On the noncommutative residue for pseudodifferential operators with log–polyhomogeneous symbols

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Abstract. We study various aspects of the noncommutative residue for an algebra of pseudodifferential operators whose symbols have an expansion

\[ a \sim \sum_{j=0}^{\infty} a_{m-j}, \quad a_{m-j}(x, \xi) = \sum_{l=0}^{k} a_{m-j,l}(x, \xi) \log^l |\xi|, \]

where \( a_{m-j,l} \) is homogeneous in \( \xi \) of degree \( m-j \). We call these symbols log–polyhomogeneous. We will explain why this algebra of pseudodifferential operators is natural.

We study log–polyhomogeneous functions on symplectic cones and generalize the symplectic residue of Guillemin to these functions. Similarly as for homogeneous functions, for a log–polyhomogeneous function this symplectic residue is an obstruction against being a sum of Poisson brackets.

For a pseudodifferential operator with log–polyhomogeneous symbol, \( A \), and a classical elliptic pseudodifferential operator, \( P \), we show that the generalized \( \zeta \)–function \( \text{Tr}(AP^{-s}) \) has a meromorphic continuation to the whole complex plane, however possibly with higher order poles.

Our algebra of operators has a bigrading given by the order and the highest log–power occurring in the symbol expansion. We construct "higher" noncommutative residue functionals on the subspaces given by the log–grading. However, in contrast to the classical case we prove that the whole algebra does not admit any nontrivial traces.

Finally we show that the analogue of the Kontsevich–Vishik trace also exists on our algebra. Our method also provides an alternative approach to the Kontsevich–Vishik trace.

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1. Introduction and summary of the results

Let $M$ be a compact smooth Riemannian manifold without boundary. We denote by $\text{CL}^*(M)$ the algebra of classical (1–step polyhomogeneous) pseudodifferential operators. $\text{CL}^*(M)$ acts naturally as unbounded operators on the Hilbert space $L^2(M)$ of square integrable functions. If $m < -\dim M$ then $\text{CL}^m(M) \subset C_1(L^2(M))$, the space of trace class operators.

The $L^2$–trace does not have a continuation as a trace functional on the whole algebra $\text{CL}^*(M)$. For assume we had a trace $\tau$ on $\text{CL}^*(M)$ that extends the $L^2$–trace. For $r$ large enough we may choose an elliptic operator $T \in \text{CL}^1(M) \otimes M_r(\mathbb{C})$ of nonvanishing Fredholm index. Let $S \in \text{CL}^{-1}(M) \otimes M_r(\mathbb{C})$ be a pseudodifferential parametrix. Then $I - ST, I - TS \in \text{CL}^{-\infty}(M) \otimes M_r(\mathbb{C})$ and we arrive at the contradiction

$$0 \neq \text{ind} T = \text{Tr}_{L^2}([T, S]) = \tau([T, S]) = 0.$$  \hfill (1.1)

However, in his seminal papers [19], [20] M. Wodzicki showed that, up to a constant, the algebra $\text{CL}^*(M)$ has a unique trace which he called the noncommutative residue. The noncommutative residue was independently discovered by V. Guillemin [10] as a byproduct of his so–called ”soft” proof of the Weyl asymptotic.

A detailed account of the noncommutative residue was given by C. Kassel [12]. B. Fedosov et al. [7] generalized the noncommutative residue to the Boutet de Monvel algebra on a manifold with boundary. Furthermore, E. Schrohe [17] studied manifolds with conical singularities.

There are several ways to define the noncommutative residue. The global definition which shows the intimate relation to $\zeta$–functions and heat kernel expansions is as follows: given $A \in \text{CL}^a(M)$. Then choose any elliptic operator $P \in \text{CL}^m(M), m > 0$, whose leading symbol is positive. Then

$$\text{Res}(A) := m \text{Res}_{s=0} \text{Tr}(AP^{-s}) = -m \times \text{coefficient of } \log t \text{ in the asymptotic} \text{ expansion of } \text{Tr}(Ae^{-tP}) \text{ as } t \to 0.$$ \hfill (1.2)

$\text{Res}(A)$ is in fact independent of the $P$ chosen. This definition uses the fact that the generalized $\zeta$–function $\text{Tr}(AP^{-s})$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at \( \{ \frac{\dim M + a - i}{m} \mid j \in \mathbb{Z}_+ \} \), $\mathbb{Z}_+ := \{ 0, 1, \ldots \}$. Via the Mellin transform

$$\text{Tr}(AP^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(Ae^{-tP}) dt,$$

this is (almost) equivalent to the asymptotic expansion

$$\text{Tr}(Ae^{-tP}) \sim_{t \to 0^+} \sum_{j=0}^\infty (c_j + c'_j \log t) t^{\frac{i-a-\dim M}{m}} + \sum_{j=0}^\infty d_j t^j,$$ \hfill (1.3)

where $c'_j = 0$ if $\frac{i-a-\dim M}{m} \notin \mathbb{Z}_+$ (cf. [8], Thm 2.7).

Another approach to the noncommutative residue was also discovered independently by M. Wodzicki and V. Guillemin. This approach works in the framework of symplectic cones. It shows that there is a local formula

$$\text{Res}(A) = (2\pi)^{-\frac{\dim M}{2}} \int_{s^*M} a(x, \xi) \frac{d\xi dx}{-\dim M} \text{ | } \text{Res}(A).$$ \hfill (1.4)
for the noncommutative residue in terms of the complete symbol of $A$. It is a remarkable fact that, although the complete symbol does not have an invariant meaning, the right hand side of (1.4) is well-defined.

Asymptotic expansions like (1.3) are an essential feature in the study of elliptic operators. They can also be achieved for certain operators with singularities, for example conical singularities [5], [4], [3] or boundary value problems [9], [8], [2]. In all these situations no higher $\log t$ powers occur. However, in [1] higher $\log t$ powers show up in equivariant heat trace expansions of the Laplacian. These higher log–terms are produced by the method of proof and it is conjectured that in fact all coefficients of $t^\alpha \log^k t, k \geq 1$, vanish.

It was one of our motivations to write this paper to provide a natural algebra of pseudodifferential operators for which higher $\log t$–powers occur in the heat expansion (1.3). Although our class of operators is not really new, to the best of our knowledge it was never looked at with regard to noncommutative residue. Our point of view shows that the existence of the noncommutative residue as the unique trace depends heavily on the absence of higher $\log t$–powers.

The other motivation arose from the following problem: given $A \in \mathrm{CL}^a(M)$ then by a result of Wodzicki $A$ is a sum of commutators of operators in $\mathrm{CL}^a(M)$ if and only if $\text{Res}(A) = 0$. Now, one can ask whether there is a natural class of pseudodifferential operators $\subset \mathrm{L}^*(M)$ containing the classical ones such that $A$ is a sum of commutators in this class.

Let us look at the following simple minded analogy between the noncommutative residue and the ordinary residue for functions. Consider the algebra $A := \mathbb{C}[z, z^{-1}]$ of Laurent polynomials. Put

$$\text{Res}\left(\sum_{j \in \mathbb{Z}} a_j z^j\right) := a_{-1}. \quad (1.5)$$

Then the Laurent polynomial $f$ has a primitive in $A$ if and only if $\text{Res}(f) = 0$. However, if we adjoin the primitive of $z^{-1}$, $\log z$, we obtain the algebra $B := A[\log z] = \mathbb{C}[z, z^{-1}, \log z]$ which is filtered by

$$B^k := \left\{ \sum_{j=0}^k f_j \log^j z \mid f_j \in A \right\}. \quad (1.6)$$

We define the ”higher” residue

$$\text{Res}_k : B^k \longrightarrow \mathbb{C}, \quad \sum_{j=0}^k f_j \log^j z \mapsto \text{Res}(f_k). \quad (1.7)$$

Then $f \in B^k$ has a primitive in $B^k$ if and only if $\text{Res}_k(f) = 0$. However, $f \in B^k$ always has a primitive in $B^{k+1}$. This picture can be transferred to the noncommutative residue for pseudodifferential operators.

The idea is quite simple: instead of looking at classical pseudodifferential operators we consider pseudodifferential operators whose symbol have an asymptotic expansion

$$a \sim \sum_{j=0}^{\infty} a_{m-j}, \quad (1.8)$$
where $a_{m-j}$ is log–polyhomogeneous i.e.

$$a_{m-j}(x,\xi) = \sum_{l=0}^{k} a_{m-j,l}(x,\xi) \log^l |\xi|,$$

(1.9)

and $a_{m-j,l}(x,\xi)$ is homogeneous of degree $m-j$ (Definition 3.1). We denote the class of pseudodifferential operators having this symbol expansion by $\text{CL}^{m,k}(M)$, where $m$ denotes the order and $k$ the highest log–power occuring in the symbol expansion (Definition 3.3).

In fact, this class of operators is not new. It was considered before by Schrohe in his thesis [16] where he constructed the complex powers for elliptic operators in this class.

We had some difficulties to find an appropriate name for the functions of the form (1.9). First we tried ”polylogarithmic” but then we were informed by several people that the polylogarithm has a completely different meaning. R. Melrose suggested to us to use ”polyhomogeneous” instead. But this would conflict with the use of this word in [8]. So we took log–polyhomogeneous as a compromise. But still, we also find this term only suboptimal.

In the present paper we show the heat expansion $\text{Tr}(A e^{-tP})$ for $A$ with log–polyhomogeneous symbol and classical $P$. We use the method of Grubb and Seeley [8]. In contrast to (1.3) there occur higher log $t$–powers (Theorem 3.7).

As a consequence of the heat expansion for $A \in \text{CL}^{m,k}(M)$ we can define the ”higher” noncommutative residue, $\text{Res}_k(A)$, as the coefficient of the highest log $t$–power in the expansion of $\text{Tr}(A e^{-tP})$ (Definition 4.1). It turns out that this residue has similar properties as the noncommutative residue of Wodzicki, in particular it is independent of the $P$ chosen. It vanishes on appropriate commutators, i.e.

$$\text{Res}_{k+1}([A, B]) = 0$$

if $A \in \text{CL}^{m,k}(M)$ and $B \in \text{CL}^{n,l}(M)$ (Theorem 4.4). There is also a local formula for $\text{Res}_k$ (Corollary 4.8).

In the context of spectral triples higher noncommutative residues were also discovered and investigated by A. Connes and H. Moscovici [3 Chap. II]. In fact, our algebra of pseudodifferential operators provides examples of spectral triples with a discrete dimension spectrum of infinite multiplicity.

As for the Wodzicki residue $\text{Res}_k$ is an obstruction for being a sum of commutators. Namely, if $A \in \text{CL}^{m,k}(M)$ then there exist $P_1, \ldots, P_N \in \text{CL}^{1,0}(M)$ and $Q_1, \ldots, Q_N \in \text{CL}^{m,k}(M)$ such that $A - \sum_{j=1}^{N} [P_j, Q_j]$ is smoothing if and only if $\text{Res}_k(A) = 0$ (Proposition 4.7). By a result of Wodzicki [19] (cf. also [11] for a generalization to Fourier integral operators) any smoothing operator is in fact a sum of commutators of classical pseudodifferential operators. Hence $A \in \text{CL}^{m,k}(M)$ is a sum of commutators if and only if $\text{Res}_k(A) = 0$ (Proposition 4.9). However, $\text{Res}_{k+1}(A)$ is always zero for $A \in \text{CL}^{m,k}(M)$, such that $A$ can always be written as a sum of commutators if one increases the log–degree by one. As a consequence, there does not exist any trace functional on the algebra $\text{CL}^{*}(M)$ (Corollary 4.11, but see (4.20)ff.).
For proving the result about commutators we generalize a result due to Guillemin [10] about homogeneous functions on symplectic cones. Namely, we generalize the notion of symplectic residue, which is closely related to the noncommutative residue, to functions on symplectic cones of the form

\[ \sum_{j=0}^{k} f_j \log^j p, \]  

where \( f_j \) is homogeneous and \( p \) is positive and homogeneous of degree 0. We call these functions log–polyhomogeneous.

Guillemin’s result says that a homogeneous function on a symplectic cone is a sum of Poisson brackets of homogeneous functions if and only if its symplectic residue vanishes. We define a residue for functions of the form (1.10) and prove an analogue of Guillemin’s result for these functions.

Finally we generalize the Kontsevich–Vishik trace functional. At the beginning of this section we remarked that the \( L^2 \)–trace does not have an extension as a trace functional to classical pseudodifferential operators. The proof uses integer order operators. Furthermore, the (higher) noncommutative residues, which are in some sense the obstructions against the extendability of the \( L^2 \)–trace as a trace, are nontrivial only for integer order operators.

