ON THE SLOPES OF THE LATTICE OF SECTIONS OF HERMITIAN LINE BUNDLES

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Abstract. In this paper we study the distribution of successive minima of global sections of powers of a metrized ample line bundle on a variety over a number field. We develop criteria for there to exist a measure on the real line describing the limiting behavior of this distribution as one considers increasing powers of the bundle. When this measure exists, we develop methods for determining it explicitly. We present applications to the distribution of Petersson norms of cusp forms of increasing weight for $\text{SL}_2(\mathbb{Z})$ and to the minimal sup norm of algebraic functions on adelic subsets of curves arising in capacity theory.

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1. INTRODUCTION

This paper is motivated by a number of connected problems concerning the size of sections of powers of an ample metrized line bundle $L$ on an integral projective variety $X$ over a number field $K$.

In §2 we recall the definition of the height $\lambda(s)$ of a non-zero section $s$ of $H^0(X, L^\otimes n)$ with respect to an adelic metric on $L$. A basic application of arithmetic intersection theory is to show the existence of “small” non-zero sections $s$, these being those of large height. For example, arithmetic amplitude theorems produce such sections via
Minkowski’s theorem in the geometry of numbers concerning the smallest non-zero vector in a Euclidean lattice having a known co-volume and rank.

A more precise question arising in the geometry of numbers is to study the distribution of the successive minima of a lattice. The \(i^{th}\) successive minima is the smallest real number such that a ball of this radius contains \(i\) linearly independent elements of the lattice. In the context of the metrized line bundles \(L\) as above, the \(i^{th}\) naive successive minimal \(\lambda_{i,n}\) of \(H^0(X, L^\otimes n)\) is the largest real number such that the space \(H^0(X, L^\otimes n)^{\geq \lambda_{i,n}}\) of all sections \(0 \neq s \in H^0(X, L^\otimes n)\) for which \(\lambda(s) \geq \lambda_{i,n}\) contains \(i\) linearly independent sections.

The goal of this paper is to study the distribution of the successive minima \(\{\lambda_{i,n}\}_i\) as \(n \to \infty\) via a number of techniques. This will be done by considering the sequence of probability measures

\[\nu_n = \frac{1}{r_n} \sum_{i=1}^{r_n} \delta_{\lambda_{i,n}}\]

where \(\delta_r\) is the Dirac measure supported on the real number \(r\) and \(r_n\) is the rank of \(H^0(X, L^\otimes n)\). The main questions are to study when the \(\nu_n\) converge weakly as \(n \to \infty\) to a measure \(\nu\) and to determine this \(\nu\) when it exists. If \(\nu\) is itself a Dirac measure, one says the metrics on the \(H^0(X, L^\otimes n)\) as \(n \to \infty\) are asymptotically semi-stable. Heuristically, this means that the successive minima tend to cluster around a particular number when properly normalized as \(n \to \infty\).

We will describe two applications of the general theory. The first is to study the distribution of the Petersson inner products of cusp forms of increasing weight for \(SL_2(\mathbb{Z})\). The second concerns the sup norms of rational functions on adelic subsets of curves arising in adelic capacity theory.

The paper is organized in the following way.

In \(\S 2\) we recall some results of Chen [14] concerning various kinds of successive minima associated to the global sections of metrized line bundles.

In \(\S 3\) we consider slopes associated to lattices of cusp forms of increasing weight for \(SL_2(\mathbb{Z})\) which have integral \(q\)-expansions. We begin by recalling work of Kuhn and Bost concerning the interpretation of Petersson norms of such cusp forms via Arakelov theory. A key issue is that the adelic metrics which arise on the line bundle \(L\) appropriate to this application are singular at infinity. Thus one cannot apply Chen’s work directly. Instead we consider forms which vanish to at least prescribed orders at infinity, and then let these orders tend to 0. An interesting conclusion in our main result, Theorem 3.2.2, is that the probability measure \(\nu\) which results in limit of large weights has support bounded above but not bounded below. In Definition 3.2.1 we define a nonzero cusp form \(f\) to not arise from a congruence between Hecke eigenforms if when we write \(f\) as a linear combination \(\sum c_i f_i\) of distinct normalized eigenforms \(f_i\), the \(c_i\) are algebraic integers divisible in the ring of all algebraic integers by the g.c.d. of the Fourier coefficients of \(f\). We will show that Petersson norms of such \(f\) are very large and contribute a vanishingly small proportion of successive minima as the weight tends to infinity. The measure \(\nu\) thus has to do with non-trivial congruences between
eigenforms which give rise to forms with integral $q$ expansions having much smaller Petersson norms.

In §4 we consider smooth projective curves $X$ of positive genus. Building on work of one of us in [31], we deduce an explicit upper bound on the largest minimum of $H^0(X, L^\otimes n)$ in terms of arithmetic intersection numbers. However, this result falls short of proving the asymptotic semi-stability of the metrics on $H^0(X, L^\otimes n)$.

In §5 we will apply the theory of Okundov bodies to study successive minima for $X$ of any dimension. We introduce local and global Chebyshev transforms which are maps from the Okounkov body of $X$ to the real numbers. The global Chebyshev transform is the sum of the local ones. We prove in Corollary 5.1.0.1 that, if the global Chebyshev transform is a constant function, the limit distribution $\nu$ is a Dirac measure. We compute explicitly the local Chebyshev transforms in some particular cases when $X$ is a projective space. The main strength of this technique is that in some cases one can compute explicitly the limit distributions of successive minima.

In §7 we consider applications to adelic capacity theory. This has to do with the possible sup norms of rational functions on adelic subsets of curves. We will apply work of Rumely to show that in the case of capacity metrics associated to adelic sets of capacity one, the associated metrized bundles are asymptotically semi-stable, and the measure $\nu$ is the Dirac measure supported at 0. We will also study the locations of the zeros of sections which arise in this case. This involves a measure theoretic result in §6 inspired by work of Serre on the decomposition of limits of weighted sums of Dirac measures.

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2. Semistability, successive minima, slopes and prior results

2.1. Measures associated to successive minima à la Chen. Let $\mathcal{E} = (E,(||\cdot||_v)_v)$ an hermitian adelic vector bundle of rank $r = \text{rank}(\mathcal{E})$ over a number field $K$ of degree $k$ over $\mathbb{Q}$ (see [20], Definition 3.1). We consider three sequences of slopes for $\mathcal{E}$:

- The (unnormalized) Harder-Narasimhan-Grayson-Stuhler slopes $(\hat{\lambda}_i)_{i=1}^r = (\hat{\lambda}_i(\mathcal{E}))_{i=1}^r$, as defined in [20], Definition 5.10. One has
  \[
  \sum_{i=1}^r \hat{\lambda}_i(\mathcal{E}) = \hat{\deg}(\mathcal{E}) = r\lambda(\mathcal{E}),
  \]
  where $\hat{\deg}(\mathcal{E})$ is the adelic degree of $\mathcal{E}$ ([20], Definition 4.1), and $\lambda(\mathcal{E}) = \frac{1}{r}\hat{\deg}(\mathcal{E})$ is the slope of $\mathcal{E}$.

- The naive adelic successive maxima $(\lambda_i)_{i=1}^r = (\lambda_i(\mathcal{E}))_{i=1}^r$ of $\mathcal{E}$, where $\lambda_i(\mathcal{E})$ is the largest real number $\lambda$ such that the set $E^{\geq \lambda}$ of elements of $E$ satisfying
  \[
  \lambda(s) := -\sum_v k_v \log ||s||_v \geq \lambda,
  \]
generates a $K$-vector space of dimension at least $i$. Here, $k_v$ is defined as follows, for each valuation of $v$ of $K$. When $v$ is finite of residual characteristic $p$, if $K_v$ is the completion of $K$ at $v$, $k_v$ is the degree of $K_v$ over $\mathbb{Q}_p$. When $v$ is real $k_v = 1$, and when $v$ is complex $k_v = 2$.

- The adelic successive maxima $(\hat{\lambda}_i(\mathcal{E}))_{i=1}^r$ of $\mathcal{E}$ (see [20], Definition 5.19) : the number $\lambda_i(\mathcal{E})$ is the supremum of the quantities $-\sum v k_v \log r_v$, where $(r_v)_v$ ranges over all families of positive real numbers such that the set of elements $s \in E$ satisfying
  
  $$\forall v, \ |s|_v \leq r_v,$$
  
  generates a $K$-vector space of dimension at least $i$.

By [20], Theorem 5.20, one has

$$\sum_{i=1}^r \lambda_i(\mathcal{E}) = \deg(\mathcal{E}) + O_K(r \log(2r)).$$

Since the same holds for the slopes $(\hat{\lambda}_i(\mathcal{E}))_{i=1}^r$, the inequalities $\lambda_i(\mathcal{E}) \leq \hat{\lambda}_i(\mathcal{E}) \leq \lambda_i(\mathcal{E})$ ensure that the same estimate also holds for the slopes $(\hat{\lambda}_i(\mathcal{E}))_{i=1}^r$. From this one deduce the following:

**Proposition 2.1.1.** Let $(\mathcal{E}_n)_{n \geq 1}$ be a sequence of hermitian adelic vector bundle of ranks $(r_n)_{n \geq 1}$ over $K$, such that $\log r_n = o(n)$. Assume that the sequence of probability measures

$$\check{\nu}_{\mathcal{E}_n} = \frac{1}{r_n} \sum_{i=1}^{r_n} \delta_{\lambda_i(\mathcal{E}_n)}$$

weakly converges to some probability measure $\nu$ with compact support on $\mathbb{R}$. Then the sequence

$$\nu_{\mathcal{E}_n} = \frac{1}{r_n} \sum_{i=1}^{r_n} \delta_{\hat{\lambda}_i(\mathcal{E}_n)},$$

weakly converges to $\nu$.

**Proof.** Let $f$ be a bounded continuous function on $\mathbb{R}$. It suffices to show that the quantity

$$e_n = \frac{1}{r_n} \sum_{i=1}^{r_n} \left( f \left( \frac{1}{n} \hat{\lambda}_i(\mathcal{E}_n) \right) - f \left( \frac{1}{n} \lambda_i(\mathcal{E}_n) \right) \right),$$

converges to 0 as $n$ tends to infinity. Let $[a, b]$ be a segment that contains the support of $\nu$. Given $\epsilon > 0$, there is a $\delta \in [0, 1]$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $x, y \in [a-2, b+2]$ satisfy $|x - y| \leq \delta$. The inequality

$$|f(x) - f(y)| \leq 2||f||_\infty 1_{x \in \mathbb{R} \setminus [a-1, b+1]} + \epsilon + \frac{2||f||_\infty}{\delta} |x - y|$$

then holds for any $x, y \in \mathbb{R}$. This yields

$$|e_n| \leq 2||f||_\infty \check{\nu}_{\mathcal{E}_n} \left( \{x \in \mathbb{R} \setminus [a-1, b+1] \} \right) + \epsilon + \frac{2||f||_\infty}{nr_n \delta} \sum_{i=1}^{r_n} \left( \hat{\lambda}_i(\mathcal{E}_n) - \lambda_i(\mathcal{E}_n) \right).$$
However,
\[ \frac{1}{nr} \sum_{i=1}^{rn} \left( \lambda_i(E_n) - \lambda_i(E_n) \right) \ll K \log \frac{r_n}{n} = o(1) \]
by the discussion preceding the statement of the proposition, and
\[ \hat{\nu}_{E_n}(\{ x \in \mathbb{R} \setminus [a-1, b+1] \}) = o(1) \]
since the limit measure \( \nu \) is supported on \([a, b]\). One thus obtain \( |e_n| \leq 2\varepsilon \) for \( n \) large enough, hence the result since \( \varepsilon > 0 \) was arbitrary. \( \square \)

From now on, let \( X \) be a projective variety of dimension \( d \) over a number field \( K \), and let \( L \) be an ample line bundle on \( X \), endowed with a continuous adelic metric \((| \cdot |_{L,v})_v\), in the sense of \[36\]. We assume that for all but a finite number of places, the metrics \((| \cdot |_{L,v})_v\) come from a single integral model of \((X, L)\) over \( \mathcal{O}_K \). The \( K \)-vector space \( H^0(X, L^{\otimes n}) \) is an adelic vector bundle, in the sense of \[20\], if equipped with the family of norms
\[ ||s||_{L^{\otimes n},v} = \sup_{x \in X(\mathbb{C}_v)} |s(x)|_{L,v}^{\otimes n}. \]
Even if the adelic vector bundle \( H^0(X, L^{\otimes n}) \) is not hermitian, one can still define its naive successive maxima \((\lambda_{i,n})_{i=1}^{rn} = (\lambda_i(H^0(X, L^{\otimes n}))_{i=1}^{rn}\) . We will rely on the following fundamental theorem of Chen.

**Theorem 2.1.2. (Chen)** Under the above hypotheses, the sequence of probability measures
\[ \nu_n = \frac{1}{rn} \sum_{i=1}^{rn} \delta_{\frac{1}{n} \lambda_{i,n}} \]
converges weakly to a compactly supported probability measure on \( \mathbb{R} \).

Indeed, replacing the \( L^\infty \)-norms at archimedean places by \( L^2 \)-norms with respect to a fixed volume form only change the normalized successive maxima \( \frac{1}{n} \lambda_{i,n} \) by a negligible amount of \( O\left( \frac{\log(n)}{n} \right) \), so that one is left with a sequence of hermitian adelic vector bundles over \( K \) which satisfies the hypotheses of Proposition 2.1.1 by Theorem 4.1.8 of \[14\].

**Remark 2.1.3.** The weak convergence of the sequence \((\nu_n)_n\) also holds when the adelic metric has mild singularities. More precisely, let us assume that \( d = 1 \), let \( (| \cdot |_{L,v})_v \) be an adelic metric coming for all but a finite number of places from a single integral model of \((X, L)\) over \( \mathcal{O}_K \), and let us allow \((| \cdot |_{L,v})_v\) to have logarithmic singularities at finitely many places. Namely, if \( L' \) is the line bundle \( L \) endowed with a continuous adelic metric \((| \cdot |_{L',v})_v\) as above then we require \( | \cdot |_{L,v} = \phi_v \cdot | \cdot |_{L',v} \) where \( \phi_v \) is a non-negative continuous function such that
\[ |1_D|_{\mathcal{O}(D)}^M \leq \phi_v \]
for some integer \( M \) and some continuous metric \(| \cdot |_{\mathcal{O}(D)}\) on \( \mathcal{O}(D) \). The homomorphism \( H^0(X, L^{\otimes n}) \rightarrow H^0(X, L'(MD)^{\otimes n}) \) then has operator norm at most 1 at each place.
Thus \( n^{-1} \lambda_1(H^0(X, L^\otimes n)) \) is bounded. With the help of Corollary 4.1.4 and Remark 4.1.5 of [14] one concludes that the corresponding sequence of measures \((\nu_n)_n\) is weakly convergent.

3. Modular forms and Petersson norms

In §3.1 we recall some work of Bost [7] and Kuhn [24] concerning the interpretation of holomorphic modular forms of weight \(12k\) for \(\text{SL}_2(\mathbb{Z})\) as sections of the \(k\)th power for \(k \geq 1\) of a particular metrized line bundle on \(\mathbb{P}^1_\mathbb{Z}\). We then study in §3.2 the successive maxima \(\{\lambda_{i,k}\}_{i=1}^k\) associated to the lattice \(S_{12k}(\Gamma, \mathbb{Z})\) of cusp forms of weight \(12k\) with integral Fourier coefficients with respect to the Petersson inner product, where \(k = \text{rank}_\mathbb{Z} S_{12k}(\Gamma, \mathbb{Z})\).

3.1. Modular forms as sections of a metrized line bundle. Let \(\mathbb{H}\) be the upper half plane and let \(\Gamma = \text{PSL}(2, \mathbb{Z})\) be the modular group. Then \(X = \Gamma \backslash (\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))\) has a natural structure as a Riemann surface. The classical \(j\) function of \(z \in \mathbb{H}\) has expansion

\[
j(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} a_n q^n \quad \text{in} \quad q = e^{2\pi i z}.
\]

The map \(z \to j(z)\) defines an isomorphism \(X \to \mathbb{P}^1_\mathbb{C}\).

The volume form of the hyperbolic metric on \(\mathbb{H}\) is

\[
\mu = \frac{dx \wedge dy}{y^2} = \frac{i}{2} \frac{dz \wedge d\overline{z}}{\text{Im}(z)^2}
\]

This form has singularities at the cusp and at the elliptic fixed points of \(\Gamma\), as described in [24, §4.2].

Define

\[
\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^2 = q + \sum_{n>1} b_n q^n
\]

to be the normalized cusp form of weight 12 for \(\Gamma\). Let \(S_{12\infty}\) be the unique cusp of \(X\), so that \(S_{12\infty}\) is associated with the orbit of \(\mathbb{P}^1(\mathbb{Q})\) under \(\Gamma\).

Suppose \(k\) is a positive integer. In [24, Def. 4.6] the line bundle \(M_{12k}(\Gamma) = O_X(S_{12\infty})^{\otimes k}\)

is defined to be the line bundle of modular forms of weight \(12k\) with respect to \(\Gamma\). This is shown to be compatible with the usual classical definition of modular forms. In particular, there is an isomorphism

\[
M_{12k}(\Gamma) \to H^0(X, O_X(S_{12\infty})^{\otimes k})
\]

between the space \(M_{12k}(\Gamma)\) of classical modular forms \(f = f(z)\) of weight \(12k\) and \(H^0(X, O_X(S_{12\infty})^{\otimes k})\) which sends \(f\) to the element \(f/\Delta^k\) of the function field \(\mathbb{C}(X) = \mathbb{C}(j)\).
The Petersson metric $| |_{\infty}$ on $\mathcal{M}_{12k}(\Gamma)_{\infty}$ is defined in [24] Def. 4.8 by

\begin{equation}
|f|_{\infty}^2(z) = |f(z)|^2(4\pi \text{ Im}(\tau))^{12k}
\end{equation}

if $f$ is a meromorphic section of $\mathcal{M}_{12k}(\Gamma)_{\infty}$. It is shown in [24] Prop. 4.9] that this metric is logarithmically singular with respect to the cusp and elliptic fixed points of $X$. See [24] p. 227-228] for the reason that the factor $4\pi$ is used on the right side of [4.4]

As in [24] §4.11], we define an integral model of $X$ to be

$$X = \text{Proj}(\mathbb{Z}[Z_0, Z_1])$$

with $Z_0$ and $Z_1$ corresponding to the global sections $j \cdot \Delta$ and $\Delta$ of the ample line bundle $\mathcal{M}_{12k}(\Gamma)_{\infty}$. The point $S_{1,\infty}$ defines a section $S_{1,\infty}$ of $X = \mathbb{P}^1_{\mathbb{Z}} \rightarrow \mathbb{Z}$. We extend $\mathcal{M}_{12k}(\Gamma)_{\infty}$ to the line bundle

$$\mathcal{M}_{12k}(\Gamma) = \mathcal{O}_X(S_{1,\infty})^{\otimes k}$$

on $X$. This model then gives natural metrics $| |_{v}$ at all non-archimedean places $v$ for the induced line bundle $\mathcal{M}_{12k}(\Gamma)_{\mathbb{Q}}$ on the general fiber $X_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} X_{\mathbb{Z}}$. When $v$ is the infinite place of $\mathbb{Q}$, we let $| |_{v}$ be the Petersson metric $| |_{\infty}$.

