E-detectors: a nonparametric framework for online changepoint detection

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Abstract

Sequential changepoint detection is a classical problem with a variety of applications. However, the majority of prior work has been parametric, for example, focusing on exponential families. We develop a fundamentally new and general framework for changepoint detection when the pre- and post-change distributions are nonparametrically specified (and thus composite). Our procedures come with clean, nonasymptotic bounds on the average run length (frequency of false alarms). In certain nonparametric cases (like sub-Gaussian or sub-exponential), we also provide near-optimal bounds on the detection delay following a changepoint. The primary technical tool that we introduce is called an e-detector, which is composed of sums of e-processes—a fundamental generalization of nonnegative supermartingales—that are started at consecutive times. We first introduce simple Shiryaev-Roberts and CUSUM-style e-detectors, and then show how to design their mixtures in order to achieve both statistical and computational efficiency. We demonstrate their efficacy in detecting changes in the mean of a bounded random variable without any i.i.d. assumptions, with an application to tracking the performance of a sports team over multiple seasons.

1. Introduction

Suppose we have a sequence of observations $X_1, X_2, \ldots$, whose distribution may change at some unknown time $\nu$. To take one concrete example that we generalize later, denote the data stream by $X_1, \ldots, X_\nu \sim P_{\mu_0}$ and $X_{\nu+1}, X_{\nu+2}, \ldots \sim P_{\mu}$, where $P_{\mu_0}$ and $P_{\mu}$ are some probability distributions with parameters $\mu_0, \mu \in \mathbb{R}$, respectively. Let $P_\nu$ and $E_\nu$ denote probability measure and expectation operator over the entire infinite data stream when the change occurs at time $\nu$. If there is no change, we think of $\nu$ as being equal to $\infty$, and we let $P_\infty$ and $E_\infty$ refer to the corresponding probability and expectation.

An online changepoint detection algorithm consists of a data-dependent stopping rule $N^* \geq 1$ adapted to the natural filtration generated by the data stream $X_1, X_2, \ldots$. If the stopping time $N^*$ is finite, then we declare that a changepoint has been detected, and if the algorithm does not stop, we deem $N^* = \infty$ and no changepoint is proclaimed. To evaluate the performances of detection algorithms, we can check how quickly the algorithm can detect a change in the distribution while controlling the frequency of false alarms.

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A widely-used measure of the speed of detection after a changepoint is the worst average delays conditioned on the least favorable observations before the change [Lorden, 1971] and conditioned on the event that the algorithm stops after the changepoint [Pollak and Siegmund, 1975]. These are defined by

$$\mathcal{J}_L(N^*) := \sup_{\nu \geq 0} \text{esssup}_{\nu} \mathbb{E}_\nu [[N^* - \nu]_+]$$,

(1)

$$\mathcal{J}_P(N^*) := \sup_{\nu \geq 0} \mathbb{E}_\nu [N^* - \nu | N^* > \nu]$$,

(2)

where the subscripts indicate the authors, and it is known that $\mathcal{J}_P(N^*) \leq \mathcal{J}_L(N^*)$ [Moustakides, 2008].

To control false alarms, we adopt the average run length (ARL) [Page, 1954] metric, defined by

$$\text{ARL} := \mathbb{E}_\infty N^*.$$  (3)

We say that the “ARL is controlled at level $\alpha$” if $\mathbb{E}_\infty N^* \geq 1/\alpha$. An equivalent error metric is the False Alarm Rate (FAR), which is the reciprocal of the ARL, and we would like to ensure that the FAR is at most $\alpha$, and we call a changepoint detection procedure as “valid” if it satisfies the above constraint.

Our implicit objective is to (approximately) minimize $\mathcal{J}_L(N^*)$ or $\mathcal{J}_P(N^*)$ while guaranteeing that the ARL is controlled at a prespecified level $\alpha \in (0, 1)$. We differ from other work in that our focus is on nonparametric and composite pre- and post-change distributions, as well as on deriving nonasymptotic guarantees. In several such settings, it is apriori unclear how to define any valid changepoint detection algorithm, let alone an optimal one. Accordingly, we first address the design problem, and then move on to questions involving (approximate) optimality for average delays.

1.1. Prior work and our contributions

If both pre- and post-change parameters $\mu_0, \mu$ are known and the distributions have densities $p_{\mu_0}$ and $p_\mu$ with respect to some common reference measure, then the CUSUM procedure [Page, 1954] has been known to achieve the optimal worst average delay (exactly for $\mathcal{J}_L(N^*)$ and asymptotically for $\mathcal{J}_P(N^*)$) as $\alpha \to 0$ among all procedures controlling ARL at the same level [Lorden, 1971, Moustakides, 1986, Ritov, 1990, Lai, 1998]. Recall that the CUSUM procedure is defined by the stopping time $N_{CS}^* := \inf \{ n \geq 1 : M_n^{CS} \geq c_{\alpha}^{CS} \}$, where $c_{\alpha}^{CS} > 0$ is a constant chosen so that $\mathbb{E}_\infty (N_{CS}^*) = 1/\alpha$, and the test statistic $M_n^{CS}$ is defined by the recursive formula

$$M_n^{CS} = \frac{p_\mu(X_n)}{p_{\mu_0}(X_n)} \cdot \max \{ M_{n-1}^{CS}, 1 \}, \quad M_0^{CS} := 0.$$  (4)

The Shiryaev–Roberts (SR) procedure [Shiryaev, 1963, Roberts, 1966] is another widely used changepoint detection procedure, and is defined by the stopping time $N_{SR}^* := \inf \{ n \geq 1 : M_n^{SR} \geq c_{\alpha}^{SR} \}$, where $c_{\alpha}^{SR} > 0$ is a constant chosen so that $\mathbb{E}_\infty (N_{SR}^*) = 1/\alpha$, and the test statistic $M_n^{SR}$ is obtained recursively as

$$M_n^{SR} = \frac{p_\mu(X_n)}{p_{\mu_0}(X_n)} \cdot [M_{n-1}^{SR} + 1], \quad M_0^{SR} := 0.$$  (5)

Unlike the CUSUM procedure, the SR procedure does not achieve exact minimax optimality for $\mathcal{J}_L(N^*)$. However, the SR procedure and its generalized versions enjoy strong asymptotic optimality guarantees [Pollak, 1985, Polunchenko and Tartakovsky, 2010, Tartakovský et al., 2012, 2014].

The literature sometimes assumes that the pre-change distribution is known or can be approximated with a high precision by using the previous history. However, the post-change distribution is typically unknown and is assumed to belong to a family of distributions $\mathcal{P} := \{ p_\mu : \mu \in \Theta \}$. In this case, one natural approach would be to replace the unknown parameter $\mu$ with an estimator $\hat{\mu}$. If we use the maximum likelihood estimator (MLE) then we have the CUSUM procedure based on the generalized likelihood ratio (GLR) rule [e.g., see Barnard, 1959, Willsky and Jones, 1976, Siegmund and Venkatraman, 1995].

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Limitations of prior work. Though the usage of GLR statistic for the changepoint detection problem has a long history and it often yields good empirical performance, the current literature has two main limitations.

First, most existing methodology relies on parametric assumptions on the family of distributions (e.g., exponential families), both pre-change and post-change. There have been attempts to move away from this setting, and we will discuss these later. However, a general framework for deriving changepoint procedures in nonparametric, composite settings has not been previously presented in the generality that we do here.

Second, the study of statistical properties has been typically focused on the asymptotic regime of $\alpha \to 0$, unless the GLR statistic is defined on a well-separated post-change parameter space. In this paper, we prefer to guarantee nonasymptotic control on the ARL at a prespecified level (such as $\alpha = 0.001$). In fact, in many settings considered, we do not know of any existing way to guarantee asymptotic control on the ARL.

(We think that in theory and practice, the first is a bigger issue than the second. Luckily, our solution for the first automatically handles the second. Indeed, in the composite and nonparametric settings we consider, it is quite unclear how to control the ARL in any asymptotic sense.)

Finally, a direct online implementation of the GLR rule is infeasible since the memory and computation time both increase at least linearly as $n \to \infty$. One natural approach to tackle this online implementation issue is to use window-limited versions of the GLR rule [Willsky and Jones, 1976, Lai, 1995]. For instance, a simple form of the window-limited GLR rule can be defined by, at each time $n$, computing $\hat{\mu}$ over only times $n - W$ to $n$ for a properly chosen window size $W > 0$. However, the optimal choice of window size $W$ has been studied only in the asymptotic setting ($\alpha \to 0$). For a fixed $\alpha$, the optimal window size depends on the difference between the pre- and post-change distributions, which are unknown.

Our contributions. We present a general framework for changepoint detection, focusing on nonparametric settings with composite pre- and post-change distributions, and nonasymptotic guarantees on the ARL:

1. We introduce the concept of an e-detector that underlies our construction of changepoint procedures. E-detector utilizes a generalization of the underlying martingale structure of likelihood ratios in classical changepoint procedures. This e-detector framework is applicable in nonparametric settings including sub-Gaussian, sub-exponential, and bounded random variables, among many others. In such settings, there is no common reference measure and likelihood ratios cannot be defined, thus composite nonnegative supermartingales, or more generally “e-processes”, must be employed in their place.

2. Despite handling composite pre- and post-change distributions, even without an i.i.d. assumption on the data, our e-CUSUM and e-SR changepoint procedures based on e-detectors can always nonasymptotically control the ARL at level $\alpha$.

3. Nonasymptotic bounds on the worst average delay are derived in special cases for classes of nonparametric distributions with exponential tail decay (such as sub-Gaussian or sub-exponential), and they match the rate of known lower bounds for exponential family distributions as $\alpha \to 0$.

4. Computationally feasible algorithms are presented, in order to run our procedures in an online fashion without windowing. Practical strategies to choose hyperparameters are discussed. These are based on an adaptive mixture method, with the number of mixture components growing slowly over time.

Even though we will not explore this aspect very much in the current paper, we note that our procedures have natural gambling interpretations, and our work can be viewed as setting the foundations for a game-theoretic approach to changepoint detection. Before discussing the general framework in detail, in the following subsection, we present a motivating real-world example involving bounded random variables to illustrate how our nonparametric framework can be easily used in settings in which it is nontrivial to apply other commonly used methods.
1.2. Example: A changepoint in Cleveland Cavaliers 2011 - 2018

The Cleveland Cavaliers are an American professional basketball team. We use the Cavaliers’ game point records over 2010-11 to 2017-18 NBA seasons to illustrate how our proposed nonparametric changepoint detection algorithm can be applied to detect an interesting changepoint in the Cavaliers’ recent history.

The left plot in Figure 1 shows the difference between the scores of the Cavaliers and those of their opposing teams (also known as Plus-Minus) in all the games from the 2010-11 to the 2017-18 regular seasons. Each red line refers to the yearly average difference score in each season. Roughly, this value shows how well the Cavaliers performed against their opponent in terms of scoring. Typically, if a seasonal average is positive (or negative) then we may say that the Cavaliers showed a strong (or poor) performance in the corresponding season. The right plot shows a changepoint detected in early 2015; NBA fans may recall one major cause of the sharp improvement — LeBron James returned to the Cavaliers in 2014. However, how can we detect such a change on the fly by only tracking the Plus-Minus for each game?

This type of question can be well-fitted into the online changepoint detection framework. Let $X_1, X_2, \ldots$ be the sequence of Plus-Minus stats we observe sequentially. After observing a poor performance of the Cavaliers in 2010-11 season, we define the Cavaliers’ pre-change distribution on the Plus-Minus stats as follows: the average Plus-Minus of the team is less than or equal to $\mu_0 := -1$. Now, we are interested in detecting a meaningful performance improvement on the fly by defining the post-change distribution as follows: the average Plus-Minus of the team is greater than $\mu_1 := 1$. Here, the gap $|\mu_1 - \mu_0|$ between averages of Plus-Minus in pre- and post-changes refers to the degree of improvement we consider as a significant one.

Although the formulation of the problem is simple as described above, it is still nontrivial to fit this problem into commonly used changepoint detection procedures for the following reasons. First, it is not easy to choose a proper parametric model to fit observed Plus-Minus stats since they are integer-valued samples with varying mean and variance over seasons as Figure 1 illustrates. Second, even if we can choose a proper model, it is difficult to find a threshold to detect the changepoint since we are interested in detecting any changes larger than $|\mu_1 - \mu_0|$ instead of a fixed post-change. Many common methods have been relying on a high-quality simulator or large enough sample history for pre-change observations to compute a valid threshold. For this example, however, it is hard to access such tools since the Cavaliers’ overall Plus-Minus

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Figure 1: Left: Plus-Minus of the Cavaliers from 2010-11 to 2017-18 seasons. Each red line corresponds to the seasonal average. Right: Path of log of e-detectors. The horizontal red line is the threshold equal to $\log(1/\alpha)$ controlling the ARL by $1/\alpha = 1000$. In this example, the procedure detects the changepoint in the Plus-Minus of the Cavaliers at the end of 2014-15 season.
stats are difficult to be modeled directly, and it is tricky to justify to use existing records to get a valid threshold as the team’s overall performance varies a lot around the 2010-11 season.

Based on the introduced framework of the changepoint detection procedure using e-detectors, we detour difficulties in the commonly used methods illustrated above as follows. First, due to the nonparametric nature of the new framework, we do not need to choose any parametric model to fit the data. Instead, we simply assume the absolute value of each Plus-Minus stat is bounded by a large number - in this example, we set 80 as the boundary. Though we set a boundary conservatively large, our detection procedure is variance-adaptive so that we can detect the changepoint efficiently without specifying the variance of observations. Second, the nonasymptotic analysis of the new framework makes it possible to choose an explicit detection boundary, which is equal to $\log(1/\alpha)$ to build a changepoint detection procedure controlling the ARL by $1/\alpha$ for any given $\alpha \in (0, 1)$. In this example, we choose $\alpha = 10^{-3}$ to make ARL is larger than at least 10 regular seasons of game dates. The right plot in Figure 1 shows the log of e-detectors on which we build the changepoint detection procedure. The red horizontal line corresponds to the detection boundary given by $\log(1/\alpha)$. We can check the procedure detects the changepoint of the Plus-Minus stat of the Cavaliers in the middle of 2014-15 season. See Section 5.2 for the detailed explanation about how we construct the online changepoint detection procedure based on the general framework we introduce throughout the paper.

**Paper outline.** The rest of the paper is organized as follows. In Section 2, we introduce a general framework about how to build composite, nonparametric, and nonasymptotic changepoint detection procedures using e-detectors. Section 3 extends the previous framework into the case where we have a set of detection procedures and explain how to use a mixture method to combine multiple procedures effectively. In Section 4, we introduce an exponential structure of e-detectors that makes it possible to design a near-optimal detection procedure with an explicit upper bound on worst average delays. Based on the proposed framework, Section 5 presents two canonical examples of Bernoulli and bounded random variables cases with real data applications of the Cavaliers 2011-2018 statistics. We conclude with a brief summary of our contribution and discussions on future directions. In the interest of space, we defer proofs to the supplement.

## 2. Nonparametric changepoint detection using e-detectors

### 2.1. Problem Setup

Let $\mathcal{P}$ denote the set of possible pre-change distributions, which could in general be a nonparametric class. Each $P \in \mathcal{P}$ is a distribution over an infinite sequence of random variables. We do not assume the observations in the sequence to be independent.

We will assume throughout that the observations $X_1, \ldots, X_\nu$ up to the unknown changepoint time $\nu$ follows a distribution $P \in \mathcal{P}$. The remaining observations $X_{\nu+1}, X_{\nu+2}, \ldots$ are drawn from a distribution $Q$ in a class of post-change distributions $\mathcal{Q}$. In this case, we let $\mathbb{P}_{P,\nu,Q}, \mathbb{E}_{P,\nu,Q}$ and $\mathbb{V}_{P,\nu,Q}$ denote probability, expectation and variance operators over the entire data stream.

If there never is a change, we will use the notation $\mathbb{P}_{P,\infty}, \mathbb{E}_{P,\infty}$ and $\mathbb{V}_{P,\infty}$. Also, if a change occurs at the beginning ($\nu = 0$) then we use $\mathbb{P}_{0,Q}, \mathbb{E}_{0,Q}$ and $\mathbb{V}_{0,Q}$. Note that technically $\mathbb{P}_{0,Q} = \mathbb{P}_{Q,\infty}$, but we use the former to denote that $Q$ is a post-change distribution and a changepoint has occurred at the very start, and the latter to denote that $Q$ is a pre-change distribution and a changepoint never occurs.

Let $\mathcal{F} := \{\mathcal{F}_n\}_{n \geq 0}$ be a filtration where we let $\mathcal{F}_0 := \{\emptyset, \Omega\}$ for simplicity. Let $M := \{M_n\}_{n \geq 0}$ be a nonnegative adapted process with respect to the filtration $\mathcal{F}$. If required, we define $M_\infty := \limsup_{n \to \infty} M_n$ and $\mathcal{F}_\infty := \sigma(\bigcup_{n \geq 0} \mathcal{F}_n)$ [see, for example, Durrett, 2019]. It is common to consider the natural filtration $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$, but there are situations where restricting the filtration could be advantageous (for example, when there are nuisance parameters).
Let $T$ denote the set of all stopping times with respect to $\mathcal{F}$, but we will later see that it typically suffices to consider finite stopping times, or those with finite expectation.

**Remark 2.1.** All of our results on ARL do not require any assumptions on the changepoint or post-change distribution of the data. However, when analyzing detection delay, we will typically assume that $X_{\nu+1}, X_{\nu+2}, \ldots$ are independent of the pre-change data, and are drawn from $Q$ as if the time $\nu$ was reset to zero. But for much of the paper, it is also okay to assume that the post-change data is drawn from $Q \mid \mathcal{F}_\nu$.

With the appropriate definitions and setup in place, we can now define our central concept, an e-detector.

### 2.2. What is an e-detector?

**Definition 2.2** ($P$-e-detector). The process $M$ is called an e-detector with respect to the class of pre-change distributions $\mathcal{P}$ if it satisfies the property

$$
E_{P,\infty}[M_\tau] \leq E_{P,\infty}[\tau], \quad \forall \tau \in T, \quad \forall P \in \mathcal{P}.
$$

(6)

If a stopping time $\tau$ has a nonzero probability of being infinite, then inequality (6) is trivially satisfied. Thus, the condition is only really required to hold for stopping times with finite expectation under some $P$. This latter set of stopping times depends on $P$, and so in order to not complicate notation, we continue to simply consider all stopping times $T$. (Also note that the condition can only be satisfied if process $M$ is $\mathcal{P}$-integrable, so this is implicitly assumed to be the case in what follows.)

