Weyl–Schrödinger Representations of Heisenberg Groups in Infinite Dimensions

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Abstract. We investigate the group $\mathcal{H}_C$ of complexified Heisenberg matrices with entries from an infinite-dimensional complex Hilbert space $H$. Irreducible representations of the Weyl–Schrödinger type on the space $L^2_\chi$ of quadratically integrable $\mathbb{C}$-valued functions are described. Integrability is understood with respect to the projective limit $\chi = \lim \chi_i$ of probability Haar measures $\chi_i$ defined on groups of unitary $i \times i$-matrices $U(i)$. The measure $\chi$ is invariant under the infinite-dimensional group $U(\infty) = \bigcup U(i)$ and satisfies the abstract Kolmogorov consistency conditions. The space $L^2_\chi$ is generated by Schur polynomials on Paley–Wiener maps. The Fourier-image of $L^2_\chi$ coincides with the Hardy space $H^2_\beta$ of Hilbert–Schmidt analytic functions on $H$ generated by the correspondingly weighted Fock space $\Gamma_\beta(H)$. An application to heat equation over $H$. An application to heat equation over $H_C$ is considered.

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1. Introduction

An aim of this work is to investigate irreducible Weyl–Schrödinger representations of the complexified Heisenberg group $\mathcal{H}_C$ (see [17, n.9]), consisting of matrix elements $X(a,b,t)$ with any $a,b \in H$ and $t \in \mathbb{C}$ such that
\[
X(a, b, t) = \begin{bmatrix}
1 & a & t \\
0 & 1 & b \\
0 & 0 & 1
\end{bmatrix},
\]
\[
X(a, b, t) \cdot X(a', b', t') = \begin{bmatrix}
1 & a + a' & t + t' + \langle a | b' \rangle \\
0 & 1 & b + b' \\
0 & 0 & 1
\end{bmatrix}
\]
(1)

where \( H \) is an infinite-dimensional complex Hilbert space and \( 1 \) is its identity map.

The group \( \mathcal{H}_C \) has the unit \( X(0, 0, 0) \) and inverse elements of the form \( X(a, b, t)^{-1} = X(-a, -b, -t + \langle a | b \rangle) \).

In what follows, we consider the infinite-dimensional unitary group \( U(\infty) = \bigcup U(i) \), containing all subgroups \( U(i) \) of unitary \( i \times i \)-matrices, which acts irreducibly on a complex Hilbert space \( \{ H, \langle \cdot | \cdot \rangle \} \) with an orthonormal basis \((e_i)_{i \in \mathbb{N}}\).

To find the desired representation, we use the space \( L^2_\chi \) of \( \mathbb{C} \)-valued functions that are quadratically integrable with respect to the probability measure \( \chi \). Wherein, according to our assumption \( \chi \) has a structure of the projective limit \( \chi = \lim \leftarrow \chi_i \) of probability Haar’s measures \( \chi_i \) on \( U(i) \), satisfying the Kolmogorov consistency conditions in an abstract Bochner’s formulation (see [23, 27]).

In [21, 24] it was shown that the projective limit \( \chi = \lim \leftarrow \chi_i \) is well defined over the projective limit \( \mathfrak{U} = \lim \leftarrow U(i) \) with respect to the Livšic transforms \( \pi_{i+1}^{i+1} : U(i+1) \to U(i) \) such that \( \chi_i = \pi_{i+1}^{i+1}(\chi_{i+1}) \). In this paper, we prove that for such \( \chi \) each function from \( L^2_\chi \) admits a superposition (linearization in the sense of [5]) on Paley–Wiener maps associated with \( U(\infty) \). As a result, it is shown that Schur polynomials form an orthonormal basis in \( L^2_\chi \) and the Fourier-image of \( L^2_\chi \) consists of Hilbert-Schmidt analytic functions on \( H_\phi \).

Note also that projective limits of probability measures over various infinite-dimensional manifolds with similar properties were investigated in [25, 34, 35].

If instead of the unitary group \( U(\infty) \) we take the infinite-dimensional linear space with a Gaussian measure \( \gamma \), a similar construction of the appropriate space \( L^2_\gamma \) can be found in the well-known works [1, 2]. In this case, the Fourier-image of \( L^2_\gamma \) coincides with the Segal–Bargmann space of entire analytic functions over which the Schrödinger type irreducible representations of Heisenberg groups are well defined. In the present paper, we change \( \gamma \) by the unitarily-invariant projective limit \( \chi = \lim \chi_i \) and, as a result, we obtain another irreducible representation, called to be the Weyl–Schrödinger type.

Infinite-dimensional Heisenberg groups over \( \mathbb{R} \) was considered in [19] by using the reproducing kernel Hilbert spaces. The Schrödinger representation of such groups using Gaussian measures over a real Hilbert space was described in [3]. Since the group \( \mathcal{H}_C \) in the case of matrix entries \( a, b, t \in \mathbb{R} \) coincides with the classical Heisenberg group over \( \mathbb{R} \) (see, e.g. [11]), the results of the
The present paper can be considered as a complexification of previous studies. The Weyl–Schrödinger representation obtained here is not equivalent to that was described earlier.

Further, let us briefly describe the main results. Consider the following mapping \( \phi: H \ni h \mapsto \phi_h \in L^2_\chi \) defined by Paley–Wiener maps

\[
\phi_h(u) := \sum \phi_i(u) \psi_i^*(h) \quad \text{with} \quad \phi_i(u) := \langle u_i(e_i) | e_i \rangle, \quad u_i = \pi_i(u),
\]

where \( \psi_i^*(\cdot) := \langle \cdot | e_i \rangle \) and the projections \( \pi_i: \mathfrak{U} \ni u \to u_i \in U(i) \) are uniquely defined by \( \pi_i^{+1} \). Every function \( \phi_h \) of variable \( u \in \mathfrak{U} \) satisfies the equality (Corollary 3)

\[
\int \exp \{ \Re \phi_h \} \, d\chi = \exp \left\{ \frac{1}{4} \|h\|^2 \right\}, \quad h \in H.
\]

The space \( L^2_\chi \) can be generated by two orthonormal bases, consisting of Schur polynomials and power polynomials of variables \( \phi_s = (\phi_{i_1}, \ldots, \phi_{i_n}) \), respectively,

\[
s_i^\lambda(u) := \frac{\det \left[ \phi_i^{\lambda_i + \eta - j}(u) \right]_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} \left( \phi_i(u) - \phi_j(u) \right)} \quad \text{and} \quad \phi_i^\lambda := \phi_i^{\lambda_1} \ldots \phi_i^{\lambda_n}.
\]

These bases are indexed by tabloids \( \nu^\lambda \) with strictly ordered \( \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n \) where \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \) is a partition of \( n \in \mathbb{N} \) and \( \eta = \eta(\lambda) \) stands for the length of \( \lambda \). Then we write briefly \( \nu^\lambda \vdash n \). The orthogonal expansion \( L^2_\chi = \bigoplus L^2_\chi^n \) holds (Theorem 1) where \( L^2_\chi^n \) are formed by \( n \)-homogeneous polynomials \( \phi_i^\lambda \), normed as follows

\[
\|\phi_i^\lambda\|^2_\chi = \int |\phi_i^\lambda|^2 \, d\chi = \beta_\lambda \lambda!, \quad \beta_\lambda := \frac{(\eta - 1)!}{(\eta - 1 + n)!}, \quad \lambda! := \lambda_1! \ldots \lambda_n!.
\]

It is also shown that the surjective linear isometry \( \Psi: H^2_\beta \ni \psi_j^* \mapsto f \in L^2_\chi \) holds (Lemma 5), where \( H^2_\beta = \sum P^\beta_n(H) \) means the Hardy space of entire analytic functions \( \psi_j^*(h) \) of variable \( h \in H \) and \( P^\beta_n(H) \) is generated by the \( n \)-homogeneous Hilbert–Schmidt polynomials \( e_i^\lambda := e_i^{\lambda_1} \ldots e_i^{\lambda_n} \), normed as

\[
\|e_i^\lambda\|^2_{H^2_\beta} = (\beta_\lambda \lambda!)^{1/2}.
\]

If the basis of symmetric tensor elements \( e_i^{\circ \lambda} := e_i^{\lambda_1} \otimes \ldots \otimes e_i^{\lambda_n} \) (associated with \( e_i^\lambda \)) in the correspondingly weighted Fock space \( \Gamma_\beta(H) \) is normed as \( \|e_i^{\circ \lambda}\|_{\Gamma_\beta} = \|e_i^\lambda\|_{H^2_\beta} \) then each function \( f \in L^2_\chi \) admits the superposition

\[
f = \Psi \circ \psi_j^*; \quad \psi_j^*(h) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\nu^\lambda \vdash n} \frac{n!}{\lambda!} e_i^{\circ \lambda}(h) \langle e_i^{\circ \lambda} | \psi_j \rangle_{\Gamma_\beta}, \quad h \in H,
\]

where the Taylor expansion on the right-hand side of any analytic function \( \psi_j^* \in H^2_\beta \) on \( H \) is uniquely determined by the corresponding element \( \psi_j \in \Gamma_\beta(H) \).
Our further goal is to analyze the inverse isomorphism $\Psi^{-1}$ which can be described by the Fourier transform under the measure $\chi$ in following way

$$\hat{f}(h) = \int \exp(\bar{\phi}_h) f \, d\chi$$

where $F = \Psi^{-1} : L^2_\chi \ni f \mapsto \hat{f} := \psi^*_f \in H^2_\beta$.

The Fourier transform $F$ acts isometrically on the Hardy space of analytic functions $H^2_\beta$ (Theorem 2). So, $F$ acts as an analytic extension of the mapping $\phi$.

Applying the superposition with $\Psi$, we describe two different representations of the additive group $(H, +)$ over $L^2_\chi$ defined by shift and multiplicative groups (Lemma 7). Using this we show (in Theorem 3) that an irreducible representation of the Heisenberg group $H_C$ can be realized on $L^2_\chi$ in the Weyl–Schrödinger form

$$X(a, b, z) \mapsto \exp(z) W^\dagger(a,b), \quad W^\dagger(a,b) := \exp \left\{ \frac{1}{2} (a \mid b) \right\} T^\dagger_b M^\dagger_a,$$

for all $a, b \in H$ and $z \in \mathbb{C}$, where $T^\dagger_b$ and $M^\dagger_a$ are defined by shift and multiplicative groups, respectively. It is also proved that the Weyl system $W^\dagger(a,b)$ has the densely-defined generator $p^\dagger_{a,b}$ which satisfies the commutation relation

$$W^\dagger(a,b)W^\dagger(a',b') = \exp \left\{ -[p^\dagger_{a,b}, p^\dagger_{a',b'}] \right\} W^\dagger(a',b')W^\dagger(a,b)$$

where the groups $M^\dagger_a$ and $T^\dagger_b$ are generated by $\bar{\phi}_a$ and $\partial^\dagger_b$, respectively.

Applying the Weyl–Schrödinger representation to the associated with $H_C$ heat equation, we prove (Theorem 4) that the following Cauchy problem with $\partial^\dagger_\tau := \partial^\dagger_{\tau_1}$,

$$\frac{dw(r)}{dr} = -\sum (\partial^\dagger_\tau + \bar{\phi}_\tau)^2 w(r), \quad w(0) = f, \quad r > 0,$$

has the unique solution $w(r) = \mathcal{G}^\dagger_r f$ for any function $f$ from a finite sum $\bigoplus L^2_\chi^{n} n$, where the 1-parameter Gaussian semigroup $\mathcal{G}^\dagger_r$ has the form

$$\mathcal{G}^\dagger_r f = \frac{1}{\sqrt{4\pi r}} \int_{c_0} \exp \left\{ - \frac{\|\tau\|^{2}_{w_0}}{4r} \right\} W^\dagger f \, d\mathcal{N}(\tau),$$

$$W^\dagger f := \lim_{n \to \infty} \exp \left\{ - \frac{\|p^\dagger_{n, \tau}(\tau)\|^{2}_{w_0}}{2} \right\} \prod_{i=1}^{n} T^\dagger_{\tau_i} M^\dagger_{-\tau_i \epsilon_i}.$$

Here $\tau = (\tau_i)$ belongs to the abstract Wiener space $\{w_0, \| \cdot \|_{w_0}\}$ defined by the injections $l_2 \ni w_0 \ni c_0$ of real Banach spaces and endowed with the Wiener measure $\mathcal{N}$ in according to the known Gross’ theorem [10], whereas the sequence of projectors $(p^\dagger_{n, \tau})$ onto $\mathbb{R}^n$ is convergent to the identity map on $w_0$.

Finally, note that this work is a continuation of previous publications [16,17]. The novelty results from the observation that the system of Schur
polynomials with variables on Paley–Wiener maps form an orthonormal basis in $L^2_\lambda$. This allowed us to investigate irreducible Weyl–Schrödinger representations and Weyl systems of the Heisenberg group $\mathcal{H}_\mathbb{C}$ on the whole space $L^2_\lambda$.

2. Invariant Probability Measure

Consider the unitary group $U(\infty) = \bigcup U(m)$ with $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{1} = U(0)$, irreducibly acting on a separable Hilbert space $H$, where subgroups $U(m)$ are identified with ranges of injections $U(m) \ni u_m \mapsto \begin{bmatrix} u_m & 0 \\ 0 & 1 \end{bmatrix} \in U(\infty)$. Following to [21,24], we use the Livšic transforms $\pi_m^{m+1} : U(m+1) \to U(m)$ of the form

$$\pi_m^{m+1} : u_{m+1} := \begin{bmatrix} z_m & a \\ b & t \end{bmatrix} \mapsto u_m := \begin{cases} z_m - [a(1+t)b] : t \neq -1 \\ z_m & : t = -1 \end{cases}$$

with $z_m \in U(m)$ defined by excluding $x_1 = y_1 \in \mathbb{C}$ from $\begin{bmatrix} y_m \\ y_1 \end{bmatrix} = \begin{bmatrix} z_m & a \\ -b & -t \end{bmatrix} \begin{bmatrix} x_m \\ x_1 \end{bmatrix}$ for $x_m, y_m \in \mathbb{C}^m$ and $a, b \in \mathbb{C}$ [24, Lem. 3.1]. It is surjective (not continuous) Borel mapping [24, Lem. 3.11].

The projective limit $\Xi := \varprojlim U(m)$ under $\pi_m^{m+1}$ has surjective Borel (not group homomorphisms) projections

$$\pi_m : \Xi \ni u \mapsto u_m \in U(m) \quad \text{such that} \quad \pi_m = \pi_u^{m+1} \circ \pi_{m-1}.$$ Their elements $u \in \Xi$ are called the virtual unitary matrices. The right action

$$\Xi \ni u \mapsto u.g \in \Xi \quad \text{with} \quad g = (v, w) \in U(\infty) \times U(\infty)$$

is defined to be $\pi_m(u.g) = w^{-1}\pi_m(u)v$, where $m$ is large enough that $v, w \in U(m)$. On $\Xi$ the involution $u \mapsto u^* = (u_k^*)$ is well defined, where $u_k^* = u_k^{-1}$ is adjoint to $u_k \in U(k)$. Thus, $[\pi_m(u.g)]^* = \pi_m(u^*.g^*)$ for all $g^* = (w^*, v^*) \in U(\infty) \times U(\infty)$.

There exists the dense embedding $U(\infty) \ni \Xi$ (see [24, n.4]) which assigns the stabilized sequence $u = (u_k)$ to each $u_m \in U(m)$ such that

$$U(m) \ni u_m \mapsto (u_k) \in \Xi,$$

$$u_k = \left\{ \begin{array}{ll} \pi_k^{m}(u_m) = (\pi_k^{k+1} \circ \ldots \circ \pi_1^m)(u_m) : k < m, \\ u_m & : k \geq m. \end{array} \right.$$ (5)

We always assume that the group $U(m)$ is endowed with the probability Haar measure $\chi_m$. Using the Kolmogorov consistency theorem (see, e.g. [24, Lem.4.8], [27, Thm 2.2], [30, Cor.4.2]), we determine the probability measure on $\Xi$ to be the projective limit

$$\chi := \varprojlim \chi_m \quad \text{under} \quad \chi_m = \pi_m^{m+1}(\chi_{m+1})$$
where $\pi_{m+1}^{m+1}(\chi_{m+1})$ means an image-measure and $\chi_0 = 1$. As is known [30, Thm 2.5], the measure $\chi$ is Radon. We now describe the necessary properties of $\chi$.

