DRINFELD REALISATIONS OF QUANTUM AFFINE SUPERALGEBRAS

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ABSTRACT. We construct Drinfeld realisations for the quantum affine superalgebras associated with the $\mathfrak{osp}(1|2n)^{(1)}$, $\mathfrak{sl}(1|2n)^{(2)}$ and $\mathfrak{osp}(2|2n)^{(2)}$ series of affine Lie superalgebras.

1. Introduction

The Drinfeld realisation of a quantum affine algebra [6] is a quantum analogue of the loop algebra realisation of an affine Lie algebra. It is indispensable for studying vertex operator representations [8, 10] and finite dimensional representations [3, 4] of the quantum affine algebra. The equivalence between the Drinfeld realisation and usual Drinfeld-Jimbo presentation in terms of Chevalley generators was known to Drinfeld [6], and has been investigated in a number of papers, see, e.g., [1, 5, 9, 11].

We construct Drinfeld realisations for a class of quantum affine superalgebras in this paper. Quantum supergroups associated with simple Lie superalgebras and their affine analogues were introduced [2, 14, 24] and extensively studied (see, e.g., [19, 22] and references therein) in the 90s. They have important applications in a variety of areas such as topology of knots and 3-manifolds [17, 20], and the theory of integrable models of Yang-Baxter type [2, 23]. In recent years there has been a resurgence of interest in these algebraic structures. Previously Drinfeld realisations were only known for the untwisted quantum affine superalgebras of types $A$ [15] and $D(2, 1; \alpha)$ [7] in the standard root systems. The realisation in type $A$ formed the launching pad for the study of integrable representations of the quantum affine special linear superalgebra in [12, 16].

In this paper, we will focus on the quantum affine superalgebras $U_q(\mathfrak{g})$ associated with the following series of affine Lie superalgebras $\mathfrak{g}$:

$$\mathfrak{osp}(1|2n)^{(1)}, \quad \mathfrak{sl}(1|2n)^{(2)}, \quad \mathfrak{osp}(2|2n)^{(2)}, \quad n \geq 1,$$

which are the affine Lie superalgebras that do not have isotropic odd roots. We construct a Drinfeld realisation $U^D_q(\mathfrak{g})$ (see Definition 2.3) for each $U_q(\mathfrak{g})$, and establish a superalgebra isomorphism between $U^D_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$ in Theorem 2.5. As explained in Remark 2.6, the isomorphism can in fact be interpreted as an isomorphism of Hopf superalgebras.

We prove Theorem 2.5 by relating the Drinfeld realisations of the quantum affine superalgebras to Drinfeld realisations of some ordinary quantum affine algebras, and then applying Drinfeld’s theorem [6]. This makes essential use of the notion of quantum correspondences introduced in [13]. A quantum correspondence between a pair $(\mathfrak{g}, \mathfrak{g}')$ of (affine) Lie superalgebras is a Hopf superalgebra isomorphism between the corresponding quantum supergroups. Here we regard the category of vector superspaces as a braided tensor category, and a Hopf superalgebra as a Hopf algebra over this category. Lemma 3.3 gives a concise description of the quantum correspondences used in this paper.

2010 Mathematics Subject Classification. 81R10,17B37.

Key words and phrases. affine Lie superalgebras; quantum affine superalgebras; Drinfeld realisations.
Throughout the paper, \( q^{1/2} \) is an indeterminate and \( t^{1/2} = \sqrt{-1}q^{1/2} \). Let \( K := \mathbb{C}(q^{1/2}) \) be the field of rational functions in \( q^{1/2} \).

## 2. Drinfeld Realisations of Quantum Affine Superalgebras

Given any affine Lie superalgebra \( \mathfrak{g} \) in \((1.1)\), we denote by \( A = (a_{ij}) \) its Cartan matrix, and realise \( A \) in terms of the set of simple roots \( \Pi = \{\alpha_i \mid i = 0, 1, 2, \ldots, n\} \) with \( a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \). Let \( \tau \subset \{0, 1, \ldots, n\} \) be the labelling set of the odd simple roots. We recall the following Dynkin diagrams of the affine Lie superalgebras.

\[
\begin{align*}
\mathfrak{osp}(1|2n)^{(1)} & \quad \begin{array}{c}
\circ \quad \cdots \quad \circ \\
\alpha_0 & \alpha_1 & \cdots & \alpha_n
\end{array} \\
\mathfrak{sl}(1|2n)^{(2)} & \quad \begin{array}{c}
\circ \quad \cdots \quad \circ \\
\alpha_0 & \alpha_1 & \cdots & \alpha_n
\end{array} \\
\mathfrak{osp}(2|2n)^{(2)} & \quad \begin{array}{c}
\circ \quad \cdots \quad \circ \\
\alpha_0 & \alpha_1 & \cdots & \alpha_n
\end{array}
\end{align*}
\]

The black nodes in the Dynkin diagram denote the odd simple roots, while the white ones are even simple roots. The bilinear form can always be normalised such that \((\alpha_i, \alpha_i) = 1\).

