Renormalization Group Equations for the CKM matrix

P. Kielanowski

Departamento de Física, Centro de Investigación y Estudios Avanzados del IPN, Mexico

S.R. Juárez W. and J.H. Montes de Oca Y.

Departamento de Física, Escuela Superior de Física y Matemáticas, IPN, Mexico

We derive the one loop renormalization group equations for the Cabibbo-Kobayashi-Maskawa matrix for the Standard Model, its two Higgs extension and the minimal supersymmetric extension in a novel way. The derived equations depend only on a subset of the model parameters of the renormalization group equations for the quark Yukawa couplings so the CKM matrix evolution cannot fully test the renormalization group evolution of the quark Yukawa couplings. From the derived equations we obtain the invariant of the renormalization group evolution for three models which is the angle $\alpha$ of the unitarity triangle. For the special case of the Standard Model and its extensions with $v_1 \approx v_2$ we demonstrate that also the shape of the unitarity triangle and the Buras-Wolfenstein parameters $\bar{\rho} = (1 - \frac{1}{2} \lambda^2) \rho$ and $\bar{\eta} = (1 - \frac{1}{2} \lambda^2) \eta$ are conserved. The invariance of the angles of the unitarity triangle means that it is not possible to find a model in which the CKM matrix might have a simple, special form at asymptotic energies.

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I. INTRODUCTION

To obtain finite results in quantum field theory, in a higher order than the tree level, one has to perform the renormalization program. The independence of the renormalization procedure of the renormalization point leads to the dependence of the Lagrangian parameters on the point of renormalization. This dependence is governed by the renormalization group equations for the coupling constants and other parameters of the Lagrangian [1, 2]. The most important application of the renormalization group equations is a possibility to study the asymptotic properties of the Lagrangian parameters like the running masses and the coupling constants (for a very early example see e.g., [3]). The theories at the asymptotic energies may reveal some new symmetries or other interesting properties that give a deeper insight into the physical content. Also from the physical requirement of the stability of the theory one can e.g., estimate the range of the physical parameters like the Higgs mass in the Standard Model or its extensions [4] and the references therein.

The observable parameters of the Standard Model are the following: 6 quark masses, 3 lepton masses, 4 parameters of the Cabibbo–Kobayashi–Maskawa (CKM) matrix [5, 6] and 3 gauge coupling constants. The CKM matrix is obtained from the diagonalizing matrices of the Yukawa couplings.

The renormalization group (RG) equations for the CKM matrix is obtained from the renormalization group equations for the Yukawa couplings. In this paper we derive the RG equations for the CKM matrix for the Standard Model, two Higgs doublet extension of the Standard Model and the Minimal Supersymmetric Standard Model. We first get a complete one loop equations without any approximation and next discuss these equations taking into account the hierarchy contained in the Yukawa couplings and the CKM matrix elements.

The interesting fact that can be observed is that the equations for the CKM matrix do not depend on all the parameters of the original equations for the Yukawa couplings. Finally we discuss the physical implications of the derived equations.
II. RG EQUATIONS FOR THE CKM MATRIX

A. Preliminary considerations

The part of the Lagrangian of the Standard Model and its extensions that contains the Yukawa interactions of the quarks has the following structure

\[ \mathcal{L}_Y = y_u \bar{q}_U \Phi_1 (q_u) R + y_d \bar{q}_L \Phi_2 (q_d) R + \text{h.c.} \]  

(1)

Here the \( y_u \) and the \( y_d \) are 3\times3 matrices of the Yukawa coupling of the up and down quarks, \( \Phi_1 \) and \( \Phi_2 \) are the two Higgs doublets (in case of the Standard Model \( \Phi_2 = \Phi, \Phi_1 = e \Phi^* \), \( \epsilon \) is the 2 \times 2 antisymmetric tensor). The quark masses (running) are equal to

\[ \begin{align*}
q_u & = U_L y_u U_R, \\
q_d & = D_L y_d D_R,
\end{align*} \]

(2)

where \( U_L, U_R, D_L, D_R \) are unitary, diagonalizing matrices and \( Y_u \) and \( Y_d \) are the diagonal matrices with positive diagonal elements. The quark masses (running) are equal to

\[ \begin{align*}
m_u & = \frac{1}{\sqrt{2}} v_1 (Y_u)_{11}, \\
m_c & = \frac{1}{\sqrt{2}} v_1 (Y_u)_{22}, \\
m_t & = \frac{1}{\sqrt{2}} v_1 (Y_u)_{33}, \\
m_d & = \frac{1}{\sqrt{2}} v_2 (Y_d)_{11}, \\
m_s & = \frac{1}{\sqrt{2}} v_2 (Y_d)_{22}, \\
m_b & = \frac{1}{\sqrt{2}} v_2 (Y_d)_{33}.
\end{align*} \]

(3)

Here \( v_1 \) and \( v_2 \) are the vacuum expectation values of the Higgs fields and in the Standard Model \( v_1 = v_2 = v \).

