The phase transition in random Horn satisfiability and its algorithmic implications

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ABSTRACT
Let $c > 0$ be a constant, and $\Phi$ be a random Horn formula with $n$ variables and $m = c \cdot 2^n$ clauses, chosen uniformly at random (with repetition) from the set of all nonempty Horn clauses in the given variables. By analyzing PUR, a natural implementation of positive unit resolution, we show that $\lim_{n \to \infty} \Pr(\Phi \text{ is satisfiable}) = 1 - F(e^{-c})$, where $F(x) = (1 - x)(1 - x^2)(1 - x^4)(1 - x^8) \cdots$. Our method also yields as a byproduct an average-case analysis of this algorithm.

1. INTRODUCTION

Phase transitions in combinatorial problems were first displayed in the seminal work of Erdős and Rényi [ER60] on random graphs. Working with the constant probability model $G(n, p)$ they showed that the probability that the graph has a “large” connected component exhibits a sharp increase at some “threshold” value of $p$. The empirical observation from [CKT91], that for a number of NP–complete problems the “hardest on the average” instances are located near such threshold points has attracted considerable interest in such threshold phenomena from several communities, such as Theory of Computing, Artificial Intelligence and Statistical Mechanics. Recent studies [MZK+99a, MZK+99b] have provided further evidence that (at least some) phase transitions have indeed an impact on algorithmic complexity, and have offered additional insight on the cases when this happens.
It turns out that there are two different notions of phase transition in a combinatorial problem $P$. One definition applies to optimization problems and directly parallels the approach from Statistical Mechanics. Potential solutions for an instance of $P$ are viewed as “states” of a system. One defines an abstract Hamiltonian (energy) function, that measures the “quality” of a given solution, and apply methods from the theory of spin glasses [MPV87] to make predictions on the typical structure of optimal solutions. In this setting a phase transition is defined as non-analytical behavior of a certain “order parameter” called free energy, and a discontinuity in this parameter, manifest by the sudden emergence of a backbone of constrained “degrees of freedom” [MZK99a] is responsible for the exponential slow-down of many natural algorithms.

The second definition is combinatorial and pertains to decision problems. It is the concept of threshold property from random graph theory, more precisely a restricted version of this notion, called sharp satisfiability threshold. A satisfiability threshold always exists for monotone problems [BT86], but may or may not be sharp (we speak of a coarse threshold in the latter case). It is this notion of phase transition that we are concerned with in this paper.

From the practical perspective of [CKT91] phase transitions are most appealing in problems that are thought to be “hard”, in particular, in NP-complete problems. Therefore a lot of recent work has been directed towards locating phase transitions in such problems. In some cases, the most proeminent of which is Hamiltonian cycle [KS83]), a complete analysis has been obtained. In other (e.g., 3-SAT [FS96, KKK97, Ach, JSV] and graph-coloring [Chv91, AM97]), obtaining such an analysis is hard, and indeed not yet accomplished task: for these problems there exists a fairly large gap between the best rigorous lower and upper bounds, and the methods that were used to obtain these bounds do not seem to be capable to yield a tight analysis.

Understanding the reasons that make problems with similar computational complexity differ so much with respect to their “mathematical tractability” is clearly a topic worth investigating. A natural intuitive explanation of this discrepancy is that problems that are easy to analyze “coincide with high probability” with problems with a simple “local” structure, while problems that are “hard to analyze” lack such an approximation. Such is the case, for instance, of the above mentioned Hamiltonian cycle, that “coincides with high probability” with the graph property “having minimum degree two” [AKS85]. Support in favor of this intuition also comes from Friedgut’s result on the existence of a sharp threshold for 3-SAT [Fri99]: his proof relies on showing that problems with coarse thresholds can be well approximated by some simple “local” property, and then proving that 3-SAT lacks such an approximation. While his result sheds no light on the “mathematical tractability” of Hamiltonian cycle, it is tempting to speculate that there might be a suitable generalization of the concept of “coarse threshold”, that 3-SAT still lacks, and that encompasses all known “mathematically tractable cases”.

A natural testbed for the above intuition is the case of polynomial time solvable problems. In these cases the hypothesis predicts that one should be able to obtain a complete analysis: often tractability arises from the existence of a “local”
characterization, that circumvents the need for exhaustively searching the exponentially large space of potential solutions. Another reason is methodological: studying tractable problems usually amounts to probabilistic analyses of decision algorithms for these problems using a methodology based on Markov chains, a task that can often be accomplished.

Such an approach was successful for some tractable versions of propositional satisfiability: out of the six maximally tractable cases of SAT that Schaefer identified in his celebrated Dichotomy Theorem [Sch78], two are trivially satisfiable and two have completely analyzed phase transitions. The transition for 2-SAT, the satisfiability problem for CNF formulas with clauses of size two, has been studied in [CR92, Goe96] and that for XOR-SAT, the satisfiability problem for linear systems of equations with boolean variables, has been studied in [CD96]). The remaining two cases are the Horn formulas and the negative Horn formulas (which are, of course, dual).

In this paper we deal with these two cases. Unlike the other two nontrivial cases, we show that Horn satisfiability has a coarse threshold. In the “critical region” the number of clauses is exponential in the number of variables, hence from a practical perspective, our results show that if do not restrict clause length, random Horn formulas of practical interest are almost certainly satisfiable (we have subsequently analyzed the bounded clause length case in [Ist]). Also, we obtain our result by modeling PUR, a natural implementation of positive unit resolution, by a Markov chain, and our method yields as a byproduct an average-case analysis of this algorithm.

2. RESULTS

A Horn clause is a disjunction of literals containing at most one positive literal. It will be called positive if it contains a positive literal and negative otherwise. A Horn formula is a conjunction of Horn clauses. Horn satisfiability (denoted by HORN-SAT) is the problem of deciding whether a given Horn formula has a satisfying assignment.

Since our main interest is in phase transitions in decision problems in the class NP, we will discuss the notion of satisfiability threshold in the framework of NP–decision problems. Our definition is slightly different from the standard one (e.g. [Pap94]), and accommodates the fact that legal encodings of instances of a problem have in general lengths from a restricted set of values.

Definition. An NP–decision problem is a five-tuple \( P = (\Sigma, D, f, g) \) such that

1. \( \Sigma \) is a finite alphabet.
2. \( f, g : \mathbb{N} \to \mathbb{N} \) are polynomial time computable, polynomially bounded functions. In addition \( f \) has range \( \{0, 1\} \). A length \( n \) is called admissible if \( f(n) = 1 \).
3. \( D \subseteq \Sigma^* \times \Sigma^* \) is a polynomial time computable relation.
for every pair \((x, y) \in \Sigma^* \times \Sigma^*\), if \((x, y) \in D\) then the length of \(x\) is acceptable and \(|y| \leq g(|x|)\).

A string \(x\) having an admissible length will be called an instance of \(P\). A string \(y\) such that \((x, y) \in D\) is called a witness for \(x\), and we write \(x \in P\) to state the fact that there exists a witness for the instance \(x\). Finally problem \(P\) is monotonically decreasing if for every instance \(x\) of \(P\) and every witness \(y\) for \(x\), \(y\) is a witness for every instance \(z\) obtained by turning some bits of \(x\) from 1 to 0. Monotonically increasing problems can be similarly defined.

The three standard probabilistic models from random graph theory [Bol85], the constant probability model, the counting model, the multiset model extend directly to any NP–decision problem, and are equivalent under fairly liberal conditions. For the purposes of this paper we recall the definition of the multiset model:

**Definition.** Let \(P\) be an NP–decision problem. The random multiset model \(\Omega(n, m)\) has two parameters, an admissible length \(n\) and an instance density \(1 \leq m \leq n\). A random sample \(x\) from \(\Omega(n, m)\) is an instance of \(P\) obtained by first setting \(x = 0^n\), then choosing, uniformly at random and with repetition, \(m\) bits of \(x\) and switching them to 1.

