Critical Exponents For Schwarzschild-Kerr and BTZ Systems

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Regarding the spin-up of Schwarzschild-Kerr and Bañados-Teitelboim-Zanelli systems as a symmetry breaking phase transition, critical exponents are evaluated and compared with classical Landau predictions. We suggest a definition of isothermal compressibility which is independent of spin direction. We find identical exponents for both systems, and possible universality in the phase transitions of these systems.

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I. INTRODUCTION

Black holes are perfect absorbers classically but do not emit anything; their classical physical temperature is absolute zero. In the semi-classical approximation black holes emit Hawking radiation with a perfect thermal spectrum. This allows a consistent interpretation of the laws of black hole mechanics as physically corresponding to the ordinary laws of thermodynamics. The classical laws of black hole mechanics together with the formula for the temperature of Hawking radiation allow one to identify the general relativistic horizon area, $\mathcal{A}/4$, as playing the mathematical role of entropy. The geometric thermodynamics of black holes has provided many useful insights to the relation between geometric quantities and physics. These analogies are particularly useful when applied to constant mass black hole metric transformations involving a symmetry reduction. The Schwarzschild to Kerr (SCH-K) transform is an example. In thermodynamics, these transitions involve order parameters and critical exponents linked to response functions (such as specific heat or magnetic susceptibility) describing phase transitions. Existing thermodynamic analogies [1], [2] suggest that the SCH-K transform can be viewed as a phase transition. The Schwarzschild solution is spherically symmetric and has higher symmetry than axially symmetric Kerr. We interpret the SCH-K transition, wherein Schwarzschild ”spins up”, as symmetry reduction, and thus assign $J$ the role of order parameter.

Three elements of critical phenomena, power law behavior, universality, and scaling, have become increasingly important in general relativity and have been identified in some general relativistic systems. Davies [1] and Lousto [3] investigated the possibility of a phase transition in the SCH-K system at $J \simeq 0.68m^2$, a point where the specific heat has an infinite discontinuity. The identification of this point with a phase transition has been further discussed by Katz et al [4], [5], and Kaburaki [6], [7]. The three elements have recently been shown to single out a critical solution for some general relativistic solution sets. In 2+1
dimensions Cai, Lu, and Zhang [8] have considered the transition to extremal black hole solutions as a critical transition and discussed this behavior in the 2+1 solution sets. Cai et al point out that Curir [9], [10] claims a critical point exists at the extremal limit of the hole and a phase transition occurs from extremal to non-extremal Kerr black hole. She has also investigated the spin interaction of the inner and outer Kerr horizons [11], [12].

In 3+1 dimensions Choptuik [13] discovered universal power law scaling of black hole mass for critical data in the gravitational collapse of a minimally coupled scalar field. Critical collapse for other matter distributions is reviewed by Gundlach [14], with an extensive list of references. Including angular momentum in families of collapsing matter solutions allows the study of angular momentum behavior at the black hole threshold in critical collapse. For fluids with equation of state \( p = k \rho \), the angular momentum vector has a critical exponent proportional to the mass critical exponent found by Choptuik.

In this paper we evaluate the critical exponents for \( J = 0 \to J \neq 0 \) transition in two systems: the 3+1 asymptotically flat SCH-K system and the 2+1 Bañados-Teitelboim-Zanelli (BTZ) black hole. In each case we take the Hawking temperature of the \( J = 0 \) horizon as the critical temperature. Surprisingly, the results are the same for both systems and are also similar to those derived from the classical Landau theory and mean field theory [15]. This is a different approach to studying the effects of adding angular momentum than during the formation of the black hole solutions in critical collapse. The black holes we consider already exist, and angular momentum is added. We do not study the role of angular momentum in the collapse process. Even so, there are some similarities between critical collapse and the critical behavior we find in the black hole spin-up process.

