Analytic expressions for the second-order scalar perturbations in the ΛCDM Universe

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Abstract

We study the second-order scalar perturbations in the conventional ΛCDM Universe within the cosmic screening approach. We get the analytic expressions for the position-dependent scalar perturbations. The small-scale limit of these perturbations is also obtained. In the era of precision cosmology, these analytic formulas play an important role since they enable to reveal and investigate different nonlinear effects, including backreaction.

1. Introduction

According to the cosmological principle [1, 2], our Universe is isotropic and homogeneous at large enough scales. This follows from the natural assumption that the laws of physics should be the same wherever in the Universe we are. Starting from a certain scale, the distribution of inhomogeneities (e.g., galaxies and groups of galaxies) should be statistically homogeneous. As a result, such statistically homogeneous Universe has the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, and its dynamics is described by the Friedmann equations. This is the zero-order/background approach. Obviously, inhomogeneities disturb this background resulting in first- and higher-order perturbations. They play a crucial role in the investigation of the large scale structure formation. Evidently (by definition!), the average values of the first-order perturbations should be equal to zero (see, e.g., [3]). On the other hand, the average values of the second-order perturbations are different from zero and can affect the background spacetime and matter. This effect is called backreaction (see, e.g., [4, 5, 6, 7, 8, 9] and references therein). It is important to determine how strong the backreaction is and to what extent we may use the standard FLRW metric as a background one. The second-order perturbations can also result in other nonlinear effects [3].

Within the cosmic screening approach, the theory of the first-order perturbations was developed in the papers [10, 11, 12, 13, 14]. The main distinctive features of this approach are the following ones. First, the perturbations (in particular, the gravitational potential) are characterized by exponential Yukawa screening at large cosmological scales. Second, the obtained formulas work at all scales including regions with large contrast of matter density $\delta \rho/\rho \gg 1$. Within this approach, the theory of the second-order perturbations was developed in [15]. Here, the system of equations for the second-order scalar, vector and tensor perturbations was derived. This system has rather complicated form. However, it is remarkable that these perturbations do not mix, and we can study them separately. In the present paper we study the second-order scalar perturbations. Even in this case the corresponding equations are still very complicated. So, we restrict ourselves to the analysis of the position-dependent second-order scalar perturbations. In other words, we exclude the velocity-dependent terms as sources of the second-order scalar perturbations. We can do it since
both the position-dependent and velocity-dependent source terms enter the equations additively. Therefore, we can study them separately. As a result, we find analytic expressions for the position-dependent second-order scalar perturbations.

The paper is structured as follows. In section 2 we briefly describe the background model and the first-order scalar perturbations. Then, we present the equations for the position-dependent second-order scalar perturbations. These equations are solved in section 3. In section 4 we study the Newtonian and post-Newtonian approximations for the found analytic solutions. In the concluding section 5 we summarize and discuss the obtained results. In Appendix A we collect formulas which we use to solve the equations for the second-order scalar perturbations.

2. From the background model to the second-order scalar perturbations: basic equations

2.1. Background model

We start with the unperturbed FLRW metric
\[ ds^2 = a^2 \left( d\eta^2 - \delta_{\alpha\beta} dx^\alpha dx^\beta \right), \quad \alpha, \beta = 1, 2, 3, \] (2.1)
where \( a(\eta) \) is the scale factor, \( \eta \) is the conformal time, and \( x^\alpha, \alpha = 1, 2, 3, \) represent the comoving coordinates. It is supposed that the spatial curvature is zero. Let us write down the corresponding Friedmann equations in the framework of the ΛCDM model:
\[ \frac{3H^2}{a^2} = \kappa \varepsilon + \Lambda, \quad \frac{2H'}{a^2} + \frac{H^2}{a^2} = \Lambda, \] (2.2)
where \( H \equiv a'/a \equiv (da/d\eta)/a, \) with the prime standing for the derivative with respect to \( \eta, \) while \( \kappa \equiv 8\pi G_N/c^4 \) (with \( c \) denoting the speed of light and \( G_N \) being the Newtonian gravitational constant). In addition, \( \varepsilon \) stands for the energy density of nonrelativistic pressureless matter, \( \Lambda \) denotes the cosmological constant, and the overline indicates the average. Obviously, the average energy density is defined by the constant average comoving mass density \( \bar{\rho} \) as follows: \( \varepsilon = \bar{\rho}c^2/a^3. \) From Eqs. (2.2) we can easily get a useful auxiliary equation
\[ H' - H^2 = -\frac{\kappa \bar{\rho}c^2}{2a}. \] (2.3)

2.2. First-order scalar perturbations

The described above background Universe is perturbed by inhomogeneities in the form of discrete point-like masses with mass density
\[ \rho(\eta, \mathbf{r}) = \sum_n m_n \delta(\mathbf{r} - \mathbf{r}_n). \] (2.4)
These masses/particles may represent galaxies and their groups. The mass density fluctuation is
\[ \delta \rho = \rho - \bar{\rho}. \] (2.5)
It is important to note that we do not assume the smallness of the mass density contrast, i.e. \( \delta \rho/\bar{\rho} \) can be much larger than unity. Hence, our scheme is valid at both superhorizon and subhorizon scales. The inhomogeneities result in scalar perturbations of the metric (2.1). In the conformal Newtonian gauge and in the first-order approximation, the perturbed metric is
\[ ds^2 = a^2 \left[ (1 + 2\Phi) d\eta^2 - (1 - 2\Phi) \delta_{\alpha\beta} dx^\alpha dx^\beta \right], \] (2.6)
where the first-order scalar perturbation $\Phi(\eta, r)$ satisfies the inequality $|\Phi| \ll 1$. This means that we work in the weak gravitational field limit. It is well known that, e.g., in the vicinity of galaxies the mass density contrast is much larger than unity, however the gravitational field is weak. Additionally, we assume that the particle peculiar velocities are much less than the speed of light: $|v_n| = |c\,dr_n/d\eta| \ll c$. For example, the today’s typical values are $(250 \div 500) \text{ km/s}$.

