The Laplacian and normalized Laplacian spectra of Möbius polyomino networks and their applications

Zhi-Yu Shi 1, Jia-Bao Liu 1,2,* , Sakander Hayat 3

1School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China
2School of Mathematics, Southeast University, Nanjing 210096, China
3Faculty of Engineering Sciences, GIK Institute of Engineering Sciences and Technology, Topi 23460, Pakistan

Abstract. Spectral theory has widely used in complex networks and solved some practical problems. In this paper, we investigated the Laplacian and normalized Laplacian spectra of Möbius polyomino networks by using spectral theory. Let $M_n$ denote Möbius polyomino networks ($n \geq 3$). As applications of the obtained results, the Kirchhoff index, multiplicative degree-Kirchhoff index, Kemeny’s constant and spanning trees of $M_n$ are obtained. Moreover, it is surprising to find that the multiplicative degree-Kirchhoff index of $M_n$ is nine times as much as the Kirchhoff index.

Keywords: Laplacian spectrum; Normalized Laplacian spectrum; Möbius polyomino networks; Topological indices

1. Introduction

In 1964, Heilbronner [1] proposed Möbius aromatic based on Huckel’s molecular orbital theory. Compared with Huckel’s system, the Möbius system is stable because of its closed shell structure. In recent years, compounds with Möbius aromatic have been synthesized. In particular, Ma et al. [2] studied the normalized Laplacian spectrum for the hexagonal Möbius graphs. Then, Geng et al. [3] obtained the normalized Laplacian spectrum of Möbius phenylene chain. For other graphs, see [5–11]. Motivated by these, we investigate the Laplacian network of Möbius phenylenes Chain. In 2019, Lei et al. [4] studied the normalized Laplacian spectrum for the hexagonal Möbius graphs. Then, Geng et al. [3] obtained the

The traditional concept of distance is the length of the shortest path between vertices $i$ and $j$, represented by $d_{ij}$. The Wiener index [14] was proposed as $W(G) = \sum_{i<j} d_{ij}$. Wiener index has been extensively studied in chemistry. In the past two decades, the research on the Wiener index are shown in references [15]. In 1994, the Gutman index [19] was proposed as $Gut(G) = \sum_{i<j} d_{ij}d_{ij}$. Klein and Randic [20] was the first to put forward the concept of resistance distance, and the resistance distance between vertices $i$ and $j$ is denoted by $r_{ij}$. One famous resistance distance-based parameter called the Kirchhoff index [21], namely $Kf(G) = \sum_{i<j} r_{ij}$. In 2007, Chen et al. [21] defined the multiplicative

E-mail address: shizhiyuah@163.com, liujiabaoaad@163.com, sakander1566@gmail.com

* Corresponding author.
degree-Kirchhoff index as $Kf^*(G) = \sum_{i<j} d_id_j r_{ij}$. For convenience, Gutman and Zhu et al. \[22, 23\] introduced the Kirchhoff index as

$$Kf(G) = n \sum_{k=2}^{n} \frac{1}{\mu_k}, \quad (1.1)$$

where $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n$ are the Laplacian eigenvalues of $G$.

Chen et al. \[21\] proposed the multiplicative degree-Kirchhoff index as

$$Kf^*(G) = 2m \sum_{k=2}^{n} \frac{1}{\lambda_k}, \quad (1.2)$$

where $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ are the normalized Laplacian eigenvalues of $G$.

In Section 2 we mainly introduce some notations and theorems. Then, the Laplacian spectrum of $M_n$ is investigated in Section 3. In Section 4 we obtained the normalized Laplacian spectrum of $M_n$ in the same way as in Section 3. The conclusion and discussion are summarized in Section 5.

2. Preliminary

Given an $n \times n$ matrix $B$, submatrix of $B$ is represented by $B[i_1, \cdots, i_k]$, where $B[i_1, \cdots, i_k]$ is formed by removing the $i_1$-th, $i_2$-th rows and columns of $B$. Let $P_B(x) = det(xI - B)$ represent characteristic polynomial of $B$.

