Nonlocal symmetries of Lax integrable equations: a comparative study

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Abstract. We continue here the study of Lax integrable equations. We consider four three-dimensional equations: (1) the rdDym equation \( u_{ty} = u_x u_{xy} - u_y u_{xx} \), (2) the 3D Pavlov equation \( u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy} \); (3) the universal hierarchy equation \( u_{yy} = u_{txy} - u_y u_{tx} \), and (4) the modified Veronese web equation \( u_{ty} = u_{txy} - u_y u_{tx} \). For each equation, using the known Lax pairs and expanding the latter in formal series in spectral parameter, we construct two infinite-dimensional differential coverings [13] and give a full description of nonlocal symmetry algebras associated to these coverings. For all the four pairs of coverings, the obtained Lie algebras of symmetries manifest similar (but not the same) structures: they are (semi) direct sums of the Witt algebra, the algebra of vector fields on the line, and loop algebras; all of them contain a component of finite grading. We also discuss actions of recursion operators on shadows of nonlocal symmetries.

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Introduction and notation

In [3], we began a systematic study of symmetry and integrability properties of Lax integrable 3D equations, i.e., equations that admit a Lax pair with non-removable parameter. All the 2D symmetry reductions of

- the rdDym equation $u_{ty} = u_xu_{xy} - u_yu_{xx}$;
- the 3D Pavlov equation $u_{yy} = u_{tx} + u_yu_{xx} - u_xu_{xy}$;
- the universal hierarchy equation $u_{yy} = u_tu_{xy} - u_yu_{tx}$;
- the modified Veronese web equation $u_{ty} = u_tu_{xy} - u_yu_{tx}$,

were described. In [4], we studied the behavior of the Lax operators admitted by these equations under symmetry reductions and showed that in a number of cases the 2D reductions (one of them is equivalent to the Gibbons-Tsarev equation [12]) inherit the Lax integrability property. We also constructed infinite series of (nonlocal) conservation laws for these reductions. Finally, in the recent paper [5] we used expansion of the Lax pair for the rdDym equation in formal series of the spectral parameter to construct two infinite-dimensional differential coverings over this equation and gave a full description of nonlocal symmetries in this covering. All these equations are linearly degenerate in the sense of [11], where such equations were classified.

In the current paper, we apply the same techniques to describe the Lie algebra structure of nonlocal symmetries for the rest three equations. Section 1 is just a brief introduction to the terminology used below. In Section 2, to make the exposition self-contained, we briefly recall the results obtained in [5]. Section 3 is devoted to the 3D Pavlov equation. The results on the universal hierarchy equation are discussed in Section 4, while the symmetries of the modified Veronese web equation are described in Section 5. In each case, recursion operators and their action on the shadows of nonlocal symmetries for the rest three equations are also discussed. We also describe an Bäcklund auto-transformation for this equation and gave a full description of nonlocal symmetries in this covering. All these equations are linearly degenerate in the sense of [11], where such equations were classified.

All the symmetry algebras below have similar (but not the same) structure and are direct or semi-direct sums of the following Lie algebras (see Table 13 on p. 22 where the main results are aggregated):

- the Witt algebra $\mathfrak{W}$ of vector fields $e_i = z^{i+1}\partial/\partial z$, $i \in \mathbb{Z}$;
- its subalgebras $\mathfrak{W}_k$ spanned by $e_i$ with $i \leq k \leq 0$ and $\mathfrak{W}_k^+$ spanned by $e_i$ with $i \geq k \geq 0$;
- the algebra $\mathfrak{W}[\rho]$ of vector fields $R(\rho)\partial/\partial \rho$ on $\mathbb{R}^1$ with a distinguished coordinate $\rho$;
- the loop algebra $\mathfrak{L}[\rho]$ spanned by the elements $z^i \otimes X$, $i \in \mathbb{Z}$, $X \in \mathfrak{W}[\rho]$, with the commutator $[z^i \otimes X, z^j \otimes Y] = z^{i+j} \otimes [X, Y]$;
- the algebra $\mathfrak{L}_k^+[\rho]$ spanned by the elements $p(z) \otimes X$, where $X \in \mathfrak{W}[\rho]$ and $p(z) \in \mathbb{R}[z]/(z^k)$ is a truncated polynomial. In a similar way, we define $\mathfrak{L}_k^-[\rho]$ with $p(z) \in \mathbb{R}[z^{-1}]/(z^{-k})$.

Semi-direct sums in the algebras of symmetries arise due to the natural actions of $\mathfrak{W}$ on $\mathfrak{L}[\rho]$, $\mathfrak{W}_k$ on $\mathfrak{L}_k^+[\rho]$, and $\mathfrak{W}_k^+$ on $\mathfrak{L}_k^-[\rho]$. 


All the equations under consideration admit scaling symmetries that allow to introduce natural weights (grading) to the space of polynomial functions on the equation. This graded structure is inherited by the symmetry algebras in all cases except for the modified Veronese web equation. Perhaps, this is the reason why the Lie algebra structure of symmetries for this equation is a bit different from the other ones.

1. Preliminaries

Everywhere below we deal with second order scalar differential equations in three independent variables \(x, y, \) and \(t\). For a general coordinate-free exposition see \[8\]. To this end, we consider the space \(J^\infty(\mathbb{R}^3, \mathbb{R})\) of infinite jets of smooth functions \(u = u(x, y, t)\) on \(\mathbb{R}^3\). This space is endowed with the coordinates

\[
x, y, t, u_{i,j,k} = \frac{\partial^{i+j+k} u}{\partial x^i \partial y^j \partial t^k}, \quad i, j, k \geq 0,
\]

and its geometric structure is determined by the Cartan distribution spanned by the total derivatives

\[
D_x = \frac{\partial}{\partial x} + \sum_{i,j,k \geq 0} u_{i+1,j,k} \frac{\partial}{\partial u_{i,j,k}}, \quad D_y = \frac{\partial}{\partial y} + \sum_{i,j,k \geq 0} u_{i,j+1,k} \frac{\partial}{\partial u_{i,j,k}},
\]

\[
D_t = \frac{\partial}{\partial t} + \sum_{i,j,k \geq 0} u_{i,j,k+1} \frac{\partial}{\partial u_{i,j,k}}.
\]

An equation \(\mathcal{E} = \{F = 0\} \subset J^\infty(\mathbb{R}^3)\) is the subset defined by the infinite system of relations \(D_\sigma(F) = 0\), where \(F = F(x, y, t, u, u_x, u_y, u_t, u_{xx}, u_{xy}, \ldots, u_{tt})\) is a smooth function and \(D_\sigma\) denotes all possible compositions of the total derivatives. Total derivatives and any differential operators in total derivatives can be restricted to \(\mathcal{E}\), i.e., expressed in terms of internal coordinates on \(\mathcal{E}\).

A symmetry of \(\mathcal{E}\) is a vector field

\[
S = \sum S_{i,j,k} \frac{\partial}{\partial u_{i,j,k}}
\]
on \(\mathcal{E}\) that commutes with the total derivatives (here and below summation is taken over all internal coordinates on \(\mathcal{E}\)). Any symmetry is an evolutionary vector field of the form

\[
\mathbf{E}_\varphi = \sum D^i_x D^j_y D^k_t (\varphi) \frac{\partial}{\partial u_{i,j,k}},
\]

where \(\varphi\) is an arbitrary smooth function on \(\mathcal{E}\) that satisfy the equation \(\ell_\mathcal{E}(\varphi) = 0\) and \(\ell_\mathcal{E}\) is the restriction of the linearization operator

\[
\ell_F = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial u_x} D_x + \ldots + \frac{\partial F}{\partial u_{tx}} D_{tx} + \frac{\partial F}{\partial u_{xy}} D_{xy} + \ldots + \frac{\partial F}{\partial u_{tt}} D_{tt}
\]
to \(\mathcal{E}\). The function \(\varphi\) is the generating function (or the characteristic) of a symmetry. Symmetries form a Lie algebra \(\text{sym}(\mathcal{E})\) with respect to commutator and the commutator induces the Jacobi bracket on the space of generating functions:

\[
\{\varphi_1, \varphi_2\} = \mathbf{E}_{\varphi_1}(\varphi_2) - \mathbf{E}_{\varphi_2}(\varphi_1).
\]
In what follows, we do not distinguish between symmetries and their generating functions.

A symmetry of the form \(s = \delta u + \alpha u_x + \beta u_y + \gamma u_t, \alpha, \beta, \gamma, \delta \in \mathbb{Z}\), is called a scaling symmetry of \(\mathcal{E}\). If an equation admits such a symmetry one can
introduce weights to polynomial functions on $\mathcal{E}$ by $|x| = -\alpha, \ |y| = -\beta, \ |t| = -\gamma, \ |u_{i,j,k}| = \delta - i\alpha - j\beta - k\gamma$, with respect to which the space $\mathcal{P}(\mathcal{E})$ of such functions becomes graded: $\mathcal{P}(\mathcal{E}) = \oplus_{\mathbb{Z}^2} \mathcal{P}_r(\mathcal{E})$. If $\mathcal{E}_p$ is a symmetry and $\varphi \in \mathcal{P}(\mathcal{E})$ we set $|\mathcal{E}_p| = |\varphi| - |u|$. Then $\mathcal{E}_p(\mathcal{P}_r(\mathcal{E})) \subset \mathcal{P}_{r+|\mathcal{E}_p|}(\mathcal{E})$ and $[\mathcal{E}_p, \mathcal{E}_q] = |\mathcal{E}_p| + |\mathcal{E}_q|$ and thus the space of polynomial symmetries becomes a $\mathbb{Z}$-graded Lie algebra.

Let $\mathcal{E}$ be an equation. A differential covering over $\mathcal{E}$ (see [13]) is an extension $\tilde{\mathcal{E}}$ of $\mathcal{E}$ by a system of first order equations

\[
\begin{align*}
\tilde{D}_x &= D_x + \sum X^j \frac{\partial}{\partial w^j}, \\
\tilde{D}_y &= D_y + \sum Y^j \frac{\partial}{\partial w^j}, \\
\tilde{D}_t &= D_t + \sum T^j \frac{\partial}{\partial w^j}
\end{align*}
\]

and consequently any differential operator $D$ in total derivatives can be lifted to $\tilde{D}$ as well. We say that a covering is Abelian if the right-hand sides of its defining equation do not depend on nonlocal variables. In the case when system [11] may be written in the form of two equations it is referred to as a Lax pair.

