On Powers of the Catalan Number Sequence

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Abstract. The Catalan number sequence is one of the most famous number sequences in combinatorics and is well studied in the literature. In this paper we further investigate its fundamental properties related to the moment problem and prove for the first time that it is an infinitely divisible Stieltjes moment sequence in the sense of S.-G. Tyan. Besides, any positive real power of the sequence is still a Stieltjes determinate sequence. Some more cases including (a) the central binomial coefficient sequence (related to the Catalan sequence), (b) a double factorial number sequence and (c) the generalized Catalan (or Fuss–Catalan) sequence are also investigated. Finally, we pose two conjectures including the determinacy equivalence between powers of nonnegative random variables and powers of their moment sequences, which is supported by some existing results.

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1. Introduction

The Catalan number sequence was introduced by Catalan (1838) and defined by $C_n = \binom{2n}{n}/(n+1)$, $n = 0, 1, 2, \ldots$. It is one of the most famous and frequently encountered number sequences in combinatorics (Koshy 2009) and has the generating function

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \frac{2}{1 + \sqrt{1 - 4x}} = \sum_{n=0}^{\infty} C_n x^n, \ |x| < 1/4$$

(Binet 1839). In the literature, there are more than 200 interpretations of the sequence so far (Stanley 2015, Roman 2015). It worths mentioning that An-Tu Ming (1692 – 1763) also used this sequence in the approximation theory, e.g., he derived the following identity of the sine function in terms of $C_n$ (Larcombe 1999, Luo 1988, 2013):

$$\left(\sin \frac{\alpha}{2}\right)^2 = \sum_{n=1}^{\infty} C_{n-1} \left(\frac{\sin \alpha}{2}\right)^{2n}, \ \alpha \in [0, \pi/2].$$

On the other hand, let us recall that a sequence $\{m_n\}_{n=0}^{\infty}$ of real numbers is called a Stieltjes moment sequence if there exists a nonnegative measure $\mu$ on $\mathbb{R}_+ \equiv [0, \infty)$ having it as its moment sequence:

$$m_n = \int_{[0,\infty)} x^n d\mu(x), \ n = 0, 1, 2, \ldots \tag{1}$$

If, in addition, such a measure $\mu$ in (1) is unique, then $\{m_n\}_{n=0}^{\infty}$ is called a Stieltjes determinate (S-det) sequence; otherwise, $\{m_n\}_{n=0}^{\infty}$ is called a Stieltjes indeterminate (S-indet) sequence. Note that if $m_0 = 1$, then (1) can be rewritten as

$$m_n = \int_{0}^{\infty} x^n dF(x), \ n = 0, 1, 2, \ldots, \tag{2}$$

where $F$ is the distribution function of a random variable $X \geq 0$, denoted $X \sim F$. If $F$ in (2) is unique by moments (namely, there is no other distribution having the same moment sequence as $F$), we also say that $F$ (or $X$) is moment-determinate (M-det) on $\mathbb{R}_+$; otherwise, $F$ (or $X$) is moment-indeterminate (M-indet) on $\mathbb{R}_+$.

Since both the Hankel matrices $\Delta_n = (C_{i+j})_{0 \leq i, j \leq n}$ and $\overline{\Delta}_n = (C_{i+j+1})_{0 \leq i, j \leq n}$ of the Catalan sequence have nonnegative determinants for all $n = 0, 1, 2, \ldots$, $\{C_n\}_{n=0}^{\infty}$ is a Stieltjes moment sequence by the remarkable characterization result due to Stieltjes (1894/1895) (see
also Shohat and Tamarkin 1943, or Akhiezer 1965). More precisely, we have
\[ \det(\Delta_n) = \det(\overline{\Delta}_n) = 1 \ \forall n \]
and this happens to be a characteristic property of the Catalan number sequence (Stanley 1999, Chapter 6). Actually, the measure \( \mu \) corresponding to \( \{C_n\}_{n=0}^\infty \) has a density \( f_C \), called Catalan density and described below:
\[ C_n = \int_0^\infty x^n d\mu(x) = \int_0^\infty x^n f_C(x) dx \equiv \int_0^\infty x^n \left( \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} \right) I_{(0,4]}(x) dx, \ \forall n, \quad (3) \]
where \( I_A \) is the indicator function of the set \( A \) (Stanley 2015, Amdeberhan et al. 2013). Namely, \( \{C_n\}_{n=0}^\infty \) is exactly the moment sequence of a bounded nonnegative random variable \( X_C \sim F_C \) with density \( f_C \), and \( X_C \) is M-det on both \( \mathbb{R}_+ \) and \( \mathbb{R} \equiv (-\infty, \infty) \), because its moment generating function exists. (See, e.g., Cramér’s and Hardy’s criteria in Lin 2017, Theorems 1 and 2.) Moreover, the random variable \( X_C/4 \) has Beta distribution \( B_{\alpha,\beta} \) with parameters \( \alpha = 1/2, \ \beta = 3/2 \) and moment sequence \( \{2^{-2n}C_n\}_{n=0}^\infty \). The Catalan distribution \( F_C \) is also called the Marchenko-Pastur (or the free Poisson) distribution in the literature (see, e.g., Banica et al. 2011).

Recently, Liang et al. (2016) studied the conditions under which the Catalan-like number sequences are Stieltjes moment sequences, while Berg (2005, 2007) investigated the moment (in)deteminacy property of the powers of the factorial sequence \( \{n!\}_{n=0}^\infty \). Mimicking Berg’s approach (Lemma 3 below), we have some new findings on the topic. We first prove that for any real \( c > 0 \), the \( c \)-th power of the Catalan sequence, \( \{C_c^n\}_{n=0}^\infty \), is a Stieltjes moment sequence, namely, \( \{C_n\}_{n=0}^\infty \) is an infinitely divisible Stieltjes moment sequence (Tyan 1975), denoted \( \{C_n\}_{n=0}^\infty \in I \), and then prove that each power sequence \( \{C_c^n\}_{n=0}^\infty \) is even S-det.