This gives some evidence that the \( L^2 \)–trace has an extension to non–integer order operators. Indeed, this is true and the corresponding functional for classical pseudodifferential operators was discovered by Kontsevich and Vishik [13, 14]. The proof of loc. cit. uses the theory of homogeneous distributions.

We present two alternative approaches to the Kontsevich–Vishik trace in the generalized context of our algebra \( \text{CL}^{*,*} \). The first one is completely analogous to the definition (1.2) of the noncommutative residue (4.20). This definition however does not show that the Kontsevich–Vishik trace is given by integration of a canonical density. This canonical density is constructed in our second approach. If a pseudodifferential operator is locally given by

\[ Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{U} a(x, y, \xi) u(y) e^{i<x-y, \xi>} dyd\xi, \]  

then it is natural to try (we are bit sloppy with notation here)

\[ \text{Tr}(A) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{M} a(x, x, \xi) d\xi dx, \]  

and the density we are looking for is

\[ (2\pi)^{-n} \int_{\mathbb{R}^n} a(x, x, \xi) d\xi |dx|. \]  

Now the integral (1.13) in general only makes sense if the order of the operator is \(< - \dim M \), i.e. if \( A \) is trace class.

It is possible to define a regularized integral for symbol functions of the form (1.9) (cf. (5.9)). However, to give (1.13) a coordinate invariant meaning this regularized integral must satisfy the usual transformation rule, at least with respect to invertible
log–polyhomogeneous functions

2.2 log–polyhomogeneous functions

2.2.1 log–polyhomogeneous functions on $\mathbb{R}^n \setminus \{0\}$

In this section we follow in part [7, Sec. 1].

Definition 2.1 A function $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is called log–polyhomogeneous of degree $(a, k)$ if

$$f(x) = \sum_{j=0}^{l} g_j(x) P_j(\log h_j(x)),$$

where the $P_j \in \mathbb{C}[t]$ are polynomials, $\deg P_j \leq k$, and $g_j, h_j \in C^\infty(\mathbb{R}^n \setminus \{0\}), h_j > 0$, are homogeneous functions, $g_j$ of degree $a$ and $h_j$ of degree $b_j$.

We denote the set of all log–polyhomogeneous functions of degree $(a, k)$ by $\mathcal{P}^{a,k} = \mathcal{P}^{a,k}(\mathbb{R}^n)$ and put

$$\mathcal{P} := \bigoplus_{a \in \mathbb{C}, k \in \mathbb{Z}_+} \mathcal{P}^{a,k}.$$  

More generally, if $M$ is a manifold we denote by $\mathcal{P}^{a,k}(M, \mathbb{R}^n)$ the set of functions $f \in C^\infty(M \times (\mathbb{R}^n \setminus \{0\}))$ such that for each $x \in M$ we have $f(x, \cdot) \in \mathcal{P}^{a,k}$.

Lemma 2.2 Each $f \in \mathcal{P}^{a,k}$ has a unique representation

$$f(x) = \sum_{j=0}^{k} f_j(x) \log^j |x|$$

with $f_j \in \mathcal{P}^{a,0}$.

In the sequel for $f \in \mathcal{P}^{a,k}$ the coefficient of $\log^j |x|$ will always be denoted by $f_j$. 

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Proof The uniqueness is clear. It suffices to prove the existence for

\[ f(x) = g(x) \log^l h(x) \]

with \( g \in \mathcal{P}^{a,0}, h \in \mathcal{P}^{b,0}, l \leq k \). Then we write

\[ h(x) = |x|^b h(x/|x|) =: |x|^b h_1(x), \]

where \( h_1 \in \mathcal{P}^{0,0} \). Thus \( \log h_1 \in \mathcal{P}^{0,0} \) and hence

\[ g(x) \log^l h(x) = \sum_{j=0}^{l} \binom{l}{j} g(x) (\log^{l-j} h_1(x)) b^j \log^j |x|. \]

An immediate consequence of Lemma 2.2 is the inclusion

\[ \mathcal{P}^{a,k} \mathcal{P}^{b,l} \subset \mathcal{P}^{a+b,k+l}, \tag{2.3} \]

i.e. \( \mathcal{P} \) is a bigraded algebra.

There is an analogue of Euler’s theorem for log–polyhomogeneous functions. Namely, consider

\[ f(x) = g(x) \log^l |x|, \quad g \in \mathcal{P}^{a,0}. \]

Then

\[ \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} x_j = \frac{d}{dt} |x|^l f(tx) = a f(x) + l g(x) \log^{l-1} |x|, \tag{2.4} \]

from which one derives the identity

\[ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (x_j F(x)) = f(x), \quad \text{if} \quad a \neq -n, \tag{2.5} \]

where

\[ F(x) = g(x) \sum_{j=0}^{l} \frac{(-1)^{l-j}!}{j!(n+a)^{l-j+1}} \log^j |x|, \quad a \neq -n. \tag{2.6} \]

Thus we have proved

**Lemma 2.3** If \( a \neq -n \) then for \( f \in \mathcal{P}^{a,k} \) there exists \( F \in \mathcal{P}^{a,k} \) such that

\[ f = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (x_j F). \]

We turn to the case \( a = -n \) which is slightly more subtle. We denote by

\[ \Delta := -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \tag{2.7} \]

the (positive) Laplacian in \( \mathbb{R}^n \).
Definition 2.4 Let $f \in \mathcal{P}^{-n,k}$,

$$f(x) = \sum_{j=0}^{k} f_j(x) \log^j |x|.$$ 

Then we put

$$\text{res}_j(f) := \int_{S^{n-1}} f_j(x) d\text{vol}_S(x),$$

where $\text{vol}_S$ denotes the volume form with respect to the standard metric on $S^{n-1}$.

Lemma 2.5 Let

$$f(x) = \sum_{j=0}^{k} f_j(x) \log^j |x| \in \mathcal{P}^{-n,k}. \quad (2.8)$$

If $n \neq 2$ then there exists a $F \in \mathcal{P}^{2-n,k}$ with $\Delta F = f$ if and only if $\text{res}_k(f) = 0$.

If $n = 2$ then there exists a $F \in \mathcal{P}^{0,k}$ with $\Delta F = f$ if and only if $\text{res}_k(f) = \text{res}_{k-1}(f) = 0$.

Proof We identify $\mathbb{R}^n \setminus \{0\}$ with $\mathbb{R}_+ \times S^{n-1}$ via

$$\mathbb{R}_+ \times S^{n-1} \to \mathbb{R}^n \setminus \{0\}, \quad (r, \theta) \mapsto r\theta.$$ 

Then we have

$$f(r\theta) = \sum_{j=0}^{k} g_j(\theta) r^{-n} \log^j r.$$ 

The Laplacian is given by

$$\Delta = -r^{1-n} \frac{\partial}{\partial r} (r^{n-1} \frac{\partial}{\partial r}) + r^{-2} \Delta_S.$$ 

For $F$ we make an Ansatz

$$F(r\theta) = \sum_{j=0}^{k} F_j(\theta) r^{2-n} \log^j r$$

and applying $\Delta$ to $F$ we obtain the system of equations

$$\Delta_S F_k = g_k, \quad (2.9a)$$

$$\Delta_S F_{k-1} = g_{k-1} - k(n-2)F_k, \quad (2.9b)$$

$$\Delta_S F_j = g_j - (j+1)(n-2)F_{j+1} + (j+1)(j+2)F_{j+2}, \quad j \leq k-2. \quad (2.9c)$$

The equation $\Delta_S F_k = g_k$ has a solution if and only if $g_k$ is orthogonal to $\ker \Delta_S$ which consists of the constants. Hence the first equation has a solution if and only if $\text{res}_k(f) = 0$. If $n = 2$ by the same reasoning the second equation has a solution if and only if $\text{res}_{k-1}(f) = 0$. Hence we have proved the ’only if’ part of the assertion.
Now let \( n \neq 2 \) and assume \( \text{res}_k(f) = 0 \). Let \( F^0_k \) be the unique solution of (2.9a) with \( \int_{S^{n-1}} F^0_k = 0 \). Put

\[
F_k := F^0_k + \frac{1}{k(n-2)} \int_{S^{n-1}} g_{k-1}(x) d\text{vol}_S(x).
\]

Then \( \Delta F_k = g_k \) and

\[
k(n-2) \int_{S^{n-1}} F_k(x) d\text{vol}_S(x) = \int_{S^{n-1}} g_{k-1}(x) d\text{vol}_S(x).
\]

Hence there exists a unique \( F^0_{k-1} \) with \( \Delta F^0_{k-1} = g_{k-1} - k(n-2)F_k \). Proceeding in this way we obtain a solution of (2.9a)–(2.9c).

The case \( n = 2 \) is treated similar. \( \square \)

**Lemma 2.6** For \( f \in \mathcal{P}^{1-n,k} \) we have \( \text{res}_k(\partial f/\partial x_j) = 0 \).

See [7, Lemma 1.2] for another proof of this lemma in the case \( k = 0 \).

**Proof** Let

\[
f(x) = \sum_{i=0}^k f_i(x) \log^i |x|, \quad f_i \in \mathcal{P}^{1-n,0}.
\]

Then

\[
\frac{\partial f}{\partial x_j}(x) = \sum_{i=0}^k \frac{\partial f_i}{\partial x_j}(x) \log^i |x| + \sum_{i=0}^{k-1} \frac{x_j f_{i+1}}{|x|^2} (i+1) \log^i |x|,
\]

hence

\[
\text{res}_k(\frac{\partial}{\partial x_j} f) = \int_{S^{n-1}} \frac{\partial f_k}{\partial x_j} d\text{vol}_S.
\]

We consider the vector field

\[
X_j(x) := (1 - x_j^2) \frac{\partial}{\partial x_j} - \sum_{i \neq j} x_j x_i \frac{\partial}{\partial x_i}.
\]

Obviously, \( X_j|_{S^{n-1}} \) is tangential to \( S^{n-1} \). One checks by direct calculation

\[
\text{div}_S X_j = (1 - n)x_j.
\]

On the other hand, by Euler’s identity,

\[
X_j f_k = \frac{\partial f_k}{\partial x_j} - x_j \sum_{i=1}^n x_i \frac{\partial f_k}{\partial x_i} = \frac{\partial f_k}{\partial x_j} + (n-1)x_j f_k,
\]

hence in view of (2.10)

\[
\int_{S^{n-1}} \frac{\partial f_k}{\partial x_j} d\text{vol}_S = \int_{S^{n-1}} X_j f_k + (1 - n)x_j f_k d\text{vol}_S = \int_{S^{n-1}} (-\text{div}_S(X_j) + (1 - n)x_j) f_k d\text{vol}_S = 0. \]
Proposition 2.7 Let \( f \in \mathcal{P}^{a,k} \). Then there exist \( f_j \in \mathcal{P}^{a,k} \) such that \( f = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j \) if and only if \( a \neq -n \) or \( \text{res}_k(f) = 0 \).

Proof The case \( a \neq -n \) follows from Lemma 2.3. Let \( a = -n \). If \( \text{res}_k(f) = \text{res}_{k-1}(f) = 0 \) then by Lemma 2.4 there exists \( F \in \mathcal{P}^{-n,k} \) such that \( f = \Delta F \) and we reach the conclusion in this case.

Thus it suffices to show that the function \(|x|^{-n} \log^{k-1} |x|\) is a sum of derivatives. But
\[
\sum_{j=1}^{n} \frac{\partial}{\partial x_j} (x_j |x|^{-n} \log^{k} |x|) = k |x|^{-n} \log^{k-1} |x|
\]
does the job.

We will also need the results of [7, (1.1)–(1.5), Lemma 1.1 and 1.3] in our context. We briefly summarize the facts. Let
\[
\sigma := \sum_{j=0}^{n} (-1)^{j+1} \xi_j d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \in \Omega^{n-1}(\mathbb{R}^n). \tag{2.11}
\]
\(\sigma|S^{n-1}\) is the volume form. Moreover, for \( f \in \mathcal{P}^{-n,0} \) the form \( f\sigma \) is closed by Euler’s theorem (2.4). Thus for any bounded domain \( D \subset \mathbb{R}^n \), \( 0 \in D \), with smooth boundary we have for \( f \in \mathcal{P}^{-n,0} \)
\[
\text{res}_k(f) = \int_{S^{n-1}} f_k(x) d\text{vol}_{S^{n-1}}(x) = \int_{S^{n-1}} f_k \sigma = \int_{\partial D} f_k \sigma. \tag{2.12}
\]
Moreover, if \( T \in \text{GL}(n, \mathbb{R}) \) then
\[
\text{res}_k(T^*f) = \int_{S^{n-1}} f_k \circ T \sigma = \text{sgn} (\det T) \int_{T(S^{n-1})} f_k T^{-1}\sigma
= \frac{1}{|\det T|} \int_{T(S^{n-1})} f_k \sigma = \frac{1}{|\det T|} \text{res}_k(f). \tag{2.13}
\]
Finally we note

Lemma 2.8 [7, Lemma 1.3] Let \( f \in \mathcal{P}^{a,k} \). Then \( \text{res}_k(\xi^\alpha \partial^\beta f) = 0 \) if \(|\beta| > |\alpha|\).

