Condensation of ‘composite bosons’ in a rotating BEC

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We provide evidence for several novel phases in the dilute limit of rotating BECs. By exact calculation of wavefunctions and energies for small numbers of particles, we show that the states near integer angular momentum per particle are best considered condensates of composite entities, involving vortices and atoms. We are led to this result by explicit comparison with a description purely in terms of vortices. Several parallels with the fractional quantum Hall effect emerge, including the presence of the Pfaffian state.

In rotating superfluid $^4$He a vortex lattice forms which, on scales large compared to the vortex lattice parameter, has a velocity field indistinguishable from a rigid body co-rotating with the container. The vortices only perturb the fluid density significantly over a region of order the coherence length, $\xi$, around the core of each vortex (of the order of an Ångström). Hence the arrangement of the vortex lattice is governed by minimising the kinetic energy of the fluid in the rotating frame. One may say that the potential energy is ‘quenched’ by the incompressibility of the fluid.

In the Bose condensed alkali gases although so far it has proved difficult experimentally to investigate the rotational properties of the condensates, there has been a vigorous theoretical debate about the stability (or otherwise) of vortices in the condensates. At a mean field level (appropriate for moderate density), the inhomogeneity of the condensate density and the existence of surface waves due to the harmonic well makes the description difficult. Nevertheless the interparticle potential energy is still largely unaffected by the presence of vortices in the limit where the coherence length is small compared to the extent of the condensate: it is the kinetic energy (and the single-particle trap potential) which determines the vortex positions.

In this Letter we show that when the coherence length is comparable to the extent of the condensate, completely new effects occur. This is due to the kinetic (and single particle trap) energy being quenched, by a combination of spherical symmetry and the special properties of the harmonic well. Hence the ground state in the rotating frame is determined by the interparticle interactions alone, reminiscent of the fractional quantum Hall effect. Indeed we find stable states that are related to those found in the Hall effect (albeit in the less familiar regime of filling fraction, $\nu \gtrsim 1$). These include ‘condensates’ of composite bosons of the atoms attached to an integral number of quanta of angular momenta, as well as the Laughlin and Pfaffian states.

In a rotating reference frame, the standard Hamiltonian for $N$ weakly interacting atoms in a trap is:

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^{N} (-\nabla_i^2 + r_i^2) + \eta \sum_{j=1, j\neq i}^{N} \delta(r_i - r_j) - 2\omega \cdot L_i \quad (1)$$

where we have used the trap energy, $\hbar\sqrt{K/m} = \hbar\omega_0$ as the unit of energy and the extent, $(\hbar^2/MK)^{1/4}$, of the harmonic oscillator ground state as unit of length. (Here $M$ is the mass of an atom and $K$ the spring constant of the harmonic trap.) The coupling constant is defined as $\eta = 4\pi\bar{n}a(h^2/MK)^{-1/2}$ where $\bar{n}$ is the average atomic density and $a$ the scattering length. The angular velocity of the trap, $\omega$, is measured in units of the trap frequency.

In the dilute limit $\eta \ll 1$, which implies in existing experimental traps that the number of atoms, $N$, would be $10 \lesssim N \lesssim 1000$. Then the average coherence (or healing) length is $\xi \sim 1/\sqrt{\bar{n}} \rightarrow \infty$. It has been shown previously that in this limit, the problem becomes two-dimensional and the Hilbert space may be truncated to the ‘lowest Landau level’ states $| \psi_m(z) \rangle \propto z^m e^{-|z|^2/2}$, where $m \geq 0$ and $z = x + iy$ in the plane normal to $\omega$. Indeed at $\omega = 1$ the problem is identical to the Quantum Hall problem, with $\omega$ replacing the magnetic field.

We have determined the exact ground state, its energy, $E_0(\omega)$, and excitation gap, $\Delta$, for the Hamiltonian Eq. (1) using Mathematica for $N \leq 8$ and $\omega \leq 1$. In addition we have determined numerically the lowest eigenvalues for $N \leq 10$ as a function of $\omega$. $L_0(\omega)$, the angular momentum of the ground state, is plotted in Fig. for $N = 6$ with $\eta = 1/N$. Angular momentum remains a good quantum number as we have made no symmetry breaking ansatz.

A corresponding plot for $^4$He in a rotating container would show jumps in the expectation value of $L_0(\omega)$ as successive vortices enter the system. The inhomogeneous density of the condensate in a trap leads to more complex, but similar, behaviour in a mean field treatment (appropriate in the high density limit). There are a number of important features in Fig. which are common to all values of $N$ which we have studied.