The five cases (i) the central binomial coefficient sequence, \( \{(\binom{2n}{n})\}_{n=0}^\infty \), (ii) the double factorial number sequence, \( \{(2n - 1)!!\}_{n=0}^\infty \), (iii) the generalized Catalan (or Fuss–Catalan) sequence, \( \{(k+1)n)/(kn + 1)\}_{n=0}^\infty \), (iv) \( \{((k+1)n)/n\}_{n=0}^\infty \) and (v) \( \{(kn)!\}_{n=0}^\infty \) are also investigated.

The main results are stated in the next section, and their proofs are given in Section 4. Section 3 provides some necessary lemmas. Finally, in Section 5 we pose two conjectures including the determinacy equivalence between powers of nonnegative random variables and powers of their moment sequences, which is supported by some existing results.
2. Main results

We start with a complete study in the simplest cases.

**Theorem 1.** (a) The Catalan sequence is an infinitely divisible Stieltjes moment sequence, i.e., \( \{C_n\}_{n=0}^\infty \in \mathcal{I} \).
(b) For each real \( c > 0 \), the power sequence \( \{C_n^c\}_{n=0}^\infty \) is S-det.
(c) For each real \( c > 0 \), the measure \( \mu_c \) (with \( \mu_1 = \mu \) in (3)) corresponding to the moment sequence \( \{C_n^c\}_{n=0}^\infty \) has the Mellin transform
\[
\mathcal{M}_c(s) \equiv \int_0^\infty t^s d\mu_c(t) = \left( \frac{2^{2s} \cdot \Gamma(s + 1/2)}{\sqrt{\pi} \cdot \Gamma(s + 2)} \right)^c = \left( \frac{1}{s + 1} \cdot \frac{\Gamma(2s + 1)}{(\Gamma(s + 1))^2} \right)^c, \ s \geq 0.
\]

In the next theorem, we treat the central binomial coefficient sequence by a new approach, which is interesting in itself. See also Lemma 6 below, in which the equality in distribution happens to be a characteristic property of the density in (4), as shown in Remark 1.

**Theorem 2.** Let \( B_0 = 1, \ B_n = \binom{2n}{n} = (n + 1)C_n, \ n = 1, 2, \ldots \). Then we have
(a) the central binomial coefficient sequence \( \{B_n\}_{n=0}^\infty \in \mathcal{I} \);
(b) if \( X \sim F \) has the moment sequence \( \{B_n\}_{n=0}^\infty \), then it has density function
\[
f(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(4-x)}}, \ x \in (0, 4), \quad (4)
\]
and moment generating function (mgf)
\[
M(t) = E[\exp(tX)] = \sum_{n=0}^\infty B_n \frac{t^n}{n!} = \exp(2t)I_0(2t), \ \forall \ t \in \mathbb{R}, \quad (5)
\]
where \( I_\alpha(t) = \sum_{k=0}^\infty (t/2)^{2k+\alpha}/(k!\Gamma(k+\alpha+1)) \), \( t \in \mathbb{R} \), is the modified Bessel function of the first kind;
(c) for each real \( c > 0 \), the power sequence \( \{B_n^c\}_{n=0}^\infty \) is S-det;
(d) for each real \( c > 0 \), the measure \( \mu_c \) corresponding to the moment sequence \( \{B_n^c\}_{n=0}^\infty \) has the Mellin transform
\[
\mathcal{M}_c(s) = \int_0^\infty t^s d\mu_c(t) = \left( \frac{2^{2s} \cdot \Gamma(s + 1/2)}{\sqrt{\pi} \cdot \Gamma(s + 2)} \right)^c = \left( \frac{\Gamma(2s + 1)}{(\Gamma(s + 1))^2} \right)^c, \ s \geq 0.
\]

Here are some interesting observations. It follows from (4) that \( Y = X/4 \) has the arcsine density: \( g(y) = \pi^{-1}(y(1-y))^{-1/2} \), \( y \in (0, 1) \). According to Example 3.5 in Berg and Durán (2005), the sequence \( \{1/(n + 1)\}_{n=0}^\infty \) belongs to \( \mathcal{I} \), so Theorem 1(a) is a consequence of
Theorem 2(a) (by using Lemma 2 below). On the other hand, the next variant of Theorem 2 is used in the proof of Theorem 3 below.

**Theorem 2'.** Let \( B_0 = 1, B_n = \frac{(2n)}{2^n}B_n = \frac{n+1}{2^n}C_n, \ n = 1, 2, \ldots \). Then we have

(a) the real sequence \( \{B_n\}_{n=0}^{\infty} \in \mathcal{I} \);
(b) if \( Y \sim G \) has the moment sequence \( \{B_n\}_{n=0}^{\infty} \), then it has density function

\[
g(y) = \frac{1}{\pi} \frac{1}{\sqrt{y(2-y)}}, \ y \in (0, 2),
\]

and mgf \( M(t) = E[\exp(tY)] = \exp(tI_0(t)), \ \forall \ t \in \mathbb{R}; \)
(c) for each real \( c > 0 \), the power sequence \( \{B^c_n\}_{n=0}^{\infty} \) is S-det;
(d) for each real \( c > 0 \), the measure \( \mu_c \) corresponding to the moment sequence \( \{B^c_n\}_{n=0}^{\infty} \) has the Mellin transform

\[
\mathcal{M}_c(s) = \int_0^\infty t^s d\mu_c(t) = \left( \frac{2^s}{\sqrt{\pi}} \frac{\Gamma(s+1/2)}{\Gamma(s+1)} \right)^c = \left( \frac{1}{2^s} \frac{\Gamma(2s+1)}{(\Gamma(s+1))^2} \right)^c, \ s \geq 0.
\]

