Operations on distributions: regular and irregular

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Abstract. In the frame of the Mikusiński sequential theory of L. Schwartz’s distributions, regular and irregular operations on distributions are considered with applications to some formulas which may be of interest to physicists, e.g. \( \sqrt{\delta} = 0 \), \( \sqrt{1 + \delta^2} = 1 + \delta \), \( \log(1 + \delta) = 0 \), \( \sin \delta = 0 \).

1. Introduction

J. Mikusiński in his sequential theory of L. Schwartz’s distributions (see [6], [4], [2]) introduced a general method of extending operations defined for (smooth, continuous, locally integrable) functions to distributions. Operations which are regular (see the definition in section 2) can easily be extended to the class of all distributions. That the operation of substitution of a fixed smooth diffeomorphism into a given function (distribution) is regular is not so easy to prove as the regularity of other operations. We will show a proof of the regularity of this operation in the multidimensional case (see section 2). If a given operation is irregular, J. Mikusiński has also proposed (see [5], [2]) a general method of extending it to distributions, but not to all of them, in general. The method is based on the notion of a delta-sequence, i.e. a sequence \((\delta_n)\) of smooth functions convergent to the distribution \(\delta\), satisfying special conditions, and the notion of regular sequence for a given distribution \(f\), i.e. a sequence \((f_n)\) of smooth functions of the form \(f_n := f * \delta_n\), which is fundamental and distributionally convergent to \(f\), where the symbol \(f * \delta_n\) denotes the convolution of \(f\) with \(n\)-th element of a certain delta-sequence \((\delta_n)\) and is meant as the result of a regular operation (see [2], p. 153). The class of all regular sequences for a distribution \(f\) is narrower than the class of all fundamental sequences for \(f\). Consequently, irregular operations are feasible for a quite large class of distributions.

Irregular operations, e.g. the two argument operations of product and composition of distributions, and the one argument operation of the substitution of distributions into a given function, may be particularly interesting to physicists. We are going to justify mathematically some formulas concerning the distribution \(\delta\) which cannot be explained on the base of the classical approach to the theory of distributions. P. Antosik obtained in [1] some important results on composition of distributions in \(\mathbb{R}\) in case the external distribution is a continuous function and the internal one is a measure or a quasi-measure. The results enabled him to receive several interesting formulas, e.g. \(\sqrt{1 + \delta^2} = 1 + \delta\). However, the proofs of a few main theorems in [1] are incorrect. The incorrectness is entirely removed in [3], due to some results concerning measure theory.

In this note we present without proofs the generalizations of certain theorems in [1] and [3] from the real line \(\mathbb{R}\) to an arbitrary open subset or an arbitrary open (bounded or not) interval
in \( \mathbb{R} \). This makes possible to extend applications of the main theorem (e.g. to prove the formula \( \log (1 + \delta) = 0 \)).

2. Regular operations

In the whole paper distributions will be considered on open subsets of a given multidimensional Euclidean space. By intervals in \( \mathbb{R}^d \), we always mean \( d \)-dimensional finite (open or closed) intervals. Let us briefly recall the notation and definitions concerning the sequential approach to the theory distributions (see [2] and [3]). If the closure \([a, b]\) of a given interval \(I = (a, b)\), with \(a, b \in \mathbb{R}^d\) and \(a < b\), is contained in a given open set \(U \subset \mathbb{R}^d\), we write \(I \subset U\). By \(C^\infty(U)\) we denote the space of smooth (i.e. \(C^\infty\)) functions on \(U\). A sequence \((\varphi_n)\) of functions in \(C^\infty(U)\) is said to be fundamental on \(U\) if for every \(I \subset U\) there exist a multi-index \(k \in \mathbb{N}^d_0\) and functions \(\Phi_n \in C^\infty(U) (n \in \mathbb{N})\) such that \(\Phi_n^{(k)} = \varphi_n\) on \(I\) and \(\Phi_n \to \text{locally uniformly} \) (shortly: l.u.) on \(I\), i.e. uniformly on each compact set \(K \subset I\) to a function on \(I\) (see [2] and [3]). It is worth noting that locally uniformly convergent sequences are fundamental (for \(k = 0\), but in general, fundamental are sequences of smooth functions which are locally derivatives of a finite order of locally uniformly convergent sequences. Obviously, the multi-index \(k\) can be always increased. We call two fundamental sequences \((\varphi_n)\) and \((\psi_n)\) equivalent on \(U\), if \((\varphi_n) \sim (\psi_n)\), if the sequence \(\varphi_1, \varphi_2, \varphi_3, \ldots\) is fundamental on \(U\). Clearly, \(\sim\) is an equivalence relation. Equivalence classes of the relation \(\sim\) are called distributions on \(U\). The distributions represented by the fundamental sequences \((\varphi_n), (\psi_n), \ldots\) will be denoted by \(f, g, \ldots\), respectively, i.e. \(f = [(\varphi_n)], g = [(\psi_n)], \ldots\). The space of all distributions on \(U\) will be denoted by \(\mathcal{D}'(U)\).

