Application of the hypercomplex fractional integro-differential operators to the fractional Stokes equation*

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Abstract

We present a generalization of several results of the classical continuous Clifford function theory to the context of fractional Clifford analysis. The aim of this paper is to show how the fractional integro-differential hypercomplex operator calculus can be applied to a concrete fractional Stokes problem in arbitrary dimensions which has been attracting recent interest (cf. [1,6]).

1 Basics on fractional calculus and special functions

For \( a, b \in \mathbb{R} \) with \( a < b \) and \( \alpha > 0 \), the left and right Riemann-Liouville fractional integrals \( I_{a+}^\alpha \) and \( I_{b-}^\alpha \) of order \( \alpha \) are defined by (see [5])

\[
(I_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} \, dt, \quad x > a,
\]

\[
(I_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} \, dt, \quad x < b.
\]

(1)

By \( RL_{a+}^\alpha \) and \( RL_{b-}^\alpha \), we denote the left and right Riemann-Liouville fractional derivatives of order \( \alpha > 0 \) on \( [a, b] \subset \mathbb{R} \) (see [3]):

\[
(RL_{a+}^\alpha f)(x) = (D^m I_{a+}^{m-\alpha} f)(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}} \, dt, \quad x > a
\]

(2)

\[
(RL_{b-}^\alpha f)(x) = (-1)^m (D^m I_{b-}^{m-\alpha} f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(t-x)^{\alpha-m+1}} \, dt, \quad x < b.
\]

(3)

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Here, \( m = [\alpha] + 1 \) and \([\alpha]\) means the integer part of \( \alpha \). The symbols \( C_{D_{a_+}^\alpha} \) and \( C_{D_{b_-}^\alpha} \) denote the left (resp. right) Caputo fractional derivative of order \( \alpha > 0 \):

\[
(C_{D_{a_+}^\alpha} f) (x) = (I_{a_+}^{\alpha - m} D^m f)(x) = \frac{1}{\Gamma(m - \alpha)} \int_a^x \frac{f^{(m)}(t)}{(x - t)^{\alpha - m + 1}} \, dt, \quad x > a
\]

\[
(C_{D_{b_-}^\alpha} f) (x) = (-1)^m \left( I_{b_-}^{\alpha - m} D^m f\right)(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b \frac{f^{(m)}(t)}{(t - x)^{\alpha - m + 1}} \, dt, \quad x < b.
\]

We denote by \( I_{a_+}^\alpha (L_1) \) the class of functions \( f \) that are represented by the fractional integral \( [ ] \) of a summable function, that is \( f = I_{a_+}^\alpha \varphi \), with \( \varphi \in \mathcal{L}_1(a, b) \). The space \( AC^m([a, b]) \) contains all functions that are continuously differentiable over \([a, b]\) up to the order \( m - 1 \) and \( f^{(m-1)} \) is supposed to be absolutely continuous over \([a, b]\).

To explicitly describe the integral kernels that are used in the sequel we need to introduce the two-parameter Mittag-Leffler function \( E_{\mu, \nu}(z) \) (cf. [3]) as \( E_{\mu, \nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \nu)} \), \( \mu > 0 \), \( \nu \in \mathbb{R} \), \( z \in \mathbb{C} \). Let us now turn to the treatment of the higher dimensional setting. We consider bounded open rectangular domains in \( \mathbb{R}^n \) of the form \( \Omega = \prod_{i=1}^{\alpha} [a_i, b_i] \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_i \in [0, 1] \), \( i = 1, \ldots, n \). The \( n \)-parameter fractional Laplace operators \( RL_{\alpha_a} \) and \( C_{\alpha_a} \), and the associated fractional Dirac operators \( RL_{D_{a_+}^\alpha} \) and \( C_{D_{a_+}^\alpha} \) acting on the variables \((x_1, \ldots, x_n)\) are defined in \( \Omega \) by

\[
RL_{\alpha_a} = \sum_{i=1}^{\alpha} RL_{\alpha_a}^i, \quad C_{\alpha_a} = \sum_{i=1}^{\alpha} C_{\alpha_a}^i, \quad RL_{D_{a_+}^\alpha} = \sum_{i=1}^{\alpha} e_i RL_{D_{a_+}^\alpha}^i, \quad C_{D_{a_+}^\alpha} = \sum_{i=1}^{\alpha} e_i C_{D_{a_+}^\alpha}^i. \quad (6)
\]

