Five-loop anomalous dimension at critical wrapping order in $\mathcal{N} = 4$ SYM

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Abstract

We compute the anomalous dimension of a length-five operator at five-loop order in the $SU(2)$ sector of $\mathcal{N} = 4$ SYM theory in the planar limit. This is critical wrapping order at five loops. The result is obtained perturbatively by means of $\mathcal{N} = 1$ superspace techniques. Our result from perturbation theory confirms explicitly the formula conjectured in arXiv:0901.4864 for the five-loop anomalous dimension of twist-three operators. We also explicitly obtain the same result by employing the recently proposed Y-system.
1 Introduction

Finding the anomalous dimension of short operators in $\mathcal{N} = 4$ SYM theory is a subject that received great attention in the recent past. The computation of the anomalous dimension of long operators is now an easily accessible task, thanks to the asymptotic Bethe ansatz techniques [1–4]. For short operators, however, a well established approach allowing us to similarly compute their anomalous dimensions does not exist yet, because of the so-called wrapping effects [5, 6].

In the last year several approaches on how to manage this problem have been proposed. One of them is based on a generalization of Lüscher formula [7–12], while others are based on the Y-system [13–16] and on the thermodynamic Bethe ansatz [17–20].

It is clearly desirable that the anomalous dimensions obtained from these approaches are tested by means of explicit perturbative standard field theory calculations. This has been done in a few cases. In particular, the anomalous dimension of the Konishi operator has been computed perturbatively up to four loops [21–23] and there is agreement with the value obtained with the proposals of [8] and of [13]. Moreover, the anomalous dimensions of twist-two operators at four loops were predicted from Lüscher formula approach [10] and then confirmed from perturbative field theory [24].

More computations are needed at the perturbative level in order to check the various proposed approaches. In this paper we perform a new test: we compute the anomalous dimension of the operator $\text{tr}(\phi Z\phi ZZ - \phi Z ZZ)$ at five-loop order by means of $\mathcal{N} = 1$ superspace techniques. This is a length-five operator, so we make a computation at critical wrapping order. This operator belongs to a supermultiplet which has a representative also in the $SL(2)$ sector [4], namely the length-three operator with two derivatives. The anomalous dimension of this twist-three operator has been predicted in [11] from a general formula conjectured on the basis of the maximal transcendentality principle [25]. Our result matches the prediction of [11], thus providing a confirmation from perturbative field theory of the general formulas proposed there.

To perform our five-loop perturbative calculation we make use of the same strategy adopted at four loops in our papers [21, 22]. We first construct the complete five-loop asymptotic dilatation operator, from which we extract the contribution of all non-wrapping graphs by subtracting the range six contributions. In this way we avoid the computation of all the graphs with interaction range from one to five. The explicit Feynman graph computation is then reduced to the consideration of only wrapping diagrams. We compute them by making use of $\mathcal{N} = 1$ superspace techniques and by

\footnote{The five-loop anomalous dimension of Konishi operator, computed recently in [12], would require a calculation one step beyond critical wrapping order.}
taking advantage of all the cancellations between supergraphs discussed in [22, 26]. The integrals are then computed with the Gegenbauer polynomial x-space technique [22, 27, 28].

We then also explicitly compare our result with the one obtained by applying the Y-system proposed in [13, 14], and we find agreement.

The paper is organized as follows. In section 2 we build the complete five-loop asymptotic dilatation operator in the $SU(2)$ sector. Section 3 contains the main result of the paper, namely the computation of the anomalous dimension of the operator $\text{tr}(\phi Z \phi Z Z - \phi \phi Z Z Z)$ at five loops. The strategy adopted for this calculation is a straightforward generalization of the one used at four loops in our previous papers [21, 22]. We then compute the same five-loop anomalous dimension in section 4 by using the approach introduced in [13, 14] and based on the Y-system technique. We then conclude with some final remarks in section 5. Details about the Feynman diagrams and loop integrals can be found in the appendices.

2 The dilatation operator at five loops

In this section we want to calculate the complete five-loop dilatation operator in the $SU(2)$ sector. Such operator was already computed in [3] but only in the case without the dressing phase [29–33], which according to [29] should have two relevant components at five loops, $\beta_{2,3,3}$ and $\beta_{2,3,4}$. We thus have to repeat the steps of [3] in order to restore the dependence on the components of the dressing phase. Moreover, we need to find the most general expression for the dilatation operator by taking similarity transformations into account. We follow the same strategy adopted in [3], and we rely on the assumption of integrability.

Consider the perturbative expansion in terms of the rescaled ‘t Hooft coupling constant
\[
\lambda = \frac{g^2 N}{(4\pi)^2}.
\]
The asymptotic dilatation operator expands as
\[
D(\lambda) = Q_2 = L + \sum_{k=1}^{\infty} \lambda^k D_k.
\]
We also assume that a conserved charge $Q_3$ exists, with expansion
\[
Q_3(\lambda) = \sum_{k=1}^{\infty} \lambda^k Q_3^{(k)}.
\]
The dilatation operator and the charge \( Q_3 \) can be written in a basis of operators built using permutations

\[
\{a_1, \ldots, a_n\} = \sum_{r=0}^{L-1} P_{a_1+r} a_1+r+1 \cdots P_{a_n+r} a_n+r+1
\]  

(2.4)

with range

\[
R = 2 + \max_{a_1 \ldots a_n} - \min_{a_1 \ldots a_n} .
\]  

(2.5)

Some rules valid in the asymptotic case for the manipulation of these structures can be found in [34].

The operators \( D_k \) and \( Q_3^{(k)} \) contain structures with range up to \( k+1 \) and \( k+2 \) respectively. We write \( D_5 \) and \( Q_3^{(5)} \) as linear combinations of the relevant basis operators with unknown coefficients, which are fixed by requiring the following constraints [3]:

- \( D \) and \( Q_3 \) respectively have even and odd parity
- \( D \) is symmetric, \( Q_3 \) is antisymmetric,
- \( D \) and \( Q_3 \) have the right BMN scaling on single-impurity states,
- \( D \) and \( Q_3 \) are perturbatively commuting up to order \( \lambda^6 \), i.e.

\[
\sum_{k=1}^{5} [D_k, Q_3^{(6-k)}] = 0 ,
\]  

(2.6)

- the spectrum of \( D \) agrees with the asymptotic Bethe equations up to five loops.

The components \( D_1 \) to \( D_4 \) and thus the dilatation operator up to four loops are known. The same holds for the charge components \( Q_3^{(1)} \), \( Q_3^{(2)} \) and \( Q_3^{(3)} \) [3, 30]. In our case we also need the expression for \( Q_3^{(4)} \), which is not present in the literature. So we applied the described procedure also at four loops. This yields \( Q_3^{(4)} \), which depends on a single undetermined coefficient. The result is shown in Table 1. Simultaneously the procedure also reproduces the known expression for \( D_4 \) as a check.

After applying the procedure at five loops, some of the coefficients in \( D_5 \) remain undetermined. They are related to similarity transformations and do not enter the spectrum of the asymptotic dilatation operator [30]. Other such coefficients can be found by applying the most general similarity transformation

\[
D \to D' = e^{-i\chi} De^{i\chi} ,
\]  

(2.7)
$$Q_3^{(4)} = -2i \left( 373 + 2\beta_{2,3,3} + \epsilon_{3a} \right) \{(1,2) - \{2,1\}\}
+ 2i \left( 180 + \beta_{2,3,3} + 2\epsilon_{3a} \right) \{(1,2,3) - \{3,2,1\}\}
+ i \left( 40 + 3\beta_{2,3,3} \right) \{(1,2,4) - \{1,4,3\} + \{1,3,4\} - \{2,1,4\}\}
+ 2i \left( (1,2,5) - \{1,5,4\} + \{1,4,5\} - \{2,1,5\}\right)
- 2i \left( 57 + \epsilon_{3a} \right) \{(1,2,3,4) - \{4,3,2,1\}\}
+ i \left( 23 - 2\beta_{2,3,3} - \epsilon_{3a} \right) \{(1,2,4,3) - \{1,4,3,2\} + \{2,1,3,4\} - \{3,2,1,4\}\}
+ 4i \left( 8 - \beta_{2,3,3} \right) \{(1,3,2,4) - \{2,1,4,3\}\}
- 4i \left( \{(1,2,3,5) - \{1,5,4,3\} + \{1,2,4,5\} - \{2,1,5,4\} + \{1,3,4,5\} - \{3,2,1,5\}\right)
+ i \left( 1 + \beta_{2,3,3} + \epsilon_{3a} \right) \{(1,3,2,4,3) - \{2,1,4,3,2\} + \{2,1,3,2,4\} - \{3,2,1,4,3\}\}
+ i \left( 1 + \epsilon_{3a} \right) \{(1,2,3,5,4) - \{1,5,4,3,2\} + \{2,1,3,4,5\} - \{4,3,2,1,5\}\}
- i \left( 7 + \epsilon_{3a} \right) \{(1,2,4,3,5) - \{2,1,5,4,3\} + \{1,3,2,4,5\} - \{3,2,1,5,4\}\}
+ 2i \left( 4 + \epsilon_{3a} \right) \{(1,4,3,2,5) - \{2,1,3,5,4\}\}
+ 14i \{(1,2,3,4,5) - \{5,4,3,2,1\}\}
$$

Table 1: Four-loop component of the first higher conserved charge $Q_3$ before the application of similarity transformations.

where the generating function $\chi$ can be expanded perturbatively as

$$\chi = \sum_{k=0}^{\infty} \lambda^k \chi_k . \quad (2.8)$$

The components of $D'$ are given explicitly by

$$D'_0 = D_0 ,$$
$$D'_1 = D_1 ,$$
$$D'_2 = D_2 ,$$
$$D'_3 = D_3 + i[D_1, \chi_2] + i[D_2, \chi_1] ,$$
$$D'_4 = D_4 + i[D_1, \chi_3] + i[D_2, \chi_2] + i[D_3, \chi_1] + \frac{1}{2}[\chi_1, [D_1, \chi_2] + [D_2, \chi_1]] , \quad (2.9)$$
$$D'_5 = D_5 + i[D_1, \chi_4] + i[D_2, \chi_3] + i[D_3, \chi_2] + i[D_4, \chi_1]$$
$$+ \frac{1}{2} [\chi_1, [D_3, \chi_1] + [D_2, \chi_2] + [D_1, \chi_3]] + \frac{1}{2} [\chi_2, [D_1, \chi_2] + [D_2, \chi_1]]$$
$$- \frac{i}{6} [\chi_1, [\chi_1, [D_1, \chi_2] + [D_2, \chi_1]]] .$$

