VERTEX-FLAMES IN COUNTABLE ROOTED DIGRAPHS
PRESERVING AN ERDŐS-MENGER SEPARATION FOR EACH
VERTEX

ATTLA JOÓ*

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It follows from a theorem of Lovász that if $D$ is a finite digraph with $r \in V(D)$, then there is a spanning subdigraph $E$ of $D$ such that for every vertex $v \neq r$ the following quantities are equal: the local connectivity from $r$ to $v$ in $D$, the local connectivity from $r$ to $v$ in $E$ and the indegree of $v$ in $E$.

In infinite combinatorics cardinality is often an overly rough measure to obtain deep results and it is more fruitful to capture structural properties instead of just equalities between certain quantities. The best known example for such a result is the generalization of Menger’s theorem to infinite digraphs. We generalize the result of Lovász above in this spirit. Our main result is that every countable $r$-rooted digraph $D$ has a spanning subdigraph $E$ with the following property. For every $v \neq r$, $E$ contains a system $R_v$ of internally disjoint $r \rightarrow v$ paths such that the ingoing edges of $v$ in $E$ are exactly the last edges of the paths in $R_v$. Furthermore, the path-system $R_v$ is “big” in $D$ in the Erdős-Menger sense, i.e., one can choose from each path in $R_v$ either an edge or an internal vertex in such a way that a resulting set separates $v$ from $r$ in $D$.

1. Introduction

Small subgraphs witnessing some kind of connectivity property play an important role in graph theory. Let us recall a result of L. Lovász of this manner. Consider a finite digraph $D$ with a given root $r \in V(D)$. We are looking for a spanning subdigraph $E$ of $D$ that preserves the local vertex-connectivities from $r$ (i.e., $\kappa_D(r,v) = \kappa_E(r,v)$ holds for every $v \in V(D) - r$) with a minimal possible number of edges. For every $v \in V(D) - r$, we need

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to keep at least $\kappa_D(r, v)$ ingoing edges hence $\sum_{v \in V(D) - r} \kappa_D(r, v)$ is a trivial lower bound for the number of edges of $E$. Surprisingly this lower bound is always sharp.

**Theorem 1.1 (L. Lovász).** If $D$ is a finite digraph and $r \in V(D)$, then there is a spanning subdigraph $E$ of $D$ such that for every $v \in V(D) - r$

$$\kappa_D(r, v) = \kappa_E(r, v) = |\text{in}_E(v)|,$$

where $\text{in}_E(v)$ is the set of the ingoing edges of $v$ in $E$.

The main result of this paper is a generalization of Theorem 1.1 to countable digraphs. The equations in Theorem 1.1 make sense in infinite digraphs as well. Even so, cardinality is an overly rough measure to give a satisfactory generalization. Instead of the equation $\kappa_E(r, v) = |\text{in}_E(v)|$ we demand the existence of a system $P$ of internally disjoint directed paths from $r$ to $v$ in $E$ such that the set of the last edges of the paths in $P$ is $\text{in}_E(v)$. If $E$ satisfies this for each vertex $v$, then $E$ is called a *vertex-flame* with respect to the root $r$ (the name “flame” was given by G. Calvillo-Vives who rediscovered Theorem 1.1 independently in his Ph.D. thesis [3]).

The equation $\kappa_E(r, v) = \kappa_D(r, v)$ means that some maximal-sized internally disjoint $r \rightarrow v$ path-system $P$ of $D$ lies in $E$. We want $P$ to be “big” in $D$ not just cardinality-wise but in the Erdős-Menger sense. More precisely, we demand that one can choose from every $P \in P$ either one internal vertex or an edge such that the resulting set separates $v$ from $r$, i.e., meets every $r \rightarrow v$ path of $D$. Note that the separation can be chosen as a vertex set $S$ whenever $rv \notin D$ and in the form $S \cup \{rv\}$ otherwise. The set of internally disjoint $r \rightarrow v$ path-systems $P$ admitting such a separation is denoted by $\mathcal{I}_D(v)$. It is easy to see that the Aharoni-Berger theorem ensures that $\mathcal{I}_D(v) \neq \emptyset$ (see Theorem 3.1 and the paragraph after it). We define a spanning subdigraph $L$ of $D$ to be $D$-vertex-large if for each vertex $v \neq r$ of $D$ there is a $P \in \mathcal{I}_D(v)$ that lies in $L$. The main result of this paper is the following theorem.

**Theorem 1.2.** If $D$ is a countable digraph with a given root vertex $r$, then there exists a $D$-vertex-large spanning subdigraph $E$ of $D$ which is a vertex-flame.

The paper is organized as follows. In the next subsection we discuss proof methods that work for finite digraphs but fail or insufficient for infinite ones and we present our proof strategy for countable digraphs. At the end of the first section we introduce some further notation. In the second section we state our key lemmas without proofs and derive the main result from
them in a single page. The third section is devoted to the proofs of the key lemmas. In the last section we discuss some open problems.

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Remark 1.3. Lovász proved Theorem 1.1 originally for edge-connectivity instead of vertex-connectivity (see Theorem 2 of [4]) from which the vertex version follows. Indeed, let $D'$ be the digraph that we obtain from $D$ by splitting every $v \in V(D) - r$ to an edge $t_vh_v$, where $t_v$ inherits the ingoing and $h_v$ the outgoing edges of $v$. Observe that the systems of edge-disjoint $r \rightarrow t_v$ paths of $D'$ and the systems of internally disjoint $r \rightarrow v$ paths in $D$ are in a natural correspondence. Let $E'$ be that we obtain by applying the edge version of Theorem 1.1 to the digraph $D'$. We define $E$ to be the spanning subdigraph of $D$ consists of the common edges of $D$ and $E'$. It is easy to check that $E$ satisfies the conditions of Theorem 1.1.

Remark 1.4. Seemingly we promised a stronger property for $E$ in the abstract than in Theorem 1.2. Namely, an $R_v$ in $E$ for $v \in V(D) - r$ that witnesses simultaneously the $D$-vertex-largeness and the vertex-flame property at $v$. If $P_v$ exemplifies the $D$-vertex-largeness and $Q_v$ shows the vertex-flame property at $v$ in $E$, then an easy application of Pym’s theorem (Theorem 2.4) results in an $R_v$ that witnesses both. Hence, the property of $E$ given in the abstract is equivalent with two properties demanded in Theorem 1.2.

