Hawking radiation from cubic and quartic black holes via tunneling of GUP corrected scalar and fermion particles

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We analyzed the effect of the generalized uncertainty (GUP) principle on the Hawking radiation from the hairy black hole in U(1) gauge-invariant scalar-vector-tensor theory by utilizing the semi-classical Hamilton-Jacobi method. To do so, we evaluate the tunneling probabilities and Hawking temperature for scalar and fermion particles for the given black hole with cubic and quartic interactions. For this purpose, we utilize the modified Klein-Gordon equation for bosons and also Dirac equations for fermions, respectively. Next, we examine that the Hawking temperature of the scalars and fermions only depend upon the properties of tunneling particles, i.e., angular momentum, energy and mass. Moreover, we present the corrected Hawking temperature of scalar and fermion particles looks similar in both interactions, but there are different mass and momentum relationships for scalar and fermion particles in cubic and quartic interactions.

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I. INTRODUCTION

As indicated by Einstein’s theory of general relativity, black holes (BHs) are very intense transcendental objects that twist spacetime so unequivocally that no light or matter can get out of their grasp. According to NASA, some primordial BHs have shaped before long after the Big Bang and may be of the estimate of a single molecule, anyhow, as enormous as a mountain. In 1960’s, John Wheeler and his collaborators propounded that BH “have no hair”, an allegory meaning that BHs trim of all complex features. In general relativity, no-hair theorem postulates that BHs are surprisingly simple objects, they can only describe by three exceptional classical parameters for them: their mass, electromagnetic charge and angular momentum [1]-[5]. The Wheelers hypothesis of no hair theorem was proven wrong in Einstein-Yang-Mills theory [6]. It is familiar that BHs have hair within the existence of Yang-Mills fields [7]-[9].

The expansions of general relativity may include scalar fields. In these theories no-hair theorem still exists. The primary no-scalar-hair theorem has connected to the massless scalar [10] and to the impartial scalar fields with a monotonous increasing self-inducting potential [11]. Heisenberg and Sujikawa [12] investigated the solution of hairy BH under U(1) gauge-invariant scalar-vector-tensor (SVT) theories for cubic and quartic order scalar-vector interactions. Herdeiro and Radu [13] have introduced a new family of BHs with scalar hair, that’s persistently associated to the Kerr family and gave a subjectively new case of hairy BHs entitled Kerr BHs with scalar hair. It is familiar that BHs can grow scalar hair in the presence of matter in their proximity [14, 15], complex scalar with time dependent stages [16]-[18], or on the off chance that when the asymptotics are cosmological or anti-de Sitter [19]-[21].

General relativity extensions to generalized gravitation theories extensively presents new parametric quantities apart from two tensor polarization [22]. The development of SVT theories with Horndeski theories was also carried out for both the U(1) gauge and non-gauge invariant cases [23]. The new parametric quantities emerging in SVT theories are important for the thermodynamical properties of BHs and expansion of universe.

Stephen Hawking (1970), presented his idea about black body radiations, which is known as Hawking radiation. All BHs losses their mass as apparitional quantum particles [24, 25]. Due to the continuous process of Hawking radiation a BH evaporates [26, 27]. Hawking evaluated the BH physics in a curved spacetime, under quantum field theory depending upon the Heisenberg uncertainty principle (GUP). The Hawking radiation through quantum
tunneling strategy of the radiated particles from a BH has been investigated [28]-[31]. Moreover, the Hawking radiation through tunneling phenomenon for numerous BHs has been investigated, abundantly, in (2+1) and (3+1)-dimensional [32-86]. Many authors have studied quantum tunneling approach for particles with different spins such as vectors (bosons), scalar and fermions through the horizons of various BHs, wormholes and other celestial objects, they also calculated their corresponding Hawking temperatures.

