Pion mass dependence of the nucleon mass in the chiral quark soliton model

K. Goeke,1 J. Ossmann,1 P. Schweitzer,1 and A. Silva2

1Institut für Theoretische Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany
2Departamento de Física and Centro de Física Computacional, Universidade de Coimbra, P-3000 Coimbra, Portugal

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The dependence of the nucleon mass on the mass of the pion is studied in the framework of the chiral quark-soliton model. A remarkable agreement is observed with lattice data from recent full dynamical simulations. The possibility and limitations to use the results from the chiral quark soliton model as a guideline for the chiral extrapolation of lattice data are discussed.

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I. INTRODUCTION

The formulation of QCD on a discrete, finite Euclidean lattice [1] is at present the only strict and model independent approach allowing to solve QCD in the low energy regime and to study, e.g., the hadronic spectrum from first principles [2]. Numerical lattice QCD simulations face technical problems, such as discretization errors or finite size effects, which are attacked and minimized with increasing success by employing improved versions of discretized actions, or by working on larger lattices available thanks to the steadily growing computer power. Still, present lattices are too small to accommodate the pion as light as it appears in nature [3–11].

The tool needed to extrapolate lattice data from the region of nowadays typically \( m_\pi \gtrsim 400 \text{ MeV} \) down to the physical value of the pion mass is, in principle, provided by the chiral perturbation theory (\( \chi \text{PT} \)). The \( \chi \text{PT} \) is an effective but rigorous approach to the description of low energy phenomena of strong interactions [12–14]. It is based on the concept of spontaneous chiral symmetry breaking with the pion as the Goldstone boson which acquires a small mass only due to explicit chiral symmetry breaking by the small current masses of the light quarks. \( \chi \text{PT} \) allows to address such questions like, e.g., how much do baryon masses change if one “switches” on the masses of light quarks and varies their values.

In order to extrapolate reliably lattice data by means of \( \chi \text{PT} \) it is important to ensure the convergence of the chiral expansion up to large values of \( m_\pi \). A first and promising matching of \( \chi \text{PT} \) and lattice results was reported, and it was established that the chiral expansion is well under control up to \( m_\pi^2 < 0.4 \text{ GeV}^2 \) [15–18], see also [7]. More conservative estimates, however, indicate that the chiral expansion is reliable only up to \( m_\pi^2 < 0.1 \text{ GeV}^2 \) [19]. The progress in computing power promises future lattice data at still lower pion masses and eventually at the physical point, which will improve the situation and make disappear this problem. In the meantime, however, it would be desirable to have a description of an intermediate region of pion masses, that would provide a safe overlap between the regime of the validity of \( \chi \text{PT} \) and the lattice data.

In this situation it is interesting to consider studies in chiral models — in particular, if they allow to go beyond the range of \( m_\pi \) where \( \chi \text{PT} \) is applicable. However, the inevitable prize to pay for the extended range of applicability compared to \( \chi \text{PT} \) is model-dependence, which introduces hardly controlable systematic uncertainties. Keeping this point critically in mind, such studies may nevertheless provide helpful insights.

In Refs. [20–24] the concept was introduced and developed to regularize chiral loops by means of suitable vertex form factors, referred to as “finite range regulators” (FRR) and intended to simulate physical effects of the pion cloud which has a finite range due to \( m_\pi \neq 0 \). As argued in [24], the FRR method corresponds in some sense to a (though model-dependent) chiral resummation reliable up to \( m_\pi^2 < 1 \text{ GeV}^2 \). While being physically intuitive and appealing, the approach was criticized to be unsatisfactory from a field-theoretic point of view, since it gives preference of one (“finite range”) regularization scheme to another (e.g. “dimensional”) regularization scheme [15].

In this work we address the question how the nucleon mass depends (implicitly) on the pion mass in another effective approach, namely in the chiral quark soliton model (\( \chi \text{QSM} \)) [25, 26]. This model was derived under certain assumptions from the instanton model of the QCD vacuum [27, 28], which provides a dynamical picture of the chiral symmetry breaking mechanism [29]. In this model the nucleon appears as a soliton of the chiral pion mean field in the limit of a large number of colours \( N_c \). The model provides a theoretically consistent description of numerous baryonic quantities ranging from static properties [30, 31] over “usual” [32] till “generalized” parton distribution functions [33], which — as far as these quantities are known — agree with phenomenology to within (10 – 30)% at the physical point.

Here — focussing on the nucleon mass \( M_N \) — we present the first study in the \( \chi \text{QSM} \) at non-physical pion masses \( m_\pi \) covering the wide range \( 0 \leq m_\pi \leq 1500 \text{ MeV} \). We make several remarkable observations. First, stable soliton solutions do exist in this range of pion masses. Second, we demonstrate that the model correctly describes also the heavy quark limit. In the opposite limit
$m_{\pi} \to 0$, which does not commute with large-$N_c$ limit [12, 36], the χQSM is known to exhibit a chiral behaviour and to incorporate leading non-analytic terms, which are at variance with the real world QCD with a finite number of colours $N_c = 3$, but in agreement with its formulation in the limit $N_c \to \infty$ [34, 35, 37]. This is consistent as the model is defined in this limit. Third, we show that the χQSM provides a satisfactory description of the variation of the lattice data on $M_N$ with $m_{\pi}$ in the considered range of pion masses.

Partly we provide explanations for these observations. Partly, however, they shall remain puzzles to be resolved upon further studies in the model.

This note is organized as follows. In Sect. II the χQSM is introduced and model results for $M_N(m_{\pi})$ presented, which we compare to lattice QCD in Sect. III. In Sect. IV we discuss the limitations for using the model quantitatively to extrapolate lattice data, and compare in Sect. V to χPT and the FRR approach. Sect. VI contains the conclusions. Technical details and a digression on the continuum, distorted by the pion field as compared to the free Dirac-Hamiltonian

$$H = \hat{H}[\bar{\psi}\psi] = \sum_{n=0}^{\infty} E_n|n\rangle \langle n|,$$

where the states $|n\rangle$ correspond to the eigenstates of the free Dirac Hamiltonian

$$H_0 = \sum_{n=0}^{\infty} E_n|n\rangle \langle n|,$$

and to incorporate leading non-analytic terms, which are at variance with the real world QCD with a finite number of colours. In the leading order of the large-$N_c$ limit the pion field is static, and one can determine the spectrum of the one-particle Hamiltonian of the effective theory (1)

$$\hat{H}[\bar{\psi}\psi] = \sum_{n=0}^{\infty} E_n|n\rangle \langle n|,$$

with, for small current quark masses $m$, corresponds to the Gell-Mann–Oakes–Renner relation

$$m_{\pi} f_{\pi}^2 = -m \langle \bar{\psi}\psi \rangle + O(m^2).$$

The χQSM is an application of the effective theory (1) to the description of baryons [25, 26]. The large-$N_c$ limit allows to solve the path integral over pion field configurations in Eq. (1) in the saddle-point approximation. In the leading order of the large-$N_c$ limit the pion field is static, and one can determine the spectrum of the one-particle Hamiltonian of the effective theory (1)

$$\hat{H}[\bar{\psi}\psi] = \sum_{n=0}^{\infty} E_n|n\rangle \langle n|,$$

and of a discrete bound state level of energy $E_{lev}$, if the pion field is strong enough. By occupying the discrete level and the states of the lower continuum each by $N_c$ quarks in an anti-symmetric colour state, one obtains a state with unity baryon number. The soliton energy $E_{sol}$ is a functional of the pion field

$$E_{sol}[U] = N_c \left[ E_{lev} + \sum_{E_n < E_{lev}} (E_n - E_{n_0}) \right]_{reg},$$

