Massey Products and Deformations

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1. Introduction

It is common knowledge that the construction of one-parameter deformations of various algebraic structures, like associative algebras or Lie algebras, involves certain conditions on cohomology classes, and that these conditions are usually expressed in terms of Massey products, or rather Massey powers. The cohomology classes considered are those of certain differential graded Lie algebras (DGLAs). It is also known that the Massey products arising in deformation theory are not precisely the products considered in the general theory of DGLAs (see [R2]). Actually, different natural problems of deformation theory give rise to different kinds of Massey products. The definitions of these Massey products involve certain equations whose coefficients turn out to be, quite unexpectedly, the structure constants of a graded commutative associative algebra. Thus to define Massey products in the cohomology of a differential graded Lie algebra one should begin by choosing a graded commutative associative algebra. It is interesting that, dually, different kinds of Massey products in the cohomology of a differential commutative associative algebra correspond in a similar way to Lie algebras. In particular, the classical Massey products correspond to the Lie algebra of strictly upper triangular matrices, while May’s matric Massey products [Ma] correspond to the Lie algebra of block strictly upper triangular matrices.

The article is organized as follows. Section 2 contains a list of various Massey-like products which arise in the cohomology of DGLAs. Most of them are related to deformations of Lie algebras. In section 3 we give a general construction of Massey products in the cohomology of DGLAs. The main result of this section is Proposition 3.1, which shows the necessity of the associativity of the auxiliary algebra. Section 4 contains an application of the construction of Section 3 to Lie algebra deformations, and in Section 5 we consider Massey products in the cohomology of graded commutative associative differential algebras.

2. Examples.

2.1. First we recall the standard definition of Massey products in the cohomology of a DGLA (see [R2]). A DGLA is a vector space $L$ over a field $\mathbb{K}$ of characteristic zero with a $\mathbb{Z}$ or $\mathbb{Z}_2$ grading $L = \bigoplus L^i$ and with a commutator operation $\mu : L \otimes L \to L$, $\mu(\alpha, \beta) = [\alpha, \beta]$ of degree 0 and a differential $\delta : L \to L$ of degree +1 satisfying the conditions

\[
[\alpha, \beta] = -(-1)^{\alpha \beta} [\beta, \alpha],
\]
\[
\delta [\alpha, \beta] = [\delta \alpha, \beta] + (-1)^\alpha [\alpha, \delta \beta],
\]
\[
[[\alpha, \beta], \gamma] + (-1)^{\alpha (\beta + \gamma)} [[\beta, \gamma], \alpha] + (-1)^{\gamma (\alpha + \beta)} [[\gamma, \alpha], \beta] = 0,
\]

where the degree of a homogeneous element is denoted by the same letter as this element. For any graded (Lie or co)algebra $A$, define the maps $S : A \otimes A \to A \otimes A$ and $C : A \otimes A \otimes A \to$
A \otimes A \otimes A$, by the formulas $S(\alpha \otimes \beta) = (-1)^{\alpha \beta} \beta \otimes \alpha$, $C(\alpha \otimes \beta \otimes \gamma) = (-1)^{\alpha(\beta + \gamma)} \beta \otimes \gamma \otimes \alpha$. Then the conditions on $\mu$ and $\delta$ may be rewritten as

$$\begin{align*}
\mu \circ S &= -\mu, \\
\delta \circ \mu &= \mu \circ (\delta \otimes 1 + 1 \otimes \delta), \\
\mu \circ (\mu \otimes 1) \circ (1 + C + C^2) &= 0.
\end{align*}$$

Note that the tensor product of homomorphisms between graded spaces is understood in the graded sense, $(f \otimes g)(x \otimes y) = (-1)^{g x} f(x) \otimes g(y)$; in particular, $(1 \otimes \delta)(\alpha \otimes \beta) = (-1)^{\alpha} \alpha \otimes \delta(\beta)$.

We denote by $H = \bigoplus_i H^i$ the cohomology of $L$ with respect to the differential $\delta$.

Let $a_1 \in H^{q_1}, \ldots, a_r \in H^{q_r}$ be cohomology classes. If $K$ and $L$ are proper subsets of $R = \{1, \ldots, r\}$, then define

$$\varepsilon(K, L) = \sum (q_k + 1)(q_l + 1),$$

where the sum is taken over all $k$ and $l$ such that $k \in K, l \in L$ and $k > l$. Also, for any subset $I$ of $R$, let

$$P(I) = \{(K, L) \mid K \subset I, K \cup L = I, K \cap L = \emptyset, K \neq \emptyset, L \neq \emptyset\}.$$  

We say that $b \in \langle a_1, \ldots, a_r \rangle$ if there exists a function assigning to each proper subset $I = \{i_1, \ldots, i_s\}$ of $R = \{1, \ldots, r\}$ an element $\alpha_I \in L^{q_{i_1} + \ldots + q_{i_s} - (s - 1)}$, such that

$$\alpha_{i} \in a_i,$$

$$\delta \alpha_I = \frac{1}{2} \sum_{(K, L) \in P(I)} (-1)^{\varepsilon(K, L) + \alpha K + 1}[\alpha_K, \alpha_L];$$

and

$$b \ni \frac{1}{2} \sum_{(K, L) \in P(R)} (-1)^{\varepsilon(K, L) + \alpha K + 1}[\alpha_K, \alpha_L].$$

Hence $\langle a_1, \ldots, a_r \rangle$ is a subset, maybe empty, of $H^{q_1 + \ldots + q_r - (r - 1)}$. It is called the Massey product of $a_1, \ldots, a_r$. Note that $\langle a_1 \rangle = 0$ and $\langle a_1, a_2 \rangle = [a_1, a_2]$. Also note that $\langle a_1, \ldots, a_r \rangle$ is nonempty if and only if for any proper subset $\{i_1, \ldots, i_s\}$ of $R$ the Massey product $\langle a_{i_1}, \ldots, a_{i_s} \rangle$ contains zero.

