Optimal control problem for coupled time-fractional diffusion systems with final observations

G. M. Bahaa and A. Hamiaz

*Department of Mathematics and Computer Sciences, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

ABSTRACT

In this paper, fractional optimal control problem for two-dimensional coupled diffusion systems with final observation is investigated. The fractional time derivative is considered in Atangana–Baleanu sense. Constraints on controls are imposed. First, by means of the classical control theory, the existence and uniqueness of the state for these systems is proved. Then, the necessary and sufficient optimality conditions for the fractional Dirichlet problems with the quadratic performance functional are derived. Finally we give some examples to illustrate the applicability of our results. The optimization problem presented in this paper constitutes a generalization of the optimal control problem of diffusion equations with Dirichlet boundary conditions considered in recent papers to coupled systems with Atangana–Baleanu time derivatives.

1. Introduction

Let $n \in \mathbb{N}$ and $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary $\Gamma$ of class $C^2$. For a time $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$.

For $y_{0,1}, y_{0,2} \in H_0^1(\Omega)$ and $f_1, f_2 \in L^2(0, T; H^{-1}(\Omega))$, let us consider the following fractional problem for coupled diffusion system:

$$0_{ABC}\mathcal{D}^\alpha_0 y_1(t) - \Delta y_1(t) + y_1(t) - y_2(t) = f_1(t),$$

$a.e. t \in ]0, T[$, (1.1)

$$0_{ABC}\mathcal{D}^\alpha_0 y_2(t) - \Delta y_2(t) + y_2(t) + y_1(t) = f_2(t),$$

$a.e. t \in ]0, T[$, (1.2)

$$y_1(x, 0) = y_{0,1}(x), \quad x \in \Omega,$$ (1.3)

$$y_2(x, 0) = y_{0,2}(x), \quad x \in \Omega,$$ (1.4)

$$y_1(x, t) = 0, \quad y_2(x, t) = 0, \quad (x, t) \in \Sigma,$$ (1.5)

where $\alpha \in (0, 1)$, $0_{ABC}\mathcal{D}^\alpha_0$ is the Atangana–Baleanu fractional derivatives in the Caputo sense.

The study of fractional calculus with the non-singular kernel is gaining more and more attention. Compared with classical fractional calculus with singular kernel, non-singular kernel models can describe reality more accurately, which has been shown recently in a variety of fields such as physics, chemistry, biology, economics, signal and image processing, control, porous media, aerodynamics and so on. For example, extensive treatment and various applications of the fractional calculus with the non-singular kernel are discussed in the works of Atangana et al. [1–3], Baleanu et al. [4, 5], Caputo [6], Djida et al. [7, 8], Gomez-Aguilar et al. [9–11]. It has been demonstrated that Fractional Order Differential Equations (FODEs) with the non-singular kernel models dynamic systems and processes more accurately than FODEs with singular kernel do, and fractional controllers perform better than integer order controllers.

There are many works on fractional diffusion equations and fractional diffusion wave equations. For instance, Agrawal [12, 13] studied the solutions for a fractional diffusion-wave equation defined in a bounded domain when the fractional time derivative is described in the Caputo sense. Using Laplace transform and finite sine transform technique, the author obtained the general solution in terms of Mittag–Leffler functions. In [14], Mophou et al. studied by means of eigenfunctions the control problems for fractional diffusion wave equation involving Riemann–Liouville fractional derivative or order $\alpha \in (\frac{1}{2}, 2)$.

Integer order optimal control problems for evolution equations have been extensively studied by many authors, for comprehensive treatment of this topic we refer to the classical monograph by Lions [15] and to [16].

CONTACT G. M. Bahaa Bahaa_gm@yahoo.com Department of Mathematics and Computer Sciences, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

© 2018 The Author(s). Published by Informa UK Limited, trading as Taylor & Francis Group

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
Extensive treatment and various applications of the fractional calculus are discussed in the works of Agrawal et al. [12, 13], Ahmad and Nieto [17], Bahaa et al. [18–22], Mophou [23–25], Debouche and Nieto [26,27], Wang and Zhou [28], Tang and Ma [29], etc. It has been demonstrated that FODE models dynamic systems and processes more accurately than integer order differential equations do, and fractional controllers perform better than integer order controllers.