The GUP plays very important role to evaluate the quantum gravity effects (quantum corrections). To acquire the GUP effects on Hawking temperature, we will utilize corrections of Klein-Gordan and Dirac equations by considering quantum effects. Black holes are the main experimentation field to investigate the quantum gravity effects and lots of literature on BH thermodynamical properties to study the quantum gravity effects under GUP. The BH thermodynamics has moreover been explored within the system of GUP [87]-[93]. Nozari and Mehdipour [94] have investigated the tunneling phenomenon under GUP effects for Schwarzschild BH and also evaluated its modified tunneling rate. Övgün and Jusufi [95] have calculated the tunneling of massive spin-1 and spin-0 particles for a warped Dvali-Gabadadze-Porrati (DGP) gravity BH and also discussed the effects of GUP for both type of particles. Övgün, Javed and Ali [96] have found the tunneling rate of charged massive bosons for various types of BHs surrounded by the perfect fluid in Rastall theory. Sharif and Javed [97] have considered tunneling phenomenon for fermion particles through the event horizons for the study of Hawking temperature. They also discussed the Hawking radiation through tunneling phenomenon for fermion particles from a pair of charged accelerating and rotating BH around NUT parameter [98]. Further, they analyzed the corresponding Hawking temperature of these BHs. Sharif and Javed [99] have also investigated the quantum corrections for regular BHs, i.e., Bardeen and ABGB.

In the continuation of the previous work, we will investigate the quantum corrections of massive spin-0 and fermion particles by using GUP effects. A brief outline of paper is given as follows: In Section II, we provide the detail of hairy BH in the existence of U(1) gauge scalar-vector-tensor theories. In Section III, we explore the GUP-corrected Klein-Gordan equation for cubic and quartic interactions and examine the quantum corrections of Hawking temperature for massive scalar particles for BH solution surrounded by SVT theory. Whereas, Section IV is devoted to the same analysis with Dirac equation for fermion particles. Finally, the results of this paper are summarized in Section V.

### II. HAIRY BLACK HOLE UNDER GAUGE-ININVARIANT SCALAR-VVECTOR-TENSOR THEORY

Black holes plays a very essential role in general relativity. The Gibbons solution which depicts a Reissner-Nordstarm BH with a non-trivial expansion field is considered as the first hairy BH. Whereas numerous ways run out scalar field hair in asymptotically flat spacetime [100]. However, more hairy BHs have gotten in models motivated by string theory and counting expansion, curvature correction and many other. By considering positive or negative, cosmological constant asymptotically (anti)-de-Sitter hairy BH is obtained. Hairy BH generally occur in natural models. However, large hairy BHs are regularly unstable and during perturbation they loose their hair, although the stable one are exceptionally little. The hairy BHs yields as cosmologically large, which conflicts the observation [101]. It will be very interesting to evaluate the tunneling probability and corrected Hawking temperature from such a BH. The metric of hairy BH for cubic and quartic interactions is defined as [12]

\[
ds^2 = -Z(r)dt^2 + h^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

where for cubic interaction

\[
Z(r) = 1 - \frac{2M}{r} + \frac{Q^2}{2M_{pl}^2r^2} + \frac{3\beta_3^2 Q^4}{14M_{pl}^4r^8} + O\left(\frac{1}{r^9}\right), \tag{1}
\]

\[
h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{2M_{pl}^2r^2} - \frac{2\beta_3^2 Q^4}{7M_{pl}^4r^8} + O\left(\frac{1}{r^9}\right),
\]

For quartic interaction

\[
Z(r) = 1 - \frac{2M}{r} + \frac{Q^2}{2M_{pl}^2r^2} - \frac{2\beta_4^2 Q^2}{M_{pl}^2r^4} + \frac{2\beta_4^2 MQ^2}{M_{pl}^2r^5} - \frac{3\beta_4^2 Q^4}{5M_{pl}^4r^6} + \frac{256\beta_2^2 M^2 Q^2}{7M_{pl}^2r^7} + \frac{3Q^2(M_{pl}^6 Q^2 \beta_3^2 - 28\beta_2^2 Q^2 - 256\beta_2^2 M^2 M_{pl}^2)}{14M_{pl}^4r^8},
\]

\[
h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{2M_{pl}^2r^2} - \frac{2\beta_4^2 MQ^2}{M_{pl}^2r^5} + \frac{2\beta_4^2 Q^4}{5M_{pl}^4r^6} - \frac{2Q^2(\beta_3^2 Q^2 - 64\beta_2^2 M^2)}{7M_{pl}^2r^8}. \tag{2}
\]
III. SCALAR PARTICLE TUNNELING VIA MODIFIED KLEIN-GORDAN EQUATION

This section is based on the effects of GUP on the tunneling of massive scalar particles from the hairy BH solutions for cubic and quartic interactions.