Firstly, at $L = N$ there is a state which corresponds to one vortex,
\[ \psi^{1\nu}(\{z_i\}) = \prod_{i=1}^{N} (z_i - z_c) e^{-|z|^2/2} \]  

where \( z_c = (\sum_{i=1}^{N} z_i)/N \) is the centre of mass coordinate and \( |z|^2 = \sum_{i=1}^{N} |z_i|^2 \). From this point we will omit normalisation factors and the ubiquitous \( e^{-|z|^2/2} \). This state has an interparticle interaction energy \( E = \eta N(2-N)/4 \) and becomes stable at \( \omega_1 = (1-N\eta/4) \). (In addition at \( \omega_1 \) all \( N \geq L > 1 \) states are metastable.)

\[ L=nN. \]  

As \( L \) is reduced \( N-m \) particles remain with all \( n \) quanta and \( m \) have all the angular momentum removed. This is our main result, which occurs at small accessible angular momenta. This is reminiscent of the ‘bound state’ composite fermions [12]. We will return to this point.

We will now attempt to reconcile the composite boson states to the vortex states found in the Nonlinear Schrödinger equation [2]. The following argument indicates a connection. Consider incompressible irrotational fluid (‘Helium’) in a two-dimensional circular container, of radius \( R \), with \( n \) point vortices at radial coordinates \( r_\alpha \). There the angular momentum of the fluid is [13]:

\[ L(\{r_\alpha\}) = N(n-\sum_{\alpha=1}^{n} (r_\alpha/R)^2) \]  

i.e. the angular momentum is reduced from \( L = nN \) by the vortices being off-centre.

To test this notion, we firstly localise the vortices (resulting in a non-rotationally invariant state) by superposing states with different \( L \). Using \( L = 10 \) and \( L = 8 \) for \( N = 5 \) (the Pfaffian state rules out \( N = 6 \)) the contour plot of probability density, Fig. 3. The two ‘dimples’ might be interpreted as two off-centre vortices (hence the angular momentum is lower than \( L = 2N \)). The figure is reminiscent of the figures in [8], although the changes in density are rather small by comparison. Note however, the superposition is certain to create features periodic with \( \cos 2\theta \), where \( \theta \) is the polar angle.

![FIG. 1. Stable states for \( N = 6 \) in the rotating frame with \( \eta = 1/N \)](image)

Almost all of the other stable states can be labelled by \( L = n(N-m) \) where \( n \) and \( m \) are nonnegative integers (this includes the Laughlin state \( n = N, m = 1 \). These values are close to ‘\( n \)-vortex states’ \( (L=nN) \), a possibility we will return to. However the actual wavefunctions for these states most closely resemble some \([10]\) used in the theory of composite fermions \([11]\) in the Hall effect.

We define

\[ Q_n(z_i) = \frac{\partial^{(N-1-n)}}{\partial z_1^{(N-1-n)}} \prod_{j=1, j \neq i}^{N} (z_i - z_j) \]  

(Note: \( \psi^{\text{Landau}} = \prod_{i=1}^{N} Q_{N-1}(z_i) \) and \( \psi^{1\nu} = \prod_{i=1}^{N} Q_1(z_i) \).) Then the states of high overlap with the true states at \( L = n(N-m) \) may be written as:

\[ \psi_{n,m}(\{z_i\}) = \sum_{j_1 < j_2 < \cdots < j_{(N-m)}}^{N} Q_n(z_{j_1})Q_n(z_{j_2}) \cdots Q_n(z_{j_{(N-m)}}) \]

Table 1 shows the overlaps of \( \psi_{n,m} \) with the true ground states for those \( L \). Their construction ensures that angular momentum is used economically to lower the energy: any given particle pair, \( i \) and \( j \), will at most be associated with two factors of \( (z_i - z_j) \).

The interpretation of the states at \( L = n(N-m) \) is that a particle in association with \( n \) quanta of angular momentum is a particularly stable entity in the vicinity of \( L = nN \). As \( L \) is reduced \( N-m \) particles remain with all \( n \) quanta and \( m \) have all the angular momentum removed. This is our main result, which occurs at small accessible angular momenta. This is reminiscent of the ‘bound state’ composite fermions [12]. We will return to this point.

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To test this notion, we firstly localise the vortices (resulting in a non-rotationally invariant state) by superposing states with different \( L \). Using \( L = 10 \) and \( L = 8 \) for \( N = 5 \) (the Pfaffian state rules out \( N = 6 \)) the contour plot of probability density, Fig. 3. The two ‘dimples’ might be interpreted as two off-centre vortices (hence the angular momentum is lower than \( L = 2N \)). The figure is reminiscent of the figures in [8], although the changes in density are rather small by comparison. Note however, the superposition is certain to create features periodic with \( \cos 2\theta \), where \( \theta \) is the polar angle.

