Lagrangian perturbation theory for a superfluid immersed in an elastic neutron star crust

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ABSTRACT
The inner crust of mature neutron stars, where an elastic lattice of neutron-rich nuclei coexists with a neutron superfluid, impacts on a range of astrophysical phenomena. The presence of the superfluid is key to our understanding of pulsar glitches, and is expected to affect the thermal conductivity and hence the evolution of the surface temperature. The coupling between crust and superfluid must also be accounted for in studies of neutron star dynamics, discussions of global oscillations and associated instabilities. In this paper, we develop Lagrangian perturbation theory for this problem, paying attention to key issues like superfluid entrainment, potential vortex pinning, dissipative mutual friction and the star’s magnetic field. We also discuss the nature of the core–crust interface. The results provide a theoretical foundation for a range of interesting astrophysical applications.

Key words: asteroseismology – dense matter – gravitational waves – hydrodynamics – stars: neutron.

1 MOTIVATION

The crust of a mature neutron star shields the high-density fluid core from the thin atmosphere. It provides a heat blanket that determines the link between the neutrino-driven cooling of the core and the evolution of the observed surface temperature (Gudmundsson, Pethick, Epstein 1983). The physics of the crust is also of key importance for neutron star dynamics. The presence of a superfluid component, and the associated rotational vortices, is central to our understanding of pulsar glitches (Espinoza et al. 2011). The coupling between the neutron-rich nuclei in the crust and the superfluid neutrons also affects global oscillations involving the crust, as in the case of the quasi-periodic oscillations observed in the tails of magnetar flares (Andersson, Glampedakis & Samuelsson 2009). On the one hand, our understanding of the crust physics is quite good and the equation of state for matter has been modelled in detail. In particular, we have a clear idea of how the composition changes with density which allows us to work out the elastic properties of the lattice of nuclei (Chamel & Haensel 2009). This also provides us with the fraction of neutrons that have dripped out of nuclei and which may be considered free to move relative to the lattice. On the other hand, our understanding of the dynamics of the coupled crust–superfluid system is quite poor. This is somewhat surprising given the relevance of the involved issues (e.g. in the context of glitches), but it is nevertheless the case.

To make progress we need to develop a theoretical framework that allows us to model the dynamics of the inner-crust region. Such a model has to consider the key physics, like elasticity, the magnetic field and superfluidity. It must account for the presence of superfluid vortices, potential pinning and mutual friction. It is also important that the model is adaptable, in order to facilitate further developments concerning, for example, finite temperature effects and heat propagation. This is obviously a major challenge. The aim of the present work is to develop a ‘complete’ model for linear perturbations of a (moderately) realistic neutron star crust. Working within the Lagrangian perturbation framework (the natural way to view the perturbed crust, and a necessity if one wants to consider the stability of the system from the formal point of view; Friedman & Schutz 1978), we provide a two-component model that accounts for the key physics. The model complements previous work on the outer-core region (Andersson, Comer & Grosart 2004) by discussing the role of the superfluid entrainment and the magnetic field (Glampedakis & Andersson 2007). We also consider the conditions that need to be implemented at the crust–core transition. The theory is developed to the point where the results can be applied to a range of interesting astrophysics problems. Having said that, there is massive room for future refinements as our understanding of the relevant physics improves.

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2 THE TWO-FLUID MODEL

The conditions in the inner crust, with an elastic lattice of increasingly neutron-rich nuclei immersed in a neutron superfluid, have (at least) two dynamical degrees of freedom; the superfluid neutrons may flow relative to the lattice. In essence, this is an example of a two-‘fluid’ system, although in this case one of the components is also affected by elastic restoring forces. Our model for this system is based on the variational approach to multifluid dynamics developed by Prix (2004) and Andersson & Comer (2006) (see also Haskell, Andersson & Comer 2011), which represents the Newtonian limit of the fully relativistic convective variational model designed by Carter (1987) (see also Andersson & Comer 2007). The elastic sector builds on the relativistic model developed by Karlovini & Samuelsson (2003), and the inclusion of superfluidity follows the strategy set out by Carter & Samuelsson (2006) and Carter, Chachoua & Chamel (2006).

2.1 Hydrodynamics, magnetic fields and elasticity

In order to model the conditions in the inner crust of a neutron star, we take as the starting point the equations for two-fluid hydrodynamics (Prix 2004; Andersson & Comer 2006; Haskell et al. 2011). Assuming that the individual components are conserved (and working in a coordinate basis where vectors are represented by their components, with indices $i, j, k$ as usual), we have the usual conservation laws for the number densities $n_x$, where $x$ is a constituent index labelling the components,

$$\partial_t n_x + \nabla \cdot \{ n_x v_x \} = 0. \tag{1}$$

Note that a repeated species index $x$ does not imply summation, in contrast to the spatial indices like $i$ for which the Einstein summation convention applies. In the outer core of a neutron star, the distinction between the two components is fairly clear. On the one hand, we have the superfluid neutrons. On the other hand, we have a conglomerate of (most likely superconducting) protons and electrons. On scales larger than the electron screening length and time-scales longer than the inverse of the plasma frequency (Mendell 1991; Glampedakis, Andersson & Samuelsson 2011), these form a single, charge-neutral, fluid. The two-fluid model simply distinguishes the neutrons (represented by $n_n$, say) from the protons/electrons (given by $n_p$). The situation in the crust is similar yet different. At densities beyond neutron drip, some neutrons remain bound in nuclei, but there is also a ‘gas’ of free neutrons. The assignment of neutrons to each component follows from the equation of state, once the nature of the ions in the lattice is established (Chamel & Haensel 2009). However, if we consider the dynamics of the system it is not clear to what extent the ‘confined’ neutrons are able to move (Carter, Chamel & Haensel 2006). This depends on how strongly bound they are, to what extent they can tunnel through the relevant interaction potentials, etc. The upshot is that one can choose to work in different chemical ‘gauges’. This issue has been discussed in detail by Carter et al. (2006) (and we will return to it later). The choice of chemical gauge affects the interpretation of the involved quantities (number densities, etc.), but the two-fluid model remains unchanged conceptually. To make a distinction from the outer core problem, we will refer to the two components in the crust as ‘free’ neutrons, with density $n_n$ and ‘confined’ baryons, represented by $n_p$. This notation may represent a slight bias towards the description advocated by Carter et al. (2006) and Carter & Samuelsson (2006), but at this point we basically want to keep the options open by not linking the discussion too much to established results for the outer core.

The ‘free’ neutrons can flow relative to the crust lattice once the system cools below the transition to $^{1}S_0$ neutron superfluidity (see Andersson, Comer & Glampedakis 2005, for typical transition temperatures). We then have two coupled equations of momentum balance. Assuming that the large-scale system comprises a sufficient number of quantized vortices so that macroscopic averaging is meaningful (this should, indeed, be the case for all astrophysical systems of interest), the two momentum equations can be written (Prix 2004) as

$$\left( \partial_t + v_x^i \nabla_i \right) \left( v_x^i + \varepsilon_x w_x^i \right) + \nabla \cdot \left( \mu_x + \Phi \right) + \varepsilon_x w_x^i \nabla_i v_x^i = f_x^i / \rho_x, \tag{2}$$

where $x = \{ f, c \}$ and $y \neq x$, the velocities are $v_x^i$, the relative velocity is defined as $w_x^i = v_x^i - v_x^i$ and $\mu_x = \mu_x / m_x$ represents the chemical potential scaled to the nucleon mass (we will assume that the neutron and proton masses are equal, $m_n = m_p = m_\pi$). The mass densities are given by $\rho_x = m_n n_n$, $\Phi$ represents the gravitational potential, which means that we have

$$\nabla^2 \Phi = 4\pi G (\rho_f + \rho_c), \tag{3}$$

and, finally, the parameter $\varepsilon_x$ encodes the entrainment. The entrainment can be expressed in terms of a single parameter $\alpha$ such that (Prix 2004)

$$\rho_x \varepsilon_x = 2\alpha. \tag{4}$$

To close the system, we need to provide an equation of state. In the present formalism, the equation of state takes the form of an energy functional $E$, the functional form of which determines the chemical potentials

$$\mu_x = \left( \frac{\partial E}{\partial n_x} \right)_{n_y, \rho_x}, \tag{5}$$

and the entrainment parameter

$$\alpha = \left( \frac{\partial E}{\partial \rho_x} \right)_{n_x, \rho_y}. \tag{6}$$

The forces on the right-hand side of (2) can be used to represent various other interactions, including dissipative terms (Andersson & Comer 2006; Haskell et al. 2011). If we focus on the conditions in the neutron star crust, then we need to account for elasticity and the
large-scale magnetic field. The latter is described by the usual electromagnetic Lorentz force, i.e. we have
\begin{equation}
 f_i^r = \frac{1}{c} \epsilon_{ijk} J^j B^k . \tag{7}
\end{equation}
Eliminating the total current with the help of Ampère’s law, i.e. \( J^i = (c/4\pi) \epsilon^{ijk} \nabla_j B_k \), this becomes
\begin{equation}
 f_i^r = \frac{B_i^r}{4\pi} (\nabla_i B_j - \nabla_j B_i) . \tag{8}
\end{equation}
In order to use this in (2) we also need to know \( n_v \), i.e. to what extent the baryons are affected by the magnetic field. It is natural to assume that all baryons that are confined to nuclei are involved, but the exact meaning of this is (as we will discuss later) somewhat fuzzy in a dynamical situation in regions of the stars where some of the neutrons are free.

