On Hermite-Hadamard Type Inequalities Via Fractional Integral Operators

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Abstract. In this paper, we give new definitions related to fractional integral operators for two variables functions using the class of integral operators. We are interested to give the Hermite–Hadamard inequality for a rectangle in plane via convex functions on co-ordinates involving fractional integral operators.

1. Introduction

The most well-known inequality related to the integral mean of a convex function is the Hermite-Hadamard inequality. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be convex function defined on the interval \( I \) of real numbers and \( a, b \in I \), with \( a < b \). Then the following double inequality is known in the literature as the Hermite-Hadamard’s inequality for convex functions [8]:

\[
 f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]  

(1)

The inequalities (1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Many generalizations and extensions of the Hermite-Hadamard inequality exist in the literature; (see [3],[4] and [30]) and references therein.

Let us consider a bidimensional interval \( \Delta =: [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). A function \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) is said to be convex on \( \Delta \) if for all \( (x,y),(z,w) \in \Delta \) and \( t \in [0,1] \), it satisfies the following inequality:

\[
 f(tx + (1-t)z, ty + (1-t)w) \leq tf(x,y) + (1-t)f(z,w).
\]

A modification for convex function on \( \Delta \) was defined by Dragomir [29], as follows:

A function \( f : \Delta \to \mathbb{R} \) is said to be convex on the co-ordinates on \( \Delta \) if the partial mappings \( f_y : [a,b] \to \mathbb{R} \), \( f_y(u) = f(u,y) \) and \( f_z : [c,d] \to \mathbb{R} \), \( f_z(v) = f(x,v) \) are convex where defined for all \( x \in [a,b] \) and \( y \in [c,d] \).

A formal definition for co-ordinated convex function may be stated as follows:

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Definition 1.1. A function $f : \Delta \to \mathbb{R}$ is called co-ordinated convex on $\Delta$, for all $(x, u), (y, v) \in \Delta$ and $t, s \in [0, 1]$, if it satisfies the following inequality:

\[
f(tx + (1-t)y, su + (1-s)v) \leq ts f(x, u) + t(1-s)f(x, v) + s(1-t)f(y, u) + (1-t)(1-s)f(y, v).
\] (2)

Note that every convex function $f : \Delta \to \mathbb{R}$ is co-ordinated convex but the converse is not generally true (see, [29]).

In [29], Dragomir proved the following inequality which is Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2. Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the following inequalities:

\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) dx + \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy \right] \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx \leq \frac{1}{4} \left[ \frac{1}{b - a} \int_a^b f(x, c) dx + \frac{1}{b - a} \int_a^b f(x, d) dx + \frac{1}{d - c} \int_c^d f(a, y) dy + \frac{1}{d - c} \int_c^d f(b, y) dy \right] \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\]

The above inequalities are sharp.

For recent developments about Hermite-Hadamard’s inequality for some convex functions on the co-ordinates, please refer to ([2],[5],[6],[11]-[13] and [15]-[23]). Also for several inequalities for convex functions on the co-ordinates see the references ([9],[10],[14],[24] and [25]).

In [28], Raina defined the following results connected with the general class of fractional integral operators:

\[
F_{\rho, \lambda}^\sigma (x) = F_{\rho, \lambda}^{\sigma(0), \sigma(1), \ldots} (x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(pk + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < 2), \quad (3)
\]

where the coefficients $\sigma(k)$ ($k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) is a bounded sequence of positive real numbers and $\mathbb{R}$ is the set of real numbers. With the help of (3), Raina and Agarwal et al. defined the following left-sided and right-sided fractional integral operators respectively, as follows:

\[
J_{\rho, \lambda, a, b}^\sigma \phi(x) = \int_a^x (x-t)^{\lambda-1} F_{\rho, \lambda}^{\sigma} \left[ \omega (x-t)^\sigma \right] \phi(t) dt, \quad x > a,
\] (4)
\[ J_{\rho,\lambda, b^{-\omega}, \omega}^{\sigma, \rho, \lambda} \psi(x) = \int_{x}^{b} (t - x)^{\lambda - 1} F_{\rho,\lambda}^{\sigma} \left[ \omega (t - x)^{\rho} \right] \psi(t) dt, \quad x < b, \]

where \( \lambda, \rho > 0, \omega \in \mathbb{R} \), and \( \psi(t) \) is such that the integrals on the right side exists.

It is easy to verify that \( J_{\rho,\lambda, a^{+\omega}, \omega}^{\sigma, \rho, \lambda} \psi(x) \) and \( J_{\rho,\lambda, b^{-\omega}, \omega}^{\sigma, \rho, \lambda} \psi(x) \) are bounded integral operators on \( L(a, b) \), if

\[ M := F_{\rho,\lambda+1}^{\sigma, \rho, \lambda+1} \left[ \omega (b - a)^{\rho} \right] < \infty. \]

In fact, for \( \psi \in L(a, b) \), we have

\[ \left\| J_{\rho,\lambda, a^{+\omega}, \omega}^{\sigma, \rho, \lambda} \psi(x) \right\| \leq M (b - a)^{\lambda} \left\| \psi \right\| \]

and

\[ \left\| J_{\rho,\lambda, b^{-\omega}, \omega}^{\sigma, \rho, \lambda} \psi(x) \right\| \leq M (b - a)^{\lambda} \left\| \psi \right\|, \]

where

\[ \left\| \psi \right\|_{p} := \left( \int_{a}^{b} |\psi(t)|^{p} dt \right)^{\frac{1}{p}}. \]

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient \( \sigma(k) \). Here, we just point out that the classical Riemann-Liouville fractional integrals \( I_{a^{+\omega}}^{\alpha} \) and \( I_{b^{-\omega}}^{\alpha} \) of order \( \alpha \) defined by (see, [1, 27, 31])

\[ (I_{a^{+\omega}}^{\alpha} \psi)(x) := \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (t - x)^{\alpha - 1} \psi(t) dt, \quad (x > a; \alpha > 0) \]

and

\[ (I_{b^{-\omega}}^{\alpha} \psi)(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t - x)^{\alpha - 1} \psi(t) dt, \quad (x < b; \alpha > 0) \]

follow easily by setting

\[ \lambda = \alpha, \quad \sigma(0) = 1, \quad \text{and} \quad \omega = 0 \]

in (4) and (5), and the boundedness of (9) and (10) on \( L(a, b) \) is also inherited from (7) and (8), (see, [26]).

