Component versus superspace approaches to $D = 4$, $N = 1$ conformal supergravity

Taichiro Kugo$^1$, Ryo Yokokura$^{2,*}$, and Koichi Yoshioka$^3$

$^1$Department of Physics and Maskawa Institute for Science and Culture, Kyoto Sangyo University, Kyoto 603-8555, Japan
$^2$Department of Physics, Keio University, Yokohama 223-8522, Japan
$^3$Osaka University of Pharmaceutical Sciences, Takatsuki 569-1094, Japan
*E-mail: ryokokur@rk.phys.keio.ac.jp

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The superspace formulation of $N = 1$ conformal supergravity in four dimensions is demonstrated to be equivalent to the conventional component field approach based on the superconformal tensor calculus. The detailed correspondence between the two approaches is explicitly given for various quantities: superconformal gauge fields, curvatures and curvature constraints, general conformal multiplets and their transformation laws, and so on. In particular, we carefully analyze the curvature constraints leading to the superconformal algebra and also the superconformal gauge fixing leading to Poincaré supergravity, since they look rather different between the two approaches.

1. Introduction

$N = 1$ supergravity (SUGRA) in four dimensions has been important for giving a boundary theory around the unification scale for constructing viable phenomenological models beyond the standard model. It has also come to have increasing importance as a low-energy effective theory for superstrings and as a tool for analyzing supersymmetric gauge theories on curved backgrounds.

However, various explicit calculations, e.g., the construction of the SUGRA Lagrangian, are complicated and nontrivial. The simplest and most convenient method is presumably the superconformal tensor calculus, which was developed by Kaku, Townsend, van Nieuwenhuizen, Ferrara, Grisaru, de Wit, van Holten, and Van Proeyen [1–6]. It is a set of rules for constructing invariant actions under local superconformal transformations, i.e., superconformal gauge fields including gravity and gravitinos and various types of matter multiplets, their transformation laws, multiplication rules, and superconformal invariant action formulas. The power of the superconformal tensor calculus comes from a larger symmetry than the usual Poincaré SUGRA. Indeed, its power as a practical computational tool was clearly demonstrated in Ref. [7] for computing the action for the general Yang–Mills (YM)–matter-coupled SUGRA system.

Kugo and Uehara (KU) have presented [8] the superconformal tensor calculus in its most complete form, and discussed the spinorial derivative $\partial_{\alpha}$ for the first time in the component field approach. They found that a special condition on an operand multiplet $\mathcal{V}_1$ must be satisfied so that its spinorial derivative $\partial_{\alpha} \mathcal{V}_1$ exists and gives a conformal multiplet. The condition depends on the spinor index.
α of $\mathcal{G}_\alpha$ and the Lorentz index $\Gamma$ of the operand $\mathcal{V}_\Gamma$, and KU implicitly suspected that the superspace formulation might not exist for the conformal SUGRA.

Nevertheless, Butter [9] has recently presented a superspace formalism of the conformal SUGRA. In contrast with the previous expectation, his formalism realizes a simpler algebra of covariant derivatives than any other superspace Poincaré SUGRA:

$$\{\nabla_\alpha, \nabla_\beta\} = \{\bar{\nabla}_\alpha, \bar{\nabla}_\beta\} = 0, \quad \{\nabla_\alpha, \bar{\nabla}_\beta\} = -2i\nabla_\alpha\bar{\nabla}_\beta.$$  \hspace{1cm} (1.1)

Requiring this algebra together with several constraints on curvatures in the vector–spinor direction, he succeeded in constructing a superspace counterpart of the conformal SUGRA in the component approach. The covariant derivatives $\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}_\dot{\alpha})$ can be freely applied on any superfield with no restriction and are identified with the transformations $P_A = (P_a, Q_\alpha, \bar{Q}_{\dot{\alpha}})$ of the superconformal group. The reason why KU’s spinorial derivatives could not be freely applied turns out to be because KU required an extraneous condition that the derivative again give a primary multiplet.

Since the superspace formalism manifests supersymmetry in a geometrically clear way, it gives a transparent and powerful means to treat systems in new situations, such as finding nonlinear realization, brane worlds, decomposition of higher-$N$ supersymmetry, partial breaking of local supersymmetry, massive SUGRA, etc. On the other hand, one needs to write down the action explicitly in terms of component fields, which could be done most easily and efficiently with the tensor calculus. That is, we have two approaches to the conformal SUGRA; one is the superspace approach based on the conformal superspace and superfields, and the other is the component approach based on the superconformal tensor calculus. Both approaches have their own strong and weak points. In order to use the advantages of both approaches, it is desirable to see the correspondence between them. The purpose of this paper is to show the equivalence of the two approaches by making the detailed correspondences manifest.

This paper is organized as follows. In Sect. 2, we recapitulate the essential parts, first of the superconformal tensor calculus in the component approach, and then of the conformal superspace approach. We use individual notation for each of these approaches and separately provide a “dictionary” to translate between them for the convenience of reading the references.

In Sect. 3 we explicitly present the correspondences of various quantities. We first discuss gauge fields and curvatures in Sect. 3.1 and show how all the curvature constraints in the component approach are satisfied in the superspace approach, although the constraints look rather different from each other. The same superconformal transformation algebras are realized in both approaches under these curvature constraints. We then discuss the component fields and transformation rules for a conformal multiplet with arbitrary external Lorentz index in Sect. 3.2, and the chiral projection and the invariant action formulas in Sect. 3.3. We analyze in Sect. 3.4 the compensated (or $u$-associated) derivative that maps a primary superfield to another primary superfield. There we also discuss KU’s restriction on the spinorial derivatives.

In Sect. 4, we investigate the matter-coupled SUGRA system and the superconformal gauge fixing to Poincaré SUGRA, mainly from the superspace viewpoint. We discuss the superspace counterpart of KU’s gauge fixing, which leads directly to the canonically normalized Einstein–Hilbert (EH) and Rarita–Schwinger (RS) terms. The correspondence to the component approach is nontrivial since the gauge invariance in the superspace approach is much larger than the component approach, and the gauge fixing written in terms of superfields give more fixing conditions than the component case. One remarkable fact is that the covariant spinor derivatives remaining after the gauge fixing automatically reproduce the complicated supersymmetry transformation in Poincaré SUGRA.
final section is devoted to the summary. We add three appendices. The notations in the component and superspace approaches are summarized separately and the “dictionary” to translate between them is given in Appendix A. The standard form of the supersymmetry transformation law for the general conformal multiplet with arbitrary external Lorentz index is cited for convenience in Appendix B. We present in Appendix C some explicit computations that are necessary in deriving the results in the text.

2. Conformal SUGRA

We first briefly review the component and superspace approaches for $D = 4, \mathcal{N} = 1$ conformal SUGRA. In both approaches the conformal SUGRA is constructed as the gauge theory of the superconformal group. The Lie algebra of the superconformal group contains the following elements: translation $P_a$, supersymmetry $Q$, Lorentz transformation $M_{ab}$, conformal boost $K_a$, supersymmetry of conformal boost $S$, dilatation $D$, and chiral rotation $A$.

2.1. Component approach

In this subsection we review the component approach. For any part of component approach in this paper, we use the notations and conventions of Ref. [8], which are the same as those of Ref. [10] except for two-component spinors and the dual of antisymmetric tensors. The details of the notations are summarized in Appendix A. The superconformal algebra consists of 15 bosonic and 8 fermionic generators, which obey the following graded commutation relations:

\[
\begin{align*}
[M_{ab}, M_{cd}] &= -M_{ad}\delta_{bc} + M_{bd}\delta_{ac} + M_{ac}\delta_{bd} - M_{bc}\delta_{ad}, \\
[M_{ab}, P_c] &= -P_a\delta_{bc} + P_b\delta_{ac}, \\
[M_{ab}, K_c] &= -K_a\delta_{bc} + K_b\delta_{ac}, \\
[D, P_a] &= P_a, \quad [D, K_a] = -K_a, \quad \{K_a, P_b\} = 2\delta_{ab}D + 2M_{ab}, \\
\{Q, Q^T\} &= -\frac{1}{2}(\gamma_a C^{-1})P_a, \quad \{S, S^T\} = \frac{1}{2}(\gamma_a C^{-1})K_a, \\
[M_{ab}, Q] &= \sigma_{ab}Q, \quad [M_{ab}, S] = \sigma_{ab}S, \\
[D, Q] &= \frac{1}{2}Q, \quad [D, S] = -\frac{1}{2}S, \quad [A, Q] = -\frac{3}{4}i\gamma_5Q, \quad [A, S] = \frac{3}{4}i\gamma_5S, \\
[K_a, Q] &= -\gamma_aS, \quad [S, P_a] = -\gamma_aQ, \\
\{Q, S^T\} &= -\frac{1}{2}C^{-1}D + \frac{1}{2}\sigma^{ab}C^{-1}M_{ab} + i\gamma_5C^{-1}A. \quad (2.1)
\end{align*}
\]

All other commutation relations vanish. The generators are generically denoted as $X_A$ and the above commutation relations are written as

\[
[X_A, X_B] = -f_{AB}^C X_C. \quad (2.2)
\]

Note that these generators represent the active operators transforming fields, not the representation matrices. The commutation relations change signs if written for representation matrices instead of active operators. In the conformal SUGRA, the superconformal symmetry is treated as local.
symmetry. The corresponding gauge fields and transformation parameters are given by

\[ h_\mu^A X_A = e_\mu^a P_a + \bar{\psi}_\mu Q + \frac{1}{2} \omega_\mu^{ab} M_{ab} + b_\mu D + A_\mu A + \bar{\varphi}_\mu S + f_\mu^a K_a, \]  
\[ \epsilon^A X_A = \xi^a P_a + \bar{\xi} Q + \frac{1}{2} \lambda^{ab} M_{ab} + \rho D + \theta A + \bar{\xi} S + \bar{\xi}^a K_a. \] 

In the component approach, the Greek letters \( \mu, \nu, \ldots \) denote curved vector indices and the Roman letters \( a, b, \ldots \) flat Lorentz indices. The group transformation laws of the gauge fields under the superconformal symmetry are

\[ \delta_{\text{group}}^B (\epsilon^B) h_\mu^A = \partial_\mu \epsilon^A + h_\mu^B \epsilon^C f^{CB}_A. \]  

The curvature of the superconformal algebra (before the deformation below) is

\[ R^A_{\mu \nu} = \partial_\nu h_\mu^A - \partial_\mu h_\nu^A + h_\nu^B h_\mu^C f^{CB}_A. \]  

The \( P_a \) translation is deformed so as to be related to the general coordinate (GC) transformation \( \delta_{\text{GC}} \) as

\[ \delta_{\tilde{P}} (\xi^a) := \delta_{\text{GC}} (\xi^\mu) - \sum_{A \neq P} \delta_A (\xi^\mu h_\mu^A), \] 

where \( \xi^\mu = \xi^a e_\mu^a \), and \( \xi^a \) is a field-independent parameter. In order to have \( [\delta_Q, \delta_Q] \sim \delta_{\tilde{P}} \), several constraints on the curvatures are imposed:

\[ R^A_{\mu \nu} (P^a) = 0, \]  
\[ R^A_{\mu \nu} (Q) = 0, \]  
\[ R^A_{\nu \lambda} (M_{ab}) e^{a \mu} e^{b \nu} - \frac{1}{2} R_{\lambda \nu} (Q) \gamma_5 \psi_\lambda + \frac{1}{2} \tilde{R}_{\mu \nu} (A) = 0, \]  

where \( \tilde{R}_{\mu \nu} \) is the dual of \( R_{\mu \nu} \). By these constraints, the \( M_{ab} \), \( S \), and \( K_a \)-gauge fields (\( \omega_\mu^{ab}, \varphi_\mu, \) and \( f_\mu^a \), respectively) become dependent fields expressed by other independent gauge fields. The \( Q \) transformations \( \delta_Q (\epsilon) \) of the dependent gauge fields are determined by those of independent gauge fields, and they deviate from the original group transformation \( \delta_{\text{group}}^Q (\epsilon) \) as

\[ \delta_Q (\epsilon) = \delta_{\text{group}}^Q (\epsilon) + \delta_Q' (\epsilon). \]  

The deviation part \( \delta_Q' (\epsilon) \) is given by

\[ \delta_Q' (\epsilon) \omega_\mu^{ab} = \frac{1}{2} R^{ab} (Q) \gamma_\mu \epsilon, \]  
\[ \delta_Q' (\epsilon) \varphi_\mu = \frac{1}{4} i \gamma^\nu (\gamma_5 R_{\nu \mu} (A) + \tilde{R}_{\nu \mu} (A)) \epsilon, \]  
\[ \delta_Q' (\epsilon) f_\mu^a = -\frac{1}{2} R^{\text{cov}}_{\nu \mu} (S) \sigma^{a \nu} \epsilon - \frac{1}{4} \epsilon^{a \nu} \tilde{R}^{\text{cov}}_{\nu \mu} (S) \gamma_5 \epsilon, \]  

where

\[ R^{\text{cov}}_{\mu \nu} = R_{\mu \nu}^A + \delta_Q (\psi_\mu) h_\nu^A - \delta_Q' (\psi_\nu) h_\mu^A. \]
Note that the RHS of Eq. (2.12) are given by \( \varepsilon^c f_P p_c^X \) with \( X = M_{ab}, S, K_a \). So they can be regarded as the deformation of the algebra by changing the structure constant of the \([Q, P_c]\) commutator from (originally) zero to the nonvanishing \( f_P p_c^X \) for \( X = M_{ab}, S, K_a \).

The resultant commutation relations are the same as the original ones

\[
[\delta_A(\varepsilon^A), \delta_B(\varepsilon^B)] = \sum_C \delta_C(\varepsilon^A f_{BA} C),
\]

for all \( A \) and \( B \), if \( P_a \) on the RHS of the \( Q-Q \) commutator is understood to be \( \tilde{P}_a \):

\[
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_p \left( \frac{1}{2} \tilde{\varepsilon}_2 \gamma_a \varepsilon_1 \right).
\]

Moreover, the definition of \( \tilde{P}_a \) transformation leads to

\[
[\delta_p(\xi^a), \delta_Q(\varepsilon)] = \sum_{A=MSK} \delta_A(\xi^a f^a_Q c^A), \quad \left( \rightarrow f_P p_a^A = -R^A_{ab} \right)
\]

where \( a = (\alpha, \dot{\alpha}) \). The superconformally covariant derivative on fields carrying only flat Lorentz indices is defined through the \( \tilde{P}_a \) transformation as

\[
\xi^a D_a \phi := \delta_p(\xi^a) \phi = \xi^a \gamma^a \partial_\mu \phi - \sum_{A \neq P} \delta_A(\gamma_{a}^A) \phi.
\]

Next, we introduce superconformal multiplets. A general conformal multiplet \( \mathcal{V}_\Gamma \) is a set of \((8 + 8) \times \text{dim} \Gamma\) complex fields,

\[
\mathcal{V}_\Gamma = [C_\Gamma, Z_{a\Gamma}, \mathcal{H}_\Gamma, K_\Gamma, B_{a\Gamma}, A_{a\Gamma}, D_\Gamma],
\]

where \( \Gamma \) represents arbitrary spinor indices \( \Gamma = (\alpha_1, ..., \alpha_m; \dot{\beta}_1, ..., \dot{\beta}_n) \) and \( \text{dim} \Gamma \) is the dimension of the Lorentz representation of \( \Gamma \). The first component \( C_\Gamma \) is defined as having the lowest Weyl weight in the multiplet so that its transformation law is given by

\[
\delta_Q(\varepsilon) C_\Gamma = \frac{1}{2} i \tilde{\varepsilon}_5 \gamma_5 Z_\Gamma, \quad \delta_M(\lambda^{ab}) C_\Gamma = \frac{1}{2} \lambda^{ab} (\Sigma_{ab})_\Gamma^\Gamma \Sigma C_\Gamma = : \frac{i}{2} \lambda^{ab} (\Sigma_{ab})_\Gamma^\Gamma :.
\]

\[
(\delta_D(\rho) + \delta_A(\theta)) C_\Gamma = \left( w_\rho + \frac{1}{2} i m_\theta \right) C_\Gamma, \quad \delta_S(\xi) C_\Gamma = \delta_K(\xi^a_K) C_\Gamma = 0.
\]

Here \( \Sigma^{ab} \) is the representation matrix of the Lorentz generator that \( C_\Gamma \) belongs to, and \( w \) and \( n \) are the Weyl and chiral weights of \( C_\Gamma \). The \( S \) and \( K_a \) transformations must annihilate the lowest-weight component \( C_\Gamma \) since they lower the Weyl weights of operands. The \( Q \) transformation law \( \delta_Q(\varepsilon) C_\Gamma = \frac{1}{2} i \tilde{\varepsilon}_5 \gamma_5 Z_\Gamma \) simply defines the second component \( Z_\Gamma \). All the higher components in the multiplet and their superconformal transformation laws are determined by requiring the superconformal algebra to hold on them, aside from some arbitrariness in defining higher-component fields. The \( Q \) transformation laws of all component fields are summarized in (B1), which also fixes the definition of higher-component fields. We call the transformation laws (B1) the standard form. Since the first component \( C_\Gamma \) specifies the whole multiplet, we denote the conformal multiplet \( \mathcal{V}_\Gamma \) using the first component as

\[
\mathcal{V}_\Gamma = [C_\Gamma].
\]
A constrained-type multiplet also exists as a conformal multiplet if some conditions are met on Weyl and chiral weights and also on its Lorentz representation. The chiral multiplet \( \Sigma^{(w,n)}_\Gamma \), for instance, exists only when the Weyl and chiral weights \((w, n)\) satisfy \( w = n \) and the Lorentz index \( \Gamma \) is made of purely undotted spinor indices; then the chiral multiplet has \((2 + 2) \times \text{dim } \Gamma \) complex components denoted by

\[
\Sigma^{(w=n)}_\Gamma = [A_\Gamma, \mathcal{P}_R \chi_\Gamma, \mathcal{F}_\Gamma].
\]  

(2.21)

These three components of a chiral multiplet are embedded into a general conformal multiplet in the form \( \mathcal{V}(\Sigma_\Gamma) = [A_\Gamma, -i\mathcal{P}_R \chi_\Gamma, -\mathcal{F}_\Gamma, i\mathcal{F}_\Gamma, iD_aA_\Gamma, 0, 0] \), so that their \( Q \) and \( S \) transformation laws are given by

\[
\delta Q_\Sigma A_\Gamma = \left( \delta Q(\varepsilon) + \delta S(\zeta) \right) A_\Gamma = \frac{1}{2} \delta R \mathcal{P}_R \chi_\Gamma,
\]

\[
\delta Q_\Sigma \mathcal{P}_R \chi_\Gamma = (-1)^\Gamma \left( \gamma^a D_a A_\Gamma \varepsilon_L + \mathcal{F}_\Gamma \varepsilon_R + \left( 2w A_\Gamma - (\Sigma^{ab} A_\Gamma) \sigma_{ab} \right) \varepsilon_R \right),
\]

\[
\delta Q_\Sigma \mathcal{F}_\Gamma = \frac{1}{2} \delta L \gamma^a D_a \mathcal{P}_R \chi_\Gamma + \delta R \left( (1 - w) \mathcal{P}_R \chi_\Gamma - \frac{1}{2} \sigma_{ab} (\Sigma^{ab} \mathcal{P}_R \chi_\Gamma) \right).
\]  

(2.22)

For the multiplet \( \mathcal{V}^{(w,n)}_\Gamma \) with purely undotted spinor \( \Gamma \) satisfying \( w = n + 2 \), the chiral projection operator \( \Pi \) exists and

\[
\Pi \mathcal{V}^{(w,n=w-2)}_\Gamma = \left[ \frac{1}{2}(\mathcal{H}_\Gamma - i\mathcal{K}_\Gamma), i\mathcal{P}_R(\gamma^a D_a \mathcal{Z}_\Gamma + \Lambda_\Gamma), -\frac{1}{2}(D_\Gamma + \Box_\Gamma + iD^a B_{a\Gamma}) \right]
\]  

(2.23)

gives a chiral multiplet with the Weyl and chiral weights \((w + 1, w + 1)\). Here \( \Box = D^a D_a \) is the superconformal d’Alembertian.

