Diffusion along chains of
normally hyperbolic cylinders

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Abstract

The present paper is part of a series of articles dedicated to the existence of Arnold
diffusion for cusp-residual perturbations of Tonelli Hamiltonians on $\mathbb{A}^3$. Our goal
here is to construct an abstract geometric framework that can be used to prove the
existence of diffusing orbits in the so-called a priori stable setting, once the preliminary
geometric reductions are performed. Our framework also applies, rather directly, to the a priori unstable setting.

The main geometric objects of interest are 3–dimensional normally hyperbolic in-
variant cylinders with boundary, which in particular admit well-defined stable and
unstable manifolds. These enable us to define, in our setting, chains of cylinders, i.e.,
finite, ordered families of cylinders in which each cylinder admits homoclinic connec-
tions, and any two consecutive elements in the family admit heteroclinic connections.

Our main result is the existence of diffusing orbits drifting along such chains,
under precise conditions on the dynamics on the cylinders, and on their homoclinic
and heteroclinic structure.

1 Introduction and main result

1. The present paper is part of a series of articles devoted to the existence of Arnold
diffusion in the so-called a priori stable setting for three-degree-of-freedom Hamiltonian
systems (see [Mar2]). It is however self-contained and our results can also be applied in
other contexts.

Let us first informally recall the setting introduced by Mather ([Ma04]) for Arnold
diffusion. Given $n \geq 1$, we denote by $\mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n$ the cotangent bundle of the torus
$\mathbb{T}^n$, equipped with its natural angle-action coordinates $(\theta, r)$ and its exact-symplectic form
$\Omega = \sum_{i=1}^{n} dr_i \wedge d\theta_i$. Consider a Hamiltonian of class $C^\infty$ on $\mathbb{A}^3$, of the form

$$H(\theta, r) = h(r) + f(\theta, r), \quad (\theta, r) \in \mathbb{A}^3,$$

(1)
where $\kappa$ is large enough and $h$ is strictly convex with superlinear growth at infinity.\footnote{the strict convexity assumption is not present in Arnold’s original formulation of the problem, while the superlinear growth is assumed only to get compact energy levels}

Fix a regular unperturbed energy level $h^{-1}(e)$ in the action space $\mathbb{R}^3$, consider an arbitrary family $(\tilde{O}_i)_{1 \leq i \leq m}$ of small open subsets in $\mathbb{R}^3$ which intersect $h^{-1}(e)$, and set $O_i = \mathbb{T}^3 \times \tilde{O}_i$. The diffusion problem, as stated in \cite{Ma04}, is to prove that for $f$ in a “cusp-residual” subset of a small enough ball centered at 0 in $C^\kappa(\mathbb{A}^3)$, the system $H$ admits orbits intersecting each $O_i$ – which we will call here diffusing orbits for short.

Three independent approaches of the diffusion question in the Mather setting were developed recently, see \cite{C, KZ, Mar2}. We shall also acknowledge the fundamental role played by Mather in the development of the field, and in providing many ideas and inspiration \cite{Ma04, Ma10, Ma12}. A common feature of the works \cite{C, KZ, Mar2} is the use of “hyperbolic cylinders” which form “chains” intersecting the open sets $O_i$. Once the existence of such chains is shown (under appropriate nondegeneracy conditions on $f$), proving the existence of diffusing orbits amounts to proving the existence of orbits “drifting along the cylinders as well as from one cylinder to the next”, possibly under additional nondegeneracy conditions on $f$.

The very definition of hyperbolic cylinders and chains is not the same in the three approaches, mainly regarding the invariance condition of the cylinders under the Hamiltonian flow. Here we consider \textit{genuinely invariant, compact 3-dimensional cylinders with boundary}, to which the usual notion of normal hyperbolicity applies and yields the existence of well-defined asymptotic manifolds. This enables us to define homoclinic and heteroclinic connections between cylinders, and to set out a natural geometric definition for a chain.

The upshot of our approach is that, modulo an adapted shadowing process, proving the existence of diffusing orbits for (1) along chains of 3-dimensional cylinders amounts to proving the existence of pseudo-orbits, generated by certain polysystems, which drift along chains of 2-dimensional annuli. These polysystems can be described as random iteration of a family of maps that are associated to the homoclinic and heteroclinic connections and to the dynamics restricted to the annuli. Thus, our method of proving the existence of diffusing orbits can be viewed as a generalization of Moeckel’s method on generic drift on Cantor sets of annuli (see \cite{M02}), which itself generalizes the Birkhoff theory of twist maps to polysystems.

The present paper considers the problem of generic drift in an abstract setting, while \cite{Mar2} implements our present results to prove the existence of diffusing orbits for Hamiltonians (1) under cusp-residual conditions on the perturbation.

The conditions we impose to our chains are also satisfied by the usual examples of \textit{a priori} unstable systems (see \cite{Mar3, LM}). In this respect, the present work is also an abstract approach of the problems considered in \cite{CG94, Bes96, BBE03, B08, BKZ13, BT99, CY09, DLS00, DLS06a, DLS06b, FM03, GT14, GL06, GR13, GLS, M02, T04}, amongst others. In particular, our approach has the following distinguished features:

- it uses homoclinic and heteroclinic “maps” that are only locally defined in general;
- we refer to \cite{Ma04} for this notion which we will not use here
- polysystems are also called Iterated Function Systems by several authors
– it does not require any precise knowledge on the invariant objects other than invariant tori for the dynamics restricted to the cylinders;
– it does not require any information on the angle of the splitting of the stable and unstable manifolds of the invariant tori, which can be arbitrarily small;
– our construction can be made “algorithmic” from a numerical point of view, which could help detect diffusion orbits in specific examples.

In addition, it is worth mentioning that the mechanism of diffusion described in this paper is similar to that observed numerically, in a model for the diffusion problem considered in [GSV13].

2. Let us be more precise. We consider $C^2$ Hamiltonian functions $H$ on $\mathbb{A}^3$. Although it is not absolutely necessary, we find it convenient to restrict our study to regular levels $H^{-1}(e)$ on which the Hamiltonian vector field is complete. In this case we say that $e$ (or the level $H^{-1}(e)$) is completely regular.

In this paper, a cylinder is a compact submanifold with boundary of $\mathbb{A}^3$, diffeomorphic to $\mathbb{T}^2 \times [0,1]$. We consider cylinders invariant under the flow generated by a Hamiltonian function $H$ on $\mathbb{A}^3$, and contained in a given, completely regular energy level of $H$. For such invariant cylinders, a notion of normal hyperbolicity relative to the energy level can be well-defined. The normal conditions we impose to the cylinders in this paper will in fact be slightly more general that the usual normal hyperbolicity, which can prove to be useful in some natural contexts. To avoid confusion, we call them tame cylinders; details will be given in Section 2.1.

A chain for $H$ is a finite ordered sequence $(C_k)_{1 \leq k \leq k_0}$ of tame cylinders contained in the same energy level, such that each cylinder $C_k$, for $1 \leq k \leq k_0$, admits homoclinic connections, that is

$$W^{-}(C_k) \cap W^{+}(C_k) \neq \emptyset,$$

and each consecutive pair of cylinders $C_k$ and $C_{k+1}$ in the chain, for $1 \leq k \leq k_0 - 1$, admits heteroclinic connections, that is

$$W^{-}(C_k) \cap W^{+}(C_{k+1}) \neq \emptyset.$$

To ensure the existence of orbits drifting along a chain, we will require additional conditions on the dynamics on the cylinders, their homoclinic structure and the heteroclinic connections between. In this paper, a cylinder (resp. a chain) satisfying those additional properties is called a good cylinder (resp. a good chain); see Section 2.2.

Given a good cylinder $C$, we define an essential subtorus of $C$ as a 2–dimensional torus (not necessarily differentiable) contained in $C$ and invariant under the Hamiltonian flow, which intersects a certain Poincaré section $\Sigma \sim \mathbb{T} \times [a,b]$ along an essential circle (see Definition 4) inside $C$. Examples of essential subtori are the components of the boundary $\partial C$; any essential subtorus is homotopic in $C$ to each of these components. The family of essential subtori will serve us as a guide to build our drifting orbits.

4that is, homotopic to $\mathbb{T} \times \{a\}$ inside $\Sigma$
Definition 1. Consider a good chain \((C_k)_{1 \leq k \leq k_*}\) contained in some completely regular energy level \(H^{-1}(e)\). Given \(\delta > 0\), we say that an orbit of the Hamiltonian flow is \(\delta\)-admissible for the chain when it intersects the \(\delta\)-neighborhood in \(H^{-1}(e)\) of any essential subtorus of \(C_k\), \(1 \leq k \leq k_*\).

The notion of \(\delta\)-admissible orbits will enable us (in [Mar2]) to obtain the existence of diffusing orbits for systems (1) once a chain that intersects each open set \(O_i\) is constructed. Indeed, for small enough perturbations \(f\), it can be proved that a cylinder intersecting \(O_i\) necessarily contains an essential torus which is itself contained in \(O_i\). So any \(\delta\)-admissible orbit for the chain intersects each set \(O_i\) too, provided that \(\delta\) is small enough.

The first main result of this paper is the following.

**Theorem A.** Let \(H\) be a \(C^2\) Hamiltonian on \(\mathbb{R}^3\) and let \(e\) be a completely regular value of \(H\). Fix \(\delta > 0\). Then, for any \(\delta\)-good chain of cylinders contained in \(H^{-1}(e)\), there exists a \(\delta\)-admissible orbit for the chain.

The notion of \(\delta\)-good chain involves a quantitative control on the homoclinic and heteroclinic connections of the cylinders, and it will be introduced in Definitions 10 and 14.

The proof of Theorem A is divided into two parts. In the first part, we introduce a polysystem of maps and homoclinic or heteroclinic correspondences associated with the chain, and prove the existence of “drifting pseudo-orbits” for this polysystem. This is the content of Theorem B which is stated and proved in Section 3 (with a constructive variation in Section 5). In the second part we derive a shadowing process (similar to but simpler than that in [GLS]) to prove the existence of genuine orbits of the Hamiltonian system which intersect arbitrarily small neighborhoods of each point of the pseudo-orbits of Theorem A see also the related shadowing results in [BT99, DLS00, DGR13, GR13, GT14].

This is the content of Theorem C stated and proved in Section 4. In the rest of this introduction we describe informally both theorems.

3. We begin with Theorem B and first describe the main objects involved in the construction. Here we call polysystem a dynamical system formed by a finite family of maps \((f_i)_{i \in I}\) – or, more generally, correspondences – from a space into itself, which are iterated in arbitrary order. A finite orbit of the polysystem \((f_i)_{i \in I}\) is a sequence \(x_0, \ldots, x_{n*}\) such that for \(0 \leq n \leq n_* - 1\), there exists \(i(n) \in I\) with \(x_{n+1} \in f_{i(n)}(x_n)\).

Consider a good chain \((C_k)_{1 \leq k \leq k_*}\) contained in a completely regular level \(H^{-1}(e)\). The shadowing process developed in Section 4 will enable us to reduce the dynamics in the neighborhood of the cylinders \(C_k\) and along their homoclinic and heteroclinic connections to that of a polysystem on the disjoint union of \(k_*\) 2-dimensional annuli diffeomorphic to \(T \times [0,1]\). Let us describe this polysystem.

In the neighborhood of a single cylinder \(C := C_k\) of the chain, the polysystem we will consider is given by a pair \((\varphi, \psi)\), where \(\varphi\) is a diffeomorphism and \(\psi\) is a homoclinic correspondence, whose existence relies on additional conditions imposed to good cylinders. To define \(\varphi\) and \(\psi\), we first require the existence of a global section \(\Sigma\) of the Hamiltonian flow restricted to \(C\), endowed with symplectic coordinates \((\theta_1, \tau_1) \in T \times [a, b]\) for some

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\[5\text{recall that a correspondence from a set } A \text{ to a set } B \text{ is a map from } A \text{ to the set } \mathcal{P}(B) \text{ of (possibly empty) subsets of } B\]
The diffeomorphism $\varphi$ is the Poincaré return map associated with $\Sigma$, so $\varphi$ is symplectic, and we assume $\varphi$ to twist all vertical lines $\theta_1 = \text{const.}$ to the right (or to the left). Other mild conditions on $\varphi$ will also be required (see Definition [3]). We define the aforementioned essential subtori of $\mathcal{C}$ as those which intersect $\Sigma$ along essential invariant circles of $\varphi$.

The correspondence $\psi$ comes from the homoclinic connections. A first natural homoclinic correspondence is defined on $\mathcal{C}$ rather than $\Sigma$, and associates with each element $x \in \mathcal{C}$ the subset of all $y \in \mathcal{C}$ such that $W^-(x)$ intersects $W^+(y)$ (with additional transversality requirements).

To deduce our correspondence $\psi$ from the previous one we then have to perform a “reduction” to the section $\Sigma$, which is achieved by a suitable transport by the Hamiltonian flow inside $\mathcal{C}$. We will distinguish between two cases: the simplest one, where $\psi$ is a globally defined diffeomorphism of the section (we then say that $\mathcal{C}$ is an $F$-cylinder), and the more difficult one, where the correspondence $\psi$ is formed by a collection of local diffeomorphisms of $\Sigma$, whose domains may intersect one another (we then say that $\mathcal{C}$ is a $P$-cylinder).

This way, each cylinder $\mathcal{C}_k$ of the chain is equipped with a polysystem $(\varphi_k, \psi_k)$. Finally, the complete polysystem attached to the chain $(\mathcal{C}_k)_{1 \leq k \leq k_0}$ is formed by the collection of the polysystems attached to each cylinder, together with additional local diffeomorphisms corresponding to the heteroclinic transitions between consecutive cylinders.

4. **Theorem [1]** asserts the existence (for any $\delta > 0$) of a $\delta$-admissible pseudo-orbit for the previous polysystem, that is, an orbit of the polysystem which passes $\delta$-close to every essential invariant circle contained in each of the sections $\Sigma_k \subset \mathcal{C}_k$. In particular, such pseudo-orbits drift along the chain in a natural sense.

Let us restrict ourselves first to a single cylinder $\mathcal{C} := \mathcal{C}_k$ of the chain, equipped with a section $\Sigma \subset \mathcal{C}$ and a polysystem $(\varphi, \psi)$. To prove the existence of $\delta$-admissible pseudo-orbits, we have to introduce additional compatibility conditions between $\varphi$ and $\psi$.

In the case of an $F$-cylinder, we use the approach introduced by Moeckel in [M02] and developed by Le Calvez in [LC07]. Their main result is that “generically”, any polysystem formed by an area-preserving twist map $\varphi$ on $\mathbb{A} = \mathbb{T} \times [a, b]$ and a globally defined area-preserving diffeomorphism $\psi$ of $\mathbb{A}$ admits a finite “connecting pseudo-orbit” whose first point is arbitrarily close to $\mathbb{T} \times \{a\}$ and whose last point is arbitrarily close to $\mathbb{T} \times \{b\}$. The single map $\varphi$ would not in general admit a connecting orbit, due to the existence of essential invariant circles. The role of the diffeomorphism $\psi$ is to allow the pseudo-orbits of the polysystem to “jump” over the essential invariant circles of $\varphi$ – under the crucial assumption that such an invariant circle is not invariant under $\psi$. To obtain $\delta$-admissible pseudo-orbits, we require the map $\psi$ to satisfy an additional condition of $\delta$-bounded oscillation.

In the case of a $P$-cylinder, the main difficulty here is that $\psi$ is not everywhere defined (and, in general, is multivalued). The compatibility conditions that we require generalize in some natural sense the previous “no simultaneous invariant circle condition.” One first crucial property is that the domain of $\psi$ intersects every essential invariant circle $\Gamma$ of $\varphi$.

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6this type of correspondence was previously studied in [DLS06a], where the authors use an appropriate restriction of the correspondence, which they refer to as the scattering map.
and that $\psi$ “breaks” this circle, in the sense that $\psi(\Gamma)$ admits a “topologically transverse” intersection\(^7\) with $\Gamma$. This first topological “splitting condition” enables one to construct pseudo-orbits of $(\varphi, \psi)$ which drift over the parts of $A$ where the essential circles are “dense enough”. Still, this is not sufficient to deal with the boundaries of the Birkhoff zones of $\varphi$. To overcome this difficulty\(^8\), we have to impose a more stringent condition for the behavior of $\psi$ on these boundaries, namely, the existence of right or left splitting arcs which will be introduced in Definition \(^7\). Under both compatibility conditions, we prove that the polysystem $(\varphi, \psi)$ admits connecting pseudo-orbits similar to those of [M02] and [LC07]. Again, a further restriction on the oscillation of the correspondence $\psi$ guarantees that connecting pseudo-orbits are $\delta$-admissible.

Once the existence of $\delta$-admissible pseudo-orbits along a single cylinder is proved, the existence of $\delta$-admissible pseudo-orbits for the full polysystem along the complete chain involves in addition the quantitative control of the heteroclinic transitions between consecutive sections.

We will give two different proofs of Theorem \(^B\) in the case of $P$-cylinders. The first one is non-constructive and extends the methods introduced in [M02] to the case of polysystems of correspondences. It allows for very mild nondegeneracy assumptions which facilitates the applications to occurrence of diffusion under cusp-residual conditions on the perturbation, as in the Mather setting.

The second proof necessitates slightly more stringent assumptions which enable us to use an iterative “Birkhoff procedure,” which makes our results more constructive with a view on quantitative results such as diffusion times. We expect this method to be effective in specific examples, due to its “algorithmic” nature.

5. We now pass to Theorem \(^C\) and briefly describe our shadowing process for pseudo-orbits, first in the case of a single good cylinder. Consider a pseudo-orbit $(x_n)_{1 \leq n \leq n_a}$ of the previous polysystem $(\varphi, \psi)$ on $\Sigma$, where $\psi = (\psi_i)_{i \in I}$, and $\psi_i$ is a local diffeomorphism of $\Sigma$ for $i \in I$, possibly reduced to a single global diffeomorphism. So, by definition, for $1 \leq n \leq n_a - 1$, either $x_{n+1} = \varphi(x_n)$ or there exists $i_n \in I$ such that $x_{n+1} = \psi_i(x_n)$.

In the first case, since $\varphi$ is a flow-induced return map, there is a time $T_n \geq 0$ such that $\Phi_H^{T_n}(x_n) = x_{n+1}$ (where $\Phi_H^T$ stands for the time-$T$ map of the Hamiltonian flow.)

In the second case, using the definition of $\psi$ as a reduced homoclinic correspondence, together with the Poincaré recurrence theorem applied to $\varphi$, it turns out that there exist a point $\xi \in H^{-1}(e)$ and a time $T_n > 0$ such that $\xi$ is arbitrarily close to $x_n$ and $\Phi_H^{T_n}(\xi)$ is arbitrarily close to $x_{n+1}$. The possibility of using the recurrence properties of $\varphi$ is a key idea, first used in this context by [GLS] (see also [NPT2]), which follows directly from the symplectic nature of our setting and the compactness of $\Sigma$ (hence, its finite measure) and is therefore intimately related to our definition of cylinders.

One can expect that slightly perturbing and "gluing together" the previous pieces of Hamiltonian orbits $\Phi_H([0, T_n] \times \{x_n\})$ yield a genuine orbit of the Hamiltonian flow which admits points located arbitrarily close to each point $x_n$ of the initial orbit. This is the main statement of Theorem \(^C\) which uses the inclination property (a $\lambda$-lemma for arcs in the setting of tame cylinders) to construct a positively invariant sequence of balls $B_n$.

\(^7\) one can think of $\psi(\Gamma)$ as a union of arcs, since $\psi$ is a collection of local diffeomorphisms

\(^8\) similar to the "large gap problem" in [DLS06a]
centered on the center-unstable manifolds of the points $x_n$ (and arbitrarily close to them) such that $\Phi_{H}^{T_n}(B_n) \subset B_{n+1}$ for a suitable $T_n > 0$. This proves our claim; for related approaches, see also [BT99, DLS00, FM03, GLS].

Finally, the shadowing process for the complete polysystem along the chain is based on the same idea and involves similar considerations regarding the heteroclinic transitions.

With the constructive method of our second proof of Theorem B one can expect to get a quantitative control on the recurrence times of the Hamiltonian flow (in the specific zones under consideration), and avoid the use of the Poincaré recurrence theorem, at least in generic situations, in view of estimating diffusion times.