Consider the Hilbert space $L^2_\chi$ of functions $f: \mathcal{U} \to \mathbb{C}$ with the following norm and inner product
\[
\|f\|_\chi = \langle f, f \rangle_\chi^{1/2}, \quad \langle f_1, f_2 \rangle_\chi := \int f_1 \overline{f_2} \, d\chi.
\]
Let $L^\infty_\chi$ be the space of $\chi$-essentially bounded functions $f: \mathcal{U} \to \mathbb{C}$ with the norm $\|f\|_\infty = \text{ess sup}_{u \in \mathcal{U}} |f(u)|$. The embedding $L^\infty_\chi \hookrightarrow L^2_\chi$ holds and $\|f\|_\infty \leq \|f\|_\chi$.

**Lemma 1.** For any $f \in L^\infty_\chi$ there exists the limit
\[
\int f \, d\chi = \lim \int f(\chi_m \circ \pi_m) \, d\chi_m. \tag{6}
\]
Moreover, the measure $\chi$ is invariant under the right action, which means that
\[
\int f(u, g) \, d\chi(u) = \int f(u) \, d\chi(u), \quad g \in U(\infty) \times U(\infty), \tag{7}
\]
\[
\int f \, d\chi = \int d\chi(u) \int f(u, g) \, d(\chi_m \otimes \chi_m)(g). \tag{8}
\]

**Proof.** The sequence $\{(\chi_m \circ \pi_m)(K)\}$ is decreasing for any compact set $K$ in $\mathcal{U}$, since $\pi_m = \pi_{m+1}^m \circ \pi_{m+1}$ yields $\pi_{m+1}(K) \subseteq (\pi_{m+1}^m)^{-1}[\pi_m(K)]$. It follows
\[
(\chi_m \circ \pi_m)(K) = \pi_{m+1}^m(\chi_{m+1})[\pi_m(K)]
\]
\[
= \chi_{m+1}(p_{m+1}^{-1}[\pi_m(K)]) \geq (\chi_{m+1} \circ \pi_{m+1})(K). \tag{9}
\]
This ensures that the necessary and sufficient conditions of the Prokhorov theorem [4, Thm IX.52] and its modification from [30, Thm 4.2] are satisfied.

Indeed, let $\tilde{U}(m) \subset U(m)$ be the set of matrices with no eigenvalue $\{1\}$ for $m \geq 1$. As is known [24, n.3], $\tilde{U}(m)$ is open in $U(m)$ and $\chi_m(\mathcal{U}(m) \backslash \tilde{U}(m)) = 0$. In virtue of [24, Lem. 3.11] the restrictions $\pi_{m+1}^m: \tilde{U}(m+1) \to \tilde{U}(m)$ are continuous and surjective. The projective limit $\lim \tilde{U}(m)$ under these restrictions has continuous surjective projections $\pi_m: \lim \tilde{U}(m) \to \tilde{U}(m)$. Restrict $\chi_m$ to $\tilde{U}(m)$. By [30, Thm 6], a probability measure $\tilde{\chi}$ satisfying conditions $\pi_m(\tilde{\chi}) = \chi_m|_{\tilde{U}(m)}$ is well defined if for every $\varepsilon > 0$ there exists a compact set $K \subset \lim \tilde{U}(m)$ such that
\[
(\chi_m \circ \pi_m)(K) \geq 1 - \varepsilon \quad \text{for all} \quad m \in \mathbb{N}.
\]

Then by the Prokhorov theorem $\tilde{\chi}$ is uniquely determined as
\[
\tilde{\chi}(K) = \inf(\chi_m \circ \pi_m)(K) \quad \text{for all} \quad K \subset \lim \tilde{U}(m). \tag{10}
\]

Let $\varepsilon > 0$ and $K_1 \subset \tilde{U}(1)$ be a compact set such that $\chi_1(K_1) > 1 - \varepsilon$. Let a compact sets $K_m \subset \tilde{U}(m)$ be defined inductively such that
\[
\pi_{m+1}^m(K_{m+1}) \subset K_m \quad \text{and} \quad \chi_{m+1}(K_{m+1}) > 1 - \varepsilon \quad \text{for all} \quad m \geq 1.
\]
Assume that $K_1, \ldots, K_m$ are constructed. Since $\chi_m = \pi_m^{m+1}(\chi_{m+1})$, we get

$$\chi_m(K_m) = \chi_{m+1}[]\pi_m^{m+1}(K_m)] > 1 - \varepsilon.$$ 

By regularity of $\chi_{m+1}|U(m)$, there exists a compact set $K_{m+1} \subset (\pi_m^{m+1})^{-1}(K_m)$ such that $\chi_{m+1}(K_{m+1}) > 1 - \varepsilon$.

The induction is complete. Then $K = \lim K_m$ with $K_0 = 1$ is compact.

By virtue of (10), we have $\tilde{\chi} = \chi$. So, $\chi = \lim \chi_m$ is well defined and by (9) and (10) we get

$$\chi(K) = \inf(\chi_m \circ \pi_m)(K) = \lim(\chi_m \circ \pi_m)(K)$$

for all compact $K \subset \Omega$.

By the known Portmanteau theorem [14, Thm 13.16] it follows that the limit (6) exists. Whereas, the property (7) is a consequence of the equalities

$$\chi(K,g) = \lim \chi_m(K,g) = \chi_m(K,m) = \chi(K)$$

for all $g = (v, w) \in U(\infty) \times U(\infty)$ where $m$ is large enough that $v, w \in U(m)$.

Finally, the function $(u,g) \mapsto f(u,g)$ with any $f \in L^\infty_X$ is integrable over $\Omega \times U(m) \times U(m)$, hence

$$\int d\chi(u) \int f(u,g) d(\chi_m \otimes \chi_m)(g) = \int d(\chi_m \otimes \chi_m)(g) \int f(u,g) d\chi(u)$$

by the Fubini theorem. It yields (8) since the internal integral on the right-hand side is independent of $g$ by (7) and $\int d(\chi_m \otimes \chi_m)(g) = 1$. The proof is complete. \hfill $\Box$

We now note the concentration property of Haar measures sequence $(\chi_m)$ satisfying the Kolmogorov conditions $\chi_m = \pi_m^{m+1}(\chi_{m+1})$ if each group $U(m)$ is endowed with the normalized Hilbert–Schmidt metric

$$d_{HS}(u,v) = \sqrt{m^{-1}tr|u - v|_{HS}} \quad \text{where} \quad |u - v|_{HS} = \sqrt{(u - v)^*(u - v)}.$$ 

As is well known (see [9,31]), $(U(m), d_{HB}, \chi_m)$ is a Lévy family. Namely, the following sequence of isoperimetric constants dependent on $\varepsilon > 0$

$$\alpha(U(m), \varepsilon) = 1 - \inf \{ \chi_m([\Omega_m]_\varepsilon) : \Omega_m \text{ be Borel set in } U(m), \chi_m(\Omega_m) > 1/2 \}$$

with $([\Omega_m]_\varepsilon) = \{ u_m \in U(m) : d_{HS}(u_m, \Omega_m) < \varepsilon \}$ is such that

$$\alpha(U(m), \varepsilon) \to 0 \quad \text{as} \quad m \to \infty.$$ 

Taking into account the Lemma 1, we can formulate the following conclusion.

**Corollary 1.** For any Borel set $\Omega_\varepsilon = \lim(\Omega_m)_\varepsilon$ with $\chi_m(\Omega_m) > 1/2$ in the projective limit $\Omega = \lim U(m)$ the equality

$$\chi(\Omega_\varepsilon) = \lim_{m \to \infty} \chi_m([\Omega_m]_\varepsilon) = 1$$
holds. Consequently, all Borel sets \( \Omega \setminus \Omega_\varepsilon \) with \( \chi_m(\Omega_m) > 1/2 \) and any \( \varepsilon > 0 \) are \( \chi \)-measure zero, i.e., the measure \( \chi = \lim \chi_m \) is concentrated outside these sets.

3. Polynomials on Paley–Wiener Maps

Let \( \mathcal{I}_\eta := \{ i = (i_1, \ldots, i_\eta) \in \mathbb{N}^\eta : i_1 < i_2 < \ldots < i_\eta \} \) be an integer alphabet of length \( \eta \) and \( \mathcal{I} = \bigcup \mathcal{I}_\eta \). Let \( \lambda = (\lambda_1, \ldots, \lambda_\eta) \in \mathbb{N}^\eta \) with \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\eta \) be a partition of an \( n \)-letter word. Let \( \mathcal{N} = \{ \square_{ij} : 1 \leq i \leq \eta, j = 1, \ldots, \lambda_i \} \) with \( i \in \mathcal{I}_\eta \). A Young \( \lambda \)-tableau with a partition \( \lambda \) is a result of filling the word \( \square_{11} \ldots \square_{1\lambda_1} \) onto the matrix \([i^\lambda]\) with \( n \) nonzero entries in \( \square_{\eta 1} \ldots \square_{\eta \lambda_\eta} \) some way without repetitions. So, each \( \lambda \)-tableau \([i^\lambda]\) can be identified with a bijection \([i^\lambda] \rightarrow \lambda^T\). The conjugate partition \( \lambda^T \) corresponds to the transpose matrix \([i^\lambda]^T\).

A Young tableau \([i^\lambda]\) is called standard (semistandard) if its entries are strictly (weakly) ordered along each row and strictly ordered down each column. Let \( \mathcal{Y} \) denote all Young tabloids \([i^\lambda]\) and \( \mathcal{Y}_n \) be its subset such that \( [i^\lambda] \vdash n \). Assume that \( \mathcal{Y}_0 = \{ \emptyset \in \mathcal{Y} : |\emptyset| = 0 \} \) and \( \eta(\emptyset) = 0 \).

As before, \( \{H, \langle \cdot, | \cdot \rangle\} \) is a separable complex Hilbert space with an orthonormal basis \( \{e_i : i \in \mathbb{N}\} \) and \( \| \cdot \| = \langle \cdot, | \cdot \rangle \}^{1/2} \). For its adjoint space \( H^* \) the conjugate-linear isometry \( * : H^* \rightarrow H^{**} = H \) is defined via \( a^*(h) = \langle h | a \rangle \) for all \( a, h \in H \). The Fourier expansion \( h = \sum e_i^* \langle h | e_i \rangle e_i \) with \( e_i^* (h) := \langle h | e_i \rangle \) holds. The tensor power \( H^{\otimes n} \), spanned by elements \( \psi_n = h_1 \otimes \ldots \otimes h_n \) with \( h_i \in H \) \((i = 1, \ldots, n)\), is endowed with the norm \( \| \psi_n \| = \langle \psi_n | \psi_n \rangle^{1/2} \) where \( \langle \psi_n | \psi'_n \rangle := \langle h_1 | h'_1 \rangle \ldots \langle h_n | h'_n \rangle \).

Let \( S_n \) be the group of \( n \)-elements permutations \( \sigma(\psi_n) := h_{\sigma(1)} \otimes \ldots \otimes h_{\sigma(n)} \). An orthogonal basis in \( H^{\otimes n} \) is formed by elements \( \sigma(i_1^{\otimes \lambda_1} \otimes \ldots \otimes i_\eta^{\otimes \lambda_\eta}) \) with \( i^\lambda \vdash n \) and \( \eta = \eta(\lambda) \), additionally indexed by all \( \sigma \in S_n \). The symmetric tensor power \( H^{\otimes n} \subset H^{\otimes n} \) is defined to be a range of the orthogonal projector \( S_n : H^{\otimes n} \ni \psi_n \mapsto h_1 \odot \ldots \odot h_n := (n!)^{-1} \sum_{\sigma \in S_n} \sigma(\psi_n) \). We assume that \( H^{\otimes n} \) is completed and that \( H^{\otimes 0} = \mathbb{C} \). Let \( \psi_n := h^{\otimes n} \) for \( h = h_i \). The embedding \( \{h^{\otimes n} : h \in H\} \subset H^{\otimes n} \) is total by the polarization formula \([7, \text{n.1.5}]\)

\[
h_1 \odot \ldots \odot h_n = \frac{1}{2^n n!} \sum_{\theta_1, \ldots, \theta_n = \pm 1} \theta_1 \ldots \theta_n h^{\otimes n}, \quad h = \sum_{i=1}^n \theta_i h_i. \tag{11}
\]

Let \( H_\eta \subset H \) be spanned by \( \{e_{i_1}, \ldots, e_{i_\eta}\} \). We can uniquely assign to any semistandard tableau \([i^\lambda]\) with \( i^\lambda \vdash n \) the element in \( H_\eta^{\otimes n} \) for which there exists the permutation \( \sigma' \in S_n \) such that \( \sigma'(i_1^{\otimes \lambda_1} \otimes \ldots \otimes i_\eta^{\otimes \lambda_\eta}) = e_{i_1}^{\otimes \lambda_1} \otimes \ldots \otimes e_{i_\eta}^{\otimes \lambda_\eta} \)
$\in H^{\odot n}$.

Taking all $\iota \in \mathcal{I}$, we conclude that the system indexed by semistandard $\lambda$-tabloids

$$\mathbf{e}^\mathcal{Y}_n = \left\{ e_{i_1}^{\odot \lambda_1} \odot \ldots \odot e_{i_n}^{\odot \lambda_n} : \lambda^\iota \vdash n, \lambda \in \mathcal{Y}_n, \iota \in \mathcal{I} \right\}, \quad e_{i}^{\odot \emptyset} = 1$$

where

$$\langle e_{i}^{\odot \lambda} \mid e_{j}^{\odot \lambda'} \rangle = \begin{cases} \lambda! / n! : \lambda = \lambda' \text{ and } \iota = \iota' \\ 0 : \lambda \neq \lambda' \text{ or } \iota \neq \iota' \end{cases}$$

forms an orthogonal basis in the symmetric tensor power $H^{\odot n}$.

The system $\{ e_{i}^{\odot \lambda} := S_n(e_{i_1}^{\odot \lambda_1} \odot \ldots \odot e_{i_n}^{\odot \lambda_n}) : \lambda^\iota \vdash n, \lambda \in \mathcal{Y}_n, \iota \in \mathcal{I} \}$, additionally indexed by all $\sigma \in S_n$, forms an orthonormal basis in the whole tensor power $H^{\odot n}$.

As usually, the symmetric Fock space is defined to be the Hilbertian orthogonal sum $\Gamma(H) = \bigoplus_{n \geq 0} H^{\odot n}$ with the orthogonal basis $\mathbf{e}^\mathcal{Y}_n := \bigcup\{ \mathbf{e}^\mathcal{Y}_n : n \in \mathbb{N}_0 \}$ of elements $\psi = \bigoplus \psi_n$ with $\psi_n \in H^{\odot n}$ endowed with the inner product and norm

$$\langle \psi \mid \psi' \rangle_R = \sum_n n! \langle \psi_n \mid \psi'_n \rangle, \quad \| \psi \|_R = \langle \psi \mid \psi \rangle_R^{1/2}.$$ 

Note that by tensor multinomial theorem the Fourier expansion under $\mathbf{e}^\mathcal{Y}_n$

$$h^{\odot n} = \sum_{\iota \lambda \vdash n} \frac{n!}{\lambda!} e_{i}^{\odot \lambda} e_{i}^{\ast \lambda}(h), \quad \left\| h^{\odot n} \right\|^2 = \sum_{\iota \lambda \vdash n} \frac{n!}{\lambda!} | e_{i}^{\ast \lambda}(h) |^2, \quad e_{i}^{\ast \lambda} := e_{i_1}^{\ast \lambda_1} \ldots e_{i_n}^{\ast \lambda_n},$$

holds in $H^{\odot n}$ for all $h \in H$. Consequently, the linearly independent, so-called, coherent states $\{ \exp(h) : h \in H \}$ in $\Gamma(H)$ have the expansion under the basis $\mathbf{e}^\mathcal{Y}_n$

$$\exp(h) := \bigoplus_{n \geq 0} \frac{h^{\odot n}}{n!} = \bigoplus_{n \geq 0} \frac{1}{n!} \left( \sum_{i \geq 0} e_{i} e_{i}^{\ast}(h) \right)^\odot n = \bigoplus_{n \geq 0} \frac{1}{n!} \sum_{\iota \lambda \vdash n} \frac{n!}{\lambda!} e_{i}^{\odot \lambda} e_{i}^{\ast \lambda}(h)$$

with $h^{\odot 0} = 1$, that is convergent, since $\left\| e_{i}^{\odot \lambda} \right\|^2_R = n! \left\| e_{i}^{\ast \lambda} \right\|^2$ and

$$\left\| \exp(h) \right\|^2_R = \sum_{n \geq 0} \frac{1}{n!} \sum_{\iota \lambda \vdash n} \left( \frac{n!}{\lambda!} \right)^2 \left\| e_{i}^{\odot \lambda} \right\|^2 \left\| e_{i}^{\ast \lambda}(h) \right\|^2 = \sum_{n \geq 0} \frac{1}{n!} \sum_{\iota \lambda \vdash n} \frac{n!}{\lambda!} \left\| e_{i}^{\ast \lambda}(h) \right\|^2$$

$$= \sum_{n \geq 0} \frac{1}{n!} \left( \sum \left\| e_{i}^{\ast}(h) \right\|^2 \right)^n = \sum_{n \geq 0} \frac{1}{n!} \left\| h \right\|^2 = \exp \left\| h \right\|^2.$$ 

**Definition 1.** For any $h \in H$ and $u \in \mathcal{U}$ the Paley–Wiener maps are defined to be

$$\phi_h(u) := \sum \phi_i(u) e_{i}^{\ast}(h) \quad \text{with} \quad \phi_i(u) := \langle u_i(e_i) \mid e_i \rangle, \quad u_i = \pi_i(u)$$

where projections $\pi_i : \mathcal{U} \ni u \to u_i \in U(i)$ are uniquely defined by $\pi_i^{i+1}$. 
These maps satisfy the orthogonal conditions $\phi_{\xi_i} = \phi_i$ and have the natural extension $\phi_{h^*} = \phi_h$ onto the adjoint space $H^*$. Note that, as in the case of linear spaces (see e.g. [12, n.4.4], [29]), the Paley–Wiener maps uniquely determine the embedding $\phi: H \ni h \mapsto \phi_h \in L^2_{\lambda}$.

