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Blood flow problem in the presence of magnetic particles through a circular cylinder using Caputo-Fabrizio fractional derivative

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Abstract. In this paper, the flow of blood mixed with magnetic particles subjected to uniform transverse magnetic field and pressure gradient in an axisymmetric circular cylinder is studied by using a new trend of fractional derivative without singular kernel. The governing equations are fractional partial differential equations derived based on the Caputo-Fabrizio time-fractional derivatives NFDt. The current result agrees considerably well with that of the previous Caputo fractional derivatives UFDt.

1. Introduction

Blood is transported within the circulatory system in a body. Blood can be categorized based on its functions such as arterial and venous. Arterial blood transports oxygen and nutrients to tissues while venous blood transports carbon dioxide and metabolic by-products to the lungs and kidneys for removal. During the blood flow circulation, parameters such as elasticity, wall porosity and external magnetic field could be used to regulate the pulsatile nature of blood flow (hence blood velocity). Studies related to blood flow in the rigid and plane arteries subjected to external magnetic field are very popular. Some relevant findings reported by the previous researchers will be reviewed.

The result obtained by Shit and Roy [1] is applicable for the clinical treatment of hemodynamic diseases such as hypertension and atherosclerosis. Moreover, external magnetic field has the potential in reducing the blood flow. Meanwhile, other parameters such as Reynold number and Darcian porosity parameter would affect the pulsating blood flow significantly. Mohan et al [2] studied the viscous elastic blood flow in the cylindrical artery through porous medium. He argued that axial velocity was affected by the magnetic field. Sattar and Wahedullah [3] used finite element method to deal with complicated domain and non-linearities. Sabzikar et al [4] highlighted that tempered fractional calculus should be used to solve tempered fractional differential equation because this operator is useful for time series analysis model where semi long range dependence can be incorporated by ARTFIMA in a natural way. Another attempt was made by [5] to solve a few examples by using the new definition of local fractional derivative for $0 \leq \alpha < 1$ which may coincide with the classical definition ($\alpha = 1$ for polynomials). It is interesting to note that there is no delay effect for the newly presented local fractional derivatives compared to the other fractional ones that are having kernel inside the definition of integral. A fractional derivative with smooth kernel has been reported by Caputo [6] which was applied on two different variables by taking Laplace transform on the time
variables and Fourier transform on spatial variables to define a non-local system which could elucidate the material properties and changes of different stages that could not be well explained by the previous local classical theories containing the singular kernel in the fractional model. In the fractional TS process [7], the governing equation of TSS has been derived, which was expressed in the form of shifted fractional derivative with order $\alpha$ lies between 0 and 1 coinciding with the stability parameter. It was then generalized by introducing another order $\beta$ lying between 0 and 1. A fractional extension defined as Brownian motion by independent fractional TS processes of order $\beta$ with a random time argument was presented. The Soret-Dufour-Hall effects [8] on the peristaltic flow of couple-stressfluid were taken into account and it was found that pressure gradient increased near the wall and decreased at the centre as Hall parameter increased. Meanwhile, the temperature increased with respect to Hall parameter and Froude number. On the other hand, the size of the trapped bolus was negatively correlated to Froude number and Hall parameter. However, it was directly related to the inclination angle of the channel.

As reported in [9, 10], artificial blood was made by adding 75% water with 25% glycerol and iron oxide magnetic particles. The fluid was then transported through a glass tube. It was observed that the velocities of blood and magnetic particles decreased significantly in the presence of magnetic field which was applied perpendicular to the cylindrical glass tube. In one of their previous studies where magnetic nano-particles were injected upstream into a blood vessel from a malignant tissue, all particles were captured either before or at the centre of the magnetic field ($z \leq 0$) when the magnet was very close to the blood vessel ($d=2.5$). As the magnet was kept away from the blood vessel (above 4.5), the magnetic nano-particles became free and they travelled downwards. The dimensionless non-linear momentum and energy equations were used by Adesanya [11] to model the non-Newtonian hydro-magnetic fluid flow under the influences of couple stress and Joule dissipation. The temperature and velocity profiles were obtained and the solutions were found to be convergent, showing the strength of the semi-analytical Adomian Decomposition Method in solving non-linear coupled differential equations.

