From Symmetry Breaking to Topology Change I

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Abstract
By means of an analogy with Classical Mechanics and Geometrical Optics, we are able to reduce Lagrangians to a kinetic term only. This form enables us to examine the extended solution set of field theories by finding the geodesics of this kinetic term’s metric. This new geometrical standpoint sheds light on some foundational issues of QFT and brings to the forefront core aspects of field theory.

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1 Introduction

Classical gauge theories are well described through differential geometry, where a gauge field is represented by a connection on a principal fibre bundle $P$ for which the structure group is the symmetry group of the theory; see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9] and references therein.

The phenomenon of spontaneous symmetry breaking also has its own geometric formulation [4, 5, 6, 7, 8, 9, 10]: the reduction of the principal bundle $P$. To this classical picture, the question that arises is that of how does a quantum field theory select its vacua given by the different possible reductions of the principal bundle $P$. One canonical answer to this question is given by the so-called “Spontaneous Symmetry Breaking” mechanism, i.e., radiative corrections (beyond the tree-level approximation) cause the QFT to “jump” from the symmetric phase to the broken-symmetric one(s) [13].

However, [13] already expresses concerns about some issues (e.g., the last paragraph of page 1894 and its continuation on page 1895) that are more explicitly treated in [14]. (For more examples about these issues, see [15] — pages 27 and 28 discuss the asymptotic nature of the series expansion, their maximum accuracy and issues regarding Borel summation — and [16].)

The analysis of the boundary conditions of the Schwinger-Dyson equation being responsible for the different phases of the theory done in [14] is more in the spirit of the self-adjoint extension of the associated Hamiltonian, as done in [17, 18]. Although it may not seem so at first, [14] is equivalent to a latticized approach to the QFT and, as such, requires 2 types of infinite limits in order to give rise to its continuum version: the thermodynamic (number of particles in the box) and the volume (size of the box) limit.

These limits do not commute and fiddling with them (as done in [14]) is analogous to resumming the perturbative series (as done in [13]).

The present work has the goal of developing a geometrical method that brings to the foreground issues of phase structure in QFT and clearly showing that these different phases are topologically inequivalent. The fact, that the different configuration spaces for the distinct solutions of the equation of motion of a given Lagrangian have different topologies, shows that expansions must be performed separately for each solution of a theory, i.e., each phase has to be regarded and treated as a separate theory.

In this sense, this work streamlines how the parameters of a theory (mass, coupling constants, etc.) determine the topology of each phase which, combined with the picture presented in [14], gives a prescription of how to examine the solution space of a field theory: The boundary conditions of the Schwinger-Dyson equations determine the parameters of the theory which, in turn, determine its phase (and topology). To this we can add the Renormalization Group framework and see how the phases of a theory (and their topologies) change: as one flows the Renormalization Group, the theory moves through its phases (and its inequivalent topologies). Note that just like in [14], the number of different solutions, found in this work, to a theory whose potential energy is given by a polynomial of degree $n$ in the fields (with no derivative couplings) is $n - 1$ times infinity, i.e., $n - 1$ solutions per spacetime point.

Moreover, when applied to String Theory, under the Landscape paradigm, this result explicitly says that each different minima of the Landscape potential has a different spacetime topology. Therefore, spontaneous symmetry breaking would lead to topological change in spacetime, a key ingredient in String Theory — doing away with the use of a Coleman-De Luccia [19] type of mechanism.

In section 2 we will put the method forward and demonstrate it on the free field. In section 3 we will apply this method for 3 different theories: $\lambda \phi^4$, Landau-Ginzburg and Seiberg-Witten.
(Note that the main difference between $\lambda \phi^4$ and Landau-Ginzburg and Seiberg-Witten is that the first one is a scalar field theory without gauge symmetry, while the latter two do have gauge fields.) Then, in section 4, we will conclude summarizing our results.

2 Quantum Field Theory and the Jacobi Metric

A typical action for a scalar field in QFT has the form,

$$ S[\phi] = \int K(\pi, \pi) - V_p(\phi) \, d^nx ; $$

$$ K(\pi, \pi) = \frac{1}{2} g(\pi, \pi) = \frac{1}{2} g_{\mu \nu} \pi^\mu \pi^\nu . $$

where $K$ is the kinetic quadratic form (bilinear and, in case $\pi \in \mathbb{C}$, hermitian; defining the inner product $g$) and $V_p$ is the potential, the index $p$ denoting coupling constants, mass terms, etc.

We want to reparameterize our field using the arc-length parameterization such that $\tilde{g}(\tilde{\pi}, \tilde{\pi}) = \tilde{g}_{\mu \nu} \tilde{\pi}^\mu \tilde{\pi}^\nu = 1$, where $\tilde{g}$ is the new metric and $\tilde{\pi}^\mu$ is the momentum field redefined in terms of $g$ (assuming $V_p(\phi)$ contains no derivative couplings). That is, we want to dilate or shrink our coordinate system in order to obtain a conformal transformation of the metric that normalizes our momentum field.