1.1. Proof strategies informally

One possible proof strategy, the original approach of Lovász, for Theorem 1.1 is “trimming” $D$ while keeping vertex-largeness. One can show, for example, that if $P$ is a system of internally disjoint $r \rightarrow u$ paths of size $\kappa_D(r,u)$ and we delete those $e \in \text{in}_D(u)$ that are unused by $P$, then $\kappa_L(r,v) = \kappa_D(r,v)$ holds for $v \in V(D) - r$ and of course $\kappa_L(r,u) = |\text{in}_L(u)|$ holds as well. Theorem 1.1 follows by applying this for each vertex one by one.

Our approach for the finite case was adding new edges repeatedly having a vertex-flame in each step. We will see that if a vertex-flame $F$ is not $D$-vertex-large, then one can properly extend $F$ with a suitable edge of $D$ such that the result is still a vertex-flame. By iterating this, we can extend any vertex-flame of $D$ to a $D$-vertex-large vertex-flame whenever $D$ is finite (actually the assumption “$\kappa_D(r,v) < \aleph_0$ for every $v \in V(D) - r$” is enough).

It will turn out that the key ideas of the proof sketches above still work in the general case, and moreover, they are compatible with our stronger
definitions. For example, if \( P \in I_D(v) \), then the \( L \) that we obtain from \( D \) by the deletion of those ingoing edges of \( v \) that are unused by \( P \) is \( D \)-vertex-large (not just \( \kappa_L(r,v) = \kappa_D(r,v) \) holds for \( v \in V(D) - r \)). The main difficulty is that by iterating the deletions or extensions infinitely many times we may lose at a limit step the property we intended to keep. In the case of the deletions, the situation is actually worse. In the finite case, vertex-largeness is transitive in the sense that if \( L_1 \) is \( D \)-vertex-large and \( L_2 \) is \( L_1 \)-vertex-large, then \( L_2 \) is \( D \)-vertex-large. (Indeed, if \( \kappa_D(r,v) = \kappa_{L_1}(r,v) \) and \( \kappa_{L_2}(r,v) = \kappa_{L_1}(r,v) \) for every \( v \), then \( \kappa_D(r,v) = \kappa_{L_1}(r,v) \) for every \( v \).) Examples show that this transitivity of vertex-largeness does not hold in general, thus applying twice a vertex-largeness preserving edge-deletion may already be problematic.

Our proof strategy for the countable case is a mixture of the two approaches above. In every step we fix some edges and delete some others. In a general step we fix the edges of a \( P \in I_D(v) \) for the next \( v \) with respect to a fixed enumeration in such a way that \( P \) covers all the (finitely many) ingoing edges of \( v \) that are already fixed. Right after this we delete all the ingoing edges of \( v \) that are unused by \( P \). It turns out that this way we keep \( D \)-vertex-largeness, and furthermore, \( P \) witnesses that in the final digraph we do not violate the vertex-flame property at \( v \). A critical part of the proof is to guarantee that each step we are really able to cover a finite subset of ingoing edges of a given vertex \( v \) by a system of internally disjoint \( r \rightarrow v \) paths. A rooted digraph \( F \) is called a quasi-vertex-flame if for each vertex \( v \in V(D) - r \) every finite subset of \( \text{in}_F(v) \) can be covered in \( F \) by an internally disjoint \( r \rightarrow v \) path-system. It turns out that our iterative process maintains the quasi-vertex-flame property assuming we have it at the beginning. We show that a maximal quasi-vertex-flame \( F \) in \( D \) has a very strong property (it is “vertex-largeness faithful” with respect to \( D \)) that allows us to replace \( D \) by \( F \) before starting the process described above.

1.2. Notation

We apply some standard notation from set theory. Variables \( \alpha, \beta, \gamma \) stand for ordinals, the smallest limit ordinal (i.e., the set of the natural numbers) is \( \omega \). For a family of sets \( \mathcal{X} \), the union of the elements of \( \mathcal{X} \) is denoted by \( \bigcup \mathcal{X} \). For an ordered pair \( \{\{u\},\{u,v\}\} \), we write simply \( uv \). We use the abbreviations \( X - x \) and \( X + x \) for \( X \setminus \{x\} \) and \( X \cup \{x\} \), respectively.

Let an infinite vertex set \( V \) and a “root vertex” \( r \in V \) be fixed through the paper. A digraph is a subset of \( V \times V \). The vertex set \( V \) and hence the digraphs may have arbitrary large infinite size (except in the proof of the
main result in section 2 where we will restrict it to countably infinite). The set of the ingoing and outgoing edges of a \( v \in V \) with respect to \( D \) is denoted by \( \text{in}_D(v) \) and \( \text{out}_D(v) \), respectively. For the set of the in-neighbours of a vertex \( v \) we write \( N_D^{\text{in}}(v) \), and \( N_D^{\text{out}}(v) \) stands for the out-neighbours. In several definitions and statements the root \( r \) will play a special role while the ingoing edges of \( r \) are irrelevant. This motivates to define a rooted digraph as a digraph \( D \) with \( \text{in}_D(r) = \emptyset \).

A \( v_0 \to v_n \) path for \( v_0 \neq v_n \in V \) is a digraph \( P = \{ v_0v_1, v_1v_2, \ldots, v_{n-1}v_n \} \), where \( v_i \in V \) are pairwise distinct. The singleton \( \{ v \} \) is considered a \( v \to v \) path. We say that \( P \) is an \( X \to Y \) path for some \( X,Y \subseteq V \) if exactly the first vertex of \( P \) is in \( X \) and exactly the last is in \( Y \). Let \( D \) be a digraph and let \( X,Y,S \subseteq V \). If every \( X \to Y \) path of \( D \) meets \( S \subseteq V \), then we say that \( S \) separates \( Y \) from \( X \) (or \( S \) is an \( XY \)-separation) in \( D \). A system \( P \) of \( u \to v \) paths is internally disjoint if the common vertices of any \( P \neq Q \in P \) are exactly \( u \) and \( v \). A path-system \( P \) is an \( r \)-fan if any two distinct paths in \( P \) have only their initial vertex \( r \) in common. If the terminal vertex \( v \) is the only common vertex, then we call the path-system a \( v \)-infan. Let \( V_{\text{first}}(P) \) be the set of the first vertices of the paths in \( P \) and we define \( V_{\text{last}}(P) \) analogously.