Habu et al [12, 13] derived a frequency equation and found that the roots depended on $\sigma e B_0^2$ due to the transverse magnetic field. In other words, the blood flow can be controlled by the external magnetic field. The oscillatory motion of viscous fluid in a thin wall tube subjected to elasticity and magnetic field has been studied as well for treating localized diseases such as cancer. To model complex problems, the fractional Atangana-Baleanu-Liouville-Caputo sense operator is more suitable than other existing fractional operators [14] because it contains all properties of fractional order derivative. Both blood and magnetic particle velocities were reduced under the influence of external magnetic field according to the predictions made by using Caputo fractional derivatives, Laplace and finite Hankel transforms [15]. Ali et al [16] employed a fractional model of Walters-B fluid. Caputo-Fabrizio time fractional derivative was used to model the MHD free convection fluid flow over a static vertical plate and the variations in fluid velocity were observed.

Recently, Shah et al. [15] adopted Caputo time fractional derivative approach proposed by [17] to solve the blood flow inside an artery subjected to external magnetic field. In the current work, we employ the new definitions of fractional derivative without singular kernel (NFDt) as proposed by Caputo and Fabrizio [6] to solve the similar problem in Shah et al. [15].
2. Mathematical modelling and solution

Similar work by Shah et al. [15] will be adopted in current work. Blood flow mixed with uniformly distributed iron particles inside a cylindrical glass tube subjected to an external perpendicular magnetic field is shown in Figure 1. Blood flows in the axial direction denoted as z. In the present study, the induced magnetic field is neglected as the Reynolds number is very small. At $t = 0$, the blood and the magnetic particles are stagnant. An external magnetic field is activated opposing the blood flow direction. Therefore, there are interactions between components such as electrically conducting fluid, magnetic field and fluid current. The fluid motion and the intensity of magnetic flux may affect the induced electromagnetic force. The flow motion, magnetic particle motion and magnetic field are governed by Navier-Stokes equations, Newton's second law of motion and Maxwell's equations, respectively. Maxwell equations are

$$\nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = \mu_0 \vec{J}, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

where $\vec{J} = \sigma(\vec{E} + \vec{V} \times \vec{B})$, is the density of the current, $\mu_0$ is the permeability of an external magnetic field, $\vec{E}$ is the intensity of an electric field and $\vec{B}$ is the magnetic flux intensity, $\sigma$ is an electrical conductivity and $\vec{V}$ is the velocity field.

The electromagnetic force can be defined as follows:

$$\vec{F} = \vec{J} \times \vec{B} = -\sigma B_0^2 u(r,t) \vec{k}. $$

(2)

Here $\vec{k}$ is the unit vector along $z$-direction and $\vec{V} = u(r,t)\vec{k}$ is the axial velocity of the blood. In momentum equation the electromagnetic force will be included. The pressure gradient applying on the unsteady blood flow in an axisymmetric glass cylindrical tube with radius $R_0$ and in the presence of external magnetic field is as follows

$$-\frac{\partial p}{\partial z} = A_0 + A_1 \cos(\omega t), A0 > 0.$$  

(3)

where $A_0$ and $A_1$ are the amplitudes of the pressure gradient. They are the components of pulsatile giving rise to the systolic and diastolic pressure. In cylindrical coordinates $(r, \theta, z)$ the governing equation for fluid flow momentum is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + \frac{KN}{\rho} (v - u) - \frac{\sigma B_0^2}{\rho} u,$$

(4)