**Proposition 3.1.1.** The global sections $H^0(X, \mathcal{M}_{12k}(\Gamma))$ are identified with the $\mathbb{Z}$-lattice of all modular forms $f$ of weight 12$k$ with respect to $\Gamma$ which have integral $q$-expansions at $S_{1,\infty}$. These are the $f$ of $H^0(X, S_{1,\infty}(\mathbb{Q})^\vee)$ such that for all finite places $v$ of $\mathbb{Q}$ one has

\begin{equation}
\|f\|_{\mathcal{M}_{12k}(\Gamma),v} = \sup_{z \in X_{\mathbb{Q}(v)}} |f(z)| \leq 1.
\end{equation}

If $f$ is not in $B \cdot H^0(X, \mathcal{M}_{12k}(\Gamma))$ for any integer $B > 1$ then

$$\|f\|_{\mathcal{M}_{12k}(\Gamma),v} = 1 \text{ for all finite } v.$$

The sublattice $S_{1,\infty}(\Gamma, \mathbb{Z})$ of all cusp forms in $H^0(X, \mathcal{M}_{12k}(\Gamma))$ has corank 1 and rank $k$. If $f \in S_{1,\infty}(\Gamma, \mathbb{Z})$, the $L^2$ Hermitian norm at the infinite place $v = \infty$ of $f$ is the usual Petersson norm

\begin{equation}
\|f\|_{\mathcal{M}_{12k}(\Gamma),\infty,\text{herm}}^2 = \int_{X(\mathbb{C})} |f|_{\infty}^2(z) \mu(z) = \int_{X(\mathbb{C})} |f(z)|^2(4\pi y)^{12k} \frac{dx dy}{y^2}
\end{equation}

associated to $f$, where $\mu(z)$ is the the volume form of the hyperbolic metric given in [4.1].

**Proof.** The first statement is a consequence of the fact that the $q$ expansions of $j$ and $\Delta$ have integral coefficients and begin with $1/q$ and $q$, respectively. The statements concerning finite places $v$ is just the definition of the metrics at such places which are associated to integral models of line bundles. The rank of $H^0(X, \mathcal{M}_{12k}(\Gamma))$ over $\mathbb{Z}$ is the dimension over $\mathbb{C}$ of $H^0(X, \mathcal{M}_{12k}(\Gamma)_{\infty}) = H^0(X, \mathcal{O}_X(S_{1,\infty})^{\otimes k})$, which equals $k + 1$ by Riemann Roch. The last statement concerning cusp forms is the definition of the Petersson norm when this is normalized as in (3.4). \qed
3.2. Successive minima and modular forms. To state our main result we need a definition.

**Definition 3.2.1.** A non-zero form \( f \in S_{12k}(\Gamma, \mathbb{Z}) \) does not arise from a congruence between eigenforms if when we write \( f \) as a linear combination \( \sum c_i f_i \) of distinct normalized eigenforms \( f_i \), the \( c_i \) are algebraic integers divisible in the ring of all algebraic integers by the g.c.d. of the Fourier coefficients of \( f \).

This terminology arises from the fact that if the \( c_i \) are integral but the last requirement in the definition fails, there is a non-trivial congruence modulo the g.c.d. of the Fourier coefficients of \( f \) between the forms \( f_i \).

**Theorem 3.2.2.** Let \( \{\lambda_{1,12k}\}_{i=1}^{k} \) be the naive successive maxima associated to \( S_{12k}(\Gamma, \mathbb{Z}) \) with respect to the \( L^2 \) Hermitian norm defined by the Petersson norm in (3.21).

1. The sequence of probability measures
   \[ \nu_{12k} = \frac{1}{k} \sum_{i=1}^{k} \delta_{\lambda_{1,12k}} \]
   converges weakly as \( k \to \infty \) to a probability measure \( \nu \).
2. The support of the measure \( \nu \) is bounded above by \( 2\pi + 6(1 - \log(12)) = -2.62625 \ldots \). The support of \( \nu \) is not bounded below.
3. As \( k \to \infty \), the proportion of successive maxima which are produced by \( f \in S_{12k}(\Gamma, \mathbb{Z}) \) which do not arise from a congruence between eigenforms goes to 0.

This result shows that in Remark 2.1.3 the limit measure need not have compact support when the metrics involved are allowed to have mild singularities. We will prove in §3.3 more quantitative results about the successive minima \( \lambda_{1,12k} \) in this Theorem.

**Remark 3.2.3.** Consider the divisors \( \text{zer}(f) \) of complex zeros of elements \( f \) of \( S = \cup_{k>0} S_{12k}(\Gamma, \mathbb{Z}) \). Recall that each such \( \text{zer}(f) = \sum_{x \in \mathbb{P}^1(\mathbb{C})} m_x x \) defines a Dirac measure \( \mu(\text{zer}(f)) = \frac{1}{\deg(f)} \sum_{x} m_x \delta_x \). It follows from work of Holowinsky and Soundararajan [23, Remark 2] and Rudnick [26] that as \( f \) ranges over any sequence of non-zero Hecke eigen cusp forms of weights going to infinity, the corresponding dirac measures \( \mu(\text{zer}(f)) \) converge weakly to the Petersson measure \( \mu \) in (3.1). However, due to part (iii) of Theorem 3.2.2 we cannot conclude from this much information about the measures associated to the zeros of forms with large height which are discussed in Remark 6.3.3.

It would be interesting to know whether cusp forms with integral \( q \)-expansions which have small Petersson norms must vanish at particular points in the upper half plane.

3.3. Petersson norms and Fourier expansions.

**Lemma 3.3.1.** Suppose that \( 0 \neq f = \sum_{n=1}^{\infty} a_n q^n \in S_{12k}(\Gamma, \mathbb{C}) \). Let \( N = \text{ord}_\infty(f) \). Then \( 1 \leq N \leq k \), and the \( L^2 \) Hermitian norm at the infinite place \( v = \infty \) of \( f \) in
has the property that
\[
\|f\|_{M_{12k}(\Gamma), \infty, \text{herm}}^2 = \int_{X(\mathbb{C})} |f(z)|^2 (4\pi y)^{12k} \frac{dxdy}{y^2} \\
\geq \sum_{n=1}^{\infty} |a_n|^2 4\pi e^{-4\pi n} \frac{(12k - 2)!}{n^{12k-1}} \\
\geq |a_N|^2 \cdot 4\pi e^{-4\pi N} \frac{(12k - 2)!}{N^{12k-1}}
\]
(3.7)

\textbf{Proof.} Since the } a_n \text{ are in } \mathbb{C}, \text{ we have } f(q) = \sum_{n=1}^{\infty} a_n q^n. \text{ For a fixed } y \geq 1 \text{ we have (as in [30, p. 786]) that}
\[
\int_{-1/2}^{1/2} |f(x + iy)|^2 \, dx = \int_{-1/2}^{1/2} f(q)\overline{f(q)} \, dx \\
= \int_{-1/2}^{1/2} \sum_{n,m=1}^{\infty} a_n \overline{a_m} q^n \overline{q}^m \\
= \sum_{n,m=1}^{\infty} a_n \overline{a_m} \int_{-1/2}^{1/2} e^{2\pi i((n-m)x+(n+m)y)} \, dx \\
= \sum_{n=1}^{\infty} |a_n|^2 e^{-4\pi ny}
\]
(3.8)

The standard fundamental domain for the action of } SL_2(\mathbb{Z}) \text{ on } \mathbb{H} \text{ contains the set } T = \{ z = x + iy : -1/2 \leq x < 1/2 \text{ and } y \geq 1 \}. \text{ Therefore}
\[
\int_{X(\mathbb{C})} |f(z)|^2 (4\pi y)^{12k} \frac{dxdy}{y^2} \geq \int_{T} |f(z)|^2 (4\pi y)^{12k} \frac{dxdy}{y^2} \\
= \int_{y=1}^{\infty} \int_{x=-1/2}^{x=1/2} |f(x + iy)|^2 \, dx \, (4\pi y)^{12k} y^{12k-2} \, dy \\
= (4\pi)^{12k} \sum_{n=1}^{\infty} |a_n|^2 \int_{y=1}^{\infty} e^{-4\pi ny} y^{12k-2} \, dy
\]
(3.9)

For all constants } c \neq 0 \text{ and all integers } \ell \geq 0, \text{ one has the indefinite integral
\[
\int e^{-cy} y^{\ell} \, dy = -e^{-cy} \cdot \sum_{j=0}^{\ell} \frac{y^{\ell-j} \, \ell!}{c^{j+1}(\ell - j)!}
\]
as one sees by differentiating the right side. Setting } c = 4\pi n \text{ and } \ell = 12k - 2 \text{ and then integrating the left hand side from } y = 1 \text{ to } \infty \text{ gives
\[
\int_{y=1}^{\infty} e^{-4\pi ny} y^{12k-2} \, dy = e^{-4\pi n} \sum_{j=0}^{12k-2} \frac{(12k - 2)!}{(4\pi n)^{j+1}(12k - 2 - j)!} \geq e^{-4\pi n} \frac{(12k - 2)!}{(4\pi n)^{12k-1}}
\]
(3.11)

Substituting this back into (3.9) shows the claimed inequalities.
Lemma 3.3.2. There exists a constant $c > 0$ such that for any integers $k, \ell$ with $1 \leq \ell \leq k$, we have

$$\lambda(\Delta^k j^{k-\ell}) \geq 6k \log \left(\frac{\ell}{k}\right) - ck.$$ 

Proof. Since $\Delta^k j^{k-\ell}$ has integral $q$-expansion and unit leading coefficient, we have

$$||\Delta^k j^{k-\ell}||_{M_{12k}(\Gamma),v} = 1,$$

for any finite place $v$. In particular, we have

$$\lambda(\Delta^k j^{k-\ell}) = -\log ||\Delta^k j^{k-\ell}||_{M_{12k}(\Gamma),\infty,\text{herm}}.$$ 

Let $F = \{z = x + iy : -1/2 \leq x \leq 1/2, x^2 + y^2 \geq 1\}$ be the standard fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. There is a constant $c \geq 1$ such that for any $z = x + iy$ in $F$, we have $|\Delta(z)| \leq ce^{-2\pi y}$ and $|j(z)| \leq ce^{2\pi y}$. We thus have

$$||\Delta^k j^{k-\ell}||_{M_{12k}(\Gamma),\infty,\text{herm}}^2 = \int_F |\Delta(z)|^{2k} |j(z)|^{2k-2\ell} (4\pi y)^{12k} \frac{dxdy}{y^2} \leq c^{4k-2\ell} \int_0^\infty e^{-4\pi t y}(4\pi y)^{12k} \frac{dy}{y^2} \leq 4\pi c^{12k}(12k - 1)\ell^{1-12k} = \left(\frac{k}{\ell}\right)^{12k} e^{O(k)},$$

giving the result by taking the logarithms of both sides of this inequality. 

Lemma 3.3.3. Let $(\lambda_{j,k})_{j=1}^k$ be the successive minima of $S_{12k}(\Gamma, \mathbb{Q})$. We have

$$\frac{\lambda_{j,k}}{k} = 6 \log \left(1 - \frac{j - 1}{k}\right) + O(1),$$

where the implicit constant in $O(1)$ is absolute.

Proof. The inequality

$$\frac{\lambda_{j,k}}{k} \geq 6 \log \left(1 - \frac{j - 1}{k}\right) + O(1)$$

follows from Lemma 3.3.2 by using the $j$ linearly independent sections $(\Delta^k j^{k-\ell})_{k-j+1 \leq \ell \leq k}$. We now prove the converse inequality. Let $s_1, \ldots, s_j$ be linearly independent elements of $S_{12k}(\Gamma, \mathbb{Q})$ such that $\lambda(s_i) \geq \lambda_{j,k}$ for any $i$. Up to multiplying the $s_i$'s by appropriate non zero rational numbers, we can assume that

$$||\Delta^k j^{k-\ell}||_{M_{12k}(\Gamma),v} = 1,$$

for any finite place $v$ and for any $i$, so that

$$\lambda(s_i) = -\log ||s_i||_{M_{12k}(\Gamma),\infty,\text{herm}}.$$ 

The linear subspace of $S_{12k}(\Gamma, \mathbb{Q})$ consisting of forms vanishing at $\infty$ at order at least $k - j + 2$ has dimension $j - 1$, and therefore can not possibly contain all $s_i$'s. Thus
there exists an index \(i\) such that \(s_i\) vanishes at \(\infty\) at some order \(N \leq k - j + 1\). By Lemma [3.3.1] we have
\[
||s_i||_{\mathcal{M}_{12k}(\Gamma),\infty,\text{herm}} \geq e^{-2\pi N \frac{(12k - 2)! \frac{1}{2}}{N^{6k - \frac{7}{2}}} \geq \left( \frac{k}{N} \right)^{12k} e^{O(k)} \geq \left( \frac{k}{k - j + 1} \right)^{12k} e^{O(k)}.
\]
We therefore obtain
\[
\frac{\lambda_{j,k}}{k} \leq \frac{\lambda(s_i)}{k} \leq 6 \log \left( 1 - \frac{j - 1}{k} \right) + O(1).
\]

**Lemma 3.3.4.** There exists constants \(c_1, c_2 > 0\) such that for any element \(f\) of \(\mathcal{S}_{12k}(\Gamma, \mathbb{R})\), the quantity \(||f||_{\mathcal{M}_{12k}(\Gamma),\infty,\sup} = \sup_{z \in X(\mathbb{C})} |f|_{\infty}(z)\) satisfies the inequalities
\[
c_1 ||f||_{\mathcal{M}_{12k}(\Gamma),\infty,\text{herm}} \leq ||f||_{\mathcal{M}_{12k}(\Gamma),\infty,\sup} \leq c_2 k^2 \log(3k) ||f||_{\mathcal{M}_{12k}(\Gamma),\infty,\text{herm}}.
\]

**Proof.** One can take \(c_1 = \text{Vol}(X(\mathbb{C}))^{-\frac{1}{4}}\), and we therefore focus on the second inequality. Let \(f\) be an element of \(\mathcal{S}_{12k}(\Gamma, \mathbb{R})\), and let \(\bar{T} = \{z = x + iy : -1/2 \leq x \leq 1/2, x^2 + y^2 \geq 1\}\) be the (closure of the) standard fundamental domain for the action of \(\text{SL}_2(\mathbb{Z})\) on \(\mathbb{H}\). Since \(|f|_{\infty}(z)\) tends to 0 as the imaginary part of \(z \in \bar{T}\) goes to infinity, there exists a point \(z_0 = x_0 + iy_0\) of \(\bar{T}\) such that \(||f||_{\mathcal{M}_{12k}(\Gamma),\infty,\sup}\) is equal to \(|f|_{\infty}(z_0)\). Writing \(f(z) = \sum_{n=1}^{\infty} a_n q^n\) with \(q = e^{2\pi inz}\), we obtain
\[
|f(z_0)| \leq \sum_{n \geq 1} |a_n| e^{-2\pi n y_0},
\]
and then the Cauchy-Schwarz inequality yields
\[
|f(z_0)|^2 \leq \left( \sum_{n \geq 1} a_n^2 \frac{e^{-4\pi n}}{n^{12k - 1}} \right) \left( \sum_{n \geq 1} n^{12k - 1} e^{4\pi n(1 - y_0)} \right) \leq \frac{||f||^2_{\mathcal{M}_{12k}(\Gamma),\infty,\text{herm}}}{4\pi (12k - 2)!} \left( \sum_{n \geq 1} n^{12k - 1} e^{4\pi n(1 - y_0)} \right),
\]
where the last inequality follows from Lemma [3.3.1]. Let us assume that \(y_0 \geq 3\). We have
\[
n^{12k - 1} e^{4\pi n(1 - y_0)} \leq e^{-4\pi y_0 + O(k \log k)} e^{-4\pi n}
\]
and therefore
\[
\frac{1}{4\pi (12k - 2)!} \left( \sum_{n \geq 1} n^{12k - 1} e^{4\pi n(1 - y_0)} \right) \leq e^{-4\pi y_0 + O(k \log k)}.
\]
We thus obtain
\[
|f(z_0)|^2 (4\pi y_0)^{12k} \leq e^{-4\pi y_0 + O(k \log(k y_0))} ||f||^2_{\mathcal{M}_{12k}(\Gamma),\infty,\text{herm}}.
\]
In particular, there exists an absolute constant \(c_3\) such that if \(y_0 \geq c_3 k \log(3k)\), then
\[
||f||^2_{\mathcal{M}_{12k}(\Gamma),\infty,\sup} = |f|_{\infty}(z_0) \leq ||f||^2_{\mathcal{M}_{12k}(\Gamma),\infty,\text{herm}}.
\]
It remains to handle the case \( y_0 \leq c_3 k \log(3k) \). We first claim that there exists a real number \( R > 0 \) such that for any \( z \) in \( T \), the projection from the disc \( D(z, R) \) with center \( z \) and radius \( R \) to \( X(\mathbb{C}) \) is injective. By a standard compactness argument, there exists a real number \( R \in [0, \frac{1}{2}] \) such that this property holds for any \( z \) in \( T \) with imaginary part at most 2. For an element \( z \) of \( T \) with imaginary part at least 2, we notice that the disc \( D(z, R) \) is contained in the interior of the fundamental domain

\[
\{ \tau \in T + \mathbb{Z} \mid |\text{Re}(\tau - z)| \leq \frac{1}{2} \},
\]

so that \( R \) has the required property.

Let \( R_k = k^{-1} R \). With \( z_0 \) as above, we have

\[
\pi R_k^2 |f(z_0)|^2 (4\pi y_0)^{12k} \leq (4\pi y_0)^{12k} \int_{D(z_0, R_k)} |f(z)|^2 dxdy,
\]

\[
\leq \frac{y_0^{12k}}{(y_0 - R_k)^{12k - 2}} \int_{D(z_0, R_k)} |f|_{\infty}(z)^2 \frac{dxdy}{y^2}
\]

\[
\leq \frac{y_0^{12k}}{(y_0 - R_k)^{12k - 2}} \|f\|^2_{M_{12k}(\Gamma), \infty, \text{herm}}.
\]

We therefore obtain

\[
\|f\|^2_{M_{12k}(\Gamma), \infty, \text{sup}} = |f|_{\infty}(z_0) \leq c_4 k^2 y_0^2 \|f\|^2_{M_{12k}(\Gamma), \infty, \text{herm}},
\]

for some absolute constant \( c_4 > 0 \). Since \( y_0 \leq c_3 k \log(3k) \), this yields

\[
\|f\|_{M_{12k}(\Gamma), \infty, \text{sup}} \leq c_4 c_3 k^2 \log(3k) \|f\|_{M_{12k}(\Gamma), \infty, \text{herm}}.
\]

We thus obtain the claimed inequality with \( c_2 = \max(1, c_4 c_3) \).

**Lemma 3.3.5.** There exists a real number \( c \) such that for any elements \( f_1, f_2 \) of \( S_{12k_1}(\Gamma, \mathbb{R}) \) and \( S_{12k_2}(\Gamma, \mathbb{R}) \) respectively, we have

\[
\|f_1 f_2\|_{M_{12k_1 + k_2}(\Gamma), \infty, \text{herm}} \leq e^{\psi(k_1) + \psi(k_2)} \|f_1\|_{M_{12k_1}(\Gamma), \infty, \text{herm}} \|f_2\|_{M_{12k_2}(\Gamma), \infty, \text{herm}},
\]

where \( \psi(k) = 2 \log(k) + \log \log(3k) + c \).

**Proof.** Let \( c_1, c_2 \) be as in Lemma 3.3.4. We have

\[
\|f_1 f_2\|_{M_{12k_1 + k_2}(\Gamma), \infty, \text{herm}} \leq c_1^{-1} \|f_1 f_2\|_{M_{12k_1 + k_2}(\Gamma), \infty, \text{sup}} \leq c_1^{-1} \|f_1\|_{M_{12k_1}(\Gamma), \infty, \text{sup}} \|f_2\|_{M_{12k_2}(\Gamma), \infty, \text{sup}} \leq c_1^{-1} c_2 k_1^2 \log(3k_1)^2 k_2^2 \log(3k_2)^2 \|f_1\|_{M_{12k_1}(\Gamma), \infty, \text{herm}} \|f_2\|_{M_{12k_2}(\Gamma), \infty, \text{herm}},
\]

and the result follows with \( c = \log(c_2) - \frac{1}{2} \log(c_1) \).
3.4. Modified logarithmic heights. To apply Chen’s work in [14] on the distribution of successive minima, we will need some estimates for the behavior of a modification of the logarithmic height of cusp forms.