**Definition 1.** Let \(R : C^\infty(U)^j \to C^\infty(U)\), \(j \in \mathbb{N}\), be an operation defined on all \(j\)-tuples of smooth functions whose values are smooth functions (or numbers treated as constant functions on \(U\)). The operation \(R\) is called regular if the sequence \(\{R(\varphi_1^n, \ldots, \varphi_j^n)\}\) is fundamental on \(U\) for arbitrary fundamental sequences \((\varphi_1^n), \ldots, (\varphi_j^n)\) on \(U\). Every regular operation \(R\) can be defined for arbitrary distributions \(f_1 = [(\varphi_1^n)], \ldots, f_j = [(\varphi_j^n)]\) on \(U\) by means of the formula \(R(f_1, \ldots, f_j) := [(R(\varphi_1^n, \ldots, \varphi_j^n))]\). Clearly, the result of every regular operation for given distributions does not depend on the choice of fundamental sequences.

Examples of regular operations are: addition, differentiation, multiplication by a fixed smooth function, tensor product, convolution with a given smooth function of bounded support and multidimensional substitution.

**Definition 2.** Let \(U\) and \(V\) be open sets in \(\mathbb{R}^d\) and \(\mathbb{R}^m\), respectively, where \(m \in \mathbb{N}, m \leq d\). Let \(\sigma_j \in C^\infty(U)\) for \(j \in \{1, \ldots, m\}\), \(\sigma := (\sigma_1, \ldots, \sigma_m) : U \to V\). Assume that for every \(x \in U\) at least one of the Jacobians:

\[
(J_{j_1, \ldots, j_m}) \sigma(x) := \frac{\partial(\sigma_1, \ldots, \sigma_m)}{\partial(\xi_{j_1}, \ldots, \xi_{j_m})} = \det A(x)
\]

is not equal to 0, where \(j_1, \ldots, j_m \in \mathbb{N}, j_1 < \ldots < j_m \leq d\) and the elements of the matrix \(A(x)\) are \(\frac{\partial \sigma_k}{\partial \xi_{j_k}}(x)\) for \(i, k \in \{1, \ldots, m\}\). The operation \(S\) of the multidimensional substitution of the function \(\sigma\) into \(\varphi \in C^\infty(V)\) is now defined by the formula: \(S(\varphi) := \varphi \circ \sigma\) on \(U\), i.e. \(S(\varphi)(x) := \varphi(\sigma(x)), x \in U\).

To prove that \(S\) is a regular operation we need three lemmas, of which the first is very well known.
Lemma 1 Let $U$ and $V$ be open sets in $\mathbb{R}^d$ and $\mathbb{R}^m$, respectively, where $m \in \mathbb{N}, m \leq d$. Let $\sigma_j \in C^\infty(U)$ for $j \in \{1, \ldots, m\}$ and $\sigma := (\sigma_1, \ldots, \sigma_m) : U \to V$. If $\varphi \in C^\infty(V)$, then $\varphi \circ \sigma \in C^\infty(U)$ and

$$ (\varphi \circ \sigma)(e_i)(x) = \sum_{j=1}^{m} (\varphi(e_j) \circ \sigma)(x) \frac{\partial \sigma_j}{\partial e_i}(x), \quad i \in \{1, \ldots, d\}, $$

for $x = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, where $\psi(e_i)(x) := \frac{\partial}{\partial e_i} \psi(x)$ for a given function $\psi$ and $x \in \mathbb{R}^d$.

