A Scalable and Statistically Robust Beam Alignment Technique for mm-Wave Systems

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Abstract—Millimeter-Wave (mm-Wave) frequency bands provide an opportunity for much wider channel bandwidth compared with the traditional sub-6 GHz band. Communication at mm-Waves is, however, quite challenging due to the severe propagation path loss. To cope with this problem, directional beamforming both at the Base Station (BS) side and at the user side is necessary in order to establish a strong path conveying enough signal power. Finding such beamforming directions is referred to as the Beam Alignment (BA) and is known to be a challenging problem. This paper presents a new scheme for efficient BA, based on the estimated second order channel statistics. As a result, our proposed algorithm is highly robust to variations of the channel time-dynamics compared with other proposed approaches based on the estimation of the channel coefficients, rather than of their second-order statistics. In the proposed scheme, the BS probes the channel in the Downlink (DL) letting each user to estimate its own path direction. All the users within the BS coverage are trained simultaneously, without requiring “beam refinement” with multiple interactive rounds of Downlink/Uplink (DL/UL) transmissions, as done in other schemes. Thus, the training overhead of the proposed BA scheme is independent of the number of users in the system. We pose the channel estimation at the user side as a Compressed Sensing (CS) of a non-negative signal and use the recently developed Non-Negative Least Squares (NNLS) technique to solve it efficiently. The performance of our proposed algorithm is assessed via computer simulation in a relevant mm-Wave scenario. The results illustrate that our approach is superior to the state-of-the-art BA schemes proposed in the literature in terms of training overhead in multi-user scenarios and robustness to variations in the channel dynamics.

Index Terms—Millimeter-Wave, Beam Alignment, Compressed Sensing, Non-Negative Least Squares (NNLS).

I. INTRODUCTION

Communication at millimeter-waves (mm-Waves for short) provides an opportunity to fulfill the demand for high data rates in the next generation communication networks because of the large available bandwidth [1]. A critical challenge to signaling at mm-Waves compared with sub-6 GHz spectrum is the severe propagation loss [2]. Fortunately, due to the small wavelength, it is possible to package a large number of antenna elements in a small form factor. Such large arrays can be used to provide high-gain directional beamforming at both the Base Station (BS) side and the User Equipment (UE) side in order to boost the Signal-to-Noise Ratio (SNR) to sufficiently high levels, such that small-cell outdoor communication is possible. Moreover, it has been observed experimentally and modeled mathematically that the propagation channel at mm-Waves is formed by a very sparse collection of scatterers in the angle domain [3–6]. This implies that, to establish reliable communication, the BS and the UE need to focus their beams in the direction of a strong path. For example, in the case of Line-of-Sight (LoS) propagation, the beams must point at each other since the LoS path is typically the strongest one. More in general, we refer to the problem of finding a narrow beam direction at both the BS and the user side yielding an SNR after beamforming above a desired threshold as the Beam Alignment (BA) problem. This problem is quite well studied in the literature [4–15]. In particular, it is known to be a challenging problem since in mm-Waves the SNR before beamforming is typically very low, especially in outdoor non-LoS conditions. Moreover, although the number of array antennas may be very large, the number of Radio Frequency (RF) chains is limited, due to the difficulty of implementing a full RF chain (including A/D conversion, modulation, and PA/LNA amplification) for each array element in a very small form factor and for a very large bandwidth. The small number of RF chains prevents the implementation of classical digital beamforming schemes in the baseband domain. In contrast, Hybrid-Digital-Analog (HDA) beamforming must be considered [7] [16]. In particular, a naive sequential scanning of the Angle-of-Departure (AoD) and Angle-of-Arrival (AoA) domains with narrow beams in order to find an alignment to a strongly connected propagation path is very time consuming and unfeasible in practice.

A. Related State-of-the-Art

The inefficiency of naive alignment search has motivated BA algorithms based on hierarchical adaptive search, interactive search, and Compressed Sensing (CS) techniques [8–15].

The fundamental idea of hierarchical methods is to use wider beam patterns at the start of the search and to refine them in several consecutive stages. In [11], for example, the authors develop a bisection algorithm in which the range of AoDs and AoAs are divided by a factor of 2 at each step and is refined by probing the resulting $2 \times 2$ sections and identifying the section with the maximum received power. A similar idea using overlapped beam patterns is used in [12]. Such hierarchical techniques, however, require the interaction of the BS with each individual user, since the training is bi-directional and involves both Downlink (DL) probing and Uplink (UL) feedback for each iterative round. Therefore, it is not obvious how to extend/adapt these approaches to a multiuser scenario, where a BS has to train potentially many users at a time.

In [13], a method is proposed where the BS and the UE iteratively and collaboratively identify the dominant
eigenvector of their channel matrix via the well-known power method. However, this approach requires to demodulate the signal at each antenna both at the BS and at the UE side. Therefore, this method is essentially incompatible with the HDA beamforming architecture.

More recently, considering the natural channel sparsity in the AoA-AoD domain [3,6], CS-based algorithms have been proposed for BA in mm-Waves [14,15,17,19]. These algorithms are efficient and particularly attractive for multi-user scenarios, but they are based on the assumption that the instantaneous channel remains invariant during the whole probing/measurement stage (the same assumption is also adopted in [11,12]). This assumption is typically not satisfied in practice due to the large Doppler spread at mm-Waves, implying fast time-variations of the channel coefficients [20,21]. It should be noticed here that the channel time-variations are greatly reduced after BA is achieved. Therefore, this method is essentially incompatible with the HDA beamforming architecture.

In our scheme, each user needs to estimate the channel from its received measurements, thus, all the users can be individually tailored to each specific user, depending on its hardware (how many RF chains) and on its pre-beamforming SNR conditions, without impacting the overall system functions.

3) Robustness to Variations in Channel Statistics: As explained before, most of the existing works in the literature use the assumption that the instantaneous channel coefficients remain invariant during the whole BA phase. This is difficult to meet in mm-Waves due to the large carrier frequency, large Doppler spread, and consequently fast channel variations [20,21].

Our scheme makes use of the channel second order statistics and is highly robust to variations in channel time-dynamics. We also illustrate via numerical simulations that CS-based algorithms fail to estimate the channel strong path direction when the channel is significantly time-varying, i.e., it undergoes several fading cycles during the estimation period, whereas our scheme performs well for a wide range of channel dynamics.

4) Low-complexity Channel Estimation and Protocol Simplicity: In our scheme, each user needs to estimate the channel from its received measurements, thus, all the computation is done at the user side. We show that the resulting channel estimation boils down to a Non-Negative Least Squares (NNLS) problem, which can be solved very efficiently via standard techniques in the literature. After estimating its best beam direction, each user communicates the corresponding beam index through a random access UL slot (see Section III), which can already benefit from full beamforming gain at the user side and from some (limited) beamforming gain at the BS side. At this point, communication can take place with full beamforming gain at both sides on regular data slots.

