Reflected forward-backward stochastic differential equations driven by \( G \)-Brownian motion with continuous monotone coefficients

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Abstract In this paper, we prove that there exists at least one solution for the reflected forward-backward stochastic differential equation driven by \( G \)-Brownian motion satisfying the obstacle constraint with monotone coefficients.

Key words reflected equation; forward-backward SDE; \( G \)-Brownian motion; monotone coefficients.

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1 Introduction

Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng systemically established a time-consistent fully expectation theory (see [14]). As a typical and important case, Peng introduced the \( G \)-expectation theory (see [15][16]). In the \( G \)-expectation
framework, the notion of G-Brownian motion and the corresponding stochastic calculus of Itô’s type were established. On that basis, many properties and applications of the G-expectation, G-Brownian motion and the G-stochastic calculus are studied (see [3, 4]).

On that basis, some authors are interested in the forward stochastic differential equation driven by G-Brownian motion (FGSDE), which has a similar form as its counterpart in the classical framework, however, holds in a q.s. sense:

\[
X_t = x + \int_0^t b(s, X_s)ds + \int_0^t h(s, X_s)d(B)_s + \int_0^t \sigma(s, X_s)dB_s, \ 0 \leq t \leq T, \text{ q.s.,}
\]

where \((B)\) is the quadratic variation of the G-Brownian motion \(B\). Under the Lipschitz assumptions on the coefficients \(b, h\) and \(\sigma\), Peng [15] and Gao [4] have proved the wellposedness of such equation with the fixed-point iteration. Moreover, Bai and Lin [2] have studied the case when coefficients are integral-Lipschitz, Lin [10] considered the reflected GSDEs with some good boundaries, Ren et al. [18] studied stochastic functional differential equation with infinite delay driven by G-Brownian motion.

On the basis of a series of studies by Hu et al. [6] and Soner et al. [19] for the G-expectation, Peng et al. [17] obtained the complete representation theorem for G-martingale. Due to this contribution, Hu et al. [7] obtained the existence, uniqueness, time consistency and a priori estimates of fully nonlinear backward stochastic differential equation driven by a given G-Brownian motion (BGSDE) under standard Lipschitz conditions. Very recently, Li and Peng [12] study the reflected solution of the following backward stochastic differential equations driven by G-Brownian motion (RBGSDE) via penalization:

\[
\begin{align*}
Y_t &= \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds + \int_t^T g(s, X_s, Y_s, Z_s)d(B)_s - \int_t^T Z_sdB_s + (A_T - A_t), \\
Y_t &\geq L_t, \ (-\int_0^t (Y_s - L_s)dA_s)_{t \in [0, T]} \text{ is a non-increasing G-martingale.}
\end{align*}
\]

Under standard Lipschitz conditions on \(f(y, z), g(y, z)\) in \(y, z\) and the \(L_G^\beta(\Omega_T)(\beta > 1)\) integrability condition on \(\xi\), there exists a triplet of processes \((Y, Z, A) \in S_G^\alpha(0, T)\) satisfy above equation for \(2 \leq \alpha < \beta\). Here, \(S_G^\alpha(0, T)\) denote the collection of process \((Y, Z, A)\) such that \(Y \in S_G^\alpha(0, T), Z \in H_G^\alpha(0, T), A\) is a continuous nondecreasing process with \(A_0 = 0\) and \(A \in S_G^\alpha(0, T)\).

It is known that forward-backward equations are encountered when one applies the stochastic maximum principle to optimal stochastic control problems. Such equations are also encountered in the probabilistic interpretation of a general type of systems quasilinear PDEs, as well as in finance (see [5] for example). In the linear expectation framework, Antonelli et al. [1] and Huang et al. [8] proved the existence of the solutions for backward-forward SDEs and reflected forward-backward SDEs respectively. However, in the G-framework, as far as we know, there is no result about the reflected forward-backward stochastic differential equations driven by G-Brownian motion (RF-BGSDEs) in which the solution of the BSDE stays above a given barrier. One of the differences is that the classical Skorohod condition should be substituted by a G-martingale condition. Moreover, the comparison theorem with respect to the increasing process \(\{A_t\}_{t \in [0, T]}\) may not hold in G-framework. So some mathematical properties of this RFBGSDE should be developed.
In this paper, we consider the solvability of the following RFBGSDEs with continuous monotone coefficients:

\[
\begin{aligned}
X_t &= x + \int_0^t b(s,X_s,Y_s)ds + \int_0^t h(s,X_s,Y_s)dB_s + \int_0^t \sigma(s,X_s)dB_s, \\
Y_t &= \xi + \int_t^T f(s,X_s,Y_s,\xi)ds + \int_t^T g(s,X_s,Y_s,\xi)dB_s - \int_t^T Z_sdB_s + (A_T - A_t), \\
Y_t &\geq L_t, \quad \{-\int_0^T (Y_s - L_s)dA_s\}_{t \in [0,T]} \text{ is a non-increasing } G-\text{martingale.}
\end{aligned}
\]  

(1.1)

We notice that the coefficients of the forward GSDE contain the solution of the backward GSDE, so the forward GSDE and the backward GSDE are coupled together. Moreover, the coefficients only need to satisfy the linear growth condition, but do not need to satisfy the Lipschitz condition.

The rest of this paper is organized as follows. In section 2, we introduce some notions and results in the G-framework which are necessary for what follows. In section 3, the existence theorem is provided.

## 2 Preliminaries

In this section, we introduce some notations and preliminary results in G-framework which are needed in the following sections. More details can be found in [3][4][15].

Let \( \Omega_T = C_0([0,T]; \mathbb{R}) \), the space of real valued continuous functions on \([0,T] \) with \( w_0 = 0 \), be endowed with the distance

\[
d(w^1, w^2) := \sum_{N=1}^{\infty} 2^{-N} \left( \max_{0 \leq t \leq T} |w^1_t - w^2_t| \right)^{\wedge 1},
\]

and let \( B_t(w) = w_t \) be the canonical process. Denote by \( F := \{ F_t \}_{0 \leq t \leq T} \) the natural filtration generated by \( B \), \( L^0(\Omega_T) \) be the space of all \( \mathbb{F} \)-measurable real functions. Let \( L_{ip}(\Omega_T) := \{ \varphi(B_{t_1},...,B_{t_n}) : \forall n \geq 1, t_1,...,t_n \in [0,T], \forall \varphi \in C_b,\text{Lip}(\mathbb{R}^n) \} \), where \( C_b,\text{Lip}(\mathbb{R}^n) \) denotes the set of bounded Lipschitz functions on \( \mathbb{R}^n \). A sublinear functional on \( L_{ip}(\Omega_T) \) satisfies: for all \( X,Y \in L_{ip}(\Omega_T) \),

(i) Monotonicity: \( E[X] \geq E[Y] \) if \( X \geq Y \).

(ii) Constant preserving: \( E[C] = C \) for \( C \in \mathbb{R} \).

(iii) Sub-additivity: \( E[X + Y] \leq E[X] + E[Y] \).

(iv) Positive homogeneity: \( E[\lambda X] = \lambda E[X] \) for \( \lambda \geq 0 \).

The triple \(( \Omega, L_{ip}(\Omega_T), E)\) is called a sublinear expectation space and \( E \) is called a sublinear expectation.

**Definition 2.1.** A random variable \( X \in L_{ip}(\Omega_T) \) is G-normal distributed with parameters \((0, [\sigma^2, \sigma^2])\), i.e., \( X \sim N(0, [\sigma^2, \sigma^2]) \), if for each \( \varphi \in C_b,\text{Lip}(\mathbb{R}) \), \( u(t,x) := E[\varphi(x + \sqrt{X})] \) is a viscosity solution to the following PDE on \( \mathbb{R}^+ \times R \):

\[
\begin{aligned}
\frac{\partial u}{\partial t} + G\left(\frac{\partial u}{\partial x}\right) &= 0, \\
u_{t_0} &= \varphi(x),
\end{aligned}
\]  

(2.1)
where $G(a) := \frac{1}{2}(a^+ \sigma^2 - a^- \sigma^2), a \in R$.

