ENTIRE SOLUTIONS TO NONLINEAR SCALAR FIELD EQUATIONS WITH INDEFINITE LINEAR PART

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Abstract. We consider the stationary semilinear Schrödinger equation
\[-\Delta u + a(x)u = f(x, u), \quad u \in H^1(\mathbb{R}^N),\]
where $a$ and $f$ are continuous functions converging to some limits $a_\infty > 0$ and $f_\infty = f_\infty(u)$ as $|x| \to \infty$. In the indefinite setting where the Schrödinger operator $-\Delta + a$ has negative eigenvalues, we combine a reduction method with a topological argument to prove the existence of a solution of our problem under weak one-sided asymptotic estimates. The minimal energy level need not be attained in this case. In a second part of the paper, we prove the existence of ground-state solutions under more restrictive assumptions on $a$ and $f$. We stress that for some of our results we also allow zero to lie in the spectrum of $-\Delta + a$.

1. Introduction and main results

Consider the semilinear elliptic equation
\[-\Delta u + a(x)u = f(x, u), \quad u \in H^1(\mathbb{R}^N)\] (1)
where $a$ and $f$ are continuous functions, $f$ being superlinear and subcritical. We are interested in the existence of nontrivial solutions in the case where
(A1) \[\lim_{|x| \to \infty} a(x) = a_\infty \quad \text{and} \quad \lim_{|x| \to \infty} f(x, u) = f_\infty(u)\]
hold uniformly for $u$ in bounded sets, for some $a_\infty > 0$ and $f_\infty \in C(\mathbb{R})$.

In the definite case, $\inf_{x \in \mathbb{R}^N} a(x) > 0$, the existence of solutions to Problem (1) has been extensively studied over the past twenty-five years, see e.g. \[4, 7, 8, 9, 12, 16, 29\], and most attention has been given to nonlinearities of the type
\[f(x, u) = q(x)|u|^{p-2}u\] (2)
with $p > 2$, $p < \frac{2N}{N-2}$ in case $N \geq 3$ and a positive function $q$ on $\mathbb{R}^N$ converging to some positive limit $q_\infty$ as $|x| \to \infty$. The main issue in studying this equation is to overcome the lack of compactness of the problem. For example, the associated energy functional
\[J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + a(x)u^2 \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx,\]
where $F(x, u) = \int_0^u f(x, s) \, ds$, does not satisfy the Palais-Smale condition, since the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is not compact. Furthermore, the set of solutions

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of the limit problem
\[
\begin{aligned}
-\Delta u + a_\infty u &= f_\infty(u), \\
u &\in H^1(\mathbb{R}^N),
\end{aligned}
\]
is invariant under translations and hence not compact. On the other hand, the
concentration-compactness principle of P.-L. Lions \[17, 18\] provides a tool to un-
derstand the nature of the lack of compactness. Using this principle, Ding and Ni
\[9\] established the existence of a ground-state solution in the special case \(2\), \(a \equiv 1\) and assuming \(q_\infty = \inf_{\mathbb{R}^N} q\). Here, by a ground-state solution, we mean a solution
with least possible energy value. It is easy to see that such a solution, which can
be obtained by constrained minimization, does not exist in the special case where
\(a \equiv 1\), \(2\) holds and \(q_\infty > q(x)\) for all \(x \in \mathbb{R}^N\). On the other hand, assuming \(2\),
\(a \equiv 1\) and only the weak one-sided estimate
\[
q(x) \geq q_\infty - C e^{-(2+\delta)\sqrt{|x|}},
\]
for some \(C, \delta > 0\), Bahri and Li \[3\] (see also \[4\]) still could prove the existence
of a positive solution of \(1\) by topological arguments combined with a minimax
principle. This solution is – in general – not a ground-state solution.

The main purpose of the present article is to extend the two kinds of results
mentioned above to the (possibly) indefinite case, i.e., to the case where \(\inf \sigma(-\Delta + a) < 0\). Here and in the following, \(\sigma(-\Delta + a)\) denotes the spectrum of the operator
\(-\Delta + a\). Since it follows from (A1) that the essential spectrum of \(-\Delta + a\) is given as
the interval \([a_\infty, \infty)\), the nonpositive part of \(\sigma(-\Delta + a)\) may only consist of finitely
many isolated eigenvalues. In particular, the operator \(-\Delta + a\) is negative (semi-
definite on a finite-dimensional subspace. In order to obtain results in this setting,
one has to control the effect of this negative spectral subspace. Our approach to this
problem uses the generalized Nehari manifold \(M\) corresponding to \(1\), which – in a
different setting – was introduced by Pankov \[19\] and studied further in \[26, 27\] (see
also \[21, 22\] for a related approach). The set \(M\) – which will be defined in Section 2
below – contains all solutions of \(1\), and minimizers of \(J\) on \(M\) are solutions of \(1\).
Therefore it is natural to call these minimizers ground-state solutions of \(1\). As in
the definite case, one may therefore distinguish between ground-state solutions and
further solutions obtained, e.g., by minimax principles on \(M\) relying on topological
arguments. We note that some existence results in the indefinite case have already
been obtained by Huang and Wang \[13\] using a classical linking theorem instead
of the generalized Nehari manifold. We will show that, under weaker assumptions
than in \[13\], a ground-state solution of \(1\) exists. Moreover, we will also treat
asymptotic conditions on \(a\) and \(f\) where – similarly as in the paper \[3\] for the
definite case – no ground-state solution can be expected to exist.

Another aim of this paper is to allow for more general nonlinearities \(f\) than in
previous papers. In order to state our first main result, we list assumptions on \(f \in C(\mathbb{R}^N \times \mathbb{R})\).

\begin{itemize}
\item[(F1)] \(|f(x, u)| \leq C_0 (1 + |u|^{p-1})\) for all \((x, u) \in \mathbb{R}^N \times \mathbb{R}\) with some constant \(C_0 > 0\)
\item[(F2)] \(|f(x, u)| = \alpha(|u|)\) as \(|u| \to 0\), uniformly in \(x\);
\item[(F3)] \(\frac{F(x, u)}{|u|^2} \to \infty\), uniformly in \(x\), as \(|u| \to \infty\), where \(F(x, u) = \int_0^u f(x, s) \, ds;\)
\end{itemize}
(F4) The mappings $u \mapsto \frac{f(x, u)}{|u|}$ and $u \mapsto \frac{f_\infty(u)}{|u|}$ are strictly increasing in $(-\infty, 0) \cup (0, \infty)$ for all $x \in \mathbb{R}^N$.

(F5) $f_\infty$ is odd, and for some $\theta > 0$ the mapping $u \mapsto \frac{f_\infty(u)}{u^{1+\theta}}$ is decreasing on $(0, \infty)$.

We also need the following stronger variant of (F2).

(F2$'$) There exists $\nu > 0$ such that $f(x, u) = o(|u|^{1+\nu})$ as $|u| \to 0$, uniformly in $x$;

Our first result reads as follows.

**Theorem 1.1.** Suppose (A1), (F1), (F2$'$) and (F3)-(F5) hold and $0 \notin \sigma(-\Delta + a)$. If $N \geq 2$ and the limit problem (3) admits a unique positive solution (up to translations), then (1) has at least one nontrivial solution provided there exists $C_1, C_2 \geq 0$ and $\alpha > 2$ such that

$$a(x) \leq a_\infty + C_1 e^{-\alpha \sqrt{a_\infty}|x|} \text{ and } F(x, u) \geq F_\infty(u) - C_2 e^{-\alpha \sqrt{a_\infty}|x|}(u^2 + u^p)$$

(4)

holds for all $x \in \mathbb{R}^N$, $u > 0$.

We point out that nonlinearities of the type (2) satisfy (F1)-(F5), as well as the weakly growing superlinear nonlinearity $f(x, u) = q(x) u \log(1 + |u|^s)$ with $s > 0$, provided $q(x) > 0$ for all $x \in \mathbb{R}^N$ and $\lim_{|x| \to \infty} q(x) = q_\infty > 0$ hold. The positive ground-state associated to the limit problem is unique for these nonlinearities, as follows from Theorem 1.1 in [14]. Moreover, the asymptotic estimate (4) is fulfilled if $q(x) \geq q_\infty - C_2 e^{-\alpha \sqrt{a_\infty}|x|}$ for all $x \in \mathbb{R}^N$.

For weakly growing superlinear nonlinearities of the type $f(x, u) = q(x) u \log(1 + |u|^s)$, Theorem 1.1 is even new in the definite case where the function $a$ is positive in $\mathbb{R}^N$. We point out that these weakly growing superlinear nonlinearities do not satisfy the usual Ambrosetti-Rabinowitz growth condition [2] which guarantees the boundedness of Palais-Smale sequences. We also note that the assumption (4) is weaker than the corresponding assumption in the paper [3] of Bahri and Li, where only the case $a \equiv 1$ was considered. In the indefinite case where $\inf \sigma(-\Delta + a) < 0$, we are not aware of any existence result under assumption (4). Our approach to prove Theorem 1.1 is strongly inspired by the work of Bahri and Li [3] in the definite case, but there are crucial differences. Most importantly, while the topological minimax argument of [3] is carried out on a unit sphere in a weighted $L^p$-space, we have to use projection maps onto the generalized Nehari manifold. Therefore the required asymptotic estimates are much harder to derive. In particular, we need to deal with eigenfunctions of $-\Delta + a$ corresponding to negative eigenvalues and their asymptotic decay. In [3], these difficulties were avoided by assuming $a \equiv 1$ and therefore dealing with the most simple spectral theoretic situation.

As in the definite case treated in [3], the solution obtained by Theorem 1.1 is not a ground-state solution in general. In the following result, we show that, strengthening the condition on $a$ or $f$ in the spirit of [3], the problem (1) admits a ground-state solution even without the condition (F5) and with (F2) instead of (F2$'$).
Theorem 1.2. Suppose (A1) and (F1)–(F4) hold. If there exists \( \theta > 0 \) and \( r_1 > 0 \) such that
\[
\inf_{x \in \mathbb{R}^N, 0 < |u| \leq r_1} \frac{|f(x,u)|}{|u|^{1+\theta}} > 0
\]
holds, then (1) admits a (nontrivial) ground-state solution, provided one of the following sets of conditions is satisfied.

(a) There exists \( S_0, C_1 > 0 \) and \( 0 < \alpha < \frac{2+\theta}{1+\theta} \) such that
\[
a(x) \leq a_{\infty} - C_1 e^{-\alpha \sqrt{a_{\infty}} |x|} \quad \text{for all } |x| \geq S_0,
\]
and there exists \( \mu > \alpha, C_2 \geq 0 \) for which
\[
F(x,u) \geq F_{\infty}(u) - C_2 e^{-\mu \sqrt{a_{\infty}} |x|} (u^2 + |u|^p) \quad \text{holds for all } x \in \mathbb{R}^N, u \in \mathbb{R}.
\]

(b) There exists \( S_0 > 0 \) for which \( a(x) \leq a_{\infty} \) holds for all \( |x| \geq S_0 \), and for every \( \eta > 0 \), there exists \( 0 < \alpha < \frac{2+\theta}{1+\theta}, C_\eta, S_\eta > 0 \) such that
\[
F(x,u) \geq F_{\infty}(u) + C_\eta e^{-\alpha \sqrt{a_{\infty}} |x|}, \quad \text{for all } |x| \geq S_\eta, \eta \leq |u| \leq \frac{1}{\eta}.
\]

Furthermore, if \( 0 \notin \sigma(-\Delta + a) \) the conclusion also holds without (5), and every \( 0 < \alpha < 2 \) is admissible in (a) and (b) above.

To our knowledge, Theorem 1.2 is the first result in this (noncompact) setting yielding existence of solutions in the case where 0 is an eigenvalue of \(-\Delta + a\). Since the eigenfunctions associated the eigenvalue 0 exhibit a slower decay rate than the ones corresponding to negative eigenvalues, we cannot expect, in general, to allow every value \( \alpha \in (0, 2) \) in Theorem 1.2. Nevertheless, any \( \alpha \in (0, 1] \) is allowed, since \( \frac{2+\theta}{1+\theta} = 1 + \frac{1}{1+\theta} > 1 \) for \( \theta > 0 \). In the case where \( 0 \notin \sigma(-\Delta + a) \) is considered, Theorem 1.2 is a generalization of results of Huang and Wang [13]. More precisely, we obtain the existence of solutions under weaker assumptions upon the potential \( a \). In particular, we only need to control the behavior of \( a_{\infty} - a(x) \) for large \( x \). Moreover, Theorem 1.2 provides the additional information that ground-state solutions exist.

The paper is organized as follows. We first state and prove some basic properties of the energy functional and the generalized Nehari manifold. Some crucial energy estimates are then derived before the actual proof of Theorem 1.1 is given. In the last section we give some further energy estimates under the assumptions of Theorem 1.2 and conclude by proving the latter. Finally, in the appendix we prove a nonlinear splitting property for weakly converging sequences in \( H^1(\mathbb{R}^N) \), which is necessary for the decomposition of Palais-Smale sequences of \( J \). Here we adapt a result in [1] Appendix] which was stated for the periodic setting. In contrast to earlier results of this type (see e.g. [15]), no Lipschitz continuity of \( f \) or bounds on \( f' \) are required here.

2. Preliminaries

Throughout this paper we shall use the following notation. For a function \( u \) on \( \mathbb{R}^N \) and an element \( y \in \mathbb{R}^N \), we write \( y \ast u \) for the translate of \( u \), i.e.,
\[
(y \ast u)(x) := u(x - y), \quad x \in \mathbb{R}^N.
\]
Let $X$ be any normed space, we will denote by $B_r(u)$ the open ball in $X$ centered at $u \in X$ with radius $r > 0$.

According to (A1), the essential spectrum of $-\Delta + a$ is equal to $[\alpha, +\infty)$ (see e.g. [24, Theorem 3.15]), and $\sigma(-\Delta + a) \cap (-\infty, \alpha_1)$ consists of isolated eigenvalues of finite multiplicity. Let

$$\inf \sigma(-\Delta + a) = \lambda_1 \leq \ldots \leq \lambda_n < 0 = \lambda_{n+1} = \ldots = \lambda_{n+l} (< \lambda_{n+l+1})$$

denote the nonpositive eigenvalues (repeated according to multiplicity), and consider a corresponding orthonormal set of eigenfunctions $e_1, \ldots, e_{n+l} \in H^2(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$. Setting $E^- = \text{span}\{e_1, \ldots, e_n\}$, $E^0 = \text{span}\{e_{n+1}, \ldots, e_{n+l}\}$ (with the convention that $E^0 = \{0\}$ if $l = 0$) and $E^0 = (E^- \oplus E^0)^\perp$, we have the so-called spectral decomposition

$$E := H^1(\mathbb{R}^N) = E^+ \oplus E^0 \oplus E^-$$

(9)

corresponding to $-\Delta + a$. Moreover, the eigenfunctions satisfy the following exponential decay estimates (see [24, Theorem 3.19]) which play a crucial role in the sequel.

If $1 \leq i \leq n$, then

$$\lim_{|x| \to \infty} |e_i(x)|e^{(1+\delta)\sqrt{\lambda_i}|x|} = 0 \quad \text{for every } 0 < \delta < \sqrt{1 + \frac{\lambda_i}{\alpha}} - 1. \quad (10)$$

On the other hand, if $n + 1 \leq i \leq n + l$, then $\lambda_i = 0$ and

$$\lim_{|x| \to \infty} |e_i(x)|e^{(1-\delta)\sqrt{\lambda_i}|x|} = 0 \quad \text{for every } \delta > 0. \quad (11)$$

2.1. Energy functional and generalized Nehari manifold. We assume for the remainder of this section that (A1), (F1)--(F4) hold, and denote by $\| \cdot \|$ an equivalent norm on $E = H^1(\mathbb{R}^N)$ which satisfies

$$\int_{\mathbb{R}^N} |\nabla u|^2 + a(x)u^2 \, dx = \|u^+\|^2 - \|u^-\|^2, \quad \text{for } u \in E.$$ 

Here and in the sequel, we let $u^\pm$ and $u^0$, respectively, be the projections of $u \in E$ onto $E^\pm$ and $E^0$, respectively, according to the decomposition (9).