In the present paper we investigate perturbations which are defined by the positions of the particles but not their peculiar velocities. As demonstrated in \cite{10}, the contribution of the velocity-dependent corrections into the total expression for the first-order scalar perturbation $\Phi$ is negligible. In this case the gravitational potential $\Phi$ satisfies the following Helmholtz-type equation \cite{10}:

$$\Delta \Phi - \frac{a^2}{\lambda^2} \Phi = \frac{\kappa c^2}{2a} \delta \rho,$$

(2.7)

where $\Delta \equiv \delta_{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta}$ is the Laplace operator in comoving coordinates. The time-dependent parameter

$$\lambda \equiv \left[ \frac{3\kappa \bar{\rho} c^2}{2a^3} \right]^{-1/2},$$

(2.8)

defines the characteristic Yukawa screening length of the gravitational interaction.

The position-dependent solution of Eq. (2.7) is \cite{10}

$$\Phi(\eta, r) = \frac{1}{3} \frac{\kappa c^2}{8\pi a} \sum_n \frac{m_n}{|r - r_n|} e^{-\mu |r - r_n|},$$

(2.9)

where

$$\mu = \frac{a}{\lambda} = \left[ \frac{3\kappa \bar{\rho} c^2}{2a} \right]^{1/2}.$$  

(2.10)

2.3. Second-order scalar perturbations

Let us turn now to the second-order scalar perturbations $\Phi^{(2)}(\eta, r)$ and $\Psi^{(2)}(\eta, r)$. The corresponding metric reads

$$ds^2 = a^2 \left[ \left( 1 + 2\Phi + 2\Phi^{(2)} \right) d\eta^2 - \left( 1 - 2\Phi - 2\Psi^{(2)} \right) \delta_{\alpha\beta} dx^\alpha dx^\beta \right].$$

(2.11)

The main aim of the present paper consists in determination of these perturbations. According to \cite{15}, the functions $\Phi^{(2)}(\eta, r)$ and $\Psi^{(2)}(\eta, r)$ satisfy the following system of master equations:

$$\Delta \Psi^{(2)} - \frac{a^2}{\lambda^2} \Psi^{(2)} = \frac{a^2}{2} Q_{00} + \frac{3a^2}{2} H Q^{(\parallel)},$$

(2.12)

$$\Phi^{(2)} - \Psi^{(2)} = a^2 Q^{(S)},$$

(2.13)

where

$$Q_{00} = - \left( \frac{3\kappa \bar{\rho} c^2}{2a^3} + \frac{15}{a^2} H^2 \right) \Phi^2 - \frac{2}{a^2} \Phi \Delta \Phi - \frac{3}{a^2} (\nabla \Phi)^2,$$

(2.14)

$$\Delta Q^{(\parallel)} = \Delta \left( \frac{5}{a^2} H \Phi^2 \right),$$

(2.15)

$$\Delta \Delta Q^{(S)} = - \frac{1}{2} \Delta Q_{00} + \frac{3}{2} \frac{\partial^2 Q_{00}}{\partial x^\alpha \partial x^\beta},$$

(2.16)
\[ Q_{\alpha\alpha} \equiv Q_{11} + Q_{22} + Q_{33} = \left( \frac{12\kappa \rho_c^2}{a^3} - \frac{15}{a^2} H^2 \right) \Phi^2 - \frac{8}{a^2} \Phi \Delta \Phi - \frac{7}{a^2} (\nabla \Phi)^2, \quad (2.17) \]

\[ Q_{\alpha\beta} = 2 \frac{\partial \Phi}{\partial x^\alpha} \frac{\partial \Phi}{\partial x^\beta} + \frac{4}{a^2} \Phi \frac{\partial^2 \Phi}{\partial x^\alpha \partial x^\beta}, \quad \alpha \neq \beta, \quad (2.18) \]

\[ Q_{\beta\beta} = \left( \frac{4\kappa \rho_c^2}{a^3} - \frac{5}{a^2} H^2 \right) \Phi^2 - \frac{4}{a^2} \Phi \Delta \Phi - \frac{3}{a^2} (\nabla \Phi)^2 + \frac{4}{a^2} \Phi \frac{\partial^2 \Phi}{\partial x^\beta \partial x^\beta} + \frac{2}{a^2} \left( \frac{\partial \Phi}{\partial x^\beta} \right)^2, \quad \beta = 1, 2, 3, \quad (2.19) \]

where \((\nabla \Phi)^2 \equiv \nabla \Phi \cdot \nabla \Phi = \delta^{\alpha\beta} \frac{\partial \Phi}{\partial x^\alpha} \frac{\partial \Phi}{\partial x^\beta}\). These expressions follow from Eqs. (3.30), (3.45), (3.34), (3.42), (3.35), (3.33) and (3.32) in [15] where we disregard the peculiar velocities and vector perturbations. Now, we should solve these equations.