The vector space \( V = S_{12k}(\Gamma, \mathbb{Q}) \) has a filtration defined by letting \( V_a \) for \( a \in \mathbb{R} \) be the \( \mathbb{Q} \)-span of all \( 0 \neq f \in S_{12k}(\Gamma, \mathbb{Q}) \) for which \( \lambda(f) \geq a \). Lemma 3.3.1 shows that \( V_a = \{0\} \) if \( a \) is sufficiently large. Following Chen in [14, p. 15, eq. (2)], we define a modified logarithmic height by

\[
\tilde{\lambda}(f) = \sup\{a \in \mathbb{R} : f \in V_a\}
\]

The proof of [14, Prop. 1.2.3] now shows \( \tilde{\lambda}(f) \) has the following properties:

**Lemma 3.4.1.** Suppose \( f \) and \( g \neq -f \) are non-zero elements of \( S_{12k}(\Gamma, \mathbb{Q}) \).

1. \( \tilde{\lambda}(rf) = \tilde{\lambda}(f) \) for \( r \in \mathbb{Q} - \{0\} \).
2. \( \tilde{\lambda}(f + g) \geq \min(\tilde{\lambda}(f), \tilde{\lambda}(g)) \), with equality if \( \tilde{\lambda}(f) \neq \tilde{\lambda}(g) \).

**Lemma 3.4.2.** Let \( \psi \) be as in Lemma 3.3.5. For any elements \( f_1, f_2 \) of \( S_{12k_1}(\Gamma, \mathbb{Q}) \) and \( S_{12k_2}(\Gamma, \mathbb{Q}) \) respectively, we have

\[
\tilde{\lambda}(f_1 f_2) \geq \tilde{\lambda}(f_1) + \tilde{\lambda}(f_2) - \psi(k_1) - \psi(k_2).
\]

**Proof.** Let us write \( f_i = \sum_j g_{i,j} \), where \( \lambda(g_{i,j}) \geq \tilde{\lambda}(f_i) \). For any \( j_1, j_2 \) and any non-archimedean place \( v \), we have

\[
\|g_{1,j_1}g_{2,j_2}\|_{M_{12k_1+k_2}(\Gamma),v} \leq \|g_{1,j_1}\|_{M_{12k_1}(\Gamma),v}\|g_{2,j_2}\|_{M_{12k_2}(\Gamma),v},
\]

and by Lemma 3.3.5 we also have

\[
\|g_{1,j_1}g_{2,j_2}\|_{M_{12k_1+k_2}(\Gamma),\infty,\operatorname{herm}} \leq e^{\psi(k_1)+\psi(k_2)}\|g_{1,j_1}\|_{M_{12k_1}(\Gamma),\infty,\operatorname{herm}}\|g_{2,j_2}\|_{M_{12k_2}(\Gamma),\infty,\operatorname{herm}}.
\]

This implies

\[
\lambda(g_{1,j_1}g_{2,j_2}) \geq \lambda(g_{1,j_1}) + \lambda(g_{2,j_2}) - \psi(k_1) - \psi(k_2)
\]

\[
\geq \tilde{\lambda}(f_1) + \tilde{\lambda}(f_2) - \psi(k_1) - \psi(k_2),
\]

hence the result, since \( f_1f_2 = \sum_{j_1,j_2} g_{1,j_1}g_{2,j_2} \). \( \square \)

3.5. Cusp forms vanishing to increasing orders at infinity. We study in this section the successive minima associated to cusp forms \( f \in S_{12k}(\Gamma, \mathbb{Z}) \) for which \( \operatorname{ord}_\infty(f) \) is at least a certain positive constant times \( k \).

**Lemma 3.5.1.** Suppose \( 1 \leq L, k \in \mathbb{Z} \). The \( \mathbb{Z} \)-lattice \( B(12k, L) \) all \( f \in S_{12k}(\Gamma, \mathbb{Z}) \) for which \( \operatorname{ord}_\infty(f) \geq k/L \) is the free \( \mathbb{Z} \)-module with basis \( \{\Delta^k j^{k-\ell} : k/L \leq \ell \leq k\} \). One has

\[
k(1 - 1/L) \leq \operatorname{rank}_\mathbb{Z}(B(12k, L)) = k + 1 - \lfloor k/L \rfloor \leq k(1 - 1/L) + 1
\]

**Proof.** This is clear from the fact that \( \Delta^k j^{k-\ell} \) lies in \( S_{12k}(\Gamma, \mathbb{Z}) \) and its first non-zero term in its Fourier expansion at \( \infty \) is \( q^\ell \). \( \square \)
Lemma 3.5.2. Let \( \{\lambda_{i,12k,L}\}_{i=1}^{k+1-[k/L]} \) be the naive successive maxima associated to \( B(12k,L) \) with respect to the \( L^2 \) Hermitian norm defined by the Petersson norm in The sequence of probability measures

\[
\nu_{12k,L} = \frac{1}{k + 1 - \left\lfloor \frac{k}{L} \right\rfloor} \sum_{i=1}^{k+1-[k/L]} \frac{1}{\lambda_{i,12k,L}}
\]

converges weakly as \( k \to \infty \) to a probability measure \( \nu_{\infty,L} \) having compact support.

Proof. For \( 0 < r < L \), let \( B_L(r) = \oplus_{q=0}^{\infty} B(12(qL+r),L) \). If \( r = 0 \), then \( 12(qL+r)/L = 12q \) is an integer for all \( q \geq 0 \) and \( B_L(0) \) is a graded algebra. It follows from Lemma 3.5.1 that the subgroup \( B(12qL,L) \cdot B(12q',L) \) of \( B(12(q+q')L,L) \) generated by all products of elements of \( B(12qL,L) \) and \( B(12q',L) \) is equal to \( B(12(q+q')L,L) \). The work in [3.4] now shows that \( B_L(0) \) is integral and \( \psi \)-quasifiltered in the sense of [14] Def. 3.2.1] with respect to the modified logarithmic heights \( \tilde{\lambda} \) on the summands of \( B_L(0) \), where \( \psi \) is the function from Lemma 3.4.2. Part (i) of Theorem 3.7.1 shows that there is an upper bound independent of \( q \) on \( \tilde{\lambda}_{\max}(B(12qL,L))/(12qL) \) when \( \tilde{\lambda}_{\max}(B(12qL,L)) \) is the maximal value of \( \tilde{\lambda} \) on \( B(12qL,L) \). One can now apply [14 Thm. 3.4.3] to conclude that

\[
\nu_{\infty,L} = \lim_{q \to \infty} \nu_{12qL,L}
\]

exists and has compact support when \( \nu_{12k,L} \) is defined as in (3.15).

Suppose now that \( 0 < r < L \). When \( k = qL + r \) and \( 0 \leq q \in \mathbb{Z} \), \( B(12k,L) \) has \( \mathbb{Z} \)-basis \( b(12k,L) = \{\Delta^k j^k : k/L \leq \ell \leq k\} \). Here \( k/L = (qL + r)/L = q + r/L \) and \( 0 < r/L < 1 \), so \( k/L \leq \ell \leq k \) is the same as \( q+1 \leq \ell \leq k = qL + r \). We have

\[
(\Delta^{L-r} j^{L-r-1}) \cdot (\Delta^k j^{k-\ell}) = \Delta^{L(q+1)} j^{L(q+1)-\ell-1}
\]

since \( k = qL + r \), where \( 0 \neq \Delta^{L-r} j^{L-r-1} \in S_{12(L-r)}(\Gamma,\mathbb{Z}) \). Taking the description of bases for \( B(12k,L) \) and \( B(12(q+1)L,L) \) in Lemma 3.5.1 into account, we see from (3.17) that multiplication by \( (\Delta^{L-r} j^{L-r-1}) \) defines an injective homomorphism from \( B(12k,L) \) to \( B(12(q+1)L,L) \). The dimension of the cokernel of this homomorphism is

\[
(q + 1)L + 1 - (q + 1) - (k + 1 - (q + 1)) = L - r
\]

which is bounded independently of \( q \). From Lemma 3.4.2 we have

\[
\tilde{\lambda}(\Delta^{L(q+1)} j^{L(q+1)\ell-1}) \geq \tilde{\lambda}(\Delta^k j^{k-\ell}) + \tilde{\lambda}(\Delta^{L-r} j^{L-r-1}) + c_1 \ln(k)
\]

(3.18)

where the constants \( c_1 \) and \( c_2 \) depend only on \( L \). It follows that for any bounded increasing continuous function \( f : \mathbb{R} \to \mathbb{R} \), one has

\[
\nu_{12(qL+r),L}(f) \leq \nu_{12(q+1)L,L}(f) + o(1)
\]

where \( o(1) \to 0 \) as \( q \to \infty \). Hence

\[
\limsup_{q \to \infty} \nu_{12(qL+r),L}(f) \leq \nu_{\infty,L}(f).
\]
From (3.17) we also have
\[(\Delta^k j^{k-1}) = \Delta^j Lq^j Lq^{-j} \cdot (\Delta^r j^{r-1})\]
In a similar way, this shows that multiplication by \((\Delta^r j^{r-1}) \in S_{12r}(\Gamma, \mathbb{Z})\) defines an injection from \(B(Lq, L)\) to \(B(k, L)\). The dimension of the cokernel of this injection is \(r\), which is bounded independently of \(q\). By arguments similar to the one above, we obtain from (3.19) that
\[
\liminf_{q \to \infty} \nu_{12(qL+\ell), L}(f) \geq \nu_{\infty, L}(f).
\]
This completes the proof of Lemma 3.5.2. □

3.6. Proof of part (i) of Theorem 3.2.2. In order to prove the weak convergence of the \(\nu_{12k}\) stated in part (i) of the Proposition, we will use the limit measures \((\nu_{\infty, L})_L\) introduced in Lemma 3.5.2.

Lemma 3.6.1. Let \(f\) be a bounded Lipschitz function from \(\mathbb{R}\) to \(\mathbb{R}\), and let \(\varepsilon\) be a positive real number. Then there exists \(L_0 = L_0(\varepsilon, f)\) such that for any \(L \geq L_0\), there exists \(k_0 = k_0(L, \varepsilon, f)\) such that for any \(k \geq k_0\), we have
\[|\nu_{12k}(f) - \nu_{12k, L}(f)| \leq \varepsilon.\]

Proof. The numbers \(L_0\) and \(k_0\) will depend on \(f\) only through the Lipschitz norm \(M\) of \(f\), namely
\[M = \sup_x |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.
\]
Let \(k \geq L \geq 2\) be integers, and let \(k' = k + 1 - \lfloor k/L \rfloor\) be the rank of \(B(12k, L)\). We denote by \((\lambda_{j, k})_{1 \leq j \leq k}\) and \((\lambda_{j, k, L})_{1 \leq j \leq k'}\) the successive minima of \(S_{12k}(\Gamma, \mathbb{Q})\) and \(B(12k, L)\) respectively. Let us write
\[\nu_{12k}(f) - \nu_{12k, L}(f) = S_1 + S_2 + S_3 + S_4,
\]
where we have set
\[S_1 = \frac{1}{k} \sum_{k' < j \leq k} f \left( \frac{\lambda_{j, k}}{k} \right),
\]
\[S_2 = \frac{1}{k} \sum_{j \leq k'} \left( f \left( \frac{\lambda_{j, k}}{k} \right) - f \left( \frac{\lambda_{j, k, L}}{k} \right) \right),
\]
\[S_3 = \frac{1}{k} \sum_{j \leq k'} \left( f \left( \frac{\lambda_{j, k, L}}{k} \right) - f \left( \frac{\lambda_{j, k, L}}{k'} \right) \right),
\]
\[S_4 = \left( \frac{1}{k} - \frac{1}{k'} \right) \sum_{j \leq k'} f \left( \frac{\lambda_{j, k, L}}{k'} \right).
\]
For \(S_1\), we have the simple estimate
\[|S_1| \leq \frac{1}{k} \sum_{k' < j \leq k} M \leq \frac{M}{L}.
\]
A similar estimate holds for $S_4$:
\[ |S_4| \leq \frac{k - k'}{kk'} \sum_{j \leq k'} M \leq \frac{M}{L}. \]

In order to estimate $S_3$, we first notice that an argument similar to the proof of Lemma 3.3.3 yields
\[ \frac{\lambda_{j,k,L}}{k} = O \left( |\log(1 - \frac{k'}{k})| + 1 \right) = O(\log(3L)). \]

Thus there exists an absolute constant $c_1$ such that $|\lambda_{j,k,L}| \leq c_1 k \log(3L)$. This implies
\[ |S_3| \leq M \frac{k - k'}{kk'} \sum_{j \leq k'} |\lambda_{j,k,L}| \leq \frac{c_1 M \log(3L)}{L}. \]

It remains to estimate $S_2$. The inclusion homomorphism of $B(12k, L)$ into $S_{12k}(\Gamma, \mathbb{Q})$ preserves slopes, hence $\lambda_{j,k,L} \leq \lambda_{j,k}$ for any $j \leq k'$. We therefore have
\[ |S_2| \leq M \frac{k - k'}{kk'} \sum_{j \leq k'} (\lambda_{j,k} - \lambda_{j,k,L}) \]
\[ = M \frac{1}{k^2} \left( \sum_{j \leq k'} \lambda_{j,k} \right) - M \left( \frac{k'}{k} \right)^2 \nu_{12k,L}(id) \]
\[ \leq M \frac{1}{k^2} \left( \sum_{j \leq k'} \lambda_{j,k} \right) - M \nu_{12k,L}(id) + \frac{2c_1 M \log(3L)}{L}. \]

Let us consider the isomorphism $S_{12k}(\Gamma, \mathbb{Q}) \to B(12(k+s), L)$ induced by multiplication by $\Delta^s$, where $s = \lfloor \frac{k-1}{L} \rfloor \geq 1$. It does not always preserve slopes, but Lemma 3.3.5 yields
\[ \lambda(\Delta^s g) \geq \lambda(g) + s \lambda(\Delta) - \psi(k) - \psi(s), \]
for any non zero element $g$ of $S_{12k}(\Gamma, \mathbb{Q})$. Correspondingly, we have
\[ \lambda_{j,k+s,L} \geq \lambda_{j,k} + s \lambda(\Delta) - \psi(k) - \psi(s), \]
for any $j \leq k$. We thus have
\[ \frac{1}{k^2} \sum_{j \leq k'} \lambda_{j,k} \leq \frac{1}{k^2} \sum_{j \leq k'} \lambda_{j,k+s,L} + \frac{\psi(k) + \psi(s) - s \lambda(\Delta)}{k} \]
\[ \leq \left( \frac{1}{k^2} \sum_{j \leq k'} \lambda_{j,k+s,L} \right) + \frac{2c_1 \log(3L)}{L} + \frac{\psi(k) + \psi(s) - s \lambda(\Delta)}{k}. \]

This implies
\[ |S_2| \leq M \left( \nu_{12(k+s),L(id)} - \nu_{12k,L(id)} \right) + \frac{c_2 M \log(3L)}{L}, \]
for some absolute constant $c_2$. Gathering our estimates, we obtain the existence of $L_0 = L_0(\varepsilon, M)$ such that for any $k \geq L \geq L_0$, we have

$$|S_1| + |S_3| + |S_4| \leq \frac{\varepsilon}{2},$$

and

$$|S_2| \leq M (\nu_{12(k+s),L}(\text{id}) - \nu_{12k,L}(\text{id})) + \frac{\varepsilon}{4}.$$ 

Since the sequence $(\nu_{12k,L}(\text{id}))_k$ is convergent by Lemma 3.5.2, we further obtain the existence, for any $L \geq L_0$ of $k_0 = k_0(L, \varepsilon, M)$ such that for any $k \geq k_0$, we have $|S_2| \leq \frac{\varepsilon}{2}$, hence the result.

**Corollary 3.6.2.** Let $f$ be a bounded Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$. Then the sequences $(\nu_{12k}(f))_k$ and $(\nu_{\infty,L}(f))_L$ are convergent and have the same limit.

**Proof.** Let $\varepsilon > 0$ be a positive real number. Let $L_0 = L_0(\varepsilon, f)$ be as in Lemma 3.6.1. For any $L \geq L_0$, we have

$$\limsup_{k \to \infty} \nu_{12k}(f) \leq \nu_{\infty,L}(f) + \varepsilon,$$

and

$$\liminf_{k \to \infty} \nu_{12k}(f) \geq \nu_{\infty,L}(f) - \varepsilon.$$ 

In particular, we have

$$\limsup_{k \to \infty} \nu_{12k}(f) \leq \liminf_{k \to \infty} \nu_{12k}(f) + 2\varepsilon.$$ 

Since $\varepsilon$ is arbitrary small, this yields the convergence of the sequence $(\nu_{12k}(f))_k$. Moreover, for any $L \geq L_0$ we have

$$|\nu_{\infty,L}(f) - \lim_{k \to \infty} \nu_{12k}(f)| \leq \varepsilon,$$

hence the convergence of the sequence $(\nu_{\infty,L}(f))_L$ to the limit $\lim_{k \to \infty} \nu_{12k}(f)$. □

We know from [] that there is a finite Borel measure $\nu$ on $\mathbb{R}$ such that for any continuous function $f$ with compact support,

$$\nu(f) = \lim_{k \to \infty} \nu_{12k}(f) = \lim_{L \to \infty} \lim_{k \to \infty} \nu_{12k,L}(f).$$ 

Part (i) of Theorem 3.2.2 now follows from:

**Lemma 3.6.3.** The equalities (3.20) hold for every bounded continuous function $f$. In particular, $\nu$ is a probability measure, and the sequences of probability measures $(\nu_{\infty,L})_L$ and $(\nu_{12k})_k$ converge weakly to $\nu$.