Theorem A immediately follows from Theorem B and Theorem C.

6. The structure of the paper is the following. Section 2 is devoted to the description of the general setting and the definition good cylinders and good chains of cylinders. In Section 3 we state and prove Theorem B while Theorem C and Theorem A are proved in Section 4. Section 5 is devoted to the more constructive method and to the second proof of Theorem B. Finally, we recall in Appendix A some basic results on twist maps, and in particular a strong form of the Birkhoff connecting lemma through a Birkhoff zone.

2 The setting: good chains of cylinders

In this section we make precise the dynamical features of good chains and good cylinders.

2.1 Tame cylinders

The main objects in [Mar1, Mar2] are normally hyperbolic cylinders with boundary, satisfying some additional conditions. In this paper we adopt a slightly more general setting, relaxing the normal hyperbolicity assumption but preserving all essential features of this situation (existence of stable and unstable manifolds foliated by center-stable and center-unstable manifolds, λ-lemma for arcs and existence of an invariant measure). We call here tame cylinders the resulting objects, for which we give a formal definition gathering together their various properties. Particular examples of tame and non normally hyperbolic cylinders, with applications to the problem of wandering domains, can be found in [LMS]. Such examples also appear in concrete systems from Celestial Mechanics, see [GMSS].

1. Let $X$ be a vector field on a smooth manifold $M$. We say that $C \subset M$ is an invariant cylinder for $X$ if $C$ is a $C^1$-submanifold of $M$, $C^1$-diffeomorphic to $\mathbb{T}^2 \times [0, 1]$, such that $X$ is everywhere tangent to $C$ and is moreover tangent to $\partial C$ at each point of $\partial C$. Note that $C$ is compact and invariant under the flow of $X$.

In the following we consider only Hamiltonian vector fields on $\mathbb{A}^3$ and the manifold $M$ is a completely regular energy level (that is, a regular level in restriction to which the Hamiltonian vector field is complete). The Hamiltonian vector field associated with a $C^2$ function $H$ is denoted by $X_H$ and its Hamiltonian flow is denoted by $\Phi_H$.
Given a topological space $X$ and two subsets $A \subset B$ of $X$ with $A$ connected, we denote by

$$cc(B, A)$$

the connected component of $B$ which contains $A$. Given any manifold with boundary $L$, we write $\text{int } L = L \setminus \partial L$.

**Definition 2.** Let $H$ be a $C^2$ Hamiltonian function on $\mathbb{R}^3$ and fix a completely regular value $e$. We denote by $d$ the usual distance on $\mathbb{R}^3$. We define a tame cylinder at energy $e$ for $H$ as an invariant cylinder $C$ contained in $H^{-1}(e)$, which satisfies the following properties.

- There exists a 5-dimensional manifold with boundary $U$ of $C$ in $H^{-1}(e)$, whose boundary contains $\partial C$, such that the subsets

$$W^+_U(C) = \left\{ x \in U \mid \Phi^t_H(x) \in U, \forall t \in \mathbb{R}^\pm, \text{ and } \lim_{t \to \pm \infty} d(x, C) = 0 \right\}$$

are 3-dimensional $C^1$ embedded submanifolds of $U$, with boundaries contained in $\partial U$. We fix $U$ once and for all and set

$$W^-(C) = \bigcup_{t \geq 0} \Phi^t_H(W^-_U(C)), \quad W^+(C) = \bigcup_{t \leq 0} \Phi^t_H(W^+_U(C)),$$

which are $\Phi_H$-invariant $C^1$ immersed submanifolds of $H^{-1}(e)$.

- There exist bi-Hölder homeomorphisms

$$J^\pm : C \times ]-1,1[ \to W^\pm_U(C)$$

such that $J^\pm(x, 0) = x$ for $x \in C$, and, setting

$$W^\pm_U(x) = J^\pm([x] \times ]-1,1[), \quad x \in C,$$

then $J^\pm(x, \cdot) : ]-1,1[ \to W^\pm_U(x)$ is a $C^1$ diffeomorphism, and

$$\forall y \in W^\pm_U(x), \lim_{t \to \pm \infty} d(\Phi^t_H(x), \Phi^t_H(y)) = 0.$$

We set

$$W^-(x) = \bigcup_{t \geq 0} \Phi^t_H(W^-_U(\Phi^{-t}_H(x))), \quad W^+(x) = \bigcup_{t \leq 0} \Phi^t_H(W^+_U(\Phi^{-t}_H(x))),$$

which are $C^1$ immersed 1-dimensional submanifolds of $H^{-1}(e)$.

- The manifolds $W^\pm(x)$ satisfy the equivariance property

$$\Phi^t_H(W^\pm(x)) = W^\pm(\Phi^t_H(x)), \quad \forall x \in C, \forall t \in \mathbb{R}.$$

- There is a negatively invariant neighborhood $\mathcal{N}^- \subset W^-_U(C)$ of $C$ in $W^-(C)$ such that for $x \in C$ and $y \in \mathcal{N}^- \cap W^-(x)$,

$$d(\Phi^{-t}_H(x), \Phi^{-t}_H(y)) \leq d(x, y), \quad \forall t \geq 0.$$
• The $\lambda$-property. For $x \in \mathcal{C}$, set $J^-_x = J^-(x, \cdot) : [1,1] \to W^-_U(x)$. Fix $x \in \mathcal{C}$. Then for any $C^1$ 1-dimensional submanifold $\Delta$ of $H^{-1}(e)$ which intersects $W^+(\mathcal{C})$ transversely in $H^{-1}(e)$ at $\xi \in W^+(x)$, there exist a family $(\Delta_t)_{t \geq t_0}$ of submanifolds of $\Delta$ containing $\xi$ and $C^1$ parametrizations $\ell_t : [1,1] \to H^{-1}(e)$ of the images $\Phi^t_H(\Delta_t)$ such that:
\[
\lim_{t \to +\infty} \| \ell_t - J^+_{\Phi^t_H(x)} \|_{C^0([1,1])} = 0.
\]

(10)

• There exists a $\Phi_H$-invariant Borel measure $\mu$ on $\mathcal{C}$ such that $\mu(O) > 0$ for any nonempty open subset $O$ of $\mathcal{C}$.

• The cylinder $\mathcal{C}$ is contained in the interior $\text{int} \, \hat{\mathcal{C}}$ of a cylinder $\hat{\mathcal{C}}$ which satisfies the previous five conditions, with natural continuation conditions (that is, with obvious notation, $U \subset \hat{U}$, $\hat{J}_U^\pm = J^\pm$, $N^- \subset \hat{N}^-$.) Any such cylinder $\hat{\mathcal{C}}$ is said to be a continuation of $\mathcal{C}$.

By analogy with the normally hyperbolic case, we say that $W^\pm(\mathcal{C})$ are the stable and unstable manifolds of $\mathcal{C}$ and that the leaves $W^\pm(x)$ are the center-stable and center-unstable manifolds of the points of $\mathcal{C}$. In the case of the (normally hyperbolic) cylinders or singular cylinders of [Mar1], the fact that the parametrizations $J^\pm$ are bi-Hölder comes from regularity of the system and adequate assumptions on the normal spectral gap, while the existence of the invariant measure is a consequence of the symplectic setting together with the Liouville theorem. The existence of continuations for the cylinders in [Mar1] directly comes from the usual normal hyperbolicity, while the $\lambda$-property is the content of the lambda-lemma for normally hyperbolic manifolds.

In the setting of Definition 2, the global manifolds $W^\pm(\mathcal{C})$ could depend on the choice of $U$, but this will cause no trouble in the following.

2. We set out now some remarks and conventions. Given a subset $A$ of a tame cylinder $\mathcal{C}$, we set
\[
W^\pm(A) = \bigcup_{x \in A} W^\pm(x),
\]
so that $W^\pm(A)$ are invariant when $A$ is invariant. This will be the case in particular when $A = \text{int} \, \mathcal{C}$.

Observe that since the parametrizations $J^\pm$ are bi-Hölder, the characteristic projections
\[
\Pi^\pm : W^\pm(\mathcal{C}) \to \mathcal{C}
\]
which to a point $y \in W^\pm(\mathcal{C})$ associate the unique point $x$ of $\mathcal{C}$ such that $y \in W^\pm(x)$, are Hölder too.

Convection. In the following, given a tame cylinder $\mathcal{C}$, we will choose once and for all one continuation of $\mathcal{C}$, which we always denote by $\hat{\mathcal{C}}$.

3. An example. We consider a system similar to that in [LMS]. Consider the following variant of the Arnold example on $\mathbb{K}^3$:

\[
H_\mu(\theta, r) = \frac{1}{2}(r_0^2 + r_1^2 + r_2^2) - (\cos(2\pi\theta_2) - 1)^2 + \mu f(\theta), \quad (\theta, r) \in \mathbb{K}^3,
\]

(13)
with $\mu \geq 0$ and where $\text{Supp} f \subset \{ \theta_2 \in [1/4, 3/4] \}$. When $\mu = 0$, the system is the uncoupled product of the integrable system generated by $\frac{1}{2}(r_0^2 + r_1^2)$ on $\mathbb{A}^2$ and the pendulum-like system $\frac{1}{2}r_2^2 - (\cos(2\pi \theta_2) - 1)^2$ on $\mathbb{A}$. This latter system admits the fixed point $(\theta_2, r_2) = O := (0, 0)$. Clearly $O$ is degenerate since the expansion of the Hamiltonian in the neighborhood of $O$ reads $\frac{1}{2}r_2^2 - 4\pi^2 \theta_2^2 + \cdots$. Note that $O$ admits well-defined stable and unstable manifolds, and also satisfies the usual $\lambda$-lemma by easy 2-dimensional arguments.

The annulus $\mathcal{A} = \mathbb{A}^2 \times \{O\}$ is invariant under the flow, but not normally hyperbolic. This is also the case when $\mu > 0$, thanks to the condition on the support of $f$.

For $\mu$ small enough, the level $H^{-1}_\mu(1)$ is compact and regular, and its intersection with $\mathcal{A}$ is the 3-dimensional torus

$$\mathcal{T} = \{ (\theta, r) \in \mathbb{A}^3 \mid (\theta_2, r_2) = O, \frac{1}{2}(r_0^2 + r_1^2) = 1 \},$$

which is invariant in $H^{-1}_\mu(1)$ but not normally hyperbolic. However, it clearly admits global invariant manifolds of $\mathcal{T}$, which are the product of $\mathcal{T}$ with the stable and unstable manifolds of $O$. In the same way, the stable and unstable manifolds of the points of $\mathcal{T}$ are the product of those points with the stable and unstable manifolds of $O$. They are uniquely defined.

Moreover, given a point $a = (a_0, a_1)$ on the circle $C$ of equation $\frac{1}{2}(r_0^2 + r_1^2) = 1$, the 2-dimensional torus

$$T_a = \{ (\theta, r) \in \mathbb{A}^2 \mid (\theta_2, r_2) = O, (r_0, r_1) = a \} \subset \mathcal{T}$$

is invariant under the flow. Varying $a$ yields a foliation of $\mathcal{T}$ by invariant tori. Given now any compact connected arc $C_a^0 \subset C$ with extremities $a$ and $a'$, the subset

$$\mathcal{C} = \{ (\theta, r) \in \mathbb{A}^2 \mid (\theta_2, r_2) = O, (r_0, r_1) \in C_a^0 \}$$

immediately satisfies our definition of a tame cylinder at energy 1 for $H_\mu$: the two components of its boundary are the invariant tori $T_a$ and $T_{a'}$. The existence of continuations comes from the choice of larger compact arcs $C_b^0 \supset C_a^0$. The $\lambda$-property comes for instance from the properties of $O$ and the invariant measure $\mu$ is the Liouville measure on $H^{-1}(0) \cap \mathcal{A}$ is deduced from the induced symplectic form on $\mathcal{A}$.

### 2.2 Good cylinders

1. We first introduce the notion of a twist section, which is the main tool to reduce our study to the Birkhoff theory. Given $a < b$ we set:

$$\mathbf{A}(a, b) = \mathbb{T} \times [a, b], \quad \Gamma(a) = \mathbb{T} \times \{a\}, \quad \Gamma(b) = \mathbb{T} \times \{b\}.$$  

We often abbreviate $\mathbf{A}(a, b)$ in $\mathbf{A}$ when there is no risk of confusion.

We refer to Appendix $\mathbf{A}$ for the usual definitions on twist maps, which here indifferently (uniformly) tilt the vertical to the right or the left. We denote by $\text{Ess}(\varphi)$ the set of essential invariant circles of an area-preserving twist map $\varphi : \mathbf{A} \to \mathbb{R}$. Elements $\Gamma \in \text{Ess}(\varphi)$ are graphs of uniformly Lipschitz functions $f_1 : \mathbb{T} \to [a, b]$, by the Birkhoff theorem. We endow $\text{Ess}(\varphi)$ with the Hausdorff topology or, equivalently, with the uniform $C^0$ topology on the corresponding functions.
Definition 3. We say that an area-preserving twist map $\varphi$ of $A$ is special if

- $\varphi$ does not admit any essential invariant circle with rational rotation number,
- the boundaries $\Gamma(a)$ and $\Gamma(b)$ are both dynamically minimal,
- each boundary $\Gamma(a)$, $\Gamma(b)$ is accumulated by a sequence of dynamically minimal elements of $\text{Ess}(\varphi)$.

In the following we will crucially use the following result, which is proved in Appendix A. Given an invariant essential circle $\Gamma \subset (A \setminus \Gamma(a))$, $\Gamma^-$ is the connected component of $A \setminus \Gamma$ located below $\Gamma$ in $A$.

Lemma 1. Let $\varphi$ be a special area-preserving twist map $\varphi$ of $A$. Then the following properties hold true:

(i) Any two distinct elements of $\text{Ess}(\varphi)$ are disjoint, so that the set $\text{Ess}(\varphi)$ admits a natural order $\leq$ given by $\Gamma_1 \leq \Gamma_2$ if $\ell_{\Gamma_1}(\theta) \leq \ell_{\Gamma_2}(\theta)$ for all $\theta \in \mathbb{T}$, where $\Gamma_j$ is the graph of $\ell_{\Gamma_j}, j = 1, 2$.

(ii) Given a nonempty subset $E \subset \text{Ess}(\varphi)$, $\inf E$ and $\sup E$ exist.

(iii) Every invariant essential circle $\Gamma \subset (A \setminus \Gamma(a))$ is either the upper boundary of a Birkhoff zone of $\varphi$, or is accumulated by a sequence of elements of $\text{Ess}(\varphi)$ located in $\Gamma^-$. 

(iv) Every invariant essential circle $\Gamma \subset (A \setminus \Gamma(b))$ is either the lower boundary of a Birkhoff zone of $\varphi$, or it is accumulated by a sequence of elements of $\text{Ess}(\varphi)$ located in $\Gamma^+$.

2. We consider a $C^2$ Hamiltonian function $H$ on $A^3$, we fix a completely regular value $e$ and a tame cylinder $C$ at energy $e$.

Definition 4. A twist section for $C$ is a quadruple $(\Sigma, A, \chi, \varphi)$ such that:

- $\Sigma$ is a global Poincaré section for the flow $(\Phi_H)|_C$, with return map $\varphi$;
- $\chi$ is an embedding of $A$ in $C$ with image $\Sigma = \chi(A)$;
- $\chi^{-1} \circ \varphi \circ \chi$ is a special area-preserving twist map of $A$.

Given such a twist section on $C$, we set

$$\partial_\star \Sigma = \chi(\Gamma(a)), \quad \partial^\star \Sigma = \chi(\Gamma(b)).$$

These two circles are contained in (distinct) boundary components of $\partial C$. We set

$$\partial_\star C = \text{cc}(\partial C, \partial_\star \Sigma), \quad \partial^\star C = \text{cc}(\partial C, \partial^\star \Sigma).$$

We finally define a continuation of $(\Sigma, A, \chi, \varphi)$ for the continuation $\hat{C}$ as a twist section $(\hat{\Sigma}, A, \hat{\chi}, \hat{\varphi})$ for $\hat{C}$ which continues the previous one in the natural way.

3. We can now define homoclinic correspondences attached to a tame cylinder. The correspondences we will have to consider will always be families of locally defined diffeomorphisms (with possibly intersecting domains), so we will not use here the complete formalism and give the simplest possible definitions. We begin with the transversality condition we impose to the center-stable and center-unstable leaves. Recall that given a manifold with boundary $L$, we denote its interior by $\text{int } L$. 

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Definition 5. We consider a $C^2$ Hamiltonian function $H$ on $\mathbb{R}^3$, we fix a completely regular value $\mathbf{e}$ and consider a tame cylinder $\mathcal{C}$ at energy $\mathbf{e}$. We define the transverse homoclinic intersection of the continuation $\hat{\mathcal{C}}$ as the set

$$\text{Homt}(\hat{\mathcal{C}}) \subset W^+(\text{int} \, \hat{\mathcal{C}}) \cap W^-(\text{int} \, \hat{\mathcal{C}})$$

formed by the points $\xi$ such that

$$W^-(\Pi^-(\xi)) \circ \phi_\xi W^+(\hat{\mathcal{C}}) \quad \text{and} \quad W^+(\Pi^+(\xi)) \circ \phi_\xi W^-(\hat{\mathcal{C}}),$$

where $\Pi^\pm$ are the characteristic projections defined in [12] and where $\phi_\xi$ stands for “intersects transversely at $\xi$ relatively to $H^{-1}(\mathbf{e})$.”

Note that $W^\pm(\text{int} \, \hat{\mathcal{C}}) = \text{int} W^\pm(\hat{\mathcal{C}})$, following our definitions. To relate our further definition with that of the so-called scattering map or scattering correspondence (see [DLS08] and references therein), let us state a lemma which is an immediate consequence of the implicit function theorem.

Lemma 2. Assume that the projections $\Pi^\pm$ are $C^1$. Let us fix $\xi \in \text{Homt}(\hat{\mathcal{C}})$ and write $x^\pm = \Pi^\pm(\xi) \in \hat{\mathcal{C}}$. Then $W^+(\hat{\mathcal{C}})$ and $W^-(\hat{\mathcal{C}})$ intersect transversely at $\xi$ in $H^{-1}(\mathbf{e})$. Moreover, there exists a 3-dimensional open neighborhood $\mathcal{O}$ of $\xi$ in $W^+(\text{int} \, \hat{\mathcal{C}}) \cap W^-(\text{int} \, \hat{\mathcal{C}})$, and open neighborhoods $\mathcal{O}^\pm$ of $x^\pm$ in $\text{int} \, \hat{\mathcal{C}}$ such that the restrictions $\Pi^\pm|_{\mathcal{O}}$ are $C^1$ diffeomorphisms from $\mathcal{O}$ onto $\mathcal{O}^\pm$.

Under the assumptions of the previous lemma, the “scattering correspondence” is the collection of all local diffeomorphisms

$$S = \Pi^+ \circ (\Pi^-|_{\mathcal{O}^-})^{-1} : O^- \to O^+.$$

We will first generalize this approach and assume the transversality condition to hold only on a full measure subset (the possibility of this coming from the construction of [Mar2]). In particular, we will be able to deal with points $x^- \in \mathcal{C}$ and $S(x^-) = x^+ \in \mathcal{C}$ with $\xi \in W^-(x^-) \cap W^+(x^+)$ and $W^+(\hat{\mathcal{C}}), W^-(\hat{\mathcal{C}})$ non transverse at $\xi$ in $H^{-1}(\mathbf{e})$. Our second and main variation consists in a reduction of the scattering correspondence by transport on the section $\Sigma$ by the Hamiltonian flow. This yields the following definition.

Definition 6. We consider a $C^2$ Hamiltonian function $H$ on $\mathbb{R}^3$, we fix a completely regular value $\mathbf{e}$ and consider a tame cylinder $\mathcal{C}$ at energy $\mathbf{e}$, with continuation $\hat{\mathcal{C}}$ and continued twist section $\hat{\Sigma}$. A homoclinic correspondence associated with these data is a family of $C^1$ local diffeomorphisms of $\text{int} \, \hat{\Sigma}$:

$$\psi = (\psi_i)_{i \in I}, \quad \psi_i : \text{Dom} \psi_i \to \text{Im} \psi_i,$$

where $\text{Dom} \psi_i$ and $\text{Im} \psi_i$ are open subsets of $\text{int} \, \hat{\Sigma}$, for which there exists a family of $C^1$ local diffeomorphisms of $\hat{\mathcal{C}}$:

$$S = (S_i)_{i \in I}, \quad S_i : \text{Dom} S_i \to \text{Im} S_i,$$

where $\text{Dom} S_i$ and $\text{Im} S_i$ are open subsets of $\text{int} \, \hat{\mathcal{C}}$, with $\text{Im} S_i \cap \text{int} \, \hat{\Sigma} \neq \emptyset$, such that for all $i \in I$. 

