LIOUVILLE TYPE THEOREMS, A PRIORI ESTIMATES AND
EXISTENCE OF SOLUTIONS FOR NON-CRITICAL HIGHER ORDER
LANE-EMDEN-HARDY EQUATIONS

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Abstract. In this paper, we are concerned with the non-critical higher order Lane-Emden-Hardy equations
\[ (-\Delta)^m u(x) = \frac{u^p(x)}{|x|^a} \quad \text{in} \quad \mathbb{R}^n \]
with \( n \geq 3, 1 \leq m < \frac{n}{2}, 0 \leq a < 2m, 1 < p < \frac{n+2m-2a}{n-2m} \) if \( 0 \leq a < 2 \), and \( 1 < p < +\infty \) if \( 2 \leq a < 2m \). We prove Liouville theorems for nonnegative classical solutions to the above Lane-Emden-Hardy equations (Theorem 1.1), that is, the unique nonnegative solution is \( u \equiv 0 \). As an application, we derive a priori estimates and existence of positive solutions to non-critical higher order Lane-Emden equations in bounded domains (Theorem 1.6 and 1.7). The results for critical order Hardy-Hénon equations have been established by Chen, Dai and Qin [5] recently.

Keywords: Lane-Emden-Hardy equations; Liouville theorems; Nonnegative solutions; Super poly-harmonic properties; Method of moving planes in local way; Blowing-up analysis.

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1. Introduction

In this paper, we first investigate the Liouville property of nonnegative solutions to the following non-critical higher order Lane-Emden-Hardy equations
\[
\begin{aligned}
(-\Delta)^m u(x) &= \frac{u^p(x)}{|x|^a} \quad \text{in} \quad \mathbb{R}^n, \\
u(x) &\geq 0, \quad x \in \mathbb{R}^n,
\end{aligned}
\]
where \( u \in C^{2m}(\mathbb{R}^n) \) if \(-\infty < a \leq 0\), \( u \in C^{2m}(\mathbb{R}^n \setminus \{0\}) \) \( \cap \) \( C^{2m-2}(\mathbb{R}^n) \) if \( 0 < a < 2m, n \geq 3, 1 < p < \frac{n+2m-2a}{n-2m} \) if \( 0 \leq a < 2 \), and \( 1 < p < +\infty \) if \( 2 \leq a < 2m \).

For \( 0 < \alpha \leq n \), PDEs of the form
\[ (-\Delta)^\alpha u(x) = \frac{u^p(x)}{|x|^a} \]
are called the fractional order or higher order Hardy (Lane-Emden, Hénon) equations for \( \alpha > 0 \) (\( \alpha = 0, a < 0 \), respectively), which have many important applications in conformal geometry and Sobolev inequalities. We say equations (1.2) have critical order if \( \alpha = n \) and non-critical order if \( 0 < \alpha < n \). Liouville type theorems for equations (1.2) (i.e., nonexistence of nontrivial nonnegative solutions) have been quite extensively studied (see [2, 4, 5, 8, 10, 13, 16, 20, 30, 32, 35, 37, 38, 40, 45] and the references therein). It is crucial in establishing
a priori estimates and existence of positive solutions for non-variational boundary value problems of a class of elliptic equations (see [3, 5, 11, 12, 31, 39]).

In the special case \( a = 0 \), equation (1.2) becomes the well-known Lane-Emden equation, which also arises as a model in astrophysics. For \( \alpha = 2 \) and \( 1 < p < p_* := \frac{n+2}{n-2} (:= \infty \) if \( n = 2 \), Liouville type theorem was established by Gidas and Spruck in their celebrated article [30]. Later, the proof was simplified to a large extent by Chen and Li in [10] using the Kelvin transform and the method of moving planes (see also [13]). For \( n > \alpha = 4 \) and \( 1 < p < \frac{n+2}{n+4} \), Lin [35] proved the Liouville type theorem for all the nonnegative \( C^4(\mathbb{R}^n) \) smooth solutions of (1.2). When \( \alpha \in (0, n) \) is an even integer and \( 1 < p < \frac{n+\alpha}{n-\alpha} \), Wei and Xu established Liouville type theorem for all the nonnegative \( C^\alpha(\mathbb{R}^n) \) smooth solutions of (1.2) in [45]. For general \( a \in \mathbb{R} \), \( 0 < \alpha \leq n \), \( 0 < p < \min\{\frac{n+\alpha-2n}{n-\alpha}, \frac{n+\alpha-\alpha}{n-\alpha}\} \) \( (1 < p < +\infty \) if \( \alpha = n \), there are also lots of literatures on Liouville type theorems for general fractional order or higher order Hardy-Hénon equations (1.2), for instance, Bidaut-Véron and Giacomini [2], Chen, Dai and Qin [5], Chen and Fang [6], Cheng and Liu [16], Dai and Qin [25], Gidas and Spruck [30], Lei [32], Mitidieri and Pohozaev [37], Phan [38], Phan and Souplet [40] and many others. For Liouville type theorems on systems of PDEs of type (1.2) with respect to various types of solutions (e.g., stable, radial, nonnegative, sign-changing, \( \cdots \)) please refer to [2, 26, 27, 30, 38, 39, 41, 43] and the references therein.

For the critical nonlinearity cases \( p = \frac{n+\alpha}{n-\alpha} \) with \( a = 0 \) and \( 0 < \alpha < n \), the quantitative and qualitative properties of solutions to fractional order or higher order conformally invariant equations (1.2) have also been widely studied. In the special case \( n > \alpha = 2 \), equation (1.2) becomes the well-known Yamabe problem (for related results, please see Gidas, Ni and Nirenberg [28, 29], Caffarelli, Gidas and Spruck [9] and the references therein). For \( n > \alpha = 4 \), Lin [35] classified all the positive \( C^4 \) smooth solutions of (1.2). In [45], among other things, Wei and Xu proved the classification results for all the positive \( C^\alpha \) smooth solutions of (1.2) when \( \alpha \in (0, n) \) is an even integer. For \( n > \alpha = 3 \), Dai and Qin [25] classified the positive \( C^3 \cap L_1 \) classical solutions of (1.2). In [19], by developing the method of moving planes in integral forms, Chen, Li and Ou classified all the positive \( L_{n-\alpha}^{2n} \) solutions to the equivalent integral equation of the PDE (1.2) for general \( \alpha \in (0, n) \), as a consequence, they obtained the classification results for positive weak solutions to PDE (1.2). Subsequently, Chen, Li and Li [17] developed a direct method of moving planes for fractional Laplacians \((-\Delta)^{\frac{\alpha}{2}}\) with \( 0 < \alpha < 2 \) and classified all the \( C^{1,1} \cap L_1 \) positive solutions to the PDE (1.2) directly as an application, where the function space

\[
\mathcal{L}_\alpha(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R} \left| \int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+\alpha}} \, dx < \infty \right. \right\}.
\]

In the limiting (i.e., critical order) cases \( \alpha = n \), there are also a large amount of literatures on classification results for positive solutions to the following critical order conformally invariant equations with exponential nonlinearities

\[
(-\Delta)^{\frac{n}{2}} u = (n - 1)! e^{nu},
\]

for instance, Chen and Li [13], Chang and Yang [21], Chen and Zhang [22], Lin [35], Wei and Xu [45] and Zhu [46]. For more literatures on the quantitative and qualitative properties of solutions to fractional order or higher order conformally invariant PDE and IE problems, please refer to [14, 13, 22, 23, 24, 46] and the references therein.
In this paper, we will establish Liouville type theorem for nonnegative classical solutions of (1.1) in the cases $1 < p < \frac{n+2m-2a}{n-2m}$ if $0 \leq a < 2$ and $1 < p < +\infty$ if $2 \leq a < 2m$. Lei [32] has proved the nonexistence of positive solutions to (1.1) for $0 \leq a < 2$ and $1 < p < \frac{n-a}{n-2m}$. One should note that, our results extend the range $p \in (1, \frac{n-a}{n-2m})$ and $0 \leq a < 2$ in [32] to the full range $1 < p < \frac{n+2m-2a}{n-2m}$ if $0 \leq a < 2$ and $1 < p < +\infty$ if $2 \leq a < 2m$.

Our Liouville type result for (1.1) is the following theorem.

**Theorem 1.1.** Assume $n \geq 3$, $1 \leq m < \frac{n}{2}$, $0 \leq a < 2m$, $1 < p < \frac{n+2m-2a}{n-2m}$ if $0 \leq a < 2$, $1 < p < +\infty$ if $2 \leq a < 2m$, and $u$ is a nonnegative solution of (1.1). If one of the following two assumptions

$$0 \leq a \leq 2 + 2p \quad \text{or} \quad u(x) = o(|x|^2) \quad \text{as } |x| \to +\infty$$

holds, then $u \equiv 0$ in $\mathbb{R}^n$.

**Remark 1.2.** In [16], Cheng and Liu proved Liouville type theorem for (1.1) in the cases $a < 0$ and $1 < p < \frac{n+2m-a}{n-2m}$ (there is actually an extra assumption $p > \frac{n}{n-2m}$ in [16], but it is clear from their proof that the assumption $p > \frac{n}{n-2m}$ is redundant and unnecessary). Among other things, Lei [32] established the nonexistence of positive solutions to (1.1) for $0 \leq a < 2$ and $1 < p < \frac{n-a}{n-2m}$. However, we found a few technical mistakes in their proof, more precisely, in their proof of super poly-harmonic properties (see Theorem 2 in [16] and Theorem 2.1 in [32]). For instance, the possibility that constant $C_0$ have to be ruled out in the proof of Theorem 2 in [16], and a factor $R^{-a}$ should be added to the last inequality in the proof of Theorem 2.1 in [32] since $R$ is sufficiently large (thus the assumption $a < 2$ is needed therein). In this paper, we will prove the super poly-harmonic properties in Theorem 2.1 via a unified approach for both $a < 0$ and $a \geq 0$, as a consequence, we repair the proof in [16] and extend the results in [32].

**Remark 1.3.** For $0 < a < 2m$, if we consider the nonnegative solutions $u \in C^{2m}(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$, then it is clear from our proof of Theorem 1.1 that Liouville theorem as Theorem 1.1 also holds for $1 < p < \frac{n+2m-2a}{n-2m}$ (see Section 2). The main difference is, instead of Theorem 2.1, we will show super poly-harmonic properties except the origin $0 \in \mathbb{R}^n$, that is, $(-\Delta)^ju \geq 0$ in $\mathbb{R}^n \setminus \{0\}$ for $i = 1, \cdots, m - 1$ (see remark 2.2).

**Remark 1.4.** In Theorem 1.1 and Remark 1.3, the smoothness assumption on $u$ at $x = 0$ is necessary. Equation (1.1) admits a distributional solution of the form $u(x) = C|x|^{-\sigma}$ with $\sigma = \frac{2m-a}{p-1} > 0$.

We also consider the following higher order Navier problem

\begin{equation}
(-\Delta)^mu(x) = u^p(x) + t \quad \text{in } \Omega,
\end{equation}

\begin{equation}
u(x) = -\Delta u(x) = \cdots = (-\Delta)^{m-1}u(x) = 0 \quad \text{on } \partial \Omega,
\end{equation}

where $n \geq 3$, $1 \leq m < \frac{n}{2}$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^{2m-2}$ boundary $\partial \Omega$ and $t \geq 0$.

Theorem 6 in Chen, Fang and Li [7] implies immediately the following a priori estimates for any positive solution $u$ to (1.5).

**Theorem 1.5.** (7) Assume $\frac{n}{n-2m} < p < \frac{n+2m}{n-2m}$. Then, for any positive solution $u \in C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega})$ to the higher order Navier problem (1.5), we have

$$\|u\|_{L^\infty(\overline{\Omega})} \leq C(n, m, p, \Omega).$$
As an application of the Liouville theorems (Theorem 1.1), we can prove the following a priori estimates for any positive solution \( u \) to (1.5) via the method of moving planes in local way and blowing-up methods (for related literatures on these methods, please see \([1, 3, 5, 11, 12, 20, 33, 44]\)). Our a priori estimates extend the range of \( p \) in Theorem 1.5 remarkably.

**Theorem 1.6.** Assume \( 1 < p < \frac{n+2m}{n-2m} \) If one of the following two assumptions

\[
i) \ \text{\Omega is strictly convex}, \quad 1 < p < \frac{n+2m}{n-2m}, \quad \text{or} \quad ii) \ 1 < p \leq \frac{n+2}{n-2}
\]

holds. Then, for any positive solution \( u \in C^{2m}(\Omega) \cap C^{2m-2}(\Omega) \) to the higher order Navier problem (1.5), we have

\[
\|u\|_{L^\infty(\Omega)} \leq C(n, m, p, t, \lambda_1, \Omega),
\]

where \( \lambda_1 \) is the first eigenvalue for \((-\Delta)^m \) in \( \Omega \) with Navier boundary conditions.

As a consequence of the a priori estimates (Theorem 1.5 and Theorem 1.6), by applying the Leray-Schauder fixed point theorem, we can derive the following existence result for positive solution to the following Navier problem for higher order Lane-Emden equations (1.6)

\[
\begin{cases}
(-\Delta)^m u(x) = u^p(x) & \text{in } \Omega, \\
u(x) = -\Delta u(x) = \cdots = (-\Delta)^{m-1} u(x) = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \( n \geq 3, \ 1 \leq m < \frac{n}{2} \) and \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^{2m-2} \) boundary \( \partial\Omega \).

**Theorem 1.7.** Assume \( 1 < p < \frac{n+2m}{n-2m} \) If one of the following two assumptions

\[
i) \ \text{\Omega is strictly convex}, \quad 1 < p < \frac{n+2m}{n-2m}, \quad \text{or} \quad ii) \ p \in \left(1, \frac{n+2}{n-2}\right) \cup \left(\frac{n}{n-2m}, \frac{n+2m}{n-2m}\right)
\]

holds. Then, the higher order Navier problem (1.6) possesses at least one positive solution \( u \in C^{2m}(\Omega) \cap C^{2m-2}(\Omega) \). Moreover, the positive solution \( u \) satisfies

\[
\|u\|_{L^\infty(\Omega)} \geq \left(\frac{\sqrt{2n}}{diam \Omega}\right)^{\frac{2m}{p-1}}.
\]

It’s well known that the super poly-harmonic properties of solutions are crucial in establishing Liouville type theorems and the representation formulae for higher order or fractional order PDEs (see e.g. \([5, 6, 7, 14, 45]\)). In Section 2, we will first prove the super poly-harmonic properties of solutions for both \( a < 0 \) and \( a \geq 0 \) via a unified approach (see Theorem 2.1). As a consequence, we can show the equivalence between the PDE (1.1) and the corresponding integral equation (2.60). Then, by applying the method of moving planes in integral forms and Pohozaev identity, we prove the Liouville theorem (Theorem 1.1) for (1.1). In Sections 3 and 4, we will prove a priori estimates and existence of positive solutions to non-critical higher order Lane-Emden equations in bounded domains \( \Omega \), using the arguments from Chen, Dai and Qin \([5]\) for critical order Hardy-Hénon equations and results from Chen, Fang and Li \([7]\). In Section 3, we will derive a priori estimates for any positive solutions to the higher order Naiver problem (1.5) (Theorem 1.6) by applying the method of moving planes in local way and Kelvin transforms. We will first establish the boundary layer estimates (Theorem 3.1), in which the properties of the boundary \( \partial\Omega \) play a crucial role. The global a priori
estimates follows from the boundary layer estimates, blowing-up analysis and the Liouville theorem (Theorem 1.1). Section 4 is devoted to the proof of Theorem 1.7. The existence of positive solutions to the higher order Lane-Emden equations (1.6) with Navier boundary conditions will be established via the a priori estimates (Theorem 1.5 and Theorem 1.6) and the Leray-Schauder fixed point theorem (Theorem 1.1).

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by using contradiction arguments. Now suppose on the contrary that $u \geq 0$ satisfies equation (1.1) but $u$ is not identically zero, then there exists some $\bar{x} \in \mathbb{R}^n$ such that $u(\bar{x}) > 0$.

In the following, we will use $C$ to denote a general positive constant that may depend on $n, m, a, p$ and $u$, and whose value may differ from line to line.

2.1. Super poly-harmonic properties. The super poly-harmonic properties of solutions are closely related to the representation formulae and Liouville type theorems (see [5, 6, 7, 14, 45] and the references therein). Therefore, in order to prove Theorem 1.1, we need the following theorem about the super poly-harmonicity.

