On Balder’s Existence Theorem for Infinite-Horizon Optimal Control Problems*

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Balder’s well-known existence theorem (1983) for infinite-horizon optimal control problems is extended to the case when the integral functional is understood as an improper integral. Simultaneously, the condition of strong uniform integrability (over all admissible controls and trajectories) of the positive part $\max\{f_0, 0\}$ of the utility function (integrand) $f_0$ is relaxed to the requirement that the integrals of $f_0$ over intervals $[T, T']$ be uniformly bounded from above by a function $\omega(T, T')$ such that $\omega(T, T') \to 0$ as $T, T' \to \infty$. This requirement was proposed by A.V. Dmitruk and N.V. Kuz’kina (2005); however, the proof in the present paper does not follow their scheme but is instead derived in a rather simple way from the auxiliary results of Balder himself. An illustrative example is also given.

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One of the most general and well-known results on the existence of solutions to infinite-horizon optimal control problems was proved by Balder [6]. Almost all conditions of his theorem are local in time (i.e., they must hold only at each separate instant of time or on each finite time interval) and ensure the existence of solutions to similar problems on finite time intervals. The only condition that regulates the behavior of the system at infinity is the requirement of strong uniform integrability of the positive part of the integrand in the objective functional over all admissible controls and corresponding trajectories. Later several authors achieved some progress in weakening this condition.

The present paper also contributes to this direction. As an alternative to Balder’s uniform integrability, we use the condition of “uniform boundedness of pieces of the objective functional” proposed by Dmitruk and Kuz’kina [10]. Note that they considered a significantly narrower class of optimal control problems, while for the general case only a scheme was outlined (without statement of particular results that can be obtained by following this scheme[1]). So the present paper is in a sense a logical completion of the paper [10]. However,

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1 The absence of exact statements to which one could refer when solving particular optimal control problems was one of the reasons for writing the present note.
we do not follow the scheme proposed in [10] but rather show that the result can be derived from those of Balder himself [5, 6] in a fairly simple way.

Recently Bogusz [9] also obtained an existence theorem in the case when the integral functional is understood as an improper integral. However, one of the hypotheses in her theorem is the existence of a locally integrable function \( \lambda : \mathbb{R}_+ \to \mathbb{R} \) that has a finite improper integral
\[
\int_0^\infty \lambda(t) \, dt
\]
and bounds from above (from below in the case of minimization problem) the integrand in the objective functional for all admissible controls and corresponding trajectories. Such a condition is essentially stronger (although formally this is not so) than the strong uniform integrability, because subtracting (adding in the case of minimization problem) the function \( \lambda \) from (to) the integrand reduces the problem to the one with negative (positive) integrand in the objective functional.

Some results on the existence of optimal solutions under conditions of different kind and/or in different statements of the problem were obtained in [2, 11].

Note that existence theorems are an inherent part of the method for solving optimal control problems based on applying necessary optimality conditions (see, e.g., [4, 1, 3, 7, 8]). Therefore, it is important to have an existence theorem under hypotheses maximally close to those under which necessary optimality conditions are valid. At present it is the condition of “uniform boundedness of pieces of the objective functional” that is often required for necessary optimality conditions to be valid (see, e.g., [4, 7]).

Let us proceed to the statement of the problem and formulate the conditions under which we will study it.

The main object of our study is the optimal control problem
\[
I(x, u) := \int_0^\infty f_0(t, x(t), u(t)) \, dt \to \max, \tag{1}
\]
\[
\dot{x}(t) = f(t, x(t), u(t)) \quad \text{for a.e. } t \in \mathbb{R}_+ := [0, +\infty), \tag{2}
\]
\[
x(t) \in A(t), \quad u(t) \in U(t, x(t)) \quad \text{for a.e. } t \in \mathbb{R}_+, \tag{3}
\]
for which the following conditions hold (where \( m, n \in \mathbb{N} \) are fixed dimensions of the control and state vectors, respectively):

(i) \( A : \mathbb{R}_+ \rightrightarrows \mathbb{R}^n \) is a set-valued map with \((\mathcal{L} \times \mathcal{B}^n)\)-measurable\(^2\) graph \( A \);
(ii) \( U : A \rightrightarrows \mathbb{R}^m \) is a set-valued map with \((\mathcal{L} \times \mathcal{B}^{n+m})\)-measurable graph \( U \);
(iii) the functions \( f : U \to \mathbb{R}^n \) and \( f_0 : U \to \mathbb{R} \cup \{-\infty\} \) are \((\mathcal{L} \times \mathcal{B}^{n+m})\)-measurable\(^3\).