To determine $\chi$, we require for consistency that each $\chi_k$ is writable in terms of operators with range up to $k + 1$, so that the dilatation operator will maintain its range after the
transformation. Therefore, $\chi_0$ is irrelevant being proportional to the identity $\{}$, while

$$\chi_1 = \tilde{\epsilon}_1 \{1\} .$$  \hspace{1cm} (2.10)$$

To explicitly preserve the invariance of the dilatation operator under parity, we demand that $\chi$ is parity-invariant. As far as Hermiticity is concerned, it is possible to choose a $\chi$ which is Hermitian for real values of the coefficients multiplying the permutation operators, so that the transformed Hamiltonian is still Hermitian. In general, however, the explicitly computation in a given renormalization scheme will produce complex values for the similarity coefficients $\tilde{\epsilon}_x$. This can be seen already at four loops: if we consider the most general Hermitian form for $\chi_2$ and $\chi_3$ as in (2.10)

$$\chi_2 = \tilde{\epsilon}_{2a} (\{1, 2\} + \{2, 1\}) + \tilde{\epsilon}_{2b} \{1\} ,$$

$$\chi_3 = i \tilde{\epsilon}_{3a} (\{2, 1, 3\} - \{1, 3, 2\}) + \tilde{\epsilon}_{3b} (\{1, 2, 3\} + \{3, 2, 1\}) + \tilde{\epsilon}_{3c} \{1\} ,$$  \hspace{1cm} (2.11)$$

the transformed dilatation operator will be in turn Hermitian only for real values of the $\tilde{\epsilon}_x$ coefficients, but a Feynman diagram computation in a generic scheme will typically produce complex values for them [22, 30]. The same happens at five loops. Anyway, the non-Hermiticity of the dilatation operator after a similarity transformation depends on the renormalization scheme and does not constitute a problem as explained in [30].

For $\chi_4$, we look for the most general parity-invariant, Hermitian operator built using operators with range up to five:

$$\chi_4 = \sum_{\alpha \in \{a, b, \ldots, n\}} \tilde{\epsilon}_{4\alpha} \chi_{4\alpha} ,$$  \hspace{1cm} (2.12)$$

we thereby use linear combinations of the permutation structures which are eigenstates with eigenvalue one under parity transformation and Hermitian conjugation. The transformation $\chi_4$ then preserves parity and Hermiticity if all its coefficients $\tilde{\epsilon}_{4\alpha}$ are real, and imaginary parts of $\tilde{\epsilon}_{4\alpha}$ are responsible for breaking the Hermiticity. The combinations $\chi_{4\alpha}$ are given in Table 2. Using (2.9) we find the most general expression for the five-loop asymptotic dilatation operator comprehensive of the dressing phase and similarity coefficients, which is shown in Table 3. The $\epsilon_x$ coefficients are redefinitions of the original $\tilde{\epsilon}_x$ ones which simplify the final result.
\[
\begin{align*}
\chi_{4a} &= \{1, 2, 3, 4\} + \{4, 3, 2, 1\} \\
\chi_{4b} &= i(\{1, 2, 4, 3\} + \{1, 4, 3, 2\} - \{2, 1, 3, 4\} - \{3, 2, 1, 4\}) \\
\chi_{4c} &= \{1, 2, 4, 3\} + \{1, 4, 3, 2\} + \{2, 1, 3, 4\} + \{3, 2, 1, 4\} \\
\chi_{4d} &= \{1, 3, 2, 4\} + \{2, 1, 4, 3\} \\
\chi_{4e} &= \{2, 1, 3, 2\} \\
\chi_{4f} &= i(\{1, 2, 4\} + \{1, 4, 3\} - \{1, 3, 4\} - \{2, 1, 4\}) \\
\chi_{4g} &= \{1, 2, 4\} + \{1, 4, 3\} + \{1, 3, 4\} + \{2, 1, 4\} \\
\chi_{4h} &= i(\{1, 3, 2\} - \{2, 1, 3\}) \\
\chi_{4i} &= \{1, 3, 2\} + \{2, 1, 3\} \\
\chi_{4j} &= \{1, 2, 3\} + \{3, 2, 1\} \\
\chi_{4k} &= \{1, 2\} + \{2, 1\} \\
\chi_{4l} &= \{1, 4\} \\
\chi_{4m} &= \{1, 3\} \\
\chi_{4n} &= \{1\}
\end{align*}
\]

Table 2: Components of the generating function

3 Computation of the anomalous dimension at five loops

In this section we are going to describe the computation of the anomalous dimension of the operator \(\text{tr}(\phi Z\phi ZZ - \phi\phi ZZ Z)\) at five loops. As already mentioned, the strategy which we adopt here is the same as the one used in our previous papers [21, 22]. We first compute the contribution from the diagrams with range from one to five by taking advantage of the asymptotic five-loop dilatation operator \(D_5\) computed in the previous section. Indeed, we exploit the fact that these diagrams are relevant also in the asymptotic case, and therefore all the information on them is encoded in \(D_5\). However, in addition to these diagrams the dilatation operator also gets contributions from range-six diagrams, which must be subtracted. The second step will be the explicit calculation of the wrapping contributions by means of \(\mathcal{N} = 1\) superspace techniques [35].

3.1 Subtraction of range-six diagrams from the asymptotic dilatation operator

Since our procedure is a straightforward generalization of the one used in [21, 22], we do not repeat here all the steps, and we refer the reader to those papers. First of all, we need to write the five-loop Hamiltonian in terms of chiral structures, which are directly related to the chiral structures of the underlying Feynman supergraphs (see [22] for a discussion of these functions). The transformation between the chiral structure basis and the one made of permutation operators is easily performed by means of the rules shown in Table 4.
Table 3: Asymptotic five-loop dilatation operator in the permutation basis

