THIRTY-TWO EQUIVALENCE RELATIONS ON KNOT PROJECTIONS

NOBORU ITO AND YUSUKE TAKIMURA

Abstract. We consider 32 homotopy classifications of knot projections (images of generic immersions from a circle into a 2-sphere). These 32 equivalence relations are obtained based on which moves are forbidden among the five type of Reidemeister moves. We show that 32 cases contain 20 non-trivial cases that are mutually different. To complete the proof, we obtain new tools, i.e., new invariants.

1. Introduction

A knot projection is the image of a generic immersion from a circle into a 2-sphere. In particular, every self-intersection is a transverse double point, which is simply called a double point. Every double point consists of two branches and thus, if the two branches are given over/under information for every double point of a knot projection, we can then obtain a knot diagram.

Several interesting homotopy classes have been considered by restricting the Reidemeister moves that consist of three types of local replacements of knot projections, as shown in Fig. 1 [1, 2, 5, 7, 8, 9, 10, 11] (for other works, see [4]). Because a knot projection is a single component, we consider five types of Reidemeister moves, namely, RI, strong RII, weak RII, strong RIII, and weak RIII, which are the local replacements defined in Fig. 2.

We can consider 32 (= 2^5) equivalence classes by restricting the Reidemeister moves. Naturally, we have Problem 1 as below.

Problem 1. (1) Which equivalence classes of knot projections are non-trivial? (2) Which two equivalence relations on knot projections are independent?

Theorem 1 described in this paper solves Problem 1.

Key words and phrases. Knot projections; spherical curves; Reidemeister moves; homotopy.

The work of N. Ito was partially supported by a Waseda University Grant for Special Research Projects (Project number: 2015K-342) and Japanese-German Graduate Externship. N. Ito was a project researcher of Grant-in-Aid for Scientific Research (S) 24224002 (April 2016 – March 2017).
2. **Main Result**

**Definition 1.** Let \( R = \{ \text{RI}, \text{strong RI}, \text{weak RI}, \text{strong RI I}, \text{weak RI I}, \text{strong RI I I}, \text{weak RI I I} \} \) and let \( S \) be a subset of \( R \). We say that two knot projections \( P \) and \( P' \) are \( S \)-equivalent if they can be related by a finite sequence consisting of the elements of \( S \). We denote this equivalence by \( \sim_S \). There are \( 32 = 2^5 \) possibilities of type

\[
\{ \text{knot projections} \} / \sim_S
\]

that is denoted by \( C_S \).

**Theorem 1.** Let \( C_S \) be as defined in Definition 1. Among the 32 possible sets, 8 sets are trivial (i.e., all knot projections are equivalent to the trivial knot projection) and the remaining 24 sets are equivalent to the following 20 sets that are non-trivial and mutually different:

\[
S = \{ \text{strong RI}, \text{weak RI}, \text{strong RI I}, \text{weak RI I}, \text{strong RI I I}, \text{weak RI I I} \}, \{ \text{strong RI I}, \text{weak RI I}, \text{strong RI I I}, \text{weak RI I I} \}, \{ \text{strong RI I I}, \text{weak RI I I} \}, \{ \text{RI}, \text{strong RI}, \text{weak RI} \}, \{ \text{RI}, \text{strong RI I}, \text{weak RI I} \}, \{ \text{RI}, \text{strong RI I I}, \text{weak RI I I} \}, \{ \text{RI}, \text{weak RI I I} \}, \{ \text{RI}, \text{weak RI I}, \text{weak RI I I} \}, \{ \text{RI}, \text{weak RI} \}, \{ \text{RI}, \text{weak RI I} \}, \{ \text{RI}, \text{weak RI I I} \}, \{ \text{RI}, \text{weak RI I I} \}, \{ \text{RI}, \text{weak RI I I} \}, \{ \text{RI}, \text{weak RI I I} \}.
\]

or \( \emptyset \).