A. Cubic Interaction for Scalar Particles

This section is devoted to study the tunneling phenomenon of scalar particles for given BH with cubic interaction and analyze the corresponding corrected Hawking temperature. For this purpose, the modified Klein-Gordan equation is given by [102]

\[-(\hbar^2)^2 \partial^2 \phi + [(-\hbar^2)^2 \partial^2 \phi + m_0^2] [1 - 2\beta(\hbar^2)^2 \partial^2 \phi] \Phi, \quad (3)\]

The wave function \(\Phi\) for scalar field is given as

\[\Phi(t, r, \theta, \varphi) = \left[ \frac{1}{\hbar} I(t, r, \theta, \varphi) \right], \quad (4)\]

we consider just first order term in \(\hbar\), so the above Eq.(3) becomes

\[
\frac{1}{Z(r)} \left( \partial_t I \right)^2 = \left[ h(r) (\partial_t I)^2 + \frac{1}{r^2} (\partial_\theta I)^2 + \frac{1}{r^2 \sin^2 \theta} (\partial_\phi I)^2 + m_0^2 \right] \times \\
\left[ 1 - 2\beta \left( h(r) (\partial_t I)^2 + \frac{1}{r^2} (\partial_\theta I)^2 + \frac{1}{r^2 \sin^2 \theta} (\partial_\phi I)^2 + m_0^2 \right) \right]. \quad (5)\]

To solve this equation, the particle’s action is defined as

\[I = -Et + W(r, \theta) + j\varphi. \quad (6)\]

Here, \(W(r, \theta)\) cannot be parted as \(W(r)\Theta(\theta)\). For sake of simplicity, we can fix the angle \(\theta\) at a particular value of \(\theta_0\).

At \(\theta = \theta_0\), Eq.(5) gets the following form

\[\bar{A}(\partial_t W)^4 + \bar{B}(\partial_t W)^2 + \bar{C} = 0, \quad (7)\]

Here

\[
\bar{A} = -2\beta h^2(r), \quad \bar{B} = h^2(r) \left( 1 - \frac{4\beta j^2}{r^2 \sin^2 \theta} - 4\beta m_0^2 \right), \\
\bar{C} = m_0^2 + \frac{j^2}{r^2 \sin^2 \theta} - \frac{2\beta j^4}{r^4 \sin^4 \theta} + \frac{4\beta m_0^2 j^2}{r^2 \sin^2 \theta} - 2\beta m_0^4 - \frac{E^2}{Z(r)}. \quad (8)\]

After solving Eq.(7), we get

\[W_{\pm}(r) = \pm \int \frac{dr}{\sqrt{Z(r)}} \left[ 1 + \beta \left( m_0^2 + \frac{E^2}{Z(r)} + \frac{j^2}{r^2 \sin^2 \theta} \right) \right] \times \\
\sqrt{E^2 - Z(r) \left( m_0^2 - \frac{j^2}{r^2 \sin^2 \theta} \right) - 2\beta Z(r) \left( \frac{j^4}{r^4 \sin^4 \theta} + 2m_0^2 Z(r) - \frac{j^2}{r^2 \sin^2 \theta} + m^4 \right)}. \quad (8)\]

After ignoring higher order terms of \(\beta\), we solve above integral to calculate the imaginary part of function at event horizon

\[\text{Im} W(r_+) = \pm \pi \left( \frac{E r_+^2}{\Delta_\pi (r_+)} \right) \left( 1 + \beta \Xi \right), \quad (9)\]
where
\[ \Xi = \left( m^2 + \frac{E^2}{F'(r_+)} + \frac{\hbar^2 \csc^2\theta}{r^2_+} \right), \]

