CĂLDĂRARU’S CONJECTURE AND TSYGAN’S FORMALITY

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ABSTRACT. In this paper we complete the proof of Căldăraru’s conjecture on the compatibility between the module structures on differential forms over poly-vector fields and on Hochschild homology over Hochschild cohomology. In fact we show that twisting with the square root of the Todd class gives an isomorphism of precalculi between these pairs of objects.

Our methods use formal geometry to globalize the local formality quasi-isomorphisms introduced by Kontsevich and Shoikhet (the existence of the latter was conjectured by Tsygan). We also rely on the fact - recently proved by the first two authors - that Shoikhet’s quasi-isomorphism is compatible with cap product after twisting with a Maurer-Cartan element.

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1. Introduction and statement of the main results

Throughout \( k \) is a ground field of characteristic 0. In this introduction \((X, \mathcal{O})\) is a ringed site\(^1\) such that \( \mathcal{O} \) is a sheaf of commutative \( k \)-algebras. We fix in addition a Lie algebroid \( \mathcal{L} \) over \((X, \mathcal{O})\).

Roughly speaking a Lie algebroid is a sheaf of \( \mathcal{O} \)-modules which is also a sheaf of Lie algebras which acts on \( \mathcal{O} \) by derivations. See \S3.1. Standard examples of Lie algebroids are the tangent bundle on a smooth manifold and the holomorphic tangent bundle on a complex manifold. Readers not familiar with Lie algebroids are advised to think of \( \mathcal{L} \) as a tangent bundle (holomorphic or not) for the rest of this introduction. Concepts like “connection” take their familiar meaning in this context. In fact: our main reason for working in the setting of Lie algebroids is that these allow us to treat the algebraic, holomorphic and \( C^\infty \)-cases in a uniform way.

1.1. The Atiyah and Todd class of a Lie algebroid. From now on we make the additional assumption that the Lie algebroid \( \mathcal{L} \) is locally free of rank \( d \) as an \( \mathcal{O} \)-module.

The Atiyah class \( A(\mathcal{L}) \in \text{Ext}^1(\mathcal{L}, \mathcal{L}^* \otimes \mathcal{L}) = H^1(X, \mathcal{L}^* \otimes \text{End}_\mathcal{O}(\mathcal{L})) \) of \( \mathcal{L} \) may for example be defined as the obstruction against the existence of a global \( \mathcal{L} \)-connection on \( \mathcal{L} \). See \S6 for more details.

The \( i \)-th scalar Atiyah class \( a_i(\mathcal{L}) \) of \( \mathcal{L} \) is defined as

\[
a_i(\mathcal{L}) = \text{tr}(\bigwedge^i A(\mathcal{L})) \in H^i(X, \bigwedge^i \mathcal{L}^*),
\]

\(^1\)We work over sites instead of spaces to cover some additional cases which are important for algebraic geometry (like algebraic spaces and Deligne Mumford stacks). Readers not interested in such generality may assume that \((X, \mathcal{O})\) is just a ringed space.
where \( \wedge^i \) is the map
\[
\bigwedge^i : (L^* \otimes \text{End}(L))^{\otimes i} \to \bigwedge^i L^* \otimes \text{End}(L)
\]
given by composition on \( \text{End}(L)^{\otimes i} \) and the exterior product on \( (L^*)^{\otimes i} \) and where \( \text{tr} \) is the usual trace on \( \text{End}(L) \), extended linearly to a map \( \wedge^i L^* \otimes \text{End}(L) \to \wedge^i L^* \).

The Todd class \( \text{td}(L) \) of \( L \) is derived from the Atiyah class \( A(L) \) by the following familiar formula:
\[
\text{td}(L) = \det \left( \frac{A(L)}{1 - \exp A(L)} \right) \in \bigoplus_{i \geq 0} H^i(X, \bigwedge^i L^*),
\]
where the function
\[
q(x) = \frac{x}{1 - \exp x}
\]
is extended to \( \bigwedge^i L^* \otimes O \text{End}(L) \) via its formal Taylor expansion. In this way the Todd class \( \text{td}(L) \) of \( L \) can be expressed in terms of the scalar Atiyah classes of \( L \).

1.2. Gerstenhaber algebras and precalculi. By definition a Gerstenhaber algebra is a graded vector space equipped with a Lie bracket \( \{-, -\} \) of degree zero and a commutative, associative cup product \( \cup \) of degree one such that the Leibniz rule is satisfied
\[
\{a, b \cup c\} = \{a, b\} \cup \{a, c\} - (-1)^{|a||b|+1} b \cup \{a, c\}.
\]
If \( A \) is a Gerstenhaber algebra then a precalculus \([11]\) over \( A \) is a quadruple \((A, M, \iota, L)\) where \( M \) is a graded vector space and \( \iota : A \otimes M \to M \) and \( L : A \otimes M \to M \) are linear maps of degree 1 and 0 respectively such that \( \iota \) makes \( M \) into an \( (A[-1], \cup) \)-module and \( L \) makes \( M \) into an \( (A, \{-, -\}) \)-Lie module and such that the following compatibilities hold for \( a, b \in A \)
\[
\begin{align*}
\iota_a L_b &- (-1)^{|a||b|+1} [a, b]^L b = \iota_{[a, b]} \\
L_a b &- (-1)^{|a|+1} \iota_a L_b = L_{[a, b]}
\end{align*}
\]
A precalculus is not the same as a Gerstenhaber module. The second equation in the previous display is not correct for a Gerstenhaber module.

Below \( \iota \) will be referred to as “contraction” and \( L \) as the “Lie derivative”. Furthermore will often write \( a \cap m \) for \( \iota_a(m) \) and as such refer to it as the “cap product”.

1.3. Poly-vector fields, poly-differential operators, differential forms and Hochschild chains in the Lie algebroid framework. For a Lie algebroid \( L \) the sheaves of \( L \)-poly-vector fields and \( L \)-differential forms are defined as
\[
T_{\text{poly}}^L(X) = \bigoplus_{n \geq -1} \bigwedge^{n+1} L, \quad \Omega^L(X) = \bigoplus_{n \leq 0} \bigwedge^n L^*.
\]
where the wedge products are taken over \( \mathcal{O}_X \).

The sheaf \( T_{\text{poly}}^L(X) \) becomes a sheaf of Gerstenhaber algebras when endowed with the trivial differential, the Lie algebroid version of the Schouten–Nijenhuis Lie bracket and the exterior product. Our grading convention is such that the Lie bracket and wedge product are of degree 0 and 1 respectively.

\(^2\)Note that our grading conventions are shifted with respect to the usual ones.
We equip $\Omega^\mathcal{L}(X)$ with the trivial(!) differential\(^3\), and also with the contraction operator and Lie derivative with respect to $\mathcal{L}$-poly-vector fields. In this way the pair $(T_{\text{poly}}^\mathcal{L}(X), \Omega^\mathcal{L}(X))$ becomes a sheaf of precalculi. In our conventions the contraction operator and Lie derivative have degrees 1 and 0 respectively.

The Lie algebroid generalization of the sheaf of $\mathcal{L}$-poly-differential operators is denoted by $D_{\text{poly}}^\mathcal{L}(X)$ [1, 24]. It is the tensor algebra over $\mathcal{O}$ of the universal enveloping algebra of $\mathcal{L}$ (see §3.3 below).

The sheaf $D_{\text{poly}}^\mathcal{L}(X)$ has similar properties as the standard sheaf of poly-differential operators on $X$ (see e.g. [15]). In particular it is a differential graded Lie algebra (shortly, from now on, a DG-Lie algebra) and also a Gerstenhaber algebra up to homotopy. For the definition of the differential, the Lie bracket (of degree 0) and the cup product (of degree 1) see §3.3.

The sheaf of $\mathcal{L}$-Hochschild chains $C_{\text{poly}}^\mathcal{L}(X)$ may be defined as the $\mathcal{O}$-dual of $D_{\text{poly}}^\mathcal{L}(X)$ (although we use a slightly different approach). Furthermore there is a differential $b_H$ as well as actions $\cap, L$ of $D_{\text{poly}}^\mathcal{L}(X)$ on $C_{\text{poly}}^\mathcal{L}(X)$ which make the pair $(D_{\text{poly}}^\mathcal{L}(X), C_{\text{poly}}^\mathcal{L}(X))$ into a precalculus up to homotopy. We refer to §3.4.

Finally, we recall that there is a Hochschild–Kostant–Rosenberg (HKR for short) quasi-isomorphism from $T_{\text{poly}}^\mathcal{L}(X)$ to $D_{\text{poly}}^\mathcal{L}(X)$; dually, there is a HKR quasi-isomorphism from $C_{\text{poly}}^\mathcal{L}(X)$ to $\Omega^\mathcal{L}(X)$. As in the classical case where $\mathcal{L}$ is the tangent bundle neither of these HKR quasi-isomorphisms is compatible with the Gerstenhaber and precalculus structures up to homotopy.

1.4. Main results. Now we consider the derived category $D(X)$ of sheaves of $k$-vector spaces over $X$. When equipped with the derived tensor product this becomes a symmetric monoidal category. Furthermore, viewed as objects in $D(X)$, both $T_{\text{poly}}^\mathcal{L}(X)$ and $D_{\text{poly}}^\mathcal{L}(X)$ are honest Gerstenhaber algebras and their combination with $\Omega^\mathcal{L}(X)$ and $C_{\text{poly}}^\mathcal{L}(X)$ yields precalculi.

Our first main result relates the Todd class of a Lie algebroid (as discussed in §1.1) to the failure of the HKR isomorphisms to preserve these precalculus structures.

**Theorem 1.1.** Let $\mathcal{L}$ be a locally free Lie algebroid of rank $d$ over the ringed site $(X, O_X)$. Then we have the following commutative diagram of precalculi in the category $D(X)$:

\[
\begin{array}{ccc}
T_{\text{poly}}^\mathcal{L}(X) & \xrightarrow{\text{HKR}_{\text{poly}} \sqrt{\operatorname{coker}\mathcal{L}}} & D_{\text{poly}}^\mathcal{L}(X) \\
\downarrow & & \downarrow \\
\Omega^\mathcal{L}(X) & \xrightarrow{(\sqrt{\text{id}(\mathcal{L})} \cap -) \circ \text{HKR}} & C_{\text{poly}}^\mathcal{L}(X),
\end{array}
\]

\(^3\)The De Rham differential $d_{\mathcal{L}}$ on $\Omega^\mathcal{L}(X)$ is not part of the precalculus structure. In the operadic setting of [11], $d_{\mathcal{L}}$ appears as a unary operation and not as a differential.
where the vertical arrows indicate actions and the horizontal arrows are isomorphisms. Here $\wedge$ denotes the left multiplication in $\Omega^L(X)$ and $i$ denotes the contraction action of $\Omega^L(X)$ on $T^L(X)$.\footnote{Note that normally we view $\Omega^L(X)$ as a module over $T^L(X)$. In the definition of the horizontal arrows in the diagram (1.5) the opposite actions appear for reasons that are mysterious to the authors.}

The convention that wavy arrows indicate actions will be used throughout below.

If we consider only the Lie brackets and the Lie algebra actions then the horizontal isomorphisms in the commutative diagram (1.5) are obtained from the horizontal arrows in diagram (1.6) below, which is part of our second main result:

Theorem 1.2. Assume that $k$ contains $\mathbb{R}$. Let $\mathcal{L}$ be a locally free Lie algebroid of rank $d$ over the ringed site $(X, \mathcal{O})$. There exist sheaves of differential graded Lie algebras $(\mathfrak{g}^L_1, d_1, [\ , \ ]_1)$ and sheaves of DG-Lie modules $(\mathfrak{m}^L_1, b_1, L_1)$ over them as well as $L_\infty$-quasi-isomorphisms $\mathfrak{g}^L_1 \to \mathfrak{g}^L_2$ and $\mathfrak{m}^L_1 \to \mathfrak{m}^L_2$, which fit into the following commutative diagram:

\[
\begin{array}{ccc}
T^L_{\text{poly}}(X) & \xrightarrow{\mathfrak{g}^L_2} & \mathfrak{g}^L_2 \\
\downarrow L_1 & & \downarrow L_2 \\
\Omega^L(X) & \xrightarrow{\mathfrak{m}^L_2} & \mathfrak{m}^L_2 \\
\end{array}
\]

where the hooked arrows are strict quasi-isomorphisms.

The proof of Theorems 1.1 and 1.2 depends on the simultaneous globalization of a number of local formality results due to Kontsevich [15] (see also [16]), Tsygan [22], Shoikhet [19] and the first two authors [3–5]. This globalization is performed by a functorial version of formal geometry [6] (see also [25]).

The existence of the upper horizontal isomorphism in (1.5) has been proved independently in [6, 10], while its explicit form has been computed in [6]. The existence of the lower horizontal isomorphism has been shown in [11] by operadic methods. Our approach via Kontsevich’s and Shoikhet’s local formality formula allows us to compute it explicitly.

If we apply the hypercohomology functor $\mathbb{H}^\bullet(X, -)$ to the commutative diagram (1.5) then we obtain the following commutative diagram of precalculi:

\[
\begin{array}{ccc}
\bigoplus_{m,n\geq 0} H^m(X, \wedge^n \mathcal{L}) & \xrightarrow{\text{HKR} \circ (\sqrt{\text{det}(\mathcal{L})})^{-1}} & \mathbb{H}^\bullet(X, D^L_{\text{poly}}(X)) \\
\downarrow \downarrow & & \downarrow \downarrow \\
\bigoplus_{m,n\geq 0} H^m(X, \wedge^n \mathcal{L}^*) & \xrightarrow{(\text{id}(\mathcal{L}) \wedge -) \circ \text{HKR}} & \mathbb{H}^\bullet(X, C^L_{\text{poly}}(X)).
\end{array}
\]

1.5. Căldărușu’s conjecture. Assume now that $X$ is a smooth algebraic or complex variety. Căldărușu’s conjecture (stated originally in the algebraic case) asserts the existence of various compatibilities between the Hochschild (co)homology and tangent (co)homology of $X$ (see below). For the full statement we refer to [8]. The results in this paper complete the proof of Căldărușu’s conjecture.

\[
\text{(1.7)}
\]
We now explain this in more detail. The Hochschild (co)homology \([20]\) of \(X\) is defined as
\[
\begin{align*}
\HH^n(X) &= \Ext^n_{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \quad (n \geq 0) \\
\HH_n(X) &= \Tor_n^{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \quad (n \leq 0)
\end{align*}
\]
where \(\Delta \subset X \times X\) is the diagonal. From these definitions it is clear that \(\HH^\bullet(X)\) has a canonical algebra structure (by the Yoneda product) and \(\HH_\bullet(X)\) is a module over it.

Furthermore if we put \(\mathcal{L} = T_X\) then it is proved in [7] (and partially in [26]) that there are isomorphisms of algebras and modules
\[
\begin{align*}
\HH^\bullet(X) &\longrightarrow \HH^\bullet(X, D^\mathcal{L}_\text{poly}(X)) \\
\HH_\bullet(X) &\longrightarrow \HH_\bullet(X, C^\mathcal{L}_\text{poly}(X))
\end{align*}
\]
where on the right-hand side we consider only the part of the precalculus given by the cup and cap product.

We define the tangent (co)homology of \(X\) by
\[
\begin{align*}
HT^\bullet(X) &= \bigoplus H^\bullet(X, \bigwedge T_X), \quad H\Omega^\bullet(X) = \bigoplus H^\bullet(X, \Omega_X^{-\bullet}).
\end{align*}
\]
where now \(\Omega_X^\bullet\) denotes the graded sheaf of differential forms on \(X\).

The commutative diagram (1.7) then yields the following

**Theorem 1.3 ("Căldărușan’s conjecture").** For a smooth algebraic or complex variety \(X\) over \(k\) there is a commutative diagram of \(k\)-algebras and modules
\[
\begin{align*}
HT^\bullet(X) &\xrightarrow{\text{HKR}^{\tau_X, \sqrt{\text{td}(X)}}} \HH^\bullet(X) \\
\Omega^\bullet(X) &\xleftarrow{\text{HKR}^{\tau_X, \sqrt{\text{td}(X)}}^{-1}} \HH_\bullet(X)
\end{align*}
\]
where \(\text{td}(X)\) is the Todd class for \(\mathcal{L} = T_X\).

Theorem 1.3 completes the proof of the parts of Căldărușan’s conjecture [8] which do not depend on \(X\) being proper. The cohomological part (the upper row in the above diagram) had already been proved in [6] and is also an unpublished result of Kontsevich.

In the proper case there is an additional assertion in Căldărușan’s conjecture which involves the natural bilinear form on \(\HH_\bullet(X)\). We do not consider this assertion in the present paper as it has already been proved by Markarian [17] and Ramadoss [18]. If we combine Theorem 1.8 with the results of Markarian and Ramadoss we obtain a full proof of Căldărușan’s conjecture. Let us also mention that in the compact Calabi-Yau case Căldărușan’s conjecture has been proved in [14].

2. **Notation and conventions**

As stated already we always work over a ground field \(k\) of characteristic 0; unadorned tensor products are over \(k\).
Most objects we consider are equipped with a topology which will be explicitly specified when needed. However if an object is introduced without a specific topology, or if the topology is not clear from the context, then it is assumed to be equipped with the discrete topology.

Many objects we will encounter are \( \mathbb{Z} \)-graded. Koszul’s sign rule is always assumed in this context. For a double or higher complex we apply the sign rule with respect to total degree.

3. Some recollections on Lie algebroids and related topics

3.1. Generalities on Lie algebroids. In this section \( R \) is a commutative \( k \)-algebra.

**Definition 3.1.** A Lie algebroid \( L \) over \( R \) is a Lie algebra over \( k \) which is in addition an \( R \)-module and is endowed with a so-called anchor map \( \rho : L \to \text{Der}_k(R) \) satisfying the compatibility

\[
[l_1, r l_2] = \rho(l_1(r)) l_2 + r[l_1, l_2], \quad r \in R, \ l_i \in L, \ i = 1, 2.
\]

The basic example of a Lie algebroid over \( R \) is \( L = \text{Der}_k(R) \) with the identity anchor map and the commutator Lie bracket.

If \( L \) is a Lie algebroid then \( R \oplus L \) is a Lie algebra with Lie bracket \([ (r, l), (r', l') ] = (\rho(l)(r') - \rho(l')(r), [r, r'] ) \). We define the **universal enveloping algebra** \( U_R(L) \) of \( L \) to be the quotient of the enveloping algebra associated to the Lie algebra \( R \oplus L \) by the relations \( r \otimes l = rl \) \( (r \in R, l \in R \oplus L) \).

For the sake of simplicity, below we will usually omit the anchor map \( \rho \) from the notations.

The universal enveloping algebra of a Lie algebroid satisfies a universal property similar to that of an ordinary enveloping algebra. This implies for example that the anchor map \( \rho \) uniquely extends to an algebra morphism from \( U_R(L) \) to \( \text{End}_k(R) \), or equivalently: it yields a left \( U_R(L) \)-module structure on \( R \).

For reasons which will become clear later we assume that our Lie algebroids are free of rank \( d \) over \( R \).

3.1.1. \( L \)-poly-vector fields and \( L \)-differential forms over \( R \). To a Lie algebroid \( L \) over \( R \) we associate

\[
T^L_{\text{poly}}(R) = \bigoplus_{n \geq -1} \wedge_R^{n+1} L,
\]

\[
\Omega^L(R) = \bigoplus_{n \leq 0} \wedge_R^{-n} L^*, \ L^* = \text{Hom}_R(L, R).
\]

We refer to (3.2) and (3.3) as the spaces of **\( L \)-poly-vector fields** and **\( L \)-forms** on \( R \).

As an exterior algebra \( T^L_{\text{poly}}(R) \) has a wedge product which we denote by \( \cup \) (“the cup product”). The extension of the Lie bracket on \( L \) to a bi-derivation on \( T^L_{\text{poly}}(R) \) defines a Lie bracket which is called the Schouten–Nijenhuis bracket and is denoted by \( [\cdot, \cdot] \). Note that with our grading conventions the cup product has degree one and the Lie bracket has degree zero. The cup product and the Lie bracket make \( T^L_{\text{poly}}(R) \) into a (shifted) Gerstenhaber algebra with trivial differential.

On the other hand, \( \Omega^L(R) \) is obviously a graded algebra with respect to the wedge product. In addition there is an analogue \( d_L \) of the De Rham differential on
\( \Omega^L(R) \), which is given on generators by
\[
d_L(r)(l) = l(r)
d_L(l_1^*(l_2)) = l_1(l_2^*(l_1)) - l_2(l_1^*(l_1)) - l_1^*(l_2^*(l_1)),
\]
for \( r \in R, l, l_i \in L, i = 1, 2, l^* \in L^* \), and is extended uniquely by Leibniz’s rule.