By the linearity of expectation and Tonelli’s theorem, a mixture of e-detectors is also an e-detector. More formally, if $\{M^a\}_{a \in A}$ is a set of e-detectors (where $a$ is a tuning parameter), then so is $\int M^a d\mu(a)$ for any fixed distribution $\mu$ over $A$. Later in this paper, we will in particular use finite mixtures of the form $(M^1 + M^2 + \cdots + M^K)/K$ in order to adapt to the unknown post-change distribution. (In fact, we will develop more sophisticated mixtures whose number of components grows slowly with time.) For later reference, we state the above as a proposition:

**Proposition 2.3.** Let $\{M^a\}_{a \in A}$ be a set of e-detectors. Then for any probability measure $\mu$ on $A$, the mixture of e-detectors $\int M^a d\mu(a)$ forms a valid e-detector.

The following proposition shows how one can immediately obtain a changepoint detection procedure from an e-detector process $M$. An e-detector is a real-valued measure of evidence for whether a changepoint has occurred or not, and may be continuously monitored, stopped and interpreted without thresholding. However, a simple thresholding rule controls the ARL.

**Proposition 2.4.** For any $\alpha \in (0,1)$ and e-detector $M$, if we declare a changepoint at the stopping time

$$N^* := \inf \{ n \geq 1 : M_n \geq 1/\alpha \},
$$

(7)

then we have

$$
\inf_{P \in \mathcal{P}} E_{P,\infty} N^* \geq 1/\alpha,
$$

(8)

That is, the online changepoint detection procedure in (7) controls the ARL at level $\alpha$.

The informal proof is one line long: dropping subscripts, the definition of an e-detector implies that $E N^* \geq EM_{N^*}$, but by definition of $N^*$, we know that $M_{N^*} \geq 1/\alpha$ if $N^*$ is finite. (If $N^*$ is not almost surely finite, the claim holds anyway.) The full proof is provided for completeness in Appendix A.1.
2.3. Constructing an e-detector based on a sequence of e-processes

The central building block of our e-detector is called an e-process. E-processes are newly-developed tools that have been shown to play a fundamental role in sequential hypothesis testing, especially in composite, nonparametric settings. E-processes are generalizations of nonnegative martingales and supermartingales, and in particular, e-processes are nonparametric and composite generalizations of likelihood ratios. They have strong game-theoretic roots, and have found utility in the meta-analysis, as well as for the purposes of anytime-valid inference in the presence of continuous monitoring [Ramdas et al., 2020, 2021, Grünwald et al., 2019, Howard et al., 2020, 2021]. The properties of e-processes have not yet been explored in changepoint analysis, and we undertake this effort here.

To understand their definition, we briefly forget about changepoint detection and consider testing the null hypothesis that \( X_1, X_2, \ldots \sim P \) for some \( P \in \mathcal{P} \). An e-process for \( \mathcal{P} \), called a \( \mathcal{P} \)-e-process, is a sequence of nonnegative random variables \( (E_t)_{t \geq 1} \) such that for any \( P \in \mathcal{P} \) and any stopping time \( \tau \) with respect to the natural filtration, we have \( \mathbb{E}_P[|E_{\tau}] \leq 1 \). The value of \( E_t \) measures evidence against the null (larger values, more evidence). A level-\( \alpha \) sequential test can be obtained by rejecting the null as soon as \( E_t \) reaches \( 1/\alpha \); this is a consequence of Ville’s inequality [Ville, 1939, Howard et al., 2020].

Clearly, nonnegative \( \mathcal{P} \)-martingales (i.e., the process is a nonnegative \( \mathcal{P} \)-martingales simultaneously for every \( P \in \mathcal{P} \)) and \( \mathcal{P} \)-supermartingales are examples of e-processes. However, e-processes are a distinct and more general class of processes. In fact, there exist natural classes \( \mathcal{P} \) for which the only \( \mathcal{P} \)-martingales are constants, and the only \( \mathcal{P} \)-supermartingales are decreasing sequences, but there are e-processes for \( \mathcal{P} \) that can increase to infinity when the data are not from \( \mathcal{P} \). See Ramdas et al. [2021] for one such example, arising from sequentially testing exchangeability of a binary sequence.

In this subsection, we show how to leverage e-processes to build e-detectors.

**Definition 2.5 (e\(_j\)-process).** For each integer \( j \geq 1 \), we define an \( e_j \)-process, \( \Lambda^{(j)} := \{\Lambda_n^{(j)}\}_{n \geq 1} \) for a class of distributions \( \mathcal{P} \) to be a nonnegative adapted process such that \( \Lambda_1^{(j)} = \cdots = \Lambda_{j-1}^{(j)} = 1 \), and

\[
\sup_{P \in \mathcal{P}} \mathbb{E}_{P,\infty} \left[ \Lambda^{(j)}_\tau \mid \mathcal{F}_{j-1} \right] \leq 1, \quad \forall \tau \in \mathcal{T}. \tag{9}
\]

When \( j = 1 \), the \( e_j \)-process is simply a standard e-process, which tests whether the data distribution is different from the proclaimed null (pre-change) distribution. Similarly, for \( j > 1 \), each \( e_j \)-process can be viewed as an e-process that begins at time \( j \), and tests the null hypothesis of a distributional change having occurred at (or around) time \( j \).

**Definition 2.6 (SR and CUSUM e-detectors).** Based on a sequence of e-processes \( \{\Lambda^{(j)}\}_{j \geq 1} \), define SR and CUSUM e-detectors \( M^{SR} \) and \( M^{CS} \), respectively by \( M_0^{SR} = M_0^{CS} := 0 \) and for each \( n \geq 1 \),

\[
M_n^{SR} := \sum_{j=1}^{n} \Lambda_n^{(j)}, \quad \text{and} \quad M_n^{CS} := \max_{j \in [n]} \Lambda_n^{(j)}. \tag{10}
\]

It is not hard to check that the above processes are indeed e-detectors.

2.4. Constructing computationally efficient e-detectors using baseline increments

In general, it may take \( O(n) \) time to update the aforementioned e-detectors at time \( n \). In order to construct e-detectors that can be updated online in sublinear time and memory (or even near-constant time and memory), it turns out to be computationally convenient to use a common “baseline” increment in order to build the underlying \( e_j \)-processes, as we do below.
**Definition 2.7 (Baseline increment).** A nonnegative, adapted process \( L := \{L_n\}_{n \geq 1} \) is called a baseline increment if for each \( n \geq 1 \), we have
\[
\sup_{P \in \mathcal{P}} \mathbb{E}_{P,\infty}[L_n | \mathcal{F}_{n-1}] \leq 1. \tag{11}
\]

It is easy to check that if \( L^1 \) and \( L^2 \) are baseline increments, and \( A^1 \) and \( A^2 \) are nonnegative and predictable processes (meaning that \( A^1_n \) and \( A^2_n \) are both \( \mathcal{F}_{n-1} \)-measurable) such that \( A^1 + A^2 \) is strictly positive, then the mixture \( (A^1 L^1 + A^2 L^2)/(A^1 + A^2) \) also forms a baseline increment. In short, predictable mixtures retain the baseline increment property.

Comparing (11) to (9), we see that a baseline increment \( L \) is not itself an \( e_j \)-process, because the expectation in (11) applies only at fixed times \( n \), with the conditioning being on the previous step \( n - 1 \), but (9) calculates expectations at any stopping time, and conditions on \( j - 1 \). It is best to think about the baseline increment as the multiplicative increment that forms the \( e_j \)-process, as follows.

**Definition 2.8 (Baseline \( e_j \)-process).** For a given baseline increment \( L := \{L_n\}_{n \geq 1} \), we define the corresponding “baseline \( e_j \)-process” \( \Lambda(j) \), for each \( j, n \in \mathbb{N} \), as below:
\[
\Lambda_n^{(j)} := \begin{cases} 1 & \text{if } n < j \\ \prod_{i=j}^{n} L_i & \text{otherwise}, \end{cases} \tag{12}
\]

Under any pre-change distribution \( P \in \mathcal{P} \), each \( \Lambda(j) \) is a nonnegative supermartingale by definition of the baseline increment \( L_i \). Therefore, a straightforward application of the optional stopping theorem implies that each \( \Lambda(j) \) satisfies condition (9), and thus is a valid \( e_j \)-process.

As an example of a baseline \( e_j \)-process, consider the case where we have i.i.d. observations \( X_1, X_2, \ldots \) from a distribution \( p_\theta \) parameterized by \( \theta \in \Theta \), and the pre-change distribution is given by \( \theta_0 \). Then, for any post-change distribution \( p_{\theta^1} \) with \( \theta_1 \neq \theta_0 \), the likelihood ratio between two distributions, \( L_n := p_{\theta_1}(X_n)/p_{\theta_0}(X_n) \) yields a baseline increment process with the inequality in (11) being replaced by the equality. Then, each \( \Lambda_n^{(j)} \) is the likelihood ratio based on \( X_j, \ldots, X_n \). What is more, instead of using a fixed post-change parameter \( \theta_1 \), we can also plug-in a running MLE or any other online nonanticipating estimator that is based on the previous history \( \mathcal{F}_{n-1} \) only, say \( \hat{\theta}_{n-1} \), into the likelihood ratio. In this case, although the value at time \( n \) \( L_n := p_{\hat{\theta}_{n-1}}(X_n)/p_{\theta_0}(X_n) \) of the resulting process may depend on the previous history \( \mathcal{F}_{n-1} \), the inequality in (12) will be satisfied as an equality, yielding a valid baseline \( e_j \)-processes.

**Remark 2.9.** While baseline increment processes provide a natural and computationally convenient way to construct \( e_j \)-process, we emphasize that any \( e \)-detector, even one that does not use baseline increments, will automatically control the ARL; see Proposition 2.4. To elaborate, baseline \( e_j \)-processes are composite \( \mathcal{P} \)-supermartingales (meaning \( \mathcal{P} \)-supermartingales for every \( P \in \mathcal{P} \)), but there exist other \( \mathcal{P} \)-processes—which are not \( \mathcal{P} \)-supermartingales—that naturally arise and these can be used to form \( e \)-detectors. For two examples that we do not pursue here, the reader may check the recent works on universal inference [Wasserman et al., 2020, Section 8] and testing exchangeability [Ramdas et al., 2021] for instances where no powerful supermartingales exist with respect to the natural filtration, but powerful \( e \)-processes can be constructed.

**Definition 2.10 (Baseline SR and CUSUM \( e \)-detectors).** When an SR or CUSUM \( e \)-detector is constructed using a sequence of baseline \( e_j \)-processes (12), we call it a “baseline SR or CUSUM \( e \)-detector”.

Each baseline SR or CUSUM \( e \)-detector can be computed recursively like their classical analogs:
\[
M_n^{SR} = L_n \cdot [M_{n-1}^{SR} + 1], \tag{13}
M_n^{CS} = L_n \cdot \max \{M_{n-1}^{CS}, 1\}. \tag{14}
\]
with $M_0^{SR} = M_0^{CS} = 0$ for each $n \in \mathbb{N}$. The above computational benefit is the primary reason to consider baseline e-detectors, but as mentioned in Remark 2.9, more general e-detectors are sometimes necessary.

The processes $M^{SR}_n$ and $M^{CS}_n$ defined above are valid e-detectors. Indeed, for any stopping time $\tau$ and pre-change distribution $P \in \mathcal{P}$, if $\mathbb{P}_{P,\infty}(\tau = \infty) > 0$ then the condition of the e-detector in (6) holds trivially. If not, then $\tau$ is finite almost surely, and we have by linearity of expectation and the tower rule:

$$
\mathbb{E}_{P,\infty}M^{CS}_n \leq \mathbb{E}_{P,\infty}M^{SR}_n = \mathbb{E}_{P,\infty} \sum_{j=1}^{\infty} \Lambda^{(j)}_\tau \mathbb{1}(j \leq \tau) = \sum_{j=1}^{\infty} \mathbb{E}_{P,\infty} \Lambda^{(j)}_\tau \mathbb{1}(j \leq \tau)
$$

$$
= \sum_{j=1}^{\infty} \mathbb{E}_{P,\infty} \left[ \mathbb{1}(j \leq \tau) \mathbb{E}_{P,\infty} \Lambda^{(j)}_\tau | \mathcal{F}_{j-1} \right] \leq \sum_{j=1}^{\infty} \mathbb{E}_{P,\infty} \mathbb{1}(j \leq \tau) = \mathbb{E}_{P,\infty} \tau,
$$

where the first inequality comes from the nonnegativity of e-processes, and the second inequality comes from the definition of the e-process $\Lambda^{(j)}_\tau$ for each $j \geq 1$. Note that this proof is also applicable to general SR and CUSUM e-detectors.

### 2.5. Changepoint procedures by thresholding e-detectors

The value of any e-detector process, like $M^{SR}_n$ or $M^{CS}_n$, is directly interpretable without thresholding (a larger value indicates more evidence of a changepoint). These can be monitored and adaptively stopped. Nevertheless, to explicitly control the ARL at level $\alpha$, we define SR and CUSUM-style online changepoint detection procedures, called “e-SR and e-CUSUM procedures” as follows.

**Definition 2.11 (e-SR and e-CUSUM procedures).** Given SR and CUSUM e-detectors $M^{SR}_n$ and $M^{CS}_n$, define e-SR and e-CUSUM procedures by the stopping times

$$
N^{*}_{SR} := \inf \left\{ n \geq 1 : M^{SR}_n \geq 1/\alpha \right\}, \quad \text{and} \quad N^{*}_{CS} := \inf \left\{ n \geq 1 : M^{CS}_n \geq c_\alpha \right\},
$$

where $c_\alpha$ is a constant chosen to control the ARL of the e-CUSUM procedure by $1/\alpha$ for some $\alpha \in (0, 1)$.

By Proposition 2.4, $1/\alpha$ is a valid choice for $c_\alpha$.

We note $c_\alpha = 1/\alpha$ may be a very conservative choice for the e-CUSUM procedure. Indeed, suppose we use the trivial e-processes, that is, we set $\Lambda^{(j)}_n := 1$ for all $j, n$. In this setting, the SR e-detector is given by $M^{SR}_n = n$ for each $n$. In contrast, the CUSUM e-detector, $M^{CS}_n$, is equal to 1 for all $n$. Therefore, any valid threshold we can choose for the e-SR procedure must be larger than $\lfloor 1/\alpha \rfloor$. On the other hand, any threshold above 1 makes $N^{*}_{CS} = \infty$, which of course controls ARL by $1/\alpha$, but the true ARL is much above the target. Building from this trivial example, it is possible to construct nontrivial examples in which letting $\alpha \to 0$ makes the gap between tight thresholds of e-SR and e-CUSUM procedures arbitrarily different.

**Remark 2.12.** Unless we assume the pre-change distribution is time-stationary, known and parametric, or we can access a good sample of the pre-change distribution or large enough historical data, computing a tight or even valid threshold $c_\alpha$ can be a challenging task. In the application sections below, we will mainly deal with non-stationary pre-change distributions where pre-change observations may not be identically distributed and thus all observations before the changepoint may follow different distributions. In this case, setting $c_\alpha = 1/\alpha$ seems to be the only reasonable choice, and we recommend using the e-SR procedure rather than e-CUSUM since if we use the same threshold for both procedures, the former always detects the changepoint faster than the latter while provably controlling the ARL at the same level.
Special case 1: when likelihood ratios are well-defined. We briefly mention two special cases of our framework. The first is parametric, when likelihood ratios are well-defined. When the pre- and post-change distributions are known, then the standard CUSUM and SR processes from (4) and (5) are both e-detectors. If the post-change distribution is only known to lie in some set, then taking a mixture likelihood ratio yields e-detectors (using either a non-anticipating predictable mixture or using a fixed mixture distribution). If the pre-change distribution is also not known, but maximum likelihood estimation is efficient under the null, then e-processes can be constructed that take the ratio of mixture likelihoods over the alternative to maximum likelihood under the null, as done in universal inference [Wasserman et al., 2020] or in Tartakovsky [2014].

Special case 2: conformal changepoint detection. The conformal changepoint detection procedures by Vovk et al. [2005], Vovk [2021] are designed to detect deviations from exchangeability. Formally, the pre-change distributions \( \mathcal{P} \) is the set of all i.i.d. product distributions over infinite sequences, or their convex hull, which is the set of all exchangeable laws. The methodology is based on designing exchangeability martingales using conformal prediction, and their procedure fits neatly into our framework. Importantly, their filtration is restricted, and is smaller than the natural filtration of the data. This allows nonparametric martingales to exist, but the e-detector property only holds with respect to a smaller class of stopping times. Nevertheless, thresholding the martingales at \( 1/\alpha \) still controls the ARL at level \( \alpha \), and the latter property is independent of the filtration used to construct the e-detector. As a side note, if one wanted to construct an e-detector that was valid at all stopping times with respect to the natural filtration of the data, e-detectors based on martingales provably do not suffice, but e-detectors based on e-processes can be constructed using the techniques from Ramdas et al. [2021].

Reflecting on the above two special cases, it is worth noting that it is possible to define e-detectors in cases where there is no common reference measure amongst the pre-change and post-change distributions, and thus no well-defined likelihood ratio process, and also when there is no nontrivial martingale can be constructed. This is precisely the utility of e-processes, which are nonparametric and composite generalizations of likelihood ratios. We delve into such cases soon.

2.6. Warm-up: upper bounds on worst average delays for baseline e-detectors

Recall that our objective is to minimize worst average delays \( J_L(N^*) \) or \( J_P(N^*) \), given above in (1) and (2) respectively, while controlling the ARL \( \mathbb{E}_{P,\infty}(N^*) \geq 1/\alpha \). To derive bound on the worst average delays, we further assume that the post-change observations \( X_{\nu+1}, X_{\nu+2}, \ldots \) are independent of the pre-change observations and form a strongly stationary process. That is, we assume that, for any finite subset \( I \subset \mathbb{N} \) and any \( j \in \mathbb{N} \), the joint distributions of \( \{X_{\nu+i}\}_{i \in I} \) and \( \{X_{\nu+i+j}\}_{i \in I} \) are equal to each other. About the underlying baseline increment, we further assume that there exist a function \( f \) and an integer \( m \geq 0 \) such that \( L_n = f(X_n, X_{n-1}, \ldots, X_{n-m}) \) for each \( n \). Here, if \( m \) is a strictly positive number then we implicitly assume that we can access \( m \) observations \( X_0, X_1, \ldots, X_{1-m} \) from the pre-change distribution in order to build changepoint detection procedures. Under these conditions, the following theorem provides analytically more tractable upper bounds on worst average delays for e-SR and e-CUSUM procedures.

**Proposition 2.13.** For a given \( \alpha \in (0,1) \), let \( N_{SR}^* \) and \( N_{CS}^* \) be e-SR and e-CUSUM procedures using baseline e-detectors. Under the settings described above, their worst average delays are upper bounded as

\[
J_P(N_{SR}^*) \leq J_L(N_{SR}^*) \leq \mathbb{E}_{0,Q} N_1/\alpha + m, \tag{16}
\]

\[
J_P(N_{CS}^*) \leq J_L(N_{CS}^*) \leq \mathbb{E}_{0,Q} N_{c_0} + m, \tag{17}
\]

respectively, where \( N_c \) is the stopping time defined for any \( c > 1 \) as \( N_c := \inf \{ n \geq 1 : \sum_{i=1}^{n} \log L_i \geq \log(c) \} \), and \( c_0 \leq 1/\alpha \) is any threshold that ensures the e-CUSUM procedure (15) has an ARL no smaller than \( 1/\alpha \).
Furthermore, if each $L_n$ is a function of $X_n$ only (i.e. $m = 0$) and $\mathbb{E}_{0,Q} \log L_1 > 0$, then

$$
\mathbb{E}_{0,Q} N_c \leq \frac{\log(c)}{\mathbb{E}_{0,Q} \log L_1} + \frac{\mathbb{V}_{0,Q} \log L_1}{[\mathbb{E}_{0,Q} \log L_1]^2} + 1. 
$$

(18)

The proof of upper bounds on worst average delays in Proposition 2.13 can be found in Appendix A.1.

