Negative Energy Solutions and Symmetries

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Abstract

We revisit the negative energy solutions of the Dirac equation, which become relevant at very high energies and study several symmetries which follow therefrom. The consequences are briefly examined.

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1 Introduction

It is well known that in relativistic Quantum Mechanics, we encounter negative energy solutions, be it for the Dirac equation or for the Klein-Gordon equation. Such negative energy solutions have no counterpart, indeed interpretation in non relativistic or classical theory. For the Klein-Gordon (K-G) equation, this could be attributed to the second time derivative, which leads to an extra degree of freedom. Pauli and Weiskoef interpreted the negative energy solutions in the context of Quantum Field Theory, but what is less well known is that these negative energy solutions of the Klein-Gordon equation were successfully interpreted thereafter by Feshbach and Villars [1] in the context of the usual single particle theory.
It could have been expected that these difficulties would be bypassed in the Dirac theory which restores the single time derivative – but here too the negative energies surfaced, because ultimately it was the same energy momentum
dispersion relation that was invoked. Dirac then had to take recourse to the negative energy sea and the hole theory to overcome the difficulty [2]. Interestingly a different explanation was given by the author several years ago in the context of Quantum Mechanical Kerr Newman Black Holes [3]. We will now study the negative energy solutions for both the Dirac and Klein-Gordon equations and examine some symmetries and also their consequences.

2 The Negative Energy Solutions

Let us write the Dirac wave function as

\[ \psi = \left( \begin{array}{c} \phi \\ \chi \end{array} \right), \tag{1} \]

where \( \phi \) and \( \chi \) are each two spinors. As is well known [4], we can then deduce

\[
\begin{align*}
\dot{\hbar}(\partial \phi / \partial t) &= c \tau \cdot (p - e/cA)\chi + (mc^2 + e\phi)\phi, \\
\dot{\hbar}(\partial \chi / \partial t) &= c \tau \cdot (p - e/cA)\phi + (-mc^2 + e\phi)\chi. \tag{2}
\end{align*}
\]

We recapitulate that at low energies \( \chi \) is small and \( \phi \) dominates, whereas it is the reverse at high energies. We also note that while sensible wave packets can be formed with the positive energy solutions alone, in general we require both signs of energy for a localized particle. In fact the Compton wavelength is the minimum extension, below which both positive and negative solutions will have to be considered. Well outside the Compton wavelength, we can continue with the usual positive energy description. More formally the positive energy solutions alone do not form a complete set of eigen functions of the Hamiltonion.

The following symmetry can be seen from (2) (with \( e = 0 \), or the absence of an external electromagnetic field for simplicity):

\[ t \rightarrow -t, \phi \rightarrow -\chi \tag{3} \]

We must remember that we are dealing with intervals at the Compton scale, so that the negative energy solutions are relevant. So the time reversal given in (3) is at the Compton scale.

We next observe that such a \((t, -t)\) behaviour in this microscopic interval has been described in detail in terms of a double Weiner process. Furthermore
this can be used in the context of two state systems to go over from the non relativistic Schrodinger theory to the relativistic theory (Cf. [5, 6, 7] for details). To see this briefly, we first define a complete set of base states by the subscript \(ı\) and \(\mathit{U}(t_2,t_1)\) the time elapse operator that denotes the passage of time between instants \(t_1\) and \(t_2\), \(t_2> t_1\). We denote by, \(C_ı(t) = \langle ı|\psi(t)\rangle\), the amplitude for the state \(|\psi(t)\rangle\) to be in the state \(|ı\rangle\) at time \(t\). We have \([3, 8, 9]\)

\[
\langle ı|U|j > \equiv U_{ıj}, \mathit{U}_{ıj}(t + \Delta t, t) \equiv \delta_{ıj} - \frac{i}{\hbar} \mathit{H}_{ıj}(t) \Delta t.
\]

