THE PRIMITIVE SPECTRUM OF A SEMIGROUP OF MARKOV OPERATORS

HENRIK KREIDLER

Abstract. The primitive spectrum is a useful object in the theory of C*-algebras. In this paper we consider a dynamical version of this notion and discuss its applications to the ergodic theory of Markov operators and to topological dynamics.

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1. Introduction

The primitive spectrum is a useful tool in the study of C*-algebras (see, e.g., Chapter IV of [Dix77], Section 4.3 of [Ped79] or Section II.6.5 of [Bla06]) and plays a crucial role in representation theory (cf. [Hof11]). Given a C*-algebra $A$ it is defined as

$$\text{Prim}(A) := \{ \ker \pi \mid 0 \neq \pi \text{ irreducible representation of } A \}.$$ 

Equipped with the hull-kernel topology (also called Jacobson topology) it becomes a quasi-compact $T_0$-space. A nice application is the so called Dauns-Hofmann Theorem asserting that – in the unital case – the center of $A$ is canonically isomorphic to $C(\text{Prim}(A))$.

In this note we study a dynamical version of the primitive spectrum in the commutative and unital case. Starting from a right amenable semigroup $S$ of Markov operators on the space of continuous functions $C(K)$ on some compact space $K$ we introduce the primitive spectrum $\text{Prim}(S)$ of $S$ as the set of absolute kernels of ergodic measures. Again we equip the primitive spectrum with a hull-kernel topology and obtain a quasi-compact $T_0$-space. We then describe the space $C(\text{Prim}(S))$ and give applications to topological dynamics and ergodic theory.

We now give a more detailed description of the results.

Based on two papers of H. H. Schaefer (see [Sch67] and [Sch68]) as well as Paragraph III.8 of [Sch74] we consider $S$-invariant ideals and measures in the first section recalling some basic definitions and facts.

In the subsequent sections we introduce and study radical $S$-ideals. The main result establishes a close connection between radical $S$-ideals, centers of attraction appearing in topological dynamics and stability conditions of the semigroup (see Theorem 4.1 and Theorem 4.7).

The last three sections are devoted to the primitive spectrum of $S$ as a topological space (cf. Proposition 5.3), the continuous functions thereon (see

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Theorem 6.3 and Theorem 6.5) and – as an application – a new description of mean ergodicity of semigroups of Markov operators (see Theorem 7.1). We finally look at some examples illustrating these results (cf. Example 7.4).

In the following we always assume $K$ to be a compact (Hausdorff) space and identify its dual space $C(K)'$ with the Banach lattice of complex regular Borel measures on $K$. Moreover, we refer to [Sch74] and [MN91] for Banach lattices and their ideal structure.

Recall that a positive operator $T \in L(C(K))$ is called Markov if $T1 = 1$. We now fix a semigroup $S \subseteq L(C(K))$ of Markov operators which is right amenable (cf. Section 2.3 of [BJM89]) if endowed with the strong operator topology, i.e., the space $C^b(S)$ of bounded continuous functions on $S$ has a right invariant mean.

2. Ergodic Measures and Primitive Ideals

In this section we introduce primitive $S$-ideals adapting concepts from the theory of $C^*$-algebras and start with the following definition going back to H. H. Schaefer (see [Sch67]).

**Definition 2.1.** A closed proper ideal $I \subseteq C(K)$ is an $S$-ideal if it is $S$-invariant, i.e., $SI \subseteq I$. It is called maximal if it is maximal among all $S$-ideals with respect to inclusion.

**Remark 2.2.** It is a standard application of Zorn's lemma that each $S$-ideal is contained in a maximal $S$-ideal (cf. Proposition 1 in [Sch67]).

In [Sin68] R. Sine used the concept of a self-supporting set of a Markov operator. Generalizing this to our setting, a nonempty closed set $L \subseteq K$ is called self-supporting if the measure $S'\delta_x \in C(K)'$ has support in $L$ for each $x \in L$ and $S \in S$.

Each self-supporting set $L$ defines an $S$-ideal $I_L := \{ f \in C(K) \mid f|_L = 0 \}$.

Conversely, each $S$-ideal is an $I_L$ for some self-supporting set $L$ and the mapping $L \mapsto I_L$ is bijective. Moreover, each maximal $S$-ideal corresponds to a minimal self-supporting set.

Given an $S$-ideal $I$ we call the unique self-supporting set $L$ with $I_L = I$ the support of $I$ and write $L = \text{supp} I$.

**Remark 2.3.** For each $S$-ideal $I$, the semigroup $S$ induces a semigroup $S_I$ of Markov operators on $C(\text{supp} I)$ given by

$$S_I := \{ S_I \mid S \in S \}$$

with $S_I f := SF|_{\text{supp} I}$ for $S \in S$ and $f \in C(\text{supp} I)$ where $F \in C(K)$ is any extension of $f$ to $K$. It is readily checked that $I$ is maximal if and only if $S_I$ is irreducible, i.e., there are no non-trivial $S_I$-ideals (see the corollary to Proposition III.8.2 in [Sch74]).

We are primarily interested in $S$-ideals defined by measures. The absolute kernel of a measure $0 \leq \mu \in C(K)'$ is

$$I_\mu := \{ f \in C(K) \mid \langle |f|, \mu \rangle = 0 \}.$$
Moreover we write \( P_S(K) \subseteq C(K)' \) for the space of invariant probability measures on \( K \) equipped with the weak* topology. By right amenability of \( S \) this is always a nonempty compact convex set (this is a simple consequence of Day’s fixed point theorem, see [Day61]).

We recall that for each \( \mu \in P_S(K) \) we obtain a semigroup of bi-Markov operators

\[
S_{\mu} = \{ S_{\mu} \mid S \in S \}
\]
on \( L^1(K, \mu) \), i.e., each \( S_{\mu} \) is a positive operator on \( L^1(K, \mu) \) with \( S_{\mu}1 = 1 \) and \( S_{\mu}'1 = 1 \). Indeed the space \( L^1(K, \mu) \) is the completion of \( C(K)/I_\mu \) with respect to the \( L^1 \)-norm and \( I_\mu \) is invariant.

**Definition 2.4.** A measure \( \mu \in P_S(K) \) is called ergodic if the fixed space \( \text{fix}(S_{\mu}) \) in \( L^1(K, \mu) \) is one-dimensional.

The following characterization of ergodicity generalizes a result of M. Rosenblatt (cf. [Ros76]) and is well-known for single operators. We give a short proof in case of semigroup actions based on Proposition 10.4 of [EFHN15].

**Proposition 2.5.** A measure \( \mu \in P_S(K) \) is ergodic if and only if \( \mu \in \text{ex}P_S(K) \).

**Proof.** Assume that \( \text{fix}(S_{\mu}) \) is not one-dimensional. Since \( \text{fix}(S_{\mu}) \) is an \( \Lambda \)-sublattice of \( L^1(K, \mu) \) with weak order unit \( 1 \), the set

\[
B = \{ f \in \text{fix}(S_{\mu}) \mid f > 0 \text{ and } \sup(f, 1 - f) = 0 \}
\]
is total in \( \text{fix}(S_{\mu}) \) (cf. page 115 of [Sch74]). But \( B \) is just the set of characteristic functions in \( \text{fix}(S_{\mu}) \). Thus there is a measurable set \( A \subseteq K \) with \( S_{\mu}1_A = 1_A \) and \( 0 < \mu(A) < 1 \). Now consider the measures \( \mu_1, \mu_2 \) defined by

\[
\mu_1 := \frac{1}{\mu(A)}1_A \mu \text{ and } \mu_2 := \frac{1}{1 - \mu(A)}1_{A'} \mu.
\]

For every \( g \in C(K) \) with \( 0 \leq g \leq 1 \) and each \( S \in S \) we obtain

\[
\int_A g \, d\mu = \int g \wedge 1_A \, d\mu = \int S_{\mu}(g \wedge 1_A) \, d\mu \leq \int Sg \wedge 1_A \, d\mu = \int_A Sg \, d\mu
\]

and

\[
\int_{A'} g \, d\mu \leq \int_{A'} Sg \, d\mu,
\]

which implies \( \mu_i \in P_S(K) \) for \( i = 1, 2 \). Moreover,

\[
\mu = \mu(A)\mu_1 + (1 - \mu(A))\mu_2,
\]

so \( \mu \notin \text{ex}P_S(K) \).

