Spin Angular Momentum Imparted by Gravitational Waves

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Abstract

Following the demonstration that gravitational waves impart linear momentum, it is argued that if they are polarized they should impart angular momentum to appropriately placed 'test rods' in their path. A general formula for this angular momentum is obtained and used to provide expressions for the angular momentum imparted by plane and cylindrical gravitational waves.

While the plane [1] and cylindrical [2] gravitational wave metrics are exact solutions of Einstein's field equations (and hence have a zero stress-energy tensor) one would expect them to carry energy. To prove that they do so it was shown [2,3,4] that they impart linear momentum to rest particles in their path. By the corresponding argument [5] for the electromagnetic waves, if they are polarized and possess a net spin, they could be expected to impart angular momentum to 'test rods' (the 1-dimensional extension of test particles) in their path. A formula for the linear momentum imparted to test particles in an arbitrary spacetime had been obtained [6] by using the extended pseudo-Newtonian \((e\psi^N)\)-formalism. It was seen that the formula gave the same result as the previous analysis for the plane gravitational wave [3] and an exact expression for the earlier approximate results for cylindrical gravitational waves [2,4]. Here we use the \(e\psi^N\)-formalism to obtain a formula for the angular momentum imparted to test rods in arbitrary spacetimes, which is then applied to the plane and cylindrical gravitational wave metrics.

The essence of the \(\psi^N\)-formalism [7] was that the tidal acceleration vector for a test rod, represented by the separation vector \(l^\mu\) moving along a world
line with timelike tangent vector \( t^\nu \),

\[
A^\mu = -R^\mu_{\nu\rho\pi} t^\nu l^\rho l^\pi, \quad (\mu, \nu, ... = 0, 1, 2, 3)
\]

defines a preferred frame along which \( A^\mu \) is maximized. In this frame the quantity whose directional derivative along the preferred direction is \( mA^\mu \) (where \( m \) is the rest-mass of a test particle) gives an operationally well defined relativistic generalization of the gravitational force, \( F^\mu \):

\[
l^\nu F^\mu_{,\nu} = mA^\mu.
\]

In the preferred frame one could replace the spacetime indices by purely spatial indices \( i, j, ... = 1, 2, 3 \).

Whereas the original formalism depended on the existence of a timelike Killing vector, the \( e\psi N \)-formalism [8] allowed for time varying spacetimes. In this case \( F^\mu \) can have a zero component as well. The quantity whose proper time derivative is \( F^\mu \) is the 4-vector momentum for the test particle. The spatial components of this vector give the momentum imparted to test particles as defined in the preferred frame (in which \( g_{0i} = 0 \)),

\[
p_i = m \int (\ln \sqrt{g_{00}})_i d\tau
\]

Thus the total energy of the test particle would be \( (m^2 - p_i p^i g_{ij})^{1/2} \). As such the significance of \( p_0 \), given by

\[
p_0 = m \int [(\ln A f)_{,0} - g^{ij} g_{ij,0} / 4A] d\tau,
\]

\[
A = (\ln \sqrt{-g})_{,0}, \quad g = \det(g_{ij}), \quad f = (g_{00})^{1/2},
\]

was not clear from the analysis.

Consider a test rod of length \( \lambda \) in the path of a gravitational wave whose preferred direction (in the above sense) is given by \( l^i \) in the preferred reference frame. Clearly the rod will acquire maximum angular momentum from the wave if it lies in the plane given by \( e_{ijkl} l^i \), where \( e_{\mu\nu\rho\pi} \) is the totally skew fourth rank tensor. Thus the spin vector will be given by [9].

\[
S^\mu = \frac{1}{2} e^{\mu jk\nu} e_{jkl} l^l p_\nu.
\]

In the preferred frame the spin vector will be proportional to \( l^i \), so that

\[
S^i = p_0 l^i.
\]
(Here the spin can be taken to be negative if the sign of the preferred direction is reversed.) Taking the magnitude of the spin vector the angular momentum can be defined. Thus the maximum angular momentum imparted to the test rod, obtained when it lies in the plane perpendicular to the preferred direction, is

\[ s = p_0 \lambda = m \lambda \int [(\ln A f),_0 - g^{ij}_0 g_{ij,0}/4A] d\tau. \]  

(7)

Hence the physical significance of the zero component of the momentum 4-vector would be that it provides an expression for the spin imparted to a test rod in an arbitrary spacetime.