In [7], Yaldız and Sarıkaya proved the following inequality which is Hermite-Hadamard inequality for fractional integral operators:

**Theorem 1.3.** Let \( \psi : [a, b] \to \mathbb{R} \) be a convex function on \( [a, b] \) with \( a < b \), then the following inequalities for fractional integral operators hold:

\[ \psi \left( \frac{a + b}{2} \right) \leq \frac{1}{2(b - a)^{\lambda}} F_{\rho,\lambda+1}^{\sigma, \rho, \lambda+1} \left[ \omega (b - a)^{\rho} \right] \left[ J_{\rho,\lambda,a^{+\omega}}^{\sigma, \rho, \lambda} \psi(b) + J_{\rho,\lambda,b^{-\omega}}^{\sigma, \rho, \lambda} \psi(a) \right] \leq \frac{\psi(a) + \psi(b)}{2} \]

with \( \lambda > 0 \).
Now, we establish new definitions related to fractional integral operators for two variables functions:

**Definition 1.4.** Let $f \in L^1([a, b] \times [c, d])$. The fractional integral operators for two variables functions with $p = (p_1, p_2)$, $\lambda = (\lambda_1, \lambda_2)$, $\rho, \lambda, w, \sigma, \alpha, \alpha_2 \in \mathbb{R}^2$; $a, b, c, d \geq 0$ defined by

$$
\mathcal{J}_{p, \lambda, a, x, p_1, \lambda_1, \rho, \lambda, w} f(x, y) := \int_a^x \int_c^y \frac{(x-t)^{\lambda_1-1} (y-t)^{\lambda_2-1} \mathcal{F}_{p_1, \lambda_1} \left[ \omega_1 (x-t)^{p_1} \right] \mathcal{F}_{p_2, \lambda_2} \left[ \omega_2 (y-t)^{p_2} \right]}{f(t,s)dsdt}, (x > a, y > c);
$$

$$
\mathcal{J}_{p, \lambda, a, b-\rho, \lambda_1, \rho, \lambda, w} f(x, y) := \int_a^x \int_y^d \frac{(x-t)^{\lambda_1-1} (s-y)^{\lambda_2-1} \mathcal{F}_{p_1, \lambda_1} \left[ \omega_1 (x-t)^{p_1} \right] \mathcal{F}_{p_2, \lambda_2} \left[ \omega_2 (s-y)^{p_2} \right]}{f(t,s)dsdt}, (x > a, y < d);
$$

$$
\mathcal{J}_{p, \lambda, b-\rho, \lambda_2, \rho, \lambda, w} f(x, y) := \int_b^x \int_c^y \frac{(t-x)^{\lambda_1-1} (y-s)^{\lambda_2-1} \mathcal{F}_{p_1, \lambda_1} \left[ \omega_1 (t-x)^{p_1} \right] \mathcal{F}_{p_2, \lambda_2} \left[ \omega_2 (y-s)^{p_2} \right]}{f(t,s)dsdt}, (x < b, y > c)
$$

and

$$
\mathcal{J}_{p, \lambda, b-\rho, \lambda_2, \rho, \lambda, w} f(x, y) := \int_b^x \int_y^d \frac{(t-x)^{\lambda_1-1} (s-y)^{\lambda_2-1} \mathcal{F}_{p_1, \lambda_1} \left[ \omega_1 (t-x)^{p_1} \right] \mathcal{F}_{p_2, \lambda_2} \left[ \omega_2 (s-y)^{p_2} \right]}{f(t,s)dsdt}, (x < b, y < d).
$$

Similar the above definition, we introduce the following integrals:

$$
\mathcal{J}_{p_1, \lambda_1, a, x, p_1, \lambda_1, \rho, \lambda, w} f\left(\frac{x + c + d}{2}\right) := \int_a^x \frac{(x-t)^{\lambda_1-1} \mathcal{F}_{p_1, \lambda_1} \left[ \omega_1 (x-t)^{p_1} \right]}{f(t,s)dsdt}, x > a;
$$

$$
\mathcal{J}_{p_1, \lambda_1, b-\rho, \lambda_1, \rho, \lambda, w} f\left(\frac{x + c + d}{2}\right) := \int_a^b \frac{(t-x)^{\lambda_1-1} \mathcal{F}_{p_1, \lambda_1} \left[ \omega_1 (t-x)^{p_1} \right]}{f(t,s)dsdt}, x < b;
$$

$$
\mathcal{J}_{p_2, \lambda_2, c, x, p_2, \lambda_2, \rho, \lambda, w} f\left(\frac{a + b + c}{2}\right) := \int_c^y \frac{(y-s)^{\lambda_2-1} \mathcal{F}_{p_2, \lambda_2} \left[ \omega_2 (y-s)^{p_2} \right]}{f(t,s)dsdt}, y > c
$$

and

$$
\mathcal{J}_{p_2, \lambda_2, b-\rho, \lambda_2, \rho, \lambda, w} f\left(\frac{a + b + c}{2}\right) := \int_c^y \frac{(s-y)^{\lambda_2-1} \mathcal{F}_{p_2, \lambda_2} \left[ \omega_2 (s-y)^{p_2} \right]}{f(t,s)dsdt}, y < d.
$$

In this paper, we are interested to give the Hermite–Hadamard inequality for a rectangle in plane via convex functions on co-ordinates involving fractional integral operators. We also study some properties of mappings associated with the Hermite–Hadamard inequality for convex functions on co-ordinates.

