Rayleigh’s Stretched String

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Abstract

We obtain a rigorous \textit{a priori} upper and lower bounds to the exact period of the celebrated \textsc{Rayleigh} stretched-string differential equation.

1 Introduction

In his monumental treatise \textit{The Theory of Sound} (see pages 45-46 \cite{2}) Lord \textsc{Rayleigh} proposes the following problem:

\begin{quote}
\textit{A horizontal elastic steel wire having length }2L_0\textit{ and spring constant }\sigma\textit{ is stretched to length }2L. \textit{A body, having a mass }m\textit{ much greater than the mass of the wire, is tied to the center of the wire and is put in motion on a vertical line through the center of the wire. The only forces on the body are produced by tension in the wire (there is no gravity, no damping). It is required to study the motion of the body.}
\end{quote}

(We have used Agnew’s formulation \cite{1}, p. 17-18.) Let \(y(t) \equiv y\) be the vertical displacement of the mass \(m\) in time \(t\), where \(y\) is positive, 0, or negative according as the body is above, or at, or below the wire in equilibrium or neutral position.

By the law of \textsc{Hooke}, the magnitude of the elastic force, or the \textit{tension}, \(T(y)\), exerted by each \textit{half} of the stretched string is equal to:

\[
T(y) = \sigma \cdot \frac{\text{stretch}}{\text{original length}} = \sigma \cdot \frac{\sqrt{L^2 + y^2} - L_0}{L_0}
\]

and therefore the \textit{vertical component} of the \textit{total} force, exerted by both \textit{halves}, which is the only component acting to move the mass \(m\) is given by

\[
\text{vertical component of } T(y) = -2\sigma \cdot \frac{\sqrt{L^2 + y^2} - L_0}{L_0} \cdot \frac{y}{\sqrt{L^2 + y^2}}, \quad (1.1)
\]
where we have employed our sign conventions.

But, by NEWTON’s second law of motion, that same force, with opposite sign, is given by \( m \frac{d^2y}{dt^2} \).

Therefore, equating the two forces, and bringing the expression for the elastic force on the same side as the force given by NEWTON’s second law, we find that the algebraic sum of the forces is zero and dividing by \( m \), we conclude that the equation of motion of the mass \( m \) is given by:

\[
\frac{d^2y}{dt^2} + \left( \frac{2\sigma}{m} \sqrt{L^2 + y^2 - L_0} \right) y = 0
\]

This differential equation \( 1.2 \) is nonlinear and very complicated, and we have never seen an exact treatment of it in the literature.

RAYLEIGH, himself, simplifies the equation by the following physical reasoning;

“The tension of the string in the position of equilibrium depends on the amount of the stretching to which it has been subjected. In any other position the tension is greater; but we limit ourselves to the case of vibrations so small that the additional stretching is a negligible fraction of the whole. On this condition, the tension may be treated as a constant.”

(Our italics.) We just saw that the magnitude of the tension, \( T(y) \), (or the force) from each half of the string on \( m \) is given by

\[
T(y) = \sigma \sqrt{L^2 + y^2 - L_0}
\]

provided the stretching is well within elastic limits. So, RAYLEIGH effectively takes \( y = 0 \) in \( 1.3 \) so that the equation \( 1.2 \) becomes

\[
\frac{d^2y}{dt^2} + \frac{2\sigma L - L_0}{mL_0L} y = 0.
\]

or, using the notation \( 1.3 \) (with \( y = 0 \), i.e., \( T := T(0) = \sigma \frac{L - L_0}{L_0} \)),

\[
\ddot{y} + \frac{2T}{mL} y = 0
\]

where the “dots” refer to time derivatives. If we assume that the initial displacement at time \( t = 0 \) is \( y_0 \), then the solution to \( 1.5 \) is

\[
y(t) = y_0 \cos \left( \sqrt{\frac{2T}{mL}} t \right)
\]
which is simple harmonic motion with period $T$ given by:

$$T = \frac{2\pi}{\sqrt{\frac{2\sigma}{mL}}} \quad (1.7)$$

Rayleigh does not discuss how close his approximate period $T$ (1.7) is to the exact period $P$. Indeed, we have been unable to find any error analysis of Rayleigh’s solution in the literature. Yet, such a famous and classical differential equation merits an investigation into the accuracy of its approximate solution.