Here, \( W_+ \) and \( W_- \) are the radial functions for the outgoing and incoming particles, respectively. The tunneling probability for scalar particles at \( r = r_+ \) is given as follows
\[ \tilde{\Gamma} = \frac{\tilde{\Gamma}_{(out)}}{\tilde{\Gamma}_{(in)}} = \frac{\exp\left(-\frac{2}{\hbar}(1mW_+)\right)}{\exp\left(-\frac{2}{\hbar}(1mW_-)\right)} = \exp\left[-\frac{4}{\hbar}(1mW_+)\right], \]
\[ = \exp\left[-\frac{4\pi r^2_+}{\hbar\Delta_{r_+}}(E) \times (1 + \tilde{\Xi}) \right]. \tag{10} \]

For \( \hbar = 1 \) and utilizing Boltzmann factor \( \tilde{\Gamma} = \exp\left(\frac{E}{T_H}\right) \), the modified temperature can be obtained as
\[ T_H' = \frac{\Delta_{r_+}(1)}{4\pi r^2_+(1 + \tilde{\Xi})} = T_H(1 - \tilde{\Xi}), \tag{11} \]

where
\[ T_H = \frac{1}{4\pi r^2_+} \left[ \frac{2M}{r^2_+} - \frac{Q^2}{M^2plr^3_+} - \frac{12Q^4\beta^3_3}{7M^2plr^9_+} - O\left(\frac{1}{r^1_+}\right) \right], \]
which is the original Hawking temperature of a corresponding BH.

B. Quartic Interaction for Scalar Particles-I

Following the same procedure given in the preceding subsection A, for this line element (1), we can calculate the corrected Hawking temperature with the effects of quantum gravity for this BH. The modified Hawking temperature for quartic interactions can be deduced as
\[ T_H' = \frac{\Delta_{r_+}(1)}{4\pi r^2_+(1 + \tilde{\Xi})} = T_H(1 - \tilde{\Xi}), \tag{12} \]

where the original Hawking temperature is
\[ T_H = \frac{1}{4\pi r^2_+} \left[ \frac{2M}{r^2_+} - \frac{Q^2}{M^2plr^3_+} - \frac{256MQ^2\beta^3_3}{M^2plr^9_+} - \frac{10\beta_4MQ^2}{M^2plr^6} + \frac{8\beta_4Q^2}{M^2plr^5} + \frac{18\beta_4Q^4}{5M^4plr^7} - \frac{12Q^2(M^2plQ^2\beta^3_3 - 28\beta^2_3Q^2 - 256\beta_4M^2M^2pl)}{7M^4plr^9} \right]. \]

C. Quartic Interaction for Scalar Particles-II

The line element of BH for Quartic interactions is defined as
\[ ds^2 = -Z(r)dt^2 + h^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{13} \]

where
\[ Z(r) = (1 - \mu) \left( \frac{r}{r_+} - 1 \right) - \frac{1 - 2\mu + 12\beta^2_3\mu^2(1 - \mu) + 8\beta_4\mu(1 - 2\mu)}{1 + 8\beta_4} \left( \frac{r}{r_+} - 1 \right)^2, \]
\[ h(r) = (1 - \mu) \left( \frac{r}{r_+} - 1 \right) - \frac{1 - 2\mu - 4\beta^2_3\mu^2(1 - \mu) + 8\beta_4\mu(1 - 2\mu)}{1 + 8\beta_4} \left( \frac{r}{r_+} - 1 \right)^2. \]
where \( \tilde{\beta}_3 = \frac{\beta_3 M^2_{pl}}{r^+_1} \) and \( \tilde{\beta}_4 = \frac{\beta_4 M^2_{pl}}{r^+_1} \).

