A PHYSICAL BIAS IN COSMOLOGICAL SIMULATIONS

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ABSTRACT

Numerical simulations play an important role in the study of the structure formation of the universe. However, the mass resolution in current simulations is still poor. Due to technical difficulties, it is necessary to use both a greatly reduced number density of particles and a greatly raised unit particle mass. Consequently, the particle masses used in cosmological simulations are about $10^{20}$ times larger than the GeV candidates in particle physics. This is a huge physical bias that cannot be neglected when interpreting the results of the simulations. Here we discuss how such a bias affects cold dark matter (CDM) cosmological simulations. We find that the small-scale properties of the CDM particle system are changed in two aspects. (1) An upper limit is imposed on the spatial resolution of the simulation results. (2) Most importantly, an unexpected short mean free path is produced, and the corresponding two-body scattering cross section is close to the value expected in the self-interaction dark matter model. Since the mean free path of real CDM particle systems is much longer than that in the simulations, our results imply that (1) there is probably no “cusp problem” in real CDM halos and (2) a much longer time is needed to form new virialized halos in real CDM particle systems than in the simulations. The last result can help us understand the “substructure problem.” Our discussion can also explain why the massive halos in the simulations may have smaller concentration coefficients.

Subject headings: dark matter — galaxies: halos — gravitation — methods: N-body simulations

1. INTRODUCTION

Thanks to the rapid development of numerical computation technology, cosmological N-body simulations have become an increasingly important tool in our understanding of the gravitational clustering process in the universe. Suto (2003) noted that the number of particles used in simulations has risen exponentially with time, from about 400 in 1975 to about $10^8$ today. This means that current N-body simulations are available to give reliable results on some complex cosmological issues such as large-scale structure formation (Einasto 2001) and the properties of dark matter halos (Cooray & Sheth 2002; Navarro et al. 1996). Sometimes such numerical methods are the only way to get a qualitative picture of the system for which there is no mature theory. This is especially the case when strong nonlinear evolution on small scales is involved, such as in the central region of dark matter halos (Primack 2003) and in the processes of galaxy formation and merging (Kauffmann et al. 1993), etc.

High-resolution simulations based on the ΛCDM cosmology model have shown nice agreement with the large-scale structure observations in several cases (Freedman & Turner 2003). The predicted power spectrum in the simulations agrees well with recent redshift surveys like the Sloan Digital Sky Survey (Tegmark et al. 2004) and the Two-Degree Field Galaxy Redshift Survey (Percival 2001), but the simulations based on cold dark matter (CDM) models do not work so well on small scales. One aspect is that the central density profiles of halos predicted in these simulations are too cuspy (Reed et al. 2003) to be consistent with the observed results of dwarf galaxies and low surface brightness galaxies (Primack 2003) or to be consistent with the X-ray and gravitational lensing study of the distribution of dark matter (DM) in galaxy clusters (Haruyoshi & Kiyoshi 2004; Tyson et al. 1998). Another aspect is that 10 times more substructures are predicted than observed (Willman et al. 2002); this is called the “substructure problem” or the “dwarf galaxy problem” (Primack 2003; Klypin et al. 1999). Some believe that these problems come from deficiencies in the CDM model and therefore introduce some alternative models, such as the self-interaction dark matter (SIDM) model, warm dark matter model, etc. (Götz & Sommer-Larsen 2003; Boehm et al. 2002; Spergel & Steinhardt 2000).

Here we find that studying a common difficulty in current N-body simulations will help us understand the above problems. Due to the technical difficulties, the total number of particles $N$ we can use even in recent simulations is still somewhat limited. The most powerful “high-resolution” simulation today with the largest total number contains only about $10^{12} - 10^{13}$ particles. But, at the same time, we have to keep the mean density of the universe at a preset level. So the mass of each particle in these simulations has to be as high as about $10^{10} M_\odot$ (for a large-scale simulation), comparable to the mass of a small galaxy. Now, when we study the large-scale structure formation process, it is an acceptable assumption that each particle in the simulation represents one galaxy. However, when we study the small-scale properties of dark matter particle systems, such as the CDM halo density profile, the merging process of halos, and the formation of galaxies, the limitations arising from an inadequate $N$ will emerge. For example, the number of particles in one CDM halo ($\sim 10^{13} M_\odot$) in current simulations ranges from $10^5$ to $10^8$, and so the mass of each particle is greater than $10^6 M_\odot$. Therefore, what we can learn from such simulations will concern halos made up of such “huge mass particles.”