In the next two sections we briefly review the SCH-K thermodynamics needed to write the response functions and examine critical behavior. The BTZ solution and its critical exponents are discussed in the fourth section. Rotational scaling inequalities are derived in section V. In the final section we discuss the similarity to classical Landau theory and universality, and the relation to critical collapse phenomena.
II. SCH-K THERMODYNAMICS

The Kerr solution, characterized by mass \( m \) and specific angular momentum \( a = J/m \), is given in Boyer-Lindquist \((t, r, \theta, \varphi)\) coordinates by

\[
\begin{align*}
    ds^2 &= \Psi dt^2 - \left( \frac{\Sigma}{\Delta} \right) dr^2 - \Sigma (d\theta^2 + \sin^2 \theta \ d\varphi^2) \\
    &\quad + (1 - \Psi) a \sin^2 \theta \ 2dt \ d\varphi - (2 - \Psi) a^2 \sin^4 \theta \ d\varphi^2,
\end{align*}
\]

where \( \Sigma = r^2 + a^2 \cos^2 \theta \), \( \Psi = 1 - 2mr/\Sigma \), \( \Delta = r^2 - 2mr + a^2 \). When \( a = 0 \), \( \Psi = 1 - 2m/r \) and the Schwarzschild solution is recovered.

It is well known that the SCH-K family of metrics is conformally scaled by the mass parameter such that

\[
g_{\mu\nu}^{Kerr}(t, r, m, a) = \left( \frac{m}{M} \right)^2 g_{\mu\nu}^{Kerr}(T, R, M, A)
\]

where \( T/M = t/m \), \( R/M = r/m \), \( A/M = a/m \). We will consider a thermodynamic system of SCH-K metrics with energies labelled by \( m \), and focus our attention on the horizons of this system, i.e. the "black hole" states \([10], [17]\). The thermodynamic parameters are \( m, T_H, S_{BH}, \Omega_h, J \).

The horizons are 2-surfaces of constant \( t \) and \( r \). The outer one is located at \( \Delta(r_+) = 0 \) with \( r_+ = m + \sqrt{m^2 - a^2} \). The horizon area is

\[
    \mathcal{A} = \oint_{r=r_+} \sqrt{g_{\theta\theta}g_{\varphi\varphi}} \ d\theta d\varphi = 4\pi(r_+^2 + a^2).
\]

The angular velocity of the outer horizon, \( \Omega_h := \frac{d\varphi}{dt}_{r_+} \), is given by

\[
    \Omega_h = \frac{a}{r_+^2 + a^2}.
\]

The Bekenstein-Hawking entropy is \( S_{BH} = (\frac{k_B c^3}{\hbar G})\mathcal{A}/4 = \mathcal{A}/4 \) in units with \( k_B = \hbar = c = G = 1 \).

\[
    S_{BH} = \pi(r_+^2 + a^2) = 2\pi m^2 \left[ 1 + \sqrt{1 - J^2/m^4} \right].
\]
Hawking radiation is the energy flux from the black hole and considered as black body radiation in a thermal bath at temperature $T_H$. Using angular momentum as an extensive thermodynamic parameter, the temperature of the black hole at constant $J$ provides the relations:

\[ T_H = \frac{\partial m}{\partial S_{BH}} J \]  
\[ = \frac{1}{4\pi m} \left[ \frac{(1 - J^2/m^4)^{1/2}}{1 + (1 - J^2/m^4)^{1/2}} \right] \]  
\[ = \frac{1}{8\pi m} [1 - (2\pi J/S_{BH})^2]. \]

$T_{sch} = 1/(8\pi m)$ is the Hawking temperature of the Schwarzschild horizon \[18\].

The rotational analog of $p = p(V, T)$ is $\Omega_h = \Omega_h(J, T_H)$ or $J = J(\Omega_h, T_H)$. To find the equation of state we rewrite Eq.(3) as

\[ m\Omega_h = \frac{J/m^2}{2[1 + \sqrt{1 - (J/m^2)^2}]}. \]

Dividing Eq.(3) by Eq.(5b) yields

\[ \phi = \frac{J/m^2}{\sqrt{1 - (J/m^2)^2}}, \quad \phi := \Omega_h/(2\pi T_H) \]

with inversion

\[ J = m^2 \frac{\phi}{(1 + \phi^2)^{1/2}}. \]

Using Eq.(5b) to eliminate $m^2$, we obtain the equation of state for Kerr black holes:

\[ J = \frac{\phi}{4(2\pi T_H)^2} \left[ (1 + \phi^2)^{-1/2} [1 + (1 + \phi^2)^{1/2}]^{-2} \right]. \]

These thermodynamic functions will be used to generate the response functions which, in turn, lead to the critical exponents.