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there exists a non-negative $C^1$ function $\tau_i : \text{Dom} \psi_i \to \mathbb{R}$ such that
\[ \forall x \in \text{Dom} \psi_i, \quad \Phi_H^{\tau_i(x)}(x) \in \text{Dom} S_i \quad \text{and} \quad \psi_i(x) = S_i(\Phi_H^{\tau_i(x)}(x)); \] (20)

there is an open subset $\text{Dom} tS_i \subset \text{Dom} S_i$, with full measure in $\text{Dom} S_i$, such that
\[ \forall y \in \text{Dom} tS_i, \quad W^-(y) \cap W^+(S_i(y)) \cap \text{Hom}t(\hat{\mathcal{C}}) \neq \emptyset. \] (21)

We say that a family $S$ satisfying the previous properties is associated with $\psi$.

Homoclinic correspondences are not uniquely defined and the domains $\text{Dom} \psi_i$ (resp. $\text{Dom} S_i$) are not necessarily pairwise disjoint (in particular, the index set $I$ could be non-countable). In the following we indifferently consider our homoclinic correspondences as defined on $\hat{\Sigma} = \hat{\chi}(\hat{A})$ or on $\hat{A}$.

4. An arc of $\hat{A}$ is a continuous map $\zeta : [0, 1] \to \hat{A}$, and we write $\tilde{\zeta} = \zeta([0, 1]) \subset \hat{A}$ for its image. Given two distinct points $\theta, \theta'$ of $T$, we write $[\theta, \theta'] \subset T$ for the unique segment bounded by $\theta$ and $\theta'$ according to the natural orientation of $T$. We let $\pi : \hat{A} \to T$ be the natural projection onto the $\theta$-coordinate.

**Definition 7.** We consider a tame cylinder $\mathcal{C}$ with section $(\Sigma, A, \chi, \varphi)$ and continuations $\hat{\mathcal{C}}$, $(\hat{\Sigma}, \hat{A}, \hat{\chi}, \hat{\varphi})$, respectively. We let $\psi = (\psi_i)_{i \in I} : \hat{\Sigma} \to \hat{A}$ be a homoclinic correspondence. We identify $\hat{\Sigma}$ with $\hat{A} = [\hat{a}, \hat{b}] \times T$ by $\hat{\chi}$. Fix $\Gamma \in \text{Ess}(\hat{\varphi})$ contained in $\hat{A} \setminus \Gamma(\hat{a})$ and let $\alpha \in \Gamma$.

- A **splitting arc** based at $\alpha$ for these data is an arc $\zeta$ of $\hat{A}$ for which
  \[ \zeta(0) = \alpha, \quad \zeta([0, 1]) \subset \Gamma^-; \quad \exists i \in I, \ z_i(0, 1]) \subset \text{Dom} \psi_i, \quad \psi_i(\zeta([0, 1])) \subset \Gamma. \]

- A **right splitting arc** based at $\alpha = (\theta, r)$ is a splitting arc $\zeta$ based at $\alpha$, which admits a derivative $\zeta'(0) = (u, v)$ with $u > 0$, and such that $\pi(\zeta) = [\theta, \theta + r]$ with $0 < r < \frac{1}{2}$.

- A **left splitting arc** based at $\alpha = (\theta, r)$ is a splitting arc $\zeta$ based at $\alpha$, which admits a derivative $\zeta'(0) = (u, v)$ with $u < 0$, and such that $\pi(\zeta) = [\theta - r, \theta]$ with $0 < r < \frac{1}{2}$.
We say that a splitting arc is based on \( \Gamma \) when it is based at some point of \( \Gamma \).

One main remark is that if \( \zeta \) is a right (resp. left) splitting arc, then (up to reparametrization) the restriction \( \zeta\big|_{[0,s]} \) with \( 0 < s \leq 1 \) is also a right (resp. left) splitting arc.

Given a point \( \alpha = (\theta_0, r_0) \) in \( \hat{A} \), we denote by

\[
V^-(\alpha) = \{ (\theta, r) \mid r \in [\hat{a}, r_0] \}
\]

the vertical below \( \alpha \) in \( \hat{A} \).

**Definition 8.** Let \( \Gamma \in \text{Ess}(\hat{\varphi}) \) contained in \( \hat{A} \setminus \Gamma(\hat{a}) \), be the graph of the continuous function \( \gamma : T \to [\hat{a}, \hat{b}] \). Let \( \zeta \) be a right splitting arc based on \( \Gamma \) and let \( [\theta_0, \theta_0 + \tau] \) be its projection on \( T \). For any \( \theta \in ]\theta_0, \theta_0 + \tau[ \), let \( \alpha(\theta) = (\theta, \gamma(\theta)) \). Then the intersection \( V^- (\alpha(\theta)) \cap \zeta \) is compact, let \( \beta(\theta) = (\theta, r_\beta) \) be the point in this intersection with maximal coordinate \( r_\beta \).

We define the domain associated with \( \xi \) as the open subset

\[
D(\zeta) = \bigcup_{\theta \in ]\theta_0, \theta_0 + \tau[} ]\beta(\theta), \alpha(\theta)[ \subset \Gamma^-.
\]

We define the domain associated with a left splitting arc similarly.

See Figure 2. The first main property of the domains defined above is the following.

**Lemma 3.** For any \( x \in D(\zeta) \), the vertical \( V^-(x) \) below \( x \) intersects \( \zeta(]0,1[) \).

The proof is obvious. The second property is stated in the following lemma (see Appendix A for a proof.)

**Lemma 4.** Consider an essential circle \( \Gamma \in \text{Ess}(\hat{\varphi}) \) contained in \( \hat{A} \setminus \Gamma(\hat{a}) \), and a right (resp. left) splitting arc \( \zeta \) based on \( \Gamma \), with domain \( D(\zeta) \). Consider an essential circle \( \Gamma_* \subset \hat{A} \) such that \( \zeta \) is contained in the domain \( \Gamma_*^+ \) above \( \Gamma_* \). Then for \( x \in D(\zeta) \) there exists a positively (resp. negatively) tilted arc \( \gamma \) with \( \gamma(0) \in \Gamma_* \) and \( \gamma(1) = x \), whose image does not intersect the union \( \Gamma \cup \zeta \).

In the previous lemma, the circle \( \Gamma_* \) is not assumed to be invariant by the return map.

5. We can now define the good cylinders, for which we distinguish between two cases.
Definition 9. Consider a tame cylinder $\mathcal{C}$ equipped with a twist section $(\Sigma, \varphi)$, with continuations $\hat{\mathcal{C}}$, $(\hat{\Sigma}, \hat{\varphi})$, and homoclinic correspondence $\psi$.

- We say that $\mathcal{C}$ is a good F-cylinder when the homoclinic correspondence reduces to a single $C^1$ diffeomorphism $\psi : \text{Dom} \psi \to \hat{\mathbb{A}}$ which is exact-symplectic and satisfies
  - $\text{Dom} \psi$ contains $\mathbb{A}$;
  - for each $\Gamma \in \text{Ess}(\varphi)$, $\psi(\Gamma) \neq \Gamma$.

- We say that $\mathcal{C}$ is a good P-cylinder when
  - for any element $\Gamma \subset \text{int} \hat{\Sigma} \setminus \text{Ess}(\hat{\varphi})$ which is not the upper boundary of a Birkhoff zone, there exists a splitting arc based on $\Gamma$;
  - if $\Gamma$ is the upper boundary of a Birkhoff zone, then there exists a right or left splitting arc based on $\Gamma$.

The notions of good P-cylinders and F-cylinders are not exclusive, however there are examples of good P-cylinders which are not good F-cylinders.

An additional definition will be necessary to produce $\delta$-admissible orbits. We denote by $\text{cl}(E)$ the closure of a subset $E \subset \hat{\Sigma}$.

Definition 10. With the same assumptions as in the previous definition, fix $\delta > 0$. We say that $\psi = (\psi_i)_{i \in I}$ is $\delta$-bounded when for each essential circle $\Gamma \in \text{Ess}(\varphi)$

$$\sup \{ \text{dist}(x, \Gamma) \mid x \in \Gamma^- \land \psi^{-1}(\text{cl} \Gamma^+) \} < \delta,$$

(22)

where $\text{dist}$ is the point-set distance on $\hat{\Sigma}$ induced by the canonical distance on $\mathbb{A}^3$ and where $\psi^{-1}(\Gamma^+)$ is the set of all points $x \in \hat{\Sigma}$ such that there exists $i \in I$ with $x \in \text{Dom} \psi_i$ and $\psi_i(x) \in \Gamma^+$. We say that $\mathcal{C}$ is $\delta$-good when $\psi$ is $\delta$-bounded.

The following remark is an immediate consequence of the possibility of “restricting the arcs.”

Remark 1. Consider a good P-cylinder $\mathcal{C}$ equipped with a twist section $(\Sigma, \varphi)$, with continuations $\hat{\mathcal{C}}$, $(\hat{\Sigma}, \hat{\varphi})$, and homoclinic correspondence $\psi = (\tilde{\psi}_i)_{i \in I}$. Then given $\rho > 0$, there exists a homoclinic correspondence $\tilde{\psi} = (\tilde{\psi}_i)_{i \in I}$ such that

$$\text{diam} \text{Dom} \tilde{\psi}_i < \rho, \quad \forall i \in I,$$

and such that $\mathcal{C}$ is still a good P-cylinder when endowed with $\tilde{\psi}$. As a consequence, given $\delta > 0$, one can always assume the homoclinic correspondence of a good P-cylinder to be $\delta$-bounded.

Remark 2. The notion of right or left splitting arc is introduced here for the sake of clarity. For practical reasons (such as the proof of generic existence of diffusion), it may be useful to replace it by a more general notion. One says that an arc $\gamma : [0, 1] \to \mathbb{A}$ is non-vertical when there exists a sequence $(s_n)_{n \in \mathbb{N}}$ in $[0, 1]$ such that $\lim_{n \to \infty} s_n = 0$ with

$$\pi(\gamma(s_n)) \neq \pi(\gamma(0)), \quad \forall n \in \mathbb{N}.$$ 

Note in particular that the image of a vertical arc by a twist map is non-vertical, see [Mar2] for explicit examples where this notion proves to be useful.
Convention. In the following, given a good cylinder $\mathcal{C}$, we implicitly choose once and for all a section $(\Sigma, \mathbf{A}, \chi, \varphi)$, a continuation $\tilde{\mathcal{C}}$, a continuation $(\tilde{\Sigma}, \tilde{\mathbf{A}}, \tilde{\chi}, \tilde{\varphi})$ and a homoclinic correspondence $\psi$, which will always be denoted this way.

2.3 Good chains

We consider a $C^2$ Hamiltonian function $H$ on $\mathbb{R}^3$, with completely regular value $e$.

Definition 11. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be good cylinders at energy $e$ for $H$, with continuations $\tilde{\mathcal{C}}_i$ and characteristic projections $\Pi_i^\pm : W^\pm(\tilde{\mathcal{C}}_i) \to \tilde{\mathcal{C}}_i$. We define the transverse heteroclinic intersection of $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ as the set

$$\text{Hett}(\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2) \subset W^-(\text{int} \tilde{\mathcal{C}}_1) \cap W^+(\text{int} \tilde{\mathcal{C}}_2)$$

(23)

formed by the points $\xi$ such that

$$W^-(\Pi_1^-(\xi)) \cap W^+(\mathcal{C}_2) \quad \text{and} \quad W^+(\Pi_2^+(\xi)) \cap W^-(\mathcal{C}_1).$$

(24)

Again, when the characteristic projections are $C^1$, as in Lemma 2, for $\xi \in \text{Hett}(\tilde{\mathcal{C}})$ we set $x^- = \Pi_1^-(\xi) \in \tilde{\mathcal{C}}_1$ and $x^+ = \Pi_2^+(\xi) \in \tilde{\mathcal{C}}_2$, so $W^+(\tilde{\mathcal{C}}_1)$ and $W^-(\tilde{\mathcal{C}}_2)$ intersect transversely at $\xi$ in $H^{-1}(e)$. Then there exist a 3-dimensional open neighborhood $\mathcal{O}$ of $\xi$ in $W^+(\text{int} \tilde{\mathcal{C}}_1) \cap W^-(\text{int} \tilde{\mathcal{C}}_2)$, and open neighborhoods $O_1^-$ of $x^-$ in $\text{int} \tilde{\mathcal{C}}_1$ and $O_2^+$ of $x^+$ in $\text{int} \tilde{\mathcal{C}}_2$, such that the restrictions $(\Pi_1^-|_{\mathcal{O}})$ and $(\Pi_2^+|_{\mathcal{O}})$ are $C^1$ diffeomorphisms from $\mathcal{O}$ onto their images $O_1^-, O_2^+$, respectively. Hence one can define the local diffeomorphisms $S_1^2 = \Pi_2^+ \circ (\Pi_1^-|_{\mathcal{O}})^{-1} : O_1^- \to O_2^+.

This motivates the following definition for the notion of heteroclinic correspondences.

Definition 12. A heteroclinic map associated with $(\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2)$ is a $C^1$ diffeomorphism

$$\psi_1^2 : \text{Dom} \psi_1^2 \to \text{Im} \psi_1^2$$

(25)

where $\text{Dom} \psi_1^2$ is open in $\text{int} \tilde{\Sigma}_1$ and $\text{Im} \psi_1^2$ is open in $\text{int} \tilde{\Sigma}_2$, for which there exists a $C^1$ diffeomorphism

$$S_1^2 : \text{Dom} S_1^2 \to \text{Im} S_1^2$$

(26)

where $\text{Dom} S_1^2$ and $\text{Im} S_1^2$ are open in $\text{int} \tilde{\mathcal{C}}_1$ and $\text{int} \tilde{\mathcal{C}}_2$, with $\text{Im} S_1^2 \cap \text{int} \tilde{\Sigma}_2 \neq \emptyset$, which satisfies the following conditions:

- there exists a non-negative $C^1$ function $\tau : \text{Dom} \psi_1^2 \to \mathbb{R}$ such that

$$\forall x \in \text{Dom} \psi_1^2, \quad \Phi_H^{\tau(x)}(x) \in \text{Dom} S_1^2 \quad \text{and} \quad \psi_1^2(x) = S_1^2 \left( \Phi_H^{\tau(x)}(x) \right);$$

(27)

- there is an open subset $\text{Dom} t S_1^2 \subset \text{Dom} S_1^2$, with full measure in $\text{Dom} S_1^2$, such that

$$\forall y \in \text{Dom} t S_1^2, \quad W^-(y) \cap W^+(S_1^2(y)) \cap \text{Hett}(\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2) \neq \emptyset.$$  

(28)

\footnotetext[10]{It turns out that in the subsequent constructions of [Mar2] the projections are locally $C^1$ in a neighborhood of the heteroclinic points we are interested in.
For technical reasons which appear in [Mar1], we finally have to consider the case where the cylinders $C_i$ have one boundary component in common (and their union itself is a cylinder). In this case the transition maps between the sections will be natural flow-induced maps.

**Definition 13.** Assume that the cylinders $C_1$ and $C_2$ are contained in a cylinder $C$ and satisfy $\partial^* C_1 = \partial^* C_2$, in which case we say they are adjacent. A transition map from $pC_1$ to $pC_2$ is a flow-induced map

$$\psi^2_1 : \text{Dom} \psi^2_1 \to \text{Im} \psi^2_1$$

where $\text{Dom} \psi^2_1$ is an open neighborhood of $\partial^* \Sigma_1$ in $\hat{\Sigma}_1$, and $\text{Im} \psi^2_1$ is an open neighborhood of $\partial^* \Sigma_2$ in $\hat{\Sigma}_2$. More precisely, there exists a non-negative function $\tau : \text{Dom} \psi^2_1 \to \mathbb{R}$ such that

$$\psi^2_1(x) = \Phi_H^{\tau(x)}(x), \quad \forall x \in \text{Dom} \psi^2_1.$$

We are now in a position to define the main objects of our study.

**Definition 14.** Fix $\delta > 0$. A $\delta$-good chain of cylinders at energy $e$ is a finite ordered family $(C_k)_{1 \leq k \leq k_e}$ of $\delta$-good cylinders at energy $e$, such that for $1 \leq k \leq k_e - 1$ there exists a heteroclinic map or a transition map $\psi^k_{k+1}$ from $\hat{C}_k$ to $\hat{C}_{k+1}$ which satisfies the following condition:

- for any open neighborhood $O$ of $\partial^* \Sigma_k$ in $\hat{\Sigma}_k$, the image $\psi^k_{k+1}(O)$ intersects a dynamically minimal circle of $\text{Ess}(\varphi_{k+1})$ located in the $\delta$-neighborhood of $\partial^* \Sigma_{k+1}$ in $\hat{\Sigma}_{k+1}$.

Again, the $\delta$-neighborhoods are defined relatively to the induced distance on the section $\hat{\Sigma}_{k+1}$.

### 3 Pseudo orbits along a good chain of cylinders

Let us first make the definition of an orbit of a polysystem more precise (see [M02] [Mar08] for a formal definition.) Let $A$ be some set and consider a set $f = \{f_i | i \in I\}$ of locally defined maps $f_i : \text{Dom} f_i \to A$. We say that a finite sequence $(x_n)_{0 \leq n \leq n_\ast - 1}$ of points of $A$ is a finite orbit of $f$, of length $n_\ast \geq 1$, when there exists a sequence $\omega = (i_n)_{0 \leq n \leq n_\ast - 1} \in I^{n_*}$ such that for $0 \leq n \leq n_\ast - 1$:

$$x_{n+1} = f_{i_n}(x_n),$$

and we write

$$x_{n_\ast} = f^\omega(x_0).$$

We formally consider the point $x_0$ as being the 0-length orbit of $x_0$.

Given a subset $B \subset A$, we set

$$f^\omega(B) = \bigcup_{x \in B_\omega} f^\omega(x)$$

where $B_\omega$ is the subset of $B$ formed by the points $x$ such that $f^\omega(x)$ is well-defined.
The full orbit of \( B \subset A \) under \( f \) is the subset of \( A \) formed by the union of all \( f^{\omega}(B) \) for all sequences (of any length) \( \omega \) (so that in particular \( B \) is contained in its full orbit under \( f \)).

**Conventions.** Given locally defined maps \( f_i : \text{Dom} f_i \to \text{Im} f_i \) on \( A \), we write \( f_i \circ f_j \) for the map defined by composition on the subset \( \text{Dom} f_i \cap \text{Im} f_j \), and for a subset \( B \subset A \), we write \( f_i^{-1}(B) \) for \( f_i(\text{Dom} f_i \cap B) \). Given a finite set of polysystems \( \varphi, \psi, \ldots \) on \( A \), we write \( (\varphi, \psi, \ldots) \) for the polysystem formed by their union. Given a polysystem \( f = \{ f_i | i \in I \} \) on a set \( A \), and a subset \( A_* \) of \( A \), we define the restriction \( f|_{A_*} \) of \( f \) to \( A_* \) as the polysystem formed by the maps

\[
f_i^* : \text{Dom} f_i \cap A_* \cap f_i^{-1}(A) \to A_*. \]

**Definition 15.** Let \( H \) be a \( C^2 \) Hamiltonian on \( \mathbb{R}^3 \) and fix a completely regular value \( e \). Let \((\mathcal{C}_k)_{1 \leq k \leq k_*}\) be a good chain of cylinders in \( H^{-1}(e) \). We define the polysystem associated with these data as the following set of (globally or locally) defined diffeomorphisms:

\[
F = \{ \hat{\varphi}_k \mid 1 \leq k \leq k_* \} \cup \{ \psi_{k,i} \mid 1 \leq k \leq k_* \}, \quad i \in I_k \cup \{ \psi_{k+1} \mid 1 \leq k \leq k_* - 1 \}, \tag{30}
\]

where \( \hat{\varphi}_k \) is the continuation of \( \varphi_k \) and \( \psi_{k+1} \) is either a heteroclinic or a transition map from \( \hat{\varphi}_k \) to \( \hat{\varphi}_{k+1} \). This polysystem will be indifferently considered to be defined on \( \bigcup_{1 \leq k \leq k_*} \mathcal{A}_k \), or on \( \bigcup_{1 \leq k \leq k_*} \mathcal{S}_k \). The finite orbits of that polysystem will be called pseudo-orbits along the chain.