The solutions of (11) are critical points of the energy functional $J: E \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^N} F(x, u) \, dx, \quad u \in E.$$ 

Considering the generalized Nehari manifold (see e.g. [27, Chapter 4])

$$\mathcal{M} = \{u \in E \setminus (E^- \oplus E^0) : J'(u)(tu + h) = 0, \quad t \geq 0, \quad h \in E^- \oplus E^0\},$$

we set

$$c = \inf_{u \in \mathcal{M}} J(u).$$

For the limit problem (3), we set

$$J_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + a_\infty u^2 \, dx - \int_{\mathbb{R}^N} F_\infty(u) \, dx \quad \text{for } u \in E,$$

consider the associated Nehari manifold $\mathcal{M}_\infty = \{u \in E \setminus \{0\} : J_\infty'(u)u = 0\}$ and let $c_\infty = \inf_{u \in \mathcal{M}_\infty} J_\infty(u)$. 

We recall that, since $a_\infty > 0$ holds and since $f_\infty$ satisfies the conditions (F1)–(F4), Problem 6 admits a ground-state solution $u_\infty \in E\{0\}$ (see 27 Theorem 3.13.) There holds $J_\infty(u_\infty) = c_\infty > 0$ and $J'_\infty(u_\infty) = 0$.

Before we give some properties of $M$, let us point out a few facts concerning the functions $f$, $f_\infty$ and their primitive $F$, $F_\infty$.

**Lemma 2.1.**

(i) For every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|f(x, u)| \leq \varepsilon |u| + C_\varepsilon |u|^{p-1} \quad \text{and} \quad F(x, u) \leq \varepsilon |u|^2 + C_\varepsilon |u|^p.$$  

(ii) For all $x \in \mathbb{R}^N$, $u \in \mathbb{R}$.

(iii) The inequality is obviously satisfied if $u = 0$ or $v = 0$. Moreover, for $0 < v \leq u$, we deduce from (13) and (F2')

$$F_\infty(u + v) - F_\infty(u) - F_\infty(v) - f_\infty(u)v - f_\infty(v)u \geq -F_\infty(v) - f_\infty(v)u$$

$$- \int_0^u \frac{f_\infty(t)}{t^{1+v}} t^{1+v} dt - \frac{f_\infty(v)}{v^{1+v}} u^{1+v}$$

$$\geq - \frac{\hat{C}_\rho}{(2+v)^2} u^{2+v} - \hat{C}_\rho uv^{1+v} \geq - \frac{3}{2} \hat{C}_\rho u^{1+\frac{2}{p}} v^{1+\frac{2}{p}},$$

where $\hat{C}_\rho := \sup_{0 < \rho \leq \rho} \frac{\rho^2}{\rho^{1+v}} < +\infty$. Since (13) and (14) are symmetric in $u$ and $v$, the same estimate holds for $0 < u \leq v$, and the proof is complete. \hfill \Box

We now study more closely the set $M$ and the behavior of $J$ on it.

**Lemma 2.2 (Properties of $M$).** There holds

(i) $\frac{1}{2} \int_{\mathbb{R}^N} f(x, u)u \, dx > \int_{\mathbb{R}^N} F(x, u) \, dx$ for all $u \in E\{0\}$, and the functional $u \mapsto \int_{\mathbb{R}^N} F(x, u) \, dx$ is weakly lower semicontinuous.

(ii) For each $w \in E\{E^- \oplus E^0\}$ let $\hat{E}(w) := \{tw + h : t \geq 0, h \in E^0 \oplus E^-\}$. Then there exists a unique nontrivial critical point $\hat{m}(w)$ of $J_{\hat{E}(w)}$. Moreover, $\hat{m}(w)$ is the unique global maximum of $J_{\hat{E}(w)}$.

(iii) There exists $\delta > 0$ such that $\|\hat{m}(w)\| \geq \delta$ for all $w \in E\{E^- \oplus E^0\}$, and for each compact subset $W \subset E\{E^- \oplus E^0\}$ there is a constant $C_W > 0$ such that $\|\hat{m}(w)\| \leq C_W$ for all $w \in W$.

**Proof.**

(i) The first assertion follows from 26 Lemma 2.1, and 25 Theorem 1.6] gives the second one.
Lemma 2.3

(ii) Similar to the proof of [26] Proposition 2.3], we see that for all $u \in \mathcal{M}$, there holds $J(u) > J(v)$ for every $v \in \tilde{E}(u) \setminus \{u\}$.

Let now $w \in E \setminus (E^- \oplus E^0)$. It is enough to prove that $\mathcal{M} \cap \tilde{E}(w) \neq \emptyset$ holds. With $w^+ \neq 0$, we set $v := \frac{w^+}{\|w^+\|}$ and claim that for $t \geq 0$, $h \in E^- \oplus E^0$, we have $J(tv + h) \leq 0$ if $\|tv + h\|$ is large. Indeed, suppose, by contradiction, let $(v_k)_k$ be a Palais-Smale sequence for $J$. Setting $v_k := \frac{t_k v + h_k}{\|t_k v + h_k\|} = s_k v + z_k$ for all $k$, we first note that $(s_k)_k$ and $(z_k)_k$ are bounded sequences, since $1 = \|v_k\|^2 = s_k^2 + \|z_k\|^2 + \|z_0\|^2$ holds for all $k$. Thus, up to a subsequence, we can assume $v_k \rightharpoonup s v$ for some $s \geq 0$, and $v_k \rightarrow v$ as $k \rightarrow \infty$. In particular, $s v + z \neq 0$, and therefore, $|t_k v(x) + h_k(x)| \rightarrow \infty$ as $k \rightarrow \infty$, for a.e. $x \in \mathbb{R}^N$ such that $sv(x) + z(x) \neq 0$. Condition (F3) together with Fatou’s Lemma now gives

$$
\frac{1}{2} s_k^2 - \frac{1}{2} \|z_k\|^2 - \frac{J(t_k v + h_k)}{\|t_k v + h_k\|^2} = \int_{\mathbb{R}^N} \frac{F(x, t_k v + h_k)}{(t_k v + h_k)^2} v_k^2 \, dx \rightarrow \infty,
$$

which contradicts the assumption $J(t_k v + h_k) \geq 0$ for all $k$ and thus proves the claim.

Next, we notice that (12) implies $J(tv) > 0$ for $t > 0$ small. Consequently $0 < \sup_{u \in \mathcal{E}(w)} J(u) < +\infty$, and we conclude as in the proof of [26] Lemma 2.6.

(iii) A similar proof as [26] Lemma 2.4] gives the first assertion. For the second one, we simply note that $\mathfrak{m}(w)$ has the form $tw^+ + h$ with $t \geq 0$ and $h \in E^- \oplus E^0$. Hence, the same argument as in the proof of (ii), together with the fact that $J(\mathfrak{m}(w)) = \int_{\mathbb{R}^N} \frac{1}{2} f(x, w) \mathfrak{m}(w) - F(x, \mathfrak{m}(w)) \, dx > 0$, implies that $\mathfrak{m}(w)$ is uniformly bounded for $w \in \mathcal{W}$, since this set is compact.

Lemma 2.3 (Coercivity). Every sequence $(u_k)_k \subset \mathcal{M}$ with $\lim_{k \to \infty} \|u_k\| = \infty$ satisfies $\lim_{k \to \infty} J(u_k) = \infty$. In particular, all Palais-Smale sequences for $J$ in $\mathcal{M}$ are bounded.

Proof. By contradiction, let $(u_k)_k \subset \mathcal{M}$ satisfy $d := \sup_{k \in \mathbb{N}} J(u_k) < \infty$ and $\lim_{k \to \infty} \|u_k\| = \infty$. Let $v_k := \frac{u_k}{\|u_k\|}$ for $k \in \mathbb{N}$. We first claim that

$$
\|v_k^+\| \neq 0 \quad \text{as } k \to \infty. \tag{15}
$$

Indeed, suppose that $\|v_k^+\| \rightarrow 0$. Then, since $0 < J(u_k) \leq \frac{1}{2} (\|u_k^+\|^2 - \|u_k^-\|^2)$, we have $\|u_k^-\| \leq \|u_k^+\|$ for all $k$ and therefore

$$
\|v_k^-\| \leq \|v_k^+\| \rightarrow 0 \quad \text{as } k \to \infty.
$$

As a consequence, since $E^0$ is finite-dimensional, we may pass to a subsequence such that $v_k \rightharpoonup v$, where $v \in E^0$ satisfies $\|v\| = 1$. Since $|u_k(x)| \to \infty$ for a.e. $x \in \mathbb{R}^N$ with $v(x) \neq 0$, it follows from (F3) and Fatou’s lemma that

$$
\int_{\mathbb{R}^N} F(x, u_k)_{u_k^+} v_k^2 \, dx \to \infty \quad \text{as } k \to \infty,
$$
and therefore
\[
0 \leq \frac{J(u_k)}{\|u_k\|^2} = \frac{1}{2} (\|v_k^+\|^2 - \|v_k^-\|^2) - \int_{\mathbb{R}^N} \frac{F(x,u_k)}{u_k^2} v_k^2 \, dx \to -\infty
\]
as \(k \to \infty\), a contradiction. Hence (15) holds, and therefore we may pass to a subsequence such that
\[
\sigma := \inf_{k \in \mathbb{N}} \|v_k^+\| > 0.
\]

Next we claim that
\[
v_k^+ \not\to 0 \in L^p(\mathbb{R}^N),
\]
where \(p > 2\) is as in (F1). Indeed, suppose by contradiction that \(v_k^+ \to 0\) in \(L^p(\mathbb{R}^N)\), and let \(s > 0\). Then (12) yields \(\lim_{k \to \infty} \int_{\mathbb{R}^N} F(x,s v_k^+) \, dx = 0\). Moreover, since \(s v_k^+ \in \dot{E}(u_k)\), Lemma 2.2(ii) implies that
\[
d \geq J(u_k) \geq J(sv_k^+) = \frac{1}{2} \|sv_k^+\|^2 - \int_{\mathbb{R}^N} F(x,sv_k^+) \, dx
\]
\[
\geq \frac{(s\sigma)^2}{2} - \int_{\mathbb{R}^N} F(x,sv_k^+) \, dx \to \frac{(s\sigma)^2}{2}
\]
for \(k \to \infty\). Since \(s > 0\) was arbitrary, we get a contradiction.

By (17) and Lions’ Lemma [18, Lemma I.1], there exists a sequence \((y_k)_k\) in \(\mathbb{R}^N\) such that, after passing to a subsequence, \(\inf_{k \in \mathbb{N}} \int_{B_{1}(0)} (y_k \ast v_k^+)^2 \, dx > 0\) and therefore, passing again to a subsequence, \(y_k \ast v_k^+ \to v\) as \(k \to \infty\), where \(v \in H^1(\mathbb{R}^N) \setminus \{0\}\).

Since \(\dim(E^- \oplus E^0) < +\infty\), we can find \(z \in E^- \oplus E^0\) such that, up to a subsequence, \(v_k^+ + v_k^- \to z\) holds, as \(k \to \infty\). If the sequence \((y_k)_k\) is bounded, we even have \(v_k^+ \to w\) for some \(w \in E^+ \setminus \{0\}\), up to a subsequence, and therefore we obtain \(v_k \to w + z \neq 0\) as \(k \to \infty\). Passing again to a subsequence, we may then also assume
\[
v_k \to w + z \quad \text{a.e. in } \mathbb{R}^N.
\]

On the other hand, if \((y_k)_k\) is unbounded, we may pass to a subsequence satisfying \(|y_k| \to \infty\) and, consequently, \(y_k \ast (v_k^- + v_k^0) \to 0\) as \(k \to \infty\). This gives \(y_k \ast v_k \to v \neq 0\), and we may pass to a subsequence satisfying
\[
y_k \ast v_k \to v \quad \text{a.e. in } \mathbb{R}^N.
\]

Now, we remark that
\[
\int_{\mathbb{R}^N} \frac{F(x,u_k)}{u_k^2} v_k^2 \, dx = \int_{\mathbb{R}^N} \frac{F(x-y_k,y_k \ast u_k)}{(y_k \ast u_k)^2} (y_k \ast v_k)^2 \, dx.
\]
Moreover, (18) implies \(|u_k| \to \infty\) pointwise a.e. where \(w + z \neq 0\), while (19) implies \(|y_k \ast u_k| \to \infty\) a.e. where \(v \neq 0\). Hence (F3) and Fatou’s Lemma again imply that
\[
\int_{\mathbb{R}^N} \frac{F(x,u_k)}{u_k^2} v_k^2 \, dx = \int_{\mathbb{R}^N} \frac{F(x-y_k,y_k \ast u_k)}{(y_k \ast u_k)^2} (y_k \ast v_k)^2 \, dx \to \infty \quad \text{as } k \to \infty,
\]
and therefore
\[
0 \leq \frac{J(u_k)}{\|u_k\|^2} = \frac{1}{2} (\|v_k^+\|^2 - \|v_k^-\|^2) - \int_{\mathbb{R}^N} \frac{F(x,u_k)}{u_k^2} v_k^2 \, dx \to -\infty.
\]
This contradiction finishes the proof. \(\square\)

From Propositions 4.1, 4.2 and Corollary 4.3 in [27], it follows that
Lemma 2.4. (a) The map \( m : E \setminus (E^{-} \oplus E^{0}) \to \mathcal{M} \) given by Lemma 2.2(ii) is continuous and its restriction \( m \) to \( \mathcal{S}^{+} := \{ u \in E^{+} : \| u \| = 1 \} \) is a homeomorphism with inverse given by \( m^{-1}(v) = \frac{v^{+}}{\| v^{+} \|}, v \in \mathcal{M} \).

(b) The functional \( \Psi : E^{+} \setminus \{ 0 \} \to \mathbb{R} \) defined by \( \Psi(w) = J(m(w)) \) is of class \( \mathcal{C}^{1} \).

Furthermore, \( \Psi := \Psi|_{\mathcal{S}^{+}} \) is also \( \mathcal{C}^{1} \) with \( \Psi'(w)z = \| m(w) \|^{+} \| J'(m(w)) \| z \) for every \( z \in T_{w} \mathcal{S}^{+} = \{ v \in E^{+} : \langle w, v \rangle = 0 \} \).

(c) \( (w_{k})_{k} \subset \mathcal{S}^{+} \) is a Palais-Smale sequence for \( \Psi \) if and only if \( (m(w_{k}))_{k} \subset \mathcal{M} \) is a Palais-Smale sequence for \( J \).

(d) \( \inf_{\mathcal{S}^{+}} \Psi = \inf_{\mathcal{M}} J = c \).

Proposition 2.5. Suppose \((A1)\) and \((F1)-(F4)\) are satisfied. If in addition \( c < c_{\infty} \) holds, then \( \{ u \} \) has a nontrivial ground-state solution.