### 3. Analytic solutions for the second-order potentials \(\Phi^{(2)}\) and \(\Psi^{(2)}\)

#### 3.1. Potential \(\Psi^{(2)}(\eta, r)\)

The potential \(\Psi^{(2)}\) is the solution of Eq. (2.12). To solve this equation, we should determine its right-hand side. First, from Eq. (2.15) we obtain the function \(Q_{\parallel}\):

\[ Q_{\parallel} = \frac{5}{a^2} H \Phi^2. \quad (3.1) \]

Taking into account Eq. (2.7) for \(\Delta \Phi\), the function \(Q_{00}\) (2.11) can be rewritten in the form

\[ Q_{00} = - \left( \frac{9\kappa \rho_c^2}{2a^3} + \frac{15}{a^2} H^2 \right) \Phi^2 - \frac{\kappa c^2}{a^3} \Phi \delta \rho - \frac{3}{a^2} (\nabla \Phi)^2. \quad (3.2) \]

Then, Eq. (2.12) takes the form

\[ \Delta \Psi^{(2)} - \frac{3\kappa \rho_c^2}{2a} \Psi^{(2)} = - \frac{9\kappa \rho_c^2}{4a} \Phi^2 - \frac{\kappa c^2}{2a} \Phi \delta \rho - \frac{3}{2} (\nabla \Phi)^2 \]

\[ = - \frac{9\kappa \rho_c^2}{4a} \Phi^2 - \frac{\kappa c^2}{2a} \Phi \delta \rho - \frac{3}{4} \Delta (\Phi^2) + \frac{3}{2} \Phi \Delta \Phi = - \frac{3}{4} \Delta (\Phi^2) + \frac{\kappa c^2}{4a} \Phi \delta \rho, \quad (3.3) \]

where we used Eq. (2.7) and the auxiliary equality

\[ (\nabla \Phi)^2 = \frac{1}{2} \Delta (\Phi^2) - \Phi \Delta \Phi. \quad (3.4) \]

It makes sense to define a new function

\[ \chi = \Psi^{(2)} + \frac{3}{4} \Phi^2, \quad (3.5) \]

which satisfies the equation

\[ \Delta \chi - \frac{3\kappa \rho_c^2}{2a} \chi = - \frac{9\kappa \rho_c^2}{8a} \Phi^2 + \frac{\kappa c^2}{4a} \Phi \delta \rho. \quad (3.6) \]

To solve this equation analytically, we resort to a supplementary simplification. Namely, we concentrate on those sources of the second-order perturbations, which dominate at sufficiently small distances where the mass density contrast is typically large. In the case of Eq. (3.6) this means dropping the term \(\sim \Phi^2\) while keeping the term \(\sim \Phi \delta \rho\) in the right-hand side. Such a simplification implies failing to take into account all sources at large enough distances, but this failure is insignificant since the disregarded sources (such as the term \(\sim \Phi^2\)) are much less than the corresponding first-order ones in the considered large-scale spatial
region, where the linear relativistic perturbation theory works very well (see the argumentation in [20] and Refs. therein). Returning to Eq. (3.6), we have

$$\Delta \chi - \frac{3 \kappa \rho c^2}{2a} \chi = \frac{\kappa c^2}{4a} \Phi \delta \rho .$$

(3.7)

This is the Helmholtz-type equation with the Green’s function

$$G_{\Pi}(r, r') = -\frac{1}{4\pi} \frac{e^{-|r-r'|/\lambda}}{|r-r'|} .$$

(3.8)

Therefore, we look for its solution in the form

$$\chi = \int dr' \left( -\frac{1}{4\pi} \frac{e^{-|r-r'|/\lambda}}{|r-r'|} \right) \left( \frac{\kappa c^2}{4a} \Phi(\eta, r') \delta \rho(\eta, r') \right)$$

$$= -\frac{1}{4\pi} \frac{\kappa c^2}{4a} \int dr' e^{-|r-r'|/\lambda} \left( \frac{1}{3} - \frac{\kappa c^2}{8\pi a} \sum_k m_k e^{-|r'-r_k|/\lambda} \right) \left( \sum_{k'} m_{k'} \delta(r' - r_{k'}) - \delta \rho \right)$$

$$= -\frac{\kappa c^2}{16\pi a} \left\{ -\frac{\rho}{3} \frac{1}{3} + \frac{1}{3} \sum_k m_k e^{-|r-r_k|/\lambda} - \frac{\kappa c^2}{8\pi a} \sum_{k,k'} m_k m_{k'} e^{-|r-r_{k'}|/\lambda} e^{-|r'-r_{k'}|/\lambda} \right\}$$

$$+ \frac{\kappa \rho c^2}{8\pi a} \sum_k m_k I_2 ,$$

(3.9)

where the integrals $I_1$ and $I_2$ are given by the formulas (A.1) and (A.2), respectively. The prime over the double sum indicates that the summation indices must not coincide. Substituting these integrals and taking into account Eq. (3.5), we finally get

$$\psi^{(2)} = \frac{3}{4} \Phi^2 + \chi = \frac{3}{4} \Phi^2 + \phi + \frac{\pi \rho \lambda}{a} \left( \frac{\kappa c^2}{8\pi a} \right)^2 \sum_k m_k e^{-|r-r_k|/\lambda}$$

$$+ \frac{1}{2} \left( \frac{\kappa c^2}{8\pi a} \right)^2 \sum_{k,k'} m_k m_{k'} e^{-|r-r_{k'}|/\lambda} e^{-|r'-r_{k'}|/\lambda} \frac{|r-r_k|}{|r-r_{k'}|} \frac{|r'-r_k|}{|r'-r_{k'}|} .$$

