Tribimaximal Neutrino Mixing and a Relation Between Neutrino- and Charged Lepton-Mass Spectra

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Abstract

Brannen has recently pointed out that the observed charged lepton masses satisfy the relation

$$m_e + m_\mu + m_\tau = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2,$$

while the observed neutrino masses satisfy the relation

$$m_{\nu 1} + m_{\nu 2} + m_{\nu 3} = \frac{2}{3}(-\sqrt{m_{\nu 1}} + \sqrt{m_{\nu 2}} + \sqrt{m_{\nu 3}})^2.$$  

It is discussed what neutrino Yukawa interaction form is favorable if we take the fact pointed out by Brannen seriously.

1 Introduction

It is well-known that the observed charged lepton mass spectrum [1] satisfies the relation

$$m_e + m_\mu + m_\tau = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2,$$  

with remarkable precision, while the observed masses \(m_{f1}, m_{f2}, m_{f3}\) of the other fundamental particles \(f_i\) (quarks and neutrinos) do not satisfy [4] a relation similar to (1.1) straightforwardly. However, Brannen [5] has recently pointed out a possibility that the observed neutrino masses can satisfy the relation

$$m_{\nu 1} + m_{\nu 2} + m_{\nu 3} = \frac{2}{3}(-\sqrt{m_{\nu 1}} + \sqrt{m_{\nu 2}} + \sqrt{m_{\nu 3}})^2.$$  

Of course, we cannot extract the values of the neutrino mass ratios \(m_{\nu 1}/m_{\nu 2}\) and \(m_{\nu 2}/m_{\nu 3}\) from the neutrino oscillation data \(\Delta m^2_{\text{solar}}\) and \(\Delta m^2_{\text{solar}}\) unless we have more information on the neutrino masses, so that we cannot judge whether the observed neutrino masses satisfy the relation (1.2) or not. Nevertheless, it is worthwhile to examine the Brannen’s speculation seriously, because it seems to bring a new view into a lepton mass matrix model if his conjecture is correct.

The Brannen’s conjecture (1.2) is somewhat unforeseen, because since we have known the existence of the lepton flavor mixing, we expect that the neutrino mass matrix structure will be quite different from that of the charged leptons, so that the neutrino mass spectrum will also be different from that of the charged leptons. If the charged lepton masses satisfy the relation (1.1), the neutrino masses will not be able to satisfy the relation (1.2) even if we allow the replacement \(\sqrt{m_{\nu i}} \rightarrow -\sqrt{m_{\nu i}}\). In order to understand the formula (1.1), for example, a model [3, 6, 7] with a seesaw-type mass generation mechanism [8] has been proposed:

$$M_f = m_L M_F^{-1} m_R,$$  

(1.3)
where $M_F$ are hypothetical heavy fermion mass matrices and, for the charged lepton sector, the structure $M_E \propto \text{diag}(1,1,1)$ is assumed. (For neutrino sector, Eq. (1.3) reads $M_\nu = m_L M_N^{-1} m_L^T$.) Here, the mass matrices $m_L$ and $m_R$ are given by $m_R \propto m_L \propto \text{diag}(v_1, v_2, v_3)$, where $v_i$ are vacuum expectation values (VEVs) of 3-family scalars $\phi_{Li}$ ($\phi_{Ri}$) and the VEVs $v_i$ satisfy the relation
\[ v_1^2 + v_2^2 + v_3^2 = \frac{2}{3} (v_1 + v_2 + v_3)^2. \] (1.4)
Of course, we do not rule out a possibility that the values of $v_i$ are negative. (Such a Higgs potential model which gives the relation (1.4) is proposed in Ref.[3, 7, 9, 10].) In such a seesaw-type model with diagonal $m_L$ and $m_R$, mixings among fermions $f_i$ are caused by the non-diagonal structure $M_F$. If $M_E$ in the charged lepton sector is proportional to a unit matrix $1$, the observed neutrino mixings tell us that the structure of the heavy neutrino matrix (the Majorana mass matrix of the right-handed neutrinos $N_R$) $M_N$ cannot be unit matrix, so that the eigenvalues of the matrix $m_L M_N^{-1} m_L^T$ will not satisfy the relation (1.2) even if we allow the replacement $\sqrt{m_{\nu i}} \rightarrow - \sqrt{m_{\nu i}}$.

If we take the Brannen’s speculation seriously, we must abandon such a model (1.3) (with a universal $m_L$ structure). For example, we may modify the model as
\[ M_f = m_L^f M_F^{-1} m_R^f, \] (1.5)
where $M_F$ have a unit matrix structure universally, at least, for the charged lepton and neutrino sectors, and $m_L^f$ (and $m_R^f$) have flavor-dependent structures. Then, in order to give the relations (1.1) and (1.2), it is required that the eigenvalues $v_{fi}$ of the matrices $m_L^f$ ($m_R^f$) always satisfy the relation (1.4). We relax the constraint on $m_L$ from a rigid ansatz of the universal structure into the constraint (1.4).