**Definition 2.2.** We call a sublinear expectation $\hat{E} : L_{ip}(\Omega_T) \to R$ a G-expectation if the canonical process $B$ is a $G$-Brownian motion under $\hat{E}[\cdot]$, that is, for each $0 \leq s \leq t \leq T$, the increment $B_t - B_s \sim N(0, [\sigma^2(t - s), \sigma^2(t - s)])$ and for all $n > 0$, $0 \leq t_1 \leq \ldots \leq t_n \leq T$ and $\varphi \in L_{ip}(\Omega_T)$,

$$\hat{E}[\varphi(B_{t_1}, \ldots, B_{t_{n-1}}, B_{t_n} - B_{t_{n-1}})] = \hat{E}[\psi(x_1, \ldots, x_{n-1}, \sqrt{t_n - t_{n-1}}B_1)]$$

where $\psi(x_1, \ldots, x_{n-1}) := \hat{E}[\varphi(x_1, \ldots, x_{n-1}, \sqrt{t_n - t_{n-1}}B_1)]$ and $B_1$ is $G$-normal distributed.

For $p \geq 1$, we denote by $L^p_G(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under the natural norm $\|X\|_{p,G} := (\hat{E}[|X|^p])^{\frac{1}{p}}$. $\hat{E}$ is a continuous mapping on $L_{ip}(\Omega_T)$ endowed with the norm $\|\cdot\|_{1,G}$. Therefore, it can be extended continuous to $L^2_G(\Omega_T)$ under the norm $\|X\|_{1,G}$.

Next, we introduce the Itô integral of $G$-Brownian motion.

Let $M^0_G(0, T)$ be the collection of processes in the following form: for a given partition $\pi_T = \{t_0, t_1, \ldots, t_N\}$ of $[0, T]$, set

$$\eta_{\pi}(w) = \sum_{k=0}^{N-1} \xi_k(w)I_{(t_k, t_{k+1})}(t)$$

where $\xi_k \in L_{ip}(\Omega_{t_k})$ for $k = 0, 1, \ldots, N - 1$ are given. For $p \geq 1$, we denote by $H^p_G(0, T), M^p_G(0, T)$ the completion of $M^p_G(0, T)$ under the norm $\|\eta\|_{H^p_G(0, T)} = (\hat{E}[(\int_0^T |\eta|^p dt)^\frac{1}{p}])^{\frac{1}{p}}$, $\|\eta\|_{M^p_G(0, T)} = (\hat{E}[(\int_0^T |\eta|^p dt)^\frac{1}{p}])^{\frac{1}{p}}$ respectively. It is easy to see that $H^2_G(0, T) = M^2_G(0, T)$. Following Li and Peng [11], for each $\eta \in H^p_G(0, T)$ with $p \geq 1$, we can define Itô integral $\int_0^T \eta_s dB_s$. Moreover, the following B-D-G inequality hold.

**Lemma 2.1.** (B) Let $p \geq 2$ and $\eta \in M^p_G(0, T)$, then we have

$$\frac{1}{p(p-1)}(\int_0^T |\eta|^2 ds)^\frac{1}{p} \leq \hat{E}[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^p] \leq \frac{1}{p(p-1)}(\int_0^T |\eta|^2 ds)^\frac{1}{p},$$

where $0 < c_p < C_p < \infty$ are constants.

Let $S^0_G(0, T) = \{h(t, B_{t_1}, \ldots, B_{t_n}) : t_1, \ldots, t_n \in [0, T], h \in C_b, (R^{n+1})\}$. For $p \geq 1$ and $\eta \in S^0_G(0, T)$, set $\|\eta\|_{S^p_G(0, T)} = (\hat{E}[\sup_{0 \leq t \leq T} |\eta(t)|]^p) \hat{E}$. Denote by $S^p_G(0, T)$ the completion of $S^0_G(0, T)$ under the norm $\|\eta\|_{S^p_G(0, T)}$.

**Definition 2.3.** Quadratic variation process of $G$-Brownian motion defined by

$$(B)_t := B^2_t - 2 \int_0^t B_s dB_s$$

is a continuous, nondecreasing process.

For $\eta \in M^0_G(0, T)$, define $\int_0^T \eta_s dB(B)_s = \sum_{j=0}^{N-1} \xi_j((B)_{t_{j+1}} - (B)_{t_j}) : M^0_G(0, T) \to L^1_G(\Omega_T)$. The mapping is continuous and can be extended to $M^1_G(0, T)$.
Lemma 2.2. \( \text{(15)} \) Let \( p \geq 1 \) and \( \eta \in M^p_D(0,T) \), then we have
\[
\mathbb{E}^{2}[\int_{0}^{T} \eta_s ds] \leq \mathbb{E}^{2}[\int_{0}^{T} \eta_s d(B)_s] \leq a^2 \mathbb{E}[\int_{0}^{T} \eta_s ds],
\]
where \( a > 0 \) is a constant independent of \( \eta \).

Theorem 2.1. \( \text{(3)} \) There exists a weakly compact subset \( P \subset \mathcal{M}(\Omega_T) \), the set of probability measures on \((\Omega_T, \mathcal{F}_T)\), such that
\[
\mathbb{E}[\eta] = \max_{P \in P} \mathbb{E}_P(\eta) \text{ for all } \eta \in L^1_P(\Omega_T).
\]
\( P \) is called a set that represents \( \mathbb{E} \).

Let \( P \) be a weakly compact set that represents \( \mathbb{E} \). For this \( P \), we define capacity
\[
c(A) = \sup_{P \in P} P(A), A \in \mathcal{F}_T.
\]
A set \( A \subset \Omega_T \) is a polar set if \( c(A) = 0 \). A property holds quasi-surely (q.s.) if it holds outside a polar set.

Lemma 2.3. \( \text{(3)} \) Let \( \{X^n\}_{n \in \mathbb{N}} \subset L^1_D(\Omega_T) \) be such that \( X^n \downarrow X \) q.s., then \( \mathbb{E}[X^n] \downarrow \mathbb{E}[X] \). In particular, if \( X \in L^1_D(\Omega_T) \), then \( \mathbb{E}[X^n - X] \downarrow 0 \), as \( n \to \infty \).

Lemma 2.4. \( \text{(13)} \) For any \( \alpha \geq 1 \), \( \delta > 0 \) and \( 1 < \gamma < \beta := \left( \frac{\alpha + \delta}{\alpha} \right), \gamma \leq 2 \), we have
\[
\mathbb{E}[\sup_{t \in [0,T]} |\xi|^\alpha] \leq \gamma^\alpha \{ (\mathbb{E}[|\xi|^\alpha + \delta])^{\frac{\alpha}{\alpha + \delta}} + 14^\frac{1}{\delta} C_\beta \mathbb{E}[|\xi|^{\alpha + \delta}] \\}, \quad \forall \xi \in L^p(\Omega_T),
\]
where \( C_\beta = \sum_{i=1}^{\infty} i^{-\frac{\beta}{\gamma - 1}} \), \( \gamma = \frac{\alpha}{\gamma - 1} \).

To get the main result of this paper, we need the following Lemma used by Lepeltier-San Martin.

