Masses and electric charges: gauge anomalies and anomalous thresholds

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We study the quantum consistency of a theory with massless electrically charged particles i.e. of massless charged leptons. We show, working in the forward limit, that the KLN cancellation of collinear divergences can only be consistently implemented using quantum interference with processes contributing to the gauge anomaly. This rules out the possibility of a quantum consistent and anomaly free theory with massless charges and long range interactions. We use the anomalous threshold singularities to derive a gravitational lower bound on the mass of the lightest charged fermion.

For theories with long range gauge forces as QED the IR completion problem goes around the quantum consistency of a quantum field theory with massless charged particles in the physical spectrum. This is an old problem that has been considered from different angles along the years (see [1–4] for an incomplete list). As a matter of fact in Nature we don’t have any example of massless charged particles. In the Standard Model this is the case both for spin 1/2 as well as for the spin 1 charged vector bosons. In the particular case of charged leptons the potential inconsistency of a massless limit should imply severe constraints on the consistency of vanishing Yukawa couplings.

Technically the infrared (IR) origin of the problem is easy to identify. In the case of massive charged particles radiative corrections due to loops of virtual photons lead to two types of infrared problem. One can be solved, in principle, using the standard Bloch-Nordsiek-recipe [5] that leads to infrared finite inclusive cross sections at each order in perturbation theory depending on an energy resolution cutoff. In this case the infrared finite cross section is defined taking into account soft radiation. In the massless case we have in addition collinear divergences that contribute logarithmically to Weinberg’s $B$ factor [5, 6]. The standard recipe used to cancel these divergences requires to include in the definition of the inclusive cross section not only soft emission but also collinear hard emission and absorption (i.e. photons with energy bigger than the energy resolution scale) and to set an angular resolution scale.

In [9] a unifying picture to the problem was suggested on the basis of degenerations. The idea is to define, for a given amplitude $S_{i,f}$ associated with a given scattering process $i \rightarrow f$ an inclusive cross section formally defined as

$$
\sum_{i' \in D(i), f' \in D(f)} |S_{i',f'}|^2 , \tag{1}
$$

where $D(i)$ is the set of asymptotic states degenerate with the asymptotic state $i$.

For the case of massless electrically charged particles the degeneration used in [9] for the case where the asymptotic state is a charged massless lepton with momentum $p$ is a state with the lepton having momentum $p - k$ and an additional on-shell photon with 4-momentum $k$ collinear to $p$. At each order in perturbation theory the KLN recipe demands us to sum over all contributions at the same order in perturbation theory that we can build using degenerate incoming and/or outgoing states. Among these are specially interesting the quantum interference effects with disconnected diagrams where the additional photon entering into the definition of a degenerate incoming state is not interacting. In particular these interference effects play a crucial role to cancel collinear divergences in processes where the incoming electron emits a collinear photon, see for instance [9, 10].

The main target of this paper is to study the quantum consistency of the KLN prescription to define a quantum field theory of $U(1)$ massless charged particles. Our findings can be summarize in two main claims. On one side we shall argue that the consistency of the KLN prescription in the forward regime implies the existence of a non vanishing gauge anomaly for the $U(1)$ gauge theory. This rules out the possibility of the existence of a theory with massless charges and long range interactions satisfying both KLN cancellation and being anomaly free. Secondly combining the weak gravity conjecture [29] and anomalous thresholds for form factors, we derive a gravitational lower bound on the mass of the lighter massless charged fermion and a qualitative upper bound on the total number of fermionic species with the same charge as the electron.

I. THE KLN-THEOREM: DEGENERACIES AND ENERGY DRESSING

Let us briefly review the key aspect of the KLN theorem [9]. In order to do that let us consider scattering theory for a given Hamiltonian $H = H_0 + gH_I$ and let us

1 For more recent discussions on IR divergences see for instance [7, 8] and references therein.

2 For a recent discussion of the KLN theorem for QED see [10] and references therein.
assume the Hamiltonian depends on a parameter \( m \). Assuming a well defined scattering theory, the Hamiltonian \( H \) can be diagonalized using the corresponding Møller operators \( U \). Let us denote \( E_i(g, m) \) the corresponding eigenvalues. If for some value \( m_e \) of the mass parameter we have degenerations i.e. \( E_i(g, m_e) = E_j(g, m_e) \) then the perturbative expansion of \( U_{ij} \) becomes singular at each order in perturbation theory. However at the same order in perturbation theory the quantity \( \sum_a U_{ai}U_{ja} \) where we sum over the set of states degenerate with the state \( a \) is free of singularities in the limit \( m = m_e \) leading to the prescription \( (1) \) for finite cross sections. The former result is true provided \( \Delta_n(g, m) = (H_0 - E)_{aa} \) has a good finite limit for \( m = m_e \).

