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Obstructions to deforming space curves lying on a smooth cubic surface

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Abstract. In this paper, we study the deformations of curves in the projective 3-space \( \mathbb{P}^3 \) (space curves), one of the most classically studied objects in algebraic geometry. We prove a conjecture due to J. O. Kleppe (in fact, a version modified by Ph. Ellia) concerning maximal families of space curves lying on a smooth cubic surface, assuming the quadratic normality of their general members. We also give a sufficient condition for curves lying on a cubic surface to be obstructed in \( \mathbb{P}^3 \) in terms of lines on the surface. For the proofs, we use the Hilbert-flag scheme of \( \mathbb{P}^3 \) as a main tool and apply a recent result on primary obstructions to deforming curves on a 3-fold developed by S. Mukai and the author.

1. Introduction

Space curves, i.e., curves embedded into \( \mathbb{P}^3 \), are one of the most classically studied objects in algebraic geometry (cf. \([7,24]\)). Among all space curves, curves lying on a smooth cubic surface were intensively studied by virtue of a beautiful geometry endowed with the surface. For example, Mumford \([17]\) found an example of a generically non-reduced component of the Hilbert scheme, whose general point corresponds to a space curve lying on a smooth cubic surface. This example was beautifully generalized by Kleppe in his systematic study \([11]\) on 3-maximal families of space curves. (See also e.g. \([2–6,9,12,13,15,16,18–21,23,27]\) for further studies related to Mumford’s example.) Let \( H(d, g)^{sc} \) denote the Hilbert scheme of smooth connected curves in \( \mathbb{P}^3 \) of degree \( d \) and genus \( g \). Let \( W \) be an irreducible closed subset of \( H(d, g)^{sc} \). Then the least degree \( s(C) \) of surfaces containing a general member \( C \) of \( W \) is a basic invariant of \( W \) and denoted by \( s(W) \). In this paper, \( W \) is called a \( s \)-maximal family (or subset) for \( s \in \mathbb{Z} \), if \( s(W) = s \) and if \( W \) is maximal with respect to \( s \), i.e., \( s(V) > s(W) \) for any irreducible closed subset \( V \) containing \( W \) properly. Every irreducible component \( V \) of \( H(d, g)^{sc} \) is a \( s(V) \)-maximal family, but the converse is not true. Let \( W \subset H(d, g)^{sc} \) be a 3-maximal family and suppose that its general member \( C \) lies on a smooth cubic surface. Kleppe \([11]\) showed that if \( d > 9 \) then \( \dim W = d + g + 18 \), and if moreover \( g \geq 3d - 18 \) and \( H^{2}(\mathbb{P}^3, I_C(3)) = 0 \), then \( W \) is a generically smooth component of \( H(d, g)^{sc} \). Here and later, \( I_C \) denotes the sheaf of ideals defining

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C in \(\mathbb{P}^3\) and \(\mathcal{I}_C(n) := \mathcal{I}_C \otimes_{\mathcal{O}_{\mathbb{P}^3}} \mathcal{O}_{\mathbb{P}^3}(n)\) for \(n \in \mathbb{Z}\). Moreover, he originated the following conjecture, but here it is presented by modifications proposed by Ellia [4].

**Conjecture 1.1.** (Kleppe (a version modified by Ellia)) Suppose that \(d > 9\), \(g \geq 3d - 18\) and \(C\) is linearly normal. If \(H^1(\mathbb{P}^3, \mathcal{I}_C(3)) \neq 0\), then

1. \(W\) is an irreducible component of \((H(d, g)^{sc})_{\text{red}}\) and
2. \(H(d, g)^{sc}\) is generically non-reduced along \(W\).

Thus the 3-maximal families \(W\) in Conjecture 1.1 are expected to give rise to generically non-reduced components of \(H(d, g)^{sc}\). The conclusion (1) of this conjecture is equivalent to that

\[
\dim_{[C]} H(d, g)^{sc} = d + g + 18. \tag{1.1}
\]

The conclusion (2) follows from (1) because if \(d > 9\), then

\[
h^0(C, N_{C/\mathbb{P}^3}) = \dim W + h^1(\mathbb{P}^3, \mathcal{I}_C(3)), \tag{1.2}
\]

where \(h^0(C, N_{C/\mathbb{P}^3})\) represents the tangential dimension of \(H(d, g)^{sc}\) at the point \([C]\) corresponding to \(C\). Ellia pointed out that (1) is false if we drop the assumption of the linear normality of \(C\) by counterexample (see also Dolcetti-Pareschi [3] for more counterexamples). The condition that \(g \geq 3d - 18\) is also necessary for (1) by dimension reason (cf. Remark 2.13). Conjecture 1.1 is related to a problem of classifying all irreducible components \(V\) of \(H(d, g)^{sc}\) with \(s(V) = s\), so far this problem has been solved for \(s \leq 2\) (cf. [26], see also [18, Prop. 4.11]) and a very few (but partial) results are obtained for \(s \geq 4\) (cf. [12,13,20]). Several papers, e.g. [4,11–13,18] contributed to Conjecture 1.1. It is known that if \(g\) is sufficiently large, then the conjecture holds to be true (see Remark 3.3). Mumford’s example appears in a region of \((d, g)\)-plane for which the conjecture is known to be true, and attains the minimal degree and the minimal genus in the region \((d, g) = (14, 24)\).

The main purpose of this paper is to settle down this conjecture assuming further that \(C\) is quadratically normal.

**Theorem 1.2.** Conjecture 1.1 is true if \(C\) is quadratically normal, i.e., \(H^1(\mathbb{P}^3, \mathcal{I}_C(2)) = 0\).

Note that if \(d > 9\) then the 3-maximal families \(W \subset H(d, g)^{sc}\) are in one-to-one correspondence with the 7-tuples \((a; b_1, \ldots, b_6)\) of integers satisfying certain numerical conditions (see (2.7) in §2.2). Then for every \(W\) (and \(C\)) in Conjecture 1.1, we have either \(b_6 = 1\) or \(b_6 = 2\) (cf. Lemmas 2.9 and 3.2). Theorem 1.2 shows that Conjecture 1.1 is always true if \(b_6 = 2\).

Another purpose of this paper is to give a sufficient condition for curves \(C\) lying on a smooth cubic surface to be obstructed in \(\mathbb{P}^3\). Here we say \(C\) is (un)obstructed in \(\mathbb{P}^3\) if the Hilbert scheme of \(\mathbb{P}^3\) is (non)singular at \([C]\). Let \(S\) be a smooth cubic surface in \(\mathbb{P}^3\) and \(C\) a smooth connected curve on \(S\). Then since \(-K_S\) is ample, we can easily see that \(H^1(C, N_{C/S}) = 0\) by adjunction. Then it follows from the exact sequence \(0 \rightarrow N_{C/S} \rightarrow N_{C/\mathbb{P}^3} \rightarrow N_{S/\mathbb{P}^3}|_C \rightarrow 0\) that \(H^1(C, N_{C/\mathbb{P}^3}) \simeq \).
$H^1(C, N_{S/P^3}|_C)$ and hence every obstruction to deforming $C$ in $\mathbb{P}^3$ is contained in $H^1(C, N_{S/P^3}|_C)$ (cf. Remark 2.16). Let $L$ denote the class in $\text{Pic} \ S$ of the invertible sheaf $\mathcal{O}_S(-1) \otimes S_{-1}^{P_3}$ on $S$. Then we have isomorphisms $H^1(C, N_{S/P^3}|_C) \simeq H^2(S, -L)$ and $H^1(\mathbb{P}^3, T_{C}(3)) \simeq H^1(S, -L)$ (cf. (2.6) and (3.1)). It follows from a general theory that the Hilbert-flag scheme of $\mathbb{P}^3$ is nonsingular at $(C, S)$ (cf. Lemma 2.14) and the first projection $pr_1: (C, S) \mapsto [C]$ from the scheme is smooth at $(C, S)$ if $H^1(S, -L) = 0$. This implies that $C$ is unobstructed in $\mathbb{P}^3$ if $H^2(S, -L) = 0$ for either $i = 1$ or $i = 2$ (cf. [11]). Otherwise it follows from the Serre duality and a vanishing theorem that $L + K_S$ is effective and $L$ is not nef (cf. Lemma 2.3).

**Theorem 1.3.** Suppose that $L + K_S \geq 0$ and there exists a (-1)-curve (i.e. a line) $E$ on $S$ such that $m := -L.E > 0$. Then $C$ is obstructed in $\mathbb{P}^3$ if either

1. $m = 1$, or
2. $2 \leq m \leq 3$ and the restriction map

$$\varrho : H^0(S, \Delta) \to H^0(E, \Delta|_E) \quad (1.3)$$

is surjective, where $\Delta := L + K_S - 2mE$ is a divisor on $S$.

Some special cases of Theorem 1.3 were also proved in [3] (for $m = 3$) and [18] (for $m = 1$) (cf. Remark 3.5).

In the proof of Theorems 1.2 and 1.3, we apply a recent result in [20,22] (cf. Theorem 2.22) and prove that a part of the first order deformations $\tilde{C}$ of $C$ in $\mathbb{P}^3$ do not lift to any deformations $\tilde{C}$ of $C$ over $k[t]/(t^3)$, where $k$ is the ground field. (Then $H(d, g)^{sc}$ is singular at $[C]$.) In the case where $h^2(S, -L) = 1$ (and hence $h^1(C, N_{C/P^3}) = 1$), we are even able to determine the dimension of $H(d, g)^{sc}$ at $[C]$ (cf. Proposition 4.6). It is not easy to determine the dimension of the Hilbert scheme at a given singular point. Nevertheless, Theorems 2.22 and Lemma 2.17 make this determination possible by a help of a geometry of lines on cubic surfaces.

The organization of this paper is as follows. In §2.1 we recall basic results on linear systems on del Pezzo surfaces, and prove a vanishing theorem (cf. Lemma 2.7), which is crucial to our proof of Theorem 1.2. In §2.2 we get more specialized into cubic surfaces and recall a well known correspondence between curves on a smooth cubic surface and 7-tuples of integers. In §2.3 and §2.4 we recall some results on Hilbert-flag schemes and primary obstructions to deforming subschemes. We prove Theorems 1.2 and 1.3 in §3 and give some examples in §4. Throughout the paper, we work over an algebraically closed field $k$ of characteristic 0.

## 2. Preliminaries

### 2.1. Linear systems on del Pezzo surfaces

In this section, we collect some results on linear systems on del Pezzo surfaces. We refer to e.g. [14,19] for the proofs.
A del Pezzo surface is a smooth projective surface $S$ with ample anticanonical divisor $-K_S$. Let $S$ be a del Pezzo surface over $k$. Since $k$ is algebraically closed, $S$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow-up of $\mathbb{P}^2$ at $r$ points ($r < 9$) in general positions, i.e., no three are on a line, no six are on a conic and any cubic containing eight points is smooth at each of them (cf. [14, §24]). The self-intersection number $K_S^2$ is called the degree of $S$ and denoted by $\text{deg} \ S$. We have $\text{deg} \ S = 8$ if $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\text{deg} \ S = 9 - r$ otherwise. The anticanonical linear system $|-K_S|$ on $S$ is base point free if $\text{deg} \ S \geq 2$, and very ample if and only if $\text{deg} \ S \geq 3$. A curve $C$ on $S$ is called a line if $K_S \cdot C = -1$ and $C^2 = -1$ and a conic if $K_S \cdot C = -2$ and $C^2 = 0$. Thus there exist no lines on $S$ if $S \simeq \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

We recall some properties of divisors on a del Pezzo surface. Let $D$ be a divisor on $S$. We say that $D$ is nef if $D \cdot C \geq 0$ for all curves $C$ on $S$. Then we have the following:

1. $D$ is nef if and only if $D \geq 0$ and $D \cdot \ell \geq 0$ for all lines $\ell$ on $S$.
2. If $D$ is nef, then $D^2 \geq 0$, where the equality holds if and only if there exists a conic $q$ on $S$ and an integer $m \geq 0$ such that $D \sim mq$. Then we say that $D$ is composed with pencils.

