SOME POLYNOMIAL IDENTITIES INVOLVING BINOMIAL
COEFFICIENTS, DOUBLE AND RISING FACTORIALS AND
THEIR PROBABILISTIC INTERPRETATIONS AND PROOFS.

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Abstract. We formulate several polynomial identities. The one side of these
identities has a nice simple form. While the other has a form of a polynomial
whose coefficients contain binomial coefficients double factorials or (and) rising
factorials. The origins and the proofs of these identities are probabilistic.
However, their form suggests universal applications in simplifying expressions.
Many useful simplifying formulae are presented in the sequel.

1. Introduction

In this note, we will present some identities involving factorials, binomial coefficients,
and the so-called rising factorials (sometimes called Pochhammer symbols).
Many of them have the forms of polynomials whose coefficients often have a form of
binomial coefficients and (or) of rising factorials of some additional variables. These
variables appear on both sides of the identity. The domains of these variables can
be extended to all complex numbers. All these identities have their descent in com-
mon, apart from the sometimes similar form. Namely, the origins of all of them are
some, rather deep probabilistic interpretation and following it sometimes nontrivial
calculation.

The paper is organized as follows. The next section is dedicated to the presenta-
tion of the identities and some of the particular, interesting particular cases. The
next section is devoted to the presentation of the probabilistic background of the
results presented in the previous section and then, finally, the presentation of the
calculations leading to the identities.

2. Identities

Let us set $(-1)!! = (0)!! = 1$ and $\binom{n}{k} = 0$ when $n < k$. In order to sim-
pify notation let us introduce the so-called rising factorial (sometimes called also
Pochhammer symbol), which is the following function:

$$(x)^{(n)} = x(x + 1)\ldots(x + n - 1),$$
defined for all complex \( x \). Notice that we have for all \( x \neq 0 \):

\[
(x)^{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)}.
\]

To learn more on Pochhammer symbols or binomial coefficients see, e.g., [6]. Notice only that, e.g., \((1)^{(n)} = n!\) or \((1/2)^{(n)} = (2n - 1)!!/2^n\). We will use these values below.

**Theorem 1.** For all natural \( k \) and all complex \( \rho \) and \( \beta \) we have:

**i)**

\[
(2.1) \quad \frac{(2k)!}{k!} (1 - \rho)^k = \sum_{j=0}^{k} \binom{2k}{2j} (1 - \rho)^{2k-2j} (1 - \rho^2)^j (2j - 1)!!(2k - 2j - 1)!!.
\]

**ii)**

\[
(2.2) \quad (1 - \rho)^k = \frac{1}{2^k} \sum_{j=0}^{2k} (-1)^j \sum_{m=0}^{[j/2]} (2\rho)^{j-2m} \binom{k}{j-2m} \binom{k-j+2m}{m}.
\]

**iii)** For all integer \( m \) we have

\[
(2.3) \quad \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(\beta)^{(j+m)}}{(\beta)^{(j)}} = (-1)^n m! \frac{(\beta)^{(m)}}{n!}.
\]

**iv)**

\[
(2.4) \quad \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{(\beta)^{(m)}}{(\beta)^{(n)}} \sum_{k=0}^{m} \binom{m}{k} \binom{n-m}{k} \frac{\rho^kk!}{(\beta)^{(k)}} = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
\frac{n!}{(n/2)!} (\beta)^{(n/2)} (1 - \rho)^{n/2} & \text{if } n \text{ is even}
\end{cases}.
\]

**v)**

\[
(2.5) \quad \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \sum_{j=0}^{m} \binom{m}{j} (1 - \rho)^j \rho^{m-j} (\beta)^{(n-j)} (\beta + m - j)^{(j)} = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
\frac{n!}{(n/2)!} (\beta)^{(n/2)} (1 - \rho)^{n/2} & \text{if } n \text{ is even}
\end{cases}.
\]

Below we present some, believed to be important, particular cases, remarks and corollaries.

