CONVERGENCE OF HYBRID SLICE SAMPLING VIA SPECTRAL GAP

KRZYSZTOF LATUSZYŃSKI,∗ University Warwick
DANIEL RUDOLF,** Universität Passau

Abstract

It is known that the simple slice sampler has robust convergence properties, however the class of problems where it can be implemented is limited. In contrast, we consider hybrid slice samplers which are easily implementable and where another Markov chain approximately samples the uniform distribution on each slice. Under appropriate assumptions on the Markov chain on the slice we show a lower bound and an upper bound of the spectral gap of the hybrid slice sampler in terms of the spectral gap of the simple slice sampler. An immediate consequence of this is that spectral gap and geometric ergodicity of the hybrid slice sampler can be concluded from spectral gap and geometric ergodicity of its simple version which is very well understood. These results indicate that robustness properties of the simple slice sampler are inherited by (appropriately designed) easily implementable hybrid versions. We apply the developed theory and analyse a number of specific algorithms such as the stepping-out shrinkage slice sampling, hit-and-run slice sampling on a class of multivariate targets and an easily implementable combination of both procedures on multidimensional bimodal densities.

Keywords: slice sampler; spectral gap; geometric ergodicity

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1. Introduction

Slice sampling algorithms are designed for Markov chain Monte Carlo (MCMC) sampling from a distribution given by a possibly unnormalised density. They belong to the class of auxiliary variable algorithms that define a suitable Markov chain on an extended state space. Following [41] and [6] a number of different versions have been discussed and proposed in [4, 9, 20, 21, 22, 25, 29, 30, 37]. We refer to these papers for details of algorithmic design and applications in Bayesian inference and statistical physics. Here let us first focus on the appealing simple slice sampler setting in which no further algorithmic tuning or design by the user is necessary: Assume that $K \subseteq \mathbb{R}^d$ and let the unnormalised density be $\varrho; K \rightarrow (0, \infty)$. The goal is to sample approximately

∗ Postal address: Department of Statistics, CV47AL Coventry, United Kingdom
∗ Email address: K.G.Latuszynski@warwick.ac.uk
** Postal address: Universität Passau, Innstraße 33, 94032 Passau, Germany
** Email address: daniel.rudolf@uni-passau.de
with respect to (w.r.t.) the distribution $\pi$ determined by $\varrho$, i.e.

$$\pi(A) = \frac{\int_A \varrho(x) \, dx}{\int_K \varrho(x) \, dx}, \quad A \in \mathcal{B}(K),$$

where $\mathcal{B}(K)$ denotes the Borel $\sigma$-algebra. Given the current state $X_n = x \in K$, the *simple slice sampling* algorithm generates the next Markov chain instance $X_{n+1}$ by the following two steps:

1. choose $t$ uniformly at random from $(0, \varrho(x))$, i.e. $t \sim U(0, \varrho(x))$;
2. choose $X_{t+1}$ uniformly at random from $K(t) := \{x \in K \mid \varrho(x) > t\}$, the level set of $\varrho$ determined by $t$.

The above defined *simple slice sampler* transition mechanism is known to be reversible w.r.t. $\pi$ and possesses very robust convergence properties that have been observed empirically and established formally. For example Mira and Tierney [21] proved that: If $\varrho$ is bounded and the support of $\varrho$ has finite Lebesgue measure, then the simple slice sampler is *uniformly ergodic*. Roberts and Rosenthal provide in [29] criteria for *geometric ergodicity*. Moreover, in [29, 50] the authors prove explicit estimates of the total variation distance of the distribution of $X_n$ to $\pi$. In the recent work [24], depending on the volume of the level sets, an explicit lower bound of the *spectral gap* of simple slice sampling is derived.

Unfortunately, the applicability of the simple slice sampler is limited. In high dimensions sampling uniformly from the level set of $\varrho$ is in general infeasible and thus the second step of the algorithm above can not be performed. Consequently, the second step is replaced by sampling a Markov chain on the level set, which has the uniform distribution as the invariant one. Following the terminology of [28] we call such algorithms *hybrid slice samplers*. We refer to [26] where various procedures and designs for the Markov chain on the slice are suggested and insightful expert advice is given.

Although being easy to implement, *hybrid slice sampling* in general has not been analyzed theoretically and little is known about its convergence properties. The present paper is aimed at closing this gap by providing statements about the inheritance of convergence from the simple to the hybrid setting.

To this end we study the absolute spectral gap of hybrid slice samplers. The absolute spectral gap of a Markov operator $P$ or a corresponding Markov chain $(X_n)_{n \in \mathbb{N}}$ is given by

$$\text{gap}(P) = 1 - \|P\|_{L^0_{2,\pi} \to L^0_{2,\pi}},$$

where $L^0_{2,\pi}$ is the space of functions $f : K \to \mathbb{R}$ with zero mean and finite variance (i.e. $\int_K f(x) \, d\pi(x) = 0$; $\|f\|_2^2 = \int_K |f(x)|^2 \, d\pi(x) < \infty$) and $\|P\|_{L^0_{2,\pi} \to L^0_{2,\pi}}$ denotes the operator norm. We refer to [32] for the functional analytic background. From the computational point of view, existence of the spectral gap (i.e. $\text{gap}(P) > 0$) implies a number of desirable and well studied robustness properties. In particular

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A notable exception is elliptical slice sampling [22], that has been recently investigated in [23], where a geometric ergodicity statement is provided.
• the spectral gap implies geometric ergodicity [15] [28] and the variance bounding property [31];
• for reversible Markov chains the spectral gap implies that a CLT holds for all functions $f \in L_{2,\pi}$, c.f. [8] [14];
• furthermore, consistent estimation of the CLT asymptotic variance is well established for geometrically ergodic chains (c.f. [27] [10] [11]).

Additionally, quantitative information on the spectral gap allows the formulation of precise non-asymptotic statements. In particular, it is well known, see e.g. [27] Lemma 2], that if $\nu$ is the initial distribution of the reversible Markov chain in question, i.e. $\nu = \mathbb{P}_{X_1}$, then
\[
\|\nu P^n - \pi\|_{TV} \leq (1 - \text{gap}(P))^n \left\| \frac{d\nu}{d\pi} - 1 \right\|_2,
\]
where $\nu P^n = \mathbb{P}_{X_{n+1}}$. See [1] Section 6 for a related $L_{2,\pi}$ convergence result. Moreover, when considering the sample average, one obtains
\[
E \left[ \frac{1}{n} \sum_{j=1}^{n} f(X_j) - \int_{K} f(x) d\pi(x) \right]^2 \leq \frac{2}{n \cdot \text{gap}(P)} + \frac{c_p \left\| \frac{d\nu}{d\pi} - 1 \right\|_{\infty}}{n^2 \cdot \text{gap}(P)},
\]
for any $p > 2$ and any function $f : K \to \mathbb{R}$ with $\|f\|_p^p = \int_K |f(x)|^p \pi(dx) \leq 1$, where $c_p$ is an explicit constant which depends only on $p$. One can also take a burn-in into account, for further details see [33] Theorem 3.41. This indicates that the spectral gap of a Markov chain is central to robustness and a crucial quantity in both asymptotic and non-asymptotic analysis of MCMC estimators.

The route we endeavour is to conclude the spectral gap of the hybrid slice sampler from the more tractable spectral gap of the simple slice sampler. So what is known about the spectral gap of the simple slice sampler? For saying more on this we require the following notation. Define $v_\theta : [0, \infty) \to [0, \infty]$ by $v_\theta(t) := \text{vol}_\theta(K(t))$, which for level $t$ returns the volume of the level set. We say for $m \in \mathbb{N}$ that $v_\theta \in \Lambda_m$ if

- $v_\theta$ is continuously differentiable and $v_\theta'(t) < 0$ for any $t \geq 0$; and
- the mapping $t \mapsto tv_\theta'(t)/v_\theta(t)^{1-1/m}$ is decreasing on the support of $v_\theta$.

Recently, in [24] Theorem 3.10] it has been shown that, if $v_\theta \in \Lambda_m$, then $\text{gap}(U) \geq 1/(m + 1)$. This provides a criterion for the existence of a spectral gap as well as a quantitative lower bound, essentially depending on whether $t \mapsto tv_\theta'(t)/v_\theta(t)^{1-1/m}$ is decreasing or not.

Now we are in a position to explain our contributions. Let $H$ be the Markov kernel of the hybrid slice sampler determined by a family of transition kernels $H_t$, where each $H_t$ is a Markov kernel with uniform limit distribution, say $U_t$, on the level determined by $t$. Consider
\[
\beta_k := \sup_{x \in K} \left( \int_0^{v_\theta(x)} \left\| H_t^k - U_t \right\|_{L_{2,\pi} \to L_{2,\pi}}^2 \frac{dt}{v_\theta(x)} \right)^{1/2},
\]
and note that the quantity $\left\| H_t^k - U_t \right\|_{L_{2,\pi} \to L_{2,\pi}}^2$ measures how fast $H_t$ gets close to $U_t$. Thus $\beta_k$ is the supremum over expectations of a function which measures the speed of
convergence of $H^k_t$ to $U_t$. The main result is stated in Theorem 1 and it is as follows: Assume that $\beta_k \to 0$ for increasing $k$ and assume $H_t$ induces a positive semi-definite Markov operator for every level $t$. Then

$$ \frac{\text{gap}(U) - \beta_k}{k} \leq \text{gap}(H) \leq \text{gap}(U), \quad k \in \mathbb{N}. $$

(1)

The first inequality implies that whenever there exists a spectral gap of the simple slice sampler and $\beta_k \to 0$, then there is a spectral of the hybrid slice sampler. The second inequality of (1) verifies a very intuitive result, namely that the simple slice sampler is always better than the hybrid one.

We demonstrate how to apply our main theorem in different settings. First, we consider a stepping-out shrinkage slice sampler, suggested in [26], in a simple bimodal 1-dimensional setting. Next we turn to the $d$-dimensional case and on each slice perform a single step of the hit-and-run algorithm, studied in [3, 16, 38]. Using our main theorem we prove equivalence of the spectral gap (and hence geometric ergodicity) of this hybrid hit-and-run on the slice and the simple slice sampler. Let us also mention here that in [30] the hit-and-run algorithm, hybrid hit-and-run on the slice and simple slice sampler are compared, according to covariance ordering [19], to a random walk Metropolis algorithm. Finally, we combine the stepping-out shrinkage and hit-and-run slice sampler. The resulting algorithm is practical and easily implementable in multidimensional settings. For this version we again show equivalence of the spectral gap and geometric ergodicity with the simple slice sampler for multidimensional bimodal targets.