The set \( \Omega \) of admissible pairs \((x, u)\) consists by definition of pairs of vector functions \( x, u \) such that \( x \in \text{AC}_{\text{loc}}^n(\mathbb{R}_+) \), \( u : \mathbb{R}_+ \to \mathbb{R}^m \) is a Lebesgue measurable function and conditions (2) and (3) hold. Here \( \text{AC}_{\text{loc}}^n(\mathbb{R}_+) \) is the space of locally absolutely continuous (i.e., absolutely continuous on each finite interval) functions \( x : \mathbb{R}_+ \to \mathbb{R}^n \) with the topology indicated in [6].

The integral in (1) is understood in [6] in the following sense:
\[
\int_0^\infty g(t) \, dt := \int_0^\infty g^+(t) \, dt - \int_0^\infty g^-(t) \, dt, \quad \text{where} \quad g^\pm := \max\{\pm g, 0\}, \tag{4}
\]

\(^2\) That is, lying in the \( \sigma \)-algebra generated in \( \mathbb{R}_+ \times \mathbb{R}^n \) by the Cartesian products of Lebesgue measurable subsets in \( \mathbb{R}_+ \) and Borel subsets in \( \mathbb{R}^n \).

\(^3\) That is, the preimages of Borel sets are \((\mathcal{L} \times \mathcal{B}^{n+m})\)-measurable.
with the convention\(^{4}\) that \((+\infty)−(+\infty)=−\infty\). Thus, the value of the functional \((\text{I})\) (equal to a finite number or ±∞) is defined for any admissible pair.

Fix an \(\alpha\in\mathbb{R}\) and put \(\Omega_\alpha:=\{(x,u)\in\Omega\mid I(x,u)\geq\alpha\}\). The existence of a solution to problem \((\text{I})\)–\((\text{III})\) is proved in [6] under the following assumptions:

(i) the function \(f(t,\cdot,\cdot)\) is continuous on \(\mathcal{U}(t):=\{(\chi,v)\in\mathbb{R}^m\times\mathbb{R}^m\mid (t,\chi,v)\in\mathcal{U}\}\) for every \(t\in\mathbb{R}_+\);

(ii) the function \(f_0(t,\cdot,\cdot)\) is upper semicontinuous on \(\mathcal{U}(t)\) for every \(t\in\mathbb{R}_+\);

(iii) the sets \(A(t)\) and \(\mathcal{U}(t)\) are closed for every \(t\in\mathbb{R}_+\);

(iv) the function \(g\) is bounded;

(v) the set \(Q(t,\chi) := \{(z^0,z)\in\mathbb{R}\times\mathbb{R}^m\mid z^0\leq f_0(t,\chi,v), \ z=f(t,\chi,v), \ v\in\mathcal{U}(t,\chi)\}\) is convex for all \((t,\chi)\in\mathcal{A}\);

(vi) the sets \(Q_t(\cdot,\cdot) := \{f_0^+(\cdot,\cdot,\cdot)\mid (x,u)\in\Omega_\alpha\}\) is strongly uniformly integrable on \(\mathbb{R}_+\), i.e., \(\inf_{t\in\mathbb{R}_+}\sup_{g\in\mathcal{F}_t^T}\int_{C_{g,c}}|g(t)|\,dt = 0\), where \(C_{g,c} = \{t\in[0,T]\mid |g(t)|>c\}\);

(vii) the set \(\{x(0)\mid (x,u)\in\Omega_\alpha\}\) is bounded;

(viii) for every \(T > 0\), the set of functions \(\mathcal{F}_t^T := \{f(\cdot,\cdot,\cdot)\mid f_{0,T} := \{(0,T)\mid (x,u)\in\Omega_\alpha\}\} is uniformly integrable on the interval \([0,T]\), i.e., \(\inf_{c>0}\sup_{g\in\mathcal{F}_t^T}\int_{C_{g,c}}|g(t)|\,dt = 0\), where \(C_{g,c} = \{t\in[0,T]\mid |g(t)|>c\}\);

