RATE OF CONVERGENCE TOWARDS MEAN-FIELD EVOLUTION FOR WEAKLY INTERACTING BOSONS WITH SINGULAR THREE-BODY INTERACTIONS

JINYEOP LEE

ABSTRACT. In this paper, we investigate the dynamics of a system of $N$ weakly interacting bosons with singular three-body interactions in three dimensions. By assuming factorized initial data $\Psi_{N,0} = \psi_0^N$ and triple collisions, we prove that in the many-particle limit, its mean-field approximation converges to quintic Hartree dynamics. Moreover, we prove that the rate of convergence towards the mean-field quintic Hartree evolution is of $O\left(N^{-\frac{1+4a}{3+2a}}\right)$ for $\varphi_0 \in H^{\frac{3}{2}}(\mathbb{R}^3)$ where $0 \leq a < \frac{1}{2}$ and $O(N^{-1})$ for $a > \frac{1}{2}$. Our proof is based on and extends the Fock space approach.

1. Introduction

We are consider in the time-evolution of a condensate of $N$-particles under the presence of a three-body interaction in the mean-field regime. We prove that, in the many-particle limit, the solution of the many-particle Schrödinger equation with a three-body interaction can be approximated by its mean-field limit (quintic Hartree equation) in the trace norm sense.

We consider a system of $N$ bosonic particles interacting with three-body interaction having factorized initial data, i.e.,

$$i\partial_t \Psi_{N,t} = \left(\sum_{j=1}^{N} (-\Delta x_j) + \frac{1}{N^2} \sum_{1 \leq i < j < k \leq N} V(x_i - x_j, x_i - x_k)\right) \Psi_{N,t}$$

$$\Psi_{N,|t=0} = \prod_{j=1}^{N} \varphi_0(x_j).$$

Note that $\Psi_{N,t} \in L^2_{\text{sym}}(\mathbb{R}^{3N})$. The mean-field evolution of the above system is given by the following quintic Hartree system

$$i\partial_t \varphi_t = -\Delta \varphi_t + \frac{1}{2} \left(\int \int V(x - y, x - z)|\varphi_t(y)|^2|\varphi_t(z)|^2\right) \varphi_t$$

$$\varphi_{t|t=0} = \varphi_0.$$

To generalize the Coulomb interactions, it is natural to assume the three-body interaction potential $V(x - y, x - z)$ is related to the distance between $x$, $y$, and $z$. Moreover, it should be symmetric up to the variables $x$, $y$, and $z$. Hence, we will assume

$$V(x - y, x - z) = \lambda(v(x - y)v(x - z) + v(y - z)v(y - x) + v(z - x)v(z - y))$$

for some constant $\lambda > 0$. We assume $v$ to be the Coulomb potential, i.e., $v(x) = 1/|x|$. Note that $v$ satisfies the following operator inequality

$$v(x) \leq C(1 - \Delta)$$

by Hardy inequality.

To understand our system rigorously at time $t \geq 0$, we proceed as follows. First, we consider the density matrix $\gamma_{N,t} = |\Psi_{N,t}\rangle \langle \Psi_{N,t}|$ associated with $\Psi_{N,t}$, which can be understood as the orthogonal projection onto $\Psi_{N,t}$. More precisely, the kernel of $\gamma_{N,t}$ is given by

$$\gamma_{N,t}(x_N; x_N') := \Psi_{N,t}(x_N) \overline{\Psi_{N,t}(x_N')}$$
where we denote \( x_m \in \mathbb{R}^{3m} \) for any \( m \in \mathbb{N} \). The \( k \)-particle marginal density is then defined through its kernel
\[
\gamma^{(k)}_{N,t}(x_k; x'_k) = \int dx_{N-k} \gamma_{N,t}(x_k, x_{N-k}; x'_k, x_{N-k}).
\]

We now focus on the trace-norm distance between the one-particle marginal density \( \gamma^{(1)}_{N,t} \) and the projection operator \( |\varphi_t\rangle \langle \varphi_t| \). In particular, we will prove that
\[
\text{Tr} \left| \gamma^{(1)}_{N,t} - |\varphi_t\rangle \langle \varphi_t| \right| \leq \frac{C e^{Kt}}{N}
\]
and find \( C \) according to the conditions on \( V \) and \( ||\varphi_0|| \). The norm \( || \cdot || \) of \( \varphi_0 \) will be discussed later.

**Theorem 1.1.** Let the three-body interaction potential \( V(x-y, x-z) \) such that
\[
V(x-y, x-z) = \lambda(|x-y|^{-1}|x-z|^{-1} + |y-z|^{-1}|y-x|^{-1} + |z-x|^{-1}|z-y|^{-1})
\]
where \( \lambda > 0 \). For \( \varphi_0 \in H^{(3/2)+\alpha}(\mathbb{R}^3) \) such that \( ||\varphi_0||_{L^2(\mathbb{R}^3)} = 1 \), and \( \varphi_t \) be the solution of the quintic Hartree equation
\[
i\partial_t \varphi_t = -\Delta \varphi_t + \frac{1}{2} \left( \int dydz V(x-y, x-z)|\varphi_t(y)|^2 |\varphi_t(z)|^2 \right) \varphi_t
\]
with initial data \( \varphi_{t=0} = \varphi_0 \). Let \( \psi_{N,t} = e^{-iH_{N,t}/\gamma} \otimes N \) and \( \gamma^{(1)}_{N,t} \) be the one-particle reduced density associated with \( \psi_{N,t} \) as defined in [14]. Then there exist constants \( C \) and \( K \), depending only on \( ||\varphi_0||_{H^{(3/2)+\alpha}} \), and \( \lambda \) such that
1. if \( 0 \leq a < 1/2 \), then
\[
\text{Tr} \left| \gamma^{(1)}_{N,t} - |\varphi_t\rangle \langle \varphi_t| \right| \leq \frac{C e^{Kt}}{N^{1+4a/(3+2a)}},
\]
2. if \( a > 1 \), then
\[
\text{Tr} \left| \gamma^{(1)}_{N,t} - |\varphi_t\rangle \langle \varphi_t| \right| \leq \frac{C e^{Kt}}{N}.
\]

**Remark 1.2.** One can obtain the same result by using the other three-body interaction potentials such that
\[
V(x-y, x-z) = V_\infty(x-y, x-z)V_C(x-y, x-z)
\]
where \( V_\infty \in L^\infty(\mathbb{R}^3) \) and \( V_C(x-y, x-z) = \lambda(|x-y|^{-1}|x-z|^{-1} + |y-z|^{-1}|y-x|^{-1} + |z-x|^{-1}|z-y|^{-1}) \) or
\[
V(x-y, x-z) = V_3(x-y, x-z) + V_\infty(x-y, x-z)
\]
where \( V_3(x-y, x-z) = v_3(x-y)v_3(x-z) + v_3(y-z)v_3(y-x) + v_3(z-x)v_3(z-y) \) with \( v_3 \in L^3(\mathbb{R}^3) \) and \( V_\infty \in L^\infty(\mathbb{R}^3) \).

**Remark 1.3.** In [14], the author assume that
\[
V(x-y, x-z) = v(x-y)v(x-z) + v(y-x)v(y-z) + v(z-x)v(z-y)
\]
where, for \( \varepsilon \in (0, 1/2) \),
\[
v(x) := \frac{\chi(|x|)}{|x|^{1-\varepsilon}} \quad \text{or} \quad G_{2+\varepsilon}(x),
\]
where \( \chi \in C_0^\infty(\mathbb{R}^+ \cup \{0\}) \) is nonnegative decreasing function and \( G_{\alpha} \), which is the kernel of the Bessel potential. Succeeding that work, through out this paper we consider the case where \( \varepsilon = 0 \) without assuming the fast decay. Moreover, we do not assume the initial data to have finite variance.

Before we describe the main ideas used in the proof of Theorem 1.1, we give a brief summary of the related known results. The derivation of the mean-field limit of a dilute Bose gas has been actively studied. First, profound results [20, 21, 26, 28] give us, using the BBGKY hierarchy, that the convergence,
\[
\text{Tr} \left| \gamma^{(1)}_{N,t} - |\varphi_t\rangle \langle \varphi_t| \right| \to 0 \quad \text{as} \quad N \to \infty.
\]
Form Erdős and Yau [16], the convergence is also proven for a singular potential (including the Coulomb case). Rodnianski and Schlein in [30], developed a coherent state approach to obtain a bound for the trace norm difference
\[ |\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t| | \leq O(N^{-1/2}) \]
for singular potentials (including Coulomb potential). The proof is based on the works of Fröhlich [17, 18, 19]. The rate of convergence \( O(N^{-1}) \) is known to be optimal [22, 23, 24, 25]. The optimal rate of convergence \( O(N^{-1}) \) was obtained in [9] for the Coulomb case. A similar approach has been applied to many-body semi-relativistic Schrödinger equations with gravitational interaction [28]. Methods in Fock space have been studied for the dynamical properties of a BEC [11, 18, 17, 18, 19, 28, 31, 30]. The same rate of convergence can be obtained by counting the number of particles in the condensate state [27, 32, 34].

In [10], a system with three-body interactions is considered. Chen and Pavlović proved that, in the Gross-Pitaevskii regime, the limiting dynamics is governed by the quintic nonlinear Schrödinger equation using the BBGKY hierarchy method in dimension \( d = 1, 2 \), i.e.,
\[ \text{Tr} |\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t| | \to 0 \quad \text{as} \quad N \to \infty. \]
For \( d = 3 \), Nam and Salzmann derived quintic nonlinear Schrödinger equation from Gross-Pitaevskii scaling, i.e.,
\[ V_N(x - y, x - z) = N^\beta V(N^\beta(x - y), N^\beta(x - z)) \]
with \( 0 < \beta < 1/6 \) with the initial data in higher Sobolev space, to be more specific, \( \varphi_0 \in H^4(\mathbb{R}^3) \). Moreover, Chen and Holmer provide the derivation of the energy critical nonlinear Schrödinger equation in [13].

In [14], Chen provides a rigorous proof for Hartree dynamics under the presence of triple (repulsive) collisions with singular interaction potential in the mean-field limit. He provided the rate in the Fock space norm sense for GMM type approximation. The provided rate is \( O(N^{-1/2}) \).

In the two-body interaction case, as in [9, 28, 30], we first embed the initial state to the Fock space replacing it by the coherent state. For the evolution of the coherent state, we need to control the fluctuation \( U(t; s) \), which is defined in [21], around the quintic Hartree dynamics. Then one can utilize the evolution of the coherent state to estimate the fluctuations for the dynamics of the factorized state. A technical difficulty here was overcome by using the method in Rodnianski and Schlein [30], which is equivalent to Lemma A.3 in this paper. It was possible to overcome the difficulty by controlling the fluctuation \( U(t; s) \) first by comparing it with an approximate dynamics \( \tilde{U}(t; s) \), whose generator is \( \tilde{L}(t) \) (see [30]). The idea was introduced, for two-body interaction, by Ginibre and Velo [20] as a limiting dynamics.

The main difficulty of this paper arise from three-body interaction. Since we are dealing with three-body interaction
\[ V(x - y, x - z) = \lambda (v(x - y)v(x - z) + v(y - z)v(y - x) + v(z - x)v(z - y)) \]
with \( v(\cdot) := 1/|x| \), the interaction \( V \) is twice more singular than Coulomb potential. Hence, we regularize each \( v \) so that the fluctuation of \( \tilde{U}(t; s) \). For this, we need to prove the wellposedness of \( \varphi_t \) in \( H^1(\mathbb{R}^3) \) and obtain the global (in time) bound of \( \|\varphi_t\|_{H^1(\mathbb{R}^3)} \). For that, we generalize the Hardy-Littlewood-Sobolev inequality in Section 13.

In this paper, we follow the approaches employed in [28, 30]. Since we want to use the approach for three-body interaction potential, we regularize interaction potential by using \( \alpha_N \leq N^{-\eta} \) for some \( 1 \leq \eta < 3/2 \). By applying and extending the techniques developed in [9, 28, 30] for the regularized potential, we obtain the optimal factor of order \( N^{-1} \). Here, the regularization has a key role, controlling the time evolution due to the three-body interaction in the Fock space. While this gives us the optimal bound for Fock states, it turns out that we lose the rate of convergence between the original evolution and the regularized evolution. Similar technique can be found in the BBGKY hierarchy approach by Chen and Holmer [11, 12] and in the physical space in the Fock space approach by Lewin, Nam, and Rougerie [30]. Another version of such behavior can also be found in [29]. To overcome this difficulty and to obtain the rate of convergence \( O(N^{-(1+4\alpha)/(3+2\alpha)}) \) or the optimal rate of convergence \( O(N^{-1}) \), we assume the initial data \( \varphi_0 \) to be in \( H^{(3/2)+\alpha}(\mathbb{R}^3) \) or \( H^{(5/2)+\epsilon}(\mathbb{R}^3) \), respectively.

