On a class of bipartite graphs of girth eight

Ming Xu, Yuan-sheng Tang

Abstract: Let $\mathbb{F}$ be an arbitrary field, for polynomials $f_2, f_3 \in \mathbb{F}[x, y]$, we define a bipartite graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ with vertex partition $P \cup L$ where $P = \mathbb{F}^3$ and $(p_1, p_2, p_3) \in P$ is adjacent to $(l_1, l_2, l_3) \in L$ if and only if $p_2 + l_2 = f_2(p_1, l_1)$ and $p_3 + l_3 = f_3(p_1, l_1)$. It is conjectured that $\Gamma_3 = \Gamma_{\mathbb{F}}(xy, xy^2)$ is the unique (up to isomorphism) girth eight graph of form $\Gamma_{\mathbb{F}}(f_2, f_3)$. The main result of this paper is that for finite field $F = \mathbb{F}_q$ where $q$ is an odd prime power and some polynomials $f(x), g(y), h(x, y) \in \mathbb{F}_q[x, y]$, there exist a positive integer $M$ such that every graph $G = \Gamma_{\mathbb{F}_q^M}(f(x)g(y), h(x, y))$ with girth at least eight is isomorphic to $\Gamma_3(\mathbb{F}_q^M) = \Gamma_{\mathbb{F}_q^M}(xy, xy^2)$. Moreover, we characterize all polynomials $f(x), g(y)$ and $h(x, y)$ for which $G$ has girth at least eight. We also prove that over algebraically closed field $F_\infty$ with characteristic zero, every graph $\Gamma_{F_\infty}(f(x)g(y), h(x, y))$ with girth at least eight is isomorphic to $\Gamma_3(F_\infty)$.

Keywords: Bipartite graph; Cycle; Girth eight; Generalized quadrangle

1 Introduction

The graphs we consider in this paper are undirected, with no loops or multiple edges. For a graph $G$, its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. The order of $G$ is the number of vertices in $V(G)$. The degree of a vertex $v \in G$ is the number of the vertices that are adjacent to it. A graph is said to be $r$-regular if the degree of every vertex is equal to $r$. A sequence $v_1v_2 \cdots v_n$ of vertices of $G$ is called a path of length $n$ if $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, 2, \ldots, n - 1$ and $v_j \neq v_{j+2}$ for $j = 1, 2, \ldots, n - 2$. A path $v_1v_2 \cdots v_n$ is called a cycle further if its length $n$ is not smaller than 3 and $v_1v_4 \cdots v_nv_1v_2$ is still a path. If $G$ contains at least one cycle, then the girth of $G$, denoted by $g(G)$, is the length of a shortest cycle in $G$. If the shortest path in $G$ with given vertices $v, v'$ as ends is of length $n$, then we say the distance between $v, v'$ in $G$ is $n - 1$. The largest distance between distinct vertices of $G$ is called diameter of $G$. Graphs $G, G'$ are called isomorphic to each other if there is a bijection $\phi$ from $V(G)$ to $V(G')$ such that $\phi(v), \phi(v')$ are adjacent in $G'$ if and only if $v, v'$ are adjacent in $G$. Other
standard graph theory definitions can be found in Bollobás [1].

For \( k \geq 2 \), let \( g_k(n) \) be the greatest number of edges in a graph of order \( n \) and girth at least \( 2k + 1 \). Bondy and Simonovits [11] showed that \( g_k(n) = O(n^{1+1/k}) \), \( n \to \infty \). This upper bound is known to be sharp in magnitude only for \( k = 2, 3, 5 \). For some graphs which provide such extremal magnitude when \( k = 2 \) or 5, we refer the reader to Lazebnik and Woldar [6], Lazebnik and Thomason [7], Payne [13], Ustimenko [16, 17], Wenger [12], Lazebnik and Ustimenko [9].

For \( k = 3 \), the lower bound of \( g_3(n) \) is of magnitude \( n^{1+1/3} \) which comes from generalized quadrangles.

For \( t \geq 2 \), the finite regular generalized quadrangle of order \( t \) is an \( (t + 1) \)-regular bipartite graph of diameter four and girth eight, denoted by \( GQ(t) \). For \( t = q \) where \( q \) is an odd prime power, only two nonisomorphic \( GQ(q) \) are known, usually denoted by \( Q(4, q) \) and \( W(q) \). Since they are dual to each other (see Benson [3]), their point-line incidence graphs are isomorphic. Thus for odd prime power \( q \), only one (up to graph isomorphism) \( GQ(q) \) is known. For more informations on generalized quadrangles, see Payne and Thas [15], Payne [13] or Van Maldeghem [10]. For a fixed edge \( \{u, v\} \) of \( W(q) \), let \( \tau \) be the subgraph of \( W(q) \) induced by the set of all vertices within distance 2 from \( u \) or \( v \). Dmytrenko [18] showed that the graph obtained from \( W(q) \) by deleting \( \tau \) is isomorphic to the graph \( \Gamma_3(F_q) = \Gamma_{3,q}(xy, xy^2) \) which was defined as follow.

Let \( F \) be a field, for polynomials \( f_2, f_3 \in \mathbb{F}[x, y] \), the graph \( \Gamma_\mathbb{F}(f_2, f_3) \) is a bipartite graph with vertex partition \( P \cup L \) where \( P = \mathbb{F}^3 = L \). A vertex \( (p) = (p_1, p_2, p_3) \) is adjacent to a vertex \( [l] = [l_1, l_2, l_3] \), denoted \( (p) \sim [l] \), if and only if

\[
p_2 + l_2 = f_2(p_1, l_1) \quad \text{and} \quad p_3 + l_3 = f_3(p_1, l_1).
\]

Then, as defined, the graph \( \Gamma_3(F_q) \) which is isomorphic to the induced subgraph of \( W(q) \) is the case when \( F = \mathbb{F}_q \) is a finite field where \( q \) is an odd prime power and \( f_2 = xy, f_3 = xy^2 \). \( \Gamma_3(F_q) \) is a \( q \)-regular bipartite graph of order \( 2q^3 \) and girth eight.

The central reason of studying the graph \( \Gamma_{F_q}(f_2, f_3) \) is to construct a new \( GQ(q) \) of order \( q \). As we pointed out, for odd prime power \( q \), \( W(q) \) is the only (up to graph isomorphism) generalized quadrangle of order \( q \) and \( \Gamma_3(F_q) \) is isomorphic to an induced subgraph of \( W(q) \). Thus if there were two polynomials \( f_2, f_3 \in \mathbb{F}_q[x, y] \) such that the graph \( \Gamma_{\mathbb{F}_q}(f_2, f_3) \) had girth at least eight and was not isomorphic to \( \Gamma_3(F_q) \), then as mentioned, one could obtain a new \( GQ(q) \) by “attaching” \( \tau \) to it. \( \Gamma_\mathbb{F}(f_2, f_3) \) is called a monomial graph if \( f_2 \) and \( f_3 \) are both monomials. For finite field \( F = \mathbb{F}_q \) where \( q \) is an odd prime power and monomial graphs \( \Gamma_{\mathbb{F}_q}(f_2, f_3) \), Dmytrenko, Lazebnik and Williford [19] investigated the viability of the
way to construct a new $GQ(q)$ and proposed the following conjecture because no such graphs had been found.

**Conjecture 1** For any odd prime power $q$, every monomial graph $\Gamma_{F_q}(f_2, f_3)$ of girth at least eight is isomorphic to $\Gamma_3(F_q) = \Gamma_{F_q}(xy, xy^2)$.

Conjecture 1 was then studied by Kronenthal [2] and was finally been proved by Hou, Lappano and Lazebnik [20]. Kronenthal and Lazebnik [3] proposed further the following two conjectures.

**Conjecture 2** For any odd prime power $q$, every graph $\Gamma_{F_q}(f_2, f_3)$ of girth at least eight is isomorphic to $\Gamma_3(F_q)$.

**Conjecture 3** Let $F_\infty$ be an algebraically closed field. Then, every graph $\Gamma_{F_\infty}(f_2, f_3)$ of girth at least eight is isomorphic to $\Gamma_3(F_\infty)$.

In [3], Kronenthal et al. confirmed Conjecture 3 for the case $f_2 = xy$, and in [4] for the case that $f_2 = x^k y^m$ is monomial. In [4], the author also obtained the following result: Let $p$ be an odd prime and $q$ a power of $p$. If $f \in F_q[x, y]$ has degree at most $p - 2$ with respect to each of $x$ and $y$. Then for any positive integers $k, m$ with $p \nmid k, \nmid m$ there exists a positive integer $M = M(k, m, q)$ such that for all integer $r > 0$, every graph $\Gamma_{F_q^M}(x^k y^m, f)$ of girth $\geq 8$ is isomorphic to $\Gamma_3(F_q^M)$.