The set of the last edges of the paths in \( P \) is denoted by \( A_{\text{last}}(P) \). For a rooted digraph \( D \) and \( v \in V - r \), let us denote by \( G_D(v) \) the set of those \( I \subseteq \text{in}_D(v) \) for which there is a system \( P \) of internally disjoint \( r \to v \) paths in \( D \) with \( A_{\text{last}}(P) = I \). The rooted digraph \( F \) is a vertex-flame if \( \text{in}_F(v) \subseteq G_F(v) \) for every \( v \in V - r \). If for every \( v \in V - r \) and for every finite \( I \subseteq \text{in}_F(v) \) we have \( I \subseteq G_F(v) \), then \( F \) is defined to be a quasi-vertex-flame. To improve the flow of words, we write simply flame, quasi-flame and large instead of vertex-flame, quasi-vertex-flame and vertex-large (except in Theorems, Lemmas etc.). The edge version of these concepts appear only among the open problems in the last section hence it will not lead to confusion.

2. The proof of the main result

In this section we state our key lemmas without proofs and derive our main result from them.

**Lemma 2.1.** For every rooted digraph \( D \), there is a quasi-vertex-flame \( F \subseteq D \) such that whenever an \( L \subseteq F \) is \( F \)-vertex-large it is \( D \)-vertex-large as well.

**Lemma 2.2.** Let \( L \subseteq D \) be rooted digraphs such that for every \( v \in V - r \) with \( \text{in}_L(v) \subseteq \text{in}_D(v) \) there is a \( P \in \mathcal{I}_D(v) \) that lies in \( L \). Then \( L \) is \( D \)-vertex-large.
Lemma 2.3. If $D$ is a quasi-vertex-flame and $L$ is $D$-vertex-large, then $L$ is a quasi-vertex-flame as well.

Theorem 2.4 (Pym, [5]). Let $D$ be a digraph and let $\mathcal{P}, \mathcal{Q}$ be systems of disjoint $X \to Y$ paths for some $X, Y \subseteq V$. Then there is a system $\mathcal{R}$ of disjoint $X \to Y$ paths for which $V_{\text{first}}(\mathcal{R}) \supseteq V_{\text{first}}(\mathcal{P})$ and $V_{\text{last}}(\mathcal{R}) \supseteq V_{\text{last}}(\mathcal{Q})$.

Proof of Theorem 1.2. Let $V = \{v_n\}_{n<\omega}$. We may assume by Lemma 2.1 that $D$ is a quasi-flame.

We construct by recursion a sequence $(\mathcal{P}_n)_{n<\omega}$ such that for every $n<\omega$:

1. $\mathcal{P}_n \in \mathcal{I}_D(v_n)$,
2. $A_{\text{last}}(\mathcal{P}_n) \supseteq \bigcup_{m<n} \text{in}_{\mathcal{P}_m}(v_n)$,
3. $\text{in}_{\mathcal{P}_n}(v_m) \subseteq A_{\text{last}}(\mathcal{P}_m)$ for $m < n$.

Let us show first that if the construction is done, then the union $E$ of the edge sets of the path-systems $\mathcal{P}_n$ form a $D$-large flame. Indeed, largeness of $E$ follows immediately from property 1. Properties 2 and 3 state together that $\text{in}_{\mathcal{P}_n}(v_n) \subseteq A_{\text{last}}(\mathcal{P}_n)$ for $m, n < \omega$ thus $\mathcal{P}_n$ ensures $\text{in}_E(v_n) \in \mathcal{G}_E(v_n)$ which means that $E$ is a flame.

Let $\mathcal{P}_0 \in \mathcal{I}_D(v_0)$ be arbitrary. Suppose that $\mathcal{P}_m$ is defined for $m < n$, where $n > 0$ and so far the conditions hold. Delete those ingoing edges of $v_0, v_1, \ldots, v_{n-1}$ from $D$ that we cannot use in the construction of $\mathcal{P}_n$ according to property 3 and let us denote the remaining digraph by $D_n$. Since properties 2 and 3 hold so far, for $\ell, m < n$ we have $\text{in}_{\mathcal{P}_\ell}(v_m) \subseteq A_{\text{last}}(\mathcal{P}_m)$. Therefore $D_n$ contains the path-systems $\mathcal{P}_0, \ldots, \mathcal{P}_{n-1}$. Thus, we may conclude by Lemma 2.2 that $D_n$ is $D$-large. Hence, Lemma 2.3 guarantees that $D_n$ is a quasi-flame. Take a $\mathcal{P} \in \mathcal{I}_D(v_n)$ that lies in $D_n$ and take an $S$ consisting of exactly one internal vertex from each path in $\mathcal{P} - \{rv_n\}$ that separates $v_n$ from $r$ in $D - rv_n$. Let $\mathcal{P}'$ consist of the segments of paths in $\mathcal{P} - \{rv_n\}$ from $S$ to $N^\text{in}_{D - rv_n}(v_n)$ (see Figure 1). Let us denote $(\bigcup_{m<n} \text{in}_{\mathcal{P}_m}(v_n)) - rv_n$ by $J$. Note that $|J| \leq n$ since each of $\mathcal{P}_0, \ldots, \mathcal{P}_{n-1}$ uses at most one ingoing edge of $v_n$. Then $J \in \mathcal{G}_{D_n}(v_n)$ because $D_n$ is a quasi-flame. Take a $\mathcal{Q}$ that witnesses $J \in \mathcal{G}_{D_n}(v_n)$ and let $\mathcal{Q}'$ be the set of the segments of the paths in $\mathcal{Q}$ from the last intersection with $S$ to $N^\text{in}_{D - rv_n}(v_n)$.

By applying Pym’s theorem (see Theorem 2.4) with $\mathcal{P}'$ and $\mathcal{Q}'$, we obtain a system $\mathcal{R}'$ of disjoint $S \to N^\text{in}_{D - rv_n}(v_n)$ paths where $V_{\text{first}}(\mathcal{R}') = S$ and $V_{\text{last}}(\mathcal{R}')$ contains the tails of the edges in $J$. We extend $\mathcal{R}'$ to a $v_n$-inflame $\mathcal{R}$ that uses all the edges in $J$. Finally, we build $\mathcal{P}_n$ by joining the initial segments of the paths in $\mathcal{P} - \{rv\}$ up to $S$ with the paths in $\mathcal{R}$ and by adding the path $\{rv_n\}$ if $rv_n \in D$. The construction ensures that $\mathcal{P}_n \in \mathcal{I}_D(v_n)$.
3. Proof of the lemmas

3.1. Preliminaries

Theorem 3.1 (R. Aharoni, E. Berger; Theorem 1.6 of [1]). For every digraph $D$ and $X, Y \subseteq V$, there is a system $\mathcal{P}$ of disjoint $X \rightarrow Y$ paths in $D$ such that one can choose exactly one vertex from each path in $\mathcal{P}$ in such a way that the resulting vertex set $S$ separates $Y$ from $X$ in $D$.