**Notation** We denote vectors by boldface small (e.g., \( \mathbf{a} \)) and matrices by boldface capital (e.g., \( \mathbf{A} \)) letters. Scalars are denoted by non-boldface letters (e.g., \( a \)). We represent sets by calligraphic letter \( \mathcal{A} \) and their cardinality with \(|\mathcal{A}|\). We denote the empty set by \( \emptyset \). We use \( \mathbb{E} \) for the expectation, \( \otimes \) for the Kronecker product of two matrices, \( \mathbf{A}^\dagger \) for transpose, \( \mathbf{A}^* \) for conjugate, and \( \mathbf{A}^H \) for conjugate transpose of a matrix \( \mathbf{A} \). The output of an optimization problem such as \( \arg\min_{x\in\mathcal{X}} f(x) \) is denoted by \( x^* \). The complex circularly symmetric Gaussian distribution with a mean \( \mu \) and a variance \( \gamma \) is denoted by \( \mathcal{CN}(\mu, \gamma) \). For an integer \( k \in \mathbb{Z} \), we use the shorthand notation \( [k] \) for the set of non-negative integers \( \{1, ..., k\} \).
II. Basic Setup

A. Channel Model

We consider a mm-Wave system including a BS equipped with a Uniform Linear Array (ULA) with \(M\) antennas and \(m\) RF chains where typically \(m \ll M\). We consider a generic UE, also equipped with an ULA with \(N\) antennas and \(n \ll N\) RF chains. We assume that both the BS and UE arrays have the antenna spacing \(d = \frac{\lambda}{2}\), where \(\lambda\) is the wavelength given by \(\lambda = \frac{c_0}{f_0}\), where \(c_0\) is the speed of the light and where \(f_0\) is the carrier frequency. We denote by \(\theta, \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]\) the steering angles with respect to the BS and UE arrays. We represent the array response of the BS and UE arrays to a planar wave coming from the angles \(\theta\) and \(\phi\) with respect to the BS and UE with the \(M\)-dim and \(N\)-dim array vectors \(\mathbf{a}(\theta) \in \mathbb{C}^M\) and \(\mathbf{b}(\phi) \in \mathbb{C}^N\) respectively, with elements

\[
|\mathbf{a}(\theta)|_k = e^{j(k-1)\pi \sin(\theta)}, \quad k \in [M],
\]

\[
|\mathbf{b}(\phi)|_l = e^{j(l-1)\pi \sin(\phi)}, \quad l \in [N].
\]

We assume that the communication between the BS and the UE occurs via a collection of sparse multi-path components (MPCs) in the AoA-AoD and delay domain, where the \(N \times M\) low-pass equivalent impulse response of the channel at a symbol time \(s\) is given by

\[
H_s(\tau) = \sum_{l=1}^{L} \rho_{s,l} |\mathbf{b}(\phi_l)|\mathbf{a}(\theta_l)^H \delta(\tau - \tau_l),
\]

where \(\rho_{s,l}\) is the random channel gain of the \(l\)-th MPC at AoA-AoD-delay \((\theta_l, \phi_l, \tau_l), l \in [L]\). Typically the number of significant MPCs satisfies \(L \ll \max\{M, N\}\). In practice there may be a large number of MPCs that convey such a small amount of signal power that can be simply neglected since in any case they will not be useful for signal transmission even after the BA is achieved. Note that in [1], [2], and [3] we made the implicit assumption (very common in most beamforming and array processing literature) that the communication bandwidth denoted by \(B\) is much smaller than the carrier frequency \(f_0\) (i.e., \(B \ll f_0\)) such that the array response is essentially frequency-invariant, i.e., the relative change of the wavelength \(\lambda\) over the frequency interval \(f \in [f_0-B/2, f_0+B/2]\) is negligible. We adopt a block fading model, where the channel gains \(\rho_{s,l}, l \in [L]\), remain invariant over the coherence time of the channel of duration \(\Delta t_c\) but change randomly across different coherence times according to a given wide-sense stationary process with given Doppler power spectral density. We also assume that each MPC is formed by a cluster of a large number of micro-scatterers corresponding (roughly) to the same delay and AoA/AoD, such that the channel gains \(\rho_{s,l} \sim \mathcal{CN}(0, \sigma_i^2)\) have a zero-mean complex Gaussian distribution. The channel model (3) can be extended to the case where there is a continuum of MPCs connecting the BS and the UE, where the channel model is given by

\[
H_s(\tau) = \rho_s(d\theta, d\phi, d\tau) |\mathbf{b}(\phi)|\mathbf{a}(\theta)^H.
\]

where \(\rho_s(d\theta, d\phi, d\tau)\) denotes the angle-delay random impulse response of the channel. The BA algorithm developed in this paper holds for the general case as long as the angular scattering function (i.e., the channel power distribution) has a small support over the AoA-AoD domain. The small support of the channel angular scattering function is motivated by experimental observations, suggesting that the propagation at mm-waves occurs along MPCs with small AoA-AoD spread, as illustrated in Fig. 1. For the sake of simplicity of exposition, we will use the discrete MPC model (3) in the sequel.

We also assume that the angle coherence time, i.e., the time scale over which the AoA-AoDs of the scatterers \((\theta_i, \phi_i)\) change significantly, is much longer that the channel coherence time \(\Delta t_c\). Hence, the angles can be treated as locally constant (but unknown) during the BA phase. This local stationarity of the scattering geometry is widely used in the literature and confirmed by channel sounding measurements (e.g., see [21] [23]).

B. Signaling Model

Consider the communication between the BS and a generic UE. Since the BS has \(m\) RF chains, it can transmit up to \(m\) different data streams. For a given signaling interval \(t_0\), let \(x_s,i(t), t \in [s t_0, (s+1) t_0]\), be the continuous-time baseband equivalent signal corresponding to the \(i\)-th data stream. We assume that the channel is time-invariant over each symbol, i.e., \(t_0 < \Delta t_c\). To transmit the \(i\)-th data stream, the BS applies a beamforming vector \(\mathbf{u}_{s,i} \in \mathbb{C}^M\). Without loss of generality, the beamforming vectors are normalized such that \(\|\mathbf{u}_{s,i}\| = 1\). The transmitted signal at symbol time \(s\) is given by

\[
x_s(t) = \sum_{i=1}^{m} x_{s,i}(t) \mathbf{u}_{s,i}.
\]

The received baseband equivalent signal at the UE array is

\[
r_s(t) = \int H_s(\tau) x_s(t - \tau) d\tau = \sum_{i=1}^{L} \rho_{s,i} |\mathbf{b}(\phi_i)|\mathbf{a}(\theta_i)^H x_s(t - \tau_i)
\]

Also, note that here we are assuming that the beamforming vectors \(\mathbf{u}_{s,i}, i \in [m]\) are implemented in the RF domain via an analog beamforming network and therefore they are frequency flat, i.e., they are constant over the whole signal bandwidth.
\[
\begin{align*}
\hat{h}_s(f) &= \sum_{l=1}^{L} \sum_{i=1}^{m} \rho_{s,i} b(\phi_i) a(\theta_i)^H \mathbf{h}_{s,i}, \\
&= \sum_{l=1}^{L} \sum_{i=1}^{m} \rho_{s,l} g_{s,l,i} x_{s,i}(t - \tau_l) b(\phi_l) \\
&= \sum_{l=1}^{L} \sum_{i=1}^{m} \rho_{s,l} g_{s,l,i} x_{s,i}(t + \tau_l) b(\phi_l)
\end{align*}
\]

where \(g_{s,l,i} := a(\theta_i)^H \mathbf{h}_{s,i}\) denotes the beamforming gain along the \(l\)-th MPC at the BS side for the \(i\)-th RF chain. The analog RF signal received at the UE antenna array is distributed to the \(n\) RF chain for demodulation. This is achieved by signal splitters that divide the signal power by a factor of \(n\). The noise in the receiver is mainly introduced by the RF chain electronics (filter, mixer, and A/D conversion). It follows that the noisy received signal at the output of the \(j\)-th RF chain at the UE side is given by

\[
y_{s,j}(t) = \frac{1}{\sqrt{n}} \mathbf{v}_{s,j}^H r_s(t) + z_{s,j}(t)
\]

where \(\mathbf{v}_{s,j} \in \mathbb{C}^N\) denotes the normalized beamforming vector of the \(j\)-th RF chain at the UE side, where \(g_{s,l,j} := \mathbf{v}_{s,j}^H b(\phi_l)\) denotes the array gain of the \(j\)-th RF chain along the \(l\)-th MPC, where \(r_{s,j}(t) = \sum_{l=1}^{L} \rho_{s,l} g_{s,l,i} x_{s,i}(t + \tau_l)\) denotes the signal contribution relative to the \(i\)-th transmitted data stream of the BS received at the output of the \(j\)-th RF chain of the UE, and where \(z_{s,j}(t)\) is the continuous-time complex Additive White Gaussian Noise (AWGN) at the output of the \(j\)-th RF chain, with Power Spectral Density (PSD) of \(N_0\) Watt/Hz. The factor \(1/\sqrt{n}\) in \([7]\) takes into account the power split said above.