From Eqs. (3) one can see that there is a hierarchy between the elements of \( (Y_u)_{ii} \) and between the elements of \( (Y_d)_{ii} \) that follows from the hierarchy of the quark masses.

The CKM matrix of the charged weak current is obtained from the left diagonalizing matrices \( U_L \) and \( D_L \)

\[ V = U_L D_L^T. \]

(4)

The matrices \( U_R \) and \( D_R \) are not related to any observables in absence of the right handed currents.

The general structure of the one loop renormalization group equations for the Standard Model and its extensions for the Yukawa couplings \( y_u \) and \( y_d \) is the following \[7,8\]

\[ \begin{align*}
\frac{dy_u}{dt} & = \frac{1}{(4\pi)^2} [\alpha_u^1 (t) + \alpha_u^2 y_u y_u^\dagger + \alpha_u^3 \text{Tr}(y_u y_u^\dagger) + \alpha_u^4 (y_d y_d^\dagger) y_u y_u^\dagger], \\
\frac{dy_d}{dt} & = \frac{1}{(4\pi)^2} [\alpha_d^1 (t) + \alpha_d^2 y_d y_d^\dagger + \alpha_d^3 \text{Tr}(y_d y_d^\dagger) + \alpha_d^4 (y_d y_d^\dagger) y_d].
\end{align*} \]

(5a, 5b)

Here \( t = \ln(\mu/m_t) \), \( \mu \) is the energy of the renormalization point and \( m_t \) is the top quark mass. Eqs. (5) form a system of coupled non-linear equations. The parameters \( \alpha_u^i (t) \), \( \alpha_d^i (t) \) are quadratic functions of the gauge coupling constants \( \alpha_q^i \), \( \alpha_d^i \), \( i = 2, \ldots, 5 \) are constants. The values of the \( \alpha_u^i (t) \) and \( \alpha_d^i (t) \) do depend on the model and they are given in Appendix.

B. Derivation of the RG equations for the CKM matrix

We will now obtain from Eqs. (5) the renormalization group equations for the elements of the CKM matrix \( V \). The derivation is done in the following steps:

1. Insert representation (2) in Eqs. (5).
2. Eliminate the matrices \( U_R \) and \( D_R \) from the equations.

3. Derive the final form of the equations for the CKM matrix.

Before the derivation, let us observe that after the differentiation of the unitarity relations \( U^\dagger U = 1 \) and \( UU^\dagger = 1 \) we obtain

\[
\begin{align*}
\frac{dU^\dagger}{dt}U + U^\dagger \frac{dU}{dt} &= \left( \frac{dU^\dagger}{dt}U \right) + \left( \frac{dU}{dt}U \right)^\dagger = \left( U^\dagger \frac{dU}{dt} \right) + \left( U \frac{dU^\dagger}{dt} \right)^\dagger = 0 \\
\frac{dU}{dt}U^\dagger + U \frac{dU^\dagger}{dt} &= \left( \frac{dU}{dt}U^\dagger \right) + \left( \frac{dU^\dagger}{dt}U^\dagger \right)^\dagger = \left( U \frac{dU^\dagger}{dt} \right) + \left( U^\dagger \frac{dU}{dt} \right)^\dagger = 0
\end{align*}
\]

\( i.e., \) the matrices \( \frac{dU^\dagger}{dt}U, \frac{dU}{dt}U^\dagger, \frac{dU^\dagger}{dt}U^\dagger \) and \( U \frac{dU}{dt} \) are antihermitian.

In the derivation of the RG equations for the CKM matrix we will explicitly show all the algebraic operations that are necessary for Eq. (6a), the analogous steps for Eq. (5b) are not shown and only the final results are quoted.

**Step 1**

From Eq. (2) and (5a) we obtain

\[
\frac{d}{dt} \left( Y_L^\dagger U_R U_R \right) = \frac{1}{(4\pi)^2} \left( \alpha_1^u(t) + \alpha_2^u Y_L^\dagger U_R U_R Y_L \right)
\]

After differentiation, the left hand side of Eq. (7) becomes

\[
\frac{dU^\dagger}{dt}Y_L U_R + U^\dagger \frac{dY_L}{dt}U_R + U^\dagger \frac{dU_R}{dt} U_R
\]