Optimal control of fractional diffusion equations has also been studied by several authors. For instance, in [30], Agrawal considered two problems, the simplest fractional variation problem and fractional variational problem of Lagrange. For both problems, the author developed the Euler–Lagrange type necessary conditions which must be satisfied for the given functional to be extremum. In [31], a general formulation and solution scheme for fractional optimal control problems are defined in sense of Riemann–Liouville and Caputo.

In [17], Ahmad et al. investigated the existence of solutions for fractional differential inclusions with four-point non-local Riemann–Liouville type integral boundary conditions. In [32], Bahaa presented the fractional optimal control problem for variational inequalities with control constraints. The author showed that the considered optimal control problem has a unique solution. In [33], Bahaa proposed necessary conditions for optimality in optimal control problems by differential equations of fractional order. In [20], Bahaa proposed the fractional optimal control problem for infinite order system with control constraints. Following the same technique [21], Bahaa presented the formulation of fractional optimal control problem when the dynamic constraints of the system are given by a fractional differential system and the performance index is described with a state and a control function. In [34], Baleanu et al. gave formulation for a fractional optimal control problem for coupled diffusion systems which can be used to describe many physical, chemical, mathematical and biological models. First, by means of the classical control theory, the existence and uniqueness of the state for these systems is proved. Fractional optimal control is characterized by the adjoint problem. By using this characterization, particular properties of fractional optimal control are proved.

This paper is organized as follows. In Section 2, we introduce some basic definitions and preliminary results. In Section 3, we give some properties of Atangana–Baleanu fractional derivatives and integration by parts. In Section 4, we formulate the fractional Dirichlet problem for diffusion equations. In Section 5, we show that our fractional optimal control problem holds and we give the optimality conditions for the optimal control. In Section 6, some illustrated examples are stated. In Section 7, we state our conclusions.

2. Preliminaries

Many definitions have been given for a fractional derivative, which include Riemann–Liouville, Grünwald–Letnikov, Weyl, Caputo, Marchaud and Riesz fractional derivatives. We will formulate the problem in terms of the left and right Caputo fractional derivatives which will be given later.

Definition 2.1 ([2]): For a given function \( x(t) \in H^1(a, b), b > a, \alpha \in (0, 1) \), the left Atangana–Baleanu fractional derivative (AB derivative) of \( x(t) \) of order \( \alpha \) in Caputo sense \( _a^\alpha D^\alpha_B x(t) \) (where \( A \) denotes Atangana, \( B \) denotes Baleanu and \( C \) denotes Caputo type) with base point \( a \) is defined at a point \( t \in (a, b) \) by

\[
_{a}^{AB}D_{t}^{\alpha}x(t) = \frac{B(\alpha)}{1-\alpha} \int_{a}^{t} \frac{d}{ds}x(s)E_{\alpha}[\gamma(t-s)^{\alpha}]ds,
\]

(2.1)

where \( \gamma = \alpha/(1-\alpha) \), and \( B(\alpha) \) being a normalization function satisfying

\[
B(\alpha) = (1-\alpha) + \frac{\alpha}{Gamma(\alpha)},
\]

(2.2)

where \( B(0) = B(1) = 1 \), \( E_{\alpha}(\cdot) \) stands for the Mittag–Leffler function defined by

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad E_{\alpha,1}(z) = E_{\alpha}(z), \quad z \in C,
\]

(2.3)
which is an entire function on the complex plane and \( \Gamma(.) \) denotes the Euler’s gamma function defined as
\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad \Re(z) > 0.
\]

The Mittag–Leffler function \( E_{\alpha,\beta}(z) \) is a two-parameter family of entire functions of \( z \) of order \( \alpha^{-1} \).

Furthermore, we recall the following Lemma.

**Lemma 2.1 ([2]):** Let \( \alpha, \beta \in \mathbb{C} \) such that \( \Re(\alpha) > 0 \) and \( \Re(\beta) > 0 \). Then we have that
\[
\left( \frac{d}{dz} \right) E_{\alpha,\beta}(z) = \frac{1}{\alpha} \left( (1 + \alpha - \beta)E_{\alpha,\beta}(z) + E_{\alpha,\beta-1}(z) \right),
\]
\( z \in \mathbb{C} \).