(ix) the set \(\mathcal{Q}(t,\chi) := \{(z^0,\chi)\in\mathbb{R}\times\mathbb{R}^m\mid z^0\leq f_0(t,\chi,v), \ z=f(t,\chi,v), \ v\in\mathcal{U}(t,\chi)\}\) is convex for all \((t,\chi)\in\mathcal{A}\);

(x) \(\mathcal{Q}(t,\chi) = \bigcap_{\delta>0}\text{cl}(\bigcup_{\chi'\in A(t)\cap B_\delta(\chi))} \mathcal{Q}(t,\chi'))\), where \(B_\delta(\chi)\) is the ball of radius \(\delta\) centered at \(\chi\);

(xi) the set of functions \(\mathcal{F}_t^\omega := \{f_\omega^+(\cdot,\cdot)\mid (x,u)\in\Omega_\alpha\}\) is strongly uniformly integrable on \(\mathbb{R}_+\), i.e., \(\inf_{h\in L_1(\mathbb{R}_+)}\sup_{g\in\mathcal{F}_t^\omega}\int_{C_{g,h}}|g(t)|\,dt = 0\), where \(C_{g,h} := \{t\in\mathbb{R}_+\mid |g(t)|>h\}\).

**Theorem A** ([6 Theorem 3.6]). If there is an \(\alpha\in\mathbb{R}\) such that \(\Omega_\alpha \neq \emptyset\) and conditions (i)–(xi) hold, then in problem \((\text{I})\)–\((\text{III})\) there exists an admissible pair \((x_*,u_*)\in\Omega)\) such that \(I(x_*,u_*) = \sup_{(x,u)\in\Omega} I(x,u)\).

As was already mentioned, the only condition in Theorem A that regulates the behavior of system \((\text{I})\)–\((\text{III})\) at infinity is condition (xi). At the same time, in many optimal economic growth problems it seems more natural to define the value of the objective functional not in the sense of \((\text{I})\) but rather in the limit sense

\[
J(x,u) := \lim_{T\to+\infty} \int_0^T f_0(t,x(t),u(t))\,dt
\]

provided that the limit exists (see, e.g., [1, 9]). We will also follow this definition, in which case problem \((\text{I})\)–\((\text{III})\) is replaced by the problem

\[
J(x,u) \to \max
\]

subject to conditions (2) and (3).

**Remark 1.** It is clear that if the value of the functional \(I(x,u)\) is finite for an admissible pair \((x,u)\), then \(J(x,u) = I(x,u)\).

As noticed in [10], instead of condition (xi) one can consider the condition

\[
\lim_{T\to+\infty} \sup_{T'>T} \sup_{(x,u)\in\Omega} \int_T^{T'} f_0(t,x(t),u(t))\,dt \leq 0.
\]

\(^{4}\) Here and below, without further mention, we reformulate all the results of [6, 10] obtained for minimization problems as applied to similar maximization problems.
It is easy to see that for admissible pairs in $\Omega$ condition (xii) is weaker than condition (xi). Indeed,

$$
\lim_{T \to +\infty} \sup_{T' > T} \sup_{(x, u) \in \Omega} \int_T^{T'} f_0(t, x(t), u(t)) \, dt \leq \lim_{T \to +\infty} \sup_{T' > T} \sup_{g \in F_0^+} \int_T^{T'} g(t) \, dt
$$

\begin{equation}
\leq \inf_{h \in L_1(\mathbb{R}^+)} \lim_{T \to +\infty} \sup_{T' > T} \sup_{g \in F_0^+} \left( \int_{[T, T'] \cap C_{g,h}} g(t) \, dt + \int_{[T, T'] \setminus C_{g,h}} h(t) \, dt \right)
\end{equation}

\begin{equation}
\leq \inf_{h \in L_1(\mathbb{R}^+)} \lim_{T \to +\infty} \sup_{T' > T} \sup_{g \in F_0^+} \int_{C_{g,h}} g(t) \, dt + 0 = \inf_{h \in L_1(\mathbb{R}^+)} \sup_{g \in F_0^+} \int_{C_{g,h}} g(t) \, dt.
\end{equation}

However, below we will still need a local version of condition (xi), namely,

(xi') for every $T > 0$, the set of functions $F_0^{T,+} := \{ f_0^+(\cdot, x(\cdot), u(\cdot)) | (0, T) \cap (x, u) \in \Omega \}$ is uniformly integrable on $[0, T]$.