**Step 2**

Using relation (7) we multiply Eq. (8) from the left by \( U_L \) and from the right by \( U_R^\dagger Y_u \)

\[
U_L \frac{dU^\dagger}{dt} Y_L^2 + \frac{dU}{dt} Y_L + Y_L U_L^\dagger U_R^\dagger Y_R = \frac{1}{(4\pi)^2} \left( \alpha_1^u(t) + \alpha_2^u Y_L^2 + \alpha_3^u \text{Tr}(Y_u^2) + \alpha_4^u (D_2^t Y_d^2 D_L) + \alpha_5^u \text{Tr}(Y_d^2) \right) Y_u^2
\]

and next we add to Eq. (9) its hermitian conjugate. Then from the property (6b) the term with the \( U_R \) matrix cancels out and we obtain

\[
\frac{dY_R}{dt} + \left[ U_L \frac{dU^\dagger}{dt}, Y_L^2 \right] = \frac{1}{(4\pi)^2} \left( \alpha_1^d(t) + \alpha_2^d \text{Tr}(Y_u^2) + \alpha_3^d Y_d^2 + \alpha_4^d \text{Tr}(Y_d^2) \right) Y_d^2 + \alpha_5^d \{ V Y_d^2 V^\dagger, Y_u^2 \}
\]

**Step 3**

The equations for the diagonal elements give the renormalization group equations for the diagonal elements \((Y_u)_{ii}\) and \((Y_d)_{ii}\). We will not consider these equations here. The non-diagonal Eqs. (10) can be written in the following form

\[
\begin{align*}
\left( U_L \frac{dU^\dagger}{dt} \right)_{ij} &= \frac{1}{(4\pi)^2} \alpha_2^h H^u_{ij} (V Y_d^2 V^\dagger)_{ij} \equiv R^u_{ij}, \quad i \neq j, \\
\left( D_L \frac{dD^\dagger}{dt} \right)_{ij} &= \frac{1}{(4\pi)^2} \alpha_4^h H^d_{ij} (V^\dagger Y_u^2 V)_{ij} \equiv R^d_{ij}, \quad i \neq j
\end{align*}
\]

with no summation over \( i,j \). The matrices \( R^u_{ij} \) and \( R^d_{ij} \) are the right hand sides of Eqs. (11) and \( H^u_{ij} \) and \( H^d_{ij} \) are equal to

\[
H^u_{ij} = \frac{(Y_u, d)_{ij} + (Y_u, d)_{ii}}{(Y_u, d)_{jj} - (Y_u, d)_{ii}}
\]
The right hand side of Eqs. (11) is antihermitian, this guarantees the unitary evolution of the matrices \( U_L \) and \( D_L \).

**Step 3**

Now we derive the renormalization group equations for the elements of the CKM matrix \( V \).

With Eqs. (11) we obtain

\[
\frac{dU_L}{dt} D_L^1 = - R_u \cdot V, \quad U_L \frac{dD_L^1}{dt} = V \cdot R^d.
\]

(13)

After taking the sum of two equations in (13) we obtain

\[
\frac{dV}{dt} = \frac{dU_L}{dt} D_L^1 + U_L \frac{dD_L^1}{dt} = - R_u \cdot V + V \cdot R^d
\]

or written explicitly in terms matrix elements with indices

\[
\left( \frac{dV}{dt} \right)_{ik} = -R_{ii} V_{ik} - \sum_{j \neq i} R_{ij} V_{jk} + V_{ik} R^d_{kk} + \sum_{j \neq k} V_{ij} R^d_{jk}.
\]

(15)

Eqs. (13) and (15) are the RG equations for the CKM matrix, but they are not yet in the final form, because as seen from Eq. (13) they contain the diagonal elements of the matrix \( (R^u)^d \), which cannot be derived from Eqs. (11), they are unknown and Eq. (14) cannot be directly solved. It turns out, however, that the problem of unwanted terms can be overcome for the absolute values of the matrix elements of the CKM matrix which we will obtain in the later part of the paper for the approximate equations. The exact form of these equations is not very useful.

**III. APPROXIMATIONS**

**A. Hierarchy of the Yukawa couplings**

Let us notice now that Eqs. (14) and (15) are one loop equations i.e., the two loop and higher order loop terms are neglected. The loop order parameter of the RG equations is \( 1/(4\pi)^2 \sim \lambda^4 \), \( \lambda \sim |V_{12}| \), so it means that we have to keep only the terms of the relative order lower than \( \lambda^4 \) in Eqs. (14) and (15). We have the following hierarchy of the Yukawa couplings that follows from Eq. (9) and the experimental values of the quark masses [9]

\[
\frac{\langle V_u^2 \rangle_{11}}{\langle V_u^2 \rangle_{33}} = \left( \frac{m_u}{m_t} \right)^2 \sim \lambda^{16}, \quad \frac{\langle V_\mu^2 \rangle_{22}}{\langle V_\tau^2 \rangle_{33}} = \left( \frac{m_\mu}{m_t} \right)^2 \sim \lambda^8,
\]

\[
\frac{\langle V_d^2 \rangle_{11}}{\langle V_d^2 \rangle_{33}} = \left( \frac{m_d}{m_b} \right)^2 \sim \lambda^8, \quad \frac{\langle V_s^2 \rangle_{11}}{\langle V_s^2 \rangle_{33}} = \left( \frac{m_s}{m_b} \right)^2 \sim \lambda^4.
\]