Given a one-dimensional covering $\tau$ (i.e., a covering [11] with $w^\alpha = w, X^\alpha = X, Y^\alpha = Y$, and $T^\alpha = T$) that smoothly depends on $\lambda \in \mathbb{R}$, one can consider the expansion $w = \sum_{-\infty}^{\infty} \lambda^i w_i$ and also expand the defining equations of the covering in formal series of the parameter. Then an infinite-dimensional covering with the nonlocal variables $w_i$ arises. If $w_i = 0$ for $i < 0$ we say this is the positive covering associated to $\tau$; if $w_i = 0$ for $i > 0$ then we have the negative covering.

A symmetry of $\tilde{\mathcal{E}}$ is a nonlocal symmetry of $\mathcal{E}$. Nonlocal symmetries are vector fields

\[
\mathcal{E}_\varphi + \sum_j \Phi^j \frac{\partial}{\partial w^j},
\]

where $\varphi, \Phi^j$ are smooth functions on $\tilde{\mathcal{E}}$ that satisfy $\tilde{\mathcal{E}}(\varphi) = 0$ together with the system

\[
\begin{align*}
\tilde{D}_x(\phi^\alpha) &= \tilde{\ell}_{X^\alpha}(\varphi) + \sum \frac{\partial X^\alpha}{\partial w^j} \Phi^j, \\
\tilde{D}_y(\phi^\alpha) &= \tilde{\ell}_{Y^\alpha}(\varphi) + \sum \frac{\partial Y^\alpha}{\partial w^j} \Phi^j, \\
\tilde{D}_t(\phi^\alpha) &= \tilde{\ell}_{T^\alpha}(\varphi) + \sum \frac{\partial T^\alpha}{\partial w^j} \Phi^j.
\end{align*}
\]

A nonlocal symmetry is called invisible if $\varphi = 0$. Solutions of the equation $\tilde{\mathcal{E}}(\varphi) = 0$ are called shadows. We say that a shadow $\varphi$ is lifted (or reconstructed) if there exists a nonlocal symmetry $\Phi = (\varphi, \Phi^1, \ldots, \Phi^j, \ldots)$. Of course, lifts (if they exist) are defined up to an invisible symmetry.

Let $\mathcal{E}_1, \mathcal{E}_2$ be equations and $\tau_1: \tilde{\mathcal{E}} \to \mathcal{E}_1$ be coverings. Then one has the diagram

\[
\begin{array}{c}
\mathcal{E}_1 \\
\cap \tau_1 \\
\mathcal{E}_2 \\
\cap \tau_2 \end{array}
\]
Table 1. The rdDym equation: commutators of local symmetries.

|         | $\psi_0$ | $v_0(\bar{Y})$ | $\theta_0(T)$ | $\theta_{-1}(\bar{T})$ | $\theta_{-2}(T)$ |
|---------|----------|----------------|---------------|------------------------|------------------|
| $\psi_0$ | 0        | 0              | 0             | $\theta_{-1}(T)$        | $2\theta_{-2}(T)$ |
| $v_0(Y)$ | $\ldots$ | $v_0([Y,\bar{Y}])$ | 0             | 0                      | 0                |
| $\theta_0(T)$ | $\ldots$ | $\ldots$ | $\theta_0([\bar{Y},\bar{T}])$ | $\theta_{-1}(T,\bar{T})$ | $\theta_{-2}(T,\bar{T})$ |
| $\theta_{-1}(T)$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | 0                |
| $\theta_{-2}(T)$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | 0                |

which is called a Bäcklund transformation between $\mathcal{E}_1$ and $\mathcal{E}_2$. If $\mathcal{E}_1 = \mathcal{E}_2$ then it is a Bäcklund auto-transformation. To any equation $\mathcal{E}$ which is called the tangent equation of $\mathcal{E}$ in Table 1. The corresponding evolutionary vector fields have the weights $\frac{\partial E}{\partial u_\ell}$ denotes the corresponding derivative. Commutators of symmetries are presented $\mathcal{E}$, consequently the equation becomes homogeneous with respect to these weights. Local symmetries are solutions of the equation $E_x \equiv \frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial u_t^2} = 0$, which is called the tangent equation of $\mathcal{E}$. A Bäcklund auto-transformation of $\mathcal{E}$ is a recursion operator for shadows of symmetries of $\mathcal{E}$; see [18].

2. The rdDym equation: a synopsis

See [4, 24, 25] for more information about the equation. A detailed discussion of coverings, nonlocal symmetries, and recursion operators for this equation can be found in [5]. Nevertheless, for the sake of completeness, we present a short overview of the results obtained earlier.

The equation reads

$$u_{ty} = xu_{xxy} - uy_{uxx}. \tag{2}$$

We assign the weights $|x| = 1, |y| = 2, |t| = 0$ to the variables $x, y, t, \text{ and } u$; consequently the equation becomes homogeneous with respect to these weights. Local symmetries are solutions of the equation $E_x \equiv D_y D_y (\varphi) - u_x D_x D_y (\varphi) + uy D_x^2 (\varphi) - u_{xx} D_x (\varphi) + u_{xxy} D_y (\varphi) = 0$. The space of solutions sym ($\mathcal{E}$) is spanned by the functions $\psi_0 = -x u_x + 2u, \quad v_0(Y) = Y uy$, $\theta_0(T) = T uy + T'(xu_x - u) + \frac{1}{2} T'' x^2$, $\theta_{-1}(T) = T uy + T' u_x, \quad \theta_{-2}(T) = T$, where $T = T(t), Y = Y(y)$ are arbitrary functions of their arguments and ‘prime’ denotes the corresponding derivative. Commutators of symmetries are presented in Table 1. The corresponding evolutionary vector fields have the weights $|E_{\psi_0}| = |E_{v_0(Y)}| = 0, |E_{\theta_0(T)}| = i, i = 0, -1, -2$.

The system

$$w_t = (u_x - \lambda)w_x \quad w_y = \lambda^{-1}uyw_x, \tag{3}$$

is a Lax pair for Equation (2). Setting $w = \sum_{i=-\infty}^{+\infty} \lambda_i^i w_i$ and inserting this expansion into (3), we obtain $w_{i,t} = u_x w_{i, x} - w_{i-1,x}$ and $w_{i,y} = uy w_{i+1,x}$. The corresponding
positive covering is defined by the system
\[
q_{1,t} = \frac{\partial}{\partial u_y}, \quad q_{1,x} = \frac{1}{u_y},
\]
\[
q_{i,t} = \frac{\partial}{\partial u_y} q_{i-1,y} - q_{i-1,x}, \quad q_{i,x} = \frac{q_{i-1,y}}{u_y},
\]
where \(i \geq 2\), with the additional nonlocal variables \(q_{i}^{(j)}\) defined by the equalities
\[
q_{i}^{(0)} = q_{i} \quad \text{and} \quad q_{i}^{(j+1)} = \left(q_{i}^{(j)}\right)^{y}.\]
The weights assigned to the nonlocal variables are \(q_{i}^{(j)} = -i, \ i \geq 1, \ j \geq 0\). The negative covering is defined by the system
\[
r_{1,x} = u_{x}^{2} - u_{t}, \quad r_{1,y} = u_{x}u_{y},
\]
\[
r_{i,x} = u_{x}r_{i-1,x} - r_{i-1,t}, \quad r_{i,y} = u_{y}r_{i-1,x},
\]
enriched by additional nonlocal variables \(r_{i}^{(j)}\) defined in the obvious way by \(r_{i}^{(0)} = r_{i}\), \(r_{i}^{(j+1)} = \left(r_{i}^{(j)}\right)^{x}\). One has \(|r_{i}^{(j)}| = i + 2, \ i \geq 1, \ j \geq 0\).

All the local symmetries of the rdDym equation can be lifted both to \(\tau^{+}\) and to \(\tau^{-}\) and we denote the lifts by the corresponding capital letters: \(\Psi_{0}\) for the lift of \(\psi_{0}, \Theta_{i}(T)\) for \(\theta_{i}(T)\), etc.

Three families of nonlocal symmetries are admitted in \(\tau^{+}\). The first one consists of invisible symmetries
\[
\Phi_{k}^{\text{inv}}(Y) = (0, \ldots, 0, \varphi_{1}^{\text{inv}}, \ldots, \varphi_{1}^{\text{inv}}, \ldots)
\]
\(k\) times
where \(\varphi_{1}^{\text{inv}} = Y(y)\), and another two are generated by the lifts \(\Psi_{-1}\) and \(\Psi_{-2}\) of the nonlocal shadows \(\psi_{-1} = q_{1}u_{y} + x\) and \(\psi_{-2} = (2q_{2} - q_{1}q_{11})u_{y}\) using the relations \(\Psi_{-k} = [\Psi_{-k+1}, \Psi_{-1}], \ k \geq 3\), and \(\Upsilon_{k}(Y) = [\Psi_{k-1}, \Phi_{k}^{\text{inv}}(Y)]\). The constructed nonlocal symmetries have the weights \(|\Psi_{k}| = |\Upsilon_{-i}(Y)| = i, \ i \leq 0, \ |\Theta_{j}(T)| = j, \ j \geq 0, \ -1, -2, |\Phi_{k}^{\text{inv}}(Y)| = k, \ k \geq 1\).

Then the following result is valid:

**Theorem 1** There exist a basis in \(\text{sym}_{\tau^{+}}(\mathcal{E})\) consisting of the elements \(\{w_{i}, v_{j}(T), v_{k}(Y)\}, \ i \leq 0, \ j = 0, -1, -2, k \in \mathbb{Z}\), such that they commute as it is indicated in Table 2. So, the algebra \(\text{sym}_{\tau^{+}}(\mathcal{E})\) is isomorphic to \(\mathfrak{w}_{0}^{\text{inv}} \times (\mathfrak{u}_{0}^{\text{inv}}[t] \oplus \mathfrak{u}_{0}[y])\) with the natural action of \(\mathfrak{w}_{0}^{\text{inv}}\) on \(\mathfrak{u}_{0}^{\text{inv}}[t] \oplus \mathfrak{u}_{0}[y]\).