(3.10)

In spite of the presence of the first-order term in the right-hand side, this function in total is of the second order. We clearly demonstrate it for the case of the small-scale limit in the next section (see the formula (4.8)).

3.2. Potential $\Phi^{(2)}(\eta, r)$

According to Eq. (2.13), to get the potential $\Phi^{(2)}$, we need to determine the function $Q^{(S)}$. This function satisfies Eq. (2.16) where $Q_{\alpha\alpha}$ and $Q_{\alpha\beta}$ are defined by Eqs. (2.17)-(2.19). The function $Q_{\alpha\alpha}$ can be rewritten as follows:

$$Q_{\alpha\alpha} = -\frac{15}{a^2} \mathcal{H} \Phi^2 - \frac{4\kappa c^2}{a^3} \Phi \delta \rho - \frac{7}{a^2} (\nabla \Phi)^2 ,$$

(3.11)

where we used Eq. (2.7). After lengthy calculations one can also derive

$$\frac{\partial^2 Q_{\alpha\beta}}{\partial x^\alpha \partial x^\beta} = -\frac{\kappa c^2}{a^3} \nabla (\delta \rho \nabla \Phi) - \frac{5}{a^2} \mathcal{H}' \Delta (\Phi^2) .$$

(3.12)
Therefore, the function $Q^{(S)}$ satisfies the following equation:

$$\Delta \Delta Q^{(S)} = -\frac{1}{2} \Delta Q_{\alpha \alpha} - \frac{15}{2a^2} H' \Delta (\Phi^2) + \frac{3\kappa \rho c^2}{2a^3} \Delta \Phi - \frac{3\kappa c^2}{2a^3} \nabla (\rho \nabla \Phi).$$

(3.13)

To solve this equation, we introduce a new function

$$f = \frac{1}{4\pi} \sum_k m_k \frac{1}{|r - r_k|} (r - r_k) \cdot \nabla \Phi(\eta, r_k),$$

(3.14)

which satisfies the equation

$$\Delta f = \nabla (\rho \nabla \Phi).$$

(3.15)

Now, applying the inverse Laplace operator $\Delta^{-1}$ to Eq. (3.13), we get

$$\Delta Q^{(S)} = -\frac{1}{2} Q_{\alpha \alpha} - \frac{15}{2a^2} H' \Phi^2 + \frac{3\kappa \rho c^2}{2a^3} \Phi - \frac{3\kappa c^2}{2a^3} f$$

$$= \frac{15\kappa \rho c^2}{4a^3} \Phi^2 + \frac{2\kappa c^2}{a^3} \Phi \rho - \frac{\kappa \rho c^2}{2a^3} \Phi + \frac{7}{2a^3} (\nabla \Phi)^2$$

$$- \frac{3\kappa c^2}{8\pi a^3} \sum_k m_k \frac{1}{|r - r_k|} (r - r_k) \cdot \nabla \Phi(\eta, r_k),$$

(3.16)

where the function $Q_{\alpha \alpha}$ is given by (3.31). With the help of Eqs. (2.7) and (3.4) as well as the formula $1/|r - r_k| = \frac{1}{2} \Delta |r - r_k|$ (see, e.g., § 106 in [21]) we finally arrive at the equation

$$\Delta Q^{(S)} = \frac{7}{4a^2} \Delta (\Phi^2) + \frac{3\kappa c^2}{16\pi a^3} \sum_k m_k \Delta \left( \frac{(r - r_k) \cdot \nabla \Phi(\eta, r_k)}{|r - r_k|} \right) + \frac{\kappa c^2}{4a^3} F_1 + \frac{5\kappa \rho c^2}{4a^3} F_2 - \frac{3\kappa c^2}{2a^3} \Phi^2.$$

(3.17)

Using the same reasoning as for Eq. (3.10), we drop the term $\sim \Phi^2$ in the right-hand side. The solution of the resulting equation can be found with the help of the Laplace operator Green’s function

$$G_L(r, r') = -\frac{1}{4\pi} \frac{1}{|r - r'|}.$$

(3.18)

Let us introduce two new functions $F_1$ and $F_2$:

$$\Delta F_1 = \Phi \rho \quad \Rightarrow \quad F_1 = \int d\eta' G_L(r, r') \Phi(\eta, r') \rho(\eta, r'),$$

(3.19)

$$\Delta F_2 = \Phi \quad \Rightarrow \quad F_2 = \int d\eta' G_L(r, r') \Phi(\eta, r').$$

(3.20)