Let $\chi(S, D)$ denote the Euler characteristic of the invertible sheaf $\mathcal{O}_S(D)$ associated to $D$. The following results are well-known.

**Lemma 2.1.** Let $D$ be a divisor on a del Pezzo surface $S$. Then

1. (Zariski decomposition) Suppose that $D \geq 0$. Then for the complete linear system $|D|$ on $S$, there exists a unique decomposition

$$|D| = |D'| + F,$$

where $F$ is the fixed part of $|D|$ (then $F$ is a 1-dimensional subscheme of $S$). Here $D'$ is nef and $F$ is given by

$$F = - \sum_{D \cdot \ell < 0} (D \cdot \ell) \ell,$$

where the sum is taken over all lines $\ell$ on $S$ such that $D \cdot \ell < 0^2$. In particular, $|D|$ is base point free if and only if $D$ is nef, except for the case where $\text{deg} \ S = 1$ and $D \sim -K_S$.

2. Suppose that $D \geq 0$ and $D^2 > 0$. Then $H^1(S, -D) = 0$ if and only if $D$ is nef. Otherwise, we have $h^1(S, -D) = h^0(F, \mathcal{O}_F)$, where $F = \text{Fix} \ |D|$.

3. If $D$ is nef and $\chi(S, -D) \geq 0$, then $H^1(S, -D) = 0$.

**Proof.** (1) follows from [19, Lemma 2.2], (2) from [18, Lemma 2.4] and (3) from [19, Lemma 2.1]. □

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1. For $S$ isomorphic to neither $\mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, $D$ is nef if and only if $D \cdot \ell \geq 0$ for all lines $\ell$ on $S$.
2. The lines $\ell$ are mutually disjoint, i.e., if $(D \cdot \ell) < 0$ and $(D \cdot \ell') < 0$, then $\ell \cap \ell' = \emptyset$. Thus the number of lines contained in the support of $F$ is at most $r$. 
We apply the above lemma to the linear systems of the canonical adjunctions and obtain the following corollary.

**Corollary 2.2.** (cf. [18, Corollary 2.5]) Let $n$ be an integer and suppose that $D + nK_S \geq 0$. Let $F$ be the fixed part of $|D + nK_S|$. Then

1. $F = \sum_{D, \ell \leq n}(n - (D, \ell))\ell$.
2. Suppose that $(D + nK_S)^2 > 0$. Then $H^1(S, -D - nK_S) = 0$ if and only if $D + nK_S$ is nef. Otherwise, $h^1(S, -D - nK_S) = h^0(F, O_F)$.

We will consider in §3 the problem of determining the obstructedness of space curves for curves $C \subset \mathbb{P}^3$ lying on a smooth cubic surface $S \subset \mathbb{P}^3$. Due to the following lemma, we will be able to restrict ourselves to the case where $\mathcal{O}_S(C) \otimes S^{-1}$ is not nef (cf. Theorem 1.3).

**Lemma 2.3.** Let $D$ be a divisor on a del Pezzo surface $S$. If $H^1(S, -D) \neq 0$ for both $i = 1$ and $i = 2$, then $D + K_S$ is effective and $D$ is not nef.

**Proof.** By Serre duality, we have $H^0(S, D + K_S) \cong H^2(S, -D)^\vee$, which implies that $D + K_S \geq 0$. Suppose that $D$ is nef for a contradiction. Then $D^2 \geq 0$. Since $H^1(S, -D) \neq 0$, this implies that $D^2 = 0$ by Lemma 2.1. Then $D$ is composed with pencils, i.e., there exists a conic $q$ on $S$ and an integer $m$ such that $D \sim mq$. Since $q$ is nef, we see that $0 \leq q.(D + K_S) = mq^2 + q.K_S = -2$, thus a contradiction. □

We next setup some notations concerning the coordinates of divisors on a del Pezzo surface. Let $S$ be a blow-up of $\mathbb{P}^2$ at $r$ points in general position. Then the Picard group Pic $S$ of $S$ has a $\mathbb{Z}$-free basis $l, e_1, \ldots, e_r$ and we have Pic $S \cong \mathbb{Z}^{r+1}$. Here and later, $l$ and $e_i$ ($1 \leq i \leq r$) represent the class of the pullback of lines in $\mathbb{P}^2$ and $r$ exceptional curves on $S$, respectively. Thus every divisor $D \sim a l - \sum_{i=1}^r b_i e_i$ on $S$ corresponds to a $(r + 1)$-tuple $(a; b_1, \ldots, b_r)$ of integers by coordinates. For examples, the anticanonical class $-K_S$ ($\cong 3l - \sum_{i=1}^r e_i$) corresponds to $(3; 1, \ldots, 1)$.

**Lemma 2.4.** (cf. [8, V,Theorem 4.9]) Suppose that $\deg S \geq 3$ and $D \sim a l - \sum_{i=1}^r b_i e_i$ in Pic $S$.

1. The class of lines on $S$ are represented by $[i] e_i$ ($1 \leq i \leq r$), $[ii] l - e_i - e_j$ for $1 \leq i < j \leq r$ and $[iii] 2l - \sum_{k=1}^5 e_{i_k}$ for $\{i_1, \ldots, i_5\} \subset \{1, \ldots, r\}$.
2. $D$ is nef if and only if
   - $[i] b_i \geq 0$ for any integer $1 \leq i \leq r$,
   - $[ii] a - b_i - b_j \geq 0$ for any integers $1 \leq i < j \leq r$ and
   - $[iii] 2a - \sum_{k=1}^5 b_{i_k} \geq 0$ for any subset $\{i_1, \ldots, i_5\} \subset \{1, \ldots, r\}$.

We next take actions of Weyl groups into account. For each $r \geq 2$, there exists a Weyl group $W_r \subset \text{Aut}(\text{Pic } S)$. Here $W_r$ is generated by the permutations of $e_i$ ($1 \leq i \leq r$) and by the Cremona transformation $\sigma$ on $\mathbb{P}^2$ (only for $r \geq 3$), where $\sigma$ is defined by $\sigma(l) = 2l - e_1 - e_2 - e_3$ and $\sigma(e_i) = 1 - \sum_{1 \leq j \leq 3, j \neq i} e_j$ if $i \in \{1, 2, 3\}$.
and \( \sigma(e_i) = e_i \) otherwise. Then by virtue of this action of \( W_r \) on \( \text{Pic} \, S \), there exists a suitable blow-up \( S \to \mathbb{P}^2 \) such that

\[
    b_1 \geq \cdots \geq b_r \quad \text{and} \quad a \geq b_1 + b_2 + b_3 \tag{2.2}
\]

(see e.g. (cf. [14]) and [19, §5.3]). We say that the basis \( \{ l, e_1, \ldots, e_r \} \) is standard for \( D \) if (2.2) is satisfied. Under this basis, it is easily seen that \( D \) is nef if and only if \( b_r \geq 0 \). If \( D \) is nef, then \( |D| \) contains an irreducible curve as a member, whose degree \( d \) and (arithmetic) genus \( g \) are respectively obtained by the formulas

\[
    d = 3a - \sum_{i=1}^{r} b_i \quad \text{and} \quad g = \frac{(a - 1)(a - 2)}{2} - \sum_{i=1}^{r} b_i(b_i - 1) \frac{1}{2}. \tag{2.3}
\]

Here the latter formula follows from the adjunction formula \( 2g - 2 = D.(K_S + D) \). In particular, if \( g > 0 \) then we have \( a - b_1 \geq 2 \). On the other hand, by the Hodge index theorem (cf. [8, Chap. V, Theorem 1.9]), we have \((-K_S)^2D^2 - (-K_S.D)^2 = (9 - r)(d + 2g - 2) - d^2 \leq 0 \), which implies

\[
    0 \leq g \leq 1 + (d - 3)d/2 \tag{2.4}
\]

if \( r = 6 \) (i.e, \( \deg S = 3 \)).

We need Lemma 2.5 below to prove Lemma 2.7. For simplicity, we assume \( \deg S \geq 3 \) in these lemmas. In what follows, the genus \( g(D) \) of a divisor \( D \) on \( S \) (or an invertible sheaf \( L \) on \( S \)) is defined by the adjunction formula, which agrees with the genus of the curve \( D \) if \( \deg D \geq 0 \) (or \( L \simeq O_S(D) \)).

**Lemma 2.5.** Let \( S \) be a del Pezzo surface, \( F \) a divisor on \( S \) that is a sum of mutually disjoint (single) \( k \) lines on \( S \). Let \( \varepsilon : S \to S' \) be the blow-down of \( F \) from \( S \) and \( D' \) a divisor on \( S' \) of genus \( g(D') \geq k \). If \( D' \) is nef and \( \deg S \geq 3 \), then \( \varepsilon^*D' - 2F \) is nef.

**Proof.** Let \( E_i \) (\( 1 \leq i \leq k \)) be mutually disjoint lines on \( S \) and put \( F := \sum_{i=1}^{k} E_i \) and \( D := \varepsilon^*D' - 2F \). Then we note that \( g(D) = g(D') - k \). Therefore, it suffices to prove the lemma for \( k = 1 \) by induction. Suppose now that \( F \) is a line on \( S \) and \( g(D') \geq 1 \). We put \( r := 9 - K^2_S \), i.e., the number of points of \( \mathbb{P}^2 \) blown-up to obtain \( S \). Then since \( \deg S \geq 3 \), we have \( 1 \leq r \leq 6 \). By virtue of the action of the Weyl group \( W_r \) on \( \text{Pic} \, S \), we may assume that \( e_r \) is the class of \( F \) and moreover, \( D \) is linearly equivalent to \( a1 - \sum_{i=1}^{r-1} b_i e_i - 2e_r \) with \( a \geq b_1 + b_2 + b_3 \) (only for \( r \geq 4 \)) and \( b_1 \geq \cdots \geq b_{r-1} \). Since \( D' \) is nef, we have \( b_{r-1} \geq 0 \), which implies \( D.\varepsilon e_i \geq 0 \) for all \( i \). It follows from \( g(D') \geq 1 \) that \( D.l - e_i - e_r = a - b_i - 2 \geq a - b_1 - 2 \geq 0 \) for all \( 1 \leq i \leq r - 1 \). Then we also see that \( D.(2l - e_{i_1} - \cdots - e_{i_4} - e_r) = 2a - b_{i_1} - \cdots - b_{i_4} - 2 \geq 2a - b_1 - \cdots - b_4 - 2 \geq 0 \) for all \( 1 \leq i_1 < \cdots < i_4 \leq r - 1 \). Thus we have proved the lemma by Lemma 2.4. \( \square \)

**Remark 2.6.** The conclusion of Lemma 2.5 is not true if \( \deg S < 3 \). In fact, suppose that \( \deg S = 2 \) and \( F \) is a line on \( S \). Then \( S' \) is a cubic surface. We consider the anticanonical class \( D' = -K_{S'} \) on \( S' \), whose genus is equal to one. Since \( D = \varepsilon^*D' - 2F \) represents the class of a line on \( S \), \( D \) is clearly not nef.
The following lemma is a generalization of Lemma 2.1 and plays an important role in our proof of Theorem 1.2.