### 2.2. Now we will consider deformations of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ with bracket $[\cdot, \cdot]$. A formal deformation of $\mathfrak{g}$ is defined as a power series

$$[g, h]_t = [g, h] + \sum_{k=1}^{\infty} t^k \gamma_k(g, h),$$

which makes $\mathfrak{g}[[t]]$, the space of formal power series with coefficients in $\mathfrak{g}$, a Lie algebra. This deformed bracket is antisymmetric and satisfies the Jacobi identity exactly when

$$\gamma_k(g, h) = -\gamma_k(h, g),$$
The problem can be stated as follows: for two given cohomology classes $\gamma_1, \gamma_2 \in H^2(g; g)$, and the product $[\gamma_i, \gamma_{k-i}]$ is taken with respect to the standard graded Lie algebra structure in $C^*(g; g)$; if $\gamma' \in C^q(g; g), \gamma'' \in C^q(g; g)$, then $[\gamma', \gamma''] \in C^q+q''-2(g; g)$ (see, for example, [Fu]). If we set $C^q(g; g) = Lq-1$, then $L = \bigoplus L^i$ is a DGLA.

For $k = 1$ condition (2) means that $\delta \gamma_1 = 0$ and elements of $H^2(g; g)$ are called infinitesimal deformations of $g$. The cohomology class of $\gamma_1$ is called the differential of the formal deformation (1), or the infinitesimal deformation defined by the formal deformation (1). A given infinitesimal deformation $c \in H^2(g; g)$ is the differential of some formal deformation if and only if there exist cochains $\gamma_i \in C^2(g; g)$ such that $\delta \gamma_1 = 0, \gamma_1 \in c$, and the whole sequence $\{\gamma_i\}$ satisfies the conditions in (2). These conditions are usually referred to as $\langle c, \ldots, c \rangle \equiv 0$ for all $r$; but there exists a difference between this $\langle c, \ldots, c \rangle$ and the Massey product defined in 2.1. (This difference was pointed out by Retakh in [R1].) Indeed, in 2.1 we chose cochains $a_I$ for all $I \subset \{1, \ldots, r\}$, while here we take we take just one cochain $\gamma_k$ for each $k$, which means, from the point of view of 2.1, that all cochains $a_I$ with sets $I$ of the same cardinality are chosen equal (the products are then multiples of each other).

2.3. Let $g$ be as above. A singular deformation of $g$ is a formal deformation defined as in (1) with $\gamma_1 = 0$, that is

$$[g, h]_t = [g, h] + \sum_{k=2}^{\infty} t^k \gamma_k(g, h).$$

It is known (see [FF]) that some Lie algebras have essentially singular deformations (those which cannot be approximated by nonsingular ones). For singular deformations the first two conditions from (2) become

$$\delta \gamma_2 = 0, \delta \gamma_3 = 0.$$

The problem can be stated as follows: for two given cohomology classes $c_2, c_3 \in H^2(g; g)$ does there exist a singular deformation (3) with $\gamma_2 \in c_2, \gamma_3 \in c_3$? The equations in (2) provide necessary and sufficient conditions for that. The first three conditions, $\delta \gamma_4 = -\frac{1}{2}[\gamma_2, \gamma_2], \delta \gamma_5 = -[\gamma_2, \gamma_3], \delta \gamma_6 = -[\gamma_2, \gamma_4] - \frac{1}{2}[\gamma_3, \gamma_3]$, can be expressed in terms of Massey products and powers as $[c_2, c_2] = 0, [c_2, c_3] = 0$, and $-\frac{1}{2}[c_3, c_3] \in \langle c_2, c_2, c_2 \rangle$. The fourth condition, $\delta \gamma_7 = -[\gamma_2, \gamma_5] - [\gamma_3, \gamma_4]$, uses a Massey-like product which is simply a restricted form of $\langle c_2, c_2, c_3 \rangle$; however, the next equation, $\delta \gamma_8 = -[\gamma_2, \gamma_6] - [\gamma_3, \gamma_5] - \frac{1}{2}[\gamma_4, \gamma_4]$, not only is related to a combination of two Massey products, $\langle c_2, c_2, c_2, c_2 \rangle$ and $\langle c_2, c_3, c_3 \rangle$, but these products may both be empty (we do not have either the equality $[c_3, c_3] = 0$, or the inclusion $\langle c_2, c_2, c_2 \rangle \ni 0$). The subsequent conditions also refer to Massey-like products, (also similar to combinations of empty Massey products) which are essentially different from those of 2.1 and 2.2.
2.4. Let $\mathfrak{g}$ be as above. Consider two formal deformations,

$$[g, h]_t = [g, h] + \sum_{k=1}^{\infty} t^k \gamma_k(g, h), \quad [g, h]'_t = [g, h] + \sum_{k=1}^{\infty} t^k \gamma'_k(g, h)$$

with $\gamma'_1 = \gamma_1$. For $i \geq 2$ set $\beta_i = \gamma'_i - \gamma_i$. The two deformations in (4) satisfy the conditions (2), that is

$$\delta \gamma_k = -\frac{1}{2} \sum_{i=1}^{k-1} [\gamma_i, \gamma_{k-i}], \quad \delta \gamma'_k = -\frac{1}{2} \sum_{i=1}^{k-1} [\gamma'_i, \gamma'_{k-i}].$$

If we substitute $\gamma_i + \beta_i$ for $\gamma'_i$ and then subtract the first equality from the second, we get the following pair of relations:

$$\delta \gamma_k = -\frac{1}{2} \sum_{i=1}^{k-1} [\gamma_i, \gamma_{k-i}], \quad \delta \beta_k = -\sum_{i=1}^{k-2} [\gamma_i, \beta_{k-i}] - \frac{1}{2} \sum_{i=2}^{k-2} [\beta_i, \beta_{k-i}].$$