The superconformal tensor calculus gives the superconformally invariant action in simple forms. The F-type invariant action formula is applied only to the chiral multiplet \( \Sigma = [A = \frac{1}{2}(A + iB), \mathcal{P}_R \chi, \mathcal{F} = \frac{1}{2}(F + iG)] \) satisfying \( w = n = 3 \) and carrying no external Lorentz index. The action is given by

\[
\int d^4x \left[ \Sigma^{(w=n=3)} \right]_F = \int d^4x e \left( F + \frac{i}{2} \tilde{\psi}_a \gamma^a \chi + \frac{i}{2} \tilde{\psi}_a \sigma^{ab} (A - i\gamma_5 B) \psi_b \right).
\]  

(2.24)

The D-type invariant action formula is applied only to the real and Lorentz-scalar multiplet \( V = [C, Z, H, K, B_a, \lambda, D] \) with \( w = 2 \) and \( n = 0 \). The action is derived from the F-type formula with the chiral projection operator \( \Pi \) as

\[
\int d^4x \left[ I^{(w=2)} \right]_D = \int d^4x \left[ -\Pi V^{(w=2)} \right]_F
\]

\[
= \int d^4x e \left( D + \Box C - \frac{1}{2} \tilde{\psi}_a \gamma^a \gamma_5 (\gamma^b D_b Z + \lambda) - \frac{1}{2} \tilde{\psi}_a \sigma^{ab} (H + i\gamma_5 K) \psi_b \right)
\]

\[
= \int d^4x \left( \frac{1}{2} i \bar{\psi}_a \gamma^a \gamma_5 \lambda - i \bar{\psi}_a \gamma^a \gamma_5 Z + \frac{1}{3} C \left( R + \frac{1}{e} \bar{\psi}_\mu \epsilon_{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu \left( \partial_\rho \psi_\sigma + \frac{i}{4} \omega_\rho \sigma^{ab} \sigma_{ab} \psi_\sigma \right) \right)
\]

\[
+ \frac{1}{4} i \epsilon_{abcd} \tilde{\psi}_a \gamma_\beta \psi_\epsilon \left( B_d - A_d C - \frac{1}{2} \tilde{\psi}_d Z \right) \right).
\]  

(2.25)
For the general YM–matter-coupled SUGRA system, the action is given by

\[
\mathcal{L} = -\frac{1}{2} \left[ \phi(S, \bar{S}e^{2V_{G}}) \Sigma_{c} \bar{\Sigma}_{c} \right]_{D} + \left[ \Sigma_{c}^{\delta} g(S) \right]_{F} - \frac{1}{4} \left[ f_{\alpha \beta} \bar{W}^{\alpha} W^{\beta} \right]_{F}
\]

where \( S_{i} = [z_{i}, P_{R} \chi_{i}, h_{i}] \) are the chiral matter multiplets with vanishing weights \( w = n = 0 \) and \( \bar{S}^{i} \) are their conjugate. In the first term, \( V_{G} \) means the YM vector multiplet of internal symmetry. The field \( \Sigma_{c} \) is a chiral compensator carrying weights \( (w, n) = (1, 1) \). For the field possessing nonvanishing superpotential \( g(S) \), it is convenient to redefine the compensator as \( \Sigma_{c} \to \Sigma_{0} = g^{1/3}(S) \Sigma_{c} = [z_{0}, P_{R} \chi_{0}, h_{0}] \) so that \( \phi \) becomes the combination of \( \bar{\phi} \) and superpotential: \( \phi(S, \bar{S}e^{2V_{G}}) = \bar{\phi}(S, \bar{S}e^{2V_{G}}) |g(S)|^{-2/3} \). In the third term, \( f_{\alpha \beta} \) is a holomorphic function of \( S_{i} \), symmetric under the exchange \( \alpha \leftrightarrow \beta \), and \( W^{\alpha} \) is the gaugino multiplet (field-strength supermultiplet) of internal symmetry. For the YM vector multiplet, the Wess–Zumino (WZ) gauge is imposed, and then the gaugino multiplet is constructed by the \( Q \) transformation that preserves the WZ gauge. We denote such \( Q \) transformation as \( \delta_{Q}^{YM}(\epsilon) \).

To go down to the Poincaré SUGRA, we fix the extraneous \( D-, A-, S-, K_{\alpha} \)-gauge symmetries. The so-called improved gauge-fixing conditions adopted in Ref. [7] are

\[
D-, A\text{-}gauge: \ z_{0} = z_{0}^{0} = \sqrt{3} \phi^{-1/2}(z, z^{*}),
\]

\[
S\text{-}gauge: \ 1/2 \mathcal{P}_{R} \chi_{0} = -z_{0} \phi^{-1}(\chi_{R}), \quad K_{\alpha} \text{-}gauge: \ b_{\mu} = 0,
\]

where \( \chi_{R0} = 1/2 \mathcal{P}_{R} \chi_{0} \) and \( \chi_{Ri} = 1/2 \mathcal{P}_{R} \chi_{i} \). These gauge conditions set the first and second components of the vector multiplet \( \phi \Sigma_{0} \bar{\Sigma}_{0} \) to 3 and 0, respectively, in the D-type action formula. As a result, the canonically normalized EH and RS terms are obtained directly.

The relation between the \( Q \) transformation \( \delta_{Q}^{P}(\epsilon) \) in the resultant Poincaré SUGRA and the gauge-fixed conformal \( Q \) transformation is given by

\[
\delta_{Q}^{P}(\epsilon) = \delta_{Q}^{YM}(\epsilon) + \delta_{A}(\theta(\epsilon)) + \delta_{S}(\zeta(\epsilon)) + \delta_{K}(\xi^{a}(\epsilon)),
\]

where

\[
\theta(\epsilon) = -\frac{i}{3} \left( G^{i} \bar{e}_{R} \chi_{Rj} - G_{i} \bar{e}_{L} \chi^{i}_{L} \right),
\]

\[
\zeta_{R}(\epsilon) = -\frac{1}{2} \left( h_{0} \phi^{-1} + \frac{1}{3} h_{i} G^{i} \right) \bar{e}_{R} - \frac{1}{3} \left( (G^{i} - \frac{1}{3} G^{i} G^{j}) \bar{e}_{R} \chi_{Rj} + G^{j} \bar{e}_{L} \chi^{j}_{L} \right) \chi_{Ri}
\]

\[
- \frac{1}{12} \left( G^{i} \gamma^{a} \nabla_{a} z_{i} - G_{i} \gamma^{a} \nabla_{a} z^{*i} \right) e_{R} + \frac{1}{2} i \gamma^{a} A_{a} e_{L},
\]

\[
\xi^{a}(\epsilon) = \frac{i}{4} \left( \bar{\psi}_{a} e - \bar{\psi}_{a} \bar{e} \right).
\]

In this expression \( \nabla_{a} z_{i} \) is the covariant derivative of the internal symmetry, and the \( G \) are given by

\[
G = 3 \log \frac{1}{2} \phi(z, z^{*}).
\]

The indices of \( G \) represent the differentiation with respect to \( z_{i} \) and \( z^{*i} \), e.g., \( G^{i}_{j} = \partial^{2} G / \partial z_{i} \partial z^{*j} \).

2.2. Conformal superspace

Next we review the conformal superspace approach [9]. In superspace, the supersymmetry transformation can be treated as a translation in the direction of the Grassmannian spinor coordinate on
the same footing as the usual translation $P_a$. The (anti)commutation relations between the spinor
covariant derivatives become complicated in Poincaré SUGRA, whereas, in conformal superspace,
they are as simple as in global supersymmetry. In any part of the superspace approach in this paper,
we use the notations and conventions of Butter [9], with a few exceptions, which will be explained
below. The details of the notations are summarized in Appendix A.

The superconformal algebra is the same as (2.1) given in the component approach, if we perform
a suitable translation of the generators between the two approaches (see Table 1). Here we refer to
only a few characteristic commutation relations:

\[
\{Q_a, \bar{Q}_a\} = -2i(\sigma^a)_{\alpha\dot{\alpha}} P_a, \quad \{S_a, \bar{S}_a\} = 2i(\sigma^a)_{\alpha\dot{\alpha}} K_a,
\]

\[
[S_a, P_a] = i(\sigma_a)_{\alpha\beta} \bar{Q}_{\dot{\beta}}, \quad \{S_a, Q_{\dot{\beta}}\} = (2D - 3i\lambda)\epsilon_{\alpha\beta} - 2M_{a\beta},
\]

\[
[\bar{S}_{\dot{a}}, P_a] = i(\bar{\sigma}_{\dot{a}})_{\dot{\alpha}\dot{\beta}} \bar{Q}_{\dot{\beta}}, \quad \{\bar{S}_{\dot{a}}, \bar{Q}_{\dot{\beta}}\} = (2D + 3i\lambda)\epsilon_{\dot{\alpha}\beta} - 2M_{a\dot{\beta}},
\]

with \(M_{a\beta} = (\sigma^{ba}\epsilon)_{a\beta} M_{ab}, \quad M_{a\dot{\beta}} = (\bar{\sigma}^{ba}\epsilon)_{a\dot{\beta}} M_{ab}.\) (2.30)

Note that the normalizations of \(Q, S, A\) are different from the component approach. The gauge
superfields corresponding to the superconformal group are denoted as

\[
h^A_M X_A = E^A_M P_A + \frac{1}{2} \phi^a_M b_{aM} + B_M D + A_M A + f_M^A K_A,
\]

(2.31)

where we use the calligraphic index \(A\) for the total superconformal algebra, the Roman uppercase
index \(A\) for the Lorentz vector and spinor set as \(P_A = (P_a, \bar{Q}_a, \bar{Q}_{\dot{\beta}})\) and \(K_A = (K_a, S_a, \bar{S}_{\dot{a}})\), and the
index \(M\) for the set of curved indices, e.g., \(A_M = (A_m, A_{\mu}, A_{\dot{\mu}})\). We assume that the vielbein \(E^A_M\) is
invertible:

\[
E^A_M E^N_A = \delta^N_M, \quad E^A_M E^B_M = \delta^B_A.
\]

(2.32)

The gauged superconformal transformations are taken by real parameter superfields. These parameter
superfields are denoted as

\[
\xi^A X_A = \xi (P)^A P_A + \frac{1}{2} \xi (M)^{ab} M_{ba} + \xi (D) D + \xi (A) A + \xi (K)^A K_A.
\]

(2.33)

The gauge fields receive the superconformal transformation \(\delta_G(\xi^A X_A)\) as

\[
\delta_G(\xi^A X_A) h^A_M = \partial_M \xi^B \delta_B^A + h^C_B \xi^B \phi^C_M,
\]

(2.34)

where the primed calligraphic index \(A'\) means all the superconformal generators other than \(P_A\),
namely, \(X_A' = (P_A, X_A')\). Note that Ref. [9] used different notation in which \(X_A\) was expressed as
\(X_a\) with no distinction from \(A\) for \((a, \alpha, \dot{\alpha})\), and our \(h^A_M\) and \(h^a_M\) were denoted by \(W^A_M\) and \(h^a_M\),
respectively.

In the same spirit as the component approach, the \(P_A\) transformation is defined as being related to
the general coordinate transformation \(\delta_{GC}\) using the field-independent parameter superfield \(\xi^A\) as

\[
\delta_G(\xi^A P_A) = \delta_{GC}(\xi^M := \xi^A E_A^M) - \delta_G(\xi^M h^B_M X^B) ,
\]

(2.35)

where \(\xi (P)^A\) is abbreviated to \(\xi^A\). The \(P_A\) transformation acting on a superfield \(\Phi\) with no curved
index defines the covariant derivative as

\[
\delta_G(\xi^A P_A) \Phi = \xi^A P_A \Phi = \xi^M \nabla_M \Phi = \xi^M (\partial_M - h^A_M X_A') \Phi.
\]

(2.36)
That is, \( P_A = \nabla_A = E_A^M \nabla_M \) on superfields with flat indices. The curvature \( R_{MN}^A \) is defined as
\[
R_{MN}^A = \partial_M h_N^{-A} - \partial_N h_M^{-A} - (E_N^C h_M^{B'} - E_M^C h_N^{B'}) f_{B'C}^{-A} - h_N^C h_M^{B'} f_{B'C}^{-A}.
\] (2.37)

Here and hereafter, we use the convention of “implicit grading”. In superspace, we generally treat both bosonic and fermionic quantities at the same time by the index \( A \) or \( M \), and should be careful with the grading of fermionic objects such as \( X_{AB} = (-)^{a(b+n)} E_B^N E_A^M X_{MN} \). \([\nabla_A, \nabla_B] = \nabla_A \nabla_B - (-)^{ab} \nabla_B \nabla_A \), and \( Z = (-)^a Y_A^A \). The grading is uniquely determined if the standard order of indices is specified. For example, the standard order of \( X_{AB} \) is \( AB \) and hence \( E_B^N E_A^M X_{MN} \) should be accompanied by the grading factor \((-)^{a(b+n)}\) since one jumps the index \( A \) over two indices \( B \) and \( N \) of \( E_B^N \) in order to recover the standard order \( AB \). Implicit grading means to assume the omission of such unique grading factors from everywhere. In other words, we can treat the indices \( A, M \) as if they were bosonic ones. The same implicit grading convention is also used for the index \( A \) of superconformal generators. In the definition of curvatures, the commutation relation of \( P_A \) is as follows:
\[
[P_A, P_B] = - R_{AB}^C X_C = - R(P)_{AB}^C P_C - \frac{1}{2} R(M)_{AB}^{cd} M_{cd}
- R(D)_{AB} D - R(A)_{AB} A - R(K)_{AB}^C K_C,
\] (2.38)
where \( R_{AB}^C = E_B^N E_A^M R_{MN}^C \) in terms of \( R_{MN}^C \) given in (2.37). In Ref. [9], \( R(P)_{AB}^C \) is expressed as \( T_{AB}^C, R(M)_{AB}^{cd} \) is \( R_{AB}^{cd} \), \( R(D)_{AB} \) is \( H_{AB} \), and \( R(A)_{AB} \) is \( F_{AB} \).

Several constraints are imposed on the curvature superfields to eliminate the redundant degrees of freedom. First, the constraints on \( R_{\alpha \beta}^A \) are as follows:
\[
R_{\alpha \beta}^A = 0, \quad R_{\bar{\alpha} \bar{\beta}}^A = 0, \quad R(P)_{\alpha \beta}^c = 2i(\sigma^c)_{\alpha \beta},
R_{\bar{\alpha} \bar{\beta}}^A = 0 \quad \text{(otherwise)},
\] (2.39)

which guarantees that the commutation relations of covariant spinor derivatives take the simple form (as in the global supersymmetry case):
\[
[\nabla_\alpha, \nabla_\beta] = 0, \quad [\bar{\nabla}_\alpha, \bar{\nabla}_\beta] = 0, \quad [\nabla_\alpha, \bar{\nabla}_\beta] = -2i\nabla_{\alpha \beta}.
\] (2.40)

Secondly, the following constraints on \( R_{\alpha a} \) are imposed:
\[
R(P)_{\gamma b}^A = 0, \quad R(D)_{\beta a} = 0, \quad R(A)_{\beta a} = 0.
\] (2.41)

By solving the Bianchi identities
\[
[\nabla_\alpha, [\nabla_B, [\nabla_C, [\nabla_B, [\nabla_C, [\nabla_A, \nabla_A]]]] + [\nabla_B, [\nabla_C, [\nabla_A, \nabla_B]]] = 0
\] (2.42)
under these constraints (with implicit grading understood), one finds that all other nonvanishing curvatures can be expressed by a single superfield \( W_{\alpha \beta \gamma} \) with totally symmetric undotted spinor indices \( \alpha, \beta, \gamma \) as seen below.