This paper is organized as follows: First, in Section 2, we introduce the idea and strategy of the proof. Since we are using regularized potential, we provide more details about it. Then, we will compare the time evolution between before and after regularization. Note that we are considering a very singular potential. We prove our main theorem by proving the theorem for regularized potential in Section 3. The main strategy is...
to embed our state into the Fock space and compare the time evolution of our stated and coherent state. For that, we use Propositions 3.1 and 3.2 which will be proved in Section 5. In Section 4 we prepare lemmas describing comparison dynamics to prove Propositions 3.1 and 3.2. The lemmas are similar to the lemmas in previous works, for example see [39, 28, 30]. Since it has been well-known about Fock space through out many papers, we review bosonic Fock space formalism in Appendix A. For now, want to argue that the evolution of the potential is singular in both $x$ and $y$, one should avoid or remove this singularity. Thus, we detour the problem caused by singularity by using regularized potential. Then, for the time evolution with regularized potential, in Proposition 2.1 we obtain the optimal rate of convergence with its mean-field dynamics with $H^3$-initial data. The distance between of the original time evolution and the time evolution with regularized potential will be provided as follows.

As we have talked in Section 1, for the Fock space analysis, the potential $V$ is more singular than we can utilize the property of the solution of quintic Hartree equation. Hence, we are going to remove the singularity of the potential $V$.

Let

$$
\nabla(x - y, x - z) = \lambda \left( (x - y)\nabla(x - z) + (y - z)\nabla(y - x) + (z - x)\nabla(z - y) \right)
$$

with

$$\nabla(x - y) = \text{sgn}(v(x - y)) \min \{ |v(x - y)|, \alpha_N^{-1} \}$$

where $\text{sgn}(v(x))$ denotes the sign of $v(x)$. We also define the regularized Hamiltonian

$$
H_N = \sum_{j=1}^N (\Delta_{x_j} - \frac{1}{N^2} \sum_{i<j<k} \nabla(x_i - x_j, x_i - x_k)).
$$

Then we have the following proposition for regularized Hamiltonian, which give the optimal rate of convergence.

**Proposition 2.1.** Let $\nabla(x - y, x - z)$ as in (6) with $\alpha_N = N^{-\eta}$ for $1 \leq \eta < 3/2$. Let $\varphi_0 \in H^1(\mathbb{R}^3)$ with $\|\varphi_0\| = 1$, and $\varphi_t$ be the solution of the quintic Hartree equation

$$
i\partial_t \varphi_t = -\Delta \varphi_t + \frac{1}{2} \left( \int dydz \nabla(x - y, x - z)|\varphi_t(y)|^2|\varphi_t(z)|^2 \right) \varphi_t
$$

with initial data $\varphi_{t=0} = \varphi_0$. Let $\overline{\psi}_{N,t} = e^{-iH_N t} \varphi_0^N$ and $\overline{\psi}_{N,t}^{(1)}$ be the one-particle reduced density associated with $\overline{\psi}_{N,t}$. Then there exist constants $C$ and $K$, depending only on $\|\varphi\|_{H^1}$, $D$, and $\lambda$ such that

$$
\text{Tr} \left| \overline{\psi}_{N,t}^{(1)} - \varphi_t \right| \leq \frac{C e^{Kt}}{N^{2-\eta}}.
$$

The proof of Proposition 2.1 is given in Section 5. It makes use of a representation of the problem on the bosonic Fock space, detail for the Fock space, see Appendix A. For now, want to argue that the evolution governed by $H_N$ and $\overline{H}_N$ are similar enough.

**Lemma 2.2.** Let $\psi_{N,0} = \varphi_0^N$, $\varphi_0 \in H^{3/2+\alpha}(\mathbb{R}^3)$, and $\|\varphi_0\|_{L^2(\mathbb{R}^3)} = 1$. Let $\psi_{N,t} = e^{-iH_N t} \varphi_0^N$ and $\overline{\psi}_{N,t} = e^{-i\overline{H}_N t} \psi_{N,0}$. Then there exist a constant $C > 0$ such that

1. If $0 \leq a < 1/2$, then $\|\psi_{N,t} - \overline{\psi}_{N,t}\|^2 \leq C N \alpha_N^{1+2a} |t|$ and
2. If $a > 1$, then $\|\psi_{N,t} - \overline{\psi}_{N,t}\|^2 \leq C N \alpha_N^3 |t|$ and
for all \( N \in \mathbb{N}, \ t \in \mathbb{R} \).

**Proof.** We consider the derivative
\[
\frac{d}{dt} \| \psi_{N,t} - \bar{\psi}_{N,t} \|^2 = -2 \text{Re} \frac{d}{dt} \langle \psi_{N,t}, \bar{\psi}_{N,t} \rangle
\]
\[
= 2 \text{Im} \langle (H_N - \bar{H}_N) \psi_{N,t}, \bar{\psi}_{N,t} \rangle
\]
\[
= \frac{2}{N^2} \sum_{i<j<k} \text{Im} \langle (V(x_i - x_j, x_i - x_k) - \nabla \sum_{i<j<k} (x_i - x_j, x_i - x_k)) \psi_{N,t}, \bar{\psi}_{N,t} \rangle.
\]

Observe that (8) gives us that
\[
\begin{align*}
|V(x - y, x - z) - \nabla (x - y, x - z)| \\
\leq |\lambda| \left( |v(x - y)v(x - z) - \nabla(x - y)\nabla(x - z)| + |v(y - x)v(y - z) - \nabla(y - z)\nabla(y - x)| + |v(z - x)v(z - y) - \nabla(z - x)\nabla(z - y)| \right)
\end{align*}
\]
\[
\leq |\lambda| \left( |v(x - y)v(x - z) - v(x - y)\nabla(x - z)| + |v(x - y)\nabla(x - z) - \nabla(x - y)v(x - z)| + |v(y - x)v(y - z) - v(y - z)\nabla(y - x)| + |v(y - x)\nabla(y - x) - \nabla(y - z)v(y - z)| + |v(z - x)v(z - y) - v(z - x)\nabla(z - y)| + |v(z - x)\nabla(z - y) - \nabla(z - x)v(z - y)| \right)
\]
\[
\leq |\lambda| \left( |v(x - y)||v(x - z) - \nabla(x - z)| + |v(x - y) - \nabla(x - y)||\nabla(x - z)| + |v(y - z)||v(y - x) - \nabla(y - x)| + |v(y - z) - \nabla(y - z)||\nabla(y - x)| + |v(z - x)||v(z - y) - \nabla(z - y)| + |v(z - x) - \nabla(z - x)||\nabla(z - y)| \right).
\]

Note that, for any \( a \geq 0 \),
\[
(8) \quad |v - \nabla| \leq |v| \cdot 1(|v| \geq \alpha_N^{-1}) \leq |v|^{2+2a} \alpha_N^{1+2a}.
\]

Note that
\[
|V(x - y, x - z) - \nabla (x - y, x - z)| \\
\leq 2|\lambda| \alpha_N^{1+2a} \left( |v(x - y)||v(x - z)|^{2+2a} + |v(y - x)||v(y - z)|^{2+2a} + |v(z - x)||v(z - y)|^{2+2a} \right).
\]

Now, we set any \( \varepsilon := a - 1 > 0 \). Using Hardy inequality and Sobolev embedding, we obtain
\[
\int_{\mathbb{R}^3} dx \ |v(x) - \nabla(x)| \ |\varphi(x)|^2 = \int_{B(0,\alpha_N)} dx \ |v(x)| \ |\varphi(x)|^2
\]
\[
\leq C \int_{B(0,\alpha_N)} dx \ |(1 - \Delta)^{1/4} \varphi(x)|^2 \leq C \int_{\mathbb{R}^3} dx \ |(1 - \Delta)^{1/4} \varphi(x)|^2 \chi_{B(0,\alpha_N)}
\]
\[
\leq C \| (1 - \Delta)^{1/4} \varphi \|_{L^\infty}^2 \int_{\mathbb{R}^3} dx \ \chi_{B(0,\alpha_N)} \leq C \| (1 - \Delta)^{1/4} \varphi \|_{L^{(3/2)+\alpha_N}}^2 \chi_{B(0,\alpha_N)}
\]
\[
\leq C \| \varphi \|_{L^{2+\alpha_N}}^2 \chi_{B(0,\alpha_N)}.
\]

In short, we have
\[
(9) \quad |v - \nabla| \leq C (1 - \Delta)^{2+\varepsilon} \alpha_N^3.
\]
Then
\[
\left| \frac{d}{dt} \| \psi_{N,t} - \bar{\psi}_{N,t} \|^2 \right| 
\leq CN \left| \langle (V(x - y, x - z) - \nabla(x - y, x - z)) \psi_{N,t}, \bar{\psi}_{N,t} \rangle \right|
\leq C N \alpha_N^3 \left( \psi_{N,t} (1 - \Delta_y)^{1/2} (1 - \Delta_z)^{2+\varepsilon} \psi_{N,t} \right)^{1/2} \left( \psi_{N,t} (1 - \Delta_y)^{1/2} (1 - \Delta_z)^{2+\varepsilon} \bar{\psi}_{N,t} \right)^{1/2}
\]
\[
+ \langle \psi_{N,t}, (1 - \Delta_x)^{1/2} (1 - \Delta_y)^{2+\varepsilon} \psi_{N,t} \rangle^{1/2} \left( \psi_{N,t} (1 - \Delta_x)^{1/2} (1 - \Delta_y)^{2+\varepsilon} \bar{\psi}_{N,t} \right)^{1/2}
+ \langle \psi_{N,t}, (1 - \Delta_x)^{1/2} (1 - \Delta_y)^{2+\varepsilon} \psi_{N,t} \rangle^{1/2} \left( \psi_{N,t} (1 - \Delta_x)^{1/2} (1 - \Delta_y)^{2+\varepsilon} \bar{\psi}_{N,t} \right)^{1/2} \right).
\]

Note that
\[
N^{(5/2)+\varepsilon} \langle \psi_{N,t}, (1 - \Delta_y)^{1/2} (1 - \Delta_z)^{2+\varepsilon} \psi_{N,t} \rangle \leq C \langle \psi_{N,t}, (H_N + N)^{(5/2)+\varepsilon} \psi_{N,t} \rangle
\leq C \langle \varphi_0^{\otimes N}, (H_N + N)^{(5/2)+\varepsilon} \psi_0^{\otimes N} \rangle \leq CN^{(5/2)+\varepsilon} \| \varphi_0 \|^2_{H^{(5/2)+\varepsilon}}.
\]

Therefore, we have that
\[
\frac{d}{dt} \| \psi_{N,t} - \bar{\psi}_{N,t} \|^2 \leq CN \alpha_N^3
\]
where the constant $C$ depends on $\| \varphi_0 \|_{H^{(5/2)+\varepsilon}} = \| \varphi_0 \|_{H^{(3/2)+\varepsilon}}$.

This concludes the proof of the desired lemma.

**Corollary 2.3.** Let $\gamma_{N,t}^{(k)}$ and $\bar{\gamma}_{N,t}^{(k)}$ be the $k$-particle reduced densities associated with $\psi_{N,t} = e^{-i H_N t} \varphi_0^{\otimes N}$ and $\bar{\psi}_{N,t} = e^{-i H_N t} \psi_0^{\otimes N}$. Suppose $\varphi \in H^{(3/2)+\varepsilon}(\mathbb{R}^3)$ and $\alpha_N = N^{-\eta}$. Then there exist a constant $C > 0$ such that

1. If $0 \leq a < 1/2$, then
   \[
   \text{Tr} \left| \gamma_{N,t}^{(k)} - \bar{\gamma}_{N,t}^{(k)} \right| \leq CN^{1-\eta(1+2a)} |t|^{1/2}.
   \]

2. If $a > 1$, then
   \[
   \text{Tr} \left| \gamma_{N,t}^{(k)} - \bar{\gamma}_{N,t}^{(k)} \right| \leq CN^{1-3a} |t|^{1/2}.
   \]

**Proof.** See [1] Corollary 2.1.

**Lemma 2.4.** Let $\varphi_t$ be the solution of the quintic Hartree equation [3] and $\phi_t$ the solution of the quintic Hartree equation
\[
i \partial_t \phi_t = -\Delta \phi_t + \frac{1}{2} \left( \int dydz \nabla(x, y, z) |\phi_t(y)|^2 |\phi_t(z)|^2 \right) \phi_t
\]
with regularized potential $\nabla$ with the same initial data $\varphi_{t=0} = \phi_{t=0} = \varphi_0$, $\phi_t = \varphi_0^{\otimes N}$. Suppose $\alpha_N \leq N^{-\eta}$. Then

1. If $0 \leq a < 1/2$, then
   \[
   \| \varphi_t - \phi_t \| \leq CN^{(1-\eta(1+2a))/2} e^{K|t|},
   \]

   Therefore
   \[
   \text{Tr} \left| \varphi_t \langle \phi_t | \otimes k \| \phi_t \rangle - \| \phi_t \| \langle \phi_t | \phi_t \rangle \| \right| \leq C k N^{(1-\eta(1+2a))/2} e^{K|t|}
   \]

2. If $a > 1$, then
   \[
   \| \varphi_t - \phi_t \| \leq CN^{(1-3a)/2} e^{K|t|}
   \]

Therefore
\[
\text{Tr} \left| \varphi_t \langle \phi_t | \otimes k \| \phi_t \rangle - \| \phi_t \| \langle \phi_t | \phi_t \rangle \| \right| \leq C k N^{(1-3a)/2} e^{K|t|}
\]
for any $k \in \mathbb{N}$.