Proof This follows from Lemma 2.6 by induction since if \( \partial^\beta = \frac{\partial}{\partial \xi_j} \partial^\gamma \) then
\[
\xi^\alpha \partial^\beta f = \frac{\partial}{\partial \xi_j} (\xi^\alpha \partial^\gamma f) - \frac{\partial \xi^\alpha}{\partial \xi_j} \partial^\gamma f. \tag*{\blacksquare}
\]
2.2 log–polyhomogeneous functions on symplectic cones

In this section we study log–polyhomogeneous functions on an arbitrary symplectic cone. Our exposition parallels [10, Sec. 6].

Let $Y$ be a symplectic cone. This is a principal bundle

$$\pi : Y \to X$$

with structure group $\mathbb{R}_+$. Denote by $\varrho_a : Y \to Y$ the action of $a \in \mathbb{R}_+$. That $Y$ is a symplectic cone means that $Y$ is symplectic, with symplectic form $\omega$, and

$$\varrho_a^* \omega = a \omega.$$

We assume furthermore that $Y$ is connected and $X$ is compact.

The main example of course is the cotangent bundle with the zero section removed, $T^*M \setminus 0$, over a compact connected manifold $M$ of dimension $\dim M > 1$.

**Definition 2.9** A function $f \in C^\infty(Y)$ is called log–polyhomogeneous of degree $(a,k)$ if

$$f = \sum_{j=0}^l g_j P_j(\log h_j)$$

with $g_j, h_j \in C^\infty(Y), P_j \in \mathbb{C}[t]$, where $g_j$ is homogeneous of degree $a$, $h_j$ is homogeneous of degree $b_j$, and $h_j > 0$ everywhere. Furthermore, $\deg P_j \leq k$.

Again, we denote the set of all log–polyhomogeneous functions of degree $(a,k)$ by $\mathcal{P}^{a,k}$. Then (2.2), (2.3) hold similarly.

We fix, for once and for all, $p \in \mathcal{P}^{1,0}$ such that $p$ is everywhere positive. Then by Euler’s identity we have $dp \neq 0$ everywhere. We put

$$Z := \{y \in Y | p(y) = 1\}.$$

$p$ plays the role of $| \cdot |$ and $Z$ plays the role of $S^{n-1}$ in the preceding section.

**Lemma 2.10** Each $f \in \mathcal{P}^{a,k}$ has a representation

$$f = \sum_{j=0}^k f_j \log^j p.$$

Furthermore, $f_k$ is independent of the choice of $p$.

**Proof** Consider

$$g \log^l h$$

with $g \in \mathcal{P}^{a,0}, h \in \mathcal{P}^{b,0}$. Then $h_1 := p^{-b}h$ is $0$–homogeneous and positive, hence $\log h_1 \in \mathcal{P}^{0,0}$. Thus

$$g \log^l h = \sum_{j=0}^l \binom{l}{j} g(x)(\log^{l-j} h_1)b^j \log^j p,$$

from which we see that the coefficient of $\log^l p$ is independent of $p$. \qed
Definition 2.11 We put
\[ \text{res}_k(f) := \text{Res}_{f_k}, \]
where \( \text{Res}_{f_k} \) is the symplectic residue of Guillemin. By the preceding Lemma \( \text{res}_k \) is well defined.

For the convenience of the reader and since we have to introduce some notation anyway we briefly recall the definition of the symplectic residue (cf. [10, Sec. 6]).

Via \( \Phi_t := \varrho e^t \) we obtain a one parameter group of diffeomorphisms of \( Y \). Let \( \Xi \in C^\infty(TY) \) be the infinitesimal generator of this group. Put \( \alpha := i \Xi \omega \in \Omega^1(Y) \), and let
\[ \mu := \alpha \wedge \omega^{n-1}. \]
If \( f \in \mathcal{P}^{-n,0} \) then the form \( f \mu \) is horizontal and invariant, hence there is a unique \((2n-1)\)-form \( \mu_f \) such that \( f \mu = \pi^* \mu_f \). Then
\[ \text{Res}(f) := \int_X \mu_f. \]
One can show (cf. [10, Proof of Lemma 6.3]) that also
\[ \text{Res}(f) = \int_Z f \mu. \]

We denote by \( \{ \cdot, \cdot \} \) the Poisson bracket associated with the symplectic structure.

Lemma 2.12 If \( f \in \mathcal{P}^{a,k}, g \in \mathcal{P}^{1,l} \) then \( \{ f, g \} \in \mathcal{P}^{a,k+l} \) and \( \text{res}_{k+l} \{ f, g \} = 0 \).

Proof W.l.o.g. we may assume
\[ f = \phi \log^k p, \quad g = \psi \log^l p, \]
with \( \phi \in \mathcal{P}^{a,0}, \psi \in \mathcal{P}^{1,0} \). Then
\[ \{ f, g \} = \{ \phi, \psi \} \log^{k+l} p + k \{ p, \psi \} \phi p^{-1} \log^{k+l-1} p \]
\[ + \{ \phi, p \} l \psi p^{-1} \log^{k+l-1} p, \]
hence \( \{ f, g \} \in \mathcal{P}^{a,k+l} \) and \( \text{res}_{k+l} \{ f, g \} = \text{Res} \{ \phi, \psi \} = 0 \) by [10, Prop. 6.1].

Now we can state the generalization of [10, Thm. 6.2] to log–polyhomogeneous functions.

Theorem 2.13 1. If \( a \neq -n \) then \( \{ \mathcal{P}^{1,0}, \mathcal{P}^{a,k} \} = \mathcal{P}^{a,k} \).

2. If \( a = -n \) then \( \{ \mathcal{P}^{1,0}, \mathcal{P}^{a,k} \} = \ker \text{res}_k \subset \mathcal{P}^{a,k} \).

Proof We follow the proof of loc. cit. and choose functions \( g_1, \ldots, g_N \in \mathcal{P}^{1,0} \) such that their differentials span the cotangent space of \( Y \) at every point. Let
\[ D_i : \mathcal{P}^{a,k} \to \mathcal{P}^{a,k}, \quad D_i f := \{ g_i, f \}. \]
We introduce a pre–Hilbert space structure on $P^{a,k}$ as follows: we identify $P^{a,k}$ with $C^\infty(Z, \mathbb{C}^{k+1})$ via
\[
\sum_{j=0}^{k} f_j \log^j p \mapsto (f_0, \ldots, f_k)|Z.
\] (2.14)

Next let $\nu$ be the restriction of $\mu$ to $Z$. This is a volume form and hence it defines a $L^2$–structure on $C^\infty(Z, \mathbb{C}^{k+1}) \simeq P^{a,k}$.

Now consider $f \log^j p, f \in P^{a,0}$. Then
\[
D_i(f \log^j p) = (D_i f) \log^j p + jfp^{-1}(D_i p) \log^{j-1} p.
\]
Thus putting
\[
q_i := p^{-1}D_i p \in P^{0,0}
\]
and $d_i := D_i|P^{a,0}$ the identification (2.14) transforms $D_i$ into
\[
D_i \cong \begin{pmatrix}
    d_i & q_i & 0 & \ldots & 0 \\
    0 & d_i & 2q_i & \ldots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & \ldots & 0 & d_i & kq_i \\
\end{pmatrix} : C^\infty(Z, \mathbb{C}^{k+1}) \to C^\infty(Z, \mathbb{C}^{k+1}).
\] (2.15)

Now we consider
\[
\Delta := \sum_{j=1}^{N} D_i D_i^t.
\] (2.16)
$\Delta$ is a self–adjoint elliptic differential operator on $C^\infty(Z, \mathbb{C}^{k+1})$. Hence
\[
\text{Im } \Delta^\perp = \ker \Delta.
\]

Therefore, consider
\[
f = \sum_{j=0}^{k} f_j \log^j p \in \ker \Delta.
\]
In view of (2.13)
\[
d_i^t f_j + jq_i f_{j-1} = 0, \quad i = 1, \ldots, N; \quad j = 0, \ldots, k,
\]
where we have put $f_{-1} := 0$.

By [10, (6.15)] we have
\[
d_i^t = -d_i + (2a + n)q_i.
\]
Abbreviating $r := -(n+2a)$ we obtain
\[
-\{g_i, f_j\} - rp^{-1}f_j \{g_i, p\} + jfp^{-1}\{g_i, p\} = 0,
\]
or
\[
\{g_i, p^r f_j\} = jfp^{-1}p^{-1}\{g_i, p\}.
\] (2.17)
Since the differentials of the $g_i$ span the cotangent space at every point we conclude that $p^r f_0 = c_0$ is constant.
By induction we assume that $p^r f_j' = 0$ for $j' < j - 1$ and $p^r f_{j-1} = c_{j-1}$ is constant. Then, in view of (2.17)
\[ \{g_i, p^r f_j\} = j c_{j-1} \{g_i, \log p\}, \]
thus
\[ \{g_i, p^r f_j - j c_{j-1} \log p\} = 0, \]
(c.f. [10, Lemma 6.6]). Again since the differentials of the $g_i$ span the cotangent space this implies
\[ f_j = c_j p^{-r} + j c_{j-1} p^{-r} \log p. \]
But since $f_j \in P^{a,0}$ the constant $c_{j-1}$ must be 0.

Summing up we have proved
\[ f_j = 0, \quad j < k, \quad f_k = c p^{-r}. \]
This implies $c = 0$ or $r = -a$. Since $r = -(n + 2a)$ this is equivalent to $c = 0$ or $r = n$.
Since $\int_Z p^{-r} \nu \neq 0$ we reach the conclusion. $\square$

3. Pseudodifferential operators with log–polyhomogeneous symbols

Let $M$ be a smooth manifold of dimension $n$. We denote by $L^*(M)$ the algebra of pseudodifferential operators with complete symbols of Hörmander type $(1,0)$. I.e. if $U$ is an open chart then $A \in L^m(U)$ can be written
\[ Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_U a(x, y, \xi) u(y) e^{i<x-y,\xi>} dyd\xi, \quad (3.1) \]
where $a \in C^\infty(U \times U \times \mathbb{R}^n)$ satisfies
\[ |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha,\beta,\gamma,K}(1 + |\xi|)^{m-|\gamma|}, \quad (3.2) \]
for $\xi \in \mathbb{R}^n, (x, y) \in K$, and $K \subset U$ compact. The class of these symbols is as usual denoted by $S^m(U \times U, \mathbb{R}^n)$.

We denote by $L^*(M, E)$ the algebra of pseudodifferential operators acting on sections of the $C^\infty$ vector bundle $E$.

We are now going to introduce a subclass of $L^*(M)$ which generalizes the classical pseudodifferential operators.

We proceed locally and introduce log–polyhomogeneous symbols:

**Definition 3.1** For an open set $U \subset \mathbb{R}^n$ we denote by $CS^{m,k}(U, \mathbb{R}^n)$ the set of symbols $a \in \cap_{\varepsilon > 0} S^{m+\varepsilon}(U, \mathbb{R}^n)$ having an asymptotic expansion
\[ a \sim \sum_{j=0}^{\infty} \psi(\xi) a_{m-j}(x, \xi) \]
where $a_{m-j} \in P^{m-j,k}(U, \mathbb{R}^n)$ and $\psi \in C^\infty(\mathbb{R}^n)$ with $\psi(\xi) = 0$ for $|\xi| \leq 1/4$ and $\psi(\xi) = 1$ for $|\xi| \geq 1/2$. 
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Remark 3.2 1. $\text{CS}^{m,0}(U, \mathbb{R}^n)$ are just the classical (1-step polyhomogeneous) symbols.
2. One could more generally consider symbols having expansions
   \[ a \sim \sum_{j=0}^{\infty} a_j, \quad (3.3) \]
   where $a_j \in \mathcal{D}^{m_j,k}(U, \mathbb{R}^n), \lim_{j \to \infty} m_j = -\infty$. These symbols were already considered by Schröhe \([10]\) who constructed the complex powers for the corresponding class of elliptic pseudodifferential operators.