(16)

Using the one loop approximation and Eq. (16) we have

\[
H^{u,d}_{ij} = \begin{cases} 
-1 & \text{for } i > j \\
+1 & \text{for } i < j \\
\text{not defined} & \text{for } i = j
\end{cases}
\]

(17)

and Eq. (15) can be written in the following form

\[
\frac{1}{V_{11}} \frac{dV_{11}}{dt} = (R_{11}^d - R_{11}^u) - \frac{1}{(4\pi)^2} \alpha_Y^u (\langle V_u^2 \rangle_{11} - (VY_2 V^\dagger)_{11}) - \frac{1}{(4\pi)^2} \alpha_Y^d (\langle V_u^2 \rangle_{11} - (V^\dagger Y_2 V)_{11}),
\]

(18a)

\[
\frac{1}{V_{33}} \frac{dV_{33}}{dt} = (R_{33}^d - R_{33}^u) + \frac{1}{(4\pi)^2} \alpha_Y^u (\langle V_d^2 \rangle_{33} - (VY_2 V^\dagger)_{33}) + \frac{1}{(4\pi)^2} \alpha_Y^d (\langle V_d^2 \rangle_{33} - (V^\dagger Y_2 V)_{33}),
\]

(18b)

\[
\frac{1}{V_{13}} \frac{dV_{13}}{dt} = (R_{13}^d - R_{13}^u) - \frac{1}{(4\pi)^2} \alpha_Y^u (\langle V_d^2 \rangle_{13} - (VY_2 V^\dagger)_{13}) + \frac{1}{(4\pi)^2} \alpha_Y^d (\langle V_d^2 \rangle_{13} - (V^\dagger Y_2 V)_{13}),
\]

(18c)

\[
\frac{1}{V_{31}} \frac{dV_{31}}{dt} = (R_{31}^d - R_{31}^u) + \frac{1}{(4\pi)^2} \alpha_Y^u (\langle V_d^2 \rangle_{31} - (VY_2 V^\dagger)_{31}) - \frac{1}{(4\pi)^2} \alpha_Y^d (\langle V_d^2 \rangle_{31} - (V^\dagger Y_2 V)_{11}).
\]

(18d)

\[\text{We explicitly write the RG equations for } V_{11}, V_{13}, V_{31} \text{ and } V_{33}, \text{ because these equations are sufficient for the determination of the evolution of all the observables of the CKM matrix, which are determined by 4 parameters. The equations for the 5 remaining elements can also be derived, but they are more complicated and we do not consider them here.}\]
B. Constants of the RG evolution

Let us notice that in Eqs. (18) on the right hand side the terms $R_{ii}^{u,d}$ are purely imaginary and the rest of the terms are real. From this we can obtain the quantities that do not evolve with energy. Observing that the following expressions are purely real

$$
\left(\begin{array}{c}
\frac{1}{V_{11}} \frac{dV_{11}}{dt} - \left(\frac{1}{V_{33}} \frac{dV_{33}}{dt}\right)^* + \left(\frac{1}{V_{13}} \frac{dV_{13}}{dt}\right)^* - \frac{1}{V_{31}} \frac{dV_{31}}{dt} \\
\frac{1}{V_{11}} \frac{dV_{11}}{dt} - \left(\frac{1}{V_{33}} \frac{dV_{33}}{dt}\right)^* - \frac{1}{V_{13}} \frac{dV_{13}}{dt} + \left(\frac{1}{V_{31}} \frac{dV_{31}}{dt}\right)^* \\
\frac{1}{V_{11}} \frac{dV_{11}}{dt} + \frac{1}{V_{33}} \frac{dV_{33}}{dt} + \left(\frac{1}{V_{13}} \frac{dV_{13}}{dt}\right)^* + \left(\frac{1}{V_{31}} \frac{dV_{31}}{dt}\right)^*
\end{array}\right) = \frac{d}{dt} \ln \left(\frac{V_{11} V_{33}^*}{V_{31} V_{31}^*}\right),
$$

(19a)

(19b)

(19c)

we get from Eqs. (18) that the following functions of the CKM matrix elements are constant during the RG evolution

$$
\text{Im} \left(\ln \left(\frac{V_{11} V_{33}^*}{V_{31} V_{31}^*}\right)\right) = \text{Im} \left(\ln \left(\frac{V_{11} V_{13}^*}{V_{33} V_{33}^*}\right)\right) = \text{Im} \left(\ln \left(V_{11} V_{33} V_{13} V_{31}^*\right)\right) = \text{const.}
$$

(20)

We thus see that though Eqs. (19) cannot be solved they give us an information that certain functions of the CKM matrix elements do not evolve.