Further note that we consider single auxiliary variable methods to keep the arguments simple. We believe that a similar analysis can also be done if one considers multi auxiliary variable methods.

The structure of the paper is as follows. In Section 2 the notation and preliminary results are provided. These include a necessary and sufficient condition for reversibility of hybrid slice sampling in Lemma 1 followed by a useful representation of slice samplers in Section 2.1 which is crucial in the proof of the main result. In Section 3 we state and prove the main result. For example in Corollary 1 a lower bound of the spectral gap of a hybrid slice sampler is provided which performs several steps w.r.t. $H_t$ on the chosen level. In Section 4 we apply our result to analyse a number of specific hybrid slice sampling algorithms in different settings that include multidimensional bimodal distributions.

### 2. Notation and basics

Recall that $\varrho : K \to (0, \infty)$ is an unnormalised density on $K \subseteq \mathbb{R}^d$ and denote the level set of $\varrho$ as

$$ K(t) = \{ x \in K \mid \varrho(x) > t \}. $$

Hence the sequence $(K(t))_{t \geq 0}$ of subsets of $\mathbb{R}^d$ satisfies

1. $K(0) = K$;
2. $K(s) \subseteq K(t)$ for $t < s$;
3. $K(t) = \emptyset$ for $t \geq \|\varrho\|_\infty$. 
Let \( \text{vol}_d \) be the \( d \)-dimensional Lebesgue measure and let \( \{ U_t \}_{t \in (0, \| \varrho \|_\infty)} \) be a sequence of distributions, where \( U_t \) is the uniform distribution on \( K(t) \), i.e.

\[
U_t(A) = \frac{\text{vol}_d(A \cap K(t))}{\text{vol}_d(K(t))}, \quad A \in \mathcal{B}(K).
\]

Further let \( \{ H_t \}_{t \in (0, \| \varrho \|_\infty)} \) be a sequence of transition kernels, where \( H_t \) is a transition kernel on \( K(t) \subseteq \mathbb{R}^d \). For convenience we extend the definition of the transition kernel \( H_t(\cdot, \cdot) \) on the measurable space \( (K, \mathcal{B}(K)) \). We set

\[
\bar{H}_t(x, A) = \begin{cases} 
0 & x \notin K(t), \\
H_t(x, A \cap K(t)) & x \in K(t).
\end{cases}
\] (2)

In the following we write \( H_t \) for \( \bar{H}_t \) and consider \( H_t \) as extension on \( (K, \mathcal{B}(K)) \). The transition kernel of the hybrid slice sampler is given by

\[
H(x, A) = \frac{1}{\varrho(x)} \int_0^{\varrho(x)} H_t(x, A) \, dt, \quad x \in K, A \in \mathcal{B}(K).
\]

If \( H_t = U_t \) we have the simple slice sampler studied in [21, 24, 29, 30]. The transition kernel of this important special case is given by

\[
U(x, A) = \frac{1}{\varrho(x)} \int_0^{\varrho(x)} U_t(A) \, dt, \quad x \in K, A \in \mathcal{B}(K).
\]

We provide a criterion for reversibility of \( H \) w.r.t. \( \pi \). Therefore let us define the density

\[
\ell(s) = \frac{\text{vol}_d(K(s))}{\int_0^{\| \varrho \|_\infty} \text{vol}_d(K(r)) \, dr}, \quad s \in (0, \| \varrho \|_\infty),
\]

of the distribution of the level sets on \( ((0, \| \varrho \|_\infty), \mathcal{B}((0, \| \varrho \|_\infty))) \).

**Lemma 1.** The transition kernel \( H \) is reversible w.r.t. \( \pi \) iff

\[
\int_0^{\| \varrho \|_\infty} \int_B H_t(x, A) U_t(dx) \, \ell(t) \, dt = \int_0^{\| \varrho \|_\infty} \int_A H_t(x, B) U_t(dx) \, \ell(t) \, dt, \quad A, B \in \mathcal{B}(K).
\]

(3)

In particular, if \( H_t \) is reversible w.r.t. \( U_t \) for almost all \( t \) (concerning \( \ell \)), then \( H \) is reversible w.r.t. \( \pi \).

Equation (3) is the detailed balance condition of \( H_t \) w.r.t. \( U_t \) in average sense, i.e.

\[
\mathbb{E}_t[H(x, dy)U_t(dx)] = \mathbb{E}_t[H(y, dx)U_t(dy)], \quad x, y \in K.
\]

Now we prove Lemma 1.

**Proof.** First, note that

\[
\int_K \varrho(x) \, dx = \int_0^{\| \varrho \|_\infty} \int_K 1_{(0, \varrho(x))}(s) \, dx \, ds
\]

\[
= \int_0^{\| \varrho \|_\infty} \int_K 1_{K(s)}(x) \, dx \, ds = \int_0^{\| \varrho \|_\infty} \text{vol}_d(K(s)) \, ds.
\]
By this, we obtain for any \( A, B \in B(K) \) that
\[
\int_B H(x, A) \pi(dx) = \int_B \int_0^{\|g\|_\infty} H_t(x, A) \frac{dt}{\text{vol}_d(K(s))} ds dx
= \int_B \int_0^{\|g\|_\infty} 1_{K(t)}(x) H_t(x, A) \frac{\ell(t)}{\text{vol}_d(K(t))} dt dx = \int_0^{\|g\|_\infty} \int_B H_t(x, A) U_t(dx) \ell(t) dt.
\]

As an immediate consequence from the previous equation we have the claimed equivalence of reversibility and \( \varpi \). By the definition of the reversibility of \( H_t \) according to \( U_t \) holds
\[
\int_B H_t(x, A) U_t(dx) = \int_A H_t(x, B) U_t(dx).
\]

This, combined with \( \varpi \), leads to the reversibility of \( H \). \( \square \)

We always want to have that \( H \) is reversible w.r.t. \( \pi \). Therefore we formulate the following assumption.

**Assumption 1.** Let \( H_t \) be reversible w.r.t. \( U_t \) for any \( t \in (0, \|g\|_\infty) \).

Now we define Hilbert spaces of square integrable functions and Markov operators. Let \( L_{2,\pi} = L_2(K, \pi) \) be the space of functions \( f: K \to \mathbb{R} \) which satisfy \( \|f\|_{2,\pi}^2 := \langle f, f \rangle_\pi < \infty \), where
\[
\langle f, g \rangle_\pi := \int_K f(x) g(x) \pi(dx)
\]
denotes the corresponding inner-product of \( f, g \in L_{2,\pi} \). For \( f \in L_{2,\pi} \) and \( t \in (0, \|g\|_\infty) \) define
\[
H_t f(x) = \int_{K(t)} f(y) H_t(x, dy), \quad x \in K. \tag{4}
\]

Note that, if \( x \notin K(t) \) we have \( H_t f(x) = 0 \) by the convention on \( H_t \), see \( \varpi \). The Markov operator \( H: L_{2,\pi} \to L_{2,\pi} \) is defined by
\[
H f(x) = \frac{1}{\varrho(x)} \int_0^{\varrho(x)} H_t f(x) dt,
\]
and similarly \( U: L_{2,\pi} \to L_{2,\pi} \) by
\[
U f(x) = \frac{1}{\varrho(x)} \int_0^{\varrho(x)} U_t(f) dt,
\]
where \( U_t(f) = \int_{K(t)} f(x) U_t(dx) \) is a special case of \( \varpi \). Further, for \( t \in (0, \|g\|_\infty) \) let \( L_{2,t} = L_2(K(t), U_t) \) be the space of functions \( f: K(t) \to \mathbb{R} \) with \( \|f\|_{2,t}^2 := \langle f, f \rangle_t < \infty \), where
\[
\langle f, g \rangle_t := \int_{K(t)} f(x) g(x) U_t(dx)
\]
denotes the corresponding inner-product of \( f, g \in L_{2,t} \). Then, \( H_t: L_{2,t} \to L_{2,t} \) can also be considered as Markov operator. Define the functional
\[
S(f) = \int_K f(x) \pi(dx), \quad f \in L_{2,\pi},
\]
as operator $S : L^2_{2,\pi} \to L^2_{2,\pi}$ which maps functions to constant functions, given by their mean value. We say $f \in L^0_{2,\pi}$ iff $f \in L^2_{2,\pi}$ and $S(f) = 0$. Now the absolute spectral gap of a Markov kernel or Markov operator $P : L^2_{2,\pi} \to L^2_{2,\pi}$ is given by

$$\text{gap}(P) = 1 - \|P - S\|_{L^2_{2,\pi} \to L^2_{2,\pi}} = 1 - \|P\|_{L^0_{2,\pi} \to L^0_{2,\pi}}.$$

For details of the last equality we refer to [33, Lemma 3.16]. Moreover, for the equivalence of $\text{gap}(P) > 0$ and (almost sure) geometric ergodicity we refer to [15, Proposition 1.2]. For any $t > 0$ the norm $\|f\|_{2,t}$ can also be considered for $f : K \to \mathbb{R}$.

**Lemma 2.** For any $f : K \to \mathbb{R}$, with the notation from above, we obtain

$$S(f) = \int_0^{\|\|\infty} U_t(f) \ell(t) \, dt. \quad (5)$$

In particular,

$$\|f\|^2_{2,\pi} = \int_0^{\|\|\infty} \|f\|^2_{2,t} \ell(t) \, dt. \quad (6)$$

**Proof.** The assertion of (6) is a special case of (5), since $S(|f|^2) = \|f\|^2_{2,\pi}$. By $\int_K \varphi(x) \, dx = \int_0^{\|\|\infty} \text{vol}_d(K(s)) \, ds$, see in the proof of Lemma 1, one obtains

$$S(f) = \int_K \frac{f(x) \varphi(x) \, dx}{\int_0^{\|\|\infty} \text{vol}_d(K(s)) \, ds} = \int_K \frac{f(x)}{\int_0^{\|\|\infty} \text{vol}_d(K(s)) \, ds} \, dx \int_0^{\|\|\infty} \varphi(x) \, dx \int_0^{\|\|\infty} \text{vol}_d(K(s)) \, ds$$

$$= \int_0^{\|\|\infty} \int_K(f(x) \frac{dx}{\text{vol}_d(K(t))}) \ell(t) \, dt = \int_0^{\|\|\infty} U_t(f) \ell(t) \, dt,$$

which proves (5). \hfill \Box

### 2.1. A useful representation

As in [33, Section 3.3] we derive a suitable representation of $H$ and $U$. We define a $d+1$-dimensional auxiliary state space. Let

$$K_\varphi = \{(x,t) \in \mathbb{R}^{d+1} \mid x \in K, t \in (0, \varphi(x))\}$$

and let $\mu$ be the uniform distribution on $(K_\varphi, \mathcal{B}(K_\varphi))$, i.e.