**Remark 1.** Taking first \( \rho = \frac{2}{3} \) then \( \rho = \frac{1}{3} \) and finally \( \rho = \frac{4}{3} \) in (2.1) we get:

\[
\sum_{j=0}^{k} \binom{2k}{2j} 5^j (2j - 1)!!(2k - 2j - 1)!! = \frac{(2k)!3^k}{k!},
\]

\[
\sum_{j=0}^{k} \binom{2k}{2j} 2^j (2j - 1)!!(2k - 2j - 1)!! = \frac{(2k)!3^k}{k!2^k},
\]

\[
\sum_{j=0}^{k} \binom{2k}{2j} (-7)^j (2j - 1)!!(2k - 2j - 1)!! = (-1)^k \frac{(2k)!3^k}{k!}.
\]
Remark 2. Taking $\rho = 1/2$ in (2.2) we get the following identity:

$$1 = \sum_{j=0}^{\lfloor j/2 \rfloor} \sum_{m=0}^{k} \binom{k - j + 2m}{m}.$$ 

Remark 3. i) Taking $\beta = 1$ and integer $k$ in (2.3) we get

$$\sum_{j=0}^{n} (-1)^{j} \binom{n}{j} (j + k) = (-1)^{n} \binom{k}{n}.$$ 

ii) Taking $\beta = 1/2$ and integer $k$ in (2.3) we get:

$$\sum_{j=0}^{n} (-1)^{j} \frac{(2n - 1)!!(2j + 2k - 1)!!}{j!(n - j)!(2j - 1)!!} = (-1)^{n} 2^{n} \binom{k}{n}(2k - 1)!!.$$ 

Remark 4. i) Taking $\beta = 1/2$ and integer $k$ in (2.4) we get:

$$\sum_{m=0}^{n} (-1)^{m} \sum_{k=0}^{m} \binom{m}{k} \binom{n - m}{k} \rho^{k} = \left\{ \begin{array}{ll} 0 & \text{if } n \text{ is odd} \\ \binom{n/2}{k} & \text{if } n \text{ is even} \end{array} \right.$$ 

ii) Changing the order of summation in (2.6) and comparing the coefficients in expansions in powers of $\rho$ we get:

$$\sum_{m=0}^{n} (-1)^{m-k} \binom{m}{k} \binom{n - m}{k} = \left\{ \begin{array}{ll} 0 & \text{if } n \text{ is odd} \\ \binom{n/2}{k} & \text{if } n \text{ is even} \end{array} \right.$$ 

iv) Taking additionally $\rho = 1$ in we get, for all, $n \geq 0$:

$$\sum_{m=0}^{n} (-1)^{m} \sum_{k=0}^{m} \binom{m}{k} \binom{n - m}{k} = 0.$$ 

v) Taking $\beta = 1/2$ in (2.4) we get after denoting $2\rho = x$ and canceling out $n!$:

$$\sum_{m=0}^{n} (-1)^{m} (2m - 1)!!(2n - 2m - 1)!! \sum_{k=0}^{m} \frac{1}{k!(m - k)!(n - m - k)!(2k - 1)!!} x^{k} = \left\{ \begin{array}{ll} 0 & \text{if } n \text{ is odd} \\ \frac{(n-1)!!}{(n/2)!} (2 - x)^{n/2} & \text{if } n \text{ is even} \end{array} \right.$$ 

vi) By taking additionally in the above-mentioned identity $x = 0$ we get:

$$\sum_{m=0}^{n} (-1)^{m} \binom{n}{m} (2m - 1)!!(2n - 2m - 1)!! = \left\{ \begin{array}{ll} 0 & \text{if } n \text{ is odd} \\ \frac{n!(n-1)!!}{(n/2)!} 2^{n/2} & \text{if } n \text{ is even} \end{array} \right.$$ 

vii) Taking $\rho = 1$ in (2.4) we get for all complex $\beta \neq 0$ and $n \geq 0$:

$$\sum_{m=0}^{n} (-1)^{m} \binom{n}{m} (\beta)^{m} (\beta^{n-m}) \sum_{k=0}^{m} \binom{m}{k} \binom{n - m}{k} \frac{k!}{(\beta)^{k}} = 0.$$
Now notice also that we have \( \phi \) "golden ratio". Take expansions in powers of \( \rho \) Hence we have:

Corollary 1. Let us denote by \( \phi \) the following number \((\sqrt{5} - 1)/2\) that is called "golden ratio". Take \( \rho = -\phi \) in (2.1). We get then

\[
2 \frac{(2k)!}{k!} L_k = \sum_{j=0}^{k} (-1)^j \binom{2k}{2j} (2j - 1)!!(2k - 2j - 1)!!(L_{2k-2j} - 5F_{2k-2j}),
\]

\[
2 \frac{(2k)!}{k!} F_k = \sum_{j=0}^{k} (-1)^j \binom{2k}{2j} (2j - 1)!!(2k - 2j - 1)!!(F_{2k-2j} - L_{2k-2j}).
\]

Proof. We know that for all \( n \geq 0 \):

\[
(1 + \phi)^n = ((1 + \sqrt{5})/2)^n = L_n/2 + F_n \sqrt{5}/2.
\]

Now notice also that we have \( \phi = 1/(1 + \phi) \) and that

\[
L_n^2 - 5F_n^2 = (-1)^n 4.
\]

Hence we have:

\[
\phi^n = (-1)^n (L_n/2 - F_n \sqrt{5}/2).
\]

Now we take \( \rho = -\phi \) and insert it in (2.1). Notice that \( 1 - \rho = (1 + \phi) \), hence

\[
(1 - \rho)^{2k-2j} = L_{2k-2j}/2 + F_{2k-2j} \sqrt{5}/2 \quad \text{and} \quad 1 - \rho^2 = \phi, \quad \text{consequently} \quad (1 - \rho^2)^{j} = \]

\[
\frac{n}{m} \binom{n}{m} (\beta)^{(n-m)} (\alpha)^{(m)} = (\alpha + \beta)^{(n)}.
\]
\((-1)^j(L_j/2 - F_j\sqrt{5}/2)\). Now it remains to perform multiplication and separation of terms with and without \(\sqrt{5}\).

\[\square\]

3. Probabilistic Background and the Proofs

3.1. Probabilistic background. All identities presented in Theorem \(\square\) stem from calculating the following moments \(E(X - Y)^n, n \geq 0\), where \(X\) and \(Y\) are two normalized random variables. The joint distribution of these random variables has one parameter and is either bivariate Normal (Gaussian) or bivariate Gamma distribution. More precisely, in the first case, the joint distribution of \((X, Y)\) has the following well-known density, valid for all \(x, y \in \mathbb{R}\) and \(|\rho| < 1\):

\[f_N(x, y|\rho) = \frac{1}{2\pi(1 - \rho^2)} \exp \left(-\frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)}\right),\]

while in the second case valid for all \(x, y, \beta > 0\) and \(|\rho| < 1\):

\[f_G(x, y|\rho) = f_g(x|\beta) f_g(y|\beta) \frac{\exp \left(-\frac{\rho(x+y)}{1-\rho}\right)}{(1 - \rho)(xy\rho)^{(\beta-1)/2}} I_{\beta-1} \left(\frac{2\sqrt{xy\rho}}{1 - \rho}\right),\]

where

\[f_g(x|\beta) = x^{\beta-1} \exp(-x)/\Gamma(\beta),\]

for \(x > 0\) and 0 otherwise. \(I_\alpha\) denotes the modified Bessel function of the first kind. The most important property of these joint densities is that they allow the so-called Lancaster expansions. To learn more about Lancaster expansions, their probabilistic interpretations and convergence problems associated with them, see, e.g., [1], [8], [9], [13], [12], [11]. The orthogonal polynomials that we are using are well presented, for example, in two well-known monographs [5], and [13].