Theorem 2.1. (Super poly-harmonic properties). Assume $n \geq 3$, $1 \leq m < \frac{n}{2}$, $-\infty < a < 2m$, $1 < p < +\infty$ and $u$ is a nonnegative solution of (1.1). If one of the following two assumptions

$$-\infty < a \leq 2 + 2p \quad \text{or} \quad u(x) = o(|x|^2) \quad \text{as} \quad |x| \to +\infty$$

holds, then

$$(-\Delta)^i u(x) \geq 0$$

for every $i = 1, 2, \cdots, m - 1$ and all $x \in \mathbb{R}^n$.

Proof. Let $u_i := (-\Delta)^i u$. We want to show that $u_i \geq 0$ for $i = 1, 2, \cdots, m - 1$. Our proof will be divided into two steps.

Step 1. We first show that

$$u_{m-1} = (-\Delta)^{m-1} u \geq 0.$$ 

If not, then there exists $0 \neq x^1 \in \mathbb{R}^n$, such that

$$u_{m-1}(x^1) < 0.$$ 

Now, let

$$\bar{f}(r) = \bar{f}(|x - x^1|) := \frac{1}{|\partial B_r(x^1)|} \int_{\partial B_r(x^1)} f(x) d\sigma$$

be the spherical average of $f$ with respect to the center $x^1$. Then, by the well-known property $\Delta \bar{u} = \Delta \bar{u}$ and $-\infty < a < 2m < n$, we have, for any $r \geq 0$ and $r \neq |x^1|$, 

$$\begin{cases} 
-\Delta \bar{u}_{m-1}(r) = \frac{u_p(x)}{|x|^a}(r), \\
-\Delta \bar{u}_{m-2}(r) = \bar{u}_{m-1}(r), \\
\vdots \\
-\Delta \bar{u}(r) = \bar{u}(r).
\end{cases}$$

$$\begin{cases} 
-\Delta \bar{u}_{m-1}(r) = \frac{u_p(x)}{|x|^a}(r), \\
-\Delta \bar{u}_{m-2}(r) = \bar{u}_{m-1}(r), \\
\vdots \\
-\Delta \bar{u}(r) = \bar{u}(r).
\end{cases}$$
From the first equation in (2.4), by Jensen’s inequality, we get, for any $r \geq 0$ and $r \neq |x^1|,$
\[
-\Delta u_{m-1}(r) = \frac{1}{|\partial B_r(x^1)|} \int_{\partial B_r(x^1)} \frac{u^p(x)}{|x|^a} d\sigma
\]
\[
\geq (r + |x^1|)^{-a} \frac{1}{|\partial B_r(x^1)|} \int_{\partial B_r(x^1)} u^p(x) d\sigma
\]
\[
\geq (r + |x^1|)^{-a} \left( \frac{1}{|\partial B_r(x^1)|} \int_{\partial B_r(x^1)} u(x) d\sigma \right)^p
\]
\[
= (r + |x^1|)^{-a} \bar{u}^p(r) \geq 0 \quad \text{if} \quad 0 \leq a < 2m,
\]
and
\[
-\Delta u_{m-1}(r) \geq |r - |x^1||^{-a} \bar{u}^p(r) \geq 0 \quad \text{if} \quad -\infty < a < 0.
\]
From (2.5) and (2.6), one has
\[
-\frac{1}{r^{n-1}} \left( r^{n-1} u_{m-1}'(r) \right) \geq 0.
\]
Since $-\infty < a < 2m < n,$ we can integrate both sides of (2.7) from 0 to $r$ and derive
\[
u_{m-1}'(r) \leq 0, \quad \frac{u_{m-1}(r)}{u_{m-1}(0)} = u_{m-1}(x^1) =: -c_0 < 0
\]
for any $r \geq 0.$ From the second equation in (2.4), we deduce that
\[
u_{m-2}'(r) = \frac{u_{m-2}(r)}{u_{m-2}(0)} \leq -c_0, \quad \forall r \geq 0,
\]
integrating from 0 to $r$ yields
\[
u_{m-2}(r) \geq \frac{c_0}{n} r, \quad \frac{u_{m-2}(r)}{u_{m-2}(0)} \geq \frac{c_0}{2n} r^2, \quad \forall r \geq 0.
\]
Hence, there exists $r_1 > 0$ such that
\[
u_{m-2}(r_1) > 0.
\]
Next, take a point $x^2$ with $|x^2 - x^1| = r_1$ as the new center, and make average of $\bar{f}$ at the new center $x^2,$ i.e.,
\[
\bar{f}(r) = \frac{1}{|\partial B_r(x^2)|} \int_{\partial B_r(x^2)} \bar{f}(x) d\sigma.
\]
One can easily verify that
\[
u_{m-2}(0) = \bar{u}_{m-2}(x^2) =: c_1 > 0.
\]
Then, from (2.5) and Jensen’s inequality, we deduce that $(u, \bar{u}, \cdots, \bar{u}_{m-1})$ satisfies
\[
\left\{ \begin{array}{l}
-\Delta \bar{u}_{m-1}(r) = \frac{u_{m-1}(r)}{|x^1|} \geq 0, \\
-\Delta \bar{u}_{m-2}(r) = \frac{u_{m-2}(r)}{u_{m-1}(r)}, \\
\quad \cdots \cdots \\
-\Delta \bar{u}(r) = \bar{u}(r)
\end{array} \right.
\]
for any $r \geq 0.$ Using the same method as obtaining the estimate (2.10), we conclude that
\[
u_{m-2}(r) \geq \bar{u}_{m-2}(0) + \frac{c_0}{2n} r^2, \quad \forall r \geq 0.
\]
Thus we infer from (2.18), (2.13), (2.14) and (2.16) that

\[(2.16) \quad \overline{u}_{m-1}(r) \leq \overline{u}_{m-1}(0) < 0, \quad \overline{u}_{m-2}(r) \geq \overline{u}_{m-2}(0) > 0, \quad \forall r \geq 0.\]

From the third equation in (2.14) and integrating, we infer that

\[(2.17) \quad \overline{u}_{m-3}(r) \leq -\frac{c_1}{n} r \quad \text{and} \quad \overline{u}_{m-3}(r) \leq \overline{u}_{m-3}(0) - \frac{c_1}{2n} r^2, \quad \forall r \geq 0.\]

Hence, there exists \(r_2 > 0\) such that

\[(2.18) \quad \overline{u}_{m-3}(r_2) < 0.\]

Next, we take a point \(x^3\) with \(|x^3 - x^2| = r_2\) as the new center and make average of \(\bar{f}\) at the new center \(x^3\), i.e.,

\[(2.19) \quad \overline{f}(r) = \overline{f}(|x - x^3|) := \frac{1}{|\partial B_r(x^3)|} \int_{\partial B_r(x^3)} f(x) d\sigma.\]

It follows that

\[(2.20) \quad \overline{u}_{m-3}(0) = \overline{u}_{m-3}(x^3) =: -c_2 < 0.\]

One can easily verify that \(\overline{u}\) and \(\overline{u}_i\) (\(i = 1, \ldots, m - 1\)) satisfy entirely similar equations as \((\overline{u}, \overline{u}_1, \ldots, \overline{u}_{m-1})\) (see (2.14)). Using the same method as deriving (2.16), we arrive at

\[(2.21) \quad \overline{u}_{m-1}(r) \leq \overline{u}_{m-1}(0) < 0, \quad \overline{u}_{m-2}(r) \geq \overline{u}_{m-2}(0) > 0, \quad \overline{u}_{m-3}(r) \leq \overline{u}_{m-3}(0) < 0\]

for any \(r \geq 0\). Continuing this way, after \(m\) steps of re-centers (denotes the centers by \(x^1, x^2, \ldots, x^m\), the \(m\) times averages of \(f\) by \(\bar{f}\) and the resulting functions coming from taking \(m\) times averages by \(\bar{u}\) and \(\bar{u}_i\) for \(i = 1, 2, \ldots, m - 1\)), we finally obtain that

\[(2.22) \quad -\Delta \overline{u}_{m-1}(r) \geq \frac{\mu^2(x)}{|x|^a}(r) \geq 0,\]

and for every \(i = 1, \ldots, m - 1\),

\[(2.23) \quad (-1)^i \overline{u}_{m-i}(r) \geq (-1)^i \overline{u}_{m-i}(0) > 0, \quad (-1)^m \overline{u}(r) \geq (-1)^m \overline{u}(0) > 0, \quad \forall r \geq 0.\]

Moreover, in the above process, we may choose \(|x^m|\) sufficiently large, such that

\[(2.24) \quad |x^m - x^{m-1}| \geq |x^m - x^{m-2}| + \cdots + |x^2 - x^1| + |x^1| + 2.\]

Now, if \(m\) is odd, estimate (2.23) implies immediately that

\[(2.25) \quad \overline{u}(r) \leq \overline{u}(0) < 0,\]

which contradicts the fact that \(u \geq 0\). Therefore, we only need to deal with the cases that \(m\) is an even integer hereafter.

Since \(m\) is even, we have \(\overline{u}(r) \geq \overline{u}(0) > 0\) for any \(r \geq 0\), furthermore, one can actually observe from the above “re-centers and iteration” process that

\[(2.26) \quad \overline{u}(0) \geq \frac{c}{2n} |x^m - x^{m-1}|^2\]

for some constant \(c > 0\). Thus we may choose \(|x^m|\) larger, such that both (2.24) and the following

\[(2.27) \quad \overline{u}(0) \geq (2p)^{\frac{2m}{(p-1)^2}} \left(1 + \frac{2n}{p}\right)^{\frac{2m}{p}}\]
hold.

For arbitrary \( \lambda > 0 \), define the re-scaling of \( u \) by

\[
(2.28) \quad u_\lambda(x) := \lambda^{\frac{2m}{p-1}} u(\lambda x).
\]

Then one can easily verify that equation (1.1) is invariant under this re-scaling. After \( m \) steps of re-centers for \( u_\lambda \), we denote the centers for \( u_\lambda \) by \( x_\lambda^1, x_\lambda^2, \ldots, x_\lambda^m \) and the resulting function coming from taking \( m \) times averages by \( \tilde{u}_\lambda \) and \( \tilde{u}_{\lambda,i} \) for \( i = 1, 2, \ldots, m - 1 \). Then (2.22) and (2.23) still hold for \((\tilde{u}_\lambda, \tilde{u}_{\lambda,1}, \ldots, \tilde{u}_{\lambda,m-1})\) and \( x_\lambda^k = \frac{1}{\lambda} x_k \) for \( k = 1, \ldots, m \), thus one has the following estimate

\[
(2.29) \quad |x_\lambda^m - x_\lambda^{m-1}| + \cdots + |x_\lambda^2 - x_\lambda^1| + |x_\lambda^1| \leq |x^m - x^{m-1}| + \cdots + |x^2 - x^1| + |x^1| =: M
\]

holds uniformly for every \( \lambda \geq 1 \).

Since we have (2.23) and \( m \) is even, it follows that

\[
(2.30) \quad \tilde{u}(r) \geq \tilde{u}(0) \geq (2p)^{\frac{2m}{p-1}} \left( 1 + \frac{2n}{p} \right)^{\frac{2m}{p-1}} > 0, \quad \forall \ r \geq 0,
\]

and hence

\[
(2.31) \quad \tilde{u}_\lambda(r) \geq \tilde{u}_\lambda(0) = \lambda^{\frac{2m}{p-1}} \tilde{u}(0) \geq \lambda^{\frac{2m}{p-1}} (2p)^{\frac{2m}{p-1}} \left( 1 + \frac{2n}{p} \right)^{\frac{2m}{p-1}} > 0, \quad \forall \ r \geq 0.
\]

For \( 0 \leq a < 2m \), by the estimate (2.31), we may assume that, we already have

\[
(2.32) \quad \tilde{u}(0) \geq (1 + M)^{\frac{a}{p-1}} (2p)^{\frac{2m}{p-1}} \left( 1 + \frac{2n}{p} \right)^{\frac{2m}{p-1}},
\]

or else we may replace \( u \) by \( u_\lambda \) with \( \lambda = (1 + M)^{\frac{a}{2m-a}} \) (still denoted by \( u \)).

For any \( 0 \leq r \leq 1 \), we have

\[
(2.33) \quad \tilde{u}(r) \geq \tilde{u}(0) \geq l_0 r^{\alpha_0},
\]

where

\[
(2.34) \quad l_0 := \tilde{u}(0) = \max \left\{ (1 + M)^{\frac{a}{p-1}}, 1 \right\} (2p)^{\frac{2m}{p-1}} \alpha_0^{\frac{2m}{p}}, \quad \alpha_0 := \max \left\{ 1, \frac{2n}{p} \right\} \geq 1.
\]

As a consequence, we infer from (2.22), (2.24), (2.29) and (2.33) that, for any \( 0 \leq r \leq 1 \),

\[
-\Delta \tilde{u}_{m-1}(r) \geq \left( r + |x^m - x^{m-1}| + \cdots + |x^2 - x^1| + |x^1| \right)^{-a} \tilde{u}^p(r)
\]

\[
(2.35) \quad \geq (1 + M)^{-a} l_0^p r^{\alpha_0 p}
\]

and

\[
-\Delta \tilde{u}_{m-1}(r) \geq \left( |x^m - x^{m-1}| - |x^{m-1} - x^{m-2}| - \cdots - |x^2 - x^1| - |x^1| - r \right)^{-a} \tilde{u}^p(r)
\]

\[
(2.36) \quad \geq l_0^p r^{\alpha_0 p}
\]

\[
\geq C_0 l_0^p r^{\alpha_0 p}, \quad \text{if} \quad -\infty < a < 0,
\]

where

\[
(2.37) \quad C_0 := \min \left\{ (1 + M)^{-a}, 1 \right\} \in (0, 1].
\]
Integrating both sides of (2.35) and (2.36) from 0 to \( r \) twice and taking into account of (2.23) yield

\[
\tilde{u}_{m-1}(r) < - \frac{C_0 l_0^{p}}{(\alpha_0 p + n)(\alpha_0 p + 2)} r^{\alpha_0 p + 2}, \quad \forall \ 0 \leq r \leq 1.
\]

This implies

\[
\frac{1}{r^{n-1}} (r^{n-1} \tilde{u}_{m-2}'(r))' < - \frac{C_0 l_0^{p}}{(\alpha_0 p + n)(\alpha_0 p + 2)} r^{\alpha_0 p + 2},
\]

and consequently,

\[
\tilde{u}_{m-2}(r) > \frac{C_0 l_0^{p}}{(\alpha_0 p + n)(\alpha_0 p + 2)(\alpha_0 p + n + 2)(\alpha_0 p + 4)} r^{\alpha_0 p + 4}, \quad \forall \ 0 \leq r \leq 1.
\]

Continuing this way, since \( m \) is an even integer, by iteration, we can finally arrive at

\[
\tilde{u}(r) > \frac{C_0 l_0^{p}}{(2\alpha_0 p)^{2m}} r^{\alpha_0 p + 2m}, \quad \forall \ 0 \leq r \leq 1.
\]

Now, define

\[
\alpha_{k+1} := 2\alpha_k p \geq \alpha_k p + 2n \quad \text{and} \quad l_{k+1} := \frac{C_0 l_0^{p}}{(2\alpha_k p)^{2m}}
\]

for \( k = 0, 1, \ldots \). Then (2.41) implies

\[
\tilde{u}(r) > \frac{C_0 l_0^{p}}{(2\alpha_0 p)^{2m}} r^\alpha l_1^\alpha_1 = l_1^\alpha_1 r^\alpha_1, \quad \forall \ r \in [0, 1].
\]

Suppose we have \( \tilde{u}(r) \geq l_k r^{\alpha_k} \), then go through the entire process as above, we can derive \( \tilde{u}(r) \geq l_{k+1} r^{\alpha_{k+1}} \) for any \( 0 \leq r \leq 1 \). Therefore, one can prove by induction that

\[
\tilde{u}(r) \geq l_k r^{\alpha_k}, \quad \forall \ r \in [0, 1], \quad \forall \ k \in \mathbb{N}.
\]

Through direct calculations, we have

\[
l_k = \frac{C_0^{k-1} l_0^{p}}{(2p)^{2m(k+(k-1)p+(k-2)p^2+\cdots+p^{k-1})\alpha_0}} \frac{2^{2m(\alpha_0 p + 1)}}{p-1} \frac{1}{\alpha_0}
\]

\[
= \frac{C_0^{k-1} l_0^{p}}{(2p)^{2m(k+(k-1)p+(k-2)p^2+\cdots+p^{k-1})\alpha_0}} \frac{2^{2m(\alpha_0 p + 1)}}{p-1} \frac{1}{\alpha_0}
\]

\[
\geq (2p)^{\frac{2m}{p-1}} \left( \frac{C_0 l_0}{(2p)^{2m-1} \alpha_0} \right)^{p-1}
\]

for \( k = 0, 1, 2, \ldots \). From (2.34), (2.37), (2.44) and (2.45), we deduce that

\[
\tilde{u}(1) \geq (2p)^{\frac{2m}{p-1}} \rightarrow +\infty, \quad \text{as} \quad k \rightarrow \infty.
\]

This is absurd. Therefore, (2.1) must hold, that is, \( u_{m-1} = (-\Delta)^{m-1} u \geq 0 \).