Proof. From Lemma [3.3] we see that \( \| \varphi_t \|_{H^1}, \| \phi_t \|_{H^1} < C \), for some constant \( C \) which only depends on \( \| \varphi_0 \|_{H^1} \). We calculate
\[
\frac{d}{dt} \| \varphi_t - \phi_t \|^2 \\
= 2 \operatorname{Im} \left( \varphi_t, \left[ \int dydz V(x - y, x - z) |\varphi_t(y)|^2 |\varphi_t(z)|^2 - \int dxdydz V(x - y, x - z) |\phi_t(y)|^2 |\phi_t(z)|^2 \right] \right) \\
= 2 \operatorname{Im} \left( \varphi_t, \left[ \int dydz \left( V(x - y, x - z) - V(x - y, x - z) \right) |\varphi_t(y)|^2 |\varphi_t(z)|^2 \right] \right) \\
= 2 \operatorname{Im} \left( \varphi_t, \left[ \int dydz \left( V(x - y, x - z) - V(x - y, x - z) \right) |\varphi_t(y)|^2 |\varphi_t(z)|^2 \right] \right) \\
+ \frac{d}{dt} \| \varphi_t - \phi_t \|^2.
\]
Then, using (8), we obtain
\[
\left| \frac{d}{dt} \| \varphi_t - \phi_t \|^2 \right| \\
\leq 2\alpha_N^2 \| \varphi_t \| \| \phi_t \| \sup_x \int dy, dz |v(x - y)|^2 |v(x - z)|^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2 \\
+ 2 \| \varphi_t - \phi_t \| \| \varphi_t \| \sup_x \int dydz |V(x - y, x - z)| \left| \| \varphi_t(y) \| \varphi_t(z) \| - |\phi_t(y)\phi_t(z)\| \right| \\
\times \left( \| \varphi_t(y)\| \varphi_t(z) \| + |\phi_t(y)\| \phi_t(z) \| \right)
\]
\[
\leq C\alpha_N^2 + C \| \varphi_t - \varphi_t \|^2.
\]
In the last inequality, we have used that
\[
\int dydz |V(x - y, x - z)| \left| \| \varphi_t(y) \| \varphi_t(z) \| - |\phi_t(y)\phi_t(z)\| \right| \left( \| \varphi_t(y)\| \varphi_t(z) \| + |\phi_t(y)\| \phi_t(z) \| \right)
\]
\[
\leq C \left( \int dydz |\varphi_t(y)\varphi_t(z) - \phi_t(y)\phi_t(z)|^2 \right)^{1/2} \\
\times \left( \int dydz |V(x - y, x - z)|^2 \left( |\varphi_t(y)|^2 |\varphi_t(z)|^2 + |\phi_t(y)|^2 |\phi_t(z)|^2 \right)^{1/2} \\
\leq C \| \varphi_t - \varphi_t \|.
\]
From (14) we obtain by Grönwall inequality
\[
\| \varphi_t - \phi_t \|^2 \leq C N^{(1 - \eta(1 + 2a))/2} (e^{C|t|} - 1)
\]
if \( 0 \leq a < 1/2 \), and
\[
\| \varphi_t - \phi_t \|^2 \leq C N^{(1 - 3\eta)/2} (e^{C|t|} - 1)
\]
if \( a > 1 \). This concludes the proof by following the proof of [7] Lemma 2.2. \( \square \)

3. PROOF OF THE MAIN RESULT

Now, we are ready prove the main result of the paper, Theorem 1.1. To have that, we first prove the Proposition 2.1

3.1. Unitary operators and their generators. As we define bosonic Fock space in Section A the new Hamiltonian for the Fock space evolution can be written as
\[
\mathcal{H}_N = \int dx a_x^* (-\Delta) a_x + \frac{1}{6N^2} \int dx dy dz V(x - y, x - z) a_x^* a_y^* a_z^* a_y a_z a_x.
\]
Since we have \( \mathcal{H}_N \psi^{(N)} = \mathcal{H}_N \psi^{(N)} \) for \( \psi \in \mathcal{F} \), (15) can be justified as a proper generalization of (11). Since we are going to use regularized potential, we define
\[
\overline{\mathcal{H}}_N = \int dx a_x^* (-\Delta) a_x + \frac{1}{6N^2} \int dx dy dz \overline{V}(x - y, x - z) a_x^* a_y^* a_z^* a_y a_z a_x
\]
which is also a generalization of \( \overline{\mathcal{H}}_N \) for the Fock space.
The one-particle marginal density $\gamma^{(1)}_\psi$ associated with $\psi$ is

$$
\gamma^{(1)}_\psi(x;y) = \frac{1}{\langle \psi, N\psi \rangle} \langle \psi, a_y^* a_x \psi \rangle.
$$

Note that $\gamma^{(1)}_\psi$ is a trace class operator on $L^2(\mathbb{R}^3)$ and $\text{Tr} \, \gamma^{(1)}_\psi = 1$.

Let $\gamma^{(1)}_{N,t}$ be the kernel of the one-particle marginal density associated with the time evolution of the factorized state $\varphi_0 \otimes N$ for Hamiltonian $\mathcal{H}_N$. By definition,

$$
\gamma^{(1)}_{N,t} = \frac{1}{N} \left( \frac{a_y^* (\varphi)}{\sqrt{N!}} \right),
$$

where

$$
\mathcal{H}_N = \int dx dy dz \nabla (x - y, x - z)|\varphi_t(x)|^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2.
$$

If we put the coherent states instead of the factorized initial data in (18) and expand $a_y^* a_x$ around $N\varphi_t(y)\varphi_t(x)$, then it is enough to consider the operator

$$
W^* (\sqrt{N}\varphi_s) e^{-i\mathcal{H}_N (t-s)} W(\sqrt{N}\varphi_s) = W^* (\sqrt{N}\varphi_t) e^{-i\mathcal{H}_N (t-s)} W(\sqrt{N}\varphi_s).
$$

Now we lead to understand the operator $W^* (\sqrt{N}\varphi_t)e^{-i\mathcal{H}_N (t-s)} W(\sqrt{N}\varphi_s)$. For the understanding, since we know that for $t = s$, we investigate the time evolution of the operator by differentiate it with respect to $t$. One can compute directly such that

$$
i\partial_t W^* (\sqrt{N}\varphi_t)e^{-i\mathcal{H}_N (t-s)} W(\sqrt{N}\varphi_s) = \mathcal{L} W^* (\sqrt{N}\varphi_t) e^{-i\mathcal{H}_N (t-s)} W(\sqrt{N}\varphi_s),
$$

where $\mathcal{L} := \sum_{k=0}^6 \mathcal{L}_k(t)$ and the exact formulas for $\mathcal{L}_k$ are as follows:

We consider evolution

$$
i\partial_t \mathcal{U} = \mathcal{L} \mathcal{U}
$$

with

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_6$$

where

$$
\mathcal{L}_0 = \frac{N}{6} \int dx dy dz \nabla (x - y, x - z)|\varphi_t(x)|^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2
$$

$$
\mathcal{L}_1 = 0
$$

$$
\mathcal{L}_2 = \int dx a_y^* (-\Delta) a_x
$$

$$
+ \frac{1}{6} \int dx dy dz \nabla (x - y, x - z) \left[ 3|\varphi_t(x)|^2 (\varphi_t(y)\varphi_t(z) a_y^* a_x + \varphi_t(y)\varphi_t(z) a_x a_y) + 3|\varphi_t(x)|^2 (\varphi_t(y)\varphi_t(z) a_x a_y + \varphi_t(y)\varphi_t(z) a_y^* a_x) \right]
$$

$$
\mathcal{L}_3 = \frac{1}{6\sqrt{N}} \int dx dy dz \nabla (x - y, x - z) \left[ (\varphi_t(x)\varphi_t(y)\varphi_t(z) a_y^* a_x + \varphi_t(x)\varphi_t(y)\varphi_t(z) a_x a_y) + 3 (\varphi_t(x)\varphi_t(y)\varphi_t(z) a_x a_y + \varphi_t(x)\varphi_t(y)\varphi_t(z) a_y^* a_x) \right]
$$

$$
\mathcal{L}_4 = \frac{1}{6N} \int dx dy dz \nabla (x - y, x - z) \left[ 3 (\varphi_t(x)\varphi_t(y) a_y^* a_x a_y + \varphi_t(x)\varphi_t(y) a_x a_y a_x) \right]
$$

$$
\mathcal{L}_5 = \frac{1}{6N} \int dx dy dz \nabla (x - y, x - z) \left[ 6\varphi_t(x)\varphi_t(y) a_z a_y a_x a_y + 3|\varphi_t(x)|^2 a_y^* a_x a_y \right]
$$

$$
\mathcal{L}_6 = \frac{1}{6N} \int dx dy dz \nabla (x - y, x - z) \left[ 6\varphi_t(x)\varphi_t(y) a_z a_y a_x a_y + 3|\varphi_t(x)|^2 a_y^* a_x a_y \right]
$$
\[ L_5 = \frac{1}{2N\sqrt{N}} \int dx dy dz \nabla(x-y, x-z, x-z) a_x^* a_y^* \left( \phi_t(z) a_z^* + \overline{\phi_t(z)} a_z \right) a_y a_x \]

\[ L_6 = \frac{1}{6N^2} \int dx dy dz \nabla(x-y, x-z, x-z) a_x^* a_y^* a_z^* a_z a_y a_x. \]

3.2. **Proof of Theorem 1.1.** As explained in Section 1, we use the technique developed in \[9, 14, 28, 36]. The proof of Theorem 1.1 is a consequence of Corollary 2.3 and Proposition 2.1.

Let \( \overline{\gamma} \) be the solution of cut-offed Hartree equation. Then we have

\[
\text{Tr} \left| \gamma^{(1)}_N - |\phi_t\rangle \langle \phi_t| \right| \leq \text{Tr} \left| \gamma^{(1)}_N - \overline{\gamma}^{(1)}_N \right| + \text{Tr} \left| \overline{\gamma}^{(1)}_N - |\overline{\gamma}\rangle \langle \overline{\gamma}| \right| + \text{Tr} \left| |\overline{\gamma}\rangle \langle \overline{\gamma}| - |\phi_t\rangle \langle \phi_t| \right|
\]

Hence, it is enough to prove Proposition 2.1. For \( \varphi \in H^{(3/2)+a}(\mathbb{R}^3) \), we put \( \eta = 5/4 \) when \( 0 \leq a < 1/2 \), and we put \( \eta = 1 \) when \( a > 1 \). This conclude the main theorem. The proof of Proposition 2.1 consists of the following two propositions. Recall the definition of \( d_N \) in \[14\]. In this section, we use \( \mathcal{U}(t) \) instead of \( \mathcal{U}(t; s) \) for notional simplicity.

**Proposition 3.1.** Suppose that the assumptions in Theorem 1.1 hold. For a Hermitian operator \( J \) on \( H^1(\mathbb{R}^3) \), let

\[
E^1_t(J) := \frac{d_N}{2N} \left< W^* (\sqrt{N} \varphi) \frac{(a^*(\varphi))^N}{\sqrt{N}!} \Omega, \mathcal{U}^*(t) e^{J} \mathcal{U}(t) \Omega \right>
\]

Then, there exist constants \( C \) and \( K \), depending only on \( \lambda \) and \( \| \varphi_0 \|_{H^1} \), such that

\[
|E^1_t(J)| \leq C \| J \| e^{Kt} \frac{1}{N^{2-\eta}}.
\]

**Proposition 3.2.** Suppose that the assumptions in Theorem 1.1 hold. For a Hermitian operator \( J \) on \( H^1(\mathbb{R}^3) \), let

\[
E^2_t(J) := \frac{d_N}{\sqrt{N}} \left< W^* (\sqrt{N} \varphi) \frac{(a^*(\varphi))^N}{\sqrt{N}!} \Omega, \mathcal{U}^*(t) \varphi(J \varphi_t) \mathcal{U}(t) \Omega \right>
\]

Then, there exist constants \( C \) and \( K \), depending only on \( \lambda \) and \( \| \varphi_0 \|_{H^1} \), such that

\[
|E^2_t(J)| \leq C \| J \| e^{Kt} \frac{1}{N^{2-\eta}}.
\]

Proof of Propositions 3.1 and 3.2 will be given later in section 3. With Propositions 3.1 and 3.2, we now prove Proposition 2.1.

