Waldhausen structures arise from assemblers; generators and relations for $K_0$ and $K_1$ of an assembler follow from this in [Zak17B, Theorem 3.8]. The map $\phi$ in the statement of the theorem arises from the biexact functor $SC(C) \times SC(D) \to SC(C \land D)$ induced by $\nu_C, D$. Applying the multiplicative structure given by Muro and Tonks to the presentations of $K_0$ and $K_1$ gives the desired result. \hfill $\Box$

Remark 8.4. The map on $\pi_1$ also induces a map

$$\text{Tor}_1(K_0(C), K_0(D)) \to K_1(C \land D).$$

If this term is nontrivial it may be possible to find an interesting interpretation for its image in $K_1(C \land D)$.

Corollary 8.5. Let $\mathcal{C}$ and $\mathcal{D}$ be closed assemblers. For any objects $A \in \mathcal{C}$, $B \in \mathcal{D}$, $\alpha \in \text{Aut}(A)$ and $\beta \in \text{Aut}(B)$,

$$\phi_\ast([A] \otimes [B, \beta]) = [(A, B), (1, \beta)] \in K_1(C \land D)$$

and

$$\phi_\ast([A, \alpha] \otimes [B]) = [(A, B), (\alpha, 1)] \in K_1(C \land D).$$

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