WEAKLY CONNES AMENABLE DUAL BANACH ALGEBRAS

AMIN MAHMOODI

ABSTRACT. We shall develop a notion of amenability for dual Banach algebras, namely weak Connes amenability, which will play the role that weak amenability does for usual Banach algebras.

1. INTRODUCTION

Amenable Banach algebras were introduced by B. E. Johnson in [11]. There are several variants of amenability, two of the most notable are weak amenability and Connes amenability. The concept of weak amenability for Banach algebras was introduced by B. E. Johnson in [10], it generalizes that introduced by W. G. Bade, P. C. Curtis, and H. G. Dales for commutative Banach algebras in [1]. The notion of Connes amenability systematically introduced by V. Runde in [13]. We recall the definitions in Definitions 2.1 and 2.2 below.

The purpose of this paper is to study a new notion of amenability for dual Banach algebras. The organization of the paper is as follows. In Section 2, we recall some background notations and definitions.

In section 3, weak Connes amenability for dual Banach algebras is introduced and some basic and hereditary properties are given. It is shown that the corresponding class of such algebras includes all Connes amenable dual Banach algebras (Theorem 3.5), as well as all weakly amenable dual Banach algebras (Theorem 3.6). It is proved that commutative, pseudo Connes amenable dual Banach algebras are always weakly Connes amenable (Corollary 3.12). We study weak Connes amenability of direct sums of dual Banach algebras (Theorem 3.15). Weak Connes amenability of the enveloping dual Banach algebras is also discussed (Corollary 3.17).

In section 4 we verify this new notion for some certain algebras. Examples are given to distinguish between the new notion and the classical concepts of amenability. In particular, we present some weakly Connes amenable dual Banach algebras which are neither Connes amenable nor weakly amenable (Theorems 4.5 and 4.7).

The author thanks an anonymous referee for suggestions, in response to a previous version of this work, which improved the paper.

2. PRELIMINARIES

Suppose that $\mathfrak{A}$ is a Banach algebra. It is known that the projective tensor product $\mathfrak{A} \hat{\otimes} \mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule in the canonical way. There is a continuous linear $\mathfrak{A}$-bimodule homomorphism $\pi : \mathfrak{A} \hat{\otimes} \mathfrak{A} \to \mathfrak{A}$ such that $\pi(a \otimes b) = ab$ for $a, b \in \mathfrak{A}$. If $E$ is a Banach $\mathfrak{A}$-bimodule, then so is the dual space $E^*$. A continuous linear map $D : \mathfrak{A} \to E$ is a derivation if it satisfies $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in \mathfrak{A}$. We call $D$ inner if there is $x \in E$ such that $D(a) = ad_x(a) := a \cdot x - x \cdot a$ for every $a \in \mathfrak{A}$.

2010 Mathematics Subject Classification. Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

Key words and phrases. dual Banach algebra, Connes amenability, weak amenability, weak Connes amenability.
Definition 2.1. A Banach algebra $\mathfrak{A}$ is weakly amenable if every derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ is inner.

Let $X$ be a Banach space. We simply denote by $wk$ and $w^*$, the $\sigma(X,X^*)$-topology and the $\sigma(X^*,X)$-topology on $X$ and $X^*$, respectively.

Let $\mathfrak{A}$ be a Banach algebra. A Banach $\mathfrak{A}$-bimodule $E$ is dual if there is a closed submodule $E_*$ of $E^*$ such that $E = (E_*)^*$. We call $E_*$ the predual of $E$. A Banach algebra $\mathfrak{A} = (\mathfrak{A}_*)^*$ is dual if it is dual as a Banach $\mathfrak{A}$-bimodule. Equivalently, a Banach algebra $\mathfrak{A}$ is dual if it is a dual Banach space such that its multiplication is separately continuous in the $w^*$-topology.

Let $\mathfrak{A}$ be a dual Banach algebra, and let $E$ be a dual Banach $\mathfrak{A}$-bimodule. Then we say $E$ is normal if the module actions of $\mathfrak{A}$ on $E$ are $w^*$-$w^*$ continuous.

Definition 2.2. A dual Banach algebra $\mathfrak{A}$ is Connes amenable if for every normal, dual Banach $\mathfrak{A}$-bimodule $E$, every $w^*$-$w^*$ continuous derivation $D : \mathfrak{A} \to E$ is inner.

Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a dual Banach algebra and let $E$ be a Banach $\mathfrak{A}$-bimodule. We write $\sigma wc(E)$ for the set of all elements $x \in E$ such that the maps
\[
\mathfrak{A} \to E, \quad a \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases},
\]
are $w^*$-$wk$ continuous. It is well known that $\sigma wc(E)$ is a closed submodule of $E$, and $\sigma wc(E)^*$ is always normal. It is shown in [15, Corollary 4.6], that $\pi^*(\mathfrak{A}_*) \subseteq \sigma wc(\mathfrak{A} \hat{\otimes} \mathfrak{A})^*$. Taking adjoints, we can extend $\pi$ to an $\mathfrak{A}$-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc((\mathfrak{A} \hat{\otimes} \mathfrak{A})^*)$ to $\mathfrak{A}$. A $\sigma wc$-virtual diagonal for a dual Banach algebra $\mathfrak{A}$ is an element $M \in \sigma wc((\mathfrak{A} \hat{\otimes} \mathfrak{A})^*)$ such that $a \cdot M = M \cdot a$ and $a \pi_{\sigma wc}(M) = a$ for $a \in \mathfrak{A}$. It is known that Connes amenability of $\mathfrak{A}$ is equivalent to existence of a $\sigma wc$-virtual diagonal for $\mathfrak{A}$ [13].

Our comprehensive references on amenability of Banach algebras are [2, 16].

3. Weak Connes amenability

Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a dual Banach algebra, and let $E$ be a Banach $\mathfrak{A}$-bimodule. We denote by $j_E : E^* \to \sigma wc(E)^*$ the adjoint of the inclusion map $\sigma wc(E) \hookrightarrow E$. It is clear that $j_E$ is an $\mathfrak{A}$-bimodule homomorphism and $\langle x, j_E(f) \rangle = f(x)$ for all $x \in \sigma wc(E)$ and $f \in E^*$.

We start with the following observation.

Lemma 3.1. Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a dual Banach algebra, let $E$ be a Banach $\mathfrak{A}$-bimodule, and let $D : \mathfrak{A} \to E^*$ be a derivation. Then:

(i) The map $j_E \circ D : \mathfrak{A} \to \sigma wc(E)^*$ is a derivation;
(ii) If $D$ is inner, then so is $j_E \circ D$. Furthermore, $j_E \circ ad_f = ad_{j_E(f)}$ for each $f \in E^*$;
(iii) For every $f \in E^*$, $j_E \circ ad_f : \mathfrak{A} \to \sigma wc(E)^*$ is $w^*$-$w^*$ continuous.

Proof. (i) For every $a, b \in \mathfrak{A}$ and $x \in \sigma wc(E)$ we have
\[
\langle x, a \cdot (j_E \circ D)(b) + (j_E \circ D)(a) \cdot b \rangle = \langle x, j_E(a \cdot D(b)) + j_E(D(a) \cdot b) \rangle = \langle x, a \cdot D(b) + D(a) \cdot b \rangle = \langle x, D(ab) \rangle = \langle x, (j_E \circ D)(ab) \rangle,
\]
and hence $j_E \circ D$ is a derivation.

(ii) This is a simple calculation.

(iii) It follows from (ii) and normality of $\sigma wc(E)^*$. \qed
**Definition 3.2.** A dual Banach algebra $\mathfrak{A} = (\mathfrak{A},*)$ is weakly Connes amenable if for every derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ such that $j_{\mathfrak{A}} \circ D : \mathfrak{A} \to \sigma wc(\mathfrak{A})^*$ is $w^*-w^*$ continuous, derivation $j_{\mathfrak{A}} \circ D$ is inner.

**Remark 3.3.** For a dual Banach algebra $\mathfrak{A}$, $\sigma wc(\mathfrak{A})$ is a closed (two-sided) ideal of $\mathfrak{A}$. To see this, take $a \in \sigma wc(\mathfrak{A})$, $b \in \mathfrak{A}$, and let $c_a \xrightarrow{wk} c$ in $\mathfrak{A}$. Because $c_a \xrightarrow{wk} ca$, for every $\varphi \in \mathfrak{A}^*$ we have

$$\lim_{\alpha} \langle \varphi, c_a(ab) \rangle = \lim_{\alpha} \langle b \cdot \varphi, ca \rangle = \langle b \cdot \varphi, ca \rangle = \langle \varphi, c(ab) \rangle$$

so that $ab \in \sigma wc(\mathfrak{A})$. Next, by $w^*$-continuity of the multiplication, $c_a \xrightarrow{wk} cb$. Then $c_a(ba) = (c_a b)a \xrightarrow{wk} (cb)a = c(ba)$, which means $ba \in \sigma wc(\mathfrak{A})$.

For a given dual Banach algebra $\mathfrak{A}$, it would be interesting to determine the set $\sigma wc(\mathfrak{A})$, see for instance Proposition 4.8 and Example 4.9 below. However, special care should be taken with the trivial cases $\sigma wc(\mathfrak{A}) = \{0\}$ and $\sigma wc(\mathfrak{A}) = \mathfrak{A}$, as follows.