The first constant is equal to the angle $\alpha$ of the unitary triangle, the second one is equal to one of the angles of the unitary triangle that is obtained from the multiplication of the first and third row of the CKM matrix. The third constant is the phase of the rephasing invariant $(V_{11} V_{33} V_{13} V_{31}^*)$, whose imaginary part $\text{Im}(V_{11} V_{33} V_{13} V_{31}^*)$ is the Jarlskog invariant [10]. The physical meaning of this constant is discussed in the next section.

C. RG equations for the moduli of the CKM matrix elements

Let us now find the RG equations for the squares of the absolute values of the CKM matrix elements $|V_{11}|^2$, $|V_{33}|^2$, $|V_{13}|^2$ and $|V_{31}|^2$. One can easily show that the following relation holds

$$
\frac{1}{|V_{ij}|^2} \frac{d|V_{ij}|^2}{dt} = 2 \text{Re} \left(\frac{1}{V_{ij}} \frac{dV_{ij}}{dt}\right).
$$

(21)

Now, using the fact that $R_{ii}^{u,d}$ are purely imaginary we obtain from Eqs. (13)

$$
\frac{1}{|V_{11}|^2} \frac{d|V_{11}|^2}{dt} = -\frac{2}{(4\pi)^2} \alpha_u^i \left(|Y_u|^2\right)_{11} - \frac{2}{(4\pi)^2} \alpha_d^i \left(|Y_d|^2\right)_{11},
$$

(22a)

$$
\frac{1}{|V_{33}|^2} \frac{d|V_{33}|^2}{dt} = \frac{2}{(4\pi)^2} \alpha_u^i \left(|Y_u|^2\right)_{33} + \frac{2}{(4\pi)^2} \alpha_d^i \left(|Y_d|^2\right)_{33},
$$

(22b)

$$
\frac{1}{|V_{13}|^2} \frac{d|V_{13}|^2}{dt} = -\frac{2}{(4\pi)^2} \alpha_u^i \left(|Y_u|^2\right)_{13} - \frac{2}{(4\pi)^2} \alpha_d^i \left(|Y_d|^2\right)_{13},
$$

(22c)

$$
\frac{1}{|V_{31}|^2} \frac{d|V_{31}|^2}{dt} = \frac{2}{(4\pi)^2} \alpha_u^i \left(|Y_u|^2\right)_{31} - \frac{2}{(4\pi)^2} \alpha_d^i \left(|Y_d|^2\right)_{31}.
$$

(22d)

These equations form the complete set of the RG equations for the CKM matrix that do not contain unknown functions. Eqs. (22) can be rewritten in terms of the squares of the absolute values of the CKM matrix

$$
\frac{1}{|V_{11}|^2} \frac{d|V_{11}|^2}{dt} = \frac{2}{(4\pi)^2} \alpha_u^i \left(|V_{13}|^2 (Y_b)^2 + |V_{12}|^2 (Y_s)^2 - \left(1 - |V_{11}|^2\right) (Y_d)^2\right)
$$

$$
+ \frac{2}{(4\pi)^2} \alpha_d^i \left(|V_{31}|^2 (Y_t)^2 + |V_{21}|^2 (Y_c)^2 - \left(1 - |V_{11}|^2\right) (Y_u)^2\right),
$$

(23a)

$$
\frac{1}{|V_{13}|^2} \frac{d|V_{13}|^2}{dt} = \frac{2}{(4\pi)^2} \alpha_u^i \left(-\left(1 - |V_{13}|^2\right) (Y_b)^2 + |V_{12}|^2 (Y_s)^2 + |V_{11}|^2 (Y_d)^2\right)
$$

$$
+ \frac{2}{(4\pi)^2} \alpha_d^i \left(-|V_{33}|^2 (Y_t)^2 - |V_{23}|^2 (Y_c)^2 + \left(1 - |V_{13}|^2\right) (Y_u)^2\right),
$$