Lemma 2 Under the assumption of Definition 2, we have

$$ (\varphi(e_j) \circ \sigma) = \frac{1}{J} \sum_{j_1 < \ldots < j_m} J_{j_1, \ldots, j_m} \sigma \cdot J_{\varphi,j_1, \ldots, j_m} \sigma, \quad j \in \{1, \ldots, m\}, $$

where

$$ J := \sum_{j_1 < \ldots < j_m} (J_{j_1, \ldots, j_m} \sigma)^2 $$

and

$$ (J_{\varphi,j_1, \ldots, j_m} \sigma)(x) := \frac{\partial (\sigma_1, \ldots, \sigma_{j-1}, \varphi \circ \sigma, \sigma_{j+1}, \ldots, \sigma_m)(x)}{\partial (\xi_1, \ldots, \xi_{j_m})}. $$

Proof. Fix $j \in \{1, \ldots, m\}$ and indices $j_1, \ldots, j_m \in \{1, \ldots, d\}$ such that $j_1 < \ldots < j_m$. To prove that

$$ (J_{j_1, \ldots, j_m} \sigma)(x) \cdot (\varphi(e_j) \circ \sigma)(x) = (J_{\varphi,j_1, \ldots, j_m} \sigma)(x), \quad x \in U. $$

Consider the matrices $A(x) = (a_{i,k}(x))$ and $B(x) = (b_{i,k}(x))$, where

$$ a_{i,k}(x) := b_{i,k}(x) := \frac{\partial \sigma_i}{\partial e_k}(x), \quad i \in \{1, \ldots, m\}, \quad i \neq j; $$

$$ a_{j,k}(x) := (\frac{\partial \varphi}{\partial \sigma_j} \circ \sigma)(x) \frac{\partial \sigma_j}{\partial e_k}(x); \quad b_{j,k}(x) := \sum_{n=1}^{m} (\frac{\partial \varphi}{\partial \sigma_n} \circ \sigma)(x) \frac{\partial \sigma_n}{\partial e_k}(x) $$

for $k \in \{1, \ldots, m\}$. Clearly,

$$ (J_{j_1, \ldots, j_m} \sigma)(x) \cdot (\varphi(e_j) \circ \sigma)(x) = \det A(x). $$

Multiplying each $i$-th (for $i \neq j$) row of the matrix $A(x)$ by $\left(\frac{\partial \varphi}{\partial \sigma_i} \circ \sigma\right)(x)$ and adding the results to the $j$-th row, we get the matrix $B(x)$, i.e.

$$ (J_{j_1, \ldots, j_m} \sigma)(x) \cdot (\varphi(e_j) \circ \sigma)(x) = \det A(x) = \det B(x). $$

But $b_{j,k}(x) = \left(\frac{\partial \varphi}{\partial \sigma_i} \circ \sigma\right)(x)$, so $\det B(x) = (J_{\varphi,j_1, \ldots, j_m} \sigma)(x)$ and equality (5) is shown. Multiplying both sides of (5) by $(J_{j_1, \ldots, j_m} \sigma)(x)$, we get

$$ (J_{j_1, \ldots, j_m} \sigma)^2 \cdot (\varphi(e_j) \circ \sigma) = J_{j_1, \ldots, j_m} \sigma \cdot J_{\varphi,j_1, \ldots, j_m} \sigma $$

for all $m$-tuples $(j_1, \ldots, j_m) \in \mathbb{N}^m$ such that $j_1 < \ldots < j_m$. Summing the above equalities over all such $m$-tuples $(j_1, \ldots, j_m)$, we obtain

$$ \sum_{j_1 < \ldots < j_m} (J_{j_1, \ldots, j_m} \sigma)^2 \cdot (\varphi(e_j) \circ \sigma) = \sum_{j_1 < \ldots < j_m} J_{j_1, \ldots, j_m} \sigma \cdot J_{\varphi,j_1, \ldots, j_m} \sigma $$