2. Hermite Hadamard Type Inequalities for Fractional Integral Operators

In this section, we will give Hermite-Hadamard type inequalities for fractional integral operators by using co-ordinated convex functions. During the this work we use the following symbols for $m = 0, 1$

$$
\mathcal{A}_m(t) := \mathcal{F}_{p_1, \lambda_1 + m} \left[ \omega_1 (b-a)^{p_1} t^m \right], \quad \mathcal{B}_m(s) := \mathcal{F}_{p_2, \lambda_2 + m} \left[ \omega_2 (d-c)^{p_2} s^m \right].
$$
Theorem 2.1. Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a co-ordinated convex on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \) and \( f \in L_1(\Delta) \). Then the following inequalities hold:

\[
\begin{align*}
& f \left( \frac{a + b}{2} \cdot \frac{c + d}{2} \right) \\
\leq & \frac{1}{4 (b - a)^3 (d - c)^3} \int_{p_1, \lambda_1+1} \int_{p_2, \lambda_2+1} \left[ \omega_1 (b - a)^3 \right] \left[ \omega_2 (d - c)^3 \right] \\
& \times \left( \mathcal{F}_{p_1, \lambda_1+c+t} (f(b, d) + \mathcal{F}_{p_1, \lambda_1+c-t} (f(b, c) + \mathcal{F}_{p_1, \lambda_1-b-d} f(a, d) + \mathcal{F}_{p_1, \lambda_1-b-d} f(a, c)) \right) \\
\leq & \frac{f (a, c) + f (a, d) + f (b, c) + f (b, d)}{4}
\end{align*}
\]

where \( p = (p_1, p_2), \lambda = (\lambda_1, \lambda_2), p, \lambda \in [0, \infty)^2; w = (\omega_1, \omega_2) \in \mathbb{R}^2; \sigma = (\sigma_1, \sigma_2) \).

Proof. According to (2) with \( x = ta + (1 - t)b, y = (1 - t)a + tb, u = s - (1 - s)d, v = (1 - s)c + sd \) and \( t_1 = s_1 = \frac{1}{2} \), we find that

\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{4} \left[ f((1 - t)a + (1 - t)b, sc + (1 - s)d) + f(a, (1 - t)b, (1 - s)c + sd) \right.
\]

\[
+ f((1 - t)a + (1 - t)b, sc + (1 - s)d) + f((1 - t)a + (1 - t)b, (1 - s)c + sd))
\]

Multiplying both sides of (14) by \( t^{1 - \theta} C^{1 - \theta} \mathcal{A}_0(t) \mathcal{B}_0(s) \), then integrating with respect to \((t, s) \) on \([0, 1] \times [0, 1] \), we obtain

\[
\begin{align*}
& \mathcal{F}_{p_1, \lambda_1+1} \left[ \omega_1 (b - a)^3 \right] \mathcal{F}_{p_2, \lambda_2+1} \left[ \omega_2 (d - c)^3 \right] f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
\leq & \frac{1}{4} \left\{ \int_0^1 \int_0^1 t^{1 - \theta} C^{1 - \theta} \mathcal{A}_0(t) \mathcal{B}_0(s) \left[ f((1 - t)a + (1 - t)b, sc + (1 - s)d) + f(a, (1 - t)b, (1 - s)c + sd) \right] ds dt \\
& + \int_0^1 \int_0^1 t^{1 - \theta} C^{1 - \theta} \mathcal{A}_0(t) \mathcal{B}_0(s) \left[ f((1 - t)a + (1 - t)b, sc + (1 - s)d) + f((1 - t)a + (1 - t)b, (1 - s)c + sd) \right] ds dt \right\}
\]

Using the change of variable in the last integrals, we have

\[
\begin{align*}
& 4 \mathcal{F}_{p_1, \lambda_1+1} \left[ \omega_1 (b - a)^3 \right] \mathcal{F}_{p_2, \lambda_2+1} \left[ \omega_2 (d - c)^3 \right] f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
\leq & \frac{1}{(b - a)^3 (d - c)^3} \\
\times & \left\{ \int_a^b \int_c^d (b - x)^{1 - \theta} (d - y)^{1 - \theta} \mathcal{F}_{p_1, \lambda_1} \left[ \omega_1 (b - x)^3 \right] \mathcal{F}_{p_2, \lambda_2} \left[ \omega_2 (d - y)^3 \right] f(x, y) dy dx \\
+ & \int_a^b \int_c^d (b - y)^{1 - \theta} (d - x)^{1 - \theta} \mathcal{F}_{p_1, \lambda_1} \left[ \omega_1 (b - x)^3 \right] \mathcal{F}_{p_2, \lambda_2} \left[ \omega_2 (d - y)^3 \right] f(x, y) dy dx \right\}
\]
For this purpose we first note that if \( f \) which gives the left hand side of inequality in (13). Now we prove the right hand side of inequality in (13).

In this way the proof is completed. ∎
Theorem 2.2. Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be a co-ordinated convex on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \) and \( f \in L_2(\Delta) \). Then the following inequalities hold:

\[
f\left(\frac{a + b + c + d}{2}\right) \leq \frac{1}{4(b - a)^{\lambda_1} \int_{p_1,\lambda_1 + 1}^{p_2,\lambda_2 + 1} [\omega_1 (b - a)^{\lambda_1}] \left[ T_{p_1,\lambda_1,\lambda_1 + 1} \left( f\left(\frac{b - a + \lambda_1}{2}\right) + T_{p_1,\lambda_1,\lambda_1 + 1} f\left(\frac{a + \lambda_1}{2}\right) \right) \right]}
\]

\[
+ \frac{1}{4(d - c)^{\lambda_2} \int_{p_2,\lambda_2 + 1}^{p_1,\lambda_1 + 1} [\omega_2 (d - c)^{\lambda_2}] \left[ T_{p_2,\lambda_2,\lambda_2 + 1} \left( f\left(\frac{a + b + c + d}{2}\right) + T_{p_2,\lambda_2,\lambda_2 + 1} f\left(\frac{a + b + c + d}{2}\right) \right) \right]}
\]

\[
\leq \frac{1}{4(b - a)^{\lambda_1} \int_{p_1,\lambda_1 + 1}^{p_2,\lambda_2 + 1} [\omega_1 (b - a)^{\lambda_1}] \left[ T_{p_1,\lambda_1,\lambda_1 + 1} \left( \varphi(a, c) + T_{p_1,\lambda_1,\lambda_1 + 1} \varphi(a, d) + T_{p_1,\lambda_1,\lambda_1 + 1} f(b, d) + T_{p_1,\lambda_1,\lambda_1 + 1} f(b, c) \right) \right]}
\]