The elastic force is different in that it only serves to restore deviations from a relaxed state of the lattice. This means that it is natural to discuss elasticity at the linear perturbation level. We will do this later, in Section 4.2. In that discussion, we will assume that the background configuration is relaxed, in which case there is no leading order elastic force in (2). This assumption is not quite realistic; the crust of an astrophysical neutron star is likely to be strained due to the regular electromagnetic spin-down of the system. It is, in principle, straightforward to account for this strain, but in the interests of clarity we have chosen not to do so here.

Before we move on, it is worth noting that the incorporation of elasticity requires us to track given ‘fluid’ elements relative to the relaxed configuration. This motivates us to use a Lagrangian framework. This approach is, of course, advantageous for a number of reasons, in particular if we are interested in considered rotating neutron stars. Our model for the inner crust builds on the two-fluid perturbation framework developed by Andersson et al. (2004), adds the magnetic field according to the analysis of Glampedakis & Andersson (2007) and provides a model for elasticity which represents the Newtonian limit of the theory developed by Carter & Samuelsson (2006).

### 2.2 Superfluidity and vortex dynamics

Let us turn our attention to the superfluid aspects of the problem. Doing this, we note that (2) accounts for the presence of a (macroscopically averaged) vortex array. In order to discuss issues concerning, for example, vortex pinning and mutual friction, it is useful to consider (2) in more detail. Following Glampedakis et al. (2011), we introduce the momentum
\begin{equation}
 p_i^r = m \left( v_i^r + \epsilon_{ijk} \mathbf{u}_{i}^j \right) . \tag{9}
\end{equation}
In a superfluid, the momentum arises as the gradient of the condensate wavefunction. The upshot of this is that the superfluid is irrotational. However, this is only true on the microscopic scale. On the scale of hydrodynamics, the superfluid can rotate by forming vortices and when these are averaged over the system mimics bulk rotation (as evidenced by equation 2). The rotation is, however, quantized and the vorticity is given by
\begin{equation}
 \mathbf{W}_i^r = \frac{1}{m} \epsilon^{ijk} \nabla_j p_i^r = n_v \kappa_i^r , \tag{10}
\end{equation}
where \( n_v \) is the number of vortices per unit surface area and \( \kappa_i^r = \kappa \hat{k}_i \) (with \( \hat{k}_i \) a unit vector along the direction of the vortex array and \( \kappa = \hbar/2m \approx 2 \times 10^{-3} \, \text{cm}^2 \, \text{s}^{-1} \) the quantum of circulation).

From equation (10) one can derive the equation that governs the vorticity. Assuming that the vortex density is conserved, we have
\begin{equation}
 (\partial_t + \mathbf{u}_i^r) n_v = 0 \quad (n_v v_i^r) = 0 , \tag{11}
\end{equation}
where \( v_i^r \), in fact, defines the macroscopically averaged vortex velocity. The fact that the vortices move with \( v_i^r \) also means that (in terms of the Lie derivative \( \mathcal{L}_{\mathbf{v}_i^r} \) along \( v_i^r \)) we have
\begin{equation}
 (\partial_t + \mathcal{L}_{\mathbf{v}_i^r}) \kappa_i^r = 0 . \tag{12}
\end{equation}
Given these relations, it follows that
\begin{equation}
 (\partial_t + \mathcal{L}_{\mathbf{v}_i^r}) (\epsilon^{ijk} \mathbf{W}_{i}^r v_j^r ) = 0 . \tag{13}
\end{equation}
In order to make contact with the macroscopic description, we now rewrite the relevant Euler equation (cf. equation 2) as
\begin{equation}
 \partial_t p_i^r + \nabla_j \left( \mu_t - \frac{m}{2} v_i^r v_j^r + v_i^r p_j^r \right) - m \epsilon_{ijk} v_j^r \mathbf{W}_{i}^r = f_i^r / n_t . \tag{14}
\end{equation}
This expression makes it clear that, in the absence of vortices and external forces, the superfluid motion follows from the gradient of a scalar potential. Moreover, it is now easy to compare (13) and (11). As discussed by Glampedakis et al. (2011), we then find that the models are consistent provided we account for the ‘Magnus force’ on the right-hand side of equation (14). This force takes the form
\begin{equation}
 \frac{f_i^r}{\rho_t} = n_v \epsilon_{ijk} \kappa_j^r \left( v_k^r - v_i^r \right) , \tag{15}
\end{equation}
and an equal and opposite force will act on the vortex array. In the simple case of a single condensate at zero temperature, force balance on the vortices requires them to flow with \( v_i^r \). [Here and in the following we ignore the inertia of the vortices (Mendell 1991).] In a more general situation, we can still make use of the above strategy to account for the forces that act on the vortices. We ‘simply’ solve the force balance equation for the vortices for \( v_i^r \) and use the result in (15). In the case of resistive scattering off of the vortex cores, e.g. by phonons, this leads to the usual representation of the vortex-mediated mutual friction (Andersson, Sidery & Comer 2006). We will now consider this problem in the context of the crust.
3 MUTUAL FRICTION AND VORTEX PINNING

To discuss the various vortex forces, we take as our starting point the equation of force balance for a single vortex:

\[ \epsilon^{ijk} \dot{\mathbf{v}}_i (v^k_j - v^k_j) + R (v^i_j - v^i_j) + F^j = 0. \]  

(16)

This accounts for (i) the Magnus force, (ii) a resistive friction associated with the normal component (e.g. nuclei, electrons and phonons in the crust), with coefficient \( R \) and (iii) a general force which we leave unspecified at this point. This force will later be taken to represent the ‘pinning’ of vortices to the nuclei in the crust. Note that the different terms all have the dimension of velocity. In order to obtain an expression for the force per unit length of a vortex, as required in the macroscopic Euler equations, we need to multiply by \( \rho_i \kappa_n \). This step assumes that the vortices form a rectilinear array (the usual Abrikosov lattice). It is the simplest set-up, essentially since it makes the averaging over vortices trivial, but there is no guarantee that this situation prevails in a real system. In particular, it may be relevant to worry about the formation of vortex tangles and superfluid turbulence (Peralta et al 2006; Andersson, Sidery & Comer 2007). We will not consider this problem here.

Neither will we account for contributions like the vortex tension or the elasticity of the vortex array. These effects are readily incorporated in our framework (see Haskell 2011, for a recent discussion of the elasticity of the vortex lattice), but we leave them out in the interest of clarity.

Finally, we assume that there are no force contributions along \( \hat{v}_i \), the direction of the vortex axis.

Solving for the vortex velocity in the standard way, we find

\[ v'_i = v'_i + \frac{1}{1 + R^2} (R f^i + \epsilon^{ijk} \dot{\mathbf{v}}_j f_k), \]  

(17)

where

\[ f^i = \epsilon^{ijk} \dot{\mathbf{v}}_j w^j_i + F^i. \]  

(18)

Now, the reaction force acting on the normal component will be

\[ \frac{f^i}{R \kappa_k} = -R (v^i_j - v^i_j) - F^j = \frac{R}{1 + R^2} (R f^i + \epsilon^{ijk} \dot{\mathbf{v}}_j f_k) + \frac{1}{1 + R^2} \epsilon^{ijk} \dot{\mathbf{v}}_j (F^i - F^i). \]  

(19)

where we have defined the projection orthogonal to the vortices:

\[ \perp_J = \delta^i_J - \hat{v}^i \dot{\mathbf{v}}_j. \]  

(20)

The force acting on the neutrons will naturally be equal and opposite:

\[ \frac{f^i}{R \kappa_k} = -\epsilon^{ijk} \dot{\mathbf{v}}_j (v^i_j - v^i_j) = -\frac{f^i}{R \kappa_k}. \]  

(21)

For later convenience, it is natural to introduce basis vectors along the macroscopic relative flow (in the plane orthogonal to the vortex). That is, we use

\[ \hat{w}^i = w^i / w, \]  

(22)

where

\[ w^i = \perp_J w^i_{\perp} \quad \text{and} \quad w^2 = (\perp_J w^i_{\perp}) w^i_{\perp}, \]  

(23)

together with the decomposition

\[ F^i = a_i \hat{w}^i + a_\perp \epsilon^{ijk} \dot{\mathbf{v}}_j \hat{w}_k. \]  

(24)

This means that we get

\[ (1 + R^2) f^i = [R^2 w + a_i R - a_\perp] \epsilon^{ijk} \dot{\mathbf{v}}_j \hat{w}_k - \frac{R w + a_\perp R + a_i}{w} \hat{w}^i. \]  