Set \( q_i = q^{(\alpha_i, \alpha_i)} \) for all \( \alpha_i \in \Pi \). For any nonzero \( z \in K \), denote
\[
\begin{align*}
[N]_z & = \frac{[N]_1!}{[N-k]_1! [k]_1!}, \quad [N]_1 = \prod_{j=1}^{N} [j]_1, \quad [j]_1 = \frac{z^j - z^{-j}}{z - z^{-1}} \quad \text{with} \ [0]_z = 1.
\end{align*}
\]

For any superalgebra \( A = A_0 \bigoplus A_1 \), we define the parity functor \([ ] : A \rightarrow \mathbb{Z}_2 = \{0, 1\}\) on homogeneous elements of \( A \) as follows: \([a] = 0\) if \( a \in A_0 \) and \([a] = 1\) if \( a \in A_1 \).

**Definition 2.1** \((21)\). Let \( \mathfrak{g} = \mathfrak{osp}(1|2n)^{(1)}, \mathfrak{sl}(1|2n)^{(2)} \) or \( \mathfrak{osp}(2|2n)^{(2)} \). The quantum affine superalgebra \( U_q(\mathfrak{g}) \) over \( K \) is an associative superalgebra with identity generated by the homogeneous elements \( e_i, f_i, k_\pm^\alpha \) \((0 \leq i \leq n)\), where \( e_s, f_s, (s \in \tau) \), are odd and the other generators are even, with the following defining relations:

\[
\begin{align*}
& k_i k_j^{-1} = k_j k_i^{-1} = 1, \quad k_i k_j = k_j k_i, \\
& k_i e_j k_j^{-1} = q_i^{a_{ij}} e_j, \quad k_i f_j k_j^{-1} = q_i^{-a_{ij}} f_j, \\
& e_i f_j - (-1)^{[\alpha_i][\alpha_j]} f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i^{\zeta_i} - q_i^{-\zeta_i}}, \quad \forall i, j, \\
& (Ad_{e_i})^{1-a_{ij}} (f_j) = (Ad_{f_j})^{1-a_{ij}} (f_j) = 0, \quad \text{if} \ i \neq j.
\end{align*}
\]

Here \( Ad_{e_i}(x) \) and \( Ad_{f_i}(x) \) are respectively defined by
\[
\begin{align*}
Ad_{e_i}(x) &= e_i x - (-1)^{[\alpha_i][\alpha_i]} k_i^{-1} x k_i e_i, \\
Ad_{f_i}(x) &= f_i x - (-1)^{[\alpha_i][\alpha_i]} k_i^{-1} x k_i f_i,
\end{align*}
\]

and \( \zeta_i = 2 \) if \( i = n \), and 1 otherwise. For any \( x, y \in U_q(\mathfrak{g}) \) and \( a \in K \), we shall write
\[
[x, y]_a = xy - (-1)^{[\alpha_i][\alpha_j]} ayx, \quad [x, y] = [x, y]_1.
\]
Then $\text{Ad}_e(e_j) = [e_i, e_j]_{q^{i,j}}$ and $\text{Ad}_f(f_j) = [f_i, f_j]_{q^{i,j}}$.

The superalgebra $U_q(g)$ has a Hopf superalgebra structure [21].

**Remark 2.2.** Note that the standard definition of the quantum affine superalgebra corresponds to $\xi = 1$ for all $i$. However, Definition 2.1 can be transformed to the standard one by an automorphism which, say, sends $e_i$ to $[\xi_i]_q e_i$ for all $i$ and leaves the other generators intact. The advantage of Definition 2.1 is that $q^{1/2}$ never appears in the defining relations.

To construct the Drinfeld realisation for the quantum affine superalgebra $U_q(g)$, we let $\mathcal{I} = \{(i, r) \mid 1 \leq i \leq n, r \in \mathbb{Z}\}$. Define the set $\mathcal{I}_0$ by $\mathcal{I}_0 := \mathcal{I}$ if $g = \mathfrak{osp}(1|2n)^{(1)}$ or $\mathfrak{sl}(1|2n)^{(2)}$, and $\mathcal{I}_0 := \mathcal{I}\setminus((i, 2r + 1) \mid 1 \leq i < n, r \in \mathbb{Z})$ if $g = \mathfrak{osp}(2|2n)^{(2)}$. Let $\mathcal{I}_0^* = \{(i, s) \in \mathcal{I}_0 \mid s \neq 0\}$. Also, for any expression $f(x_{r_1}, \ldots, x_{r_k})$ in $x_{r_1}, \ldots, x_{r_k}$, we use $\text{sym}_{r_1, \ldots, r_k} f(x_{r_1}, \ldots, x_{r_k})$ to denote $\sum_{\sigma} f(x_{\sigma(r_1)}, \ldots, x_{\sigma(r_k)})$, where the sum is over the permutation group of the set $\{r_1, r_2, \ldots, r_k\}$.

**Definition 2.3.** For $g = \mathfrak{osp}(1|2n)^{(1)}, \mathfrak{sl}(1|2n)^{(2)}$ or $\mathfrak{osp}(2|2n)^{(2)}$, we let $U_q^D(g)$ be the associative superalgebra over $K$ with identity, generated by

$$\xi_{i,r}, \gamma_{i,1}, \kappa_{i,s}, \gamma_{i}^{1/2}, \text{ for } (i, r) \in \mathcal{I}_0, (i, s) \in \mathcal{I}_0^*, 1 \leq i \leq n,$$

where $\xi_{i,r}, \gamma_{i}^{1/2}$ are odd and the other generators are even, with the following defining relations.

