On a choice of the Bondi radial coordinate and news function for the axisymmetric two-body problem.

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Abstract

In the Bondi formulation of the axisymmetric vacuum Einstein equations, we argue that the “surface area” coordinate condition determining the “radial” coordinate can be considered as part of the initial data and should be chosen in a way that gives information about the physical problem whose solution is sought. For the two-body problem, we choose this coordinate by imposing a condition that allows it to be interpreted, near infinity, as the (inverse of the) Newtonian potential. In this way, two quantities that specify the problem – the separation of the two particles and their mass ratio – enter the equations from the very beginning. The asymptotic solution (near infinity) is obtained and a natural identification of the Bondi “news function” in terms of the source parameters is suggested, leading to an expression for the radiated energy that differs from the standard quadrupole formula but agrees with recent non-linear calculations. When the free function of time describing the separation of the two particles is chosen so as to make the new expression agree with the classical result, closed-form analytic expressions are obtained, the resulting metric approaching the Schwarzschild solution with time. As all physical quantities are defined with respect to the flat metric at infinity, the physical interpretation of this solution depends strongly on how these definitions are extended to the near-zone and, in particular, how the “time” function in the near-zone is related to Bondi’s null coordinate.

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I. INTRODUCTION

The characteristic initial value formulation \([1]\) of the axisymmetric vacuum Einstein equations, first proposed by Bondi and co-workers \([2]\), has two distinct advantages over the initial value formulation in terms of a space-like hypersurface \([3]\): (i) there are no constraint equations to be satisfied by the initial data and (ii) the unknown functions can be determined, one at a time, by performing an hierarchical series of quadratures over the initial data and “known” quantities (including functions of integration from previous quadratures) \([2, 4, 5]\). Thus there are no partial differential equations in three independent variables to be solved!

This is achieved by writing the equations in terms of coordinates adapted to a series of null hypersurfaces \(u = \text{const}\), one of which is the initial hypersurface (assuming that the spacetime, outside a world tube containing the source, can be foliated by such hypersurfaces), and requiring that all but one coordinate be constant on the null geodesics (generators) in these null hypersurfaces. Then the “main” equations are ordinary differential equations that propagate the initial data along these geodesics (“rays” or bicharacteristics). The radial coordinate – the one that varies along the rays – must be specified by imposing one additional condition.

There are two standard choices for this additional condition, one of which is always made, resulting in the so-called “area” or “affine” radial coordinate, respectively. After either choice is made and the equations integrated, the dynamics of the two-body problem must be deduced by interpreting the properties of the two bodies (masses, separation) in terms of the functions of integration in the chosen coordinates. This is a non-trivial matter involving a lot of approximations \([2, 6]\).

Now, for a unique solution, additional (boundary) data must be given on a timelike (a world tube) or null hypersurface intersecting the initial \(u = \text{const}\) hypersurface. Such a timelike boundary hypersurface is used in the “Cauchy Characteristic Matching” (CCM) approach \([3, 7]\) and is conventionally defined as having constant “radial coordinate”, while in the interior (Cauchy) problem it is identified with a coordinate sphere (in a different coordinate system).

In this paper we propose that the choice of radial coordinate should be made in a way that is appropriate to the geometry of the physical problem, i.e., that the surfaces of constant radial coordinate must be identified with the level surfaces of a physical quantity.
In this way, the definition of the radial coordinate implicitly defines the physical problem. Applying this idea to the two-body problem, we impose a condition that follows from the requirement that the two-surfaces of constant radial coordinate in any null hypersurface share a geometric property with the surfaces of constant Newtonian gravitational potential of two point particles in Euclidean 3-space. This choice has the advantage that important physical parameters, such as the separation\(^1\) of the two bodies and their mass ratio, appear in the equations from the very beginning. They thus act as “source” terms for this problem. We then integrate the equations, assuming a series expansion near infinity and asymptotically flat boundary conditions. By considering the angular dependence of the “mass” parameter of the solution, the arbitrary function of two variables generating the solution is then naturally identified with the first time derivative of the quadrupole moment per unit mass of the source.

Section II introduces the notation we will be using and section III describes the choice of radial coordinate. In section IV we integrate the Einstein equations assuming a series expansion of the unknowns near infinity. In section V we use physical arguments to determine the arbitrary function in the solution, arriving at a result which differs from Bondi’s. We argue that this discrepancy is due to our asymptotic definitions of physical quantities. In section VI we discuss the validity of our results, pointing out that a similar choice for the radial coordinate might be convenient for investigating the near field also. Finally, in section VII, we state explicitly all assumptions in our approach, compare our derivation to Bondi’s and clarify the conditions under which our calculations can be interpreted as modelling the real physical problem.

II. FORMULATION OF THE PROBLEM - NOTATION

We will restrict our considerations to the the case where the axial Killing vector is hypersurface orthogonal (no particle spin). We will use the symbols \(u, \xi, \eta, \varphi\) for the four coordinates. The assumption that the Killing vector trajectories (parametrized by \(\varphi\)) are hypersurface orthogonal implies that \(g^{u \varphi} = g^{\xi \varphi} = g^{\eta \varphi} = 0\), while the requirements that \(u\) is a null coordinate and the curves along which only \(\xi\) varies are null geodesics imply

\(^1\) The word “separation” does not mean “physical distance”. For the sense in which physical terms are used in this paper, see the Remark at the end of section III.
that \( g^{uu} = g^{u\eta} = 0 \) also. Thus there are 5 remaining metric functions of \( u, \xi, \eta, \) denoted by the letters \( B, V, K, R, U \). We will write the line element in terms of a canonical Newman-Penrose null frame\(^2\) as 
\[ ds^2 = l \otimes n + n \otimes l - m \otimes m - m \otimes m, \]
where:
\[ l = \frac{1}{B} du, \quad n = d\xi + \frac{V}{2} du, \quad m = -\frac{1}{\sqrt{2}} \left( \frac{K^2}{R} (d\eta - U du) + i R d\varphi \right), \]
\[ \Delta = B \left( \frac{\partial}{\partial u} - \frac{V}{2} \frac{\partial}{\partial \xi} + U \frac{\partial}{\partial \eta} \right), \quad D = \frac{\partial}{\partial \xi}, \quad \delta = \frac{1}{\sqrt{2}} \left( \frac{R}{K^2} \frac{\partial}{\partial \eta} + i \frac{\partial}{\partial \varphi} \right). \] (1)