$$\mu(dx, dt) = \frac{dt \, dx}{\text{vol}_{d+1}(K_\varphi)}.$$

Note that $\text{vol}_{d+1}(K_\varphi) = \int_K \varphi(x) \, dx$. By $L^2_{2,\mu} = L^2(K_\varphi, \mu)$ we denote the space of functions $f : K_\varphi \to \mathbb{R}$ that satisfy $\|f\|^2_{2,\mu} := (f, f)_{\mu} < \infty$, where

$$\langle f, g \rangle_{\mu} := \int_{K_\varphi} f(x,s) \varphi(x,s) \mu(dx,s).$$
denotes the corresponding inner-product for \( f, g \in L_{2, \mu} \). Here, similar to (6), we have
\[
\|f\|_{2, \mu}^2 = \int_0^\|\| \| f(\cdot, s) \|_{2, \mu}^2 \ell(s) ds.
\]

Let \( T : L_{2, \mu} \to L_{2, \pi} \) and \( T^* : L_{2, \pi} \to L_{2, \mu} \) be given by
\[
Tf(x) = \frac{1}{g(x)} \int_0^{g(x)} f(x, s) ds, \quad \text{and} \quad T^* f(x, s) = f(x).
\]

Then \( T^* \) is the adjoint operator of \( T \), i.e. for all \( f \in L_{2, \pi} \) and \( g \in L_{2, \mu} \) we have
\[
\langle f, Tg \rangle_\pi = \langle T^* f, g \rangle_\mu.
\]

Then, for \( f \in L_{2, \mu} \) define
\[
\tilde{H} f(x, s) = \int_{K(s)} f(y, s) H_s(x, dy).
\]

By the stationarity of \( U_s \) according to \( H_s \) it is easily seen that
\[
\|\tilde{H} f\|_{2, \mu}^2 = \int_K \int_0^{g(x)} \left| \tilde{H} f(x, s) \right|^2 \frac{ds dx}{\int_K g(y) dy} = \int_0^{\|\|} \int_{K(s)} \left| \tilde{H} f(x, s) \right|^2 U_s(dx) \ell(s) ds
\]
\[
\leq \int_0^{\|\|} \int_{K(s)} \int_{K(s)} |f(y, s)|^2 H_s(x, dy) U_s(dx) \ell(s) ds
\]
\[
= \int_0^{\|\|} \int_{K(s)} |f(x, s)|^2 U_s(dx) \ell(s) ds = \|f\|_{2, \mu}^2.
\]

Further, define
\[
\tilde{U} f(x, s) = \int_{K(s)} f(y, s) U_s(dy).
\]

Then, \( \tilde{H} : L_{2, \mu} \to L_{2, \mu}, \tilde{U} : L_{2, \mu} \to L_{2, \mu} \) as well as
\[
\|\tilde{H}\|_{L_{2, \mu} \to L_{2, \mu}} = 1, \quad \|\tilde{U}\|_{L_{2, \mu} \to L_{2, \mu}} = 1.
\]

By the construction we have the following.

**Lemma 3.** Let \( H, U, T, T^*, \tilde{H} \) and \( \tilde{U} \) as above. Then
\[
H = T \tilde{H} T^* \quad \text{and} \quad U = T \tilde{U} T^*.
\]

Here \( TT^* : L_{2, \pi} \to L_{2, \pi} \) satisfies \( TT^* f(x) = f(x) \), i.e. \( TT^* \) is the identity operator, and \( T^* T : L_{2, \mu} \to L_{2, \mu} \) satisfies
\[
T^* T f(x, s) = T f(x),
\]
i.e. it returns the average of the function \( f(x, \cdot) \) over the second variable.
3. On the spectral gap of hybrid slice samplers

We start with a relation between the convergence on the slices and the convergence of \( T \bar{H}^k T^* \) to \( T \bar{U} T^* \) for increasing \( k \).

**Lemma 4.** Let \( k \in \mathbb{N} \). Then
\[
\left\| T(\bar{H}^k - \bar{U})T^* \right\|_{L_2,\pi \rightarrow L_2,\pi} \leq \sup_{x \in K} \left( \int_0^{\phi(x)} \left\| \frac{H_t^k - U_t}{\phi(x)} \right\|_{L_2,\pi \rightarrow L_2,\pi} dt \right)^{1/2}.
\]

**Proof.** First, note that \( \|f\|_{2,\pi} < \infty \) implies \( \|f\|_{2,t} < \infty \) for \( \ell \)-a.e. \( t \). For any \( k \in \mathbb{N} \) and \( f \in L_2,\pi \) we have
\[
(\bar{H}^k T^* f)(x,t) = (H_t^k f)(x) \quad \text{and} \quad (\bar{U} T^* f)(x,t) = U_t(f),
\]
such that
\[
T(\bar{H}^k - \bar{U})T^* f(x,t) = \int_0^{\phi(x)} (H_t^k - U_t) f(x) \frac{1_{K(t)}(x)}{\phi(x)} dt.
\]
It follows that
\[
\left\| T(\bar{H}^k - \bar{U})T^* f \right\|_{2,\pi}^2 = \int_K \int_0^{\phi(x)} \left( H_t^k - U_t \right) f(x) \frac{1_{K(t)}(x)}{\phi(x)} \frac{\phi(x)}{\int_K \phi(y) dy} dx dt
\]
\[
\leq \int_K \int_0^{\phi(x)} \left( H_t^k - U_t \right) f(x) \left( \frac{1_{K(t)}(x)}{\phi(x)} \right)^2 \frac{\phi(x)}{\int_K \phi(y) dy} dx dt
\]
\[
= \int_0^{\phi(x)} \int_K \left( H_t^k - U_t \right) f(x) \left( \frac{1_{K(t)}(x)}{\phi(x)} \right)^2 \frac{dx}{\text{vol}_d(K(t))} \frac{\text{vol}_d(K(t))}{\int_0^{\phi(x)} \text{vol}_d(K(s)) ds} dt
\]
\[
= \int_0^{\phi(x)} \left\| (H_t^k - U_t) f \right\|_{2,t}^2 \ell(t) dt
\]
\[
\leq \int_0^{\phi(x)} \left\| H_t^k - U_t \right\|_{L_2,\pi \rightarrow L_2,\pi}^2 \left\| f \right\|_{2,t}^2 \ell(t) dt
\]
\[
= \int_K \int_0^{\phi(x)} \left\| H_t^k - U_t \right\|_{L_2,\pi \rightarrow L_2,\pi}^2 \frac{dx}{\text{vol}_d(K(t))} \frac{\text{vol}_d(K(t))}{\int_0^{\phi(x)} \text{vol}_d(K(s)) ds} dt
\]
\[
= \int_K \int_0^{\phi(x)} \left\| H_t^k - U_t \right\|_{L_2,\pi \rightarrow L_2,\pi}^2 \frac{dt}{\phi(x)} \frac{\phi(x)}{\int_K \phi(y) dy} dx
\]
\[
\leq \|f\|_{2,\pi}^2 \sup_{x \in K} \int_0^{\phi(x)} \left\| H_t^k - U_t \right\|_{L_2,\pi \rightarrow L_2,\pi}^2 \frac{dt}{\phi(x)}.
\]
\[
\square
\]

**Remark 1.** If there exists a number \( \beta \in [0,1] \) such that \( \|H_t - U_t\|_{L_2,\pi \rightarrow L_2,\pi} \leq \beta \) for any \( t \in (0,\|\phi\|_{\infty}) \), then one obtains (as a consequence from the former lemma) that
\[
\left\| T \bar{H}^k T^* - S \right\|_{L_2,\pi \rightarrow L_2,\pi} \leq \left\| T \bar{U} T^* - S \right\|_{L_2,\pi \rightarrow L_2,\pi} + \beta^k.
\]
Here we employed the triangle inequality and the fact that \( \|H_t^k - U_t\|_{L_2,\pi \rightarrow L_2,\pi} \leq \|H_t - U_t\|_{L_2,\pi \rightarrow L_2,\pi} \leq \beta^k \), see for example [33] Lemma 3.16.
Now a corollary follows which provides a lower bound for $\text{gap}(T\tilde{H}^kT^*)$.

**Corollary 1.** Let us assume that $\text{gap}(U) > 0$, i.e. $\|U - S\|_{L_{2,\pi} \to L_{2,\pi}} < 1$, and let us denote

$$\beta_k = \sup_{x \in K} \left( \int_0^{\varrho(x)} \|H^k_t - U_t\|_{L_{2,\pi} \to L_{2,\pi}}^2 \frac{dt}{\varrho(x)} \right)^{1/2}.$$  

Then

$$\text{gap}(T\tilde{H}^kT^*) \geq \text{gap}(U) - \beta_k. \quad (7)$$

**Proof.** It is enough to prove

$$\|T\tilde{H}^kT^* - S\|_{L_{2,\pi} \to L_{2,\pi}} \leq \|U - S\|_{L_{2,\pi} \to L_{2,\pi}} + \beta_k.$$  

By $\tilde{H}^k = \tilde{U} + \tilde{H}^k - \tilde{U}$ and Lemma 4 we have

$$\|T\tilde{H}^kT^* - S\|_{L_{2,\pi} \to L_{2,\pi}} = \|T\tilde{U}T^* - S + T(\tilde{H}^k - \tilde{U})T^*\|_{L_{2,\pi} \to L_{2,\pi}} \leq \|U - S\|_{L_{2,\pi} \to L_{2,\pi}} + \beta_k.$$  

**Remark 2.** If one can sample w.r.t. $U_t$ for every $t \geq 0$, then $H_t = U_t$ and in the estimate of Corollary 1 we obtain $\beta_k = 0$ and equality in (7).  

Now let us state the main theorem.

**Theorem 1.** Let us assume that for almost all $t$ (w.r.t. $\ell$) $H_t$ is positive semi-definite on $L_{2,t}$ and let

$$\beta_k = \sup_{x \in K} \left( \int_0^{\varrho(x)} \|H^k_t - U_t\|_{L_{2,\pi} \to L_{2,\pi}}^2 \frac{dt}{\varrho(x)} \right)^{1/2}.$$  

Further assume that $\lim_{k \to \infty} \beta_k = 0$. Then

$$\frac{\text{gap}(U) - \beta_k}{k} \leq \text{gap}(H) \leq \text{gap}(U), \quad k \in \mathbb{N}. \quad (8)$$

Several conclusions can be drawn from the theorem: First, under the assumption that $\lim_{k \to \infty} \beta_k = 0$, the LHS of (8) implies that in the setting of the theorem, whenever the simple slice sampler has a spectral gap, so does the hybrid version. See Section 4 for examples. Second, it also provides a quantitative bound on $\text{gap}(H)$ given appropriate estimates on $\text{gap}(U)$ and $\beta_k$. Third, the RHS of (8) verifies the intuitive result that the simple slice sampler is better than the hybrid one (in terms of the spectral gap).