The quantum field theory meaning of \( \Delta \) is the difference of energy between the bare and the dressed state. The theorem works if for fixed and finite UV cutoff the limit of this dressing effect is finite in the degeneration limit.

For the case of QED and for \( m \) the mass of the electron, degeneracies appear in the limit \( m_e \to 0 \). As stressed in [9] in this case the limit of \( \Delta \) for \( m_e \to 0 \) and fixed UV cutoff is not finite. The problem is associated with the well known behavior of the renormalization constant \( Z \) for the photon field which goes as

\[
Z = 1 - \frac{e^2}{6\pi^2} \log \frac{\Lambda}{m_e}. \tag{2}
\]

The origin of the problem is well understood. Using Källen-Lehmann-representation to extract the value of \( Z \) from the imaginary part of the bubble amplitude i.e. \( \text{Im}D(k^2) \) for the photon propagator \( D(k^2) \), we get for massless electrons a branch cut singularity in the physical sheet for the threshold \( k^2 = 0 \) where the on-shell photon can go into a collinear pair of on-shell electron and positron.

In [9] this problem was explicitly addressed and the suggested solution was to keep \( m_e = 0 \) but to add a mass scale in the definition of \( Z \) (see [10]) associated with some IR resolution scale let us say \( \delta \). The logic of this argument is to assume an IR correction of \( (2) \) where effectively \( m_e \) is replaced by \( \delta \) and to use this corrected \( Z \) to define a \( \Delta \) non singular in the limit \( m_e \to 0 \).

Note that the singular limit of \( Z \) in the massless limit is the IR version of the famous Landau pole problem for QED. In this case we are not considering the limit where we send the UV cutoff to infinity but instead the limit \( m_e \to 0 \). In the massless limit there are contributions to the Källen-Lehmann-function coming from processes in which the on-shell photon with energy \( \omega \) produces a pair of electron positron both collinear and on-shell. Incidentally note that in principle we have contributions of amplitudes where the on-shell photon decays into a set of a large number \( n \) of electron-positron-pairs and photons where all of them are on-shell and collinear. The approach of the KLN program is to assume that after taking all these IR contributions into account the resulting \( Z \), for fixed UV cutoff, is finite in the limit \( m_e \to 0 \).

This does not imply solving the UV problem or avoiding the standard Landau pole, that depends on the sign in \( (2) \) and that now will become dependent on the added resolution scale \( \delta \). It simply means that for fixed UV cutoff the limit \( m_e \to 0 \) could be non singular. In section IV we will revisit the consistency of the limit \( m_e \to 0 \) from a different point of view.

Can we check the consistency of the KLN proposal perturbatively? To the best of our knowledge the KLN program of finding a redefinition of \( Z \) where the cancellation can be defined in an effective way has not been developed. Thus we should expect that perturbative violations of the KLN theorem could appear whenever we work in the kinematical regime where originally appears the singularity responsible for the former behaviour of \( Z \), namely in the forward regime \( q^2 = 0 \).

II. DEGENERACIES AND ANOMALOUS THRESHOLDS

In scattering theory the existence of anomalous thresholds for form factors of bound states is well known (see [12–14]). The idea is simply to consider the triangular contribution to the form factor of a particle \( A \) by some external potential. If the particle \( A \) can decay into a pair of particles \( N \) and \( B \) where only \( N \) interacts with the external potential we get the triangular amplitude depicted in figure 1. If we now impose all the internal lines to be on-shell we can find a critical transfer momentum for which the corresponding amplitude has a leading Landau singularity in the physical sheet. This transfer momentum defines the anomalous threshold. This leads to a logarithmic contribution to the amplitude and to a non vanishing absorptive part forbidden by standard unitarity. The simplest way to set when this singularity is physical is using the Coleman-Grossman-theorem [13] that dictates that the singularity is physical if the triangular diagram can be interpreted as a space-time
physical process with energy momentum conservation in all vertices and with the internal lines on-shell i.e. as a Landau-Cutkosky-diagram.