The Bianchi identities with the first constraints (2.39) imply that the curvatures \( R_{a b} \) and \( R_{\alpha a} \beta \gamma \) can be expressed by a “gaugino” superfield \( W_{\bar{\alpha}} \), which is superconformal algebra valued:
\[
R_{\alpha \beta \gamma} = -[\nabla_\alpha, \nabla_\beta \gamma] = 2i\epsilon_{\alpha \beta} W_\gamma, \quad R_{\bar{\alpha} \gamma} = -[\bar{\nabla}_{\bar{\alpha}}, \nabla_\gamma] = 2i\epsilon_{\bar{\alpha} \gamma} W_\gamma,
R_{\alpha \beta, \bar{a} \bar{\beta}} = -\epsilon_{\bar{a} \bar{\beta}} [\nabla_\alpha, W_\beta] - \epsilon_{a \beta} [\bar{\nabla}_{\bar{a}}, W_\bar{\beta}],
\] (2.43)
(2.44)
where \( R_{a\beta\gamma} = (\sigma^b)_{\beta\gamma} R_{ab} \), which is \( R_{a(\beta\gamma)} \) in Ref. [9]. The brackets ( ) on the indices imply symmetrisation with the weight one, i.e., \( \psi_{(\alpha \chi\beta)} = (1/2) (\psi_{\alpha \chi \beta} + \psi_{\beta \chi \alpha}) \). This algebra-valued superfield \( \mathcal{W}_a \) satisfies
\[
\{ \nabla_\alpha, \mathcal{W}_\gamma \} = \{ \tilde{\nabla}_\alpha, \mathcal{W}_\gamma \} = 0, \quad \text{(chirality)} \tag{2.45}
\]

\[
\{ \nabla^\alpha, \mathcal{W}_a \} = \{ \tilde{\nabla}_{\beta}, \mathcal{W}^\beta \}, \quad \text{(reality)} \tag{2.46}
\]

\[
[M_{bc}, \mathcal{W}_a] = (\sigma_{bc})_a^\beta \mathcal{W}_\beta, \quad [D, \mathcal{W}_a] = \frac{3}{2} \mathcal{W}_a, \quad [A, \mathcal{W}_a] = i \mathcal{W}_a, \quad [K_A, \mathcal{W}_a] = 0. \tag{2.47}
\]

The further input of the second constraints (2.41) implies that \( \mathcal{W}_a \) has no \( P_A, D, A \) components, \( \mathcal{W}(P)_{a\alpha}^A = \mathcal{W}(D)_{a\alpha} = \mathcal{W}(A)_{a\alpha} = 0 \), so that
\[
\mathcal{W}_a = \frac{1}{2} \mathcal{W}(M)_{a\alpha}^\beta M_{\beta c} + \mathcal{W}(K)_{a\alpha}^\beta K_{\beta c}. \tag{2.48}
\]

With the help of the superconformal algebra, the chirality and reality conditions (2.45) and (2.46) lead to the final expression
\[
\mathcal{W}_a = (\varepsilon R^{bc} \gamma_{\beta \gamma} W_{a\beta \gamma} M_{\gamma c} + \frac{1}{2} (\nabla^\gamma W_{\gamma a}^\beta) S_{\beta} - \frac{1}{2} (\nabla^\gamma W_{\gamma a}^\beta) K_{\beta \gamma}, \tag{2.49}
\]

\[
\mathcal{W}_\tilde{a} = (\tildesig^{bc})_{\tilde{a}}^\gamma \tilde{W}_{\beta \gamma} M_{\gamma c} - \frac{1}{2} (\tilde{\nabla}_{\gamma} W_{\gamma a}^\beta) S_{\beta} - \frac{1}{2} (\tilde{\nabla}_{\gamma} W_{\gamma a}^\beta) K_{\beta \gamma}. \tag{2.50}
\]

with which the reality condition is written:
\[
\{ \nabla^\alpha, \mathcal{W}_a \} = \{ \tilde{\nabla}_\alpha, \mathcal{W}_\tilde{a} \} = -\frac{1}{2} (\nabla^\alpha \nabla^\beta W_{\gamma a}^\beta) K_{\beta \gamma} = -\frac{1}{2} (\tilde{\nabla}_\alpha \tilde{\nabla}_{\gamma} W_{\gamma a}^\beta) K_{\beta \gamma}. \tag{2.51}
\]

In this way, the gaugino superfield \( \mathcal{W}_a \) is expressed by the totally symmetric superfield \( W_{a\beta \gamma} \), which satisfies
\[
\tilde{\nabla}_\alpha W_{a\beta \gamma} = 0, \quad D W_{a\beta \gamma} = \frac{3}{2} W_{a\beta \gamma}, \quad A W_{a\beta \gamma} = i W_{a\beta \gamma}, \quad K_A W_{a\beta \gamma} = 0. \tag{2.52}
\]

Owing to Eqs. (2.43) and (2.44), all the curvatures \( R_{AB} \) can also be written in terms of \( W_{a\beta \gamma} \), its conjugate, and their covariant derivatives. In particular, the \( R_{ab} \) component is expressed as
\[
R_{a\tilde{a},b\tilde{b}} = \varepsilon_{a\tilde{a}} \left( 2 W_{a\beta}^\gamma Q_\gamma + \nabla^\gamma W_{a\beta}^\delta M_\delta \gamma + \nabla^\gamma W_{\gamma a\beta} D - \frac{3}{2} i \nabla^\gamma W_{\gamma a\beta} A 
+ \frac{1}{4} \nabla^2 W_{a\beta}^\gamma S_\gamma - i \nabla^\gamma W_{\gamma a\beta} \tilde{S}_\gamma + \frac{1}{2} \nabla_\alpha \nabla^\beta \gamma W_{\gamma \beta}^\delta K_{\delta \gamma} \right) 
+ \varepsilon_{ab} \left( -2 W_{a\beta}^\gamma \tilde{Q}_\gamma + \tilde{\nabla}_\gamma W_{a\tilde{a}}^\beta \tilde{M}_\gamma \beta + \tilde{\nabla}_\gamma W_{\tilde{a} \beta}^\gamma \tilde{D} + \frac{3}{2} i \tilde{\nabla}_\gamma W_{\tilde{a} \beta}^\gamma A 
- \frac{1}{4} \tilde{\nabla}^2 W_{\tilde{a} \beta}^\gamma \tilde{S}_\gamma - i \nabla^\gamma W_{\tilde{a} \beta}^\gamma S_\gamma + \frac{1}{2} \tilde{\nabla}_\alpha \nabla^\beta \gamma W_{\tilde{a} \beta}^\delta K_{\beta \gamma} \right). \tag{2.53}
\]

Now the concept of primary superfield is introduced to describe matter superfields, invariant action over the superspace, and so on. A primary superfield \( \Phi_\Gamma \) is defined as the superfield on which the action of the superconformal group is
\[
M_{bc} \Phi_\Gamma = (S_{bc})_{\Gamma,\Sigma} \Phi_\Sigma, \quad D \Phi_\Gamma = \Delta \Phi_\Gamma, \quad A \Phi_\Gamma = i w \Phi_\Gamma, \quad K_A \Phi_\Gamma = 0. \tag{2.54}
\]
The D-type integration is related to the F-type one as

\[ \int d^4x d^4\theta EV, \]

(2.55)

where \( E = \det(E_M^A) \). Here we are using implicit grading and omitting to write the superdeterminant “sdet”. The superconformal transformation law of the density \( E \) is

\[ \delta_G(\xi^A X_A)E = E E_A^M \delta_G(\xi^A X_A)E_M^A = E E_A^M h_M^C \xi^B f_B^C A = E \xi^B f_B^C A \]

\[ \rightarrow DE = -2E, \quad M_{ab}E = AE = K_A E = 0, \]

(2.56)

since the superconformal generators \( X_{E} \) other than \( P_A \) carry nonpositive Weyl weights so that the commutator \([X_{E}, X_{C}]\) yields a positive Weyl weight \( P_A \) only when \( X_{C} = P_C \) (and \( X_{E} = D \) or \( A \)), in which case \( E_A^M h_M^C = E_A^M E_M^C = \delta_A^C \). From (2.56), the invariance conditions for the action \( S_D \) become

\[ DV = 2V, \quad AV = M_{ab}V = K_A V = 0. \]

(2.57)

That is, \( V \) must be a \((\Delta, w) = (2, 0)\) primary real superfield with no Lorentz index. The invariance of \( S_D \) under the GC transformation in superspace is manifest and hence invariant under the \( P_A \) transformation. Thus the action \( S_D \) is fully superconformal invariant and is called D-type integration.

The superconformal counterpart of the \( d^2\theta \) integral in global supersymmetry is

\[ S_F = \int d^4x d^2\theta E W. \]

(2.58)

The chiral density \( E \) is given by the superdeterminant of the vielbein in the chiral subspace with dotted spinor directions being omitted from \( E_M^A \), i.e., \( E = \det E_m^a \) with \( E_m^a = E_m^a \), \( a = (a, \alpha) \), and \( m = (m, \mu) \). In (2.58), \( W \) is a covariantly chiral superfield defined by \( \bar{\nabla}_a W = 0 \). The invariance of the action \( S_F \) requires that \( W \) must be a \((\Delta, w) = (3, 2)\) primary chiral superfield with no Lorentz index. Since the integral \( S_F \) does not depend on \( \bar{\theta} \), it is supposed to be executed at \( \bar{\theta} = 0 \), which is called F-type integration. Performing the \( d^2\theta \) integration in (2.58), we obtain the component expression of the F-type integration as

\[ \int d^4x d^2\theta E W = \int d^4x e \left(-\frac{1}{4} \nabla^2 W + \frac{i}{2} \bar{\psi}_{a\dot{\alpha}} (\bar{\sigma}^a)_{\dot{\alpha}\dot{\beta}} \nabla_{\dot{\beta}} W - \left( \bar{\psi}_{a} \bar{\sigma}^{ab} \bar{\psi}_b \right) W \right)_{\theta = \bar{\theta} = 0}. \]

(2.59)

The D-type integration is related to the F-type one as

\[ \int d^4x d^4\theta EV = \frac{1}{2} \int d^4x d^2\theta E \mathcal{P}[V] + \frac{1}{2} \int d^4x d^2\bar{\theta} \bar{E} \mathcal{P}[V], \]

(2.60)
where

\[ \mathcal{P}[V] = -\frac{1}{4} \tilde{\nabla}^2 V \]  

(2.61)
is the chiral projection operator. The component expression of the D-type integration is obtained using Eq. (2.60).

The action of the matter-coupled SUGRA system is given in conformal superspace as

\[ S = -3 \int d^4 x d^4 \theta E \Phi^c \tilde{\Phi}^c e^{-K/3} + \left( \int d^4 x d^2 \theta E (\Phi^c)^3 W + \text{h.c.} \right), \]  

(2.62)

where \( \Phi^c \) is the compensator chiral superfield carrying Weyl and chiral weights \((\Delta, w) = (1, 2/3)\).

The Kahler potential \( K \) and the superpotential \( W \) are the functions of chiral matter superfields \( \Phi_i \) with weights \((\Delta_i, w_i) = (0, 0)\). In addition, \( K \) is a real function and \( W \) is holomorphic. The gauge-fixing conditions leading to Poincaré SUGRA with the canonically normalized EH term are given in Ref. [9]:

\[ D, A\text{-gauge} : \Phi^c = \tilde{\Phi}^c = e^{K/6}, \quad K_A\text{-gauge : } B_M = 0. \]  

(2.63)

The \( K_A \)-gauge condition \( B_M = 0 \) together with the curvature constraints imply the restriction of the form of the \( K_A \)-gauge field \( f_M^{\alpha} \). Recall that the curvature \( R(D)_{AB} \) is written as

\[ R(D)_{AB} = E_A^M E_B^N (\partial_M B_N - \partial_N B_M) + 2f_{AB}(-)^a - 2f_{BA}(-)^b. \]  

(2.64)

We see that the constraints \( R(D)_{\alpha\beta} = 0 \) and \( B_M = 0 \) constrain \( f_{\alpha\beta} \) in the form

\[ f_{\alpha\beta} = -\epsilon_{\alpha\beta} \bar{R}, \quad f_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} R, \quad f_{\dot{\alpha}\dot{\beta}} = -f_{\dot{\beta}\dot{\alpha}} = -\frac{1}{2} G_{\dot{\alpha}\dot{\beta}}. \]  

(2.65)

The constraint \( R(D)_{\alpha\dot{\beta}} = 0 \) implies

\[ f_{\alpha\dot{\beta}} = -f_{\dot{\beta}\alpha}. \]  

(2.66)

Further, the constraints \( R(K)_{\alpha\beta, \dot{\gamma}} = 0 \) and \( R(K)_{\alpha\dot{\gamma}, \beta} = 0 \) and their conjugates give \((3.42), (3.43)\) of Ref. [9])

\[ 3if_{\alpha\beta} = \frac{1}{2} D_{\dot{\alpha}} G_{\beta\dot{\beta}} + D_{\beta} G_{\alpha\dot{\beta}} + \epsilon_{\alpha\beta} \bar{D}_{\dot{\beta}} \bar{R}, \]  

(2.67)

\[ -3if_{\dot{\alpha}\dot{\beta}} = \frac{1}{2} \bar{D}_{\dot{\alpha}} G_{\beta\dot{\beta}} + \bar{D}_{\dot{\beta}} G_{\alpha\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} D_{\alpha} R, \]  

(2.68)

where \( D_A \) is the covariant derivative after \( K_A \)-gauge fixing; \( D_A = [\nabla_A + f_A^{\beta} K_B]_{B_M = 0} \). Finally, the constraint \( R(K)_{\alpha\beta} = 0 \) means \((3.49)\) in Ref. [9])

\[ f_{\alpha\dot{\alpha}\beta, \dot{\beta}} = \frac{i}{2} (D_A f_{\alpha\dot{\beta}} + \bar{D}_{\dot{\alpha}} f_{\alpha\beta}) + 2\epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{R} \bar{R} + \frac{1}{2} \bar{G}_{\beta\dot{\alpha}} G_{\alpha\dot{\beta}}. \]  

(2.69)

Since the conformal curvatures are written in terms of \( W_{\alpha\beta\gamma} \), the curvatures after gauge fixing are written in terms of \( R, G_{\alpha\dot{\beta}}, W_{\alpha\beta\gamma} \), and the derivative \( D_A \).
It is noted that the gauge-fixing conditions (2.63) also fix the $A$-gauge superfield $A_M$. The covariantly chiral condition of $\Phi^c$ is $0 = \bar{\nabla}^{\dot{\alpha}} \Phi^c = E^{\dot{a}M} \partial_M \Phi^c - B^{\dot{a}} \Phi^c - \frac{2}{3} i A^{\dot{a}} \Phi^c$, and further imposing the gauge conditions $\Phi^c = e^{K^6} \Phi^c$ and $B^{\dot{a}} = 0$ leads to

$$A^{\dot{a}} = -\frac{i}{4} K_i^{\dot{a}} \bar{\Phi}^{\dot{a}i}, \quad (2.70)$$

where $\bar{\nabla}^{\dot{a}} \Phi^i = E^{\dot{a}M} \partial_M \Phi^i$ and $K_i^{\dot{a}} = \partial K / \partial \Phi^{\dot{a}i}$. The chirality condition for matter superfields $\Phi^i$ is used; $0 = \bar{\nabla}^{\dot{a}} \Phi^i = E^{\dot{a}M} \partial_M \Phi^i = \bar{\nabla}^{\dot{a}} \Phi^i$. In the same way, from $\nabla_\alpha \bar{\Phi}^c = 0, A_\alpha$ is fixed as

$$A_\alpha = \frac{i}{4} K^{\dot{a}} \nabla_\alpha \Phi^i. \quad (2.71)$$

Similarly, from the relation $\bar{\nabla}^{\dot{a}} \nabla_\alpha \Phi^c = -2i \nabla_\alpha \Phi^c$, we obtain

$$A_{\alpha}^{\dot{a}} = \frac{i}{4} (K^{\dot{a}} \nabla_\alpha \Phi^i - K_i^{\dot{a}} \nabla_\alpha \Phi^{\dot{a}i}) - \frac{3}{2} G_{\alpha}^{\dot{a}} + \frac{1}{4} g_{ij}^{\dot{a}} (\nabla_\alpha \Phi^i)(\bar{\nabla}^{\dot{a}} \Phi^j), \quad (2.72)$$

where $G_{\alpha}^{\dot{a}} = -2 f_{\alpha}^{\dot{a}}$ and $g_{ij}^{\dot{a}} = \partial^2 K / \partial \Phi^i \partial \Phi^j$.

### 3. Correspondence between component and superspace approaches

In this section we present the correspondence between the component and superspace formulations. The objects that we deal with are the superconformal algebra, gauge fields, curvatures and their constraints, conformal multiplets with external Lorentz indices, chiral projection, and invariant actions. Note that the notations and conventions are different in the two approaches and a “dictionary” to translate between them is given in Appendix A for spinors, vectors, gamma matrices, and tensors.

#### 3.1. Superconformal algebra, gauge fields, and curvatures

As discussed in Sect. 2.1, the $Q$ and $P_a$ transformations in the component approach are deformed from the original group laws. In the following, we use only the final form of them and the deformed $\tilde{P}_a$ transformation is simply denoted as $P_a$.

Let us begin with the dictionary for the normalization of superconformal generators and Weyl and chiral weights. The correspondence is shown in Table 1. The correspondence of gauge parameters is set to satisfy $e^{A}X_A \leftrightarrow \xi^{A}|X_A$ and is shown in Table 2. The vertical bar “|” means the $\theta = \bar{\theta} = 0$ projection, i.e., the lowest component of the superfield. Since gauge fields $\times$ generators essentially represent the common quantity in both approaches, $h^{\mu A}X_A \leftrightarrow h_m^{A}|X_A$, the correspondence of gauge fields appears with inverse normalizations of the generators shown in Table 3. In the table, the curved index $\mu$ of the component approach corresponds to the index $m$ of superspace.

Table 1. The generators and the Weyl and chiral weights.

| component | superspace |
|-----------|------------|
| $P_a$, $2Q_a$, $2\bar{Q}^{\dot{a}}$ | $(P_a$, $Q_a$, $\bar{Q}^{\dot{a}}) = P_A$ |
| $-M_{ab}$, $D_s$, $\frac{4}{3} A$ | $M_{ab}$, $D$, $A$ |
| $K_a$, $-2S_a$, $-2\bar{S}^{\dot{a}}$ | $(K_a$, $S_a$, $\bar{S}^{\dot{a}}) = K_A$ |
| $w$, $\frac{2}{3} n$ | $\Delta$, $w$ |
Table 2. The gauge transformation parameters.

| component | superspace |
|-----------|------------|
| $P, Q$    | $(\xi^a, \frac{1}{2} \tilde{\xi})$ | $(\xi(P)^a, \xi(P)^a, \tilde{\xi}(P)_a) = \xi(P)^a|$
| $M, D, A$ | $\lambda^{ab}, \rho, -\frac{1}{2} \tilde{\theta}$ | $\xi(M)^{ab}, \xi(D), \xi(A)|$
| $K, S$    | $(\xi_K^a, -\frac{1}{2} \tilde{\xi})$ | $(\xi(K)^a, \xi(K)^a, \tilde{\xi}(K)_a) = \xi(K)^a|$

Table 3. The gauge fields.

| component | superspace |
|-----------|------------|
| $P, Q$    | $(e^\mu_a, \frac{1}{2} \bar{\psi}_\mu)$ | $E_m^\alpha| = (E^a_m, E_{ma}, E_{m}^\dot{a})| = (e^a_m, \frac{1}{2} \bar{\psi}_{ma}, \frac{1}{2} \bar{\psi}_m^\dot{a})$
| $M, D, A$ | $\omega^{ab}_m, b_\mu, \frac{1}{2} A_\mu$ | $\phi^{ab}_m| = \omega^{ab}_m, B_m|, A_m|$
| $K, S$    | $(f^\mu_a, -\frac{1}{2} \bar{\psi}_\mu)$ | $f_m^\alpha| = (f^a_m, f_{ma}, f_m^\dot{a})|$

Table 4. The covariant curvatures.

| component | superspace |
|-----------|------------|
| $-\left( R_{ab}(P^\alpha), \frac{1}{2} R_{ab}(Q) \right)$ | $\left( R(P)^{\alpha\beta}, R(P)^{ab\gamma}, R(P)^{ab\dot{c}} \right)| = \xi(P)^{\alpha\beta}|$
| $-R_{ab}^{N\alpha}(M^{cd}), -R_{ab}(D), -\frac{1}{2} R_{ab}(A)$ | $R(M)^{\alpha\beta cd}, R(D)^{ab\gamma}, R(A)^{ab}\dot{c}|$
| $-\left( R_{ab}^{S\gamma}(K), -\frac{1}{2} R_{ab}^{S\dot{c}}(S) \right)$ | $\left( R(K)^{\alpha\beta}, R(K)^{ab\gamma}, R(K)^{ab\dot{c}} \right)| = \xi(K)^{\alpha\beta}|$

The curvature in superspace, $R_{mn}^{C}$, with curved tensor indices was defined in Eq. (2.37). The lowest component of the flat indexed curvature superfield $R_{ab}^{C}$ is given by

$$R_{ab}^{C}| = E_b^N E_a^M R_{MN}^{C}| = e_a^m e_b^n R_{mn}^{C}| = -\frac{i}{2} \left( \psi_a^\alpha (\sigma_b)_{\alpha\beta} - \psi_b^\alpha (\sigma_a)_{\alpha\dot{\beta}} \right) \mathcal{W}^{\beta\dot{c}|} + \frac{i}{2} \left( \bar{\psi}_{\alpha\dot{c}} (\bar{\sigma}_b)_{\dot{\alpha}\dot{\beta}} - \bar{\psi}_{\alpha\dot{c}} (\bar{\sigma}_a)_{\dot{\alpha}\dot{\beta}} \right) \mathcal{W}^{\beta\dot{c}|} + \frac{1}{4} \psi_a^\alpha \psi_b^\beta R_{ab}^{C}|. \quad (3.1)$$

Using the correspondence of gauge fields given in Table 3, we find that the curvatures coincide with (the negative of) the covariant curvatures with the algebra deformation of the component approach, up to the normalization of generators shown in Table 4. The “covariantization” is necessary only for the $M, S, K_a$ curvatures in the component approach, which correspond in superspace to the fact that the gaugino superfield $\mathcal{W}_a$ has nonvanishing components only for the $M, S, K_a$ generators. In obtaining Table 4, we have used the relations of $\mathcal{W}_a$ shown in Table 5 to the curvatures in the component approach. Note that these quantities stand for the spinor–vector components of the superspace curvature $R_{ab}^{C}$ because of the relations (2.43).