Remark 2.14. The stopping time $N_{1/\alpha}$ delivers a level-$\alpha$ sequential test for the null hypothesis $H_0: P \in \mathcal{P}$. On the other hand, the stopping time $N_{c_\alpha}$ may not necessarily control the type-1 error by $\alpha$ since the threshold $c_\alpha$ can be significantly smaller than $1/\alpha$ as discussed earlier.

3. Combining baseline e-detectors using the method of mixtures

In the previous section, we discussed how one can construct a valid e-detector and derive upper bounds on worst average delays. However, in most composite and nonparametric changepoint detection scenarios, there is no single optimal e-detector but instead there are often several applicable e-detectors to choose from. In this section, we introduce a practicable and computationally efficient strategy to construct a good e-detector for minimizing the upper bound on worst average delays. However, in most composite and nonparametric changepoint detection scenarios, there is no single optimal e-detector but instead there are often several applicable e-detectors to choose from. In this section, we introduce a practicable and computationally efficient strategy to construct a good e-detector for minimizing the upper bound on worst average delays in Proposition 2.13, especially for the upper bound (18) in the $m = 0$ case.

To be specific, suppose we have a set of baseline increments $\{L^\lambda\}_{\lambda \in \Pi}$ parametrized by $\lambda \in \Pi$. Then, under the additional condition in Section 2.6 with $m = 0$, an ideal choice of the parameter $\lambda^*$ for a post-change distribution $Q$ is given by

$$
\lambda^*(Q) = \arg \max_{\lambda \in \Pi} \mathbb{E}_{0,Q} \log L_1(\lambda),
$$

which minimizes the first term of the upper bound of worst average delays in (18). The first term of the upper bound, which often becomes a leading term especially for small enough $\alpha$, is inversely proportional to

$$
\mathbb{E}_{0,Q} \log L_1^{(\lambda^*)} := D(Q||\mathcal{P}),
$$

(20)

where the second argument $\mathcal{P}$ in $D(Q||\mathcal{P})$ explicitly refers to the dependency of the baseline increment $L^{\lambda^*}$ to the class of pre-change distributions $\mathcal{P}$. For the rest of the paper, we will assume that the set of baseline increments, $\{L^\lambda\}_{\lambda \in \Pi}$ is rich enough such that $D(Q||\mathcal{P}) > 0$ for all $Q \in \mathcal{Q}$. As we observe later, in many canonical cases, we have $D(Q||\mathcal{P}) = \inf_{P \in \mathcal{P}} KL(Q||P)$ where $KL(Q||P)$ is the Kullback-Leibler (KL) divergence from $Q$ to $P$.

Generally, computing $\lambda^*$ is not feasible since it depends on the unknown post-change distribution $Q$. Next, we show how to build a mixture of baseline e-detectors that can detect the changepoint nearly as quickly as the one with $\lambda^*$, when known lower and upper bounds $\lambda_L$ and $\lambda_U$ on $\lambda^*$ are available.

Recall that an average of e-detectors is also a valid e-detector, in the sense of satisfying condition (6). Therefore, for any mixing distribution $W$ supported on $[\lambda_L, \lambda_U]$, we can define mixtures of e-SR and e-CUSUM procedures by following stopping times:

$$
N^*_m\text{SR} := \inf \left\{ n \geq 1 : \int \sum_{j=1}^{n} \prod_{i=j}^{n} L_i(\lambda) \ dW(\lambda) \geq 1/\alpha \right\},
$$

(21)

$$
N^*_m\text{CS} := \inf \left\{ n \geq 1 : \int \max_{j \in [n]} \prod_{i=j}^{n} L_i(\lambda) \ dW(\lambda) \geq c_\alpha \right\},
$$

(22)
where \( c_\alpha > 1 \) is a fixed constant which controls the ARL for some \( \alpha \in (0, 1) \). Since the mixture of e-CUSUM procedure is based on a valid e-detector, we can always set the threshold \( c_\alpha \) to be equal to \( 1/\alpha \) as same as the threshold of the mixture of e-SR procedures.

**Remark 3.1.** Instead of using mixtures, one may be tempted to consider swapping the above integral with a supremum over \( \lambda \in [\lambda_L, \lambda_U] \). However, this does not in general yield a valid e-detector.

### 3.1. Computational and analytical aspects of mixtures of baseline e-detectors

Though any mixing distribution yields a valid e-detector, for computational efficiency, we only consider discrete mixtures where the support of mixing distribution has at most countably many elements. To be specific, let \( \{\omega_k\}_{k \geq 1} \) be a set of nonnegative mixing weights with \( \sum_{k \geq 1} \omega_k = 1 \) and let \( \{\lambda_k\}_{k \geq 1} \) be the corresponding supporting set. For ease of notation, we denote \( L^{(\lambda_k)} := L(k) \) for each \( k \geq 1 \). Based on the set of nonnegative mixing weights and the corresponding set of baseline increments, we define mixtures of SR and CUSUM e-detectors as \( M_n^{mSR} = M_n^{mCS} := 1 \), and for each \( n \in \mathbb{N} \),

\[
\begin{align*}
M_n^{mSR} &:= \sum_{k=1}^{\infty} \omega_k \sum_{i=1}^{n} \prod_{j=i}^{k} L_i(k) := \sum_{k=1}^{\infty} \omega_k M_n^{SR}(k), \\
m_{n}^{mCS} &:= \sum_{k=1}^{\infty} \omega_k \max_{j \in [n]} \prod_{i=j}^{n} L_i(k) := \sum_{k=1}^{\infty} \omega_k M_n^{CS}(k).
\end{align*}
\]

Let \( K := | \{k : \omega_k > 0\} | \) be the number of nonzero mixing weights.

**Finite mixtures.** If \( K < \infty \), we may for simplicity assume that the first \( K \) weights \( \omega_1, \ldots, \omega_K \) are the only nonzero values. In this case, we can compute mixtures of SR and CUSUM e-detectors by

\[
\begin{align*}
M_n^{mSR} &= \sum_{k=1}^{K} \omega_k M_n^{SR}(k), \quad \text{and} \quad M_n^{mCS} = \sum_{k=1}^{K} \omega_k M_n^{CS}(k),
\end{align*}
\]

where each \( M_n^{SR}(k) \) and \( M_n^{CS}(k) \) are computed recursively as

\[
\begin{align*}
M_n^{SR}(k) &= L_n(k) \cdot \left[ M_{n-1}^{SR}(k) + 1 \right], \\
M_n^{CS}(k) &= L_n(k) \cdot \max \left\{ M_{n-1}^{CS}(k), 1 \right\}
\end{align*}
\]

with \( M_0^{SR}(k) = M_0^{CS}(k) = 0 \) for each \( k \in [K] \) and \( n \in \mathbb{N} \). Therefore, if each computation of \( L_n(k) \) has constant time and space complexities, then the evaluation of mixtures of SR and CUSUM e-detectors at each time \( n \) requires \( O(K) \) time and space complexity.

**Infinite mixtures, scheduling functions and adaptive re-weighting.** If \( K = \infty \) or if \( K \) is to be chosen adaptively as an increasing function of \( n \) we modify our strategy as follows. We first choose an increasing function \( K : \mathbb{N} \to \mathbb{N} \), and let \( K^{-1} : \mathbb{N} \to \mathbb{N} \) be the generalized inverse function of \( K \) defined by \( K^{-1}(k) := \inf \{j \geq 1 : K(j) \geq k\} \) for each \( k \in \mathbb{N} \). Note that \( K^{-1} \) is also an increasing function. We call such function \( K \) as a scheduling function. We intentionally overload notation: in what follows, \( K(n) \) plays the same role as the constant \( K \) in the case of finite support. Note that \( K^{-1}(k) \leq n \) for any \( k \leq K(n) \); we will use this simple fact below when defining nested summations.
Based on a scheduling function $K$ and its generalized inverse $K^{-1}$, we define \textit{adaptive SR and CUSUM e-detectors}, $M_n^{\text{aSR}}$ and $M_n^{\text{aCS}}$, respectively, as

$$M_n^{\text{aSR}} = \sum_{k=1}^{K(n)} \omega_k \sum_{j=K^{-1}(k)}^{n} \gamma_j \prod_{i=j}^{n} L_i(k) := \sum_{k=1}^{K(n)} \omega_k M_n^{\text{SR}}(k),$$

(28)

$$M_n^{\text{aCS}} = \sum_{k=1}^{K(n)} \omega_k \max_{1 \leq j \leq n} \gamma_j \prod_{i=j}^{n} L_i(k) := \sum_{k=1}^{K(n)} \omega_k M_n^{\text{CS}}(k),$$

(29)

where $\gamma_j := 1/\sum_{k=1}^{K(j)} \omega_k \geq 1$ is the \textit{adaptively re-weighting factor} at time $j$, which ensures that the mixing weights always sum to one at each time. Here, we restrict not only the space over the index $k$ from $[1, \infty]$ to $[1, K(n)]$ but also the space over the index $j$ from $[1, n]$ to $[K^{-1}(k), n]$. This is because it makes possible to compute both $M_n^{\text{aSR}}$ and $M_n^{\text{aCS}}$ efficiently since each $M_n^{\text{SR}}(k)$ and $M_n^{\text{CS}}(k)$ have following recursive representations

$$M_n^{\text{SR}}(k) = L_n(k) \cdot \left[ M_{n-1}^{\text{SR}}(k) + \gamma_n \right],$$

(30)

$$M_n^{\text{CS}}(k) = L_n(k) \cdot \max \left\{ M_{n-1}^{\text{CS}}(k), \gamma_n \right\},$$

(31)

for each $n \geq K^{-1}(k)$ and $M_n^{\text{aSR}}(k) = M_n^{\text{aCS}}(k) = 0$ for all $n = 0, 1, \ldots, K^{-1}(k) - 1$. Therefore, if each computation of $L_n(k)$ has constant time and space complexities then the computations of adaptive SR and CUSUM e-detectors at each time $n$ have $O(K(n))$ time and space complexities as well. For the purpose of implementing an online algorithm, we are typically interested in the case $K(n) = O(\log(n))$.

\textbf{Remark 3.2.} Both mixtures of SR and CUSUM e-detectors can be viewed as special cases of their adaptive counterparts where the scheduling function $K$ is understood as a constant function. In this case, we have $\gamma_j = \sum_{k=1}^{K} \omega_k = 1$ for each $j$, and thus $M_n^{\text{aSR}} = M_n^{\text{SR}}$ and $M_n^{\text{aCS}} = M_n^{\text{CS}}$ for each $n \in \mathbb{N}$.

Unlike finite mixtures, adaptive SR and CUSUM e-detectors rely on scheduling function and adaptive re-weighting factor, which makes the mixing distribution to vary over time. Hence, we cannot simply apply Proposition 2.3 to check whether this adaptive scheme yields valid e-detectors. The following proposition formally states the validity of adaptive SR and CUSUM e-detectors. For simple presentation, we defer the proof to Appendix A.2.

\textbf{Proposition 3.3.} For any mixing weights \{\omega_k\}_{k \in \mathbb{N}} and a scheduling function $K$, adaptive SR and CUSUM e-detectors defined in (28) and (29) form valid e-detectors satisfying the condition (6).

Now, based on $M_n^{\text{aCS}}$ and $M_n^{\text{aSR}}$, the adaptive e-SR and e-CUSUM procedures are defined by the following stopping times:

$$N_n^{\text{aSR}} := \inf \left\{ n \geq 1 : M_n^{\text{aSR}} \geq 1/\alpha \right\},$$

(32)

$$N_n^{\text{aCS}} := \inf \left\{ n \geq 1 : M_n^{\text{aCS}} \geq c_\alpha \right\},$$

(33)

where $\alpha \in (0, 1)$ is a fixed constant and $c_\alpha$ is a positive value controlling ARL of the adaptive e-CUSUM procedure by $1/\alpha$. Similar to the usual e-CUSUM procedure case we discussed before, we can always set $c_\alpha = 1/\alpha$. In this case, from the fact $N_n^{\text{aSR}} \leq N_n^{\text{aCS}}$, which is implied by $M_n^{\text{aSR}} \geq M_n^{\text{aCS}}$, we have

$$\mathbb{E}_{P, \infty} N_n^{\text{aCS}} \geq \mathbb{E}_{P, \infty} N_n^{\text{aSR}} \geq 1/\alpha,$$

(34)

where the last inequality comes from Proposition 2.4 with the fact that $M_n^{\text{aSR}}$ is a valid e-detector. However, the threshold $c_\alpha$ for the adaptive e-CUSUM procedure can be chosen to be a significantly smaller value if we have enough knowledge about the pre-change distribution, as discussed in Section 2.5.
3.2. Worst average delay analysis for adaptive mixtures of e-detectors

We now derive general upper bounds on worst average delays for adaptive e-SR and e-CUSUM procedures. As we did before, we further assume that post-change observations $X_{v+1}, X_{v+2}, \ldots$ are independent to pre-change observations and form a strong stationary process. Also, we further assume that there exist a function $f_k$ and an integer $m \geq 0$ such that $L_n(k) = f_k(X_n, X_{n-1}, \ldots, X_{n-m})$ for each $k$ and $n$. Again, if $m$ is a strictly positive number then we implicitly assume that there exist $m$ observations $X_0, X_1, \ldots, X_{1-m}$ from the pre-change distribution we can use to build changepoint detection procedures. Under this additional condition for worst average delay analysis, the following theorem provides analytically more tractable upper bounds on worst average delays for $N_{aSR}^*$ and $N_{aCS}^*$.

**Theorem 3.4.** Under additional conditions described above, worst average delays for $N_{aSR}^*$ and $N_{aCS}^*$ can be upper bounded as follows:

\[
J_P(N_{aSR}^*) \leq J_L(N_{aSR}^*) \leq \min_{j > m} \left[ E_{0,Q} N_{1/\alpha}(j) + j - 1 \right],
\]

\[
J_P(N_{aCS}^*) \leq J_L(N_{aCS}^*) \leq \min_{j > m} \left[ E_{0,Q} N_{e_\alpha}(j) + j - 1 \right],
\]

where, for $j \in \mathbb{N}$ and $c > 0$, $N_{e}(j)$ is stopping times defined by

\[
N_{e}(j) := \inf \left\{ n \geq 1 : \sum_{k=1}^{K(j)} \omega_k \prod_{i=1}^{n} L_i(k) \geq c \right\}.
\]

Here, $c_\alpha$ is the same threshold used to build the adaptive e-CUSUM procedure in (33). Note that for mixtures of the SR and CUSUM e-detectors where the scheduling function $K$ is a constant function, the stopping times in the upper bounds do not depend on the index $j$, and thus the upper bounds can be reduced as

\[
J_P(N_{mSR}^*) \leq J_L(N_{mSR}^*) \leq E_{0,Q} N_{1/\alpha} + m,
\]

\[
J_P(N_{mcS}^*) \leq J_L(N_{mcS}^*) \leq E_{0,Q} N_{e_\alpha} + m,
\]

where, for $c > 0$, $N_{c}$ is the stopping time

\[
N_c := \inf \left\{ n \geq 1 : \sum_{k=1}^{K} \omega_k \prod_{i=1}^{n} L_i(k) \geq c \right\}.
\]

The proof of upper bounds on worst average delays can be found in Appendix A.2.

Unlike the baseline e-detector case, however, due to mixing weights, it is nontrivial to get further simplified upper bounds on worst average delays as we did in Section 2.6. Next, we present specific adaptive e-SR and e-CUSUM procedures based on exponential baseline increments where we can compute both procedures efficiently and derive upper bounds on worst average delays with explicit forms.

4. Exponential baseline e-detectors and their mixtures

Building upon recent advances in time uniform concentration inequalities and sequential testing developed in Howard et al. [2020], Shin et al. [2021], below we impose an exponential structure on baseline e-detectors and present how one can approximate oracle e-SR and e-CUSUM procedures based on the optimal $\lambda^*$ by ones based on a mixture of properly chosen set of baseline increments $\{L^k\}_{k \geq 1}$ with mixing weights $\{\omega_k\}_{k \geq 1}$. 
To be specific, let assume there exists an extended real-valued convex function $\psi$ on $\mathbb{R}$ that is finite and strictly convex on a supporting set $\Pi \subset \mathbb{R}$ which contains 0 in nonempty interior $\Pi^o$. Furthermore, we assume that $\psi$ is continuously differentiable on $\Pi^o$ with $\psi(0) = \nabla \psi(0) = 0$. Based on the function $\psi$, we define the “exponential baseline increment” as follows.

**Definition 4.1** (Exponential baseline increment). For each $n \in \mathbb{N}$ and $\lambda \in \Pi$, define

$$L_n^\lambda = \exp \{ \lambda s(X_n) - \psi(\lambda)v(X_n) \},$$

where $s$ is a real-valued function and $v$ is a positive function on the sample space. $L_n^\lambda = \{ L_n^\lambda \}_{n \geq 1}$ is called an exponential baseline increment if it satisfies condition (11) in Definition 2.7.

Above, $s$ and $v$ are mnemonics for sum and variance. For each $Q \in \mathcal{Q}$, define real numbers $\mu(Q), \sigma^2(Q)$ and $\Delta^*(Q)$ by

$$\mu(Q) := \mathbb{E}_{0,Q}s(X_1), \quad \sigma^2 := \mathbb{E}_{0,Q}v(X_1) \quad \text{and} \quad \Delta^*(Q) := \frac{\mu(Q)}{\sigma^2(Q)},$$

where we assume that all expectations are finite. The following proposition provides an explicit expression for $D(Q||P) := \max_{\lambda \in \Pi} \mathbb{E}_{0,Q} \log L_1^{(\lambda)}$ and a sufficient condition to have $D(Q||P) > 0$ when the underlying baseline increments have the form specified in (41).

**Proposition 4.2.** For a fixed $Q \in \mathcal{Q}$, suppose there exist $\lambda^* \in \Pi^o$ such that $\Delta^*(Q) = \nabla \psi(\lambda^*)$. Then we have

$$D(Q||P) = \mathbb{E}_{0,Q} \log L_1^{(\lambda^*)} = \psi^*(\Delta^*(Q)) \sigma^2(Q) \geq 0,$$

where $\psi^*$ is the convex conjugate of $\psi$. Thus, if $\Delta^*(Q) \neq 0$, we have $D(Q||P) > 0$.