In a similar way, local symmetries are lifted to \(\tau^{-}\) and three families of nonlocal symmetries arise in this covering. They are \(\Psi_{k}, \ k \geq 1, \ \Theta_{i}(T), \ i \geq -2, \ \Phi_{l}^{\text{inv}}\), and have the following weights: \(|\Psi_{k}| = k, \ k \geq 0, \ |\Phi_{l}^{\text{inv}}| = -l - 2, \ l \geq 1, \ |\Theta_{i}(T)| = i, \ i \geq -3, \ |\Upsilon_{i}(Y)| = 0\).
The Lie algebra structure is then described by

**Theorem 2** There exist a basis in \( \text{sym}_{-(E)} \) consisting of the elements \( \{w_i, v_j(T), v(Y)\} \), \( i \geq 0, j \in \mathbb{Z} \), that satisfy the commutator relations presented in Table 3. Hence, the Lie algebra \( \text{sym}_{-(E)} \) is isomorphic to \( \mathfrak{m}_0^+ \otimes \mathcal{L}[t] \oplus \mathcal{B}[y] \) with the natural action of \( \mathfrak{m}_0^+ \) on \( \mathcal{L}[t] \).

**Remark 1** Note that the components of the invisible symmetries are constructed using the operator

\[
\mathcal{V} = q_i \frac{\partial}{\partial y} + \sum_{i=1}^{\infty} (i + 1)q_{i+1} \frac{\partial}{\partial q_i}.
\]

Similar operators will arise in the study of other equations below.

The algebra \( \text{sym}(E) \) admits a recursion operator \( \hat{\chi} = R_+(\chi) \) defined by the system

\[
\begin{align*}
D_t(\hat{\chi}) &= u_y^{-1}(u_y D_x(\chi) - u_x D_y(\chi)) + (u_x u_{xy} - u_y u_{xx})\hat{\chi}, \\
D_x(\hat{\chi}) &= u_y^{-1}(u_{xy} \hat{\chi} - D_y(\chi)),
\end{align*}
\]

see [20]. This means that \( \hat{\chi} \) is a nonlocal shadow whenever \( \chi \) is. Another recursion operator \( \chi = R_-(\hat{\chi}) \) is given by the system

\[
\begin{align*}
D_x(\chi) &= D_t(\hat{\chi}) - u_x D_x(\hat{\chi}) + u_{xx} \hat{\chi}, \\
D_y(\chi) &= -u_y D_x(\hat{\chi}) + u_{xy} \hat{\chi}.
\end{align*}
\]

The operators \( R_+ \) and \( R_- \) are mutually inverse.

The actions of \( R_+ \) and \( R_- \) on \( \text{sym}(E) \) may be prolonged to the shadows of nonlocal symmetries from \( \text{sym}(E^+) \) and \( \text{sym}(E^-) \) if we replace the derivatives \( D_t, D_x \) and \( D_y \) in [3] and [5] by the total derivatives \( D_t, D_x \) and \( D_y \) in the Whitney product of the coverings \( \tau^+ \) and \( \tau^- \) in the sense of [13]. The resulting operators will be also denoted by \( R_+ \) and \( R_- \).

Note that the operators act nontrivially on ‘vacuum’: \( R_+(0) = \theta_{-2}(T), R_-(0) = v_0(Y) \), which immediately follows from Equations [3] and [5]; thus the actions are reasonable to consider modulo \( \theta_{-2}(T) \) for \( R_+ \) and \( v_0(Y) \) for \( R_- \). Taking into account this remark, we have the following

**Proposition 1** Modulo the images of the trivial symmetry, the action of recursion operators is of the form

\[
\begin{align*}
R_+(\theta_i(T)) &= \begin{cases} 
\alpha^+_i \theta_{i-1}(T), & i > -2, \\
0, & i = -2
\end{cases}, & R_-(\theta_i(T)) &= \begin{cases} 
\alpha^-_i \theta_{i+1}(T), & i \geq -2,
\end{cases} \\
R_+(v_j(T)) &= \beta^+_j v_{i+1}(Y), & i \leq 0, & R_-(v_j(T)) &= \begin{cases} 
\beta^-_j v_{i+1}(Y), & i < 0, \\
0, & i = 0,
\end{cases} \\
R_+(\psi)_i &= \gamma^+_i \psi_{i-1}, & R_-(\psi)_i &= \gamma^-_i \psi_{i+1}, & i \in \mathbb{Z},
\end{align*}
\]

---

**Table 3.** The \( \text{rdDym} \) equation: commutators in \( \text{sym}_{-(E)} \).

| \( w_i \) | \( v_j(T) \) | \( v(Y) \) |
|---|---|---|
| \( (j-i)w_{i+j} \) | \( jv_{i+j}(T) \) | 0 |
| \( v_j(T) \) | \( \ldots \) | \( v_{i+j}(T,T) \) | 0 |
| \( v(Y) \) | \( \ldots \) | \( \ldots \) | \( v([Y,Y]) \) |
We choose the following internal coordinates on $E$ notation and write $\theta$ scalar multipliers and modulo the image of the trivial shadow. We also ‘compress’ the shadows that a shadow lives in $\tau$ that of $\psi_m$ of nonlocal symmetries in coverings $\dot{E}^+$ and $\dot{E}^-$ and ‘tunnel’ from the series of $\theta_k(T)$ to that of $v_k(Y)$. See Figure 1.

Remark 2 In all the figures here and below straight straight arrows denote actions up to scalar multipliers and modulo the image of the trivial shadow. We also ‘compress’ the notation and write $\upsilon$, instead of $\theta(T)$, $v_k$ instead of $v_k(Y)$, etc. Notation $(\cdot)^+$ means that a shadow lives in $\tau^+$, $(\cdot)^-$ is for those who live in $\tau^-$; shadows marked by $(\cdot)^\pm$ live in both coverings.

3. The 3D Pavlov equation

This equation was discussed in [9, 26], for example.

3.1. The equation

The 3D Pavlov equation is of the form

$$u_{yy} = u_{xx} + u_y u_{xx} - u_x u_{xy}, \quad (6)$$

We choose the following internal coordinates on $E$:

$$u^0_{k,l} = u_{xx\ldots x t\ldots t}^k, \quad u^1_{k,l} = u_{xx\ldots x t\ldots t}^k, \quad k, l \geq 0.$$ 

Then the total derivatives read

$$D_x = \frac{\partial}{\partial x} + \sum_{k,l} \left( u^0_{k+1,l} \frac{\partial}{\partial u^0_{k,l}} + u^1_{k+1,l} \frac{\partial}{\partial u^1_{k,l}} \right),$$

$$D_y = \frac{\partial}{\partial y} + \sum_{k,l} \left( u^1_{k,l} \frac{\partial}{\partial u^0_{k,l}} + D_x^k D_t l \left( u^0_{11} + u^0_{00} u^0_{20} - u^0_{01} u^0_{10} \right) \frac{\partial}{\partial u^1_{k,l}} \right),$$

$$D_t = \frac{\partial}{\partial t} + \sum_{k,l} \left( u^0_{k,l+1} \frac{\partial}{\partial u^0_{k,l}} + u^1_{k,l+1} \frac{\partial}{\partial u^1_{k,l}} \right)$$

in these coordinates. We assign the weights $|t| = 0, |y| = 1, |x| = 2, |u| = 3$ and hence $|u^0_{k,l}| = 3 - 2k, |u^1_{k,l}| = 3 - 2k - 1$.

Symmetries of $\dot{E}$ are solutions to the equation

$$\ell_E(\varphi) \equiv D_y^2(\varphi) - D_x D_x(\varphi) - u_y D_x^2(\varphi) + u_x D_x D_y(\varphi) - u_{xx} D_y(\varphi) + u_{xy} D_x(\varphi). \quad (7)$$
The space sym (\(E\)) of solutions to Equation (7) is spanned by the functions
\[
\varphi_1 = 2x - y u_x, \quad \varphi_2 = 3u - 2x u_x - y u_y,
\]
\[
\theta_0(T) = T u_t + T'(x u_x + y u_y - u) + \frac{1}{2} T''(y^2 u_x - 2xy) - \frac{1}{6} T''' y^3,
\]
\[
\theta_1(T) = T u_y + T'(y u_x - x) - \frac{1}{2} T'' y^2, \quad \theta_2(T) = T u_x - T' y, \quad \theta_3(T) = T,
\]
where \(T\) is a function of \(t\) and ‘prime’ denotes the \(t\)-derivatives. Commutators of these symmetries are presented in Table 4. The corresponding vector fields have the weights \(|E_{\varphi_1}| = -1, |E_{\varphi_2}| = 0, |E_{\theta_j}| = -i, i = 0, \ldots, -3\).

3.2. The Lax pair and hierarchies

The Lax pair for the 3D Pavlov equation is \(q_t = (\lambda^2 - \lambda u_x - u_y)q_x, q_y = (\lambda - u_x)q_x\). Expanding \(q\) in integer powers of \(\lambda\), we arrive to the covering \(q_{i,t} = q_{i-2, x} - u_x q_{i-1, x} - u_y q_{i, x}, q_{i, y} = q_{i-1, x} - u_x q_{i, x},\) for all \(i \in \mathbb{Z}\).

The positive covering corresponding to this system is
\[
q_{0, t} + u_y q_{0, x} = 0, \quad q_{0, y} + u_x q_{0, x} = 0; \quad q_{1, t} + u_y q_{1, x} = -u_x q_{0, x}, \quad q_{1, y} + u_x q_{1, x} = q_{0, x}; \quad q_{i, t} + u_y q_{i, x} = q_{i-2, x} - u_x q_{i-1, x}, q_{i, y} + u_x q_{i, x} = q_{i-1, x},
\]
where \(i \geq 2\), to which nonlocal variables \(q_i^{(j)}\) defined by \(q_i^{(0)} = q_i, q_i^{(j+1)} = q_i^{(j)}\) are added. One has \(q_i^{(j)} = -i - 2j\). This covering is not Abelian.

The negative covering is given by
\[
r_{1, y} = u_t + u_x u_y, \quad r_{1, x} = u_y + u_x, \quad r_{i, y} = r_{i-1, t} + u_y r_{i-1, x}, \quad r_{i, x} = r_{i-1, y} + u_x r_{i-1, x},
\]
i \geq 2, with additional nonlocal variables \(r_i^{(j)}\) defined by \(r_i^{(0)} = r_i, r_i^{(j+1)} = r_i^{(j)}\). One has \(r_i^{(j)} = i + 3\).