Therefore, for $Q^{(S)}$ we get

$$Q^{(S)} = \frac{7}{4a^2} \Phi^2 + \frac{3\kappa c^2}{16\pi a^3} \sum_k m_k \left( \frac{(r - r_k) \cdot \nabla \Phi(\eta, r_k)}{|r - r_k|} \right) + \frac{\kappa c^2}{4a^3} F_1 + \frac{5\kappa \rho c^2}{4a^3} F_2,$$

(3.21)

where

$$F_1 = \frac{1}{4\pi} \sum_k \frac{m_k}{|r - r_k|} \left( \frac{1}{3} - \frac{\kappa c^2}{8\pi a^3} \sum_l m_l e^{-\mu |r_k - r_l|} \right) = -\frac{1}{4\pi} \sum_k \frac{m_k}{|r - r_k|} \Phi(\eta, r_k)$$

(3.22)

and

$$F_2 = \frac{1}{4\pi} \int d\eta' \frac{m_k}{|r - r'|} \left( \frac{1}{3} - \frac{\kappa c^2}{8\pi a^3} \sum_k \frac{m_k}{|r' - r_k|} e^{-\mu |r' - r_k|} \right).$$

(3.23)
The function $F_2$ can be also expressed as follows:

$$F_2 = \frac{1}{12\pi\rho} \left[ \int d\mathbf{r}' \frac{\delta\rho(\eta, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \sum_k m_k e^{-\mu|\mathbf{r} - \mathbf{r}_k|} \right],$$  \hspace{0.5cm} (3.24)

where we used the integral (A.3).

Finally, the potential $\Phi^{(2)}$ reads:

$$\Phi^{(2)} = \Psi^{(2)} + a^2 Q^{(S)},$$  \hspace{0.5cm} (3.25)

where $\Psi^{(2)}$ and $Q^{(S)}$ are given by Eqs. (3.10) and (3.21), respectively.

4. Newtonian and post-Newtonian cosmological approximations

In the present section we consider the derived above formulas at distances much smaller than the screening length: $\mu r = ar/\lambda = r_{ph}/\lambda \ll 1$, where $r_{ph} = ar$ is the physical distance. Thereby we analyze the Newtonian and post-Newtonian approximations. We call them cosmological since the obtained expressions depend on the scale factor $a$. To study these limits, we consider an auxiliary model. In this model a sphere of comoving radius $R$ contains $N$ discrete particles. Outside this sphere, the rest of the Universe is uniformly filled with matter with constant comoving mass density $\rho$. For such a geometrical configuration, we will get the first- and second-order perturbations in a point with radius-vector $\mathbf{r}$ inside the sphere: $r < R$. Additionally, we assume that the physical radius of the sphere is much less than the screening length: $aR/\lambda = \mu R \ll 1$.

4.1. Gravitational potential

It is well known (see, e.g., [21]) that the first-order perturbation $\Phi(\eta, \mathbf{r})$ corresponds to the gravitational potential. In the cosmic screening approach this function is given by Eq. (2.9). In the case of the described above model, the function (2.9) takes the form

$$\Phi = \frac{1}{3} - \frac{\kappa c^2}{8\pi a} \sum_{n=1}^{N} \frac{m_n}{|\mathbf{r} - \mathbf{r}_n|} e^{-\mu|\mathbf{r} - \mathbf{r}_n|} - \frac{\kappa \bar{\rho} c^2}{8\pi a} \int_{r'R > R} d\mathbf{r}' e^{-\mu|\mathbf{r} - \mathbf{r}'|} |\mathbf{r} - \mathbf{r}'|, \hspace{0.5cm} (4.1)$$

where the sum is taken over all discrete masses inside the sphere (i.e., $r_n < R$). Since we consider the point inside the sphere ($r < R$), the integral in (4.1) coincides with $\mathcal{I}_1(r < R)$ given by (A.3). Therefore,

$$\Phi = \frac{1}{3} - \frac{\kappa c^2}{8\pi a} \sum_{n=1}^{N} \frac{m_n}{|\mathbf{r} - \mathbf{r}_n|} e^{-\mu|\mathbf{r} - \mathbf{r}_n|} - \frac{1}{3} e^{-\mu R} (1 + \mu R) \frac{\sinh(\mu r)}{\mu r}.$$  \hspace{0.5cm} (4.2)

In the limit $\mu R \to 0 \Rightarrow \mu r \to 0$, $\mu r_n \to 0$ we obtain

$$\Phi \approx -\frac{\kappa c^2}{8\pi a} \sum_{n=1}^{N} \frac{m_n}{|\mathbf{r} - \mathbf{r}_n|} \equiv \Phi_N,$$  \hspace{0.5cm} (4.3)

where $\Phi_N$ is the Newtonian potential at the position $\mathbf{r}$ inside the sphere, produced by all $N$ discrete masses. It is worth reminding that $\kappa c^2/(8\pi) = G_N/c^2$ and the physical distance $r_{ph} = ar$. The formula (4.3) clearly demonstrates that for the considered model and in the given approximation the gravitational potential is determined by the particles from the nearest environment, and the term $1/3$ is exactly compensated by the contribution of an infinite number of remote particles. As a result, the discussed expression is truly of the first order of smallness.
4.2. Second-order perturbations