Note that the graphs considered in these results all contains at least a monomial, it implies that for the construction of a new generalized quadrangle to succeed for all $q$, we must go beyond the families of $\Gamma_{F_q}(x^k y^m, f)$. Namely, $f_2$ and $f_3$ must be both polynomials. For this purpose, in this paper, we extend the results above by $f_2$ with a class of special polynomials rather than monomials. More specifically, we let $f_2$ be a product of two univariate polynomials. First, we make some definitions. For any field $F$, let $F^* = F \setminus \{0\}$ denote the set of nonzero elements in $F$. For positive integer $k$, let $F[x]_k$ be the set of polynomials in $F[x]$ of order at most $k$ and $[1, k] = \{1, 2, \ldots, k\}$. Assume that $p$ is an odd prime, let $K_p = \{p^j \mid j \geq 0\}$, $q \in K_p \setminus \{1\}$, then $F_q$ is a finite field of size $q$. For positive integers $m, n$, let $M = M(mn)$ be the least common multiple of integers $2, 3, \ldots, mn$. Clearly, any polynomials $T(x) \in F_q[x]_{mn}$ can be decomposed completely over $F_q^M$. For any $a \in F_q$, let $\rho_a(x) = x(x - a) \in F_q[x]$. For $u, v \in K_p$, let $\Phi_p(u, v) = \{(i, j) \in K_p^2 : iv = ju\}$. The main results of this paper are as follows:

**Theorem 1.1** Let $q$ be an odd prime power. $m, n$ are positive integers such that $q > \max\{mn + 3n + 1, 2mn + 4\}$ and $M = M(mn)$. Let $f(x), g(x) \in F_q[x]_m$ be monic polynomials with $f(0) = g(0) = 0$ and
Let $h(x, y) = \sum_{1 \leq i,j \leq n} h_{i,j} x^i y^j \in \mathbb{F}_q[x, y]$ be a nonzero polynomial of order at most $n$ with respect to both $x$ and $y$. Suppose the graph $G = \Gamma_{\mathbb{F}_q^*}(f(x)g(y), h(x, y))$ has girth at least eight, then we have

$$G \cong \Gamma_3(\mathbb{F}_q^*)$$

In particular, there are some $a \in \mathbb{F}_q, \zeta \in \mathbb{F}_q^*, u, v \in K_p \cap [1, m]$ and $s, r \in K_p \cap [1, n]$ such that one of the following is valid.

(i) $f(x) = \rho_u^v(x), g(y) = y^v$ and $h(x, y) = \zeta x^{su/v} y^s + \sum_{(i,j) \in \Phi_p(u,v)} h_{i,j} \rho_i^j(x) y^j$.

(ii) $f(x) = x^v, g(y) = \rho_u^v(y)$ and $h(x, y) = \zeta x^r y^{ra/v} + \sum_{(i,j) \in \Phi_p(v,u)} h_{i,j} x^i y^j$.

(iii) $f(x) = x^v, g(y) = y^v$ and $h(x, y) = h_{2su/v,s} x^{2su/v} y^s + \sum_{(i,j) \in \Phi_p(v,u)} h_{i,j} x^i y^j$.

(iv) $f(x) = x^v, g(y) = y^v$ and $h(x, y) = h_{r,2rv/u} x^{r} y^{2rv/u} + \sum_{(i,j) \in \Phi_p(u,v)} h_{i,j} x^i y^j$.

The following theorem is an analog of Theorem 1.1 for algebraically closed field $\mathbb{F}_\infty$ with characteristic zero.

**Theorem 1.2** Suppose that $f(x), g(x) \in \mathbb{F}_\infty[x]$ are monic polynomials with $f(0) = g(0) = 0$ and $h(x, y) = \sum_{i,j \geq 1} h_{i,j} x^i y^j \in \mathbb{F}_\infty[x, y]$ is a nonzero polynomial such that $\Gamma_{\mathbb{F}_\infty}(f(x)g(y), h(x, y))$ has girth at least eight. Then we have

$$\Gamma_{\mathbb{F}_\infty}(f(x)g(y), h(x, y)) \cong \Gamma_3(\mathbb{F}_\infty)$$

In particular, there are some $a \in \mathbb{F}_\infty, \zeta \in \mathbb{F}_\infty^*$ such that one of the following is valid.

(i) $f(x) = \rho_u(x), g(y) = y$ and $h(x, y) = \zeta xy + h_{2,1} \rho_u(x) y$.

(ii) $f(x) = x, g(y) = \rho_u(y)$ and $h(x, y) = \zeta xy + h_{1,2} x \rho_u(y)$.

(iii) $f(x) = x, g(y) = y$ and $h(x, y) = \zeta \rho_u(x) y$.

(iv) $f(x) = x, g(y) = y$ and $h(x, y) = \zeta x \rho_u(y)$.

The paper is organized as follows. The description about 4- and 6-cycles in graph $\Gamma_{\mathbb{F}_p}(f_2, f_3)$ and some isomorphisms about these graphs are given in section 2. In section 3 and section 4, we show some useful lemmas and discuss the forms of $f(x)g(y)$ and $h(x, y)$, respectively. In section 5, we devote to prove Theorem 1.1 and Theorem 1.2. In section 6, we make some concluding remarks and state a conjecture.
2 Cycles and isomorphisms of graphs $\Gamma_F(f_2, f_3)$

Let $\Gamma_F(f_2, f_3)$ be the graph defined as before. Let $S$ be a 4-cycle

$$(a_1, a_2, a_3) \sim [x_1, x_2, x_3] \sim (b_1, b_2, b_3) \sim [y_1, y_2, y_3] \sim (a_1, a_2, a_3),$$

hereafter, we denote $S$ by the first coordinates of its vertices, i.e. $S \triangleq (a_1, b_1; x_1, y_1)$. Similarly, for a 6-cycle $S'$:

$$(a_1, a_2, a_3) \sim [x_1, x_2, x_3] \sim (b_1, b_2, b_3) \sim [y_1, y_2, y_3] \sim (c_1, c_2, c_3) \sim [z_1, z_2, z_3] \sim (a_1, a_2, a_3),$$

we denote $S' \triangleq (a_1, b_1, c_1; x_1, y_1, z_1)$.

Define the functions

$$\Delta_2 : \mathbb{F}[s_1, s_2] \to \mathbb{F}[t_1, t_2, t_3, t_4]$$

$$f(s_1, s_2) \mapsto f(t_1, t_3) - f(t_2, t_3) + f(t_2, t_4) - f(t_1, t_4),$$

and

$$\Delta_3 : \mathbb{F}[s_1, s_2] \to \mathbb{F}[t_1, t_2, t_3, t_4, t_5, t_6]$$

$$f(s_1, s_2) \mapsto f(t_1, t_4) - f(t_2, t_4) + f(t_2, t_5) - f(t_3, t_5) + f(t_3, t_6) - f(t_1, t_6).$$

These functions help us to formulate the necessary and sufficient conditions for the existence of 4- and 6-cycles in $\Gamma_F(f_2, f_3)$ (see Dmytrenko [18], [3]).

**Proposition 2.1** Graph $\Gamma_F(f_2, f_3)$ contains a 4-cycle if and only if there exist $a, b, x, y \in \mathbb{F}$ such that:

$$\begin{cases} 
\Delta_2(f_2)(a, b; x, y) = \Delta_2(f_3)(a, b; x, y) = 0 \\
  a \neq b, x \neq y 
\end{cases}$$

Similarly, $\Gamma_F(f_2, f_3)$ contains a 6-cycle if and only if there exist $a, b, c, x, y, z \in \mathbb{F}$ such that:

$$\begin{cases} 
\Delta_3(f_2)(a, b, c; x, y, z) = \Delta_3(f_3)(a, b, c; x, y, z) = 0 \\
  a \neq b, b \neq c, c \neq a \\
  x \neq y, y \neq z, z \neq x 
\end{cases}$$

From this proposition we can see that $\Gamma_F(f_2, f_3)$ contains the 4-cycle $S$ if and only if $\Delta_2(f_2)(S) = \Delta_2(f_3)(S) = 0$. And it contains the 6-cycle $S'$ if and only if $\Delta_3(f_2)(S') = \Delta_3(f_3)(S') = 0$.