Since $M_F \propto 1$ do not play any essential role in the modified model (1.5), we may rather consider a Frogatt-Nielsen [11] type model with six dimensional operators $\bar{f}_L \phi_{f} H_L \phi_{f} f_R$, where $H_L$ is the conventional SU(2)$_L$-doublet Higgs scalar, and $\phi_{f}$ are 3-family SU(2)$_L$-singlet scalars. However, in the present paper, we do not have interest in whether the effective mass matrix form originates in a seesaw model or in a Frogatt-Nielsen model. Our interest is only in the neutrino Yukawa interaction form. For convenience, in the present paper, we will discuss a possible model in the framework of a seesaw mass matrix model.

The purpose of the present paper is not to give a model which leads to the Brannen’s conjecture. The purpose is to investigate what Yukawa interaction form is required when we accept the Brannen’s conjecture and when we take the observed neutrino mixing into consideration. In the present paper, it is assumed that the eigenvalues $v_{fi}$ of the Dirac mass matrices $m_L^f$ of the charged leptons and neutrinos satisfy the relation (1.4). In the next section, we will introduce a useful parametrization for describing charged lepton and neutrino masses in terms of a permutation symmetry $S_3$ [12]. In Sec. 3, we will investigate a possible neutrino mass spectrum within the framework of the $S_3$ symmetry. In Sec. 4, we will propose a phenomenological neutrino Yukawa coupling form, where the $S_3$ symmetry is explicitly broken. Finally, Sec. 5 is devoted in the concluding remarks.
2 Mass spectrum parameters \( z_i \)

It is convenient to use parameters \( z_i \) which are defined by 
\[
\langle \phi^0_{L_i} \rangle \equiv v_i = vz_i \text{ with the normalization condition } z_1^2 + z_2^2 + z_3^2 = 1.
\]

The relation (1.1) \([1.4]\) requires that the parameters \( z_i \) satisfy the relation
\[
z_1^2 + z_2^2 + z_3^2 = \frac{2}{3} (z_1 + z_2 + z_3)^2. \tag{2.1}
\]

The values of \( z_i \) are obtained from the observed charged lepton masses \([1]\) as
\[
z_1 \sqrt{m_e} = z_2 \sqrt{m_\mu} = z_3 \sqrt{m_\tau} = \frac{1}{\sqrt{m_e + m_\mu + m_\tau}}, \tag{2.2}
\]
and the explicit numerical values are \( z_1 = 0.016473 \), \( z_2 = 0.236869 \) and \( z_3 = 0.971402 \).

Since the relation (2.1) is invariant under any exchange \( z_i \leftrightarrow z_j \), it is useful to use the language of the permutation symmetry \( S_3 \): We define the singlet \( \phi_\sigma \) and doublet \( (\phi_\pi, \phi_\eta) \) of \( S_3 \) for the fields \( \phi_i \) (and also for \( f_\alpha \)):
\[
\phi_\pi = \frac{1}{\sqrt{2}} (\phi_3 - \phi_2), \quad \phi_\eta = \frac{1}{\sqrt{6}} (2\phi_1 - \phi_2 - \phi_3), \quad \phi_\sigma = \frac{1}{\sqrt{3}} (\phi_3 + \phi_2 + \phi_1), \tag{2.3}
\]

where we have taken the basis \((e_1, e_2, e_3)\) as \((e_1, e_2, e_3) = (e, \mu, \tau)\) corresponding to the definition (2.2). The definition (2.3) is different from the conventional one. This definition will become useful for description of the neutrino mixing as we show later [for example, see Eq. (3.3)].

In order that the VEVs \( v_i \) satisfy the relation (1.4), it needs that the Higgs potential leads to the relation
\[
z_\pi^2 + z_\eta^2 = z_\sigma^2, \tag{2.4}
\]
where \( \langle \phi^0_\pi \rangle = vz_\pi \), \( \langle \phi^0_\eta \rangle = vz_\eta \) and \( \langle \phi^0_\sigma \rangle = vz_\sigma \). The explicit form of the Higgs potential which leads to the relation (2.4) has been given in Ref.\([9, 10]\).