Let us study now the second-order perturbation $\Psi^{(2)}(\eta, \mathbf{r})$ given by the formula \textbf{(4.10)}. According to the previous subsection, in the Newtonian limit $\Phi^2 \to \Phi_0^2$. Therefore, we only need to investigate the function $\chi$. For the considered model with $N$ discrete particles inside the sphere and uniformly distributed matter outside the sphere, the function $\chi$ takes the form

$$
\chi = \frac{1}{18} - \frac{\mu^2}{72\pi^2} \sum_{k=1}^{N} m_k e^{-\mu|\mathbf{r} - \mathbf{r}_k|} - \frac{\mu^3}{144\pi^2} \sum_{k=1}^{N} m_k e^{-\mu|\mathbf{r} - \mathbf{r}_k|} + \frac{\mu^4}{288\pi^2 \rho^2} \sum_{k,k'=1}^{N,} m_k m_{k'} e^{-\mu|\mathbf{r} - \mathbf{r}_k|} e^{-\mu|\mathbf{r}_k' - \mathbf{r}_k|} \int_{r' > R} dr' e^{-\mu|\mathbf{r} - \mathbf{r}'|} \frac{e^{-\mu|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mu^4}{288\pi^2 \rho^2} \sum_{k=1}^{N} m_k \int_{r' > R} dr' e^{-\mu|\mathbf{r} - \mathbf{r}'|} \frac{e^{-\mu|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \frac{e^{-\mu|\mathbf{r}_k' - \mathbf{r}_k|}}{|\mathbf{r}_k' - \mathbf{r}_k|} + \frac{\mu^4}{288\pi^2 \rho^2} \sum_{k=1}^{N} m_k \int_{r' > R} dr' e^{-\mu|\mathbf{r} - \mathbf{r}'|} \frac{e^{-\mu|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \frac{e^{-\mu|\mathbf{r}_k' - \mathbf{r}_k|}}{|\mathbf{r}_k' - \mathbf{r}_k|}
$$

Here the introduced term $S_N$ incorporates pure sums. The integrals $\mathcal{I}_1(r < R)$, $\mathcal{I}_2(r < R)$ and $\mathcal{I}_3(r < R)$ are given by Eqs. \textbf{(A.3)} , \textbf{(A.7)} and \textbf{(A.9)}, respectively. In the last line of \textbf{(4.4)} $\mathbf{r}_k$ are the radius-vectors of discrete particles inside the sphere. To evaluate the expression in this line, we suppose that the discrete masses are concentrated in the central part of the sphere, i.e. $r_k \ll R \Rightarrow \mathbf{r}_k \ll \mathbf{r}'$. Hence,

$$
\frac{e^{-\mu|\mathbf{r} - \mathbf{r}_k|}}{|\mathbf{r} - \mathbf{r}_k|} \int_{r' > R} dr' \frac{e^{-\mu|\mathbf{r}' - \mathbf{r}_k|}}{|\mathbf{r}' - \mathbf{r}_k|} + \int_{r' > R} dr' \frac{e^{-\mu|\mathbf{r} - \mathbf{r}_k|}}{|\mathbf{r} - \mathbf{r}_k|} \frac{e^{-\mu|\mathbf{r}' - \mathbf{r}_k|}}{|\mathbf{r}' - \mathbf{r}_k|} \approx \frac{e^{-\mu|\mathbf{r} - \mathbf{r}_k|}}{|\mathbf{r} - \mathbf{r}_k|} \int_{r' > R} dr' e^{-\mu|\mathbf{r} - \mathbf{r}'|} + \int_{r' > R} dr' \frac{e^{-\mu|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \frac{e^{-\mu|\mathbf{r}_k' - \mathbf{r}_k|}}{|\mathbf{r}_k' - \mathbf{r}_k|} \mathcal{I}_3(r < R),
$$

where the integrals $\mathcal{I}_4$ and $\mathcal{I}_5(r < R)$ are given by Eqs. \textbf{(A.10)} and \textbf{(A.12)}, respectively. Substituting this expression into \textbf{(4.4)} and taking into account the integrals \textbf{(A.5)}, \textbf{(A.7)}, \textbf{(A.9)}, \textbf{(A.10)} and \textbf{(A.12)}, we get

$$
\chi = \frac{1}{18} + \frac{1}{36} e^{-\mu R} \left[ (1 + \mu R) \cosh(\mu R) \frac{\sinh(\mu R)}{\mu R} \right] - \left( 3 + 3 \mu R + \mu^2 R^2 + \mu R e^{-\mu R} \cosh(\mu R) - e^{-\mu R} \sinh(\mu R) \right) e^{-\mu R} - \frac{\mu^2}{144\pi^2 \rho} \sum_{k=1}^{N} m_k e^{-\mu|\mathbf{r} - \mathbf{r}_k|} \left[ (1 + \mu R) e^{-\mu R} - 1 \right] + \mu^3 \frac{144\pi^2 \rho}{288\pi^2 \rho^2} \sum_{k,k'=1}^{N} m_k m_{k'} e^{-\mu|\mathbf{r} - \mathbf{r}_k|} e^{-\mu|\mathbf{r}_k' - \mathbf{r}_k|}.
$$