Assume that $p$ is an odd prime. Let $K_p = \{p^j \mid j \geq 0\}, q \in K_p \setminus \{1\}$ and $\mathbb{F}_q$ be a finite field of size $q$. The following isomorphisms will be useful for the further work.
Lemma 2.2 Assume $f_2(x, y), f_3(x, y) \in \mathbb{F}[x, y]$. Then

(i) \ $\Gamma_{\mathbb{F}}(f_2(x, y), f_3(x, y)) \cong \Gamma_{\mathbb{F}}(f_3(x, y), f_2(x, y)).$

(ii) \ $\Gamma_{\mathbb{F}}(f_2(x, y), f_3(y, x)) \cong \Gamma_{\mathbb{F}}(f_2(y, x), f_3(y, x)).$

(iii) For any $\alpha \in \mathbb{F}^*$, $\Gamma_{\mathbb{F}}(f_2(x, y), f_3(x, y)) \cong \Gamma_{\mathbb{F}}(f_2(x, y), \alpha f_3(x, y)).$

(iv) For any $\beta \in \mathbb{F}^*$, $\Gamma_{\mathbb{F}}(f_2(x, y), f_3(x, y)) \cong \Gamma_{\mathbb{F}}(f_2(x, y), f_3(x, y) + \beta f_2(x, y)).$

(v) For any $t_1(x) \in \mathbb{F}[x], t_2(y) \in \mathbb{F}[y]$, $\Gamma_{\mathbb{F}}(f_2(x, y), f_3(x, y)) \cong \Gamma_{\mathbb{F}}(f_2(x, y), f_3(x, y) + t_1(x) + t_2(y)).$

Furthermore, if $\mathbb{F} = \mathbb{F}_q$, then, for any $u \in K_p$,

(vi) \ $\Gamma_{\mathbb{F}_q}(f_2(x, y), f_3(x, y)) \cong \Gamma_{\mathbb{F}_q}(f_2(x^u, y), f_3(x^u, y)).$

(vii) \ $\Gamma_{\mathbb{F}_q}(f_2(x, y), f_3(x, y)) \cong \Gamma_{\mathbb{F}_q}(f_2^u(x, y), f_3(x, y)).$

Proof We refer the reader to [3] for the proofs of (i) \sim (v).

For (vi), take $\pi : (p_1, p_2, p_3) \mapsto (p_1^u, p_2, p_3)$ and $\pi : (l_1, l_2, l_3) \mapsto (l_1^u, l_2, l_3)$

For (vii), take $\pi : (p_1, p_2, p_3) \mapsto (p_1, p_2^u, p_3)$ and $\pi : (l_1, l_2, l_3) \mapsto (l_1, l_2^u, l_3)$

It’s easy to see that $\pi$ is indeed an isomorphism.

From Lemma 2.2, we see the graph $\Gamma_3(\mathbb{F})$ is isomorphic to graph $\Gamma_{\mathbb{F}}(xy, x^2y)$. Hereafter, we let $\Gamma_3(\mathbb{F}) = \Gamma_{\mathbb{F}}(xy, x^2y)$.

Lemma 2.3 (i) The girth of $\Gamma_3(\mathbb{F}) = \Gamma_{\mathbb{F}}(xy, x^2y)$ is 8.

(ii) The girth of $\Gamma_{\mathbb{F}}(x^3y, x^2y)$ is 6 if $|\mathbb{F}| > 5$.

(iii) The girth of $\Gamma_{\mathbb{F}}(xy, x^3y)$ is 6 if $|\mathbb{F}| > 3$.

(iv) $\Gamma_{\mathbb{F}_q}(x^3y, x^2y) \cong \Gamma_{\mathbb{F}_q}(xy, x^2y)$.

(v) $\Gamma_{\mathbb{F}_q}(x^3y, x^2y) \cong \Gamma_{\mathbb{F}_q}(xy, x^3y) \cong \Gamma_{\mathbb{F}_q}(xy, x^2y)$.

Proof Suppose $a, b \in \mathbb{F}$ are distinct and so are $c, d \in \mathbb{F}$. From $\Delta_2(xy)(a,b;c,d) = (a - b)(c - d) \neq 0$, we see that $\Gamma_3(\mathbb{F})$ and $\Gamma_{\mathbb{F}}(xy, x^2y)$ contain no 4-cycle and thus have girth at least 6.

(i) Let $S_0 = (a, b; c, d, f)$, where $e \in \mathbb{F}\{a, b\}$ and $f \in \mathbb{F}\{c, d\}$. If $\Delta_3(xy)(S_0) = (a - b)(c - d) - (a - c)(f - d) = 0$, then we have

$$\Delta_3(x^2y)(S_0) = (a^2 - b^2)(c - d) - (a^2 - c^2)(f - d) = (a - b)(c - d)(c - e) \neq 0.$$ 

Hence, $\Gamma_3(\mathbb{F})$ contains no 6-cycle. Therefore, $\Gamma_3(\mathbb{F})$ has girth 8 since it contains the 8-cycle $(1, 0, 1, 0; 1, 0, -1, 0)$.

(ii) If $\Delta_2(x^2y)(a, b; c, d) = (a^2 - b^2)(c - d) = 0$, then we have $a^2 = b^2 \neq 0$ and thus

$$\Delta_2(x^3y)(a, b; c, d) = (a^3 - b^3)(c - d) = a^2(a - b)(c - d) \neq 0.$$
Hence, $\Gamma_F(x^3y, x^2y)$ has no 4-cycle and has girth at least 6. Furthermore, the girth of $\Gamma_F(x^3y, x^2y)$ is equal to 6 if $|F| > 5$ since, for any $t \in F \setminus \{0, 1, -1, 2, 1/2\}$,

$$(t, 1 - t, t(t - 1); t^2(t - 1)^2, t^2, (t - 1)^2)$$

is a 6-cycle of $\Gamma_F(x^3y, x^2y)$.

(iii) We have already pointed out that $\Gamma_F(xy, x^2y^3)$ has no 4-cycle. If $|F| > 3$, then $\Gamma_F(xy, x^2y^3)$ has girth 6 since, for any $t \in F \setminus \{0, 1, -1\}$,

$$(0, 1 - 2t^2, t + 2; 0, 1, t)$$

is a 6-cycle of $\Gamma_F(xy, x^2y^3)$.

(iv) Since $x^3$ is a permutation in $F_5$, from $(x^3)^2 \equiv x^2 (\text{mod} x^5 - x)$ we have

$$\Gamma_F(x^3y, x^2y) \cong \Gamma_F(x^3y, (x^3)^2y) \cong \Gamma_F(xy, x^2y).$$

(v) What needed proof follows simply from $x^4 \equiv x (\text{mod} x^3 - x)$.

Part (v) of Lemma 2.2 shows that for any graph $\Gamma_F(f_2, f_3)$, we can assume $f_2$ and $f_3$ contain only mixed terms, i.e. $f_i(x, 0)$ and $f_i(0, y)$ are zero polynomials, $i = 2, 3$. Thus, for bipartite graph $G = \Gamma_{F_{q^M}}(f(x)g(y), h(x, y))$ defined as before, we can assume $f(0) = g(0) = 0$ and $h(x, y) = \sum_{1 \leq i, j \leq n} h_{i, j} x^i y^j$. In the next section, we discuss the forms of $f(x)$ and $g(y)$.

$\bf{3}$ \hspace{1em} The forms of $f(x)$ and $g(y)$

Let $q, m, n$ be positive integers defined as before i.e. $q \in K_p \setminus \{1\}$ where $p$ is an odd prime and

$$q > \max\{mn + 3n + 1, 2mn + 4\}, \quad (3.1)$$

$M$ is the least common multiple of integers $2, 3, \cdots, mn$. Then any polynomials $T(x) \in F_q[x]_{mn}$ can be decomposed completely over $F_q[M]$. Suppose that $f(x), g(x) \in F_q[x]_m$ are monic polynomials with $f(0) = g(0) = 0$,

$$h(x, y) = \sum_{1 \leq i, j \leq n} h_{i, j} x^i y^j \in F_q[x, y]$$

is a nonzero polynomial of $x$-order and $y$-order at most $n$, and the bipartite graph $G = \Gamma_{F_{q^M}}(f(x)g(y), h(x, y))$ has girth at least 8. The following lemmas give some restrictions of $f(x)$ and $g(y)$.
Lemma 3.1 (i) If \( a, b \in \mathbb{F}_q^M \) are distinct with \( f(a) = f(b) \), then the polynomial

\[
\theta_{a,b}(y) = h(a, y) - h(b, y) \in \mathbb{F}_q^M[y]
\]

is injective in \( \mathbb{F}_q^M \).

(ii) If \( c, d \in \mathbb{F}_q^M \) are distinct with \( g(c) = g(d) \), then the polynomial

\[
\phi_{c,d}(x) = h(x, c) - h(x, d) \in \mathbb{F}_q^M[x]
\]

is injective in \( \mathbb{F}_q^M \).

Proof Since (ii) is symmetrical to (i), we only give proof for (i).