Lemma 2.5. \( \text{(13)} \) Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a continuous function with linear growth, that is, there exist a constant \( M < \infty \) such that \( \forall x \in \mathbb{R}^m, |f(x)| \leq M(1 + |x|) \). Then the sequence of functions
\[
f_n(x) = \inf_{y \in Q} \{ f(y) + n|x - y| \}
\]
is well defined for \( n \geq M \) and satisfies
\begin{enumerate}[(i)]  
  \item linear growth: \( \forall x \in \mathbb{R}^m, |f_n(x)| \leq M(1 + |x|) \);
  \item monotonicity in \( n \): \( \forall x \in \mathbb{R}^m, f_n(x) \leq f_{n+1}(x) \);
  \item Lipschitz condition: \( \forall x, y \in \mathbb{R}^m, |f_n(x) - f_n(y)| \leq n|x - y| \);
  \item strong convergence: if \( x_n \to x \), then \( f_n(x_n) \to f(x) \).
\end{enumerate}
3 main result

Definition 3.1. A quadruple of processes $(X, Y, Z, A)$ is called a solution of reflected FBGSDEs (1.1) if the following properties are satisfied:

(i) $X \in M^2_G(0, T), (Y, Z, A) \in S^2_G(0, T)$;

(ii) 
$$
\begin{align*}
X_t &= x + \int_0^t b(s, X_s, Y_s) \, ds + \int_0^t \sigma(s, X_s) \, dB_s, \\
Y_t &= \xi + \int_t^T f(s, X_s, Y_s, Z_s) \, ds + \int_t^T g(s, X_s, Y_s, Z_s) \, dB_s - \int_t^T Z_s \, dB_s + (A_T - A_t),
\end{align*}
$$

(iii) $Y_t \geq L_t, \quad \{-\int_0^t (Y_s - L_s) \, dA_s\}_{t \in [0, T]}$ is a non-increasing $G$-martingale.

In the sequel, we will work under the following assumptions: for any $s \in [0, T], w \in \Omega, x, x', y, z \in R, \beta > 2$:

(H1) $b(\cdot, x, y, z), h(\cdot, x, y, z), \sigma(\cdot, x) \in M^2_G(0, T), f(\cdot, x, y, z), g(\cdot, x, y, z) \in M^\beta_G(0, T)$;
(H2) $b, h$ are increasing in $y$ and $f, g$ are increasing in $x$;
(H3) there exists a constant $M > 0$, such that 
$$
|b(s, x, y)| + |h(s, x, y)| \leq M(1 + |x| + |y|), \quad |f(s, x, y, z)| + |g(s, x, y, z)| \leq M(1 + |y| + |z|);
$$
$$
|\sigma(s, x)| \leq M(1 + |x|), \quad |\sigma(s, x) - \sigma(s, x')| \leq M|x - x'|;
$$
(H4) $\xi \in L^\beta_G(\Omega_T)$ and $\xi \geq L_T, \text{ q.s.}$
(H5) $(L_t)_{t \in [0, T]} \in S^2_G(0, T)$ and there exists a constant $c$ such that $L_t \leq c$, for each $t \in [0, T]$.

For notational simplification, by Lemma 2.2, we only consider the case $h = 0$ and $g = 0$. I.e., we consider the following equation:

$$
\begin{align*}
X_t &= x + \int_0^t b(s, X_s, Y_s) \, ds + \int_0^t \sigma(s, X_s) \, dB_s, \\
Y_t &= \xi + \int_t^T f(s, X_s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dB_s + (A_T - A_t),
\end{align*}
$$

(3.1)

But the results still hold for other case. In the following, $C$ always denote a positive constant which may change from line to line.

Theorem 3.1. Suppose that $\xi, b, f, \sigma$ satisfy (H1)-(H4), $L$ satisfies (H5). Then the RFBGSDE (3.1) has at least one solution $(X, Y, Z, A)$.

Proof. In order to construct a solution of (3.1), our basic idea is to consider the following iteration:

$$
\begin{align*}
X^n_t &= x + \int_0^t b(s, X^n_s, Y^n_s) \, ds + \int_0^t \sigma(s, X^n_s) \, dB_s, \\
Y^n_t &= \xi + \int_t^T f(s, X^n_s, Y^n_s, Z^n_s) \, ds - \int_t^T Z^n_s \, dB_s + (A^n_T - A^n_t),
\end{align*}
$$

(3.2)

$Y^n_t \geq L_t, \quad \{-\int_0^t (Y^n_s - L_s) \, dA^n_s\}_{t \in [0, T]}$ is a non-increasing $G$-martingale.
We will show that the limit of the sequence \( \{X^n, Y^n, Z^n, A^n\} \) for \( n \in \mathbb{N} \) verifies equations (3.1).

Step 1: Construction of the starting point.

Let us consider the following two standard reflected backward \( G \)-stochastic differential equations:
\[
\begin{aligned}
Y_t^0 &= \xi - K \int_t^T (1 + |Y_s^0| + |Z_s^0|)ds - \int_t^T Z_s^0 dB_s + (A_T^0 - A_t^0), \\
Y_t^0 &\geq L_t, \left\{ - \int_0^t (Y_s^0 - L_s)dA_s^0 \right\}_{t \in [0,T]} \text{ is a non-increasing } G-\text{martingale}
\end{aligned}
\] (3.3)

and
\[
\begin{aligned}
U_t &= |\xi| + K \int_t^T (1 + |U_s| + |V_s|)ds - \int_t^T V_s dB_s + (N_T - N_t), \\
U_t &\geq L_t, \left\{ - \int_0^t (U_s - L_s)dN_s \right\}_{t \in [0,T]} \text{ is a non-increasing } G-\text{martingale},
\end{aligned}
\]

where \( K > 0 \) is a constant. By virtue of the Lipschitz property of the coefficients \( \pm K(1 + |y| + |z|) \), thanks to Theorem 5.1 in [12], each one has a unique solution denoted by \((Y_t^0, Z_t^0, A_t^0)\) and \((U_t, V_t, N_t)\) respectively. More precisely, \( (Y_t^0, Z_t^0, A_t^0), (U_t, V_t, N_t) \in S_G^2(0, T) \) for \( 2 \leq \alpha < \beta \), such that \( \left\{ - \int_0^t (Y_s^0 - L_s)dA_s^0 \right\}_{t \in [0,T]}, \left\{ - \int_0^t (U_s - L_s)dN_s \right\}_{t \in [0,T]} \) are non-increasing \( G \)-martingale. By the comparison theorem in [12], we know that for all \( t \in [0,T], Y_t^0 \leq U_t \), q.s..

Step 2: Construction of \( X^0 \).

Now, we consider the forward equation
\[
X_t^0 = x + \int_0^t b(s, X_s^0, Y_s^0)ds + \int_0^t \sigma(s, X_s^0)dB_s,
\] (3.4)

where \( Y^0 \) is the solution of (3.3).

Let \( \{b_k(s, x, y)\}_{k \geq 0} \) be the sequence defined in Lemma 2.5. then we can conclude that the following GSDE has a unique solution \( X_t^{0,k} \in M_G^2(0, T) \) by the lipschitz property of \( b_k \), i.e.,
\[
X_t^{0,k} = x + \int_0^t b_k(s, X_s^{0,k}, Y_s^0)ds + \int_0^t \sigma(s, X_s^{0,k})dB_s.
\] (3.5)

Moreover, by the comparison theorem in [9] and Lemma 2.5, we know that for \( t \in [0,T], X_t^{0,k} \leq X_t^{0,k+1} \leq S_t \), where \( S_t \in M_G^2(0, T) \) is the unique solution of the following GSDE:
\[
S_t = x + K \int_0^t (1 + |S_s| + |U_s|)ds + \int_0^t \sigma(s, S_s)dB_s.
\]

Actually, by Corollary 3.2 in chapter V of [15], we know that \( X_t^{0,k}, S_t \in S_G^2(0, T) \). So there exists a lower semi-continuous process \( X_t^0 \in S_G^2(0, T) \) such that \( X_t^{0,k} \uparrow X_t^0 \) as \( k \to \infty \), q.s.. Notice that \( X_t^{0,k}, X_t^0 \in S_G^2(0, T) \), which obviously belong to a larger space \( L_G^2(0, T) \). Then by the downward monotone convergence theorem (Lemma 2.3), we have \( \mathbb{E} \|[X_t^{0,k} - X_t^0]_7^2 \|_7 \downarrow 0 \) as \( k \to \infty \).