Namely, in the first case, we have the so-called Poisson-Mehler expansion

\[f_N(x, y|\rho) = f_N(x) f_N(y) \sum_{n \geq 0} \rho^n h_n(x) h_n(y),\]

where we denoted for simplicity

\[f_N(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}},\]

and

\[h_n(x) = H_n(x)/\sqrt{n!}.\]

\(h_n(x)\) is the orthonormal modification of the so-called probabilistic Hermite polynomials \(\{H_n(x)\}\), i.e., polynomials defined by the following three-term recurrence:

\[H_{n+1}(x) = xH_n(x) - nH_{n-1}(x),\]

with \(H_{-1}(x) = 0\) and \(H_0(x) = 1\). It is also known, that

\[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_n(x) h_m(x) \exp(-x^2/2) dx = \delta_{nm},\]

where \(\delta_{nm}\) denotes Kronecker’s delta.

In the second case we have

\[f_G(x, y|\rho) = f_g(x|\beta) f_g(y|\beta) \sum_{j \geq 0} \rho^j l_n(x|\beta) l_n(y|\beta),\]
where \( l_n(x|\beta) = \sqrt{n!/(\beta)^n} L_n(x|\beta) \) is the orthonormal modification of the so-called generalized Laguerre polynomials \( \{L_n(x|\beta)\} \), defined by the following expansions:

\[
L_n(x|\beta) = \sum_{k=0}^{n} (-1)^k \frac{(\beta)^{(n)}}{(n-k)!(\beta)^k} x^k / k!
\]

Let us remark that the expansion (3.4) is known under the name Hardy-Hille (see, e.g. [13], p.102) formula. One knows that:

\[
\int_0^\infty l_n(x|\beta) l_m(x|\beta) f_g(x|\beta) \, dx = \delta_{n,m}.
\]

Our aim is to calculate \( \{E(X - Y)^n\}_{n \geq 0} \). We will do it in two ways.

The first way is to calculate the generating function of the set of these numbers i.e. the function

\[
g(t) = \sum_{n \geq 0} t^n E(X - Y)^n / n!.
\]

It can be calculated in the following way since it is known that all moments exist, hence one can exchange integration and summation. Namely, we have

\[
g(t) = \int \int \exp (tx - ty) f(x, y|\rho) \, dx \, dy.
\]

Obviously here and below the integration is depending on the case either over whole \( \mathbb{R}^2 \) or over \( \mathbb{R}^+ \times \mathbb{R}^+ \). The second way is to calculate the number using an expansion:

\[
E(-X + Y)^n = \sum_{m=0}^{n} (-1)^m E X^m Y^{n-m}.
\]

Now

\[
E X^m Y^{n-m} = \int \int x^m y^{n-m} f(x, y|\rho) \, dx \, dy.
\]

where \( f \) is either \( f_N \) or \( f_G \) presented above. Using one of the expansions (3.2) or (3.4) we will find these moments using the numbers

\[
H_{m,j} = E x^m h_j(X),
\]

where \( h_i(x) \) is either \( h_j(x) \) if we consider the Normal case or \( l_j(x) \) if we consider Gamma case. Since both expansions (3.2) and (3.4) have similar structure we have:

\[
E X^m Y^{n-m} = \sum_{j=0}^{\min(m,n-m)} \rho^j H_{m,j} H_{n-m,j}.
\]

This is so since \( X \) and \( Y \) have the same distributions and since \( H_{m,j} = 0 \) if \( j > m \) because of the orthogonality of polynomials \( h \) or \( l \). We will calculate this function assuming either expansion (3.2) or (3.4).

**Case 1.** So let us start with expansion (3.2). First let us calculate the auxiliary numbers \( H_{j,n} \). We have:

\[
H_{j,n} = E x^j h_n(X) = \frac{1}{\sqrt{n!}} E x^j H_n(X) = \begin{cases} 0 & \text{if } n > j \text{ or } j - n \text{ odd} \\ \frac{j!}{2^{(j-n)/2}(j-n)/2)!\sqrt{n!}} & \text{if } j - n \text{ is even} \end{cases}
\]
This is so since polynomials $h_n$ are orthogonal and also since it is common knowledge, that $\forall n \geq 0$:

$$x^j = j! \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{1}{2^m m!(j - 2m)!} H_{j - 2m}(x).$$

Secondly, let us calculate the following auxiliary function that will simplify many further calculations.