By the way, in general, the eigenvalues \( \lambda_i \) of any Hermitian matrix \( M \) \((MM^\dagger \text{ if } M \text{ is not Hermitian})\) can be expressed by the following form:
\[
\lambda_1 = \frac{1}{3} a - \frac{1}{3} b \sin \theta, \quad 
\lambda_2 = \frac{1}{3} a - \frac{1}{3} b \sin \left( \theta + \frac{2}{3} \pi \right), \quad 
\lambda_3 = \frac{1}{3} a - \frac{1}{3} b \sin \left( \theta + \frac{1}{3} \pi \right), \tag{2.5}
\]

where
\[
a = \text{Tr} M, \quad b = \sqrt{2} \sqrt{3 \text{Tr}(M^2) - (\text{Tr} M)^2}. \tag{2.6}
\]

Therefore, any three-flavor mass spectrum can always be expressed by the two parameters \( b/a \) and \( \theta \) independently of the structure of \( M \). Only for the case with \( \text{Tr}(M^2) = \frac{2}{3} (\text{Tr} M)^2 \), it gives the relation \( b = \sqrt{2} a \). Inversely, if we take \( b = \sqrt{2} a \) in (2.5), the matrix \( M \) satisfies the relation
\[ \text{Tr}(M^2) = \frac{3}{2} (\text{Tr}M)^2. \] (Although the author \[13\] and Brannen \[5\] have discussed a specific mass matrix form

\[
\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & e^{i\theta} & e^{-i\theta} \\ e^{-i\theta} & 0 & e^{i\theta} \\ e^{-i\theta} & e^{i\theta} & 0 \end{pmatrix}, \tag{2.7}
\]

in connection with the mass formula (1.1), the constraint \( b = \sqrt{2}a \) has been assumed (not derived) in their models, so that it does not mean that the relation (1.1) was derived in their models.)

In the present paper, since we take the three-flavor Higgs potential model \[10\] which gives the relation (2.1) [i.e. (2.4)], the parameters \( z_i \) can be expressed as

\[
\begin{align*}
z_1 &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \theta_e, \\
z_2 &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \left( \theta_e + \frac{2}{3} \pi \right), \\
z_3 &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \left( \theta_e + \frac{4}{3} \pi \right),
\end{align*}
\tag{2.8}
\]

where the numerical value of \( \theta_e \) is given by

\[
\theta_e = \frac{\pi}{4} - \varepsilon = 42.7324^\circ \quad (\varepsilon = 2.2676^\circ),
\tag{2.9}
\]

from the observed charged lepton masses.

Since we assume that the neutrino masses \( m_{\nu i} \) satisfy the relation (1.2), we can also define the \( z^\nu_i \) parameters in the neutrino sector as

\[
\begin{align*}
 z_1^\nu &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \theta_\nu, \\
z_2^\nu &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \left( \theta_\nu + \frac{2}{3} \pi \right), \\
z_3^\nu &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \left( \theta_\nu + \frac{4}{3} \pi \right),
\end{align*}
\tag{2.10}
\]

with \( |z_1^\nu| < |z_2^\nu| < |z_3^\nu| \). Then, Brannen \[5\] has also empirically found that if the observed neutrino mass values are given by

\[
\theta_\nu = \theta_e + \frac{\pi}{12} = 57.7324^\circ,
\tag{2.11}
\]

which gives

\[
z_1^\nu = -0.079938, \quad z_2^\nu = 0.385404, \quad z_3^\nu = 0.9192788.
\tag{2.12}
\]

The Brannen’s empirical relation (2.11) [i.e. the \( z^\nu_i \)-values (2.12)] predicts

\[
R \equiv \frac{|m_{\nu 2}^2 - m_{\nu 1}^2|}{|m_{\nu 3}^2 - m_{\nu 2}^2|} = \frac{(z_1^\nu)^4 - (z_2^\nu)^4}{(z_3^\nu)^4 - (z_2^\nu)^4} = 0.0318,
\tag{2.13}
\]

which is in good agreement with the observed value of \( R \) \[14\] \[15\]

\[
R = \frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2} = \frac{(7.9^{+0.6}_{-0.5}) \times 10^{-5}}{(2.72^{+0.38}_{-0.25}) \times 10^{-3}} = (2.9 \pm 0.5) \times 10^{-2}.
\tag{2.14}
\]
However, note that we cannot predict the ratio $R$ only under the assumption (1.2), i.e. unless we also assume the value of $\theta_\nu$. In Fig. 1, we illustrate the predicted value of $R$ versus $\theta_\nu$ under the assumption (1.2). The present observed values of $\Delta m^2_{ij}$ are consistent with the value of $\theta_\nu \simeq 55^\circ - 58^\circ$. In other words, if we want to consider a model in which the neutrino masses satisfy the relation (1.2), we must build a model which gives $\theta_\nu \simeq 55^\circ - 58^\circ$.

In the present paper, we take the Brannen’s conjecture (2.11) seriously, and we investigate what Yukawa interaction from in the neutrino sector can lead to the relation (2.11). Brannen[5] has tried to build a new mass model with an algebraic approach. However, in the present paper, we will discuss a possible mass matrix form within the framework of a conventional mass matrix model, i.e. based on a Higgs mechanism and an extended seesaw mechanism.