Label Möbius polyomino networks as shown in the Figure 1(1). Evidently, $|V(M_n)| = 2n$, $|E(M_n)| = 3n$ and $V_1 = \{1, 2, \cdots, n\}, V_2 = \{1', 2', \cdots, n'\}$ is an automorphism of $M_n$.

Then $L(M_n)$ and $\mathcal{L}(M_n)$ can be expressed by

$$L(M_n) = \begin{pmatrix} \mathcal{L}_{V_1 V_1} & \mathcal{L}_{V_1 V_2} \\ \mathcal{L}_{V_2 V_1} & \mathcal{L}_{V_2 V_2} \end{pmatrix}, \quad \mathcal{L}(M_n) = \begin{pmatrix} \mathcal{L}_{V_1 V_1} & \mathcal{L}_{V_1 V_2} \\ \mathcal{L}_{V_2 V_1} & \mathcal{L}_{V_2 V_2} \end{pmatrix},$$
where
\[ L_{V_1V_2} = L_{V_2V_1}, \quad L_{V_1V_2} = L_{V_2V_1}, \quad \mathcal{L}_{V_1V_1} = \mathcal{L}_{V_2V_2}, \quad \mathcal{L}_{V_1V_2} = \mathcal{L}_{V_2V_1}. \]

Let
\[ T = \begin{pmatrix} \frac{1}{\sqrt{2}} I_n & \frac{1}{\sqrt{2}} I_n \\ \frac{1}{\sqrt{2}} I_n & -\frac{1}{\sqrt{2}} I_n \end{pmatrix}, \]
then
\[ TL(M_n)T = \begin{pmatrix} L_A & 0 \\ 0 & L_S \end{pmatrix}, \quad TL(M_n)T = \begin{pmatrix} L_A & 0 \\ 0 & L_S \end{pmatrix}, \]
where
\[ L_A = L_{V_1V_1} + L_{V_1V_2}, \quad L_S = L_{V_1V_1} - L_{V_1V_2}, \quad L_A = L_{V_1V_1} + L_{V_1V_2}, \quad L_S = L_{V_1V_1} - L_{V_1V_2}. \]

In what follows, the theorems that we present will be used throughout the Section 3 and Section 4.

**Theorem 2.1.** [24] If \( L_A, L_S, L_A, L_S \) are defined as above, the following formula can be obtained. Then
\[ P_{L(M_n)}(x) = P_{L_A}(x)P_{L_S}(x), \quad P_{L(M_n)}(x) = P_{L_A}(x)P_{L_S}(x). \]

**Theorem 2.2.** [25, 26] For a cycle with \( n \) vertices \( C_n \):

1. The Kirchhoff index of \( C_n \) is
\[ Kf(C_n) = \frac{n^3 - n}{12}. \]
2. The Laplacian eigenvalues of \( C_n \) is
\[ \alpha_i = 2 - 2\cos \frac{2\pi i}{n}, \quad i \in [1, n]. \]

**Theorem 2.3.** [27] For a connected graph \( G \) with vertices \( n \). The number of spanning trees of \( G \) is
\[ \tau(G) = \frac{1}{n} \prod_{i=2}^{n} \mu_i, \]
where \( \mu_i \) is the Laplacian eigenvalue of \( G \).

**Theorem 2.4.** [28] For a connected graph \( G \) with vertices \( n \). Kemeny’s constant of \( G \) is
\[ Kc(G) = \sum_{i=2}^{n} \frac{1}{\lambda_i}, \]
where \( \lambda_i \) is the normalized Laplacian eigenvalue of \( G \).

Evidently, from Theorem 2.4 and (1.4), the relation between the Kemeny’s constant and multiplicative degree-Kirchhoff index is
\[ Kf^*(G) = 2mKc(G). \]

**Theorem 2.5.** [27] If graph \( G \) with \( |V_G| = n \) and \( |E_G| = m \). The number of spanning trees of \( G \) is
\[ 2m\tau(G) = \prod_{i=1}^{n} d_i \cdot \prod_{i=2}^{n} \lambda_i, \]
where \( \lambda_i \) is the normalized Laplacian eigenvalue of \( G \).