(25)

Expressing the result in the usual form (Andersson et al. 2006), we have

\[ \frac{f^i}{R \kappa_k} = w B^i_{\text{eff}} \epsilon^{ijk} \dot{\mathbf{v}}_j \hat{w}_k - w B^i_{\text{eff}} \hat{w}^i, \]  

(26)

where

\[ B^i_{\text{eff}} = \frac{1}{1 + R^2} \left[ R + a_i R - a_\perp \frac{R}{w} \right] \]  

(27)

and

\[ B^i_{\text{eff}} = \frac{1}{1 + R^2} \left[ R^2 + a_i R - a_\perp \frac{R}{w} \right]. \]  

(28)

The first term in each bracket represents the standard mutual friction. The second terms illustrate how a pinning force may be accounted for in the macroscopic multifluids model. It is worth noting that this model can also be applied to the problem of (potentially strong) interaction between neutron vortices and proton flux tubes in the outer core of a neutron star (cf. Glampedakis et al. 2011).
3.1 Perfect pinning

Having discussed the general model, we are equipped to consider the limiting case of perfect ‘pinning’. The interaction between the vortex lines and the crustal nuclei may be strong enough to ‘pin’ the vortices and force them to move along with the crust (Donati & Pizzochero 1978). This has profound implications for the macroscopic dynamics of the system. Given that the vortex lines are no longer free to move, the superfluid neutrons cannot spin down (or up). Hence, a lag will build up between the two components as the crust slows down due to magnetic braking. When this lag develops, the Magnus force will tend to push the vortices out (or in, if the torque is reversed) (cf. equation 15). Eventually, the force will be strong enough to overcome the pinning and break the vortices free. This leads to a transfer of angular momentum, which could explain large pulsar glitches (see Sidery, Passamonti & Andersson 2010, for a recent discussion). Vortex pinning may also have a severe effect on neutron star precession. By acting as a gyroscope, the pinned vortices are expected to lead to extremely short period precession, of the order of the rotation period (rather than the several months to years period expected from a typical crustal deformation) (Jones & Andersson 2001; Link & Epstein 2001). While this general picture is supported by a range of theoretical models, we are still quite far from a detailed understanding of the nature and strength of vortex pinning. However, for the present study it is sufficient to assume that a pinning force is acting.

Let us assume that there is ‘perfect’ pinning, \( v_c = v_c \). In this case, the equation of force balance for a single vortex takes the form

\[
\epsilon^{ijk} \dot{v}_j \left( v_i^c - v_i \right) + \mathcal{F}^c = 0. \tag{29}
\]

As long as the system remains below the unpinning limit, this provides us with the required pinning force \( \mathcal{F}^c \). Given this, we find that the force acting on the neutrons is

\[
\frac{\mathcal{F}^c}{\rho \kappa} = -\epsilon^{ijk} \dot{v}_j \left( v_i^c - v_i \right), \tag{30}
\]

while the reaction force on the charged component will be

\[
\frac{\mathcal{F}^\parallel}{\rho \kappa} = \epsilon^{ijk} \dot{v}_j \left( v_i^c - v_i \right). \tag{31}
\]

The main conclusion from this exercise is that, if the lag between the two components is sufficiently small to allow us to consider the vortices as perfectly ‘pinned’, the exact form of the pinning force does not appear explicitly in the equations of motion.

3.2 Vortex creep

Let us now consider the situation where the lag between the two components is close to the critical value for unpinning. This regime is of great physical interest as it is at the heart of many theoretical models for pulsar glitches. If one assumes that parts of the system are always slightly subcritical, one may allow for a population of thermally excited vortices to unpin randomly and transfer angular momentum to the crust. This is usually referred to as ‘vortex creep’ (Alpar et al. 1984). Describing this behaviour is clearly a challenging task, both from the microscopic and the macroscopic point of view. On the one hand, there have been efforts to calculate the pinning ‘force’ and the barrier that the thermally excited vortices would have to overcome to unpin (Link 2009). On the other hand, there have been attempts to incorporate the concept of vortex creep in a macroscopic hydrodynamical description, by assuming that only a fraction of the vortices, on average, participates in the dynamics (Jahan-Miri 2006).

Here we adopt a phenomenological approach aimed at exploring the hydrodynamics of the creep regime. We start by noting that vortex creep would correspond to motion such that \( v'_c \approx v'_c \). Assuming that (24) describes the ‘pinning force’ completely, i.e. that there is no component along \( \hat{k} \), we can rewrite (17) as

\[
v'_c = v'_c - v'_c = \frac{1}{1 + \mathcal{R}^2} \left[ a_1 \left( \mathcal{R} - w_{\text{cr}} - a_\perp \right) \hat{w}_r^c \right]
+ \frac{1}{1 + \mathcal{R}^2} \left[ a_1 + (w_{\text{cr}} + a_\perp) \mathcal{R} \right] \epsilon^{ijk} \dot{v}_j \hat{w}_k^c. \tag{32}
\]

From this expression we see that there are two ways of enforcing vortex creep. One would be to let \( \mathcal{R} \to \infty \). This model has recently been used in studies of precession (Glampedakis, Andersson & Jones 2008) and the unstable r-modes (Glampedakis & Andersson 2009). However, with the ‘pinning’ force explicitly in the problem we have another option. Focusing on the \( \mathcal{R} \ll 1 \) case, which should be relevant in the crust (Feibelman 1979; Bildsten & Epstein 1989), we can demand that

\[
a_1 \left( \mathcal{R} - w_{\text{cr}} - a_\perp \right) \equiv v_1 \ll v_c \tag{33}
\]

and

\[
a_1 + (w_{\text{cr}} + a_\perp) \mathcal{R} \equiv v_\perp \ll v_c. \tag{34}
\]

With these definitions, the creep velocity is given by

\[
v'_c = v_1 \hat{w}_r + v_\perp \epsilon^{ijk} \dot{v}_j \hat{w}_k, \tag{35}
\]

and we have

\[
v_c^2 = v_1^2 + v_\perp^2. \tag{36}
\]
This model is, obviously, more complicated since we now have three coefficients to specify: we need $R$, $a_i$ and $a_\perp$. However, this provides more flexibility and could allow us to, for example, consider a specific form for the ‘pinning’ force or, indeed, the creep rate $v_c$. An important point is that in this model you do not have to have strong drag, $R \gg 1$, to effect pinning. This may be particularly relevant if we want to make our neutron star precession models more realistic (see Link 2003).

Let us conclude by writing down the force that enters in the hydrodynamics. After some straightforward algebra, we have

$$a_\parallel \approx v_\perp + v_i R$$

(37)

and

$$a_\perp \approx -w - v_i + v_\perp R.$$  

(38)

This means that the force becomes

$$f^i_\rho = -v_\perp \tilde{u}^i + (w + v_i)\epsilon^{ijk} \tilde{K}_j \tilde{u}_k,$$

(39)

which means that

$$f^2 = v_\perp^2 + (w + v_i)^2 \approx w^2.$$  

(40)

The required hydrodynamical force follows once we multiply by $\rho \nabla \cdot \delta n$.

It is important to remember that the ‘pinning’ force discussed here does not describe the realistic interaction between the vortices and the nuclei. We have discussed a purely phenomenological model which allows us to describe the motion when vortices are not yet completely free so that the drag force cannot be approximated as linear in the velocities (Link 2009). Our simple ‘pinning’ model allows us to consider the dynamical implications of this regime.

4 LAGRANGIAN PERTURBATIONS

So far, we have discussed the general conditions that prevail in the inner neutron star crust and a two-fluid ‘hydrodynamics’ model that accounts for the key features. We will now take an important step towards astrophysical applications by developing a framework for Lagrangian perturbations of this system. The aim is to provide a model that can be applied to a range of important problems in neutron star dynamics, from pulsar glitches to the gravitational-wave-driven instability of the r-modes and magnetar oscillations. These problems are all naturally approached within perturbation theory. Moreover, they all represent scenarios where the sensitive interplay between the crust, the superfluid and the magnetic field is expected to be important.