In order to do that, we will borrow an idea from classical mechanics called Jacobi’s metric [20]. In the Newtonian setting, Jacobi’s metric gives an intrinsic geometry for the configuration space, where dynamical orbits become geodesics (i.e., it maps every Hamiltonian flow into a geodesic one; therefore, solving the equations of motion implies finding the geodesics of Jacobi’s metric and vice-versa). This can be done for any closed non-dissipative system (with total energy $E$), regardless of the number of degrees of freedom.

Before we proceed any further, let us take a look at a simple example in order to motivate our coming definition of Jacobi’s metric. Let us consider a model similar to the harmonic oscillator (without further considerations, we simply allow for the analytic continuation of the frequency, $\omega^2 = \mu$ or $\omega^2 = i \mu$) in Quantum Mechanics (1-dimensional QFT). Its Lagrangian is given by $L = \frac{1}{2} \dot{q}^2 \mp \frac{\mu}{2} q^2$ ($\mu > 0$), from which we conclude that, for a fixed $E$ such that $E = \frac{1}{2} (\dot{q}^2 \pm \mu q^2)$, we have $\dot{q} = \frac{dq}{dt} = \sqrt{2 \left( E \mp \frac{\mu}{2} q^2 \right) .}$

So, in the spirit of what was said above, we want to find a reparameterization of the time coordinate in order to have the normalization $\dot{q} = 1$, where the dot represents a derivative with respect to this new time variable.

Thus,

$$ \frac{dq}{dt} = \frac{dq}{ds} \frac{ds}{dt} = \sqrt{2 \left( E \mp \frac{\mu}{2} q^2 \right) } \quad \text{if} \quad \frac{ds}{dt} = \sqrt{2 \left( E \mp \frac{\mu}{2} q^2 \right) } \quad \Rightarrow \frac{dq}{ds} \equiv 1 . $$

$$ \therefore \quad ds = \sqrt{2 \left( E \mp \frac{\mu}{2} q^2 \right) } \, dt ; $$

$$ \Rightarrow \quad \tilde{g}_E = 2 \left( E \mp \frac{\mu}{2} q^2 \right) g . $$
As expected, the new metric, $\tilde{g}_E$, is a conformal transformation of the original one, $g$; and under this new metric, our original Lagrangian is simply written as,

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^2 \mp \frac{\mu}{2} q^2 ;$$

$$= \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j \mp \frac{\mu}{2} q^2 ;$$

$$= \frac{1}{2} \tilde{g}(\dot{q}, \dot{q}) \mp \frac{\mu}{2} q^2 ;$$

$$\therefore L(q, \dot{q}) = \tilde{g}_E(\dot{q}, \dot{q}) .$$

Now, as we said above [20], the Euler-Lagrange equations for this conformally transformed Lagrangian are simply the geodesics of the metric $\tilde{g}_E$. The task of finding the geodesics $\gamma(s)$ of the $\tilde{g}_E$ metric can be more easily accomplished with the help of the normalization condition,

$$\tilde{g}_E \left( \frac{d\gamma}{ds}, \frac{d\gamma}{ds} \right) = 1,$$

and the initial condition $\gamma(s = 0) = 0$, given that, in this fashion, we only need to solve a first order differential equation:

$$\tilde{g}_E \left( \frac{d\gamma}{ds}, \frac{d\gamma}{ds} \right) = 1 ;$$

$$\Rightarrow 2 \left( E \mp \frac{\mu}{2} \gamma^2 \right) g \left( \frac{d\gamma}{ds}, \frac{d\gamma}{ds} \right) = 1 ;$$

$$\therefore \frac{d\gamma}{ds} = \sqrt{\frac{1}{2 \left( E \mp \frac{\mu}{2} \gamma^2 \right) }} ;$$

with the initial condition $\gamma(0) = 0$.

Analytically solving the equation above yields 2 possible answers, depending on the particular form of the potential (note that $\mu > 0$ in both cases):

1. $V_+ = +\mu \gamma^2/2$: For the case of a positive pre-factor, we get that $\gamma \sqrt{\mu (2E - \mu \gamma^2)} + 2E \arcsin(\gamma \sqrt{\mu/2E}) - 2s \sqrt{\mu} = 0$; and

2. $V_- = -\mu \gamma^2/2$: For the case of a negative pre-factor, we find that $\gamma \sqrt{\mu (2E + \mu \gamma^2)} + 2E \arcsinh(\gamma \sqrt{\mu/2E}) - 2s \sqrt{\mu} = 0$.