To prove the theorem we need some further results.

**Lemma 5.** 1. For any $t \in (0, \|\varrho\|_\infty)$ assume that $H_t$ is reversible with respect to $U_t$. Then $\tilde{H}$ is self-adjoint on $L_{2,\mu}$.

2. Assume that for almost all $t$ (w.r.t. $\ell$) $H_t$ is positive semi-definite on $L_{2,t}$, i.e. for all $f \in L_{2,1}$ holds $\langle H_t f, f \rangle_t \geq 0$. Then $\tilde{H}$ is positive semi-definite on $L_{2,\mu}$.  


Proof. Note that \( \| f \|_{L^2, \mu} < \infty \) implies \( \| f(\cdot, t) \|_{L^2, t} < \infty \) for almost all \( t \) (w.r.t. \( t \)).

To 1. Let \( f, g \in L^2, \mu \) then we have to show that
\[
\langle \tilde{H}f, g \rangle_\mu = \langle f, \tilde{H}g \rangle_\mu.
\]

Note that for \( f, g \in L^2, \mu \) we have for almost all \( t \), by the reversibility of \( H_t \), that
\[
\langle H_t f(\cdot, t), g(\cdot, t) \rangle_t = \langle f(\cdot, t), H_t g(\cdot, t) \rangle_t.
\]

By
\[
\langle \tilde{H}f, g \rangle_\mu = \int_{K_0} \tilde{H}f(x, t)g(x, t) \mu(dx, t)
\]
\[
= \int_{K_0} \int_{0}^{g(x)} \int_{K(t)} f(y, t) H_t(x, dy)g(x, t) \frac{dx dt}{\text{vol}_d(K_0)}
\]
\[
= \int_{0}^{\infty} \int_{K(t)} \int_{K(t)} f(y, t) H_t(x, dy)g(x, t) U_t(dy) \ell(t) dt
\]
\[
= \int_{0}^{\infty} \langle H_t f(\cdot, t), g(\cdot, t) \rangle_t \ell(t) dt
\]
the assertion of 1. is proven.

To 2. We have to prove for all \( f \in L^2, \mu \) that
\[
\langle \tilde{H}f, f \rangle_\mu = \int_{K_0} \tilde{H}f(x, t)f(x, t) \mu(dx, t) \geq 0.
\]

Note that for \( f \in L^2, \mu \) we have for almost all \( t \) that
\[
\langle H_t f(\cdot, t), f(\cdot, t) \rangle_t \geq 0.
\]

By the same computation as in 1 we obtain that the positive semi-definiteness of \( H_t \) carries over to \( \tilde{H} \).

The statement and proofs of the following lemmas follow closely the line of arguments of [39, 40] and essentially use [40, Lemma 12 and Lemma 13]. We provide alternative proofs of the aforementioned employed lemmas in Appendix A.

**Lemma 6.** Let \( \tilde{H} \) be positive semi-definite on \( L^2, \mu \). Then
\[
\left\| T \tilde{H}^{k+1} T^* - S \right\|_{L^2, \pi \rightarrow L^2, \pi} \leq \left\| T \tilde{H}^k T^* - S \right\|_{L^2, \pi \rightarrow L^2, \pi}, \quad k \in \mathbb{N}.
\] (9)

Further, if
\[
\beta_k = \sup_{x \in K} \left( \int_{0}^{g(x)} \left\| H_t^k - U_t \right\|_{L^2, t \rightarrow L^2, t}^2 \frac{dt}{g(x)} \right)^{1/2}
\]
and \( \lim_{k \to \infty} \beta_k = 0 \). Then
\[
\left\| U - S \right\|_{L^2, \pi \rightarrow L^2, \pi} \leq \left\| H - S \right\|_{L^2, \pi \rightarrow L^2, \pi}.
\]
Proof. Let $S_1 : L_{2,\mu} \to L_{2,\pi}$ and the adjoint $S_1^* : L_{2,\pi} \to L_{2,\mu}$ be given by

\[
S_1(f) = \int_{K} f(x,s) \mu(dx,s) \quad \text{and} \quad S_1^*(g) = \int_{K} g(x) \pi(dx).
\]

Thus, $\langle S_1 f, g \rangle_\pi = \langle f, S_1^* g \rangle_\mu$. Furthermore observe that $S_1 S_1^* = S$. Let $R = T - S_1$ and note that $RR^* = I - S$, with identity $I$, and $RR^* = (RR^*)^2$. Since $RR^* \neq 0$ and the projection property $RR^* = (RR^*)^2$ one gets $\|RR^*\|_{L_{2,\pi}\to L_{2,\pi}} = 1$. We have

\[
R\tilde{H}^k R^* = (T - S_1)\tilde{H}^k (T^* - S_1^*)
= T\tilde{H}^k T^* - T\tilde{H}^k S_1^* - S_1 \tilde{H}^k T^* + S_1 \tilde{H}^k S_1^* = T\tilde{H}^k T^* - S.
\]

Further $\|S_1 \tilde{H} S_1^*\|_{L_{2,\mu}\to L_{2,\mu}} \leq 1$. Then, by Lemma 12 it follows that

\[
\|R\tilde{H}^{k+1} R^*\|_{L_{2,\pi}\to L_{2,\pi}} \leq \|R\tilde{H}^k R^*\|_{L_{2,\pi}\to L_{2,\pi}}
\]

and the proof of (9) is completed.

By Lemma 3 we obtain $\|T(\tilde{H}^k - \tilde{U})T^*\|_{L_{2,\pi}\to L_{2,\pi}} \leq \beta_k$ and by (9) as well as Lemma 3 we obtain

\[
\|T\tilde{H}^k T^* - S\|_{L_{2,\pi}\to L_{2,\pi}} \leq \|H - S\|_{L_{2,\pi}\to L_{2,\pi}}, \quad k \in \mathbb{N}.
\]

This implies by triangle inequality that

\[
\lim_{k \to \infty} \|T\tilde{H}^k T^* - S\|_{L_{2,\pi}\to L_{2,\pi}} = \|U - S\|_{L_{2,\pi}\to L_{2,\pi}}
\]

and the assertion is proven. \hfill \Box

**Lemma 7.** Let $\tilde{H}$ be positive semi-definite on $L_{2,\mu}$. Then

\[
\|H - S\|_{L_{2,\pi}\to L_{2,\pi}}^k \leq \|T\tilde{H}^k T^* - S\|_{L_{2,\pi}\to L_{2,\pi}}, \quad (10)
\]

for any $k \in \mathbb{N}$.

**Proof.** As in the proof of Lemma 6 we use $R\tilde{H}^k R^* = T\tilde{H}^k T^* - S$ to reformulate the assertion. It remains to prove that

\[
\|R\tilde{H}^k R^*\|_{L_{2,\pi}\to L_{2,\pi}}^k \leq \|R\tilde{H}^k R^*\|_{L_{2,\pi}\to L_{2,\pi}}.
\]

Recall that $RR^*$ is a projection and satisfies $\|RR^*\|_{L_{2,\pi}\to L_{2,\pi}} = 1$. By Lemma 3 the assertion is proven. \hfill \Box

Now we turn to the proof of Theorem 1.

**Proof of Theorem 1.** By Lemma 5 we know that $\tilde{H}: L_{2,\mu} \to L_{2,\mu}$ is self-adjoint and positive semi-definite. By Lemma 6 we have

\[
\|U - S\|_{L_{2,\pi}\to L_{2,\pi}} \leq \|H - S\|_{L_{2,\pi}\to L_{2,\pi}}.
\]
By Theorem 1 we have for any $k \in \mathbb{N}$ that
\[ \| \tilde{T}T^k \bar{H} - S \|_{L_2, \pi \to L_2, \pi} \leq \| U - S \|_{L_2, \pi \to L_2, \pi} + \beta_k. \] (11)

Then
\[ \| U - S \|_{L_2, \pi \to L_2, \pi} \geq \| \tilde{T}T^k \bar{H} - S \|_{L_2, \pi \to L_2, \pi} - \beta_k \]
\[ \geq \| H - S \|_{L_2, \pi \to L_2, \pi}^k - \beta_k \]
\[ \geq 1 - k (1 - \| H - S \|_{L_2, \pi \to L_2, \pi}) - \beta_k \]
\[ = 1 - k \text{gap}(H) - \beta_k, \]

where we applied a version of Bernoulli’s inequality, i.e. $1 - x^n \leq n(1 - x)$ for $x \geq 0$ and $n \in \mathbb{N}$. Thus,
\[ \frac{\text{gap}(U) - \beta_k}{k} \leq \text{gap}(H) \]
and the proof is completed. \( \square \)

4. Applications

In this section we apply Theorem 1 under different assumptions with different Markov chains on the slices. We provide a criterion of geometric ergodicity of these hybrid slice samplers by showing that there is a spectral gap whenever the simple slice sampler has a spectral gap.

First we consider a class of bimodal densities in a 1-dimensional setting. We study a stepping-out shrinkage slice sampler, suggested in [26], which is explained in Algorithm 1.

Then we consider a hybrid slice sampler which performs a hit-and-run step on the slices in a $d$-dimensional setting. Here we impose very weak assumptions on the unnormalised densities. The drawback is that an implementation of this algorithm might be difficult.

Motivated by this difficulty we study a combination of the previous sampling procedures on the slices. The resulting hit-and-run stepping-out shrinkage slice sampler is presented in Algorithm 2. Here we consider a class of bimodal densities in a $d$-dimensional setting.

4.1. Stepping-out and shrinkage procedure

Let $w > 0$ be a parameter and $\varrho: \mathbb{R} \to (0, \infty)$ be an unnormalised density. We say $\varrho \in \mathcal{R}_w$ if the following conditions are satisfied: There exist $t_1, t_2 \in (0, \| \varrho \|_{\infty})$ with $t_1 \leq t_2$ such that

1. for all $t \in (0, t_1] \cup [t_2, \| \varrho \|_{\infty})$ the level set $K(t)$ is an interval;
2. for all $t \in [t_1, t_2]$ there are disjoint intervals $K_1(t), K_2(t)$ with strictly positive Lebesgue measure, such that
\[ K(t) = K_1(t) \cup K_2(t) \]
and for all $\varepsilon > 0$ holds $K_i(t + \varepsilon) \subseteq K_i(t)$ for $i = 1, 2$. For convenience we set $K_i(t) = \emptyset$ for $t \notin [t_1, t_2]$. 
3. for all \( t \in (0, \|g\|_{\infty}) \) we assume \( \delta_t < w \) where
\[
\delta_t := \begin{cases} 
\inf_{r \in K_{g_1}(t), s \in K_{g_2}(t)} |r - s| & t \in [t_1, t_2) \\
0 & \text{otherwise.}
\end{cases}
\]

The next result shows that certain bimodal densities belong to \( \mathcal{R}_w \).