3 S₃ symmetry and Yukawa interaction in the neutrino sector

We have assumed that the Yukawa interaction in the charged lepton sector is given by

$$H_e = y_e \left( \bar{\nu} L_1 E R_1 \phi^d_{L1} + \bar{\nu} L_2 E R_2 \phi^d_{L2} + \bar{\nu} L_3 E R_3 \phi^d_{L3} \right),$$

with a universal coupling constant $y_e$, where $\ell_{Li} = (\nu_{Li}, e_{Li})$ and $\phi^d_{L_i} = (\phi^0_{Li}, \phi^-_{Li})$. (In the previous section, the neutrino $\nu_i$ ($i = 1, 2, 3$) denoted the mass eigenstates. However, in the present section, we will use the same notation $\nu_i$ as the SU(2)$_L$ partners of $e_{Li} = (e_L, \mu_L, \tau_L)$, respectively.) We also assume the same structure for $L_{R_i} e_{R_i} e_{R_i}$, where $L_{R_i} = (E^c_{R_i}, N^c_{R_i})$ and $\phi^d_{R_i} = (\phi^+_{R_i}, \phi^0_{R_i})$. The explicit quantum number assignments are found, for example, in Refs.[7, 10] for SU(2)$_L \times$ SU(2)$_R \times$ U(1) model, and in Ref.[17] for SO(10) × SO(10) model.

The Yukawa interaction (3.1) is invariant under the S₃ symmetry, but, of course, it is not a general form which is invariant under the S₃ symmetry. Only when we consider that the Yukawa interaction (3.1) with the VEV $\langle \phi_{L1} \rangle = v_i = v z_i$ gives the mass matrix $m^c_L$ in the seesaw model (1.5) with $M = M_0 I$, the predicted charged lepton masses satisfy the formula (1.1).

The present observed neutrino data strongly suggest that the neutrino mixing is almost described by the so-called tribimaximal mixing [18]

$$U = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}. \quad (3.2)$$

Under the definition (2.3), the neutrino states $(\nu_e, \nu_\mu, \nu_\tau)$ are represented by

$$\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix} \begin{pmatrix}
\nu_\eta \\
\nu_\sigma \\
\nu_\pi
\end{pmatrix}. \quad (3.3)$$

Therefore, it is useful to express the neutrino (Dirac) matrix $M^D_{\nu}$ on the basis $(\nu_\eta, \nu_\sigma, \nu_\pi)$, not on the basis $(\nu_\pi, \nu_\eta, \nu_\sigma)$. Here, the neutrino states $\nu_\pi$, $\nu_\eta$ and $\nu_\sigma$ are defined by a relation similar
to Eq.(2.3) with \((\nu_1, \nu_2, \nu_3) = (\nu_e, \nu_\mu, \nu_\tau)\). When the mass matrix \(M_\nu\) becomes diagonal on the basis \((\nu_\eta, \nu_\sigma, \nu_\pi)\), the mixing matrix is given by the tribimaximal mixing (3.2) exactly.

If we require the \(S_3\) symmetry, the Yukawa interaction is generally given by the form

\[
H_\nu = y_1 \frac{\bar{\ell}_\pi N_\pi + \bar{\ell}_\eta N_\eta + \bar{\ell}_\sigma N_\sigma}{\sqrt{3}} \phi_u^a + y_2 \left[ \frac{\bar{\ell}_\pi N_\eta + \bar{\ell}_\eta N_\pi}{\sqrt{2}} \phi_u^\pi + \frac{\bar{\ell}_\pi N_\sigma - \bar{\ell}_\eta N_\pi}{\sqrt{2}} \phi_u^\eta \right] + \frac{y_3}{\sqrt{2}} \left[ \bar{\ell}_\pi \phi_u^u \right. \\
+ \left. \bar{\ell}_\eta \phi_u^u \right] + y_4 \frac{\bar{\ell}_\pi N_\eta + \bar{\ell}_\eta N_\pi}{\sqrt{6}} - \frac{2\bar{\ell}_\sigma N_\sigma}{\sqrt{6}} \phi_u^u,
\]

(3.4)

where the heavy neutrinos \(N_i\) denote \(\nu_{Ri}\), and \(\phi^u = (\bar{\phi}_L^+, \bar{\phi}_L^0)\). In the present investigation, the \(S_3\) symmetry is a necessary condition, but it is not a sufficient condition. For the charged lepton sector, we have assumed the form (3.1), which corresponds to the case with \(y_1 = \sqrt{2}y_2 = y_3 = y_e/\sqrt{3}\) and \(y_4 = 0\) in the \(S_3\) invariant general form (3.4) with \(\bar{\ell}_a N_\pi \rightarrow \bar{\ell}_a e R_a\) and \(\phi_u^a \rightarrow \phi_u^d\) \((a = \pi, \eta, \sigma)\). Therefore, in order to obtain the form (3.1), we must a further selection rule in addition to the \(S_3\) invariance, for example, a cyclic permutation symmetry, and so on. For the neutrino sector, in order to reduce the number of parameters, we assume the following selection rules:

(i) The VEVs of the up-type Higgs scalars \(\phi_u^a\) satisfy the relation

\[
\langle \phi_u^u \rangle^2 + \langle \phi_u^\eta \rangle^2 = \langle \phi_u^\pi \rangle^2,
\]

(3.5)
as well as Eq.(2.4). (We assume a similar Higgs potential structure for the up-type Higgs scalars \(\phi_u^a\) as well as \(\phi_u^d\).)

