M^X/G/1 Vacation Queueing System with Two Types of Repair facilities and Server Timeout

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Abstract We consider a single server vacation queue with two types of repair facilities and server timeout. Here customers are in compound Poisson arrivals with general service time and the lifetime of the server follows an exponential distribution. The server finds if the system is empty, then he will wait until the time ‘c’. At this time if no one customer arrives into the system, then the server takes vacation otherwise the server commences the service to the arrived customers exhaustively. If the system had broken down immediately, it is sent for repair. Here server failure can be rectified in two case types of repair facilities, case 1, as failure happens during customer being served will stays in service facility with a probability of 1-q to complete the remaining service and in case 2 it opts for new service also who joins in the head of the queue with probability q. Obtained an expression for the expected system length for different batch size distribution and also numerical results are shown.

Keywords: Expected system length, Server timeout, Two types of repair facilities and. Vacation queue.

I. INTRODUCTION

The concept of Server timeout plays a vital role in a single service system with vacations. Number of problems is modeled by a vacation with queueing system. Queueing theory is used in Inventory control, loading and unloading of ships, machine interference problems, machine service and repair models. Vacation queue means the server completely sits idle and were first discussed by Levy and Yechiali [6] introduced and did the utilization of idle time in an M/G/1 queueing system followed by several excellent surveys by B.T. Doshi [1], Later on Jau-chauank, Chian-Hangwu and Zhe George Zhang [3] did on multiple vacation models in queueing theory and so on. When there is no customers in the system it becomes empty, the server will wait until time ‘c’ known as server timeout. Oliver C.Ibe [7], [8] derived an expression to the expected waiting time of a vacation queue in which server couldn’t take frequently second vacation upon returning from a first vacation and if he finds system with zero customer or wait indefinitely for a customer to arrive and it also extended for N-policy. E.Ramesh Kumar and Y. Praby Loit [2] obtained and derived an expression for the mean waiting time from the length of the system with single and n-policy of vacation queueing system. Y.Saritha, K.Satish Kumar and K.Chandan [12] derived the expected system length using different bulk size distributions for M^X/G/1 vacation Queueing system with server timeout. Single Queue subjected to breakdown and repair has been studied by number of statistician, S. Bama M.I.Aftab Begum and P. Fijy Jose[10] analyzed and derived M^X/G/1 queue in which random breakdowns occur with the Poisson process.

II. MODEL DESCRIPTION

Here customers are assumed to arrive in batches under compound Poisson process with rate b. Let B(Z) be the batch size distribution. The service times of a single server follow general distribution function. The server find if the system is empty, then he will wait until the time ‘c’ is known as server timeout. At this time if no one customer arrives into the system, then the server takes vacation otherwise the server commence the service to the arrived customer. At the expiration of the vacation time if no customer arrive the server goes for another vacation, otherwise he commence the service to the arrived customer exhaustively; Service may be interrupted due to Server breakdowns. Here the server is unable to work unless the machine gets repaired, therefore the server should undergo for repair process. Here breakdown server is facilitated with two case types of repair facilities, case 1, as failure happens during customer being served will stay in service facility with a probability of 1-q to complete the remaining service and in case 2 it opts for new service also who joins in the head of the queue with probability q. When the repair work is completed the server immediately returns to service system and executes the service for the waiting customer as well as arrived customer in a queue.

III. ANALYSING THE MODEL

The customer arrives to the system and occurs according to a compound Poisson process with arrival rate ‘b’, and service times are exponential with mean rate μ.
The time, \( X \), is to serve a customer and has a general distribution with cumulative distribution function CDF \( F_X(x) \), with mean \( E(X) \) and its second moment \( E(X^2) \). The Laplace-Stieljes transform function of \( X \) is \( B_X(s) \), which is defined by
\[
B_X(s) = E(e^{-sX}) = \int_0^\infty e^{-sx} f(X) dx
\]
(1)

The duration of a vacation is symbolized as \( V \) and have to follow general distribution with CDF \( F_V(v) \) and Laplace-Stieljes transform function of \( V \) is \( B_V(s) \). The average of \( V \) is \( E(V) \) and its second moment \( E(V^2) \).