\[
+ \frac{1}{4(d - c)^{\lambda_2} \int_{p_2,\lambda_2 + 1}^{p_1,\lambda_1 + 1} [\omega_2 (d - c)^{\lambda_2}] \left[ T_{p_2,\lambda_2,\lambda_2 + 1} \left( \varphi(a, c) + T_{p_2,\lambda_2,\lambda_2 + 1} \varphi(a, d) + T_{p_2,\lambda_2,\lambda_2 + 1} f(b, d) + T_{p_2,\lambda_2,\lambda_2 + 1} f(b, c) \right) \right]}
\]

\[
\leq f(a, c) + f(a, d) + f(b, c) + f(b, d)
\]

where \( p = (p_1, p_2), \lambda = (\lambda_1, \lambda_2), p, \lambda \in [0, \infty)^2; w = (w_1, w_2) \in \mathbb{R}^2; \sigma = (\sigma_1, \sigma_2). \)

Proof. Since \( f : \Delta \to \mathbb{R} \) is co-ordinated convex on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \), it follows that the mapping \( g_x : [c, d] \to \mathbb{R}, g_x(y) = f(x, y) \) is convex on \([c, d]\) for all \( x \in [a, b] \). Then by using inequalities (12), we can write for \( x \in [a, b] \)

\[
g_x\left(\frac{c + d}{2}\right) \leq \frac{1}{2(d - c)^{\lambda_2} \int_{p_2,\lambda_2 + 1}^{p_1,\lambda_1 + 1} [\omega_2 (d - c)^{\lambda_2}] \left[ T_{p_2,\lambda_2,\lambda_2 + 1} g_x(d) + T_{p_2,\lambda_2,\lambda_2 + 1} g_x(c) \right]}
\]

\[
\leq \frac{g_x(c) + g_x(d)}{2}.
\]

That is for \( x \in [a, b], \)

\[
f\left(x, \frac{c + d}{2}\right) \leq \frac{1}{2(d - c)^{\lambda_2} \int_{p_2,\lambda_2 + 1}^{p_1,\lambda_1 + 1} [\omega_2 (d - c)^{\lambda_2}] \left[ \int_c^d (d - y)^{\lambda_2 - 1} T_{p_2,\lambda_2} \left[ \omega_2 (d - y)^{\lambda_2} f(x, y) \right] dy \right]
\]

\[
+ \int_c^d (y - c)^{\lambda_2 - 1} T_{p_2,\lambda_2} \left[ \omega_2 (y - c)^{\lambda_2} f(x, y) \right] dy \right]}
\]

\[
\leq \frac{f(x, c) + f(x, d)}{2}.
\]
Then multiplying both sides of (17) by \( \frac{1 - \alpha}{2(b-a)^{\alpha_1 + 1}}\) and \( \frac{1}{2(b-a)^{\alpha_1 + 1}}\) respectively and then integrating with respect to \( x \) over \([a, b]\), we get

\[
\frac{1}{2(b-a)^{\alpha_1}} \int_{a}^{b} (b - x)^{\alpha_1 - 1} F_{\alpha_1, \alpha_2}^{\alpha_1} \left( \alpha_1 (b - x)^{\alpha_1} \right) f \left( x, \frac{c + d}{2} \right) dx
\]

(18)

\[
\leq \frac{1}{4(b-a)^{\alpha_1}} \frac{1}{(d-c)^{\alpha_1}} \int_{a}^{b} (b - x)^{\alpha_1 - 1} F_{\alpha_1, \alpha_2}^{\alpha_1} \left( \alpha_1 (b - x)^{\alpha_1} \right) f \left( x, \frac{c + d}{2} \right) dx
\]

\[
\leq \frac{1}{4(b-a)^{\alpha_1}} \int_{a}^{b} (b - x)^{\alpha_1 - 1} F_{\alpha_1, \alpha_2}^{\alpha_1} \left( \alpha_1 (b - x)^{\alpha_1} \right) f \left( x, \frac{c + d}{2} \right) dx
\]

and

\[
\frac{1}{2(b-a)^{\alpha_1}} \int_{a}^{b} (x - a)^{\alpha_1 - 1} F_{\alpha_1, \alpha_2}^{\alpha_1} \left( \alpha_1 (x - a)^{\alpha_1} \right) f \left( x, \frac{c + d}{2} \right) dx
\]

(19)
In a similar way applying for the mapping \( g_y : [a, b] \to \mathbb{R} \), \( g_y(x) = f(x, y) \), we have

\[
\frac{1}{2(d - c)^{\lambda_2}F_{p_2, \lambda_2 + 1}^{\alpha_2}(d - c)^{\alpha_2}} \int_{c}^{d} (d - y)^{\lambda_2 - 1}F_{p_2, \lambda_2}^{\alpha_2} \left[ \alpha_2 (d - y)^{\alpha_2} \right] f \left( \frac{a + b}{2}, y \right) dy
\]

(20)

\[
\leq \frac{1}{4(b - a)^{\lambda_1}(d - c)^{\lambda_2}F_{p_1, \lambda_1 + 1}^{\alpha_1}(d - c)^{\alpha_1}} \int_{c}^{d} (d - y)^{\lambda_2 - 1}F_{p_2, \lambda_2}^{\alpha_2} \left[ \alpha_2 (d - y)^{\alpha_2} \right] f(a, y) dy
\]

\[
\times \left[ \int_{a}^{b} \int_{c}^{d} (b - x)^{\lambda_1 - 1}(d - y)^{\lambda_2 - 1}F_{p_1, \lambda_1}^{\alpha_1} \left[ \alpha_1 (b - x)^{\alpha_1} \right] F_{p_2, \lambda_2}^{\alpha_2} \left[ \alpha_2 (d - y)^{\alpha_2} \right] f(x, y) dy dx \right]
\]

\[
+ \int_{a}^{b} \int_{c}^{d} (x - a)^{\lambda_1 - 1}(d - y)^{\lambda_2 - 1}F_{p_1, \lambda_1}^{\alpha_1} \left[ \alpha_1 (x - a)^{\alpha_1} \right] F_{p_2, \lambda_2}^{\alpha_2} \left[ \alpha_2 (d - y)^{\alpha_2} \right] f(x, y) dy dx
\]