For \( i = 1, \ldots, n \) the partial derivatives \( RL_{\alpha_a}^i \) and \( C_{\alpha_a}^i \) are the left Riemann-Liouville and Caputo fractional derivatives of orders \( 1 + \alpha_i \) and \( 1 + \alpha_i \), with respect to the variable \( x_i \in [a_i, b_i] \).

Under certain conditions we have that \( RL_{\alpha_a} = -RL_{D_{a_+}^\alpha} \) and \( RL_{D_{a_+}^\alpha} \) and \( C_{\alpha_a} = -C_{D_{a_+}^\alpha} \) (see [2]). Due to the nature of the eigenfunctions and the fundamental solution of these operators we additionally need to consider the variable \( \tilde{x} = (x_2, \ldots, x_n) \) in \( \tilde{\Omega} = \prod_{i=2}^{\alpha} [a_i, b_i] \), and the fractional Laplace and Dirac operators acting on \( \tilde{x} \) defined by

\[
RL_{\alpha_a} = \sum_{i=2}^{\alpha} RL_{\alpha_a}^i, \quad C_{\alpha_a} = \sum_{i=2}^{\alpha} C_{\alpha_a}^i, \quad RL_{D_{a_+}^\alpha} = \sum_{i=2}^{\alpha} e_i RL_{D_{a_+}^\alpha}^i, \quad C_{D_{a_+}^\alpha} = \sum_{i=2}^{\alpha} e_i C_{D_{a_+}^\alpha}^i. \quad (7)
\]

Next recalling from [2] we know that a family of fundamental solutions of the fractional Dirac operator \( C_{D_{a_+}^\alpha} \) can be represented in the way \( \mathcal{G}_1^\alpha (x) = \sum_{i=1}^{\alpha} e_i \mathcal{G}_1^\alpha (x) \), where

\[
(C_1^\alpha)^{-1} (x) = (x_1 - a_1)^{-\alpha_1} E_{1+\alpha_1, 1-\alpha_1} \left( -(x_1 - a_1)^{1+\alpha_1} C_{\alpha_a}^1 g_0(\tilde{x}) \right) + (x_1 - a_1)^{-\alpha_1} E_{1+\alpha_1, 1-\alpha_1} \left( -(x_1 - a_1)^{1+\alpha_1} C_{\Delta_{a_+}^\alpha}^1 g_1(\tilde{x}) \right),
\]

and for \( i = 2, \ldots, n \)

\[
(C_2^\alpha)^{-1} (x) = \left( E_{1+\alpha_1, 1} \left( -(x_1 - a_1)^{1+\alpha_1} C_{\Delta_{a_+}^\alpha}^1 g_0(\tilde{x}) \right) \right) + (x_1 - a_1) \left( E_{1+\alpha_1, 1} \left( -(x_1 - a_1)^{1+\alpha_1} C_{\Delta_{a_+}^\alpha}^1 g_1(\tilde{x}) \right) \right),
\]

with \( g_0(\tilde{x}) = v(a_1, \tilde{x}) \) and \( g_1(\tilde{x}) = v'_{x_1} (a_1, \tilde{x}) \). The functions \( v \) and \( v'_{x_1} \) are defined in Corollary 3.5 of [2].

