Smooth Centrally Symmetric Polytopes in Dimension 3 are IDP

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Abstract. In 1997 Oda conjectured that every smooth lattice polytope has the integer decomposition property. We prove Oda’s conjecture for centrally symmetric 3-dimensional polytopes, by showing they are covered by lattice parallelepipeds and unimodular simplices.

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1. Introduction

A lattice polytope in $\mathbb{R}^d$ is the convex hull of finitely many points in the integer lattice $\mathbb{Z}^d$. All polytopes in this paper will be assumed to be lattice polytopes. They appear naturally in a variety of different fields, such as combinatorics, commutative algebra, toric geometry and optimization, where their geometric and arithmetic behavior has been intensively studied in recent decades. In [5], Oda posed the following fundamental problem:

Problem 1.1. Given two lattice polytopes $P, Q \subseteq \mathbb{R}^d$, when can every lattice point $p$ in the Minkowski sum $P + Q := \{x + y: x \in P, y \in Q\}$ be written as the sum of two lattice points $p_1 \in P$ and $p_2 \in Q$, i.e., $p = p_1 + p_2$?

In general, for arbitrary lattice polytopes, not every lattice point in $P + Q$ is the sum of a lattice point in $P$ and a lattice point in $Q$, not even in the special case $P = Q$. For example, let $P$ be the convex hull of $(0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 2, 1)$ and consider $2P$. Then $2P$ contains the lattice point $(1, 1, 1)$ but this point cannot be written as the sum of any two lattice points in $P$. Of particular interest in this context are so-called IDP polytopes — a lattice polytope has the Integer Decomposition Property (or is IDP for
short) if for every integer \( n \geq 1 \) and every lattice point \( p \in nP \cap \mathbb{Z}^d \), there are lattice points \( p_1, \ldots, p_n \in P \cap \mathbb{Z}^d \) such that \( p = p_1 + \cdots + p_n \). IDP polytopes are of great interest when studying the arithmetic behavior of dilated polytopes (Ehrhart theory) as well as in commutative algebra and toric geometry. The following basic fact will play a crucial role in this note:

**Proposition 1.2** (See, e.g., Bruns and Gubeladze [1]). *Unimodular simplices, parallelepipeds, and zonotopes are IDP.*

A natural notion in toric geometry is that of a smooth polytope: a lattice polytope \( P \) is **smooth** if it is simple and if its primitive edge directions at every vertex form a basis of the lattice \( (\text{aff} \ P) \cap \mathbb{Z}^d \). In particular, every face of a smooth lattice polytope is itself smooth.

Due to its relation with projective normality of projective toric varieties, the following specialization of Problem 1.1 was also asked by Oda [5]. It has since become known as *Oda’s Conjecture*.

**Problem 1.3** (Oda’s Conjecture). *Is every smooth lattice polytope IDP?*

The purpose of this note is to prove the following case of Oda’s conjecture.

**Theorem 1.4.** *Every centrally symmetric 3-dimensional smooth polytope is IDP.*

We have organized the paper as follows. In Section 2, we recall some basic facts about smooth lattice polytopes which we will apply in the proof of Theorem 1.4. In Section 3, we provide a proof of Theorem 1.4. We have structured the crucial steps of the proof into subsequent subsections. Finally in Section 4, we conclude the paper with some open questions which might help to settle Problem 1.3 for the 3-dimensional case.

### 2. Preliminaries

The following lemma is an immediate consequence of having IDP.

**Lemma 2.1** (Bruns and Gubeladze [2, Page 65]). Let \( P, P_1, \ldots, P_m \subseteq \mathbb{R}^d \) be lattice polytopes such that \( P = P_1 \cup \cdots \cup P_m \). If \( P_1, \ldots, P_m \) are IDP, then so is \( P \).

From the definition of a smooth lattice polytope, the following fact straightforwardly follows.

**Lemma 2.2.** Let \( P \subseteq \mathbb{R}^d \) be a smooth \( d \)-dimensional lattice polytope. Let \( v \) be a vertex of \( P \) and let \( p_1, \ldots, p_d \) denote the primitive ray generators on the edges on \( v \). Then the parallelepiped spanned by \( p_1, \ldots, p_d \) from \( v \) does not contain any lattice points aside from its vertices.

The following two lemmas are known to the experts—we include them for the sake of completeness. We start by introducing some notation.
Definition 2.3. Let $P$ be a polytope and $a$ a linear function. For a real number $c$, let $P_c$ be the hyperplane cut of $P$:
$$P_c := \{ x \in P \mid a(x) = c \}.$$ We call $c$ special if $P_c$ contains a vertex of $P$. For fixed $P$ and $a$, the set of special $c$’s is finite.