**Theorem 3.** Let \( D_0 = 1, D_n = (2n-1)!! = (2n-1)(2n-3) \cdots 3 \cdot 1 \). Then we have

(a) the double factorial sequence \( \{D_n\}_{n=0}^{\infty} \in \mathcal{I} \);
(b) for real \( c > 0 \), the power sequence \( \{D^c_n\}_{n=0}^{\infty} \) is S-det iff \( c \leq 2 \);
(c) for each real \( c \in (0, 2] \), the measure \( \mu_c \) corresponding to the moment sequence \( \{D^c_n\}_{n=0}^{\infty} \) has the Mellin transform

\[
\mathcal{M}_c(s) = \int_0^\infty t^s d\mu_c(t) = \left( \frac{2^s}{\sqrt{\pi}} \frac{\Gamma(s+1/2)}{\Gamma(s+1)} \right)^c = \left( \frac{1}{2^s} \frac{\Gamma(2s+1)}{(\Gamma(s+1))^2} \right)^c, \ s \geq 0.
\]

In the next result we consider the sequence \( \{(2n)!\}_{n=0}^{\infty} \) instead of the moment sequence \( \{n!\}_{n=0}^{\infty} \) in Lemma 5 below.

**Theorem 4.** (a) The moment sequence \( \{(2n)!\}_{n=0}^{\infty} \in \mathcal{I} \).
(b) For real \( c > 0 \), the power sequence \( \{(2n)!^c\}_{n=0}^{\infty} \) is S-det iff \( c \leq 1 \).
(c) For each real \( c \in (0, 1] \), the measure \( \mu_c \) corresponding to the moment sequence \( \{(2n)!^c\}_{n=0}^{\infty} \) has the Mellin transform \( \mathcal{M}_c(s) = \int_0^\infty t^s d\mu_c(t) = (\Gamma(2s+1))^c, \ s \geq 0 \).

For a fixed positive integer \( k \) and the generalized Catalan numbers

\[
C_{k,n} = \frac{1}{kn+1} \binom{k+1}{n}, \ n = 0, 1, 2, \ldots
\]

(6)
(or Fuss–Catalan numbers of order \(k\)), we have the following results.

**Theorem 5.** Let \(k \geq 2\) be a fixed positive integer. Then

(a) the sequence \(\{C_{k,n}\}_{n=0}^{\infty} \in \mathcal{I}\);

(b) for each \(c > 0\), the power sequence \(\{C_{k,n}^c\}_{n=0}^{\infty}\) is S-det;

(c) for each real \(c > 0\), the measure \(\mu_{k,c}\) corresponding to the moment sequence \(\{C_{k,n}^c\}_{n=0}^{\infty}\) has the Mellin transform

\[
M_{k,c}(s) = \int_0^\infty t^s d\mu_{k,c}(t) = \left(\frac{1}{ks + 1} \cdot \frac{\Gamma((k + 1)s + 1)}{\Gamma(s + 1)\Gamma(ks + 1)}\right)^c, \ s \geq 0.
\]

**Theorem 6.** Let \(k \geq 2\) be a fixed positive integer. Then

(a) the sequence \(\{(k+1)n\}_n^{\infty} \in \mathcal{I}\);

(b) for each \(c > 0\), the power sequence \(\{(k+1)n\}_n^{\infty}c\) is S-det;

(c) for each real \(c > 0\), the measure \(\mu_{k,c}\) corresponding to the moment sequence \(\{(k+1)n\}_n^{\infty}c\) has the Mellin transform

\[
M_{k,c}(s) = \int_0^\infty t^s d\mu_{k,c}(t) = \left(\frac{\Gamma((k + 1)s + 1)}{\Gamma(s + 1)\Gamma(ks + 1)}\right)^c, \ s \geq 0.
\]

**Theorem 7.** Let \(k \geq 3\) be a fixed positive integer. Then

(a) \(\{(kn)!\}_n^{\infty}\) is an infinitely divisible Stieltjes moment sequence;

(b) for real \(c > 0\), the power sequence \(\{(kn)!c\}_n^{\infty}\) is S-det iff \(kc \leq 2\).

3. **Lemmas**

To prove the above main results, we need some notations and lemmas. For a positive and infinitely differentiable function \(f\) defined on \((0, \infty)\), we say that \(f\) is a completely monotone function if \((-1)^nf^{(n)}(x) \geq 0\) for all \(x \in (0, \infty)\) and \(n \geq 1\), denoted \(f \in \mathcal{CM}\), and that \(f\) is a Bernstein function, denoted \(f \in \mathcal{B}\), if its derivative \(f' \in \mathcal{CM}\). The following useful Lemmas 3–5 are due to Berg (2005, 2007), Berg and Durán (2004) as well as Berg and López (2015), in which Lemma 3 extends a result of Bertoin and Yor (2001, Proposition 1).

**Lemma 1.** Let \(\{m_n\}_{n=0}^{\infty} \in \mathcal{I}\) be the moment sequence of a bounded nonnegative random variable \(X\). Then for each \(c > 0\), the power sequence \(\{m_n^c\}_{n=0}^{\infty}\) is S-det.

**Proof.** Assume \(X \sim F\) and \(X \leq M\) a.s. (almost surely), where \(M\) is a positive constant, and let \(X_c \sim F_c\) be a nonnegative random variable with moment sequence \(\{m_n^c\}_{n=0}^{\infty}\). Then
\[ m_n \leq M^n \text{ and } m_n^c \leq M^{cn} \text{ for all } n \geq 1 \text{ and } c > 0. \] The latter in turn implies that \( X_c \leq M^c \text{ a.s.} \forall c > 0, \) because \( \lim_{n \to \infty} (E[X^n_c])^{1/n} = \lim_{n \to \infty} (m_n^c)^{1/n} \leq M^c. \) (See, e.g., Rudin 1987, p.71, and note that \( E[X^n_c] = \int_0^1 (F_c^{-1}(t))^n dt, \) where the quantile function \( F_c^{-1}(t) = \inf\{x : F_c(x) \geq t\}, \ t \in (0,1). \) Therefore, \( X_c \) is M-det on \( \mathbb{R}_+ \) by Hardy’s or Cramér’s criterion. Another approach is to calculate the Carleman quantity for \( X_c : \)

\[ C[X_c] = \sum_{n=1}^{\infty} (E[X^n_c])^{-1/(2n)} = \sum_{n=1}^{\infty} (m_n^c)^{-1/(2n)} \geq \sum_{n=1}^{\infty} M^{-c/2} = \infty \]

and apply Carleman’s criterion (see, e.g., Lin 2017, Theorem 2). The proof is complete.