Let us now consider the degenerations as formally representing the massless electron as a composite state of electron and collinear photon. In this case we can consider the triangular contribution in figure 2 to the form factor where the electron in the triangle interacts with the external potential. In this case it is easy to see that an anomalous threshold can appear only in the forward limit when the transfer momentum \( q^2 \) is zero (see Appendix A).

From the KLN theorem point of view we can associate these kinematical conditions to the degeneration defined by the absorption and emission process of a collinear photon with the same value of the 4-momentum \( k \) and with \( k \) collinear to \( q \). In this case the logarithmic divergence \( \log(m_e) \) of the anomalous threshold can be canceled with the corresponding KLN sum.

However the KLN prescription in this forward limit allows us to have different 4-momentum \( k \) and \( k' \) for the absorbed and emitted photon. If this amplitude is logarithmically divergent it cannot be trivially canceled by a one loop contribution to the form factor. Next we shall see that this is indeed the case and that the only possible cancellation leading to a consistent theory of massless charged particles is using quantum interference with processes controlled by the triangular graph defining the gauge anomaly of the underlying gauge theory.

### III. THE KLN ANOMALY

In this section we shall consider the absorption emission process in the forward limit with \( k \neq k' \). Let us fix as data of the form factor scattering process the 4-momentum of the initial electron \( p \) and the exchanged energy-momentum that will denote \( q \). Let us denote the amplitude \( S(p, q) \). For these data the KLN prescription requires to define the sum

\[
\sum_{n_i, n_f} |S(p, q; n_i, n_f)|^2, \tag{3}
\]

where \( n_i \) and \( n_f \) denote the different degenerate states contributing to the process that are characterized by the number \( n_i \) of absorbed collinear photons attached to the incoming line and the number \( n_f \) of emitted collinear photons attached to the outgoing line. All of these photons are assumed to have energies bigger than the IR energy resolution scale set by the Bloch-Nordiesk-recipe. Generically each term in the sum involves the integral over the 3-momentum of the collinear photons within a given angular resolution scale. The amplitudes contain internal lines with the corresponding propagators being on-shell.

In what follows we shall be interested in the *forward corner* of phase space characterized by vanishing transfer momentum, i.e.

\[
q^2 = 0. \tag{4}
\]

In the forward regime the first absorption emission process contributing to the sum contains one absorbed photon and one emitted photon. This process is characterized by the following set of kinematical conditions

\[
pq \approx 0 \tag{5}
\]

and \( kp \approx k'p' \approx 0 \). This implies that in this corner of phase space the two propagators entering into the amplitude are on-shell. This after integration leads to a collinear divergence. Moreover in these kinematical conditions we have

\[
k' - k = q \tag{6}
\]

and, as mentioned, in the forward limit the outgoing electron has the same momentum as the in-coming one, i.e. \( p = p' \). Since for this amplitude both the absorbed and the emitted photons are collinear to the incoming and outgoing electron respectively, the KLN recipe indicates that this divergence should be canceled by the

![FIG. 2. Landau Cutkosky diagram associated with the anomaly.](image-url)
The interesting part of the integral is the one over the collinear contribution of virtual photons running in the loop.

In what follows we shall show that in the forward limit emission absorption processes with \( k \neq k' \) lead to logarithmic divergences. The diagrams that lead to the collinear term are given in figure 3. We work in the chiral basis and choose the kinematics for the electron to run in z-direction. In the appendix we explain the details and the notations used in the calculation. We omit all terms that will not lead to a collinear divergence. In these conditions we get for the amplitude

\[
iM = \sum_{n=1}^{4} iM_n \approx -ie^3 \frac{2\sqrt{2} \theta}{E \omega q (\omega + \omega_q) \left( \frac{m^2}{2\tau^2} + \theta^2 \right)} \left[ 4E^2 + \omega_q^2 (1 - \lambda \lambda_q) + (\omega + \omega_q)^2 (1 + \lambda \lambda') \right]. \tag{7}\]

In order to perform the integration over photon momenta \( \frac{d^4 k}{(2\pi)^4 2\omega \omega' 2\omega q} \) we shall use the constraint \( k' = q + k \), coming from the conservation of energy and momentum. The interesting part of the integral is the one over the small angle \( \theta \), since there the collinear divergence shows up. Thus, we omit the other integrations for simplicity. In the collinear limit \( \omega' = \omega_q + \omega \) and \( \theta' = \frac{\omega}{\omega + \omega_q} \theta + \frac{\omega \theta_q}{\omega + \omega_q} \). Including these constraints, and integrating over the small angle \( \theta \) gives

\[
\int_0^\delta |iM|^2 d\theta = \frac{8e^6}{E^2 \omega_q^2 (\omega + \omega_q)^2} \left[ 4E^2 + \omega_q^2 (1 - \lambda \lambda_q) + (\omega + \omega_q)^2 (1 + \lambda \lambda') \right] \log \left( \frac{E \delta}{m_e} \right), \tag{8}\]

where \( \delta \) is a small angular resolution scale.