In particular, we have $\delta \gamma_1 = 0$ and $\delta \beta_2 = 0$. The following problem arises. For two given cohomology classes $c, b \in H^2(\mathfrak{g}; \mathfrak{g})$ does there exist a pair of formal deformations (4) with $\gamma'_1 = \gamma_1 \in c$, $\gamma'_2 = \gamma_2 - \beta_2 \in b$? The conditions for this are provided by the relations in (5). First of all, the conditions $\delta \gamma_2 = -\frac{1}{2} [\gamma_1, \gamma_1]$ and $\delta \beta_3 = -[\gamma_1, \beta_2]$ mean that $[c, c] = 0$ and $[c, b] = 0$ in order for $\gamma_2$ and $\beta_3$ to exist. The next pair of relations says that $\delta \gamma_3 = -[\gamma_1, \gamma_2]$ and $\delta \beta_4 = -[\gamma_1, \beta_3] - [\gamma_2, \beta_2] - \frac{1}{2} [\beta_2, \beta_2]$, which implies that $\langle c, c, c \rangle \equiv 0$ and $-\frac{1}{2} [b, b] \in \langle c, c, b \rangle$. The third pair, $\delta \gamma_4 = -[\gamma_1, \gamma_3] - \frac{1}{2} [\gamma_2, \gamma_2]$ and $\delta \beta_5 = -[\gamma_1, \beta_4] - [\gamma_2, \beta_3] - [\gamma_3, \beta_2] - [\beta_2, \beta_3]$, may be interpreted separately as $\langle c, c, c, c \rangle \equiv 0$ (in the sense of 2.2) and a product related to a combination of $\langle c, c, c, b \rangle$ and $\langle c, b, b \rangle$ (empty Massey products) contains zero. However, the cochains $\gamma_2$ and $\gamma_3$ are shared by the equations and so in order for $\gamma_4$ and $\beta_5$ to exist, $\langle c, c, c, c \rangle$ must contain zero using cochains which also make the second product contain zero. The subsequent relations mean that certain Massey-like products involving $c$ and $b$ contain zero with cochains that put $0 \in \{c, \ldots, c\}$ for all $r$ (in the sense of 2.2). These examples show that very natural questions about formal deformations of Lie algebras lead to different kinds of Massey-like products of cohomology classes in $H^2(\mathfrak{g}; \mathfrak{g})$. In the next section we will show that all these products are actually described by a very simple general construction, which involves an almost arbitrary auxiliary (graded) associative commutative algebra.

3. General construction

3.1. Inductive definition. Let $L = \bigoplus L^q$, $H = \bigoplus H^q$ be as above. Suppose that we are given a finite or infinite sequence of integers $\{q_k \in \mathbb{Z}\}_{k=1}^{n}$ or $\infty$ and an indexed set $\{c^{ij}_k \in \mathbb{K}\}_{i,j,k=1}^{n}$ or $\infty$, which satisfy the following condition:

$$c^{ij}_k = 0, \text{ if } i \geq k, j \geq k \text{ or } q_k \neq q_i + q_j - 1.$$ 

Also we suppose that some integer $r$ is fixed, and $c^{ij}_k = 0$ for $k \leq r$. 


Let \( a_1 \in H^{q_1}, \ldots, a_r \in H^{q_r} \), and let \( b \in H^{q_{r+s}+1} \) for some \( s \geq 1 \). We say that

\[
b \in \langle a_1, \ldots, a_r \rangle_s,
\]

if there exists a sequence \( \alpha_k \in L^{q_k}, k = 1, \ldots, r+s-1 \), such that

(i) \( \delta \alpha_k = \sum_{i,j} c^{ij}_k [\alpha_i, \alpha_j] \) for \( 1 \leq k \leq r+s-1 \); in particular, \( \delta \alpha_1 = 0, \ldots, \delta \alpha_r = 0 \);

(ii) \( \alpha_1 \in a_1, \ldots, \alpha_r \in a_r \);

(iii) \( \delta \beta = 0 \) and \( \beta \in b \), where \( \beta = \sum_{i,j} c^{ij}_{r+s} [\alpha_i, \alpha_j] \).

Note that in order for \( \langle a_1, \ldots, a_r \rangle_s \) to be nonempty it is necessary that \( \langle a_1, \ldots, a_r \rangle_t \ni 0 \)

All the examples of Section 2 are particular cases of this definition. But in all these examples the coefficients \( c^{ij}_k \) have two more properties, which make the definition particularly convenient.

First, for \( i \neq j \), the two terms \( c^{ij}_k [\alpha_i, \alpha_j] \) and \( c^{ji}_k [\alpha_j, \alpha_i] \) in the sum \( \sum_{i,j} c^{ij}_k [\alpha_i, \alpha_j] \) both contribute the same cochain, \( [\alpha_i, \alpha_j] = \pm [\alpha_j, \alpha_i] \) with possibly different coefficients. It is convenient to make the the two terms equal with the condition

\[
c^{ij}_k = (-1)^{q_i q_j -1} c^{ji}_k.
\]

Second, in all the examples above the coefficients \( c^{ij}_k \) were chosen in such a way that the sum \( \sum_{i,j} c^{ij}_k [\alpha_i, \alpha_j] \) is automatically a cocycle (belongs to Ker \( \delta \)).