### 2 Fractional Hypercomplex Integral Operators

In this section we recall the definitions and the main properties of the fractional versions of the Teodorescu and Cauchy-Bitsadze operators that are going to be used in the sequel to treat the fractional Stokes problem. For all the detailed proofs and calculations we refer to our paper [2]. First we recall the following fractional Stokes formula
Theorem 2.5. Let \( f, g \in C^{0,n}(\Omega) \cap AC^1(\Omega) \cap AC(\Omega) \), then we have
\[
\int_{\Omega} \left[ -f \left( C^{\alpha}_a(\cdot) \right)(x) + f(x) \right] \left( RL^{\alpha}_a + g \right)(x) \, dx = \int_{\partial \Omega} f(x) \, d\sigma(x) \left( I^\alpha_a + g \right)(x),
\]
where \( d\sigma(x) = n(x) \, d\Omega \), with \( n(x) \) being the outward pointing unit normal vector at \( x \in \partial \Omega \), where \( d\Omega \) is the classical surface element, and where \( dx \) represents the \( n \)-dimensional volume element.

Replacing \( f \) by \( C^\alpha g(x-y) \) in (10) we now may obtain the following fractional Borel-Pompeiu formula (a detailed proof is presented in [2]).

Theorem 2.6. Let \( g \in C^{0,n}(\Omega) \cap AC^1(\Omega) \cap AC(\Omega) \). Then the following fractional Borel-Pompeiu formula holds
\[
-\int_{\Omega} C^\alpha g(x + a - y) \left( RL^{\alpha}_a + g \right)(y) \, dy + \int_{\partial \Omega} C^\alpha g(x + a - y) \, d\sigma(y) \left( I^\alpha_a + g \right)(y) = g(x).
\]

From (11) we may introduce the following definition.

Definition 2.3. Let \( g \in AC^1(\Omega) \). Then the linear integral operators
\[
(T^\alpha g)(x) = -\int_{\Omega} C^\alpha g(x + a - y) \, dy,
\]
\[
(F^\alpha g)(x) = \int_{\partial \Omega} C^\alpha g(x + a - y) \, d\sigma(y) \left( I^\alpha_a + g \right)(y)
\]
are called the fractional Teodorescu and Cauchy-Bitsadze operator, respectively.

The previous definition allows us to rewrite (11) in the alternative form \( (T^\alpha RL^{\alpha}_a + F^\alpha g)(x) = g(x) \), with \( x \in \Omega \). For the regularity and mapping properties of (12) we refer to [2]. Again, in [2] we proved the following result:

Theorem 2.4. The fractional operator \( T^\alpha \) is the right inverse of \( C^{\alpha}_a \), i.e., for \( g \in L_p(\Omega) \), with \( p \in ]1, \frac{2}{1-\alpha} [ \) and \( \alpha^* = \min_{1 \leq i \leq n} \{ \alpha_i \} \), we have \( (C^{\alpha}_a, T^\alpha g)(x) = g(x) \).

All these tools in hand allow us to obtain the following Hodge-type decomposition which is our key tool to treat boundary value problems related to the fractional Dirac operator, such as presented with a small example in the next section (see [2] for a detailed proof):

Theorem 2.5. Let \( q = \frac{2p}{2-(1-\alpha^*)p} \), \( p \in ]1, \frac{2}{1-\alpha^*} [ \), and \( \alpha^* = \min_{1 \leq i \leq n} \{ \alpha_i \} \). The space \( L_q(\Omega) \) admits the following direct decomposition
\[
L_q(\Omega) = L_q(\Omega) \cap \ker(C^{\alpha}_a) \oplus C^{\alpha}_a(W^{\alpha,p}_{a+}(\Omega)),
\]
where \( W^{\alpha,p}_{a+}(\Omega) \) is the space of functions \( g \in W^{\alpha,p}_{a+}(\Omega) \) such that \( \text{tr}(g) = 0 \). Moreover, we can define the following projectors
\[
P^\alpha : L_q(\Omega) \to L_q(\Omega) \cap \ker(C^{\alpha}_a),
\]
\[
Q^\alpha : L_q(\Omega) \to C^{\alpha}_a(W^{\alpha,p}_{a+}(\Omega)).
\]

In the previous theorem the fractional Sobolev space \( W^{\alpha,p}_{a+}(\Omega) \) has the following norm:
\[
\|f\|_{W^{\alpha,p}_{a+}(\Omega)} := \|f\|_{L^p_{\alpha}(\Omega)} + \sum_{k=1}^n \left\| C^{\alpha+(1+\alpha_k)}_{a+k} f \right\|_{L^p_{\alpha}(\Omega)},
\]
where \( \| \cdot \|_{L^p_{\alpha}(\Omega)} \) is the usual \( L_p \)-norm in \( \Omega \), and \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_k \in ]0,1], k = 1, \ldots, n \).