For convenience, all \( p \) represents \( 2 + \sqrt{3} \) and all \( q \) represents \( 2 - \sqrt{3} \) throughout the paper.
3. Laplacian spectrum of $M_n$

In this section, we mainly obtain the Laplacian spectrum of $M_n$. According to Laplacian matrix of $M_n$, we can get $L_{V_1V_1}$ and $L_{V_1V_2}$:

$$L_{V_1V_1} = \begin{pmatrix}
3 & -1 & 0 & \cdots & 0 \\
-1 & 3 & -1 & \cdots & \vdots \\
0 & -1 & 3 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 3 \\
0 & \cdots & 0 & 0 & -1
\end{pmatrix}_{n \times n},$$

$$L_{V_1V_2} = \begin{pmatrix}
-1 & -1 & 0 & \cdots & 0 \\
-1 & -1 & 0 & \cdots & \vdots \\
0 & -1 & -1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & -1 \\
0 & \cdots & 0 & 0 & -1
\end{pmatrix}_{n \times n}.$$

Based on Theorem 2.1, Laplacian spectrum consists of the eigenvalues of $L_A$ and $L_S$ of $M_n$ can be obtained.

$$L_A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & \vdots \\
0 & -1 & 2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 2 \\
0 & \cdots & 0 & 0 & -1
\end{pmatrix}_{n \times n},$$

$$L_S = \begin{pmatrix}
4 & -1 & 0 & \cdots & 0 \\
-1 & 4 & -1 & \cdots & \vdots \\
0 & -1 & 4 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 4 \\
0 & \cdots & 0 & 0 & -1
\end{pmatrix}_{n \times n}.$$

Assume that $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n$ are the roots of $P_{L_A}(x) = 0$, and $0 < \beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \leq \beta_n$ are the roots of $P_{L_S}(x) = 0$. Noticing that $L_A$ is the Laplacian matrix of cycle $C_n$, the following lemma can be obtained from (1.1) and Theorem 2.2(1).

**Lemma 3.1.** For Möbius polyomino networks $M_n$ ($n \geq 3$),

$$K_f(M_n) = \frac{n^3 - n}{6} + 2n \sum_{i=1}^{n} \frac{1}{\beta_i},$$

where $\beta_i$ is the eigenvalue of $L_S$.

Next, we first determine $\sum_{i=1}^{n} \frac{1}{\beta_i}$. 
Lemma 3.2. Let \( 0 < \beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \leq \beta_n \) be the eigenvalues of \( L_S \). Then
\[
\sum_{i=1}^{n} \frac{1}{\beta_i} = \frac{n}{2\sqrt{3}} \cdot \frac{p^n - q^n}{p^n + q^n + 2}.
\]

Proof. Let
\[
P_{L_S}(x) = \det(xI - L_S) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.
\]

According to Vieta’s theorem of \( P_{L_S}(x) \), one obtains
\[
\sum_{i=1}^{n} \frac{1}{\beta_i} = (-1)^{n-1}a_{n-1} \frac{1}{\det(L_S)}.
\]

Since \((-1)^{n-1}a_{n-1}\) and \( \det(L_S) \), we focus on \( i \)-th order principal submatrix \( F_i \), which consists of the first \( i \) rows and columns of the following matrix \( L'_S \), \( i \in [1,n] \). Let \( f_i = \det(F_i) \).
\[
L'_S = \begin{pmatrix}
4 & -1 & & \\
-1 & 4 & -1 & \\
& -1 & 4 & -1 \\
& & \ddots & \\
& & & -1 & 4 \\
& & & & -1 & 4
\end{pmatrix}
\]

It is easy to get that \( f_1 = 4, f_2 = 15, f_3 = 56 \) and for \( 3 \leq i \leq n \), \( f_i = 4f_{i-1} - f_{i-2} \). For convenience, we let \( f_0 = 1 \). Furthermore, one can verify that
\[
f_i = \frac{p^{i+1} - q^{i+1}}{2\sqrt{3}}.
\]

Fact 1. \((-1)^{n-1}a_{n-1} = \frac{n}{2\sqrt{3}}(p^n - q^n)\).

