Applications of Borel distribution series on holomorphic and bi-univalent functions

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ABSTRACT. In present manuscript, we introduce and study two families $B_\Sigma(\lambda, \delta; \alpha)$ and $B^*_\Sigma(\lambda, \delta; \beta)$ of holomorphic and bi-univalent functions which involve the Borel distribution series. We establish upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in each of these families. We also point out special cases and consequences of our results.

1. Introduction

We indicate by $A$ the family of functions which are holomorphic in the open unit disk
$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$
and have the following normalized type:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1)$$

We also indicate by $S$ the subclass of $A$ consisting of functions which are also univalent in $\mathbb{U}$. According to the Koebe one-quarter theorem [8], every function $f \in S$ has an inverse $f^{-1}$ defined by

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U}),$$
and

$$f(f^{-1}(w)) = w, \quad \text{quad } |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4},$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2a_3 + a_4) w^4 + \cdots. \quad (2)$$

A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ stand for the class of normalized bi-univalent functions

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Applications of Borel distribution series on holomorphic functions in $U$ given by (1). For a brief historical account and for several interesting examples of functions in the class $\Sigma$, see the pioneering work on this subject by Srivastava et al. [18], which actually revived the study of bi-univalent functions in recent years. From the work of Srivastava et al. [18], we choose to recall here the following examples of functions in the class $\Sigma$:

$$\frac{z}{1 - z}, \quad -\log(1 - z) \quad \text{and} \quad \frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right).$$

We notice that the class $\Sigma$ is not empty. However, the Koebe function is not a member of $\Sigma$.

In a considerably large number of sequels to the aforementioned work of Srivastava et al. [18], several different subclasses of the bi-univalent function class $\Sigma$ were introduced and studied analogously by the many authors (see, for example, [1–7, 9–11, 13, 14, 16, 17, 19–28, 30, 31]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin expansion (1) were obtained in many of these recent papers. The problem to find the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n|, \quad (n \in \mathbb{N} \setminus \{1, 2\}; \mathbb{N} := \{1, 2, 3, \ldots\})$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subclasses of the bi-univalent function class $\Sigma$ (see, for example, [14, 19, 21]).

Recently, Srivastava [12] in his survey-cum-expository review article, explored the mathematical application of $q$-calculus, fractional $q$-calculus and fractional $q$-differential operators in Geometric Function Theory.

A discrete random variable $x$ is said to have a Borel distribution, if it takes the values $1, 2, 3, \ldots$, with the probabilities

$$\frac{e^{-\delta}}{1!}, \quad \frac{2\delta e^{-2\delta}}{2!}, \quad \frac{9\delta^2 e^{-3\delta}}{3!}, \ldots,$$

respectively, where $\delta$ are called the parameters. Hence

$$\text{Prob}(x = r) = \frac{(\delta r)^{r-1} e^{-\delta r}}{r!}, \quad (r = 1, 2, 3, \ldots).$$

Wanas and Khuttar [29] introduced the following power series whose coefficients are probabilities of the Borel distribution:

$$\mathcal{M}(\delta, z) = z + \sum_{k=2}^{\infty} \frac{(\delta(k - 1))^{k-2} e^{-\delta(k-1)}}{(k-1)!} z^k, \quad (z \in \mathbb{U}; \, 0 < \delta \leq 1).$$

We note by the familiar Ratio Test that the radius of convergence of the above series is infinity.

Now, we considered the linear operator $\mathcal{B}_\delta : \mathcal{A} \rightarrow \mathcal{A}$ which is defined as follows:

$$\mathcal{B}_\delta f(z) = \mathcal{M}(\delta, z) * f(z) = z + \sum_{k=2}^{\infty} \frac{(\delta(k - 1))^{k-2} e^{-\delta(k-1)}}{(k-1)!} a_k z^k, \quad z \in \mathbb{U},$$
where (*) indicate the Hadamard product (or convolution) of two series.

Very recently, Srivastava and El-Deeb [15] have introduced some applications of the Borel distribution.

We now recall the following lemma that will be used to prove our main results.