The proof of Proposition 4.2 can be found in Appendix A.3. For the rest of the section, we assume

$$\{ \lambda \in \mathbb{R} : \Delta^*(Q) = \nabla \psi(\lambda), Q \in \mathcal{Q} \} \subset \Pi \quad \text{and} \quad \Delta^*(Q) \neq 0, \quad \forall Q \in \mathcal{Q}.$$

Then, Proposition 4.2 implies $D(Q||P) > 0$ for all $Q \in \mathcal{Q}$. Also, for ease of notation, we will drop the dependency of $Q$ from related parameters and simply write $\lambda^*, \mu, \sigma^2$ and $\Delta$.

The exponential structure of the baseline increment in (41) results in a simple form of $\lambda^*$ such that

$$\lambda^* = (\nabla \psi)^{-1}(\Delta^*) = \nabla \psi^*(\Delta^*),$$

where the second equality comes from the fact $\lambda = \nabla \psi^* \circ \nabla \psi(\lambda)$ for each $\lambda \in \Pi$. Although $\lambda^*$ still depends on the unknown post-change distribution $Q$ via $\Delta^*$, in many cases, we can find upper and lower bounds on $\Delta^*$. In this section, we explain how to use the knowledge of the range of $\Delta^*$ to build a mixture of exponential baseline e-detectors that has explicit upper bounds on worst average delays.

### 4.1. Well-separated pre- and post-change distributions

Suppose we have knowledge of upper and lower bounds on the parameter $\Delta^*$ given in (42), i.e. we know a pair $(\Delta_L, \Delta_U)$ such that $\Delta_L < \Delta^* < \Delta_U$. It then follows that $\lambda_L < \lambda^* < \lambda_U$, where $\lambda_L = \nabla \psi^*(\Delta_L)$, $\lambda^* = \nabla \psi^*(\Delta^*)$ and $\lambda_U = \nabla \psi^*(\Delta_U)$. To simplify our presentation, we only consider the one-sided and well-separated case: $0 < \lambda_L < \lambda_U$.

Let $1/\alpha$ be the target level of the ARL control for a fixed $\alpha \in (0, 1)$. Let $\{ L(k) \}_{k \in [K]}$ and $\{ \omega_k \}_{k \in [K]}$ be $K$ exponential baseline increments and mixing weights whose specific values will be defined later in this subsection. Since each $L_n(k)$ is a function of the $n$-th observation $X_n$ for each $k \in [K]$, Theorem 3.4 implies that, if the post-change observations form a strong stationary process then the worst average delays
for mixtures of e-SR and e-CUSUM procedures, \( N_{\text{mSR}}^* \) and \( N_{\text{mCS}}^* \) can be upper bounded by \( \mathbb{E}_{0,Q}N_{1/\alpha} \) and \( \mathbb{E}_{0,Q}N_{c_\alpha} \), respectively, where \( N_c \) is the stopping time defined in (40). Furthermore, as we can always set the threshold for the e-CUSUM procedure to be \( c_{\alpha} \leq 1/\alpha \), we have \( \mathbb{E}_{0,Q}N_{c_\alpha} \leq \mathbb{E}_{0,Q}N_{1/\alpha} \). Therefore, in this subsection, we focus on constructing a set of baseline increments on which we can derive a tight upper bound on \( \mathbb{E}_{0,Q}N_{1/\alpha} \).

Algorithm 1 provides the details of our methodology for computing mixtures of e-SR procedures. The inputs to the algorithm are the upper and lower bounds \( \Delta_U \) and \( \Delta_L \) on \( \Delta^* \) and the maximal number of baselines processes \( K_{\text{max}} \). Mixture of e-CUSUM procedures can be executed similarly by replacing the update in Line 9 with the updating rule

\[
M_n^{CS}(k) \leftarrow \exp \{ \lambda_k s(X_n) - \psi_k v(X_n) \} \cdot \max \{ M_{n-1}^{CS}(k), 1 \}.
\]  

Also, for the mixture of e-CUSUM procedures, we can replace the threshold \( 1/\alpha \) with a smaller value \( c_{\alpha} \) if we have enough information about the pre-change distribution. For both e-SR and e-CUSUM procedures, at each time \( n \), updates of mixtures of e-detectors have \( O(K_{\alpha}) \) time and space complexities, which do not depend on \( n \).

Algorithm 1 relies critically on the function \texttt{computeBaseline} in Line 1, which returns a set of parameters and weights to compute a mixture of e-detectors along with a threshold value \( g_{\alpha} > 0 \) that will appear in the upper bound on worst average delays given in Theorem 4.3 that will be explained below. The details of the function \texttt{computeBaseline} are fairly technical and are given in Algorithm 3 in Appendix A.3.

In the main result of this section, we provide bounds on ARL and worst average delays for the mixtures of e-CP procedures obtained with Algorithm 1 that is a function of the parameter \( \lambda^* \) and the threshold value \( g_{\alpha} \). The proof can be found in Appendix A.3.

\textbf{Theorem 4.3.} Let \( N_{\text{mSR}}^* \) and \( N_{\text{mCS}}^* \) be the stopping times corresponding to the mixtures of e-SR and e-CUSUM procedures in Algorithm 1 and its variant, respectively. Then, both procedures control the ARL by \( 1/\alpha \). If we further assume that the post-change observations \( X_{\nu+1}, X_{\nu+2}, \ldots \) are i.i.d. samples from a post-change distribution \( Q \), then the worst average delays for \( N_{\text{mSR}}^* \) and \( N_{\text{mCS}}^* \) can be upper bounded as

\[
\max \{ J_L(N_{\text{mSR}}^*), J_L(N_{\text{mCS}}^*) \} \leq \frac{g_{\alpha}}{D(Q||P)} + \frac{\mathbb{V}_{0,Q} \log L_1^{(\lambda^*)}}{[D(Q||P)]^2} + 1. \tag{46}
\]

The same bound holds also for \( J_P(N_{\text{mSR}}^*) \) and \( J_P(N_{\text{mCS}}^*) \).

In Proposition A.3 in Appendix A.3, we show that if the number of baseline processes \( K_{\text{max}} \) in Algorithm 1 is chosen large enough then the quantity \( g_{\alpha} \) returned by the \texttt{computeBaseline} can be upper bounded by the expression

\[
\inf_{\eta > 1} \eta \left[ \log(1/\alpha) + \log \left( 1 + \frac{\log \psi^{(\Delta_U)}}{\log \psi^{(\Delta_L)}} \right) \right], \tag{47}
\]

which can be easily evaluated numerically. Expression (97) in Appendix A.3 provides a precise formula for how large \( K_{\text{max}} \) needs to be in order for the above bound to be in effect. In most practical cases, \( K_{\text{max}} = 1000 \) is a large enough choice. Also, in many canonical examples we will present later, if we choose large enough \( K_{\text{max}} \) satisfying the condition (97) then the first term \( \frac{g_{\alpha}}{D(Q||P)} \) of the upper bound of worst average delays in Theorem 4.3 become a leading term. In this case, from the inequality (47), we can check that this leading term is \( O \left( \log(1/\alpha)/D(Q||P) \right) \) as \( \alpha \to 0 \).
The bound on worst average delays in Theorem 4.3 is obtained by first defining the auxiliary stopping time
\[
\bar{N}_g := \inf \left\{ n \geq 1 : \sup_{\lambda \in (\lambda_L, \lambda_U)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g \right\}, \quad g > 1.
\] (48)

Using the same arguments as in the proof of Proposition 2.13, we immediately have that if the post-change observations are i.i.d. from \( Q \), then for any \( g > 1 \),
\[
\mathbb{E}_{0, Q} \bar{N}_g \leq \frac{g}{D(Q || P)} + \mathbb{V}_{0, Q} \left[ \log L_1^{(\lambda^*)} \right] \frac{1}{[D(Q || P)]^2} + 1.
\] (49)

The bound (46) is finally established by showing that the stopping time \( \bar{N}_{g_\alpha} \) obtained by using the threshold \( g_\alpha \) produced by Algorithm 3 is a deterministic upper bound to the stopping times \( N_{c_\alpha} \) and \( N_{1/\alpha} \) corresponding to mixtures of SR and CUSUM e-detectors. In detail, it holds that for any stream of observations \( X_1, X_2, \ldots, \)
\[
N_{c_\alpha} \leq N_{1/\alpha} \leq \bar{N}_{g_\alpha}.
\] (50)

This nontrivial result is formally stated in Lemma A.2 in Appendix A.3. Its proof leverages geometric arguments used in Shin et al. [2021, Theorem 2] to analyze the properties of sequential generalized likelihood ratio tests.

\begin{algorithm}
\caption{Pseudo-code of the mixture of e-SR procedures}
\begin{algorithmic}[1]
\Require ARL parameter \( \alpha \in (0, 1) \), Boundary values \( 0 < \Delta_L < \Delta_U \), Maximum number of baselines \( K_{\text{max}} \in \mathbb{N} \).
\Ensure Stopping time \( N^*_{\text{mSR}} \) of the mixture of e-SR procedures.
\Data Data stream \( X_1, X_2, \ldots \) (observed sequentially)
\For {\( k = 0, 1, \ldots, K_\alpha \)}
\State \( M^0_{\text{SR}}(k) \leftarrow 0 \), \( \psi_k \leftarrow \psi(\lambda_k) \)
\EndFor
\State \( M^0_{\text{mSR}} \leftarrow 0 \), \( n \leftarrow 0 \)
\While {\( M^\text{mSR}_n < 1/\alpha \)}
\State \( n \leftarrow n + 1 \)
\State Observe \( X_n \)
\For {\( k = 0, 1, \ldots, K_\alpha \)}
\State \( M^\text{SR}_n(k) \leftarrow \exp \{ \lambda_k s(X_n) - \psi_k v(X_n) \} \cdot [M^\text{SR}_{n-1}(k) + 1] \)
\State \( M^\text{mSR}_n \leftarrow \sum_{k=0}^{K_\alpha} \omega_k M^\text{SR}_n(k) \)
\EndFor
\State \( N^*_{\text{mSR}} \leftarrow n \)
\EndWhile
\State \Return The stopped time \( N^*_{\text{mSR}} \)
\end{algorithmic}
\end{algorithm}

4.2. Non-separated pre- and post-change distributions

The previous subsection discussed how to build mixtures of e-SR and e-CUSUM procedures with an explicit upper bound on worst average delays when we have known and positive boundary values, \( \lambda_L \) and \( \lambda_U \) on the unknown \( \lambda^* \) via the knowledge of \( \Delta_L < \Delta^* < \Delta_U \). However, in many cases, we may not be fully certain about the boundary values. In this subsection, we discuss how we can generalize the previous argument to the no separation case whereby we only know the sign of \( \lambda^* (> 0) \) but do not have specific boundary values.
Recall that, for the well-separated case, we calibrated the mixtures of finitely many exponential baseline e-detectors using the stopping time $\bar{N}_{g_\alpha}$ in (48), which is in turn based on the maximum of underlying baseline increments over the known upper and lower bounds of $\lambda^*$. Since we no longer have knowledge of the boundary values $\lambda_L$ and $\lambda_U$, we may use similar stopping times where the range of maximum and the threshold slowly increase over time. In this case, we need an infinite sequence of baseline procedures $\{L(k)\}_{k \in \mathbb{N}}$ and mixing weights $\{\omega_k\}_{k \in \mathbb{N}}$ to build adaptive e-SR and e-CUSUM procedures.

The bound in Theorem 3.4 along with the fact $\gamma_j \geq 1$ for all $j \in \mathbb{N}$ implies that, for any given scheduling function $K : \mathbb{N} \to \mathbb{N}$, if the post-change observations form a strong stationary process then worst average delays for adaptive e-SR and e-CUSUM procedures can be upper bounded by $\min_{j \geq 1} \left[ \mathbb{E}_{0,Q} N_{1/\alpha}(j) + j - 1 \right]$ and $\min_{j \geq 1} \left[ \mathbb{E}_{0,Q} N_{c_\alpha}(j) + j - 1 \right]$, respectively, where we recall that $N_c(j)$ is defined for $c > 0$ by

$$N_c(j) := \inf \left\{ n \geq 1 : \sum_{k=1}^{K(j)} \omega_k \prod_{i=1}^{n} L_i(k) \geq c \right\}. \quad (51)$$

Again, since we can set the threshold for the e-CUSUM procedure in such a manner that $c_\alpha \leq 1/\alpha$ (so that $\mathbb{E}_{0,Q} N_{c_\alpha}(j) \leq \mathbb{E}_{0,Q} N_{1/\alpha}(j)$), in this subsection, we focus on constructing a set of baseline increments on which we can derive a tight upper bound on $\min_{j \geq 1} \left[ \mathbb{E}_{0,Q} N_{1/\alpha}(j) + j - 1 \right]$.

To derive the set of baseline increments, we use a time-varying boundary function $g$. Here, we intentionally overload notation: the constant $g$ in the previous subsection for the well-separation case can be viewed as a constant function $g$ in what follows. Let $g : [1, \infty) \to [0, \infty)$ be a nonnegative and nondecreasing continuous function such that the mapping $t \mapsto g(t)/t$ is nonincreasing and $\lim_{t \to \infty} g(t)/t = 0$. For a chosen positive number $\Delta_0 > 0$, let

$$D_0 := \psi^*(\Delta_0) \quad \text{and} \quad V_0 := \inf \{ t \geq 1 : D_0 \geq g(t)/t \}. \quad \text{(52)}$$

Now, for any fixed $\eta > 1$ and $j \in \mathbb{N}$, define $\Delta_1 > \Delta_2 > \cdots$ as positive solutions of the equations

$$\psi^*(\Delta_k) = \frac{g(V_0 \eta^k)}{V_0 \eta^k}, \quad k = 0, 1, 2, \ldots. \quad (53)$$

Finally, based on the sequence $\{\Delta_k\}_{k \geq 0}$, define

$$\lambda_k := \nabla \psi^*(\Delta_k),$$

and set $\omega_0 := \alpha^{-1} e^{-g(V_0)} \mathbb{1}(g(V_0) > v_{\min} D_0), \omega_k := \alpha^{-1} e^{-g(V_0 \eta^k)/\eta}$ for each $k \in \mathbb{N}$ where $v_{\min} := \min_x v(x)$, recalling the function $v$ from Definition 4.1.

Based on the quantities defined above, we can construct the stopping time $N_{1/\alpha}(j)$ for each $j$. The following lemma shows that we can upper bound the stopping time $\bar{N}_{g}(j)$ with another stopping time $\bar{N}_{g}(j)$ from which we can derive an explicit upper bound on its expected stopping time.

**Lemma 4.4.** For any fixed $j \geq 1$, $\Delta_0 > 0$, and tuning parameter $\eta > 1$, let $N_{1/\alpha}(j)$ be the stopping time based on the parameters defined above. Then, we have

$$N_{1/\alpha}(j) \leq \bar{N}_{g}(j), \quad \text{(54)}$$

where $\bar{N}_{g}(j)$ is a stopping time defined by

$$\bar{N}_{g}(j) := \inf \left\{ n \geq 1 : \sup_{\lambda \in (\lambda_K(j), \lambda_0)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g(V_0 \eta^{K(j)}) \right\}. \quad (55)$$
Note that the chosen set of weights \( \{ \omega_k \}_{k \geq 0} \) yields valid adaptive e-SR and e-CUSUM procedures if
\[
e^{-g(V_0)} \mathbb{1}(g(V_0) > v_{\min}D_0) + \sum_{k=1}^{\infty} e^{-g(V_0^k)/\eta} \leq \alpha.
\] (55)

Once the above condition is satisfied, we can use the worst average delay analysis in Section 3.2 with the bound in Lemma 4.4 to get an explicit upper bound on the worst average delay of the adaptive e-SR and e-CUSUM procedures.

In detail, let \( j^* \) be the smallest integer satisfying \( \lambda_{K(j^*)} < \lambda^* \) and set \( K^* := K(j^*) \). If we also have \( \lambda^* < \lambda_0 \) then Lemma 4.4 implies
\[
\mathbb{E}_{0,Q} N_1/\alpha(j^*) \leq \mathbb{E}_{0,Q} \bar{N}_g(j^*) \leq \mathbb{E}_{0,Q} N_\star,
\] (56)
where the stopping time \( N_\star \) is defined by
\[
N_\star := \inf \left\{ n \geq 1 : \sum_{i=1}^{n} \log L_i^{(\lambda^*)} \geq g \left( V_0 \eta^{K^*} \right) \right\},
\] (57)
and the expectation \( \mathbb{E}_{0,Q} N_\star \) is typically on the order of \( g \left( V_0 \eta^{K^*} \right) / D(Q\|P) \). Based on this observation, in the rest of this subsection, we introduce a practical and interpretable way to choose a boundary function \( g \) and related tuning parameters which minimize the leading term \( g \left( V_0 \eta^{K^*} \right) \) while satisfying the condition (55) on the set of mixing weights.

First note that, although we have no bounds on \( \Delta^\star \) in the no separation case, we can still choose \( \Delta_L \) and \( \Delta_0 \) with \( \Delta_L < \Delta_0 \) as tuning parameters that represent our initial guess on the range of the unknown \( \Delta^\star \). Since it is possible that the unknown parameter \( \Delta^\star \) of the post-change distribution is outside of the boundary \( (\Delta_L, \Delta_0) \), instead of assigning the entire \( \alpha \) to the inside of the guessed interval, we split it into two parts by \( r\alpha \) and \( (1-r)\alpha \), respectively where \( r \in (0, 1) \) is another tuning parameter called the importance weight. Roughly speaking, larger \( r \) implies we make a higher bet on that the unknown \( \Delta^\star \) is inside of our chosen boundaries \( (\Delta_L, \Delta_0) \).