The left Atangana–Baleanu fractional derivative in Riemann–Liouville sense with parameter family of entire functions of \( \alpha \) in (2.6) we consider the usual classical derivative \( \partial_t \).

The associated left AB fractional integral is also defined as
\[
\begin{align*}
A_{\beta}B_{\alpha}^{\alpha}x(t) &= \frac{1 - \alpha}{B(\alpha)} x(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}
\times \int_0^t x(s)(t - s)^{\alpha - 1}ds, \quad \text{(left AB)}
\end{align*}
\]
where
\[
\alpha^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1}ds
\]
is the classical left Riemann–Liouville integral.

Notice that if \( \alpha = 0 \) in (2.6), we recover the initial function and if \( \alpha = 1 \) in (2.6) we consider the usual ordinary integral. So from the definition on [2], we recall the following definition.

**Definition 2.2 ([2]):** For a given function \( x(t) \in H^1(a, b) \), \( b > t > a \), the right Atangana–Baleanu fractional derivative of \( x(t) \) of order \( \alpha \) in Caputo sense with base point \( b \) is defined at a point \( t \in (a, b) \) by

\[
\begin{align*}
&\left( R_{\alpha}^{\alpha} \right)^{\alpha} x(t) = -\frac{B(\alpha)}{1 - \alpha} \int_0^b E_{\alpha}(t-s)^{\alpha - 1}ds,
\end{align*}
\]
and in Riemann–Liouville sense with
\[
\begin{align*}
&\left( R_{\alpha}^{\alpha} \right)^{\alpha} x(t) = -\frac{B(\alpha)}{1 - \alpha} \int_0^b x(s)E_{\alpha}(t-s)^{\alpha - 1}ds,
\end{align*}
\]
where
\[
\begin{align*}
&\left( R_{\alpha}^{\alpha} \right)^{\alpha} x(t) = \frac{B(\alpha)}{1 - \alpha} \int_0^t x(s)(t - s)^{\alpha - 1}ds,
\end{align*}
\]
is the classical right Riemann–Liouville integral.

### 3. Some properties of AB derivatives and integration by parts

In this section, we state some important lemmas for properties of AB derivatives and integration by parts which can be found in [4]. Some recent results and properties concerning this operator can be found [2] and therein.

**Lemma 3.1 ([4]):** The left AB Caputo fractional derivatives and the left AB Riemann–Liouville derivative are related by the identity:
\[
A_{\beta}B_{\alpha}^{\alpha}x(t) = \frac{B(\alpha)}{1 - \alpha} x(a) E_{\alpha}[-\gamma(t)^{\alpha}].
\]

The right AB Caputo fractional derivatives and the right AB Riemann–Liouville derivative are related by the identity:
\[
A_{\beta}B_{\alpha}^{\alpha}x(t) = \frac{B(\alpha)}{1 - \alpha} x(b) E_{\alpha}[-\gamma(b)^{\alpha}].
\]

There are useful relations between the left and right AB FDs in the Riemann–Liouville and Caputo senses and the associated AB fractional integrals as the following formulas state.

\[
\begin{align*}
&\left( A_{\alpha}^{\alpha} \right)^{\alpha} x(t) = A_{\alpha}^{\alpha} x(t),
\end{align*}
\]
and
\[
\begin{align*}
&\left( A_{\alpha}^{\alpha} \right)^{\alpha} x(t) = A_{\alpha}^{\alpha} x(t).
\end{align*}
\]

As a consequence, the backwards in time with the fractional-time derivative with non-singular Mittag–Leffler kernel at the based point \( T \) is equivalently written as a forward in time operator with the fractional-time derivative with non-singular Mittag–Leffler kernel \( A_{\alpha}^{\alpha} \).
Lemma 3.2 ([4]): The AB integral operators and ABR differential operators form a commutative family of differential operators:

\[ \text{ABR} D^\alpha_{a+} \left( \text{ABR} D^\beta_{a+} f(t) \right) = \text{ABR} D^{\alpha+\beta}_{a+} f(t), \]  
\[ \text{ABR} D^\alpha_{a+} \left( \text{ABR} D^\beta_{a+} g(t) \right) = f(t) = \text{ABR} D^{\alpha+\beta}_{a+} g(t). \]  

(3.7)

for \( \alpha, \beta \in (0, 1) \) and \( a, x \) satisfying the conditions from Definition (2.1).