**III. SCH-K CRITICAL BEHAVIOR**
A. Response Functions

Two of the systems where critical behavior has been studied are fluid/gas and magnetic systems. We will examine response functions: specific heat, compressibility, susceptibility. Critical exponents are defined with the use of a dimensionless temperature deviation $\epsilon := (T - T_c)/T_c$ where $T_c$ is the critical temperature. The most common critical exponents are

1. Fluid/Gas Systems

specific heat at constant volume $C_V \approx |\epsilon|^{-\alpha}$

fluid/gas density difference $\rho_{fl} - \rho_{gas} \approx (-\epsilon)^\beta$

isothermal compressibility $\kappa_T \approx |\epsilon|^{-\gamma}$

critical isotherm ($\epsilon = 0$) $p - p_c \approx |\rho_{fl} - \rho_{gas}|^\delta \text{ sgn}(\rho_{fl} - \rho_{gas})$.

2. Magnetic Systems

zero-field specific heat $C'_H \approx |\epsilon|^{-\alpha}$

zero-field magnetization $M \approx |\epsilon|^{\beta}$

zero-field isothermal susceptibility $\chi_T \approx |\epsilon|^{-\gamma}$

critical isotherm ($\epsilon = 0$) $H \approx |M|^\delta \text{ sgn}(M)$.

B. Rotating Systems

Rotating systems display thermodynamic similarity to magnetic systems. The angular momentum of the Kerr system provides an interesting analogy. Smarr’s formula

$m = 2[((\kappa/2\pi)(A/4) + \Omega_h J)]$

is obtained here using Eqs.(3), and (4). Equation (3) is analogous to the Euler relation $U = TS - MH$, where $MH$ is magnetic energy for magnetization $M$. The analogy is more
striking if one writes the magnetic energy in quantum form as \(-g\mu_B HJ\), with \(g\) the Landé g-factor, \(\mu_B\) the Bohr magneton, and \(J = m\hbar\) (here \(m\) is the magnetic quantum number).

Just as \(M\) and \(H\) form an extensive/intensive pair, so do \(J\) and \(\Omega_h\).

Critical exponents for rotating systems are

- specific heat: \(C_\Omega \approx |\epsilon|^{-\alpha}\)
- angular momentum: \(J \approx |\epsilon|^{\beta}\)
- isothermal compressibility: \(\kappa_T \approx |\epsilon|^{-\gamma}\)
- critical isotherm (\(\epsilon = 0\)): \(\Omega_h \approx |J|^\delta \text{ sgn}(J)\).

For critical temperature \(T_c = T_{sch}\), the temperature deviation is

\[
\epsilon = \frac{T_H - T_c}{T_c} = \frac{(1 - J^2/m^4)^{1/2} - 1}{(1 - J^2/m^4)^{1/2} + 1},
\]

with the inversion

\[
(a/m)^2 = -\frac{4\epsilon}{(1 - \epsilon)^2}, \quad a\Omega_h = \frac{\epsilon}{\epsilon - 1}.
\]

C. Response Functions for SCH-K

1. Specific Heat

Given the entropy and temperature in Eq.(5c), the specific heat at constant \(J\) is

\[
C_J = \left[ T_H \frac{\partial S_{BH}}{\partial T_H} \right]_{J} = \frac{4\pi m T_H S_{BH}}{1 - 8\pi m T_H - 4\pi T_H^2 S_{BH}}.
\]

Equation (13) agrees with the form given by Davies [1] upon using \(T_H \rightarrow T_H/8\pi\) and \(S_{BH} \rightarrow 8\pi S_{BH}\). In the Schwarzschild limit \(C_{J=0} = -8\pi m = -m/T_{sch}\), where the negative sign indicates the well known idea that Schwarzschild black holes get hotter as they lose mass.