(21)

and

\[
\frac{1}{2(d - c)^{\lambda_2}F_{p_2, \lambda_2 + 1}^{\alpha_2}(d - c)^{\alpha_2}} \int_{c}^{d} (y - c)^{\lambda_2 - 1}F_{p_2, \lambda_2}^{\alpha_2} \left[ \alpha_2 (y - c)^{\alpha_2} \right] f \left( \frac{a + b}{2}, y \right) dy
\]

(21)

\[
\leq \frac{1}{4(b - a)^{\lambda_1}(d - c)^{\lambda_2}F_{p_1, \lambda_1 + 1}^{\alpha_1}(d - c)^{\alpha_1}} \int_{c}^{d} (y - c)^{\lambda_2 - 1}F_{p_2, \lambda_2}^{\alpha_2} \left[ \alpha_2 (y - c)^{\alpha_2} \right] f(a, y) dy
\]

\[
\times \left[ \int_{a}^{b} \int_{c}^{d} (b - x)^{\lambda_1 - 1}(y - c)^{\lambda_2 - 1}F_{p_1, \lambda_1}^{\alpha_1} \left[ \alpha_1 (b - x)^{\alpha_1} \right] F_{p_2, \lambda_2}^{\alpha_2} \left[ \alpha_2 (y - c)^{\alpha_2} \right] f(x, y) dy dx \right]
\]

\[
+ \int_{a}^{b} \int_{c}^{d} (x - a)^{\lambda_1 - 1}(y - c)^{\lambda_2 - 1}F_{p_1, \lambda_1}^{\alpha_1} \left[ \alpha_1 (x - a)^{\alpha_1} \right] F_{p_2, \lambda_2}^{\alpha_2} \left[ \alpha_2 (y - c)^{\alpha_2} \right] f(x, y) dy dx
\]

(21)
Adding the inequalities (18)-(21), we obtain

\[
\frac{1}{2(b - a)^{\lambda_{1}}f^{\nu_{1}}_{p_{1},\lambda_{1}+1}[\omega_{1}(b - a)^{\nu_{1}}]} \left[ J_{p_{1},\lambda_{1},d+\nu_{1}}^{\nu_{1}} f \left( \frac{b + c + d}{2} \right) + J_{p_{1},\lambda_{1},b-\nu_{1}}^{\nu_{1}} \left( \frac{a + c + d}{2} \right) \right] \\
+ \frac{1}{2(d - c)^{\lambda_{2}}f^{\nu_{2}}_{p_{2},\lambda_{2}+1}[\omega_{2}(d - c)^{\nu_{2}}]} \left[ J_{p_{2},\lambda_{2},c+\nu_{2}}^{\nu_{2}} f \left( \frac{a + b}{2} \right) + J_{p_{2},\lambda_{2},d-\nu_{2}}^{\nu_{2}} \left( \frac{a + b}{2} \right) \right]
\]

\[
\leq \frac{1}{4(b - a)^{\lambda_{1}}(d - c)^{\lambda_{2}}f^{\nu_{1}}_{p_{1},\lambda_{1}+1}[\omega_{1}(b - a)^{\nu_{1}}]} J_{p_{1},\lambda_{1},d+\nu_{1}}^{\nu_{1}} f(b, d) + J_{p_{1},\lambda_{1},d+\nu_{1}}^{\nu_{1}} f(b, c) + J_{p_{1},\lambda_{1},b-\nu_{1}}^{\nu_{1}} f(a, d) + J_{p_{1},\lambda_{1},b-\nu_{1}}^{\nu_{1}} f(a, c)
\]

\[
+ \frac{1}{4(d - c)^{\lambda_{2}}f^{\nu_{2}}_{p_{2},\lambda_{2}+1}[\omega_{2}(d - c)^{\nu_{2}}]} J_{p_{2},\lambda_{2},c+\nu_{2}}^{\nu_{2}} f(a, d) + J_{p_{2},\lambda_{2},c+\nu_{2}}^{\nu_{2}} f(a, c) + J_{p_{2},\lambda_{2},d-\nu_{2}}^{\nu_{2}} f(a, d) + J_{p_{2},\lambda_{2},d-\nu_{2}}^{\nu_{2}} f(a, c)
\]

where \( p = (p_{1}, p_{2}) \), \( \lambda = (\lambda_{1}, \lambda_{2}) \), \( p_{1}, \lambda_{1} \in [0, \infty)^{2} \); \( w = (w_{1}, w_{2}) \in \mathbb{R}^{2} \); \( \sigma = (\sigma_{1}, \sigma_{2}) \). Thus, we proved the second and the third inequalities in (16).

Now, using the left side of inequality in (12), we also have

\[
f \left( \frac{a + b + c + d}{2} \right)
\]

\[
\leq \frac{1}{2(b - a)^{\lambda_{1}}f^{\nu_{1}}_{p_{1},\lambda_{1}+1}[\omega_{1}(b - a)^{\nu_{1}}]} \int_{a}^{b} (b - x)^{\lambda_{1} - 1} f^{\nu_{1}}_{p_{1},\lambda_{1}} [\omega_{1}(b - x)^{\nu_{1}}] f \left( \frac{x + c + d}{2} \right) dx
\]

\[
+ \int_{a}^{b} (x - a)^{\lambda_{1} - 1} f^{\nu_{1}}_{p_{1},\lambda_{1}} [\omega_{1}(x - a)^{\nu_{1}}] f \left( \frac{x + c + d}{2} \right) dx
\]

and

\[
f \left( \frac{a + b + c + d}{2} \right)
\]

\[
\leq \frac{1}{2(d - c)^{\lambda_{2}}f^{\nu_{2}}_{p_{2},\lambda_{2}+1}[\omega_{2}(d - c)^{\nu_{2}}]} \int_{c}^{d} (d - y)^{\lambda_{2} - 1} f^{\nu_{2}}_{p_{2},\lambda_{2}} [\omega_{2}(d - y)^{\nu_{2}}] f \left( \frac{a + b + c + d}{2} \right) dy
\]

\[
+ \int_{c}^{d} (y - c)^{\lambda_{2} - 1} f^{\nu_{2}}_{p_{2},\lambda_{2}} [\omega_{2}(y - c)^{\nu_{2}}] f \left( \frac{a + b + c + d}{2} \right) dy.
\]
By adding these inequalities, we get

\[
\frac{1}{2(b-a)^{\alpha_1}} \left[ \int_a^b (x-a)^{\alpha_1-1} f(x)dx \right] \leq \frac{1}{4(b-a)^{\alpha_1}} \left[ \int_a^b (x-a)^{\alpha_1} f(x)dx \right]
\]

which gives the first inequality in (16).