The natural contraction operation of \( L \)-forms on \( R \) with respect to \( L \)-poly-vector fields is denoted by \( \cap \) (the “cap product”). The Lie derivative \( L \)-forms on \( R \) with respect to \( L \)-poly-vector fields is specified in the usual way via Cartan’s homotopy formula as the commutator of \( d_L \) and the contraction. The pair
\[
((T_{\text{poly}}^L(R), [-,-], \cup), (\Omega^L(R), \cap, L))
\]
forms a precalculus (see §3.2).

3.1.2. \textit{L-connections.} As usual \( L \) is a Lie algebroid over \( R \).

\textbf{Definition 3.2.} Let \( M \) be an \( R \)-module. An \textit{\( L \)-connection} on \( M \) is a \( k \)-linear map \( \nabla \) from \( M \) to \( L^* \otimes_R M \), which satisfies Leibniz’s rule
\[
\nabla(rm) = d_L(r) \otimes m + r \nabla m, \quad r \in R, \quad m \in M.
\]

The \( L \)-connection \( \nabla \) is said to be \textit{flat}, if \( \nabla^2 = 0 \). Equivalently, the assignment \( l \mapsto \nabla_l \), where \( \nabla_l \) denotes the composition of \( \nabla \) followed by contraction with respect to \( l \), defines a Lie algebra morphism from \( L \) to \( \text{End}_k(M) \).

If we let \( l \in L \) act as \( \nabla_l \) then a flat \( L \)-connection on \( M \) extends to a left \( U_R(L) \)-module structure on \( M \).

Furthermore a flat \( L \)-connection \( \nabla \) on \( M \) can be extended to a differential (denoted by the same symbol) on the graded \( R \)-module \( \Omega^L(R) \otimes_R M \) via Leibniz’s rule
\[
\nabla(\omega \otimes_R m) = d_L \omega \otimes_R m + (-1)^{\lvert \omega \rvert} \omega \wedge \nabla m, \quad \omega \in \Omega^L(R), \quad m \in M.
\]

3.1.3. \textit{\( L \)-differential operators over \( R \).} In this section we define the algebra of poly-differential operators of a Lie algebroid and we list some of its properties. We give some explicit formulæ along the lines of [5].

As in the case of ordinary Lie algebras \( U_R(L) \) (see §3.1) may be naturally filtered by giving \( R \) filtered degree 0 and \( L \) filtered degree 1. In particular
\[
F_0 U_R(L) = R, \quad F_1 U_R(L) = R \oplus L,
\]
We view \( U_R(L) \) as an \textit{\( R \)-central} bimodule via the natural embedding of \( R \) into \( U_R(L) \). Explicitly, if we denote this embedding by \( i \) then\(^5\)
\[
(rD) \overset{\text{def}}{=} Dr \overset{\text{def}}{=} i(r)D, \quad r \in R, \quad D \in U_R(L)
\]

Moreover \( U_R(L) \) is an \( R \)-coalgebra [24], i.e. \( U_R(L) \) possesses an \( R \)-linear coproduct \( \Delta : U_R(L) \to U_R(L) \otimes_R U_R(L) \) and an \( R \)-linear counit, satisfying the usual axioms. The comultiplication actually takes values in
\[
(U_R(L) \otimes_R U_R(L))^t = \left\{ \sum_j D_j \otimes E_j \in U_R(L) \otimes_R U_R(L) \mid \forall r \in R : \sum_j D_j i(r) \otimes E_j = \sum_j D_j \otimes E_j i(r) \right\}
\]
\(^5\)Note that there is an at first sight more natural right \( R \)-module structure on \( U_R(L) \) given by the formula \( Dr = Di(r) \). This alternative right module structure will not be used in this paper.
which is an $R$-algebra even though $U_R(L) \otimes_R U_R(L)$ is not.

The comultiplication $\Delta$ and counit $\epsilon$ are given by similar formulæ as in the Lie algebra case

$$\Delta(r) = r \otimes_R 1 + 1 \otimes_R r \quad r \in R$$
$$\Delta(l) = l \otimes_R 1 + 1 \otimes_R l \quad l \in L$$
$$\Delta(\textit{DE}) = D(\textit{E}(1)) \otimes D(\textit{E}(2)) \quad D, E \in U_R(L)$$
$$\epsilon(D) = D(1)$$

(3.6)

In the third formula we have used Sweedler’s convention. The expression on the right side is well defined because it is the product inside the algebra $(U_R(L) \otimes_R U_R(L))'$. In the fourth formula we have used the natural action of $U_R(L)$ on $R$ (see §3.1).

The algebra (better: in the terminology of [1, 24] “the Hopf algebroid”) $U_R(L)$ may be thought of as an algebra of $L$-differential operators on $R$: in the case $L = \text{Der}_k(R)$ and $R$ is smooth over $k$ then $U_R(L)$ coincides with the algebra of differential operators on $R$.

3.1.4. $L$-jets. Let $(U_R(L))_{\leq n}$ be the elements of degree $\leq n$ with respect to the canonical filtration on $U_R(L)$ introduced in §3.1.3. The $L$-$n$-jets are defined as

$$J^n L = \text{Hom}_R(U_R(L)_{\leq n}, R)$$

(this is unambiguous, as the left and right $R$-modules structures on $U_R(L)$ are the same, see (3.5)). We also put

$$J^0 L = \text{Hom}_R(U_R(L), R) = \text{proj}\lim_n J^n L \quad (\text{as } U_R(L) = \text{inj}\lim_n(U_R(L))_{\leq n}).$$

$J^0 L$ has a natural commutative algebra structure obtained from the comultiplication on $U_R(L)$. Thus for $\phi_1, \phi_2 \in J^0 L, D \in U_R(L)$ we have

$$(\phi_1 \phi_2)(D) = \phi_1(D(1)) \phi_2(D(2)),$$

and the unit in $J^0 L$ is given by the counit on $U_R(L)$.

In addition $J^0 L$ has two commuting left $U_R(L)$-module structures which we now elucidate. First of all there are two distinct monomorphisms of $k$-algebras

$$\alpha_1 : R \to J^0 L : r \mapsto (D \mapsto r \epsilon(D)), $$
$$\alpha_2 : R \to J^0 L : r \mapsto (D \mapsto D(r)).$$

It will be convenient to write $R_i = \alpha_i(R)$ and to view $J^0 L$ as an $R_1 - R_2$-bimodule.

There are also two distinct commuting actions by derivations of $L$ on $J^0 L$. Let $l \in L, \phi \in J^0 L, D \in U_R(L)$.

$$1\nabla l(\phi)(D) = l(\phi(D)) - \phi(lD)$$
$$2\nabla l(\phi)(D) = \phi(Dl)$$

Again it will be convenient to write $L_i$ for $L$ acting by $1\nabla$. Then $1\nabla$ defines a flat $L_i$-connection on $J^0 L$, considered as an $R_i$-module. The connection $1\nabla$ is the well-known Grothendieck connection. It follows that $J^0 L$ is a $U_R(L)_1 - U_R(L)_2$-bimodule (with both $U_R(L)_1$ and $U_R(L)_2$ acting on the left).

The $U_R(L)_2$ action on $J^0 L$ takes the very simple form

$$(D \cdot \phi)(E) = \phi(ED)$$

(for $D, E \in U_R(L)_2, \phi \in J^0 L$).
Define $\epsilon : JL \to R$ by $\epsilon(\phi) = \phi(1)$ and put $J^cL = \ker \epsilon$. Then $JL$ is complete for the $J^cL$-adic topology and the filtration on $JL$ induced by (3.7) coincides with the $J^cL$-adic filtration. If we filter $JL$ with the $J^cL$-adic filtration then we obtain

$$\text{gr } JL = S_R L^*$$

and the $R_1$ and $R_2$-action on the r.h.s. of this equation coincide (here and below the letter S stands for “symmetric algebra”).

The induced actions on $\text{gr } JL = S_R L^*$ of $l \in L$, considered as an element of $L_1$ and $L_2$, are given by the contractions $i_{-l}$ and $i_l$.

In case $R$ is the coordinate ring of a smooth affine algebraic variety and $L = \text{Der}_k(R)$ then we may identify $JL$ with the completion $\hat{R} \otimes R$ of $R \otimes R$ at the kernel of the multiplication map $R \otimes R \to R$. The two actions of $R$ on $JL$ are respectively $\hat{R} \otimes 1$ and $1 \otimes \hat{R}$.

Similarly a derivation on $R$ can be extended to $\hat{R} \otimes R$ in two ways by letting it act respectively on the first and second factor. Since derivations are continuous they act on adic completions and hence in particular on $JL$. This provides the two actions of $L$ on $JL$.

In the sequel we will view the action labelled by “1” as the default action. I.e. we will usually not write the 1 explicitly.

3.2. Relative poly-vector fields, poly-differential operators. We need relative poly-differential operators and poly-vector fields. So assume that $A \to B$ is a morphism of commutative $k$-algebras. Then

$$T_{\text{poly}, A}(B) = \bigoplus_{n \geq -1} T^n_{\text{poly}, A}(B)$$

$$D_{\text{poly}, A}(B) = \bigoplus_{n \geq -1} D^n_{\text{poly}, A}(B)$$

where $T^n_{\text{poly}, A}(B) = \bigwedge^{n+1}_B \text{Der}_A(B)$. Similarly $D^n_{\text{poly}, A}(B)$ is the set of maps with $n+1$ arguments $B \otimes_A \cdots \otimes_A B \to B$ which are differential operators when we equip $B$ with the diagonal $B \otimes_A \cdots \otimes_A B$-algebra structure.

It is easy to see that $T_{\text{poly}, A}(B)$ is a Gerstenhaber algebra when equipped with the Schouten bracket and the exterior product. Similarly $D_{\text{poly}, A}(B)$ is a graded subspace of the relative Hochschild complex $C^*_A(B)$ and since differential operators are closed under composition one easily sees that it is in fact a sub-$B_\infty$-algebra.

If $A$ and $B$ are DG-algebras then we equip $T_{\text{poly}, A}(B)$, $D_{\text{poly}, A}(B)$ with the total differentials $[d_B, -]$ and $[d_B, -] + d_H$ where $d_H$ denotes the Hochschild differential. Similar results now apply.

3.3. The sheaf of $L$-poly-differential operators.

Definition 3.3. For a Lie algebroid $L$ over $R$ we define the graded vector space $D^{L}_{\text{poly}}(R)$ of $L$-poly-differential operators on $R$ as the tensor algebra over $R$ of $U_R(L)$ with shifted degree, i.e.

$$D^{L}_{\text{poly}}(R) = \bigoplus_{n \geq -1} U_R(L)^{\otimes_R(n+1)}.$$
The action of $U_R(L)$ on $R$ extends to a map
\[(3.9)\quad D^L_{\text{poly}}(R) \to \text{Hom}_k(R^\otimes n+1, R)\]
defined by
\[(D_1 \otimes \cdots \otimes D_{n+1})(r_1 \otimes \cdots \otimes r_{n+1}) \mapsto D_1(r_1) \cdots D_{n+1}(r_{n+1})\]
whose image lies in the space $D_{\text{poly}}(R)$ of poly-differential operators on $R$.

$D^L_{\text{poly}}(R)$ is a $B_\infty$-algebra. In particular it is a DG-Lie algebra and furthermore it is a Gerstenhaber algebra up to homotopy. In Appendix A we give the formulæ for the full $B_\infty$-structure. Here we content ourselves by reminding the reader of the basic operations.

The **Gerstenhaber bracket** on $D^L_{\text{poly}}(R)$ is defined by
\[(3.10)\quad [D_1, D_2] = D_1\{D_2\} - (-1)^{|D_1||D_2|} D_2\{D_1\}, \quad D_i \in D^L_{\text{poly}}(R), \ i = 1, 2,\]
where
\[D_1\{D_2\} = \sum_{i=0}^{[D_1]} (-1)^{|D_2|} (\text{id} \otimes \Delta D_2 \otimes \text{id} \otimes |D_1|^{-i})(D_1) \cdot (1 \otimes |D_2| \otimes 1 \otimes |D_1|^{-i}).\]
It is a Lie bracket of degree 0. The special element $\mu = 1 \otimes R 1 \in D^L_{\text{poly}}(R) = U_R(L) \otimes_R U_R(L)$ satisfies $[\mu, \mu] = 0$. The **Hochschild differential** is defined as the operator $d_\text{H} = [\mu, -]$.

The **cup product** on $D^L_{\text{poly}}(R)$ is defined by
\[(3.11)\quad D_1 \cup D_2 = (-1)^{|D_1|-1(|D_2|-1)} D_1 \otimes_R D_2\]
(See also Appendix A for an explicit derivation of the previous formulæ).

One may now show that these operations make the 4-tuple $(D^L_{\text{poly}}(R), d_\text{H}, [\ , \ ], \cup)$ into a Gerstenhaber algebra up to homotopy (see Lemma A.1). Indeed if $R$ is smooth over $k$ and $L = \text{Der}_k(R)$ is the tangent bundle then the operations we have defined are the same as those one obtains from the identification $D^L_{\text{poly}}(R) = D_{\text{poly}}(R)$ where we view the right-hand side as a sub-$B_\infty$-algebra of the Hochschild complex $C^\bullet(R)$ of $R$ (cfr. §3.2).

It is in fact, as we explain now, not necessary to verify that we have defined a homotopy Gerstenhaber structure on $D^L_{\text{poly}}(R)$. Indeed the results can be obtained directly from the known results for the Hochschild complex (see [12, 13]). Similarly it is not necessary to write explicit formulæ for $[-,-]$ and $\cup$ (or for the whole $B_\infty$-structure for that matter). This point of view will be useful when we consider Hochschild chains as in that case the formulæ become more complicated.

The $L_2$-action on $JL$ commutes with the $R_1$-action (see §3.1.4) so we obtain a ring homomorphism
\[U_{R_2}(L_2) \to D_{R_1}(JL) : D \mapsto (\theta \mapsto D(\theta)).\]
and hence a map
\[(3.12)\quad D^L_{\text{poly}}(R_2) \to D_{\text{poly}, R_1}(JL).\]
of Gerstenhaber algebras up to homotopy. The right-hand side has an $R_1$-connection given by $[\nabla, -]$ and it follows from [6, Prop. 4.2.4, Lemma 4.3.4] that the left-hand side of (3.12) is given by the horizontal sections for this connection.
Now as discussed in §3.2, we know that $D_{\text{poly}, R}(\mathcal{J}L)$ is a $B_\infty$-algebra and it is an easy verification that the braces and the differential, which make up the $B_\infty$-structure, are horizontal for $[\nabla, -]$. Hence the $B_\infty$-structure on $D_{\text{poly}, R}(\mathcal{J}L)$ descends to $D_{\text{poly}}^L(R)$ and one verifies that its basic operations are indeed given by the formulæ we gave earlier.

3.4. The Hochschild complex of $L$-chains over $R$. We start with the following definition.

**Definition 3.4.** For a Lie algebroid $L$ over $R$, the graded $R$-module

\[
C_{\text{poly}, p}^L(R) = \begin{cases} \mathcal{J}L_{\mathcal{R}-p}, & p < 0 \\ R, & p = 0, \end{cases}
\]

is called the space of **Hochschild $L$-chains over $R$**.

Our aim in this section will be to show that the pair

$$(D_{\text{poly}}^L(R), C_{\text{poly}}^L(R))$$

is a precalculus up to homotopy. We will do this without relying on explicit formulæ (as they are quite complicated). Instead we will reduce to a relative version of [5] which discusses Hochschild (co)homology. Explicit formulæ are given in Appendix B.

Let us first remind the reader that if $A$ is a $k$-algebra then the pair $(C^*(A), C_*(A))$ consisting of the spaces of Hochschild cochains and chains is a precalculus up to homotopy. For $C^*(A)$ this is just the (shifted) homotopy Gerstenhaber structure which we have already mentioned in §3.3 and which was introduced in [12, 13].

The full precalculus structure up to homotopy on $(C^*(A), C_*(A))$ is a more intricate object. A complete treatment in a very general setting has been given in [5]. It is shown that the precalculus structure can be obtained from two interacting $B_\infty$-module structures on $C_*(A)$. These $B_\infty$-module structures are obtained from brace type operations. For more operadic approaches see [11].

Although we do not really use them for the benefit of the reader we state the well-known formulæ for the contraction, the Lie derivative and the differential. If $P \in C^{m-1}(A) = \text{Hom}(A^\otimes m, A)$ and $(a_0| \cdots |a_t) \in C_{-t}(A) = A^\otimes t+1$ then we have

\[
\begin{align*}
\iota_P(a_0| \cdots |a_t) &= (a_0P(a_1, \ldots, a_m)|a_{m+1}||a_t) \\
\mathcal{L}_P(a_0| \cdots |a_t) &= \sum_{i=0}^{t-m+1} (-1)^{(m-1)i} (a_0| \cdots |a_{i-1})P(a_i, \ldots, a_{i+m-1})|a_{i+m}|| \cdots |a_t \\
&\quad + \sum_{l=t-m+2}^{t+1} (-1)^lP(a_l, \ldots, a_t, a_0, \ldots, a_{m-t+l-2})|a_{m-t+l-1}|| \cdots |a_{t-1})
\end{align*}
\]

The differential $b_{\mu}$ is defined as $\mathcal{L}_{\mu}$ where $\mu$ is the multiplication, considered as an element of $\text{Hom}(A^\otimes 2, A)$.

To construct the precalculus structure up to homotopy on $(D_{\text{poly}}^L(R), C_{\text{poly}}^L(R))$ we proceed as in §3.3. We first define an object that is larger than $C_{\text{poly}}^L(R)$.
**Definition 3.5.** The space of $L$-poly-jets over $R$ is the completed space of relative Hochschild chains $\hat{C}_{R_1,\bullet}(JL)$. Explicitly

$$\hat{C}_{R_1,\bullet}(JL) = \bigoplus_{p \leq 0} JL^\otimes_{R_1} -p - 1$$

The Grothendieck connection $^1\nabla$ on $JL$ (see §3.1.4) yields a connection on $\hat{C}_{R_1,\bullet}(JL)$ by Leibniz’s rule to which we also refer as to the Grothendieck connection. The following result was proved in [2].

**Proposition 3.6.** For a Lie algebroid $L$ over a commutative ring $R$ as above, there is an isomorphism of graded vector spaces

$$\hat{C}_{R_1,\bullet}(JL) \overset{^1\nabla}{\to} C_L^{poly}(R)$$

which sends $\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_p \in \hat{C}_{R_1,-p}(JL) \overset{^1\nabla}{\to}$ to $\epsilon(\phi_1)\phi_2 \otimes \cdots \otimes \phi_p \in C_{poly,1-p}(R)$

**Proof.** The arguments of the proof of [2, Prop. 1.11], can be repeated almost verbatim. \hfill $\square$

The formulæ from [5] for the Hochschild complexes now yield that

$$(D_{poly,R_1}(JL),\hat{C}_{R_1,\bullet}(JL)) \subset (C_{cont,\bullet}(JL),\hat{C}_{R_1,\bullet}(JL))$$

is a precalculus up to homotopy. Furthermore one verifies that the formulæ in [5] are compatible with the Grothendieck connection $^1\nabla$. Hence the precalculus descends to one on

$$\quad (D_{poly,R_1}(JL) \overset{^1\nabla}{\to} \hat{C}_{R_1,\bullet}(JL)) = (D_L^{poly}(R), \hat{C}_L^{poly}(R))$$

where we use (3.12) as well as Proposition 3.6.

It remains to check a compatibility. Namely, if $R/k$ is smooth and $L$ is the tangent bundle then we have

$$D_L^{poly}(R) = D_{poly}(R)$$

We also have $JL = R \otimes R$ (see §3.1.4) and in this way we obtain an isomorphism

$$C_L^{poly,-p}(R) = (R \otimes R)^{\otimes_{R_1} p} \to R^{\otimes_{R_1} p+1} : (r_1 \otimes s_1) \otimes \cdots \otimes (r_p \otimes s_p) \mapsto (r_1 \cdots r_p) \otimes s_1 \otimes \cdots \otimes s_p$$

which yields an isomorphism of graded vector spaces

$$C_L^{poly}(R) = \hat{C}_\bullet(R)$$

Thus we have an isomorphism of pairs of graded vector spaces

$$(D_L^{poly}(R), C_L^{poly}(R)) = (D_{poly}(R), \hat{C}_\bullet (R))$$

The right-hand side is a precalculus up to homotopy (as it is basically a pair of spaces of Hochschild chains/cochains).