For every $h \in H$ the $l_2$-valued function $\phi_h(u)$ of variable $u \in \mathcal{U}$ is well-defined, since $(\varepsilon_i^+(h)) \in l_2$ and $|\langle u_i(\varepsilon_i) | \varepsilon_i \rangle| \leq 1$. We show that $\phi_h \in L^2_{\lambda}$. Assign for any partition $\lambda = (\lambda_1, \ldots, \lambda_\eta) \in \mathbb{N}^\eta$ of the weight $|\lambda| = \lambda_1 + \ldots + \lambda_\eta$ the constant

$$\beta_\lambda := \frac{(\eta - 1)!}{(\eta - 1 + |\lambda|)!} \leq 1, \quad \eta = \eta(\lambda).$$

**Lemma 2.** To every semistandard tableau $[\iota^\lambda]$ one can uniquely assign the function

$$\phi_{\iota^\lambda}(u) := \phi_{\iota_1^\lambda}(u) \cdots \phi_{\iota_\eta^\lambda}(u), \quad \phi_0^\lambda \equiv 1$$

of variable $u \in \mathcal{U}$ belonging to $L^\infty_{\lambda}$. The system of $\chi$-essentially bounded functions

$$\phi_\iota^\lambda := \bigcup \{ \phi_\iota^\lambda_n : n \in \mathbb{N}_0 \} \quad \text{with} \quad \phi_\iota^\lambda_n := \bigcup \{ \phi_{\iota_\iota^\lambda} : \iota \vdash n, \iota \in \mathcal{I}_\eta \}$$

is orthogonal in the space $L^2_{\lambda}$ and is normed as follows

$$\|\phi_{\iota^\lambda}\|_{\lambda}^2 = \int |\phi_{\iota^\lambda}|^2 d\chi = \lambda! \beta_\lambda, \quad \iota \vdash n, \quad \lambda! := \lambda_1! \cdots \lambda_\eta!.$$

**Proof.** According to (4), we have $(\pi_m \circ \pi_{m+1}^{-1})u_{m+1}(\varepsilon_m) = u_m(\varepsilon_m)$ for $t = -1$ and $(\pi_m \circ \pi_{m+1}^{-1})u_{m+1}(\varepsilon_m) = u_m(\varepsilon_m) - [a(1+t)^{-1}b] e_m$ for $t \neq -1$ for any integer $l \geq 1$. This means that $(\phi_k \circ \pi_m^{-1})(u_m) = \langle u_m(\varepsilon_m) | \varepsilon_k \rangle \neq 0$ for all $k \leq m$ and that

$$(\phi_m \circ \pi_{m+1}^{-1})(u_{m+1}) = \langle u_m(\varepsilon_m) | \varepsilon_m \rangle \quad \text{for} \quad t = -1,$n \text{ and that}

$$(\phi_m \circ \pi_{m+1}^{-1})(u_{m+1}) = \langle u_m(\varepsilon_m) | \varepsilon_m \rangle - a(1+t)^{-1}b \langle \varepsilon_m | \varepsilon_m \rangle \quad \text{for} \quad t \neq -1.$$ (17)

Let $U(\eta)$ with $\eta = \eta(\lambda)$ be the unitary group acting over the linear complex span $\{\varepsilon_1, \ldots, \varepsilon_\eta\}$ in $H$. Let $\chi_\eta$ be the probability Haar measure on $U(\eta)$ and $\pi_\eta: \mathcal{U} \to U(\eta)$ be the corresponding projector. Using (6) and (17), we obtain

$$\int |\phi_{\iota^\lambda}(u)|^2 d\chi(u) = \lim \int |(\phi_i \circ \pi_{m+1}^{-1})(u_m)|^2 d\chi_m(u_m)$$

$$= \lim \int |(\phi_{\xi_1} \circ \pi_{m+1}^{-1})(u_m) \cdots (\phi_{\xi_\eta} \circ \pi_{m+1}^{-1})(u_m)|^2 d\chi_m(u_m)$$

$$= \int |(\phi_{\xi_1} \circ \pi_{\eta-1}^{-1})(u_\eta) \cdots (\phi_{\xi_\eta} \circ \pi_{\eta-1}^{-1})(u_\eta)|^2 d\chi_\eta(u_\eta).$$ (18)
By (18) and the known integral formula for unitary groups \( U(\eta) \) [28, 1.4.9], we get
\[
\int |\phi_\lambda^\eta|^2 d\chi = \int \prod_{k=1}^{\eta(\lambda)} |(u_\eta(e_n) | e_{t_k})|^2 d\chi_{\eta}(u_\eta) = \frac{(\eta(\lambda) - 1)! \lambda!}{(\eta(\lambda) - 1 + |\lambda|)!}.
\]

On the other hand, the invariant property (8) provides the formula
\[
\int f d\chi = \frac{1}{2\pi} \int d\chi(u) \int_{-\pi}^{\pi} f[\exp(i\theta)u] d\theta, \quad f \in L^\infty_\chi.
\]
From (19) it follows the orthogonality relations \( \phi_\lambda^\eta \perp \phi_\lambda^k \) with \( |\lambda'| \neq |\lambda| \), since
\[
\int \phi_\lambda^\eta \bar{\phi}_\lambda^k d\chi = \frac{1}{2\pi} \int \phi_\lambda^\eta \bar{\phi}_\lambda^k d\chi \int_{-\pi}^{\pi} \exp[i(|\lambda'| - |\lambda|)\theta] d\theta = 0
\]
for any \( \lambda', \lambda \in \mathbb{Y}\backslash\{\emptyset\} \). Let \( |\lambda'| = |\lambda| \) and \( \eta(\lambda') > \eta(\lambda) \) for definiteness. Then there exists an index \( k \) with a nonzero integer \( \lambda'_k \) in \( \lambda' = (\lambda'_1, \ldots, \lambda'_k, \ldots, \lambda'_{\eta(\lambda')}) \) \( \in \mathbb{Y}\backslash\{\emptyset\} \) such that \( \eta(\lambda) < k \leq \eta(\lambda') \). In this case \( \phi_\lambda^\eta \perp \phi_\lambda^k \) because (19) yields
\[
\int \phi_\lambda^\eta \bar{\phi}_\lambda^k d\chi = \frac{1}{2\pi} \int \phi_\lambda^\eta \bar{\phi}_\lambda^k d\chi \int_{-\pi}^{\pi} \exp[i(\lambda'_k - \lambda_k)\theta] d\theta = 0.
\]
Consider the case \( |\lambda'| = |\lambda| \) and \( \eta(\lambda') = \eta(\lambda) \). If \( \phi_\lambda^\eta \neq \phi_\lambda^k \) then \( \lambda' \neq \lambda \). There exists an index \( 0 < k \leq \eta(\lambda) \) such that \( \lambda'_k \neq \lambda_k \). As above, \( \phi_\lambda^\eta \perp \phi_\lambda^k \), because
\[
\int \phi_\lambda^\eta \bar{\phi}_\lambda^k d\chi = \frac{1}{2\pi} \int \phi_\lambda^\eta \bar{\phi}_\lambda^k d\chi \int_{-\pi}^{\pi} \exp[i(\lambda'_k - \lambda_k)\theta] d\theta = 0.
\]
This proves that the system \( \phi^\eta \) is orthogonal. \( \square \)

4. Orthonormal Basis of Schur Polynomials
Let \( ^\lambda n, \eta = \eta(\lambda) \) and \( t_i = (t_{i_1}, \ldots, t_{i_n}) \) be a complex variable. Let \( t^\lambda_i := \prod t^\lambda_{i_j} \). The \( n \)-homogenous Schur polynomial is defined (see, e.g. [18]) to be
\( s^\lambda_i(t_i) := D\lambda(t_i)/\Delta(t_i) \) where \( D\lambda(t_i) = \det \left[t^\lambda_j + \eta - j\right] \) with \( \lambda_j = 0 \) for \( j > \eta \), \( \Delta(t_i) = \prod_{1 \leq i < j \leq n} (t_{i_j} - t_{j_i}) \) is Vandermonde’s determinant. It can be written as \( s^\lambda_i(t_i) = \sum_{|\lambda'|} t^\lambda_{i} \) with summation over all semistandard Young tabloids \( [8, 1.2.2] \).

We construct an orthonormal basis in \( L^2 \chi \) consisting of Schur polynomials on Paley–Wiener maps. Assign (uniquely) to \( i \in \mathcal{I}_\eta \) the vector \( \phi_i := (\phi_{i_1}, \ldots, \phi_{i_n}) \). Let \( s^\lambda_i(u) = (s^\lambda_i \circ \phi)(u) \) be \( n \)-homogeneous functions of variable \( u \in \mathfrak{U} \) with \( \lambda \in \mathbb{N}^n \), defined by the formulas (3). Denote
\[
s^\eta_n := \bigcup \{ s^\lambda_i : ^\lambda n \}, \quad s^\eta := \bigcup \{ s^\eta_n : n \in \mathbb{N}_0 \} \quad \text{with} \quad s_0 = s^\emptyset \equiv 1.
\]
Theorem 1. The system of Schur polynomials $s^Y$ forms an orthonormal basis in $L^2_\chi$ and $s^Y_n$ is the same basis in $L^{2,n}_\chi$. The following orthogonal decomposition holds,

$$L^2_\chi = C \oplus L^2_{\chi,1} \oplus L^2_{\chi,2} \oplus \ldots.$$  \hspace{1cm} (20)

For any $h \in H$ the equality (2) uniquely defines the conjugate-linear embedding

$$\phi: H \ni h \mapsto \phi_h \in L^2_\chi$$

such that $\|\phi_h\|_\chi = \|h\|$.  \hspace{1cm} (21)

Proof. Let $U(\eta)$ be the unitary group over the linear complex span $\{e_{i_1}, \ldots, e_{i_\eta}\}$ with $\eta = \eta(\lambda)$. Taking into account (17) similarly as (18), we obtain

$$\int s^\lambda_i \bar{s}^\mu_i \, d\chi = \int s^\lambda(z_\eta) \, \bar{s}^\mu(z_\eta) \, d\chi_\eta = \delta_{\lambda\mu}$$

for all $[\lambda]$, $[\mu]$ with $i = (i_1, \ldots, i_\eta)$ and $\lambda, \mu \in \mathbb{N}^\eta$. In fact, the corresponding Schur polynomials $\{s^\lambda_i: \lambda \in \mathbb{N}^\eta\}$ are characters of the group $U(\eta)$. Hence, by the Weyl integration formula, the right-hand side integral is equal to Kronnecker’s delta $\delta_{\lambda\mu}$ [26, Thm 8.3.2 & Thm 11.9.1].

The family of finite alphabets $i \in \mathcal{F}$ is directed and for any $i, i'$ there exists $i''$ such that $i \cup i' \subset i''$. This means that the whole system $s^\chi_n$ is orthonormal in $L^2_\chi$.

The property $s^\mu_i \perp s^\lambda_i$ with $|\mu| \neq |\lambda|$ for any $i, j \in \mathcal{F}$ follows from (19), since

$$\int s^\mu_i \bar{s}^\lambda_i \, d\chi = \frac{1}{2\pi} \int s^\mu_i \bar{s}^\lambda_i \, d\chi \int_{-\pi}^{\pi} \exp \left(i(|\mu| - |\lambda|) \vartheta\right) \, d\vartheta = 0$$

for all $\lambda \in \mathcal{Y}$ and $\mu \in \mathcal{Y}\setminus\{0\}$. This yields $L^{2,|\mu|}_\chi \perp L^{2,|\lambda|}_\chi$ in the space $L^2_\chi$. Taking $\lambda = 0$ with $|\vartheta| = 0$, we get $1 \perp L^{2,|\mu|}_\chi$ for all $\mu \in \mathcal{Y}\setminus\{0\}$. Hence, (20) is proved.

By Lemma 2 the subsystem $\phi_k = s^1_k$ is orthonormal in $L^2_\chi$, hence by Definition 1 it instantly follows that $\|\phi_h\|_\chi^2 = \sum |s^1_k(h)|^2 \int |\phi_k|^2 \, d\chi = \|h\|^2$. It follows the isometric embedding (21).

The set $\tilde{U}(m)$ of matrices with no eigenvalue $\{-1\}$ has Stone–Čech compactification $\tilde{U}(m)$ such that the mapping $\tilde{\pi}_m^{m+1}$ has a continuous $U(m)$-valued extension

$$\tilde{\pi}_m^{m+1}: \tilde{U}(m + 1) \rightarrow U(m).$$

This fact follows from [33, Thm 19.5] by virtue of that $U(m)$ is compact. Hence, the projective limit $\hat{\mathfrak{U}} := \varprojlim \tilde{U}(m)$, determined by $\tilde{\pi}_m^{m+1}$, is a compact set in $\mathfrak{U}$ with continuous $U(m)$-valued projections $\tilde{\pi}_m: \hat{\mathfrak{U}} \rightarrow U(m)$.

Since $U(\infty)$ on $H$ acts irreducibly, for any $u' \neq u''$ there is $m$ such that

$$\phi_m(u') = \langle \pi_m(u') \mid e_m \rangle \neq \langle \pi_m(u'') \mid e_m \rangle = \phi_m(u''),$$

i.e., $\phi^Y$ separates $\mathfrak{U}$ and so $\hat{\mathfrak{U}}$. Hence, the system of Schur polynomials $s^Y$ also separates $\mathfrak{U}$. Moreover, each complex-conjugate function $\bar{\phi}_m(u) = \langle e_m \mid \pi_m(u) (e_m) \rangle = \langle \pi_m(u^*) (e_m) \mid e_m \rangle$ belongs to $\phi^Y$. Thus, by the Stone–Weierstrass
approximation theorem the complex linear span of polynomials $\phi^Y$, as well as, of $s^Y$, forms a dense subspace in the Banach space of all continuous functions $C(\hat{U})$.

Let $\tilde{\chi}_m$ means the image of $\chi_m$ under $\hat{U}(m) \rightarrow U(m)$. In Lemma 1 it inductively was shown that for every $\varepsilon > 0$ there exists a compact set $\lim K_m \subset \hat{U}$ such that

$$\tilde{\chi}_m(K_m) \geq 1 - \varepsilon$$

for all $m$.

where $\tilde{\chi}_m(K_m) = \chi_m(K_m) = \chi_m(K_m)$, by definition of the measure $\tilde{\chi}_m$ as an image. Hence, by the Prokhorov theorem the projective limit $\tilde{\chi} = \lim \tilde{\chi}_m$, defined by mappings $\tilde{\chi}_m^{m+1}$, possesses the properties

$$\tilde{\chi}(\Omega) = \inf \tilde{\chi}_m(\Omega) = \inf \chi_m(\Omega) = \lim \chi_m(\Omega) = \chi(\Omega)$$

for all Borel $\Omega$ in $\hat{U}$ or otherwise $\tilde{\chi}|_{\hat{U}} = \chi|_{\hat{U}}$. Consequently,

$$\tilde{\chi}|_{\hat{U}} = \chi|_{\hat{U}} = \chi|_{\hat{U} \cup (U \setminus \hat{U})} = \chi|_{\hat{U}}$$

since $\chi(\hat{U} \setminus \hat{U}) = 0$.