Proof. Let \( (v_{k}) \subset \mathcal{S}^{+} \) be a minimizing sequence for \( \Psi \). By Ekeland’s variational principle [28, Theorem 8.5], we can find a sequence \( (w_{k}) \subset \mathcal{S}^{+} \) such that \( \| w_{k} - v_{k} \| \to 0, \Psi(w_{k}) \to \Psi = c \) and \( \| \Psi'(w_{k}) \| \to 0 \) as \( k \to \infty \). Setting \( u_{k} := m(w_{k}) \) for all \( k \), we obtain that \( (u_{k}) \subset \mathcal{M} \) is a Palais-Smale sequence for \( J \) at level \( c \). By Lemma 2.3, \( (u_{k}) \) is a bounded sequence. Thus, up to a subsequence, we may assume that \( u_{k} \rightharpoonup u \), weakly in \( E \), for some \( u \in E \), and the weak sequential continuity of \( J' \) gives \( J'(u) = 0 \). In particular, if \( u \neq 0 \) then \( u \in \mathcal{M} \) holds, and since \( J \) is lower semicontinuous on \( \mathcal{M} \) we obtain

\[
c \leq J(u) \leq \liminf_{k \to \infty} J(u_{k}) = c.
\]

On the other hand, if \( u = 0 \) holds, then we can find \( (y_{k}) \subset \mathbb{R}^{N} \) and \( \delta > 0 \) such that

\[
\liminf_{k \to \infty} \int_{B_{1}(0)} (y_{k} \ast u_{k})^{2} \, dx \geq \delta > 0.
\]

Indeed, if this were false, the concentration-compactness Lemma [18, Lemma 1.1] would imply \( \| u_{k} \|_{L^{p}} \to 0 \) as \( k \to \infty \) and, since

\[
c = \lim_{k \to \infty} J(u_{k}) = \lim_{k \to \infty} \int_{\mathbb{R}^{N}} \frac{1}{2} f(x, u_{k})u_{k} - F(x, u_{k}) \, dx \leq \varepsilon \sup_{k \in \mathbb{N}} \| u_{k} \|_{L^{2}} + C_{\varepsilon} \lim_{k \to \infty} \| u_{k} \|_{L^{p}}, \]

holds for all \( \varepsilon > 0 \), this would contradict the fact that \( c > 0 \).

Now, we also remark that \( (y_{k}) \) must be unbounded, since we are assuming \( u_{k} \to 0 \). Hence, passing to a subsequence if necessary, we may suppose \( |y_{k}| \to \infty \) and \( y_{k} \ast u_{k} \to w \) as \( k \to \infty \). The compact embedding \( H^{1}(B_{1}(0)) \to L^{2}(B_{1}(0)) \) then implies \( w \neq 0 \). For every \( t_{k} > 0 \), we have by Lemma 2.2(ii)

\[
J(u_{k}) \geq J(t_{k}u_{k}) = J_{\infty}(t_{k}(y_{k} \ast u_{k})) + \frac{t_{k}^{2}}{2} \int_{\mathbb{R}^{N}} (a(x + y_{k}) - a_{\infty}) (y_{k} \ast u_{k})^{2} \, dx
\]

\[
+ \int_{\mathbb{R}^{N}} F(x + y_{k}, t_{k}(y_{k} \ast u_{k})) - F_{\infty}(t_{k}(y_{k} \ast u_{k})) \, dx.
\]

Choosing \( t_{k} > 0 \) such that \( t_{k}(y_{k} \ast u_{k}) \in \mathcal{M}_{\infty} \) holds, it follows that \( (t_{k})_{k} \) is a bounded sequence, since \( y_{k} \ast u_{k} \rightharpoonup w \neq 0 \) (compare with [26, Proposition 2.7].)

Since \( a(x + y_{k}) \to a_{\infty} \) and \( y_{k} \ast u_{k} \to w \) as \( k \to \infty \), the dominated convergence theorem gives \( \int_{\mathbb{R}^{N}} (a(x + y_{k}) - a_{\infty})(y_{k} \ast u_{k})^{2} \, dx \to 0 \), as \( k \to \infty \). Moreover since \( t_{k}u_{k} \to 0 \) as \( k \to \infty \), \((A1)\) and \((F2)\) imply \( \int_{\mathbb{R}^{N}} F(x, t_{k}u_{k}) - F_{\infty}(t_{k}u_{k}) \, dx \to 0 \), as
$k \to \infty$. Hence, we conclude that
\[
 c = \lim_{k \to \infty} J(u_k) \geq \limsup_{k \to \infty} J(\kappa_k u_k) \geq c_\infty
\]
holds, which contradicts our assumption $c < c_\infty$ and gives the desired conclusion.　□

3. THE EXISTENCE OF A NONTRIVIAL SOLUTION

We now consider the case where $0 \notin \sigma(-\Delta + a)$ holds, and wish to prove the existence of a solution to (1) under the conditions of Theorem 1.1. We shall assume throughout this section that $N \geq 2$, (A1), (F1)–(F5) and (F2′) are satisfied for some $\nu > 0$. In addition, the ground-state solution of (3) will be required to be unique (up to translations).

The proof of Theorem 1.1 will rely on a topological degree argument applied to a barycenter type map. In order to set up a corresponding minimax principle which avoids noncompactness of Palais-Smale sequences, we first need some asymptotic estimates.

3.1. Asymptotic estimates. The properties (A1), (F1)–(F4) and the oddness of $f_\infty$ ensure the existence of a positive ground-state solution to the limit problem (3). More precisely, according to [5, Théorème 1], [6, Theorem 1] and [11, Theorem 2], there exists a ground-state solution $u_\infty \in C^2(\mathbb{R}^N)$ of (3), positive, radially symmetric, radially decreasing and satisfying the following exponential decay property:
\[
 \lim_{|x| \to \infty} u_\infty(x)|x|^\beta e^{\sigma |x|^2} \quad \text{exists and is positive (21)}
\]

We recall a result of [3] which we shall use repeatedly in the sequel.

Proposition 3.1. ([3, Proposition 1.2]) Let $\varphi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\psi \in C(\mathbb{R}^N)$ radially symmetric, satisfy for some $\sigma \geq 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$
\[
 \varphi(x)|x|^\beta e^{\sigma |x|} \longrightarrow \gamma \quad \text{as} \quad |x| \to \infty
\]
and
\[
 \int_{\mathbb{R}^N} |\psi(x)|(1 + |x|^\beta)e^{\sigma |x|} \, dx < \infty.
\]
Then, as $|y| \to \infty$, there holds
\[
 \left( \int_{\mathbb{R}^N} (y * \varphi)\psi \, dx \right) |y|^\beta e^{\sigma |y|} \longrightarrow \gamma \int_{\mathbb{R}^N} \psi(x) \exp(-\sigma x_1) \, dx.
\]

An immediate consequence of this proposition and the estimate (10) on the eigenfunctions $e_i$ is the existence of some constant $\kappa_1 > 0$ such that
\[
 \int_{\mathbb{R}^N} (y * u_\infty)|e_i| \, dx \leq \kappa_1 |y|^{-\frac{N-1}{2}} e^{-\sqrt{\kappa_1} |y|} \quad \text{for all} \ i = 1, \ldots, n \text{ and } |y| \geq 1. \quad (22)
\]

In the next result, we consider a convex combination of two translates of the ground-state solution $u_\infty$ and its projection on $\mathcal{M}$. We derive estimates concerning its behavior as these translates are moved far apart from each other and far away from the origin. The outcome of this study will be used to show that the energy of such a convex combination can be made smaller than $2c_\infty$ under suitable conditions (see Lemma 3.3). We introduce the following notation which will be used in the next two lemmata. For $y, z \in \mathbb{R}^N$, we let $|y, z| := \min\{|y|, |z|, |y - z|\}$, where $| \cdot |$ denotes the Euclidean norm on $\mathbb{R}^N$. 
Lemma 3.2.  
(i) There exists $S_1 \geq 1$ such that $(1-s)(y*u_\infty)+s(z*u_\infty) \notin E^-$, for all $y, z \in \mathbb{R}^N$ with $|y, z| \geq S_1$ and all $s \in [0, 1]$.
(ii) For $y, z \in \mathbb{R}^N$ with $|y, z| \geq S_1$, and $s \in [0, 1]$, let $t_\infty = t_\infty(s, y, z) > 0$ and $h_\infty = h_\infty(s, y, z) \in E^-$ be chosen such that
\[
\mathfrak{m}((1-s)(y*u_\infty)+s(z*u_\infty)) = t_\infty((1-s)(y*u_\infty)+s(z*u_\infty)+h_\infty)
\]
holds. Then
\[
0 < \inf_{|y, z| \geq S_1} t_\infty(s, y, z) \leq \sup_{|y, z| \geq S_1} t_\infty(s, y, z) < +\infty \tag{23}
\]
and there exists $\kappa_2 > 0$ such that
\[
\sup_{s \in [0, 1]} \|h_\infty(s, y, z)\| \leq \kappa_2 \max\{|y|^{-N-1}, |z|^{-N-1}\} e^{-\sqrt{\kappa_2} \min\{|y|, |z|\}} \tag{24}
\]
for $y, z \in \mathbb{R}^N$ with $|y, z| \geq S_1$.
Furthermore, if $((s_k, y_k, z_k))_k \subset [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N$ satisfies $\lim_{k \to \infty} s_k = s \in [0, 1]$ and $\lim_{k \to \infty} |y_k, z_k| = \infty$, there exists some $T = \lim_{k \to \infty} t_\infty(s_k, y_k, z_k) > 0$ such that
\[
J(\mathfrak{m}((1-s_k)(y_k*u_\infty)+s_k(z_k*u_\infty))) \rightarrow J_\infty((1-s)Tu_\infty)+J_\infty(sTu_\infty), \tag{25}
\]
as $k \to \infty$. Moreover, $T = T(s)$ is uniquely determined by the relation
\[
\int_{\mathbb{R}^N} \left[\frac{(1-s)^2}{2}+\frac{s^2}{2}\right] T |h_\infty(u_\infty)-((1-s)Tu_\infty)-s f_\infty(sTu_\infty)| u_\infty \ dx = 0. \tag{26}
\]

Proof.  
(i) Since $a(x) \to a_\infty$ as $|x| \to \infty$ and $((z-y)*u_\infty)(x) \to 0$ as $|z-y| \to \infty$ for all $x \in \mathbb{R}^N$, the dominated convergence theorem implies that
\[
\int_{\mathbb{R}^N} \left|\nabla((1-s)(y*u_\infty)+s(z*u_\infty))\right|^2 + a(x)\left|((1-s)(y*u_\infty)+s(z*u_\infty))\right|^2 \ dx
\]
\[
= \left[\frac{(1-s)^2}{2}+\frac{s^2}{2}\right] \int_{\mathbb{R}^N} \|\nabla u_\infty\|^2 + a_\infty u_\infty^2 \ dx \tag{27}
\]
as $|y, z| \to \infty$. Since $\int_{\mathbb{R}^N} |\nabla h|^2 + a(x) h^2 \ dx = -\|h\|^2 \leq 0$ for all $h \in E^-$, the conclusion follows from (27) and the fact that $(1-s)^2 + s^2 \geq \frac{1}{2}$ for all $s \in [0, 1]$.
(ii) We set $w_\infty := (1-s)(y*u_\infty)+s(z*u_\infty)$. Since $J'(t_\infty|w_\infty+h_\infty|)(t_\infty|w_\infty+h_\infty|) = 0$, we find
\[
0 < 2c \leq 2J(t_\infty|w_\infty+h_\infty|) + 2 \int_{\mathbb{R}^N} F(x, t_\infty|w_\infty+h_\infty|) \ dx
\]
\[
= t_\infty^2 \int_{\mathbb{R}^N} \|\nabla(w_\infty+h_\infty)\|^2 + a(x)\|w_\infty+h_\infty\|^2 \ dx
\]
\[
\leq t_\infty^2 \left\{ \int_{\mathbb{R}^N} |\nabla w_\infty|^2 + a(x)w_\infty^2 \ dx + 4C\|u_\infty\| \|h_\infty\| - \|h_\infty\|^2 \right\}.
\]
Using (27), we deduce that $\sup_{|y, z| \geq S_1} \|h_\infty(s, y, z)\| < +\infty$ and
\[
\inf_{|y, z| \geq S_1} t_\infty(s, y, z) > 0. \text{ Now suppose by contradiction, that } t_\infty \text{ is not bounded above, and let } ((s_k, y_k, z_k))_k \subset [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N \text{ satisfy } |y, z| \geq S_1
\]
for all $k$ as well as $\lim_{k \to \infty} t_\infty(s_k, y_k, z_k) = \infty$. Up to a subsequence, we may assume $s_k \to s \in [0, 1]$, and the continuity of $\bar{m}$ then implies $\lim_{k \to \infty} |y_k| = \infty$ and $s \neq 1$, or $\lim_{k \to \infty} |z_k| = \infty$ and $s \neq 0$, up to a subsequence.

We consider the case $s \neq 1$ and $|y_k| \to \infty$ as $k \to \infty$; the other case follows similarly. Writing $h_\infty = \sum_{i=1}^{n} A_i^\infty e_i$ with $A_i^\infty, \ldots, A_n^\infty \in \mathbb{R}$, the upper bound on $\|h_\infty\|$ and the decay property (10) of the eigenfunctions $e_1, \ldots, e_n$ imply $h_\infty(x + y_k) \to 0$ as $k \to \infty$ for all $x \in \mathbb{R}^N$. Since $u_\infty > 0$ holds on $\mathbb{R}^N$, we have for all $x \in \mathbb{R}^N$

$$t_\infty[w_\infty(x + y_k) + h_\infty(x + y_k)] \geq t_\infty[(1 - s_k)u_\infty(x) + h_\infty(x + y_k)] \to +\infty,$$

as $k \to \infty$. The assumption (F3) and Fatou’s Lemma then give

$$\int_{\mathbb{R}^N} |\nabla [w_\infty(x) + h_\infty(x)]|^2 + a(x)[w_\infty(x) + h_\infty(x)]^2 \, dx$$

$$= \frac{1}{t_\infty} \int_{\mathbb{R}^N} f(x, t_\infty[w_\infty(x) + h_\infty(x)])(w_\infty(x) + h_\infty(x)) \, dx$$

$$> 2 \int_{\mathbb{R}^N} \frac{F(x + y_k, t_\infty[w_\infty(x + y_k) + h_\infty(x + y_k)])}{(t_\infty[w_\infty(x + y_k) + h_\infty(x + y_k)])} (w_\infty(x + y_k) + h_\infty(x + y_k))^2 \, dx$$

$$\to +\infty,$$

as $k \to \infty$,

which contradicts the boundedness of $w_\infty + h_\infty$ and concludes the proof of (23).

Now the property $J'(t_\infty[w_\infty + h_\infty])h_\infty = 0$ and the inequality $f(x, u + v) \geq f(x, u)v$ for all $x \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, which follows from (F4), together give

$$\|h_\infty\|^2 \leq \int_{\mathbb{R}^N} \nabla w_\infty \cdot \nabla h_\infty + a(x)w_\infty h_\infty \, dx + t_\infty^{-1} \int_{\mathbb{R}^N} [f(x, t_\infty w_\infty) h_\infty] \, dx. \quad (28)$$

We write again $h_\infty = \sum_{i=1}^{n} A_i^\infty e_i$, so that

$$|\lambda_n| \sum_{i=1}^{n} (A_i^\infty)^2 \leq \|h_\infty\|^2 = - \sum_{i=1}^{n} \lambda_i(A_i^\infty)^2 \leq |\lambda_1| \sum_{i=1}^{n} (A_i^\infty)^2. \quad (29)$$

Then, using (14) and the facts that $u_\infty \in L^\infty(\mathbb{R}^N)$ and $w_\infty$ is positive, we infer from (23) and (29) that

$$\|h_\infty\|^2 \leq \int_{\mathbb{R}^N} w_\infty \left| \sum_{i=1}^{n} \lambda_i A_i^\infty e_i \right| \, dx + C' \int_{\mathbb{R}^N} w_\infty \left| \sum_{i=1}^{n} A_i^\infty e_i \right| \, dx$$

$$\leq C'' \|h_\infty\| \max_{1 \leq i \leq n} \int_{\mathbb{R}^N} w_\infty |e_i| \, dx,$$

with constants $C', C'' > 0$, and hence

$$\|h_\infty\| \leq C'' \max_{1 \leq i \leq n} \int_{\mathbb{R}^N} w_\infty |e_i| \, dx \leq C'' \max_{1 \leq i \leq n} \int_{\mathbb{R}^N} (y * u_\infty + z * u_\infty)|e_i| \, dx$$

$$\leq 2C'' \kappa_1 \max \{|y|^{-\frac{N-1}{2}}, |z|^{-\frac{N-1}{2}} \} e^{-\sqrt{\kappa_2} \min\{|y|, |z|\}}$$

by (22). This proves (24) with $\kappa_2 = 2C'' \kappa_1$. 