   Our class of operators, which is more closely to classical operators, is large enough for our purposes. However, our results (with suitable modifications) remain true for the slightly more general symbols $a \sim \sum_{j=0}^{\infty} a_j, \quad (3.3)$.

Definition 3.3 We denote by $\text{CL}^{m,k}(U)$ the class of pseudodifferential operators which can be written in the form $a \sim \sum_{j=0}^{\infty} a_j, \quad (3.3)$ with $a \in \text{CS}^{m,k}(U \times U, \mathbb{R}^n)$.

It is fairly straightforward to check that this class of pseudodifferential operators satisfies the usual rules of calculus. We summarize the results for the convenience of the reader.

Proposition 3.4 1. If $A \in \text{CL}^{m,k}(U)$ is properly supported then the complete symbol $\sigma_A$ is in $\text{CS}^{m,k}(U, \mathbb{R}^n)$ and
   \[ \sigma_A \sim \sum_{\alpha \in \mathbb{Z}_+^n} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha a(x, y, \xi)|_{y=x} \]
   \[ \sim \sum_{j=0}^{\infty} \sum_{|\alpha|+l=j} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha a_{m-l}(x, y, \xi)|_{y=x}. \quad (3.4) \]

2. If $A \in \text{CL}^{m,k}(U), B \in \text{CL}^{m',k'}(U)$ are properly supported then $AB \in \text{CL}^{m+m',k+k'}(U)$ and
   \[ \sigma_{AB} \sim \sum_{\alpha \in \mathbb{Z}_+^n} \frac{i^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha \sigma_A(x, \xi)) \partial_x^\alpha \sigma_B(x, \xi) \]
   \[ \sim \sum_{j=0}^{\infty} \sum_{|\alpha|+l'+l=j} \frac{i^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha a_{m-l}(x, \xi)) \partial_x^\alpha b_{m'-l'}(x, \xi), \quad (3.5) \]
   if $\sigma_A \sim \sum a_{m-j}, \sigma_B \sim \sum b_{m'-j}$.

3. If $A \in \text{CL}^{m,k}(U)$ then $A^t$ in $\text{CL}^{m,k}(U)$ and
   \[ \sigma_{A^t} \sim \sum_{\alpha \in \mathbb{Z}_+^n} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha a^t(x, \xi) \]
   \[ \sim \sum_{j=0}^{\infty} \sum_{|\alpha|+l=j} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha a_{m-l}(x, \xi)^t. \quad (3.6) \]
Proposition 3.5 Let $\kappa : U \rightarrow V$ be a diffeomorphism and $A \in CL^{m,k}(U)$ properly supported. Then $\kappa_*A$ given by $(\kappa_*A)u := (A(u \circ \kappa)) \circ \kappa^{-1}$ is in $CL^{m,k}(V)$ and
\[
\sigma_{\kappa_*A}(y, \eta)|_{y=\kappa(x)} \sim \sum_{\alpha \in \mathbb{Z}^n_+} \frac{1}{\alpha!} (\partial^\alpha_\xi \sigma_A)(x, D\kappa^l(x)\eta) D^\alpha_x e^{i<\kappa''(z),\eta>}|_{z=x},
\]
where
\[
\kappa''(z) = \kappa(z) - \kappa(x) - D\kappa(x)(z - x).
\]
Furthermore,
\[
\Phi_\alpha(x, \eta) := D^\alpha_x e^{i<\kappa''(z),\eta>}|_{z=x}
\]
is a polynomial in $\eta$ of degree not higher than $|\alpha|/2$.

Proof Except the assertion $\kappa_*A \in CL^{m,k}(V)$ this is [18] Thm. 4.2]. $\kappa_*A \in CL^{m,k}(V)$ now follows from (3.7). \qed

As usual, this result allows to define operators of the class $CL^{m,k}$ on manifolds. For a smooth manifold $M$ we denote by $CL^{m,k}(M)$ the corresponding space of pseudodifferential operators. We note that (3.7) shows that the leading symbol of $A \in CL^{m,k}(M)$ can be considered as an element of $\mathcal{P}^{m,k}(T^*M)$. Because of its importance we single out this observation:

Proposition 3.6 The leading symbol which is locally defined by $\sigma_L^m(A) := a_m$ induces a surjective linear map
\[
\sigma_L : CL^{m,k}(M) \rightarrow \mathcal{P}^{m,k}(T^*M) \cong C^\infty(S^*M, \mathbb{C}^{k+1})
\]
with $\ker \sigma_L = CL^{m-1,k}(M)$. Furthermore, for $A \in CL^{a,k}(M), B \in CL^{b,l}(M)$ we have
\[
\sigma_L^{a+b}(AB) = \sigma_L^a(A)\sigma_L^b(B).
\]

Proof That $\sigma_L$ is well defined follows immediately from Proposition 3.3. Similar to 2.14 the isomorphism $\mathcal{P}^{m,k}(T^*M) \cong C^\infty(S^*M, \mathbb{C}^{k+1})$ is given by
\[
\sum_{j=0}^k f_j \log^j |.| \mapsto (f_0, \ldots, f_k)|S^*M,
\]
where $S^*M$ denotes the cosphere bundle of $M$. Surjectivity of $\sigma_L$ is proved by the standard construction of gluing together local pseudodifferential data. \qed

For defining the log–polyhomogeneous residue of an operator $A \in CL^{a,k}(M)$ we need the meromorphic continuation of the function $\text{Tr}(AP^{-s})$ for a classical elliptic pseudodifferential operator $P$. In principle, this could be done for any elliptic $P \in CL^{m,k}(M)$. The meromorphic continuation of $\text{Tr}(P^{-s}), P \in CL^{m,k}(M)$, was already proved in [13]. However, for treating $\text{Tr}(AP^{-s})$ one has to modify Sections 1 and 2 of [8]. Therefore, we decided to content ourselves to classical elliptic $P \in CL^{m,0}(M)$, which is enough for our purposes. Then the method of loc. cit. directly applies. Nevertheless, as mentioned on p. 488 of loc. cit. the results there could be generalized to operators of class $CL^{m,k}$.

First we state the expansion result for the resolvent (cf. [3, Thm. 2.7]). For the definition of ellipticity with parameter see loc. cit. Def. 2.7.
Theorem 3.7 Let $M$ be a compact manifold of dimension $n$, $E$ a $C^\infty$ vector bundle over $M$, and $A \in \text{CL}^{a,k}(M,E)$. Assume furthermore that $P \in \text{CL}^{m,0}(M,E)$ is elliptic with parameter $\mu \in \Gamma$. Then for $\lambda \in -\Gamma^m := \{-\mu^m | \mu \in \Gamma\}$ and $N \in \mathbb{N}$ with $-Nm + a < -n$ the kernel $(A(P - \lambda)^{-N})(x,y)$ of $A(P - \lambda)^{-N}$ satisfies on the diagonal

$$(A(P - \lambda)^{-N})(x,x) \sim \sum_{j=0}^{\infty} \sum_{l=0}^{k+1} c_{jl}(x)\lambda^{\frac{a+n-j}{m} - N} \log^j \lambda + \sum_{j=0}^{\infty} d_j(x)\lambda^{-j-N},$$

as $|\lambda| \to \infty$, uniformly in closed subsectors of $\Gamma$.

Furthermore, $c_{j,k+1} = 0$ if $\frac{j-a-n}{m} \notin \mathbb{Z}_+$.

Proof This is proved similar to [8, Thm 2.7]. For the convenience of the reader we indicate the steps of the proof of loc. cit. that have to be modified. During this proof we will use freely the notation of [8].

Let $\mathcal{P}$ be a pseudodifferential parametrix of $(P - \lambda)^{-N}$ as constructed in [8, p. 503]. Then

$$A(P - \lambda)^{-N} = A\mathcal{P}^N + \sum_{l \geq 1} A\mathcal{P}R_{(l)}$$

(3.8)

where $\mathcal{P}R_{(l)} \in \text{OP}(S^{-\infty,-(l+1)m})$ and thus also $A\mathcal{P}R_{(l)} \in \text{OP}(S^{-\infty,-(l+1)m})$. Since the symbol expansion of $A\mathcal{P}R_{(l)}$ depends on $\mu$ only as a rational function of $\lambda = -\mu^m$ the kernel of $A\mathcal{P}R_{(l)}$ has the asymptotic expansion

$$K_{A\mathcal{P}R_{(l)}}(x,x,\lambda) \sim_{\lambda \to \infty} \sum_{\sigma=0}^{\infty} c_{l,\sigma}(x)\lambda^{-l-N-\sigma}$$

(3.9)

([8, p. 504]).

To expand the kernel of $A\mathcal{P}^N$ we note that $A\mathcal{P}^N = \text{OP}(q)$ where $q \in S^{a+\varepsilon,-Nm,0} \cap S^{a+\varepsilon,-Nm}$ for every $\varepsilon > 0$. Note that although $A$ is not weakly polyhomogeneous in the sense of [8] we have $A \in \cap_{\varepsilon > 0} L^{a+\varepsilon}(M)$.

Now $q$ has an expansion $q \sim \sum_{j \geq 0} q_j$, where

$$q_j = \sum_{|\alpha|+l+l' = j} \frac{j-|\alpha|}{\alpha!} (\partial_x^\alpha a_{a-l}) \partial_x^{a-l'} b_{-mN-l'},$$

(3.10)

$a_{a-l} \in \mathcal{P}^{a-l,k}(T^*M), b_{-mN-l'}$ is $-mN-l'$-homogeneous in $(\xi, \mu)$ for $|\xi| \geq 1$.

The kernels of the remainders $r_j = q - \sum_{0 \leq j < J} q_j$ are expanded as in [8, (2.3)] which gives again integer powers $\lambda^{-N-1}$ in the expansion of the kernel. Here we have to note again that $b_{-mN-l'}(x,\xi,\mu)$ is a rational function in $-\mu^m$.

Picking one of the summands of $q_j$ we are finally facing the problem of expanding the integral

$$\int_{\mathbb{R}^n} a(x,\xi) \log^t |\xi| b(x,\xi,\mu) d\xi, \quad t \leq k,$$

(3.11)

where $a(x,\xi)$ is $(a-l-|\alpha|)$-homogeneous for $|\xi| \geq 1$ and $b(x,\xi,\mu)$ is $(-mN-l')$-homogeneous in $(\xi, \mu)$ for $|\xi| \geq 1$. Furthermore, $b(x,\xi,\mu)$ is a rational function of $-\mu^m$ with coefficients homogeneous in $\xi$. 

Now we split the integral (3.11) into the three pieces $|\xi| \geq |\mu|$, $1 \leq |\xi| \leq |\mu|$, $|\xi| \leq 1$ as in [8, (2.6)].

By homogeneity for $|\xi| \geq 1$ we find

$$\int_{|\xi| \geq 1} a(x, \xi) \log^t |\xi| b(x, \xi, \mu) d\xi$$

$$= |\mu|^{a+n+j-mN} \int_{|\xi| \geq 1} a(x, \xi) (\log |\xi\mu|)^t b(x, \xi, \frac{\mu}{|\mu|}) d\xi$$

$$= \sum_{\sigma=0}^t \left( \begin{array}{l} t \\ \sigma \end{array} \right) \int_{|\xi| \geq 1} a(x, \xi) \log^{t-\sigma} |\xi| b(x, \xi, \frac{\mu}{|\mu|}) d\xi |\mu|^{a+n+j-mN} \log^\sigma |\mu|.$$  

(3.12)

Since $b$ is holomorphic in $\mu$ by [8, Lemma 2.3] this is actually an expansion in terms of the functions $\mu^{a+n+j-mN} \log^\sigma \mu$.