In particular, $\tilde{\chi} = \lim \tilde{\chi}_m$ is regular on $\hat{U}$ by the Riesz–Markov theorem [20, 1.1].

As a consequence, the space $L^2_\chi$ coincides with the completion of $C(\hat{U})$ and for any $f \in L^2_\chi$ there exists a sequence $(f_n) \subset \text{span}(s^Y)$ such that $\int |f - f_n|^2 d\chi \rightarrow 0$. Hence, the system $s^Y$ forms an orthogonal basis in $L^2_\chi$.

Finally, $s^Y_n \cap L^2_\chi$ is total in $L^2_\chi$ and $s^Y_n \bot s^Y_m$ if $n \neq m$. This yields (20).

\section{5. Unitarily-Weighted Symmetric Fock Space}

Define on the tensor power $H^{\otimes n}$ the unitarily-weighted norm $\| \cdot \|_{H^{\otimes n}_\beta} = \sqrt{\langle \cdot | \cdot \rangle_{H^{\otimes n}_\beta}}$ where the inner product $\langle \cdot | \cdot \rangle_{H^{\otimes n}_\beta}$ is determined by the relations

$$\langle \epsilon^{\otimes \lambda}_i | \epsilon^{\otimes \lambda'}_{i'} \rangle_{H^{\otimes n}_\beta} = \begin{cases} \frac{(\eta - 1)!}{(\eta - 1 + \delta_i)!} & : \lambda = \lambda' \text{ and } \iota = \iota' \\ 0 & : \lambda \neq \lambda' \text{ or } \iota \neq \iota'. \end{cases} \quad (22)$$

Here $\epsilon^{\otimes \lambda}_i := \sigma'(\epsilon^{\otimes \lambda_1}_{i_1} \otimes \ldots \otimes \epsilon^{\otimes \lambda_n}_{i_n})$ with $\eta = \eta(\lambda)$ and $\sigma' \in S_n$ is fixed. Let $H^{\otimes n}_\beta$ be the completion of $\{H^{\otimes n}_\beta, \| \cdot \|_{H^{\otimes n}_\beta} \}$. Its closed subspace, defined by the projection

$$S_n : H^{\otimes n}_\beta \ni \epsilon^{\otimes \lambda}_i \mapsto \epsilon^{\otimes \lambda}_i = (n!)^{-1} \sum_{\sigma \in S_n} \sigma(\epsilon^{\otimes \lambda}_i)$$

forms an unitarily-weighted symmetric tensor power $H^{\otimes n}_\beta \subset H^{\otimes n}_\beta$ with the inner product determined by relations $\langle \epsilon^{\otimes \lambda}_i | \epsilon^{\otimes \lambda'}_{i'} \rangle_{H^{\otimes n}_\beta} = \beta_\lambda \langle \epsilon^{\otimes \lambda}_i | \epsilon^{\otimes \lambda'}_{i'} \rangle$ or more specific
\begin{equation}
\langle e_i^{\lambda} | e_i^{\lambda'} \rangle_{H_\beta^\otimes n} = \begin{cases} \\
\lambda! \frac{(\eta - 1)!}{n! (\eta - 1 + n)!} : \lambda = \lambda' \text{ and } i = i' \\
0 : \lambda \neq \lambda' \text{ or } i \neq i'. 
\end{cases} \tag{23}
\end{equation}

**Definition 2.** The *unitarily-weighted symmetric Fock space* is defined to be the Hilbertian orthogonal sum \( \Gamma_\beta(\mathcal{H}) = \bigoplus_{n \geq 0} H_\beta^\otimes n \) of elements \( \psi = \bigoplus \psi_n \), \( \psi_n \in H_\beta^\otimes n \) with the orthogonal basis \( e^\psi = \bigcup \{ e^\psi_n : n \in \mathbb{N}_0 \} \) and the following inner product and norm
\[
\langle \psi | \psi' \rangle_\beta = \sum n! \langle \psi_n | \psi_n' \rangle_{H_\beta^\otimes n}, \quad \|\psi\|_\beta = \langle \psi | \psi \rangle_\beta^{1/2}.
\]

We immediately notice that \( \|h\|_\beta^2 = \sum |e^*_i(h)|^2 = \|h\|^2 \) for all \( h = \sum e_i e^*_i(h) \in H_\beta^\otimes n \).

**Lemma 3.** The set of coherent states \( \{ \exp(h) : h \in \mathcal{H} \} \) is total in \( \Gamma_\beta(\mathcal{H}) \) and the expansion (13) is convergent in \( \Gamma_\beta(\mathcal{H}) \). The injections
\[
\Gamma(\mathcal{H}) \hookrightarrow \Gamma_\beta(\mathcal{H}) \quad \text{and} \quad H^\otimes n \hookrightarrow H_\beta^\otimes n
\]
are contractive and dense. The \( \Gamma_\beta(\mathcal{H}) \)-valued function \( H \ni h \mapsto \exp(h) \) is entire analytic. The shift group, defined to be
\[
\mathcal{T}_a \exp(h) := \exp(h + a) = \exp(\partial_a) \exp(h) \quad \text{with} \quad \partial_a \exp(h) = \frac{d \exp(h + za)}{dz} \bigg|_{z=0}
\]
for \( a, h \in \mathcal{H} \), has a unique linear extension \( \Gamma_\beta(\mathcal{H}) \ni \psi \mapsto \mathcal{T}_a \psi \in \Gamma_\beta(\mathcal{H}) \) such that
\[
\|\mathcal{T}_a \psi\|_\beta^2 \leq \exp(\|a\|^2)\|\psi\|_\beta^2 \quad \text{and} \quad \mathcal{T}_{a+b} = \mathcal{T}_a \mathcal{T}_b = \mathcal{T}_b \mathcal{T}_a, \quad a, b \in \mathcal{H}. \tag{24}
\]

**Proof.** Taking into account that \( \beta \lambda \leq 1 \), we get the following inequalities
\[
\|h^\otimes n\|_{H_\beta^\otimes n}^2 = \sum_{i^{\lambda \geq n}} \left( \frac{n!}{\lambda!} \right)^2 \|e_i^{\otimes \lambda}\|_{H_\beta^\otimes n}^2 |e_i^{\lambda}(h)|^2 = \sum_{i^{\lambda \geq n}} \beta^n \frac{n!}{\lambda!} |e_i^{\lambda}(h)|^2 \leq \|h^\otimes n\|^2 = \|h\|^2,
\]
\[
\|\exp(h)\|_\beta^2 = \sum_{n \geq 0} \frac{1}{n!} \sum_{i^{\lambda \geq n}} \beta^n \frac{n!}{\lambda!} |e_i^{\lambda}(h)|^2 \leq \exp(\|h\|^2) \leq \exp(\|h\|^2) \leq \|h\|^2 \tag{14}.
\]

Hence, (12), (13) are convergent in \( \Gamma_\beta(\mathcal{H}) \). This implies that \( h \mapsto \exp(h) \) is analytic and inclusions \( \Gamma(\mathcal{H}) \hookrightarrow \Gamma_\beta(\mathcal{H}) \) and \( H^\otimes n \hookrightarrow H_\beta^\otimes n \) are contractive. By the polarization formula (11) their ranges are dense.

Using the binomial formula \( (h + za)^\otimes n = \bigoplus_{m=0}^n (m)(za)^\otimes m \odot h^\otimes (n-m) \), we find
\[
\partial_a^m \exp(h) = \frac{d^m \exp(h + za)}{dz^m} \bigg|_{z=0} = \bigoplus_{n \geq m} S_{n/m}[a^{\otimes m} \odot h^\otimes (n-m)] \odot (n-m)! = \odot (n-m)! \in \mathbb{C}
\]
with the orthogonal projector \( S_{n/m} \) defined as \( \psi_m \odot \psi_{n-m} = S_{n/m} (\psi_m \odot \psi_{n-m}) \in H_\beta^\otimes n \) for all \( \psi_m \in H_\beta^\otimes m \) and \( \psi_{n-m} \in H_\beta^\otimes (n-m) \). By orthogonality \( \|S_{n/m}\| \leq 1 \).
Applying the expansions (12) to $a^{\otimes m}$ and $h^{\otimes(n-m)}$, by (22), we get

$$\|a^{\otimes m} \otimes h^{\otimes(n-m)}\|_{H^2_{\beta}}^2 = \sum_{\lambda \vdash n\, m} \binom{m!}{\lambda!} \binom{m!(n-m)!}{\mu!} \|\epsilon_{i}^{\otimes \lambda} \otimes \epsilon_{j}^{\otimes \mu}\|_{H^2_{\beta}}^2 \|\epsilon_{i}^{\star} \lambda(a)\|^{2} \|\epsilon_{j}^{\star} \mu(h)\|^2$$

with summations over semistandard tableaux $[\lambda]$, $[\mu]$ and $i, j \in \mathcal{S}$. Let $(\lambda, \mu) \in \mathbb{N}^{\eta(\lambda, \mu)}$ be the smallest partition of number $n$ with the length $\eta(\lambda, \mu)$ containing the partitions $\lambda$ for $m$ and $\mu$ for $n-m$. Then $\eta(\lambda, \mu) \geq \max\{\eta(\lambda), \eta(\mu)\}$ and so

$$\|\epsilon_{i}^{\otimes \lambda} \otimes \epsilon_{j}^{\otimes \mu}\|_{H^2_{\beta}}^2 = \frac{(\eta(\lambda, \mu) - 1)!}{(\eta(\lambda, \mu) - 1 + n)!} \leq \min\{\gamma_{\lambda}, \gamma_{\mu}\},$$

since $\frac{(\eta-1)!}{(\eta-1+n)!}$ is decreasing in variable $\eta$. Thus, the following inequality

$$\|a^{\otimes m} \otimes h^{\otimes(n-m)}\|_{H^2_{\beta}}^2 \leq \sum_{\lambda \vdash n\, m} \binom{m!}{\lambda!} \binom{m!(n-m)!}{\mu!} \min\{\eta_{\lambda, \mu}\} \|\epsilon_{i}^{\star} \lambda(a)\|^{2} \|\epsilon_{j}^{\star} \mu(h)\|^2$$

holds. Using this inequality and that $\|S_{n/m}\| \leq 1$, we find

$$\|\partial_{a}^{m} \exp(h)\|_{\beta}^2 = \sum_{n \geq m} \|S_{n/m}[a^{\otimes m} \otimes h^{\otimes(n-m)}]\|_{\beta}^2 \leq \sum_{n \geq m} \|S_{n/m}\|^{2} \|a^{\otimes m} \otimes h^{\otimes(n-m)}\|_{\beta}^2$$

$$\leq \|a^{\otimes m}\|^{2} \sum_{n \geq m} \frac{\|S_{n/m}\|^{2} \|h^{\otimes(n-m)}\|_{\beta}^2}{(n-m)!} \leq \|a\|^{2m} \|\exp(h)\|_{\beta}^2.$$  

Summing with coefficients $1/m!$, we get $\|T_{a} \exp(h)\|_{\beta}^2 \leq \exp\left(\|a\|^{2}\right) \|\exp(h)\|_{\beta}^2$. This inequality and totality of $\{\exp(x) : h \in H\}$ in $\Gamma_{\beta}(H)$ yield the required inequality (24). It also follows that $\Gamma_{\beta}(H)$ is invariant under $T_{a}$ and that the group property (24) holds, since $\partial_{a+b} = \partial_{a} + \partial_{b}$ for all $a, b \in H$ by linearity.

\[\square\]

**Lemma 4.** The mapping $\phi : H \ni h \mapsto \phi_{h} \in L^{2}_{\chi}$, extended onto $T_{a} \exp(h)$ as

$$\Phi : T_{a} \exp(h) \mapsto \sum_{n \geq 0} \frac{1}{n!} \sum_{\lambda \vdash n} \frac{n!}{\lambda!} \phi_{\lambda}^{\otimes \lambda}(h + a), \quad a \in H,$$

has the unique isometric conjugate-linear extension

$\Phi : \Gamma_{\beta}(H) \ni \psi \mapsto \Phi \psi \in L^{2}_{\chi}$ with the adjoint mapping $\Phi^{*} : L^{2}_{\chi} \to \Gamma_{\beta}(H)$ defined to be $\langle \Phi \epsilon_{i}^{\otimes \lambda} | f \rangle_{\chi} = \langle \epsilon_{i}^{\otimes \lambda} | \Phi^{*} f \rangle_{\beta}$ for all $f \in L^{2}_{\chi}$ in such way that

$$\tilde{\Phi} : \epsilon_{i}^{\otimes \lambda}/\|\epsilon_{i}^{\otimes \lambda}\|_{\beta} \mapsto \phi_{\lambda}/\|\phi_{\lambda}\|_{\chi} \quad \text{for all} \quad \lambda \in \mathcal{Y}, \quad i \in \mathcal{S}_{\eta(\lambda)}.$$

As a result, the conjugate-linear isometries $\Gamma_{\beta}(H) \hat{\Phi} \simeq L^{2}_{\chi}$ and $H^{\otimes n} \hat{\Phi} \simeq L^{2n}_{\chi}$ hold.
Proof. By Lemma 3 the $\Gamma_\beta(H)$-valued function $H \ni h \mapsto \tau_a \exp(h)$ is well defined for all $a \in H$. Let us use the expansion $\phi_{h+a} = \sum \chi_i^* (h + a) \phi_i$. By Lemma 2 and Theorem 1, $\phi: H \ni h \mapsto \phi_h \in L^2_\chi$ may be extended to $\Phi$ in the following way

$$\Phi \tau_a \exp(h) = \sum_{n \geq 0} \frac{1}{n!} \sum_{r \vdash n} \frac{n!}{\lambda!} \phi_i^* \chi_i^\lambda (h + a) = \prod_{i \geq 0} \sum_{n \geq 0} \frac{\phi_i^n}{n!} \chi_i^\lambda (h + a)$$

$$= \prod \exp(\phi_i^* (h + a)) = \exp(\phi_{h+a})$$

where

$$\Phi[(h + a)^\otimes n] = \phi_{h+a}^n = \sum_{r \vdash n} \frac{n!}{\lambda!} \phi_i^* \chi_i^\lambda (h + a), \quad a \in H$$

is an orthogonal component of $\Phi \tau_a \exp(h)$ in $L^2_\chi$. It follows that

$$\| \exp(\phi_{h+a}) \|^2_\chi = \sum_{n \geq 0} \frac{1}{n!^2} \sum_{r \vdash n} \| \phi_i^* \|^2_\chi \frac{n!^2}{\lambda!^2} |\chi_i^\lambda (h + a)|^2$$

$$= \sum_{n \geq 0} \frac{1}{n!^2} \sum_{r \vdash n} \frac{n!^2}{\lambda!} |\chi_i^\lambda (h + a)|^2 \leq \sum_{n \geq 0} \frac{1}{n!} \sum_{r \vdash n} \frac{n!}{\lambda!} |\chi_i^\lambda (h + a)|^2$$

$$= \prod \exp |\chi_i^* (h + a)|^2 = \exp \| h + a \|^2.$$

Hence, the composition $\mathfrak{U} \ni u \mapsto \{ \Phi \exp(h + a)(u) \}$ is well defined in $L^2_\chi$.