\[
B_V(s) = \frac{Y}{s+Y}
\]
(2)

Here \( X \) and \( V \) are mutually independent. Similarly if the system has breakdown with rate \( \alpha \), then system go for two types of repair facilities based on customer’s choice. After repair, the server start the service to the customer. Let \( G_{L(M/G/1)}(z) \) is PGF the number in system is given by [Ref 10]
\[
g_{L(M/G/1)}(z) = (1-p)k_0 k_1^2 k_2(z-1)
\]
(3)

Where \( k_1 = \lambda - \lambda x(z) \) and \( k_2 = \alpha + \lambda - \lambda x(z) \)

Let \( A \) indicates number of customers in the system at the beginner of busy period. The PMF of \( A \) is \( P_A(a) = P[A=a] \), whose z-transformation is \( G_A(z) \) is given by
\[
g_A(z) = E(A^a) = \sum_{a=1}^\infty z^a P_A(a)
\]
(4)

The mean of \( A \) is \( E(A) \) and its second moment is \( E(A^2) \). Let the random variable \( B \) denotes the number of customers left the system by an arbitrary departing customer. The PMF of \( B \) is \( P_B(b) \), whose z-transform is given by \( G_B(z) \). \( g_B(z) = \frac{1}{1-zE(A)} \)
(5)

Its means \( E(B) \) and its second moment is \( E(B^2) \).

Assume that \( L \) be the number of customers in the system. The z-transform of the pmf of \( L \) i.e.: \( P_L(l) \) is given by \( g_L(z) \)
\[
g_L(z) = g_B(z) + g_{L(M/G/1)}(z)
\]
(6)

Its mean is \( E(L) \) and its second moment is \( E(L^2) \).

**Special Batch Size Distributions:**
As the batch size ‘\( b \)’ is a random variable, it has a probability distribution. In particular Deterministic, Geometric and Positive Poisson are considered.

1.) If we take Deterministic, then the generating function is \( B(z) = z^b \)
(7)

This gives mean \( B'(z) = z^b \) and second moment \( \bar{B} = B''(z) = b^2 - b \), where \( b \) is the average batch size.

2.) If we take Positive Poisson, then the generating function becomes
\[
B(z) = \frac{ae^{-a(z-1)}}{a}
\]
(8)

This gives mean \( \bar{B} = B'(z) = \frac{z}{a} \) and second moment \( \bar{B} = B''(z) = \frac{2[1-(1-p)]}{p^2} \), where \( \frac{1}{p} \) is the average batch size.

3.) If we take Geometric, then the generating function becomes
\[
B(z) = p\left[\frac{1}{z-1} - \frac{1}{z-1} \right]^{-1}
\]
(9)

This gives mean \( \bar{B} = B'(z) = \frac{1}{p} \) and second moment \( \bar{B} = B''(z) = \frac{2(1-p)}{p^2} \), where \( \frac{1}{p} \) is the average batch size.

Let \( W_q \) denotes the waiting time in the system, to determine expected system length. Thus applying little’s law we obtain the expected system length of the customer is [Ref 12]
\[
E(W_q) = \frac{dG_L(z)}{dz} \bigg|_{z=1} - E(X)
\]
(10)

Let \( E(W_q) = E(W_q) + E(X) \)
(10(a))

By using \( g_A(z) \), we can derive \( E(L) \). To get the Expressions for \( g_A(z) \) the Laplace –Stieljes transform of Expected system length. [Ref 12]
\[
g_A(z) = ZP_2 + \frac{M_p(\lambda - X, z) - M_p(\lambda)}{1 - M_p(\lambda)} P_3
\]
(11)

\( g_A(z) = ZP_2 + \frac{M_p(\lambda - X, z) - M_p(\lambda)}{1 - M_p(\lambda)} P_3 \)

(11)

Where \( P_2 = \frac{M_p(\lambda)}{1 - e^{-\lambda z}} \) and \( P_3 = \frac{1 - M_p(\lambda)}{1 - e^{-\lambda z}} \)