1. $\gamma_{i}^{1/2}$ are central, and $\gamma_i^{1/2} \gamma^{-1/2} = 1$,

$$\gamma_i \gamma_i^{-1} = \gamma_i^{-1} \gamma_i = 1, \quad \gamma_i \gamma_j = \gamma_j \gamma_i,$$

$$\gamma_i \xi_{i,r} \gamma_i^{-1} = q_{i,r} \xi_{i,r}, \quad [\kappa_{i,r}, \xi_{i,r}] = \frac{u_{i,r} \gamma_i^{q_{i,r}/2}}{r(q-q^{-1})} \xi_{i,r}, \quad (2.3)$$

$$[\kappa_{i,r}, \kappa_{i,s}] = \delta_{r+s, 0} \frac{u_{i,r} (\gamma_i - \gamma^{-1})}{r(q-q^{-1})(q-q^{-1})}, \quad (2.4)$$

$$[\xi_{i,r}, \xi_{i,s}] = \delta_{i,j} \frac{\gamma_i^{q_{i,s}} - \gamma_i^{q_{i,s}}}{q-q^{-1}}, \quad (2.5)$$

$$[\xi_{i,r}^{\pm}, \xi_{i,s}^{\pm}]_{q_{i,j}}^{\pm} = 0, \quad (\text{if } (i, j) \neq (n, n), (2.6))$$

where $\theta = 2$ if $g = \mathfrak{osp}(2|2n)^{(2)}, (i, j) \neq (n, n), \text{ and } 1$ otherwise; the $u_{i,r}$ are given in (2.3), and $\hat{k}_{i,r}$ are defined by

$$\sum_{r \in \mathbb{Z}} \hat{k}_{i,r} u^{-r} = \gamma_i \exp \left( (q-q^{-1}) \sum_{r \geq 0} k_{i,r} u^{-r} \right), \quad (2.7)$$

2. Serre relations

(A) $n \neq i \neq j, \quad \ell = 1 - a_{ij}$,

$$\text{sym}_{r_1, \ldots, r_\ell} \left( (-1)^k \left[ \begin{array}{c} \ell \\ k \end{array} \right]_{q_{i,j}} \xi_{i,r_1} \cdots \xi_{i,r_k} \xi_{i,r_{k+1}} \cdots \xi_{i,r_\ell} = 0; \right.$$
Remark 2.4. Consider the automorphism which leaves the other generators intact but
maps $\kappa_{i,s} \mapsto \frac{q_i^{-1}}{q_j^{-1}} \kappa_{i,s}$ and $\xi_{i,s}^+ \mapsto \frac{q_i^{-1}}{q_j^{-1}} \xi_{i,s}^+$ for all $i$. It transforms the relations (2.3)-(2.4) into the more standard form

\[
[k_{i,s}, \xi_{i,s}^\pm] = \frac{u_{i,j,r} q_i^{j/r} r}{(q_i - q_j^{-1})^{j+s+r}} \xi_{i,j,s+r}^\pm,
\]

\[
[k_{i,s}, \kappa_{j,s}] = \delta_{r+s,0} \frac{u_{i,j,r} (q_i - q_j^{-1})}{(q_i - q_j^{-1})^2} (q_j - q_i^{-1}),
\]

\[
[\xi_{i,s}^+, \xi_{j,s}^-] = \delta_{i,j} \frac{q_i^{j/r} \kappa_{i,s}^+ - \kappa_{i,s}^+ q_j j/s}{q_i - q_j^{-1}},
\]

\[
\sum_{r \in \mathbb{Z}} \hat{k}_{i,r} u^{-r} = \gamma_i \exp \left( (q_i - q_i^{-1}) \sum_{r=0}^{\infty} \kappa_{i,r} u^{-r} \right),
\]

\[
\sum_{r \in \mathbb{Z}} \hat{k}_{i,-r} u^r = \gamma_i^{-1} \exp \left( (q_i^{-1} - q_i) \sum_{r=0}^{\infty} \kappa_{i,-r} u^r \right).
\]

The following theorem is the main result of this paper; its proof will be given in the next section.
Theorem 2.5 (Main Theorem). Let $\mathfrak{g} = \mathfrak{osp}(1|2n)^{(1)}$, $\mathfrak{sl}(1|2n)^{(2)}$ or $\mathfrak{osp}(2|2n)^{(2)}$. There exists a superalgebra isomorphism $\Psi : \mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q^D(\mathfrak{g})$ such that

for $\mathfrak{g} = \mathfrak{osp}(1|2n)^{(1)}$:

\[
e_i \mapsto \xi^+_i, \quad f_i \mapsto \xi^-_i, \quad k_i \mapsto \gamma_i, \quad k_i^- \mapsto \gamma_i^-, \quad \text{for } 1 \leq i \leq n,
\]