Figure 1. Blood Flow Model
where \( p \) is the pressure, \( \nu \) is the kinetic viscosity, \( \rho \) is the fluid density, \( N \) is the number of magnetic particles per unit volume, \( K \) is the Stokes constants and \( u \) and \( v \) are fluid and magnetic particles (iron rich particles) velocity, respectively. Due to the relative motion between fluid and magnetic particles there is a force term as well in the above mentioned equation. Here Newton's second law of motion is describing the motion of iron rich particles, which is as follows

\[
m \frac{\partial v}{\partial t} = K(u - v),
\]

where \( m \) is the magnetic particles average mass. In order to obtain a time fractional model with governing equations which has the dimension for \( t \) as well, multiplying Eqs. (4) and (5) by

\[
\lambda = \frac{R_0}{A_0},
\]

we obtain

\[
\lambda^\alpha D_t^\alpha u = \frac{\lambda}{\rho} (A_0 + A_1 \cos(\omega t)) + \lambda \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + \frac{K \lambda}{\rho} (v - u) - \frac{\sigma B_0^2 \lambda}{\rho} u,
\]

(6)

\[
\lambda^\alpha D_t^\alpha = \frac{K \lambda}{m} (u - v),
\]

(7)

\[
D_t^{(\alpha)} f(t) = \frac{\Gamma(1 - \alpha)}{1 - \alpha} \int_a^t f(\tau) e^{-\frac{\alpha}{1 - \alpha} \tau} d\tau,
\]

(8)

with \( \alpha \in [0,1], \ A \in [-\infty,t] \) and \( f \in L^1(a,b) \), where \( L^1(a,b) \) be the class of all integrable functions \( f \) on \([a,b]\), which is called the new fractional time derivative NFDt and it is obtained by changing the kernel \((t - \tau)^{\alpha}\) with the function \( e^{-\frac{\alpha}{1 - \alpha} \tau} \) and \( \frac{1}{\Gamma(1 - \alpha)} \) with \( \frac{M(\alpha)}{1 - \alpha} \). Moreover normalization function is denoted by \( M(\alpha) \) with the condition \( M(0) = M(1) = 1 \). When \( f(t) \) is a constant, then NFDt turns to zero like UFDt, but, on the opposite side, for \( t = \tau \) the kernel does not have singularity. NFDt method can also be applied to the \( f(t) \) such that \( f(t) \notin H^1(a,b) \). In fact, the new definition of fractional derivative can also be defined for \( f \in L^1(-\infty,b) \) for any \( \alpha \in [0,1] \) as

\[
D_t^{\alpha} f(t) = \frac{\alpha M(\alpha)}{1 - \alpha} \int_a^t (f(t) - f(\tau)) e^{-\frac{\alpha}{1 - \alpha} \tau} d\tau.
\]

(9)

If we assume \( A = 1 - \alpha \in [0,\infty], \alpha = \frac{1}{1 + \alpha} \in [0,1] \), then NFDt in Eq. (8) changes to the form as follows

\[
D_t^\alpha f(t) = \frac{\lambda}{\sigma} \int_a^t \tilde{f}(\tau) e^{-\frac{\omega}{1 - \alpha} \tau} d\tau.
\]

(10)

The associated initial and boundary conditions are as follows:

\[
u(r,0) = 0, v(r,0) = 0, r \in [0, R_0],
\]

(11)

\[
u(R_0, t) = 0, v(R_0, t) = 0, t > 0.
\]

(12)

We introduce the following dimensionless variables

\[
r' = \frac{r}{R_0}, t' = \frac{t}{\lambda}, u' = \frac{u}{u_0}, A_0' = \frac{\lambda A_0}{\rho u_0}, A_1' = \frac{\lambda A_1}{\rho u_0}, \omega' = \frac{\lambda \omega}{\omega},
\]

(13)

where \( u_0 \) is the characteristic velocity.
Applying the above mentioned variables into Eqs. (6), (7), (9) and (10), the dimensionless governing equations are given as follows (after dropping the dashes)

\[ D_r \frac{\partial^2 u}{\partial r^2} = A_0 + A_1 \cos(\omega r) + \frac{1}{\text{Re}} \left( \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2} \right) + R(v - u) - Ha^2 u, \]