Let now \((s_k, y_k, z_k)\) be such that \(\lim_{k \to \infty} s_k = s\) and \(\lim_{k \to \infty} |y_k, z_k| = \infty\). By (28), we can assume that, up to a subsequence, \(\lim_{k \to \infty} t_{k \infty}(s_k, y_k, z_k) = T > 0\) holds. Consequently, we find

\[
0 = \lim_{k \to \infty} J'(t_{k \infty}[w_{k \infty} + h_{k \infty}]) t_{k \infty}[w_{k \infty} + h_{k \infty}]
= ((1 - s)^2 + s^2)T^2 \int_{\mathbb{R}^N} |\nabla u_{k \infty}|^2 + a_{k \infty} u_{k \infty}^2 \, dx
- \int_{\mathbb{R}^N} f_{k \infty}((1 - s)Tu_{k \infty})(1 - s)Tu_{k \infty} \, dx - \int_{\mathbb{R}^N} f_{k \infty}(sTu_{k \infty}) sTu_{k \infty} \, dx.
\]

Since \(J'_{k \infty}(u_{k \infty})u_{k \infty} = 0\), we conclude that \(T\) satisfies (26). The strict monotonicity in (F4) and the fact that \(u_{k \infty} > 0\) on \(\mathbb{R}^N\) give the uniqueness of \(T\) (recall that \(s \in [0, 1]\) is fixed). Hence the whole sequence \((t_{k \infty}(s_k, y_k, z_k))\) converges towards \(T\) and the latter is uniquely determined by \(s = \lim_{k \to \infty} s_k\).

To prove (25), remark that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^N} F(x, t_{k \infty}[w_{k \infty} + h_{k \infty}]) \, dx = \int_{\mathbb{R}^N} F_{\infty}((1 - s)Tu_{\infty}) \, dx + \int_{\mathbb{R}^N} F_{\infty}(sTu_{\infty}) \, dx
\]

holds. Hence, similar arguments as above imply \(\lim_{k \to \infty} J(t_{k \infty}[w_{k \infty} + h_{k \infty}]) = J_{\infty}((1 - s)Tu_{\infty}) + J_{\infty}(sTu_{\infty})\) which concludes the proof.

\[\square\]

**Remark 3.1.** Taking \(s = 0\) in the above lemma, we find \(\lim_{|y| \to \infty} J(\hat{m}(y * u_{\infty})) = c_{\infty}\). In particular, \(c \leq c_{\infty}\) holds.

The following result is crucial for the construction of the min-max value below.

We now work under the additional assumption (4) of Theorem 1.1. Furthermore, we may assume that

\[2 < \alpha < p\]

holds in (4).

**Lemma 3.3 (Energy estimate).** There exists \(S_2 \geq \frac{3}{4}S_1\) such that

\[J(\hat{m}((1 - s)(y * u_{\infty}) + s(z * u_{\infty}))) < 2c_{\infty}\]

holds for all \(s \in [0, 1]\), \(R \geq S_2\) and \(y, z \in \mathbb{R}^N\) with \(|y| \geq R\), \(|z| \geq R\) and \(\frac{2}{3}R \leq |y - z| \leq 2R\).

**Proof.** Throughout the proof, we consider

\[R \geq \frac{3}{2}S_1\]

and \(y, z \in \mathbb{R}^N\) with \(|y| \geq R\), \(|z| \geq R\) and \(\frac{2}{3}R \leq |y - z| \leq 2R\).

For such \(y, z\) and \(s \in [0, 1]\), we set \(w_{\infty} = (1 - s)(y * u_{\infty}) + s(z * u_{\infty})\) and choose \(t_{\infty}, h_{\infty}\) as in Lemma 3.2 (ii). We emphasize that \(w_{\infty}, t_{\infty}\) and \(h_{\infty}\) depend in a crucial way on \(y, z\), but we suppress this dependence in our notation. All constants in the following will neither depend on \(R\) nor on \(s, y, z\). In the sequel, we will bound terms relative to the asymptotic exchange energy

\[d_{y, z} := \int_{\mathbb{R}^N} f_{\infty}(y * u_{\infty})(z * u_{\infty}) \, dx = \int_{\mathbb{R}^N} f_{\infty}(u_{\infty})(z - y) * u_{\infty} \, dx.\]
We first collect a few easy consequences of Proposition 3.1. First, since by (F2′)
and (21) we have
\[ \int_{\mathbb{R}^N} f_\infty(u_\infty(x))e^{\sqrt{\sigma_\infty}|x|}(1 + |x|^\frac{N-1}{2}) \, dx < \infty, \]

Proposition 3.1 implies that there is \( \kappa_3 > 0 \) such that
\[ \frac{1}{\kappa_3}|y - z|^{-\frac{N-1}{2}}e^{-\sqrt{\sigma_\infty}|y - z|} \leq \kappa_3|y - z|^{-\frac{N-1}{2}}e^{-\sqrt{\sigma_\infty}|y - z|} \]
for \( R, y, z \) satisfying (31). Moreover, by making \( \kappa_3 \) larger if necessary, we may also assume that
\[ \max\{|y|^{1-N}, |z|^{1-N}\}e^{-2\sqrt{\sigma_\infty}\min\{|y|, |z|\}} \leq \kappa_3 R^{-\frac{N-1}{2}}d_{y,z} \]
for \( R, y, z \) satisfying (31). Now let \( \alpha > 2 \) be as in assumption (I). Applying Proposition 3.1 to \( \varphi = u_\infty^2 \), \( \psi(x) = e^{-\alpha\sqrt{\sigma_\infty}|x|} \), \( \sigma = 2\sqrt{\sigma_\infty} \) and \( \beta = N - 1 \), we obtain
\[ \int_{\mathbb{R}^N} e^{-\alpha\sqrt{\sigma_\infty}|x|}\{(y * u_\infty)^2 + (z * u_\infty)^2\} \, dx \leq C \max\{|y|^{1-N}, |z|^{1-N}\}e^{-2\sqrt{\sigma_\infty}\min\{|y|, |z|\}} \]
for all \( y, z \in \mathbb{R}^N \) with some constant \( C > 0 \). Therefore by (33) we have, making \( \kappa_3 \) larger if necessary,
\[ \int_{\mathbb{R}^N} e^{-\alpha\sqrt{\sigma_\infty}|x|}\{(y * u_\infty)^2 + (z * u_\infty)^2\} \, dx \leq \kappa_3 R^{-\frac{N-1}{2}}d_{y,z} \]
for all \( R, y, z \) satisfying (31). Moreover, taking (31) into account, and applying Proposition 3.1 to \( \varphi(x) = e^{-\alpha\sqrt{\sigma_\infty}|x|}, \psi = u_\infty^p, \sigma = \alpha\sqrt{\sigma_\infty} \) and \( \beta = \alpha\frac{N-1}{2} \), we find that
\[ \int_{\mathbb{R}^N} e^{-\alpha\sqrt{\sigma_\infty}|x|}\{(y * u_\infty)^p + (z * u_\infty)^p\} \, dx = \int_{\mathbb{R}^N} [(-y) * \varphi + (-z) * \varphi] \psi \, dx \]
\[ \leq C' \max\{|y|^{-\frac{N-1}{2}}, |z|^{-\frac{N-1}{2}}\}e^{-2\sqrt{\sigma_\infty}\min\{|y|, |z|\}} \]
\[ \leq C' \max\{|y|^{1-N}, |z|^{1-N}\}e^{-2\sqrt{\sigma_\infty}\min\{|y|, |z|\}} \]
for all \( y, z \in \mathbb{R}^N, |y|, |z| \geq 1 \), with some constant \( C' > 0 \). Therefore by (33) we have, making \( \kappa_3 \) again larger if necessary,
\[ \int_{\mathbb{R}^N} e^{-\alpha\sqrt{\sigma_\infty}|x|}\{(y * u_\infty)^p + (z * u_\infty)^p\} \, dx \leq \kappa_3 R^{-\frac{N-1}{2}}d_{y,z} \]
for all \( R, y, z \) satisfying (31). Finally, let \( \nu > 0 \) be as in assumption (F2′). Then applying Proposition 3.1 to \( \varphi = \psi = u_\infty^{1+\frac{\nu}{2}}, \alpha = \sqrt{\sigma_\infty} \) and \( \beta = N - 1 \) yields
\[ \int_{\mathbb{R}^N} (y * u_\infty)^{1+\frac{\nu}{2}}(z * u_\infty)^{1+\frac{\nu}{2}} \, dx = \int_{\mathbb{R}^N} ((y-z) * u_\infty)^{1+\frac{\nu}{2}}u_\infty^{1+\frac{\nu}{2}} \, dx \leq C''|y-z|^{1-N}e^{-\sqrt{\sigma_\infty}|y-z|} \]
for all \( y, z \in \mathbb{R}^N \) with some constant \( C'' > 0 \). Therefore, making \( \kappa_3 \) again larger if necessary,
\[ \int_{\mathbb{R}^N} (y * u_\infty)^{1+\frac{\nu}{2}}(z * u_\infty)^{1+\frac{\nu}{2}} \, dx \leq \kappa_3 R^{-\frac{N-1}{2}}d_{y,z} \]
holds for all \( R, y, z \) satisfying (31). We now have all the tools to estimate
\[ J(\tilde{\mathbf{m}}(w_\infty)) = \frac{\nu^2}{2} \int_{\mathbb{R}^N} |\nabla(w_\infty + h_\infty)|^{2} + \alpha(x)|w_\infty + h_\infty|^{2} \, dx - \int_{\mathbb{R}^N} F(x, t_\infty[w_\infty + h_\infty]), \, dx. \]
We start by estimating the first integral on the right-hand side which we split in the following way.

\[ \int_{\mathbb{R}^N} |\nabla(w_\infty + h_\infty)|^2 + a(x)|w_\infty + h_\infty|^2 \, dx \]

\[ = ((1 - s)^2 + s^2) \int_{\mathbb{R}^N} |\nabla u_\infty|^2 + a_\infty u_\infty^2 \, dx \]

\[ + 2(1 - s) s \int_{\mathbb{R}^N} \nabla(y * u_\infty) \cdot \nabla(z * u_\infty) + a_\infty(y * u_\infty)(z * u_\infty) \, dx \]

\[ + \int_{\mathbb{R}^N} (a(x) - a_\infty)w_\infty^2 \, dx + 2 \int_{\mathbb{R}^N} \nabla w_\infty \cdot \nabla h_\infty + a(x)w_\infty h_\infty \, dx - \|h_\infty\|^2. \]

The property \( J'_\infty(u_\infty) = 0 \) implies that

\[ \int_{\mathbb{R}^N} \nabla(y * u_\infty) \cdot \nabla(z * u_\infty) + a_\infty(y * u_\infty)(z * u_\infty) \, dx = \int_{\mathbb{R}^N} f_\infty(y * u_\infty)(z * u_\infty) = d_{y,z}. \quad (37) \]

We deduce from (34) and condition (4) that

\[ \int_{\mathbb{R}^N} (a(x) - a_\infty)u_\infty^2 \, dx \leq 2C_1 \int_{\mathbb{R}^N} [(y * u_\infty)^2 + (z * u_\infty)^2] e^{-\alpha \sqrt{\|u_\infty\|}} \, dx \]

\[ \leq 2C_1 \kappa_3 R^{-\frac{N-1}{2}} d_{y,z} \quad \text{for all } s \in [0, 1] \text{ and } R, y, z \text{satisfying (31)}. \]

As in the proof of Lemma 8.2 we write \( h_\infty = \sum_{i=1}^n A_i^\infty e_i \). Using (22), (24), (29) and (33), we obtain

\[ \int_{\mathbb{R}^N} \nabla w_\infty \cdot \nabla h_\infty + a(x)w_\infty h_\infty \, dx \leq \left( \sum_{i=1}^n (A_i^\infty)^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n \lambda_i^2 \left( \int_{\mathbb{R}^N} w_\infty |e_i| \, dx \right)^2 \right)^{\frac{1}{2}} \]

\[ \leq \kappa_4 \max\{|y|^{1-N}, |z|^{1-N}\} e^{-2 \sqrt{\|u_\infty\|} \min(|y|, |z|)} \leq \kappa_4 \kappa_3 R^{-\frac{N-1}{2}} d_{y,z} \]

for \( s \in [0, 1] \) and \( R, y, z \) satisfying (31). Here \( \kappa_4 := \frac{2 \sqrt{\|\Lambda\|}}{\sqrt{|\lambda_n|}} \kappa_1 \kappa_2. \) Turning to the second integral, we write

\[ \int_{\mathbb{R}^N} F(x, t_\infty[w_\infty + h_\infty]) \, dx = \int_{\mathbb{R}^N} F_\infty(st_\infty u_\infty) + F_\infty((1-s)t_\infty u_\infty) \, dx \]

\[ + \int_{\mathbb{R}^N} F_\infty(t_\infty w_\infty) - [F_\infty(st_\infty(y * u_\infty)) + F_\infty((1-s)t_\infty(z * u_\infty))] \, dx \]

\[ + \int_{\mathbb{R}^N} F(x, t_\infty w_\infty) - F_\infty(t_\infty w_\infty) \, dx + \int_{\mathbb{R}^N} F(x, t_\infty[w_\infty + h_\infty]) - F(x, t_\infty w_\infty) \, dx. \]

From (14) and (36), it follows that
\[
\int_{\mathbb{R}^N} F_\infty(t_\infty w_\infty) - [F_\infty(st_\infty(y * u_\infty)) + F_\infty((1 - s)t_\infty(z * u_\infty))] \, dx
\]
\[
\geq (1 - s)t_\infty \int_{\mathbb{R}^N} f_\infty(st_\infty(y * u_\infty))(z * u_\infty) \, dx
\]
\[
+ st_\infty \int_{\mathbb{R}^N} f_\infty((1 - s)t_\infty(z * u_\infty))(y * u_\infty) \, dx
\]
\[
- C t_\infty^{2+\nu} s^{1+\frac{2}{\nu}} (1 - s) \int_{\mathbb{R}^N} (y * u_\infty)^{1+\frac{2}{\nu}} (z * u_\infty)^{1+\frac{2}{\nu}} \, dx
\]
\[
\geq (1 - s)t_\infty \int_{\mathbb{R}^N} f_\infty(st_\infty(y * u_\infty))(z * u_\infty) \, dx
\]
\[
+ st_\infty \int_{\mathbb{R}^N} f_\infty((1 - s)t_\infty(z * u_\infty))(y * u_\infty) \, dx
\]
\[
- \kappa_5 R^{-\frac{N-1}{2}} d_{y,z}
\]
for \( s \in [0, 1] \) and \( R, y, z \) satisfying (31). Here \( \kappa_5 := C \kappa_3 \sup_{|y, z| \geq S_1} [t_\infty(s, y, z)]^{2+\nu} \) and \( C = C_\rho \) is the constant from (14) corresponding to the value
\[
\rho = \|u_\infty\|_\infty \sup_{|y, z| \geq S_1} t_\infty(s, y, z),
\]
which is finite by (23). Moreover, condition (4) as well as (24), (35) and (26) imply that
\[
\int_{\mathbb{R}^N} F(x, t_\infty w_\infty) - F_\infty(t_\infty w_\infty) \, dx \geq -2C_2 t_\infty^2 \int_{\mathbb{R}^N} e^{-\alpha \sqrt{|\nabla f|}} \{(y * u_\infty)^2 + (z * u_\infty)^2\} \, dx
\]
\[
- 2^{n-1} C_2 t_\infty^p \int_{\mathbb{R}^N} e^{-\alpha \sqrt{|\nabla f|}} \{(y * u_\infty)^p + (z * u_\infty)^p\} \, dx \geq -\kappa_6 R^{-\frac{N-1}{2}} d_{y,z},
\]
for \( s \in [0, 1] \) and \( R, y, z \) satisfying (31), where \( \kappa_6 > 0 \) is a constant. From (13), (23), (24) and (29), we finally obtain a constant \( \kappa_7 > 0 \) such that
\[
\int_{\mathbb{R}^N} F(x, t_\infty (w_\infty + h_\infty)) - F(x, t_\infty w_\infty) \, dx \geq \int_{\mathbb{R}^N} f(x, t_\infty w_\infty) h_\infty \, dx
\]
\[
= \sum_{i=1}^n A_i \int_{\mathbb{R}^N} f(x, t_\infty w_\infty) e_i \, dx \geq -\sup_{x \in \mathbb{R}^N} \|u_\infty\|_\infty \|x\|_\infty \sum_{i=1}^n |A_i| \int_{\mathbb{R}^N} w_\infty |e_i| \, dx
\]
\[
\geq -\kappa_7 R^{-\frac{N-1}{2}} d_{y,z}
\]
for \( s \in [0, 1] \) and \( R, y, z \) satisfying (31). Summarizing, we can write
\[
J(\hat{m}(w_\infty)) \leq J_\infty(st_\infty u_\infty) + J_\infty((1 - s)t_\infty u_\infty)
\]
\[
+ (s(1 - s)t_\infty^2 + \kappa_8 R^{-\frac{N-1}{2}}) d_{y,z}
\]
\[
- (1 - s)t_\infty \int_{\mathbb{R}^N} f_\infty(st_\infty(y * u_\infty))(z * u_\infty) \, dx
\]
\[
- st_\infty \int_{\mathbb{R}^N} f_\infty((1 - s)t_\infty(z * u_\infty))(y * u_\infty) \, dx.
\]
Lemma 3.4. Proposition II.1 (see also [28, Theorem 8.4]) holds in our context. Smale sequences taken from \(M\) and \(<\delta \kappa\) with \(s\) for all \(R\). We now claim that there is some \(c\) for these values of \(R,y,z\) and \(s\).