In the limit $\mu R \to 0 \Rightarrow \mu R \to 0$, $\mu R \to 0$ we obtain

$$
\chi \approx \frac{k_c^2}{16\pi a} \sum_{k=1}^{N} \frac{m_k}{|\mathbf{r} - \mathbf{r}_k|} \frac{k_c^2}{8\pi a} \sum_{k'=1}^{N} \frac{m_{k'}}{|\mathbf{r}_k' - \mathbf{r}_k|} = - \frac{k_c^2}{16\pi a} \sum_{k=1}^{N} \frac{m_k}{|\mathbf{r} - \mathbf{r}_k|} \Phi_N|_{\mathbf{r}=\mathbf{r}_k}.
$$
Therefore, taking into account Eqs. (3.10) and (4.3), the potential $\Psi(2)(\eta, r)$ tends to

$$
\Psi_N^{(2)} = -\frac{3}{4} \Phi_N^2 - \frac{\kappa c^2}{16\pi a} \sum_{k=1}^{N} \frac{m_k}{|r - r_k|} \Phi_N|_{r=r_k},
$$

(4.8)
in full agreement with the formula (3.73) in [13]. This expression clearly demonstrates that $\Psi(2)$ is really of the second order of smallness.

To investigate the same limit of the second-order potential $\Phi_N^{(2)}(\eta, r)$, we need to substitute (4.8) as well as the limit of $Q(S)$ into Eq. (3.25). One can show that

$$
Q_N^{(S)} = \frac{7}{4a^2} \Phi_N^2 - \frac{\kappa c^2}{16\pi a^3} \sum_{k=1}^{N} \frac{m_k}{|r - r_k|} \Phi_N|_{r=r_k} + \frac{3\kappa c^2}{16\pi a^3} \sum_{k=1}^{N} m_k \left( \frac{(r - r_k) \cdot \nabla \Phi_N|_{r=r_k}}{|r - r_k|} \right).
$$

(4.9)

Consequently, we find

$$
\Phi_N^{(2)} = \Phi_N^2 - \frac{\kappa c^2}{8\pi a} \sum_{k=1}^{N} \frac{m_k}{|r - r_k|} \Phi_N|_{r=r_k} + \frac{3\kappa c^2}{16\pi a^3} \sum_{k=1}^{N} m_k \left( \frac{(r - r_k) \cdot \nabla \Phi_N|_{r=r_k}}{|r - r_k|} \right).
$$

(4.10)

The expressions (4.9) and (4.10) agree with the formulas (3.74) and (3.75) in [13], respectively.

It is worth mentioning that the position-dependent second-order scalar perturbation $\Phi_N^{(2)}$ coincides with the corresponding expression for the post-Newtonian metric correction presented in the textbook [21] if one chooses the proper gauge [15].

5. Conclusion

In the present paper we have studied the second-order scalar perturbations for the ΛCDM cosmological model. We have found the analytic expressions for the position-dependent perturbations (see Eqs. (3.10) and (3.25)). We have also studied the Newtonian and post-Newtonian approximations for these solutions. In the small-scale limit, the obtained formulas conform with the corresponding expressions in [13, 21]. These analytic formulas play an important role since they enable to reveal different nonlinear effects, including backreaction, and to determine how strong the backreaction is and to what extent we may use the standard FLRW metric as a background one. Obviously, the perturbative approach is robust if the second-order corrections are much smaller than the first-order ones. If this is the case, it is usually enough to be limited to the first order. However, how can we know it from numerical simulations? Performing a numerical simulation, which takes into account the second-order perturbations, is not an easy task. Instead of such a complicated procedure, we suggest the following test. The simulation can be performed on the basis of the first-order approach. After that one can calculate the first-order perturbation $\Phi$ at a number of points. On the other hand, using our analytic expressions, one can calculate the second-order perturbations $\Phi^{(2)}$ and $\Psi^{(2)}$ at the same points. Then, if $\Phi \gg \Phi^{(2)}, \Psi^{(2)}$, the perturbative scheme is robust and the backreaction is apparently negligible. In the opposite case the backreaction should be certainly taken into account. Even if the backreaction is small, in the era of precision cosmology, the analytic expressions for the second-order perturbations can be very helpful in revealing other observable nonlinear effects [5].
Appendix A. Integrals

In this appendix we present the integrals used for our calculations. First, we list three integrals where integration over the radial coordinate runs from zero to infinity:

\[ I_1 = \int dr e^{-a|r-r'|/\lambda} \frac{1}{|r-r'|} = \frac{4\pi\lambda^2}{a^2}; \quad (A.1) \]

\[ I_2 = \int dr e^{-a|r-r'|/\lambda} \frac{e^{-a|r'-r_k|/\lambda}}{|r'-r_k|} = \frac{2\pi\lambda e^{-a|r-r_k|/\lambda}}{a}; \quad (A.2) \]

\[ I_3 = \int dr e^{-\mu|r-r_k|} \frac{1}{|r-r_k|} = \frac{4\pi\mu^2}{|r-r_k|}; \quad (A.3) \]