We can now deduce from the super position of states principle that,

\[
C_ı(t + \Delta t) = \sum_j \left[ \delta_{ıj} - \frac{i}{\hbar} \mathit{H}_{ıj}(t) \Delta t \right] C_j(t)
\]

and finally, in the limit,

\[
\hbar \frac{dC_ı(t)}{dt} = \sum_j \mathit{H}_{ıj}(t) C_j(t)
\]

where the matrix \(\mathit{H}_{ıj}(t)\) is identified with the Hamiltonian operator. We have argued earlier at length that \([5]\) leads to the Schrodinger equation \([3, 9]\). In the above we have taken the usual unidirectional time to deduce the non relativistic Schrodinger equation. If however we consider a Weiner process in \([6]\) that is, allow \(t\) to fluctuate between \((t - \Delta t, t + \Delta t)\), (to which we will return shortly), then we will have to consider instead of \([5]\)

\[
C_ı(t - \Delta t) - C_ı(t + \Delta t) = \sum_j \left[ \delta_{ıj} - \frac{i}{\hbar} \mathit{H}_{ıj}(t) \right] C_j(t)
\]

Equation \([6]\) in the limit can be seen to lead to the relativistic Klein-Gordon equation rather than the Schrodinger equation with the second time derivative \([9, 7]\). In other words the symmetry in \([3]\) is in-built at the Compton scale in the relativistic description, be it for the Klein-Gordon equation or the Dirac equation, and Zitterbewegung is a manifestation of this (Cf. [1, 10]). We can push these considerations further. We have already seen the symmetry given in \([3]\): In case of a charged particle, in addition, \(e \rightarrow -e\) and vice versa (with complexification). This apart it suggests that the coordinate \(\vec{x}\), as it were splits into the coordinate \(\vec{x}_1\) and \(\vec{x}_2\) which mimic the wave function.
in (1) at low and high energies, in the sense that the former dominates at low energies while the latter dominates at high energies, following the wave function as in (1). The fact that these go into each other following (3) as $t \to -t$ can be explained in terms of the development of a two Weiner process see briefly above (Cf.[7]). Let us elaborate.

In this case there are two derivatives, one for the usual forward time and another for a backward time given by

$$\frac{d_+}{dt}x(t) = b_+,$$ $\frac{d_-}{dt}x(t) = b_-$$ (7)

where we are considering for the simplicity, a single dimension $x$. This leads to the Fokker-Planck equations

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho b_+) = V \Delta \rho,$$

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho b_-) = -U \Delta \rho$$ (8)

defining

$$V = \frac{b_+ + b_-}{2}; U = \frac{b_+ - b_-}{2}$$ (9)

We get on addition and subtraction of the equations in (8) the equations

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho V) = 0$$ (10)

$$U = \nu \nabla \ln \rho$$ (11)

It must be mentioned that $V$ and $U$ are the statistical averages of the respective velocities and their differences. We can then introduce the definitions

$$V = 2\nu \nabla S$$ (12)

$$V - iU = -2\nu \nabla (\ln \psi)$$ (13)

We will not pursue this line of thought here but refer the reader to Smolin [11] for further details. We now observe that the complex velocity in (13) can be described in terms of a positive or uni directional time $t$ only, but a complex coordinate

$$x \to x + ix'$$ (14)

To see this let us rewrite (9) as

$$\frac{dX_r}{dt} = V, \quad \frac{dX_i}{dt} = U,$$ (15)
where we have introduced a complex coordinate $X$ with real and imaginary parts $X_r$ and $X_i$, while at the same time using derivatives with respect to time as in conventional theory.

We can now see from (9) and (15) that

$$W = \frac{d}{dt}(X_r - iX_i)$$

That is we can alternatively use derivatives with respect to the usual uni directional time derivative to introduce the complex coordinate (14) (Cf.ref.[5]. Let us now generalize (14), which we have taken in one dimension for simplicity, to three dimensions. Then as discovered by Hamilton, we end up with not three but four dimensions,

$$(1, i) \rightarrow (I, \tau),$$

where $I$ is the unit $2 \times 2$ matrix and $\tau$s are the Pauli matrices. We get the special relativistic Lorentz invariant metric at the same time. (In this sense, as noted by Sachs [12], Hamilton would have hit upon Special Relativity, if he had identified the new fourth coordinate with time).

That is,

$$x + iy \rightarrow Ix_1 + ix_2 + jx_3 + kx_4,$$

where $(i, j, k)$ momentarily represent the Pauli matrices; and, further,

$$x_1^2 + x_2^2 + x_3^2 - x_4^2$$

is invariant.