Next, conditions (F4) and (F5) give
\[
\int_{\mathbb{R}^N} f_\infty(s \infty(y \ast u_\infty))(z \ast u_\infty)dx \geq s \infty \min\{(s \infty)^0, 1\}d_{y,z}
\]
and
\[
\int_{\mathbb{R}^N} f_\infty((1 - s)\infty(y \ast u_\infty))(z \ast u_\infty)dx \geq (1 - s)\infty \min\{(1 - s)\infty)^0, 1\}d_{y,z},
\]
and therefore (38) yields
\[
J(\hat{m}(w_\infty)) - 2c_\infty \leq J(\hat{m}(w_\infty)) - J(\infty(s \infty u_\infty) - J(\infty((1 - s)\infty u_\infty)
\]
\[
\leq \left[s(1 - s)\infty^\prime \left(1 - \min\{(s \infty)^0, 1\} - \min\{(1 - s)\infty)^0, 1\}\right) + \kappa_8 R^{-\frac{N+1}{2}}\right]d_{y,z}.
\]
We now claim that there is some \(R_1 \geq \frac{3}{4} S_1\) and some \(\kappa_9 > 0\) such that
\[
s(1 - s)\infty^\prime \left(1 - \min\{(s \infty)^0, 1\} - \min\{(1 - s)\infty)^0, 1\}\right) < -\kappa_9
\]
for all \(s \in [\delta_0, 1 - \delta_0], R \geq R_1\) and \(y,z\) satisfying (31). For this we consider an arbitrary sequence \((s_k, y_k, z_k)_k \subset [\delta_0, 1 - \delta_0] \times \mathbb{R}^N \times \mathbb{R}^N\) such that \([y_k, z_k] \to \infty\) and \(s_k \to s \in [\delta_0, 1 - \delta_0]\) as \(k \to \infty\). According to Lemma 3.2, we have \(\lim_{k \to \infty} t_{\infty}(s_k, y_k, z_k) = T\) with \(T = T(s)\) given by (20). Note that \(T > 0\) by (20), and \(\min\{sT, (1 - s)T\} \geq \delta_0 T > 0\). Moreover, \((s(1 - s)T^2 \geq (\delta_0 T)^2 > 0, and (20) implies max\{sT, (1 - s)T\} \geq 1\). Consequently
\[
s(1 - s)T^2 \left(1 - \min\{(sT)^0, 1\} - \min\{(1 - s)T)^0, 1\}\right) < 0,
\]
and this shows that (11) holds for all \(s \in [\delta_0, 1 - \delta_0], R \geq R_1\) and \(y, z\) satisfying (31), where \(R_1 \geq \frac{3}{4} S_1\) and \(\kappa_9 > 0\) are suitable constants. Going back to (11), we conclude that
\[
J(\hat{m}(w_\infty)) \leq 2c_\infty - [\kappa_9 - \kappa_8 R^{-\frac{N+1}{2}}]d_{y,z},
\]
for these values of \(R, y, z\) and \(s\), and the right hand side of this inequality is smaller than \(2c_\infty\) for \(R\) large enough. Together with (39) this finishes the proof. \(\square\)

We conclude this preparatory section by describing the behavior of the Palais-Smale sequences taken from \(M\), and show that the result of Bahri and Lions [4 Proposition II.1] (see also [28, Theorem 8.4]) holds in our context.

Lemma 3.4. Let \((u_k)_k \subset M\) be a sequence for which \((J(u_k))_k\) is bounded and \(J'(u_k) \to 0\) as \(k \to \infty\) holds. Then, there exist \(\ell \in \mathbb{N} \cup \{0\}, (x_i^k)_k \subset \mathbb{R}^N, 1 \leq i \leq \ell,\) and \(\overline{w}, w_1, \ldots, w_\ell \in E\) satisfying (up to a subsequence)
(i) \(J'(\overline{w}) = 0,\)
(ii) \(J'(w_i) = 0, i = 1, \ldots, \ell,\)
(ii) \(|x_k^i| \to \infty\) and \(|x_k^i - x_k^j| \to \infty\) as \(k \to \infty\) for \(1 \leq i \neq j \leq \ell\),

(iii) \(|u_k - [\varpi + \sum_{i=1}^{\ell} x_k^i \ast w_i]| \to 0\) as \(k \to \infty\),

(iv) \(J(u_k) \to J(\varpi) + \sum_{i=1}^{\ell} J_\infty(w_i)\), as \(k \to \infty\).

**Proof.** From Lemma 2.3 \((u_k)_k\) is a bounded sequence in \(E\). Up to a subsequence, we may assume \(u_k \rightharpoonup \varpi\) for some \(\varpi \in E\) and \(J(u_k) \to d\) as \(k \to \infty\). Since \(J'\) is weakly sequentially continuous we obtain \(J'(\varpi) = 0\).

**Step 1:** Let \(v_k^1 := u_k - \varpi\) for all \(k \in \mathbb{N}\). Since \(v_k^1 \to 0\) in \(E\) and \(a(x) \to a_\infty\) for \(|x| \to \infty\), the compactness of the embedding \(H^1(B_R(0)) \hookrightarrow L^2(B_R(0))\) for all \(R > 0\) implies

\[
\int_{\mathbb{R}^N} (a(x) - a_\infty) |v_k^1|^2 \, dx \to 0, \quad \text{as } k \to \infty.
\] (42)

Moreover, from (A1) and (F2), it follows that

\[
\int_{\mathbb{R}^N} F(x, v_k^1) - F_\infty(v_k^1) \, dx \to 0, \quad \text{as } k \to \infty.
\] (43)

Consequently, as \(k \to \infty\), there holds \(J_\infty(v_k^1) = J(v_k^1) + o(1)\)

\[= J(u_k) - J(\varpi) + \int_{\mathbb{R}^N} [F(x, u_k) - F(x, \varpi) - F(x, v_k^1)] \, dx + o(1) = J(u_k) - J(\varpi) + o(1),\]

where the last step follows from Proposition A.1. (Remark that \(|\varpi(x)| \to \infty\) as \(|x| \to \infty\), since \(J'(\varpi) = 0\). See [20] Lemma 1.) For every \(\varphi \in E\), we have furthermore

\[
J'(v_k^1)\varphi = \int_{\mathbb{R}^N} \nabla v_k^1 \cdot \nabla \varphi + a_\infty v_k^1 \varphi \, dx - \int_{\mathbb{R}^N} f_\infty(v_k^1) \varphi \, dx
\]

\[= J'(u_k)\varphi - J'(\varpi)\varphi + \int_{\mathbb{R}^N} (a_\infty - a(x)) v_k^1 \varphi \, dx + \int_{\mathbb{R}^N} [f(x, v_k^1) - f_\infty(v_k^1)] \varphi \, dx
\]

\[+ \int_{\mathbb{R}^N} [f(x, u_k) - f(x, \varpi) - f(x, v_k^1)] \varphi \, dx.
\]

Since \(\lim_{k \to \infty} J'(u_k) = 0\) in \(H^{-1}\) and \(J'(\varpi) = 0\), similar arguments as above (using again Proposition A.1) imply

\[J'(v_k^1) \to 0\] in \(H^{-1}\), as \(k \to \infty\).

**Step 2:** Let

\[
\zeta := \limsup_{k \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_k^1|^2 \, dx \right).
\]

If \(\zeta = 0\), then [18] Lemma II.1] gives \(\|v_k^1\|_{L^p} \to 0\) as \(k \to \infty\), and from

\[
\|v_k^1\|^2 \leq C \int_{\mathbb{R}^N} |\nabla v_k^1|^2 + a_\infty(v_k^1)^2 \, dx = C(J'_\infty(v_k^1)v_k^1 + \int_{\mathbb{R}^N} f_\infty(v_k^1) \, dx),
\]

\[\leq C(J'_\infty(v_k^1)v_k^1 + \varepsilon \|v_k^1\|^2_{L^4} + C_\varepsilon \|v_k^1\|^p_{L^p})\] for all \(\varepsilon > 0\),

we obtain \(\|v_k^1\| \to 0\), and hence \(u_k \rightharpoonup \bar{u}\) in \(E\), as \(k \to \infty\), and the proof is complete.
On the other hand, if $\zeta > 0$, then, passing to a subsequence, we can find a sequence $(x^n_k)_k \subset \mathbb{R}^N$ satisfying $|x^n_k| \to \infty$ as $k \to \infty$ and

$$\int_{B_1(0)} [(-x^n_k) \cdot v^n_k]^2 dx = \int_{B_1(x^n_k)} (v^n_k)^2 dx > \frac{\zeta}{2}$$

for all $k$. Since $((-x^n_k) \cdot v^n_k)_k$ is bounded in $E$, passing to a further subsequence, we find $(-x^n_k) \cdot v^n_k \to w_1 \neq 0$, using the compactness of the embedding $H^1(B_1(0)) \hookrightarrow L^2(B_1(0))$. Since $J'_\infty$ is weakly sequentially continuous and invariant under translation, there holds $J'_\infty(w_1) = \lim_{k \to \infty} J'_\infty((-x^n_k) \cdot v^n_k) = \lim_{k \to \infty} J'_\infty(v^n_k) = 0 \in H^{-1}$.

Setting $v^n_k := v^n_k - (x^n_k \cdot w_1)$, we obtain $v^n_k \to 0$, and the same arguments as above, applied to $J'_\infty$, give

$$J'_\infty(v^n_k) = J'_\infty(v^n_k - J_\infty(w_1) + o(1) = J(u_k) - J(\bar{u}) - J_\infty(w_1) + o(1),$$

$J'_\infty(v^n_k) \to 0$ in $H^{-1}$ and $\|v^n_k\|^2 = \|u_k\|^2 - \|\bar{u}\|^2 - \|w_1\|^2 + o(1)$ as $k \to \infty$.

Iterating this procedure, we construct sequences $(x^n_k)_k \subset \mathbb{R}^N$ such that $|x^n_k| \to \infty$ and $|x^n_k - x^n_i| \to \infty$ for all $i \neq j$, as $k \to \infty$, and critical points $w_i$ of $J_\infty$ such that $J_\infty(u_k - \bar{w} - \sum_{j=1}^i x^n_j \cdot w_j) = J(u_k) - J(\bar{u}) - \sum_{j=1}^i J_\infty(w_j) + o(1)$, and $J'_\infty(u_k - \bar{w} - \sum_{j=1}^i x^n_j \cdot w_j) \to 0$ in $H^{-1}$ as $k \to \infty$. Since $J_\infty(w) \geq c_\infty > 0$ holds for every critical point $w$ of $J_\infty$, and since $(J(u_k))_k$ is bounded, the procedure has to stop after a finite number of steps.

### 3.2. The proof of Theorem 1.1

Suppose all the assumptions of Theorem 1.1 are satisfied. We shall prove the existence of a nontrivial solution to (1) in three steps. First note that $c \leq c_\infty$ holds, by Remark 3.1. If $c < c_\infty$ then Proposition 2.5 gives the desired conclusion. Hence, we can assume that $c = c_\infty$ holds.

Now, consider the barycenter function $\beta: E \setminus \{0\} \to \mathbb{R}^N$ given by

$$\beta(u) = \frac{1}{\|u\|_p} \int_{\mathbb{R}^N} \frac{x}{|x|} |u(x)|^p dx, \quad u \in E \setminus \{0\}.$$

This function is continuous on $E \setminus \{0\}$ and uniformly continuous on the bounded subsets of $E \setminus \{u \in E : \|u\|_p < r\}$ for every $r > 0$. Moreover, $|\beta(u)| < 1$ for every $u \neq 0$. For $b \in B_1(0) \subset \mathbb{R}^N$, we set

$$I_b := \inf_{u \in M} J(u) = \inf_{\beta(u) = b} \inf_{v \in S^+} \Psi(v) \geq c.$$

We claim that if $c = I_b$ for some $|b| < 1$, then $J$ has a nontrivial critical point, i.e., (1) has a nontrivial solution.

Indeed, let $(v_k)_k \subset S^+$ with $\beta(m(v_k)) = b$ for all $k \in \mathbb{N}$ be a minimizing sequence for $I_b$, i.e., $\lim_{k \to \infty} \Psi(v_k) = I_b$. For each $k \in \mathbb{N}$, choose $\delta_k > 0$ such that $|\beta(m(v)) - \beta(m(v_k))| < \frac{1-|b|}{2}$ holds for every $v \in S^+$ with $\|v - v_k\| \leq 2\delta_k$. According to Ekeland’s variational principle (see [28, Theorem 8.5]), we can find some $w_k \in S^+$ satisfying $c = I_b \leq \Psi(w_k) \leq I_b + \|w_k - v_k\| \leq 2\delta_k$ and $\|\Psi'(w_k)\|_* \leq \frac{c}{\delta_k}$. Setting $u_k := m(w_k)$ for all $k \in \mathbb{N}$, we obtain a Palais-Smale sequence $(u_k)_k \subset M$ for $J$ at level $I_b = c$ with the additional property that $|\beta(u_k)| \leq \frac{1+|b|}{2} < 1$ for all $k \in \mathbb{N}$. Remark that by Lemma 2.3 $(u_k)_k$ is a bounded sequence. Hence, the estimates (12), together with the fact that $u_k \in M$ for all $k$, imply $\inf_{k \in \mathbb{N}} \|u_k\|_p > 0$. 