**Proof.** By the Portmanteau theorem, it is sufficient to prove that (3.20) holds for any bounded Lipschitz function on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be such a function. Let $a, b > 0$ be real numbers such that the supports of the measures $(\nu_{12k})_k$ and $(\nu_{12k,L})$ are all contained in the interval $[-\infty, b]$, and let $\chi : \mathbb{R} \to [0, 1]$ be a continuous function with
compact support, whose restriction to the interval \([-a, b]\) is equal to 1. Lemma 3.3.3 implies that we have
\[ \nu_{12k,L}(\cdot) \leq c e^{-\frac{\pi}{h}}, \]
for all \(k \geq L \geq 1\), where \(c\) is an absolute constant. The same estimate holds as well for the measure \(\nu\). In particular, we have
\[ |\nu_{12k,L}(f) - \nu_{12k,L}(\chi f)| = |\nu_{12k,L}((1 - \chi)f)| \leq c||f||_\infty e^{-\frac{\pi}{h}}. \]
Letting \(k\) and then \(L\) tend to infinity, we obtain by corollary 3.6.2 that
\[ \lim_{k \to \infty} \nu_{12k}(f) - \nu(\chi f) \leq c||f||_\infty e^{-\frac{\pi}{h}}. \]
Since we also have
\[ |\nu(f) - \nu(\chi f)| \leq c||f||_\infty e^{-\frac{\pi}{h}}, \]
this yields
\[ |\nu(f) - \lim_{k \to \infty} \nu_{12k}(f)| \leq 2c||f||_\infty e^{-\frac{\pi}{h}}. \]
Letting \(a\) tend to infinity, we obtain that the common limit of the sequences \((\nu_{12k}(f))_k\) and \((\nu_{\infty,L}(f))_L\) is \(\nu(f)\), hence the result.

\[ \square \]

3.7. Proof of part (ii) of Theorem 3.7.1. We will show the following sharper result.

Theorem 3.7.1. The rank of \(S_{12k}(\Gamma, \mathbb{Z})\) over \(\mathbb{Z}\) is \(k\), and \(S_{12k}(\Gamma, \mathbb{Z})\) has \(\{\Delta^k j^{k-\ell} : 1 \leq \ell \leq k\}\) as a basis over \(\mathbb{Z}\). Suppose \(0 \neq f = \sum_{n=1}^{\infty} a_n q^n \in S_{12k}(\Gamma, \mathbb{Z})\). Let \(\text{ord}_\infty(f)\) be the measure of \(\nu\) with respect to the metrics of Proposition 3.1.1. Let \(h : \mathbb{R}_0^+ \to \mathbb{R}\) be the monotonically increasing function defined by
\[ h(c) = 2\pi c + 6(\log(c) + 1 - \log(12)). \]

i. For \(\epsilon > 0\), there are only finitely many \(k\) and \(f\) for which
\[ \lambda(f)/k - h(\text{ord}_\infty(f)/k) \geq \epsilon \]
up to replacing \(f\) by non-zero rational multiple of itself (which does not change \(\lambda(f)\) or \(\text{ord}_\infty(f)\)).

ii. Suppose \(r_0 > h(1) = 2\pi + 6(1 - \log(12)) = -2.62625\ldots\). Then for all sufficiently large \(k\) and all \(f \in S_{12k}(\Gamma, \mathbb{Z})\) one has \(\lambda(f)/k \leq r_0\).

iii. Suppose \(c > 0\) and \(\epsilon > 0\). For all sufficiently large \(k\), there are at least \(e^k\) successive maxima \(\lambda_{i,12k}\) among the total of \(k\) successive maxima associated to \(S_{12k}(\Gamma, \mathbb{Z})\) for which
\[ \frac{\lambda_{i,12k}}{k} \leq h(c) + \epsilon. \]

By Proposition 3.1.1 \(S_{12k}(\Gamma, \mathbb{Z})\) has corank 1 in \(H^0(\mathcal{X}, \mathcal{M}_{12k}(\Gamma))\). The rank of \(H^0(\mathcal{X}, \mathcal{M}_{12k}(\Gamma))\) is \(k+1\), so \(S_{12k}(\Gamma, \mathbb{Z})\) has rank \(k\). The form \(\Delta^k j^{k-\ell}\) lies in \(S_{12k}(\Gamma, \mathbb{Z})\) for \(0 < \ell \leq k\), and its first non-zero term in its Fourier expansion at \(\infty\) is \(q^\ell\). Hence the set of these forms is a \(\mathbb{Z}\)-basis for \(S_{12k}(\Gamma, \mathbb{Z})\), and \(1 \leq \text{ord}_\infty(f) \leq k\) for \(0 \neq f \in S_{12k}(\Gamma, \mathbb{Z})\).
The logarithmic height of \( f \) with respect to the metrics \( || \cdot ||_{L,v} \) we have defined on \( L = \mathcal{M}_{12k}(\Gamma) \) for each place \( v \) of \( \mathbb{Q} \) is

\[
\lambda(f) = -\sum_v \log||f||_{L,v}.
\]

By the product formula, multiplying \( f \) by a non-zero rational number does not change \( \lambda(f) \). We now replace \( f \) by a rational multiple of itself without changing \( \lambda(f) \) to be able to assume \( f \in S_{12k}(\Gamma,\mathbb{Z}) \) is not in \( B \cdot M_{12k}(\Gamma,\mathbb{Z}) \) for any integer \( B > 1 \).

Proposition \[3.1.1\] shows \( ||f||_{L,v} = 1 \) for each finite \( v \), while if \( v = v_\infty \) is the infinite place,

\[
(3.21) \quad ||f||_{L,v_\infty}^2 = \int_{X(\mathbb{C})} |f(z)|^2 (4\pi y)^{12k} \frac{dx\,dy}{y^2}
\]

is the Petersson norm. As in Lemma \[3.3.1\], the above computations now give

\[
2\lambda(f) = -\log\left(\int_{X(\mathbb{C})} |f(z)|^2 (4\pi y)^{12k} \frac{dx\,dy}{y^2}\right)
\]

\[
\leq -\log(4\pi e^{-4\pi N \frac{(12k-2)!}{N^{12k-1}}})
\]

\[
(3.22) \quad = -\log(4\pi) + 4\pi N - \log((12k-2)! + (12k-1)\log(N)).
\]

Suppose \( N = ck \) for some constants \( r \) and \( c \). Since \( \log(N) \geq 0 \), \((3.22)\) gives

\[
2\frac{\lambda(f)}{k} \leq -\frac{\log(4\pi)}{k} + 4\pi c - \frac{\log((12k-2)!)}{k} + (12 - 1/k) \cdot (\log(c) + \log(k))
\]

\[
\leq 4\pi c + \frac{\log(12k-1) + \log(12k)}{k} - \frac{(12k)!}{k} + 12 \cdot (\log(c) + \log(k))
\]

\[
\leq 4\pi c + \frac{\log(12k-1) + \log(12k)}{k} - \frac{12k \log(12k) - 12k}{k} + 12 \cdot (\log(c) + \log(k))
\]

\[
\leq 4\pi c + \frac{2\log(12k)}{k} - \frac{12k \log(12) + \log(k) - 12k}{k} + 12 \cdot (\log(c) + \log(k))
\]

\[
\leq 4\pi c + \frac{2\log(12k)}{k} - 12(\log(12) - 1) + 12\log(c)
\]

\[(3.23)\]

We conclude from \((3.23)\) that

\[
(3.24) \quad \frac{\lambda(f)}{k} - h(c) \leq \log(12k)/k
\]

when \( h(c) = 2\pi c + 6(\log(c) + 1 - \log(12)) \). Thus \((7.39)\) implies that if \( \frac{\lambda(f)}{k} - h(c) \geq \epsilon > 0 \) then \( k \) is bounded above by a function of \( \epsilon \). For each fixed \( k \), we have \( c = N/k \geq 1/k \) so \( h(c) \) is bounded below. Thus \( \lambda(f)/k - h(c) \geq \epsilon > 0 \) implies the Petersson norm of \( f \) is bounded from above. So there are only finitely many possibilities for \( f \) up to multiplication by a non-zero rational number, as claimed in part (i) of Theorem \[3.7.1\].

Part (ii) of Theorem \[3.7.1\] now follows from part (i).

To prove part (iii), suppose \( 1 \leq j \leq k \). By part (i), if \( M(k,j) \) is the submodule of forms \( f \in S_{12k}(\Gamma,\mathbb{Z}) \) for which \( \text{ord}_\infty(f) > j \), the corank of \( M(k,j) \) in \( S_{12k}(\Gamma,\mathbb{Z}) \) is \( j \).
So at least \(j\) successive maxima of \(S_{12k}(\Gamma, \mathbb{Z})\) do not arise from forms in \(N(k,j)\). If \(f\) is not in \(N(k,j)\), then (7.9) shows
\[
\frac{\lambda(f)}{k} \leq h(\text{ord}_\infty(f)/k) + \log(12k)/k \leq h(j/k) + \log(12k)/k
\]
since \(h(c)\) is monotonically increasing with \(c\). Therefore at least \(j\) of the successive minima \(\{\lambda_i, 12k\}_{i=1}^k\) associated to \(S_{12k}(\Gamma, \mathbb{Z})\) satisfy the bound
\[
\frac{\lambda_i, 12k}{k} \leq h(j/k) + \log(12k)/k
\]

Since \(h(c) \to -\infty\) as \(c = j/k \to 0^+\) and \(\log(12k)/k \to 0\) as \(k \to \infty\), this proves part (iii) of Theorem 3.7.1.

3.8. **Proof of part (iii) of Theorem 3.7.1.** We suppose \(0 \neq f \in S_{12k}(\Gamma, \mathbb{Z})\) and that \(f\) does not arise from a congruence between eigenforms, in the sense of Definition 3.2.1. We will develop an upper bound on \(\lambda(f)\). We have \(\lambda(f) = \lambda(f/m)\) when \(m\) is the g.c.d. in \(\mathbb{Z}\) of the Fourier coefficients of \(f\). In view of Definition 3.2.1 we can replace \(f\) by \(f/m\) in order to be able to assume that
\[
f = \sum_i c_i f_i
\]
in which the \(c_i\) are non-zero algebraic algebraic integers and the \(f_i\) are distinct normalized Hecke eigenforms in \(S_{12k}(\Gamma, \mathbb{C})\). The Fourier coefficients of each \(f_i\) are algebraic integers. Since \(f\) is fixed by \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\), the terms on the right side of (3.25) break into orbits under \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) in the following sense. If \(\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) and \(f_i\) is given, then \(\sigma(f_i) = f_j\) and \(c_j = \sigma(c_i)\) for a unique \(j\).

Since the g.c.d. of the Fourier coefficients of \(f\) is now 1, we have
\[
2\lambda(f) = -\ln(\langle f, f \rangle)
\]
where \(\langle f, f \rangle\) is the Petersson norm. The Petersson inner product \(\langle f_i, f_j \rangle\) is 0 if \(i\) is not \(j\) since then \(f_i\) and \(f_j\) have distinct Hecke eigenvalues and the Petersson inner product is Hermitian with respect to Hecke operators. So
\[
\langle f, f \rangle = \sum_i |c_i|^2 \langle f_i, f_i \rangle.
\]
Since each \(0 \neq c_i\) is by assumption an algebraic integer, and we have shown that every Galois conjugate of \(c_i\) arises as \(c_j\) for some \(j\), we conclude there must be an \(i\) for which \(|c_i| \geq 1\). Thus (3.27) gives
\[
\langle f, f \rangle \geq \langle f_i, f_i \rangle
\]
Recall now that since \(f_i\) is a normalized eigenform, \(f_i = \sum_{n=1}^\infty a_n q^n\) has \(a_1 = 1\). So \(N = 1\) in (3.22). Combining Lemma 3.3.1 with (3.26), (3.27) and (3.28) gives
\[
2\lambda(f) = -\ln(\langle f, f \rangle) \leq -\ln(\langle f_i, f_i \rangle) \leq -\ln(4\pi e^{4\pi}(12k - 2)!) 
\]
It follows that \(\lambda(f)/k\) is bounded above by \(-c\ln(k)\) for some constant \(c\). Since the measure \(\nu\) in part (i) of Theorem 3.7.1 is a probability measure on the real line, it
follows that as $k \to \infty$ the proportion of successive minima arising from $f$ of the above kind among all the successive minima associated to $S_{12k}(\Gamma, \mathbb{Z})$ must go to 0.

4. A RESULT FROM ARAKELOV THEORY.

Let $X$ be a projective smooth curve over $K$. We assume that $X$ is geometrically irreducible, of positive genus $g$. Let $L$ be a line bundle on $X$ of degree $d \geq 2g + 1$. Assume $\mathcal{X}$ is a regular model of $X$, and $\mathcal{L}$ a line bundle extending $L$ to $\mathcal{X}$. Denote by $\omega$ the relative dualizing sheaf of $\mathcal{X}$ over Spec($\mathbb{Z}$). Choose a positive metric $h$ on the restriction $L_C$ of $L$ to the Riemann surface $X(\mathbb{C})$. We equip $X(\mathbb{C})$ with the Kähler form $c_1(L_C, h)$, and $\omega$ with the associated metric.

Fix a positive integer $n$. We endow $E = H^0(X, L \otimes n)$ with the $L^2$-norm $h_{L^2}$. Let $E = H^0(X, L \otimes n)$.

Given two hermitian line bundles $\bar{L}_1$ and $\bar{L}_2$ over $\mathcal{X}$, we denote by $\bar{L}_1 \cdot \bar{L}_2 \in \mathbb{R}$ the arithmetic intersection number of the first Chern classes of $\bar{L}_1$ and $\bar{L}_2$. Let $\Delta_K$ be the absolute discriminant of $K$, and $r = [K : \mathbb{Q}]$ its absolute degree. We let $BV \in \mathbb{R}$ be the functional on $X(\mathbb{C})$ defined by Bismut and Vasserot in [5, Theorem 8 (see [21, p. 536)].

Theorem 1. Let $s \in H^0(\mathcal{X}, \mathcal{L}^\otimes n)$ be a global section of $\mathcal{L}^\otimes n$. Then

$$\lambda(s) \leq n \log(n) \frac{3rd}{4g} + n \frac{\bar{L}^2 + d \bar{\omega} \bar{L}}{2gd} + nd \frac{\log(|\Delta_K|)}{2g} - n \frac{BV}{4g} + \frac{nr d}{g} (r_1 + r_2) \log(2) + \frac{nr d}{2g} (1 + \log(2\pi)) + \varepsilon,$$

(4.1)

where $\varepsilon$ is a function of $n$, $\bar{L}_1^2$, $\bar{\omega}_1 \bar{L}_1$, $\bar{\omega}_2$, $L_C$, and of the metric $h$. When $\bar{L}_1^2$, $\bar{\omega}_1 \bar{L}_1$, $\bar{\omega}_2$, $L_C$ and $h$ are fixed, if $n$ tends to infinity, $\varepsilon/n$ goes to zero.

4.1. To prove Theorem 1, we let $\mathbb{P}(E)$ be the (Grothendieck) projective space of $E$ and $X \subset \mathbb{P}(E)$ the canonical embedding of $X$ in $\mathbb{P}(E)$. Denote by $h(X)$ the projective height of $X$. Let $N = r_n = nd + 1 - g$ be the rank of $E$ and, for every $k$ between 1 and $N$, let $\mu_k = -\lambda_{k,n}$ be the $k$-th successive minimum of $(\mathcal{E}, h_{L^2})$. Define

$$\mu = \frac{\mu_1 + \ldots + \mu_N}{N}.$$

If $C$ is the constant

(4.2)

$$C = \frac{2dn g (nd - 2g)}{n^2d^2 + nd - 2g^2}$$

it is proved in [31, Theorem 4, that

(4.3)

$$\frac{h(X)}{r} + 2nd \mu \geq C(\mu - \mu_1).$$
4.2. If $O(1)$ is the restriction to $X$ of the canonical hermitian line bundle on $\mathbb{P}(E)$, the height $h(X)$ is, by definition, the number

$$h(X) = \overline{O(1)} \cdot \overline{O(1)}.$$  

We denote by $h_{FS}$ the metric on $L \otimes n$ induced by the canonical isomorphism $L \otimes n \simeq O(1)$. Let $s_1, \ldots, s_N$ be an orthonormal basis of $(E, h_L)$, and let

$$B(x) = \sum_{j=1}^{N} \|s_j(x)\|^2$$

be the Bergman kernel. For any global section $s \in H^0(X, L \otimes n)$ we have

$$\|s\|^2 = B(x) \|s\|_{FS}^2.$$ 

Therefore, if $\varphi(x) = \log B(x)$, we get

$$n^2 \bar{L}^2 = \overline{O(1)}^2 - \frac{1}{2} \int_{X(\mathbb{C})} \varphi(c_1(O(1)) + nc_1(\bar{L}))$$

(see, for example, [6] (3.2.3)). Bouche [8] and Tian [34] proved that, when $n$ goes to infinity,

$$B(x) = n + \epsilon,$$

where $\epsilon$ is bounded by a constant depending on the restriction to $X(\mathbb{C})$ of $\overline{O(1)}$ and $\bar{L}$. Therefore $\varphi(x) = \log(n) + O\left(\frac{1}{\sqrt{n}}\right)$.

Using (4.4) and (4.5) we conclude that

$$h(X) = n^2 \bar{L}^2 + rdn \log(n) + O(1).$$

4.3. Let $\widehat{\deg}(\widehat{E})$ be the arithmetic degree of $\widehat{E} = (\mathcal{E}, h_L)$, and $r_1$ (resp. $r_2$) the number of real (resp. complex) places of $K$. The second Minkowski theorem, extended to number fields by Bombieri and Vaaler, says that

$$r \mu \leq -\frac{\widehat{\deg}(\widehat{E})}{N} + C(N, K),$$

where

$$C(N, K) = \frac{\log(|\Delta_K|)}{2} + (r_1 + r_2) \log(2) - \frac{1}{N} (r_1 \log V_N + r_2 \log(V_{2N}))$$

$V_N$ being the volume of the standard unit ball in $\mathbb{R}^N$.

By the Stirling formula, when $N$ goes to infinity,

$$\log(V_N) = -\frac{N}{2} \log(N) + \frac{N}{2}(1 + \log(2\pi)) + O(\log(N)).$$

Therefore

$$-\frac{1}{N} (r_1 \log(V_N) + r_2 \log(V_{2N})) = r \left(\frac{\log(N)}{2} - \frac{1 + \log(2\pi)}{2} + O(\log(N)/N)\right).$$
with an absolute constant appearing in $O(\log(N)/N)$. Thus

$$
C(N, K) = \frac{\log(|\Delta_K|)}{2} + (r_1 + r_2) \log(2)
+ r \left( \frac{\log(N)}{2} - \frac{1 + \log(2\pi)}{2} + o(N) \right).
$$

(4.8)

4.4. According to [21], Theorem 8, as $n$ goes to infinity

$$
\hat{\deg}(\bar{\mathcal{E}}) = \frac{n^2 \tilde{L}^2}{2} + n \left( -\bar{\omega} \cdot \bar{L} + \frac{BV}{4} \right) + \frac{rd}{4} n \log(n) + o(n),
$$

where $o(n)$ depends only on the restriction to $X(\mathbb{C})$ of $O(1)$ and $\bar{L}$. Using (4.7) we get

$$
n^2 \tilde{L}^2 + 2 nd r \mu \leq n^2 \tilde{L}^2 - \frac{2 nd}{nd + 1 - g} \left( \frac{n^2 \tilde{L}^2}{2} + \frac{rd}{4} n \log(n) + n \left( -\bar{\omega} \cdot \bar{L} + \frac{BV}{4} \right) + o(n) \right) + 2 nd C(N, K)
+ 2 nd C(N, K) + o(n).
$$

(4.9)

From (4.2) we deduce that, when $n \geq g$,

$$
C \geq 2g(1 - (2g + 1)/(nd)).
$$

Therefore (4.3) and (4.6) imply that

$$
r \mu_1 \geq -\frac{n^2 \tilde{L}^2}{2 nd} - \frac{n^2 \tilde{L}^2}{2g} - \frac{rd n \mu}{g} + \frac{rd}{2g} n \log(n) + o(n),
$$

and, using (4.9) and (4.8), we deduce that

$$
r \mu_1 \geq n \frac{\tilde{L}^2 - d\bar{\omega} \cdot \bar{L}}{2gd} - nd \frac{\log(|\Delta_K|)}{2g} - \frac{3rd}{4g} n \log(n) + \frac{n \cdot BV}{4g}
+ \frac{rd n}{2g} \left( r_1 + r_2 \right) \log(2) - \frac{rd n}{2g} \left( 1 + \log(2\pi) \right) + o(n).
$$

(4.10)

Theorem 1 follows.
4.5. One can also get an upper bound for $r_{\mu_1}$ as follows. Since $\mu_k \leq \mu_{k+1}$ we have
\[ r_{\mu_1} \leq r_{\mu} \leq -\frac{\deg(\mathcal{E})}{N} + C(N, K) = -n \frac{\tilde{L}^2}{2d} + O(\log n). \]
The difference between this upper bound of $r_{\mu_1}$ with its lower bound (4.10) is bounded from below because of the following lemma.

**Lemma 1:**
\[ -n \frac{\tilde{L}^2}{2d} + n \frac{\tilde{L}^2 + d\tilde{\omega}\tilde{L}}{2gd} \geq \frac{nd}{8g(g-1)} \tilde{\omega}^2. \]

4.6. To prove Lemma 1 we note that the hermitian line bundle $d\tilde{\omega} - 2(g - 1)\tilde{L}$ has degree zero on $X$. Therefore, by the Hodge index theorem of Faltings and Hiljač, its square is non positive:
\[ 0 \geq (d\tilde{\omega} - 2(g - 1)\tilde{L})^2, \]
i.e.
\[ -(g - 1)\tilde{L}^2 + d\tilde{\omega}\tilde{L} \geq \frac{d^2}{4(g - 1)} \tilde{\omega}^2 \]
and Lemma 1 follows.

4.7. When $X$ is semi-stable, $X$ has genus at least two, and $\tilde{\omega}$ is equipped with the Arakelov metric [1], it is a result of Ullmo [35] that $\tilde{\omega}^2 > 0$. So, in that case, both sides of the inequality of Lemma 1 are strictly positive.