(14)

\[ GD_r^2 v = u - v, \]

(15)

where \( R \) is the particle concentration parameter, \( \text{Re} \) is the Reynolds number, and \( Ha \) and \( G \) are the Hartmann and mass number of the particle, respectively. The dimensionless initial and boundary conditions are as follows:

\[ u(r,0) = 0, v(r,0) = 0, r \in [0,1], \]

(16)

\[ v(1,t) = 0, v(1,t) = 0, t > 0. \]

(17)

3. Solution to the problem

Applying Laplace transformation and finite Hankel transform of order zero to the Eqs. (14) and (15) subject to initial and boundary conditions (16) and (17) with respect to the temporal variable \( t \) and radial coordinate \( r \), the fluid velocity becomes

\[ \frac{s \bar{u}_H(r_n,s)}{s + \alpha(1 - s)} = \frac{J_1(r_n)}{r_n} \left[ \frac{A_0}{s} + \frac{A_1 s}{s^2 + \omega^2} \right] - \frac{1}{\text{Re}} r_n^2 \bar{u}_H(r_n,s) + R s \bar{u}_H(r_n,s) - (R + Ha^2) \bar{u}_H(r_n,s), \]

(18)

and the magnetic particles velocity becomes

\[ \bar{v}_H(r_n,s) = \frac{J_1(r_n)}{s + \alpha(1 - s)} \left( \frac{Gs}{s} + \alpha(1 - s) \right), \]

(19)

where \( J_1(r_n) \) and \( r_n \) are the Bessel function and the positive roots of the Bessel function of first kind respectively. Moreover

\[ -r_n^2 u_H(r_n,s) = \frac{\partial^2 u(r_n,s)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r_n,s)}{\partial r}, \]

(20)

and the finite Hankel transform of the function \( \bar{u}(r,s) \) is given by

\[ \bar{u}_H(r_n,s) = \int_0^1 r \bar{u}(r,s) J_0(r r_n) dr. \]

(21)

Writing the fluid velocity in Eq. (18), explicitly in the following way

\[ \bar{u}_H(r_n,s) = \frac{G(s + \alpha(1 - s))}{4 s h_n^2 (s + g_n)(s + h_n)} \left[ A_0 \left[ \frac{A_1 s}{s^2 + \omega^2} \right] J_1(r_n) \right], \]

(22)

solving the coefficient which is involved in Eq. (22) for the fluid velocity, we have

\[ \bar{u}_H(r_n,s) = j_n \left[ \frac{k_n}{s + g_n} + \frac{l_n}{s + h_n} \right] \left[ \frac{A_0}{s} + \frac{A_1 s}{s^2 + \omega^2} \right] J_1(r_n), \]

(23)

where the parameters which are used in Eq. (22) were introduced to reduce the step size, areas follows

\[ a_n = \frac{r_n^2}{\text{Re}} + R + Ha, \]

(24)

\[ d_n = a_n \alpha^2 - R \alpha^2, \]

(25)

\[ c_n = \alpha + 2 a_n \alpha + a_n \alpha G - 2 a_n \alpha^2 + 2 R \alpha^2 - 2 R \alpha, \]

(26)

\[ b_n = G + 1 - \alpha + a_n G + a_n - 2 a_n \alpha - a_n \alpha G + a_n \alpha^2 - R - R \alpha^2 + 2 R \alpha, \]

(27)
\[ e_n = c_n - \sqrt{c_n^2 - 4b_n d_n}, \quad (28) \]
\[ f_n = c_n + \sqrt{c_n^2 - 4b_n d_n}, \quad (29) \]
\[ g_n = \frac{e_n}{2b_n}, \quad (30) \]
\[ h_n = \frac{f_n}{2b_n}, \quad (31) \]
\[ j_n = \frac{1}{4b_n^2} (h_n - g_n), \quad (32) \]
\[ k_n = (\alpha(1 + g_n) - g_n)^2 - G g_n (\alpha(1 + g_n) - g_n), \quad (33) \]
\[ l_n = -(\alpha(1 + h_n) - h_n)^2 + G h_n (\alpha(1 + h_n) - h_n). \quad (34) \]