for every $j \in \{1, \ldots, m\}$. Since $J > 0$ on $U$, our assertion follows, by (3).
Lemma 3 Let \( \sigma \) be a given function satisfying the assumption of Definition 2. If \( \Phi_n \in C^\infty(V) \) (\( n \in \mathbb{N} \)) and the sequence \((\Phi_n \circ \sigma)\) is fundamental on \( U \), then the sequence \((J_{\Phi_n,j_1j_2\ldots j_m}\sigma)\) is fundamental on \( U \).

Proof. Assume that the sequence \((\Phi_n \circ \sigma)\) is fundamental on \( U \). By (4), due to the Laplace theorem, we have

\[
(J_{\Phi_n,j_1j_2\ldots j_m}\sigma)(x) = \sum_{k=1}^{m}(-1)^{j+k} \frac{\partial(\Phi_n \circ \sigma)}{\partial \xi_k}(x) \cdot \det \left( \frac{\partial \sigma_s}{\partial \xi_{j_l}}(x) \right),
\]

where \( s \in \{1, \ldots, j-1, j+1, \ldots, m\} \) and \( t \in \{1, \ldots, k-1, k+1, \ldots, m\} \).

Since \( \sigma_i \in C^\infty(U) \) for \( i \in \{1, \ldots, m\} \), all elements of the above determinants are smooth functions on \( U \). Consequently, the determinants themselves are smooth functions on \( U \). But differentiation is a regular operation and we infer from the assumptions that the sequence \((\frac{\partial(\Phi_n \circ \sigma)}{\partial \xi_k})\) is fundamental on \( U \) for \( k \in \{1, \ldots, m\} \). Finally, the sequence \((J_{\Phi_n,j_1j_2\ldots j_m}\sigma)\) is fundamental on \( U \), as a finite iteration of the regular operations.

Theorem 1 Multidimensional substitution is a regular operation.

Proof. Suppose that \( \sigma \) satisfies the assumptions of Definition 2 and the sequence \((\varphi_n)\) is fundamental on \( V \subset \mathbb{R}^m \), i.e. for each interval \( I' \subset V \) there exist a multi-index \( k \in \mathbb{N}_0^m \) and functions \( \Phi_n \in C^\infty(V) \) (\( n \in \mathbb{N} \)) such that \( \Phi_n^{(k)} = \varphi_n \) and \( \Phi_n \to \text{l.u.} \) to a certain function on \( I' \). Fix an interval \( I \subset U \) and a compact set \( K \subset I \). Since the map \( \sigma \) is continuous, the image \( \sigma(K) \) of the compact set \( K \) is a compact set contained in \( V \subset \mathbb{R}^m \). Hence \( \Phi_n \rightharpoonup \sigma(K) \) and thus \( \Phi_n \circ \sigma \rightharpoonup \sigma \) on \( K \), which implies that \( \Phi_n \circ \sigma \to \text{l.u.} \) on \( U \).

We will prove, by induction, that the sequence \((\Phi_n^{(k)} \circ \sigma)\) is fundamental on \( U \) for each \( k \in \mathbb{N}_0^m \). For \( k = 0 \) our assertion is obvious, because \( \Phi_n \circ \sigma \to \text{l.u.} \) on \( U \) and every locally uniformly convergent sequence is fundamental. Suppose that the sequence \((\Phi_n^{(k)} \circ \sigma)\) is fundamental on \( U \) for \( k \in \mathbb{N}_0^m \). We will show that the sequence \((\Phi_n^{(k+\epsilon_j)} \circ \sigma)\) is fundamental on \( U \) for \( j \in \{1, \ldots, m\} \). By Lemma 2, we have

\[
(\Phi_n^{(k+\epsilon_j)} \circ \sigma) = (\Phi_n^{(k)} \circ \sigma_j) = \frac{1}{J} \sum_{j_1 < \ldots < j_m} J_{j_1,\ldots,j_m} \cdot \Phi_n^{(k)} \circ J_{\Phi_n^{(k)};j_1\ldots j_m} \sigma
\]