In this paper we consider OFDM signaling with given subcarrier separation \(\Delta f\). Each symbol \(s\) in the general model defined before corresponds here to an OFDM symbol. The number of subcarriers is given by \(B = 2/\Delta f\), where \(B\) denotes the channel bandwidth as defined before. We make the standard assumption that the duration \(\tau_{CP}\) of the Cyclic Prefix (CP) of the OFDM modulation is longer than the channel delay spread, implying \(\tau_0 = 1/\Delta f + \tau_{CP}\) with \(\tau_{CP} \geq \max \{\tau_l\} - \min \{\tau_l\}\). Hence, after OFDM demodulation, the Inter-Block Interference is completely removed and we can focus on a per-symbol model in the frequency domain \([22]\).

Applying the Fourier transform to the matrix-valued channel impulse response \([3]\), the frequency-domain channel matrix at symbol interval \(s\) is given by

\[
\hat{H}_s(f) = \sum_{l=1}^{L} \rho_{s,l} b(\phi_l) a(\theta_l)^H e^{-j2\pi f \tau_l}.
\]

We denote the OFDM subcarriers as \(f_{\omega} = \frac{\omega}{\pi} : \omega \in [F]\). The channel matrix at subcarrier \(\omega\) is given by \(H_s[\omega] := \hat{H}_s(f_{\omega})\).

Let \(\hat{x}_{s,j}[\omega]\) denote the frequency-domain data symbol for the \(i\)-th stream. Applying OFDM demodulation to the received signal \([7]\), we obtain the corresponding frequency-domain received signal at the \(j\)-th receiver RF chain, with transmit beamforming vector \(\mathbf{u}_{s,i}\) and receive beamforming vector \(\mathbf{v}_{s,j}\) in the form

\[
y_{s,i,j}[\omega] = \frac{1}{\sqrt{n}} \mathbf{v}_{s,j}^H \mathbf{H}_s[\omega] \mathbf{u}_{s,i} \hat{x}_{s,i}[\omega] + \tilde{z}_{s,j}[\omega]
\]

where \(\tilde{z}_{s,j}[\omega] \sim \mathcal{CN}(0, \sigma^2)\) denotes the noise at \(j\)-th RF chain of UE at subcarrier \(\omega\), with variance \(\sigma^2 = \Delta f N_0\).

### C. Beam Alignment

During the DL probing slots (see frame structure discussed in Section \([III]\), we assume that the signal corresponding to different data streams \(x_{s,i}(t)\) are orthogonal, i.e.,

\[
\langle x_{s,i}, x_{s,i'} \rangle := \int_{t_0}^{t_0+T} x_{s,i}(t) x_{s,i'}(t) dt = E_i \delta_{i,i'},
\]

where \(E_i\) is the energy per symbol for the \(i\)-th data stream and \(\delta_{i,i'}\) is the Kronecker delta symbol (equal to 1 for \(i = i'\) and 0 otherwise). For example, this can be obtained in the frequency domain by using OFDM and mapping the different streams onto sets of non-overlapping subcarriers. We define the SNR after beamforming (ABF) for the \(i\)-th data stream received at the \(j\)-th RF chain at the UE by

\[
\text{SNR}_{ABF}^{i,j} := \frac{1}{E_i} \mathbb{E} \left[ \frac{1}{n N_0 B_i} \| r_{s,j}(t) \|^{2} dt \right] = \frac{E_i}{n N_0 B_i} \sum_{l=1}^{L} \gamma_{g_{s,l,1,j}} |g_{s,l,1,j}|^2
\]

\[
= \frac{P_i \sum_{l=1}^{L} \| g_{s,l,1,j} \|^2 |g_{s,l,1,j}|^2}{n N_0 B_i},
\]

where \(B_i\) and \(P_i\) denote the bandwidth and the average power of \(x_{s,i}(t)\), respectively. We have \(\sum_{i=1}^{m} P_i = P_{\text{tot}}\), where \(P_{\text{tot}}\) is the overall transmit power of the BS. In particular, for equal power allocation \(P_i = P_{\text{tot}}/m\) over the streams, we have

\[
\text{SNR}_{ABF}^{i,j} = \frac{P_{\text{tot}} \sum_{l=1}^{L} \| g_{s,l,1,j} \|^2 |g_{s,l,1,j}|^2}{mn N_0 B_i},
\]

For later use, we also define the SNR before beamforming (BBF) by

\[
\text{SNR}_{BBF} := \frac{P_{\text{tot}} \sum_{l=1}^{L} \| g_{s,l,1,j} \|^2}{N_0 B_i}.
\]

This is the SNR obtained when a single data stream \((m = 1)\) is transmitted through a single BS antenna and is received in a single UE antenna (isotropic transmission) over a single RF chain \((n = 1)\) with full-band spreading.

A challenge in mm-Wave communication is that the SNR before beamforming \(\text{SNR}_{BBF}\) in \([13]\) is typically very low. This cannot be increased by simply boosting the transmit power \(P_{\text{tot}}\) because of hardware and regulation.
Illustration of the sparsity of the channel matrix $\mathbf{H}_s[\omega]$ at an arbitrary subcarrier $\omega$ consisting of 3 off-grid AoA-AoDs with increasing number of antennas for $M = N = 4$ (a), $M = N = 8$ (b), $M = N = 16$ (c), $M = N = 32$ (d).

limitations, also because, in general, we would like to design energy-efficient systems. An option to increase the SNR consists of communicating over a smaller bandwidth $B' < B$. However it is well-known that this strategy is suboptimal\(^{2}\). In fact, assuming a Gaussian channel with SNR equal to $\text{SNR}_{\text{BBF}}$, Shannon's capacity formula yields that the achievable rate in bit/s when communicating over a bandwidth $B'$ is given by $R = B' \log (1 + (B/B') \text{SNR}_{\text{BBF}})$, which is increasing for $0 < B' < B$. Hence, by using a bandwidth smaller than the available channel bandwidth $B$, the achievable rate is reduced.

It follows that the only viable alternative consists of using antenna arrays with a large number of antennas both at the BS and at the UE. The goal of BA is to find good beamforming vectors $\mathbf{u}_m$ and $\mathbf{v}_s$ at the BS and the UE, respectively, in order to boost the SNR by a factor $\approx M$ at the BS side and a factor $\approx N$ at the UE side. This is achieved by aligning the beamforming vectors along the AoA-AoD of a strong MPC of the channel.