In summary for generic \( q \) and for emission absorption processes we get a double pole for \( k = k' \) that can interfere with a disconnected diagram where the photon is not interacting. For \( q^2 = 0 \) we have a double pole on the kinematical sub manifold defined by \( k - k' = q \) that leads, for fixed \( q \) and after integration over \( k \), to a collinear divergence that don’t interfere with disconnected diagrams where the photon is not interacting. Thus we have obtained an additional collinear divergent contribution from the KLN-theorem \([1]\), which is not canceled by any known loop factors. We will refer to this contributions as a KLN anomaly.

### IV. THE KLN ANOMALY AND THE TRIANGULAR ANOMALY

From a perturbative point of view a crucial ingredient of anomalies in four dimensions are triangle Feynman diagrams with currents inserted at the vertices. This is the case for the original ABJ anomaly \([16, 17]\) as well as for gauge anomalies. The difference lies in the type of currents we insert in the vertices.

The analytic properties of triangular graph amplitudes were extensively studied in the early 60’s using Landau equations \([18, 19]\) and Cutkosky rules \([20]\). As already mentioned it was first observed in \([21]\) the existence of anomalies in four dimensions are triangle Feynman diagrams with currents inserted at the vertices. This is of anomalies in four dimensions are triangle Feynman diagrams with currents inserted at the vertices. This is known loop.

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Landau singularity. In the language of Landau equations the normal threshold corresponds to the reduced graph where the Feynman parameter $\alpha$ of one of the three lines is equal to zero. In what follows we shall discuss the anomalous threshold. This corresponds to put the three lines of the triangle on-shell. The threshold is determined by the value of transfer momentum $q^2$ at which the corresponding diagram with all the internal lines on-shell and with external real photons is kinematically allowed. For massless particles running in the triangle this anomalous threshold exists and it is given by $q^2 = 0$. The corresponding discontinuity is determined by Cutkosky rules as

$$
\int d^4p \prod \theta(p^2_i) \delta(p_i^2) \prod C_i,
$$

where the $C_i$ are the physical values of the three amplitudes determined by the non unitary cut (see figure 4). As shown in [19] the discontinuity of the triangular amplitude goes as

$$q \delta(q^2)$$

and it is non vanishing. Let us now look at this discontinuity as an anomalous threshold. The physical process associated with this discontinuity can be understood as

a real incoming photon that for massless charges decays into a pair of collinear on-shell electron and positron. One piece of the pair interacts with the external potential with some transfer momentum and finally the pair annihilates giving rise to a massless photon. Note that the discontinuity for the anomaly graph relies on the fact that for massless charges the photon can decay into a pair of on-shell collinear charged particles. If we fix one chirality for the running electron this discontinuity gives us the anomaly. To cancel the gauge anomaly for $U(1)$ we need to have real representations i.e. to add both chiralities in the loop.

The decay of the photon into a pair of collinear massless fermions can be formally interpreted as a degeneracy between the photon and a pair of collinear massless charged particles. From this point of view the anomaly is just the anomalous threshold associated with this formal compositeness of the photon. In more precise terms what makes the anomaly anomalous is the existence of an absorptive part of the triangular amplitude that is expected, from standard unitarity (only one cut), to vanish.

Let us now relate the KLN anomaly and the triangular anomaly. As discussed the KLN anomaly appears whenever $k \neq k'$ with zero transfer momentum (4). From the KLN theorem point of view the contribution computed in the former section should cancel with some contribution to the form factor of the electron. From the KLN theorem point of view the contribution computed in the former section should cancel with some contribution to the form factor of the electron.

FIG. 3. These diagrams lead to a mass divergence once (1) is applied, with the kinematics $k' = k + q$.

3 Normal thresholds are relevant for the study of chiral anomalies in two dimensions. In this case the leading singularity for the corresponding two point diagram represents the $\eta'$ [22].

4 In [11] we have formally included in the $C_i$ the propagator factors distinguishing bosons from fermions in the cuted lines.

FIG. 4. The anomaly diagram with the non unitary cuts.