Lemma 3.3 (The semigroup property [4]): The semigroup property for AB fractional differintegrals is not satisfied in general. For example, taking \( B(\alpha) = 1 \) we get

\[ \text{AB}^{2/3} \left( \frac{1}{3} + \frac{2}{3} \right) = \frac{1}{3} t + \frac{2}{3} \]  
and yet

\[ \text{AB}^{1/3} \left( \frac{1}{3} \right) = \frac{1}{3} t + \frac{1}{3} \]  
\[ = \frac{4}{9} t + \frac{1}{9} \]  
\[ = \frac{4}{9} t + \frac{1}{9} \]  

– two entirely different expressions.

Can we find conditions for when the semigroup property does hold?

First, note that it will be sufficient to consider fractional integrals only. Any function which satisfies the semigroup property for AB fractional derivatives generates one which satisfies it for AB fractional integrals and vice versa. This is because

\[ \text{ABR} D^\alpha_{a+} \left( \text{ABR} D^\beta_{b+} f(t) \right) = g(t) = \text{ABR} D^{\alpha+\beta}_{a+} f(t) \]  

(3.10)

is exactly equivalent to

\[ \text{ABR} D^\beta_{a+} \left( \text{ABR} D^\alpha_{b+} g(t) \right) = f(t) = \text{ABR} D^{\alpha+\beta}_{a+} g(t). \]  

(3.11)

This is good to know, because the definition of AB fractional integrals is much simpler and easier to work with than that of ABR fractional derivatives.

Next we state the following proposition which gives the integration by parts.

Lemma 3.4 (Integration by parts, see [38]): Let \( \alpha > 0, p \geq 1, q \geq 1, \) and \( (1/p) + (1/q) \leq 1 + \alpha(p \neq 1) \) and \( q \neq 1 \) in the case \( (1/p) + (1/q) = 1 + \alpha. \) Then for any \( \phi(x) \in L^p(a, b), \phi(x) \in L^q(a, b), \) we have

\[ \int_a^b \phi(x) \text{ABR} D^\alpha_{a+} f(x) \, dx = \int_a^b \psi(x) \text{ABR} D^\alpha_{b+} f(x) \, dx, \]  

(3.12)

\[ \int_a^b \phi(x) \text{ABR} D^\alpha_{a+} f(x) \, dx = \int_a^b \psi(x) \text{ABR} D^\alpha_{b+} f(x) \, dx, \]  

(3.13)

if \( \phi(x) \in \text{ABR} L^p(a, b) \) and \( \psi(x) \in \text{ABR} L^q(a, b), \) then

\[ \int_a^b \psi(x) \text{ABR} D^\alpha_{a+} f(x) \, dx = \int_a^b \phi(x) \text{ABR} D^\alpha_{b+} f(x) \, dx, \]  

(3.14)

\[ \int_a^b \phi(x) \text{ABR} D^\alpha_{a+} f(x) \, dx = \int_a^b \psi(x) \text{ABR} D^\alpha_{b+} f(x) \, dx, \]  

(3.15)

\[ \int_a^b \phi(x) \text{ABR} D^\alpha_{a+} f(x) \, dx = \int_a^b \psi(x) \text{ABR} D^\alpha_{b+} f(x) \, dx, \]  

(3.16)

where the left generalized fractional integral operator

\[ E^\alpha_{y, a+} x(t) = \int_a^t (t - \tau)^{\alpha-1} E^\alpha_{y, \tau} \times \omega(t - \tau)^{\gamma} \]  

(3.17)

and the right generalized fractional integral operator

\[ E^\alpha_{y, a+} x(t) = \int_t^b (t - \tau)^{\alpha-1} E^\alpha_{y, \tau} \times \omega(t - \tau)^{\gamma} \]  

(3.18)

Next we state the following proposition which gives the weak formulation of the problem (1.1)–(1.5), that will be fundamental in our analysis.