For a rooted digraph $D$ and $v \in V - r$, let $S_D(v)$ be the set of those $S \subseteq V \setminus \{r, v\}$ that separates $v$ from $r$ in $D - rv$ and for which $D$ admits a system $\mathcal{P}$ of internally disjoint $r \rightarrow v$ paths such that $S$ consists of choosing exactly one internal vertex from each path in $\mathcal{P}$. We call $S_D(v)$ the set of the Erdős-Menger separations corresponding to $v$ in the rooted digraph $D$. By applying Theorem 3.1 with $X := N_{D - rv}^\text{out}(r)$ and $Y := N_{D - rv}^\text{in}(v)$ in $D - rv$, we conclude that $S_D(v) \neq \emptyset$ for every rooted digraph $D$ and $v \in V - r$. Note that if $S \in \mathcal{S}_D(v)$ is witnessed by $\mathcal{P}$, then either $\mathcal{P} \in \mathcal{I}_D(v)$ or $\mathcal{P} + \{rv\} \in \mathcal{I}_D(v)$ depending on if $rv \in D$. Therefore $\mathcal{I}_D(v) \neq \emptyset$. Furthermore, if $S \in \mathcal{S}_G(v) \cap \mathcal{S}_D(v)$, where $G \subseteq D$, then any path-system showing $S \in \mathcal{S}_G(v)$ exemplifies $S \in \mathcal{S}_D(v)$ as well.

Corollary 3.2. The rooted digraph $L \subseteq D$ is $D$-vertex-large if and only if $L \supseteq \text{out}_D(r)$ and $\mathcal{S}_L(v) \cap \mathcal{S}_D(v) \neq \emptyset$ for every $v \in V - r$.

Since for a flame $F \subseteq D$, $F \cup \text{out}_D(r)$ remains a flame, finding a $D$-large flame is equivalent with finding a flame that preserves an Erdős-Menger separation for each $v \in V - r$.

An augmenting walk for a system $\mathcal{P}$ of disjoint $X \rightarrow Y$ paths in a digraph $D$ is a finite $W \subseteq D$ such that the symmetric difference of $W$ and $\bigcup \mathcal{P}$ is (the
edge set of) a system of disjoint $X \to Y$ paths $\mathcal{Q}$ covering one more vertex from $X$ and from $Y$ than $\mathcal{P}$. The name comes from the fact that if such a $W$ exist, then it is possible to find one as a walk in a certain auxiliary digraph.

**Lemma 3.3 (Augmenting walk).** Let $D$ be a digraph and let $\mathcal{P}$ be a system of disjoint $X \to Y$ paths in $D$ for some $X,Y \subseteq V$. There is either an $XY$-separation $S$ consisting of exactly one vertex from each path of $\mathcal{P}$ or there is a system $\mathcal{Q}$ of disjoint $X \to Y$ paths in $D$ such that $|\mathcal{P} \setminus \mathcal{Q}| + 1 = |\mathcal{Q} \setminus \mathcal{P}| < \aleph_0$, $V_{\text{first}}(\mathcal{Q}) \supseteq V_{\text{first}}(\mathcal{P})$ and $V_{\text{last}}(\mathcal{Q}) \supseteq V_{\text{last}}(\mathcal{P})$.

For more details about the Augmenting walk lemma and its role in the proof of the Aharoni-Berger theorem we refer to Lemmas 3.3.2. and 3.3.3. and Theorem 8.4.2. of [2] (it is discussed for undirected graphs but the directed case involves no additional ideas).

It is worth mentioning an alternative characterisation of $\mathcal{I}_D(v)$ (follows from Theorem 4.7 of [1]). A system $\mathcal{P}$ of internally disjoint $r \to v$ paths in $D$ is called **strongly maximal** if for every internally disjoint system $\mathcal{Q}$ of $r \to v$ paths in $D$, $|\mathcal{Q} \setminus \mathcal{P}| \leq |\mathcal{P} \setminus \mathcal{Q}|$.

**Proposition 3.4.** For every rooted digraph $D$ and $v \in V - r$, $\mathcal{I}_D(v)$ is the set of the strongly maximal internally disjoint $r \to v$ path-systems of $D$.

**Proof.** Assume first $rv \notin D$. Let $\mathcal{P} \in \mathcal{I}_D(v)$ and pick an $S$ that separates $v$ from $r$ and consists of choosing one internal vertex from each path in $\mathcal{P}$. Let $\mathcal{Q}$ be a system of internally disjoint $r \to v$ paths. Then the paths $\mathcal{P} \setminus \mathcal{Q}$ use exactly $|\mathcal{P} \setminus \mathcal{Q}|$ vertices from $S$ and each path in $\mathcal{Q} \setminus \mathcal{P}$ goes through at least one of these vertices. Since the paths $\mathcal{Q} \setminus \mathcal{P}$ are internally disjoint, $|\mathcal{Q} \setminus \mathcal{P}| \leq |\mathcal{P} \setminus \mathcal{Q}|$ follows. To show the other direction, let $\mathcal{P}$ be a strongly maximal system of internally disjoint $r \to v$ paths in $D$. Let $X := N^{\text{out}}_D(r)$ and $Y := N^{\text{in}}_D(v)$. It is easy to check that the set of the $X \to Y$ segments $\mathcal{P}'$ of the paths in $\mathcal{P}$ form a strongly maximal system of disjoint $X \to Y$ paths. By applying Lemma 3.3 with $\mathcal{P}'$, we obtain an $S$ which exemplifies $\mathcal{P} \in \mathcal{I}_D(v)$.

If $rv \in D$, then $\mathcal{P}$ is strongly maximal if and only if $\{rv\} \in \mathcal{P}$ and $\mathcal{P} - \{rv\}$ is strongly maximal in $D - rv$. Since $\mathcal{P} \in \mathcal{I}_D(v)$ if and only if $\{rv\} \in \mathcal{P}$ and $\mathcal{P} - \{rv\} \in \mathcal{I}_{D - rv}(v)$, we are done by applying the proved case in $D - rv$. 

### 3.2. Uniting bubbles

Let $D$ be a rooted digraph. The entrance $\text{ent}_D(X)$ of an $X \subseteq V - r$ with respect to $D$ is $\{v \in X: \exists uv \in D \text{ with } u \notin X\}$. We write $\text{int}_D(X)$ for $X \setminus \text{ent}_D(X)$. A set $B \subseteq V - r$ is a $v$-bubble with respect to $D$ if there exists
a \( v \)-infan \( \mathcal{P} = \{ P_u : u \in \text{ent}_D(B) \} \) in \( D \cap (B \times B) \) where \( P_u \) starts at \( u \). Let us denote the set of the \( v \)-bubbles in \( D \) by \( \text{bubb}_D(v) \). Clearly, \( \{ v \} \in \text{bubb}_D(v) \) since either the \( v \rightarrow v \) path or the empty set is a witness for it depending on if \( v \in \text{ent}_D(\{ v \}) \).