Both of these conditions assume a compact form if we consider the numbers \( c^{ij}_k \), or rather \( (-1)^{q_i -1} c^{ij}_k \), as the structure constants of a certain algebra, that is if we consider a \( \mathbb{K} \)-algebra \( G \) with a basis \( g^k, k = 1, 2, \ldots \) and with \( g^i g^j = \sum_k (-1)^{q_i -1} c^{ij}_k g^k \). Then the first condition on the \( c^{ij}_k \) becomes evidently that of graded commutativity (we put \( \deg g^k = q_k - 1 \)), while the second condition turns out to be that of associativity. To demonstrate this it is convenient to have the previous construction in a more general form; in this compact construction we will use rather an auxiliary coassociative coalgebra than an associative algebra.

### 3.2. Direct definition.

Let \( L, \mu, \delta, S, C \) and \( H \) be the same as in 2.1. Now let \( F \) be a graded cocommutative coassociative coalgebra, that is a \( (\mathbb{Z} \text{ or } \mathbb{Z}_2) \)-graded vector space \( F \) with a degree 0 mapping (comultiplication) \( \Delta: F \to F \otimes F \) satisfying the conditions \( S \circ \Delta = \Delta \) and \( (1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta \).

Suppose also that in \( F \) a filtration \( F_0 \subset F_1 \subset F \) is given, such that \( F_0 \subset \ker \Delta \) and \( \text{Im} \Delta \subset F_1 \otimes F_1 \). The next proposition is, technically, the central fact of this article.

**Proposition 3.1.** Suppose a linear mapping \( \alpha: F_1 \to L \) of degree 1 satisfies the condition

\[
\delta \circ \alpha = \mu \circ (\alpha \otimes \alpha) \circ \Delta.
\]

Then

\[
\mu \circ (\alpha \otimes \alpha) \circ \Delta(F) \subset \ker \delta.
\]
Remark. The left-hand side of the last formula is well defined because $\Delta(F)$ is contained in $F_1 \otimes F_1$, the domain of $\alpha \otimes \alpha$.

Proposition 3.1 will be proved in Subsection 3.3 below.

Definition. Let $a: F_0 \to H, b: F/F_1 \to H$ be two linear maps of degree 1. We say that $b$ is contained in the Massey $F$-product of $a$, and write $b \in \langle a \rangle_F$, or $b \in \langle a \rangle_F^1$, if there exists a degree 1 linear mapping $\alpha: F_1 \to L$ satisfying condition (6), and such that the diagrams

\[
\begin{array}{ccc}
F_0 & \xrightarrow{\alpha \otimes r_0} & \ker \delta \\
\parallel & \downarrow \pi & \downarrow \pi \\
F_0 & \xrightarrow{a} & H \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
F & \xrightarrow{\mu \circ (\alpha \otimes \alpha) \otimes \Delta} & \ker \delta \\
\downarrow \pi & & \downarrow \pi \\
F/F_1 & \xrightarrow{b} & H \\
\end{array}
\]

are commutative, where the vertical maps labeled $\pi$ denote the projections of each space onto the quotient space.

Note that the upper horizontal maps of the diagrams in (7) are well defined, since $\alpha(F_0) \subset \alpha(\ker \Delta) \subset \ker \delta$ by virtue of (6), and $\mu \circ (\alpha \otimes \alpha) \circ \Delta(F) \subset \ker \delta$ by Proposition 3.1.

The inductive definition of 3.1 is a particular case of this direct definition in the following way. One can take for $F$ the vector space spanned by $f_1, \ldots, f_{r+s}$, define $F_0$ and $F_1$ as subspaces spanned by $f_j$ with $j \leq r$ and $j < r+s$ respectively, and let $\deg f_k = q_k - 1$, $\Delta f_k = \sum_{i,j} (-1)^{q_i-1} c_{ij}^{kj} f_i \otimes f_j$. The maps $a: F_0 \to H$ and $b: F/F_1 \to H$ are defined by the formulas $f_k \mapsto a_k$ ($k \leq r$) and $f_{r+s} + F_1 \mapsto b$. The map $\alpha$ takes $f_k$ into $\alpha_k$; condition (6) and the commutativity of diagrams (7) become conditions (i), (ii), and (iii) of 3.1.

Note in conclusion that our definition makes sense even in the case, when $F_1 = F$. In this case we do not need to specify any $b$, and we will simply say that $a$ satisfies the condition of triviality of Massey $F$-products.

3.3. Proof of Proposition 3.1. We need to prove that

$$\delta \circ \mu \circ (\alpha \otimes \alpha) \circ \Delta = 0.$$ 

The definition of a DGLA and condition (6) imply:

$$\delta \circ \mu \circ (\alpha \otimes \alpha) \circ \Delta = \mu \circ (\delta \otimes \Delta + \Delta \otimes \delta) \circ (\alpha \otimes \alpha) \circ \Delta$$

$$= \mu \circ ((\delta \circ \alpha) \otimes \alpha - \alpha \otimes (\delta \circ \alpha)) \circ \Delta$$

$$= \mu \circ ((\delta \circ \alpha) \otimes \alpha) \circ \Delta - \mu \circ (\alpha \otimes (\delta \circ \alpha)) \circ \Delta;$$

$$\mu \circ ((\delta \circ \alpha) \otimes \alpha) \circ \Delta = \mu \circ ((\mu \circ (\alpha \otimes \alpha) \circ \Delta) \otimes \alpha) \circ \Delta$$

$$= \mu \circ ((\mu \circ (\alpha \otimes \alpha) \circ \Delta) \otimes (1 \otimes \alpha \otimes 1)) \circ \Delta$$

$$= \mu \circ (\mu \otimes 1) \circ (\alpha \otimes \alpha \otimes \alpha) \circ (\Delta \otimes 1) \circ \Delta;$$

and similarly

$$\mu \circ (\alpha \otimes (\delta \circ \alpha)) \circ \Delta = \mu \circ (1 \otimes \mu) \circ (\alpha \otimes \alpha \otimes \alpha) \circ (1 \otimes \Delta) \circ \Delta.$$
The following two lemmas hold even if the coalgebra $F$ is neither cocommutative, nor coassociative.