Now we expand using [8, Theorem 1.12]

$$b(x, \xi, \mu) = \sum_{0 \leq \nu < M} b_\nu(x, \xi) \mu^{-m\nu-Nm} + R_M(x, \xi, \mu)$$  

(3.13)

where $b_\nu$ is homogeneous in $|\xi|$ for $|\xi| \geq 1$ and

$$R_M(x, \xi, \mu) = O( < \xi >^{M-l'} \mu^{-m(M+N)}),$$  

(3.14)

thus

$$\int_{|\xi| \leq 1} a(x, \xi) \log^t |\xi| b(x, \xi, \mu) d\xi$$

$$= \sum_{0 \leq \sigma < M} \mu^{-m\nu-Nm} \int_{|\xi| \leq 1} a(x, \xi) \log^t |\xi| b_\nu(x, \xi) d\xi + O(\mu^{-m(M+N)}).$$  

(3.15)

Furthermore,

$$\int_{1 \leq |\xi| \leq |\mu|} a(x, \xi) \log^t |\xi| b_\nu(x, \xi) d\xi \mu^{-m\nu-Nm}$$

$$= \mu^{-m\nu-Nm} c_\nu(x) \int_{|\mu|}^{||\mu||} r^{a-|a|+m\nu-l'-n-1} \log^t r dr$$

$$= \begin{cases} 
\sum_{\sigma=0}^t \frac{c_\nu(x)(\log |\mu|)\mu^{-m\nu-Nm}|\mu|^{a-j+m\nu+n}}{t+1}, & a-j+m\nu+n \neq 0, \\
\frac{c_\nu(x)(\log^t |\mu|)\mu^{-m\nu-Nm}}{t+1}, & a+m\nu+n-j = 0. 
\end{cases}$$  

(3.16)

Again invoking [8, Lemma 2.3] we end up with the desired expansion. The remainder term coming from $R_M$ is estimated exactly as in [8, p. 498]. (3.16) also shows that $c_{j,k+1} = 0$ if $\frac{l-a-n}{m} \notin \mathbb{Z}_+$. 

As a consequence of Theorem 3.7 we have an asymptotic expansion

$$\text{Tr}(A(P - \lambda)^{-N}) \sim \sum_{j=0}^{\infty} \sum_{l=0}^{k+1} c_{jl} \lambda^{\frac{n+a-j-N}{m}} \log^l \lambda + \sum_{j=0}^{\infty} d_j \lambda^{-j-N},$$  

(3.17)
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where \( c_{j,k+1} = 0 \) if \( \frac{j-a-n}{m} \notin \mathbb{Z}_+ \).

If the eigenvalues of the leading symbol of \( P \) lie in \( \text{Re } \lambda > 0 \) then via the appropriate Cauchy integral we obtain the heat expansion

\[
\text{Tr}(e^{-tP}) \sim_{t \to 0+} \sum_{j=0}^{\infty} t^{\frac{j-a-n}{m}} \tilde{c}_j(\log t) + \sum_{j=0}^{\infty} \tilde{d}_j t^j
\]

with a polynomial \( \tilde{c}_j \in \mathbb{C}[x] \) of degree

\[
\deg \tilde{c}_j \leq \begin{cases} 
  k, & \frac{j-a-n}{m} \notin \mathbb{Z}_+, \\
  k+1, & \frac{j-a-n}{m} \in \mathbb{Z}_+. 
\end{cases}
\]

Furthermore, the generalized \( \zeta \)–function

\[
\text{Tr}(AP^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-tP}) dt
\]

has a meromorphic continuation to \( \mathbb{C} \) with poles in \( \{ \frac{a+n-j}{m} | j \in \mathbb{Z}_+ \} \) of order \( k+1 \). Cf. e.g. [3, Lemma 2.1], [9], [15, Sec 2.1], for a discussion of the relation between the expansions (3.17), (3.18) and the poles of (3.20).

Using the notation \( f(s) =: \sum_k \text{Res}_k f(s_0)(s - s_0)^{-k} \) for Laurent expansions we note explicitly

\[
\text{Res}_{k+1} \text{Tr}(AP^{-s})|_{s = \frac{a+n-j}{m}} = \frac{(-1)^k}{\Gamma\left(\frac{a+n-j}{m} + 1\right)} \frac{d^k}{dt^k} \tilde{c}_j(t)|_{t=0}, \quad \frac{j-a-n}{m} \notin \mathbb{Z}_+, \quad (3.21)
\]

\[
\text{Res}_{k+1} \text{Tr}(AP^{-s})|_{s = -j} = (-1)^{k+j+1} j! \frac{d^{k+1}}{dt^{k+1}} \tilde{c}_{a+n+mj}(t)|_{t=0}, \quad \frac{j-a-n}{m} \in \mathbb{Z}_+. \quad (3.22)
\]

4. The log–polyhomogeneous noncommutative residue

In this section we consider a compact closed manifold \( M \) of dimension \( n \). Let \( E \) be a \( C^\infty \) vector bundle over \( M \) and \( A \in \text{CL}^{a,k}(M,E) \) a pseudodifferential operator with log–polyhomogeneous symbol.

We pick an elliptic classical pseudodifferential operator \( P \in \text{CL}^{m,0}(M,E) \) of order \( m > 0 \) whose leading symbol is scalar and positive. For instance we could take a generalized Laplace operator for \( P \). The assumption that the leading symbol of \( P \) is scalar guarantees that the commutator \([P,A]\) lies in \( \text{CL}^{a+m-1,k}(M,E) \), which makes life easier in the sequel. We emphasize however that this assumption is not really important.

We put

\[
\nabla^0_P A := A, \quad \nabla^{j+1}_P A := [P, \nabla^j A],
\]

and by induction we have

\[
\nabla^j_P A \in \text{CL}^{a+j(m-1),k}(M,E).
\]
Lemma 4.2 \(\) If \(\text{Definition 4.1}\) We put
\[
\text{Proposition 4.3} \quad \left(\begin{array}{l}
\text{Lemma 4.2} \quad \text{If} \quad \text{Definition 4.1} \quad \text{We put} \\
\text{Proposition 4.3} \end{array}\right)
\]

\[\begin{align*}
\Delta_N &: = \{ (t_1, \ldots, t_N) \in \mathbb{R}^N \mid 0 \leq t_1 \leq \cdots \leq t_N \leq 1 \}. \quad (4.3)
\end{align*}\]

Then we have the well–known formula\[\]
\[
e^{-tP}A = \sum_{j=0}^{N-1} (-t)^j j! (\nabla_P^j A) e^{-tP} + R_N(A, P, t), \quad (4.4)
\]

where
\[
R_N(A, P, t) = (-1)^N \int_{t\Delta_N} e^{-t_1 P} (\nabla_P^N A) e^{-(t-t_1)P} dt_1 \cdots dt_N \\
= \frac{(-t)^N}{(N-1)!} \int_0^1 (1-s)^{N-1} e^{-stP} (\nabla_P^N A) e^{-(1-s)tP} ds. \quad (4.5)
\]

Lemma 4.2 \(\) If \(p, \varepsilon, N > 0\) such that \((N-a)/m - p - \varepsilon > 0\) then we have the estimate
\[
\|R_N(A, P, t)(P + c)^p\| \leq Ct^{\frac{N-a}{m} - p - \varepsilon}, \quad 0 < t \leq 1.
\]

Here \(c > 0\) is any constant such that \(P + c\) is invertible.

Proof \(\) Since the leading symbol of \(P\) is positive the operator \(P + c\) is invertible for some \(c > 0\). \(\nabla_P^N A \in \text{CL}^{a+N(m-1), k}(M, E) \subset L^{a+N(m-1)+m\varepsilon}(M, E)\), hence \((P + c)^{\frac{N-a}{m} - N - \varepsilon} \nabla_P^N A\) is bounded and thus we obtain the estimate
\[
\|e^{-t_1 P} (\nabla_P^N A) e^{-(t-t_1)P} (P + c)^p\| \leq C\| (P + c)^{\frac{N-a}{m} + N + p + \varepsilon} e^{-(t-t_1)P}\| \\
\leq C(t - t_1)^{\frac{N-a}{m} - N - p - \varepsilon}, \quad 0 \leq t_1 < t \leq 1.
\]

Integrating this inequality gives the desired estimate. \(\square\)

Proposition 4.3 \(\) Let \(A \in \text{CL}^{a,k}(M, E)\) and let \(P \in \text{CL}^{m,0}(M, E)\) be a classical elliptic pseudodifferential operator of order \(m > 0\) whose leading symbol is scalar and positive. Then
\[
1. \quad \text{Res}_k(A, P^{\alpha}) = \text{Res}_k(A, P) \text{ for any } \alpha > 0.
\]

\(^1\)Actually it was not well–known to the author; he has learned the formula and the following lemma from Henri Moscovici
2. If $P_u$ is a smooth 1–parameter family of such operators then $\text{Res}_k(A, P_u)$ is independent of the parameter $u$.

3. If $B \in \text{CL}^{b,l}(M, E)$ then $\text{Res}_{k+l}([A, B], P) = 0$.

**Proof**

1. is straightforward.

2. In view of Lemma 4.2 we have for $N$ large enough

\[
\frac{d}{du} A e^{-tP_u} = - \int_0^t A e^{-(t-t_1)P_u} \left( \frac{d}{du} P_u \right) e^{-t_1P_u} dt_1
\]

\[
= \sum_{j=0}^{N-1} \frac{(-t)^{j+1}}{(j+1)!} (A \nabla_P^j \frac{d}{du} P_u) e^{-tP_u} + R_N(t),
\]

where

\[
\|R_N(t)\|_{tr} = O(t), \quad t \to 0.
\]

Here $\| \cdot \|_{tr}$ denotes the trace norm and this trace norm estimate follows from Lemma 4.2 since $(P + c)^{-p}$ is trace class if $p > \dim M/m$.

Thus we conclude

\[
\frac{d}{du} \text{Tr}(A e^{-tP_u}) = \sum_{j=0}^{N-1} \frac{(-t)^{j+1}}{(j+1)!} t^{j+1} \text{Tr}(A \nabla_P^j \frac{d}{du} P_u) e^{-tP_u} + O(t), \quad t \to 0.
\]

Since $\text{Tr}(A \nabla_P^j \frac{d}{du} P_u) e^{-tP_u}$ has no $t^{-j-1} \log^{k+1} t$ term in its asymptotic expansion (cf. (3.18)) we reach the conclusion.

3. We use again Lemma 4.2 and we find for $N$ large enough

\[
A e^{-tP} B = \sum_{j=0}^{N-1} \frac{(-t)^j}{j!} A (\nabla_P^j B) e^{-tP} + R_N(t),
\]

where $\|R_N(t)\|_{tr} = O(t), \quad t \to 0$. Thus

\[
\text{Tr}([A, B] e^{-tP}) = - \sum_{j=1}^{N-1} \frac{(-1)^j}{j!} t^j \text{Tr}(A (\nabla_P^j B) e^{-tP}) + O(t), \quad t \to 0.
\]

Since $A \nabla_P^j B \in \text{CL}^{a+b+j(m-1), k+l}(M, E)$ the highest log $t$ power occuring as coefficient of $t^{-j}$ in the asymptotic expansion of $\text{Tr}(A (\nabla_P^j B) e^{-tP})$ as $t \to 0$ is $\leq k + l$ and hence $\text{Res}_{k+l}([A, B], P) = 0$.

This proposition immediately implies:

**Theorem 4.4** For $A \in \text{CL}^{a,k}(M, E)$ choose $P \in \text{CL}^{m,0}(M, E)$ as in the preceding proposition. Then

\[
\text{Res}_k(A) := \text{Res}_k(A, P)
\]

is well–defined independent of the particular $P$ chosen. $\text{Res}_k$ is a linear functional on $\text{CL}^{a,k}(M, E)$ which vanishes on appropriate commutators. More precisely,

\[
\text{Res}_{k+l}([A, B]) = 0
\]

for $A \in \text{CL}^{a,k}(M, E), B \in \text{CL}^{b,l}(M, E)$.
4. The log–polyhomogeneous noncommutative residue

**Proof** Given $P_j \in \text{CL}^{m_j,0}(M, E)$, $j = 1, 2$, in view of 1. of the preceding proposition we may replace $P_j$ by $P_j^{1/m_j}$ to obtain operators of order 1. Then $P_u := uP_1 + (1-u)P_2$, $0 \leq u \leq 1$, is a smooth 1–parameter family to which 2. of the preceding proposition applies. Hence $\text{Res}_k(A, P_1) = \text{Res}_k(A, P_2)$. (4.11) now follows from 3. of loc. cit. $\Box$

We note that $\text{Res}_0$ coincides with the noncommutative residue of Wodzicki. This follows directly from the definition and [12, Thm. 1.4].

Although (4.11) holds $\text{Res}_k$ is not a trace on the full algebra $\text{CL}^*\ast(M, E)$. We will see in Theorem 4.11 below that this algebra does not have any nontrivial traces. However, as we will see at the end of this section the sequence of functionals $(\text{Res}_k)_{k \in \mathbb{Z}^+}$ corresponds to a trace on a graded algebra constructed from the filtration $(\text{CL}^*_{k})_{k \in \mathbb{Z}^+}$.

Note furthermore that $\text{Res}_k(A) = 0$ if $\text{ord}(A) < -\dim M$ because then $A$ is of trace class and hence $\text{Tr}(AP^{-s})$ is regular at $s = 0$.

