Chern-Simons theory, matrix integrals, and perturbative three-manifold invariants

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**Abstract:** The universal perturbative invariants of rational homology spheres can be extracted from the Chern-Simons partition function by combining perturbative and nonperturbative results. We spell out the general procedure to compute these invariants, and we work out in detail the case of Seifert spaces. By extending some previous results of Lawrence and Rozansky, the Chern-Simons partition function with arbitrary simply-laced group for these spaces is written in terms of matrix integrals. The analysis of the perturbative expansion amounts to the evaluation of averages in a Gaussian ensemble of random matrices. As a result, explicit expressions for the universal perturbative invariants of Seifert homology spheres up to order five are presented.

**Keywords:** Chern-Simons theory
1. Introduction

Chern-Simons theory [44] has been at the heart of the developments in three-manifold topology and knot theory for the last ten years. The partition function of Chern-Simons theory defines a topological invariant of three-manifolds, sometimes known as the Witten-Reshetikhin-Turaev invariant, that can be studied from many different points of view. In general, the invariant thus obtained contains information about the three-manifold itself but also about the gauge theory group that one uses to define the theory.

However, from a perturbative point of view it is clear that one can extract numerical invariants of the three-manifold which are intrinsic to it and do not depend on the gauge group. This goes as follows: if we compute the partition function in perturbation theory, the contribution at a given order consists of a sum of terms associated to Feynman diagrams. Each term is the product of a group dependent factor (the group weight of the diagram), and a factor involving multiple integrals.
of the propagators over the three-manifold. This last factor does not depend on the gauge group one started with, and in this sense it is universal. Therefore, one can extract from perturbation theory an infinite series of invariants, the so-called universal perturbative invariants of three-manifolds.

The idea of looking at the perturbative expansion of Chern-Simons theory in order to extract numerical invariants that “forget” about the gauge group was first implemented in the context of knot invariants, leading to the theory of Vassiliev invariants and the Kontsevich integral (see [7, 28]). The perturbative approach to the study of the partition function of Chern-Simons theory has a long story, starting in [14]. This has been pursued from many points of view. On the one hand, the structure of the perturbative series has been analyzed in detail (see for example [3, 5] and [16] for a nice review), leading to the graph homology of trivalent graphs as a systematic tool to organize the expansion. On the other hand, the asymptotic expansion of the nonperturbative results has also been studied [26, 35, 36, 37, 29, 30], although so far all the analysis have focused on theories with gauge group SU(2). Finally, a mathematically rigorous theory of universal perturbative invariants of three-manifolds has been constructed starting from the Kontsevich integral: the so-called LMO invariant [31] and its Aarhus version [8].

The main goal of the present paper is to elaborate on the topological field theory approach to universal perturbative invariants. The point of view presented here is very similar to the one advocated in [1, 2] to extract Vassiliev invariants from Chern-Simons perturbation theory: first, one analyzes the structure of the perturbative series of an observable in the theory. This means in practical terms choosing a basis of independent group factors and compute its value for various gauge groups. In a second step, one computes the corresponding invariant nonperturbatively for those gauge groups, performs an asymptotic expansion, and extracts the universal invariants by comparing to the perturbative result. This program was applied successfully in [1, 2] to compute Vassiliev invariants of many knots. It turns out that, in the case of the Chern-Simons partition function, the first step is relatively easy, but the calculation of the partition function for arbitrary gauge groups in a way that is suitable for an asymptotic expansion turns out to be trickier, except in very simple cases.

In this paper, some well-known results concerning the structure of the perturbative series are put together, and we carry out a detailed analysis up to order five. The focus is on a rather general class of rational homology spheres, Seifert spaces. The partition function of Chern-Simons theory with gauge group SU(2) on these spaces and its asymptotic expansion have been studied in [20, 33, 36, 37, 29, 30]. The extension to higher rank gauge groups has also been considered [11, 23] but in forms that are not useful for a systematic perturbative expansion. In [31], Lawrence and Rozansky found a beautiful expression for the SU(2) partition function on Seifert spaces in terms of a sum of integrals and residues. It turns out that their result can be generalized to any simply-laced group and written in terms of integrals over the Cartan subalgebra of the gauge group (these kind of integrals already appeared in a related context in [36]). Interestingly, they are closely related to models of random matrices, and one can use matrix model technology to study the Chern-Simons partition function on these spaces. The resulting expressions can be expanded in series in a fairly systematic way, and by comparing the result with the general structure of the perturbative expansion, the universal perturbative invariants can be extracted. It should be mentioned that the full LMO invariant of Seifert spaces has been computed by Bar-Natan and Lawrence [9] by using techniques from the
theory of the Aarhus integral. However, their result is rather implicit and involves a complicated graphical calculus.

This paper is organized as follows: in section 2, we review the computation of the Chern-Simons partition function starting from a surgery presentation. In section 3, we analyze in some detail the structure of the Chern-Simons perturbation series. In section 4 we compute the exact partition function of Seifert spaces for simply-laced gauge groups, generalizing the results of Lawrence and Rozansky, and we make the connection to matrix models. In section 5, we analyze the asymptotic expansion of the exact result, explain how to evaluate the matrix integrals, and present the results for universal perturbative invariants up to order five. In section 6, we comment on the possible relevance of these results to other physical contexts, and some avenues for future research are suggested. The Appendix collects the explicit expressions for the group factors and the matrix integrals, together with a summary of the properties of symmetric functions that are used in the paper.

2. The partition function of Chern-Simons theory

In this section we review some well-known results about the computation of the Chern-Simons partition function in terms of surgery presentations. An excellent summary, that we follow quite closely, is given in [36].

We consider Chern-Simons theory on a three-manifold \( M \) and for a simply-laced gauge group \( G \), with action

\[
S(A) = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),
\]

(2.1)

where \( A \) is a \( G \)-connection on \( M \). We will be interested in framed three-manifolds, i.e. a three-manifold together with a trivialization of the bundle \( TM \oplus TM \). As explained in [4], for every three-manifold there is a canonical choice of framing, and the different choices are labeled by an integer \( s \in \mathbb{Z} \) in such a way that \( s = 0 \) corresponds to the canonical framing. Unless otherwise stated, we will always work in the canonical framing, and we will explain below how to incorporate this in the calculations, following [26, 20, 30].

As shown by Witten in [14], the partition function of Chern-Simons theory

\[
Z_k(M) = \int \mathcal{D}A e^{iS_{CS}(A)}.
\]

(2.2)

defines an invariant of framed manifolds. There is a very nice procedure to evaluate (2.2) in a combinatorial way which goes as follows. By Lickorish theorem (see for example [32]), any three-manifold \( M \) can be obtained by surgery on a link \( \mathcal{L} \) in \( S^3 \). Let us denote by \( \mathcal{K}_i, i = 1, \cdots, L \), the components of \( \mathcal{L} \). The surgery operation means that around each of the knots \( \mathcal{K}_i \) we take a tubular neighborhood \( \text{Tub}(\mathcal{K}_i) \) that we remove from \( S^3 \). This tubular neighborhood is a solid torus with a contractible cycle \( \alpha_i \) and a noncontractible cycle \( \beta_i \). We then glue the solid torus back after performing an \( SL(2, \mathbb{Z}) \) transformation given by the matrix

\[
U(p_i, q_i) = \begin{pmatrix} p_i & r_i \\ q_i & s_i \end{pmatrix}.
\]

(2.3)
This means that the cycles $p_i\alpha_i + q_i\beta_i$ and $r_i\alpha_i + s_i\beta_i$ on the boundary of the complement of $K_i$ are identified with the cycles $\alpha_i, \beta_i$ in $\text{Tub}(K_i)$.

This geometric description leads to the following prescription to compute the invariants in Chern-Simons theory. By canonical quantization, one associates a Hilbert space to any two-dimensional compact manifold that arises as the boundary of a three-manifold, so that the path-integral over a manifold with boundary gives a state in the corresponding Hilbert space. As it was shown in [44], the states of the Hilbert space of Chern-Simons theory associated to the torus are in one to one correspondence with the integrable representations of the WZW model with gauge group $G$ at level $k$. We will use the following notations in the following: $r$ denotes the rank of $G$, and $d$ its dimension. $y$ denotes the dual Coxeter number. The fundamental weights will be denoted by $\lambda_i$, and the simple roots by $\alpha_i$, with $i = 1, \cdots, r$. The weight and root lattices of $G$ are denoted by $\Lambda_w$ and $\Lambda_r$, respectively. Finally, we put $l = k + y$.