However, in the view of particle physicists, it is hard to accept that the mass of a dark matter particle is some $10^{65}$ times larger than that of usual GeV candidates. And it is also hard for astrophysicists to accept, because a point particle with such a huge mass is somehow a massive black hole, which will affect the galaxy seriously. If we do not believe the mass of the CDM particle should be the size of a small galaxy, then the particles in the simulations are something different from real CDM particles. In other words, we have unavoidably introduced a physical bias in current simulations by having to simultaneously reduce the particle number density $n$ and raise...
the mass of each particle \( m \) by a huge factor \( f_{AE} \). The bias is like trying to study the clustering properties of a huge society of ants by simulating a colony of elephants: we call it the “ant-elephant (AE) bias.” Because CDM particle systems with AE bias can have different dynamical and statistical properties, the clustering and evolution scenario of CDM halos that we get from the simulations may be very different from the truth. One might ask, since such a physical bias is present in the current cosmological simulations, are the simulation results reliable? To what extent will these results reflect the actual situation? Is there any connection between the bias and the “cusp problem” and the “substructure problem” mentioned above? Learning how the AE bias affects the simulation results and identifying the circumstances when the effect is serious can help us re-construct a correct scenario of dark matter clustering and evolution from the simulation results.

When the factor \( f_{AE} \) is less than about \( 10^4 \), it is possible to test the effects of the AE bias in the numerical simulation directly. The number of particles used has already increased about 100 times in the past 30 years, which means the factor \( f_{AE} \) has already decreased by a large amount, yet no serious changes in large-scale properties have been reported. However, the factor \( f_{AE} \) can still be as big as \( 10^5 \) in the simulations, and therefore it is indeed necessary to study how the simulation results will be affected by such a physical bias.

2. PHYSICAL DIFFERENCE CAUSED BY THE AE BIAS

In this Letter, we study the physical difference the AE bias makes on the cold dark matter systems and find that the simulation results are affected in two aspects.

2.1. Constraint on the Spatial Resolution of Simulation Results

It is easy to see that the AE bias can impose an upper limit \( L_{\text{min}} \) on the spatial resolution of the simulation results. The reason is that the AE bias has greatly reduced the particle number density. To get reliable statistical results from simulation data, the size of the statistical sample boxes cannot be too small, for they must include enough simulation particles for consideration. For example, when discussing the galaxy formation process at the center region of a CDM halo, one needs to calculate the local density distribution \( \rho \) of the CDM particles. Yet, when estimating the local density at a certain position, the size of the sample box \( L \) cannot be too small. A very small sample box will include very limited simulation particles, and the statistical result will be unreliable.

Consider an extreme case: if we use a ultrasmall sample box with volume \( V \), \( m_{sim}/\rho \), then we can only find about one simulation particle in it. Here the one simulation particle with huge mass \( m_{sim} \) actually represents all the real CDM particles within the sample box. This is not a good approximation, but it is unavoidable because of technical difficulties; i.e., we are unable to deal with so many particles. Consequently, the simulation results cannot provide reliable information when the physical processes occur on scales smaller than \( L_{\text{min}} \). We have to be aware of this upper limit \( L_{\text{min}} \) when using the simulation results to discuss the galaxy formation process and the merging process of substructures.

\( L_{\text{min}} \) can be estimated in this way: the simulation particle number within a sample box with size \( L \) is about \( N = \rho L^3/m_{\text{sim}} \), where \( m_{\text{sim}} \) is the mass of each simulation particle and \( \rho \) is the local density; set \( N = 1 \), and we get

\[
L_{\text{min}} = \left( \frac{m_{\text{sim}}}{\rho} \right)^{1/3} = \left( \frac{m_{\text{sim}}}{100 M_\odot} \right)^{1/3} \left( \frac{\rho}{\rho_c \Omega_{\text{cdm}}/h} \right)^{-1/3}\text{ (kpc} h^{-1}).
\]

Here we represent \( m_{\text{sim}} \) and \( \rho \) with the solar mass \( (M_\odot) \) and the critical density of the universe \( \rho_c = 3H_0^2/8\pi G = 1.88 \times 10^{-29} \text{m}^{-3} \text{h}^2 \text{kg} \). Equation (1) gives a physical limitation on the spacial resolution of the simulation results.