5. Chebyshev transforms

5.1. **Definition of Chebychev transforms.** In this section, $X$ is projective, of arbitrary dimension $d$. Following Nystrom [], and Yuan [], we take a regular point $x \in X(K)$, and $t_1, \ldots, t_d \in \mathcal{O}_x$ a system of parameters of the regular local ring $\mathcal{O}_{X,x}$, which identifies the completion $\hat{\mathcal{O}}_{X,x}$ to the ring of power series $K[[t_1, \ldots, t_d]]$ in $d$ variable over $K$. We also choose a local trivialization $\sigma \in L_x$ of $L$ around $x$.

Any section $s \in H^0(X, L^\otimes n)$ has a germ at $x$ in $\hat{L}_x^\otimes n = L_x^\otimes n \otimes_{\mathcal{O}_{X,x}} \hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}}_{X,x}^\otimes n$, which can be uniquely written as a a power series
\[ s_x = \left( \sum_{\alpha \in \mathbb{N}^d} a_\alpha t_\alpha \right) \sigma^\otimes n, \]
with $a_\alpha \in K$. Here we have set $t_\alpha = t_1^{\alpha_1} \ldots t_d^{\alpha_d}$. The **order of vanishing** of $s$ at $x$ is defined by the formula
\[ \text{ord}_{x,t}(s) = \min\{ \alpha \in \mathbb{N}^d \mid a_\alpha \neq 0 \}, \]
where the minimum is taken with respect to the lexicographic order on \( \mathbb{N}^d \): this does not depend on \( \sigma \). Likewise, we define the leading coefficient of \( s \) at \( x \) as
\[
\text{lead}_{x,\sigma,t}(s) = a_{\text{ord}_{x,t}(s)} \neq 0.
\]
This depends in general on the choices of \( t \) and \( \sigma \).

One strategy for upper bounding the height \( \lambda(s) \) of a section \( s \) is to apply the product formula
\[
1 = \prod_v |\text{lead}_{x,\sigma,t}(s)|_v^{d_v},
\]
and to give an upper bound of \( |\text{lead}_{x,\sigma,t}(s)|_v \) in terms of \( ||s||_{L_v^n,\sigma} \), which is a problem of local nature; namely it only depends on the \( v \)-adic metric on \( L \). This motivates the introduction of the local quantities
\[
F^{x,\sigma,t}_{L,v}(\alpha) = \sup_{s \in H^0(X,L)_v} \frac{|\text{lead}_{x,\sigma,t}(s)|_v}{||s||_{L,v}},
\]
where \( \alpha \) belongs to the finite set \( \text{ord}_{x,t}(H^0(X,L)_v \setminus \{0\}) \). It is shown in [] that the quantity
\[
c^{x,\sigma,t}_{L,v}(\alpha) = \lim_{n \to \infty} \frac{1}{n} \log F^{x,\sigma,t\otimes n}_{L,v}(\alpha_n),
\]
where \( (\alpha_n)_n \) is a sequence such that \( \alpha_n \in \text{ord}_{x,t}(H^0(X,L^{\otimes n}) \setminus \{0\}) \), and such that \( \frac{1}{n} \alpha_n \) converges to \( \alpha \), is well-defined for any \( \alpha \) in the interior of the closure \( \Delta_{x,t}(L) \) of the set
\[
\bigcup_{n \geq 1} \frac{1}{n} \text{ord}_{x,t}(H^0(X,L^{\otimes n}) \setminus \{0\}).
\]
The set \( \Delta_{x,t}(L) \) is a convex body in \( \mathbb{R}^d \): this is the Okounkov body of \( L \), which depends on the choice of \( t = (t_1, \ldots, t_d) \). For example, if \( X \) is a curve then \( \Delta_{x,t}(L) \) is the interval \([0, \deg(L)]\). Also, if \( (X, L) = (\mathbb{P}^d_K, \mathcal{O}(1)) \), then \( \Delta_{x,t}(L) \) is a \( d \)-dimensional simplex.

The concave function
\[
c^{x,\sigma,t}_{L,v} : \alpha \in \Delta_{x,t}^*(L) \mapsto c^{x,\sigma,t}_{L,v}(\alpha) \in \mathbb{R}
\]
is called the local Chebychev transform of \( L \) at \( x \). The domain \( \Delta_{x,t}^*(L) \) of \( c_{L,v} \) does not depend on the metric on \( L \), but \( c^{x,\sigma,t}_{L,v} \) itself does.

**Example** 5.1.1. Consider the particular case \((X, L) = (\mathbb{P}^1_K, \mathcal{O}(1))\), with the line bundle metric
\[
||s([x_0 : x_1])||_{L,v} = \frac{|s(x_0, x_1)|_v}{\max(|x_0|_v, r^{-1}_v |x_1|_v)},
\]
for some \( r_v > 0 \). The maximum modulus principle if \( v \) is archimedean, and a direct computation otherwise, shows that we have
\[
\|s\|_{L^\otimes_n,\infty} = \sup_{|z|_v = r_v} |s(1, z)|_\infty.
\]
Let us consider the regular point \( x = [1 : 0] \) with a local parameter \( t = X_1 X_0 \), and a local trivialization \( \sigma = X_0 \). We have
\[
F_{x,\sigma}^{\otimes n, t}(\alpha) = \sup_{s \in \mathbb{C}[X_0, X_1]_{n-\alpha}} \|X_1^\alpha s\|_{L^\otimes_n, \nu} \leq e^{nc_{x,\sigma, t}(1/\alpha)} \|s\|_{L^\otimes_n, \nu},
\]
so that \( F_{L^\otimes_n, \nu}(\alpha) \) equals \( r_v^{-\alpha} \). In particular, we have
\[
c_{x, \sigma, t}(\alpha) = -\alpha \log r_v
\]
for \( \alpha \in [0, 1] = \Delta_{x, t}(L) \).

We now define the **global Chebychev transform** as the sum
\[
c_{x, t}^{\otimes n} = \sum_v k_v c_{x, \sigma, t}^{\otimes n},
\]
which still depends on \( t \), but not on the choice of the local trivialization \( \sigma \) any more. While this global Chebychev transform breaks down into a sum of local components, it allows to control global invariants, such that the heights of nonzero sections:

**Proposition 5.1.2.** The height of a nonzero global section \( s \) of \( L^\otimes n \) satisfies
\[
\lambda(s) \leq n \sup_{\alpha \in \Delta_{x, t}(L)} c_{x, t}^{\otimes n}(\alpha).
\]

**Proof.** If a section \( s \in H^0(X, L^\otimes n) \) vanishes at order \( \alpha \) at \( x \), then one has
\[
|\text{lead}_{x, \sigma, t}(s)|_v \leq F_{L^\otimes_n, \nu}(\alpha) \|s\|_{L^\otimes_n, \nu} \leq e^{nc_{x, \sigma, t}^{\otimes n}(1/\alpha)} \|s\|_{L^\otimes_n, \nu}.
\]
Raising this inequality to the power \( k_v \), and taking the product over all places \( v \) yields
\[
1 = \prod_v |\text{lead}_{x, \sigma, t}(s)|_v \leq e^{nc_{x, \sigma, t}^{\otimes n}(1/\alpha)} \prod_v \|s\|_{L^\otimes_n, \nu}^{k_v} = e^{nc_{x, \sigma, t}^{\otimes n}(1/\alpha)-\lambda(s)},
\]
so that \( \lambda(s) \leq nc_{x, t}^{\otimes n}(1/\alpha) \leq n \sup_{\beta \in \Delta_{x, t}(L)} c_{x, t}^{\otimes n}(\beta). \)

Likewise, a theorem of Yuan \([\ldots]\) ensures that the mean value of \( c_{x, t}^{\otimes n} \) computes the expectation of the limit distribution \( \nu : \)
\[
\frac{1}{\text{vol}(\Delta_{x, t}(L))} \int_{\Delta_{x, t}(L)} c_{x, t}^{\otimes n}(\alpha) d\alpha = \int_{\mathbb{R}} x d\nu.
\]
In particular, if \( c_{x, t}^{\otimes n} \) is a constant function, then by the preceding lemma, the left hand side is an upper bound for the support of \( \nu \), so that the expectation of \( \nu \) is an upper bound for its support. This proves:
Corollary 5.1.0.1. If the global Chebychev transform $c_{L}^{x,t}$ is a constant function, then the limit distribution $\nu$ is a dirac measure supported at one point.

Intuitively, the limit distribution $\nu$ is expected to be completely described by $c_{L}^{x,t}$ when the zeroes of sections of large heights concentrates at the point $x$. Since this is not the case in general (see for instance Example [??]), we should obtain better results by considering

$$\sup_{s \in H^0(X,L) \nu} \frac{\text{lead}_{x_1, \sigma, t} (s)}{|s|_{L^2}},$$

where $x_1, \ldots, x_r$ are distinct rational regular points (with a choice of local parameters at each of these points). In section [?], we will compute this quantity for $r = 2$, and a particular choice of $x_1, x_2$, in the context of example [??].

5.2. Computation of Chebychev local transforms at archimedean places: the $L^2$ method. Here we assume for simplicity that $X$ is a curve, i.e. $d = 1$, so that $\Delta_{x,t}(L) = [0, D]$ where $D = \deg(L)$, and we focus on a particular archimedean place $v$. We choose a volume form $dV$ on $X(\mathbb{C}_v)$, so that $H^0(X,L \otimes n)_{\nu}$ is endowed with the hermitian norm

$$||s||_{L^2}^2 = \int_{X(\mathbb{C}_v)} |s(x)|_{L^2}^2 \nu dV(x).$$

One can show using Gromov lemma (see [??] or [??]) that the Chebychev local transform $c_{L}^{x,t}(\alpha)$ can be computed using

$$F_{L^2, \nu, \text{herm}}(\alpha) = \sup_{s \in H^0(X,L \otimes n)_{\nu}} \frac{\text{lead}_{x_1, \sigma, t} (s)}{|s|_{L^2}},$$

instead of $F_{L^2, \nu, \text{herm}}(\alpha)$. Let us denote by $[\alpha]$ the linear form on $H^0(X,L \otimes n(-\alpha x))_{\nu}$ which takes a section $s$ to the coefficient of $t^n$ in its Taylor series expansion around $x$, so that

$$F_{L^2, \nu, \text{herm}}(\alpha) = \sup_{s \in H^0(X,L \otimes n(-\alpha x))_{\nu}} \frac{||[\alpha](s)||_{L^2}}{|s|_{L^2}}.$$

is the operator norm of $[\alpha]$ on the hermitian space $H^0(X,L \otimes n(-\alpha x))_{\nu}$. In particular, if $(s_{x,\alpha,j})_j$ is an orthonormal basis of $H^0(X,L \otimes n(-\alpha x))_{\nu}$, then we have

$$F_{L^2, \nu, \text{herm}}(\alpha)^2 = \sum_j |[\alpha](s_{x,\alpha,j})|^2.$$ 

For $\alpha = 0$, this equals the value of the $n$-th Bergman kernel at $x$, for which precise asymptotics are known (see [??]). The case $\alpha > 0$ is much more elusive in general, but we will see in the remaining of this section how to handle completely the case of the Fubini-Study metric, and partially the case of the capacity metric of a disc on the projective line, by computing $F_{L^2, \nu, \text{herm}}(\alpha)$ with an explicit orthonormal basis.
5.3. The $L^2$ method in use: the Chebychev local transform of the capacity metric of a disc. Let us consider $(X(\mathbb{C}_v), L) = (\mathbb{P}^1(\mathbb{C}_v), \mathcal{O}(1))$, with the line bundle metric
\[
|s(x_0 : x_1)|_{L,v} = \frac{|s(x_0, x_1)|_v}{\max(|x_0|_v, r_v^{-1}|x_1|_v)},
\]
at an archimedean place $v$, which is the capacity metric associated to a disc of radius $r_v$ in the complex projective line, just as in example 5.11. Contrary to the situation considered in 5.11, we choose the point $x = \frac{1}{2}(x_1)$ with a local parameter $t = \frac{x_1 - x_0}{x_0}$, and a local trivialization $\sigma = X_0$. Instead of considering a volume form $dV$ as above, we rather use the distribution $dL$ defined by
\[
\int_{\mathbb{P}^1(\mathbb{C}_v)} f dV = \frac{1}{4} \int_{-\pi}^\pi f([1 : r_v e^{i\theta}]) |\sin(\theta)| d\theta,
\]
since one can check that the corresponding $F_{L \otimes n,v,\text{herm}}$ still computes $c_{L,v}$. We now show that we have the formula
\[
F_{L \otimes n,v,\text{herm}}(\alpha)^2 = 4^{-2\alpha} r_v^{-4\alpha} \sum_{j=0}^{n-\alpha} (2j + 2\alpha + 1) \binom{j + 2\alpha}{j}^2.
\]
Using Stirling’s formula, this will imply the following:

**Proposition 5.3.1.** With $x, \sigma, t$ as above, the local Chebychev transform of the capacity metric associated to a disc of radius $r_v$ on the complex projective line, as defined above, with respect to a point on the boundary of the disc, is given by the formula
\[
\psi_{L,v}(\alpha) = -\alpha \log(4r_v) + \frac{1}{2} (1 + \alpha) \log(1 + \alpha) - \frac{1}{2} (1 - \alpha) \log(1 - \alpha) - \alpha \log(\alpha).
\]
for $\alpha \in [0, 1]$.

In order to compute $F_{L \otimes n,v,\text{herm}}(2\alpha)$, let us consider the orthogonal decomposition
\[
\mathbb{C}_v[X_0, X_1]_{2n-2k} = V^+ \oplus V^-,
\]
where $V^\pm$ is the space of polynomials $s \in \mathbb{C}_v[X_0, X_1]_{2n-2k}$ such that $s(X_0, X_1) = \pm s(r_v^{-1}X_1, r_vX_0)$. Since any $s$ in $V^-$ satisfies $s(1, r_v) = 0$, we get
\[
F_{L \otimes n,v,\text{herm}}(2\alpha) = \sup_{s \in V^+} \frac{|s(1, r_v)|_v}{||(X_1 - r_vX_0)^{2\alpha} s||_{L \otimes n,v,\text{herm}}}.
\]
However, the linear map
\[
\Psi : T = T(Y_0, Y_1) \in \mathbb{C}_v[Y_0, Y_1]_{n-k} \mapsto T(r_vX_0X_1, r_v^2X_0^2 + X_1^2) \in V^+
\]
is an isomorphism, with
\[
||(X_1 - r_vX_0)^{2\alpha} \Psi(T)||_{L \otimes n,v,\text{herm}}^2 = \frac{r_v^{4n}}{4} \int_{-\pi}^\pi |T(1, 2 \cos(\theta))|^2 |\sin(\theta)| |e^{i\theta} - 1|^{4\alpha} d\theta
\]
\[
= \frac{r_v^{4n}}{4} \int_{-2}^2 |T(1, y)|^2 (2 - y)^{2\alpha} dy,
\]
We proceed as in section 5.3, using the Fubini-Study volume form $u$ is strictly positive. Again, the functions

$$\text{Jac}_{\alpha,j}(Y_0, Y_1) = (2Y_0 - Y_1)^{-2\alpha} \frac{\partial^j}{\partial Y_1^j} Y_0^{n-\alpha-j} (2Y_0 + Y_1)^j (2Y_0 - Y_1)^{j+2\alpha}$$

for $0 \leq j \leq n - \alpha$. The explicit formulae

$$\Psi(\text{Jac}_{\alpha,j})(1, r_v) = r_v^{2n-2\alpha} (-4)^j \binom{j + 2\alpha}{j},$$

$$|| (X_1 - r_vX_0)^{2\alpha} \Psi(\text{Jac}_{\alpha,j}) ||_{L^\otimes n,v,\text{herm}} = r_v^{2n} 4(j + 2\alpha + 1)^{-\frac{1}{2}},$$

yield

$$P_{L^\otimes 2n,v,\text{herm}}^x(s^{\otimes 2n}) (2\alpha)^2 = \sum_{j=0}^{n-\alpha} \frac{|| (X_1 - r_vX_0)^{2\alpha} \Psi(\text{Jac}_{\alpha,j}) ||_{L^\otimes n,v,\text{herm}}^2}{||(X_1 - r_vX_0)^{2\alpha} \Psi(\text{Jac}_{\alpha,j}) ||_{L^\otimes n,v,\text{herm}}} = 4^{-2\alpha} r_v^{-4\alpha} \sum_{j=0}^{n-\alpha} (2j + 2\alpha + 1)^2 \binom{j + 2\alpha}{j}^2,$$

hence the result.

5.4. The $L^2$ method in use: the Chebychev local transform of the Fubini-Study metric. Let us consider the complex projective space $(X(\mathbb{C}_v), L) = (\mathbb{P}^d(\mathbb{C}_v), \mathcal{O}(1))$, with the Fubini-Study metric

$$|s(x_0 : x_1)|_{L,v} = \frac{|s(x_0, x_1)|_v}{\sqrt{|x_0|_v^2 + \ldots + |x_d|_v^2}},$$

at an archimedean place $v$. We pick a point $x = [x_0 : \cdots : x_d]$ of $\mathbb{P}^d(\mathbb{C}_v)$. We have a natural identification

$$T_x \mathbb{P}^d(\mathbb{C}_v) = \{ T \in \mathbb{C}_v[X_0, \ldots, X_d]_1 \mid T(x_0, \ldots, x_d) = 0 \}.$$

Let $T_1, \ldots, T_{d+1}$ be a linear basis of $\mathbb{C}_v[X_0, \ldots, X_d]_1$, such that $T_1, \ldots, T_d$ span $T_x \mathbb{P}^d(\mathbb{C}_v)$ under the identification above. The functions $t_j = \frac{T_j}{T_{d+1}}$, for $j = 1, \ldots, d$, then form a system of local parameters at $x$, while $\sigma = T_{d+1}$ is a local trivialization of $L$ around $x$.

We proceed as in section 5.3 using the Fubini-Study volume form $dV = \frac{\omega_{FS}}{d}$, where

$$\omega_{FS} = i \sigma \bar{\sigma} \log(||X_0|| + \ldots + |X_d||).$$

Let $U_1, \ldots, U_{d+1}$ be the output of the Gram-Schmidt orthonormalization process applied to the basis $T_1, \ldots, T_{d+1}$. In particular, $U_1, \ldots, U_{d+1}$ form an orthonormal basis of $\mathbb{C}_v[X_0, \ldots, X_d]_1$, and each coefficient

$$\gamma_j = \langle T_j | U_j \rangle^{-1}$$

is strictly positive. Again, the functions $u_j = \frac{U_j}{U_{d+1}}$, for $j = 1, \ldots, d$, form a system of local parameters at $x$, and $\tau = U_{d+1}$ is a local trivialization of $L$ around $x$. One can
check the formulae

\[ \text{ord}_{x,t}(s) = \text{ord}_{x,u}(s) \]

\[ \text{lead}_{x,\sigma \otimes n, t}(s) = \gamma^\alpha \text{ord}_{x,\sigma \otimes n, u}(s) \text{ if } \alpha = \text{ord}_{x,t}(s) \in \mathbb{N}^d, \]

with \( \alpha_{d+1} = 1 - \sum_{j=1}^{d} \alpha_j \). In particular, we have,

\[ F_{x,\sigma \otimes n, t}^{x, \tau \otimes n, u}(\alpha) = \alpha_{d+1} \log(\gamma) + \frac{1}{2} h_d(\alpha) \]

on the Okounkov body

\[ \Delta_{x,t}(L) = \{ \alpha \in \mathbb{R}_+^d \mid \alpha_1 + \ldots + \alpha_d \leq 1 \}, \]

where \( h_d \) is the entropy functional, defined by

\[ h_d : \alpha \in \Delta_{x,t}(L) \mapsto \sum_{j=1}^{d+1} \alpha_j \log \left( \frac{1}{\alpha_j} \right) \text{ where } \alpha_{d+1} = 1 - \sum_{j=1}^{d} \alpha_j. \]

6. Measures associated to zeros of sections.

We develop in this section a counterpart for the zeros of the non-zero elements of \( \bigcup_{n=1}^{\infty} H^0(X, L^{\otimes n})^{\geq \lambda} \) of a theory of Serre [29] concerning measures arising from Weil numbers associated to abelian varieties of arbitrary dimension over a given finite field. To motivate the study of zeros of high slope we will first discuss an example.