\[
D_5 = +4(1479 + 14\beta_{2,3,3} - \beta_{3,2,4})\{1\} - 4(2902 + 42\beta_{2,3,3} - 3\beta_{2,3,4} + 32\epsilon_{4k})\{1\}
- 4(-816 - 32\epsilon_{4b} - 13\beta_{2,3,3} + \beta_{2,3,4} + 16\epsilon_{4b})\{1, 2\} + \{2, 1\}
+ 2(512 + 48\beta_{2,3,3} - 3\beta_{2,3,4} + 4\epsilon_{2a} + 4\epsilon_{3a} + 64\epsilon_{4f} + 32\epsilon_{4h})\{1, 3\}
+ 8(20 + \beta_{2,3,3} + 2\epsilon_{2a} + 2\epsilon_{3a} - 16\epsilon_{4b} - 16\epsilon_{4f})\{1, 4\} + 4\{1, 5\}
- 4(324 + 16\epsilon_{4h} - \beta_{2,3,3} + 2\epsilon_{2a} + 2\epsilon_{3a} + 32\epsilon_{4b})\{1, 2, 3\} + \{3, 2, 1\}
+ 4(94 - 15\beta_{2,3,3} + \beta_{2,3,4} + 16\epsilon_{4h} - 16\epsilon_{4k} + 16\epsilon_{4k})\{1, 3, 2\}
+ 4(94 - 15\beta_{2,3,3} + \beta_{2,3,4} + 16\epsilon_{4h} - 16\epsilon_{4h} - 16\epsilon_{4k})\{2, 1, 3\}
- 8(12 + \beta_{2,3,3} + \epsilon_{2a} + \epsilon_{3a} - 8\epsilon_{4b} + 4i\epsilon_{4m})\{1, 3, 4\} + \{2, 1, 4\}
- 8(12 + \beta_{2,3,3} + \epsilon_{2a} + \epsilon_{3a} - 8\epsilon_{4b} - 4i\epsilon_{4m})\{1, 2, 4\} + \{1, 4, 3\}
- 4(1 - 8i\epsilon_{4f})\{1, 2, 5\} + \{1, 5, 4\}
- 4(1 + 8i\epsilon_{4f})\{1, 4, 5\} + \{2, 1, 5\} - 8\{1, 3, 5\}
- 2(40 - 12\beta_{2,3,3} + \beta_{2,3,4} + 4\epsilon_{2a} + 4\epsilon_{3a} - 32\epsilon_{4b})\{2, 1, 3, 2\}
+ 8(35 + \epsilon_{2a} + \epsilon_{3a} - 8\epsilon_{4b})\{1, 2, 3, 4\} + \{4, 3, 2, 1\}
+ 8(-21 + 2\beta_{2,3,3} + 4\epsilon_{4f} - 8\epsilon_{4h})\{1, 3, 2, 4\} + \{2, 1, 4, 3\}
+ 4(\epsilon_{2a} + \epsilon_{3a} - 8\epsilon_{4b} + \epsilon_{4f} - \epsilon_{4h} + i\epsilon_{4f})\{1, 2, 3, 4\} + \{3, 2, 1, 4\}
+ 4(\epsilon_{2a} + \epsilon_{3a} + 8\epsilon_{4b} + \epsilon_{4f} - \epsilon_{4h} + i\epsilon_{4f})\{1, 2, 4, 3\} + \{1, 4, 3, 2\}
+ 32(1 - 2i\epsilon_{4f})\{1, 2, 4, 5\} + \{2, 1, 5, 4\}
- 8(3 - 8\epsilon_{4f} - 8i\epsilon_{4f})\{1, 2, 4, 5\} - 8(3 - 8\epsilon_{4f} + 8i\epsilon_{4f})\{2, 1, 4, 5\}
+ 2(-2i\epsilon_{2a} - i\epsilon_{3a} + 16\epsilon_{4f} + 16i\epsilon_{4g})\{1, 2, 3, 5\} + \{1, 5, 4, 3\}
+ 2(2i\epsilon_{2a} + i\epsilon_{3a} + 16\epsilon_{4f} - 16i\epsilon_{4g})\{1, 3, 2, 4\} + \{3, 2, 1, 5\}
+ 2(4 - 2i\epsilon_{2a} - i\epsilon_{3a} - 16\epsilon_{4f} - 16i\epsilon_{4g})\{1, 4, 3, 5\} + \{2, 1, 3, 5\}
+ 2(4 + 2i\epsilon_{2a} + i\epsilon_{3a} - 16\epsilon_{4f} + 16i\epsilon_{4g})\{1, 3, 2, 5\} + \{1, 3, 5, 4\}
+ 2(10 - \beta_{2,3,3} + 16\epsilon_{4b} + 16i\epsilon_{4c})\{1, 3, 2, 4\} + \{2, 1, 4, 3, 2\}
+ 2(10 - \beta_{2,3,3} + 16\epsilon_{4b} - 16i\epsilon_{4c})\{2, 1, 3, 2, 4\} + \{3, 2, 1, 4, 3\}
+ 4(4 + \epsilon_{2a} + \epsilon_{3a} - 16i\epsilon_{4d})\{2, 1, 4, 3, 5\}
+ 4(4 + \epsilon_{2a} + \epsilon_{3a} + 16i\epsilon_{4d})\{1, 3, 2, 5, 4\}
+ 4(2 + \epsilon_{2a} + 2i\epsilon_{2a} + \epsilon_{3a} - 8i\epsilon_{4b} - 8i\epsilon_{4c} + 8i\epsilon_{4d})\{1, 2, 4, 3, 5\} + \{2, 1, 5, 4, 3\}
+ 4(2 + \epsilon_{2a} - 2i\epsilon_{2a} + \epsilon_{3a} + i\epsilon_{4b} - 8i\epsilon_{4b} - 8i\epsilon_{4c} + 8i\epsilon_{4d})\{1, 3, 2, 4, 5\} + \{3, 2, 1, 5, 4\}
+ 4(2 - \epsilon_{2a} - \epsilon_{3a} + 16\epsilon_{4b} + 16i\epsilon_{4c})\{1, 2, 5, 4, 3\}
+ 4(2 - \epsilon_{2a} - \epsilon_{3a} + 16\epsilon_{4b} - 16i\epsilon_{4c})\{3, 2, 1, 4, 5\}
- 4(16\epsilon_{4b} + 7)\{1, 4, 3, 2, 5\} + \{2, 1, 3, 5, 4\}
- 4(\epsilon_{2a} + \epsilon_{3a} - 8i\epsilon_{4a} - 8i\epsilon_{4b})\{1, 2, 3, 5, 4\} + \{1, 5, 4, 3, 2\}
- 4(\epsilon_{2a} + \epsilon_{3a} + 8i\epsilon_{4a} - 8i\epsilon_{4b})\{2, 1, 3, 4, 5\} + \{4, 3, 2, 1, 5\}
- 28\{1, 2, 3, 4, 5\} + \{5, 4, 3, 2, 1\}
\]
\( \chi(a, b, c, d, e) = -\{ \} + 5\{1\} - \{a, b\} - \{a, c\} - \{a, d\} - \{a, e\} - \{b, c\} - \{b, d\} - \{b, e\} \)
\( - \{c, d\} - \{c, e\} - \{d, e\} + \{a, b, c\} + \{a, b, d\} + \{a, b, e\} + \{a, c, d\} \)
\( + \{a, c, e\} + \{a, d, e\} + \{b, c, d\} + \{b, c, e\} + \{b, d, e\} + \{c, d, e\} \)
\( - \{a, b, c, d\} - \{a, b, c, e\} - \{a, b, d, e\} - \{a, c, d, e\} - \{b, c, d, e\} \)
\( + \{a, b, c, d, e\} , \)
\( \chi(a, b, c, d) = \{\} - 4\{1\} + \{a, b\} + \{a, c\} + \{a, d\} + \{b, c\} + \{b, d\} + \{c, d\} \)
\( - \{a, b, c\} - \{a, b, d\} - \{a, c, d\} - \{b, c, d\} + \{a, b, c, d\} , \)
\( \chi(a, b, c) = -\{\} + 3\{1\} - \{a, b\} - \{a, c\} - \{b, c\} + \{a, b, c\} , \)
\( \chi(a, b) = \{\} - 2\{1\} + \{a, b\} , \)
\( \chi(1) = -\{\} + \{1\} , \)
\( \chi() = \{\} . \)

\{a, b, c, d, e\} = \chi(a, b, c, d, e) + \chi(b, c, d, e) + \chi(a, c, d, e) + \chi(a, b, d, e) + \chi(a, b, c, e) 
\ + \chi(a, b, c, d) + \chi(c, d, e) + \chi(b, d, e) + \chi(b, c, e) + \chi(b, c, d) + \chi(a, d, e) 
\ + \chi(a, c, e) + \chi(a, c, d) + \chi(a, b, e) + \chi(a, b, d) + \chi(a, b, c) + \chi(d, e) 
\ + \chi(c, e) + \chi(c, d) + \chi(b, e) + \chi(b, d) + \chi(b, c) + \chi(a, e) + \chi(a, d) 
\ + \chi(a, c) + \chi(a, b) + 5\chi(1) + \chi() , \)
\{a, b, c, d\} = \chi(a, b, c, d) + \chi(a, b, c) + \chi(a, b, d) + \chi(a, c, d) + \chi(b, c, d) 
\ + \chi(a, b) + \chi(a, c) + \chi(a, d) + \chi(b, c) + \chi(b, d) + \chi(c, d) 
\ + 4\chi(1) + \chi() , \)
\{a, b, c\} = \chi(a, b, c) + \chi(a, b) + \chi(a, c) + \chi(b, c) + 3\chi(1) + \chi() , \)
\{a, b\} = \chi(a, b) + 2\chi(1) + \chi() , \)
\{1\} = \chi(1) + \chi() , \)
\{\} = \chi() . \)

Table 4: Rules for the conversion between permutation operators and chiral structures
The dilatation operator written in terms of chiral structures is given in Table 5. As discussed in the previous section, the $\epsilon_x$ coefficients in Table 5 are related to similarity transformations. Note that even if these coefficients (which are scheme-dependent) do not enter the asymptotic spectrum, some of them may (and actually do) enter the spectrum of short operators once the subtraction of range-six contributions is performed. From computations at three and four loops we found in our scheme \[22\]

$$
\epsilon_{3a} = -4, \quad \epsilon_{3b} = -\frac{4}{3}i, \quad \epsilon_{3c} = \frac{4}{3}i, \quad \epsilon_{2a} = -\frac{i}{2}, \quad (3.1)
$$

while all the other $\epsilon_x$ which are relevant here can be computed from range-six diagrams, as explained in appendix A.

Now we can subtract from $D_5$ the contributions of range-six diagrams. As in the four-loop case, these can be divided into two classes:

- the diagrams whose chiral structure has range six can be subtracted simply by erasing the corresponding structures from $D_5$,
- the diagrams with a chiral structure of range not greater than five, which become range-six because of vector interactions, sum up to zero thanks to the general argument described in \[22\].

So we obtain the five-loop subtracted dilatation operator, shown in Table 6, simply by removing all the range-six chiral structures from the full asymptotic operator of Table 5.

Now we can apply this operator to the two length-five states of the $SU(2)$ sector, which mix under renormalization:

$$
O_1 = \text{tr}(\phi Z \phi ZZ), \quad O_2 = \text{tr}(\phi \phi ZZZ). \quad (3.2)
$$

All the short-range chiral structures have an action on these states which is proportional to the mixing matrix

$$
M = \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}. \quad (3.3)
$$

After the range-six subtraction, the matrix expression of the subtracted dilatation operator on the length-five subsector still depends on a subset of the $\epsilon_x$ coefficients (more precisely, it depends on $\epsilon_{4b}, \epsilon_{4f}$). Using the values given in appendix A we find

$$
D_{5}^{\text{sub}} \rightarrow 2(1665 + 104\beta_{2,3,3} - 8\beta_{2,3,4})M. \quad (3.4)
$$

The $\beta_{2,3,3}$ component of the dressing phase is known to be equal to $4\zeta(3)$ \[30\], while for $\beta_{2,3,4}$ we use the value $-40\zeta(5)$ conjectured in \[29\]. So we have