3.3. Nonlocal symmetries in the positive covering

3.3.1. Lifts of local symmetries

All the local symmetries can be lifted to \(\tau^+\). In more detail, we have the following results.

The lift of \(\varphi_1 = y u_x - 2x\) is \(\Phi_1 = (\varphi_1, \varphi_0^1, \ldots, \varphi_1^i, \ldots)\), where \(\varphi_1^i = y q_{i, x} + (i + 1)q_{i+1}\). The symmetry \(\varphi_2 = 2x u_x + y u_y - 3u\) is lifted by \(\Phi_2 = (\varphi_2, \varphi_0^0, \varphi_0^1, \varphi_1^i, \ldots)\),

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\varphi_1 & \varphi_2 & \theta_0(T) & \theta_1(T) & \theta_2(T) & \theta_3(T) \\
\hline
\varphi_1 & 0 & \varphi_1 & -2\theta_2(T) & 2\theta_3(T) & 0 \\
\hline
\theta_0(T) & \ldots & 0 & \theta_1(T) & -2\theta_2(T) & -3\theta_3(T) \\
\hline
\theta_1(T) & \ldots & \theta_0[T, T)] & \theta_1[T, T)] & \theta_2[T, T)] & \theta_3[T, T)] \\
\hline
\theta_2(T) & \ldots & \ldots & \theta_2[T, T)] & \theta_3[T, T)] & 0 \\
\hline
\theta_3(T) & \ldots & \ldots & \ldots & \ldots & 0 \\
\hline
\end{array}
\]
where \( \psi^0 = -\varphi_1 q_0, \varphi_x \) and \( \psi^i = -\varphi_1 q_i, y q_i, i \geq 1 \). The lift of \( \theta_2(T) = T u_x - T y \) is \( \Theta_2(T) = (\theta_1, T q_0, \ldots, T q_i, \ldots) \). The symmetry

\[
\theta_1(T) = T u_y + T(y u_x - x) - \frac{1}{2} T'' y^2
\]

admits the lift \( \Theta_1(T) = (\theta_1, \theta_1, \theta_1, \ldots, \theta_1, \ldots) \), where \( \theta_1 = -\theta_1(T) q_0, \theta_1 = -\theta_2(T) q_i, x + T q_i-1, x, i \geq 1 \). The lift of

\[
\theta_0(T) = T u_t + T'(x u_x + y u_y - u) + T'' \left( \frac{1}{2} y^2 u_x - x y \right) - \frac{1}{6} T''' y^3
\]

is \( \Theta_0(T) = (\theta_0, \theta_0, \theta_0, \theta_0, \ldots, \theta_0, \ldots) \), where \( \theta_0 = -\theta_1(T) q_0, \theta_0 = -\theta_1(T) q_0, \theta_0 + \theta_2(T) q_i, x - \theta_2(T) q_i, x + T q_i-1, x, i \geq 2 \). Finally, for \( \theta_3(T) = T \) one has \( \Theta_3(T) = (\theta_3, 0, 0, 0, \ldots) \).

3.3.2. Nonlocal symmetries Three families of nonlocal symmetries exist for the Pavlov equation in \( \tau^+ \). The first one consists of invisible symmetries

\[
\Phi^k_{inv}(Y) = (0, 0, 0, \varphi^k_{inv}, \varphi^k_{inv}, \ldots, \varphi^k_{inv}, \ldots), \quad k = 1, 2, \ldots,
\]

where for every \( i \geq 1 \) it holds \( \varphi^k_{inv} = R_{i-1}(Q) \). Here \( R_0(Q) = Q(q_0) \) is an arbitrary function of \( q_0 \) and for \( n \geq 1 \) we define

\[
R_n(Q) = \frac{1}{n} \mathcal{Y}(R_{n-1}(Q)),
\]

where \( \mathcal{Y} \) is the vector field

\[
\mathcal{Y} = \sum_{i=0}^{\infty} (i + 1) q_i + \frac{\partial}{\partial q_i}.
\]

Now we define explicitly the nonlocal symmetry \( \Psi_{-1} = (\psi_1, \psi_1, \ldots, \psi_1, \ldots) \) by setting

\[
\psi_1 = \frac{q_1}{q_0}, \psi_1 = y
\]

and

\[
\psi_1^{i} = -(i + 2) q_i + \frac{q_1 q_{i+1}}{q_0}, i \leq -1, \psi_1^{i} = -(i + 2) q_i + \frac{q_1 q_{i+1}}{q_0}.
\]

Then the elements of the second nonlocal family are \( \Psi_{-k} = [\Phi_1, \Psi_{-1}], k \geq 2 \). One has \( |\Psi_{-k}| = -k - 1 \). Finally, we define \( \Xi_j(Q) = [\Psi_{-j}, \Phi^2_{inv}(Q)], l \geq 1 \). Distribution of symmetries along weights is \( |\Psi_{i}| = -l - 1, l \geq 1 \), \( |\Phi_2| = 0, |\Theta_k(T)| = k - 2, k = 0, 1, 3 \), \( |\Xi_j(Q)| = -j + 1, j \geq 1 \), \( |\Phi^2_{inv}(Q)| = l, l \geq 1 \).

3.3.3. Lie algebra structure Consider the spaces \( W \) spanned by \( \Phi_1, \Phi_2, \) and \( \Psi_{i}, i \leq -1, V[T] \) spanned by \( \Theta_1(T), i = 0, 3 \), \( V[q_0] \) spanned by \( \Phi^2_{inv}(Q) \) and \( \Xi_j(Q) \), \( i, j \geq 1 \). Then the following result holds:

**Theorem 3** There exist bases \( w_i \) in \( W, i \leq 0 \), \( v_i(T) \) in \( V[T], i = 0, 1, \ldots, 3 \), and \( v_i(Q) \) in \( V[q_0], i \in \mathbb{Z} \), such that their commutators satisfy the relations presented in Table 5. In other words, sym \( \cdot (E) \) is isomorphic to \( \mathfrak{w}_6^- \times (\mathfrak{l}[q_0] \oplus \mathfrak{l}_4^- [t]) \) with the natural action of the Witt algebra \( \mathfrak{w}_6^- \) on \( \mathfrak{l}[q_0] \oplus \mathfrak{l}_4^- [t] \).
3.4. Nonlocal symmetries in the negative covering

3.4.1. Lifts of local symmetries  Similar to the case of $\tau^+$, all the local symmetries are lifted to the covering $\tau^-$. Namely, the symmetry $\varphi_1 = yu_x - 2x$ has the lift $\Phi_1 = (\varphi_1, \varphi_1^1, \varphi_1^2, \varphi_1^3, \ldots)$, where $\varphi_1^1 = yr_1, x - 3u$ and $\varphi_1^i = yr_{i-1} - (i + 2)r_{i-1}$, $i \geq 2$. The symmetry $\varphi_2 = 2xu_x + yu_y - 3u$ has the lift $\Phi_2 = (\varphi_2, \varphi_2^1, \varphi_2^2, \varphi_2^3, \ldots)$, where $\varphi_2^i = 2xu_y + yr_{i-1} - (i + 3)r_{i-1}$, $i \geq 1$.

To describe the lift $\Theta_3(T) = (\theta_3, \theta_3^1, \theta_3^2, \ldots)$ of $\theta_3(T) = T$, consider the operator

$$\mathcal{Y} = y \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial y} + 3u \frac{\partial}{\partial x} + 4q_1 \frac{\partial}{\partial q_1} + \sum_{i=1}^{\infty} (i + 4)q_{i+1} \frac{\partial}{\partial q_i}$$

and set $\theta_3^i = yT'$ and $\theta_3^i = \frac{1}{i!}\mathcal{Y}(\varphi_0^{i-1})$, $i \geq 2$.

To describe the lifts of $\theta_2(T) = Tu_x - T'y$, $\theta_1(T) = Tu_y + T'(yu_x - x) - \frac{1}{4}T''y^2$, and $\theta_0(T) = Tu_t + T'(xu_x + yu_y - u) + T''(\frac{1}{2}y^2u_x - xy) - \frac{1}{6}T'''y^3$, we shall need the nonlocal symmetry $\Psi_0$, (see Equation (9) below). Namely, we set

$$\Theta_2(T) = \frac{1}{3}[\Psi_0, \Theta_3(T)], \quad \Theta_1(T) = -\frac{1}{2}[\Psi_0, \Theta_2(T)], \quad \Theta_0(T) = -[\Psi_0, \Theta_1(T)]$$

3.4.2. Nonlocal symmetries  The invisible symmetries in $\tau^-$ are of the form

$$\Phi_{\text{inv}}^k(T) = (0, \ldots, 0, \varphi_{\text{inv}}^{k,k+1}, \ldots, \varphi_{\text{inv}}^{k,k+i}, \ldots),$$

where for every $i \geq 1$ it holds $\varphi_{\text{inv}}^{k,k+i} = R_{i-1}(T)$ and the sequence of functions $R_n$, $n \geq 0$ is defined as follows

$$R_0(T) = T, \quad R_{n+1}(T) = \frac{1}{n+1} \mathcal{Y}(R_n(T))$$

with the operator $\mathcal{Y}$ being defined by Equation (8).

Let us now introduce the nonlocal symmetries

$$\Psi_0 = (\psi_0, \psi_0^1, \ldots, \psi_0^i, \ldots)$$

and $\Psi_1 = (\psi_1, \psi_1^1, \ldots, \psi_1^i, \ldots)$ by setting $\psi_0 = 4r_1 - 3uu_x - 2xu_y - yu_t$, $\psi_1 = 5r_2 - 4u_xr_1 - yr_{1,t} - 3uu_y - 2xu_t + yu_uu_x$, and $\psi_0^i = (i+4)r_{i+1} - 3ur_{i,x} - 2xr_{i,y} - yr_{i,t}$, $\psi_1^i = (i+5)r_{i+2} - yr_{i+1,t} - 3ur_{i,y} - 2xr_{i,t} - (4r_1 - yu_t)r_{i,x}$ for $i \geq 1$.

Using the symmetries $\Psi_0$ and $\Psi_1$, we define by induction two new families of nonlocal symmetries by $\Psi_k = [\Psi_0, \Psi_{k-1}]$, $k \geq 2$, and $\Omega_l(T) = [\Psi_l, \Theta_1(T)]$.