Proof of Fact 1. Obviously, we obtain \((-1)^{n-1}a_{n-1}\) is the sum of all the principal minors of order \( n-1 \) of \( L_S \). According to the property of \( L_S \), we know that
\[
det(L_S[i]) = \begin{cases} 
    f_{n-1}, & i = 1 \text{ or } n; \\
    f_{i-1}f_{n-i} - f_{i-2}f_{n-i-1}, & i \in [2,n-1].
\end{cases}
\]

Thus, we can obtain
\[
(-1)^{n-1}a_{n-1} = \sum_{i=1}^{n} det(L_S[i])
\]
\[
= det(L_S[1]) + det(L_S[n]) + \sum_{i=2}^{n-1} det(L_S[i])
\]
\[
= \frac{n}{2\sqrt{3}}(p^n - q^n).
\]

This result as desired.

Fact 2. \( \det(L_S) = p^n + q^n + 2 \).

Proof of Fact 2. By expanding the last row of \( L_S \), we can arrive at
\[
det(L_S) = f_n - f_{n-2} + 2
\]
\[
= p^n + q^n + 2.
\]
Thus Fact 2 holds. 
Substituting Facts 1 and 2 into (3.3) yields lemma 3.2. 
In view of lemmas 3.1 and 3.2, the following theorem can be obtained.

**Theorem 3.3.** For Möbius polyomino networks $M_n$ $(n \geq 3)$,

\[ Kf(M_n) = \frac{n^3 - n}{6} + \frac{n^2}{\sqrt{3}} \cdot \frac{p^n - q^n}{p^n + q^n + 2}. \]

**Remark.** We found that Theorem 3.3 can be obtained by different methods in [29] and [30].

In view of the Laplacian spectrum of $M_n$, the following theorem can be obtained.

**Theorem 3.4.** For Möbius polyomino networks $M_n$ $(n \geq 3)$,

\[ \tau(M_n) = \frac{n}{2}(p^n + q^n + 2). \]

**Proof.** Based on Theorem 2.2(2), we know that the eigenvalues of $L_A$ are $\alpha_i = 2 - 2 \cos \frac{2\pi i}{n}$ $(1 \leq i \leq n)$ and the product of $\alpha_2, \alpha_3, \cdots, \alpha_n$ are $\prod_{i=2}^{n} \alpha_i = n^2$. By Fact 2, we get

\[ \prod_{i=1}^{n} \beta_i = \det(L_S) = p^n + q^n + 2. \]

Together with Fact 2 and Theorem 2.3, Theorem 3.4 follows immediately.

4. Normalized Laplacian spectrum of $M_n$

In this section, we mainly obtain the normalized Laplacian spectrum of $M_n$. Moreover, we find that the multiplicative degree-Kirchhoff index of $M_n$ is nine times of Kirchhoff index.

According to normalized Laplacian matrix of $M_n$, we can get $\mathcal{L}_{V_1V_1}$ and $\mathcal{L}_{V_1V_2}$:

\[
\mathcal{L}_{V_1V_1} = \begin{pmatrix}
1 & -\frac{1}{3} & \cdots & -\frac{1}{3} \\
-\frac{1}{3} & 1 & \cdots & -\frac{1}{3} \\
\vdots & \ddots & \ddots & \vdots \\
-\frac{1}{3} & \cdots & 1 & -\frac{1}{3}
\end{pmatrix}_{n \times n},
\]

\[
\mathcal{L}_{V_1V_2} = \begin{pmatrix}
-\frac{1}{3} & \cdots & -\frac{1}{3} \\
-\frac{1}{3} & \cdots & -\frac{1}{3} \\
\vdots & \ddots & \vdots \\
-\frac{1}{3} & \cdots & -\frac{1}{3}
\end{pmatrix}_{n \times n}.
\]

Based on Theorem 2.1, we can get normalized Laplacian spectrum which is obtained by the eigenvalues of $\mathcal{L}_A$ and $\mathcal{L}_S$ of $M_n$. 
\[
\mathcal{L}_A = \begin{pmatrix}
\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{pmatrix} \quad n \times n,
\]

\[
\mathcal{L}_S = \begin{pmatrix}
\frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3}
\end{pmatrix} \quad n \times n.
\]

Assume that \(0 = \gamma_1 < \gamma_2 \leq \gamma_3 \leq \cdots \leq \gamma_n\) are the roots of \(P_{\mathcal{L}_A}(x) = 0\), and \(0 < \delta_1 \leq \delta_2 \leq \delta_3 \leq \cdots \leq \delta_n\) are the roots of \(P_{\mathcal{L}_S}(x) = 0\). Next, we derive the formulas of \(\sum_{i=2}^{n} \frac{1}{\gamma_i}\) and \(\sum_{i=1}^{n} \frac{1}{\delta_i}\).