A representation given by a highest weight $\Lambda$ is integrable if the weight $\rho + \Lambda$ is in the fundamental chamber $F_l$ ($\rho$ denotes as usual the Weyl vector, given by the sum of the fundamental weights). The fundamental chamber is given by $\Lambda_w/l\Lambda_r$ modded out by the action of the Weyl group. For example, in $SU(N)$ a weight $p = \sum_{i=1}^r p_i\lambda_i$ is in $F_l$ if

$$\sum_{i=1}^r p_i < l, \quad \text{and} \quad p_i > 0, \quad i = 1, \cdots, r.$$  

In the following, the basis of integrable representations will be labeled by the weights in $F_l$.

In the case of simply-laced gauge groups, the $\text{Sl}(2, \mathbb{Z})$ transformation given by $U(p, q)$ has the following matrix elements in the above basis [26, 36]:

$$U_{\alpha\beta}(p, q) = \frac{|\text{sign}(q)|^{\Delta_+}}{(l|q|)^{r/2}} \exp\left[ -\frac{i\pi}{12} \Phi(U(p, q)) \left( \frac{\text{Vol} \Lambda_w}{\text{Vol} \Lambda_r} \right) \frac{l}{2} \right]$$

$$\cdot \sum_{n \in \Lambda_r/l\Lambda_r} \sum_{w \in \mathcal{W}} \epsilon(w) \exp\left\{ i\pi lq (p\alpha^2 - 2\alpha(n + w(\beta)) + s(ln + w(\beta))^2) \right\}.$$  

In this equation, $|\Delta_+|$ denotes the number of positive roots of $G$, and the second sum is over the Weyl group $\mathcal{W}$ of $G$. $\Phi(U(p, q))$ is the Rademacher function:

$$\Phi \begin{bmatrix} p & r \\ q & s \end{bmatrix} = \frac{p + s}{q} - 12s(p, q),$$  

where $s(p, q)$ is the Dedekind sum

$$s(p, q) = \frac{1}{4q} \sum_{n=1}^{q-1} \cot \left( \frac{\pi n}{q} \right) \cot \left( \frac{\pi np}{q} \right).$$  

With these data we can already present Witten’s result for the Chern-Simons partition function of $M$. As before, suppose that $M$ is obtained by surgery on a link $\mathcal{L}$ in $S^3$. Then, the partition
function of $M$ is given by:

$$Z(M, l) = e^{i\phi_{fr}} \sum_{\alpha_1, \ldots, \alpha_L \in F_l} Z_{\alpha_1, \ldots, \alpha_L}(L) U^{(p_1, q_1)} \cdots U^{(p_L, q_L)}. \quad (2.8)$$

In this equation, $Z_{\alpha_1, \ldots, \alpha_L}(L)$ is the invariant of the link $L$ with representation $\alpha_i - \rho$ attached to its $i$-th component (recall that the weights in $F_l$ are of the form $\rho + \Lambda$). The phase factor $e^{i\phi_{fr}}$ is a framing correction that guarantees that the resulting invariant is in the canonical framing for the three-manifold $M$. Its explicit expression is:

$$\phi_{fr} = \frac{\pi k d}{12l} \left( \sum_{i=1}^{L} \Phi(U^{(p_i, q_i)}) - 3\sigma(L) \right), \quad (2.9)$$

where $\sigma(L)$ is the signature of the linking matrix of $L$.

3. Chern-Simons perturbation theory

The expression (2.8) gives the nonperturbative result for the partition function of $M$, and allows an explicit evaluation for many three-manifolds for any gauge group $G$ and level $k$. However, from the point of view of Chern-Simons perturbation theory, the partition function can be also understood as an asymptotic series in $l^{-1}$, whose coefficients can be computed by evaluating Feynman diagrams. In this section we review some known facts about the perturbative expansion of Chern-Simons theory and we state our strategy to compute the universal perturbative invariants.

We are interested in the perturbative evaluation of the partition function (2.2). Let us assume (as we will do in this paper) that $M$ is a rational homology sphere. The classical solutions of the Chern-Simons action are just flat connections on $M$, and for a rational homology sphere these are a finite set of points. Therefore, in the perturbative evaluation one expresses $Z_k(M)$ as a sum of terms associated to stationary points:

$$Z_k(M) = \sum_c Z_k^{(c)}(M), \quad (3.1)$$

where $c$ labels the different flat connections $A^{(c)}$ on $M$. Each of the terms in this sum has a perturbative expansion as an asymptotic series in $l^{-1}$. The structure of the perturbative series was analyzed in various papers [44, 37, 3] and is given by the following expression:

$$Z_k^{(c)}(M) = Z_{1\text{-loop}}^{(c)}(M) \cdot \exp \left\{ \sum_{\ell=1}^{\infty} S_\ell^{(c)} x^\ell \right\}. \quad (3.2)$$

In this equation, $x$ is the effective expansion parameter:

$$x = \frac{2\pi i}{l}. \quad (3.3)$$
The one-loop correction $Z^{(c)}_{1\text{-loop}}(M)$ was first analyzed in [44], and has been studied in big detail since then. It has the form,

$$Z^{(c)}_{1\text{-loop}}(M) = \frac{(2\pi x)^{\frac{1}{2}(\dim H^0 - \dim H^1)}}{\text{vol}(H_c)} e^{-\frac{1}{4}S_{\text{CS}}(A^{(c)}) - \frac{i}{4} \varphi \sqrt{|\tau^{(c)}_R|}},$$  \hspace{1cm} (3.4)

where $H^{0,1}_c$ are the cohomology groups with values in the Lie algebra of $G$ associated to the flat connection $A^{(c)}$, $\tau^{(c)}_R$ is the Reidemeister-Ray-Singer torsion of $A^{(c)}$, $H_c$ is the isotropy group of $A^{(c)}$, and $\varphi$ is a certain phase. More details about the structure of this term can be found in [44, 20, 26, 35, 36].

Our main object of concern in this paper are the terms in the exponential of (3.2) corresponding to the trivial connection, which we will simply denote by $S_\ell$. In order to make a precise statement about the structure of these terms, we have to explain in some detail what is the appropriate set of diagrams we want to consider. In principle, in order to compute $S_\ell$ we just have to consider all the connected bubble diagrams with $\ell$ loops. To each of these diagrams we will associate a group factor times a Feynman integral. However, not all these diagrams are independent, since the underlying Lie algebra structure imposes the Jacobi identity:

$$\sum_e (f_{abc} f_{edc} + f_{dae} f_{ecb} + f_{ace} f_{edb}) = 0.$$  \hspace{1cm} (3.5)

This leads to the diagram relation known as IHX relation. Also, antisymmetry of $f_{abc}$ leads to the so-called AS relation (see for example [7, 16, 28, 39]). The existence of these relations between diagrams suggests to define an equivalence relation in the space of connected trivalent graphs by quotienting by the IHX and the AS relations, and this gives the so-called graph homology. The space of homology classes of connected diagrams will be denoted by $\mathcal{A}(\emptyset)^{\text{conn}}$. This space is graded by half the number of vertices, and this number gives the degree of the graph. The space of homology classes of graphs at degree $\ell$ is then denoted by $\mathcal{A}(\emptyset)_\ell^{\text{conn}}$. For every $\ell$, this is a finite-dimensional vector space of dimension $d(\ell)$. The dimensions of these spaces are explicitly known for low degrees (see for example [39]), and we have listed some of them in Table 1. Finally, notice that, given any group $G$, we have a map

$$\mathcal{A}(\emptyset)^{\text{conn}} \longrightarrow \mathbb{R}$$  \hspace{1cm} (3.6)

that associates to every graph $\Gamma$ its group theory factor $r_T(G)$. This map is an example of a weight system for $\mathcal{A}(\emptyset)^{\text{conn}}$. Every gauge group gives a weight system for $\mathcal{A}(\emptyset)^{\text{conn}}$, but one may in principle find weight systems not associated to gauge groups, although so far the only known example is the one constructed by Rozansky and Witten in [38], which uses instead hyperKähler manifolds.

| $\ell$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------|---|---|---|---|---|---|---|---|---|----|
| $d(\ell)$ | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 8 |

**Table 1:** Dimensions $d(\ell)$ of $\mathcal{A}(\emptyset)_\ell^{\text{conn}}$ up to $\ell = 10$. 


We can now state very precisely what is the structure of the $S_\ell$ appearing in (3.2): since the Feynman diagrams can be grouped into homology classes, we have

$$S_\ell = \sum_{\Gamma \in \mathcal{A}(\emptyset)^{\text{conn}}_\ell} r_\Gamma(G) I_\Gamma(M). \quad (3.7)$$