Conversely, take an ergodic measure \( \mu \in P_S(K) \) and suppose that \( \mu = \frac{1}{2}(\mu_1 + \mu_2) \) for some \( \mu_1, \mu_2 \in P_S(K) \). Since

\[
|\langle f, \mu_1 \rangle| \leq 2|\langle f, \mu \rangle| \leq 2\|f\|_{L^1(K, \mu)}^2,
\]

\( \mu_1 \) extends uniquely to a continuous functional \( \tilde{\mu}_1 \in L^\infty(K, \mu) = L^1(K, \mu)' \). The semigroup \( S_{\mu} \) is mean ergodic and \( \text{fix}(S_{\mu}) \) is one dimensional, so \( \text{fix}(S_{\mu}) \) is also one-dimensional. Consequently we obtain \( \tilde{\mu}_1 = 1 \in L^\infty(K, \mu) \) which implies \( \mu_1 = \mu \).

\( \square \)
We are now ready to introduce primitive $S$-ideals.

**Definition 2.6.** An $S$-ideal $p$ is called primitive if there is $\mu \in \text{ex} \, P_S(K)$ with $p = I_\mu$.

The set of all primitive $S$-ideals is called the **primitive spectrum of $S$**, denoted by $\text{Prim}(S)$.

**Remark 2.7.** The supports of primitive $S$-ideals are precisely the supports of ergodic measures. Instead of looking at the ideal space it is therefore justified (and sometimes helpful) to see the primitive spectrum as a subset of the power set of $K$.

**Remark 2.8.** If $S \subseteq \mathcal{L}(C(K))$ is irreducible, i.e., there are no non-trivial $S$-ideals, then $\text{Prim}(S)$ is a singleton. Other examples are given below (cf. Example 5.6).

We need the following result which relates invariant measures for quotient systems to invariant measures on $K$.

**Proposition 2.9.** Let $L \subseteq K$ be the support of an $S$-ideal and consider the semigroup $S_L$ of Markov operators on $C(L)$ induced by $S$. The canonical continuous embedding

$$i : C(L)' \longrightarrow C(K)'$$

with $i(\mu)(f) := \langle f|_L, \mu \rangle$ for each $f \in C(K)$ restricts to continuous embeddings

$$i : P_{S_L}(L) \longrightarrow P_S(K),$$

$$i : \text{ex} \, P_{S_L}(L) \longrightarrow \text{ex} \, P_S(K).$$

with $i(\text{ex} \, P_{S_L}(L)) = \{\tilde{\mu} \in \text{ex} \, P_S(K) \mid \text{supp} \, \tilde{\mu} \subseteq L\}$.

**Proof.** It is obvious that images of invariant measures remain invariant. Now assume that $\mu \in \text{ex} \, P_S(L)$ and suppose that $i(\mu) = \frac{1}{2}(\tilde{\mu}_1 + \tilde{\mu}_2)$ for measures $\tilde{\mu}_1, \tilde{\mu}_2 \in P_S(K)$. Then $\text{supp} \, \tilde{\mu}_i \subseteq \text{supp} \, \mu$ for $i = 1, 2$ and therefore $\tilde{\mu}_1$ and $\tilde{\mu}_2$ restrict to measures $\mu_1, \mu_2 \in P_{S_L}(L)$ with $\mu = \frac{1}{2}(\mu_1 + \mu_2)$, so $\mu_1 = \mu = \mu_2$ since $\mu$ is ergodic. □

**Corollary 2.10.** Each maximal $S$-ideal is primitive.

**Proof.** Take a maximal $S$-ideal $I = I_L$. Then the induced semigroup $S_I$ on $C(L)$ is irreducible and consequently every $S_I$-invariant measure $\mu$ is strictly positive, i.e., $\text{supp} \, \mu = L$. □

3. **Radical Ideals**

The Jacobson topology on the primitive spectrum of C*-algebras can be defined using the notions of hull and kernel (see Section 4.3 of [Ped79] or Section II.6.5 of [Bla06]). In our context they also yield a natural correspondence between closed subsets of $\text{Prim}(S)$ and so-called **radical $S$-ideals**.

**Definition 3.1.** For subsets $A \subseteq \text{Prim}(S)$ and $I \subseteq C(K)$ we set

$$\ker(A) := \bigcap_{p \in A} p,$$

$$\text{hull}(I) := \{p \in \text{Prim}(S) \mid I \subseteq p\}.$$
(i) For a subset $I \subseteq C(K)$ the $\mathcal{S}$-radical of $I$ is
\[
\text{rad}_\mathcal{S}(I) := \ker(\text{hull}(I)) = \bigcap_{p \in \text{Prim}(\mathcal{S}) \mid I \subseteq p} p.
\]

(ii) An $\mathcal{S}$-ideal $I$ is a radical $\mathcal{S}$-ideal if $I = \text{rad}_\mathcal{S}(I)$.

(iii) The semigroup $\mathcal{S}$ is radical free if the zero ideal is a radical $\mathcal{S}$-ideal, i.e., if $\text{rad}_\mathcal{S}(0) = 0$.

We denote the set of all radical $\mathcal{S}$-ideals by $\text{Rad}(\mathcal{S})$ and write $R(\mathcal{S}) := \text{rad}_\mathcal{S}(0)$.

**Remark 3.2.** Our definition of a radical free semigroup is more general than the one of Schaefer (using maximal $\mathcal{S}$-ideals, see [Sch68]).

**Remark 3.3.** The $\mathcal{S}$-radical of a subset $I \subseteq C(K)$ is either $C(K)$ or an $\mathcal{S}$-radical ideal. Moreover, we always have $\text{hull}(I) = \text{hull}(\text{rad}_\mathcal{S}(I))$.

**Remark 3.4.** Clearly, $\mathcal{S}$ is radical free if and only if the union of all supports of invariant ergodic measures is dense in $K$. Note that the latter set is generally not closed (see Example 5.6 (iii) below). For the Markov semigroup induced by the shift on $K = \beta\mathbb{N} \setminus \mathbb{N}$ this set is nowhere dense (cf. Corollary 1.5 in [Cho67]).

We need the following lemma which relates radical and primitive ideals of quotient systems to the corresponding $\mathcal{S}$-ideals of $C(K)$.

**Lemma 3.5.** Let $I = I_L \subseteq C(K)$ be an $\mathcal{S}$-ideal and $\mathcal{S}_I$ the semigroup of Markov operators on $C(L)$ induced by $\mathcal{S}$. Then the mappings
\[
\begin{align*}
\{ p \in \text{Prim}(\mathcal{S}) \mid I \subseteq p \} &\rightarrow \text{Prim}(\mathcal{S}_I), \quad p \mapsto p|_L, \\
\{ J \in \text{Rad}(\mathcal{S}) \mid I \subseteq J \} &\rightarrow \text{Rad}(\mathcal{S}_I), \quad J \mapsto J|_L,
\end{align*}
\]
where $J|_L := \{ f|_L \mid f \in J \}$ for $J \subseteq C(K)$, are inclusion preserving bijections. Moreover, $R(\mathcal{S}_I) = \text{rad}_\mathcal{S}(I)|_L$.