$$
D_{5}^{\text{sub}} \rightarrow 2(1665 + 416\zeta(3) + 320\zeta(5))M. \quad (3.5)
$$
\[ D_5 = -1960 \chi(1) + 1568(\chi(1, 2) + \chi(2, 1)) + 16(\epsilon_{2a}^2 + \epsilon_{3a} + 8\epsilon_{4f} - 8\epsilon_{4b} + \beta_{2,3,3} - 40)\chi(1, 3) + 16(\epsilon_{2a} + \epsilon_{3a} - 16\epsilon_{4f} + 8\epsilon_{4f} + 4)\chi(1, 4) - 4\chi(1, 5) - 784(\chi(1, 2, 3) + \chi(3, 2, 1)) + 2(64 - 8\beta_{2,3,3} + 8\epsilon_{2a}^2 + 4i\epsilon_{2a} + 8\epsilon_{3a} + 2i\epsilon_{3c}) - 32(\epsilon_{4f} + 4\chi(1, 3, 2) + 2(64 - 8\beta_{2,3,3} + 8\epsilon_{2a}^2 - 4i\epsilon_{2a} + 8\epsilon_{3a} - 2i\epsilon_{3c}) - 32(\epsilon_{4f} + 4i\epsilon_{3c}) - 2(30 + \beta_{2,3,3} + 4\epsilon_{2a}^2 - 4i\epsilon_{2a} + 4\epsilon_{3a} + 4i\epsilon_{3b} + 2i\epsilon_{3c}) - 48\epsilon_{4b} - 16i(2\epsilon_{4a} + 2\epsilon_{4c} + 2\epsilon_{4d} + 2\epsilon_{4e} + 2\epsilon_{4f} + 2\epsilon_{4g} + 2\epsilon_{4h} + 4\epsilon_{4m})\chi(1, 3, 4) + \chi(2, 1, 4)) + 2(30 + \beta_{2,3,3} + 4\epsilon_{2a}^2 - 4i\epsilon_{2a} + 4\epsilon_{3a} + 4i\epsilon_{3b} + 2i\epsilon_{3c}) - 48\epsilon_{4b} + 16i(2\epsilon_{4a} + 2\epsilon_{4c} + 2\epsilon_{4d} + 2\epsilon_{4e} + 2\epsilon_{4f} + 2\epsilon_{4g} + 2\epsilon_{4h} + 4\epsilon_{4m})\chi(1, 2, 4) + \chi(1, 4, 3)) - 4(1 - 8i(2\epsilon_{4a} + 2\epsilon_{4c} + 2\epsilon_{4d} + 4\epsilon_{4g} + 4\epsilon_{4h}))(\chi(1, 2, 5) + \chi(1, 5, 4)) - 4(1 + 8i(2\epsilon_{4a} + 2\epsilon_{4c} + 2\epsilon_{4d} + 4\epsilon_{4g} + 4\epsilon_{4h}))(\chi(1, 4, 5) + \chi(2, 1, 5)) - 8\chi(1, 3, 5) + 2(8\beta_{2,3,3} - 3 - 2\epsilon_{2a}^2 - 4\epsilon_{3a} + 64\epsilon_{4b} + 32\epsilon_{4h})\chi(2, 1, 3, 2) + 224(\chi(1, 2, 3, 4) + \chi(4, 2, 3, 1)) - 4(20 + 3\beta_{2,3,3} - 2\epsilon_{2a} - 3\epsilon_{3a} - 16\epsilon_{4f} + 16\epsilon_{4h})(\chi(1, 3, 2, 4) + \chi(2, 1, 4, 3)) + 2(2 - \beta_{2,3,3} - 4i\epsilon_{2a} + 2i\epsilon_{3b} - 32\chi(1, 3, 2, 4) + 16i(\epsilon_{4a} + \epsilon_{4c} + \epsilon_{4d} + \epsilon_{4e} + \epsilon_{4f}))(\chi(2, 1, 3, 4) + \chi(3, 2, 1, 4)) + 2(2 - \beta_{2,3,3} - 4i\epsilon_{2a} + 2i\epsilon_{3b} - 32\chi(1, 3, 2, 4) + 16i(\epsilon_{4a} + \epsilon_{4c} + \epsilon_{4d} + \epsilon_{4e} + \epsilon_{4f}))(\chi(2, 1, 3, 4) + \chi(3, 2, 1, 4)) + 2(2 - \beta_{2,3,3} - 4i\epsilon_{2a} + 2i\epsilon_{3b} - 32\chi(1, 3, 2, 4) + 16i(\epsilon_{4a} + \epsilon_{4c} + \epsilon_{4d} + \epsilon_{4e} + \epsilon_{4f}))(\chi(2, 1, 3, 4) + \chi(3, 2, 1, 4))
\[ D_{\text{sub}}^5 = -1960 \chi(1) + 1568(\chi(1,2) + \chi(2,1)) + 16(\varepsilon_{2a}^2 + \varepsilon_{3a} + 8\varepsilon_{4f} - 8\varepsilon_{4b} + \beta_{2,3,3} - 40)\chi(1,3) \\
+ 16(\varepsilon_{2a}^2 + \varepsilon_{3a} - 16\varepsilon_{4b} - 8\varepsilon_{4f} + 4)\chi(1,4) - 784(\chi(1,2,3) + \chi(3,2,1)) \\
+ 2(64 - 8\beta_{2,3,4} + 8\varepsilon_{2a}^2 + 4i\varepsilon_{2a} + 8\varepsilon_{3a} + 2i\varepsilon_{ac} \\
- 32(\varepsilon_{4b} + \varepsilon_{4d}) + 32i(\varepsilon_{4a} + \varepsilon_{4c} + \varepsilon_{4d} + \varepsilon_{4e} + \varepsilon_{4f})\chi(1,3,2) \\
+ 2(64 - 8\beta_{2,3,4} + 8\varepsilon_{2a}^2 - 4i\varepsilon_{3a} + 8\varepsilon_{3a} - 2i\varepsilon_{ac} \\
- 32(\varepsilon_{4b} + \varepsilon_{4d}) - 32i(\varepsilon_{4a} + \varepsilon_{4c} + \varepsilon_{4d} + \varepsilon_{4e} + \varepsilon_{4f} + \varepsilon_{4g} + \varepsilon_{4h})\chi(2,1,3) \\
+ 2(30 + \beta_{2,3,4} + 4\varepsilon_{2a}^2 - 4i\varepsilon_{2a} + 4\varepsilon_{3a} + 4i\varepsilon_{3b} + 2i\varepsilon_{ac} \\
- 48\varepsilon_{4b} - 16i(2\varepsilon_{4a} + 2\varepsilon_{4d} + \varepsilon_{4e} + 2\varepsilon_{4g} + 2\varepsilon_{4j} + \varepsilon_{4m})\chi(1,3,4) + \chi(2,1,4)) \\
+ 2(30 + \beta_{2,3,4} + 4\varepsilon_{2a}^2 + 4i\varepsilon_{2a} + 4\varepsilon_{3a} - 4i\varepsilon_{3b} - 2i\varepsilon_{ac} \\
- 48\varepsilon_{4b} + 16i(2\varepsilon_{4a} + 2\varepsilon_{4d} + \varepsilon_{4e} + 2\varepsilon_{4g} + 2\varepsilon_{4j} + \varepsilon_{4m})\chi(1,2,4) + \chi(1,4,3)) \\
+ 2(8\beta_{2,3,4} - \beta_{2,3,4} - 4\varepsilon_{2a}^2 - 4\varepsilon_{3a} + 64\varepsilon_{4b} + 32\varepsilon_{4b})\chi(2,1,3,2) \\
+ 224(\chi(1,2,3,4) + \chi(4,3,2,1)) \\
- 4(20 - 3\beta_{2,3,4} - 4\varepsilon_{2a}^2 - 4\varepsilon_{3a} - 16\varepsilon_{4f} + 16\varepsilon_{4b})\chi(1,3,2,4) + \chi(2,1,4,3)) \\
+ 2(4 - \beta_{2,3,4} - 4i\varepsilon_{2a} + 2i\varepsilon_{3b} \\
- 32\varepsilon_{4b} - 16\varepsilon_{4f} + 16\varepsilon_{4b} - 16i(\varepsilon_{4a} + \varepsilon_{4c} + \varepsilon_{4d} + \varepsilon_{4e} + \varepsilon_{4j}))\chi(2,1,3,4) + \chi(3,2,1,4) \\
+ 2(4 - \beta_{2,3,4} + 4i\varepsilon_{2a} - 2i\varepsilon_{3b} \\
- 32\varepsilon_{4b} - 16\varepsilon_{4f} + 16\varepsilon_{4b} + 16i(\varepsilon_{4a} + \varepsilon_{4c} + \varepsilon_{4d} + \varepsilon_{4e} + \varepsilon_{4j})\chi(1,2,4,3) + \chi(1,4,3,2)) \\
+ 2(10 - \beta_{2,3,4} + 16\varepsilon_{4b} + 16i\varepsilon_{4e})\chi(1,3,2,4,3) + \chi(2,1,4,3,2)) \\
+ 2(10 - \beta_{2,3,4} + 16\varepsilon_{4b} - 16i\varepsilon_{4e})\chi(2,1,3,2,4) + \chi(3,2,1,4,3))
\]

Table 6: Subtracted five-loop dilatation operator

### 3.2 Wrapping diagrams

We now consider the contribution from wrapping diagrams. First of all, we must list all the possible wrapping graphs and classify them according to their chiral structures. Then, thanks to the general argument described in [22], several pairs of diagrams can be immediately cancelled. For the remaining diagrams, we apply the standard D-algebra procedure and obtain momentum integrals which can be computed using the GPXT [22, 27, 28]. The relevant diagrams after pair cancellations are listed in appendix [B].

There are three completely chiral wrapping structures, shown in figure [B.1]. By identifying the sixth and the first lines of the diagrams, these structures can be written as $\chi(2,1,3,4,5)$, $\chi(1,2,3,4,5)$ and $\chi(1,3,2,5,4)$, respectively. All the other wrapping diagrams can be obtained from range-five graphs by adding a wrapping vector interaction, and so have chiral structures of range not greater than five.

A minimal set of *independent* chiral structures, with up to four loops and range less...
than or equal to five, can be chosen as $\chi(2, 4, 1, 3), \chi(3, 2, 1, 4), \chi(1, 2, 3, 4), \chi(1, 4, 3, 2), \chi(1, 3, 2), \chi(2, 1, 3), \chi(1, 2, 3), \chi(2, 1, 4), \chi(2, 1), \chi(1, 4)$ and $\chi(1)$. All the other structures with the required range, which appear in the expansion of the subtracted dilatation operator, are either a reflection or simply a different way of writing one element of the minimal set. In particular, when acting on a length-five state, $\chi(1, 2, 4)$ and $\chi(1, 3)$ are equivalent to $\chi(2, 1, 4)$ and $\chi(1, 4)$, respectively. The wrapping diagrams for all the independent structures, together with the results of D-algebra and color factors for the length-five states, are shown in Figs. B.2–B.12. For non-symmetric structures, the corresponding reflection is indicated.