While the usual Minkowski four vector transforms as the basis of the four dimensional representation of the Poincare group, the two dimensional representation of the same group, given by the right hand side of (17) in terms of Pauli matrices, obeys the quaternionic algebra of the second rank spinors (Cf.Ref.[5, 13, 12] for details).

In fact one representation of the two dimensional form of the quaternion basis elements is the set of Pauli matrices above. Thus a quaternion may be expressed in the form

$$Q = -i\tau_\mu x^\mu = \tau_0 x^4 - i\tau_1 x^1 - i\tau_2 x^2 - i\tau_3 x^3 = (\tau_0 x^4 + i\vec{\tau} \cdot \vec{r})$$
This can also be written as
\[ Q = -i \begin{pmatrix} ix^4 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & ix^4 - x^3 \end{pmatrix}. \]

As can be seen from the above, there is a one to one correspondence between a Minkowski four-vector and \( Q \). The invariant is now given by \( \bar{Q}Q \), where \( \bar{Q} \) is the complex conjugate of \( Q \).

In this description we would have from (17), returning to the usual notation,
\[ [x^i \tau^i, x^j \tau^j] \propto \epsilon_{ijk} \tau^k \neq 0 \quad (19) \]

In other words, as (19) shows, the coordinates no longer follow a commutative geometry. It is quite remarkable that the noncommutative geometry (19) has been studied by the author in some detail (Cf. [7]), though from the point of view of Snyder’s minimum fundamental length, which he introduced to overcome divergence difficulties in Quantum Field Theory. Indeed we are essentially in the same situation, because as we have seen, for our positive energy description of the universe, there is the minimum Compton wave length cut off for a meaningful description [14, 15, 16].

Proceeding further we could think along the lines of \( SU(2) \) and consider the transformation [17]
\[ \psi(x) \rightarrow \exp\left[\frac{1}{2}ig \tau \cdot \omega(x)\right] \psi(x). \quad (20) \]

This leads as is well known to a covariant derivative
\[ D_\lambda \equiv \partial_\lambda - \frac{1}{2}ig \tau \cdot W_\lambda, \quad (21) \]

remembering that \( \omega \) in this theory is infinitessimal. We are thus lead to vector Bosons \( W_\lambda \) and an interaction like the strong interaction, described by
\[ W_\lambda \rightarrow W_\lambda + \partial_\lambda \omega - g\omega \Lambda W_\lambda. \quad (22) \]

However, we are this time dealing, not with iso spin, but between positive and negative energy states as in (1). Also we must bear in mind that this non-electromagnetic force between particles and anti particles would be valid only within the Compton time, inside this Compton scale Quantum Mechanical "bridge" [18].
These considerations are also valid for the Klein-Gordon equation in the two component notation developed by Feshbach and Villars [11] [19]. There too, we get equations like (2). We would like to re-emphasize that our usual description in terms of positive energy solutions is valid above the Compton scale.

3 A Further Symmetry

As we consider both signs of the energy, we denote the expectation of an operator by the equation (Cf. also ref.[1])

$$\int \psi^* \tau_3 \Omega \psi d^3x$$  \hspace{1cm} (23)

where $\tau_3$ is the usual Pauli matrix is given by

$$\psi^* = (\phi, \chi) = \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (24)

We use (23) for the observable: $\Omega = x^i x_i$. Then we can easily see the following. Let us first consider (23) for the two cases: First the negative energy component $\chi$ is vanishingly small, as in our usual description and second where the negative energy component dominates and $\phi$ is vanishingly small, that is for the very high energy case. Then we can easily verify that

$$\Omega \rightarrow -\Omega$$

This has the following consequence. The Minkowski invariant

$$x^\mu x_\mu$$  \hspace{1cm} (25)

of the Lorentz group goes over to the invariant of the four dimensional rotation group

$$x_0^2 + x^i x_i$$  \hspace{1cm} (26)

for negative energies and vice versa.