**Lemma 1.**

$$\Delta \otimes 1 = C \circ (1 \otimes \Delta) \circ S : F \otimes F \rightarrow L \otimes L \otimes L.$$  

**Proof.** Let $f, g \in F$. Then

$$C \circ (1 \otimes \Delta) \circ S (f \otimes g) = (-1)^{fg} C \circ (1 \otimes \Delta) (g \otimes f)$$

$$= (-1)^{fg} C \circ g \otimes \Delta(f)$$

$$= \Delta(f) \otimes g$$

$$= (\Delta \otimes 1)(f \otimes g).$$

**Lemma 2.**

$$C \circ (\alpha \otimes \alpha \otimes \alpha) = (\alpha \otimes \alpha \otimes \alpha) \circ C : F \otimes F \otimes F \rightarrow L \otimes L \otimes L.$$  

**Proof.** Let $f, g, h \in F$. Then

$$C \circ (\alpha \otimes \alpha \otimes \alpha)(f \otimes g \otimes h) = (-1)^{g+h} C \circ \alpha(f) \otimes \alpha(g) \otimes \alpha(h)$$

$$= (-1)^{g+h} C \circ \alpha(g) \otimes \alpha(h) \otimes \alpha(f),$$

$$(\alpha \otimes \alpha \otimes \alpha) \circ C (f \otimes g \otimes h) = (-1)^{f+g+h} (\alpha \otimes \alpha \otimes \alpha)(g \otimes h \otimes f)$$

$$= (-1)^{f+g+h} (\alpha \otimes \alpha \otimes \alpha)(f \otimes g \otimes h).$$

Using the Lemmas, the cocommutativity of $F$, and the coassociativity of $F$:

$$\mu \circ (\mu \otimes 1) \circ (\alpha \otimes \alpha \otimes \alpha) \circ (\Delta \otimes 1) \circ \Delta$$

$$= \mu \circ (\mu \otimes 1) \circ (\alpha \otimes \alpha \otimes \alpha) \circ C \circ (1 \circ \Delta) \circ S \circ \Delta$$

$$= \mu \circ (\mu \otimes 1) \circ C \circ (\alpha \otimes \alpha \otimes \alpha) \circ (1 \circ \Delta) \circ S \circ \Delta$$

$$= \mu \circ (\mu \otimes 1) \circ C \circ (\alpha \otimes \alpha \otimes \alpha) \circ (1 \circ \Delta) \circ \Delta$$

$$= \mu \circ (\mu \otimes 1) \circ C \circ (\alpha \otimes \alpha \otimes \alpha) \circ (\Delta \otimes 1) \circ \Delta.$$  

Similarly,

$$\mu \circ (\mu \otimes 1) \circ C \circ (\alpha \otimes \alpha \otimes \alpha) \circ (\Delta \otimes 1) \circ \Delta$$

$$= \mu \circ (\mu \otimes 1) \circ C^2 \circ (\alpha \otimes \alpha \otimes \alpha) \circ (\Delta \otimes 1) \circ \Delta.$$  

Hence,

$$\mu \circ (\mu \otimes 1) \circ (\alpha \otimes \alpha \otimes \alpha) \circ (\Delta \otimes 1) \circ \Delta$$

$$= \frac{1}{3} \mu \circ (\mu \otimes 1) \circ (1 + C + C^2) \circ (\alpha \otimes \alpha \otimes \alpha) \circ (\Delta \otimes 1) \circ \Delta = 0.$$  

In the same way,

$$\mu \circ (1 \otimes \mu) \circ (\alpha \otimes \alpha \otimes \alpha) \circ (1 \otimes \Delta) \circ \Delta = 0.$$
and hence,
\[ \delta \circ \mu \circ (\alpha \otimes \alpha) \circ \Delta = 0. \]

3.4. **Survey of examples of Section 2.** Our goal is to show that all the Massey-like products considered in Section 2 are covered by the previous construction.

For the definition of Massey products in 2.1 one should set
\[ F = \text{span}\{f_I|\emptyset \not\subseteq I \subseteq R = \{1,\ldots,r\}\}, \]
\[ F_0 = \text{span}\{f_1,\ldots,f_r\}, \quad F_1 = \text{span}\{f_I|I \not\subseteq R\}, \]
\[ \deg f_I = \sum_{i \in I} (q_i - 1), \]
\[ \Delta f_I = \frac{1}{2} \sum_{K \subseteq I, \emptyset \neq K \neq I} (-1)^{\varepsilon(K,I-K)} f_K \otimes f_{I-K}. \]

The dual algebra \( G = F^* \) is spanned by \( g_I, \emptyset \not\subseteq I \subseteq R \), and the multiplication in \( G \) is defined by the formula
\[ g^K g^L = \begin{cases} 0 & \text{if } K \cap L = \emptyset, \\ \frac{1}{2} (-1)^{\varepsilon(K,L)} g^{K \cup L} & \text{if } K \cap L \neq \emptyset, \end{cases} \]

If we put \( x_i = g^{(i)} \), then for \( I = \{i_1,\ldots,i_s\} \)
\[ g^I = 2^{1-s} x_{i_1} \cdots x_{i_s}. \]

Hence \( G \) is the maximal ideal in the quotient of the free graded commutative algebra with the generators \( x_1,\ldots,x_r \) with \( \deg x_i = q_i - 1 \) over the ideal generated by \( x_1^2,\ldots,x_r^2 \). By the way, \( F_0^* = G/G^2 \), and \( F_1^* = G/G^r \).