**Lemma 4.1.** Let \(0 = \gamma_1 < \gamma_2 \leq \gamma_3 \leq \cdots \leq \gamma_n\) be the eigenvalues of \(\mathcal{L}_A\). Then

\[
\sum_{i=2}^{n} \frac{1}{\gamma_i} = \frac{n^2 - 1}{4}.
\]

**Proof.** Suppose that

\[
P_{\mathcal{L}_A}(x) = \det(xI - \mathcal{L}_A) = x^n + b_1x^{n-1} + \cdots + b_{n-2}x^2 + b_{n-1}x.
\]

Applying Vieta’s theorem, one can get

\[
\sum_{i=2}^{n} \frac{1}{\gamma_i} = \frac{(-1)^{n-2}b_{n-2}}{(-1)^{n-1}b_{n-1}}. \quad (4.4)
\]

Before calculating \((-1)^{n-1}b_{n-1}\) and \((-1)^{n-2}b_{n-2}\), we focus on \(i\)-th order principal submatrix \(G_i\), which consists of the first \(i\) rows and columns of the following matrix \(\mathcal{L}_A'\), \(i \in [1, n]\). Let \(g_i = \det(G_i)\).

\[
\mathcal{L}_A' = \begin{pmatrix}
\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{pmatrix} \quad n \times n.
\]

It is easy to get that \(g_1 = \frac{2}{3}, g_2 = \frac{4}{9}, g_3 = \frac{4}{27}\) and for \(3 \leq i \leq n\), \(g_i = \frac{2}{3}g_{i-1} - \frac{1}{3}g_{i-2}\). For convenience, we let \(g_0 = 1\). Furthermore, we can get

\[
g_i = \frac{i + 1}{3^i}.
\]

**Fact 3.** \((-1)^{n-1}b_{n-1} = \frac{n^2}{3^{n-1}}.\)
Proof of Fact 3. Evidently, we obtain \((-1)^{n-1}b_{n-1}\) is the sum of all the principal minors of order \(n-1\) of \(\mathcal{L}_A\). According to the property of \(\mathcal{L}_A\), we know that

\[
det(\mathcal{L}_A[i]) = \begin{cases} 
g_{n-1}, & i = 1 \text{ or } n; 
g_{i-1}g_{n-i} - \frac{1}{3}g_{i-2}g_{n-i-1}, & i \in [2, n-1]. 
\end{cases}
\]

Therefore, one can get

\[
(-1)^{n-1}b_{n-1} = \sum_{i=1}^{n} det(\mathcal{L}_A[i])
= det(\mathcal{L}_A[1]) + det(\mathcal{L}_A[n]) + \sum_{i=2}^{n-1} det(\mathcal{L}_A[i])
= \frac{n^2}{3^{n-1}},
\]

which is the desired result.

\begin{flushright}
\textbf{Fact 4.} \((-1)^{n-2}b_{n-2} = \frac{n^2(n^2-1)}{4 \cdot 3^{n-1}}.
\end{flushright}

Proof of Fact 4. Obviously, we obtain \((-1)^{n-2}b_{n-2}\) is the sum of all the principal minors of order \(n-2\) of \(\mathcal{L}_A\). In a similar way, we know that

\[
det(\mathcal{L}_A[i, j]) = \begin{cases} 
g_{n-2}, & i = j = n; 
g_{i-2}g_{n-i}, & j = 1 \text{ or } n, \text{ and } i \in [2, n-1]; 
g_{i-1}g_{j-i}g_{n-j} - \frac{1}{3}g_{i-2}g_{j-i-1}g_{n-j-1}, & i < j \text{ and } i,j \in [2, n-1]. 
\end{cases}
\]

Thus, one can obtain

\[
(-1)^{n-2}b_{n-2} = \sum_{1 \leq i < j \leq n} det(\mathcal{L}_A[i, j])
= det(\mathcal{L}_A[1, n]) + \sum_{i=2}^{n-1} det(\mathcal{L}_A[1, i]) + \sum_{i=2}^{n-1} det(\mathcal{L}_A[i, n]) + \sum_{2 \leq i < j \leq n-1} det(\mathcal{L}_A[i, j])
= \frac{n^2(n^2-1)}{4 \cdot 3^{n-1}}.
\]

This completes the proof.