**Lemma 2.7.** Suppose that \( \deg S \geq 3 \) and \( D \) is effective. If

1. \( \chi(S, -D) \geq 0 \)
2. \( D \cdot E \geq -1 \) for any line \( E \) on \( S \),

then we have \( H^1(S, 3F - D) = 0 \), where \( F \) is the fixed part of \( |D| \).

**Proof.** If \( D \) is nef, then we have \( F = 0 \) and hence the lemma follows from the first condition and Lemma 2.1. Suppose now that \( D \) is not nef. Then by the same lemma, \( F \) is non-empty. It follows from the second condition that \( F \) is a sum of mutually disjoint lines \( E_i \) \((1 \leq i \leq k)\) on \( S \). Let \( \varepsilon : S \to S' \) be the blow-down of \( F \) from \( S \). Then \( \Delta := D - F \) is the pull-back of a nef divisor \( \Delta' \) on \( S' \), i.e. \( \varepsilon^* \Delta' \sim \Delta \). Since \( D \sim \varepsilon^* \Delta' + F \), \( K_S = \varepsilon^* K_{S'} + F \) and \( F^2 = -k \), we compute that

\[
2\chi(S, -D) - 2 = D.(D + K_S) = \Delta'.(\Delta' + K_{S'}) - 2k = 2\chi(S', -\Delta') - 2 - 2k.
\]

Thus \( g(\Delta') = \chi(S', -\Delta') = \chi(S, -D) + k \geq k \). Moreover, since \( \Delta' \) is nef, so is \( D - 3F \sim \varepsilon^* \Delta' - 2F \) by Lemma 2.5. Then we note that \( D.F = F^2 = -k \) and hence we see that \( \chi(S, 3F - D) \geq 0 \) by

\[
\chi(S, 3F - D) - \chi(S, -D) = \chi(S, 3F) - \chi(S, O_S) - 3D.F = -3k + 3k = 0.
\]

Thus we have completed the proof by Lemma 2.1. \( \Box \)

Finally, we prepare a lemma concerning the bigness of divisors.

**Lemma 2.8.** If \( \deg S \geq 3 \), \( D + K_S \geq 0 \) and \( \chi(S, -D) \geq 0 \) then \( D \) is big, i.e., \( D^2 > 0 \).

**Proof.** It follows from the Riemann-Roch theorem that

\[
D^2 = 2\chi(S, -D) - 2 - D.K_S \geq -2 - K_S.D.
\]

Then since \(-K_S\) is ample, we have

\[
-2 - K_S.D = -2 - K_S.(D + K_S) + K_S^2 \geq -2 + K_S^2 > 0.
\]

\( \Box \)
2.2. 3-maximal families

In this section, we consider cubic del Pezzo surfaces. First we recall some properties of curves on the surfaces. Let $S$ be a smooth cubic surface. Then $S$ is a blow-up of $\mathbb{P}^2$ at six points in general position. As we have seen in the previous section, for every divisor $D$ on $S$, Pic $S$ has a standard basis $\{l, e_1, \ldots, e_6\}$ and $D$ corresponds to a 7-tuple $(a; b_1, \ldots, b_6)$ of integers satisfying

$$b_1 \geq \cdots \geq b_6 \quad \text{and} \quad a \geq b_1 + b_2 + b_3. \tag{2.5}$$

Then $D$ is nef if and only if $b_6 \geq 0$. Moreover, $|D|$ contains a smooth connected curve not a line nor a conic if and only if $a > b_1$ and $b_6 \geq 0$. Its degree $d$ and genus $g$ are respectively given by (2.3) and they satisfy the inequality (2.4).

We next consider the projective (ab)normality of curves on a smooth cubic surface. Given an integer $n$, a projective variety $V \subset \mathbb{P}^d$ is said to be $n$-normal if $H^1(\mathbb{P}^d, I_V(n)) = 0$. $V$ is called projectively normal if $V$ is $n$-normal for all $n \in \mathbb{Z}$.

**Lemma 2.9.** Let $C$ be a curve on a smooth cubic surface $S$. We assume that $C + nK_S \geq 0$ and $(C + nK_S)^2 > 0$. Then

$$h^1(\mathbb{P}^3, I_C(n)) = h^0(F, \mathcal{O}_F),$$

where $F$ is the fixed part of $|C + nK_S|$. In particular, $C$ is $n$-normal if and only if $C + nK_S$ is nef.

**Proof.** We note that $I_S(n) \simeq \mathcal{O}_{\mathbb{P}^3}(n-3)$, whose $i$-th cohomology groups vanish for $i = 1, 2$. It follows from an exact sequence $0 \to I_S(n) \to I_C(n) \to \mathcal{O}_S(-nK_S - C) \to 0$ on $\mathbb{P}^3$ that

$$H^1(\mathbb{P}^3, I_C(n)) \simeq H^1(S, -nK_S - C). \tag{2.6}$$

Therefore, the lemma follows from Corollary 2.2.

**Remark 2.10.** Since $-K_S$ is effective and ample, if $D$ is nef and big, then so is $D - K_S$. This implies that in the setting of Lemma 2.9, if $C$ is $n$-normal then $C$ is $m$-normal for all $m < n$.

Let $C$ be a curve on $S$ with coordinate $(a; b_1, \ldots, b_6)$ under a standard basis. We note that every element of the Weyl group preserves the class $K_S$. Therefore, if $\{l, e_1, \ldots, e_6\}$ is a standard basis for $C$, then so is for $D := C + nK_S$ with any integer $n$. Therefore, provided that $D \geq 0$ and $D^2 > 0$, $C$ is $n$-normal if and only if $b_6 \geq n$ by Lemma 2.9. One should be careful in applying Lemma 2.9 to a computation of the $n$-abnormality $h^1(\mathbb{P}^3, I_C(n))$ of $C$. The support of the fixed part $F$ of $|C + nK_S|$ consists of any set of mutually disjoint lines, whose number is at most 6. The next example shows that even under the standard basis, the support of $F$ does not necessarily consist of $e_1, \ldots, e_6$. This fact corresponds to the fact that every blow-down of a cubic surface along 5 lines is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$ blown up at a point.
**Example 2.11.** Let \( C \) be a curve (of \( d = 18 \) and \( g = 31 \)) on a smooth cubic surface \( S \) corresponding to the 7-tuple \((12; 5, 5, 2, 2, 2, 2)\). Here and later, we abuse notations and identify divisor classes on \( S \) with 7-tuples of integers corresponding to them. We put \( D := C + 3K_S \) in \( \text{Pic} \ S \). Then we see that

\[
D = (2; 1, 1, 0, 0, 0, 0) + (1; 1, 1, -1, -1, -1, -1) = D' + F,
\]

where \( D' = 2l - e_1 - e_2 \) is nef and \( F \) is the fixed part of \(|C + 3K_S|\). We note that \( F \) consists of 5 disjoint lines \( l - e_1 - e_2 \) and \( e_i \) (\( 3 \leq i \leq 6 \)). Since \( D' \) is also big by \( D'^2 = 2 \), we have \( H^1(S, -D') = 0 \). Then it follows from (2.6) and the exact sequence \( 0 \to O_S(-D) \to O_S(-D') \to O_F \to 0 \) that \( h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = h^1(S, -D) = h^0(F, O_F) = 5 \).

Let \( d > 0 \) and \( g \) be two integers satisfying (2.4) and \((a; b_1, \ldots, b_6)\) a 7-tuple of integers satisfying a set of conditions

\[
(2.5), \ (2.3) \ \text{(with} \ r = 6), \ a > b_1 \ \text{and} \ b_6 \geq 0. \quad (2.7)
\]

Then according to [11], we can associate to \((a; b_1, \ldots, b_6)\) a closed subset of the Hilbert scheme. Let \( H(d, g)^{sc} \) be the Hilbert scheme of smooth connected curves of degree \( d \) and genus \( g \) in \( \mathbb{P}^3 \).

**Definition 2.12.** We define a closed subset \( W(a; b_1, \ldots, b_6) \subset H(d, g)^{sc} \) by taking the closure in \( H(d, g)^{sc} \) of the family of curves \( C \subset \mathbb{P}^3 \) lying on a smooth cubic surface \( S \subset \mathbb{P}^3 \) and such that

\[
C \sim a l - \sum_{i=1}^{6} b_i e_i
\]
on \( S \) for some (standard) basis \( \{l, e_1, \ldots, e_6\} \) of \( \text{Pic} \ S \).

Let \( W = W(a, b_1, \ldots, b_6) \). If \( d > 9 \) then every general member \( C \) of \( W \) is contained in a unique cubic surface \( S \), and hence \( W \) is birationally equivalent to \( \mathbb{P}^{d+g-1} \)-bundle over \(|O_{\mathbb{P}^3}(3)| \simeq \mathbb{P}^{19} \), where the numbers \( d + g - 1 \) and 19 are equal to the dimensions of the linear systems \(|O_S(C)| \) on \( S \) and \(|O_{\mathbb{P}^3}(3)| \) on \( \mathbb{P}^3 \), respectively. In particular, \( W \) is irreducible and of dimension \( d + g + 18 \). It is known that if \( d > 2 \) then every 3-maximal family in \( H(d, g)^{sc} \) (see §1 for its definition) can be obtained as \( W(a; b_1, \ldots, b_6) \) for some \((a; b_1, \ldots, b_6)\) satisfying (2.7), provided that its general member is contained in a smooth cubic surface (cf. [9,11]). Conversely, if \( d > 9 \) then \( W(a; b_1, \ldots, b_6) \) becomes a 3-maximal family.

**Remark 2.13.** By deformation theory, every irreducible component of \( H(d, g)^{sc} \) is of dimension at least \( 4d \) (\( = \chi(C, N_C/\mathbb{P}^3) \)). Therefore, if \( d > 9 \) and \( W = W(a; b_1, \ldots, b_6) \subset H(d, g)^{sc} \) is an irreducible component of \( (H(d, g)^{sc})_{\text{red}} \), then we have \( g \geq 3d - 18 \) by dimension.
2.3. Hilbert-flag schemes and Primary obstructions

In this section, we briefly recall the definition of Hilbert-flag schemes and their infinitesimal properties (cf. [11,25]). Given a projective scheme $X$ and a pair of Hilbert polynomials $P$ and $Q$, there exists a contravariant functor $HF_{P,Q} : (\text{schemas}) \to (\text{sets})$ that to each base scheme $B$ assigns a pair of closed subschemes $C \subset S \subset X \times_k B$, both flat over $B$, and where the fibers of $C$ (resp., $S$) have the Hilbert polynomial $P$ (resp. $Q$). This functor is represented by a projective scheme $HF_{P,Q} X$, so called the Hilbert-flag scheme of $X$. Let $(C, S)$ be a pair of closed subschemes of $X$ with Hilbert polynomials $(P, Q)$, respectively and such that $C \subset S \subset X$. Then the normal sheaf $N_{(C,S)/X}$ of $(C, S)$ is a sheaf of $O_X$-module and defined by the fiber product

$$N_{(C,S)/X} := N_{C/X} \times_{N_{S/X}} N_{S/X}$$

of the projection $N_{C/X} \to N_{S/X}|_C$ and the restriction $N_{S/X} \to N_{S/X}|_C$ of normal sheaves of $C$ and $S$ in $X$, respectively (cf. [21, §2.2]).