Proposition 3.5 ([8]): Let \( \phi, \psi \in C^\infty(\bar{Q}). \) Then, we have

\[ \int_0^T \int_\Omega \left( \text{ABR} D^{\alpha}_{t+} \phi(x, t) - \Delta y(x, t) \right) \psi(x, t) \, dx \, dt \]  

\[ = \int_0^T \int_\Omega \left[ \frac{\partial}{\partial \sigma} y(x, t) - \frac{\partial}{\partial \sigma} \phi(x, t) \right] \, dx \, dt \]  

\[ + \int_0^T \int_\Omega \left[ \frac{\partial}{\partial \sigma} y(x, t) - \frac{\partial}{\partial \sigma} \phi(x, t) \right] \, dx \, dt \]  

\[ - \frac{B(\alpha)}{1 - \alpha} \int_0^T \int_\Omega \left[ \frac{\partial}{\partial \sigma} y(x, t) - \frac{\partial}{\partial \sigma} \phi(x, t) \right] \, dx \, dt \]  

\[ + \frac{B(\alpha)}{1 - \alpha} \int_0^T \int_\Omega \left[ \frac{\partial}{\partial \sigma} y(x, t) - \frac{\partial}{\partial \sigma} \phi(x, t) \right] \, dx \, dt \]  

\[ \times \left[ \frac{\partial}{\partial \sigma} \right] \, dx \, dt. \]  

(3.19)
We also introduce the space
\[ \mathcal{W}(0, T) := \{ y : y \in L^2(0, T; H^1_0(\Omega)), \quad _0^{ABC}D_t^\alpha y(t) \in L^2(0, T; H^{-1}(\Omega)) \}, \]
which is contained. The spaces considered in this paper are assumed to be real.

**Lemma 2.4** holds for

In view of Lemma 2.4, the Atangana–Baleanu derivatives

**Lemma 4.1:** Let \( 0 < \alpha < 1, \mathcal{X} \) be a Banach space and \( f \in C([0, T], \mathcal{X}) \). Then for all \( t_1, t_2 \in [0, T] \)
\[ _0f^\alpha f(t_1) - _0f^\alpha f(t_2) \parallel X \leq \frac{\| f \|_{C([0, T]; \mathcal{X})}}{\Gamma(\alpha + 1)} |t_1 - t_2|^\alpha. \]

**Remark 3.7:** Since \( C([0, T], \mathcal{X}) \subset L^\infty([0, T]; \mathcal{X}) \subset L^2((0, T); \mathcal{X}) \) because \([0, T]\) is a bounded subset of \( \mathbb{R} \), Lemma 2.4 holds for \( f \in L^2((0, T); \mathcal{X}) \) and we have that \( _0f^\alpha f \in C([0, T], \mathcal{X}) \subset L^2((0, T); \mathcal{X}) \).

### 4. Coupled diffusion system with Atangana–Baleanu derivatives

For \( y_{0,1}, y_{0,2} \in H^1_0(\Omega) \) and \( f_1, f_2 \in L^2(0, T; H^{-1}(\Omega)) \), let us consider the fractional problem for coupled evolution system: Find

\[ y = (y_1, y_2) \in \mathcal{W}(0, T) \times \mathcal{W}(0, T) \]

such that
\[ _0^{ABC}D_t^\alpha y_1(t) - \Delta y_1(t) + y_1(t) - y_2(t) = f_1(t), \quad a.e. t \in [0, T], \]
\[ _0^{ABC}D_t^\alpha y_2(t) - \Delta y_2(t) + y_2(t) + y_1(t) = f_2(t), \quad a.e. t \in [0, T], \]
\[ y_1(x, 0) = y_{0,1}(x), \quad x \in \Omega, \]
\[ y_2(x, 0) = y_{0,2}(x), \quad x \in \Omega, \]
\[ y_1(x, t) = 0, \quad y_2(x, t) = 0, \quad (x, t) \in \Sigma. \]

We also need some trace results.

**Lemma 4.1:** Let \( f_1, f_2 \in L^2(Q) \) and \( y_1, y_2 \in L^2((0, T); H^1_0(\Omega)) \) be such that \( _0^{ABC}D_t^\alpha y_1(t) \) and \( _0^{ABC}D_t^\alpha y_2(t) \) be such that \( _0^{ABC}D_t^\alpha y_1(t) - \Delta y_1(t) + y_1(t) - y_2(t) = f_1(t) \) and \( _0^{ABC}D_t^\alpha y_2(t) - \Delta y_2(t) + y_2(t) + y_1(t) = f_2(t) \). Then we have:

(i) \( y_1|_{\Sigma}, y_2|_{\Sigma} \) exists and belongs \( L^2((0, T); H^{-1}(\Gamma)) \),
(ii) \( y_1(0), y_2(0) \) belongs to \( L^2(\Omega) \).