$$m_n(t) = Eh_n(X) \exp(tX) = \frac{1}{\sqrt{2\pi n!}} \int_{-\infty}^{\infty} H_n(x) \exp(-x^2/2)dx$$

$$= \frac{\exp(t^2/2)}{\sqrt{2\pi n!}} \int_{-\infty}^{\infty} H_n(x) \exp(-(x-t)^2/2)dx.$$ 

Now, let us change the variable under the integral by setting $y = (x-t)$. Then, we get:

$$m_n(t) = \frac{\exp(t^2/2)}{\sqrt{2\pi n!}} \int_{-\infty}^{\infty} H_n(y + t) \exp(-y^2/2)dy.$$ 

Next, we utilize the following the well-known expansion

$$H_n(x + y) = \sum_{j=0}^{n} \binom{n}{j} H_j(x)y^{n-j}.$$ 

Now, since we have (3.3), we see that $m_n(t) = t^n \frac{\exp(t^2/2)}{\sqrt{\pi n!}}$ and we get:

$$E \exp(tX - tY) = \sum_{n \geq 0} E \exp(tX - tY)$$

$$= \frac{1}{2\pi} \sum_{n \geq 0} \rho^n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(tx - ty) \exp\left(-\frac{(x^2 + y^2)}{2}\right) \exp\left(-\frac{n^2}{n!}\right)\rho^n t^n/n! = \exp(t^2(1 - \rho))$$

Thus, we deduce that:

$$E(X - Y)^n = \begin{cases} \frac{n!}{(n/2)!} (1 - \rho)^{n/2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$ 

**Case 2.** Now let us consider expansion (3.4). We need to recall some simple, well-known facts: The Gamma distribution with rate parameter zero and shape parameter $\beta > 0$, is the distribution with the density $f_g$ presented above. It is easy to see, recalling the definition of the Euler’s Gamma function that:

$$EX^n = \int_{0}^{\infty} x^n f_g(x|\beta)dx = (\beta)^{(n)}.$$ 

As before, let us calculate the set of auxiliary quantities and functions: We start with the numbers:

$$H_{j,n} = EX^j l_n(X).$$
We have:

\[(3.11) \quad H_{j,n} = \frac{\sqrt{n!}}{\Gamma(\beta) \sqrt{\Gamma(\beta)^{(n)}}} \int_0^\infty x^j L_n^{(\beta)}(x)x^{\beta-1} \exp(-x)dx.\]

Now, we use the well-known expansion of \(x^j\) in powers of Laguerre polynomials:

\[(3.12) \quad x^j = j! \sum_{k=0}^j (-1)^k \frac{(\beta)^{(j)}}{(j-k)! (\beta)^{(k)}} L_k^{(\beta)}(x),\]

and then we use the orthogonality of the Laguerre polynomials. Hence we have:

\[(3.13) \quad H_{j,n} = (-1)^n \frac{j!}{(\beta)^{(n)}} \sqrt{\frac{n!}{\Gamma(\beta)^{(n)}}}.\]

Let us calculate now, the following auxiliary functions:

\[m_n(t) = E l_n(X) \exp(tX) = \sqrt{\frac{n!}{(\beta)^{(n)}}} E \exp(tX)L_n(X|\beta)\]

\[= \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{1}{\Gamma(\beta)} \int_0^\infty \exp(tx)L_n(x|\beta)x^{\beta-1} \exp(-x)dx\]

\[= \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{1}{\Gamma(\beta)} \int_0^\infty L_n(x|\beta)x^{\beta-1} \exp(-x(1-t))dx.\]

Now, let us change variables under the integral by considering \(y = x(1-t)\). Then we get:

\[m_n(t) = \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{1}{\Gamma(\beta)} \frac{1}{1-t} \int_0^\infty L_n(y \frac{1}{1-t}) \left( \frac{y}{1-t} \right)^{\beta-1} \exp(-y)dy.\]

We will now utilize the following well-known recurrence relation for Laguerre polynomials (see, e.g., [2]):

\[L_n(y|\beta) = \sum_{j=0}^n L_{n-j}(x|\beta+j)(y-x)^j/j! .\]