The correspondences of the curvatures (Table 4) are summarized in a simple expression:

$$R_{ab}^{C} X_C \quad \text{(component)} \leftrightarrow -R_{ab}^{C}| X_C \quad \text{(superspace)}. \quad (3.2)$$

We emphasize that such identification holds for the flat indexed curvatures, as it does for the curved indexed gauge fields:

$$h_{\mu}^{C} X_C \quad \text{(component)} \leftrightarrow h_{m}^{C}| X_C \quad \text{(superspace)}. \quad (3.3)$$
Table 5. $W_\xi$ and the curvatures.

| component | superspace |
|-----------|------------|
| $R_{\alpha\beta}(Q)\gamma_\xi$ | $(W(M)^{\alpha\beta}, W(M)_{\dot{\alpha}\dot{\beta}}) = -2 (R(P)_{\alpha\beta}, -R(P)_{\dot{\alpha}\dot{\beta}})$ |
| $-\frac{i}{4} \sigma^{\alpha\beta} \gamma_5 R_{\alpha\beta}(A)$ | $(W(K)_{\dot{\alpha}}, 0) (W(K)_{\dot{\beta}}, 0)$ |
| $\frac{1}{4} R^{\alpha\beta\gamma}(S)_{\gamma\xi} = -\frac{i}{3} \left( \begin{array}{cc} 0 & (\tilde{\alpha}^{\alpha\beta})_{\dot{\gamma}} \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) R_{\alpha\beta}$ |
| $\frac{1}{4} R^{\alpha\beta\gamma}(S)_{\gamma\xi} = -\frac{i}{2} \left( \begin{array}{cc} (R(K)_{\dot{\alpha}^\beta}, (R(K)_{\dot{\beta}^\alpha}) & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$ |

Table 6. $W_{a\beta\gamma}$, $W_{a\alpha\dot{\gamma}}$, and the curvatures $R(X)_{\alpha\beta\gamma}$ in superspace.

For instance, the component approach counterpart of the flat indexed gauge field in superspace is found through the expression

$$h_\alpha^{\dot{c}} = E_a^M h_M^{\dot{c}} = e_\alpha^m h_m^{\dot{c}} - \frac{1}{2} \bar{c}_a^{\dot{c}} h_\alpha^{\dot{c}}.$$  \hspace{1cm} (3.4)

The constraints on curvatures also have a correspondence, though the constraints in superspace are directly imposed on the spinor–spinor or spinor–vector components of curvatures. The restricted form of the vector–vector component $R_{\alpha\beta}$ in superspace is derived from other constraints and is explicitly written in terms of the primary chiral superfield $W_{a\beta\gamma}$. That is, Eq. (2.53) implies the expressions shown in Table 6 for the curvatures $R(X)_{\alpha\beta\gamma}$ in superspace. The chiral decomposition of the antisymmetric tensor is defined in Eq. (A19).

We can see the correspondence of curvature constraints using the fact that all the curvature components $R(X)_{\alpha\beta\gamma}$ with vector–vector indices are expressed by $W_{a\beta\gamma}$ in superspace. First, the constraint (2.8) in the component approach is equivalent to $R_{\alpha\beta}(P^c) = 0$ and hence corresponds to $R(P)_{\alpha\beta} = 0$ in superspace, as seen in Tables 4 and 6. Secondly, the constraint (2.9), equivalent to $R_{\alpha\beta}(Q)_{\gamma\beta} = 0$, corresponds to the equation $R(P)_{\alpha\beta} |(\sigma^h)_{a\delta} = 0$ and its conjugate in superspace. It is found from Table 6 that $R(P)_{\alpha\beta}$ has only a chiral component $R(P)_{\gamma\dot{\gamma}, \beta^\alpha = 2 \bar{c}_\beta^{\dot{c}} W_{\gamma\alpha}^\alpha$ so that

$$R(P)_{\alpha\beta} (\sigma^h)_{a\delta} \propto (\tilde{\sigma}^h)_{\dot{\delta}} W_{\gamma\beta}^\alpha (\sigma^h)_{a\delta} \propto \bar{c}_\beta^{\dot{c}} W_{\gamma\alpha}^\alpha.$$  \hspace{1cm} (3.5)
Table 7. The superconformal group transformations.

| component | superspace |
|-----------|------------|
| $\delta_{\varepsilon}(\xi^a) + \delta_{Q}(\varepsilon)$ | $\delta_{G}(\xi(P)^a|P_a) + \delta_{G}(\xi(P)^a|Q_a) = \delta_{G}(\xi(P)^a|P_a)$ |
| $\delta_{\lambda}(\lambda^{ab}) + \delta_{D}(\rho) + \delta_{A}(\theta)$ | $\delta_{G}(\xi(M)^{ab}|M_{ab}) + \delta_{G}(\xi(D)|D) + \delta_{G}(\xi(A)|A)$ |
| $\delta_{\xi}(\xi^a) + \delta_{S}(\zeta)$ | $\delta_{G}(\xi(K)^a|K_a) + \delta_{G}(\xi(K)^a|S_{a}) = \delta_{G}(\xi(K)^a|K_a)$ |

which vanishes since $W_{ab\gamma}$ is a totally symmetric superfield. The final constraint (2.10) in the component approach, which is equivalently rewritten as

$$R_{ac}^{\text{cov}}(M^{cb}) + \frac{1}{2}i\tilde{R}_{a}(A) = 0,$$

(3.6)

corresponds to (the lowest of) the relation between $R(M)^{cd}_{ab}$ and $R(A)_{ab}$ as

$$R(M)^{cb}_{ac} + \frac{2}{3}(\ast R(A))^{b}_{a} = 0.$$  

(3.7)

This also follows from Table 6, which says that both $R(M)^{cb}_{ac}$ and $R(A)_{ab}$ are given by $\nabla^{\beta}W_{\beta\alpha\gamma}$ and its conjugate.

The correspondence of the superconformal group transformations is shown in Table 7. The correspondence of transformation parameters is given in Table 2. These can be shown by examining the commutation relations in both approaches. The correspondence is trivial for the $M_{ab}, D, A, S, K_{a}$ transformations, but slightly nontrivial for the commutation relations of $P_{A} = (P_{a}, Q_{a}, \bar{Q}^{\dot{A}})$. In particular, the supercharge $Q_{a}$ is treated differently in both approaches. In the superspace approach, it is the spinor part of the translation in superspace so that it is defined as a combination of the general coordinate and gauge transformations. In the component approach, the $Q$ transformation is defined as the YM group law of the superconformal group though it is deformed by the curvature constraints.

Let us examine the commutation relations of the $P_{A}$ transformation, which is defined in Eq. (2.35) as

$$\delta_{G}(\xi^{A}P_{A}) = \delta_{GC}(\xi^{M} := \xi^{A}E_{A}^{M}) - \delta_{G}(\xi^{M} h^{B}_{M}E_{B}^{A}X_{B}).$$

(3.8)

We thus need the commutation relations between two GC transformations in superspace and the GC and group transformations $X_{B'}$ other than $P_{A}$. Noting that the field-independent pieces are the flat indexed parameters $\xi^{A}$ and $\eta^{A}$, we find with a straightforward calculation the following commutation relations:

$$[\delta_{GC}(\xi^{B}E_{B}^{N}), \delta_{GC}(\eta^{C}E_{C}^{L})] = \delta_{GC}(\xi^{N} \eta^{L} (\partial_{L}E_{N}^{A} - \partial_{N}E_{L}^{A})E_{A}^{M}),$$

$$[\delta_{G}(\xi^{A}h_{A}^{A'}X_{A'}), \delta_{GC}(\eta^{4}E_{4}^{A})] = \delta_{G}(\eta^{L} \xi^{N}h_{N}^{B'}X_{B'}) + \delta_{G}(\eta^{4} \xi^{N}(\partial_{L}E_{N}^{A})h_{A}^{B'}X_{B'}) - \delta_{GC}(\eta^{L}E_{L}^{C}h_{N}^{D}f_{B'C}^{B'}D_{D}^{M}),$$

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\[ [\delta_G(\xi^A h_A^A X_{A'}), \delta_G(\eta^B h_B^B X_{B'})] \]
\[ = \delta_G(\xi^L \eta^N (h_N^E E_L^E - h_L^E E_N^E) h_{E'E'}^E h_{F'A'}^F X_{A'}) \]
\[ + \delta_G((\eta^N (\partial_N \xi^L) - \xi^N (\partial_N \eta^L)) h_L^A X_{A'}) + \delta_G(\eta^N \bar{\delta} R_LN^A X_{A'}) \]  
\hspace{1in} \text{(3.9)}

Using these relations and the definition of the \( P_a \) transformation, we obtain
\[ [\delta_G(\xi^A P_a), \delta_G(\eta^B P_B)] = -\delta_G(\xi^A \eta^B R_{AB} \tilde{C} X_C). \]  
\hspace{1in} \text{(3.10)}

The parameter \( \xi^A \) is either vector \( \xi^a \) or spinor \( \xi^\alpha \). When we take both \( \xi^A \) and \( \eta^B \) to be spinors, Eq. (3.10) implies the following \( Q - Q \) commutation relation by using the constraints on \( R_{ab}^\alpha \) :
\[ [\delta_G(\xi^a Q_a), \delta_G(\eta^b Q_b)] = 2\delta_G(\eta^\beta \bar{\eta}_\beta \xi_a) P_a \]  
\hspace{1in} \text{(3.11)}

This agrees with the \( Q - Q \) commutation relation in the component approach,
\[ [\delta_Q(\bar{e}_1), \delta_Q(\bar{e}_2)] = \delta_P\left( \frac{1}{2} \bar{e}_2 \gamma^a e_1 \right), \]  
\hspace{1in} \text{(3.12)}

if \( \frac{1}{2} \bar{e}_1 \leftrightarrow (\xi^a \bar{\eta}_\alpha) \) and \( \frac{1}{2} \bar{e}_2 \leftrightarrow (\eta^\beta \bar{\xi}_a) \), as given in Table 2. Next, if we consider the vector parameter \( \xi^a \) and the spinor parameter \( \eta^\beta \), Eq. (3.10) becomes
\[ [\delta_G(\xi^a P_a), \delta_G(\eta^b Q_b)] \]
\[ = -\left( \delta_G\left( \frac{1}{2} \xi^a \eta^b R(M)_{ab} \right)_{cd} M_{cd} + \delta_G\left( \xi^a \eta^b R(K)_{ab} \tilde{S}_{22} \right) + \delta_G\left( \xi^a \eta^b R(K)_{ab} \tilde{C} \right) \right) \]  
\hspace{1in} \text{(3.13)}

Using the curvature expression \( R_{ab} = -i(\sigma_a)_{\beta\gamma} \gamma \), this commutation relation corresponds to
\[ [\delta_P(\xi^a), \delta_Q(\bar{e})] = \sum_{A=M,S,K} \delta_A(\xi^b \delta_Q(\bar{e}) h^A_b) \]
\[ = \delta_M\left( \frac{1}{2} \xi^a R^{ac} (Q)_{a'c} \right) \bar{e} + \delta_S\left( \frac{1}{4} i \xi^a R^{ab} (S)_{a'b} \right)\bar{e} \]
\[ + \delta_K\left( -\frac{1}{2} \xi^b R_{cb} (S)_{ac} - \frac{1}{4} \xi^b \delta_{ac} \tilde{R}_{cb} (S)_{a'b} \right) \]  
\hspace{1in} \text{(3.14)}

in the component approach when \( \frac{1}{2} \bar{e} = (\eta^\beta \bar{\eta}_\beta) \). Finally, setting both \( \xi^A \) and \( \eta^B \) to be vectors, we have
\[ [\delta_G(\xi^a P_a), \delta_G(\eta^b P_b)] = -\delta_G(\xi^a \eta^b R_{ab}^A X_A), \]  
\hspace{1in} \text{(3.15)}

which reproduces
\[ [\delta_P(\xi^a), \delta_P(\xi^b)] = \sum_{A \neq P} \delta_A(\xi^a \xi^b R_{ab}^{PV(A)}) \]  
\hspace{1in} \text{(3.16)}

in the component approach with the correspondence \( \xi^a_1 \leftrightarrow \xi^a \) and \( \xi^b_2 \leftrightarrow \eta^b \). Note that both \( R_{ab}(P^C) \) in the component approach and \( R(P)_{ab}^c \) in superspace vanish.

We comment on the geometrical meaning of the correspondence of commutation relations. In particular, the commutation relation \([\delta_P, \delta_Q] \), which is algebraically determined by some constraints in the component formulation, is understood as a vector–spinor curvature in superspace.
Table 8. Conformal multiplet with the Weyl weight \( w \) and Lorentz index \( \Gamma \).

| Weyl weight \( w \) | Component \( \mathcal{C}_\Gamma \) | Superspace \( \Phi_\Gamma \) |
|----------------------|----------------------|----------------------|
| \( w \)             | \( \mathcal{Z}_\Gamma \) | \( (\Phi_\Gamma) \) |
| \( w + \frac{1}{2} \) | \( \mathcal{H}_\Gamma \) | \( \frac{1}{2}((\Phi_\Gamma) + \tilde{\Phi}_\Gamma) \) |
| \( w + 1 \)         | \( \mathcal{K}_\Gamma \) | \( \frac{1}{4}((\Phi_\Gamma) - \tilde{\Phi}_\Gamma) \) |
| \( w + \frac{3}{2} \) | \( \Lambda_\Gamma \) | \( \frac{i}{4}((\Phi_\Gamma) + \tilde{\Phi}_\Gamma) \) |
| \( w + 2 \)         | \( \mathcal{D}_\Gamma \) | \( \frac{i}{2}((\Phi_\Gamma) - \tilde{\Phi}_\Gamma) \) |

3.2. Conformal multiplet

We have shown that the superconformal transformations in both approaches satisfy exactly the same algebra. Once the algebra is fixed, the transformation rule for a general conformal multiplet is uniquely determined in the component approach. That is, if the component with the lowest Weyl weight is specified, all other components in the multiplet and their transformation rules are found, up to some ambiguity in field definitions. So we are led to the exact correspondence of superconformal multiplets:

Conformal multiplet \( \mathcal{V}_\Gamma \) in (2.18) \( \leftrightarrow \) Primary superfield \( \Phi_\Gamma \) in (2.54).

In the component approach, the first component \( \mathcal{C}_\Gamma \) in \( \mathcal{V}_\Gamma \) is defined as having the lowest Weyl weight in the multiplet so that its \( S \) and \( K_a \) transformations, which lower the Weyl weight, must vanish. In the superspace approach, a primary superfield is defined as being \( K_a \) invariant. As discussed before, \( \mathcal{C}_\Gamma \) and \( \Phi_\Gamma \) satisfy the same form of superconformal transformations, Eqs. (2.19) and (2.54), respectively. Further, if they have the same Weyl weight \( w = \Delta \) and chiral weight \( n = (3/2)w \) as well as the same representation matrices for the Lorentz group \( \Sigma_{ab} = -S_{ab} \), the multiplets in both approaches coincide with each other. The higher components are determined successively by \( Q \) transformations and some ambiguities in the field definitions are fixed by the standard form (B1) in the component approach [8].

Thus, in the superspace approach, higher components in a superfield can be found by applying \( Q_\alpha \) (= \( \nabla_\alpha \)) successively and comparing them with the transformation laws in the component approach. The details are given in Appendix C.1, from which we find the superfield expressions shown in Table 8 for the correspondence of a conformal multiplet with the Weyl weight \( w \) and the Lorentz index \( \Gamma \). In this correspondence, the overall factor is fixed by the identification of the first components \( \mathcal{C}_\Gamma \leftrightarrow \Phi_\Gamma \). In the last line, we have used an identity

\[
\nabla^\alpha \tilde{\nabla}^2 \nabla_\alpha - \tilde{\nabla}_\alpha \nabla^2 \tilde{\nabla}^\alpha = 8(W_\alpha \tilde{\nabla}^\alpha + \nabla^\alpha \nabla_\alpha + \{\tilde{\nabla}_\alpha, W^\alpha\}),
\]

which is the conformal superspace counterpart of the identity \( D^\alpha \tilde{D}^2 D_\alpha - \tilde{D}_\alpha D^2 \tilde{D}^\alpha = 0 \) in global supersymmetry. The RHS in (3.17) comes from nonzero vector–spinor curvatures and depends on
the gaugino superfield \( \mathcal{W}_\alpha \). Noticing that \( \mathcal{W}_\alpha \) has only the \( M \) and \( K_4 \) components, and \( \{ \nabla^\alpha, \mathcal{W}_\alpha \} = \{ \tilde{\nabla}_\dot{\alpha}, \mathcal{W}_{\dot{\alpha}} \} \) have only the \( K_\alpha \) component (see Eqs. (2.49), (2.50), and (2.51)), the above superfield expressions of \( \Lambda \) and \( D \) for a multiplet with no Lorentz index reduce to

\[
\Lambda \leftrightarrow \frac{i}{4} \left( -\tilde{\nabla}^2 \nabla_\alpha \Phi \right), \quad D \leftrightarrow \frac{1}{8} \tilde{\nabla}_{\dot{\alpha}} \nabla^2 \tilde{\nabla}_{\dot{\alpha}} \Phi = \frac{1}{8} \nabla^\alpha \nabla^2 \nabla_\alpha \Phi, \quad (3.18)
\]

which are the same forms as in global supersymmetry.

### 3.3. Chiral projection and invariant actions

In this subsection we discuss the correspondences of chiral multiplets, the chiral projection, and the superconformally invariant actions.