**Lemma 3.5.** Let \( D \) be a rooted digraph and let \( \alpha \) be an ordinal number. Suppose that \( \langle B_\beta : \beta < \alpha \rangle \) is a sequence where \( B_\beta \in \text{bubb}_D(v_\beta) \) for some \( v_\beta \in V - r \). Let us denote \( \bigcup_{\gamma < \beta} B_\gamma \) by \( B_{< \beta} \). If for each \( \beta < \alpha \) either \( v_\beta = v_0 \) or \( v_\beta \in \text{int}_D(B_{< \beta}) \), then \( B_{< \alpha} \in \text{bubb}_D(v_0) \).

**Proof.** For every \( u \in \bigcup_{1 \leq \beta \leq \alpha} \text{ent}_D(B_{< \beta}) \), we construct a \( u \rightarrow v_0 \) path \( P_u \) in such a way that for each \( \beta \) the paths \( \{ P_u : u \in \text{ent}_D(B_{< \beta}) \} \) exemplify \( B_{< \beta} \in \text{bubb}_D(v_0) \).

Let \( \{ P_u : u \in \text{ent}_D(B_{< 1}) \} \) be an arbitrary path-system witnessing \( B_0 \in \text{bubb}_D(v_0) \). Suppose that \( \beta > 1 \) and \( P_u \) is defined whenever there is a \( \gamma < \beta \) for which \( u \in \text{ent}_D(B_{< \gamma}) \). If \( \beta \) is a limit ordinal, then \( P_u \) is defined for \( u \in \text{ent}_D(B_{< \beta}) \) as well and the conditions hold.

Assume that \( \beta = \gamma + 1 \). Fix a path-system \( Q = \{ Q_u : u \in \text{ent}_D(B_\gamma) \} \) which shows \( B_\gamma \in \text{bubb}_D(v_\gamma) \). Note that \( \text{ent}_D(B_{< \gamma + 1}) \setminus \text{ent}_D(B_{< \gamma}) \subseteq \text{ent}_D(B_\gamma) \) (see Figure 2). Since \( v_\gamma \in \{ v_0 \} \cup \text{int}_D(B_{< \gamma}) \), all the paths in \( Q \) meet \( B_{< \gamma} \). Furthermore, if two paths in \( Q \) have the same vertex as first meeting with \( B_{< \gamma} \), then it must be \( v_\gamma \) and hence \( v_\gamma = v_0 \) holds since in this case \( v_\gamma \notin \text{int}_D(B_{< \gamma}) \). For \( u \in \text{ent}_D(B_{< \gamma + 1}) \setminus \text{ent}_D(B_{< \gamma}) \), consider the initial segment \( Q'_u \) of \( Q_u \) up to the first vertex \( w \) that is in \( B_{< \gamma} \). Join \( Q'_u \) and \( P_w \) to obtain \( P_u \). 

![Figure 2. The construction of the path-system witnessing \( B_{< \gamma + 1} \in \text{bubb}_D(v_0) \). (Vertex \( v_0 \) can be in \( \text{ent}_D(B_{< \gamma}) \)](image)

**Corollary 3.6.** For every rooted digraph \( D \) and \( v \in V - r \), \( \text{bubb}_D(v) \) is closed under arbitrary large union.
Corollary 3.7. For every rooted digraph $D$ and $v \in V - r$, there is a $\subseteq$-largest element of $\text{bubb}_D(v)$, namely $\bigcup \text{bubb}_D(v) =: B_{v,D}$.

3.3. Preserving largeness

For $S \in \mathcal{S}_D(v)$, we denote by $B_{S,v,D}$ the set of those $u \in V$ for which every $r \rightarrow u$ path in $D - rv$ meets $S$.

Proposition 3.8. Assume that $D$ is a rooted digraph, $v \in V - r$ and $S \in \mathcal{S}_D(v)$. Then $B_{S,v,D} \in \text{bubb}_D(v)$ with $\text{ent}_{D - rv}(B_{S,v,D}) = S$ and $N^\text{in}_{D - rv}(v) \subseteq B_{S,v,D}$.

Proof. The inclusions $B_{S,v,D} \supseteq S \supseteq \text{ent}_{D - rv}(B_{S,v,D})$ are clear from the definition of $B_{S,v,D}$ as well as $r \notin B_{S,v,D}$ and $v \in B_{S,v,D}$. Suppose for a contradiction that there is a $u \in S \setminus \text{ent}_{D - rv}(B_{S,v,D})$. By the choice of $S$, there is a system $\mathcal{P}$ of internally disjoint $r \rightarrow v$ paths in $D - rv$ for which $S$ consists of choosing one internal vertex from each path in $\mathcal{P}$. The unique path $P_u \in \mathcal{P}$ which goes through $u$ enters $B_{S,v,D}$ and therefore meets $\text{ent}_{D - rv}(B_{S,v,D})$. But then $P_u$ has at least two vertices in $S$ since $S \supseteq \text{ent}_{D - rv}(B_{S,v,D})$ which is a contradiction. Hence $S = \text{ent}_{D - rv}(B_{S,v,D})$. The terminal segments of the paths in $\mathcal{P}$ from $S$ witnessing $B_{S,v,D} \in \text{bubb}_D(v)$. Finally, if $w \in N^\text{in}_{D - rv}(v) \setminus B_{S,v,D}$, then $w$ is reachable from $r$ without touching $S$ and hence $v$ as well (since $v \notin S$) which is impossible.