**Proof.** The natural projection $P: C(K) \rightarrow C(L)$ is a Banach lattice epimorphism. Thus, if $J \subseteq C(L)$ is a closed ideal, then $J := P^{-1}(\tilde{J})$ is a closed ideal containing $I$ with $\tilde{J} = P(P^{-1}(\tilde{J})) = J|_L$. It is readily checked that $J$ is the unique closed ideal containing $I$ with $J|_L = \tilde{J}$. Clearly $\tilde{J}$ is $\mathcal{S}_I$-invariant if and only if $J$ is $\mathcal{S}$-invariant.

We therefore obtain mutually inverse and inclusion preserving mappings
\[
\{ J \subseteq C(K) \mid J \text{ $\mathcal{S}$-ideal with } I \subseteq J \} \leftrightarrow \{ \tilde{J} \subseteq C(L) \mid \tilde{J} \text{ $\mathcal{S}_I$-ideal} \},
\]
\[
J \mapsto J|_L,
\]
\[
P^{-1}(\tilde{J}) \leftrightarrow \tilde{J}.
\]

Now assume that $I \subseteq J = I_\mu$ for some $\mu \in \text{ex } P_\mathcal{S}(K)$. Then $\text{supp } \mu \subseteq L$ and we thus find $\nu \in \text{ex } P_\mathcal{S}_I(L)$ with $i(\nu) = \mu$ (cf. Proposition 2.3). Moreover we obtain for every $f \in C(L)$ that
\[
\langle |f|, \nu \rangle = \int_L |f| \, d\mu,
\]
for each extension $F \in C(K)$ of $f$ to $K$. Thus $f \in I_\nu$ if and only if $f \in I_\mu |_L$.
If, on the other hand, $\tilde{J} = I_\nu$ for some $\nu \in \text{ex} P_{S_f}(L)$, then Equation (1) holds for $\mu = i(\nu)$ and thus $\tilde{J} = I_\mu |_L$. We have thus shown that

$$\{ p \in \text{Prim}(S) \mid I \subseteq p \} \rightarrow \text{Prim}(S_f), \quad p \mapsto p|_L$$

is bijective.

Before proceeding with the remaining assertions, we make the following two observations.

- For a family $(J_\alpha)_{\alpha \in A}$ of $S$-ideals with $I \subseteq J_\alpha$ for every $\alpha \in A$

$$\left( \bigcap_{\alpha \in A} J_\alpha \right)|_L = \bigcap_{\alpha \in A} (J_\alpha|_L).$$

- For two $S$-ideals $J_1, J_2$ with $I \subseteq J_1, J_2$ the inclusion $J_1|_L \subseteq J_2|_L$ implies $J_1 \subseteq J_2$.

Now take an $S$-ideal $J \subseteq C(K)$ with $I \subseteq J$. Then $J$ is radical if and only if

$$J = \bigcap_{p \in \text{Prim}(S)} p$$

which is – by the observations above – equivalent to

$$J|_L = \bigcap_{p \in \text{Prim}(S)} p|_L = \bigcap_{p \in \text{Prim}(S_f)} p|_{L|_L} = \bigcap_{p \in \text{Prim}(S_f)} p,$$

i.e., $J|_L$ being a radical $S_f$-ideal.

The identity $R(S_f) = \text{rad}_S(I)|_L$ now follows from the fact that $R(S_f)$ is the smallest radical $S_f$-ideal and $\text{rad}_S(I)$ is the smallest radical $S$-ideal containing $I$.

Our main class of examples for radical ideals are the absolute kernels of (possibly non-ergodic) invariant measures. The following result generalizes Proposition 12 of [Sch68] using similar arguments.

**Proposition 3.6.** The following assertions are valid.

(i) For each $\mu \in P_S(K)$ the $S$-ideal $I_\mu$ is a radical $S$-ideal.

(ii) If $K$ is metrizable, then for each radical $S$-ideal $I$ there is $\mu \in P_S(K)$ with $I = I_\mu$.

**Proof.** For (i) let $\mu \in P_S(K)$. By the Choquet-Bishop-de Leeuw Theorem (see Section 4 of [Phe01]) there is a probability measure $\nu \in C(P_S(K))'$ with

$$\langle f, \mu \rangle = \int_{P_S(K)} \langle f, \lambda \rangle \, d\nu(\lambda)$$

for each $f \in C(K)$ such that $\nu(A) = 0$ for each Baire measurable $A \subseteq P_S(K)$ with $A \cap \text{ex} P_S(K) = \emptyset$. We define $M := \text{supp} \nu \cap \text{ex} P_S(K)$ and show that

$$\text{rad}(I_\mu) \subseteq \bigcap_{\lambda \in M} I_\lambda \subseteq I_\mu$$

which implies the claim since $I_\mu \subseteq \text{rad}(I_\mu)$ always holds.
Let $\lambda \in M$ and take $f \in I_\mu$ with $f \geq 0$. This implies

$$0 = \langle f, \mu \rangle = \int_{\text{supp}\, \nu} \langle f, \tilde{\lambda} \rangle \, d\nu(\tilde{\lambda})$$

and in particular $\langle f, \lambda \rangle = 0$. Consequently $I_\mu \subseteq I_\lambda$ which implies $\text{rad}(I_\mu) \subseteq I_\lambda$ and therefore

$$\text{rad}(I_\mu) \subseteq \bigcap_{\lambda \in M} I_\lambda.$$ 

Now consider $f \geq 0$ with $\langle f, \lambda \rangle = 0$ for each $\lambda \in M$. Then

$$U := \{ \tilde{\lambda} \in P_S(K) \mid \langle f, \tilde{\lambda} \rangle > 0 \} \subseteq P_S(K)$$

is open and $U \cap M = \emptyset$. This implies $\nu(U) = 0$ and thus

$$\langle f, \mu \rangle = \int_{\text{supp}\, \nu \setminus U} \langle f, \tilde{\lambda} \rangle \, d\nu(\tilde{\lambda}) = 0,$$

i.e., $f \in I_\mu$.

Now assume that $K$ is metrizable and $I$ a radical $S$-ideal. We may assume that $I = 0$ (otherwise we pass to $C(\text{supp}\, I)$, cf. Lemma 3.3). Take a countable base of the topology consisting of nonempty open sets $U_n, n \in \mathbb{N}$. Since the supports of ergodic measures are dense in $K$ we find $\mu_n \in \text{ex}\, P_S(K)$ with $\text{supp}\, \mu_n \cap U_n \neq \emptyset$ for each $n \in \mathbb{N}$. For

$$\mu := \sum_{n=1}^{\infty} 2^{-n} \mu_n \in P_S(K)$$

we obtain $\mu(U_n) > 0$ for each $n \in \mathbb{N}$, hence $\mu(U) > 0$ for each non-empty open set $U \subseteq K$. \qed

Remark 3.7. Part (ii) of Proposition 3.6 is wrong in the non-metric case. If $K = \beta\mathbb{N} \setminus \mathbb{N}$ and $S$ is the Markov semigroup induced by the shift, then

$$\bigcap_{n \in \mathbb{N}} I_{\mu_n} \nsubseteq \text{rad}(0)$$

for every sequence of probability measures $(\mu_n)_{n \in \mathbb{N}} \subseteq C(K)'$ (cf. Corollary 1.10 of [Cho67]).