In the superspace approach, a primary chiral superfield \( \Phi_\Gamma \) is defined as being a primary superfield satisfying the chirality condition

\[
\tilde{\nabla}_\dot{\alpha} \Phi_\Gamma = 0. \quad (3.19)
\]

Since “primary” means the \( K_4 \) invariant, a consistency for such a multiplet to exist requires

\[
0 = \{ \tilde{S}_\dot{\alpha}, \tilde{\nabla}_{\dot{\beta}} \} \Phi_\Gamma = \left( (2D + 3i4)\epsilon^{\dot{\alpha}\dot{\beta}} - 2M^{\dot{\alpha}\dot{\beta}} \right) \Phi_\Gamma, \quad (3.20)
\]

which requires \( \Phi_\Gamma \) to have the Weyl and chiral weights \((\Delta, w)\) satisfying \( 2\Delta - 3w = 0 \) and to carry only undotted spinor indices \( \Gamma = (\alpha_1 \alpha_2 \cdots) \). These conditions for weights and Lorentz index are exactly the same as given in Eq. (2.21) in the component approach. The component fields in a conformal multiplet with the chirality condition are found from Table 8:

\[
(\mathcal{C}_\Gamma, \mathcal{Z}_\Gamma, \mathcal{H}_\Gamma, \mathcal{K}_\Gamma, \mathcal{E}_{\alpha}, \Lambda_\Gamma, D_\Gamma) \leftrightarrow (\Phi_\Gamma |, -i\nabla_\alpha \Phi_\Gamma |, \frac{1}{4} \nabla^2 \Phi_\Gamma |, \frac{i}{4} \nabla^2 \Phi_\Gamma |, i\nabla_\alpha \Phi_\Gamma |, 0, 0), \quad (3.21)
\]

by using the equations

\[
\tilde{\nabla}_{\dot{\beta}} \nabla_{\alpha} \Phi_\Gamma = \{ \tilde{\nabla}_{\dot{\beta}}, \nabla_{\alpha} \} \Phi = -2i\nabla_{\alpha\dot{\beta}} \Phi_\Gamma, \quad \nabla^\dot{\alpha} \Phi_\Gamma = 0. \quad (3.22)
\]

The last equation follows from \( M^{\dot{\beta}\dot{\gamma}} \Phi_\Gamma = K_\alpha \Phi_\Gamma = 0 \) for a primary superfield \( \Phi_\Gamma \) with purely undotted \( \Gamma \). Comparing the expression (3.21) with the embedding formula referred to above Eq. (2.22) in the component approach, we find the following correspondence between a conformal chiral multiplet in the component approach and a primary chiral superfield \( \Phi_\Gamma \),

\[
[ \mathcal{A}_\Gamma, \mathcal{P}_R \chi_\Gamma, \mathcal{F}_\Gamma ] \quad \text{(component)} \leftrightarrow \quad [ \Phi_\Gamma, \nabla_\alpha \Phi_\Gamma |, -\frac{i}{4} \nabla^2 \Phi_\Gamma | ] \quad \text{(superspace)} \quad (3.23)
\]

The algebra \( \{ \tilde{\nabla}_\dot{\alpha}, \tilde{\nabla}_{\dot{\beta}} \} = 0 \) in Eq. (2.40) implies that the equation

\[
\tilde{\nabla}_\dot{\alpha} \nabla^2 \Phi_\Gamma = 0 \quad (3.24)
\]

identically holds for any superfield \( \Psi_\Gamma \). So \( \tilde{\nabla}^2 \Phi_\Gamma \) formally seems a chiral superfield. However, if \( \tilde{\nabla}^2 \Phi_\Gamma \) is not primary, it still has to contain \( 8 + 8 \) components, in contrast with the fact that a chiral superfield has only \( 2 + 2 \) components. This odd property happens in the superconformal case since \( \tilde{S}_\dot{\alpha} \) acts as an inverse operator of \( \tilde{\nabla}_\dot{\alpha} \).

If \( \tilde{\nabla}^2 \Phi_\Gamma \) is primary, it contains only \( 2 + 2 \) components for a primary chiral superfield. For \( \Psi_\Gamma \) with the Weyl and chiral weights \((\Delta, w)\), \( \tilde{\nabla}^2 \Phi_\Gamma \) has \((\Delta + 1, w + 2)\) and becomes chiral and primary.
if \(2(\Delta + 1) - 3(w + 2) = 0\) and \(\Gamma\) is purely undotted. This means that \(\nabla^2\) gives a chiral projection operator if it acts on a primary superfield \(\Psi_\Gamma\) whose weights and index satisfy those conditions. This agrees with the conditions for the chiral projection operator in the component approach given in Eq. (2.23). Taking care with the coefficients, we find the correspondence between the chiral projection operators \(\Pi\) in the component approach and \(\mathcal{P}\) in superspace:

\[
\Pi \leftrightarrow -\mathcal{P} = \frac{1}{4} \bar{\nabla}^2. \tag{3.25}
\]

We show in Appendix C.2 that the component fields of a projected superfield \(\mathcal{P}\Psi_\Gamma\) are identified with those of \(\Pi\Psi_\Gamma\) in Eq. (2.23) in the component approach. In this identification, the following equations are useful:

\[
\begin{align*}
\nabla^2 \bar{\nabla}^2 &= \nabla_\alpha \nabla^2 \bar{\nabla}^\alpha + 8 \nabla^a \nabla_\alpha - 2i \nabla_a (\bar{\sigma}^a)^\dot{\alpha}\alpha [\nabla_\alpha, \bar{\nabla}_\dot{\alpha}] - 8 W_\dot{\alpha} \bar{\nabla}^\dot{\alpha}, \\
\bar{\nabla}^2 \nabla^2 &= \nabla_\alpha \bar{\nabla}^2 \nabla^{\alpha} + 8 \nabla^a \nabla_\alpha + 2i \nabla_a (\bar{\sigma}^a)^\dot{\alpha}\alpha [\nabla_\alpha, \bar{\nabla}_\dot{\alpha}] + 8 W^{a\alpha} \nabla_\alpha. \tag{3.26}
\end{align*}
\]

We remark that the sum of these yields

\[
\nabla^2 \bar{\nabla}^2 + \bar{\nabla}^2 \nabla^2 - \nabla_\alpha \nabla^2 \nabla^{\alpha} - \bar{\nabla}_\dot{\alpha} \bar{\nabla}^2 \bar{\nabla}^{\dot{\alpha}} = 16 \nabla^a \nabla_\alpha + 8 W^{a\alpha} \nabla_\alpha - 8 W_\dot{\alpha} \bar{\nabla}^\dot{\alpha}. \tag{3.27}
\]

which is the conformal superspace counterpart of the global supersymmetry identity

\[
D^2 \bar{D}^2 + \bar{D}^2 D^2 - 2D^{a\alpha} \bar{D}^2 D_\alpha = 16 \Box. \tag{3.28}
\]

Finally, we discuss the superconformally invariant actions. First is the correspondence of the F-type invariant action for the conformal chiral multiplet \(\Sigma\) without external Lorentz index. The component expansion of the F-type integration (2.59) is coincident with the expression (2.24) of F-type invariant action in the component approach, if account is taken of the correspondences of gauge fields (Table 3) and chiral multiplet components (3.23):

\[
\int d^4x \left[ \Sigma \right]_F \leftrightarrow \int d^4x d^2\theta E \Sigma + \int d^4x d^2\bar{\theta} \bar{E} \bar{\Sigma}. \tag{3.29}
\]

The other is the correspondence of the D-type invariant action for the general real conformal multiplet \(V\) without external Lorentz index. Since the D-type formula is obtained from the F-type one by using the chiral projection operator, the correspondences of (3.25) and (3.29) directly lead to the following correspondence of the D-type invariant actions (2.25) in the component approach and the D-type integral (2.60) in the superspace approach:

\[
\int d^4x \left[ V \right]_D \leftrightarrow 2 \int d^4x d^4\theta E V. \tag{3.30}
\]

3.4. \textit{u-associated derivatives}

3.4.1. \textit{Restriction on the existence of conformal spinor derivatives?}

We first mention a historical puzzle about the conformal spinor derivative. In Ref. [8], KU constructed the spinor derivative in the component approach and claimed that such a spinor derivative \(\partial_\alpha\) exists only when some special conditions are met on an operand multiplet \(\mathcal{V}_\Gamma\). On the other hand, Butter defined [9] in the superspace formalism the conformally covariant derivatives \(\nabla_A\), which can act on any superfield with no restriction. What is the difference?
The point is that KU defined in their component approach a conformal multiplet $\mathcal{V}_\Gamma$ by its first component $C_\Gamma$, denoting $\mathcal{V}_\Gamma = \left[C_\Gamma\right]$ which has the lowest Weyl weight in the multiplet. Therefore, the $S$ and $K_a$ transformations of $C_\Gamma$ must vanish, since $S$ and $K_a$ lower the Weyl weight. In superspace terminology, such a multiplet is arranged in a primary superfield $\Phi_\Gamma$:

$$\mathcal{V}_\Gamma = \left[C_\Gamma\right] \leftrightarrow \Phi_\Gamma, \quad \delta_S C_\Gamma = \delta_K C_\Gamma = 0 \leftrightarrow K_a \Phi_\Gamma = 0. \quad (3.31)$$

KU looked for the spinor derivative $\mathcal{D}_a$ as a mapping of a conformal multiplet $\mathcal{V}_\Gamma$ to another conformal multiplet whose first component is $Z_{a\Gamma}$, which is the second component of $\mathcal{V}_\Gamma$:

$$\mathcal{D}_a : \mathcal{V}_\Gamma = \left[C_\Gamma\right] \to \mathcal{D}_a \mathcal{V}_\Gamma = \left[Z_{a\Gamma}\right]. \quad (3.32)$$

That is crucial and is the only difference from the superspace covariant derivatives $\nabla_a$, which generally do not bring a primary superfield into primary. This freedom employed in the superspace formulation is consistent with the freedom of the $Q$ transformation in the component approach, because the $S$ transformation of $Z_{a\Gamma}$ is not generally required to vanish. Thus, the conformal covariant spinor derivative that corresponds to the $Q$ transformation is $\nabla_a$, not $\mathcal{D}_a$. Conversely speaking, once the image $\nabla_a \Phi_\Gamma$ is required to be primary, $S_\beta \nabla_a \Phi_\Gamma = 0$ leads to the same conditions for $\Phi_\Gamma$ as KU found in the component approach.

### 3.4.2. $u$-associated derivative

We need the $S$ and $K_a$ invariance of multiplets, for instance, in constructing the invariant actions by the D-type and F-type formulas. Reference [8] has shown that, if one has a compensating multiplet $u$ (or any conformal multiplet whose first component is guaranteed to be nonvanishing, like the compensator used for gauge fixing), the covariant derivative $\mathcal{D}_a(u)$ is constructed, which maps a conformal multiplet into another conformal one without any restriction.

Consider a conformal multiplet $u$ with the Weyl and chiral weights $(w_0, n_0)$ and no external Lorentz index. The component fields are denoted as

$$u = \left[C_u, Z_u, \mathcal{H}_u, K_u, B_{u}, A_{u}, D_{u}\right]. \quad (3.33)$$

Assuming that the first component $C_u$ is nonvanishing, we construct the following spinor:

$$\chi^S := \frac{iZ_u}{w_0 + n_0 C_u}, \quad (3.34)$$

which is nonlinearly shifted under the $S$ transformation as $\delta_S(\zeta)\chi^S = \zeta$. Then the $u$-associated spinor derivative $\mathcal{D}_a(u)$ is defined by

$$\mathcal{D}_a(u) \mathcal{V}_\Gamma = \left[Z_{a\Gamma} + i(w+n)\lambda^a \sigma_{ab} \lambda^b S C_\Gamma - (\sigma_{ab})^a_{\beta} \lambda^b (\Sigma^{ab} C_\Gamma)\right], \quad (3.35)$$

where $w$ and $n$ are the Weyl and chiral weights of $C_\Gamma$. Since $\delta_S(\zeta) Z_{a\Gamma} = -i(w+n)\zeta C_\Gamma + \sigma_{ab} \zeta (\Sigma^{ab} C_\Gamma)$, the quantity in the brackets on the RHS is invariant under the $S$ transformation, so that it defines a conformal multiplet $\mathcal{D}_a(u) \mathcal{V}_\Gamma$. The barred derivative $\bar{\mathcal{D}}^{\bar{a}(u)} \mathcal{V}_\Gamma$ is given by $\bar{\mathcal{D}}^{\bar{a}(u)} \mathcal{V}_\Gamma = (\mathcal{D}^{a(u)} \mathcal{V}_\Gamma)^*.$

Similarly, the $u$-associated vector derivative is constructed as follows. We define a vector $V^aK$ and a spinor $\chi^S$ by

$$V^aK := \frac{1}{4w_0} \left( \frac{D_a C_u}{C_u} + \frac{D_a C_u^*}{C_u^*} \right), \quad \chi^S := \frac{1}{2w_0} i\gamma_5 \left( \frac{Z_u}{C_u} + \frac{Z_u^*}{C_u^*} \right), \quad (3.36)$$

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so that $V_a^K$ and $\chi^S$ are shifted under the $K_a$ and $S$ transformations, respectively, as $\delta_K(\xi^K)V_a^K = \xi^K_a$ and $\delta_S(\xi)\chi^S = \xi$. The $S$ transformation of the vector $V_a^K$ yields the spinor $\chi^S$ as $\delta_S(\xi)V_a^K = -\frac{1}{4}\xi_a\chi^S$. By adding appropriate terms containing these fields, the superconformally covariant derivative $D_aC_G$ defined in (2.17) can be made $S$-invariant, and the $u$-associated vector derivative is defined by

$$D_a(u)V_G = \left[ D_aC_G - 2wV_a^KC_G + 2V^bK(\Sigma_{ab}C)_G + \frac{1}{2}\tilde{\chi}^S \gamma^a \gamma_5 Z_G \right. \\
\left. + \frac{1}{4}(\tilde{\chi}^S \gamma_5 \gamma^b \chi^S)(\delta_{ab}nC_G + (\tilde{\Sigma}_{ab}C)_G) \right],$$

(3.37)

so that $D_a(u)V_G$ is a conformal multiplet.

We now show the superspace expression for the $u$-associated derivatives using the correspondences given in the previous section. First we introduce the primary superfield $X_u$ that corresponds to $u$:

$$u \leftrightarrow X_u,$$

(3.38)

where $X_u$ has the Weyl and chiral weights $(\Delta_0, w_0) = (w_0, \frac{2}{3}n_0)$. From the correspondences of weights and component fields (Table 8), $\lambda^S$ is identified as

$$\lambda^S \leftrightarrow \frac{2}{(2\Delta_0 + 3w_0)X_u} \nabla_a X_u |.$$

(3.39)

By reading the correspondence of $D_a(u)V_G$ (3.35), we find the following superspace expression for the $u$-associated spinor derivative:

$$D_a(u)V_G \leftrightarrow -i\left( \nabla_a + \frac{1}{(2\Delta_0 + 3w_0)X_u} (\nabla^\beta X_u) (S_\beta, Q_\alpha) \right) \Phi_G,$$

(3.40)

and similarly for the dotted spinor derivative. When $X_u$ is a real superfield with special weights $\Delta_0 = 2$ and $w_0 = 0$, this expression reduces to the compensated spinor derivatives discussed in Ref. [11]. So, (3.40) stands for the generalization to $X_u$ with arbitrary weights.

We also construct the superspace expression for the $u$-associated vector derivative. For this purpose, we consider the real superfield $Y_u$ defined by

$$Y_u = \log X_u + \log \bar{X}_u.$$

(3.41)

Using the component field correspondence (Table 8), we identify $V_a^K$ and $\chi^S$ as

$$V_a^K \leftrightarrow \frac{1}{4\Delta_0} \nabla_a Y_u |,$$

$$\chi^S \leftrightarrow \frac{1}{2\Delta_0} i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left( -i \nabla_a Y_u + i \bar{\nabla}_a Y_u \right).$$

(3.42)

1 Precisely speaking, this $Y_u$ itself is not a proper primary superfield unless $\Delta_0 = w_0 = 0$ since $\log X_u$ has no definite values of Weyl and chiral weights. In the following expressions, however, only its derivative $\nabla_a Y_u = \nabla_a X_u / X_u + \nabla_a \bar{X}_u / \bar{X}_u$ appears, which is a proper superfield with the Weyl and chiral weights of the operator $\nabla_a$. 
Then the superspace expression for the $u$-associated vector derivative is found by translating (3.37):

$$
\begin{align*}
D_a^{(u)} v_\Gamma & \leftrightarrow \nabla_a \Phi_\Gamma - \frac{1}{2\Delta_0} (\nabla_a Y_u) D \Phi_\Gamma - \frac{1}{2\Delta_0} (\nabla_b Y_u) M_{ab} \Phi_\Gamma \\
& + \frac{1}{4\Delta_0} \left( \nabla^a Y_u \tilde{\nabla}_a Y_u \right) i \begin{pmatrix} 0 & (\sigma_a)_{\alpha\beta} \\ (\sigma^b)_{\alpha\beta} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left( -i \nabla_\beta \Phi_\Gamma + i \tilde{\nabla}_\beta \Phi_\Gamma \right) \\
& + \frac{1}{8\Delta_0} \left( \nabla^a Y_u \tilde{\nabla}_a Y_u \right) \frac{1}{2\Delta_0} \left( \nabla_\beta Y_b \frac{1}{2\Delta_0} \left( \nabla_\beta Y_u \right) \\
& \times \left( -\frac{3i}{2} \eta_{ab} A + \frac{1}{2} ( -i \bar{e}_{abcd} ) ( -M^{cd} ) \right) \Phi_\Gamma.
\end{align*}
$$

(3.43)

When $X_u$ is a real primary superfield $X$ with the weights $(\Delta_0, w_0) = (2, 0)$, this reduces to the compensated vector derivative with parameter $\lambda = 1$ given in Ref. [11], if we replace $Y_u \rightarrow 2 \log X$.

4. Superconformal gauge fixing to Poincaré SUGRA

The superconformal group is larger than the super Poincaré and the extra $D$, $A$, $S$, $K$-gauge symmetry should be fixed to have Poincaré SUGRA, which is useful, e.g., for phenomenological applications. In this section, we examine the superconformal gauge fixing of the matter-coupled conformal SUGRA to Poincaré SUGRA, and give the correspondence of gauge-fixing conditions between the superspace and component approaches. In this paper we focus on the chiral superfield matter system. The gauge fixing for the system containing YM gauge fields of internal symmetry will be discussed elsewhere (T. Kugo et al., manuscript in preparation).

4.1. Gauge fixing in the superspace approach

The matter superfields $\Phi^i (i = 1, 2, \ldots, n)$ are introduced to be primary and covariantly chiral with respect to the superconformal symmetry, $\tilde{\nabla}^{\alpha} \Phi^i = 0$. They have the Weyl and chiral weights $(\Delta, w) = (0, 0)$. The matter-coupled SUGRA action in conformal superspace is given by

$$
S = -3 \int d^4 x d^4 \theta \, E \Phi^c \Phi^c e^{-K/3} + \left( \int d^4 x d^2 \theta \, E (\Phi^c)^3 W + \text{h.c.} \right),
$$

(4.1)

where the Kähler potential $K = K(\Phi^i, \tilde{\Phi}^i)$ is a real function of matter superfields, and the superpotential $W = W(\Phi^i)$ is a holomorphic one. In the first term (the D-type action), the superconformal gauge invariance leads to the conditions that the compensator chiral superfield $\Phi^c$ is primary and has the weights $(\Delta, w) = (1, 2/3)$. In the second term (the F-type action), the compensator dependence is also fixed by the superconformal gauge invariance.