(ii) We have assumed the universality of the Yukawa coupling constants for the \((e_1, e_2, e_3)\) basis in the charged lepton sector as shown in Eq.(3.1). We also assume the universality of the Yukawa coupling constants in the neutrino sector [however, not for the \((\nu_1, \nu_2, \nu_3)\) basis, but for the \((\nu_\pi, \nu_\eta, \nu_\sigma)\) basis].

Considering that the observed mixing is almost given by the tribimaximal mixing (3.2), i.e. the neutrino Dirac mass matrix \(M_\nu^D\) is almost diagonal on the \((\nu_\eta, \nu_\sigma, \nu_\pi)\) basis, we confine ourselves to investigating a specific case with \(y_1 = y_2 = y_\nu\) and \(y_3 = y_4 = 0\):

\[
H_\nu = y_\nu \frac{\bar{\ell}_\pi N_\pi + \bar{\ell}_\eta N_\eta + \bar{\ell}_\sigma N_\sigma}{\sqrt{3}} \phi_u^a + \frac{y_\nu}{\sqrt{2}} \left[ \bar{\ell}_\pi N_\eta + \bar{\ell}_\eta N_\pi \right] \phi_u^\pi + \frac{y_\nu}{\sqrt{2}} \left[ \bar{\ell}_\pi N_\sigma - \bar{\ell}_\eta N_\pi \right] \phi_u^\eta.
\]

(3.6)

The interaction (3.6) yields the neutrino Dirac mass matrix \(M_\nu^D\)

\[
M_\nu^D = m_\nu^D \begin{pmatrix}
\frac{z^u}{\sqrt{3}} - \frac{z^\pi}{\sqrt{2}} & 0 & \frac{z^u}{\sqrt{2}} \\
0 & \frac{z^u}{\sqrt{3}} & 0 \\
\frac{z^u}{\sqrt{2}} & 0 & \frac{z^u}{\sqrt{3}} + \frac{z^\pi}{\sqrt{2}}
\end{pmatrix},
\]

(3.7)
where \( m_0^\nu = y_\nu v_u, \langle \phi_u^\nu \rangle = v_u z^u_\pi, \langle \phi_\eta^\nu \rangle = v_u z^u_\eta, \langle \phi_\sigma^\nu \rangle = v_u z^u_\sigma \), and \( (z^u_\pi)^2 + (z^u_\eta)^2 + (z^u_\sigma)^2 = 1 \). The mass eigenvalues are given by

\[
(m_\nu') = \left( \frac{1}{\sqrt{6}} z^u_\pi - \frac{1}{\sqrt{2}} \right) m_0^\nu,
\]

\[
m_\nu^\nu = \frac{1}{\sqrt{3}} z^u_\sigma m_0^\nu,
\]

\[
(m_\pi') = \left( \frac{1}{\sqrt{6}} z^u_\pi + \frac{1}{\sqrt{2}} \right) m_0^\nu.
\]

When we use the relation (3.5), i.e.

\[
(z^u_\pi)^2 + (z^u_\eta)^2 = (z^u_\sigma)^2 = \frac{1}{2},
\]

we obtain

\[
(m_\eta') = \left( \frac{1}{\sqrt{6}} - \frac{1}{2} \right) m_0^\nu,
\]

\[
m_\sigma = \frac{1}{\sqrt{6}} m_0^\nu,
\]

\[
(m_\pi') = \left( \frac{1}{\sqrt{6}} + \frac{1}{2} \right) m_0^\nu,
\]

independently of the parameter value of \( z^u_\pi / z^u_\eta \) (in other words, independently of the magnitude of the \( \nu_\pi \leftrightarrow \nu_\eta \) mixing). The result (3.10) means

\[
\theta_\nu = \frac{\pi}{3},
\]

from the definition of \( \theta_\nu \), Eq.(2.11).

As seen in Fig. 1, although the present interaction form could not give the Brannen’s relation (2.11), the value \( \theta_\nu = 60^\circ \) is very close to the Brannen’s conjecture \( \theta_\nu = 57.73^\circ \). In fact, the result (3.10) predicts

\[
R = \frac{m_\nu^2 - m_\mu^2}{m_\nu^3 - m_\mu^2} = \frac{4\sqrt{6} - 9}{4\sqrt{6} + 9} = 0.04245.
\]

Although the value (3.12) is somewhat large comparing with the Brannen’s prediction (2.14), the prediction(3.11) is, at present, not ruled out within three sigma.