Suppose by contradiction that there is no $\pi \in E \setminus \{0\}$ such that $J'(\pi) = 0$. According to Lemma 5.4, and the assumption $c = c_\infty$, we can find a sequence $(x_k)_k \subset \mathbb{R}^N$ such that $|x_k| \to \infty$ and $\|u_k - (x_k * u_\infty)\| \to 0$, as $k \to \infty$. Noticing further that $\|x_k * u_\infty\|_{L^p} = \|u_\infty\|_{L^p} > 0$ holds for all $k$, and $|\beta(x_k * u_\infty)| \to 1$ as $k \to \infty$, the uniform continuity of $\beta$ gives

$$1 = \lim_{k \to \infty} |\beta(x_k * u_\infty)| \leq \limsup_{k \to \infty} |\beta(u_k)| \leq \frac{1 + |b|}{2}.$$  

This contradicts our assumption $|b| < 1$, and shows that there must exist some $\tilde{\pi} \in E \setminus \{0\}$ such $J'(\tilde{\pi}) = 0$. From Lemma 5.4, it follows that $\lim_{k \to \infty} \|m(w_k) - \tilde{\pi}\| = 0$, and thus, $J(\tilde{\pi}) = c$ holds with $\beta(\tilde{\pi}) = b$. This proves the claim.

It remains to see what happens when $c = c_\infty < I_b$ holds for every $|b| < 1$. For $R > 0$, let $y = (0, \ldots, 0, R) \in \mathbb{R}^N$ and consider the open ball

$$\Omega_R := B_{\frac{1}{2}R}(\frac{R}{2}) = \{(1-s)y + sz \in \mathbb{R}^N : 0 \leq s < 1, z \in \partial \Omega_R\}.$$

It has the following properties.

(i) $0, y \in \Omega_R$, $\frac{3}{4}R \leq |y - z| \leq 2\min\{|y|, |z|\} = 2R$ for all $z \in \partial \Omega_R$, and $|y - z| = 2R$ if and only if $z = -y$.

(ii) For every $x \in \overline{\Omega}_R \setminus \{y\}$ there exists a unique $(s, z) \in (0, 1) \times \partial \Omega_R$ satisfying $x = (1-s)y + sz$.

(iii) For every $x \in \overline{\Omega}_R \setminus \{0\}$ there exists exactly one $(\tau, \zeta) \in (0, 1] \times \partial \Omega_R$ satisfying $x = \tau \zeta$. Moreover, $\tau$ is given by

$$\tau(x) = \frac{1}{5R} \left[\sqrt{15|x|^2 + x_N^2} - x_N\right],$$

where $x = (x_1, \ldots, x_N) \in \overline{\Omega}_R \setminus \{0\} \subset \mathbb{R}^N$.

The function $g: \overline{\Omega}_R \to B_1(0)$ given by

$$g(x) = \begin{cases} \tau(x) \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a continuous bijection which satisfies $g(\partial \Omega_R) = \partial B_1(0)$. Furthermore, $g$ is smooth on $\Omega_R \setminus \{0\}$. For $b = (0, \ldots, 0, |b|)$ with $0 < |b| < 1$, we have $g(\frac{\sqrt{3}}{2}b) = b$ and $g'(\frac{\sqrt{3}}{2}b) = \frac{\sqrt{3}}{2}id$. Thus $\deg(g, \Omega_R, b) = 1$.

We now define a min-max value as follows. Let $R \geq S_2$ where $S_2$ is given in Lemma 3.3 and consider $\gamma_0: \partial \Omega_R \to \mathcal{M}$ given by

$$\gamma_0(z) := \tilde{m}(z * u_\infty)$$

for all $z \in \partial \Omega_R$.

We set $\Gamma_R := \{\gamma: \overline{\Omega}_R \to \mathcal{M} : \gamma$ continuous and $\gamma|_{\partial \Omega_R} = \gamma_0\}$ and

$$c_0 := \inf_{\gamma \in \Gamma_R} \max_{x \in \overline{\Omega}_R} J(\gamma(x)).$$

We claim that for $b = (0, \ldots, 0, |b|)$ with $0 < |b| < 1$ fixed, there holds $I_b \leq c_0 < 2c_\infty$ for $R$ large enough.

To show the left-hand inequality, consider for each $\gamma \in \Gamma_R$ the homotopy $\eta: [0, 1] \times \overline{\Omega}_R \to B_1(0)$ given by $\eta(\xi, x) = \xi \beta(\gamma(x)) + (1 - \xi)g(x)$, $0 \leq \xi \leq 1$, $x \in \overline{\Omega}_R$. Since $\gamma|_{\partial \Omega_R} = \gamma_0$ and $\xi \beta(\gamma_0(z)) + (1 - \xi)g(z) \to \frac{\sqrt{3}}{2}z$ uniformly for $z \in \partial \Omega_R$ and $0 \leq \xi \leq 1$, as $R \to \infty$, we obtain $b \notin \eta([0, 1] \times \partial \Omega_R)$ for $R$ large enough. The homotopy invariance of the degree then implies $\deg(\beta \circ \gamma, \Omega_R, b) = \deg(g, \Omega_R, b) = 1$.

Using the existence property, we can therefore find some $x_b \in \Omega_R$ for which
\[ \beta(\gamma(x_b)) = b, \] and this gives \( I_b \leq J(\gamma(x_b)) \). Since \( \gamma \in \Gamma_R \) was arbitrarily chosen, we obtain \( I_b \leq c_0 \). Lemma 3.3 gives the second inequality, when we consider \( \gamma_2 \in \Gamma_R \) given by
\[ \gamma_2((1 - s)y + sz) = \hat{m}((1 - s)(y \ast u_\infty) + s(z \ast u_\infty)) \quad s \in [0, 1], \ z \in \partial \Omega_R. \]

In particular, the min-max level \( c_0 \) satisfies
\[ c = c_\infty < c_0 < 2c_\infty \tag{46} \]
for \( R \) large enough.

We now wish to prove that \( J \) has a (nontrivial) critical point at level \( c_0 \). For this, we note that \( S^+ := \{ u \in E^+ : \| u \| = 1 \} \) is a complete connected \( C^1 \)-Finsler manifold, as a closed connected \( C^1 \)-submanifold of the Banach space \( E^+ \). Moreover, as \( R \geq S_2 \), we have a family \( \mathcal{F}_R = \{ (m^{-1} \circ \gamma)(\Omega_R) : \gamma \in \Gamma_R \} \) of compact subsets of \( S^+ \) which is a homotopy-stable family with boundary \( B_R := (m^{-1} \circ \gamma_0)(\partial \Omega_R) \subset S^+ \), in the sense of Ghoussoub [10] Definition 3.1. Since \( J(\gamma_0(z)) \) converges to \( c_\infty \) as \( R \to \infty \), uniformly for \( z \in \partial \Omega_R \), we have further
\[ \sup \Psi(B_R) = \max_{z \in \partial \Omega_R} J(\gamma_0(z)) < c_0 = \inf \max_{\gamma \in \Gamma_R} J(\gamma(x)) = \inf_{A \in \mathcal{F}_R} \sup_{v \in A} \Psi(v) \]
for large \( R \). Using the min-max principle [11] Theorem 3.2, we can find a sequence \( (v_k)_k \subset S^+ \) such that \( \Psi(v_k) \to c_0 \) and \( \| \Psi'(v_k) \|_* \to 0 \) as \( k \to \infty \). Consequently, the sequence \( (m(v_k))_k \subset M \) is a Palais-Smale sequence for \( J \) at level \( c_0 \). Now, any sign-changing critical point \( \omega \) of \( J_\infty \) satisfies \( J_\infty(\omega) \geq 2c_\infty \) (see e.g. [11] Lemma 2.4).

Hence, the estimates (46), the uniqueness of the positive solution of (3) and the closedness of the positive solution of (3) together with Lemma 3.4 imply that, up to a subsequence, \( m(v_k) \to \pi \) as \( k \to \infty \) for some \( \pi \in E \setminus \{0\} \) which satisfies \( J'(\pi) = 0 \) and \( J(\pi) = c_0 \). This concludes the proof. \[\square\]

4. Existence of a ground-state solution

This section is devoted to the existence of a ground-state solution of (1) under the conditions of Theorem 1.2. Therefore, we assume from now on, that \( a \) and \( f \) satisfy (A1), (F1)-(F4). If \( E^0 \neq \{0\} \) we suppose, in addition, that (5) holds. The proof of Theorem 1.2 relies upon Proposition 2.4 and a similar energy estimate as before (compare Lemma 3.2 and Lemma 1.2). This time we shall consider the translate of one ground-state of the limit equation (3) together with a cutoff argument. The latter is well-suited to our setting, since through (6) and (8) we only control the behavior of \( a \) and \( F \) respectively, for large \( |x| \). We do not have (nor do we require) any information about what happens elsewhere.

For the remainder of this section, we choose some ground-state solution \( u_\infty \) of (3). Our hypotheses ensure that \( u_\infty \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) satisfies either \( u_\infty > 0 \) or \( u_\infty < 0 \) on \( \mathbb{R}^N \). Furthermore, for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[ |u_\infty(x)| \leq C_\varepsilon e^{-(1-\varepsilon)\sqrt{\frac{1}{\varepsilon}}|x|} \quad \text{for all } x \in \mathbb{R}^N \tag{47} \]
(see [20] and [23] Theorem C.3.5). Taking \( \theta \) as in (5) if \( E^0 \neq \{0\} \) and setting \( \theta = 0 \) otherwise, we fix some \( 0 < \varepsilon < \min\left\{ 1 - \frac{\alpha(1+\theta)}{(2+\theta)}, 1 - \sqrt{\frac{\alpha}{\theta}} \right\} \) and set \( A_\varepsilon := (1-\varepsilon)\sqrt{\frac{1}{\varepsilon}} \). Moreover, we consider a cut-off function \( \chi \in C^\infty(\mathbb{R}^N), 0 \leq \chi \leq 1 \), such that \( \chi(x) = 1 \) if \( |x| \leq 1 - \varepsilon \) and \( \chi(x) = 0 \) if \( |x| \geq 1 \).

For \( R > 0 \), we set
\[ u^R_\infty(x) := (\frac{x}{R})u_\infty(x), \quad x \in \mathbb{R}^N. \]
Similar to [8] Lemma 2, we have the following estimates as $R \to \infty$.

\[
\int_{\mathbb{R}^N} |\nabla u_\infty|^2 - |\nabla u_\infty^R|^2 \, dx = O(e^{-2(1-\epsilon)A_\epsilon R})
\]

\[
\int_{|x| \geq R} |u_\infty|^s \, dx = O(e^{-sA_\epsilon R}) \quad \text{for } s > 0.
\]

In the following it will be helpful to consider

\[ R_y := \gamma |y| \quad \text{for } y \in \mathbb{R}^N \setminus \{0\} \text{ with some fixed } \gamma \in (0, 1). \]

We shall estimate the asymptotic behavior of some integrals involving the eigenfunctions and translates of the ground-state $u_\infty$.

**Lemma 4.1.** For $\sigma, \nu > 0$ and $0 < \beta < \min\{\nu, \nu(1-\gamma) + \sigma\gamma\}$, there exists $C > 0$ such that

\[
\int_{\mathbb{R}^N} |y * u_\infty^R|^\sigma e^{-\nu A_\epsilon |x|} \, dx \leq C e^{-\beta A_\epsilon |y|}
\]

holds for all $y \in \mathbb{R}^N$. Moreover, if $\sigma > \nu$, the conclusion also holds when $\beta = \nu$.

**Proof.** Let $0 < \beta < \min\{\nu, \nu(1-\gamma) + \sigma\gamma\}$ and consider $y \in \mathbb{R}^N$. There holds

\[
e^{\beta A_\epsilon |y|} \int_{\mathbb{R}^N} |y * u_\infty^R|^\sigma e^{-\nu A_\epsilon |x|} \, dx \leq C \int_{\{|x| \leq \gamma |y|\}} e^{(\beta-\sigma)A_\epsilon |x|} e^{-(\nu-\beta)A_\epsilon |x+y|} \, dx,
\]

using (47). Now, if $\beta \geq \sigma$, we can write

\[
\int_{\{|x| \leq \gamma |y|\}} e^{(\beta-\sigma)A_\epsilon |x|} e^{-(\nu-\beta)A_\epsilon |x+y|} \, dx \leq C |y|^N e^{(\beta-\sigma)A_\epsilon |y|} e^{-(\nu-\beta)A_\epsilon (1-\gamma)|y|}
\]

\[
= C |y|^N e^{-(\nu(1-\gamma) + \sigma\gamma - \beta)A_\epsilon |y|}.
\]

The assumption $\beta < \nu(1-\gamma) + \sigma\gamma$ then gives the desired result.

On the other hand, if $\beta < \sigma$, we have

\[
\int_{\{|x| \leq \gamma |y|\}} e^{(\beta-\sigma)A_\epsilon |x|} e^{-(\nu-\beta)A_\epsilon |x+y|} \, dx \leq e^{-(\nu-\beta)A_\epsilon (1-\gamma)|y|} \int_{\mathbb{R}^N} e^{-(\sigma-\beta)A_\epsilon |x|} \, dx,
\]

and the conclusion follows. $\square$

As a consequence of the preceding lemma, we obtain the following estimates for the integrals below.

\[
\int_{\mathbb{R}^N} |y * u_\infty^R| |e_i| \, dx \leq C e^{-A_\epsilon |y|} \quad \text{for all } y \in \mathbb{R}^N, 1 \leq i \leq n + l,
\]

(51)

with some constant $C = C(\epsilon, \gamma) > 0$. Moreover, for any $0 < \delta < (\sqrt{1 + \frac{1}{\lambda_{n+1}}} - 1)(1-\gamma)$, there exists $C = C(\epsilon, \gamma, \delta) > 0$ satisfying

\[
\int_{\mathbb{R}^N} (y * u_\infty^R) |e_i| \, dx \leq C e^{-(1+\delta)A_\epsilon |y|} \quad \text{for all } y \in \mathbb{R}^N, 1 \leq i \leq n.
\]

(52)

**Lemma 4.2.** (i) There exists $S_3 > 0$ such that $y * u_\infty^R \notin H^- \oplus H^0$ for all $|y| \geq S_3$. 

(ii) For $|y| \geq S_3$, let $t_y > 0$, \(h_y^1 \in H^-\) and \(h_y^2 \in H^0\) satisfy
\[
\hat{m}(y \ast u_R^\infty) = t_y[y \ast u_R^\infty + h_y^1 + h_y^2].
\]
Then, as $|y| \to \infty$, there holds $t_y \to 1$ and $\|h_y^1\| + \|h_y^2\| \to 0$. More precisely, there exists $C = C(\varepsilon, \gamma) > 0$ such that
\[
\|h_y^1\| \leq Ce^{-\frac{\varepsilon + \gamma}{2\varepsilon + \gamma}} A_\varepsilon|y|
\]
and
\[
\|h_y^2\| \leq Ce^{-\frac{\varepsilon}{\varepsilon + \gamma}} A_\varepsilon|y|
\]
hold for all $|y| \geq S_3$.

**Proof.**

(i) Since $a(x) \to a_\infty$ as $|x| \to \infty$, and since $|x| \leq R_y$ implies $|x + y| \geq (1 - \gamma)|y|$, the dominated convergence theorem, together with (55) and (49), gives
\[
\int_{\mathbb{R}^N} |\nabla (y \ast u_R^\infty)|^2 + a(x)|y \ast u_R^\infty|^2 \, dx = \int_{\mathbb{R}^N} |\nabla u_R^\infty|^2 + a(x)|u_R^\infty|^2 \, dx
\]
\[
\quad \to \int_{\mathbb{R}^N} |\nabla u_\infty|^2 + a_\infty|u_\infty|^2 \, dx \quad \text{as } |y| \to \infty.
\]
Since $\int_{\mathbb{R}^N} |\nabla h^2 + a(x)|h^2| \, dx = -\|h\|^2 \leq 0$ for all $h \in H^- \oplus H^0$, the conclusion follows from (55) and the fact that $0 < 2c_\infty \leq \int_{\mathbb{R}^N} |\nabla u_\infty|^2 + a_\infty|u_\infty|^2 \, dx$.