Suppose \( a, b \in \mathbb{F}_q^M \) are distinct with \( f(a) = f(b) \), Let \( S_1 = (a, b, c, d) \). Since \( G \) has no 4-cycle, from \( \Delta_2(f(x)g(y))(S_1) = 0 \) we see

\[
\Delta_2(h(x, y))(S_1) = \theta_{a,b}(c) - \theta_{a,b}(d) \neq 0
\]

for any \( c, d \in \mathbb{F}_q^M \) with \( c \neq d \), and thus \( \theta_{a,b}(y) \) must be injective in \( \mathbb{F}_q^M \).

\( \square \)

Lemma 3.2 At least one of \( f(x) \), \( g(y) \) is injective in \( \mathbb{F}_q^M \).

Proof Suppose that neither \( f(x) \) nor \( g(y) \) is injective in \( \mathbb{F}_q^M \). Let \( a, b, c, d \) be elements in \( \mathbb{F}_q^M \) such that \( a \neq b , c \neq d , f(a) = f(b) \) and \( g(c) = g(d) \). According to Lemma 3.1, the polynomial \( \theta_{a,b}(y) \in \mathbb{F}_q^M[y] \) and \( \phi_{c,d}(x) \in \mathbb{F}_q^M[x] \) are injective in \( \mathbb{F}_q^M \). Let \( S_2 = (a, b, t', t, c, d) \). Then we have \( \Delta_3(f(x)g(y))(S_2) = 0 \) and

\[
\Delta_3(h(x, y))(S_2) = h(b, c) - h(a, d) + \theta_{a,b}(t) - \phi_{c,d}(t')
\]

Since \( \theta_{a,b}(y) \) is injective in \( \mathbb{F}_q^M \), one can take a \( t \in \mathbb{F}_q^M \setminus \{c, d\} \) such that

\[
h(b, c) - h(a, d) + \theta_{a,b}(t) \notin \{\phi_{c,d}(a), \phi_{c,d}(b)\}.
\]

Moreover, from that \( \phi_{c,d}(x) \) is injective in \( \mathbb{F}_q^M \), for such \( t \) one can take a \( t' \in \mathbb{F}_q^M \setminus \{a, b\} \) such that \( \Delta_3(h(x, y))(S_2) = 0 \) and thus \( G \) has a 6-cycle of form \( S_2 \), contradicts to the assumption. \( \square \)

Lemma 3.3 There are no distinct \( x_0, x_1, x_2 \in \mathbb{F}_q^M \) satisfying \( f(x_0) = f(x_1) = f(x_2) \) or \( g(x_0) = g(x_1) = g(x_2) \).

Proof Assume distinct \( x_0, x_1, x_2 \in \mathbb{F}_q^M \) satisfy \( f(x_0) = f(x_1) = f(x_2) \). According to Lemma 3.1 the polynomials \( \theta_{x_0,x_1}(y), \theta_{x_1,x_2}(y), \theta_{x_2,x_0}(y) \) are injective in \( \mathbb{F}_q^M \).

Let \( S_3 = (x_0, x_1, x_2, y_0, y_1, y_2) \). Then, we have \( \Delta_3(f(x)g(y))(S_3) = 0 \) and

\[
\Delta_3(h(x, y))(S_3) = \theta_{x_0,x_1}(y_0) + \theta_{x_1,x_2}(y_1) + \theta_{x_2,x_0}(y_2).
\]
Clearly, there are distinct $y_0, y_1, y_2 \in \mathbb{F}_{q^m}$ satisfying $\Delta_3(h(x, y))(S_3) = 0$, and thus $G$ contains a 6-cycle of form $S_3$, contradicts to the assumption.

Similarly, one can show that there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^m}$ satisfying $g(x_0) = g(x_1) = g(x_2)$.

\[\cdotp\]

**Lemma 3.4** Let $T(x) \in \mathbb{F}_q[x]_{mn}$ be a monic polynomial with $T(0) = 0$.

(i) If $T(x)$ is injective in $\mathbb{F}_{q^m}$, then there is some $u \in K_p$ such that

$$T(x) = x^u.$$ (3.2)

(ii) If $T(x)$ is not injective in $\mathbb{F}_{q^m}$ and there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^m}$ with $T(x_0) = T(x_1) = T(x_2)$, then there are some $v \in K_p$ and $a \in \mathbb{F}_q$ such that

$$T(x) = x^v(x - a)^r.$$ (3.3)

**Proof** Assume first that $T(x) \in \mathbb{F}_q[x]_{mn}$ has no root in $\mathbb{F}_{q^m}^*$. Clearly, we have $T(x) = x^\deg(T(x))$. Let $u$ be the largest integer in $K_p$ with $u \mid \deg(T(x))$. Then, we have $k = \frac{\deg(T(x))}{u} \in [1, mn]$. We note that $x^k - 1$ has no repeated roots in $\mathbb{F}_{q^m}^*$ since its derivative $kx^{k-1}$ has no root in $\mathbb{F}_{q^m}^*$. Therefore, the polynomial $x^k - 1$ has $k$ distinct roots in $\mathbb{F}_{q^m}^*$. If $T(x)$ is injective in $\mathbb{F}_{q^m}$, we have $k = 1$ (note that $x^u$ is indeed injective in $\mathbb{F}_{q^m}$ since $u \in K_p$), thus (3.2) follows. If $T(x)$ is not injective in $\mathbb{F}_{q^m}$ and there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^m}$ with $T(x_0) = T(x_1) = T(x_2)$, then we must have $k = 2$ and thus (3.3) is valid for $a = 0$.

Assume now that $T(x) \in \mathbb{F}_q[x]_{mn}$ has roots in $\mathbb{F}_{q^m}^*$ and there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^m}$ with $T(x_0) = T(x_1) = T(x_2)$. Clearly, $T(x)$ has just one root in $\mathbb{F}_{q^m}^*$. Hence, we have $T(x) = x^r(x - a)^s$ for some $a \in \mathbb{F}_{q^m}^*$ and positive integers $r, s$. Let $v \in K_p$ be the largest one which divides both $r$ and $s$. Then, the positive integers $k = r/v$ and $l = s/v$ satisfy $2 \leq k + l \leq mn$ and that the $x$-polynomial $(k + l)x - ka \in \mathbb{F}_{q^m}[x]$ is not the zero polynomial.

Furthermore, we assume $k + l > 2$. Let $R(x) = x^k(x - a)^l$. From $v \in K_p$ and $R''(x) = T(x) \in \mathbb{F}_q[x]_{mn}$, we see that there are no distinct $x_0, x_1, x_2 \in \mathbb{F}_{q^m}$ with $R(x_0) = R(x_1) = R(x_2)$. Therefore, for any $b \in \mathbb{F}_q \setminus \{0, a\}$ we have $R(b) \in \mathbb{F}_{q^m}^*$ and there are some $\alpha(b) \in \mathbb{F}_{q^m} \setminus \{0, a, b\}$ and integers $u(b) \geq 1, v(b) \geq 0$ such that

$$R(x) - R(b) = (x - b)^{u(b)}(x - \alpha(b))^{v(b)}.$$ (3.4)

For any $b \in \mathbb{F}_q \setminus \{0, a\}$ with $(k + l)b \neq ka$, since the derivative of $R(x)$ is

$$R'(x) = x^{k-1}(x - a)^{l-1}((k + l)x - ka),$$
we see $R'(b) \neq 0$ and thus from (3.4) we have $u(b) = 1, v(b) = k + l - 1 \geq 2$ and $(k + l)\alpha(b) - ka = 0$. Hence, $\alpha = \alpha(b)$ is independent of $b$. Then, for any $b \in \mathbb{F}_q \setminus \{0, a, \alpha\}$, from (3.4) we have

$$x^k(x - a)^l - b^k(b - a)^l - (x - b)(x - \alpha)^{k+l-1} = 0. \tag{3.5}$$

Therefore, from (3.1) we have

$$|\mathbb{F}_q \setminus \{0, a, \alpha\}| \geq q - 3 > mn \geq k + l, \tag{3.6}$$

and thus from (3.5) we see that the $y$-polynomial

$$x^k(x - a)^l - y^k(y - a)^l - (x - y)(x - \alpha)^{k+l-1}$$

is the zero polynomial, which is impossible since its $y$-order is $k + l > 1$. Hence, we must have $k + l = 2, a \in \mathbb{F}^{*}_q$ and thus (3.3) follows. $\square$

The results of Lemma 3.2, Lemma 3.3 and Lemma 3.4 imply that $f(x)$ and $g(y)$ must have one of the following forms.

1. $f(x) = x^u(x - a)^u, g(y) = y^v$ where $a \in \mathbb{F}_q, 2u \leq m$ and $u, v \in K_p \cap [1, m]$.
2. $f(x) = x^v, g(y) = y^v(y - a)^u$ where $a \in \mathbb{F}_q, 2v \leq m$ and $u, v \in K_p \cap [1, m]$.
3. $f(x) = x^u, g(y) = y^v$ where $u, v \in K_p \cap [1, m]$.