Substituting Facts 3 and 4 into (4.4) yields lemma 4.1.

\begin{flushright}
\textbf{Lemma 4.2.} Let \(0 < \delta_1 \leq \delta_2 \leq \delta_3 \leq \cdots \leq \delta_n\) be the eigenvalues of \(\mathcal{L}_S\). Then

\[
\sum_{i=1}^{n} \frac{1}{\delta_i} = \frac{\sqrt{3n}}{2} \cdot \frac{p^n - q^n}{p^n + q^n + 2}.
\]

Proof. Let

\[
P_{\mathcal{L}_S}(x) = det(xI - \mathcal{L}_S) = x^n + k_1x^{n-1} + \cdots + k_{n-1}x + k_n.
\]

Applying Vieta’s theorem, one can get

\[
\sum_{i=1}^{n} \frac{1}{\delta_i} = \frac{(-1)^{n-1}k_{n-1}}{det(\mathcal{L}_S)}.
\]

In order to obtain \((-1)^{n-1}k_{n-1}\) and \(det(\mathcal{L}_S)\), we focus on \(i\)-th order principal submatrix \(H_i\), which consists of the first \(i\) rows and columns of the following matrix \(\mathcal{L}_S'\), \(i \in [1, n]\). Let \(h_i = det(H_i)\).
\[
L_S' = \begin{pmatrix}
\frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{4}{3}
\end{pmatrix}_{n \times n}.
\]

It is easy to get that 
\[h_1 = \frac{4}{3}, h_2 = \frac{5}{3}, h_3 = \frac{56}{27}\] and for \(3 \leq i \leq n\), 
\[h_i = \frac{4}{3}h_{i-1} - \frac{1}{3}h_{i-2}.\] For convenience, we let \(h_0 = 1\). Furthermore, one can verify that 
\[h_i = p^{i+1} q^{i+1} - \frac{p^{n-i+1}}{2\sqrt{3} \cdot 3^i}.\]

**Fact 5.** \((-1)^{n-1} k_{n-1} = \frac{n}{2\sqrt{3} \cdot 3^{n-1}}.\)**

**Proof of Fact 5.** Similarly, we obtain \((-1)^{n-1} k_{n-1}\) is the sum of all the principal minors of order \(n - 1\) of \(L_S\). Thus, we know that 
\[
det(L_S[i]) = \begin{cases} h_{n-1}, & i = 1 \text{ or } n; \\ h_{i-1}h_{n-i} - \frac{1}{9}h_{i-2}h_{n-i-1}, & i \in [2, n-1]. \end{cases}
\]

Therefore, one can obtain 
\[
(-1)^{n-1} k_{n-1} = \sum_{i=1}^{n} det(L_S[i]) \\
= det(L_S[1]) + det(L_S[n]) + \sum_{i=2}^{n-1} det(L_S[i]) \\
= \frac{n}{2\sqrt{3} \cdot 3^{n-1}}.
\]

This completes the proof.

**Fact 6.** \(det(L_S) = p^n q^n + 2.\)**

**Proof of Fact 6.** By expanding the last row of \(L_S\), we can arrive at 
\[
det(L_S) = h_n - \frac{1}{9}h_{n-2} + \frac{2}{3^n} \\
= \frac{p^n q^n + 2}{3^n}.
\]

Thus Fact 6 holds.

Substituting Facts 5 and 6 into (4.5) yields lemma 4.2.

Based on Theorem 2.4, lemmas 4.1 and 4.2, the following theorem can be obtained.