$E_{sol}[U]$ is logarithmically divergent, see App. A for the explicit expression in the proper-time regularization. By

$$\langle \bar{\psi}\psi \rangle = -\int \frac{d^4p_E}{(2\pi)^4} \left. \frac{8N_c M'}{p_E^2 + M'^2} \right|_{reg} = -8N_c M' I_1(m),$$

where $I_1$ is its proper-time regularized version, see App. A, and $M' \equiv M + m$. Note that in QCD strictly speaking $\langle \bar{\psi}\psi \rangle$ is well-defined only in the chiral limit. The pion is not a dynamical degree of freedom in the theory (1). Instead the dynamics of the pion field appears only after integrating out the quark fields which yields the effective action

$$S_{eff}[U] = \frac{f^2}{4} \int d^4x \, tr \, \partial^\mu U \partial_\mu U^\dagger + \ldots$$

where the dots denote the four-derivative Gasser-Leutwyler terms with correct coefficients, the Wess-Zumino term, terms $\propto m$ (and an infinite series of higher-derivative terms) [26]. The pion decay constant $f_{\pi} = 93$ MeV in Eq. (3) is given in the effective theory by the logarithmically UV-divergent loop integral (whose regularized version we denote by $I_2$, see App. A)

$$f_{\pi}^2 = \int \frac{d^4p_E}{(2\pi)^4} \left. \frac{4N_c M'^2}{(p_E^2 + M'^2)^2} \right|_{reg} = 8N_c M'^2 I_2(m).$$

The mass of the pion can be determined from the position of the pole of the pion propagator in the effective theory

$$m_{\pi}^2 = m \frac{I_1(m)}{I_2(m)},$$

which, for small current quark masses $m$, corresponds to the quark mass $m$.

II. PION AND NUCLEON IN THE EFFECTIVE THEORY

Let us consider the effective theory which was derived from the instanton model of the QCD vacuum [27, 28] and is given by the partition function [39, 40]

$$Z_{eff} = \int D\bar{\psi} D\psi DU \exp \left\{ i \int d^4x \, \bar{\psi} \left( i \gamma \cdot \partial - M U^\dagger U - m \right) \psi \right\},$$

(1)

where $U = \exp(i \tau^a \pi^a)$ denotes the SU(2) chiral pion field with $U^\dagger = \exp(i \tau^a \pi^a)$, and $M$ is the dynamical (“constituent”) quark mass due to spontaneous breakdown of chiral symmetry, and $m = m_u = m_d$ is the current quark mass. We neglect throughout isospin breaking effects.

The effective theory (1) is valid at low energies below a scale set by the inverse of the average instanton size $\rho_{av}^{-1} \approx 600$ MeV. The dynamical mass is momentum dependent, i.e. $M = M(p)$, and goes to zero for $p \gg \rho_{av}^{-1}$. In practical calculations it is convenient to replace $M(p)$ by a constant mass $M$, and to regularize the effective theory within an appropriate regularization scheme with a cutoff $\Lambda_{cut} = O(\rho_{av}^{-1})$. In the present work we use $M = 350$ MeV from the instanton vacuum model [29].

In the effective theory (1) chiral symmetry is spontaneously broken and a non-zero quark–vacuum condensate $\langle \bar{\psi}\psi \rangle \equiv \langle \psi_u \bar{\psi}_u + \psi_d \bar{\psi}_d \rangle$ appears which is given in leading order of the large-$N_c$ limit by the quadratically UV-divergent Euclidean loop integral

$$\langle \bar{\psi}\psi \rangle = -\int \frac{d^4p_E}{(2\pi)^4} \left. \frac{8N_c M'}{p_E^2 + M'^2} \right|_{reg} = -8N_c M' I_1(m),$$

(2)

where $I_1$ is its proper-time regularized version, see App. A, and $M' \equiv M + m$. Note that in QCD strictly speaking $\langle \bar{\psi}\psi \rangle$ is well-defined only in the chiral limit. The pion is not a dynamical degree of freedom in the theory (1). Instead the dynamics of the pion field appears only after integrating out the quark fields which yields the effective action

$$S_{eff}[U] = \frac{f^2}{4} \int d^4x \, tr \, \partial^\mu U \partial_\mu U^\dagger + \ldots$$

(3)

where the dots denote the four-derivative Gasser-Leutwyler terms with correct coefficients, the Wess-Zumino term, terms $\propto m$ (and an infinite series of higher-derivative terms) [26]. The pion decay constant $f_{\pi} = 93$ MeV in Eq. (3) is given in the effective theory by the logarithmically UV-divergent loop integral (whose regularized version we denote by $I_2$, see App. A)

$$f_{\pi}^2 = \int \frac{d^4p_E}{(2\pi)^4} \left. \frac{4N_c M'^2}{(p_E^2 + M'^2)^2} \right|_{reg} = 8N_c M'^2 I_2(m).$$

(4)

The mass of the pion can be determined from the position of the pole of the pion propagator in the effective theory (1). Its relation to the current quark mass is given by the equation (for the $I_1$ see App. A)

$$m_{\pi}^2 = m \frac{I_1(m)}{I_2(m)},$$

(5)

which, for small current quark masses $m$, corresponds to the Gell-Mann–Oakes–Renner relation

$$m_{\pi} f_{\pi}^2 = -m \langle \bar{\psi}\psi \rangle + O(m^2).$$

(6)
minimizing $E_{\text{sol}}[U]$ one obtains the self-consistent solitonic pion field $U_c$. This procedure is performed for symmetry reasons in the so-called hedgehog ansatz

$$\pi^a(x) = e^a_r \, P(r), \quad U(x) = \cos P(r) + i\tau^a e^a_r \sin P(r),$$

with the radial (soliton profile) function $P(r)$ and $r = |x|$, $e_r = x/r$. The nucleon mass $M_N$ is given by $E_{\text{sol}}[U_c]$. The self-consistent profile satisfies $P_c(0) = -\pi$ and decays in the chiral limit as $1/r^2$ at large $r$. For finite $m$ it exhibits a Yukawa tail $\propto \exp(-m_\pi r)/r$ with the pion mass $m_\pi$ connected to $m$ by the relation (5).

For the following discussion we note that the soliton energy can be rewritten as an expansion in powers of $\partial^2 U$ as follows

$$E_{\text{sol}}[U] = \sum_{k=1}^{\infty} F_k[(\partial U)^k]$$

where $F_k[(\partial U)^k]$ symbolically denotes a functional in which $\partial^2 U$ appears $k$-times (appropriately contracted). Note that in leading order of the large-$N_c$ limit the soliton field is time-independent, i.e. $\partial U$ is just $\nabla U$. For some observables the lowest orders in expansions analogous to (11) were computed [30–33].

Let us also remark that in the case $m_u = m_d = m_s$ the above given formulae for the soliton energy in the SU(2) version of the model coincide with those from the SU(3) version [31].

In order to describe further properties of the nucleon, it is necessary to integrate over the zero modes of the soliton solution in the path integral, which assigns a definite momentum, and spin and isospin quantum numbers to the baryon. Corrections in the $1/N_c$-expansion can be included by considering time dependent pion field fluctuations around the soliton solution. The $\chi$QSM allows to evaluate baryon matrix elements of local and non-local QCD quark bilinear operators like $\langle B' | \Psi \Gamma \Psi | B \rangle$ (with $\Gamma$ denoting some Dirac- and flavour-matrix) with no adjustable parameters. This provides the basis for the wide range of applicability of this model [30–33].