\[
e_0 \mapsto \text{Ad}_{\xi^+_0} \cdots \text{Ad}_{\xi^+_{n-1}} \text{Ad}_{\xi^-_{n-1}} \cdots \text{Ad}_{\xi^-_0} (\xi^{(1)}_{1,1}) \gamma_\mathfrak{g}^{-1},
\]

\[
f_0 \mapsto c_0 \gamma^+ \gamma_\mathfrak{g} \text{Ad}_{\xi^+_1} \cdots \text{Ad}_{\xi^+_{n-1}} \cdots \text{Ad}_{\xi^-_{n-1}} (\xi^{(1)}_{1,-1}), \quad k_0 \mapsto \gamma\gamma_\mathfrak{g}^{-1},
\]

for $\mathfrak{g} = \mathfrak{sl}(1|2n)^{(2)}$:

\[
e_i \mapsto \xi^+_i, \quad f_i \mapsto \xi^-_i, \quad k_i \mapsto \gamma_i, \quad k_i^- \mapsto \gamma_i^-, \quad \text{for } 1 \leq i \leq n,
\]

\[
e_0 \mapsto \text{Ad}_{\xi^+_2} \cdots \text{Ad}_{\xi^+_{n-1}} \cdots \text{Ad}_{\xi^-_{n-1}} (\xi^{(1)}_{1,1}) \gamma_\mathfrak{g}^{-1},
\]

\[
f_0 \mapsto c_0 \gamma^+ \gamma_\mathfrak{g} \text{Ad}_{\xi^+_1} \cdots \text{Ad}_{\xi^+_{n-1}} (\xi^{(1)}_{1,-1}), \quad k_0 \mapsto \gamma\gamma_\mathfrak{g}^{-1},
\]

where $k_{\mathfrak{g}}$ is defined by

\[
\gamma_\mathfrak{g} = \begin{cases} 
\gamma^1 \gamma^2 \cdots \gamma_n^2, & \mathfrak{g} = \mathfrak{osp}(1|2n)^{(1)}, \\
\gamma^1 \gamma^2 \cdots \gamma_n^2, & \mathfrak{g} = \mathfrak{sl}(1|2n)^{(2)}, \\
\gamma^1 \gamma^2 \cdots \gamma_n, & \mathfrak{g} = \mathfrak{osp}(2|2n)^{(2)},
\end{cases}
\]

and $c_\mathfrak{g} \in K$ is determined by (2.1).

Remark 2.6. We can transcribe the Hopf superalgebra structure of $\mathcal{U}_q(\mathfrak{g})$ to $\mathcal{U}_q^D(\mathfrak{g})$ using $\Psi$. For example, if $\Delta$ is the co-multiplication of $\mathcal{U}_q(\mathfrak{g})$, the co-multiplication of $\mathcal{U}_q^D(\mathfrak{g})$ is given by $(\Psi \otimes \Psi) \circ \Delta \circ \Psi^{-1}$. Then clearly $\Psi$ is an isomorphism of Hopf superalgebras.

3. Proof of the main theorem

We prove Theorem 2.5 in this section.

3.1. Smash products. Let $\mathfrak{g}$ be any of the affine Lie superalgebras in the first row of Table 1 (which is (1.1)), and let $\mathfrak{g}'$ be the ordinary affine Lie algebra corresponding to $\mathfrak{g}$ in the second row. We will speak about the pair $(\mathfrak{g}, \mathfrak{g}')$ of affine Lie (super)algebras in the table. Now $\mathfrak{g}'$ has the same Cartan matrix $A = (a_{ij})$ as $\mathfrak{g}$. We let $\Pi' = \{\alpha'_0, \alpha'_1, \ldots, \alpha'_{n}\}$ be the set of simple roots of $\mathfrak{g}'$ which realises the Cartan matrix, and take $(\alpha'_n, \alpha'_n) = 1$. Recall $t^{1/2} = \sqrt{-1} q^{1/2}$ and let $t_i = t^{(\alpha'_i, \alpha'_i)/2}$ for all $i$. 

| $\mathfrak{g}$ | $\mathfrak{osp}(1|2n)^{(1)}$ | $\mathfrak{sl}(1|2n)^{(2)}$ | $\mathfrak{osp}(2|2n)^{(2)}$ |
|-------------|-----------------|-----------------|-----------------|
| $\mathfrak{g}'$ | $A_{2n}^{(2)}$ | $B_{n}^{(1)}$ | $D_{n+1}^{(2)}$ |

Table 1
The quantum affine algebra $U_q(g')$ is an associative algebra over $K$ with identity generated by the elements $e_i', f_i', k_i'^{\pm 1}$ ($0 \leq i \leq n$) with the following defining relations:

$$
\begin{align*}
  k_i'^1k_i'^{-1} &= k_i'^{-1}k_i'^1 = 1, \quad k_i'k_j' = k_j'k_i', \\
  k_i'e_i'k_i'^{-1} &= t_i^{\alpha_i}e_i', \quad k_i'f_i'k_i'^{-1} = t_i^{-\alpha_i}f_i', \\
  e_i'f_j' - f_j'e_i' &= \delta_{ij} \frac{k_i' - k_j'^{-1}}{t_i - t_i^{-1}}, \quad \forall i, j;
\end{align*}
$$

(3.1)

Here $\zeta_i = 2$ if $i = n$, and $1$ otherwise; $\text{Ad}_{e_i'}(x)$ and $\text{Ad}_{f_i'}(x)$ are respectively defined by

$$
\text{Ad}_{e_i'}(x) = e_i'x - k_i'^{-1}e_i',  \\
\text{Ad}_{f_i'}(x) = f_i'x - k_i'^{-1}xk_i'^{-1}f_i'.
$$

It is well known that $U_q(g')$ is a Hopf algebra.