5 The anomaly matching [24] reflects that the discontinuity of the triangular graph is the same for the IR and UV physical spectrum running in the triangle.
can be only canceled effectively adding a mass for the theory is anomalous. In fact once we sum over all chiralities in the this is only possible if the corresponding gauge theory is non vanishing value for the triangular graph. However, the only possibility to cancel the KLN anomaly is to assume where we need to include the triangular anomaly in the photon line. The second comment concerns the recent discussion of symmetries in massless QED [26]. The first thing to be noticed is that in the collinear case the corresponding dressing using coherent states [27] is ill defined (see [8] for a brief discussion). In the symmetry language this could be interpreted as indicating that KLN recipe is violating these symmetries. Actually a potential way to interpret our result is that in the massless case the collinear dressing in the forward limit is incompatible with non anomalous gauge invariance [6].

In case the origin of the transfer momentum is gravitational the situation is more interesting and richer. In fact in this case although we keep the same electromagnetic degeneration due to collinear electromagnetic radiation the external field, once it is assumed to be gravitational, can contribute to the form factor due to the graviton photon vertex. The analysis of this case is postponed to a future work.

V. A LOWER BOUND FOR THE ELECTRON MASS

In the former section we have argued that a quantum theory of massless charged fermions is inconsistent. The core of the argument is that consistency requires to cancel the KLN anomaly and that is only possible if the theory has non vanishing $U(1)$ gauge anomaly i.e. if the theory is by itself inconsistent.

In what follows we shall put forward the following conjecture:

In a theory with minimal length scale $L$ the minimal mass of a $U(1)$ charged fermion, for instance the electron, is given by

$$m_e \geq \frac{\hbar}{L} e^{-\sqrt{2\nu}},$$

(13)

6 In [28] it is argued that non vanishing gravitational topological susceptibility implies the absence of massless fermions.
where $e^2$ is the corresponding coupling and $\nu$ is the number of fermionic species with charge equal to the electron charge.

Before sketching the argument let us make explicit the logic underlying this conjecture. The bound (13) can be naively obtained from the perturbative expression (2) as the minimal mass of the electron consistent with pushing the perturbative Landau pole to $\frac{\hbar}{L}$. To argue in that way will force us to assume that the perturbative result for $Z$ already rules out the consistency of a theory of massless electrons. This will contradict the basic assumption of the KLN theorem of the potential redefinition of $Z$ with a well defined $m_e \to 0$ limit. Thus our approach to set a bound on the electron mass will consist in looking for some anomalous threshold singularity depending on the electron mass and to set the bound by analyzing the limit $m_e \to 0$ of these contributions to form factors.

In order to look for the appropriated form factor we shall use the constraints on the charged spectrum coming from the weak gravity conjecture [29]. This conjecture is equivalent to say that in absence of SUSY extremal electrically charged black holes are unstable. This leads to the existence in the spectrum of a particle with mass satisfying

$$m_e^2 \leq e^2 M_P^2.$$  

Once we accept the instability of charged black holes in absence of SUSY we can compute the effect of this instability to the form factor of the charged black hole in the presence of an external electric potential. Denoting $m_e$ the mass of the minimally charged particle we have again the anomalous threshold contribution where the black hole interaction with the external potential is mediated by the charged particle through the corresponding triangular graph. Assuming all the particles in the process to be on-shell the anomalous threshold is given by

$$t_0 = 4m_e^2 - \frac{(M_{bh}^2 - M_{bh}^2 - m_e^2)^2}{M_{bh}^2},$$

where we think the instability as the decay of a black hole of mass $M$ and charge $Q$ into a smaller black hole of mass $M'$ and a particle with mass $m$ and minimal charge $e$ that we will call the electron (see figure 6).

As shown in appendix A for the typical gravitational binding energy that we expect for a black hole the anomalous threshold contribution to the corresponding amplitude will go as

$$\log \left( \frac{M_P}{m_e} \right),$$

where we have used as UV cutoff the Planck mass.

The imaginary part of this amplitude can be interpreted as an anomalous threshold to the absorptive part of the form factor of the charged black hole in the presence of an external electromagnetic field. Now we have what we were looking for, namely a physical amplitude that depends on the electron mass in a way that is singular in the massless limit. In order to avoid the singular limit $m_e \to 0$ we can impose, on the basis of unitarity, that the corresponding amplitude is smaller than one. If we do that we get

$$\nu C^2 \log \left( \frac{M_P}{m_e} \right) \leq 1,$$  

where $C$ represents the physical decay amplitude of the black hole to emit an electron. If we assume this amplitude to be proportional to the electromagnetic coupling we get the lower bound above. Here $\nu$ is the number of charged fermionic species with equal charge to the electron.