3. **Invariants of knot projections**

3.1. **New tools—new invariants.** In this section, we introduce a new invariant \( \text{Coh}_{\text{odd}}(P) \) for a knot projection \( P \) under weak RI and strong RI I I to detect one of the two cases: \( C_{\{ \text{weak RI, strong RI I I} \}} \) and \( C_{\{ \text{weak RI I I, strong RI I I} \}} \).

Let \( P \) be a knot projection with an arbitrary orientation. If the orientation induces an orientation of an \( n \)-gon of \( P \) (i.e., the orientations of \( n \) edges are coherent), the \( n \)-gon is called a **coherent** \( n \)-gon. An \( n \)-gon that is not coherent is called an **incoherent** \( n \)-gon. The sum of the number of coherent \((2m + 1)\)-gons \((m \in \mathbb{Z}_{\geq 0})\) is called the **odd coherent number**. We set the function \( \text{Coh}_{\text{odd}} \) from the set of knot projections to \( \{0, 1\} \) such that

\[
\text{Coh}_{\text{odd}}(P) = \begin{cases} 
0 & \text{if odd coherent number is 0}, \\
1 & \text{if odd coherent number is not 0}.
\end{cases}
\]

By definition, \( \text{Coh}_{\text{odd}}(P) \) does not depend on the choice of the orientation of \( P \).
Theorem 2. Let $P$ be a knot projection. Then, $\text{Coh}^{\text{odd}}(P)$ is invariant under weak RI II and strong RI III.

Proof. (Invariance under strong RI III) A single strong RI III between two knot projections $P$ and $P'$ preserves the condition $\text{Coh}^{\text{odd}}(P) = 1 = \text{Coh}^{\text{odd}}(P')$ (see Fig. 2).

• (Invariance under weak RI II) Suppose that $a(\geq 2)$, $b(\geq 1)$, $c(\geq 2)$, and $d(\geq 1)$ are integers. The left (resp. right) part of Fig. 3 represents the local situation of a knot projection $P$ (resp. $P'$). Assume that $\text{Coh}^{\text{odd}}(P) = 0$. Then, a-gon (resp. c-gon) is an incoherent a-gon (resp. c-gon) or coherent even a-gon (resp. c-gon), i.e., a coherent a-gon (resp. c-gon) where $a$ (resp. $c$) is an even integer. Any other polygon of $P'$ is an incoherent or coherent even $n$-gon for some $n$. This is because $a \equiv a - 2 \pmod{2}$ and $c \equiv c - 2 \pmod{2}$. If the $(b + d)$-gon in $P'$ is a coherent $(b + d)$-gon, which is a boundary of a disk, we cannot apply a single weak RI II to obtain $P$; this indicates that there is a contradiction. Thus, $(b + d)$-gon is an incoherent $(b + d)$-gon. Then, $\text{Coh}^{\text{odd}}(P') = 0$. Similarly, if $\text{Coh}^{\text{odd}}(P') = 0$, it can be easily seen that $\text{Coh}^{\text{odd}}(P) = 0$. As a result, $\text{Coh}^{\text{odd}}(P) = 0 \Leftrightarrow \text{Coh}^{\text{odd}}(P') = 0$ and then, $\text{Coh}^{\text{odd}}(P) = \text{Coh}^{\text{odd}}(P')$. □

Next, we introduce an invariant $C(P)$ in a similar fashion. For a knot projection $P$, $c(P)$ denotes the number of double points. We define a function from the set of all knot projections to $\{0, 1\}$ such that

$$C(P) = \begin{cases} 0 & \text{if } c(P) = 0, \\ 1 & \text{if } c(P) \neq 0. \end{cases}$$

Theorem 3. $C(P)$ is invariant under weak RI II, weak RI III, and strong RI III.