Finally, using the right hand side of inequality in (12), we can state

\[
\frac{1}{2(b-a)^{\alpha_1}} \left[ \int_a^b (x-a)^{\alpha_1-1} f(x)dx \right] \leq \frac{1}{2(b-a)^{\alpha_1}} \left[ \int_a^b (x-a)^{\alpha_1} f(x)dx \right]
\]
and

\[
\frac{1}{2(d-c)^{\lambda_2}\mathcal{F}^{\alpha_2}_{\rho_2,\lambda_2+1}[\omega_2(d-c)^{\beta_2}]}
\int_c^d (d-y)^{\lambda_2-1}\mathcal{F}^{\alpha_2}_{\rho_2,\lambda_2}[\omega_2(y-c)^{\beta_2}]f(b,y)dy
\]

\[
+ \int_c^d (y-c)^{\lambda_2-1}\mathcal{F}^{\alpha_2}_{\rho_2,\lambda_2}[\omega_2(y-c)^{\beta_2}]f(a,y)dy
\]

\[
\leq \frac{f(b,c) + f(b,d)}{2}
\]

which give by addition (23)-(26), the last inequality in (16). \[\square\]

3. Fractional Integral Operators for Co-ordinated Convex Functions

Firstly, we give the following lemma for our results.

**Lemma 3.1.** Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, \ 0 \leq c < d \). If \( \frac{\partial f}{\partial a} \in L_1(\Delta) \), then the following inequalities hold:

\[
\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}
\]

\[
+ \frac{1}{4(b-a)^{\lambda_2}(d-c)^{\beta_2}\mathcal{F}^{\alpha_2}_{\rho_2,\lambda_2+1}[\omega_2(b-a)^{\beta_2}]\mathcal{F}^{\alpha_2}_{\rho_2,\lambda_2+1}[\omega_2(d-c)^{\beta_2}]}
\]

\[
\times \left\{ \mathcal{F}^{\alpha}_{\rho,\lambda,a+c+d}f(b,d) + \mathcal{F}^{\alpha}_{\rho,\lambda,a+d-c}f(b,c) + \mathcal{F}^{\alpha}_{\rho,\lambda,c-a+d}f(a,d) + \mathcal{F}^{\alpha}_{\rho,\lambda,b-a-c}f(a,c) - A \right\}
\]

\[
= \frac{(b-a)(d-c)}{4\mathcal{F}^{\alpha_2}_{\rho_2,\lambda_2+1}[\omega_2(b-a)^{\beta_2}]}
\]

\[
\times \left\{ \int_0^1 \int_0^1 t^{\lambda_1} s^{\beta_1 \mathcal{A}(t) \mathcal{B}(s)} \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt - \int_0^1 \int_0^1 (1-t)^{\lambda_1} s^{\beta_1 \mathcal{A}(1-t) \mathcal{B}(s)} \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt 
\]

\[
- \int_0^1 \int_0^1 t^{\lambda_1} (1-s)^{\beta_1 \mathcal{A}(t) \mathcal{B}(1-s)} \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt
\]

\[
+ \int_0^1 \int_0^1 (1-t)^{\lambda_1} (1-s)^{\beta_1 \mathcal{A}(1-t) \mathcal{B}(1-s)} \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt \right\},
\]
where \( p = (p_1, p_2), \lambda = (\lambda_1, \lambda_2), p, \lambda \in [0, \infty)^2; w = (w_1, w_2) \in \mathbb{R}_+^2; \sigma = (\sigma_1, \sigma_2) \) and

\[
A = \frac{1}{4(b - a)^2} F_{p_1, \lambda_1}^{p_1, \lambda_1 + 1} \left[ \omega_1 (b - a)^{p_1} \right] \left[ F_{p_1, \lambda_1}^{\sigma_1, \lambda_1 + \omega_1} f(b, c) + F_{p_1, \lambda_1}^{\sigma_1, \lambda_1 + \omega_1} f(b, d) 
+ F_{p_1, \lambda_1}^{\sigma_1, \lambda_1 + \omega_1} f(a, c) + F_{p_1, \lambda_1}^{\sigma_1, \lambda_1 + \omega_1} f(a, d) \right] + \frac{1}{4(d - c)^2} F_{p_2, \lambda_2}^{p_2, \lambda_2 + 1} \left[ \omega_2 (d - c)^{p_2} \right] 
\times \left[ F_{p_2, \lambda_2}^{\sigma_2, \lambda_2 + \omega_2} f(b, d) + F_{p_2, \lambda_2}^{\sigma_2, \lambda_2 + \omega_2} f(a, d) + F_{p_2, \lambda_2}^{\sigma_2, \lambda_2 + \omega_2} f(a, c) + F_{p_2, \lambda_2}^{\sigma_2, \lambda_2 + \omega_2} f(b, c) \right].
\]