**Remark 3.4.** Let $\mathfrak{A}$ be a dual Banach algebra. Then:

(i) If $\sigma wc(\mathfrak{A}) = \{0\}$, then $j_{\mathfrak{A}}$ is the zero map. Therefore, for every derivation $D : \mathfrak{A} \to \mathfrak{A}^*$, $j_{\mathfrak{A}} \circ D = 0$. So, $\mathfrak{A}$ is weakly Connes amenable.

(ii) If $\sigma wc(\mathfrak{A}) = \mathfrak{A}$ (or equivalently $\mathfrak{A}^*$ is normal by [15, Proposition 4.4]), then $j_{\mathfrak{A}}$ is the identity map on $\mathfrak{A}^*$ and then $j_{\mathfrak{A}} \circ D = D$. In particular, Definition 3.2 becomes: $\mathfrak{A}$ is weakly Connes amenable if and only if every $w^*-w^*$ continuous derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ is inner.

As the name suggests, weak Connes amenability is weaker than Connes amenability.

**Theorem 3.5.** Every Connes amenable dual Banach algebra is weakly Connes amenable.

**Proof.** Let $\mathfrak{A} = (\mathfrak{A},*)$ be a Connes amenable dual Banach algebra, and let $D : \mathfrak{A} \to \mathfrak{A}^*$ be a derivation such that $j_{\mathfrak{A}} \circ D : \mathfrak{A} \to \sigma wc(\mathfrak{A})^*$ is $w^*-w^*$ continuous. By the assumption, $j_{\mathfrak{A}} \circ D$ is inner, as required. □

For dual Banach algebras, weak amenability implies weak Connes amenability as follows.

**Theorem 3.6.** Every weakly amenable dual Banach algebra is weakly Connes amenable.

**Proof.** Let $\mathfrak{A} = (\mathfrak{A},*)$ be a weakly amenable dual Banach algebra, and let $D : \mathfrak{A} \to \mathfrak{A}^*$ be a derivation such that $j_{\mathfrak{A}} \circ D : \mathfrak{A} \to \sigma wc(\mathfrak{A})^*$ is $w^*-w^*$ continuous. By the assumption, $D$ is inner. Then by Lemma 3.1(ii), $j_{\mathfrak{A}} \circ D$ is inner. □

Suppose that $\mathfrak{A}$ is a Banach algebra and that $E$ is a Banach $\mathfrak{A}$-bimodule. A derivation $D : \mathfrak{A} \to E^*$ is $w^*$-approximately inner if there exists a net $(f_\alpha) \subseteq E^*$ such that $D(a) = w^* - \lim_{\alpha} ad_{f_\alpha}(a)$ for all $a \in \mathfrak{A} [7]$. The concept of $w^*$-approximate Connes amenability were introduced in [12]. A dual Banach algebra $\mathfrak{A}$ is $w^*$-approximately Connes amenable if for every normal, dual Banach $\mathfrak{A}$-bimodule $E$, every $w^*$-$w^*$ continuous derivation $D : \mathfrak{A} \to E$ is $w^*$-approximately inner.

**Definition 3.7.** A dual Banach algebra $\mathfrak{A} = (\mathfrak{A},*)$ is $w^*$-approximately weakly Connes amenable if for every derivation $D : \mathfrak{A} \to \mathfrak{A}^*$ such that $j_{\mathfrak{A}} \circ D : \mathfrak{A} \to \sigma wc(\mathfrak{A})^*$ is $w^*-w^*$ continuous, derivation $j_{\mathfrak{A}} \circ D$ is $w^*$-approximately inner.

**Proposition 3.8.** Every $w^*$-approximately Connes amenable dual Banach algebra is $w^*$-approximately weakly Connes amenable.
Every pseudo Connes amenable dual Banach algebra is weakly Connes amenable.

**Theorem 3.11.**

Let $\mathfrak{A}$ be a dual Banach algebra. Composing the canonical inclusion $\mathfrak{A} \otimes \mathfrak{A} \rightarrow (\mathfrak{A} \otimes \mathfrak{A})^*$ with the quotient map $(\mathfrak{A} \otimes \mathfrak{A})^* \rightarrow \sigma_{wc}(\mathfrak{A} \otimes \mathfrak{A})^*$, we obtain an $\mathfrak{A}$-bimodule homomorphism $\zeta : \mathfrak{A} \otimes \mathfrak{A} \rightarrow \sigma_{wc}(\mathfrak{A} \otimes \mathfrak{A})^*$ with $w^*$-dense range.

**Definition 3.9.** ([12]) A dual Banach algebra $\mathfrak{A}$ is **pseudo Connes amenable** if there exists a net $(m_\alpha)_\alpha$ in $\mathfrak{A} \otimes \mathfrak{A}$, called an approximate $\sigma_{wc}$-diagonal for $\mathfrak{A}$, such that $a : \zeta(m_\alpha) = \zeta(m_\alpha) \cdot a \rightarrow 0$, and $a \pi_{\sigma_{wc}}(m_\alpha) \rightarrow a$ for every $a \in \mathfrak{A}$.