Now, given tuning parameters \( \Delta_L, \Delta_0 \) and \( r \), we compute the set of \( \{ g_{r\alpha}, K_L, \eta \} \) by executing the function computeBaseline, just like in Algorithm 1, except that \( \alpha \) is replaced replaced by \( r\alpha \). Then, we can extend the boundary function \( g \) to accommodate the case in which the unknown \( \Delta^\star \) is not inside the initial interval we had guessed. To be specific, we use the boundary function
\[
t \in [1, \infty) \mapsto g(t) := g_{r\alpha} + s\eta \log \left( 1 + \log \eta \left( \frac{t}{V_0 \eta^{K_L} \lor 1} \right) \right),
\] (58)
where \( V_0 := g_{r\alpha}/D_0 \) and \( s > 1 \) is a constant obtained as the solution of the equation
\[
\zeta(s) - 1 := \sum_{k=1}^{\infty} \frac{1}{(1+k)^s} = e^{g_{r\alpha}/\eta} \left[ \alpha - \left\{ e^{-g_{r\alpha}} \mathbb{1}(g_{r\alpha} > D_0) + K_L e^{-g_{r\alpha}/\eta} \right\} \right].
\] (59)
Note that the right hand side of the above equation is approximately equal to \( (1-r)\alpha e^{g_{r\alpha}/\eta} \). Therefore,
\[
s \approx \zeta^{-1} \left( 1 + \left[ 1 - r \right] \alpha e^{g_{r\alpha}/\eta} \right).
\] (60)

In Algorithm 2, we provide the detailed steps for the adaptive e-SR procedure based on the boundary function in (58). The algorithm can be easily modified for the adaptive e-CUSUM procedure by replacing the update in Line 14 with the rule
\[
M_{n}^{CS}(k) := \exp \left\{ \lambda_k s(X_n) - \psi_k v(X_n) \right\} \cdot \max \left\{ M_{n-1}^{CS}(k), \gamma \right\}.
\] (61)
and its e-CUSUM variant, respectively. Then, both procedures control ARL by

\[ m \geq 1, \text{Number of baselines for the well-separated regime } K_0 \in \mathbb{N}. \]

**Output:** Stopping time \( N_{\text{aSR}}^* \) of the adaptive e-SR procedure.

**Data:** Data stream \( X_1, X_2, \ldots \) (observed sequentially).

1. Obtain \( \{\lambda_0, \lambda_1, \ldots, \lambda_{K_L}\}, \{\omega_0, \omega_1, \ldots, \omega_{K_L}\}, \{g_{\alpha}, K_L, \eta, W\} \) by executing `computeBaseline(\( r\alpha, \Delta_L, \Delta_0, K_0 \))` in Algorithm 3.
2. \( s \leftarrow \zeta^{-1}(1 + [\alpha - W] e^{g_{\alpha}/\eta}) \) /* \( \zeta(s) \approx 1 + [1 - r] e^{g_{\alpha}/\eta} \) */
3. \( M_0^{\text{SR}}(k) \leftarrow 0, \psi_k \leftarrow \psi(\lambda_k), \omega_k \leftarrow \omega_k \frac{W}{\alpha}, \forall k = 0, 1, \ldots, K_L \)
4. \( M_0^{\text{aSR}} \leftarrow 0, \gamma \leftarrow 1/\sum_{k=0}^{K_L} \omega_k, n \leftarrow 0 \)
5. **while** \( M_n^{\text{aSR}} < 1/\alpha \) **do**
6. \( n \leftarrow n + 1 \)
7. /* Occasionally add a new baseline increment. */
8. \( K_n \leftarrow K_L + \lfloor m \log_\eta n \rfloor \)
9. **if** \( K_n > K_{n-1} \) **then**
10. **for** \( k = K_{n-1} + 1, \ldots, K_n \) **do**
11. Compute \( \Delta_k \) as the solution of \( \psi^*(z) = \frac{g_k}{\gamma V_0} \) with respect to \( z \geq 0 \), where
12. \( V_0 := g_{\alpha}/D_0 \) and \( g_k := g_{\alpha} + s\eta \log(1 + k - K_L) \).
13. \( M_{n-1}^{\text{SR}}(\mu_k) \leftarrow 0, \psi_k \leftarrow \psi(\lambda_k), \omega_k \leftarrow \alpha^{-1} e^{-g_k/\eta} \)
14. \( \gamma \leftarrow \left(1/\gamma + \sum_{k=K_{n-1}}^{K_n} \omega_k\right)^{-1} \)
15. Observe \( X_n \)
16. \( M_n^{\text{SR}}(k) \leftarrow \exp\{\lambda_k s(X_n) - \psi_k v(X_n)\} \cdot [M_{n-1}^{\text{SR}}(k) + \gamma], \forall k = 0, 1, \ldots, K_n \)
17. \( M_n^{\text{aSR}} \leftarrow \sum_{k=0}^{K_n} \omega_k M_n^{\text{SR}}(k) \)
18. \( N_{\text{aSR}}^* \leftarrow n \)
19. **return** The stopped time \( N_{\text{aSR}}^* \)

Also, for the adaptive e-CUSUM procedure, we can replace the threshold \( 1/\alpha \) with a smaller value \( c_\alpha \) if we have enough information about the pre-change distribution.

In term of computational complexity, in Algorithm 2, we set the scheduling function \( K : \mathbb{N} \rightarrow \mathbb{N} \) as

\[ K(n) := K_L + \lfloor m \log_\eta n \rfloor, \quad (62) \]

where \( m \geq 1 \) is a tuning parameter. Therefore, for both adaptive e-SR and e-CUSUM procedures, updates of statistics have \( O(m \log_\eta n) \) time and space complexities at each time \( n \). Although it is not a fully online algorithm, logarithm time and space complexities make it feasible to run adaptive e-SR and e-CUSUM procedures in most practical online settings.

From Section 3.1, we know that both procedures control the ARL by \( 1/\alpha \). The following theorem introduces explicit upper bound on the worst average delays for both procedures.

**Corollary 4.5.** Let \( N_{\text{aSR}}^* \) and \( N_{\text{aCS}}^* \) be stopping times corresponding to the adaptive e-SR procedures in Algorithm 2 and its and e-CUSUM variant, respectively. Then, both procedures control ARL by \( 1/\alpha \). If we further assume that post-change observations \( X_{\nu+1}, X_{\nu+2}, \ldots \) are i.i.d. samples from a post-change
distribution then the worst average delays for \( N_{aSR}^* \) and \( N_{aCS}^* \) can be upper bounded as

\[
\max \{ J_L(N_{aSR}^*), J_L(N_{aCS}^*) \} \\
\leq \left\{ \begin{array}{ll}
g_{\alpha} g_{\alpha} \left[ \log L_{\alpha}(\Delta^*) + \frac{\nu_0, r}{D(\eta)} \right] \left[ \frac{\psi^*(\Delta^*)}{\psi^*(\Delta_0)} \right]^2 + 1 & \text{if } \Delta^* \geq \Delta_0 \\
g_{\alpha} g_{\alpha} + s \eta \log(1 + K^* - K_L) + \frac{\nu_0, r}{D(\eta)} \left[ \frac{\psi^*(\Delta)}{\psi^*(\Delta^*)} \right] + \left[ \left( \frac{\psi^*(\Delta)}{\psi^*(\Delta^*)} \right)^2 \right]^{1/m} & \text{if } \Delta^* \leq \Delta_L \\
\end{array} \right.
\]

(63)

Note that \( \eta, s > 1 \) and \( r \in (0, 1) \) do not depend on the unknown \( \Delta^* \).

5. Examples

5.1. Bernoulli random variables with dependent, time-varying means

Winning rates of the Cavaliers. To illustrate how changepoint detection procedures based on e-detectors work, we revisit the example of the Cleveland Cavaliers, an American professional basketball team introduced in Section 1.2. Instead of using Plus-Minus stats, in this example, we are monitoring the performance of the Cavaliers by keeping track of wins and losses over all the games. Let \( X_1, X_2, \ldots \in \{0, 1\} \) be the sequence of win indicators during 2010-11 to 2017-18 regular seasons, where \( X_i = 1 \) if the Cavaliers won game \( i \). Though Figure 2 presents monthly and seasonal averages for the purpose of visualization, we use the underlying binary sequence to build a changepoint detection procedure.

Modeling winning probabilities as a dependent sequence of Bernoullis. To detect a significant improvement of the performance of the Cavaliers, we assume that before an unknown changepoint \( \nu \in \mathbb{N} \cup \{\infty\} \), the conditional average of winning probability given the sample history is less than or equal to \( p_0 := 0.49 \). That is, under any pre-change distribution \( P \) we have \( p_n := \mathbb{E}_P[X_n | \mathcal{F}_{n-1}] \leq p_0 \). (For simplicity, \( \mathcal{F} \) is taken to be the natural filtration of the data.) Thus, the pre-change class of distributions is

\[
\mathcal{P} := \{(p_1, p_2, \ldots) : p_i \leq p_0, \forall i \geq 1\},
\]

where we parameterize each distribution \( P \) over binary sequences by the sequence of conditional probabilities.

Our objective is to build mixtures of e-SR and e-CUSUM procedures tuned to quickly detect any significantly improved win rate larger than \( q_0 := 0.51 \) after the changepoint. This can be modeled by assuming that after some changepoint \( \nu \), the distribution \( Q \) is such that \( \mathbb{E}_{P_{\nu}, Q}[X_n | \mathcal{F}_{n-1}, n > \nu] := q_n \geq q_0 \). Thus, we may think of the post-change class of distributions as being

\[
Q := \{(q_1, q_2, \ldots) : q_i \geq q_0, \forall i \geq 1\}.
\]

In particular, this formalization allows for the winning probabilities to fluctuate over time before and after the changepoint (accounting for factors like form, injuries, etc.).

Deriving exponential baseline processes. For each \( \lambda > 0 \), define a baseline increment process \( L^{(\lambda)} := \{L_n^{(\lambda)}\}_{n \geq 1} \) as

\[
L_n^{(\lambda)} := \exp \{\lambda (X_n - p_0) - \psi_B(\lambda)\}, \tag{64}
\]

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where $\psi_B(\lambda) := \log \left(1 - p_0 + p_0 e^\lambda\right) - \lambda p_0$ is the Bernoulli cumulant generating function. Note that each $L(\lambda)$ is a valid baseline increment as it satisfies the inequality (11) in Definition 2.7. That is, under any pre-change distribution $P$, we have

$$E_{P,\infty} \left[ L_n(\lambda) \mid F_{n-1} \right] = E_{P,\infty} \left[ \exp \left\{ \lambda X_n - \log \left(1 - p_0 + p_0 e^\lambda\right) \right\} \mid F_{n-1} \right]$$

$$= E_{P,\infty} \left[ \frac{e^{\lambda X_n}}{1 - p_0 + p_0 e^\lambda} \right] = \frac{1 - p_0 + p_0 e^\lambda}{1 - p_0 + p_0 e^\lambda} \leq 1, \forall \lambda \geq 0.$$

To derive exponential baseline processes, we first consider a simplified post-change distribution $Q$ where each post-change observation is identically distributed with $E_{0,Q} \left[ X_1 \right] := q \geq q_0 > p_0$. In this case, the optimal choice of $\lambda \geq 0$ given by

$$\lambda^* := \underset{\lambda \geq 0}{\arg \max} E_{0,Q} \exp \left\{ \lambda \left( X_1 - p_0 \right) - \psi_B(\lambda) \right\}. \quad (65)$$

Since the baseline increment has the exponential structure, by Proposition 4.2, we have that

$$D(Q||P) := E_{0,Q} \log L_1(\lambda^*) = \psi_B(q - p_0) = KL(q||p_0), \quad (66)$$

where $KL(q||p_0)$ is the Kullback-Leibler (KL) divergence of Bernoulli distributions with parameters $q$ and $p_0$ written as

$$KL(q||p_0) := q \log \frac{q}{p_0} + (1 - q) \log \frac{1 - q}{1 - p_0}. \quad (67)$$

for $q,p_0 \in (0,1)$. The appearance of the KL divergence in (66) is not a coincidence as the baseline increment can be viewed as a re-parametrized likelihood ratios between two Bernoulli processes. However, the simple geometric structure of the baseline increment make it possible to utilize a prior knowledge about the post-change distribution via Algorithm 1 and 2.

For instance, suppose we know upper and lower bounds of conditional means of the post-change distribution as $q_n \in (q_L, q_U), \forall n > \nu$. Let $N^*_nSR$ and $N^*_nCS$ be stopping times of mixtures of e-SR and e-CUSUM procedures in Algorithm 1. In this case, derived changepoint detection procedures do not rely on a specific choice of a post-change distribution $Q \in Q$. However, these procedures can still perform almost as well as the one optimized to a specific choice of the post-change distribution within the same range $(q_L, q_U)$. Typically, if the post-change observations are i.i.d. samples from a post-change distribution $Q$ with $E_{0,Q} \left[ X_1 \right] := q \in (q_L, q_U)$, then, by Theorem 4.3, the worst average delays have the following explicit upper bound:

$$\max \left\{ \mathcal{J}_L(N^*_nSR), \mathcal{J}_L(N^*_nCS) \right\} \leq \frac{q_0}{KL(q||p_0)} + \frac{q(1 - q)}{KL(q||p_0)^2} \left[ \log \left( \frac{1 - p_0}{p_0} \frac{q}{1 - q} \right) \right]^2 + 1.$$

Typically for small $\alpha \ll 1$, from Proposition A.3, we can simplify the above upper bound as

$$\max \left\{ \mathcal{J}_L(N^*_nSR), \mathcal{J}_L(N^*_nCS) \right\} \approx \inf_{\eta > 1} \eta \left[ \log \left( \frac{1}{\alpha} \right) + \log \left( 1 + \left[ \log_{KL(q||p_0)} \left( \frac{KL(q||p_0)}{KL(q||p_0)} \right) \right] \right) \right],$$

which matches the rate of the worst average delays, $O \left( \log \left( \frac{1}{\alpha} / KL(q||p_0) \right) \right)$ of the oracle changepoint detection procedure as $\alpha \to 0$. 

22
Monthly win rates with seasonal averages

| Date | win rate |
|------|----------|
| 2012 | 0.0      |
| 2014 | 0.2      |
| 2016 | 0.4      |
| 2018 | 0.6      |

CP detected at 2015–02–26 (SR) / 2015–04–05 (CUSUM)

Figure 2: Left: Monthly win rates of the Cavaliers from 2010-11 to 2017-18 seasons (the raw data is Bernoulli, which is harder to visualize). Each red line corresponds to the seasonal average. Right: Paths of log e-detectors (SR: red; CUSUM: green). The horizontal line is the threshold (common to both procedures) equal to \( \log(1/\alpha) \), ensuring that the ARL is at least \( 1/\alpha = 10^3 \), larger than the number of games in 12 seasons (82 per season). The e-SR procedure detects a changepoint during the 2014-15 season.

Implementation of Algorithm 1 and its results. The lower bound \( q_L \) can be chosen as \( q_0 = 0.51 \) since it is the minimum winning rate we consider as a significant improvement from before the changepoint, when the rates are upper bounded by \( p_0 = 0.49 \). We can also safely assume that the win rate cannot be too high given the competitiveness of the NBA, so that the improved win rates cannot be larger than 0.9. In our framework, these considerations can be encoded by setting \( \Delta_L := q_0 - p_0 = 0.02 \) and \( \Delta_U := 0.41 \) as input parameters of Algorithm 1. As in Section 1.2, we set \( \alpha := 10^{-3} \) to ensure that the ARL is at least \( 1/\alpha := 10^3 \), which is more than the total number of games over 12 years of regular seasons. Finally, we set the maximum number of baselines \( K_{\text{max}} := 1000 \). In fact, the computeBaseline function of Algorithm 3 returns only 69 baseline processes, and thus the resulting mixtures of e-SR and e-CUSUM procedures of Algorithm 1 can be computed efficiently in an online fashion.

The right plot in Figure 2 presents the log e-detector values using mixtures of e-SR (red) and e-CUSUM (green) procedures. Although there were a few months in which monthly win rates were higher than \( p_0 \), overall log e-detectors remained at a stable level over the first four seasons. However, after the 2014-15 season starts, the log e-detectors increase rapidly and both procedures detect a changepoint during the 2014-15 season, which is the first season that marked the return of LeBron James to the Cavaliers.

5.2. Mean-shift detection in general bounded random variables

Plus-Minus of the Cavaliers revisited. We return to the Cavaliers 2011-2018 example from Section 1.2. Let \( \tilde{X}_1, \tilde{X}_2, \ldots \) be the sequence of Plus-Minus stats from each game. We assume that the average Plus-Minus of the team is less than or equal to \( \mu_< := -1 \) before the changepoint (if any), while after the changepoint it is greater than \( \mu_> := 1 \). Here, the gap \( |\mu_> - \mu_<| \) between averages of Plus-Minus in pre- and post-changes refers to the degree of improvement we consider as significant.

For convenience, we first normalize the observed sequence. We assume that the absolute value of each Plus-Minus is bounded by 80, meaning that no team beats another by over 80 points (such an extreme game has never happened in NBA history). Accordingly, define the normalized Plus-Minus, \( X_n := (\tilde{X}_n + 80)/160 \in [0,1] \) for each \( n \). Then, the pre-change observations have conditional mean at most \( m := \)
\[ (\mu_\text< + 80)/160 = 0.494 \text{ and the minimum gap to detect is equal to } \delta := |\mu_\text> - \mu_\text<|/160 = 0.0125. \]

**Modeling plus-minus stats as a bounded sequence with time-varying, dependent means.** After the normalization above, the Plus-Minus stats form sequence of bounded random variables \( X_1, X_2, \ldots \) on \([0, 1]\). Each observation may have different distribution (due to seasonal effects, injuries, form, etc.), but we assume that all observations before an unknown changepoint \( \nu \) have a mean less than or equal to a known boundary \( m \in (0, 1) \), when conditioned on the past sample history. That is, under any pre-change distribution \( P \), we have \( \mu_n := \mathbb{E}_{P,\infty} [X_n \mid F_{n-1}] \leq m, \ \forall n \geq 1 \). In other words, we use

\[ \mathcal{P} := \{ P : \mu_n \leq m, \forall n \geq 1 \}, \]

where other characteristics about \( P \) (outside of its sequence of conditional means) are irrelevant. But after the changepoint, all observations have (conditional) mean larger than the boundary \( m \) with the minimum gap equal to \( \delta \). Thus,

\[ \mathcal{Q} := \{ P : \mu_n \geq m + \delta, \forall n \geq 1 \}, \]

To build an e-SR or e-CUSUM procedure, we need to choose a baseline increment. To derive it, we first consider a simplified setting where both pre- and post-change observations are independently and identically distributed with \( \mathbb{E}_{P,\infty} [X] \leq m \) and \( \mathbb{E}_{0,Q} [X] \geq m + \delta \), respectively. In this simplified case, we simply refer \( P \) and \( Q \) to marginal pre- and post-change distributions and \( \mathcal{P} \) and \( \mathcal{Q} \) to their collections. Then, define \( \text{KL}^\text{inf}(Q; m) := \inf_{P \in \mathcal{P}} \text{KL}(Q, P) \) to be the smallest KL divergence between \( Q \) and \( \mathcal{P} \). It is known (see, e.g., Honda and Takemura [2010, 2015]) that \( \text{KL}^\text{inf} \) has the following variational representation:

\[ \text{KL}^\text{inf}(Q, m) = \sup_{\lambda \in (0, 1)} \mathbb{E}_{0,Q} \log \left( 1 + \lambda \left( \frac{X}{m} - 1 \right) \right) =: D(Q||\mathcal{P}). \quad (68) \]

Accordingly, for each \( \lambda \in (0, 1) \), define the baseline increment \( L_\lambda := \{ L_n \}_{n \geq 1} \) as

\[ L_\lambda := 1 + \lambda \left( \frac{X_n}{m} - 1 \right), \quad (69) \]

for each \( n \in \mathbb{N} \). Though the baseline increment above has been derived in the simplified i.i.d. setting, it can be checked that \( L_\lambda \) is also a valid baseline increment for the general time-varying, dependent means case since it is nonnegative whenever \( X_n, m \in [0, 1] \) as assumed in our setup, and for each pre-change distribution \( P \in \mathcal{P} \), we have

\[ \mathbb{E}_{P,\infty} [L_\lambda \mid F_{n-1}] = 1 + \lambda \left( \mathbb{E}_{P,\infty} [X_n \mid F_{n-1}] - 1 \right) \leq 1, \]

where the inequality comes from the condition \( \mu_n \leq m \) for any pre-change distribution.