Proposition 3.9. For every rooted digraph $D$ and $v \in V - r$,

$$\text{ent}_{D - rv}(B_{v,D}) \in \mathcal{S}_D(v).$$

Proof. Let $S \in \mathcal{S}_D(v)$ be arbitrary. It follows from Proposition 3.8 that $B_{v,D} \supseteq S$. Thus $\text{ent}_{D - rv}(B_{v,D})$ separates $S$ from $r$ in $D - rv$ and therefore it separates $v$ from $r$ in $D - rv$ as well. It remains to show a system $\{P_u : u \in \text{ent}_{D - rv}(B_{v,D})\}$ of internally disjoint $r \rightarrow v$ paths where $P_u$ goes through $u$. Since $B_{v,D} \in \text{bubb}_D(v)$, it is enough to prove that there is an $r$-fan $\{Q_u : u \in \text{ent}_{D - rv}(B_{v,D})\}$ in $D - rv$ where $Q_u$ terminates at $u$. To do so, apply the Aharoni-Berger theorem (Theorem 3.1) in $D - rv$ with $X := N^\text{out}_{D - rv}(r)$ and $Y := \text{ent}_{D - rv}(B_{v,D})$ (see Figure 3 without the paths). Observe that the resulting separation $T$ is in $\mathcal{S}_D(v)$. By Proposition 3.8, $B_{T,v,D} \in \text{bubb}_D(v)$, which implies $B_{T,v,D} \subseteq B_{v,D}$. It follows that $T \subseteq \text{ent}_{D - rv}(B_{v,D})$. Suppose for a contradiction that $w \in \text{ent}_{D - rv}(B_{v,D}) \setminus T$. Since $w \notin \text{ent}_{D - rv}(B_{v,D})$, we can pick a $uw \in D - rv$ with $u \notin B_{v,D}$. Since $T$ separates $w \in \text{ent}_{D - rv}(B_{v,D})$ from $X$ in $D - rv$ and $w \notin T$, we may conclude that $T$ separates $u$ from $X$ in $D - rv$ as well. But then $u \in B_{T,v,D} \subseteq B_{v,D}$ contradicts $u \notin B_{v,D}$. $\square$
Lemma 3.10 (Characterisation of largeness). Let $L \subseteq D$ be rooted digraphs. Then $L$ is $D$-vertex-large if and only if $u \in B_{v,L}$ for every $uv \in D \setminus L$. Furthermore, if $L$ is $D$-vertex-large and $v \in V - r$, then $\text{ent}_{D-rv}(B_{v,L}) = \text{ent}_{L-rv}(B_{v,L}) \in \mathcal{S}_D(v)$.

Proof. Assume that $L$ is $D$-large and let $uv \in D \setminus L$. By applying the reformulation of largeness in Corollary 3.2 we know that $u \neq r$ and there is some $S \in \mathcal{S}_L(v) \cap \mathcal{S}_D(v)$. Proposition 3.8 guarantees $u \in B_{S,v,D}$. Since $L \subseteq D$, it follows directly from the definition that $B_{S,v,D} \subseteq B_{S,v,L}$. By combining these, we obtain $u \in B_{S,v,D} \subseteq B_{S,v,L} \subseteq B_{v,L}$.

For the “if” direction, take an arbitrary $v \in V - r$. If $rv \in D$, then $rv \in L$ because $r \in B_{v,L}$ is impossible by the definition of bubbles. By Proposition 3.9, we know that $\text{ent}_{L-rv}(B_{v,L}) \in \mathcal{S}_L(v)$. Suppose for contradiction that $\text{ent}_{L-rv}(B_{v,L}) \notin \mathcal{S}_D(v)$. The only possible reason for this is that $\text{ent}_{L-rv}(B_{v,L})$ does not separate $v$ from $r$ in $D-rv$ (just in $L-rv$). It follows that there is a $w \in \text{ent}_{D-rv}(B_{v,L}) \setminus \text{ent}_{L-rv}(B_{v,L})$ witnessed by some edge $uw \in D \setminus L$. Then $w \in \text{int}_{L-rv}(B_{v,L})$ and by assumption $u \in B_{w,L}$. Hence, by Lemma 3.5, $B_{v,L} \cup B_{w,L} \in \mathcal{bubb}_L(v)$ which is a contradiction because $u$ witnesses $B_{v,L} \subsetneq (B_{v,L} \cup B_{w,L})$.

Proof of Lemma 2.2. To show the largeness of $L$, we use the characterization of largeness in Lemma 3.10. Let $uv \in D \setminus L$ be arbitrary. By assumption there is a $\mathcal{P} \in \mathcal{D}(v)$ that lies in $L$. We pick an $S$ witnessing $\mathcal{P} \in \mathcal{D}(v)$. By Proposition 3.8, $u \in B_{S,v,D}$. Therefore $u \in B_{S,v,D} \subseteq B_{S,v,L} \subseteq B_{v,L}$. Since $uv \in D \setminus L$ was arbitrary, we may conclude by Lemma 3.10 that $L$ is large.

3.4. A “largeness-faithful” quasi-flame

Claim 3.11. For every rooted digraph $D$, $v \in V - r$ and $u \in (V \setminus B_{v,D}) - r$, there is an $r$-fan $\mathcal{P}$ in $D-rv$ with $V_{\text{last}}(\mathcal{P}) = \text{ent}_{D-rv}(B_{v,D}) + u$. 
Proof. It follows from Proposition 3.9 that there is an \( r \)-fan \( Q \) with \( V_{\text{last}}(Q) = \text{ent}_{D-rv}(B_v,D) \). Let \( Q' := Q \) if \( Q \) does not meet \( u \) otherwise replace the unique \( Q \in Q \) through \( u \) by its initial segment up to \( u \) to obtain \( Q' \) from \( Q \). Either way \( Q' \) is an \( r \)-fan such that \( V_{\text{last}}(Q') \) consists of all but one element of \( \text{ent}_{D-rv}(B_v,D) + u =: Y \) and the paths \( Q' \) can be extended forward to get an internally disjoint system of \( r \to v \) paths. Let \( X := N^\text{out}_{D-rv}(r) \) and let \( Q'' \) consist of the \( X \to Y \) (terminal) segments of the paths \( Q' \) (see Figure 3). We apply the Augmenting walk method (Lemma 3.3) with \( Q'' \) in \( D-rv \). If the augmentation is possible, then the resulting path-system together with the corresponding edges from \( r \) is appropriate for \( P \). Suppose for a contradiction that the augmentation is impossible. Then we obtain an \( S \in \mathcal{G}_D(v) \) that separates \( \text{ent}_{D-rv}(B_v,D) + u \) from \( N^\text{out}_{D-rv}(r) \) in \( D-rv \). By the choice of \( u \), \( B_{S,v,D} \supset B_{v,D} \) which is a contradiction.

Lemma 3.12. Let \( D \supseteq G \supseteq H \) be rooted digraphs and let \( v \in V - r \). Suppose that there is a \( uw \in D \setminus G \) with \( u \notin B_{v,H} \) and \( w \in \text{int}_{H-rv}(B_{v,H}) \). If \( \text{ent}_{H-rc}(B_{v,H}) = \text{ent}_{G-rc}(B_{v,H}) \), then \( (I+uw) \in \mathcal{G}_{G+uw}(w) \) for all \( I \in \mathcal{G}_G(w) \).