**Lemma 1** (see [8]). If \( h \in \mathcal{P} \), then

\[
|c_k| \leq 2, \quad (\forall k \in \mathbb{N}),
\]

where \( \mathcal{P} \) is the family of all functions \( h \), holomorphic in \( U \), for which

\[
\Re(h(z)) > 0, \quad (z \in U),
\]

with

\[
h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in U).
\]

2. **Coefficient estimates for the bi-univalent function class \( \mathcal{B}_\Sigma(\lambda, \delta; \alpha) \)**

In this section, we first define the bi-univalent function class \( \mathcal{B}_\Sigma(\lambda, \delta; \alpha) \).

**Definition 1.** A function \( f \in \Sigma \), given by (1), in said to be the bi-univalent function class \( \mathcal{B}_\Sigma(\lambda, \delta; \alpha) \) if it satisfies the following conditions:

\[
\left| \arg \left( 1 + \frac{z (B_\delta f(z))'}{B_\delta f(z)} + \frac{z (B_\delta f(z))''}{(B_\delta f(z))'} - \frac{\lambda z^2 (B_\delta f(z))'' + z (B_\delta f(z))'}{\lambda (B_\delta f(z))'} + (1 - \lambda) (B_\delta f(z))' \right) \right| < \frac{\alpha \pi}{2}
\]

and

\[
\left| \arg \left( 1 + \frac{w (B_\delta g(w))'}{B_\delta g(w)} + \frac{w (B_\delta g(w))''}{(B_\delta g(w))'} - \frac{\lambda w^2 (B_\delta g(w))'' + w (B_\delta g(w))'}{\lambda (B_\delta g(w))'} + (1 - \lambda) (B_\delta g(w))' \right) \right| < \frac{\alpha \pi}{2},
\]

where \( z, w \in U \), \( 0 < \alpha \leq 1 \), \( 0 \leq \lambda \leq 1 \) and \( 0 < \delta \leq 1 \).

If we choose \( \lambda = 1 \) in Definition 1, the family \( \mathcal{B}_\Sigma(\lambda, \delta; \alpha) \) reduces to the family \( \mathcal{S}_\Sigma(\delta; \alpha) \) of bi-starlike functions which satisfying the following conditions

\[
\left| \arg \left( \frac{z (B_\delta f(z))'}{B_\delta f(z)} \right) \right| < \frac{\alpha \pi}{2}
\]

and

\[
\left| \arg \left( \frac{w (B_\delta g(w))'}{B_\delta g(w)} \right) \right| < \frac{\alpha \pi}{2}.
\]

If we choose \( \lambda = 0 \) in Definition 1, the family \( \mathcal{B}_\Sigma(\lambda, \delta; \alpha) \) reduces to the family \( \mathcal{K}_\Sigma(\delta; \alpha) \) of bi-convex functions which satisfying the following conditions:

\[
\left| \arg \left( 1 + \frac{z (B_\delta f(z))''}{(B_\delta f(z))'} \right) \right| < \frac{\alpha \pi}{2}
\]
and
\[ \left| \arg \left( 1 + \frac{w(B_\delta g(w))''}{(B_\delta g(w))'} \right) \right| < \frac{\alpha \pi}{2}. \]

Our first main result is asserted by Theorem 1 below.

**Theorem 1.** Let the function \( f \in B_\Sigma(\lambda, \delta; \alpha) \ (0 < \alpha \leq 1; 0 \leq \lambda \leq 1; 0 < \delta \leq 1) \) be given by (1). Then
\[ |a_2| \leq \frac{2\alpha}{\sqrt{2\alpha e^{-2\delta} \left[ (\lambda + 1)^2 + 2\delta(3 - 2\lambda) - 5 \right] + (1 - \alpha)(2 - \lambda)^2 e^{-2\delta}}} \]
and
\[ |a_3| \leq \frac{4\alpha^2 e^{2\delta}}{(2 - \lambda)^2} + \frac{\alpha e^{2\delta}}{\delta(3 - 2\lambda)}. \]