Step 2. Next, we will show that all the other \( u_i (i = 1, \cdots, m-2) \) must be nonnegative, that is,

\[
u_{m-1}(x) \geq 0, \quad \forall \ i = 2, 3, \cdots, m-1, \quad \forall \ x \in \mathbb{R}^n.
\]

Suppose on the contrary that, there exists some \( 2 \leq i \leq m-1 \) and some \( x^0 \in \mathbb{R}^n \) such that

\[
u_{m-1}(x) \geq 0, \quad u_{m-2}(x) \geq 0, \cdots, \quad u_{m-i+1}(x) \geq 0, \quad \forall \ x \in \mathbb{R}^n,
\]
Then, repeating the similar “re-centers and iteration” arguments as in Step 1, after \( m - i + 1 \) steps of re-centers (denotes the centers by \( \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{m-i+1} \)), the signs of the resulting functions \( \tilde{u}_{m-j} (j = i, \ldots, m - 1) \) and \( \tilde{u} \) satisfy
\[
(2.50) \quad (-1)^{j-i+1} \tilde{u}_{m-j}(r) \geq (-1)^{j-i+1} \tilde{u}_{m-j}(0) > 0, \quad (-1)^{m-i+1} \tilde{u}(r) \geq (-1)^{m-i+1} \tilde{u}(0) > 0
\]
for any \( r \geq 0 \). Since \( u \geq 0 \), it follows immediately from (2.50) that \( m - i + 1 \) is even and
\[
(2.51) \quad \tilde{u}(r) \geq \tilde{u}(0) > 0, \quad \forall r \geq 0.
\]
Furthermore, since \( m - i \) is odd, we infer from (2.50) that
\[
(2.52) \quad -\Delta \tilde{u}(r) = \tilde{u}_1(r) \leq \tilde{u}_1(0) =: -\tilde{c} < 0, \quad \forall r \geq 0,
\]
and hence, by integrating, one has
\[
(2.53) \quad \tilde{u}(r) \geq \tilde{u}(0) + \frac{\tilde{c}}{2n} r^2 > \frac{\tilde{c}}{2n} r^2, \quad \forall r \geq 0.
\]
Therefore, if we assume that \( u(x) = o(|x|^2) \) as \( |x| \to +\infty \), we will get a contradiction from (2.53).

Or, if we assume that \( -\infty < a \leq 2 + 2p \), combining (2.54) with the estimate (2.22), we get that, for \( r \geq r_0 \) sufficiently large,
\[
(2.54) \quad -\Delta \tilde{u}_{m-1}(r) \geq (r + \bar{x}^{m-i+1} - \bar{x}^m) + \cdots + (\bar{x}^2 - \bar{x}^1)|^{-a} \tilde{u}'(r)
\]
\[
\geq \left( \frac{\tilde{c}}{4n} \right)^p r^{2p-a} \quad \text{if } 0 \leq a \leq 2 + 2p,
\]
and
\[
(2.55) \quad -\Delta \tilde{u}_{m-1}(r) \geq (r - \bar{x}^{m-i+1} - \bar{x}^m) - \cdots - (\bar{x}^2 - \bar{x}^1)|^{-a} \tilde{u}'(r)
\]
\[
\geq \left( \frac{\tilde{c}}{4n} \right)^p r^{2p-a} \quad \text{if } -\infty < a < 0.
\]
Now, by a direct integration on (2.54) and (2.55), we get, if \( -\infty < a < 2 + 2p \), then
\[
(2.56) \quad \tilde{u}_{m-1}(r) \leq \tilde{u}_{m-1}(r_0) - \left( \frac{\tilde{c}}{4n} \right)^p \frac{r^{2+2p-a} - r_0^{2+2p-a}}{(n + 2p - a)(2 + 2p - a)} \to -\infty, \quad \text{as } r \to \infty;
\]
if \( a = 2 + 2p \), then
\[
(2.57) \quad \tilde{u}_{m-1}(r) \leq \tilde{u}_{m-1}(r_0) - \left( \frac{\tilde{c}}{4n} \right)^p \frac{\ln r - \ln r_0}{n - 2} \to -\infty, \quad \text{as } r \to \infty.
\]
This contradicts \( u_{m-1} \geq 0 \) and thus (2.47) must hold. This concludes the proof of Theorem 2.1. \( \square \)

**Remark 2.2.** For \( 0 < a < 2m \), if we consider the nonnegative solutions \( u \in C^{2m}(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n) \), then it is clear from our proof of Theorem 2.1 that we can show super poly-harmonic properties except the origin \( 0 \in \mathbb{R}^n \), that is, \( (-\Delta)^iu \geq 0 \) in \( \mathbb{R}^n \setminus \{0\} \) for \( i = 1, \ldots, m - 1 \).
2.2. Equivalence between PDE and IE. By applying Theorem 2.1 for \( a \geq 0 \), we can deduce from \(-\Delta u \geq 0, u \geq 0, u(\bar{x}) > 0\) and maximum principle that
\[
\tag{2.58}
 u(x) > 0, \quad \forall x \in \mathbb{R}^n.
\]

Then, by maximum principle, Lemma 2.1 from Chen and Lin [15] and induction, we can also infer further from \((-\Delta)^i u \geq 0\) \((i = 1, \cdots, m - 1)\), \(u > 0\) and equation (1.1) that
\[
\tag{2.59}
(-\Delta)^i u(x) > 0, \quad \forall i = 1, \cdots, m - 1, \quad \forall x \in \mathbb{R}^n.
\]

Next, we will show that the positive solution \(u\) to (1.1) also satisfies the following integral equation
\[
\tag{2.60}
 u(x) = \int_{\mathbb{R}^n} \frac{C}{|x - \xi|^{n-2m}} \cdot \frac{u^p(\xi)}{|\xi|^a} d\xi.
\]

Indeed, we have the following theorem on the equivalence between PDE (1.1) and IE (2.60).

**Theorem 2.3.** Assume \( n \geq 3, 1 \leq m < \frac{n}{2}, 0 \leq a < 2m \) and \( 1 < p < \infty \). Suppose \( u \) is nonnegative classical solution to (1.1), then it also solves the integral equation (2.60), and vice versa.

**Proof.** Let \( \delta(x - \xi) \) be the Dirac Delta function and \( \phi_r(x - \xi) \) be the solution of the following equation
\[
\tag{2.61}
\begin{cases}
(-\Delta)^m \phi_r(x - \xi) = \delta(x - \xi), & \xi \in B_r(x), \\
\phi_r(x - \xi) = (-\Delta) \phi_r(x - \xi) = \cdots = (-\Delta)^{m-1} \phi_r(x - \xi), & \xi \in \partial B_r(x).
\end{cases}
\]

One can easily verify that, \((-\Delta)^i \phi_r(x - \xi)\) must take the following form
\[
\tag{2.62}
(-\Delta)^i \phi_r(x - \xi) = \frac{c_i}{|x - \xi|^{n+2i-2m}} + \sum_{k=1}^{m-i} c_{i,k} \frac{|x - \xi|^{2m-2i-2k}}{r^{n-2k}}
\]
for \( i = 0, 1, \cdots, m - 1 \), where the coefficients satisfy \( c_i + \sum_{k=1}^{m-i} c_{i,k} = 0 \) \((i = 0, 1, \cdots, m - 1)\). In particular, when \( i = m - 1 \), by (2.62), we have
\[
\tag{2.63}
(-\Delta)^{m-1} \phi_r(x - \xi) = \frac{C_{m-1}}{|x - \xi|^{n-2}} - \frac{C_{m-1}}{r^{n-2}}, \quad \xi \in \overline{B_r(x)},
\]
and hence
\[
\tag{2.64}
\frac{\partial}{\partial v_\xi} [(-\Delta)^{m-1} \phi_r(x - \xi)] \leq 0, \quad \xi \in \partial B_r(x),
\]
where \( v_\xi \) denotes the unit outer normal vector at \( \xi \in \partial B_r(x) \). Next we define function \( f(\xi) \) by
\[
\tag{2.65}
(-\Delta)^{m-1} \phi_r(x - \xi) = \frac{C_{m-1}}{|x - \xi|^{n-2}} - \frac{C_{m-1}}{r^{n-2}} =: f(\xi) \geq 0.
\]
It is obvious that \( f \in L^1(B_r(x)) \), thus \((-\Delta)^{m-2} \phi_r(x - \xi)\) is super-harmonic in the sense of distribution in \( B_r(x) \), and hence we derive
\[
\tag{2.66}
\inf_{\xi \in B_r(x)} (-\Delta)^{m-2} \phi_r(x - \xi) \geq \inf_{\xi \in \partial B_r(x)} (-\Delta)^{m-2} \phi_r(x - \xi) = 0
\]
and
\[
\frac{\partial \langle -\Delta \rangle^{m-2} \phi_r(x - \xi)}{\partial \nu_\xi} \leq 0, \quad \xi \in \partial B_r(x).
\]
Continuing this way, we conclude that, for \(i = 0, 1, 2, \ldots, m-1\),
\[
\inf_{\xi \in B_r(x)} \langle -\Delta \rangle^i \phi_r(x - \xi) \geq \inf_{\xi \in \partial B_r(x)} \langle -\Delta \rangle^i \phi_r(x - \xi) = 0
\]
and
\[
\frac{\partial \langle -\Delta \rangle^i \phi_r(x - \xi)}{\partial \nu_\xi} \leq 0, \quad \xi \in \partial B_r(x).
\]
From (2.63), we can get \(\langle -\Delta \rangle^{m-1} \phi_r(x - \xi)\) monotone increases about \(r\) and tends to \(\frac{C_{m-1}}{|x - \xi|^{n-2}}\) as \(r \to +\infty\). As a consequence, we arrive at, for any \(r_2 > r_1 > 0\),
\[
\langle -\Delta \rangle \left[ \langle -\Delta \rangle^{m-2} \phi_{r_2}(x - \xi) - \langle -\Delta \rangle^{m-2} \phi_{r_1}(x - \xi) \right] \geq 0, \quad \xi \in B_{r_1}(x),
\]
and
\[
0 = \langle -\Delta \rangle^{m-2} \phi_{r_1}(x - \xi) \leq \langle -\Delta \rangle^{m-2} \phi_{r_2}(x - \xi), \quad \xi \in \partial B_{r_1}(x).
\]
By maximum principle, we deduce that
\[
\langle -\Delta \rangle^{m-2} \phi_{r_2}(x - \xi) \geq \langle -\Delta \rangle^{m-2} \phi_{r_1}(x - \xi), \quad \forall \xi \in \mathbb{R}^n.
\]
So \(\langle -\Delta \rangle^{m-2} \phi_r(x - \xi)\) also monotone increases about \(r\) and tends to \(\frac{C_{m-2}}{|x - \xi|^{n-4}}\) as \(r \to +\infty\). Continuing this way, we can derive
\[
\langle -\Delta \rangle^i \phi_r(x - \xi) \uparrow \frac{C_i}{|x - \xi|^{n-2m+2i}}, \quad \text{as} \quad r \to +\infty.
\]
By Lemma 1 in [34] and equation (1.1), we have \(\langle -\Delta \rangle^{m-1} u\) solves the following equation
\[
\langle -\Delta \rangle^m u = \frac{u^p(x)}{|x|^a} + m\delta(0) \quad \text{in} \quad B_\rho(0)
\]
in the sense of distributions for arbitrary \(\rho > 0\), where \(m \geq 0\) and \(\delta(0)\) is the Delta distribution concentrated at the origin. Since \(u \in C(\mathbb{R}^n)\), it follows that \(m = 0\). Therefore, multiplying both sides of (2.74) by \(\phi_r(x - \xi)\) and integrating by parts on \(B_r(x)\), by Theorem 2.1 and (2.69), one has
\[
\int_{B_r(x)} \phi_r(x - \xi) \frac{u^p(\xi)}{|\xi|^a} d\xi = u(x) + \sum_{i=0}^{m-1} \int_{\partial B_r(x)} (-\Delta)^i u(\xi) \cdot \frac{\partial \langle -\Delta \rangle^{m-i-1} \phi_r(x - \xi)}{\partial \nu_\xi} d\sigma 
\leq u(x)
\]
for any $x \neq 0$. At the same time, multiplying $(-\Delta)^i u$ by $(-\Delta)^{m-i} \phi_r(x - \xi)$ ($i = 1, \ldots, m-1$) and integrating by parts on $B_r(x)$, by Theorem 2.1 and (2.69), one also has

\begin{equation}
\int_{B_r(x)} (-\Delta)^{m-i} \phi_r(x - \xi) \cdot (-\Delta)^i u(\xi) d\xi
= u(x) + \sum_{j=0}^{i-1} \int_{\partial B_r(x)} (-\Delta)^j u(\xi) \cdot \frac{\partial [(-\Delta)^{m-j-1} \phi_r(x - \xi)]}{\partial v_\xi} d\sigma
\leq u(x).
\end{equation}

Thus, by letting $r \to +\infty$ in (2.75), (2.76) and using Levi's monotone convergence theorem, we obtain

\begin{equation}
\int_{\mathbb{R}^n} \frac{1}{|x - \xi|^{n-2m}} \cdot \frac{w^p(\xi)}{|\xi|^a} d\xi < \infty
\end{equation}

and

\begin{equation}
\int_{\mathbb{R}^n} \frac{(-\Delta)^i u(\xi)}{|x - \xi|^{n-2i}} < \infty
\end{equation}

for $i = 1, \ldots, m-1$. Therefore, there exists a sequence $\{r_k\}$ such that, as $r_k \to \infty$,

\begin{equation}
\frac{1}{r_k^{n-2m-1}} \int_{\partial B_{r_k}(x)} \frac{w^p(\xi)}{|\xi|^a} d\sigma \to 0,
\end{equation}

and

\begin{equation}
\frac{1}{r_k^{n-2i-1}} \int_{\partial B_{r_k}(x)} (-\Delta)^i u(\xi) d\sigma \to 0 \quad \text{for } i = 1, 2, \ldots, m-1.
\end{equation}

From (2.79), it follows that, as $r_k \to +\infty$,

\begin{equation}
\frac{1}{r_k^{n-2m-1+a}} \int_{\partial B_{r_k}(x)} w^p(\xi) d\sigma = \frac{1}{r_k^{n-1-(2m-a)}} \int_{\partial B_{r_k}(x)} w^p(\xi) d\sigma \to 0
\end{equation}

Then, by Jensen’s inequality, we have

\begin{equation}
\left( \frac{1}{r_k^{n-1-(2m-a)}} \int_{\partial B_{r_k}(x)} w^p(\xi) d\sigma \right)^{\frac{1}{p}} \geq \frac{1}{r_k^{2m-a}} \int_{\partial B_{r_k}(x)} u(\xi) d\sigma,
\end{equation}

and hence

\begin{equation}
\frac{1}{r_k^{n-1}} \int_{\partial B_{r_k}(x)} u(\xi) d\sigma \to 0.
\end{equation}

Combining this with (2.62) and (2.80) implies

\begin{equation}
\sum_{i=0}^{m-1} \int_{\partial B_{r_k}(x)} (-\Delta)^i u(\xi) \cdot \frac{\partial [(-\Delta)^{m-i-1} \phi_{r_k}(x - \xi)]}{\partial v_\xi} d\sigma \to 0,
\end{equation}

inserting (2.84) into (2.75) and letting $r_k \to +\infty$, we derive immediately

\begin{equation}
u(x) = \int_{\mathbb{R}^n} \frac{C}{|x - \xi|^{n-2m}} \cdot \frac{w^p(\xi)}{|\xi|^a} d\xi,
\end{equation}

that is, $u$ satisfies the integral equation (2.60).
Conversely, assume that \( u \) is a nonnegative classical solution of integral equation (2.60), then

\[
(-\Delta)^m u(x) = \int_{\mathbb{R}^n} \left[ (-\Delta)^m \left( \frac{C}{|x - \xi|^{n-2m}} \right) \right] \frac{u^p(\xi)}{|\xi|^a} d\xi = \int_{\mathbb{R}^n} \delta(x - \xi) \frac{u^p(\xi)}{|\xi|^a} d\xi = \frac{u^p(x)}{|x|^a},
\]

that is, \( u \) also solves the PDE (1.1). This completes the proof of equivalence between PDE (1.1) and IE (2.60). \( \square \)

For \( 2 \leq a < 2m \) and \( 1 < p < \infty \), one can easily observe that the regularity at \( 0 \) of \( u \) indicated by the integral equation (2.60) contradicts with \( u \in C^{2m-2}(\mathbb{R}^n) \), thus we must have \( u \equiv 0 \) in \( \mathbb{R}^n \).