**Proof of Proposition 2.1.** Formally, the proof is the same with previous results but for the sake of completeness, we include the proof of this theorem. Recall that

\[
\gamma^{(1)}_N, t = \frac{1}{N} \left< \frac{(a^*(\varphi))^N}{\sqrt{N}!} \Omega, e^{-\overline{\Pi}_N t} a_y a_x e^{-i \overline{\Pi}_N t} \frac{(a^*(\varphi))^N}{\sqrt{N}!} \Omega \right>
\]

From the definition of the creation operator in \[35\], we can easily find that

\[
\{0, 0, \ldots, 0, \varphi^{\otimes N}, 0, \ldots\} = \frac{(a^*(\varphi))^N}{\sqrt{N}!} \Omega,
\]

where the \( \varphi^{\otimes N} \) on the left-hand side is in the \( N \)-th sector of the Fock space. Recall that \( P_N \) is the projection onto the \( N \)-particle sector of the Fock space. From \[13\], we find that

\[
\frac{(a^*(\varphi))^N}{\sqrt{N}!} \Omega = \frac{\sqrt{N}}{N^{2N/2} N^{2-2N/2}} P_N W(\sqrt{N} \varphi) \Omega = d_N P_N W(\sqrt{N} \varphi) \Omega.
\]

Since \( \overline{\Pi}_N \) does not change the number of particles, we also have that

\[
\gamma^{(1)}_{N, t}(x; y) = \frac{1}{N} \left< \frac{(a^*(\varphi))^N}{\sqrt{N}!} \Omega, e^{-\overline{\Pi}_N t} a_y a_x e^{-i \overline{\Pi}_N t} \frac{(a^*(\varphi))^N}{\sqrt{N}!} \Omega \right> = d_N \left< \frac{(a^*(\varphi))^N}{\sqrt{N}!} \Omega, e^{-i \overline{\Pi}_N t} P_N W(\sqrt{N} \varphi) \Omega \right>
\]
Thus, from Propositions 3.1 and 3.2, we find that
\[ | \frac{d}{dt} \langle \psi | = 0 \rangle \]

To simplify it further, we use the relation
\[ e^{\sqrt{\lambda} t} \phi = (\sqrt{\lambda} t)^N \Omega \]

(and an analogous result for the creation operator) to obtain that
\[ \langle \psi \rangle = \langle \psi \rangle \]

Recall the definition of \( E_1^1(J) \) and \( E_1^2(J) \) in Propositions 3.1 and 3.2. For any compact one-particle Hermitian operator \( J \) on \( L^2(\mathbb{R}^3) \), we have
\[ \text{Tr} J \langle \psi \rangle \langle \psi \rangle = \int dxdy J(x,y) \left( \langle \psi \rangle \langle \psi \rangle - \langle \psi \rangle \langle \psi \rangle \right) \]

Thus, from Propositions 3.1 and 3.2 we find that
\[ \langle \psi \rangle \langle \psi \rangle = \langle \psi \rangle \langle \psi \rangle \]

Since the space of compact operators is the dual to that of the trace class operators, and since \( \langle \psi \rangle \langle \psi \rangle \) and \( \langle \psi \rangle \langle \psi \rangle \) are Hermitian,
\[ \text{Tr} \langle \psi \rangle \langle \psi \rangle = \langle \psi \rangle \langle \psi \rangle \]

which concludes the proof of Proposition 2.1.
4. Comparison dynamics

As briefly mentioned in Section 1, the key technical estimate is the upper bound on the fluctuation of the expected number of particles under the evolution $\mathcal{U}(t; s)$, which is the following lemma. This section is to provide useful comparison dynamics.

**Lemma 4.1.** Suppose that the assumptions in Theorem 1.1 hold. Let $\mathcal{U}(t; s)$ be the unitary evolution defined in (21). Then for any $\psi \in \mathcal{F}$ and $j \in \mathbb{N}$, there exist constants $C \equiv C(j)$ and $K \equiv K(j)$ such that

$$\langle \mathcal{U}(t; s) \psi, \mathcal{N}^j \mathcal{U}(t; s) \psi \rangle \leq C e^{Kt} \left\langle \psi, (\mathcal{N} + 1)^{2j+3} \psi \right\rangle.$$

We now begin the proof of Lemma 4.1. First, we introduce a truncated time-dependent generator with fixed $M > 0$ as follows:

$$L^{(M)}_N(t) = \int dx a_x^* (-\Delta_x) a_x + \frac{1}{6} \int dx dy dz V(x - y, x - z) \left[ 3|\varphi_t(x)|^2 \left( \varphi_t(y) \varphi_t(z) a^*_y a^*_z + \overline{\varphi_t(y)} \overline{\varphi_t(z)} a_y a_z \right) + 3|\varphi_t(x)|^2 \left( \varphi_t(y) \overline{\varphi_t(z)} a^*_y a_z + \overline{\varphi_t(y)} \varphi_t(z) a^*_y a_y \right) \right]$$

$$+ \frac{1}{6\sqrt{N}} \int dx dy dz V(x - y, x - z) \chi(N \leq M) \left[ \varphi_t(x) \varphi_t(y) \varphi_t(z) a^*_x a^*_y a^*_z + \overline{\varphi_t(x)} \overline{\varphi_t(y)} \overline{\varphi_t(z)} a_x a_y a_z \right]$$

$$+ \frac{1}{6N} \int dx dy dz V(x - y, x - z) \chi(N \leq M) \left[ 3 \left( \varphi_t(x) \varphi_t(y) a^*_x a^*_y a^*_z + \overline{\varphi_t(x)} \overline{\varphi_t(y)} a_x a_y a_z \right) \right]$$

$$+ \frac{1}{6N} \int dx dy dz V(x - y, x - z) \chi(N \leq M) \left[ 6 \varphi_t(x) \overline{\varphi_t(y)} a^*_x a^*_z a_y a_y + 3|\varphi_t(x)|^2 a^*_x a^*_y a_y \right]$$

$$+ \frac{1}{2N^{3/2}} \int dx dy dz V(x - y, x - z) \chi(N \leq M) a^*_x a^*_y a^*_z a_y a_x$$

$$+ \frac{1}{N^2} \int dx dy dz V(x - y, x - z) a^*_x a^*_y a^*_z a_y a_x.$$

We remark that $M$ will be chosen to be $M = N^{1/3}$ later in the proof of Lemma 4.1. Define a unitary operator $\mathcal{U}^{(M)}$ by

$$\mathcal{U}^{(M)}(t; s) = L^{(M)}_N(t) \mathcal{U}^{(M)}(t; s) \text{ and } \mathcal{U}^{(M)}(s; s) = 1.$$

We use a three-step strategy.

**Step 1.** Truncation with respect to $\mathcal{N}$ with $M > 0$.

**Lemma 4.2.** Suppose that the assumptions in Theorem 1.1 hold and let $\mathcal{U}^{(M)}$ be the unitary operator defined in (24). Then, there exist constants $C$ and $K$ such that, for all $N \in \mathbb{N}$ and $M > 0$, $\psi \in \mathcal{F}$, and $t, s, \in \mathbb{R}$,

$$\left\langle \mathcal{U}^{(M)}(t; s) \psi, (\mathcal{N} + 1)^j \mathcal{U}^{(M)}(t; s) \psi \right\rangle$$

$$\leq \left\langle \psi, (\mathcal{N} + 1)^j \psi \right\rangle C \exp \left( 4jK|t-s| \left( 1 + \sqrt{\frac{M}{N}} + \frac{M}{N} + C^{-1} \left( \frac{M}{N} \right)^{3/2} \right) \right).$$
Proof. Following the proof of Lemma 3.5 in [36], we get

\[
\frac{d}{dt} \left\langle U^{(M)}(t; 0) \psi, (N + 1)^2 U^{(M)}(t; 0) \psi \right\rangle \\
= \left\langle U^{(M)}(t; 0) \psi, [iL^{(M)}(t), (N + 1)^2] U^{(M)}(t; 0) \psi \right\rangle \\
= \text{Im} \int \text{d}x \text{d}y \text{d}z \nabla(x - y, x - z) |\varphi_t(x)|^2 \varphi_t(y) \varphi_t(z) \left\langle U^{(M)}(t; 0) \psi, [a_z^* a_y^*, (N + 1)^2] U^{(M)}(t; 0) \psi \right\rangle \\
+ \frac{1}{3 \sqrt{N}} \text{Im} \int \text{d}x \text{d}y \text{d}z \nabla(x - y, x - z) \varphi_t(x) \varphi_t(y) \varphi_t(z) \left\langle U^{(M)}(t; 0) \psi, [a_z^* a_y^* a_z^* \chi(N \leq M), (N + 1)^2] U^{(M)}(t; 0) \psi \right\rangle \\
+ \frac{1}{\sqrt{N}} \text{Im} \int \text{d}x \text{d}y \text{d}z \nabla(x - y, x - z) \varphi_t(x) \varphi_t(y) \varphi_t(z) \left\langle U^{(M)}(t; 0) \psi, [a_z^* a_y^* a_z^* \chi(N \leq M), (N + 1)^2] U^{(M)}(t; 0) \psi \right\rangle \\
+ \frac{1}{N \sqrt{N}} \text{Im} \int \text{d}x \text{d}y \text{d}z \nabla(x - y, x - z) \varphi_t(x) \varphi_t(y) \varphi_t(z) \left\langle U^{(M)}(t; 0) \psi, [a_z^* a_y^* a_z^* \chi(N \leq M), (N + 1)^2] U^{(M)}(t; 0) \psi \right\rangle.
\]

Using the pull-through formulae \(a_x N = (N + 1) a_x \) and \(a_x^* N = (N - 1) a_x^* \), we find

\[
[a_x^*, (N + 1)^2] = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k (N + 1)^k a_x^*, \quad [a_x, (N + 1)^2] = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k (N + 1)^k a_x.
\]

As a consequence,

\[
[a_x^* a_y^*, (N + 1)^2] = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k a_x^* (N + 1)^k a_y^* + (N + 1)^k a_x^* a_y^*,
\]

\[
= \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k \left( N^{k/2} a_x^* a_y^* (N + 1)^k/2 + (N + 1)^{k/2} a_x^* a_y^* (N + 3)^{k/2} \right),
\]

\[
[a_x, (N + 1)^2] = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k (N + 1)^k a_x = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k (N + 1)^k/2 a_x N^{k/2}.
\]

Moreover,

\[
[a_x^* a_y^* a_z^*, (N + 1)^2] = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k \left( (N - 1)^{k/2} a_y^* a_z^* a_z^* (N + 2)^{k/2} + (N + 1)^{k/2} a_y^* a_z^* a_z^* (N + 3)^{k/2} \right),
\]

\[
[a_x^* a_y^* a_z^*, (N + 1)^2] = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k \left( (N - 1)^{k/2} a_y^* a_y^* a_z^* a_z^* (N + 2)^{k/2} + (N - 1)^{k/2} a_y^* a_y^* a_z^* a_z^* (N + 1)^{k/2} \right),
\]

\[
[a_y^* a_z^* a_z^*, (N + 1)^2] = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k \left( (N - 2)^{k/2} a_z^* a_z^* a_z^* N^{k/2} + (N - 1)^{k/2} a_z^* a_z^* a_z^* (N + 1)^{k/2} \right),
\]

\[
+ (N + 1)^{k/2} a_y^* a_y^* a_z^* a_z^* (N + 2)^{k/2},
\]

\[
[a_y^* a_z^* a_z^*, (N + 1)^2] = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k \left( (N - 2)^{k/2} a_z^* a_z^* a_z^* N^{k/2} + (N - 1)^{k/2} a_z^* a_z^* a_z^* (N + 1)^{k/2} \right),
\]

\[
+ (N + 1)^{k/2} a_y^* a_y^* a_z^* a_z^* (N + 3)^{k/2}.
\]
\[ [a_x^* a_y^* a_x, (N + 1)^2] \]
\[ = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k \Big((N - 3)^{k/2} a_x^* a_y^* a_x a_y a_x (N - 2)^{k/2} + (N - 2)^{k/2} a_x^* a_y^* a_x a_y a_x (N - 1)^{k/2} \]
\[ + (N - 1)^{k/2} a_x^* a_y^* a_x a_y a_x N^{k/2} + N^{k/2} a_x^* a_y^* a_x a_y a_x (N + 1)^{k/2} \]
\[ + (N + 1)^{k/2} a_x^* a_y^* a_x a_y a_x (N + 2)^{k/2} \]
To control the contribution from the terms in the right-hand side of (25), we use the bounds of the form

$$I_1 = \left| \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \nabla (x - y, x - z) |\varphi_t(x)|^2 \varphi_t(y) \varphi_t(z) \langle U^{(M)}(t; 0) \psi, (\mathcal{N} + 1) \frac{\delta}{\mathcal{N}} a_x^* a_y a_z (N + 2)^{k/2} U^{(M)}(t; 0) \psi \rangle \right|$$

$$\leq \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \nabla (x - y, x - z) |\varphi_t(x)|^2 |\varphi_t(y)| a_y (N + 1) \frac{\delta}{\mathcal{N}} a_x^* a_y a_z (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \| a^* (\nabla (x - \cdot) \varphi_t) (\mathcal{N} + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \|$$

$$\leq K \| (\mathcal{N} + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \|^2,$$