Proof. A single weak RI II, weak RI III, or strong RI III between two knot projections $P$ and $P'$ preserves the condition $C(P) = 1 = C(P')$, as can be seen in Fig. 2. □

Theorem 4. Let $P$ be a knot projection. A knot projection $P^{2s}r$ (resp. $P^{2w}r$) with no coherent (resp. incoherent) 2-gons is obtained from $P$ only decreasing double points by a finite sequence consisting of strong RI (resp. weak RI). Two knot projections, $P^{2sr}$ (resp. $P^{2w}r$) and $P'^{2sr}$ (resp. $P'^{2w}r$), are sphere isotopic if and only if $P$ and $P'$ are related by a finite sequence consisting of strong RI (resp. weak RI).

Proof. We can prove the claim by considering the general RI restricted to only strong RI (resp. weak RI) in [5, Proof of Theorem 2.2 (2)]. □
3.2. Known invariants and facts. Let $P$ be a knot projection. For every double point of $P$, we provide over/under information, as shown in Fig. 4; as a result, we obtain a knot diagram $K_P$, with over/under information for every double point (cf. [11, Sec. 6.6]).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4}
\caption{A double point to a crossing of a knot diagram}
\end{figure}

**Fact 1.** If $P$ and $P'$ are related by a finite sequence consisting of RI and weak RI II, then a knot with knot diagram $K_P$ is isotopic to the knot with knot diagram $K_{P'}$. As a corollary, a knot projection $P$ and trivial knot projection $O$ are related by a finite sequence consisting of RI and weak RI II if and only if $P$ and $O$ are related by a finite sequence consisting of RI [5, Page 13, Corollary 4.1].

**Fact 2.** Let $P$ be a knot projection and $O$ be the trivial knot projection. If $P$ and $O$ are related by a finite sequence consisting of RI and strong RI II, then $P$ is a connected sum of knot projections, each of which is $O$, the curve with the shape of $\infty$, or the trefoil projection $3_1$, as shown in Fig. 5 [7, Page 621, Theorem 4].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5}
\caption{The curve with the shape of $\infty$ (left) and trefoil projection $3_1$ (right)}
\end{figure}

For every double point, we locally replace the two branches of a double point with two simple arcs, as shown in Fig. 6. After applying the replacements, called

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6}
\caption{Seifert resolution}
\end{figure}
Seifert resolution, to all the double points of a knot projection $P$, we obtain an arrangement of circles, called the Seifert circle arrangement, denoted by $S(P)$. The number of circles in $S(P)$ is called Seifert circle number and is denoted by $s(P)$.

**Fact 3** (a well-known fact). Let $P$ be a knot projection. Then, the Seifert circle arrangement $S(P)$ and the Seifert circle number $s(P)$ are invariant under weak RI I and weak RI II.

**Fact 4.** (1) Let $P^r$ be a knot projection obtained from a knot projection $P$ with no 1-gons and 2-gons only decreasing double points by a finite sequence consisting of RI and RI I. Two knot projections, $P^r$ and $P^{r'}$, are sphere isotopic if and only if $P$ and $P'$ are related by a finite sequence consisting of RI and RI I [10, Page 2302, Lemma 2.1] (See also [5, Page 5, Theorem 2.2 (3)] and [6]).

(2) Let $P^{1r}$ (resp. $P^{2r}$) be a knot projection obtained from a knot projection $P$ with no 1-gons (resp. 2-gons) only decreasing double points by a finite sequence consisting of RI (resp. RI II). Two knot projections, $P^{1r}$ and $P^{1r'}$ (resp. $P^{2r}$ and $P^{2r'}$), are sphere isotopic if and only if $P$ and $P'$ are related by a finite sequence consisting of RI (resp. RI II) [5, Page 5, Theorem 2.2 (1) and (2)].

**Fact 5.** Let $P^{sr}$ (resp. $P^{wr}$) be a knot projection obtained from a knot projection $P$ with no 1-gons and coherent (resp. incoherent) 2-gons only decreasing double points by a finite sequence consisting of RI and strong RI I (resp. weak RI I). Two knot projections, $P^{sr}$ and $P^{sr'}$, are sphere isotopic if and only if $P$ and $P'$ are related by a finite sequence consisting of RI and strong RI I (resp. weak RI I) [9, Page 2, Theorem 1].