4. Centers of Attraction

Radical $S$-ideals can also be described via an ergodic stability condition. To formulate our theorem we recall that a net $(T_\alpha)_{\alpha \in A} \subseteq \mathfrak{c}S \subseteq \mathcal{L}(C(K))$ of operators is right ergodic if

$$\lim_\alpha T_\alpha(\text{Id} - S) = 0$$

for each $S \in S$ with respect to the strong operator topology. We note that there always are right ergodic operator nets for $S$ (see Corollary 1.5 of [Sch13]). The following result generalizes Theorem 4 of [Sch68].
Theorem 4.1. For each support $L \subseteq K$ of an $S$-ideal
\[
\text{rad}_S(I_L) = \left\{ f \in C(K) \left| \lim_{\alpha} \int_{L} T_{\alpha} |f| \, d\mu = 0 \text{ for each } \mu \in C(L)' \right. \right\}
\]
\[
= \left\{ f \in C(K) \left| \lim_{\alpha} (T_{\alpha} |f|)_{L} = 0 \text{ in the norm of } C(L) \right. \right\}
\]
where $(T_{\alpha})_{\alpha \in A}$ is any right ergodic operator net for $S$.

In particular, if $(T_{\alpha})_{\alpha \in A}$ is any right ergodic operator net for $S$, then an $S$-ideal $I_L$ is a radical $S$-ideal if and only if every $f \in C(K)$ satisfying
\[
\lim_{\alpha} (T_{\alpha} |f|)_{L} = 0
\]
in the norm of $C(L)$ vanishes on $L$.

Proof. By Lemma 3.5 we may assume $L = K$. Take $f \in R(S)$ and any right ergodic operator net $(T_{\alpha})_{\alpha \in A}$ for $S$.

Let $\mu \in C(K)'$ and observe that each subnet of $(T'_{\alpha} \mu)_{\alpha \in A}$ has a subnet converging to some $\nu \in P_S(K)$. By Proposition 3.6 the ideal $I_{\nu}$ is an intersection of primitive ideals, hence $\langle |f|, \nu \rangle = 0$. We therefore obtain that each subnet of $(\langle T_{\alpha} |f|, \mu \rangle)_{\alpha \in A}$ has a subnet converging to zero which implies
\[
\lim_{\alpha} (\langle T_{\alpha} |f|, \mu \rangle) = 0.
\]

Now let $f \in C(K)$ with $\lim_{\alpha} T_{\alpha} |f| = 0$ weakly for some right ergodic operator net $(T_{\alpha})_{\alpha \in A}$ for $S$. Then for $\mu \in P_S(K)$
\[
0 = \lim_{\alpha} (\langle T_{\alpha} |f|, \mu \rangle) = \langle |f|, \mu \rangle
\]
which proves $f \in R(S)$ and thus the first equation.

By Theorem 1.7 of \cite{Sch13} the semigroup $S$ is mean ergodic on $R(S)$ with mean ergodic projection $P = 0$ and therefore
\[
R(S) \subseteq \left\{ f \in C(K) \left| \lim_{\alpha} T_{\alpha} |f| = 0 \text{ in the norm of } C(K) \right. \right\}.
\]

The converse inclusion is obvious. \hfill $\square$

If $S$ has a right ergodic operator sequence (for example if it has a Følner sequence (as defined in Assumption 4.4 below), then Lebesgue’s Theorem yields the following result.

Corollary 4.2. Suppose that $(T_n)_{n \in \mathbb{N}}$ is a right ergodic operator sequence for $S$. Then
\[
\text{rad}_S(I) = \left\{ f \in C(K) \left| \lim_{n \to \infty} T_n |f|(x) = 0 \text{ for each } x \in \text{supp } I \right. \right\}
\]
for each $S$-ideal $I$.

In the case where $S$ is the semigroup generated by a Markov lattice homomorphism $T \in L(C(K))$ (i.e., a Koopman operator), the radical $R(S)$ of the zero ideal is the almost weakly stable part of $C(K)$ with respect to $T$ (cf. (9.4) on page 176 of \cite{EFHN13}).
Corollary 4.3. Assume that \( \varphi : K \to K \) is a continuous mapping, \( T_\varphi f := f \circ \varphi \) for \( f \in C(K) \) and \( S = \{ T_n^\varphi \mid n \in \mathbb{N} \} \). Then

\[
R(S) = \left\{ f \in C(K) \left| \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |(T_n^\varphi f, \mu)| = 0 \text{ for each } \mu \in C(K)' \right. \right\}.
\]

For Koopman semigroups \( S \) we obtain a dynamical characterization of \( R(S) \).

Assumption 4.4. For the rest of this section \( S \) is a semigroup acting on \( K \) such that

\( K \to K, \ x \mapsto sx \)

is continuous for each \( s \in S \). Let \( S \) be the associated Koopman semigroup, i.e., \( S = \{ T_s \mid s \in S \} \) with \( T_s f(x) := f(sx) \) for \( f \in C(K), s \in S \) and \( x \in K \).

Moreover, we assume that \((F_n)_{n \in \mathbb{N}}\) is a Følner sequence for \( S \), i.e., each \( F_n \) is a finite subset of \( S \) satisfying

\[
\lim_{n \to \infty} \frac{|F_n \Delta sF_n|}{|F_n|} = 0
\]

for each \( s \in S \).

Remark 4.5. Under Assumption 4.4, \( S \) is left amenable (as a discrete semigroup) and thus \( S \) is right amenable. Moreover we obtain an ergodic operator sequence \((T_n)_{n \in \mathbb{N}}\) for \( S \) by setting

\[
T_n f := \frac{1}{|F_n|} \sum_{s \in F_n} T_s f
\]

for \( f \in C(K) \) and \( n \in \mathbb{N} \).

We now introduce certain “attractors” of the dynamical system \((K; S)\).

Definition 4.6. A closed non-empty set \( L \subseteq K \) is a (global) center of attraction if for each open set \( U \supseteq L \) we have

\[
\lim_{n \to \infty} \frac{1}{|F_n|} |\{ s \in F_n \mid sx \in U \}| = 1
\]

for every \( x \in K \).

Global as well as point-dependent centers of attraction for \( \mathbb{N}_0 \)- and \( \mathbb{R}_{\geq 0} \)-actions have been examined by several authors (see, e.g., [Hil30], [Ber51], [JR72], [Sig77], Exercise I.8.3 in [Man87] and [Dai16]). In a recent paper Z. Chen and X. Dai study the chaotic behaviour of minimal centers of attraction with respect to a point for discrete amenable group actions (cf. [CD17]). It is known that in case of \( \mathbb{N}_0 \)- and \( \mathbb{R}_{\geq 0} \)-actions on metric compact spaces there always is a unique minimal (global) center of attraction given by the closure of the union of the supports of ergodic measures. For the action of a semigroup with a Følner sequence we obtain the following result.

Theorem 4.7. Under Assumption 4.4 the definition of a center of attraction does not depend on the Følner sequence. Moreover, for a closed non-empty set \( L \subseteq K \) the following assertions are equivalent.

(a) \( L \) is a center of attraction.
(b) \( I_L := \{ f \in C(K) \mid f|_L = 0 \} \subseteq R(S) \).