4 The form of $h(x, y)$

Now we obtained the three forms of $f(x)$ and $g(y)$. Before discussing the form of $h(x, y)$, we would like to show the following lemma which will be effectively used.

**Lemma 4.1** Let $W \subseteq \mathbb{F}_q^2$ be a set with

$$\min\{|\{(c \in \mathbb{F}_q : (a, c) \in W)\}, |\{(d \in \mathbb{F}_q : (d, b) \in W)\}| > 2N \tag{4.1}$$

For any $(a, b) \in W$. Suppose for each $i \in [1, mn]$ the $x$-order and $y$-order of the polynomial $f_i(x, y) \in \mathbb{F}_q[x, y]$ are at most $N$, and for any $(a, b) \in W$ the $t$-polynomial $\sum_{1 \leq i \leq \min} f_i(a, b)t^i \in \mathbb{F}_q[t]_{mn}$ has no root in $\mathbb{F}^{*}_q$. Then, there is an integer $s \in [1, mn]$ such that $f_s(a, b) \neq 0$ for any $(a, b) \in W$ and $f_i(x, y)$ is the zero polynomial for any $i \neq s$. 

Proof. Let \( s \in [1, mn] \) be an integer such that \( f_s(x, y) \) is not the zero polynomial. Let \( (a, b) \in W \) be an arbitrary pair with \( f_s(a, b) \neq 0 \). Let

\[
A = \{ d \in \mathbb{F}_q : (d, b) \in W, f_s(d, b) \neq 0 \},
\]

\[
B = \{ c \in \mathbb{F}_q : (a, c) \in W, f_s(a, c) \neq 0 \}.
\]

From (4.1) and that the polynomials \( f_s(a, y) \in \mathbb{F}_q[y] \) and \( f_s(x, b) \in \mathbb{F}_q[x] \) are of order at most \( N \), we see

\[
\min(\{|A|, |B|\}) > N. \tag{4.2}
\]

Since for any \( d \in A \) the \( t \)-polynomial \( \sum_{1 \leq i \leq mn} f_i(d, b) t^i \in \mathbb{F}_q[t]_{mn} \) has no root in \( \mathbb{F}_q^* \), for any \( i \neq s \) we have \( f_i(d, b) = 0 \). Hence, for any \( i \neq s \) from (4.2) the \( x \)-polynomial \( f_i(x, b) \in \mathbb{F}_q[x]_N \) is the zero polynomial. Similarly, one can conclude that, for any \( i \neq s \), the \( y \)-polynomial \( f_i(a, y) \in \mathbb{F}_q[y]_N \) is the zero polynomial. Therefore, for any \( i \neq s \) from (4.2) we see that the polynomial \( f_i(x, y) \in \mathbb{F}_q[x, y] \) is the zero polynomial. Clearly, we have \( f_s(a, b) \neq 0 \) for any \( (a, b) \in W \). \( \square \)

For any \( a \in \mathbb{F}_q \), let

\[
\rho_a(x) = x(x - a) \in \mathbb{F}_q[x]
\]

and \( Q_{\mathbb{F}_q}(a) \) denote the set of pairs \((c, d)\) such that \( \rho_a(c), \rho_a(d) \) are distinct elements in \( \mathbb{F}_q^* \), i.e.

\[
Q_{\mathbb{F}_q}(a) = \{(c, d) \in (\mathbb{F}_q \setminus \{0, a\})^2 : c \neq d, c + d \neq a\}.
\]

For \( u, v \in K_p \), let

\[
\Phi_p(u, v) = \{(i, j) \in K_p^2 : iv = ju\}.
\]

Lemma 4.2. Suppose \( u, v \in K_p \cap [1, m] \) and \( a, b \in \mathbb{F}_q \) satisfy \( 2u \leq m \) and \( b^v = a \).

(i) If \( f(x) = \rho_a^n(x) \) and \( g(y) = y^v \), then there is an \( s \in K_p \cap [1, n] \) such that

\[
h_s(d^v) \rho_b^{sm}(c) - h_s(c^v) \rho_b^{sm}(d) \neq 0, \text{ for any } (c, d) \in Q_{\mathbb{F}_q}(b), \tag{4.3}
\]

\[
h_s(c) - h_s(a - c) \neq 0, \text{ for any } c \in \mathbb{F}_q^* \setminus \{a/2\}, \tag{4.4}
\]

\[
h(x, y) = h_s(x)y^a + \sum_{(i, j) \in \Phi_p(u, v), j \neq s} h_{2i,j} \rho^a_i(x)y^j, \tag{4.5}
\]

where

\[
h_j(x) = \sum_{1 \leq i \leq n} h_{i,j} x^i \in \mathbb{F}_q[x], \tag{4.6}
\]
(ii) If $f(x) = x^n$ and $g(y) = \rho^n_v(y)$, then there is an $r \in K_\rho \cap [1, n]$ such that

$$h_r^*(d^n)\rho^n_v(c) - h_r^*(c^n)\rho^n_v(d) \neq 0, \text{ for any } (c, d) \in Q_{q^m}(b), \quad (4.7)$$

$$h_r^*(c) - h_r^*(a - c) \neq 0, \text{ for any } c \in F_{q^m}\setminus\{a/2\}, \quad (4.8)$$

$$h(x, y) = h_r^*(y)x^r + \sum_{(i,j) \in \Phi_{u,v}, i \neq r} h_{i,j}x^i\rho_i^v(y), \quad (4.9)$$

where

$$h_r^*(y) = \sum_{1 \leq j \leq n} h_{i,j}y^j \in F_q[y]. \quad (4.10)$$

**Proof** Since (ii) is symmetrical to (i), we only prove (i).

Assume $f(x) = \rho^n_u(x)$ and $g(y) = y^v$. For $(c, d) \in Q_{q^m}(b)$ and $t \in F_{q^m}^*$, let

$$S_4 = (0, c^v, d^v; t\rho^n_v(d), 0, t\rho^n_v(c)).$$

Then, from $v \in K_\rho$ and $b^v = a$ we see

$$\Delta_3(f(x)g(y))(S_4) = f(d^n)g(t\rho^n_v(c)) - f(c^n)g(t\rho^n_v(d))$$

$$= t^v(cd)^v((d^v - a)^v(c - b)^u - (c^v - a)^v(d - b)^u)$$

$$= 0,$$

and thus the $t$–polynomial

$$\Delta_3(h(x, y))(S_4) = h(d^n, t\rho^n_v(c)) - h(c^n, t\rho^n_v(d))$$

$$= \sum_{1 \leq j \leq n} (h_j(d^n)\rho^n_v(c) - h_j(c^n)\rho^n_v(d))t^j \quad (4.11)$$

$$= \sum_{1 \leq j \leq n} H_j(d, c)t^j$$

has no root in $F_{q^m}^*$, where

$$H_j(x, y) = h_j(x^v)\rho^n_v(y) - h_j(y^v)\rho^n_x(x). \quad (4.12)$$

Clearly, for any $j \in [1, n]$, the $x$–order and $y$–order of $H_j(x, y) \in F_q[x, y]$ are at most $mn$. By applying Lemma \[3.1\] for $W = Q_{q}(b), f_j(x, y) = H_j(x, y)$ and $N = mn$, according to \[3.1\] we see that there is an integer $s \in [1, n]$ such that

$$H_j(x, y) = 0, \text{ for } j \neq s. \quad (4.13)$$
For any \((c, d) \in Q_{F_q}(b)\), from (4.13) we have \(\Delta_3(h(x, y))(S_4) = H_s(d, c)t^s\) and thus (4.3) follows.

If \(j \neq s\) and \(h_j(x)\) is not the zero polynomial, from (4.12) and (4.13) we see

\[
h_j(x) = \frac{h_j(c^e)}{\rho_b^{ju}(c)}\rho_b^{ju}(x),
\]

and then \(v\) divides \(ju\) and, moreover, from \(v \in K_p\) and \(b^e = a\) we have

\[
h_j(x) = h_{2ju/v,j}\rho_a^{ju/v}(x).
\]

Hence, we have

\[
h(x, y) = h_s(x)y^s + \sum_{j \neq s, v/ju} h_{2ju/v,j}\rho_a^{ju/v}(x)y^j.
\]

For any \(c \in F_q^*\), from \(u \in K_p\) we see

\[
f(c) = \rho_a^u(c) = \rho_a^u(a - c) = f(a - c)
\]

and thus from Lemma [3.1] and (4.15) the \(y\)-polynomial

\[
h(c, y) - h(a - c, y) = \sum_{1 \leq j \leq n} (h_j(c) - h_j(a - c))y^j = (h_s(c) - h_s(a - c))y^s
\]

is injective in \(F_q^*\). Therefore, we have (4.14) and according to Lemma [3.3] we see \(s \in K_p \cap [1, n]\).