(In [10], due to the continuity and compact-valuedness of the functions and set-valued maps considered there, condition (xi') holds automatically.)

Let us make the following important observation.

**Proposition 1.** Under condition (xi'), condition (xii) is equivalent to each of the following conditions:

(xii') there is a continuous function $\omega: \mathbb{R}_+^2 \to \mathbb{R}_+$ such that $\omega(T, T') \to 0$ as $T, T' \to \infty$ and

$$
\sup_{(x, u) \in \Omega} \int_T^{T'} f_0(t, x(t), u(t)) \, dt \leq \omega(T, T') \quad \forall T, T': T' > T \geq 0;
$$

(xii'') there is a continuous function $\tilde{\omega}: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\tilde{\omega}(T) \to 0$ as $T \to \infty$ and

$$
\sup_{T' > T} \sup_{(x, u) \in \Omega} \int_T^{T'} f_0(t, x(t), u(t)) \, dt \leq \tilde{\omega}(T) \quad \forall T \geq 0.
$$

**Proof.** Clearly, condition (xii'') implies condition (xi') (it suffices to take $\omega(T, T') := \tilde{\omega}(T)$) and condition (xi') implies condition (xii) (for $\lim_{T \to \infty} \sup_{T' > T}$ is the same as $\lim_{T, T' \to \infty, T' > T}$, while the latter does not exceed $\lim_{T,T' \to \infty}$). Let us show that condition (xii) implies condition (xii''). Put

$$
\tilde{\omega}(T) := \left( \sup_{T' > T} \sup_{(x, u) \in \Omega} \int_T^{T'} f_0(t, x(t), u(t)) \, dt \right)^+, \quad T \geq 0.
$$

Due to condition (xii) we have $\lim_{T \to \infty} \tilde{\omega}(T) = 0$. Therefore, there is a $T_1$ such that $\tilde{\omega}(T) \leq 1$ for $T \geq T_1$. Let us show that this function is bounded for all $T \geq 0$. For $T < T_1$ we have

$$
\tilde{\omega}(T) \leq \sup_{(x, u) \in \Omega} \int_0^{T_1} f_0^+(t, x(t), u(t)) \, dt + \tilde{\omega}(T_1) \leq \inf_{c > 0} \sup_{g \in F_0^+} \left( \int_{C_{g,c}} g(t) \, dt + c T_1 \right) + 1.
$$

On the whole it would be incorrect to say that condition (xii) is weaker than condition (xi), because condition (xii) is considered for the set $\Omega$ while condition (xi) is considered only for the subset $\Omega_\alpha \subset \Omega$. Therefore, formally none of the conditions implies the other.
Due to condition (xii′) there is a constant $c_1 > 0$ such that

$$\sup_{g \in C^{T_1}} \int_{C^{T_1}} g(t) \, dt \leq 1.$$ 

Then $\tilde{\omega}(T) \leq c_1 T_1 + 2$ for all $T \geq 0$.

Put $\tilde{\omega}_1(T) := \sup_{T' \geq T} \tilde{\omega}(T')$ for $T \geq 0$. Then $\tilde{\omega}_1$ is a bounded monotonically nonincreasing function on $\mathbb{R}_+$ that tends to zero as $T \to \infty$.

Finally, put $\tilde{\omega}(T) := \int_{T-1}^{T} \tilde{\omega}_1(t^+) \, dt$ (recall that $t^+ = \max\{t, 0\}$). It is clear that $\tilde{\omega}$ is a continuous function on $\mathbb{R}_+$ that satisfies all requirements formulated in condition (xii′). □

One of the important corollaries to condition (xii) is that the value of the functional $J(\cdot, \cdot)$ is defined on any admissible pair. For completeness, we will give a slightly shorter proof of this fact than in [10].

**Proposition 2.** Under conditions (xii′) and (xii), the value of the functional $J(x, u)$ is defined for every admissible pair $(x, u) \in \Omega$ and is equal to either a finite number or $-\infty$.