In particular there is a unique minimal center of attraction \( M(S) \) given by the closure of the union of the supports of ergodic measures, i.e.,

\[ M(S) = \text{supp } R(S). \]

**Proof.** Take a non-empty and closed set \( L \subseteq K \). The mapping

\[ I_L : C_0(K \setminus L) \to \mathbb{C}, \quad f \mapsto f|_{K \setminus L} \]

is an isomorphism of Banach lattices. Now \( L \) is a center of attraction if and only if

\[ \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{s \in F_n} \mathbbm{1}_A(sx) = 0 \]

for each compact set \( A \subseteq K \setminus L \) and each \( x \in K \). By Lebesgue’s Theorem this equivalent to

\[ \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{s \in F_n} T_s' \mu(A) = 0 \]

for each \( \mu \in C_0(K \setminus L)' \) and each compact set \( A \subseteq K \setminus L \). But this is the case if and only if

\[ \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{s \in F_n} T_s' \mu = 0 \]

with respect to the weak* topology of \( C_0(K \setminus L)' \) which means that

\[ \lim_{n \to \infty} \mathcal{T}_n[f] = 0 \]

with respect to the weak topology for each \( f \in I_L \), i.e., \( I_L \subseteq R(S) \). \( \Box \)

## 5. The Primitive Spectrum as a Topological Space

In this section we return to a general right amenable Markov semigroup \( S \subseteq \mathcal{L}(C(K)) \) and analyze the topology of \( \text{Prim}(S) \). It turns out that it basically has the same properties as the (non-dynamical) primitive spectrum of \( C^* \)-algebras and the topology of affine schemes of algebraic geometry (cf. Section (2.2) of [GW10]). We employ methods as in Chapter IV of [Dix77] and Section 4.3 of [Ped79] and first prove two technical lemmas before introducing a topology on \( \text{Prim}(S) \). Recall that given \( \mu \in P_S(K) \) we write \( S_\mu \) for the induced semigroup on \( L^1(K, \mu) \).

**Lemma 5.1.** If \( \mu \in P_S(K) \) and \( L \subseteq K \) is the support of an \( S \)-ideal, then \( \mathbbm{1}_L \in \text{fix}(S_\mu) \).

**Proof.** We fix \( S \in S \). For each open set \( O \supseteq L \) take a continuous function \( f_O \) with \( f_O(K) \subseteq [0, 1], f|_L = 1 \) and \( f_O|_{(K \setminus O)} = 0 \). The set \( A \) of open sets containing \( L \) is directed and thus we obtain a net \( (f_O)_{O \in A} \) with

\[ \| \mathbbm{1}_L - f_O \|_{L^1(K, \mu)} \leq \mu(O \setminus L) \to 0 \]
by regularity of $\mu$. By definition of $S_\mu$

$$S_\mu \mathbb{1}_L = \lim_{O} Sf_O.$$

in $L^1(K, \mu)$. Moreover, 

$$Sf_O(x) = \langle Sf_O, \delta_x \rangle = \langle f_O, S'\delta_x \rangle = 1$$

for each $x \in L$ since $L$ is the support of an $S$-ideal. This implies 

$$0 = \lim_{O}(1 - Sf_O) \cdot \mathbb{1}_L = \mathbb{1}_L - S_\mu \mathbb{1}_L \cdot \mathbb{1}_L,$$

where the limit is taken in $L^1(K, \mu)$. Thus 

$$\mathbb{1}_L = S_\mu \mathbb{1}_L \cdot \mathbb{1}_L$$

which shows $\mathbb{1}_L \subseteq S_\mu \mathbb{1}_L$ and consequently $\mathbb{1}_L \in \text{fix}(S_\mu)$ by Theorem 13.2 (d) of [EFHN15].

\underline{Lemma 5.2}. Consider two $S$-ideals $I_1, I_2$. If $p$ is a primitive $S$-ideal with $I_1 \cap I_2 \subseteq p$, then $I_1 \subseteq p$ or $I_2 \subseteq p$.

\textbf{Proof}. Let $p = I_\mu$ for some $\mu \in \text{ex P}_S(K)$ and let $L_j := \text{supp } I_j$ for $j = 1, 2$. By Lemma 5.1, $\mathbb{1}_{L_j} \in \text{fix}(S_\mu)$ for $j = 1, 2$ and therefore $\mu(L_j) \in \{0, 1\}$ by Proposition 2.5. Since $\text{supp } \mu \subseteq L_1 \cup L_2$ we know that $\mu(L_1 \cup L_2) = 1$, so there is $j \in \{1, 2\}$ with $\mu(L_j) = 1$. But this means $\text{supp } \mu \subseteq L_j$ and consequently $I_j \subseteq p$. \hfill $\Box$

We are now ready to equip $\text{Prim}(S)$ with a topology and remind the reader that $\text{hull}$ and $\text{kernel}$ were introduced in Definition 3.1.

\underline{Proposition 5.3}. The mapping 

$$\text{hull}(\text{ker}(A)) = \{ p \in \text{Prim}(S) \mid \text{ker}(A_1) \cap \text{ker}(A_2) \subseteq p \}$$

defines a Kuratowski closure operator.

\textbf{Proof}. It is readily checked that 

$$\emptyset = \emptyset, A \subseteq \overline{A} \text{ and } \overline{A} = \overline{A}$$

for each $A \subseteq \text{Prim}(S)$. It remains to show that $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$ for all $A_1, A_2 \subseteq \text{Prim}(S)$. Applying Lemma 5.2 to the ideals $I_j := \text{ker}(A_j)$ for $j = 1, 2$ yields 

$$\overline{A_1 \cup A_2} = \text{hull}(\text{ker}(A_1 \cup A_2)) = \text{hull}(\text{ker}(A_1) \cap \text{ker}(A_2)) = \{ p \in \text{Prim}(S) \mid \text{ker}(A_1) \cap \text{ker}(A_2) \subseteq p \} = \overline{A_1} \cup \overline{A_2}.$$

\underline{Definition 5.4}. The topology on $\text{Prim}(S)$ induced by the closure operator of Proposition 5.3 is called the $\text{hull}$-$\text{kernel}$-$\text{topology}$.

We from now on equip $\text{Prim}(S)$ with the hull-kernel-topology.

\underline{Proposition 5.5}. The following assertions are valid.

(i) The mappings 

$$\{ \emptyset \neq A \subseteq \text{Prim}(S) \text{ closed} \} \leftrightarrow \text{Rad}(S)$$

$$A \mapsto \text{ker}(A) \quad \text{hull}(I) \leftrightarrow I$$

are mutually inverse bijections.
(ii) The sets
\[ U_f := \{ p \in \text{Prim}(\mathcal{S}) \mid f \notin p \} \]
for \( f \in C(K) \) define a base for the hull-kernel-topology of \( \text{Prim}(\mathcal{S}) \).

(iii) If \( K \) is metrizable, then \( \text{Prim}(\mathcal{S}) \) has a countable base.

(iv) The space \( \text{Prim}(\mathcal{S}) \) is \( T_0 \). Given \( p \in \text{Prim}(\mathcal{S}) \), the set \( \{ p \} \) is closed if and only if \( p \) is a maximal \( S \)-ideal.

(v) The space \( \text{Prim}(\mathcal{S}) \) is quasi-compact.

(vi) The mapping 
\[ \pi : \text{ex P}_S(K) \to \text{Prim}(\mathcal{S}), \quad \mu \mapsto I_\mu \]
is continuous and surjective.

**Proof.** Assertion (i) is obvious. Clearly we have \( \text{Prim}(\mathcal{S}) = U_1 \). Now take a closed set \( \emptyset \neq A \subseteq \text{Prim}(\mathcal{S}) \). Then \( A = \ker(I) \) for some \( S \)-ideal \( I \) and we obtain
\[ \text{Prim}(\mathcal{S}) \setminus A = \bigcup_{f \in I} \{ p \in \text{Prim}(\mathcal{S}) \mid f \notin p \} = \bigcup_{f \in I} U_f. \]
Moreover, each \( U_f \) is open since \( \text{Prim}(\mathcal{S}) \setminus U_f = \text{hull}(\{ f \}) \). This proves (ii) and then (iii) is a direct consequence.