For any \( Q \in \mathcal{Q} \), let \( \lambda^* \) be the optimal choice of \( \lambda \in (0, 1) \) given by

\[ \lambda^* = \arg \max_{\lambda \in (0, 1)} \mathbb{E}_{0,Q} \log \left( 1 + \lambda \left( \frac{X}{m} - 1 \right) \right). \quad (70) \]

Unfortunately, it is typically difficult to compute the optimal \( \lambda^* \) since it depends on the unknown post-change distribution \( Q \) in a complicated way. In this case, we use a sub-exponential lower bound from Fan et al. [2015], Howard et al. [2021], given by

\[ \tilde{L}_n := \exp \left\{ \lambda \left( \frac{X_n}{m} - 1 \right) - \psi_E(\lambda) \left( \frac{X_n}{m} - 1 \right)^2 \right\} \leq 1 + \lambda \left( \frac{X_n}{m} - 1 \right) = L_\lambda, \quad (71) \]
where \( \psi_E(\lambda) := -\log(1 - \lambda) - \lambda \) for \( \lambda \in (0, 1) \). Note that for each \( \lambda \in (0, 1) \), the process \( \tilde{L}^\lambda \) is itself a valid exponential baseline increment with \( s(x) := x/m - 1 \) and \( v(x) := (x/m - 1)^2 \).

The lower bound in (71) also implies the lower bound on \( \text{KL}_{\text{inf}}(Q, m) \)

\[
\text{KL}_{\text{inf}}(Q, m) \geq \sup_{\lambda \in (0, 1)} \{ \lambda \mu - \psi_E(\lambda) \sigma^2 \} = \sigma^2 \psi^*_E(\Delta^*),
\]

where \( \psi^*_E(u) := u - \log(1 + u) \) is the convex conjugate of \( \psi_E \), while \( \mu, \sigma^2 \) and \( \Delta^* \) from Section 4 are given by

\[
\mu := \mathbb{E}_{0,Q} s(X) = \frac{\mathbb{E}_{0,Q} X - m}{m}, \\
\sigma^2 := \mathbb{E}_{0,Q} v(X) = \frac{\mathbb{E}_{0,Q} (X - m)^2}{m^2}, \\
\Delta^* := \frac{\mu}{\sigma^2} = \frac{m [\mathbb{E}_{0,Q} X - m]}{\mathbb{E}_{0,Q} (X - m)^2}.
\]

Noting that \( \psi^*_E(u) \approx u^2/2 \) for small \( u \), we see that for small \( \Delta^* \ll 1 \), one has

\[
\text{KL}_{\text{inf}}(Q, m) \gtrsim \frac{[\mathbb{E}_{0,Q} X - m]^2}{2\mathbb{E}_{0,Q} (X - m)^2}.
\]

Note that the oracle \( \Delta^* \) depends on the unknown post-change distribution only via first and second moments. Therefore, in contrast to the original set of baseline increments \( \{ L^\lambda \}_{\lambda \in (0,1)} \), the exponential baseline increments \( \{ \tilde{L}^\lambda \}_{\lambda \in (0,1)} \) that lower bound it, allow us to more easily set a range \( \langle \Delta_L, \Delta_U \rangle \) to build mixtures of e-SR and e-CUSUM procedures. For example, if we assume that the post-change distribution has mean at least \( m + \delta \) for a positive \( \delta \) then we can upper and lower bound \( \Delta^* \) by

\[
\Delta_L := \frac{m \delta}{(1 - m)^2} \leq \Delta^* \leq \frac{m (1 - m)}{\delta^2} =: \Delta_U.
\]

Now, given \( \Delta_L \) and \( \Delta_U \), we can use Algorithm 1 to run the mixture of e-SR or e-CUSUM procedure to detect the changepoint based on the exponential baseline baseline increments \( \{ \tilde{L}^\lambda \}_{\lambda \in (0,1)} \). It is also straightforward to build corresponding mixtures of e-SR and e-CUSUM procedures for the original baseline increment \( \{ L^\lambda \}_{\lambda \in (0,1)} \) which is always more sample-efficient.

**Implementation of Algorithm 1 and its Results.** Recall that in the plus-minus stats running example, we use pre-change mean \( m = 0.494 \) and the minimum gap \( \delta = 0.0125 \), which bounds \( \Delta^* \) by \( \Delta_L := 0.024 \) and \( \Delta_U := \frac{m (1 - m)}{\delta^2} \) = 1600. As before, we choose \( \alpha = 10^{-3} \) to make ARL larger than 12 regular seasons, and set the maximum number of baselines \( K_{\text{max}} = 1000 \). Based on these parameters, we can build mixtures of e-SR and e-CUSUM procedures. Though the difference between \( \Delta_L \) and \( \Delta_U \) may seem to be large, the actual number of baselines returned by the function \( \text{computeBaseline} \) in Algorithm 1 is 190, which is small enough to compute detection procedures efficiently on the fly.

Figure 3 shows e-detectors (left) and their logarithms (right). The horizontal line corresponds to the detection boundary given by \( 1/\alpha \) (left) and \( \log \alpha^{-1} \) (right). Similar to the winning rate example, the log e-detectors remained stable during the first four regular seasons, although the difference between SR and CUSUM e-detectors is larger than before. After 2014-15 season started, both e-detectors increased rapidly, and the e-SR procedure detected a changepoint during the 2014-15 season, but the e-CUSUM procedure detected the changepoint only in the following season (as expected, since both procedures use the same threshold).
6. Discussion

Summary. We have presented a general framework for changepoint detection based on a new concept called e-detectors. The proposed framework is nonparametric as it does not rely on a parametric assumption on the data-generating distribution (though, when such assumptions are made, we recover well-known parametric methods as special cases). Also, the framework comes with nonasymptotic guarantees, since every component of the framework can be chosen and analyzed explicitly without any asymptotic approximations. By introducing additional structures such as baseline increments and exponential e-detectors on the top of the general framework, we can construct computationally and statistically efficient online algorithms that have explicit upper bounds on worst average delays. Finally, through examples involving Bernoulli and bounded random variables, we explained how one can apply the presented framework in practical settings, with NBA data serving as a running case study.

Game-theoretic interpretation of an e-detector. We briefly mention here a game-theoretic interpretation of an e-detector along the lines of the game-theoretic interpretations of martingales and supermartingales as the wealth of a gambler playing a fair game (well known since the time of Ville [1939]). We first summarize the game-theoretic interpretation of a \( P \)-e-process, as described in Ramdas et al. [2021].

The standard game-theoretic setup of Shafer and Vovk [2019] involves three players: a forecaster, a skeptic, and reality. The forecaster claims at the start that \( P \) is a plausible model for the yet-to-be-observed data; meaning that the observations are in accordance with (or generated by) some \( P \in P \). The skeptic plays (in parallel) a family of games indexed by \( P \in P \) against nature, and begins with one dollar in each game. The skeptic’s objective in the \( P \)-th game is to sequentially test whether the \( P \) is a good explanation for the data by betting against \( P \). At each time step, the skeptic places fair bets (relative to \( P \), in the \( P \)-th game) about the next outcome. Then nature reveals the next outcome, and the skeptic’s wealth in every game is updated. The magnitude of the skeptic’s wealth in the \( P \)-th game is direct evidence against \( P \) being a good explanation; the larger the wealth, the more unlikely the data came from \( P \). Thus in each game, the gambler places different bets, but nature’s moves (the outcomes) are identical across all games. The skeptic’s overall
evidence against $\mathcal{P}$ is measured by their worst wealth across all the games. If this evidence exceeds $1/\alpha$, it means that the skeptic multiplied their initial capital by at least $1/\alpha$ in every game, and if we reject $\mathcal{P}$ when this happens, Ville’s inequality implies that we have a valid level-$\alpha$ sequential test.

Since our e-detectors are constructed to be cumulative sums of e-processes started at consecutive times, their game-theoretic interpretation builds on the aforementioned one. Informally, the forecaster not only claims that the data sequence follows $\mathcal{P}$ from the start, but that this will not change after some amount of time. The skeptic now wishes to detect a change, if one occurs, as soon as possible. To accomplish this task, the skeptic is provided with one extra dollar every day that they invest (using a $\mathcal{P}$ e-process) into testing whether the data from that day onwards is still explained well by $\mathcal{P}$. E-detectors use the wealth in all these games (one against each $P \in \mathcal{P}$, starting at each time) as a measure of evidence against the forecaster’s claims. The SR e-detector uses the sum (across time) of the minimum wealth (across $P$ at each time), though it could use the amount that this wealth exceeds $n$, which is the total dollar amount invested up to time $n$. The CUSUM e-detector uses the max-min wealth; the maximum (across time) of the minimum wealth (across $P$). These are, of course, only two ways of constructing e-detectors, and we leave other constructions to future work.

**Future directions.** There remain a whole host of follow-up directions; we mention only a few below. First, our online changepoint detection framework can be straightforwardly generalized to the multi-stream setting where we are monitoring a large number of data streams. In the classical parametric setting, minimum or summation of local CUSUM statistics for multi-stream data were proposed and their asymptotic optimality was studied [Hadjiliadis et al., 2009, Mei, 2010]. Since either minimum or scaled summation (average) of e-detectors also forms a valid e-detector, we can apply the framework in this paper to the multi-stream setting seamlessly. It is interesting to investigate how the framework can be even further generalized to structural multi-stream settings [Xie and Siegmund, 2013, Zou et al., 2019, Chen et al., 2020].

The kernel changepoint detection is an important class of changepoint detection methods [Desobry et al., 2005, Li et al., 2015]. Due to the higher-order dependency among samples, however, it is nontrivial to construct valid and efficient e-detectors based on kernel-based statistics. It is an interesting open direction how to generalize existing kernel-based methods into our general framework to make it possible to analyze kernel changepoint detection algorithms in a nonasymptotic way. As it currently stands, neither framework is more general than the other, because the kernel methods often assume i.i.d. data before the changepoint, while we abstain from such strong assumptions.

Last, throughout this paper, we have only focused on detecting whether a changepoint happened or not but have not dealt with inferential questions surrounding when the changepoint occurred. Future work could study how to perform such inference with e-detectors in our nonparametric settings, either online or post-hoc.

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A. Proofs

A.1. Proofs for statements in Section 2

Proof of Proposition 2.4 (ARL control). For any given $\alpha \in (0, 1)$ let $N^*$ be the stopping time defined by

$$N^* := \inf \{ n \geq 1 : M_n \geq 1/\alpha \}.$$  \hfill (75)

For any pre-change distribution $P \in \mathcal{P}$, we may assume $N^* < \infty$ with probability one under $P$, without loss of generality. (If not, we have $E_{P,\infty}N^* = \infty$, which immediately proves the claim.) Then, from the definition of an e-detector, we have

$$E_{P,\infty}[M_{N^*}] \leq E_{P,\infty}N^*,$$  \hfill (76)

which implies that, for any $P \in \mathcal{P}$,

$$E_{P,\infty}N^* \geq E_{P,\infty}[M_{N^*}]
= E_{P,\infty}[M_{N^*} \mathbb{1}(N^* < \infty)] \quad \text{(since $P_{P,\infty}(N^* < \infty) = 1$)}
\geq E_{P,\infty}\left[ \frac{1}{\alpha} \mathbb{1}(N^* < \infty) \right] \quad \text{(by the definition of $N^*$)}
= \frac{1}{\alpha} P_{P,\infty}(N^* < \infty) = \frac{1}{\alpha},$$

as desired. \hfill \Box

Proof of Proposition 2.13 (Upper bounds on worst average delays). We first prove (16). Note that

$$N^*_{SR} = \inf \left\{ n \geq 1 : \sum_{j=1}^{n} \prod_{i=j}^{n} L_i \geq 1/\alpha \right\}
\leq \min_{j \geq 1} \inf \left\{ n \geq j : \prod_{i=j}^{n} L_i \geq 1/\alpha \right\} := \min_{j \geq 1} N_j,$$
Since $N_j \parallel \mathcal{F}_\nu$ for all $j > \nu + m$, we have
\[
E_{P,\nu,\nu}[[N_{SR}^* - \nu]_+ | \mathcal{F}_\nu] \\
\leq \min_{j \geq 1} E_{P,\nu,\nu}[[N_j - \nu]_+ | \mathcal{F}_\nu] \\
\leq \min_{j > \nu + m} E_{P,\nu,\nu}[[N_j - \nu]_+ | \mathcal{F}_\nu] \quad \text{(Since $N_j \geq j > \nu + m$)} \\
= \min_{j > \nu + m} E_{P,\nu,\nu}[[N_j] - \nu \quad \text{(Since $N_j \parallel \mathcal{F}_\nu, \forall j > \nu + m$)}
\]
\[
= \min_{j > \nu + m} E_{P,\nu,\nu} \left[ \inf \left\{ n \geq j : \prod_{i=j}^n L_i \geq 1/\alpha \right\} \right] - \nu \\
= \min_{j - \nu > m} E_{0,\nu} \left[ \inf \left\{ n \geq j - \nu : \prod_{i=j-\nu}^n L_i \geq 1/\alpha \right\} \right]
\]
\[
= \min_{j' > m} E_{0,\nu} \left[ \inf \left\{ n \geq j' : \prod_{i=j'}^n L_i \geq 1/\alpha \right\} \right] \quad \text{(by setting $j' := j - \nu$)}
\]
\[
= \min_{j' > m} E_{0,\nu} N_{1/\alpha} + j' - 1 \\
= E_{0,\nu} N_{1/\alpha} + m,
\]
where the third equality comes from the fact that the distribution of $X_{j-m}, X_{j-m+1}, \ldots$ under $P_{P,\nu,\nu}$ is equal to the one of $X_{j-\nu-m}, X_{j-\nu-m+1}, \ldots$ under $P_{0,\nu}$ provided by $j - m > \nu$, and the fifth equality is based on the strong stationarity of post-change observations. Since the last term depends on neither $\nu$ nor $\mathcal{F}_\nu$, we have the claimed result:
\[
J_P(N_{SR}^*) \leq J_L(N_{SR}^*) = \sup_{\nu \geq 0} \text{esssup} E_{P,\nu,\nu}[[N_{SR}^* - \nu]_+ | \mathcal{F}_\nu] \leq E_{0,\nu} N_{1/\alpha} + m,
\]
as desired.

To prove (17), first note that
\[
N^*_{CS} = \inf \left\{ n \geq 1 : \max_{j \in [n]} \prod_{i=j}^n L_i \geq c_\alpha \right\}
\]
\[
\leq \min \inf_{j \geq 1} \left\{ n \geq j : \prod_{i=j}^n L_i \geq c_\alpha \right\}
\]
\[
:= \min_{j \geq 1} N_j.
\]
The remaining part of the proof for (17) is followed by the same argument for (16).

Finally, to prove (18), it is enough to show the following inequality holds:
\[
E_{0,\nu} N(g) \leq \frac{g}{E_{0,\nu} \log L_1} + \frac{\nu \log L_1}{E_{0,\nu} \log L_1} + 1, \quad \forall g > 0,
\]
where $N(g) := \inf \{ n \geq 1 : \sum_{i=1}^n \log L_i \geq g \}$ for each $g > 0$. The proof of the above upper bound (78) is based on the Lorden’s inequality Lorden [1970] which can be stated as follows:
Fact A.1 (Lorden’s inequality [Lorden (1970)]). Suppose $X_1, X_2, \ldots$ are i.i.d. samples with $\mathbb{E}X_1 = \mu > 0$ and $\mathbb{E}X_1^2 = \sigma^2 < \infty$. For each $g > 0$, set $N(g) := \inf \{ n : S_n := \sum_{i=1}^{n} X_i \geq g \}$ and $R_g := S_{N(g)} - g$. Then, the following inequality holds:

$$\sup_{g > 0} \mathbb{E}[R_g] \leq \frac{\mathbb{E}X_1^2}{\mu} = \mu + \frac{\sigma^2}{\mu}. \quad (79)$$

Now, to prove the upper bound (78), fix a constant $g > 0$. Since $\mathbb{E}_{0,Q} \log L_1 > 0$, we have $\mathbb{E}_{0,Q} N(g) < \infty$. Therefore, by Wald’s equation,

$$\mathbb{E}_{0,Q} \log L_1 \mathbb{E}_{0,Q} N(g) = \mathbb{E}_{0,Q} \left[ N(g) \sum_{i=1}^{\infty} \log L_i \right]. \quad (80)$$

For each $g > 0$, set $R_g := \sum_{i=1}^{N(g)} \log L_i - g$. Then, from the Lorden’s inequality, we have

\[
\begin{align*}
\mathbb{E}_{0,Q} \log L_1 \mathbb{E}_{0,Q} N(g) &= \mathbb{E}_{0,Q} \left[ N(g) \sum_{i=1}^{\infty} \log L_i \right] \\
&\leq c + \sup_{g > 0} \mathbb{E}_{0,Q}[R_g] \\
&\leq g + \frac{\mathbb{E}_{0,Q} \log L_1}{\mathbb{E}_{0,Q} \log L_1} + \mathbb{E}_{0,Q} \log L_1,
\end{align*}
\]

where the first inequality comes from the definition of $N(g)$. By multiplying $1/\mathbb{E}_{0,Q} \log L_1$ on both sides of the inequality (81), we have the claimed upper bound as desired. \qed