The pseudo-orbits above are therefore generated by successive applications of the maps \( \hat{\varphi}_k, \psi_{k,i}, \psi_{k+1} \) in any possible order. Our aim in this section is to prove the following result.

**Theorem B.** Let \( H \) be a \( C^2 \) Hamiltonian on \( \mathbb{R}^3 \) and let \( e \) be a completely regular value. Let \((\mathcal{C}_k)_{1 \leq k \leq k_*}\) be a \( \delta \)-good chain of cylinders at energy \( e \) for some \( \delta > 0 \). Then there exists a pseudo-orbit \( (x_n)_{0 \leq n \leq n_*} \) along the chain such that for any essential invariant circle \( \Gamma \) in \( \bigcup_{1 \leq k \leq k_*} \text{Ess}(\varphi_k) \), there is a \( \nu \in \{0, \ldots, n_*\} \) with \( d(x_{\nu}, \Gamma) < \delta \).

We first prove in Section 3.1 the existence of pseudo-orbits in the case where the chain is reduced to a single cylinder. **Theorem B** is easily deduced from this latter result in Section 3.2. The main remark in our proof comes from Lemma 14 of Appendix C; it is enough to prove the existence of pseudo-orbits for the “symmetrized” polysystem

\[
G = \{ \hat{\varphi}_k, \hat{\varphi}_k^{-1} \mid 1 \leq k \leq k_* \} \cup \{ \psi_{k,i} \mid 1 \leq k \leq k_* \}, \quad i \in I_k \cup \{ \psi_{k+1} \mid 1 \leq k \leq k_* - 1 \} \tag{31}
\]

and get the conclusion by density.

### 3.1 Pseudo-orbits in the case of a single cylinder

Given \( \nu > 0 \), we define a \( \nu \)-ball of \( \mathbb{T} \times \mathbb{R} \) as a subset \( B = B_\theta \times B_r \) where \( B_\theta \) and \( B_r \) are intervals of \( \mathbb{T} \) and \( \mathbb{R} \) respectively, such that

\[
\text{length } B_r > \nu \text{ length } B_\theta. \tag{32}
\]

The center of \( B \) is \((a_\theta, a_r)\), where \( a_\theta, a_r \) are the mid-points of \( B_\theta \) and \( B_r \). We keep the notation and conventions of the last section for the good cylinders and their continuations.
Proposition 1. Let $H$ be a $C^2$ Hamiltonian on $\mathbb{A}^3$ and let $\mathbf{e}$ be a completely regular value. Let $\mathcal{C}$ be a good cylinder at energy $\mathbf{e}$ for $H$. Let $g = (\hat{\varphi}, \hat{\varphi}^{-1}, \psi)$ be the associated symmetrized) polysystem on $\mathbb{A}$, with $\psi = (\psi_i)_{i \in I}$. Fix $\Gamma_\bullet \in \text{Ess}(\hat{\varphi})$, $\Gamma_\bullet \subset \mathbb{A} \setminus \Gamma(b)$. Fix a neighborhood $V$ of $\Gamma_\bullet$ in $\mathbb{A}$ (see Figure 3.1). Then the full orbit of $V$ under $g$ contains $\Gamma(b)$.

Figure 3: The setting of Proposition 1

We will examine the P-cylinder and F-cylinders separately, the simpler case of F-cylinders being an easy generalization of [M02]. The assumption $\Gamma_\bullet \subset \mathbb{A} \setminus \Gamma(b)$ is a matter of simplification for the case of F-cylinders and can easily be relaxed in practical cases.

Proof of Proposition 1 in the case of P-cylinders. We assume for example that $\hat{\varphi}$ tilts the vertical to the right, the other case being exactly similar. We will prove a slightly stronger property than our initial statement. Namely, let $\mathbb{A}_\ast$ be the $\hat{\varphi}$–invariant subannulus of $\mathbb{A}$ limited by the (disjoint) circles $\Gamma_\bullet$ and $\Gamma(b)$. We consider the restricted polysystem $g|_{\mathbb{A}_\ast}$ instead of $g$ and we consider a neighborhood $V$ of $\Gamma_\bullet$ in $\mathbb{A}_\ast$.

1. For simplicity, we still denote by $\hat{\varphi}$ and $\psi$ the restrictions of $\hat{\varphi}$ and $\psi$ to $\mathbb{A}_\ast$. We also assume without loss of generality that $V$ is open in $\mathbb{A}_\ast$, and connected. Let $U$ be the full orbit of the open set $V$ under the polysystem $g = (\hat{\varphi}, \hat{\varphi}^{-1}, \psi = (\psi_i)_{i \in I})$ on $\mathbb{A}_\ast$. Note that $\hat{\varphi}(U) = U$ and $\psi_i(U) \subset U$. Set $U_c = \text{cc}(U, \Gamma_\bullet)$. Then $U_c$ is open and contains $V$, so $\hat{\varphi}(U_c) = U_c$. It is therefore enough to prove that $U_c$ intersects $\Gamma(b)$: since $\Gamma(b)$ is dynamically minimal, this yields $\Gamma(b) \subset U_c \subset U$.

Let us assume by contradiction that $U_c \cap \Gamma(b) = \emptyset$, so that $U_c$ is contained in the lower connected component of $\mathbb{A}_\ast \setminus \Gamma(b)$.

2. Set $O = \mathbb{A}_\ast \setminus U_c$, so that $O$ is open, contains $\Gamma(\hat{b})$, and $O \cap V = \emptyset$. Moreover, since $\hat{\varphi}(U_c) = U_c$, $\hat{\varphi}(O) = \mathbb{A}_\ast \setminus \hat{\varphi}(U_c) = \mathbb{A}_\ast \setminus U_c = O$.

Then $\hat{\varphi}(\text{cc}(O, \Gamma(\hat{b}))) = \text{cc}(O, \Gamma(\hat{b}))$ and so $\hat{\varphi}(\text{cc}(O, \Gamma(\hat{b}))) = \text{cc}(O, \Gamma(\hat{b}))$. Let

$\hat{U} = \mathbb{A}_\ast \setminus \text{cc}(O, \Gamma(\hat{b}))$,
so that  is open and , and set finally
\[\mathcal{U} = \text{cc}\left(\hat{U}, \Gamma_*\right),\]

hence is open, connected and \(\hat{\varphi}(\mathcal{U}) = \mathcal{U}\). Moreover clearly
\[\overline{\mathcal{U}} \subset A_* \setminus \text{cc}(O, \Gamma(\hat{b})),\]

and
\[U_c \subset \mathcal{U},\]

since \(\overline{O} = A_* \setminus \text{Int}(\overline{U_c}) \subset A_* \setminus U_c\), so \(\text{cc}(O, \Gamma(\hat{b})) \subset A_* \setminus U_c\) and \(U_c \subset A_* \setminus \text{cc}(O, \Gamma(\hat{b})) = \hat{U}\), which proves our claim since \(\Gamma_* \subset U_c\).

3. Let us now prove that \(\Gamma := \text{Fr}\mathcal{U}\) is a Lipschitz graph over \(T\), invariant under \(\hat{\varphi}\), by the Birkhoff theorem (see Appendix [A]). By local connectedness of \(A_*\), one readily proves that \(\text{Int}\overline{\mathcal{U}} = \mathcal{U}\), since \(\mathcal{U}\) is a connected component of the complement of the closure of an open set. Moreover \(\hat{\varphi}(\mathcal{U}) = \mathcal{U}\). Let now \(S\) be the quotient of \(A_*\) by the identification of each boundary circle to one point, so that \(S\) is homeomorphic to \(S^2\). Up to this quotient, \(\mathcal{U}\) is a connected component of the complement in \(S\) of a compact connected subset, so is homeomorphic to a disk. Going back to the initial space \(A_*\) proves that \(\mathcal{U}\) is homeomorphic to \(T \times [0,1]\). So by the Birkhoff theorem, \(\Gamma = \partial \mathcal{U}\) is a Lipschitz graph over \(T\), invariant under \(\hat{\varphi}\) (see [M02] for more details).

4. Let us now prove that \(\Gamma \subset \overline{U_c}\), and so \(\Gamma \subset \text{Fr}\mathcal{U} = \text{cl}(U_c) \setminus U_c\). Assume that \(x \in \Gamma\) is not in \(\overline{U_c}\), so that there exists a small ball \(B(x, \varepsilon)\) with \(B(x, \varepsilon) \cap \overline{U_c} = \emptyset\). Let \(z\) be some point on the vertical through \(x\), located under \(\Gamma\) and inside \(B(x, \varepsilon)\). Let us show that the semi-vertical \(\sigma\) over \(z\) in \(A_*\) is disjoint from \(\overline{U_c}\). First \(\Gamma \cap \sigma = \{x\}\), since \(\Gamma\) is a graph, so that \(\sigma = \{z, x\} \cup \{x, \xi\}\), with \(\xi \in \Gamma(\hat{b})\). Clearly \(\{z, x\} \subset B(x, \varepsilon)\) so \([z, x] \cap \overline{U_c} = \emptyset\), and \([x, \xi]\) \cap \overline{U} = \emptyset since \(\Gamma = \partial \mathcal{U}\) is a graph. Since \(U_c \subset \mathcal{U}\), this proves that \(\sigma \cap \overline{U_c} = \emptyset\).

As a consequence \(\sigma \cup \Gamma(\hat{b})\) is a connected set which satisfies \((\sigma \cup \Gamma(\hat{b})) \cap \overline{U_c} = \emptyset\). Therefore \((\sigma \cup \Gamma(\hat{b})) \subset \text{cc}(O, \Gamma(\hat{b}))\) and thus \((\sigma \cup \Gamma(\hat{b})) \cap \overline{U} = \emptyset\) by (34). This is a contradiction since \(x \in \Gamma \subset \overline{U}\). Therefore \(\Gamma \subset \overline{U_c}\).

5. Since \(\Gamma\) is an invariant essential circle for the special twist map \(\hat{\varphi}\), there are only two possibilities:
- either \(\Gamma\) is the upper boundary of a Birkhoff zone,
- or \(\Gamma\) is accumulated from below by essential invariant circles in the Hausdorff topology.

We will prove that both possibilities yield a contradiction with the initial assumption that \(U_c \cap \Gamma(\hat{b}) = \emptyset\).

6. Assume first that \(\Gamma\) is the upper boundary of a Birkhoff zone \(\mathcal{U}\) and let \(\Gamma_*\) be the lower boundary of \(\mathcal{U}\). Let \(\nu\) be the Lipschitz constant of \(\Gamma_*\). Since \(\Gamma_*\) is a graph and \(U_c\) is open, connected, contains \(V\) and \(\overline{U_c} \cap \Gamma \neq \emptyset\), then \(U_c \cap \Gamma_* \neq \emptyset\). So there exists a \(\nu\)-ball \(B \subset U_c\) centered on \(\Gamma_*\).

Since \(\mathcal{U}\) is a good \(P\)-cylinder, there exists a right or left splitting arc \(\zeta\) based at some point \(\alpha\) of \(\Gamma\) (see Section [2]). Let \(D\) be its associated domain. By restricting \(\zeta\) if necessary, one can moreover assume without loss of generality that \(D \subset \mathcal{U}\). We introduce the closed
connected set $X = \Gamma \cup \bar{\zeta}$, where $\bar{\zeta} = \zeta([0,1])$, which disconnects the annulus $\hat{A}$ since it contains $\Gamma$.

Assume first that $\zeta$ is a right splitting arc. By Proposition 3 there exist $z_0 \in B$ and $n \in \mathbb{N}$ such that $z_n := \hat{\varphi}^n(z_0) \in D$. Then, by Lemma 4 there exists a positively tilted arc emanating from $\Gamma_*$ and ending at $z_n$ which does not intersect $X$.

By Lemma 9 there exists a negatively tilted arc $\gamma$ with image in $B$ emanating from $\Gamma_*$ and ending at $z_0$. Therefore, by Lemma 11 $\gamma_n := f^n \circ \gamma$ is a negatively tilted arc emanating from $\Gamma_*$ and ending at $z_n$.

Assume that the image $\bar{\gamma}_n$ does not intersect $X$, then by Lemma 12 the vertical $V^-(z_n)$ does not intersect $X$, which contradicts Lemma 3. Therefore $\bar{\gamma}_n \cap X \neq \emptyset$, thus $\bar{\gamma}_n \cap \bar{\zeta} \neq \emptyset$.

If now $\zeta$ is a left splitting arc, we use $\hat{\varphi}^{-1}$ instead of $\hat{\varphi}$. This first yields a $z_0 \in B$ such that $z_{-n} := \hat{\varphi}^{-n}(z_0) \in D$, then a negatively tilted arc emanating from $\Gamma_*$ and ending at $z_{-n}$ which does not intersect $X$, and a positively tilted arc, still denoted by $\gamma_n$, emanating from $\Gamma_*$ and ending at $z_{-n}$. As above, this proves that $\bar{\gamma}_n \cap \bar{\zeta} \neq \emptyset$.

As a consequence, $U_c \cap \bar{\zeta} \neq \emptyset$ since $\bar{\gamma}_n \subset U_c$, and therefore there is a small open ball $B \subset U_c$ centered on $\zeta([0,1])$ and, by definition, an index $i \in I$ such that $B \subset \text{Dom} \psi_i$. Thus $\psi_i(B)$ is an open set which intersects $\Gamma$, and therefore also $U_c$ since $\Gamma \subset \overline{U_c}$. This proves that $\psi_i(B) \subset U_c$ by connectedness, so that $U_c$ contains points strictly above the circle $\Gamma$. This is a contradiction with the construction of $\Gamma = \text{Fr} U$ and the inclusion $U_c \subset U$, which ensures that all points of $U_c$ are located below $\Gamma$.

**7.** Assume now that $\Gamma$ is accumulated from below by an increasing sequence $(\Gamma_m)_{m \geq 1}$ of essential invariant circles for $\hat{\varphi}$. Let $\zeta$ be a splitting arc based on $\Gamma$. Let $S_m$ be the closed strip limited by $\Gamma_m$ and $\Gamma_{m+1}$. For $m$ large enough, $S_m \cap \bar{\zeta}$ contains a $C^0$ curve $\ell$ which intersects both $\Gamma_m$ and $\Gamma_{m+1}$. Now $\Gamma \subset \overline{U_c}$, so that $U_c \cap S_m$ contains a $C^0$ curve $\ell'$ which also intersects both $\Gamma_m$ and $\Gamma_{m+1}$. Therefore, by Lemma 10 there exists an integer $n$ such that $\hat{\varphi}^n(U_c) \cap \ell \neq \emptyset$, and so by invariance of $U_c$ under $\hat{\varphi}$, $U_c \cap \ell \neq \emptyset$. Since $\ell \subset \bar{\zeta} \subset \text{Dom} \psi_i$ for some $i \in I$, there exists a ball $B \subset U_c$ centered on $\ell \subset \bar{\zeta}$ and contained in $\text{Dom} \psi_i$. This yields the same contradiction as in the previous paragraph.

**8.** In the case of a $P$-cylinder, we have therefore proved that the two possibilities of our alternative yield a contradiction, which proves that our initial assumption $U_c \cap \Gamma(b) = \emptyset$ is false. This concludes the proof of the proposition for $P$-cylinders.

---

**Proof of Proposition 4 in the case of $F$-cylinders.** Now $\psi : \text{Dom} \psi \rightarrow \text{Im} \psi$ is an exact-symplectic diffeomorphism, with $A \subset \text{Dom} \psi$. A main difference with [M02] is that the circles $\Gamma_*$ and $\Gamma(b)$ are not invariant under $\psi$. However, $\psi(\Gamma) \neq \Gamma$ and $\psi(\Gamma) \cap \Gamma \neq \emptyset$ for any $\Gamma \subset \text{Ess}(\varphi)$ contained in $\text{Dom} \psi$, by our assumptions on $\psi$.

1. To take advantage of this simple setting, we do not restrict the polysystem as in the proof for $P$-cylinders. We let $V$ be a neighborhood of $\Gamma_*$ in $A$ and let $U$ be the full orbit of $V$ under $g = (\varphi, \varphi^{-1}, \psi)$. So clearly $\varphi(U) = U$ and $\psi(U \cap \text{Dom} \psi) \subset U$. Set now $U_c = cc(U \cap A_*, \Gamma_*)$

where, as above, $A_*$ is the annulus limited by $\Gamma_*$ and $\Gamma(b)$. Clearly $\varphi(U_c) = U_c$.

2. We assume as above that $U_c$ is contained in the lower connected component of $A_* \setminus \Gamma(b)$. Therefore $U_c \subset A \subset \text{Dom} \psi$ so $\psi(U_c) \subset U$ and since $\psi(\Gamma_*) \cap \Gamma_* \neq \emptyset$, $\psi(U_c) \subset U_c$. Define
\( \mathcal{U} \) as in (33). Then \( \hat{\varphi}(\mathcal{U}) = \mathcal{U} \), and \( \Gamma = \text{Fr}\mathcal{U} \) is a Lipschitz graph invariant by \( \hat{\varphi} \), so \( \Gamma \in \text{Ess}(\varphi) \) since \( \Gamma \subset A \). Now necessarily \( \psi(\Gamma) \subset \Gamma \), otherwise \( U_c \setminus \psi(U_c) \) would contain a nonempty open set, which would contradict the fact that \( \psi \) is exact-symplectic. Therefore \( \psi(\Gamma) = \Gamma \) by compactness and connectedness. This is a contradiction with the assumption that no element of \( \text{Ess}(\varphi) \) is invariant under \( \psi \). This concludes the proof of the proposition for \( F \)-cylinders. \( \square \)

As a consequence of Lemma 14 under the assumptions of Proposition 1 for any \( \delta > 0 \), the polysystem \( \tilde{f} = (\hat{\varphi}, \psi) \) admits orbits which intersect the \( \delta \)-neighborhoods of \( \Gamma \), and \( \Gamma(b) \). We now take into account the additional assumption that \( \psi \) is \( \delta \)-bounded.

**Corollary 1.** With the same assumptions and notation as in Proposition 1 if \( \psi \) is \( \delta \)-bounded, any orbit of \( f = (\hat{\varphi}, \psi) \) which intersects the \( \delta \)-neighborhoods of \( \Gamma \) and \( \Gamma(b) \) intersects the \( \delta \)-neighborhood of every element of \( \text{Ess}(\varphi) \) contained in \( A \).

**Proof.** Consider an orbit \( (x_n)_{0 \leq n \leq n_*} \) of \( f \) with \( x_0 \) in the \( \delta \)-neighborhood of \( \Gamma \), and \( x_{n_*} \) in the \( \delta \)-neighborhood of \( \Gamma(b) \). Let \( \Gamma \in \text{Ess}(\varphi) \) be contained in the complement of these neighborhoods. Then there exists \( \nu \in \{0, \ldots, n_*\} \) such that \( x_\nu \in \Gamma^- \cap \psi^{-1}(\text{cl}\Gamma^+) \) (if the orbit has no point in \( \Gamma^- \cap \psi^{-1}(\text{cl}\Gamma^+) \)), an immediate induction shows that \( x_n \in \Gamma^- \) for \( 0 \leq n \leq n_* \). Now, since \( \psi \) is \( \delta \)-bounded,

\[
\text{Sup} \{ \text{dist}(x, \Gamma) \mid x \in \Gamma^- \cap \psi^{-1}(\text{cl}\Gamma^+) \} < \delta,
\]

which proves that \( \text{dist}(x_\nu, \Gamma) < \delta \). \( \square \)

### 3.2 End of proof of Theorem 3 and a remark on pseudo-orbits

Let \( A = [a_k, b_k] \) and \( \hat{A} = [\hat{a}_k, \hat{b}_k] \). Fix \( \delta > 0 \). Let \( V_1 = \mathbb{T} \times [a_1, a_1 + \delta] \subset A_1 \). We proceed by induction. Given \( k \in \{1, \ldots, k_*\} \), we set the following condition:

- \( (H_k) \): the full orbit \( O := O(V_1) \) of \( V_1 \) under the polysystem \( G \) contains \( \Gamma(b_k) \subset A_k \),

(where \( G \) was defined in (31).)