We need the following, which is [12, Proposition 5.6].

**Proposition 3.10.** Let $\mathfrak{A}$ be a pseudo Connes amenable dual Banach algebra, and let $E$ be a normal, dual Banach $\mathfrak{A}$-bimodule such that each $w^*$-approximate identity of $\mathfrak{A}$ is also a one-sided $w^*$-approximate identity for $E$. Then every $w^*$-approximately inner derivation $D : \mathfrak{A} \rightarrow E$ is $w^*$-approximately inner.

We now discuss relations between pseudo Connes amenability and these new notions.

**Theorem 3.11.** Every pseudo Connes amenable dual Banach algebra is $w^*$-approximately weakly Connes amenable.

**Proof.** Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a pseudo Connes amenable dual Banach algebra, and let $D : \mathfrak{A} \rightarrow \mathfrak{A}_*$ be a derivation such that $j_D \circ D : \mathfrak{A} \rightarrow \sigma_{wc}(\mathfrak{A})^*$ is $w^*$-$w^*$ continuous. We use Proposition 3.10 with $\sigma_{wc}(\mathfrak{A})^*$ in place of $E$. Thus $j_D \circ D$ is $w^*$-approximately inner, as required.

**Corollary 3.12.** Every commutative, pseudo Connes amenable dual Banach algebra is weakly Connes amenable.

**Proof.** Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a pseudo Connes amenable dual Banach algebra, and let $D : \mathfrak{A} \rightarrow \mathfrak{A}_*$ be a derivation such that $j_D \circ D : \mathfrak{A} \rightarrow \sigma_{wc}(\mathfrak{A})^*$ is $w^*$-$w^*$ continuous. By Theorem 3.11 there is a net $(\phi_\alpha)_\alpha$ in $\sigma_{wc}(\mathfrak{A})^*$ that $j_D \circ D(a) = w^* - \lim_\alpha ad_{\phi_\alpha}(a)$ for $a \in \mathfrak{A}$. As $\mathfrak{A}$ is commutative, $ad_\phi = 0$ for each $\phi \in \sigma_{wc}(\mathfrak{A})^*$. Hence $j_D \circ D = 0$, and then $\mathfrak{A}$ is weakly Connes amenable.

**Proposition 3.13.** Every commutative, $w^*$-approximately Connes amenable dual Banach algebra is weakly Connes amenable.

**Proof.** The proof is similar to that of Corollary 3.12.

Suppose that $\mathfrak{A} = (\mathfrak{A}_*)^*$ and $\mathfrak{B} = (\mathfrak{B}_*)^*$ are dual Banach algebras. We consider the $\ell^1$-direct sum $\mathfrak{A} \oplus \mathfrak{B}$ with norm $\|(a, b)\| = \|a\| + \|b\|$ for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. This is a dual Banach algebra under pointwise-defined operations and with predual the $\ell^\infty$-direct sum $\mathfrak{A}_* \oplus \mathfrak{B}_*$, where the norm $|||\cdot|||$ is defined through $|||\cdot|||_\infty = \max(||\phi||, ||\psi||)$ for $\phi \in \mathfrak{A}_*$ and $\psi \in \mathfrak{B}_*$. The duality is given by $\langle (a, b), (\phi, \psi) \rangle = \langle a, \phi \rangle + \langle b, \psi \rangle \quad (a \in \mathfrak{A}, \ b \in \mathfrak{B}, \ \phi \in \mathfrak{A}_*, \ \psi \in \mathfrak{B}_*)$.

We write $i_\mathfrak{A} : \mathfrak{A} \rightarrow \mathfrak{A} \oplus \mathfrak{B}$ for the natural injective homomorphism. It is obvious that $i_\mathfrak{A}(\phi, \psi) = \phi$ for $(\phi, \psi) \in \mathfrak{A}_* \oplus \mathfrak{B}_* = (\mathfrak{A} \oplus \mathfrak{B})^*$. It follows from [12, Lemma 2.6] that $\sigma_{wc}(\mathfrak{A} \oplus \mathfrak{B}) = \sigma_{wc}(\mathfrak{A}) \oplus \sigma_{wc}(\mathfrak{B})$. We also denote by $\nu_\mathfrak{A} : \sigma_{wc}(\mathfrak{A} \oplus \mathfrak{B})^* = \sigma_{wc}(\mathfrak{A})^* \oplus \sigma_{wc}(\mathfrak{B})^* \rightarrow \sigma_{wc}(\mathfrak{A})^*$ the adjoint of the natural embedding $\sigma_{wc}(\mathfrak{A}) \hookrightarrow \sigma_{wc}(\mathfrak{A} \oplus \mathfrak{B}) = \sigma_{wc}(\mathfrak{A} \oplus \mathfrak{B})$, so that $\nu_\mathfrak{A}(\phi, \psi) = \phi$ for $(\phi, \psi) \in \sigma_{wc}(\mathfrak{A})^* \oplus \sigma_{wc}(\mathfrak{B})^*$. Similarly we define $i_\mathfrak{B}$ and $\nu_\mathfrak{B}$. 