**Lemma 8.** Let \( g_1 : \mathbb{R} \to (0, \infty) \) and \( g_2 : \mathbb{R} \to (0, \infty) \) be unnormalised density functions. Let us assume that \( g_1, g_2 \) are lower semi-continuous and quasi-concave, i.e. the level sets are open intervals, and
\[
\inf_{r \in \arg \max_g, s \in \arg \max_{g_2}} |r - s| < w.
\]
Then \( g_{\max} := \max \{g_1, g_2\} \in \mathcal{R}_w \).

**Proof.** For \( t \in (0, \|g_{\max}\|_{\infty}) \) let \( K_{g_{\max}}(t), K_{g_1}(t) \) and \( K_{g_2}(t) \) be the level sets of \( g_{\max}, g_1 \) and \( g_2 \) of level \( t \). Note that
\[
K_{g_{\max}}(t) = K_{g_1}(t) \cup K_{g_2}(t).
\]

With the choice
\[
t_1 = \inf \{t \in (0, \|g_{\max}\|_{\infty}) : K_{g_1}(t) \cap K_{g_2}(t) = \emptyset\}, \\
t_2 = \min \{\|g_1\|_{\infty}, \|g_2\|_{\infty}\}
\]
we have \([1] \text{ and } [2] \). Observe that \( \arg \max_{g_i} \subseteq K_{g_i}(t) \) for \( i = 1, 2 \), which yields
\[
\inf_{r \in K_{g_1}(t), s \in K_{g_2}(t)} |r - s| \leq \inf_{r \in \arg \max_{g_1}, s \in \arg \max_{g_2}} |r - s| < w.
\]

In [26] a stepping-out and shrinkage procedure is suggested for the transitions on the level sets. The procedures are explained in Algorithm [1] where a single transition from the resulting hybrid slice sampler from \( x \) to \( y \) is presented.

**Algorithm 1.** A hybrid slice sampling transition of the stepping-out and shrinkage procedure from \( x \) to \( y \), i.e. input \( x \) and output \( y \). The stepping-out procedure has input \( x \) (current state), \( t \) (chosen level), \( w > 0 \) (step size parameter from \( \mathcal{R}_w \)) and outputs an interval \([L, R]\). The shrinkage procedure has input \([L, R]\) and output \( y \):

1. Choose a level \( t \sim \mathcal{U}(0, g(x)) \);
2. Stepping-out with input \( x, t, w \) outputs an interval \([L, R]\):
   (a) Choose \( u \sim \mathcal{U}[0, 1] \). Set \( L = x - uw \) and \( R = L + w \);
   (b) Repeat until \( t \geq g(L) \), i.e. \( L \notin K(t) \): Set \( L = L - w \);
   (c) Repeat until \( t \geq g(R) \), i.e. \( R \notin K(t) \): Set \( R = R + w \);
3. Shrinkage procedure with input \([L, R]\) outputs \( y \):
   (a) Set \( \bar{L} = L \) and \( \bar{R} = R \);
(b) Repeat:

i. Choose \( v \sim U[0,1] \) and set \( y = L + v(R - L) \);

ii. If \( y \in K(t) \) then return \( y \) and exit the loop;

iii. If \( y < x \) then set \( L = y \), else \( R = y \).

For short we write \( |K(t)| = \text{vol}_1(K(t)) \) and for \( t \in (0,\|\varrho\|_\infty) \) we set

\[
\gamma_t := \frac{(w - \delta_t) |K(t)|}{w (|K(t)| + \delta_t)}.
\]

Now we provide useful results to apply Theorem 1.

**Lemma 9.** Let \( \varrho \in \mathcal{R}_w \) with \( t_2 > 0 \) satisfying \([\ref{condition1}]\) and \([\ref{condition2}]\) of the definition of \( \mathcal{R}_w \). Moreover, let \( t \in (0,\|\varrho\|_\infty) \).

1. Then, the transition kernel \( H_t \) of the stepping-out and shrinkage slice sampler from Algorithm \([\ref{algorithm1}]\) takes the form

\[
H_t(x, A) = \gamma_t U_t(A) + (1 - \gamma_t) \left[ 1_{K_1(t)}(x)U_{t,1}(A) + 1_{K_2(t)}(x)U_{t,2}(A) \right],
\]

with \( x \in \mathbb{R}, A \in \mathcal{B}(\mathbb{R}) \) and

\[
U_{t,i}(A) = \begin{cases} \frac{|K_i(t)\cap A|}{|K_i(t)|}, & t \in [t_1,t_2), \\ 0, & t \in (0,t_1) \cup [t_2,\|\varrho\|_\infty), \end{cases}
\]

for \( i = 1,2 \), i.e. in the case \( t \in [t_1,t_2) \) we have that \( U_{t,i} \) denotes the uniform distribution in \( K_i(t) \). (For \( t \in (0,t_1) \cup [t_2,\|\varrho\|_\infty) \) we have \( H_t = U_t \) since \( \delta_t = 0 \) yields \( \gamma_t = 1 \).)

2. The transition kernel \( H_t \) is reversible and induces a positive semi-definite operator, i.e. for any \( f \in L_{2,1} \) holds \( \langle H_t f, f \rangle_t \geq 0 \).

3. Then \( \|H_t - U_t\|_{L_{2,1} \to L_{2,1}} = 1 - \gamma_t \) and

\[
\beta_k \leq \left( \frac{1}{t_2} \int_0^{t_2} (1 - \gamma_t)^{2k} \, dt \right)^{1/2}, \quad k \in \mathbb{N}.
\]

**Proof.** To \([\ref{theorem1}]\): For \( t \in (0,t_1) \cup [t_2,\|\varrho\|_\infty) \) the stepping-out procedure returns an interval that contains \( K(t) \) entirely. Then, since \( K(t) \) is also an interval, the shrinkage scheme returns a sample w.r.t. \( U_t \) in \( K(t) \).

For \( t \in [t_1,t_2), i \in \{1,2\} \) and \( x \in K_i(t) \), within the stepping-out procedure with probability \( (w - \delta_t)/w \) an interval that contains \( K(t) = K_1(t) \cup K_2(t) \) and with probability \( 1 - (w - \delta_t)/w \) an interval that contains \( K_i(t) \) but not \( K(t) \setminus K_i(t) \) is returned. We distinguish these cases:

Case 1: ‘\( K(t) \) contained in the stepping-out output’:

Then, the shrinkage scheme returns with probability \( |K(t)|/(|K(t)| + \delta_t) \) a sample w.r.t. \( U_t \) and with probability \( 1 - |K(t)|/(|K(t)| + \delta_t) \) a sample w.r.t. \( U_{t,i} \).

Case 2: ‘\( K_i(t) \), but not \( K(t) \setminus K_i(t) \), contained in the stepping-out output’:

Then, the shrinkage scheme returns with probability 1 a sample w.r.t. \( U_{t,i} \).
In total, for \( x \in K(t) \) we obtain
\[
H_t(x, A) = \frac{(w - \delta_t)}{w} \left[ \frac{|K(t)|}{|K(t)| + \delta_t} U_t(A) + \left(1 - \frac{|K(t)|}{|K(t)| + \delta_t}\right) U_{t,i}(A) \right] + \left(1 - \frac{(w - \delta_t)}{w}\right) U_{t,i}(A)
= \gamma_t U_t(A) + (1 - \gamma_t) U_{t,i}(A),
\]
where we emphasize that for \( t \in (0, t_1) \cup [t_2, \|\varrho\|_\infty) \) follows \( \gamma_t = 1 \) (since \( \delta_t = 0 \)), such that \( H_t(x, A) \) coincides with \( U_t(A) \).

To 2: For \( A, B \in B(\mathbb{R}) \) we have
\[
\int_A H_t(x, A) U_t(dx) = \gamma_t U_t(B) U_t(A)
+ (1 - \gamma_t) \int_A \left[ \chi_{K_i(t)}(x) U_{t,i}(B) + \chi_{K_{i2}(t)}(x) U_{t,2}(B) \right] U_t(dx)
= \gamma_t U_t(B) U_t(A) + (1 - \gamma_t) \left[ \frac{|K_1(t)|}{|K(t)|} U_{t,1}(A) U_{t,1}(B) + \frac{|K_2(t)|}{|K(t)|} U_{t,2}(A) U_{t,2}(B) \right],
\]
which is symmetric in \( A, B \) and therefore implies the claimed reversibility w.r.t. \( U_t \).

Similarly, we have
\[
\langle H_t f, f \rangle_t = \gamma_t U_t(f)^2 + (1 - \gamma_t) \left[ \frac{|K_1(t)|}{|K(t)|} U_{t,1}(f)^2 + \frac{|K_2(t)|}{|K(t)|} U_{t,2}(f)^2 \right] \geq 0,
\]
where \( U_{t,i}(f) \) denotes the integral of \( f \) w.r.t. \( U_{t,i} \) for \( i = 1, 2 \), which proves the positive semi-definiteness.