So we get:

\[m_n(t) = \sqrt{\frac{n!}{\Gamma(\beta)^{(n)}}} \frac{1}{(1-t)^\beta} \frac{1}{\Gamma(\beta)} \times \int_0^\infty \left( \sum_{j=0}^n \left( \frac{-ty}{1-t} \right)^j L_{n-j}(x|\beta+j)/j! \right) y^{\beta-1} \exp(-y)dy.\]
Further, we have:

\[ m_n(t) = \sum_{j=0}^{n} \frac{n!}{(\beta)^{(n)}} \frac{1}{(1-t)^j \Gamma(\beta)} \times \]

\[ \sum_{j=0}^{n} \int_{0}^{\infty} (-1)^{j} \frac{t^{j}}{(1-t)^j} L_{n-j}(y | (\beta + j)) y^{j+\beta-1} \exp(-y) dy \]

\[ = (-1)^n \sqrt{n!} \frac{t^n \Gamma(\beta + n)}{(\beta)^{(n)} \Gamma(\beta) n!(1-t)^{\beta+n}} = (-1)^n \sqrt{n!} \frac{t^n (\beta)^{(n)}}{(\beta)^{(n)} n!(1-t)^{\beta+n}}. \]

Hence we have now:

\[ g(t) = E \exp(tX - tY) = \sum_{n \geq 0} \rho^n \int_{0}^{\infty} \exp(tx - ty) L_n(x|\beta) L_n(y|\beta) \frac{f_g(x|\beta) f_g(y|\beta)}{\rho^2} dx dy \]

\[ = \sum_{n \geq 0} \rho^n m_n(t)m_n(-t) = \sum_{n \geq 0} \rho^n \frac{n!(-1)^n t^{2n} \Gamma(\beta)^{(n)} (\beta)^{(n)}}{n! n!(1-t^2)^{n+\beta}} \]

\[ = \frac{1}{(1-t^2)^\beta} \sum_{n \geq 0} \rho^n \frac{n!(-1)^n t^{2n} \Gamma(\beta)^{(n)}}{n! n!(1-t^2)^{n}} \]

\[ = \frac{1}{(1-t^2)^\beta} \sum_{n \geq 0} (-1)^n \left( \frac{\rho t^2}{1-t^2} \right)^n \frac{(\beta)^{(n)}}{n!}, \]

and further we have:

\[ g(t) = \frac{1}{(1-t^2)^\beta} \left( 1 + \frac{\rho t^2}{1-t^2} \right)^{-\beta} \]

\[ = \frac{1}{(1-t^2(1-\rho))^\beta} = \sum_{n \geq 0} t^{2n} (1-\rho)^n (\beta)^{(n)} / n!. \]

We use here twice the well-known formula for the so-called binomial series. Thus we deduce

\[ (3.14) \quad E(X - Y)^n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n!}{(n/2)!} (\beta)^{(n/2)} (1-\rho)^{n/2} & \text{if } n \text{ is even} \end{cases}. \]

3.2. Proofs. All proofs are based on considering different ways of calculating numbers \{E(X - Y)^n\}_{n \geq 1} when assuming one of the discussed above joint distributions.

Proof of assertion i) of Theorem \[ \square \]

Let \((X,Y)\) have bivariate Gaussian distribution given by \[ \square \].

We have

\[ E(X - Y)^{2k} = E(X - \rho Y - (1-\rho)Y)^{2k} = \]

\[ \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (1-\rho)^{2k-j} E(X - \rho Y)^j Y^{2k-j} = \]

\[ \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (1-\rho)^{2k-j} E((X - \rho Y)^j | Y) Y^{2k-j}. \]
Now, we use the fact that random variables \(X - \rho Y\) and \(Y\) are independent, hence \(E(X - \rho Y)^j\) and \(EY^j = \sum_{m=0}^{n} (-1)^m \binom{n}{m} EX^m Y^{n-m} \). Thus we get

\[
E(X - Y)^{2k} = \sum_{j=0}^{k} \binom{2k}{2j}(1 - \rho)^{2k-2j}(1 - \rho^2)^j(2j - 1)!!(2k - 2j - 1)!!.\]

Proof of assertion ii) of Theorem 1.