The weights of the obtained symmetries are $|\Phi_1| = -1$, $|\Phi_2| = 0$, $|\Psi_k| = k + 1$, $k \geq 0$, $|\Omega_l(T)| = l$, $l \geq 1$, $|\Phi_{\text{inv}}^k(T)| = -l - 3$, $l \geq 1$, $|\Theta_1(T)| = -i$, $i = 0, \ldots, 3$.

### Table 5. The Pavlov equation: commutators in $\text{sym}_{\tau^-}(\mathcal{L})$.

| $\mathcal{L}$ | $w_j$ | $v_j(T)$ | $v_j(Q)$ |
|---------------|-------|-----------|-----------|
| $w_i$ | $(j-i)w_{i+j}$ | $jv_{i+j}(T)$, $-3 \leq i+j \leq 0$ | $jv_{i+j}(Q)$ |
| $v_i(T)$ | $\ldots$ | $(j-i)v_{i+j}([T,T])$, $-3 \leq i+j \leq 0$ | $0$ |
| $v_i(Q)$ | $\ldots$ | $\ldots$ | $v_{i+j}([Q,Q])$ |
3.5. Recursion operators

We have the following result (see [20]):

Proposition 2 Equation (9) admits the recursion operator for symmetries \( \psi = R_+(\varphi) \) defined by the following system:

\[
D_t(\psi) = -u_y D_x(\psi) + u_{xy} \psi + D_y(\varphi), \quad D_y(\psi) = -u_x D_x(\psi) + u_{xx} \psi + D_x(\varphi). \quad (10)
\]

The inverse operator \( \varphi = R_-(\psi) \) is defined by system

\[
D_x(\varphi) = u_x D_x(\psi) + D_y(\psi) - u_{xx} \psi, \quad D_y(\varphi) = D_t(\psi) + u_y D_x(\psi) - u_{xy} \psi. \quad (11)
\]

The action of the recursion operators on shadows is schematically shown in Figure 2. See Remark 2 for notation. Here \( \xi_i^+ \) and \( \omega_i^- \) are the shadows of \( \Xi_i(T) \) and \( \Omega_i(T) \), respectively.

4. The universal hierarchy equation

The universal hierarchy equation (UHE) was discussed in [16, 17].
The UHE admits the following Lax representation

\[
\begin{array}{c|c|c|c|c}
\nu & \theta_0(X) & \theta_1(X) & \varphi_0(T) & \varphi_1(T) \\
\hline
\nu & 0 & 0 & -\theta_1(X) & 0 \\
\theta_0(X) & \ldots & \theta_0([X,X]) & \theta_1([X,X]) & 0 \\
\theta_1(X) & \ldots & \ldots & 0 & 0 \\
\varphi_0(T) & \ldots & \ldots & \varphi_0([\bar{T},T]) & \varphi_1([\bar{T},T]) \\
\varphi_1(T) & \ldots & \ldots & \ldots & 0 \\
\end{array}
\]

4.1. The equation

The UHE reads

\[
u_{yy} = u_t u_{xy} - u_y u_{tx}.
\]

We assign the weights \(|x| = 0, |y| = 1, |t| = 0, |u| = -1\) to the variables \(x, y, t, u\).

Similar to Section 3.1, we consider the internal coordinates

\[
u^0_{k,l} = \underbrace{x_x \ldots x_t \ldots x_t}_{k \text{ times}}, \quad u^1_{k,l} = \underbrace{x_x \ldots x_t \ldots x_t}_{k \text{ times}} \quad k, l \geq 0.
\]

on \(\mathcal{E}\). Consequently, \(|u^0_{k,l}| = -1, |u^1_{k,l}| = -2\).

The total derivatives in the chosen coordinates are

\[
\begin{align*}
D_x &= \frac{\partial}{\partial x} + \sum_{k,l} \left( u^0_{k+1,l} \frac{\partial}{\partial u^0_{k,l}} + u^1_{k+1,l} \frac{\partial}{\partial u^1_{k,l}} \right), \\
D_y &= \frac{\partial}{\partial y} + \sum_{k,l} \left( u^1_{k,l} \frac{\partial}{\partial u^0_{k,l}} + D_x^2 D_t^1 \left( u^0_{01} u^1_{10} - u^0_{00} u^1_{11} \right) \frac{\partial}{\partial u^0_{k,l}} \right), \\
D_t &= \frac{\partial}{\partial t} + \sum_{k,l} \left( u^1_{k,l+1} \frac{\partial}{\partial u^0_{k,l}} + u^1_{k,l+1} \frac{\partial}{\partial u^1_{k,l}} \right).
\end{align*}
\]

Local symmetries of \(\mathcal{E}\) are solutions to the equation \(\ell_\mathcal{E}(\varphi) \equiv D^2_x D^1_t(\varphi) - u_t D_x D_y(\varphi) + u_y D_x D_t(\varphi) - u_{xy} D_t(\varphi) = 0\). The space sym (\(\mathcal{E}\)) is spanned by the functions \(\theta_0(X) = X u_x - X' u, \theta_1(X) = X, \varphi_0(T) = T u_t + T' u_y, \varphi_1(T) = T u_y, v = y u_y + u\), where \(X\) is a function of \(x\) and \(T\) is a function of \(t\), while ‘prime’ denotes the corresponding derivatives. The commutators are presented in Table 7.

Weights of the evolutionary vector fields are \(|E_v| = |E_{\theta_0(X)}| = |E_{\varphi_0(T)}| = 0, |E_{\theta_1(X)}| = 1, |E_{\varphi_1(T)}| = -1\).

4.2. The Lax pair and hierarchies

The UHE admits the following Lax representation \(q_t = \lambda^2 (\lambda u_t - u_y) q_x, q_y = \lambda^{-1} u_y q_x\).

Expansion in powers of \(\lambda\) leads to the system \(q_{t,t} = u_t q_{t+1,x} - u_y q_{t+2,x}, q_{t,y} = u_y q_{t+1,x}\).

The corresponding positive covering is of the form

\[
q_{t,y} = \frac{u_t}{u_y} q_{t-1,y} - q_{t-1,t}, \quad q_{t,x} = \frac{q_{t-1,y}}{u_y}.
\]
\[ i > 1, \text{ with the additional variables } q_i^{(j)} \text{ that satisfy the relations } q_i^{(0)} = q_i, \quad q_i^{(j+1)} = q_i^{(j)}, \\text{.} \]  
One has \(|q_i^{(j)}| = i + 1\).

The equations defining the negative covering are
\[
\begin{align*}
 r_{1,y} & = u_x u_y, \\
 r_{i,y} & = u_y r_{i-1,x}, \\
 r_{1,t} & = u_x u_t - u_y, \\
 r_{i,t} & = u_t r_{i-1,x} - r_{i-1,y}.
\end{align*}
\]
i > 1, with \( r_i^{(j)} \) defined by \( r_i^{(j+1)} = r_i^{(j)} x_t. \) The weights are \(|r_i^{(j)}| = -i - 1\).

4.3. Nonlocal symmetries in the positive covering
4.3.1. Lifts of local symmetries  
The local symmetries of the UHE are lifted as follows.

The symmetry \( v = y u_x + u \) is lifted to \( Y = (v, v^1, v^2, \ldots, v^i, \ldots) \), where \( v^i = -(i+1)q_i + y q_i y \). The lift \( \Theta_0(X) = (\theta_0, \theta_1^0, \theta_2^0, \ldots, \theta_0^0, \ldots) \) of \( \theta_0(X) = X u_x - X' u \) is defined by
\[
\theta_0^0 = \frac{X}{u_y} q_i - 1 y.
\]

Let us now introduce the operator
\[
Y = -y \frac{\partial}{\partial y} + 2q_1 \frac{\partial}{\partial y} + \sum_{k=1}^{\infty} (k + 2) q_{k+1} \frac{\partial}{\partial q_k}
\]
and define by induction the quantities \( R_i(T) \) as follows:
\[
R_1(T) = -T' y, \quad R_i(T) = \frac{1}{i} Y(R_{i-1}(T)), \quad i \geq 2.
\]
Then the lift of \( \varphi_0(T) = T u_t + T' y u_y \) is \( \Phi_0(T) = (\varphi_0, \varphi_0^1, \varphi_0^2, \varphi_0^3, \ldots) \), where \( \varphi_0^0 = T q_{i,t} + T' y q_{i,y} + R_{i+1}(T) \), while the symmetry \( \varphi_1(T) = T u_y \) is lifted by \( \Phi_1(T) = (\varphi_1, \varphi_1^1, \varphi_1^2, \varphi_1^3, \ldots) \) with \( \varphi_1^1 = T q_{i,y} - R_i(T) \). Finally, the lift of \( \theta_1(X) = X \) is \( \Theta_1(X) = (\theta_1, 0, \ldots, 0, \ldots) \).

4.3.2. Nonlocal symmetries  
There exists a family of invisible symmetries
\[
\Phi_{\text{inv}}^i(T) = (0, \ldots, 0, \varphi_{\text{inv}}^{i+1}, \varphi_{\text{inv}}^{i+2}, \ldots),
\]
where \( \varphi_{\text{inv}}^1 = T \) and \( \varphi_{\text{inv}}^i = R_{i-1}(T), \ i > 1 \), \( R_{i-1}(T) \) being defined by Equation (13).