A. The mass of the nucleon in the large-$N_c$ limit

If one managed to solve QCD in the limit $N_c \to \infty$ one would in principle obtain for the mass of the nucleon an expression of the form (let the $M_i$ be independent of $N_c$)

$$M_N = N_c M_1 + N_c^0 M_2 + N_c^{-1} M_3 + \ldots .$$

The $\chi$QSM provides a practical realization of the large-$N_c$ picture of the nucleon [41], and respects the general large-$N_c$ counting rules. In leading order of the large-$N_c$ limit one approximates in the $\chi$QSM the nucleon mass by the expression for the soliton energy in Eq. (9), i.e. one considers only the first term in the expansion (12)

$$M_N \approx N_c M_1 = E_{\text{sol}}.$$ 

We obtain numerically

$$M_N = 1254 \text{ MeV}$$ 

where the cutoff in the regulator function $R$, see App. A, is chosen such that for $m_\pi = 140 \text{ MeV}$ the physical value of the pion decay constant $f_\pi$ is reproduced. We observe an overestimate of the physical value $M_N = 940 \text{ MeV}$ by about 30%. This is not surprising, given the fact that we truncate the series in Eq. (12) after the first term and thus neglect corrections which are generically $O(1/N_c)$, i.e. of the order of magnitude of the observed overestimate.

In fact, the soliton approach is known to overestimate systematically the physical values of the baryon masses because of – among others – spurious contributions due to the soliton center of mass motion [38]. Taking into account the corresponding corrections, which are $O(N_c^0)$, reduces the soliton energy by the right amount of about 300 MeV. (Note that there are also other sources of corrections at $O(N_c^0)$, see Ref. [38].) We shall keep in mind this systematic overestimate, when we will discuss lattice data below.

B. The chiral limit

In the following we will be interested in particular in the pion mass dependence of the nucleon mass. From $\chi$PT we know that

$$M_N(m_\pi) = M_N(0) + A m_\pi^2 + B m_\pi^3 + \ldots ,$$

where the dots denote terms which vanish faster than $m_\pi^3$ with $m_\pi \to 0$. The constants $M_N(0)$ and $A$ (which is related to the pion-nucleon sigma-term) serve to absorb infinite counter-terms in the process of renormalization in $\chi$PT – in the sense of renormalizability in $\chi$PT. However, the constant $B$, which accompanies the so-called leading non-analytic (in $m$, since $m_\pi^3 \propto m^3/2$) contribution, is finite. For this constant $\chi$PT, as well as any theory which correctly incorporates chiral symmetry, yields [12, 36]

$$B = -k \frac{3g^2}{32\pi f_\pi^2} .$$

with $k = 1$ for finite $N_c$. However, the limits $m_\pi \to 0$ and $N_c \to \infty$ do not commute. If we choose first to take $N_c \to \infty$ while keeping $m_\pi$ finite, and only then we consider the chiral limit, then $k = 3$. The reason for that is the special role played by the $\Delta$-resonance. In the

\footnote{In this work we quote numerical results to within an accuracy of 1 MeV. However, one should keep in mind that we neglect isospin breaking effects (and electromagnetic corrections). Therefore we shall round off the physical masses as $m_\pi = 140 \text{ MeV}$ and $M_N = 940 \text{ MeV}$, and the same is understood for our numerical results.}
large-$N_c$ limit the nucleon and the $\Delta$-resonance become mass-degenerated\footnote{Though in the $\chi$QSM one cannot handle $O(N_c^0)$ corrections to $M_N$, still one is able to describe consistently the mass difference $(M_\Delta - M_N)$ and to reproduce the large-$N_c$ counting rule in Eq. (16) by considering a class of particular (so-called “rotational”) $1/N_c$-corrections [28]. In the soliton picture $\Delta$ and nucleon are just different rotational states of the same classical soliton. In this note we do not consider rotational corrections and work to leading-order in the large-$N_c$ expansion.}

\[
M_\Delta - M_N = O(N_c^{-1}).
\] (16)

Taking $N_c \to \infty$ while $m_\pi$ is kept finite, one has $M_\Delta - M_N \ll m_\pi$ and has to consider the contribution of the $\Delta$-resonance as intermediate state in chiral loops on equal footing with the contribution of the nucleon. The contribution of the $\Delta$-resonance appears (in quantities which do not involve polarization) to be twice the contribution of the nucleon, hence $k = 3$ [37]. This is the situation in the $\chi$QSM and, in fact, in this model one recovers [35] the correct (in the large-$N_c$ limit) leading non-analytic term in Eq. (15).

\section{C. The heavy quark limit}

The $\chi$QSM is based on chiral symmetry which consideration makes sense only when explicit chiral symmetry breaking effects are small. Nevertheless one can consider the model, in principle, for any value of $m$. In this context let us first note that taking the limit of a large current quark mass $m \to m_Q$ in Eqs. (2, 4, 5) yields, see Table I

\[
\lim_{m \to m_Q} m_\pi = 2 \pi Q
\] (17)

which is the correct heavy quark limit for the mass of a meson (see Sec.II D on details of parameter fixing).

What happens to the mass of the soliton in this limit? Consider the eigenvalue problem in Eq. (7). With increasing $m \to m_Q$ the “potential term” $MU^7\pi$ is less and less important and the spectrum of the full Hamiltonian (7) becomes more and more similar to the spectrum of the free Hamiltonian (8). In this limit the eigenvalues of the full and free Hamiltonian nearly cancel in the sum over energies in Eq. (9). Only the contribution of the discrete level is not compensated and approaches $E_{lev} \approx m_Q$. Thus, we recover the correct heavy quark limit

\[
\lim_{m \to m_Q} M_N = N_c m_Q.
\] (18)

However, the above considerations are formal, since the crucial step consists in demonstrating that a stable soliton solution indeed exists, i.e. that a self-consistent profile can be found for which the soliton energy (9) takes a minimum. In practice, we find that stable soliton solutions exist at least up to $m = O(700 \text{ MeV})$, which is sufficient for a comparison to lattice QCD results. In this range we observe a tendency to approach the limit (18) from below as expected, see Table I. Thus, we find that both the pion and the nucleon mass are correctly described in the effective theory when the current mass of the quarks becomes large.

\section{D. Fixing of parameters}

When studying the nucleon mass as function of $m_\pi$ we must specify what is fixed and what varies in the chiral limit. Here we make the following choice. We keep the dynamical constituent quark mass $M = 350 \text{ MeV}$ and $f_\pi = 93 \text{ MeV}$ fixed. In this way we obtain the results shown in Table I, and plotted as solid line in Fig. 1. The meaning of the dashed curve in Fig. 1 is explained in App. B where a digression is given on the pion-nucleon sigma-term related to the slope of $M_N(m_\pi)$.

It should be noted that keeping $f_\pi$ fixed in the chiral limit is a choice often considered in literature. However, at this point other choices could be considered as well, e.g. one could fix the pion decay constant to its value $F = 88 \text{ MeV}$ in the chiral limit, or allow $f_\pi$ to be $m_\pi$ dependent, which strictly speaking is the case in $\chi$PT and in lattice QCD. In $\chi$PT

\[
f_\pi(m_\pi) = F \left(1 + \frac{m_\pi^2}{4\pi F^2} \bar{l}_4 + O \left( \frac{m_\pi^4}{F^4} \right) \right)
\] (19)

where $\bar{l}_4$ is a low energy constant [13, 14]. In lattice calculations $f_\pi$ increases with larger $m_\pi$, exceeding its physical value by about 40% at $m_\pi \sim 1 \text{ GeV}$, e.g. [42]. A more consistent way of fixing model parameters could consists in choosing $\Lambda_{\text{cut}}$ (and/or $M$) such that in the model $f_\pi(m_\pi)$ satisfies (19) with the correct value for $\bar{l}_4$, and agrees in each case with lattice results at large $m_\pi$.

Remarkably, Eq. (19) holds in the model and parameters can be fixed to reproduce $\bar{l}_4$ correctly [34]. However,
it is a subtle issue, how to simulate in a chiral model the lattice situation in a realistic way. There a common procedure is to keep fixed all bare lattice parameters but the bare current quark mass (the hopping parameter). In some sense, the dimensionful quantity, which is kept fixed in lattice calculations while \(m_\pi\) “is varied”, is the Sommer scale \([43]\) defined by the notion of the heavy quark potential — absent in chiral models.