Now we consider the integrals which correspond to the model described in section [4]. In this case integration over the radial coordinate runs from the radius \( R \) of the sphere to infinity:

\[ I_1(r) = \int_{r>R} dr' e^{-\mu|r-r'|} \frac{1}{|r-r'|} = -\frac{2\pi}{\mu^r} \int_{R}^{\infty} dr' e^{1+\mu R} \left( e^{-\mu|r-r'|} - e^{-\mu|r'-r|} \right); \quad (A.4) \]

\[ I_1(r < R) = \frac{4\pi}{\mu^2} e^{-\mu R} (1 + \mu R) \frac{\sinh(\mu r)}{\mu r}; \quad (A.5) \]

\[ I_2(r) = \int_{r'>R} dr' e^{-\mu|r-r'|}, \quad (A.6) \]

\[ I_2(r < R) = \frac{4\pi}{\mu^3} e^{-\mu R} \left[ \frac{\sinh(\mu r)}{\mu r} \left( 3 + 3\mu R + \mu^2 R^2 \right) - \cosh(\mu r) (1 + \mu R) \right]; \quad (A.7) \]

\[ I_3(r) = \int_{r'>R} dr' \int_{r''>R} dr'' \frac{e^{-\mu|r-r'|} e^{-\mu|r''-r'|}}{|r-r'| |r'-r''|}, \quad (A.8) \]

\[ I_3(r < R) = \frac{8\pi^2}{\mu^4} \frac{\sinh(\mu r)}{\mu r} e^{-\mu R} \left[ 2(1 + \mu R) + (-\mu R \cosh(\mu R) + \sinh(\mu R)) e^{-\mu R} \right]; \quad (A.9) \]

\[ I_4 = \int_{r'>R} dr' \frac{e^{-\mu r'}}{r'} = \frac{4\pi}{\mu^2} (1 + \mu R) e^{-\mu R}; \quad (A.10) \]

\[ I_5(r) = \int_{r'>R} dr' \frac{e^{-\mu r'|}}{|r'|} = \frac{4\pi}{\mu^2} \frac{e^{-\mu R}}{|r'|}; \quad (A.11) \]

\[ I_5(r < R) = \frac{2\pi}{\mu} e^{-2\mu R} \frac{\sinh(\mu r)}{\mu r}. \quad (A.12) \]

References

[1] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, John Wiley and Sons, Inc., New York (1972).
[2] G.F.R. Ellis, Cosmology and verifiability, Quarterly Journal of the Royal Astronomical Society 16 (1975) 245.
[3] M. Eingorn, M. Brilenkov and B. Vlahovic, Zero average values of cosmological perturbations as an indispensable condition for the theory and simulations, Eur. Phys. J. C 75 (2015) 381: [arXiv:1407.3244].
[4] S. Räisänen, Backreaction: directions of progress, Class. Quant. Grav. 28 (2011) 164008; [arXiv:1102.0408].
[5] C. Clarkson, G. Ellis, J. Larena and O. Umeh, Does the growth of structure affect our dynamical models of the universe? The averaging, backreaction and fitting problems in cosmology, Rept. Prog. Phys. 74 (2011) 112901; [arXiv:1109.2314].

[6] T. Buchert and S. Rasanen, Backreaction in late-time cosmology, Annual Review of Nuclear and Particle Science 62 (2012) 57; [arXiv:1112.6339].

[7] S.R. Green and R.M. Wald, Examples of backreaction of small scale inhomogeneities in cosmology, Phys. Rev. D 87 (2013) 124037; [arXiv:1304.2318].

[8] T. Buchert et al., Is there proof that backreaction of inhomogeneities is irrelevant in cosmology? Class. Quant. Grav. 32 (2015) 215021; [arXiv:1505.07800].

[9] K. Bolejko and M. Korzyński, Inhomogeneous cosmology and backreaction: Current status and future prospects, Int. J. Mod. Phys. D 26 (2017) 1730011; [arXiv:1612.08222].

[10] M. Eingorn, First-order cosmological perturbations engendered by point-like masses, Astrophys. J. 825 (2016) 84; [arXiv:1509.03835].

[11] M. Eingorn and R. Brilenkov, Perfect fluids with $\omega = \text{const}$ as sources of scalar cosmological perturbations, Phys. Dark Univ. 17 (2017) 63; [arXiv:1509.08181].

[12] M. Eingorn, C. Kiefer and A. Zhuk, Scalar and vector perturbations in a universe with discrete and continuous matter sources, JCAP 09 (2016) 032; [arXiv:1607.03594].

[13] M. Eingorn, Cosmological law of universal gravitation, Int. J. Mod. Phys. D 26 (2017) 1750121; [arXiv:1709.02264].

[14] M. Eingorn, C. Kiefer and A. Zhuk, Cosmic screening of the gravitational interaction, Int. J. Mod. Phys. D 26 (2017) 1743012; [arXiv:1711.01759].

[15] R. Brilenkov and M. Eingorn, Second-order cosmological perturbations engendered by point-like masses, Astrophys. J. 845 (2017) 153; [arXiv:1704.10282].

[16] J.M. Bardeen, Gauge-invariant cosmological perturbations, Phys. Rev. D 22 (1980) 1882.

[17] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, Theory of cosmological perturbations, Physics Reports 215 (1992) 203.

[18] R. Durrer, The Cosmic Microwave Background, Cambridge University Press, Cambridge (2008).

[19] D.S. Gorbunov and V.A. Rubakov, Introduction to the Theory of the Early Universe: Cosmological Perturbations and Inflationary Theory, World Scientific, Singapore (2011).

[20] J. Adamek, C. Clarkson, R. Durrer and M. Kunz, Does small scale structure significantly affect cosmological dynamics? Phys. Rev. Lett. 114 (2015) 051302; [arXiv:1408.2741].

[21] L.D. Landau and E.M. Lifshitz, The classical theory of fields (Course of theoretical physics series, volume 2, fourth edition), Oxford Pergamon Press, Oxford (2000).