Therefore we offer the following theorem to fill this gap.

**Theorem 1.** Let $P$ be the true period of oscillation of the mass $m$. Then Rayleigh’s approximate period $T$ overestimates the true period, $P$. Indeed, the following inequalities are valid:

$$\frac{T}{\sqrt{1 + \frac{\sigma m}{2TL_0} \cdot y_0^2}} \leq P \leq T \quad (1.8)$$

Our lower bound shows the dependence of the period on the reciprocal, of the initial displacement, $y_0$, and tends to the upper bound as $y_0$ tends to 0.

### 2 The Formula for the Period

The differential equation (1.2) does not have the variable $t$ nor the first derivative, $\dot{y}$, appearing explicitly. Therefore, we can transform it into a separable equation for the velocity, $v := \dot{y}$

$$v := \dot{y} \quad (2.1)$$

since

$$\ddot{y} = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = v \frac{dv}{dy} \quad (2.2)$$

Substituting the right-hand side of (2.2) into (1.2), transposing the expression for the tension to the right-hand side, separating the variables and forming the indefinite integral of both sides we obtain

$$\frac{v^2}{2} = -\frac{\sigma}{2m} \left( \frac{y_0^2}{2L_0} - \sqrt{L^2 + y^2} \right) + C \quad (2.3)$$

where $C$ is the constant of integration. At time $t = 0$, the conditions are $y(0) = y_0$ and $v(0) = 0$. Therefore, evaluating the constant $C$, recalling that $v = \frac{dy}{dt}$, taking the square root of both sides, and using a negative sign because the displacement, $y(t)$, is a decreasing function during the first quarter oscillation, we obtain:

$$\frac{dy}{dt} = -\sqrt{\frac{2\sigma}{m} (y_0^2 - y^2) \left( \frac{1}{L_0} - \frac{2}{L^2 + y^2 + \sqrt{L^2 + y_0^2}} \right)} \quad (2.4)$$

Multiplying the quarter-period by 4, we obtain the following formula for the full period, $P$, of oscillation of the mass:
Theorem 2. The true period, $P$, of oscillation of the mass $m$ is given by:

$$P = 4 \sqrt{\frac{m^2}{2\sigma}} \int_0^{y_0} \frac{1}{\sqrt{y_0^2 - y^2} \sqrt{\frac{1}{L_0} - \frac{1}{\sqrt{L^2 + y^2 + \sqrt{L^2 + y_0^2}}}}} \, dy.$$  \hfill (2.5)

This integral is very complicated for exact computation. Indeed, if we make the change of variable

$$z^2 := L^2 + y^2, \quad z_0^2 := L^2 + y_0^2$$

we obtain the formula

$$P = 4 \sqrt{\frac{m^2}{2\sigma}} \int_L^{z_0} \frac{z}{\sqrt{-\frac{1}{2L_0} z^4 + z^3 + \left( \frac{L^2}{2L_0} - z_0 \right) z^2 - L^2 z + L^2 z_0 + \frac{L^2}{2L_0}}} \, dz.$$  \hfill (2.6)

If we write

$$\sqrt{-\frac{1}{2L_0} z^4 + z^3 + \left( \frac{L^2}{2L_0} - z_0 \right) z^2 - L^2 z + L^2 z_0 + \frac{L^2}{2L_0}}$$

as

$$\sqrt{-\frac{1}{2L_0}} \sqrt{(z - a)(z - b)(z - c)(z - d)}$$

(2.7)

and

$$P(z) := 4 \sqrt{\frac{m^2}{2\sigma}} \int \frac{z}{\sqrt{-\frac{1}{2L_0} \sqrt{(z - a)(z - b)(z - c)(z - d)}}} \, dz$$

(2.9)

then

$$P = P(z_0) - P(L).$$

(2.10)

The integral in (2.9) is a very complicated algebraic combination of an elliptic integral of the third kind and an elliptic integral of the first kind, and we will not pursue this line of inquiry any further.