The frame transformations preserving the \( l \) direction and keeping \( m - m \) parallel to the Killing vector are a boost in the \( l - n \) plane and a null rotation around \( l \) parametrized by \( S \) and \( T \), respectively:
\[ l \rightarrow l/S, \quad n \rightarrow S[n + T(m + \overline{m}) + T^2 l], \quad m \rightarrow m + T l. \] (2)

If one chooses \( S = \Xi, \xi, T = -R \Xi, \eta/(\sqrt{2} K^2 \Xi, \xi) \) and redefines \( B, V \) and \( U \) appropriately, the new frame is the canonical frame that is obtained by a redefinition of the radial coordinate \( \xi \rightarrow \Xi(u, \xi, \eta) \). Thus an additional condition must be imposed to eliminate this freedom, effectively reducing the number of unknown functions of three variables from 5 to 4. Bondi’s standard notation (and choice of radial coordinate, \( K = \xi \)) is obtained by making the substitutions:
\[ \xi \rightarrow r, \quad \eta \rightarrow -\cos \theta, \quad B \rightarrow \exp(-2\beta), \quad V \rightarrow V_B/r, \quad K \rightarrow r, \quad U \rightarrow \sin \theta U_B, \quad R \rightarrow r \sin \theta \exp(-\gamma). \]

The reason for using \( \xi, \eta \) instead of the more usual \( r, x (= \cos \theta) \) is that we want to reserve the latter symbols for the radial and angular coordinates of a Euclidean prolate spheroidal coordinate system, which we will use to impose a condition on the function \( K(u, \xi, \eta) \) in the next section. Here we note that, in such a coordinate system (with singularities at \( \pm a \) on the z-axis), the Schwarzschild solution for a mass \( M \) at \( z = a \) on the symmetry axis is given by the substitutions:
\[ \xi \rightarrow r - ax, \quad \eta \rightarrow \frac{r x - a}{r - ax}, \quad B \rightarrow 1, \quad V \rightarrow 1 - \frac{2M}{r - ax}, \quad K \rightarrow r - ax, \quad U \rightarrow 0, \quad R \rightarrow \sqrt{(r^2 - a^2) \sqrt{1 - x^2}}. \] (3)

In the limit \( a \rightarrow 0 \), we obtain the standard form in spherical \( r, x \) coordinates. What is more, \( B \) remains a solution if we allow \( a \) to be a function, \( a(u) \), as it is simply a change of

\(^2\) We use the standard NP notation \( \# \): the complex null-tetrad basis \( \{ l, n, m, \overline{m} \} \) is normalized to \( l \cdot n = m \cdot \overline{m} = 1 \). We will restrict the symbols \( \{ l, n, m, \overline{m} \} \) to denote the basis co-vectors (differential forms) while the vectors (differential operators) will be denoted by the standard symbols \( \{ D, \Delta, \delta, \bar{\delta} \} \).
coordinates applied to the Schwarzschild solution in “spherical” $\xi, \eta$ coordinates:

$$B = 1, \quad V = 1 - \frac{2M}{\xi}, \quad K = \xi, \quad U = 0, \quad R = \xi \sqrt{1 - \eta^2}. \quad (4)$$

The 2-surfaces of constant $u, \xi$ play an important role in the characteristic initial value formulation of the Einstein equations $[1, 5, 7]$. In particular, the induced metric on them,

$$ds^2 = \frac{K^4}{R^2} d\eta^2 + R^2 d\varphi^2, \quad (5)$$

is known as part of the initial data. Thus imposing conditions on this part of the metric, in a sense, defines the physical problem whose evolution is determined by Einstein’s equations. Of course, only the $\eta$ dependence of $K$ is determined by the initial data, and this can be eliminated by a redefinition of the $\eta$ coordinate. This is the approach usually taken in imposing the “spherical area” radial coordinate condition $K = \xi$. We will instead impose a different condition that will allow these 2-surfaces to have different shapes depending on the ratio $a(u)/\xi$, where $a(u)$ is a measure of the separation of the two particles.

III. THE GEOMETRY OF THE SURFACES OF CONSTANT NEWTONIAN POTENTIAL

Prolate spheroidal coordinates (in Euclidean 3-space) is the appropriate coordinate system to use for the two particle problem. The Euclidean line element in such a coordinate system with singularities at $\pm a$ on the z-axis is

$$ds^2_{FLAT} = \left( r^2 - a^2 x^2 \right) \left( \frac{dr^2}{r^2 - a^2} + \frac{dx^2}{1 - x^2} \right) + (r^2 - a^2)(1 - x^2) d\varphi^2, \quad (6)$$

and the solution of the Laplace equation for the Newtonian potential of two point particles with masses $m_1$ at $z = a$ and $m_2$ at $z = -a$ is

$$V_N = -\frac{m_1}{r - a x} - \frac{m_2}{r + a x}. \quad (7)$$

For the 1-particle problem (say $m_2 = 0$), Bondi’s geometrical coordinate choice $K = \xi = r - a x$, admits several interpretations in terms of properties of the surfaces of constant Newtonian potential:

- $K$ is constant on the surfaces of constant Newtonian potential
- $2/K^2$ is the Gaussian curvature of the surfaces of constant Newtonian potential
- $1/K^4$ is the flat-space norm of the gradient $(\eta^{ab}V_{N,a}V_{N,b})$ of the Newtonian potential of a unit mass.

It is tempting to impose a condition, following from such an interpretation of $K$, on the metric functions in (1), with $(V - 1)/2$ playing the role of the Newtonian potential. However, apart from ambiguities in relating the metric function $V$ to the “Newtonian potential”, involving the function $B$, one must remember that the function $V$ is not a part of the data on the initial $u$ hypersurface and, therefore, relating it to $K$ for all $\xi$ (and $u$) may unduly restrict the evolution. We will instead impose a condition that will determine $K(u, \xi, \eta)$ without reference to the other metric functions.