It is clear from the expression for $V_+$ that $\gamma^2 \leq 2E/\mu$, i.e., the length of the [classical] geodesic is bounded; this does not happen with $V_-$. These geodesics, $\gamma_{\pm}$, clearly depend on the parameters $E$ and $\mu$; therefore, in order to plot $\gamma(s)$, we have to make 2 distinct choices: $E/\mu = 1$ (left plot) and $E/\mu = -1$ (right plot). The last plot comparatively depicts both geodesics.
This “harmonic oscillator” model already portrays the features that we want to identify in forthcoming applications of this technique: changing the values of the parameters of the potential we can identify 2 distinct types of geodesics which will be related to different phases on the coming examples.

Note that we do not need to engage on discussions of the stability (boundedness) of the potential for all we wanted to show was that the parameters of the potential will give rise to qualitatively and quantitatively different geodesic solutions. We could have used any other [bounded] potential, e.g., \( V(q) = -\frac{\mu}{2} q^2 + \frac{\lambda}{4} q^4 \) and done an analogous analysis. However, we just wanted to keep the number of parameters \( (E, \mu, \lambda) \) low, hence our choice.

At this point, the generalization of Jacobi’s metric to the field theoretical setting can be accomplished via the definition of the following conformally transformed metric:

\[
L = \frac{1}{2} \tilde{g}(\pi, \pi) - V_p(\phi) ;
\]

\[
\equiv \tilde{g}_E(\pi, \pi) ;
\]

where,

\[
\tilde{g}_E = 2 \left( E - V_p(\phi) \right) g ;
\]

and \( E \) is the total energy of the system.

### 3 Applications

In what follows, we will apply this technique for finding distinct geodesics to different potentials, showing that the different geodesics pertain to different phases of the QFT at hand.

We will see from the examples below that the presence of gauge symmetry does not affect our results, as expected, once all we are doing is conformally changing the metric of the phase space.
3.1 The $\lambda \phi^4$ Potential

This theory is defined by the Lagrangian $L = \frac{1}{2} (g(\pi, \pi) - \mu \phi^2 - \frac{\lambda}{2} \phi^4)$, i.e., the potential is given by $V(\phi) = \frac{1}{2} (\mu \phi^2 + \frac{\lambda}{2} \phi^4)$, with reflection symmetry $\mathbb{Z}_2, \phi \mapsto -\phi$.

Canonically, the reasoning is that the mass parameter ($\mu$) has to change sign so that the fields go from the symmetric phase to the broken symmetric one. Mathematically speaking, this represents a change in the solution space given by $\mathcal{F} = \ker\{\Box + \mu + \lambda \phi^2\} \equiv \{\phi \mid (\Box + \mu + \lambda \phi^2) \phi = 0\}$, i.e., different parameters (in this case, $\mu$ and $\lambda$) will give rise to equivalence classes of solutions: a symmetric phase equivalence class and an equivalence class for each broken symmetric phase. This division of the solution space into equivalence classes labelled by the parameters of the potential is true for classic and for quantum field theories.

In this particular case, these equivalence classes are labeled by the parameter $\frac{\mu}{\lambda}$ and there are, essentially, 2 of them: The symmetric equivalence class is parameterized by $\frac{\mu}{\lambda} > 0$ and the broken symmetric equivalence class is parameterized by $\frac{\mu}{\lambda} < 0$. Note that $\lambda \neq 0$, otherwise we would be treating the free scalar field QFT.

However, QFTs can undergo a dynamical process, Spontaneous Symmetry Breaking, whereby the quantum fields move from one solution space (equivalence class) to another, namely from the symmetric phase to the broken symmetric one (note that this is not possible in classical field theories, given that the origin of this symmetry breaking dynamics is due to quantum effects).

Therefore, the solution space equivalence classes (labeled by $p = \frac{\mu}{\lambda}$) change from the symmetric $\mathcal{S}_{p > 0} = \ker\{\Box + \lambda \phi^2 + \mu\}$ to the broken symmetric $\mathcal{B}_{p < 0} = \ker\{\Box + \lambda \phi^2 - \mu\}$. Thence, their geometry changes accordingly, a fact that can be measured by the procedure outlined above, using $\tilde{g}_E = 2 \left( E \pm \frac{\mu}{2} \gamma^2 + \frac{\lambda}{4} \gamma^4 \right) g$ and solving the normalization condition $\tilde{g}_E(\gamma', \gamma') = 1$, where $\gamma(s)$ is the geodesic we want to compute and $\gamma' = \frac{ds}{ds}$, $s$ being the arc-length parameter.

However, this naïve, albeit common, reasoning is just part of the story. The full picture presents itself only upon a more detailed analysis of $\int \sqrt{E - \mu \gamma^2/2 - \lambda \gamma^4/4} \, d\gamma$, which is the [elliptic] integral that needs to be solved in order to find $\gamma(s)$. Therefore, it is useful to consider the polynomial $P(\gamma) = (\gamma^2 - r_1)(\gamma^2 - r_2)$, where $(-\lambda/4) P(\gamma) = E - \mu \gamma^2/2 - \lambda \gamma^4/4$ and $r_{1,2} = -\left( \mu \pm \sqrt{4 E \lambda + \mu^2} \right)/\lambda$. It is the discriminant of this polynomial $P(\gamma)$ that will, ultimately, determine the different phases of the theory: $\Delta = (r_1 - r_2)^2 = \lambda E + \mu^2/4 \leq 0$.