\subsection*{D. Sparse Beamspace Representation}

The AoA-AoDs $(\theta_t, \phi_t)$ in (8) take continuous values. For later applications in the paper, we need a finite-dimensional representation of the channel. Following the well-known approach of\(^{3, 28, 29}\), known as \textit{beamspace representation}, we obtain such a representation by quantizing the matrix-valued channel response (9) with respect to a discrete dictionary in the AoA-AoD domain. We consider the discrete set of AoA-AoDs

$$\Theta := \{ \hat{\theta} : \frac{1}{2} \sin(\hat{\theta}) = \frac{k}{M}, k \in [M] \},$$

$$\Phi := \{ \hat{\phi} : \frac{1}{2} \sin(\hat{\phi}) = \frac{k'}{N}, k' \in [N] \},$$

and use the corresponding array responses $\mathcal{A} := \{ \mathbf{a}(\hat{\theta}) : \hat{\theta} \in \Theta \}$ and $\mathcal{B} := \{ \mathbf{b}(\hat{\phi}) : \hat{\phi} \in \Phi \}$ as a discrete dictionary to represent the channel response. For the ULAs considered in this paper, the dictionary $\mathcal{A}$ and $\mathcal{B}$, after suitable normalization, yield orthonormal bases corresponding to the columns of the $M \times N$ and $N \times N$ DFT matrices $\mathbf{F}_M$ and $\mathbf{F}_N$\(^{5}\), where

$$(\mathbf{F}_M)[k,k'] = \frac{1}{\sqrt{M}} e^{j2\pi(k-1)(\frac{k}{M} - \frac{1}{2})}, k,k' \in [M],$$

$$(\mathbf{F}_N)[k,k'] = \frac{1}{\sqrt{N}} e^{j2\pi(k-1)(\frac{k}{N} - \frac{1}{2})}, k,k' \in [N].$$

Hence, we obtain the exact representation of the channel matrix $\mathbf{H}_s[\omega] = \mathbf{F}_N \mathbf{H}_s[\omega] \mathbf{F}_M^\dagger$, where

$$\mathbf{H}_s[\omega] = \sum_{m=1}^{L} \rho_m e^{-j2\pi \frac{\omega}{\omega_0}} \mathbf{b}(\phi_m) \mathbf{a}(\theta_m)^\dagger,$$

and where $\mathbf{a}(\theta_m) := \mathbf{F}_M^\dagger \mathbf{a}(\theta_m)$, and $\mathbf{b}(\phi_m) := \mathbf{F}_N^\dagger \mathbf{b}(\phi_m)$ denote the coefficient vectors of the array responses $\mathbf{a}(\theta_m)$ and $\mathbf{b}(\phi_m)$ with respect to the DFT bases, respectively. The $m$-th entry of $\mathbf{a}(\theta_m)$ is given by

$$[\mathbf{a}(\theta_m)]_{m'} = \frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} e^{-j \pi \sin(\theta_i)} e^{j \pi \sin(\theta_i)} e^{-j \pi \sin(\theta_i)} e^{-j \pi \sin(\theta_i)} e^{-j \pi \sin(\theta_i)}$$

where $\psi_i = \frac{\sin(\psi_i M)}{\sin(\psi_i)} - \frac{1}{2} \sin(\theta_i) - \frac{1}{2}$. A similar expression holds for $\mathbf{b}(\phi_m)$. It is seen from (19) that $[\mathbf{a}(\theta_m)]_{m'} = \frac{1}{\sqrt{M}} [\sin(\psi_i M)]_{m'}$ is a localized kernel around $\theta_i = \sin^{-1} \left( \frac{2(\pi - 1)}{2(\pi - 1)} \right)$ with a resolution of $\frac{1}{M}$. In general, the AoA-AoDs of the MPCs are not aligned with the discrete grid $\mathcal{G} = \Theta \times \Phi$. However, as the number of antennas $M$ at the BS and $N$ at the UE increases, the DFT basis provide a good sparsification of the channel matrix $\mathbf{H}_s[\omega]$. This is qualitatively illustrated in Fig. 2 for a channel with $L = 3$ discrete MPCs and off-grid AoA-AoD components. It is seen that, as $M$ and $N$ increase, the resulting representation $\mathbf{H}_s[\omega]$ is more and more sparse.

\section{III. Proposed Beam-Alignment Algorithm}

\subsection{A. High-Level Overview}

In this section, we provide a high-level overview of our scheme, the details of which are presented in the next sections. Fig. 3 shows the proposed frame structure. During the DL
Pseudo-random beam sweeping (BS beacon)
Random access slot (random access control channel)
Data slots

Fig. 3: (Top) Frame structure of the proposed BA scheme. (Bottom) Different beacon signals are orthogonally multiplexed over disjoint sets of subcarriers \( \omega \in \mathcal{F}_i, i \in [m] \). In the figure’s example we have two orthogonal beacon signals on the “blue” and on the “orange” combs of subcarriers.

beacon slots, the BS broadcasts a probing signal to all the UEs, while the UEs stay in listening mode. The probing signal is formed by a sequence of pseudo-random beam patterns (referred to as the transmit beamforming codebook), repeated periodically, and a priori known to all users. Each UE makes measurements of the beacon transmission by applying its own (individual) sequence of receive beam patterns (referred to as the receive beamforming codebook) during the beacon slots. The number of measurements may differ from user to user, depending on the individual pre-beamforming SNR and on the number of receiver RF chains. When a UE has gathered enough measurements, it estimates the AoA-AoD information for its strongest MPC (using the method detailed in the following), and transmits over the UL random access slot a beamformed control packet using a beamforming vector in the direction of its best estimated AoA (UE side). The control packet contains the user ID, the index of the estimated best AoD (BS side), and possibly some additional protocol information (e.g., rate information which can be derived from the estimated SNR after beamforming). During the random access slots, the BS stays in listening mode and uses its \( m \) RF chains to form \( m \) coarse beam patterns (sectors), covering the whole BS angle domain, in order to provide some receiver beamforming gain. We assume that, when the UE has correctly estimated its best MPC, with high probability, the beamforming gain at the BS side and the sectorization at the BS side are sufficient to successfully decode the control packet. This assumption is justified by the fact that the control packet can be encoded at low rate. After sending the UL control packet, the UE puts itself in listening mode with the same beamforming vector used to send the control packet in the UL. At this point, the BS knows the beamforming vectors to be used for all UEs whose control packet was successfully decoded. Each time, after BS decodes the control packet of a new connecting user, the BS responds with a beamformed Acknowledgment (ACK) packet sent in one of the data slots. It follows that a beamforming gain at both the BS and the UE side is achieved for the ACK packet and for all subsequent data packets, both in the DL and in the UL. Therefore, data communication can take place at a rate that depends on the SNR after beamforming. In the case that the control packet is received in error (e.g., this may be due to an estimation error in the AoA/AoD information, or to a collision in the random access slot), the BS can not respond with a correct beamformed ACK, and the UE, after a given time-out, will realize that something went wrong. Then, the UE keeps gathering more measurements from the beacon slots, and the BA procedure restarts.

B. BS Channel Probing and UE Sensing

Without loss of generality, we focus on the BA procedure for a generic UE and omit the UE index, since the scheme applies in parallel and virtually without mutual interaction for all connecting UEs (apart from possible collisions in the random access UL slot). Consider the channel matrix \( \mathbf{H}_s(\omega) \) between the BS and the UE arrays, as defined in Section II-D and its beamspace representation \( \mathbf{H}_s(\omega) \) at beacon slot \( s \in [T] \) and subcarrier \( \omega \in [F] \), where \( T \) is the effective period of beam training.

For simplicity of exposition, we assume that the beacon slot contains a single OFDM symbol interval. At each beacon slot, the BS uses its \( m \) RF chains to probe the channel along \( m \) beamforming vectors \( \mathbf{u}_{s,i}, i \in [m] \), by transmitting an OFDM symbol \( x_{s,i}(t) \) along each \( \mathbf{u}_{s,i} \). We design the beacon OFDM symbols \( x_{s,i}(t) \) such that they are mutually orthogonal in the frequency domain. In particular, for each \( i \in [m] \) we define a subset \( \mathcal{F}_i \subset [F] \) of size \( |\mathcal{F}_i| \leq F \) such that \( \mathcal{F}_i \cap \mathcal{F}_{i'} = \emptyset \) for \( i \neq i' \) (see Fig. 3). We make the additional assumption that each subset \( \mathcal{F}_i \) forms a “comb” of equal size \( |\mathcal{F}_i| = F' \), with widely separated subcarriers such that the frequency separation is larger than the channel coherence bandwidth. Hence, the corresponding channel matrices \( \mathbf{H}_s(\omega) \) for the different subcarriers \( \omega \in \mathcal{F}_i \) are mutually uncorrelated.