To do this we write the flat-space metric (6) in terms of an orthogonal system of coordinates $\xi, \eta, \varphi$, where the $\xi$ coordinate is the (inverse of the) Newtonian potential per unit mass:

$$m_1 + m_2 = \frac{m_1}{r - a x} + \frac{m_2}{r + a x}, \quad (m_1 + m_2) \eta = m_1 \frac{r x - a}{r - a x} + m_2 \frac{r x + a}{r + a x}. \quad (8)$$

Let us denote by $N_2(\phi, \psi)$ the inner product of the gradients of the functions $\phi, \psi$ with respect to the flat metric (6):

$$N_2(\phi, \psi) \equiv \frac{(r^2 - a^2) \phi, r \psi, r + (1 - x^2) \phi, x \psi, x}{r^2 - a^2 x^2}. \quad (9)$$

Then it is easy to verify that $\xi, \eta$ as functions of $r, x$ given by (8) satisfy $N_2(\xi, \eta) = 0$. It is remarkable that orthogonality is maintained when the “radial” and “angular” coordinates of the individual 1-particle problems [see (3)] are combined linearly in this simple way. This is undoubtedly due to the appropriateness of prolate spheroidal coordinates for describing this problem. In terms of a parameter $\lambda$ depending on the ratio of the masses, the coordinate transformation (8) can be written

$$\xi = \frac{r^2 - a^2 x^2}{r + \lambda a x}, \quad \eta = x - a \frac{(\lambda r + a x)(1 - x^2)}{r^2 - a^2 x^2}, \quad \text{where} \quad \lambda \equiv \frac{m_1 - m_2}{m_1 + m_2}. \quad (10)$$

We note that, when $x \to \pm 1$, also $\eta \to \pm 1$, while as $r \to \infty$, $\xi \to r$ and $\eta \to x$. Thus, for $r \gg a$, the coordinates $\xi, \eta$ behave as prolate spheroidal (or spherical) coordinates $r, x$.

In terms of the new coordinates defined in (10), the flat metric (6) takes the form

$$ds_{FLAT}^2 = \frac{d\xi^2}{N_2(\xi, \xi)} + \frac{d\eta^2}{N_2(\eta, \eta)} + (r^2 - a^2)(1 - x^2) d\varphi^2. \quad (11)$$
Now, it is easy to verify that 

\[ N_2(\eta, \eta) = N_2(\xi, \xi)(r^2 - a^2)(1 - x^2)/\xi^4, \]

so that the metric on the \( \xi = \text{const} \) surfaces in Euclidean 3-space can be put in the form (5) with

\[
R^2 = (r^2 - a^2)(1 - x^2), \quad K^4 = \frac{\xi^4}{N_2(\xi, \xi)} = \frac{\xi^4}{1 + S}, \quad \text{where}
\]

\[
S = \frac{a^2(1 - \lambda^2)(-r^2 + 3r^2x^2 + 4\lambda arx^3 + a^2x^2 + \lambda^2a^2x^4)}{(r + \lambda ax)^4},
\]

(13)

The denominator of \( S \) (or \( \xi \)) cannot vanish as the coordinate ranges are \( r \geq a, -1 \leq x \leq 1 \), and the parameter \( \lambda \) is by definition less than one in absolute value.

We are now ready to make our choice for the Bondi “radial” coordinate \( \xi \) that is appropriate for the two particle problem: we will require that \( K \) will be given by (12) for any value of \( u \), i.e., that the parameter \( a \) in (13), determining the separation of the two particles, will be allowed to be a function of \( u \).

To be used in Einstein’s equations \( K \) must, of course, be given as a function of \( u, \xi, \eta \) by inverting the coordinate transformation (10). This cannot be done in closed form, as it depends on the roots of a 5th order polynomial in \( \xi, \eta \). However, it is easy to convert the series expansion of \( K \) for \( r \to \infty \) to one for \( \xi \to \infty \), making use of the limits noted after (10). This is most easily done by replacing, in the series expansion of \( S \) with respect to \( r \), the symbols \( r, x \) by \( \xi, \eta \), respectively, one order at a time and expanding again:

\[
S \simeq \frac{(1 - \lambda^2)(-1 + 3x^2)a(u)^2}{r^2} - \frac{4\lambda(1 - \lambda^2)x(-1 + 2x^2)a(u)^3}{r^3} + O\left(\frac{1}{r}\right),
\]

\[
S - \frac{(1 - \lambda^2)(-1 + 3\eta^2)a(u)^2}{\xi^2} \simeq -\frac{4\lambda(1 - \lambda^2)x(-3 + 5x^2)a(u)^3}{r^3} + O\left(\frac{1}{r}\right),
\]

\[
S - \frac{(1 - \lambda^2)(-1 + 3\eta^2)a(u)^2}{\xi^2} + \frac{4\lambda(1 - \lambda^2)\eta(-3 + 5\eta^2)a(u)^3}{\xi^3} \simeq O\left(\frac{1}{r}\right).
\]

(14)

Substituting in (12) we obtain, to this order,

\[
K \simeq \xi - \frac{(1 - \lambda^2)(-1 + 3\eta^2)a(u)^2}{4\xi} + \frac{\lambda(1 - \lambda^2)\eta(-3 + 5\eta^2)a(u)^3}{\xi^2} + O\left(\frac{1}{\xi}\right).
\]

(15)

In this way we can compute the series expansion of \( S \), and therefore of \( K \), as a function of \( u, \xi, \eta \) to any desired order. We note that the angular dependence of the first two “correction terms” in \( K \) are proportional to the Legendre polynomials \( P_2, P_3 \) and therefore vanish when integrated over angles. Thus we can interpret our coordinate \( \xi \), alternatively, as an approximate “luminosity distance”: the area of the surfaces of constant \( u \) and \( \xi \) equals \( 4\pi\xi^2 \) to a good approximation. To find out how good this approximation is we must carry
the expansion to 5 more orders! We then find (see (5)):

\[ \int dS = \int K^2 d\eta d\varphi = 4\pi \xi^2 \left( 1 - \frac{4}{35} (1 - \lambda^2)^2 (1 - 6 \lambda^2) \frac{a(u)^6}{\xi^6} + O\left(\frac{1}{\xi^7}\right) \right). \quad (16) \]

Thus our radial coordinate \( \xi \) is also a “luminosity distance” to a very good approximation.

It should be pointed out that the reflection symmetry, \( \eta \rightarrow -\eta \), assumed in many investigations [6, 9], holds here only if, at the same time, \( \lambda \rightarrow -\lambda \). This is to be expected, as the interchange of the positive and negative \( z \)-directions will not give the same problem unless the two particles are also interchanged.

Finally, we compute the flat-space quadrupole moment of the two particles relative to their center of mass. In terms of cylindrical coordinates \( \rho, z, \varphi \) with origin at the center of mass, the mass density distribution is

\[ \mu = \frac{\delta(\rho)}{2\pi \rho} \{ m_1 \delta[z - (1 - \lambda) a(u)] + m_2 \delta[z + (1 + \lambda) a(u)] \}. \quad (17) \]

Then the quadrupole moment tensor can be written

\[ D^{ab} = \int \mu (3 x^a x^b - x^2 \delta^{ab}) dV = Q(u) (3 \delta_z^a \delta_z^b - \delta^{ab}), \quad (18) \]

where

\[ Q(u) = \int \{ m_1 \delta[z - (1 - \lambda) a(u)] + m_2 \delta[z + (1 + \lambda) a(u)] \} z^2 dz = (m_1 + m_2)(1 - \lambda^2) a(u)^2, \quad (19) \]

and we have used the definition of \( \lambda \) to write \( m_1, m_2 \) in terms of \( m_1 + m_2 \) and \( \lambda \).