We use (3.7) with (3.8) getting:

\[
\frac{(2k)!}{k!}(1 - \rho)^2 = 2 \sum_{j=0}^{k-1} \binom{2k}{2j} \frac{j^2}{n!} H_{j,n} H_{2k-j,n} + (-1)^j \binom{2k}{k} \frac{k!}{n!} H_{2k,n},
\]

where \(H_{j,n} = \left\{ \begin{array}{ll}
0 & \text{if } n > j \text{ or } j - n \text{ odd} \\
\frac{\beta^n}{(n-j)!} & \text{if } j - n \text{ is even}
\end{array} \right. \). Notice that if \(j - n\) is even then \(\binom{2k}{j} H_{j,n} H_{2k-j,n} = 2^n \binom{2k}{k} \frac{k!}{(j-n)!} \frac{(j-n)!}{2^n} \) and when \(k - n\) is even \(H_{k,n} = 2^n \binom{k}{k-n} \frac{(k-n)!}{2^n}\), hence (2.2) can be reformulated in the following way:

\[
\frac{1}{2k-1} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \sum_{n=0}^{\infty} \binom{2k}{j} \frac{(2\rho)^n}{n!} m_{h_j,n,k} + \frac{(-1)^k}{2^k} \sum_{n=0}^{\infty} \binom{2k}{k} \frac{k!}{n!} m_{h_k,n,k} = (1 - \rho)^k,
\]

where \(m_{h_j,n,k} = \left\{ \begin{array}{ll}
0 & \text{if } j > k - 1 \text{ or } j - n \text{ is odd} \\
\frac{\beta^{(n-j)!}}{(j-n)!} & \text{if } j - n \text{ is even}
\end{array} \right. \). Now it remains to change the order of summation in the internal sums.

Proof of assertion iii) of Theorem 1.

Let us recall (3.5) and (3.11) and let us calculate \(H_{j,n}\) directly, getting:

\[
H_{j,n} = \frac{\sqrt{n!}}{\Gamma(\beta) \Gamma(n + \beta)} \sum_{k=0}^{n} \frac{(-1)^k \frac{\beta^n}{k!}}{k! (n-k)!} (\frac{\beta}{n})^{k+1} \int_0^{\infty} x^{k+j} \exp(-x)dx
\]

\[
= \frac{\sqrt{n!}}{\Gamma(n + \beta)} \sum_{k=0}^{n} \frac{(-1)^k \frac{\beta^n}{k!}}{n-k)!} (\frac{\beta}{n})^{k+1} (\frac{\beta}{n})^{k+1}(\frac{\beta}{n})^{k+1}.
\]

Now, we compare it with the calculated already \(H_{j,n}\) for Gamma distribution i.e.,

\[
(-1)^n \binom{\frac{1}{2}}{n} (\beta) \frac{\Gamma(n/2)}{\sqrt{\pi}}.
\]

Proof of assertion iv) of Theorem 1.

Recall (3.13). We have:

\[
\sum_{m=0}^{n} \frac{n!}{(m/2)! \Gamma(n/2)} (1 - \rho)^{n/2} \frac{(-1)^m}{m! \Gamma(m)} EX^m Y^{n-m}.
\]
Now we use (3.7) with $H_{j,m}$ given by (3.13). By (3.14) we have

$$E(-X + Y)^n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \min(m, n-m) \rho^j H_{m,j} H_{n-m,j}$$

$$= \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \min(m, n-m) \rho^j \binom{m}{j} \binom{n-m}{j} \binom{(β)(m)(β)(n-m)}{(β)(j)} j!$$

$$= \sum_{m=0}^{n} \binom{n}{m} \binom{(β)(m)}{(β)(j)} \binom{(β)(n-m)}{(β)(j)} \sum_{j=0}^{m} \binom{m}{j} \binom{n-m}{j} j^j \rho^j$$

$$= \left\{ \begin{array}{ll}
\frac{n!}{(n/2)!} (1-\rho)^{n/2} & \text{if } n \text{ is odd} \\
0 & \text{if } n \text{ is even}
\end{array} \right.$$

This is so, since $\binom{n}{m}(n-m)$ is zero whenever $m > j$ or $n - m > j$.