Next we define out threshold properties for monotonically decreasing problems under the multiset model. Similar definitions can be given for monotonically increasing problems, or when using one of the two other random models.

**Definition.** Let \(P\) be any monotonically decreasing decision problem under the multiset random model \(\Omega(n, m)\). A function \(\theta\) is a threshold function for \(P\) if for every function \(m\), defined on the set of admissible instances and taking integer values, we have

1. if \(m(n) = o(\theta(n))\) then \(\lim_{n \to \infty} \Pr_{x \in \Omega(n, m)}[x \in P] = 1\), and
2. if \(m(n) = \omega(\theta(n))\) then \(\lim_{n \to \infty} \Pr_{x \in \Omega(n, m)}[x \in P] = 0\),

\(\theta\) is called a sharp threshold if in addition the following property holds:

3. For every \(\epsilon > 0\) define the two functions \(\mu_1(n), \mu_2(n)\) by

\[
\mu_1(n) = \min\{m \in \mathbb{N} : \Pr_{x \in \Omega(n, m)}[x \in P] \leq 1 - \epsilon\},
\]

\[
\mu_2(n) = \min\{m \in \mathbb{N} : \Pr_{x \in \Omega(n, m)}[x \in P] \leq \epsilon\}.
\]

Then we have

\[
\lim_{n \to \infty} \frac{\mu_2(n) - \mu_1(n)}{\theta(n)} = 0.
\]

If, on the other hand, for some \(\epsilon > 0\) the amount \(\frac{\mu_2(n) - \mu_1(n)}{\theta(n)}\) is bounded away from 0 as \(n \to \infty\), \(\theta\) is called a coarse threshold. These two cases are not exhaustive.
as the above quantity could in principle oscillate with \( n \). Nevertheless they are so for most “natural” problems.

A useful modification of the above framework has the set of admissible lengths specified by an increasing function \( N : \mathbb{N} \rightarrow \mathbb{N} \). We correspondingly redefine the random model as \( \Omega(n, m) = \overline{\Omega}(N(n), m) \) and the threshold function by \( \theta(n) = \overline{\theta}(N(n)) \). Such will be the case of random Horn satisfiability, for which a random formula from \( \Omega(n, m) \) is obtained by choosing \( m \) clauses independently, uniformly at random and with repetition from the set of all \( N(n) = (n + 2) \cdot 2^n - 1 \) Horn clauses over variables \( x_1, \ldots, x_n \).

The following is our main result:

**Theorem 2.1.** \( \theta(n) = 2^n \) is a threshold function for random Horn satisfiability. Moreover, for every constant \( c > 0 \)

\[
\lim_{n \to \infty} \Pr_{\Phi \in \Omega(n, c \cdot 2^n)}[\Phi \text{ is satisfiable}] = 1 - F(e^{-c}), \tag{2.1}
\]

where

\[
F(x) = (1 - x)(1 - x^2)(1 - x^4) \cdots (1 - x^{2^k}) \cdots .
\]

The result makes clear that random Horn satisfiability has a coarse threshold.

The algorithm PUR, employed in the proof Theorem 2.1 is displayed in Fig. 1. PUR is a natural implementation of positive unit resolution, which is complete for HORN-SA T [HW74].

As a byproduct, our analysis yields the following two results, which provide an average-case analysis of PUR:

**Fig. 1.** Algorithm PUR
Theorem 2.2. Let $X_n \in [0, n]$ be the r.v. denoting the number of iterations of PUR on a random satisfiable formula $\Phi \in \Omega(n, c \cdot 2^n)$. Then $X_n$ converges in distribution to a distribution $\rho$ on $[0, n]$ having support on the nonnegative integers, $\rho = (\rho_k)_{k \geq 0}$, $\rho_k = \text{Prob}[\rho = k]$, given by

$$\rho_k = \frac{e^{-2^k c}}{1 - F(e^{-c})} \cdot \prod_{i=1}^{k-1} (1 - e^{-2^i c}).$$

The case of unsatisfiable formulas displays one feature not present in the previous result: fluctuations due to the nature of the binary expansion of $n$, wobbles in the terminology of P. Flajolet [Fla].

Theorem 2.3. Let $Y_n$ be the r.v. denoting the number of iterations of PUR on a random formula $\Phi \in \Omega(n, c \cdot 2^n)$, and, for $k \in [0, n]$, possibly a function of $n$, let $\eta_{n,k}$ be the probability that $Y_n = \lfloor \log_2(n) \rfloor + k$, conditional on $\Phi$ being unsatisfiable. Then

- $\lim_{n \to \infty} |k - \log_2(n)| = \infty$ implies that $\lim_{n \to \infty} \eta_{n,k} = 0$
- for every $k \in \mathbb{Z}$

$$\eta_{n,k} = G(k - 1, c_n) - G(k, c_n) + o(1),$$

where

$$G(k, c) = e^{-c \sum_{j=-\infty}^{k} 2^j},$$

$$c_n = \frac{e}{2^{(\log_2(\sqrt{n}))}}.$$

3. NOTATION AND USEFUL RESULTS

For $n \in \mathbb{N}$ and $0 \leq p \leq 1$, we denote by $B(n, p)$ a random variable having a Bernoulli distribution with parameters $n, p$. For $\lambda \in \mathbb{R}$, $P\text{o}(\lambda)$ will denote a Poisson distribution with expected value $\lambda$.

We will use “with high probability” (w.h.p.) as a substitute for “with probability $1 - o(1)$.” We also say that a sequence $(p_n)_{n \in \mathbb{N}}$ of real numbers is exponentially small (written $o(1/poly)$) if for every polynomial $Q$, $p_n = o(1/Q(n))$. We will measure, as usual, the distance between two probability distributions with integer values $P = (p_i)$ and $Q = (q_i)$ by their total variation distance $d_{TV}(P, Q) = \frac{1}{2} \sum |p_i - q_i|$, and recall the following inequalities from [She84] and [BHJ92] (page 2 and Remark 1.4):
Lemma 3.1. If \( n, p, \lambda, \mu > 0 \) then
\[
\text{d}_{TV}(B(n, p), P_0(np)) \leq \min \left\{ np^2, \frac{3p^2}{2} \right\}
\]
\[
\text{d}_{TV}(P_0(\lambda), P_0(\mu)) \leq |\mu - \lambda|.
\]

Definition. Given two probability distributions \( D \) and \( D' \), we say that \( D' \) stochastically dominates \( D \) if for every \( x \), \( \Pr[D \geq x] \leq \Pr[D' \geq x] \), and write \( D \prec D' \) when this holds.

The following are two conditional probability tricks.