We write Eq.(13) as

\[
C_J = -\frac{4\pi m^2 J(J - 2\Omega_h m^3)}{J^2 - 12\Omega_h^2 m^6}.
\]
A discontinuity in $C_J$ occurs at $J_\infty = \sqrt{12} \Omega_h m^3$, or $a_\infty = 2\sqrt{3} \Omega_h m^2$. With $\Omega_h$ expressed in Eq. (3) we have

$$a_\infty = \frac{\sqrt{3} a_\infty}{1 + \sqrt{1 - (a_\infty/m)^2}}. \quad (15)$$

For $a_\infty = x m$, $x^2 = 2\sqrt{3} - 1$, or $a_\infty \simeq .68 m$. The temperature of the discontinuity is

$$4\pi T_H(a_\infty) \simeq .42/m. \quad (16)$$

Lousto [3] has used this temperature as the critical temperature for deriving critical exponents. The presence of this discontinuity in $C_J$ means that the Helmholtz free energy (as well as the Gibbs potential) does not have a valid Taylor expansion [15] in term of $J$. Such an expansion must exist in order for Landau theory to predict critical exponents.

In the following, we use $4\pi T_{Sch} = .5/m$ as the critical temperature since symmetry reduction occurs at the $J = 0$ to $J \neq 0$ transition. There is no discontinuity in the specific heat at $T_{Sch}$.

The specific heat at constant $\Omega_h$ is

$$C_\Omega = -\frac{16\pi m T_H S_{BH}}{1 + 4\Omega_h^2 m^2} \right. \left.(17) \quad = -\frac{\pi J^2 (J - 2\Omega_h m^3)}{4\Omega_h^2 m^4}.\right.$$

2. Isothermal Compressibility

Davies [1] indicates that $C_\Omega - C_J$ measures how the black hole can be spun up at constant temperature, with the difference in specific heats depending on $\kappa_T$. The form suggested by Davies is spin direction dependent:

$$\kappa_T^{Dav} = -\frac{1}{J} \left[ \frac{\partial J}{\partial \Omega_h} \right]_{T_H} \quad (18a)$$

$$= -(1/\Omega_h) + 8(J/m) - 4\Omega_h (J/m)^2. \quad (18b)$$

Since $\Omega_h(-a) = -\Omega_h$, and $J(-a) = -J$, it follows that $\kappa_T^{Dav}(-a) = -\kappa_T^{Dav}$. There are other possibilities. We could follow the definition of magnetic susceptibility $\chi_T$ as the fluid/gas
analogy of isothermal compressibility including no powers of $J$ or we could include even powers of $n$ where

$$\kappa_T^{\text{rot}(n)} = -\frac{1}{J^n} \left[ \frac{\partial J}{\partial \Omega_h} \right]_{T_H}. \tag{19}$$

Clearly $\kappa_T^{\text{rot}(n)}(-a) = \kappa_T^{\text{rot}(n)}$ for $n$ even.

D. Critical Exponents for SCH-K

The critical exponents to be found are $\alpha$ from the specific heat, $\beta$ from the angular momentum, $\gamma$ from the compressibility, and $\delta$ from the critical isotherm.

a. $\alpha = 0$ from the expansion of $C_\Omega$ in Eq.(17). $C_\Omega = -8\pi m^2 (1 + 4\epsilon + \ldots)$. The value is unchanged if we expand $C_J$ in Eq.(13) in analogy with $C_V$ since $C_J = -8\pi m^2 (1 - 5\epsilon + \ldots)$.

b. $\beta = 1/2$ from the inversion of Eq.(11). $J = 2m^2 \left| \epsilon \right|^{1/2} e^{\epsilon}$.

c. $\gamma = n/2 = 0, 1/2, \text{ or } 1$. The value $1/2$ is computed from Davies’ formula (18b)

$$\frac{1}{\kappa_T^{\text{Dav}}} = \frac{\Omega_h}{1 - 8a\Omega_h + 4(a\Omega_h)^2} \left| \epsilon \right|^{1/2} \approx \frac{\Omega_h}{2m(1 + 7\epsilon + 4\epsilon^2)}.$$  

The other values follow from $n = 0$ and $n = 2$ in Eq.(19).

d. $\delta = 1$. The critical isotherm is given by $\Omega_h = J(1 - \epsilon)/(4m^3)$.