By Lemma 2.5 and the dominated convergence theorem with respect to \( t \), we have
\[
\begin{aligned}
\mathbb{E} \int_0^T |b_k(s, X_s^{0,k}, Y_s^0) - b(s, X_s^0, Y_s^0)|^2 ds \\
\leq \mathbb{E} \int_0^T |b_k(s, X_s^{0,k}, Y_s^0) - b_k(s, X_s^{0,k}, Y_s^0)|^2 ds + \mathbb{E} \int_0^T |b_k(s, X_s^{0,k}, Y_s^0) - b(s, X_s^0, Y_s^0)|^2 ds
\end{aligned}
\]
\[
C \int_0^T \mathbb{E}[[X_{t}^{0,k} - X_{s}^{0}]^2] ds + \int_0^T \mathbb{E}[[b_k(s, X_t^0, Y_s^0) - b(s, X_t^0, Y_s^0)]^2] ds \to 0, \quad (3.6)
\]

as \( k \to \infty \).

Moreover, by the comparison theorem in \([12]\), we have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\sigma(s, X_{s}^{0,k}) - \sigma(s, X_{s}^0)|^2 ds \right] \leq C \int_0^T \mathbb{E} |X_{s}^{0,k} - X_{s}^{0,2}]^2 ds \to 0,
\]

as \( k \to \infty \).

On the other hand, since \(|\sigma(s, X_{s}^{0,k}) - \sigma(s, X_{s}^0)| \leq M|X_{s}^{0,k} - X_{s}^0|\), then according to Lemma 2.1, we have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\sigma(s, X_{s}^{0,k}) - \sigma(s, X_{s}^0)|^2 ds \right] \leq C \int_0^T \mathbb{E} |X_{s}^{0,k} - X_{s}^{0,2}]^2 ds \to 0,
\]

as \( k \to \infty \).

Now taking limit on both side of (3.5), then we obtain that the continuous process \( X_t^0 \in \mathcal{S}_G^\alpha(0, T) \) satisfies (3.4).

Step 3: Construction of \((X^n, Y^n, Z^n, A^n)\).

We focus on \((X^1, Y^1, Z^1, A^1)\). First, based on \( X^n \), we can construct \( Y^n \). In fact, denote \( f^n(s, w, y, z) = f(s, X^n(s,w), y, z) \), then one can easily check that \(|f^n(s, w, y, z)| \leq K(1 + |y| + |z|)\). Define once again \( f^n(s, w, y, z) \) the approximating sequence in Lemma 2.5, then by Theorem 5.1 in \([12]\), for \( 2 \leq \alpha < \beta \), we have a unique triple \((Y^{1,k}, Z^{1,k}, A^{1,k}) \in \mathcal{S}_G^\alpha(0, T) \) satisfying

\[
\begin{cases}
Y_{t}^{1,k} = \xi + \int_t^T f^n_k(s, w, Y_{s}^{1,k}, Z_{s}^{1,k}) ds - \int_t^T Z_{s}^{1,k} dB_s + (A_{T}^{1,k} - A_{t}^{1,k}), \\
Y_{t}^{1,k} \geq L_t, \{ - \int_0^t (Y_{s}^{1,k} - L_s) dA_{s}^{1,k} \} \in [0, T] \text{ is a non-increasing } G\text{-martingale.}
\end{cases}
\]

Moreover, by the comparison theorem in \([12]\), we have

\[
Y_{t}^0 \leq Y_{t}^{1,k} \leq Y_{t}^{1,k+1} \leq U_t, \quad t \in [0, T], \text{ q.s.}
\]

Then it is easy to see that there exists a constant \( C \) independent of \( k \), such that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_{t}^{1,k}|^\alpha \right] \leq C.
\]

By Proposition 3.1 in \([12]\), we have

\[
\mathbb{E}[(\int_0^T |Z_{s}^{1,k}|^2 ds)^{\frac{\alpha}{2}}] \leq C'(\mathbb{E}[\sup_{0 \leq t \leq T} |Y_{t}^{1,k}|^\alpha] + M \mathbb{E} [\sup_{0 \leq t \leq T} |Y_{t}^{1,k}|^\alpha]) \leq C
\]

and

\[
\mathbb{E}[[A_{T}^{1,k}]^\alpha] \leq C_\alpha \mathbb{E} [\sup_{0 \leq t \leq T} |Y_{t}^{1,k}|^\alpha + (MT)^\alpha] \leq C,
\]

where \( C' \) is a constant.

As \( \{Y_{t}^{1,k}\}_{k \geq 1} \) is an increasing sequence, we denote the limit by \( Y^1 \). It is easy to see that \( Y_{t}^1 \geq Y_{t}^0 \) for each \( t \in [0, T] \). Moreover, by Fatou's lemma, we have \( Y_{t}^1 \in \mathcal{S}_G^\alpha(0, T) \). Hence, by the dominated convergence theorem with respect to \( t \), we have

\[
\int_0^T \mathbb{E}|Y_{s}^{1,k} - Y_{s}^{1}|^\alpha ds \to 0.
\]
as $k \to \infty$. i.e., $Y^{1,k} \to Y^1$ in $M^2_s(0,T)$. In the following, we can show that this convergence holds in $S^2_s(0,T)$.

Applying G-Itô's formula to $(Y_{t}^{1,k} - Y_{t}^{1,j}|^2)$, we know that

$$\begin{align*}
|Y_{t}^{1,k} - Y_{t}^{1,j}|^\alpha &+ \alpha \int_t^T (|Y_{t}^{1,k} - Y_{t}^{1,j}|^2)^{\frac{\alpha - 1}{2}} |Z_{t}^{1,k} - Z_{t}^{1,j}|^2 d(B)_s \\
&= \alpha(1 - \frac{\alpha}{2}) \int_t^T (|Y_{t}^{1,k} - Y_{t}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{t}^{1,k} - Y_{t}^{1,j})^2 |Z_{t}^{1,k} - Z_{t}^{1,j}|^2 d(B)_s \\
&+ \alpha \int_t^T (|Y_{s}^{1,k} - Y_{s}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{s}^{1,k} - Y_{s}^{1,j}) [f^1_k(s, w, Y_{s}^{1,k}, Z_{s}^{1,k}) - f^1_j(s, w, Y_{s}^{1,j}, Z_{s}^{1,j})] ds \\
&+ \alpha \int_t^T (|Y_{s}^{1,k} - Y_{s}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{s}^{1,k} - Y_{s}^{1,j}) d(A_{s}^{1,k} - A_{s}^{1,j}) \\
&- \alpha \int_t^T (|Y_{s}^{1,k} - Y_{s}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{s}^{1,k} - Y_{s}^{1,j}) (Z_{s}^{1,k} - Z_{s}^{1,j}) dB_s \\
&\leq \alpha \int_t^T |Y_{s}^{1,k} - Y_{s}^{1,j}|^{\alpha - 1} f^1_k(s, w, Y_{s}^{1,k}, Z_{s}^{1,k}) - f^1_j(s, w, Y_{s}^{1,j}, Z_{s}^{1,j}) | ds \\
&+ \alpha \int_t^T (|Y_{s}^{1,k} - Y_{s}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{s}^{1,k} - Y_{s}^{1,j})^+ dA_{s}^{1,k} \\
&+ \alpha \int_t^T (|Y_{s}^{1,k} - Y_{s}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{s}^{1,k} - Y_{s}^{1,j})^- dA_{s}^{1,j} \\
&- \alpha \int_t^T (|Y_{s}^{1,k} - Y_{s}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{s}^{1,k} - Y_{s}^{1,j}) (Z_{s}^{1,k} - Z_{s}^{1,j}) dB_s. 
\end{align*}$$