Notice that the \(\chiQSM\) describes numerous observables to within an accuracy of typically \((10 – 30)\%\) \([31]\), but cannot be expected to be much more precise than that. Overestimating the value of \(f_\pi\) in the chiral limit by 5% or underestimating its lattice values by 30% (or 40%) lies within the accuracy of the model. Note that changing \(f_\pi\) (at non-physical \(m_\pi\)) would alter in particular values of the cutoff \(\Lambda_{\text{cut}}\) and consequently change \(M_N\) in Table I. Since \(f_\pi\) and \(M_N\) depend on \(\Lambda_{\text{cut}}\) logarithmically and thus weakly, these changes may be expected to be small and within the accuracy of the model.

From a practical point of view, our choice to keep \(f_\pi\) fixed to its physical value may be considered as one effective prescription, whose consequences one may think absorbed in the unavoidable model dependence of the results.

It would be interesting to consider other effective prescriptions which would provide more insights into the unavoidable model dependence of the results.

| \(m_\pi\) | \(m\) | \(\Lambda_{\text{cut}}\) | \(\langle \bar{\psi}\psi \rangle^{1/3}\) | \(N_c E_{\text{lev}}\) | \(N_c E_{\text{cont}}\) | \(E_{\text{sol}}\) |
|---|---|---|---|---|---|---|
| 0  | 649 | 220 | 681 | 530 | 1211 |
| 10 | 649 | 220 | 682 | 530 | 1212 |
| 50 | 649 | 220 | 692 | 526 | 1218 |
| 140 | 649 | 220 | 744 | 510 | 1254 |
| 300 | 635 | 211 | 872 | 494 | 1366 |
| 600 | 666 | 204 | 1202 | 484 | 1686 |
| 1200 | 799 | 205 | 2006 | 465 | 2471 |

### III. COMPARISON TO LATTICE DATA

In this Section we compare \(M_N(m_\pi)\) from \(\chiQSM\) with lattice results on the nucleon mass from Refs. \([4–11]\). Wherever possible we will use only such lattice data, where all bare lattice parameters where kept fixed apart from the current quark mass (or the hopping parameter)\(M_N(m_\pi)\) from the model overestimates the lattice data by about 300 MeV which is expected, if we recall the discussion in Sec. II A. Remarkably we observe that it is possible to introduce an \(m_\pi\)-independent subtraction constant \(C = O(300 \text{ MeV})\) depending on the lattice data, such that \([M_N(m_\pi) – C]\) agrees well with the lattice data.

In Fig. 2a we compare the \(\chiQSM\) result for \(M_N(m_\pi)\) to full lattice data by the UKQCD and QCDSF Collaborations \([6, 7]\) obtained from simulations on \((16 – 24)^3 \times 48\) lattices using the standard Wilson plaquette action for gauge fields and the non-perturbatively \(O(a)\) improved action for fermions. The fat points in Fig. 2a were extracted from simulations at \(5.20 \leq \beta \leq 5.29\) on lattices of the physical size \(L \geq 1.96\) fm with \(m_\pi L > 6\) and lattice spacings \(a = 0.09\) fm to \(0.12\) fm fixed by the Sommer method with \(r_0 = 0.5\) fm \([43]\). Other lattice spacings quoted below were also determined by means of this popular method which, however, is not free of criticism \([45]\).

A best fit yields \(C = 329\) MeV with a \(\chi^2\) per degree of freedom of \(\chi^2_{\text{dof}} = 0.7\) and is shown in Fig. 2a. The uncertainty of the constant \(C\) due to the statistical error of the lattice data is of the order of magnitude of few MeV, see Table II and thus negligible, see the remarks in footnote 1.

For comparison in Fig. 2a also data by the UKQCD and QCDSF Collaborations \([6, 7]\) are shown from smaller lattices \(L = 1.5\) fm to 1.7 fm. Finite size effects are clearly visible, and were subject to a detailed study in Ref. \([7]\). Since the present \(\chiQSM\) calculation by no means is able to simulate finite volume effects, we restrict our study to data from lattices with \(L \geq 2\) fm. The study of Ref. \([7]\) indicates that for lattices of this size finite volume effects can be assumed to be small. Also, in this work we consider data obtained from lattices with spacings \(a \leq 0.13\) fm and assume discretization effects to be negligible. Such effects are difficult to control systematically \([46]\).

In Fig. 2b the \(\chiQSM\) result for \(M_N(m_\pi)\) is compared to lattice data by the JLQCD Collaboration from Ref. \([5]\), where dynamical simulations with two flavours were performed on a \(20^3 \times 48\) lattice using the plaquette gauge action and the non-perturbatively \(O(a)\) improved Wilson quark action at \(\beta = 5.20\) with lattice spacings \(0.10\) fm \(\leq a \leq 0.13\) fm. The physical size of the lattice was \(1.96\) fm \(\leq L \leq 2.53\) fm and the range \(500\) MeV \(\leq m_\pi \leq 1\) GeV was covered corresponding to \(m_\pi L > 5\). The best fit yields \(C = 341\) MeV with a \(\chi^2_{\text{dof}} = 0.3\) and is shown in Fig. 2b.

In Figs. 2c and d we compare \(M_N(m_\pi)\) from the model to lattice results by the CP-PACS Collaboration \([4]\), which were obtained on \(24^3 \times 48\) lattices from...
FIG. 2: $M_N(m_\pi)$ vs. $m_\pi^2$ from the chiral quark-soliton model vs. lattice data on $M_N(m_\pi)$ from full simulations by (2a) the UKQCD-QCDSF Collaborations [6, 7], (2b) the JLQCD Collaboration [5], (2c,d) the CP-PACS Collaboration [4], (2e) the RBC Collaboration [8], (2f) the MILC Collaboration [9, 10]. All data were obtained from large lattices $L \gtrsim 2$ fm, with the exception of the data marked by crosses in (2a), see text. Unless error bars are shown, here and in the following figures the statistical error of the lattice data is comparable to or smaller than the size of the points in the plot.

FIG. 3: $M_N(m_\pi)$ vs. $m_\pi^2$ from the chiral quark-soliton model vs. lattice data on $M_N(m_\pi)$ obtained in the quenched approximation by 3a the CSSM Lattice Collaboration [11] and 3b the MILC Collaboration [9].
a renormalization-group improved gauge action and a
mean field improved clover quark action at \( \beta = 2.1 \) (and
\( \beta = 2.2 \)) with dynamical \( u \)- and \( d \)-quarks and quenched
\( s \)-quarks. For hadrons with no valence \( s \)-quarks this prac-
tically means a full two-flavour simulation. The physical
lattice spacings and sizes were \( a = 0.09 \) fm to 0.13 fm
and \( L = 2.22 \) fm to 3.12 fm with pion masses in the
range \( 500 \) MeV \( \leq m_\pi \leq 1 \) GeV, such that \( m_\pi L \geq 7 \).
We observe that \( M_N(m_\pi) \) from the \( \chi \)QSM with a sub-
traction constant \( C = 331 \) MeV with \( \chi^2_{\text{dof}} = 2.2 \) (and
\( C = 338 \) MeV with \( \chi^2_{\text{dof}} = 0.4 \)) describes well the lattice
data of Ref. [4], Fig. 2c (and Fig. 2d).