For \(j \notin K_p\) and \(c \in F_q^*\), let \(Z_j(c)\) denote the set of roots of the \(y\)-polynomial \(c^j - y^j + (y - c)^j\). Let \(Z(c) = \cup_{j \in [1, n] \cap K_p} Z_j(c)\). Clearly, we have

\[
|Z(c)| < \frac{n(n + 1)}{2}.
\]

Let \((c, d)\) be a pair in \(Q_{F_q}(b)\) satisfying \(\rho_b(d) \notin Z(\rho_b(c))\). We note that such \((c, d)\) do exist because, for any \(c \in F_q^*\), from (3.1) and (4.16) we have

\[
|\rho_b(F_q \setminus [0, b, c, b - c])| = \frac{q - 3}{2} > |Z(\rho_b(c))|.
\]

Then, it is clear that, for any \(j \in [1, n] \setminus K_p\),

\[
\rho_b^j(c) - \rho_b^j(d) + (\rho_b(d) - \rho_b(c))^j \neq 0.
\]

For \(t \in F_q^*\), let

\[
S_5 = (0, c^e, d^w; 0, t\rho_b^u(d), t(\rho_b^u(d) - \rho_b^u(c))).
\]
Then, from \( s \in K_p \) we have
\[
\Delta_3(h_s(x)y^s)(S_5)
\]
\[
= (h_s(c^v) - h_s(d^v))(t\rho_b^v(d))^s + h_s(d^v)(t(\rho_b^v(d) - \rho_b^v(c)))^s
\]
(4.19)
\[
= (h_s(c^v)\rho_b^v(d) - h_s(d^v)\rho_b^v(c))^s.
\]

For any \( j \in [1,n] \) with \( j \neq s \) and \( v \mid ju \), from \( v \in K_p \) and \( b^v = a \) we see
\[
\rho_b^{ju}(x) = (x(x - b))^{ju} = (x^v(x^v - a))^{ju/v} = \rho_a^{ju/v}(x^v)
\]
and thus from \( u \in K_p \) we have
\[
\Delta_3(\rho_b^{iu/v}(x)y^j)(S_5)
\]
\[
= (\rho_a^{iu/v}(c^v) - \rho_a^{iu/v}(d^v))(t\rho_b^v(d))^j + \rho_a^{iu/v}(d^v)(t(\rho_b^v(d) - \rho_b^v(c)))^j
\]
(4.20)
\[
= t^j\rho_b^v(d)(\rho_b^v(c) - \rho_b^v(d) + (\rho_b(d) - \rho_b(c))^j).
\]

If \( h_{2ju/v,j} \neq 0 \) for some \( j \in [1,n] \setminus K_p \) with \( j \neq s \) and \( v \mid ju \), from (4.13), (4.15), (4.18), (4.19) and (4.20) we see that the \( t \)–polynomial \( \Delta_3(h(x,y))(S_5) \in \mathbb{F}_q[t,u] \) has at least two nonzero coefficients, and thus there is some \( t \in \mathbb{F}_q^* \) such that \( \Delta_3(h(x,y))(S_5) = 0 \). By taking \( j = v \) in (4.20), we get \( \Delta_3(f(x)y^j)(S_5) = 0 \) and thus \( G \) has a 6-cycle of form \( S_5 \), contradicts to the assumption.

Hence, we have \( h_{2ju/v,j} = 0 \) for any \( j \in [1,n] \setminus K_p \) with \( j \neq s \) and \( v \mid ju \), and thus (4.15) follows from (4.19).

Though Lemma 4.2 shows the form of \( h(x,y) \), the polynomial \( h_j(x) \) contained in it is still not clear. The following lemma gives the expression of \( h_j(x) \) in different conditions. For brevity, we denote \( h_j(x) \) by \( z(x) \).

**Lemma 4.3** Let \( z(x) = \sum_{1 \leq i \leq n} z_i x^i \in \mathbb{F}_q[x] \), and
\[
R(x, y) = z(x^v)\rho_b^v(y) - z(y^v)\rho_b^w(x),
\]
(4.21)
where \( b \in \mathbb{F}_q, w \in K_p \cap [1, mn/2] \) and \( v \in K_p \cap [1, m] \). Suppose
\[
z(c) - z(b^v - c) \neq 0, \text{ for any } c \in \mathbb{F}_q \setminus \{b^v/2\},
\]
(4.22)
and
\[
R(d, c) \neq 0, \text{ for any } (c, d) \in Q_{\mathbb{F}_q}\{b\}.
\]
(4.23)
If \( b \neq 0 \), then we have \( w \geq v, z_{w/v} + z_{2w/v}b^w \neq 0 \) and
\[
z(x) = z_{w/v}x^{w/v} + z_{2w/v}x^{2w/v}.
\]
(4.24)
If \( b = 0 \) and \( w \geq v \), then there are some \( w_1 \in K_p \) and \( \sigma \in \{1, -1\} \) with \( z_{2w/v + \sigma w_1} \neq 0 \) such that
\[
z(x) = z_{2w/v}x^{2w/v} + z_{2w/v + \sigma w_1}x^{2w/v + \sigma w_1}.
\] (4.25)

If \( b = 0 \) and \( w < v \), then \( p = 3, v = 3w, z_1 \neq 0 \) and
\[
z(x) = z_1x.
\] (4.26)

**Proof**  Let \( [1, n]_v = \{v, 2v, \ldots, nv\} \) and \( R(x, y) = \sum_{i=1} r_i(y)x^i \), where \( r_i(y) \in \mathbb{F}_q[y] \), from \(4.21\) we have
\[
r_i(y) = \begin{cases} 
-z(y^v), & \text{if } i = 2w \notin [1, n]_v, \\
b^wz(y^v), & \text{if } i = w \notin [1, n]_v, \\
z_{i/v}r_i^v(y), & \text{if } i \in [1, n]_v \setminus \{w, 2w\}, \\
z_{jv}y^{2w} - b^v y^v)^j - z(y^v), & \text{if } i = 2w = 2jv \in [1, n]_v, \\
z_{jv}(y^{2w} - b^v y^v)^j + b^wz(y^v), & \text{if } i = w = jv \in [1, n]_v, \\
0, & \text{else.}
\end{cases}
\] (4.27)

Let \( k \) denote the largest integer with \( r_k(y) \neq 0 \) and \( l \) the least integer with \( r_l(y) \neq 0 \). Clearly, we have
\[
1 \leq l \leq k \leq mn.
\] (4.28)

For any \( c \in \mathbb{F}_q \setminus \{0, b, b/2\} \), from \(4.22\) we have
\[
R(b, c) = z(b^v)r_i^v(c) - z(c^v)r_i^v(b) = z(b^v)r_i^v(c) \neq 0, \text{ if } b \neq 0,
\]
\[
R(b - c, c) = z((b-c)^v)r_i^v(c) - z(c^v)r_i^v(b-c) = (z(b^v - c^v) - z(c^v))r_i^v(c) \neq 0,
\]
and thus, from \(4.23\) and \( R(x, c) \in \mathbb{F}_q[x]_{mn} \), there are \( \alpha(c) \in \mathbb{F}_q^* \) and positive integers \( l(c) < k(c) \) such that
\[
R(x, c) = \alpha(c)x^{l(c)}(x - c)^{k(c) - l(c)}.
\] (4.29)

Let \( \Theta = \{c \in \mathbb{F}_q \setminus \{0, b, b/2\} : r_k(c) \neq 0, r_l(c) \neq 0\} \). Then, for any \( c \in \Theta \), we have \( l(c) = l \) and \( k(c) = k \). From \(4.24\) and \(4.29\) we have \( k - l \in K_p \), then \( R(x, c) = \alpha(c)(x^k - c^{k-l}x^l) \), from \(4.27\) we have \( w = jv \in [1, n]_v \). Since for any \( j \) with \( 1 \leq 2j \leq n \) the polynomial \( z_{2j}(x^2 - b^vx)^j - z(x) \in \mathbb{F}_q[x] \) is
of order at most \( n \) and for any \( j' \in [1, n] \) the polynomial \( z_{j'}(x^2 - b^w x)^{j'} + b^w z(x) \in \mathbb{F}_q[x] \) is of order at most \( 2n \), from (4.30), (4.27), and (4.28) we see easily

\[
|\Theta| \geq q - 1 - 3n > mn \geq k. \tag{4.30}
\]