We could expect that the Foldy-Wouthuysen transformation goes over to a Lorentz transformation in the negative energy realm. A simple way of seeing this is as follows: The Foldy-Wouthuysen transformation is given by

$$S = e^{\beta \vec{\alpha} \cdot \vec{p} \Theta}$$
\[ \tan 2|p|\Theta = \frac{|p|}{m} \]  

while the Lorentz transformation is described by

\[ S = -e^{-i\vec{\alpha} \cdot \vec{p}(\mu)} \]

\[ \tan h\mu|p| = \frac{pc}{E + mc^2} \]  

(Cf. ref. [4]). Comparison of (25) and (26) show that effectively \( x^j \to ix^j \) or \( p_j \to ip_j \). Under this transformation (27) and (28) get interchanged.

### 4 Remarks

i) As mentioned, the above considerations for the Dirac equation all apply for the positive and negative energy solutions of the Klein-Gordon equation (Cf. [19]).

ii) We make the following remark about the negative and positive energy solutions of the Dirac equation. We consider for simplicity the free particle solutions [4]. The solutions are of the type

\[ \psi = \psi_A + \psi_S \]  

where

\[ \psi_A = e^{\frac{E}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } e^{\frac{E}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]  

and

\[ \psi_S = e^{-\frac{E}{\hbar}t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } e^{-\frac{E}{\hbar}t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

denote respectively the negative energy and positive energy solutions. From (29) the probability of finding the particle in a small volume about a given point is given by

\[ |\psi_A + \psi_S|^2 = |\psi_A|^2 + |\psi_S|^2 + (\psi_A^* \psi_S + \psi_S^* \psi_A) \]  

8
Equations (30) and (31) show that the negative energy and positive energy solutions form a coherent Hilbert space and so the possibility of transition to negative energy states exists. This difficulty however can be overcome by the well known Hole theory which uses the Pauli exclusion principle, and is described in many standard books on Quantum Mechanics.

However the last or interference term on the right side of (31) is like the zitterbewegung term. When we remember that we really have to consider averages over space time intervals of the order of $\frac{h}{mc}$ and $\frac{h}{mc^2}$ as Dirac himself pointed out (Cf.[2]), this term disappears and effectively the negative energy solutions and positive energy solutions stand decoupled in what is now the physical universe.

A more precise way of looking at this is[15] that as is well known, for the homogeneous Lorentz group, $p_0$ commutes with all operators and yet it is not a multiple of the identity as one would expect according to Schur’s lemma: The operator has the eigen values $\pm 1$ corresponding to positive and negative energy solutions. This is a super selection principle or "spin" referred to in [20] pointing to the two incoherent or decoupled Hilbert spaces or universes [20] now represented by states $\psi_A$ and $\psi_S$ which have been decoupled owing to the averaging over the Compton wavelength space-time intervals which eliminates the interference term in (31). But all this refers to energies such that our length scale is greater than the Compton wavelength.

Thus once again we see that outside the Compton wavelength region we recover the usual physics.

iii) It is worth recapitulating that we have identified the negative energy solutions with anti particles and via the mechanism described by (31), that is based on the fact that physical measurements are time averages over intervals of the order of the Compton scale. We conclude that the anti particles are very short lived, because outside the Compton wavelength that is in our physical world, we are in the manifold of positive solutions. Further these considerations also show (Cf.refs.[19] [21]) that there is an asymmetry between particles and anti particles. Indeed this prediction has since been suggested through experiment: Firstly there is the observed neutrino and anti neutrino asymmetry that violates CP, observed in the MiniBooNE experiment at Fermilab recently. Specifically the oscillation patterns for the neutrino and anti neutrino appear to be different with a confidence level of about 99.7% This in fact corroborates an earlier LSND experiment report at the Los Alamos National Laboratory in 1990, but since not taken seriously because it appeared too sensational.
The other CP violation has been found in the so-called B factories at SLAC, US and KEK Lab in Japan. This collaboration has calculated that the parameter associated with CP violation – \( \sin 2\beta \) – is 0.74 ± 0.07, compared with its earlier estimate of 0.99±0.14. The increased accuracy stems from the larger number of decay events observed this time – 88 million in total. The BELLE collaboration puts the value of \( \sin 2\beta \) – which they call \( \sin 2\psi_1 \) – at 0.79 ± 0.10 [22, 23].

The new estimates established beyond doubt that CP violation exists.

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