**Proof.** In light of the conditions (3) and (4), we have
\[ 1 + \frac{z(B_\delta f(z))'}{B_\delta f(z)} + \frac{z(B_\delta f(z))''}{(B_\delta f(z))'} - \frac{\lambda z^2 (B_\delta f(z))'' + z (B_\delta f(z))'}{\lambda z (B_\delta f(z))'} + (1 - \lambda)B_\delta f(z) = [p(z)]^\alpha \]
and
\[ 1 + \frac{w(B_\delta g(w))'}{B_\delta g(w)} + \frac{w(B_\delta g(w))''}{(B_\delta g(w))'} - \frac{\lambda w^2 (B_\delta g(w))'' + w (B_\delta g(w))'}{\lambda w (B_\delta g(w))'} + (1 - \lambda)B_\delta g(w) = [q(w)]^\alpha, \]
where \( g = f^{-1} \) and the functions \( p, q \in \mathcal{P} \) have the following series representations:
\[ p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \]
and
\[ q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \]
By comparing the corresponding coefficients of (5) and (6), we find that
\[ (2 - \lambda)e^{-\delta} a_2 = \alpha p_1, \]
\[ 2\delta(3 - 2\lambda)e^{-2\delta} a_3 - \left( 5 - (\lambda + 1)^2 \right) e^{-2\delta} a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, \]
\[ -(2 - \lambda)e^{-\delta} a_2 = \alpha q_1 \]
and
\[ 2\delta(3 - 2\lambda)e^{-2\delta} (2a_2^2 - a_3) - \left( 5 - (\lambda + 1)^2 \right) e^{-2\delta} a_2^3 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2. \]
Thus, by using (9) and (11), we conclude that
\[ p_1 = -q_1 \]
and
\[ 2(2 - \lambda)^2 e^{-2\delta} a_2^2 = \alpha^2 (p_1^2 + q_1^2). \]
If we add (10) to (12), we obtain
\[(15)\quad 2e^{-2\delta} \left[ (\lambda + 1)^2 + 2\delta(3 - 2\lambda) - 5 \right] a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} \left( p_1^2 + q_1^2 \right).\]

Substituting the value of \( p_1^2 + q_1^2 \) from (14) into the right-hand side of (15), and after some computations, we deduce that
\[(16)\quad a_2^2 = \frac{\alpha^2(p_2 + q_2)}{2\alpha e^{-2\delta} \left[ (\lambda + 1)^2 + 2\delta(3 - 2\lambda) - 5 \right] + (1 - \alpha)(2 - \lambda)^2 e^{-2\delta}}.\]

By taking the moduli of both sides of (16) and applying the Lemma 1 for the coefficients \( p_2 \) and \( q_2 \), we have
\[|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha e^{-2\delta} \left[ (\lambda + 1)^2 + 2\delta(3 - 2\lambda) - 5 \right] + (1 - \alpha)(2 - \lambda)^2 e^{-2\delta}}}.\]

Next, in order to determine the bound on \( |a_3| \), by subtracting (12) from (10), we get
\[(17)\quad 4\delta(3 - 2\lambda)e^{-2\delta} (a_3 - a_2^2) = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2} \left( p_1^2 - q_1^2 \right).\]

Now, upon substituting the value of \( a_2^2 \) from (14) into (17) and using (13), we deduce that
\[(18)\quad a_3 = \frac{\alpha^2 (p_1^2 + q_1^2)}{2(2 - \lambda)^2 e^{-2\delta}} + \frac{\alpha(p_2 - q_2)}{4\delta(3 - 2\lambda)e^{-2\delta}}.\]

Finally, by taking the moduli on both sides of (18) and applying the Lemma 1 once again for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), it follows that
\[|a_3| \leq \frac{4\alpha^2 e^{2\delta}}{(2 - \lambda)^2} + \frac{\alpha e^{2\delta}}{\delta(3 - 2\lambda)}.
\]

This completes the proof of Theorem 1. \(\square\)

Putting \( \lambda = 1 \) in Theorem 1, we state:

**Corollary 1.** For \( 0 < \alpha \leq 1 \) and \( 0 < \delta \leq 1 \), let the function \( f \in S_\Sigma(\delta; \alpha) \) be given by (1). Then
\[|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(2\delta - 1)e^{-2\delta} + (1 - \alpha)e^{-2\delta}}},\]
and
\[|a_3| \leq 4\alpha^2 e^{2\delta} + \frac{1}{\delta} \alpha e^{2\delta}.\]

Putting \( \lambda = 0 \) in Theorem 1, we state:
Corollary 2. For $0 < \alpha \leq 1$ and $0 < \delta \leq 1$, let the function $f \in K_\Sigma(\delta; \alpha)$ be given by (1). Then

$$|a_2| \leq \frac{\alpha}{\sqrt{\alpha e^{-2\delta}(3\delta - 2) + (1 - \alpha)e^{-2\delta}}}$$

and

$$|a_3| \leq \alpha^2 e^{2\delta} + \frac{1}{3\delta} \alpha e^{2\delta}.$$

3. Coefficient estimates for the bi-univalent function class $B^*_\Sigma(\lambda, \delta; \beta)$

In this section, we first define the bi-univalent function class $B^*_\Sigma(\lambda, \delta; \beta)$.