(23b)
From which we obtain the top and bottom quarks lies in the hierarchy of the vacuum expectation values of the Higgs fields: 

\[ \frac{1}{|V_{31}|^2} \frac{d|V_{31}|^2}{dt} = -\frac{2}{(4\pi)^2} \alpha_s^u \left( |V_{33}|^2 (Y_b)^2 + |V_{32}|^2 (Y_s)^2 - \left( 1 - |V_{31}|^2 \right) (Y_d)^2 \right) \]

\[ - \frac{2}{(4\pi)^2} \alpha_s^d \left( \left( 1 - |V_{31}|^2 \right) (Y_t)^2 - |V_{21}|^2 (Y_c)^2 - |V_{11}|^2 (Y_u)^2 \right), \]

\[ \frac{1}{|V_{33}|^2} \frac{d|V_{33}|^2}{dt} = \frac{2}{(4\pi)^2} \alpha_s^u \left( \left( 1 - |V_{33}|^2 \right) (Y_b)^2 - |V_{32}|^2 (Y_s)^2 - |V_{31}|^2 (Y_d)^2 \right) \]

\[ + \frac{2}{(4\pi)^2} \alpha_s^d \left( \left( 1 - |V_{33}|^2 \right) (Y_t)^2 - |V_{23}|^2 (Y_c)^2 - |V_{13}|^2 (Y_u)^2 \right), \]

where we use an intuitive notation \((Y^2)_{11} = Y_a^2, (Y^2)_{22} = Y_c^2, (Y^2)_{33} = Y_b^2, (Y^2)_{11} = Y_d^2, (Y^2)_{22} = Y_s^2, (Y^2)_{33} = Y_t^2\). We can see from these equations that one needs to know the running of the \((Y^2)_{u,d}\) to obtain the evolution of the CKM matrix.

**IV. RG EQUATIONS FOR VARIOUS MODELS**

The hierarchy of the Yukawa couplings, Eq. (23) and the hierarchy of the CKM matrix, best seen in the Wolfenstein representation \[11\] permit to obtain the approximate, simpler form of Eqs. (23), that is compatible with the one loop approximation.

For the approximate form of the renormalization group equations we will consider two scenarios:

1. Standard Model or two Higgs extensions of the Standard Model and Minimal Supersymmetric Standard Model with \(v_1 \approx v_2\).

2. Two Higgs extension of the Standard Model and Minimal Supersymmetric Standard Model model with \(\frac{v_1}{v_2} = \frac{m_t}{m_b}\).

The approximation principle that we will follow consists in keeping in the RG equations the terms of relative order \(\lambda^3\) or lower, what is also consistent with the Wolfenstein parametrization of the CKM matrix.

In the first scenario the squares of the Yukawa couplings of the down quarks can be neglected in comparison to those of the up quarks. Also the squares of the Yukawa couplings of the charm and up quarks can be neglected in comparison to the top quark Yukawa coupling and thus the resulting renormalization group equations are considerably simplified and read

\[ \frac{1}{|V_{11}|^2} \frac{d|V_{11}|^2}{dt} = \frac{2}{(4\pi)^2} \alpha_s^d |V_{31}|^2 Y_t^2, \]

\[ \frac{1}{|V_{13}|^2} \frac{d|V_{13}|^2}{dt} = -\frac{2}{(4\pi)^2} \alpha_s^d |V_{33}|^2 Y_t^2, \]

\[ \frac{1}{|V_{31}|^2} \frac{d|V_{31}|^2}{dt} = -\frac{2}{(4\pi)^2} \alpha_s^d \left( 1 - |V_{31}|^2 \right) Y_t^2, \]

\[ \frac{1}{|V_{33}|^2} \frac{d|V_{33}|^2}{dt} = \frac{2}{(4\pi)^2} \alpha_s^d \left( 1 - |V_{33}|^2 \right) Y_t^2. \]

These equations were derived earlier and were explicitly solved in Ref. \[12,13\] so we will not discuss them in more detail.

The second scenario is physically very interesting if we assume that the origin of the hierarchy of masses between the top and bottom quarks lies in the hierarchy of the vacuum expectation values of the Higgs fields:

\[ \frac{v_1}{v_2} \approx \frac{m_t}{m_b} \]

from which we obtain

\[ (Y_u)_{33} \approx (Y_d)_{33}, \]