**Proof.** The existence of a limit in (3) follows from the estimate

$$
\lim_{T \to +\infty} \int_{0}^{T} f_0(t, x(t), u(t)) \, dt = \lim_{T_1 \to +\infty} \lim_{T \to +\infty} \left( \int_{0}^{T_1} f_0(t, x(t), u(t)) \, dt + \int_{T_1}^{T} f_0(t, x(t), u(t)) \, dt \right) 
\leq \lim_{T_1 \to +\infty} \int_{0}^{T_1} f_0(t, x(t), u(t)) \, dt + \lim_{T_1 \to +\infty} \sup_{T \geq T_1} \int_{T_1}^{T} f_0(t, x(t), u(t)) \, dt
\leq \lim_{T \to +\infty} \int_{0}^{T} f_0(t, x(t), u(t)) \, dt,
$$

where we have used condition (xii) at the last step. At the same time, the limit does not exceed $\tilde{\omega}(0)$ for a continuous function $\tilde{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$.

Now we formulate our main result. By analogy with the set $\Omega_{\alpha}$, we introduce the set $\tilde{\Omega}_{\alpha} := \{(x, u) \in \Omega \mid J(x, u) \geq \alpha\}$ for $\alpha \in \mathbb{R}$.

**Theorem 1.** If there is an $\alpha \in \mathbb{R}$ such that $\tilde{\Omega}_{\alpha} \neq \emptyset$ and conditions (i)–(x), (xii′) and (xii) (or (xii′), or (xii′′)) hold with $\Omega_{\alpha}$ replaced by $\tilde{\Omega}_{\alpha}$, then in problem [6], (2), (3) there exists an admissible pair $(x_s, u_s) \in \Omega$ such that $J(x_s, u_s) = \sup_{(x, u) \in \Omega} J(x, u)$.

The main role in the proof is played by another result of Balder.

**Theorem B** ([6 Theorem 3.2]). Suppose conditions (i)–(vi), (ix) and (x) hold. Suppose also that $\{(x_k, u_k)\}_{k=1}^{\infty}$ is a sequence in $\Omega$ such that the sequence $\{x_k\}_{k=1}^{\infty}$ converges weakly to a function $x_0 \in AC_{\text{loc}}^{0}(\mathbb{R}_+)$ and the set of functions $\{f_0(x_k, \cdot, u_k(\cdot))\}_{k=1}^{\infty}$ is strongly uniformly integrable on $\mathbb{R}_+$. Then there exists a Lebesgue measurable function $u_\ast : \mathbb{R}_+ \to \mathbb{R}_+^m$ such that $(x_0, u_\ast) \in \Omega$ and

$$I(x_0, u_\ast) \geq \lim_{k \to \infty} I(x_k, u_k).$$

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Footnote 6: For a definition of the weak convergence in $AC_{\text{loc}}^{0}(\mathbb{R}_+)$, see [6].
Proof of Theorem 1. Let \( \{(x_k, u_k)\}_{k=1}^{\infty} \) be a maximizing sequence for \( J(\cdot, \cdot) \) from \( \tilde{\Omega}_\alpha \). Due to conditions (vii), (viii) (with \( \Omega_\alpha \) replaced by \( \tilde{\Omega}_\alpha \)) and Theorem 2.1 from [8], the sequence \( \{x_k\}_{k=1}^{\infty} \) contains a subsequence that converges weakly to some function \( x_0 \in \text{AC}_{\text{loc}}(\mathbb{R}^+) \). Let us pass to this subsequence, denoting it again by \( \{(x_k, u_k)\}_{k=1}^{\infty} \).

For \( N \in \mathbb{N} \) introduce the function

\[
f_0^N(t, \chi, v) := \begin{cases} f_0(t, \chi, v), & t \in [N-1, N), \ (t, \chi, v) \in U, \\ 0, & t \in \mathbb{R}^+ \setminus [N-1, N), \ (t, \chi, v) \in U, \end{cases}
\]

and consider problem (11)–(3) with \( f_0^N \) instead of \( f_0 \). Denote the corresponding functional (in which the integral is actually taken over the interval \([N-1, N)\) by \( I_N \). Assume first that for every \( N \in \mathbb{N} \) all hypotheses of Theorem 3 hold for this truncated problem (with the objective functional \( I_N \)) and for our sequence \( \{(x_k, u_k)\}_{k=1}^{\infty} \). Then there exists a Lebesgue measurable function \( u_{N*} : \mathbb{R}^+ \to \mathbb{R}^n \) such that \((x_0, u_{N*}) \in \Omega \) and

\[
I_N(x_0, u_{N*}) \geq \lim_{k \to \infty} I_N(x_k, u_k).
\]