![FIG. 2. The probability density for the superposition of the states \( L = 2N \) and \( L = 2N-2 \) for \( N = 5 \) with ‘dimples’ reminiscent of vortex cores.](image)
symmetric polynomials, $C_r$, $0 \leq r \leq n$:

$$C_r(\{z_i\}) = \sum_{i_1 < i_2 < \cdots < i_r} z_{i_1} z_{i_2} \cdots z_{i_r}.$$

Although there is no need to write wavefunctions of both the particle, $z_i$, and the vortex coordinates, $\zeta$, it will be convenient.

It will be useful to note the form of the resulting particle wave function, $\psi$, when there is one vortex. The construction implies the natural one-vortex states are $\psi_{N-1} = \sum_{i=1}^{N} z_i = z_c$; and $\psi_0 = \prod_{i=1}^{N} z_i$, i.e. a ‘simple’ single vortex.

It is tempting to relate the stable states at $\zeta = 0$ to $\zeta = \frac{1}{2}$, consistent with the special cases:

Moreover, this purely vortex description, $\phi(\zeta)$, requires more vortices than $n$ (in $L = n(N-m)$). For example $L = N$: expanding the product in Eq.2 we see there is a term $z_c^N = C_1(\{z_i\})^N$ whose generation requires $N$ vortex factors (even more for larger $L$). In addition, the number of vortices is not fixed, as the number in the vortex state $\zeta^N$ is indeterminate since they do not affect the particle wave function (in a sense it is the vortex vacuum state). This is in stark contrast to the incompressible ($\xi \to 0$) case where the number of vortices is fixed and they are classical entities.

If the displaced vortex picture were correct, one might expect a factor in the vortex wavefunction of the form $(\zeta - \zeta')^2$ corresponding to the vortices rotating around the centre of the trap. This factor alone would lead to the following eigenvalues, $\rho_{nm}^v$ (with corresponding eigenvectors $\zeta^m$), of $\rho^v$: $\rho_{00} = \frac{1}{3}$, $\rho_{10} = \frac{1}{2}$ and $\rho_{20} = \frac{1}{4}$. As can be seen from Fig.3 this is not the case. The most pronounced feature is a maximum at $m = 0$. The vortices tend to condense, in the $m = 0$ state, not to separate in $|\zeta_1 - \zeta_2|$. (Further evidence comes from evaluating the particle density matrix $\rho^v$).

These difficulties in describing the $\psi_{n,m}$ states purely in vortex variables occur because the particles are binding to the vortices. This leads to a strongly correlated state whose description requires additional vortex variables if they are used alone. One interpretation uses ideas from the quantum Hall effect [12]. At the centre of each vortex there is a decrease in the particle density. Thus in terms of interparticle interactions, this is a low energy region for an additional particle.

Mathematically this is described most easily for the Laughlin state, using $N$ vortices, $\zeta_\alpha$, with a factor $\prod_{\alpha \neq \beta} (\zeta_\alpha - z_i)$ where the $\alpha$-th particle experiences no suppression of its amplitude: it is ‘bound’. This can be expressed as

$$\psi_{n,m}(\{z_i\}) = \int \prod_{\beta=1}^{N} d^3\zeta_{\beta} e^{-|\zeta|^2} e^{\sum_{\alpha \neq j} \zeta_\alpha - z_j}$$

(Noting $e^{\eta^n}$ and $(-1)^n \eta^n e^{\eta^n}$ respectively play the roles of a delta function and its $n$th derivative within the lowest Landau level.) i.e. the delta function factors bind the $i$th particle to the $i$th vortex.

To generate the states $\psi_{n,m}$ we use the derivatives of the Lowest Landau level delta function so that:

$$\psi_{n,m}(\{z_i\}) = \int \prod_{\beta=1}^{N} d^3\zeta_{\beta} e^{-|\zeta|^2} e^{\sum_{\alpha \neq j} \zeta_\alpha - z_j}$$

where

$$\phi_{n,m}(\zeta_{\alpha}) = \prod_{\beta=1}^{N} \zeta_{\beta}^{(N-1-n)} \sum_{\gamma_1 < \gamma_2 < \cdots < \gamma_m} \zeta_{\gamma_1}^{*n} \zeta_{\gamma_2}^{*n} \cdots \zeta_{\gamma_m}^{*n}$$

which can be interpreted as a ‘condensate’ of $(N - m)$ composites (each consisting of an atom and a vortex) in the state, $\zeta^r$ with $r = (N - 1 - n)$. The remaining unbound atoms remain condensed in the single particle ground state. (The states $L = n(N-m)$ are also selected using a composite fermion approach [17].)