The 2 inequalities, $\Delta > 0$ and $\Delta < 0$, are related by the analytic continuation (and, therefore, their boundary conditions for their respective Schwinger-Dyson equations are different) of the mass parameter, i.e., $\mu \mapsto -\mu$; meanwhile, the case $\Delta = 0$ has to be computed separately. This whole framework is potentially a very interesting result, once discriminants are closely related and contain information about ramifications (“branching out” or “branches coming together”) in Number Theory. In this sense, it is more appropriate to label the different solution space equivalence classes using $\Delta$ rather than $p = \mu/\lambda$. The symmetric one is given by $\mathcal{S}_{\Delta > 0}$, while the broken symmetric one is now seen to be split into 2, $\mathcal{B}_{\Delta = 0}$ and $\mathcal{B}_{\Delta < 0}$.

The analytical answers for the geodesic $\gamma(s)$ are found to be given by the following implicit equations:

$$\Delta = 0:\qquad \frac{1}{3} \gamma^3 + \frac{\mu}{\lambda} \gamma + s = 0;$$

(2)
\[ \Delta > 0 : \]
\[
3 s + \gamma \sqrt{(\gamma^2 - r_1)(\gamma^2 - r_2)} + 2 r_1 \sqrt{r_2} F\left(\frac{\gamma}{\sqrt{r_1}}; \frac{\gamma}{\sqrt{r_2}}\right) - \frac{2}{3} \mu \sqrt{r_2} \left[ F\left(\frac{\gamma}{\sqrt{r_1}}; \frac{\gamma}{\sqrt{r_2}}\right) - E\left(\frac{\gamma}{\sqrt{r_1}}; \frac{\gamma}{\sqrt{r_2}}\right) \right] = 0 ;
\]

\[ \Delta < 0 : \]
\[
3 s + \gamma \sqrt{(\gamma^2 - r_1)(\gamma^2 - r_2)} + 2 r_1 \sqrt{r_2} F\left(\frac{\gamma}{\sqrt{r_1}}; \frac{\gamma}{\sqrt{r_2}}\right) - \frac{2}{3} \mu \sqrt{r_2} \left[ F\left(\frac{\gamma}{\sqrt{r_1}}; \frac{\gamma}{\sqrt{r_2}}\right) - E\left(\frac{\gamma}{\sqrt{r_1}}; \frac{\gamma}{\sqrt{r_2}}\right) \right] = 0 .
\]

Note that, in the equations above, \( F(z; k) \) is the incomplete elliptic integral of the first kind, while \( E(z; k) \) is the incomplete elliptic integral of the second kind. Moreover, \( r_{1,2} \) are defined (as shown above) as the solutions to the polynomial equation \( P(\gamma) = 0 : r_{1,2} = \mu \pm \sqrt{\mu^2 + 4 \lambda E / (-\lambda)} \). On top of this, although (3) and (4) have the same form, they will yield distinct solutions, once the relation given by \( \Delta = \lambda E + \mu^2 / 4 \) will either be positive or negative, which, in turn, affects the outcome of \( r_{1,2} \).

The graphical results are shown below, where \( \frac{E}{\lambda} = 1, \gamma(0) = 0 \) and \( \Delta \), respectively, assumes positive (\( \Delta > 0 \)), null (\( \Delta = 0 \)) and negative (\( \Delta < 0 \)) values.

We can clearly see that, for positive \( \Delta \) (leftmost graph), we have a smooth geodesic \( \gamma(s) \) representing the symmetric phase. Then, when \( \Delta \) vanishes (middle graph), there is a clear change which delimits these 2 different phases of the theory. Moreover, when \( \Delta \) is negative (rightmost graph), we have the third [broken-symmetric] phase of the theory (which has a finite geodesic). Note that just as mentioned before, this construction is made at every point in spacetime; therefore, the number of solutions is 3 times infinity.

### 3.2 The Landau-Ginzburg Functional

To start off, let us consider the case where the base manifold is a compact Riemann surface \( \Sigma \) equipped with a conformal metric and the vector bundle is a Hermitian line bundle \( L \) (\( i.e. \), with fiber \( \mathbb{C} \) and a Hermitian metric \( \langle \cdot, \cdot \rangle \) on the fibers).