$$N^{1/2} I_2 = \left| \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \nabla (x - y, x - z) \varphi_t(x) \varphi_t(y) \varphi_t(z) \langle U^{(M)}(t; 0) \psi, (\mathcal{N} + 1) \frac{\delta}{\mathcal{N}} a_x^* a_y a_z (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \rangle \right|$$

$$\leq \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \nabla (x - y, x - z) \varphi_t(x) |\varphi_t(y)| a_y (N \leq M) (N + 1) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \| a^* (\nabla (x - \cdot) \varphi_t) (\mathcal{N} \leq M) (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \|$$

$$\leq K M^{1/2} \| (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \|^2,$$

$$N^{1/2} I_3 = \left| \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \nabla (x - y, x - z) \varphi_t(x) \varphi_t(y) \varphi_t(z) \langle U^{(M)}(t; 0) \psi, (\mathcal{N} + 1) \frac{\delta}{\mathcal{N}} a_x^* a_y a_z (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \rangle \right|$$

$$\leq \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \nabla (x - y, x - z) \varphi_t(x) |\varphi_t(y)| a_y (N \leq M) (N + 1) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \| a^* (\nabla (x - \cdot) \varphi_t) (\mathcal{N} \leq M) (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \|$$

$$\leq K M^{1/2} \| (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \|^2,$$

and

$$I_4 = \left| \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \nabla (x - y, x - z) \varphi_t(x) \varphi_t(y) \langle U^{(M)}(t; 0) \psi, (\mathcal{N} + 1) \frac{\delta}{\mathcal{N}} a_x^* a_y a_z (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \rangle \right|$$

$$\leq \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \nabla (x - y, x - z) \varphi_t(x) |\varphi_t(y)| a_y a_x a_z (N \leq M) (N + 1) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \| a_x (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \|$$

$$\leq K M \| (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \|^2.$$

On the other hand, to control contribution arising from $I_5$ in the right-hand side of (25), we use that

$$N^{3/2} I_5 = \left| \int \mathrm{d}x \mathrm{d}y \mathrm{d}z \nabla (x - y, x - z) \varphi_t(z) \langle U^{(M)}(t; 0) \psi, (\mathcal{N} + 1) \frac{\delta}{\mathcal{N}} a_x^* a_y a_z \chi (N \leq M) (N + 2) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \rangle \right|$$

$$\leq \int \mathrm{d}x \alpha^{-1} (N) \left| a_x a_y a_z (N \leq M) (N + 1) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \right| \left| a (v(x - \cdot) \varphi_t) (\mathcal{N} \leq M) \right| \left| a_x \chi (N + 3) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \right|$$

$$\leq K \alpha^{-1} M^{3/2} \sup_x (v(x - \cdot) \varphi_t) \left| (N + 1) \frac{\delta}{\mathcal{N}} U^{(M)}(t; 0) \psi \right|^2.$$
This implies
\[ \frac{d}{dt} \left\langle U^{(M)}(t; 0)\psi, (N + 1)^j U^{(M)}(t; 0)\psi \right\rangle \]
\[ \leq K \left( 1 + \sqrt{\frac{M}{N}} + \frac{M}{N} + \alpha_N^{-1} \left( \frac{M}{N} \right)^{3/2} \right) \sum_{k=0}^{j} \binom{j}{k} \left\langle U^{(M)}(t; 0)\psi, (N + 3)^j U^{(M)}(t; 0)\psi \right\rangle \]
\[ \leq 4^j K \left( 1 + \sqrt{\frac{M}{N}} + \frac{M}{N} + \alpha_N^{-1} \left( \frac{M}{N} \right)^{3/2} \right) \left\langle U^{(M)}(t; 0)\psi, (N + 1)^j U^{(M)}(t; 0)\psi \right\rangle. \]

Applying the Grönwall Lemma with Lemma B.5, we get the desired result. \( \square \)

**Step 2: Weak bounds on the \( U \) dynamics.**

**Lemma 4.3.** For arbitrary \( t, s \in \mathbb{R} \) and \( \psi \in \mathcal{F} \), we have
\[ \left\langle \psi, U(t; s)NU(t; s)^* \psi \right\rangle \leq 6 \left\langle \psi, (N + N + 1)^\ell \right\rangle. \]
Moreover, for every \( \ell \in \mathbb{N} \), there exists a constant \( C(\ell) \) such that
\[ \left\langle \psi, U(t; s)N^2U(t; s)^* \psi \right\rangle \leq C(\ell) \left\langle \psi, (N + N)^{2\ell} \right\rangle, \]
\[ \left\langle \psi, U(t; s)N^{2\ell+1}U(t; s)^* \psi \right\rangle \leq C(\ell) \left\langle \psi, (N + N)^{2\ell+1}(N + 1)^\ell \right\rangle \]
for all \( t, s \in \mathbb{R} \) and \( \psi \in \mathcal{F} \).

**Proof.** We may follow the proof of Lemma 3.6 in [36] without any change. \( \square \)

**Step 3: Comparison between the \( U \) and \( U^{(M)} \) dynamics.**

**Lemma 4.4.** Suppose that the assumptions in Theorem 1.1 hold. Then, for every \( j \in \mathbb{N} \), there exist constants \( C \equiv C(j) \) and \( K \equiv K(j) \) such that
\[ \left| \left\langle \psi, N^j \left(U(t; s) - U^{(M)}(t; s)\right) \psi \right\rangle \right| \]
\[ \leq C(j) \left( \frac{\left((N/M)^j + (N/M)^j-1 + (N/M)^j-2\right) \left((N+1)^j+3\left((N+1)^j+3/2\right)\right)^2}{1 + \sqrt{\frac{M}{N}} + \frac{M}{N} + \alpha_N^{-1} \left( \frac{M}{N} \right)^{3/2}} \right) \times \exp \left( K(j) |t - s| \left( 1 + \sqrt{\frac{M}{N}} + \frac{M}{N} + \alpha_N^{-1} \left( \frac{M}{N} \right)^{3/2} \right) \right) \]
and
\[ \left| \left\langle \psi, N^j \left(U^{(M)}(t; s) - U^{(M)}(t; s)\right) \psi \right\rangle \right| \]
\[ \leq C(j) \left( \frac{\left((1/M)^j + (1/M)^j-1 + (1/M)^j-2\right) \left((N+1)^j+3\left((N+1)^j+3/2\right)\right)^2}{1 + \sqrt{\frac{M}{N}} + \frac{M}{N} + \alpha_N^{-1} \left( \frac{M}{N} \right)^{3/2}} \right) \times \exp \left( K(j) |t - s| \left( 1 + \sqrt{\frac{M}{N}} + \frac{M}{N} + \alpha_N^{-1} \left( \frac{M}{N} \right)^{3/2} \right) \right). \]

**Proof.** To simplify the notation we consider the case \( s = 0, t > 0 \) only; other cases can be treated in a similar manner. To prove the first inequality of the lemma, since \( \alpha_N = N^{-\eta} \), we expand the difference of the two evolution as follows:
\[ \left\langle \psi, N^j \left(U(t; 0) - U^{(M)}(t; 0)\right) \psi \right\rangle = \left\langle \psi, \frac{1}{t} \int_0^t ds \left( U(t; 0)\psi, N^j U(t; s) \left( \mathcal{L}_N(s) - \mathcal{L}_N^{(M)}(s) \right) U^{(M)}(s; 0)\psi \right) \right\rangle \]
\[ = J_1 + J_2 + J_3 + J_4 + J_5 \]
where
\[ J_1 := \]
- \frac{i}{3\sqrt{N}} \int_0^t ds \int dx dy dz \nabla(x-y, x-z) \\
\times \left\langle \mathcal{U}(t;0)\psi, N^2\mathcal{U}(t;s)\chi(N > M) \left( \varphi_t(x)\varphi_t(y)\varphi_t(z)a_x^*a_y^*a_z^* + \varphi_t(x)\varphi_t(y)\varphi_t(z)a_xa_ya_z \right) \mathcal{U}^{(M)}(s;0)\psi \right\rangle,
J_2 := \\
- \frac{i}{\sqrt{N}} \int_0^t ds \int dx dy dz \nabla(x-y, x-z) \\
\times \left\langle \mathcal{U}(t;0)\psi, N^3\mathcal{U}(t;s)\chi(N > M) \left( \varphi_t(x)\varphi_t(y)\varphi_t(z)a_x^*a_y^*a_z^* + \varphi_t(x)\varphi_t(y)\varphi_t(z)a_xa_ya_z \right) \mathcal{U}^{(M)}(s;0)\psi \right\rangle,
J_3 := \\
- \frac{i}{\sqrt{N}} \int_0^t ds \int dx dy dz \nabla(x-y, x-z) \\
\times \left\langle \mathcal{U}(t;0)\psi, N^3\mathcal{U}(t;s)\chi(N > M) \left( 6\varphi_t(x)\varphi_t(y)a_x^*a_y^*a_z^* + 3|\varphi_t(x)|^2a_x^*a_y^*a_z \right) \mathcal{U}^{(M)}(s;0)\psi \right\rangle,
and
J_5 := - \frac{i}{N\sqrt{N}} \int_0^t ds \int dx dy dz \nabla(x-y, x-z)\varphi_t(z) \\
\times \left\langle \mathcal{U}(t;0)\mathcal{U}(t;s)\psi, a_xa_y^*a_z^* \left( a_x\chi(N > M) + \chi(N > M)a_x^* \right) a_xa_ya_z \mathcal{U}^{(M)}(s;0)\psi \right\rangle.

Note that \(\chi(N > M) \leq (N/M)^k\) for any \(k \geq 1\) and also note from Lemma 13.20 that
\begin{equation}
\|N\mathcal{U}(t;s)^*N^2\mathcal{U}(t;0)\psi\|^2 \leq 6(N^2\mathcal{U}(t;0)\psi, (N + N + 1)^2N^2\mathcal{U}(t;0)\psi) \\
\leq C(j)\langle \psi, (N + N)^{2j+2}(N + 1)\psi \rangle \leq C(j)N^{2j+2}(\psi, (N + 1)^{2j+3}\psi).
\end{equation}

Then
\[
|J_2| \leq \frac{C}{\sqrt{N}} \int_0^t ds \int dx\varphi_t(x)||a_x\mathcal{U}(t;s)^*N^2\mathcal{U}(t;0)\psi|| \\
\times \|a(v(x-\cdot)\varphi_t)a(v(x-\cdot)\varphi_t)\chi(N > M + 1)\mathcal{U}^{(M)}(s;0)\psi\| \\
\leq \frac{C}{\sqrt{N}} \sup_x \|v(x-\cdot)\varphi_t\|^2 \int_0^t ds \int dx\varphi_t(x)||a_x\mathcal{U}(t;s)^*N^2\mathcal{U}(t;0)\psi|| \|\chi(N > M + 1)\mathcal{U}^{(M)}(s;0)\psi\| \\
\leq \frac{C}{\sqrt{N}} \sup_x \|v(x-\cdot)\varphi_t\|^2 \int_0^t ds \|N^2\mathcal{U}(t;s)^*N^2\mathcal{U}(t;0)\psi\| \|\chi(N > M + 1)\mathcal{U}^{(M)}(s;0)\psi\| \\
\leq C(j)N^{2j}\|\chi(N + 1)^j\psi\| \int_0^t ds \left\langle \mathcal{U}^{(M)}(s;0)\psi, \frac{N^{2j}}{M^2} \mathcal{U}^{(M)}(s;0)\psi \right\rangle^{1/2} \\
\leq C(j)(N/M)^2\|\chi(N + 1)^j\psi\| \int_0^t ds \left\langle \mathcal{U}^{(M)}(s;0)\psi, \mathcal{U}^{(M)}(s;0)\psi \right\rangle^{1/2}.
\]

For \(J_1\), using similar approach, we can obtain the same bound. Then we consider \(J_3\).
\[
|J_3| \leq \frac{C}{\sqrt{N}} \int_0^t ds \int dx \|a_x\mathcal{U}(t;s)^*N^3\mathcal{U}(t;0)\psi\| \\
\times \|a(v(x-\cdot)\varphi_t)a(v(x-\cdot)\varphi_t)a_z\chi(N > M + 1)\mathcal{U}^{(M)}(s;0)\psi\| \\
\leq \frac{C}{\sqrt{N}} \sup_x \|v(x-\cdot)\varphi_t\|^2 \int_0^t ds \int dx \|a_x\mathcal{U}(t;s)^*N^3\mathcal{U}(t;0)\psi|| \|\chi(N > M + 1)a_x\mathcal{U}^{(M)}(s;0)\psi\|.
\]
\[ \begin{aligned}
&\leq \frac{C}{N} \sup_x \|v(x - \cdot)\varphi_t\|^2 \int_0^t ds \|N^{1/2}U(t; s)^*N^2U(t; 0)\psi\| \chi(N > M + 1)N^{1/2}U(M)(s; 0)\psi
\\&\leq C(j)N^{j-1/2}((N + 1)^{j+1}\psi \int_0^t ds \left\langle U(M)(s; 0)\psi, \frac{N^{2j}}{M^{2j-1}}U(M)(s; 0)\psi \right\rangle^{1/2}
\\&\leq C(j)(N/M)^{j-1/2}((N + 1)^{j+1}\psi \int_0^t ds \left\langle U(M)(s; 0)\psi, N^{2j}U(M)(s; 0)\psi \right\rangle^{1/2}.
\end{aligned} \]