Proof of assertion iv) of Theorem 1

One can easily notice, basing on (3.4), that $E(l_n(X)|Y) = \rho^n l_n(Y)$. Let us hence calculate first, the conditional moments $\eta_n(y|β, ρ) = E(X^n|Y = y)$. We have by (3.12)

$$\eta_j(y|β, ρ) = E(X^j|Y = y) = j! \sum_{k=0}^{j} (-1)^k \binom{(β)(j)}{(β)(k)} \rho^k L_k(β)(y)$$

$$= j! \sum_{k=0}^{j} (-1)^k \frac{(β)(j)}{(β)(k)} \rho^k \sum_{m=0}^{k} (-1)^m \binom{(β)(k)}{(k-m)!} y^m/m!$$

$$= j! \sum_{m=0}^{j} \frac{(β)(j)}{(β)(m)!} \frac{(β)(j)}{(β)(m)!} \frac{(β)(m)}{(j-m)!} \frac{(β)(j)}{(j-m)!} \frac{(ρy)^m (j-m)!}{(k-m)!} \frac{(ρy)^m (j-m)!}{(k-m)!}$$

$$= j! \sum_{m=0}^{j} \frac{1}{m!} (ρy)^m (1-\rho)^{j-m} = j! (1-\rho)^{j} L_j(-\frac{ρy}{1-ρ}).$$

Further we get:

$$E ((X - Y)^n|Y = y) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \eta_j(y|β, ρ) y^{n-j}$$

$$= \sum_{j=0}^{n} (-1)^{n-j} y^{n-j} \binom{n}{j} \binom{j}{m} \frac{(β)(j)}{(β)(m)} \frac{(ρy)^m (1-\rho)^{j-m}}{(n-j)! (j-m)!}$$

$$= \sum_{j=0}^{n} \frac{n!}{m!(n-m)!} \frac{(β)(m)}{(β)(m)} \frac{(ρy)^n}{(n-m)!} \frac{(-1)^{n-j} (n-m)! (β)(j) y^{n-j} (1-\rho)^{j-m}}{(n-j)! (j-m)!}$$

$$= \sum_{j=0}^{n} \frac{n!}{m!(n-m)!} \frac{(ρy)^n}{(n-m)!} \sum_{s=0}^{n-m} (-1)^{n-m-s} \binom{n-m}{s} y^{n-m-s} (1-\rho)^s (β + m)^s$$

$$= \sum_{t=0}^{n} (-1)^t y^{n-t} \frac{n!}{t!(n-t)!} \sum_{s=0}^{t} \binom{t}{s} (ρy)^{t-s} (1-\rho)^s (β + t - s)^s$$
true for all complex \( \beta \) we use the formula:

\[
E(X - Y)^n = \int_0^{\infty} \sum_{s=0}^{n} \binom{n}{s} y^{-s} (1 - \rho)^s \times \\
\sum_{m=0}^{n-s} (-1)^{n-m-s} \binom{n-s}{m} \rho^m (\beta + m)^{(s)} f_g(x|\beta) dx \\
= \sum_{s=0}^{n} \binom{n}{s} (\beta)^{(n-s)} (1 - \rho)^s \sum_{m=0}^{n-s} (-1)^{n-m-s} \binom{n-s}{m} \rho^m (\beta + m)^{(s)} \\
= \sum_{t=0}^{n} (-1)^{n-t} t! (\beta)^{(n-t)} (1 - \rho)^s \rho^{-s} (\beta + n - t)(t-s) (\beta + t - s)^{(s)}.
\]

In the last line, we have changed the order of summation. To get the assertion we use the formula:

\[
(\beta)^{(n)} (\beta + n)^{(m)} = (\beta)^{(n+m)}
\]

true for all complex \( \beta \).

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