By using Robotnov/Hartley and Miller-Ross functions which are defined respectively as follows
\[
L^{-1}\left(\frac{1}{s^q - a}\right) = F_q(a, t) = \sum_{n=0}^{\infty} a_n t^{(n+1)q-1} \Gamma((n+1)q),
\]
\[
L^{-1}\left(\frac{s^{-v}}{s - a}\right) = E_v(a, s) = \sum_{k=0}^{\infty} a_k s^{v+k} \Gamma(v+k+1),
\]
the inverse Laplace transform of \( u_H(r; t; s) \) of Eq. (23), is given as follows
\[
\begin{align*}
  u_H(r, s) & = \left[ A_0 j_n \left( k_n \frac{1 - e^{-g_n t}}{g_n} + l_n \frac{1 - e^{-h_n t}}{h_n} \right) + A_1 j_n \left[ \cos \omega t \left( k_n e^{-g_n t} + l_n e^{-h_n t} \right) \right] \right] \frac{J_1(r_n)}{r_n}.
\end{align*}
\]

### 3.1. Fluid Velocity

By using the inverse Hankel transform given in the following relation
\[
 u(r, t) = 2 \sum_{n=1}^{\infty} \frac{J_n(r_n)}{J^2_1(r_n)} u_H(r, t), \quad (38)
\]

The fluid velocity in the presence of an external magnetic field, which is applied perpendicular to the blood flow and by using NFDt for \( \alpha \in [0,1] \) is given as follows
\[
\begin{align*}
  u(r, s) = 2 \sum_{n=1}^{\infty} \frac{J_n(r_n)}{J_1(r_n)} \left[ A_0 j_n \left( k_n \frac{1 - e^{-g_n t}}{g_n} + l_n \frac{1 - e^{-h_n t}}{h_n} \right) + A_1 j_n \left[ \cos \omega t \left( k_n e^{-g_n t} + l_n e^{-h_n t} \right) \right] \right].
\end{align*}
\]
3.2. Magnetic particles velocity
The magnetic particles velocity can be obtained from Eq. (19) by using the inverse Hankel and Laplace transforms and is given as follows

$$v(r,t) = \frac{\alpha - p_n + \alpha p_n}{G+1-\alpha} \int_0^t e^{-p_n \tau} u(r,t-\tau) d\tau, \alpha \in [0,1],$$

where

$$p_n = \frac{\alpha}{G+1-\alpha}.$$

4. Numerical results and discussion
The equations have been solved by using Mathematica software. The effect of fractional parameter $\alpha = 1$ at time $t = 2$ on the fluid velocity $u(r,t)$ and the magnetic particle (iron rich particles) velocity $v(r,t)$ was examined. All flow results were obtained by using NFDt defined by Caputo and Fabrizio [6]. It is interesting to note that these results agree considerably well with those of UFDt in [15] as shown in Fig. [2] and [3] for $R = 0.5$, $Ha = 2$, $\omega = \pi/4$, $A_u = 0.5$, $A_i = 0.6$, $G = 0.8$, $Re = 5$.

5. Conclusion
The flow of blood mixed with magnetic particles in a circular cylinder has been modeled with Caputo-Fabrizio Fractional Derivative without Singular Kernel (NFDt). Both Laplace and finite Hankel transforms have been used to obtain the analytical solution of the governing fractional partial differential non-linear equations. Numerical models have been generated by using Mathematica...
software package. The results have been compared with those of the previous Caputo fractional derivative (UFDt) and good agreement has been found in terms of the flow characteristics. The study is helpful and supportive for other practical non-linear problems. It is interesting to note that both fluid and magnetic particle velocities exhibit the same response in the presence of an external magnetic field. However, the magnetic particle velocity is lower than the fluid velocity due to the drag force and other resistive forces.

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