The factors $I_\Gamma(M)$ appearing in (3.7) are certain (complicated) integrals of propagators over $M$. It was shown in [5] that these are differentiable invariants of the three-manifold $M$, and since the dependence on the gauge group has been factored out, they only capture topological information of $M$, in contrast to $Z_k(M)$, which also depends on the choice of the gauge group. These are the *universal perturbative invariants* defined by Chern-Simons theory. Notice that, at every order $\ell$ in perturbation theory, there are $d(\ell)$ independent perturbative invariants. Of course, these invariants inherit from $\mathcal{A}(\emptyset)^{\text{conn}}_\ell$ the structure of a finite-dimensional vector space, and it is convenient to pick a basis once and for all. Here we will study these invariants up to order 5, and we choose the basis presented by Sawon in [39]:

$$\begin{align*}
\ell = 1 : & \quad \includegraphics[scale=0.5]{ell1_graph} \\
\ell = 2 : & \quad \includegraphics[scale=0.5]{ell2_graph} \\
\ell = 3 : & \quad \includegraphics[scale=0.5]{ell3_graph} \\
\ell = 4 : & \quad \includegraphics[scale=0.5]{ell4_graph} \\
\ell = 5 : & \quad \includegraphics[scale=0.5]{ell5_graph}
\end{align*} \quad (3.8)$$

As in [39], we will denote the graphs with $k$ circles joined by lines by $\theta_k$. Therefore, the graph corresponding to $\ell = 1$ will be denoted by $\theta$, the graph corresponding to $\ell = 2$ will be denoted $\theta_2$, and so on. The second graph for $\ell = 4$ will be denoted by $\omega$, and the second graph in $\ell = 5$ by $\omega \theta$. The group factors associated to these diagrams can be easily computed by using the techniques of [12] (see also [3, 7]). Explicit results for all classical gauge groups are presented in the Appendix.

Remarks:

1. It is interesting to understand the framing dependence of the universal perturbative invariants (see [5] for a discussion of this issue). As shown in [44], the full partition theory $Z_k(M)$ changes as follows under a change of framing:

$$Z \rightarrow e^{\frac{\pi i s}{k}} Z, \quad (3.9)$$

where $s \in \mathbb{Z}$ labels the choice of framing and

$$c = \frac{kd}{k+y} \quad (3.10)$$
is the central charge of the WZW model with group $G$. Using now that (see Appendix A)

$$r_\theta(G) = 2yd,$$

(3.11)

we find that under a change of framing one has

$$I_\theta(M) \rightarrow I_\theta(M) - \frac{s}{48},$$

(3.12)

while the other universal perturbative invariants remain the same. Since we will work in the canonical framing of $M$, this will produce a canonical value of $I_\theta(M)$.

2. Notice that Chern-Simons theory detects the graph homology through the weight system associated to Lie algebras. Unfortunately it is known [43] that there is an element of graph homology at degree 16 that it is not detected by any weight system associated to simple Lie algebras. However, there is a very elegant mathematical definition of the universal perturbative invariant of a three-manifold that works directly in the graph homology. This is called the LMO invariant [31] and it is a formal linear combination of homology graphs with rational coefficients:

$$\omega(M) = \sum_{\Gamma \in \mathcal{A}(\emptyset)^{\text{conn}}} I^{\text{LMO}}_\Gamma(M) \Gamma \in \mathcal{A}(\emptyset)^{\text{conn}}[Q].$$

(3.13)

It is believed that the universal invariants extracted from Chern-Simons perturbation theory agree with the LMO invariant. More precisely, since the LMO invariant $\omega(M)$ is taken to be 0 for $S^3$, we have:

$$I^{\text{LMO}}_\Gamma(M) = I_\Gamma(M) - I_\Gamma(S^3),$$

(3.14)

as long as the graph $\Gamma$ is detected by Lie algebra weight systems. In that sense the LMO invariant is more refined than the universal perturbative invariants extracted from Chern-Simons theory.

3. The Chern-Simons approach to the theory of universal perturbative invariants is very similar to the approach to Vassiliev invariants based on the analysis of vevs of Wilson loops in perturbation theory [1, 2]. The role of graph homology is played there by the homology of chord diagrams (see for example [3, 28]).

4. The Chern-Simons partition function on Seifert spaces

In this section we write the partition function of Chern-Simons theory on Seifert homology spheres as a sum of integrals over the Cartan subalgebra and a set of residues, by extending results of Lawrence and Rozansky [30] for $SU(2)$. We also show that these integrals can be interpreted in terms of matrix integrals associated to a random matrix model.

4.1 Seifert homology spheres

Seifert homology spheres can be constructed by performing surgery on a link $L$ in $S^3$ with $n + 1$ components, consisting of $n$ parallel and unlinked unknots together with a single unknot whose linking number with each of the other $n$ unknots is one. The surgery data are $p_j/q_j$ for the unlabeled
unknots, \( j = 1, \cdots, n \), and 0 on the final component. \( p_j \) is coprime to \( q_j \) for all \( j = 1, \cdots, n \), and the \( p_j \)'s are pairwise coprime. After doing surgery, one obtains the Seifert space \( M = X(\frac{m_1}{q_1}, \cdots, \frac{m_n}{q_n}) \). This is rational homology sphere whose first homology group \( H_1(M, \mathbb{Z}) \) has order \(|H|\), where

\[
H = P \sum_{j=1}^{n} \frac{q_j}{p_j}, \quad \text{and} \quad P = \prod_{j=1}^{n} p_j. \tag{4.1}
\]

Another topological invariant that will enter the computation is the signature of \( L \), which turns out to be \( \sigma(L) = \sum_{i=1}^{n} \operatorname{sign}(\frac{q_i}{p_i}) - \operatorname{sign}\left(\frac{H}{P}\right). \tag{4.2}\)

For \( n = 1, 2 \), Seifert homology spheres reduce to lens spaces, and one has that \( L(p, q) = X(\frac{q}{p}) \). For \( n = 3 \), we obtain the Brieskorn homology spheres \( \Sigma(p_1, p_2, p_3) \) (in this case the manifold is independent of \( q_1, q_2, q_3 \)). In particular, \( \Sigma(2, 3, 5) \) is the Poincaré homology sphere. Finally, the Seifert manifold \( X(\frac{2}{-1}, \frac{m}{(m+1)/2}, \frac{t-m}{1}) \), with \( m \) odd, can be obtained by integer surgery on a \((2, m)\) torus knot with framing \( t \).

### 4.2 Computation of the partition function

In order to compute the partition function of \( M \), we first have to compute the invariant of \( L \) for generic representations \( \beta - \rho, \Lambda_1, \cdots, \Lambda_n \) of the gauge group \( G \), where \( \beta - \rho \) is the irreducible representation coloring the unknot with surgery data 0, and \( \Lambda_i \) are irreducible representations coloring the unknots with surgery data \( p_i/q_i, i = 1, \cdots, n \). This can be easily done by using the formula of [44] for connected sums of knots, and one obtains:

\[
Z_{\beta, \rho}^{\Lambda_1, \cdots, \Lambda_n}(L) = \prod_{i=1}^{n} \sum_{\rho + \Lambda_i \in \mathcal{F}_i} S_{\beta \rho + \Lambda_i} \mathcal{U}_{\rho + \Lambda_i \rho}^{(p_i/q_i)}, \tag{4.3}
\]

where the framing correction is given by the general formula \( (2.9) \). Seifert homology spheres can be also obtained by doing surgery on \( n \) strands parallel to \( S^1 \) in \( S^2 \times S^1 \) \[33\], and then \( (4.4) \) follows from Verlinde’s formula \[12\].

This expression is not suitable for an asymptotic expansion in \( 1/l \), since it involves a sum over integrable representations that depends itself on \( l \). In order to obtain a useful expression, we follow a series of steps generalizing the procedure in \[30, 34\]. First of all, we perform the matrix multiplication \( \sum_{\rho + \Lambda_i \in \mathcal{F}_i} S_{\beta \rho + \Lambda_i} \mathcal{U}_{\rho + \Lambda_i \rho}^{(p_i/q_i)} \). This gives

\[
\sum_{\rho + \Lambda_i \in \mathcal{F}_i} S_{\beta \rho + \Lambda_i} \mathcal{U}_{\rho + \Lambda_i \rho}^{(p_i/q_i)} = \exp\left(\frac{\pi i k d}{4l} \operatorname{sign}\left(\frac{q_i}{p_i}\right)\right) U_{\beta \rho}^{(-q_i/p_i)}, \tag{4.5}
\]