A main ingredient of our proposed BA scheme is the pseudo-random beamforming codebook transmitted by the BS during the beacon slots, defined as the collection of sets \( \mathcal{C}_{\text{BS}} := \{ \mathcal{U}_{s,i} : s \in [T], i \in [m] \} \), where \( \mathcal{U}_{s,i} \) is the angle-domain support (i.e., the subset of quantized angles in the virtual beamspace representation) defining the directions to which the transmit beam patterns \( \mathbf{u}_{s,i} \) sends the signal power. We let \( |\mathcal{U}_{s,i}| = \kappa_u \leq M \) for all \((s,i)\). The beamforming vectors are given by \( \mathbf{u}_{s,i} = \mathbf{F}_M \tilde{\mathbf{u}}_{s,i} \), where \( \tilde{\mathbf{u}}_{s,i} = \frac{1}{\sqrt{\kappa_u}} \), and where \( \mathbf{1}_{\kappa_u} \) denotes a vector with 1 at components in the support set \( \mathcal{U}_{s,i} \) and 0 elsewhere. An example of such patterns with the corresponding vector \( \tilde{\mathbf{u}}_{s,i} \) is shown in Fig. 4 (a). The

\footnote{Once BA is achieved and regular data communication takes place, the BS and the UE can keep using the beacon slot in addition to the data slots to enable some beam tracking algorithm and keep the alignment over smooth variations of the AoA-AoDs. In this paper we are only concerned with the initial BA, i.e., when the UE needs to connect to the BS without a priori information on the direction of its strong MPCs.}

\footnote{The generalization to multiple OFDM symbols per slot is immediate and slots of \( S \geq 1 \) OFDM symbols shall be used in the numerical results.}
pseudo-random nature of the codebook is due to the fact that the sequences of angular support sets \( \{ \mathcal{U}_{s,i} : i \in [m], s \in [T] \} \) are generated in a pseudo-random manner.

The second ingredient of our proposed BA algorithm is a local receive codebook at each UE, through which the UE makes measurements in order to estimate the AoA-AoD information of its strong MPCs. Each UE can customize (locally) its own receive beamforming codebook defined by the collection of sets \( \mathcal{C}_{\text{UE}} := \{ \mathcal{V}_{s,j} : s \in [T], j \in [n] \} \), where \( \mathcal{V}_{s,j} \) is the angle-domain support defining the frequency domain. This receiver beam patterns \( \mathbf{v}_{s,j} \) collect signal power. We let \( |\mathcal{V}_{s,j}| = \kappa_v \leq N \) for all \(( s, j) \). The beamforming vectors are given by \( \mathbf{v}_{s,j} = \mathbf{F}_N \mathbf{h}_{s,j}, \) where \( \mathbf{h}_{s,j} = \frac{\mathbf{1}_{\kappa_v}}{\sqrt{\kappa_v}} \). Similar to the parameter \( \kappa_u \) at the transmitter side, the parameter \( \kappa_v \) controls the spread of the sensing window at the UE side. This is illustrated again in Fig 4(a).

During the \( s \)-th beacon slot, the UE applies the receive beamforming vector \( \mathbf{v}_{s,j} \) to its \( j \)-th RF chain, obtaining the frequency-domain received signal (after OFDM demodulation) given by (9) for \( i \in [m] \) and \( \omega \in \mathcal{F}_i \). Note that the \( m \) probing signals \( x_{s,i}(t) \) are orthogonal in the frequency domain and therefore can be perfectly separated at the receiver. It is convenient to write (9) directly in terms of the beamspace representation as

\[
y_{s,i,j}[\omega] = \frac{1}{\sqrt{n}} \mathbf{v}_{s,j}^H \mathbf{H}_s[\omega] \mathbf{u}_{s,i} \bar{x}_{s,i}[\omega] + \tilde{z}_{s,j}[\omega], \quad \omega \in \mathcal{F}_i. \tag{20}
\]

The BS total transmit power \( P_{\text{tot}} \) is allocated equally to all the probing slots \( i \in [m] \), all the subcarriers in \( \mathcal{F}_i \), and all the \( \kappa_u \) beamspace directions. Hence, the symbols \( \{ \bar{x}_{s,i}[\omega] : \omega \in \mathcal{F}_i \} \) have uniform power distribution with \( \mathbb{E}[|\bar{x}_{s,i}[\omega]|^2] = \frac{P_{\text{tot}}}{m \mathcal{F}_i} := P_{\text{dim}} \) (power per transmit signal dimension). In fact, without loss of generality, we choose the frequency-domain probing symbols to be constant and given by \( \bar{x}_{s,i}[\omega] = \sqrt{P_{\text{dim}}} \).

For the class of beamforming patterns defined by \( \mathcal{C}_{\text{BS}} \) and by \( \mathcal{C}_{\text{UE}} \), from (19) it is easy to show that \( |g_{s,i,j}|^2 \leq \frac{M}{\kappa_u} \) and \( |g_{s,i,j}|^2 \leq N/\kappa_v \). It follows that

\[
\mathbb{E}[|\mathbf{H}_s[\omega] \mathbf{u}_{s,i}|^2] \leq \frac{MN \sum_{l=1}^L \gamma_l}{\kappa_u \kappa_v}. \tag{21}
\]

Using this bound in (20), we obtain the maximum possible SNR for channel estimation in the per-subcarrier observation (20), given by

\[
\text{SNR}_{\text{ABF}} := \frac{P_{\text{dim}}}{n} \frac{MN \sum_{l=1}^L \gamma_l}{\kappa_u \kappa_v \mathcal{F}_i} = \frac{BMN}{\kappa_u \kappa_v \mathcal{F}_i} \times \left( \frac{F' \Delta f}{N} \right) \times \text{SNR}_{\text{BBF}}. \tag{22}
\]

Expression (22) puts in evidence the role of the different factors: the first term is the ratio of the maximal available beamforming gain \( MN \), divided by the total signal dimensions in the spatial multiplexing domain \( \kappa_u \kappa_v \mathcal{F}_i \). The second term corresponds to the power concentration in the frequency domain, and the third term is the SNR before beamforming, defined in (15).

The frequency spreading factor \( F' \) and angle spreading factors \( \kappa_u, \kappa_v \) can be optimized depending on the specific cell topology (e.g., on the size of the cell, which in turns determines the worst-case SNR before beamforming). Clearly, by making \( \kappa_u \) (resp., \( \kappa_v \)) larger, each beam pattern probes (resp., sense) simultaneously more directions, but the total power is spread over all such directions. In contrast, by making \( \kappa_u \) (resp., \( \kappa_v \)) smaller, the beam pattern explores less directions but obtains better power concentration in the angle domain. It is also important to notice the effect of \( F' \); as we shall see in Section III-C the AoA-AoD estimator builds some sample-mean statistics by averaging over a sufficiently large number of uncorrelated channel fading realization over the frequency domain. Hence, larger \( F' \) provide better averaging at the cost of spreading the total power over more subcarriers.

**Remark 1:** The proposed BS pseudo-random beamforming codebook can be regarded as a generalization of the classical “coarse beam sweeping”, where the probing vectors pack all directions in the given coarse angle intervals (sectors) \( \mathcal{F}_i \). Our results show that using the proposed method with random beam patterns instead of the traditional coarse beam sweeping, we can achieve a very good beam alignment without the need of the interactive beam refinement phase as usually considered in current practical approaches (e.g., see [4]).