In the following, we will also obtain a contradiction for \( 1 < p < \frac{n+2m-2a}{n-2m} \) and \( 0 \leq a < 2m \) by applying the method of moving planes and Pohozaev identity to the equivalent integral equation (2.60) (see subsection 2.3 and 2.4). The proof still works for \( u \in C^2(\mathbb{R}^n) \).

2.3. Radial symmetry of positive solution. From Theorem 2.3, we know that the positive classical solution \( u \) to PDE (1.1) is also a positive solution to the equivalent integral equation (2.60).

If \( u \) is a nonnegative solution to IE (2.60), we must have either \( u \equiv 0 \) or \( u > 0 \) in \( \mathbb{R}^n \).

The next Theorem says that all the locally integrable positive solutions to IE (2.60) must be radially symmetric and monotone decreasing about the origin.

**Theorem 2.4.** Assume \( n \geq 3, 1 \leq m < \frac{n}{2}, 0 \leq a < 2m \) and \( 1 < p < \frac{n+2m-n}{n-2m} \). Suppose \( u \) is a positive solution to IE (2.60) satisfying \( \frac{u^{p-1}}{|x|^a} \in L^\infty_{loc}(\mathbb{R}^n) \), then \( u \) is radially symmetric and monotone decreasing about the origin.

**Proof.** We define the Kelvin transform of \( u \) by

\[
\bar{u}(x) = \frac{1}{|x|^{n-2m}} u \left( \frac{x}{|x|^2} \right), \quad x \neq 0.
\]

Since \( u \) satisfies the integral equation

\[
(2.87) \quad u(x) = C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x - y|^{n-2m}|y|^a} dy,
\]
it follows that, for \( x \neq 0 \),
\[
(2.88) \quad \bar{u}(x) = \frac{C}{|x|^{n-2m}} \int_{\mathbb{R}^n} \frac{u^p(y)}{|x|^2 - y^2} \frac{|y|^{n-2m}}{|y|^a} dy
\]
\[
= \frac{C}{|x|^{n-2m}} \int_{\mathbb{R}^n} \frac{u^p\left(\frac{y}{|y|^2}\right)}{|x|^2 - y^2} \frac{1}{|y|^{n-2m}} \frac{1}{|y|^a} dy
\]
\[
= \frac{C}{|x|^{n-2m}} \int_{\mathbb{R}^n} \frac{|x|^{n-2m}}{|x|^2 - y^2} \frac{|y|^{n-2m}}{|y|^{2n-a}} dy
\]
\[
= \frac{C}{|x|^{n-2m}} \int_{\mathbb{R}^n} \frac{1}{|x|^2 - y^2} \frac{u^p\left(\frac{y}{|y|^2}\right)}{|y|^{n-2m}} dy
\]
\[
= \frac{C}{|x|^{n-2m}} \int_{\mathbb{R}^n} \frac{1}{|x|^2 - y^2} \frac{u^p(y)}{|y|^a} dy,
\]

where \( \tau := n + 2m - a - p(n - 2m) > 0 \).

We will apply the method of moving planes in integral forms to the integral equation (2.88) and carry out the process of moving plane in the \( x_1 \) direction. For this purpose, we need some definitions.

Let \( \lambda \leq 0 \) be an arbitrary non-positive real number and let the moving plane be
\[
T_\lambda := \{ x \in \mathbb{R}^n : x_1 = \lambda \}.
\]
We denote
\[
(2.90) \quad \Sigma_\lambda := \{ x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 < \lambda \},
\]
and let
\[
(2.91) \quad x^\lambda := (2\lambda - x_1, x_2, \cdots, x_n)
\]
be the reflection of \( x \) about the plane \( T_\lambda \), and define
\[
(2.92) \quad \bar{u}_\lambda(x) := \bar{u}(x^\lambda), \quad \omega_\lambda(x) := \bar{u}_\lambda(x) - \bar{u}(x).
\]

By properly exploiting some global properties of the integral equations, we will show that, for \( \lambda \) sufficiently negative,
\[
(2.93) \quad \omega_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}.
\]
Then, we start moving the plane \( T_\lambda \) from near \( x_1 = -\infty \) to the right as long as (2.93) holds, until its limiting position and finally derive symmetry and monotonicity. Therefore, the moving plane process can be divided into two steps.

**Step 1.** Start moving the plane from near \( x_1 = -\infty \). Define the set
\[
(2.94) \quad \Sigma^-_\lambda := \{ x \in \Sigma_\lambda \setminus \{0^\lambda\} : \omega_\lambda(x) < 0 \}. 
\]
We can deduce from (2.3) that, for $x \in \Sigma_\lambda \setminus \{0\}$,

\[(2.95)\]

\[
\omega_\lambda(x) = \bar{u}_\lambda(x) - \bar{u}(x)
\]

\[
= C \int_{\mathbb{R}^n} \frac{1}{|x^\lambda - y|^{n-2m}} \frac{\bar{u}^p(y)}{|y|^\tau} dy - C \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2m}} \frac{\bar{u}^p(y)}{|y|^\tau} dy
\]

\[
= C \int_{\Sigma_\lambda} \frac{1}{|x^\lambda - y|^{n-2m}} \frac{\bar{u}^p(y)}{|y|^\tau} dy + C \int_{\Sigma_\lambda} \frac{1}{|x^\lambda - y|^{n-2m}} \frac{\bar{u}^p(y^\lambda)}{|y^\lambda|^\tau} dy
\]

\[
-C \int_{\Sigma_\lambda} \frac{1}{|x - y|^{n-2m}} \frac{\bar{u}^p(y)}{|y|^\tau} dy - C \int_{\Sigma_\lambda} \frac{1}{|x - y|^{n-2m}} \frac{\bar{u}^p(y^\lambda)}{|y^\lambda|^\tau} dy
\]

\[
\geq C \int_{\Sigma_\lambda} \frac{1}{|x - y|^{n-2m}} \frac{\bar{u}^p(y^\lambda)}{|y^\lambda|^\tau} dy
\]

\[
\geq C \int_{\Sigma_\lambda} \frac{p\bar{u}^{p-1}(y^\lambda)}{|y^\lambda|^\tau} \cdot \omega_\lambda(y) \frac{\omega_\lambda(y)}{|y^\lambda|^\tau} dy
\]

In particular, for $x \in \Sigma_\lambda$, we have

\[(2.96)\]

\[
0 > \omega_\lambda(x) \geq C \int_{\Sigma_\lambda} \frac{p\bar{u}^{p-1}(y^\lambda)}{|x - y|^{n-2m}} \cdot \omega_\lambda(y) \frac{\omega_\lambda(y)}{|y|^\tau} dy.
\]

By Hardy-Littlewood-Sobolev inequality, one gets, for arbitrary $\frac{n}{n-2m} \leq q < \infty$,

\[(2.97)\]

\[
\|\omega_\lambda\|_{L^q(\Sigma^-_\lambda)} \leq C \left\| \int_{\Sigma_\lambda} \frac{p\bar{u}^{p-1}(y)}{|x - y|^{n-2m}} \cdot \omega_\lambda(y) \frac{\omega_\lambda(y)}{|y|^\tau} dy \right\|_{L^q(\Sigma^-_\lambda)}
\]

\[
\leq C \left\| \frac{\bar{u}^{p-1}(x)}{|x|^\tau} \cdot \omega_\lambda(x) \right\|_{L^{q_0}(\Sigma^-_\lambda)}
\]

\[
\leq C \left\| \frac{\bar{u}^{p-1}}{|x|^\tau} \right\|_{L^{2/n}(\Sigma^-_\lambda)} \cdot \|\omega_\lambda\|_{L^q(\Sigma^-_\lambda)}
\]

Since $\frac{u^{p-1}}{|x|^{\tau}} \in L^{\frac{n}{2m}}(\mathbb{R}^n)$, we have, for any $r > 0$,

\[(2.98)\]

\[
\int_{|x| \geq r} \frac{\bar{u}^{(p-1)\frac{n}{2m}}(x)}{|x|^\tau \frac{n}{2m}} dx = \int_{|x| \geq r} \frac{1}{|x|^{\tau+(p-1)(n-2m)/2m}} u^{(p-1)\frac{n}{2m}} \left( \frac{x}{|x|^2} \right) dx
\]

\[
= \int_{|x| \leq \frac{R}{2}} \frac{1}{|x|^{2(n-4m-\alpha)/2m}} u^{(p-1)\frac{n}{2m}}(x) dx
\]

\[
= \int_{|x| \leq \frac{R}{2}} \frac{u^{(p-1)\frac{n}{2m}}(x)}{|x|^{\frac{2n}{2m}}} dx < +\infty.
\]

Therefore, there exists a $\Lambda_0$ sufficiently large, such that, for any $\lambda \leq -\Lambda_0$,

\[(2.99)\]

\[
C \left\| \frac{\bar{u}^{p-1}(x)}{|x|^\tau} \right\|_{L^{2/n}(\Sigma^-_\lambda)} \leq C \frac{1}{2}.
\]
Thus, we must have, for any \( \frac{n}{n-2m} < q < \infty \),

\[
(2.100) \quad \| \omega_\lambda \|_{L^q(S_\lambda^-)} = 0,
\]

Combining this with (2.95) implies \( \Sigma_\lambda^- = \emptyset \), and hence

\[
(2.101) \quad \omega_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}.
\]

**Step 2. Move the plane to the limiting position to derive symmetry and monotonicity.** Now we move the plane \( T_\lambda \) to the right as long as (2.93) holds. Define

\[
(2.102) \quad \lambda_0 := \sup \{ \lambda \in \mathbb{R} \mid \omega_\rho \geq 0 \text{ in } \Sigma_\rho \setminus \{0^\rho\}, \forall \rho \leq \lambda \}.
\]

By applying a entirely similar argument as in Step1, we can also start moving the plane from near \( x_1 = +\infty \) to the left, thus we must have \( \lambda_0 < +\infty \). Now, we will show that \( \lambda_0 = 0 \).

Suppose on the contrary that \( \lambda_0 < 0 \), we will show that

\[
(2.103) \quad \omega_{\lambda_0}(x) \equiv 0, \quad \forall x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}.
\]

We prove (2.103) by contradiction arguments. Suppose on the contrary that \( \omega_{\lambda_0}(x) \geq 0 \), but \( \omega_{\lambda_0}(x) \) is not identically zero in \( \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\} \). We will obtain a contradiction with (2.102) via showing that the plane \( T_\lambda \) can be moved a little bit further to the right, more precisely, there exist an \( 0 < \varepsilon < |\lambda_0| \) small enough, such that \( w_\lambda \geq 0 \) in \( \Sigma_{\lambda} \setminus \{0^\lambda\} \) for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon) \).

It can be clearly seen from (2.97) and (2.99) in Step 1 that, our goal is to prove that, one can choose \( \varepsilon > 0 \) sufficiently small such that, for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon) \),

\[
(2.104) \quad \left\| \frac{u^{p-1}(x)}{|x|^\tau} \right\|_{L^2m(S_\lambda^-)} \leq \frac{1}{2C},
\]

where the constant \( C \) is the same as in (2.97) and (2.99).

In fact, by (2.98), we can choose \( R > 0 \) large enough, such that

\[
(2.105) \quad \left( \int_{|x| \geq R} \frac{u^{p-1}(x)}{|x|^\tau} \frac{dx}{n} \right)^{\frac{n}{2m}} < \frac{1}{4C}.
\]

Now fix this \( R \), in order to derive (2.104), we only need to show

\[
(2.106) \quad \lim_{\lambda \to \lambda_0^+} \mu(S_{\lambda}^- \cap B_R(0)) = 0.
\]

To this end, we define \( E_\delta := \{ x \in (\Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}) \cap B_R(0) \mid w_{\lambda_0}(x) > \delta \} \) and \( F_\delta := (\Sigma_{\lambda_0} \cap B_R(0)) \setminus E_\delta \) for any \( \delta > 0 \), and let \( D_\lambda := (\Sigma_{\lambda} \setminus \Sigma_{\lambda_0}) \cap B_R(0) \) for any \( \lambda > \lambda_0 \). Then, one can easily verify that

\[
(2.107) \quad \lim_{\delta \to 0^+} \mu(F_\delta) = 0, \quad \lim_{\lambda \to \lambda_0^+} \mu(D_\lambda) = 0,
\]

\[
(2.108) \quad \Sigma_{\lambda}^- \cap B_R(0) = \Sigma_{\lambda}^- \cap (E_\delta \cup F_\delta \cup D_\lambda) \subset \Sigma_{\lambda}^- \cap E_\delta \cup F_\delta \cup D_\lambda.
\]

For an arbitrary fixed \( \eta > 0 \), one can choose a \( \delta > 0 \) small enough, such that \( \mu(F_\delta) \leq \eta \). For this fixed \( \delta \), we are to prove

\[
(2.109) \quad \lim_{\lambda \to \lambda_0^+} \mu(\Sigma_{\lambda}^- \cap E_\delta) = 0.
\]

Indeed, one can observe that \( \bar{u}(x^{\lambda_0}) - \bar{u}(x^\lambda) = \omega_{\lambda_0}(x) - \omega_{\lambda}(x) > \delta \) for all \( x \in \Sigma_{\lambda}^- \cap E_\delta \). It follows that

\[
(2.110) \quad (\Sigma_{\lambda}^- \cap E_\delta) \subset G_\delta^{\lambda} := \{ x \in B_R(0) \cap \left( (\Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}) \setminus \{0^\lambda\} \right) \mid \bar{u}(x^{\lambda_0}) - \bar{u}(x^\lambda) > \delta \}.\]
By Chebyshev’s inequality, we get

\[
\mu(G_\delta^\lambda) \leq \frac{1}{\delta^r} \int_{G_\delta^\lambda} \left| \bar{u}(x^\lambda) - \bar{u}(x^\lambda) \right|^r dx \\
\leq \frac{1}{\delta^r} \int_{B_R(0)} \left| \bar{u}(x) - \bar{u}(x + 2(\lambda - \lambda_0)e_1) \right|^r dx
\]

for any \(1 \leq r < \frac{n}{n-2m}\), where \(e_1 = (1,0,\ldots,0) \in \mathbb{R}^n\), and hence

\[
\lim_{\lambda \to \lambda_0^+} \mu(G_\delta^\lambda) = 0,
\]

from which (2.109) follows immediately.

Therefore, by (2.107), (2.108) and (2.109), we have

\[
\lim_{\lambda \to \lambda_0^+} \mu(\Sigma_\lambda^\theta \cap B_R(0)) \leq \mu(F_\delta) \leq \eta.
\]

Since \(\eta > 0\) is arbitrarily chosen, (2.106) follows immediately from (2.113). Combining (2.105) and (2.106), we finally arrive at (2.104).

From the last inequality of (2.97), we have, for any \(\frac{n}{n-2m} < q < \infty\),

\[
\|\omega_\lambda\|_{L^q(\Sigma_\lambda^\theta)} \leq C \left\| \frac{\bar{u}^{p-1}}{|x|^q} \right\|_{L^\infty(\Sigma_\lambda^\theta)} \cdot \|\omega_\lambda\|_{L^q(\Sigma_\lambda^\theta)}.
\]

By (2.104) and the above estimate, we deduce that, there exists an \(\varepsilon > 0\) sufficiently small, such that, for all \(\lambda \in (\lambda_0, \lambda_0 + \varepsilon)\), \(\|\omega_\lambda\|_{L^q(\Sigma_\lambda^\theta)} = 0\), thus \(\mu(\Sigma_\lambda^\theta) = 0\). Furthermore, by (2.95), we have \(\Sigma_\lambda^- = \emptyset\), and hence \(\omega_\lambda(x) \geq 0\) in \(\Sigma_\lambda \setminus \{0^\lambda\}\) for all \(\lambda \in (\lambda_0, \lambda_0 + \varepsilon)\). This contradicts with the definition of \(\lambda_0\). Therefore, (2.103) must hold. By (2.95) and (2.103), we get, for any \(x \in \Sigma_{\lambda_0}\),

\[
0 = \omega_{\lambda_0}(x) = \bar{u}(x^{\lambda_0}) - \bar{u}(x)
\]

\[
= C \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|x - y|^{n-2m}} - \frac{1}{|x - y^{\lambda_0}|^{n-2m}} \right) \left( \frac{1}{|y^{\lambda_0}|^q} - \frac{1}{|y^q|} \right) \bar{u}^p(y) dy
\]

\[
> 0.
\]

That is a contradiction! Thus we must have \(\lambda_0 = 0\), and hence

\[
u(-x_1, x_2, \ldots, x_n) \geq u(x_1, x_2, \ldots, x_n), \quad \forall \ x \in \Sigma_0.
\]

We can also move the plane from \(x_1 = +\infty\) to the left and the limiting position is also \(\lambda_0 = 0\), so one has

\[
u(-x_1, x_2, \ldots, x_n) \leq u(x_1, x_2, \ldots, x_n), \quad \forall \ x \in \Sigma_0.
\]

Therefore,

\[
u(-x_1, x_2, \ldots, x_n) \equiv u(x_1, x_2, \ldots, x_n), \quad \forall \ x \in R^n
\]

that is, \(u(x)\) is symmetric with respect to the plane \(T_0 = \{x \in \mathbb{R}^n | x_1 = 0\}\).