Therefore, the partition function of \( M \) will be given by

\[
Z_k(M) = e^{i \phi_0} \sum_{\beta \in \mathcal{F}} \frac{\prod_{i=1}^{n} \sum_{\rho + \Lambda_i \in \mathcal{F}_i} S_{\beta \rho + \Lambda_i} \mathcal{U}_{\rho + \Lambda_i \rho}^{(p_i/q_i)}}{S_{\beta \rho}^{n-2}}. \tag{4.4}
\]
where the $\text{SL}(2, \mathbb{Z})$ transformation in the right hand side is given by

$$S \cdot U(p_i, q_i) = \left( \begin{array}{cc} -q_i & -s_i \\ p_i & r_i \end{array} \right) = U(-q_i, p_i), \quad (4.6)$$

and the phase factor is needed in order to keep track of the framing. The partition function is then, up to a multiplicative constant, given by:

$$\sum_{\beta \in \mathcal{F}_i} \frac{1}{\prod_{\alpha > 0} \left( \sin \frac{\pi}{l} (\beta \cdot \alpha) \right)^{n-2}} \prod_{i=1}^{n} \sum_{n_i \in \Lambda_i/p_i \Lambda_i} \sum_{w_i \in \mathcal{W}} \epsilon(w_i) \exp \left\{ \frac{i\pi}{lp_i} \left( -q_i \beta^2 - 2\beta(ln_i + w(\rho)) + r_i(ln_i + w(\rho))^2 \right) \right\}. \quad (4.7)$$

If $G$ is simply-laced, the summand is invariant under the simultaneous shift,

$$\beta \rightarrow \beta + l\alpha, \quad n_i \rightarrow n_i - q_i \alpha, \quad (4.8)$$

and also under

$$n_i \rightarrow n_i + p_i \alpha. \quad (4.9)$$

In these equations, $\alpha$ is any element in the root lattice. This invariance allows us put $n_i = 0$ in the above sum by extending the range of $\beta$: $\beta = p + l\alpha$, where $p \in \mathcal{F}_i$, and $\alpha = \sum_i a_i \alpha_i$, $0 \leq a_i < P$. It is easy to see that the resulting summand is invariant under the Weyl group $\mathcal{W}$ acting on $\beta$, and by translations by $lP\alpha$, where $\alpha$ is any root. We can then sum over Weyl reflections and divide by the order of $\mathcal{W}$, denoted by $|\mathcal{W}|$, and use the translation symmetry to extend the sum over $\beta$ in the above set to a sum over $\beta \in \{\Lambda_w/lP\Lambda_i\} \backslash \mathcal{M}$. Here $\mathcal{M}$ denotes the set given by the wall of $\mathcal{F}_i$ together with its Weyl reflections and translations by $lP\alpha$ inside $\Lambda_w/lP\Lambda_i$ (for $SU(N)$, the wall of $\mathcal{F}_i$ is given by the weights with $\sum_i p_i = l$). We won’t need a precise description of the points of $\mathcal{M}$ in the following, since they only enter in the contribution of irreducible flat connections to the path integral $[30]$. After performing all these changes, and using the Weyl denominator formula

$$\prod_{\alpha > 0} 2 \sinh \frac{\alpha}{2} = \sum_{w \in \mathcal{W}} \epsilon(w) e^{\rho(\alpha)}, \quad (4.10)$$

we can write (4.7) as:

$$\frac{1}{|\mathcal{W}|} e^{\frac{i\pi}{lP} \beta^2 \sum_{i=1}^{n} \frac{r_i^2}{p_i}} \sum_{\beta \in \{\Lambda_w/lP\Lambda_i\} \backslash \mathcal{M}} \frac{1}{\prod_{\alpha > 0} \left( \sin \frac{\pi}{l} (\beta \cdot \alpha) \right)^{n-2}} \cdot e^{\frac{i\pi}{lP} \beta^2 \sum_{i=1}^{n} \frac{r_i^2}{p_i}} \prod_{\alpha > 0} \left( \sin \frac{\pi}{l} (\beta \cdot \alpha) \right)^{n-2} \cdot e^{-i\frac{\pi}{lP} \sum_{i=1}^{n} (2i) \sin \frac{\pi}{l} (\beta \cdot \alpha)}. \quad (4.11)$$
The last step involves transforming the above sum in a sum over integrals and residues. To do that, we generalize slightly [30] and we introduce a holomorphic function of \( \beta_1, \ldots, \beta_r \) and \( x_1, \ldots, x_r \) given by:

\[
    h(\beta, x) = \frac{e^{-i\pi l P \beta} \prod_{\alpha>0} (e^{\pi i (\beta-\alpha)} - e^{-\pi i (\beta-\alpha)})^{n-2} \prod_{i=1}^r (1 - e^{-2\pi i \beta_i})}{\prod_{i=1}^r (1 - e^{-2\pi i \beta_i})}.
\]

(4.12)

where \( \beta = \sum_{i=1}^r \beta_i \lambda_i \in \Lambda_w \otimes \mathbb{C} \), \( x = \sum_{i=1}^r x_i \alpha_i \in \Lambda_r \otimes \mathbb{C} \). This function satisfies:

\[
    h(\beta + l P \alpha, x) = e^{2\pi i P \alpha \cdot x} h(\beta, x - l H \alpha),
\]

(4.13)

for any \( \alpha \in \Lambda_r \). Notice also that \( h(\beta, x) \) has poles at the points of \( \Lambda_w \), the weight lattice. Introduce now the integral over \( C^r \):

\[
    \Theta(x) = \int_{C^r} h(\beta, x) d\beta
\]

(4.14)

where \( C^r = C \times \cdots \times C \) is a multiple contour in \( \mathbb{C}^r \), and \( C \) is the contour considered in [30]: a line through the origin from \((-1+i)\infty\) to \((1-i)\infty\) for \( \text{sign}(H/P) > 0 \) (if \( \text{sign}(H/P) < 0 \), we rotate \( C \) by \( \pi/2 \) in the clockwise direction). This contour is chosen to guarantee good convergence properties as \( \beta_i \to \infty \).

Let us now shift the contour in such a way that it crosses all the poles corresponding to the weights in the chamber \( \Lambda_w/l P \Lambda_r \). Using (4.13) it is easy to see that, if \( P \alpha \cdot x \in \mathbb{Z} \) for any root \( \alpha \), the resulting integral can be written as

\[
    \sum_{i=1}^r \Theta(x - l H \alpha_i) - \sum_{1 \leq i < j \leq r} \Theta(x - l H (\alpha_i + \alpha_j)) + \cdots + (-1)^{r-1} \Theta(x - l H \sum_{i=1}^r \alpha_i).
\]

(4.15)

The difference between the original integral and the shifted integral (4.15) can be written as

\[
    \sum_{t \in \Lambda_r/H \Lambda_r} \int_{C^r} f(\beta, x) e^{-2\pi i t \beta} .
\]

(4.16)

On the other hand, the effect of shifting the contour is to pick the residues corresponding to all the weights in the chamber \( \Lambda_w/l P \Lambda_r \). Here the residue is understood as \( \lim_{\beta_i \to n_i} \prod \beta_i - n_i h(\beta, x) \), and the residues for the weights that are not in \( \mathcal{M} \) are simply given by \( (2\pi i)^{-r} f(\beta, x) \). Putting everything together we find,

\[
    \sum_{n \in (\Lambda_w/l P \Lambda_r) \setminus \mathcal{M}} f(n, x) = \sum_{t \in \Lambda_r/H \Lambda_r} \int_{C^r} f(\beta, x) e^{-2\pi i t \beta} d\beta - (2\pi i)^r \sum_{n \in \mathcal{M}} \text{Res}(h(\beta, x), \beta = n),
\]

(4.17)

whenever \( P \alpha \cdot x \in \mathbb{Z} \). We can apply this formula to (4.12), since what we have there is just a sum of expressions of the form \( f(\beta, x) \) in (4.12), with \( x \) of the form \( \alpha/P \), \( \alpha \in \Lambda_r \). In this context, the sum over \( t \in \Lambda_r/H \Lambda_r \) is interpreted as a sum over reducible flat connections on the Seifert sphere, and of course \( t = 0 \) corresponds to the trivial connection. In the remaining of this paper we will
focus on these contributions, \textit{i.e.} we will not deal with the residue terms in (4.17), that should give the contribution of irreducible flat connections \cite{37, 30}. In fact, we will only analyze in detail the contribution of the trivial connection in order to make contact with the universal perturbative invariants.