**Remark:** All statements regarding physical quantities (particle separation, mass density, quadrupole moment, etc.) in this paper refer to an unphysical background Minkowski space, which is the limit as \( \xi \rightarrow \infty \) of the sought solution of Einstein’s equations. In the physical, curved, spacetime they are to be understood only as names referring to the combination of variables in their definition. Thus, for example, “particle separation” means \( 2 a(u) \) and not physical distance between the two particles.

**IV. SERIES SOLUTION OF THE VACUUM EINSTEIN EQUATIONS**

Knowing \( K \) and assuming a formal series expansion for the function \( R \), valid near \( \xi \rightarrow \infty \), where \( R \) approaches the flat-space limit \( 4 \xi \), we can carry out the well-known hierarchical
series of $\xi$ integrations to obtain the other metric functions. Thus, we assume that as $\xi \to \infty$, $R$ has the series expansion

$$R \simeq \xi \sqrt{(1 - \eta^2)} \left( 1 + \frac{c_1(u, \eta)}{\xi} + \frac{c_2(u, \eta)}{\xi^2} + \frac{c_3(u, \eta)}{\xi^3} + O\left(\frac{1}{\xi^4}\right) \right).$$

Then the Ricci component $\Phi_{00}$ in the frame defined in (1) can be solved for $B_\xi/B$, giving

$$\frac{B_\xi}{B} = -\frac{K_\xi}{K} + 2 \frac{R_\xi}{K} \frac{K_\xi}{R} - \frac{K R_\xi}{R^2 K_\xi}.$$  

Integrating the series expansion of the rhs when $K, R$ are given by equations (15), (20), and choosing the function of $u, \eta$ of integration to satisfy the boundary condition (4) at infinity, we find

$$\mathcal{B} \simeq 1 + \frac{(1 - 3\eta^2)(1 - \lambda^2) a(u)^2 + 2 c_1(u, \eta)^2}{4 \xi^2} + \ldots$$

We next compute the series expansion of the Weyl tensor component $\Psi_0$, obtaining

$$\Psi_0 \simeq -\frac{(1 - 3\eta^2)(1 - \lambda^2) a(u)^2 + 2 c_1(u, \eta)^2 - 4 c_2(u, \eta)}{2 \xi^4} + \ldots$$

Now, the condition for asymptotic flatness (absence of incoming radiation at infinity) is that

$$\Psi_0 \simeq O\left(\frac{1}{\xi}\right),$$

so that we must require that

$$c_2(u, \eta) = \frac{(1 - 3\eta^2)(1 - \lambda^2) a(u)^2 + 2 c_1(u, \eta)^2}{4}.$$  

Next, the Ricci component $\Phi_{01}$ can be written

$$\left[\frac{B K^6 U}{R^2}\right]_{\xi} = K^2 \left[\frac{B_\xi B_\eta}{B^2} - \frac{B_{\xi u}}{B} + 2 \frac{R_\xi R_\eta}{R^2} + 2 \frac{R_{\xi \eta}}{R} \right] + 2 K K_\xi \frac{B_\eta}{B} - 2 \frac{R_\eta}{R},$$

so that, with the series expansions of $K, R, B$ known, $U$ can be obtained. We find

$$U \simeq \frac{(1 - \eta^2) c_{1, \eta}(u, \eta) - 2 \eta c_1(u, \eta)}{\xi^2} + \frac{(1 - \eta^2)[4 c_1(u, \eta) c_{1, \eta}(u, \eta) - x_0(u, \eta)] - 8 \eta c_1(u, \eta)^2}{3 \xi^3} + \ldots,$$
where \( x_0(u, \eta) \) is an arbitrary function of integration. The second such function, to be added to \( U \), must be set equal to zero for asymptotic flatness.

Proceeding in the same way, equation \( \Phi_{11} + 3 \Lambda = 0 \) can be solved for \( (V K K_\xi)_\xi \) in terms of known quantities, and, with \( V \) known, equation \( \Phi_{02} = 0 \) gives \( (K R_u/R)_\xi \). Denoting by \( y_0(u, \eta), z_0(u, \eta) \) the functions of integration, \( V \) and \( R_u \) are given by

\[
V \simeq 1 + \frac{y_0(u, \eta)}{\xi} + \ldots \tag{27}
\]
\[
R_u \simeq \sqrt{(1 - \eta^2)[z_0(u, \eta) + \frac{2 c_1(u, \eta) z_0(u, \eta) + (1 - 3 \eta^2) (1 - \lambda^2) a(u) \ddot{a}(u)}{2 \xi}]} + \ldots \tag{28}
\]

Finally the requirement that this agrees with the \( u \) derivative of (20) determines the \( u \) derivatives of the coefficients \( c_i(u, \eta) \) (except for \( c_2(u, \eta) \) which is given by (24)):

\[
c_{1, u}(u, \eta) = z_0(u, \eta), \quad c_{3, u}(u, \eta) = \ldots \tag{29}
\]

This completes the integration of the so-called “main” equations. Of the remaining equations, \( \Phi_{11} = 0 \) is satisfied identically, while the vanishing of \( \Phi_{12}, \Phi_{22} \) impose the following two conditions – conservation laws – on the three functions of integration \( x_0(u, \eta), y_0(u, \eta) \) and \( z_0(u, \eta) = c_{1, u}(u, \eta) \):

\[
x_0\eta = -y_0\eta + c_1 c_{1, u} \eta - 3 c_{1, u} c_{1, \eta} + 3 \eta (1 - \lambda^2) a(u) \dot{a}(u)
\]
\[
y_0\eta = 2 c_{1, u}^2 - 2 c_{1, u} + (1 - \eta^2) c_{1, u, \eta} \eta - 4 \eta c_{1, u, \eta}
\]
\[
+ (1 - 3 \eta^2)(1 - \lambda^2) [\dot{a}(u)^2 + a(u) \dddot{a}(u)]. \tag{30}
\]