We now collect all the contributions to find the leading wrapping correction to the dilatation operator:

$$D^w_5 = -10 \left( -\left(\frac{1}{6} - \frac{4}{5} \zeta(3)\right) (\chi(2, 1, 3, 4) + \chi(4, 5, 3, 2, 1)) + \left(\frac{14}{5} - 4\zeta(5)\right) (\chi(1, 2, 3, 4) + \chi(5, 4, 3, 2, 1)) - \left(\frac{19}{30} - \frac{4}{5} \zeta(3)\right) (\chi(1, 3, 2, 5, 4) + \chi(5, 3, 4, 1, 2)) - \left(\frac{1}{3} + \frac{12}{5} \zeta(3) - 4\zeta(5)\right) (\chi(2, 4, 1, 3) + \chi(1, 3, 2, 4)) + \left(\frac{1}{3} - \frac{12}{5} \zeta(3) + 4\zeta(5)\right) (\chi(3, 2, 1, 4) + \chi(2, 1, 3, 4)) + (8\zeta(5) - 14\zeta(7)) (\chi(1, 2, 3, 4) + \chi(4, 3, 2, 1)) - \left(\frac{2}{5} + \frac{12}{5} \zeta(3) - 4\zeta(5)\right) (\chi(1, 4, 3, 2) + \chi(1, 2, 4, 3)) + \left(\frac{13}{10} + \frac{8}{5} \zeta(3)\right) \chi(1, 3, 2) + \left(\frac{19}{10} + \frac{8}{5} \zeta(3)\right) \chi(2, 1, 3) + \left(\frac{18}{5} + \frac{44}{5} \zeta(3) - 12\zeta(5)\right) (\chi(2, 1, 4) + \chi(1, 3, 4)) - (8\zeta(5) - 14\zeta(7)) (\chi(2, 1) + \chi(1, 2)) - \left(\frac{18}{5} + 8\zeta(3) + 8\zeta(5) - 28\zeta(7)\right) \chi(1, 4) + 8\zeta(5) \chi(1) \right).$$

In the basis (3.2) this expression reads

$$D^w_5 \rightarrow 2(1 - 128\zeta(3) + 640\zeta(5) - 560\zeta(7)) M.$$  

The correct five-loop dilatation operator for the length-five states is obtained by adding this wrapping contribution to the subtracted operator of Table 6. The matrix form of
the result is
\[ D_4^{\text{sub}} + D_4^{w} \rightarrow 4(833 + 144\zeta(3) + 480\zeta(5) - 280\zeta(7))M. \] (3.8)

Since the asymptotic dilatation operator in the \(SU(2)\) subsector is proportional to the mixing matrix \([3.3]\) also at all the lower loop orders, the five-loop part of the anomalous dimension of the non-protected eigenstate is simply the non-zero eigenvalue of \(D_5^{\text{sub}} + D_5^{w}\):
\[ \gamma_5 = 6664 + 1152\zeta(3) + 3840\zeta(5) - 2240\zeta(7). \] (3.9)

Including the lower orders, the anomalous dimension up to five loops thus reads
\[ \gamma = 8\lambda - 24\lambda^2 + 136\lambda^3 - 8[115 + 16\zeta(3)]\lambda^4 + [6664 + 1152\zeta(3) + 3840\zeta(5) - 2240\zeta(7)]\lambda^5. \] (3.10)

This result coincides with the one presented in \([11]\) for the anomalous dimension of the twist-three operator in the \(SL(2)\) sector. This last operator and our operator \(\text{tr}(\phi Z\phi ZZ - \phi \phi ZZ Z)\) belong indeed to the same supermultiplet.\(^2\) Our result gives then a direct field theoretical confirmation of the conjectures and assumptions made in \([11]\).

### 4 Computation with the Y-system

In this section we want to compare the result of our direct, field-theoretical computation with the value of the anomalous dimension which is obtained using the Y-system technique \([13, 14, 19]\) when applied to the twist-three operator of the \(SL(2)\) sector.

To compute the leading wrapping correction to the anomalous dimension, we extend to five loops the explicit computation of \([13]\) for the four-loop case. In order to restore the required dependence of \(Y_{a,0}^{*}(u)\) on \(L\) and on the Bethe root \(u_{4,1}\), we need to repeat all the steps of that computation.

We start from the general expression for \(Y_{a,0}^{*}(u)\) \([13]\):
\[ Y_{a,0}^{*}(u) = \left( \frac{x[-a]}{x[a]} \right)^{L} f^{[-a]} \phi^{[a]} T_{a,-1}^{L} T_{a,1}^{R}, \] (4.1)

where \(f^{[\pm a]}(u) = f(u \pm ia/2)\), \(x\) is a function of \(u\) defined by \(u/\sqrt{\lambda} = x + 1/x\), \(\phi\) is a fixed function of \(u\) whose expression is given in \([13]\) and \(T_{a,-1}^{L}\) and \(T_{a,1}^{R}\) are the transfer

\(^2\)All two-impurity states belong to supermultiplets which have representatives both in \(SU(2)\) and \(SL(2)\) \([4]\). In particular, the two-impurity, length-\(L\) subsector of \(SU(2)\) is mapped onto the two-impurity, length-(\(L - 2\)) subsector of \(SL(2)\). So in our case we have to consider \(L = 3\).
matrix eigenvalues of anti-symmetric irreducible representations of the $SU(2|2)_L$ and $SU(2|2)_R$ subgroups of the full $SU(2,2|4)$ symmetry.

Since for a state in the $SL(2)$ sector the only non-zero Bethe roots are those of type $u_{4,j}$, at leading order the term involving the function $\phi$ simplifies. In the notation of [13] it becomes

$$\frac{\phi_{[-a]}}{\phi_{[+a]}} = \frac{B_{[-a][+a]}R_{[-a][-a]}R_{[-a][+a]}(+)}{B_{[-a][-a]}R_{[-a][+a]}(+)} .$$

(4.2)

The contribution from the first two factors of (4.1) hence yields

$$\frac{x_{[+a]}}{x_{[-a]}} \frac{\phi_{[-a]}}{\phi_{[+a]}} \rightarrow 92^{2L} \lambda^L (4u_{4,1}^2 + 1)^2 \frac{2}{(a^2 + 4u^2)L \bar{y}_{-a}(u)} ,$$

(4.3)

where

$$\bar{y}_{a}(u) = 9[((1 - a)^2 + 4u^2)^2 + 8u_{4,1}^2((1 - a)^2 - 4u^2 + 2u_{4,1}^2)].$$

(4.4)

The action of the two $SU(2|2)$ subgroups of the full symmetry group is the same on the $SL(2)$ sector and therefore $T_{a,-1}^L$ and $T_{a,1}^R$ can be computed from the same generating functional

$$W = \left[ 1 - \frac{B^+(+)R^-(+)}{B^(-)R^(-)D} \right] \left[ 1 - \frac{R^+(+)R^-(+)}{R^(-)D} \right]^{-2} \left[ 1 - D \right] , \quad D = e^{-i\partial_a} ,$$

(4.5)

using

$$W^{-1} = \sum_{a=0}^{\infty} (-1)^a T_{a,1}^{[-a]} D^a .$$

(4.6)

In this way we obtain the contribution from the third factor of (4.1) as

$$T_{a,-1}^L T_{a,1}^R \rightarrow 2^{10} \lambda^2 \frac{[12a(u^2 - u_{4,1}^2) + 3a(a^2 - 1)]^2}{(4u_{4,1}^2 + 1)^2 (a^2 + 4u^2)^2 \bar{y}_{a}(u)} .$$

(4.7)

Putting all the factors together, we find

$$Y_{a,0}(u) = 9\lambda^L + 2^{10} + 2L \frac{[12a(u^2 - u_{4,1}^2) + 3a(a^2 - 1)]^2}{(a^2 + 4u^2)^L + 2 \bar{y}_{a}(u)\bar{y}_{-a}(u)} .$$

(4.8)

Here, the root $u_{4,1}$ is the solution of the two-impurity Bethe equations for the $SL(2)$ sector, which at order $\lambda^0$ read

$$\left( \frac{u_{4,1} + i/2}{u_{4,1} - i/2} \right)^{L+1} = 1 .$$

(4.9)
For $L = 3$ we get $u_{4,1} = 1/2$. Using the expression for $Y_{a,0}^*(u)$, the leading wrapping contribution to the anomalous dimension can be found with the help of the formula [13]

$$
\delta \gamma_{L+2} = \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial Y_{a,0}^*}{\partial u} \log[1 + Y_{a,0}^* (u)] . \tag{4.10}
$$

At the lowest order in $\lambda$, we find

$$
\frac{\partial Y_{a,0}^*}{\partial u} = -2i . \tag{4.11}
$$

Moreover, since $Y_{a,0}^*(u) \sim \lambda^{L+2}$, we can approximate $\log[1 + Y_{a,0}^* (u)] = Y_{a,0}^* (u) + o(\lambda^{L+2})$, so that at leading order we have

$$
\delta \gamma_{L+2} = -\frac{1}{\pi} \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} Y_{a,0}^* (u) . \tag{4.12}
$$

The integral can now be computed by using the residue method, closing the integration path at infinity in the upper half of the complex plane. In the end, the following result is found at five loops

$$
\delta \gamma_5 = -\lambda^5 [128 + 512 \zeta(3) - 2560 \zeta(5) + 2240 \zeta(7)] . \tag{4.13}
$$

This is the correction which must be added to the asymptotic five-loop anomalous dimension computed using the Bethe equations:

$$
\gamma_5^{\text{as}} = \lambda^5 [6792 + 1664 \zeta(3) + 1280 \zeta(5)] . \tag{4.14}
$$

The five-loop contribution to the anomalous dimension of the two-impurity, length-five operator is

$$
\gamma_5 = \gamma_5^{\text{as}} + \delta \gamma_5 = \lambda^5 [6664 + 1152 \zeta(3) + 3840 \zeta(5) - 2240 \zeta(7)] , \tag{4.15}
$$

which agrees with our result.

5 Concluding remarks

In this paper we have computed perturbatively the planar anomalous dimension of a length-five operator at five loops by extending the procedure of our paper [22]. Moreover,
we have explicitly shown that our result agrees with the one obtained by applying the Y-system proposed in [13, 14].

The anomalous dimension computed here was predicted in [11] from a general formula for the five-loop anomalous dimension of twist-three operators. This formula was conjectured on the basis of the maximal transcendentality principle [25].

With our calculation we have then given a new test of these existing proposals on how to compute the anomalous dimension of short operators.

It would be important to test the recently obtained five-loop anomalous dimension of the Konishi operator [12]. However, this would require a calculation beyond critical wrapping order. The complexity of the calculation in this case increases dramatically, even in the context of the $\mathcal{N} = 1$ superspace techniques used here. In order to make this calculation manageable it would be necessary to find new cancellation patterns beyond the ones discovered in [22, 26].