Fact 3.1. Let \( A_n, B_n \), and \( C_n \) be events such that \( \Pr[C_n | B_n] = 1 - o(1) \). Then
\[
|\Pr[A_n | B_n] - \Pr[A_n | B_n \land C_n]| = o(1).
\]

Proof.
Applying the chain rule for conditional probability we get
\[
|\Pr[A_n | B_n] - \Pr[A_n | B_n \land C_n]| = \Pr[A_n | B_n \land C_n] · \Pr[C_n | B_n] + \Pr[A_n | B_n \land \overline{C_n}] · \Pr[\overline{C_n} | B_n] - \Pr[A_n | B_n \land C_n] = |\Pr[A_n | B_n \land C_n] · (1 - o(1)) + \Pr[A_n | B_n \land \overline{C_n}] · o(1) - \Pr[A_n | B_n \land C_n]| = o(1).
\]

Fact 3.2. If \( B \) is a random variable taking integer values in the interval \( I \), then for every event \( A \),
\[
\min_{\lambda \in I} \{\Pr[A | (B = \lambda)]\} \leq \Pr[A] \leq \max_{\lambda \in I} \{\Pr[A | (B = \lambda)]\}.
\]

Several “concentration of measure” results will be used in the sequel. They include:

Proposition 3.2. (Chernoff bound) Let \( X_1, \ldots, X_n \) be independent 0/1 random variables with \( \Pr(X_i = 1) = p \). Let \( X = X_1 + \ldots + X_n \), \( \mu = E[X] \) and \( \delta > 0 \). Then
\[
\Pr[|X - \mu| \geq \delta \cdot \mu] \leq \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}}\right)^\mu.
\]

A related inequality from [AES92] is:
Proposition 3.3. Let $P$ have Poisson distribution with mean $\mu$. For $\epsilon > 0$, 
\[
\Pr[P \leq \mu \cdot (1 - \epsilon)] \leq e^{\epsilon^2 \mu/2},
\]
\[
\Pr[P \geq \mu \cdot (1 + \epsilon)] \leq [e^\epsilon (1 + \epsilon)^{-(1 + \epsilon)}]^{\mu}.
\]

We regard the algorithm PUR as working in stages, indexed by the number of variables still left unassigned; thus the stage number decreases as PUR moves on. Let $\Phi$ denote an input formula over $n$ variables. For $i, 1 \leq i \leq n$, $A_i$, $R_i$, and $S_i$ respectively denote the event that PUR accepts at stage $i$, the event that PUR rejects at stage $i$, and the event that PUR reaches stage $i - 1$ (“survives stage $i$”). Also, $\Phi_i$ denotes the $\Phi$ at the beginning of stage $i$, $N_i$ denotes the number of clauses of $\Phi_i$, $HP_{1,i}$ the number of positive unit clauses of $\Phi_i$, $HP_{2,i}$ the number of positive non-unit clauses, $HN_{1,i}$ the number of negative unit clauses and $HN_{2,i}$ the number of negative non-unit clauses. Finally, for simplicity define $\Pi = F(e^{-c})$ and $\Pi_i$ to be the product of the first $i$ terms from $\Pi$.

We will assert stochastic domination via couplings of Markov chains (for an extensive treatment see [Lin92]). The framework needed for our coupling result is made precise in the following definitions (especially tailored for the context of this paper, rather than being standard).

Definition. Let $(X_n)$ be a Markov chain having state space $S$ and transition matrix $X$. A stopping rule $H$ for $X_n$ is a set $H$ of transitions of $(X_n)$ (i.e. pairs of states $(i,j) \in S \times S$ such that $X_{i,j} > 0$).

Intuition: We will use stopping rules $H$ to talk about the probability (denoted $\Pr[A|H]$) of properties $A$ of the Markov chain that only hold conditional on $(X_n)$ making only transitions from $H$.

Definition. Let $X_t = (X_{0,t}, \bar{X}_t)$ and $Y_t = (Y_{0,t}, \bar{Y}_t)$ be two Markov chains on $Z \times Z^d$ having transition matrices $X$, $Y$, respectively. Let $H_1$, $H_2$ be two stopping rules for $(X_n)$, $(Y_n)$, respectively. Let $0 \in B \subset \{0, \ldots, d\}$. A $(B, H_1, H_2)$-majorizing (Markovian) coupling of $X$ and $Y$ is a Markov chain $Z = (Z_{t,1}, Z_{t,2})$ on $(Z \times Z^d)^2$, $Z_{t,1} = (Z_{t,01}, \ldots, Z_{t,d1})$, $Z_{t,2} = (Z_{t,02}, \ldots, Z_{t,d2})$, having transition matrix $(Z_{i,j},(k,l))_{i,j,k,l} \in Z^{d+1}$ such that:

- for every $i, j \in Z^{d+1}$, $\Pr[Z_{t+1,1} = j | Z_{t,1} = i] = X_{i,j}$,
- for every $i, j \in Z^{d+1}$, $\Pr[Z_{t+1,2} = j | Z_{t,2} = i] = Y_{i,j}$,
- for every $i, j, k, l \in Z^{d+1}$, if $Z_{(i,j),(k,l)} > 0$ and $(i, k) \in H_1$ then $(j, l) \in H_2$.
- for every $t \geq 0$ and every state $(Z_{t,1}, Z_{t,2})$ of $Z_t$ reachable through moves in $H_1 \times (Z^{d+1})^2$ only, we have $Z_{t,11} = Z_{t,22}$ for all $i \in B$, and
\[ Z_{t,01} \leq Z_{t,02}. \]

**Intuition:** The first two conditions express the fact that the coupling is *Markovian*. The third condition (denoted symbolically \( H_1 \leq H_2 \)) relate the two stopping rules. Finally, the last condition allows us to compare two quantities of interest for the Markov chains \((X_n)\) and \((Y_n)\), namely \( \sum_{i \in B} X_{i,t} \) and \( \sum_{i \in B} Y_{i,t} \).

Let us now formally state this comparison result.

**Lemma 3.4.** Let \((X_t), (Y_t), H_1, H_2, B\) be as in the previous definition, and suppose it is possible to construct a \((B, H_1, H_2)\)-majorizing coupling of \((X_t)\) and \((Y_t)\). Then, for every \( a \in \mathbb{Z}, \)

\[ \Pr[ \sum_{i \in B} X_{i,t} \geq a | H_1 ] \leq \Pr[ \sum_{i \in B} Y_{i,t} \geq a | H_2 ] \]

**Proof.**
Define

\[ H_{B,a} = \{ \lambda = (\lambda_0, \ldots, \lambda_d) : \sum_{i} \lambda_i \geq a \}. \]

Then

\[ \Pr[X_t \in H_{B,a} | H_1] = \sum_{x \in H_{B,a}} \Pr[X_t = x | H_1] \] (3.1)
\[ = \sum_{x \in H_{B,a}} \Pr[Z_{t,1} = x | H_1 \times S^2] \] (3.2)
\[ = \sum_{x \in H_{B,a}} \sum_{y \in S} \Pr[(Z_{t,1} = x) \land (Z_{t,2} = y)| H_1 \times S^2] \] (3.3)
\[ = \sum_{x \in H_{B,a}} \sum_{y \in S} \Pr[(Z_{t,1} = x) \land (Z_{t,2} = y)| H_1 \times H_2] \] (3.4)
\[ = \sum_{x \in H_{B,a}} \sum_{y \in H_{B,a}} \Pr[(Z_{t,1} = x) \land (Z_{t,2} = y)| H_1 \times H_2] \] (3.5)
\[ \leq \sum_{y \in H_{B,a}} \sum_{x \in H_{B,a}} \Pr[(Z_{t,1} = x) \land (Z_{t,2} = y)| H_1 \times H_2] \] (3.6)
\[ \leq \sum_{y \in H_{B,a}} \sum_{x \in S} \Pr[(Z_{t,1} = x) \land (Z_{t,2} = y)| S^2 \times H_2] \] (3.7)
\[ \leq \sum_{y \in H_{B,a}} \sum_{x \in S} \Pr[(Z_{t,1} = x) \land (Z_{t,2} = y)| S^2 \times H_2] \] (3.8)
Procedure \texttt{PUR}_2(\Phi)

if (\Phi contains no positive unit clauses)
  first eliminate a random clause
  then independently, with probability $1/t$
  eliminate every remaining clause.
  and continue recursively
else
  choose a random positive unit clause $x$
  set $x$ to 1 in $\Phi$
  and continue recursively

Fig. 2. Second version of PUR

$$
= \sum_{y \in H_{B,a}} \Pr[Z_{t,2} = y | S^2 \times H_2] \quad (3.9)
$$

$$
= \Pr[Y_t \in H_{B,a} | H_2]. \quad (3.10)
$$

Lines 3.2, 3.10 follow from the Markovian character of the coupling. Line 3.4 follows from $H_1 \leq H_2$. The rest are simple arithmetical calculations.