Definition 2. A function $f \in \Sigma$, given by (1), is said to be in the bi-univalent function class $B^*_\Sigma(\lambda, \delta; \beta)$ if it satisfies the following conditions:

$$\Re \left\{ 1 + \frac{z (B_\delta f(z))'}{B_\delta f(z)} + \frac{z (B_\delta f(z))''}{(B_\delta f(z))'} - \frac{\lambda z^2 (B_\delta f(z))'' + z (B_\delta f(z))'}{\lambda z (B_\delta f(z))' + (1 - \lambda)B_\delta f(z)} \right\} > \beta$$

and

$$\Re \left\{ 1 + \frac{w (B_\delta g(w))'}{B_\delta g(w)} + \frac{w (B_\delta g(w))''}{(B_\delta g(w))'} - \frac{\lambda w^2 (B_\delta g(w))'' + w (B_\delta g(w))'}{\lambda w (B_\delta g(w))' + (1 - \lambda)B_\delta g(w)} \right\} > \beta,$$

where $z, w \in \mathbb{U}, \ 0 \leq \beta < 1, \ 0 \leq \lambda \leq 1$ and $0 < \delta \leq 1$, and $g = f^{-1}$ is given by (2).

In particular, if we choose $\lambda = 1$ in Definition 2, the family $B^*_\Sigma(\lambda, \delta; \beta)$ reduces to the family $S^*_\Sigma(\delta; \beta)$ of bi-starlike functions which satisfying the following conditions

$$\Re \left\{ \frac{z (B_\delta f(z))'}{B_\delta f(z)} \right\} > \beta$$

and

$$\Re \left\{ \frac{w (B_\delta g(w))'}{B_\delta g(w)} \right\} > \beta.$$

Also, if we choose $\lambda = 0$ in Definition 2, the family $B^*_\Sigma(\lambda, \delta; \beta)$ reduces to the family $K^*_\Sigma(\delta; \beta)$ of bi-convex functions which satisfying the following conditions

$$\Re \left\{ 1 + \frac{z (B_\delta f(z))''}{(B_\delta f(z))'} \right\} > \beta$$

and

$$\Re \left\{ 1 + \frac{w (B_\delta g(w))''}{(B_\delta g(w))'} \right\} > \beta.$$

Our second main result is asserted by Theorem 2 below.
Theorem 2. Let the function \( f \in \mathcal{B}_\Sigma^*(\lambda, \delta; \beta) \) \((0 \leq \beta < 1; 0 \leq \lambda \leq 1; 0 < \delta \leq 1)\) be given by (1). Then

\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{e^{-2\delta} \left[ (\lambda + 1)^2 + 2\delta(3 - 2\lambda) - 5 \right]}}
\]

and

\[
|a_3| \leq \frac{4(1 - \beta)^2 e^{2\delta}}{(2 - \lambda)^2} + \frac{(1 - \beta)e^{2\delta}}{\delta(3 - 2\lambda)}.
\]

Proof. In view of the conditions (19) and (20), there exist the functions \( p, q \in \mathcal{P} \) such that

\[
1 + \frac{z (B_\delta f(z))'}{B_\delta f(z)} + \frac{z (B_\delta f(z))''}{(B_\delta f(z))'} - \frac{\lambda z^2 (B_\delta f(z))'' + z (B_\delta f(z))'}{\lambda z (B_\delta f(z))'} + (1 - \lambda)B_\delta f(z) = \beta + (1 - \beta)p(z)
\]

and

\[
1 + \frac{w (B_\delta g(w))'}{B_\delta g(w)} + \frac{w (B_\delta g(w))''}{(B_\delta g(w))'} - \frac{\lambda w^2 (B_\delta g(w))'' + w (B_\delta g(w))'}{\lambda w (B_\delta g(w))'} + (1 - \lambda)B_\delta g(w) = \beta + (1 - \beta)q(w),
\]