By Proposition 1 the connected component of \( O \) containing \( \Gamma(a_1) \) also contains \( \Gamma(b_1) \), so that \( (H_1) \) is satisfied.

Assume that \( (H_k) \) is satisfied for some \( k \in \{1, \ldots, k_* - 1\} \). Then by our definition of a \( \delta \)-good chain, since \( O \) is a neighborhood of \( \Gamma(b_k) \), \( \psi^{k+1}(O) \) intersects some dynamically minimal circle \( \Gamma_{k+1} \in \text{Ess}(\varphi_{k+1}) \) located in the \( \delta \)-neighborhood of \( \Gamma(a_{k+1}) \). As a consequence, the invariance of \( O \) under \( \hat{\varphi}_{k+1} \) proves that \( O \) contains a neighborhood \( V_{k+1} \) of \( \Gamma_{k+1} \), and Proposition 1 applied to \( (\hat{\varphi}_{k+1}, \hat{\varphi}_{k+1}^{-1} ; \psi_{k+1}) \) proves that \( O \) contains a neighborhood of \( \Gamma(b_{k+1}) \), so that \( (H_{k+1}) \) is satisfied.

By finite induction, \( (H_k) \) is satisfied for \( k \in \{1, \ldots, k_*\} \).

The previous statement holds without any boundedness assumption on the homoclinic or heteroclinic maps. Assuming now that the full assumptions for good chains are satisfied, then Corollary 1 (and the remark before) proves the existence of an orbit of the initial polysystem \( F \) defined in (30) which intersects the \( \delta \)-neighborhood of each element of \( \text{Ess}(\varphi_k) \), \( 1 \leq k \leq k_* \), which concludes the proof of Theorem 3. \( \square \)
We can finally take advantage of the Poincaré recurrence theorem and prescribe the form of the pseudo-orbits. This is the content of the next remark, whose proof follows the same lines as that of Lemma 14 and is omitted.

**Remark 3.** With the same assumptions as in Theorem $\text{[B]}$, one can choose the orbit $(x_n)_{1 \leq n \leq n^*}$ so that it is $2\delta$-admissible and is the concatenation of segments of the form

$$\psi \circ \varphi^m(x)$$

with $m > 0$, where

$$\varphi \in \{\varphi_k \mid 1 \leq k \leq k_*\} \quad \text{and} \quad \psi \in \left\{\psi_{k,i} \mid 1 \leq k \leq k_*, \ i \in I_k\right\} \cup \left\{\psi_{k+1}^i \mid 1 \leq k \leq k_* - 1\right\}.$$

In other words, the occurrences of homoclinic, heteroclinic or transition maps are isolated.

### 3.3 An addendum for singular cylinders

We now state an additional result which will enable us in $\text{[Mar2]}$ to deal with “singular cylinders” exactly in the same way as if they were good cylinders. More precisely, singular cylinders are 3-dimensional manifolds diffeomorphic to the product of $T^1$ with a 2-sphere with three open disks with disjoint closures cut off$^{11}$. Our way to deal with singular cylinders is to “fill-in” one of the cut-off disks, and extend the dynamics to a tame 3-dimensional cylinder. Since the extended dynamics on the filled disk does not correspond to “true” orbits of the system, one is interested in obtaining pseudo-orbits that avoid that disk. See $\text{[Mar2]}$ for details.

**Corollary 2.** We keep the assumptions of Proposition $\text{[4]}$ and we moreover assume that $\mathcal{C}$ is a $P$-cylinder. Assume that $K$ is a compact subset of $\mathcal{A}$, invariant under $\varphi$ and contained in the interior of a Birkhoff zone $Z$ of $\varphi$. Let $\rho = \text{dist}(K, \partial Z)$. Assume that for each $i \in I$,

$$\text{diam Dom } \psi_i < \rho/2.$$  \hspace{1cm} (36)

and that $\psi$ is $\delta$-bounded. Then $f = (\varphi, \psi)$ admits a $\delta$-admissible pseudo-orbit which does not intersect $K$.

**Proof.** First modify $\psi$ by suppressing all maps whose domain intersect $K$, and denote by $\tilde{\psi}$ the resulting polysystem. Then, since the interior of $Z$ contains no essential invariant circle, by $\text{[35]}$, $\mathcal{C}$ is still a good cylinder when endowed with the new polysystem $(\varphi, \psi)$. Therefore there exists a $\delta$-admissible pseudo-orbit $(x_n)_{0 \leq n \leq n^*}$ for $(\varphi, \psi)$. Assume that $x_0$ is located below $Z$ and that $x_{n^*}$ is located above $Z$. Then an immediate induction proves that the orbit $(x_n)$ does not intersect $K$. \hfill $\square$

Assumption (36) is not a restriction: one easily sees that starting with a good $P$-cylinder, one can always “reduce the domains” of the homoclinic correspondence without altering the properties relative to the splitting arcs.

$^{11}$such singular cylinders appear near double-resonances in near-integrable systems
4 Shadowing of pseudo-orbits and proof of Theorem [A]

The aim of this section is to prove that given an arbitrary pseudo-orbit along a good chain (as introduced in Definition [15]), there exists an orbit of the Hamiltonian system which passes arbitrarily close to any of its points. Theorem [A] is then an immediate consequence of the existence of pseudo-orbits intersecting arbitrarily small neighborhoods of any essential invariant circle in the twist sections of a good chain, as proved in Theorem [B]. The main result of this section is the following.

**Theorem C.** Let $H$ be a Hamiltonian of class $C^2$ on $\mathbb{R}^3$ and fix a completely regular value $e$ of $H$. Let $(\mathcal{C}_k)_{1 \leq k \leq k_*}$ be a good chain of cylinders at energy $e$. Then given a pseudo-orbit $(x_n)_{0 \leq n \leq n_*}$ along this chain and any $\delta > 0$, there exists a solution $\gamma$ of the Hamiltonian vector field $X_H$ and a finite sequence of times $(\tau_n)_{0 \leq n \leq n_*}$, with $\tau_0 = 0$, such that $\text{dist}(\gamma(\tau_n), x_n) < \delta$ for $0 \leq n \leq n_*$.\[\text{Proof.}\] The proof generalizes those of [BT99] [DLS00]. We keep the notation of Section [3] for the polysystem $\{\hat{\mathcal{C}}_k, \hat{\psi}_k, \hat{\psi}_k^{k+1}\}$, with $\hat{\psi}_k = (\hat{\psi}_k, i_{I_k})$, which we consider to be defined on the union \[\bigcup_{1 \leq k \leq k_*} \hat{\mathcal{C}}_k.\]

1. Let $(x_n)_{0 \leq n \leq n_*}$ be a pseudo-orbit along the chain. Our proof relies on the existence of a sequence $(y_j)_{0 \leq j \leq j_*}$ of points of $\hat{\mathcal{C}} = \bigcup_{1 \leq k \leq k_*} \hat{\mathcal{C}}_k$, which satisfies the following two properties.

- $(P_1)$ For each $n \in \{0, \ldots, n_*\}$, there exists $j_n \in \{0, \ldots, j_*\}$ such that $x_n = y_{j_n}$.
- $(P_2)$ Fix $j \in \{0, \ldots, j_* - 1\}$ and let $k$ be such that $y_j \in \hat{\mathcal{C}}_k$. Then one of the following three conditions is satisfied:
  
  (i) there exists $\tau_j \geq 0$ such that $y_{j+1} = \Phi_H^{\tau_j}(y_j)$,
  
  (ii) $y_{j+1} = S_{k,i}(y_j)$,
  
  (iii) $y_{j+1} = S_k^{k+1}(y_j)$.

The existence of the sequence $(y_j)$ is proved by induction, starting with $y_0 = x_0$. Assume that $y_{j_n} = x_n$. By definition

\[x_{n+1} = \varphi_k(x_n) \quad \text{or} \quad x_{n+1} = \psi_{k,i}(x_n) \quad \text{or} \quad x_{n+1} = \psi_k^{k+1}(x_n).\]

In the first case, set $y_{j_{n+1}} = x_{n+1}$. In the second case, by definition of $\psi_{k,i}$, there exists $\tau \geq 0$ such that $S_{k,i}(\Phi_H(x_n)) = x_{n+1}$. Set $y_{j_{n+1}} = \Phi_H(x_n)$ and $y_{j_{n+2}} = x_{n+1}$. The last case is similar to the latter one when $\psi_k^{k+1}$ is a heteroclinic map: there exists $\tau \in \mathbb{R}$ such that $S_k^{k+1}(\Phi_H(x_n)) = x_{n+1}$, so set $y_{j_{n+1}} = \Phi_H^{\tau}(x_n)$ and $y_{j_{n+2}} = x_{n+1}$. When $\psi_k^{k+1}$ is a transition map (that is, $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ are adjacent cylinders, see Definition [13]), by definition there exists $\tau \geq 0$ such that $x_{k+1} = \Phi_H^{\tau}(x_n)$, so we set $y_{j_{n+1}} = x_{n+1}$.

The induction stops after a finite number of steps and yields a sequence $(y_j)_{0 \leq j \leq j_*}$ with $j_* \leq 2n$, which satisfies $(P_1)$ and $(P_2)$.
2. This provides us with a sequence \( g := (g_0, \ldots, g_{j_* - 1}) \) of elements from the set
\[
\{ \Phi_H^\tau \mid \tau \in [0, +\infty[ \cap \{ S_k \mid 1 \leq k \leq k_* \} \cap \{ S_{k+1}^* \mid 1 \leq k \leq k_* - 1 \},
\]
such that
\[
y_{j+1} = g_j(y_j), \quad 0 \leq j \leq j_* - 1. \tag{37}\n\]
We then say for short that the \( g \)-orbit of a point \( z \) of \( \hat{C} \), close enough to \( y_0 \), is the ordered sequence of points
\[
(z, g_0(z), g_1 \circ g_0(z), \ldots, g_{j_* - 1} \circ \cdots \circ g_0(z)), \tag{38}\n\]
and we similarly define the \( g \)-orbit of a subset of \( \hat{C} \), close enough to \( y_0 \). These \( g \) orbits are well-defined, thanks to the continuity of the various maps and the openness of their domains.

3. For each \( k \), we fix a negatively invariant neighborhood \( \mathcal{M}_k \subset W^-(\hat{C}_k) \) of \( \hat{C}_k \) in \( W^-(\hat{C}_k) \) such that, given \( z \in \hat{C}_k \) and \( w \in \mathcal{M}_k \cap W^-(z) \), then
\[
d(\Phi_H^\tau(z), \Phi_H^\tau(w)) \leq d(z, w), \quad \forall \tau \geq 0. \tag{39}\n\]
(see Definition \[2, \[40\]). In the case of adjacent cylinders \( \hat{C}_k \) and \( \hat{C}_{k+1} \), one can moreover assume
\[
\mathcal{M}_{k+1} \cap W^-(\hat{C}_k) \subset \mathcal{M}_k. \tag{40}\n\]
To see this, note that the union \( \hat{C}_k \cup \hat{C}_{k+1} \) is itself a tame cylinder \( C \), and fix a negatively invariant neighborhood \( N \subset W^-(C) \) which satisfies (39). Then set \( \mathcal{M}_k = N \cap W^-(\hat{C}_k) \) and \( \mathcal{M}_{k+1} = N \cap W^-(\hat{C}_{k+1}) \).

4. We say that a point \( z \in \hat{C}_k \) is recurrent when it is positively and negatively recurrent under the restriction of \( \Phi_H \) to \( \hat{C}_k \). Recall that \( \hat{C}_k \) admits a Borel measure which is positive on the open sets and invariant under \( \Phi_H \). The set of recurrent points has full measure in \( \hat{C}_k \), by the Poincaré recurrence theorem.

5. We write \( B(u, \rho) \) for the ball in \( H^{-1}(e) \) centered at \( u \in H^{-1}(e) \) with radius \( \rho \). Fix \( \delta > 0 \). Let \( \delta_0 > 0 \) be so small that the \( g \)-orbit \( (D_j)_{0 \leq j \leq j_*} \) of \( D_0 := B(y_0, \delta_0) \cap \hat{C} \) is well-defined (and so \( D_{j+1} = g_j(D_j) \) for \( 0 \leq j \leq j_* - 1 \)) and satisfies
\[
D_j \subset B(y_j, \delta/2), \quad 0 \leq j \leq j_. \tag{41}\n\]
We will prove the existence of a sequence \( (z_j)_{1 \leq j \leq j_*} \), where the point \( z_j \) is in \( D_j \) and is recurrent, together with a sequence of balls \( (B_j)_{0 \leq j \leq j_*} \), such that, for \( 0 \leq j \leq j_* \):
- \( B_j \) is centered at a point in \( W^-(z_j) \cap \mathcal{M} \) and \( B_j \subset B(z_j, \delta/2) \), \hspace{1cm} (C_j)
and such that there moreover exists a time \( \sigma_j \geq 0 \) with:
- \( \Phi_H^{\sigma_j}(B_j) \subset B_{j+1}, \quad 0 \leq j \leq j_* - 1. \) \hspace{1cm} (T_j)

We will construct these sequences backwards, by finite induction. More precisely, given some recurrent point \( z_{j+1} \in D_{j+1} \) together with a ball \( B_{j+1} \) satisfying \( (C_{j+1}) \), we will find a recurrent point \( z_j \in D_j \), a ball \( B_j \) satisfying \( (C_j) \) and a time \( \sigma_j \geq 0 \) which satisfies \( (T_j) \).
6. Assume first that \( g_j = \Phi_H^j \), in which case the sets \( D_j \) and \( D_{j+1} \) are either contained in the same cylinder, say \( \mathcal{C}_k \), or consecutive adjacent cylinders \( \mathcal{C}_k, \mathcal{C}_{k+1} \).

In the first case, let \( z_j := \Phi_H^{-\tau_j}(z_{j+1}) \), so \( z_j \in D_j \) and \( z_j \) is a recurrent point of \( \widehat{\mathcal{C}}_k \). Let \( w_{j+1} \) be the center of \( B_{j+1} \), and set \( w_j = \Phi_H^{-\tau_j}(w_{j+1}) \). Then \( w_j \in W^-(z_j) \cap \mathcal{M}_k \) by equivariance of the unstable foliation, and satisfies \( d(w_j, z_j) \leq d(w_{j+1}, z_{j+1}) < \delta/2 \), so that there exists a ball \( B_j \) centered at \( w_j \) and contained in \( B(z_j, \delta/2) \), which satisfies

\[
\Phi_H^\tau(B_j) \subset B_{j+1}
\]

Hence Conditions \((C_j)\) and \((T_j)\) with \( \sigma_j = \tau_j \) are satisfied.

In the case of adjacent cylinders, \( z_j := \Phi_H^{-\tau_j}(z_{j+1}) \) and \( z_{j+1} \) both belong to the intersection \( \widehat{\mathcal{C}}_k \cap \widehat{\mathcal{C}}_{k+1} \) and \( z_j \) is again recurrent in \( \widehat{\mathcal{C}}_k \) by invariance of \( \widehat{\mathcal{C}}_k \) and the definition of adjacent cylinders. Let again \( w_{j+1} \) be the center of \( B_{j+1} \), so that \( w_{j+1} \in \mathcal{M}_{k+1} \cap W^-(\widehat{\mathcal{C}}_k) \). By \((40)\), this proves that \( w_j = \Phi_H^{-\tau_j}(w_{j+1}) \) is in \( \mathcal{M}_k \), since this latter neighborhood is negatively invariant. As above, \( d(w_j, z_j) \leq d(w_{j+1}, z_{j+1}) < \delta/2 \) and the conclusion follows.

7. Assume now that \( g_j = S_{\ell,i} \), so that \( z_{j+1} \in \widehat{\mathcal{C}}_\ell \). Let \( R_j \) and \( R_{j+1} \) be the full-measure subsets of \( D_j \) and \( D_{j+1} \) formed by the recurrent points. Since \( S_{\ell,i} \) is \( C^1 \) and hence preserves the full-measure property:

\[
R_{j+1} \cap S_{\ell,i} \left( R_j \cap \text{Dom} S_{\ell,i} \right)
\]

is a full measure subset of \( D_{j+1} \). Therefore, there exists a point

\[
\bar{z}_j \in R_j \cap \text{Dom} S_{\ell,i}
\]

such that \( \bar{z}_{j+1} := S_{\ell,i}(\bar{z}_j) \) is recurrent and so close to \( z_{j+1} \) that (by continuity of the center-unstable foliation) \( W^-(\bar{z}_{j+1}) \) intersects the ball \( B_{j+1} \).

By definition of \( \text{Dom} S_{\ell,i} \), the submanifold \( W^-(\bar{z}_j) \) intersects \( W^+(\text{int} \widehat{\mathcal{C}}_{\ell}) \) transversely at some point \( \xi \in W^+(\bar{z}_{j+1}) \). Apply the \( \lambda \)-property (Definition \(2\) to \( W^-(\bar{z}_j) \) in the neighborhood of \( \xi \), together with the positive recurrence property of \( \bar{z}_{j+1} \); there exists a (large) time \( \tau \) such that \( \Phi_H^\tau(W^-(\bar{z}_j)) = W^-(\Phi_H^\tau(\bar{z}_j)) \) intersects \( B_{j+1} \). Fix

\[
\zeta \in W^-(\Phi_H^\tau(\bar{z}_j)) \cap B_{j+1}
\]

and note that \( \Phi_H^\tau(\bar{z}_j) \) is negatively recurrent as \( \bar{z}_j \) is. Hence there exist arbitrarily large times \( \tau' \) such that \( z_j := \Phi_H^{-\tau' + \tau}(\bar{z}_j) \in D_j \). In particular, one can choose \( \tau' \) such that this latter condition is satisfied and

\[
\Phi_H^{\tau'}(\zeta) \in W^-(z_j) \cap B(z_j, \delta/2) \cap \mathcal{N}.
\]

Therefore, there exists a ball \( B_j \) centered at \( \Phi_H^{\tau'}(\zeta) \) and contained in \( B(z_j, \delta/2) \) such that \( \Phi_H^\tau(B_j) \subset B_{j+1} \). This proves \((C_j)\) and \((T_j)\) with \( \sigma_j = \tau' \).

8. The case where \( g_j = S_{\ell-1}^\ell \) is completely similar to the previous one.
9. This yields our sequences \((z_j)_{0 \leq j \leq j_\ast}\) and \((B_j)_{0 \leq j \leq j_\ast}\). To end the proof, observe that for any point \(a_0 \in B_0\), the point \(a_j = \Phi_H^{\sigma_j-1 + \cdots + \sigma_0}(a_0)\) is in \(B_j\), \(1 \leq j \leq j_\ast\). Therefore
\[
\text{dist}(a_j, y_j) \leq \text{dist}(a_j, z_j) + \text{dist}(z_j, y_j) \leq \delta, \quad 0 \leq j \leq j_\ast.
\]
In particular, by \((P_1)\)
\[
\text{dist}(a_{jn}, x_n) \leq \delta, \quad 0 \leq n \leq n_\ast.
\]
Therefore the solution \(\gamma = \Phi_H(\cdot, a_0)\) satisfies our requirement, with \(\tau_n = \sigma_{jn-1} + \cdots + \sigma_0\) for \(1 \leq n \leq n_\ast\).

5 A more constructive point of view

In this section we provide an alternative proof, of algorithmic nature, of Proposition \(\mathbb{I}\) under slightly different conditions. First, we introduce some compatibility conditions between the homoclinic correspondence and the twist map, and we state a version of Proposition \(\mathbb{I}\) that assumes these conditions; it also provides pseudo-orbits of the form introduced in Remark \(\mathbb{I}\). At the end of the section we discuss some other ways to prove this result which do not require the additional compatibility conditions.