Proof. Integrating by parts, we obtain

\[
I_1 = \int_0^1 \int_0^1 t^1 (a + t) \mathcal{A}_1(t) \mathcal{B}_1(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1 - t)b, sc + (1 - s)d) ds dt
= \int_0^1 s^1 \mathcal{B}_1(s) \left\{ \int_0^1 t^1 \mathcal{A}_1(t) \frac{1}{a - b} \frac{\partial f}{\partial s} (ta + (1 - t)b, sc + (1 - s)d) \right\} ds
- \frac{\lambda_1}{a - b} \int_0^1 t^{h-1} \mathcal{A}_0(t) \frac{\partial f}{\partial s} (ta + (1 - t)b, sc + (1 - s)d) dt \right\} ds
= \int_0^1 s^1 \mathcal{B}_1(s) \left\{ \frac{1}{b - a} F_{p_1, \lambda_1}^{\omega_1, \lambda_1 + 1} (b - a)^{p_1} \right\} \frac{\partial f}{\partial s} (a, sc + (1 - s)d) ds
+ \frac{\lambda_1}{b - a} \int_0^1 I_1^{h-1} s^1 \mathcal{B}_1(s) \frac{\partial f}{\partial s} (ta + (1 - t)b, sc + (1 - s)d) ds
= \frac{1}{b - a} (d - c) F_{p_1, \lambda_1}^{\omega_1, \lambda_1 + 1} (b - a)^{p_1} F_{p_2, \lambda_2}^{\omega_2, \lambda_2 + 1} (d - c)^{p_2} f(a, c)
- \frac{1}{b - a} (d - c) \int_0^1 t^{h-1} \mathcal{A}_0(t) F_{p_2, \lambda_2}^{\omega_2, \lambda_2 + 1} (d - c)^{p_2} f(ta + (1 - t)b, c) dt
- \frac{1}{b - a} (d - c) \int_0^1 s^{h-1} F_{p_1, \lambda_1}^{\omega_1, \lambda_1 + 1} (b - a)^{p_1} \mathcal{B}_0(s) f(a, sc + (1 - s)d) dt
+ \frac{1}{b - a} (d - c) \int_0^1 I_1^{h-1} s^{h-1} \mathcal{A}_0(t) \mathcal{B}_0(s) f(ta + (1 - t)b, sc + (1 - s)d) ds dt.
In this way by integration by parts, we get

\[
I_2 = \int_0^1 \int_0^1 (1-t)^{\lambda_2} \mathcal{A}_1(1-t) \mathcal{B}_1(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt \\
= - \int_0^1 \int_0^1 \mathcal{F}_{\rho_1}^{\alpha_1} [\omega_1 (b-a)^{\rho_1}] \mathcal{F}_{\rho_2}^{\alpha_2} [\omega_2 (d-c)^{\rho_2}] f(b, c) ds dt \\
+ \int_0^1 \int_0^1 \mathcal{F}_{\rho_1}^{\alpha_1} [\omega_1 (b-a)^{\rho_1}] \mathcal{F}_{\rho_2}^{\alpha_2} [\omega_2 (d-c)^{\rho_2}] f(ta + (1-t)b, c) ds dt \\
+ \int_0^1 \int_0^1 (1-t)^{\lambda_1-1} \mathcal{A}_0(1-t) \mathcal{B}_0(s) f(b, sc + (1-s)d) ds dt \\
- \int_0^1 \int_0^1 (1-t)^{\lambda_1-1} \mathcal{A}_0(1-t) \mathcal{B}_0(s) f(ta + (1-t)b, sc + (1-s)d) ds dt,
\]

\[
I_3 = \int_0^1 \int_0^1 t^{\lambda_1} (1-s)^{\lambda_2} \mathcal{A}_3(t) \mathcal{B}_1(1-s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt \\
= - \int_0^1 \int_0^1 \mathcal{F}_{\rho_1}^{\alpha_1} [\omega_1 (b-a)^{\rho_1}] \mathcal{F}_{\rho_2}^{\alpha_2} [\omega_2 (d-c)^{\rho_2}] f(a, d) ds dt \\
+ \int_0^1 \int_0^1 t^{\lambda_1-1} \mathcal{A}_0(t) \mathcal{F}_{\rho_2}^{\alpha_2} [\omega_2 (d-c)^{\rho_2}] f(ta + (1-t)b, d) ds dt \\
+ \int_0^1 \int_0^1 (1-s)^{\lambda_2-1} \mathcal{F}_{\rho_1}^{\alpha_1} [\omega_1 (b-a)^{\rho_1}] \mathcal{B}_0(1-s) f(a, sc + (1-s)d) ds dt \\
- \int_0^1 \int_0^1 t^{\lambda_1-1} (1-s)^{\lambda_2-1} \mathcal{A}_0(t) \mathcal{B}_0(1-s) f(ta + (1-t)b, sc + (1-s)d) ds dt,
\]

and

\[
I_4 = \int_0^1 \int_0^1 (1-t)^{\lambda_1} (1-s)^{\lambda_2} \mathcal{A}_4(1-t) \mathcal{B}_1(1-s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt \\
= - \int_0^1 \int_0^1 \mathcal{F}_{\rho_1}^{\alpha_1} [\omega_1 (b-a)^{\rho_1}] \mathcal{F}_{\rho_2}^{\alpha_2} [\omega_2 (d-c)^{\rho_2}] f(b, d) ds dt \\
+ \int_0^1 \int_0^1 (1-t)^{\lambda_1} \mathcal{A}_0(1-t) \mathcal{F}_{\rho_2}^{\alpha_2} [\omega_2 (d-c)^{\rho_2}] f(ta + (1-t)b, d) ds dt \\
- \int_0^1 \int_0^1 (1-t)^{\lambda_1} \mathcal{A}_0(1-t) \mathcal{F}_{\rho_2}^{\alpha_2} [\omega_2 (d-c)^{\rho_2}] f(ta + (1-t)b, d) ds dt
\]
Using the change of variables for $t, s \in [0, 1], x = ta + (1 - t)b, y = sc + (1 - s)d$ from (27)-(30), we get

\[
I_1 - I_2 - I_3 + I_4 = \frac{1}{(b-a)(d-c)} \int_0^1 (1-s)F_{p_1, \lambda_1+1}^{\alpha_1} [\omega_1 (b-a)^{\alpha_1}] B_0(1-s)f(b, sc + (1-s)d)ds - \frac{1}{(b-a)(d-c)} \int_0^1 (1-t)\left(1-(1-t)\right){\mathcal A}_0(1-t)B_0(1-s)f(ta + (1-t)b, sc + (1-s)d)ds dt.
\]