The metric for plane fronted gravitational waves [4] is

\[ ds^2 = dt^2 - dx^2 - L^2(t, x)[(\cos h2\beta + \sin h2\beta \cos 2\vartheta)dy^2 + (\cosh 2\beta - \sinh 2\beta \cos \vartheta)dz^2 - 2 \sinh 2\beta \sin 2\vartheta d\vartheta dy dz], \]  

(8)

where \( L, \beta \) and \( \vartheta \) are arbitrary functions of \( u = t - x \). In the case \( \vartheta = 0, \pi/2 \) we get linearly polarized waves, while in the case \( \beta' = 0 \neq \vartheta' \) we get circularly polarized waves. In general they are called elliptically polarized [1]. The zero component of the momentum 4-vector for this line element is \( p_0 = \text{constant} \). Hence the spin angular momentum imparted to a test rod by it is also constant. Unfortunately the formalism does not tell us the value of this constant. It would have to be determined by extraneous physical considerations. Thus it would have to be zero for linearly polarized waves and non-zero for circularly polarized waves on physical grounds. Thus only consistency of our analysis has so far been demonstrated. The fact that \( \dot{p}_0 \) above is a direct consequence of the vacuum Einstein equations. It is to be noted that even with the vacuum Einstein equations holding, it is not necessary that \( p_0 = \text{constant} \). This is seen in the case of cylindrical gravitational waves which we discuss next.

The metric for cylindrical gravitational waves is

\[ ds^2 = e^{2(\gamma-\psi)} dt^2 - e^{2(\gamma-\psi)} d\rho^2 - \rho^2 e^{-2\psi} d\phi^2 - e^{2\psi} dz^2, \]

(9)

where \( \gamma \) and \( \psi \) are arbitrary functions of the time and radial coordinates, \( t \) and \( \rho \) and are given by

\[ \psi = AJ_0(\omega \rho) \cos \omega t + BY_0(\omega \rho) \sin \omega t, \]

\[ \gamma = \frac{1}{2} \omega \rho [(A^2 J_0 J'_0 - B^2 Y_0 Y'_0) \cos 2\omega t - AB(J_0 Y'_0 + Y_0 J'_0) \sin 2\omega t - 2(J_0 Y'_0 - Y_0 J'_0) \omega t], \]

(10)
where $A$ and $B$ are arbitrary constants corresponding to the strength of the cylindrical gravitational waves, $J_0$ and $Y_0$ are the Bessel function and the Neumann function of zero orders respectively. Here prime denotes differentiation with respect to $\omega \rho$, $\omega$ being the angular frequency. To avoid singularities at the source, following Weber and Wheeler [2,4], we take $B = 0$.

The zero component of the momentum 4-vector is

$$p_0 = -m[1 + AJ_0/\omega \rho J'_0] \ln |1 - 2\omega \rho AJ'_0 \cos \omega t| \tag{11}$$

Hence the spin angular momentum imparted to a test rod by it is

$$s = -m\lambda[(1 + AJ_0/\omega \rho J'_0) \ln |1 - 2\omega \rho AJ'_0 \cos \omega t|] + \text{constant.} \tag{12}$$

It is clear that for a zero amplitude wave ($A = 0$) there would be no angular momentum imparted to the test rod, as one would naturally require. Hence the constant of integration must be zero here. Notice that we must require that $A$ be such that $|2\omega \rho J'_0 A| < 1$ so that $p_0$ and $s$ remain non-singular.

There being only two exact gravitational wave solutions available we are limited in exploring the consequences of our analysis. The plane wave case highlighted the shortcoming of our analysis, in that it can not determine the constant of integration without reference to extraneous physical considerations. On the other hand, the cylindrical wave analysis demonstrates that physical consequences can be deduced using the $e\psi N$-formalism.

If there is a linear superposition of spins in the gravitational wave we would have to take the corresponding linear superposition of the resulting expressions for the angular momentum imparted by the waves to test rods. Thus, for a random superposition the net angular momentum transferred would be zero. However, if the waves have only one polarization then they must carry a net spin along the preferred direction (positive helicity) or opposite to it (negative helicity). In this case the spin would become apparent through the angular momentum it imparts to the test rod.

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