The $\langle A^2 \rangle$ Asymmetry and Longitudinal Propagator in Lattice SU(2) Gluodynamics at $T \simeq T_c$.

V. G. Bornyakov  
NRC “Kurchatov Institute” - IHEP, 142281 Protvino, Russia  
School of Biomedicine, Far East Federal University, 690950 Vladivostok, Russia

V. V. Bryzgalov  
NRC “Kurchatov Institute” - IHEP, 142281 Protvino, Russia

V. K. Mitrjushkin  
Joint Institute for Nuclear Research, 141980 Dubna, Russia  
NRC “Kurchatov Institute” - ITEP, 117218 Moscow, Russia

R. N. Rogalyov  
NRC “Kurchatov Institute” - IHEP, 142281 Protvino, Russia

We study numerically the chromoelectric-chromomagnetic asymmetry of the dimension two gluon condensate as well as the longitudinal gluon propagator at $T \simeq T_c$ in the Landau-gauge SU(2) lattice gauge theory. We show that substantial correlation between the asymmetry and the Polyakov loop as well as the correlation between the longitudinal propagator and the Polyakov loop pave the way to studies of the critical behavior of the asymmetry and the longitudinal propagator. The respective values of critical exponents and amplitudes are evaluated.

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I. INTRODUCTION

The gluon propagators were studied intensively in lattice gauge theories. Recently also the asymmetry of the chromoelectric-chromomagnetic gluon condensate have received considerable attention. Motivation for these studies can be found, e.g. in [1–4] and references therein. We only mention the relation of the zero-momentum longitudinal and transverse propagators to the chromoelectric and chromomagnetic screening masses and, therefore, to the properties of strongly interacting quark-gluon matter. Our attention here is concentrated on the critical behavior of these quantities in the Landau-gauge SU(2) lattice gauge theory. It is well known that the second-order phase transition of the 3D Ising universality class occurs in this model and the Polyakov loop provides the order parameter [5, 6]; its nonzero value is associated with the spontaneous breaking of the $Z_2$ center symmetry.

Though the behavior of the asymmetry and the gluon propagators at $T \sim T_c$ have received much attention in the literature, the situation with their temperature and volume dependence in a close vicinity of $T_c$ is far from being clear.

It was shown in Ref. [7] that the phase transition both for SU(2) and SU(3) can be clearly identified by a peak in the temperature derivative of the chromoelectric screening mass

$$ m_e = \frac{1}{\sqrt{D_L(0)}} \tag{1} $$

where $D_L(0)$ is the zero-momentum longitudinal gluon propagator. It was assumed that the critical behavior of $D_L(0)$ stems from the interference of the Gribov-copy effects and the singular critical behavior

$$ D_L(0) \sim |\tau|^{-\gamma_D}, \tag{2} $$

where $\tau = \frac{T - T_c}{T_c}$. Some reasoning was given in favor of the equality $\gamma_D = \gamma$, where $\gamma$ is the critical exponent characterizing the behavior of the standard order-parameter correlator,

$$ G(0) = \int d\vec{x} \langle P(\vec{x}) P(0) \rangle \sim |\tau|^{-\gamma}, \tag{3} $$

$P$ is the Polyakov loop.

Here we suggest a new approach to the studies of the critical behavior of $D_L(0)$, which makes it possible to overcome the difficulties associated with the need to combine the Gribov mass with the singular critical behavior. We argue that the critical exponent $\gamma$ is unrelated to the critical behavior of $D_L(0)$.
Our approach is based on the study of correlations between $P$ and $D_L(0)$ and between $P$ and the asymmetry $A$ which will be defined later in this section. In the studies of these correlations we employ well-established properties of the Polyakov loop.

In particular, the leading term of the asymptotic expansion of the Polyakov loop in the infinite-volume limit has the form

$$ P = B \tau^\beta + \varepsilon(\tau^\beta) , $$

where the critical exponent $\beta$ coincides with that evaluated in the 3D Ising model if the universality hypothesis is satisfied. Recently, the critical exponents of the 3D Ising model were successfully employed in the studies of the Polyakov loop.

In the $SU(2)$ lattice gauge theory, the critical amplitude $B = 0.825(1)$ was evaluated in [7] in the case of $N_t = 4$.