**Lemma 2.** If the two Stieltjes moment sequences \( \{s_n\}_{n=0}^{\infty} \) and \( \{t_n\}_{n=0}^{\infty} \) are infinitely divisible, then so is the product sequence \( \{s_n t_n\}_{n=0}^{\infty}. \)

**Proof.** For each \( c > 0, \) let \( X_c \) and \( Y_c \) be two independent nonnegative random variables having the moment sequences \( \{s_n^c\}_{n=0}^{\infty} \) and \( \{t_n^c\}_{n=0}^{\infty}, \) respectively. Then the product \( Z_c = X_c Y_c \) has the moment sequence \( \{s_n^c t_n^c\}_{n=0}^{\infty}, \) because \( E[Z^n_c] = E[X^n_c] E[Y^n_c] = s_n^c t_n^c \forall n \) due to the independence of \( X_c \) and \( Y_c. \)

**Lemma 3.** Let \( \alpha, \beta > 0 \) and \( f \in \mathcal{B} \) with \( f(\alpha) > 0. \) Define the sequence \( \{s_n\}_{n=0}^{\infty} \) by \( s_0 = 1 \) and \( s_n = f(\alpha) f(\alpha + \beta) \cdots f(\alpha + (n-1)\beta), \ n = 1, 2, \ldots. \) Then \( \{s_n\}_{n=0}^{\infty} \in \mathcal{I}, \) and the power sequence \( \{s_n^c\}_{n=0}^{\infty} \) is S-det if \( c \in (0,2]. \) Moreover, for each real \( c > 0, \) let \( X_c \sim F_c \) be a nonnegative random variable having the moment sequence \( \{s_n^c\}_{n=0}^{\infty}. \) If \( X_c \) is M-det, then the Mellin transform of \( F_c \) is of the form \( M_c(z) = \int_0^\infty t^z dF_c(t) = \exp[-c\psi(z)], \ Re(z) \geq 0, \) where the exponent function

\[ \psi(z) = -z \log f(\alpha) + \int_0^\infty \left[ (1 - e^{-z\beta x}) - z(1 - e^{-\beta x}) \right] \frac{e^{-\alpha x}}{x(1 - e^{-\beta x})} d\kappa(x), \]

and the measure \( \kappa \) has the Laplace transform

\[ \frac{f'(s)}{f(s)} = \int_0^\infty e^{-sx} d\kappa(x), \ s > 0. \]

**Lemma 4.** Assume that a Stieltjes moment sequence \( \{m_n\}_{n=0}^{\infty} \) is the product \( m_n = s_n t_n \) of two Stieltjes moment sequences \( \{s_n\}_{n=0}^{\infty} \) and \( \{t_n\}_{n=0}^{\infty}. \) If \( t_n > 0 \) for all \( n \) and \( \{s_n\}_{n=0}^{\infty} \) is S-indet, then also \( \{m_n\}_{n=0}^{\infty} \) is S-indet.

**Lemma 5.** (a) The moment sequence \( \{n!\}_{n=0}^{\infty} \in \mathcal{I}. \)
(b) For real \( c > 0 \), the power sequence \( \{(n!)^c\}_{n=0}^\infty \) is S-det iff \( c \leq 2 \).

(c) For each real \( c \in (0, 2] \), the measure \( \mu_c \) corresponding to the moment sequence \( \{(n!)^c\}_{n=0}^\infty \) has the Mellin transform \( M_c(s) = \int_0^\infty t^s d\mu_c(t) = (\Gamma(s+1))^c, \ s \geq 0 \).

**Lemma 6.** Let \( X \sim F \) have the density \( f \) defined in (4), and let \( Z = X - 2 \). Then

(a) the distribution of \( Z \) is symmetric and \( Z^2 \overset{d}{=} X \), where \( \overset{d}{=} \) means ‘equal in distribution’;

(b) \( Z \) has the mgf \( M_Z(t) = E[\exp(tZ)] = I_0(2t), \ t \in \mathbb{R} \), where \( I_0 \) is the modified Bessel function of the first kind and is defined in (5);

(c) the mgf of \( X \) is equal to \( M_X(t) = e^{2t}I_0(2t), \ t \in \mathbb{R} \).

**Proof.** Clearly, \( Z \) has a symmetric density function

\[
    h(z) = \frac{1}{\pi} \frac{1}{\sqrt{4 - z^2}}, \quad z \in (-2, 2).
\]

(7)

We now carry out the distribution of \( Z^2 \) : for \( x \in (0, 4) \),

\[
    \Pr(Z^2 \leq x) = \Pr(-\sqrt{x} < Z < \sqrt{x}) = 2 \Pr(0 \leq Z < \sqrt{x}) = \frac{2}{\pi} \int_0^{\sqrt{x}} \frac{1}{\sqrt{4 - z^2}} dz.
\]

By changing variables \( z = \sqrt{u} \), we have

\[
    \Pr(Z^2 \leq x) = \frac{1}{\pi} \int_0^x \frac{1}{\sqrt{u(4-u)}} du = F(x), \ x \in (0, 4).
\]