To 3: By virtue of [33] Lemma 3.16, the reversibility (equivalently self-adjointness) of \( H_t \) and [12] Theorem V.5.7 we have
\[
\|H_t - U_t\|_{L_{2,t} \rightarrow L_{2,t}} = \sup_{\|f\|_{L_{2,t}} \leq 1} |\langle H_t f, f \rangle| = \sup_{\|f\|_{L_{2,t}} \leq 1, U_t(f) = 0} \langle H_t f, f \rangle,
\]
where the last equality follows by the positive semi-definiteness. Observe that for any \( f \in L_{2,s} \) with \( s \in [t_1, t_2] \) we have by \( U_{s,i}(f)^2 \leq U_{s,i}(f^2) \) for \( i = 1, 2 \) that
\[
\frac{|K_1(s)|}{|K(s)|} U_{s,1}(f)^2 + \frac{|K_2(s)|}{|K(s)|} U_{s,2}(f)^2 \leq \frac{|K_1(s)|}{|K(s)|} U_{s,1}(f^2) + \frac{|K_2(s)|}{|K(s)|} U_{s,2}(f^2) = \|f\|^2_{L_{2,s}}.
\]
Therefore, the equality in (13) yields
\[
\|H_t - U_t\|_{L_{2,t} \rightarrow L_{2,t}} \leq \sup_{\|f\|_{L_{2,t}} \leq 1, U_t(f) = 0} (1 - \gamma_t) \|f\|^2_{L_{2,t}} = 1 - \gamma_t.
\]
For \( t \in (0, t_1) \cup [t_2, \|\varrho\|_\infty) \) by \( H_t = U_t \) and \( 1 - \gamma_t = 0 \) we have an equality. For \( t \in [t_1, t_2] \) with
\[
h(x) = \frac{|K(t)|}{|K_1(t)|} \chi_{K_1(t)}(x) - \frac{|K(t)|}{|K_2(t)|} \chi_{K_2(t)}(x)
\]
the upper bound of (15) is attained for \( f = \|h\|_{L_{2,t}} \) in the supremum expression of (15).
We turn to the verification of (12): For $t \in (t_2, \|\varrho\|_{\infty}]$ we have $1 - \gamma_t = 0$ and for $t \in (0, t_2)$ the function $1 - \gamma_t$ is increasing which also yields that $t \mapsto (1 - \gamma_t)^{2k}$ is increasing on $(0, t_2)$ for any $k \in \mathbb{N}$. By [33, Lemma 3.16] we obtain
\[ \|H_t^k - U_t\|_{L_2, t \mapsto L_2, t} \leq (1 - \gamma_t)^k. \]
Consequently, we have
\[ \beta_k \leq \sup_{r \in (0, t_2)} \frac{1}{r} \int_0^r (1 - \gamma_t)^{2k} \, dt. \tag{16} \]
Further, note that for $a \in (0, \infty)$, any increasing function $g: (0, a) \to \mathbb{R}$ and $p, q \in (0, a)$ with $p \leq q$ holds
\[ \frac{1}{p} \int_0^p g(t) \, dt \leq \frac{1}{q} \int_0^q g(t) \, dt. \tag{17} \]
The former inequality can be verified by showing that the function $p \mapsto G(p)$ for $p \geq 0$ with $G(p) = \frac{1}{p} \int_0^p g(t) \, dt$ satisfies $G'(p) \geq 0$. Applying (17) with $g(t) = (1 - \gamma_t)^{2k}$ in combination with (16) yields (12). □

By Theorem 1 and the previous lemma we have the following result.

**Corollary 2.** For any $\varrho \in \mathcal{R}_w$ the stepping-out and shrinkage slice sampler has a spectral gap if and only if the simple slice sampler has a spectral gap.

**Remark 3.** We want to discuss two extremal situations:

- Consider densities $\varrho: \mathbb{R} \to (0, \infty)$ that are lower semi-continuous and quasi-concave, i.e. the level sets are open intervals. Loosely formulated we assume to have uni-modal densities. Then, for any $w > 0$ we have $\varrho \in \mathcal{R}_w$ (just take $t_1 = t_2$ arbitrarily) and $\delta_t = 0$ for all $t \in (0, \|\varrho\|_{\infty})$. Hence, $H_t = U_t$ for all $t \in (0, \|\varrho\|_{\infty})$ and Algorithm 1 provides an effective implementation of simple slice sampling.

- Assume that $\varrho: \mathbb{R} \to (0, \infty)$ satisfies (1) and (2) from the definition of $\mathcal{R}_w$ for some $w > 0$, but for any $t \in (0, \|\varrho\|_{\infty})$ we have $\delta_t \geq w$. In this setting our theory is not applicable and it is clear that the corresponding Markov chain does not work well, since it is not possible to get from one part of the support to the other one.

### 4.2. Hit-and-run slice sampler

The idea is to combine the hit-and-run algorithm with slice sampling. We ask whether a spectral gap of simple slice sampling implies a spectral gap of this combination. The hit-and-run algorithm was proposed by Smith [38]. It is well studied, see for example [3, 5, 13, 12, 16, 17, 35], and used for numerical integration, see [33, 34]. We define the setting and the transition kernel of hit-and-run.

We consider a $d$-dimensional state space $K \subseteq \mathbb{R}^d$ and $\varrho: K \to (0, \infty)$ is an unnormalised density. We denote the diameter of a level set by
\[ \text{diam}(K(t)) = \sup_{x, y \in K(t)} |x - y| \]
with the Euclidean norm $|\cdot|$. We impose the following assumption.
Assumption 2. The limit $\kappa := \lim_{t \to 0} \frac{\text{vol}_d(K(t))}{\text{diam}(K(t))^d}$ exists and there are numbers $c, \varepsilon \in (0, 1)$, such that
\[\inf_{t \in (0, \varepsilon)} \frac{\text{vol}_d(K(t))}{\text{diam}(K(t))^d} = c > 0.\] (18)

Note that under Assumption 2 we always have $\kappa \geq c$. If $K$ is bounded, has positive Lebesgue measure and $\inf_{x \in K} \varphi(x) > 0$, then Assumption 2 is satisfied with $\kappa = c$. Moreover, for instance, the density of a standard normal distribution satisfies Assumption 2 with unbounded $K$ where also $c = \kappa$. However, the following example indicates that this is not always the case.

Example 1. Let $K = (0, 1)^2$ and $\varphi(x_1, x_2) = 2 - x_1 - x_2$. Then, for $t \in (0, 1]$ we have
\[K(t) = \{(x_1, x_2) \in (0, 1)^2 : x_2 \in [0, \min\{1, 2 - t - x_1\}]\},\]
such that $\text{vol}_2(K(t)) = 1 - t^2/2$. Moreover, the fact that $\{(\alpha, 1-\alpha) : \alpha \in (0, 1)\} \subseteq K(t)$ yields $\text{diam}(K(t)) = \sqrt{2}$, such that for $\varepsilon = 1$ we have $c = 1/4$ and $\kappa = 1/2$.

In general, we consider Assumption 2 as weak regularity requirement, since there is no condition on the level sets and also no condition on the modality.

Let $S_{d-1}$ be the Euclidean unit sphere and $\sigma_d = \text{vol}_{d-1}(S_{d-1})$. A transition from $x$ to $y$ by hit-and-run on the level set $K(t)$ works as follows:

1. Choose $\theta \in S_{d-1}$ uniformly distributed;
2. Choose $y$ according to the uniform distribution on the line $x + r\theta$ intersected with $K(t)$.

This leads to
\[H_t(x, A) = \int_{S_{d-1}} \int_{L_t(x, \theta)} 1_A(x + s\theta) \frac{ds}{\text{vol}_1(L_t(x, \theta))} \frac{d\theta}{\sigma_d}\]
with
\[L_t(x, \theta) = \{r \in \mathbb{R} \mid x + r\theta \in K(t)\}.\]
The hit-and-run algorithm is reversible and induces a positive-semidefinite operator on $L_{2,t}$, see [35]. The following property is well known, see for example [5].

Proposition 1. For $t \in (0, \|\varphi\|_\infty)$, $x \in K(t)$ and $A \in \mathcal{B}(K)$ we have
\[H_t(x, A) = \frac{2}{\sigma_d} \int_A \frac{dy}{|x - y|^{d-1} \text{vol}_1(L(x, \frac{y-x}{|x-y|}))}\] (19)
and
\[\|H_t - U_t\|_{L_{2,t} \to L_{2,t}} \leq 1 - \frac{2}{\sigma_d} \frac{\text{vol}_d(K(t))}{\text{diam}(K(t))^d}.\] (20)

Proof. The representation of $H_t$ stated in (19) is well known, see for example [5]. From this we have for any $x \in K(t)$ that
\[H_t(x, A) \geq \frac{2}{\sigma_d} \frac{\text{vol}_d(K(t))}{\text{diam}(K(t))^d} \frac{\text{vol}_d(K(t) \cap A)}{\text{vol}_d(K(t))}.\]
which means that the whole state space $K(t)$ is a small set. By [18] we have uniform ergodicity and by [33, Proposition 3.24] we obtain (20).
Further, we obtain the following helpful result.

**Lemma 10.** Under Assumption 2 we have with

\[ \beta_k = \sup_{x \in K} \left( \int_0^{g(x)} \left\| H_t^k - U_t \right\|^2_{L_2(t \to L_2, t)} \frac{dt}{\rho(x)} \right)^{1/2} \]

that \( \lim_{k \to \infty} \beta_k = 0 \).

**Proof.** By (29) and [33, Lemma 3.16], in combination with reversibility (equivalently self-adjointness) of \( H_t \), holds

\[ \left\| H_t^k - U_t \right\|^2_{L_2(t \to L_2, t)} \leq \left( 1 - \frac{2}{\sigma_d} \frac{\text{vol}_d(K(t))}{\text{diam}(K(t))^d} \right)^{2k}. \quad (21) \]

Let \( g_k: [0, \|\rho\|_\infty) \to [0, 1] \) be given by

\[ g_k(t) = \begin{cases} 
(1 - \frac{2}{\sigma_d} \frac{\text{vol}_d(K(t))}{\text{diam}(K(t))^d})^{2k} & t \in (0, \|\rho\|_\infty), \\
(1 - \frac{2}{\sigma_d} \frac{\text{vol}_d(K(t))}{\text{diam}(K(t))^d})^{2k} & t = 0,
\end{cases} \]

which is the continuous extension at zero of the upper bound of (21) with \( \kappa \geq C \in (0, 1] \) from Assumption 2. Note that \( \lim_{k \to \infty} g_k(t) = 0 \) for all \( t \in [0, \|\rho\|_\infty) \) and \( \beta_k \leq \sup_{r \in (0, \|\rho\|_\infty]} \left( \frac{1}{\tau} \int_0^r g_k(t) dt \right)^{1/2} \). Considering the continuous function

\[ h_k(r) = \begin{cases} 
\frac{1}{\tau} \int_0^r g_k(t) dt & r \in (0, \|\rho\|_\infty], \\
g_k(0) & r = 0,
\end{cases} \]

the supremum can be replaced by a maximum over \( r \in [0, \|\rho\|_\infty] \) which is attained, say for \( r^{(k)} \in [0, \|\rho\|_\infty] \), i.e. \( \beta_k \leq h_k(r^{(k)})^{1/2} \). Define

\[ (r_0^{(k)})_{k \in \mathbb{N}} := \{ r^{(k)} | r^{(k)} = 0, k \in \mathbb{N} \} \subseteq (r^{(k)})_{k \in \mathbb{N}}, \]

\[ (r_1^{(k)})_{k \in \mathbb{N}} := \{ r^{(k)} | r^{(k)} \in (0, \varepsilon), k \in \mathbb{N} \} \subseteq (r^{(k)})_{k \in \mathbb{N}}, \]

\[ (r_2^{(k)})_{k \in \mathbb{N}} := \{ r^{(k)} | r^{(k)} \geq \varepsilon, k \in \mathbb{N} \} \subseteq (r^{(k)})_{k \in \mathbb{N}}. \]