Let $M_{t}^{j,k} = -\alpha \int_0^t (|Y_{s}^{1,k} - Y_{s}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{s}^{1,k} - Y_{s}^{1,j})^+ dA_{s}^{1,k} - \alpha \int_0^t (|Y_{s}^{1,k} - Y_{s}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{s}^{1,k} - Y_{s}^{1,j})^- dA_{s}^{1,j} + \alpha \int_0^t (|Y_{s}^{1,k} - Y_{s}^{1,j}|^2)^{\frac{\alpha - 1}{2}} (Y_{s}^{1,k} - Y_{s}^{1,j}) (Z_{s}^{1,k} - Z_{s}^{1,j}) dB_s$. Since

$$0 \geq -(Y_{s}^{1,k} - Y_{s}^{1,j})^+ \geq -(Y_{s}^{1,k} - L_s),$$

then

$$0 \geq \int_0^t -(Y_{t}^{1,k} - Y_{t}^{1,j})^+ dA_{t}^{1,k} \geq - \int_0^t (Y_{t}^{1,k} - L_s) A_{t}^{1,k},$$

which implies $\{M_{t}^{j,k}\}_{t \in [0,T]}$ is a G-martingale. We rewrite (3.11) as

$$\begin{align*}
M_{t}^{j,k} - M_{t}^{l,k} + |Y_{t}^{1,k} - Y_{t}^{1,j}|^\alpha &+ \alpha \int_t^T |Y_{t}^{1,k} - Y_{t}^{1,j}|^{\alpha - 2} |Z_{t}^{1,k} - Z_{t}^{1,j}|^2 d(B)_s \\
&\leq \alpha \int_t^T |Y_{s}^{1,k} - Y_{s}^{1,j}|^{\alpha - 1} f^1_k(s, w, Y_{s}^{1,k}, Z_{s}^{1,k}) - f^1_j(s, w, Y_{s}^{1,j}, Z_{s}^{1,j}) | ds.
\end{align*}$$

Taking conditional expectation on both side, we obtain

$$\begin{align*}
|Y_{t}^{1,k} - Y_{t}^{1,j}|^\alpha &+ \alpha \mathbb{E}_t \int_t^T |Y_{t}^{1,k} - Y_{t}^{1,j}|^{\alpha - 2} |Z_{t}^{1,k} - Z_{t}^{1,j}|^2 d(B)_s \\
&\leq \alpha \mathbb{E}_t \int_t^T |Y_{s}^{1,k} - Y_{s}^{1,j}|^{\alpha - 1} f^1_k(s, w, Y_{s}^{1,k}, Z_{s}^{1,k}) - f^1_j(s, w, Y_{s}^{1,j}, Z_{s}^{1,j}) | ds.
\end{align*}$$

(3.12)
By Lemma 2.4, for $0 < \varepsilon < \frac{4 - \alpha}{3\alpha - 4}$, we have
\[
\mathbb{E}[\sup_{0 \leq t \leq T} |Y_{t_{1}}^{1,k} - Y_{t_{1}}^{1,j}|^\alpha] \\
\leq \alpha \mathbb{E}\left[ \int_{0}^{T} |Y_{s}^{1,k} - Y_{s}^{1,j}|^{\frac{\alpha}{2}} |Y_{s}^{1,k} - Y_{s}^{1,j}|^{\frac{\alpha}{2}} \left| f_{k}^{1}(s, w, Y_{s}^{1,k}, Z_{s}^{1,k}) - f_{k}^{1}(s, w, Y_{s}^{1,j}, Z_{s}^{1,j}) \right| ds \right]^{1 + \varepsilon} \\
\leq \alpha (\mathbb{E}\left[ \int_{0}^{T} |Y_{s}^{1,k} - Y_{s}^{1,j}|^{\alpha} ds \right]^{\frac{2}{1 + \varepsilon}} \\
\cdot \mathbb{E}\left[ \sup_{t \in [0, T]} |Y_{t_{1}}^{1,k} - Y_{t_{1}}^{1,j}|^{\frac{2}{1 + \varepsilon}} \left( \int_{0}^{T} |f_{k}^{1}(s, w, Y_{s}^{1,k}, Z_{s}^{1,k}) - f_{k}^{1}(s, w, Y_{s}^{1,j}, Z_{s}^{1,j})|^{2} ds \right)^{\frac{1}{1 + \varepsilon}} \right]^{\frac{1}{1 + \varepsilon}} \\
\leq \alpha (\mathbb{E}\left[ \int_{0}^{T} |Y_{s}^{1,k} - Y_{s}^{1,j}|^{\alpha} ds \right]^{\frac{2}{1 + \varepsilon}} \\
\cdot \left\{ (\mathbb{E}\left[ \sup_{t \in [0, T]} |Y_{t_{1}}^{1,k} - Y_{t_{1}}^{1,j}|^{\alpha} \right]^{\frac{2}{1 + \varepsilon}} \mathbb{E}\left[ \int_{0}^{T} |f_{k}^{1}(s, w, Y_{s}^{1,k}, Z_{s}^{1,k}) - f_{k}^{1}(s, w, Y_{s}^{1,j}, Z_{s}^{1,j})|^{2} ds \right]^{\frac{\alpha}{2(1 + \varepsilon)}} \right\}^{\frac{1}{1 + \varepsilon}} \\
\leq \alpha \mathbb{E}\left[ \int_{0}^{T} |Y_{s}^{1,k} - Y_{s}^{1,j}|^{\alpha} ds \right]^{\frac{2}{1 + \varepsilon}} \\
\cdot \left\{ (\mathbb{E}\left[ \sup_{t \in [0, T]} |Y_{t_{1}}^{1,k} - Y_{t_{1}}^{1,j}|^{\alpha} \right]^{\frac{2}{1 + \varepsilon}} \mathbb{E}\left[ \int_{0}^{T} |f_{k}^{1}(s, w, Y_{s}^{1,k}, Z_{s}^{1,k}) - f_{k}^{1}(s, w, Y_{s}^{1,j}, Z_{s}^{1,j})|^{2} ds \right]^{\frac{\alpha}{2(1 + \varepsilon)}} \right\}^{\frac{1}{1 + \varepsilon}}.
\]
\[
\leq C\{\mathbb{E}\left[\sup_{t \in [0,T]} |Y_s^{1,k} - Y_s^{1,j}|^\alpha\right]\}^{\frac{1}{\alpha}}\mathbb{E}\left[\left(\int_0^T |f^k(s, w, Y_s^{1,k} , Z_s^{1,k} ) - f^j(s, w, Y_s^{1,j} , Z_s^{1,j} )|^2 ds\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}
+ \mathbb{E}\left[\sup_{t \in [0,T]} |Y_s^{1,k} - Y_s^{1,j}|^\alpha\right]^{\frac{1}{\alpha}}\mathbb{E}[A_{T}^{1,k}]^{\frac{1}{2}} + \mathbb{E}[A_{T}^{1,j}]^{\frac{1}{2}} + \mathbb{E}[\sup_{t \in [0,T]} |Y_s^{1,k} - Y_s^{1,j}|^\alpha]\}
\leq C\{\mathbb{E}\left[\sup_{t \in [0,T]} |Y_s^{1,k} - Y_s^{1,j}|^\alpha\right] + (\mathbb{E}\left[\sup_{t \in [0,T]} |Y_s^{1,k} - Y_s^{1,j}|^\alpha\right])^{\frac{1}{2}}\}.
\]

It is straightforward to show that
\[
\lim_{j,k \to \infty} \mathbb{E}[\int_0^T |Z_s^{1,k} - Z_s^{1,j}|^2 ds] = 0.
\]

Then there exists a process \(Z_t^1 \in H_{\alpha}^2(0,T)\) such that \(\mathbb{E}[\int_0^T |Z_s^{1,k} - Z_s^{1,j}|^2 ds] \to 0\), as \(k \to \infty\).