Also for \(J_4\), we can obtain the same bound. Thus

\[ |J_5| \leq \frac{C}{N\sqrt{N}} \int_0^t ds \int dydz \|a_ya_zU(t; s)^*N^2U(t; 0)\psi\|
\\times \|a(yz - \cdot)\varphi_t\| \int_0^t ds \int dydz \|a_ya_zU(t; s)^*N^2U(t; 0)\psi\| \|a_ya_z\chi(N > M + 1)U(M)(s; 0)\psi\|
\\\leq \frac{C}{N\sqrt{N}} \sup_x \|v(x - \cdot)\varphi_t\| \int_0^t ds \int dydz \|NU(t; s)^*N^2U(t; 0)\psi\| \|N\chi(N > M + 1)U(M)(s; 0)\psi\|
\\\leq C(j)N^{j-1}((N + 1)^{j+3/2}\psi \int_0^t ds \left\langle U(M)(s; 0)\psi, \frac{N^{2j}}{M^{2j-2}}U(M)(s; 0)\psi \right\rangle^{1/2}
\\\leq C(j)(N/M)^{j-1}((N + 1)^{j+3/2}\psi \int_0^t ds \left\langle U(M)(s; 0)\psi, N^{2j}U(M)(s; 0)\psi \right\rangle^{1/2}.
\]

We thus conclude that

\[ \left| \left\langle U(t; 0)\psi, N^j \left( U(t; 0) - U(M)(t; 0) \right) \psi \right\rangle \right| \]

\[ \leq C(j) \left( \frac{(N/M)^j + (N/M)^{j-1/2} + (N/M)^{j-1}}{(1 + \sqrt{M/N} + \alpha^{-1}M^3/N^3/2))} \right)
\\\times \exp \left( K(j)|t-s| \left( 1 + \sqrt{M/N} + \alpha^{-1}M^3/N^3/2 \right) \right) \]

(27)

The proof of the second part of the lemma is similar and we omit it. \(\square\)

We now prove Lemma 4.1 by combining the three steps above.

Proof of Lemma 4.1. From Lemmas 4.2, 4.3 and 4.4 with the choice \(\alpha_N = N^{-\eta}\) and \(M = N^{1-(2\eta/3)}\). Since \(N > 1\), we have that

\[ \left\langle U(t; s)\psi, N^jU(t; s)\psi \right\rangle
\\= \left\langle U(t; s)\psi, N^j(U - U(M))(t; s)\psi \right\rangle + \left\langle (U - U(M))(t; s)\psi, N^jU(M)(t; s)\psi \right\rangle
\\+ \left\langle U(M)(t; s)\psi, N^jU(M)(t; s)\psi \right\rangle
\\\leq Ce^{K|t-s|} \left\langle \psi, (N + 1)^{2j+3}\psi \right\rangle. \]

(28)

\(\square\)

Recall the definition of \(\tilde{U}(t; s)\) in (30). In the next lemma, we prove an estimate similar to Lemma 4.1 for the evolution with respect to \(\tilde{U}\).

Lemma 4.5. Suppose that the assumptions in Theorem 4.1 hold. We consider another evolution

\[ \tilde{U}(t; s) = (L_2 + L_1^* + L_0)\tilde{U}. \]

Then

\[ \left\langle \tilde{\psi}, (N + 1)^j\tilde{\psi} \right\rangle \leq Ce^{Kt}. \]

(29)
Let $\tilde{L} = L_2 + L_4^* + L_6$ and define the unitary operator $\tilde{U}(t; s)$ by

$$i\partial_t \tilde{U}(t; s) = \tilde{L}(t) \tilde{U}(t; s) \quad \text{and} \quad \tilde{U}(s; s) = 1.$$  

(30)

Since $\tilde{L}$ does not change the parity of the number of particles,

$$\left\langle \Omega, \tilde{U}^n(t; 0) a_x^* \tilde{U}(t; 0) \Omega \right\rangle = \left\langle \Omega, \tilde{U}^n(t; 0) a_x^* \tilde{U}(t; 0) \Omega \right\rangle = 0.$$  

(31)

Proof. For (29), we derive this with respect to time. Then

$$\frac{d}{dt} \left\langle \tilde{\psi}, (N + 1)^j \tilde{\psi} \right\rangle = \left\langle \tilde{\psi}, [i(L_2 + L_4^* + L_6), (N + 1)^j] \tilde{\psi} \right\rangle$$

$$= 2 \text{Im} \int dxdydz V_N(x - y, x - z) |\varphi_t(x)|^2 |\varphi_t(y)| |\varphi_t(z)| \left\langle \tilde{\psi}, [a_y^* a_z^*, (N + 1)^j] \tilde{\psi} \right\rangle$$

$$= 2 \text{Im} \int dxdydz V_N(x - y, x - z) |\varphi_t(x)|^2 |\varphi_t(y)| |\varphi_t(z)|$$

$$\times \left( (N + 3)^{j-1} a_y^* a_z^* \psi, (N + 3)^{1-j} (N + 1)^j \tilde{\psi} \right)$$

Then

$$\left| \frac{d}{dt} \left\langle \tilde{\psi}, (N + 1)^j \tilde{\psi} \right\rangle \right|$$

$$\leq C \int dxdydz V_N(x - y, x - z) |\varphi_t(x)|^2 |\varphi_t(y)| |\varphi_t(z)|$$

$$\times \left\| (N + 3)^{j-1} a_y^* a_z^* \psi \right\| \left\| (N + 1)^j \tilde{\psi} \right\|$$

$$\leq C \left( \int dxdydz |V_N(x - y, x - z)|^2 |\varphi_t(x)|^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2 \right)^{1/2}$$

$$\times \left( \int dxdydz |\varphi_t(x)|^2 \left\| (N + 3)^{j-1} a_y^* a_z^* \psi \right\|^2 \right)^{1/2} \left\| (N + 1)^j \tilde{\psi} \right\|$$

$$\leq C \left\| (N + 1)^j \tilde{\psi} \right\|^2.$$  

Hence by Grönwall’s inequality, we obtain (29). \hfill \Box

Lemma 4.6. Suppose that the assumptions in Theorem 1.1 hold. Then, for any for any $\psi \in \mathcal{F}$ and $j \in \mathbb{N}$, there exist a constant $C \equiv C(j)$ such that

$$\left\| (N + 1)^{j/2} L_3 \psi \right\| \leq \frac{Ce^{Kt}}{\sqrt{N}} \left\| (N + 1)^{(j+3)} \psi \right\|.$$  

Proof. Let

$$A_3 = \int dxdydz V(x - y, x - z) \varphi_t(x) \varphi_t(y) \varphi_t(z) a_x^* a_y^* a_z^*$$

and

$$B_3 = \int dxdydz V(x - y, x - z) \varphi_t(x) \varphi_t(y) \varphi_t(z) a_x^* a_y^* a_z^*$$

Then

$$L_3 = \frac{1}{6\sqrt{N}} \left( \left( A_3 + A_3^* \right) + 3(B_3 + B_3^*) \right).$$

Take any $\xi \in \mathcal{F}$. Then

$$\left\langle \xi, (N + 1)^{j/2} A_3^* \psi \right\rangle$$

$$= \int dxdydz \nabla(x - y, x - z) \varphi_t(x) \varphi_t(y) \varphi_t(z) \left\langle \xi, (N + 1)^{j/2} a_x^* a_y^* a_z^* \psi \right\rangle.$$
\[
\begin{align*}
\leq & \left( \int dxdydz \, |\nabla(x - y, x - z)|^2 |\varphi_t(x)|^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2 \left\| (N+1)^{-1/2} \xi \right\|^2 \right)^{1/2} \\
& \times \left( \int dxdydz \, \left\| a_x a_y a_x (N+1)^{j/2} \psi \right\|^2 \right)^{1/2} \\
& \leq C \|\varphi_t\|_{H^1(\mathbb{R}^3)} \|\xi\| \left\| (N+1)^{(j+3)/2} \psi \right\|.
\end{align*}
\]

Similarly, this holds for $A_3$, $B_3$, and $B_3^*$. It leads us the lemma.

\textbf{Lemma 4.7.} Suppose that the assumptions in Theorem 4.7 hold. Then, for any any $\psi \in \mathcal{F}$ and $j \in \mathbb{N}$, there exist a constant $C \equiv C(\psi)$ such that

\[
\left\| (N+1)^{j/2} \mathcal{L}_4^c \psi \right\| \leq \frac{C e^{K_j}}{N} \left\| (N+1)^{(j+4)/2} \psi \right\|.
\]

\textbf{Proof.} Let

\[
A_4 = \int dxdydz \, \nabla(x - y, x - z) \varphi_t(x) \varphi_t(y) a_x a_y a_x a_z.
\]

Then

\[
\mathcal{L}_4^c = \frac{1}{2N} (A_4 + A_4^*).
\]

Take any $\xi \in \mathcal{F}$. Then

\[
\begin{align*}
& \left\langle \xi, (N+1)^{j/2} A_x^* \psi \right\rangle \\
& = \int dxdydz \, \nabla(x - y, x - z) \varphi_t(x) \varphi_t(y) \left\langle (N+1)^{-1/2} \xi, (N+1)^{(j+1)/2} a_x^* a_y a_x \psi \right\rangle \\
& \leq \left( \int dxdydz \, |\nabla(x - y, x - z)|^2 |\varphi_t(x)|^2 |\varphi_t(y)|^2 \left\| a_x (N+1)^{-1/2} \xi \right\|^2 \right)^{1/2} \\
& \quad \times \left( \int dxdydz \, \left\| a_x a_y a_x (N+1)^{(j+1)/2} \psi \right\|^2 \right)^{1/2} \\
& \leq C \|\varphi_t\|_{H^1(\mathbb{R}^3)} \|\xi\| \left\| (N+1)^{(j+4)/2} \psi \right\|.
\end{align*}
\]

Similarly for $A_4^*$, we obtain desired lemma.

\textbf{Lemma 4.8.} Suppose that the assumptions in Theorem 4.7 hold. Then, for any any $\psi \in \mathcal{F}$ and $j \in \mathbb{N}$, there exist a constant $C \equiv C(\psi)$ such that

\[
\left\| (N+1)^{j/2} \mathcal{L}_5^c \psi \right\| \leq \frac{C e^{K_j}}{\alpha N \sqrt{N}} \left\| (N+1)^{(j+5)/2} \psi \right\|.
\]

\textbf{Proof.} Let

\[
A_5 = \int dxdydz \, \nabla(x - y, x - z) \varphi_t(z) a_x^* a_x a_y a_x
\]

Then

\[
\mathcal{L}_5^c = \frac{1}{2N \sqrt{N}} (A_5 + A_5^*).
\]

Take any $\xi \in \mathcal{F}$. Then

\[
\begin{align*}
& \left\langle \xi, (N+1)^{j/2} A_x^* \psi \right\rangle \\
& = \int dxdydz \, \nabla(x - y, x - z) \varphi_t(z) \left\langle (N+1)^{-1/2} \xi, (N+1)^{(j+1)/2} a_x^* a_y a_x \psi \right\rangle \\
& \leq \left( \int dxdydz \, \left\| a_x a_y a_x (N+1)^{-3/2} \xi \right\|^2 \right)^{1/2} \\
& \quad \times \left( \int dxdydz \, |\nabla(x - y, x - z)|^2 |\varphi_t(z)|^2 \left\| a_y a_x (N+1)^{(j+3)/2} \psi \right\|^2 \right)^{1/2}
\end{align*}
\]
From Lemmata 4.6, 4.7, and 4.8, we have

\[
\leq C_{\alpha_N}^{-1} \left( \int_{\Omega} \omega(x,y) (N+1)^{-3/2} \right)^{1/2}
\]

\[
\times \left( \int_{\Omega} \omega(x,y) \left( (1+y^2) + (1+y^2) \right) \right)^{1/2}
\]

\[
\leq C_{\alpha_N}^{-1} \left\| \varphi_0 \right\|_{H^s(\mathbb{R}^3)} \left\| (N+1)^{1/2} \right\|.
\]

**Lemma 4.9.** Suppose that the assumptions in Theorem 1.1 hold. Let \( \alpha_N = N^{-\eta} \). Then, for all \( j \in \mathbb{N} \), there exist constants \( C = C(j) \) and \( K = K(j) \) such that, for any \( f \in L^2(\mathbb{R}^3) \),

\[
\left\| (N+1)^{1/2} \left( \mathcal{U}^*(t) \phi(f) \mathcal{U}(t) - \mathcal{U}^*(t) \phi(f) \tilde{\mathcal{U}}(t) \right) \right\| \leq \frac{C e^{K_t}}{N^{2-\eta} \| f \|_{L^2(\mathbb{R}^3)}}.
\]