For the condition in 2.2 that \( \langle c,\ldots,c \rangle \not\supseteq 0 \) for all \( r \) one should set
\[ F = \text{span}\{f_1,f_2,\ldots\}, \]
\[ F_0 = \mathbb{K} f_1, \quad F_1 = F, \]
\[ \deg f_i = 0, \]
\[ \Delta f_i = -\frac{1}{2} \sum_{k=1}^{i-1} f_k \otimes f_{i-k}. \]

The dual algebra \( G \) is formed by finite or infinite linear combinations of \( g^k, k = 1,2,\ldots \) with the multiplication
\[ g^k g^l = -\frac{1}{2} g^{k+l}. \]
If we put \( g^1 = t \), then \( g^k = (-2)^{1-k} t^k \) (in the left hand side \( k \) is a superscript, while in the right hand side it is the exponent). Hence \( G \) is the maximal ideal in \( \mathbb{K}[[t]] \), the algebra of formal power series in one variable \( t \) of degree 0. Again \( F^* = G/G^2 \).

For the definition of 2.3 one has

\[
F = \text{span}\{f_2, f_3, \ldots\},
\]
and all the formulas are obvious modifications of the formulas of the previous paragraph. In particular, \( G \) is the square of the maximal ideal of \( \mathbb{K}[[t]] \). If we set \( u = t^2, v = t^3 \), we may interpret \( G \) as the maximal ideal in \( \mathbb{K}[[u, v]]/(u^3 - v^2) \). Again \( F^* = G/G^2 \).

For the problem in 2.4 of the existence of a pair of deformations one should take

\[
F = \text{span}\{f_1, f_2, f_3, \ldots; \varphi_2, \varphi_3, \ldots\},
\]

\[
F_0 = \text{span}\{f_1, \varphi_2\}, F_1 = F,
\]

\[
\text{deg}(f_i) = 0, \text{deg}(\varphi_i) = 0,
\]

\[
\Delta f_i = -\frac{1}{2} \sum_{k=1}^{i-1} f_k \otimes f_{i-k}, \quad \Delta \varphi_i = -\frac{1}{2} \sum_{k=1}^{i-2} f_k \otimes \varphi_{i-k} - \frac{1}{2} \sum_{k=2}^{i-2} \varphi_k \otimes \varphi_{i-k}.
\]

The dual algebra \( G \) is formed by finite or infinite linear combinations of \( g^k, k = 1, 2, \ldots \), and \( \psi^k, k = 2, 3, \ldots \) with the multiplication

\[
g^k g^l = -\frac{1}{2} g^{k+l}, \quad g^k \psi^l = -\frac{1}{2} \psi^{k+l}, \quad \psi^k \psi^l = -\frac{1}{2} \psi^{k+l}.
\]

If we put \( g^1 = t, \psi^2 = u \), then \( g^k = (-2)^{1-k} t^k, \psi^l = (-2)^{2-l} t^{l-2} u \), and \(-2u^2 = t^2 u \). Hence \( G \) is the maximal ideal in \( \mathbb{K}[[t, u]]/(2u^2 + t^2 u) \), and again we have \( F^* = G/G^2 \).

4. An application to Lie algebra deformations

4.1. Deformations. We consider the algebraic version of deformation theory (see, for example, [Fi]). Let \( S \) be a commutative associative \( \mathbb{K} \)-algebra with an identity element, and with a distinguished maximal ideal \( \mathfrak{m} \subset S \) with \( S/\mathfrak{m} \cong \mathbb{K} \); let \( \varepsilon: S \to S/\mathfrak{m} = \mathbb{K} \) be the projection with \( \varepsilon(1) = 1 \). Consider a Lie algebra \( \mathfrak{g} \) over \( S \). A deformation of \( \mathfrak{g} \) with base \( S \) is, by definition, a structure of a Lie algebra over

\[
\mathfrak{K} \text{ on the } S \text{-module } \mathfrak{g} \otimes S \text{, such that } 1 \otimes \varepsilon: \mathfrak{g} \otimes S \to \mathfrak{g} \otimes \mathbb{K} = \mathfrak{g} \text{ is a Lie algebra homomorphism.}
\]

Example. Let \( \dim \mathfrak{g} < \infty, \mathbb{K} = \mathbb{R}, S = C^\infty X \), the algebra of smooth functions on a smooth compact manifold \( X \), and \( \mathfrak{m} \) be the ideal of functions vanishing at a point \( x_0 \in X \). Then a deformation of \( \mathfrak{g} \) with base \( S \) is the same as a smooth family of Lie algebra structures on \( \mathfrak{g} \), parametrized by points \( x \in X \) and reducing to the initial Lie algebra structure on \( \mathfrak{g} \) for \( x = x_0 \).
Suppose that \( \dim S < \infty \). Let \( \tau: (g \otimes S) \times (g \otimes S) \to g \otimes S \) be a skew-symmetric \( S \)-bilinear binary operation on \( g \otimes S \) which satisfies the condition of \( 1 \otimes \varepsilon \) being a homomorphism, that is \( (1 \otimes \varepsilon) \tau(g \otimes s, h \otimes t) = [g, h] \varepsilon(st) \), but not necessarily satisfying the Jacobi identity. For a linear functional \( \varphi: m \to K \) define a map \( \alpha_{\varphi}: g \otimes g \to g \) by the formula
\[
\alpha_{\varphi}(g, h) = (1 \otimes \varphi)(\tau(g \otimes 1, h \otimes 1) - [g, h] \otimes 1)
\]
(it follows from the fact that \( 1 \otimes \varepsilon \) is a homomorphism that \( \tau(g \otimes 1, h \otimes 1) - [g, h] \otimes 1 \in g \otimes m \)). Evidently, \( \alpha_{\varphi} \) is bilinear and skew-symmetric, which means that \( \alpha_{\varphi} \in C^2(g; g) \). Moreover, if we put \( m^* = F \), then \( \varphi \mapsto \alpha_{\varphi} \) is a linear map \( \alpha: F \to C^2(g; g) \), and it is clear that \( \tau \) with the properties listed above and \( \alpha \) determine each other.