Let \( X = \mathbb{P}^1_\mathbb{Q} \) and \( L = \mathcal{O}(1) \). We endow \( L \) with the non archimedean metrics coming from the integral model \( (\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(1)) \), and the archimedean metric given in affine coordinates by

\[ |s(z)|_{L, \infty} = \frac{|s(z)|}{\max(1, |4z - 1|)}. \]
This is the capacity metric associated to the disc of center $\frac{1}{4}$ and radius $\frac{1}{4}$. For the sake of the computation, we rather use the $L^2$ metric on the boundary of this disc, rather than the supremum norm: this does not affect the asymptotic slopes.

Let $s_n$ denote a degree $n$ nonzero integer polynomial of smallest norm. A computation performed with Magma yields a small list of explicit irreducible integer polynomials $f_1, f_2, f_3, \ldots$, starting with

\[
\begin{align*}
    f_1 &= z, \\
    f_2 &= 2z - 1, \\
    f_3 &= 5z^2 - 4z + 1, \\
    f_4 &= 29z^4 - 44z^3 + 27z^2 - 8z + 1,
\end{align*}
\]

such that

\[
\begin{align*}
    s_{50} &= \pm f_1^{34} f_2^6 f_3 f_4, \\
    s_{100} &= \pm f_1^{63} f_2^{11} f_3^4 f_4 f_5 f_6, \\
    s_{200} &= \pm f_1^{127} f_2^{23} f_3^8 f_4^3 f_5 f_6 f_7, \\
    s_{300} &= \pm f_1^{190} f_2^{34} f_3^{12} f_4^4 f_5^2 f_6 f_7.
\end{align*}
\]

The polynomials $f_5, f_6, f_7, f_8$ have degree 6, 8, 8 and 2 respectively. Numerically, the quantity $\frac{1}{n} \text{ord}_{f_1}(s_n)$ seems to converge to a limit (close to 0.63) as $n$ grows. Similarly, $\lim_{n \to \infty} \frac{1}{n} \text{ord}_{f_j}(s_n)$ appears to exist for higher $j$. This suggests the existence of a limit distribution of zeros associated to sections of minimal norm which is discrete.

However, replacing the disc of center $\frac{1}{4}$ and radius $\frac{1}{4}$ by a the disc of center 0 and radius 1, the corresponding lattices become asymptotically semistable, and one doesn’t expect such a discreteness result, but rather a uniform distribution of the zeros of small sections along the boundary of the unit disk.

In this section, we give support to the intuition that the general case must interpolate between these two situations.

6.1. Measures and distributions. We begin by recalling from [22,9,10] some basic definitions concerning Borel measures on a locally compact Hausdorff topological space $Z$ whose topology has a countable base.

Following [22, p. 219] we define $c(Z)$ to be the class of compact subsets of $Z$. The $\sigma$-algebra $b(Z)$ of Borel subsets of $Z$ is the smallest $\sigma$-algebra that contains $c(Z)$. A Borel measure $\mu$ is defined on $[22, p. 30, 219]$ to be an extended real valued non-negative countably additive function on $b(Z)$ such that $\mu(\emptyset) = 0$. We will say that $\mu$ is bounded if its values are bounded, and that $\mu$ is a probability measure if $\mu(Z) = 1$. Let $C_c(Z)$ be the real vector space of continuous functions on $Z$ that have compact support. By [9, Cor. to Thm. 5, p. IV.58] and [9, Def. 2, p. III.7], the integral $\int f \mu$ is a well defined real number if $f \in C_c(Z)$ and $\mu$ is a bounded Borel measure.

Recall now that $X$ is a smooth projective curve over a global field $F$. For each place $v$ of $F$ we let $\overline{C}_v$ be the completion of an algebraic closure $\overline{F}_v$ of $F_v$. 


Suppose first that $v$ is archimedean. Then $C_v = \mathbb{C}$, and $X(C_v)$ is compact. We let $M_v$ be the set of all bounded Borel measures on $X(C_v)$. If $v$ is non-archimedean, $X(C_v)$ is not locally compact, so that one cannot simply apply the usual theory of measures on Borel sets to $X(C_v)$. A natural alternative is to work instead with the Berkovich space $X_{Berk,C_v}$ defined in [3]; see also [33, 2, 32, 19, 11, 12, 13]. The space $X_{Berk,C_v}$ contains $X(C_v)$ in a natural way and is Hausdorff, locally compact, path connected and has a countable base. For non-archimedean $v$ we let $M_v$ be the set of all bounded Borel measures on $X_{Berk,C_v}$.

For archimedean (resp. non-archimedean) $v$, will say that a sequence $\{\mu_i\}_i^{\infty}$ of elements of $M_v$ converges to an element $\mu \in M_v$ if for all $f \in C_c(X(C_v))$ (resp. $f \in C_c(X_{Berk,C_v})$) one has

$$\lim_{i \to \infty} \int f \mu_i = \int f \mu.$$  

This notion of convergence is equivalent to weak convergence as defined in [4, Chapter 1, §1].

6.2. Dirac, atomic and continuous measures. Let $v$ be a place of $F$ and suppose $x \in X(C_v)$. If $v$ is archimedean, let $\delta_x \in M_v$ be the dirac measure associated to $x$. If $v$ is non-archimedean, we view $x$ as a point of $X_{Berk,C_v}$ and we again let $\delta_x \in M_v$ be the associated dirac measure. Suppose $D = \sum_{x \in X(C_v)} m_x x$ is a non-zero effective divisor of $X(C_v)$ that is stable under the action of $\text{Aut}(C_v/F)$, so that $m_x = 0$ for almost all $x$. We define the dirac measure of $D$ to be

$$\mu(D) = \frac{1}{\deg(D)} \sum_{x \in X(C_v)} m_x \delta_x.$$

Let $T$ be a countable collection of effective non-zero elements of $\text{Div}_F(X(C_v))$. Define the set $M_v(T)$ of atomic measures associated to $T$ to be the set of all measures of the form

$$\nu = \sum_{j=1}^{\infty} \lambda_j \cdot \mu(D_j) = \lim_{N \to \infty} \sum_{j=1}^{N} \lambda_j \cdot \mu(D_j)$$  

for an ordered set $\{D_j\}_{j=1}^{\infty}$ of elements of $T$ and a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of non-negative numbers such that $\sum_{j=1}^{\infty} \lambda_j = 1$. Let $\overline{M_v(T)}$ be the sequential closure of $M_v(T)$ in $M_v$. Thus $\overline{M_v(T)}$ consists of those $\mu \in M_v$ for which $\mu = \lim_{i \to \infty} \mu_i$ for some countable ordered subset $\{\mu_i\}_{i=1}^{\infty} \subset M_v(T)$.

We write $T - S$ for the complement in $T$ of a finite subset $S$ of $T$. We will call the elements of

$$\overline{M_{\text{cont},v}(T)} = \cap_S \overline{M_v(T - S)}$$

the continuous elements of $\overline{M_v(T)}$, where in this intersection $S$ ranges over all finite subsets of $T$. Thus $\overline{M_{\text{cont},v}(T)}$ is the set of elements of $M_v$ that can be realized as limits of atomic measures associated to elements of $T - S$ for each finite subset $S$ of $T$. 

6.3. **Decompositions of measures.** The following result was inspired by a talk by J. P. Serre in November of 2015 [29].

**Proposition 6.3.1.** Fix a choice of a countable collections $T$ and $T'$ of effective non-zero divisors $D$ of $X(\mathbb{C}_v)$ that are stable under the action of $\text{Aut}(\mathbb{C}_v/F)$ such that $T \subset T'$. Each element $\mu$ of $\overline{\mathcal{M}}(T)$ can be written as

$$\mu = \alpha \cdot \mu_{\text{atomic}} + \beta \cdot \mu_{\text{cont}}$$

for some real constants $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, some atomic measure $\mu_{\text{atomic}}$ in $\mathcal{M}(T')$, and some measure $\mu_{\text{cont}}$ that is in $\overline{\mathcal{M}}_{\text{cont},v}(T')$ if $\beta \neq 0$.

**Remark 6.3.2.** In the following examples, we let $F = \mathbb{Q}$ and $X = \mathbb{P}^1 \mathbb{F}_k$ and we take $v$ to be the archimedean place.

i. The decomposition in (6.2) need not be unique. For example, let $D_r$ be the divisor associated to the rational point $r \in \mathbb{Q}$ relative to some affine coordinate $z$ on $X$. Define $T = \{D_r : r \in \mathbb{Q}\} = T'$. Then $\mu(D_0)$ is atomic, and $\mu(D_0) = \lim_{r \to 0+} \mu(D_r)$ is also continuous.

ii. The decomposition in (6.2) is sometimes unique. This is the case if $T = \{D_r : r \in \mathbb{Z}\} = T'$. Then $\lim_{r \to \infty} \mu(D_r)$ is the Dirac measure $\mu(D_\infty)$ associated to the point at infinity. This is the unique element of $\overline{\mathcal{M}}_{\text{cont},v}(T')$, and no real multiple of $\mu(D_\infty)$ is atomic.

iii. The decomposition in (6.2) depends on the choice of $T'$. For instance, suppose $T = \{D_0 + D_r : r \in \mathbb{Z}\}$ and $T' = \{D_r : r \in \mathbb{Z}\}$. Then $\lim_{r \to \infty} \mu(D_0 + D_r) = \frac{1}{2}(\mu(D_0) + \mu(D_\infty))$ is in $\overline{\mathcal{M}}_{\text{cont},v}(T)$, since this limit does not change if we let $r$ range over all arbitrarily large integers. But $\frac{1}{2}(\mu(D_0) + \mu(D_\infty))$ is not in $\overline{\mathcal{M}}_{\text{cont},v}(T')$.

iv. The choice of $T$ specifies the kind of input data one wants to consider when taking limits of dirac measures. The choice of $T'$ controls how such limits can be decomposed into atomic and continuous parts. Increasing $T'$ enlarges the possible atomic measures. Example (iii) above shows some measures may cease to be continuous with respect to a larger $T'$.

**Remark 6.3.3.** In Theorem (6.1.3 c) we will apply these notions to the zeros of sections of a metrized line bundle $L$ on $X$. Suppose $1 \leq n \in \mathbb{Z}$ and that $f$ is a non-constant section of $f \in H^0(X, L^\otimes n)$. Let zer$(f)$ be the effective non-zero divisor of zeros of $f$. Suppose that $\lambda \in \mathbb{R} \cup \{-\infty\}$. Let $T = T(\lambda)$ be the set of divisors zer$(f)$ associated to non-constant elements $f$ of $\cup_{n \geq 1} H^0(X, L^\otimes n)_{\geq \lambda}$. We define $\mathcal{M}_v^\geq_{\lambda}$ to be the resulting set $\mathcal{M}_v(T(\lambda))$ of atomic measures for this choice of $T = T(\lambda)$. In this case, we will set $T' = T(\lambda)'$ to be the set of irreducible components of elements of $T$. We define $\overline{\mathcal{M}}_v^\geq_{\lambda} = \overline{\mathcal{M}}_{\text{cont},v}(T(\lambda))$ and $\overline{\mathcal{M}}_v^\geq_{\lambda} = \overline{\mathcal{M}}_{\text{cont},v}(T(\lambda)' )$. One reason for allowing $T'$ to be larger in this case is to be better able to detect, via the atomic parts of limit measures, whether small sections must all vanish on particular irreducible divisors.
To begin the proof of Proposition 6.3.1, write

\[(6.3) \quad \mu = \lim_{i \to \infty} \mu_i \]

for some countable sequence \(\{\mu_i\}_{i=0}^{\infty}\) of elements of \(\mathcal{M}_v(T)\). Fix an ordering \(\{D_j\}_{j=1}^{\infty}\) of the elements of \(T'\). We can then write each \(\mu_i\) as a convergent infinite sum

\[
\mu_i = \sum_{j=1}^{\infty} a_{i,j} \cdot \mu(D_j)
\]

in which \(a_{i,j} \geq 0\) and \(\sum_{j=1}^{\infty} a_{i,j} = 1\).

**Lemma 6.3.4.** To prove Proposition 6.3.1, we can (and will) assume from now on that \(T = T'\). There is a sequence \(\{\tilde{\mu}_i\}_{i=1}^{\infty}\) of measures

\[(6.4) \quad \tilde{\mu}_i = \sum_{j \geq 1} \tilde{a}_{i,j} \cdot \mu(D_j) \]

in \(\mathcal{M}_v(T)\) with the following properties:

a. \(\lim_{i \to \infty} \tilde{\mu}_i = \mu\).

b. For each \(j \geq 1\), the limits \(c_j = \lim_{i \to \infty} \tilde{a}_{i,j}\) and \(c = \sum_{j=1}^{\infty} c_j\) exist, and \(0 \leq c \leq 1\).

c. If \(c = \sum_{j=1}^{\infty} c_j < 1\) we can require \(\tilde{a}_{i,j} \geq c_j\) for all integers \(i\) and \(j\).

**Proof.** The first statement is clear from the fact that \(\mathcal{M}_v(T) \subset \mathcal{M}_v(T')\). The sequences \((a_{i,j})_{j=1}^{\infty}\) lie in the space \(\prod_{j=1}^{\infty} [0, 1]\), and this space is sequentially compact for the product topology. We can thus pass to a subsequence of the \(\mu_i\) to be able to assume that \(c_j = \lim_{i \to \infty} a_{i,j}\) exists for all \(i\). We assume in what follows that the \(a_{i,j}\) have this property.

Since \(a_{i,j} \geq 0\) for all \(i, j\) we have \(0 \leq c_j\) for all \(j\). For each \(1 \leq N \in \mathbb{Z}\) and \(\epsilon > 0\), there is an integer \(m(N, \epsilon) \geq 1\) such that \(\sum_{j=1}^{N} c_j \leq (\sum_{j=1}^{N} a_{i,j}) + \epsilon\) if \(i \geq m(N, \epsilon)\). Since \(\sum_{j=1}^{\infty} a_{i,j} = 1\) for all \(i\), we conclude that \(\sum_{j=1}^{N} c_j \leq 1\) for all \(N\). It follows that \(c = \sum_{j=1}^{\infty} c_j\) converges and \(0 \leq c \leq 1\).

If \(c = 1\), we can let \(\tilde{a}_{i,j} = a_{i,j}\) for all \(i\) and \(j\) to prove the Lemma.

Suppose now that \(c = \sum_{j=1}^{\infty} c_j < 1\). We define \(T_i^+ = \{j \mid 1 \leq j \in \mathbb{Z}\} \text{ and } (a_{i,j} - c_j) \geq 0\) and \(T_i^- = \{j \mid 1 \leq j \in \mathbb{Z}\} \text{ and } j \not\in T_i^+\}. By the dominated convergence theorem, we have

\[
\epsilon_i = \frac{1}{1 - c} \sum_{j \in T_i^-} (c_j - a_{i,j}) \to 0,
\]
as \( i \) tends to infinity. After discarding the first terms of the sequence \( \{ \mu_i \}_{i=1}^{\infty} \), we can assume that the inequality \( \epsilon_i \leq 1 \) holds for all \( i \). We then have
\[
\sum_{j \in T_i^n} \epsilon_i (a_{i,j} - c_j) \geq \epsilon_i \sum_{1 \leq j \in \mathbb{Z}} (a_{i,j} - c_j) = \epsilon_i (1 - c) = \sum_{j \in T_i^-} (c_j - a_{i,j})
\]
This implies the existence of a sequence \( \{ b_{i,j} \}_{j \in T_i^+} \) such that \( 0 \leq b_{i,j} \leq \epsilon_i (a_{i,j} - c_j) \) for any \( j \in T_i^+ \), and such that
\[
(6.5) \sum_{j \in T_i^+} b_{i,j} = \sum_{j \in T_i^-} (c_j - a_{i,j}).
\]
For \( j \in T_i^+ \) we now set \( \tilde{a}_{i,j} = a_{i,j} - b_{i,j} \) if \( j \in T_i^+ \), and we let \( \tilde{a}_{i,j} = c_j \) if \( j \in T_i^- \). If \( j \in T_i^+ \) then
\[
\tilde{a}_{i,j} = a_{i,j} - b_{i,j} \geq a_{i,j} - \epsilon_i (a_{i,j} - c_j) \geq a_{i,j} - (a_{i,j} - c_j) = c_j \geq 0
\]
because \( \epsilon_i \leq 1 \). So \( \tilde{a}_{i,j} \geq c_j \) for all \( i,j \), proving statement (c) of the Lemma.

Because of (6.5), the measures
\[
\tilde{\mu}_i = \sum_{j \geq 1} \tilde{a}_{i,j} \mu(D_j)
\]
have the property that
\[
\sum_{j} \tilde{a}_{i,j} = \sum_{j \in T_i^+} a_{i,j} - \sum_{j \in T_i^+} b_{i,j} + \sum_{j \in T_i^-} c_j = \sum_{j \in T_i^+} a_{i,j} - \sum_{j \in T_i^+} (c_j - a_{i,j}) + \sum_{j \in T_i^-} c_j = \sum_{j} a_{i,j} = 1.
\]
Thus \( \tilde{\mu}_i \in \mathcal{M}_v(T) \).

For all \( j \) we have
\[
c_j = \lim_{i \to \infty} \tilde{a}_{i,j}
\]
because \( c_j = \lim_{i \to \infty} a_{i,j} \) and \( \lim_{i \to \infty} \epsilon_i = 0 \). Thus statement (b) of the Lemma holds.

From (6.5) we have
\[
\sum_{j \geq 1} |\tilde{a}_{i,j} - a_{i,j}| = \sum_{j \in T_i^+} b_{i,j} + \sum_{j \in T_i^-} (c_j - a_{i,j}) = 2(1 - c) \epsilon_i \to 0 \quad \text{as} \quad i \to \infty.
\]
Consequently the sequence \( \tilde{\mu}_i \) has the same limit as \( \mu_i \) as required by condition (a) of the Lemma. \( \square \)

We now continue the proof of Proposition 6.3.1 after replacing \( \{ \mu_i \}_{i=1}^{\infty} \) by the sequence \( \{ \tilde{\mu}_i \}_{i=1}^{\infty} \) produced by Lemma 6.3.4. To simplify notation we will now assume \( \mu_i = \tilde{\mu}_i \) for all \( i \). Recall that we have assumed \( T = T' \).

Suppose first that \( c_j = 0 \) for all \( j \). We claim in this case that \( \mu \in \mathcal{M}_{cont,v}(T) \), so (6.2) holds with \( \alpha = 0, \beta = 1 \), \( \mu_{cont} = \mu \) and \( \mu_{atomic} \) any element of \( \mathcal{M}_v(T) \). To prove
this, it will suffice to show that $\mu \in \overline{M_v}(T - S)$ for all finite subsets $S$ of $T$. Such an $S$ is contained in $\{D_j\}_{j=1}^N$ for some integer $N$. Define

$$h(i) = \sum_{j=N+1}^{\infty} a_{i,j} = 1 - \sum_{j=1}^{N} a_{i,j}.$$  

Since $c_j = \lim_{i \to \infty} a_{i,j} = 0$ for all $j$ we have $\lim_{i \to \infty} h(i) = 1$. So for sufficiently large $i$, one has $h(i) > 0$, and the measure $\nu_i = \frac{1}{h(i)} \sum_{j=N+1}^{\infty} a_{i,j}\mu(D_j)$ lies in $M_v(T - S)$. Write

$$\mu_i = \sum_{j=1}^{N} a_{i,j} \mu(D_j) + h(i)\nu_i.$$  

Since $\lim_{i \to \infty} a_{i,j} = 0$ for $1 \leq j \leq N$ and $\lim_{i \to \infty} h(i) = 1$, this shows $\mu_i = \lim_{i \to \infty} \nu_i$ lies in $\overline{M_v}(T - S)$, as claimed.