In order to present the final result for the contribution of reducible flat connections to the partition function of Chern-Simons theory on Seifert spaces, we have to collect the prefactors, including the phases. Define as in \cite{30}:

\[
\phi = 3 \text{sign} \left( \frac{H}{P} \right) + \sum_{i=1}^{n} \left( 12 s(q_i, p_i) - \frac{q_i}{p_i} \right). \tag{4.18}
\]

Therefore, the contribution of reducible flat connections to the Chern-Simons partition function of \( X(p_1^{q_1}, \cdots, p_n^{q_n}) \) is given by

\[
\left( -1 \right)^{\mid \Delta_+ \mid} \left( \frac{\text{Vol}\Lambda_w}{\text{Vol}\Lambda_r} \right) \left( \frac{\text{sign}(P)}{|P|^{1/2}} \right) e^{\frac{\pi i}{12} \text{sign}(H/P)} \cdot \prod_{n \in \Lambda_r/H\Lambda_r} \int d\beta e^{-\beta^2/2\hat{x}^2 - i\beta \cdot \hat{x}} \prod_{\alpha > 0}^{n} \left( 2 \sinh \frac{\beta \cdot \alpha}{2p_i} \right) \prod_{\alpha > 0} \left( 2 \sinh \frac{\beta \cdot \alpha}{Z} \right)^{n-2} \tag{4.19}
\]

In this equation, \( \phi \) is given by (4.18), and in obtaining the phase factor we have made use of the Freudenthal-De Vries formula

\[
\rho^2 = \frac{1}{12} dy. \tag{4.20}
\]

We have also introduced the hatted coupling constant

\[
\hat{x} = \frac{P x}{H}, \tag{4.21}
\]

where \( x \) is the coupling constant given in (3.3). In the evaluation of the above integral we can rotate the integration contour \( C^r \) to \( R^r \) as long as we are careful with phases in the Gaussian integral, as explained for example in \cite{44}. If we specialize (4.19) to \( G = SU(2) \), we obtain the result derived in \cite{30}. The expression (4.19) is in principle only valid for simply-laced groups, although the results for the perturbative series turn out to be valid for any gauge group.

Notice that, in the sum over \( \Lambda_r/H\Lambda_r \), the \( t \)'s that are related by Weyl transformations correspond to the same flat connection. Fortunately, each of the integrals in (4.19) is invariant under Weyl permutations of \( t \), so in order to consider the contribution of a given flat connection, one can just evaluate (4.19) for a particular representative and then multiply by the corresponding degeneracy factor (\textit{i.e.} the number of Weyl-equivalent \( t \) configurations giving the same flat connection).

If one is just interested in obtaining the contribution of the trivial connection, one can use the shorter arguments of \cite{36} and end up with (4.19) with \( t = 0 \). The contribution of the reducible connections can also be obtained by generalizing the arguments of \cite{37} to the higher rank situation.
4.3 Connection to matrix models

In (4.19) we have written the contribution of reducible connections to the Chern-Simons partition function in terms of an integral over the Cartan subalgebra, since $d\beta = \prod_{i=1}^{r} d\beta_i$ and $\beta_i$ are the Dynkin coordinates. In fact, the above expression can be interpreted as the partition function of a random matrix model (for a review of random matrices, see [24, 25]). To see this, let us consider a slight generalization of the above results to the $U(N)$ and $O(2r)$ theories. The partition function for these groups can be obtained by writing $\beta$ in terms of the orthonormal basis in the space of weights.

Let us first consider the case of $U(N)$. Denote the orthonormal basis as $\{e_k\}_{k=1, \ldots, N}$, and put $\beta = \sum_k \beta_k e_k$ (where $\beta_k$ are taken to be independent variables), $t = \sum_k t_k e_k$. It is well-known that the positive roots can be written as

$$\alpha_{kl} = e_k - e_l, \quad 1 \leq k < l \leq N. \quad (4.22)$$

Therefore, the integral in (4.19) becomes

$$\int d\beta e^{-\sum_k \beta_k^2/2 \xi - t \sum_k t_k \beta_k} \frac{\prod_{i=1}^{n} \prod_{k<l} 2 \sinh \frac{\beta_k - \beta_l}{2 p_i}}{\prod_{k<l} (2 \sinh \frac{\beta_k - \beta_l}{2})^{n-2}} \quad (4.23)$$

We can interpret the $\beta_k$ as the eigenvalues of a Hermitian matrix in a Gaussian potential and interacting through

$$\sum_{i=1}^{n} \sum_{k<l} \log \left( 2 \sinh \frac{\beta_k - \beta_l}{2 p_i} \right) + (2 - n) \sum_{k<l} \log \left( 2 \sinh \frac{\beta_k - \beta_l}{2} \right). \quad (4.24)$$

Notice moreover that for a small separation of the eigenvalues (4.24) becomes, at leading order,

$$\sum_{k<l} \log (\beta_k - \beta_l)^2 \quad (4.25)$$

which is the interaction between eigenvalues of the standard Hermitian matrix model. Therefore, the integral above can be interpreted as a nonlinear deformation of the usual Gaussian unitary ensemble (GUE). In fact, as we will see in detail in the next section, Chern-Simons perturbation theory means that we expand around the GUE, and the perturbative corrections are obtained by evaluating averages in this ensemble. Note that the non-trivial reducible flat connections, labeled by $t$, are interpreted in the matrix model language as a source term coupling linearly to the eigenvalues.

Similar considerations apply to the orthogonal group $O(2r)$. The positive roots can be written in terms of an orthonormal basis as follows:

$$\alpha_{kl}^\pm = e_k \pm e_l, \quad 1 \leq k < l \leq r, \quad (4.26)$$

and the interaction between the eigenvalues reduces again, in the limit of small separation, to

$$\sum_{k<l} \log (\beta_k^2 - \beta_l^2)^2, \quad (4.27)$$

which is the eigenvalue interaction of the orthogonal ensemble $O(N)$ for even $N$.  

13
5. Asymptotic expansion and matrix integrals

5.1 Asymptotic expansion of the exact result

In this subsection we will study the asymptotic expansion of the exact result obtained in the previous section for the contribution of the trivial connection \( t = 0 \).

The expression (4.19) is very well suited for an asymptotic expansion in powers of \( x^\ell \): we just have to expand the integrand in a power series of \( \beta \), and integrate the result term by term with the Gaussian weight. The integrand has the expansion:

\[
\prod_{n} \prod_{\alpha > 0} 2 \sinh \frac{\beta \alpha}{2p_i} \left( 2 \sinh \frac{\beta \alpha}{2} \right)^{n-2} = \frac{1}{P[\Delta]} \left( \prod_{\alpha > 0} (\beta \cdot \alpha)^2 \right)^{f(\beta)},
\]

(5.1)

where \( f(\beta) \) has the form

\[
f(\beta) = \prod_{\alpha > 0} \left( 1 + \sum_{s=1}^{\infty} a_s (\beta \cdot \alpha)^{2s} \right),
\]

(5.2)

The coefficients \( a_s \) can be obtained in a very straightforward way from (5.1). They are polynomials of degree \( s \) in \( n \) and in the power sums

\[
\pi_j = \sum_{i=1}^{n} p_i^{-2j}.
\]

(5.3)

One has, for example,

\[
a_1 = \frac{1}{24} (\pi_1 + 2 - n),
\]

\[
a_2 = \frac{1}{5760} \left( 16 + 5n^2 - 18n - 10n\pi_1 + 20\pi_1 + 5\pi_1^2 - 2\pi_2 \right).
\]

(5.4)

Let us analyze in more detail the structure of \( f(\beta) \). Define

\[
\sigma_j(\beta) = \sum_{\alpha > 0} (\beta \cdot \alpha)^{2j}.
\]

(5.5)

By taking the log of (5.2), one finds:

\[
f(\beta) = \exp \left( \sum_{k=1}^{\infty} a_k^{(c)} \sigma_k(\beta) \right),
\]

(5.6)

where the connected coefficients \( a_k^{(c)} \) are defined in the usual way: \( \log(1 + \sum_{n} a_k x^k) = \sum_{k} a_k^{(c)} x^k \). An explicit expression for \( f(\beta) \) can be obtained as follows. Let \( \vec{k} = (k_1, k_2, \cdots) \) be a vector whose components are nonnegative integers. Denote \( \ell = \sum_{j} jk_j \), and define:

\[
a_k^{(c)} = \prod_{j} (a_j^{(c)})^{k_j}, \quad \sigma_k(\beta) = \prod_{j} \sigma_{kj}^{(c)}(\beta).
\]

(5.7)
Then,
\[
f(\beta) = 1 + \sum_{\vec{k}} \frac{1}{\vec{k}!} a^{(c)}_{\vec{k}} \sigma_{\vec{k}}(\beta),
\]
with \( \vec{k}! = \prod_j k_j! \) and the sum is over all vectors \( \vec{k} \). We see that the perturbative expansion of the partition function can be written in terms of the quantities
\[
\mathcal{R}_{\vec{k}}(G) = \int d\beta \Delta^2(\beta) e^{-\beta^2/2} \sigma_{\vec{k}}(\beta),
\]
where we have denoted
\[
\Delta^2(\beta) = \prod_{\alpha>0} (\beta \cdot \alpha)^2.
\]
Notice that, when we write \( \beta \) in terms of the orthogonal basis (4.22) or (4.26), (5.10) is indeed the square of the Vandermonde determinant in the variables \( \beta_j \) (for \( U(N) \)) or \( \beta_j^2 \) (for \( O(2r) \)). Therefore, as we anticipated before, the asymptotic expansion of the integral is an expansion around the corresponding Gaussian ensemble, and the perturbative corrections can be evaluated systematically as averages in this ensemble.