Using the same formalism as defined earlier in (A) for this line element, we obtain the corrected Hawking temperature with the effect of quantum gravity for this BH. The modified Hawking temperature for quartic interactions is deduced as

\[
T'_H = \frac{\Delta_\mu_2 (r_+)}{4\pi r^2_+ (1 + \tilde{\beta}\Sigma)} = T_H (1 - \tilde{\beta} \Xi),
\]

where the original Hawking temperature is

\[
T_H = \frac{1}{r^2_+ (r^+_1 + 8 M^2_{pl} \beta_4)^2} \left[ 16 \mu M^2_{pl} r_+ \left( -3 (\mu - 1) \mu M^2_{pl} \beta^2_3 \right) + 2 (1 - 2 \mu) (r - r_+) \right] (r^4_+ + 8 M^2_{pl} \beta_4) - 32 M^2_{pl} \beta_4 r^2_+ (1 - 2 \mu) (r - r_+) - 4 \mu M^2_{pl} (r - r_+) \times (r^4_+ + 8 M^2_{pl} \beta_4)^2 - 2 r_+^2 (r^4_+ + 8 M^2_{pl} \beta_4) (1 - 2 \mu) - 4 \mu M^2_{pl} (r - r_+) \times (1 - 2 \mu) \beta_4 \right].
\]

IV. THE FERMION PARTICLES TUNNELING VIA MODIFIED DIRAC EQUATION

In this section, we will focus on studying the effects of GUP on the tunneling procedure of fermion particles from the hairy BH for cubic and quartic interactions.

A. Cubic Interactions for Fermion Particles

In order to investigate the quantum tunneling of fermion particles for hairy BH, the Dirac equation is given as follows [103]

\[
\{ i \tilde{\sigma}^\mu (x) \left [ \partial_\mu - \tilde{\Gamma}_\mu (x) \right ] \} \tilde{\Psi} (x) = \frac{m_0}{\hbar} \tilde{\Psi} (x).
\]

The Dirac spinor wave function \( \tilde{\Psi} (x) \) has two sign states, the +ive and -ive energy eigenstates. The spacetime dependent Dirac matrices \( \tilde{\sigma}^\mu \) can be defined in term of constant Dirac matrices, \( \tilde{\sigma}^i \), by utilizing triads, \( \epsilon^\mu_{(i)} (x) \), given by

\[
\tilde{\sigma}^\mu = \epsilon^\mu_{(i)} (x) \tilde{\sigma}^i,
\]

where \( \tilde{\sigma}^i \) can be defined as

\[
\tilde{\sigma}^i = \left ( \tilde{\sigma}^0, \tilde{\sigma}^1, \tilde{\sigma}^2, \tilde{\sigma}^3 \right ),
\]

with

\[
\tilde{\sigma}^0 = \sigma^3, \quad \tilde{\sigma}^1 = i \sigma^1, \quad \tilde{\sigma}^2 = i \sigma^2, \quad \tilde{\sigma}^3 = i \sigma^3,
\]

where \( \sigma^1, \sigma^2, \sigma^3 \) represent Pauli matrices and \( \tilde{\Gamma}_\mu (x) \) can be defined as

\[
\tilde{\Gamma}_\mu (x) = \frac{1}{4} g^\lambda_{\alpha \alpha} \left ( \epsilon^\lambda_{(i)} \epsilon^\alpha_{(i)} - \tilde{\Gamma}^\alpha_{(i) \mu} \right ) s^\lambda (x).
\]

Here, \( \tilde{\Gamma}^a_{(i) \mu} \) represents Christoffel symbol and \( g_{\mu \nu} (x) \) is metric tensor and can be defined with triads as follows

\[
g_{\mu \nu} (x) = \epsilon^i_{(i)} \epsilon^j_{(j)} \eta_{ij}, \quad \mu, \nu, i, j = 0, 1, 2, 3,
\]
where the tensor $\eta_{ij}$ represents Minkowski spacetime and $s^{\lambda\nu}(x)$ is spin operator given as follows

$$s^{\lambda\nu}(x) = \frac{1}{2} \left[ \tilde{\sigma}^\lambda(x), \tilde{\sigma}^\nu(x) \right].$$ \hspace{1cm} (21)