\[ proof: \] 

The proof is analogous to that of Theorem 3.5. □
Lemma 3.14. Let $\mathfrak{A}$ and $\mathfrak{B}$ be dual Banach algebras, and let $D: \mathfrak{A} \oplus^1 \mathfrak{B} \to (\mathfrak{A} \oplus^1 \mathfrak{B})^* = \mathfrak{A}^* \otimes^\Sigma^\infty \mathfrak{B}^*$ be a derivation. Then:

(i) $j_\mathfrak{A} \circ (i_\mathfrak{A}^* \circ D) = \nu_\mathfrak{A} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)$;
(ii) $j_\mathfrak{B} \circ (i_\mathfrak{B}^* \circ D) = \nu_\mathfrak{B} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)$.

Proof. We only prove (i). For every $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, and $c \in \sigmawc(\mathfrak{A})$ we have

$$
\langle c, j_\mathfrak{A} \circ (i_\mathfrak{A}^* \circ D)(a, b) \rangle = \langle c, (i_\mathfrak{A}^* \circ D)(a, b) \rangle = \langle (c, 0), D(a, b) \rangle
$$

$$
= \langle (c, 0), (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a, b) \rangle = \langle c, \nu_\mathfrak{A} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a, b) \rangle,
$$

as required. \qed

Theorem 3.15. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are weakly Connes amenable dual Banach algebras such that $\mathfrak{A}^2$ and $\mathfrak{B}^2$ are $w^*$-dense in $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then $\mathfrak{A} \oplus^1 \mathfrak{B}$ is weakly Connes amenable.

Proof. Take a derivation $D: \mathfrak{A} \oplus^1 \mathfrak{B} \to (\mathfrak{A} \oplus^1 \mathfrak{B})^* = \mathfrak{A}^* \otimes^\Sigma^\infty \mathfrak{B}^*$ for which $j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D : \mathfrak{A} \oplus^1 \mathfrak{B} \to \sigmawc(\mathfrak{A} \oplus^1 \mathfrak{B})^* = \sigmawc(\mathfrak{A})^* \otimes^\Sigma^\infty \sigmawc(\mathfrak{B})^*$ is $w^*$-$w^*$ continuous. Then $i_\mathfrak{A}^* \circ D \circ i_\mathfrak{A} : \mathfrak{A} \to \mathfrak{A}^*$ is a derivation. An easy calculation shows that $j_\mathfrak{A} \circ (i_\mathfrak{A}^* \circ D \circ i_\mathfrak{A}) : \mathfrak{A} \to \sigmawc(\mathfrak{A})^*$ is $w^*$-$w^*$ continuous. Since $\mathfrak{A}$ is weakly Connes amenable, there exists $\varphi \in \sigmawc(\mathfrak{A})^*$ such that $j_\mathfrak{A} \circ (i_\mathfrak{A}^* \circ D)(a, 0) = ad_\varphi(a)$ and then, by Lemma 3.14(i), $\nu_\mathfrak{A} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a, 0) = ad_\varphi(a)$ for every $a \in \mathfrak{A}$. We regard $\varphi = (\varphi, 0)$ as an element of $\sigmawc(\mathfrak{A})^* \otimes^\Sigma^\infty \sigmawc(\mathfrak{B})^*$. Then it is easily checked that $\nu_\mathfrak{A} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ (D - ad_\varphi))(a, 0) = 0$. So, by replacing $D$ by $D - ad_\varphi$, we may suppose that $\nu_\mathfrak{A} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a, 0) = 0$ for every $a \in \mathfrak{A}$. For each $a_1, a_2 \in \mathfrak{A}$, $a \in \sigmawc(\mathfrak{A})$ and $b \in \sigmawc(\mathfrak{B})$ we see that

$$
\langle (a, b), (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1, a_2, 0) \rangle = \langle (a, b), j_{\mathfrak{A} \oplus^1 \mathfrak{B}}((a_1, 0) \cdot D(a_2, 0) + D(a_1, 0) \cdot (a_2, 0)) \rangle
$$

$$
= \langle (a, b), (a_1, 0) \cdot (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_2, 0) + (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1, 0) \cdot (a_2, 0) \rangle
$$

$$
= \langle (aa_1, 0), (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_2, 0) + (a_2a_1, 0), (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1, 0) \rangle
$$

$$
= \langle aa_1, \nu_\mathfrak{A} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_2, 0) \rangle + \langle a_2a_1, \nu_\mathfrak{A} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1, 0) \rangle
$$

$$
= 0 + 0 = 0, \quad (aa_1, a_2a_1 \in \sigmawc(\mathfrak{A}) by Remark 3.3
$$

and whence $(j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1, a_2, 0) = 0$ for all $a_1, a_2 \in \mathfrak{A}$. Now, it follows from $w^*$-density of $\mathfrak{A}^2$ and $w^*$-continuity of $j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D$ that $(j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a, 0) = 0$ for every $a \in \mathfrak{A}$.