**Fact 6.** Let us consider the local replacement of a double point as shown in Fig. 7. After applying the replacements to all the double points of a knot projection $P$, we have an arrangement of circles on $S^2$, called circle arrangement $\tau(P)$. The number of circles in $\tau(P)$ is called the circle number $|\tau(P)|$. $\tau(P)$ and $|\tau(P)|$ are invariant under RI and strong RI I [9, Page 5, Theorem 3 and Corollary 3].

**Fact 7.** Let $P$ be a knot projection and $D_P$ a knot diagram obtained by arbitrarily information of any over/under-crossing data. Let $tr(P)$ be the trivializing number [3, Page 440, Theorem 13] and $g(P)$ the canonical genus of $D_P$. The integer $W(P) = tr(P) - 2g(P)$ is invariant under RI, weak RI II, and weak RI III [8, Page 4, Theorem 2].

**Fact 8.** Let $P$ be a knot projection. The Arnold invariant $J^+_S(P)$ is invariant under strong RI I and RI II (Polyak mentions that the result was obtained by Arnold, see [11, Sec. 2.4]. For Arnold’s serial work, see [1]).
4. Proof of Theorem (1)

(1) \( S = \emptyset \). First, we consider the case where \( S = \emptyset \). It can be easily seen that, for this case, there exists only an equivalence relation, i.e., sphere isotopy. Here, two knot projections \( P \) and \( P' \) are called sphere isotopic if there exists a smooth family of homeomorphisms \( h_t : S^2 \to S^2 \) for \( t \in [0,1] \) such that \( h_0 \) is the identity map of \( S^2 \) and \( h_1(P) = P' \). Such a family of \( h_t \) is called a sphere isotopy. Thus, the relation \( \sim_S \) is different from any other equivalence relation.

- (Contracting sets.) Next, if \( S \) satisfies the condition that every knot projection is equal to the trivial knot projection \( O \), \( S \) is called a contracting set. Eight contracting sets, each of which satisfies the condition that every knot projection \( \sim_S O \), are listed below. To avoid confusion, we list them by specifying \( S \).

- (2) \( \{ \text{RI, strong RII, weak RII, strong RIII, weak RIII} \} \)
- (3) \( \{ \text{RI, strong RII, weak RII, strong RIII} \} \)
- (4) \( \{ \text{RI, strong RII, weak RII, weak RIII} \} \)
- (5) \( \{ \text{RI, strong RII, strong RIII, weak RIII} \} \)
- (6) \( \{ \text{RI, weak RII, strong RIII, weak RIII} \} \)
- (7) \( \{ \text{RI, strong RII, strong RIII} \} \)
- (8) \( \{ \text{RI, strong RII, weak RIII} \} \)
- (9) \( \{ \text{RI, weak RII, strong RIII} \} \)

Recall that every knot projection can be related to the trivial knot projection \( O \) by a finite sequence consisting of RI, RII, and RIII. To show that all the eight sets are contracting sets, we notice that it is sufficient to show that for cases (7), (8), and (9), \( S \) generates \( \{ \text{RI, RII, RIII} \} \) (= \( \{ \text{RI, strong RII, weak RII, strong RIII, weak RIII} \} \)). However, the last three cases have been already obtained from [8, Page 7, Proposition 1] using Fig. 8, the proof of which is simple; hence, we describe the proof here again.

![Figure 8. Key sequences of figures. Upper line represents a single strong RII (resp. weak RIII), which consists of two RIs, a weak RII (resp. strong RII), and a strong RIII (resp. weak RIII). Lower line represents a single strong RIII (resp. weak RIII) consists of two strong RIIIs and a weak RIII (resp. strong RIII).](image)

(7) \( \{ \text{RI, strong RII, strong RIII} \} \) generates \( \{ \text{RI, RII, RIII} \} \). This is because weak RIII is generated by strong RII and strong RIII and weak RII is generated by RI, strong RII, and weak RIII.