The series solution obtained in this section reduces to the one with Bondi’s coordinate choice \( K = \xi \) in the limit \( \lambda^2 \to 1 \). In fact it can be obtained as a coordinate transformation from the Bondi solution if one redefines the \( c_i \) to be given in terms of the corresponding \( c_i B \) (= coefficients in the series expansion of \( R \) in powers \( K \)) by the expressions obtained when \( K \) is replaced by its series expansion (15). One then finds that \( c_1 = c_{1, B} \) and \( c_2 \) is given by (24). To make the conservation laws (30) take the Bondi form, one must also redefine the functions of integration \( x_0(u, \eta), y_0(u, \eta) \) as follows:

\[
x_0(u, \eta) \to x_{0, B}(u, \eta) + \frac{9}{2} \eta (1 - \lambda^2) a(u)^2, \tag{31}
\]
\[
y_0(u, \eta) \to y_{0, B}(u, \eta) + (1 - 3 \eta^2)(1 - \lambda^2) a(u) \ddot{a}(u).
\]

Despite the fact that, formally, the solution obtained here is but a coordinate transformation of the Bondi solution, the explicit appearance of the “source” terms involving \( a(u) \) and \( \lambda \)}
in the conservation laws (30) makes the choice of the arbitrary function \( c_1 \) describing the two-body problem and, consequently, the physical interpretation of the solution particularly simple and transparent.

V. PARTICULAR SOLUTION DESCRIBING THE TWO-BODY PROBLEM

The solution obtained in the previous section is a “general” solution of the equations in that it depends on the arbitrary function \( c_1(u, \eta) \). To obtain the solution for a particular problem, an appropriate choice for this function must be made. With the coordinates chosen to fit the two-body problem, it is reasonable to expect that the required arbitrary function will have a simple form. First, non-singular behavior on the axis (\( \eta = \pm 1 \)) requires that \( c_1(u, \eta) = q(u, \eta) (1 - \eta^2) \) for some \( q(u, \eta) \) that is well behaved at \( \eta = \pm 1 \). Making this substitution in the second conservation equation (30) and replacing \( y_0(u, \eta) \) by \(-2M(u, \eta)\) (Bondi’s “mass aspect” definition – see equation (27)), we obtain

\[
2 M_u = -2 (1 - \eta^2)^2 q_u^2 \\
-(1 - 3 \eta^2)\{(1 - \lambda^2) [\dot{a}(u)^2 + a(u) \ddot{a}(u)] - 4 q_u\} \\
-(1 - \eta^2)[(1 - \eta^2)q_{u\eta\eta} - 8 \eta q_{u\eta}].
\]

(32)

Now, the rhs of this equation describes the energy loss of the system. On physical grounds, it must be negative definite and have the angular dependence that is appropriate to the one-dimensional motion of the two particles. This angular dependence, being independent of \( \xi \), can be identified with the angular distribution of the flow of energy at infinity obtained in the linearized theory using the Landau-Lifshitz pseudotensor (see [11] equation 110.15)

\[
\frac{dI}{d\Omega} \sim \left[ \frac{1}{4}(\tilde{D}_{ab}n^a n^b)^2 + \frac{1}{2}(\ddot{D}_{ab} - \tilde{D}_{ab} n^b \tilde{D}_{ac} n^c) \right],
\]

(33)

where \( \tilde{D}_{ab} \) is the third time derivative of the quadrupole moment tensor\(^5\) (15) and \( n^a \) are the Cartesian components of the unit vector in the direction of propagation, which in our

---

\(^5\) It is consistent to use the flat-space definition (15) for the quadrupole moment tensor here, as the L-L pseudotensor is defined in terms of the linear solution to the field equations, which is determined by a flat-space integral over the sources. Besides, only the angular dependence of \( D_{ab} \) is used in the present argument, not the exact form of \( Q(u) \).
coordinates equal \( (\sqrt{1 - \eta^2}) \cos \varphi, \sqrt{1 - \eta^2}) \sin \varphi, \eta \) at infinity. We find

\[
\frac{dI}{d\Omega} \sim \left[ \frac{1}{4} (3 \eta^2 - 1)^2 + \frac{1}{2} (6 - 3 \eta^2 + 1) \right] = \left[ \left( \frac{3 \eta^2 - 1}{2} \right)^2 + 1 - (3 \eta^2 - 1) \right] = \frac{9}{4} (\eta^2 - 1)^2. \tag{34}
\]

Thus the angular dependence of the mass-loss equation \( (32) \) will be \( \sim (1 - \eta^2)^2 \), as appropriate for this system, if we choose

\[
q(u, \eta) = \frac{1}{4} (1 - \lambda^2) a(u) \dot{a}(u), \quad \text{so that} \tag{35}
\]

\[
c_1(u, \eta) = \frac{1}{4} (1 - \lambda^2) a(u) \dot{a}(u) (1 - \eta^2) = \frac{1}{8} \dot{Q} (1 - \eta^2), \tag{36}
\]

where \( \dot{Q} \equiv (1 - \lambda^2) a(u)^2 \) is the quadrupole moment per unit mass – see \( (19) \). The same result follows from the requirement that the rhs of \( (32) \) be negative for all \( u, \eta \), so that energy flows \( \text{out} \) of the system in all directions and at all times: \( (35) \) is the unique solution that is regular on the axis and makes the linear (and “source” terms) on the rhs of \( (32) \) vanish.

With this \( c_1(u, \eta) \), the first conservation equation can be integrated giving

\[
x_0(u, \eta) = \eta (1 - \lambda^2) a(u)^2 \left[ \frac{3}{2} + \frac{(1 - \eta^2)(1 - \lambda^2) \dot{a}(u)^2}{8} \right] + 2 \int M, \eta \, d\tau, \tag{37}
\]

where \( M(u, \eta) \) is determined by the equation

\[
M, u = -(1 - \eta^2)^2 \left( \frac{(1 - \lambda^2) [\dot{a}(u)^2 + a(u) \ddot{a}(u)]}{4} \right)^2 = -(1 - \eta^2)^2 \left( \frac{\dot{Q}}{8} \right)^2 \tag{38}
\]

once the function \( a(u) \) is known.