Now, we consider the ordinary irreducible representation of permutation group $S_n$ on the Specht $\lambda$-module $S^\lambda_i$ that is corresponded to the standard Young tableau $[\lambda]$. The following known hook formula (see [8, 1.4.3]) holds,

$$h_\lambda := n! \left( \prod_{i \leq \lambda_j} h(i,j) \right)^{-1} \quad \text{where} \quad h_\lambda = \dim S^\lambda_i,$$

(25)

with $h(i,j) = \# \{ \square_{i',j'} \in [\lambda]: i' \geq i, j' \geq j \} = \# \{ \square_{i',j'} \in [\lambda]: i' = i, j' \geq j \}$ independent of $i \in \mathcal{I}$. Assign to $i \in \mathcal{I}_\eta$ the vectors

$$\left( \phi_{\lambda_i}(u) \chi_i^* (h), \ldots, \phi_{\lambda_n}(u) \chi_n^* (h) \right) := t_i(u, h).$$

Let $s^\lambda_i(u, h) := s^\lambda_i(t_i)$ with $t_i = t_i(u, h)$ for all $u \in \mathcal{U}$, where polynomial terms are $\phi_i^\lambda(u) \chi_i^\lambda (h) = \phi_{i_1}^\lambda(u) \chi_i^\lambda_1 (h) \ldots \phi_{i_n}^\lambda(u) \chi_i^\lambda_n (h)$. Applying the Frobenius formula [18, 1.7] and taking into account (2), (3), (25), we obtain

$$\phi_{\beta}^n(u) = \sum_{r \vdash \lambda} h_\lambda s^\lambda_i(u, h), \quad h \in H$$

where $s^\lambda_i = 0$ if $\lambda_i > l_\lambda$ and the summation is over all standard tabloids. Hence, the $\{ \phi_{\beta}^n: h \in H \}$ is total in $L^2_\chi$ by Theorem 1. In consequence, $\{ \exp(\phi_h): h \in H \}$ is total in $L^2_\chi$. This yields surjectivity of $\Phi$ and of all its restrictions to $H_\beta^\otimes n$.

Corollary 2. The sets $\{ \phi_{\beta}^n: h \in H \}$ in $L^2_\chi$ and $\{ \exp(\phi_h): h \in H \}$ in $L^2_\chi$ are total.
6. Fourier Analysis on Virtual Unitary Matrices

Consider the isometry \( H^\otimes n \cong P^n_\beta(H) \) (see e.g., [7, 1.6]), where the space \( P^n_\beta(H) \) of unitarily-weighted \( n \)-homogeneous Hilbert–Schmidt polynomials of variable \( h \in H \) is defined to be a restriction to the diagonal in \( H \times \ldots \times H \) of the \( n \)-linear forms \( \mathcal{P} \circ \psi_n \) endowed with the norm \( \| \psi_n^* \|_{P^n_\beta} = \| \psi_n \|_{H^\otimes n} \) where

\[
\psi_n^*(h) := \langle h^{\otimes n} | \psi_n \rangle_{H^\otimes n} \cong (\langle h, \ldots, h \rangle | \mathcal{P} \circ \psi_n), \quad \psi_n \in H^\otimes n.
\]

Let \( H^2_\beta = \sum_{n \geq 0} P^n_\beta(H) \) be the direct sum of functions \( \psi^*(h) = \sum \psi_n^*(h) \) of variable \( h \in H \) with summands \( \psi_n^* = \mathcal{P} \circ \psi_n \in P^n_\beta(H) \) where \( \psi = \sum \psi_n \in \Gamma_\beta(H) \). Since the set \( \{\exp(h): h \in H\} \) is total in \( \Gamma_\beta(H) \), elements of \( H^2_\beta \) can be written as

\[
H^2_\beta = \left\{ \psi^*(h) = \langle \exp(h) | \psi \rangle_\beta : \psi = \sum \psi_n \in \Gamma_\beta(H) \right\}.
\]

The analyticity of \( H \ni h \mapsto \psi^*(h) \) is a result of the composition \( \exp(\cdot) \) and \( \psi^*(\cdot) \).

**Definition 3.** Let \( H^2_\beta \) be defined as a Hardy space of unitarily-weighted Hilbert–Schmidt analytic functions \( \psi^*(h) \) of variable \( h \in H \) endowed with the inner product

\[
\langle \psi^*(\cdot) | \varphi^*(\cdot) \rangle_{H^2_\beta} := \langle \varphi | \psi \rangle_\beta \quad \text{where} \quad \| \psi^* \|^2_{H^2_\beta} = \langle \psi^*(\cdot) | \psi^*(\cdot) \rangle_{H^2_\beta} = \sum n! \| \psi_n^* \|^2_{P^n_\beta}.
\]

The conjugate-linear surjective isometry from \( H^2_\beta \) onto \( \Gamma_\beta(H) \) is realized by the conjugate-linear mapping

\[
*: \Gamma_\beta(H) \ni \psi \longmapsto \psi^* \in H^2_\beta, \quad \psi = \sum \psi_n.
\]

On the other hand, the correspondence \( \Phi: \mathcal{E}_i^{\circ \lambda} \cong \phi_i^\lambda \) with \( \lambda \in \mathbb{Y} \) and \( i \in \mathcal{J}_{\eta(\lambda)} \) allows us to determine the conjugate-linear isometry from \( \Gamma_\beta(H) \) onto \( L^2_\chi \). As a result, the mapping

\[
\Psi: H^2_\beta \ni \mathcal{E}_i^{\circ \lambda} / \| \mathcal{E}_i^{\circ \lambda} \|_\beta \longmapsto \phi_i^\lambda / \| \phi_i^\lambda \|_\chi \in L^2_\chi
\]

defines the surjective isometry

\[
\Psi: H^2_\beta \longrightarrow L^2_\chi \quad \text{and its adjoint} \quad \Psi^*: L^2_\chi \longrightarrow H^2_\beta.
\]

**Lemma 5.** The systems of Hilbert–Schmidt polynomials of variable \( h \in H \),

\[
\mathcal{E}_i^{\circ \lambda} := \bigcup \{ \mathcal{E}_i^{\circ \lambda}: i \vdash n, i \in \mathcal{J}\} \quad \text{and} \quad \mathcal{E}_i^\lambda := \bigcup \{ \mathcal{E}_i^\lambda: n \in \mathbb{N}_0 \}
\]

where \( \mathcal{E}_i^0 = 1 \), form orthogonal bases in \( P^n_\beta(H) \) and \( H^2_\beta \), respectively, such that

\[
\| \mathcal{E}_i^{\circ \lambda} \|^2_{P^n_\beta} = \beta_\lambda \| \mathcal{E}_i^{\circ \lambda} \|^2 = \frac{(\eta(\lambda) - 1)! \lambda!}{(\eta(\lambda) - 1 + n)! n!}, \quad i \vdash n.
\]
Every function $\psi^* \in H^2_\beta$ with $\psi \in \Gamma_\beta(H)$ has the expansion with respect to $\mathfrak{e}^* \mathfrak{Y}$

$$\psi^*(h) = \langle \exp(h) \mid \psi \rangle_\beta = \sum_{n \geq 0} \frac{1}{n!} \sum_{\nu \vdash n} \frac{n!}{\lambda!} \epsilon^*_{i^\lambda}(h) \langle \mathfrak{e}^{\circ \lambda} \mid \psi_n \rangle_\beta$$

(26)

with summation in the inner sum over all semistandard tabloids $[\nu^\lambda]$ such that $\nu^\lambda \vdash n$. Each function $\psi^* \in H^2_\beta$ is entire Hilbert–Schmidt analytic and can be also written as

$$\psi^*(h) = \langle \psi^*(\cdot) \mid \exp(\cdot \mid h) \rangle_{H^2_\beta} = \langle \psi^*(\cdot) \mid E(\cdot, h) \rangle_{H^2_\beta}, \quad \psi \in \Gamma_\beta(H)$$

where $E(h', h) := |\exp(h' \mid h)|^2/|\exp(h \mid h)|$ for all $h \in H$. 

(27)

The following linear isometries, defined by linearization via coherent states, hold

$$H^2_\beta \xrightarrow{\psi} L^2_\chi, \quad P^n_\beta(H) \xrightarrow{\psi} L^2_{\chi^n}. \quad (28)$$

Proof. Taking into account (13) and (23), we conclude that every $\psi^* \in H^2_\beta$ such that $\psi = \bigoplus \psi_n \in \Gamma_\beta(H)$ with $\psi_n \in H^{\circ n}_\beta$ has the following expansion

$$\psi^*(h) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\nu \vdash n} \frac{n!}{\lambda!} \epsilon^*_{i^\lambda}(h) \epsilon^{\circ \lambda}(\mathfrak{e}_n) \mid \psi_n \rangle_\beta \quad \text{where} \quad \psi = \bigoplus_{n \geq 0} \sum_{\nu \vdash n} \frac{\epsilon^{\circ \lambda} \mid \psi_n \rangle_\beta}{\| \epsilon^{\circ \lambda} \|_\beta^2}.$$ 

On the other hand, in relative to the inner product $\langle \cdot \mid \cdot \rangle_\Gamma$, we have

$$\exp(h' \mid h) = \bigoplus_{n \geq 0} \frac{1}{n!} \sum_{\nu \vdash n} \frac{n!}{\lambda!} \epsilon^*_{i^\lambda}(h') \epsilon^*_{i^\lambda}(h) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\nu \vdash n} \epsilon^*_{i^\lambda}(h') \epsilon^*_{i^\lambda}(h) \| \epsilon^{\circ \lambda} \|_\beta^2.$$ 

Verify the first equality in (27) by substituting (26) into the formula (27). We get

$$\psi^*(h) = \left\langle \sum_{n \geq 0} \sum_{\nu \vdash n} \frac{\epsilon^{\circ \lambda} \mid \psi_n \rangle_\beta}{\| \epsilon^{\circ \lambda} \|_\beta^2} \epsilon^*_{i^\lambda}(h') \mid \sum_{n \geq 0} \frac{1}{n!} \sum_{\nu \vdash n} \frac{\epsilon^*_{i^\lambda}(h') \epsilon^*_{i^\lambda}(h)}{\| \epsilon^{\circ \lambda} \|_\beta^2} \right\rangle_{H^2_\beta}$$

$$= \sum_{n \geq 0} \frac{1}{n!} \sum_{\nu \vdash n} \frac{n!}{\lambda!} \epsilon^*_{i^\lambda}(h) \epsilon^{\circ \lambda}(\mathfrak{e}_n) \mid \psi_n \rangle_\beta = \langle \exp(h) \mid \psi \rangle_\beta.$$ 

If $\omega(h') := \psi^*(h) \exp(h \mid h')\exp(h \mid h')^{-1}$ then $\omega^*(h) = \psi^*(h)$ for $h = h' \in H$. Now, putting $\omega^*(h') := \langle \psi^*(\cdot) \mid \exp(h' \mid \cdot) \exp(h' \mid h')^{-1} \exp(\cdot \mid h') \rangle_{H^2_\beta}$ we obtain

$$\psi^*(h) = \omega^*(h) = \langle \omega^* \mid \exp(\cdot \mid h) \rangle_{H^2_\beta}$$

$$= \langle \psi^*(\cdot) \mid \exp(h \mid \cdot) \exp(h \mid h) \rangle_{H^2_\beta} = \langle \psi^*(\cdot) \mid E(\cdot, h) \rangle_{H^2_\beta}.$$ 

Hence, the second equality in (27) holds. Lemma 4 yields (28).

□

Remark 1. Since $\phi_h = \sum \epsilon^*_i(h) \phi_i$ for all $h = \sum \epsilon^*_i(h) \epsilon_i$, a range of the embedding (21) coincides with $L^2_{\chi^1}$. 


Lemma 6. Denote \( \exp\langle h' \mid h \rangle := K(h', h) \). The functions

\[ H \ni h \mapsto \Psi(\Psi \circ K)(u, h) \quad \text{and} \quad H \ni h \mapsto \Psi(\Psi \circ E)(u, h) \]

with \( u \in \mathcal{U} \) take values in \( L^2_\chi \) and can be represented as follows

\[ \Psi(\Psi \circ K)(u, h) = \exp(\phi_h(u)) \quad \text{and} \quad \Psi(\Psi \circ E)(u, h) = \exp(2 \Re \phi_h(u) - \|h\|^2) \]

where the last exponential function has the power series expansion

\[ \exp \{ 2 \Re \phi_h - \|h\|^2 \} = \sum_{m,n \geq 0} \frac{\|h\|^{m+n}}{m!n!} h_{n,m}(\phi_h/\|h\|, \bar{\phi}_h/\|h\|) \]

which we take values in \( H \ni f \mapsto C \). The functions \( h_{n,m}(z, \bar{z}) \) for \( z \in \mathbb{C} \).

Proof. Applying the transform \( \Psi \) to \( K(h', h) \) in variable \( h' \in H \), we obtain

\[ \Psi(\Psi \circ K)(u, h) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\lambda \vdash n} \frac{n!}{\lambda!} \phi_{\lambda}(u) \epsilon_{\lambda}(h) = \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{\lambda \vdash n} \phi_{\lambda}(u) \epsilon_{\lambda}(h) \right)^n = \exp(\phi_h(u)). \]

Similarly, applying \( \Psi \) to \( E(h', h) \) in variable \( h' \in H \), we obtain

\[ \Psi(\Psi \circ E)(u, h) = \left | \sum_{n \geq 0} \frac{1}{n!} \sum_{\lambda \vdash n} \frac{n!}{\lambda!} \phi_{\lambda}(u) \epsilon_{\lambda}(h) \right |^2 \left( \sum_{n \geq 0} \frac{1}{n!} \sum_{\lambda \vdash n} \frac{n!}{\lambda!} |\epsilon_{\lambda}(h)|^2 \right)^{-1} = \exp(2 \Re \phi_h(u) - \|h\|^2). \]

By Lemma 4, \( \Psi(\Psi \circ K)(\cdot, h) \) and \( \Psi(\Psi \circ E)(\cdot, h) \) with \( h \in H \) take values in \( L^2_\chi \). The expansion (29) follows from [13, n.12] where polynomials \( h_{n,m}(z, \bar{z}) \) were introduced.

Theorem 2. For any \( f = \sum f_n \in L^2_\chi \) with \( f_n \in L^2_\chi \) the entire function

\[ \hat{f}(h) := \langle \exp(h) \mid \Phi^* f \rangle \beta \quad \text{of variable} \quad h \in H \]

and its Taylor coefficients at zero \( d_0^n \hat{f} \) have the integral representations

\[ \hat{f}(h) = \int \exp(\phi_h) f \, d\chi = \int \exp(2 \Re \phi_h - \|h\|^2) f \, d\chi, \]

\[ d_0^n \hat{f}(h) = \int \bar{\phi}_h^n f_n \, d\chi, \]

respectively. The Fourier transform \( F : L^2_\chi \ni f \mapsto \hat{f} \in H^2_\beta \) provides the isometries

\[ L^2_\chi \overset{E}{=} H^2_\beta \quad \text{and} \quad L^2_\chi^n \overset{E}{=} P^n_\beta(H). \]
Proof. Since $\Psi = \Phi \circ \ast^{-1}$, we obtain $\Psi^* = \ast \circ \Phi^*$. From (27) it follows that

$$\hat{f}(h) = \langle \exp(h) \mid \Phi \ast f \rangle_\beta = \langle (\Psi^* \circ f)(\cdot) \mid K(\cdot, h) \rangle_{H^2_\beta} = \langle (\Psi^* \circ f)(\cdot) \mid E(\cdot, h) \rangle_{H^2_\beta}. $$

Thus,

$$\hat{f}(h) = \langle (\Psi^* \circ f)(\cdot) \mid K(\cdot, h) \rangle_{H^2_\beta} = \langle (\Psi^* \circ f)(\cdot) \mid E(\cdot, h) \rangle_{H^2_\beta}$$

$$= \langle f(\cdot) \mid (\Psi \circ E)(\cdot, h) \rangle_{\chi} = \int \exp \left( 2 \Re \phi_h - \|h\|^2_{H^2_\beta} \right) f \, d\chi$$

by Lemma 6. On the other hand, according to the same claim

$$\hat{f}(h) = \langle (\Psi^* \circ f)(\cdot) \mid K(\cdot, h) \rangle_{H^2_\beta} = \langle f(\cdot) \mid (\Psi \circ K)(\cdot, h) \rangle_{\chi} = \int \exp \left( \phi_h \right) f \, d\chi.$$

It particularly follows that for all $h = \alpha x$ with $x \in H$,

$$\hat{f}(\alpha x) = \int \exp \left( \phi_{\alpha x} \right) f \, d\chi = \sum \alpha^n \int \frac{\phi_{\alpha x}^n}{n!} f_n \, d\chi, \quad \alpha \in \mathbb{C}.$$ 

Using the $n$-homogeneity of derivatives, we find

$$d_0^n \hat{f}(\alpha x) = \frac{d^n}{d\alpha^n} \sum \alpha^n \int \frac{\phi_{\alpha x}^n}{n!} f_n \, d\chi |_{\alpha = 0} = \int \phi_{\alpha x}^n f_n \, d\chi.$$

Finally, we notice that the isometry $L^2_\chi \sim H^2_\beta$ holds, since the isometry $\Phi^*$ is surjective by Lemma 5. Similarly, we get $L^2_\chi \sim P^\beta (H).$ \hfill $\Box$

Corollary 3. For any $h \in H$ the Paley–Wiener map $\phi_h$ satisfies the equality

$$\int \exp \left\{ \Re \phi_h \right\} d\chi = \exp \left\{ \frac{1}{4} \|h\|^2 \right\}.$$

Proof. It is enough to put $f \equiv 1$ and to replace $h$ by $h/2$ in the formula (30). \hfill $\Box$

Corollary 4. The isometry $\ast : H^2_\beta \rightarrow H^2_\beta$ has the factorization $\ast = F \circ \Phi$.