The couplings we need are very simple, and employ the following idea: suppose the recurrences describing $X_{t+1} - X_t$ and $Y_{t+1} - Y_t$ are identical, except for one term, which is $B(m_1, \tau)$ for $(X_t)$ and $B(m_2, \tau)$ for $(Y_t)$, where $m_1 \leq m_2$ are positive integers and $\tau \in (0, 1)$. Obtain a coupling by identifying $B(m_1, \tau)$ with the outcome of the first $m_1$ Bernoulli experiments in $B(m_2, \tau)$.

4. THE UNIFORMITY LEMMA

The crux of our analysis relies on the observation that the behavior of PUR on a random Horn instance can be described by a stochastic recurrence (Markov chain).

Lemma 4.1. (“The Uniformity Lemma” :)

1. Suppose PUR does not halt before stage $t$. Then, conditional on $N_t$, the clauses of $\Phi_t$ are random and independent.

2. Consider \texttt{PUR}_2, the modified version of the algorithm PUR from Figure 2 (that does not check for accepting/rejecting, but may produce empty clauses).
Let $E_i$ represent the number of empty clauses at stage $i$. Then for every stage $t$, conditional on $\Gamma_t = (HN_{1,t}, HN_{2,t}, HP_{1,t}, HP_{2,t}, E_t)$ the clauses of $\Phi_t$ are chosen uniformly at random and are independent.

3. Consider again the original version of PUR. Suppose now that we condition $\Gamma_t$ and on the fact that $\Phi$ survives Stage $t$ as well. Then we have

$$N_{t-1} = N_t - \Delta_1 P(t) - \Delta_2 P(t),$$

where

- $\Delta_1 P(t)$, the number of positive clauses that are satisfied at stage $t$, has the distribution $1 + B (HP_{1,t} - 1, \frac{1}{2})$.
- $\Delta_2 P(t)$, the number of positive non-unit clauses that are satisfied at stage $t$, has the binomial distribution $B (HP_{2,t}, \frac{1}{2})$.

Proof.

The proof is based on the method of deferred decisions [KMP90]. The crux of this method is to consider the random formula $\Phi$ as being disclosed gradually as the algorithm proceeds, rather than as being completely determined at the very beginning of the algorithm. Following a suggestion of Achlioptas [Ach] the process can be conveniently imagined as having the occurrences of each literal in the formula represented by a card that has the literal as its value. The cards corresponding to each clause are arranged in separate piles, and are all initially face down (to reflect the fact that initially we don’t know anything about the formula). Part of the unveiling process will consist of dealing (turning face up) the cards from each pile that contain a specific literal. We also assume that (unless otherwise specified by the unveiling process) the still undealt parts of each pile is “hidden”, so that we don’t know its height.

1. For the first part of the lemma (that conditions only on $N_t$) the disclosure process consists of first unveiling, at each stage greater than $t$, the location of a random positive unit clause of $\Phi_t$ (guaranteed to exist). We fill it with a random variable among those left. The process continues by providing

- all the occurrences of this variable.
- the locations and complete contents of clauses that contain this variable in positive form, and
- the locations of the clauses that have been completely filled.

We refer to the clauses in the latter two cases as blocked, since we have complete information about them, and they will no longer be involved in the unveiling process.

Suppose PUR arrives at stage $t$ on $\Phi$. Then in stages $i = n, n-1, \ldots, t+1$, $\Phi_i$ should have contained a unit clause consisting of a positive literal but not its
complement. This information does not condition in any way the structure of the clauses of $\Phi_t$, that correspond to the non-blocked piles, counted by $N_t$. In fact that the only information we have at Stage $t$ about these piles is their number $N_t$.

For each such pile all disclosed literals appear only in negative form, since otherwise the clause would have been satisfied and blocked. Hence the residual (hidden) part still obeys the Horn restriction. Given the uniformity in the choice of the initial clauses of $\Phi$, it follows that the clauses of $\Phi_t$ are chosen uniformly at random (and independently) among all nonempty Horn clauses in the remaining variables.

2. We will prove the result inductively, starting with Stage $n$ (where it certainly is true) and working downwards. At each stage, the disclosure process will offer some information on the type of the hidden portion of the clause, namely whether it is a positive unit, positive non-unit, negative or empty.

Definition. For notational convenience define $p_1(t) = \frac{1}{t}$, $p_2(t) = \frac{1}{t-1}$, $p_3(t) = \frac{1}{2}$, $p_4(t) = \frac{t-1}{(2^{t-1})}$.

If $HP_{1,t} > 0$, to carry on the disclosure process:

a. choose a random positive unit clause, fill it with a random variable $x$ among those left, and block.

b. independently with probability $1/t$ fill any of the remaining positive unit clauses with $x$ and block.

c. for any positive non-unit clause:
   (i) with probability $p_1(t)$ fill one entry of the clause with $x$, fill the rest of the clause with a random, non-empty, combination of negated remaining literals and block.
   (ii) if the first case did not happen then, with probability $p_2(t)$, fill one entry with $\overline{x}$ and set the type of the remaining clause to “positive unit”.
   (iii) if the first two cases did not happen then, with probability $p_3(t)$, fill one entry with $\overline{x}$ (but do nothing else).
   (iv) otherwise do nothing.

d. for any negative unit clause:
   (i) with probability $p_1(t)$ fill one entry of the clause with $\overline{x}$, set the type of the remaining clause to “empty”.
   (ii) otherwise do nothing.

e. for any negative non-unit clause:
   (i) with probability $p_4(t)$ fill one entry of the clause with $\overline{x}$ and set the type of the remaining clause to “negative unit”.
(ii) if the first case did not happen then, with probability $p_3(t)$, fill one entry of the clause with $\ pall$ (but do nothing else).

(iii) otherwise do nothing.

In the opposite case, $HP_{1,t} = 0$, the disclosure process consists of performing the procedure described in the algorithm, and additionally filling every eliminating clause with a random Horn clause in the remaining variables that is not a positive unit clause.

By a tedious but straightforward case analysis it is easy to see that in both cases the uniformity property carries through to the next stage. The reason is that in all cases the only information we disclose about each remaining clause is its type, but not its content. Moreover, we get the following recurrences for the case $HP_{1,t} > 0$:

\[
\begin{align*}
HP_{1,t-1} &= HP_{1,t} - 1 - \Delta_{1,P}(t) + \Delta_{12,P}(t), \\
HP_{2,t-1} &= HP_{2,t} - \Delta_{2,P}(t) - \Delta_{12,P}(t), \\
HN_{1,t-1} &= HN_{1,t} - \Delta_{E}(t) + \Delta_{12,N}(t), \\
HN_{2,t-1} &= HN_{2,t} - \Delta_{12,N}(t), \\
E_{t-1} &= E_t + \Delta_{E}(t),
\end{align*}
\]

where

\[
\begin{align*}
\Delta_{1,P}(t) &= B(HP_{1,t} - 1, p_1(t)), \\
\Delta_{2,P}(t) &= B(HN_{2,t}, p_1(t)), \\
\Delta_{12,P}(t) &= B(HP_{2,t} - \Delta_{2,P}(t), p_2(t)), \\
\Delta_{E}(t) &= B(HN_{1,t}, p_1(t)), \\
\Delta_{12,N}(t) &= B(HN_{2,t}, p_4(t)).
\end{align*}
\]

3. The conditioning on PUR surviving Stage $t$ implies that up to Stage $t - 1$ the algorithm PUR and its modified version $PUR_2$ work in the same way. With respect to $PUR_2$ it gives us one additional piece of information with respect to merely conditioning on $\Gamma_1$: that $\Delta_{E}(t) = 0$. The desired recurrence follows from the previous point.