The UHE also admits another two families of nonlocal symmetries in \( \tau^+ \) defined as follows. Let us set \( \Psi_0 = (\psi_0, \psi_0^1, \psi_0^2, \psi_0^3, \ldots) \), where \( \psi_0 = 2 q_1 u_y - y u_t \) and \( \psi_0^i = -(i+2) q_{i+1} - y q_{i+1, t} + 2 q_i q_{i, y} \). We also introduce \( \Psi_1 = (\psi_1, \psi_1^0, \psi_1^1, \psi_1^2, \psi_1^3, \ldots) \) with \( \psi_1 = -3 q_2 u_y + 2 q_1 u_t - y q_{i+1, t} \) and \( \psi_1^i = (i + 3) q_{i+2} + y q_{i+1, t} + 2 q_{i+1} - (3 q_2 + y q_{i, t}) q_{i, y} \). Then we set \( \Psi_k = (\Psi_0, \Psi_{k-1}), \ k \geq 2, \) and \( \Xi_l(T) = (\Psi_l, \Phi_1(T)), \ l \geq 1 \). Distribution of the constructed symmetries along weights is given by \( |Y| = |\Theta_0(X)| = |\Phi_0(T)| = 0, \ |\Theta_1(X)| = 1, \ |\Phi_1(T)| = -1, \ |\Psi_k| = k + 1, k \geq 0, \ |\Phi_{\text{inv}}^i(T)| = -l - 1, \ |\Xi_l(T)| = l, l \geq 1 \).
4.4. Nonlocal symmetries in the negative covering

4.4.1. Lifts of local symmetries

The symmetry \( v = yu_y + u \) is lifted to \( \Upsilon = (v, v^1, v^2, v^3, \ldots) \), where \( v^i = (i + 1)r_i + yu_y r_{i-1,x} \) and \( r_0 \) denotes \( u \).

The lift of \( \Theta_0(X) = Xu_x - X'u \) is \( \Theta_0(X) = (\theta_0(X), \theta_1^0, \theta_2^0, \ldots, \theta_i^0, \ldots) \) with \( \theta_0^0 = Xr_{i,x} - R_{i+1}(X) \), \( R_{i+1} \) being defined by Equation (13). For the symmetry \( \varphi_0(T) = Tu_t + T'yu_y \), one has \( \Phi_0(T) = (\varphi_0(T), \varphi_1^0, \varphi_2^0, \ldots, \varphi_i^0, \ldots) \), where \( \varphi_0^i = Tr_{i,t} + T'yu_y r_{i-1,x} \). The symmetry \( \varphi_1(T) \) is lifted to \( \Phi_1(T) = (\varphi_1(T), \varphi_1^1, \varphi_1^2, \ldots, \varphi_i^1, \ldots) \), where \( \varphi_1^i = Tr_{i,y} \). Finally, for \( \theta_1(X) = X \) one has \( \Theta_1(X) = (\theta_1(X), R_1(X), \ldots, R_i(X), \ldots) \), where again \( R_i \) is defined by (13).

4.4.2. Nonlocal symmetries

In the \( \tau^- \) covering of the UHE there exists a family of invisible symmetries of the form

\[ \Phi^k_{\text{inv}}(X) = (0, \ldots, 0, \varphi^1_{\text{inv}}, \ldots, \varphi^j_{\text{inv}}, \ldots), \]

where \( \varphi^1_{\text{inv}} = X \) and \( \varphi^j_{\text{inv}} = R_{i-1}(X) \) (see Equation (13) for the definition of \( R_i \)).

Consider now two nonlocal symmetries \( \Psi_j = (\psi_j^1, \psi_j^2, \psi_j^3, \ldots, \psi_j^i, \ldots) \), \( j = -1, -2 \), defined by \( \psi_{-1} = 2r_1 - uu_{x,xx}, \psi_{-2} = (i + 2)(i + 1)r_{i+1} - ur_{i,xx} \), and \( \psi_{-2} = 3r_2 - 2r_1 u_x - ur_{1,x} + uu_{x,xx}, \psi_{-2} = (i + 3)r_{i+2} - ur_{i+1,x} + uu_x - 2ru_{r_{i,x}}. \) We now introduce two families of nonlocal symmetries by setting \( \Psi_{-k} = [\Psi_{-1}, \Psi_{-k+1}] \), \( k \leq -3 \), \( \Omega_l(X) = [\Psi_l, \Phi_5(X)] \), \( l \leq -1 \).

The \( \tau^- \)-nonlocal symmetries are distributed along weights as follows: \( |\Upsilon| = |\Theta_0(X)| = |\Phi_0(T)| = 0, \ |\Theta_1(X)| = 1, \ |\Phi_1(T)| = -1 = |\Theta_0(X)|, \ |\Phi_k| = -k, \ k \leq -1 \), \( |\Phi^k_{\text{inv}}(X)| = i + 1, \ i \geq 1 \), \( |\Omega_l(X)| = j, \ j \leq -1 \).

4.4.3. Lie algebra structure

Consider the following subspaces in \( \text{sym}_{\tau^{-}}(E) \): \( W \) spanned by \( \Upsilon, \Psi_k, k \leq -1 \), \( V[x] \) spanned by \( \Omega_l(X), l \geq 1 \), \( \Theta_0(X), \Theta_1(X) \), and \( \Phi^k_{\text{inv}}(X), k \geq 1 \), \( V[t] \) spanned by \( \Phi_0(T), \Phi_1(T) \). Then the following result holds:

| \( w_i \) | \( v_i(X) \) | \( v_i(T) \) |
|---|---|---|
| \( w_i \) | \( (j-i)w_{i+j} \) | \( jv_{i+j}(X), 0 \leq i + j \leq 1 \), otherwise \( jv_{i+j}(T) \) |
| \( v_i(X) \) | \( \ldots \) | \( v_{i+j}(\{X,X\}), 0 \leq i + j \leq 1 \), otherwise \( 0 \) |
| \( v_i(T) \) | \( \ldots \) | \( \ldots \) \( v_{i+j}(\{T,T\}) \) |

**Table 8.** The UHE: commutators in \( \text{sym}_{\tau^{-}}(E) \).
The modified Veronese web equation (mVWE) was studied in [1] and is related to the 

\[ 0 \leq i + j \leq 0, \quad \text{otherwise} \]

Thus, \( \text{sym}^+ (\mathcal{E}) \) is isomorphic to \( \mathcal{M}_0 \times (\mathcal{S}_z^+ [t] \oplus \mathcal{L}[x]) \) with the natural action of \( \mathcal{M}_0 \) on \( \mathcal{S}_z^+ [t] \oplus \mathcal{L}[x] \).

### 4.5. Recursion operators

The following proposition describes recursion operators for the symmetries of the UHE (see [21]):

**Proposition 3**

Equation (12) admits the recursion operator for symmetries \( \psi = \mathcal{R}_+ (\varphi) \) defined by the following system:

\[
\begin{align*}
D_x (\psi) &= u_y^{-1} (-D_y (\varphi) + u_{xy} \psi), \\
D_y (\psi) &= D_t (\varphi) - u_y^{-1} (u_t D_y (\varphi) + (u_y u_{tx} - u_t u_{xy}) \psi). 
\end{align*}
\]

The inverse operator \( \varphi = \mathcal{R}_- (\psi) \) is defined by system

\[
D_t (\varphi) = D_y (\psi) - u_t D_x (\psi) + u_{tx} \psi; \quad D_y (\varphi) = -u_y D_x (\psi) + u_{xy} \psi. \tag{15}
\]

The action of the recursion operators on local symmetries and shadows is schematically shown in Figure 3.

### 5. The modified Veronese web equation

The modified Veronese web equation (mVWE) was studied in [1] and is related to the Veronese web equation, [24] [10], by the Bäcklund transformation [18].

\[
\begin{align*}
&\cdots \mathcal{R}_- \mathcal{R}_+ \psi^+ \mathcal{R}_+ \psi_1^+ \mathcal{R}_+ \psi_0^+ \mathcal{R}_+ \psi^{-} \mathcal{R}_- \mathcal{R}_+ \psi^{-}_1 \mathcal{R}_+ \psi^{-}_2 \mathcal{R}_- \mathcal{R}_+ \psi^{-}_3 \mathcal{R}_+ \cdots \\
&\cdots \mathcal{R}_- \mathcal{R}_+ \xi^+ \mathcal{R}_+ \xi_1^+ \mathcal{R}_+ \xi_0^+ \mathcal{R}_+ \varphi^+ \mathcal{R}_+ \varphi^{-} \mathcal{R}_- \mathcal{R}_+ \theta^+ \mathcal{R}_+ \theta_1^+ \mathcal{R}_- \mathcal{R}_+ \theta_0^+ \mathcal{R}_+ \omega^+ \mathcal{R}_- \mathcal{R}_+ \omega_1^+ \mathcal{R}_+ \omega_2^+ \mathcal{R}_- \mathcal{R}_+ \cdots
\end{align*}
\]

\[
\begin{align*}
&\cdots \mathcal{R}_- \mathcal{R}_+ \psi^+ \mathcal{R}_+ \psi_1^+ \mathcal{R}_+ \psi_0^+ \mathcal{R}_+ \psi^{-} \mathcal{R}_- \mathcal{R}_+ \psi^{-}_1 \mathcal{R}_+ \psi^{-}_2 \mathcal{R}_- \mathcal{R}_+ \psi^{-}_3 \mathcal{R}_+ \cdots \\
&\cdots \mathcal{R}_- \mathcal{R}_+ \xi^+ \mathcal{R}_+ \xi_1^+ \mathcal{R}_+ \xi_0^+ \mathcal{R}_+ \varphi^+ \mathcal{R}_+ \varphi^{-} \mathcal{R}_- \mathcal{R}_+ \theta^+ \mathcal{R}_+ \theta_1^+ \mathcal{R}_- \mathcal{R}_+ \theta_0^+ \mathcal{R}_+ \omega^+ \mathcal{R}_- \mathcal{R}_+ \omega_1^+ \mathcal{R}_+ \omega_2^+ \mathcal{R}_- \mathcal{R}_+ \cdots
\end{align*}
\]

**Figure 3.** The UHE: action of recursion operators [11], [15].
\[ \text{Table 10. The mVwe: commutators of local symmetries.} \]

| \( \varphi(T) \) | \( \theta_0(X) \) | \( \theta_1(X) \) | \( v(Y) \) |
|----------------|----------------|----------------|-----------|
| \( \varphi(T) \) | \( \varphi([T,T]) \) | 0 | 0 | 0 |
| \( \theta_0(X) \) | \( \theta_0([X,X]) \) | \( \theta_1([X,X]) \) | 0 |
| \( \theta_1(X) \) | \( \theta_0([X,X]) \) | 0 | 0 |
| \( v(Y) \) | \( v([Y,Y]) \) | \( v([Y,Y]) \) | \( v([Y,Y]) \) |

5.1. The equation

The mVWE has the form

\[ u_{ty} = u_t u_{xy} - u_y u_{tx}. \]  \hfill (16)