The Landau-Ginzburg functional is defined for a section \( \varphi \) and a unitary connection \( D_A = d + A \) of \( L \) as \( (\sigma \in \mathbb{R} \) is a real scalar),
\[ L(\varphi, A) = \int_{\Sigma} |F_A|^2 + |D_A \varphi|^2 + \frac{1}{4} (\sigma - |\varphi|^2)^2. \]

Thus, its Euler-Lagrange equations [of motion] are given by:

\[ D_A^* D_A \varphi = \frac{1}{2} (\sigma - |\varphi|^2) \varphi; \]
\[ D_A^* F_A = - \text{Re}(D_A \varphi, \varphi); \]

where, \( D_A^* \) is the dual of \( D_A \), i.e., \( D_A^* = -\ast D_A \ast = -\ast (d + A) \ast \). Note that the equation for \( F_A \) (the second one above) is linear in \( A \). Since \( D_A \) is a unitary connection, \( A \) is a \( u(1) \)-valued 1-form. This Lie algebra (of the group \( U(1) \)) will sometimes be identified with \( i\mathbb{R} \) — in other words, our \( A \) corresponds to \(-iA \) in the standard physics literature (where \( A \) is real-valued).

Before we go any further, some notational remarks are in order. We decompose the space of 1-forms, \( \Omega^1 \), on \( \Sigma \) as \( \Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1} \), with \( \Omega^{1,0} \) spanned by 1-forms of the type \( dz \) and \( \Omega^{0,1} \) by 1-forms of the type \( d\bar{z} \). Here \( z = x + iy \) is a local conformal parameter on \( \Sigma \) and \( \bar{z} = x - iy \). Therefore, \( dz = dx + idy, d\bar{z} = dx - idy \), \( \partial_x = \frac{1}{2}(\partial_x - i\partial_y) \) and \( \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \). Furthermore, if \( \partial_x \) and \( \partial_y \) are an orthonormal basis of the tangent space of \( \Sigma \) at a given point, we have that \( \langle dz, dz \rangle = 2, \langle d\bar{z}, d\bar{z} \rangle = 2 \) and \( \langle dz, d\bar{z} \rangle = 0 \). Given that the decomposition \( \Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1} \) is orthogonal, we may also decompose \( D_A \) accordingly: \( D_A = \partial_A + \bar{\partial}_A \), where \( \partial_A \varphi \in \Omega^{1,0}(\mathcal{L}) \) and \( \bar{\partial}_A \varphi \in \Omega^{0,1}(\mathcal{L}) \) for all sections \( \varphi \) of \( \mathcal{L} \) (holomorphic, \( \bar{\partial}_A f(z, \bar{z}) = 0 \Leftrightarrow f(z, \bar{z}) = f(z) \), and anti-holomorphic, \( \partial_A f(z, \bar{z}) = 0 \Leftrightarrow f(z, \bar{z}) = f(\bar{z}) \), parts; i.e., the space of 1-forms and the space of connections is decomposable into a direct sum of its holomorphic and anti-holomorphic parts: \( \partial_A = \partial + A^{1,0} \) and \( \bar{\partial}_A = \bar{\partial} + A^{0,1} \); while the exterior derivative is given by \( d = \partial + \bar{\partial} \). As expected, we have that \( \partial_A \partial_A = 0 = \partial_A \bar{\partial}_A \) and \( F_A = - (\partial_A \bar{\partial}_A + \bar{\partial}_A \partial_A) \).

It is not difficult to show [24] that:

\[ L(\varphi, A) = \int_{\Sigma} |F_A|^2 + |D_A \varphi|^2 + \frac{1}{4} (\sigma - |\varphi|^2)^2; \]
\[ = 2\pi \deg(\mathcal{L}) + \int_{\Sigma} 2 |\partial_A \varphi|^2 + \left( *(-iF_A) - \frac{1}{2} (\sigma - |\varphi|^2) \right)^2; \]

where \( \deg(\mathcal{L}) = c_1(\mathcal{L}) = \frac{i}{2\pi} \text{tr}(F_A); \)

i.e., the degree of the line bundle is given by the 1st Chern class.

Therefore, a very useful consequence from the above is that if \( \deg(\mathcal{L}) \geq 0 \) the lowest possible value for \( L(\varphi, A) \) is realized if \( \varphi \) and \( A \) satisfy

\[ \bar{\partial}_A \varphi = 0; \]
\[ *(iF_A) = \frac{1}{2} (\sigma - |\varphi|^2). \]

These equations are just the expression of the self-duality of the Landau-Ginzburg functional. If \( \deg(\mathcal{L}) < 0 \), then these equations cannot have any solution, thus one has to consider the self-duality equations arising from the Landau-Ginzburg functional where the term \( +2\pi \deg(\mathcal{L}) \) is substituted by \( -2\pi \deg(\mathcal{L}) \). Therefore, without loss of generality, we shall assume \( \deg(\mathcal{L}) \geq 0 \). A necessary condition for the solvability of (6) is that,
\[2 \pi \deg(L) = \int i F_A = \frac{1}{2} \int_\Sigma (\sigma - |\varphi|^2) \leq \frac{\sigma}{2} \text{Area}(\Sigma) ;\]
\[
\therefore \sigma \geq \frac{4 \pi \deg(L)}{\text{Area}(\Sigma)} ; \tag{7}
\]
and the equality only occurs if, and only if, \(\varphi \equiv 0\).