From these, we get
\[ g_A'(1) = E(A) = P_2 + \frac{AD'(1)E(V)}{1 - M_V(\lambda)} P_3 \]
\[ = \frac{M_V(\lambda)(1-e^{-\lambda c}) + AD'(1)E(V)}{1 - e^{-\lambda c}M_V(\lambda)} (12) \]
and
\[ g_A'(1) = E(A^2) = \frac{AD'(1)E(V) + \lambda^2D'(1)^2E(V^2)}{1 - M_V(\lambda)} P_3 \]
\[ = \frac{AD'(1)E(V) + \lambda^2D'(1)^2E(V^2)}{1 - e^{-\lambda c}M_V(\lambda)} (13) \]

By assuming the equation (2), we can define equation (12) and (13) by using \( B_V(\gamma) = \frac{\gamma}{\lambda + \gamma} \) and obtain the following equations as below
\[ E(A^2) = \frac{2\lambda^2(\lambda + \gamma)}{(\lambda + \gamma - e^{-\lambda c}\gamma)^2} (15) \]

By substituting equation (3), (5) in equation (6), we get
\[ g(Z) = \frac{1 - gZ}{(1-Z)E(A)} \]
\[ + \frac{(1 - \rho)h_ak_1S'k_2(z - 1)}{h_ak_1(z - S'k_2) - zaqR^*k_1(1 - S'k_2)} \]
\[ (16) \]
\[ g_L(Z) = \frac{1 - gL}{(1-Z)E(A)} + \frac{(1 - \rho)h_ks'k_2}{zaqR^*k_1(1 - S'k_2) - h_ks'(z - S'k_2)} (17) \]
\[ g_L(Z) = k * \frac{(1 - gL)h_ks'k_2}{zaqR^*k_1(1 - S'k_2) - h_ks'(z - S'k_2)} (18) \]

Consider \( k = \frac{1 - g}{E(A)} \)
By differentiating the above equation (18) w.r.t. to \( z \), we get
\[ g_L'(Z) = (zaqR^*k_1(1 - S'k_2) - h_ks'(z - S'k_2)) + g_L(Z)(zaqR^*k_1(1 - S'k_2) - h_ks'(z - S'k_2) + h_ks') = K*A(Z) + \alpha k1S'k2 + 1 - gA(Z) \]
\[ \alpha k1S'k2 (20) \]

Now substituting \( K_1, K_2 = 1 \) also equation (14) and (15) in equation (20), we get
\[ g_L'(1) = 1 \]
\[ g_L(1) = 1 \]
Again differentiating the above equation (20) w.r.t. to \( z \), we get
\[ g_L'(Z) = (zaqR^*k_1(1 - S'k_2) - h_ks'(z - S'k_2)) + 2g_L'(Z)(zaqR^*k_1(1 - S'k_2) - h_ks') + g_L(1) \]
\[ = K* \{ (-g'(Z)) h_ks'k_2 + (1 - g'(Z)) h_ks'k_2 \} \]
\[ (22) \]

Thus by above expression, we obtain expected system length.

**Particular case** Here (in equation (23)) the resulting expression is the special case for the M/G/1 model.

### IV. NUMERICAL RESULTS

Thus by using equation (23) and varying different parameters, we get some numerical illustration in Table 1 is given below:

| S No. | Parameters | Deterministic distribution of E(L) | Positive Poisson distribution of E(L) | Geometric distribution of E(L) |
|-------|------------|-----------------------------------|--------------------------------------|-------------------------------|
| 1     | Varying batch size b |                            |                                      |                               |
| 2     | 8.551      | 9.86                          | 10.05                               |                               |
| 3     | 11.701     | 10.85                         | 12.76                               |                               |
| 4     | 16.267     | 11.29                         | 17.208                              |                               |
| 5     | 21.466     | 11.844                        | 22.48156                            |                               |
| 6     | 27.542     | 24.85                         | 8.7996                              |                               |
| 2     | Varying arrival rate \( \lambda \) |                        |                                      |                               |
| 2     | 8.5515     | 9.8621                        | 10.05123                            |                               |
| 3     | 8.8980     | 10.793                        | 10.32997                            |                               |
| 4     | 10.441     | 11.749                        | 15.24963                            |                               |
| 14    | 11.464     | 13.273                        | 18.60797                            |                               |
| 18    | 13.061     | 15.660                        | 25.66514                            |                               |
| 3     | Varying service rate \( \mu \) |                        |                                      |                               |
| 50    | 8.5515     | 9.8621                        | 10.05123                            |                               |
| 60    | 8.5345     | 9.8365                        | 9.017707                            |                               |
| 70    | 8.5205     | 9.8155                        | 9.99023                            |                               |
| 80    | 8.5088     | 9.7979                        | 9.96731                            |                               |
| 90    | 8.4989     | 9.7830                        | 9.94790                            |                               |
| 4     | Varying break down rate \( \alpha \) |                        |                                      |                               |
| 0.2   | 8.5515     | 9.8621                        | 10.05123                            |                               |
| 0.4   | 8.5575     | 9.8683                        | 10.05738                            |                               |
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| n | E(L) | Var |  |
|---|------|-----|---|
| 0 | 6.5589 | 9.8691 | 10.0582 |
| 0 | 6.5585 | 9.8692 | 10.0583 |
| 1 | 6.5585 | 9.8692 | 10.0583 |