Recall that a fan $\Sigma$ is said to coarsen another fan $\Sigma'$ if any $\sigma' \in \Sigma'$ is contained in some cone $\sigma \in \Sigma$. We refer to Bruns and Gubeladze [2, Section 1] for details and references on fans.

In the following lemma, we assume the notation as in Definition 2.3.

Lemma 2.4. For $c_1 < c_2$, the normal fans of $P_{c_1}$ and $P_{c_2}$ coincide if the interval $[c_1, c_2]$ does not contain any special values. If $c_2$ is the only special value in this interval, then the normal fan of $P_{c_2}$ coarsens that of $P_{c_1}$ (see Fig. 1).

Proof. This is a consequence of Rambau [6, Lemma 2.2.2], where we regard the hyperplane cuts $P_c$ as fibers of a projection defined by $a$, from the polytope $P$ to the line. See also Schneider [7, Lemmas 2.4.12 and 13].

Lemma 2.5. Let $P \subseteq \mathbb{R}^d$ be a smooth $d$-dimensional lattice polytope, $F$ a facet of $P$ and $a: \mathbb{R}^d \rightarrow \mathbb{R}$ the primitive linear functional defining $F$, i.e., $a(\mathbb{Z}^d) = \mathbb{Z}$, $F = \{ x \in P \mid a(x) = c \}$ for some $c \in \mathbb{Z}$ and $a(x) \geq c$ for all $x \in P$. Then $F' := P_{c+1}$ is a lattice polytope whose normal fan coarsens that of $F$.

Proof. As $P$ is simple all but one of the edge directions from each vertex of $F$ lie in $F$. Further the smoothness condition implies that there is a lattice point on any edge adjacent to a vertex in $F$ but not contained in $F$ at lattice distance 1 from the affine hull of $F$. Hence, $F'$ is the convex hull of primitive ray generators of edges adjacent to the vertices in $F$, but not belonging to $F$.

The statement about the normal fan is a general fact about simple polytopes. Let $P' \supset P$ be a (not necessarily lattice) polytope with the same normal fan as $P$ constructed as follows: The supporting hyperplanes of $P'$ coincide with those of $P$, apart from the hyperplane supporting $F$, which is shifted parallelly by $1 \gg \epsilon > 0$ in the outer direction. As $P$ is simple, there are no vertices of
$P'$ in $P'_c$ (recall that a vertex is contained in at least $d$ facets). The values in $[c, c + 1)$ are nonspecial for $P'$, as $a$ is primitive. Further, for $l \in [c, c + 1]$ we have $P'_l = P_l$. By Lemma 2.4, $P'_{c+1} = P_{c+1}$ may only have a fan that coarsens that of $F = P'_c$.

\[\square\]

3. Proof of the Main Result

3.1. Covering of Lattice Polygons

**Lemma 3.1.** Let $F \subseteq \mathbb{R}^2$ be a smooth lattice polygon. Every unimodular simplex $\Delta \subset F$ can be extended to a lattice unit square in $F$.

**Proof.** After a unimodular transformation, we may assume that $\Delta$ is the standard simplex, i.e., the central triangle in Fig. 2. Assume to the contrary that $\Delta$ cannot be extended to a unit square. This means that the three points $q_1, q_2$ and $q_3$ in Fig. 2 are not contained in $F$. By convexity, it follows that $F$ does not contain any lattice point in the three shaded regions. On the other hand, we assumed that $\Delta \neq F$, so $F$ has to contain at least one further lattice point besides $v_1, v_2$ and $v_3$. Without loss of generality, we may assume that there is another lattice point in the region $A$. Further, by symmetry, we may even assume that there is a lattice point in $A$ that is strictly to the left (and possibly below) of $v_1$ with respect to Fig. 2.

This implies that all further lattice points in region $B$ have to lie on the vertical line through $v_3$, as otherwise $q_2$ would lie in $F$. Let $v$ be the point furthest up on this line, where $v = v_3$ is possible. This is a vertex of $F$, and we consider the parallelepiped spanned by the two primitive ray generators on the edges on it. One of the edges goes down and leftwards into region $A$, but misses $v_1$. The other one goes down and rightwards into region $C$, possibly

![Figure 2. Illustration of the proof of Lemma 3.1](image-url)
hitting $v_2$. Hence, $v_1$ lies in the interior of the parallelepiped, contradicting Lemma 2.2.

3.2. Pushing Facets

**Lemma 3.2.** Let $P \subseteq \mathbb{R}^3$ be a 3-dimensional, smooth lattice polytope with a facet $F$ that is a unimodular triangle. Then (up to translation) the section of $P$ defined in Lemma 2.5 coincides with $rF$ for some integer $r \geq 0$. If $P$ has interior lattice points (in particular, if $P = -P$), then $r \geq 2$.