The value \( \langle \phi^\nu_{\pi} \rangle \neq 0 \) yields the \( \nu_\pi-\nu_\eta \) mixing:

\[
\tan 2\theta_{\pi\eta} = \frac{z^u_\pi}{z^u_\eta}.
\]

If we take \( z^u_\pi / z^u_\eta = z^d_\pi / z^d_\eta \), the case yields a large deviation from the tribimaximal mixing. Therefore, we cannot choose the same values \( z^u_\pi \) as \( z^d_\pi \). We must consider \( \langle \phi^\nu_{\pi} \rangle \sim 0 \) differently from the case of the down-type Higgs scalars \( \phi^d_i \).

4 Phenomenological neutrino Yukawa interaction form
In the previous section, we have required the $S_3$ invariance for the neutrino Yukawa interaction $H_\nu$, and, in order to obtain the result (3.11) near to the Brannen’s relation (2.11), we have found that we need a VEV structure $\langle \phi^u \rangle \simeq 0$ differently from the VEV structure of $\phi^d$. In the present section, we assume the same structure of $\langle \phi^u \rangle$ as $\langle \phi^d \rangle$ (i.e. the same parameter values of $z_i$). Alternatively, we abandon the $S_3$ invariance of $H_\nu$, although we still use the notation $(\pi, \eta, \sigma)$ in the $S_3$ symmetry. The purpose of the present section is not to derive the Brannen’s relation (2.11) from a model with some symmetry, but to investigate what Yukawa interaction form is required if we take the Brannen’s relation (2.11) and the tribimaximal mixing (3.2) seriously. Of course, if we introduce several adjustable parameters, it is always possible. Therefore, we still adhere to the universality of the coupling constants.

We consider only terms which keep the mass matrix diagonal on the $(\nu_\eta, \nu_\sigma, \nu_\pi)$ basis:

$$H_\nu = y_\nu \bar{\ell}_\pi N_\pi + \bar{\ell}_\eta N_\eta + \bar{\ell}_\sigma N_\sigma \phi_\sigma + y_\nu \bar{\ell}_\pi N_\pi - \bar{\ell}_\eta N_\eta \phi_x$$

$$+ y_\nu \bar{\ell}_\sigma N_\sigma - 2\bar{\ell}_\sigma N_\sigma \phi_y. \quad (4.1)$$

Here, we have assumed that the first term in (4.1) is still invariant under the $S_3$ symmetry. The three $(\bar{\ell}N)$ terms in (4.1) are linearly independent of each other. Similarly, we assume that the scalars $\phi_x$ and $\phi_y$ are linearly independent of $\phi_\sigma$, i.e. $\phi_x$ and $\phi_y$ are given by linearly independent combinations of $\phi_\pi$ and $\phi_\eta$.

In order to fix $\phi_x$ and $\phi_y$, we assume that the interaction $H_\nu$ is invariant under the exchange $\nu_\pi \leftrightarrow \nu_\eta (\phi_\pi \leftrightarrow \phi_\eta)$, i.e. we assume

$$\phi_x = \frac{\phi_\pi - \phi_\eta}{\sqrt{2}}, \quad \phi_y = \frac{\phi_\pi + \phi_\eta}{\sqrt{2}}. \quad (4.2)$$

Although this symmetry is analogous to the so-called $2 \leftrightarrow 3$ flavor symmetry [19] in the neutrino mass matrix, the present $\pi-\eta$ symmetry does not mean $\nu_\mu \leftrightarrow \nu_\tau$ symmetry.

As a result, we obtain

$$H_\nu = y_\nu \bar{\ell}_\pi N_\pi \left[ \frac{1}{\sqrt{3}} \phi_\sigma - \sqrt{\frac{2}{3}} \left( -\sqrt{\frac{3+1}{2\sqrt{2}}} \phi_\pi + \sqrt{\frac{3-1}{2\sqrt{2}}} \phi_\eta \right) \right]$$

$$+ y_\nu \bar{\ell}_\eta N_\eta \left[ \frac{1}{\sqrt{3}} \phi_\sigma - \sqrt{\frac{2}{3}} \left( \sqrt{\frac{3-1}{2\sqrt{2}}} \phi_\pi - \sqrt{\frac{3+1}{2\sqrt{2}}} \phi_\eta \right) \right]$$

$$+ y_\nu \bar{\ell}_\sigma N_\sigma \left[ \frac{1}{\sqrt{3}} \phi_\sigma - \sqrt{\frac{2}{3}} \left( \frac{1}{\sqrt{2}} \phi_\pi + \frac{1}{\sqrt{2}} \phi_\eta \right) \right].$$