A. Comments on the Uniformity Lemma

A few comments on the contents of the uniformity lemma are in order. Although (as shown by Lemma 4.1 (i)) it would seem that we can characterize the state of PUR at Stage $t$ by a single number, $N_t$, this is not so, for two reasons:

- first, the above uniformity result is conditional (on PUR surviving Stage $t + 1$) and does not hold throughout the whole evolution of the algorithm. For instance it is not true at stages before stage $t + 1$, since unit clauses that
are the negation of the variable being set cannot appear. An unconditional uniformity result is provided by Lemma 4.1 (ii). However, it applies to a modified algorithm, which is no longer complete for HORN-SAT, and cannot be used to obtain an exact result (rather than just a lower bound on the threshold, as is done e.g. in [FS96] for $k$-SAT).

- second, as shown by Lemma 4.1 (iii), a stochastic recurrence for $N_{t-1}$ cannot be determined by only using the value of $N_t$; instead we need additional information on the structure of $\Phi_t$ captured by the five-tuple $\Gamma_t$.

Fortunately it is possible to circumvent both these problems. On one hand it will turn out that all we need for the analysis is the conditional uniformity result (i), as long as we can “control” the value $N_t$. On the other hand, this value can be indirectly estimated throughout the “most interesting regime of PUR “.

B. A coupling result

The following result makes a first step towards estimating $N_t$, by showing that we can “approximate” this value by the value of a Markov chain with a simpler structure. The intuitive idea is simple: by Lemma 4.1 (iii) the “net decrease” $N_{t-1} - N_t$ is approximately $1 + B(HP_{1,t} + HP_{2,t} - 1, \frac{1}{t})$ which is intuitively less than $1 + B(N_t - 1, \frac{1}{t})$.

Lemma 4.2. Consider the modified version of PUR from Figure 3. Then

---

**Procedure $PUR_3(\Phi)$**

if ($\Phi$ contains no positive unit clauses)
  first eliminate a random clause
  then independently, with probability $1/t$
  eliminate every remaining clause
  and continue recursively
else
  first, independently with probability $1/t$
  eliminate every negative non-unit clause
  then
  choose a random positive unit clause $x$
  set $x$ to 1 in $\Phi$
  and continue recursively

Fig. 3. Third version of PUR
1. Conditional on \( \Gamma^{(2)}_t = (HN^{(2)}_{1,t}, HN^{(2)}_{2,t}, HP^{(2)}_{1,t}, HP^{(2)}_{2,t}, E_t) \) (the same quantities as in Lemma 4.1 (ii); we only use the superscript to indicate the fact that we are dealing with a different algorithm) the clauses of \( \Phi_t \) denote their number by \( N^{(2)}_t \) are uniform and independent.

2. Define \( S_0 = \{ (a, b, c, d, e) \rightarrow (a_1, b_1, c_1, d_1, e_1) : (c > 0) \& \& (e_1 = 0) \} \). Define the stopping rules \( H_2, H_3 \) for \( \Gamma_t, \Gamma^{(2)}_t \) to be respectively the set of legal transitions of \( \Gamma_t, \Gamma^{(2)}_t \) that are in \( S_0 \). Finally, define \( B = \{ 0, 1, 2, 3 \} \). Then it is possible to construct a \( (B, H_2, H_3) \)-majorizing coupling of the Markov chains \( \Gamma_t \) and \( \Gamma^{(2)}_t \).

3. If \( HP^{(2)}_{1,t} > 0 \) then \( N^{(2)}_{t-1} = N^{(2)}_t - 1 - \Delta_{1,P}(t) - \Delta_{2,P}(t) - \Delta_{1,N}(t) - \Delta_{2,N}(t) \), where

\[
\begin{align*}
\Delta_{1,P}(t) &= B \left( HP_{1,t} - 1, \frac{1}{t} \right), \\
\Delta_{2,P}(t) &= B \left( HN_{2,t}, \frac{1}{t} \right), \\
\Delta_{1,N}(t) &= B \left( HN_{1,t}, \frac{1}{t} \right), \\
\Delta_{2,N}(t) &= B \left( HN_{2,t}, \frac{1}{t} \right).
\end{align*}
\]

Consequently, irrespective of the value of \( HP^{(2)}_{1,t} \),

\[
N^{(2)}_t - N^{(2)}_{t-1} \overset{D}{=} 1 + B(N^{(2)}_t - 1, \frac{1}{t}).
\]

Proof.

1. The proof is identical to the one of Lemma 4.1 (ii), and thus omitted.

2. The intuition behind the definition of the set \( S_0 \) is simple, and displays the connection with the desired analysis of the algorithm \( PUR \): we restrict the set of legal transitions of \( \Gamma_t, \Gamma^{(2)}_t \) to those for which \( HP_{1,t} > 0 \) and \( E_{t-1} = 0 \) (in other words those for which \( PUR \) survives stage \( t \), and thus works like \( PUR_2 \)).

The coupling can be described in a very intuitive way. Suppose that we carry on the disclosure process corresponding to the algorithm \( PUR_2 \), but the blocking of a clause is accomplished by placing a red pebble on the corresponding pile, rather than physically eliminating it. We modify this process to also place, at each stage \( j \) such that \( HP_{1,j} > 0 \), some blue pebbles on the piles corresponding to negative non-unit clauses, at follows: each such clause that has no pebble on it independently receives a blue pebble with probability \( 1/j \). It is easy to see that the new pebbling process (red and blue) simulates the algorithm \( PUR_3 \). The coupling easily follows.

3. The result follows from point 1, by separately considering the behavior of \( PUR_3 \) in the two cases, \( HP^{(2)}_{1,t} > 0, HP^{(2)}_{2,t} > 0 \).
5. THE PROOF OUTLINE

We will prove only the second part of the theorem, since the first part directly follows from it. By the proof of Lemma 4.1 the behavior of the algorithm can be described (with the above mentioned caveats) by a stochastic recurrence involving $N_t$. Proposition 6.1 below proves the important fact that with high probability $N_t$ stays close to its expected value, which is $N_n(1 - o(1))$ for $t = n - O(n^{1/2})$.

So, intuitively, the number of clauses of $\Phi_t$ stays (almost) the same, while the number of variables decreases by one. The net effect of one iteration is thus to “double the constant $c$”. We build the proof on three technical lemmas, Lemmas 6.2, 6.5, and 6.6. Intuitively, these lemmas show the following:

- Lemma 6.2 states that with probability $1 - o(1)$ PUR rejects “in the first $\log n + \Theta(1)$ stages” (if at all; we will make this more precise in Theorem 2.3).
- Lemma 6.5 states that with probability $1 - o(1)$ PUR does not reject in any fixed number of steps.
- Lemma 6.6 obtains a coarse inequality for the satisfaction probability

$$e^{-c} - o(1) \leq \Pr[\Phi \in \text{HORN-SAT}] \leq \frac{e^{-c/4}}{1 - e^{-c/4}} + o(1).$$

A consequence of this result is that a constant number, say $k$, of iterations “blows up” $c$ so that the resulting constant $2^k c$ is so large that $\Phi_{n-k}$ is unsatisfiable with probability arbitrarily close to 1.

Next we obtain a relation between the probability that PUR rejects $\Phi_n$ and the probability that PUR rejects $\Phi_{n-1}$ ($\Phi_{n-1}$ is defined with probability $1 - o(1)$ in the case when $c = \Theta(1)$ due to Lemma 6.5): the former is equal to the latter multiplied by the probability that PUR survives stage $n$. This latter term is one minus the probability that PUR accepts at stage $n$, which is asymptotically equal $e^{-c}$, and minus the probability that PUR rejects at step $n$, which is $o(1)$ and can be asymptotically neglected. Iterating this relation for a large enough (but constant) number of steps $k$ that make $\Pr[\Phi_{n-k} \text{ is unsatisfiable}]$ “close enough to 1” and the partial product $\Pi_k$ “close enough to $\Pi$” allows us to argue that, for every $\epsilon > 0$, the probability that PUR rejects is, for sufficiently large $n$, within $\epsilon$ of the value $\Pi$ prescribed by the theorem.