4.1 The unentrained two-fluid problem revisited

Since the two-fluid problem has two dynamical degrees of freedom, it is natural to introduce two distinct Lagrangian displacement vectors $\xi_s^i$ (Andersson et al. 2004). In order to distinguish between these displacements, we use variations $\Delta x$ such that (for any, scalar or vectorial, quantity $Q$)

$$\Delta x Q = \delta Q + L_n Q,$$

(41)

where $\delta$ represents an Eulerian perturbation. The perturbed continuity equations (cf. equation 1) then take the form (Friedman & Schutz 1978)

$$\Delta x n_x = -n_s \nabla \cdot \delta n_x = -\nabla_i \left( n_s \xi_{s}^{i} \right).$$

(42)

This means that the equation that describes the perturbed gravitational potential is

$$\nabla^2 \delta \Phi = 4\pi \rho_{mb} G (\delta n_x + \delta n_y) = -4\pi \rho_{mb} GV_i \left( n_s \xi_{s}^{i} + n_y \xi_{y}^{i} \right).$$

(43)

Considering the simplest case of vanishing entrainment and no ‘external’ forces, $f^i_\rho = f^i_c = 0$, the results of Andersson et al. (2004) show that

$$(\partial_t + L_n) \Delta x \xi_s + \nabla_i \left( \Delta x \Phi + \Delta x \mu_s - \frac{1}{2} \Delta x v_s^2 \right) = 0.$$  

(44)

After some algebra, this leads to

$$\partial_t^2 \xi_s^i + 2v_s^i \nabla_j \partial_s \xi_s + \left( v_s^j \nabla_j \right)^2 \xi_s^i + \nabla_i \delta \Phi + \xi_s^i \nabla_j \Phi - (\nabla_i \xi_s^j) \nabla_j \mu_s + \nabla_i \mu_s = 0.$$  

(45)

Here, the Lagrangian perturbation of the chemical potential can be written (with $y \neq x$) as

$$\Delta x \mu_s = \Delta \mu_s + \xi_s^i \nabla_j \mu_s$$

$$= \left( \frac{\partial \mu_s}{\partial n_x} \right)_{n_y} \delta n_x + \left( \frac{\partial \mu_s}{\partial n_y} \right)_{n_x} \delta n_y + \xi_s^i \nabla_j \mu_s$$

$$= -\left( \frac{\partial \mu_s}{\partial n_x} \right)_{n_y} \nabla_i \left( n_s \xi_{s}^{i} \right) - \left( \frac{\partial \mu_s}{\partial n_y} \right)_{n_x} \nabla_i \left( n_y \xi_{y}^{i} \right) + \xi_s^i \nabla_j \mu_s$$

(46)

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using the fact that $\tilde{\mu}_x = \tilde{\mu}_x(n, n_x)$. Hence, we arrive at the following final form for the perturbed Euler equations:

$$
\partial_t^j \xi^i + 2n_x \nabla_j \partial_t^i \xi^i + (\nabla_j \partial_t^i) \xi^i + \nabla_j \delta \Phi + \nabla_j (\Phi + \tilde{\mu}_x) - \nabla_j \left[ \left( \frac{\partial \tilde{\mu}_x}{\partial n_x} \right) \nabla_j (n_x \xi^i) + \left( \frac{\partial \tilde{\mu}_x}{\partial n_y} \right) \nabla_j (n_y \xi^i) \right] = 0.
$$

(47)

Andersson et al. (2004) demonstrated how one can proceed further and derive useful conserved quantities, extending the single-fluid analysis of Friedman & Schutz (1978) to the two-fluid arena. The importance of the obtained canonical energy $E_c$ stems from the fact that it can be used to assess the stability of the system. In order for the evolution to be dynamically unstable, i.e. for a perturbation to blow up in the absence of additional forces, we must have $E_c = 0$. A secular (viscosity- or radiation-driven) instability requires $E_c < 0$, provided that the energy lost through dissipation is positive (which makes sense). A particularly nice feature of the analysis of Andersson et al. (2004) is the proof that the standard instability criterion for gravitational-wave instabilities, that a normal mode of oscillation becomes unstable when its pattern speed changes sign (e.g. when an originally retrograde mode in a rotating star becomes prograde; Andersson 2004), holds also in the two-fluid problem. This had previously been assumed to be the case, but there was no formal proof. In the following, we will not attempt to address the issue of the canonical energy for more complex systems; our focus is entirely on the perturbed equations of motion. A stability analysis for these equations would be interesting, but as this may well be prohibitively complicated we leave the problem for future consideration.

4.2 Accounting for elasticity and the magnetic field

The equation that represents the crust dynamics must account for both elastic and magnetic contributions. The former leads to the well-known contribution

$$\Delta_c \left( \frac{f^i_j}{\rho_c} \right) = \frac{1}{\rho_c} \nabla^j \sigma_{ij},
$$

(48)

where the shear tensor is given by

$$\sigma_{ij} = \tilde{\mu} \left( \nabla_i \xi^j + \nabla_j \xi^i \right) - \frac{2}{3} \tilde{\mu} \left( \nabla \xi^k \right) \delta_{ij}
$$

(49)

(here one should not confuse the shear modulus $\tilde{\mu}$ with the chemical potentials $\mu_x$). It is important to keep in mind that these expressions are only valid for unstrained background configurations.

In the case of the magnetic field, we need the Lagrangian perturbation of (8). The required results have already been derived by Glampedakis & Andersson (2007), so we simply restate them here. The perturbations are determined from

$$\Delta_c \left( \frac{B^i}{\rho_c} \right) = 0,
$$

(50)

which leads to

$$\Delta_c B^i = -B^j \nabla_j \xi^i
$$

(51)

and

$$\Delta_c B_i = B_j \nabla_j \xi^i - B_i \left( \nabla_j \xi^j \right) + B^i \nabla_j \xi^j.
$$

(52)

Finally, using

$$\Delta_c (\nabla_j B_i) = \nabla_j (\Delta_c B_i) - B_i \nabla_j \nabla_j \xi^i,
$$

(53)

we obtain from (8)

$$\Delta_c \left( \frac{f^i_j}{\rho_c} \right) = \frac{B^j}{4\pi \rho_c} \left[ \nabla_j (\Delta_c B_i) - \nabla_i (\Delta_c B_j) \right].
$$

(54)

It is straightforward to express the magnetic perturbations in terms of the displacement $\xi^i_x$, but since the expressions that result are quite involved we have decided not to do so here.

4.3 Entrainment

The elastic and magnetic forces take relatively simple forms. However, the fluid part of the problem becomes much more complex if we consider the generic situation where the entrainment is not vanishing. In the unentrained case, the two equations of motion are coupled chemically through the equation of state (e.g. through various interactions) and gravitationally since variations in the number density of one fluid affect the gravitational potential and hence the motion of the other fluid. In contrast, the entrainment parameter $\alpha$ encodes how the internal energy of the system depends on the relative velocity of the two fluids. This usually leads to a stronger coupling of the components. Since the entrainment encodes the effective dynamical mass of each matter constituent, it will affect most scenarios that involve the superfluid regions of the star.

Including the entrainment, the Euler equations take the form (2). However, since

$$w^i \nabla_j \left( \varepsilon_x w^j_{\varepsilon x} \right) = \mathcal{L}_{w_{\varepsilon x}} \left( \varepsilon_x w^j_{\varepsilon x} \right) - \varepsilon_x w^j_{\varepsilon x} \nabla_i w^i_{\varepsilon x},
$$

(55)
these can be rewritten as
\begin{equation}
(\partial_t + L_{\alpha}) (v^i + \varepsilon_x w^{i3}) + \nabla_i \left( \Phi + \bar{\mu}_x - \frac{v^i}{2} \right) = 0.
\end{equation}

We want to consider Lagrangian perturbations of this system. The derivation follows the same approach as in the unentrained case, described by Andersson et al. (2004) and summarized in Section 4.1.

First, we note that the continuity equations and the Poisson equation are not affected by entrainment, so we can still use equations (42) and (43). Secondly, perturbing the Euler equations we have
\begin{equation}
(\partial_t + L_{\alpha}) \left[ \Delta_x v_i^i + \Delta_x \left( \varepsilon_x w^{i3} \right) \right] + \nabla_i \left( \Delta_x \Phi + \Delta_x \bar{\mu}_x - \frac{\Delta_x v^2}{2} \right) = 0.
\end{equation}

This follows immediately since the Lagrangian variation commutes with $\partial_t + L_{\alpha}$. Most of the terms in this equation were already considered in the unentrained problem. A key difference now is that $\delta \bar{\mu}_x$ depends on the entrainment. This is obvious since the chemical potential, $\mu_x$, is the partial derivative of the energy functional, $E$, with respect to the number density, $n_x$. Since $E$ depends on the entrainment, the Eulerian variation of the chemical potential also depends on the entrainment. In general, we have
\begin{equation}
\delta \bar{\mu}_x = -\left( \frac{\partial \bar{\mu}_x}{\partial n_x} \right) n_x w \left( v \right) \nabla_i \left( n_x \xi^i \right) + \left( \frac{\partial \bar{\mu}_x}{\partial w^2} \right) n_x w \partial w^2.
\end{equation}

where
\begin{equation}
\left( \frac{\partial \bar{\mu}_x}{\partial w^2} \right)_{n_x, n_y} = \frac{1}{m_B} \left( \frac{\partial \alpha}{\partial n_x} \right)_{n_y, w^2} = \frac{1}{m_B} A_x
\end{equation}

and
\begin{equation}
\partial w^2 = 2 w^{i3} \delta w_{ij},
\end{equation}

giving
\begin{equation}
\delta \bar{\mu}_x = -\left( \frac{\partial \bar{\mu}_x}{\partial n_x} \right) n_x w \left( v \right) \nabla_i \left( n_x \xi^i \right) + \left( \frac{\partial \bar{\mu}_x}{\partial w^2} \right) n_x w \partial w^2.
\end{equation}