**Proof.** The normal fan of $F$ has no proper coarsenings. Hence, by Lemma 2.5, $F$ and $F'$ are similar, and since $F$ is a unimodular triangle and $F'$ is a lattice polytope, $F' = rF$ for some integer $r \geq 0$. We note that if $r = 0$ or $r = 1$ then $P$ does not contain interior lattice points. □

**Lemma 3.3.** Let $\Delta \subseteq \mathbb{R}^2$ be a unimodular triangle and $r \geq 1$ an integer. Then the Cayley polytope of $\Delta$ and its $r$-th dilate, i.e., $Q = \text{conv}((\Delta,1),(r\Delta,0)) \subseteq \mathbb{R}^3$, can be covered by unimodular simplices. In particular, it is IDP.

**Proof.** The following straightforward argument shows that $Q$ can be covered by lattice polytopes isomorphic to either $\text{conv}((\Delta,1),(-\Delta,0))$ or $\text{conv}((\Delta,1),(\Delta,0))$ as illustrated by Fig. 3.

The statement is clear when $r = 1$. Let $r \geq 2$. Every dilate $r\Delta$ can be triangulated by translates of $\Delta$ and $-\Delta$. Let $v$ be a point in $Q$ and let $S$ be the center of similarity of $\Delta$ and $r\Delta$, i.e., the center of the scaling transformation which in our case is $S = (0,0,r/(r-1)) \in \mathbb{R}^3$. Let $v'$ be the intersection of the straight line connecting $S$ and $v$ with the hyperplane $\{x_3 = 0\}$ and let $R$ be a triangle in the triangulation containing $v'$. Then $v$ is contained in $\text{conv}((\Delta,1),(R,0))$.

The polytopes $\text{conv}((\Delta,1),(-\Delta,0))$ and $\text{conv}((\Delta,1),(\Delta,0))$ in turn are easily seen to have a unimodular triangulation since every 3-dimensional lattice

![Figure 3. Illustration of the proof of Lemma 3.3](attachment:image.png)
simplex contained in \( \text{conv}((\Delta, 1), (\Delta, 0)) \) and \( \text{conv}((\Delta, 1), (\Delta, 0)) \) is unimodular. One can say much more on triangulations of such polytopes e.g. by the Cayley trick, see Huber, Rambau and Santos [4] or Sturmfels [8]. □

3.3. Conclusion

Proof of Theorem 1.4. By Lemma 2.1, it suffices to cover \( P \) by parallelepipeds and unimodular simplices. Let \( v \in P \) be distinct from 0. Let \( v' \) be the intersection of the half ray \( \mathbb{R}_{\geq 0}v \) with a facet \( F \) of \( P \).

1. If \( F \) is not a unimodular simplex, then by Lemma 3.1 there exists a unit square \( D \) such that \( v' \in D \subseteq F \). Hence, \( v \in \text{conv}(D, -D) \), which is a parallelepiped since it is unimodularly equivalent to the parallelepiped spanned by \((1, 0, 0), (0, 1, 0), (2a + 1, 2b + 1, 2\ell)\), where \( \ell \) is the lattice distance of \( D \) from the origin and \( a, b \) are two integers.

2. If \( F \) is a unimodular simplex, let \( F' \) be as in Lemma 3.2. If \( v \in \text{conv}(F, F') \), we are done by Lemma 3.3. Otherwise, let \( \tilde{v} \) be the intersection of the half ray \( \mathbb{R}_{\geq 0}v \) with \( F' \). We proceed as in case 1 replacing \( v' \) by \( \tilde{v} \). □

Example 3.4. Let \( C_d = [-1, 1]^d \subset \mathbb{R}^d \) and consider its \( n \)-th dilate \( nC_d \). Then \( nC_d \) is a centrally symmetric smooth polytope. By chiseling off antipodal vertices of \( nC_d \) at distance 1, there appear two unimodular facets and the smoothness is preserved. (See, e.g., Castillo, Liu, Nill and Paffenholz [3] for details on chiselings.) Successive chiselings give us various examples of centrally symmetric smooth polytopes containing unimodular facets.

4. Summary

We have proved that any centrally symmetric 3-dimensional smooth polytope \( P \) is covered by parallelepipeds and unimodular simplices. It would be desirable to strengthen the statement to show that \( P \) admits a unimodular covering. This would follow from a positive answer to one of the following questions.

Question 4.1. Do 3-dimensional parallelepipeds admit a unimodular covering? Do centrally symmetric parallelepipeds of the form \( \text{conv}(D, -D) \), where \( D \) is a unit square, admit a unimodular covering?

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