(ii) Since $J'(t_y[y \ast u_R^\infty + h_y^1 + h_y^2])(t_y[y \ast u_R^\infty + h_y^1 + h_y^2]) = 0$ holds, we find
\[
0 < 2c \leq 2J(t_y[y \ast u_R^\infty + h_y^1 + h_y^2]) + 2 \int_{\mathbb{R}^N} F(x, t_y[y \ast u_R^\infty + h_y^1 + h_y^2]) \, dx
\]
\[
= t_y^2 \int_{\mathbb{R}^N} |\nabla [y \ast u_R^\infty + h_y^1]|^2 + a(x)|y \ast u_R^\infty + h_y^1|^2 \, dx
\]
\[
\leq t_y^2 \left( \int_{\mathbb{R}^N} |\nabla y \ast u_R^\infty|^2 + a(x)|y \ast u_R^\infty|^2 \, dx + 2C\|u_\infty\| \|h_y^1\|^2 \right).
\]
From (55), we deduce that
\[
\sup_{|y| \geq S_3} \|h_y^1\| < +\infty \quad \text{and} \quad \inf_{|y| \geq S_3} t_y > 0.
\]
Furthermore, we claim that \(\{t_y(y \ast u_R^\infty + h_y^1 + h_y^2) : |y| \geq S_3\}\) is bounded, so that
\[
\sup_{|y| \geq S_3} t_y < +\infty \quad \text{and} \quad \sup_{|y| \geq S_3} \|h_y^2\| < +\infty
\]
holds. Indeed, suppose by contradiction that (up to a subsequence) $|t_y(y \ast u_R^\infty + h_y^1 + h_y^2)| \to \infty$ as $|y| \to \infty$. Setting $w_y := \frac{y \ast u_R^\infty + h_y^1 + h_y^2}{\|y \ast u_R^\infty + h_y^1 + h_y^2\|}$, we first note that $s_y$ stays bounded as $|y| \to \infty$, since $\|s_y(y \ast u_R^\infty)^+\| \leq \|w_y\|$ and $\|(y \ast u_R^\infty)^+\| \not\to 0$ as $|y| \to \infty$, according to (55). Hence, $\dim(H^- \oplus H^0) < \infty$ implies that, up to a subsequence, $s_y \to s$, $\frac{s_y}{\|s_y\|} \to z_1$ and $\frac{z_y}{\|z_y\|} \to z_2$ as $|y| \to \infty$ for some $s \geq 0$, $z_1 \in H^-$ and $z_2 \in H^0$.

If $s \neq 0$, we obtain that
\[
\|(-y) \ast w_y - s u_R^\infty - (-y) \ast z_1 - (-y) \ast z_2\| \to 0 \quad \text{as } |y| \to \infty.
\]
Hence, \((-y) * w_y \to s u_{\infty} \neq 0\) as \(|y| \to \infty\), which implies that (up to a subsequence) \(w_y(x + y) \to s u_{\infty}(x)\) a.e. on \(\mathbb{R}^N\), and since \(w_y = \frac{t_y(y * u_{R_y} + h_y^1 + h_y^2)}{||t_y(y * u_{R_y} + h_y^1 + h_y^2)||}\) and \(|u_{\infty}| > 0\), we obtain that for a.e. \(x \in \mathbb{R}^N\), \(|t_y(u_{R_y}^N(x) + h_y^1(x + y) + h_y^2(x + y))| \to \infty\) as \(|y| \to \infty\). But then, Fatou’s Lemma gives

\[
\int_{\mathbb{R}^N} \frac{\|\nabla(y * u_{R_y}^N + h_y)^2\|^2}{\| [y * u_{R_y}^N + h_y^1 + h_y^2]\|^2} dx
\]

\[
= \int_{\mathbb{R}^N} \frac{f(x, t_y[y * u_{R_y}^N + h_y^1 + h_y^2]) t_y[y * u_{R_y}^N + h_y^1 + h_y^2]}{\| t_y[y * u_{R_y}^N + h_y^1 + h_y^2]\|^2} dx
\]

\[
\geq 2 \int_{\mathbb{R}^N} \frac{F(x + y, t_y[u_{R_y}^N(x) + h_y^1(x + y) + h_y^2(x + y)])}{\{t_y[u_{R_y}^N(x) + h_y^1 + h_y^2(x + y)]\}^2} (w_y(x + y))^2 dx
\]

\[
\to \infty \quad \text{as} \quad |y| \to \infty,
\]

which contradicts the boundedness of \(y * u_{R_y}^N + h_y^1\) and the fact that \(||(y * u_{R_y}^N)^+|| \to 0\) as \(|y| \to \infty\). On the other hand, if \(s = 0\), then \(w_y \to z_1 + z_2\) strongly in \(H\), and hence \(||z_1 + z_2|| \neq 1\) holds. The same argument as above gives (up to a subsequence) \(|t_y[u_{R_y}^N(x - y) + h_y^1(x) + h_y^2(x)]| \to \infty\) as \(|y| \to \infty\) and, using Fatou’s Lemma, we obtain again a contradiction. Therefore, the claim is proved and (57) holds.

Now \(J'(t_y[y * u_{R_y}^N + h_y^1 + h_y^2])(h_y^1 + h_y^2) = 0\) implies

\[
0 \leq \|h_y^1\|^2 + \int_{\mathbb{R}^N} \frac{f(x, t_y[y * u_{R_y}^N + h_y^1 + h_y^2])}{t_y[y * u_{R_y}^N + h_y^1 + h_y^2]} (h_y^1 + h_y^2)^2 dx
\]

\[
= \int_{\mathbb{R}^N} \nabla(y * u_{R_y}^N) \cdot \nabla h_y^1 + a(x)(y * u_{R_y}^N) h_y^1 dx
\]

\[
- \int_{\mathbb{R}^N} \frac{f(x, t_y[y * u_{R_y}^N + h_y^1 + h_y^2])}{t_y[y * u_{R_y}^N + h_y^1 + h_y^2]} (y * u_{R_y}^N)(h_y^1 + h_y^2) dx.
\]

Writing \(h_y = \sum_{i=1}^{n+1} A_i^y e_i\) and \(h_y^2 = \sum_{i=n+1}^{n+n+1} A_i^y e_i\) and using the fact that \(t_y[y * u_{R_y}^N + h_y^1 + h_y^2]\) is bounded in \(L^\infty(\mathbb{R}^N)\), we obtain

\[
\|h_y^1\|^2 + \int_{\mathbb{R}^N} \frac{f(x, t_y[y * u_{R_y}^N + h_y^1 + h_y^2])}{t_y[y * u_{R_y}^N + h_y^1 + h_y^2]} (h_y^1 + h_y^2)^2 dx
\]

\[
\leq C \sum_{i=1}^{n+n+1} |A_i^y| \int_{\mathbb{R}^N} |y * u_{R_y}^N| |e_i| dx \leq C \|h_y^1 + h_y^2\| e^{-A_i|y|},
\]

for some \(C > 0\), by (31), (55), (57) and since all norms are equivalent on \(H^- \oplus H^0\). In the case \(E^0 = \{0\}\), the lemma is proved, since \(h_y^2 = 0\) holds. If \(E^0 \neq \{0\}\), we notice that (F4) and (5) imply

\[
\inf \left\{ \frac{|f(x, u)|}{|u|^{1+y}} : x \in \mathbb{R}^N, 0 < |u| \leq r \right\} > 0
\]
Proof of Theorem 1.2. As before, let 0 < \( \mu < 2 \) and 0 < \( \alpha < 2 \) and, if \( \ker(-\Delta + a) = E^0 \neq \{0\} \) holds, \( \|h^1_y\| \) be satisfied.

**Remark 4.1.** In the case \( E^0 = \{0\} \), we have \( h^2_y = 0 \), and using (52) instead of (51) in (59) in the proof above, we obtain a better estimate for the decay of \( h^1_y \).

Namely, for every 0 < \( \delta < \min\{\sqrt{1 + \frac{\|h^1_y\|}{a_{\infty}}}, 1 - \gamma\} : 1 \leq i \leq n\}

\[\|h^1_y\| \leq Ce^{-(1+\delta)A_{\epsilon}|y|} \]  

holds for all \( |y| \geq S_3 \) with some \( C = C(\epsilon, \gamma, \delta) > 0 \).

We are now ready to prove Theorem 1.2. Let therefore conditions (A1), (F1)–(F4) and, if \( \ker(-\Delta + a) = E^0 \neq \{0\} \) holds, (5) be satisfied.

**Proof of Theorem 1.2.** As before, let 0 < \( \varepsilon < \min\{1 - \frac{\alpha(1+\gamma)}{(2+\sigma)}, 1 - \sqrt{\frac{\sigma}{2}}\} \). In case (7), we may also assume

\[\alpha < \mu < 2 \quad \text{and} \quad \varepsilon < \min\left\{1 - \frac{\alpha}{\mu}, 1 - \frac{\mu}{2}\right\}.\]  

for all \( r > 0 \). Choosing \( r = \sup_{|y| \geq S_3} \|t_y[y * u^R_{\infty} + h^1_y + h^2_y]\|_\infty < \infty \), we obtain

\[
\int_{\mathbb{R}^N} |h^1_y + h^2_y|^{2+\theta} \, dx \\leq C \int_{\mathbb{R}^N} |y * u^R_{\infty} + h^1_y + h^2_y|^{(1+\theta)2} \, dx \\
+ \max\{1, 2^{\theta-1}\} \int_{\mathbb{R}^N} |y * u^R_{\infty}|^\theta (h^1_y + h^2_y)^2 \, dx
\]

since 0 < \( \frac{\theta}{1+\theta} < \min\{2, 2(1-\gamma) + \theta \gamma\} \), using Lemma 4.1 and (59).

Since all norms are equivalent on \( E^{-} \oplus E^{0} \), the preceding estimate gives

\[\|h^1_y + h^2_y\| \leq Ce^{-\frac{(2+\theta)}{2(1+\theta)}A_\epsilon|y|} \] for all \( |y| \geq S_3 \).

Combining this estimate with (59), we find

\[\|h^1_y\| \leq Ce^{-(\frac{2+\theta}{1+\theta})A_\epsilon|y|} \] for all \( |y| \geq S_3 \).

Hence, \( \|h^1_y + h^2_y\| \to 0 \) as \( |y| \to \infty \) which, in turn, implies that

\[
\int_{\mathbb{R}^N} |\nabla|y * u^R_{\infty} + h^1_y||^2 + a(x)|y * u^R_{\infty} + h^1_y||^2 \, dx \to \int_{\mathbb{R}^N} |\nabla u_{\infty}|^2 + a_{\infty} u_{\infty}^2 \, dx,
\]

as \( |y| \to \infty \). From (50) and (57), we can assume that (up to a subsequence) \( t_y \to T > 0 \) as \( |y| \to \infty \). Then, we find

\[0 = J'(t_{\infty}[y * u^R_{\infty} + h^1_y + h^2_y])t_{\infty}[y * u^R_{\infty} + h^1_y + h^2_y] \to J'(T u_{\infty}) T u_{\infty},\]

as \( |y| \to \infty \), i.e., \( T u_{\infty} \in M_{\infty} \). Since \( u_{\infty} \in M_{\infty} \), it follows that \( T = 1 \). This concludes the proof. \( \square \)
We fix $\gamma > 0$ such that $\frac{\alpha}{2(1-\gamma^2)} < \gamma < 1$ and consider the corresponding radii $R_y$ as defined in (50). We claim that we can find $S \geq S_3$ such that

$$J(\tilde{m}(y * u_{R_y}^\infty)) < c_\infty \quad \text{for all } |y| \geq S,$$  

(64)

where $S_3$ is given by Lemma 4.2. The conclusion of the theorem then follows from Proposition 2.5. Let us prove (64). With the notation of Lemma 4.2, consider $|y| \geq S_3$ and $(t_y, h_y^1, h_y^2) \in (0, \infty) \times E^- \times E^0$ with $\tilde{m}(y * u_{R_y}^\infty) = t_y [y * u_{R_y}^\infty + h_y^1 + h_y^2]$. There holds

$$J(\tilde{m}(y * u_{R_y}^\infty)) = \frac{t_y^2}{2} \int_{\mathbb{R}^N} \left| \nabla u_{R_y}^\infty \right|^2 + a_\infty |u_{R_y}^\infty|^2 dx - \int_{\mathbb{R}^N} F_\infty(t_y u_{R_y}^\infty) dx$$

$$+ \frac{t_y^2}{2} \left[ \int_{\mathbb{R}^N} |\nabla h_y^1|^2 + a(x) |h_y^1|^2 dx + 2 \int_{\mathbb{R}^N} \nabla (y * u_{R_y}^\infty) \cdot \nabla h_y^1 + a(x)(y * u_{R_y}^\infty)h_y^1 dx 
+ \int_{\mathbb{R}^N} (a(x + y) - a_\infty) |u_{R_y}^\infty|^2 dx \right] - \left[ \int_{\mathbb{R}^N} F(x + y, t_y u_{R_y}^\infty) - F_\infty(t_y u_{R_y}^\infty) dx 
+ \int_{\mathbb{R}^N} \{ F(x, t_y y * u_{R_y}^\infty + h_y^1 + h_y^2) - F(x, t_y y * u_{R_y}^\infty) \} dx \right].$$

In the following, we let $K_1, K_2, \ldots$ denote positive constants depending possibly on $S_3$, $\alpha$, $\theta$, $\varepsilon$, $\gamma$, $\mu$ and $u_\infty$ but not on $y$. From (18), (19) and the fact that $t_y$ remains bounded as $|y| \to \infty$, it follows that

$$\frac{t_y^2}{2} \int_{\mathbb{R}^N} \left| \nabla u_{R_y}^\infty \right|^2 + a_\infty |u_{R_y}^\infty|^2 dx - \int_{\mathbb{R}^N} F_\infty(t_y u_{R_y}^\infty) dx$$

$$\leq J_\infty(t_y u_\infty) + \frac{t_y^2}{2} \int_{\mathbb{R}^N} \left( |\nabla u_{R_y}^\infty|^2 - |\nabla u_\infty|^2 \right) dx + \int_{\{|y| \geq (1-\varepsilon) R_y\}} \left| F_\infty(t_y u_{R_y}^\infty) \right| dx$$

$$\leq c_\infty + K_1 e^{-2(1-\varepsilon)A_y R_y}$$

for $y \in \mathbb{R}^N \setminus \{0\}$, using the fact that $J_\infty(t_y u_\infty) \leq J_\infty(u_\infty) = c_\infty$, since $u_\infty \in \mathcal{M}_\infty$ holds.

We now consider the case where assumption (a) of Theorem 1.2 is satisfied. Then (63) implies

$$\int_{\mathbb{R}^N} (a(x + y) - a_\infty) |u_{R_y}^\infty|^2 dx = \int_{\{|x| \leq R_y\}} (a(x + y) - a_\infty) |u_{R_y}^\infty|^2 dx$$

$$\leq -C_1 e^{-\alpha \sqrt{\alpha^2 - |y|}} \int_{\{|x| \leq 1\}} e^{-\alpha \sqrt{\alpha^2 - |x|}} u_\infty^2 dx \leq -K_2 e^{-\alpha \sqrt{\alpha^2 - |y|}}$$

for $|y| \geq \max \{ \frac{1}{\gamma}, \frac{1}{2\gamma} \}$. Furthermore, (7) and Lemma 4.1 together with (63) yield

$$\int_{\mathbb{R}^N} F(x + y, t_y u_{R_y}^\infty) - F_\infty(t_y u_{R_y}^\infty) dx \geq -C_2 \int_{\mathbb{R}^N} e^{-\mu \sqrt{\alpha^2 - |x+y|}} \left| t_y u_{R_y}^\infty \right|^2 + \left| t_y u_{R_y}^\infty \right|^p dx$$

$$= -C_2 \int_{\mathbb{R}^N} e^{-\mu \sqrt{\alpha^2 - |x|}} \left| t_y (-y) \right| u_{R_y}^\infty \left| u_{R_y}^\infty \right|^2 + \left| t_y (-y) \right| u_{R_y}^\infty \left| u_{R_y}^\infty \right|^p dx \geq -K_3 e^{-\mu A_y \left| y \right|}$$

for $|y| \geq S_3$. If $E^0 \neq \{0\}$, we obtain moreover

$$\int_{\mathbb{R}^N} \nabla (y * u_{R_y}^\infty) \cdot \nabla h_y^1 + a(x)(y * u_{R_y}^\infty)h_y^1 dx$$

$$\leq \| \alpha \|_{H_y^1} \max_{1 \leq i \leq n} \int_{\mathbb{R}^N} |y * u_{R_y}^\infty| \left| e_i \right| dx \leq K_4 e^{-\frac{44 + 3\theta}{6} \alpha \sqrt{\alpha^2 - |y|}},$$
and
\[
\int_{\mathbb{R}^N} \{F(x, t_y[y * u_{\infty}^R + h_y^1 + h_y^2]) - F(x, t_y(y * u_{\infty}^R))\} \, dx \\
\geq - \int_{\mathbb{R}^N} |f(x, t_y(y * u_{\infty}^R))t_y(h_y^1 + h_y^2)| \, dx \\
\geq -K_5\|h_y^1 + h_y^2\| \max_{1 \leq i \leq n+1} \int_{\mathbb{R}^N} |y * u_{\infty}^R| |e_i| \, dx \geq -K_6 e^{-(2+\delta)A_\varepsilon|y|},
\]
for \(|y| \geq S_3\) as a consequence of (13), (51), (53) and (54).