Instead, we turn our attention to the upper and lower bounds for the true period, $P$.

### 3 Upper and Lower Bounds for the True Period

Proof of the bounds on the true period. We begin with the exact formula (3.1) for the true period, $P$:

$$P = 4 \sqrt{\frac{m^2}{2\sigma}} \int_0^{y_0} \frac{1}{\sqrt{y_0^2 - y^2} \sqrt{\frac{1}{L_0} - \frac{1}{\sqrt{L^2 + y^2 + \sqrt{L^2 + y_0^2}}}}} \, dy.$$  \hfill (3.1)
Taking \( y = y_0 = 0 \) in the sum \( \sqrt{L^2 + y^2} + \sqrt{L^2 + y_0^2} \) only makes the integrand larger and we conclude that

\[
P < 4 \sqrt{\frac{m}{2\sigma}} \int_0^{y_0} \frac{1}{\sqrt{y_0^2 - y^2} \sqrt{\frac{1}{L_0} - \frac{1}{y^2}}} \, dy
\]

\[
= 4 \sqrt{\frac{m \pi}{2\sigma}} \frac{\pi}{\sqrt{\sigma L}} 2
\]

\[
= \frac{2\pi}{\sqrt{\frac{2T}{mL}}}
\]

and we have proved the validity of the upper bound:

\[
P \leq \frac{2\pi}{\sqrt{\frac{2T}{mL}}}
\]

(3.2)

For the lower bound, we proceed similarly, and we start by observing that if we take \( y = y_0 \) in the sum \( \sqrt{L^2 + y^2} + \sqrt{L^2 + y_0^2} \) the integrand becomes smaller and we conclude that

\[
P \geq 4 \sqrt{\frac{m}{2\sigma}} \int_0^{y_0} \frac{1}{\sqrt{y_0^2 - y^2} \sqrt{\frac{1}{L_0} - \frac{1}{\sqrt{L^2 + y_0^2}}} \, dy}
\]

\[
= 4 \sqrt{\frac{m \pi}{2\sigma}} \frac{\pi}{\sqrt{\sigma L}} \frac{1}{\sqrt{L_0 \sqrt{L^2 + y_0^2}}}
\]

\[
\geq \frac{2\pi}{\sqrt{\frac{2T}{mL} + \frac{\sigma y_0^2}{L_0} L_0}}
\]

where, twice, we have used the inequality

\[
\sqrt{L^2 + y_0^2} \leq L + \frac{y_0^2}{2L}.
\]

Therefore, we have proved the validity of the lower bound:

\[
P \geq \frac{2\pi}{\sqrt{\frac{2T}{mL} + \frac{\sigma y_0^2}{L_0} L_0}}
\]

(3.3)

This completes the proof of the theorem. □
Corollary 1. If $\overline{P}$ denotes Rayleigh’s approximate period (1.7), then, the relative error

$$R := \frac{P - \overline{P}}{P}$$

in the approximation $P \approx \overline{P}$, satisfies the inequality

$$-\frac{m}{4(L - L_0)} \cdot y_0^2 \leq R \leq 0$$

(3.4)

This shows that the lower bound to the relative error is proportional to the square of the initial displacement and tends to zero with a “quadratic” rate of convergence.

Acknowledgment

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References

[1] Ralph Palmer Agnew *Differential Equations*, second edition McGraw-Hill Inc., New York, 1960.

[2] John William Strutt *Theory of Sound, Vol 1*, MacMillan and Co., New York, 1877.