However, it is still not enough, for it is hard to do statistical analysis using a sample box that can include only about one particle. In general, when we discuss the statistical results of a given simulation sample, the statistical fluctuation must be taken into account. For example, when we calculate the local density \( \rho \) at the halo center from the simulation result, the volume of the sample box must be big enough to be able to contain adequate simulation particles. The least particle number that one sample box must contain depends on the requirement of a reliable statistical result. A sample box with a size round \( L_{\text{min}} \) is still not big enough, and it can hardly give a reliable statistical result of \( \rho \).

Let us consider the statistical result of \( \rho \) being proportional to the particle number \( N \) found in the sample box with given volume \( V \); then the statistical fluctuation of \( \rho \) is about \( \delta \rho \rho \approx \delta N/N \). For a Gauss random field, \( \delta N/N \sim 1/\sqrt{N} \). We suggest that a reliable statistical result requires \( \delta \rho \rho \ll \delta N/N \sim 1/\sqrt{N} \), where \( N \) is the particle number found in the sample box with given volume \( V \); then we get \( N > 10^4 \text{a}^3 \). This result means that the sample box must include at least \( N > 10^4 \text{a}^3 \) particles in order to give a reliable statistical result of \( \rho \). Thus, we get another limitation on the size of the sample box

\[
L_{\text{min}1} = \left( \frac{V}{10^5} \right)^{1/3} \left( \frac{N}{10^4} \right)^{1/3} \approx L_{\text{min}} \left( \frac{100 \text{a}^3}{10^5 \text{a}^3} \right)^{1/3}.
\]

\( L_{\text{min}2} \) gives a statistical limitation on the spatial resolution of the simulation results. Here the factor \( a \) varies between 0 and 100 based on different problems; \( L_{\text{min}2} \) is commonly larger than \( L_{\text{min}1} \). Only when \( a > 100 \) is \( L_{\text{min}2} \approx L_{\text{min}1} \). On scales smaller than \( L_{\text{min}2} \), not only can the statistical error be serious but the physical process can be affected as well. Let us take the study of the galaxy formation process at the halo center region as an example again; one can find few massive particles in simulation using a sample box smaller than \( L_{\text{min}2} \), but not like the numerous real CDM particles that can be smoothly distributed in this field. Therefore, the simulation CDM particle system will offer a much more asymmetric and instable gravitational background for the study of the galaxy formation process on such scales.

\( L_{\text{min}1} \) and \( L_{\text{min}2} \) can help us estimate a lower bound to the scale of applicability of the simulation results. For example, let us suggest that we have a \( \Lambda \text{CDM} \) cosmology simulation with \( m_{\text{sim}} = 10^4 M_\odot \). For the center region of one CDM halo, \( \rho \rho_c \approx 10^4 \), set \( a = 1 \), and we get \( L_{\text{min}1} \approx 1 \text{ kpc} h^{-1} \), \( L_{\text{min}2} > 10 \text{ kpc} h^{-1} \). When using these simulation results to discuss the central properties of the halo, such as the density profile or the galaxy formation process, the AE bias must be taken into account. This is not an exciting conclusion, and we have to accept the fact that by raising the mass resolution 10 times, the spatial resolution will only increase by a factor of 10.

This AE bias effect comes from the statistical error \( \delta N/N \), which only occurs on small scales and can be neglected on
large scales where plenty of simulation particles are considered to reach a reliable statistical result.

2.2. Variation of the Mean Free Path

The second effect of the AE bias is a variation in the mean free path. Since gravitation is a long-range interaction, there are three different factors that can change the CDM particle velocity. One is the gravitation of the existing halo, i.e., after the halo has formed. The mean free path of the particles will then be about the size of the halo, \( R \). Another factor is the scattering of two CDM particles. Suppose that particle 1 passes by a stable particle 2 with velocity \( v \). If particle 1 gains a scattering angle larger than 45\(^\circ\) (in this case \(|\delta v| \sim |v|\), and we call it a “scattering”), the impact parameter \( b \) should satisfy \( b < 2Gm/v^2 \), where \( m \) is the particle mass. Let \( n \) be the particle number density and set the \( v^2 \) equal to the velocity dispersion \( \sigma_v^2 \); then the mean free path of scattering \( L_s \) should satisfy \( n(L, \pi b^2) \geq 1 \), and we get

\[
L_s = \sigma_v/(4\pi G^2 m \rho).
\]

If we define \( T_\sigma = L_s \), we can also expect the two-body relaxation time \( T_\sigma \) to be proportional to \( \alpha_i^2 v^2 / G^2 m \rho \propto 1/m \propto N \) (for a given \( \sigma_i \) and density \( \rho \)). The third resulting proportionality comes from summing the velocity deflections from distant particles. Binney & Tremaine (1987) estimated the relaxation time that was due to this cause to be proportional to \( N! / N \ln N \). But analytical and simulation results (Huang et al. 1993; Diemand et al. 2004) show that the relaxation time is proportional to \( N \), not \( N! / N \ln N \), if the softening parameter is taken into account in the simulations. Here we use equation (2) to discuss the effect of the AE bias.