The following result is useful when studying the solutions of the above functionals \cite{24}: Let \(\Sigma\) be a compact Riemann surface with a conformal metric and \(L\) as before. For any solution of \((5)\), we have that \(|\varphi| \leq \sigma\) on \(\Sigma\). That is, this maximum principle states that the amplitude of the field cannot exceed the height of the potential (see the first plot below).

Let us now construct the Jacobi metric for this potential and find its possible geodesics:
\[
\tilde{g}_E = 2 \left( E - V_{\sigma}(\gamma) \right) g ;
\]
\[
= 2 \left( E + \frac{1}{4} (\sigma - |\gamma|^2)^2 \right) g .
\]

In a complete analogy to what was previously done, we consider the \(P(\gamma) = E - V(\gamma)\) polynomial. However, in order to make this analysis clearer, let us, in fact, use a slightly different polynomial given by \(P'(\gamma) = 4 \left( P(\gamma) - E \right)\). It is straightforward to see that \(P'(\gamma) = (|\gamma|^2 - \sigma) (|\gamma|^2 - \sigma)\), which means that \(\sigma\) is the only root of \(P'(\gamma)\), with double multiplicity (the 2 roots coalesce into 1).

This situation implies that the discriminant of \(P'(\gamma)\) vanishes, \(i.e., \Delta = 0\), and the different phases of the theory are labelled by \(\sigma = 0\) and \(\sigma > 0\).

The plots below have, respectively, the following values for \(\sigma\): 0.0, 3.0, 5.0, 7.0, 11.0 and 31.0. As they show, when \(\sigma = 0\) we have \(c_1(L) = \deg(L) = \frac{i}{2 \pi} \text{tr}(F_A) = 0\) and its geodesic has a clear character which is quite different otherwise:

\[3.3 \quad \text{The Seiberg-Witten Functional}\]

In this case, the base manifold, \(\mathcal{M}\), is a compact, oriented, 4-dimensional Riemannian manifold endowed with a spin\(^c\) structure, \(i.e.,\) a spin\(^c\) manifold. The determinant line of this spin\(^c\) structure
will be denoted by $\mathcal{L}$ and the Dirac operator determined by a unitary connection $A$ on $\mathcal{L}$ will be denoted by $\mathcal{D}_A$. Recalling the half spin bundle $S^\pm$ defined by the spin$^c$ structure, we see that $\mathcal{D}_A$ maps sections of $S^\pm$ into sections of $S^\mp$ [21, 22, 23, 24].

In this fashion the Seiberg-Witten functional for a unitary connection $A$ on $\mathcal{L}$ and a section $\varphi$ of $S^+$ is given by,

$$SW[\varphi, A] = \int_{\mathcal{M}} |\nabla_A \varphi|^2 + |F_A^+|^2 + \frac{R}{4} |\varphi|^2 + \frac{1}{8} |\varphi|^4;$$

where $\nabla_A$ is the spin$^c$ connection induced by $A$ and the Levi-Civita connection of $\mathcal{M}$. $F_A^+$ is the self-dual part of the curvature of $A$ and $R$ is the scalar curvature of $\mathcal{M}$. Its Euler-Lagrange equations are given by,

$$\nabla_A^* \nabla_A \varphi = - \left( \frac{R}{4} + \frac{1}{4} |\varphi|^2 \right) \varphi;$$

$$d^* F_A^+ = - \text{Re} \langle \nabla_A \varphi, \varphi \rangle.$$

Using a spin frame, it is not difficult [24] to show that the Seiberg-Witten functional can be written in the following form:

$$SW[\varphi, A] = \int_{\mathcal{M}} |\mathcal{D}_A \varphi|^2 + \left| F_A^+ - \frac{1}{4} \langle e_j \cdot e_k \cdot \varphi, \varphi \rangle e^j \wedge e^k \right|^2;$$

where $e^j$ are 1-forms dual to the tangent vectors $e_j$, $e^j(e_k) = \delta^j_k$; $j, k = 1, \ldots, 4$. As a corollary of the above, the lowest possible value of the Seiberg-Witten functional is achieved if $\varphi$ and $A$ are solutions of the Seiberg-Witten equations:

$$\mathcal{D}_A \varphi = 0;$$

$$F_A^+ = \frac{1}{4} \langle e_j \cdot e_k \cdot \varphi, \varphi \rangle e^j \wedge e^k.$$

Thus, self-duality is at work yet again: the absolute minima of the Seiberg-Witten functional satisfy not only the second order equations (9) and (10), but also the first order Seiberg-Witten equations (11) and (12).