**Remark 3.1.** The definition of $U_q(g')$ given here is related to the standard one by an automorphism analogous to that given in Remark 2.2.

Let $I_{g'} = I_g$ and $I_{g'}^s = I_{g}^s (I_g$ and $I_{g}^s$ are defined before Definition 2.3). The Drinfeld realisation $U_{g'}(q')$ of $U_q(g')$ is an associative algebra over $K$ with identity generated by the generators

$$
\xi_{i,r}^{\pm}, \gamma_i'^{\pm 1}, \kappa_{i,r}', \gamma^{\pm 1/2}, \quad \text{for } (i, r) \in I_{g'}', (i, s) \in I_{g'}^s, 1 \leq i \leq n,
$$

with the defining relations [6]:

(1) $[\gamma'^{\pm 1/2}, \xi_{i,r}^{\pm}] = [\gamma'^{\pm 1/2}, \kappa_{i,r}'], = [\gamma'^{\pm 1/2}, \gamma_i'] = 0,$

$$
\begin{align*}
  \gamma_i'y_i'^{-1} &= y_i'^{-1}y_i' = 1, \quad \gamma_i'y_j' = y_j'y_i', \\
  \gamma_i'\xi_{j,r}^{\pm}y_i'^{-1} &= t_i^{\delta_{ij}}\xi_{j,r}^{\pm}, \quad [\kappa_{i,r}', \xi_{j,s}^{\pm}] = \frac{u_{i,j,r}}{r(t - t^{-1})}\gamma^{\pm |r|/2}\xi_{j,s+r}^{\pm}, \\
  [\kappa_{i,r}', \kappa_{j,s}'] &= \delta_{rs,0}\frac{u_{i,j,r}}{r(t - t^{-1})(t - t^{-1})}, \\
  [\xi_{i,r}^{\pm}, \xi_{j,s}^{\pm}] &= \delta_{rs}\frac{\gamma^{\pm|r-s|}k_{i,r+s}^{\pm} - \gamma^{\pm|s-r|}k_{j,r+s}^{\pm}}{q^{1-q}}, \\
  \text{sym}_{r,s}[\xi_{i,r}^{\pm}, \xi_{j,s}^{\pm}]_{_{i,j}} &= 0, \quad \text{for } (g', i, j) \neq (A_{2n}^{(2)}, n, n),
\end{align*}
$$

(3.2)

where $\theta = 2$ if $g = D_{n+1}^{(2)}, (i, j) \neq (n, n)$, and $1$ otherwise; $k_{i,r}'^{\pm}$ are defined by

$$
\begin{align*}
  \sum_{r \in \mathbb{Z}} k_{i,r}'^{+}u^{-r} &= \gamma_i'\exp \left( (t - t^{-1}) \sum_{r > 0} \kappa_{i,r}'u^{-r} \right), \\
  \sum_{r \in \mathbb{Z}} k_{i,-r}'^{-1}u^r &= \gamma_i'^{-1}\exp \left( (t^{-1} - t) \sum_{r > 0} \kappa_{i,-r}'u^r \right),
\end{align*}
$$

(3.3)

and the scalars $u_{i,j,r}'$ are given in (3.4);

(2) Serre relations
(A) \( n \neq i \neq j \), or \( g' \neq D_{n+1}^{(2)}, j + 1 < i = n, \ell = 1 - a_{ij} \),

\[
sym_{r_1, \ldots, r_i} \sum_{k=0}^{i} (-1)^k \left[ \left( \sum_{j=1}^{r_i} \xi_{i,j}^{+} \xi_{i,j}^{-} \xi_{i,j-1}^{+} \ldots \xi_{i,1}^{+} \xi_{i,1}^{-} \xi_{i,1}^{+} \ldots \xi_{i,1}^{+} \right) = 0; \right.
\]

(B) For \( g = A_{2n}^{(2)} \),

\[
sym_{r_1, r_2, r_3}[[\xi_{n,r_1}^{\pm}, \xi_{n,r_2}^{\pm}], \xi_{n,r_3}^{\pm}] = 0;
\]

\[
sym_{r_1}([\xi_{n,r_2}^{\pm}, \xi_{n,r_3}^{\pm}]) - t_n^{\ell} [\xi_{n,r_1}^{\pm}, \xi_{n,s_1}^{\pm}] = 0;
\]

\[
sym_{r_1}([t_n^{\ell} [\xi_{n,r_2}^{\pm}, \xi_{n,r_3}^{\pm}], \xi_{n-1,k}^{\pm}])
+ (t_n^{2} + t_n^{-2})[[\xi_{n-1,k}^{\pm}, \xi_{n,r_1}^{\pm}], \xi_{n,s_1}^{\pm}] = 0;
\]