**Lemma 3.7.** The precalculus up to homotopy on the right-hand side of (3.18) is the same one as the one we have constructed on the left-hand side.
Proof. Note that going from the pair \((k, R)\) to \((R, JL)\) is a base extension by \(R\) (since \(JL = R \hat{\otimes} R\)). Since the formulae in [5] are clearly compatible with base extension we have that the precalculus structure on

\[
(D_{\text{poly}}(R), \hat{C}_{\bullet}(R)) = (R \hat{\otimes} D_{\text{poly}}(R), R \hat{\otimes} \hat{C}_{\bullet}(R))
\]

is obtained by base extension from the one on

\[
(D_{\text{poly}}(R), \hat{C}_{\bullet}(R))
\]

Furthermore one checks that the Grothendieck connections on \(D_{\text{poly}}(R)\) and \(\hat{C}_{\bullet}(R)\) under the isomorphism (3.19) act by the standard Grothendieck connection on the copy of \(R\) appearing on the left of \(\hat{\otimes}\) and trivially on \(D_{\text{poly}}(R), \hat{C}_{\bullet}(R)\). Hence its invariants are precisely \(D_{\text{poly}}(R), \hat{C}_{\bullet}(R)\). This finishes the proof. □

3.5. The Hochschild–Kostant–Rosenberg Theorem in the Lie algebroid framework. We recall the Lie algebroid version of the famous cohomological Hochschild–Kostant–Rosenberg (shortly, HKR) quasi-isomorphism; for a proof, we refer to [1].

**Theorem 3.8.** We consider a Lie algebroid \(L\) over \(R\) in the sense of Definition 3.1, which is assumed to be free of rank \(d\) over \(R\).

Then, the map

\[
HKR(l_1 \land \cdots \land l_p) = (-1)^{\sum_{i<j} l_i \cdot l_j} \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (-1)^{\sigma} l_{\sigma(1)} \otimes_R \cdots \otimes_R l_{\sigma(p)}
\]

defines a quasi-isomorphism of complexes from \((T^L_{\text{poly}}(R), 0)\) to \((D^L_{\text{poly}}(R), d_R)\).

There is a dual version of Theorem 3.8, which will also be needed.

**Theorem 3.9.** The quasi-isomorphism (3.20) induces the quasi-isomorphism

\[
HKR(a) = a \circ HKR
\]

of complexes from \((C^L_{\text{poly}}(R), b_H)\) to \((\Omega^L(R), 0)\).

4. Fedosov resolutions in the Lie algebroid framework

We consider a Lie algebroid \(L\) over a ring \(R\) in the sense of Definition 3.1. As always we assume that \(L\) is free of rank \(d\) over \(R\).

The aim of this section is to discuss the relevant Fedosov resolutions [9] in the Lie algebroid framework, which are needed to formulate and prove the globalization result, which in turn leads to the main results: for this purpose, we first give a brief review of algebraic formal geometry [6, 23, 25], then we discuss Fedosov resolutions of \(L\)-poly-vector fields and \(L\)-poly-differential operators as Gerstenhaber algebras up to homotopy, referring to [6] for details. Finally, we discuss Fedosov resolutions of \(\Omega^L(R)\) (see (3.3)) and \(C^L_{\text{poly}}(R)\) (see (3.13)) which are compatible with the precalculus structure up to homotopy.

4.1. The (affine) coordinate space of a Lie algebroid. For a Lie algebroid \(L\) over \(R\) as above, its so-called coordinate space \(R^{\text{coord},L}\) has been introduced and discussed in details in [6, 23], to which we refer for a more extensive treatment.

The construction of \(R^{\text{coord},L}\) owes to formal geometry: it may be thought of as the coordinate ring of the (formal) variety of (local) trivializations of the Lie
algebroid $L$ over $R$. As such, its main property is the existence of an isomorphism of $R^{\text{coord}}_L$-algebras

$$(4.1) \quad t : R^{\text{coord}}_L \otimes R_1, JL \to R^{\text{coord}}_L [x_1, \ldots, x_d] = R^{\text{coord}}_L \hat{\otimes} F, \quad F = k[x_1, \ldots, x_d]$$

which is universal with respect to the existence of such isomorphisms. Note that $R^{\text{coord}}_L$ is not a topological ring. It is equipped with the discrete topology (like $R$).

As a consequence of the universal property of $R^{\text{coord}}_L$, $R^{\text{coord}}_L$ admits an action of $GL_d(k)$, such that the following identity holds on $R^{\text{coord}}_L \hat{\otimes} F$ for $A \in GL_d(k)$

$$(A^{-1} \hat{\otimes} A)|_{JL} = \text{Id}_{JL}$$

where $JL$ is considered as a subalgebra of $R^{\text{coord}}_L \hat{\otimes} F$ through (4.1).

By means of $R^{\text{coord}}_L$, we consider the graded algebra $C^{\text{coord}}_L = \Omega^{R^{\text{coord}}_L} \otimes \Omega_{R_1}$ $\Omega^{L_1}(R_1)$. It has the structure of a DG-algebra with differential $d_{C^{\text{coord}}_L} = d_{\Omega^{R^{\text{coord}}_L}} \otimes \Omega_{R_1}$ $1 + 1 \otimes \Omega_{R_1} d_{L_1}$, and inherits from $R^{\text{coord}}_L$ a rational $GL_d(k)$-action.

The universal isomorphism (4.1) extends to an isomorphism

$$(4.2) \quad t : C^{\text{coord}}_L \otimes R_1, JL \to C^{\text{coord}}_L [x_1, \ldots, x_d],$$

where we used the respective obvious identifications

$$C^{\text{coord}}_L \otimes R^{\text{coord}}_L (R^{\text{coord}}_L \otimes R_1, JL) \cong \Omega^{R^{\text{coord}}_L} \otimes \Omega_{R_1} (\Omega^{L_1}(R_1) \otimes R_1, JL) \cong C^{\text{coord}}_L \otimes R_1, JL,$$

$$C^{\text{coord}}_L \otimes R^{\text{coord}}_L (R^{\text{coord}}_L \hat{\otimes} F) \cong C^{\text{coord}}_L [x_1, \ldots, x_d].$$

We endow the graded algebra on the left-hand, resp. right-hand, side of (4.2) with the following natural differential

$$(4.3) \quad 1 \nabla^{\text{coord}} = d_{\Omega^{R^{\text{coord}}_L}} \otimes \Omega_{R_1} 1 + 1 \otimes \Omega_{R_1} 1 \nabla, \quad \text{resp.}$$

$$(4.4) \quad d = d_{\Omega^{R^{\text{coord}}_L}} \otimes 1,$$

where $1 \nabla$ has been introduced in Subsection 3.4. Both (4.3) and (4.4) are, by construction, flat $C^{\text{coord}}_L$-connections on the respective spaces, and the obvious inductions from $C^{\text{coord}}_L$ into $C^{\text{coord}}_L \otimes R_1, JL$ and $C^{\text{coord}}_L [x_1, \ldots, x_d]$ are morphisms of DG-algebras.

The main property of the connections (4.3) and (4.4) lies in the existence of the **Maurer–Cartan form on $C^{\text{coord}}_L$**: namely, according to [23, Subsection 1.6] and [6, Subsection 5.2], there exists a unique element $\omega$ of $C^{\text{coord}}_L \hat{\otimes} \text{Der}(F)$ of degree 1, satisfying

$$t \circ 1 \nabla^{\text{coord}} \circ t^{-1} - d = \omega,$$

where the expression on the left-hand side is naturally viewed as a $C^{\text{coord}}_L$-linear derivation of $F$. Furthermore, $\omega$ satisfies the Maurer–Cartan equation in the DG-Lie algebra $C^{\text{coord}}_L \hat{\otimes} \text{Der}(F)$, i.e.

$$d\omega + \frac{1}{2} [\omega, \omega] = 0,$$

which implies that $d + [\omega, \bullet]$ is a flat connection on $C^{\text{coord}}_L [x_1, \ldots, x_d]$ and the verticality condition

$$(4.5) \quad t_v \omega = 1 \otimes v, \quad v \in gl_d(k)$$

(here, $t_v$ on the left-hand side denotes the contraction operation on $C^{\text{coord}}_L$ with respect to $v$, coming from the infinitesimal action of $gl_d(k)$ on $R^{\text{coord}}_L$; $v$ on the right-hand side denotes the linear vector field associated to $v$, acting on $F$).
Finally, we consider the affine coordinate space $R^{\text{aff},L}$ of a Lie algebroid $L$ over $R$: it is simply the $GL_d(k)$-invariant ring
\[ R^{\text{aff},L} = (R^{\text{coord},L})^{GL_d(k)} \]
(it may be viewed, in the standard terminology of formal geometry, as the coordinate ring of the (formal) quotient variety of (local) trivializations of $L$; it may be viewed, in the standard terminology of formal geometry, as the coordinate ring of the (formal) quotient variety of (local) trivializations of $L$ over $R$ with respect to linear changes of coordinates on $L$); it is an $R$-algebra in an obvious way, and enjoys a universal property similar to the one satisfied by $R^{\text{coord},L}$, for which we refer to [6, Subsection 5.4].

Similarly, we have the DG-algebra $C^{\text{aff},L} = \Omega_{R^{\text{aff},L}} \otimes_{\Omega_{R_1}} \Omega^{L_1}(R_1)$, with differential $d_{C^{\text{aff},L}} = d_{\Omega_{R^{\text{aff},L}}} \otimes_{\Omega_{R_1}} 1 + 1 \otimes_{\Omega_{R_1}} d_{L_1}$. We may further consider the graded algebra
\[ C^{\text{aff},L} \otimes_{R^{\text{aff},L}} (R^{\text{aff},L} \otimes_{R_1} JL) \cong \Omega_{R^{\text{aff},L}} \otimes_{\Omega_{R_1}} (\Omega^{L_1}(R_1) \otimes_{R_1} JL) \cong C^{\text{aff},L} \otimes_{R_1} JL, \]
endowed with the natural differential
\[ 1 \nabla^{\text{aff}} = d_{\Omega_{R^{\text{aff},L}}} \otimes_{\Omega_{R_1}} 1 + 1 \otimes_{\Omega_{R_1}} 1 \nabla, \]
making the natural inclusion $C^{\text{aff},L} \hookrightarrow C^{\text{aff},L} \otimes_{R_1} JL$ into a morphism of DG-algebras. Obviously, $1 \nabla^{\text{coord}}$ descends by its very construction to $C^{\text{aff},L} \otimes_{R_1} JL$ and identifies with $1 \nabla^{\text{aff}}$.

**Lemma 4.1.** $R^{\text{aff},L}$ is of the form $S \otimes R$ where $S$ is an (infinitely generated) polynomial ring.

**Proof.** See [6, §5.3].

Note that the decomposition $R^{\text{aff},L} = S \otimes R$ is not canonical.

### 4.2. Fedosov resolutions of $L$-poly-vector fields and $L$-poly-differential operators on $R$.

In this section, we recall briefly the main results of [6, §4.3], to which we refer for more details. We consider relative poly-differential operators and poly-vector fields (see §3.2) in the following situation: $(A, d_A) = (C^{\text{aff},L}, d_{C^{\text{aff},L}})$ and $(B, d_B) = (C^{\text{aff},L} \otimes_{R_1} JL, 1 \nabla^{\text{aff}})$.

**Theorem 4.2.** For a Lie algebroid $L$ over $R$ as above, there exist quasi-isomorphisms of Gerstenhaber algebras up to homotopy
\begin{align*}
& (T^{L_2}_{\text{poly}}(R), 0, [\ , \ ], \cup) \hookrightarrow (T_{\text{poly}, C^{\text{aff},L}}(C^{\text{aff},L} \otimes_{R_1} JL), 1 \nabla^{\text{aff}}, [\ , \ ], \cup), \\
& (D^{L_2}_{\text{poly}}(R), d_H, [\ , \ ], \cup) \hookrightarrow (D_{\text{poly}, C^{\text{aff},L}}(C^{\text{aff},L} \otimes_{R_1} JL), 1 \nabla^{\text{aff}} + d_H, [\ , \ ], \cup). \tag{4.7}
\end{align*}

**Proof.** We refer to [6] for details. For example the map (4.7) is derived by suitable base extension from (3.12). For the fact that the maps are quasi-isomorphisms we refer to [6, Prop. 7.3.1].

### 4.3. The Fedosov resolution of $L$-forms on $R$.

We consider the precalculus $(\Omega^{L}(R), 0, L, \cap)$ of $L$-forms over the Gerstenhaber algebra $(T^{L}_{\text{poly}}(R), 0, [\ , \ ], \cup)$, described in §3.1: we describe now a well-suited resolution of $(\Omega^{L}(R), 0, L, \cap)$ which is compatible with the Fedosov resolution $(T_{\text{poly}, C^{\text{aff},L}}(C^{\text{aff},L} \otimes_{R_1} JL), 1 \nabla^{\text{aff}}, [\ , \ ], \cup)$ from Theorem 4.2.
Theorem 4.3. For a Lie algebroid $L$ over $R$ as above, there exists a quasi-isomorphism of precalculi as in the following commutative diagram $^6$:

$$(T_{\text{poly}}^L(R), 0, [\ ,\ ], \cup) = (T_{\text{poly}}^{L_2}(R_2), 0, [\ ,\ ], \cup) \cong (T_{\text{poly}, \mathcal{C}^{\text{aff}}, L} (\mathcal{C}^{\text{aff}, L} \otimes_R \mathcal{C}^{L}, 1^{\mathcal{C}^{\text{aff}}, L}, [\ ,\ ], \cup),$$

$$(\Omega^L(R), 0, L, \cap) = (\Omega^{L_2}(R_2), 0, L, \cap) \cong (\Omega_{\mathcal{C}^{\text{aff}}, L} \otimes_R \mathcal{C}^{L} / \mathcal{O}_{\mathcal{C}^{\text{aff}}, L}, 1^{\mathcal{C}^{\text{aff}}, L}, L, \cap)$$



the vertical arrows denoting the contraction and Lie derivative.

Proof. We refer to [6, §4.3.3]: we observe that the construction of the quasi-isomorphism uses a dualization of the construction of the quasi-isomorphism (4.6), and that contraction operations and differentials are preserved by the above quasi-isomorphism, whence all algebraic structures are preserved. $\square$

4.4. The Fedosov resolution of $L$-chains on $R$. We consider the DG-algebra $(\mathcal{C}^{\text{aff}, L} \otimes_R \mathcal{C}^{L}, 1^{\mathcal{C}^{\text{aff}}, L})$, and to it we associate the $\mathcal{C}^{\text{aff}, L}$-relative Hochschild chain complex, i.e.

$$\widehat{C}_{\mathcal{C}^{\text{aff}, L}, \bullet} (\mathcal{C}^{\text{aff}, L} \otimes_R \mathcal{C}^{L}) = \bigoplus_{p \leq 0} (\mathcal{C}^{\text{aff}, L} \otimes_R \mathcal{C}^{L}) \hat{\otimes}_{\mathcal{C}^{\text{aff}, L}} (-p+1)$$

$$\cong \bigoplus_{p \leq 0} (\mathcal{C}^{\text{aff}, L} \otimes_R \mathcal{C}^{L}) \hat{\otimes}_{\mathcal{C}^{\text{aff}, L}} (-p+1) = \mathcal{C}^{\text{aff}, L} \otimes_R \widehat{C}_{\mathcal{C}^{\text{aff}, L}, \bullet} (\mathcal{C}^{L})$$

Further, we have the identification

$$\mathcal{C}^{\text{aff}, L} \otimes_R \widehat{C}_{\mathcal{C}^{\text{aff}, L}, \bullet} (\mathcal{C}^{L}) \cong \Omega_{\mathcal{C}^{\text{aff}, L}} \otimes_{\Omega_R^1} \left( \Omega^{L_1} \otimes_R \widehat{C}_{\mathcal{C}^{\text{aff}, L}, \bullet} (\mathcal{C}^{L}) \right).$$

and one checks that the differentials coming from the Grothendieck connection on each side are the same, i.e.

$$1^{\mathcal{C}^{\text{aff}}, L} = d_{\Omega_{\mathcal{C}^{\text{aff}}, L}} \otimes 1 + 1 \otimes 1^{\mathcal{C}^{\text{aff}}, L}$$

Proposition 4.4. For a Lie algebroid $L$ over $R$ as above, the cohomology of $(\widehat{C}_{\mathcal{C}^{\text{aff}, L}, \bullet} (\mathcal{C}^{\text{aff}, L} \otimes_R \mathcal{C}^{L}), 1^{\mathcal{C}^{\text{aff}}, L})$ is concentrated in degree 0, where

$$H^0 \left( \widehat{C}_{\mathcal{C}^{\text{aff}, L}, \bullet} (\mathcal{C}^{\text{aff}, L} \otimes_R \mathcal{C}^{L}), 1^{\mathcal{C}^{\text{aff}}, L} \right) \cong C^L_{\text{poly}} (R).$$

Proof. Taking the inverse of (3.15) we obtain a morphism

$$C^L_{\text{poly}} (R) \cong \widehat{C}_{\mathcal{C}^{\text{aff}, L}, \bullet} (\mathcal{C}^{L}) \otimes_{\mathcal{C}^{\text{aff}, L}} \Omega^1 \cong \widehat{C}_{\mathcal{C}^{\text{aff}, L}, \bullet} (\mathcal{C}^{L})$$

which extends to a morphism

$$\left( C^L_{\text{poly}} (R), 0 \right) \to (\Omega_{\mathcal{C}^{\text{aff}, L}} \otimes_{\Omega_R^1} \left( \Omega^{L_1} \otimes_R \widehat{C}_{\mathcal{C}^{\text{aff}, L}, \bullet} (\mathcal{C}^{L}) \right), d_{\Omega_{\mathcal{C}^{\text{aff}, L}}} \otimes 1 + 1 \otimes 1^{\mathcal{C}^{\text{aff}, L}}$$

$^6\Omega_A$, for a topological $k$-algebra $A$, denotes the continuous De Rham complex. A similar convention holds for an extension of topological algebras $B/A$. 


We will show that it is a quasi-isomorphism. To this end we make use of the identification $R^{\text{aff}, L} = S \otimes R$ given in Lemma 4.1. The right-hand side of the extended morphism becomes

$$\Omega_S \widehat{\otimes} \left( \Omega^{L_1}(R_1) \widehat{\otimes} R_1, C_{R_1, \bullet}(JL) \right), d_S \otimes 1 + 1 \otimes 1 \nabla$$

Using a filtration argument together with a suitable version of Poincaré’s Lemma for $S$, the previous complex is quasi-isomorphic to

$$\left( \Omega^{L_1}(R_1) \otimes R_1, \nabla \right)$$

It remains to show that for each $p \leq 0$

$$\left( \Omega^{L_1}(R_1) \otimes R_1, \nabla \right)$$

has cohomology in degree 0. Filtering this complex with respect to the $J$-adic filtration and taking the associated graded complex one verifies that one obtains

$$\left( \Omega^L(R) \otimes R_1, S(L^*)^{\otimes -p-1}, d \right)$$

where the differential $d$ is obtained from the action of $L$ on $S(L^*)^{\otimes p+1}$ by contraction. Using again a suitable version of Poincaré’s Lemma one finds that the resulting complex is indeed exact in degrees $< 0$. $\square$

**Theorem 4.5.** For a Lie algebroid $L$ over $R$ as above, there is a quasi-isomorphism of precalculi up to homotopy as in the following commutative diagram:

$$\begin{array}{cccc}
(D_{\text{poly}}^L(R), d_H, [ , ], \cup) & \longrightarrow & (D_{\text{poly}, C^{\text{aff}, L}}^L(C^{\text{aff}, L} \widehat{\otimes} R_1, JL), \nabla^{\text{aff}} + d_H, [ , ], \cup), \\
\downarrow & & \downarrow \\
(C^L_{\text{poly}}(R), b_H, L, \cap) & \longrightarrow & \left( C^L_{C^{\text{aff}, L}}(C^{\text{aff}, L} \widehat{\otimes} R_1, JL), \nabla^{\text{aff}} + b_H, L, \cap \right)
\end{array}$$

the vertical arrows denoting the contraction and Lie derivative.

5. **Globalization of Tsygan’s formality in the Lie algebroid framework**

The present section is devoted to the proof of Theorem 1.2. We first briefly review some basic facts on $L_\infty$-algebras, $L_\infty$-modules and related morphisms. This is discussed in [6, §6] for $L_\infty$-morphisms. Here we add a discussion on the descent procedure for $L_\infty$-modules over $L_\infty$-algebras and related morphisms.

Then, we add a short *excursus* on Kontsevich’s and Shoikhet’s formality theorems: we focus on the main properties of both formality morphisms, without delving into the technical details of their respective constructions.

Finally, we give the main lines, along which the globalization of Tsygan’s formality can be proved: the proof is a combination of the properties of Kontsevich’s and Shoikhet’s $L_\infty$-morphisms with the Fedosov resolutions from §4.