The modified Dirac equation is defined as

$$-i\gamma^0(x)\partial_0 \Psi = \left( i\gamma^i(x)\partial_i - i\gamma^\mu(x)\Gamma_\mu - \frac{m_0}{\hbar} \right) \left( 1 + \gamma^2 \partial_i \partial^i - \gamma^2 m_0^2 \right) \Psi,$$ \hspace{1cm} (22)

the above equation can be rewrite in the following form as

$$\left[ i\gamma^0(x)\partial_0 + i\gamma^i(x)(1 - \gamma^2 m_0^2)\partial_i + i\gamma^2 \gamma^i(x)\partial_i \partial^i - \frac{m_0}{\hbar} \left( 1 + \gamma^2 \partial_i \partial^i - \gamma^2 m_0^2 \right) \right] \Psi,$$ \hspace{1cm} (23)

where $\Psi$ is generalized Dirac spinor. The wave function can be defined as

$$\Psi(x) = \exp \left( i\frac{\gamma^0}{\hbar} \tilde{I}(t, r, \theta, \phi) \right) \left( A(t, r, \theta, \phi) B(t, r, \theta, \phi) \right),$$ \hspace{1cm} (24)

where $A(t, r, \theta, \phi)$ and $B(t, r, \theta, \phi)$ are arbitrary functions of spacetime coordinates. Using Eq. (24) in Eq. (23), we get the following set of equations for first order in $\hbar$ and $\beta$, i.e.,

$$A \left[ \frac{1}{\sqrt{Z(r)}} \left( \frac{\partial I}{\partial t} \right) + m_0 \left( 1 - \gamma^2 m_0^2 \right) + h(r)\beta m_0 \left( \frac{\partial I}{\partial r} \right)^2 + \frac{\gamma^2 m_0}{r} \left( \frac{\partial I}{\partial \theta} \right)^2 + \frac{\gamma^2 m_0}{r^2 \sin^2 \theta} \left( \frac{\partial I}{\partial \phi} \right)^2 \right]$$

$$B \left[ i\sqrt{h(r)} \left( 1 - \gamma^2 m_0^2 \right) \left( \frac{\partial I}{\partial r} \right)^2 + \frac{\beta h(r)}{r} \left( \frac{\partial I}{\partial \theta} \right)^2 + \frac{\beta h(r)}{r \sin \theta} \left( \frac{\partial I}{\partial \phi} \right)^2 \right]$$

$$+ i\beta \sqrt{h(r)} \frac{\partial I}{\partial r} \left( \frac{\partial I}{\partial \theta} \right)^2 + \frac{\beta \sqrt{h(r)}}{r^2 \sin^2 \theta} \left( \frac{\partial I}{\partial \phi} \right)^2 + \frac{\beta \sqrt{h(r)}}{r^2 \sin^2 \theta} \left( \frac{\partial I}{\partial \phi} \right)^2$$

$$+ \frac{\beta}{r^2 \sin^2 \theta} \left( \frac{\partial I}{\partial \phi} \right)^2 = 0,$$ \hspace{1cm} (25)

$$A \left[ -i\sqrt{h(r)} \left( 1 - \gamma^2 m_0^2 \right) \left( \frac{\partial I}{\partial r} \right)^2 + \frac{\beta h(r)}{r} \left( \frac{\partial I}{\partial \theta} \right)^2 + \frac{\beta h(r)}{r \sin \theta} \left( \frac{\partial I}{\partial \phi} \right)^2 \right]$$

$$- i\beta \sqrt{h(r)} \left( \frac{\partial I}{\partial r} \right)^2 + \frac{\beta \sqrt{h(r)}}{r^2 \sin^2 \theta} \left( \frac{\partial I}{\partial \phi} \right)^2 + \frac{\beta \sqrt{h(r)}}{r^2 \sin^2 \theta} \left( \frac{\partial I}{\partial \phi} \right)^2$$

$$+ \frac{\beta}{r^2 \sin^2 \theta} \left( \frac{\partial I}{\partial \phi} \right)^2 - i\beta h(r) \sqrt{h(r)} \left( \frac{\partial I}{\partial r} \right)^3 \right] + B \left[ \frac{1}{\sqrt{Z(r)}} \left( \frac{\partial I}{\partial t} \right) - m_0 \left( 1 - \gamma^2 m_0^2 \right) \right.$$