Put \( u_\ast(t) := u_{N*}(t) \) for \( t \in [N-1, N) \), \( N \in \mathbb{N} \). Clearly, \((x_0, u_\ast) \in \Omega \) and

\[
J(x_0, u_\ast) = \lim_{K \to \infty} \sum_{N=1}^{K} I_N(x_0, u_\ast) = \lim_{K \to \infty} \sum_{N=1}^{K} I_N(x_0, u_{N*}) \geq \lim_{K \to \infty} \sum_{N=1}^{K} \lim_{k \to \infty} I_N(x_k, u_k) \geq \lim_{k \to \infty} \lim_{K \to \infty} \sum_{N=1}^{K} I_N(x_k, u_k) \geq \lim_{k \to \infty} \lim_{K \to \infty} (J(x_k, u_k) - \tilde{\omega}(K)) = \lim_{k \to \infty} J(x_k, u_k),
\]

where \( \tilde{\omega} \) is a function from condition (xii\( \prime \)).

Thus, \((x_0, u_\ast)\) is the required admissible pair. It remains to explain why the conclusion of Theorem 3 holds for the truncated problem with functional \( I_N \). Among all hypotheses of Theorem 3 only conditions (ix) and (x) need to be checked for \( t \notin [N-1, N) \). The validity of condition (ix) follows from the fact that a projection of a convex set is convex. However, the validity of condition (x) is fairly difficult to prove (moreover, we are even not sure that it takes place).

To overcome this difficulty, we proceed as follows. Note that in the above reasoning the values of \( u_{N*}(t) \) are used only for \( t \in [N-1, N) \). Therefore, we can arbitrarily vary the sequence \( \{(x_k, u_k)\}_{k=1}^{\infty} \) and the parameters of problem (11)–(3) outside the interval \([N-1, N) \). In particular, we can set \( f(t, \cdot, \cdot) = 0 \), \( A(t) = \mathbb{R}^n \), \( U(t, \cdot) = \{0\} \) and \( u_k(t) = 0 \) for \( t \notin [N-1, N) \), as well as \( x_k(t) = x_k(N-1) \) for \( 0 \leq t < N-1 \) and \( x_k(t) = x_k(N) \) for \( t \geq N \). For the problem thus modified (still with the functional \( I_N \)) the validity of all hypotheses of Theorem 3 is undoubtable, and we get the desired function \( u_{N*} \) on \([N-1, N) \). \[ \square \]

Remark 2. From the formal point of view, Theorem 1 cannot be said to strengthen Theorem A not only for reasons explained in footnote 9 but also in view of the following important remark. Theorems 1 and A deal with problems in which the objective functionals are defined differently. In particular, it may happen that for the same parameters of the problem, an optimal solution exists in one problem and does not exist in the other, or that optimal solutions exist in both problems but are different. Nevertheless, the hypothesis in Theorem 1 concerning the behavior of the control system at infinity seems to be essentially weaker than that in Theorem A. As an illustration, we give the following example.
Example 1. Consider the problem

\[
\int_0^\infty \frac{u(t)}{t+1} \, dt \to \text{max},
\]

(7)

\[\dot{x}(t) = u(t) \quad \text{for a.e. } t \in \mathbb{R},\]

(8)

\[x(t) \in [-t, t] \cap [-1, 1], \quad u(t) \in [-1, 1] \quad \text{for a.e. } t \in \mathbb{R}^+.\]

(9)

It is clear that \(x(0) = 0\) and the absolute value of the integrand in (7) is bounded by \(1/(t+1)\) for every admissible pair \((x, u)\). All local conditions (i)–(x) and (xi′) are satisfied. Let us show that condition (xi′′) also holds:

\[
\int_{T'}^T u(t) \frac{dt}{t+1} = \int_{T'}^T \dot{x}(t) \frac{dt}{t+1} = \frac{x(T')}{T'+1} - \frac{x(T)}{T+1} + \int_T^{T'} \frac{x(t)}{(t+1)^2} \, dt \leq \frac{1}{T'+1} + \frac{1}{T+1} + \int_T^{T'} \frac{dt}{(t+1)^2} = \frac{2}{T+1} \quad \forall T > 0. \tag{10}
\]

Thus, if we consider the functional (7) as an improper integral, i.e., in the sense of (5), then we can apply Theorem 1 which guarantees the existence of an optimal solution.