Similar to that in (3.6), it is easy to see that
\[
\mathbb{E}\left[\left|\int_0^T f^k_1(s, w, Y_s^{1,k} , Z_s^{1,k} ) - f^j_1(s, w, Y_s^{1,j} , Z_s^{1,j} )\right| ds\right] \to 0, \quad \text{as} \quad k \to \infty.
\]

Meanwhile, since
\[
A_t^{1,k} - A_t^{1,j} = Y_t^{1,k} - Y_t^{1,j} - (Y_t^{1,k} - Y_t^{1,j})
- \int_0^t [f^k_1(s, w, Y_s^{1,k} , Z_s^{1,k} ) - f^j_1(s, w, Y_s^{1,j} , Z_s^{1,j} )] ds + \int_0^t (Z_s^{1,k} - Z_s^{1,j}) dB_s.
\]

Then by Lemma 2.1, (3.13), (3.16) and (3.17), we have
\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} |A_t^{1,k} - A_t^{1,j}|^\alpha\right]
\leq C\{\mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_s^{1,k} - Y_s^{1,j}|^\alpha\right] + \mathbb{E}[\int_0^T |Z_s^{1,k} - Z_s^{1,j}|^2 ds]^{\frac{1}{2}}\} \to 0,
\]
as \(k, j \to \infty\), which implies that \(\{A_t^{1,k}\}_{k \geq 0}\) is a Cauchy sequence in \(S_{\alpha}^2(0,T)\). We denote its limit as \(A_t^1\), i.e., \(\mathbb{E}\left[\sup_{0 \leq t \leq T} |A_t^{1,k} - A_t^1|^\alpha\right] \to 0\) as \(k \to \infty\). It is easy to see that \(A_0^1 = 0\) and \(A_t^1\) is a non-decreasing process, since the sequence \(\{A_t^{1,k}\}_{k \geq 0}\) have the property.

Taking limit on both side of (3.7), then we have
\[
Y_t^1 = \xi + \int_0^T f^1(s, w, Y_s^1 , Z_s^1 ) ds - \int_0^T Z_s^1 dB_s + A_t^1 - A_t^1.
\]

Moreover, \((Y^1, Z^1, A^1) \in S_{\alpha}^2(0,T), \ 2 \leq \alpha < \beta\).

In the following it remains to prove that \(\{-\int_0^t (Y_s^1 - L_s) dA_s^1\}_{t \in [0,T]}\) is a non-increasing \(G\)-martingale. Notice that \(\{-\int_0^t (Y_s^1 - L_s) dA_s^1\}_{t \in [0,T]}\) is a non-increasing \(G\)-martingale. By H"{o}lder inequality, (3.10), (3.13) and (3.18), it follows that
\[
\hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} | -\int_0^t (Y_s^1 - L_s) dA_s^1 - (-\int_0^t (Y_s^1 - L_s) dA_s^1)|\right]
\leq \hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} |\int_0^t (Y_s^1 - Y_s^{1,k}) dA_s^1|\right] + \hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} |\int_0^t (Y_s^1 - L_s) d(A_s^1 - A_s^{1,k})|\right].
\]
Next, using the same method as that on a martingale. Indeed, note that

\[ \left( \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^1 - Y_{t-1}^1|^2 \right] \right)^{1/2} \left( \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^1 - L_t|^2 \right] \right)^{1/2} \rightarrow 0, \]

as \( k \to \infty \), which implies that \( \{ - \int_0^T (Y_t^1 - L_t) dA_t \}_{t \in [0, T]} \) is a non-increasing \( G \)-martingale.

Finally we obtain a triple \( (Y_t^1, Z_t^1, A_t^1) \) is satisfying the following equation:

\[
\begin{align*}
Y_t^1 &= \xi + \int_t^T f^1(s, w, Y_s^1, Z_s^1) ds - \int_t^T Z_s^1 dB_s + (A_t^1 - A_t), \\
Y_t^1 &\geq L_t, \{ - \int_0^T (Y_s^1 - L_s) dA_s^1 \}_{t \in [0, T]} \text{ is a non-increasing } G \text{-martingale.}
\end{align*}
\]

Next, using the same method as that on \( X^0 \), we can construct \( X^1 \) based on \( Y^1 \). Moreover, since \( b \) is monotonic on \( y \) and \( Y^0 \leq Y^1 \), we have through the comparison on GSDEs that \( X^0 \leq X^1 \).

Repeating the same procedure, we get the existence of a sequence \( (X^n, Y^n, Z^n, A^n) \) which is a solution of (3.2) and for any \( t \leq T, n \in \mathbb{N}, \)

\[ X_t^n \leq X_{t+1}^n \leq S_t, \quad L_t \leq Y_t^n \leq Y_{t+1}^n \leq U_t, \text{q.s.} \]

Moreover, \( X_t^n \in M^\beta(0, T), (Y_t^n, Z_t^n, A_t^n) \in S^\beta(0, T), A_t^n \) is a nondecreasing process with \( A^n_0 = 0 \), where in each step of iteration, we denote \( f^n(s, w, y, z) = f(s, X^n_{s-1}, y, z) \).

By the G-Itô’s formula to \( e^{\lambda t}|Y_t^n - c|^2 \), here \( c \) is the constant in (H5) and \( \lambda > 0 \) is a constant, we have

\[
\begin{align*}
M_t^n - M_t^0 + e^{\lambda t}|Y_t^n - c|^2 &+ \lambda \int_t^T e^{\lambda s}|Y_s^n - c|^2 ds + \int_t^T e^{\lambda s}|Z_s^n|^2 d(B)_s \\
&\leq e^{\lambda T}|\xi - c|^2 + 2 \int_t^T e^{\lambda s}(Y_s^n - c)f(s, X_s^n, Y_s^n, Z_s^n) ds
\end{align*}
\]

where \( M_t^n = \int_0^t e^{\lambda s}(Y^n_s - c)Z^n_s dB_s - \int_0^t e^{\lambda s}|Y_t^n - c|dA_t^n \). We claim that \( \{M_t^n\}_{t \in [0, T]} \) is a \( G \)-martingale. Indeed, note that

\[ 0 \geq - \int_t^T |Y_s^n - c|dA_s^n \geq - \int_t^T (Y_s^n - L_t)dA_s^n. \]

Thus we can conclude that

\[ 0 \geq \mathbb{E}[ - \int_t^T |Y_s^n - c|dA_s^n ] \geq \mathbb{E}[ - \int_t^T (Y_s^n - L_t)dA_s^n ] = 0. \]

It follows that \( \{M_t^n\}_{t \in [0, T]} \) is a \( G \)-martingale.

Since \( f \) satisfies the linear growth condition (H3), we have

\[
\begin{align*}
2 \int_t^T e^{\lambda s}|Y_t^n - c|f(s, X_s^n, Y_s^n, Z_s^n) ds \\
&\leq M \int_t^T e^{\lambda s}(c + 1)^2 ds + \left( M + \frac{2M^2}{\sigma^2} \right) \int_t^T e^{\lambda s}|Y_s^n - c|^2 ds + \frac{1}{2} \int_t^T e^{\lambda s}|Z_s^n|^2 d(B)_s.
\end{align*}
\]

Let \( \lambda = M + \frac{2M^2}{\sigma^2} + 1 \), then

\[
M_t^n - M_t^0 + e^{\lambda t}|Y_t^n - c|^2 + \frac{1}{2} \int_t^T e^{\lambda s}|Z_s^n|^2 d(B)_s
\]
\[
\leq e^{\lambda T}|\xi - c|^2 + M \int_t^T e^{\lambda s}(c + 1)^2 ds \tag{3.22}
\]

Taking conditional expectation on both side of (3.22), we have
\[
\mathbb{E}_t[\int_t^T e^{\lambda s}|Z^n_s|^2 d(B)_s] \leq 2\mathbb{E}_t[e^{\lambda T}|\xi - c|^2] + 2\mathbb{E}_t[M \int_t^T e^{\lambda s}(c + 1)^2 ds]. \tag{3.23}
\]

Taking expectation on both side of (3.23), we have
\[
2\mathbb{E}_t[\int_t^T e^{\lambda s}|Z^n_s|^2 ds] \leq 2\mathbb{E}_t[e^{\lambda T}|\xi - c|^2] + 2\mathbb{E}_t[M \int_t^T e^{\lambda s}(c + 1)^2 ds] < C, \tag{3.24}
\]

where \(C\) is a constant independent of \(n\), which implies that \(\{Z^n_t\}_{t \in [0,T]}\) is a bounded process in \(L^2_G(0,T)\) independent of \(n\).