At this point, it is worth noting that
\begin{align}
\delta w_{ij} &= \partial \xi^i \xi^j + \varepsilon_x \nabla_i \xi^j - \varepsilon_x \nabla_j \xi^i - \partial \xi^i \nabla_j \xi^j + \xi^i \nabla_i \xi^j - \partial \xi^j \nabla_i \xi^i + \xi^j \nabla_i \xi^j \\
&= \partial \xi^i \xi^j + \varepsilon_x \nabla_i \xi^j + \varepsilon_x \nabla_j \xi^i - \xi^i \nabla_j \xi^j - \xi^j \nabla_i \xi^i + \xi^j \nabla_i \xi^j \\
&= \partial \xi^i \xi^j + \varepsilon_x \nabla_i \xi^j - \varepsilon_x \nabla_j \xi^i - \xi^i \nabla_j \xi^j + \xi^j \nabla_i \xi^i.
\end{align}

The only other piece of equation (57) that was not present in the unentrained problem can be written as
\begin{equation}
(\partial_t + L_{\alpha}) \Delta_x \left( \varepsilon_x w^{i3} \right) = \epsilon_x (\partial_t + L_{\alpha}) \Delta_x w^{i3} + w^{i3} (\partial_t + L_{\alpha}) \Delta_x \varepsilon_x.
\end{equation}

The first term follows easily from
\begin{equation}
\Delta_x w^{i3} = \delta w^{i3} + \xi^i \nabla_i w^{i3} + w^{i3} \nabla_i \xi^i.
\end{equation}

The second term in (63) is worth discussing in more detail. We first consider
\begin{equation}
\Delta_x \varepsilon_x = \delta \varepsilon_x + \xi^i \nabla_i \varepsilon_x.
\end{equation}

Using the definition for $\varepsilon_x$, and the perturbed continuity equation, we find
\begin{equation}
\Delta_x \varepsilon_x = \frac{2}{\rho_x} \left[ \delta \alpha + \nabla_i \left( \alpha \xi^i \right) \right].
\end{equation}

In the general case, the entrainment parameter $\alpha$ is a function of the two number densities, e.g. $n_t$ and $n_c$, and $w^2$. This means that
\begin{equation}
\delta \alpha = A_t \delta n_t + A_c \delta n_c + 2 A_w \delta w^{i3} \delta w_{ij},
\end{equation}

where we have defined
\begin{equation}
A_w = \left( \frac{\partial \alpha}{\partial w^2} \right)_{n_t, n_c}.
\end{equation}

We can also use
\begin{equation}
\nabla_i \alpha = A_t \nabla n_t + A_c \nabla n_c + 2 A_w \delta w^{i3} \nabla_i \delta w_{ij}.
\end{equation}

This means that we can write $\delta \alpha$ as
\begin{align}
\delta \alpha &= -A_t \nabla_i \left( n_t \xi^i \right) - A_c \nabla_i \left( n_c \xi^i \right) + 2 A_w \delta w^{i3} \left[ \partial_i \xi^i - \partial_i \xi^i + \varepsilon_x \nabla_i \xi^j - \varepsilon_x \nabla_j \xi^i + \xi^j \nabla_i \xi^j - \xi^j \nabla_j \xi^i \right].
\end{align}

After some algebra, we finally find that
\begin{align}
\Delta_x \varepsilon_x = \frac{2}{\rho_x} \left\{ \left( \alpha - A_t n_t \right) \nabla_i \xi^i - A_c n_c \nabla_i \xi^i + 2 A_w \delta w^{i3} \left[ \partial_i \left( \xi^i - \xi^i \right) - v^i \nabla i \xi^j - v^i \nabla j \xi^i + \xi^j \nabla i \xi^j - \xi^j \nabla j \xi^i \right] \right\}.
\end{align}
We can now combine the above results to get a general expression for the right-hand side of (63) in terms of the two displacements. However, this expression will be rather lengthy and may not be particularly useful. In most situations of interest, a reduced version should suffice. In principle, one may consider different simplifying assumptions. The most drastic would be to consider the entrainment parameter to be uniform. The natural way to achieve this would be to take $\alpha = \text{constant}$. Then we have $\mathcal{A}_c = \mathcal{A}_w = 0$, and the equations simplify greatly. In fact, from (71) we are only left with

$$\Delta x = \frac{2\alpha}{\rho c} \nabla n_\parallel = \nabla n_\parallel.$$  

(72)

A more realistic model would be based on an expansion for small relative velocities (Comer & Joynt 2003). One would expect $w_\parallel$ to be small in most cases, so it makes sense to use the approximate equation of state

$$E(n_\parallel, n_\perp, w^2) \approx E_0(n_\parallel, n_\perp) + E_1(n_\parallel, n_\perp)w^2.$$  

(73)

In this case, we simply have

$$\alpha = E_1,$$  

(74)

and it is obviously the case that $\mathcal{A}_w = 0$. Hence, we have

$$\Delta x = \frac{2}{\rho c} \left[ (\alpha - \mathcal{A}_c n_\parallel) \nabla n_\perp - \mathcal{A}_c n_\perp \nabla n_\perp \right].$$  

(75)

This expression completes our analysis of the perturbations of the inviscid problem for the coupled crust–superfluid system.

### 4.4 Perturbing the mutual friction

The various contributions associated with the superfluid vortices add further complexity to the problem. As an illustration of this, we will focus on the mutual friction, which follows from equation (26). To complete the perturbation equations, we need

$$\Delta x \left( \frac{f_{\parallel}^t}{\rho c} \right) = -\Delta x \left( \frac{f_{\parallel}^t}{\rho c} \right) = \Delta x \left( B_{\text{eff}}^t \epsilon^{i j k} n_{\parallel} n_{\perp} \nabla n_{\parallel} - \kappa n_{\perp} B_{\text{eff}}^t \xi_{\perp}^i \right),$$  

(76)

where it is worth recalling that we defined $\mathcal{W}_{\parallel}^t = n_{\parallel} \nabla_{\parallel}$ in Section 2.2.

From equations (27) and (28), we see that the perturbations of the coefficients $B_{\text{eff}}^t$ and $B_{\text{eff}}^\parallel$ will (in general) require knowledge of $\mathcal{R}$, $a_{\parallel}$ and $a_{\perp}$. The perturbations of these quantities can, obviously, be treated in the same way as $\alpha$ in the previous section. However, in this case any simplifications would rely on an understanding of the detailed microphysics. In addition to these quantities, we need the variation of the equation of state, $\rho$. Determining this quantity is straightforward.

The main new piece of information required for the mutual friction is the perturbed vorticity. In general, when the vortices are not moving with either of the macroscopic fluids, we need to perturb (10), after solving for the vortex velocity as in Section 3. The procedure is relatively straightforward, but as the final expressions are messy, and not very instructive, we will not work out the details here. Instead we consider the two extremes of free and pinned vortices.

In the first case, when the vortices are free so that $v_{\parallel}^t = v_{\perp}^\parallel$, we see that (10) leads to

$$\left( \partial_t + L_{n_{\parallel}} \right) \mathcal{W}_{\parallel}^t + \mathcal{W}_{\perp}^t \left( \nabla v_{\perp}^t \right) = 0.$$  

(77)

Perturbing this, it is quite easy to show that (in the case of a stationary and axisymmetric background)

$$\left( \partial_t + L_{n_{\parallel}} \right) \left[ \Delta t \mathcal{W}_{\parallel}^t + \mathcal{W}_{\perp}^t \left( \nabla v_{\perp}^t \right) \right] = 0.$$  

(78)

We need the trivial solution to this equation, which means that we have

$$\Delta t \mathcal{W}_{\parallel}^t = -\mathcal{W}_{\perp}^t \left( \nabla v_{\perp}^t \right).$$  

(79)

Finally, we perturb (11) (which is completely analogous to the continuity equation for $n_{\parallel}$) to get

$$\Delta_t n_{\parallel} = -n_{\parallel} \nabla v_{\perp}^t.$$  

(80)

Given these results, and the discussion in Section 3.1, it is easy to work out what happens when the vortices are (perfectly) pinned. In that case, we have $v_{\parallel}^t = v_{\perp}^\parallel$, and as a result we find that

$$\Delta x \mathcal{W}_{\parallel}^t = -\mathcal{W}_{\perp}^t \left( \nabla v_{\perp}^t \right)$$  

(81)

and

$$\Delta_x n_{\parallel} = -n_{\parallel} \nabla v_{\perp}^t.$$  

(82)

\(^1\) It is worth noting that $\mathcal{E}_c = \text{constant}$ is only consistent for a uniform density model.
5 THE CRUST–CORE INTERFACE

In order for the developed perturbation framework to be useful for neutron star astrophysics, we need to consider the crust–core interface. This region is known to be important for a range of problems, especially since the associated viscous boundary layer may provide efficient dissipation of large-scale flows in the core. We will not consider the viscous problem here, but it is worth keeping in mind that a key issue concerns to what extent the velocity perturbations are continuous across the interface. If they are not, then viscosity works to smooth out the discontinuities (over some relatively short length-scale), leading to damping of the bulk motion. The magnetic field may play a similar role. When a magnetic field penetrates the interface, discontinuities would induce Alfvén waves which would effect an efficient coupling between the crust and the core.