This optimal solution can easily be found explicitly. Indeed, since

\[
\lim_{T \to \infty} \int_0^T u(t) \frac{dt}{t+1} = \lim_{T \to \infty} \left( \frac{x(T)}{T+1} + \int_0^T \frac{x(t)}{(t+1)^2} \, dt \right) = \lim_{T \to \infty} \int_0^T \frac{x(t)}{(t+1)^2} \, dt,
\]

it suffices to maximize \(x(t)\) at every \(t\) (which is possible here), i.e., set \(u_*(t) = 1\) for \(t < 1\) and \(u_*(t) = 0\) for \(t \geq 1\). The corresponding optimal trajectory is \(x_*(t) = \min\{t, 1\}\).

Since the integrand is positive, by Remark 1 this solution is also optimal in the case when the objective functional is understood in the sense of (4). Let us show that nevertheless Theorem A is inapplicable in this case for any \(\alpha\) (except for \(\alpha\) equal to the exact value \(\alpha_* (= \ln 2)\) of the functional on the optimal solution, but in this case the theorem is almost worthless, because the set \(\Omega_{\alpha_*}\) consists of a single admissible pair). The reason is that condition (xi) of strong uniform integrability does not hold for \(\alpha < \alpha_*\). Let us demonstrate this.

Consider first an admissible pair with \(u(t) = \cos t\), i.e., the pair

\[u_0(t) = \cos t, \quad x_0(t) = \sin t, \quad t \geq 0.\]

Then

\[
\int_0^\infty \left( \frac{u_0(t)}{t+1} \right)^+ \, dt = \int_0^\infty \max\{\cos t, 0\} \frac{dt}{t+1} = +\infty; \tag{11}
\]

i.e., no family of functions containing the integrand in (11) can be strongly uniformly integrable.

To show that condition (xi) is violated even for admissible pairs for which the value of the functional (in any sense) is close to the optimal value, it suffices to construct such an admissible pair from pieces:

• first, on a sufficiently large interval \([0, T_1]\) with \(T_1 = \pi/2 + 2\pi k, k \in \mathbb{N}\), use the optimal control \(u_*\) and follow the optimal trajectory \(x_*\);
second, on a sufficiently large interval \([T_1, T_2]\), use the control \(u_0\) and follow the trajectory \(x_0\) (since \(x_0(T_1) = 1 = x_*(T_1)\), we can switch from one trajectory to the other);

- for \(t > T_2\), use the control \(u = 0\).

Due to the vanishing control on the last interval, the value of the functional (in any sense) on such an admissible pair is finite. By virtue of estimate (11) (note that replacing \(u\) with \(-u\) changes the trajectory \(x\) to \(-x\), so estimate (11) also holds for the absolute value of the integral on the left-hand side), the value of the functional (in any sense) on such a pair differs from the optimal value by at most \(4/(T_1 + 1)\). Choosing a sufficiently large \(T_2\) (depending on \(T_1\)), we can make the integral analogous to (11) as large as desired. This means that condition (xi) of strong uniform integrability does not hold for \(\Omega_\alpha\) for any \(\alpha < \alpha_*\).

**Remark 3.** A similar example can be constructed without state constraints. For example, it suffices to replace \(u(t)\) with \(u(t)(1 - x(t)^2)\) in (7) and (8) and introduce the initial condition \(x(0) = 0\) in (9) instead of the state constraint.

**Remark 4.** For the problem considered in Example 1 the existence result from [9, Theorem 7.9] is also inapplicable, because it requires that there should be a locally integrable function \(\lambda: \mathbb{R}_+ \to \mathbb{R}\) with finite improper integral \(\int_0^\infty \lambda(t) \, dt\) that would majorize the integrand in the objective functional for all admissible pairs in \(\tilde{\Omega}_\alpha\). It is clear that there is no such a function in our problem for \(\alpha < \alpha_*\) (while for \(\alpha = \alpha_*\), as mentioned above, the set \(\tilde{\Omega}_\alpha\) consists of a single pair and the theorem becomes almost worthless).

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