Now suppose $c_j > 0$ for some $j$. We have shown $0 \leq c = \sum_{j=1}^{\infty} c_j \leq 1$, so in fact $0 < c \leq 1$ and

(6.6) $\nu = \frac{1}{c} \sum_{j=1}^{\infty} c_j \mu(D_j)$

is an atomic measure, i.e. an element of $M_v(T)$.

Suppose first that $c = \sum_{j=1}^{\infty} c_j = 1$. For any $\epsilon > 0$, we can then find an integer $N \geq 1$ such that $\sum_{j=1}^{N} c_j \geq 1 - \epsilon$ and an integer $i(N) \geq N$ such that $|a_{i(N),j} - c_j| \leq \epsilon/N$ for $1 \leq j \leq N$. Now

$$1 \geq \sum_{j=1}^{N} a_{i(N),j} \geq \sum_{j=1}^{N} c_j - \epsilon \geq 1 - 2\epsilon.$$  

Hence

$$0 \leq \sum_{j=N+1}^{\infty} a_{i(N),j} \leq 2\epsilon \quad \text{and} \quad 0 \leq \sum_{j=N+1}^{\infty} c_j = 1 - \sum_{j=1}^{N} c_j \leq \epsilon.$$  

It follows that

$$\sum_{j=1}^{\infty} |a_{i(N),j} - c_j| \leq \sum_{j=1}^{N} |a_{i(N),j} - c_j| + \sum_{j=N+1}^{\infty} (a_{i(N),j} + c_j) \leq N \cdot \epsilon/N + 2\epsilon + \epsilon = 4\epsilon.$$  

Now

$$\mu_i(N) = \sum_{j=1}^{\infty} a_{i(N),j} \mu(D_j) \quad \text{and} \quad \nu = \sum_{j=1}^{\infty} c_j \mu(D_j)$$  

with

$$\sum_{j=1}^{\infty} a_{i(N),j} = 1 = \sum_{j=1}^{\infty} c_j.$$
We now let $f$ be continuous real valued function with compact support on $X(\mathbb{C}_v)$ if $v$ is archimedean and on $X_{Berk,\mathbb{C}_v}$ if $v$ is non-archimedean. The above computations show

$$|\int f\mu_i(N) - \int f\nu| \leq 4 \cdot \epsilon \cdot \sup_z |f(z)|.$$  

Since $\epsilon$ was arbitrary and $\lim_{i \to \infty} \mu_i$ was assumed to exist, it follows from the definition of convergence of measures that

$$\mu = \lim_{i \to \infty} \mu_i = \nu$$

when $c = 1$. So we can let $\alpha = 1$, $\beta = 0$, $\mu_{\text{atomic}} = \mu = \nu$ and we can pick $\mu_{\text{cont}}$ to be any measure since we put no conditions on $\mu_{\text{cont}}$ when $\beta = 0$.

Finally, suppose $0 < c = \sum_{j=1}^{\infty} c_j < 1$. We arranged by means of Lemma 6.3.4 that $b_{i,j} = a_{i,j} - c_j \geq 0$ for all $i$ and $j$. Since

$$(6.7) \sum_{j=1}^{\infty} b_{i,j} = \sum_{j=1}^{\infty} a_{i,j} - \sum_{j=1}^{\infty} c_j = 1 - c > 0$$

the measure

$$(6.8) \nu_i = \sum_{j=1}^{\infty} \frac{b_{i,j}}{1-c} \cdot \mu(D_j)$$

is in $\mathcal{M}_v(T)$. Set $\alpha = c$, $\beta = 1 - c$ and $\mu_{\text{atomic}} = \nu$ as in (6.6). We have

$$(6.9) \mu_i = \alpha \cdot \mu_{\text{atomic}} + \beta \cdot \nu_i$$

for all $i$. Hence prove that there is an equality of the form (6.2), it will suffice by (6.9) to show that $\lim_{i \to \infty} \nu_i$ exists and is an element $\mu_{\text{cont}}$ of $\mathcal{M}_{\text{cont},v}(T)$.

Convergence is clear from (6.9), $\beta > 0$ and the fact that the $\mu_i$ converge. To show that $\mu_{\text{cont}}$ lies in $\mathcal{M}_{\text{cont},v}(T)$ it is enough to show for each finite subset $S$ of $T$ that $\mu_{\text{cont}} \in \mathcal{M}_v(T - S)$. Given $S$, there will an integer $N$ so $\{D_j\}_{j=1}^N$ contains $S$. Since

$$(6.10) \lim_{i \to \infty} b_{i,j} = 0 \quad \text{for} \quad 1 \leq j \leq N$$

we conclude from (6.8) that

$$(6.11) \mu_{\text{cont}} = \lim_{i \to \infty} \nu_i = \lim_{i \to \infty} \sum_{j=N+1}^{\infty} \frac{b_{i,j}}{1-c} \cdot \mu(D_j).$$

Here

$$d_i \sum_{j=N+1}^{\infty} \frac{b_{i,j}}{1-c} \cdot \mu(D_j)$$

is an atomic measure that involves no $D_j$ in $S$ when

$$\frac{1}{d_i} = \sum_{j=N+1}^{\infty} \frac{b_{i,j}}{1-c}.$$
We have
\[
\lim_{i \to \infty} \frac{1}{d_i} = \lim_{i \to \infty} \left( \sum_{i=1}^{\infty} \frac{b_{i,j}}{1 - c} - \sum_{j=1}^{N} \frac{b_{i,j}}{1 - c} \right) = 1
\]
because of (6.7) and (6.10). We conclude from (6.11) that
\[
\mu_{\text{cont}} = \lim_{i \to \infty} d_i \sum_{j=N+1}^{\infty} \frac{b_{i,j}}{1 - c} \cdot \mu(D_j)
\]
is an element of $\mathcal{M}_{\text{cont},v}(T)$.

7. Adelic sets of capacity one

7.1. Statement of results. In this section we assume $X$ is a smooth projective geometrically irreducible curve over a number field $K$. Let $\overline{K}$ be an algebraic closure of $K$, and let $\mathcal{X}$ be a finite Gal($\overline{K}/K$)-stable subset of $X(\overline{K})$. By an adelic subset of $X$ we will mean a product $\mathcal{E} = \prod_v E_v$ over all the places $v$ of $K$ of subsets $E_v$ of $X(\overline{K}_v)$ in $\mathcal{X}$ when $\overline{K}_v$ is an algebraic closure of $K_v$. As noted in at the beginning of [27, §4.1], subsets of $X(\overline{K}_v)$ are better suited for global capacity theory than those of $X(C_v)$.

We will assume that the $E_v$ satisfy the standard hypotheses described in [27, Def. 5.1.3] relative to $\mathcal{X}$. In particular, each $E_v$ is algebraically capacitifiable with respect to $\mathcal{X}$. We will assume each $E_v$ has positive inner capacity $\gamma(\mathcal{E}, \mathcal{X})$ with respect to every point $\zeta \in X(\overline{K}_v) - E_v$ in the sense of [27, p. 134-135, 196].

In [27, Def. 5.1.5], Rumely defined a capacity $\gamma(\mathcal{E}, \mathcal{X})$ of such an $\mathcal{E}$ relative to $\mathcal{X}$. For each ample effective divisor $D = \sum_{\zeta \in \mathcal{X}} a_{\zeta} \cdot \zeta$ supported on $\mathcal{X}$ one has the sectional capacity $S_\gamma(\mathcal{E}, D)$ of $\mathcal{E}$ relative to $D$ (13.1, 25). We will show in Lemma 7.3.1 below that Rumely’s results in [28] imply that $\gamma(\mathcal{E}, \mathcal{X})$ is the infimum of $S_\gamma(\mathcal{E}, D)^{1/\deg(D)^2}$ as $D$ ranges over all ample effective divisors supported on $\mathcal{X}$ provided $\gamma(\mathcal{E}, \mathcal{X}) \geq 1$.

We will recall in the next section Rumely’s definition in [27] of the Green’s function $G(z, \zeta; E_v) \in \mathbb{R} \cup \{\infty\}$ of pairs $z, \zeta \in X(\overline{K}_v)$. Define $G(z, D; E_v) = \sum_{\zeta \in \mathcal{X}} a_{\zeta} \cdot G(z, \zeta; E_v)$. We will regard meromorphic sections of powers of $L = \mathcal{O}_X(D)$ as elements of the function field $K(X)$. Then 1 is an element of $H^0(X, L)$ with divisor $D$. Define a $v$-adic metric on $L$ via
\[
|1|_v(z) = \exp(-G(z, D; E_v)) \quad \text{for} \quad z \in X(\overline{K}_v)
\]
We will call these the Green’s metrics on $L$ associated to $\mathcal{E}$.

We will show the following result.

Theorem 7.1.1. Suppose that $D$ is an ample effective divisor with support $\mathcal{X}$ such that
\[
\gamma(\mathcal{E}, \mathcal{X}) = S_\gamma(\mathcal{E}, D)^{1/\deg(D)^2} = 1
\]
Give $L = \mathcal{O}_X(D)$ the Green’s metrics associated to $\mathcal{E}$, and suppose $E_v$ is compact if $v$ is archimedean. Let $\{\lambda_{n,v}\}_{v, n}$ be the set of successive maxima of $H^0(X, L^n)$. Let $v$ be
the limiting distribution associated the sets \( \{ \lambda_n, i / n \} \) as \( n \to \infty \). Then \( \nu \) is the dirac measure supported on 0.

Thus lattices associated to metrized line bundles associated to adelic sets of capacity one are asymptotically semi-stable, in the sense that all of their successive maxima are approximately equal.

**Corollary 7.1.2.** Suppose there is a non-constant morphism \( h : X \to \mathbb{P}^1 \) over \( K \) all of whose poles are at one point \( \zeta \in X(K) \). Write \( \mathbb{P}^1 = \mathbb{A}^1 \cup \{ \infty \} \) and \( N = \deg(h) \), and let \( D = N\zeta = h^*(\infty) \). Suppose \( E_v = \{ z \in X(\overline{K}) : |h(z)|_v \leq 1 \} \) for all \( v \). Then the hypotheses of Theorem 7.1.3 hold, so that \( \nu \) is the Dirac measure supported at 0.

**Proof.** The equality (7.2) in this case is a consequence of Rumely’s pullback formula [27, Thm. 5.1.14] together with the computation of capacities of adelic disks in \( \mathbb{P}^1 \) given in [27, §5.2].

We will discuss zeros of successive minima in the case described in Corollary 7.1.2.

**Theorem 7.1.3.** Identify the morphism \( h : X \to \mathbb{P}^1 \) of Corollary 7.1.2 with an element of the function field \( K(X) \). Let \( z \) be the affine coordinate for \( \mathbb{P}^1 \) which has image \( h \) under the induced map \( K(\mathbb{P}^1) = K(z) \to K(X) \) of function fields. Let \( w \) be an archimedean place of \( K \). Let \( \mu_0 \) be the uniform measure on the boundary of the unit disk \( B_w = \{ z \in \mathbb{P}^1(\overline{K}) = \mathbb{P}^1(\mathbb{C}) : |z|_w = 1 \} \). Then \( \frac{1}{N} h^{-1}(\mu_0) \) is the equilibrium measure \( \mu(E_w, D) = E_w = h^{-1}(B_w) \) in the sense of [27, p. 214-215] with respect to the polar divisor \( D = N\zeta \) of \( h \). The measure \( \mu(E_w, D) \) is an element of \( \cap_{\lambda < 0} \overline{\mathcal{M}}_{\text{cont,} v}^{\lambda} \) where \( \overline{\mathcal{M}}_{\text{cont,} v}^{\lambda} \) is the set of continuous measures on \( X(\overline{K}) \) that are associated in Remark 6.3.3 to the zeros of elements of \( \cup_{n \geq 1} H^0(X, L \otimes n)^{\geq \lambda} \).

**Theorem 7.1.4.** Let \( h \) and \( E_v \) for all \( v \) be as in Corollary 7.1.2. Suppose \( w \) is a non-archimedean place of \( K \). Define \( E_w \) to be the closure of \( E_w \) in \( X_{\text{Berk,} C_w} \). Let \( h : X \to \mathbb{P}^1_{O_K} \) be the minimal regular model of \( h : X \to \mathbb{P}^1_K \) (see [17]). Let \( \infty \) be the section of \( \mathbb{P}^1_{O_K} \to \text{Spec}(O_K) \) defined by the point at infinity. Then \( h^*(\infty) = N\zeta + J \) for some vertical divisor \( J \) when \( \overline{\zeta} \) is the closure in \( X \) of the point \( \zeta \in X(K) \). Let \( \{ Y_i \}_{i=1}^{\ell} \) be the set of reduced irreducible components of the special fiber \( X_w \) of \( X \), and let \( m_i \) be the multiplicity of \( Y_i \) in \( X_w \). There is a unique point \( \xi_i \in X_{\text{Berk,} C_w} \) whose reduction is the generic point of \( Y_i \). Let \( \delta_i \) be the delta measure supported on \( \xi_i \) on \( X_{\text{Berk,} C_w} \), and let \( (h^*(\infty), Y_i) \) be the intersection number of \( h^*(\infty) \) and \( Y_i \). Writing \( D = N\zeta \), the measure

\[
(7.3) \quad \mu(E_w, D) = \frac{1}{N} \sum_{i=1}^{\ell} m_i(h^*(\infty), Y_i) \delta_i
\]

is a probability measure on \( X_{\text{Berk,} C_w} \). The measure \( \mu(E_w, D) \) is an element of \( \cap_{\lambda < 0} \overline{\mathcal{M}}_{\text{cont,} v}^{\lambda} \) where \( \overline{\mathcal{M}}_{\text{cont,} v}^{\lambda} \) is the set of continuous measures on \( X_{\text{Berk,} C_w} \) that are associated in Remark 6.3.3 to the zeros of elements of \( \cup_{n \geq 1} H^0(X, L \otimes n)^{\geq \lambda} \).
Thus under the hypotheses of Theorems 7.1.3 and 7.1.4 to achieve sections that demonstrate semi-stability, one can use sections whose zeros approach the measures \( \mu(E_w, D) \) while avoiding any prescribed finite set of points. The measure in Theorem 7.1.4 was defined by Chambert-Loir in [11], and we will use his results in the proof.

7.2. Green’s functions in Rumely’s capacity theory. Following [27], let \( q_v \) be the order of the residue field of a finite place \( v \) of \( K \). If \( v \) is a real place, let \( q_v = e \), while if \( v \) is complex let \( q_v = e^2 \). Define a \( v \)-adic log by \( \ln_v(r) = \ln(r)/\ln(q_v) \) for \( 0 < r \in \mathbb{R} \). We let \( || \) \( v \) be the standard absolute value \( | | \) \( v \) if \( v \) is finite, and we let \( || \) \( v \) be the Euclidean absolute value if \( v \) is archimedean. The product formula then becomes

\[
\sum_v \ln_v ||\alpha||_v \cdot \ln(q_v) = 0
\]

for \( \alpha \in K - \{0\} \).

Suppose now that \( \zeta \in X(\overline{K}_v) - E_v \). In [27, §3 - §4] Rumely defines a real valued canonical distance function \( [z, w]_\zeta \) of pairs of points \( z, w \in X(\overline{K}_v) - \{\zeta\} \). He then defines a Green’s function \( G(z, \zeta; E_v) \) in the following way.

Suppose first that \( E_v \) is compact. Rumely shows that there is a unique positive Borel measure \( \mu_v = \mu_v(E_v, \zeta) \) supported on \( E_v \) that minimizes the energy integral

\[
V_{\zeta}(E_v) = -\int_{E_v \times E_v} \ln_v [x, w]_\zeta \mu_v(x) \mu_v(w)
\]

One then has a conductor potential

\[
u_E(z, \zeta) = -\int_{E_v} \ln_v [x, w]_\zeta \mu_v(w).
\]

This function vanishes at almost all \( z \in E_v \). One lets

\[
G(z, \zeta; E_v) = \begin{cases} V_{\zeta}(E_v) - u_E(z, \zeta) & \text{if } z \notin E_v \cup \{\zeta\} \\ \infty & \text{if } z = \zeta \\ 0 & \text{if } z \in E_v \end{cases}
\]

(7.6)

Suppose now that \( v \) is a finite place. A PL\(\zeta\) domain (see [27, Def. 4.2.6]) is a subset of the form

\[
U_v = \{z \in X(\overline{K}_v) : |f(z)|_v \leq 1\}
\]

for a non-constant function \( f(z) \in \overline{K}_v(X) \) having poles only at \( \zeta \). Define

\[
G(z, \zeta; U_v) = \begin{cases} \frac{1}{\deg(f(z))} \ln_v |f(z)|_v & \text{if } z \notin U_v \cup \{\zeta\} \\ \infty & \text{if } z = \zeta \\ 0 & \text{if } z \in U_v \end{cases}
\]

(7.8)

Suppose now that \( v \) is finite and that \( E_v \) is an arbitrary algebraically capacitifiable subset of \( X(\overline{K}_v) - \{\zeta\} \) in the sense of [27]. In [27, §3 - §4], Rumely shows that there is there exist an infinite increasing sequence \( \{E_{v, i}\}_{i=1}^\infty \) of compact subsets of \( E_v \) and an
infinite decreasing sequence \( \{U_{v,j}\}_{j=1}^{\infty} \) of \( PL_\zeta \) domains containing \( E_v \) with the property that

\[
\lim_{i \to \infty} \gamma_\zeta(E'_{v,i}) = \gamma_\zeta(E_v) = \lim_{j \to \infty} \gamma_\zeta(U_{v,j})
\]

when \( \gamma_\zeta(E_v) \) is the local capacity of \( E_v \) with respect to \( \zeta \). It is shown in [27, Thm. 4.4.4] that the fact that \( E_v \) is algebraically capacitable implies

\[
\lim_{i \to \infty} G(z,\zeta;E'_{v,i}) = \lim_{j \to \infty} G(z,\zeta;U_{v,j})
\]

except for a set \( \Delta \) of \( z \) of inner capacity zero contained in \( E_v \), and the left hand limit in (7.10) is 0 for all \( z \in E_v \). By [27, Prop. 4.4.1], \( G(z,\zeta;E'_{v,i}) \) is non-increasing with \( i \), \( G(z,\zeta;U_{v,j}) \) is non-decreasing with \( j \), \( G(z,\zeta;E'_{v,i}) \geq G(z,\zeta;U_{v,j}) \) for all \( i \) and \( j \). The convergence in (7.10) is uniform over \( z \) in compact subsets of \( X(K_v) - \{\zeta\} - \Delta \).

We now define

\[
G(z,\zeta;E_v) = \begin{cases} 
\lim_{i \to \infty} G(z,\zeta;E'_{v,i}) & \text{if } z \neq \zeta \\
\infty & \text{if } z = \zeta
\end{cases}
\]

Suppose now that \( D = \sum_\zeta n_\zeta \zeta \) an effective divisor of degree \( \deg(D) = \sum_\zeta n_\zeta > 0 \). Let

\[
G(z,D;E_v) = \sum_\zeta n_\zeta G(z,\zeta;E_v)
\]

and

\[
\mu_v(E'_{v,i},D) = \frac{1}{\deg(D)} \sum_\zeta n_\zeta \mu_v(E'_{v,i},\zeta)
\]

7.3. Successive maxima for adelic sets of capacity one. The object of this section is to prove Theorem 7.1.1. We must first make a slight extension of Lemma 4.9 of [16].

**Lemma 7.3.1.** Suppose \( \gamma(\mathcal{E},\mathcal{X}) \geq 1 \). Then \( \gamma(\mathcal{E},\mathcal{X}) \) is the infimum of \( S_\gamma(\mathcal{E},D')^{1/\deg(D')^2} \) as \( D' \) ranges over all ample effective divisors supported on \( \mathcal{X} \).