Another issue arising in this context is the potential appearance of nuclear pasta, that the nucleons form non-trivial topological clusters (e.g. rods or plates) rather than spherical nuclei arranged in a Coulomb lattice. Accounting for these structures is, in principle, straightforward once the properties of the various phases are understood. In most cases, one would expect the system to remain ‘isotropic’ on macroscopic scales owing to the fact that the pasta structures will ‘freeze’ in a random fashion on some smaller scale, and the ‘fluid model’ arises from a larger scale average. However, there are cases where these structures may be aligned on macroscopic scales (for instance due to a strong magnetic field or the existence of an ordered array of vortices). Then we may need to consider non-isotropic elasticity. The computation of the microscopic input parameters (equation of state, entrainment, shear modulus, etc.) is very complicated in the pasta phase. It may, for instance, be that the densities exhibit discontinuous jumps across the crust–core interface, which could lead to discontinuities in the velocities (see e.g. equation 97) and thus to enhanced viscous damping as discussed above. These are very important issues, but for simplicity we will ignore them in the following, assuming an isotropic solid and continuous densities across the crust–core interface.

5.1 Chemical gauge

As before, we consider a system where a charge-carrying component is coupled to a neutral superfluid. Furthermore, we assume that the superfluid extends across the interface. At this point, we have to return to the issue of the ‘chemical gauge’, i.e. we have to discuss the physical meaning of \( n_f \) and \( n_c \). In developing the model, we have taken the view that \( n_f \) represents the neutrons that are not confined to nuclei in the crust, while \( n_c \) represents all protons as well as the confined neutrons (making up the ions in the lattice). This view leads to a natural description of the elastic and magnetic forces. However, it does not lead to a straightforward connection to the core, where one would usually distinguish all the neutrons, \( n_n \), from the protons, \( n_p \). Problems arise from the fact that the analysis requires variables that are ‘meaningful’ across the crust–core interface. In principle, the problem would be more straightforward if we were to use a two-fluid model based on \( n_n \) and \( n_p \) also in the crust. The downside to this would be that we would then have to reconsider the Lorentz force and the elasticity contributions. After all, some of the neutrons will be associated with the nuclei and hence should be affected by the crust motion.

Focusing on the generic case, we will connect the standard two-fluid model for the core with the crust model we have developed. This forces us to consider the relevance of the chemical gauge and serves to clarify some of the key issues.

The chemical gauge choice relates to the neutrons that are considered ‘free’. The issue is subtle since, in a dynamic situation, even the neutrons that are associated with the nuclei may be able to tunnel through the relevant interaction potential. This makes concepts like the atomic number somewhat hazy. In general, one may introduce a new basis such that

\[
n'_f = n'_c + (1 - a_c) n'_p,
\]

where \( a_c \) (which we will take to be constant in the following, a good approximation at the level of the individual fluid elements) accounts for the fact that some of the neutrons move with the (crust) protons. We also have

\[
n'_c = a_c n'_p.
\]

Given these relations, it is easy to show that the neutron momentum is independent of the chemical gauge (Carter et al. 2006). This follows immediately from the definition of the momentum (Andersson & Comer 2006):

\[
\rho_i = \frac{\partial L}{\partial m'_i}.
\]

where \( L \) represents the relevant Lagrangian. Hence, we have

\[
\rho_i^b = \rho_i^f.
\]

It also follows that

\[
\mu_n = \mu_f.
\]

However, these results also show that we must, in general, have \( v'_i \neq v'_n \) and \( \varepsilon_i \neq \varepsilon_n \). Finally, in order to consider the vortices across the interface, it is natural to assume that

\[
\Omega_i^f = \Omega_n^b.
\]

The behaviour of the vortices may, of course, be more complicated than this, but it makes sense to first consider the simplest ‘reasonable’ model.

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5.2 The background configuration

Following Glampedakis & Andersson (2007), we represent the moving interface by a level set of a scalar function \( f \) which can be extended in a smooth fashion. Expecting the interface to move with the charged component, we require

\[
[\partial_t + \mathcal{L}_v] f = 0,
\]

from which it is easy to show that

\[
[\partial_t + \mathcal{L}_v] \nabla f = 0.
\]

(89)

In other words, the gradient \( \nabla f \) is constant in the frame moving with \( v_i' \). From this, it follows that the perturbation \( \Delta f \) satisfies

\[
[\partial_t + \mathcal{L}_v] \Delta f = 0.
\]

(90)

The trivial solution to this equation is \( \Delta f = 0 \), which essentially means that a fluid element at the original surface remains at the perturbed surface. The normal to the surface can obviously be taken to be \( \hat{N}_i = \nabla f \), and hence we have the unit normal

\[
\hat{N}_i = \nabla f / N, \quad \text{where} \quad N = |\nabla f| = (g^{ij} \nabla_i f \nabla_j f)^{1/2}.
\]

(91)

This means that

\[
[\partial_t + \mathcal{L}_v] \hat{N}_i = N [\partial_t + \mathcal{L}_v] N.
\]

(92)

which shows that, even though \( \hat{N}_i \) is not preserved by the flow, any change in the unit normal is parallel to the normal itself. We will use this fact later.

These considerations are quite general. However, in the problem of interest we may restrict ourselves to configurations (at the unperturbed level) that are stationary and axisymmetric. These assumptions mean that we have \( \partial_t \hat{N}_i = 0 \) and \( N \hat{N}_i v_i' = 0 \). The latter represents a no-penetration condition, simply stating that (in the background configuration) the core fluids do not migrate into the crust.

In order to obtain the interface conditions, we identify a small cylinder of fluid aligned with the normal to the interface, \( \hat{N}_i \). Integrating the various equations over this small volume, we will be able to deduce the relevant conditions to impose. While carrying out this exercise we need to make sure that the equations we consider are valid in both the crust and the core. Given this, it is natural to take as our starting point the conservation of baryon number. Assuming that the problem is stationary, the core equation

\[
\nabla_i \left( n_s v_n' + n_p v_p' \right) = 0
\]

(93)

matches to the crust result:

\[
\nabla_i \left( n_f v_f' + n_c v_c' \right) = 0.
\]

(94)

Integrating these over the small volume, we see that we should impose

\[
\hat{N}_i \left( n_s v_n' + n_p v_p' \right) = \hat{N}_i \left( n_f v_f' + n_c v_c' \right)
\]

(95)

at the interface. Noting that there are no chemical gauge issues concerning the protons, the corresponding conservation law leads to (assuming that the densities are continuous across the interface)

\[
\hat{N}_i \left( v_n' - v_f' \right) = 0.
\]

(96)

Given this, and the fact that the total number density is given by \( n = n_f + n_c = n_t + n_c \), we can rewrite the first condition as

\[
n_s \hat{N}_i \left( v_n' - v_f' \right) = n_t \hat{N}_i \left( v_n' - v_c' \right).
\]

(97)

The final conditions (97) and (98) are, of course, trivially satisfied for an axisymmetric system since \( \hat{N}_i v_i' = 0 \).

Moving on to the momentum equations, it is natural to work in the frame moving with the protons/crust. After all, the volume that we integrate over is fixed in this frame. We can also safely treat any relative flow as planar, since the volume we consider is arbitrarily small. These assumptions simplify the analysis greatly.

Let us first introduce the total momentum flux:

\[
\pi_i = \rho_s v_n^i + \rho_p v_c^i,
\]

(98)

where \((x, y)\) is either \((n, p)\) or \((f, c)\), depending on whether we consider the core or the crust. Combining the Euler equations in the appropriate way, we find that (Andersson & Comer 2006)

\[
\partial_t \pi_i + \nabla_j \left( v_i' \pi_j^s + v_i' \pi_j^p \right) + \nabla_i p + \rho \nabla_i \Phi = \nabla / T_{ij},
\]

(99)

where the pressure \( p \) is defined such that

\[
\nabla_i p = \sum x_n \nabla_i \mu_s - \alpha \nabla_i w_{ny}^2.
\]

(100)

The elastic and magnetic stresses are accounted for in \( T_{ij} \).