The remaining stable states are consistent with the bosonic Pfaffian state [17], at $L = \frac{1}{2} N(N-2)$ for even $N$ and $L = \frac{1}{4} (N-1)^2$ for odd $N$.

$$\psi^{PF}(\{z_i\}) = \prod_{i<j} (z_i - z_j) Pf \left( \frac{1}{z_i - z_j} \right)$$

where the Pfaffian is defined

$$Pf \left( \frac{1}{z_i - z_j} \right) = A \left[ \frac{1}{(z_1 - z_2) (z_3 - z_4) \cdots (z_{N-1} - z_N)} \right]$$

FIG. 3. Eigenvalues ($\rho_{nm}^v$) of the single vortex density matrix for eigenfunctions $\zeta^m$ for $L = 2N-2$ for $N = 6$. The trace is normalised to unity, having suppressed the weight associated with the vortex ‘vacuum’ state, $m = N$. 

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where the Pfaffian is defined

$$Pf \left( \frac{1}{z_i - z_j} \right) = A \left[ \frac{1}{(z_1 - z_2) (z_3 - z_4) \cdots (z_{N-1} - z_N)} \right]$$
where $A$ denotes antisymmetrisation of the following product. (This is generalised for odd $N$ by omitting one of the particles in each term of the antisymmetrisation.) The overlaps of $\psi^{Pf}$ with the exact ground state for $N = 5$ and $L = 8$, $N = 6$ and $L = 12$ and $N = 7$ and $L = 18$ are: 0.91$^2$, 0.90$^2$ and 0.80$^2$.

Some nearby stable states, e.g. $L = 10$ and $L = 14$, are well described as simple modifications of the Pfaffian state. This uses the conjecture (which has been demonstrated by direct evaluation for $4 \leq N \leq 8$) that the Pfaffian state may be represented by a product of two Laughlin states for $N/2$ particles (or for odd $N$, a cluster of $(N-1)/2$ and one of $(N+1)/2$):

$$\psi^{Pf}(\{z_i\}) = S \prod_{i<j \in \sigma_1} (z_i - z_j)^2 \prod_{k<l \in \sigma_2} (z_k - z_l)^2$$

where the two subsets, $\sigma_1$ and $\sigma_2$, each have $N/2$ particles ($(N-1)/2$ and $(N+1)/2$ for odd $N$). $S$ indicates that the wave function is symmetrised over the distribution of the particles into these subsets. These two well-correlated clusters appear to be ‘dual’ to the clusters of Halperin [8] which have a high internal energy, due to the lack of nodal factors.

For example the state $N = 6$, $L = 14$ has overlap 0.96$^2$ with a state with two quanta of angular momenta in the centre of mass motion of the clusters (defining $Z_b = \sum_{i \in \sigma_b} z_i$, $b = 1$ or 2):

$$\psi^{L=14}(\{z_i\}) = S(Z_1 - Z_2)^2 \psi^{Pf}$$

The state $N = 6$, $L = 10$ has overlap 0.97$^2$ with a state where there is one factor of centre of mass motion and one vortex has been ‘removed’ from one of the clusters:

$$\psi^{L=10}(\{z_i\}) = S(Z_1 - Z_2) \psi^{Pf} \prod_{\sigma_p \neq \sigma_q \neq \sigma_1} \frac{1}{z_p - z_q}$$

(The apparent asymmetry of the last factor only involving the first cluster, $\sigma_1$, is illusory due to the overall symmetrisation.)

In conclusion, this Letter provides evidence that the weak coupling limit of rotating BEC’s contains some novel phenomena. These occur even in the regime where one would anticipate small numbers of vortices and hence should be open to experimental investigation in the near future.

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| 2N−4 | 2N−2 | 2N | 3N−6 | 3N−3 | 3N | 4N−8 | 4N−4 | 4N | 5N−10 | 5N−5 | 5N | 6N−6 | 6N |
|------|------|-----|------|------|----|------|------|----|-------|------|----|------|----|
| 5    | 8    | 10  | 0.98$^2$ | 0.87$^2$ | 12 | 15  | 0.96$^2$ | 20 | 1 |
| 6    | 10   | 12  | 0.86$^2$ | 0.69$^2$ | 12 | 15  | 0.95$^2$ | 20 | 24 | 0.94$^2$ | 30 | 1 |
| 7    | 12   | 0.83$^2$ | 15  | 0.49$^2$ | 18 | 0.85$^2$ | 24 | 30 | 35 | 42 | 1 |
| 8    | 12   | 0.66$^2$ | 14  | 0.83$^2$ | 18 | 0.85$^2$ | 24 | 30 | 35 | 42 | ... |

TABLE 1. Stable states for $N \leq 8$: the upper number is their angular momentum and the lower is their overlap with the $Q$ wavefunctions. ♠ indicates that the wavefunction can also be written (or derived from) a Pfaffian state.

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