If \( p_1, p_2 \in \text{Prim}(\mathcal{S}) \) with \( p_1 \neq p_2 \), then \( M_1 \neq M_2 \) for the supports \( M_i := \text{supp} p_i, \ i = 1, 2 \). We may assume that there is \( x \in M_2 \setminus M_1 \) and find \( f \in C(K) \) with \( f|_{M_1} = 0 \) and \( f(x) = 1 \). Then \( p_1 \notin U_f \) and \( p_2 \in U_f \) which shows that \( \text{Prim}(\mathcal{S}) \) is a \( T_0 \)-space.

Take a maximal \( S \)-ideal and assume that \( p \in \overline{\{ m \}} \). Then \( m \subseteq p \) and thus \( m = p \) by maximality of \( m \).

Conversely, suppose that \( \{ m \} \) is closed and take a maximal \( S \)-ideal \( p \) with \( m \subseteq p \). Then \( \ker(\{ m \}) = m \subseteq p \) and thus 
\[ p \in \text{hull}(\ker(\{ m \})) = \overline{\{ m \}}, \]
i.e., \( p = m \).

Take closed subsets \( A_j \subseteq \text{Prim}(\mathcal{S}) \) for \( j \in J \) with 
\[ \bigcap_{j \in J} A_j = \emptyset \]
and let \( I_j := \ker(A_j) \) be the corresponding radical ideals for \( j \in J \). We show that 
\[ \sum_{j \in J} I_j = C(K). \]
Denote the ideal on the left side by \( I \) and assume that it is a proper invariant ideal. Since there are no dense ideals in \( C(K) \), the closure \( \overline{I} \) is contained in a maximal \( S \)-ideal \( p \). But then \( p \in A_j \) for each \( j \in J \) since the sets \( A_j \) are closed, a contradiction.

Take \( j_1, \ldots, j_k \) with \( 1 \in I_{j_1} + \ldots + I_{j_k} \) for some \( k \in \mathbb{N} \). Then 
\[ \sum_{m=1}^{k} I_{j_m} = C(K) \]
and consequently
\[ \bigcap_{m=1}^{k} A_{j_m} = \emptyset. \]
To see that the mapping \( \pi \) of (vi) is continuous observe that for \( f \in C(K) \) the set
\[ \pi^{-1}(U_f) = \{ \mu \in \text{ex}P_S(K) \mid \langle |f|, \mu \rangle \neq 0 \} \]
is open in \( \text{ex}P_S(K) \).

**Example 5.6.**

(i) For the trivial semigroup \( S = \{ \text{Id} \} \) every ideal is invariant and \( \text{Prim}(S) \) coincides with the maximal ideal space of the commutative \( C^\ast \)-algebra \( C(K) \), i.e., it is homeomorphic to \( K \).

(ii) Consider the torus \( K = T := \{ z \in \mathbb{C} \mid |z| = 1 \} \) and the rotation \( \varphi_a(z) := az \) for \( z \in T \) and some fixed \( a \in T \) with \( a^k = 1 \). Denote the group of \( k \)th roots of unity by \( G_k \). The supports of ergodic measures are then given by
\[ M_b := bG_k = \{ bz \in T \mid z^k = 1 \} \]
for \( b \in T \) and we obtain a homeomorphism
\[ T/G_k \rightarrow \text{Prim}(\{ T_n^\varphi \mid n \in \mathbb{N} \}), \quad bG_k \mapsto I_{M_b} \]
if we endow the factor group \( T/G_k \) with the quotient topology.

(iii) Consider the space \( L := \{0,1\}^\mathbb{N} \) and the shift \( \varphi: L \rightarrow L \) given by
\[ \varphi((x_n)_{n \in \mathbb{N}}) := (x_{n+1})_{n \in \mathbb{N}} \]
for each \( (x_n)_{n \in \mathbb{N}} \in L \). For each \( k \in \mathbb{N} \) consider the minimal set
\[ M_k := \{ \varphi^n(x^k) \mid n \in \{0, \ldots, 2k - 1\} \} \]
with \( x^k = (x_m)_{m \in \mathbb{N}} \) defined by
\[ x_m := \begin{cases} 0 & \text{if } m \in \{1, \ldots, k\} + 2k\mathbb{N}_0, \\ 1 & \text{else}. \end{cases} \]
Now take the subsystem \((K; \varphi)\) given by the closure
\[ K := \bigcup_{k \in \mathbb{N}} M_k. \]
Then it is readily seen that \( K \) is the set
\[ \bigcup_{k \in \mathbb{N}} M_k \cup \{(x_m)_{m \in \mathbb{N}} \in L \mid (x_m)_{m \in \mathbb{N}} \text{ increasing or decreasing}\}. \]
We claim that \( \text{Prim}(\{ T_n^\varphi \mid n \in \mathbb{N} \}) \) is not Hausdorff. It suffices to show that the sequence \((m_n)_{n \in \mathbb{N}} \) in \( \text{Prim}(\{ T_n^\varphi \mid n \in \mathbb{N} \}) \) with \( m_n := I_{M_n} \) converges to two different points.
To this end, consider \( k \in \mathbb{N} \) and the open subset
\[ U := \left( \prod_{i=1}^{k} \{1\} \times \prod_{i=k+1}^{\infty} \{0,1\} \right) \cap K. \]
Then \( U \) is a neighborhood of \((1)_{n \in \mathbb{N}}\) and \( M_l \cap U \neq \emptyset \) for each \( l \geq k \). By Proposition 5.5 (ii) this implies \( m_l \to (1)_{n \in \mathbb{N}} \) and a similar argument shows \( m_l \to (0)_{n \in \mathbb{N}} \).

6. Continuous Functions on the Primitive Spectrum

It is our goal to describe the continuous functions on \( \text{Prim}(S) \). As above we write \( M(S) \) for the support of \( R(S) \), i.e., the closure of the union of all supports of invariant ergodic measures, and recall that the semigroup on \( C(M(S)) \) induced by \( S \) is denoted by \( S_{R(S)} \). Now consider the following functions.

**Definition 6.1.** For a function \( f \in \text{fix}(S_{R(S)}) \) we define

\[
\hat{f} : \text{Prim}(S) \to C, \quad I_\mu \mapsto \int_{M(S)} f \, d\mu.
\]

Since continuous fixed functions are constant on supports of ergodic measures, the function \( \hat{f} \) is in fact well-defined for each \( f \in \text{fix}(S) \) and the next lemma shows that it is also continuous.

**Lemma 6.2.** If \( f \in \text{fix}(S_{R(S)}) \) then \( \hat{f} \in C(\text{Prim}(S)) \).

**Proof.** Let \( p = I_\mu \in \text{Prim}(S) \) and \( \varepsilon > 0 \). We set

\[
f_\varepsilon := \sup \left( \varepsilon \cdot 1 - \left| f - \int_{M(S)} f \, d\mu \cdot 1 \right|, 0 \right) \in C(M(S)).
\]

Then \( U := U_{f_\varepsilon} \) is an open neighborhood of \( p \). Moreover, for each \( q = I_\nu \in U \)

\[
\varepsilon - \left| \int_{M(S)} f \, d\nu - \int_{M(S)} f \, d\mu \right| > 0
\]

which means \( |\hat{f}(p) - \hat{f}(q)| < \varepsilon \).

In the radical free case (i.e., \( M(S) = K \)) we now obtain a linear mapping from the fixed space \( \text{fix}(S) \) to \( C(\text{Prim}(S)) \). It turns out that this is actually an isomorphism.