Proof. In fact, $\Phi : H^2_\beta \ni \psi \mapsto \Phi \psi = f \in L^2_\chi$ and $F : L^2_\chi \ni f \mapsto \hat{f} \in H^2_\beta$. \hfill $\Box$

Corollary 5. For every $f \in L^2_\chi$ the Taylor expansion at zero of the function

$$\hat{f}(h) = \sum \frac{1}{n!} d_0^n \hat{f}(h)$$

with $f = \sum f_n \in L^2_\chi$, $f_n \in L^{2,n}_\chi$ has the coefficients

$$d_0^n \hat{f}(h) = \int f_n \phi_n^h \, d\chi = \sum_{\nu \vdash n} h_\nu s^\lambda [f, \epsilon^\dagger (h)], \quad f_i := \int f \phi_i \, d\chi$$

with summation over all standard Young tabloids $[\nu^\lambda]$ such that $\nu \vdash n$ where $s^\lambda = 0$ if the conjugate partition $\lambda^\dagger$ has $\lambda^\dagger_1 > \eta(\lambda)$ and $s^\lambda [f, \epsilon^\dagger (h)] := s^\lambda (t_i)$ with $t_i = f_i, \epsilon^\dagger (h)$. 

Remark 2. In the finite-dimensional case where the space $L^2(U)$, the multiplicative group onto the total set $\{U\}$, the shift group on $L^2(U)$ provides the surjective isometry $F$.

By the Frobenius formula [18, I.7] we find that $\phi_h^n(u) = \sum_{\lambda \vdash n} h_{\lambda} s_\lambda^h(u, h)$, where $s_\lambda^h = 0$ if $\lambda^T > \eta(\lambda)$, and $s_\lambda^h(u, h)$ is defined by (3), whereas $h_{\lambda}$ by (25). Thus,

$$\exp \phi_h(u) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\lambda \vdash n} h_{\lambda} s_\lambda^h(u, h) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\lambda \vdash n} \frac{n!}{\lambda!} \phi_h^n(u) \epsilon^\lambda(h).$$  \hspace{1cm} (32)

Using (32) in combination with Theorem 1, we find

$$\hat{f}(h) = \int f(u) \exp \phi_h(u) d\chi(u) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\lambda \vdash n} h_{\lambda} s_\lambda^h[f, \epsilon^\lambda(h)]$$

where the derivative at zero may be defined as

$$d_0^n \hat{f}(h) = \sum_{\lambda \vdash n} h_{\lambda} s_\lambda^h[f, \epsilon^\lambda(h)] \text{ with } s_\lambda^h[f, \epsilon^\lambda(h)] := \int f(u) s_\lambda^h(u, h) d\chi(u).$$

In fact, for $zh$ with $z \in \mathbb{C}$ and $\lambda \vdash n$ with $\lambda^T > \eta(\lambda)$ we find

$$s_\lambda^h[f, \epsilon^\lambda(zh)] = z^n s_\lambda^h[f, \epsilon^\lambda(h)].$$

Hence, the derivative $d_0^n \hat{f}(h) = (d^n/dz^n) \hat{f}(zh)|_{z=0}$ is a Taylor coefficient of $\hat{f}$.

Now, the Frobenius formula and Theorem 1 yield the first equality in (31). By Lemmas 5 and 6 the second formula in (31) also holds. \hspace{1cm} $\square$

Remark 2. In the finite-dimensional case $\mathfrak{U} = U(m)$, the Hardy space $H^2_\beta$ of entire analytic functions of variable $h \in \mathbb{C}^m$ has the following orthogonal basis $\{e^{*\lambda} = e^{*\lambda_1} \ldots e^{*\lambda_m} : \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Y}\}$. The Fourier transform

$$\hat{f}(h) = \int \exp(\phi_h)f d\chi_m = \int \exp(2 \text{Re} \phi_h - \|h\|^2)f d\chi_m, \hspace{1cm} h \in \mathbb{C}^m$$

provides the surjective isometry $F: L^2_{\chi_m} \ni f \mapsto \hat{f} \in H^2_\beta$, defined by mappings

$$F: e^{*\lambda} \mapsto \phi^{*\lambda} \text{ such that } \|e^{*\lambda}\|_{H^2_\beta}^2 = \|\phi^{*\lambda}\|_{\chi_m}^2 = \frac{(m-1)!\lambda!}{(m-1+|\lambda|)!}$$

where the space $L^2_{\chi_m}$ with the Haar measure $\chi_m$ on $U(m)$ has the orthogonal basis $\{\phi^{*\lambda} = \phi^{*\lambda_1} \circ \pi_m^{-1} \ldots \phi^{*\lambda_m} \circ \pi_m^{-1} : \lambda \in \mathbb{Y}\}$.

7. Intertwining Properties of Fourier Transform

The shift group on $H^2_\beta$ is defined as $T_a \psi^*(h) := \langle T_a \exp(h) | \psi \rangle_\beta$ for all $\psi \in \mathcal{I}_\beta(H)$, $a, h \in H$. By (27), $\langle T_a \exp(h) | \psi \rangle_\beta = T_a \psi^*(h) = \langle T_a \psi^*(\cdot) | \exp(\cdot | h) \rangle_{H^2_\beta}$. Hence,

$$T_a \psi^*(h) = \langle T_a \exp(h) | \psi \rangle_\beta = \langle \psi^*(\cdot) | \exp(\cdot | h + a) \rangle_{H^2_\beta} = \langle \psi^*(\cdot) | M_{a^*} \exp(\cdot | h) \rangle_{H^2_\beta}$$

where $M_{a^*} \exp(\cdot | h) := \exp a^*(\cdot) \exp(\cdot | h) = \exp(\cdot | h + a)$ is defined to be the multiplicative group onto the total set $\{\exp(\cdot | h) : h \in H\}$ in $H^2_\beta$. 

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Comparing the above formulas, we obtain that $M_a^*$ is adjoint to $T_a$ on $H^2_\beta$. By virtue of adjoint relations, $\|T_a \psi^*\|_{H^2_\beta} = \|M_a^* \psi^*\|_{H^2_\beta}$. The isometry $H^2_\beta \simeq \Gamma_\beta(H)$ yields $\|T_a \psi^*\|_{H^2_\beta} = \|T_a \psi\|_\beta$. According to (24), we have

$$\|T_a \psi^*\|_{H^2_\beta}^2 \leq \exp(\|a\|^2)\|\psi^*\|_{H^2_\beta}^2$$

and $T_{a+b} = T_a T_b = T_b T_a$

$$\|M_a^* \psi^*\|_{H^2_\beta}^2 \leq \exp(\|a\|^2)\|\psi^*\|_{H^2_\beta}^2$$

and $M_{a+b}^* = M_a^* M_b^* = M_b^* M_a^*$

(33)

for $a, b \in H$. Thus, these groups are strongly continuous with densely defined closed generators $\partial_a^* \psi^* := \lim_{z \to 0}(T_{za} \psi^* - \psi^*)/z$ and $a^* \psi^* := \lim_{z \to 0}(M_{za} \psi^* - \psi^*)/z$.

Hence, the additive group $(H, +)$ on $H^2_\beta$ is represented by $M_{a^*} : H^2_\beta \to H^2_\beta$ and the generator $dM_{a^*}/dz \big|_{z=0} = a^*$ of its 1-parameter subgroup $M_{a^*}$ is strongly continuous with the dense domain $\mathcal{D}(a^*) = \{ \psi^* \in H^2_\beta : a^* \psi^* \in H^2_\beta \}$.

On the other hand, the group $(H, +)$ can be represented as $M^\dagger_{a^*} = \Psi M_a \Psi^* : L^2_\chi \to L^2_\chi$. The generator of its strongly continuous subgroup

$$\mathbb{C} \ni z \longmapsto M^\dagger_{za^*}, \quad dM^\dagger_{za^*}/dz \big|_{z=0} = \partial_a^\dagger \quad \text{with} \quad \partial_a^\dagger = \Psi \partial_a^* \Psi^*$$

has the dense domain $\mathcal{D}(\partial_a^\dagger) = \{ f \in L^2_\chi : \partial_a^\dagger f \in L^2_\chi \}$ and is closed, since $a^*$ is closed.

The group $(H, +)$ on $L^2_\chi$ can be also represented by $T^\dagger_a := \Psi T_a \Psi^* : L^2_\chi \to L^2_\chi$. From Lemmas 3 and 5 it follows that the generator of strongly continuous subgroup

$$\mathbb{C} \ni z \longmapsto T^\dagger_{za}, \quad dT^\dagger_{za}/dz \big|_{z=0} = \partial_a^\dagger \quad \text{with} \quad \partial_a^\dagger := \Psi \partial_a^* \Psi^*$$

has the dense domain $\mathcal{D}(\partial_a^\dagger) = \{ f \in L^2_\chi : \partial_a^\dagger f \in L^2_\chi \}$ and is closed, since $\partial_a^*$ is closed. By (27) $\hat{f}(h) = \langle \exp(h) \mid \Phi^*f \rangle_{\beta}^* = \langle (\Phi^* \circ f)(\cdot) \mid \exp(\langle \cdot \mid h \rangle) \rangle_{H^2_\beta}$. Hence, by Lemma 6,

$$T^\dagger_a \hat{f}(h) = \langle (\Phi^* \circ f)(\cdot) \mid T_a \exp(\langle \cdot \mid h \rangle) \rangle_{H^2_\beta} = \int f \exp(\langle \phi_{h+a} \rangle) \; d\chi.$$

**Lemma 7.** The additive group $(H, +)$ on $L^2_\chi$ has two representations $a \mapsto M^\dagger_{a^*}$ and $a \mapsto T^\dagger_a$ which are adjoint, strongly continuous with closed densely defined generators $\partial_a^\dagger$ and $\partial_a^\dagger$, respectively. For every $f \in \mathcal{D}(\partial_a^{\dagger m}) = \{ f \in L^2_\chi : \partial_a^{\dagger m} f \in L^2_\chi \}$ with $m \in \mathbb{N}_0$,

$$\partial_a^{\dagger m} T_a F(f) = F(\bar{\partial}_a^{\dagger m} M^\dagger_{a^*} f) \quad a \in H.$$  

(34)

For every $f \in \mathcal{D}(\partial_a^{\dagger m}) = \{ f \in L^2_\chi : \partial_a^{\dagger m} f \in L^2_\chi \}$ with $m \in \mathbb{N}_0$,

$$a^{\dagger m} M_a \Psi^* F(f) = F(\bar{\partial}_a^{\dagger m} T_a^\dagger f), \quad a \in H.$$  

(35)

As a conclusion, $\partial_a^{\dagger} = -i \partial_a^{\dagger}$. Moreover, the following commutation relations hold,$$
M^\dagger_a T^\dagger_b = \exp(a \mid b) T^\dagger_b M^\dagger_a, \quad (\bar{\partial}_a^\dagger b - \partial_b^\dagger \phi_a) f = \langle a \mid b \rangle f,$$

(36)
for all $f$ from the dense subspace $\mathcal{D}(\partial_{\alpha}^{2}) \cap \mathcal{D}(\partial_{b}^{2}) \subset L_{x}^{2}$ and nonzero $a, b \in H$.

**Proof.** Using that $T_{a}$ and $M_{\alpha}$ are adjoint, we find that

$$
\partial_{a}^{m}T_{a}\hat{f}(h) = \int \frac{d^{m}M_{za}\hat{f}}{dz^{m}}|_{z=0} \exp \hat{\phi}_{h} d\chi = \int (\partial_{a}^{m} f) \exp \hat{\phi}_{h} d\chi, \quad m \geq 0
$$

for all $f \in L_{x}^{2}$. This gives (34). Since $M_{\alpha}\psi^{*}(h) = \langle \psi^{*}(\cdot) | M_{\alpha} \exp(\cdot | h) \rangle_{H_{\beta}^{\infty}} = \exp a^{*}(h) \psi^{*}(h)$, we obtain

$$
a^{*}M_{a}\hat{f}(h) = \int (\partial_{a}^{m} f) \exp \hat{\phi}_{h} d\chi \quad \text{with} \quad f \in \mathcal{D}(\partial_{a}^{m}), \quad \psi^{*} = \Psi^{*} f.
$$

This together with the group property by applying $F$ and $F^{-1}$ yields (35).

Now, we prove the commutation relations. For any $f \in L_{x}^{2}$ and $h \in H$, we have

$$
M_{b}T_{a}\hat{f}(h) = \exp(h | b)\hat{f}(h + a),
$$

$$
T_{a}M_{b}\hat{f}(h) = \exp(h + a | b)\hat{f}(h + a) = \exp(a | b)M_{b}T_{a}\hat{f}(h).
$$

For each $\hat{f} \in \mathcal{D}(b^{*}) \cap \mathcal{D}(\partial_{a}^{2})$ and $t \in \mathbb{C}$ by differentiation, we obtain

$$
(d^{2}/dt^{2})T_{ta}M_{tb}\hat{f}|_{t=0} = (\partial_{a}^{2} + 2\partial_{a}^{*}b^{*} + b^{*2})\hat{f}.
$$

Subsequently, taking into account (38) together with $(d/dt)[\exp(ta | i\bar{b})M_{tb}\hat{T}_{ta}] = [(d/dt)\exp(ta | i\bar{b})]M_{tb}\hat{T}_{ta} + \exp(ta | i\bar{b})[(d/dt)M_{tb}\hat{T}_{ta}],$ we find

$$
(\partial_{a}^{2} + 2\partial_{a}^{*}b^{*} + b^{*2})\hat{f} = (d/dt)[(d/dt)\exp(ta | i\bar{b})M_{tb}\hat{T}_{ta}]|_{t=0} = 2\langle a | b \rangle\hat{f} + (\partial_{a}^{2} + 2b^{*}\partial_{a}^{*} + b^{*2})\hat{f}.
$$

Hence, for each $\hat{f}$ from the dense subspace $\mathcal{D}(b^{*}) \cap \mathcal{D}(\partial_{a}^{2}) \subset H_{\beta}^{\infty}$, which includes all polynomials generated by finite sums $\Psi^{*}(f) = \bigoplus \psi_{n} \in T_{\beta}(H)$ with $\psi_{n} \in H_{\beta}^{\infty}$,

$$
T_{a}M_{b} = \exp(a | b)M_{b}T_{a}, \quad (\partial_{a}^{*}b^{*} - b^{*}\partial_{a}^{*})\hat{f} = \langle a | b \rangle\hat{f}. \quad (39)
$$

Corollary 4 yields $F = \ast \circ \Phi^{*}$ and $F^{-1} = \Phi \circ \ast^{-1}$. The equality (37) for $m = 0$ can be rewritten as $M_{b}T_{a}\hat{f}(a) = \langle \exp(a) | T_{b}\Phi^{*}f \rangle_{\beta}$ with $f \in L_{x}^{2}$ or in another way $\ast \circ T_{b} = M_{b} \circ \ast$. Hence, $T_{b} = \Phi T_{b}\Phi^{*} = \Phi \circ \ast^{-1} \circ M_{b} \circ \ast \circ \Phi^{*} = F^{-1}M_{b} F$ and $\partial_{b}^{\dagger} = F^{-1}b^{*} F$. Similarly, $M_{a}^{\dagger} = F^{-1}T_{a} F$ and $\partial_{a}^{\dagger} = F^{-1}\partial_{a} F$. Finally,

$$
M_{a}^{\dagger}T_{b}^{\dagger} - F^{-1}T_{a}M_{b}^{\dagger} F = \exp(a | b)F^{-1}M_{b} \circ T_{a} F = \exp(a | b)T_{b}^{\dagger}M_{a}^{\dagger},
$$

$$
(\partial_{a}^{\dagger}\partial_{b}^{\dagger} - \partial_{b}^{\dagger}\partial_{a}^{\dagger})f = F^{-1}(\partial_{a}^{*}b^{*} - b^{*}\partial_{a}^{*}) F f = \langle a | b \rangle f
$$

for all $f$ from the dense subspace $\mathcal{D}(\partial_{a}^{2}) \cap \mathcal{D}(\partial_{b}^{2}) \subset L_{x}^{2}$, which includes all functions generated by finite sums $\Phi(\bigoplus \psi_{n})$ with $\psi_{n} \in H_{\beta}^{\infty}$. \qed
8. Infinite-Dimensional Heisenberg Group

Our goal is to describe an irreducible representation on the space $L^2_\chi$ of the group $\mathcal{H}_\mathbb{C}$, defined by (1). We will use the appropriate generalization of Weyl’s system which in our case is written in the form of $L^2_\chi$-valued function of variable $h \in H$

$$W^\dagger(h) := W^\dagger(a, b) = \exp \left\{ \frac{1}{2} \langle a | b \rangle \right\} T^\dagger_b M^\dagger_{a, \ast}. \quad \text{(39)}$$

For convenience, we will use the quaternion algebra $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ of numbers $\zeta = (\alpha_1 + \alpha_2i) + (\alpha_1' + \alpha_2'i)j = \alpha + \alpha'j$ such that $i^2 = j^2 = k^2 = ij = k = -1$, $k = i = -j$, $ki = -ik = j$, where $(\alpha, \alpha') \in \mathbb{C}^2$ with $\alpha = \alpha_1 + \alpha_2i, \alpha' = \alpha_1' + \alpha_2'i \in \mathbb{C}$ and $\alpha_1, \alpha'_1 \in \mathbb{R}$ ($i = 1, 2$) [26, 5.5.2]. Let us denote $\chi' := \mathbb{H}\zeta$ for all $\zeta = \alpha + \alpha'j \in \mathbb{H}$.