Proof. The statement is trivial for \( u = r \) thus let \( u \neq r \). We may assume that \( rw \notin I \) otherwise we apply the Lemma first with \( I-rw \) and then adding \( rw \) cannot ruin anything. By Claim 3.11, there is an \( r \)-fan \( P \) in \( H-rv \) with \( V_{\text{last}}(P) = \text{ent}_{H-rc}(B_{v,H}) + u \). Then \( P \) is an \( r \)-fan in \( G-rv \) for which \( V_{\text{last}}(P) = \text{ent}_{G-rc}(B_{v,H}) + u \) because \( G \supseteq H \) and \( \text{ent}_{H-rc}(B_{v,H}) = \text{ent}_{G-rc}(B_{v,H}) \) by assumption. Let \( Q \) be a path-system witnessing \( I \in \mathcal{G}_G(w) \). Continue forward the paths in \( P \) (see Figure 4) using the terminal segments \( Q' \) of the paths in \( Q \) from the last intersection with \( \text{ent}_{G-rc}(B_{v,H}) \) and edge \( uw \) to obtain a path-system witnessing \( (I+uw) \in \mathcal{G}_{G+uw}(w) \).
Lemma 3.13. Let $D \supseteq G$ be rooted digraphs. If for every $uv \in D \setminus G$ there is an $I \in \mathcal{G}_G(v)$ such that $(I + uv) \notin \mathcal{G}_{G+uv}(v)$, then whenever an $H$ is $G$-vertex-large it is $D$-vertex-large as well.

Proof. Suppose for a contradiction that $H$ is $G$-large but not $D$-large. By applying Lemma 3.10 with $D$ and $H$, we obtain that there is some $uv \in D \setminus H$ with $u \notin B_{v,H}$. Since $H$ is assumed to be $G$-large, we may conclude by Lemma 3.10 that $uv \notin G$ and $\text{ent}_{G-rv}(B_{v,H}) = \text{ent}_{H-rv}(B_{v,H})$. Finally, $v \in \text{int}_{H-rv}(B_{v,H})$ by Proposition 3.9. We use Lemma 3.12 with $w := v$ to obtain $(I + uv) \notin \mathcal{G}_{G+uv}(v)$ for every $I \in \mathcal{G}_G(v)$ which is a contradiction. □

Proof of Lemma 2.1. By applying Zorn’s lemma, we may pick a $\subseteq$-maximal quasi-flame $F \subseteq D$. Lemma 3.13 ensures that whenever an $L$ is $F$-large it is $D$-large as well. □

3.5. Preserving the quasi-flame property via preserving largeness

Claim 3.14. Let $D$ be a rooted digraph and let $X \subseteq V - r$ be finite. Suppose that there is an $r$-fan $P$ in $D$ with $V_{\text{last}}(P) = X$. If $L$ is $D$-vertex-large, then there is an $r$-fan $Q$ in $L$ with $V_{\text{last}}(Q) = X$.

Proof. Suppose for a contradiction that $L$ does not contain a desired $r$-fan $Q$. We may assume that there is a $v \in V$ which is an isolated vertex in $D$ (otherwise we consider an isomorphic copy of $D$ on a proper subset of $V$). Extend $D$ and $L$ with the edges $xv$ ($x \in X$) to obtain $D'$ and $L'$, respectively. Clearly $\kappa_{L'}(r, v) < \kappa_{D'}(r, v) = |X|$. Since $rv \notin D'$, Proposition 3.9 ensures $\kappa_{L'}(r, v) = |\text{ent}_{L'}(B_{v,L'})|$. By combining these, $\kappa_{L'}(r, v) = \kappa_{L'}(r, v) = |\text{ent}_{L'}(B_{v,L'})| < |X| < \aleph_0$ (see Figure 5). It follows that there is a $uw \in D \setminus L$ with $u \notin B_{v,L'}$ and $w \in \text{int}_{L'}(B_{v,L'})$ otherwise $\text{ent}_{L'}(B_{v,L'})$ would separate $v$ from $r$ in $D'$ as well.

Figure 5. $B_{v,L'}$ and $uw \in D \setminus L$ ($X$ may have common vertices with $\text{ent}_{L'}(B_{v,L'})$)
By applying 3.12 with $D'$ and $G := H := L'$, we may conclude that 
$(I + uw) \in G_{L'+uw}(w)$ for all $I \in G_{L'}(w)$. From the construction it is clear 
that $v$ has no outgoing edges and $w \neq v$ therefore $(I + uw) \in G_{L'+uw}(w)$ holds 
for all $I \in G_L(w)$ as well. The finite separation $\text{ent}_{L'}(B_v,L')$ witnesses that 
$\kappa_L(r,v) < \aleph_0$. Since $L$ is $D$-large, $\kappa_D(r,v) = \kappa_L(r,v) < \aleph_0$. Take a system 
$\mathcal{P}$ of internally disjoint $r \rightarrow w$ paths in $L$ with $|\mathcal{P}| = \kappa_D(r,w)$. Then for 
$I = A_{\text{last}}(\mathcal{P})$, we obtain $(A_{\text{last}}(\mathcal{P}) + uw) \in G_{L'+uw}(w)$, which implies that $L$ 
contains a system of internally disjoint $r \rightarrow w$ paths of size $|A_{\text{last}}(\mathcal{P})| + 1 = \kappa_D(r,v) + 1 < \aleph_0$ 
which is a contradiction.

Proof of Lemma 2.3. Suppose that $D$ is a quasi-flame and $L$ is $D$-large. 
Let $v \in V - r$ be arbitrary. Assume first that $\kappa_D(r,v) < \aleph_0$. Then $\kappa_D(r,v) = |\text{in}_D(v)|$ follows from the fact that $D$ is a quasi-flame. By the largeness 
of $L$, we have $\kappa_L(r,v) = \kappa_D(r,v)$. By combining these, we may conclude 
$\kappa_L(r,v) = |\text{in}_D(v)| < \aleph_0$ which means $\text{in}_D(v) = \text{in}_L(v) \in G_L(v)$.