We study critical behavior of the quantities

$$ \mathcal{Y} = A - A_C $$

and

$$ \mathcal{D} = D_L(0) - D_L^C(0) , $$

where $D_L^C(0)$ is the average value of the longitudinal gluon propagator at the critical temperature (referred to as $m_G^{\text{Gribov}}$ in [22]) and $A_C$ is the average value of the asymmetry at the critical temperature in the infinite-volume limit. Based on the analysis of correlations, we argue that the leading term of the asymptotic expansion of $A$ in $\tau$ at $\tau \to 0_+$ has the form

$$ \mathcal{Y} \simeq C_\mathcal{Y} \tau^\beta , $$

where $\beta$ is the critical exponent of the 3D Ising model. We also show that $\mathcal{D}$ has a similar behavior

$$ \mathcal{D} \simeq C_\mathcal{D} \tau^\beta , $$

and evaluate the critical amplitudes $C_\mathcal{Y}$ and $C_\mathcal{D}$.

### A. Definitions and simulation details

At nonzero temperatures, there are electric

$$ \langle A_E^2 \rangle = g^2 \langle A^a_1(x)A_1^a(x) \rangle , $$

and magnetic

$$ \langle A_M^2 \rangle = g^2 \langle A^a_1(x)A_1^a(x) \rangle , $$

dimension two condensates. The quantity of particular interest is the (color)electric-magnetic asymmetry

$$ A \equiv \frac{1}{2} \left( \langle A_E^2 \rangle - \frac{1}{3} \langle A_M^2 \rangle \right) , $$

it peaks at the phase transition and monotonically decreases in the deconfinement phase [8]. Expressions for $A$, definitions of the propagators $D_L(p)$ and $D_T(p)$, and relations between them are given in [10, 11], and [9].

We study these quantities in the $SU(2)$ lattice gauge theory with the standard Wilson action in the Landau gauge.

We generated ensembles of $O(1000)$ independent Monte Carlo lattice gauge-field configurations on the $N_t \times N_s^3$ lattice (in our study, $N_t = 8$ and $N_s = 32 \times 72$). Consecutive configurations (considered as independent) were separated by $100 \div 450$ sweeps, each sweep consisting of one local heatbath update followed by $N_t/2$ microcanonical updates.

Following Ref. [12], we use the gauge-fixing algorithm that combines $Z(2)$ flips for space directions with the simulated annealing (SA) algorithm followed by overrelaxation. It is referred to as the ‘FSA’ algorithm. The other details of simulations and a more thorough description of the gauge-fixing procedure can be found in [11].

Here we do not consider details of the approach to the continuum limit and renormalization considering that the lattices with $N_t = 8$ (corresponding to spacing $a \simeq 0.08$ fm) are sufficiently fine. Thus we consider a bare gluon propagator and the respective asymmetry.

### II. $A^2$ ASYMMETRY IN THE DECONFINEMENT PHASE

Details of the $A$ behavior near $T_c$ were omitted in the pioneering work [3] because of low statistics. Critical behavior of the asymmetry was first investigated in [13], however, that analysis was based on configurations with positive Polyakov loop only. Moreover, the distribution of the configurations in $A$ was not considered though the distributions in the Polyakov loop were successfully employed in the studies of the confinement-deconfinement phase transition.

Here we overcome the limitations of the approach used in [13]. Firstly, we compute the average value of the asymmetry using simulation data in both $Z_2$ sectors and, secondly, focus our attention on the distribution in $A$ and the correlation between the asymmetry $A$ and the Polyakov loop $P$.

To decide whether the average value of the asymmetry depends on $\tau$, we fit it to a constant function,

$$ \langle A \rangle \simeq c . $$

The results are shown in Table I. In contrast to the conclusions made in [13], the average value of the asymmetry depends neither on the temperature nor on the lattice size.

| $N_s$ | $c$     | $p$-value |
|-------|---------|-----------|
| 32    | 4.67(4) | 0.911     |
| 48    | 4.61(3) | 0.995     |
| 72    | 4.60(2) | 0.883     |

**TABLE I:** Results of the fit (13) over the range $-0.01 < \tau < 0.03$.

In order to investigate the temperature dependence of the asymmetry in more detail, we consider also the distribution of generated configurations in the asymmetry. First we notice that the distribution for $T < T_c$ (Fig. 1) differs significantly from that for $T > T_c$ (Fig. 2); at $T > T_c$ two peaks emerge similar to the case of the Polyakov loop.