Let \( \Delta: F \to F \otimes F \) be the comultiplication in \( F \) dual to the multiplication in \( m \), \( \delta: C^2(g; g) \to C^3(g; g) \) be the Lie algebra cochain differential, \( \mu: C^2(g; g) \otimes C^2(g; g) \to C^2(g; g) \) be the Lie product.

**Proposition 4.1.** The operator \( \tau \) satisfies the Jacobi identity if and only if \( \alpha \) satisfies the Maurer-Cartan equation
\[
\delta \circ \alpha + \frac{1}{2} \mu \circ (\alpha \otimes \alpha) \circ \Delta = 0.
\]

This proposition may be regarded as well known, but we will prove it below (see 4.3) for the sake of completeness.

Let \( F_0 = (m/m^2)^* \subset F \). This is the tangent space of \( S \) at \( m \). Obviously, \( F_0 \subset \ker \Delta \), and Proposition 4.1 shows that if \( \tau \) is a deformation of \( g \), that is if \( \tau \) satisfies the Jacobi identity, then \( \alpha(F_0) \subset \ker \delta \). Consider the composition
\[
\begin{align*}
a: F_0 &\xrightarrow{\alpha} \ker \delta \xrightarrow{\pi} H^2(g; g), \\
\end{align*}
\]
where \( \pi \) is the projection map. An arbitrary linear map \( F_0 \to H^2(g; g) \) is called an infinitesimal deformation of \( g \) with base \( S \), and \( a \) is called the infinitesimal deformation determined by the deformation \( \tau \), or the differential of the deformation \( \tau \). Proposition 4.1 implies

**Theorem 4.2.** An infinitesimal deformation \( a: F_0 \to H^2(g; g) \) is a differential of some deformation with base \( S \) if and only if \(-\frac{1}{2}a\) satisfies the condition of triviality of Massey \( S \)-products.

### 4.2. Formal deformations

Let \( S \) be a local commutative associative algebra over \( \mathbb{K} \) with maximal ideal \( m \) and canonical projection \( \varepsilon: S \to \mathbb{K} \). Suppose that \( \dim m^{k-1}/m^k < \infty \) for all \( k \). For a vector space \( V \) we define \( V \otimes S \) as \( \lim_{\leftarrow} (V \otimes (S/m^k)) \). A formal deformation of a Lie algebra \( g \) with base \( S \) is defined in the same way as a deformation of \( g \) with base \( S \), but with \( g \otimes S \) instead of \( g \otimes S \).

**Example.** Let \( S = \mathbb{K}[t] \). Then a formal deformation of \( g \) with base \( S \) is the same as a 1-parameter formal deformation of \( g \) (see 2.2).

Everything that was said in 4.1, including Proposition 4.1 and Theorem 4.2 may be repeated for formal deformations with two changes: a functional \( \varphi: m \to \mathbb{K} \) is assumed to
be continuous, that is \( \varphi(m^k) = 0 \) for some \( k \); the space \( F \) is the space of all continuous functionals \( m \to K \).

Note that the formal version of Theorem 4.2 generalizes the results of 2.2, 2.3, and 2.4. In these cases \( S \) corresponds to the straight line \( (K[[t]]) \), the semicubic parabola \( (K[[u, v]]/(u^3 - v^2)) \) and the union of two smooth curves with a first order tangency \( (K[[t, u]]/(u(t^2 + 2u))) \).

4.3. Proof of Proposition 4.1. We prove here Proposition 4.1 as it was stated in 4.1. The formal version from 4.2 is proved in a similar way.

Let \( \{m_i\} \) be a basis of \( m \). Suppose that \( \tau: (g \otimes S) \times (g \otimes S) \to g \otimes S \) satisfies the conditions listed in 4.1. Then for \( g, h \in g \)

\[
\tau(g \otimes 1, h \otimes 1) = [g, h] \otimes 1 + \sum_i \alpha_i(g, h) \otimes m_i
\]

with some \( \alpha_i \in C^2(g; g) \). Hence for \( g, h, k \in g \)

\[
\tau(\tau(g \otimes 1, h \otimes 1), k \otimes 1) = \tau([g, h] \otimes 1 + \sum_i \alpha_i(g, h) \otimes m_i, k \otimes 1)
\]

\[
= [[g, h], k] \otimes 1 + \sum_i \alpha_i([g, h], k) \otimes m_i + \sum_i [\alpha_i(g, h), k] \otimes m_i
\]

\[
+ \sum_{i,j} \alpha_{ij}(\alpha_i(g, h), k) \otimes m_im_j, \tag{10}
\]

and the Jacobi identity for \( \tau \) means that the cyclic sum of this expression with respect to \( g, h, k \) is equal to 0.

Let \( \varphi: m \to K \) be a linear functional, and let \( \varphi(m_i) = \varphi_i \in K \). Then \( \alpha \varphi = \sum \varphi_i \alpha_i \).

Let \( \Delta \varphi = \sum_p \psi'_p \otimes \psi''_p \); we suppose that this sum is symmetric, that is with each \( \psi'_p \otimes \psi''_p \) it contains also \( \psi''_p \otimes \psi'_p \). Let \( \psi'_p(m_i) = \psi'_{p,i}, \psi''_p(m_i) = \psi''_{p,i} \). Then

\[
\varphi(m_im_j) = \Delta \varphi(m_i \otimes m_j) = \sum_p \psi'_p(m_i) \psi''_p(m_j) = \sum_p \psi'_{p,i} \psi''_{p,j}
\]

Apply \( \varphi \) to the last term of (10):

\[
\varphi \left( \sum_{i,j} \alpha_{ij}(\alpha_i(g, h), k) \otimes m_im_j \right) = \sum_{i,j} \varphi(m_im_j) \alpha_{ij}(\alpha_i(g, h), k)
\]

\[
= \sum_{i,j,p} \psi'_{p,i} \psi''_{p,j} \alpha_{ij}(\alpha_i(g, h), k)
\]

\[
= \sum_p \alpha \psi'_{p,i} (\alpha \psi''_{p,i}(g, h), k).
\]