Since the equation is invariant under rotation, the \(x_1\) direction can be chosen arbitrarily. We conclude that the positive solution \(u(x)\) must be radially symmetric and monotone decreasing about the origin \(0 \in \mathbb{R}^n\). This finishes our proof of Theorem 2.4.
2.4. Pohozaev identity and nonexistence of positive radially symmetric solutions. By Theorem 2.4, we deduce that the positive classical solution \( u \) to PDE (1.1) is a positive radially symmetric solution to IE (2.60), i.e., \( u(x) = u(|x|) > 0 \). Next, we will show that there is no positive radially symmetric classical solutions to (2.60), which leads to a contradiction.

**Theorem 2.5.** Assume \( n \geq 3, 1 \leq m < \frac{n}{2}, 0 \leq a < 2m \) and \( 1 < p < \frac{n+2m-2a}{n-2m} \), then (2.60) has no positive radially symmetric classical solutions.

**Proof.** Suppose \( u(x) = u(|x|) > 0 \) is a positive radially symmetric classical solution to (2.60), that is,

\[
(2.119) \quad u(x) = C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-2m}|y|^a} dy.
\]

Then, for any \( \mu > 0 \),

\[
(2.120) \quad u(\mu x) = C \int_{\mathbb{R}^n} \frac{u^p(y)}{|\mu x-y|^{n-2m}|y|^a} dy.
\]

Take the derivatives on both sides of (2.120) with respect to \( \mu \) and let \( \mu = 1 \), we get

\[
(2.121) \quad x \cdot \nabla u(x) = C \frac{d}{d\mu} \left[ \int_{\mathbb{R}^n} \frac{u^p(y)}{|\mu x-y|^{n-2m}|y|^a} dy \right]_{\mu=1} = (n-2m)C \int_{\mathbb{R}^n} \frac{u^p(y)(\mu x-y) \cdot x}{|\mu x-y|^{n-2m-2}|y|^a} dy |_{\mu=1} = (n-2m)C \int_{\mathbb{R}^n} \frac{u^p(y)(x-y) \cdot x}{|x-y|^{n-2m-2}|y|^a} dy.
\]

Multiply both sides of (2.121) by \( \frac{u^p(x)}{|x|^a} \) and integrate on \( B_r(0) \) for any \( r > 0 \), one has

\[
(2.122) \quad LHS = \int_{B_r(0)} (x \cdot \nabla u(x)) \frac{u^p(x)}{|x|^a} dx = \int_0^r \int_{\partial B_s(0)} s \frac{d(u(s))}{ds} \cdot \frac{u^p(s)}{s^a} d\sigma ds = \int_0^r \frac{u_n s^{n-a}}{p+1} d(u^{p+1}(s)) = \frac{u_n}{p+1} r^{n-a} u^{p+1}(r) - \frac{(n-a) u_n}{p+1} \int_0^r u^{p+1}(s) s^{n-a-1} ds = \frac{r^{1-a}}{p+1} \int_{\partial B_r(0)} u^{p+1}(x) d\sigma - \frac{n-a}{p+1} \int_{B_r(0)} \frac{u^{p+1}(x)}{|x|^a} dx,
\]

and

\[
(2.123) \quad RHS = -(n-2m)C \int_{B_r(0)} \frac{u^p(x)}{|x|^a} \int_{\mathbb{R}^n} \frac{(x-y) \cdot xu^p(y)}{|x-y|^{n-2m-2}|y|^a} dy,
\]
where \( w_n \) denotes the area of the unit sphere. Since \( u \) is a positive radially symmetric classical solution to (2.60), we have \( u(r) \) monotone decreases about \( r \geq 0 \) and

\[
(2.124) \quad u(x) = u(r) = C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x - y|^{n-2m}} |y|^s dy.
\]

\[
= \frac{C}{r^{n-2m}} \int_{\mathbb{R}^n} \frac{u^p(y)}{|x| - \frac{y}{r} |y|^a} dy.
\]

\[
\geq \frac{C}{r^{n-2m}} \int_0^r \int_{\partial B_r(0)} \frac{u^p(s)}{|x| - \frac{s \sigma}{r}} |y|^n s^a d\sigma(ds,
\]

where \( r = |x| \) and \( \sigma \) is an arbitrary unit vector on \( \partial B_1(0) \). Observe that

\[
(2.125) \quad \frac{1}{|x|} \geq \frac{1}{2n-2m}, \quad \forall \ s \in [0, r] \hspace{1em} \text{and} \hspace{1em} \forall \ \sigma \in \partial B_1(0),
\]

thus we infer from (2.124) that

\[
u(x) = u(r) \geq \frac{C}{r^{n-2m}} \int_0^r w_n s^{n-1-a} u^p(s) \frac{u^p(r)}{2n-2m} ds
\]

\[
\geq \frac{Cw_n}{(2r)^{n-2m} u^p(r)} \int_0^r s^{n-1-a} ds
\]

\[
= \frac{Cw_n}{2n-2m(n-a)} \cdot \frac{u^p(r)}{r^{n-2m} r^{n-a}}
\]

\[
= \frac{Cw_n r^{n-2m-a} u^p(r)}{2n-2m(n-a)}.
\]

Therefore,

\[
(2.126) \quad u^{p-1}(r) \leq \frac{2n-2m(n-a)}{Cw_n r^{2m-a}}.
\]

Let \( \tilde{C} = \left( \frac{2n-2m(n-a)}{Cw_n} \right)^{\frac{1}{p-1}} > 0 \), then one has the following decay estimate

\[
(2.127) \quad u(r) \leq \frac{\tilde{C}}{r^{\frac{2m-a}{p-1}}}, \quad \forall \ r > 0.
\]

Note that \( 1 < p < \frac{n+2m-2a}{n-2m} \), we derive from the decay estimate (2.127) that

\[
(2.128) \quad \int_{\mathbb{R}^n} \frac{u^{p+1}(x)}{|x|^a} dx < \infty,
\]

and hence

\[
(2.129) \quad \int_0^\infty r^{-a} \left( \int_{\partial B_r(0)} u^{p+1}(x)d\sigma \right) dr < \infty.
\]

Thus there exists a sequence \( \{ r_j \} \), such that \( r_j \to +\infty \) as \( j \to \infty \) and

\[
(2.130) \quad r_j^{1-a} \left( \int_{\partial B_{r_j}(0)} u^{p+1}(x)d\sigma \right) \to 0, \quad \text{as} \ j \to \infty.
\]
By letting \( r = r_j \to +\infty \) in (2.122) and (2.123), we conclude from (2.128) and (2.130) that

\[
-(n - a) p + 1 \int_{\mathbb{R}^n} \frac{u^{p+1}(x)}{|x|^a} \, dx = -(n - 2m) C \int_{\mathbb{R}^n} \frac{u^p(x)}{|x|^a} \int_{\mathbb{R}^n} \frac{u^p(y) x \cdot (x - y)}{|x - y|^{n-2m+2}|y|^a} \, dydx.
\]

At the same time, by direct calculations, we have

\[
\frac{2m - n}{2} \int_{\mathbb{R}^n} \frac{u^{p+1}(x)}{|x|^a} \, dx
= \frac{2m - n}{2} \int_{\mathbb{R}^n} \frac{u^p(x)}{|x|^a} C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x - y|^{n-2m}|y|^a} \, dydx
= (2m - n) C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|x - y|^2 u^p(x) u^p(y)}{|x|^a |y|^a |x - y|^{n-2m+2}} \, dydx
= (2m - n) C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left[ (x - y) \cdot x + (y - x) \cdot y \right] u^p(x) u^p(y) \, dydx
= -(n - 2m) C \int_{\mathbb{R}^n} \frac{u^p(x)}{|x|^a} \int_{\mathbb{R}^n} \frac{u^p(y) x \cdot (x - y)}{|x - y|^{n-2m+2}|y|^a} \, dydx.
\]

Combining (2.131) and (2.132), we deduce further that

\[
\left( \frac{2m - n}{2} + \frac{n - a}{p + 1} \right) \int_{\mathbb{R}^n} \frac{u^{p+1}(x)}{|x|^a} \, dx = 0.
\]

Since \( 1 < p < \frac{n+2m-2a}{n-2m} \), it is easy to see that

\[
\frac{2m - n}{2} + \frac{n - a}{p + 1} > 0,
\]

thus we must have

\[
\int_{\mathbb{R}^n} \frac{u^{p+1}(x)}{|x|^a} \, dx = 0,
\]

which is a contradiction with \( u > 0 \)! Therefore, (2.60) does not have any positive radially symmetric classical solutions.

Since we have proved the positive classical solution \( u \) to PDE (1.1) is also a positive radially symmetric solution to IE (2.60), Theorem 2.5 leads to a contradiction. Therefore, we must have \( u \equiv 0 \) in \( \mathbb{R}^n \), that is, the unique nonnegative solution to PDE (1.1) is \( u \equiv 0 \) in \( \mathbb{R}^n \).

This concludes the proof of Theorem 1.1.

3. Proof of Theorem 1.6

In this section, we will prove Theorem 1.6 via the method of moving planes in local way and blowing-up techniques.

3.1. Boundary layer estimates. In this subsection, we will first establish the following boundary layer estimates by applying Kelvin transform and the method of moving planes in local way. The properties of the boundary \( \partial \Omega \) will play a crucial role in our discussions.
Theorem 3.1. Assume one of the following two assumptions

i) \( \Omega \) is strictly convex, \( 1 < p < \frac{n + 2m}{n - 2m} \), or

ii) \( 1 < p \leq \frac{n + 2}{n - 2} \)

holds. Then, there exists a \( \delta > 0 \) depending only on \( \Omega \) such that, for any positive solution \( u \in C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega}) \) to the higher order Navier problem (1.5), we have

\[
\|u\|_{L^\infty(\overline{\Omega}_\delta)} \leq C(n, m, p, \lambda_1, \Omega),
\]

where the boundary layer \( \overline{\Omega}_\delta := \{ x \in \overline{\Omega} \mid \text{dist}(x, \partial\Omega) \leq \delta \} \).

Remark 3.2. When \( m = 1 \), Theorem 3.1 still holds for \( p = \frac{n + 2}{n - 2} \).

Proof. We will carry out our proof of Theorem 3.1 by discussing the two different assumptions i) and ii) separately.

Case i) \( \Omega \) is strictly convex and \( 1 < p < \frac{n + 2m}{n - 2m} \). For any \( x^0 \in \partial\Omega \), let \( \nu^0 \) be the unit internal normal vector of \( \partial\Omega \) at \( x^0 \), we will show that \( u(x) \) is monotone increasing along the internal normal direction in the region

\[
\Sigma_{\delta_0} = \{ x \in \overline{\Omega} \mid 0 \leq (x - x^0) \cdot \nu^0 \leq \delta_0 \},
\]

where \( \delta_0 > 0 \) depends only on \( x^0 \) and \( \Omega \).

To this end, we define the moving plane by

\[
T_\lambda := \{ x \in \mathbb{R}^n \mid (x - x^0) \cdot \nu^0 = \lambda \},
\]

and denote

\[
\Sigma_\lambda := \{ x \in \Omega \mid 0 < (x - x^0) \cdot \nu^0 < \lambda \}
\]

for \( \lambda > 0 \), and let \( x^\lambda \) be the reflection of the point \( x \) about the plane \( T_\lambda \).

Let \( u_i := (-\Delta)^i u \) for \( 1 \leq i \leq m - 1 \). By maximum principle, we have

\[
u_i(x) > 0 \quad \text{in } \Omega \quad \text{for } 1 \leq i \leq m - 1. \tag{3.4}\]

Define

\[
W^\lambda_i(x) := u(x^\lambda) - u(x) \quad \text{and} \quad W_i^\lambda(x) := u_i(x^\lambda) - u_i(x) \tag{3.5}
\]

for \( 1 \leq i \leq m - 1 \). Then we can deduce from (1.5) that, for any \( \lambda \) satisfying the reflection of \( \Sigma_\lambda \) is contained in \( \Omega \),

\[
\begin{aligned}
-\Delta W_{m-1}^\lambda(x) &= u(x^\lambda) - u(x) = p \eta_{\lambda}^{p-1}(x) W^\lambda(x), \quad x \in \Sigma_\lambda, \\
-\Delta W_{m-2}^\lambda(x) &= W_{m-1}^\lambda(x), \quad x \in \Sigma_\lambda, \\
& \quad \ldots \ldots \\
-\Delta W^\lambda(x) &= W_1^\lambda(x), \quad x \in \Sigma_\lambda, \\
W^\lambda(x) \geq 0, \quad W_1^\lambda(x) \geq 0, \ldots, \quad W_{m-1}^\lambda(x) \geq 0, \quad x \in \partial \Sigma_\lambda,
\end{aligned}
\]

where \( \eta_{\lambda}(x) \) is valued between \( u(x^\lambda) \) and \( u(x) \) by mean value theorem. Now, we will prove that there exists some \( \delta > 0 \) sufficiently small (depending on \( m, p, \|u\|_{L^\infty(\overline{\Omega})} \) and \( \Omega \)), such that

\[
W^\lambda(x) \geq 0 \quad \text{in } \Sigma_\lambda \quad \text{for all } 0 < \lambda \leq \delta. \quad \text{(3.7)}
\]

This provides a starting point to move the plane \( T_\lambda \).
Indeed, suppose on the contrary that there exists a $0 < \lambda \leq \delta$ such that
\begin{equation}
W^{\lambda}(x) < 0 \quad \text{somewhere in } \Sigma_{\lambda}.
\end{equation}
Let
\begin{equation}
\zeta(x) := \cos \left(\frac{(x - x^0) \cdot \nu^0}{\delta}\right),
\end{equation}
then it follows that $\zeta(x) \in [\cos 1, 1]$ for any $x \in \Sigma_{\lambda}$ and $-\frac{\Delta \zeta(x)}{\zeta(x)} = \frac{1}{\delta^2}$. Define
\begin{equation}
\overline{W}^{\lambda}(x) := \frac{W^{\lambda}(x)}{\zeta(x)} \quad \text{and} \quad \overline{W}^{\lambda}_i(x) := \frac{W^{\lambda}_i(x)}{\zeta(x)}
\end{equation}
for $i = 1, \cdots, m - 1$ and $x \in \Sigma_{\lambda}$. Then there exists a $x_0 \in \Sigma_{\lambda}$ such that
\begin{equation}
\overline{W}^{\lambda}(x_0) = \min_{\Sigma_{\lambda}} \overline{W}^{\lambda}(x) < 0.
\end{equation}
Since
\begin{equation}
-\Delta W^{\lambda}(x_0) = -\Delta \overline{W}^{\lambda}(x_0) \zeta(x_0) - 2 \nabla \overline{W}^{\lambda}(x_0) \cdot \nabla \zeta(x_0) - \overline{W}^{\lambda}(x_0) \Delta \zeta(x_0),
\end{equation}
one immediately has
\begin{equation}
W^{\lambda}_1(x_0) = -\Delta W^{\lambda}(x_0) \leq \frac{1}{\delta^2} \overline{W}^{\lambda}(x_0) < 0.
\end{equation}
Thus there exists a $x_1 \in \Sigma_{\lambda}$ such that
\begin{equation}
\overline{W}^{\lambda}_1(x_1) = \min_{\Sigma_{\lambda}} \overline{W}^{\lambda}_1(x) < 0.
\end{equation}
Similarly, it follows that
\begin{equation}
W^{\lambda}_2(x_1) = -\Delta W^{\lambda}_1(x_1) \leq \frac{1}{\delta^2} \overline{W}^{\lambda}_1(x_1) < 0.
\end{equation}
Continuing this way, we get $\{x_i\}_{i=1}^{m-1} \subset \Sigma_{\lambda}$ such that
\begin{equation}
\overline{W}^{\lambda}_i(x_i) = \min_{\Sigma_{\lambda}} \overline{W}^{\lambda}_i(x) < 0,
\end{equation}
(3.16)
\begin{equation}
W^{\lambda}_{i+1}(x_i) = -\Delta W^{\lambda}_i(x_i) \leq \frac{1}{\delta^2} \overline{W}^{\lambda}_i(x_i) < 0
\end{equation}
for $i = 1, 2, \cdots, m - 2$, and
\begin{equation}
\overline{W}^{\lambda}_{m-1}(x_{m-1}) = \min_{\Sigma_{\lambda}} \overline{W}^{\lambda}_{m-1}(x) < 0,
\end{equation}
(3.18)
\begin{equation}
p_{\lambda}^{p-1}(x_{m-1})W^{\lambda}(x_{m-1}) = -\Delta W^{\lambda}_{m-1}(x_{m-1}) \leq \frac{1}{\delta^2} \overline{W}^{\lambda}_{m-1}(x_{m-1}) < 0.
\end{equation}
(3.19)
Therefore, we have
\begin{equation}
W^\lambda(x_0) \geq \delta^2 W^\lambda_1(x_0) \geq \delta^2 W^\lambda_1(x_1) \frac{\zeta(x_0)}{\zeta(x_1)} \geq \delta^4 W^\lambda_2(x_1) \frac{\zeta(x_0)}{\zeta(x_1)}
\end{equation}
\begin{equation}
\geq \delta^4 W^\lambda_2(x_2) \frac{\zeta(x_0)}{\zeta(x_2)} \geq \delta^6 W^\lambda_3(x_2) \frac{\zeta(x_0)}{\zeta(x_2)} \geq \delta^6 W^\lambda_3(x_3) \frac{\zeta(x_0)}{\zeta(x_3)}
\end{equation}
\begin{equation}
\geq \cdots \cdots \geq \delta^{2m-2} W^\lambda_{m-1}(x_{m-1}) \frac{\zeta(x_0)}{\zeta(x_{m-1})}
\end{equation}
\begin{equation}
\geq \delta^{2m} \eta^p_\lambda(x_{m-1}) W^\lambda_{m-1} \frac{\zeta(x_0)}{\zeta(x_{m-1})}
\end{equation}
\begin{equation}
\geq p \delta^{2m} \|u\|_{L^p(\Omega)}^{-p} W^\lambda(x_0),
\end{equation}
that is,
\begin{equation}
1 \leq p \delta^{2m} \|u\|_{L^p(\Omega)}^{-p},
\end{equation}
which is absurd if we choose \( \delta > 0 \) small enough such that
\begin{equation}
0 < \delta < \left( p\|u\|_{L^p(\Omega)}^{-p} \right)^{-\frac{1}{2m}}.
\end{equation}
So far, our conclusion is: the method of moving planes can be carried on up to \( \lambda = \delta \).