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A Range-six diagrams

In this appendix we consider the range-six diagrams relevant for the computation of the $\epsilon_x$ coefficients, which enter the subtracted dilatation operator on length-five states. To this end, we compute all the range-six diagrams which are either completely chiral (Fig. A.1) or contain a single vector interaction (Figs. A.3-A.9), and the two completely chiral range-five graphs (Fig. A.2). In all the figures containing lists of diagrams, the symmetry factor is explicitly shown when it differs from 1. By comparing the results from these diagrams with the corresponding coefficients in the five-loop asymptotic dilatation operator, we find a set of relations for the coefficients. Some of these can be used to determine the needed $\epsilon_x$ coefficients, while all the other ones reduce to identities which allow us to perform consistency checks.

Let us define $C[\chi(\ldots)]$ as the coefficient of the chiral structure $\chi(\ldots)$ in the asymptotic dilatation operator $D_5$ of Table 5. $C[\chi(\ldots)]$ is computed from the coefficient of the $1/\epsilon$ pole of the sum of all the diagrams with structure $\chi(\ldots)$, multiplied by a factor $-10$ according to the definition of the anomalous dimension which, in the case of a multiplicatively renormalized operator, is simply given by

$$\gamma(\mathcal{O}) = \lim_{\epsilon \to 0} \left[ \epsilon g \frac{d}{dg} \log Z_O(g, \epsilon) \right], \quad (A.1)$$

where

$$\mathcal{O}_{\text{ren}} = Z_O \mathcal{O}_{\text{bare}} \quad (A.2)$$

and $\epsilon$ is the dimensional regularization parameter. The $(g^2 N)^5$ factor coming from color and the $1/(4\pi)^{10}$ from the momentum integrals combine into the $\lambda^5$ coupling (2.1) which multiplies the five-loop Hamiltonian, and is not shown explicitly.

The constraints from diagrams are summarized in Table A.1 where the results are written in terms of the momentum integrals $J_i$, which are listed in Appendix C. The partial contributions coming from the single diagrams with vector interactions are given in Table A.2 where the final results for the chiral structures already contain all the symmetry factors and contributions from possible reflected structures.

Using the conditions on the coefficients of the dilatation operator, several equations relating a subset of the $\epsilon_{4x}$ coefficients can be found:

$$\begin{align*}
\epsilon_{4a} &= \frac{13}{64} i, \\
\epsilon_{4b} &= -\frac{85}{192} i, \\
\epsilon_{4c} &= \frac{i}{96}, \\
\epsilon_{4d} &= -\frac{5}{192} i, \\
\epsilon_{4e} &= \frac{\pi^4 - 75}{960} i, \\
\epsilon_{4f} &= \frac{35}{96}, \\
\epsilon_{4g} &= -\frac{3}{32} i.
\end{align*} \quad (A.3)$$

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$C[\chi(1, 2, 3, 4, 5)] = -10 J_1 = -28$
$C[\chi(2, 1, 4, 3, 5)] = -10 J_2 = -8/3$
$C[\chi(1, 3, 2, 5, 4)] = -10 J_3 = 2/3$
$C[\chi(1, 2, 5, 4, 3)] = -10 J_4 = -4$
$C[\chi(1, 2, 4, 3, 5)] = -10 J_5 = 5$
$C[\chi(3, 2, 1, 4, 5)] = -10 J_6 = -8/3$
$C[\chi(1, 3, 2, 4, 5)] = -10 J_7 = 16/3$
$C[\chi(1, 2, 3, 5, 4)] = -10 J_8 = -11/3$
$C[\chi(2, 1, 3, 4, 5)] = -10 J_9 = 28/3$
$C[\chi(1, 4, 3, 2, 5)] = -10 J_{10} = 1/3$
$C[\chi(1, 5, 4, 3)] = -10(-2 J_{11}) = -46/3$
$C[\chi(1, 3, 4, 5)] = -10(-2 J_{12}) = -8$
$C[\chi(1, 2, 4, 5)] = -10(-2 J_{13}) = -26/3$
$C[\chi(1, 2, 5, 4)] = -10(-2 J_{14}) = -16/3$
$C[\chi(2, 1, 4, 5)] = -10(-2 J_{15}) = 20/3$
$C[\chi(1, 4, 3, 5)] = -10(-2 J_{16}) = 6$
$C[\chi(1, 3, 2, 5)] = -10(-2 J_{17}) = 4/3$
$C[\chi(1, 3, 2, 4, 3)] = -10 J_{18} = 25/3 - \pi^4/30 - 8\zeta(3)$
$C[\chi(2, 1, 3, 2, 4)] = -10 J_{19} = 10/3 + \pi^4/30 - 8\zeta(3)$

Table A.1: Constraints on the coefficients of the dilatation operator

These relations are sufficient to completely determine the action of the subtracted dilatation operator on the length-five states.
Figure A.1: Completely chiral range-six diagrams

Figure A.2: Completely chiral range-five diagrams

Figure A.3: Range-six diagrams with structure $\chi(1, 2, 4, 5)$
Figure A.4: Range-six diagrams with structure $\chi(1, 5, 4, 3)$

Figure A.5: Range-six diagrams with structure $\chi(1, 3, 4, 5)$

Figure A.6: Range-six diagrams with structure $\chi(1, 4, 3, 5)$

Figure A.7: Range-six diagrams with structure $\chi(1, 3, 2, 5)$

Figure A.8: Range-six diagrams with structure $\chi(1, 2, 5, 4)$
Table A.2: Results of D-algebra for diagrams with vector interactions

| Diagram Structure | Relations | Resultant Diagrams |
|-------------------|-----------|--------------------|
| $\chi(1, 2, 4, 5)$ | $G_1 \rightarrow J_{10}$, $G_3 \rightarrow -(J_{10} + J_1 + 2J_{13})$ | $G_2 \rightarrow -J_7$, $G_4 \rightarrow J_7 + J_1$ |
| $\chi(1, 5, 4, 3)$ | $G_1 \rightarrow 0$, $G_3 \rightarrow J_4$ | $G_2 \rightarrow J_1$, $G_4 \rightarrow -(J_1 + J_4 + 2J_{11})$ |
| $\chi(1, 3, 4, 5)$ | $G_1 \rightarrow J_3$, $G_3 \rightarrow J_1 + J_9$ | $G_2 \rightarrow -J_9$, $G_4 \rightarrow -(J_3 + J_1 + 2J_{12})$ |
| $\chi(1, 4, 3, 5)$ | $G_1 \rightarrow -(J_9 + J_5 + 2J_{16})$, $G_3 \rightarrow J_9$ | $G_2 \rightarrow J_5$, $G_4 \rightarrow 0$ |
| $\chi(1, 2, 5, 4)$ | $G_1 \rightarrow -2(J_8 + J_{14})$, $G_3 \rightarrow 0$ | $G_2 \rightarrow J_8$ |
| $\chi(2, 1, 4, 5)$ | $G_1 \rightarrow -2(J_9 + J_{15})$, $G_3 \rightarrow 0$ | $G_2 \rightarrow J_9$ |
| $\chi(1, 3, 2, 5)$ | $G_1 \rightarrow -(J_7 + J_8 + 2J_{17})$, $G_3 \rightarrow J_8$ | $G_2 \rightarrow J_7$, $G_4 \rightarrow 0$ |

Figure A.9: Range-six diagrams with structure $\chi(2, 1, 4, 5)$
B Wrapping

In this appendix we list all the relevant wrapping diagrams. For the non-symmetric chiral structures, the corresponding reflections are indicated in the figure captions. Note that for the length-five states, the structures $\chi(1, 2, 4)$ and $\chi(1, 4, 3)$ are the same as $\chi(2, 1, 4)$ and its reflection $\chi(1, 3, 4)$. Similarly, $\chi(1, 3)$ is the same structure as $\chi(1, 4)$.

The wrapping diagrams we need to compute and the corresponding contributions are listed in Figs. B.1-B.11. In each case, the final result for the whole structure already contains all the symmetry factors and the contribution for the possible reflected structure. The symmetry factor of a diagram is explicitly shown if different from 1. In each diagram, the color factor $(g^2N)^5$ combines with the $1/(4\pi)^{10}$ from the momentum integral to produce the coupling $\lambda^5$ as defined in (2.1), and thus it is not shown explicitly.

![Wrapping Diagrams](image)

| Diagram | Contribution |
|---------|--------------|
| $G_1 \to \chi(2, 1, 3, 4, 5) J_{22} \to 4M J_{22}$ |
| $G_2 \to \chi(1, 2, 3, 4, 5) J_{20} \to M J_{20}$ |
| $G_3 \to \chi(1, 3, 2, 5, 4) J_{21} \to M J_{21}$ |

$\chi_{\text{chiral}} \to 2J_{22} \chi(2, 1, 3, 4, 5) + 2J_{20} \chi(1, 2, 3, 4, 5) + 2J_{21} \chi(1, 3, 2, 5, 4) - 2M(J_{20} + J_{21} + 4J_{22})$

Figure B.1: Wrapping diagrams with completely chiral structure
Figure B.2: Wrapping diagrams with structure $\chi(2, 4, 1, 3)$ or $\chi(1, 3, 2, 4)$

\[
G_1 \rightarrow \chi(2, 4, 1, 3)(-J_{20} - J_{21} - 2J_{24}) \rightarrow M(-J_{20} - J_{21} - 2J_{24}) \\
G_2 \rightarrow \chi(2, 4, 1, 3)J_{20} \rightarrow M J_{20} \\
G_3 \rightarrow \chi(2, 4, 1, 3)J_{21} \rightarrow M J_{21} \\
G_4 \rightarrow 0
\]

$\chi(2, 4, 1, 3) \rightarrow -4J_{24} \chi(2, 4, 1, 3) \rightarrow -4M J_{24}$

Figure B.3: Wrapping diagrams with structure $\chi(3, 2, 1, 4)$ or $\chi(2, 1, 3, 4)$

\[
G_1 \rightarrow \chi(3, 2, 1, 4)(-J_{20} - J_{22} - 2J_{25}) \rightarrow -2M(-J_{20} - J_{22} - 2J_{25}) \\
G_2 \rightarrow -\chi(3, 2, 1, 4)(-J_{20}) \rightarrow 2M(-J_{20}) \\
G_3 \rightarrow -\chi(3, 2, 1, 4)(-J_{22}) \rightarrow 2M(-J_{22}) \\
G_4 \rightarrow 0
\]