Therefore, \( Z^2 \) has the same distribution as \( X \). This proves part (a). To prove part (b), we note that \( E[Z^k] = 0 \) for odd \( k \), and \( E[Z^{2n}] = E[X^n] = B_n = \binom{2n}{n}, \ n \geq 0 \). Hence,

\[
    M_Z(t) = E[\exp(tZ)] = \sum_{k=0}^\infty E[Z^k] \frac{t^k}{k!} = \sum_{n=0}^\infty E[Z^{2n}] \frac{t^{2n}}{(2n)!} = \sum_{n=0}^\infty \frac{t^{2n}}{(n!)^2} = I_0(2t), \ t \in \mathbb{R}.
\]

Alternatively, we can consider the random variable \( Z_* = Z/2 \) which has the symmetric Beta distribution with characteristic function

\[
    \phi_*(t) \equiv E[\exp(itZ_*)] = \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2} \left( \frac{t}{2} \right)^{2n}, \ t \in \mathbb{R}
\]

(see Kingman 1963, Theorem 1). This in turn implies that its mgf \( M_{Z_*}(t) = \phi_*(-it) = I_0(t), \ t \in \mathbb{R} \), and hence \( M_Z(t) = M_{Z_*}(2t) = I_0(2t), \ t \in \mathbb{R} \). Finally, we have the mgf of \( X \) :

\[
    M_X(t) = E[\exp(tX)] = E[\exp(t(Z + 2))] = e^{2t}E[\exp(tZ)] = e^{2t}I_0(2t), \ t \in \mathbb{R}.
\]
This completes the proof.

For simplicity, we denote by $X \sim F \sim \{m_n\}_{n=0}^{\infty} \sim \mathcal{M}$ the random variable $X$ having distribution $F$, moment sequence $\{m_n\}_{n=0}^{\infty}$ and Mellin transform $\mathcal{M}$.

**Lemma 7.** Let $a$ and $c$ be two positive real constants. Consider the random variables $X_1$ and $X_c$ satisfying

$$X_1 \sim F_1 \sim \{m_n\}_{n=0}^{\infty} \sim \mathcal{M}_1, \quad X_c \sim F_c \sim \{m_n\}_{n=0}^{\infty} \sim \mathcal{M}_c,$$

and $\mathcal{M}_c(s) = \mathcal{M}_1^c(s)$, $s \geq 0$. Then the Mellin transform $\mathcal{M}_{c,a}(s) = \mathcal{M}_{1,a}^c(s)$, $s \geq 0$.

**Proof.** For $s \geq 0$, we have

$$\mathcal{M}_{c,a}(s) = E[(X_c/a)^s] = \frac{1}{a^{cs}} E[X_c^s] = \frac{1}{a^{cs}} \mathcal{M}_c(s) = \frac{1}{a^{cs}} \mathcal{M}_1^c(s) = \left(\frac{1}{a^s} E[X_1^s]\right)^c = E[(X_1/a)^s]^c = \mathcal{M}_{1,a}^c(s).$$

The proof is complete.

4. Proofs of main results

**Proof of Theorem 1.** Rewrite first

$$C_n = \frac{(2n)!}{n!(n+1)!} = \frac{2n \cdot 2n-1 \cdot 2n-2 \cdot 2n-3 \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{n \cdot n+1 \cdot n-1 \cdot n \cdots 4 \cdot 3 \cdot 2} = 2^n \cdot \frac{2n-1 \cdot 2n-3 \cdots 5 \cdot 3 \cdot 1}{n+1 \cdot n \cdots 4 \cdot 3 \cdot 2} = 2^n \left(2 - \frac{3}{n+1}\right) \left(2 - \frac{3}{n}\right) \cdots \left(2 - \frac{3}{4}\right) \left(2 - \frac{3}{3}\right) \left(2 - \frac{3}{2}\right).$$

Then set the functions $h(x) = 2(2-3/(x+1))$, $x > 1/2$, and $f(x) = h(x+1/2)$, $x > 0$. Next, take $\alpha = 1/2$ and $\beta = 1$. We have $f(\alpha) = h(1) = 1 > 0$, $f \in \mathcal{B}$, and the Catalan number $C_n = f(\alpha)f(\alpha+\beta)\cdots f(\alpha+(n-1)\beta) = \Pi_{k=1}^{n} h(k)$ for all $n \geq 1$. Lemma 3 applies and hence the Catalan sequence $\{C_n\}_{n=0}^{\infty} \in \mathcal{I}$. This proves part (a). Part (b) follows from Lemma 1 and the fact (3) immediately. To prove part (c), we write, by Lemma 3,

$$\mathcal{M}_c(s) = \int_0^{\infty} t^s d\mu_c(t) = \left[\int_0^{\infty} t^s d\mu_1(t)\right]^c = \mathcal{M}_1^c(s), \quad s \geq 0.$$
Therefore, it remains to carry out $\mathcal{M}_1$ and we have, by (3), that, for $s \geq 0$,

$$\mathcal{M}_1(s) = \int_0^\infty t^s d\mu_1(t) = \int_0^4 t^s \left( \frac{1}{2\pi} \sqrt{4-t} \right) dt$$

$$= \frac{2^{2s+1}}{\pi} \int_0^1 x^{s-1/2} (1-x)^{1/2} dx = \frac{2^{2s+1}}{\pi} \cdot \frac{\Gamma(s+1/2)\Gamma(3/2)}{\Gamma(s+2)}$$

$$= \frac{2^s}{\sqrt{\pi}} \cdot \frac{\Gamma(s+1/2)}{\Gamma(s+2)} = \frac{1}{s+1} \cdot \frac{\Gamma(2s+1)}{(\Gamma(s+1))^2}.$$  

The last equality is due to the duplication formula for the gamma function:

$$\Gamma(x)\Gamma(x+1/2) = 2^{1-2x} \sqrt{\pi} \Gamma(2x), \; x > 0$$

(see, e.g., Gradshteyn and Ryzhik 2014, Section 8.335). The proof is complete.