We will denote
\[
Z_0 = \int d\beta \Delta^2(\beta) e^{-\beta^2/2},
\]
so that the partition function on Seifert spaces can be written, using (4.19), as
\[
\log \frac{Z_k(M)}{Z_{1\text{-loop}}} = -\frac{1}{24} dy \phi x + \log \left( 1 + \sum_{\ell=1}^{\infty} \left( \sum_{\vec{k}} \frac{1}{\vec{k}!} a^{(c)}_{\vec{k}} \mathcal{R}_{\vec{k}}(G) \right) \hat{x}^{\ell} \right).
\]
In this equation \( Z_{1\text{-loop}} \) is given by
\[
Z_{1\text{-loop}} = \frac{(-1)^{|\Delta_+|}}{|W| (2\pi i)^r} \left( \frac{\text{Vol}_w \Lambda_w}{\text{Vol}_r \Lambda_r} \right) e^{\pi i d \text{sign}(H/P) / |P| d/2} Z_0 \hat{x}^{d/2},
\]
and indeed gives the one-loop contribution around the trivial connection. This follows by comparing the exact result with the perturbative expansion
\[
\log \frac{Z_k(M)}{Z_{1\text{-loop}}} = \sum_{\ell=1}^{\infty} \left( \sum_{\Gamma \in \mathcal{A}(\emptyset)_{\text{conn}}} r_{\Gamma}(G) I_{\Gamma}(M) \right) \hat{x}^{\ell}.
\]
We also see that, by comparing (5.11) and (5.12), we can extract the value of the universal perturbative invariants \( I_{\Gamma}(M) \) at each order \( x^{\ell} \). In order to do that we just have to evaluate \( \mathcal{R}_{\vec{k}}(G) \) for all vectors \( \vec{k} \) with \( \sum_j j k_j \leq \ell \), and also the group factors \( r_{\Gamma}(G) \) for graphs \( \Gamma \) with \( 2\ell \) vertices. Of course, from a mathematical point of view it is not obvious that the asymptotic expansion of the exact partition function has the structure predicted by the perturbation theory analysis. The fact that this is the case provides an important consistency check of the procedure.
5.2 Evaluating the integrals

We now address the problem of computing the integrals in (5.9). As we explained in section 4, the partition function of Chern-Simons theory on Seifert spaces can be interpreted as a matrix model with an interaction between eigenvalues of the form \( \log(\sinh(\beta_i - \beta_j)) \). In the perturbative approach we have to expand the sin in power series, and the integrals \( \mathcal{R}_k(G) \) are nothing but averages of symmetric polynomials in the eigenvalues in a Gaussian matrix model. We will present two methods to compute these averages.

The first method gives the complete answer only up to \( \ell = 5 \), but it has the advantage of providing general expressions for any simply-laced gauge group. The starting point is the following identity:

\[
\int d\beta e^{-\frac{1}{2} \beta^2} \prod_{\alpha > 0} 4 \sinh \left( \frac{t(\beta \cdot \alpha)}{2} \right) \sinh \left( \frac{s(\beta \cdot \alpha)}{2} \right) = \left( \frac{2\pi}{a} \right)^{r/2} |\mathcal{W}| (\det (C))^{1/2} 2^{\ell+2} \prod_{\alpha > 0} 2 \sinh \left( \frac{ts(\rho \cdot \alpha)}{2a} \right),
\]

(5.15)

where \( C \) is the Cartan matrix of the group. This formula is easily proved by using (4.10). Another useful fact is that \( \sigma_1(\beta) \) can be written as (see [15], pp. 519-20)

\[
\sum_{\alpha > 0} (\beta \cdot \alpha)^2 = y\beta^2.
\]

(5.16)

One can easily show that, by expanding (5.15) in \( s, t \), and by using (5.16), it is possible to determine the integrals \( \mathcal{R}_k(G) \) for any gauge group up to \( \ell = 5 \), therefore this is enough for the computational purposes of the present paper. The answer is given in terms of \( y, d \), and the quantities

\[
\alpha_k = \prod_{\alpha > 0} (\alpha \cdot \rho)^{2k}.
\]

(5.17)

For example, one finds:

\[
\mathcal{R}_{(0,1,0,\ldots)}(G) = 5dy^2.
\]

(5.18)

The answers obtained by this method are listed in the Appendix.

In order to evaluate the integrals (5.9) for arbitrary \( \sigma_k \), it is important to have a more general and systematic method. Here is where the connection to matrix integrals becomes computationally useful. It is easy to see that, since the integrals \( \mathcal{R}_k(G) \) are normalized, one can evaluate them in \( U(N) \) and \( O(2r) \) instead of \( SU(N) \) and \( SO(2r) \). Therefore, one has

\[
\mathcal{R}_k(SU(N)) = \frac{1}{Z_0} \int d\beta e^{-\sum_j \beta_j^2/2} \prod_{i<j} (\beta_i - \beta_j)^2 \sigma_k(\beta),
\]

(5.19)

where

\[
\sigma_n(\beta) = \sum_{i<j} (\beta_i - \beta_j)^{2n}.
\]

(5.20)
In (5.19) one integrates over $N$ independent variables $\beta_1, \cdots, \beta_N$. It is clear that the $\sigma_k(\beta)$ are symmetric polynomials in these $N$ variables. One can for example write (5.20) in terms of power sum polynomials $P_j(\beta)$ (defined in (A.8)) as follows,

$$
\sigma_n(\beta) = NP_{2n}(\beta) + \frac{1}{2} \sum_{s=1}^{2n-1} (-1)^s \binom{2n}{s} P_s(\beta)P_{2n-s}(\beta).
$$

(5.21)

The averages of symmetric polynomials in the Gaussian unitary ensemble can be evaluated in principle by using the Selberg integral [34], or the results of [10]. A more effective way is the following: any symmetric polynomial in the $\beta_i$’s can be written as a linear combination of Schur polynomials $S_\lambda(\beta)$, which are labeled by Young tableaux associated to a partition $\lambda$ (see (A.6)). Therefore, if we know how to compute the normalized average of a Schur polynomial,

$$
\langle S_\lambda(\beta) \rangle = \frac{1}{Z_0} \int d\beta e^{-\sum_i \beta_i^2/2} \prod_{i<j} (\beta_i - \beta_j)^2 S_\lambda(\beta),
$$

(5.22)

we can compute all $R_\beta$. An explicit expression for (5.22) has been presented in [14]. The result is the following: let $|\lambda|$ be the total number of boxes in the tableau labeled by $\lambda$, and let $\lambda_i$ denote the number of boxes in the $i$-th row of the Young tableau. Define now the $|\lambda|$ integers $f_i$ as follows

$$
f_i = \lambda_i + |\lambda| - i, \quad i = 1, \cdots, |\lambda|.
$$

(5.23)

Following [14], we will say that the Young tableau associated to $\lambda$ is even if the number of odd $f_i$’s is the same as the number of even $f_i$’s. Otherwise, we will say that it is odd. If $\lambda$ is odd, the normalized average $\langle S_\lambda(\beta) \rangle$ vanishes. Otherwise, it is given by:

$$
\langle S_\lambda(\beta) \rangle = (-1)^{A(A-1)} \frac{\prod_{f \text{ odd}} f!! \prod_{f' \text{ even}} f''!!}{\prod_{f \text{ odd}, f' \text{ even}} (f - f')} \dim \lambda,
$$

(5.24)

where $A = \ell/2$ (notice that $\ell$ has to be even in order to have a non vanishing result). Here $\dim \lambda$ is the dimension of the irreducible representation of $SU(N)$ associated to $\lambda$, and can be computed by using the hook formula. This expression solves the problem of computing the averages (5.19) in the general case: we express the product of power sums appearing in (5.21) in terms of Schur polynomials by using Frobenius formula (A.9), and then we compute the averages of these with (5.24). As an example of this procedure, let us compute $R_{(0,1,0,\cdots)}(SU(N))$. Using (5.21) and Frobenius formula (A.9), we find:

$$
\sigma_2 = (N-1)S_1 - (N+1)S_3 + (N-3)S_5 - (N+3)S_7 + 10S_9.
$$

(5.25)