W.l.o.g. we assume that \( (r_0^{(k)})_{k \in \mathbb{N}} \neq \emptyset, (r_1^{(k)})_{k \in \mathbb{N}} \neq \emptyset \) and \( (r_2^{(k)})_{k \in \mathbb{N}} \neq \emptyset \). Then, \( \lim_{k \to \infty} h_k(r_0^{(k)}) = 0 \) and using Assumption 2 we have

\[ 0 \leq \lim_{k \to \infty} h_k(r_1^{(k)}) \leq \lim_{k \to \infty} \sup_{s \in (0, r_1)} g_k(s) \leq \lim_{k \to \infty} \left( 1 - \frac{2C}{\sigma_d} \right)^{2k} = 0. \]

Moreover, by the definition of \( (r_2^{(k)})_{k \in \mathbb{N}} \) note that \( 1/(r_2^{(k)}) \cdot 1_{(0, r_2^{(k)})}(t) \leq \varepsilon^{-1} \) for \( t \in (0, \infty) \), such that

\[ \lim_{k \to \infty} h_k(r_1^{(k)}) = \lim_{k \to \infty} \int_0^{\|\rho\|_\infty} \frac{1_{(0, r_1^{(k)})}(t)}{r_1^{(k)}} g_k(t) dt = \int_0^{\|\rho\|_\infty} \lim_{k \to \infty} \frac{1_{(0, r_1^{(k)})}(t)}{r_1^{(k)}} g_k(t) dt = 0. \]

Consequently \( \lim_{k \to \infty} h_k(r^{(k)}) = 0 \), such that \( \lim_{k \to \infty} \beta_k \leq \lim_{k \to \infty} h_k(r^{(k)})^{1/2} = 0. \)
This observation leads by Theorem 1 to the following result.

**Corollary 3.** Let $\rho: K \to (0, \infty)$ and let Assumption 2 be satisfied. Then, the hit-and-run slice sampler has an absolute spectral gap if and only if the simple slice sampler has an absolute spectral gap.

We stress that we do not know whether the level sets of $\rho$ are convex, star-shaped or have any additional structure. In this sense the imposed assumptions on $\rho$ can be considered as weak. This also means that it might be difficult to implement hit-and-run in this generality. In the next section we consider a combination of hit-and-run, stepping-out and shrinkage procedure, where we provide a concrete implementable algorithm.

### 4.3. Hit-and-run, stepping-out and shrinkage slice sampler

We combine hit-and-run, stepping-out and shrinkage procedure. Let $w > 0$, let $K \subseteq \mathbb{R}^d$ and assume that $\rho: K \to (0, \infty)$. We say $\rho \in \mathcal{R}_{d,w}$ if the following conditions are satisfied:

1. there are not necessarily normalised lower semi-continuous and quasi-concave densities $\rho_1, \rho_2: K \to (0, \infty)$, i.e. the level sets are open and convex, with
   \[ \inf_{y \in \arg \max \rho_1, z \in \arg \max \rho_2} |z - y| \leq \frac{w}{2} \]
such that $\rho(x) = \max\{\rho_1(x), \rho_2(x)\}$.
2. the limit $\kappa := \lim_{t \downarrow 0} \frac{\text{vol}(K(t))}{\text{diam}(K(t))^d}$ exists and there are numbers $c, \varepsilon \in (0, 1]$ such that
   \[ \inf_{t \in (0, \varepsilon)} \frac{\text{vol}(K(t))}{\text{diam}(K(t))^d} = c. \]

For $i = 1, 2$ let the level set of $\rho_i$ be denoted by $K_i(t)$ for $t \in [0, \|\rho_i\|_\infty)$ and set $K_i(t) = \emptyset$ for $t \geq \|\rho_i\|_\infty$. Then, by $\rho = \max\{\rho_1, \rho_2\}$ follows that $K(t) = K_1(t) \cup K_2(t)$. If $K$ is bounded and has positive Lebesgue measure, then 2. is always satisfied. For $K = \mathbb{R}^d$ one has to check 2. For example $\rho: \mathbb{R}^d \to (0, \infty)$ with
\[ \rho(x) = \max\{\exp(-\alpha |x|^2), \exp(-\beta |x - x_0|^2)\} \]
and $2\beta > \alpha$ satisfies 1. and 2. for $w = 2|x_0|$. The rough idea for a transition from $x$ to $y$ of the combination of the different methods on the level set $K(t)$ is as follows: Consider a line/segment of the form
\[ L_i(x, \theta) = \{ r \in \mathbb{R} \mid x + r\theta \in K(t) \}. \]
Then, run the stepping out and shrinkage procedure on $L_i(x, \theta)$ and return $y$. In detail, we present a single transition from $x$ to $y$ of the hit-and-run, stepping-out, shrinkage slice sampler in Algorithm 2.

**Algorithm 2.** A hybrid slice sampling transition of hit-and-run, stepping-out and shrinkage procedure from $x$ to $y$, i.e. input $x$ and output $y$. The stepping-out procedure on $L_i(x, \theta)$ (line of hit-and-run on level set) has inputs $x$, $w > 0$ (step size parameter from $\mathcal{R}_{d,w}$) and outputs an interval $[L, R]$. The shrinkage procedure has input $[L, R]$ and output $y = x + s\theta$.
1. Choose a level $t \sim \mathcal{U}(0, \varrho(x))$;

2. Choose a direction $\theta \in S_{d-1}$ uniformly distributed;

3. Stepping-out on $L_t(x, \theta)$ with $w > 0$ outputs an interval $[L, R]$:
   
   (a) Choose $u \sim \mathcal{U}[0,1]$. Set $L = uw$ and $R = L + w$;
   
   (b) Repeat until $t \geq \varrho(x + L \theta)$, i.e. $L \notin L_t(x, \theta)$: Set $L = L - w$;
   
   (c) Repeat until $t \geq \varrho(x + R \theta)$, i.e. $R \notin L_t(x, \theta)$: Set $R = R + w$;

4. Shrinkage procedure with input $[L, R]$ outputs $y$:
   
   (a) Set $\bar{L} = L$ and $\bar{R} = R$;
   
   (b) Repeat:

   i. Choose $v \sim \mathcal{U}[0,1]$ and set $s = \bar{L} + v(\bar{R} - \bar{L})$;
   
   ii. If $s \in L_t(x, \theta)$ return $y = x + s \theta$ and exit the loop;
   
   iii. If $s < 0$ then set $\bar{L} = s$, else $\bar{R} = s$.

Now we present the corresponding transition kernel on $K(t)$. Since $\varrho \in \mathcal{R}_{d,w}$ we can define for $i = 1, 2$ the open intervals

$$L_{t,i}(x, \theta) = \{s \in \mathbb{R} \mid x + s \theta \in K_i(t)\}$$

and have $L_t(x, \theta) = L_{t,1}(x, \theta) \cup L_{t,2}(x, \theta)$. Let

$$\delta_{t,\theta,x} = \inf_{r \in L_{t,1}(x, \theta), s \in L_{t,2}(x, \theta)} |r - s|.$$ 

and note that if $\delta_{t,\theta,x} > 0$ then $L_{t,1}(x, \theta) \cap L_{t,2}(x, \theta) = \emptyset$.

We also write for short $|L_t(x, \theta)| = \text{vol}_1(L_t(x, \theta))$ and for $A \in \mathcal{B}(K)$, $x \in K$, $\theta \in S_{d-1}$ let $A_{x,\theta} = \{s \in \mathbb{R} \mid x + s \theta \in A\}$. With this notation, for $t > 0$, the transition kernel $H_t$ on $K(t)$ is given by

$$H_t(x, A) = \int_{S_{d-1}} \left[ \gamma_t(x, \theta) \frac{|L_t(x, \theta) \cap A_{x,\theta}|}{|L_t(x, \theta)|} \right. + (1 - \gamma_t(x, \theta)) \sum_{i=1}^2 1_{K_i}(x) \frac{|L_{t,i}(x, \theta) \cap A_{x,\theta}|}{|L_{t,i}(x, \theta)|} \left. \right] \frac{d\theta}{\sigma_d},$$

with

$$\gamma_t(x, \theta) = \frac{(w - \delta_{t,x,\theta})}{w} \cdot \frac{|L_t(x, \theta)|}{|L_t(x, \theta)| + \delta_{t,x,\theta}}.$$ 

The following result is helpful.

**Lemma 11.** For $\varrho \in \mathcal{R}_{d,w}$ and for any $t \in (0, \|\varrho\|_{\infty})$ holds:

1. The transition kernel $H_t$ is reversible and induces a positive semi-definite operator on $L_{2,t}$, i.e. for $f \in L_{2,t}$ holds $(H_t f, f)_t \geq 0$. 

2. We have
\[ \|H_t - U_t\|_{L_2,t \to L_2,t} \leq 1 - \frac{\text{vol}_d(K(t))}{\sigma_d \text{diam}(K(t))}, \] (22)

in particular \( \lim_{k \to \infty} \beta_k = 0 \) with \( \beta_k \) defined in Theorem 7.

Proof. First, note that \( L_t(x + s\theta, \theta) = L_t(x, \theta) - s, \ |L_t(x + s\theta, \theta)| = |L_t(x, \theta)| \) and \( \gamma_t(x + s\theta, \theta) = \gamma_t(x, \theta) \) for any \( x \in \mathbb{R}^d, \ \theta \in S_{d-1} \) and \( s \in \mathbb{R} \).

To 1: The reversibility of \( H_t \) w.r.t. \( U_t \) (in the setting of \( w \in \mathcal{R}_{d,w} \)) is inherited by the reversibility of hit-and-run and the reversibility of the combination of the stepping-out and shrinkage procedure, see Lemma 9.