Next, we will move the plane \( T_\lambda \) further along the internal normal direction at \( x^0 \) as long as the property
\begin{equation}
W^\lambda(x) \geq 0 \quad \text{in } \Sigma_\lambda
\end{equation}
holds. One can conclude that the moving planes process can be carried on (with the property (3.23)) as long as the reflection of \( \Sigma_\lambda \) is still contained in \( \Omega \).

In fact, let \( T_{\lambda_0} \) be a plane such that (3.23) holds and the reflection of \( \Sigma_{\lambda_0} \) about \( T_{\lambda_0} \) is contained in \( \Omega \). Then there exists a \( \kappa > 0 \) such that, the reflection of \( \Sigma_{\lambda_0+\kappa} \) about \( T_{\lambda_0+\kappa} \) is still contained in \( \Omega \). By (3.6), (3.23) and strong maximum principles, one actually has
\begin{equation}
W^{\lambda_0}(x) > 0, \quad W_i^{\lambda_0}(x) > 0 \quad \text{in } \Sigma_{\lambda_0},
\end{equation}
thus there exists a constant \( c_\delta > 0 \) such that
\begin{equation}
W^{\lambda_0}(x) \geq c_\delta > 0, \quad W_i^{\lambda_0}(x) \geq c_\delta > 0 \quad \text{in } \Sigma_{\lambda_0-\frac{\delta}{2}}.
\end{equation}
By the continuity of \( u \), we infer that, there exists a \( 0 < \epsilon < \min\{\kappa, \frac{\delta}{2}\} \) such that, for any \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \),
\begin{equation}
W^\lambda(x) > 0, \quad W_i^\lambda(x) > 0 \quad \text{in } \Sigma_{\lambda_0-\frac{\delta}{2}}.
\end{equation}
Suppose there exists a \( \lambda_0 < \lambda \leq \lambda_0 + \epsilon \) such that
\begin{equation}
W^\lambda(x) < 0 \quad \text{somewhere in } \Sigma_\lambda \setminus \Sigma_{\lambda_0-\frac{\delta}{2}}.
\end{equation}
Let
\begin{equation}
\zeta(x) := \cos \left( \frac{x - x^0 - (\lambda_0 - \frac{\delta}{2}) \nu^0}{\delta} \right) \cdot \nu^0 \quad \text{and} \quad \overline{W}^\lambda(x) := \frac{W^\lambda(x)}{\zeta(x)}
\end{equation}
for \( x \in \Sigma_{\lambda} \setminus \Sigma_{\lambda_0 - \frac{\delta}{4}} \). Then there exists a \( x_0 \in \Sigma_{\lambda} \setminus \Sigma_{\lambda_0 - \frac{\delta}{4}} \) such that

\[
\tilde{W}_\lambda(x_0) = \min_{\Sigma_{\lambda} \setminus \Sigma_{\lambda_0 - \frac{\delta}{4}}} W_\lambda(x) < 0,
\]

by using similar arguments as proving (3.20), one can also arrive at

\[
W_\lambda(x_0) \geq p\delta_{2m} p^{-1} L_\infty(\Omega) W_\lambda(x_0),
\]

which contradicts with the choice of \( \delta \). Therefore, we have proved that

\[
W_\lambda(x) \geq 0 \quad \text{in} \quad \Sigma_{\lambda}
\]

for any \( \lambda \in (\lambda_0, \lambda_0 + \epsilon] \), that is, the plane \( T_\lambda \) can be moved forward a little bit from \( T_{\lambda_0} \).

Therefore, there exists a \( \delta_0 > 0 \) depending only on \( x_0 \) and \( \Omega \) such that, \( u(x) \) is monotone increasing along the internal normal direction in the region

\[
\Sigma_{\delta_0} := \{ x \in \overline{\Omega} \mid 0 \leq (x - x_0) \cdot \nu_0 \leq \delta_0 \}.
\]

Since \( \partial \Omega \) is \( C^{2m-2} \), there exists a small \( 0 < r_0 < \frac{\delta_0}{8} \) depending on \( x_0 \) and \( \Omega \) such that, for any \( x \in B_{r_0}(x_0) \cap \partial \Omega \), \( u(x) \) is monotone increasing along the internal normal direction at \( x \) in the region

\[
\Sigma_x := \{ z \in \overline{\Omega} \mid 0 \leq (z - x) \cdot \nu_x \leq \frac{3}{4} \delta_0 \}.
\]

where \( \nu_x \) denotes the unit internal normal vector at the point \( x \) \( (\nu_{x_0} := \nu_0) \). Since \( \Omega \) is strictly convex, there also exists a \( \theta > 0 \) depending on \( x_0 \) and \( \Omega \) such that

\[
I := \{ \nu \in \mathbb{R}^n \mid ||\nu|| = 1, \nu \cdot \nu_0 \geq \cos \theta \} \subset \{ \nu_x \mid x \in B_{r_0}(x_0) \cap \partial \Omega \},
\]

and hence, we have, for any \( x \in B_{r_0}(x_0) \cap \partial \Omega \) and \( \nu \in I \),

\[
u_{x_0} := \nu_0 \]

(3.35) \( u(x + s\nu) \) is monotone increasing with respect to \( s \in \left[ 0, \frac{\delta_0}{2} \right] \).

Let

\[
D := \{ x + r_0 \nu_0 \mid x \in B_{r_0}(x_0) \cap \partial \Omega \},
\]

one can easily verify that

\[
\max_{B_{r_0}(x_0) \cap \partial \Omega} u(x) \leq \max_D u(x).
\]

For any \( x \in \overline{D} \), let

\[
\overline{V}_x := \left\{ x + \nu \mid \nu \cdot \nu_0 \geq ||\nu|| \cos \theta, ||\nu|| \leq \frac{\delta_0}{4} \right\}
\]

be a piece of cone with vertex at \( x \), then it is easy to see that

\[
u_{x_0} := \nu_0 \]

(3.39) \( u(x) = \min_{z \in \overline{V}_x} u(z) \).

Now we need the following Lemma to control the integral of \( u \) on \( \overline{V}_x \).
Lemma 3.3. Let $\lambda_1$ be the first eigenvalue for $(-\Delta)^m$ in $\Omega$ with Navier boundary condition, and $0 < \phi \in C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega})$ be the corresponding eigenfunction (without loss of generality, we may assume $\|\phi\|_{L^\infty(\Omega)} = 1$), i.e.,

$$\begin{cases}
(-\Delta)^m \phi(x) = \lambda_1 \phi(x) & \text{in } \Omega,
\phi(x) = -\Delta \phi(x) = \cdots = (-\Delta)^{m-1} \phi(x) = 0 & \text{on } \partial \Omega.
\end{cases}$$

Then, we have

$$\int_{\Omega} u^p(x)\phi(x)dx \leq C(\lambda_1, p, |\Omega|).$$

Proof. Multiply both side of (1.5) by the eigenfunction $\phi(x)$ and integrate by parts, one gets

$$\int_{\Omega} u^p(x)\phi(x)dx \leq \int_{\Omega} (u^p(x) + t)\phi(x)dx = \int_{\Omega} (-\Delta)^m u(x) \cdot \phi(x)dx$$
$$= \int_{\Omega} u(x) \cdot (-\Delta)^m \phi(x)dx = \lambda_1 \int_{\Omega} u(x)\phi(x)dx.$$

By Hölder’s inequality, we have

$$\int_{\Omega} u^p(x)\phi(x)dx \leq \lambda_1 \left( \int_{\Omega} u^p(x)\phi(x)dx \right)^{\frac{1}{p}} \left( \int_{\Omega} \phi(x)dx \right)^{\frac{1}{p}},$$

and hence

$$\int_{\Omega} u^p(x)\phi(x)dx \leq \lambda_1^{\frac{1}{p}} \int_{\Omega} \phi(x)dx \leq \lambda_1^{\frac{1}{p}} |\Omega|.$$

This completes the proof of Lemma 3.3.

By (3.39) and Lemma 3.3 we see that, for any $x \in \overline{D}$,

$$C(\lambda_1, p, \Omega) \geq \int_{\Omega} u^p(x)\phi(x)dx \geq \int_{\overline{\Omega}} u^p(z)\phi(z)dz$$
$$\geq u^p(x)|\overline{\Omega}| \cdot \min_{\overline{\Omega}} \phi =: u^p(x) \cdot C(n, m, x^0, \Omega),$$

where $\overline{\Omega} := \{x \in \Omega | \text{dist}(x, \partial \Omega) \geq r_0\}$, and hence

$$u(x) \leq C(n, m, p, x^0, \lambda_1, \Omega), \quad \forall x \in \overline{D}.$$

Therefore, we arrive at

$$\max_{\overline{B_{r_0}(x^0) \cap \Omega}} u(x) \leq \max_{\overline{D}} u(x) \leq C(n, m, p, x^0, \lambda_1, \Omega).$$

Since $x^0 \in \partial \Omega$ is arbitrary and $\partial \Omega$ is compact, we can cover $\partial \Omega$ by finite balls $\{B_{r_k}(x^k)\}_{k=0}^K$ with centers $\{x^k\}_{k=0}^K \subset \partial \Omega$ ($K$ depends only on $\Omega$). Therefore, there exists a $\delta > 0$ depending only on $\Omega$ such that

$$\|(u)\|_{L^\infty(\Omega_{\delta})} \leq \max_{0 \leq k \leq K} \max_{B_{r_k}(x^k) \cap \Omega} u(x) \leq \max_{0 \leq k \leq K} C(n, m, p, x^k, \lambda_1, \Omega) =: C(n, m, p, \lambda_1, \Omega),$$

where the boundary layer $\overline{\Omega_{\delta}} := \{x \in \overline{\Omega} | \text{dist}(x, \partial \Omega) \leq \delta\}$. This completes the proof of boundary layer estimates under assumption i).

Case ii) $1 < p \leq \frac{n+2}{n-2}$. Under this assumption, we do not require the convexity of $\Omega$ anymore. Since $\partial \Omega$ is $C^{2m-2}$, there exists a $R_0 > 0$ depending only on $\Omega$ such that, for any
$x^0 \in \partial \Omega$, there exists a $\overline{x}^0$ satisfying $B_{R_0}(\overline{x}^0) \cap \Omega = \{x^0\}$. For any $x^0 \in \partial \Omega$, we define the Kelvin transform centered at $\overline{x}^0$ by

$$x \mapsto x^* := \frac{x - x^0}{|x - x^0|^2} + x^0, \quad \Omega \mapsto \Omega^* \subset B_{\frac{1}{R_0}}(\overline{x}^0),$$

and hence there exists a small $0 < \varepsilon_0 < \frac{1}{100R_0}$ depending on $x^0$ and $\Omega$ such that $B_{\varepsilon_0}((\overline{x}^0)^*) \cap \partial \Omega^*$ is strictly convex.

Now we define

$$\overline{u}(x^*) := \frac{1}{|x^* - x^0|^{n-2}} u \left( \frac{x^* - x^0}{|x^* - x^0|} + x^0 \right),$$

$$\overline{u}_i(x^*) := \frac{1}{|x^* - x^0|^{n-2}} u_i \left( \frac{x^* - x^0}{|x^* - x^0|} + x^0 \right)$$

for $i = 1, \ldots, m-1$. Then, we have

$$\overline{u}(x^*) > 0, \quad \overline{u}_i(x^*) > 0 \quad \text{in } \Omega^*, \quad \text{and from (1.5), we infer that } \overline{u}(x^*) \text{ and } \overline{u}_i(x^*) \text{ satisfy}$$

$$-\Delta \overline{u}_{m-1}(x^*) = \frac{1}{|x^* - x^0|^{n-2}} \overline{u}^{(m-1)}(x^*) + \frac{1}{|x^* - x^0|^{n+2}}, \quad x^* \in \Omega^*, \quad \text{for } i = 1, \ldots, m-1,$$

and $\overline{u}_i(x^*) = 0$ for $i = 1, \ldots, m-1$. Let $\nu^0$ be the unit internal normal vector of $\partial \Omega^*$ at $(x^0)^*$, we will show that $\overline{u}(x^*)$ is monotone increasing along the internal normal direction in the region

$$\Sigma_{\delta^*} = \left\{ x^* \in \Omega^* \mid 0 \leq (x^* - (x^0)^*) \cdot \nu^0 \leq \delta^* \right\},$$

where $\delta^* > 0$ depends only on $x^0$ and $\Omega$.