Finally, using \( \dot{Q} \), the quadrupole moment per unit mass of the system, the leading terms in the components of the Weyl tensor in the frame defined in \( (1) \), with \( c_1(u, \eta) \) given by \( (36) \), are:

\[
\Psi_0 = \frac{\alpha}{\xi} [c_3(u, \eta) + \lambda \eta (3 - 5 \eta^2) \dot{Q} a(u)] + \ldots,
\]

\[
\Psi_1 = \frac{\sqrt{(1 - \eta^2)}}{\sqrt{2} \xi} \left( \int M, \eta \, d\tau - \frac{3}{2} \eta (Q + \frac{(1 - \eta^2) \dot{Q}^2}{16}) \right) + \ldots,
\]

\[
\Psi_2 = \frac{1}{\xi^3} \left( M + \frac{(1 - 3 \eta^2) \dot{Q}}{4} + \frac{(1 - \eta^2)^2 \dot{Q} \ddot{Q}}{64} \right) + \ldots,
\]

\[
\Psi_3 = \frac{\eta \sqrt{(1 - \eta^2)} \ddot{Q}}{2 \sqrt{2} \xi} + \ldots, \quad \Psi_4 = \frac{(1 - \eta^2) \ddot{Q}}{8 \xi} + \ldots. \tag{39}
\]

Using the evolution equations \( (29) \), it can be shown that \( \Psi_0 \) is proportional to \( (1 - \eta^2) \).
We note that equation (38), giving the radiated energy directly in terms of the parameters λ and a(υ) describing the source, is an *exact* result following from a particular, physically motivated, choice of the arbitrary function c₁(υ, η) (or q(υ, η)). And this choice of q(υ, η), which eliminates all linear terms on the rhs of the energy-conservation equation (32), is made possible by the existence of the extra term introduced by our choice of the function K, which fortuitously has the correct angular dependence (1 − 3 η²). The final expression for the mass loss (38), relating the radiated energy to the square of the second “time” derivative of the “quadrupole moment per unit mass” differs from the standard quadrupole formula (third time derivative of the quadrupole moment) obtained in the linearized theory. However, it closely resembles the results of approximate non-linear calculations, where the radiated energy is found to be proportional to the square of an integral with respect to u of Ω/M weighted by exp(2υ/M) [12, 13], where M is a large constant “background” mass. Thus our choice of c₁(υ, η), equation (36), implying that M,υ ∼ ˙Q² in the fully non-linear case, is in remarkably good agreement with these results. The discrepancy with the classical result must be attributed to the different definitions of “time” and “quadrupole moment per unit mass” (see the Remark at the end of section III). And the following considerations reinforce this conclusion.

Bondi’s approximate linear calculation for the news function, which agrees with the conclusions of linearized theory [11] and post-Newtonian calculations [14, 15], gives c₁(υ, η) = −½ ˙Q(u)(1 − η²) (in our notation Bondi’s c equals −c₁). If we want to reconcile the two results we must require that, as a consequence of the equations of motion⁶, the quadrupole moment satisfies the equation ˙Q(u) = −½ ˙Q(u)/(m₁ + m₂) so that,

\[
Q(u) = Q₀ \exp\left[-\frac{u - u₀}{4(m₁ + m₂)}\right], \quad \text{and therefore,}
\]

\[
a(u) = a₀ \exp\left[-\frac{u - u₀}{8(m₁ + m₂)}\right].
\]

(40)

Of course, a(u) should be determined by the equations of motion following from the vanishing of the divergence of the energy momentum tensor of the two particles. But this requires knowledge of the field in the vicinity of the particles, which is well beyond the scope of the approximate calculations near null infinity carried out here.

⁶ The equations of motion are used in the derivation of the result in linearized or post-Newtonian theory [11, 14], and, implicitly, in Bondi’s derivation.
Nevertheless, we point out that, with \( a(u) \) given by (40), all remaining \( u \) integrations (see equations (29), (37), (38)) can be evaluated analytically in terms of elementary functions (exponentials); and choosing the functions (of \( \eta \)) of integration to vanish (except for the final mass), the resulting solution will smoothly approach the Schwarzschild solution as \( a(u) \to 0 \) exponentially with \( u \to \infty \). Now, if we assume that the Minkowskian definitions of “separation”, “velocity” and “time” used here approximate adequately the corresponding physical quantities, this solution has the unphysical feature that the two particles approach each other at a diminishing rate (both \( a(u) \) and \( \dot{a}(u) \) decrease with time): the collision seems to end with hardly a whimper rather than a bang! However, the physical interpretation of “velocity” depends on the definition of “time”: near the source the time function, which can be taken to equal \( u + \xi \) at infinity, is expected to have a logarithmic singularity, as in the Schwarzschild case \([16, 17]\). A time function of the form \( t = u + \xi - (1 + k)a(u) + 8(m_1 + m_2)\log(\xi/a_0) \) (where \( k=\text{const} \)) evaluated on the curve \( \xi = a(u) \) gives \( t = u_0 - k a(u) \), so that \( \frac{da}{dt} = -1/k \). Thus, with a proper definition of time in the near-zone, the rate of approach implied by (40) might be physically acceptable. It seems too speculative to discuss further this possibility. But this example shows that one cannot simply identify \( u \)-derivatives at infinity with time-derivatives in the near-zone, and that the mass-loss formula (38) may be consistent with the standard quadrupole formula!

In any case, this particular solution, being an exact analytic solution (in the form of a series expansion which can be continued to any order) of the Einstein equations, including the evolution equations, which approaches the Schwarzschild solution as \( u \to \infty \), can be useful in testing the accuracy of numerical codes. For this reason, explicit expressions for the metric functions and the Weyl tensor components, together with the verification that the metric is Ricci-flat to the appropriate order and becomes Schwarzschild as \( u \to \infty \), are included in the Mathematica notebook “NewRadialCoord.nb” mentioned in footnote 3.

VI. DISCUSSION

The freedom in the choice of coordinates inherent in general relativity is invariably used to simplify the equations. This is due to the complexity of the equations. However, a coordinate choice that best simplifies the equations may not describe the physical problem in the most natural way. For example, in the static, axially symmetric problem, Weyl’s
canonical $\rho, z$ coordinates require that a physically spherical source be described as a linear distribution of mass.

In the Bondi formulation of the axisymmetric vacuum equations, the essential simplification comes from the use of a null coordinate $u$ and angular coordinates $\eta, \varphi$ which are constant on the null rays. The choice of parametrization on these rays does not simplify the mathematical problem any further. In this paper we propose that it should be made in a way that gives information about the particular physical problem whose solution is sought. For the asymptotic solution near infinity of the two-body problem, imposing a condition that follows from interpreting the surfaces on which $u, \xi$ are constant as surfaces of constant Newtonian potential, allows us to relate the arbitrary function of two variables generating the solution to two parameters describing the source, $\lambda$ and $a(u)$, by a symmetry argument without making any approximations. This should be compared to Bondi’s approximate and, in his own words, “distinctly crude” derivation of the corresponding result.