**Proof of Theorem 2.** Rewrite first

$$B_n = \frac{(2n)!}{(n!)^2} = \frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdot \frac{2n-2}{n-1} \cdot \frac{2n-3}{n-1} \cdots \frac{5}{3} \cdot \frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{1} \cdot \frac{1}{1}$$

$$= 2^n \cdot \frac{2n-1}{n} \cdot \frac{2n-3}{n-1} \cdots \frac{3}{2} \cdot \frac{1}{1}$$

$$= 2^n \left( 2 - \frac{1}{n} \right) \left( 2 - \frac{1}{n-1} \right) \cdots \left( 2 - \frac{1}{3} \right) \left( 2 - \frac{1}{2} \right) \left( 2 - \frac{1}{1} \right).$$

Set the functions $h(x) = 2(2 - 1/x), \; x > 1/2,$ and $f(x) = h(x+1/2), \; x > 0.$ It is seen that $f \in \mathcal{B}.$ Next, take $\alpha = 1/2$ and $\beta = 1,$ then we have $f(\alpha) = 2 > 0$ and $B_n = f(\alpha) f(\alpha + \beta) \cdots f(\alpha + (n-1)\beta) = \Pi_{k=1}^n h(k), \; n \geq 1.$ Lemma 3 applies and this proves part (a). To prove part (b), assume that $U$ is a uniform random variable with values in $(0, 1)$ and is independent of $X.$ Then the product $XU$ has the moment sequence $\{C_n\}_{n=0}^\infty$ and hence has the density $f_C$ defined in (3) because the Catalan sequence is $S$-det. Therefore, we have

$$\Pr(XU \leq y) = \int_0^y f_C(x) dx, \; y \geq 0,$$

or, equivalently,

$$\int_0^1 \Pr(X \leq y/u) du = \int_0^y \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} dx, \; y \in (0, 4). \quad (8)$$

By changing variables $y/u = x,$ we rewrite (8) as

$$\int_y^\infty F(x) \frac{y}{x^2} dx = \int_y^0 \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} dx, \; y \in (0, 4). \quad (9)$$
Solving Eq. (9) by differentiating both sides twice with respect to \( y \), we prove the density function \( f \) of \( X \) to be the one given in (4). The identity in (5) is proved in Lemma 6.

Part (c) follows from parts (a) and (b) and Lemma 1 immediately. Finally, we carry out the Mellin transform of \( X \) (or \( F \)). For each \( s \geq 0 \), \( E[(XU)^s] = E[X^s]E[U^s] = E[X^s]/(s+1) \) happens to be the Mellin transform of \( \mu_1 \) corresponding to the Catalan sequence, namely, \( \frac{1}{s+1} \frac{\Gamma(2s+1)}{(1+(s+1))^2} \), from which the required \( E[X^s] \) follows. This completes the proof.

**Proof of Theorem 2’**. (Method I) Note that if \( \{s_n\}_{n=0}^\infty \in \mathcal{I} \), then so is \( \{a^n s_n\}_{n=0}^\infty \) for each constant \( a > 0 \). Therefore, Theorem 2’ follows from Theorem 2, and vice versa.

(Method II) Define the functions \( h(x) = 2 - 1/x, \ x > 1/2 \), and \( f(x) = h(x+1/2), \ x > 0 \). Take \( \alpha = 1/2 \) and \( \beta = 1 \), then \( f \in \mathcal{B} \) and \( f(\alpha) = 1 > 0 \). The remaining proof is similar to that of Theorem 2. Note also that \( Y \) has the same distribution as \( X/2 \), and hence the required results follow from Theorem 2 and Lemma 7 as well. The proof is complete.

**Proof of Theorem 3.** Consider the functions \( h(x) = 2x - 1, \ x > 1/2 \), and \( f(x) = h(x+1/2), \ x > 0 \). It is seen that \( f \in \mathcal{B} \). Next, take \( \alpha = 1/2 \) and \( \beta = 1 \), then we have \( f(\alpha) = 1 > 0 \) and \( D_n = f(\alpha) f(\alpha + \beta) \cdots f(\alpha + (n-1)\beta) = \prod_{k=1}^n h(k), \ n \geq 1 \). Lemma 3 applies and this proves both (a) and the sufficient part of (b). To prove the necessary part of (b), we write first \( D_n = n! \cdot (2n)!/[2^n (n!)^2] = n! \cdot B_n \equiv s_n \cdot t_n \) for all \( n \geq 1 \) and \( D_n^c = s_n^c \cdot t_n^c \) for \( c > 0 \) (see Theorem 2’). Namely, the Stieltjes moment sequence \( \{D_n^c\}_{n=0}^\infty \) is the product of two Stieltjes moment sequences \( \{s_n^c\}_{n=0}^\infty \) and \( \{t_n^c\}_{n=0}^\infty \). If \( c > 2 \), \( \{s_n^c\}_{n=0}^\infty \) is S-indet (Lemma 5), and hence \( \{D_n^c\}_{n=0}^\infty \) is also S-indet by Lemma 4.

To prove (c), let us recall that \( \{D_n\}_{n=0}^\infty \) is in fact the moment sequence of the chi-square random variable \( \chi_1^2 \) (the square of the standard normal random variable) with density function \( f(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} \exp(-x/2), \ x > 0 \), and Mellin transform

\[
\mathcal{M}_1(s) = \int_0^\infty t^s f(t)dt = \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{s-1/2} e^{-t/2} dt = \frac{2^s}{\sqrt{\pi}} \Gamma(s+1/2), \ s \geq 0.
\]

The proof is complete.