A.2. Proofs for statements in Section 3

Proof of Proposition 3.3 (Validity of adaptive SR and CUSUM e-detectors). To see adaptive SR and CUSUM e-detectors are actually valid e-detectors, first note that $M_{n}^{\text{aCS}} \leq M_{n}^{\text{aSR}}$ for each $n \in \mathbb{N}$. Therefore, it is enough to show that $\mathbb{P}_{P,\infty} M_{\tau}^{\text{aSR}} \leq \mathbb{P}_{P,\infty} \tau$ for any stopping time $\tau$ and pre-change distribution $P \in \mathcal{P}$. If $\mathbb{P}_{P,\infty}(\tau = \infty) > 0$ then the above inequality holds trivially. Otherwise if $\mathbb{P}_{P,\infty}(\tau = \infty) = 0$, we have that

\[
\begin{align*}
\mathbb{P}_{P,\infty} M_{\tau}^{\text{aSR}} &= \mathbb{P}_{P,\infty} \sum_{k=1}^{K(\tau)} \omega_k \sum_{j=K^{-1}(k)}^{\tau} \gamma_j \prod_{i=j}^{\tau} L_i(k) \\
&= \mathbb{P}_{P,\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{K(j)} \gamma_j \omega_k \Lambda_{\tau}^{(j)}(k) \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{K(j)} \gamma_j \omega_k \mathbb{E}_{P,\infty} \mathbb{I}(j \leq \tau) \mathbb{E}_{P,\infty} \left[ \Lambda_{\tau}^{(j)}(k) \mid F_{j-1} \right] \\
&\leq \sum_{j=1}^{\infty} \sum_{k=1}^{K(j)} \gamma_j \omega_k \mathbb{E}_{P,\infty} \mathbb{I}(j \leq \tau) = \sum_{j=1}^{\infty} \mathbb{E}_{P,\infty} \mathbb{I}(j \leq \tau) = \mathbb{P}_{P,\infty} \tau,
\end{align*}
\]

as desired. Above, the sole inequality follows since each $\Lambda_{\tau}^{(j)}(k)$ is an $e_j$-process, and the following equality invokes the definition of $\gamma_j$ for each $j$. \qed
Proof of Theorem 3.4 (Upper bounds on worst average delays for adaptive SR and CUSUM). We first prove the upper bound for the adaptive e-SR procedure in (35). Note that

\[ N_{aSR}^* = \inf \left\{ n \geq 1 : \sum_{k=1}^{K(n)} \omega_k \sum_{j=K^{-1}(k)+1}^{n} \gamma_j \prod_{i=j}^{n} L_i(k) \geq 1/\alpha \right\} \]

\[ \leq \inf \left\{ n \geq 1 : \sum_{k=1}^{K(n)} \omega_k \prod_{i=j}^{n} L_i(k) \geq 1/\alpha \right\} \] (since \( \gamma_j \geq 1 \) for each \( j \))

\[ = \inf \left\{ n \geq j : \sum_{k=1}^{K(j)} \omega_k \prod_{i=j}^{n} L_i(k) \geq 1/\alpha \right\} \]

\[ \leq \min_{j \geq 1} \inf \left\{ n \geq j : \sum_{k=1}^{K(j)} \omega_k \prod_{i=j}^{n} L_i(k) \geq 1/\alpha \right\} \]

\[ := \min_{j \geq 1} N_j. \]

Since \( N_j \parallel \mathcal{F}_\nu \) for all \( j > \nu + m \), we have

\[ \mathbb{E}_{P,\nu,Q} \left[ [N_{aSR}^* - \nu]_+ \mid \mathcal{F}_\nu \right] \]

\[ \leq \min_{j > \nu + m} \mathbb{E}_{P,\nu,Q} [N_j - \nu \mid \mathcal{F}_\nu] \] (Since \( N_j \geq j > \nu + m \))

\[ = \min_{j > \nu + m} \mathbb{E}_{P,\nu,Q} \left[ \inf \left\{ n \geq j : \sum_{k=1}^{K(j)} \omega_k \prod_{i=j}^{n} L_i(k) \geq 1/\alpha \right\} \right] - \nu \] (\( N_j \parallel \mathcal{F}_\nu, \forall j > \nu + m \))

\[ = \min_{j > \nu + m} \mathbb{E}_{0,Q} \left[ \inf \left\{ n \geq j - \nu : \sum_{k=1}^{K(j)} \omega_k \prod_{i=j-\nu}^{n} L_i(k) \geq 1/\alpha \right\} \right] \]

\[ = \min_{j' > m} \mathbb{E}_{0,Q} \left[ \inf \left\{ n \geq j' : \sum_{k=1}^{K(j'+\nu)} \omega_k \prod_{i=j'}^{n} L_i(k) \geq 1/\alpha \right\} \right] \] (by setting \( j' := j - \nu \))

\[ \leq \min_{j' > m} \mathbb{E}_{0,Q} \left[ \inf \left\{ n \geq j' : \omega_k \prod_{i=1}^{n} L_i(k) \geq 1/\alpha \right\} \right] + j' - 1 \]

\[ \leq \min_{j' > m} \mathbb{E}_{0,Q} N_{1/\alpha}(j') + j' - 1, \]

where the second equality comes from the fact that the distribution of \( X_{j-m}, X_{j-m+1}, \ldots \) under \( \mathbb{P}_{P,\nu,Q} \) is equal to the one of \( X_{j-\nu-m}, X_{j-\nu-m+1}, \ldots \) under \( \mathbb{P}_{0,Q} \) provided by \( j - m > \nu \) and the forth equality is based on the strong stationarity of post-change observations. Since the last term depends on \( \mathcal{F}_\nu \), we have the claimed result:

\[ \mathcal{J}_P(N_{aSR}^*) \leq \mathcal{J}_L(N_{aSR}^*) = \sup_{\nu \geq 0} \text{esssup} \mathbb{E}_{P,\nu,Q} \left[ [N_{aSR}^* - \nu]_+ \mid \mathcal{F}_\nu \right] \leq \min_{j > m} \left[ \mathbb{E}_{0,Q} N_{1/\alpha}(j) + j - 1 \right], \] (82)

as desired.
To prove the adaptive e-CUSUM procedure case in (36), first note that

\[ N_{aCS}^* = \inf \left\{ n \geq 1 : \sum_{k=1}^{K(n)} \omega_k \max_{K^{-1}(k) \leq j \leq n} \gamma_j \prod_{i=j}^{n} L_i(k) \geq c_\alpha \right\} \]

\[ \leq \inf \left\{ n \geq 1 : \max_{j \leq n} \sum_{k=1}^{K(j)} \omega_k \prod_{i=j}^{n} L_i(k) \geq c_\alpha \right\} \]

\[ \leq \min_{j \geq 1} \inf \left\{ n \geq j : \sum_{k=1}^{K(j)} \omega_k \prod_{i=j}^{n} L_i(k) \geq c_\alpha \right\} \]

:= \min_{j \geq 1} N_j,

where the first inequality comes from \( \gamma_j \geq 1 \) with the fact \( K^{-1}(k) \leq j \) for any \( k \leq K(j) \). The remaining part of the proof of (36) follows the same argument used to obtain (35).

\[ \square \]

### A.3. Proofs for statements in Section 4

**Proof of Proposition 4.2.** To simplify the notation, we drop the subscripts \( \{0, Q\} \) and \( 1 \). The claim reduces to proving the inequality

\[ \mathbb{E} \log L^{(\lambda^*)} = \psi^*(\Delta^*) \sigma^2, \]  

(83)

where \( \log L^{(\lambda^*)} := \lambda^* s(X) - \psi(\lambda^*) v(X) \) and \( \Delta^* = \nabla \psi(\lambda^*) \). To prove the quality, first note that from the definition of the convex conjugate \( \psi^* \) of \( \psi \), we have

\[ \psi^*(\Delta^*) := \sup_{\lambda \in \Pi} \left\{ \lambda \Delta^* - \psi(\lambda) \right\} = \lambda^* \Delta^* - \psi(\lambda^*), \]  

(84)

which implies

\[ \mathbb{E} \log L^{(\lambda^*)} = \lambda^* \mu - \psi(\lambda^*) \sigma^2 \]

\[ = \sigma^2 \left[ \lambda^* \Delta^* - \psi(\lambda^*) \right] \]

\[ = \sigma^2 \psi^*(\Delta^*), \]

as desired.

\[ \square \]

**Proof of Theorem 4.3.** We first recall the definition of the stopping time in (48):

\[ \tilde{N}_g := \inf \left\{ n \geq 1 : \sup_{\lambda \in (\lambda_L, \lambda_U)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g \right\}, \quad g > 0. \]  

(85)

Then, the same argument used in the proof Proposition 2.13 immediately implies that, if the post-change observations are i.i.d. from \( Q \), then, for any \( g > 0 \),

\[ \mathbb{E}_{0,Q} \tilde{N}_g \leq \frac{g}{D(Q||P)} + \frac{\mathbb{V}_{0,Q} \left[ \log L_1^{(\lambda^*)} \right]}{[D(Q||P)]^2} + 1. \]  

(86)

The claim of the theorem follows from Lemma A.2, whose statement and proof are given below.

\[ \square \]
Lemma A.2. Let $N_{1/\alpha}$ and $N_{c\alpha}$ be stopping times where the underlying mixing weights $\{\omega_k\}$ and parameters of baseline increments $\{\lambda_k\}$ are chosen via Algorithm 3. Let $\bar{N}_{g\alpha}$ be the stopping time defined in (48) with the threshold given by Algorithm 3. Then, for any stream of observations $X_1, X_2, \ldots$, 

$$N_{c\alpha} \leq N_{1/\alpha} \leq \bar{N}_{g\alpha},$$

(87)
deterministically, provided that $1 < c\alpha < 1/\alpha$.

Proof of Lemma A.2 and Algorithm 3. Throughout this proof, we set $D_L := \psi^*(\Delta_L) < \psi^*(\Delta_U) =: D_U$. The first inequality $N_{c\alpha} \leq N_{1/\alpha}$ follows directly from the definition of the stopping time in (40) along with the condition that $c\alpha \leq 1/\alpha$. To prove the second inequality $N_{1/\alpha} \leq \bar{N}_{g\alpha}$, we will exploit on general geometric construction introduced in Shin et al. [2021] to analyze the performance of sequential generalized likelihood ratio tests. To that effect, set $\hat{\mu}_n := S_n/V_n$. Then, for each fixed $\lambda > 0$ such that $\Delta = \nabla \psi^*(\lambda)$, Proposition 4.2 and the identity $\lambda = \nabla \psi^*(\Delta)$ imply that the function $\hat{\mu}_n \mapsto V_n^{-1} \sum_{i=1}^n \log L_i(\lambda) = \lambda \hat{\mu}_n - \psi(\lambda) = \lambda (\hat{\mu}_n - \Delta) + \psi^*(\Delta)$ is a mapping of $\hat{\mu}_n$ into the tangent line of the function $z \mapsto \psi^*(z)$ at $z = \Delta$.

Next, set $V_U := g\alpha/D_U$ and $V_L := g\alpha/D_L$, and define the set

$$R := \{(z, y) \in [0, \infty)^2 : y \leq \psi^*(z)\}.$$  

(88)

Then, the stopping event of $\bar{N}_{g\alpha}$ can be expressed as

$$\left\{ \exists n \geq 1 : \sup_{\lambda \in (\Delta_L, \Delta_U)} \sum_{i=1}^n \log L_i(\lambda) \geq g\alpha \right\}$$

$$= \left\{ \exists n \geq 1 : V_n < V_U, \left(\hat{\mu}_n, \frac{g\alpha}{V_n}\right) \in H(\Delta_U) \right\} \cup \left\{ \exists n \geq 1 : V_n \geq V_U, \left(\hat{\mu}_n, \frac{g\alpha}{V_n}\right) \in R \setminus H(\Delta_L) \right\}$$

$$= \left\{ \exists n \geq 1 : V_n < V_U, \sum_{i=1}^n \log L_i(\lambda_U) \geq g\alpha \right\} \cup \left\{ \exists n \geq 1 : V_n \geq V_U, \sup_{\lambda \geq \lambda_U} \sum_{i=1}^n \log L_i(\lambda) \geq g\alpha \right\},$$

$H(\Delta_U)$ and $H(\Delta_L)$ are half spaces contained in and tangent to $R$ at $(\Delta_U, g\alpha/V_U)$ and $(\Delta_L, g\alpha/V_L)$, respectively. See Figure 4 for an illustration of the stopping event of $\bar{N}_{g\alpha}$.

Figure 4: Illustration of the stopping event of $\bar{N}_{g\alpha}$ defined in (48), and related regions $H(\Delta_U)$, $H(\Delta_L)$ and $R$. The stopping time $\bar{N}_{g\alpha}$ is the first time when $(\hat{\mu}_n, g\alpha/V_n)$ is located in one of the colored areas.

Note that the first decomposition part $\left\{ \exists n \geq 1 : V_n < V_U, \sum_{i=1}^n \log L_i(\lambda_U) \geq g\alpha \right\}$ is nonempty only if $V_U > \min_x y(x) := v_{\min}$, which is equivalent to $g\alpha > v_{\min} D_U$. For the second part, a straightforward
extension of Lemma 1 in the appendix of Shin et al. [2021] implies that, for any fixed $\eta > 1$, the second part can be further decomposed by sets of simple events as follows:

$$\left\{ \exists n \geq 1 : V_n \geq V_U, \sup_{\lambda > \lambda_L} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g_\alpha \right\}$$

$$\bigcup_{k=1}^{K(\eta)-1} \left\{ \exists n \geq 1 : V_n \in \left[ V_U \eta^{k-1}, V_U \eta^{k} \right), \sum_{i=1}^{n} \log L_i^{(\lambda_k)} \geq g_\alpha / \eta \right\}$$

$$\bigcup \left\{ \exists n \geq 1 : V_n \geq V_U \eta^{K(\eta)-1}, \sum_{i=1}^{n} \log L_i^{(\lambda_{K(\eta)})} \geq g_\alpha / \eta \right\},$$

where $K(\eta)$ is a positive integer defined by

$$K(\eta) = \left\lceil \log_\eta \left( \frac{D_U}{D_L} \right) \right\rceil,$$

and, for $k = 1, \ldots, K(\eta) - 1$, $\lambda_k$ is given by $\lambda_k := \psi^* (\Delta_k)$, with $\Delta_k$ the solution with respect to $z > 0$ of the equation

$$\psi^* (z) = \frac{D_U}{\eta^k},$$

while $\Delta_{K(\eta)} := \Delta_L$. It can be checked that $\lambda_U := \lambda_0 > \lambda_1 > \lambda_2 > \cdots > \lambda_{K(\eta)} = \lambda_L$. Decomposing the stopping event of $N_{g_\alpha}$, we can lower bound the stopping time $N_{g_\alpha}$ for any $\eta > 1$ as:

$$\bar{N}_{g_\alpha} = \inf \left\{ n \geq 1 : \sup_{\lambda \in (\lambda_L, \lambda_U)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g_\alpha \right\}$$

$$\geq \inf \left\{ n \geq 1 : e^{-g_\alpha} \prod_{i=1}^{n} L_i^{(\lambda_0)} 1 (g_\alpha > v_{\min} D_U) + \sum_{k=1}^{K(\eta)} e^{-g_\alpha / \eta} \prod_{i=1}^{n} L_i^{(\lambda_k)} \geq 1 \right\}$$

$$= \inf \left\{ n \geq 1 : e^{-g_\alpha} 1 (g_\alpha > v_{\min} D_U) \exp \left\{ \lambda_0 S_n - \psi (\lambda_0) V_n \right\} + \sum_{k=1}^{K(\eta)} e^{-g_\alpha / \eta} \exp \left\{ \lambda_k S_n - \psi (\lambda_k) V_n \right\} \geq 1 / \alpha \right\}$$

$$:= \inf \left\{ n \geq 1 : \sum_{k=0}^{K(\eta)} \omega_k (\eta) \exp \left\{ \lambda_k S_n - \psi (\lambda_k) V_n \right\} \geq 1 / \alpha \right\} := \bar{N}(\eta),$$

where $\omega_0 (\eta) := \alpha^{-1} e^{-g_\alpha} 1 (g_\alpha > v_{\min} D_U)$ and $\omega_k (\eta) := \alpha^{-1} e^{-g_\alpha / \eta}$ for each $\eta > 1$ and $k = 1, \ldots, K(\eta)$. As a quick remark, note that $\omega_0 (\eta)$ does not depend on $\eta$ and each $\omega_k (\eta)$ in fact does not depend on the index $k$ but we use this notation just for consistency.

Finally, set $\eta_\alpha := \left( \frac{D_U}{D_L} \right)^{1/K_{\alpha}}$ where $K_{\alpha}$ is the integer defined in (93) of Algorithm 3. It can be easily checked that $K_{\eta_\alpha} = K_{\alpha}$. Therefore, once we choose $\eta = \eta_\alpha$ then, from the definition of the mixing weights in Algorithm 3, we conclude that $\bar{N}(\eta_\alpha) \geq \bar{N}_{1/\alpha}$ provided that

$$W := e^{-g_\alpha} 1 (g_\alpha > v_{\min} D_U) + K_{\alpha} e^{-g_\alpha / \eta_\alpha} \leq \alpha,$$

(95)
Algorithm 3: Pseudo-code of computeBaseline function

**Input:** ARL parameter $\alpha \in (0, 1)$, Boundary values $0 < \Delta_L < \Delta_U$, Maximum number of baselines $K_{\text{max}} \in \mathbb{N}$.

**Output:** Parameters of baseline increments $\lambda_U = \lambda_0 > \lambda_1 > \cdots > \lambda_{K_{\alpha}} = \lambda_L$, Mixing weights $\omega_0, \omega_1, \ldots, \omega_{K_{\alpha}} \in [0, 1]$, Auxiliary values used to compute baseline increments $\{g_{\alpha}, K_{\alpha}, \eta_{\alpha}, W\}$.

1. Compute parameters for boundary values by $\lambda_L := \nabla \psi^\ast(\Delta_L)$ and $\lambda_U := \nabla \psi^\ast(\Delta_U)$.

   /* If the separation is large enough, use a single baseline. */

2. if $\log(1/\alpha) \leq v_{\text{min}} \psi^\ast(\Delta_L)$ then
   
   Set $K_{\alpha} := 1$, $\lambda_1 := \lambda_L$ and $\omega_1 := 1$

   return Parameter $\lambda_1$ and mixing weight $\omega_1$

3. Compute the threshold $g_{\alpha} > \log(1/\alpha)$ given by

   $$g_{\alpha} := \inf \left\{ g > \log(1/\alpha) : e^{-g} \mathbb{1}(g > v_{\text{min}} D_U) + \min_{k \in [K_{\text{max}}]} k \exp \left\{ -g \left( \frac{D_U}{D_L} \right)^{-1/k} \right\} \leq \alpha \right\}. \quad (92)$$

   (See Algorithm 4 for an explicit way to compute it)

4. Compute the number of baselines $K_{\alpha} \in \mathbb{N}$ by

   $$K_{\alpha} = \arg \min_k k \exp \left\{ -g_{\alpha} \left( \frac{D_U}{D_L} \right)^{-1/k} \right\}, \quad (93)$$

   where $D_L := \psi^\ast(\Delta_L) < \psi^\ast(\Delta_U) =: D_U$.

5. Compute the spacing parameter $\eta_{\alpha} := \left( \frac{D_U}{D_L} \right)^{1/K_{\alpha}}$.

   /* Compute parameters of baseline increments and mixing weights */

6. Set $\lambda_0 := \lambda_U$ and $\lambda_{K_{\alpha}} := \lambda_L$.

7. if $K_{\alpha} \geq 2$ then

   for $k = 1, \ldots, K_{\alpha} - 1$ do

   Compute $\Delta_k$ as the solution of the equation $\psi^\ast(z) = D_U \eta^{-k}$ with respect to $z > 0$.

   Compute the $k$-th parameter as $\lambda_k := \nabla \psi^\ast(\Delta_k)$.