**FIG. 1:** Probability density function for the distribution in the asymmetry at $\tau = -0.00765$, $L = 6.0$ fm.

For this reason, we consider correlation between the asymmetry and the Polyakov loop; this correlation is clearly seen, for example, in the scatter plot in Fig. 3. In view of such a substantial correlation it is natural to consider the asymmetry as the function of the Polyakov loop.

The regression analysis deals with the conditional distribution described by the cumulative distribution function $F(A|P)$. $F(A|P)$ describes the distribution of generated configurations in the asymmetry for a fixed value $P$ of the Polyakov loop. We are interested in the conditional expectation

$$
\langle A \rangle_P = E(A|P) = \int \frac{dF(A|P)}{dA} A dA.
$$

(14)

In the neighborhood of the critical temperature (that is, in the neighborhood of $P = 0$) it can be fitted to a polynomial as follows

$$
E(A|P) \simeq A_C + \sum_{j=1}^{n} A_j P^j
$$

(15)

For fixed value of $N_s$ we combined the data obtained for different values of $4g^2 \in [2.508, 2.518]$ ($g$ is the coupling) and determined the coefficients $A_j$ using the method of least squares for the regression model with $n = 3$. The results are presented in Table II, see also Fig. 4.

| $N_s$ | $A_0$    | $A_1$    | $A_2$    | $A_3$    |
|-------|----------|----------|----------|----------|
| 32    | 4.4380(39) | -66.19(14) | 141(2)   | 364(50)  |
| 48    | 4.4950(16) | -65.96(7)  | 136(1)   | 380(33)  |
| 72    | 4.545(12)  | -65.80(6)  | 139(1)   | 400(37)  |

**TABLE II:** Results of the fit (15) over the range $-0.07 < P < 0.07$. The values $2.508 \leq 4/g^2 \leq 2.518$ are taken into account.

Now it is well to employ our knowledge of the critical behavior of the Polyakov loop for the investigation of the critical behavior of the asymmetry. First we note that the width of the distribution of field configurations in the Polyakov loop tends to zero as the
lattice size tends to infinity: \( P = \langle P \rangle \). Thus the expectation value

\[
E(A \mid P = P(\tau))
\]

determines the asymmetry \( \langle A(\tau) \rangle \) in the infinite-volume limit.

\[
\langle A(\tau) \rangle = A_C + A_1 P + o(P).
\]

At \( \tau > 0 \) spontaneous breaking of the center symmetry occurs and we choose the positive Polyakov-loop sector. In this sector, in the infinite-volume limit,

\[
P(\tau) = B \tau^\beta + \sigma(\tau^\beta).
\]

Combining this relation with formula (15), we arrive at the leading term of the asymptotic expansion of the asymmetry at \( \tau \to 0_+ \),

\[
\langle A(\tau) \rangle = A_C + A_1 B \tau^\beta + \sigma(\tau^\beta);
\]

therefore,

\[
C_Y = A_1 B.
\]

It should be emphasized that the only assumption used in the derivation of this formula is that the relation

\[
E(A \mid P) = A_C + A_1 P + o(P)
\]

is valid at \( P > 0 \) with some values of \( A_C \) and \( A_1 \).

The critical amplitude \( B \) in the case under consideration (\( N_t = 8 \)) can be determined following the ideas outlined in [9] and making use of the critical coupling \( 4g^2 = 2.5104(2) \) reported in [14]. With our statistics only a rough estimate can be obtained,

\[
B = 0.147(35),
\]

giving \( C_Y = -9.6(2.3) \), the error is almost completely accounted for by poor precision in the determination of \( B \), \( A_1 = -65.5(2) \) is determined much more precisely (here we discard errors associated with the choice of the regression model etc).

\section*{III. CRITICAL BEHAVIOR OF \( D_L(0) \)}

We begin with the observation that the zero-momentum longitudinal propagator (and, therefore, the electric screening mass \( m_e = \frac{1}{\sqrt{D_L(0)}} \)) is strongly correlated with the Polyakov loop, see the scatter plot in Fig.5. Thus we will study the dependence of \( D \equiv D_L(0) \) on the Polyakov loop \( P \) near the criticality on the basis of the regression analysis.