**Theorem 4.3.** For Möbius polyomino networks \(M_n\) \((n \geq 3)\), 
\[Kc(M_n) = \frac{n^2 - 1}{4} + \frac{\sqrt{3}n}{2} \cdot \frac{p^n - q^n}{p^n + q^n + 2}.\]

In view of (1.2), lemmas 4.1 and 4.2, one get the multiplicative degree-Kirchhoff index.

**Theorem 4.4.** For Möbius polyomino networks \(M_n\) \((n \geq 3)\), 
\[Kf^*(M_n) = \frac{3}{2}(n^3 - n) + \frac{3\sqrt{3}n^2(p^n - q^n)}{p^n + q^n + 2}.\]
According to Theorem 3.3 and Theorem 4.4, we find that the multiplicative degree-Kirchhoff index of $M_n$ is nine times of Kirchhoff index. Since the degree of each point in the Möbius polyomino networks is three, $K^*(M_n) = d_djKf(M_n) = 9Kf(M_n)$ can be obtained from the definition of Kirchhoff index and multiplicative degree-Kirchhoff index index. Therefore, it is not difficult to obtain and prove the correctness of results.

In view of the normalized Laplacian spectrum of $M_n$, the following theorem can be obtained.

**Theorem 4.5.** For Möbius polyomino networks $M_n$ $(n \geq 3)$,

$$
\tau(M_n) = \frac{n}{2}(p^n + q^n + 2).
$$

**Proof.** Based on Theorem 2.5, one can get $\prod_{i=1}^{2n} d_i \prod_{i=2}^{n} \frac{1}{\gamma_i} \prod_{i=1}^{n} \frac{1}{\delta_i} = 2 \cdot 3n \cdot \tau(M_n)$, where

$$
\prod_{i=1}^{2n} d_i = 3^{2n},
$$

$$
\prod_{i=2}^{n} \frac{1}{\gamma_i} = (-1)^{n-1} b_{n-1} = \frac{n^2}{3^{n-1}},
$$

$$
\prod_{i=1}^{n} \frac{1}{\delta_i} = \text{det} (L_S) = \frac{p^n + q^n + 2}{3^n}.
$$

Hence,

$$
\tau(M_n) = \frac{n}{2}(p^n + q^n + 2).
$$

The result as desired.

5. Conclusion and discussion

In this paper, based on the Laplacian matrix and normalized Laplacian matrix of Möbius polyomino networks, the Kirchhoff index, multiplicative degree-Kirchhoff index, Kemeny’s constant and spanning trees of Möbius polyomino networks are determined through the decomposition theorem and Vieta’s theorem.

Spectral theory has important applications in many fields. Wu [6] and Ding [31] obtained the mean first-passage time of Koch networks and 3-prism graph according to the Laplacian spectrum, respectively. Thus, we can explore the mean first-passage time of Möbius polyomino networks.

Funding

This work was supported in part by National Natural Science Foundation of China Grant 11601006.

References

[1] E. Heilbronner, Huckel molecular orbitals of Möbius-type conformations of annulenes, Tetrahedron Letters 5(29) (1964) 1923-1928.

[2] X. Ma, H. Bian, The normalized Laplacians, degree-Kirchhoff index and the spanning trees of hexagonal Möbius graphs, Applied Mathematics and Computation 355 (2019) 33-46.
[3] X. Geng, P. Wang, L. Lei, S. Wang, On the Kirchhoff indices and the number of spanning trees of Möbius phenylenes chain and Cylinder phenylenes chain, Polycyclic Aromatic Compounds (2019) https://doi.org/10.1080/10406638.2019.1693405.

[4] L. Lei, X. Geng, S.C. Li, Y.J. Peng, Y. Yu, On the normalized Laplacian of Möbius phenylene chain and its applications, International Journal of Quantum Chemistry 119 (24) (2019) e26044.

[5] S. Li, W. Yan, T. Tian, The spectrum and Laplacian spectrum of the dice lattice, Journal of Statistical Physics 164 (2016) 449-462.

[6] B. Wu, Z.Z. Zhang, G.R. Chen, Properties and applications of Laplacian spectra for Koch networks, Journal of Physics A: Mathematical and Theoretical 45(2) (2012) 025102.

[7] Y.J. Yang, D.J. Klein, Two-point resistances and random walks on stellated regular graphs, Journal of Physics A: Mathematical and Theoretical 52(7) (2019) 075201.