(C) For \( g = D_{n+1}^{(2)} \),

\[
sym_{r_1}([[\xi_{n-1,k}^{\pm}, \xi_{n,r_1}^{\pm}], \xi_{n,s_1}^{\pm}] = 0.
\]

The scalars \( u'_{i,j,r} \) in the above equations are defined by

\[
A_{2n}^{(2)}: \quad u'_{i,j,r} = \begin{cases} (t_n^{2r} - t_n^{-2r})(t_n^{2r} + t_n^{-2r} + (-1)^{r-1}), & \text{if } i = j = n, \\ (t_i^{r_a} - t_i^{-r_a}), & \text{otherwise}; \end{cases}
\]

\[
B_n^{(1)}: \quad u'_{i,j,r} = t_i^{r_a} - t_i^{-r_a};
\]

\[
D_{n+1}^{(2)}: \quad u'_{i,j,r} = \begin{cases} (t_n^{2r} - t_n^{-2r}), & \text{if } i = j = n, \\ (1 + (-1)^{r})(t_i^{r_a/2} - t_i^{-r_a/2}), & \text{otherwise}. \end{cases}
\]

Remark 3.2. The above is the Drinfeld realisation \([6]\) of the quantum affine algebra \( U_r(\g') \) for each \( g' \), taking into account the variation of the definition of \( U_r(\g') \) discussed in Remark 3.2.

Applied to the quantum affine algebras under consideration, Drinfeld’s theorem \([6]\) gives the following algebra isomorphism

\[
\rho : U_r(\g') \rightarrow U_r^D(\g');
\]
where

\[ \sigma \cdot \xi^+_{i,j} = (-1)^{(\alpha_i,\alpha_j)} \xi^+_{i,j}, \quad \sigma \cdot f_j = (-1)^{(\alpha_i,\alpha_j)} f_j, \quad \sigma \cdot k_j = k_j, \quad i \neq 0, \]  

(3.7)

which preserves the multiplication of \( U_q(\mathfrak{g}) \). This defines a left \( KG \)-module algebra structure on \( U_q(\mathfrak{g}) \). Similarly, let \( G \) act on \( U_q^D(\mathfrak{g}) \) by

\[ \sigma \cdot \xi^\pm_{j,r} = (-1)^{(\alpha_i,\alpha_j)} \xi^\pm_{j,r}, \quad \sigma \cdot k_{j,r} = k_{j,r}, \quad \sigma \cdot \gamma_j = \gamma_j, \quad \sigma \cdot \gamma = \gamma, \]  

(3.8)

for all \( i, j \geq 1 \) and \( r \in \mathbb{Z} \). This again preserves the multiplication of \( U_q^D(\mathfrak{g}) \).

By using a standard construction in the theory of Hopf algebras, we manufacture the smash product superalgebras

\[ U_q(\mathfrak{g}) := U_q(\mathfrak{g}) \# KG, \quad U_q^D(\mathfrak{g}) := U_q^D(\mathfrak{g}) \# KG, \]  

(3.9)

which have underlying vector superspaces \( U_q(\mathfrak{g}) \otimes KG \) and \( U_q^D(\mathfrak{g}) \otimes KG \) respectively, where \( KG \) is regarded as purely even. The multiplication of \( U_q(\mathfrak{g}) \) (resp. \( U_q^D(\mathfrak{g}) \)) is defined, for all \( x, y \in U_q(\mathfrak{g}) \) (resp. \( U_q^D(\mathfrak{g}) \)) and \( \sigma, \tau \in G \), by

\[ (x \otimes \sigma)(y \otimes \tau) = x\sigma y \otimes \sigma\tau. \]

We will write \( x \sigma \) and \( \sigma x \) for \( x \otimes \sigma \) and \( (1 \otimes \sigma)(x \otimes 1) \) respectively.
In exactly the same way, we introduce a group \( \mathbb{Z}_2 \) corresponding to each simple root \( \alpha'_i \) of \( g' \) with \( i \neq 0 \). The group is generated by \( \sigma'_i \) such that \( \sigma'^2_i = 1 \). Let \( G' \) be the direct product of all such groups, and define a \( G' \)-action on \( U_l(g') \) by

\[
\sigma'_i \cdot e'_j = (-1)^{(\alpha'_i, \alpha'_j)} e'_j, \quad \sigma'_i \cdot f'_j = (-1)^{- (\alpha'_i, \alpha'_j)} f'_j, \quad \sigma'_i \cdot k'_j = k'_j, \quad i \neq 0.
\] (3.10)

This induces a \( G' \)-action on \( U^D_l(g') \) analogous to (3.8). Now we introduce the smash product algebras

\[
U_l(g') = U_l(g') \# K G', \quad U^D_l(g') = U^D_l(g') \# K G'.
\]

Clearly we can extend equation (3.5) to the algebra isomorphism

\[
U_l(g') \xrightarrow{\sim} U^D_l(g'),
\] (3.11)

which is the identity on \( KG' \).

3.2. Quantum correspondences. We classified the quantum correspondences in [13, Theorem 4.9]; the following ones are relevant to the present paper, which were first established in [21].