For this purpose, we define the moving plane by

$$T_\lambda^* := \left\{ x^* \in \mathbb{R}^n \mid (x^* - (x^0)^*) \cdot \nu^0 = \lambda \right\},$$

and denote

$$\Sigma_\lambda^* := \left\{ x^* \in \Omega^* \mid 0 < (x^* - (x^0)^*) \cdot \nu^0 < \lambda \right\}$$

for $\lambda > 0$, and let $x^*_\lambda$ be the reflection of the point $x^*$ about the plane $T_\lambda^*$. Define

$$U^\lambda(x^*) := \overline{u}(x^*_\lambda) - \overline{u}(x^*) \quad \text{and} \quad U^\lambda_i(x^*) := \overline{u}_i(x^*_\lambda) - \overline{u}_i(x^*)$$
for $1 \leq i \leq m - 1$. Then we can deduce from (3.51) that, for any $\lambda$ satisfying the reflection of $\Sigma^*_\lambda$ is contained in $\Omega^*$,

$$
\begin{align*}
-\Delta U^\lambda_{m-1}(x^*) &= \frac{\nabla^2(x^*)}{|x^* - x^0|^m} - \frac{\nabla^2(x^*)}{|x^* - x^0|^m} + \frac{t}{|x^* - x^0|^{n+2}} - \frac{t}{|x^* - x^0|^{n+2}}, \quad x^* \in \Sigma^*_\lambda, \\
-\Delta U^\lambda_{m-2}(x^*) &= \frac{\nabla^2(x^*)}{|x^* - x^0|^m} - \frac{\nabla^2(x^*)}{|x^* - x^0|^m}, \quad x^* \in \Sigma^*_\lambda, \\
& \quad \ldots \ldots \\
-\Delta U^\lambda(x^*) &= \frac{\nabla^2(x^*)}{|x^* - x^0|^m} - \frac{\nabla^2(x^*)}{|x^* - x^0|^m}, \quad x^* \in \Sigma^*_\lambda, \\
U^\lambda(x^*) &\geq 0, \quad U^\lambda_1(x^*) \geq 0, \quad \ldots, U^\lambda_{m-1}(x^*) \geq 0, \quad x^* \in \partial \Sigma^*_\lambda.
\end{align*}
$$

(3.56)

Notice that for any $x^* \in \Sigma^*_\lambda$ with $\lambda < \frac{1}{R_0}$, one has

$$
0 < |x^* - x^0| < |x^* - \overline{x}^0| < \frac{1}{R_0},
$$

(3.57)

and hence, by direct calculations, it follows from (3.56) and $t \geq 0$ that

$$
\begin{align*}
-\Delta U^\lambda_{m-1}(x^*) &\geq \frac{p R_0^p \xi^p}{|x^* - x^0|^m} U^\lambda(x^*) \geq p R_0^p \xi^p \cdot (x^*) U^\lambda(x^*), \quad x^* \in \Sigma^*_\lambda, \\
-\Delta U^\lambda_{m-2}(x^*) &\geq R_0^4 U^\lambda_{m-1}(x^*), \quad x^* \in \Sigma^*_\lambda, \\
& \quad \ldots \ldots \\
-\Delta U^\lambda(x^*) &\geq R_0^4 U^\lambda_1(x^*), \quad x^* \in \Sigma^*_\lambda, \\
U^\lambda(x^*) &\geq 0, \quad U^\lambda_1(x^*) \geq 0, \quad \ldots, \quad U^\lambda_{m-1}(x^*) \geq 0, \quad x^* \in \partial \Sigma^*_\lambda.
\end{align*}
$$

(3.58)

where $\xi^\lambda(x^*)$ is valued between $\overline{\nabla}(x^*_\lambda)$ and $\overline{\nabla}(x^*)$ by mean value theorem, and thus

$$
\|\xi^\lambda\|_{L^\infty(\Sigma^*_\lambda)} \leq (diam \Omega + R_0)^{n-2} \|u\|_{L^\infty(\Omega)}.
$$

(3.59)

Now, we will prove that there exists some $\delta > 0$ sufficiently small (depending on $m$, $p$, $\|u\|_{L^\infty(\Omega)}$ and $\Omega$), such that

$$
U^\lambda(x^*) \geq 0 \quad \text{in} \quad \Sigma^*_\lambda
$$

for all $0 < \lambda \leq \delta$. This provides a starting point to move the plane $T^*_\lambda$.

In fact, suppose on the contrary that there exists a $0 < \lambda \leq \delta$ such that

$$
U^\lambda(x^*) < 0 \quad \text{somewhere in} \quad \Sigma^*_\lambda.
$$

(3.61)

Let

$$
\psi(x^*) := \cos \left( \frac{(x^* - (x^0)^*) \cdot \nu^0}{\delta} \right),
$$

(3.62)

then $\psi(x^*) \in [\cos 1, 1]$ for any $x^* \in \Sigma^*_\lambda$ and $-\Delta \psi = \frac{1}{\delta^2}$. Define

$$
\overline{U}^\lambda(x^*) := \frac{U^\lambda(x^*)}{\psi(x^*)} \quad \text{and} \quad \overline{U}^\lambda_i(x^*) := \frac{U^\lambda_i(x^*)}{\psi(x^*)}
$$

for $i = 1, \ldots, m - 1$ and $x^* \in \Sigma^*_\lambda$. Then there exists a $x^*_0 \in \Sigma^*_\lambda$ such that

$$
\overline{U}^\lambda(x^*_0) = \min_{\Sigma^*_\lambda} \overline{U}^\lambda(x^*) < 0.
$$

(3.64)
Since
\[ -\Delta U^\lambda(x_0^*) = -\Delta U^\lambda(x_0^*) \psi(x_0^*) - 2\nabla U^\lambda(x_0^*) \cdot \nabla \psi(x_0^*) - U^\lambda(x_0^*) \Delta \psi(x_0^*), \]
one immediately has
\[ R_0^4 U_1^\lambda(x_0^*) \leq -\Delta U^\lambda(x_0^*) \leq \frac{1}{\delta^2} U^\lambda(x_0^*) < 0. \]
Thus there exists a \( x_1^* \in \Sigma_\lambda^* \) such that
\[ \overline{U_1^\lambda}(x_1^*) = \min_{\Sigma_\lambda^*} \overline{U_1^\lambda}(x^*) < 0. \]
Similarly, it follows that
\[ R_0^4 U_2^\lambda(x_1^*) \leq -\Delta U_1^\lambda(x_1^*) \leq \frac{1}{\delta^2} U_1^\lambda(x_1^*) < 0. \]
Continuing this way, we get \( \{x_i^*\}_{i=1}^{m-1} \subset \Sigma_\lambda^* \) such that
\[ \overline{U_i^\lambda}(x_i^*) = \min_{\Sigma_\lambda^*} \overline{U_i^\lambda}(x^*) < 0, \]
\[ R_0^4 U_{i+1}^\lambda(x_i^*) \leq -\Delta U_i^\lambda(x_i^*) \leq \frac{1}{\delta^2} U_i^\lambda(x_i^*) < 0 \]
for \( i = 1, 2, \ldots, m-2 \), and
\[ \overline{U_{m-1}^\lambda}(x_{m-1}) = \min_{\Sigma_\lambda^*} \overline{U_{m-1}^\lambda}(x^*) < 0, \]
\[ p R_0^\tau \xi_\lambda^{p-1}(x_{m-1}^*) U^\lambda(x_{m-1}^*) \leq -\Delta U_{m-1}^\lambda(x_{m-1}^*) \leq \frac{1}{\delta^2} U_{m-1}^\lambda(x_{m-1}^*) < 0. \]
Therefore, we have
\[ U^\lambda(x_0^*) \geq (\delta R_0^2)^2 U_1^\lambda(x_1^*) \geq (\delta R_0^2)^2 U_1^\lambda(x_1^*) \frac{\psi(x_0^*)}{\psi(x_1^*)} \]
\[ \geq (\delta R_0^2)^4 U_2^\lambda(x_1^*) \frac{\psi(x_0^*)}{\psi(x_1^*)} \geq (\delta R_0^2)^4 U_2^\lambda(x_2^*) \frac{\psi(x_0^*)}{\psi(x_2^*)} \]
\[ \geq (\delta R_0^2)^6 U_3^\lambda(x_2^*) \frac{\psi(x_0^*)}{\psi(x_2^*)} \geq (\delta R_0^2)^6 U_3^\lambda(x_3^*) \frac{\psi(x_0^*)}{\psi(x_3^*)} \]
\[ \geq \cdots \cdots \geq (\delta R_0^2)^{2m-2} U_{m-1}^\lambda(x_{m-1}^*) \frac{\psi(x_0^*)}{\psi(x_{m-1}^*)} \]
\[ \geq p\delta^{2m} R_0^{4m-(p-1)(n-2)} \xi_\lambda^{p-1}(x_{m-1}^*) U^\lambda(x_{m-1}^*) \frac{\psi(x_0^*)}{\psi(x_{m-1}^*)} \]
\[ \geq p\delta^{2m} R_0^{4m-(p-1)(n-2)} (\text{diam } \Omega + R_0)^{(p-1)(n-2)} \|u\|_{L^\infty(\Omega)}^{p-1} U^\lambda(x_0^*), \]
that means,
\[ 1 \leq p\delta^{2m} (\text{diam } \Omega + R_0)^{4m} \|u\|_{L^\infty(\Omega)}^{p-1}, \]
which is absurd if we choose \( \delta > 0 \) small enough such that
\[ 0 < \delta < (\text{diam } \Omega + R_0)^{-2} \left( p\|u\|_{L^\infty(\Omega)}^{p-1} \right)^{-\frac{1}{2m}}. \]
So far, we have proved that the plane \( T^*_\lambda \) can be moved on up to \( \lambda = \delta \).

Next, we will move the plane \( T^*_\lambda \) further along the internal normal direction at \((x^0)^*\) as long as the property

\[
U^\lambda(x^*) \geq 0 \quad \text{in } \Sigma^*_\lambda
\]

holds. Completely similar to the proof of Case i), one can actually show that the method of moving planes can be carried on (with the property (3.76)) as long as the reflection of \( \Sigma^*_\lambda \) is still contained in \( \Omega^* \). We omit the details here.

Therefore, there exists a \( \delta_* > 0 \) depending only on \( x^0 \) and \( \Omega \) such that, \( \overline{u}(x^*) \) is monotone increasing along the internal normal direction in the region

\[
\Sigma_{\delta_*} := \left\{ x^* \in \overline{\Omega}^* \mid 0 \leq (x^* - (x^0)^*) \cdot \nu_0 \leq \delta_* \right\}.
\]

Since \( \partial \Omega^* \) is \( C^{2m-2} \), there exists a small \( 0 < \varepsilon_1 < \min\{\delta_/8, \varepsilon_0\} \) depending on \( x^0 \) and \( \Omega \) such that, for any \( x^* \in B_{\varepsilon_1}((x^0)^*) \cap \partial \Omega^* \), \( \overline{u}(x^*) \) is monotone increasing along the internal normal direction at \( x^* \) in the region

\[
\Sigma_\theta^* := \left\{ z^* \in \overline{\Omega}^* \mid 0 \leq (z^* - x^*) \cdot \nu_{x^*} \leq \frac{\delta_*}{4} \right\}.
\]

where \( \nu_{x^*} \) denotes the unit internal normal vector at the point \( x^* \) \( (\nu_{(x^0)^*} := \nu^0) \). Since \( B_{\varepsilon_1}((x^0)^*) \cap \partial \Omega^* \) is strictly convex, there exists a \( \theta > 0 \) depending on \( x^0 \) and \( \Omega \) such that

\[
S := \{ \nu^* \in \mathbb{R}^n \mid \nu^* = 1, \nu^* \cdot \nu^0 \geq \cos \theta \} \subset \{ \nu_{x^*} \mid x^* \in B_{\varepsilon_1}((x^0)^*) \cap \partial \Omega^* \},
\]

and hence, it follows that, for any \( x^* \in B_{\varepsilon_1}((x^0)^*) \cap \partial \Omega^* \) and \( \nu^* \in S \),

\[
\overline{u}(x^* + s\nu^*) \quad \text{is monotone increasing with respect to } s \in \left[ 0, \frac{\delta_*}{2} \right].
\]

Now, let

\[
D^* := \left\{ x^* + \varepsilon_1 \nu^0 \mid x^* \in B_{\varepsilon_1}((x^0)^*) \cap \partial \Omega^* \right\},
\]

one immediately has

\[
\max_{B_{\varepsilon_1}((x^0)^*) \cap \partial \Omega} \overline{u}(x^*) \leq \max_{D^*} \overline{u}(x^*).
\]

For any \( x^* \in \overline{D^*} \), let

\[
\overline{V}_{x^*} := \left\{ x^* + \nu^* \mid \nu^* \cdot \nu^0 \geq |\nu^*| \cos \theta, \ |\nu^*| \leq \frac{\delta_*}{4} \right\}
\]

be a piece of cone with vertex at \( x^* \), then it is obvious that

\[
\overline{u}(x^*) = \min_{z^* \in \overline{V}_{x^*}} \overline{u}(z^*).
\]
Therefore, by (3.84) and Lemma 3.3, we get, for any $x^* \in \overline{D}$,

\begin{equation}
C(\lambda_1, p, \Omega) \geq \int_{\Omega} u^p(x) \phi(x) dx
\end{equation}

\begin{equation}
= \int_{\Omega^*} \frac{\overline{\nabla}^p(x^*)}{|x^* - x^0|^{2n - p(n - 2)}} \phi \left( \frac{x^* - x^0}{|x^* - x^0|^2} + x^0 \right) dx^*
\end{equation}

\begin{equation}
\geq \int_{\Omega^*} \frac{\overline{\nabla}^p(z^*)}{|z^* - x^0|^{2n - p(n - 2)}} \phi \left( \frac{z^* - x^0}{|z^* - x^0|^2} + x^0 \right) dz^*
\end{equation}

\begin{equation}
\geq \overline{\nabla}^p(x^*) R_0^{2n - p(n - 2)} |V_{x^*}^*| \cdot \min_{\overline{\Omega}} \phi =: \overline{\nabla}^p(x^*) \cdot C(n, m, p, x^0, \Omega),
\end{equation}

where $\overline{\Omega} := \{ x \in \Omega \mid dist(x, \partial \Omega) \geq r_1 \}$ with $r_1 = \varepsilon_1 R_0^2$, and hence

\begin{equation}
\overline{\nabla}(x^*) \leq C(n, m, p, x^0, \lambda_1, \Omega), \quad \forall x \in \overline{D}.
\end{equation}

As a consequence, we derive that

\begin{equation}
\max_{B_{r_1}(x^0) \cap \Omega^*} \overline{\nabla}(x^*) = \max_{\overline{\Omega}} \overline{\nabla}(x^*) \leq C(n, m, p, x^0, \lambda_1, \Omega).
\end{equation}

There exists a small $r_0 > 0$ depending only on $x^0$ and $\Omega$ such that, for each $x \in B_{r_0}(x^0) \cap \overline{\Omega}$, one has $x^* \in B_{r_1}((x^0)^*) \cap \overline{\Omega}^*$. Therefore, (3.87) yields

\begin{equation}
\max_{B_{r_0}(x^0) \cap \Omega} u(x) = \max_{x \in B_{r_0}(x^0) \cap \overline{\Omega}} |x^* - x^0|^{n-2} \overline{\nabla}(x^*)
\end{equation}

\begin{equation}
\leq \frac{1}{R_0^{n-2}} \max_{B_{r_1}(x^0) \cap \overline{\Omega}} \overline{\nabla}(x^*) \leq C(n, m, p, x^0, \lambda_1, \Omega).
\end{equation}

Since $x^0 \in \partial \Omega$ is arbitrary and $\partial \Omega$ is compact, we can cover $\partial \Omega$ by finite balls $\{ B_{r_k}(x^k) \}_{k=0}^K$ with centers $\{ x^k \}_{k=0}^K \subset \partial \Omega$ ($K$ depends only on $\Omega$). Therefore, there exists a $\delta > 0$ depending only on $\Omega$ such that

\begin{equation}
\| u \|_{L^\infty(\overline{\Omega}_\delta)} \leq \max_{0 \leq k \leq K} \max_{B_{r_k}(x^k) \cap \overline{\Omega}} u(x) \leq \max_{0 \leq k \leq K} C(n, m, p, x^k, \lambda_1, \Omega) =: C(n, m, p, \lambda_1, \Omega),
\end{equation}

where the boundary layer $\overline{\Omega}_\delta := \{ x \in \overline{\Omega} \mid dist(x, \partial \Omega) \leq \delta \}$. This completes the proof of boundary layer estimates under assumption ii).

This concludes our proof of Theorem 3.1. 

3.2. Blowing-up analysis and interior estimates. In this subsection, we will obtain the interior estimates (and hence, global a priori estimates) via the blowing-up analysis arguments (for related literatures on blowing-up methods, please refer to [1, 3, 5, 11, 12, 20, 33, 44]).