We assign zero weights to all the variables under consideration. Internal coordinates are chosen similar to the previous cases, i.e.,

\[ u_k = u_x \ldots x, \quad u_{k,l} = u_x \ldots x \ldots y, \quad u_{k,l}^y = u_x \ldots y \ldots y, \]

where \( k \geq 0, \ l > 0 \). Then the total derivatives read

\[ D_x = \frac{\partial}{\partial x} + \sum_k u_{k+1} \frac{\partial}{\partial u_k} + \sum_{k,l} \left( u_{k,l+1}^y \frac{\partial}{\partial u_{k,l}} + u_{k+1,l}^y \frac{\partial}{\partial u_{k,l}} \right), \]

\[ D_y = \frac{\partial}{\partial y} + \sum_k u_{k,1}^y \frac{\partial}{\partial u_k} + \sum_{k,l} \left( u_{k,l+1}^y \frac{\partial}{\partial u_{k,l}} + D_k^y D_{l-1}((u_{01}^0 u_{11}^1 - u_{01}^0 u_{11}^1)) \frac{\partial}{\partial u_{k,l}} \right), \]

\[ D_t = \frac{\partial}{\partial t} + \sum_{r} u_{r,1}^y \frac{\partial}{\partial u_k} + \sum_{k,l} \left( D_x^r D_y ((u_{01}^0 u_{11}^1 - u_{01}^0 u_{11}^1)) \frac{\partial}{\partial u_{k,l}} + u_{k,l+1}^y \frac{\partial}{\partial u_{k,l}} \right). \]

Symmetries are defined by the equation

\[ \ell_F(\varphi) = D_t D_y(\varphi) - u_t D_x D_y(\varphi) + u_y D_t D_x(\varphi) - u_{xy} D_t(\varphi) + u_{tx} D_y(\varphi) = 0. \]  \hfill (17)

The space of solutions is generated by the functions \( \varphi(T) = T u_t, \ v(Y) = Y u_y, \theta_0(X) = X u_x - X' u, \theta_1(X) = X, \) where \( X = X(x) \), \( Y = Y(y) \), and \( T = T(t) \) are arbitrary functions of their arguments. The commutators of the symmetries are presented in Table \([10]\)

5.2. The Lax pair and hierarchies

The mVwe admits the Lax pair

\[ q_t = (\lambda + 1)^{-1} u_t q_x, \quad q_y = \lambda^{-1} u_y q_x. \]  \hfill (18)

Expanding in powers of \( \lambda \), one obtains \( q_{i-1,t} + q_{i,t} = u_t q_{i,x}, \ q_{i-1,y} = u_y q_{i,x} \). Then the positive covering acquires the form

\[ q_{1,t} = \frac{u_t}{u_y}, \quad q_{1,x} = \frac{1}{u_y}, \]

\[ q_{i,x} = \frac{q_{i-1,y}}{u_y}, \quad q_{i,t} = \frac{u_t}{u_y} q_{i-1,y} - q_{i-1,t}. \]
$i > 1$, the additional variables being $q_i^{(j)}$ defined as usual: $q_i^{(0)} = q_i$, $q_i^{(j+1)} = q_i^{(j)}$ with $\left\{ q_i^{(j)} \right\} = 0$.

The defining equations for the negative covering are

\[
\begin{align*}
    r_{1,t} &= u_t(u_x - 1), & r_{1,y} &= u_x u_y; \\
    r_{i,t} &= u_t r_{i-1,x} - r_{i-1,t}, & r_{i,y} &= u_y r_{i-1,x},
\end{align*}
\]

$i > 1$. The auxiliary variables are $r_i^{(j)}$, defined by $r_i^{(0)} = r_i$, $r_i^{(j+1)} = r_i^{(j)}$. Similar to the positive case, their weights are trivial.

5.3. Nonlocal symmetries in the positive covering

5.3.1. Lifts of local symmetries All the local symmetries can be lifted to the $\tau^+$ covering. Namely, the lift of $\varphi_i(T) = T u_t$ is $\Phi(T) = (\varphi(T), \varphi', \ldots, \varphi')$, where $\varphi = T q_i$. The lift of $\theta_0(X) = X u_x - X' u$ is given by $\Theta_0(X) = (\theta_0(X), \theta'_0, \ldots, \theta'_0)$, where $\theta_0 = X q_i$. To lift the symmetry $\psi(Y) = Y u_y$, consider the operator

\[
\mathcal{Y} = q_i \frac{\partial}{\partial y} + \sum_{k=1}^{\infty} q_{k+1} \frac{\partial}{\partial q_k}
\]

and set recursively

\[
R_1(Y) = Y' q_1, \quad R_n(Y) = \frac{1}{n} R_{n-1}(Y).
\]

Then $\Upsilon(Y) = (\psi(Y), \psi_1, \ldots, \psi_i, \ldots)$, where $\psi^i = Y q_i u_y - R_i(Y)$. Finally, for $\theta_1(X) = X$ one has $\Theta_1 = (\theta_1(X), 0, \ldots, 0, \ldots)$ for the lift of $\theta_1(X) = X$.

5.3.2. Nonlocal symmetries There exist three families of ‘purely nonlocal’ symmetries in $\tau^+$. The first consists of the invisible symmetries which are of the form

\[
\Phi^k_{\text{inv}}(Y) = \left( 0, \ldots, 0, \varphi^1_{\text{inv}}, \ldots, \varphi^i_{\text{inv}}, \ldots \right)
\]

where $\varphi^1_{\text{inv}} = Y$ and $\varphi^i_{\text{inv}} = R_{i-1}(Y)$, $i > 1$, $R_i(Y)$ being defined by $^{[15]}$.

The second family is constructed as follows: symmetries $\Psi_0$ and $\Psi_1$ are defined by $\Psi_0 = (\psi_0^0, \psi_1^0, \ldots, \psi_i^0, \ldots)$, where $\psi_0^0 = q_1 u_y + u$, $\psi_0^i = -(i + 1) q_{i+1} - i q_i + q_1 q_{i+1}$, $i > 0$, and $\Psi_1 = (\psi_1^1, \psi_1^i, \ldots, \psi_i^i, \ldots)$, where $\psi_1^0 = (-2 q_2 - q_1 + q_1 q_{i+1}) u_y$, $\psi_1^i = (i + 2) q_{i+2} + (i + 1) q_{i+1} - q_1 q_{i+1} + (-2 q_2 - q_1 + q_1 q_{i+1}) q_{i+1}$, $i > 0$. Let us also set by induction $\Psi_k = [\Psi_0, \Psi_{k-1}] + k k_k - 1, k > 1$.

The third family consists of the symmetries $\Xi_k(Y) = [\Psi_k, \Phi^1_{\text{inv}}(Y)] - (k - 1)! Y(Y)$, $k = 0, 1, \ldots$

5.3.3. Lie algebra structure Consider the following subspaces in $\text{sym}^+ (\mathcal{E})$: $V[x]$ spanned by $\Theta_0(X)$, $\Theta_1(X)$, $V[t]$ spanned by $\Phi(T)$, $V[y]$ spanned by $\Xi_k(Y)$, $\Upsilon(Y)$, and $\Phi^1_{\text{inv}}(Y)$, $W$ spanned by $\Psi_k$. The following result holds:

**Theorem 7** There exist bases $w_i$, $i \geq 1$, in $W$, $v_i(X), i = 0, 1$, in $V[x]$, $v_i(Y), i \in \mathbb{Z}$, in $V[y]$, $v(T)$ in $V[t]$, such that their commutators satisfy the relations presented in Table $^{[1]}$. In other words, $\text{sym}^+(\mathcal{E})$ is isomorphic to $\mathfrak{w}_0^+ \times (\mathfrak{e}[y] \oplus \mathfrak{e}_0^+ [x]) \oplus \mathfrak{e}[t]$ with the natural action of the Witt algebra $\mathfrak{w}_0^+$ on $\mathfrak{e}[y] \oplus \mathfrak{e}_0^+ [x]$. Here $\mathfrak{w}_0^+$ denotes the subalgebra in $\mathfrak{w}_0^+$ generated by the elements $e_i - e_0$, $i \geq 1$. 

Nonlocal symmetries of Lax integrable equations

5.4. Nonlocal symmetries in the negative covering

5.4.1. Lifts of local symmetries

The symmetry \( \varphi(T) = Tu \) is lifted to \( \Phi(T) = (\varphi(T), \varphi^1, \ldots, \varphi^i, \ldots) \), where \( \varphi^i = Tr_{i,t} \). To define the lift of \( \theta_0(X) = Xu_x - x' u \), consider the operator

\[
\mathcal{Y} = u \frac{\partial}{\partial x} + 2r_1 \frac{\partial}{\partial u} + \sum_{k=1}^{\infty} (k + 2)r_{k+1} \frac{\partial}{\partial r_k}
\]

and define by induction the quantities \( R_n(X) \) by setting

\[
R_1(X) = Xu_x, \quad R_n(X) = \frac{1}{n} \mathcal{Y}(R_{n-1}).
\]

Then \( \Theta_0(X) = (\theta_0(X), \theta_1^0, \ldots, \theta_1^0, \ldots) \), where \( \theta_1^0 = Xr_{i,x} - R_{i+1}(X) \), \( R_n \) being defined by \( \Phi(T) = (\varphi(T), \varphi^1, \ldots, \varphi^i, \varphi^i, \ldots) \).

5.4.2. Nonlocal symmetries

Similar to the positive case, three families of nonlocal symmetries arise in \( \tau^- (\mathcal{E}) \). The first consists of invisible symmetries

\[
\Phi^k_{inv}(X) = \left( 0, \ldots, 0, \varphi^1_{inv}, \ldots, \varphi^i_{inv}, \ldots \right),
\]

where \( \varphi^1_{inv} = X \) and \( \varphi^i_{inv} = R_{i-1}(X) \), \( i \geq 2 \).