The averages of the different Schur polynomials can be computed from (5.24), and we obtain, after some simple algebra:

$$
R_{(0,1,0,\cdots)}(SU(N)) = 5N^2(N^2 - 1),
$$

(5.26)

in agreement with (5.18).
Let us now consider the orthogonal ensemble. The averages that we want to compute are given by
\[
\frac{1}{Z_0} \int d\beta e^{-\sum_{1 \leq i < j \leq r} (\beta_i^2 - \beta_j^2)^2} \sigma_k(\beta),
\]  
(5.27)
where
\[
\sigma_n(\beta) = \sum_{i<j} \left\{ (\beta_i + \beta_j)^{2n} + (\beta_i - \beta_j)^{2n} \right\}
= (2r - 2^{2n-1}) P_n(\beta_1^2) + \sum_{s=1}^{n-1} \binom{2n}{2s} P_s(\beta_1^2) P_{n-s}(\beta_1^2).
\]  
(5.28)
The functions \(\sigma_k(\beta)\) are now symmetric polynomials in the \(\beta_i^2\), so we can write them in terms of Schur polynomials \(S_\lambda(\beta_i^2)\). This allows to express the integrals (5.27) in terms of the integrals
\[
\int_0^\infty \cdots \int_0^\infty dy \Delta^2(y)(y_1 \cdots y_r)^{n-1} e^{-(y_1 + \cdots + y_r)/2} S_\lambda(y),
\]  
(5.29)
which are a special case of a generalization of the Selberg integral studied by Kadell [27], see also [33]. Their value is given by
\[
r! \prod_{i=1}^r \Gamma(\lambda_i + \alpha + r - i) \prod_{i<j} (\lambda_i - \lambda_j + j - i).
\]  
(5.30)
In our case, \(\alpha = 1/2\). The normalized average of a Schur polynomial is then:
\[
\langle S_\lambda(\beta_i^2) \rangle = 2^{\lambda} \dim \lambda \prod_{i=1}^r \frac{\Gamma(\lambda_i + 1/2 + r - i)}{\Gamma(1/2 + r - i)}
\]  
(5.31)
In this equation, \(\dim \lambda\) denotes the dimension of the representation of \(SU(r)\) associated to \(\lambda\). This solves the problem of computing the averages (5.27) in the orthogonal ensemble. As a simple example, let us consider again \(k = (0, 1, 0, \cdots)\). It is easy to see that
\[
\sigma_2 = (2r - 2) S_{\Box} + (14 - 2r) S_{\downarrow},
\]  
(5.32)
and one finds
\[
R_{(0,1,0,\cdots)}(SO(2r)) = 20r(2r-1)(r-1)^2,
\]  
(5.33)
in agreement with (5.18).

### 5.3 Universal perturbative invariants up to order 5

Using the above ingredients, it is easy to find the universal perturbative invariants of Seifert spaces up to order 5. Although the coefficients \(a_s\) in (5.2) are functions of \(n\) and the Newton polynomials \(P_k(p_i^2)\), the answer turns out to be more compact when written in terms of elementary symmetric
polynomials $E_k$ in the variables $p_i^{-2} - 1$ by using (A.10) (so for example $E_1 = -n + \sum_{i=1}^n p_i^{-2}$). One finds,

$$I_\theta = -\frac{1}{48} \left( \phi - \frac{P}{H} (2 + E_1) \right),$$

$$I_{\theta_2} = \frac{1}{1152} \left( \frac{P}{H} \right)^2 (1 + E_1 + E_2),$$

$$I_{\theta_3} = \frac{1}{13824} \left( \frac{P}{H} \right)^3 E_3,$$

$$I_{\theta_4} = \frac{1}{11059200} \left( \frac{P}{H} \right)^4 \left( 82E_4 - 46E_3 - 18(1 + E_2 + E_3)E_1 - 9E_2^2 - 18E_2 - 9E_1^2 - 9 \right),$$

$$I_\omega = \frac{1}{1382400} \left( \frac{P}{H} \right)^4 \left( 2E_4 - 6E_3 + 2E_1(1 + E_2 - E_3) + E_2^2 + 2E_2 + E_1^2 + 1 \right),$$

$$I_{\theta_5} = \frac{1}{66355200} \left( \frac{P}{H} \right)^5 \left( 55E_5 + E_4(27E_1 - 56) - E_3(9E_2 + 36E_1 + 8) \right),$$

$$I_{\omega_\theta} = \frac{1}{8294400} \left( \frac{P}{H} \right)^5 \left( 5E_5 - E_4(3E_1 + 16) + E_3(E_2 + 4E_1 + 12) \right).$$

(5.34)

Note that the first universal invariant is given by

$$I_\theta = \frac{\lambda(M)}{2},$$

(5.35)

where $\lambda(M)$ is the Casson invariant of $M$, in accord with the general result of [36] and with the result for the LMO invariant [31]. We have also checked that the value for $I_{\theta_2}$ listed above agrees with the value obtained in [3] for the LMO invariant using the Aarhus integral. It would be interesting to see if these invariants have some nice integrality properties. The perturbative $SU(2)$ invariants of integral homology spheres do exhibit some integrality properties (discussed for example in [30]), but they include extra factors coming from the $SU(2)$ group weights. In general, the invariants listed above are rational even for integral homology spheres. For example, $I_{\theta_4} \in \mathbb{Z}/4$ for Brieskorn integral homology spheres. Also, $I_{\theta_2} - \frac{1}{1152} \in \mathbb{Z}/2$ for those spaces.

6. Open problems

Besides the original motivation of understanding universal perturbative invariants from a field theory point of view, the results presented here also provide a computationally feasible framework to study the Chern-Simons partition function with higher rank gauge groups. There are various avenues to explore in the context of Chern-Simons theory and the theory of three-manifold invariants. It would be interesting for example to understand the structure of perturbation theory in the background of a reducible nontrivial connection, and work out the asymptotic expansion starting from (4.19). One could also consider other three-manifolds (not necessarily rational homology spheres) and see in particular if the matrix model representation provided here can be generalized to other cases.
There are also various interesting physical contexts in which the results of this paper might be relevant. Let us end by mentioning a few of them:

1) The computation of Rozansky-Witten invariants \([38]\) involves the universal perturbative invariants of three-manifolds that are extracted from Chern-Simons theory, but the weight system is now associated to a hyperKähler manifold (see \([33]\) for a very nice review). Therefore, the universal perturbative invariants of Chern-Simons theory are relevant in Rozansky-Witten theory. On the other hand, this theory is an essential ingredient in the worldvolume theory of M2 membranes in manifolds of \(G_2\) holonomy \([24]\), and the computation of the Rozansky-Witten partition function should play a role in understanding membrane instanton effects in \(G_2\) compactifications of M theory.

2) Chern-Simons theory on a three-manifold \(M\) describes topological A branes wrapping the Lagrangian submanifold \(M\) in the Calabi-Yau \(T^*M\) \([45]\). Moreover, the perturbative invariants of Chern-Simons theory correspond to topological open string amplitudes on that target. Having a systematic procedure to compute Chern-Simons perturbative invariants may prove to be useful in further understanding topological open strings in those backgrounds.

3) Another consequence of our results is that topological A branes in \(T^*M\) are described by a matrix model, when \(M\) is a Seifert sphere. It has been recently shown \([17]\) that topological B branes on some noncompact Calabi-Yau spaces are described by a Hermitian matrix model characterized by a potential \(W(\Phi)\) with multiple cuts. It would be interesting to know if there is a relation between the matrix models of \([17]\) and the ones presented here. Notice that according to our results the partition function of \(U(N)\) Chern-Simons theory on \(S^3\) can be written as

\[
Z = e^{\frac{-x}{N(N^2-1)}} \frac{N!}{\prod_{i=1}^N d\beta_i} \int e^{-\sum_i \beta_i^2/2x} \prod_{i<j} (2 \sinh(\beta_i - \beta_j/2))^2, \tag{6.1}
\]

This describes open topological A strings on \(T^*S^3\) with \(N\) branes wrapping \(S^3\), or equivalently (after the geometric transition of \([22]\)) closed topological strings on the resolved conifold. In the limit \(x \to 0\), (6.1) gives the standard Gaussian model, as we have argued at length in this paper. On the other hand, it is shown in \([17]\) that the Gaussian model (which corresponds to \(W(\Phi) = \Phi^2\)) describes type topological B strings in the deformed conifold geometry. This is consistent with the fact that, as explained in \([11]\), the deformed conifold geometry gives the mirror of the resolved conifold only at small 't Hooft coupling \(t = Nx\), which for fixed \(N\) means precisely small \(x\). This also suggests to consider multicut matrix models with a potential \(W(\Phi)\), as in \([17]\), but where the eigenvalue interaction is not the usual one \(\prod_{i<j} (\beta_i - \beta_j)^2\) but \(\prod_{i<j} (2 \sinh((\beta_i - \beta_j)/2))^2\). In view of the above observation, these deformed models might be relevant to understand the mirror of the geometric transition studied in \([11,17]\). Indeed, a compact version of this, corresponding to a unitary matrix model where the \(\beta_i\) are periodic variables, has been considered in \([22]\) in order to describe the stringy realization of the \(\mathcal{N} = 2\) Seiberg-Witten geometry.