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The imaginary part of this amplitude can be interpreted as an anomalous threshold to the absorptive part of the form factor of the charged black hole in the presence of an external electromagnetic field. Now we have what we were looking for, namely a physical amplitude that depends on the electron mass in a way that is singular in the massless limit. In order to avoid the singular limit $m_e \to 0$ we can impose, on the basis of unitarity, that the corresponding amplitude is smaller than one. If we do that we get

$$\nu C^2 \log \left( \frac{M_P}{m_e} \right) \leq 1,$$  

where $C$ represents the physical decay amplitude of the black hole to emit an electron. If we assume this amplitude to be proportional to the electromagnetic coupling we get the lower bound above. Here $\nu$ is the number of charged fermionic species with equal charge to the electron.

As shown in appendix A for the typical gravitational binding energy that we expect for a black hole the anomalous threshold contribution to the corresponding amplitude will go as

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entropy $N$ we should expect that the threshold for an absorptive part should be $t_0 \sim O(1/N)$ in Planck units i.e. absorption of one information bit. The existence of massless charged particles pushes down this threshold to the anomalous value $O(m_e^2)$ and therefore we could expect a lower information bound for the mass of the electron $m_e \sim 1/N$ in Planck units for the largest possible black hole. Thus and using a cosmological bound for the largest black hole we could conclude that the lower bound on the mass of electically charged fermions is given, in Planck units, by $\sqrt{\frac{7}{8} \pi}$ with $N_H$ determined by the Hubble radius of the Universe as $\frac{N_H^2}{\pi^2}$.

To end let us make a comment on the criticality condition. In [24] the mass of Yukawa couplings $e^2 \sim \frac{\pi}{2}$ typical of classicalization.

Before ending we would like to make a very general comment on black hole physics intimately related with the former discussion. In [33] we put forward a constituent portrait of black holes. The most obvious consequence of this model is the prediction of anomalous thresholds in the corresponding form factors at small angle. On the other hand these anomalous thresholds define a canonical example of in principle observable quantum hair.

VI. FINAL COMMENT

It looks like that nature abhors massless charged particles whenever the charge is associated with a long range force as electromagnetism. This is not a serious problem for confined particles but it is certainly a problem for charged leptons. Taken seriously, it will means that the limit with vanishing Yukawa couplings should be quantum mechanically inconsistent. In string theory we count with a geometrical interpretation of Yukawa couplings in terms of intersections and in some constructions based on brane configurations in terms of world sheet instanton contributions. It looks like that a consistency criteria for string compactifications should prevent the possibility of massless charged leptons and consequently of vanishing Yukawa couplings. The problem of a consistent massless limit of leptons is on the other hand related with the problem of naturalness in t’Hooft’s sense. Naively the symmetry enhancement that will make natural the massless limit is chiral symmetry. What we have observed in this note can be read from this point of view.

The IR collinear divergences, if canceled in the way suggested by the KLN-theorem, prevent the realization of this chiral symmetry indicating the unnatural condition of the massless limit of charged leptons. A hint in that direction was the observation of [32] about the existence for massless QED of non vanishing helicity changing amplitudes in the absence of any supporting instanton like topology. Thus, it looks that the existence of a fundamental lower bound on the mass of charged leptons is inescapable.

Appendix A: Anomalous threshold kinematics

Let us consider the leading Landau singularity for the diagram in figure 1. This corresponds to have all the internal lines of the diagram on-shell satisfying energy momentum conservation in the three vertices. Following [14] the diagram is presented in Breit frame. The transfer momentum is given by $-4p^2$, the normal threshold is given by $4M^2$ where $M$ is the mass of the particle in the triangle interacting with the external source. The anomalous threshold associated with the leading Landau singularity is given by

$$t_0 = 4M^2 - \frac{M^2_0 - M^2_1 - M^2_2}{M^4_0},$$

(A1)

where $t_0 = -4p^2_0$. This is the minimum momentum where all the particles in figure 1 can be on-shell. Here also the scattering angles have to be below a small threshold which in our case refer to the resolution scale angle $\delta$. Note that this anomalous threshold is independent on the energy of the process. The reason for calling it anomalous is that it is smaller than the normal threshold given by standard unitarity.