6. THE KEY LEMMAS

Proposition 6.1. For every $c > 0$ and every $t, n - c\sqrt{n} \leq t \leq n$, the conditional probability that the inequality

$$N_n - (n-t) \left[1 + \frac{2(N_n - 1)}{t}\right] \leq N_j \leq N_n$$

(6.1)
holds for all \( t \leq j \leq n \), in the event that \( \text{PUR} \) reaches stage \( t \), is \( 1 - o(1) \).

**Proof.**

For ease of notation, define \( E_t \) to be the event that Relation 6.1 holds, and the sequences \( y_t = N_n - (n - t) \left[ 1 + \frac{2(N_n - 1)}{t+1} \right] \) and \( z_t = N_n \). By the Lemma 4.2 (ii) and Lemma 3.4 we have:

\[
\Pr[N_t^{(2)} \geq y_t | H_3] \leq \Pr[N_t \geq y_t | H_2].
\]

But conditioning on \( H_3 \), \( H_2 \) is the same thing as conditioning on the algorithms not remaining without unit clauses, and not producing empty clauses, in other words working like \( \text{PUR} \). So \( \Pr[E_t | S_{t+1}] \geq \Pr[N_t^{(2)} \geq y_t | H_3] \).

\( H_3 \) implies that \( N_{j+1}^{(2)} - N_j^{(2)} \overset{D}{=} B(N_{j+1}^{(2)} - 1, \frac{1}{j+1}) \) for every \( j \geq t \).

So, defining the Markov chain \( U_t \) by \( U_n = N_n \) and \( U_t - U_{t-1} \overset{D}{=} 1 + \eta_t \), where the \( \eta_t \) are independent variables having the Bernoulli distribution \( B(N_j - 1, \frac{1}{j}) \), it follows that

\[
\Pr[U_t \geq y_t] = \Pr[N_t^{(2)} \geq y_t | H_3] \leq \Pr[E_t | S_{t+1}]
\]

By the Chernoff bound, and reasoning inductively, we infer that with probability \( 1 - o(1) \) we have \( \eta_t \leq \frac{2(U_t - 1)}{t+1} \leq \frac{2(N_n - 1)}{t+1} \) for every \( t \leq j \leq n \). Plugging this inequality in the definition of \( U_t \) and using equation 6.2 proves the lemma.

**Lemma 6.2.** Let \( p = p(n) \) such that \( \lim_{n \to \infty} [n - \log_2 n - p(n)] = \infty \). Then \( \Pr[R_p | S_{p+1}] \), i.e., the conditional probability that \( \text{PUR} \) rejects at stage \( p(n) \) in the event that \( \text{PUR} \) reaches stage \( p(n) \), is \( 1 - o(1) \).

To prove this lemma we need the following trivial combinatorial result:

**Lemma 6.3.**

Let \( a(n) \) white balls and \( b(n) \) black balls be thrown independently into \( n \) bins. Pick a random bin among those containing a white ball, and let \( X_n \) be the event that the chosen bin contains a black ball as well. Then \( \Pr[X_n] = 1 - \left( 1 - \frac{1}{n} \right)^{b(n)} \).

**Proof.**

It is easy to see that the bin we choose can be seen as the result of choosing a random bin among all \( n \) bins. So \( \Pr[X] \) is simply the probability that a randomly chosen bin gets a black ball. But this is \( 1 - \left( 1 - \frac{1}{n} \right)^{b(n)} \).

**Proof of Lemma 6.2:**
Let $T$ denote the event $E_n \land E_{n-1} \land \cdots \land E_p$. It follows from Proposition 6.1 that $\Pr[T|S_j] = 1 - o(1)$. Then, by Fact 3.1, $\Pr[R_p|S_{p+1}] = \Pr[R_p|S_{p+1} \land T] + o(1)$. Since $T$ implies $N_p \in I = [y_p, z_p]$,

$$\Pr[R_p|S_{p+1} \land T] \geq \min_{\lambda \in I} \{\Pr[R_p|S_{p+1} \land T \land (N_p = \lambda)]\}. $$

Thus, the claim holds if we show that $\max_{\lambda \in I} \Pr[R_p|S_{p+1} \land T \land (N_p = \lambda)] = o(1)$.

Suppose that $N_p = \lambda$, the events $T$, $S_{p+1}$ hold, and we further condition on the number of negative unit clauses. The event $R_p$ can be mapped into $X_p$ of the previous “balls into bins” experiment, with the positive unit clauses representing the white balls, the negative unit clauses being the black balls, and the remaining $p$ variables being the bins.

From Lemma 4.1 it follows that the number of negative unit clauses of $\Phi_p$ has a binomial distribution $B(\lambda, \frac{p}{N_p})$. Since $\lambda \frac{p}{N_p} \geq y_p \frac{p}{N_p} = (1 + o(1))c \cdot 2^{\log_2(n+1)} = \omega(n)$, it follows easily by the Chernoff bound that with probability $1 - o(1)$ the number of both positive and negative unit clauses of $\Phi_p$ is larger than $\frac{p_y}{2N_p}$. Since this amount is $\omega(n)$ the claim is a consequence of Lemma 6.3.

**Proposition 6.4.** With probability $1 - o(1)$ PUR does not reject $\Phi$ at stage $n$.

**Proof.**

Let $U$ be the number of unit clauses in $\Phi$. The variable $U$ has a binomial distribution with parameters $2^n c$ and $\frac{2^n}{(n+2)^{2^n-1}}$, so it is asymptotically a Poisson distribution with parameter $2c$. In fact Proposition 3.1 and Proposition 3.3 together imply that with probability $1 - o(1)$, $U \leq 2c(1 + n^{1/3}) \leq 4cn^{1/3}$.

Consider the $U$ unit clauses of $\Phi$ as being balls to be tossed into $n$ bins. The probability that two of them end up in the same bin is at most $(\frac{U}{2n})^2 \cdot \frac{1}{n}$, which, in view of the above upper bound on $U$, is $o(1)$. So with probability $1 - o(1)$ no variable appears more than once in a unit clause of $\Phi$, and thus, PUR does not reject.

**Lemma 6.5.** For every $k > 0$, with probability $1 - o(1)$, PUR does not reject in any of the stages $n, n - 1, \ldots, n - k + 1$.

**Proof.**

A simple induction on $k$, coupled with the fact that, conditioned on $N_t$, $\Phi_t$ is a random formula, and Proposition 6.1.

**Lemma 6.6.** For every positive constant $c$, $e^{-c} - o(1) \leq \Pr[\Phi \in \text{HORN-SAT}] \leq e^{-c/4} + o(1)$.

**Proof.**

Let $c > 0$ be a constant.

$$\Pr[\Phi \in \text{HORN-SAT}] \geq \Pr[\text{PUR accepts at the first step}]$$
\[
= \Pr[\Phi \text{ contains no positive unit clauses}]
= \left(1 - \frac{n}{(n + 2) \cdot 2^n - 1}\right)^{2^n c}
= e^{-\frac{n^2 n}{(n+2)2^n-1} - o(1)}
\geq e^{-c - o(1)},
\]
since \[\frac{n^2 n}{(n+2)2^n-1} \leq 1\]. This proves the lower bound.