Definition 16. Let \(H\) be a \(C^2\) Hamiltonian on \(\mathbb{R}^3\) with completely regular value \(e\). Let \(\mathcal{C}\) be a good \(P\)-cylinder at energy \(e\). Let \(\varphi: \hat{A} \supset \mathcal{C}\) be its return map. We say that \(\mathcal{C}\) is very good if the following conditions hold:

\[
(i) \text{ for every element } \Gamma \in \text{Ess}(\varphi) \text{ contained in } \mathbb{T} \times [\hat{a}, \hat{b}] \text{ which is not the upper boundary of a Birkhoff zone, there exists a splitting arc based on } \Gamma;
\]

\[
(ii) \text{ when } \varphi \text{ tilts the verticals to the right (resp. left), for each element } \Gamma \in \text{Ess}(\varphi) \text{ which is the upper boundary of a Birkhoff zone in } \mathbb{T} \times [\hat{a}, \hat{b}], \text{ there exists a right (resp. left) splitting arc based on } \Gamma.
\]

The compatibility between the orientation of the splitting arcs and the direction of the twist of the map \(\varphi\) will enable us to avoid any use of the Poincaré recurrence theorem, opening up the possibility for applications to explicitly estimate drifting times (possibly using computer-assisted methods.)

Proposition 2. Let \(\mathcal{C}\) be a very good \(P\)-cylinder. Assume the setting of Proposition \(\mathbb{I}\). Let \(\Gamma_\ast \subseteq A \setminus \Gamma(b)\) in \(\text{Ess}(\varphi)\). Fix a neighborhood \(V\) of \(\Gamma_\ast\) in \(A\). Then, there exists a point \(z_0 \in V\), an integer \(N \geq 1\), and a pair \(i_n \in I\) and \(m_n > 0\) for each \(0 \leq n \leq N - 1\), such that the orbit given by
\[
z_{n+1} = \psi_{i_n} \circ \varphi^{m_n}(z_n), \quad n = 0, \ldots, N - 1,
\]
has the following properties: (i) \(z_N \in \Gamma(b)^+\), and, (ii) for every essential invariant circle \(\Gamma\) between \(\Gamma_\ast\) and \(\Gamma(b)\), there exists a point \(z_n\) that is \(\delta\)-close to \(\Gamma\).
Proof of Proposition 1. In the sequel we will assume that \( \hat{\varphi} \) tilts the verticals to the right (the case when \( \hat{\varphi} \) tilts the verticals to the left can be dealt with similarly.) For the sake of simplicity, we assume that \( \Gamma_\bullet = \Gamma(a) \) and we denote \( \mathbf{A}_s = \mathbb{T} \times [a, b] \). We still write \( \hat{\varphi}, \psi \) for the restrictions of the corresponding maps to \( \mathbf{A}_s \). The proof immediately extends to the case where \( \Gamma_\bullet \) is any element of \( \text{Ess}(\hat{\varphi}) \). Following Remark 1, we will also assume that \( \text{diam \ Dom } \psi_i < \delta \) for \( i \in I \).

As in the proof of Proposition 1 in the case of \( \text{P-cylinders} \), we want to prove that given any connected open neighborhood \( V \) of \( \Gamma(a) \) in \( \mathbf{A}_s \), the orbit \( U \) of \( V \) under the restricted polysystem \( (\hat{\varphi}, \psi)|_{\mathbf{A}_s} \) satisfies

\[
\Gamma(b) \subset \text{cc}(U, \Gamma(a)).
\]

1. The Birkhoff procedure. Let us first give a formal definition, inspired by the first step of the proof of Proposition 1. Consider any connected positively \( \hat{\varphi} \)-invariant open subset \( U \subset \mathbf{A}_s \) containing \( \Gamma(a) \) and such that \( \overline{U} \cap \Gamma(b) = \emptyset \). Set \( O = \mathbf{A}_s \setminus U \). We define the filled subset \( \mathcal{U} \) associated with \( U \) as the open set

\[
\mathcal{U} = \text{cc}(\mathbf{A}_s \setminus \text{cc}(O, \Gamma(b)), \Gamma(a)).
\]

One proves as in Proposition 1 that \( \text{Int \ cl} \left( \mathcal{U} \right) = \mathcal{U} \), and that \( \mathcal{U} \) is homeomorphic to \( \mathbb{T} \times [0, 1[ \) and satisfies \( \hat{\varphi} \left( \mathcal{U} \right) = \mathcal{U} \). As a consequence, by the Birkhoff theorem, the frontier \( \text{Fr} \mathcal{U} \) is in \( \text{Ess}(\hat{\varphi}) \), and moreover one easily deduces from the proof of Proposition 1 that \( \text{Fr} \mathcal{U} \subset \text{Fr} U \).

We can now introduce our procedure. Let \( \nu \) be a uniform Lipschitz constant for the circles of \( \text{Ess}(\hat{\varphi}) \). We define \( \nu \)-balls of \( \hat{\mathbf{A}} \) as the intersections with \( \hat{\mathbf{A}} \) of open rectangles satisfying (32). For two elements \( \Gamma_1, \Gamma_2 \in \text{Ess}(\hat{\varphi}) \), we write \( \Gamma_1 \sqsupset \Gamma_2 \) (resp. \( \Gamma_1 \sqsupseteq \Gamma_2 \)) if \( \gamma_1(\theta) > \gamma_2(\theta) \) (resp. \( \gamma_1(\theta) \geq \gamma_2(\theta) \)) for all \( \theta \in \mathbb{T} \), where \( \gamma_i \) is the Lipschitz function whose graph is \( \Gamma_i \), for \( i = 1, 2 \).

Definition 17. Denote by \( \mathcal{P} \) the set of pairs \((\Gamma, V), \) where \( \Gamma \in \text{Ess}(\hat{\varphi}) \) and \( \Gamma \subset \mathbb{T} \times [a, \hat{\mathbf{b}}[ \), where

- if \( \Gamma = \Gamma(a) \), then \( V \) is a connected open neighborhood of \( \Gamma(a) \) in \( \mathbf{A}_s \),
- if \( \Gamma > \Gamma(a) \), then \( V \) is a \( \nu \)-ball centered on \( \Gamma \).

We define the Birkhoff procedure as the map

\[
\mathbb{B} : \mathcal{P} \longrightarrow \text{Ess}(\hat{\varphi}), \quad \mathbb{B}(\Gamma, V) = \text{Fr} \mathcal{U},
\]

where \( \mathcal{U} \) is the filled subset associated with the connected \( \hat{\varphi} \)-invariant open set

\[
U = \bigcup_{n \geq 0} \hat{\varphi}^n(\Gamma^- \cup V) = \Gamma^- \cup \bigcup_{n \geq 0} \hat{\varphi}^n(V).
\]

Recall that \( \Gamma^- \) stands for the connected component of \( \mathbf{A}_s \setminus \Gamma \) located below \( \Gamma \). Observe that since \( \hat{\varphi} \) is a special twist map, \( \mathbb{B}(\Gamma, V) > \Gamma \). Note that the region between \( \Gamma \) and \( \mathbb{B}(\Gamma, V) \) is not necessarily a Birkhoff zone. One main property of \( \mathbb{B} \) that we will use is the following easy transition lemma.
Lemma 5. Fix \((\Gamma, V) \in \mathcal{P}\) and let \(\Gamma' = \mathbb{B}(\Gamma, V)\). Then, for any open set \(V'\) intersecting \(\Gamma'\), there exists \(n \geq 0\) such that \(\hat{\varphi}^n(V) \cap V' \cap (\Gamma')^- \neq \emptyset\).

Proof. By definition, \(\mathbb{B}(\Gamma, V) = \operatorname{Fr} U\) so \(\mathbb{B}(\Gamma, V) \subset \operatorname{Fr} U \subset \operatorname{cl} U\). Hence \(V' \cap U \neq \emptyset\), so that there exists \(n \geq 0\) with \(\hat{\varphi}^n(V) \cap V' \neq \emptyset\). Moreover \(V \subset (\Gamma')^-\) by construction, so \(\hat{\varphi}^n(V) \subset (\Gamma')^-\).

The following result is the crucial step in our subsequent construction of diffusing orbits. Recall that we write \(\psi = (\psi_i)_{i \in I}\) for the homoclinic correspondence.

Lemma 6. Consider a pair \((\Gamma, V) \in \mathcal{P}\) and let \(\Gamma' = \mathbb{B}(\Gamma, V)\). Then there exist \(n \geq 0, i \in I\) and a \(\nu\)-ball \(V'\) centered on \(\Gamma'\) such that

\[
V' \subset \psi_i(\hat{\varphi}^n(V)).
\]

Proof. By the assumption that \(\hat{\varphi}\) is a special twist map, either \(\Gamma'\) is the upper boundary of a Birkhoff zone, or is accumulated from below by a sequence of essential circles.

First, assume that \(\Gamma'\) is the upper boundary of a Birkhoff zone. By the very good cylinder assumption, there exists a right splitting arc \(\zeta\) based at a point \(\alpha \in \Gamma'\) such that \(\tilde{\zeta} = \zeta([[0, 1]]) \subset \Gamma^+ \cap (\Gamma')_\alpha^-,\) and \(i \in I\) such that \(\tilde{\zeta}\{\alpha\} \subset \operatorname{Dom} \psi_i\). Introduce the closed connected set \(X = \Gamma' \cup \tilde{\zeta},\) which disconnects the annulus \(A_s\). Let \(D\) be the domain associated with \(\tilde{\zeta}\).

By Lemma 5, there exist \(z_0 \in V\) and \(n \geq 0\) such that \(z_n = \hat{\varphi}^n(z_0) \in D\). Since \(V\) is a \(\nu\)-ball, by Lemma 9 one can moreover find a negatively tilted arc \(\gamma\) contained in \(V\), emanating from some point of \(\Gamma\) and ending at \(z_0\). Therefore, by Lemma 11, \(\gamma_n = \hat{\varphi}^n \circ \gamma\) is a negatively tilted arc emanating from \(\Gamma\) and ending at \(z_n\). Now, by Lemma 14 there exists a positively tilted arc emanating from \(\Gamma\) and ending at \(z_n\), whose image does not intersect \(X\).

Assume that \(\tilde{\gamma}_n\) does not intersect \(X\). Then by Lemma 12 the vertical \(V^{-}(z_n)\) does not intersect \(X\), which contradicts the definition of a domain. Therefore \(\tilde{\gamma}_n \cap X \neq \emptyset\), thus \(\tilde{\gamma}_n \cap (\tilde{\zeta}\\{\alpha\}) \neq \emptyset\).

As a consequence, \(\hat{\varphi}^n(V) \cap (\tilde{\zeta}\\{\alpha\}) \neq \emptyset\), and therefore there exists a ball \(B\) centered on \(\tilde{\zeta}\) satisfying

\[
B \subset (\operatorname{Dom} \psi_i \cap \hat{\varphi}^n(V)).
\]

So, by definition of \(\tilde{\zeta}\), \(\psi_i(B) \subset \psi_i(\hat{\varphi}^n(V))\) contains a \(\nu\)-ball \(V'\) centered on \(\Gamma'\), which proves our claim.

Second, assume that \(\Gamma'\) is accumulated from below by a sequence of essential circles \(\Gamma_n,\) with \(\Gamma_n < \Gamma_{n+1}\) for all \(n\). Let \(\zeta\) be a splitting arc based at a point in \(\Gamma'\). There exists \(\Gamma_{n_0} < \Gamma\) such that \(\tilde{\zeta} \supseteq \Gamma^+ \cap (\Gamma')^-\) intersects both \(\Gamma_{n_k}\) and \(\Gamma_{n_{k+1}}\). There exists a segment \(\zeta[\sigma, \tau]\) of \(\tilde{\zeta}\) such that \(\zeta[\sigma] \in \Gamma_{n_{k+1}},\ z(\tau) \in \Gamma_{n_k},\) and \(\zeta|\sigma, \tau|\) is contained in the region between \(\Gamma_{n_k}\) and \(\Gamma_{n_{k+1}}\). Lemma 5 applied to \(\Gamma_{n_{k+1}}\), which is a neighborhood of \(\Gamma'\), implies that there exists \(m \geq 0\) such that \(\hat{\varphi}^m(V)\) intersects both \(\Gamma_{n_k}\) and \(\Gamma_{n_{k+1}}\); moreover, there exists an arc \(\tilde{\xi} = [\xi, \beta] \subseteq \hat{\varphi}^m(V)\) that is contained in the region between \(\Gamma_{n_k}\) and \(\Gamma_{n_{k+1}}\), with \(\xi(\alpha) \in \Gamma_{n_{k+1}}\) and \(\xi(\beta) \in \Gamma_{n_k}\). By Lemma 14 for some \(m' \geq m\) sufficiently large we have \(\hat{\varphi}^{m'-m}(\tilde{\xi}) \cap \zeta|\sigma, \tau| \neq \emptyset\). As \(\hat{\varphi}^{m'-m}(\tilde{\xi}) \subseteq \hat{\varphi}^{m'}(V)\), it follows that
$\varphi^m(V) \cap \zeta] \sigma, \tau[ \neq \emptyset$. Since $\zeta$ is a splitting arc based at a point in $\Gamma'$, there exists a homoclinic correspondence $\psi_i$ such that $\psi_i(\varphi^m(V))$ contains a $\nu$-ball centered on $\Gamma'$. This ends the proof of the lemma.

\[ \square \]

2. Coherent sequences of circles and the existence of pseudo-orbits. We now prove (43). We endow $\text{Ess}(\hat{\varphi})$ with the natural order on the graphs. We say that an increasing sequence $(\Gamma_n)_{0 \leq n \leq n_\ast}$ of elements of $\text{Ess}(\hat{\varphi})$, with $\Gamma_0 = \Gamma(a)$, is coherent if there exists a sequence $(V_n)_{0 \leq n \leq n_\ast}$, where $V_0$ is a neighborhood of $\Gamma_0$ in $A_\ast$ and, for $n \geq 1$, $V_n$ is $\nu$-ball centered on $\Gamma_n$ satisfying for every $n = 0, \ldots, n_\ast - 1$:

$$V_{n+1} \subset \psi_i_n(\varphi^m(V_n)), \quad (45)$$

for some $i_n \in I$ and $m_n \geq 0$.

We say that $(V_n)_{0 \leq n \leq n_\ast}$ is the system of neighborhoods associated with $(\Gamma_n)_{0 \leq n \leq n_\ast}$ and that $\Gamma_{n_\ast}$ is the upper circle of the coherent sequence.

**Lemma 7.** Fix a neighborhood $V$ of $\Gamma(a)$ in $A_\ast$. Choose an essential invariant circle $\Gamma(a) < \Gamma_\ast \leq \Gamma(\hat{b})$. Then there exists a coherent sequence $(\Gamma_n)_{0 \leq n \leq n_\ast}$, with associated neighborhoods $(V_n)_{0 \leq n \leq n_\ast}$, such that

$$\Gamma_0 = \Gamma(a), \quad V_0 = V, \quad \Gamma_{n_\ast} = \Gamma_\ast.$$

Moreover, we can choose $(\Gamma_n)_{0 \leq n \leq n_\ast}$ with the additional property that if for $\Gamma_n < \Gamma_{n+1}$ there exists $\Gamma \in \text{Ess}(\hat{\varphi})$ with $\Gamma_n < \Gamma < \Gamma_{n+1}$, then $d_H(\Gamma_n, \Gamma_{n+1}) < \delta$, where $d_H$ denotes the Hausdorff distance. That is, any two consecutive circles in the sequence are either $\delta$-close, or they are at the boundary of a Birkhoff zone.

**Proof.** We recall that, since $\hat{\varphi}$ is special, for any $\Gamma \leq \Gamma_\ast$, either $\Gamma < \Gamma_\ast$ or $\Gamma = \Gamma_\ast$. Let $\text{Coh}$ be the set of all coherent sequences of circles originating at $\Gamma_0$, with $V_0 = V$, and such that $\Gamma_n \leq \Gamma_\ast$ for all $n = 0, \ldots, n_\ast$. This set is nonempty by Lemma 6. Let $\text{Coh}^*$ the set of upper circles of all elements of $\text{Coh}$. Let

$$\Gamma^* = \text{Sup} \, \text{Coh}^*,$$

(see Lemma 6) so $\Gamma^* > \Gamma(a)$. Assume by contradiction that $\Gamma^* < \Gamma_\ast$.

Since $\hat{\varphi}$ is special, one proves as in Lemma 6 that either $\Gamma^* \in \text{Coh}^*$, or $\Gamma^* \notin \text{Coh}^*$ and is accumulated from below in the Hausdorff topology by elements of $\text{Coh}^*$.

In the former case, there exists a coherent sequence $(\Gamma_n)_{1 \leq n \leq n_\ast}$ with $\Gamma_{n_\ast} = \Gamma^*$, and with associated neighborhoods $(V_n)_{0 \leq n \leq n_\ast}$. Let $\Gamma := \mathbb{B}(\Gamma_{n_\ast}, V_{n_\ast})$. By Lemma 6 there exist $m \geq 0$, $i \in I$ and a $\nu$-ball $V$ centered on $\Gamma := \mathbb{B}(\Gamma_{n_\ast}, V_{n_\ast})$ such that

$$V \subset \psi_i(\varphi^m(V_{n_\ast})).$$

We can choose the neighborhood $V_{n_\ast}$ to be small enough so that $V_{n_\ast} \cap \Gamma_\ast = \emptyset$, in which case, since $\Gamma_{n_\ast} = \Gamma^* < \Gamma_\ast$, we also have $\Gamma = \mathbb{B}(\Gamma_{n_\ast}, V_{n_\ast}) \leq \Gamma_\ast$. Therefore one can extend the sequence $(\Gamma_n)_{1 \leq n \leq n_\ast}$ to a longer coherent sequence, by adjoining to the previous sequence $\Gamma_{n_\ast + 1} = \Gamma$ and $V_{n_\ast + 1} = V$. The resulting coherent sequence is in $\text{Coh}$, and has its upper circle located in $(\Gamma^*)^+$, which is a contradiction.
Assume now that $\Gamma^* \not= \text{Coh}^*$ and is accumulated from below by elements of $\text{Coh}^*$. In particular, $\Gamma^*$ is not the upper boundary of a Birkhoff zone. By condition (i) on very good cylinders, there exists a splitting arc $\zeta$ based on $\Gamma^*$, with $\psi_i(\zeta([0,1])) \subset \Gamma^*$ for some $i \in I$. So, by accumulation, there exists a coherent sequence $(\Gamma_n)_{1 \leq n \leq n_a}$ such that

$$\zeta(1) \in (\Gamma_{n_a})^-.$$ 

Consider the circle $\Gamma = \mathbb{B}(\Gamma_{n_a}, V_{n_a})$, which satisfies $\Gamma_{n_a} < \Gamma < \Gamma^*$ (otherwise $\Gamma = \Gamma^*$, which contradicts our initial assumption $\Gamma^* \not= \text{Coh}^*$). The set $K = \zeta^{-1}(\Gamma)$ is compact and contained in $]0,1[$. Set $\delta = \text{Max} \, K$. So $\zeta(\sigma) \in \Gamma$ and $\zeta([\sigma,1]) \subset \Gamma^-$ by continuity. Moreover, there exists $\sigma' > \sigma$ such that $\zeta(\sigma') \in \Gamma_{n_a}$; let $\sigma'$ be the smallest one with such property. Thus $\zeta([\sigma,\sigma'])$ is an arc contained in the closed annulus bounded by $\Gamma_{n_a}$ and $\Gamma$, and has one endpoint on $\Gamma_{n_a}$ and the other one on $\Gamma$.

By Lemma 8 below, there exists a point $z \in V_{n_a}$ with the property that for the vertical segment $V^{-}(z)$ below $z$ there exists $m \geq 0$ such that $\varphi^m(V^{-}(z))$ intersects $\zeta([\sigma,\sigma'])$. Since $V_{n_a}$ is a $\nu$-ball, the vertical segment $V^{-}(z)$ is contained in $V_{n_a}$. It follows that $\varphi^m(V_{n_a}) \cap \zeta([\sigma,\sigma']) \neq \emptyset$.

Therefore, since $\psi_i(\zeta) \subset \Gamma^*$, the image $\psi_i(\varphi^m(V_{n_a}))$ contains a $\nu$-ball $V^*$ centered on $\Gamma^*$. We can choose the neighborhood $V^*$ small enough so that $V^* \cap \Gamma_a = \emptyset$, hence $\mathbb{B}(\Gamma^*, V^*) \subseteq \Gamma_a$. Thus one can construct a coherent sequence in $\text{Coh}$ given by

$$\Gamma_1, \ldots, \Gamma_{n_a}, \Gamma^*, \mathbb{B}(\Gamma^*, V^*)$$

whose upper circle is located in $(\Gamma^*)^+ \cap (\Gamma_a)^-$, which is a contradiction. This proves the first claim of the lemma.