**Remark 2:** In the proposed scheme, the channel probing takes place only in the DL, where BS actively probes the channel while all the UEs are in the receiving mode. Thus, our scheme is markedly different from interactive channel estimation schemes in which both the UE and the BS take turns and probe the channel in a bi-directional way (e.g., see [4] [13] [20]). In general, interactive schemes require some coordination among the UEs, which is difficult to establish in the initial acquisition mode. In the absence of such a coordination, the simultaneous transmission of those UEs interested in joining the system might create severe multi-access interference to the UEs already in the system.

### C. Channel Estimation at the UE Side

The strong MPCs of the channel correspond to the components \( (k, k') \) in the matrix \( \mathbf{H}_s[\omega] \) with large second moment. Notice also that the element second moments \( \mathbb{E}[|\mathbf{H}_s[\omega] \mathbf{u}_{s,i}|^2] \) are invariant both with respect to \( s \) (time) and with respect to \( \omega \) (frequency). This follows immediately from the channel model definition and the assumption of uncorrelated MPCs. If we had direct access to measurements of the elements of \( \mathbf{H}_s[\omega] \), a naive approach would build estimators for the second moments (as sample mean), and try to identify the largest. However, this would require a number of RF chains equal to the number of antenna elements. In contrast, we have only access to the projections \( \bar{y}_{s,j}^H \mathbf{H}_s[\omega] \mathbf{u}_{s,i} \) from the observation in (20).

Using \( \bar{x}_{s,i}[\omega] = \sqrt{P_{\text{dim}}} \) in (20), we can write the received beacon symbol observation at the UE as

\[
y_{s,i,j}[\omega] = \sqrt{\frac{P_{\text{dim}}}{n}} \bar{y}_{s,j}^H \mathbf{H}_s[\omega] \mathbf{u}_{s,i} + \tilde{z}_{s,j}[\omega]
\]

\[
= \sqrt{\frac{P_{\text{dim}}}{n}} (\mathbf{u}_{s,i}^T \otimes \bar{y}_{s,j}^H) \tilde{h}_s[\omega] + \tilde{z}_{s,j}[\omega]
\]

\(^3\text{This tradeoff in the choice of the spreading parameters } F' \text{ and } \kappa_u, \kappa_v \text{ can be seen as an instance of the well-known exploration-exploitation tradeoff in statistics.}\)
where $\tilde{h}_s[\omega] = \text{vec}(\mathbf{H}_s[\omega])$ denotes the vectorized beamspace representation of the channel matrix at subcarrier $\omega \in \mathcal{F}_s$, where we used the well-known identity $\text{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})\text{vec}(\mathbf{B})$, and where we define the combined probing and sensing beamforming pattern as $\mathbf{g}_{s,i,j} = \mathbf{u}_{s,i} \otimes \mathbf{v}_{s,j} \in \mathbb{C}^{MN}$, which is common across all the subcarriers $\omega \in \mathcal{F}_s$ but differs for different pairs of RF chains $(i,j)$ at the BS and the UE.

In practice, each beacon slot is formed by a block of $S \geq 1$ OFDM symbols. With a slight abuse of notation, we index the symbols belonging to the $(s+1)$-th slot as $sS+s'$, for $s' \in [S]$. The instantaneous received power at the $j$-th RF chain of the UE from the signal transmitted by the $i$-th RF chain of the BS on the $s$-th beacon slot is given by

$$
\hat{q}_{s,i,j} = \frac{1}{SF'} \sum_{s' \in [S]} \sum_{\omega \in \mathcal{F}_s} |\tilde{h}_{sS+s',i,j}[\omega]|^2
+ \frac{1}{SF'} \sum_{s' \in [S]} \sum_{\omega \in \mathcal{F}_s} |\tilde{\zeta}_{sS+s',j}[\omega]|^2 + \frac{1}{SF'} \sum_{s' \in [S]} \sum_{\omega \in \mathcal{F}_s} \xi_{sS+s',j}[\omega]^2
= \frac{P_{\text{dim}}}{n} \mathbf{g}_{s,i,j}^H \left( \frac{1}{SF'} \sum_{s' \in [S]} \sum_{\omega \in \mathcal{F}_s} \tilde{h}_{sS+s',i,j}[\omega]\tilde{h}_{sS+s',i,j}[\omega]^H \right) \mathbf{g}_{s,i,j}
+ \frac{P_{\text{dim}}}{n} \sum_{s' \in [S]} \sum_{\omega \in \mathcal{F}_s} |\tilde{\zeta}_{sS+s',j}[\omega]|^2 + \frac{1}{SF'} \sum_{s' \in [S]} \sum_{\omega \in \mathcal{F}_s} \xi_{sS+s',j}[\omega]^2
,$$

where the first and the second term correspond to the signal contribution and to the noise contribution, and where

$$\xi_{sS+s',j}[\omega] = 2 \sqrt{\frac{P_{\text{dim}}}{n}} \text{Re} \left\{ \mathbf{g}_{s,i,j}^H \tilde{h}_{sS+s',i,j}[\omega] \tilde{\zeta}_{sS+s',j}[\omega]^H \right\}$$

denotes the signal-noise cross term. The key idea underlying our method follows from the fact that, when the number of dimensions $S \times F'$ over which the averaging of the instantaneous power is large, such that the signal-noise term becomes negligible and the empirical covariance matrix of the channel vector converges as

$$\frac{1}{SF'} \sum_{s' \in [S]} \sum_{\omega \in \mathcal{F}_s} \tilde{h}_{sS+s',i,j}[\omega]\tilde{h}_{sS+s',i,j}[\omega]^H \rightarrow E[\mathbf{h}_s[\omega]\mathbf{h}_s[\omega]^H] =: \Sigma_h.$$  

Meanwhile the noise term converges to

$$\frac{1}{SF'} \sum_{s' \in [S]} \sum_{\omega \in \mathcal{F}_s} |\tilde{\zeta}_{sS+s',j}[\omega]|^2 \rightarrow \sigma^2.$$  

The received power in (24) gives a 1-dimensional noisy projection of the covariance matrix $\Sigma_h$ with respect to the combined probing and sensing vector $\mathbf{g}_{s,i,j}$. It is important to note that $\Sigma_h$ is independent of the subcarrier and time slot indices $\omega$ and $s$, respectively, due to the channel stationarity in the frequency and time domain.

In order to derive our proposed estimation method, we approximate (24) as

$$\hat{q}_{s,i,j} \approx \frac{P_{\text{dim}}}{n} \mathbf{g}_{s,i,j}^H \Sigma_h \mathbf{g}_{s,i,j} + \sigma^2.$$  

When all the AoA-AoDs lie on a discrete grid, $\tilde{h}_s[\omega]$ is a sparse vector with i.i.d. components with only a few nonzero coefficients corresponding to the scatterers. Due to the independence of the channel gain of the scatterers, the covariance matrix of $\tilde{h}_s[\omega]$ would be a diagonal $MN \times MN$ matrix with only a few nonzero diagonal elements corresponding to the scatterers. In practice, $\Sigma_h$ is still sparse and approximately diagonal for sufficiently large $M$ and $N$ (as illustrated in Fig. 4), even if the AoA-AoDs of the scatterers do not lie on the discrete grid.