Let us discuss the gauge fixing of superconformal symmetry to go down to Poincaré SUGRA. For a nonvanishing superpotential, it is useful to redefine the compensator $\Phi^c$ as

$$
\Phi^c \rightarrow \Phi^0 = \Phi^c W^{1/3}.
$$

(4.2)

---

2 The redefinition (4.2) is possible when $W \neq 0$. For $W = 0$, a convenient gauge choice may be $\Phi^c = e^{K/6}$ (and $B_M = 0$), which is the same condition as the one given in Ref. [9] and mentioned in Sect. 2.2.
The new chiral compensator $\Phi^0$ has the weights $(\Delta, w) = (1, 2/3)$. The action in terms of $\Phi^0$ is given by

$$S = -3 \int d^4x d^4\theta \, E \, \Phi^0 \Phi^0 e^{-G/3} + \left( \int d^4x d^2\theta \, \mathcal{E} \, (\Phi^0)^3 + \text{h.c.} \right)$$  \hspace{1cm} (4.3)

with

$$G = K + \ln |W|^2.$$  \hspace{1cm} (4.4)

One of the virtues of using $\Phi^0$ and $G$ is revealed in introducing YM gauge fields, i.e., $\Phi^0$ and $G$ are invariant under possible internal symmetry, while $\Phi^0$ and $K$ are not invariant. This invariant property of $\Phi^0$ and $G$ makes it simple to fix the superconformal gauge symmetry irrespective of internal ones (T. Kugo et al., manuscript in preparation).

In the component approach, Ref. [7] discussed the superconformal gauge-fixing conditions that realize the canonically normalized EH and RS terms and also give a real gravitino mass, given in (2.27). We find that its superspace counterparts are

$$D, A\text{-gauge : } \Phi^0 = \Phi^0 = e^{G/6}, \quad K_A\text{-gauge : } B_M = 0.$$  \hspace{1cm} (4.5)

The second condition is imposed by an appropriate $K_A$-gauge transformation of the $D$-gauge superfield: $\delta_G(\xi(K)A)B_M = -2E_M\xi(K)^iA$. On the other hand, the first condition seems peculiar since the chiral superfield $\Phi^0$ does not have enough numbers of independent components that can be set equal to the general real superfield $e^{G/6}$. It is, however, noticed that the gauge fixing (4.5) is given in conformal superspace where all gauge transformations have real superfield parameters. Therefore, the finite $D$- and $A$-gauge transformations $\Phi^0 \mapsto e^{\xi(D)} + \frac{2}{3} \xi(A) \Phi^0$ are possible with the real superfield parameters $\xi(D) = G/6 - (1/2) \ln(\Phi^0)\Phi^0$ and $\xi(A) = (3/4i) \ln(\Phi^0)\Phi^0$, which brings $\Phi^0$ to $e^{G/6}$.

The gauge-fixing conditions (4.5) imply several other equations for superfield components. We here focus on the chiral compensator $\Phi^0$ and the $A$-gauge superfield $A_M$. Recall that the covariant derivative takes the following form for a primary superfield $\Phi^{(\Delta, w)}$ with the Weyl and chiral weights $(\Delta, w)$ and no external Lorentz index:

$$\nabla_A \Phi^{(\Delta, w)} = D_A^P \Phi^{(\Delta, w)} - (\Delta B_A + iwA_A) \Phi^{(\Delta, w)},$$  \hspace{1cm} (4.6)

$$\nabla_A \nabla_\alpha \Phi^{(\Delta, w)} = D_A^p \nabla_\alpha \Phi^{(\Delta, w)} - \left( \left( \Delta + \frac{1}{2} \right) B_A + i(w - 1)A_A \right) \nabla_\alpha \Phi^{(\Delta, w)}$$

$$+ (2\Delta + 3w)f_{A\alpha} \Phi^{(\Delta, w)},$$  \hspace{1cm} (4.7)

where the last term $(2\Delta + 3w)f_{A\alpha} \Phi^{(\Delta, w)}$ stands for $-f_{A\alpha} \Phi^{(\Delta, w)}$ and we have used the equation $K_B \nabla_\alpha \Phi^{(\Delta, w)} = [K_B, \nabla_\alpha] \Phi^{(\Delta, w)} = 0$ for $B = b, \beta$, which is satisfied if $\Phi^{(\Delta, w)}$ is primary. The derivative $D_A^P$ is defined by

$$D_A^P = E_A^M \partial_M - \frac{1}{2} \phi_A^{bc} M_{cb},$$  \hspace{1cm} (4.8)

which is the covariant derivative in Poincaré SUGRA and different from the derivative in Ref. [9] ($D_A$, discussed in Sect. 2.2). Plugging the gauge-fixing conditions (4.5) into the RHS of (4.6) and...
We then find that (4.14) and (4.15) are given by

\[ \Phi^0 = e^{G/6}, \]  
\[ \nabla_{\dot{a}} \Phi^0 = \left( \mathcal{D}^a_{\dot{a}} - \frac{2}{3} i A_{\dot{a}} \right) e^{G/6}, \]  
\[ \nabla^2 \Phi^0 = \left( \mathcal{D}^a_{\dot{a}} + \frac{1}{3} i A_{\dot{a}} \left( \mathcal{D}^a_{\dot{a}} - \frac{2}{3} i A_{\dot{a}} \right) e^{G/6} \right) - 4 f_{\alpha}^a e^{G/6}. \]  

Note that \( \nabla_{\dot{a}} \Phi^0 \neq \nabla_{\dot{a}} e^{G/6} \) but \( \mathcal{D}^a_{\dot{a}} \Phi^0 = \mathcal{D}^a_{\dot{a}} e^{G/6} \) since the gauge-fixing condition \( \Phi^0 = e^{G/6} \) violates the \( D \) and \( A \) symmetries but preserves \( M_{ab} \).

After the gauge fixing, the chirality condition of the compensator \( \Phi^0 \) turns out to determine the \( A \)-gauge superfield. Applying (4.6) to \( \tilde{\nabla}_{\dot{a}} \Phi^0 = 0 \) and using the gauge-fixing condition (4.5), we obtain

\[ 0 = \tilde{\nabla}_{\dot{a}} \Phi^0 = \bar{D}^a_{\dot{a}} \Phi^0 - \frac{2}{3} i A_{\dot{a}} \Phi^0 \quad \rightarrow \quad A_{\dot{a}} = \frac{-i}{4} G_{J^a} \bar{D}^a_{\dot{a}} \Phi^0, \]  

where the field derivatives of \( G \) are denoted as \( G_i = \partial G/\partial \Phi^i \) and \( G_{i^*} = \partial G/\partial \Phi^{i^*} \). Similarly, the condition \( \nabla_{\dot{a}} \Phi^0 = 0 \) fixes \( A_{\dot{a}} \) as

\[ A_{\dot{a}} = \frac{i}{4} G_j \mathcal{D}^a_{\dot{a}} \Phi^j, \]  

with which the components of the chiral compensator, (4.10) and (4.11), are rewritten as

\[ \nabla_{\dot{a}} \Phi^0 = \frac{1}{3} e^{G/6} G_i \mathcal{D}^a_{\dot{a}} \Phi^i, \]  
\[ \nabla^2 \Phi^0 = \frac{1}{3} e^{G/6} \left( G_i \mathcal{D}^2 \Phi^i + \left( G_{ij} + \frac{1}{12} G_i G_j \right) (\mathcal{D}^a_{\dot{a}} \Phi^j) (\mathcal{D}^a_{\dot{a}} \Phi^i) - 24 \bar{R} \right). \]  

We have used the relation \( f_{\alpha \beta} = -\epsilon_{\alpha \beta \bar{R}} \) in Eq. (2.65), which comes from the curvature constraint in the \( B_M = 0 \) gauge. Equation (4.15) relates the compensator \( F \) component to the auxiliary field \( \bar{R} \), the undetermined part of the \( S \)-gauge field \( f_{\alpha \beta} \). It is noticed that the superfield components are not given by the covariant derivative of Poincaré SUGRA (\( \mathcal{D}^a \)) but should be defined by the conformal one (\( \nabla \)). For the matter superfields \( \Phi^i \) with vanishing weights \( (\Delta, w) = (0, 0) \), these two derivatives give the same results for the first derivatives (spinor components), but different for the second ones (\( F \) components). For the comparison with the component approach, we rewrite the above results with the conformally covariant derivative \( \nabla \). Equations (4.6), (4.7), and (4.13) imply \( \mathcal{D}^a_{\dot{a}} \Phi^i = \nabla_{\dot{a}} \Phi^i \) and

\[ \mathcal{D}^2 \Phi^i = \nabla^2 \Phi^i - i A_{\alpha} \mathcal{D}^a_{\dot{a}} \Phi^i = \nabla^2 \Phi^i + \frac{1}{4} G_j \nabla_{\alpha} \Phi^i \nabla_{\dot{a}} \Phi^i. \]  

We then find that (4.14) and (4.15) are given by

\[ A_{\alpha} = \frac{i}{4} G_i \nabla_{\alpha} \Phi^i, \quad A_{\dot{a}} = \frac{-i}{4} G_{i^*} \tilde{\nabla}_{\dot{a}} \Phi^{i^*}, \]  
\[ \nabla_{\dot{a}} \Phi^0 = \frac{1}{3} e^{G/6} G_i \nabla_{\dot{a}} \Phi^i, \]  
\[ \nabla^2 \Phi^0 = \frac{1}{3} e^{G/6} \left( G_i \nabla^2 \Phi^i + \left( G_{ij} + \frac{1}{3} G_i G_j \right) \nabla^\alpha \Phi^i \nabla_{\dot{a}} \Phi^j \nabla_{\dot{a}} \Phi^i - 24 \bar{R} \right). \]
The chirality condition of the compensator also fixes the vector part of the $A$-gauge field. The chirality condition and the algebra $\{\nabla_\alpha, \bar{\nabla}^\beta\} = -2i\nabla_\alpha \bar{\nabla}^\beta$ imply

$$\bar{\nabla}^\beta \nabla_\alpha \Phi^0 = -2i\nabla_\alpha \Phi^0. \quad (4.20)$$

After the gauge fixing, the vector derivative on the RHS becomes $\nabla_\alpha \bar{\nabla}^\beta \Phi^0 = \mathcal{D}_\alpha \bar{\nabla}^\beta \Phi^0 - \frac{2}{3} i A_\alpha \bar{\nabla}^\beta \Phi^0$, which is used to determine the vector part $A_\alpha \bar{\nabla}^\beta$. Evaluating the LHS of $(4.20)$ by using $(4.7)$ with the gauge-fixing conditions $(4.5)$ and Eq. $(4.14)$, we find

$$A_\alpha \bar{\nabla}^\beta = -\frac{i}{4} \mathcal{D}_\alpha \bar{\nabla}^\beta G - \frac{1}{4} e^{-G/6} \left( \mathcal{D}^{\alpha \beta} + \frac{1}{3} i A_\beta \bar{\nabla}^\alpha \right) e^{G/6} G_i \mathcal{D}_\alpha \Phi^i - 3 f_\alpha^\beta \Phi^0 - \frac{3}{2} G_\alpha \bar{\nabla}^\beta \Phi^0. \quad (4.21)$$

In going to the second line, we have used $D_\alpha \Phi^i = \nabla_\alpha \Phi^i$ and $f_\alpha^\beta = -G_{\alpha \beta}/2$, which reads from the curvature constraints after the gauge fixing. The second-order derivative is modified by using

$$\mathcal{D}^{\alpha \beta} \mathcal{D}_\alpha \Phi^i + i A_\alpha \mathcal{D}^{\alpha \beta} \mathcal{D}_\alpha \Phi^i = \bar{\nabla}^\beta \nabla_\alpha \Phi^i = -2i\nabla_\alpha \bar{\nabla}^\beta \Phi^i = -2i\mathcal{D}^{\alpha \beta} \bar{\nabla}^\beta \Phi^i, \quad (4.22)$$

which also follows from $(4.7)$ and the gauge-fixing conditions. Equation $(4.21)$ is regarded as determining $G_\alpha \bar{\nabla}^\beta$ in terms of the auxiliary $A$-gauge field $A_\alpha$.

**4.2. Correspondence to the component approach**

We here show the correspondence of superconformal gauge fixing between the superspace and component approaches. First, note that the correspondences of the potentials and compensators are as shown in Table 9. The symbols in the component approach are explained in Sect. 2.1.

Let us see that the gauge-fixing conditions $(4.5)$ in superspace are equivalent to the improved $D_\ast$, $A_\ast$, $S_\ast$, $K_\ast$-gauge conditions $(2.27)$ in the component approach. As discussed in Sect. 3.3, the component correspondence between the compensator multiplet $\Sigma_0$ and the compensator superfield $\Phi^0$ is

$$\Sigma_0 = [z_0, \mathcal{P}_R \chi_0, h_0] \leftrightarrow [\Phi^0, \nabla_\alpha \Phi^0, -\frac{1}{4} \nabla^2 \Phi^0]. \quad (4.23)$$

The gauge conditions $(4.5)$ or their consequences $(4.9)$ directly mean the correspondence of the gauge-fixed lowest components:

$$z_0 = \sqrt{3} \phi^{-\frac{1}{2}} (z, z^n) = e^{-\bar{G}/6} \leftrightarrow \Phi^0 = e^{G/6}. \quad (4.24)$$

For the spinor components, the $S$-gauge condition in the component approach and Eq. $(4.18)$ in superspace exactly agree with each other:

$$2 \chi_{R0} = -2z_0 \bar{\phi}^{-1} \Phi^i \chi_{Ri} = -\frac{1}{3} e^{-G/6} G_i (2 \chi_{Ri}) \leftrightarrow \nabla_\alpha \Phi^0 = \frac{1}{3} e^{G/6} G_i \nabla_\alpha \Phi^i. \quad (4.25)$$
The correspondence of the $K_a$ gauge is trivial. Note that the $S$-gauge condition in the superspace approach, $B_a = 0$, is used in deriving the spinor component of $\Phi^0$ (4.18), which leads to the correspondence (4.25). For the $F$ components, the auxiliary field $h_0$ in $\Sigma_0$ is not gauge-fixed in the component approach. This corresponds to the fact that, in superspace, the $F$ component of $\Phi^0$ contains the auxiliary part $\bar{R}$ after the gauge fixing, as given in (4.19).

The virtue of gauge fixing in conformal superspace is twofold. The first is that it leads to the superspace Poincaré SUGRA directly and easily. The second is that it finds the supersymmetry transformation in the resultant Poincaré SUGRA in a straightforward way. Namely, the remaining Poincaré supersymmetry is just given by the covariant spinor derivatives $D^P_a$ and $\bar{D}^P_a$. In the component approach, however, the remaining supersymmetry is deformed from the original one by the requirement that it keeps the $D$, $A$, $S$, $K_a$-gauge conditions intact; this is explicitly found in Ref. [7] by adding a complicated combination of the $A$, $S$, $K_a$-gauge transformations with nontrivial field-dependent parameters (2.28). We finally show this correspondence of the Poincaré supersymmetry after the gauge fixing; in particular, the covariant spinor derivative $D^P_a$ reproduces the deformed supersymmetry in the component approach.

The Poincaré spinor derivative (4.8) is related to the conformal one as $D^P_a = \nabla_a + A_a^A + f_a^A K_A$ after the gauge fixing. The supersymmetry transformation in Poincaré superspace defined by $\eta^a D^P_a$ is then given by the following linear combination of superconformal $A, K_A$ transformations:

$$\eta^a D^P_a = \eta^a \nabla_a + \xi(A)'(\eta) A + \xi(K)'(\eta) S_a + \xi(K)'(\eta) \Phi^0 K_a,$$

$$\xi(A)'(\eta) = \eta^a A_a + \bar{\eta}_\dot{a} \dot{A}_\dot{a} = \frac{i}{4} G_{\dot{a}} \eta^a D^P_a \Phi^f - \frac{i}{4} G_{\dot{a}} \eta^a D^P_{\bar{a}} \bar{\Phi}^{\dot{f}}$$

$$\xi(K)'(\eta)_a = \eta^a f_\beta A^\beta_a + \bar{\eta}_{\dot{a}} \dot{f}_{\dot{\beta}} a = \eta^a R + \frac{i}{2} G^\beta_a \bar{\eta}_{\dot{a}} \dot{f}_{\dot{\beta}} a$$

$$\xi(K)'(\eta)_{\dot{a}} = \eta^a f_\beta A^\beta_{\dot{a}} + \bar{\eta}_{\dot{a}} \dot{f}_{\dot{\beta}} a = -\bar{\eta}_{\dot{a}} R - \frac{1}{2} G^\beta \bar{\eta}_{\dot{a}} \dot{f}_{\dot{\beta}} a,$$

$$\xi(K)'(\eta)^a = \eta^a f_\beta \alpha + \bar{\eta}_{\dot{a}} \dot{f}_{\dot{\beta}} \dot{\alpha}.$$  (4.26)

Similarly, the Poincaré $Q$ transformation after the gauge fixing is written as

$$\delta_G(\eta^a Q^P_a) = \delta_G(\eta^a Q_a) + \delta_G(\xi(A)'(\eta) A) + \delta_G(\xi(K)'(\eta) B K_B)$$  (4.27)

with the same parameters given in (4.26). We show that this transformation is exactly the same as the $Q$ transformation (2.28) in the component approach by examining the correspondence between the transformation parameters (4.26) and (2.29). For the $A$ transformation, the parameter in superspace is

$$\xi(A)'(\eta) = \frac{3}{4} \left( \frac{i}{3} G_{\dot{a}} (2 \eta^a) (\frac{1}{2} \nabla_\dot{a} \Phi^f) - \frac{i}{3} G_{\dot{a}} (2 \bar{\eta}_{\dot{a}}) (\frac{1}{2} \bar{\nabla}_\dot{a} \bar{\Phi}^{\dot{f}}) \right).$$  (4.28)

Noticing the parameter correspondences $\frac{3}{2} \theta \leftrightarrow \xi(A)'(\eta)$ and $\bar{\theta} \leftrightarrow 2 \left( \eta^a \bar{\eta}_{\dot{a}} \right)$ given in Table 2, we find that (4.28) agrees with $\theta(\varepsilon)$ of (2.29) in the component approach. For the $S$ transformation, the above parameter $\xi(K)'(\eta)_a$ in superspace is rewritten by the other auxiliary fields with (4.19) and
(4.21), and is given by

\[
\xi(K)'(\eta)_{\alpha} = \left(\eta_{\alpha} \bar{R} + \frac{1}{2} \bar{\eta}_{\beta} G_{\alpha}^{\beta}\right) \\
= -\frac{1}{2} \left\{ -\frac{1}{2} \left(\left(-\frac{1}{4} \nabla^2 \Phi^0 \right) e^{-G/6} - \frac{1}{3} \left(-\frac{1}{4} \nabla^2 \Phi^i \right) G_i \right) (2\eta_{\alpha}) \\
+ \frac{1}{3} \left( (G_{ij} + \frac{1}{3} G_i G_j) (2\eta^j) \left(\frac{1}{2} \nabla \Phi^i \right) + G_{ij} (2\bar{\eta}_{\beta}) \left(\frac{1}{2} \bar{\nabla} \bar{\Phi}^i \right) \right) \left(\frac{1}{2} \nabla \Phi^j \right) \\
+ \frac{1}{12} e^c \left( G_i \nabla_m \Phi^j - G_{ij} \nabla_m \Phi^m \right) \left(i\sigma_{\alpha}^c (2\bar{\eta}_{\beta}) \right) + \frac{i}{4} e^m \left(\frac{1}{4} A_m \left(i\sigma_{\alpha}^c (2\bar{\eta}_{\beta}) \right) \right) \right\},
\]

(4.29)

which is the same as \(z_R(\epsilon)\) of (2.29) in the component approach with the correspondences of parameters and gauge fields given in Tables 2 and 3, especially \(\bar{\xi} \leftrightarrow -2(\xi(K)^{\alpha}, \bar{\xi}(K)_{\alpha})\) and \(\frac{1}{4} A_{\alpha} \leftrightarrow A_m\).