$$- h(r)\beta m_0 \left( \frac{\partial I}{\partial r} \right)^2 - \frac{\beta m_0}{r^2} \left( \frac{\partial I}{\partial \theta} \right)^2 - \frac{\beta m_0}{r^2 \sin^2 \theta} \left( \frac{\partial I}{\partial \phi} \right)^2 \right] = 0.$$ \hspace{1cm} (26)
When determinant of coefficient matrix equals to zero, we get the following non-trivial solution

\[
\frac{1}{Z(r)} \left( \frac{\partial I}{\partial t} \right)^2 - h(r) \left( \frac{\partial I}{\partial r} \right)^2 - \frac{1}{r^2} \left( \frac{\partial I}{\partial \theta} \right)^2 - \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial I}{\partial \phi} \right)^2 = m_0^2
\]

\[+ \tilde{\beta} \left[ - \frac{2}{r^4} \left( \frac{\partial I}{\partial \theta} \right)^2 - 2h^2(r) \left( \frac{\partial I}{\partial r} \right)^4 - \frac{2}{r^2 \sin^4 \theta} \left( \frac{\partial I}{\partial \phi} \right)^4 - \frac{4h(r)}{r^2} \left( \frac{\partial I}{\partial r} \right)^2 \left( \frac{\partial I}{\partial \phi} \right)^2 \right] = 0.
\]  

(27)

By using separation of variables technique, we assume

\[
I(t, r, \theta, \phi) = -Et + W(r) + \Theta(\theta, \phi),
\]

(28)

where $E$ represents energy and $j$ represents angular momentum of the particle, and $J_\theta = \partial_\theta \Theta, J_\phi = \partial_\phi \Theta$.

\[
W_{\pm}(r) = \pm \int \frac{1}{\sqrt{Z(r)h(r)}} \sqrt{E^2 - Z(r) \left( \frac{\partial^2 I}{\partial r^2} + \frac{J_\theta^2 + J_\phi^2 \csc^2 \theta}{r^2} \right)}
\]

\[\times \left[ 1 + \frac{\tilde{\beta}}{Z(r)} \left( \frac{2E^2 m_0^2 Z(r) - E^4}{E^2 - Z(r) \left( \frac{\partial^2 I}{\partial r^2} + \frac{J_\theta^2 + J_\phi^2 \csc^2 \theta}{r^2} \right)} \right) \right] dr.
\]

(29)

Neglecting the higher terms of $\tilde{\beta}$ and obtain the result at $r = r_+$ after solving the above integral

\[
\text{Im} W(r_+) = \pm \pi \left( \frac{E r_+^2}{\Delta_\tau (r_+)} \right) (1 + \tilde{\beta} \Sigma),
\]

(30)

where

\[
\Sigma = \frac{1}{Z'(r_+)} \left( \frac{2E^2 m_0^2 Z'(r_+) - E^4}{E^2 - Z'(r_+) \left( \frac{\partial^2 I}{\partial r^2} + \frac{J_\theta^2 + J_\phi^2 \csc^2 \theta}{r_+} \right)} \right).
\]

The tunneling rate of scalar particles at $r = r_+$ can be calculated as

\[
\Gamma = \frac{\Gamma_{(out)}}{\Gamma_{(in)}} = \exp \left[ -\frac{2}{\hbar} (\text{Im} W_+) \right] = \exp \left[ -\frac{4}{\hbar} \text{Im} W_+ \right],
\]

(31)

\[
= \exp \left[ -\frac{4\pi r_+^2}{\hbar \Delta_\tau (r_+)} (E) \times (1 + \tilde{\beta} \Sigma) \right].
\]

The modified Hawking temperature can be obtained as

\[
T'_H = \frac{\Delta_\tau (r_+)}{4\pi r_+^2 (1 + \tilde{\beta} \Sigma)} = T_H (1 - \tilde{\beta} \Sigma),
\]

(32)

where

\[
T_H = \frac{1}{4\pi r_+^2} \left[ \frac{2M}{r_+^2} - \frac{Q^2}{M^2 r_+^2} + \frac{12Q^4 \beta_3^2}{7M^2 r_+^9} - O \left( \frac{1}{r_+^{10}} \right) \right],
\]

which is the original Hawking temperature of a corresponding BH.
B. Quartic Interaction for Fermion Particles-I