Next we confront the model results for \( M_N(m_\pi) \) to
lattice data from dynamical two-flavour simulations with
domain wall fermions by the RBC Collaboration [8],
which have the virtue of preserving chiral invariance.
In Ref. [8] a renormalization group improved (“doubly
blocked Wilson”) gauge action with \( \beta = 0.80 \) was used on a
16\(^3\) \times 32 lattice with the physical spacing of 0.12 fm
and a lattice size about 2 fm. The range of pion masses
was \( m_\pi = 470 \) MeV to 650 MeV. A best fit yields for the
constant \( C = 306 \) MeV with \( \chi^2_{\text{dof}} = 0.02 \), and provides a
very good description of the data, see Fig. 2c.

In Fig. 2f we confront \( M_N(m_\pi) \) from the \( \chi \)QSM with
2001 lattice data by the MILC Collaboration [9] ob-
tained from simulations with three dynamical quarks
using a one-loop Symanzik improved gauge action and
an improved Kogut-Susskind quark action. The phys-
ical lattice size was tuned to \( L = 2.6 \) fm with a lat-
tice spacing of 0.13 fm and the range of pion masses
340 MeV \( \leq m_\pi \leq 2.2 \) GeV was covered. The best fit
yields \( C = 256 \) MeV for the subtraction constant (with
\( \chi^2_{\text{dof}} = 1.2 \)).

One could worry whether the SU(2) model results
so be compared to three-flavour lattice simula-
tions, though in [9] for the nucleon mass no significant dif-
ferences were noticed between two- and three-flavour runs.
However, as noted in Sec. II, the nucleon mass is the
same in the SU(2) and SU(3) versions of the \( \chi \)QSM if
\( m_u = m_d = m_s \), which is the case for the lattice data [9]
for \( m_\pi > 700 \) MeV. Restricting the fitting procedure to
this range of \( m_\pi \) (the last three points in Fig. 2f) we ob-
tain \( C = 260 \) MeV (with a \( \chi^2_{\text{dof}} = 1.9 \)). This fit is shown in
Fig. 2f. Note that it is practically indistinguishable, c.f.
footnote 1, from the fit where the whole \( m_\pi \)-range (i.e.
also data with \( m_u = m_d < m_s \)) was used, and equally
well describes the region of lower \( m_\pi \). In any case we
observe a good agreement with the lattice data [9] up to
\( m_\pi = 1.5 \) GeV.

We include in Fig. 2f also the recent small-\( m_\pi \) (2004-)
MILC data [10] from the “coarse” lattices with \( L = 20 \)
(or \( L = 24 \) for the lowest \( m_\pi \)-value) which have a lattice
spacing \( a \approx 0.12 \) fm comparable to [9]. These data are
not compatible with the \( \chi \)QSM result. In this context it
is important to note that the simulation for the highest
\( m_\pi \)-value of the 2004 MILC data [10] is an extended run
of the simulation for the lowest \( m_\pi \) of the 2001 data [9].
One would therefore expect that they coincide in the plot
in Fig. 2f, which is not the case. In fact, the MILC data
[9, 10] for this particular simulation are well consistent
with each other in lattice units: \( a m_\pi \) and \( a M_N \) from
these runs agree within statistical error bars.

The discrepancy of the results from [9, 10] in Fig. 2f
is due to the different values for \( r_1 \) — a parameter de-
fined similarly to the Sommer scale \( r_0 \) and used to fix the
physical units in [9, 10]. Different methods were used to
determine the physical unit of \( r_1 \) — resulting in the value
\( r_1 = 0.35 \) fm in [9], and \( r_1 = 0.324(4) \) fm in [10]. Had we
used the \( r_1 \)-value from [9] to give physical units to the di-

dimensionless lattice numbers for \( a m_\pi \) and \( a M_N \) from [10],
then the two data sets would perfectly agree in Fig. 2f,
and the 2004 data would be compatible with the \( \chi \)QSM
results. The precise determination of the physical units
of lattice data is a difficult issue [43, 45], see also [9, 10].

It is instructive to compare the model results also to
lattice data obtained from simulations performed in the
quenched approximation, e.g., by the CSSM Lat-
tice Collaboration [11], where a mean-field improved
gauge action and a fat-link clover fermion (“FLIC”) ac-
tion was used on a \( 16^3 \times 32 \) lattice with a lattice spac-
ing of \( a = 0.125 \) fm. The calculation covers the range
540 MeV \( \leq m_\pi \leq 920 \) MeV. A best fit to the quenched
data gives \( C = 345 \) MeV (with a \( \chi^2_{\text{dof}} = 1.3 \)) which yields
a good agreement with the lattice data, see Fig. 3a.

For the quenched data by the MILC Collaboration [9],
however, we observe that a fit would work much worse.
In this case we refrain from fitting and show instead in
Fig. 3b the fit to the full MILC data from Fig. 2f, which
nicely illustrates how results from full and quenched cal-
culations differ. Interestingly, at large \( m_\pi \) the full and
quenched data of Ref. [9] agree well with eachother. In
fact, it is not surprising that differences between full
and quenched simulations become less pronounced with in-
creasing \( m_\pi \), i.e. with increasing fermion masses.

Thus, we observe that the \( \chi \)QSM-result for \( M_N(m_\pi) \)
supplemented by an \( m_\pi \)-independent subtraction con-
stant \( C \) (whose precise value follows from a best fit to the
respective lattice data) is able to describe the lattice
data [4–9, 11] over a wide range of pion masses
350 MeV \( \leq m_\pi \leq 1500 \) MeV. The results are shown in
Figs. 2 and 3 and are summarized in Table II.

IV. \( \chi \)QSM AS TOOL FOR EXTRAPOLATION?

From the \( [M_N(m_\pi) - C] \) at \( m_\pi = 140 \) MeV with the
constant \( C \) fitted to the respective data, we can read
off in principle the physical value of the nucleon mass –
“extrapolated” from the respective lattice data by means
of the \( \chi \)QSM as a guideline. These extrapolated values
are indicated by arrows in Figs. 2a-f and 3a, and are
summarized in Table II.

It is worthwhile stressing that the extrapolated values
in Table II agree within an accuracy of \( \pm 5\% \) with the
physical nucleon mass. (Though at the same time, lat-
tice data for \( f_\pi \) are underestimated by up to 40% at large
m_π \sim 1\text{GeV}, \text{see Sect. II.D.}) The uncertainty of the extrapolated values for the nucleon mass is the same as for the fit constant C as quoted in Table II, i.e., it is of the order of few MeV and thus negligible, see footnote 1.

However, what does not need to be negligible is the systematic error of such an extrapolation. First of all, this extrapolation is done within the \( \chi \) QSM, and the results are model-dependent. E.g., handling the model parameters in the chiral limit differently would change our results — though one may expect a qualitatively similar picture, see the discussion in Sec. II.D. Apart from this source of model-dependence, which will be subject to future numerical studies [44], there are principal difficulties to estimate reliably the systematic error within the model, as we shall see in the following.

Let us first address the question of the range of reliability of the \( \chi \) QSM-description of \( M_N(m_π) \). For that it is instructive to compare to \( \chi \) PT and the effective FRR approach. \( \chi \) PT to \( O(p^4) \) was argued to provide a reliable expansion for \( M_N(m_π) \) up to \( m_π^2 < 0.4 \text{GeV}^2 \) [15–18] (\( p \) is to be identified with the generic small expansion parameter in \( \chi \) PT, e.g., in our context pion mass). A more conservative bound \( m_π^2 < 0.1 \text{GeV}^2 \) was given in [19]. The effective FRR approach was argued to correspond to a partial resummation of the chiral expansion, and to be valid up to \( m_π^2 < 1 \text{GeV}^2 \) [24] (which comes, of course, at the price of introducing model dependence, see the discussion in Sec. I.)