A similar argument, with $\mathfrak{B}$ in place of $\mathfrak{A}$, shows that $(j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(0, b) = 0$ for every $b \in \mathfrak{B}$. It follows that $j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D = 0$, and hence $\mathfrak{A} \oplus^1 \mathfrak{B}$ is weakly Connes amenable. \qed

Proposition 3.16. Let $\mathfrak{A}$ be a commutative Banach algebra, let $\mathfrak{B}$ be a dual Banach algebra, and let $\theta: \mathfrak{A} \to \mathfrak{B}$ be a (continuous) homomorphism with $w^*$-dense range. If $\mathfrak{A}$ is weakly amenable, then $\mathfrak{B}$ is weakly Connes amenable.

Proof. Take a derivation $D: \mathfrak{B} \to \mathfrak{B}^*$ such that $j_\mathfrak{B} \circ D : \mathfrak{B} \to \sigmawc(\mathfrak{B})^*$ is $w^*$-$w^*$ continuous. Then $(j_\mathfrak{B} \circ D)(a) : \mathfrak{A} \to \sigmawc(\mathfrak{B})^*$ is a derivation. By [2, Theorem 2.8.63 (iii)], $(j_\mathfrak{B} \circ D) \circ \theta = 0$. From $w^*$-continuity of $j_\mathfrak{B} \circ D$ and $w^*$-density of the range of $\theta$, we conclude that $j_\mathfrak{B} \circ D = 0$, as required. \qed

Let $\mathfrak{A}$ be a Banach algebra. We write $WAP(\mathfrak{A}^*)$ for the space of all weakly almost periodic functionals on $\mathfrak{A}$. It is noted by V. Runde that $WAP(\mathfrak{A}^*)^*$ is a dual Banach algebra with a universal property [15, Theorem 4.10]. Later, M. Daws called $WAP(\mathfrak{A}^*)^*$ the dual Banach algebra enveloping algebra of $\mathfrak{A}$ [6, Definition 2.10]. There is a (continuous) homomorphism $\kappa : \mathfrak{A} \to -\to \mathfrak{A}$.
Let $\mathfrak{A}$ be a commutative, weakly amenable Banach algebra. Then $WAP(\mathfrak{A}^*)^*$ is weakly Connes amenable.

**Proof.** We apply Proposition 3.16 for the canonical homomorphism $\kappa : A \to WAP(\mathfrak{A}^*)^*$.

**Corollary 3.18.** Let $\mathfrak{A}$ be a commutative $C^*$-algebra. Then $WAP(\mathfrak{A}^*)^*$ is weakly Connes amenable.

**Proof.** Since $C^*$-algebras are weakly amenable [9], it is immediate by Proposition 3.16.

4. Examples

In this section we give some examples to show difference between weak Connes amenability and some older notions such as Connes amenability, weak amenability and pseudo (Connes) amenability.

**Example 4.1.** It is well known that $\ell^1 = \ell^1(\mathbb{N})$ with pointwise multiplication is a commutative, weakly amenable dual Banach algebra. It follows from Proposition 3.16 that $\ell^1$ is weakly Connes amenable. Further, $WAP((\ell^1)^*)^* = WAP(\ell^\infty)^*$ is also weakly Connes amenable, by Corollary 3.17. However, they fail to be Connes amenable because of the lack of an identity.

**Example 4.2.** Consider the semigroup $\mathbb{N}_\lor$ which is $\mathbb{N}$ with the operation $m \lor n := \max\{m, n\}$, $(m, n \in \mathbb{N})$. It is known that $\ell^1(\mathbb{N}_\lor)$ is a commutative, weakly amenable dual Banach algebra. An argument similar to Example 4.1 shows that both $\ell^1(\mathbb{N}_\lor)$ and $WAP(\ell^1(\mathbb{N}_\lor)^*)^*$ are weakly Connes amenable. Notice that none of them are Connes amenable by [5, Theorem 5.13] and [6, Page 262], respectively.

**Example 4.3.** We denote by $\mathbb{N}_\land$ the semigroup $\mathbb{N}$ with the operation $m \land n := \min\{m, n\}$, $(m, n \in \mathbb{N})$. It is known that $\ell^1(\mathbb{N}_\land)$ is a commutative, weakly amenable Banach algebra. By Corollary 3.17, $WAP(\ell^1(\mathbb{N}_\land)^*)^*$ is weakly Connes amenable. It was shown in [5, Theorem 7.6] that $WAP(\ell^1(\mathbb{N}_\land)^*)^*$ is not Connes amenable.