Now we show the second claim of the lemma. We will restrict ourselves to coherent sequences with the property that the neighborhood $V$ of $\Gamma(a)$ is contained in a $\delta$-neighborhood of $\Gamma(a)$, and that each neighborhood $V_n$ associated to the coherent sequence is contained in a $\delta$-neighborhood of the corresponding $\Gamma_n$. Restricting to coherent sequences satisfying this condition is possible due to (15).

We construct recursively a finite sequence of essential circles $\Gamma^j$, $j = 0, 1, \ldots, N$, with the following properties:

(i) $\Gamma(a) = \Gamma^0 < \Gamma^1 < \ldots < \Gamma^{N-1} < \Gamma^N = \Gamma_a$;

(ii) For each $j \in \{0, \ldots, N - 1\}$, either

(ii.a) the Hausdorff distance between $\Gamma^j$ and $\Gamma^{j+1}$ is less than $\delta$, or

(ii.b) the region between $\Gamma^j$ and $\Gamma^{j+1}$ is a Birkhoff zone.

We start by setting $\Gamma^0 = \Gamma(a)$. Let $0 < \delta' < \delta$. If $\Gamma^j$ has been constructed, we define the set

$$\mathcal{F}_{\Gamma^j} = \{ \Gamma \in \text{Ess}(\varphi) \mid \Gamma^j < \Gamma < \Gamma_a, \text{ and } d_H(\Gamma^j, \Gamma) < \delta' \},$$

where $d_H$ stands for the Hausdorff distance.

If $\mathcal{F}_{\Gamma^j} \neq \emptyset$ we define

$$\Gamma^{j+1} = \text{Sup} \mathcal{F}_{\Gamma^j}.$$
By construction, $d_H(\Gamma^j, \Gamma^{j+1}) \leq \delta' < \delta$.

If $\mathcal{R}_j = \emptyset$, it follows from Lemma 1 that $\Gamma^j$ is the lower boundary of some Birkhoff zone. In this case we define $\Gamma^{j+1}$ as the upper boundary of that Birkhoff zone. In this case we must have $d_H(\Gamma^j, \Gamma^{j+1}) \geq \delta'$. Of course, it may happen that $d_H(\Gamma^j, \Gamma^{j+1}) < \delta$.

This recursive construction ends in finitely many steps at $\Gamma_*$. Otherwise, we would obtain an infinite sequence $\Gamma^j$, $j = 0, 1, \ldots$, satisfying (ii), with $\Gamma^j < \Gamma_*$ for all $j$. Take $\Gamma^* = \sup \Gamma^j$. Since the sequence $\Gamma^j$ is infinite, it means that it accumulates on $\Gamma^*$ from below. Take a $\delta'$-neighborhood of $\Gamma^*$ in the Hausdorff topology. There exists $\Gamma^{j*} < \Gamma^*$ that is contained in that neighborhood. It follows that also $\Gamma^{j*+1}$ is contained in the same neighborhood, where $\Gamma^{j*} < \Gamma^{j*+1} < \Gamma^*$. Thus, both $\Gamma^{j*+1}$ and $\Gamma^*$ are $\delta'$-close to $\Gamma^{j*}$. This is in contradiction with the definition of $\Gamma^{j*+1}$ as a supremum (17). Hence the recursive construction ends in finitely many steps, therefore it must end at $\Gamma_*$. At this point we have constructed a finite sequence of essential circles $\Gamma^j$, $j = 0, 1, \ldots, N$ satisfying properties (i) and (ii). We apply successively the first statement of the Lemma 7, constructing a coherent sequence from $\Gamma^0$ to $\Gamma^1$, then continuing with a construction of a coherent sequence from $\Gamma^1$ to $\Gamma^2$, and so on. Concatenating these coherent sequence yields a coherent sequence as in the second claim of Lemma 7.

The lemma is proved.

The following result (which can be viewed as a loose extension of Lemma 10) has been used in the above argument.

**Lemma 8.** Let $f : A \to A$ be an area-preserving special twist map. Let $\Gamma_0 < \Gamma^*$ be two nonintersecting essential invariant circles contained in $A$. Let $C$ be a continuous curve which intersect both circles $\Gamma_0, \Gamma^*$, and let $U_\epsilon$ be a neighborhood of some point in $\Gamma_0$. Then there exists $z \in U_\epsilon$, with the vertical segment $V^-(z)$ below $z$ contained in $U_\epsilon$, such that the positive orbit of $V^-(z)$ under $f$ intersects $C$.

**Proof of Lemma 8** Without loss of generality, we can assume that $U_\epsilon$ is a $\nu$-ball, and $U_\epsilon \cap C = \emptyset$.

In the covering space of $A$ there exists $\theta_1 < \theta_2$ such that the projection of $C$ onto the $\theta$-coordinate is between $\theta_1$ and $\theta_2$. Let $\rho_\epsilon$ the rotation number of $\Gamma_0$; all points in $\Gamma_0$ have the same rotation number equal to $\rho_\epsilon$. Apply the Birkhoff procedure to $\Gamma_0$ and $U_\epsilon$, obtaining $B(\Gamma_0, U_\epsilon) = \Gamma' \leq \Gamma^*$. Since $\varphi$ is a special twist map, $\Gamma' \cap \Gamma_0 = \emptyset$, and the rotation number $\rho'$ of $\Gamma'$ satisfies $\rho' > \rho_\epsilon$. Let $\varepsilon > 0$ small. There exists $N > 0$ such that, for every $\xi \in \Gamma_0$ and $\xi' \in \Gamma'$ we have, relative to the covering space, that

$$
\pi_\theta[\varphi^N(\xi')] - \varepsilon < \pi_\theta[\varphi^N(\xi)] > 1 + \theta_2 - \theta_1.
$$

(48)

Choose a point $x' \in \Gamma'$ and let

$$
V' = \bigcap_{i=0}^N \varphi^{-i}(B_{\rho'}(\varphi^i(x'), \varepsilon/2)),
$$

where $B_{\rho'}(\varphi^i(x'), \varepsilon/2)$ is a $\nu$-ball centered at $\varphi^i(x')$ with the $\theta$-component of the ball of diameter less than $\varepsilon$. For any point $\zeta \in V'$ we have that $\varphi^N(\zeta) \in B_{\rho'}(\varphi^N(x'), \varepsilon/2)$, hence

$$
\pi_\theta[\varphi^N(\zeta)] > \pi_\theta[\varphi^N(x')] - \varepsilon.
$$
By Lemma 3, there exists \( z \in U \cdot \) and \( n \geq 0 \) such that \( \varphi^n(z) \in V' \). Let \( x \) be the intersection point between the vertical line \( V'(z) \) through \( z \) and \( \Gamma \cdot \). The point \( \varphi^n(z) \) lies on the positive tilted curve \( \varphi^n(V'(x)) \) starting at \( \varphi^n(x) \) and ending at \( \varphi^n(z) \), which is the image of the vertical line \( V'(z) \) under \( \varphi^n \). The image of this curve under \( \varphi^N \) has one endpoint at \( \varphi^N(\phi^n(x)) \) and the other endpoint at \( \varphi^N(\phi^n(z)) \). Applying (15) to \( \xi = \phi^n(x) \) and \( \xi' = x' \), we have \( \pi_\theta[\varphi^N(x')] - \varepsilon - \pi_\theta[\varphi^N(x)] > 1 + \theta_2 - \theta_1 \), and since \( \varphi^n(z) \in V' \), we have \( \pi_\theta[\varphi^N(\varphi^n(z))] > \pi_\theta[\varphi^N(x')] - \varepsilon \).

We conclude
\[
\pi_\theta[\varphi^N(\varphi^n(z))] - \pi_\theta[\varphi^N(\varphi^n(x))] > 1 + \theta_2 - \theta_1. 
\]
Thus, the image of \( \varphi^{n+1}(V'(z)) \) of \( V'(z) \) under \( \varphi^{n+1} \) has the \( \theta \)-projection of its endpoints more than \( 1 + \theta_2 - \theta_1 \) apart, thus \( \varphi^{n+1}(V'(z)) \cap C \neq \emptyset \).

We can now end the proof of Proposition 2. By Lemma 7 applied to \( \Gamma \cdot = \Gamma(b) \), there exists a coherent sequence of circles \( (\Gamma_n)_{0 \leq n \leq n_\cdot} \), with associated neighborhoods \( (V_n)_{0 \leq n \leq n_\cdot} \), such that
\[
V_0 = U \cdot, \quad \Gamma_{n_\cdot} = \Gamma(b).
\]
Then, by continuity, there exists a sequence \( (W_n)_{0 \leq n \leq n_\cdot} \) of open sets with \( W_{n_\cdot} \subset V_{n_\cdot} \), such, for \( 0 \leq n \leq n_\cdot - 1 \):
\[
W_n \subset V_n, \quad \psi_i \circ \hat{\varphi}^{m_n}(W_n) = W_{n+1}
\]
for suitable \( i_n \in I \) and \( m_n \geq 0 \). As a consequence, since \( W_{n_\cdot} \cap \Gamma(b) = \emptyset \), there exists a point \( z_0 \in W_0 \) such that \( z_{n+1} = \psi_{i_n}(\hat{\varphi}^{m_n}(z_n)) \in W_{n+1} \), for \( n = 0, \ldots, N - 1 \). In particular \( z_N \in \Gamma(b) \).

At the beginning of the proof we assumed that the maps \( \psi_i \) are chosen so that the domain of each map has diameter less than \( \delta \). The coherent sequence obtained at the end of Lemma 7 has been constructed so that, for any two consecutive circles \( \Gamma_n < \Gamma_{n+1} \), either \( d_H(\Gamma_n, \Gamma_{n+1}) < \delta \), or the region between \( \Gamma_n \) and \( \Gamma_{n+1} \) is a Birkhoff zone. Each essential invariant circle is between a pair \( \Gamma_n \) and \( \Gamma_{n+1} \) as above, or is at the boundary of a Birkhoff zone between \( \Gamma_n \) and \( \Gamma_{n+1} \), for some \( n \). Hence the pseudo-orbit \( (z_n)_{0 \leq n \leq N} \) obtained above gets \( \delta \)-close to every essential invariant circle.

This concludes the proof of Proposition 2 in the case of very good \( P \)-cylinders.

The same type of argument is used for the following corollary dedicated to the “singular cylinders”, for which a constructive proof can also be deduced from the previous one.

**Corollary 3.** We keep the assumptions of Proposition 2. Assume that \( K \) is a compact subset of \( \Lambda \), invariant under \( \varphi \) and contained in the interior of a Birkhoff zone \( Z \) of \( \varphi \). Let \( \rho = \text{dist}(K, \partial Z) \). Assume that for each \( i \in I \),
\[
\text{diam Dom } \psi_i < \text{Min} \{ \rho/2, \delta \}.
\]

Then \( f = (\hat{\varphi}, \psi) \) admits a \( \delta \)-admissible pseudo-orbit which does not intersect \( K \).

**Proof.** For each of the boundary circles of \( \partial Z \), by assumption (49) the domains of the homoclinic maps \( \psi_i \) which intersect these boundaries do not intersect \( K \). Thus, in order to
cross the Birkhoff zone \( Z \) (that is, to obtain pseudo-orbits from some small ball centered on the lower boundary to some small ball centered on the upper boundary), in the argument of Proposition 2 we only use the ‘inner’ map \( \phi \). Since \( K \) is invariant under \( \phi \), it means that the resulting pseudo-orbits do not intersect \( K \).

3. Removing the compatibility condition between splitting arcs and twist. The compatibility condition between the orientation of the splitting arcs and the direction of the twist in Definition 16 seems to be non-generic, even in specific classes of examples as [LM]. To conclude this paper, we indicate below two ways to get rid of this condition, which therefore yields other proofs of Proposition 1 using the Birkhoff procedure.

**Method 1.** One can use the approach of Proposition 2 in conjunction with the Poincaré recurrence theorem, under the weaker assumption that \( \mathcal{C} \) is a good cylinder, rather than a very good one. That is, the results of Proposition 2 and of Corollary 3 remain valid under the same setting as in Proposition 1. To see this, using the first part of the proof of Proposition 1, it is enough to prove that given any connected open neighborhood \( V \) of \( \Gamma(p,a) \) in \( A_* \), the orbit \( U \) of \( V \) under the polysystem \( (\tilde{\varphi}, \tilde{\varphi}^{-1}, \psi)_|A_* \) satisfies

\[
\Gamma(b) \subset \text{cc}(U, \Gamma(a)).
\] (50)

One can then easily modify each step of the previous proof to get the same conclusion. However, we lose the benefit of the previous formulation, since the new proof strongly relies on the Poincaré recurrence theorem.

**Method 2.** We now use the equivariance properties of the homoclinic correspondences (see Appendix B).

Assume for instance that \( \varphi \) tilts the verticals to the right and assume moreover that the homoclinic correspondence \( \psi \) is equivariant. For the key part of the proof of Proposition 2 let \( (\Gamma, V) \in \mathcal{P} \) and \( \Gamma' = B(\Gamma, V) \). Assume that there is a left-splitting arc \( \tilde{\zeta} \) for \( \Gamma' \) based at a point \( \alpha \in \Gamma' \), such that \( \tilde{\zeta}(\alpha) \subset \text{Dom}(\psi_i) \) for some homoclinic map \( \psi_i \), and \( \psi_i(\tilde{\zeta}(\alpha)) \subset \Gamma \). Since the inverse map \( \varphi^{-1} \) tilts the verticals to the left, the argument in the proof of Lemma 6 implies that \( \varphi^{-n}(V) \cap (\tilde{\zeta}(\alpha)) \neq \emptyset \) for some \( n > 0 \). Choose \( m > n \). Then \( \varphi^{m-n}(V) \cap \varphi^m(\tilde{\zeta}(\alpha)) \neq \emptyset \). By the definition of an equivariant homoclinic correspondence, there exists a homoclinic map \( \psi_{i(m)} \) satisfying

\[
\text{Dom} \psi_{i(m)} = \tilde{\varphi}^m(\text{Dom} \psi_i), \quad \text{Im} \psi_{i(m)} = \tilde{\varphi}^m(\text{Im} \psi_i),
\]

and

\[
\psi_{i(m)} \circ \tilde{\varphi}^m = \tilde{\varphi}^m \circ \psi_i.
\]

Therefore

\[
\psi_{i(m)}(\varphi^m(\tilde{\zeta}(\alpha))) = \varphi^m(\psi_i(\tilde{\zeta}(\alpha))) \subset \varphi^m(\Gamma) \subset \Gamma,
\]

since \( \Gamma \) is \( \varphi \)-invariant. The rest of the proof goes as before.
A A reminder on twist maps

We refer to [LC], [H] and [HK] for surveys on twist maps. Let \( a < b \) be fixed. We set
\[
\mathbf{A} = \mathbb{T} \times [a, b], \quad \Gamma(a) = \mathbb{T} \times \{a\}, \quad \Gamma(b) = \mathbb{T} \times \{b\}.
\]
The closure of a subset \( E \subset \mathbf{A} \) will be indifferently denoted by \( \text{cl} \ E \) or \( \overline{E} \), and its interior will be denoted by \( \text{Int} \ E \). The set \( \text{Fr} \ E = \text{cl} \ E \setminus \text{Int} \ E \) is the frontier of \( E \). A disk is an open connected and simply connected subset of \( \mathbf{A} \).

Here we say that \( f : \mathbf{A} \to \mathbf{A} \) is a twist map when it is a \( C^1 \) diffeomorphism, preserves \( \Gamma(a) \) and \( \Gamma(b) \) and tilts the vertical, that is, \( f(\theta, r) = (\Theta, R) \) with
\[
\partial_r \Theta(\theta, r) > 0 \quad \text{or} \quad \partial_r \Theta(\theta, r) < 0, \quad \forall (\theta, r) \in \mathbf{A}.
\]
Then \( f \) tilts the vertical to the right in the former case and to the left in the latter one. A continuous map \( f : \mathbf{A} \to \mathbf{A} \) is said to be area-preserving when it leaves invariant a Radon measure which is positive on the open subsets of \( \mathbf{A} \). An essential circle in \( \mathbf{A} \) is a \( C^0 \) curve which is homotopic to \( \Gamma(a) \).

**Theorem (Birkhoff).** Let \( f : \mathbf{A} \to \mathbf{A} \) be an area-preserving twist map. Then there exists \( \nu > 0 \) such that any essential circle invariant under \( f \) is the graph of some \( \nu \)-Lipschitz function \( \ell : \mathbb{T} \to [a, b] \).

The second result from Birkhoff’s theory we need is the following.

**Theorem (Birkhoff).** Let \( f : \mathbf{A} \to \mathbf{A} \) be an area-preserving twist map. Assume that \( U \) is an open subset of \( \mathbf{A} \) homeomorphic to \( \mathbb{T} \times [0, 1] \), with \( \Gamma(a) \subset U \), such that \( f(U) \subset U \) and such that \( U \) is the interior of its closure. Then the frontier \( \text{Fr} \ U \) is an invariant essential circle.

One easily deduces from the first Birkhoff theorem that the set \( \text{Ess}(f) \) of essential invariant circles of \( f \), endowed with the Hausdorff topology, is compact. Given \( \Gamma \in \text{Ess}(f) \) with \( \Gamma = \text{Graph}(\ell) \), we set
\[
\Gamma^+ = \{(\theta, r) \in \mathbf{A} \mid r > \ell(\theta)\}, \quad \Gamma^- = \{(\theta, r) \in \mathbf{A} \mid r < \ell(\theta)\}.
\]
By the Poincaré theory, every \( \Gamma \in \text{Ess}(f) \) admits a rotation number in \( \mathbb{T} \) for \( f|_{\Gamma} \). One can choose a common lift to \( \mathbb{R} \) for the rotation number of all circles, which yields a function \( \rho : \text{Ess}(f) \to \mathbb{R} \). This function is continuous and increasing, in the sense that if \( \Gamma_i = \text{Graph} \ell_i, \ i = 1, 2 \) are invariant with \( \ell_1 \leq \ell_2 \), then \( \rho(\ell_1) \leq \rho(\ell_2) \). Moreover, \( \rho(\ell_1) < \rho(\ell_2) \) when \( \ell_1 < \ell_2 \).

**Definition 18.** Let \( f : \mathbf{A} \to \mathbf{A} \) be an area-preserving twist map of the annulus \( \mathbf{A} \). Let \( \ell_\bullet \) and \( \ell^\bullet \) be two functions \( \mathbb{T} \to \mathbb{R} \) whose graphs \( \Gamma_\bullet \) and \( \Gamma^\bullet \) are in \( \text{Ess}(f) \). Then one says that the set
\[
\mathcal{B} = \{(\theta, r) \mid \theta \in \mathbb{T}, \ \ell_\bullet(\theta) \leq r \leq \ell^\bullet(\theta)\}
\]
is a Birkhoff zone when that there is no element \( \Gamma = \text{Graph} \ell \in \text{Ess}(f) \) such that \( \ell_\bullet \leq \ell \leq \ell^\bullet \) and \( \ell \neq \ell_\bullet \), \( \ell \neq \ell^\bullet \).

We are now in a position to prove Lemma 1, for which we refer to Definition 3.
Proof of Lemma 1. The main property of a special twist map $f$, coming from the fact that no element of $\text{Ess}(f)$ has rational rotation, is that two distinct elements of $\text{Ess}(f)$ are disjoint (see [HK], Section 13.2). As a consequence, the rotation number $\rho : \text{Ess}(f) \to \mathbb{R}$ is a homeomorphism onto its image $\mathcal{R} = \rho(\text{Ess}(f))$, by compactness of $\text{Ess}(f)$, where $\text{Ess}(f)$ is endowed with the uniform topology. The boundaries of the Birkhoff zones are sent by $\rho$ on the boundaries of the maximal intervals in the complement $\text{Rot}\setminus \rho(\text{Ess}(f))$, where $\text{Rot} = [\rho(\Gamma(a)), \rho(\Gamma(b))]$ is the rotation interval of $f$. The other claims easily follow.