(4.3)
Since we obtain
\[ \langle \phi_\pi \rangle = v z_\pi = \frac{v}{\sqrt{2}} \cos \theta_\epsilon, \quad \langle \phi_\eta \rangle = v z_\eta = -\frac{v}{\sqrt{2}} \sin \theta_\epsilon, \quad \langle \phi_\sigma \rangle = v z_\sigma = \frac{v}{\sqrt{2}}, \]  
(4.4)

from the definitions (2.3) and (2.8), the neutrino Yukawa interactions (4.3) yields the following neutrino Dirac mass matrix \( M^D_\nu \) on the \( (\nu_\eta, \nu_\sigma, \nu_\eta) \) basis:
\[ M^D_\nu = \text{diag}(m_{\eta\eta}, m_{\sigma\sigma}, m_{\pi\pi}), \]
(4.5)
\[ m_{\eta\eta} = y_\nu v \left[ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \left( \theta_\epsilon + \frac{\pi}{12} \right) \right], \]
\[ m_{\sigma\sigma} = y_\nu v \left[ \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \sin \left( \theta_\epsilon - \frac{\pi}{4} \right) \right], \]
(4.6)
\[ m_{\pi\pi} = y_\nu v \left[ \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \cos \left( \theta_\epsilon - \frac{\pi}{12} \right) \right], \]

where we have used that \( \sin(\pi/12) = (\sqrt{3} - 1)/2\sqrt{2} \) and \( \cos(\pi/12) = (\sqrt{3} + 1)/2\sqrt{2} \). Comparing the expression (4.6) with the expression (2.10), we obtain the Brannen’s empirical relation (2.11), \( \theta_\nu = \theta_\epsilon + \pi/12 \), which can give a reasonable prediction (2.13). Note that the definition of the \( (\pi, \eta, \sigma) \) basis of \( S_3 \), (2.3), is essential for the derivation of the Brannen’s relation (2.11). If we took another conventions of the \( S_3 \) representation, we would not obtain the Brannen’s relation (2.11).

If we regard \( m_{\nu 3} \) as \( m_{\nu 3} = \sqrt{\Delta m^2_{\text{atm}}} \), from \( m_{\nu i} \propto (z_i^\nu)^2 \), we obtain
\[ m_{\nu 1} = (3.94^{+0.27}_{-0.43}) \times 10^{-4} \text{ eV}, \quad m_{\nu 2} = (9.17^{+0.62}_{-0.43}) \times 10^{-3} \text{ eV}, \quad m_{\nu 3} = (5.22^{+0.35}_{-0.25}) \times 10^{-2} \text{ eV}, \]
(4.7)
where we have used the best fit value \[ \Delta m^2_{\text{atm}} = (2.72^{+0.38}_{-0.39}) \times 10^{-3} \text{ eV}^2. \]

The predicted values (4.7) lead to
\[ \Delta m^2_{21} = (8.39^{+1.17}_{-0.77}) \times 10^{-5} \text{ eV}^2, \]
(4.8)
which is in good agreement with the observed value \[ \Delta m^2_{21} = (7.9^{+0.9}_{-0.8}) \times 10^{-5} \text{ eV}^2. \]

Of course, since the mass matrix \( M^D_\nu \) is diagonal in the \( (\nu_\eta, \nu_\sigma, \nu_\eta) \) basis, the neutrino mixing matrix \( U \) is exactly given by the tribimaximal mixing (3.2), which gives
\[ \sin^2 2\theta_{23} = 1, \]
(4.9)
\[ \tan^2 \theta_{12} = \frac{1}{2} \quad (\theta_{12} = 35.26^\circ), \]
(4.10)
\[ |U_{13}| = 0. \]
(4.11)

The prediction (4.10) is also in good agreement with the observed value \[ \tan^2 \theta_{21} = 0.45^{+0.09}_{-0.07}. \]

We would like to emphasize that, in order to obtain the tribimaximal mixing (3.2), the magnitudes of the mass eigenvalues \( |m_{\eta\eta}| < |m_{\sigma\sigma}| < |m_{\pi\pi}| \) are essential together with the definition of the \( S_3 \) basis (2.3).

5 Concluding remarks
In the present paper, it has been pointed out that if we take the phenomenological relations (1.2) and (2.11) seriously, we must consider that the Yukawa interactions in the neutrino sector
are given by the form (4.1). The Yukawa interactions (4.1) leads to the tribimaximal mixing (3.2) and the Brannen’s relation (2.11) [so that the neutrino mass spectrum (4.7)]. Although, in Sec.4, we have put a requirement of the $\pi \leftrightarrow \eta$ symmetry for the neutrino Yukawa interaction $H_\nu$ instead of the $S_3$ symmetry, the requirement is, of course, merely a phenomenological assumption. For the charged lepton sector, we have assumed the $S_3$ invariant Yukawa interaction, while, for the neutrino sector, we have assumed the $S_3$-breaking interaction (4.1). The tribimaximal mixing (3.2) has already been derived, for example, from an $A_4$ symmetry [20]. The interaction form (4.1) will be understood from an extended finite symmetry.