In order to prove the upper bound, define \( p = \log_2 n + \log \log n \), let \( Y \) be the event “PUR accepts,” and let \( Z \) the event “PUR stops in at most \( p \) iterations.” By Lemma 6.2, \( \Pr[Z] = 1 - o(1) \), so \( \Pr[Y] \leq \Pr[Y|Z] = o(1) \). However, given \( Z \), \( Y \) is equivalent to \( A_n \lor (A_{n-1} \land S_n) \lor (A_{n-p+1} \land S_n \land \cdots \land S_{n-p+2}) \). So, by the Bayes rule, \( \Pr[Y|Z] \) is at most

\[
\Pr[A_n] + \Pr[A_{n-1}|S_n] + \cdots + \Pr[A_{n-p+1}|S_n \land S_{n-1} \land \cdots \land S_{n-p+2}].
\]

We cannot apply directly Fact 3.1, because this sum has an unbounded number of terms. Instead, we will use the following simple consequence of Bayes conditioning:

\[
\Pr[A_i|S_n \land \cdots \land S_{i+1}] \leq \Pr[A_i|S_n \land \cdots \land S_{i+1} \land E_i] + \Pr[E_i|S_n \land \cdots \land S_{i+1}].
\]

From Proposition 6.1 the sum of all “second terms” is \( o(1) \). As to the first term, the conditioning implies that the clauses of \( \Phi_i \) are chosen uniformly at random and their number is between \( y_i \) and \( z_i \). Since PUR accepts \( \Phi_i \) if and only if \( \Phi_i \) contains no positive literals, we have

\[
1 - \left(1 - \frac{i}{(i + 2)2^i - 1}\right)^{y_i} - o(1) \leq \Pr[A_i|S_n \land \cdots \land S_{i+1} \land E_i] \leq 1 - \left(1 - \frac{i}{(i + 2)2^i - 1}\right)^{z_i} + o(1).
\]

in particular

\[
\Pr[A_i|S_n \land \cdots \land S_{i+1} \land E_i] \leq \left(1 - \frac{i}{(i + 2)2^i - 1}\right)^{y_i}.
\]

The right hand side is less or equal than \( e^{-\frac{i^{3/4}}{(i+2)^{2i}} - 1} \). Since \( \frac{i}{i+2} \geq \frac{1}{3} \) and \( y_i \geq N_n \cdot \left(1 - \frac{\log n + \log \log n}{n - \log n + \log \log n}\right) \geq \frac{3N(n)}{4} \) for a sufficiently large \( n \) we have, (assuming such an \( n \)) \( e^{-\frac{i^{3/4}}{(i+2)^{2i}} - 1} \leq e^{-\frac{2^{n-1}}{4}} \).

Summing up all these upper bounds for \( \Pr[A_i|S_n \land \cdots \land S_{i+1} \land E_i] \) and observing the exponents as part of the progression \( \{\frac{3}{4} \cdot j\} \), we obtain the desired upper bound

\[
\frac{e^{-c/4}}{1-e^{-c/4}} + o(1).
\]
7. PUTTING IT ALL TOGETHER

Now we complete the proof of Theorem 2.1 by proving equation (2.1).

In order to prove this result it suffices to show that

\[
\lim_{n \to \infty} \Pr_{\Phi \in \Omega(m,n)}[\text{PUR rejects } \Phi] = F(e^{-c}).
\] (7.1)

It is easy to see that \(F\) is well-defined on \((0,1)\) and has the following Taylor series expansion

\[
\tilde{F}(x) = (-1)^{b_0} + (-1)^{b_1}x + (-1)^{b_2}x^2 + \cdots (-1)^{b_i}x^i + \cdots
\]

with \(b_i\) being the number of ones in the binary representation of \(i\). Also \(F\) is monotonically decreasing, positive on \((0,1)\), and has limit 1 at 0.

Fix \(\epsilon > 0\). Let \(R\) be the event “PUR rejects \(\Phi\)”. What we need to show is that for a sufficiently large \(n\),

\[
(1 - \epsilon)\Pi \leq \Pr[R] \leq (1 + \epsilon)\Pi.
\] (7.2)

Since \(\Pi\) converges and \(\Pi > 0\), there exists some \(k_0\) such that for all \(k \geq k_0\),

\[
\sqrt{1 - \epsilon} < \frac{\Pi_k}{\Pi} < (1 + \epsilon).
\] (7.3)

By Lemma 6.6, there exist some \(n_0 > 0\) and \(c_0 > 0\) such that for every \(n > n_0\) and every \(c > c_0\), \(\Pr_{\Phi \in \Omega(n,2^n,c)}[\text{PUR rejects } \Phi] > \sqrt{1 - \epsilon}\). Keeping in mind the fact that events \(A_n, A_{n-1}, \ldots, A_{n-k+1}\) are incompatible with \(R\) we obtain the equality

\[
\Pr[R] = \Pr[R|\overline{A_n} \land \cdots \land \overline{A_k}] \cdot \Pr[\overline{A_n}] \cdot \prod_{1 \leq i \leq k} \Pr[A_{n-i} \land \overline{A_n} \land \cdots \land \overline{A_{n-i+1}}].
\]

for every fixed \(k\).

Although conceptually simple, the rest of the proof is a little bit cumbersome.

We first consider the case \(c > 4\ln 2\) (so that the upper bound in Lemma 6.6 is strictly less than one).

Choose \(k\) so that, for large enough \(n\), \(y_{n-k} > c_0 \cdot 2^n - k\). This is possible since \(y_{n-k} \geq c \cdot 2^n[1 - \frac{k}{2^n}]\).

We claim (and it is in the proof of these two relations where the assumption \(c > 4\ln 2\) will be used) that for every \(j, n - k \leq j \leq n\), that

\[
\Pr[\overline{A_j}|\overline{A_n} \land \cdots \land \overline{A_{j+1}}] = \Pr[\overline{A_j}|S_n \land \cdots \land S_{j+1}] + o(1),
\] (7.4)

and

\[
\Pr[R|\overline{A_n} \land \cdots \land \overline{A_{j+1}}] = \Pr[R|S_n \land \cdots \land S_{j+1}] + o(1).
\] (7.5)
We will postpone proving these equations and will see how the theorem can be proven from these equations.

From equations 6.3 and 7.4 it follows that

\[ 1 - \left(1 - \frac{n-i}{(n-i+2)2^{n-i} - 1}\right)^{y_{n-i}} - o(1) \]

\[ \leq \Pr[\overline{A_n} - \overline{A_n}] \leq 1 - \left(1 - \frac{n-i}{(n-i+2)2^{n-i} - 1}\right)^{z_{n-i}} + o(1). \]

This proves that, for every \( i = 1, \ldots, k, \)

\[ \lim_{n \to \infty} \Pr[\overline{A_n} - \overline{A_n}] = (1 - e^{-c^2}). \]

In a similar vein, we have, for large enough \( n, \)

\[ \sqrt{1 - \epsilon} \leq \Pr[\bar{R} - \bar{R}] \leq 1. \]

If we take a large enough \( n, \) since the second part is asymptotically equal to \( \Pi_k, \) by (7.3) we have (7.2).

For a general \( c > 0, \) define \( c^* \) to be the infimum of all \( c's \) for which the relation 7.1 holds for every \( c' > c. \) Suppose \( c^* > 0. \) The single-step version of (7.5) provides \( \Pr[R\bar{A}_n] = \Pr[R|S_n] + o(1), \) so \( \Pr[R] = \Pr[\bar{A}_n] \Pr[R|\bar{A}_n] + o(1). \) Let \( c < c^* \) and let \( n_1 \) be such that for all \( n \geq n_1, 2c(1 - \frac{1}{n})^2 > c^*. \) By Fact 3.1 and Proposition 6.1 we have \( \Pr[R|S_n] = \Pr[R|S_n \cap E_{n-1}] + o(1). \) Then by Fact 3.1 we have \( \min_{\lambda \in f} \{\Pr[R|S_n \cap E_{n-1} \cap (N_{n-1} = \lambda)]\} \leq \Pr[R|S_n \cap E_{n-1} \cap (N_{n-1} = \lambda)] \leq \max_{\lambda \in f} \{\Pr[R|S_n \cap E_{n-1} \cap (N_{n-1} = \lambda)]\}. \) Conditioned on surviving stage \( n \) and on the value of \( N_i, \) \( \Phi_{n-1} \) is a random formula. Since both \( y_{n-1} \) and \( z_{n-1} \) are asymptotically equal to \( 2^n c, \) for large \( n, \) \( \Phi_{n-1} \) is a random formula with \( n-1 \) variables and \( 2^{n-1} \cdot (2c + o(1)) \) clauses. Thus, \( \lim_{n \to \infty} \Pr[R|S_n] = \lim_{n \to \infty} \Pr[R|S_n \cap E_{n-1}] = F(e^{-2\epsilon}). \) Since \( \Pr[\bar{A}_n] \) is asymptotically equal to \( 1 - e^{-c}, \) and \( F(\epsilon) = (1 - e^{-c})F(2c), \) (7.2) holds for \( c. \) This shows that \( c^* = 0, \) hence 7.1 is true for every \( c > 0. \)