What might be the range of reliability of the \( \chi \) QSM? Recall that the \( \chi \) QSM expression for the nucleon mass (as well as for any other quantity) may be considered as a resummed infinite series in derivatives of the chiral field \( U = \exp(iσ^aκ^a) \), c.f. Eq. (11), whereby it is understood that each chiral order is evaluated in the large-N_c limit. (Keep in mind that these limits do not commute, see Sect. II.B.) Thus, one may argue that the \( \chi \) QSM corresponds to a chiral expansion, which is completely resummed – to leading order of the large-N_c expansion. If one were happy with this approximation, then \( \chi \) QSM results could be considered reliable for all \( m_π \) including the heavy quark limit, see the discussion in Sec. II.C. It must be stressed, that – as in the case of the FRR approach – this chiral resummation is performed within a particular model of the nucleon. Thus, our results and conclusions are inevitably model dependent.

The wide range of reliability of the \( \chi \) QSM we observe in practice is illustrated in Fig. 4, where we show the \( \chi \) QSM-fit to the MILC lattice data [9] and the \( \chi \) PT-fit from [17]. (For further details see Sec. V.)

The perhaps most serious restriction might be that the \( \chi \) QSM assumes the number of colours \( N_c \) to be large. What might be the effects due to 1/N_c-corrections?

![FIG. 4: Lattice data by MILC [9] on \( M_N(m_π) \) vs. \( m_π \) and the fits to these data in \( \chi \) PT from Ref. [17] (where the physical value of \( M_N \) was used as input) and in the \( \chi \) QSM, c.f. Fig. 2f. The figure illustrates the wide range of applicability of the \( \chi \) QSM.](image-url)
Let us consider the mass difference between the $\Delta$-resonance and the nucleon as a measure for such corrections. Note that this mass difference vanishes not only in the large-$N_c$ limit, see Eq. (16), but also in the heavy quark limit [36]. This is supported by lattice results [4, 9, 11], where always $\Delta \equiv M_\Delta - M_N < m_\pi$ holds. (This means that on present day lattices the $\Delta$-resonance is safe from strong decays and thus a stable particle.) In the region $m_\pi^2 > 0.32\text{GeV}^2$ one finds $\Delta \lesssim \frac{1}{2} m_\pi$ [4, 9, 11]. Thus, practically for most of the present day lattice data the condition

$$\Delta \equiv M_\Delta - M_N \ll m_\pi \text{ for } m_\pi^2 > 0.32 \text{GeV}^2$$

is satisfied, such that the $\Delta$-nucleon mass difference can be neglected to a good approximation. This may be a reason for the good description of lattice data in the $\chi$QSM up to $m_\pi^2 \lesssim 2.3 \text{GeV}^2$ in Sec. III.

However, in the physical region $\Delta = 290\text{MeV}$ is larger than $m_\pi = 140\text{MeV}$, and the leading order large-$N_c$ treatment of the nucleon mass inevitably introduces a serious systematic error. In fact, one could express the nucleon mass as a function of $m_\pi$ and $y = \Delta/m_\pi$ as

$$M_N(m_\pi, \Delta) = F(m_\pi, y), \quad y = \frac{\Delta}{m_\pi}. \quad (21)$$

Then, using the $\chi$QSM as a guideline for the extrapolation of lattice data corresponds to approximating

$$M_N = F(m_\pi, 2.1) \approx F(m_\pi, 0) \text{ at } m_\pi = 140 \text{MeV}. \quad (22)$$

It is difficult to quantify the systematic error associated with this approximation. A very rough estimate of this error within the model is given in Appendix C.

Finally, let us discuss the role of the subtraction constant $C$. As mentioned in Sec. II the appearance of such a constant is theoretically well motivated and understood in the soliton approach [38]. The comparison to lattice data indicates that this constant is about $C = \mathcal{O}(300\text{MeV})$ and $m_\pi$-independent in the covered $m_\pi$-range within the statistical accuracy of the lattice data, see Table II. Although this happens to be the magnitude for this constant needed for the model result to coincide with the physical value of the nucleon mass, see Eq. (13), it would be premature to assume the constant $C$ to be $m_\pi$-independent for all $m_\pi$. However, this is what we implicitly did when quoting the extrapolated values for $M_N$ in Table II. The question, whether or not the constant $C$ is $m_\pi$ dependent, cannot be answered rigorously within the model.

As an intermediate summary, we conclude that besides the general drawback of being model-dependent, the use of the $\chi$QSM as a guideline for an extrapolation of lattice data is limited by two major sources of systematic error, namely $1/N_c$ corrections and a possible $m_\pi$-dependence of the constant $C$. Both are not under control within the model and prevent a reliable estimate of the systematic error of the extrapolation. At this point it is instructive to compare to $\chi$PT and the FRR model, which may give us a rough idea about the size of the systematic effects.

V. COMPARISON TO $\chi$PT AND FRR

In order to test the results from the $\chi$QSM below the $m_\pi$ range available from lattice QCD we have to compare to $\chi$PT which allows to connect model-independently lattice data through the physical point to the chiral limit. In the following we shall focus on the analyses [7, 16, 17] in $\chi$PT up to $\mathcal{O}(p^4)$ (and refer to them simply as $\chi$PT). To this order the chiral expansion of the nucleon mass is characterized by 4 low-energy constants — or more precisely, by 4 linearly independent combinations of them. These constants are in principle known from studies of nucleon-nucleon or pion-nucleon low-energy scattering data. However, they can alternatively be determined from a fit to lattice data.

Fig. 5a shows the $M_N(m_\pi)$ as obtained in $\chi$PT from fits to the lattice data [4–7] satisfying $\Lambda < 0.15\text{fm}$, $m_\pi L > 5$ and respectively the constraint $m_\pi < 800\text{MeV}$ in Ref. [7], and $m_\pi < 600\text{MeV}$ in Ref. [16]. Also shown in Fig. 5a is the fit of Ref. [17] to the MILC data [9], where a simultaneous fit to lattice data on $M_\Delta(m_\pi)$ was performed. In [16, 17] the physical values of the nucleon and/or $\Delta$-mass were included as constraints to the fit. Note that low-energy constants resulting from the different fits [7, 16, 17] are compatible with the respective phenomenological values. From this point of view the different lattice data in Fig. 5a are consistent with each other. For comparison in Fig. 5a our $\chi$QSM-fits to the same data sets are shown.

Notice that the two highest-$m_\pi$ data points (marked by empty circles in Fig. 5a) by the UKQCD Collaboration [6, 7] were obtained from somehow smaller lattices of the size $L = 1.56\text{fm}$ and $1.68\text{fm}$ and clearly do not follow the tendency of the $M_N(m_\pi)$ from the $\chi$QSM. These points are therefore omitted from the fit shown in Fig. 5a. Including these points significantly worsens the $\chi^2$ of the fit, see Table III where the results are summarized. Since the $\chi$QSM-description of the lattice data effectively works also at significantly larger $m_\pi$, see Secs. III and IV and c.f. Fig. 4, we conclude that, quoting Ref. [16], “the surprisingly good (and not yet understood) agreement with lattice data [above $m_\pi > 600\text{MeV}$] even up to $m_\pi \approx 750\text{MeV}”$ is an accidental consequence of comparing $\chi$PT to $\mathcal{O}(p^4)$ at the edge (if not above) its range of applicability to lattice data where finite size effects start to play a role.

In fact, up to $m_\pi \lesssim (500 - 550)\text{MeV}$ the $\chi$PT to $\mathcal{O}(p^4)$ [7, 16, 17] and the $\chi$QSM describe $M_N(m_\pi)$ in good qualitative agreement, see Fig. 5a. Beyond this point, however, the nucleon mass as function of $m_\pi^2$ from $\chi$PT in Refs. [7, 16] changes the curvature, indicating that the range of reliability of $\chi$PT to $\mathcal{O}(p^4)$ could be $m_\pi \lesssim 500\text{MeV}$ (which, in fact, is not far from the generally assumed bound $m_\pi < 600\text{MeV}$). This contrasts the $\chi$QSM-result exhibiting in agreement with lattice data negative curvature up to the highest considered $m_\pi^2$.