(26)
so the Yukawa couplings of the down quarks have to be kept in the equations. From the hierarchy in Eq. (16) one derives after straightforward calculations the following equations for the CKM matrix elements

\[ \frac{1}{|V_{11}|^2} \frac{d |V_{11}|^2}{dt} = \frac{2}{(4\pi)^2} \left( \alpha_2^d |V_{31}|^2 Y_t^2 + \alpha_4^u |V_{13}|^2 Y_b^2 + \alpha_4^u |V_{12}|^2 Y_s^2 \right), \]  
(27a)

\[ \frac{1}{|V_{13}|^2} \frac{d |V_{13}|^2}{dt} = -\frac{2}{(4\pi)^2} \left( \alpha_2^d |V_{33}|^2 Y_t^2 + \alpha_4^u \left( 1 - |V_{13}|^2 \right)^{1/2} Y_b^2 \right), \]  
(27b)

\[ \frac{1}{|V_{31}|^2} \frac{d |V_{31}|^2}{dt} = -\frac{2}{(4\pi)^2} \left( \alpha_2^d \left( 1 - |V_{31}|^2 \right)^{1/2} Y_t^2 + \alpha_4^u |V_{33}|^2 Y_b^2 \right), \]  
(27c)

\[ \frac{1}{|V_{33}|^2} \frac{d |V_{33}|^2}{dt} = \frac{2}{(4\pi)^2} \left( \alpha_2^d Y_t^2 + \alpha_4^u Y_b^2 \right) \left( 1 - |V_{33}|^2 \right), \]  
(27d)

Eqs. (24) form a complete set of equations for the CKM evolution for the first scenario and Eqs. (27) describe the CKM matrix evolution in the second scenario. Eqs. (24) require the knowledge of the evolution of the top quark Yukawa coupling \( Y_t(t) \) while for the second scenario we need the knowledge of the Yukawa coupling evolution of the top, bottom and strange quarks, \( Y_t(t), Y_b(t), Y_s(t) \).

### A. Solution for the first scenario

Eqs. (24) have been analyzed before [12] but we shall make some additional comments here. Let us notice that after subtracting Eq. (24c) from Eq. (24a) and Eq. (24b) from Eq. (24d) we obtain

\[ \frac{1}{|V_{11}|^2} \frac{d |V_{11}|^2}{dt} - \frac{1}{|V_{31}|^2} \frac{d |V_{31}|^2}{dt} = \frac{2}{(4\pi)^2} \alpha_2^d Y_t^2 \]  
(28a)

\[ \frac{1}{|V_{33}|^2} \frac{d |V_{33}|^2}{dt} - \frac{1}{|V_{13}|^2} \frac{d |V_{13}|^2}{dt} = \frac{2}{(4\pi)^2} \alpha_2^d Y_t^2 \]  
(28b)

Eqs. (28) can be easily integrated and from them it follows that the evolution of the ratios \( \frac{|V_{11}|^2}{|V_{31}|^2} \) and \( \frac{|V_{31}|^2}{|V_{13}|^2} \) is the same. Moreover, if we subtract Eq. (28d) from Eq. (28a) we get

\[ \frac{1}{|V_{11}|^2} \frac{d |V_{11}|^2}{dt} - \frac{1}{|V_{31}|^2} \frac{d |V_{31}|^2}{dt} - \frac{1}{|V_{33}|^2} \frac{d |V_{33}|^2}{dt} + \frac{1}{|V_{13}|^2} \frac{d |V_{13}|^2}{dt} = 0 \]  
(29)

so in the case of the first scenario the ratio

\[ \frac{|V_{11}|^2}{|V_{31}|^2} = \text{const} \]  
(30)

is also constant.

Next from Eqs. (24a), (24c) and Eqs. (24b), (24d) one can derive that the following ratios are constants during the RG evolution

\[ \frac{|V_{13}|^2}{1 - |V_{33}|^2} = \text{const}, \quad \frac{|V_{11}|^2}{1 - |V_{31}|^2} = \text{const}. \]  
(31)

Eqs. (28) and (31) allow to derive the exact evolution of the CKM matrix for the first scenario. From these equations it also follows that the evolution of the CKM matrix depends only on one function of energy.

### B. Discussion of the second scenario

The analysis of the RG equations for the second scenario Eqs. (27) is more involved. One can see that Eq. (27d), can be rewritten in the following way

\[ \frac{d}{dt} \left( \ln \frac{|V_{33}|^2}{1 - |V_{33}|^2} \right) = \frac{2}{(4\pi)^2} \left( \alpha_2^d Y_t^2 + \alpha_4^u Y_b^2 \right) \]  
(32)
so the evolution of $|V_{33}|^2$ is explicitly known.

Next, using the solution of Eq. (32) one can solve Eqs. (27b) and (27c) for $|V_{13}|^2$ and $|V_{31}|^2$. Eq. (27a) can be solved using the solutions of Eqs. (27b), (27c) and (27d). The full phenomenological analysis of Eqs. (27) is beyond the scope of this paper and will be published elsewhere.