4.1 The residue density

**Proposition 4.5** (cf. [20, Prop. 3.2]) Let $M$ be a (not necessarily compact) smooth manifold. For $A \in \text{CL}^{m,k}(M, E, F)$ there exists a density

$$\omega_k(A) \in C^\infty(M, \text{Hom}(E, F) \otimes |\Omega|)$$

with the following properties:

1. $A \mapsto \omega_k(A)$ is $\mathbb{C}$–linear.

2. If $\varphi \in C_0^\infty(M)$ then $\omega_k(\varphi A) = \varphi \omega_k(A)$.

3. If $\kappa : U \to V \subset \mathbb{R}^n$ is a local chart then

$$\kappa^\ast(\omega_k(\kappa \ast A)) = \omega_k(A).$$

4. In a local coordinate chart we have

$$\omega_k(A)(x) = \frac{(k + 1)!}{(2\pi)^n} \text{res}_k(a_{-n}(x, \cdot))|dx|$$

$$= \frac{(k + 1)!}{(2\pi)^n} \left( \int_{||\xi||=1} a_{-n,k}(x, \xi)|d\xi| \right)|dx|. \quad (4.12)$$

5. If the complete symbol of $A \in \text{CL}^{m,k}(M, E, F)$ vanishes at $p \in M$ then $\omega_k(A)(p) = 0$. If $m < -\dim M$ then $\omega_k(A) = 0$.

6. Let $E = F$, $A \in \text{CL}^{a,k}(M, E)$, $B \in \text{CL}^{b,l}(M, E)$. If $A$ or $B$ have compact support then

$$\int_M \text{tr}_{E_\xi}(\omega_{k+l}([A, B])(x)) = 0.$$

**Proof** We take (4.12) as the definition of $\omega_k(A)$. To show that $\omega_k(A)$ is well defined we proceed along the lines of [11, Thm. 1.4].
Let $U \subset \mathbb{R}^n$, $A \in \mathrm{CL}^{m,k}(U, E, F)$ and $\kappa : U \to V$ a diffeomorphism. Since this consideration is local we may assume $E, F$ to be trivial bundles. By Proposition 3.5 we have

$$\sigma_{\kappa \ast}(\kappa(x), \eta) \sim \sum_{\alpha \in \mathbb{Z}^n_+} (\partial^\alpha \sigma_A)(x, D\kappa^t(x)\eta)\Phi_\alpha(x, \eta),$$

where $\Phi_\alpha(x, \eta)$ is a polynomial in $\eta$ of degree not higher than $|\alpha|/2$ and $\Phi_0 = 1$. Hence in view of (2.12), (2.13), and Lemma 2.8

$$\text{res}_k[\sigma_{\kappa \ast}(\kappa(x), \eta) - n]\left|d\kappa(x)\right| = \left|\det D\kappa(x)\right|\text{res}_k[\sigma_A(x, D\kappa^t(x)\eta) - n]\left|d\eta\right| dx \in \text{res}_k[\sigma_A(x, \eta) - n]|dx|.$$

This proves 3. and 4. Furthermore, 1., 2., and 5. are simple consequences of 4.

To prove 6., using a partition of unity we may assume that $M = U$ is a coordinate chart and $A$ has compact support. Now we proceed exactly as in the proof of [8, Thm. 1.4]: denote by $a, b$ the complete symbols of $A, B$. We obtain from the symbol expansion of a product Proposition 3.4

$$\frac{(2\pi)^n}{(k + 1)!} \text{tr}_{E_x} \omega_{k+1}([A, B])(x)$$

$$= \int_{|\xi| = 1} \left( \sum_{\alpha \in \mathbb{Z}^n_+} \frac{i^{-|\alpha|}}{\alpha!} \text{tr}_{E_x} \left( \partial^\alpha_\xi a \partial^\alpha_x b - \partial^\alpha_x b \partial^\alpha_\xi a \right) \right)_{-n,k}\left|d\xi\right| dx$$

$$= \int_{|\xi| = 1} \left( \sum_{\alpha \in \mathbb{Z}^n_+} \frac{i^{-|\alpha|}}{\alpha!} \text{tr}_{E_x} \left( \partial^\alpha_\xi a \partial^\alpha_x b - \partial^\alpha_x a \partial^\alpha_\xi b \right) \right)_{-n,k}\left|d\xi\right| dx.$$

Now, as noted in loc. cit.

$$\partial^\alpha_\xi a \partial^\alpha_x b - \partial^\alpha_x a \partial^\alpha_\xi b$$

(4.13)

can be written as a sum of derivatives

$$\sum_{j=1}^n \partial^j_\xi A_j + \partial^j_x B_j,$$

(4.14)

where $A_j, B_j$ are bilinear expressions in $a, b$ and their derivatives and $a$ or one of its derivatives occurs in every summand. Hence, since $a$ has compact support the $A_j, B_j$ have compact support. The assertion now follows from Lemma 2.8.

\[\square\]

**Lemma 4.6** Let $M$ be a compact Riemannian manifold, $\dim M = n$. Furthermore, let $Q \in \mathrm{CL}^{-n,k}(M, E)$ with leading symbol $q_n(x, \xi) = |\xi|^{-n} \log^k |\xi|$. Then

$$\text{Res}_k(Q) = \int_M \omega_k(Q) \neq 0.$$
Proof  This lemma could be derived from the proof of [8, Thm 2.1]. But since our situation is much simpler we will give an ad hoc proof.

Note first that in a coordinate system

\[
\omega_k(Q)(x) = \frac{(k + 1)!}{(2\pi)^n} \text{vol}(S^{n-1}) d\text{vol}_M(x)
\]

and thus

\[
\int_M \omega_k(Q) = \frac{(k + 1)!}{(2\pi)^n} \text{vol}(S^{n-1}) \text{vol}(M) \neq 0.
\]

We pick an elliptic operator \( P \in \text{CL}^{m,0}(M, E), m > 0 \), whose leading symbol \( p(x, \xi) \)

is scalar and positive.

Then since \( Q \) has order \(-n\) the coefficient of \( \lambda^{-N} \log^{k+1} \lambda \) in the asymptotic expansion of \( \text{Tr}(A(P - \lambda)^{-N}) \) equals the corresponding coefficient of

\[
(2\pi)^{-n} \int_{T^*M} \varphi(|\xi|)|\xi|^{-n} \log^k |\xi|(p(x, \xi) - \lambda)^{-N} d\xi dx.
\]

Here, \( \varphi \in C^\infty(\mathbb{R}) \) is a function with \( \varphi(t) = 0 \), if \( t \leq 1/2 \) and \( \varphi(t) = 1 \), if \( t \geq 1 \).

Now

\[
\int_{\mathbb{R}^n} \varphi(|\xi|)|\xi|^{-n} \log^k |\xi|(p(x, \xi) - \lambda)^{-N} d\xi
\]

\[
= \lambda^{-N} \int_{\mathbb{R}^n} \varphi(|\lambda^{1/m} \xi|)|\xi|^{-n} \log^k |\lambda^{1/m} \xi|(p(x, \xi) - 1)^{-N} d\xi
\]

\[
= \lambda^{-N} \int_{|\xi| \leq 1} \varphi(|\lambda^{1/m} \xi|)|\xi|^{-n} \log^k |\lambda^{1/m} \xi|(p(x, \xi) - 1)^{-N} d\xi + O(\lambda^{-N} \log^k \lambda)
\]

\[
= \lambda^{-N} \int_0^1 \int_{|\xi| = 1} \varphi(|\lambda^{1/m} \varphi|) \varphi^{-1} \log^k |\lambda^{1/m} \varphi|(p(x, \varphi \xi) - 1)^{-N} d\xi d\varphi + O(\lambda^{-N} \log^k \lambda)
\]

\[
= (-1)^N \lambda^{-N} \log^k \lambda^{1/m} \text{vol}(S^{n-1}) \int_0^1 \varphi(|\lambda^{1/m} \varphi|) \varphi^{-1} d\varphi + O(\lambda^{-N} \log^k \lambda)
\]

\[
= \frac{(-1)^N}{m^{k+1}} \text{vol}(S^{n-1}) \lambda^{-N} \log^{k+1} \lambda + O(\lambda^{-N} \log^k \lambda)
\]

and hence

\[
(2\pi)^{-n} \int_{T^*M} \varphi(|\xi|)|\xi|^{-n} \log^k |\xi|(p(x, \xi) - \lambda)^{-N} d\xi dx
\]

\[
= \left( \frac{(-1)^N}{m^{k+1}(k + 1)!} \right) \int_M \omega_k(Q) \lambda^{-N} \log^{k+1} \lambda + O(\lambda^{-N} \log^k \lambda)
\]

which proves the assertion in view of Definition [1.1].

Proposition 4.7 Let \( M \) be a connected compact manifold of dimension \( n > 1 \). Choose a \( Q \in \text{CL}^{-n,k}(M) \) with \( \text{Res}_k(Q) = \int_M \omega_k(Q) \neq 0 \). Then there exist \( P_1, \ldots, P_N \in \text{CL}^{1,0}(M) \) such that for any \( A \in \text{CL}^{n,k}(M) \) there exist \( Q_1, \ldots, Q_N \in \text{CL}^{n,k}(M) \) such that

\[
A - \sum_{j=1}^N [P_j, Q_j] - \frac{\int_M \omega_k(A)}{\text{Res}_k(Q)} Q \in L^{-\infty}(M).
\]
4. The log–polyhomogeneous noncommutative residue

**Proof** We choose $P_1, \ldots, P_N \in \text{CL}^{1,0}(M)$ such that the differentials of the leading symbols span $T^*M \setminus 0$ at every point. We consider the leading symbol $\sigma^a_k(A) \in \mathcal{P}^{a,k}(T^*M)$ of $A$. If $a \neq -n$ then by Theorem 2.13 there exist $Q_j^{(1)}, \ldots, Q_j^{(N)} \in \text{CL}^{a,k}(M)$ such that

$$A - \sum_{j=1}^N [P_j, Q_j^{(1)}] \in \text{CL}^{a-1,k}(M).$$

(Note that the leading symbol of a commutator is the Poisson bracket of the leading symbols).

If $a - 1 \neq -n$ we iterate the procedure. Thus if $a \not\in \{l \in \mathbb{Z} | l \geq -n\}$ then by induction we find operators $Q_j^{(l)} \in \text{CL}^{a,k}(M)$ such that

$$A^{(l)} := A - \sum_{j=1}^N [P_j, Q_j^{(l)}] \in \text{CL}^{a-l,k}(M). \quad (4.15)$$

If $a \in \{l \in \mathbb{Z} | l \geq -n\}$ then (4.15) holds for $l \leq a + n$. In view of Proposition 4.5 we have

$$\int_M \omega_k(A^{(a+n)}) - \int_M \frac{\omega_k(A)}{\text{Res}_k(Q)} Q = 0.$$

Using again Theorem 2.13 we find by induction operators $Q_j^{(l)} \in \text{CL}^{a,k}(M)$ such that

$$A^{(l)} := A - \sum_{j=1}^N [P_j, Q_j^{(l)}] - \int_M \frac{\omega_k(A)}{\text{Res}_k(Q)} Q \in \text{CL}^{a-l,k}(M). \quad (4.16)$$

Now choose $Q_j \in \text{CL}^{a,k}(M)$ with $Q_j - Q_j^{(l)} \in \text{CL}^{a-l,k}(M)$. Then we reach the conclusion. ~

**Corollary 4.8** For $A \in \text{CL}^{a,k}(M, E)$ we have

$$\text{Res}_k(A) = \int_M \text{tr}_{E_x} \omega_k(A)(x) = \frac{(k + 1)!}{(2\pi)^n} \int_{S^*M} a_{-n,k}(x, \xi) |d\xi dx|.$$}

**Proof** For $E = \mathbb{C}$ this is an immediate consequence of Theorem 4.4, Proposition 4.3 and the preceding proposition. For arbitrary $E$ it follows from the facts that there exists a bundle $F$ making $E \oplus F$ trivial and $\text{CL}^{a,k}(M, \mathbb{C}^r) \cong \text{CL}^{a,k}(M) \otimes M_r(\mathbb{C})$. □

We note explicitly as a consequence of Proposition 4.7 that if $A \in \text{CL}^{a,k}(M)$ there are always $Q_j \in \text{CL}^{a,k+1}(M)$ such that

$$A - \sum_{j=1}^N [P_j, Q_j] \in L^{-\infty}(M).$$

This can actually be improved, using the fact that every classical pseudodifferential operator with vanishing residue is a sum of commutators. This result is due to Wodzicki [19], a generalization to algebras of Fourier integral operators is due to Guillemin [11]. In particular every smoothing operator is a sum of commutators.
of classical pseudodifferential operators. Since [19] is written in Russian we briefly sketch the argument: one actually has to show that $\text{CL}^*(M)/[\text{CL}^*(M), \text{CL}^*(M)]$ is one–dimensional. Given any trace $\tau$ on $\text{CL}^*(M)$ consider first $\tau|L^{-\infty}(M)$ By [11, Appendix] $L^{-\infty}(M)/[L^{-\infty}(M), L^{-\infty}(M)] \cong \mathbb{C}$ and it is spanned by the $L^2$–trace. Thus $\tau|L^{-\infty}(M) = c \text{Tr}_{L^2}$. The same argument as in the introduction (1.1) then shows $c = 0$. Hence $\tau$ induces a trace on $\text{CL}^*(M)/L^{-\infty}(M)$ and thus in view of Proposition 4.7 it is a constant multiple of the Wodzicki residue.