IV. CRITICAL EXPONENTS AND THE BTZ SOLUTION

One of the motives for finding critical exponents is the possibility that their validity will extend beyond the single system from which they were obtained and thus be somewhat "universal". We have calculated the exponents for the SCH-K horizons. Another system
with horizon thermodynamics for which critical exponents can be calculated is the anti-de Sitter (AdS) family of black holes of varying dimensions. They differ from the SCH-K system in several respects. The AdS solutions are not asymptotically flat, there is no unique asymptotic timelike Killing norm, and they have a cosmological constant. As a first check of possible universality, we calculate the critical exponents for the 2+1 BTZ black hole.

The Lorentzian 2+1 section of the BTZ black hole [21] is

$$ds^2 = N^2dT^2 - \rho^2[d\Phi - (\frac{j}{2\rho^2})dT]^2 - (y/\rho)^2N^{-2}dy^2. \tag{20}$$

$\Phi$ is $2\pi$ periodic. The outer horizon has radius $y_+$ where

$$y_+^2 = (m/l^2)[1 - (jl/m)^2]^{1/2}, \tag{21a}$$

$$2\rho^2(y) = 2y^2 - y_+^2 + m/l^2, \tag{21b}$$

$$N^2(y) = (yl/\rho)^2(y^2 - y_+^2). \tag{21c}$$

The AdS solution as a bound state of BTZ is fixed by $m = -1$ and $j = 0$. This black hole has temperature $T$, entropy $S$, and horizon rotation speed $\Omega$ just as the Kerr system does. For the AdS black hole [21] we have

$$T = \frac{l\sqrt{2m}}{2\pi} \left[ \frac{1 - (jl/m)^2}{1 + \sqrt{1 - (jl/m)^2}} \right]^{1/2}. \tag{22}$$

$$S = \frac{\pi}{2}\rho(y_+) \tag{23}$$

$$\Omega = -\frac{j}{2\rho^2(y_+)} \tag{24}$$

We take the critical temperature for the spin-up transition as the temperature of the non-rotating hole

$$T_c = \frac{l\sqrt{m}}{2\pi}. \tag{22}$$

The temperature deviation for the critical parameters is
\[ \epsilon = \frac{T - T_c}{T_c} = \sqrt{2} \left[ \frac{1 - (jl/m)^2}{1 + \sqrt{1 - (jl/m)^2}} \right]^{1/2} - 1, \]  

(25)

with the inversion

\[ (jl/m)^2 = 1 - \frac{(1 + \epsilon)^2}{16} [1 + \epsilon + (9 + 2\epsilon + \epsilon^2)^{1/2}]^2. \]

The \( \beta \) exponent follows directly from this expansion and we find \( \beta = 1/2 \). From Eq. (24) we have

\[ \Omega \simeq -\frac{j}{m/l^2 + \sqrt{m/l^2}} \]  

(26)

so that \( \delta = 1 \). \( \alpha = 0 \) follows from the specific heat

\[ C_\Omega = \left[ T \frac{\partial S}{\partial T} \right]_\Omega \simeq \sqrt{\frac{m\pi^2}{8l^2}}. \]  

(27)

Just as in the SCH-K calculation, \( \left[ \frac{\partial \Omega}{\partial J} \right]_T \) is constant as one approaches the critical point. The critical behavior is determined by the power of \( J \) in the compressibility and we find \( \gamma = n/2 \). Evaluating the thermodynamic functions shows the critical exponents to be identical to those calculated for the 3+1 SCH-K system.

V. SCALING INEQUALITIES

The critical exponents can be constrained by inequalities based on thermodynamic arguments. Two of the most useful \[15\] are the Rushbrooke inequality

\[ \alpha + 2\beta + \gamma \geq 2 \]

and Widom’s (actually Griffiths’ \[15\])

\[ \gamma \geq \beta(\delta - 1). \]

The values predicted by the classical theory of Landau \[22\] are

\[ \alpha = 0, \beta = 1/2, \gamma = 1, \delta = 3, \]

(28)
and they satisfy both inequalities. These inequalities need to be examined for rotating systems.