On the other hand, when \(E^0 = \{0\}\), we fix
\[
0 < \delta < \min \left\{ \left( \frac{1}{\alpha_\infty} \right)^2 - 1 \right\}
\]
and use Remark 4.1 and (52) to estimate
\[
\int_{\mathbb{R}^N} \nabla(y * u_{\infty}^R) \cdot \nabla h_y^1 + a(x)(y * u_{\infty}^R)h_y^1 \, dx \\
\leq |\lambda_1| \|h_y^1\| \max_{1 \leq i \leq n} \int_{\mathbb{R}^N} |y * u_{\infty}^R| |e_i| \, dx \leq K_7 e^{-2(1+\delta)A_\varepsilon|y|},
\]
and, since \(h_y^2 = 0\) in this case,
\[
\int_{\mathbb{R}^N} \{F(x, t_y[y * u_{\infty}^R + h_y^1 + h_y^2]) - F(x, t_y(y * u_{\infty}^R))\} \, dx \\
\geq - \int_{\mathbb{R}^N} |f(x, t_y(y * u_{\infty}^R))t_yh_y^1| \, dx \\
\geq -K_8\|h_y^1\| \max_{1 \leq i \leq n} \int_{\mathbb{R}^N} |y * u_{\infty}^R| |e_i| \, dx \geq -K_9 e^{-2(1+\delta)A_\varepsilon|y|},
\]
Putting the previous estimates together and noting in addition that
\[
\int_{\mathbb{R}^N} |\nabla h_y^1|^2 + a(x)|h_y^1|^2 \, dx \leq 0,
\]
we obtain, for \(|y| \geq \max\{S_3, \frac{S_3}{1+\gamma}, \frac{S_3}{\gamma}\}\),
\[
J(\tilde{m}(y * u_{\infty}^R)) \leq c_\infty + K_1 e^{-2(1-\varepsilon)A_\varepsilon R_y} - K_2 e^{-\alpha \sqrt{a_\infty}|y|} + K_3 e^{-\mu A_\varepsilon|y|} + (K_4 + K_5) e^{-2(2+\delta)A_\varepsilon|y|}
\]
in the case where \(E^0 \neq \{0\}\), and
\[
J(\tilde{m}(y * u_{\infty}^R)) \leq c_\infty + K_1 e^{-2(1-\varepsilon)A_\varepsilon R_y} - K_2 e^{-\alpha \sqrt{a_\infty}|y|} + K_3 e^{-\mu A_\varepsilon|y|} + (K_7 + K_9) e^{-2(1+\delta)A_\varepsilon|y|}
\]
when \(E^0 = \{0\}\), respectively. Our choice of \(\gamma\) implies \(2(1-\varepsilon)A_\varepsilon > \alpha \sqrt{a_\infty}\), and our choice of \(\varepsilon\) gives \(\frac{2+\delta}{2+\delta}A_\varepsilon > \alpha \sqrt{a_\infty}\) and \(\mu A_\varepsilon > \alpha \sqrt{a_\infty}\). Hence, the conclusion follows, since \(K_2 > 0\).

Next we consider the case of assumption (b) in Theorem 1.2. By Lemma 4.2 (ii), we may choose \(\eta > 0\) such that
\[
\eta \leq \inf \{t_y|u_{\infty}(x)| : |y| \geq S_3, |x| \leq 1\} \leq \sup \{t_y|u_{\infty}(x)| : |y| \geq S_3, |x| \leq 1\} \leq \frac{1}{\eta},
\]
and we consider $C_\eta, S_\eta$ such that (3) holds with this choice of $\eta$. Since by assumption $a(x) \leq a_\infty$ for all $|x| \geq S_0$, we obtain
\[
\int_{\mathbb{R}^N} (a(x + y) - a_\infty)|u_{R_y}^\infty|^2 \, dx \leq 0 \quad \text{for} \quad |y| \geq \frac{S_0}{1 - \gamma}.
\]
Moreover,
\[
\int_{\mathbb{R}^N} F(x + y, t_y u_{R_y}^\infty) - F_\infty(t_y u_{R_y}^\infty) \, dx \geq C_\eta e^{-\alpha\sqrt{a_\infty}|y|} \int_{|x| \leq 1} e^{-\alpha\sqrt{a_\infty}|x|} \, dx
\]
\[
= C_\eta e^{-\alpha\sqrt{a_\infty}|y|} \quad \text{for} \quad |y| \geq \max\left\{\frac{1}{\gamma}, \frac{S_\eta}{(1 - \gamma)}\right\} \text{ with a constant } \tilde{C}_\eta > 0.
\]
Together with the previous estimates, we find, for $|y| \geq \max\{S_3, \frac{S_0}{(1 - \gamma)}, \frac{S_\eta}{(1 - \gamma)}, \frac{1}{\gamma}\}$,
\[
J(\tilde{m}(y * u_{R_y}^\infty)) \leq c_\infty + K_1 e^{-2(1 - \varepsilon)A_y R_y} - \tilde{C}_\eta e^{-\alpha\sqrt{a_\infty}|y|} + \langle K_4 + K_6 \rangle e^{-2\langle \alpha \rangle A_y |y|}
\]
in the case where $E^0 \neq \{0\}$, and
\[
J(\tilde{m}(y * u_{R_y}^\infty)) \leq c_\infty + K_1 e^{-2(1 - \varepsilon)A_y R_y} - \tilde{C}_\eta e^{-\alpha\sqrt{a_\infty}|y|} + \langle K_7 + K_9 \rangle e^{-2(1 + \delta)A_y |y|}
\]
when $E^0 = \{0\}$, respectively. The conclusion follows as above from the choice of $\gamma$ and $\varepsilon$. \hfill \Box

**Appendix A. A nonlinear splitting property**

The aim of this section is to prove the following result, which was needed in the proof of Lemma 3.4. The proof is an adaptation of an argument in [1] Appendix to our setting. Throughout this section, we assume (F1)-(F4).

**Proposition A.1.** Let $(u_k)_k \subset H^1(\mathbb{R}^N)$ and $\bar{u} \in H^1(\mathbb{R}^N)$ be such that $u_k \rightharpoonup \bar{u}$ weakly in $H^1(\mathbb{R}^N)$ and $|\bar{u}(x)| \to 0$ as $|x| \to \infty$. Then, as $k \to \infty$,
\[
\int_{\mathbb{R}^N} F(x, u_k) - F(x, \bar{u}) - F(x, u_k - \bar{u}) \, dx \to 0,
\]
\[
\int_{\mathbb{R}^N} |f(x, u_k) - f(x, \bar{u}) - f(x, u_k - \bar{u})| \varphi \, dx \to 0,
\]
uniformly in $\|\varphi\| \leq 1$.

**Proof.** We set $\tilde{C} := \sup_k \|u_k\| < +\infty$ and fix some $\varepsilon > 0$. According to (F2), we can choose $0 < s_0 < 1$ such that $|f(x, t)| \leq \varepsilon |t|$ and hence $|F(x, t)| \leq \frac{\varepsilon}{2} |t|^2$ holds for all $|t| \leq 2s_0$ and all $x \in \mathbb{R}^N$. Moreover, by assumption, there exists $R > 0$ such that $|\bar{u}(x)| \leq s_0$ for all $|x| \geq R$ and $\int_{\mathbb{R}^N \setminus B_R(0)} |\bar{u}|^2 \, dx < 1$.

Hence, we obtain
\[
\int_{\mathbb{R}^N \setminus B_R(0)} |f(x, u_k)| \varphi \, dx \leq \varepsilon \int_{\mathbb{R}^N \setminus B_R(0)} |\bar{u}|^2 \, dx \frac{1}{2} \|\varphi\|_{L^2}^2 < \varepsilon \|\varphi\|_{L^2}^2
\]
and
\[
\int_{\mathbb{R}^N \setminus B_R(0)} |F(x, \bar{u})| \, dx \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^N \setminus B_R(0)} |\bar{u}|^2 \, dx < \frac{\varepsilon}{2}.
\]
Setting $U_k := \{x \in \mathbb{R}^N \setminus B_R(0) : |u_k(x)| \leq s_0\}$ for $k \in \mathbb{N}$, we find
\[
\int_{U_k} |f(x, u_k) - f(x, u_k - \bar{u})| \varphi \, dx \leq \varepsilon \|u_k\|_{L^2} \|u_k - \bar{u}\|_{L^2} \|\varphi\|_{L^2} \leq 3\tilde{C}\varepsilon \|\varphi\|_{L^2}
\]
and
\[
\int_{U_k} |F(x, u_k) - F(x, u_k - \bar{u})| \, dx \leq \frac{\varepsilon}{2} (\|u_k\|_{L^2}^2 + \|u_k - \bar{u}\|_{L^2}^2) \leq \frac{5\varepsilon}{2} \tilde{C}^2.
\]
Now we consider $\mathcal{V}_k^s := \{ x \in \mathbb{R}^N \setminus B_R(0) : |u_k(x)| \geq s \}$ for $k \in \mathbb{N}$ and $s > 2$. Notice that in particular $|u_k(x) - \bar{u}(x)| > s - 1 > 1$ holds for all $x \in \mathcal{V}_k^s$. As a consequence, we can write

$$\int_{\mathcal{V}_k^s} |f(x, u_k) - f(x, u_k - \bar{u})||\varphi| \, dx \leq 2C_0 \int_{\mathcal{V}_k^s} (|u_k|^{p-1} - |u_k - \bar{u}|^{p-1})|\varphi| \, dx$$

$$\leq 2C_0(s - 1)^{p-2}\int_{\mathcal{V}_k^s} (|u_k|^{2-1} - |u_k - \bar{u}|^{2-1})|\varphi| \, dx$$

$$\leq 2C_0(s - 1)^{p-2}\|u_k\|_{L^2}^{2-1} - \|u_k - \bar{u}\|_{L^2}^{2-1})\|\varphi\|_{L^2}$$

$$\leq \hat{C}(s - 1)^{p-2}\hat{C}^{2-1}||\varphi||.$$  

Similarly, we find

$$\int_{\mathcal{V}_k^s} |F(x, u_k) - F(x, u_k - \bar{u})| \, dx \leq \hat{C}(s - 1)^{p-2}\hat{C}^{2-1}$$

for all $k \in \mathbb{N}$. Choosing $s > 2$ large enough, we can achieve $\hat{C}(s - 1)^{p-2}\hat{C}^{2-1} < \varepsilon$ and $\hat{C}(s - 1)^{p-2}\hat{C}^{2-1} < \varepsilon$.

For the next step, we remark that from the continuity of $f_\infty$ and $F_\infty$, we may choose $\delta = \delta(\varepsilon) > 0$ such that $|f_\infty(u) - f_\infty(v)| < \frac{s_0 \varepsilon}{3}$ and $|F_\infty(u) - f_\infty(v)| < \frac{s_0 \varepsilon}{3}$ hold for all $|u - v| < \delta$, $|u|, |v| \leq s + 1$. By (A1), we may now pick $R_0 = R_0(\varepsilon) \geq R$ satisfying $|\bar{u}(x)| < \delta$ for all $|x| \geq R_0$ and $|f(x, u) - f_\infty(u)| < \frac{s_0 \varepsilon}{3}$ for all $|u| \leq s + 1$ and all $|x| \geq R_0$. Setting $W_k := \{ x \in \mathbb{R}^N \setminus B_{R_0}(0) : s_0 \leq |u_k(x)| \leq s \}$ for $k \in \mathbb{N}$, we obtain

$$\int_{W_k} |f(x, u_k) - f(x, u_k - \bar{u})||\varphi| \, dx \leq \int_{W_k} |f(x, u_k) - f_\infty(x, u_k)||\varphi| \, dx$$

$$+ \int_{W_k} |f_\infty(u_k) - f_\infty(u_k - \bar{u})||\varphi| \, dx + \int_{W_k} |f_\infty(u_k - \bar{u}) - f(x, u_k - \bar{u})||\varphi| \, dx$$

$$< s_0 \varepsilon \|\varphi\|_{L^2}\|W_k\|^\frac{3}{2} \leq s_0 \varepsilon \frac{1}{s_0} \|u_k\|_{L^2}\|\varphi\|_{L^2} \leq \varepsilon \hat{C}\|\varphi\|_{L^2},$$

and

$$\int_{W_k} |F(x, u_k) - F(x, u_k - \bar{u})| \, dx \leq \int_{W_k} \int_0^1 |f(x, tu_k) - f_\infty(x, tu_k)| |u_k| \, dt \, dx$$

$$+ \int_{W_k} |F_\infty(u_k) - F_\infty(u_k - \bar{u})| \, dx$$

$$+ \int_{W_k} \int_0^1 |f_\infty(t(u_k - \bar{u}) - f(x, t(u_k - \bar{u}))| |u_k - \bar{u}| \, dt \, dx$$

$$< \frac{s_0 \varepsilon}{3} \|W_k\|^\frac{3}{2} (\|u_k\|_{L^2} + s_0 \|W_k\|^\frac{3}{2} + \|u_k - \bar{u}\|_{L^2}) \leq \varepsilon \frac{4 \hat{C}^2}{3}$$

for all $k \in \mathbb{N}$. Finally, since $u_k \to \bar{u}$ strongly in $L^r(B_{R_0}(0))$, $2 \leq r < 2^*$, we can choose $k_0 \in \mathbb{N}$ large enough such that

$$\int_{B_{R_0}(0)} |f(x, u_k) - f(x, \bar{u}) - f(x, u_k - \bar{u})||\varphi| \, dx \leq \varepsilon \|\varphi\|$$

and

$$\int_{B_{R_0}(0)} |F(x, u_k) - F(x, \bar{u}) - F(x, u_k - \bar{u})| \, dx \leq \varepsilon$$
hold for all \( k \geq k_0 \). Combining the above estimates, and remarking that \((U_k \cup V_k \cup W_k) \setminus B_{R_0}(0) = \mathbb{R}^N \setminus B_{R_0}(0)\) holds for all \( k \), we obtain

\[
\int_{\mathbb{R}^N} |f(x, u_k) - f(x, u_k - \bar{u}) - f(x, \bar{u})| |\varphi| \, dx \leq 3\varepsilon(1 + \tilde{C})\|\varphi\|
\]

and

\[
\int_{\mathbb{R}^N} |F(x, u_k) - F(x, u_k - \bar{u}) - F(x, \bar{u})| \, dx \leq \varepsilon(2 + \frac{23}{6}\tilde{C}^2)
\]

for all \( k \geq k_0 \). This concludes the proof. \( \square \)

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