**Proof.** Let

\[
\mathcal{R}_1(f) := \left( \mathcal{U}^*(t) - \mathcal{U}^*(t) \right) \phi(f) \tilde{\mathcal{U}}(t)
\]

and

\[
\mathcal{R}_2(f) := (\mathcal{U}^*(t) \phi(f)) \left( \mathcal{U}(t) - \tilde{\mathcal{U}}(t) \right)
\]

so that

\[
(32) \quad \mathcal{U}^*(t) \phi(f) \mathcal{U}(t) - \mathcal{U}^*(t) \phi(f) \tilde{\mathcal{U}}(t) = \mathcal{R}_1(f) + \mathcal{R}_2(f).
\]

We begin by estimating the first term in the right-hand side of (32). From Lemma 4.1,

\[
\left\| (N+1)^{1/2} \mathcal{R}_1(f) \right\| \leq \frac{C e^{K_t}}{N^{2-\eta} \| f \|_{L^2(\mathbb{R}^3)}}
\]

From Lemmata 4.6, 4.7, and 4.8, we have

\[
\int_0^t ds \left\| (N+1)^{1/2} \mathcal{L}_4 \mathcal{U}^*(t) \phi(f) \tilde{\mathcal{U}}(t) \right\| \leq \frac{C e^{K_t}}{N^{2-\eta} \| f \|_{L^2(\mathbb{R}^3)}}
\]

and

\[
\int_0^t ds \left\| (N+1)^{1/2} \mathcal{L}_4 \mathcal{U}^*(t) \phi(f) \tilde{\mathcal{U}}(t) \right\| \leq \frac{C e^{K_t}}{N^{2-\eta} \| f \|_{L^2(\mathbb{R}^3)}}
\]
From Lemma A.3, thus, from Lemma 4.5, we obtain for $R$

Then, since the integrand in the right-hand side does not depend on $s$, we get

$$
\| (N+1)^{j/2} R_1(f) \Omega \| \leq C e^{Kt} \left( \frac{1}{\sqrt{N}} + \frac{1}{N} + \frac{1}{\alpha_N N \sqrt{N}} \right) \| f \|_{L^2(\mathbb{R}^3)}.
$$

Thus, from Lemma 4.3, we obtain for $R_1(f)$ that

$$
\| (N+1)^{j/2} R_1(f) \Omega \| \leq C \| f \| e^{Kt} \frac{1}{N^{(3/2) - \eta}}.
$$

The study of $R_2(f)$ is similar and we can again obtain that

$$
\| (N+1)^{j/2} R_2(f) \Omega \| \leq C \| f \| e^{Kt} \frac{1}{N^{(3/2) - \eta}}.
$$

This completes the proof of the desired lemma.

\[ \square \]

5. Proof of Propositions 3.1 and 3.2

In this section, we prove Propositions 3.1 and 3.2 by applying the lemmas proved in Section 4. Even though the proofs are almost the same as previous works, we include the following proves since underlying lemmas and logic are different. The structure of the proof is given in [28]. We, however, provide this section since the exponents of $(N+1)$ are a bit different.

Proof of Proposition 3.1. Recall that Comparison dynamics

We begin by

$$
E_l^1(J) = \frac{d_N}{N} \left\langle \frac{W^*(\sqrt{N} \varphi)(a^*(\varphi))^N}{\sqrt{N!}} \Omega, U^*(t) d\Gamma(J) U(t) \Omega \right\rangle.
$$

From Lemma A.3

$$
\| (N+1)^{-1} \hat{\varphi} W^*(\sqrt{N} \varphi) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \| \leq C \frac{d_N}{N}.
$$

By successively applying Lemma 4.1 (and also using the inequality (12)), we also get

$$
\| (N+1)^{-1} \hat{\varphi} U^*(t) d\Gamma(J) U(t) \Omega \| \leq C e^{Kt} \| (N+1)^{3/2} U(t) \Omega \|
$$

Thus, from (33), (34), and (35),

$$
|E_l^1(J)| \leq \frac{C \| J \| e^{Kt}}{N^{2 - \eta}},
$$

which proves the desired result.

\[ \square \]

For the proof of Proposition 3.2, we take almost verbatim copy of the proof of Lemma 4.2 in [28]. To make the paper self-contained, we write it in detail below.

Proof of Proposition 3.2. Recall the definitions of $R_1$ and $R_2$ in the proof of Lemma 4.9. Let $R = R_1 + R_2$ so that

$$
R(f) = U^*(t) \phi(f) U(t) - \tilde{U}^*(t) \phi(f) \tilde{U}(t).
$$

From the parity conservation, (31),

$$
P_{2k} \tilde{U}^*(t) \phi(J \varphi_t) \tilde{U}(t) \Omega = 0
$$

for all $k = 0, 1, \ldots$. (See Lemma 8.2 in [28] for more detail.) Thus, (36)

$$
|E_l^2(J)|
$$
For the second term in the right-hand side of (37), we use Lemmas A.3 and 4.9, where we put \( f \) place of \( J\varphi_t \). Using Lemma 4.5, we have
\[
\left\langle \frac{\sqrt{N}}{f} \sum_{k=1}^{N+1} (N+1)^{-1/2} P_{2k-1} \right\rangle \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \leq (N+1)^{1/2} \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \leq \frac{C}{N^{1/3}}.
\]
Let \( K = \frac{1}{2} N^{1/3} \) so that Lemmas A.3 and A.4 show that
\[
\left\langle \frac{\sqrt{N}}{f} \sum_{k=1}^{K} (N+1)^{-1/2} P_{2k-1} \right\rangle \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \leq \frac{C}{N^{1/3}}.
\]
Using Lemma 4.5, we have
\[
\left\langle (N+1)^{1/2} \frac{\sqrt{N}}{f} \sum_{k=1}^{K} (N+1)^{-1/2} P_{2k-1} \right\rangle \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \leq \frac{C}{N^{1/3}}.
\]
For the second term in the right-hand side of (37), we use Lemmas A.3 and 4.9, where we put \( J\varphi_t \) in the place of \( f \) for the latter. Altogether, we have
\[
\left\langle (N+1)^{1/2} \frac{\sqrt{N}}{f} \right\rangle \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \leq \frac{C}{N^{1/3}}\cdot
\]
which is the desired conclusion. \( \square \)
Appendix A. Standard Fock space formalism

This section is devoted to introduce the standard Fock space formalism. One can see more details in many articles, for example, [2, 9, 36].

To consider the system of $N$-bosons, we want to embed our the system into a larger space so-called bosonic Fock space over $L^2(\mathbb{R}^3)$ which is defined as

$$F := \bigoplus_{n \geq 0} L^2(\mathbb{R}^3, dx) \otimes_n = \mathbb{C} \bigoplus_{n \geq 1} L^2_s(\mathbb{R}^3, dx_1, \ldots, dx_n),$$

where $L^2_s = L^2(\mathbb{R}^3, dx_1, \ldots, dx_n)$ is a symmetric subspace of $L^2(\mathbb{R}^3, dx_1, \ldots, dx_n)$ where we let $L^2(\mathbb{R}^3) \otimes^0 = \mathbb{C}$. An element $\psi \in F$ is called a state, and it can be understood as a sequence $\psi = \{\psi^{(n)}\}_{n \geq 0}$ of $n$-particle wave functions $\psi^{(n)} \in L^2_s(\mathbb{R}^3)$. The inner product on $F$ is defined by

$$\langle \psi_1, \psi_2 \rangle = \sum_{n \geq 0} \langle \psi_1^{(n)}, \psi_2^{(n)} \rangle_{L^2(\mathbb{R}^3)}$$

$$= \psi_1^{(0)} \psi_2^{(0)} + \sum_{n \geq 0} \int dx_1 \ldots dx_n \psi_1^{(n)}(x_1, \ldots, x_n) \psi_2^{(n)}(x_1, \ldots, x_n).$$

A vacuum state is defined as $\Omega := \{1, 0, 0, \ldots \} \in F$. Since a state in a Fock space can have a different number of particles, we define the creation operator $a^*(f)$ and the annihilation operator $a(f)$ on $F$ by

$$(a^*(f) \psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

and

$$(a(f) \psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int dx f(x) \psi^{(n+1)}(x, x_1, \ldots, x_n)$$

which creates a particle $f$ to the system and annihilates $f$ from the system (respectively). Note that both $a^*(f)$ and $a(f)$ are not self-adjoint. We define the self-adjoint operator $\phi(f)$ such as

$$\phi(f) = a^*(f) + a(f).$$

We also use operator-valued distributions $a^*_x$ and $a_x$ satisfying

$$a^*(f) = \int dx f(x) a^*_x, \quad a(f) = \int dx \overline{f(x)} a_x$$

for any $f \in L^2(\mathbb{R}^3)$. The canonical commutation relation between creation and annihilation operators is

$$[a(f), a^*(g)] = (f, g)_{L^2(\mathbb{R}^3)}, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0,$$

which also assumes the form

$$[a_x, a^*_y] = \delta(x - y), \quad [a_x, a_y] = [a^*_x, a^*_y] = 0.$$

Moreover, the number operator $\mathcal{N}$, which gives us the expected number of the state in Fock space, is defined by

$$(\mathcal{N} : \int dx a^*_x a_x, \quad \text{for each non-negative integer } n, \text{ we introduce the projection operator onto the } n\text{-particle sector of the Fock space,}$$

$$P_n(\psi) := (0, 0, \ldots, 0, \psi^{(n)}, 0, \ldots)$$

for $\psi = (\psi^{(0)}, \psi^{(1)}, \ldots) \in F$. For simplicity, with slight abuse of notation, we will use $\psi^{(n)}$ to denote $P_n \psi$, and it satisfies that $(\mathcal{N} \psi)^{(n)} = n \psi^{(n)}$. For an operator $J$ on the one-particle sector $L^2(\mathbb{R}^3, dx)$, we define its second quantization $d\Gamma(J)$ by

$$(d\Gamma(J) \psi)^{(n)} = \sum_{j=1}^{n} J_j \psi^{(n)}$$
where $J_j = 1 \otimes \cdots \otimes J_j \otimes \cdots \otimes 1$ is the operator $J$ acting on the $j$-th variable only. The number operator defined above can also be understood as the second quantization of the identity, i.e., $\mathcal{N} = d\Gamma(1)$. With a kernel $J(x; y)$ of the operator $J$, the second quantization $d\Gamma(J)$ can be also be written as

$$d\Gamma(J) = \int dx dy \, J(x; y) a_x^* a_y,$$

which is consistent with (31).

Since the annihilation operator and the creation operator forms the number operator, it is natural to control the operators by the number operator. To control the operators and second quantization, we provide the following lemma.

**Lemma A.1.** For $\alpha > 0$, let $D(\mathcal{N}^\alpha) = \{ \psi \in \mathcal{F} : \sum_{n \geq 1} n^{2\alpha} ||\psi^{(n)}||^2 < \infty \}$ denote the domain of the operator $\mathcal{N}^\alpha$. For any $f \in L^2(\mathbb{R}^3, dx)$ and any $\psi \in D(\mathcal{N}^{1/2})$, we have

$$\|a(f)\psi\| \leq \|f\| \|\mathcal{N}^{1/2}\psi\|,$$

$$\|a^*(f)\psi\| \leq \|f\| \|\mathcal{N}^{1/2}\psi\|,$$

$$\|\phi(f)\psi\| \leq 2\|f\| \|\mathcal{N}^{1/2}\psi\|.$$

Moreover, for any bounded one-particle operator $J$ on $L^2(\mathbb{R}^3, dx)$ and for every $\psi \in D(\mathcal{N})$, we find

$$\|d\Gamma(J)\psi\| \leq \|J\| \|\mathcal{N}\psi\|.$$

**Proof.** See [10 Lemma 2.1] for [11], and see [9 Lemma 3.1] for [12].

Heuristically, there are eigenvectors of $a_x$ with the eigenvalue $\sqrt{\mathcal{N}}f$, where $f \in L^2(\mathbb{R}^3)$. It known as the coherent states, defined by, for $f \in L^2(\mathbb{R}^3)$,

$$\psi_{\text{coh}}(f) := e^{-\|f\|^2/2} \sum_{n \geq 0} \frac{(a^*(f))^n}{n!} \Omega = e^{-\|f\|^2/2} \sum_{n \geq 0} \frac{1}{\sqrt{n!}} f^{\otimes n}.$$

Then from [17] one obtain $\gamma_{\psi_{\text{coh}}}^{(1)}(x; y) = \varphi_t(x)\varphi_t(y)$, which is exactly the one-particle marginal density associated with the factorized wave function $\varphi_t^{\otimes N}$. Note that, unlike our system with $N$-particles, such eigenvectors of the annihilation operator can have any number of particles. We, however, can utilize coherent states for our goal.

The coherent state can be generated by acting Weyl operator $W(f)$ on vacuum state $\Omega$. i.e.,

$$\psi_{\text{coh}}(f) = W(f)\Omega = e^{-\|f\|^2/2} \exp(a^*(f)) \Omega = e^{-\|f\|^2/2} \sum_{n \geq 0} \frac{1}{\sqrt{n!}} f^{\otimes n}.$$

Where the Weyl operator $W(f)$ is defined by

$$W(f) := \exp(a^*(f) - a(f))$$

and it also satisfies

$$W(f) = e^{-\|f\|^2/2} \exp(a^*(f)) \exp(-a(f)),$$

which is known as the Hadamard lemma in Lie algebra. We collect the useful properties of the Weyl operator and the coherent states in the following lemma.