To prove the Rushbrooke inequality, consider the relation between the specific heats at constant angular momentum and constant angular velocity:

\[ C_\Omega = C_J + \frac{TJ^n}{2J^2 \kappa_{\text{rot} (n)}} \left( \frac{\partial J}{\partial T} \right)_{\Omega_h}^2. \] (29)

The specific heats are negative in the region around \( J = 0 \). Explicitly including the negative signs, (29) can be rewritten as

\[ |C_\Omega| = |C_J| + \frac{TJ^n}{2J^2 \kappa_{\text{rot} (n)}} \left( \frac{\partial J}{\partial T} \right)_{\Omega_h}^2 \]

or

\[ |C_\Omega| > \frac{TJ^n}{2J^2 \kappa_{\text{rot} (n)}} \left( \frac{\partial J}{\partial T} \right)_{\Omega_h}^2. \]

Substituting the critical behavior, we find the Rushbrooke rule for rotating systems

\[ \alpha + n\beta + \gamma - 2 \geq 0. \] (30)

Proof of the Widom-Griffiths inequality starts with \( \Omega \simeq J^\delta \) and \( J \simeq \epsilon^\beta \) from Eq.(10). Thus

\[ \frac{\partial \Omega}{\partial J} \simeq \delta J^{\delta - 1} \simeq \delta \epsilon^{\beta(\delta - 1)}. \]

Eqs. (14) and (15) express

\[ \frac{1}{\kappa_{\text{rot} (n)}} = -J^n \left[ \frac{\partial \Omega}{\partial J} \right] \simeq \epsilon^\gamma \]

or

\[ \frac{\partial \Omega}{\partial J} \simeq J^{-n} \epsilon^\gamma \simeq \epsilon^{\gamma - n\beta}. \]

It follows that \( \epsilon^{\gamma + \beta(1 - \delta - n)} \simeq \delta \). As one approaches the critical point and \( \epsilon \to 0 \), the previous expression can be constant only if the exponent is zero: \( \gamma = \beta(\delta - 1) + n\beta \). Since \( n \) is defined to range over positive even integers and \( \beta \) must be positive, the inequality is expressed as
\[ \gamma \geq \beta(\delta - 1). \] (31)

The four exponent values computed here are \( \alpha = 0, \beta = 1/2, \gamma = 0, 1/2, \) or 1, \( \delta = 1. \) They satisfy the Widom-Griffiths inequality for all values. Rushbrooke’s is satisfied for \( \gamma = 1 \) and \( n = 2. \)

**VI. DISCUSSION**

We have investigated the critical behavior of the spin-up transition in the 3+1 Schwarzschild-Kerr system and the 2+1 BTZ system. One of the striking characteristics of critical phenomena is the fact that many measures of a system’s behavior near a critical point are quite independent of the details of the interactions between the particles making up the system. The universal features are not only independent of the numerical details of the interparticle interactions, but are also independent of the most fundamental aspects of the structure of the system. The critical exponents for the two systems we examined are the same. This result is interesting because of the differences in the physical dimension and geometries of the two spacetimes. One should note that the dimension of the thermodynamic space has not changed so long as the cosmological constant is not a thermodynamic variable. The agreement of the two sets of critical exponents for the spin-up transition indicates universality may exist for the \( J = 0 \rightarrow J \neq 0 \) transition but the other higher dimensional AdS spaces need to be checked as well as the effect of increasing the dimension of the thermodynamic space by considering a thermodynamic cosmological constant.

Another interesting aspect of the critical exponents calculated in this paper is their similarity to the classic Landau exponents, which are \( \alpha = 0, \beta = 1/2, \gamma = 1, \delta = 3. \)

The calculation of these numbers is based on a series expansion of the Helmholtz potential \( F = m - TS \) as a function of order parameter \( J: \)

\[ F(J, T) = L_0(T) + L_2(T)J^2 + \ldots \] (32)

where the coefficients can be expanded about the critical temperature
\begin{equation}
L_i(T) = l_{i(0)} + l_{i(1)}(T - T_c) + \ldots
\end{equation}

Some of the thermodynamic response functions examined for critical behavior are simple
derivatives of the Helmholtz free energy. Following Stanley we write

\[ \Omega = \left[ \frac{\partial F}{\partial T} \right]_J = 2L_2(T)J + 4L_4(T)J^3 + \ldots \]

\[ \left[ \frac{\partial \Omega}{\partial J} \right]_T = \frac{\partial^2 F}{\partial T^2} = 2L_2(T) + 12L_4(T)J^2 + \ldots \]