Although our discussion of the Seiberg-Witten functional, so far, has mirrored our discussion of the Landau-Ginzburg one, the parameter $\sigma$ on the latter has had no analogue in the former. This can be accomplished with the introduction of a 2-form $\mu$ and the consideration of the perturbed functional,

$$SW_\mu[\varphi, A] = \int_{\mathcal{M}} |\mathcal{D}_A \varphi|^2 + \left| F_A^+ - \frac{1}{4} \langle e_j \cdot e_k \cdot \varphi, \varphi \rangle e^j \wedge e^k + \mu \right|^2;$$

$$= \int_{\mathcal{M}} |\nabla_A \varphi|^2 + |F_A^+|^2 + \frac{R}{4} |\varphi|^2 + \left| \mu - \frac{1}{4} \langle e_j \cdot e_k \cdot \varphi, \varphi \rangle e^j \wedge e^k \right|^2 + 2 \langle F_A^+, \mu \rangle.$$

Their corresponding first order equations of motion are,
\[ D_A \varphi = 0 \]
\[ F_A^+ = \frac{1}{4} \langle e_j \cdot e_k \cdot \varphi, \varphi \rangle e^j \wedge e^k - \mu . \]

If we assume that \( \mu \) is closed and self-dual, then we see that \( \langle F_A, \mu \rangle = \langle F_A^+, \mu \rangle \), once \( \langle F_A^-, \mu \rangle = 0 \) due to the orthogonality between anti-self-dual and self-dual forms. Thus, given that \( F_A \) represents the first Chern class \( c_1(L) \) of the line bundle \( L \), and we assumed \( \mu \) to be closed (hence it represents a cohomology class \( [\mu] \)), the integral

\[ \int_M \langle F_A, \mu \rangle , \]

does not depend on the connection \( A \), thus representing a topological invariant, denoted by \( (c_1(L) \wedge [\mu])[M] \).

Just as before, we also have a maximum principle: For any solution \( \varphi \) of (9) — in particular, for any solution of (11) — on a compact 4-dimensional Riemannian manifold, we have that,

\[ \max_M |\varphi|^2 \leq \max_{x \in M} (-R(x), 0) . \]

As a direct consequence of this, if the compact, oriented, Riemannian Spin\(^c\) manifold \( M \) has nonnegative scalar curvature, the only possible solution of the Seiberg-Witten equations is,

\[ \varphi \equiv 0 ; \quad F_A^+ \equiv 0 . \]

The Jacobi metric for this Seiberg-Witten model is given by

\[ \tilde{g}_E = 2 \left( E + \frac{R}{4} |\gamma|^2 + \frac{1}{8} |\gamma|^4 \right) g ; \]

for a geodesic \( \gamma \); and, just like before, although \( P(\gamma) = E - V(\gamma) = E + R |\gamma|^2/4 + |\gamma|^4/8 \), let us consider \( P'(\gamma) = 8 \left( P(\gamma) - E \right) = \left( |\gamma|^2 - 0 \right) \left( |\gamma|^2 + 2R \right) \). It is then clear that when \( R = 0 \) the discriminant vanishes \( (\Delta = 0) \) and the 2 roots merge into 1, what constitutes one of the phases of the theory. When \( R \neq 0 \), the discriminant is either \( \Delta > 0 (R > 0) \) or \( \Delta < 0 (R < 0) \), which accounts for the other 2 phases of the theory.

The plots below were obtained with the choice of \( E = 1.0 \) and \( R \) respectively equal to 0.0, 3.0, 7.0, -3.0, -5.0, -7.0.
4 Conclusions

Historically, the conformal transformation (1) was used by Jacobi with a twofold purpose: unify Hamilton’s (mechanics) and Fermat’s (geometrical optics) action principles into a single framework, and turn every Hamiltonian flow into a geodesic one [1, 20]. The extension of the Jacobi metric from the Riemannian setting to the Lorentzian one was done in [25].

Note that this construction extends to field theory, once the space of solutions of the field equations is metrizable: fields satisfying the equations of motion are particular sections of a principal fibre bundle, $P$, that belong to $\ker\left\{\Box + V'(\phi)\right\}$; in this sense, if $P$ is endowed with a metric, its fibers inherit an induced metric and our construction of (1) is nothing but a conformal transformation of such a metric.

However, rather than dealing with the phase space of the [quantum] field theory in question, we restrain ourselves to studying its geodesic structure.

In this sense, we have shown that this geodesic structure is quite rich and its different parameters label different phases of the theory. Moreover, the technique works for both, global (subsection 3.1) and local (gauge) symmetries (subsections 3.2 and 3.3).

As mentioned in the introduction, this method can be used in order to replace the Coleman-De Luccia mechanism in string theoretical models. The way to do this is to use Jacobi’s metric in order to translate between the spacetime geometry and a potential $V$ for a Lagrangian describing the particular model in question. In this sense, the theory at hand can be thought of as a QFT, and we can repeat the analysis outlined in this paper looking for the possible [topologically inequivalent] phases of the string theoretical model being treated.

Therefore, in this string theoretical setting, the picture that emerges is that quantum effects (in dynamically breaking the symmetry) would be responsible for the generation of topologically inequivalent “baby universes”. Thus, there is no need for the use of a Coleman-De Luccia type of mechanism in order to generate bubbles of baby universes.