$\chi(3, 2, 1, 4) \rightarrow -4J_{25} \chi(3, 2, 1, 4) \rightarrow 8M J_{25}$
\[ G_1 \rightarrow \chi(1, 2, 3, 4)(-J_{22} - J_{23} - 2J_{26}) \rightarrow -2M(-J_{22} - J_{23} - 2J_{26}) \]
\[ G_2 \rightarrow -\chi(1, 2, 3, 4)(-J_{22}) \rightarrow 2M(-J_{22}) \]
\[ G_3 \rightarrow -\chi(1, 2, 3, 4)(-J_{20}) \rightarrow 2M(-J_{20}) \]
\[ G_4 \rightarrow \chi(1, 2, 3, 4)(-J_{27}) \rightarrow 8M(-J_{27}) \]

Figure B.4: Wrapping diagrams with structure \( \chi(1, 2, 3, 4) \) or \( \chi(4, 3, 2, 1) \)

\[ \chi(1, 2, 3, 4) \rightarrow -2\chi(1, 2, 3, 4)(-J_{1} + J_{20} + J_{23} + 2J_{26}) \]
\[ \rightarrow 4M(-J_{1} + J_{20} + J_{23} + 2J_{26}) \]

\[ \chi(1, 4, 3, 2) \rightarrow -4J_{27}\chi(1, 4, 3, 2) \rightarrow 8M J_{27} \]

Figure B.5: Wrapping diagrams with structure \( \chi(1, 4, 3, 2) \) or \( \chi(1, 2, 4, 3) \)
\[ G_1 \rightarrow -\chi(1, 3, 2)(2J_{20} + 2J_{29}) \rightarrow 2M(2J_{20} + 2J_{29}) \]
\[ G_2 \rightarrow \chi(1, 3, 2) J_{20} \rightarrow -2M J_{20} \]
\[ G_3 \rightarrow 0 \]
\[ \chi(1, 3, 2) \rightarrow -2J_{29} \chi(1, 3, 2) \rightarrow 4M J_{29} \]

Figure B.6: Wrapping diagrams with structure \( \chi(1, 3, 2) \)

\[ G_1 \rightarrow -\chi(2, 1, 3)(2J_{20} + 2J_{28}) \rightarrow 2M(2J_{20} + 2J_{28}) \]
\[ G_2 \rightarrow \chi(2, 1, 3) J_{20} \rightarrow -2M J_{20} \]
\[ G_3 \rightarrow 0 \]
\[ \chi(2, 1, 3) \rightarrow -2J_{28} \chi(2, 1, 3) \rightarrow 4M J_{28} \]

Figure B.7: Wrapping diagrams with structure \( \chi(2, 1, 3) \)

\[ G_1 \rightarrow -\chi(1, 2, 3)(J_{22} + J_{23} + 2J_{26}) \rightarrow -M(J_{22} + J_{23} + 2J_{26}) \]
\[ G_2 \rightarrow \chi(1, 2, 3) J_{22} \rightarrow M J_{22} \]
\[ G_3 \rightarrow \chi(1, 2, 3)(J_{22} + J_{23} + 2J_{26}) \rightarrow M(J_{22} + J_{23} + 2J_{26}) \]
\[ G_4 \rightarrow -\chi(1, 2, 3) J_{22} \rightarrow -M J_{22} \]
\[ \chi(1, 2, 3) \rightarrow 0 \]

Figure B.8: Wrapping diagrams with structure \( \chi(1, 2, 3) \) or \( \chi(3, 2, 1) \)
\[ G_1 \rightarrow \chi(2, 1, 4)(-J_{22} - J_{23} - 2J_{26}) \rightarrow M(-J_{22} - J_{23} - 2J_{26}) \]
\[ G_2 \rightarrow -\chi(2, 1, 4) J_{22} \rightarrow -M J_{22} \]
\[ G_3 \rightarrow -\chi(2, 1, 4)(-J_{22}) \rightarrow -M(-J_{22}) \]
\[ G_4 \rightarrow -\chi(2, 1, 4)(-J_{21} - J_{23} + (-2J_{32} + 2J_{33} + 2J_{34} - 2i \epsilon_{\mu\nu\rho\sigma} j_{35}^{\mu\rho\nu}) \]
\[ \rightarrow -M(-J_{21} - J_{23} + (-2J_{32} + 2J_{33} + 2J_{34} - 2i \epsilon_{\mu\nu\rho\sigma} j_{35}^{\mu\rho\nu})) \]
\[ G_5 \rightarrow \chi(2, 1, 4) J_{20} \rightarrow M J_{20} \]
\[ G_6 \rightarrow \chi(2, 1, 4)(-J_{20}) \rightarrow M(-J_{20}) \]
\[ G_7 \rightarrow -\chi(2, 1, 4)(-J_{22}) \rightarrow -M(-J_{22}) \]
\[ G_8 \rightarrow \chi(2, 1, 4)(-J_{21}) \rightarrow M(-J_{21}) \]
\[ G_9 \rightarrow -\chi(2, 1, 4)(-J_{20} - 2J_{22} - J_{23} - 2J_{25} - 4J_{26} - 2J_{27} - 4J_{30}) \]
\[ \rightarrow -M(-J_{20} - 2J_{22} - J_{23} - 2J_{25} - 4J_{26} - 2J_{27} - 4J_{30}) \]
\[ G_{10} \rightarrow \chi(2, 1, 4)(-J_{20} - J_{22} - 2J_{25}) \rightarrow M(-J_{20} - J_{22} - 2J_{25}) \]
\[ G_{11} \rightarrow \chi(2, 1, 4)(-J_{22} - J_{23} - 2J_{26}) \rightarrow M(-J_{22} - J_{23} - 2J_{26}) \]
\[ G_{12} = G_{13} = G_{14} = G_{15} \rightarrow 0 \]
\[ G_{16} \rightarrow \chi(2, 1, 4)(-J_{21}) \rightarrow M(-J_{21}) \]
\[ G_{17} = G_{18} = G_{19} \rightarrow 0 \]
\[ G_{20} \rightarrow -\chi(2, 1, 4)(-J_{21}) \rightarrow -M(-J_{21}) \]
\[ G_{21} \rightarrow \chi(2, 1, 4)(-J_{20} - J_{22} - 2J_{27}) \rightarrow M(-J_{20} - J_{22} - 2J_{27}) \]
\[ G_{22} \rightarrow -\chi(2, 1, 4)(-J_{20}) \rightarrow -M(-J_{20}) \]
\[ G_{23} \rightarrow -\chi(2, 1, 4)(-J_{22}) \rightarrow -M(-J_{22}) \]
\[ G_{24} \rightarrow 0 \]

\[ \chi(2, 1, 4) \rightarrow 4(2J_{30} + J_{32} - J_{33} - J_{34} + i \epsilon_{\mu\nu\rho\sigma} j_{35}^{\mu\rho\nu}) \chi(2, 1, 4) \]
\[ \rightarrow 4 M (2J_{30} + J_{32} - J_{33} - J_{34} + i \epsilon_{\mu\nu\rho\sigma} j_{35}^{\mu\rho\nu}) \]

Figure B.9: Wrapping diagrams with structure \( \chi(2, 1, 4) \) or \( \chi(1, 3, 4) \)
\[ G_1 \rightarrow \chi(2, 1)(-J_1) \rightarrow M(-J_1) \]
\[ G_2 \rightarrow -\chi(2, 1)(-J_{22} - J_{23} - 2J_{26}) \rightarrow -M(-J_{22} - J_{23} - 2J_{26}) \]
\[ G_3 \rightarrow -\chi(2, 1)(-J_{20}) \rightarrow -M(-J_{20}) \]
\[ G_4 \rightarrow \chi(2, 1)(-J_{22}) \rightarrow M(-J_{22}) \]

\[ \chi(2, 1) \rightarrow -2\chi(2, 1)(J_1 - J_{20} - J_{23} - 2J_{26}) \]
\[ \quad \rightarrow -2M(J_1 - J_{20} - J_{23} - 2J_{26}) \]

Figure B.10: Wrapping diagrams with structure \( \chi(2, 1) \) or \( \chi(1, 2) \)

\[ G_1 \rightarrow \chi(1) J_1 \rightarrow -2M J_1 \]
\[ G_2 \rightarrow -\chi(1) J_{20} \rightarrow 2M J_{20} \]
\[ \chi(1) \rightarrow 2\chi(1)(J_1 - J_{20}) \rightarrow -4M(J_1 - J_{20}) \]

Figure B.11: Wrapping diagrams with structure \( \chi(1) \)
\[ G_1 \rightarrow \chi(1, 4)(J_{21} + J_{23} - (-2J_{32} + 2J_{33} + 2J_{34} - 2i \epsilon_{\mu\nu\rho\sigma} J_{35}^{\mu\nu\rho\sigma})) \]
\[ \rightarrow M(J_{21} + J_{23} - (-2J_{32} + 2J_{33} + 2J_{34} - 2i \epsilon_{\mu\nu\rho\sigma} J_{35}^{\mu\nu\rho\sigma})) \]
\[ G_2 \rightarrow 0 \]
\[ G_3 \rightarrow -\chi(1, 4) J_{21} \rightarrow -M J_{21} \]
\[ G_4 \rightarrow -\chi(1, 4)(J_{22} + J_{23} + 2J_{26}) \rightarrow -M(J_{22} + J_{23} + 2J_{26}) \]
\[ G_5 \rightarrow \chi(1, 4)(-J_{22}) \rightarrow M(-J_{22}) \]
\[ G_6 \rightarrow \chi(1, 4) J_{22} \rightarrow M J_{22} \]
\[ G_7 \rightarrow 0 \]
\[ G_8 \rightarrow \chi(1, 4) J_{22} \rightarrow M J_{22} \]
\[ G_9 \rightarrow -\chi(1, 4)(J_{22} + J_{23} + 2J_{26}) \rightarrow -M(J_{22} + J_{23} + 2J_{26}) \]
\[ G_{10} \rightarrow 0 \]
\[ G_{11} \rightarrow \chi(1, 4)(2J_{22} + 2J_{23} + 8J_{26} + 4J_{31}) \]
\[ \rightarrow M(2J_{22} + 2J_{23} + 8J_{26} + 4J_{31}) \]
\[ G_{12} = G_{13} = G_{14} \rightarrow 0 \]
\[ G_{15} \rightarrow -\chi(1, 4) J_{21} \rightarrow -M J_{21} \]
\[ G_{16} \rightarrow \chi(1, 4) J_{21} \rightarrow M J_{21} \]
\[ G_{17} \rightarrow 0 \]
\[ G_{18} = G_{19} = G_{20} \rightarrow 0 \]

\[ \chi(1, 4) \rightarrow 4(J_{31} + J_{32} - J_{33} - J_{34} + i \epsilon_{\mu\nu\rho\sigma} J_{35}^{\mu\nu\rho\sigma}) \chi(1, 4) \]
\[ \rightarrow 4 M (J_{31} + J_{32} - J_{33} - J_{34} + i \epsilon_{\mu\nu\rho\sigma} J_{35}^{\mu\nu\rho\sigma}) \chi(1, 4) \]