With the above considerations in mind, we conclude that the $\chi$QSM describes lattice data [4–7] constrained
by \( a < 0.15 \text{fm} \), \( m_\pi L > 5 \) and \( L \geq 2 \text{fm} \) very well, see Fig. 5a and c.f. Fig. 4. In the range below \( m_\pi \lesssim 500 \text{MeV} \) we observe a good qualitative and quantitative agreement of the \( \chi \)QSM with \( \chi \)PT to \( \mathcal{O}(p^4) \) \cite{7, 16, 17}. In this range the curves for \( M_N(m_\pi) \) from the two approaches agree with each other to within an accuracy of 50 MeV and better, see Fig. 4 and 5a. This number may give us a flavour of the magnitude of the systematic error of the \( \chi \)QSM-extrapolation of lattice data, though \( \chi \)PT to \( \mathcal{O}(p^5) \) may also have such an intrinsic uncertainty, as indicated in Table III, due to unestimated contributions from \( \mathcal{O}(p^5) \).

Next we consider the effective approach based on the

finite range regulator (FRR) method of Ref. \cite{24}. There 5 free parameters appear, which can well be constrained by lattice data thanks to the larger range of applicability of the approach, namely up to \( m_\pi^2 = 1 \text{GeV}^2 \). In Ref. \cite{24} different shapes of regulators were exploited and shown to yield practically the same results. Fig. 5b shows the 5-parameter fit (using the dipole-type regulator) to the CP-PACS lattice data sets with \( \beta = 2.1 \) and 2.2 from Ref. \cite{4}. In the \( \chi \)QSM-approach with one free parameter only, we were able to fit both data sets separately, see Figs. 2c and d. For sake of comparison we include the combined \( \chi \)QSM-fit in Fig. 5b and summarize the results in Table IV.

As demonstrated in Fig. 5b and Table IV the results for \( M_N(m_\pi) \) from the FRR approach and the \( \chi \)QSM describe the CP-PACS data \cite{4} equally well. Having a closer look on the region \( m_\pi^2 < 0.3 \text{GeV}^2 \) we see that \( M_N(m_\pi) \) from the \( \chi \)QSM starts to deviate more and more strongly from the fit of the FRR-approach with decreasing \( m_\pi^2 \), though the curves remain very similar. The reason for the discrepancy (which apparently was of no relevance for \( m_\pi^2 > 0.3 \text{GeV}^2 \), see Fig. 5b) should be attributed to the fact, that \( M_\Delta - M_N \) is kept finite in the FRR approach, but neglected in the \( \chi \)QSM.

Thus one is lead to the conclusion, that the FRR method and the \( \chi \)QSM are completely consistent modulo \( 1/N_c \) corrections.

At the physical point the difference between the values of \( M_N \) extrapolated by means of the FRR method and the \( \chi \)QSM is about 40 MeV, which again may give us a rough idea on the theoretical uncertainty due to neglecting the \( \Delta \)-nucleon mass difference. Note that for this rough estimate we use the central value of the extrapolated \( M_N \) from the FRR approach, see Table IV, which has a substantially larger statistical uncertainty arising

![Fig. 5: (a) Selected lattice data on \( M_N(m_\pi) \) by QCDSF-UKQCD, CP-PACS, JLQCD \cite{4–7} and the corresponding fits in \( \chi \)PT to \( \mathcal{O}(p^4) \) from Refs. \cite{7, 16}, in comparison lattice data by MILC \cite{9} and the fit in \( \chi \)PT to \( \mathcal{O}(p^4) \) from Ref. \cite{17}. (b) Lattice data by CP-PACS \cite{4} and the fit in the finite range regulator approach (with a dipole regulator) \cite{24}. In both figures the respective \( \chi \)QSM-fits are shown.](image-url)
from fitting 5 parameters to the lattice data. In this respect the $\chi$QSM-fit is far more precise.

To summarize, the comparison of the $\chi$QSM-based fits and those obtained using the first-principle approach in $\chi$PT [7, 16] and the FRR approach [24] leads us to the following conclusion. The systematic uncertainty of $M_N$ due to neglecting the finite $\Delta$-nucleon mass-splitting in the $\chi$QSM is effectively of the order of magnitude of 50 MeV with the tendency to underestimate the nucleon mass. Noteworthy, a similar result -- concerning both sign and order of magnitude -- follows from a crude estimate within the model itself, see App. C.

Recall that we did not consider isospin breaking effects or electromagnetic corrections $\sim \mathcal{O}(10\text{MeV})$, see footnote 1. Including this we have to assign a systematic error to the $\chi$QSM-fits of about $(\delta M_N)_{\text{syst}} \approx 60$ MeV. Taking into account a systematic error of this magnitude we observe that the extrapolated values in Tables II, III and IV are all consistent with the physical mass of the nucleon.

We stress that we do not see any possibility to quantify the uncertainty of the $\chi$QSM-extrapolation of lattice data more quantitatively than that.

VI. CONCLUSIONS

The implicit dependence of the nucleon mass $M_N$ on the pion mass $m_\pi$ was studied in the large-$N_c$ limit in the framework of the chiral quark soliton model. The $M_N(m_\pi)$ in the model exhibits a chiral behaviour and includes leading non-analytic terms which are consistent with the large-$N_c$ formulation of QCD [35]. As was shown here, the model describes correctly also the heavy quark limit. The most remarkable observation we make here is that the model results for $M_N(m_\pi)$ well describe lattice data from full simulations [3–9] over the wide range of pion masses $0.1\text{GeV}^2 < m_\pi^2 < 2.5\text{GeV}^2$, provided one takes into account the generic overestimate of the nucleon mass in the soliton approach [38]. This is done by introducing an $m_\pi$-independent subtraction constant, i.e. one single parameter to be fitted to the respective lattice data set.

The good description of the lattice data on $M_N(m_\pi)$ can partly be understood as follows. In the $\chi$QSM, in the leading order of the large-$N_c$ limit, the $\Delta$-nucleon mass-splitting $\Delta = M_\Delta - M_N \sim \mathcal{O}(N_c^{-1})$ is neglected. That is a reasonable approximation when comparing to the present day lattice simulations where $\Delta^2 \ll m_\pi^2$ holds. However, the remarkable precision, to which the model describes the lattice data, remains a puzzle — to be clarified further by model studies.

We observe that the values for the nucleon mass "extrapolated" from the lattice data [3–9] on the basis of results from the $\chi$QSM are in good agreement with extrapolations based on the first principle approach in $\chi$PT [7, 16] or the effective FRR approach of Ref. [24], and agree with the physical nucleon mass to within 5%. (But one has to keep in mind that at the same time — for the adopted handling of model parameters in the chiral limit — the lattice values of the pion decay constant at large $m_\pi \sim 1\text{GeV}$ are underestimated by up to 40%.)

It is difficult to exactly quantify the theoretical uncertainty of this extrapolation due to the model-dependence. The main limitation for using the $\chi$QSM as a guideline for the chiral extrapolation of lattice data is due to the large-$N_c$ limit. There is no strict control within the model of the theoretical uncertainty introduced by neglecting the finite $\Delta$-nucleon mass-splitting at the physical point, and we cannot quantify this and other uncertainties due to model-dependence quantitatively. This limits the use of the model as a quantitative effective tool for the extrapolation of lattice data.

Still, the model may provide interesting qualitative insights — in particular in those cases when the matching between lattice results and $\chi$PT is difficult. A prominent example for that are (moments of) structure functions. In order to use the $\chi$QSM as a qualitative, but within its model accuracy reliable device, which is helpful for a comparison of lattice results to experimental data, further model studies are necessary — concerning the issue of handling model parameters in the chiral limit as well as addressing other observables [44].