Following the same process, given in the preceding Section (A) for line element (2) of quartic interactions, we calculate the corrected Hawking temperature under the effect of quantum gravity. The modified Hawking temperature for quartic interactions of fermion particles is deduced as

\[
T_H' = \frac{\Delta_E (r_+)}{4\pi r_+^2 (1 + \beta \Sigma)} = T_H(1 - \beta \Sigma),
\]

where

\[
T_H = \frac{1}{4\pi r_+^2} \left[ \frac{2M}{r_+^2} - \frac{Q^2}{M_{pl}^2 r_+^2} - \frac{256MQ^2 \beta_4^2}{M_{pl}^2 r_+^8} - \frac{10\beta_4 MQ^2}{M_{pl}^2 r_+^5} + \frac{8\beta_4 Q^2}{5M_{pl}^4 r_+^7} + \frac{18\beta_4 Q^4}{5M_{pl}^4 r_+^7} \right.
\]

\[
- \frac{12Q^2(M_{pl}^2 \beta_3^2 - 28\beta_3^2 Q^2 - 256\beta_4^2 M_{pl}^2 \beta_3^2)}{7M_{pl}^4 r_+^9},
\]

which is the original Hawking temperature of the corresponding BH.

C. Quartic Interaction for Fermion Particles-II

By following the same procedure, given in the preceding Section (A) for line element (13) of quartic interactions, we calculate the corrected Hawking temperature under the effect of quantum gravity. The modified Hawking temperature for quartic interactions of fermion particles is deduced as

\[
T_H' = \frac{\Delta_E (r_+)}{4\pi r_+^2 (1 + \beta \Sigma)} = T_H(1 - \beta \Sigma),
\]

where

\[
T_H = \frac{1}{r_+^2 (r_+^2 + 8M_{pl}^2 \beta_4)^2} \left[ -16\mu M_{pl}^2 r_+ \left( -3(\mu - 1)\mu M_{pl}^2 \beta_3^2 
\right. \right.
\]

\[
+ 2(1 - 2\mu) \beta_4 (r - r_+)(r_+^2 + 8M_{pl}^2 \beta_4) + 32M_{pl}^2 \beta_4 r_+^4 (1 - 2\mu)(r - r_+)
\]

\[
+ 4\mu M_{pl}^2 r_+ \left( -3(\mu - 1)\mu M_{pl}^2 \beta_3^2 + 2(1 - 2\mu) \beta_4 \right) + r_+ (\mu - 1)(r - r_+)
\]

\[
\times \left( r_+^2 + 8M_{pl}^2 \beta_4 \right)^2 + 2r_+^2 (r_+^2 + 8M_{pl}^2 \beta_4)(1 - 2\mu) + 4\mu M_{pl}^2 (r - r_+)
\]

\[
\times \left( -3(\mu - 1)\mu M_{pl}^2 \beta_3^2 + 2(1 - 2\mu) \beta_4 \right) \right].
\]

which is the original Hawking temperature of the corresponding BH.

V. CONCLUSION

In this work, we have analyzed the tunneling probability and corrected temperature \( T_H' \) of hairy BH for cubic and quartic interactions. For this purpose, we utilized Hamilton-Jacobi ansatz and WKB approximation and considered the modified Klein-Gordon and Dirac equations for scalar and fermion particles, respectively. We calculated corrected Hawking temperature for scalar and fermion particles and radiated temperature looked over preserved energy and charge. By utilizing modified Klein-Gordon and Dirac equations, the corrected Hawking temperature \( T_H' = T_H(1 - \beta \Sigma) \) and \( T_H = T_H(1 - \beta \Sigma) \) values have been computed given in Eqs.(11) and (32) by considering the effects of quantum gravity. We have concluded that the quantum gravity effects increased the Hawking temperature. We demonstrated that the corrected Hawking temperatures does not just based on the properties of the BH, yet in addition depend upon the quantum numbers, i.e., angular momentum, mass and energy of the transmitted particles. The expression of corrected Hawking temperature in Eqs.(11) and (12) for scalar particle look similar to the expression in Eq.(32) and (33) for fermion particles but the angular momentum and mass are not same.
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