Two nonlocal symmetries, \( \Psi_{-1} = (\psi_{-1}, \psi_{-1}^1, \ldots, \psi_{-1}^i, \ldots) \) and \( \Psi_{-2} = (\psi_{-2}, \psi_{-2}^1, \ldots, \psi_{-2}^i, \ldots) \), are constructed explicitly. Namely, we set \( \psi_{-1} = 2r_1 - uu_x + u, \psi_{-1}^1 = (i + 2)r_{i+2} + (i + 1)r_1 - ur_{i,x} \) and \( \psi_{-2} = 3r_2 - 2r_1 u_x - ur_{1,x} + uu_x^2 - u, \psi_{-2}^1 = (i + 3)r_{i+2} - (i + 1)r_1 - ur_{i+1,x} + (uu_x - 2r_1)r_{i,x} \). Then the second family is defined by \( \Psi_{-k-1} = [\Psi_{-1}, \Psi_{-k}] - k\Psi_{-k} + (-1)^{k+1}(k-3)! \Psi_{-1}, k > 1 \), while the third one is \( \Omega_{-l}(X) = [\Psi_{-1}, \Phi_4(X)] + (-1)^{l+1}(l-2)!\Theta_1(X), l \geq 0 \).

5.4.3. Lie algebra structure

Let \( W \) spanned by \( \Psi_k, V[x] \) spanned by \( \Omega_l(X), \Theta_0(X), \Theta_1(X) \), and \( \Phi^k_{inv}(X) \), \( V[y] \) spanned by \( \Phi(T), V[y] \) spanned by \( \mathcal{Y}(Y) \) be subspaces in \( \text{sym}_{\tau^-}(\mathcal{E}) \).
ψ defines a recursion operator ϕ yields s is more complicated than in the previous sections. It is described by

\[ D \]

The system Proposition 4 λ and s another covering for Equation (16). Note that s Now put Remark 4 Obviously, the Lie algebra \( \tilde{W}_0 \) is isomorphic to \( \tilde{W}_1 \), the isomorphism \( e_{-k} - e_0 \mapsto -(e_k - e_0), k \geq 1 \), is given by the change of variable \( z \mapsto z^{-1} \).

5.5. Recursion operators

To construct a recursion operator for Equation (16) we use the techniques of \[ \text{cf. [28, 15, 19, 23, 14]} \] also. We find a shadow for Equation (16) in the covering (18). It is

\[ s \]

Table 12. The \( \text{mVwe} \): commutators in \( \text{sym} \rangle (E) \).

| \( w_i \) | \( v_i(X) \) | \( v(Y) \) | \( v(T) \) |
| --- | --- | --- | --- |
| \( (j-t)w_{i+j} + iw_i - jw_j \) | \( j(v_{i+j}(X) - v_i(X)) \) | 0 | 0 |
| \( v_i(X) \) | \( v_{i+j}([X, X]) \) | 0 | 0 |
| \( v(Y) \) | \( \ldots \) | \( v([Y, Y]) \) | 0 |
| \( v(T) \) | \( \ldots \) | \( \ldots \) | \( v([T, T]) \) |

Theorem 8 There exist bases \( w_i, i \leq -1 \), in \( W, v_i(X), i \in \mathbb{Z} \), in \( V[x] \), \( v(T) \) in \( V[y] \), \( v(Y) \) in \( V[y] \), such that their commutators satisfy the relations presented in Table 12. Thus, \( \text{sym}_{\rangle E}(E) \) is isomorphic to \( \tilde{W}_0 \times \mathbb{L}[x] \oplus \tilde{W}[y] \oplus \tilde{W}[t] \) with the natural action of the Witt algebra \( \tilde{W} \) on \( \mathbb{L}[x] \). Here \( \tilde{W}_0 \) denotes the subalgebra in \( \tilde{W}_0 \) generated by the elements \( e_i - e_0, i \leq -1 \).

for Equation (16). Note that \( s \) is a solution to the linearization (17) of Equation (16). Now put

\[ s = \sum_{n=-\infty}^{\infty} s_n \lambda^n. \]

Since (17) is independent of \( \lambda \), each \( s_n \) is a solution to (17). Substituting (22) to (21) yields \( s_{n-1,t} + s_{n,t} = u_t s_{n,x} - u_x s_n, s_{n-1,y} = u_y, s_{n,x} - u_{xy} s_n \). Denoting \( s_{n-1} = \varphi \) and \( s_n = \psi \), we have

Proposition 4 The system

\[ D_t(\psi) = -D_t(\varphi) + u_y^{-1} (u_t D_y(\varphi) + (u_t u_{xy} - u_y u_{tx}) \psi), \]
\[ D_x(\psi) = u_y^{-1} (D_y(\varphi) + u_y \psi) \]

defines a recursion operator \( \varphi = \mathcal{R}_+(\varphi) \) for symmetries of Equation (16). The inverse operator \( \varphi = \mathcal{R}_-(\psi) \) is given by system

\[ D_t(\varphi) = -D_t(\psi) + u_t D_x(\psi) - u_{xx} \psi, \quad D_y(\varphi) = u_y D_x(\psi) - u_{xy} \psi. \]

The action of recursion operators \( \mathcal{R}_+ \) and \( \mathcal{R}_- \) on the shadows of nonlocal symmetries is more complicated than in the previous sections. It is described by
Proposition 6 Thus we have

\[
\frac{\mathcal{R}_-}{\mathcal{R}_+} \psi_i^+ \frac{\mathcal{R}_-}{\mathcal{R}_+} \psi_1^+ \mathcal{R}_- \psi_0^+ \mathcal{R}_- \psi_{-1}^+ \mathcal{R}_- \psi_{-2}^+ \mathcal{R}_- \psi_{-3}^+ \cdots \]

\[
\cdots \frac{\mathcal{R}_-}{\mathcal{R}_+} \xi_i^+ \frac{\mathcal{R}_-}{\mathcal{R}_+} \xi_0^+ \mathcal{R}_- \xi_1^+ \mathcal{R}_- \xi_{-1}^+ \mathcal{R}_- \xi_{-2}^+ \mathcal{R}_- \xi_{-3}^+ \cdots
\]

Figure 4. The mVWE: action of recursion operators (25), (26).

Proposition 5 The action of (25) and (26) on the shadows \( \psi_i^+ \), \( \xi_i^+ \), \( \omega_i^- \) is of the form

\[
\mathcal{R}_+(\psi_i^+) = \sum_{j=1}^{i+1} \alpha_{ij} \psi_j^+, \quad \alpha_{i+1} \neq 0, \quad i \geq 0, \quad (25)
\]

\[
\mathcal{R}_+(\xi_i^+) = \sum_{j=1}^{i+1} \beta_{ij} \xi_j^+, \quad \beta_{i+1} \neq 0, \quad i \geq 0, \quad (26)
\]

\[
\mathcal{R}_-(\psi_{-k}) = \sum_{j=1}^{k+1} \gamma_{kj} \psi_{-j}^-, \quad \gamma_{k+1} \neq 0, \quad k \geq 1, \quad (27)
\]

\[
\mathcal{R}_-(\omega_i^-) = \sum_{j=0}^{i+1} \delta_{ij} \omega_j^- + \varepsilon_i \theta_0^-, \quad \delta_{i+1} \neq 0, \quad i \geq 0, \quad (28)
\]

where \( \alpha_{ij}, \beta_{ij}, \gamma_{kj}, \) and \( \varepsilon_i \) are certain constants. To find the action of \( \mathcal{R}_- \) to \( \psi_i^+ \), \( \xi_i^+ \) one has to apply \( \mathcal{R}_- \) to both sides of (25), (26) and then to solve the obtained triangular systems. In the same way it is possible to find the action of \( \mathcal{R}_+ \) to \( \psi_{-i} \), \( \omega_i^- \).

These results are schematically shown in Figure 4, where the wavy arrows indicate the action (25), (26). The usage of straight arrows corresponds to Remark 2.

5.6. Bäcklund auto-transformation

Consider again the first and the second equations from the positive covering of Equation (16) and replace \( q_1 \) by \( v \) in them:

\[
v_t = \frac{u_x}{u_y}, \quad v_x = \frac{1}{u_y}.
\]

This gives the following expressions for \( u_t \) and \( u_y \):

\[
u_t = \frac{v_y}{v_x}, \quad u_y = \frac{1}{v_x}.
\]

Cross-differentiation of this system with respect to \( y \) and \( t \) gives \( v_{tx} = v_tv_{xy} - v_xv_{ty} \). This equation differs from Equation (16) just by the change of variables

\[
\begin{align*}
x &\mapsto y, \\ y &\mapsto x.
\end{align*}
\]

Thus we have

Proposition 6 The superposition of (25) and (26) defines a Bäcklund auto-transformation for Equation (16). The inverse transformation is given by the superposition of (31) and (30).
Table 13. Lie algebras of nonlocal symmetries.

|                       | \( \tau^+ \)                                      | \( \tau^- \)                                      |
|-----------------------|--------------------------------------------------|--------------------------------------------------|
| rdDym equation        | \( \mathfrak{w}_0^0 \times (L_{-3}^t \oplus L_y) \) | \( \mathfrak{w}_0^+ \times L_t \oplus \mathfrak{w}_y \) |
| 3D Pavlov equation    | \( \mathfrak{w}_0^+ \times (L_{-3}^q \oplus L_{-4}^t) \) | \( \mathfrak{w}_0^+ \times L_t \) |
| UHE                   | \( \mathfrak{w}_0^+ \times (L_{+2}^y \oplus L_t) \) | \( \mathfrak{w}_0^+ \times L_t \oplus \mathfrak{w}_y \) |
| mVwe                  | \( \tilde{\mathfrak{w}}_0^+ \times (L_{+2}^y \oplus L_{+2}^x \oplus V_t) \) | \( \tilde{\mathfrak{w}}_0^+ \times L_t \oplus \mathfrak{w}_y \oplus \mathfrak{w}_t \) |

6. Conclusions

The equations discussed above have many common features:

(i) all of them admit a differential coverings with non-removable parameter;
(ii) all of them are linearly degenerate;
(iii) each of these equation can be obtained as a symmetry reduction of the 5D equation

\[ u_{zx} + u_{yz} - u_{zs} + u_x u_{xs} - u_y u_{yx} = 0, \text{ see } [2]; \]

(iv) as it is shown in [22], they are pair-wise related by Bäcklund transformations.

This similarity manifests itself in striking resemblance of their symmetry algebra structures (see Table 13). Perhaps, the mVWE equation stands alone a bit: its symmetries are not graded in the same sense as symmetries of the other three equations.

We think that it will be extremely interesting to find out which properties of equations, besides their linearly degeneracy, are responsible for the such symmetry structures and plan to shed light on this problem in future research. We also intend to clarify the invariant meaning of the operators \( \mathcal{Y} \) that played such an important role in the above discussed constructions.

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