If we apply \( \varphi \) to the whole right hand side of (10), we get

\[
\alpha \varphi([g, h], k) + [\alpha \varphi(g, h), k] + \sum_p \alpha \psi'_{p,i} (\alpha \psi''_{p,i}(g, h), k).
\]
The cyclic sum of the last expression is equal to
\[ \delta \alpha_{ij}(g, h, k) + \frac{1}{2} \sum_p [\alpha_{ij}^p, \alpha_{ij}^p](g, h, k), \]
and the Jacobi identity for \( \tau \) means precisely that this is equal to 0. Proposition 4.1 follows.

5. Classical Massey products.

5.1. Definitions. Originally Massey products were considered not for the cohomology of DGLAs, but rather for the cohomology of graded commutative multiplicative complexes. Let \( A = \bigoplus A^i \) be a graded differential associative commutative algebra with the differential \( \delta \) of degree +1, and let \( H = \bigoplus H^i \) be its cohomology. Let \( a_1 \in H^{q_1}, \ldots, a_r \in H^{q_r}, b \in H^{q_1 + \ldots + q_r - (r-1)} \). We say that \( b \) belongs to the Massey product of \( a_1, \ldots, a_r \), and write \( b \in \langle a_1, \ldots, a_r \rangle \) if there exist a family \( \alpha_{ij} \in A^{q_i + \ldots + q_j - (j-i-1)}, 1 \leq i < j \leq r + 1, (i, j) \neq (1, r + 1) \), such that

(i) \( \delta \alpha_{ij} = \sum_{k=i+1}^{j-1} (-1)^{\alpha_{ik}} \alpha_{ik} \alpha_{kj} \); in particular, \( \delta \alpha_{i, i+1} = 0 \);

(ii) \( \alpha_{i, i+1} \in A \);

(iii) \( b \supseteq \sum_{k=2}^{r} (-1)^{\alpha_{1k}} \alpha_{1k} \alpha_{k, r+1} \). What is important here, is that the right hand side of the equality (i) belongs to \( \text{Ker} \delta \) if (i) holds for all \( \delta \alpha_{i', j'} \) with \( j' - i' < j - i \); in particular, (i) implies that the right hand side of (iii) belongs to \( \text{Ker} \delta \).

Note that \( \langle a \rangle = 0, \langle a, b \rangle = ab \), and \( \langle a_1, \ldots, a_r \rangle \) is non-empty if and only if \( \langle a_i, a_{i+1}, \ldots, a_j \rangle \neq 0 \) for all \( i, j \) such that \( 1 \leq i < j \leq r, (i, j) \neq (1, r) \).

5.2. May’s matric Massey products. The following generalization of the previous construction belongs to May [Ma]. Let \( p_1, \ldots, p_{r+1} \) be positive integers, and let \( a_i \) denote not an element of \( H^{q_i} \), but rather a \( p_i \times p_{i+1} \) matrix whose entries are elements of \( H^{q_i} \). Then the definition of 5.1 may be repeated with \( b \) being a \( p_1 \times p_{r+1} \) matrix with entries in \( H^{q_1 + \ldots + q_r - (r-1)} \), \( \alpha_{ij} \) being a \( p_i \times p_j \) matrix with entries in \( A^{q_i + \ldots + q_j - (j-i-1)} \), and the products in (i) and (iii) being considered in the matrix sense. The resulting product is called the matric Massey product.

5.3. The general construction. Let \( A \) and \( H \) denote the same as before. Let \( Q \) be a graded Lie coalgebra over \( \mathbb{K} \) with the comultiplication \( \Delta : Q \to Q \otimes Q \), and let \( Q_0 \subset Q_1 \subset Q \) be a filtration with \( \Delta(Q_0) = 0, \Delta(Q) \subset Q_1 \otimes Q_1 \). Let \( a : Q_0 \in H, b : Q/Q_1 \to H \) be linear maps of degree 1 such that \( \delta \circ \alpha = \mu \circ (\alpha \otimes \alpha) \circ \Delta \) (where \( \mu : A \otimes A \to A \) and \( \delta : A \to A \) are the multiplication and the differential in \( A \)), such that the diagrams similar to (7) (with \( Q \) and \( A \) instead of \( F \) and \( L \)) are commutative.

It is important for this definition that the fact similar to Proposition 3.1 holds: \( Q \) being a Lie coalgebra implies that \( \delta \circ \mu \circ (\alpha \otimes \alpha) \circ \Delta = 0 \). The proof of this fact is a replica of that of Proposition 3.1.

For the classical Massey product the corresponding Lie coalgebra is spanned by \( f_{ij} \) \( (1 \leq i \leq j \leq r + 1) \), \( \deg f_{ij} = \deg \alpha_{ij} - 1 \), and \( \Delta \) is defined by the formula
\[ \Delta f_{ij} = \frac{1}{2} \sum_{1<k<j} (f_{ik} f_{kj} - (-1)^{f_{ik} f_{kj}} f_{ik} f_{kj}). \]

If we ignore the coefficient \( \frac{1}{2} \) (which is the matter of the substitution \( f_{ij} = 2f'_{ij} \)), then the dual graded Lie algebra becomes the graded Lie algebra of strictly upper triangular \( (r+1) \times (r+1) \) matrices.

Similarly, the matric Massey product of 5.2 corresponds to the graded Lie algebra of block strictly upper triangular matrices with the block sizes \( p_1, \ldots, p_{r+1} \).

The authors would like to thank Michael Penkava for discussing these results with them and providing valuable suggestions.

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