for \( j \in \{1, \ldots, m\} \). Since \( \sigma_j \in C^\infty(U) \) for \( j = 1, \ldots, m \) and \( J > 0 \), we see that \( J \in C^\infty(U) \), \( J_{j_1,\ldots,j_m} \in C^\infty(U) \) for \( j_1 < \ldots < j_m \) and \( \frac{1}{J} \in C^\infty(U) \). By Lemma 3 and the induction hypothesis, the sequence \((J_{\Phi_n^{(k)};j_1\ldots j_m} \sigma)\) is fundamental on \( U \). By (6), since any finite iteration of regular operations is regular, we conclude that a sequence \((\Phi_n^{(k+\epsilon_j)} \circ \sigma)\) is fundamental on \( U \).

Consequently, the sequence \((\Phi_n^{(k)} \circ \sigma)\) is fundamental on \( U \) for each \( k \in \mathbb{N}_0^m \) and this implies the assertion.

Corollary 1 Let \( \sigma \) be a given function satisfying the assumption of Definition 2. The operation \( S \) of the multidimensional substitution of the function \( \sigma \) into a distribution on an open set \( V \subset \mathbb{R}^m \) exists for every \( f = [(\varphi_n)] \) on \( V \) in the sense of the formula: \( S(f) = f \circ \sigma := [(\varphi_n \circ \sigma)] \).
3. Irregular operations

We discuss now irregular operations based on the notion of a delta-sequence (see [5] and [2]). We consider the class $\Delta$ (see [3]) of delta-sequences $(\delta_n)$, whose elements are smooth functions defined in a neighborhood of 0, which satisfy the following conditions: 0° $\delta_n \geq 0$ for $n \in \mathbb{N}$; 1° there is a sequence of positive numbers $(\alpha_n)$, converging to 0, such that $\delta_n(x) = 0$ for $|x| \geq \alpha_n$, $n \in \mathbb{N}$; 2° $\int \delta_n = 1$ for $n \in \mathbb{N}$; 3° for every multi-index $k \in \mathbb{N}_0^n$ there is a positive constant $M_k$ such that $\alpha_n f |\delta_n^{(k)}| \leq M_k$ for $n \in \mathbb{N}$.

By a regular sequence for a given distribution $f$, we mean every sequence $(f_n)$ of the form $f_n := f * \delta_n$ for $(\delta_n) \in \Delta$ and $n \in \mathbb{N}$. Clearly, every regular sequence is a fundamental sequence.

Let $R : C^\infty(U)^j \to C^\infty(U)$, $j \in \mathbb{N}$, be an arbitrary operation as in Definition 1, but not necessarily regular. We say that the operation $R$ is feasible for given distributions $f_1, \ldots, f_k$ in the sense of irregular operation if for arbitrary regular sequences $(f_n^1), \ldots, (f_n^k)$ corresponding to $f_1, \ldots, f_k$ the sequence $(R(f_n^1, \ldots, f_n^k))$ is fundamental and we put then

$$R(f_1, \ldots, f_k) := \lim_{n \to \infty} (R(f_n^1, \ldots, f_n^k)) = \lim_{n \to \infty} R(f_n^1, \ldots, f_n^k),$$

where the limit is meant in the distributional sense.

The examples of irregular operations are: the one argument operations of the Lojasiewicz value of a distribution at a point, the integral of a distribution, the Fourier transform of a distribution, the substitution of distributions into a fixed distribution (in particular, a continuous function), and the two argument operations of product, convolution and composition of distributions (see [2]).

In particular, the last operation, the composition $g(f)$ of distributions $f$ and $g$, is meant in the sense of the distributional limit:

$$g(f) := \lim_{n \to \infty} g_n \circ f_n,$$

where the symbol $\circ$ denotes the composition of functions, $f_n := f * \delta_{1,n}$, $g_n := g * \delta_{2,n}$ and the limit is assumed to exist for all $(\delta_{1,n}), (\delta_{2,n}) \in \Delta$. This operation is studied in [1] and [3] and interesting results are obtained for $f$ and $g$ from certain classes of functions and distributions on the real line $\mathbb{R}$.