To the contrary, if we adhere the $S_3$ symmetry, as we discussed in Sec.3, we can obtain the result (3.11), which is numerically near to the Brannen’s relation (2.11). However, we must assume that $\langle \phi^u \rangle \simeq 0$ in the up-type Higgs scalars differently from the case in the down-type Higgs scalars. The case (3.6) is also interesting.

In the present seesaw model, the 3-family SU(2)$_L$-doublet scalars $\phi_i$ cause a flavor-changing neutral current (FCNC) problem. Besides, the new heavy fermions $E_{Ri}$ and $E_{Li}$ together with the scalars $\phi_i$ considerably affect the evolution of the gauge coupling constants. If we want to avoid these problems, we can take an alternative model, a Frogatt-Nielsen-like [11] model:

$$H_{\text{eff}} = y_e \bar{\ell}_L H^d_L \left( \frac{\phi}{\Lambda} \right)^2 e_R + y_\nu \bar{\nu}_L H^u_L \frac{\phi}{\Lambda} \nu_R + \bar{\nu}_R M_R \nu_R^*, \quad (5.1)$$

where $\ell_L = (\nu_L, e_L)$, $H^d_L = (H^+_d, H^0_d)$, $H^u_L = (H^0_u, H^-_u)$, $\phi$ is a 3-family SU(2)$_L$-singlet scalar, and $\Lambda$ is a scale of the effective theory. (Here, we have denoted the expression (5.1) symbolically. For example, the interaction $\bar{\ell}_\nu \phi \nu_R$ should read the interaction (4.1).)

In the present paper, we have discussed the masses and mixings only in the lepton sectors. We think that if there is a beautiful law in the masses and mixings of the fundamental fermions, we will find it just in the lepton sectors, because the mass generation mechanism seems to be simple just in the lepton sectors. If we define

$$R_f(\eta_1, \eta_2) = \frac{2}{3} \left( \eta_1 \sqrt{|m_{f1}|} + \eta_2 \sqrt{|m_{f2}|} + \sqrt{|m_{f3}|} \right)^2, \quad (5.2)$$

where $\eta_i = \pm 1 \ (i = 1, 2)$, the ratio $b/a$ in the expression (2.5) with $m_{fi} \propto \lambda_i^2$ is given by

$$\frac{b}{\sqrt{2}a} = \sqrt{\frac{2 - R}{R}}, \quad (5.3)$$

where $R$ (not $R_f$) is defined by Eq.(2.13) and $a$ and $b$ are defined by Eq.(2.5) [or Eq.(2.6)]. In the lepton sectors, we have found

$$R_e(\pm, +) = R_\nu(\pm, -) = 1, \quad (5.4)$$

so that $(b/\sqrt{2}a)_e = (b/\sqrt{2}a)_\nu = 1$. However, as seen in Table 1, in the quark sectors, there is no solution of $a/b$ which gives $R_a(\eta^u_1, \eta^u_2) = R_a(\eta^d_1, \eta^d_2)$. Therefore, the present idea in the lepton
sectors cannot be applied to the quark sectors straightforwardly. In the seesaw model (1.5), we have assumed that the heavy fermion mass matrices $M_F$ in the lepton sectors are structureless, i.e. $M \propto 1$. For quark sectors, we may consider that $M_F$ have some structure except for the unit matrix $1$, so that the quark masses will not satisfy the relation (5.2) with $R_f = 1$. Our goal is to find a unified description of the quark and lepton masses and mixings. For this purpose, the present study in the lepton sectors will provide a promising clue to the unified model.

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Table 1: Values of $R_f(\eta_1, \eta_2)$ which is defined by Eq. (4.2): For convenience, the pole mass values [1] for the charged lepton masses, the values (3.13) for the neutrino masses, and the quark mass values for the running mass values at $\mu = m_Z$ [21] are used, respectively. In the Table, the center values have been used as the input values.

| Sector       | $R(++, +)$ | $R(-, +)$ | $R(+, -)$ | $R(-, -)$ |
|--------------|------------|-----------|-----------|-----------|
| Charged lepton | 0.999998   | 0.946922  | 0.376006  | 0.34374   |
| Neutrino     | 1.28       | 1.00      | 0.251     | 0.137     |
| Up-quark     | 0.753      | 0.743     | 0.590     | 0.581     |
| Down-quark   | 0.955      | 0.834     | 0.481     | 0.397     |

Fig. 1 $R$ versus $\theta_\nu$ under the relation (1.2). The horizontal lines denote the observed values $R = (2.9 \pm 0.5) \times 10^{-2}$. 