Consider the Hilbert space $H \oplus H\bar{\jmath}$ with $\mathbb{H}$-valued inner product

$$\langle h | h' \rangle = \langle a + b\bar{\jmath} | a' + b'\bar{\jmath} \rangle = \langle a | a' \rangle + \langle b | b' \rangle + \langle [a' | b] - [a | b'] \rangle \bar{\jmath} \quad \text{(40)}$$

where $h = a + b\bar{\jmath}$ with $a, b \in H$. Hence,

$$\Im \langle h | h' \rangle = \langle a' | b \rangle - \langle a | b' \rangle, \quad \Im \langle h | h \rangle = 0. \quad \text{(41)}$$

**Theorem 3.** The representation of $\mathcal{H}_\mathbb{C}$ over $L^2_\chi$ in the Weyl–Schrödinger form

$$S^\dagger: \mathcal{H}_\mathbb{C} \ni X(a, b, t) \longmapsto \exp(t)W^\dagger(h), \quad h = a + b\bar{\jmath} \quad \text{(42)}$$

is well defined and irreducible. The Weyl system satisfies the relation

$$W^\dagger(h + h') = \exp \left\{ - \frac{\Im \langle h | h' \rangle}{2} \right\} W^\dagger(h)W^\dagger(h') \quad \text{(43)}$$

which on any real subspace $\{\tau h: \tau \in \mathbb{R}\}$ transforms to the 1-parameter group

$$W^\dagger((\tau + \tau')h) = W^\dagger(\tau h)W^\dagger(\tau' h) = W^\dagger(\tau' h)W^\dagger(\tau h) \quad \text{(44)}$$

with the densely defined generator on $L^2_\chi$ of the form $p^\dagger_h := \partial^\dagger_b + \bar{\partial}^\dagger_a$. Moreover, the following commutation relations hold,

$$W^\dagger(h)W^\dagger(h') = \exp \left\{ \Im \langle h | h' \rangle \right\} W^\dagger(h')W^\dagger(h) \quad \text{where} \quad \Im \langle h | h' \rangle = -[p^\dagger_h, p^\dagger_{h'}] \quad \text{with} \quad [p^\dagger_{h, p^\dagger_{h'}} := p^\dagger_{h}p^\dagger_{h'} - p^\dagger_{h'}p^\dagger_{h} \quad \text{(45)}$$

on the dense subspace $\mathfrak{D}(\bar{\partial}_a^2) \cap \mathfrak{D}(\partial_b^2) \subset L^2_\chi$.

**Proof.** Let us consider the auxiliary group $\mathbb{C} \times (H \oplus H\bar{\jmath})$ with multiplication $(t, h)(t', h') = (t + t' - \frac{1}{2} \Im \langle h | h' \rangle, h + h')$ for all $h = a + b\bar{\jmath}, h' = a' + b'\bar{\jmath} \in H \oplus H\bar{\jmath}$. The mapping $G: X(a, b, t) \longmapsto (t - \frac{1}{2}\langle a | b \rangle, a + b\bar{\jmath})$ is a group isomorphism, since

$$G(X(a, b, t)X(a', b', t')) = G(X(a + a', b + b', t + t' + \langle a | b' \rangle)) = (t + t' + \langle a | b \rangle - \frac{1}{2}(\langle a + a' | b + b' \rangle), (a + a') + (b + b')\bar{\jmath})$$

$$= (t + t' - \frac{1}{2}(\langle a | b \rangle + \langle a' | b' \rangle) + \frac{1}{2}(\langle a' | b' \rangle - \langle a | b \rangle), (a + a) + (b + b')\bar{\jmath})$$
Using group properties and the commutation relation (39), we obtain

\[ \exp \left\{ -\frac{3\langle h | h' \rangle}{2} \right\} W(h) W(h') = \exp \left\{ \frac{\langle a | b' \rangle}{2} - \frac{\langle a' | b \rangle}{2} \right\} W(h) W(h') \]

\[ = \exp \left\{ \frac{\langle a | b \rangle}{2} + \frac{\langle a' | b' \rangle}{2} \right\} \exp \left\{ -\frac{3\langle h' | h \rangle}{2} \right\} M_{b*} T_a M_{b'*} T_{a'} \]

\[ = \exp \left\{ \frac{1}{2} \langle a + a' | b + b' \rangle \right\} M_{b + b'*} T_{a + a'} = W(h + h'). \tag{44} \]

Hence, the mapping \( C \times (H \oplus H\hat{j}) \ni (t, h) \mapsto \exp(t)W(h) \) acts as a group isomorphism into the operator algebra over \( H^2_{\beta} \). So, the representation

\[ S: \mathcal{H}_C \ni X(a, b, t) \mapsto \exp(t)W(h) = \exp \left\{ t + \frac{1}{2} \langle a | b \rangle \right\} M_{b*} T_a \]

is also well defined over \( H^2_{\beta} \), as a composition of group isomorphisms.

Let us check the irreducibility. Suppose the contrary. Assume there exist an element \( h_0 \neq 0 \) in \( H \) and an integer \( n > 0 \) such that

\[ \exp \left\{ t + \frac{1}{2} \langle a | b \rangle \right\} \exp \langle c | a \rangle \langle c + b | h_0 \rangle^n = 0 \quad \text{for all} \quad a, b, c \in H. \]

But, this is only possible for \( h_0 = 0 \). It gives a contradiction. Finally, using that

\[ \exp \left\{ t + \frac{1}{2} \langle a | b \rangle \right\} T^\dagger_b M^\dagger_{a*} = F^{-1} \left( \exp \left\{ t + \frac{1}{2} \langle a | b \rangle \right\} M_{b*} T_a \right) F, \]

we obtain that \( S^\dagger = F^{-1} S F \) is irreducible. Applying \( F, F^{-1} \) to (44) we get (40).

Consider the Weyl system \( W^\dagger \) on the space \( L^2_{\chi} \). By (40) we obtain the equality

\[ W^\dagger(h) W^\dagger(h') = \exp \left\{ \frac{3\langle h | h' \rangle}{2} \right\} W^\dagger(h + h') = \exp \left\{ -\frac{3\langle h' | h \rangle}{2} \right\} W^\dagger(h' + h) \]

\[ = \exp \left\{ -3 \langle h' | h \rangle \right\} \exp \left\{ \frac{3\langle h' | h \rangle}{2} \right\} W^\dagger(h' + h) \]

\[ = \exp \left\{ -3 \langle h' | h \rangle \right\} W^\dagger(h') W^\dagger(h). \]

Using this equality, we get (41) for any fixed \( h = a + b\hat{j} \in H \oplus H\hat{j} \). The 1-parameter group \( W^\dagger(\tau a, \tau b) = W^\dagger(\tau h) \) with real \( \tau \) has the generator \( p^\dagger_{a,b} = \frac{d}{d\tau} W^\dagger(\tau h) \bigg|_{\tau=0} = \frac{d}{d\tau} \exp \left\{ \frac{1}{2} \langle \tau a | \tau b \rangle \right\} T^\dagger_{\tau b} M^\dagger_{\tau a*} \bigg|_{\tau=0} = \partial^\dagger + \tilde{\phi}_a. \)
Taking into account the inequalities (33) and that $F$ is isometric, we get
\[ \|W^+(\tau a, \tau b)f\|_\chi^2 \leq \exp (\|\tau a\|^2 + \|\tau b\|^2) \|f\|_\chi^2, \]
for $f \in L^2_\chi$.
Hence, the group $W^+(\tau a, \tau b)$ in variable $\tau \in \mathbb{R}$ is strongly continuous on $L^2_\chi$ and therefore has the dense domain $\mathcal{D}(p^+_h) = \{ f \in L^2_\chi : p^+_h f \in L^2_\chi \}$. Moreover, its generator $p^+_h$ is closed (see, e.g., [22, Prop. 1]). Note also that $p^+_h = \tau p^+_h$ for $\tau \in \mathbb{R}$.

Finally, applying the commutation relation (36) and commutability of group generators in different directions over the dense set $\mathcal{D}(\tilde{\phi}_a^2) \cap \mathcal{D}(\partial_b^2) \subset L^2_\chi$, we have
\[
-\Im \langle h | h' \rangle = \langle a | b' \rangle - \langle a' | b \rangle = \bar{\phi}_a \partial_{b'} - \bar{\phi}_{a'} \partial_b + \partial_{b'} \phi_a - \partial_b \phi_{a'} = (\partial_{b'}^{\dagger} + \phi_a)(\partial_b^{\dagger} + \phi_{a'}') - (\partial_b^{\dagger} + \phi_{a'})(\partial_{b'}^{\dagger} + \phi_a) = [p^+_h, p^{\dagger}_h].
\]

\[\] 9. Heat Equation Associated with Weyl System

In what follows, we will consider the real Banach space $c_0$ and let $\xi_n^*$ be the coordinate functional, i.e., $\xi_n^*(x) = \xi_n$ for $x \in c_0$. Since, the embedding $I : l_2 \rightarrow c_0$ is continuous, the Gelfand triple $l_1 \rightarrow l_2 \rightarrow c_0$ with adjoint $I^*$ holds. The mapping $Q : l_1 \rightarrow c_0$ with $Q := I \circ I^*$ is positive and $(Q\xi^* | Q\xi^*)_{l_2} := \xi^*(Q\xi^*) = \sum n \xi_n^2 = \|\xi\|^2_{l_2}$ where $\xi = Q\xi^* \in \mathcal{H}(Q)$ and $\xi^* \in l_1 = c_0^*$. By the Aronszajn-Kolmogorov decomposition theorem (see e.g., [22, Prop. 1]) the appropriate reproducing kernel Hilbert space can be determined as $\mathcal{H}(Q) = l_2$.

Consider the abstract Wiener space defined by $I : l_2 \rightarrow c_0$. Given $\xi_1^*, \ldots, \xi_n^* \in l^1 = c_0^*$, we assign the family of cylinder sets $O_n^\xi = \{ \xi \in c_0 : (\xi_n^*(\xi), \ldots, \xi_n^*(\xi)) \in \Omega_n \}$ with any Borel $\Omega_n \subset \mathbb{R}^n$ that are not a $\sigma$-field. Define the $\sigma$-additive extension $w$ of the Gaussian measure $\gamma$ onto the Borel $\sigma$-algebra $\mathcal{B}(c_0)$, called futhere the Wiener measure, such that
\[
w(O_n^\xi) := \gamma(O_n^\xi) \quad \text{with} \quad \gamma(O_n) := (2\pi)^{-n/2} \int_{\Omega_n} \exp \left\{ -\|\omega\|^2_{l_2}/2 \right\} d\omega.
\]

By Gross’ theorem [10] there exists a smaller abstract Wiener space \{w_0, \| \cdot \|_{w_0} \} such that injections $l_2 \rightarrow w_0 \rightarrow c_0$ are continuous and the increasing sequence of orthogonal projectors $p_n : l_2 \rightarrow \mathbb{R}^n$ has the extension ($p_n^\sim$) on $w_0$ that is convergent to the identity operator on $w_0$ and $w(w_0) = 1$. The integral of any cylinder function $v : c_0 \rightarrow \mathbb{R}$ such that $v = \rho \circ p_n^\sim$ is defined to be $\int_{O_n^\xi} v \, d\omega = \int_{\Omega_n} \rho \, d\gamma$. The Fernique theorem [6], [15, Thm 3.1] implies that these exist $\varepsilon, \eta > 0$ such that $\| \cdot \|_{w_0}$ satisfies the following conditions with a sufficiently large $K > 0$,
\[
\int_{c_0} \exp \left\{ \varepsilon \|\xi\|^2_{w_0} \right\} d\omega(\xi) < \infty, \quad \omega(\|\xi\|_{w_0} \geq K) \leq \exp \left\{ -\eta K^2 \right\}.
\]
Let us go back to the Weyl system $W^\dagger$. Consider in $L^2_\chi$ the dense subspace $L^+_{\chi} := n \geq 0 \bigoplus_{m=0}^n L^2_{\chi,m}$. Let $a = b = i\xi_m e_m$ with $\xi_m \in \mathbb{R}$. Then by Theorem 3

$$W^\dagger(i\xi_m e_m, i\xi_m e_m) = \exp \{ -\xi_m^2/2 \} T^\dagger_{i\xi_m} M^\dagger_{-i\xi_m}.$$

**Theorem 4.** For any $f \in L^+_{\chi}$ and $\xi = (\xi_m) \in c_0$ there exists the limit

$$W^\dagger f_\xi = \lim_{n \to \infty} W^\dagger_{p_n^\sim}(\xi) f, \quad W^\dagger_{p_n^\sim}(\xi) := \exp \left\{ - \frac{||p_n^\sim(\xi)||^2}{2} w_0 \right\} \prod_{m=1}^n T^\dagger_{i\xi_m} M^\dagger_{-i\xi_m}$$

$w$-almost everywhere on $c_0$ such that the 1-parameter Gaussian semigroup

$$\Phi^\dagger f = \frac{1}{\sqrt{4\pi r}} \int_{c_0} \exp \left\{ - \frac{||\xi||^2}{4r} \right\} W^\dagger_{\xi} f \, dw(\xi), \quad r > 0 \quad (45)$$

on the space $L^+_{\chi}$ is generated by $-\sum (\partial_m^\dagger + \phi_m)^2$ with $\partial_m^\dagger := \partial_m^\dagger_{i\xi_m}$. As a consequence, $w(r) = \Phi^\dagger f$ is unique solution of the Cauchy problem

$$\frac{dw(r)}{dr} = -\sum (\partial_m^\dagger + \phi_m)^2 w(r), \quad w(0) = f \in L^+_{\chi}.$$ \hspace{1cm} (46)

**Proof.** Note that $(M_b T_a)^* = T_a^* M_b^* = M_a T_b^*$, hence, $(\partial_m^\dagger + \phi_m)^* = \partial_m^\dagger + \phi_m$ is self-adjoint for $a = b$, as a generator of the group $W^\dagger(\tau a, \tau a) = \exp \{ ||\tau a||^2/2 \}$ $T^\dagger_{\tau a} M^\dagger_{\tau a}$ with $\tau \in \mathbb{R}$. Replacing $a = b$ by $i\tau a$ with $\tau \in \mathbb{R}$, we obtain that

$$W^\dagger(i\tau a, i\tau a) = \exp \left\{ - \frac{1}{2} \langle \tau a \mid \tau a \rangle \right\} T^\dagger_{i\tau a} M^\dagger_{-i\tau a}$$

has the generator $i(\partial_m^\dagger + \phi_m)$ with self-adjoint $\partial_m^\dagger + \phi_m$. By relations (36), $W^\dagger(i\tau a, i\tau a)$ is unitary.