In what follows, we assume that the two embeddings $C \hookrightarrow S$ and $S \hookrightarrow X$ are both regular (then so is $C \hookrightarrow X$). Then it follows from a general theory (cf. [25, Proposition 4.5.3]) that $H^0(X, N_{(C,S)/X})$ and $H^1(X, N_{(C,S)/X})$ respectively represent the tangent space and the obstruction space of $HF_{P,Q} X$ at $(C, S)$, and we have

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \leq \dim_{(C,S)} HF_{P,Q} X \leq h^0(X, N_{(C,S)/X}).$$

(2.8)

Thus if $H^1(X, N_{(C,S)/X}) = 0$ then $HF_{P,Q} X$ is nonsingular at $(C, S)$ of expected dimension, that is, the number in the left hand side of (2.8). The expected dimension coincides with $\chi(X, N_{(C,S)/X})$, provided that $H^i(X, N_{(C,S)/X}) = 0$ for all $i \geq 2$. Let $Hilb_P X$ denote the Hilbert scheme of $X$ with Hilbert polynomial $P$. Then there exist two natural projections $pr_1 : HF_{P,Q} X \to Hilb_P X$ and $pr_2 : HF_{P,Q} X \to Hilb_Q X$, i.e., the first and the second projections. Correspondingly, there exist two natural exact sequences

$$0 \rightarrow \mathcal{I}_{C/S} \otimes_S N_{S/X} \rightarrow N_{(C,S)/X} \rightarrow N_{C/X} \rightarrow 0 \quad (2.9)$$

and

$$0 \rightarrow N_{C/S} \rightarrow N_{(C,S)/X} \rightarrow N_{S/X} \rightarrow 0 \quad (2.10)$$

of sheaves on $X$, where $\pi_1$ and $\pi_2$ induce the tangent map and the map on obstruction spaces of $pr_1$ and $pr_2$, respectively (cf. [21, §2.2]).

We recall that a normal projective variety $Z$ is called Fano if $-K_Z$ is ample. The following lemma shows that if $S$ and $X$ are both Fano and if the two embeddings $C \hookrightarrow S$ and $S \hookrightarrow X$ are both of codimension one, then all the higher cohomology groups of $N_{(C,S)/X}$ vanish, and we benefit a nice property from the Hilbert-flag scheme of $X$. 


Lemma 2.14. (1) If $S$ and $X$ are both Fano, and both $C \subset S$ and $S \subset X$ are effective Cartier divisors, then we have $H^i(X, N_{(C,S)/X}) = 0$ for all $i > 0$.

(2) If $X$ is a Fano 3-fold, $S$ is a del Pezzo surface and $C$ is a curve of degree $d = (-K_S.C)$ and genus $g$, then $HFX$ is nonsingular at $(C, S)$ of expected dimension
\[ \chi(X, N_{(C,S)/X}) = \frac{(-K_X.S^2)_X}{2} + d + g, \]
where $(D_1.D_2.D_3)_X$ denotes the intersection number of divisors $D_1, D_2, D_3$ on $X$.

**Proof.** We note by adjunction that $N_{C/S} \simeq -K_S|_C + K_C$ and $N_{S/X} \simeq -K_X|_S + K_S$. Therefore, the higher cohomology groups $H^i(C, N_{C/S})$ and $H^i(S, N_{S/X})$ vanish for all $i > 0$ by the ampleness of $-K_S$ and $-K_X$, respectively. Thus (1) follows from the exact sequence (2.10). By Riemann-Roch formulas on curves and surfaces, we see that $\chi(C, N_{C/S}) = d + g - 1$ and $\chi(S, N_{S/X}) = (-K_X.S^2)_X/2 + 1$. Hence we obtain (2) by additivity on Euler characteristics. \qed

Lemma 2.15. (cf. [11,21]) If $S \subset \mathbb{P}^3$ is a smooth cubic surface, and $C$ is a smooth curve on $S$ of degree $d$ and genus $g$, then

1. $H^1(C, N_{C/S}) = H^1(S, N_{S/\mathbb{P}^3}) = 0$.
2. $HF\mathbb{P}^3$ is nonsingular at $(C, S)$ of expected dimension $\chi(\mathbb{P}^3, N_{(C,S)/\mathbb{P}^3}) = d + g + 18$, which coincides with the dimension of 3-maximal families in $H(d, g)^{sc}$ containing $C$ if $d > 9$ (cf. §2.2).

**Proof.** Since $S$ and $\mathbb{P}^3$ are both Fano, we obtain (1). Since $-K_{\mathbb{P}^3} \sim 4H$ and $S \sim 3H$, where $\text{Pic}\mathbb{P}^3 \simeq \mathbb{Z}[H]$, we see that $(-K_{\mathbb{P}^3}.S^2)_{\mathbb{P}^3} = 36$ and thus (2) follows from Lemma 2.14. \qed

We next recall the definition of primary obstructions to deforming subschemes. Let $\tilde{C}$ be a first order deformation of $C$ in $X$, that is, an infinitesimal deformation $\tilde{C}$ of $C$ in $X$ over the ring $k[t]/(t^2)$ of dual numbers. Then $\tilde{C}$ naturally corresponds to a global section $\alpha$ of $N_{C/X}$. Since the embedding $C \hookrightarrow X$ is regular, every obstruction to deforming $C$ in $X$ is contained in $H^1(C, N_{C/X})$ (cf. [25, Theorem 4.3.5]). Every $\alpha$ in $H^0(C, N_{C/X})$ defines an element $\text{ob}(\alpha)$ of $H^1(C, N_{C/X})$ such that $\text{ob}(\alpha)$ is zero if and only if $\tilde{C}$ extends to a deformation $\tilde{C}$ of $C$ over $k[t]/(t^3)$. Here $\text{ob}(\alpha)$ is called the primary obstruction of $\alpha$ (or $\tilde{C}$). It is known that $\text{ob}(\alpha)$ is expressed as a cup product of $\alpha \in \text{Hom}_X(I_C, O_C) \simeq H^0(C, N_{C/X})$ and the extension class $e = [0 \to I_C \to O_X \to O_C \to 0] \in \text{Ext}^1(O_C, I_C)$ and we have $\text{ob}(\alpha) = \alpha \cup e \cup \alpha$ (cf. [20, Theorem 2.1]). If $\text{ob}(\alpha) \neq 0$ then $\tilde{C}$ does not lift to a global deformation of $C$ in $X$ and $\text{Hilb}_P X$ is singular at $[C]$.

Remark 2.16. Here we give a remark on obstructions to deforming space curves lying on a smooth cubic surface. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface, $C$ a smooth curve on $S$. Then by Lemma 2.15, we always have $H^1(C, N_{C/S}) = H^1(S, N_{S/\mathbb{P}^3}) = 0$. Thus both of the deformations of $C$ in $S$ and the deformations of $S$ in $\mathbb{P}^3$ behave well. However, those of $C$ in $\mathbb{P}^3$ can behave badly in
general. For example, for curves $C$ considered in Mumford’s example [17], i.e., $C \sim -4K_S + 2E$ and $E$ is a line on $S$, we see that $H^1(C, N_{C/P^3}) \neq 0$. It follows from the exact sequence $0 \rightarrow N_{C/S} \rightarrow N_{C/P^3} \rightarrow N_{S/P^3} \rightarrow 0$ that $H^1(C, N_{C/P^3}) \cong H^1(C, N_{S/P^3}|_C)$ and hence every obstruction to deforming $C$ in $\mathbb{P}^3$ is contained in $H^1(C, N_{S/P^3}|_C)$, and this cohomology group does not vanish in the case of Mumford’s example. In fact, it was proved by Curtin [1] that there exists a first order deformation $\tilde{C}$ of $C$ in $\mathbb{P}^3$ whose primary obstruction is nonzero in $H^1(C, N_{C/P^3})$. Our method of computing primary obstructions, which will be explained in § 2.4, is based on a technique used in [1] and also its generalizations in [16, 18].

Let $\mathcal{W}_{C,S}$ be an irreducible component of $HF_{P,Q} \times X$ passing through $(C, S)$, and let $pr'_1 : \mathcal{W}_{C,S} \rightarrow \text{Hilb}_P X$ be the restriction of $pr_1$ to $\mathcal{W}_{C,S}$. Then it follows from a general deformation theory (cf. [11, Lemma A10], see also [9, Theorem 1.3.4]) that if $H^1(S, \mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$, then $pr_1$ is smooth at $(C, S)$. Then so is $pr'_1$. If moreover $H^1(X, N_{(C,S)/X}) = 0$ then $pr'_1$ is dominant in a neighborhood of $[C]$ (cf. [21, Theorem 2.4]). If $H^1(X, N_{(C,S)/X}) = 0$ and $H^1(S, \mathcal{I}_{C/S} \otimes_S N_{S/X}) \neq 0$, then there exists an exact sequence

$$H^0(X, N_{(C,S)/X}) \xrightarrow{p_1} H^0(C, N_{C/X}) \xrightarrow{\delta} H^1(S, \mathcal{I}_{C/S} \otimes_S N_{S/X}) \rightarrow 0,$$

(2.11)

which is deduced from (2.9). Then since $p_1$ is not surjective, there exists a first order deformation $\tilde{C}$ of $C$ in $X$ not contained in any first order deformation $\tilde{S}$ of $S$ in $X$. We need the following lemma for our proof of Theorem 1.2.