Suppose on the contrary that Theorem 1.6 does not hold. By the boundary layer estimates (Theorem 3.1), there exists a sequence of positive solutions $\{ u_k \} \subset C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega})$ to the higher order Navier problem (1.5) and a sequence of interior points $\{ x^k \} \subset \Omega \setminus \overline{\Omega}_\delta$ such that

\begin{equation}
m_k := u_k(x^k) = \| u_k \|_{L^\infty(\overline{\Omega}_\delta)} \to +\infty \quad \text{as} \quad k \to \infty.
\end{equation}
For $x \in \Omega_k := \{ x \in \mathbb{R}^n \mid \lambda_k x + x^k \in \Omega \}$, we define

$$ v_k(x) := \frac{1}{m_k} u_k(\lambda_k x + x^k) \quad \text{with} \quad \lambda_k := m_k^{-\frac{1}{m_k}} \to 0 \quad \text{as} \quad k \to \infty. $$

Then $v_k(x)$ satisfies $\|v_k\|_{L^\infty(\Omega_k)} = v_k(0) = 1$ and

$$ (-\Delta)^m v_k(x) = \frac{1}{m_k} \lambda_k^{2m} (-\Delta)^m u_k(\lambda_k x + x^k) $$

$$ = \frac{1}{m_k} \lambda_k^{2m} \left( u_k^p(\lambda_k x + x^k) + t \right) = v_k^p(x) + \frac{t}{m_k} $$

for any $x \in \Omega_k$. Since $\text{dist}(x^k, \partial \Omega) > \tilde{\delta}$, one has

$$ \Omega_k \supset \{ x \in \mathbb{R}^n \mid |\lambda_k x| \leq \tilde{\delta} \} = B_{\frac{\tilde{\delta}}{\lambda_k}}(0), $$

and hence

$$ \Omega_k \to \mathbb{R}^n \quad \text{as} \quad k \to \infty. $$

For arbitrary $x^0 \in \mathbb{R}^n$, there exists a $N_1 > 0$, such that $B_1(x^0) \subset \Omega_k$ for any $k \geq N_1$. By (3.92) and $\|v_k\|_{L^\infty(\Omega_k)} \leq 1$, we can infer from regularity theory and Sobolev embedding that

$$ \|v_k\|_{C^{2m-1,\gamma}(B_1(0))} \leq C(1 + t), $$

and further that

$$ \|v_k\|_{C^{2m-1,\gamma}(B_{\frac{\tilde{\delta}}{\lambda_k}}(0))} \leq C(1 + t) $$

for $k \geq N_1$, where $0 \leq \gamma < 1$. As a consequence, by Arzelà-Ascoli Theorem, there exists a subsequence $\{v^{(1)}_k\} \subset \{v_k\}$ and a function $v \in C^{2m}(\overline{B_1(x^0)})$ such that

$$ v^{(1)}_k \rightharpoonup v \quad \text{and} \quad (-\Delta)^m v^{(1)}_k \rightharpoonup (-\Delta)^m v \quad \text{in} \quad B_1(x^0). $$

There also exists a $N_2 > 0$ such that $B_2(x^0) \subset \Omega_k$ for any $k \geq N_2$. By (3.92) and $\|v_k\|_{L^\infty(\Omega_k)} \leq 1$, we can deduce that

$$ \|v^{(1)}_k\|_{C^{2m+1,\gamma}(B_2(0))} \leq C(1 + t) $$

for $k \geq N_2$, where $0 \leq \gamma < 1$. Therefore, by Arzelà-Ascoli Theorem again, there exists a subsequence $\{v^{(2)}_k\} \subset \{v^{(1)}_k\}$ and $v \in C^{2m}(\overline{B_2(x^0)})$ such that

$$ v^{(2)}_k \rightharpoonup v \quad \text{and} \quad (-\Delta)^m v^{(2)}_k \rightharpoonup (-\Delta)^m v \quad \text{in} \quad B_2(x^0). $$

Continuing this way, for any $j \in \mathbb{N}^+$, we can extract a subsequence $\{v^{(j)}_k\} \subset \{v^{(j-1)}_k\}$ and find a function $v \in C^{2m}(\overline{B_j(x^0)})$ such that

$$ v^{(j)}_k \rightharpoonup v \quad \text{and} \quad (-\Delta)^m v^{(j)}_k \rightharpoonup (-\Delta)^m v \quad \text{in} \quad B_j(x^0). $$

By extracting the diagonal sequence, we finally obtain that the subsequence $\{v^{(k)}_k\}$ satisfies

$$ v^{(k)}_k \rightharpoonup v \quad \text{and} \quad (-\Delta)^m v^{(k)}_k \rightharpoonup (-\Delta)^m v \quad \text{in} \quad B_j(x^0) $$

for any $j \geq 1$. Therefore, we get from (3.92) that $0 \leq v \in C^{2m}(\mathbb{R}^n)$ satisfies

$$ (-\Delta)^m v(x) = v^p(x) \quad \text{in} \quad \mathbb{R}^n. $$
By the Liouville theorem (Theorem 1.1), we must have \( v \equiv 0 \) in \( \mathbb{R}^n \), which is a contradiction with
\[
(3.103) \quad v(0) = \lim_{k \to \infty} v_k^{(k)}(0) = 1.
\]
This concludes our proof of Theorem 1.6.

### 4. Proof of Theorem 1.7

In this section, by applying the a priori estimates (Theorem 1.5 and Theorem 1.6) and the following Leray-Schauder fixed point theorem (see e.g. [18, 5]), we will prove the existence of positive solutions to the higher order Lane-Emden equations (1.6) with Navier boundary conditions.

**Theorem 4.1.** Suppose that \( X \) is a real Banach space with a closed positive cone \( P \), \( U \subset P \) is bounded open and contains \( 0 \). Assume that there exists \( \rho > 0 \) such that \( B_{\rho}(0) \cap P \subset U \) and that \( T: \overline{U} \to P \) is compact and satisfies
i) For any \( x \in P \) with \( |x| = \rho \) and any \( \lambda \in [0, 1) \), \( x \neq \lambda Tx \);
ii) There exists some \( y \in P \setminus \{0\} \) such that \( x - Tx \neq ty \) for any \( t \geq 0 \) and \( x \in \partial U \).

Then, \( T \) possesses a fixed point in \( \overline{U}_\rho \), where \( U_\rho := U \setminus B_{\rho}(0) \).

Now we let
\[
(4.1) \quad X := C^0(\Omega) \quad \text{and} \quad P := \{ u \in X \mid u \geq 0 \}.
\]
Define
\[
(4.2) \quad T(u)(x) := \int_\Omega G_2(x, y^m) \int_\Omega G_2(y^m, y^{m-1}) \int_\Omega \cdots \int_\Omega G_2(y^2, y^1) u^p(y^1) dy^1 dy^2 \cdots dy^m,
\]
where \( G_2(x, y) \) is the Green’s function for \( -\Delta \) with Dirichlet boundary condition in \( \Omega \).

Suppose \( u \in C^0(\Omega) \) is a fixed point of \( T \), i.e., \( u = Tu \), then it is easy to see that \( u \in C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega}) \) and satisfies the Navier problem
\[
(4.3) \quad \begin{cases}
(\Delta)^mu(x) = u^p(x) \quad \text{in} \ \Omega, \\
u(x) = -\Delta u(x) = \cdots = (\Delta)^{m-1}u(x) = 0 \quad \text{on} \ \partial\Omega.
\end{cases}
\]

Our goal is to show the existence of a fixed point for \( T \) in \( P \setminus B_{\rho}(0) \) for some \( \rho > 0 \) (to be determined later) by using Theorem 4.1. To this end, we need to verify the two conditions i) and ii) in Theorem 4.1 separately.

i) First, we show that there exists \( \rho > 0 \) such that for any \( u \in \partial B_{\rho}(0) \cap P \) and \( 0 \leq \lambda < 1 \),
\[
(4.4) \quad u - \lambda Tu \neq 0.
\]
For any \( x \in \overline{\Omega} \), it holds that
\[
|T(u)(x)| = \left| \int_\Omega G_2(x, y^m) \int_\Omega G_2(y^m, y^{m-1}) \cdots \int_\Omega G_2(y^2, y^1) u^p(y^1) dy^1 \cdots dy^m \right|
\leq \int_\Omega G_2(x, y^m) \int_\Omega G_2(y^m, y^{m-1}) \cdots \int_\Omega G_2(y^2, y^1) dy^1 \cdots dy^m \cdot \|u\|^p_{C^0(\overline{\Omega})}
\leq \rho^{p-1} \left( \int_\Omega G_2(x, y) dy \right)^m_{C^0(\overline{\Omega})} \cdot \|u\|_{C^0(\overline{\Omega})}.
\]
Let $g(x) := \int_\Omega G_2(x, y)dy$, then it solves
\begin{equation}
\begin{cases}
-\Delta_x g(x) = 1 & \text{in } \Omega, \\
g(x) = 0 & \text{on } \partial\Omega.
\end{cases}
\end{equation}

For a fixed point $x^0 \in \Omega$, we define the function
\begin{equation}
\beta(x) := \left(\frac{\text{diam } \Omega}{2n}\right)^2 \left(1 - \frac{|x - x^0|^2}{(\text{diam } \Omega)^2}\right)_+,
\end{equation}
then it satisfies
\begin{equation}
\begin{cases}
-\Delta_x \beta(x) = 1 & \text{in } \Omega, \\
\beta(x) > 0 & \text{on } \partial\Omega.
\end{cases}
\end{equation}

By maximum principle, we get
\begin{equation}
0 \leq g(x) < \beta(x) \leq \left(\frac{\text{diam } \Omega}{2n}\right)^2, \quad \forall \, x \in \overline{\Omega}.
\end{equation}

Therefore, we infer from (4.5) and (4.9) that
\begin{equation}
\|T(u)\|_{C^0(\overline{\Omega})} < \rho^{p-1} \frac{(\text{diam } \Omega)^{2m}}{(2n)^m} \|u\|_{C^0(\overline{\Omega})} = \|u\|_{C^0(\overline{\Omega})}
\end{equation}
if we take
\begin{equation}
\rho = \left(\frac{\sqrt{2n}}{\text{diam } \Omega}\right)^{\frac{2m}{p-1}} > 0.
\end{equation}

This implies that $u \neq \lambda T(u)$ for any $u \in \partial B_\rho(0) \cap P$ and $0 \leq \lambda < 1$.

\textbf{ii)} Now, let $\varphi \in C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega})$ be the unique positive solution of
\begin{equation}
\begin{cases}
(-\Delta)^m \varphi(x) = 1, & x \in \Omega, \\
\varphi(x) = -\Delta \varphi(x) = \cdots = (-\Delta)^{m-1} \varphi(x) = 0, & x \in \partial\Omega.
\end{cases}
\end{equation}
We will show that
\begin{equation}
 u - T(u) \neq t\varphi \quad \forall \, t \geq 0, \quad \forall u \in \partial U,
\end{equation}
where $U := B_R(0) \cap P$ with sufficiently large $R > \rho$ (to be determined later). First, observe that for any $u \in \overline{U}$,
\begin{equation}
\|(-\Delta)^m T(u)\|_{C^0(\overline{\Omega})} = \|u\|^p_{C^0(\overline{\Omega})} \leq R^p,
\end{equation}
and hence
\begin{equation}
\|T(u)\|_{C^{0,\alpha}(\Omega)} \leq CR^p \quad \forall \, 0 < \alpha < 1,
\end{equation}
thus $T : \overline{U} \to P$ is compact.

We use contradiction arguments to prove (4.13). Suppose on the contrary that, there exists some $u \in \partial U$ and $t \geq 0$ such that
\begin{equation}
 u - T(u) = t\varphi,
\end{equation}
then one has \( \|u\|_{C^0(\Omega)} = R > \rho > 0, \) \( u \in C^{2m}(\Omega) \cap C^{2m-2}(\Omega) \) and satisfies the Navier problem

\[
\begin{cases}
(-\Delta)^m u(x) = u^p(x) + t, & u(x) > 0, \quad x \in \Omega, \\
u(x) = -\Delta u(x) = \cdots = (-\Delta)^{m-1}u(x) = 0, & x \in \partial \Omega.
\end{cases}
\]

Choose a constant \( C_1 > \lambda_1 \). Since \( u(x) > 0 \) in \( \Omega \) and \( p > 1 \), it is easy to see that, there exists another constant \( C_2 > 0 \) (e.g., take \( C_2 = C_1^{\frac{p}{p-1}} \)), such that
\[
u^p(x) \geq C_1 u(x) - C_2.
\]

If \( t \geq C_2 \), then we have
\[
(-\Delta)^m u(x) = u^p(x) + t \geq C_1 u(x) - C_2 + t \geq C_1 u(x) \quad \text{in} \ \Omega.
\]
Multiplying both side of (4.19) by the eigenfunction \( \phi(x) \), and integrating by parts yield
\[
C_1 \int_\Omega u(x)\phi(x)dx \leq \int_\Omega (-\Delta)^m u(x) \cdot \phi(x)dx = \int_\Omega u(x) \cdot (-\Delta)^m \phi(x)dx
\]
\[
= \lambda_1 \int_\Omega u(x)\phi(x)dx,
\]
and hence
\[
0 < (C_1 - \lambda_1) \int_\Omega u(x)\phi(x)dx \leq 0,
\]
which is absurd. Thus, we must have \( 0 \leq t < C_2 \). Next, we carry on our proof by discussing two different assumptions.

If \( \frac{n}{n-2m} < p < \frac{n+2m}{n-2m} \), by the a priori estimates (Theorem 1.5), we derive that
\[
\|u\|_{L^\infty(\Omega)} \leq C(n, m, p, \Omega) =: C'_0.
\]

If \( \Omega \) is strictly convex, \( 1 < p < \frac{n+2m}{n-2m} \), or if \( 1 < p \leq \frac{n+2}{n-2} \), by the a priori estimates (Theorem 1.6), we know that
\[
\|u\|_{L^\infty(\Omega)} \leq C(n, m, p, t, \lambda_1, \Omega).
\]
We will show that the above a priori estimates (4.23) are uniform with respect to \( 0 \leq t < C_2 \), i.e., for \( 0 \leq t < C_2 \),
\[
\|u\|_{L^\infty(\Omega)} \leq C(n, m, p, t, \lambda_1, \Omega) =: C''_0.
\]
Indeed, it is clear from Theorem 3.1 that, the thickness \( \bar{\delta} \) of the boundary layer and the boundary layer estimates are uniform with respect to \( t \). Therefore, if (4.24) does not hold, there exist sequences \( \{t_k\} \subset [0, C_2] \), \( \{x^k\} \subset \Omega \setminus \overline{\Omega}_\delta \) and \( \{u_k\} \) satisfying
\[
\begin{cases}
(-\Delta)^m u_k(x) = u_k^p(x) + t_k, & x \in \Omega, \\
u_k(x) = -\Delta u_k(x) = \cdots = (-\Delta)^{m-1}u_k(x) = 0, & x \in \partial \Omega,
\end{cases}
\]
but \( m_k := u_k(x^k) = \|u_k\|_{L^\infty(\Omega)} \to +\infty \) as \( k \to \infty \). For \( x \in \Omega_k := \{ x \in \mathbb{R}^n \mid \lambda_k x + x^k \in \Omega \} \), we define \( v_k(x) := \frac{1}{m_k} u_k(\lambda_k x + x^k) \) with \( \lambda_k := m_k^{\frac{1-p}{p}} \to 0 \) as \( k \to \infty \). Then \( v_k(x) \) satisfies
\[
\|v_k\|_{L^\infty(\Omega_k)} = v_k(0) = 1 \quad \text{and}
\]
\[
(-\Delta)^m v_k(x) = v_k^p(x) + \frac{t_k}{m_k^p}
\]
for any \( x \in \Omega_k \). Since \( 0 \leq t < C_2 \) and \( m_k \to +\infty \), by completely similar blowing-up methods as in the proof of Theorem 1.6 in subsection 3.2, we can also derive a subsequence \( \{v_k^{(k)}\} \subset \{v_k\} \) and a function \( v \in C^{2m}(\mathbb{R}^n) \) such that
\[

(4.27) \quad \lim_{k \to \infty} v_k^{(k)} = v \quad \text{and} \quad (-\Delta)^m v_k^{(k)} = (-\Delta)^m v \quad \text{in} \quad B_j(x^0)

\]
for arbitrary \( j \geq 1 \), and hence \( 0 \leq v \in C^{2m}(\mathbb{R}^n) \) solves
\[

(4.28) \quad (-\Delta)^m v(x) = v^p(x) \quad \text{in} \quad \mathbb{R}^n.

\]
By Theorem 1.1, one immediately has \( v \equiv 0 \), which contradicts with \( v(0) = 1 \). Therefore, the uniform estimates (4.24) must hold.

Now we let
\[
C := \max\{C'_0, C''_0\} > 0 \quad \text{and} \quad R := C_0 + \rho \quad \text{and} \quad U := B_{C_0 + \rho}(0) \cap P,
\]
then (4.22) and (4.24) implies
\[

(4.29) \quad \|u\|_{L^\infty(\overline{\Omega})} \leq C_0 < C_0 + \rho,

\]
which contradicts with \( u \in \partial U \). This implies that
\[

(4.30) \quad u - T(u) \neq t\varphi

\]
for any \( t \geq 0 \) and \( u \in \partial U \) with \( U = B_{C_0 + \rho}(0) \cap P \).

From Theorem 4.1, we deduce that there exists a \( u \in \{B_{C_0 + \rho}(0) \cap P \} \setminus B_\rho(0) \) satisfies
\[

(4.31) \quad u = T(u),

\]
and hence \( \rho \leq \| u \|_{L^\infty(\overline{\Omega})} \leq C_0 + \rho \) solves the higher order Navier problem
\[

(4.32) \quad \begin{cases} \quad (-\Delta)^m u(x) = v^p(x), & u(x) > 0, \quad x \in \Omega, \\ \quad u(x) = -\Delta u(x) = \cdots = (-\Delta)^{m-1} u(x) = 0, & x \in \partial \Omega. \end{cases}

\]
By regularity theory, we can see that \( u \in C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega}) \).

This concludes our proof of Theorem 1.7.

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