5.1. **Descent for $L_\infty$-algebras and $L_\infty$-modules.** We discuss a series of descent scenarios for $L_\infty$-algebras, $L_\infty$-modules and related morphisms, which are modelled after the formalism for descent of differential forms in differential geometry. The verification of the results in this section are along the same lines as [23, §7.6, §7.7]. To clearly separate all the various cases we have numbered them.
(1) To start it is convenient to work over an arbitrary DG operad $O$ with underlying graded operad $\mathcal{O}$ (thus we forget the differential on $O$). Assume that $\mathfrak{g}$ is an algebra over $O$ and consider a set of $\mathcal{O}$-derivations $(\ell_v)_{v \in \mathfrak{s}}$ of degree $-1$ on $\mathfrak{g}$. Put $L_v = d_\mathfrak{g} \ell_v + \ell_v d_\mathfrak{g}$. This is a derivation of $\mathfrak{g}$ of degree zero which commutes with $d_\mathfrak{g}$.

Put
\begin{equation}
\mathfrak{g}^s = \{w \in \mathfrak{g} \mid \forall v \in \mathfrak{s} : \ell_v w = L_v w = 0\}
\end{equation}

It is easy to see that $\mathfrak{g}^s$ is an algebra over $O$ as well. Informally we will call such a set of derivations $(\ell_v)_{v \in \mathfrak{s}}$ an $\mathfrak{s}$-action.

(2) Assume that $M$ is a $\mathfrak{g}$-module and assume that $\mathfrak{s}$ also acts on $M$, in a way compatible with the action of $\mathfrak{s}$ on $\mathfrak{g}$, i.e. a general element $v$ of $\mathfrak{s}$ determines an operator $\ell_v$ on $M$, such that Leibniz’s rule holds true for the operations $\mathcal{O}(n) \otimes (\mathfrak{g}^{\otimes n-1} \otimes M) \to M$. Again, we set $L_v = d_M \ell_v + \ell_v d_M$, which is a derivation of degree $0$ on $M$ compatible with the derivations $L_v$ on $\mathfrak{g}$, $d_M$ being the differential on $M$.

(3) The above constructions apply in particular if $\mathfrak{g}$ is an $L_\infty$-algebra. Assume that it has Taylor coefficients $Q_n$, $n \geq 1$. Then $L_v$ is defined by means of $d_\mathfrak{g} = Q_1$, and the derivation property of $\ell_v$ reads as
\begin{equation}
\ell_v (Q_n(x_1, \ldots, x_n)) = \sum_{i=1}^n (-1)^{\sum_{j=1}^{i-1} |x_j| + i} Q_n(x_1, \ldots, \ell_v x_i, \ldots, x_n), \ x_j \in \mathfrak{g}, \ j = 1, \ldots, n.
\end{equation}

Under these conditions the $L_\infty$-structure descends to $\mathfrak{g}^s$.

(4) Similarly if $M$ is an $L_\infty$-module over $\mathfrak{g}$ defined by Taylor coefficients $R_n$ then the compatibility condition is
\begin{equation}
\ell_v (R_n(x_1, \ldots, x_n; m)) = \sum_{i=1}^n (-1)^{\sum_{j=1}^{i-1} |x_j| + i} R_n(x_1, \ldots, \ell_v x_i, \ldots, x_n; m) + (-1)^{\sum_{j=1}^{n-1} |x_j| + n-1} R_n(x_1, \ldots, \ell_v m), \ m \in M, \ x_j \in \mathfrak{g}, \ j = 1, \ldots, n.
\end{equation}

If this holds then $M^s$ becomes an $L_\infty$-module over $\mathfrak{g}^s$.

(5) We also need descent for $L_\infty$-morphisms. This does not immediately fall under the operadic framework given in (1), (2) but it is easy enough to give explicit formulae like (5.2)(5.3). Thus assume $\psi : \mathfrak{g} \to \mathfrak{h}$ is an $L_\infty$-morphism between $L_\infty$-algebras with $\mathfrak{s}$-action. Under the following compatibility condition
\begin{equation}
\ell_v (\psi_n(x_1, \ldots, x_n)) = \sum_{i=1}^n (-1)^{\sum_{j=1}^{i-1} |x_j| + (i-1)} \psi_n(x_1, \ldots, \ell_v x_i, \ldots, x_n)
\end{equation}

$x_j \in \mathfrak{g}, \ j = 1, \ldots, n, \ n \geq 1, \psi$ descends to an $L_\infty$-morphism $\psi^s : \mathfrak{g}^s \to \mathfrak{h}^s$.

(6) Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a morphism between $L_\infty$-algebras with $\mathfrak{s}$-action such that the descent condition (5.4) holds, and let $N$ be an $L_\infty$-module over $\mathfrak{h}$ equipped with a compatible $\mathfrak{s}$-action. Let $N_\psi$ be the pullback of $N$ along $\psi$. Then the $\mathfrak{s}$-action on $N_\psi$ is compatible with the $\mathfrak{s}$-action on $\mathfrak{g}$.

(7) Now assume that $\mathfrak{g}$ is an $L_\infty$-algebra and $M, N$ are $L_\infty$-modules over $\mathfrak{g}$. Assume that all objects are equipped with an $\mathfrak{s}$-action and that the descent conditions are satisfied.
Assume that $\varphi : M \to N$ is an $L_\infty$-module morphism. Then the condition for $\varphi$ to descend to an $L_\infty$-morphism $M^s \to N^s$ is

$$
\iota_v (\varphi_n (x_1, \ldots , x_n ; m)) = \sum_{i=1}^n (-1)^{\sum_{j=1}^{i-1} |x_j| + (i-1)} \varphi_n (x_1, \ldots , \iota_v x_i, \ldots , x_n ; m) + 
\sum_{i=1}^n (-1)^{\sum_{j=1}^{i} |x_j| + n} \varphi_n (x_1, \ldots , x_n ; \iota_v m),
$$

for $m \in M$, $x_j \in g$, $j = 1, \ldots , n$, $n \geq 1$.

5.2. **Twisting of $L_\infty$-algebras and $L_\infty$-modules.** We refer to [9, §2], for a very detailed exposition of $L_\infty$-algebras, $L_\infty$-modules and the associated twisting procedures. See also [27].

**Convention.** We will work with infinite sums. We assume throughout that the occurring sums are convergent and that standard series manipulations are allowed. This will be the case in our applications.

If $(g, Q)$ is an $L_\infty$-algebra then the Maurer-Cartan equation is defined as

$$
\sum_{j=1}^{\infty} \frac{1}{j!} Q_n (\underbrace{\omega, \ldots , \omega}_j) = 0,
$$

and a solution $\omega \in g_1$ is called a Maurer–Cartan element (MC element for short). Below we will only use DG-Lie algebras and in this case (5.6) reduces to the finite sum

$$
d\omega + \frac{1}{2} [\omega, \omega] = 0.
$$

A MC element defines a new “twisted” DG-Lie structure on $g$ (denoted by $g_\omega$) with Taylor coefficients

$$
Q_{\omega,n}(x_1, \ldots , x_n) = \sum_{j} \frac{1}{j!} Q_{n+j}(\underbrace{\omega, \ldots , \omega}_j, x_1, \ldots , x_n), \quad n \geq 1
$$

If $g$ is actually a DG-Lie algebra then twisting keeps the bracket but changes the differential to

$$
d_{\omega} = d_g + [\omega, -].
$$

If $h$ is another $L_\infty$ algebra, $\psi$ is an $L_\infty$-morphism from $g$ to $h$ and $\omega$ is a MC element in $g$ then

$$
\psi(\omega) = \sum_{n \geq 1} \frac{1}{n!} \psi_n (\underbrace{\omega, \ldots , \omega}_n)
$$

is a MC element in $h$.

We may also twist $\psi$ with respect to $\omega$, so as to get an $L_\infty$-morphism $\psi_\omega$ from $g_\omega$ to $g_\psi(\omega)$, where

$$
\psi_{\omega,n}(x_1, \ldots , x_n) = \sum_{j \geq 0} \frac{1}{j!} \psi_{n+j}(\underbrace{\omega, \ldots , \omega}_j, x_1, \ldots , x_n), \quad n \geq 1.
$$

If $M$ is an $L_\infty$-module over a DG-Lie algebra with Taylor coefficients $R_n$ and $\omega \in g_1$ is a MC element then we may define a twisted $L_\infty$ structure on $M_\omega$ over $g_\omega$ by the
If $g$ is a DG-Lie algebra and $M$ is a DG-Lie module over $g$ then twisting keeps the $g$-action on $M$ but changes the differential on $M$ to
\[ d_\omega = d + \omega \cdot . \]
Twisting of modules is compatible with pullback. More precisely if $\psi : g \to h$ is an $L_\infty$-morphism, $N$ is an $L_\infty$-module over $h$ and $\omega \in g_1$ is a MC element then we have
\[ (N_{\psi}(\omega))_\psi = (N_\psi)_\omega \]
If $\varphi : M \to N$ is an $L_\infty$-morphism of DG-Lie modules over the DG-Lie algebra $g$ and $\omega$ is a MC element in $g_1$ then we obtain a twisted $L_\infty$-morphism $\varphi_\omega : M_\omega \to N_\omega$ which is defined by
\[ \varphi_\omega(x_1, \ldots, x_n; m) = \sum_{j \geq 0} \frac{1}{j!} \varphi_{n+j}(\omega, \ldots, \omega, x_1, \ldots, x_n; m), \quad n \geq 1. \]

5.3. Compatibility of twisting and descent. Assume now that $g$ is a DG-Lie algebra equipped with an $s$-action and that $\omega \in g_1$ is a MC element. Then $s$ still acts on $g_\omega$ (as the concept of an $s$-action only refer to the underlying Lie algebra structure on $g$). However $g^s$ and $g^s_\omega$ will be different (as the Lie derivative $L_v$ for $v \in s$ will be different).

If $(M, R)$ is an $L_\infty$-module over $g$ which is also equipped with a compatible $s$-action then the $s$-actions on $g_\omega$ and $M_\omega$ are compatible provided the following condition holds
\[ R_{n}(t_v \omega, x_2, \ldots, x_n; m) = 0, \quad x_i \in g, \quad i = 2, \ldots, n, \quad n \geq 2, \quad m \in M. \]
This condition is automatic if $M$ is a DG-Lie module.

If $\psi : g \to h$ is an $L_\infty$-morphism of DG-Lie algebras equipped with an $s$-action and the descent condition (5.4) is satisfied for $\psi$ then an easy computation (see e.g. [23, §7.7]) shows that the same descent condition will be satisfied for $\psi_\omega$ if the following condition holds
\[ \psi_n(t_s \omega, x_2, \ldots, x_n) = 0, \quad x_i \in g, \quad i = 2, \ldots, n, \quad s \in s, \quad n \geq 2, \]
Furthermore if in this setting $N$ is an $L_\infty$-module over $h$ with compatible $s$-action such that the compatibility condition (5.10) holds then the corresponding condition will hold for $N_\psi$.

Similarly if we have an $L_\infty$-morphism $\varphi : M \to N$ between DG-Lie modules over a DG-Lie algebra $g$ such that $g, M, N$ are equipped with compatible $s$-actions in such a way that the descent condition (5.5) holds for $\varphi$ then the same descent condition will be satisfied for $\varphi_\omega$ if the following condition holds
\[ \varphi_n(t_s \omega, x_2, \ldots, x_n; m) = 0, \quad m \in M, \quad x_i \in g, \quad i = 2, \ldots, n, \quad n \geq 2, \quad s \in s. \]
5.4. Kontsevich’s and Shoikhet’s formality theorems. In this brief section,
we quote (without proofs) Kontsevich’s and Shoikhet’s formality theorems, along
with the relevant properties, which we will need later in the proof of globalization
results.

We consider the algebra \( F = k[x_1, \ldots, x_d] \) of formal power series over
a field \( k \) containing \( \mathbb{R} \).

To \( F \), we associate the DG-Lie algebras \( (T_{\text{poly}}(F), 0, [\, , \,]) \), resp. \( (D_{\text{poly}}(F), d_H, [\, , \,]) \),
of formal poly-vector fields, resp. formal poly-differential operators, on \( F \); further,
we consider the DG-Lie modules \( (\Omega_F, 0, L) \), resp. \( (\tilde{C}_\bullet(F), b_H, L) \), over \( (T_{\text{poly}}(F), 0, [\, , \,]) \),
resp. \( (D_{\text{poly}}(F), d_H, [\, , \,]) \), where \( \Omega_F \) denotes the continuous De Rham complex of \( F \)
with De Rham differential \( d \), and \( \tilde{C}_\bullet(F) \) is the continuous Hochschild chain complex
of \( F \).

The following is Kontsevich’s celebrated “formality” result.

**Theorem 5.1.** [15] There is an \( L_\infty \)-quasi-isomorphism
\[
U : (T_{\text{poly}}(F), 0, [\, , \,]) \to (D_{\text{poly}}(F), d_H, [\, , \,]),
\]

enjoying the following properties:

i) the first Taylor coefficient of \( U \) coincides with the Hochschild–Kostant–
Rosenberg quasi-isomorphism
\[
\text{HKR}(\partial_1 \wedge \cdots \wedge \partial_p) = (-1)^{\binom{p+1}{2}} \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^{\sigma} \partial_{\sigma(1)} \otimes \cdots \otimes \partial_{\sigma(p)}
\]

from the DG-vector space \( (T_{\text{poly}}(F), 0) \) to the DG-vector space \( (D_{\text{poly}}(F), d_H) \).

ii) If \( n \geq 2 \), and \( \gamma_i, i = 1, \ldots, n \), are elements of \( T^0_{\text{poly}}(F) \), then
\[
U_n(\gamma_1, \ldots, \gamma_n) = 0.
\]

iii) If \( n \geq 2 \), \( \gamma_1 \) is a linear vector field on \( F \) (i.e. an element of \( \mathfrak{gl}_d \)), \( \gamma_i \),
\( i = 2, \ldots, n \) are general elements of \( T_{\text{poly}}(F) \), then
\[
U_n(\gamma_1, \gamma_2, \ldots, \gamma_n) = 0.
\]

By composing the action \( L \) of \( D_{\text{poly}}(F) \) on \( \tilde{C}_\bullet(F) \) with the \( L_\infty \)-quasi-isomorphism \( U \)
from Theorem 5.1, \( \tilde{C}_\bullet(F) \) inherits an \( L_\infty \)-module structure over the DG-Lie algebra
\( (T_{\text{poly}}(F), 0, [\, , \,]) \).

The first part of the following theorem was a conjecture by Tsygan [22] which
has been proved by Shoikhet in [19]. The second part has been proved in [9].

**Theorem 5.2.** There is an \( L_\infty \)-quasi-isomorphism
\[
S : (\tilde{C}_\bullet(F), b_H, L \circ U) \to (\Omega_F, 0, L)
\]
of \( L_\infty \)-modules over the DG-Lie algebra \( (T_{\text{poly}}(F), 0, [\, , \,]) \), enjoying the following
properties:

i) the 0-th Taylor coefficient of \( S \) coincides with the Hochschild–Kostant–
Rosenberg quasi-isomorphism
\[
\text{HKR}((a_0 \cdots | a_p)) = \frac{1}{p!} a_0 da_1 \cdots da_p
\]

from the DG-vector space \( (\tilde{C}_\bullet(F), b_H) \) to the DG-vector space \( (\Omega_F, 0) \).
ii) If $n \geq 1$, $\gamma_1$ is a linear vector field on $F$, $\gamma_i$, $i = 2, \ldots, n$ are general elements of $T_{\text{poly}}^{\text{coord}}(F)$, and $c$ is a general element of $\hat{C}_\bullet(F)$, then

$$S_n(\gamma_1, \ldots, \gamma_n; c) = 0.$$  

5.5. Formality theorem in the ring case. This section is devoted to the proof of a Tsygan-like formality theorem in the case of a Lie algebroid $L$ over a $k$-algebra $R$, such that $L$ is free over $R$ of rank $d$: the proof combines Shoikhet’s formality theorem 5.2 with the Fedosov resolutions from §4.

**Theorem 5.3.** For a Lie algebroid $L$ over $R$ as above, there exist DG-Lie algebras $(\mathfrak{g}_i^L, d_i, [\cdot, \cdot])$ and DG-Lie modules $(\mathfrak{m}_i^L, b_i, L_i)$ over $\mathfrak{g}_i$, $i = 1, 2$, and $L_\infty$-quasi-isomorphisms $\mathcal{U}_L$ from $\mathfrak{g}_1^L$ to $\mathfrak{g}_2^L$ and $\mathcal{S}_L$ from $\mathfrak{m}_2^L$ to $\mathfrak{m}_1^L$, which fit into the following commutative diagram:

$$
\begin{array}{cccccc}
T_{\text{poly}}^{\text{coord}}(R)^{\wedge L} & \xrightarrow{\mathcal{U}_L} & \mathfrak{g}_1^L & \xrightarrow{\mathcal{S}_L} & \mathfrak{g}_2^L & \xrightarrow{D_{\text{poly}}^{\text{coord}}(R)} \\
\bigl(\mathfrak{g}_1^L\bigr) & \bigl(\mathfrak{g}_2^L\bigr) & & & & \\
\Omega^{\text{poly}}(R)^{\wedge L} & \xrightarrow{\mathcal{S}_L} & \mathfrak{m}_2^L & \xrightarrow{D_{\text{poly}}^{\text{coord}}(R)} & \mathfrak{m}_1^L & \xrightarrow{\mathcal{U}_L} \\
\bigl(\mathfrak{m}_2^L\bigr) & \bigl(\mathfrak{m}_1^L\bigr) & & & & \\
\end{array}
$$

such that the induced maps

$$T_{\text{poly}}^{\text{coord}}(R) \to \mathcal{H}^*(D_{\text{poly}}^{\text{coord}}(R), d_H), \quad \mathcal{H}^*(C_{\text{poly}}^{\text{coord}}(R), b_H) \to \Omega^{\text{coord}}(R)$$

on (co)homology coincide with the respective HKR-quasi-isomorphisms. The morphisms indicated by hooked arrows are true quasi-isomorphisms of DG-Lie algebras and DG-Lie modules respectively.

**Proof.** The first step in the proof of Theorem 5.3 may be borrowed from [6, §7.3]. Namely, we consider the following graded vector spaces:

$$C^{\text{coord}, L \otimes T_{\text{poly}}(F)} \cong T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L \otimes F}),$$

$$C^{\text{coord}, L \otimes D_{\text{poly}}(F)} \cong D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L \otimes F}),$$

$$C^{\text{coord}, L \otimes \Omega_F} \cong \Omega_{C^{\text{coord}, L \otimes F}/C^{\text{coord}, L}},$$

$$C^{\text{coord}, L \otimes \hat{C}_\bullet(F)} \cong \hat{C}_{C^{\text{coord}, L \otimes F}},$$

where the DG-algebra $C^{\text{coord}, L}$ has been introduced in §4.1, and where $\hat{C}_{C^{\text{coord}, L \otimes F}}$ denotes the $C^{\text{coord}, L}$-relative Hochschild chain complex of the DG-algebra $C^{\text{coord}, L \otimes F}$.

The Maurer–Cartan form on $C^{\text{coord}, L \otimes F}$ introduced in §4.1 defines a twisted differential $d_\omega = d + \omega$ on the listed graded vector spaces and as explained in §5.2 $d_\omega$ is compatible with the respective DG-Lie algebra and DG-Lie module structures.

Thus, formal geometry has provided us with the following DG-Lie algebras and respective DG-Lie modules:

$$\left( T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L \otimes F}), d_\omega, [\cdot, \cdot] \right), \quad \left( D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L \otimes F}), d_\omega + d_H, [\cdot, \cdot] \right).$$

$$\left( \Omega_{C^{\text{coord}, L \otimes F}/C^{\text{coord}, L}}, d_\omega, L \right), \quad \left( \hat{C}_{C^{\text{coord}, L \otimes F}}, d_\omega + b_H, L \right)$$
We repeat that, viewing all DG-Lie algebra and DG-Lie module structures above as \( L_\infty \)-structures, the differential \( d_\omega \) is the twist of the standard structures with respect to the MC element \( \omega \) of \( C^\text{coord}.L \otimes \text{Der}(F) = T^0_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F) \).

The \( L_\infty \)-quasi-isomorphism \( \mathcal{U} \) of Theorem 5.1 extends \( C^\text{coord}.L \)-linearly to an \( L_\infty \)-quasi-isomorphism

\[
\mathcal{U}_L : (T_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F), d, [\ , \ ]) \to (D_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F), d + d_H, [\ , \ ]).
\]

The composition of the DG-Lie action \( L \) of \( D_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F) \) on \( \hat{C}_{\text{coord}.L} \cdot (C^\text{coord}.L \otimes F) \) with the \( L_\infty \)-quasi-isomorphism \( \mathcal{U}_L \) endows the latter graded vector space with a structure of \( L_\infty \)-module over the DG-Lie algebra \( T_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F) \), which is obtained by \( C^\text{coord}.L \)-base extension of the corresponding \( L_\infty \)-module structure of \( \hat{C}_\cdot(F) \) over \( T_{\text{poly}}(F) \).