Despite its simplicity and directness, our derivation of the form of $c_1$ is subject to a serious objection, as the result seems to depend on a choice of coordinates. How can such an arbitrary choice lead to a physically meaningful result? There are two answers to this criticism: (i) Once the arbitrary function in the solution ($q(u, \eta)$) has been related to the arbitrary function ($Q(u)$) determining the coordinate transformation (15), the latter is no longer arbitrary. The same result could have been obtained with Bondi’s choice of radial coordinate $\xi_B = K$, had one chosen the arbitrary functions $x_0(u, \eta), y_0(u, \eta)$ to include extra source terms as in equation (31), and then transformed the radial coordinate from $K$ to $\xi$ to eliminate the extra term in the “mass aspect” (this is discussed further in the next section). In this sense, the leading term in the coordinate transformation (15) is determined by the solution. (ii) Our choice of radial gauge, following from the identification, near infinity, of the two-surfaces of constant $u, \xi$ with the surfaces of constant Newtonian potential, is not an arbitrary choice of coordinates but is closely related to the physics: the surfaces of constant $\xi$ are the wave fronts of the radiation emitted from the system, carrying information about

---

7 See comments after equation (91) in [2]. An outline of Bondi’s derivation is given in section VII.
8 This extra term in the “mass aspect” does not change the “Bondi Mass”, as its integral over angles vanishes. In fact, in obtaining his formula for $c_{1B}$, Bondi also obtains a first-order expression for $M_B$ that has precisely the form $\dot{f}(u)(1 - 3\cos^2 \theta)$, as in [31]; in going from eq. 88 to eq. 89 of reference [2], Bondi has set his function $p(u)$ equal to $2Q_{u0}$ (plus a constant $M0$)!
its properties; and identifying their constant time sections \((t = u + \xi \text{ near infinity})\) with the surfaces of constant Newtonian potential gives a description that matches the dynamics more accurately than the use of either spherical or prolate-spheroidal \([6]\) surfaces (any change in the dynamics at infinity must be manifested through changes in the Newtonian potential). In fact, the dynamics can be better approximated if one allows the parameters \(m_1, m_2\) to depend on \(u\) to reflect the relativistic velocity dependence of mass. All \(\xi\) integrations remain unchanged and only equations involving \(u\) derivatives of the unknown functions will acquire extra terms, leading to more complicated evolution equations \([29]\) and conservation laws \([30]\). Indeed, the energy loss equation \([38]\), expressed in terms of \(\ddot{Q}\), remains unchanged even when the parameter \(\lambda\) in \(Q\) is allowed to depend on \(u\).

This leads to the suggestion that, in the inner problem also, a coordinate condition based on a property of appropriately defined constant-Newtonian-potential (or constant-Newtonian-energy) surfaces should be used, as these surfaces have the topology of the “pair of pants” picture of the horizon \([5]\): near the source, the set of points with constant Newtonian potential consists of two disconnected subsets, one around each particle. This can best be seen if, near \(\xi = 0\), one switches to the \(r, x\) coordinates used in the definition of \(\xi, \eta\) \([10]\). Of course, near the source the null coordinate \(u\) will not be well behaved and a different time coordinate must be used, while care must be exercised in defining the “Newtonian potential”.

VII. SUMMARY AND CONCLUSIONS

This paper addresses the mathematical problem of solving Einstein’s equations with the symmetries (one hypersurface-orthogonal Killing vector with closed spacelike orbits) and boundary conditions (asymptotically flat, no incoming radiation) that are appropriate for the two-particle collision problem. All calculations are carried out near infinity, and physical quantities are defined with respect to the limiting Minkowski space (see the Remark at the end of section III). The Bondi formulation of this problem is used as it allows free initial data and a sequential integration of the equations. Both the solution and the coordinate system are only assumed to be valid outside some sufficiently large time-like world tube.

A particular, explicit, solution to this mathematical problem depends on two arbitrary choices: (a) the “radial” coordinate condition – the coordinate expression for the determinant
of the metric on the two-surfaces of constant \( u, \xi \) (the function \( K(u, \xi, \eta) \)). (b) the free function \( c_1(u, \eta) \). Mathematically, these choices are completely arbitrary (modulo boundary conditions).

The basic idea of this paper is to try to “guess” the appropriate form for these arbitrary functions based on the “expected” properties of the solution describing the two-particles; properties which can be calculated in the region where the solution is valid, i.e., outside the large world tube. Thus, (a) the function \( K(u, \xi, \eta) \) is chosen by the requirement that, on any surface of constant \( u \) and \( \xi \), it is identical to the corresponding function in the metric of the surfaces of constant Newtonian potential of two point particles in Euclidean 3-space. Even though the expression for \( K \) in Euclidean space is valid everywhere, we only use the asymptotic form of \( K \), valid for, say, \( \xi > 10 a(u) \). (b) The function \( c_1(u, \eta) \) (or \( q(u, \eta) \)) is chosen by the requirement that, at infinity, the angular dependence of the emitted radiation is that appropriate to quadrupole radiation. The same \( q(u, \eta) \) satisfies the physical requirement that a non-negative amount of energy is radiated in any direction and at any time.

The physical significance of allowing \( K \) to have a term \( f(u, \eta)/\xi \) is that it changes the “mass-aspect”. This is most easily seen by making the change of coordinates \( r \rightarrow \tilde{r} + p(u, \theta)/\tilde{r} \) to the Schwarzschild metric in null-spherical coordinates, \( ds^2 = (1 - 2M/r)du^2 + 2 du dr - r^2d\Omega^2 \). Then, to first order in \( 1/\tilde{r} \), the metric acquires an extra term \( 2 du d\theta p_\theta/\tilde{r} \) but the “mass aspect” (=coefficient of \( 1/\tilde{r} \) in \( g_{uu} \)) is also changed to \( M - p_u \) due to a contribution from the \( 2 du dr \) term in the metric. This is the basic reason for the existence of the extra term on the rhs of the mass-loss equation \( \dot{M} \) for the metric defined by (11), which allows the linear terms to vanish, leaving \( \dot{M} = -\dot{c}_1 \), when one chooses \( K = \xi - f(u)(3\eta^2-1)/\xi + \ldots \) and \( c_1 = \frac{1}{2}f(u)(1-\eta^2) \). One could then make the ad-hoc choice \( f(u) = \tilde{Q}(u) \) and claim that one has obtained the standard quadrupole formula as an exact result of the non-linear equations! But this ad-hoc choice lacks any physical justification for calling this \( \tilde{Q} \) the “quadrupole moment”. Our interpretation of \( \xi \) in terms of the Newtonian potential at infinity gives both the correct angular dependence to the \( 1/\xi \) term in \( K \) and provides a physical interpretation of the coefficient \( f(u) \), namely \( Q(u)/(4M) \) (with \( Q \), admittedly, defined in a flat background).