Let us consider the problem on the crust side of the interface (the core follows simply from letting \( f \rightarrow n \) and \( c \rightarrow p \)). In the frame moving with the crust, we have \( v_c' = 0 \) (obviously), and we also need to replace \( v_i' \rightarrow w_{ki}^i \). This leads to

\[
\pi_i' \rightarrow \rho_t (1 - \epsilon_t) w_{ki}^i.
\]

(101)
and the momentum equation takes the form

$$\partial_t \pi_i + \nabla_i \left[ \rho_i (1 - \delta_i) w_i^c w_i^c + \delta_i' p \right] + \rho \nabla_i \Phi = \nabla_i T_{ij}. \quad (103)$$

We now integrate this equation over the small volume straddling the interface. As long as the total density is continuous across the interface, the gravitational potential and its derivative will be smooth, and therefore the corresponding integral vanishes as we let the volume shrink. This exercise tells us that there will be no local force associated with the interface as long as

$$\hat{N}_j \left[ \rho_i (1 - \delta_i) w_i^c w_i^c + \delta_i' p - T_i^j \right]_{\text{crust}} = \hat{N}_j \left[ \rho_i (1 - \delta_i) w_i^c w_i^c + \delta_i' p - T_i^j \right]_{\text{core}}. \quad (104)$$

This condition is quite general. In particular, it needs to hold also on the perturbative level. As far as the background configuration is concerned, we obviously have $\hat{N}_j v_i = 0$, which means that [representing the change in a given quantity across the interface by $\langle \ldots \rangle$ (Glampedakis & Andersson 2007)]

$$\hat{N}_j \langle \delta_i' p - T_i^j \rangle = 0. \quad (105)$$

These are the usual traction conditions.

For the vertical component we see that

$$\langle p \rangle = \hat{N}^i \hat{N}^j \langle T_{ij} \rangle, \quad (106)$$

while the horizontal components lead to

$$\perp_{\hat{N}} \hat{N}^i \langle T_{ij} \rangle = 0, \quad (107)$$

where we have defined the projection (orthogonal to the normal)

$$\perp_{\hat{N}} = g^{ij} - \hat{N}^i \hat{N}^j. \quad (108)$$

It is relevant to note that the pressure may now be affected by the presence of a relative flow.

If we assume that the background configuration is such that the crust is relaxed, we only need to account for the magnetic stresses. Then we have

$$T_{ij} = -\delta g_{ij} + \frac{1}{4\pi} B_i B_j. \quad (109)$$

We also know that $\nabla_i B_i = 0$, which implies that we must have

$$\langle \hat{N}^i B_i \rangle = 0. \quad (110)$$

In this case, condition (106) leads to

$$\langle p \rangle + \frac{B_i}{8\pi} \perp \langle B_i \rangle = 0, \quad (111)$$

while (107) becomes

$$\langle \hat{N}^i B_i \rangle \perp_{\hat{N}} \langle B_i \rangle = 0. \quad (112)$$

Combined with (110), this shows that if the magnetic field penetrates the interface, then all components of the background field $B_i$ must be continuous.

To complete the analysis of the interface, we need one more condition. We obtain this condition from the momentum equation for the neutrons. This choice is natural since we need to consider a quantity that remains relevant on both sides of the interface, and there are no chemical gauge issues concerning the momentum of the superfluid (essentially since it follows from the phase of the macroscopic quantum wavefunction). However, we still have to be careful. Basically, the presence of vortices and potential pinning complicates the picture. To make progress, we will take the view that the irrotational condensate and the vortices can be considered separately. At the end of the day, the total momentum equation involves an average over these components. In effect, the interface must reflect this large-scale average. Our approach to the problem represents this, yet it is admittedly rather naive. A number of issues need to be better understood, in particular concerning the way in which vortices extend from the fluid core to the elastic environment of the crust. Let us simply mention two problems. First of all, we know from low-temperature laboratory superfluids that vortices connect orthogonally to solid walls. It is natural to ask if the same is true for vortices that penetrate the neutron star crust–core interface. Secondly, the entrainment in the core leads to the vortices being magnetized, due to the entrainment, but this effect relies on the protons being superconducting so is not active in the crust. The upshot is that a magnetized vortex somehow connects to an unmagnetized one. How does this work? The answer may be linked to the transition from superconducting protons to the ones locked in nuclei. In this case, one would expect the presence of a current sheet. Presumably, this may also resolve any issues concerning the magnetic vortices, but the details have not yet been considered. These and other issues need to be resolved by future work. In the following, we will adopt the pragmatic view that really difficult problems are perhaps best ignored.

Anyway, considering first the irrotational part, we have

$$\partial_t \rho_i^c + \nabla_i \left( \hat{\rho}_i - \frac{1}{2} \rho_i^c u_i^c \right) = 0, \quad (113)$$

where $x = [n, f]$, depending on whether we are in the core or the crust. Working in the crust frame, and integrating over a small volume, this leads to the interface condition

$$\hat{\rho}_i = \left( \frac{1}{2} - \varepsilon_i \right) u_i^c = \hat{\rho}_i - \left( \frac{1}{2} - \varepsilon_i \right) u_i^c. \quad (114)$$

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It should be noted that in the particular choice of chemical gauge where we consider all the neutrons in the crust we have $f \to n$. In that case, the neutron chemical potential must be continuous, as expected.

Moving on to the vortices, we still do not have to worry about chemical gauge issues. Basically, one would expect each vortex to penetrate into the crust, leading to the vortex density $n_i$ being continuous. Hence, we consider (13), where we recall that the vortex velocity $v_i'$ depends on whether there is pinning or not. Working in the crust frame (as before) and integrating, we arrive at the condition

$$\left( \hat{\nabla} \right)_i \mathcal{W}_i^c = (\hat{\nabla} \mathcal{W}_i^c)_{\text{pro}} ,$$

(115)

where we have used the fact that (in the present analysis) we do not consider pinning in the core. However, because of the chemical gauge invariance, we have $\mathcal{W}_i^c = \mathcal{W}_i^p$, which means that our final interface condition is

$$v_i^c = v_i^p .$$

(116)

This result is extremely intuitive. If the crust vortices are free, moving with $v_i'$, then we must have $v_i' = v_i^c$ at the interface. Meanwhile, if the vortices are pinned, moving with $v_i'$, then the condition should be $v_i^c = v_i'$.

These results represent two limiting cases. In the general case, we would also need to keep track of the vortex velocity. This is, in principle, straightforward, but we will not discuss the results here. It may be worth noting that such models may be considered from a three-fluid point of view, with the vortices forming a distinct ‘species’. This is an interesting strategy that could prove advantageous in some situations.

5.3 The perturbed problem

Let us now move on to the conditions that need to be imposed at the linear perturbation level. We can think of two possible strategies. Either we take the view that the conditions derived in the previous section are ‘exact’, which means that we can perturb them directly, or we start from the relevant perturbation equations and carry out the analysis for a small volume straddling the interface all over again. In principle, these two approaches should lead to the same answer. In practice, we find it useful to use a combination of them.

We begin by considering particle conservation. In general, Lagrangian variation of the continuity equations leads to

$$\left( \partial_t + \mathcal{L} \right) \left( \Delta_c n + n_\chi \nabla \xi^c_i \right) = 0. $$

(117)

We require the trivial solution, i.e. take

$$\Delta_c n + n_\chi \nabla \xi^c_i = 0. $$

(118)

In the case of the protons, we can integrate this equation over the small cylinder across the interface. Taking the volume small enough that variations in $n_p$ can be neglected (we are not allowing for density discontinuities), we then find that

$$N_i \left( \xi^c_p - \xi^c_i \right) = 0. $$

(119)

As one would have expected, the normal component of the proton displacement should be continuous. This condition follows immediately if we want to avoid there being a void (or overlap) in the proton fluid at the interface.

To get the second condition, we perturb the equation for total baryon conservation. This leads to (in the crust)

$$\Delta_c n = \Delta_c (n_i + n_\chi) = -n_\chi \nabla j_i + \xi^c_i \nabla n_i - \nabla j_i \left( n_i \xi^c_i \right) . $$

(120)

If we (again) assume that the densities are smooth, then we only need to consider

$$\Delta_c n \approx -\nabla j_i \left( n_i \xi^c_i + n_i \xi^c_i \right) . $$

(121)

After integration across the interface, we find that we must have

$$N_j \left[ n_i \left( \xi^c_i - \xi^c_i \right) - n_i \left( \xi^c_i - \xi^c_i \right) \right] = 0. $$

(122)

In the case of the ‘comprehensive’ gauge, when $n_i = n_\chi$, this reduces to

$$N_j \left( \xi^c_p - \xi^c_i \right) = 0. $$

(123)

Again, this condition seems natural, and we learn that the more complicated nature of (122) results from the fact that not all neutrons in the crust are free. We also see that we must keep careful track of the different number densities across the interface.