[8] Y.J. Yang, H.P. Zhang, Kirchhoff index of linear hexagonal chains, International Journal of Quantum Chemistry 108(3) (2008) 503-512.

[9] J. Huang, S.C. Li, X. Li, The normalized Laplacian, degree-Kirchhoff index and spanning trees of the linear polyomino chains, Applied Mathematics and Computation 289 (2016) 324-334.

[10] Z.X. Zhu, J.B. Liu, The normalized Laplacian, degree-Kirchhoff index and the spanning tree numbers of generalized phenylenes, Discrete Applied Mathematics 254 (2019) 256-267.

[11] Y. Pan, J. Li, Kirchhoff index, multiplicative degree-Kirchhoff index and spanning trees of the linear crossed hexagonal chains, International Journal of Quantum Chemistry 118 (24) (2018) e25787.

[12] Y. Pan, C. Liu, J. Li, Kirchhoff indices and numbers of spanning molecular graphs derived from linear crossed polyomino chain, Polycyclic Aromatic Compounds (2020) https://doi.org/10.1080/10406638.2020.1725898.

[13] J.A. Bondy, U.S.R. Murty, Graph theory, Springer, New York 2008.

[14] H. Wiener, Structural determination of paraffin boiling points, Journal of the American Chemical Society 69 (1947) 17-20.

[15] A.A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, Acta Applicandae Mathematicae 66 (2001) 211-249.

[16] A.A. Dobrynin, I. Gutman, S. Klavžar, P. Žigert, Wiener index of hexagonal systems, Acta Applicandae Mathematicae 72 (2002) 247-294.

[17] M. Knor, R. Skrekovski, A. Tepeh, Orientations of Graphs with maximum Wiener index, Discrete Applied Mathematics 211 (2016) 121-129.

[18] S.C. Li, Y.B. Song, On the sum of all distances in bipartite graphs, Discrete Applied Mathematics 169 (2014) 176-185.

[19] I. Gutman, Selected properties of the schultz molecular topological index, Journal of Chemical Information and Computer Sciences 34 (1994) 1087-1089.

[20] D.J. Klein, M. Randić, Resistance distances, Journal of Mathematical Chemistry 12 (1993) 81-95.

[21] H.Y. Chen, F.J. Zhang, Resistance distance and the normalized Laplacian spectrum, Discrete Applied Mathematics 155 (2007) 654-661.

[22] I. Gutman, B. Mohar, The quasi-Wiener and the Kirchhoff indices coincide, Journal of Chemical Information and Computer Sciences 36 (1996) 982-985.

[23] H.Y. Zhu, D.J. Klein, I. Lukovits, Extensions of the Wiener number, Journal of Chemical Information and Computer Sciences 36 (1996) 420-428.

[24] Y.L. Yang, T.Y. Yu, Graph theory of viscoelasticities for polymers with starshaped, multiple-ring and cyclic multiple-ring molecules, Macromolecular Chemistry and Physics 186 (1985) 609-631.

[25] Y.J. Yang, X.Y. Jiang, Unicyclic graphs with extremal Kirchhoff index, MATCH Communications in Mathematical and in Computer Chemistry 60 (2008) 107-120.

[26] R.B. Bapat, Graphs and matrices, Springer, New York 2010.

[27] F.R.K. Chung, Spectral graph theory, American Mathematical Society Providence, RI, 1997.

[28] S. Butler, Algebraic aspects of the normalized Laplacian, in: A. Beveridge, J. Griggs, L. Hogben, G. Musiker, P. Tetali (eds.), Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications, IMA, 2016.

[29] Z. Cinkir, Effective resistances and Kirchhoff index of ladder graphs, Journal of Mathematical Chemistry 54 (2016) 955-966.
[30] G.A. Baigonakova, A.D. Mednykh, Elementary formulas for Kirchhoff index of Möbius ladder and prism graphs, Siberian Electronic Mathematical Reports 16 (2019) 1654-1661.

[31] Q. Ding, W. Sun, F. Chen, Applications of Laplacian spectra on a 3-prism graph, Modern Physics Letters B 28 (2) (2014) 1450009.