We should note that, in the first inequality in (4.30), \( r_i(0) = 0 \) for any \( i \) has been taken into account. Furthermore, for any \( c, d \in \Theta \), from

\[
\alpha(c) d^k (d - c)^{k-l} = R(d, c) = -R(c, d) = \alpha(d) c^l (c - d)^{k-l}
\]

we have \( 2 \nmid (k - l) \) and that there is some \( \alpha \in \mathbb{F}_q^* \) such that \( \alpha(c) = \alpha c^l \) is true for any \( c \in \Theta \). Therefore, from (4.29) we see

\[
R(x, c) = \alpha c^l x^l (x - c)^{k-l}, \quad \text{for any } c \in \Theta,
\]

and thus from (4.30) we have

\[
R(x, y) = \alpha x^l y^l (x - y)^{k-l}. \tag{4.31}
\]

If \( b \neq 0 \), from (4.21) and (4.31) we have

\[
\alpha x^l b^l (x - b)^{k-l} = R(x, b) = -z(b^w) \rho_b^w (x) = -z(b^w) x^w (x - b)^w
\]

and thus we see \( z(b^w) = -\alpha b^l, l = k - l = w \) and, for \( c \in \mathbb{F}_q \setminus \{0, b\} \),

\[
z(x^w) = \rho_b^{-w}(c)(z(c^w) \rho_b^w (x) + \alpha c^w x^w (x - c)^w)
\]

\[
= \rho_b^{-w}(c)(z(c^w) + \alpha c^w x^w) x^w - \rho_b^{-w}(c)(z(c^w) b^w + \alpha c^w) x^w,
\]

which implies \( w \geq v \), (4.21) and \( z_{w/v} + z_{2w/v} b^w = \alpha \rho_b^{-w}(c)c^w (b^w - c^w) \neq 0 \).

Assume \( b = 0 \) now. For any \( c \in \mathbb{F}_q^* \), from (4.21) and (4.31) we have

\[
z(x^w) = c^{-2w}(z(c^w)) x^{2w} + \alpha c^l x^l (x - c)^{k-l}. \tag{4.32}
\]

As the left side of (4.32) is independent of \( c \), we see \( k - l \in K_p, 2w \in \{k, l\} \) and

\[
z(x^w) = \begin{cases} 
  c^{-2w}(z(c^w) + \alpha c^l) x^{2w} - \alpha x^l, & \text{if } l < 2w = k, \\
  c^{-2w}(z(c^w) - \alpha c^k) x^{2w} + \alpha x^k, & \text{if } l = 2w < k,
\end{cases}
\]

namely, there are integers \( w_0 \in K_p \) and \( \sigma \in \{1, -1\} \) such that
\[ z(x^v) = e^{-2w}(z(x^v) - \sigma \alpha 2w+\sigma w_0)x^{2w} + \sigma \alpha x^{2w+\sigma w_0}. \] 

(4.33)

From (4.33) and \( \sigma \alpha \neq 0 \) we see

\[ v \mid (2w + \sigma w_0), \] 

(4.34)

\[ z(2w + \sigma w_0) \mid v = \sigma \alpha \] and

\[ e^{-2w}(z(x^v) - \sigma \alpha 2w+\sigma w_0) = \begin{cases} z_{2w/v}, & \text{if } w \geq v, \\ 0, & \text{otherwise.} \end{cases} \] 

(4.35)

If \( w \geq v \), from (4.34) we have \( v \leq w_0 \) and thus from (4.33) and (4.35) we see that (4.25) is true for \( w_1 = w_0/v \in K_p \) and \( z_{2w/v+\sigma w_1} = \sigma \alpha \neq 0 \).

If \( w < v \), from (4.34) we have \( p = 3, \sigma = 1, v = 3w, w_0 = w \) and thus from (4.33) and (4.35) we see

\[ z(x) = z_1x \] \( z_1 = \sigma \alpha \neq 0 \). \hfill \Box

Lemma 4.2 and Lemma 4.3 give the form of \( h(x, y) \) in the first two cases of \( f(x), g(y) \) (which were concluded before). To prove the Theorem 1.1 we need to discuss \( h(x, y) \) in more details.

**Theorem 4.4** Let \( a \in F_q, u, v \in K_p \) with \( u \leq m/2 \) and \( v \leq m \).

(i) If \( f(x) = \rho_a^u(x) \) and \( g(y) = y^v \), then there are \( s \in K_p \cap [1, n] \) and \( \zeta \in F_q^* \) such that

\[ h(x, y) = \zeta x^{su/v}y^s + \sum_{(i,j) \in \Phi_p(u,v)} h_{2i,j} \rho_a^u(x)y^j. \] 

(4.36)

(ii) If \( f(x) = x^v \) and \( g(y) = \rho_a^u(y) \), then there are \( r \in K_p \cap [1, n] \) and \( \zeta \in F_q^* \) such that

\[ h(x, y) = \zeta x^r y^{ru/v} + \sum_{(i,j) \in \Phi_p(v,u)} h_{1i,j} x^i \rho_a^u(y). \] 

(4.37)

**Proof** Since (ii) is symmetrical to (i), we only give proof for (i).

Assume \( f(x) = \rho_a^u(x) \) and \( g(y) = y^v \). According to Lemma 4.2 and Lemma 4.3 we see that there is an \( s \in K_p \cap [1, n] \) and \( \zeta \in F_q^* \) such that (4.36) or

\[ a = 0, v \leq su, h(x, y) = \zeta x^{2su/v+\sigma w_3}y^s + \sum_{(i,j) \in \Phi_p(u,v)} h_{2i,j} \rho_a^u(x)y^j \] 

(4.38)
for some \( w_1 \in K_p \cap [1, n] \) and \( \sigma \in \{1, -1\} \), or

\[
a = 0, \ p = 3, \ v = 3su, \ h(x, y) = \zeta xy^s + \sum_{(i,j) \in \Phi_p(u,v)} h_{2i,j}p_i(x)y^j.
\] (4.39)

Assume that (4.38) is valid. From Lemma 2.2 we see that \( G \) is isomorphic to

\[
G_1 = \Gamma_{F_{q^m}}(x^{2su/v+\sigma w_1}y^s, x^{2u}y^v),
\]

and thus, from Lemmas 3.3 and 3.4 we have \( 2su/v+\sigma w_1 \in K_p \). If \( \sigma = 1 \), then we have \( p = 3, \ w_1 = su/v \) and that \( G_1 \) is isomorphic to

\[
\Gamma_{F_{q^m}}(x^{3}y^s, x^{2}y^3)
\]

whose girth is 6, contradicts to that \( G_1 \) has girth at least 8. Hence, we have \( \sigma = -1, \ w_1 = su/v \) and that (4.36) is valid for \( a = 0 \).

Assume that (4.39) is valid. Then, from Lemma 2.2 and \( p = 3 \) we see that \( G \) is isomorphic to

\[
\Gamma_{F_{q^m}}(x^{2u}y^{3su}, x^{2}y^s)
\]

whose girth is 6, contradicts to that \( G \) has girth at least 8.

The following theorem gives the form of \( h(x, y) \) when \( f(x), g(y) \) are both injective i.e. \( f(x) = x^u, \ g(y) = y^v \) where \( u, v \in K_p \cap [1, m] \).

**Theorem 4.5** Assume \( f(x) = x^u \) and \( g(y) = y^v \) for some \( u, v \in K_p \cap [1, m] \). Then, either there is some \( s \in K_p \cap [1, n] \) with \( v \leq su \) and \( h_{2su/v,s} \neq 0 \) such that

\[
h(x, y) = h_{2su/v,s}x^{2su/v}y^s + \sum_{(i,j) \in \Phi_p(u,v)} h_{i,j}x^iy^j,
\] (4.40)

or there is some \( r \in K_p \cap [1, n] \) with \( u \leq rv \) and \( h_{r,2rv/u} \neq 0 \) such that

\[
h(x, y) = h_{r,2rv/u}x^ry^{2rv/u} + \sum_{(i,j) \in \Phi_p(u,v)} h_{i,j}x^iy^j.
\] (4.41)

**Proof** For \( a, b \in \mathbb{F}_{q^m}^*, \ t \in \mathbb{F}_{q^m}\{0,1\} \), let

\[
S_0 = (0, at^v, a; 0, b(1 - t^u))
\] (4.42)

Then, from \( \Delta_3(f(x)g(y))(S_0) = \Delta_3(x^uy^v)(S_0) = 0 \) we see that, for any \( a \in \mathbb{F}_{q^m}^* \) and \( t \in \mathbb{F}_{q^m}\{0,1\} \), the \( b \)-polynomial

\[
\Delta_3(h(x, y))(S_0) = \sum_{1 \leq j \leq n} E_j(a, t)b^j
\] (4.43)
has no root in $F_{q^{M}}$, where

$$E_j(x, y) = h_j(xy^v) - h_j(x) + h_j(x)(1 - y^u)^j. \quad (4.44)$$

Since the $x$–order and $y$–order of $E_j(x, y) \in F_q[x, y]$ are at most $mn$, by applying Lemma 4.1 for $W = F_{q^{M}} \times (F_q \setminus \{0, 1\})$, $N = mn$ and $f_j(x, y) = E_j(x, y)$ according to (3.1) we see that there is an $s \in [1, n]$ such that

$$E_j(x, y) = 0, \text{ for any } j \neq s. \quad (4.45)$$

Therefore, we have $\Delta_3(h(x, y))(S_6) = E_s(a, t)b^s$ and thus

$$E_s(a, t) \neq 0, \text{ for any } a \in F_{q^{M}}^{*} \text{ and } t \in F_q \setminus \{0, 1\}. \quad (4.46)$$