Remark 2.6. We would like to remark that our results coincide with the classical ones presented in [3] when considering the limit case of \( \alpha = (1, \ldots, 1) \). However, we can notice differences in the fractional setting, for instance in the expression of the fundamental solution and in the function spaces considered.
3 A fractional Stokes problem

Recently fractional versions of the Stokes problem have attracted a fast growing interest (see for instance [6]). The application of the version of the Laplacian allows us to model sub-diffusion problems of (in our case incompressible) flows. The following system describes the simplest model of Stokes equation with sub (resp. super) dissipation. In the Riemann-Liouville case (the Caputo case is treated analogously) it has the form

\[-RL\Delta^\alpha_a u + \text{grad}^\alpha p = F \quad \text{in } \Omega \]
\[\text{div}^\alpha u = 0 \quad \text{in } \Omega \]
\[u = 0 \quad \text{on } \partial \Omega\]

Here again we suppose that \(\Omega\) is a rectangular domain, \(F\) is given, \(p\) is the unknown pressure of the flow and \(u\) its unknown velocity. As in the continuous case treated in [4], the hypercomplex fractional calculus that we proposed in the previous section, now allows us to set up closed solution formulas for \(u\) and \(p\). To proceed in this direction, remember that following [2] the fractional Laplacian can be split in the form

\[-RL\Delta^\alpha_a u + \text{grad}^\alpha p = F \]

If we now apply the fractional \(T^\alpha\)-operator from the left to this equation we get

\[T^\alpha RL\Delta^\alpha_a u + T^\alpha \text{grad}^\alpha p = T^\alpha F\]

Now we can apply our generalized fractional Borel-Pompeiu formula leading to

\[Q^\alpha RL\Delta^\alpha_a u + Q^\alpha \text{grad}^\alpha p = T^\alpha\]

Application of the projector \(Q^\alpha\) arising the fractional version of the Hodge decomposition then leads to

\[Q^\alpha RL\Delta^\alpha_a u + Q^\alpha \text{grad}^\alpha p = T^\alpha F\]

Since \(F^\alpha RL\Delta^\alpha_a u\) and \(F^\alpha p\) are in the kernel of \(RL\Delta^\alpha_a\), we get \(Q^\alpha F^\alpha RL\Delta^\alpha_a u = 0\) and \(Q^\alpha F^\alpha p = 0\) so that our original equation simplifies to

\[Q^\alpha RL\Delta^\alpha_a u + Q^\alpha \text{grad}^\alpha p = Q^\alpha T^\alpha F\]

Next we apply once more \(T^\alpha\) to the left of the equation and use that \(Q^\alpha RL\Delta^\alpha_a = RL\Delta^\alpha_a\), so that the latter equation is equivalent to

\[T^\alpha RL\Delta^\alpha_a u + T^\alpha Q^\alpha \text{grad}^\alpha p = T^\alpha Q^\alpha T^\alpha F\]

which in turn equals

\[u - F^\alpha u + T^\alpha Q^\alpha \text{grad}^\alpha p = T^\alpha Q^\alpha T^\alpha F,\]

so that we finally get the following formula for the velocity of the flow

\[u = T^\alpha Q^\alpha T^\alpha F - T^\alpha Q^\alpha \text{grad}^\alpha p\]

The pressure then can be determined by the equation

\[\text{Sc}(Q^\alpha p) = \text{Sc}(T^\alpha Q^\alpha T^\alpha F)\]

resulting from the second equation.
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