Using the expression for the probing and sensing vectors $\mathbf{u}_{s,i} = \frac{1}{\sqrt{\mathbf{u}_{s,i}^H \mathbf{u}_{s,i}}} \mathbf{u}_{s,i}$ and $\mathbf{v}_{s,j} = \frac{1}{\sqrt{\mathbf{v}_{s,j}^H \mathbf{v}_{s,j}}} \mathbf{v}_{s,j}$, respectively, (27) can be further simplified to

$$\hat{q}_{s,i,j} \approx \frac{P_{\text{dim}}}{n} \mathbf{g}_{s,i,j}^H \Sigma_h \mathbf{g}_{s,i,j} + \sigma^2.$$  

where $N_{s,i,j} = \mathbf{U}_{s,i} \times \mathbf{U}_{s,i}$ denotes the subset of AoA-AoD quantized directions probed/sensed by the beamforming vectors $\mathbf{u}_{s,i}$ and $\mathbf{v}_{s,j}$, respectively, and where $\hat{q}_{r,c} = E[|\mathbf{H}_s[\omega]_{r,c}|^2]$ denotes the second moment of the channel coefficient of the scatterer located at the discrete AoD $e$ at the BS and AoA $r$ at the UE. Notice that because of the sparsity of the channel in the virtual beam domain, most elements $\hat{q}_{r,c}$ are equal to or near zero, and only a few, corresponding to the directions $(r,c)$ strongly coupled by scatterers, are large. With reference to Fig. 4(b), letting $\Gamma$ denote the $N \times M$ matrix with elements $\Gamma_{r,c}$, we notice that the summation in (28) includes all elements $\Gamma_{r,c}$ at the index coordinates $(r,c) \in N_{s,i,j}$ at the crossing points of the probing directions (vertical lines in Fig. 4(b)) and the sensing directions (horizontal lines in Fig. 4(b)). Hence, the term $\frac{P_{\text{dim}}}{n} \sum_{(r,c) \in N_{s,i,j}} \Gamma_{r,c}$ in (28) can be further written as the inner product of $N_{s,i,j}$, $\gamma$ where $b_{s,i,j} := 1_{\mathbf{U}_{s,i}} \otimes 1_{\mathbf{U}_{s,i}}$ is a binary vector containing 1 at the AoA-AoDs probed by $\mathbf{U}_{s,i} \times \mathbf{U}_{s,i}$ and is 0 elsewhere, and where we define $\gamma := \frac{P_{\text{dim}}}{n} \text{vec}(\mathbf{I}) \in \mathbb{R}^{MN}$. In general, for a finite number of subcarriers, (27) – (28) hold only approximately since the statistical fluctuations are not negligible. We consider this as a residual noise $\tilde{w}_{s,i,j}$, such that the approximation in (28) yields the equality

$$\hat{q}_{s,i,j} = \mathbf{b}_{s,i,j}^T \gamma + \sigma^2 + \hat{w}_{s,i,j}.$$  

In each beacon slot, since the BS transmits along $m$ RF chain in each beacon slot and the UE has $n$ RF chains to sense the channel, the UE obtains $mn$ equations for the unknown vector $\gamma$ as in (29). Over $T$ beacon slots the UE obtains $mnT$ equations, which can be written in the form

$$\mathbf{q} = \mathbf{B} \gamma + \sigma^2 \mathbf{1} + \mathbf{w},$$  

where the vector $\mathbf{q} = [\hat{q}_{1,1,1, \ldots}, \hat{q}_{1,m,1, \ldots}, \ldots, \hat{q}_{T,m,n}, \ldots, \hat{q}_{T,m,n}]^T$ consists of all $mnT$ measurements calculated as in (24), where the $mnT \times MN$ matrix $\mathbf{B} = [b_{1,1,1, \ldots}, b_{1,m,1, \ldots}, \ldots, b_{T,m,n}, \ldots, b_{T,m,n}]^T$ is uniquely defined by the beamforming codebooks $\mathbf{C}_{\text{BS}}$ and $\mathbf{C}_{\text{UE}}$, and where $\mathbf{w} \in \mathbb{R}^{mnT}$ is the residual noise in the measurements. At this point, some remarks are in order.

Remark 3: In our proposed scheme, at each acquisition slot, each UE extracts its own set of measurements from its received signal. An implicit assumption, however, is that each UE is synchronized with the BS and knows the BS codebook $\mathbf{C}_{\text{BS}}$.
such that it can construct the matrix $\mathbf{B}$ in (30). This assumption is explicitly or implicitly made in virtually all works dealing with initial beam acquisition (aka, BA problem), as reviewed in Section I. Therefore, this is not a particularly restrictive assumption specific to our approach.

**Remark 4:** While the BS codebook $C_{BS}$ is broadcasted to all UEs, the sensing codebook $C_{UE}$ can be user-specific. In particular, we may imagine a system where each UE has a set of possible codebooks, characterized by a different number of (active) receive RF chains $n$ and sensing directions $\kappa_v$, and uses the most appropriate codebook depending on its hardware and value of SNR before beamforming. Intuitively, strong users (suffering from a small pathloss) can select larger $\kappa_v$ and/or $n$ to better explore the channel directions and estimate $\gamma$ faster, whereas weak users (suffering from a large pathloss) should select a smaller $\kappa_v$ and/or $n$ to attain a reasonable training SNR (see (22)). Although the weak users might need to wait longer to take more measurements before they are able to estimate their channels, (30) remains still valid since the channel gains (second order channel statistics) $\gamma$ are stable across many channel coherence blocks. This is in stark contrast with the conventional CS-based techniques used for BA via estimating the instantaneous complex channel gains, where the underlying channel might change drastically while taking the measurements, especially when only very few number of RF chains $m, n$ are available. Thus, in these schemes, $T$ is limited by the channel coherence time and this prevents from accumulating enough measurements and enough signal power. This effect is clearly visible in the results of Section IV, where we compare our method with state-of-the-art CS-based methods.

**D. Non-Negative Least Squares**

In order to identify the AoA-AoD directions of the strong scatterers, we estimate the $MN$ dimensional vector $\gamma$ from the $mnT$-dimensional observation given in (30). Because of the presence of the measurement noise $\mathbf{w}$, a standard approach consists of solving the Least-Squares (LS) problem $\min_{\gamma} \| \mathbf{B} \gamma + \sigma^2 \mathbf{1} - \hat{\mathbf{q}} \|^2$. However, in general $MN$ is significantly larger than $mnT$, such that the system of equations is heavily underdetermined and the LS solution yields meaningless results. The key observation here is that $\gamma$ is sparse (by assumption) and non-negative (by construction). Recent results in CS show that when the underlying parameter $\gamma$ is non-negative, the simple non-negative constrained LS given by

$$\gamma^* = \arg \min_{\gamma \in \mathbb{R}^{MN}} \| \mathbf{B} \gamma + \sigma^2 \mathbf{1} - \hat{\mathbf{q}} \|^2,$$

is still enough to impose sparsity of the solution $\gamma^*$ [31, 32], with no need for an explicit sparsity-promoting regularization term in the objective function as in the classical LASSO algorithm [33]. The (convex) optimization problem (31) is generally referred to as Non-Negative Least Squares (NNLS), and has been well investigated in the literature. An early reference is [34], showing that NNLS might yield a “Super-Resolution” property depending on the structure of the measurement matrix (e.g., $\mathbf{B}$ in our case). More recently, by the advent of CS [35, 36], the NNLS has reemerged in the context of sparse signal recovery, where it has been shown that the non-negativity constraint alone might suffice to recover the underlying signal $\gamma$ in the noiseless [37, 40] as well as in the noisy case [31, 32]. Moreover, [31] demonstrates that NNLS has a noisy recovery performance comparable to that of LASSO. In [31] it is also shown that NNLS along with an appropriate thresholding provides state-of-the-art performance in terms of support estimation. This property is very relevant in the context of this paper, where the identification of the support of $\gamma$ corresponds to finding the AoA-AoD directions strongly coupled by MPC.