(8) \( \{ \text{RI, strong RII, weak RIII} \} \) generates \( \{ \text{RI, RII, RIII} \} \). This is because strong RIII is generated by strong RII and weak RIII. The reminder is clear from case (7).
\( \{ \text{RI, weak RI, strong RI III} \} \) generates \( \{ \text{RI, RI II, RI III} \} \). This is because strong RI II is generated by RI, weak RI II and strong RI III. The reminder is clear from case (7).

- **23 non-trivial equivalence classes.** Only 19 of the 23 cases need to be considered because 4 cases are the same as 2 other cases (see Point 1 and Point 2). In the following points, if two sets, \( S_1 \) and \( S_2 \), of Reidemeister moves generate the same equivalence relation, we say that \( S_1 \) is equivalent to \( S_2 \) and write \( S_1 \sim S_2 \).

**Point 1:** (10) \( \{ \text{strong RI II, weak RI II, weak RI III} \} \)

\[ \sim (11) \{ \text{strong RI II, weak RI II, strong RI III, weak RI III} \} \]

\[ \sim (12) \{ \text{strong RI II, weak RI II, strong RI III} \} \]

by using Fig. 8.

**Point 2:** (13) \( \{ \text{strong RI II, strong RI III} \} \)

\[ \sim (14) \{ \text{strong RI II, strong RI III, weak RI III} \} \]

\[ \sim (15) \{ \text{strong RI II, weak RI III} \} \]

by using Fig. 8.

Now, consider Conditions 1, 2, and 3 to list the remaining 19 cases.

- **Condition 1.** \( \text{RI} \in S \).

  First, we treat 11 (resp. 8) cases with \( \text{RI} \notin S \) (resp. \( \in S \)), called the non-RI (resp. RI) group. The condition is critical and stated as follows: the trivial knot projection and the curve with the shape of \( \infty \) can be related by the elements of \( S \) if and only if \( \text{RI} \in S \).

  (11) \( \{ \text{strong RI II, weak RI II, strong RI III, weak RI III} \} \)

  (14) \( \{ \text{strong RI II, strong RI III, weak RI III} \} \)

  (16) \( \{ \text{weak RI II, strong RI III} \} \)

  (17) \( \{ \text{weak RI II, strong RI III, weak RI III} \} \)

  (18) \( \{ \text{strong RI III} \} \)

  (19) \( \{ \text{weak RI} \} \)

  (20) \( \{ \text{strong RI II, weak RI} \} \)

  (21) \( \{ \text{weak RI II, weak RI III} \} \)

  (22) \( \{ \text{strong RI III, weak RI III} \} \)

  (23) \( \{ \text{strong RI III} \} \)

  (24) \( \{ \text{weak RI III} \} \)

- **Condition 2.** An equivalent class containing the trivial knot projection consists of a single element.

  A case satisfying Condition 2 is called a single triviality case. Cases satisfying (resp. not satisfying) Condition 2 are (16), (17), (19), (21), (22), (23), and (24) (resp. (11), (14), (18), and (20)). This is because \( C(P) = 0 \Leftrightarrow P \) belongs to \( O \) under (16), (17), (19), (21), (22), (23), or (24) by Theorem 3. Thus, we obtain Table 1.

|        | non-RI group |
|--------|--------------|
| non-single triviality | (11), (14), (18), (20) |
| single triviality     | (16), (17), (19), (21), (22), (23), (24) |

On the other hand, Cases (25)–(32), each of which contains RI, are listed as follows. These cases are referred to as the RI group.

(25) \( \{ \text{RI} \} \)

(26) \( \{ \text{RI, strong RI} \} \)

(27) \( \{ \text{RI, weak RI} \} \)
Condition 3. There exists an equivalence class containing both the knot projection $P_F$, as shown in Fig. 9, and the trivial knot projection. A case satisfying Condition 3 is called flower trivial. We now classify the RI group into two subsets as shown in Table 2 (note that all cases of the RI group satisfy the non-single triviality) by applying Fact 4 (2) to (25), Fact 5 to (26), Fact 5 to (27), Fact 4 (1) to (28), Fact 2 to (29), and Fact 1 to (30).