Accordingly, the \( L_\infty \)-quasi-isomorphism \( \mathcal{S} \) of Theorem 5.2 extends to an \( L_\infty \)-quasi-isomorphism of \( L_\infty \)-modules

\[
\mathcal{S}_L : \left( \hat{C}_{\text{coord}.L} \cdot (C^\text{coord}.L \otimes F), d + b_H, L \circ \mathcal{U}_L \right) \to \left( \Omega_{C^\text{coord}.L \otimes F / C^\text{coord}.L}, d, L \right)
\]

both viewed as \( L_\infty \)-modules over \( T_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F) \).

As outlined in §5.2 we may apply the twisting procedures for \( L_\infty \)-algebras, \( L_\infty \)-modules and \( L_\infty \)-morphisms to the present case, where the MC element is the Maurer–Cartan form \( \omega \): thus, we get an \( L_\infty \)-morphism \( \mathcal{U}_{L,\omega} \)

\[
\mathcal{U}_{L,\omega} : (T_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F), d_\omega, [\ , \ ]) \to (D_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F), d_\omega + d_H, [\ , \ ]),
\]

where here and below we used Property ii) of Theorem 5.1, which yields that the MC element \( \mathcal{U}(\omega) \) equals \( \omega \).

The \( L_\infty \)-morphism \( \mathcal{U}_{L,\omega} \) yields an \( L_\infty \)-module structure on \( \hat{C}_{\text{coord}.L} \cdot (C^\text{coord}.L \otimes F) \) over the \( \omega \)-twisted DG-Lie algebra \( T_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F) \).

Translating (5.8) to the present case we have

\[
\left( \hat{C}_{\text{coord}.L} \cdot (C^\text{coord}.L \otimes F), d_\omega + b_H, L \circ \mathcal{U}_{L,\omega} \right)
= \left( \hat{C}_{\text{coord}.L} \cdot (C^\text{coord}.L \otimes F), d + b_H, L \circ \mathcal{U}_L \right)_\omega
\]

from which we get an \( L_\infty \)-quasi-isomorphism

\[
\mathcal{S}_{L,\omega} : \left( \hat{C}_{\text{coord}.L} \cdot (C^\text{coord}.L \otimes F), d_\omega + b_H, L \circ \mathcal{U}_{L,\omega} \right) \to \left( \Omega_{C^\text{coord}.L \otimes F / C^\text{coord}.L}, d_\omega, L \right)
\]

of \( L_\infty \)-modules.

Using the isomorphism (4.2) we obtain isomorphisms of DG-Lie algebras and respective DG-Lie modules

\[
(T_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F), d_\omega, [\ , \ ]) \cong (T_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes R_1JL), 1^\text{coord}, [\ , \ ]),
\]

\[
(D_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes F), d_\omega + d_H, [\ , \ ]) \cong (D_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes R_1JL), 1^\text{coord} + d_H, [\ , \ ]),
\]

\[
(\Omega_{C^\text{coord}.L \otimes F / C^\text{coord}.L}, d_\omega, L) \cong (\Omega_{C^\text{coord}.L \otimes R_1JL / C^\text{coord}.L}, 1^\text{coord}, L),
\]

\[
(\hat{C}_{\text{coord}.L} \cdot (C^\text{coord}.L \otimes F), d_\omega + b_H, L) \cong (\hat{C}_{\text{coord}.L} \cdot (C^\text{coord}.L \otimes R_1JL), 1^\text{coord} + b_H, L),
\]

an \( L_\infty \)-morphism

\[
\mathcal{U}_L^{\text{coord}} : (T_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes R_1JL), 1^\text{coord}, [\ , \ ]) \to (D_{\text{poly},C^\text{coord}.L}(C^\text{coord}.L \otimes R_1JL), 1^\text{coord} + d_H, [\ , \ ]),
\]
which yields an $L_{\infty}$-module structure on $\hat{\mathcal{C}}_{\text{coord}. L} \cdot (\mathcal{C}^{\text{coord}. L} \hat{\otimes} R, JL)$ over $T_{\text{poly}, \text{coord}. L} \left( \mathcal{C}^{\text{coord}. L} \hat{\otimes} R, JL \right)$, and finally an $L_{\infty}$-morphism

$$S_{L}^{\text{coord}} : \left( \hat{\mathcal{C}}_{\text{coord}. L} \cdot (\mathcal{C}^{\text{coord}. L} \hat{\otimes} R, JL), 1 \nabla^{\text{coord}} + b_H, L \circ U_{L}^{\text{coord}} \right) \rightarrow \left( \Omega_{\text{coord}. L} \hat{\otimes} F/\text{coord}. L, 1 \nabla^{\text{coord}}, L \right).$$

We recall from §4.1 that there is a rational action of $\text{GL}_{d}(k)$ on $\mathcal{C}^{\text{coord}. L}$, which extends in a natural way to a (topological) rational action on all DG-Lie algebras and DG-Lie modules above. The previous actions determine infinitesimally actions of $\mathfrak{gl}_{d}(k)$ on all DG-Lie algebras and DG-Lie modules considered so far in the sense of §5.2.

The $L_{\infty}$-morphism $U_{L, \omega}$ descends with respect to the action of the set $\mathfrak{s} = \mathfrak{gl}_{d}(k)$ (using the notations of §5.2), because the descent condition (5.11) is satisfied as a consequence of Property $\text{iii}$ of Theorem 5.1 and of the verticality property (4.5) of $\omega$.

Similarly, Property $\text{ii}$ of Theorem 5.2, together with the verticality property of $\omega$ implies that $S_{L, \omega}$ descends with respect to the action of $\mathfrak{gl}_{d}(k)$ (see §5.3). Summarizing all arguments so far, and because of the compatibility of the $\mathfrak{gl}_{d}(k)$-action with the isomorphism (4.2), we get $L_{\infty}$-morphisms

$$(U_{L}^{\text{coord}})^{\mathfrak{gl}_{d}(k)} : (T_{\text{poly}, \text{coord}. L} \left( \mathcal{C}^{\text{coord}. L} \hat{\otimes} R, JL \right), 1 \nabla^{\text{coord}}, \left[ \cdot, \cdot \right])^{\mathfrak{gl}_{d}(k)}$$

$$\rightarrow \left( D_{\text{poly}, \text{coord}. L} \left( \mathcal{C}^{\text{coord}. L} \hat{\otimes} R, JL \right), 1 \nabla^{\text{coord}} + d_{H}, \left[ \cdot, \cdot \right] \right)^{\mathfrak{gl}_{d}(k)}$$

and

$$(S_{L}^{\text{coord}})^{\mathfrak{gl}_{d}(k)} : \left( \hat{\mathcal{C}}_{\text{coord}. L} \cdot (\mathcal{C}^{\text{coord}. L} \hat{\otimes} R, JL), 1 \nabla^{\text{coord}} + b_H, L \circ U_{L}^{\text{coord}} \right)^{\mathfrak{gl}_{d}(k)}$$

$$\rightarrow \left( \Omega_{\text{coord}. L} \hat{\otimes} F/\text{coord}. L, 1 \nabla^{\text{coord}}, L \right)^{\mathfrak{gl}_{d}(k)},$$

Repeating almost verbatim the arguments at the end of [6, §7.3.3], there are obvious isomorphisms of DG-Lie algebras and DG-Lie modules

$$(T_{\text{poly}, \text{aff}. L} \left( \mathcal{C}^{\text{aff}. L} \hat{\otimes} R, JL \right), 1 \nabla^{\text{aff}}, \left[ \cdot, \cdot \right]) \cong (T_{\text{poly}, \text{coord}. L} \left( \mathcal{C}^{\text{coord}. L} \hat{\otimes} R, JL \right), 1 \nabla^{\text{coord}}, \left[ \cdot, \cdot \right])^{\mathfrak{gl}_{d}(k)},$$

$$(D_{\text{poly}, \text{aff}. L} \left( \mathcal{C}^{\text{aff}. L} \hat{\otimes} R, JL \right), 1 \nabla^{\text{aff}} + d_{H}, \left[ \cdot, \cdot \right]) \cong (D_{\text{poly}, \text{coord}. L} \left( \mathcal{C}^{\text{coord}. L} \hat{\otimes} R, JL \right), 1 \nabla^{\text{coord}} + d_{H}, \left[ \cdot, \cdot \right])^{\mathfrak{gl}_{d}(k)},$$

$$\left( \Omega_{\text{aff}. L} \hat{\otimes} F/\text{aff}. L, 1 \nabla^{\text{aff}}, L \right) \cong \left( \Omega_{\text{coord}. L} \hat{\otimes} F/\text{coord}. L, 1 \nabla^{\text{coord}}, L \right)^{\mathfrak{gl}_{d}(k)},$$

$$\left( \hat{\mathcal{C}}_{\text{aff}. L} \cdot (\mathcal{C}^{\text{aff}. L} \hat{\otimes} R, JL), 1 \nabla^{\text{aff}} + b_H, L \right) \cong \left( \hat{\mathcal{C}}_{\text{coord}. L} \cdot (\mathcal{C}^{\text{coord}. L} \hat{\otimes} R, JL), 1 \nabla^{\text{coord}} + b_H, L \right)^{\mathfrak{gl}_{d}(k)}.$$
The $L_\infty$-morphism $\mathfrak{S}_L$ is obtained from $S_{L,\omega}$ using the isomorphism (4.2) and by (5.9) the Taylor components of $S_{L,\omega}$ are given by

$$S_{L,\omega,n}(\gamma_1, \ldots, \gamma_n; c) = \sum_{m \geq 0} \frac{1}{m!} S_{L,n+m}(\omega_1, \ldots, \omega_m, \gamma_1, \ldots, \gamma_n; c),$$

where $\gamma_i \in T_{\text{poly}, C^{\text{coord}}, L}(C^{\text{coord}}, L \otimes F)$, $c \in \hat{C}_{\text{Coh}, L}^*(C^{\text{coord}}, L \otimes F)$. $T_{\text{poly}, C^{\text{coord}}, L}(C^{\text{coord}}, L \otimes F)$, $D_{\text{poly}, C^{\text{coord}}, L}(C^{\text{coord}}, L \otimes F)$, $\Omega_{C^{\text{coord}}, L \otimes F}$ and $\hat{C}_{\text{Coh}, L}^*(C^{\text{coord}}, L \otimes F)$ are bi-graded complexes: the first degree is the natural degree coming from poly-vector degree, (shifted) Hochschild degree, (negative) form degree and (negative) Hochschild degree respectively.

The component $S_{L,\omega,0}$ can be written into a sum

$$S_{L,\omega,0}(c) = \sum_{n \geq 0} \frac{1}{n!} S_n(\omega_1, \ldots, \omega_n; c);$$

the grading property of the $L_\infty$-quasi-isomorphism $S$ of Theorem 5.2 implies that the component $S_{L,\omega,0}$ indexed by $n$ has bi-degree $(n, -n)$.

Dualizing [6, Lemma 7.3.2], and using Property i) of Theorem 5.2, we get the following commutative diagram of graded vector spaces:

$$(5.13) \quad \begin{array}{ccc}
C_{\text{poly}}^L(R) & \xrightarrow{\text{HKR}} & C_{\text{Coh}, L}^* \otimes \hat{C}_* \otimes (F) \\
\Omega^L(R) & \xrightarrow{\text{HKR}} & \Omega_{C^{\text{aff}}, L \otimes F}
\end{array}$$

where the morphism HKR on the left vertical arrow has been defined in Theorem 3.9.

The twisting procedure and the descent procedure by the isomorphism (4.2) produce the commutative diagram

$$(5.14) \quad \begin{array}{ccc}
C_{\text{poly}}^L(R) & \xrightarrow{\text{HKR}} & \hat{C}_{\text{Coh}, L \otimes F}(C_{\text{aff}}, L \otimes R, JL) \\
\Omega^L(R) & \xrightarrow{\text{HKR}} & \Omega_{C_{\text{aff}}, L \otimes F}(\hat{C}_{\text{aff}}, L \otimes F)
\end{array}$$

out of the commutative diagram (5.13); the above bi-gradings naturally translate into bi-gradings on $\Omega_{C_{\text{aff}}, L \otimes F}$ and $\hat{C}_{\text{aff}, L \otimes F}(C_{\text{aff}}, L \otimes F)$. The component $S_{L,0}$ is a sum of terms $\mathfrak{S}_{L,0}^n$, $n \geq 0$, of bi-degree $(n, -n)$.

We now prove that the morphisms $\mathfrak{S}_{L,0}$ and $\mathfrak{S}_{L,0}^0$ coincide at the level of cohomology. For this, we consider on the double complexes $\Omega_{C_{\text{aff}}, L \otimes F}$ and $\hat{C}_{\text{aff}, L \otimes F}(C_{\text{aff}}, L \otimes F)$ the filtration with respect to the second degree: then, the corresponding spectral sequences degenerate at their first terms, because of the results of §4.3, §4.4, and the resulting complexes consist of single columns $\Omega^L(R, 0)$ and $C_{\text{poly}}^L(R, b_1)$. Thus, the respective second terms of the spectral sequences coincide with $\Omega^L(R)$ and with $H^*(C_{\text{poly}}^L(R, b_1))$. Since both spectral sequences degenerate at their first term (i.e. the cohomology with respect to the first degree is concentrated in degree 0), $\mathfrak{S}_{L,0}$ and $\mathfrak{S}_{L,0}^0$ obviously coincide at the level of cohomology, and this ends the proof. □
5.6. **Functoriality property of Theorem 5.3.** We consider two Lie algebroids \((L, R), (M, S)\) as above.

**Definition 5.4.** An algebraic morphism from \((L, R)\) to \((M, S)\) consists of a pair \((\ell, \rho)\), where \(i\) \(\rho\) is a \(k\)-algebra morphism from \(R\) to \(S\), and \(ii\) \(\ell\) is a Lie algebra morphism from \(L\) to \(M\), enjoying the following compatibility properties with respect to the corresponding anchor maps:

\[
\rho(l(r)) = \ell(l)(\rho(r)), \quad \ell(rl) = \rho(r)\ell(l), \quad r \in R, \quad l \in L.
\]

The universal property of the universal enveloping algebra of a Lie algebroid yields, for any algebraic morphism \(\varphi = (\rho, \ell)\) from \((L, R)\) to \((M, S)\), a Hopf algebra morphism \(\varphi_D : U_R(L) \rightarrow U_S(M)\). Thus, \((\ell, \rho)\) defines a morphism \(\varphi_D\) of \(B_\infty\)-algebras from \(D^L_{\text{poly}}(R)\) to \(D^M_{\text{poly}}(S)\): in particular, it restricts to a morphism of Gerstenhaber algebras up to homotopy.

Further, the algebraic morphism \(\varphi\) defines a morphism \(\varphi_T : T^L_{\text{poly}}(R) \rightarrow T^M_{\text{poly}}(S)\) by extending (via the \(S\)-linear wedge product) the assignment

\[
\varphi_T : S \otimes_R L \rightarrow M : s \otimes_R l \mapsto s\ell(l)
\]

Since \((\ell, \rho)\) preserves the anchor map and Lie bracket, we have a morphism of Gerstenhaber algebras from \(T^L_{\text{poly}}(R)\) to \(T^M_{\text{poly}}(S)\).

**Proposition 5.5.** We assume \((L, R), (M, S)\) to be Lie algebroids over \(R\) and \(S\) respectively, and \(\varphi = (\ell, \rho)\) to be an algebraic morphism between them as in Definition 5.4; we further assume that the morphism

\[
\varphi_T : S \otimes_R L \rightarrow M : s \otimes_R l \mapsto s\ell(l)
\]

is an isomorphism of \(S\)-modules.

The morphism \((\ell, \rho)\) determines a morphism of DG-algebras

\[
(\Omega^L(R), d_L) \xrightarrow{\varphi} (\Omega^M(S), d_M),
\]

which satisfies

\[
(5.14) \quad \varphi_\Omega(\gamma \cap \omega) = \varphi_T(\gamma) \cap \varphi_\Omega(\omega), \quad \gamma \in T^L_{\text{poly}}(R), \quad \omega \in \Omega^L(R),
\]

and a morphism of DG-algebras

\[
\varphi_J : JL \rightarrow JM,
\]

which satisfies

\[
(5.15) \quad \rho(\alpha(E)) = \varphi_J(\alpha)(\varphi_D(E)), \quad \alpha \in JL, \quad E \in U_R(L),
\]

\[
(5.16) \quad \varphi_J(1 \nabla l(\alpha)) = 1 \nabla \ell(l) \varphi_J(\alpha), \quad \alpha \in JL, \quad l \in L,
\]

\[
(5.17) \quad \varphi_J(\ell(2 \nabla l(\alpha)) = 2 \nabla \ell(l) \varphi_J(\alpha), \quad \alpha \in JL, \quad l \in L,
\]

and which commutes with the algebra monomorphisms \(\alpha_i, i = 1, 2\) (see §3.1.4).

**Proof.** Since \(\varphi_T\) is an isomorphism of \(S\)-modules, we define \(\varphi_\Omega\) on \(L\)-differential forms on \(R\) via

\[
\varphi_\Omega(r) = \rho(r), \quad \varphi_\Omega(l^*(s\ell(l))) = sp(l^*(l)), \quad r \in R, \quad s \in S, \quad l \in L, \quad l^* \in L^*,
\]

and we extend it to \(\Omega^R(L)\) by \(R\)-linearity and by multiplicativity with respect to the wedge product.
To prove that $\varphi_\Omega$ intertwines $d_L$ and $d_M$, it suffices to verify the claim on $R$ and $L^*$. In the first case, we have
\[
\varphi_\Omega(d_L(r))(s\ell(l)) = sp(d_L(r))(l) = sp(l(r)) = s\ell(l)(\rho(r)) = s\ell(l)(\varphi_\Omega(r)) = d_M(\varphi_\Omega(r))(s\ell(l)),
\]
for a general element $r$ of $R$, $s$ of $S$ and $l$ of $L$, while in the second case we have
\[
\varphi_\Omega(d_Ll^*)(s\ell(l_1), s\ell(l_2)) = s_1s_2\rho(s_1l_1l_2).
\]

By compatibility with wedge products, it suffices to prove (5.14) for $\alpha$ of $L$, and for a general $\omega$: we check exemplarily the claim for $\gamma$ in $L$, i.e.
\[
\varphi_\Omega(l \cap \omega)(s_1\ell(l_1), \ldots, s_p\ell(l_p)) = s_1\cdots s_p\rho((l \cap \omega)(l_1, \ldots, l_p)) = s_1\cdots s_p\rho(\omega(l_1, \ldots, l_p)) = \varphi_\Omega(\omega)(\ell(l_1), \ldots, s_p\ell(l_p)) = (\varphi_T(l) \cap \varphi_\Omega(\omega))(s_1\ell(l_1), \ldots, s_p\ell(l_p)).
\]

We now define the morphism $\varphi_J$ on $JL$: for a general element $\alpha$ of $JL$, we set
\[
\varphi_J(\alpha)(s) = sp(\alpha(1)), \quad \varphi_J(\alpha)(s\ell(l_1) \cdots \ell(l_p)) = sp(\alpha(1) \cdots l_p), \quad s \in S, \quad l_1, \ldots, l_p \in L.
\]

It is sufficient to define $\varphi_J$ on such elements of $U_S(M)$, since, being $\varphi_T$ an isomorphism of $S$-modules, a general element of $U_S(M)$ is a sum of elements of the form
\[
(s_1\ell(l_1)) \cdots (s_p\ell(l_p)) = s_1s_2\ell(l_2) \cdots s_p\ell(l_p) = s_1\ell(l_1)(s_2\ell(l_2)) \cdots s_p\ell(l_p) + s_1s_2\ell(l_1)\ell(l_2) \cdots s_p\ell(l_p) = \cdots,
\]
where the product has to be understood in $U_S(M)$.

Since $\varphi_D$ is defined by extending $p$ and $\ell$ in a way compatible with the Lie algebroid structure of $U_R(L)$, (5.15) follows immediately.