Obviously, since \(\{X^n_t\}_{n \in \mathbb{N}}\) and \(\{Y^n_t\}_{n \in \mathbb{N}}\) are increasing and bounded, then there exist two semi-continuous processes \(\{X_t\}_{t \in [0,T]}\) and \(\{Y_t\}_{t \in [0,T]}\) such that
\[
X_t = \lim_{n \to \infty} X^n_t, \quad Y_t = \lim_{n \to \infty} Y^n_t.
\]

Moreover, we have \(|X^n_t - X_t| \downarrow 0\) and \(|Y^n_t - Y_t| \downarrow 0\) as \(n \to \infty\). Notice that \(\{X^n_t\}_{n \in \mathbb{N}}\) and \(\{Y^n_t\}_{n \in \mathbb{N}}\) are bounded by \(S_t\) and \(U_t\) respectively, which imply that \(X_t, Y_t \in S^2_G(0,T)\). They also belong to \(L^2_G(\Omega_T)\), which is a larger space. Thanks to the Lemma 2.3, we have
\[
\mathbb{E}[|X^n_t - X_t|^2] \downarrow 0, \quad \mathbb{E}[|Y^n_t - Y_t|^2] \downarrow 0, \tag{3.25}
\]
as \(n \to \infty\). Therefore, by the Lebesgue dominated convergence with respect to \(t\), we have
\[
\int_0^T \mathbb{E}[|X^n_t - X_t|^2] dt \to 0, \quad \int_0^T \mathbb{E}[|Y^n_t - Y_t|^2] dt \to 0,
\]
as \(n \to \infty\), which imply that \(X^n_t \to X_t, Y^n_t \to Y_t\) in \(M^2_G(0,T)\).

Now taking the limit on the first equation in (3.2) and using the same method as that on \(X^n_t\), we conclude that \(X \in M^2_G(0,T), Y \in M^2_G(0,T)\) is a solution to the following forward GSDE:
\[
X_t = x + \int_0^t b(s, X_s, Y_s) ds + \int_0^t \sigma(s, X_s) dB_s, \tag{3.26}
\]
which also implies that \(X\) is continuous in \(t\).

Let us focus on the backward equation part of (3.2). For any \(n \geq 1\), we have
\[
Y^n_t = \xi + \int_t^T f(s, X^{n-1}_s, Y^n_s, Z^n_s) ds - \int_t^T Z^n_s dB_s + (A^n_T - A^n_t).
\]

By (3.24), there exists a real positive constant \(C\) independent of \(n\) such that \(\mathbb{E}[\int_0^T |Z^n_s|^2 ds] \leq C\). Therefore the sequence \(\{Z^n\}_{n \in \mathbb{N}}\) is a Cauchy type in \(M^2_G(0,T)\). Actually, applying G-Itô’s formula to \(|Y^n_t - Y^n|\), we can obtain
\[
|Y^n_t - Y^n| + \int_t^T |Z^n_s - Z^n_s|^2 d(B)_s
\]
\[ = 2 \int_t^T (Y^m_s - Y^n_s)[f(s, X^{m-1}_s, Y^m_s, Z^m_s) - f(s, X^{n-1}_s, Y^n_s, Z^n_s)]ds + 2 \int_t^T (Y^m_s - Y^n_s)d(A^m_s - A^n_s) - 2 \int_t^T Y^m_s - Y^n_s)(Z^m_s - Z^n_s)dB_s \]
\[ \leq 2 \int_t^T (Y^m_s - Y^n_s)[f(s, X^{m-1}_s, Y^m_s, Z^m_s) - f(s, X^{n-1}_s, Y^n_s, Z^n_s)]ds + 2 \int_t^T (Y^m_s - Y^n_s)^+dA^m_s - 2 \int_t^T (Y^m_s - Y^n_s)^-(dA^n_s - 2 \int_t^T (Y^m_s - Y^n_s)(Z^m_s - Z^n_s)dB_s. \quad (3.27) \]

Let \( M_t^{m,n} = 2 \int_0^t (Y^m_s - Y^n_s)(Z^m_s - Z^n_s)dB_s - 2 \int_0^t (Y^m_s - Y^n_s)^+dA^m_s - 2 \int_0^t (Y^m_s - Y^n_s)^-dA^n_s \), which is a \( G \)-martingale. Then we have

\[ M_t^{m,n} - M_t^{m,n} + |Y^m_t - Y^n_t|^2 + \int_t^T |Z^m_s - Z^n_s|^2d(B)_s \]
\[ \leq 2 \int_t^T (Y^m_s - Y^n_s)[f(s, X^{m-1}_s, Y^m_s, Z^m_s) - f(s, X^{n-1}_s, Y^n_s, Z^n_s)]ds. \]

Taking expectation we may conclude

\[ \mathbb{E}^2 \mathbb{E}^2 \left[ \int_0^T |Z^m_s - Z^n_s|^2ds \right] \leq \mathbb{E}^2 \left[ \int_0^T |Z^m_s - Z^n_s|^2d(B)_s \right] \]
\[ \leq \left( \int_0^T \mathbb{E} |Y^m_s - Y^n_s|^2ds \right) \mathbb{E} \left[ \int_0^T \mathbb{E} |f(s, X^{m-1}_s, Y^m_s, Z^m_s) - f(s, X^{n-1}_s, Y^n_s, Z^n_s)|^2ds \right] \mathbb{E} \left[ \int_0^T (Z^m_s - Z^n_s)d(B)_s \right]. \quad (3.28) \]

Notice that \(|f(t, x, y, z)| \leq K(1 + |y| + |z|), \{Z^n\}_{n \in \mathbb{N}} \text{ is bounded in } M^2_G(0, T), \{X^n\}_{n \in \mathbb{N}} \text{ and } \{Y^n\}_{n \in \mathbb{N}} \text{ are bounded by the process } S \text{ and } U \text{ respectively, then the second factor in above inequality is bounded by constant independent of } m \text{ and } n. \text{ Therefore, } \{Z^n\}_{n \in \mathbb{N}} \text{ is a Cauchy sequence in } M^2_G(0, T) \text{ through the convergence of } \{Y^n\}_{n \in \mathbb{N}} \text{ to } Y \text{ in } M^2_G(0, T). \text{ So let us set } Z_t := \lim_{n \to \infty} Z^n_t \text{ in } M^2_G(0, T). \]

By (3.2), we have

\[ A^n_t = Y^n_0 - Y^n_t - \int_0^t f(s, X^{n-1}_s, Y^n_s, Z^n_s)ds + \int_0^t Z^n_sdB_s. \]

Thus

\[ \mathbb{E}||A^n_t - A^n_t||^2 \leq 4\mathbb{E}||Y^n_0 - Y^n_t||^2 + |Y^n_t - Y^n_t|^2 \]
\[ + T \int_0^T \mathbb{E}|f(s, X^{n-1}_s, Y^n_s, Z^n_s) - f(s, X^{n-1}_s, Y^n_s, Z^n_s)|^2ds + \mathbb{E}||\int_0^T (Z^m_s - Z^n_s)d(B)_s||^2 \]
\[ =: 4(I_1 + I_2 + I_3). \quad (3.29) \]

By (3.25), we have

\[ I_1 \leq \mathbb{E}||Y^n_0 - Y_0|| + \mathbb{E}||Y^n_0 - Y_0||^2 + \mathbb{E}||Y^n_t - Y^n_t||^2 + \mathbb{E}||Y^n_t - Y^n_t||^2 \to 0. \quad (3.30) \]

as \( m, n \to \infty. \)
By Lemma 2.1 and \( \{Z^n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( M_G^2(0,T) \), we have

\[
I_3 \leq \sigma^2 \mathbb{E} \int_0^T |Z^n_t - Z^n_t|^2 ds \to 0,
\]

as \( m,n \to \infty \).