4) Although in this paper the focus has been on the perturbative expansion of the partition function, the matrix model is also very useful to understand its large \(N\) expansion. This is an interesting problem in itself, and it would be nice to see what is the connection to the approach of \([19]\). But of course the large \(N\) expansion of these models is particularly interesting in view of the large \(N\) dualities involving Chern-Simons theory \([21,22]\). Although these dualities are not expected
to hold for arbitrary Seifert spaces, the results presented here may be useful to understand in detail the case of lens spaces (already analyzed in [21]) and shed light on the situation for more general three-manifolds.

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A. Appendix

A.1 Group theory factors

We first present the group theory factors associated to the connected graphs that give a basis of $\mathcal{A}(\emptyset)^{\text{conn}}$ up to order 5. The evaluation of these factors is straightforward by using the graphical techniques of Cvitanović [12], and rather immediate for all of them (except for $r_\omega(G)$, that gives the quartic Casimir in the adjoint and has been computed for all gauge groups in the second reference of [12]). Our conventions are as in [12]: the Lie algebra in the defining representation has Hermitian generators $T_i$, $i = 1, \ldots, d$ satisfying the commutation relations $[T_i, T_j] = i\epsilon_{ijk}$. The generators are normalized in such a way that the quadratic Casimir of the adjoint representation $C_A$ (which is defined here by $C_{A\delta_{ij}} = \sum_{k,l} C_{ikl}C_{jkl}$) is twice the dual Coxeter number. This implies that $\text{Tr}(T_i T_j) = a\delta_{ij}$ with $a = 1$ for $SU(N)$ and $Sp(N)$, and $a = 2$ for $SO(N)$ (notice that these normalizations differ from the ones in [1, 2]).

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
 & $SU(N)$ & $SO(N)$ \\
\hline
$d$ & $N^2 - 1$ & $\frac{1}{2}N(N - 1)$ \\
\hline
$y$ & $N$ & $N - 2$ \\
\hline
$\alpha_2$ & $\frac{1}{60}N^2(N^2 - 1)(2N^2 - 3)$ & $\frac{1}{480}N(N - 1)(N - 2)(8N^3 - 45N^2 + 54N + 32)$ \\
\hline
\end{tabular}
\caption{Dimensions, dual Coxeter numbers and $\alpha_2$ for $SU(N)$ and $SO(N)$}
\end{table}

The group factor will be written in terms of the dual Coxeter number $y$, the dimension of the group $d$, and $\alpha_2$ (where $\alpha_k$ is defined in (5.17)). Their values for $SU(N)$, $SO(N)$ are listed in Table 2. The results for $Sp(N)$ follow from the $Sp(N) = SO(-N)$ relation [13], so for $Sp(N)$ one has $d = N(N + 1)/2$, $y = N + 2$ and $\alpha_2^{Sp(N)}(N) = \alpha_2^{SO(N)}(-N)$. The group theory factors for the graphs in (3.8) are listed in Table 3.
| $\ell$ | graph | group factor |
|-------|-------|-------------|
| 1     | ![Graph 1](https://example.com/graph1.png) | $2dy$       |
| 2     | ![Graph 2](https://example.com/graph2.png) | $4dy^2$     |
| 3     | ![Graph 3](https://example.com/graph3.png) | $8dy^3$     |
| 4     | ![Graph 4](https://example.com/graph4.png) | $16dy^4$    |
| 5     | ![Graph 5](https://example.com/graph5.png) | $32dy^5$    |

Table 3: Group theory factors for the Feynman graphs up to $\ell = 5$.

### A.2 Matrix integrals

We now list the results for the matrix integrals (5.9), up to order 5. The results for $k_1 = 0$ are:

\[
\begin{align*}
R_{(0,1,0,\ldots)}(G) &= 5dy^2, \\
R_{(0,0,1,0,\ldots)}(G) &= 35dy^3, \\
R_{(0,2,0,0,\ldots)}(G) &= 25d(d+12)y^4 - 2880\alpha_2, \\
R_{(0,0,0,1,0,\ldots)}(G) &= 350dy^4 - 1680\alpha_2, \\
R_{(0,1,1,0,0,\ldots)}(G) &= 35y\{5d(d+24)y^4 - 1728\alpha_2\}, \\
R_{(0,0,0,0,1,0,\ldots)}(G) &= 4620y\{dy^4 - 12\alpha_2\}. 
\end{align*}
\]  

(A.1)

The results for $k_1 > 0$ can be obtained from (A.1) very easily: insertions of $\sigma_1(\beta)$ can be reduced to insertions of $\beta^2$ by using (5.16), and these can be computed by taking derivatives with respect to $a$ in (5.15). We have for example:

\[
R_{(2,1,0,\ldots)}(G) = 5d(d+4)(d+6)y^4. 
\]  

(A.2)

Finally, the integral $Z_0$ in (5.11) is given by

\[
Z_0 = (2\pi)^{\frac{d}{2}}|W|(\det C)^{\frac{1}{2}} \prod_{\alpha > 0} (\alpha \cdot \rho). 
\]  

(A.3)
Note that this combines with the rest of the factors in (5.13) to produce, up to an overall phase

\[ Z_{1\text{-loop}} = \frac{(2\pi)|\Delta_+|}{(|l|H|)^{d/2}} \left( \frac{\text{Vol} \Lambda_w}{\text{Vol} \Lambda_r} \right)^{\frac{1}{2}} \prod_{\alpha > 0} (\alpha \cdot \rho), \tag{A.4} \]

where we have used that \( \text{Vol} \Lambda_r / \text{Vol} \Lambda_w = \det(C) \). If instead of taking the volume of the root lattice in (A.4) we take that of the coroot lattice, the resulting expression for the partition function is probably valid for any gauge group.

### A.3 Symmetric polynomials

Here we summarize some ingredients of the elementary theory of symmetric functions that are used in the paper. A standard reference is [33].

Let \( x_1, \ldots, x_N \) denote a set of \( N \) variables. The elementary symmetric polynomials in these variables, \( E_m(x) \), are defined as:

\[ E_m(x) = \sum_{i_1 < \cdots < i_m} x_{i_1} \cdots x_{i_m}. \tag{A.5} \]

The products of elementary symmetric polynomials provide a basis for the symmetric functions of \( N \) variables with integer coefficients. Another basis is given by the Schur polynomials, \( S_\lambda(x) \), which are labeled by Young tableaux. A tableau will be denoted here by a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \), where \( \lambda_i \) is the number of boxes of the \( i \)-th row of the tableau, and we have \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \). The total number of boxes of a tableau will be denoted by \( |\lambda| = \sum i \lambda_i \). The Schur polynomials are defined as quotients of determinants,

\[ S_\lambda(x) = \frac{\det x_j^{\lambda_i + N - i}}{\det x_j^{N - i}}. \tag{A.6} \]

A third set of symmetric functions is given by the Newton polynomials \( P^\mathbf{k}(x) \). These are labeled by vectors \( \mathbf{k} = (k_1, k_2, \ldots, k_p) \), where the \( k_j \) are nonnegative integers, and they are defined as

\[ P^\mathbf{k}(x) = \prod_{j=1}^p P_j^{k_j}(x), \tag{A.7} \]

where

\[ P_j(x) = \sum_{i=1}^N x_i^j, \tag{A.8} \]

are power sums. The Newton polynomials are homogeneous of degree \( \ell = \sum_j jk_j \) and give a basis for the symmetric functions in \( x_1, \ldots, x_N \) with rational coefficients. They are related to the Schur polynomials through the Frobenius formula,

\[ P^\mathbf{k}(x) = \sum_\lambda \chi_\lambda(\mathbf{k}) S_\lambda(x), \tag{A.9} \]
where the sum is over all tableaux such that $|\lambda| = \ell$, and $\chi_{\lambda}(\vec{k})$ is the character of the symmetric group in the representation associated to $\lambda$ and evaluated on the conjugacy class associated to $\vec{k}$ (this is the conjugacy class with $k_j$ cycles of length $j$). Finally, one has the following relation between elementary symmetric polynomials and Newton polynomials,

$$E_m(x) = \sum_{\vec{k}} (-1)^{\sum_j (k_j - 1)} \prod_j k_j! j^{k_j} P^{\vec{k}}(x), \quad (A.10)$$

where the sum is over all vectors with $\sum_j jk_j = m$.

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