Now what remains is to prove (7.4) and (7.5). We will prove only (7.4); proving the other is quite similar. Let \( T \) be the event that \( \bar{R} \) rejects in one of the first \( k \) stages. Note that \( \Pr[T] = o(1), \) as seen in Lemma 6.5. Note that \( T = R_n \lor (S_n \land R_{n-1}) \lor \cdots \lor (S_n \land \cdots \land S_{n-k+2} \land R_{n-k+1}), \) so the probability of each of the \( k \) terms in the disjunction is \( o(1). \)

Note that

\[ \Pr[\overline{A_n} - \overline{A_j}] = \sum_{\epsilon_{r+1} \in \{-1, 1\}} \Pr[\overline{A_n} \land \overline{A_j} \land R_{n+1}^\epsilon \land \cdots \land R_{j+1}^\epsilon]. \]
where $X^{-1}$ denotes the opposite of the event $X$. All terms in the sum, other than
$Pr[\mathcal{A}_n \land \cdots \land \mathcal{A}_j \land \mathcal{R}_n^{-1} \land \cdots \land \mathcal{R}_{j+1}^{-1}]$ are either inconsistent (the algorithm rejects twice) or imply one of the terms appearing in the disjunction of the decomposition of $T$. Thus,

$$Pr[A_n \land \cdots \land A_j] = Pr[A_n \land R_n \land \cdots \land A_{j+1} \land A_j] + o(1),$$

that is,

$$Pr[A_n \land \cdots \land A_j] = Pr[S_n \land \cdots \land S_{j+1} \land A_j] + o(1).$$

Similarly, $Pr[A_n \land \cdots \land A_{j+1}] = Pr[S_n \land \cdots \land S_{j+1}] + o(1)$.

Note that for every sequence of events $A_n$ and $B_n$ with $\liminf_{n \to \infty} Pr[B_n] > 0$,$$Pr[A_n \land o(1) Pr[B_n] + o(1) - Pr[T].$$Since $e^{-c/4} < 1$, the required condition is guaranteed.

8. PROOF OF THEOREM 2.2

From equations (7.4) and (6.3) and Proposition 6.1 it follows that the probability that the algorithm accepts exactly at Stage $k$, given that it has not stopped before, tends (as $n \to \infty$) to $e^{-2^k c}$. We have

$$Pr[A_{n-k} \land \Phi \in SAT] = Pr[A_{n-k} \land \Phi \in SAT \land S_{n-k+1}] = Pr[A_{n-k} \land \Phi \in SAT \mid S_{n-k+1}] \cdot Pr[S_{n-k+1}] \cdot Pr[S_{n-k+1}] = Pr[A_{n-k}] Pr[S_{n-k+1}].$$

Therefore

$$\rho_k = \lim_{n \to \infty} Pr[A_{n-k} \Phi \in SAT] = \lim_{n \to \infty} \frac{Pr[A_{n-k} S_{n-k+1}]}{Pr[\Phi \in SAT]} \cdot Pr[S_{n-k+1}]$$

$$= \frac{e^{-2^k c}}{1 - F(e^{-c})} \prod_{i=1}^{k-1} (1 - e^{-2^i c}).$$
9. PROOF OF THEOREM 2.3

We will only provide an outline of the proof of Theorem 2.3, since its overall philosophy is quite similar to the one used to prove Theorem 2.1.

Redefine, for the purpose of this section, the index $k$ to refer to events taking place at stage $n - \lfloor \log_2(n) \rfloor - k$. For instance $S_k$ is the same as the event $Y_n > n - \lfloor \log_2(n) \rfloor - k$.

Theorem 2.3 follows, of course, from the following claim

**Lemma 9.1.**

$$\lim_{n \to \infty} \Pr[Y_n > \lfloor \log_2(n) \rfloor + k | R] - G(k, c_n) = 0.$$  \hspace{1cm} (9.1)

To prove Lemma 9.1 we first show, using methods similar to the ones used to prove Lemma 6.6, the following result

**Lemma 9.2.**

$$\lim_{k \to -\infty} \lim_{n \to \infty} \Pr[Y_n > \lfloor \log_2(n) \rfloor + k | R] = 1.$$  

The proof of Lemma 9.1 proceeds now by observing that

$$
\Pr[(Y_n > \lfloor \log_2(n) \rfloor + k) \land R] \\
= \Pr[S_k \land R] = \Pr[S_{k-1} \land \overline{R}_k \land R] \\
= \Pr[\overline{R}_k \land R | S_{k-1}] \Pr[S_{k-1}] \\
= (\Pr[\overline{R}_k | S_{k-1}] - o(1)) \cdot (\Pr[S_{k-1} \land R] + o(1)) \\
= (\Pr[\overline{R}_k | S_{k-1} \land E_k] - o(1)) \cdot (\Pr[S_{k-1} \land R] + o(1)) \\
= \Pr[\overline{R}_k | S_{k-1} \land E_k] \cdot \Pr[(Y_n > \lfloor \log_2(n) \rfloor + k - 1) \land R] + o(1)
$$

By Lemma 6.3 the first term is approximately $e^{-c_n 2^k}$.

Iterating downwards for a constant number of steps, up to $k_0 \in \mathbb{Z}$, we infer

$$\Pr[Y_n > \lfloor \log_2(n) \rfloor + k | R] = \Pr[Y_n > \lfloor \log_2(n) \rfloor + k_0 | R] \cdot \prod_{j=k_0+1}^{k} \Pr[\overline{R}_k | S_{k-1} \land E_k] + o(1).$$

Choosing $k_0$ small enough so that, by Lemma 9.2, the first term is “close enough to 1” and the product is “close enough to $G(c_n, k)$” proves relation 9.1.
10. FURTHER DISCUSSIONS AND OPEN PROBLEMS

There are several versions of Horn satisfiability whose phase transition is worth studying. One of them is the class of extended Horn formulas [CH91, SAFS95], for which PUR is still a valid algorithm [CH91]. On the other hand, Horn-like restrictions have been employed to design tractable restrictions of various formalisms of interest in Artificial Intelligence, for example in constraint programming, temporal reasoning, spatial reasoning, etc. In many such cases positive unit resolution has natural analogs, (for instance arc-consistency in the case of ORD-HORN formulas in temporal reasoning [NB95]), and it would be interesting to see whether the ideas in this paper can inspire similar results.

Let us also remark that, as shown in [Ist], the average-case behavior of PUR as displayed in Theorem 2.2, is responsible for a physical property called critical behavior, widely studied in Statistical Mechanics and related areas (see, for instance, [Sla94], for the case of percolation), and similar to the one observed experimentally in [KS94] for the case of k-SAT.

One final issue is whether one can meaningfully define and study the existence of a “physical phase transition” in HORN-SAT. The major problem is a “degeneracy” property of our random model for Horn satisfiability: one can satisfy all but the positive unit clauses of any formula by the assignment 11...1. But under the random model employed in this paper the fraction of such clauses is o(1), a property that is not shared by any of the previously studied problems, and which makes the “physical interpretation” problematic. Whether the problem becomes meaningful under a different random model remains to be seen.

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