**Theorem 6.3.** If \( S \) is radical free, then the fixed space \( \text{fix}(S) \) is a Banach sublattice of \( C(K) \) and the mapping

\[
\hat{\cdot}: \text{fix}(S) \to C(\text{Prim}(S)), \quad f \mapsto \hat{f}
\]

is an isometric Banach lattice isomorphism.

**Proof.** We first show that \( \text{fix}(S) \) is a sublattice of \( C(K) \). Take \( f \in \text{fix}(S) \) and \( S \in S \). Since \( M(S) = K \), it suffices to prove that \( S_p |f|_{\text{supp} p} = |f|_{\text{supp} p} \) for each \( p \in \text{Prim}(S) \). However, this is true since the fixed space \( \text{fix}(S_p) \) consists only of constant functions for every \( p \in \text{Prim}(S) \).

The mapping \( \hat{\cdot} \) is clearly linear. For \( f \in \text{fix}(S) \)

\[
\|\hat{f}\| = \sup_{p \in \text{Prim}(S)} |\hat{f}(p)| = \sup_{\mu \in \text{ex P}_S(K)} |\langle f, \mu \rangle| \leq \|f\|.
\]
The set \( \{ x \in K \mid |f(x)| = \|f\| \} \) is the support of an \( \mathcal{S} \)-ideal (cf. Theorem 1.2 in [Sim68]) and thus contains a minimal support of an \( \mathcal{S} \)-ideal \( M \) which in turn supports an ergodic measure \( \mu \). This implies

\[
|\hat{f}(M)| = |\langle f, \mu \rangle| = |f(x)| = \|f\|
\]

for each \( x \in M \) and consequently \( \|\hat{f}\| = \|f\| \).

Now take \( f \in \text{fix}(\mathcal{S}) \) and \( p = I_\mu \in \text{Prim}(\mathcal{S}) \). For each \( x \in \text{supp}(\mu) \)

\[
|\hat{f}(p)| = |f(x)| = |f|(x) = |\hat{f}(p)|.
\]

It remains to show that \( ^\wedge \) is surjective. Take \( f \in C(\text{Prim}(\mathcal{S})) \) with \( 0 \leq f \leq 1 \).

We fix \( n \in \mathbb{N} \) and consider the open sets

\[
U_{k,n}:= \left\{ p \in \text{Prim}(\mathcal{S}) \mid \frac{k-1}{n} < f(p) < \frac{k+1}{n} \right\}
\]

for \( k \in \{0, ..., n\} \). Then \( U_{k,n} = \text{hull}(I_{k,n}) \) for invariant ideals \( I_{k,n} \subseteq C(K) \) and \( k \in \{0, ..., n\} \). Assume

\[
I := \sum_{k=0}^{n} I_{n,k} \neq C(K).
\]

Then \( I \) is contained in a maximal \( \mathcal{S} \)-ideal \( p \). Since \( I_{n,k} \subseteq p \) for all \( k \in \{0, ..., n\} \),

\[
p \in \bigcap_{k=0}^{n} \text{hull}(I_{n,k}) = \left( \bigcup_{k=0}^{n} U_{k,n} \right)^c = \emptyset,
\]

a contradiction.

We thus find \( 0 \leq f_{n,k} \in I_{n,k} \) for \( k \in \{0, ..., n\} \) with \( 1 = \sum_{k=0}^{n} f_{n,k} \) (cf. II.5.1.4 in [Bla06]). Now set \( g_n := \sum_{k=1}^{n} \frac{k}{n} f_{n,k} \).

Take \( p \in \text{Prim}(\mathcal{S}) \). If \( k \in \{0, ..., n\} \) with \( p \notin U_{k,n} \), then \( I_{n,k} \subseteq p \) and therefore \( f_{n,k} \in p \), i.e., \( f_{n,k}|_{\text{supp } p} = 0 \). This implies

\[
|g_n(x) - f(p)| = \left| \sum_{k : p \in U_{k,n}} \frac{k}{n} f_{n,k}(x) - \sum_{k : p \not\in U_{k,n}} f_{n,k}(x) f(p) \right| \\
\leq \sum_{k : p \in U_{k,n}} \left| \frac{k}{n} - f(p) \right| |f_{n,k}(x)| \leq \frac{1}{n}
\]

for \( x \in \text{supp } p \). In particular we obtain

\[
|g_n(x) - g_m(x)| \leq \frac{1}{n} + \frac{1}{m}
\]

for all \( x \in M := \bigcup_{p \in \text{Prim}(\mathcal{S})} \text{supp } p \) and all \( n, m \in \mathbb{N} \). Since \( \mathcal{S} \) is radical free, \( M \) is dense whence \((g_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \( C(K) \). Denoting its limit by \( g \) we obtain \( g(x) = f(p) \) for \( x \in \text{supp } p \), \( p \in \text{Prim}(\mathcal{S}) \) and, since \( M \) is dense, \( g \in \text{fix}(\mathcal{S}) \). Moreover, we clearly have \( g = f \).

\[\square\]

The first part of the proof of Theorem 6.5 is based on the proof of Theorem 5 in [Sch68], while the second part uses arguments from the Dauns-Hofmann Theorem II.6.5.10 in [Bla06].

We now focus on the general case, i.e., \( \mathcal{S} \) not being radical free.
Lemma 6.4. The mapping 
\[ \vartheta : \text{Prim}(\mathcal{S}) \rightarrow \text{Prim}(\mathcal{S}_R(\mathcal{S})), \quad p \mapsto p|_{M(\mathcal{S})} \]
is a homeomorphism.

Proof. Note first that \( \vartheta \) is well-defined and bijective by Lemma 3.5 since \( R(\mathcal{S}) \subseteq p \) for each \( p \in \text{Prim}(\mathcal{S}) \). Lemma 3.5 also implies that 
\[ \text{Rad}(\mathcal{S}) \rightarrow \text{Rad}(\mathcal{S}_R(\mathcal{S})), \quad I \mapsto I|_{M(\mathcal{S})} \]
is bijective. By Proposition 5.5 we obtain that \( A \subseteq \text{Prim}(\mathcal{S}) \) is closed if and only if \( A = \text{hull}(I) \) for some \( I \in \text{Rad}(\mathcal{S}) \), which means 
\[
\vartheta(A) = \{ q \in \text{Prim}(\mathcal{S}_R(\mathcal{S})) \mid I|_{M(\mathcal{S})} \subseteq q \} = \text{hull}(I|_{M(\mathcal{S})}),
\]
i.e., if and only if \( \vartheta(A) \) is closed. \( \Box \)

By combining Lemma 6.4 with Theorem 6.3 we obtain our main result.

Theorem 6.5. The fixed space \( \text{fix}(\mathcal{S}_R(\mathcal{S})) \) is a Banach sublattice of \( C(M(\mathcal{S})) \) and the mapping 
\[ \hat{\cdot} : \text{fix}(\mathcal{S}_R(\mathcal{S})) \rightarrow C(\text{Prim}(\mathcal{S})), \quad f \mapsto \hat{f} \]
is an isometric Banach lattice isomorphism.

7. Mean Ergodic Semigroups of Markov Operators

With the description of the space \( C(\text{Prim}(\mathcal{S})) \) we can now analyze mean ergodicity of Markov semigroups extending Theorem 2 of [Sch67].