Table 2. RI group

| non-flower triviality | RI group (non-single triviality) |
|-----------------------|---------------------------------|
| (25), (26), (27), (28), (29), (30) |
| flower triviality     | (31), (32)                      |

Because it is easier to classify the RI group than the non-RI group, we do that first. An equivalence class containing $P$ is denoted by $[P]$.

4.1. Classification of the non-flower trivial cases in the RI group: (25), (26), (27), (28), (29), and (30). The knot projections, $4_1$ and $5_1$ are defined in Fig. 10. Table 3 classifies the non-flower trivial cases in the RI group except for the pair (25) and (30). By using the circle number (Fact 6), $|\tau(3_1)| \neq |\tau(4_1)|$ for (25). However, $[3_1] = [4_1]$ for (30).

4.2. Classification of the flower trivial cases in the RI group: (31) and (32). The knot projection $7_4$ is defined in Fig. 11. We can see that $[7_4] = [O]$, under the equivalence relation of (31). However, $[7_4] \neq [O]$, under the equivalence relation of (32) because $W(7_4) = 2 \neq 0 = W(O)$ (Fact 7). This completes the classification of the RI group.
4.3. Classification of the single trivial cases in the non-RI group: (16), (17), (19), (21), (22), (23), and (24). In this section, the curve appearing as \( \infty \) is denoted by \( \infty \). First, referring to Table 4, we can see that equivalence class

![Figure 11. 7₄](image)

Table 4. Non-RI group satisfying single triviality

| Case | Formulae | Properties |
|------|----------|------------|
| (16) | \( \infty = [3₁] = [P_Y] \) | \( |P_C| \neq |P_F| \) \( \text{Coh}^{\text{odd}}(P_F) \neq \text{Coh}^{\text{odd}}(P_C) \) (by Theorem 2) |
| (17) | \( \infty = [3₁] = [P_Y] \) | \( |P_C| = |P_F| \) |
| (19) | \( \infty = [3₁] \neq [P_Y] \) | \( s(3₁) \neq s(P_Y) \) (by Fact 3) | \( c(P_F^{2\text{wr}}) = c(P_F) = 8 \) \( P_F^{2\text{wr}} = P_F \) (by Theorem 4) \( \text{If } [P] = |P_F|, c(P) \geq 8 \). |
| (21) | \( \infty \neq [3₁] \neq [P_Y] \) | \( s(3₁) \neq s(P_Y) \) (by Fact 3) | \( \exists P \text{ s.t. } [P] = |P_F| \) and \( c(P) < 8 \) |
| (22) | \( \infty \neq [3₁] = [P_Y] \) | \( |P_F| \) has at least two elements. |
| (23) | \( \infty \neq [3₁] = [P_Y] \) | \( |P_F| \) consists of only \( P_F \). |
| (24) | \( \infty \neq [3₁] \neq [P_Y] \) | \( |3₁| \) consists of only \( 3₁ \). |

[3₁] contains (resp. does not contain) the curve with the shape of \( \infty \) under (16), (17), (19), or (21) (resp. (22), (23), or (24)) because the number of double points \( c(P) \) is invariant under any type of RI. Here, let us consider the curve \( P_Y \) with no 2-gons, which is obtained from the trivial knot projection via three RIs increasing double points, as shown in Fig. 12. The curve \( P_Y \) is (resp. is not) an element in [3₁] under (16) or (17) (resp. (19) or (21)). From Fact 8 \( s(P_Y) = 4 \neq 2 = s(3₁) \), under the equivalence relation of (19) or (21).