As for (5.16), it suffices to check the identity on $R$ and on elements of $U_S(M)$ of the form $s\ell(l_1) \cdots \ell(l_p)$. In the first case, we have for $s \in S$, $l \in L$
\[
(l^\ell(\ell(l))(\varphi_J(\alpha))) = \ell(l)(\varphi_J(\alpha)(s)) - \varphi_J(\alpha)(l^\ell(l)s)
\]
\[
= \ell(l)(sp(\alpha(1))) - \varphi_J(\alpha)(l^\ell(l)s)
\]
\[
= \ell(l)(sp(\alpha(1)) + sp(l(\rho(\alpha(1)) - \varphi_J(\alpha)(l^\ell(l)s) - \varphi_J(\alpha)(s\ell(l))
\]
\[
= \ell(l)(sp(\alpha(1)) - sp(\alpha(l))
\]
\[
= sp(l^\ell(\ell(\alpha)(1)) = \varphi_J(l^\ell(\ell(\alpha)(s)).
\]
As for the second case, we have for \( \alpha \in JL, \ l_i, l, \ i = 1, \ldots, p, \ s \in S \)

\[
(1^{\nabla_l(l_1)} \varphi_J(\alpha))(s \ell(l_1) \cdots \ell(l_p)) = \ell(l)(\varphi_J(\alpha)(s \ell(l_1) \cdots \ell(l_p))) - \varphi_J(\alpha)(\ell(l)s \ell(l_1) \cdots \ell(l_p)) \\
= \ell(l)(sp(\alpha(l_1 \cdots l_p))) - \varphi_J(\alpha)(\ell(l)s \ell(l_1) \cdots \ell(l_p)) \\
= \ell(l)(s)(\rho(\alpha(l_1 \cdots l_p)) + \ell(l)(\rho(\alpha(l_1 \cdots l_p))) \\
- \varphi_J(\alpha)(\ell(l)s(\ell(l_1) \cdots \ell(l_p))) - \varphi_J(\alpha)(s(\ell(l_1) \cdots \ell(l_p)) \\
= sp(l(\alpha(l_1 \cdots l_p))) - sp(\alpha)(il_1 \cdots l_p) \\
= \varphi_J(1^{\nabla_l(\alpha)}(s \ell(l_1) \cdots \ell(l_p))).
\]

The identity (5.17) as well as the compatibility with \( \alpha_i, \ i = 1, 2 \) are verified by similar computations.

Assume now that \( \varphi = (\ell, \rho) : (L, R) \to (M, S) \) is as in the previous lemma and that \( \varphi_T : S \otimes_R L \to M \) is an isomorphism. As always we assume that \( L \) (and hence \( M \)) is free of rank \( d \). Looking at associated graded objects we see that the extended map

\[
S_1 \otimes_R JL \to JM : s \otimes \alpha \mapsto s \varphi_J(\alpha)
\]

is an isomorphism. Hence any \( R_1 \)-linear differential operator on \( JL \) can be extended to an \( S_1 \)-linear differential operator on \( JM \). We use this to define a map

\[
\varphi_D : D_{R_1}(JL) \to D_{S_1}(JM)
\]

and a corresponding map of \( B_{sa} \)-algebras

\[
\varphi_D : D_{poly, R_1}(JL) \to D_{poly, S_1}(JM)
\]

such that the following diagram is commutative

\[
\begin{array}{ccc}
D_{poly, R_1}(JL) & \xrightarrow{\varphi_D} & D_{poly, S_1}(JM) \\
\downarrow & & \downarrow \\
D_{poly, R_1}(JL) & \xrightarrow{\varphi_D} & D_{poly, S_1}(JM)
\end{array}
\]

where the vertical monomorphisms have been defined in (3.12).

An easy computation shows that \( \varphi_D \) in (3.12) commutes with the action of the Grothendieck connection \( [1^{\nabla}, -] \). It follows by the discussion in \S 3.3 that if we take the invariants for \([1^{\nabla}, -]\) of the lower line in (5.20) we obtain the upper line.

We extend \( \varphi_J \) to a map of graded vector spaces

\[
\varphi_C : \tilde{C}_{R, \bullet}(JL) \to \tilde{C}_{S, \bullet}(JM) : \alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \varphi_J(\alpha_1) \otimes \cdots \otimes \varphi_J(\alpha_n)
\]

which is again essentially just base extension over \( S/R \). This map obviously commutes with the Grothendieck connection \( 1^{\nabla} \). We obtain a map of pairs of graded vector spaces

\[
(\varphi_D, \varphi_C) : (D_{poly, R_1}(JL), \tilde{C}_{R, \bullet}(JL)) \to (D_{poly, S_1}(JM), \tilde{C}_{R, \bullet}(JL))
\]

and as this map is just base extension over \( S/R \) it is compatible with all structures defined in [5], hence in particular with the DG-Lie algebra and DG-Lie module structures and also with the precalculus structure up to homotopy.
Taking invariants for $\nabla$ and using (3.16) we obtain a commutative diagram of precalculus structure up to homotopy

\[
(D^L_{\text{poly}}(R), d_H, [\ , \ ], \cup) \xrightarrow{\varphi_D} (D^M_{\text{poly}}(S), d_H, [\ , \ ], \cup).
\]

One also obtains from Proposition 5.5 a commutative diagram of precalculi.

\[
(T^L_{\text{poly}}(R), 0, [\ , \ ], \cup) \xrightarrow{\varphi_T} (T^M_{\text{poly}}(S), 0, [\ , \ ], \cup).
\]

Furthermore from (5.18) and the universal property of coordinate spaces (see (4.1)) we obtain an $R$-algebra morphism from $R^{\text{coord}, L}$ to $S^{\text{coord}, M}$. It extends further to a morphism of DG-algebras from $C^{\text{coord}, L}$ to $C^{\text{coord}, M}$ thanks to (5.16) and the fact that $\varphi_\Omega$ is a morphism of DG-algebras from $\Omega^L(R)$ to $\Omega^M(S)$.

Finally, the algebraic morphism $(\ell, \rho)$ induces precalculus morphisms (up to homotopy) between all corresponding Fedosov resolutions, since the monomorphism $\alpha_2$ and the connection $\nabla$, which are needed in the construction of the Fedosov resolutions of §4 (we refer to [6] for more details thereabout), have been proved to be preserved by $(\ell, \rho)$.

As a consequence of these arguments, we deduce the following theorem, which expresses the functoriality properties of the commutative diagram (5.12) of Theorem 5.3.

**Theorem 5.6.** For a general algebraic morphism $\varphi = (\ell, \rho)$ from $(L, R)$ to $(M, S)$ as in Definition 5.4, which induces an isomorphism $S \otimes_R L \cong M$ of $S$-modules, and such that $L$ is free of rank $d$, there are $L_\infty$-quasi-isomorphisms $\mathfrak{U}_L$, $\mathfrak{U}_M$, $\mathfrak{S}_L$ and $\mathfrak{S}_M$ of DG-Lie algebras and DG-Lie modules, fitting in the commutative diagram

\[
\begin{array}{cccc}
T^L_{\text{poly}}(R) & \xrightarrow{\varphi_T} & \mathfrak{U}_L & \xrightarrow{\varphi_D} & D^L_{\text{poly}}(R) \\
\downarrow & & \downarrow & & \downarrow \\
T^M_{\text{poly}}(S) & \xrightarrow{\varphi_T} & \mathfrak{U}_M & \xrightarrow{\varphi_D} & D^M_{\text{poly}}(S) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{S}_M & \xrightarrow{\varphi_J} & \mathfrak{S}_L & \xrightarrow{\varphi_J} & C^M_{\text{poly}}(S) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^L(R) & \xrightarrow{\varphi_\Omega} & \mathfrak{S}_L & \xrightarrow{\varphi_\Omega} & C^L_{\text{poly}}(R)
\end{array}
\]

where we have borrowed notations from Proposition 5.5; all such morphisms are compatible with respect to the composition of algebraic morphisms between Lie algebroids.
Note that Theorem 5.6 makes no reference to the (homotopy) precalculus structures which we discussed above; we will need these below.

5.7. Proof of Theorem 1.2. We now collect the results of §5.5 and §5.6 to give the proof of Theorem 1.2, via a well-suited gluing procedure.

We consider a ringed site $(X,\mathcal{O})$, and a sheaf of Lie algebroids $\mathcal{L}$, such that $\mathcal{L}$ is locally free of rank $d$ over $\mathcal{O}$. We replace $X$ by its full subcategory of objects $U$ such that $\mathcal{L}(U)$ is free over $\mathcal{O}(U)$. This does not change the category of sheaves.

All sheaves of DG-Lie algebras and DG-Lie modules in the commutative diagram (1.6) are obtained by sheafifying the corresponding presheaves of DG-Lie algebras and DG-Lie modules, i.e.

$$
\begin{align*}
U \rightarrow T_{\text{poly}}^{\mathcal{L}(U)}(\mathcal{O}(U)), & \quad U \rightarrow D_{\text{poly}}^{\mathcal{L}(U)}(\mathcal{O}(U)), \\
U \rightarrow \Omega^{\mathcal{L}(U)}(\mathcal{O}(U)), & \quad U \rightarrow C_{\text{poly}}^{\mathcal{L}(U)}(\mathcal{O}(U)).
\end{align*}
$$

Since $\mathcal{L}$ is locally free of order $d$ over $\mathcal{O}$, for a morphism $V \rightarrow U$ in $X$, the corresponding restriction morphism $(\mathcal{O}(U), \mathcal{L}(U)) \rightarrow (\mathcal{O}(V), \mathcal{L}(V))$, yields an isomorphism

$$
\mathcal{O}(V) \otimes_{\mathcal{O}(U)} \mathcal{L}(U) \cong \mathcal{L}(V).
$$

Thus, any restriction morphism as above may be viewed as an algebraic morphism between Lie algebroids, satisfying the isomorphism property of Theorem 5.6.

If we then consider the DG-Lie algebras and DG-Lie modules

$$
U \rightarrow \mathfrak{g}^{\mathcal{L}(U)}_i, \quad U \rightarrow \mathfrak{m}^{\mathcal{L}(U)}_i, \quad i = 1, 2,
$$

Theorem 5.3 produces, for any $U$ in $X$, $L_{\infty}$-quasi-isomorphisms $\mathcal{U}_{\mathcal{L}(U)}$ and $\mathcal{S}_{\mathcal{L}(U)}$ which fit into a commutative diagram (5.12). By Theorem 5.6 these are actually morphisms of presheaves.

Sheafifying all presheaves and morphisms between presheaves concludes the proof.

6. The relationship between Atiyah classes and jet bundles

In the present section we review some technical results from [6, §8], to which we refer for more details. We need only the main notations and conventions for use in §7.

For a field $k$ of characteristic 0, we consider a sheaf $\mathcal{L}$ of Lie algebroids over a ringed site $(X, \mathcal{O})$, which is locally free of rank $d$ over $\mathcal{O}$.

We have a short exact sequence of $\mathcal{O}_1$-$\mathcal{O}_2$-bimodules

$$
0 \rightarrow \mathcal{L}^* \rightarrow J^1 \mathcal{L} \rightarrow \mathcal{O} \rightarrow 0
$$

where $\mathcal{O}_i$, $i = 1, 2$, denotes a copy of $\mathcal{O}$ embedded in $J \mathcal{L}$ via the monomorphism $\alpha_i$ and where $J^1 \mathcal{L}$ was introduced in §3.1.4.

For a general $\mathcal{O}$-module $\mathcal{E}$, tensoring over $\mathcal{O}_2$ yields a short exact sequence

$$
\begin{array}{c}
\mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E} \rightarrow J^1 \mathcal{L} \otimes_{\mathcal{O}_2} \mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{E} \\
\end{array}
$$

which we will call the $\mathcal{L}$-Atiyah sequence. The $\mathcal{L}$-Atiyah class $A_{\mathcal{L}}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{L}$ is the class of this sequence in $\text{Ext}^1_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E})$. As explained in §1.1, if $\mathcal{E}$ is a vector bundle, the $i$-th scalar Atiyah class $a_{\mathcal{L}, i}(\mathcal{E})$ of $\mathcal{E}$ is defined as

$$
a_{\mathcal{L}, i}(\mathcal{E}) = \text{tr}(\bigwedge^i A_{\mathcal{L}}(\mathcal{E})) \in \text{H}^i(X, \bigwedge^i \mathcal{L}^*),
$$

(6.1)
Below we will only consider the case \( \mathcal{E} = \mathcal{L} \). In that case we simplify the notations to
\[
A(\mathcal{L}) = A_{\mathcal{L}}(\mathcal{L}), \quad a_i(\mathcal{L}) = a_{i,\mathcal{L}}(\mathcal{L})
\]
Observe that the \( a_i(\mathcal{L}) \) are cohomology classes. We now outline how we may realize them as explicit cocycles.

By the very construction of \( C_{\text{coord}, \mathcal{L}} \) and \( C_{\text{aff}, \mathcal{L}} \), there are natural morphisms of DG-algebras
\[
\Omega^2_{\mathcal{L}}(X) \xrightarrow{\theta} C_{\text{aff}, \mathcal{L}} \otimes_{\mathcal{O}_1} \Omega_{\mathcal{J}_L/\mathcal{O}_1} \xrightarrow{C_{\text{coord}, \mathcal{L}} \otimes_{\mathcal{O}_1} \Omega_{\mathcal{J}_L/\mathcal{O}_1}} C_{\text{coord}, \mathcal{L}} \otimes_{\mathcal{O}_1} \Omega_F
\]
The differentials on the first three DG-algebras are the natural ones (see §4.1). The differential on the fourth DG-algebra is \( d + L \omega \) for a certain MC element \( \omega \in C_{\text{coord}, \mathcal{L}} \otimes_{\mathcal{O}_1} \text{Der}(F) \) and \( d \) the natural differential. See again §4.1.

The MC element \( \omega \) can be expressed as
\[
\omega = \eta_\alpha \omega_{\alpha,i} \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, d,
\]
where \( \eta_\alpha \) is in \( C_{\text{coord}, \mathcal{L}} \) and has degree 1, \( \omega_{\alpha,i} \) belongs to \( F \) and \( \partial_{x_i} = \partial/\partial x_i \).

If we define \( \Xi \) to be the matrix with entries
\[
\Xi_{ij} = \eta_\alpha \partial_x \omega_{\alpha,i} \in C_{\text{coord}, \mathcal{L}} \otimes \Omega_F,
\]
where \( \partial_x \) is the De Rham differential on \( \Omega_F \), then on the nose we have
\[
\text{Tr}(\Xi^n) \in C_{\text{coord}, \mathcal{L}} \otimes \Omega_F
\]
Furthermore it is true that
\[
(d + L \omega)(\text{Tr}(\Xi^n)) = 0
\]
It is shown in [6, §8] that \( \text{Tr}(\Xi^n) \) is actually the image of a (necessarily unique) element in \( C_{\text{aff}, \mathcal{L}} \otimes_{\mathcal{O}_1} \Omega_{\mathcal{J}_L/\mathcal{O}_1} \). Abusing notation somewhat we will still write this element as \( \text{Tr}(\Xi^n) \). It is still a cocycle and in this way represents an element of
\[
\text{H}^{2n}(X, C_{\text{aff}, \mathcal{L}} \otimes_{\mathcal{O}_1} \Omega_{\mathcal{J}_L/\mathcal{O}_1})
\]
which maps naturally to the hypercohomology
\[
\text{H}^{2n}(X, C_{\text{aff}, \mathcal{L}} \otimes_{\mathcal{O}_1} \Omega_{\mathcal{J}_L/\mathcal{O}_1})
\]
Further, we observe that the injection \( \Omega^\mathcal{L}(X) \xrightarrow{\theta} C_{\text{aff}, \mathcal{L}} \otimes_{\mathcal{O}_1} \Omega_{\mathcal{J}_L/\mathcal{O}_1} \) of DG-algebras is a quasi-isomorphism, as discussed in §4.3. Thus \( \theta \) induces an isomorphism
\[
\bigoplus_{m,n} \text{H}^m(X, \wedge^n \mathcal{L}^*) = \text{H}^*(X, \Omega^\mathcal{L}(X)) \xrightarrow{\text{H}(\theta)^{-1}} \text{H}^*(X, C_{\text{aff}, \mathcal{L}} \otimes_{\mathcal{O}_1} \Omega_{\mathcal{J}_L/\mathcal{O}_1})
\]
The following identity is [6, eq. (8.8)]
\[
a_n(\mathcal{L}) = \text{H}(\theta)^{-1} \left( \text{Tr}(\Xi^n) \right), \quad n \geq 1,
\]
which indeed expresses \( a_n(\mathcal{L}) \) in terms of the explicit cocycle \( \text{Tr}(\Xi^n) \).
7. Proof of Theorem 1.1

The aim of this Section is to prove Theorem 1.1 which implies Căldăraru’s conjecture (Theorem 1.3) as has been outlined in the introduction.

For this purpose, we first remind the reader of the main result of [5] about compatibility between cap products. We then prove a ring-theoretical globalized version of this result (compare to the proof of Theorem 5.3). By functoriality (see §5.6), we obtain the sheaf-theoretical globalization. Finally, using results of [5], we compute explicitly the isomorphism appearing in the compatibility between cap products, which we identify with the composition of the (co)homological HKR-quasi-isomorphism followed by left multiplication by the square root of the (modified) Todd class.

7.1. A memento of compatibility between cup and cap products. In this section, we present a memento of the main results of [5, 6] concerning compatibility between cup and cap products respectively.

First of all as before $F$ is the algebra of formal power series in $d$ variables over the field $k$ which we assume to contain $\mathbb{R}$ for now. We recall the existence of (homotopy) Gerstenhaber algebra structures on $T_{\text{poly}}(F)$ and $D_{\text{poly}}(F)$, which together with $\Omega_F$ and $\hat{\mathcal{C}}_\bullet(F)$ yield (homotopy) precalculi [5].

We recall also the $L_\infty$-quasi-isomorphisms $U$ introduced in Theorem 5.1 and $S$ introduced in Theorem 5.2. We denote by $U_n$, $n \geq 1$, resp. $S_n$, $n \geq 0$, the $n$-th Taylor component of $U$, resp. $S$.

We further consider a commutative DG-algebra $(m, d_m)$. The precalculus structures on $(T_{\text{poly}}(F), \Omega_F)$ and $(D_{\text{poly}}(F), \hat{\mathcal{C}}_\bullet(F))$, can be extended by $m$-linearity to precalculi $(T^{m}_{\text{poly}}(F), \Omega^m_F) = (T_{\text{poly}}(F) \hat{\otimes} m, \Omega_F \hat{\otimes} m)$ and $(D^m_{\text{poly}}(F), \hat{\mathcal{C}}^m_\bullet(F)) = (D_{\text{poly}, m}(F \hat{\otimes} m), \hat{\mathcal{C}}_{\bullet, m}(F \hat{\otimes} m))$.

Convention. Below we will work with potentially infinite series with coefficients in $m$. We make the standard assumption that we are in a setting where all these series converge and standard series manipulations are allowed. In our actual application all series will be finite for degree reasons.

A MC element $\gamma$ of $T^{m}_{\text{poly}}(F)$ can be written as a sum

$$\gamma = \gamma_{-1} + \gamma_0 + \gamma_1 + \gamma_2 + \cdots,$$

where $\gamma_i$ is an element of $T^{m}_{\text{poly}}(F)$ of poly-vector degree $i$, $i \geq -1$, which satisfies the Maurer–Cartan equation

$$d_m \gamma + \frac{1}{2}[\gamma, \gamma] = 0.$$

We denote by $U(\gamma)$ the image of a MC element $\gamma$ as above with respect to $U$ (see (5.7)). This is again a MC element. Further, we set

$$U_{\gamma, 1}(\gamma_1) = \sum_{n \geq 0} \frac{1}{n!} U_{n+1}(\gamma_1, \ldots, \gamma_n; \gamma_1), \quad \gamma_1 \in T^{m}_{\text{poly}}(F),$$

and

$$S_{\gamma, 0}(c) = \sum_{n \geq 0} \frac{1}{n!} S_n(\gamma_1, \ldots, \gamma_n; c), \quad c \in \hat{\mathcal{C}}^m_\bullet(F).$$

Since $U$ and $S$, are $L_\infty$-quasi-isomorphisms, $U_{\gamma, 1}$ and $S_{\gamma, 0}$ are both quasi-isomorphisms of DG-vector spaces.
Theorem 7.1. For a general commutative DG-algebra \((m, d_m)\) as above, and for a general MC element \(\gamma\) of \(T^m_{\text{poly}}(F)\), \(\cal U_{\gamma,1}\) and \(S_{\gamma,0}\) descend to quasi-isomorphisms of (homotopy) precalculi, fitting into the commutative diagram

\[
\begin{array}{ccc}
(T^m_{\text{poly}}(F), d_m + [\gamma, \bullet], [\ , \ ], \cup) & \xrightarrow{\cal U_{\gamma,1}} & (D^m_{\text{poly}}(F), d_m + d_H + [\cal U(\gamma), \bullet], [\ , \ ], \cup), \\
\downarrow & & \downarrow \\
(\Omega^m_{L}, d_m + L_{\gamma}, L, \cap) & \xrightarrow{S_{\gamma,0}} & (\hat C^m_{\bullet}(F), d_m + b_H + L_{\cal U(\gamma)}, L, \cap)
\end{array}
\]

in the sense that \(\cal U_{\gamma,1}\) and \(S_{\gamma,0}\) preserve Lie brackets, Lie actions, cup and cap products up to homotopy.