In terms of numerical implementations, the NNLS can be posed as an unconstrained LS problem over the positive
orthant and can be solved by several efficient techniques such as Gradient Projection, Primal-Dual techniques, etc., with an affordable computational complexity [41]. We refer to [42, 43] for the recent progress on the numerical solution of NNLS and a discussion on other related work in the literature.

IV. PERFORMANCE EVALUATION

In this section we evaluate the the performance of our proposed algorithm via numerical simulations. To run the NNLS optimization in (31), we use the implementation of NNLS in MATLAB® called lsqnonneg.m.

Channel and Signal Model. For simplicity, we consider a very sparse channel model with only one path (one scatterer). We assume \( f_0 = 70 \) GHz carrier frequency and bandwidth of \( B = 1 \) GHz. The OFDM subcarrier spacing is \( 480 \) kHz in compliance with recent 3GPP standard specifications [44]. Assuming \( \tau_c \Delta f = 0.25 \) (i.e., the CP length is 25% of the OFDM duration) we obtain \( t_0 = 2.6 \) µs and around \( F = 2048 \) subcarriers (plus some guard band). We fix the frame duration of our scheme (i.e., the repetition interval of the beacon slot) to 1 ms, consists of 384 OFDM symbols (per subcarrier). A beacon slot contains \( S = 14 \) OFDM symbols, the random access slot also contains 14 OFDM symbols, and the remaining 356 symbols are used for data transmission [44]. Unless otherwise specified, we assume that the BS has \( M = 32 \) antennas and \( m = 3 \) RF chains, and the UE has \( N = 32 \) antennas and \( n = 2 \) RF chains. The simulations in this paper consider \( \text{SNR}_{BBF} = -33 \text{ dB} \) (unless otherwise stated) [44]. We announce an individual experiment to be successful if the index of the strongest component in \( \gamma \) is correctly estimated (i.e., it coincides with the actual scatterer location, up to the discrete angle grid quantization).

Dependence on the Random BS Codebook. We generated at random 4 different probing codebooks at the BS side. Fig. 5 illustrates the detection probability for the different pseudo-random codebooks, where the power spreading factors at the BS and the user sides are set to \( \kappa_u = \kappa_v = 16 \) respectively. We repeated each experiment 200 times and plot the resulting detection probability versus training period length \( T \). As expected, increasing \( T \) significantly improves the detection probability. More importantly, different codebooks have quite similar performances. This shows the fact that the performance of our scheme is quite insensitive to the choice of the random probing codebook, as long as it is sufficiently randomized.

Dependence on the Beam Spreading Factors \( \kappa_u \) and \( \kappa_v \). The spatial spreading factors \( \kappa_u \) and \( \kappa_v \) impose a trade-off between the angle coverage of the probing/sensing matrix \( B \) (exploration) and its receive SNR at the user side (exploitation). This is illustrated in Fig. 6. It is seen that increasing the spreading factor from \( \kappa_u = \kappa_v = 4 \) to \( \kappa_u = \kappa_v = 8 \) improves the performance. However, increasing \( \kappa_u, \kappa_v \) to \( \kappa_u = \kappa_v = 16 \) slightly degrades the performance, and the degradation is severe when \( \kappa_u, \kappa_v \) are increased to \( \kappa_u = \kappa_v = 25 \).

Dependence on the number of subcarriers \( F' \). As explained in Section III-B and III-C a large number of subcarriers \( F' \) deployed for each RF chain at the BS side (together with the factor \( S \), i.e., the number of OFDM symbols contained in a slot) ensures a reliable averaging of the instantaneous power received at the UE side. This averaging scheme makes the residual noise term in (29) approximately negligible, hence, ensuring a good performance of the NNLS estimator. However, increasing \( F' \) means a larger spreading of the total power. As shown in Fig. 6, increasing the number of subcarriers from \( F' = 1 \) to \( F' = 3 \) improves the performance, but increasing \( F' \) to \( F' = 10, 30 \) degrades the performance considerably.

Dependence on the probing dimensions \( (\kappa_u \kappa_v mn) \). Note that, for a certain pre-beamforming SNR, the output of the proposed BA scheme inherently depends on the probing dimensions, i.e., the product \( \kappa_u \kappa_v mn \). This is illustrated in Fig. 7. When \( (\kappa_u \kappa_v mn) \) are constant, the performance of the proposed scheme is independent of the specific values of
For a multi-user scenario, we deploy one RF chain at the BS side, where fixing the BS and channel parameters, the BA procedure can be completed in the shortest time.

**System-Level Scalability.** For a multi-user scenario, we denote by $K$ the total number of active users in the system, and by $K(T)$ the number of users that are able to successfully detect their strong MPC direction within $T$ frames. Fig. 10 compares the fraction $\frac{K(T)}{K}$ of those users in our scheme with the corresponding fraction in the interactive bisection method proposed in [11], where we assume an ideal feedback and cost free for each iterative round in [11]. As we can see, the training overhead of interactive methods scales proportionally with the number of active users, whereas in our scheme all the users are essentially trained simultaneously, so that the overhead does not grow with the number of users. Note that in practice, the feedback scheme for each iterative round in [11] costs UL transmissions and may not be ideal since the beamforming gains are very poor at the initial rounds. In contrast, the proposed scheme needs only one UL transmission of the control packet, where the full beamforming gain at the UE side and the sectored beamforming gain at the BS side (as discussed in Section III-A) are available.

**Robustness w.r.t. Variations in Channel Statistics.** To investigate the sensitivity of the proposed scheme as well as competing CS-based schemes to the time-variation of the channel coefficients, we consider a simple Gauss-Markov model for the channel correlation in time given by

$$
\rho_{s,t} = \alpha \rho_{s-1,t} + \sqrt{1-|\alpha|^2} \nu_{s,t}, \quad s \in \mathbb{Z}_+, \tag{32}
$$

where $\rho_{0,t} \sim \mathcal{CN}(0, \gamma_t)$, where $\nu_{s,t} \sim \mathcal{CN}(0, \gamma_t)$ is an i.i.d. sequence (innovation), and where $|\alpha| \in [0, 1]$ controls the channel correlation in time. We assume that the channel is constant over each beacon slot of 14 OFDM symbols, and evolves in time according to (32) from slot to slot, i.e., $\alpha$ denotes the channel correlation coefficient at samples taken 1 ms apart. Fig. 11 illustrates the comparison of the performance of our proposed scheme with that of the CS-based technique in [15]. It is seen that our method exhibits much robust performance across a wide range of channel time-correlations whereas the algorithm in [15] is quite fragile and fails.
comparison of detection probability given by the fraction of users whose channel is estimated until a given time slot $T$ given by $\frac{K(T)}{K}$. We take $M = N = 32$, $F' = 3$, $m = 3$, $n = 2$, $\kappa_u = \kappa_v = 8$, SNR$_{BBF} = -33$ dB.

![Comparison of detection probability $P_D$ between proposed NNLS and OMP in terms of the number of slots $T$.](image1)

In the UL (uplink), in our proposed scheme all the users are trained simultaneously. Thus, the BA scales very well with the number of active users in the system. We posed the BA as the estimation of the second order statistics of the channel and proposed a novel technique based on NNLS that achieves reliable recovery with high probability. We illustrated, via numerical simulations and comparison with other competitive techniques in the literature, that our algorithm is highly robust to variations in the channel statistics.

**ACKNOWLEDGMENTS**

X.S. is sponsored by the China Scholarship Council (201604910530). G.C. is supported by the Alexander von Humboldt Foundation through a Professorship Grant, and this work was also supported in part by a Collaborative Research Grant of Intel Research.

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