**Lemma A.2.** Let $f, g \in L^2(\mathbb{R}^3)$.

1. The commutation relation between the Weyl operators is given by

$$W(f)W(g) = W(g)W(f) e^{-2i\cdot1m(f, g)} = W(f + g) e^{-i\cdot1m(f, g)}.$$

2. The Weyl operator is unitary and satisfies that

$$W(f)^* = W(f)^{-1} = W(-f).$$

3. The coherent states are eigenvectors of annihilation operators, i.e.,

$$a_x \psi(f) = f(x)\psi(f) \Rightarrow a(g)\psi(f) = (g, f)_{L^2}\psi(f).$$
The commutation relation between the Weyl operator and the annihilation operator (or the creation operator) is thus
\[ W^*(f)a_xW(f) = a_x + f(x) \quad \text{and} \quad W^*(f)a_x^*W(f) = a_x^* + \overline{f(x)}. \]

(4) The distribution of \( N \) with respect to the coherent state \( \psi(f) \) is Poisson. In particular,
\[ \langle \psi(f), N\psi(f) \rangle = \|f\|^2, \quad \langle \psi(f), N^2\psi(f) \rangle - \langle \psi(f), N\psi(f) \rangle^2 = \|f\|^2. \]

We omit the proof of the lemma, since it can be derived from the definition of the Weyl operator and elementary calculation.

For
\[ (44) \]
we note that \( C^{-1}N^{1/4} \leq d_N \leq CN^{1/4} \) for some constant \( C > 0 \) independent of \( N \), which can be easily checked by using Stirling’s formula. Then we have the following lemmas.

**Lemma A.3.** There exists a constant \( C > 0 \) independent of \( N \) such that, for any \( f \in L^2(\mathbb{R}^3) \) with \( \|f\|_{L^2(\mathbb{R}^3)} = 1 \), we have
\[ \left\| (N + 1)^{-1/2} W^*(\sqrt{N}f) \frac{(a^*(f))^N}{\sqrt{N!}} \Omega \right\| \leq \frac{C(t)}{d_N}. \]

**Proof.** See [21, Lemma 6.3]. \( \square \)

**Lemma A.4.** Let \( P_m \) be the projection onto the \( m \)-particle sector of the Fock space \( \mathcal{F} \) for a non-negative integer \( m \). Then, for any non-negative integers \( k \leq (1/2)N^{1/3} \) and for any \( f \in L^2(\mathbb{R}^3) \) with \( \|f\|_{L^2(\mathbb{R}^3)} = 1 \),
\[ \left\| P_k W^*(\sqrt{N}f) \frac{(a^*(f))^N}{\sqrt{N!}} \Omega \right\| \leq \frac{2}{d_N} \]
and
\[ \left\| P_{k+1} W^*(\sqrt{N}f) \frac{(a^*(f))^N}{\sqrt{N!}} \Omega \right\| \leq \frac{2(k+1)^{3/2}}{d_N \sqrt{N}}. \]

**Proof.** See [23, Lemma 7.2]. \( \square \)

**Appendix B. Properties of the solution of quintic Hartree equation**

In this section, our goal is to bound
\[ (45) \]
which will appear in the proofs given in Section [4]. Note that it is different from the potential energy because we have a square for \( V \).

The following lemma cannot be directly applied for our purpose. We offer it, however, to provide an intuition to the readers.

**Lemma B.1** (generalized Young’s inequality). Let \( p_j > 1 \) for each \( j = 1, \ldots, 5 \) with
\[ \sum_{j=1}^{5} \frac{1}{p_j} = 3. \]
Let \( f_j \in L^{p_j}(\mathbb{R}^n) \) for each \( j = 1, \ldots, 5 \). Then there exists a constant \( C(n, \{p_j\}_{j=1}^{5}) \), independent of \( f_j \), such that
\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} dx \, dy \, dz \, f_1(x)f_2(y)f_3(z)f_4(x-y)f_5(x-z) \leq C(n, \{p_j\}_{j=1}^{5}) \prod_{j=1}^{5} \|f_j\|_{p_j}. \]
Proof. Let
\[ I = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} dx dy dz \, f_1(x) f_2(y) f_3(z) f_4(x - y) f_5(x - z) \]
Then, we integrate \( y \) and \( z \) first so that we have
\[ I = \int_{\mathbb{R}^n} \, dx \, f_1(x) \left( f_2 \ast f_4 \right)(x) \left( f_3 \ast f_5 \right)(x) \]
Using Hölder inequality,
\[ |I| \leq \|f_1\|_{p_1} \|f_2 \ast f_4\|_q \|f_3 \ast f_5\|_r \]
where
\[ \frac{1}{p_1} + \frac{1}{q} + \frac{1}{r} = 1. \]
By Young’s convolutional inequality,
\[ \|f_2 \ast f_4\|_q \leq \|f_2\|_{p_2} \|f_4\|_{p_4} \quad \text{and} \quad \|f_3 \ast f_5\|_r \leq \|f_3\|_{p_3} \|f_5\|_{p_5} \]
with
\[ \frac{1}{q} + 1 = \frac{1}{p_2} + \frac{1}{p_4} \quad \text{and} \quad \frac{1}{r} + 1 = \frac{1}{p_3} + \frac{1}{p_5}. \]
Then combining (46) and (48) together with (47) and (49), we
\[ |I| \leq C(n, \{p_j\}_{j=1}^5) \prod_{j=1}^5 \|f_j\|_{p_j} \]
with
\[ \sum_{j=1}^5 \frac{1}{p_j} = 3. \]
If we allow \( f_4 \in L^p_{w_4} \) and \( f_5 \in L^p_{w_5} \) instead of \( L^p \), we can utilize such lemma for our Coulomb singularity. According to Lieb and Loss in [31], Hardy-Littlewood-Sobolev inequality can be understood as a weak Young’s inequality. Hence, by we provide the following lemma, which generalize Hardy-Littlewood-Sobolev inequality.

**Lemma B.2.** Let \( p_1, p_2, p_3 > 1 \) and \( 0 < \lambda_1, \lambda_2 < n \) with
\[ \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{\lambda_1 + \lambda_2}{n} = 3. \]
Let \( f_j \in L^p_{w_j}(\mathbb{R}^n) \). Then there exists a constant \( C(n, \lambda, p_1, p_2, p_3) \), independent of \( f_j \), such that
\[ \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} dx dy dz \, f_1(x) f_2(y) f_3(z) \frac{1}{|x - y|^{\lambda_1}} \frac{1}{|x - z|^{\lambda_2}} \right| \leq C(n, \lambda, p_1, p_2, p_3) \prod_{j=1}^3 \|f_j\|_{p_j}. \]

**Proof.** This proof generalize the proof of Theorem 4.3 in [31, pp.108-110]. The lemma is followed by applying twice the normal Hardy-Littlewood-Sobolev-inequality (in the first step, call \( g = |\cdot|^{-\lambda_1} \ast f_2 \)).

Using this lemma, we can prove the boundedness of \( H^1(\mathbb{R}^3) \)-norm as follows.

**Lemma B.3** (Boundedness of \( H^1(\mathbb{R}^3) \)-norm). Let \( \varphi_t \) be the solution of quintic Hartree equation with initial data \( \varphi_0 \). If \( \|\varphi_0\|_{H^1} < C \), then
\[ \|\varphi_t\|_{H^1} \leq C. \]
Proof. Define the energy $\mathcal{E}(\varphi_t)$ by
\[
\mathcal{E}(\varphi_t) := \frac{1}{2} \int dx |\nabla \varphi_t(x)|^2 + \frac{\lambda}{6} \int dx dy dz V(x - y, x - z) \varphi_t(x)^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2
\]
\[
\geq \frac{1}{2} \int dx |\nabla \varphi_t(x)|^2
\]
\[
+ \frac{\lambda}{6} \int dx dy dz (v(x - y)v(x - z) + v(y - z)v(y - x)
\]
\[
+ v(z - x)v(z - y))|\varphi_t(x)|^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2.
\]
By Lemma B.2 we obtain
\[
\int dx dy dz \frac{1}{|x - y||x - z|}|\varphi_t(x)|^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2 \leq C||\varphi||_L^{3}_{3/8}.
\]
By Riesz-Thorin interpolation theorem, one get
\[
||\varphi||^3_{L^{3/8}} = ||\varphi||^6_{L^{6/4}} \leq ||\varphi||^5_{L^{5/4}} ||\varphi||_{L^6} \leq ||\varphi||^5_{L^{5/4}} ||\varphi||_{H^1}.
\]
Hence, for sufficiently small $\varepsilon > 0$ so that
\[
\frac{1}{2} \int dx |\nabla \varphi_t(x)|^2 - C\varepsilon||\varphi_t||_{H^1}^2 > 0,
\]
we have
\[
\mathcal{E}(\varphi_0) = \mathcal{E}(\varphi_t) \geq \frac{1}{2} \int dx |\nabla \varphi_t(x)|^2 - C||\varphi_t||_{H^1}^1
\]
\[
\geq \frac{1}{2} \int dx |\nabla \varphi_t(x)|^2 - C \left( \varepsilon ||\varphi_t||_{H^1}^2 + \frac{1}{\varepsilon} \right).
\]
Thus,
\[
\mathcal{E}(\varphi_0) + \frac{C}{\varepsilon} \geq C||\varphi_t||_{H^1}^2.
\]
This leads us to the conclusion. \hfill \square

Remark B.4. For $\lambda > -1/4$, the proof of Lemma B.3 is a bit easier as follows.

If $\lambda > 0$, we are done since
\[
\mathcal{E}(\varphi_t) \geq \frac{1}{2} \int dx |\nabla \varphi_t(x)|^2
\]
implies that $||\varphi_t||_{H^1} \leq C(\mathcal{E}(\varphi_0) + ||\varphi_0||_{L^2})$.

Note that
\[
\int dx dy dz \frac{1}{|x - y||x - z|}|\varphi_t(x)|^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2 \leq 2||\varphi_t||_{H^1}^2 ||\varphi_t||_{L^2}^4
\]
implies
\[
\mathcal{E}(\varphi_0) = \mathcal{E}(\varphi_t) \geq \frac{1}{2} \int dx |\nabla \varphi_t(x)|^2 - 2\lambda ||\varphi_t||_{H^1} ||\varphi_t||_{L^2}^4 \geq \left( \frac{1}{2} - 2\lambda ||\varphi_t||_{L^2}^4 \right) ||\varphi_t||_{H^1}^2.
\]
Then, for $\lambda$ satisfying
\[
|\lambda| < \frac{1}{4},
\]
we have
\[
||\varphi_t||_{H^1} \leq C.
\]

Then we have the following lemma which is our goal of this section.

Lemma B.5. Let $\varphi_t$ be the solution of quintic Hartree equation with three-body interaction potential $V(x - y, x - z)$ having initial data $\varphi_0 \in H^1$. Then
\[
\int dx dy dz |V(x - y, x - z)|^2 |\varphi_t(x)|^2 |\varphi_t(y)|^2 |\varphi_t(z)|^2 \leq C.
\]
Proof. First, we rewrite
\[
\int dx \, dy \, |V(x - y, x - z)|^2 |\varphi_t(x)|^2 |\varphi_t(y)|^2 = \lambda^2 \int dx \, dy \left( \frac{1}{|x - y|} \frac{1}{|x - z|} + \frac{1}{|y - x|} \frac{1}{|y - z|} + \frac{1}{|z - x|} \frac{1}{|z - y|} \right)^2 |\varphi_t(x)|^2 |\varphi_t(y)|^2
\]
\[
\leq 2\lambda^2 \int dx \, dy \frac{1}{|x - y|^2} \frac{1}{|x - z|^2} |\varphi_t(x)|^2 |\varphi_t(y)|^2
+ 2\lambda^2 \int dx \, dy \frac{1}{|x - y|^2} \frac{1}{|x - z|^2} |\varphi_t(x)|^2 |\varphi_t(y)|^2
+ 2\lambda^2 \int dx \, dy \frac{1}{|x - y|^2} \frac{1}{|x - z|^2} |\varphi_t(x)|^2 |\varphi_t(y)|^2
=: I_1 + I_2 + I_3
\]
By Lemma [3:3] we can prove that
\[
I_1 = 2\lambda^2 \int dx \, dy \frac{1}{|x - y|^2} \frac{1}{|x - z|^2} |\varphi_t(x)|^2 |\varphi_t(y)|^2 \leq C ||\varphi_t||_{H^1}^4 \leq C.
\]
Similarly for other terms, we get the conclusion. \qed

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School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea
Email address: jinyeoplee@kias.re.kr

Department of Mathematics, LMU Munich,Theresienstrasse 39, 80333 Munich, Germany
Email address: lee@math.lmu.de