**Proof:** In view of Lemma 2.4, \( _0^{ABC}D_t^\alpha y_1(t) \) and \( _0^{ABC}D_t^\alpha y_2(t) \) are contained. Hence, \( y_1(0), y_2(0) \) belongs to \( L^2(\Omega) \) since \( _0^{ABC}D_t^\alpha y_1(t) = y_1(t) - y_1(0), \)

For the Laplace operator \( \Delta = \sum_{i=1}^n (\partial^2 / \partial x_i^2) \) in (4.1), (4.2), we define the bilinear form \( \pi(y, \phi) \) as follows.

**Definition 4.1:** For each \( t \in [0, T], y = (y_1, y_2) \) and \( \phi = (\phi_1, \phi_2) \), we define a family of bilinear forms \( \pi(y, \phi) \) on \((H^1_0(\Omega))^2\) by
\[ \pi(t; y, \phi) = (-\Delta y_1 + y_1 - y_2, \phi_1)_1(\Omega) + (-\Delta y_2 + y_2 + y_1, \phi_2)_1(\Omega), \]
\[ y, \phi \in (H^1_0(\Omega))^2, \]
which can be written as
\[ \pi(t; y, \phi) = \int_{\Omega} (\nabla y_1(x)\phi_1(x) + \nabla y_2(x)\phi_2(x)dx \]
\[ + \int_{\Omega} [y_1\phi_1 + y_2\phi_2 - y_2\phi_1 + y_1\phi_2]dx, \] (4.7)
where \( \nabla = \sum_{i=1}^n (\partial / \partial x_i) \) is the grad operator.

**Lemma 4.2:** The bilinear form \( \pi(t; y, \phi) \) in (4.7) is coercive on \((H^1_0(\Omega))^2\) that for \( y = (y_1, y_2) \), we have
\[ \pi(t; y, y) \geq \lambda \| y \|_{(H^1_0(\Omega))^2}^2, \quad \lambda > 0. \] (4.8)

**Proof:** It is well known that the ellipticity of \( \Delta = \sum_{i=1}^n (\partial^2 / \partial x_i^2) \) is sufficient for the coerciveness of \( \pi(t; y, \phi) \) on \((H^1_0(\Omega))^2\). Then we get
\[ \pi(t; y, y) = \int_{\Omega} (\nabla y_1(x)\phi_1(x) + \nabla y_2(x)\phi_2(x)dx \]
\[ + \int_{\Omega} [y_1\phi_1 + y_2\phi_2 - y_2\phi_1 + y_1\phi_2]dx \]
\[ \geq \beta \sum_{i=1}^n |\partial / \partial x_i y_1(x)|^2_{1(\Omega)} + |\partial / \partial x_i y_2(x)|^2_{1(\Omega)} \]
\[ + \beta \sum_{i=1}^n |\partial / \partial x_i y_1(x)|^2_{2(\Omega)} + |\partial / \partial x_i y_2(x)|^2_{2(\Omega)} \]
\[ \geq \lambda_1 |y_1|^2_{(H^1_0(\Omega))^2} + \lambda_2 |y_2|^2_{(H^1_0(\Omega))^2} \]
\[ \geq \lambda |y|^2_{(H^1_0(\Omega))^2}, \quad \lambda = \max(\lambda_1, \lambda_2) > 0. \]

Also we assume that \( \forall y, \phi \in (H^1_0(\Omega))^2 \) the function \( t \rightarrow \pi(t; y, \phi) \) is continuously differentiable in \([0, T]\) and the bilinear form \( \pi(t; y, \phi) \) is symmetric.
\[ \pi(t; y, \phi) = \pi(\phi, y) \quad \forall y, \phi \in (H^1_0(\Omega))^2. \] (4.9)

Then (4.1)–(4.5) constitute a fractional Dirichlet coupled problem. First by using the Lax–Milgram lemma, we prove sufficient conditions for the existence of a
unique solution of the mixed initial-boundary value problem (4.1)–(4.5).