Second, we distinguish between (16) and (17). Let \( P_C \) be the knot projection defined in Fig. 13. For (16), \( \text{Coh}^{\text{odd}}(P_F) = 0 \) and \( \text{Coh}^{\text{odd}}(P_C) = 1 \), where \( \text{Coh}^{\text{odd}}(P) \) is invariant under weak RI and strong RI (i.e., \( [P_F] \neq [P_C] \)). On the other hand, \( [P_F] = [P_C] \) under the equivalence relation of (17).
Third, we distinguish between (19) and (21), where (19) and (21) can be detected by the minimum number of double points in class $[P_F]$. Here, note that from Theorem 4, $P_{F}^{2\sigma r} = P_{F}$ under (19), which implies that the minimum number of double points in the equivalence class is eight. However, $[P_F]$ contains the knot projection with two (< 8) double points under (21).

Finally, consider the remaining three equivalence relations (22), (23), and (24). We notice that $[3_1]$ consists of a single element $3_1$ (resp. at least two elements) under (24) (resp. (22) or (23)) because $3_1$ does not have an incoherent (resp. has a coherent) 3-gon. Further, for the flower knot projection $P_F$, $[P_F]$ consists of a single element $P_F$ (resp. at least have two elements) under (23) (resp. (22)) because $P_F$ does not have a coherent (resp. have an incoherent) 3-gon.

4.4. Classification of the non-single trivial cases in the non-RI group: (11), (14), (18), and (20). In this section, we say that the symbol $\infty$ indicates the curve with the shape of $\infty$. Table 5 shows the claim.

| Case | Detection | Formulae | Key Fact |
|------|-----------|----------|----------|
| (11) | $\infty = 3_1 = [P_Y]$ | | |
| (14) | $\infty \neq 3_1 = [P_Y]$ | $J_{S^+}(\infty) = 0 \neq 2 = J_{S^+}(3_1)$ | Fact 8 |
| (18) | $\infty \neq 3_1 \neq [P_Y]$ | $3_1^{2\sigma r} = 3_1$, $P_{Y}^{2\sigma r} = P_{Y}$, $\infty^{2\sigma r} = \infty$ | Theorem 4 |
| (20) | $\infty = 3_1 \neq [P_Y]$ | $3_1^{2\sigma r} = \infty$, $P_{Y}^{2\sigma r} = P_{Y}$ | Fact 4 (2) |

References

[1] V.I. Arnold, Topological invariants of plane curves and caustics, American Mathematical Society, Providence, RI, 1994.
[2] T. Hagge and J. Yazinski, On the necessity of Reidemeister move 2 for simplifying immersed planar curves, Banach Center Publ. 103 (2014), 101–110.
[3] R. Hanaki, Trivializing number of knots, J. Math. Soc. Japan 66 (2014), 435–447.
[4] N. Ito, Knot projections, CRC Press, Boca Raton, 2016.
[5] N. Ito and Y. Takimura, (1, 2) and weak (1, 3) homotopies on knot projections, J. Knot Theory Ramifications 22 (2013), 1350085, 14pp.
[6] N. Ito and Y. Takimura, Addendum: “(1, 2) and weak (1, 3) homotopies on knot projections”,
J. Knot Theory Ramifications 23 (2014), 1491001, 2pp.
[7] N. Ito, Y. Takimura, and K. Taniyama, Strong and weak (1, 3) homotopies on knot projections,
Osaka J. Math. 52 (2015) 617–646.
[8] N. Ito and Y. Takimura, Strong and weak (1, 2, 3) homotopies on knot projections, Internat.
J. Math. 26 (2015) 1550069 (8 pages).
[9] N. Ito and Y. Takimura, Strong and weak (1, 2) homotopies on knot projections and new
invariants, Kobe J. Math. 33 (2016), 13–30.
[10] M. Khovanov, Doodle groups, Trans. Amer. Math. Soc. 349 (1997) 2297–2315.
[11] M. Polyak, Invariants of plane curves and fronts via Gauss diagrams, Topology 37 (1998),
989–1009.

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1, Komaba,
Meguro-ku, Tokyo, 153-8914, Japan
Email address: noboru@ms.u-tokyo.ac.jp

Gakushuin Boys’ Junior High School, 1-5-1, Mejiro, Toshima-ku Tokyo, 171-0031, Japan
Email address: Yusuke.Takimura@gakushuin.ac.jp