where \( g = f^{-1} \) and the functions \( p, q \in \mathcal{P} \) have the series expansions given by (7) and (8), respectively. Thus, by comparing the corresponding coefficients in (21) and (22), we get

\[
(2 - \lambda)e^{-\delta}a_2 = (1 - \beta)p_1,
\]

\[
2\delta(3 - 2\lambda)e^{-2\delta}a_3 - \left(5 - (\lambda + 1)^2\right)e^{-2\delta}a_2^2 = (1 - \beta)p_2,
\]

\[
-(2 - \lambda)e^{-\delta}a_2 = (1 - \beta)q_1
\]

and

\[
2\delta(3 - 2\lambda)e^{-2\delta} (2a_2^2 - a_3) - \left(5 - (\lambda + 1)^2\right)e^{-2\delta}a_2^2 = (1 - \beta)q_2.
\]

We now find from (23) and (25) that

\[
p_1 = -q_1
\]

and

\[
2 (2 - \lambda)^2 e^{-2\delta}a_2^2 = (1 - \beta)^2 (p_1^2 + q_1^2).
\]

By adding (24) and (26), we obtain

\[
2e^{-2\delta} \left[ (\lambda + 1)^2 + 2\delta(3 - 2\lambda) - 5 \right] a_2^2 = (1 - \beta)(p_2 + q_2).
\]
Consequently, we have
\[ a_2^2 = \frac{(1 - \beta)(p_2 + q_2)}{2e^{-2\delta}[(\lambda + 1)^2 + 2\delta(3 - 2\lambda) - 5]}. \]

Next, by applying the Lemma 1 for the coefficients \( p_2 \) and \( q_2 \), we deduce that
\[ |a_2| \leq \sqrt{\frac{2(1 - \beta)}{e^{-2\delta}[(\lambda + 1)^2 + 2\delta(3 - 2\lambda) - 5]}}. \]

In order to determinate the bound on \( |a_3| \), by subtracting (26) from (24), we get
\[ 4\delta(3 - 2\lambda)e^{-2\delta}(a_3 - a_2^2) = (1 - \beta)(p_2 - q_2) \]

or, equivalently,
\[ a_3 = a_2^2 + \frac{(1 - \beta)(p_2 - q_2)}{4\delta(3 - 2\lambda)e^{-2\delta}}. \]

Substituting the value of \( a_2^2 \) from (27) into (28), it follows that
\[ a_3 = \frac{(1 - \beta)^2(p_1^2 + q_1^2)}{2(2 - \lambda)^2e^{-2\delta}} + \frac{(1 - \beta)(p_2 - q_2)}{4\delta(3 - 2\lambda)e^{-2\delta}}. \]

Finally, by applying the Lemma 1 once again for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we get
\[ |a_3| \leq \frac{4(1 - \beta)^2e^{2\delta}}{(2 - \lambda)^2} + \frac{(1 - \beta)e^{2\delta}}{\delta(3 - 2\lambda)}. \]

We have thus completed the proof of Theorem 2.

Putting \( \lambda = 1 \) in Theorem 2, we state:

**Corollary 3.** For \( 0 \leq \beta < 1 \) and \( 0 < \delta \leq 1 \), let \( f \in S^*_\Sigma(\delta; \beta) \) be given by (1). Then
\[ |a_2| \leq \sqrt{\frac{2(1 - \beta)}{(2\delta - 1)e^{-2\delta}}}, \]

and
\[ |a_3| \leq 4(1 - \beta)^2e^{2\delta} + \frac{1}{\delta}(1 - \beta)e^{2\delta}. \]

Putting \( \lambda = 0 \) in Theorem 2, we state:

**Corollary 4.** For \( 0 \leq \beta < 1 \) and \( 0 < \delta \leq 1 \), let \( f \in K^*_\Sigma(\delta; \beta) \) be given by (1). Then
\[ |a_2| \leq \sqrt{\frac{1 - \beta}{(3\delta - 2)e^{-2\delta}}}, \]

and
\[ |a_3| \leq (1 - \beta)^2e^{2\delta} + \frac{1}{3\delta}(1 - \beta)e^{2\delta}. \]
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