**Lemma 2.17.** Suppose that $H^1(X, N_{(C,S)/X}) = 0$ and $H^1(S, \mathcal{I}_{C/S} \otimes_S N_{S/X}) \neq 0$. If the primary obstruction $ob(\alpha)$ is nonzero in $H^1(C, N_{C/X})$ for every global section $\alpha \in H^0(C, N_{C/X}) \setminus \text{im } p_1$, then $pr'_1$ is dominant in a neighborhood of $[C]$. If moreover $H^0(S, \mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$, then

$$\dim_{[C]} \text{Hilb}_P X = \dim_{(C,S)} HF_{(P,Q)} \times X.$$

**Proof.** The proof is essentially same as that of [19, Lemma 4.11], where $X$ and $S$ are assumed to be a smooth del Pezzo 3-fold and its smooth hyperplane section, respectively. However we repeat the proof here for the reader’s convenience. We show that every small global deformation of $C$ in $X$ is contained in that of $S$ in $X$. Let $T$ be a small neighborhood of $[C]$ in $\text{Hilb}_P X$ and $T \rightarrow \text{Hilb}_P X$ the embedding of $T$. Then by base change, there exists a family $C_T \subset X \times T$ of curves in $X$ with a point $0 \in T$ such that $C_0 = C$. Let $\text{Spec } k[t]/(t^2) \rightarrow T$ be an element of the Zariski tangent space of $T$ at $0$. Then there exist a first order deformation $\tilde{C} \rightarrow \text{Spec } k[t]/(t^2)$ of $C$ and a global section $\alpha$ of $N_{C/X}$, correspondingly. Then by assumption, $\alpha$ is contained in $\text{im } p_1$, and hence there exists a first order deformation $(\tilde{C}, \tilde{S})$ of $(C, S)$ with $\tilde{S} \supset \tilde{C}$. Since $HF X$ is nonsingular at $(C, S)$, there exists a global deformation $(C_T, S_T)$ of $(C, S)$ over $T$ as a lift of $(\tilde{C}, \tilde{S})$. Thus $pr'_1$ is dominant near $[C]$. If moreover $H^0(S, \mathcal{I}_{C/S} \otimes_S N_{S/X}) = 0$, then $pr'_1$ is locally an embedding in a neighborhood of $(C, S)$ (cf. [21, §2.2]). Thus we have

$$\dim_{(C,S)} HF_{(P,Q)} \times X = \dim \mathcal{W}_{C,S} = \dim pr'_1(\mathcal{W}_{C,S}) = \dim_{[C]} \text{Hilb}_P X.$$
Corollary 2.18. Let \( X = \mathbb{P}^3 \), \( S \) a smooth cubic surface in \( \mathbb{P}^3 \), and \( C \subset S \) a smooth curve of degree \( d > 9 \) and genus \( g \). If \( \text{ob}(\alpha) \neq 0 \) in \( H^1(C, N_{C/\mathbb{P}^3}) \) for all \( \alpha \notin \text{im} \, p_1 \), then \( \dim_{[C]} H(d, g)^{\text{re}} = d + g + 18 \).

Proof. We see that \( \mathcal{I}_{C/S} \otimes S_{\mathbb{P}^3} \simeq -C - 3K_S \) in \( \text{Pic } S \). Since \(-K_S(-C - 3K_S) = -d + 9 < 0\), we have \( H^0(S, \mathcal{I}_{C/S} \otimes S_{\mathbb{P}^3}) = 0 \). Then we have proved the corollary by Lemma 2.15.

2.4. Obstructedness criterion

In this section, we recall a result in [22] concerning primary obstructions to deforming curves on a 3-fold. Our proofs of Theorems 1.2 and 1.3 heavily depend on Theorem 2.22. We refer to [16, 19, 20, 22] for more information about exterior components, infinitesimal deformations with pole, and also the proof of Theorem 2.22.

Let \( X \) be a projective 3-fold and \( C \) an irreducible curve on \( X \). We assume that there exists an intermediate surface \( S \) such that \( C \leftarrow S \leftarrow X \) are regular embeddings. Let \( \alpha \) be a global section of \( N_{C/X} \). We consider a natural projection \( \pi_{C/S} : N_{C/X} \to N_{S/X} \big|_C \), which induces maps \( H^i(C, N_{C/X}) \to H^i(C, N_{S/X} \big|_C) \) \( (i = 0, 1) \) on their cohomology groups. The images of \( \alpha \) and \( \text{ob}(\alpha) \) in \( H^i(C, N_{S/X} \big|_C) \) \( (i = 0, 1) \) by the induced maps are called the exterior component of \( \alpha \) and \( \text{ob}(\alpha) \) and denoted by \( \pi_{C/S}(\alpha) \) and \( \text{ob}_S(\alpha) \), respectively. By definition, if \( \text{ob}_S(\alpha) \) is nonzero then so is \( \text{ob}(\alpha) \).

We recall the definition of infinitesimal deformations with poles, which was introduced in [16]. We are interested in a global section \( \gamma \) of \( N_{S/X} \big|_C \) such that \( \gamma \) does not lift to a global section of \( N_{S/X} \) but lifts to that of \( N_{S/X}(E) := N_{S/X} \otimes_S \mathcal{O}_S(E) \) after admitting a pole along a divisor \( E \geq 0 \) on \( S \).

Definition 2.19. Let \( E \) be a nonzero effective Cartier divisor on \( S \). Then a rational section \( \beta \in H^0(S, N_{S/X}(E)) \setminus H^0(S, N_{S/X}) \) is called an infinitesimal deformation with pole.

Here and later, for a sheaf \( \mathcal{F} \) and a Cartier divisor \( E \) on \( S \), we denote the sheaf \( \mathcal{F} \otimes_S \mathcal{O}_S(E) \) by \( \mathcal{F}(E) \). When \( \mathcal{F} \) is invertible, we abuse notations and denote the invertible sheaf \( \mathcal{F}(E) \) by \( \mathcal{F} + E \). If \( E \) is effective then for every integer \( i \geq 0 \) there exists a natural map

\[
H^i(S, \mathcal{F}) \to H^i(S, \mathcal{F} \otimes_S \mathcal{O}_S(E)). \tag{2.12}
\]

Given a cohomology class \( \mathcal{c} \) in \( H^i(S, \mathcal{F}) \), we denote by \( r(\mathcal{c}, E) \) the image of \( \mathcal{c} \) by this map (and similarly for \( \mathcal{c} \big|_C \) in \( H^i(C, \mathcal{F} \big|_C) \)). Let \( \mathcal{k}_C \) denote the extension class in \( \text{Ext}^1_S(\mathcal{O}_C, \mathcal{O}_S(-C)) \) of the short exact sequence

\[
0 \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{2.13}
\]

on \( S \). When \( E \geq 0 \) is prime and \( \beta \) is a global section of \( \mathcal{F}(E) \), we call the restriction \( \beta \big|_E \) of \( \beta \) to \( E \), that is a global section of \( \mathcal{F}(E) \big|_E \), the principal part of \( \beta \) along \( E \).

The following lemma is a generalization of [20, Lemma 3.1].
Lemma 2.20. Let $L$ be an invertible sheaf on $S$, $E$ a nonzero effective divisor on $S$ not containing $C$ as its component, $\gamma$ a global section of $L|_C := L \otimes_S \mathcal{O}_C$. Then

(1) $r(\gamma, E)$ lifts to a global section $\beta$ of $L + E$ on $S$ if and only if $r(\gamma, E) \cup k_C = 0$ in $H^1(S, L + E - C)$.

(2) If $H^1(S, L + E - C) = 0$ and $\gamma \cup k_C \neq 0$, then there exist a triplet $(E', E_0, \beta)$ of a subdivisor $E' \subset E$ on $S$, a prime divisor $E_0 \subset E'$, and a lift $\beta$ of $r(\gamma, E')$ in $H^0(S, L + E')$ such that the principal part $\beta|_{E_0}$ of $\beta$ along $E_0$ is nonzero and contained in the subgroup $H^0(E_0, (L + E' - C)|_{E_0})$.

Proof. (1) follows from [20, Lemma 3.1]. (Consider the first coboundary map of $(2.13) \otimes_S L + E$, which is a map taking a cup product with $k_C$.) Since $E$ is nonzero and effective, there exist positive integers $k, m_i (1 \leq i \leq k)$ and prime divisors $E_i (1 \leq i \leq k)$ such that

$$E = \sum_{i=1}^k m_i E_i.$$ 

Since the reduction map $r(\ast, E)$ (cf. (2.12)) and the cup product map $\cup k_C$ are compatible, we see that

$$r(\gamma, E') \cup k_C = r(\gamma \cup k_C, E')$$

in $H^1(S, L + E' - C)$ for any subdivisor $E' \subset E$. Thus by admitting to $\gamma$ a new pole along some $E_i$, or increasing the order of poles along $E_i$, we obtain a divisor $E' \subset E$ such that

(a) $r(\gamma, E') \cup k_C = 0$ in $H^1(S, L + E' - C)$ and

(b) $r(\gamma, E' - E_i) \cup k_C \neq 0$ in $H^1(S, L + E' - E_i - C)$.

for some $1 \leq i \leq k$. Then $r(\gamma, E')$ lifts to a global section $\beta$ of $L + E'$ by (a) but does not lift to that of $L + E' - E_i$ by (b). Then by virtue of [20, Lemma 3.1], $\beta|_{E_i}$ is nonzero in $H^0(E_i, (L + E')|_{E_i})$ and contained in the subgroup $H^0(E_i, (L + E' - C)|_{E_i})$. \hfill $\Box$

We recall a sufficient condition for $\text{ob}_S(\alpha)$ to be nonzero. Let $E_i (1 \leq i \leq k)$ be nonzero effective prime divisors on $S$ such that

(1) $E_i$ are mutually disjoint, i.e., $E_i \cap E_j = \emptyset$ if $i \neq j$, and

(2) if $D$ and $D'$ are two effective divisors on $S$ whose supports are contained in $\bigcup_{i=1}^k E_i$ and if $D \leq D'$, then the natural map

$$H^1(S, D) \rightarrow H^1(S, D') \quad (2.14)$$

is injective.
Example 2.21. If $S$ is a del Pezzo surface and $E_i (1 \leq i \leq k)$ are mutually disjoint lines on $S$, then the map (2.14) is injective. In fact, since $D$ and $D'$ have supports on $\bigcup_{i=1}^{k} E_i$, so does $E := D' - D$. If $D \subseteq D'$ then $E$ is nonzero and effective. Then since $E_i$ are $(-1)$-curves on $S$, we see that $H^0(E, \mathcal{O}_E(D')) = 0$, where $\mathcal{O}_E(D') \simeq \mathcal{O}_S(D') \otimes \mathcal{O}_E$. Thus the injectivity of (2.14) follows from the exact sequence $0 \to \mathcal{O}_S(D) \to \mathcal{O}_S(D') \to \mathcal{O}_E(D') \to 0$.

Let $\gamma = \pi_{C/S}(\alpha)$ be the exterior component of $\alpha$. We consider a divisor $E = \sum_{i=1}^{k} m_i E_i$ on $S$ with positive coefficients $m_i \in \mathbb{Z}_{>0}$ and assume that $C \not\equiv E_i$ for any $i = 1, \ldots, k$ (then $C \cap E_i$ are finitely many points). We assume furthermore that $r(\gamma, E)$ lifts to a section $\beta \in H^0(S, N_{S/X}(E)) \setminus H^0(S, N_{S/X})$ (an infinitesimal deformation with pole), i.e., we have

$$r(\pi_{C/S}(\alpha), E) = \beta|_C \quad \text{in} \quad H^0(C, N_{S/X}(E)|_C). \quad (2.15)$$

Let $\beta_i := \beta|_{E_i}$ be the principal part of $\beta$ along $E_i$. Then by assumption, $\beta_i$ is a global section of the invertible sheaf $N_{S/X}(m_i E_i)|_{E_i} (\simeq N_{S/X}(E)|_{E_i})$ on $E_i$. Moreover, by Lemma 2.20, $\beta_i$ is contained in the subgroup

$$H^0(E_i, N_{S/X}(m_i E_i - C)|_{E_i}) \subset H^0(E_i, N_{S/X}(m_i E_i)|_{E_i}).$$

We illustrate the relations among $\alpha$, $\beta$ and $\beta_i$ in Fig. 1.