**Proof of Theorem 4.** Let the random variables \( \mathcal{E} \) and \( X \) have the moment sequences \( \{n!\}_{n=0}^\infty \) and \( \{B_n\}_{n=0}^\infty = \{(2n)\}_{n=0}^\infty \), respectively. Then \( Z = \mathcal{E}^2 \) has the moment sequence \( s_n \equiv E[Z^n] = (2n)!, \ n \geq 0 \). Next, write \( s_n = (2n)(n!)^2 = B_n(n!)^2 \) and \( s_n^c = B_n^c(n!)^{2c} \) for
each $c > 0$. Therefore, $\{s_n\}_n \in \mathcal{I}$ by Theorem 2(a) and Lemma 5(a). This proves part (a). Moreover, if $c > 1$, the sequence $\{(n!)^{2c}\}$ is S-indet due to Lemma 5(b), so is $\{s_n^c\}_n$ by Lemma 4. On the other hand, if $c \in (0, 1]$, the Carleman quantity $\sum_{n=0}^{\infty} [s_n^c]^{-1/(2n)} = \infty$ by Stirling formula $\Gamma(x+1) = x\Gamma(x) \approx \sqrt{2\pi} x^{x+1/2} e^{-x}$ as $x \to \infty$, and hence $\{s_n^c\}_n$ is S-det. This completes the proof of part (b). To prove part (c), let $c \in (0, 1]$, and let $X_c, E_{2c}$ be two independent nonnegative random variables having moment sequences $\{B_n^c\}_n, \{(n!)^{2c}\}_n$, respectively. Finally, define $Z_c = X_c E_{2c}$, which has the moment sequence $\{s_n^c\}_n$. Then we have, by Theorem 2(d) and Lemma 5(c), that

$$E[Z_c^s] = E[X_c^s] E[E_{2c}^s] = \left(\frac{\Gamma(2s+1)}{(\Gamma(s+1))^2}\right)^c (\Gamma(s+1))^{2c} = (\Gamma(2s+1))^c, \ s \geq 0.$$ 

The proof is complete.

**Proof of Theorem 5.** Mimicking the proof of Theorem 1, we define for the sequence $\{C_{k,n}\}_n$ the functions:

$$h_\ell(x) = (k+1)^{1/k} \left(1 + 1/k \right) \frac{2 - (\ell - 2)/k}{kx - (\ell - 2)}, \ x > \ell/(k+1), \ \ell = 1, 2, \ldots, k.$$ 

Next, define the functions: $f_\ell(x) = h_\ell(x + \ell/(k+1)), \ x > 0, \ \ell = 1, 2, \ldots, k$. Note that all the functions $f_\ell \in \mathcal{B}$. Taking $\alpha_\ell = 1 - \ell/(k+1), \ \beta_\ell = 1, \ \ell = 1, 2, \ldots, k$, we have $f_\ell(\alpha_\ell) = h_\ell(1) = (k+1)^{1/k} \frac{k-\ell+1}{k-\ell+2} > 0, \ \ell = 1, 2, \ldots, k$, and

$$C_{k,n} = \prod_{\ell=1}^k (\prod_{i=1}^n f_\ell(\alpha_\ell + (i-1)\beta_\ell)) = \prod_{\ell=1}^k (\prod_{i=1}^n h_\ell(i)).$$ 

By Lemma 3, each sequence $\{\prod_{i=1}^n h_\ell(i)\}_n \in \mathcal{I}$ (with the first term equal to 1), so is the product sequence $\{C_{k,n}\}_n$ due to Lemma 2. This proves part (a). To prove part (b), let us recall that the measure corresponding to the moment sequence $\{C_{k,n}\}_n$ has a bounded support $(0, (k+1)^{k+1}/k^k]$ and hence Lemma 1 applies (see, e.g., Banica et al. 2011, Theorem 2.1).

Finally, to prove part (c), we recall first that the Mellin transform of the measure $\mu_{k,1}$ is

$$\mathcal{M}_{k,1}(s) = \int_0^\infty t^s d\mu_{k,1}(t) = \frac{1}{ks+1} \frac{\Gamma((k+1)s+1)}{\Gamma(s+1)\Gamma(k+1)}, \ s \geq 0$$ 

(see Penson and Życzkowski 2011, p. 2). Then Lemma 3 completes the proof.
Proof of Theorem 6. As before, mimicking the proof of Theorem 2, we define the functions:

\[ h^*_\ell(x) = (k + 1)^{1/k} \left[ (1 + 1/k) - \frac{(k - \ell + 1)/k}{kx - (\ell - 1)} \right], \quad x > \ell/(k + 1), \ \ell = 1, 2, \ldots, k, \]

and \( f^*_\ell(x) = h^*_\ell(x + \ell/(k + 1)), \quad x > 0, \ \ell = 1, 2, \ldots, k. \) All the functions \( f^*_\ell \in B. \) Taking \( \alpha^*_\ell = 1 - \ell/(k + 1), \quad \beta^*_\ell = 1, \ \ell = 1, 2, \ldots, k, \) we have \( f^*_\ell(\alpha^*_\ell) = h^*_\ell(1) = (k + 1)^{1/k} > 0, \ \ell = 1, 2, \ldots, k. \)

\[ (kn + 1)C^*_{k,n} = \left( \begin{array}{c} (k + 1)n \\ n \end{array} \right) = \prod_{\ell=1}^{k} \left( \prod_{i=1}^{n} f^*_\ell(\alpha^*_\ell + (i - 1)\beta^*_\ell) \right) = \prod_{\ell=1}^{k} \left( \prod_{i=1}^{n} h^*_\ell(i) \right). \]

Again, Lemmas 2 and 3 apply, and the sequence \( \{((kn+1)n)\infty_{n=0} \in I. \) This proves part (a).

To prove part (b), we can use Cramér’s or Carleman’s criterion again. For the rest of the proof, let \( Y \) be a nonnegative random variable having the moment sequence \( \{(kn+1)n\infty_{n=0} \), and let \( U \) be a uniform random variable on \((0,1)\) and independent of \( Y. \) Then the product \( X = YU^k \) has the moment sequence \( \{C^*_{k,n}\infty_{n=0} \) and Mellin transform \( E[X^s] = E[Y^sU^ks] = \frac{1}{ks+1}E[Y^s], \ s \geq 0. \) This implies by Theorem 5(c) that

\[ E[Y^s] = \frac{\Gamma((k + 1)s + 1)}{\Gamma(s + 1)\Gamma(k + 1)} , \quad s \geq 0, \]

and finally, Lemma 3 completes the proof of part (c).