A similar argument holds for \(\xi(K)'(\eta)_{\alpha}\). Finally, the discussion the \(K_{\alpha}\) transformation part. Using \(f_{\beta\alpha} = -f_{\alpha\beta}\), which comes from the curvature constraints after the gauge fixing, we have

\[
f_{\beta\alpha} = -e^{a}_{m} f_{m\beta} + \frac{1}{2} e^{a}_{m} \psi_{a} f_{m\beta} + \frac{1}{2} e^{a}_{m} \bar{\psi}_{m} \delta_{\beta}^{a} \mid .
\]

(4.30)

With this form at hand, the parameter \(\xi(K)'(\eta)_{\alpha}\) of (4.26) in superspace is rewritten as

\[
(\eta^{\beta} f_{\beta\alpha} + \bar{\eta}_{\beta} f^{\beta}_{\alpha}) = \frac{1}{4} \left\{ e^{m}_{a} \left( (-2f_{m\beta}) (2\eta_{\beta}) + (-2f_{m\beta}) (2\bar{\eta}_{\beta}) \right) \\
- e^{m}_{a} \psi_{a} \left( -2(\eta^{\beta} f_{\beta\alpha} + \bar{\eta}_{\beta} f^{\beta}_{\alpha}) \right) \\
- e^{m}_{a} \bar{\psi}_{m} \left( -2(\eta^{\beta} f_{\beta\alpha} + \bar{\eta}_{\beta} f^{\beta}_{\alpha}) \right) \right\} \mid .
\]

(4.31)

The RHS is the same as \(\xi_{\alpha}(\epsilon)\) of (2.29) in the component approach with the correspondences of the \(S\)-gauge field given in Table 3 and the \(S\) transformation parameter discussed above.

### 5. Summary

In this paper, we have investigated the 4D \(\mathcal{N} = 1\) conformal SUGRA in two different approaches. One is the superconformal tensor calculus, developed in the 1980s [8], which uses the ordinary 4D field theory. The other is the superspace formalism, recently constructed in Ref. [9], which uses the conformal superspace and superfields.

Though there are apparent differences in supersymmetry transformation and superconformal multiplets, we have shown that the two approaches are completely equivalent, and have clarified the correspondences of superconformal generators (Table 1), gauge fields (Table 3, Eq. (3.3)), curvatures (Table 4, Eq. (3.2)) and their constraints (read from Table 6), superconformal transformations (Table 7), multiplet fields (Table 8), chiral projection (Eq. (3.25)), and invariant actions (Eqs. (3.29), (3.30)).

The action in the superspace formalism has a much greater gauge invariance than the component approach. Therefore, the correspondence between the two approaches should also be clarified for the gauge-fixing conditions. We clarify how to obtain Poincaré SUGRA and the remaining supersymmetry by fixing the superconformal gauge symmetry in the general matter-coupled SUGRA system.
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Appendix A. Notations

In the component approach part of the text, we use the notation of KU [8], which is the same as Ref. [10] except for the two-component spinors and the dual of the second-rank antisymmetric tensor. In the superspace approach part, we use the notation of Wess and Bagger [12].

A.1. Notation in the component approach

We use Roman letters for flat Lorentz indices, Greek letters $\mu, \nu, \ldots$ for curved vectors, and Greek letters $\alpha, \beta, \ldots$ for two-component spinors. We also use the Euclidean notation (the Pauli metric). The metric and the totally antisymmetric tensor are given by

$$\delta_{ab} = \text{diag}(1, 1, 1, 1), \quad \varepsilon^{1234} = 1. \tag{A1}$$

The gamma matrices satisfy

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}, \tag{A2}$$

and $\gamma_5$ and $\sigma_{ab}$ are defined as

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_{ab} = \frac{1}{4}(\gamma_a\gamma_b - \gamma_b\gamma_a) = \begin{pmatrix} (\sigma_{ab})_\alpha^\beta & 0 \\ 0 & (\bar{\sigma}_{ab})_{\dot{\alpha}}^{\dot{\beta}} \end{pmatrix}. \tag{A3}$$

The relation between four-component and two-component spinors is

$$\psi = \begin{pmatrix} \psi_\alpha \\ \psi_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \bar{\psi} = \psi^T C = \begin{pmatrix} \psi_\alpha & \psi_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_R & \psi_L \end{pmatrix}, \tag{A4}$$

$$C = \begin{pmatrix} -\varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \tag{A5}$$

where $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor with $\varepsilon^{12} = \varepsilon_{12} = 1$. The raising and lowering rules of the spinor index are defined by

$$\psi^\alpha = \varepsilon_{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \psi^\beta\varepsilon_{\beta\alpha}. \tag{A6}$$

The dual of the antisymmetric tensor $F_{ab}$ and its self-dual and antiself-dual parts are defined as

$$\tilde{F}_{ab} := \frac{1}{2}\varepsilon_{abcd}F^{cd}, \quad F_{ab}^\pm := \frac{1}{2}(F_{ab} \pm \tilde{F}_{ab}). \tag{A7}$$
Using the relation $\tilde{\sigma}_{ab} = -\gamma^5 \sigma_{ab}$, we find
\[ \sigma_{ab} \psi_R = \sigma_{ab} \psi_R, \quad \sigma_{ab} \psi_L = \sigma_{ab} \psi_L, \quad \sigma^{ab} F^\pm_{ab} = \frac{1 + \gamma^5}{2} \sigma^{ab} F_{ab}. \] (A8)

A.2. Notation in the superspace approach

We use the indices $a, b, \ldots$ for flat Lorentz vectors, $\alpha, \beta, \ldots$ for flat Lorentz spinors, $m, n, \ldots$ for curved vectors, and $\mu, \nu, \ldots$ for curved spinors. The indices $A, B, \ldots$ are the sets of flat vectors and spinors, and $M, N, \ldots$ the sets of curved vectors and spinors. We also use the Minkowski metric. The metric and the totally antisymmetric tensor are given by
\[ \eta_{ab} = \text{diag}(-1, 1, 1, 1), \quad \varepsilon^{0123} = -\varepsilon^{0123} = 1. \] (A9)

The standard contractions of two-component spinors are
\[ \xi \psi = \xi^\alpha \psi_\alpha, \quad \tilde{\xi} \tilde{\psi} = \tilde{\xi}_{\dot{\alpha}} \tilde{\psi}_{\dot{\alpha}}, \] (A10)
and the raising and lowering rules of the index are defined by
\[ \psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \varepsilon_{\alpha\beta} \psi^\beta, \quad \tilde{\psi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \tilde{\psi}_{\dot{\beta}}, \quad \tilde{\psi}^\dot{\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}} \tilde{\psi}^\dot{\beta}, \] (A11)
where $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$ are the second-order antisymmetric tensors with $\varepsilon^{12} = \varepsilon_{21} = 1$. The Hermitian conjugate of the spinor is given by $(\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}}$, and the Hermitian conjugate rule for the spinor product is
\[ (\xi_\alpha \psi_\beta)^\dagger = \bar{\psi}_{\dot{\beta}} \bar{\xi}_{\dot{\alpha}}. \] (A12)

The 4D Pauli matrices $\sigma_a$ are defined as
\[ (\sigma_0, \sigma_1, \sigma_2, \sigma_3)_{\alpha\beta} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \] (A13)
and their Hermitian conjugates are
\[ (\tilde{\sigma}_a)^{\dot{\alpha}\dot{\beta}} = e^{\dot{\alpha}\dot{\gamma}} \varepsilon_{\dot{\gamma}\dot{\delta}} (\sigma_a)_{\delta\beta} = (\sigma_a)^{\beta\alpha}. \] (A14)

With these matrices, any flat Lorentz vector $V_{\alpha}$ can be expressed as a mixed spinor $V_{a\dot{\beta}}$ and vice versa:
\[ V_{\alpha\dot{\beta}} = (\sigma_a)_{\alpha\dot{\beta}} V_{a}, \quad V^{a} = -\frac{1}{2} (\tilde{\sigma}_{a})^{\dot{\alpha}\dot{\beta}} V_{a\dot{\beta}}. \] (A15)

The matrices $\sigma^{ab}$ and $\tilde{\sigma}^{ab}$ are defined as
\[ (\sigma^{ab})_{\alpha} = \frac{1}{4} (\sigma^a \tilde{\sigma}^b - \sigma^b \tilde{\sigma}^a)_{\alpha} \beta, \quad (\tilde{\sigma}^{ab})_{\dot{\alpha}} = \frac{1}{4} (\tilde{\sigma}^a \sigma^b - \tilde{\sigma}^b \sigma^a)_{\dot{\alpha}} \dot{\beta}, \] (A16)
and satisfy the relations
\[ \varepsilon_{abcd} (\sigma^{cd})_{\alpha} = -2i (\sigma_{ab})_{\alpha} \beta, \quad \varepsilon_{abcd} (\tilde{\sigma}^{cd})_{\dot{\alpha}} = 2i (\tilde{\sigma}_{ab})_{\dot{\alpha}} \dot{\beta}. \] (A17)
In two-component spinor notation, any antisymmetric tensor $F_{ab}$ can be decomposed into chiral and antichiral parts:

$$F_{ab} = - (\epsilon \sigma_{ab})^{\alpha \beta} F^{-}_{\alpha \beta} + (\tilde{\sigma}_{ab} \epsilon)^{\dot{\alpha} \dot{\beta}} F^{+}_{\dot{\alpha} \dot{\beta}},$$  

(A18)

where

$$F^{-}_{\alpha \beta} = \frac{1}{2} (\sigma_{ab} \epsilon)_{\alpha \beta} F_{ab}, \quad F^{+}_{\dot{\alpha} \dot{\beta}} = - \frac{1}{2} (\epsilon \tilde{\sigma}_{ab})_{\dot{\alpha} \dot{\beta}} F_{ab}. \quad (A19)$$

The dual of the antisymmetric tensor $F_{ab}$ is defined as

$$(*F)_{ab} := \frac{1}{2} \epsilon_{abcd} F^{cd} = i (\epsilon \sigma_{ab})^{\alpha \beta} F^{-}_{\alpha \beta} + i (\tilde{\sigma}_{ab} \epsilon)^{\dot{\alpha} \dot{\beta}} F^{+}_{\dot{\alpha} \dot{\beta}}. \quad (A20)$$

The self-dual and antiself-dual parts of $F_{ab}$ are

$$F^{\pm}_{ab} := \frac{1}{2} (F_{ab} \mp i (*F)_{ab}), \quad (A21)$$

which coincide with the chiral and antichiral parts, respectively:

$$F^{-}_{ab} = - (\epsilon \sigma_{ab})^{\alpha \beta} F^{-}_{\alpha \beta}, \quad F^{+}_{ab} = (\tilde{\sigma}_{ab} \epsilon)^{\dot{\alpha} \dot{\beta}} F^{+}_{\dot{\alpha} \dot{\beta}}. \quad (A22)$$

### A.3. Correspondence of notations

We summarize the correspondence of notations between the component and superspace approaches in Table A1.

| component | superspace |
|-----------|------------|
| $\delta_{ab}$ | $\eta_{ab}$ |
| $\epsilon^{a\dot{b}}, \epsilon_{a\dot{b}}$ | $\epsilon^{a\dot{b}}, -\epsilon_{a\dot{b}}$ |
| $\epsilon^{\dot{a}\dot{b}}, \epsilon_{\dot{a}\dot{b}}$ | $\epsilon^{\dot{a}\dot{b}}, -\epsilon_{\dot{a}\dot{b}}$ |
| $(\psi_{a}, \bar{\psi}_{\dot{a}})$, $(\dot{\psi}_{a}, \bar{\dot{\psi}}_{\dot{a}})$ | $(\psi_{a}, \bar{\psi}_{\dot{a}})$, $(\dot{\psi}_{a}, \bar{\dot{\psi}}_{\dot{a}})$ |
| $\gamma_{a}$ | $i \gamma_{a} = i \left( \begin{array}{cc} 0 & (\sigma_{a})_{\mu \dot{\beta}} \\ (\tilde{\sigma}_{a})_{\mu \dot{\beta}} & 0 \end{array} \right)$ |
| $\gamma_{5}$ | $\left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$ |
| $\sigma_{ab}$ | $- \left( \begin{array}{cc} (\sigma_{ab})_{\mu \dot{\beta}} & 0 \\ 0 & (\tilde{\sigma}_{ab})_{\mu \dot{\beta}} \end{array} \right)$ |
| $\epsilon_{abcd}$ | $-i \epsilon_{abcd}$ |
| $\tilde{F}_{ab}$ | $-i (*F)_{ab}$ |
| $F^{\pm}_{ab}$ | $F^{\pm}_{ab}$ |
Appendix B. $Q$ transformation of the conformal multiplet

The supersymmetry $Q$ transformation laws take the following form for the fields in a general conformal multiplet $[C_{\Gamma}, Z_{\Gamma}, H_{\Gamma}, K_{\Gamma}, B_{a\Gamma}, \Lambda_{\Gamma}, D_{\Gamma}]$:

\[
\delta_{Q}(\varepsilon) C_{\Gamma} = \frac{1}{2} i \bar{\varepsilon} \gamma_{5} Z_{\Gamma},
\]

\[
\delta_{Q}(\varepsilon) Z_{\Gamma} = (-)^{1/2} (i \gamma_{5} H_{\Gamma} - K_{\Gamma} - \gamma^{a} B_{a\Gamma} + i \gamma^{a} D_{a} C_{\Gamma} \gamma_{5}) \varepsilon,
\]

\[
\delta_{Q}(\varepsilon) H_{\Gamma} = \frac{1}{2} i \bar{\varepsilon} \gamma_{5} (\gamma^{a} D_{a} Z_{\Gamma} + \Lambda_{\Gamma}),
\]

\[
\delta_{Q}(\varepsilon) K_{\Gamma} = -\frac{1}{2} \bar{\varepsilon} (\gamma^{a} D_{a} Z_{\Gamma} + \Lambda_{\Gamma}),
\]

\[
\delta_{Q}(\varepsilon) B_{a\Gamma} = -\frac{1}{2} \bar{\varepsilon} (D_{a} Z_{\Gamma} + \gamma_{a} \Lambda_{\Gamma}) - \frac{1}{4} i R_{bc}(Q) \gamma_{5} \gamma_{a}(\Sigma_{bc} C)|, \n\]

\[
\delta_{Q}(\varepsilon) \Lambda_{\Gamma} = (-)^{1/2} (\sigma^{ab} F_{ab\Gamma} + i \gamma_{5} D_{\Gamma}) \varepsilon
\]

\[
+ \frac{1}{8} \left( \gamma_{c} \bar{\varepsilon} R_{ab}(Q) \gamma_{c} (\Sigma^{ab} Z)_{\Gamma} + \gamma_{5} \gamma_{c} \bar{\varepsilon} R_{ab}(Q) \gamma_{5} \gamma_{c}(\Sigma^{ab} Z)_{\Gamma} \right),
\]

\[
\delta_{Q}(\varepsilon) D_{\Gamma} = \frac{1}{2} i \bar{\varepsilon} \gamma_{5} \gamma^{a} D_{a} \Lambda_{\Gamma} - \frac{1}{4} \bar{\varepsilon} (R_{ab}(A) + \gamma_{5} \bar{R}_{ab}(A))(\Sigma^{ab} Z)_{\Gamma}
\]

\[
+ (-)^{1/2} \frac{1}{4} i \bar{\varepsilon} (i \gamma_{5}(\Sigma^{ab} \gamma^{c} B_{c})_{\Gamma} - (\Sigma^{ab} \gamma^{c} D_{c})_{\Gamma})(R_{ab}(Q)C^{-1})^{T}. \quad (B1)
\]

This transformation law is called the standard form. The definition of $F_{ab\Gamma}$ in the transformation law of $\Lambda_{\Gamma}$ is

\[
F_{ab\Gamma} = D_{a} B_{b\Gamma} - D_{b} B_{a\Gamma} + \frac{1}{2} i \bar{\varepsilon}_{abcd}[D^{c}, D^{d}] C_{\Gamma}. \quad (B2)
\]

Appendix C. Derivations of correspondence

C.1. Conformal multiplets with arbitrary Lorentz indices

In this subsection we explicitly derive the correspondences of conformal multiplets with arbitrary Lorentz index, i.e., between $V_{\Gamma}$ in the component approach (Eq. (2.18)) and $\Phi_{\Gamma}$ in the superspace approach (Eq. (2.54)). In the first place, the correspondence of the lowest components is obtained by the property of superconformal transformations. There is an ambiguity for the overall constant factor, which is fixed by

\[
C_{\Gamma} \leftrightarrow \Phi_{\Gamma}|. \quad (C1)
\]

We then obtain the correspondences of higher components by operating the $Q$ transformations in order. The action of $Q$ transformation is given by the covariant spinor derivative since the fields have only Lorentz indices. As given in Table 2, the correspondence of the $Q$ transformation parameters is $\bar{\varepsilon} \leftrightarrow 2 \left( \bar{\xi}(P)^{a} - \bar{\xi}(P)_{\dot{a}} \right)$. In the following, we simply denote the parameters in superspace as $\left( \xi^{a} \bar{\xi}_{\dot{a}} \right)$. 
The correspondence of the second components is obtained by the $Q$ transformations of the first components, namely,

$$\delta_Q(\varepsilon)C_T \leftrightarrow (\xi^\alpha \nabla_\alpha + \bar{\xi}_\dot{\alpha} \bar{\nabla}^\dot{\alpha}) \Phi_T = \frac{1}{2} i \left( \begin{array}{cc} 2 \xi^\alpha & 2 \bar{\xi}_\dot{\alpha} \\ 0 & -1 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left( \begin{array}{c} -i \nabla_\alpha \Phi_T \\ +i \bar{\nabla}^\dot{\alpha} \Phi_T \end{array} \right). \quad (C2)$$