Let $\partial_{E_i}$ denote the coboundary map of the short exact sequence

$$[0 \longrightarrow N_{E_i/S} \longrightarrow N_{E_i/X} \xrightarrow{\pi_{E_i/S}} N_{S/X}|_{E_i} \longrightarrow 0] \otimes E_i \mathcal{O}_{E_i}(E) \quad (2.16)$$

on $E_i$. Then $\partial_{E_i}(\beta_i)$ defines an element of $H^1(E_i, N_{E_i/S}(E)) (\simeq H^1(E_i, (m_i + 1)E_i)).$ The following theorem is a refinement of [20, Theorem 1.1], which enables us to deduce the nonzero of $\ob_S(\alpha)$ from that of the cup product of $\partial_{E_i}(\beta_i)$ with $\beta_i$.

**Theorem 2.22.** (cf. [22, Theorem 1]) Suppose that $H^1(S, N_{S/X}) = 0$. Then the exterior component $\ob_S(\alpha)$ of $\ob(\alpha)$ is nonzero in $H^1(C, N_{S/X}|_C)$ if we have the following:
(1) Let $\Delta := C + K_X|_S - 2E$ in Pic $S$ and let $E_{\text{red}} := \sum_{i=1}^{k} E_i$, i.e., the reduced part of $E$. Then the restriction map

$$H^0(S, \Delta) \xrightarrow{|E_{\text{red}}|} H^0(E_{\text{red}}, \Delta|_{E_{\text{red}}})$$

to $E_{\text{red}}$ is surjective, and

(2) There exists an integer $1 \leq i \leq k$ such that $\partial_{E_i}(\beta_i) \cup \beta_i \neq 0$, where the cup product is taken by the map

$$H^1(E_i, (m_i + 1)E_i) \times H^0(E_i, N_{S/X}(m_i E_i - C)|_{E_i}) \xrightarrow{\cup} H^1(E_i, N_{S/X}(2m_i + 1)E_i - C)|_{E_i}).$$

In the rest of this section, we assume that $X = \mathbb{P}^3$, $S \subset X$ is a smooth cubic surface, and $E_i$ are lines on $S$. The following lemma gives a sufficient condition for the cup product $\partial_{E_i}(\beta_i) \cup \beta_i$ considered in Theorem 2.22 to be nonzero.

**Lemma 2.23.** Let $Z_i := C \cap E_i$ be the scheme-theoretic intersection of $C$ with $E_i$. If we have the following:

[i] $\beta_i \neq 0$, equivalently, $\beta$ is not contained in $H^0(S, N_{S/\mathbb{P}^3}(E - E_i))$.

[ii] $(C.E_i) = 3 - m_i$, and

[iii] If $m_i = 1$ then $Z_i$ is a general member of a linear system $\Lambda := |O_{E_i}(2)|$ on $E_i \simeq \mathbb{P}^1$,

then the cup product $\partial_{E_i}(\beta_i) \cup \beta_i$ is nonzero.

**Proof.** Since $N_{S/\mathbb{P}^3} \simeq O_S(3)$ and $E_i$ is a $(-1)$-curve, we see that $N_{S/\mathbb{P}^3}(m_i E_i)|_{E_i}$ is an invertible sheaf on $E_i \simeq \mathbb{P}^1$ of degree $3 - m_i$. Then it has a nonzero section by [i], and thereby we obtain $3 - m_i \geq 0$. This implies that $m_i = 1, 2$ or 3. Then it follows from the condition [ii] that $N_{S/\mathbb{P}^3}(m_i E_i - C)|_{E_i}$ is a trivial sheaf. Therefore, taking a cup product with $\beta_i$ is just a multiplication by a nonzero scalar. Hence for the proof, it suffices to prove that $\partial_{E_i}(\beta_i) \neq 0$. We consider the map

$$\pi_{E_i/S}(E) : H^0(E_i, N_{E_i/\mathbb{P}^3}(m_i E_i)) \rightarrow H^0(E_i, N_{S/\mathbb{P}^3}(m_i E_i)|_{E_i}),$$

which is induced by a sheaf homomorphism $\pi_{E_i/S} \otimes O_{E_i}(E_i)$ in (2.16). We see that this map is zero if $m_i > 1$ and not surjective if $m_i = 1$, because $N_{E_i/\mathbb{P}^3} \simeq O_{\mathbb{P}^1}(1)^{\oplus 2}$, $N_{S/\mathbb{P}^3}|_{E_i} \simeq O_{\mathbb{P}^1}(3)$ and $O_{E_i}(E_i) \simeq O_{\mathbb{P}^1}(-1)$ on $E_i \simeq \mathbb{P}^1$. Thus if $m_i > 1$ then we are done. Suppose that $m_i = 1$. Then by the condition [iii], $Z_i$ is a finite subscheme of $E_i$ of length 2. Given an invertible sheaf $\mathcal{L}$ and its global section $\gamma$, we denote by $\text{div}_0(\gamma)$ the divisor of zero of $\gamma$. Then by [20, Lemma 3.1], $\beta_i$ is contained in the subgroup $H^0(E_i, N_{S/\mathbb{P}^3}(E_i - C)|_{E_i}) \subset H^0(E_i, N_{S/\mathbb{P}^3}(E_i)|_{E_i})$. Thereore, as a section of the sheaf $N_{S/\mathbb{P}^3}(E_i)|_{E_i} \simeq O_{\mathbb{P}^1}(2)$ on $E_i \simeq \mathbb{P}^1$, we have $\text{div}_0(\beta_i) = Z_i$. If $\partial_{E_i}(\beta_i) = 0$, then $Z_i$ is contained in the linear subsystem

$$\{\text{div}_0(\gamma) \mid \gamma \in \text{im} \pi_{E_i/S}(E_i)\} \subseteq |N_{S/\mathbb{P}^3}(E_i)|_{E_i}|$$

of codimension one, thereby contradicting the genericity of $Z_i$ as mentioned in the condition [iii]. Thus we conclude that $\partial_{E_i}(\beta_i) \neq 0$. □
The next lemma will be used to prove that the condition [iii] in Lemma 2.23 is satisfied in the case where \( m_i = 1 \), i.e., \( C.E_i = 2 \).

**Lemma 2.24.** Let \( E \) be a line and \( D \) a nef divisor on \( S \). If \( D \sim m(-K_S - E) \) for any integer \( m \), then we have \( H^1(S, D - E) = 0 \), and in particular the rational map \( |D| \to |\mathcal{O}_E(D)| \) sending a curve \( C \in |D| \) to \( Z := C \cap E \) is dominant.

**Proof.** We note that \( q := -K_S - E \) is the class of conics on \( S \) residual to \( E \). Put \( L := D + q \) in Pic \( S \). Since \( D \) is nef, so is \( L \). Since \( L \) is not composed with pencils, \( L \) is also big. Thus \( H^1(S, D - E) \simeq H^1(S, K_S + L) = 0 \) as a consequence of Kawamata-Viehweg vanishing theorem. Then the lemma follows from the exact sequence

\[
0 \longrightarrow \mathcal{O}_S(-E) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_E \longrightarrow 0| \otimes \mathcal{O}_S(D). \tag{2.17}
\]

\( \square \)

### 3. Proof of theorems 1.2 and 1.3

In this section, we prove Theorems 1.2 and 1.3. Let \( S \subset \mathbb{P}^3 \) be a smooth cubic surface, \( C \subset \mathbb{P}^3 \) a curve contained in \( S \). We define an invertible sheaf \( L \) on \( S \) by

\[
L := \mathcal{O}_S(C) \otimes S^{-1}_{S/\mathbb{P}^3}.
\]

Then \( L \sim C + 3K_S \) in Pic \( S \). We will see that the two cohomology groups \( H^1(S, -L) \) and \( H^2(S, -L) \) on \( S \) are important for studying the deformations of \( C \) in \( \mathbb{P}^3 \).

Since \( H^1(C, N_{C/S}) = 0 \), as we saw in §1, the cohomology group \( H^1(C, N_{S/\mathbb{P}^3}|_C) \) contains every obstruction to deforming \( C \) in \( \mathbb{P}^3 \). Since \( H^i(S, N_{S/\mathbb{P}^3}) = 0 \) for all \( i > 0 \), it follows from the exact sequence (2.13) \( \otimes S N_{S/\mathbb{P}^3} \) that

\[
H^1(C, N_{S/\mathbb{P}^3}|_C) \simeq H^2(S, -L). \tag{3.1}
\]

The following lemma shows that if \( H^1(S, -L) \neq 0 \) and \( \chi(S, -L) \geq 0 \), then the obstruction space (3.1) is nonzero by \( H^2(S, -L) \simeq H^0(S, L + K_S)^\vee \).

**Lemma 3.1.** Suppose that \( C \) is of degree \( d > 9 \) and genus \( g \geq 3d - 18 \). Then

(1) \( H^0(S, -L) = 0 \),
(2) \( \chi(S, -L) \geq 0 \) and
(3) If \( H^1(S, -L) \neq 0 \) then \( L + K_S \geq 0 \), \( L \) is big and not nef.

**Proof.** (1) follows from \( L.K_S = 9 - d < 0 \). By Riemann-Roch theorem on \( S \), we have

\[
\chi(S, -L) = (C + 3K_S)(C + 4K_S)/2 + 1 = g - 3d + 18
\]

and hence we obtain (2) by assumption. This implies that if \( H^1(S, -L) \neq 0 \) then \( H^2(S, -L) \neq 0 \). Then it follows from Lemma 2.3 that \( L + K_S \) is effective and \( L \) is not nef. Finally \( L \) is big by Lemma 2.8. \( \square \)
We next relate $H^1(S, -L)$ to a tangent map on the Hilbert-flag scheme. By Lemma 2.14, we note that $H^i(\mathbb{P}^3, N_{(C, S)/\mathbb{P}^3}) = 0$ for all $i > 0$. This implies that the Hilbert-flag scheme $HF\mathbb{P}^3$ of $\mathbb{P}^3$ is nonsingular at $(C, S)$ of expected dimension $\chi(\mathbb{P}^3, N_{(C, S)/\mathbb{P}^3}) = d + g + 18$ (cf. Lemma 2.15). Let $\text{Hilb}^{sc}\mathbb{P}^3$ denote the Hilbert scheme of smooth connected curves in $\mathbb{P}^3$, and let $HF^{sc}\mathbb{P}^3 \subset HF\mathbb{P}^3$ denote the subscheme parametrising pairs $(C', S')$ of a curve $C' \in \text{Hilb}^{sc}\mathbb{P}^3$ and a surface $S'$ containing $C'$, i.e., we define by $HF^{sc}\mathbb{P}^3 := pr_1^{-1}(\text{Hilb}^{sc}\mathbb{P}^3)$, where $pr_1 : HF\mathbb{P}^3 \rightarrow \text{Hilb}\mathbb{P}^3$ is the first projection. Then by (2.11) the cokernel of the tangent map

$$p_1 : H^0(\mathbb{P}^3, N_{(C, S)/\mathbb{P}^3}) \rightarrow H^0(C, N_{C/\mathbb{P}^3}) \quad (3.2)$$

of $pr_1$ at $(C, S)$ is isomorphic to $H^1(S, N_{S/\mathbb{P}^3}(-C)) = H^1(S, -L)$. The following lemma immediately follows from Corollary 2.2 and Lemma 2.9.