8. Set $W := e^{-g_{\alpha}} \mathbb{1}(g_{\alpha} > v_{\text{min}} D_U) + K_{\alpha} e^{-g_{\alpha}/\eta_{\alpha}}$, and compute mixing weights by

   $$\omega_0 = W^{-1} e^{-g_{\alpha}} \mathbb{1}(g_{\alpha} > v_{\text{min}} D_U) \quad \text{and} \quad \omega_k = W^{-1} e^{-g_{\alpha}/\eta_{\alpha}}, \forall k \in [K_{\alpha}]. \quad (94)$$

9. return $\{\lambda_0, \lambda_1, \ldots, \lambda_{K_{\alpha}}\}, \{\omega_0, \omega_1, \ldots, \omega_{K_{\alpha}}\}, \{g_{\alpha}, K_{\alpha}, \eta_{\alpha}, W\}$
where the constant $g_\alpha$ is the constant used in Algorithm 3 and given by
\[
g_\alpha := \inf \left\{ g > \log(1/\alpha) : e^{-g} \mathbb{1}(g > v_{min}D_U) + \min_{k \in [K_{\alpha\beta}]} k \exp \left\{ -g \left( \frac{D_U}{D_L} \right)^{-1/k} \right\} \leq \alpha \right\}. \tag{96}
\]
By the definition of $K_\alpha$, we can immediately check that the inequality (95) holds, which proves the claimed inequality $N_{1/\alpha} \leq N(\eta_\alpha) \leq N_{g_\alpha}$, as desired.

We conclude this section with a formal proof of the validity of the upper bound in (47).

**Proposition A.3.** Let $K_{\alpha\beta}$ be a large enough integer such that
\[
K_\alpha = \arg \min_{k \in \{1, \ldots, K_{\alpha\beta}\}} k \exp \left\{ -g_\alpha \left( \frac{D_U}{D_L} \right)^{-1/k} \right\} = \arg \min_{k \in \mathbb{N}} k \exp \left\{ -g_\alpha \left( \frac{D_U}{D_L} \right)^{-1/k} \right\}. \tag{97}
\]
Then, the quantity $g_\alpha$ Algorithm 1 specified in (92) is such that
\[
g_\alpha < \inf_{\eta > 1} \left[ \log(1/\alpha) + \log \left( 1 + \left[ \log_{\eta} \left( \frac{\psi^*(\Delta U)}{\psi^*(\Delta L)} \right) \right] \right) \right]. \tag{98}
\]
**Proof of Proposition A.3.** Once $K_{\alpha\beta}$ is large enough to satisfy (97) then the constant $g_\alpha$ can be written as
\[
g_\alpha = \inf \left\{ g > \log(1/\alpha) : e^{-g} \mathbb{1}(g > v_{min}D_U) + \inf_{\eta > 1} \left[ \log_{\eta} \left( \frac{D_U}{D_L} \right) \right] e^{-g/\eta} \leq \alpha \right\}, \tag{99}
\]
because the following equality holds for each $g > 0$ and $D_U > D_L$,
\[
\min_{k \in \mathbb{N}} k \exp \left\{ -g \left( \frac{D_U}{D_L} \right)^{-1/k} \right\} = \inf_{\eta > 1} \left[ \log_{\eta} \left( \frac{D_U}{D_L} \right) \right] e^{-g/\eta}. \tag{100}
\]
Finally, to prove the claimed upper bound in (98), first note that, from (99), we have $g_\alpha \leq g(\eta)$ where $g(\eta) > 0$ is given by
\[
g(\eta) := \inf \left\{ g > \log(1/\alpha) : e^{-g} \left[ \log_{\eta} \left( \frac{D_U}{D_L} \right) \right] e^{-g/\eta} \leq \alpha \right\} \leq \eta \left[ \log(1/\alpha) + \log \left( 1 + \left[ \log_{\eta} \left( \frac{D_U}{D_L} \right) \right] \right) \right],
\]
for each $\eta > 1$. By taking infimum over $\eta > 1$, we get the upper bound in (98), as desired.

**Proof of Lemma 4.4.** The proof of Lemma 4.4 is similar to the one of Lemma A.2 except the previous threshold $g_\alpha$ being replaced with $g \left( V_0/\eta^{K(i)} \right)$. Also note that, in this proof, the terms $\Delta_0$ and $V_0$ play a similar role of $\Delta_U$ and $V_U$ in the previous proof of Lemma A.2.

As same as the Lemma A.2 case, set $S_n := \sum_{i=1}^n s(X_i)$, $V_n := \sum_{i=1}^n v(X_i)$, and $\hat{\mu}_n := S_n/V_n$. Then, for each fixed $\lambda > 0$ with $\Delta = \nabla \psi(\lambda)$, the function $\hat{\mu}_n \mapsto V_n^{-1} \sum_{i=1}^n \log L_{ij}^{(\lambda)} = \lambda \hat{\mu}_n - \psi(\lambda) = \lambda (\hat{\mu}_n - \Delta) + \psi^*(\Delta)$ is a mapping of $\hat{\mu}_n$ to the tangent line of the function $z \mapsto \psi^*(z)$ at $z = \Delta$. Now,
since the boundary function $g$ is non-decreasing, the stopping event of $\bar{N}_g(j)$ can be bounded as

$$\left\{ \exists n \geq 1 : \sup_{\lambda \in (\lambda_K(j), \lambda_0)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g \left( V_0 \eta^{K(j)} \right) \right\}
\setminus \left\{ \exists n \geq 1 : V_n < V_0 ; \sum_{i=1}^{n} \log L_i^{(\lambda_0)} \geq g (V_0) \right\}
\cup \left\{ \exists n \geq 1 : V_n \in \left[ V_0, V_0 \eta^{K(j)} \right) , \sup_{\lambda \in (\lambda_K(j), \lambda_0)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g (V_n) \right\}
\cup \left\{ \exists n \geq 1 : V_n \geq V_0 \eta^{K(j)} , \sum_{i=1}^{n} \log L_i^{(\lambda_K(j))} \geq g \left( V_0 \eta^{K(j)} \right) \right\}
= \left\{ \exists n \geq 1 : V_n < V_0 , \left( \hat{\mu}_n , \frac{g (V_0)}{V_n} \right) \in H (\Delta_0) \right\}
\cup \left\{ \exists n \geq 1 : V_n \in \left[ V_0, V_0 \eta^{K(j)} \right) , \left( \hat{\mu}_n , \frac{g (V_n)}{V_n} \right) \in R \right\}
\cup \left\{ \exists n \geq 1 : V_n \geq V_0 \eta^{K(j)} , \left( \hat{\mu}_n , \frac{g (V_0 \eta^{K(j)})}{V_n} \right) \in H (\Delta_K(j)) \right\}
= \left\{ \exists n \geq 1 : V_n < V_0 , \sum_{i=1}^{n} \log L_i^{(\lambda_0)} \geq g (V_0) \right\}
\cup \left\{ \exists n \geq 1 : V_n \in \left[ V_0, V_0 \eta^{K(j)} \right) , \sup_{\lambda \in (\lambda_K(j), \lambda_0)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g (V_n) \right\}
\cup \left\{ \exists n \geq 1 : V_n \geq V_0 \eta^{K(j)} , \sum_{i=1}^{n} \log L_i^{(\lambda_K(j))} \geq g \left( V_0 \eta^{K(j)} \right) \right\},$$

where $V_0 := \inf \left\{ t \geq 1 : D_0 \geq g(t)/t \right\}$ and the set $R$ is defined by

$$R := \left\{ (z, y) \in [0, \infty)^2 : y \leq \psi^* (z) \right\}, \quad \text{ (101)}$$

and $H (\Delta_0)$ and $H (\Delta_K(j))$ are half spaces contained in and tangent to $R$ at $\left( \Delta_0, \frac{g (V_0)}{V_0} \right)$ and $\left( \Delta_L, \frac{g (V_0 \eta^{K(j)})}{V_0 \eta^{K(j)}} \right)$, respectively. See Figure 5 for an illustration of the upper bound of the stopping event of $\bar{N}_g(j)$.

Note that the first decomposition part $\left\{ \exists n \geq 1 : V_n < V_0 , \sum_{i=1}^{n} \log L_i^{(\lambda_0)} \geq g (V_0) \right\}$ is nonempty only if $V_0 > v_{\min}$, which is equivalent to $g (V_0) > v_{\min} D_0$. For the second part, a straightforward extension of Lemma 1 in the appendix of Shin et al. [2021] implies that, for any fixed $\eta > 1$, the second part can be further decomposed by sets of simple events as follows:

$$\left\{ \exists n \geq 1 : V_n \in \left[ V_0, V_0 \eta^{K(j)} \right) , \sup_{\lambda \in (\lambda_K(j), \lambda_0)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g (V_n) \right\}
\subset \bigcup_{k=1}^{K(j)} \left\{ \exists n \geq 1 : V_n \in \left[ V_0 \eta^{k-1} , V_0 \eta^{k} \right) , \sum_{i=1}^{n} \log L_i^{(\lambda_k)} \geq g \left( V_0 \eta^{k} \right) / \eta \right\}, \quad \text{ (102)}$$

where each $\lambda_k$ is given by $\lambda_k := \psi^* (\Delta_k)$ and each $\Delta_k$ is the solution of the equation (52) for $k = 1, \ldots, K(j)$. 

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\[ \Delta_0 H(\Delta_0) \]

\[ \Delta K(j) \]

\[ g(V_0) \]

\[ g(V_0) \eta K(j) \]

\[ \eta K(j) \]

\[ g(x) \]

\[ \psi^*(z) \]

\[ H(\Delta_0) \]

\[ H(\Delta K(j)) \]

\[ y = \psi^*(z) \]

\[ 0 \]

\[ \Delta K(j) \]

\[ \Delta_0 \]

\[ 0 \]

\[ z \]

\[ y \]

\[ \lambda \]

\[ \lambda \in (\lambda K(j), \lambda_0) \]

\[ \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g(V_0) \eta K(j) \]

\[ \sup_{\lambda \in (\lambda K(j), \lambda_0)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g(V_0) \eta K(j) \]

\[ v_{\min}^D_0 \]

\[ K(j) \]

\[ \omega_k \]

\[ L_i^{(\lambda_k)} \]

\[ \frac{1}{\alpha} \]

\[ N_{1/\alpha}(j) \]

\[ \bar{N}_g(j) \]

\[ \bar{N}_g(j) = \inf \left\{ n \geq 1 : \sup_{\lambda \in (\lambda K(j), \lambda_0)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g(V_0) \eta K(j) \right\} \]

\[ \geq \inf \left\{ n \geq 1 : e^{-g(V_0)} \prod_{i=1}^{n} L_i^{(\lambda_0)} \leq 1 \right\} \]

\[ \geq \inf \left\{ n \geq 1 : \prod_{i=1}^{n} L_i^{(\lambda_0)} \geq 1/(\alpha) \right\} \]

\[ \bar{N}_g(j) = N_{1/\alpha}(j) \]

which proves the claimed inequality.

Proof of Corollary 4.5. From Theorem 3.4 and Lemma 4.4, worst average delays of adaptive e-SR and e-CUSUM procedures are upper bounded by \( \max_{\nu \geq 0} \min_{j \geq 1} [\mathbb{E}_{0,Q} N_G(j + \nu) + j - 1] \) where \( \bar{N}_g(j) \) is a stopping time defined by

\[ \bar{N}_g(j) = \inf \left\{ n \geq 1 : \sup_{\lambda \in (\lambda K(j), \lambda_0)} \sum_{i=1}^{n} \log L_i^{(\lambda)} \geq g(V_0) \eta K(j) \right\} , \]
bound:

\[
\min_{j \geq 1} [\mathbb{E}_{0,Q} N_G(j) + j - 1] \leq \mathbb{E}_{0,Q} N_G(1)
\]

\[
= \mathbb{E}_{0,Q} \inf \left\{ n \geq 1 : \sup_{\lambda \in (\lambda_{K(1)}, \lambda_0)} \sum_{i=1}^n \log L_i^{(\lambda)} \geq g \left( V_0 \eta K(1) \right) \right\}
\]

\[
\leq \mathbb{E}_{0,Q} \inf \left\{ n \geq 1 : \sum_{i=1}^n \log L_i^{(\lambda_0)} \geq g_{r\alpha} \right\}.
\]

Since \( \mathbb{E}_{0,Q} \log L_1^{(\lambda_0)} = \sigma^2 (\lambda_0 \Delta^* - \psi(\lambda_0)) \geq \sigma^2 (\lambda_0 \Delta_0 - \psi(\lambda_0)) = \sigma^2 \psi^*(\Delta_0) \), by the same argument of Proposition 2.13, the last term above can be further upper bounded by

\[
\frac{g_{r\alpha}}{D(Q||P)} \psi^*(\Delta^*) + \mathbb{V}_{0,Q} \left[ \frac{\log L_1^{(\lambda_0)}}{[D(Q||P)]^2} \right] \left[ \frac{\psi^*(\Delta^*)}{\psi^*(\Delta_0)} \right]^2 + 1.
\]

(104)

Since we are in the case \( \lambda^* \geq \lambda_0 \), we have \( \frac{\psi^*(\Delta^*)}{\psi^*(\Delta_0)} \geq 1 \), which can be understood as a measure of inefficiency due to the misspecified upper bound of the oracle \( \lambda^* \).

Now, consider the case \( \lambda^* < \lambda_0 \) where we correctly specified the upper bound. In this case, let \( j^* \) be the smallest integer satisfying \( \lambda_{K(j^*)} := \lambda_{K^*} < \lambda^* < \lambda_0 \). Then, we can further upper bound the worst average delays by

\[
\min_{j \geq 1} [\mathbb{E}_{0,Q} N_G(j) + j - 1] \leq \mathbb{E}_{0,Q} \inf \left\{ n \geq 1 : \sum_{i=1}^n \log L_i^{(\lambda^*)} \geq g \left( V_0 \eta K^* \right) \right\} + j^* - 1
\]

\[
:= \mathbb{E}_{0,Q} N_* + j^* - 1.
\]

By Equation (49), we have the following intermediate upper bound on the worst average delays,

\[
\mathbb{E}_{0,Q} N_* + j^* - 1 \leq g \left( V_0 \eta K^* \right) + \frac{\mathbb{V}_{0,Q} \log L_1^{(\lambda^*)}}{[D(Q||P)]^2} + j^*.
\]

(105)

Note that if \( j^* = 1 \) then \( \lambda_{K(1)} = \lambda_L < \lambda^* \). Thus, in this case, we also correctly specified the lower bound, and the above bound is reduced to the same upper bound on the worst average delays in Theorem 4.3 of the well-separation case except the ARL parameter \( \alpha \) being replaced by \( r\alpha \).

Finally, to get an explicit upper bound on \( j^* \) for the case \( j^* > 1 \), fist note that, from the definition of \( \lambda_{K(j^* - 1)} \) with the fact \( \lambda^* < \lambda_{(j^* - 1)} \Leftrightarrow \Delta^* < \Delta_{K(j^* - 1)} \), we have

\[
\frac{g \left( V_0 \eta K^{(j^* - 1)} \right)}{V_0 \eta K^{(j^* - 1)}} = \psi^* \left( \Delta_{K(j^* - 1)} \right) > \psi^* (\Delta^*).
\]

(106)

Also, the condition \( K(j) \geq K_L + m \log \eta j \) implies

\[
\frac{\eta^{-K_L}}{V_0} V_0 \eta K^j \leq \left[ \frac{\eta^{-K_L}}{V_0} V_0 \eta K(j) \right]^{1/m},
\]

(107)

for each \( j \geq 1 \). By combining two inequalities above, we have

\[
\frac{g \left( V_0 \eta K^{(j^* - 1)} \right)}{V_0 \eta K^{(j^* - 1)}} \leq \left[ \psi^* (\Delta_L) \frac{g \left( V_0 \eta K^* \right)}{\psi^* (\Delta^*)} \right]^{1/m}.
\]

(108)
In sum, by combining all bounds above, we have

$$\min_{j \geq 1} \left[ \mathbb{E}_{0,Q} N_G(j) + j - 1 \right] \leq \begin{cases} \frac{g_{r\alpha}}{D(Q||P)} \left[ \log L_1^{(\lambda)} \right] + \frac{V_0, Q}{[D(Q||P)]^2} \left[ \psi^*(\Delta^*) \right] + \frac{\psi^*(\Delta^*)}{\psi^*(\Delta_0)} \left[ \log L_1^{(\lambda)} \right] + 1 & \text{if } \lambda^* \geq \lambda_0 \\ \frac{g_{r\alpha} + \eta \log(1 + K^* - K_L)}{D(Q||P)} + \frac{V_0, Q}{[D(Q||P)]^2} \left[ \log L_1^{(\lambda)} \right] + \frac{\psi^*(\Delta^*)}{\psi^*(\Delta_0)} \left[ \log \left( \frac{D_U}{D_L} \right) \right]^{1/m} & \text{if } \lambda^* \in (\lambda_L, \lambda_0) \\ \frac{g_{r\alpha} + \eta \log(1 + K^* - K_L)}{D(Q||P)} + \frac{V_0, Q}{[D(Q||P)]^2} \left[ \log L_1^{(\lambda)} \right] + \frac{\psi^*(\Delta^*)}{\psi^*(\Delta_0)} \left[ \log \left( \frac{D_U}{D_L} \right) \right]^{1/m} & \text{if } \lambda^* \leq \lambda_L \end{cases}$$

as desired.

**B. An explicit way to compute the threshold in Algorithm 3**

The following pseudo-code describes how to compute the threshold $g_\alpha$ defined by

$$g_\alpha := \inf \left\{ g > \log(1/\alpha) : e^{-g} \left[ g > v_{\min D_U} \right] + \min_{k \in [K_{\max}]} k \exp \left\{ -g \left( \frac{D_U}{D_L} \right)^{-1/k} \right\} \leq \alpha \right\}. \quad (108)$$

**Algorithm 4: Pseudo-code of computeThreshold function**

**Input:** ARL parameter $\alpha \in (0, 1)$, Boundary values $0 < \Delta_L < \Delta_U$, Maximum number of baselines $K_{\max} \in \mathbb{N}$, Tolerance $\epsilon > 0$.

**Output:** Threshold $g_\alpha > 0$ that is defined and used in Algorithm 3.

1. Set $D_L := \psi^*(\Delta_L) < \psi^*(\Delta_U) =: D_U$ and define a function $f$ on $\mathbb{R}_+$ as

$$f(g) := \min_{k \in [K_{\max}]} k \exp \left\{ -g \left( \frac{D_U}{D_L} \right)^{-1/k} \right\}. \quad (109)$$

2. if $f(v_{\min D_U}) \leq \alpha$ then

3. Compute $g_\alpha := \inf \left\{ g \in (\log(1/\alpha), v_{\min D_U}) : f(g) \leq \alpha \right\}$ by using the bisection method to the function $g \mapsto f(g) - \alpha$ with endpoints $(\log(1/\alpha), v_{\min D_U})$ and tolerance $\epsilon$.

4. else

5. Compute $g_\alpha := \inf \left\{ g \in \left( v_{\min D_U}, \frac{D_U}{D_L} \log(2/\alpha) \right) : e^{-g} + f(g) \leq \alpha \right\}$ by using the bisection method to the function $g \mapsto e^{-g} + f(g) - \alpha$ with endpoints $\left( v_{\min D_U}, \frac{D_U}{D_L} \log(2/\alpha) \right)$.

6. return $g_\alpha > 0$