For $j \notin K_p \cup \{s\}$, from (4.44) and (4.45) we see

$$h_j(x)(1 - (1 - y^u)^j) = h_j(xy^v)$$

$$= h_j(xy^v)(1 - (1 - y^u)^j)$$

$$= h_j(x)(1 - (1 - y^u)^j)^2$$

and thus we have

$$h_j(x) = 0, \text{ if } j \notin K_p \cup \{s\}. \quad (4.47)$$

For $j \in K_p \setminus \{s\}$, from (4.44) and (4.45) we see $h_j(xy^v) = h_j(x)y^j u$ and thus we have $v \leq j u$ and

$$h_j(x) = \begin{cases}h_{ju/v, s}x^{ju/v}, & \text{if } v \leq ju, \\0, & \text{otherwise.} \end{cases} \quad (4.48)$$

By following the proof of Lemma 4.1 from (4.46) and

$$E_s(x, y) = \sum_{1 \leq i \leq n} h_{i, s}(y^iv - 1 + (1 - y^u)^s)x^i,$$

one can show that there is an $r \in [1, n]$ such that $h_{r, s} \neq 0$ and

$$t^{rv} - 1 + (1 - t^u)^s \neq 0, \text{ for any } t \in F_{q^{M}} \setminus \{0, 1\}, \quad (4.49)$$

$$h_{i, s}(y^iv - 1 + (1 - y^u)^s) = 0, \text{ for any } i \neq r. \quad (4.50)$$

From (4.49), we see $su \neq rv$ if $s \in K_p$. Therefore, from (4.50) we see

$$h_s(x) = \begin{cases}h_{r, s}x^r + h_{su/v, s}x^{su/v}, & \text{if } s \in K_p \text{ and } v \leq su, \\h_{r, s}x^r, & \text{otherwise,} \end{cases}$$
and thus from (4.47) and (4.48) we have

\[ h(x, y) = h_{r,s}x^ry^s + \sum_{(i,j) \in \Phi_p(u,v)} h_{i,j}x^iy^j. \]  

(4.51)

According to (4.51) and Lemma 2.2, we see that \( G = \Gamma_{\mathbb{F}_q^m}(x^ry^s, x^ry^s) \) is isomorphic to \( G \) and thus has girth at least 8.

Now we conclude that either \( r \) or \( s \) is not in \( K_p \). Suppose \( r, s \in K_p \) in contrast. Since \( su \neq rv \), there is a \( w_2 \in K_p \setminus \{1\} \) such that either \( su = rvw_2 \) or \( rv = suw_2 \). Let \( t_0 \) be a \((w_2 - 1)\)-th root of unity with \( t_0 \neq 1 \). As \( w_2 - 1 \in [1,mn] \), from (3.1) we note that such \( t_0 \) do exist. Then, \( G_2 \) contains the 6-cycle \((0, 1, t_0^2; t_0^3, 0, 1) \), contradicts to that its girth is at least 8.

Assume that \( r \) is not in \( K_p \). According to Lemmas 3.3 and 3.4, we have \( s \in K_p, 2 \mid r \) and \( r/2 \in K_p \). Then, from (4.49) we see

\[ |rv - su| \in K_p \]  

(4.52)

If \( rv < su \), then from (4.52) we have \( p = 3, 3rv = 2su \) and thus from Lemma 2.2 we see that \( G_2 \) is isomorphic to

\[ \Gamma_{\mathbb{F}_q^m}(x^{3rv}y^{3sv}, x^uy^v) \cong \Gamma_{\mathbb{F}_q^m}(x^{2su}y^{3sv}, x^uy^v) \cong \Gamma_{\mathbb{F}_q^m}(x^2y^3, xy) \]

whose girth is of 6, contradicts to that \( G_2 \) has girth at least 8. Hence, we have \( rv > su \) and thus from (4.52) we see \( rv = 2su \) and (4.40).

Similarly, one can show that (4.41) is valid if \( s \) is not in \( K_p \).

\[ \square \]

5 Proofs of Theorem 1.1 and Theorem 1.2

From the discussions in section 3 and 4, we see that for the bipartite graph \( G = \Gamma_{\mathbb{F}_q^m}(f(x)g(y), h(x, y)) \) defined as before, there are some \( a \in \mathbb{F}_q, \zeta \in \mathbb{F}_q^*, u, v \in K_p \cap [1,m] \) and \( s, r \in K_p \cap [1,n] \) such that one of the following is valid.

(i) \( f(x) = \rho^a(x), g(y) = y^u \) and \( h(x, y) = \zeta x^{su/v}y^s + \sum_{(i,j) \in \Phi_p(u,v)} h_{2i,j}x^iy^j \).

(ii) \( f(x) = x^v, g(y) = \rho^u(y) \) and \( h(x, y) = \zeta x^r y^{ru/v} + \sum_{(i,j) \in \Phi_p(v,u)} h_{i,2j}x^iy^j \).

(iii) \( f(x) = x^v, g(y) = y^v \) and \( h(x, y) = h_{2su/v,u}x^{2su/v}y^s + \sum_{(i,j) \in \Phi_p(u,v)} h_{i,j}x^iy^j \).

(iv) \( f(x) = x^v, g(y) = y^v \) and \( h(x, y) = h_{r,2uv/u}x^{2uv/u}y^s + \sum_{(i,j) \in \Phi_p(u,v)} h_{i,j}x^iy^j \).

From the isomorphisms (Lemma 2.2), one can easily conclude the following corollary.
Corollary 5.1 The bipartite graph $G = \Gamma_{F_q} (f(x)g(y), h(x, y))$ defined as before is isomorphic to $\Gamma_3(F_q)$. Then, we have proved Theorem 1.1.

In particular, for algebraically closed field $F_\infty$ with characteristic zero. One can get the Theorem 1.2 from the above argument by replacing $q$ by $\infty$, $K_p$ by $\{1\}$, respectively.

As we mentioned in section 1, Kronenthal, Laubennik and Williford proved that for polynomial $f \in F[x, y]$, the graph $\Gamma_F(x^k y^m, f)$ with girth at least eight is isomorphic to $\Gamma_3[F]$. The authors also proved the related results over infinite families of finite fields. Note that these two results are the particular case (when $f(x) = x^k$ and $g(y) = y^m$) of Theorem 1.2 and Theorem 1.1 respectively. In this paper, we prove this case in a more concise way. Moreover, we give all relevant polynomials such that the graph has girth at least eight.

6 Conclusion

For odd prime power $q$ and monomials $f_2, f_3 \in F_q[x, y]$, the girth of monomial graph $\Gamma_{F_q}(f_2, f_3)$ has been studied extensively (see [18, 19] and [2]). Dmytrenko, Lazebnik and Williford [19] proved that any monomial graph $\Gamma_{F_q}(f_2, f_3)$ of girth at least eight is isomorphic to a graph $\Gamma_{F_q}(xy, x^k y^{2k})$ for $k$ with $(k, q) = 1$. In particular, the integer $k$ can be restricted to be 1 further if the odd prime power $q$ is not greater than $10^{10}$ or of form $q = p^{x^3} a^b$ for odd prime $p$. In [19], Conjecture 1 was proposed which was later proved by Hou, Lappano and Lazebnik [20].

Theorem 6.1 ([20]). Suppose $q$ is an odd prime power and $f_2, f_3 \in F_q[x, y]$ are monomials, then the bipartite graph $\Gamma_{F_q}(f_2, f_3)$ with girth at least eight is isomorphic to $\Gamma_3(F_q)$.

However, little is now when $f_2, f_3 \in F_q[x, y]$ are arbitrary polynomials, which brings Conjecture 2 and 3 that we mentioned in section 1. In this paper, we proved Conjecture 3 for $f_2 = f(x)g(y)$, i.e., $f_2$ is a product of two univariate polynomials. We also proved related uniqueness results over finite fields. Note that the two results we obtained in this paper are the extensions of the results in [4]. Moreover, we obtained all polynomials such that the graph we studied has girth at least eight.

Till now, Conjecture 2 and 3 are still open, as we proved Conjecture 3 when $f_2$ is a class of special polynomials, we believe the conjecture holds for any polynomials $f_2$, namely, Conjecture 3 is true.
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