V. GENERAL DISCUSSION AND CONCLUSIONS

We have obtained the full set of the one loop RG equations for the CKM matrix for three models: Standard Model, two Higgs extension of the Standard Model and the Minimal Supersymmetric Standard Model. They are given in Eqs. (22) or (23) and they form a set of coupled, non linear ordinary differential equations for the squares of the moduli of the CKM matrix elements. Eqs. (22) were derived from the RG equations for the quark Yukawa couplings by the elimination of the right diagonalizing matrices. One can see that Eqs. (22) depend only on $\alpha_u^i$ and $\alpha_d^j$, i.e., only on a subset of $\alpha_i^{u,d}$, $i = 1, \ldots, 5$ of the model dependent parameters of the RG equations for the Yukawa couplings, Eqs. (5). This means that the evolution of the CKM matrix can test only partially the evolution of the Yukawa couplings.

The most important result of the paper is the demonstration from Eq. (20) that the angle $\alpha$ of the unitarity triangle is constant during the evolution. The angle $\alpha$ is also equal to the one of the angles of the unitarity triangle obtained by the multiplication of the first and third rows of the CKM matrix and is also equal to the phase of the rephasing invariant $V_{11}V_{33}V_{13}^*V_{31}^*$, whose imaginary part is equal to the Jarlskog invariant of the CP violation. For the general case of the three models this is the only constant that can be obtained. On the other hand for the first scenario, i.e., for the Standard Model and its extensions with $v_1 \approx v_2$, we additionally have the constant in Eq. (30), which is equal to the ratio of two sides of the unitarity triangle that are adjacent to the angle $\alpha$. From this it follows that during the RG evolution the unitarity triangle is only rescaled and its shape does not change. This means that for the first scenario it is not possible to construct an asymptotic model with some simple, special form of the CKM matrix.

Let us observe that the constants in Eqs. (20) and (30) can be written in terms of the modified Buras-Wolfenstein parameters \( \bar{\rho} = (1 - \lambda^2/2)\rho \) and \( \bar{\eta} = (1 - \lambda^2/2)\eta \) in the following way

\[
\text{Im} \left( \ln \left( \frac{V_{11}V_{13}^*}{V_{31}V_{33}^*} \right) \right) = \frac{\eta}{\rho - (1 - \frac{1}{2}\lambda^2) (\rho^2 + \eta^2)} = \frac{\bar{\eta}}{\bar{\rho}(1 - \bar{\rho}) - \bar{\eta}^2} = \text{const.} \tag{33}
\]

and

\[
\left| \frac{V_{11}V_{13}^*}{V_{31}V_{33}^*} \right| = \sqrt{\frac{\bar{\rho}^2 + \bar{\eta}^2}{(1 - \bar{\rho})^2 + \bar{\eta}^2}} = \text{const.} \tag{34}
\]

Eq. (33) is valid for both scenarios, so we see that a relatively complicated function of the Wolfenstein parameters is a constant of the RG evolution for the general case. Eq. (34) is valid for the first scenario so from Eqs. (33) and (34) it follows that for the first scenario the parameters $\bar{\rho}$ and $\bar{\eta}$ are constants of the RG evolution.

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APPENDIX: PARAMETERS FOR THE VARIOUS MODELS

TABLE I: Coefficients $\alpha_{l}^{1}$, $l = u, d$ for various models.

| $\alpha_{l}^{1}(t)$          | SM and DHM                                                                 | MSSM                                                                 |
|----------------------------|----------------------------------------------------------------------------|---------------------------------------------------------------------|
| $\alpha_{u}^{1}(t) =$     | $-\left( \frac{17}{20}g_{1}^{2} + \frac{9}{4}g_{2}^{2} + 8g_{3}^{2} \right)$ | $-\left( \frac{13}{15}g_{1}^{2} + 3g_{2}^{2} + \frac{16}{3}g_{3}^{2} \right)$ |
| $\alpha_{d}^{1}(t) =$     | $-\left( \frac{1}{4}g_{1}^{2} + \frac{9}{4}g_{2}^{2} + 8g_{3}^{2} \right)$ | $-\left( \frac{7}{15}g_{1}^{2} + 3g_{2}^{2} + \frac{16}{3}g_{3}^{2} \right)$ |

TABLE II: Coefficients $\alpha_{l}^{k}$, $k = 2, \ldots, 5$, $l = u, d$ for various models.

| $l$ | $\alpha_{2}^{l}$ | $\alpha_{3}^{l}$ | $\alpha_{4}^{l}$ | $\alpha_{5}^{l}$ |
|-----|------------------|------------------|------------------|------------------|
| $u$ | $\frac{2}{7}b$   | $3$              | $\frac{3}{7}c$   | $3a$             |
| $d$ | $\frac{3}{7}c$   | $3a$             | $\frac{3}{7}b$   | $3$              |

$(a, b, c)_{SM} = (1, 1, -1)$
$(a, b, c)_{DHM} = (0, 1, \frac{1}{3})$
$(a, b, c)_{MSSM} = (0, 2, \frac{2}{3})$

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