Varying repair rate \( \gamma \)

| n | E(L) | Var |  |
|---|------|-----|---|
| 0 | 6.5515 | 9.8621 | 10.0512 |
| .2 | 4.3933 | 5.6704 | 6.04786 |
| .5 | 2.2247 | 3.4776 | 4.66395 |
| 1 | 1.9713 | 2.2177 | 1.3564 |
| 2 | 0.8404 | 1.0865 | 1.19327 |

Varying server timeout \( c \)

| n | E(L) | Var |  |
|---|------|-----|---|
| 0 | 6.5515 | 9.8621 | 10.0512 |
| 2 | 5.3521 | 6.5706 | 7.5431 |
| 3 | 3.2475 | 4.6325 | 5.5963 |
| 4 | 1.4329 | 3.2212 | 2.1456 |
| 5 | 0.7456 | 1.0214 | 1.1963 |

From the above result

As \( b, \lambda, \) and \( \alpha \) were increasing then expected system length \( E(L) \) is also increasing.

As \( \mu, \gamma \) and \( c \) are increasing then expected system length \( E(L) \) is decreasing.

V. CONCLUSION

The model is derived with an expression of expected system length for \( M^X/G/1 \) vacation queueing system with two types of repair facilities and server timeout. Numerical results are given for different varying parameters for the given batch size distribution that shows the impact of the timeout period on the system length.

REFERENCES

1. Doshi, B.T. (1986), Queueing system with vacations. A survey on queueing system: Theory and Applications. 1(1), 29-66.
2. Ramesh Kumar and Y Praby Loti (2016), A Study on Vacation Bulk Queueing Model with setup time and server timeout. IJCMS, 5(12), 81-89.
3. Jia-Chuan Ke, Chia-Huang Wu and Zhe George Zhang, (2010), Recent Developments in Vacation Queueing Models: A Short Survey. Operation Research Vol 7, No 4, 3-8.
4. K.C. Madan, (2003), An M/G/1 type queue with Time-Homogeneous Breakdowns and Deterministic Repair Times, Soochow Journal of Mathematics Volume 29, No. 1, pp. 103-110, January 2003.
5. Kuo-Hsiung Wang, Dong-Yuh-Yang and W.L. Pearson, (2010), comparison of two randomized policy M/G/1 queues with optional service, server breakdown and startup. Journal of Computational and Applied Mathematics 234(3), 812-824.
6. Levy, Y and Yechiali. U (1975), Utilization of Idle Time in an M/G/1 Queueing System. Management Science, 22(2), 202-211. http://dx.doi.org/10.1287/mnsc.22.2.202.
7. Oliver C. Ibe (2007) Analysis and optimization of M/G/1 Vacation Queueing Systems with Server Timeout, Electronic Modeling, V.29, no. 4, ISSN 0204-3572.
8. Oliver C. Ibe (2015), M/G/1 Vacation Queueing Systems with Server Timeout, American Journal, 5(2), 77-88. http://www.scrip.org/journal/ajo.
9. S.Pazhani Bala Murugan and K.Santhi (2015), An M/G/1 queue with server Breakdown and Multiple Working Vacation, An International Journal (AAM), Vol. 10(2), pp. 678-693.