Proof of Theorem 7. Rewrite

\[ (kn)! = \left( \begin{array}{c} kn \\ n \end{array} \right) \left( \begin{array}{c} (k - 1)n \\ n \end{array} \right) \cdots \left( \begin{array}{c} 2n \\ n \end{array} \right) (n!)^k \equiv A_k(n)A_{k-1}(n) \cdots A_2(n)A_1(n). \]

Recall that each \( \{A_i(n)\infty_{n=0} \in I \) by Lemma 5 and Theorems 2 and 6, where \( i = 1, 2, \ldots, k, \) so is the product sequence \( \{(kn)!\infty_{n=0} \) due to Lemma 2. This proves (a). To prove the sufficiency part of (b), we can use Carleman’s criterion and Stirling formula, while the necessity part of (b) follows from the result of (a) as well as Lemmas 4 and 5(b). The proof is complete.

5. Remarks

Remark 1. The property in Lemma 6(a) happens to be a characterization of the density (4). In other words, if the distribution of \( Z \) is symmetric and \( Z^2 \overset{d}{=} Z + 2, \) then \( Z \) has the density (7). To see this, we have from the above equality that \( Z \geq -2 \) and hence \(-2 \leq Z \leq 2 \) due to the symmetric condition. This further implies that \( Z \) is M-det. By the equality in distribution, we can carry out all the moments of \( Z \) and claim that \( Z \) has the density (7).
Remark 2. Recall that the mgf of Beta distribution $B_{\alpha,\beta}$ is

$$M_{\alpha,\beta}(t) = 1 + \sum_{n=1}^{\infty} \left( \prod_{r=0}^{n-1} \frac{\alpha + r}{\alpha + \beta + r} \right) \frac{t^n}{n!}, \quad t \in \mathbb{R}.$$ 

It can be shown that the mgf of the Catalan distribution $F_C$ with density $f_C$ in (3) is equal to

$$M_C(t) = \frac{1}{2} \frac{3}{2} \left( 1 + \sqrt{1 - 4x/27} \right)^{2/3} - \frac{2^{2/3}x^{1/3}}{2^{4/3}3^{1/2}\pi x^{2/3} \left( 1 + \sqrt{1 - 4x/27} \right)^{1/3}}, \quad x \in (0, 27/4],$$

due to Penson and Solomon (2001) (see also Młotkowski et al. 2013, Theorem 4.3). For the complicated general case, see Liu et al. (2011) or Penson and Życzkowski (2011).

Remark 3. The Fuss–Catalan density (of order 2) corresponding to the moment sequence $\{C_{2,n}\}_{n=0}^{\infty}$ in (6) is of the form

$$f_2(x) = \frac{3 \left( 1 + \sqrt{1 - 4x/27} \right)^{2/3} - 2^{2/3}x^{1/3}}{2^{4/3}3^{1/2}\pi x^{2/3} \left( 1 + \sqrt{1 - 4x/27} \right)^{1/3}}, \quad x \in (0, 27/4],$$

due to Penson and Solomon (2001) (see also Młotkowski et al. 2013, Theorem 4.3). For the complicated general case, see Liu et al. (2011) or Penson and Życzkowski (2011).

Remark 4. Recall that the powers of the standard exponential random variable $E$ have the moment (in)determinacy property: for each real $c > 0$, $E^c$ is M-det iff $c \leq 2$ (see, e.g., Targhetta 1990). This together with Lemma 5 implies that for each $c > 0$, the $c$-th power $E^c$ of $E$ is M-det iff the $c$-th power $\{n!\}_{n=0}^{\infty}$ of $\{n!\}_{n=0}^{\infty}$ is S-det. An interesting by-product in the proof of Theorem 3 is the factorization of $\chi_1^2$.

More precisely, $\chi_1^2$ has the same distribution with the product $E \cdot Y$, where the random variable $Y$ is defined in Theorem 2' and is independent of $E$.

Remark 5. Recall also that any positive real power of a bounded nonnegative random variable is M-det. In view of these existing results including Theorems 1 and 3 as well as Lemma 5, we would like to pose the following.
Conjecture 1. Let $X$ be a nonnegative random variable with finite moments $m_n = E[X^n]$ of all orders $n \geq 1$. Moreover, let $\{m_n\}_{n=0}^{\infty} \in \mathcal{I}$ and for each real $c > 0$, denote by $X_c$ a nonnegative random variable having the moment sequence $\{m_n^c\}_{n=0}^{\infty}$. Then the $c$-th power $X^c$ is M-det on $\mathbb{R}_+$ iff the $c$-th power sequence $\{m_n^c\}_{n=0}^{\infty}$ is S-det, equivalently, $X_c$ is M-det on $\mathbb{R}_+$.

Remark 7. Note that for each real $a > 0$, $\{\Gamma(an + 1)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence because $E[(E^a)^n] = \Gamma(an + 1)$ for each $n \geq 0$. In view of Lemma 5 and Theorems 4 and 7, we also pose the following.

Conjecture 2. Let $a > 0$ be a real constant and $m_n = \Gamma(an + 1)$, $n \geq 0$. Then
(a) the moment sequence $\{m_n\}_{n=0}^{\infty} \in \mathcal{I}$;
(b) for real $c > 0$, the power sequence $\{m_n^c\}_{n=0}^{\infty}$ is S-det iff $ac \leq 2$;
(c) for each real $c \in (0, 2/a]$, the measure $\mu_c$ corresponding to the moment sequence $\{m_n^c\}_{n=0}^{\infty}$ has the Mellin transform $M_c(s) = \int_0^\infty t^s d\mu_c(t) = (\Gamma(as + 1))^c$, $s \geq 0$.

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