On a forthcoming work, we want to take this construction further, elucidating its relation to Morse Theory and Deformation and Geometric Quantization, also making clearer the connection with the Renormalization Group.

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References

[1] Theodore Frankel, *The Geometry of Physics: An Introduction*, Cambridge University Press, 2003. ISBN: 0521539277.

[2] M. Nakahara, *Geometry, Topology and Physics*, Taylor & Francis, 2003. ISBN: 0750306068.

[3] R. W. R. Darling, *Differential Forms and Connections*, Cambridge University Press, 1994. ISBN: 0521466000.

[4] Y. Choquet-Bruhat, C. DeWitt-Morette, *Analysis, Manifolds and Physics, Part I and II*, North Holland. ISBN: 0444860177 and 0444504737.

[5] Yvan Kerbrat, Helene Kerbrat-Lunc, *Spontaneous symmetry breaking and principal fibre bundles*, J. Geometry and Physics Vol. III, No. 2 (1986).

[6] R. O. Fulp, L. K. Norris, *Splitting of the connection in gauge theories with broken symmetry*, J. of Mathematical Physics, Vol. 24, 1871, (1983).

[7] M. E. Mayer, *The geometry of symmetry breaking in gauge theories*, Act. Phys. Austr. (Suppl.) XXIII, 477–490 (1941).

[8] Gennadi Sardanashvily, *On the geometry of spontaneous symmetry breaking*, J. Math. Phys. 33 (4), April 1992.

[9] Carlos T. Simpson, *Higgs bundles and local systems*, Publications Mathématiques de l’IHÉS, 75 (1992), p. 5–95.

[10] Y. Ne’eman, *Geometrization of Spontaneously Broken Gauge Symmetries*, Theoretical and Mathematical Physics, 139(3): 745–750 (2004).

[11] G. S. Guralnik, C. R. Hagen, T. W. B. Kibble, *Global Conservation Laws and Massless Particles*, Phys. Rev. Lett. 13, 585–587 (1964). P. W. Higgs, *Broken symmetries, massless particles and gauge fields*, Phys. Lett. 12 132. F. Englert and R. Brout, *Broken Symmetry and the Mass of Gauge Vector Mesons*, Phys. Rev. Lett. 13, 321–323 (1964). P. W. Higgs, *Broken Symmetries and the Masses of Gauge Bosons*, Phys. Rev. Lett. 13, 508–509 (1964).

[12] F. Strocchi, *Spontaneous Symmetry Breaking in Local Gauge Quantum Field Theory; The Higgs Mechanism*, Commun. Math. Phys. 56, 57–78 (1977).

[13] Sidney Coleman, Erick Weinberg, *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*, Phys. Rev. D 7, 1888–1910 (1973). Erick Weinberg, *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*, hep-th/0507214.

[14] S. Garcia, G. Guralnik, Z. Guralnik, *Theta Vacua and Boundary Conditions of the Schwinger Dyson Equations*, hep-th/9612079.

[15] R. E. Borcherds and A. Barnard, *Lectures on Quantum Field Theory*, math-ph/0204014.

[16] Klaus Fredenhagen and Karl-Henning Rehren and Erhard Seiler, *Quantum Field Theory: Where We Are*, hep-th/0603155.
[17] Reed M. and Simon B.; *Methods of modern mathematical physics*, Volume 1: Functional Analysis. Academic Press, New York 1972.

[18] Dunford N. and Schwartz J. T.; *Linear operators*, Volume 2: Spectral Theory. Interscience, New York 1964.

[19] Sidney Coleman, Frank De Luccia, *Gravitational effects on and of vacuum decay*, Phys. Rev. D 21, 3305–3315 (1980).

[20] Detlef Laugwitz, *Differential and Riemannian Geometry*, Academic Press, 1965; ASIN: B000CSCJUY. Cornelius Lanczos, *The Variational Principles of Mechanics*, Dover Publications, 1986; ISBN: 0486650677. V. I. Arnold, *Mathematical Methods of Classical Mechanics*. Springer, 1997. ISBN: 0387968903.

[21] N. Seiberg, E. Witten, *Monopoles, Duality and Chiral Symmetry Breaking in N = 2 Supersymmetric QCD*, Nucl. Phys. B 431 (1994) 484–550, hep-th/9408099.

[22] S. K. Donaldson, *The Seiberg-Witten equations and 4-manifold topology*, Bull. Amer. Math. Soc. 33 (1996), 45–70.

[23] Liviu I. Nicolaescu, *Notes on Seiberg-Witten Theory*. American Mathematical Society, 2000. ISBN: 0821821458.

[24] Jürgen Jost, *Riemannian Geometry and Geometric Analysis*. Springer. ISBN: 3540426272.

[25] Marek Szydłowski, Michale Heller, Wiesław Sasin, *Geometry of spaces with the Jacobi metric*, J. Math. Phys. 37 (1), 1996, 346–360.