Let us remark that some of these results can be generalized, due to certain modifications in the proofs, to the case of an arbitrary open subset of $\mathbb{R}$. Namely, Theorems 2, 3 and 7 from [3] can be proved in the following more general form:

**Theorem 2** Assume that $g$ is a continuous function on $V$ such that, for some $p \in [1, +\infty)$ and $A > 0$,

$$|g(y)| \leq A(1 + |y|)^p, \quad y \in V$$

and $f \in L_{loc}^p(U)$ for the same $p$ as above. Then the composition $g(f)$ exists and is equal to the function $g \circ f \in L_{loc}^1(U)$ in the distributional sense.

**Theorem 3** Let $g$ be a continuous and bounded function on $V$ and $f$ be a distribution such that the Lojasiewicz value $f(x)$ at $x$ exists for almost all $x \in U$ and the regular part $f_r$ of $f$ is a measurable function. Then the composition $g(f)$ exists in the distributional sense, the composition $g \circ f_r$ exists a.e., $g \circ f_r \in L_{loc}^1(U)$ and $g(f) = g \circ f_r$.

**Theorem 4** Suppose that $g$ is a continuous function on $(a, b)$, where $-\infty \leq a < b \leq \infty$, $g(x) = o(|x|)$ as $x \to a+$ and $x \to b-$, $f$ is a measure on $U$, $f(U) \subseteq (a, b)$ and $f = f_r + f_s$ is the Lojasiewicz decomposition of $f$. Then the compositions $g(f)$ and $g(f_r)$ exist, $g(f_r)$ is a locally integrable function and $g(f) = g(f_r)$ on $U$. 

The main Theorem 8 of the paper [3] is also true in a more general form, with \( \mathbb{R} \) replaced by an arbitrary open (bounded or not) interval in \( \mathbb{R} \). Namely,

**Theorem 5** Assume that \( g \) is a continuous function on \( (a, b) \), where \( -\infty \leq a < b \leq \infty \) such that
\[
\lim_{x \to a^+} \frac{g(x)}{x} = \alpha, \quad \lim_{x \to b^-} \frac{g(x)}{x} = \beta.
\]
Moreover, assume that \( f \) is a measure on \( U \), \( f(U) \subset (a, b) \) and \( f = f_r + f_s \) is the Lojasiewicz decomposition of \( f \). Then the composition \( g(f) \) exists and
\[
g(f) = g \circ f_r - \alpha f_s^- + \beta f_s^+ \text{ on } U.
\]

By the Lojasiewicz decomposition of a distribution (a measure) \( f \), in the formulation of Theorems 3, 4 and 5 above, we understand the equality \( f = f_r + f_s \), where \( f_r \) is the regular part of the distribution \( f \) (see [1]), which is a locally integrable function and \( f_s := f - f_r \). By \( f^+ \) and \( f^- \) for a measure \( f \), we mean \( f^+ := \frac{|f|+f}{2} \) and \( f^- := \frac{|f|-f}{2} \) (see [1]).

Applying Theorem 5, the following formulas can be derived: 1° \( \sqrt{\delta} = 0 \); 2° \( \sin \delta = 0 \); 3° \( \sqrt{1 + \delta^2} = 1 + \delta \); 4° \( \log (1 + \delta) = 0 \). To obtain them it suffices to put \( (a, b) := (-\infty, \infty) \) in 1°–3°, \( (a, b) := (-\frac{1}{2}, \infty) \) in 4°, \( f := \delta \) in all cases 1°–4°, and
\[
g_1(x) = \begin{cases} \sqrt{x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0; \end{cases}
\]
\[
g_2(x) := \sin x; g_3(x) := \sqrt{1 + x^2}; g_4(x) := \log (1 + x) \text{ for } x \in (a, b) \text{ in cases 1°–4°, respectively.}
\]

The mentioned formulas follow from Theorem 5, because its assumptions are obviously fulfilled by \( f \) and \( g_i \) (\( i = 1, 2, 3, 4 \)) defined above.

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