**Lemma 3.2.** Suppose that $L \geq 0$ and $L$ is not nef. Then the fixed part $F$ of $|L|$ is given by

$$F = m_1E_1 + \cdots + m_kE_k,$$

where $E_i$ $(1 \leq i \leq k)$ are mutually disjoint lines on $S$ such that $L.E_i < 0$ and $m_i = -L.E_i$. Here we have $m_i \leq 3$ for all $i$. If moreover $L^2 > 0$ then we have the following

1. $h^1(S, -L) = h^0(F, \mathcal{O}_F)$, and
2. $C$ is quadratically normal (resp. linearly normal) if and only if $m_i = 1$ (resp. $1 \leq m_i \leq 2$) for all $i$.

**Proof of Theorem 1.2.** We assume that $C$ satisfies the hypothesis of Conjecture 1.1. Then $H^1(S, -L) \simeq H^1(\mathbb{P}^3, \mathcal{I}_C(3)) \neq 0$ by (2.6). Thereby the tangent map $p_1$ defined above is not surjective. Let $\alpha$ be a global section of $N_{C/\mathbb{P}^3}$. Then by lemma 2.17 (or more directly by Corollary 2.18), it suffices to prove that the primary obstruction $\text{ob}(\alpha)$ of $\alpha$ is nonzero if $\alpha$ is not contained in the image of $p_1$. It follows from Lemma 3.1 that $\chi(S, -L) \geq 0, L + K_S \geq 0, L$ is not nef and $L^2 > 0$.

Suppose now that $C$ is quadratically normal. Then Lemma 3.2 shows that the fixed part $F$ of $|L|$ is given by

$$F = E_1 + \cdots + E_k,$$

where $k = h^1(S, -L)$ and $E_i$ $(1 \leq i \leq k)$ are lines on $S$ mutually disjoint. Then $L - F$ is clearly nef, and also big by $(L - F)^2 = L^2 - F^2 > L^2 > 0$. Thus we see that $H^1(S, -L + F) = 0$. Let $k_C$ be the extension class defined by (2.13) and $\gamma := \pi_{C/S}(\alpha)$ the exterior component of $\alpha$ (see §2.4 for the definition). We note that the map $\delta$ in (2.11), which is the first coboundary map of (2.9), factors through the coboundary map $\cup k_C$ of (2.13) $\otimes_S N_{S/\mathbb{P}^3}$ (cf. [21, Lemma 2.2]). Then since $\alpha$ is not contained in im $p_1$, the cup product $\gamma \cup k_C$ is nonzero in $H^1(S, -L)$. We note that $H^1(S, N_{S/\mathbb{P}^3}(F - C)) \simeq H^1(S, -L + F) = 0$. Then by Lemma 2.20, admitting to $\gamma$ some poles along $F \cap C$, the section $\gamma$ (in fact $r(\gamma, F)$) on $C$ lifts to a global section of $N_{S/\mathbb{P}^3}(F)$ on $S$, i.e., an infinitesimal deformation.
with pole (cf. Definition 2.19). More precisely, by the same lemma, there exist a prime divisor $E_i \subset F$ on $S$ ($1 \leq i \leq k$) and a lift $\beta \in H^0(S, N_{S/P^3}(F))$ of $r(\gamma, F)$ such that the principal part $\beta_i := \beta|_{E_i}$ of $\beta$ along $E_i$ is nonzero in $H^0(E_i, N_{S/P^3}(F)|_{E_i}) \simeq H^0(E_i, N_{S/P^3}(E_i)|_{E_i})$.

Now we check that the two conditions (1) and (2) of Theorem 2.22 are both satisfied. Let us define a divisor $\Delta$ on $S$ as in the theorem. Then $\Delta = C + K_{P^3}|S - 2F \sim L + KS - 2F$. The Serre duality shows that $H^1(S, \Delta - F) \simeq H^1(S, 3F - L)^\vee$ and the last cohomology group is zero by Lemma 2.7. Therefore the restriction map $H^0(S, \Delta) \to H^0(F, \Delta|_F)$ is surjective. Thus (1) is satisfied.

To check the condition (2) of Theorem 2.22, we prove that the three conditions [i], [ii] and [iii] of Lemma 2.23 are all satisfied. [i] is clear. [ii] follows from $m_i = 1$ and $C.E_i = 2$. Since $C$ is a general member of the 3-maximal family $W$, so is $Z_i := C \cap E_i$ in $|O_{E_i}(2)|$ on $E_i \simeq P^1$ by Lemma 2.24. Thus [iii] follows. Then the cup product $\partial_E(\beta_i) \cup \beta_i$ considered in Theorem 2.22 (2) is nonzero. Thereby we have proved Theorem 1.2. 

Remark 3.3. In this remark, we collect some known results related to Conjecture 1.1. Kleppe [11] proved the conjecture is true in the range of the $(d, g)$-plane: $g > -1 + (d^2 - 4)/8$ for $14 \leq d \leq 17$ and $g > 7 + (d - 2)^2/8$ for $d \geq 18$. Later, Ellia [4] proved the conjecture in the wider range: $g > G(d, 5)$ for $d \geq 21$, where $G(d, 5)$ denotes the maximal genus of curves of degree $d$ not contained in a quartic surface and $G(d, 5) \approx d^2/10$ for $d \gg 0$ (cf. [6]). It has been proved in [18] that the conjecture is true if $h^1(P^3, \mathcal{I}_C(3)) = 1$ (that is the case $b_6 = 2$ and $b_5 \geq 3$) by a method of this paper. Recently in the appendix of [13] and more recently in [12], Kleppe has further extended the known range of $(d, g)$ where Conjecture 1.1 holds to be true by a method of [11] together with a result in [4], but his result does not cover our result. It is notable that his result shows that the conjecture is true for some classes of quadratic non-normal curves $C$ (with $b_6 = 1$, $b_5 \geq 5$ and satisfying some further assumptions) (cf. [13, Theorem A.3]). As far as we know, every proof that has been known so far is partial, and Conjecture 1.1 is still open (in the case where $C \subset P^3$ is quadratically non-normal).

In the rest of this section, we prove Theorem 1.3.

Proof of Theorem 1.3. Let $C$ satisfy the assumption of the theorem. For the proof, it suffices to show that there exists a global section $\alpha$ of $N_{C/P^3}$ such that $ob(\alpha)$ (or its exterior component $ob(\alpha)$) is nonzero.

Let $m := -L.E$ and suppose that $1 \leq m \leq 3$. If $1 \leq j \leq m$ then $(-L + jE).E = m - j \geq 0$. Therefore, there exists a sequence

$$H^1(S, -L) \to H^1(S, -L + E) \to \cdots \to H^1(S, -L + mE)$$

of natural surjective maps. Since $L + K_S$ is effective by assumption, so are $L$ and $L - mE$ by Corollary 2.2. Since $L \sim mE$, we have $H^0(S, -L + mE) = 0$. Then since the invertible sheaf $O_E(-L + mE)$ on $E \simeq P^1$ is trivial, we deduce from the exact sequence (2.17) (for $D = -L + mE$) that

$$h^1(S, -L + (m - 1)E) - h^1(S, -L + mE) = 1.$$
Therefore there exists an element $\xi$ of $H^1(S, -L)$ such that $r(\xi, (m - 1)E) \neq 0$ in $H^1(S, -L + (m - 1)E)$ and $r(\xi, mE) = 0$ in $H^1(S, -L + mE)$. Here and later, we use the same notation $r(\ast, D)$ in §2.4 for a divisor $D \geq 0$ on $S$. It follows from the exact sequence (2.11) that there exists a global section $\alpha$ of $N_{S\backslash \mathbb{P}^3}$ such that $\delta(\alpha) = \xi$, where $\delta$ is the first coboundary map of (2.9). Let $\gamma := \pi_{S\backslash S}(\alpha)$ denote the exterior component of $\alpha$ (see §2.4) and let $k_C$ be the extension class of (2.13). Then $\gamma \cup k_C = \xi$ as in the proof of Theorem 1.2. Moreover by the choice of $\xi$ and Lemma 2.20, there exists a global section $\beta$ of $N_{S\backslash S}(mE)$ such that $\beta|_C = r(\gamma, mE)$ and the principal part $\beta|_E$ of $\beta$ along $E$ defines a nonzero global section of $N_{S\backslash S}(mE - C)|_E$ on $E$.

Let us define a divisor $\Delta$ on $S$ by $\Delta := C + K_{\mathbb{P}^3}|_S - 2mE$ as in Theorem 2.22. We check that $\Delta$ and $\beta$ (or $\beta|_E$) satisfy the two assumptions (1) and (2) of the theorem. Let $\varphi$ denote the restriction map defined by (1.3). If $m = 1$ then $\Delta = L + K_S - 2E$ is effective by Corollary 2.2. Thus we see that $\varphi$ is surjective for $m = 1$ by $\Delta, E = 0$ and Lemma 3.4 below, and also for $2 \leq m \leq 3$ by assumption. Thus (1) is satisfied.

To prove that the cup product $\vartheta_E(\beta|_E) \cup \beta|_E$ considered in Theorem 2.22 is nonzero, we again apply Lemma 2.23. We have already seen that $\beta|_E \neq 0$ (cf. [ii]). It is also clear that $m(:= -L.E) = 3 - C.E$ (cf. [ii]). Finally, if $m = 1$, i.e., $C.E = 2$, replacing $C$ with a general member $C'$ of $|C|$, we can assume that the intersection $Z = C \cap E$, that is a divisor on $E \simeq \mathbb{P}^1$ of degree 2, is general in $|\mathcal{O}_E(2)|$ by Lemma 2.24 (cf. [iii]). In fact, if $C'$ is obstructed in $\mathbb{P}^3$, then so is $C$ by upper semicontinuity. Thereby we have obtained all the desired properties of $\beta|_E$ enough for proving that its cup product with $\vartheta_E(\beta|_E)$ is nonzero. Then by Theorem 2.22, we have completed the proof of Theorem 1.3. \hfill \Box

Lemma 3.4. Let $E$ be a line on $S$ and $\Delta$ a divisor on $S$ such that $n := \Delta.E \geq 0$. If there exists a conic $q$ on $S$ such that $q.E = 1$ and $\Delta - nq \geq 0$, then the restriction map $\varphi$ in (1.3) is surjective.