Our attention should be concentrated on the cumulative distribution function \( F(D \mid P) \) of the gluon propagator at a given value of the Polyakov loop \( P \). First we note that, at small \( |\tau| \), \( F(D \mid P) \) does not depend on \( \tau \) and shows an approximate scaling with \( P \):

\[
F(D \mid sP) = F(\lambda(s)D \mid P),
\]

it is interesting that the distribution under consideration is non-Gaussian.
Similar to the case of asymmetry, we fit the combined data set for all temperatures over the range $0.986T_c < T < 1.03T_c$ to a common regression function

$$\langle D \rangle \simeq f(P) = D_C + \sum_{j=1}^{4} D_j P^j,$$  \quad (23)

the zero-momentum propagator $D$ is a predicted variable (regressor), the Polyakov loop $P$ is an explanatory variable (regressand), the Polyakov loop $P$ is an explanatory variable (regressor).

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Thus in the infinite-volume limit the leading terms of the asymptotic expansion at small $\tau$ have the form

$$\langle D \rangle \simeq D_C + \theta(\tau) C_D \tau^g$$  \quad (24)

where

$$C_D = B D_1 = -330(80) \text{ GeV}^{-2},$$  \quad (25)

$B$ and $D_1$ appear in formulas (21) and (23), respectively; again $D_1 = -2200(100)$ is determined much more precisely than the critical amplitude $B$.

This result stems only from

- the fact that, in the infinite-volume limit, the distribution in $P$ becomes infinitely narrow (the Polyakov-loop susceptibility tends to infinity as $V \rightarrow \infty$);

- and the smooth dependence of $\langle D_L(0) \rangle$ on $P$ at $P = 0$.

In a series of studies [15–17], it was concluded that, in a large, however finite, volume, $D_L(0)$ has a smooth behavior at the criticality: it approaches a maximum value at some temperature $T_m < T_c$ and then gradually decreases. Such behavior is seen not only at $N_t = 16$, as was stated in the mentioned studies, but also at $N_t = 8$ [13]. This observation should have a natural explanation in our approach provided that we consider only the positive Polyakov-loop sector (which is the case for the cited works).

Firstly, we note that at the temperatures well below $T_c$, the longitudinal gluon propagator slowly increases with temperature. We consider that this growth is unrelated to the observed correlation with the Polyakov loop and we disregard reasons behind it. Secondly, the values of the Polyakov loop are concentrated near zero; however, when the temperature comes to the critical one from below, the width of the distribution in the Polyakov loop begins to rise. In view of the correlation shown in Figs. 5 and 6, this results in a decrease of $D_L(0)$ in the positive Polyakov-loop sector (in the negative Polyakov-loop sector $D_L(0)$ increases). At $T = T_m$, the decrease caused by fluctuations near criticality overwhelms the above-mentioned increase characteristic for the confinement domain.

It is not surprising that $D_L(0)$ shows a smooth behavior even at very large volumes [15–17] because an approach to the infinite-volume limit near the criticality presents a challenge, especially in the case when only the positive Polyakov-loop sector is taken into consideration [13]. In any case, for a comprehensive investigation of the critical behavior of Green functions both Polyakov-loop sectors should be taken into account in some neighborhood of the critical temperature.

IV. CONCLUSIONS

We have studied the asymmetry $A$ and the longitudinal gluon propagator in the Landau-gauge $SU(2)$ gluodynamics on lattices with $32 \leq N_t \leq 72$ and $N_t = 8$ in the range of temperatures $0.99T_c < T < 1.03T_c$.

Our findings can be summarized as follows:
• Both the asymmetry and the longitudinal propagator have a significant correlation with the Polyakov loop.

• Regression analysis reveals that $\langle A \rangle_P$ and $\langle D_L(0) \rangle_P$ are smooth functions of the Polyakov loop. Therefore, if the relation (20) is valid, the critical exponents of $A - A_C$ and $D_L(0) - D_L^C(0)$ are given by

$$\beta_A = \beta_D = \beta = 0.326419(3) \quad (26)$$

and the respective critical amplitudes are given by the formulas (19) and (25). Since the regression analysis gives only an indication to the validity of (20), more thorough investigation of smoothness of $\langle A \rangle_P$ and $\langle D_L(0) \rangle_P$ is needed.

• Volume dependence of both $D_L(0)$ and $A$ can be accounted for by the volume dependence of the Polyakov loop. Similar studies of correlations between the Ployakov loop and the gluon propagators in QCD may shed light on both critical behavior of screening masses and dynamics of gluon fields near $T_c$.

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