**Proof.** This result is shown in [16] Lemma 4.9 if \( \gamma(\mathcal{E},\mathcal{X}) > 1 \). We now suppose \( \gamma(\mathcal{E},\mathcal{X}) = 1 \), so the Green’s matrix \( \Gamma(\mathcal{E},\mathcal{X}) \) has \( \text{val}(\Gamma(\mathcal{E},\mathcal{X})) = 0 \). By [27] Prop. 5.1.8, Prop. 5.1.9], \( \Gamma(\mathcal{E},\mathcal{X}) \) is a symmetric real matrix with non-negative off diagonal entries that has all non-positive eigenvalues and at least one eigenvalue equal to 0. Let \( I \) be the identity matrix of the same size as \( \Gamma(\mathcal{E},\mathcal{X}) \). For all \( \epsilon > 0 \), the matrix \( \Gamma_\epsilon = \Gamma(\mathcal{E},\mathcal{X}) - \epsilon I \) is negative definite, symmetric and has non-negative off diagonal entries. We now apply the arguments of [16] Lemma 4.9] to \( \Gamma_\epsilon \) and let \( \epsilon \to 0 \). Since the space of probability vectors of a prescribed size is compact, this implies Lemma 7.3.1. \( \square \)

**Lemma 7.3.2.** Let \( ||_v \) be the Green’s metric (7.1) on \( L_v \), and let \( ||_v^\otimes n \) be the resulting metric on \( L_v^\otimes n \). For \( f_v \in H^0(L_v^\otimes n) \) and \( z \in X(K_v) \) let \( |f_v|_v^\otimes n(z) \) be the norm with
respect to $|\overset{\circ}{\otimes}^n_v$ of the image of $f_v$ in the fiber of $L^\otimes_n$ at $z$. Regarding $f_v$ as an element of the function field $K_v(X)$, let $f_v(z) \in \overline{K_v} \cup \{\infty\}$ be the value of $f_v$ at $z$. Then

$$\|f_v\|_{L^\otimes_n, v} = \sup_{z \in X(\overline{K_v})} |f_v|^\otimes_n(z) \text{ equals } \sup(f_v, E_v) = \sup_{z \in E_v} |f_v(z)|_v.$$

**Proof.** In view of (7.14), the Green’s metric on $f_v \in H^0(L^\otimes_n)$ is specified by

$$\ln |f_v|^\otimes_n(z) = \ln(|f_v(z)|_v) + \ln(|1|^\otimes_n(z)) = \ln |f_v(z)|_v - nG(z, D; E_v).$$

Suppose first that $v$ is archimedean. We have supposed in this case that $E_v$ is compact. Then $\ln |f_v(z)|_v - nG(z, D; E_v)$ is a well defined harmonic function on the open set $X(\overline{K_v}) - E_v = X(\mathbb{C}) - E_v$, so it achieves its maximum on the boundary of $X(\overline{K_v}) - E_v$. This boundary lies in $E_v$ and $G(z, D; E_v) = 0$ for $z \in E_v$ by [27, Def. 3.2.1], so (7.14) holds. Suppose now that $v$ is non-archimedean. By [27] p. 282, Def. 4.4.12, $G(z, D; E_v)$ is the supremum of $G(z, D; U_v)$ over RL domains $U_v \supset E_v$ defined by functions having poles in $X$. The fact that (7.14) holds is now a consequence of the formula for $G(z, D; U_v)$ when $U_v$ is RL-domain in ([27] p. 277, eq. (2)) together with the maximum modulus principle of [27] Thm. 1.4.2]. \hfill \square

**Lemma 7.3.3.** There is no section $f \in H^0(L^\otimes_n)$ that has height $\lambda(f) > 0$.

**Proof.** Suppose $f \in H^0(L^\otimes_n)$ is a section with $\lambda(f) = -\sum_v k_v \log \|f\|_{L^\otimes_n, v} > 0$. Then $f$ defines a morphism $X \to \mathbb{P}^1$ such that $f^{-1}(\infty) = D'$ is supported on $X$, since $f$ is a section of $L^\otimes_n = O_X(nD)$ and $D$ is supported on $X$. Write $r_v = \sup(f, E_v)$. We let $\mathcal{E}' = \prod_v E'_v$ be the adelic polydisc of the projective line $\mathbb{P}^1$ such that each $E'_v \subset A^1(\overline{K_v}) = P^1(\overline{K}) - \{\infty\}$ is the disc around the origin with radius $r_v > 0$ with respect to $|\cdot|_v$. By the definition of the $r_v$, we have $\mathcal{E} \subset f^{-1}(\mathcal{E}')$. This and Rumely’s pullback formula [28] Prop. 4.1 give

$$S_\gamma(\mathcal{E}, D') \leq S_\gamma(f^{-1}(\mathcal{E}'), D') = S_\gamma(\mathcal{E}', \infty)^{\deg(D')}.$$  

By Lemma [7.3.2] $\lambda(f) = -\sum_v k_v \log \|f\|_{L^\otimes_n, v} = -\sum_v [K_v : \mathbb{Q}_v] \ln(r_v)$ and this is $-\ln(S_\gamma(\mathcal{E}', \infty))$ by [28] Prop. 3.1 and [27] p. 339. Because $\lambda(f) > 0$ we conclude from (7.16) that $S_\gamma(\mathcal{E}, D') < 1$. Hence $S_\gamma(\mathcal{E}, D')^{1/\deg(D')^2} < 1$. This contradicts the hypothesis in (7.2) because of Lemma [7.3.1]. \hfill \square

**Lemma 7.3.4.** Suppose $\epsilon > 0$. There is a finite place $v_0$ of $K$ and a subset $E'_{v_0}$ of $E_{v_0}$ with the following properties:

1. $E'_{v_0}$ is capacifiable with respect to $D$, and $|G(z, D; E'_{v_0}) - G(z, D; E_{v_0})| < \epsilon$ for all $z \in X(C_v)$.
2. The set $\mathcal{E}' = E'_{v_0} \times (\prod_{v \neq v_0} E_v)$ has capacity $S_\gamma(\mathcal{E}', D) < S_\gamma(\mathcal{E}, D) = 1$.
3. Let $\lambda(s) = -\sum_v k_v \log \|s\|_{L^\otimes_n, v}$ be the height of a section $s \in H^0(X, L^\otimes_n)$ associated to the Green’s metrics for $\mathcal{E}$, and let $\lambda'(s)$ is the corresponding height for $\mathcal{E}'$. Then $|\lambda(s) - \lambda'(s)|/n < k_{v_0} \epsilon$.
4. There is a rational function $f \in K(X)$ whose divisor of poles is a positive integral multiple of $D$ with the following properties: we have $\sup(f, E_v) \leq 1$ for all finite $v \neq v_0$, $\sup(f, E'_{v_0}) \leq 1$ and $\sup(f, E_v) < 1$ for all archimedean $v$.\hfill \square
Choose a place $v_0$ where $E_{v_0}$ is $\mathcal{X}$-trivial in the sense of [27, Def. 5.1.1] By [27, Prop. 4.4.13], the Green’s function $G(z, \zeta; E_{v_0})$ for any $z \in X(K_{v_0})$ and any $\zeta \in X(K_{v_0}) - E_{v_0}$ is the infimum of $G(z, \zeta; E'_{v_0})$ over compact subsets $E'_{v_0}$ of $E_{v_0}$. Furthermore, we have $G(z, \zeta; E_{v_0}) < G(z, \zeta; E'_{v_0})$ for $z \in X(K_{v_0}) - E_{v_0}$ by the computations in [27, §5.2.B] since we took $E_{v_0}$ to be $\mathcal{X}$-trivial. So we can take $E'_{v_0}$ to be a compact subset of $E_{v_0}$ such that the global Green’s matrix $\Gamma(\mathcal{X}, \mathcal{E})$ defined in [27, Theorem 5.1.4] differs from $\Gamma(\mathcal{X}, \mathcal{E}')$ by a matrix with positive entries that are arbitrarily close to 0. Then $E'_{v_0}$ is capacitifiable by [27, Theorem 4.3.4], so (1) holds. The value of the game defined by $\Gamma(\mathcal{X}, \mathcal{E}')$ is larger than that defined by $\Gamma(\mathcal{X}, \mathcal{E})$, so we get (2); see [27, p. 327-328]. The log of the Green’s metric on $\mathcal{L}$ at $v_0$ associated with $E'_{v_0}$ and with $E_{v_0}$ differs by a constant we can make arbitrarily close to 0, so we get (3) from (7.1). To prove (4), we first note that hypothesis (7.2) in Theorem 7.1.1 implies the following. When we write $D(7.1)$, to prove (4), we first note that hypothesis (7.2) in Theorem 7.1.1 implies the following. When we write $D(7.1)$, we first note that hypothesis (7.2) in Theorem 7.1.1 implies the following. When we write $D(7.1)$, we first note that hypothesis (7.2) in Theorem 7.1.1 implies the following. When we write $D$, choose a place $v_0$ where $E_{v_0}$ is $\mathcal{X}$-trivial in the sense of [27, Def. 5.1.1] By [27, Prop. 4.4.13], the Green’s function $G(z, \zeta; E_{v_0})$ for any $z \in X(K_{v_0})$ and any $\zeta \in X(K_{v_0}) - E_{v_0}$ is the infimum of $G(z, \zeta; E'_{v_0})$ over compact subsets $E'_{v_0}$ of $E_{v_0}$. Furthermore, we have $G(z, \zeta; E_{v_0}) < G(z, \zeta; E'_{v_0})$ for $z \in X(K_{v_0}) - E_{v_0}$ by the computations in [27, §5.2.B] since we took $E_{v_0}$ to be $\mathcal{X}$-trivial. So we can take $E'_{v_0}$ to be a compact subset of $E_{v_0}$ such that the global Green’s matrix $\Gamma(\mathcal{X}, \mathcal{E})$ defined in [27, Theorem 5.1.4] differs from $\Gamma(\mathcal{X}, \mathcal{E}')$ by a matrix with positive entries that are arbitrarily close to 0. Then $E'_{v_0}$ is capacitifiable by [27, Theorem 4.3.4], so (1) holds. The value of the game defined by $\Gamma(\mathcal{X}, \mathcal{E}')$ is larger than that defined by $\Gamma(\mathcal{X}, \mathcal{E})$, so we get (2); see [27, p. 327-328]. The log of the Green’s metric on $\mathcal{L}$ at $v_0$ associated with $E'_{v_0}$ and with $E_{v_0}$ differs by a constant we can make arbitrarily close to 0, so we get (3) from (7.1). To prove (4), we first note that hypothesis (7.2) in Theorem 7.1.1 implies the following. When we write $D = \sum_{x \in \mathcal{X}} n_x$, then $n_x > 0$ for all $x$ and the probability vector $P = (n_x/\deg(D))_{x \in \mathcal{X}}$ must define an optimum strategy for the game associated to $\Gamma(\mathcal{X}, \mathcal{E})$. Furthermore, $S_{\mathcal{X}}(\mathcal{E}, D) = 0$ says that this optimum strategy achieves value 0. Since $\Gamma(\mathcal{X}, \mathcal{E}') - \Gamma(\mathcal{X}, \mathcal{E})$ has all positive entries, playing $P$ in the game defined by $\Gamma(\mathcal{X}, \mathcal{E}')$ leads to a positive value. This means that the construction of Rumely in [27, §6, Corollary 6.2.7] produces a function with the properties in (4).}

**Lemma 7.3.5.** Let $\mathcal{E}'$ be as in Lemma 7.3.4. There is a constant $c$ independent of $n$ such $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ has a basis of sections $s$ for which $\lambda'(s) \geq c$.

**Proof.** Let $f$ in part (4) of Lemma 7.3.4 have divisor $mD$ for some $m > 0$. By Riemann-Roch, we can find a finite subset $\{h_j\}_{j \in J}$ of elements of the function field $K(\mathcal{X})$ with the following properties. The poles of the $h_j$ are supported on $\mathcal{X} = \text{supp}(D)$, and the height $\lambda(h_j)$ of $h_j$ with respect to the Green’s metrics associated to $\mathcal{E}$ is finite. Further, for all $n$, the collection of functions $\{h_j f^i\}_{j \in J, 0 \leq i < n}$ contains a basis for $H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) = H^0(\mathcal{X}, O_{\mathcal{X}}(nD))$. Now Lemma 7.3.2 gives

$$\lambda'(h_j f^i) = -\sum_{v} k_v \ln(\sup(h_j f^i, E'_v)) \geq -\sum_{v} k_v \ln(\sup(h_j, E'_v)) = \lambda'(h_j)$$

because $\sup(f^i, E'_v) \leq 1$ for all $i$ by Lemma 7.3.1. Since there are finitely many $h_j$, this proves the Lemma.

**Proof of Theorem 7.1.1**

Let $c$ be as in Lemma 7.3.5. In view of Lemmas 7.3.5 and 7.3.4, for each $\epsilon > 0$, there is a basis of sections $s$ of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ such that when $\lambda(s)$ is the height function associated to the Green’s metrics coming from $\mathcal{E}$, we have $\lambda(s)/n \geq c/n - \epsilon$. Letting $n \to \infty$ shows that the limiting measure $\nu$ associated to the ratios $\lambda_i/n$ as $\lambda_i$ ranges over the successive maxima of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ can have no support on the negative real axis. On the other hand, Lemma 7.3.3 shows the support is also trivial on the positive real axis. So $\nu$ must be the Dirac measure supported at 0.
Measures associated to zeros of small sections. The object of this subsection is to prove Theorems 7.1.3 and 7.1.4. Accordingly we suppose there is a morphism $h : X \to \mathbb{P}^1$ such that $D = h^*(\infty) = N \zeta$ for some point $\zeta \in X(K)$, where $N = \deg(h)$. We also suppose
\[ E_v = \{ z \in X(K_v) : |h(z)|_v \leq 1 \} = h^{-1}(B_v) \]
for all $v$ when
\[ B_v = \{ z \in K_v : |z|_v \leq 1 \} \subset \mathbb{P}^1(K_v) - \{ \infty \}. \]
Here $h^*O_{\mathbb{P}^1}(\infty) = O_X(D)$ and for all $n > 0$ we have a global section $\Phi_n(z)$ in $H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(n\infty))$ for any integer $a$ when $\Phi_n(z)$ is the $n$th cyclotomic polynomial. The set $\text{zer}(\Phi_n(z))$ of zeros of $\Phi_n(z)$ is just the set of all primitive $n$th roots of unity.

We now fix a place $w$ of $K$. Define $\overline{B}_w = B_w$ and $\overline{E}_w = E_w$ if $w$ is archimedean. If $w$ is non-archimedean, we let $\overline{B}_w$ be the closure of $B_w$ in $\mathbb{P}^1_{\text{Berk}}$ and we let $\overline{E}_w$ be the closure of $E_w$ in $X_{\text{Berk,C_w}}$.

Suppose first that $w$ is archimedean. In Theorem 7.1.3 we let $\mu_0$ be the uniform measure on the boundary of the unit disk $B_w = \{ z \in \overline{\mathbb{P}^1(K_w)} = \mathbb{P}^1(\mathbb{C}) : |z|_w = 1 \}$, and we defined $\mu(E_w, D)$ to be $\frac{1}{N}h^{-1}(\mu_0)$, where $D = N\zeta$ is the polar divisor of $h$. By [27, Prop. 4.1.25], $\mu(E_w, D)$ is the equilibrium measure of $E_w$ with respect to $D$.

Suppose now that $w$ is non-archimedean. The probability measure $\mu(E_w, D)$ on $X_{\text{Berk,C_w}}$ described in Theorem 7.1.4 is well defined by [11, §2.3] and the paragraph following [11, Theorem 3.1].

We claim that for all $w$,
\[ \mu(\overline{E}_w, D) = \lim_{m \to \infty} \delta_{\text{zer}(h^*(\Phi_{2^m}(z)))} \]
where $h^*(\Phi_{2^m}(z))$ is a section of $H^0(X, h^*O_{\mathbb{P}^1}(2^m\infty)) = H^0(X, O_X(nD))$ and $\delta_{\text{zer}(h^*(\Phi_{2^m}(z)))}$ is the dirac measure associated to the zeros of this section.

Suppose first that $w$ is archimedean and that $h : X \to \mathbb{P}^1$ is the identity map. The zeros of $h^*(\Phi_{2^m}(z))$ are simply all the odd powers of a primitive root of unity of order $2^m$. Then (7.17) is clear from the fact that in this case, $\overline{E}_w = \overline{B}_w$ is the unit disc about the origin, so $\mu(\overline{E}_w, D)$ is the uniform measure on the boundary of the unit disc. For archimedean $w$, the case of all $h : X \to \mathbb{P}^1$ satisfying our hypotheses follows from this and the fact that $\mu(E_w, D) = \frac{1}{N}h^{-1}(\mu_0)$.

Suppose now that $w$ is non-archimedean. As in Theorem 7.1.4 let $h : X \to \mathbb{P}^1_{O_K}$ be the minimal regular model of the finite morphism $h : X \to \mathbb{P}^1_K$. We give the line bundle $O_{\mathbb{P}^1_{O_K}}(\infty) = \mathcal{L}$ on $\mathbb{P}^1_{O_K}$ the adelic metric associated to the Weil height. Then $\mathbb{P}^1_{O_K}$ and the divisors defined by the zeros of $\Phi_{2^m}(z)$ have height equal to 0. We give $h^*\mathcal{L}$ the pull back of the adelic metric of $\mathcal{L}$. For any cycle $Z$ on $X$ we have from [6, Prop. 3.2.1] that
\[ H_{h^*\mathcal{L}}(Z) = H_{\mathcal{L}}(h_*Z) \]
where $H_{\mathcal{L}}$ is the height before normalization that is defined in [6, §3.1.1]. If $Z$ is the cycle $X = h^*\mathbb{P}^1$ we have $h_*h^*\mathbb{P}^1 = N \cdot \mathbb{P}^1$ by the projection formula so we conclude $H_{h^*\mathcal{L}}(X) = 0$.
zeros of $h^*(\Phi_{2m}(z))$. Then $h_*Z$ is contained in the divisor of zeros of $\Phi_{2m}(z)$, and so $H_L(h_*Z) = 0$. Thus $H_L(h_*Z) = 0$ by (7.18). By [6, §3.1.4], the same is now true if we replace $Z$ by any cycle contained in the base change of $Z$ by a morphism $\text{Spec}(O_K') \to \text{Spec}(O_K)$ associated to a finite extension $K'$ of $K$. We conclude that the Galois conjugates of any zero of $h^*(\Phi_{2m}(z))$ have adelic height 0 with respect to the above adelic metric on $h^*(L)$, and this is also the height of $X$ with respect to this metric. So these zeros as $m \to \infty$ form a generic sequence of points of $X(K_w)$ in the sense of [11, Thm. 3.1]. Now [11, Thm. 3.1] shows that the limit on the right hand side of (7.17) equals the Berkovich measure described in just before [11, Example 3.2], and this equals the measure $\mu(E_w,D)$ defined in Theorem 7.1.4. We have now shown (7.17) in all cases.

Consider the normalized height $\lambda(h^*(\Phi_{2m}(z)))$ of $h^*(\Phi_{2m}(z))$ with respect to the Green’s metrics on $O_X(2^mD) = h^*O_{P^1}(2^m\infty)$ associated to $E = \prod_v E_v$. We have $\lambda(h^*(\Phi_{2m}(z))) \to 0$ as $m \to \infty$ because $\Phi_{2m}(z) = (z^{2m} - 1)/(z^{2m-1} - 1)$ has normalized height tending toward 0 with respect to the Green’s metrics on $O_{P^1}(\infty)$ which are associated to $B = \prod_v B_v$. Since the zero sets of the $\Phi_{2m}(z)$ are disjoint for different $m$, for sufficiently large $m$ the zeros of $h^*(\Phi_{2m}(z))$ will avoid any prescribed finite subset of $X(K_w)$. Thus we see that the limit in (7.17) lies in $\bigcap_{\lambda<0} \mathcal{M}_{\text{cont},w}^{\geq \lambda}$. This completes the proof of Theorems 7.1.3 and 7.1.4.

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