As discussed in the text the discontinuity associated with this singularity can be computed using the Cutkosky rules for the diagram. The corresponding amplitude contains a term proportional to $\log(1 - \frac{e}{\delta})$. For the diagram in figure 2 where we use the degeneration between the electron and a pair electron and collinear photon (both on-shell) the anomalous threshold gives the $\log(m_e)$ terms in the amplitude.

In order to get a clearer picture of the underlying kinematics we can compute the relative velocity $v$ between the two particles 1 and 2. This is given by the so called Kallen-function

$$v = A(M^2_2 - M^2_1 - M^2_0),$$

(A2)

with $A^2 = M^4_0 + M^4_1 + M^4_2 - 2M^2_0 M^2_1 - 2M^2_0 M^2_2 - 2M^2_1 M^2_2$. In the degenerate case with $M_0 = M_2 = m_e$ and $M_1 = m_e$ the mass of a photon we get the limit $v = \infty$ corresponding to particles 1 and 2 moving collinearly i.e. they remain coincident.

Introducing a binding energy as $M_0 + B = M_1 + M_2$ we observe that for $M_0 < M_1 + M_2$ the velocity $v$ defined above is imaginary reaching collinearity in the limit $B \to 0$. Moreover in the limit where $M_1$ is much larger than $M_2$ the anomalous threshold can be approximated by:

$$t_0 \approx 4Bm_e \left(2 - \frac{B}{m_e}\right),$$

(A3)

In the gravitational case $t_0$ goes from zero in the limit $B \to 0$ to the normal threshold $4m^2_e$ in the limit of maximal gravitational binding energy.
Appendix B: Notation and identities for the amplitudes

The Dirac spinor for the electron in chiral representation is given by

$$u^D(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix} .$$  \hfill (B1)



In the limit where the mass \( m_e \) of the electron goes to 0

$$u_L(p) = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad u_R(p) = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (B2)$$

holds, where \( E \) is the energy of the electron. Then the first diagram in figure 3 gives the amplitude

$$iM_1 = \frac{-ie^3}{(2pk)(2pk')} \left( u^R_R(p)(\varepsilon\sigma)((p + k')\tilde{\sigma})(\varepsilon\sigma)((p + k)\tilde{\sigma})(\varepsilon\sigma)u_R(p) \\
+u^L_L(p)(\varepsilon\sigma)((p + k')\tilde{\sigma})(\varepsilon\sigma)((p + k)\tilde{\sigma})(\varepsilon\sigma)u_L(p) \right) , \quad (B3)$$

where we omitted the terms proportional to the electron mass because they will give no collinear divergent term in the limit \( m \to 0 \). For the photon we choose a 4-momentum

$$k^\mu = \omega \begin{pmatrix} 1 \\ \sin(\theta) \cos(\phi) \\ \sin(\theta) \sin(\phi) \\ \cos(\theta) \end{pmatrix} , \quad (B4)$$

and a polarization vector

$$\varepsilon^\mu = \frac{1}{\sqrt{\cos^2(\theta) + 1}} \begin{pmatrix} 0 \\ \exp(-i\lambda \phi) \cos(\theta) \\ i\lambda \exp(-i\lambda \phi) \cos(\theta) \\ -\sin(\theta) \end{pmatrix} , \quad (B5)$$

where \( \omega \) is the energy and \( \lambda = \pm 1 \) corresponds to a right-/left-handed photon. In the collinear limit all \( \theta \) angles appearing in the calculations are small. So that we can approximate

$$2pk \approx E\omega \left( \frac{m_e^2}{E^2} + \theta^2 \right) \quad (B6)$$

and

$$((p + k)\tilde{\sigma})(\varepsilon\sigma)u_R(p) \approx \sqrt{2\theta} \left( E + \frac{\omega}{2}(1 + \lambda) \right) u_R(p) . \quad (B7)$$