From the expression for \( \Omega \) we see that if \( l_{2(0)} \) is non-zero then the isothermal derivative of the
horizon angular speed is constant at the critical point. This behavior agrees with the direct
calculations done in the previous sections producing \( \delta = 1 \) and a \( \gamma \) that reflects the power
of \( J \) multiplying the derivative in the compressibility definition (34). In the Landau model
for the magnetic critical exponent, \( l_{2(0)} \) is set to zero based on expecting the inverse zero
field susceptibility to be zero. The magnetic critical \( \gamma \) follows directly from the expansion
with \( l_{2(0)} = 0 \) and \( l_{2(1)} \) providing the \( \gamma = 1 \) behavior with no multiplying \( J \) factor in the
susceptibility definition. With \( l_{2(0)} = 0 \), the Landau model predicts \( \delta = 3 \). If, in these
magnetic expansions, \( l_{2(0)} \) is not set to zero, then the Helmholtz expansion produces the
same \( \gamma \) and \( \delta \) critical exponents that are found directly for the rotating systems.

The other critical exponents, \( \alpha \) and \( \beta \), agree for both the Landau model and the rotational
calculation. The \( \alpha = 0 \) exponent also follows from the Helmholtz expansion with \( l_{2(0)} \) non-
zero, even though in the magnetic case a more complicated argument is needed to produce
this behavior. The magnetic \( \beta = 1/2 \) is the single exponent that does not follow from the
expansion but from the definition of the black hole temperature. In the Landau model, it is
calculated by looking at the zero field magnetization (with \( J \) analogous to \( \mathcal{M} \)) in Eq.(34).
There is no non-zero \( J \) for zero \( \Omega \) in the rotational system when \( J \) is assumed to come from
purely kinematic rotation.

As pointed out in the introduction, investigating angular momentum during critical col-
lapse studies the effects of initial data sets which break spherical symmetry during collapse,
while here we have studied the effects of breaking spherical symmetry by adding angular momentum to already existing black hole solutions. We found from the inversion of

$$\epsilon = \frac{(1 - J^2/m^4)^{1/2} - 1}{(1 - J^2/m^4)^{1/2} + 1}$$

that

$$J = 2m^2 \left| \frac{\epsilon^{1/2}}{1 - \epsilon} \right| .$$

Choptuik [13] studied the spherically symmetric collapse of a massless scalar field and found that the mass scaled as

$$m \approx (p - p_*)^\gamma$$

where $p$ indexes a one-parameter set of initial data and $\gamma \approx 0.37$. Garfinkle et al [23] investigated the effects of breaking spherical symmetry with angular momentum during the gravitational collapse of a massless scalar field, and found that the angular momentum scaled as

$$J \approx (p - p_*)^\mu$$

with $\mu \approx 0.76$ or $\mu \approx 2\gamma$, and where $\gamma$ depends on the type of collapsing matter. This suggests that the pure thermodynamic relation between $J$ and $m$ may be reflected in critical collapse behavior. We note that the thermodynamic system is an equilibrium system and the other is not. Gundlach [24] investigated the critical collapse of a perfect fluid with equation of state $p = k\rho$ and found that the angular momentum critical exponent was

$$\mu = \gamma \frac{5(1 + 3k)}{3(1 + k)}$$

for mass exponent $\gamma$ and a range of $k$. This is not reflected in the mass dependence of the thermodynamic relation. Gundlach has emphasized three final endpoints of collapse: a black hole, a star, or a dispersive explosion with no remnant. Perfect fluid initial data sets seem too restrictive to model all of Gundlach’s possibilities. Perfect fluids have not been found
as interiors for the Kerr vacuum. The mass-angular momentum relation in thermodynamics and in critical collapse, and the relevance of perfect fluid collapse to non-black hole states should be interesting to investigate.

In conclusion, we have extended classic scaling inequalities to include rotation, and have defined an isothermal compressibility which is independent of spin direction. We have calculated four of the critical exponents for the symmetry breaking spin-up of both a 3+1 and 2+1 black hole. The critical exponents for the two cases agree with each other and with some of the values calculated from a Landau type Helmholtz expansion. The agreement may indicate a universality in spin-up behavior.

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