We turn to the positive semi-definiteness: Let \( C_t = \text{vol}_d(K(t)) \). We have
\[
\langle f, H_t f \rangle_t = \int_{S_{d-1}} \int_{K(t)} \gamma_t(x, \theta) f(x) \int_{L_t(x, \theta)} f(x + r\theta) \frac{dr}{|L_t(x, \theta)|} \frac{dx \, d\theta}{C_t \sigma_d} \\
+ \sum_{i=1}^2 \int_{S_{d-1}} \int_{K(t)} (1 - \gamma_t(x, \theta)) f(x) \int_{L_{t,i}(x, \theta)} f(x + r\theta) \frac{dr}{|L_{t,i}(x, \theta)|} \frac{dx \, d\theta}{C_t \sigma_d}.
\]

We prove positivity of the first summand. The positivity of the other two summands follows by the same arguments. For \( \theta \in S_{d-1} \) let us define the projected set
\[
P_{\theta \perp}(K(t)) = \{ \tilde{x} \in \mathbb{R}^d \mid \tilde{x} \perp \theta, \ \exists s \in \mathbb{R} \text{ s.t. } \tilde{x} + s \theta \in K(t) \}.
\]

Then
\[ \int_{S_{d-1}} \int_{K(t)} \gamma_t(x, \theta) f(x) \int_{L_t(x, \theta)} f(x + r\theta) \frac{dr}{|L_t(x, \theta)|} \frac{dx \, d\theta}{C_t \sigma_d} \\
= \int_{S_{d-1}} \int_{P_{\theta \perp}(K(t))} \int_{L_{t}(\tilde{x}, \theta)} \gamma_t(\tilde{x} + s\theta, \theta) f(\tilde{x} + s\theta) \times \\
\int_{L_{t}(\tilde{x} + s\theta, \theta)} f(\tilde{x} + (r + s)\theta) \frac{dr}{|L_{t}(\tilde{x} + s\theta, \theta)|} \frac{ds \, d\tilde{x} \, d\theta}{C_t \sigma_d} \\
= \int_{S_{d-1}} \int_{P_{\theta \perp}(K(t))} \int_{L_{t}(\tilde{x}, \theta)} \gamma_t(\tilde{x}, \theta) f(\tilde{x} + s\theta) \times \\
\int_{L_{t}(\tilde{x}, \theta) - s} f(\tilde{x} + (r + s)\theta) \frac{dr}{|L_{t}(\tilde{x}, \theta) - s|} \frac{ds \, d\tilde{x} \, d\theta}{C_t \sigma_d} \\
= \int_{S_{d-1}} \int_{P_{\theta \perp}(K(t))} \gamma_t(\tilde{x}, \theta) \left( \int_{L_{t}(\tilde{x}, \theta)} f(\tilde{x} + u\theta) du \right) \frac{2 \, ds \, d\tilde{x} \, d\theta}{C_t \sigma_d} \geq 0.
\]

This gives that \( H_t \) is positive semi-definite.
To [2]: For any \( x \in K(t) \) and measurable \( A \subseteq K(t) \) we have

\[
H_t(x, A) \geq \int_{S_{d-1}} \gamma_t(x, \theta) \int_{L_t(x, \theta)} 1_A(x + s \theta) \frac{ds}{|L_t(x, \theta)|} d\theta \\
= \int_{S_{d-1}} \int_0^\infty \gamma_t(x, \theta) 1_A(x - s \theta) \frac{ds}{|L_t(x, \theta)|} d\theta \\
+ \int_{S_{d-1}} \int_0^\infty \gamma_t(x, \theta) 1_A(x + s \theta) \frac{ds}{|L_t(x, \theta)|} d\theta \\
= \int_{\mathbb{R}^d} \frac{\gamma_t(x, \frac{x}{\|x\|})}{\|L_t(x, \frac{x}{\|x\|})\|} 1_A(x - y) |y|^{d-1} dy + \int_{\mathbb{R}^d} \frac{\gamma_t(x, \frac{x}{\|x\|})}{\|L_t(x, \frac{x}{\|x\|})\|} 1_A(x + y) |y|^{d-1} dy \\
= \frac{2}{\sigma_d} \int_{A} \frac{\gamma_t(x, \frac{x-y}{\|x-y\|})}{|x-y|^{d-1}|L_t(x, \frac{x-y}{|x-y|})|} dy \geq \frac{\text{vol}_d(K(t))}{\sigma_d \text{diam}(K(t))^{\alpha}} \cdot \frac{\text{vol}_d(A)}{\text{vol}_d(K(t))}.
\]

Here the last inequality follows by the fact that \( \delta_{t,x,\theta} \leq w/2 \) and \( |L_t(x, \theta)| + \delta_{t,x,\theta} \leq \text{diam}(K(t)) \). Thus, by [18] we have uniform ergodicity and by [33, Proposition 3.24] we obtain (22). Finally, \( \lim_{k \to \infty} \beta_k = 0 \) follows by the same arguments as in Lemma [10].

This observation leads by Theorem [1] to the following result.

**Corollary 4.** Let \( \varrho \in \mathcal{R}_{d,w} \). Then, the hit-and-run, stepping-out, shrinkage slice sampler has an absolute spectral gap if and only if the simple slice sampler has an absolute spectral gap.

5. Concluding remarks

We provide a general framework to prove convergence results of hybrid slice sampling via spectral gap arguments. More precisely, we state sufficient conditions for the spectral gap of appropriately designed hybrid slice sampler to be equivalent to the spectral gap of the simple slice sampler. Since all Markov chains we are considering are reversible, this also provides a criterion for geometric ergodicity, see [28].

To illustrate how our analysis can be applied to specific hybrid slice sampling implementations, we analyse the hit-and-run on the slice algorithm on multidimensional targets under weak conditions and the easily implementable stepping-out shrinkage hit-and-run on the slice for bimodal \( d \)-dimensional distributions. The latter analysis can be in principle extended to settings with more than two modes at the price of further notational and computational complexity.

These examples demonstrate that robustness of the simple slice sampler is inherited by its appropriately designed hybrid versions in realistic computational settings and give theoretical underpinning for their use in applications.

**Appendix A. Technical lemmas**

**Lemma 12.** Let \( H_1 \) and \( H_2 \) be two Hilbert spaces. Further, let \( R: H_2 \to H_1 \) be a bounded linear operator with adjoint \( R^*: H_1 \to H_2 \) and let \( Q: H_2 \to H_2 \) be a bounded
linear operator which is self-adjoint. Then
\[ \|RQ^{k+1}R^*\|_{H_1 \to H_1} \leq \|Q\|_{H_2 \to H_2} \|R|Q|^k R^*\|_{H_1 \to H_1}. \]

Let us additionally assume that \(Q\) is positive semi-definite. Then
\[ \|RQ^{k+1}R^*\|_{H_1 \to H_1} \leq \|Q\|_{H_2 \to H_2} \|RQ^k R^*\|_{H_1 \to H_1}. \]

**Proof.** Let us denote the inner-products of \(H_1\) by \(\langle \cdot, \cdot \rangle_1\) and \(H_2\) by \(\langle \cdot, \cdot \rangle_2\). By the spectral theorem for the bounded and self-adjoint operator \(Q: H_2 \to H_2\) we obtain
\[ \frac{\langle QR^* f, R^* f \rangle_2}{\langle R^* f, R^* f \rangle_2} = \int_{\text{spec}(Q)}^{} \lambda d\nu_{Q,R^* f}(\lambda), \]
where \(\text{spec}(Q)\) denotes the spectrum of \(Q\) and \(\nu_{Q,R^* f}\) denotes the normalized spectral measure. Thus,
\[ \|RQ^{k+1}R^*\|_{H_1 \to H_1} = \sup_{\langle f, f \rangle_1 \neq 0} \frac{\langle Q^{k+1}R^* f, R^* f \rangle_2}{\langle f, f \rangle_1} \]
\[ = \sup_{\langle f, f \rangle_1 \neq 0} \frac{\langle R^* f, R^* f \rangle_2}{\langle f, f \rangle_1} \left| \frac{\langle Q^{k+1}R^* f, R^* f \rangle_2}{\langle R^* f, R^* f \rangle_2} \right| \]
\[ \leq \|Q\|_{H_2 \to H_2} \sup_{\langle f, f \rangle_1 \neq 0} \frac{\langle R^* f, R^* f \rangle_2}{\langle f, f \rangle_1} \int_{\text{spec}(Q)}^{} |\lambda|^{k+1} d\nu_{Q,R^* f}(\lambda) \]
\[ \leq \|Q\|_{H_2 \to H_2} \sup_{\langle f, f \rangle_1 \neq 0} \frac{\langle R^* f, R^* f \rangle_2}{\langle f, f \rangle_1} \int_{\text{spec}(Q)}^{} |\lambda|^k d\nu_{Q,R^* f}(\lambda) \]
\[ = \|Q\|_{H_2 \to H_2} \|R|Q|^k R^*\|_{H_1 \to H_1}. \]

We used that the operator norm of \(Q: H_2 \to H_2\) and the operator norm of \(|Q|: H_2 \to H_2\) is the same. If \(Q\) is positive semi-definite, then \(Q = |Q|\).

**Lemma 13.** Let as assume that the conditions of Lemme 12 are satisfied. Further let \(\|R\|_{H_2 \to H_1} = \|RR^*\|_{H_1 \to H_1} \leq 1\). Then
\[ \|RQR^*\|_{H_1 \to H_1} \leq \|R|Q|^k R^*\|_{H_1 \to H_1}. \]

Let us additionally assume that \(Q\) is positive semi-definite. Then
\[ \|RQR^*\|_{H_1 \to H_1} \leq \|RQ^k R^*\|_{H_1 \to H_1}. \]
Proof. We use the same notation as in the proof of Lemma 12. Thus

\[ \| RQR^* \|_{H_1 \to H_1}^k = \sup_{(f,f) \neq 0} \left( \frac{(R^*f, R^*f)_2}{(f,f)_1} \right)^k \left( \frac{(QR^*f, R^*f)_2}{(R^*f, R^*f)_2} \right)^k \]

\[ = \sup_{(f,f) \neq 0} \left( \frac{(R^*f, R^*f)_2}{(f,f)_1} \right)^k \int_{\text{spec}(Q)} |\lambda|^k \, d\nu_{R^*f}(\lambda) \]

\[ \leq \sup_{(f,f) \neq 0} \left( \frac{(R^*f, R^*f)_2}{(f,f)_1} \right)^k \int_{\text{spec}(Q)} |\lambda|^k \, d\nu_{R^*f}(\lambda) \]

\[ = \sup_{(f,f) \neq 0} \left( \frac{(R^*f, R^*f)_2}{(f,f)_1} \right)^k \left( \frac{|Q|^k R^*f, R^*f)_2}{(R^*f, R^*f)_2} \right) \]

\[ \leq \| RR^* \|_{H_1 \to H_1}^{k-1} \| R |Q|^k R^* \|_{H_1 \to H_1} \leq \| R |Q|^k R^* \|_{H_1 \to H_1}. \]

Note that we applied Jensen inequality. Further, if \( Q \) is positive-semidefinite then \( Q = |Q| \), which finishes the proof. \( \square \)

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