Theorem 7.1. The following assertions are equivalent.
(a) \( \mathcal{S} \) is mean ergodic.
(b) The following two conditions are satisfied.
   (i) The mapping 
   \[ \text{ex } P_\mathcal{S}(K) \rightarrow \text{Prim}(\mathcal{S}), \quad \mu \mapsto I_\mu \]
is a homeomorphism. 
   (ii) For each \( f \in \text{fix}(\mathcal{S}_R(\mathcal{S})) \) there is \( F \in \text{fix}(\mathcal{S}) \) with \( f = F|_{M(\mathcal{S})} \).
(c) The following three conditions are satisfied.
   (i) The primitive spectrum \( \text{Prim}(\mathcal{S}) \) is a Hausdorff space.
   (ii) For each \( \mu \in \mathcal{P}_\mathcal{S}(K) \) the support \( \text{supp } \mu \) is uniquely ergodic, i.e., \( \mu \) is the only invariant measure having its support in \( \text{supp } \mu \).
   (iii) For each \( f \in \text{fix}(\mathcal{S}_R(\mathcal{S})) \) there is \( F \in \text{fix}(\mathcal{S}) \) with \( f = F|_{M(\mathcal{S})} \).

Proof. Assume that \( \mathcal{S} \) is mean ergodic with mean ergodic projection \( P \in \mathcal{L}'(C(K)) \). We first show that \( \text{Prim}(\mathcal{S}) \) is Hausdorff.
Consider $I_{\mu_1}, I_{\mu_2} \in \text{Prim}(S)$ with $\mu_1 \neq \mu_2$. Since $S$ is mean ergodic, we find $f \in \text{fix}(S)$ with

$$c_1 := \langle f, \mu_1 \rangle < \langle f, \mu_2 \rangle =: c_2.$$ 

Choose $c \in (c_1, c_2)$ and set $U_1 := f^{-1}((-\infty, c))$ and $U_2 := f^{-1}((c, \infty))$. The sets

$$V_i := \{ p \in \text{Prim}(S) \mid \text{supp} p \cap U_i \neq \emptyset \} \subseteq \text{Prim}(S)$$

are open by Proposition 5.5(ii) and $I_{\mu_i} \subseteq V_i$ for $i = 1, 2$.

Assume there is $p \in V_1 \cap V_2$. Then there are $x_i \in U_i \cap \text{supp} p$ for $i = 1, 2$ and thus $f(x_1) < c < f(x_2)$. Since $f$ is constant on supports of ergodic measures, this is a contradiction.

Given $\mu \in \text{ex P}_S(K)$ we know that $\mathcal{S}_{I_{\mu}} \subseteq \mathcal{L}(\text{C(supp} \mu))$ is also mean ergodic. Since $\text{fix}(\mathcal{S}_{I_{\mu}})$ is one-dimensional, we obtain that $\text{fix}(\mathcal{S}_{I_{\mu}})$ is isometric.

Finally assume that (b) is valid. The mapping $\pi : \text{ex P}_S(K) \to \text{Prim}(\mathcal{S}_R(S))$, $\mu \mapsto I_{\mu}$ is a homeomorphism since the mapping is injective by (iii), $\text{ex P}_S(K)$ is Hausdorff.

Finally assume that (b) is valid. The mapping $\Phi_1 : \text{C(Prim}(S)) \to \text{C(ex P}(K))$, $f \mapsto f \circ \pi$ is then an isometric isomorphism of Banach lattices and by Theorem 6.5 the mapping

$$^\wedge : \text{fix}(\mathcal{S}_R(S)) \to \text{C(Prim}(S))$$

is also. Now consider the map

$$\Phi_2 : \text{fix}(S) \to \text{fix}(\mathcal{S}_R(S))$$

This is an isometric Banach space embedding (isometry follows with the same arguments as in the proof of Theorem 6.5 and by (ii) it is surjective.
Thus \( \Phi := \Phi_1 \circ \Phi_2 : \text{fix}(S) \rightarrow C(\text{ex } P(K)), \quad f \mapsto \langle f, \cdot \rangle \)

is an isometric isomorphism of Banach spaces.

Now take \( \mu_1, \mu_2 \in P_S(K) \) with \( \mu_1 \neq \mu_2 \). The space \( \text{ex } P(K) \) is compact since it is homeomorphic to \( \text{Prim}(S) \). By Choquet theory (cf. Proposition 1.2 in [Phe01]) we thus find measures \( \tilde{\mu}_1, \tilde{\mu}_2 \in C(\text{ex } P(K))' \) such that

\[
\langle f, \mu_i \rangle = \int_{\text{ex } P_S(K)} \langle f, \nu \rangle \, d\tilde{\mu}_i(\nu)
\]

for each \( f \in C(K) \) and \( i = 1, 2 \). We then obtain

\[
\langle f, \mu_i \rangle = \int_{\text{ex } P_S(K)} \Phi(f)(\nu) \, d\tilde{\mu}_i(\nu) = \langle \Phi(f), \tilde{\mu}_i \rangle
\]

for each \( f \in \text{fix}(S) \) and \( i = 1, 2 \). Since \( C(\text{ex } P_S(K)) \) separates \( C(\text{ex } P_S(K))' \), this proves that \( \text{fix}(S) \) separates \( P_S(K) \). Now \( S \) consists of Markov operators and therefore \( \text{fix}(S) \) separates \( \text{fix}(S)' \). Thus \( S \) is mean ergodic by Theorem 1.7 of [Sch13].

\[ \Box \]

**Corollary 7.2.** If \( S \) is radical free, then \( S \) is mean ergodic if and only if

\[
\text{ex } P_S(K) \rightarrow \text{Prim}(S), \quad \mu \mapsto I_\mu
\]

is a homeomorphism.

The next corollary follows from Proposition 2.9 and Lemma 6.4.

**Corollary 7.3.** The semigroup \( S \) is mean ergodic if and only if \( S_{R(S)} \) is mean ergodic and for each \( f \in \text{fix}(S_{R(S)}) \) there is \( F \in \text{fix}(S) \) with \( f = F|_{M(S)} \).

Finally we discuss some examples showing that the conditions of Theorem 7.1 (c) are independent of each other.

**Example 7.4.** Consider the following continuous mappings \( \varphi : K \rightarrow K \)

and the induced semigroups \( S = \{ T^n_\varphi \mid n \in \mathbb{N} \} \subseteq \mathcal{L}(C(K)) \).

(i) If \( K = [0, 1] \) and \( \varphi(x) = x^2 \) for \( x \in K \), then \( M(S) = \{ 0, 1 \} \) and the primitive spectrum is the two point discrete space. Clearly, both fixed points define uniquely ergodic sets, so conditions (i) and (ii) of Theorem 7.1 (c) are fulfilled. However, \( S \) is not mean ergodic since the function \( f : \{ 0, 1 \} \rightarrow \mathbb{C} \) defined by \( f(0) := 0 \) and \( f(1) := 1 \) has no invariant continuous extension to \( K \).

(ii) If \( K = \mathbb{T} \) then there is a homeomorphism \( \varphi : \mathbb{T} \rightarrow \mathbb{T} \) such that \( S \) is not uniquely ergodic, but minimal (see Theorem 5.8 of [Par81]). Thus the primitive spectrum is trivial and \( S \) is radical free whence Theorem 7.1 (c) (i) and (iii) are valid. \( S \) is not mean ergodic since supports of ergodic measures are not uniquely ergodic.

(iii) If \( K \) and \( \varphi \) are defined as in Example 5.6 (iii), then supports of ergodic measures are uniquely ergodic and \( S \) is radical free. Thus Theorem 7.1 (c) (ii) and (iii) are fulfilled. The semigroup \( S \) is not mean ergodic since the primitive spectrum is not Hausdorff.
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**HENRIK KREIDLER, Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany**

*E-mail address: hekr@fa.uni-tuebingen.de*