Kontsevich [15] has first stated and proved that \(\cal U_{\gamma,1}\) defines a quasi-isomorphism of Gerstenhaber algebras up to homotopy from \(T^m_{\text{poly}}(F)\) to \(D^m_{\text{poly}}(F)\) in the sense specified above. We observe that the identity \(\cal U_{\gamma,1}([\gamma_1, \gamma_2]) = [\cal U_{\gamma,1}(\gamma_1), \cal U_{\gamma,1}(\gamma_2)]\) at the level of cohomology, for \(\gamma_i\) in \(T^m_{\text{poly}}(F)\), \(i = 1, 2\), holds true, because \(U\) is an \(L_{\infty}\)-morphism: in particular, there is a homotopy operator expressing the compatibility with Lie brackets, expressible in terms of the Taylor components of \(U\) twisted by the MC element \(\gamma\). On the other hand, the identity \(\cal U_{\gamma,1}(\gamma_1 \cup \gamma_2) = \cal U_{\gamma,1}(\gamma_1) \cup \cal U_{\gamma,1}(\gamma_2)\) at the level of cohomology comes from a more complicated identity up to homotopy: in this situation, the homotopy operator is not expressible in terms of the Taylor components of \(U\). For an explicit description of the homotopy operator, we refer to [5, 6, 16].

The actual formulation of Theorem 7.1 has been first proposed as a conjecture in the particular case, where \(\gamma\) is a (formal) Poisson structure, by Shoikhet [19]: this conjecture has been proved in [4]. A more general result has been stated and proved in [5], to which we refer for more details. The identity \(S_{\gamma,0}(L_{\cal U_{\gamma,1}(\gamma_1)}(c)) = L_{\gamma_1}(S_{\gamma,0}(c))\) at the level of cohomology, for \(\gamma_1\) in \(T^m_{\text{poly}}(F)\), \(c\) in \(\hat C^m_{\bullet}(F)\), is a consequence of the fact that \(S_{L, \gamma}\) is an \(L_{\infty}\)-morphism of \(L_{\infty}\)-modules (in particular, there is a homotopy formula involving the Taylor components of \(U\) and \(S\), twisted by \(\gamma\)). The identity \(S_{\gamma,0}(\cal U_{\gamma,1}(\gamma_1) \cap c) = (\gamma_1 \cap S_{\gamma,0}(c))\) at the level of cohomology holds true in virtue of a homotopy formula, but the corresponding homotopy operator does not involve the Taylor components of \(U\) and \(S\): such an operator has been explicitly described in [5].

We briefly review in section §7.1.1 below the construction of the homotopy operator for the compatibility between cap products.

7.1.1. The homotopy formula for the compatibility between cap products. For later computations, we write down the explicit homotopy operator for the compatibility between the \(\cap\)-actions: namely, for a MC element \(\gamma\) as in Theorem 7.1, for \(\gamma_1\) a general element of \(T^m_{\text{poly}}(F)\) and \(c\) a general element of \(\hat C^m_{\bullet}(F)\), we have the homotopy relation

\[
S_{\gamma,0}(\cal U_{\gamma,1}(\gamma_1) \cap c) - \gamma_1 \cap S_{\gamma,0}(c) = (d_m + L_{\gamma})\cal H^S_{\gamma_1}(c) + \cal H^S_{\gamma_1}(d_m \gamma_1 + [\gamma, \gamma_1], c) + (-1)^{\gamma_1}\cal H^S_{\gamma_1}(d_m c + b_H c + L_{\cal U(\gamma)} c),
\]
where
\begin{equation}
\mathcal{H}^S_{\gamma}(\gamma_1, c) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in \mathcal{G}^S_{n+1,m+1}} \circ W_{D,\Gamma} S_{\Gamma}(\gamma_1, \gamma, \ldots, \gamma, c),
\end{equation}
c being of Hochschild degree $-m$.

In (7.2), the second sum is over “$S$-admissible graphs” of type $(n+1, m+1)$: these are directed graphs with $n+2$ vertices of the first type and $m+1$ vertices of the second type and with an orientation of the outgoing edges from vertices of the first type, and with a special vertex of the first type, labelled by 0. The vertices of the second type can be only endpoints of edges, and $S$-admissible graphs do not contain edges starting and ending at the same vertex; finally, the vertex 0 has only incoming edges.

To the vertex 1 of the first type of an $S$-admissible graph $\Gamma$ is assigned the poly-vector field $\gamma_1$: the number of outgoing edges from 1 equals the poly-vector degree of $\gamma_1$ plus 1. To any other vertex of the first type, except 0, is assigned a copy of the MC element $\gamma$. To the $i$-th vertex of the second type is assigned the $i+1$-th component of the Hochschild chain $c$. Pictorially, here is an $S$-admissible graph of type $(4, 5)$, with corresponding coloring by poly-vector fields and Hochschild chains:

![Figure 1 - An S-admissible graph of type (4, 5)](image)

The differential form $S_{\Gamma}(\gamma_1, \gamma, \ldots, \gamma, c)$ is defined explicitly in [4, 5, 19].

More important for our purposes is the integral weight $\circ W_{D,\Gamma}$, for a general $S$-admissible graph of type $(n+1, m+1)$,
\begin{equation}
\circ W_{D,\Gamma} = \int_{\mathcal{Y}^+_{n+1,m+1}} \omega_{D,\Gamma}.
\end{equation}
First of all, $\mathcal{Y}^+_{n+1,m+1}$ denotes the codimension-1-submanifold (with corners) of the compactified configuration space $D^+_{n+1,m+1}$ of $n+1$ points in the punctured unit disk $D^*$ and $m+1$ cyclically oriented points in $S^1$, consisting of configurations of points, where the point labelled by 1 moves on a smooth curve from the origin to the first point $\Gamma$ (with respect to the cyclic order) in $S^1$. Graphically,
In Figure 1, the dashed line represents the curve, along which the point 1 (labelled as “◦”) moves. The differential form $\omega_{D,\Gamma}$ associated to a graph in $\mathcal{G}^S_{n+1,m+1}$ is a product of smooth 1-forms on $\mathcal{D}^+_n\times m$; the basic ingredient is a slight modification of the exterior derivative of Kontsevich’s angle function, see [5, 15] for more details.

For the globalization procedure of the compatibility between cap products, we need the following technical Lemma, which corresponds, in the present framework, to Theorem 5.2, ii).

**Lemma 7.2.** If $\Gamma$ is an $S$-admissible graph in $\mathcal{G}^S_{n+1,m+1}$, $n \geq 1$, and at least one of the poly-vector fields $\gamma_i$, $i \neq 1$, is linear on $F$, then

$$\circledast W_{D,\Gamma}^S(\gamma_1, \gamma_2, \ldots, \gamma_{n+1}, c) = 0.$$ 

**Proof.** The first point of the first type in $\mathcal{Y}^+_{n+1,m+1}$, by the very construction of $\mathcal{Y}^+_{n+1,m+1}$, moves from the origin 0 to the first point in $S^1$ with respect to the cyclic order: to the former point is associated the poly-vector field $\gamma_1$. Any other point associated to a vertex of the first type moves freely in the punctured unit disk $D^\times$.

Without loss of generality we assume $\gamma_2$ to be an $m$-valued linear vector field: the valence (i.e. the number of outgoing edges) of the corresponding vertex of the first type is 1, while the linearity of $\gamma_2$ implies that there can be at most one incoming edge to the vertex corresponding to $\gamma_2$. This follows from the construction of the differential form $\mathcal{S}(\gamma_1, \gamma_2, \ldots, \gamma_{n+1}, c)$.

Thus, we may safely restrict to $S$-admissible graphs $\Gamma$, such that the vertex 2 has valence exactly 1 and with at most one incoming edge.

If the vertex labelled by 2 does not have incoming edges, the corresponding integral weight $W_{D,\Gamma}$ vanishes by dimensional reasons: in fact, we integrate over a 1-form (corresponding to the only outgoing edge from 2) over a 2-dimensional submanifold (with corners) of $D^\times$.

If the vertex labelled by 2 has exactly one incoming and one outgoing edge, we may apply [5, Lemma 6.1], to prove the vanishing of the corresponding weight $W_{D,\Gamma}$. \qed

**7.2. The proof of Theorem 1.1 in the ring case.** We will first assume that the ground field contains $\mathbb{R}$. At the end of the section we will show how to get rid of this restriction.
We consider a Lie algebroid $L$ over $R$ as in Definition 3.1 free of rank $d$ over $R$. Then, we set $(m, d_m) = (C_{\text{coord}, L}, d)$, where $d = d_{0_{\text{coord}, L} \otimes \Omega_R, 1} + 1 \otimes \Omega_R, d_{R, 1}$ (see §4.1 for more details), and the Maurer–Cartan form $\omega$ is an $m$-valued vector field on $F$ obeying

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$ 

By Theorem 5.1, ii) we have $\mathcal{U}(\omega) = \omega$. Furthermore one checks that by degree reasons $\mathcal{U}_o$ and $\mathcal{S}_o$ yield finite sums when evaluated on specific elements. The same goes for the associated homotopies. So the results of §7.1 apply.

Combining the arguments of the proof of Theorem 5.3 with Theorem 7.1 we get the following commutative diagram of precalculi up to homotopy

\begin{equation}
(T_{\text{poly}, C_{\text{aff}, L}}(C_{\text{aff}, L} \otimes \Omega_R, JL), 1 \nabla_{\text{aff}}, [\ , \ ], \cup) \xrightarrow{\mathcal{U}_{L, 1}} (D_{\text{poly}, C_{\text{aff}, L}}(C_{\text{aff}, L} \otimes \Omega_R, JL), 1 \nabla_{\text{aff}} + d_H, [\ , \ ], \cup)
\end{equation}

\begin{equation}
\xrightarrow{\mathcal{S}_{L, 0}} (\mathcal{C}_{\text{aff}, L \otimes \Omega_R, JL} / C_{\text{aff}, L}, 1 \nabla_{\text{aff}}, L, \cap) \xrightarrow{\mathcal{H}_{L, 0}} (\mathcal{C}_{\text{aff}, L \otimes \Omega_R, JL} / C_{\text{aff}, L}, 1 \nabla_{\text{aff}} + b_H, L, \cap)
\end{equation}

The fact that $\mathcal{U}_{L, 1}$ preserves the respective Lie brackets up to homotopy is a consequence of the fact that $\mathcal{U}_L$ is an $L_{\infty}$-morphism; similarly, the fact that $\mathcal{S}_{L, 0}$ preserves the Lie module structure up to homotopy is a consequence of the fact that $\mathcal{S}_L$ is an $L_{\infty}$-morphism of $L_{\infty}$-modules.

On the other hand, $\mathcal{U}_{L, 1}$ is compatible with respect to the products labelled by $\cup$ up to homotopy by the results of [6, §10.1].

As for the compatibility between the actions labelled by $\cap$ up to homotopy, we first observe that the homotopy formula (7.1) is well-defined in the case $(m, d_m) = (C_{\text{coord}, L}, d)$ and $\gamma = \omega$, with the same notations as above: by the same arguments as in the proof of Theorem 5.3 it remains to prove that the homotopy operator (7.2) descends to a homotopy operator

$$\mathcal{S}^S_L : T_{\text{poly}, C_{\text{aff}, L}}(C_{\text{aff}, L} \otimes \Omega_R, JL) \otimes \mathcal{C}_{C_{\text{aff}, L}, \bullet}(C_{\text{aff}, L} \otimes \Omega_R, JL) \to \Omega_{C_{\text{aff}, L \otimes \Omega_R, JL} / C_{\text{aff}, L}}.$$ 

The homotopy operator $\mathcal{H}^S_L$ descends as a consequence of Lemma 7.2 together with the verticality property of the Maurer–Cartan form $\omega$, see §4.1.

If we now couple the commutative diagram (7.4) with the results of §4.2, §4.3 and §4.4, and using the same notations introduced at the end of the proof of Theorem 5.3 we get the following commutative diagram of precalculi up to homotopy

\begin{equation}
(T_{\text{poly}}(R), g_{L, 1} \cup \mathcal{U}_{L, 1} \cup g_{L, 2} \cup D_{\text{poly}}(R), \mathcal{S}_{L, 0} \cup m_{L, 1} \cup m_{L, 2} \cup \mathcal{C}_{\text{poly}}(R))
\end{equation}

The quasi-isomorphisms $\mathcal{U}_{L, 1}$ and $\mathcal{S}_{L, 0}$ are obtained from $\mathcal{U}_{L, \omega, 1}$ and $\mathcal{S}_{L, \omega, 0}$ respectively by means of the descent procedure: since $\omega$ is an $m$-valued vector field in $T^m_{\text{poly}}(F) = g_{L, 1}$, for $m = C_{\text{coord}, L}$, we can use the results of [6, §10.1], and [5, §6], to evaluate explicitly $\mathcal{U}_{L, \omega, 1}$ and $\mathcal{S}_{L, \omega, 0}$, namely

\begin{equation}
\mathcal{U}_{L, \omega, 1} = \text{HKR} \circ j(\omega), \quad \mathcal{S}_{L, \omega, 0} = j(\omega) \wedge \text{HKR},
\end{equation}
where

\[ j(\omega) = \det \sqrt{\exp\left(\frac{\Xi}{\omega}\right) - \exp\left(-\frac{\Xi}{\omega}\right)} \]  

with \( \Xi \) as defined in (6.3). To interpret (7.7) one should develop the right-hand side formally in terms of \( \text{Tr}(\Xi^n) \) and then substitute the expression for \( \Xi \) given in (6.3). This yields an element of \( C^{\text{coord},L} \otimes \Omega_F \) of degree \( 2n \). Thus \( j(\omega) \) is a sum of elements in \( C^{\text{coord},L} \otimes \Omega_F \) of even total degree.

By the discussion in §6 the element \( \text{Tr}(\Xi^n) \in C^{\text{coord},L} \otimes \Omega_F \) may be interpreted as an element in \( C^{\text{aff},L} \otimes \Omega_{L/J_0} \) via the inclusions (6.2). Hence the same holds for \( j(\omega) \). We keep the same notation for this reinterpreted version of \( j(\omega) \).

We thus get the following formulæ:

\[ \mathcal{U}_{L,1} = \text{HKR} \circ t_j(\omega), \quad \mathcal{G}_{L,0} = j(\omega) \wedge \text{HKR}. \]

We now briefly indicate how we may replace \( k \) by a general field of characteristic zero (see [6, §10.4] for more details). Our arguments depend on the existence of a number of explicit homotopies. These homotopies are constructed as scalar linear combinations of sums over certain graphs. For the arguments to work the coefficients need to satisfy certain linear equations. These equations have a solution over \( \mathbb{R} \) (given that over this field we have homotopies that work). Thus they have a solution over any field of characteristic zero.

### 7.3. Functoriality properties of the commutative diagram (7.5)

The computations in the proof of Proposition 5.5 imply the following theorem, expressing the functoriality properties of the commutative diagram (7.5).

**Theorem 7.3.** For a general algebraic morphism \((f, \rho)\) from \((L, R)\) to \((M, S)\) as in Definition 5.4, which induces an isomorphism \( S \otimes_R L \cong M \) of \( S \)-modules, and such that \( L \) is free of rank \( d \) over \( R \) there exist quasi-isomorphisms \( \mathcal{U}_{L,1}, \mathcal{U}_{M,1}, \mathcal{G}_{L,0} \) and \( \mathcal{G}_{M,0} \), fitting in the commutative diagram of precalculus up to homotopy

\[
\begin{array}{cccccccc}
T^L_{\text{poly}}(R) & \xrightarrow{\varphi_T} & \mathcal{B}_1^L & \xrightarrow{\mathcal{U}_{L,1} = \text{HKR} \circ t_j(\omega_L)} & \mathcal{B}_2^L & \xrightarrow{\mathcal{G}_{L,0} = j(\omega_L) \wedge \text{HKR}} & D^L_{\text{poly}}(R) \\
\downarrow \varphi_T & & \downarrow \varphi_T & & \downarrow \varphi_D & & \downarrow \varphi_D \\
T^M_{\text{poly}}(S) & \xrightarrow{\varphi_T} & \mathcal{B}_1^M & \xrightarrow{\mathcal{U}_{M,1} = \text{HKR} \circ t_j(\omega_M)} & \mathcal{B}_2^M & \xrightarrow{\mathcal{G}_{M,0} = j(\omega_M) \wedge \text{HKR}} & D^M_{\text{poly}}(S) \\
\downarrow \varphi_0 & & \downarrow \varphi_0 & & \downarrow \varphi_j & & \downarrow \varphi_j \\
\mathcal{O}^M(S) & \xrightarrow{\varphi_0} & \mathcal{M}_1^M & \xrightarrow{\mathcal{G}_{M,0} = j(\omega_M) \wedge \text{HKR}} & \mathcal{M}_2^M & \xrightarrow{\mathcal{G}_{M,0} = j(\omega_M) \wedge \text{HKR}} & \mathcal{C}^M_{\text{poly}}(S) \\
\downarrow \varphi_0 & & \downarrow \varphi_o & & \downarrow \varphi_j & & \downarrow \varphi_j \\
\mathcal{O}^L(R) & \xrightarrow{\varphi_0} & \mathcal{M}_1^L & \xrightarrow{\mathcal{G}_{L,0} = j(\omega_L) \wedge \text{HKR}} & \mathcal{M}_2^L & \xrightarrow{\mathcal{G}_{L,0} = j(\omega_L) \wedge \text{HKR}} & \mathcal{C}^L_{\text{poly}}(R)
\end{array}
\]

where we borrow notations from Proposition 5.5, and where \( \omega_L \) and \( \omega_M \), denote the Maurer–Cartan on \( C^{\text{coord},L} \) and \( C^{\text{coord},M} \) respectively. The precalculus structures up to homotopy on \((\mathcal{B}_i^+, \mathcal{M}_i^+)\) are defined as in §5.5.

Almost all important objects appearing in Theorem 7.3, have already appeared in Theorem 5.6, hence the functoriality properties extend to the present situation.
The commutativity of the upper and lower squares involving \( j(\omega) \) follows from the compatibility of the inclusions (6.2) with the base extension \( S/R \).

7.4. Proof of Theorem 1.1 in the global case. For a ringed site \((X, \mathcal{O})\), we consider now the derived category \( D(X) \) of sheaves of \( k \)-vector spaces over \( X \). As before, \( \mathcal{L} \) is a locally free sheaf of Lie algebroids over \( \mathcal{O} \) of rank \( d \).

According to the results of Section 3, transported to the framework of sheaves of \( k \)-vector spaces \((T^\mathcal{L}_{\text{poly}}(X), \Omega^\mathcal{L}(X))\) and \((D^\mathcal{L}_{\text{poly}}(X), C^\mathcal{L}_{\text{poly}}(X))\) are precalculi up to homotopy. Therefore, viewed as objects of \( D(X) \), they are genuine precalculi.

Additionally, the sheafification procedure can be applied to the commutative diagram (7.9), in virtue of the results of §7.3: if we further consider the resulting commutative diagram of sheaves of \( k \)-vector spaces in the derived category \( D(X) \), then using (6.4) we get the commutative diagram of precalculi

\[
\begin{array}{ccc}
T^\mathcal{L}_{\text{poly}}(X) & \xrightarrow{\mathcal{L}^1_f} & \mathcal{L}_{\text{poly}}(X) \\
\downarrow \mathcal{L}^1_{\text{HKR}} & \mathcal{L}^1_{\text{HKR}} & \downarrow \\
D^\mathcal{L}_{\text{poly}}(X) & \xrightarrow{\mathcal{L}^2_{\text{HKR}}} & C^\mathcal{L}_{\text{poly}}(X) \\
\downarrow \mathcal{L}^2_1 & \downarrow \mathcal{L}^2_{\text{HKR}} & \downarrow \\
\mathcal{L}^1_{\text{organ}} & \mathcal{L}^2_{\text{organ}} & \mathcal{L}^3_{\text{organ}} \\
\end{array}
\]

where all horizontal and vertical arrows represent isomorphisms in the derived category \( D(X) \). Here \( \mathcal{L}^1_{\text{organ}} \) is the so-called modified Todd class of \( \mathcal{L} \) which is obtained by replacing the function \( q(x) \) in the definition of the Todd class (see (1.2)) by

\[
\mathcal{L}^1_{\text{organ}}(x) = \frac{x}{e^{x/2} - e^{-x/2}}.
\]

Hence at this point we have proved Theorem 1.1 provided that we replace the Todd class by the modified one. To obtain the result for the ordinary Todd class we follow the method of [6, §10.3]. We have

\[
\begin{align*}
\mathcal{L}^1_{\text{organ}}(x) &= \mathcal{L}^1_{\text{organ}}(x) \det(e^{-A(\mathcal{L})/2}) \\
&= \mathcal{L}^1_{\text{organ}}(x) e^{-\text{Tr}(A(\mathcal{L}))/2} \\
&= \mathcal{L}^1_{\text{organ}}(x) e^{-a_1(\mathcal{L})/2}
\end{align*}
\]

In other words it is sufficient to prove that \((\mathcal{L}^1_{\text{organ}}, e^{a_1(\mathcal{L})/4} \wedge -)\) defines an automorphism of the precalculus \((T^\mathcal{L}_{\text{poly}}(X), \Omega^\mathcal{L}(X))\).