**Lemma 3.3** ([13], [21]). For each pair \((g, g')\) in Table 7 there exists an isomorphism \( \psi : U_q(g) \xrightarrow{\sim} U_l(g') \) of associative algebras given by

\[
e_0 \mapsto \tau e'_0, \quad f_0 \mapsto \tau f'_0, \quad k_0^{\pm 1} \mapsto \tau k_i^{\pm 1}, \quad \sigma_i \mapsto \sigma'_i, \quad e_i \mapsto \left( \prod_{k=i+1}^{m+n} \sigma'_k \right) e'_i, \quad f_i \mapsto \left( \prod_{k=i}^{m+n} \sigma'_k \right) f'_i, \quad k_i \mapsto \sigma'_i k'_i,
\] (3.12)

for \( i \neq 0 \), where \( \tau, \tau_f \in KG' \) are defined by

\[
\tau = \begin{cases} 1, & g' = A^{(2)}_{2n}, \\ \prod_{i=0}^{n} \sigma'_i, & g' = B^{(1)}_n, \\ \prod_{i=0}^{n} \sigma'_{2+2i}, & g' = D^{(2)}_n, \end{cases} \quad \tau_f = \begin{cases} 1, & g' = A^{(2)}_{2n}, \\ \prod_{i=1}^{n} \sigma'_i, & g' = B^{(1)}_n, \\ \prod_{i=0}^{n} \sigma'_{2+2i}, & g' = D^{(2)}_n, \end{cases}
\]

with \( \sigma'_j = 1 \) for all \( j > n \).

**Remark 3.4.** Within the context of Hopf algebras over braided tensor categories, the above associative algebra isomorphism becomes a Hopf algebra isomorphism; see [13, Section 4] for details.

Now for each \( g' \), we introduce a surjection \( o : I_n = \{1, \ldots, n\} \rightarrow \{\pm 1\} \) defined by \( o(i) = (-1)^{n-i} \). We also define \( c := c(g) \) such that \( c = 1/2 \) if \( g = \mathfrak{osp}(2|2n)^{(2)} \), and 1 otherwise.

We have the following result.

**Theorem 3.5.** For each pair \((g, g')\) in Table 7 there is an isomorphism \( \varphi : U^D_q(g) \xrightarrow{\sim} U^D_l(g') \) of associative algebras given by

\[
\gamma \mapsto \gamma', \quad k_{i,r} \mapsto -o(i)^{rc} k_{i,r}', \quad \gamma_i^{\pm 1} \mapsto \sigma'_i \gamma_i^{\pm 1}, \quad \sigma_i \mapsto \sigma'_i, \quad \xi_{i,r} \mapsto o(i)^{rc} \left( \prod_{k=i+1}^{m+n} \sigma'_k \right) \xi_{i,r}', \quad \xi_{i,r} \mapsto o(i)^{rc} \left( \prod_{k=i+1}^{m+n} \sigma'_k \right) \xi_{i,r}'.
\] (3.13)
Proof. We can show that the map $\varphi$ indeed gives rise to a homomorphism of associative algebras, then by inspecting (3.13), we immediately see that it is an isomorphism with the inverse map given by

$$\varphi^{-1} : \gamma' \mapsto \gamma, \quad \kappa'_{ij} \mapsto -o(i)^{cr} k_{ij}, \quad \gamma'_i \mapsto \sigma_i \gamma_i, \quad \sigma'_i \mapsto \sigma_i,$$

$$\xi'_{i,j} \mapsto o(i)^{cr} \left( \prod_{k=i+1}^{m+n} \sigma_k \right) \xi'_{i,j}, \quad \xi'_{i,j} \mapsto o(i)^{cr} \left( \prod_{k=i+1}^{m+n} \sigma_k \right) \xi_{i,j}.$$

(3.14)

We prove that $\varphi$ is an algebra homomorphism by showing that the elements $\varphi(\xi'_{i,j}), \varphi(\kappa_{i,j}), \varphi(\gamma^+_i), \varphi(\gamma), \varphi(\sigma_i)$ in $U_q^D (\mathfrak{g})$ satisfy the defining relations of $U_q^D (\mathfrak{g})$.

Let us start by verifying the first set of relations in Definition 2.3. Using $\xi'_{i,j} \sigma'_{ij} = (-1)^{(s_i, s_j)} \sigma' \xi'_{i,j}$ and $(-1)^{(s_i, s_j)} q_i^{s_{aij}} = q_i^{s_{aij}}$, we immediately obtain

$$\varphi(\gamma_i) \varphi(\xi'_{i,j}) \varphi(\gamma_i^{-1}) = q_i^{s_{aij}} \varphi(\xi'_{i,j}).$$

Since $u_{i,j} = o(i)^{cr} o(j)^{cr} u'_{i,j}$, we have

$$[\varphi(\kappa_{i,j}), \varphi(\xi'_{i,j})] = \frac{u_{i,j}}{r(q - q^{-1})} \varphi(\xi'_{i,j}),$$

$$[\varphi(\kappa_{i,j}), \varphi(\kappa_{j,i})] = \delta_{r,s,0} \frac{u_{i,j}}{r(q - q^{-1})(q - q^{-1})}.$$