Summing up we can improve Proposition 4.7 as follows:

**Proposition 4.9** Let $M$ be a connected compact manifold of dimension $n > 1$. Choose a $Q \in \text{CL}^{-n,k}(M)$ with $\text{Res}_k(Q) = \int_M \omega_k(Q) \neq 0$. Then there exist $P_1, \ldots, P_N \in \text{CL}^{1,0}(M)$ such that for any $A \in \text{CL}^{a,k}(M)$ there exist $Q_1, \ldots, Q_N \in \text{CL}^{a,k}(M)$ and classical pseudodifferential operators $R_1, \ldots, R_N, S_1, \ldots, S_N \in \text{CL}^{*,0}(M)$ such that

$$A = \sum_{j=1}^N [P_j, Q_j] + \sum_{j=1}^N [R_j, S_j] + \frac{\int_M \omega_k(A)}{\text{Res}_k(Q)} Q.$$

In particular, since $\text{Res}_{k+1}(A) = 0$ there exist $Q_1, \ldots, Q_N \in \text{CL}^{a,k+1}(M)$ and $R_1, \ldots, R_N, S_1, \ldots, S_N \in \text{CL}^{*,0}(M)$ such that

$$A = \sum_{j=1}^N [P_j, Q_j] + \sum_{j=1}^N [R_j, S_j].$$

**Remark 4.10** This result generalizes the corresponding result for classical pseudodifferential operators due to Wodzicki [19, Prop. 5.4]. In [11], Wodzicki’s result was generalized to certain algebras of Fourier integral operators.

An immediate consequence is the

**Corollary 4.11** There are no nontrivial traces on the algebra $\text{CL}^{*,*}(M)$.

We close this section with two further remarks. First we note that our algebra $\text{CL}^{*,*}(M, E)$ provides examples of spectral triples with discrete dimension spectrum of infinite multiplicity as defined by Connes and Moscovici [6, Def. II.1]. The algebra involved in a spectral triple consists of bounded operators. Therefore, we put

$$\mathcal{A}(M, E) := \{ A \in \text{CL}^{0,*}(M, E) \mid \sigma_L(A) \in \mathcal{P}^{0,0} \}. \quad (4.17)$$

This is a subalgebra of $\text{CL}^{*,*}$ consisting of bounded operators. If $P \in \text{CL}^{1,0}(M, E)$ is a classical elliptic pseudodifferential operator whose leading symbol is scalar and positive then (3.20) shows that

$$(\mathcal{A}(M, E), L^2(M, E), P) \quad (4.18)$$

is a spectral triple with dimension spectrum $\mathcal{S}_d := \{ k \in \mathbb{Z} \mid k \leq \text{dim} M \}$ of infinite multiplicity.
5. The Kontsevich–Vishik trace

The assumption that the leading symbol is scalar does not apply to the Dirac operator. However, this assumption is needed only to guarantee that \([P, A]\) is of order 0. Thus, given a spin manifold, \(M\), with Dirac operator, \(D\), acting on \(C^\infty(M, S)\) we denote by \(A_{\text{scal}}(M, S)\) the algebra of operators \(A \in A(M, S)\) whose complete symbol is scalar. Having a scalar complete symbol is a coordinate invariant property. Hence

\[
(A_{\text{scal}}(M, S), L^2(M, S), D)
\]

is another spectral triple with dimension spectrum \(S_d := \{k \in \mathbb{Z} | k \leq \dim M\}\).

Our second remark concerns the fact that although \(\text{Res}_{k+l}([A, B]) = 0\) if \(A \in \text{CL}^{a,k}, B \in \text{CL}^{b,l}\), the functional \(\text{Res}_k\) is not a trace on the whole algebra \(\text{CL}^{*,*}\). To shed some light on this fact we consider an arbitrary filtered algebra

\[
\mathcal{B} := \bigcup_{k=0}^\infty \mathcal{B}^k, \quad \mathcal{B}^k \subset \mathcal{B}^{k+1}.
\]

From \(\mathcal{B}\) we can construct a graded algebra

\[
\mathcal{GB} := \bigoplus_{k=0}^\infty \mathcal{B}^k,
\]

where the product is given by

\[
((a_k)_{k \geq 0} \circ (b_k)_{k \geq 0})_m := \sum_{j=0}^m a_j b_{m-j}.
\]

Then it is straightforward to see that traces on \(\mathcal{GB}\) are in one–one correspondence to sequences of linear functionals \(\tau_k : \mathcal{B}^k \rightarrow \mathbb{C}\) satisfying

\[
\tau_{k+l}([A, B]) = 0
\]

for \(A \in \mathcal{B}^k, B \in \mathcal{B}^l\).

Thus Proposition 4.9 says that if \(M\) is connected, \(\dim M > 1\), then up to a scalar factor there is exactly one trace on the algebra \(\mathcal{G}\text{CL}^{*,*}(M)\) constructed from the filtration

\[
\text{CL}^{*,*}(M) = \bigcup_{k=0}^\infty \text{CL}^{*,k}(M).
\]

This fact was communicated to the author by R. Nest.

5. The Kontsevich–Vishik trace

In this section we briefly show that the analogue of the Kontsevich–Vishik [13, 14] trace exists on \(\text{CL}^{*,*}\), too. During the whole section \(M\) will be a compact manifold without boundary. We will present two proofs which slightly differ from the method of loc. cit.

The first proof is exactly along the lines of Definition 4.4. Consider \(a \notin \mathbb{Z}\) and \(A \in \text{CL}^{a,k}(M, E)\). Furthermore, choose \(P \in \text{CL}^{m,0}(M, E), m > 0\), self–adjoint elliptic
whose leading symbol is scalar and positive. Then by (3.20) the function \( \text{Tr}(AP^{-s}) \) is regular at \( s = 0 \). Analogously to Definition 4.1 we put

\[
\text{TR}(A,P) := \text{Tr}(AP^{-s})|_{s=0} \quad (5.1)
\]

\[
= \text{coefficient of } t^0 \log^0 t \text{ in the asymptotic expansion of } \text{Tr}(Ae^{-tP})
\]

**Theorem 5.1** \( \text{TR}(A) := \text{TR}(A,P) \) is independent of the particular \( P \) chosen. \( \text{TR} \) defines a linear functional on

\[
\bigcup_{a \in \mathbb{C} \setminus \mathbb{Z}, k \geq 0} \text{CL}^{a,k}(M,E).
\]

Furthermore,

(i) \( \text{TR} \mid \text{CL}^{a,k}(M,E) = \text{Tr}_{L^2} \mid \text{CL}^{a,k}(M,E) \) if \( a < - \dim M \).

(ii) \( \text{TR}([A,B]) = 0 \) if \( A \in \text{CL}^{a,k}(M,E), B \in \text{CL}^{b,l}(M,E), a + b \notin \mathbb{Z} \).

Thus, although the \( L^2 \)–trace cannot be extended as a trace on \( \text{CL}^{*,*} \) there is an extension of the \( L^2 \)–trace to non–integer order operators.

**Proof** It suffices to prove the analogue of Proposition 4.3. Obviously, \( \text{TR}(A,P^\alpha) = \text{TR}(A,P) \). Thus w.l.o.g. we may assume \( P \) to be of integer order. The proof of loc. cit. shows

\[
\frac{d}{du} \text{Tr}(e^{-tP_a}) \sim_{t \to 0^+} \sum_{j=0}^\infty \frac{(-1)^{j+1}}{(j+1)!} t^{j+1} \text{Tr}(A(\nabla_j^P \frac{d}{du} P_a)e^{-tP_a})
\]

\[
\text{Tr}([A,B]e^{-tP}) \sim_{t \to 0^+} \sum_{j=1}^\infty \frac{(-1)^j}{j!} t^j \text{Tr}(A(\nabla_j^P B)e^{-tP}),
\]

and from (3.18) we immediately conclude that the coefficient of \( t^0 \) in these expansions is 0. \( \square \)

The **Kontsevich–Vishik** trace has another interesting property with respect to holomorphic families of operators. Furthermore, the **Kontsevich–Vishik** trace is given as the integral over a canonical density. In order to generalize these facts to our algebra we first introduce a regularized integral on the space of symbols.

Consider \( f \in \text{CS}^{m,k}(\mathbb{R}^n) \). We write

\[
f = \sum_{j=0}^{N} \psi f_{m-j} + g, \quad (5.3)
\]

with \( f_{m-j} \in \mathcal{P}^{m-j,k}(\mathbb{R}^n), g \in \text{CS}^{m-N-1,k}(\mathbb{R}^n) \). \( \psi \in C^\infty(\mathbb{R}^n), \psi(\xi) = 0 \) if \( |\xi| \leq 1/4 \) and \( \psi(\xi) = 1 \) if \( |\xi| \geq 1/2 \). In the sequel \( \psi \) will be fixed. Then

\[
g(\xi) = O(|\xi|^{m-N}), \quad |\xi| \to \infty. \quad (5.4)
\]
This implies the asymptotic expansion
\[
\int_{|\xi| \leq R} f(\xi) d\xi \sim_{R \to \infty} \sum_{j=0, j \neq m, n}^{\infty} p_{m+n-j}(\log R) R^{m-j+n} + p_0(\log R) R^0,
\] (5.5)

with polynomials \( p_{\alpha} \) of degree
\[
\deg p_{\alpha} \leq \begin{cases} 
  k, & \alpha \neq 0, \\
  k + 1, & \alpha = 0.
\end{cases} \tag{5.6}
\]

To see this we note that
\[
\int_{|\xi| \leq R} g(\xi) d\xi = \int_{R^n} g(\xi) d\xi + O(|\xi|^{m-N+n}), \quad |\xi| \to \infty, \tag{5.7}
\]

and splitting the integral over \( \psi f_{m-j} \) into \( \int_{1 \leq |\xi| \leq R} \int_{|\xi| \leq 1} f_{m-j}(\xi) d\xi r^{m-j+n-1} \log^l r dr, \tag{5.8} \)

which implies (5.5).