Via the inclusions (6.2) together with (6.4) we may as well prove that \((\mathcal{L}^1_{\text{organ}}, e^{\text{Tr}(\mathcal{L})/4} \wedge -)\) defines an automorphism of the precalculus \((C^\text{coord}_{\text{organ}} T^\mathcal{L}_{\text{poly}}(F), C^\text{coord}_{\text{organ}} \Omega^\mathcal{L}(F))\) or equivalently that \((\mathcal{L}^1_{\text{organ}}, - \text{Tr}(\mathcal{L}) \wedge -)\) act as derivations. The fact that \(\mathcal{L}^1_{\text{organ}}\) is a derivation with respect to the cup product and Lie bracket has been checked in [6, §10.3]. So it remains to show compatibility with the cap product and Lie derivative.
As $\text{Tr}(\Xi) = \sum i_{\alpha} \eta_{\alpha} d_F(\partial_i \omega_\alpha)$ we first derive some identities for $\iota_d F_b$ and $d_F b \land -$ with $b$ in $F$.

First we claim

(7.10) $d_F b \land (D \cap \sigma) = -\iota_{d_F b}(D) \cap \sigma + (-1)^{|D|+1} D \cap (d_F b \land \sigma)$

for $b \in F$, $D \in T_{\text{poly}}(F)$, $\sigma \in \Omega_F$. If $D = D_1 \cup D_2$ and (7.10) holds for $D_1, D_2$ then it holds for $D$ as well. To see this note

$$d_F b \land ((D_1 \cup D_2) \cap \sigma) = d_F b \land (D_1 \cap (D_2 \cap \sigma))$$

$$= -\iota_{d_F b}(D_1) \cap (D_2 \cap \sigma) + (-1)^{|D_1|+1} D_1 \cap (d_F b \land (D_2 \cap \sigma))$$

$$= -\iota_{d_F b}(D_1) \cap (D_2 \cap \sigma) - (-1)^{|D_1|+1} D_1 \cap \iota_{d_F b}(D_2) \cap \sigma$$

$$\quad + (-1)^{|D_1|+|D_2|} D_1 \cap D_2 \cap (d_F b \land \sigma)$$

$$= -\iota_{d_F b}(D_1 \cup D_2) \cap \sigma + (-1)^{|D_1|+|D_2|+1} (D_1 \cup D_2) \cap (d_F b \land \sigma)$$

So we only have to consider the case where $D$ is a function or a vector field. The case that $D$ is a function is trivial so assume that $D$ is a vector field. In that case we find for the right-hand side of (7.10)

$$\quad -\iota_{d_F b}(D) \cap \sigma + (-1)^{|D|+1} D \cap (d_F b \land \sigma) = Db \cap \sigma - Db \land \sigma + d_F b \land (D \cap \sigma)$$

$$= d_F b \land (D \cap \sigma)$$

which is equal to the left-hand side of (7.10).

For the Lie derivative we use $L_D = [d_F, D \cap -]$. It is clear that $d_F$ and $d_F b \land -$ commute. We then compute using (7.10)

$$d_F b \land L_D \sigma = d_F b \land (d_F(D \cap \sigma) - (-1)^{|D|+1} D \cap d_F \sigma)$$

$$= -d_F(d_F b \land (D \cap \sigma)) + (-1)^{|D|} d_F b \land (D \cap d_F \sigma)$$

$$= d_F(\iota_{d_F b}(D) \cap \sigma) + (-1)^{|D|} d_F(D \cap (d_F b \land \sigma))$$

$$\quad + (-1)^{|D|+1} \iota_{d_F b}(D) \cap d\sigma - D \cap (d_F b \land d_F \sigma)$$

$$= L_{\iota_{d_F b}} D(\sigma) + (-1)^{|D|} L_D(d_F b \land \sigma)$$

If $\eta$ is an odd element in $C_{\text{coord}}^\infty$ then $\iota_{\eta d_F b} D = \eta \iota_{d_F b} D$ and $L_{\iota_{\eta d_F b} D}(\sigma) = L_{\iota_{\eta d_F b} D}(\sigma) = -\eta L_{\iota_{d_F b}}(\sigma)$. Using this we find

$$\text{Tr}(\Xi) \land (D \cap \sigma) = -\iota_{\text{Tr}(\Xi)}(D) \cap \sigma + D \cap (\text{Tr}(\Xi) \land \sigma)$$

and

$$\text{Tr}(\Xi) \land L_D \sigma = -L_{\iota_{\text{Tr}(\Xi)} D}(\sigma) + L_D(\text{Tr}(\Xi) \land \sigma)$$

We conclude that $(\iota_{\text{Tr}(\Xi)}, -\text{Tr}(\Xi) \land -)$ does indeed define a derivation of precalculi.

**Appendix A. Explicit formulæ for the $B_\infty$-structure on poly-differential operators**

In this appendix and the next we develop the precalculus structure on $L$-chains over $L$-cochains up to homotopy. The results in these appendices are provided for background and are not essential for the results in the body of the paper.

The graded vector space $V = D^0_{\text{poly}}(R)$ is naturally a $B_\infty$-algebra. This means that the cofree coassociative coalgebra (with counit) $T(V)$ is canonically equipped with the structure of a DG bi-algebra.
The corresponding associative product \( m \) on \( T(V) \) is uniquely determined by its Taylor components \( m_{p,q} : T^p(V) \otimes T^q(V) \to V \). We have \( m_{p,q} = 0 \) if \( p \neq 1 \) and

\[
(A.1) \quad m_{1,q}(D \otimes (D_1 \otimes \cdots \otimes D_q)) = D(D_1, \ldots, D_q) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_q \leq |D| + \sum_{k=1}^q |D_k| - 1} (-1)^{\sum_{k=1}^q |D_k| (i_k - 1)} \\
\left( 1^{\otimes (i_1 - 1)} \otimes \Delta^{D_1} \otimes 1^{\otimes (i_2 - i_1 - |D_1| - 1)} \otimes \Delta^{D_2} \otimes \cdots \otimes 1^{\otimes (i_q - i_{q-1} - |D_{q-1}| - 1)} \otimes \Delta^{D_q} \otimes 1^{\otimes (|D| + \sum_{k=1}^q |D_k| - |D_q|)} \right)(D)
\]

\[
(A.2) \quad D_1 \cup D_2 = (-1)^{|D_1|+1} \mu(D_1, D_2), \quad D_i \in D_{\text{poly}}^L(R), \quad i = 1, 2.
\]

It is obvious that the cup product has (shifted) degree 1. An easy verification using Formula (A.1) shows that the previous definition of cup product coincides with the one given in Formula (3.11).

We now have the following compatibilities

**Lemma A.1.** The degree 0 operation (3.10) and the degree 1 operation (A.2) satisfy the following properties:

\[
(A.3) \quad [D_1, D_2] = -(-1)^{|D_1||D_2|}[D_2, D_1],
\]

\[
(A.4) \quad [D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{|D_1||D_2|}[D_2, [D_1, D_3]],
\]

\[
(A.5) \quad D_1 \cup D_2 = (-1)^{|D_1|-1}D_2 \cup D_1 \pm \left( d_H(D_1 \{D_2\}) - (d_H D_1)\{D_2\} - (-1)^{|D_1|} D_1\{d_H D_2\} \right),
\]

\[
(A.6) \quad D_1 \cup (D_2 \cup D_3) = (D_1 \cup D_2) \cup D_3,
\]

and

\[
(A.7) \quad [D_1, D_2 \cup D_3] = [D_1, D_2] \cup D_3 + (-1)^{|D_1||D_2|-1} D_2 \cup [D_1, D_3] + (-1)^{|D_1|} \left( d_H(D_1 \{D_2, D_3\}) - (d_H D_1)\{D_2, D_3\} - (-1)^{|D_1|} D_1\{d_H D_2, D_3\} - (-1)^{|D_1|+|D_2|} D_1\{D_2, d_H D_3\} \right),
\]

for general elements \( D_i \) of \( D_{\text{poly}}^L(R) \), \( i = 1, 2, 3 \), and where \( d_H = [\mu, \bullet] \), \( \mu = 1 \otimes_R 1 \).
Appendix B. The precalculus structure on $L$-chains

We need results from [5, 21] about algebraic structures on Hochschild (co)chains, which have to be adapted to the Lie algebroid framework.

According to [5, 21], there are two distinct, non-compatible, left $B_\infty$-module structures on the Hochschild chain complex of $A$, viewed as a $B_\infty$-algebra with respect to the brace operations (A.1). Equivalently, we view the two left $B_\infty$-module structures on the Hochschild chain complex as the data of two left actions $m_{L,i}$, $i = 1, 2$, on the left comodule cofreely cogenerated by the Hochschild chain complex of $A$ over the coalgebra cofreely cogenerated by the Hochschild cochain complex of $A$.

These results can be applied to the present situation with due changes: $	ilde{C}_{R, \bullet}(JL)$ has two left and right $B_\infty$-module structures over the $B_\infty$-algebra $D_{poly}^L(R)$.

We borrow the main notations and sign conventions from [5]. We denote by $m_{L,i}$, $i = 1, 2$ the two left $B_\infty$-module structures on $\tilde{C}_{R, \bullet}(JL)$: they are uniquely determined by their Taylor components

$$m^{(1,0)}_{L,i}(P \otimes (Q_1 \otimes \cdots \otimes Q_q) \otimes a \otimes (R_1 \otimes \cdots \otimes R_r)) \cdot (D_{-a}) = \sigma(a)(D_0 \otimes \cdots \otimes D_{-a}) = a(D_1 \otimes \cdots \otimes D_{-a} \otimes D_0), \quad D_i \in U_R(L), \quad i = 0, \ldots, -a,$$

which obviously satisfies $\sigma(-a+1) = \text{id}$, and the indices in the summation satisfy $l \leq j_1, j_i + [Q_{j_i}] + 1 \leq j_{i+1}$, $i = 1, \ldots, q - 1$, $j_q + [Q_q] \leq -[a]$, $k_i + [R_i] \leq -[a]$, $k_{i+1} + [R_{i+1}] \leq -[a]$, $i = 1, \ldots, r$, $k_r + [R_r] \leq -[a]$, $[P] + \sum_{b=1}^q [Q_b] + \sum_{j=1}^r [R_j] + l - 1$, and

$$m^{(0,0)}_{L,i}(a \otimes (R_1 \otimes \cdots \otimes R_r)) \cdot (D_{-a}) = \sum_{1 \leq i_1 \leq \cdots \leq i_p \leq -[a]} (-1)^{\sum_{b=1}^p [R_{i_b}]} a \left( \left( \otimes^{i_1} \Delta [R_1] \otimes \cdots \otimes^{i_{q-1}} [R_{q-1}] \otimes \Delta [R_q] \otimes \cdots \otimes^{i_{r-1}} [R_{r-1}] \otimes \Delta [R_r] \otimes \otimes \right) \cdot \right) \cdot (D_{-a}) \right),$$

where the summation is over indices $i_1, \ldots, i_r$, such that $1 \leq i_1, i_k + [D_k] + 1 \leq i_{k+1}$, $k = 1, \ldots, p - 1$, $i_p + [D_p] \leq -[a]$. We observe that the components of $m_{L,i}$, resp. $m_{L,i}$, are non-trivial only if $p \leq -[a]$, with no restrictions on $q, r$, resp. only if $q = r = 0$, with no restrictions on $p$.

It is easy to verify that both (B.1) and (B.2) have degree 0 and satisfy an infinite family of quadratic relations involving braces.
The Taylor components of $m_{L,i}$, $i = 1, 2$, permit to define a pairing of degree 0 between $D^L_{\text{poly}}(R)$ and $\tilde{C}_{R\bullet}(JL)$ via

\begin{equation}
L_D a = m_{L,1}^{1,0,0}(D \otimes a) + (-1)^{|D|} m_{L,2}^{0,0,1}(a \otimes D), \quad D \in D^L_{\text{poly}}(R), \ a \in \tilde{C}_{R\bullet}(JL).
\end{equation}

Similarly, we may consider two distinct pairings between $D^L_{\text{poly}}(R)$ and $\tilde{C}_{R\bullet}(JL)$:

\begin{equation}
D \cap a = (-1)^{|D|} m_{L,1}^{1,0,0}(\mu \otimes D \otimes a),
\end{equation}

\begin{equation}
\mu \circ D \cap a = (-1)^{|a|} m_{L,1}^{1,0,1}(\mu \otimes a \otimes D), \quad D \in D^L_{\text{poly}}(R), \ a \in \tilde{C}_{R\bullet}(JL).
\end{equation}

It follows from their very definition that both (B.4) and (B.5) have degree 1.

**Lemma B.1.** The pairing (B.3) of degree 0 and the pairings (B.4) and (B.5) of degree 1 satisfy the following properties:

\begin{equation}
L_{[D_1, D_2]} a = L_{D_1}(L_{D_2} a) - (-1)^{|D_1||D_2|} L_{D_2}(L_{D_1} a),
\end{equation}

\begin{equation}
D \cap a = (-1)^{|D||a|-1} a \cap D \pm \left( b_H(m_{L,1}^{1,0,0}(D \otimes a) - m_{L,1}^{1,0,0}(a \otimes D) - (-1)^{|D|} m_{L,1}^{1,0,0}(D \otimes b_H a)) \right),
\end{equation}

\begin{equation}
D_1 \cap (D_2 \cap a) = (D_1 \cup D_2) \cap a,
\end{equation}

\begin{equation}
(a \cap D_1) \cap D_2 = a \cap (D_1 \cup D_2),
\end{equation}

\begin{equation}
L_{D_1}(D_2 \cap a) = [D_1, D_2] \cap a + (-1)^{|D_1||D_2|-1} D_2 \cap L_{D_1} a + (-1)^{|D_1|} \left( b_H(m_{L,1}^{1,1,0}(D_1 \otimes D_2 \otimes a)) - m_{L,1}^{1,1,0}(d_H D_1 \otimes D_2 \otimes a) - (-1)^{|D_1|} m_{L,1}^{1,1,0}(D_1 \otimes d_H D_2 \otimes a) \right),
\end{equation}

\begin{equation}
L_{D_1}(a \cap D_2) = L_{D_1} a \cap D_2 + (-1)^{|D_1||a|-1} a \cap [D_1, D_2] + (-1)^{|D_1|} \left( b_H(m_{L,1}^{1,0,1}(D_1 \otimes a \otimes D_2)) - m_{L,1}^{1,0,1}(d_H D_1 \otimes a \otimes D_2) - (-1)^{|D_1|} m_{L,1}^{1,0,1}(D_1 \otimes b_H a \otimes D_2) \right),
\end{equation}

\begin{equation}
-(-1)^{|D_1|+|a|} m_{L,1}^{1,0,1}(D_1 \otimes a \otimes d_H D_2)),
\end{equation}

The pairing (B.3) of degree 0 and the pairings (B.4) and (B.5) of degree 1 satisfy the following properties:
and finally
(B.12) 
\[ L_{D_1 \cup D_2} a + (-1)^{|D_1|(|D_2| - 1)} L_{D_1} a =\]
\[ = \left( D_1 \cap L_{D_2} a + (-1)^{|D_1|(|D_2| - 1)} (D_1 \cap D_2 \cap L_{D_2} a \cap D_1) \right) \]
\[ + (-1)^{|D_2| - 1} \left( L_{D_1} a \cap D_2 + (-1)^{|D_1| + |a| - 1} (D_2 \cap D_1 \cap L_{D_2} a) \right) \]
\[ + (-1)^{|D_2| - 1} \left( [D_1, D_2] \cap a + (-1)^{|a| - 1} ([D_1, D_2] - 1) \cap [D_1, D_2] \right) \]
\[ + (-1)^{|D_2|} [D_2] b_H(m_{L,2}^0, 0, 2(a \otimes R_1 \otimes R_2)) - (-1)^{|D_1|} m_{L,2}^0, 0, 2(b_H a \otimes D_1 \otimes D_2) \]
\[ + (-1)^{|D_2|} m_{L,2}^0, 0, 2(a \otimes d_H D_1 \otimes D_2) + (-1)^{|D_1| + |D_2|} m_{L,2}^0, 0, 2(a \otimes D_1 \otimes d_H D_2) \]
\[ + (-1)^{|D_1|} b_H(m_{L,2}^0, 0, 2(a \otimes D_2 \otimes D_1)) - (-1)^{|D_2|} m_{L,2}^0, 0, 2(b_H a \otimes D_2 \otimes D_1) \]
\[ + (-1)^{|D_1|} m_{L,2}^0, 0, 2(a \otimes d_H D_2 \otimes D_1) + (-1)^{|D_1| + |D_2|} m_{L,2}^0, 0, 2(a \otimes D_2 \otimes d_H D_1), \]
for a general element \( a \) of \( C_{R \bullet}(JL) \) and general elements \( D, D_i, i = 1, 2, \) of \( D_{poly}^L(R) \), and where \( b_H = L_{\mu} \), for \( \mu \) as before.

As for Lemma A.1, the proof essentially makes use of the brace identities, of the fact that \( m_{L,i}, i = 1, 2, \) is a left action with respect to the brace operations, and of the fact that \( m_{L,1} \) and \( m_{L,2} \) satisfy a weak compatibility, as explained in more details in [5].

Both actions \( m_{L,1} \) and \( m_{L,2} \) are compatible with the Grothendieck connection, i.e.
\[ 1 \nabla_l \left( m_{L,1}^{q, r}(D \otimes Q_1 \otimes \cdots \otimes a \otimes R_1 \otimes \cdots) \right) = m_{L,1}^{q, r}(D \otimes Q_1 \otimes \cdots \otimes 1 \nabla_l a \otimes R_1 \otimes \cdots), \]
\[ q, r \geq 0, \]
\[ 1 \nabla_l \left( m_{L,2}^{0, p, 0}(D_1 \otimes \cdots \otimes a) \right) = m_{L,2}^{0, p, 0}(D_1 \otimes \cdots \otimes 1 \nabla_l a), \]
\[ p \geq 0, \]
for \( D, D_i \) \((i = 1, \ldots, p), Q_j \) \((j = 1, \ldots, q), R_k \) \((k = 1, \ldots, r)\) elements of \( D_{poly}^L(R) \), and \( a \) of \( C_{R \bullet}(JL) \). Both identities follow from the fact that \( 1 \nabla \) commutes with the operator \( \sigma \) and from the fact that \( U_R(L) \) is a Hopf algebroid, in particular, the comultiplication is an algebra morphism.

Then, in virtue of Lemma B.1, the pairings (B.3), (B.4) and (B.5) are compatible with the Grothendieck connection, implying in particular that the Hochschild differential is also compatible therewith. By the very same arguments, Formule (B.7), (B.8), (B.9), (B.10), (B.11) and (B.12) are compatible with the Grothendieck connection, whence \((ker(1 \nabla)) \cap C_{R \bullet}(JL), b_H, L, \cap)\), where \( \cap \) denotes here both (B.4) and (B.5), inherits a structure of precalculus up to homotopy over the Gerstenhaber algebra \((D_{poly}^L(R), d_R, [\cdot, \cdot, \cdot], \cup)\) up to homotopy.

For the sake of completeness, we write down explicit formulæ for the Hochschild differential \( b_H \) on the complex of Hochschild \( L \)-chains on \( R \) and for the pairing (B.5) between \( D_{poly}^L(R) \) and \( C_{poly}^L(R) \); in [7], we will deduce the same formulæ in the framework of homological algebra and derived functors. Explicitly, 
\[ b_H(a) = a \circ d_H, \]
\[ a \cap D = (-1)^{|a|} a(D \otimes R \bullet), a \in C_{poly}^L(R), \]
\[ D \in D_{poly}^L(R). \]

We observe that (B.6) implies that \( b_H \), the Hochschild differential on \( L \)-chains, is compatible with respect to (B.3), and that (B.10) and (B.11), in the special case
$D_1 = \mu$, imply that $b_H$ satisfies Leibniz's rule with respect to (B.4) and (B.5) respectively.

Thus, combining these arguments with Proposition 3.6, we have the following important

**Theorem B.2.** For a Lie algebroid $L$ over the ring $R$ as above, the twist of (B.3), (B.4), (B.5) and of the Hochschild differential $b_H$ with respect to the isomorphism $\chi$ endow $C^L_{poly}(R)$ with a structure of precalculus up to homotopy over the Gerstenhaber algebra $(D^L_{poly}(R), d_H, [\cdot, \cdot], \cup)$ up to homotopy.

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