Let $\Phi'_j = \prod_{k=j}^n \sigma'_k$. Then

$$\xi'_{n,j} \Phi'_j = (-1)^{\delta_{n,j}} \Phi'_j \xi'_{n,j}, \quad \xi'_{i,j} \Phi'_j = (-1)^{\delta_{i,j} + \delta_{i+1,j}} \Phi'_j \xi'_{i,j}, \quad i \neq n.$$

Using this we obtain

$$\varphi(\xi'_{i,j}) \varphi(\xi'_{i,j}) = (-1)^{|\xi'_{i,j}|} \varphi(\xi'_{i,j}) \varphi(\xi'_{i,j})$$

$$= \delta_{i,j} \varphi(\gamma'_i) \varphi(\gamma_i) \varphi(\kappa'_{i,j}) - \varphi(\gamma'_i) \varphi(\gamma_i) \varphi(\gamma_i^{-1}) \varphi(\kappa'_{i,j}),$$

where we have used $\varphi(\kappa'_{i,j}) = o(i)^{cr} \kappa'_{i,j}$ since $\varphi(\kappa_{i,j}) = -o(i)^{cr} \kappa_{i,j}$. Now we have the obvious relations

$$[\varphi(\xi'_{i,j}), \varphi(\xi'_{i,j})]_{q_{i,j}} = (-1)^{\delta_{i,j+1}} \Phi'_{j+1} [\xi'_{i,j}, \xi'_{i,j}]_{q_{i,j}},$$

$$[\varphi(\xi'_{i,j}), \varphi(\xi'_{i,j})]_{q_{i,j}} = (-1)^{\delta_{i,j+1}} \Phi'_{j+1} [\xi'_{i,j}, \xi'_{i,j}]_{q_{i,j}}, \quad i \neq n.$$

Using them together with (3.2), we obtain $\text{sym}_{r,s} \{ \varphi(\xi'_{i,j}), \varphi(\xi'_{i,j}) \}_{q_{i,j}} = 0$, if $(q, i, j) \neq (A_{2n}, n, n)$. We can similarly show that $\text{sym}_{r,s} \{ \varphi(\xi'_{i,j}), \varphi(\xi'_{i,j}) \}_{q_{i,j}} = 0$.

The Serre relations in Definition 2.3 can be verified in the same way. For example, in the case $q' = B_{2n}^{(1)}$, we have

$$\text{sym}_{r_1, r_2, r_3} \sum_{k=0}^3 \left[ \begin{array}{c} 3 \\ k \end{array} \right] \prod_{q_{i,j=1}}^r \left[ \begin{array}{c} \varphi(\xi'_{i,j}) \cdots \varphi(\xi'_{i,k}) \varphi(\xi'_{i+1,j}) \cdots \varphi(\xi'_{i+1,k}) \end{array} \right]$$

$$= \sigma'_n \text{sym}_{r_1, r_2, r_3} \sum_{k=0}^3 (-1)^k \left[ \begin{array}{c} 3 \\ k \end{array} \right] \prod_{j=n}^{r+1} \left[ \begin{array}{c} \xi'_{j+1} \cdots \xi'_{j+1} \xi'_{j+1} \cdots \xi'_{j+1} \end{array} \right] = 0.$$

We omit the proof of the other Serre relations.
3.3. **Proof of Theorem 2.5.** The Main Theorem is an easy consequence of Theorem 3.5.

**Corollary 3.6.** Theorem 2.5 holds for each pair \((g, g')\) in Table 7.

**Proof.** By composing the isomorphism (3.5) with those in Lemma 3.3 and Theorem 3.5, we immediately obtain the algebra isomorphism

\[ \Phi = \varphi \circ \rho \circ \psi : \mathcal{U}_q(g) \leftrightarrow \mathcal{U}_q^D(g). \]

Note that \(\Phi\) preserves the \(\mathbb{Z}_2\)-grading, thus is an isomorphism of superalgebras.

One can easily check that \(\Phi(U_q(g) \otimes 1) = U_q^D(g) \otimes 1\). Let \(\eta : U_q(g) \rightarrow U_q(g)\) be the embedding \(x \mapsto x \otimes 1\), and \(\nu : U_q^D(g) \otimes 1 \rightarrow U_q^D(g)\) be the natural isomorphism \(y \otimes 1 \mapsto y\). Then \(\nu \circ \Phi \circ \eta\) is the superalgebra isomorphism \(\Psi\) of Theorem 2.5. \(\square\)

We comment on a possible alternative approach to the proof of Theorem 2.5. For the affine Lie superalgebras in (1.1), the combinatorics of the affine Weyl groups of the root systems essentially controls the structures of the affine Lie superalgebras themselves. The corresponding quantum affine superalgebras have enough Lusztig automorphisms, which can in principle be used to prove Theorem 2.5 by following the approach of [1]. It will be interesting to work out the details of such a proof, although it is expected to be much more involved than the one given here.

**Acknowledgements**

This research was supported by National Natural Science Foundation of China Grants No. 11301130, No. 11431010, and Australian Research Council Discovery-Project Grant DP140103239.

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