Figure B.12: Wrapping diagrams with structure \( \chi(1, 4) \)
C  Integrals

In this appendix we list the pole parts of all the required logarithmically divergent momentum integrals. The computations have been performed using the GPXT [22, 27, 28]. The factor $1/(4\pi)^{10}$ in each integral has been omitted. Note that the explicit value of integral $J_{35}^{\mu\nu\rho\sigma}$ is not needed since this integral appears only for structures $\chi(2, 1, 4)$ and $\chi(1, 4)$, and the two contributions exactly cancel each other.

\[
J_1 = \begin{array}{c}
\includegraphics{fig1.png}
\end{array} = \frac{1}{120\varepsilon^5} - \frac{1}{12\varepsilon^4} + \frac{11}{24\varepsilon^3} - \frac{19}{12\varepsilon^2} + \frac{14}{5\varepsilon}
\]

\[
J_2 = \begin{array}{c}
\includegraphics{fig2.png}
\end{array} = \frac{2}{15\varepsilon^5} - \frac{4}{15\varepsilon^4} - \frac{1}{6\varepsilon^3} + \frac{3}{10\varepsilon^2} + \frac{4}{15\varepsilon}
\]

\[
J_3 = \begin{array}{c}
\includegraphics{fig3.png}
\end{array} = \frac{2}{15\varepsilon^5} - \frac{2}{5\varepsilon^4} + \frac{1}{5\varepsilon^3} + \frac{4}{15\varepsilon^2} - \frac{1}{15\varepsilon}
\]

\[
J_4 = \begin{array}{c}
\includegraphics{fig4.png}
\end{array} = \frac{1}{20\varepsilon^5} - \frac{3}{10\varepsilon^4} + \frac{17}{20\varepsilon^3} - \frac{14}{15\varepsilon^2} + \frac{2}{5\varepsilon}
\]

\[
J_5 = \begin{array}{c}
\includegraphics{fig5.png}
\end{array} = \frac{3}{40\varepsilon^5} - \frac{3}{10\varepsilon^4} + \frac{17}{40\varepsilon^3} + \frac{1}{30\varepsilon^2} - \frac{1}{2\varepsilon}
\]

\[
J_6 = \begin{array}{c}
\includegraphics{fig6.png}
\end{array} = \frac{1}{20\varepsilon^5} - \frac{1}{5\varepsilon^4} + \frac{17}{60\varepsilon^3} - \frac{2}{15\varepsilon^2} + \frac{4}{15\varepsilon}
\]

\[
J_7 = \begin{array}{c}
\includegraphics{fig7.png}
\end{array} = \frac{3}{40\varepsilon^5} - \frac{17}{60\varepsilon^4} + \frac{43}{120\varepsilon^3} + \frac{7}{60\varepsilon^2} - \frac{8}{15\varepsilon}
\]

\[
J_8 = \begin{array}{c}
\includegraphics{fig8.png}
\end{array} = \frac{1}{30\varepsilon^5} - \frac{7}{30\varepsilon^4} + \frac{5}{6\varepsilon^3} - \frac{3}{2\varepsilon^2} + \frac{11}{30\varepsilon}
\]

\[
J_9 = \begin{array}{c}
\includegraphics{fig9.png}
\end{array} = \frac{1}{30\varepsilon^5} - \frac{11}{60\varepsilon^4} + \frac{7}{15\varepsilon^3} - \frac{19}{60\varepsilon^2} - \frac{14}{15\varepsilon}
\]
\begin{align*}
J_{10} &= - \frac{11}{120\varepsilon^5} - \frac{1}{3\varepsilon^4} + \frac{3}{8\varepsilon^3} + \frac{1}{6\varepsilon^2} - \frac{1}{30\varepsilon} \\
J_{11} &= - \frac{1}{10\varepsilon^3} + \frac{13}{30\varepsilon^2} - \frac{23}{30\varepsilon} \\
J_{12} &= - \frac{1}{60\varepsilon^3} + \frac{1}{12\varepsilon^2} - \frac{2}{5\varepsilon} \\
J_{13} &= - \frac{1}{20\varepsilon^3} + \frac{3}{20\varepsilon^2} - \frac{13}{30\varepsilon} \\
J_{14} &= - \frac{1}{5\varepsilon^3} + \frac{2}{3\varepsilon^2} - \frac{4}{15\varepsilon} \\
J_{15} &= - \frac{1}{30\varepsilon^3} + \frac{1}{30\varepsilon^2} + \frac{1}{3\varepsilon} \\
J_{16} &= - \frac{3}{20\varepsilon^3} + \frac{11}{60\varepsilon^2} + \frac{3}{10\varepsilon} \\
J_{17} &= - \frac{1}{15\varepsilon^3} + \frac{2}{15\varepsilon^2} + \frac{1}{15\varepsilon} \\
J_{18} &= \frac{1}{24\varepsilon^5} - \frac{1}{4\varepsilon^4} + \frac{5}{8\varepsilon^3} - \frac{1}{\varepsilon^2} \left( \frac{1}{12} + \frac{4\zeta(3)}{5} \right) - \frac{1}{\varepsilon} \left( \frac{5}{6} - \frac{4\zeta(3)}{5} - \frac{\pi^4}{300} \right) \\
J_{19} &= \frac{1}{24\varepsilon^5} - \frac{1}{6\varepsilon^4} + \frac{1}{8\varepsilon^3} + \frac{1}{\varepsilon^2} \left( \frac{1}{3} - \frac{\zeta(3)}{5} \right) - \frac{1}{\varepsilon} \left( \frac{1}{3} - \frac{4\zeta(5)}{5} + \frac{\pi^4}{300} \right) \\
J_{20} &= \frac{1}{120\varepsilon^5} - \frac{1}{12\varepsilon^4} + \frac{11}{24\varepsilon^3} - \frac{19}{12\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{14}{5} - 4\zeta(5) \right)
\end{align*}
\[ J_{21} = \frac{1}{15\varepsilon^5} - \frac{1}{4\varepsilon^4} + \frac{1}{5\varepsilon^3} + \frac{29}{60\varepsilon^2} - \frac{1}{\varepsilon} \left( \frac{19}{30} - \frac{4\zeta(3)}{5} \right) \]

\[ J_{22} = \frac{1}{40\varepsilon^5} - \frac{1}{6\varepsilon^4} + \frac{61}{120\varepsilon^3} - \frac{17}{30\varepsilon^2} - \frac{1}{\varepsilon} \left( \frac{1}{6} - \frac{4\zeta(3)}{5} \right) \]

\[ J_{23} = \frac{14}{\varepsilon} \zeta(7) \]

\[ J_{24} = -\frac{1}{20\varepsilon^3} + \frac{3}{20\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{1}{6} + \frac{6\zeta(3)}{5} - 2\zeta(5) \right) \]

\[ J_{25} = -\frac{1}{10\varepsilon^3} + \frac{13}{30\varepsilon^2} - \frac{1}{\varepsilon} \left( \frac{1}{6} - \frac{6\zeta(3)}{5} + 2\zeta(5) \right) \]

\[ J_{26} = -\frac{2\zeta(5)}{\varepsilon} \]

\[ J_{27} = -\frac{1}{60\varepsilon^3} + \frac{1}{12\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{1}{5} + \frac{6\zeta(3)}{5} - 2\zeta(5) \right) \]

\[ J_{28} = \frac{1}{5\varepsilon^2} - \frac{1}{\varepsilon} \left( \frac{19}{20} + \frac{4\zeta(3)}{5} \right) \]

\[ J_{29} = \frac{1}{20\varepsilon^2} - \frac{1}{\varepsilon} \left( \frac{13}{20} + \frac{4\zeta(3)}{5} \right) \]

\[ J_{30} = \frac{1}{\varepsilon} \left( \frac{9}{10} + \frac{11\zeta(3)}{5} - 2\zeta(5) \right) \]

\[ J_{31} = -\frac{1}{\varepsilon} \left( \frac{9}{10} + 2\zeta(3) - 7\zeta(7) \right) \]
\[ J_{32} = -\frac{1}{\varepsilon} \left( \frac{1}{5} + \frac{2\zeta(3)}{5} + 2\zeta(5) - \frac{7\zeta(7)}{2} \right) \]

\[ J_{33} = -\frac{1}{\varepsilon} \left( \frac{3}{10} + \frac{3\zeta(3)}{5} - 2\zeta(5) \right) \]

\[ J_{34} = \frac{1}{\varepsilon} \left( \frac{1}{10} + \frac{\zeta(3)}{5} - 2\zeta(5) + \frac{7\zeta(7)}{2} \right) \]

\[ J_{\mu\nu\rho\sigma}^{\mu\nu\rho\sigma} = \frac{\partial_{\mu}}{\partial_{\nu}} \frac{\partial_{\rho}}{\partial_{\sigma}} \]
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