By comparing with $\delta_Q(\varepsilon)C_T$ in (B1) and using the correspondence of $\gamma_5$ in Table A1, we find

$$Z_T \leftrightarrow \left( \begin{array}{c} -i \nabla_\alpha \Phi_T \\ +i \bar{\nabla}^\dot{\alpha} \Phi_T \end{array} \right). \quad (C3)$$

The correspondences of the other components are obtained in similar ways. The $Q$ transformations of the second components are

$$\delta_Q(\varepsilon)Z_T \leftrightarrow (\xi^\alpha \nabla_\alpha + \bar{\xi}_\dot{\alpha} \bar{\nabla}^\dot{\alpha}) \left( \begin{array}{c} -i \nabla_\beta \Phi_T \\ +i \bar{\nabla}^\dot{\beta} \Phi_T \end{array} \right) \quad (C4)$$

By comparing with $\delta_Q(\varepsilon)Z_T$ in (B1) and using the correspondence of gamma matrices in Table A1, we find the correspondences of $H_T$, $\kappa_T$, $B_{a\Gamma}$ as

$$H_T \leftrightarrow \frac{1}{4} (\nabla^2 \Phi_T + \bar{\nabla}^2 \Phi_T), \quad \kappa_T \leftrightarrow -\frac{1}{4} i (\nabla^2 \Phi_T - \bar{\nabla}^2 \Phi_T), \quad B_{a\Gamma} \leftrightarrow -\frac{1}{4} (\bar{\sigma}_a)_{\dot{\beta}\dot{\rho}} [\nabla_{\dot{\beta}}, \bar{\nabla}^\dot{\rho}] \Phi_T. \quad (C5)$$

The $Q$ transformation of $H_T$ implies

$$\delta_Q(\varepsilon)H_T \leftrightarrow (\xi^\alpha \nabla_\alpha + \bar{\xi}_\dot{\alpha} \bar{\nabla}^\dot{\alpha}) \left( \frac{1}{4} (\nabla^2 \Phi_T + \bar{\nabla}^2 \Phi_T) \right) \quad (C6)$$

Here we have used the identities

$$\nabla_\alpha \bar{\nabla}^2 - \bar{\nabla}^2 \nabla_\alpha + 4i \nabla_\alpha \bar{\nabla}^\dot{\beta} \bar{\nabla}^\dot{\beta} + 8 \nabla_\alpha = 0,$$

$$\bar{\nabla}^\dot{\alpha} \nabla^2 - \nabla^2 \bar{\nabla}^\dot{\alpha} + 4i \bar{\nabla}^\dot{\alpha} \bar{\nabla}^\dot{\beta} \nabla_\beta - 8 \nabla_\dot{\alpha} = 0. \quad (C7)$$
These identities are shown by evaluating (anti)commutators of covariant derivatives. By comparing with \( \delta_Q(\varepsilon)\mathcal{H}_\Gamma \) in (B1), we find

\[
\Lambda_\Gamma \leftrightarrow \frac{i}{4} \left( -\bar{\nabla}^2 \nabla_\alpha \Phi_\Gamma \right) + 2i \left( \mathcal{W}_\alpha \nabla^{\dot{\alpha}} \right) \Phi_\Gamma. \tag{C8}
\]

The correspondence of the \( \Lambda_\Gamma \) component can also be obtained by the \( Q \) transformation of \( \mathcal{K}_\Gamma \) or \( \mathcal{B}_{a\Gamma} \), which is found to be consistent with (C8).

Finally, the \( Q \) transformation of \( \Lambda_\Gamma \) leads to the correspondence of the \( \mathcal{D}_\Gamma \) component. For the undotted spinor in \( \Lambda_\Gamma \), the \( Q \) transformation is found from (C8):

\[
\delta_Q(\varepsilon)\Lambda_{\Gamma\alpha} \leftrightarrow (\xi^\beta \nabla_\beta + \bar{\xi}^\dot{\beta} \bar{\nabla}^{\dot{\beta}}) \left( \frac{i}{4} - \bar{\nabla}^2 \nabla_\alpha \Phi_\Gamma \right) + 2i \left( \mathcal{W}_\alpha \nabla^{\dot{\alpha}} \right) \Phi_\Gamma
\]

\[
= \xi^\beta (\sigma^{ab})_{\beta\alpha} (\nabla_a B_{b\Gamma} - \nabla_b B_{a\Gamma} - i[\nabla_a, \nabla_b] \Phi_\Gamma)
\]

\[
+ i\bar{\xi}^\dot{\beta} \left( \frac{1}{8} \nabla^\beta \bar{\nabla}^2 \nabla_\dot{\beta} \Phi_\Gamma - \mathcal{W}^{\beta} \nabla_\dot{\beta} \Phi_\Gamma \right)
\]

\[
+ 2i\bar{\xi}^{\dot{\beta}} (R(P)_{cd})_{\alpha} \mathcal{M}_{\dot{d}} \bar{\nabla}^{\dot{\beta}} \Phi_\Gamma. \tag{C9}
\]

In the second line, we have introduced the superfield \( B_{a\Gamma} \) made from the original \( \Phi_\Gamma \) as

\[
B_{a\Gamma} = \frac{1}{4} (\tilde{\sigma}_a)^\dot{\beta} \nabla_\dot{\beta} \Phi_\Gamma,
\]

whose lowest component matches \( B_{a\Gamma} \) in the component approach as given in (C5). For the dotted spinor in \( \Lambda_\Gamma \), a similar expression holds. We then have the correspondence

\[
\delta_Q(\varepsilon)\Lambda_\Gamma \leftrightarrow (\xi^\beta \nabla_\beta + \bar{\xi}^\dot{\beta} \bar{\nabla}^{\dot{\beta}}) \left( \frac{i}{4} - \bar{\nabla}^2 \nabla_\alpha \Phi_\Gamma \right) + 2i \left( \mathcal{W}_\alpha \nabla^{\dot{\alpha}} \right) \Phi_\Gamma
\]

\[
= (-)^{\gamma} \frac{1}{2} (\sigma^{ab})_{\dot{\alpha}} \gamma \left( \frac{1}{2} \bar{\nabla}_\gamma \nabla^{\dot{\gamma}} \Phi_\Gamma + \mathcal{W}_{\dot{\alpha}} \nabla^{\dot{\alpha}} \Phi_\Gamma \right) \left( \frac{2\xi_{\dot{\gamma}}}{2\bar{\xi}^{\dot{\gamma}}} \right)
\]

\[
+ (-)^{\gamma} \frac{1}{2} \left( 0 \right) \left( \frac{1}{8} \bar{\nabla}_\gamma \bar{\nabla}^2 \nabla_\dot{\gamma} \Phi_\Gamma \right) \left( \frac{2\bar{\xi}^{\dot{\gamma}}}{2\xi^{\dot{\gamma}}} \right)
\]

\[
\times (-2) \left( R(P)_{ab} \right)^\delta \left( R(P)_{ab} \right)_{\dot{\delta}} \left( \frac{1}{2} \bar{\nabla}_\gamma \nabla^{\dot{\gamma}} \Phi_\Gamma + \mathcal{W}_{\dot{\alpha}} \nabla^{\dot{\alpha}} \Phi_\Gamma \right) \left( \frac{2\xi_{\dot{\gamma}}}{2\bar{\xi}^{\dot{\gamma}}} \right)
\]

\[
+ (-)^{\gamma} \frac{1}{2} \left( 0 \right) \left( \frac{1}{8} \bar{\nabla}_\gamma \bar{\nabla}^2 \nabla_\dot{\gamma} \Phi_\Gamma \right) \left( \frac{2\bar{\xi}^{\dot{\gamma}}}{2\xi^{\dot{\gamma}}} \right)
\]

\[
\times (-2) \left( R(P)_{ab} \right)^\delta \left( R(P)_{ab} \right)_{\dot{\delta}} \left( \frac{1}{2} \bar{\nabla}_\gamma \nabla^{\dot{\gamma}} \Phi_\Gamma + \mathcal{W}_{\dot{\alpha}} \nabla^{\dot{\alpha}} \Phi_\Gamma \right) \left( \frac{2\xi_{\dot{\gamma}}}{2\bar{\xi}^{\dot{\gamma}}} \right). \tag{C11}
\]

In this modification, we have used the relations (A17), the identity (3.17), and the equation \( \left\{ \bar{\nabla}_\dot{\alpha}, \mathcal{W}_\alpha \right\} \Phi_\Gamma = -\frac{1}{2} \bar{\nabla}^{\alpha} \bar{\nabla}_\dot{\gamma} \mathcal{W}_{\beta\alpha} \gamma K_{\beta\gamma \dot{\gamma}} \Phi_\Gamma = 0 \) for a primary superfield \( \Phi_\Gamma \). By comparing with
\[ \delta \Phi(\epsilon) \Lambda \] in (B1) and the definition (B2), we find

\[ \mathcal{D}_\Gamma \leftrightarrow \frac{1}{8} \tilde{\nabla}_a \tilde{\nabla}^2 \tilde{\nabla}^a \Phi_\Gamma | + \mathcal{W}_a \tilde{\nabla}^a \Phi_\Gamma | = \frac{1}{8} \tilde{\nabla}^a \tilde{\nabla}^2 \tilde{\nabla}_a \Phi_\Gamma | - \mathcal{W}^{\alpha} \tilde{\nabla}_\alpha \Phi_\Gamma |, \quad (C12) \]

and the correspondence

\[ \mathcal{F}_{ab} \Gamma \leftrightarrow (\nabla_a \Phi_\Gamma B - \nabla_b \Phi_\Gamma A) | + \frac{1}{2} \epsilon_{abcd} [\nabla^c, \nabla^d] \Phi_\Gamma |. \quad (C13) \]

That completes the correspondence of the components in a conformal multiplet, which is summarized in Table 8.

The \( Q \) transformation of \( \mathcal{D}_\Gamma \) is a nontrivial consistency check and is explicitly calculated as

\[
\delta Q(\epsilon) \mathcal{D}_\Gamma \leftrightarrow (\xi^a \nabla_a + \bar{\xi}^\dot{a} \tilde{\nabla}^\dot{a}) \left( \frac{1}{8} \tilde{\nabla}_\dot{\beta} \tilde{\nabla}^2 \tilde{\nabla}_\dot{\beta} \Phi_\Gamma + \mathcal{W}_\dot{\beta} \bar{\tilde{\nabla}}^\dot{\beta} \Phi_\Gamma \right)
= \frac{1}{2} i \left( 2 \xi^a \quad 2 \bar{\xi}^\dot{a} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) i \left( \begin{array}{cc} 0 & (\sigma^a)_{\dot{\alpha} \dot{\beta}} \\ (\bar{\sigma}^\dot{a})^{\dot{\alpha} \dot{\beta}} & 0 \end{array} \right) \nabla_a \left( \frac{i}{4} \left( -\tilde{\nabla}^2 \Phi_\Gamma + \tilde{\nabla}^\dot{\alpha} \bar{\tilde{\nabla}}^\dot{\alpha} \Phi_\Gamma \right) + 2 i \left( \frac{1}{4} \tilde{\nabla}^\dot{\alpha} \bar{\tilde{\nabla}}^\dot{\alpha} \Phi_\Gamma \right) \right)
- \frac{1}{4} \left( 2 \xi^a \quad 2 \bar{\xi}^\dot{a} \right) \left( -\frac{4}{3} \right) \left( R(A)_{ab} + \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) (-i)(\ast R(A))_{ab} \right) \left( -M_{ab} \right) \left( \frac{1}{4} \tilde{\nabla}^\dot{\alpha} \bar{\tilde{\nabla}}^\dot{\alpha} \Phi_\Gamma \right)
+ \frac{1}{4} (-\Gamma)^{\dot{e}} \left( 2 \xi^a \quad 2 \bar{\xi}^\dot{a} \right) i \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \left( -M_{ab} i \left( \begin{array}{cc} 0 & (\sigma^c)_{\dot{\alpha} \dot{\beta}} \\ (\bar{\sigma}^\dot{c})^{\dot{\alpha} \dot{\beta}} & 0 \end{array} \right) B_{c \Gamma} \right) \left( -2 \right) \left( \frac{1}{4} \tilde{\nabla}^\dot{\alpha} \bar{\tilde{\nabla}}^\dot{\alpha} \Phi_\Gamma \right)
- \frac{1}{4} (-\Gamma)^{\dot{e}} \left( 2 \xi^a \quad 2 \bar{\xi}^\dot{a} \right) \left( -M_{ab} i \left( \begin{array}{cc} 0 & (\sigma^c)_{\dot{\alpha} \dot{\beta}} \\ (\bar{\sigma}^\dot{c})^{\dot{\alpha} \dot{\beta}} & 0 \end{array} \right) \nabla_c \Phi_\Gamma \right) \left( -2 \right) \left( \frac{1}{4} \tilde{\nabla}^\dot{\alpha} \bar{\tilde{\nabla}}^\dot{\alpha} \Phi_\Gamma \right). \quad (C14) \]

In this modification, we have used the relations \( \{ \tilde{\nabla}_\dot{\beta}, \mathcal{W}_\dot{\beta} \} \nabla_a \Phi_\Gamma = -\frac{1}{2} \tilde{\nabla}^\dot{\beta} \nabla^\dot{\gamma} W_{\dot{\gamma} \dot{\beta} \dot{\alpha}} \nabla_{\dot{\alpha}} \Phi_\Gamma = 0 \) and \( \mathcal{W}(K)^{\dot{a}}_{\dot{\beta}} K_{\dot{\gamma}} \tilde{\nabla}^\dot{\beta} \nabla_{\dot{\alpha}} \Phi_\Gamma = 0 \). Noticing the correspondences of generators, curvatures, gamma matrices, and lower components given in Sect. 3 and above, we find that this form exactly agrees with \( \delta Q(\epsilon) \mathcal{D}_\Gamma \) in (B1).

C.2. Chiral projection

In this subsection, we show the correspondence of the chiral projection between the two approaches.

In the component approach, the chiral projection operator \( \Pi \) acts on a conformal multiplet \( \mathcal{V}_\Gamma \) with special weights and index, and gives a chiral multiplet \( \Pi \mathcal{V}_\Gamma \) whose component expression is explicitly given in (2.23). In the superspace approach, the chiral projection operator \( \mathcal{P} \) is defined by the superconformal covariant derivative as \( \mathcal{P} = \frac{1}{4} \tilde{\nabla}^2 \) and gives a chiral superfield \( \mathcal{P} \Phi_\Gamma \) from a primary superfield \( \Phi_\Gamma \) with special weights and index. In particular, \( \Gamma \) should be made of purely undotted spinor indices. When one matches \( \mathcal{V}_\Gamma \) with \( \Phi_\Gamma \), the correspondence of the chiral projection is

\[ \Pi \mathcal{V}_\Gamma \leftrightarrow -\mathcal{P} \Phi_\Gamma = \frac{1}{4} \tilde{\nabla}^2 \Phi_\Gamma. \quad (C15) \]

In what follows, we show this correspondence explicitly by component level; namely, each component of the chiral superfield \( \frac{1}{4} \tilde{\nabla}^2 \Phi_\Gamma \) coincides with (2.23) in the component approach.
For a general chiral superfield, its components, which should match those of the corresponding chiral multiplet, are given in (3.23). First, the lowest component of $\frac{1}{4} \tilde{\nabla}^2 \Phi_\Gamma$ is

$$\frac{1}{4} \tilde{\nabla}^2 \Phi_\Gamma = \frac{1}{2} \left( \frac{1}{4} (\nabla^2 \Phi_\Gamma + \bar{\nabla}^2 \Phi_\Gamma) - i \left( - \frac{1}{4} i(\nabla^2 \Phi_\Gamma - \bar{\nabla}^2 \Phi_\Gamma) \right) \right). \quad (C16)$$

We find from (C5) that the RHS just corresponds to $\frac{1}{2} (H_\Gamma - i K_\Gamma)$ in the component approach, which is the lowest component of the chiral multiplet $\Pi V_\Gamma$, as shown in (2.23).

The second component of the chiral superfield is given by its covariant derivative, and, for $\frac{1}{4} \tilde{\nabla}^2 \Phi_\Gamma$, it becomes

$$\nabla_\alpha \left( \frac{1}{4} \tilde{\nabla}^2 \Phi_\Gamma \right) = i \left( (\sigma^a)_{\alpha \dot{\beta}} \nabla_\alpha (i \tilde{\nabla}^\dot{\beta} \Phi_\Gamma) \right) - \frac{i}{4} \tilde{\nabla}^2 \nabla_\alpha \Phi_\Gamma + 2i \mathcal{W}_\alpha \Phi_\Gamma \right), \quad (C17)$$

where the identity (C7) has been used. By comparing with the component correspondences (C3) and (C8), we find that the RHS reads $i P_R (\gamma^a D_a \mathcal{Z}_\Gamma + \Lambda_\Gamma)$ in the component approach, which is exactly the second component of $\Pi V_\Gamma$ given in (2.23).

Finally, the highest component of the chiral superfield $\frac{1}{4} \tilde{\nabla}^2 \Phi_\Gamma$ is

$$-\frac{1}{4} \tilde{\nabla}^2 \left( \frac{1}{4} \tilde{\nabla}^2 \Phi_\Gamma \right) = -\frac{1}{2} \left( \frac{1}{8} \nabla_\alpha \tilde{\nabla}^2 \tilde{\nabla}^\dot{\alpha} \Phi_\Gamma + \nabla^a \Phi_\Gamma \right) + i \nabla_\alpha \left( -\frac{1}{4} (\tilde{\sigma}^a)_{\dot{\alpha} \alpha} \{ \nabla_\alpha, \tilde{\nabla}_\dot{\alpha} \} \Phi_\Gamma \right) \right), \quad (C18)$$

where we have used the identity (3.26) and the equation $\mathcal{W}_\alpha \tilde{\nabla}^\dot{\alpha} \Phi_\Gamma = 0$, which holds on a primary $\Phi_\Gamma$ with purely undotted $\Gamma$, namely, $\mathcal{W}_\alpha \tilde{\nabla}^\dot{\alpha} \Phi_\Gamma = (\tilde{\nabla}^\dot{\alpha}, \mathcal{W}_\alpha) \Phi_\Gamma = \frac{1}{2} \tilde{\nabla}^a \tilde{\nabla}^\dot{b} \tilde{\gamma} W_{\dot{b} \dot{a} \dot{\gamma} \Gamma} = 0$, since $\mathcal{W}_\alpha \Phi_\Gamma = 0$ for such $\Phi_\Gamma$. By comparing with the correspondence of $B_{a \Gamma}$ in (C5) and also (C12), again noting $\mathcal{W}_\alpha \tilde{\nabla}^\dot{\alpha} \Phi_\Gamma = 0$, the RHS of (C18) corresponds to $-\frac{1}{2} (D_\Gamma + \Box C_\Gamma + i D^a B_{a \Gamma})$ in the component approach, which is the highest $\mathcal{F}$ component of $\Pi V_\Gamma$, as shown in (2.23).

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