Next we need the perturbed versions of (106) and (107). To derive these, we assume (as in the standard level set method) that the general conditions can meaningfully be extended away from the interface, and be perturbed in the usual way. There may be some technical issues associated with this approach, but we will not go into the details of this here. To derive the relevant conditions we start from (104) which leads to

$$\hat{\nabla} j_i \left[ \rho_i (1 - \varepsilon_i) w_i^p \partial_t \left( \xi^c_i - \xi^c_i \right) + \delta_i \Delta_c p - \Delta_i T^c_i \right]_{\text{crust}} = \hat{\nabla} j_i \left[ \rho_i (1 - \varepsilon_i) w_i^p \partial_t \left( \xi^c_i - \xi^c_i \right) + \delta_i \Delta_p p - \Delta_i T^c_i \right]_{\text{core}} . $$

(124)

It is easy to see that, for an axisymmetric background, the first term in each expression does not contribute to the normal component. Contracting with $\hat{\nabla} i$, we are left with

$$\Delta_c p - \hat{\nabla} \nabla \Delta_i T^c_i \right]_{\text{crust}} = \left[ \Delta_p p - \hat{\nabla} \nabla \Delta_i T^c_i \right]_{\text{core}} . $$

(125)
That is, we arrive at the expected traction condition. The horizontal result is (obviously) more complicated. A projection orthogonal to \( \hat{N}_i \) leads to
\[
\hat{N}_i \left[ \rho_1 (1 - \varepsilon_1) w_\alpha^0 \partial_i (\xi_\alpha - \xi_\alpha^0) - (\delta_\alpha^i - \delta_\alpha^i) \right] = \hat{N}_i \left[ \rho_2 (1 - \varepsilon_2) w_\alpha^0 \partial_i (\xi_\alpha^0 - \xi_\alpha^0) - (\delta_\alpha^i - \delta_\alpha^i) \right] .
\]
As in previous cases, the additional complications arise from the choice of chemical gauge.

In the magnetic field case, we need (51) and (52). It is also useful to note that (110) leads to
\[
\langle \hat{N}_i \Delta_i \Delta_k \rangle = 0 .
\]

From this we see that
\[
\langle \hat{N}_i \Delta_i \Delta_k \rangle = \langle B^i \hat{N}_j \left( \nabla_i \xi_j^0 + \nabla_i \xi_j^0 \right) \rangle .
\]

We then have
\[
\hat{N}_i \Delta_i p - \hat{N}_i \hat{N}_j T_{ij} = \hat{N}_i \left( \Delta_i p + \frac{1}{8\pi} \Delta_i \Delta_k B^k \right) - \frac{1}{4\pi} \hat{N}_i \left( B^i \Delta_i B_k + B_i \Delta_i B_k \right) .
\]

The vertical component becomes
\[
\left\langle \Delta_i p + \frac{1}{8\pi} \Delta_i \Delta_k B^k \right\rangle = \frac{1}{4\pi} \langle \hat{N}_i B^i \rangle \langle \hat{N}_i \Delta_k B_k \rangle ,
\]
while the horizontal condition becomes
\[
\perp \langle \hat{N}_i \left( B^i \Delta_i B_k + B_i \Delta_i B_k \right) \rangle = 0 .
\]

At the perturbation level, we also need to consider the elastic problem. As before, we assume that the background configuration is relaxed (and the core is fluid!?), in which case we have
\[
\Delta_i T_{ij} = \hat{\mu} \left( \nabla_i \xi_j^0 + \nabla_i \xi_j^0 \right) - \frac{2}{3} \hat{\mu} g_{ij} \nabla_i \xi_j^0
\]
and
\[
\Delta_i T_{ij} = g_{ij} \Delta_i T_{ij} .
\]

In this case, the vertical condition becomes
\[
\left\langle \Delta_i p + \frac{2}{3} \hat{\mu} \nabla_i \xi_j^0 \right\rangle = 2 \langle \hat{\mu} \hat{N}_j \hat{N}_j \nabla_j \xi_j^0 \rangle ,
\]
while the horizontal one can be written as
\[
\perp \langle \hat{\mu} \hat{N}_j \left( \nabla_j \xi_j^0 + \nabla_j \xi_j^0 \right) \rangle = 0 .
\]

Combining the elastic and magnetic results to arrive at the conditions to impose in the general case is, of course, straightforward.

Finally, we consider the conditions relating to the superfluid component. Perturbing the scalar condition (114), we see that we should require
\[
\left[ \Delta_i \hat{\mu} \varepsilon_i - w_{\alpha}^2 \Delta_i \hat{\mu} \varepsilon_i + \left( \frac{1}{2} - \varepsilon_{\alpha} \right) \Delta_i \nu_{\alpha}^2 \right]_{\text{crust}} = \left[ \Delta_i \hat{\mu} \varepsilon_i - w_{\alpha}^2 \Delta_i \hat{\mu} \varepsilon_i + \left( \frac{1}{2} - \varepsilon_{\alpha} \right) \Delta_i \nu_{\alpha}^2 \right]_{\text{core}} .
\]
As far as the vorticity is concerned, it follows naturally from the discussion leading up to (116) that we should have
\[
\perp \langle \Delta_i \nu_i^0 - \Delta_i \nu_i^0 \rangle = 0 .
\]

This condition is (obviously) satisfied if we have \( \perp \langle \xi_i^0 - \xi_i^0 \rangle = 0 \) in the free vortex case and \( \perp \langle \xi_i^0 - \xi_i^0 \rangle = 0 \) when perfect pinning prevails. This ensures that there are no kinks in the vortices at the interface.

6 CONCLUSIONS

We have developed a Lagrangian perturbation framework relevant for the conditions that apply in a mature neutron star, accounting for the presence of superfluid components, the elastic crust and the magnetic field. The considered physics impacts on a wide range of astrophysical phenomena, from pulsar glitches to magnetar seismology and various gravitational-wave-emission mechanisms. Hence, the reported theoretical developments provide us with a solid foundation to consider exciting applications.

There is also significant scope for future improvements of the theory. Most importantly, the effort should be extended to relativistic gravity. This would open the door to truly quantitative considerations of realistic neutron star models, e.g. based on a modern supranuclear equation of state with composition and thermal gradients. Developments in this direction are in progress.

We also need to improve our understanding of the physics involved. Our discussion highlights the need to know a wider set of parameters, like the superfluid entrainment both in the star’s core and in the elastic crust. The various interactions involving superfluid vortices, from pinning to mutual friction, also need to be understood. While we can continue to advance our understanding of the phenomenology of these very complex systems, we need to impose realistic constraints on our models. This requires, if not a precise knowledge of the involved parameters, at least some idea of what the permissible ranges may be. To achieve this goal, we need a continued dialogue across different branches of physics, a challenging but ultimately rewarding exercise.

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REFERENCES

Alpar M. A., Pines D., Anderson P. W., Shaham J., 1984, ApJ, 276, 325
Andersson N., 2003, Classical Quantum Gravity, 20, R105
Andersson N., Comer G. L., 2006, Classical Quantum Gravity, 23, 5505
Andersson N., Comer G. L., 2007, Living Rev. Relativ., 10, 1
Andersson N., Comer G. L., Grosart K., 2004, MNRAS, 355, 918
Andersson N., Comer G. L., Glampedakis K., 2005, Nuclear Phys. A, 763, 212
Andersson N., Sidery T., Comer G. L., 2006, MNRAS, 368, 162
Andersson N., Sidery T., Comer G. L., 2007, MNRAS, 381, 747
Andersson N., Glampedakis K., Samuelsson L., 2009, MNRAS, 396, 894
Bildsten L., Epstein R. I., 1989, ApJ, 342, 951
Carter B., 1987, in Amile A., Choquet-Bruhat M., eds, Relativistic Fluid Dynamics. Springer, Heidelberg, p. 1
Carter B., Samuelsson L., 2006, Classical Quantum Gravity, 23, 5367
Carter B., Chachoua E., Chamel N., 2006, Gen. Relativ. Gravity, 38, 83
Carter B., Chamel N., Haensel P., 2006, Int. J. Modern Phys. D, 15, 777
Chamel N., Haensel P., 2009, Living Rev. Relativ., 11, 10
Comer G. L., Joynt R., 2003, Phys. Rev. D, 68, 023002
Donati P., Pizzochero P. M., 2006, Phys. Lett. B, 640, 74
Espinoza C. M., Lyne A. G., Stappers B. W., Kramer M., 2011, MNRAS, 414, 1679
Feibelman P. J., 1979, Phys. Rev. D, 4, 1589
Friedman J. L, Schutz B. F., 1978, ApJ, 221, 937
Glampedakis K., Andersson N., 2007, MNRAS, 377, 630
Glampedakis K., Andersson N., Jones D. I., 2008, Phys. Rev. Lett., 100, 081101
Glampedakis K., Andersson N., 2009, Phys. Rev. Lett., 102, 141101
Glampedakis K., Andersson N., Samuelsson L., 2011, MNRAS, 410, 805
Gudmundsson E. H., Pethick C. J., Epstein R. I., 1983, ApJ, 272, 286
Haskell B., 2011, Phys. Rev. D, 83, 043006
Haskell B., Andersson N., Comer G. L., 2011, preprint
Jahan-Miri M., 2006, ApJ, 650, 326
Jones D. I., Andersson N., 2001, MNRAS, 324, 811
Karlovini M., Samuelsson L., 2003, Classical Quantum Gravity, 20, 3613
Link B., 2003, Phys. Rev. Lett., 91, 101101
Link B., 2009, Phys. Rev. Lett., 102, 131101
Link B., Epstein R. J., 2001, ApJ, 556, 392
Mendell G., 1991, ApJ, 380, 515
Peralta C., Melatos A., Giacobello M., Ooi A., 2006, ApJ, 651, 1079
Prix R., 2004, Phys. Rev. D, 69, 043001
Sidery T., Passamonti A., Andersson N., 2010, MNRAS, 405, 1061

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