As we can see from the formula above, for certain values of \( \rho \) and \( \sigma_i \), the AE bias will raise the value of \( m \) and make \( L_s \) shorter. For a virialized halo, we can estimate the velocity dispersion from galaxy dynamics (Binney & Tremaine 1987); \( \sigma_v^2 \approx GM/R \), with \( M \) the total mass and \( R \) the radius of the halo. Substitute \( \sigma_i \) into equation (2), and we have

\[
L_s = \frac{R M \bar{\rho}}{3 m \rho},
\]

where \( \bar{\rho} = M/3 \pi R^3 \). If the AE bias causes \( m \rightarrow m_{AE} \) and \( N \rightarrow N_{AE} \), then we get

\[
L_s \rightarrow L_s / f_{AE} \text{ and } T_\sigma \rightarrow T_\sigma / f_{AE}.
\]

For example, in the central region of a typical halo in the simulation, let \( R = 1 \) Mpc \( h^{-1} \), \( M/M = N_{halo} = 10^6 \), and \( \rho \bar{\rho} = 10^5 \), and we then get \( L_s = 3.3 \) Mpc \( h^{-1} \), close to the dimension of the halo. If we apply the expression of the scattering cross section commonly used in the SIDM model, \( \sigma_{SI} \approx 1/(Lp) \), and if we let the central area density of the halo be \( \rho \approx 100 \) GeV cm\(^{-3} \), we can estimate \( \sigma_{SI} \approx 9 \times 10^{-26} \) cm\(^2 \) GeV\(^{-1} \). In the above, we only used typical data of the central area of halos. Indeed, the mean scattering cross section of the whole halo should be smaller. However, considering that the mass of the DM particle candidates is on the order of GeV, \( f_{AE} \sim 10^{26} \), from equation (4) we can see that the actual scattering cross section between two DM particles should be far less than the cross section in the simulation. In this case, the change of the particle velocity depends mostly on the deflection by the existing halo. When discussing the SIDM model, Spergel & Steinhardt (2000) and Davé et al. (2001) suggested that if the \( \sigma_{SI} \) of the DM particles is around \( 10^{-23} \) to \( 10^{-25} \) cm\(^2 \) GeV\(^{-1} \), or their mean free path \( L \sim 1 \) kpc–1 Mpc is at the solar radius level, then the halo center cusp problem and the excessive substructure problem can be solved. Now we can see that the AE bias has provided an unexpected short mean free path \( L_{AE} \) and \( \sigma_{AE} \) smaller than the SIDM model expectation.

To give a rough estimate here, we set the simulation timescale to be \( t_{AE} \sim 100 \) Gyr, the particle velocity \( \bar{v} \sim 1000 \) km s\(^{-1} \), and \( L_s \sim R \sim 1 \) Mpc. Then one particle in simulation will scatter for about \( \sigma_{AE} L_{scattering} = t_{AE} / (L_s / \bar{v}) \sim 10 \) times. Yet the real CDM particles will not experience such scattering for they have a far more smaller scattering cross section. This is the difference caused by the AE bias.

Unfortunately, at present, there is not a quantitative theory on how the mean free path \( L_s \) will affect the history of structure formation, but the qualitative picture is clear. When \( L_s \ll R \), a short \( L_s \) means a large scattering cross section and a high thermal conductivity, which might alleviate the cusp problem and make the halos more spherical. But some authors argue that a short \( L_s \) can also drain energy from the halo centers, which could lead to core collapse (Barkett 2001). When \( L_s \gg R \), a large \( L_s \) means a small scattering cross section and a large average relaxation time \( T_\sigma \). Equation (4) shows us that with improvement in the mass resolution (decrease in \( f_{AE} \)), \( T_\sigma \) will be increased. The simulation by Diemand et al. (2004) shows that \( T_\sigma \sim N \) and that it can easily rise to more than 1000 Gyr by increasing the total particle number \( N \) in the simulation. In this case, the tidal force is unable to damage small halos in one Hubble time, as was shown in the simulation by Kazantzidis et al. (2004). Here the substructures are mainly what remain of halo merging. A very small scattering cross section also means that the effects of the two-body scattering of the CDM particles can be neglected and that a very poor thermal conductivity is expected. If \( T_\sigma \) is far much longer than one Hubble time, then it is hard to make the CDM halo virialize with the poor thermal conductivity. And a new density distribution implies a different particle energy distribution, so there will be not enough time to form new substructures on scales less than \( R \), such as the \( \rho \sim r^{-2} \) cusp profile in the isothermal model.