From the model point of view the observations made in this work also contribute to a better understanding of the physics which underlies the chiral quark-soliton model, and may — that is our hope — stimulate further studies in this direction in this, and perhaps also other models.

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APPENDIX A: PROPER-TIME REGULARIZATION

The proper-time regularized versions of the integrals $I_1$ and $I_2$ appearing in Eqs. (2, 4) are given by

$$I_1(m) = \int_{\Lambda_{\text{cut}}^{-2}}^{\infty} \frac{du}{u^{3/2}} \frac{\exp(-uM')}{(4\pi)^2},$$

$$I_2(m) = \int_{\Lambda_{\text{cut}}^{-2}}^{\infty} \frac{du}{2u} \frac{\exp(-uM')}{(4\pi)^2} \int_0^1 \exp(u\beta(1-\beta)m^2) d\beta,$$

where the $m$-dependence is hidden in $M' \equiv M + m$. For a given $m_\pi$ and $f_\pi = 93$ MeV fixed and due to Eqs. (4, 5) both the current quark mass and the cutoff are (implicit) functions of $m_\pi$, i.e. $m = m(m_\pi)$ and $\Lambda_{\text{cut}} = \Lambda_{\text{cut}}(m_\pi)$; some selected values are shown in Table I.

The expression for the soliton energy (9) in the proper-time regularization is given by

$$E_{\text{sol}} = N_c \left[ E_{\text{lev}} + \sum_n (R(E_n) - R(E_{n_0})) \right];$$

$$R(\omega) = \frac{1}{4\sqrt{\pi}} \int_{\Lambda_{\text{cut}}^{-2}}^{\infty} \frac{du}{u^{3/2}} \exp(-u\omega^2).$$

APPENDIX B: THE PION-NUCLEON SIGMA-TERM $\sigma_{N\pi}$

The pion-nucleon sigma-term is an important quantity to learn about chiral symmetry breaking effects in the nucleon. The Feynman-Hellmann theorem [47] relates $\sigma_{N\pi}$ to the slope of $M_N(m)$ (with $m = m_q = m_u = m_d$ neglecting isospin breaking effects) as follows

$$\sigma_{N\pi} \equiv m \frac{\partial M_N(m)}{\partial m} = m_\pi^2 \frac{\partial M_N(m_\pi)}{\partial m_\pi^2}$$

where the second equality holds, strictly speaking, only for small $m_\pi$. Eq. (B1) offers a convenient way to learn about $\sigma_{N\pi}$ from lattice calculations of $M_N(m_\pi)$, see e.g. [22]. Direct lattice calculations are more difficult but possible, see e.g. [48].

Apart from (B1) one also can evaluate in the model directly the actual definition of $\sigma_{N\pi}$ as double commutator of the strong interaction Hamiltonian with two axial isovector charges [49, 50], or exploit a sum rule for the twist-3 distribution function $e(x)$ [51]. All three methods yield the same result in the $\chi$QSM [35].

By means of Eq. (B1) we obtain in the present study in the proper-time regularization: $\sigma_{N\pi} = 40$ MeV. This agrees well with other $\chi$QSM calculations [30, 49, 50] performed in this regularization scheme.

In the $\chi$QSM the pion-nucleon sigma-term is quadratically UV-divergent, and as such particularly sensitive to details of regularization. In Ref. [35] a way was found to compute $\sigma_{N\pi}$ in a regularization scheme independent way with the result $\sigma_{N\pi} \approx 68$ MeV. The prize to pay for the regularization scheme independence was the use of an approximation expected to work within an accuracy of $O(30\%)$ and well-justified by notions from the instanton vacuum model.

On the basis of our results for $M_N(m_\pi)$ we are now in a position to check the accuracy of this approximation in practice. For that we note that the Feynman-Hellmann relation (B1) allows to determine $M_N(m_\pi)$ from $\sigma_{N\pi}(m_\pi)$ up to an integration constant.

From the approximate (but regularization scheme independent) result for $\sigma_{N\pi}(m_\pi)$ from Ref. [35] one obtains for $M_N(m_\pi)$ the result shown as dashed line in Fig. 1, where for convenience the integration constant is chosen such that both curves coincide at a central value of $m_\pi = 1$ GeV in the plot. We observe a good agreement of our exact result and the approximation of Ref. [35], see Fig. 1.

Also our result for $\sigma_{N\pi}$ agrees with the regularization scheme-independent result for $\sigma_{N\pi}$ from Ref. [35] to within the expected accuracy of $O(30\%)$.

Thus, our results confirm that the instanton model motivated approximation advocated in [35] works well. Note, however, that here we went far beyond the study of Ref. [35]. In this work we practically demonstrated that stable soliton solutions do exist also for large values of the pion mass, while in Ref. [35] this was presumed.

The regularization scheme independent result of Ref. [35] is in good agreement with recent extractions indicating $\sigma_{N\pi} = (60 - 70)$ MeV [52], which is substantially more sizeable than the value obtained from earlier analyses [53].

In this context it is worthwhile mentioning that the spectrum of exotic (“pentaquark”) baryons which were predicted in the framework of the $\chi$QSM [54] and for which recently possible observations were reported – see [55] for recent overviews – could provide an independent mean to access information on the pion-nucleon sigma-term [56]. The presently available data on the exotic baryons favour a large value for $\sigma_{N\pi}$ as found in [52].

APPENDIX C: SYSTEMATIC UNCERTAINTY OF THE ROTATING SOLITON APPROACH

The neglect of $\Delta = M_\Delta - M_N$ in the leading order of the large-$N_c$ limit introduces a systematic uncertainty which is difficult to quantify. Here we roughly estimate this uncertainty on the basis of the soliton approach.

In this approach the nucleon and the $\Delta$-resonance are
just different rotational excitations of the same classic object, the soliton. A non-zero mass difference between the nucleon and the ∆-resonance arises due to considering a particular class of (“rotational”) \(1/N_c\)-corrections \([25, 26]\). The mass of an SU(2)-baryon \(M_B\) (\(B = N\) or ∆ in the real world with \(N_c = 3\) colours) with spin \(S_B\) is given by

\[
M_B = E_{\text{sol}} + M_2 + S_B(S_B + 1) \frac{\Delta}{3} + \ldots \tag{C1}
\]

with \(E_{\text{sol}}\) as defined in Eq. (9), \(M_2\) denoting as in Eq. (12) the \(O(N_c^0)\) correction to the baryon mass which is the same for the nucleon and the ∆-resonance, and the dots representing higher order \(1/N_c\)-corrections.

Then we obtain as a first correction to \(M_N\) in Eq. (22)

\[
M_N(m_\pi, \Delta) = M_N(m_\pi, 0) + \frac{\Delta(m_\pi)}{4} + \ldots \tag{C2}
\]

Focusing on the linear order correction in ∆ and neglecting higher orders, we see that by neglecting ∆ one underestimates the nucleon mass by about 70 MeV at the physical value of the pion mass. This is in good agreement with the systematic uncertainty roughly estimated in Sec. IV.

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$M_N(m_\pi)/\text{GeV}$ (a)

$\chi$QSM-extrapolated $M_N = 927$ MeV

$\chi$PT to $O(p^4)$ Ali Khan et al. Ref [7]

$\chi$PT to $O(p^4)$ Procura et al. Ref [14]

$\chi$QSM: $M_N(m_\pi) - 0.326$ GeV

$M_N = 1.8$ GeV

$\chi$QSM-extrapolated $M_N = 927$ MeV

$M_N = 1.6$ GeV

$M_N = 1.4$ GeV

$M_N = 1.2$ GeV

$M_N = 1.0$ GeV

$M_N = 0.8$ GeV

$M_N = 0.6$ GeV

$M_N = 0.4$ GeV

$M_N = 0.2$ GeV

$M_N = 0.0$ GeV