Proof. Let $q' := -K_S - E$. Then by $q'.E = 2$, we have $nq \approx mq'$ for any integer $m$. Then it follows from Lemma 2.24 that $H^0(S, nq) \to H^0(E, nq|_E)$ is surjective. Since $\Delta - nq$ is effective, $|\Delta|$ contains $|nq|$ as a linear subsystem. We note that $\mathcal{O}_E(nq) \simeq \mathcal{O}_E(\Delta)$ by degree. Since the restriction of $\varphi$ to $H^0(S, nq)$ is surjective, so is $\varphi$. \hfill \Box

Remark 3.5. Some special cases of Theorem 1.3 were also proved in [3] ($m = 3$) and [18] ($m = 1$). The same conclusion was proved in [18] under the assumption that $F = Bs|L|$ is a (single) line (cf. [18, Proposition 3.1]). Dolcetti and Pareschi [3] proved that if $d \geq 21$ and $G(d, S) < g \leq d^2/8 - d/2 + 1$, then every linearly non-normal curve $C \in H(d, g)^{sc}$ lying on a smooth cubic surface belongs to a non-reduced component of $H(d, g)^{sc}$ of dimension $d + g + 20$ (hence $C$ is obstructed in $\mathbb{P}^3$), whose general members are linearly non-normal curves lying on a quartic surface with a double conic (cf. [3, Theorem 2.1]). Such curves are generic projections of curves lying a smooth quartic del Pezzo surface in $\mathbb{P}^4$, and this fact was first pointed out by Ellia [4]. See [13] for examples of obstructed curves with $m = 2$ (cf. Remark 3.3).
4. Examples

In this section, we consider applications of Theorems 1.2 and 1.3. We first look at applications of Theorem 1.2. We give two series of 3-maximal families of space curves satisfying the assumption of Conjecture 1.1.

Let \((a; b_1, \ldots, b_6)\) be a 7-tuple of integers satisfying the set of conditions (2.7), and let \(W := W(a; b_1, \ldots, b_6)\) be the irreducible closed subset of \(H(d, g)^{sc}\) associated to it (cf. Definition 2.12). We denote by \(C\) a general member of \(W\). Then if \(d > 9\), \(W\) becomes a 3-maximal family of \(H(d, g)^{sc}\) (cf. §1) and \(C\) is contained in a unique smooth cubic surface \(S\). In Examples 4.1 and 4.2 below, we have \(b_6 = 2\) and \(C\) is quadratically normal by Lemma 2.9. Then by virtue of Theorem 1.2, \(W\) becomes a component of \((H(d, g))^{sc\text{red}}\) of \(d + g + 18\), and moreover \(H(d, g)^{sc}\) is generically non-reduced along \(W\). Thus in this example, the Hilbert scheme \((H(d, g))^{sc}\) is highly singular along \(W\). In fact, at the generic point \(C \in W\), the tangential dimension of the Hilbert scheme is greater than its dimension as a scheme by \(h^1(\mathbb{P}^3, \mathcal{I}_C(3))\) (cf. (1.2)). We compute the two numbers \(h^1(\mathbb{P}^3, \mathcal{I}_C(3))\) and \(h^1(C, \mathcal{O}_C(3))\), where the latter represents the dimension of the obstruction space \(H^1(C, N_C^{\mathbb{P}^3})\) of \(H(d, g)^{sc}\) at \([C]\). It can be computed by the formula

\[
h^1(C, N_C^{\mathbb{P}^3}) = h^1(C, \mathcal{O}_C(3)) = h^0(S, C + 4K_S), \tag{4.1}
\]

which is deduced from (3.1) and the Serre duality. In the following examples, \(F\) denotes the fixed part of the linear system \([C + 3K_S]\) on \(S\).

Example 4.1. Let \(\lambda\) be a non-negative integer and let

\[
W = W(\lambda + 14; 2, 2, 2, 2, 2, 2) \subset H(d, g)^{sc}.
\]

Then we have \(d = 3(\lambda + 10)\) and \(g = (\lambda + 16)(\lambda + 9)/2\), thus \(\dim W = d + g + 18 = (\lambda + 16)(\lambda + 15)/2\). By Lemma 2.1, we have \(F = (0; -1, -1, -1, -1, -1) = \sum_{i=1}^5 e_i\). Therefore, by the method of computations used in Example 2.11, we see that \(h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = h^0(F, \mathcal{O}_F) = 6\). It follows from (4.1) that \(h^1(C, \mathcal{O}_C(3)) = h^0(S, C + 4K_S - 2F) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(\lambda + 2)) = (\lambda + 4)(\lambda + 3)/2\).

Example 4.2. Let \(\lambda \geq 0\) and let

\[
W = W(\lambda + 17; \lambda + 8, 7, 2, 2, 2, 2) \subset H(d, g)^{sc}.
\]

Then \(d = 2(\lambda + 14)\) and \(g = 8\lambda + 67\), thus \(\dim W = 10\lambda + 113\). We note that \(F\) consists of 5 disjoint lines on \(S\) by \(F = (1; 1, 1, -1, -1, -1) = (1 - e_1 - e_2) + \sum_{i=3}^6 e_i\), where \(1\) is the class of the pullback of lines in \(\mathbb{P}^2\) (cf. Example 2.11). This implies that \(h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 5\). Moreover, by using the formula (4.1) again, we compute that \(h^1(C, \mathcal{O}_C(3)) = h^0(S, C + 4K_S - 2F) = h^0(S, (\lambda + 3)1 - (\lambda + 2)e_1 - e_2) = 2\lambda + 6\).

We next look at applications of Theorem 1.3. Let \(S\) be a smooth cubic surface in \(\mathbb{P}^3\). We fix a line \(E\) on \(S\) and denote by \(\varepsilon\) the blow-down \(S \to S'\) of \(E\).
Proposition 4.3. Let $0 \leq k \leq 2$ be any integer and $q$ a conic on $S$ such that $q.E = 1$ and $D'$ a nef divisor on $S'$. Let $D$ be a divisor on $S$ defined by

$$D = -4K_S + 2(3 - k)E + (2 - k)q + \epsilon^*D',$$

and let $\Lambda := |D|$ be the linear system on $S$ spanned by $D$. Then

(1) every general member $C$ of $\Lambda$ is smooth and connected,
(2) $C.E = k$ and
(3) $C$ is obstructed in $\mathbb{P}^3$.

Proof. Since $D.E = 4 - 2(3 - k) + 2 - k = k \geq 0$, $D$ is nef and hence $\Lambda$ is base point free (cf. §2.1). Then (1) follows from Bertini’s theorem, (2) from $D.E = k$. We put $L := C + 3K_S$. Then $m := -L.E = 3 - k$ and we have $1 \leq m \leq 3$. Let $\Delta := L + K_S - 2mE = C + 4K_S - 2mE = (2 - k)q + \epsilon^*D'$. Then $\Delta.E = 2 - k \geq 0$. Since $\Delta - (\Delta.E)q = \epsilon^*D' \geq 0$, it follows from Lemma 3.4 that the restriction map $q$ in (1.3) is surjective (for all $0 \leq k \leq 2$). Thus (3) follows from Theorem 1.3. □

By taking $E$ and $q$ in Proposition 4.3 as $E = e_6$ and $q = 1 - e_6$, respectively, we obtain the following example.

Example 4.4. Let $0 \leq k \leq 2$ be an integer and let $(a; b_1, \ldots, b_5)$ be a 6-tuple of integers satisfying $a \geq b_1 + b_2 + b_3$ and $b_1 \geq b_2 \geq \cdots \geq b_5 \geq 0$. Since the invertible sheaf $O_{S'}(a; b_1, \ldots, b_5)$ on $S'$ is nef (cf. Lemma 2.4), it follows from Proposition 4.3 that every general member $C$ of the linear system

$$|O_S(14 - k + a; b_1 + 4, b_2 + 4, b_3 + 4, b_4 + 4, b_5 + 4, k)|$$

on $S$ is a smooth connected curve in $\mathbb{P}^3$ with $C.e_6 = k$. Moreover, $C$ is obstructed in $\mathbb{P}^3$.

Theorem 1.3 can be applied to determinations of the dimension of the Hilbert scheme $H(d, g)^{sc}$. We first recall a result due to Kleppe.

Theorem 4.5. ([11, Theorem 1.1]) Let $W$ be a 3-maximal family in $H(d, g)^{sc}$ whose general member $C$ is contained in a smooth cubic surface. If $d > 9$ or $h^0(\mathbb{P}^3, \mathcal{I}_C(3)) = 1$, then we have

$$h^1(\mathbb{P}^3, \mathcal{I}_C(3)) - h^1(C, O_C(3)) \leq \dim[C] H(d, g)^{sc} - \dim W \leq h^1(\mathbb{P}^3, \mathcal{I}_C(3)),$$

where the inequality to the right is strict if and only if $C$ is obstructed in $\mathbb{P}^3$.

See [21, Theorem 2.4] for a generalization of this theorem. Theorems 4.5 and 1.3 allow us to determine the dimension of $H(d, g)^{sc}$ at $[C]$ in the case where $h^1(C, O_C(3)) = 1$.

Proposition 4.6. Let $C \subset \mathbb{P}^3$ be a smooth connected curve of degree $d$ and genus $g$ lying on a smooth cubic surface $S$. 
(1) If \( h^0(\mathbb{P}^3, \mathcal{I}_C(3)) = 1 \), \( h^1(C, \mathcal{O}_C(3)) = 1 \) and \( C \) is obstructed, then
\[
\dim_{[C]} H(d, g)^{sc} = d + g + 17 + h^1(\mathbb{P}^3, \mathcal{I}_C(3)).
\]
(4.2)

(2) Suppose that \( C \) is a member of the linear system
\[
|\mathcal{O}_S(12; b_1, b_2, \ldots, b_6)|
\]
on \( S \) with \( b_i \) satisfying \( 0 \leq b_i \leq 4 \) for all \( i \). If \( b_j = 2 \) for some \( 1 \leq j \leq 6 \), then we have (4.2).

Proof. (1) follows from Theorem 4.5 and \( \dim W = d + g + 18. \) We prove (2). We note that \( L + K_S = C + 4K_S = \sum_{i=1}^{6}(4 - b_i)\epsilon_i \geq 0. \) Since \( -L.\epsilon_j = 1, \) \( C \) is obstructed by Theorem 1.3. Moreover, since \( C + 4K_S \) is a sum of lines on \( S, \) we see that \( h^1(C, \mathcal{O}_C(3)) = h^0(S, C + 4K_S) = 1 \) by (4.1). Since \( -3K_S - C \) is not effective, we have \( h^0(\mathbb{P}^3, \mathcal{I}_C(3)) = 1. \) Thus (2) follows from (1).

The following example was studied in detail in [10] (see also [11]).

Example 4.7. (Kleppe) Let \( S \) be a smooth cubic surface, \( E_1 \) and \( E_2 \) two skew lines on \( S \) and \( C \) a smooth connected curve on \( S \) such that \( C \sim -4K_S + 2E_1 + 2E_2, \) i.e., \( C \sim (12; 4, 4, 4, 4, 2, 2). \) We see that \( C \) is of degree \( d = 16 \) and genus \( g = 29. \) Since \( g < 3d - 18, \) the 3-maximal family \( W := W(12; 4, 4, 4, 4, 2, 2) \) containing \( C \) is not a component of \( (H(16, 29)^{sc})_\text{red} \) (cf. Remark 2.13). \( H(16, 29)^{sc} \) has a singularity of codimension 1 along \( W. \) In fact, we see that \( h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 2 \) and \( h^1(C, \mathcal{O}_C(3)) = 1. \) Then by proposition 4.6, we have \( \dim_{[C]} H(16, 29)^{sc} = 64. \) Here this number 64 equals to the expected dimension 4\( d \) of \( H(16, 29)^{sc} \) at \([C] \).

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