Of course, the question of uniqueness remains: could a different physical interpretation of the “radial coordinate” \( \xi \) lead to a function \( K \) that (i) agrees with Bondi’s \( K = \xi \) both at
infinity and for the 1-particle limits ($\lambda \to \pm 1$, or $a(u) \to 0$), (ii) the limit $\lim_{\xi\to\infty} \xi(K - \xi)$ exists and has the form $f(u)(3\eta^2 - 1)$ but with a different physical meaning of $f(u)$?

It is appropriate, at this point, to recall how Bondi obtains his relation between $c_1$ and the quadrupole moment. He begins with the static, axially symmetric vacuum metric in Weyl coordinates

$$e^{2\psi}dt^2 - e^{-2\psi}[e^{2\sigma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2],$$

in which the function $\psi$ satisfies the flat-space Laplace equation. He then

1. Takes for $\psi$ the solution (in spherical coordinates $\rho = R \sin \Theta, \ z = R \cos \Theta$)

$$\psi = -\frac{m}{R} - \frac{D \cos \Theta}{R^2} - \frac{(Q + 1/3 m^3)(3 \cos^2 \Theta - 1)}{2 R^3},$$

$m, D$ and $Q$ being called the "mass", "dipole", "quadrupole moment" of the source. The term $1/3 m^3$ is needed to make $Q = 0$ for the Schwarzschild solution, as it is obtained from a linear mass distribution in these coordinates.

2. Finds the coordinate transformation to Bondi coordinates $(t, R, \Theta) \to (u, r, \theta)$ as a series approximation for large $r$ (or $R$).

3. By comparing coefficients of the transformed (to Bondi coordinates) Weyl static solution with the series expansion of the metric (1), he finds expressions for the unknown functions $c_3, x_0, y_0$ (equivalent to his $C, N, M$) in terms of $m, D, Q$. The function $c_1$ is also obtained as a coordinate-dependent expression.

4. Next he supposes that, for slowly varying fields, he can use the same expressions for these functions allowing the constants $m, D, Q$ to be functions of $u$. And, for weak fields, he solves the linearized form of the equations (29), (30) determining the $u$ derivative of $c_3, x_0, y_0$ (expressed now in terms of $m, D, Q$) for the $u$-dependence of $D, m, c_1$, respectively. In this way $Q$, instead of $c_1$, becomes the free function.

It will be observed that the last step involves two serious approximations. Bondi is alluding to this in calling his derivation "distinctly crude". In particular, what conclusions would have he obtained had he allowed the constants $m, D, Q$ to be time-dependent before making the coordinate transformation, and, perhaps, having found first a time-dependent first-order correction to the static solution? This crucial step (allowing the constants to
be functions of \(u\) is necessary in order to generalize the static Newtonian potential to a
dynamic one, but it is also the weakest point in the derivation: when \(\psi\) is \(u\)-dependent, the
Weyl metric is not a solution (to first order in \(\dot{m}, \dot{D}, \dot{Q}\)) of the equations!

In our approach, we make contact with a \(u\)-dependent Newtonian potential via our inter-
pretation of \(\xi\), which we are free to do as it is a coordinate condition, not an approximation
for slowly varying fields. Moreover, we do not need to integrate the linear approximations of
(29), (30) to relate the free function \(c_1\) to the properties of the source. We obtain \(c_1\) directly
by a physical argument: it is the unique, non-singular solution that makes the linear terms
on the rhs of (32) vanish, resulting in an exact expression for the mass loss (\(\dot{M} = -c_1^2\)) that
has both the correct angular dependence and is negative definite. If we accept these argu-
ments as physically reasonable, then we are obliged to conclude that the radiation depends
on the “second derivative of the quadrupole moment per unit mass” – a result which seems
to disagree with the standard quadrupole formula. We must remember, however, that our
physical quantities (and those of references [12, 13]) are defined at infinity and the “dots”
in our formula cannot be translated directly to “time derivatives” in the near-zone. On the
other hand, the quadrupole formula has been proved [19] only in a linear (in the metric
tensor) approximation to the field equations.

The entire solution depends on an arbitrary function of one variable, \(a(u)\), that is qual-
itatively related to the particle separation. This function is determined by the equation of
motion which, however, cannot be evaluated at infinity.

In conclusion, we wish to stress that the solution obtained in this paper (with the choice
(36) for \(c_1\)) is a formal asymptotic approximation (for large \(\xi\)) to an exact solution of the
full non-linear Einstein equations that has all the required properties for describing the two
particle collision problem. The solution can be continued to any desired order in the expan-
sion and appears to converge for \(\xi > a(u)\) in the sense that no large numerical coefficients
appear. However, it is safer to assume that it is only valid for, say, \(\xi > 10 a(u)\). The inter-
pretation of the solution, which was also the motivation for making the above choices for the
free functions \(K\) and \(c_1\), avoids mathematically unjustified steps in Bondi’s interpretation;
however, it involves physical quantities which are defined only in a background Minkowski
space – the limiting form of the solution as \(\xi \to \infty\). Using this solution to calculate the
outer boundary data for the inner problem in a CCM approach [5, 7], the physical definition
of the function \(a(u)\) and of “time” in the near-zone can, in principle, be obtained.
Note added in proof: It can be shown that the form of $K$ given in (15) is valid for any mass distribution. Specifically, if we use spherical coordinates $r, x$ and define $\xi(r, x)$ by the series expansion (Newtonian potential of a unit-total-mass distribution)

$$
\frac{1}{\xi} = \frac{1}{r} + \frac{M_1(u)x}{r^2} + \frac{M_2(u)(3x^2 - 1)}{2r^3} + \frac{M_3(u)x(5x^2 - 3)}{2r^4} + \ldots
$$

and $\eta(r, x)$ by the requirement that $\nabla \xi \cdot \nabla \eta = 0$, then, proceeding as in Section III, we find that

$$
K = \frac{k_1(u)(3\eta^2 - 1)}{\xi} + \frac{k_2(u)(5\eta^2 - 3)}{\xi^2} + \ldots,
$$

where $k_1(u) = [M_1(u)^2 - M_2(u)]/4$ and $k_2(u) = -M_1(u)^3 + \frac{3}{2}M_1(u)M_2(u) - \frac{1}{2}M_3(u)$. Thus, when the dipole-moment vanishes, $k_1(u) = -M_2(u)/4 = -Q/(4M)$ for any mass distribution.

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