Suppose now that $\kappa_D(r,v) \geq \aleph_0$ and let $J \subseteq \text{in}_L(v)$ be finite. Since $D$ 
is a quasi-flame we can pick a $\mathcal{P}$ witnessing $J \in G_D(v)$. Let $\mathcal{P}'$ be the $r$-fan 
that we get by the deletion of the last edges of the paths $\mathcal{P}$. Since none 
of the paths in $\mathcal{P}'$ goes through $v$ and $\kappa_D(r,v) \geq \aleph_0$, we can extend $\mathcal{P}'$ in 
$D$ by a new path to obtain an $r$-fan $\mathcal{P}''$ where $V_{\text{last}}(\mathcal{P}'') = X$ consists of the tails of the edges in $J$ and $v$. By Claim 3.14, there is an $r$-fan $\mathcal{Q}''$ in 
$L$ with $V_{\text{last}}(\mathcal{Q}'') = X$. To obtain $\mathcal{Q}'$, we delete the unique path in $\mathcal{Q}$ which 
terminates $v$. Then none of the paths in $\mathcal{Q}'$ goes through $v$ and hence by 
extending them with the edges $J$, the resulting path-system $\mathcal{Q}$ witnesses 
$J \in G_L(v)$.

4. Open problems

4.1. Beyond countability

One can replace in Theorem 1.2 the countability of $D$ by the formally weaker 
assumption that $\kappa_D(r,v) \leq \aleph_0$ for every $v \in V - r$ (it is an easy application 
of Davies-trees [6]). We believe that more is true.

Conjecture 4.1. We may omit the countability of $D$ in Theorem 1.2.

4.2. Upper and lower bounds

Question 4.2. Let $F \subseteq L \subseteq D$ be a rooted digraphs where $F$ is a vertex-
flame and $L$ is $D$-vertex-large. Is there necessarily an $E$ with $F \subseteq E \subseteq L$ such 
that $E$ is a vertex-flame which is $D$-vertex-large?
It is true if $\kappa_D(r,v) < \aleph_0$ for $v \in V - r$. We can simply take a $\subseteq$-maximal quasi-flame in $L$ which extends $F$. It is automatically a flame and its $L$-largeness (see the proof before subsection 3.5) implies that it is $D$-vertex-large as well.

4.3. Preserving all Erdős-Menger separations in a flame

Consider the reformulation of $D$-largeness in Corollary 3.2. If $\kappa_D(r,v) < \aleph_0$, then $\mathcal{S}_D(v) \cap \mathcal{S}_F(v) \neq \emptyset$ implies $\mathcal{S}_D(v) \subseteq \mathcal{S}_F(v)$. Thus, in a finite rooted digraph we preserve automatically every Erdős-Menger separation when we demand $D$-largeness but it is usually false for infinite digraphs. It seems a natural question if we can always preserve all the Erdős-Menger separations in a flame. We construct a rooted digraph $D$ of size $2^{\aleph_0}$ which witnesses that the answer is no (however, the question remains open for smaller digraphs).

Consider the digraph at Figure 6. Extend it with all the edges $v_i v_{j,k}$ ($i \leq \omega$, $j < \omega$, $k < 2$). For every $f \in \omega^2$, pick a new vertex $v_f$ with $N^\text{in}_D(v_f) := \{v_{i,f(i)} : i < \omega\}$ and $N^\text{out}_D(v_f) = \emptyset$. It is easy to check that $\{v_{i,0} v_i, v_{i,1} v_i\} \notin \mathcal{G}_D(v_i)$ for $i < \omega$ and $N^\text{out}_D(r) \in \mathcal{S}_D(v_f)$ for each $f \in \omega^2$. Let $F \subseteq D$ be a flame. Then there is an $f \in \omega^2$ such that $v_{i,1-f(i)} v_i /\notin F$ for $i < \omega$. We show that $N^\text{out}_D(r) /\notin \mathcal{S}_F(v_f)$. Suppose for a contradiction that $\mathcal{P} := \{P_w : w \in N^\text{out}_D(r)\}$ exemplifies that $N^\text{out}_D(r) \in \mathcal{S}_F(v_f)$ where $P_w$ goes through $w$. Then path $P_{u_i}$ necessarily uses vertex $v_{i,f(i)}$ because $u_i v_{i,1-f(i)} /\notin F$. But then the paths $P_{u_i}$ use already all the vertices $\{v_{i,f(i)} : i < \omega\} = N^\text{in}_D(v_f)$ and therefore $P_{v_\omega}$ has no chance to reach $v_f$ which is a contradiction.

![Figure 6. The first step of the construction of $D$](image-url)
4.4. Extending a flame by a $\mathcal{P} \in \mathcal{I}_D(v)$ keeping flame property

**Question 4.3.** Let $D$ be a rooted digraph and let $F \subseteq D$ be a vertex-flame. Is there necessarily for every $v \in V - r$ a $\mathcal{P} \in \mathcal{I}_D(v)$ such that $F \cup \bigcup \mathcal{P}$ is a vertex-flame?

One can prove that if the set of the new edges given by $\mathcal{P}$ to $F$ is $\subseteq$-minimal among the choices $\mathcal{I}_D(v)$, then $F \cup \bigcup \mathcal{P}$ is a flame. Calvillo-Vives proved (the edge version of) Theorem 1.1 based on this observation. Examples show that the premisses of the implication may fail to be satisfiable in infinite digraphs.

4.5. The edge version of flames and largeness

Let $D$ be a rooted digraph. For $v \in V - r$, we define $\mathcal{E}_D(v)$ to be the set of those edge-disjoint $r \rightarrow v$ path-systems $\mathcal{P}$ for which one can choose exactly one edge from each path in $\mathcal{P}$ in such a way that the resulting $C$ is an $rv$-cut in $D$. An $L \subseteq D$ is $D$-edge-large if for every $v \in V - r$ a $\mathcal{P} \in \mathcal{E}_D(v)$ lies in $L$. A rooted digraph $D$ is an edge-flame if for all $v \in V - r$ there is a system $\mathcal{P}$ of edge-disjoint $r \rightarrow v$ path such that $A_{\text{last}}(\mathcal{P}) = \text{in}_F(v)$.

**Question 4.4.** Does there exist a $D$-edge-large $E$ which is an edge-flame for every rooted digraph $D$?

It seems that most of the tools we developed works in the edge version as well. A major new difficulty is that a $\mathcal{P} \in \mathcal{E}_D(v)$ may give infinitely many ingoing edges to a vertex other than $v$ (not just at most one as in the vertex version) and therefore our quasi-flame approach is not sufficient itself to overcome this complication.

The edge version of the problem is stronger than the vertex version in the following sense. If the answer for Question 4.4 is yes, then one can derive the vertex version from it as we sketched in Remark 1.3. Similarly simple reduction in the other direction seems unlikely. The edge version analogues of all of our earlier open questions are also open.

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Attila Joó

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
MTA-ELTE Egerváry Research Group
jooattila@renyi.hu