It is well known that the measure algebra $M(G)$ of a locally compact group $G$, is a dual Banach algebra with predual $C_0(G)$. It was shown that $M(G)$ is Connes amenable if and only if $G$ is amenable [14]. $M(G)$ is weakly amenable if and only if $G$ is discrete, see for instance [16, Theorem 4.2.13].

**Example 4.4.** Let $G$ be a locally compact group. It follows from Theorems 3.6 and 3.5 that $M(G)$ is weakly Connes amenable if $G$ is either discrete or amenable. It would be desirable to characterize those locally compact groups $G$ for which $M(G)$ is weakly Connes amenable.

**Theorem 4.5.** Let $G$ be a non-discrete abelian locally compact group, and set $\mathfrak{A} := M(G) \oplus^1 \ell^1$. Then:

(i) $\mathfrak{A}$ is not weakly amenable;
(ii) $\mathfrak{A}$ is not Connes amenable;
(iii) $\mathfrak{A}$ is weakly Connes amenable.
The result now follows from Theorem 3.15.

Remark 4.6. Suppose that \( G \) is either a discrete or an amenable locally compact group. The same argument as in Theorem 4.5 (iii) shows that \( M(G) \oplus^1 \ell^1 \) is weakly Connes amenable.

We write \( \mathfrak{A}^\sharp \) for the unitization of an algebra \( \mathfrak{A} \).

**Theorem 4.7.** Let \( G \) be a non-discrete abelian locally compact group, and set \( \mathfrak{A} := M(G) \oplus^1 \ell^1 \). Then:

(i) \( \mathfrak{A} \) is not weakly amenable;
(ii) \( \mathfrak{A} \) is not pseudo amenable;
(iii) \( \mathfrak{A} \) is not Connes amenable;
(iv) \( \mathfrak{A} \) is not pseudo Connes amenable;
(v) \( \mathfrak{A} \) is weakly Connes amenable.

**Proof.** (i) This is essentially the same as the proof of Theorem 4.5 (i).

(ii) By \[15\] Theorem 3.1 and \[3\] Theorem 4.1, \( \ell^1 \) is not pseudo amenable. As \( \ell^1 \) is the image of \( \mathfrak{A} \) under the projection \( P_{\ell^1} : \mathfrak{A} \rightarrow \ell^1 \), \( \mathfrak{A} \) cannot be pseudo amenable by \[3\] Proposition 2.2.

(iii) Since \( \ell^1 \) is not Connes amenable, a similar argument as in Theorem 4.5 (ii) holds.

(iv) It follows from \[12\] Theorem 3.2 and Theorem 5.1 that \( \ell^1 \) is not pseudo Connes amenable. Towards a contradiction, suppose that \( \mathfrak{A} \) is pseudo Connes amenable. Then \( \ell^1 \), the image of \( \mathfrak{A} \) under \( P_{\ell^1} \), must be pseudo Connes amenable by \[12\] Proposition 4.5, which is not the case.

(v) Weak amenability of \( \ell^1 \) implies that of \( \ell^1 \). Using Proposition 3.6 and Example 4.4, weak Connes amenability of \( \mathfrak{A} \) is a consequence of Theorem 3.15.

Let \( \mathcal{V} \) be a Banach space of dimension (at least) 2, and let \( f \in \mathcal{V}^* \) be a non-zero element such that \( ||f|| \leq 1 \). Then \( \mathcal{V} \) equipped with the product defined by \( ab := f(a)b \) for \( a, b \in \mathcal{V} \), is a non-commutative Banach algebra denoted by \( \mathcal{V}_f \). Moreover, if \( \mathcal{V} \) is a dual Banach space and if \( f \) is \( w^* \)-continuous, then \( \mathcal{V}_f \) is a dual Banach algebra, see for instance \[18\].

**Proposition 4.8.** Let \( \mathcal{V} \) be a dual Banach space with \( dim \mathcal{V} \geq 2 \), and let \( 0 \neq f \in \mathcal{V}^* \) be \( w^* \)-continuous such that \( ||f|| \leq 1 \). Then:

(i) \( \sigma_{wc}(\mathcal{V}_f) = ker f \);

(ii) \( \mathcal{V}_f \) is not pseudo Connes amenable;

(iii) \( \mathcal{V}_f \) is not \( w^* \)-approximately Connes amenable;

(iv) \( \mathcal{V}_f \) is weakly Connes amenable.