Lemma 7 implies that $[M^\dagger_{-i\xi_m e_m}, T^\dagger_{i\xi_k e_k}] = 0$ and $[M^\dagger_{-i\xi_m e_m}, M^\dagger_{-i\xi_k e_k}] = 0$, as well as, $[T^\dagger_{i\xi_m e_m}, T^\dagger_{i\xi_k e_k}] = 0$ for any $m \neq k$. In view of the relations (36),

$$[\partial_m^\dagger_{i\xi_m e_m}, \partial_k^\dagger_{i\xi_k e_k}] = 0 \quad \text{if} \quad m \neq k \quad \text{and} \quad [\partial_m^\dagger_{i\xi_m e_m}, \partial_k^\dagger_{i\xi_k e_k}] = -\xi_m^2.$$ \hspace{1cm} (47)

Check that (45) holds. Denote $W^\dagger_{p_n^\sim}(\xi) := \prod_{m=1}^n W^\dagger(i\xi_m e_m, i\xi_m e_m)$ and $T^\dagger_{p_n^\sim}(\xi) := \prod_{m=1}^n T^\dagger_{i\xi_m e_m}$, as well as, $M^\dagger_{p_n^\sim}(\xi) := \prod_{m=1}^n M^\dagger_{-i\xi_m e_m}$ with $\xi = (\xi_m) \in w_0$. Using (33) with the operator norm over $H^2_{\beta}$, we get the inequality

$$\ln \prod_{m=1}^n \|T_{i\xi_m e_m}\|^2_{L^2(H^2_{\beta})} \leq \sum_{m=1}^n \langle \xi_m e_m \mid \xi_m e_m \rangle = \sum_{m=1}^n \xi_m^2 = \|p_n^\sim(\xi)\|^2_{L^2}.$$ 

The relation $T^\dagger_{i\xi_m e_m} = \Psi T_{i\xi_m e_m} \Psi^*$ implies that the left-hand side term above can be changed by $\ln \prod_{m=1}^n \|T^\dagger_{i\xi_m e_m}\|^2_{L^2(L^2_{\chi})}$. For $M^\dagger_{p_n^\sim}(\xi) := \prod_{m=1}^n M^\dagger_{-i\xi_m e_m}$ similarly.

Using the unitarity of groups $W^\dagger(i\xi_m e_m, i\xi_m e_m)$, we find by virtue of (47) that their product $W^\dagger_{p_n^\sim}(\xi) = \exp \left\{ -\|p_n^\sim(\xi)\|^2/2 \right\} T^\dagger_{p_n^\sim}(\xi) M^\dagger_{p_n^\sim}(\xi)$ is also unitary. Taking into account the continuity of $T_0 : l_2 \leftrightarrow w_0$ and that $p_n^\sim$ converges to the
identity mapping on $w_0$, as well as, that $w(w_0) = 1$, we obtain for all $f \in L^+\chi$, $n \geq 0$,
\[\|W^\dagger_{p_n^\sim}(\xi)f\|_\chi \leq \exp \{-\|p_n^\sim(\xi)\|^2_{L^2}/2\}\|f\|_\chi \leq \exp \{-\|I_0\|^2\|\xi\|^2_{w_0}/2\}\|f\|_\chi.\]

The Lebesgue dominated convergence theorem implies that there exists
\(\lim \|W^\dagger_{p_n^\sim}(\xi)f\|_\chi w\)-almost everywhere in variable $\xi \in w_0$ for all $f \in L^2_{\chi,m}$ and $m > 0$. By completeness of $L^2_{\chi,m}$, the limit $W^\dagger f$ is well defined $w$-almost everywhere and
\[\|W^\dagger f\|_\chi \leq \exp \{-\|I_0\|^2\|\xi\|^2_{w_0}/2\}\|f\|_\chi\]
for all $f \in L^+\chi, \xi \in w_0$. (48)

The $\| \cdot \|_\chi$-norm of integrand in (45) is bounded by $\exp \{\|\xi\|^2_{w_0}\}$ with any $\varepsilon > 0$. By Fernique’s theorem and (48), the integral (45) with the Wiener measure $w$ exists for all $f \in L^+\chi$. The equality $w(w_0) = 1$ implies that the integral (45) is absolutely convergent uniformly in variables $r > 0$ on the whole space $c_0$. It provides the $C_0$-property of $G_r$ in variables $r > 0$ on any finite sum $\bigoplus_{m=0}^n L^2_{\chi,m}$.

Prove that the semigroup $G_r$ is generated by $\sum p_{m}^{12}$ with $p_{m}^{1} := i(\partial_{m}^{1} + \overline{\partial}_{m})$. By differentiation of $W^\dagger(i\xi_m a, i\xi_m a)$ at $\xi_m = 0$, we get that its generator coincides with $p_{m}^{1}$. In fact, $W^\dagger(i\xi_m a, i\xi_m a)f = \exp \{\xi_m p_{m}^{1}\}f$ for all $f \in \phi^\chi$.

Applying the next formula for Gamma functions with $\alpha = (\alpha, \ldots, \alpha) \in \mathbb{N}_0^n$
\[\prod_{m=1}^n \frac{1}{\sqrt{4\pi r}} \int \exp \left\{ -\frac{\xi^2_m}{4r} \right\} \xi_m^{2\alpha_m} d\xi_m \big|_{\xi_m=2\sqrt{\tau}x_m} = \prod_{m=1}^n \frac{(2\sqrt{\tau})^2}{\sqrt{\pi}} \int \exp \left\{ -x^2_m \right\} x_m^{2\alpha_m} dx_m = 2^{2n}r^n \prod_{m=1}^n \Gamma \left( \frac{2\alpha_m + 1}{2} \right) = 2^{n}r^n \left( \frac{2\alpha_m - 1}{2\alpha_m} \right)!,\]
we find that for any $L^2_{\chi,m}$-valued cylinder function $h_n = (W^\dagger f) \circ p_{m}^\sim$ we have
\[G_r^\dagger h_n = \prod_{m=1}^n \frac{1}{\sqrt{4\pi r}} \int \exp \left\{ -\frac{\xi^2_m}{4r} \right\} \exp \{\xi_m p_{m}^{1}\} d\xi_m h_n = \sum_{\alpha \in \mathbb{N}_0^n} \prod_{m=1}^n \frac{1}{\sqrt{4\pi r}} \int \exp \left\{ -\frac{\xi^2_m}{4r} \right\} \xi_m^{2\alpha_m} d\xi_m h_n = \sum_{\alpha \in \mathbb{N}_0^n} 2^{n}r^n \prod_{m=1}^n \frac{(2\alpha_m - 1)!}{(\alpha_m - 1)!} \frac{p_{m}^{12}}{(2\alpha_m)!} h_n = \exp \left\{ r \sum_{m=1}^n \right. p_{m}^{12} \left. \right\} h_n.\]

Using (48), we obtain that $0 \leq r \mapsto G_r^\dagger$ is the 1-parameter $C_0$-semigroup on any finite sum $\bigoplus_{m=0}^n L^2_{\chi,m}$ with densely defined closed generator $\sum_{m=1}^n p_{m}^{12}$.

Applying the known relation \([32]\) between the initial problem (46) and the 1-parameter $C_0$-semigroup $G_r^\dagger$, we obtain that the function $w_n(r) = G_r^\dagger f_n$ for any $n \in \mathbb{N}$ solves this problem in the sense that $dG_r^\dagger f_n/dr|_{r=0} = \sum_{m=1}^n p_{m}^{12} f_n$ for all $f_n \in \bigoplus_{m=0}^n L^2_{\chi,m}$. The theorem is proved. \(\square\)
Taking into account the isometries \( H_\beta^2 \overset{\psi}{\sim} L_\chi^2 \) and \( P_\beta^n(H) \overset{\psi}{\sim} L_\chi^{2,n} \) from (28), defined by linearization, we can rewrite the Cauchy problem in polynomial form.

Consider the Weyl system \( W(a, b) = \exp \{ (a \mid b) / 2 \} M_{b,T_a} \) defined by (43) on the dense subspace of polynomials \( P_\beta(H) := \sum_{n \geq 0} P_\beta^n(H) \) in \( H_\beta^2 \), consisting of all finite sums of \( n \)-homogenous polynomials \( \psi^*(h) = \sum \psi^*_n(h) \) of variable \( h \in H \) with components \( \psi^*_n = \mathcal{P} \circ \psi_n \in P_\beta^n(H) \). Replacing \( a \) by \( \tau a \) and \( b \) by \( \tau b \) with real \( \tau \in \mathbb{R} \), we get that \( T_{\tau a} \) and \( M_{\tau b} \) are generated by closed generators on \( P_\beta(H) \),

\[
\partial_{\alpha}^\ast \psi^* = \lim_{\tau \to 0} (T_{\tau a} \psi^* - \psi^*) / \tau \quad \text{and} \quad a^* \psi^* = \lim_{\tau \to 0} (M_{\tau a} \psi^* - \psi^*) / \tau, \quad a, b \in H.
\]

As a consequence, the 1-parameter Weyl system \( W(\tau a, \tau b) \) has the generator

\[
\frac{d}{d\tau} \left. W(\tau a, \tau b) \right|_{\tau = 0} = \frac{d}{d\tau} \exp \left\{ \frac{1}{2} (a \mid b) \right\} \bigg|_{\tau = 0} = b^* + \partial_{a}^\ast
\]
densely defined on \( P_\beta(H) \) such that \( (\tau b)^* + \partial_{\alpha}^\ast = \tau (b^* + \partial_{\alpha}^\ast) \) for real \( \tau \). Let \( W_{p_n^-}(\xi) = \prod_{m=1}^{n} W(i\xi_m \epsilon_m, i\xi_m \epsilon_m), \quad T_{p_n^-}(\xi) = \prod_{m=1}^{n} T_i \xi_m \epsilon_m, \quad M_{p_n^-}(\xi) = \prod_{m=1}^n M - i\xi_m \epsilon_m \).

**Corollary 6.** For all \( \psi^* \in P_\beta(H) \) and \( \xi = (\xi_m) \in c_0 \) there exists the limit

\[
W_n \psi^* = \lim_{n \to \infty} W_{p_n^-}(\xi) \psi^*, \quad W_{p_n^-}(\xi) := \exp \left\{ - \frac{\| p_n^-(\xi) \|_{\mathfrak{w}}^2}{2} \right\} \prod_{m=1}^{n} M - i\xi_m \epsilon_m T_i \xi_m \epsilon_m
\]

\( \mathfrak{w} \)-almost everywhere on \( c_0 \) such that the 1-parameter Gaussian semigroup

\[
\mathfrak{G}_r \psi^* = \frac{1}{\sqrt{4\pi r}} \int_{c_0} \exp \left\{ -\frac{\| \xi \|_{\mathfrak{w}}^2}{4r} \right\} W_n \xi \psi^* d\mathfrak{w}(\xi), \quad r > 0
\]
is generated by \( -\sum (\epsilon_m^* + \partial_{m}^\ast)^2 \). Thus, \( w(r) = \mathfrak{G}_r \psi^* \) is unique solution of the problem

\[
\frac{dw(r)}{dr} = -\sum (\epsilon_m^* + \partial_{m}^\ast)^2 w(r), \quad w(0) = \psi^* \in P_\beta(H)
\]
in the space of Hilbert–Schmidt polynomials \( P_\beta(H) \).

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References

[1] Bargmann, V.: On a Hilbert space of analytic functions and a associated integral transform. Commun. Pure Appl. Math. I(14), 187–214 (1961)
[2] Bargmann, V.: On a Hilbert space of analytic functions and a associated integral transform. Commun. Pure Appl. Math. II(20), 1–101 (1967)
[3] Beltiță, I., Beltiță, D., Măntoiu, M.: On Wigner transforms in infinite dimensions. J. Math. Phys. 57, 1–13 (2016)
[4] Bourbaki, N.: Integration II. Springer, Berlin (2004)
[5] Carando, D., Zalduendo, I.: Linearization of functions. Math. Ann. 328, 683–700 (2004)
[6] Fernique, M.X.: Intégrabilité des vecteurs Gaussiens. C. R. Acad. Sci. Paris, Sér. A 270, 1698–1699 (1970)
[7] Floret, K.: Natural norms on symmetric tensor products of normed spaces. Note Mat. 17, 153–188 (1997)
[8] Fulton, W.: Young Tableaux: With Applications to Representation Theory and Geometry. Cambridge University Press, Cambridge (2008)
[9] Gromov, M., Milman, V.: A topological application of the isoperimetric inequality. Am. J. Math. 105, 843–854 (1983)
[10] Gross, L.: Abstract Wiener spaces. In: Doebner, H.D. (ed.) Proceedings of 5th Berkeley Symposium on Mathematical Statistics and Probability, part I(1). California University Press, pp. 31–42 (1965)
[11] Hall, B.: Quantum Theory for Mathematicians, Graduate Texts in Mathematics, vol. 267. Springer, Berlin (2013)
[12] Holmes, I., Sengupta, A.N.: The Gaussian Radon transform in the classical Wiener space. Commun. Stoch. Anal. 8(2), 247–268 (2014)
[13] Itô, K.: Complex multiple Wiener integral. Jpn. J. Math. 22, 63–86 (1952)
[14] Klenke, A.: Probability Theory. A Comprehensive Course. Springer, Berlin (2008)
[15] Kuo, H.: Gaussian measures in Banach spaces. Lecture Notes, vol. 463. Springer, Berlin (1975)
[16] Lopushansky, O.: The Hilbert–Schmidt analyticity associated with infinite dimensional unitary groups. Results Math. 71(1), 111–126 (2017)
[17] Lopushansky, O.: Paley–Wiener isomorphism over infinite-dimensional unitary groups. Results Math. 72(4), 2101–2120 (2017)
[18] Macdonald, I.G.: Symmetric Functions and Hall Polynomial, 2nd edn. Oxford University Press, Oxford (2015)
[19] Neeb, K.-H.: Holomorphy and Convexity in Lie Theory. De Gruyter Expositions in Mathematics, vol. 28, Berlin (2000)
[20] Nelson, E.: Regular probability measures on functions space. Ann. Math. 69, 630–643 (1959)
[21] Neretin, Y.A.: Hua-type integrals over unitary groups and over projective limits of unitary groups. Duke Math. J. 114(2), 239–266 (2002)
[22] Niemi, H., Weron, A.: Dilation theorems for positive definite operator kernels having majorants. J. Funct. Anal. 40(1), 54–65 (1981)
[23] Okada, S., Okazaki, Y.: Projective limit of infinite Radon measures. J. Aust. Math. Soc. 25(A), 328–331 (1978)
[24] Olshanski, G.: The problem of harmonic analysis on the infinite-dimensional unitary group. J. Funct. Anal. 205(2), 464–524 (2003)
[25] Pickrell, D.: Measures on infinite-dimensional Grassmann manifolds. J. Funct. Anal. 70, 323–356 (1987)
[26] Procesi, G.: Lie Groups: An Approach Through Invariants and Representations. Springer, Berlin (2007)
[27] Rao, M.M.: Projective limits of probability spaces. Multivar. Anal. 1(1), 28–57 (1971)
[28] Rudin, W.: Function Theory in the Unit Ball of $C^n$. Springer, Berlin (2008)
[29] Stroock, D.W.: Probability Theory: An Analytic View. Cambridge University Press, Cambridge (2010)
[30] Tomas, E.: On Prohorov’s criterion for projective limits. In: de Pagner, B. (ed.) Operator Theory: Advances and Applications, vol. 168, no. 15, pp. 251–261. Birkhauser, Basel (2006)
[31] Voiculescu, D.: Limit laws for Random matrices and free products. Invent. Math. 104(1), 201–220 (1991)
[32] Vrabie, I.I.: Co-semigroups and Applications, North-Holland Mathematics Studies, vol. 191. Elsevier, Amsterdam (2003)
[33] Willard, S.: General Topology. Dover, New York (2004)
[34] Yamasaki, Y.: Projective limit of Haar measures on $O(n)$. Publ. Res. Inst. Math. Sci. Kyoto Univ. 8, 141–149 (1972/73)
[35] Yamasaki, Y.: Lecture Notes on Measures on Infinite Dimensional Spaces. World Scientific Publishing, Singapore (1985)

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