By Lemma 2.5 and \( \{X^n\}_{n \in \mathbb{N}}, \{Y^n\}_{n \in \mathbb{N}} \) and \( \{Z^n\}_{n \in \mathbb{N}} \) are convergence in \( M_G^2(0,T) \)

\[
I_2 \leq 3T \int_0^T \mathbb{E}[(f(s, X^{m-1}_s, Y^n_s, Z^n_s) - f_k(s, X^{m-1}_s, Y^n_s, Z^n_s))^2] ds \\
+ \int_0^T \mathbb{E}[|f_k(s, X^{m-1}_s, Y^n_s, Z^n_s) - f_k(s, X^{n-1}_s, Y^n_s, Z^n_s)|^2] ds \\
+ \int_0^T \mathbb{E}[|f_k(s, X^{n-1}_s, Y^n_s, Z^n_s) - f(s, X^{n-1}_s, Y^n_s, Z^n_s)|^2] ds \to 0,
\]

as \( k,m,n \to \infty \), where \( f_k \) is the sequence defined in Lemma 2.5.

Combining (3.29)-(3.32), then \( \{A^n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( L_G^3(\Omega_T) \). Let us set \( A_t := \lim_{n \to \infty} A^n_t \) in \( L_G^2(\Omega_T) \). From this, we can show that \( \{Y^n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( S_G^2(\Omega_T) \).

Actually, by (3.27), we have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n_t - Y^n_t|^2 \right] \\
= 2 \mathbb{E} \left[ \int_0^T (Y^n_s - Y^n_s)(f(s, X^{m-1}_s, Y^n_s, Z^n_s) - f(s, X^{n-1}_s, Y^n_s, Z^n_s)) ds \right] \\
+ 2 \mathbb{E} \left[ \int_0^T (Y^n_s - Y^n_s)d(A^n - A^n_s) \right] + \mathbb{E} \left[ \int_0^T (Y^n_s - Y^n_s)(Z^n_s - Z^n_s) dB_s \right] \\
\leq 2(\mathbb{E} \left[ \int_0^T |Y^n_t - Y^n_t|^2 ds \right])^{\frac{1}{2}} \left( \mathbb{E} \left[ \int_0^T |f(s, X^{m-1}_s, Y^n_s, Z^n_s) - f(s, X^{n-1}_s, Y^n_s, Z^n_s)|^2 ds \right] \right)^{\frac{1}{2}} \\
+ 2 \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n_t - Y^n_t| \cdot |A^n - A^n_s| \right] + \mathbb{E} \left[ \int_0^T |Y^n_t - Y^n_t|^2 |Z^n_s - Z^n_s|^2 ds \right]^{\frac{1}{2}} \\
\leq 2(\mathbb{E} \left[ \int_0^T |Y^n_t - Y^n_t|^2 ds \right])^{\frac{1}{2}} \left( \mathbb{E} \left[ \int_0^T |f(s, X^{m-1}_s, Y^n_s, Z^n_s) - f(s, X^{n-1}_s, Y^n_s, Z^n_s)|^2 ds \right] \right)^{\frac{1}{2}} \\
+ \varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n_t - Y^n_t|^2 \right] + \frac{1}{2} \mathbb{E} \left[ |A^n - A^n_s|^2 \right] + \varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n_t - Y^n_t|^2 \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T |Z^n_s - Z^n_s|^2 ds \right],
\]

where in the above inequalities we have used the Hölder inequality, Lemma 2.1 and Young inequality. Choosing \( \varepsilon \) small enough, then we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} |Y^n_t - Y^n_t|^2 \\
\leq C(\mathbb{E} \left[ \int_0^T |Y^n_t - Y^n_t|^2 ds \right])^{\frac{1}{2}} + \frac{1}{2} \mathbb{E} \left[ |A^n - A^n_s|^2 \right] \to 0,
\]

as \( m,n \to \infty \), which implies that \( Y^n_t \to Y_t \) in \( S_G^2(0,T) \) as \( n \to \infty \). Going back to (3.29), it is easy to see that \( A^n_t \to A_t \) in \( S_G^2(0,T) \) as \( n \to \infty \).
Remark 3.1. The same proof works also when the terminal condition \(b\) is replaced by \(\Phi(x)\), where \(\Phi\) is a continuous bounded increasing function.

Remark 3.2. In our assumptions, \(f\) has sublinear growth independent of \(x\). If we assume that
\[
|f(t,x,y,z)| \leq M(1 + |x| + |y| + |z|), \forall t \in [0,T], x,y,z \in \mathbb{R},
\]
then \(b\) should have a sublinear growth independent of \(y\), i.e.
\[
b(s,x,y) \leq M(1 + |x|).
\]
We can also construct a sequence of Lipschitz-continuous functions to approximate them. However, in this case, we should first construct a solution of some forward GSDEs, which is different from that in this paper.

Theorem 3.2. Suppose that \(\xi, b, h, f, g, \sigma\) satisfy (H1)-(H4), \(L\) satisfies (H5). Then the RF-BGSDE (1.1) has at least one solution \((X,Y,Z,A)\).

Remark 3.1. The same proof works also when the terminal condition \(\xi\) is replaced by \(\Phi(\xi)\), where \(\Phi\) is a continuous bounded increasing function.

References

[1] F. Antonelli, S. Hammadène, Existence of solutions of backward-forward SDEs with continuous monotone coefficients, Statist. Probab. Lett., 76(2006), 1559-1569.
[2] X. Bai, Y. Lin, On the existence and uniqueness of solutions to stochastic differential equations driven by $G$-Brownian motion with integral-Lipschitz coefficients, Acta. Math. Appl. Sin. Engl. Ser., 30(2014), 589-610.

[3] L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: application to $G$-Brownian motion paths, Potential Anal., 34(2011), 139-161.

[4] F.Q. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by $G$-Brownian motion, Stochastic Process. Appl., 119(2009), 3356-3382.

[5] Y. Hu, N-person differential games governed by semilinear stochastic evolution systems. Appl. Math. Optim., 24(1991), 257-271.

[6] M. Hu, S. Ji, S. Peng, On representation theorem of $G$-expectation and paths of $G$-Brownian motion, Acta. Math. Appl. Sin. Engl. Ser., 25(2009), 539-546.

[7] M. Hu, S. Ji, S. Peng, Y. Song, Backward stochastic differential equation driven by $G$-Brownian motion, Stochastic Process. Appl., 124(2014), 759-784.

[8] Z. Huang, J. Lepeltier, Z. Wu, Reflected forward-backward differential equations with continuous monotone coefficients, Statist. Probab. Lett., 80(2010), 1569-1576.

[9] P. Luo, F. Wang, Stochastic differential equations driven by $G$-Brownian motion and ordinary differential equations, Stochastic Process. Appl., 124(2013), 3869-3885.

[10] Y. Lin, Stochastic differential equations driven by $G$-Brownian motion with reflecting boundary conditions, Electron. J. Probab., 18(2013), 1-23.

[11] X. Li, S. Peng, Stopping times and related Itô’s calculus with $G$-Brownian motion, Stochastic Process. Appl., 121(2011), 1492-1508.

[12] H. Li, S. Peng, Reflected solutions of BSDEs driven by $G$-Brownian motion, 2017, arXiv:1705.10973v1.

[13] J. Lepeltier, J. San Martin, Backward stochastic differential equations with continuous coefficients, Statist. Probab. Lett., 34(1997), 425-430.

[14] S. Peng, Nonlinear expectations and nonlinear Markov chains, Chinese Ann. Math., 26B(2005), 159-184.

[15] S. Peng, Nonlinear expectations and stochastic calculus under uncertainty, 2010. arXiv:1002.4546v1.

[16] S. Peng, $G$-expectation, $G$-Brownian motion and related stochastic calculus of Itô type, 2006. arXiv:math/0601035v1.

[17] S. Peng, Y. Song and J. Zhang, A complete representation theorem for $G$-martingales, Stochastics, 86(2014), 609-631.
[18] Y. Ren, Q. Bi, R. Sakthivel, Stochastic functional differential equation with infinite delay driven by $G$-Brownian motion, Mathe. Meth. Appl. Sci., 36 (2013), 1746-1759.

[19] M. Soner, N. Touzi and J. Zhang, Martingale representation theorem for the $G$-expectation, Stochastic Process. Appl., 121(2011), 265-287.

[20] Y. Song, Some properties on $G$-evaluation and its applications to $G$-martingale decomposition, Sci. China Math., 54(2011), 287-300.