The current simulations have brought in an unexpectedly short \( L_s \sim R \), not in both cases above, and it is different from the real CDM particle systems (\( L_s \gg R \)). The corresponding scattering cross section is smaller than what is needed in the SIDM model, and it cannot be neglected. The simulation results will be sensitive to \( L_s \) and to the relaxation effects of subhalos, such as the number and mass density of subhalos (Ma & Boylan-Kolchin 2004). One can expect that, with improvements to the numerical technique, the AE bias will be gradually alleviated, and the simulation results will gradually move to the \( L_s \gg R \) regime. By increasing \( L_s \), the halo center cusp problem can be alleviated. But the relaxation time will soon rise up to more than one Hubble time. As a result, halos will tend to be more triaxial, and more small halos will survive the process of merging. In other words, the AE bias decreases the number of substructures. The substructure problem in future simulations can become more serious. Actually, this is in agreement to a certain extent, with the discussion of gravitational lensing (Mao et al. 2004), in which more substructures are expected in real DM halos than in the current simulations. But if \( L_s \) increases more, the thermal conductivity in the halos will become poor, which means it will be harder to form enough virialized halos in one Hubble time.
By the way, equation (3) can help us understand the concentration properties of DM halos in simulations. In a given simulation in which the particles have a fixed mass \( m \), bigger halos will have larger total masses \( M \). Then, based on equation (3), one can predict that the big halos will have larger \( L_s/R \) than small halos. Large \( L_s/R \) mean that the particles can move more freely in the halo, which will become more dispersed, so equation (3) can give a rational explanation of why the more massive halos in the simulations have a smaller concentration coefficient \( C \). Equation (3) is available for all virialized particle systems with a pure Newton gravitation interaction. If we suppose the particles to be the stars of elliptical galaxies, then the discussion above will predict that the more massive elliptical galaxies will be more dispersed. In other words, they will have a fainter mean surface brightness, and this is consistent with the observations (Binney & Merrifield 1998, p. 205).

3. CONCLUSION AND DISCUSSION

In summary, we have pointed out that a physical bias exists in current cosmological simulations. We find that the AE bias can change the small-scale properties of the CDM particle system in two respects. (1) An upper limit is imposed on the spatial resolution of the simulation results. (2) An unexpected short mean free path of the system is generated. The corresponding two-body scattering cross section is near the value expected in the SIDM model.

What can we learn from the discussion above? The upper limit \( L_{\text{min}} \) of the first effect can help us estimate the reliable range of the simulation results. The second effect can alter the simulation results on small scales and cause the halo center to be more cuspy and more spherical. These results can help us understand the cusp problem and the substructure problem in the current simulations, and they can help us reconstruct a correct scenario for the clustering and evolution of the real dark matter in the universe. Because the AE bias factor \( J_{AE} \) is currently about \( 10^5 \), the mean free path \( L_s \) and the relaxation time \( T_s \) of real CDM particles systems should be much larger than what we see in the simulations. When \( L_s \) is far more larger than the halo diameter, the halo will be dispersed and will be hard put to form a cuspy density profile. When \( T_s \) is much longer than one Hubble time, CDM halos will not have enough time to be virialized because of the poor thermal conductivity (this is contrary to the prevailing view of the popular halo model), and nearly all the centers of the existing halos can survive the process of merging until the present time. Our results imply (1) that real CDM halos may not have the cusp problem (the cusps of the simulation halos are mainly the effect of the AE bias) and (2) that real CDM halos may not have enough time to be virialized. This result can help us understand the substructure problem. The discussion above also tells us that the theoretical analysis of the halo property, which is based on plenty of thermal conductivity, is available for simulation halos but not for the real CDM halos.

On the other hand, since it is hard to form new virialized halos in one Hubble time and since most existing small halos (coming from primary disturbances) can survive the process of merging, the number of these small halos may retain some information about the primordial disturbance. If we believe the number of dwarf galaxies is proportional to the number of small halos, then the percentage of dwarf galaxies in galaxy clusters can tell us something about the small-scale properties of the primordial disturbance.

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