**Proof.** (i) It is easy to verify that every left ideal of \( \mathcal{V}_f \) is contained in \( ker f \). In particular, \( \sigma_{wc}(\mathcal{V}_f) \subseteq ker f \). For the converse, first note that the map \( \mathcal{V}_f \rightarrow \mathcal{V}_f, a \mapsto ab = f(a)b \) is \( w^* \)-wk continuous for all \( b \in \mathcal{V}_f \). Now, take an arbitrary element \( b \in ker f \) and let \( a \xrightarrow{w^*} a \) in \( \mathcal{V}_f \). Then, for all \( \varphi \in \mathcal{V}_f^* \) and all \( \alpha \) we have \( \langle \varphi, ba \rangle = f(b)\langle \varphi, a \rangle = 0 = \langle \varphi, ba \rangle \), i.e., \( ba \xrightarrow{wk} ba \). Thus \( b \in \sigma_{wc}(\mathcal{V}_f) \), as required.
Assume that $V_f$ is pseudo Connes amenable and that $(m_\alpha) \subseteq A\hat{\otimes}A$ is an approximate $\sigma_{wc}$-diagonal for $A$. Take a non-zero element $a \in \ker f$. Then $0 = f(a)\pi_{\sigma_{wc}}(m_\alpha) = a\pi_{\sigma_{wc}}(m_\alpha) \xrightarrow{w^*} a$, which implies that $(x,a) = 0$ for each $x$ in the predual of $V_f$. It forces $a$ to be zero, a contradiction.

Towards a contradiction, suppose that $V_f$ is $w^*$-approximately Connes amenable. By [12, Proposition 2.2], $V_f$ has a right $w^*$-approximate identity, i.e., there is a net $(e_\alpha) \subseteq V_f$ for which $f(a)e_\alpha = ae_\alpha \xrightarrow{w^*} a$ for every $a \in V_f$. We take a non-zero element $a \in \ker f$, and then the rest of the proof is analogous to (ii).

It was shown that $V_f$ is weakly amenable [17, Page 507], see also [3, Proposition 2.13]. Hence, $V_f$ is automatically weakly Connes amenable.

Example 4.9. It is denoted by $\mathbb{M}_n$ the collection of all $n \times n$ matrices, $n \in \mathbb{N}$, with entries from $\mathbb{C}$. It is known that $\mathbb{M}_n = \mathbb{M}_n^*$ is a dual Banach algebra, and that $w^*$-topology and $wk$-topology are the same on $\mathbb{M}_n$. It is then straightforward that $\sigma_{wc}(\mathbb{M}_n) = \mathbb{M}_n$.

References

[1] W. G. Bade, P. C. Curtis, H. G. Dales, Amenability and weak amenability for Beurling and Lipschitz algebras. Proc. London Math. Soc. (3) 55 (1987), 359-377.

[2] H. G. Dales, Banach algebras and automatic continuity, Clarendon Press, Oxford, 2000.

[3] H. G. Dales, A. T.-M. Lau, D. Strauss, Banach algebras on semigroups and on their compactifications. Mem. Amer. Math. Soc. 205 (966) (2010).

[4] H. G. Dales, R. J. Loy, Y. Zang, Approximate amenability for Banach sequence algebras. Studia Math. 177 (2006), 81-96.

[5] M. Daws, Connes-amenability of bidual and weighted semigroup algebras, Math. Scand. 99 (2006), 217-246.

[6] M. Daws, Dual Banach algebras: representations and injectivity, Studia Math. 178 (2007), 231-275.

[7] F. Ghahramani, R. J. Loy, Generalized notions of amenability. J. Func. Anal. 208 (2004), 229-260.

[8] F. Ghahramani, Y. Zhang, Pseudo-amenable and pseudo-contractible Banach algebras. Math. Proc. Camb. Phil. Soc. 142 (2007), 111-123.

[9] U. Haagerup, All nuclear $C^*$-algebras are amenable. Invent. Math. 74 (1983), 305-319.

[10] B. E. Johnson, Derivations from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$. In: J. P. Pier (ed.), Harmonic Analysis(Luxembourg, 1987), pp. 191-198. Springer Verlag, 1988.

[11] B. E. Johnson, Cohomology in Banach algebras, Mem. Amer. Math. Soc. 127 (1972).

[12] A. Mahmoodi, Connes-amenability-like properties, Studia Math. 220 (2014), 55-72.

[13] V. Runde, Amenability for dual Banach algebras, Studia Math. 148 (2001), 47-66.

[14] V. Runde, Connes-amenability and normal, virtual diagonals for measure algebras I. J. London Math. Soc. 67 (2003), 643-656.

[15] V. Runde, Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule, Math. Scand. 95 (2004), 124-144.

[16] V. Runde, Amenable Banach Algebras: A Panorama, Springer Monographs in Mathematics, 2020.

[17] Y. Zhang, Weak amenability of a class of Banach algebras, Canad. Math. Bull. 44 (2001), 504-508.

[18] M. Ziamanesh, B. Shojaee, A. Mahmoodi, Essential amenability of dual Banach algebras, Bull. Aust. Math. Soc. 100 (2019), 479-488.

Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran, e-mail: a_mahmoodi@iauctb.ac.ir