We then define \( \int_{R^n} f(\xi) d\xi := \lim_{R \to \infty} \int_{|\xi| \leq R} f(\xi) d\xi \) to be the constant term in the asymptotic expansion (5.5), i.e.
\[
\int_{R^n} f(\xi) d\xi := \lim_{R \to \infty} \int_{|\xi| \leq R} f(\xi) d\xi := p_0(0). \tag{5.9}
\]

Note that (5.8) implies that the coefficient of \( R^0 \log^{k+1} R \) in the expansion (5.5) equals
\[
\frac{\text{res}_k(f_{-n})}{k+1}. \tag{5.10}
\]

**Proposition 5.2** (cf. [15, Lemma 2.1.4]) Let \( A \in \text{GL}(n, \mathbb{R}) \) be a regular matrix and \( f \sim \sum_{j \geq 0} f_{m-j} \in \text{CS}^{m,k}(\mathbb{R}^n) \). We have the transformation rule
\[
\int_{R^n} f(A\xi) d\xi = |\det A|^{-1} \left( \int_{R^n} f(\xi) d\xi + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{l+1} \int_{S^{n-1}} f_{-n,l}(\xi) \log^{l+1} |A^{-1}\xi| d\xi \right). \tag{5.11}
\]

**Proof** It suffices to prove the proposition for \( f \in C^\infty(\mathbb{R}^n) \) with
\[
f(\xi) = f(\xi/|\xi|)|\xi|^\alpha \log^l |\xi|, \quad |\xi| \geq 1.
\]

Then
\[
\int_{|\xi| \leq R} f(\xi) d\xi = \int_{|\xi| \leq 1} f(\xi) d\xi + \int_{1}^{R} \int_{S^{n-1}} f(\xi) d\xi r^{n+\alpha-1} \log^l r dr, \tag{5.11}
\]

and hence
\[
\int_{R^n} f(\xi) d\xi = \int_{|\xi| \leq 1} f(\xi) d\xi + \begin{cases} 
  \frac{(-1)^{l+1}l!}{(\alpha+n)^{l+1}} \int_{S^{n-1}} f(\xi) d\xi, & \alpha \neq -n, \\
  0, & \alpha = -n.
\end{cases} \tag{5.12}
\]
5. The Kontsevich–Vishik trace

On the other hand for $R$ large

$$\int_{|\xi| \leq R} f(A\xi) d\xi = |\det A|^{-1} \int_{|A^{-1}\xi| \leq R} f(\xi) d\xi$$

$$= |\det A|^{-1} \left( \int_{|\xi| \leq 1} f(\xi) d\xi + \int_{|\xi| \geq 1, |A^{-1}\xi| \leq R} f(\xi) d\xi \right),$$

$$\int_{|\xi| \geq 1, |A^{-1}\xi| \leq R} f(\xi) d\xi$$

$$= \int_{S^{n-1}} f(\xi) \int_{1}^{R/|A^{-1}\xi|} r^{\alpha+n-1} \log r dr d\xi$$

$$= \left\{ \begin{array}{ll}
\int_{S^{n-1}} f(\xi) \left( \frac{R}{|A^{-1}\xi|} \right)^{\alpha+n} \sum_{j=0}^{l} \frac{(-1)^{l-j} j!}{j!} \log j \left( \frac{R}{|A^{-1}\xi|} \right) (\alpha+n)^{j-l-1} d\xi, & \alpha \neq -n, \\
+ \frac{(-1)^{l+1} l!}{(\alpha+n)^{l+1}} d\xi, & \\
\int_{S^{n-1}} f(\xi) \frac{1}{l+1} \log^{l+1} \left( \frac{R}{|A^{-1}\xi|} \right) d\xi, & \alpha = -n,
\end{array} \right.$$ 

thus

$$\lim_{R \to \infty} \int_{|\xi| \geq 1, |A^{-1}\xi| \leq R} f(\xi) d\xi = \left\{ \begin{array}{ll}
\frac{(-1)^{l+1} l!}{(\alpha+n)^{l+1}} \int_{S^{n-1}} f(\xi) d\xi, & \alpha \neq -n, \\
\frac{(-1)^{l+1} l!}{l+1} \int_{S^{n-1}} f(\xi) \log^{l+1} |A^{-1}\xi| d\xi, & \alpha = -n,
\end{array} \right. \quad (5.13)$$

and we reach the conclusion. \hfill \Box

Next, we consider an open subset $U \subset \mathbb{R}^n$ and a pseudodifferential operator $A \in \mathcal{CL}^{a,k}(U, \mathbb{C}^p, \mathbb{C}^q)$, $a \notin \mathbb{Z}$, with amplitude $a \in \mathcal{CS}^{a,k}(U, \mathbb{R}^n) \otimes \text{Hom} \left( \mathbb{C}^p, \mathbb{C}^q \right)$, i.e.

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{U} a(x, y, \xi) u(y) e^{i<x-y, \xi>} dy d\xi. \quad (5.14)$$

For fixed $x$ we have $a(x, \cdot) \in \mathcal{CS}^{a,k}(\mathbb{R}^n) \otimes \text{Hom} \left( \mathbb{C}^p, \mathbb{C}^q \right)$ and thus we may put

$$\omega_{KV}(A) := (2\pi)^{-n} \int_{\mathbb{R}^n} a(x, x, \xi) d\xi \ |dx| \quad \omega_{KV}$$

has similar properties as the residue density:

**Lemma 5.3** We have

1. $\omega_{KV}(A) \in C^\infty(U, \text{Hom} \left( \mathbb{C}^p, \mathbb{C}^q \right) \otimes |\Omega|)$.

2. $A \mapsto \omega_{KV}(A)$ is linear.
3. If \( \varphi \in C_0^\infty(U) \) then \( \omega_{KV}(\varphi A) = \varphi \omega_{KV}(A) \).

4. If \( \kappa : U \to V \) is a diffeomorphism then

\[
\kappa^*(\omega_{KV}(\kappa_* A)) = \omega_{KV}(A).
\]

**Proof** 1.–3. are straightforward. To prove 4. we denote variables in \( U \) by \( x, y \) and variables in \( V \) by \( \tilde{x}, \tilde{y} \). \( \kappa_* A \) has the amplitude function

\[
(x, y, \xi) \mapsto a(\kappa^{-1} \tilde{x}, \kappa^{-1} \tilde{y}, \phi(\tilde{x}, \tilde{y})^{-1} \xi) \frac{|\det D\kappa^{-1}(\tilde{x}, \tilde{y})|}{|\det \phi(\tilde{x}, \tilde{y})|}
\]

(cf. [18, Sec. 4.1, 4.2]), where \( \phi(\tilde{x}, \tilde{y}) \) is smooth with \( \phi(\tilde{x}, \tilde{x}) = D\kappa^{-1}(\tilde{x})^t \). Since \( a \not\in \mathbb{Z} \) the preceding proposition gives

\[
\omega_{KV}(\kappa_* A) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(\kappa^{-1} \tilde{x}, \kappa^{-1} \tilde{y}, \phi(\tilde{x}, \tilde{x})^{-1} \xi) d\xi |d\tilde{x}|
\]

(5.16)

and thus

\[
\kappa^* \omega_{KV}(\kappa_* A) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x, x, \xi) d\xi |dx| = \omega_{KV}(A). \quad \Box
\]

By 4. of the preceding lemma \( \omega_{KV} \) can be defined globally on a manifold. So, for \( A \in \text{CL}^{a,k}(M, E, F) \), \( a \not\in \mathbb{Z} \), we obtain a well–defined density

\[
\omega_{KV}(A) \in C^\infty(M, \text{Hom}(E, F) \otimes |\Omega|).
\]

(5.18)

Furthermore, it is obvious that if \( a < - \dim M \) and \( E = F \) then we have

\[
\text{Tr}(A) = \int_M \text{tr}_{E_x}(\omega_{KV}(A)(x)).
\]

(5.19)

Following Kontsevich–Vishik [13–14] we now introduce holomorphic families. Let \( G \subset \mathbb{C} \) be a domain. A family of symbols \( a(z) \in C^{z,k}(U, \mathbb{R}^n) \), \( z \in G \), is called holomorphic if

\[
a(z) \sim \sum_{j=0}^\infty \psi_{a_{-j}},
\]

(5.20)

where

\[
a_{z-j}(z, x, \xi) = \sum_{l=0}^k a_{z-j,l}(z, x, \xi) \log^l |\xi|,
\]

(5.21)

and

\[
a_{z-j,l}(z, x, \cdot) \in \mathcal{P}^{z,0}(\mathbb{R}^n),
\]

is a holomorphic map into \( C^\infty(U \times \mathbb{R}^n) \). Furthermore,

\[
z \mapsto a_{z-j}(z, \cdot, \cdot)
\]

(5.22)

is holomorphic with values in \( C^{K(N)}(U \times \mathbb{R}^n) \), \( K(N) \to \infty \) as \( N \to \infty \).

A family \( A(z) \in \text{CL}^{z,k}(M, E) \) is called holomorphic if in every chart \( A(z) \) is given by a holomorphic amplitude \( a(z) \in C^{z,k}(U \times U, \mathbb{R}^n) \otimes \text{End}(\mathbb{C}^p) \).
Lemma 5.4 Let \( f(z) \in \text{CS}^{z,k}(\mathbb{R}^n) \), \( z \in G \), be a holomorphic family. Then the function

\[
I(z) := \int_{\mathbb{R}^n} f(z) dz
\]

is meromorphic in \( G \) with poles in \( \mathbb{Z} \cap G \) of order \( \leq k + 1 \). Furthermore,

\[
\text{Res}_{k+1} I(z)|_{z=\nu} = (-1)^{k+1} k! \text{res}_k (f_{-n}(\nu, \cdot)).
\]

Proof Writing

\[
f(z) = \sum_{j=0}^{N} f_{z-j} + g(z) \tag{5.24}
\]

with \( g(z, \xi) = O(|\xi|^{z-N}), |\xi| \to \infty \), we find that

\[
z \mapsto \int_{\mathbb{R}^n} g(z, \xi) d\xi \tag{5.25}
\]

is holomorphic for \( z \in G, \text{Re} z < -n + N \).

Next consider a function of the form

\[
f(z, \xi) = \psi(\xi) f_{z-j,l}(z, \xi) \log^l|\xi|, \quad l \leq k. \tag{5.26}
\]

Then \( \int_{|\xi| \leq 1} f(z, \xi) d\xi \) is holomorphic for \( z \in G \) and from (5.3) one derives

\[
\lim_{R \to \infty} \int_{1 \leq |\xi| \leq R} f_{z-j,l}(z, \xi) \log^l|\xi| d\xi = \frac{(-1)^{l+1} l!}{(z+n-j)^{l+1}} \int_{|\xi|=1} f_{z-j,l}(z, \xi) d\xi \tag{5.27}
\]

and we reach the conclusion. \( \square \)

Proposition 5.5 Let \( A(z) \in \text{CL}^{z,k}(M,E) \), \( z \in G \), be a holomorphic family of operators. Then the function

\[
I(z) := \int_M \text{tr}_{E_z}(\omega_{KV}(A(z))(x))
\]

is meromorphic in \( G \) with poles in \( \mathbb{Z} \cap G \) of order \( \leq k + 1 \). Moreover,

\[
\text{Res}_{k+1} I(z)|_{z=\nu} = \frac{(-1)^{k+1}}{k+1} \text{Res}_k (A(\nu)).
\]

Note that \( \text{Res}_{k+1} \) on the left hand side means the coefficient of \( (z-\nu)^{-k-1} \) in the Laurent expansion (cf. (3.21)) while \( \text{Res}_k \) on the right hand side means the noncommutative residue as defined in Definition 4.1.

Proof This is a straightforward consequence of the preceding lemma and Corollary 4.8. \( \square \)
Now we are ready to state and prove the main result of this section, which is the natural generalization of [73, Sec. 3].

**Theorem 5.6** For $a \in \mathbb{C} \setminus \mathbb{Z}$ there exists a linear functional

$$
\text{TR} : \text{CL}^{a,k}(M, E) \rightarrow \mathbb{C}
$$

with the following properties:

(i) For $A \in \text{CL}^{a,k}(M, E)$ and any self-adjoint elliptic $P \in \text{CL}^{m,0}(M, E)$, $m > 0$, with scalar and positive leading symbol we have

$$
\text{TR}(A) = \text{Tr}(AP^{-s})|_{s=0}.
$$

(ii) $\text{TR}(A) = \int_M \text{tr}_{E_x}(\omega_{KV}(A)(x))$.

(iii) $\text{TR}|_{\text{CL}^{a,k}(M, E)} = \text{Tr}_{L^2}|_{\text{CL}^{a,k}(M, E)}$ if $a < -\dim M$.

(iv) $\text{TR}([A, B]) = 0$ if $A \in \text{CL}^{a,k}$, $B \in \text{CL}^{b,l}$, $a + b \notin \mathbb{Z}$.

(v) If $A(z) \in \text{CL}^{z,k}(M, E)$, $z \in G$, is a holomorphic family then $\text{TR}(A(z))$ is meromorphic with poles in $z \in G \cap \{-\dim M + j \mid j \in \mathbb{Z}_+\}$ of order $\leq k + 1$. One has

$$
\text{Res}_{k+1}\text{TR}(A(z))|_{z=\nu} = \frac{(-1)^{k+1}}{k+1}\text{Res}_k(A(\nu)).
$$

**Proof** We take (i) as definition for TR. Then it only remains to prove (ii). Let $A \in \text{CL}^{a,k}(M, E)$. Choose $P \in \text{CL}^{1,0}(M, E)$ self-adjoint elliptic with scalar and positive leading symbol. We consider the family

$$
A(z) := AP^{-a+z}.
$$

Then $A(z)$, $z \in \mathbb{C}$, is a holomorphic family of operators. If $z < -\dim M$ then $A(z)$ is trace class and hence by (5.19) and Theorem 5.1

$$
\text{TR}(A(z)) = \text{Tr}(A(z)) = \text{Tr}(AP^{z-a})
$$

$$
= \int_M \text{tr}_{E_x}(\omega_{KV}(A(z))(x)).
$$

Since left and right hand side extend meromorphically to $\mathbb{C}$ we find in particular

$$
\text{TR}(A) = \text{TR}(A(a)) = \int_M \text{tr}_{E_x}(\omega_{KV}(A)(x)).
$$
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