Given $(u, v) \in \mathbb{R}^2$, let $\angle(u, v)$ be the oriented angle of $(u, v)$ in $[0, 2\pi]$. Let $f : A \to A$ be an area-preserving twist map. Fix a circle $\Gamma \subset \text{Ess}(f)$. An arc emanating from $\Gamma$ is a $C^0$ function $\gamma : (0, 1) \to A$ such that $\gamma(0) \in \Gamma$ and $\gamma(1) \in \Gamma^+$. We say that such an arc ends at $\gamma(1)$. An $C^1$ arc emanating from $\Gamma$ with $\gamma'(s) \neq 0$ for $s \in [0, 1]$ is said to be positively tilted (resp. negatively tilted) when $\angle((0, 1), \gamma'(0)) \in [0, \pi]$ (resp. $\angle((0, 1), \gamma'(0)) \in [-\pi, 0]$) and when the continuous lift to $\mathbb{R}$ of $s \mapsto \angle((0, 1), \gamma'(s))$ is positive (resp. negative) over $[0, 1]$.

![Figure 4: Positively and negatively tilted arcs](image)

We can now prove our second lemma on special twist maps and domains associated with right or left splitting arcs, stated in Section 2.

Proof of Lemma 4. We adopt the natural convention for the (oriented) parametrization of geometric segments and their concatenation. Let $\zeta$ be a right splitting arc based on $\Gamma$ whose image is contained in $\Gamma^+$, and let $x = (\theta, r)$ be a point in the associated domain $D$. Let $[\theta_0, \theta_0 + \tau]$ be the projection of $\zeta([0, 1])$ on $\mathbb{T}$ and let $\xi$ be a point in the image of $\zeta$ located on the vertical through $\theta_0 + \tau$. Let $y \in \Gamma$ be the intersection point of the vertical through $x$ with $\Gamma$ and let $y' \in \Gamma$ be the intersection point of the vertical through $\xi$ with $\Gamma$. Let $[y, y']_\Gamma$ be the segment of $\Gamma$ limited by $y, y'$ and above $[\theta, \theta_0 + \tau]$. Fix some point $z \in \Gamma$ such that the segment $[z, \xi]$ is positively tilted. We define a $C^0$ arc emanating from $\Gamma$ and ending at $x$ as the concatenation of the arcs $[z, \xi], [\xi, y']_\Gamma$, $-[y, y']_\Gamma$ and $[y, x]$. Approximating the segment $-[y, y']_\Gamma$ from below and “rounding the corners” then yields a positively tilted $C^1$ arc which satisfies our requirements.

The following easy result on negatively tilted arcs is used several times in our constructions.
Lemma 9. Let $\Gamma$ be an essential circle of $A$ which is the graph of a $\nu$-Lipschitz function $\ell : \mathbb{T} \to [0,1]$, and let $B$ be a $\nu$-ball centered on $\Gamma$. Then for any $z \in \Gamma^+ \cap B$, there exists a negatively tilted arc emanating from $\Gamma$ and ending at $z$, whose image is contained in $B$.

Proof. Set $B = B_\theta \times B_r$. Then since $B$ is a $\nu$-ball the graph of $\ell |_{B_\theta}$ is contained in $B$. Given $z = (\theta, r) \in \Gamma^+ \cap B$ and $\theta_0 \in B_\theta$ with $\theta_0 < \theta$ close enough to $\theta$ so that $r - \ell(\theta_0) > \nu,$ and setting $z_0 = (\theta_0, \ell(\theta_0))$, the segment $[z_0, z]$ satisfies our requirements. \qed

The proof of the following lemma is immediate.

Lemma 10. Let $f : A \to A$ be an area-reserving twist map. Let $\Gamma^\pm$ be two nonintersecting essential invariant circles contained in $A$. Then for any continuous curves $C$ and $C'$ which intersect both circles $\Gamma^\pm$, the positive orbit of $C$ under $f$ intersects $C'$.

We refer to [LC86] and [LC87] for the proofs of the following two results from Birkhoff’s theory.

Lemma 11. Let $f : A \to A$ be an area-preserving twist map and let $\Gamma$ be an essential invariant circle for $f$. The inverse image $f^{-1} \circ \gamma$ of a positively tilted arc $\gamma$ emanating from $\Gamma$ is a positively tilted arc emanating from $\Gamma$. The direct image $f \circ \eta$ of a negatively tilted arc $\eta$ emanating from $\Gamma$ is a negatively tilted arc emanating from $\Gamma$.

Given a point $x \in A$, we define the lower vertical $V^-(x)$ as the vertical segment joining a point of $\Gamma(a)$ to $x$.

Lemma 12. Let $f : A \to A$ be an area-preserving twist map. Let $\Gamma$ be an essential invariant circle for $f$. Let $X$ be a connected closed subset of $A$ which disconnects the annulus $A$ and such that $X \subset \Gamma^+$. Let $x \in A$ be such that there exists a positively tilted arc $\gamma$ and a negatively tilted arc $\eta$, both emanating from $\Gamma$ and ending at $x$, such that the images of $\gamma$ and $\eta$ do not intersect $X$. Then the vertical $V^-(x)$ does not intersect $X$.

The following strong connecting lemma appeared with a different proof in [GR13].

Proposition 3. Let $f : A \to A$ be a (not necessarily special) area-preserving twist map. Let $\Gamma_\star$ and $\Gamma^\star$ be the boundary components of some Birkhoff zone of instability for $f$. Fix a pair of open sets $V_\star, V^\star$ which intersect $\Gamma_\star$ and $\Gamma^\star$ respectively, with moreover $V_\star \subset (\Gamma^\star)^-$. Then there exist a point $z \in V_\star$ and an integer $n \geq 0$ such that $f^n(z) \in V^\star$. Moreover the integer $n$ can be chosen arbitrarily large.

Proof. Set
$$U = \bigcup_{n \geq 0} f^n(\Gamma^\star \cup V_\star) = \Gamma^- \cup \left( \bigcup_{n \geq 0} f^n(V_\star) \right),$$
so that $U$ is a connected and $f$-invariant neighborhood of $\Gamma(a)$, which satisfies $U \subset (\Gamma^\star)^-$. 


Hence the frontier $\Gamma := \text{Fr} U$ of its associated filled subset (see Section 5) is in $\text{Ess}(f)$ and satisfies $\Gamma_\bullet \leq \Gamma \leq \Gamma^\bullet$. Therefore $\Gamma = \Gamma_\bullet$ or $\Gamma = \Gamma^\bullet$. The former equality is impossible by construction, so $\Gamma = \Gamma^\bullet$.

As a consequence, $\Gamma^\bullet \subset \text{Fr} U \subset \text{Fr} U$, so there exists an integer $n \geq 0$ such that

$$f^n(V_\bullet) \cap V^\bullet \neq \emptyset,$$

which proves our claim. Finally, observe that by choosing arbitrarily small open subsets $W_\bullet \subset V_\bullet$, $W^\bullet \subset V^\bullet$ and applying the previous result to the pair $W_\bullet$, $W^\bullet$, one can ensure that the integer $n$ can be chosen arbitrarily large. \hfill \square

### B Equivariance properties for homoclinic correspondences

We describe here some equivariance properties of the homoclinic correspondences with respect to the Hamiltonian flow, see [DLS08] for related equivariance results. These properties reveal themselves to be useful in the constructive framework of Section 5.

**Definition 19.** Consider a $C^2$ Hamiltonian function $H$ on $\mathbb{R}^3$, a completely regular value $e$ and a tame cylinder $\mathcal{C}$ at energy $e$, with continuation $\tilde{\mathcal{C}}$ and continued twist section $\tilde{\Sigma}$ with return map $\tilde{\varphi}$. We say that a homoclinic correspondence $\psi = (\psi_i)_{i \in I}$ is equivariant when for each $i \in I$ and for each $m \in \mathbb{Z}$ there exists an index $i(m)$ such that

$$\text{Dom } \psi_{i(m)} = \tilde{\varphi}^m(\text{Dom } \psi_i), \quad \text{Im } \psi_{i(m)} = \tilde{\varphi}^m(\text{Im } \psi_i)$$

and

$$\psi_{i(m)} \circ \tilde{\varphi}^m = \tilde{\varphi}^m \circ \psi_i.$$  

The main result of this section is the following.

**Lemma 13.** With the same assumptions as in the previous definition, any homoclinic correspondence $(\psi_i)_{i \in I}$ can be extended to an equivariant homoclinic correspondence, in the sense that there exists an equivariant homoclinic correspondence $(\psi_i)_{i \in I}$ with $I \supset I$ and $\psi_i = \psi_i$ for $i \in I$.

**Proof.** Given a subset $A \subset \tilde{\mathcal{C}}$ and a function $T : A \to \mathbb{R}$, we denote by $\Phi^T : A \to \tilde{\mathcal{C}}$ the map defined by

$$\Phi^T(x) = \Phi_H(T(x), x), \quad x \in A.$$

1. Fix an element $\psi = \psi_i : \text{Dom } \psi \to \text{Im } \psi$ of the initial homoclinic correspondence. By definition, there exist a $C^1$ map $S : \text{Dom } S \to \text{Im } S$ and a $C^1$ function $\tau : \text{Dom } \psi \to \mathbb{R}^+$ such that the following diagram holds:

\[
\begin{array}{ccc}
\text{Dom } \psi & \xrightarrow{\psi} & \text{Im } \psi \\
\downarrow \Phi^\tau & & \downarrow j \\
\text{Dom } S & \xrightarrow{S} & \text{Im } S
\end{array}
\]
commutes (where we denoted by $j$ the canonical inclusion). The map $S$ moreover satisfies condition [21] of Definition [3].

2. Since $\text{Dom} \ S$ is an open subset of $\text{int} \mathcal{E}$ which contains $\Phi^\tau(\text{Dom} \ \psi)$, one can find $C^1$ functions $\alpha, \beta : \text{Dom} \ \psi \to \mathbb{R}$ such that $\alpha < \tau < \beta$ and

\[ D(S) := \bigcup_{x \in \text{Dom} \ \psi} \Phi_H(\{\alpha(x), \beta(x)[ \times \{x\}\}) \subset \text{Dom} \ S. \]

By equivariance of $S$ and commutativity of the previous diagram

\[ I(S) := \bigcup_{x \in \text{Im} \ \psi} \Phi_H(\{\alpha(x) - \tau(x), \beta(x) - \tau(x)[ \times \{x\}\}) \]

is an open subset of $\text{Im} \ S$. We still denote by $S$ the induced map from $D(S)$ to $I(S)$. Note that the previous diagram still commutes if $S : \text{Dom} \ S \to \text{Im} \ S$ is replaced with $S : D(S) \to I(S)$. Note moreover that if $D_I(S) := D(S) \cap \text{Homt} \mathcal{E}$, then $D_I$ is an open subset of full measure of $D(S)$ such that

\[ \forall x \in D_I(S), \quad W^-(x) \cap W^+(S(x)) \cap \text{Homt} \mathcal{E} \neq \emptyset. \quad (54) \]

3. For $m \in \mathbb{Z}$, we denote by $T^{(m)} : \hat{\Sigma} \to \mathbb{R}^+$ the $m^{th}$ return time map associated with the Hamiltonian flow, so that $\hat{\varphi}^m = \Phi^{T^{(m)}} : \hat{\Sigma} \to \hat{\Sigma}$. We still denote by $T^{(m)}$ the “fiberwise continuation” of $T^{(m)}$ to the domains $D(S)$ and $I(S)$, that is:

\[ \forall x \in \text{Dom} \ \psi, \quad \forall z \in \Phi_H(\{\alpha(x), \beta(x)[ \times \{x\}\}), \quad T^{(m)}(z) = T^{(m)}(x), \]

\[ \forall y \in \text{Im} \ \psi, \quad \forall z \in \Phi_H(\{\alpha(y) - \tau(y), \beta(y) - \tau(y)[ \times \{y\}\}), \quad T^{(m)}(z) = T^{(m)}(y). \]

Then, observe that the map $S^{(m)} : \Phi^{T^{(m)}}(D(S)) \to \Phi^{T^{(m)}}(I(S))$ defined by

\[ S^{(m)} = \Phi^{T^{(m)}} \circ S \circ \Phi^{-T^{(m)}} \]

satisfies

\[ \forall z \in \Phi^{T^{(m)}}(D_I(S)), \quad W^-(z) \cap W^+(S^{(m)}(z)) \cap \text{Homt} \mathcal{E} \neq \emptyset, \]

by equivariance of the characteristic foliations and preservation of the transversality. Moreover, $\Phi^{T^{(m)}}(D_I(S))$ is an open subset with full measure of $\Phi^{T^{(m)}}(D(S))$.

4. For $m \in \mathbb{Z}$, let $\psi^{(m)}$ be the unique map such that the following diagram

\[ \begin{array}{ccc}
\hat{\varphi}^{(m)}(\text{Dom} \ \psi) & \xrightarrow{\psi^{(m)}} & \hat{\varphi}^{(m)}(\text{Im} \ \psi) \\
\downarrow \Phi^\tau & & \downarrow j \\
\Phi^{T^{(m)}}(D(S)) & \xrightarrow{S^{(m)}} & \Phi^{T^{(m)}}(I(S))
\end{array} \]

commutes. Then clearly $\text{Dom} \ \psi^{(m)} = \hat{\varphi}^m(\text{Dom} \ \psi)$, $\text{Im} \ \psi^{(m)} = \hat{\varphi}^m(\text{Im} \ \psi)$ and

\[ \forall x \in \text{Dom} \ \psi, \quad \psi^{(m)}(x) \circ \hat{\varphi}^m = \hat{\varphi}^m \circ \psi. \quad (55) \]
5. Let us consider the new index set $I = I \times \mathbb{Z}$ and set

$$\psi_{(i,m)} = \psi_i^{(m)}.$$  

Then the previous construction proves that $(\psi_{(i,m)})_{(i,m) \in I}$ is a homoclinic correspondence, with associated family $(S_i^{(m)})_{(i,m) \in I}$.

6. It suffices now to show that it is equivariant. Fix $i = (i, k) \in I$, set $\psi := \psi_i^{(k)}$, fix $m \in \mathbb{Z}$ and set $\psi_i^{(m)} = \psi_i^{(m+k)}$. Then

$$\text{Dom} \psi_i^{(m)} = \varphi_{m+k}(\text{Dom} \psi_i) = \varphi^m(\text{Dom} \psi),$$

$$\text{Im} \psi_i^{(m)} = \varphi_{m+k}(\text{Im} \psi_i) = \varphi^m(\text{Im} \psi),$$

and, by (55):

$$\psi_i^{(m)} \circ \varphi_{m+k} = \varphi^m \circ \varphi_{m+k} \circ \psi_i = \varphi^m \circ \varphi_k \circ \psi_i = \varphi^m \circ \psi \circ \varphi_k$$

so that

$$\psi_i^{(m)} \circ \varphi^m = \varphi^m \circ \psi.$$  

This proves the equivariance condition for $\psi_1$, with $i(m) = (i, m + k)$.  

Observe that while homoclinic correspondences giving rise to good cylinders can always be modified in order to obtain $\delta$-bounded correspondences, this is no longer the case for equivariant correspondences, due to the extension process.

C Symmetrization of polysystems

The following general lemma has been used several times in specific contexts.

**Lemma 14.** Let $A$ be a metric space endowed with a finite Borel measure, positive on the nonempty open subsets of $A$. Let $\varphi$ be a measure-preserving homeomorphism of $A$ and let $(\psi_i)_{i \in I}$ be a polysystem on $A$, where $\text{Dom} \psi_i$ is open and the map $\psi_i : \text{Dom} \psi_i \rightarrow \text{Im} \psi_i$ is a homeomorphism, for all $i \in I$. Fix a nonempty open subset $V \subset A$. Let $U_f$ and $U_g$ be the full orbit of $V$ under the polysystems $f = (\varphi, \psi = (\psi_i)_{i \in I})$ and $g = (\varphi, \varphi^{-1}, \psi = (\psi_i)_{i \in I})$ respectively. Then $U_f$ is contained and dense in $U_g$.

**Proof.** Since $\varphi$, $\varphi^{-1}$ and $\psi_i$ are open maps, the full orbits $U_f$ and $U_g$ are open in $A$, and clearly $U_f \subset U_g$. We assume that the index set $I$ does not contain $\{-1, 1\}$ and we write $I = \{-1, 1\} \cup I$. Set $\tau_1 = \varphi^{-1}$, $\tau_1 = \varphi$, $\tau_i = \psi_i$ for $i \in I$, so that

$$f = (\tau_1, (\tau_i)_{i \in I}), \quad g = (\tau_1, (\tau_i)_{i \in I}).$$

Fix a nonempty open subset $W \subset U_g$. Then (by continuity of the maps) there exist a nonempty open subset $\tilde{W}$ of $V$ and a sequence $\omega = (\omega_n)_{0 \leq n \leq n_* - 1} \in I^{m*}$ such that 

$$g^\omega(\tilde{W}) \subset W.$$  

We will iteratively modify the set $\tilde{W}$ and the sequence $\omega$ so that we get a nonempty open set $\tilde{W} \subset W$ and a sequence $\tilde{\omega} \in (1 \cup I)^{m*}$ such that

$$f^{\tilde{\omega}}(\tilde{W}) = g^{\omega}(\tilde{W}) \subset W.$$
At each step of the process, the number of occurrences of the index \(-1\) in the sequence \(\omega\) will decrease by 1, and so the process stops after a finite number of steps. Let us describe the first one.

We set, for \(0 \leq n \leq n_* - 1\):

\[
W_n = g^{(\omega_0, \ldots, \omega_{n-1})}(W)
\]

so that \(W_{n_*} = g^{\pi}(W) \subset W\). Let \(\pi \geq 0\) be the smallest index such that \(\omega_\pi = -1\). Since \(\tau_1 = \varphi\) is measure-preserving, there exists a nonempty open subset \(O \subset W_\pi\) and an integer \(\nu \geq 1\) such that

\[
\tau_1^{\nu+1}(O) \subset W_\pi,
\]

by the Poincaré recurrence theorem. Consider the new sequence \(\omega' = (\omega'_0, \ldots, \omega'_{n_*+\nu-2})\) with

\[
\begin{align*}
\omega'_n &= \omega_n, & 0 \leq n \leq \pi - 1, \\
\omega'_n &= 1, & \pi \leq n \leq \pi + \nu - 1, \\
\omega'_n &= \omega_{n-\nu+1}, & \pi + \nu \leq n \leq n_* + \nu - 2,
\end{align*}
\]

where the first line has to be omitted when \(n = 0\). Set

\[
W'_n = \tau_{\omega_0}^{-1} \circ \cdots \circ \tau_{\omega_{\pi-1}}^{-1}(O) \subset W,
\]

and note that \(W' \neq \emptyset\). Set

\[
W'_0 = W', \quad W'_n = g^{(\omega'_0, \ldots, \omega'_{n-1})}(W'), \quad 1 \leq n \leq n_* + \nu - 1.
\]

Therefore, by construction:

\[
W'_n \subset W_n, \quad 0 \leq n \leq \pi - 1, \quad \text{and} \quad W'_\pi = O \subset W_\pi.
\]

Hence

\[
W'_{\pi+\nu} = \tau_1^\nu(W'_\pi) = \tau_1^{\nu-1}(\tau_1^{\nu+1}(W'_\pi)) = \tau_1^{\nu-1}(\tau_1^{\nu+1}(O)) \subset \tau_{-1}(W_\pi) = W_{\pi+1}.
\]

As a consequence, by definition of the sequence \(\omega'\):

\[
W'_n \subset W_{n-\nu+1}, \quad \pi + \nu + 1 \leq n \leq n_* + \nu - 1.
\]

In particular

\[
W'_{n_*+\nu-1} \subset W_{n_*}
\]

and if \(\ell\) is the number of occurrences of \(-1\) in \(\omega\), then \(-1\) occurs \(\ell - 1\) times in \(\omega'\). Iterating this process therefore yields a nonempty open set \(\tilde{W} \subset W\) and a sequence \(\tilde{\omega} \in \{0, 1\}^{m_*}\) such that \(f^{\tilde{\omega}}(\tilde{W}) \subset W_{n_*} \subset W\), so that the full orbit \(U_f\) of \(V\) under \(f\) intersects \(W\). Since \(W\) is arbitrary in \(U_g\), this proves that \(U_f\) is dense in \(U_g\). \(\square\)
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