Then, multiplying both sides of (31) by \( \frac{1}{4F_{p_1, \lambda_1+1}^{\alpha_1} [\omega_1 (b-a)^{\alpha_1}] F_{p_2, \lambda_2+1}^{\alpha_2} [\omega_2 (d-c)^{\alpha_2}] } {\frac{(b-a)(d-c)}{4F_{p_1, \lambda_1+1}^{\alpha_1} [\omega_1 (b-a)^{\alpha_1}] F_{p_2, \lambda_2+1}^{\alpha_2} [\omega_2 (d-c)^{\alpha_2}] } } \), thus we obtain desired result. \( \square \)

**Theorem 3.2.** Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( 0 \leq a < b, 0 \leq c < d \). If \( \frac{\partial f}{\partial a} \) is co-ordinated convex function on \( \Delta \), then the following inequalities hold:

\[
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \leq \frac{1}{4(b-a)(d-c)} F_{p_1, \lambda_1+1}^{\alpha_1} [\omega_1 (b-a)^{\alpha_1}] F_{p_2, \lambda_2+1}^{\alpha_2} [\omega_2 (d-c)^{\alpha_2}] \left[ J_{p_1, \lambda_1, \alpha_1}^a f(b, d) + J_{p_1, \lambda_1, \alpha_1}^b f(b, c) + J_{p_1, \lambda_1, \alpha_1}^a f(a, d) + J_{p_1, \lambda_1, \alpha_1}^b f(a, c) \right] - \lambda \left( \frac{\partial^2 f}{\partial a \partial s} (a, c) + \frac{\partial^2 f}{\partial b \partial s} (b, c) + \frac{\partial^2 f}{\partial a \partial s} (a, d) + \frac{\partial^2 f}{\partial b \partial s} (b, d) \right)
\]

where \( p = (p_1, p_2), \lambda = (\lambda_1, \lambda_2), p, \lambda \in [0, \infty)^2; w = (\omega_1, \omega_2) \in \mathbb{R}^2; \alpha = (\alpha_1, \alpha_2) \).
\[ A = \frac{1}{4(b-a)^{1+1/2} F_1^{\alpha_1} \left( a_1 + (b-a)^{1/2} \right)} \left[ J_{\psi_1}^{\alpha_1} f(b, c) + J_{\psi_1}^{\alpha_1} f(b, d) \right] \]

\[ + J_{\psi_1, b} f(a, c) + J_{\psi_1, b} f(a, d) + \frac{1}{4(d-c)^{1+1/2} F_2^{\alpha_2} \left( a_2 + (d-c)^{1/2} \right)} \left[ J_{\psi_2} f(b, d) + J_{\psi_2} f(a, c) + J_{\psi_2} f(a, d) + J_{\psi_2} f(a, c) \right] \]

and

\[ \alpha_3(k) := \frac{\sigma_1(k) \Gamma(p_1 k + \lambda_1 + 2)}{\Gamma(p_1 k + \lambda_1 + 1)}, \quad \alpha_4(k) := \frac{\sigma_2(k) \Gamma(p_2 k + \lambda_2 + 2)}{\Gamma(p_2 k + \lambda_2 + 1)} \]

**Proof.** From Lemma 3.1, we have

\[ \left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \]

\[ + \frac{1}{4(b-a)^{1+1/2} F_1^{\alpha_1} \left( a_1 + (b-a)^{1/2} \right)} \left[ J_{\psi_1} f(b, c) + J_{\psi_1} f(b, d) \right] \]

\[ \times \left[ J_{\psi_2} f(b, d) + J_{\psi_2} f(a, c) + J_{\psi_2} f(a, d) + J_{\psi_2} f(a, c) \right] - A \]

\[ \leq \frac{1}{4F_1^{\alpha_1} \left( a_1 + (b-a)^{1/2} \right)} \left[ J_{\psi_2} f(b, d) + J_{\psi_2} f(a, c) + J_{\psi_2} f(a, d) + J_{\psi_2} f(a, c) \right] \]

\[ \times \left( \int_0^1 \int_0^1 t^{1+1/2} A_1(t) B_1(s) \left\| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right\| ds dt \right) \]

\[ + \int_0^1 \int_0^1 (t-1)^{1+1/2} A_1(t) B_1(s) \left\| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right\| ds dt \]

\[ + \int_0^1 \int_0^1 t^{1+1/2} (1-s)^{1/2} A_1(t) B_1(1-t) \left\| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right\| ds dt \]

\[ + \int_0^1 \int_0^1 (1-t)^{1+1/2} A_1(1-t) B_1(s) \left\| \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \right\| ds dt \]

Since \( \frac{\partial^2 f}{\partial t \partial s} \) is co-ordinated convex function on \( \Delta \), we can write

\[ \left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right| \]

\[ + \frac{1}{4(b-a)^{1+1/2} F_1^{\alpha_1} \left( a_1 + (b-a)^{1/2} \right)} \left[ J_{\psi_1} f(b, c) + J_{\psi_1} f(b, d) \right] \]

\[ \times \left[ J_{\psi_2} f(b, d) + J_{\psi_2} f(a, c) + J_{\psi_2} f(a, d) + J_{\psi_2} f(a, c) \right] - A \]
\[
\begin{align*}
\frac{(b-a)(d-c)}{4\mathcal{F}_{\beta_1,\alpha_1}^1[\alpha_1^2(b-a)^2]}
\times &\left\{\int_0^1 \int_0^1 \left[t^{\alpha_1} s^{\beta_1} \mathcal{A}_1(t) \mathcal{B}_1(s) + (1-t)^{\alpha_1} s^{\beta_1} \mathcal{A}_1(t) \mathcal{B}_1(s) + t^{\alpha_1} (1-s)^{\beta_1} \mathcal{A}_1(t) \mathcal{B}_1(1-s) \right] x \left[s \left| \frac{\partial^2 f}{\partial s^2}(a, c) \right| + s(1-t) \left| \frac{\partial^2 f}{\partial s^2}(b, c) \right| \right] \right\},
\end{align*}
\]

In the above inequality calculating the integrals, we obtain desired result. \(\square\)

**Remark 3.3.** If we take \(\lambda_1 = \alpha, \lambda_2 = \beta, \tau_1(0) = 1 = \tau_2(0), w_1 = 0 = w_2\) in Lemma 3.1, Theorem 2.1, Theorem 2.2, we have the inequalities which is proved by Sarikaya et al.in [21].

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