For the out-state holds analogously

$$2pk' \approx E\omega' \left( \frac{m_e^2}{E^2} + \theta'^2 \right) \quad (B8)$$

and

$$u^R_R(p)(\varepsilon\sigma)((p + k')\tilde{\sigma}) \approx \sqrt{2\theta'} \left( E + \frac{\omega'}{2}(1 + \lambda') \right) u^R_R(p) , \quad (B9)$$

see [9]. Furthermore, for a small arbitrary angle \( \theta \) we have

$$u^R_R(p)(\varepsilon\sigma)u_R(p) \approx \sqrt{2E\theta} . \quad (B10)$$

Similar expressions hold for the left-handed part of the amplitude. The amplitude [B3] simplifies and in the same way the other 3 diagrams can be worked out, then we have

$$iM_1 \approx -ie^3 \frac{2\sqrt{2\theta'\theta_q}}{E\omega'} \left( E(2E + \omega + \omega') + \frac{\omega\omega'}{2}(1 + \lambda\lambda') \right) , \quad (B11)$$

$$iM_2 \approx -ie^3 \frac{2\sqrt{2\theta'\theta_q}}{E\omega'} \left( E(2E - \omega - \omega') + \frac{\omega\omega'}{2}(1 + \lambda\lambda') \right) , \quad (B12)$$

$$iM_3 \approx ie^3 \frac{2\sqrt{2\theta'\theta_q}}{E\omega_q} \left( E(2E + \omega + \omega_q) - \frac{\omega\omega_q}{2}(1 - \lambda\lambda_q) \right) , \quad (B13)$$

$$iM_4 \approx ie^3 \frac{2\sqrt{2\theta'\theta_q}}{E\omega_q} \left( E(2E - \omega + \omega_q) - \frac{\omega\omega_q}{2}(1 - \lambda\lambda_q) \right) . \quad (B14)$$

iM_3 and iM_4 have propagators which go with

\[ 2pq \approx E\omega_q \left( \frac{m_e^2}{E^2} + \theta_q^2 \right) \quad \text{(B15)} \]

in the denominator. The constraints energy and momentum conservation are

\[ \omega' = \omega + \omega_q \text{ and } \omega'\theta' = \omega\theta + \omega_q\theta_q \rightarrow \theta' = \alpha\theta + \beta \quad \text{(B16)} \]

where \( \alpha = \omega/(\omega + \omega_q) \) and \( \beta = \omega_q\theta_q/(\omega + \omega_q) \). Since \( \theta_q \) is indeed small but never 0, and we never integrate over this angle we can send the mass of the electron, in our computations, in the denominator of the propagators (B15) and (B16) to zero. Furthermore, for computational reasons we can effectively ignore the terms proportional to \( \alpha \), because these terms would only give a \( \log(\beta) \) contribution in the integration of the amplitude square \( \int \left| iM \right|^2 d\theta d\phi \). Thus, having in mind that the only interesting term is the collinear divergent part that goes like \( \log \left( \frac{E}{m} \right) \), we can simplify the expression (B15), (B8) and (B11) to (B14).

\[ 2pq \approx E\omega_q \theta_q^2 \quad \text{(B17)} \]

\[ 2pk' \approx E(\omega + \omega_q)\beta^2 = 2pq \frac{\omega_q}{\omega + \omega_q} \quad \text{(B18)} \]

\[ iM_{1col} \approx -ie^3 \left( \frac{2\sqrt{2}\theta}{E\omega_q} \right) \left( \frac{m_e^2}{E^2} + \theta^2 \right) \left( E(2E + \omega + \omega') + \frac{\omega\omega'}{2}(1 + \lambda') \right) \quad \text{(B19)} \]

\[ iM_{2col} \approx -ie^3 \left( \frac{2\sqrt{2}\theta}{E\omega_q} \right) \left( \frac{m_e^2}{E^2} + \theta^2 \right) \left( E(2E - \omega - \omega') + \frac{\omega\omega'}{2}(1 + \lambda') \right) \quad \text{(B20)} \]

\[ iM_{3col} \approx ie^3 \left( \frac{2\sqrt{2}\theta}{E\omega(\omega + \omega_q)} \right) \left( \frac{m_e^2}{E^2} + \theta^2 \right) \left( E(2E + \omega - \omega_q) - \frac{\omega\omega_q}{2}(1 - \lambda\lambda_q) \right) \quad \text{(B21)} \]

\[ iM_{4col} \approx ie^3 \left( \frac{2\sqrt{2}\theta}{E\omega(\omega + \omega_q)} \right) \left( \frac{m_e^2}{E^2} + \theta^2 \right) \left( E(2E - \omega + \omega_q) - \frac{\omega\omega_q}{2}(1 - \lambda\lambda_q) \right) \quad \text{(B22)} \]

respectively. This is just a computational trick in order to fish out the \( \log \left( \frac{E}{m} \right) \) part from of a much richer amplitude and we get for this the result [7].

\[ \int \left| iM \right|^2 d\theta d\phi \]

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