Lemma 4.3 ([23, 24]): (Fractional Green’s formula for evolution systems). Let \( y = (y_1, y_2) \) be the solution of system (4.1)–(4.5). Then for any \( \phi = (\phi_1, \phi_2) \in (C^\infty(\Omega))^2 \) such that \( \phi(x, T) = (\phi_1, \phi_2)(x, T) = 0 \) in \( \Omega \) and \( \phi = (\phi_1, \phi_2) = 0 \) on \( \Sigma \), we have for each \( i = 1, 2 \)

\[
\int_0^T \int_\Omega \left[ \int_0^1 \frac{\partial \phi_i}{\partial \tau} (x, t - \tau) y_i (x, \tau) d\tau - \int_0^T \int_{\Omega} \frac{\partial y_i}{\partial \tau} d\sigma d\tau \right] \phi_i (x, t) dxd\tau + \int_\Omega \int_0^T y_i (x, t) \left( - \int_0^1 \frac{\partial \phi_i}{\partial \tau} T_0 \alpha (\phi_i (x, t)) \right) d\sigma d\tau - \frac{B(\alpha)}{1 - \alpha} \int_\Omega \int_0^T y_i (x, 0) E_{\alpha, \alpha} \left[ - \gamma t^\alpha \right] \phi_i (x, t) dxd\tau = \frac{f_i (x)}{\nu}.
\]

Lemma 4.4: If (4.8) and (4.9) hold, then the problem (4.1)–(4.5) admits a unique solution \( y(t) = (y_1(t), y_2(t)) \) in \((V, (\Omega, T))^2\).

Proof: From the coerciveness condition (4.8) and using the Lax–Milgram lemma, there exists a unique element \( y(t) = (y_1(t), y_2(t)) \in (H^1_0(\Omega))^2 \) such that

\[
\left( \int_0^1 \frac{\partial \phi_i}{\partial \tau} y_i (x, t) d\tau + \phi_i (x, t) \right)_{L^2(\Omega)} + \pi (t; \gamma, \phi) = L(\phi) \quad \text{for all} \quad \phi = (\phi_1, \phi_2) \in (H^1_0(\Omega))^2,
\]

where \( L(\phi) \) is a continuous linear form on \((H^1_0(\Omega))^2\) and takes the form

\[
L(\phi) = \int_Q \left[ f_1 \phi_1 + f_2 \phi_2 \right] dxd\tau - \frac{B(\alpha)}{1 - \alpha} \left[ \int_0^1 \int_\Omega y_1 E_{\alpha, \alpha} \left[ - \gamma t^\alpha \right] \phi_1 (x, t) dxd\tau \right] + \frac{B(\alpha)}{1 - \alpha} \left[ \int_0^1 \int_\Omega y_2 E_{\alpha, \alpha} \left[ - \gamma t^\alpha \right] \phi_2 (x, t) dxd\tau \right],
\]

\[
f = (f_1, f_2) \in (L^2(\Omega))^2, \quad y_0 = (y_{0,1}(x), y_{0,2}(x)) \in (L^2(\Omega))^2.
\]

Then Equation (4.11) equivalents to there exists a unique solution \( y(t) = (y_1(t), y_2(t)) \in (H^1_0(\Omega))^2 \) for

\[
\left( \frac{\partial}{\partial \tau} y_1 (x, t) - \Delta y_1 (x, t) + y_1 + y_2, \phi_1 (x) \right)_{L^2(\Omega)} + \left( \frac{\partial}{\partial \tau} y_2 (x, t) - \Delta y_2 (x, t) + y_2 - y_1, \phi_2 (x) \right)_{L^2(\Omega)} - \frac{B(\alpha)}{1 - \alpha} \left[ \int_0^1 \int_\Omega y_1 E_{\alpha, \alpha} \left[ - \gamma t^\alpha \right] \phi_1 (x, t) dxd\tau \right] + \frac{B(\alpha)}{1 - \alpha} \left[ \int_0^1 \int_\Omega y_2 E_{\alpha, \alpha} \left[ - \gamma t^\alpha \right] \phi_2 (x, t) dxd\tau \right] = L(\phi).
\]

Then Equation (4.13) is equivalent to the fractional evolution equations

\[
\frac{\partial}{\partial \tau} y_1 (x, t) - \Delta y_1 (x, t) + y_1 + y_2 = f_1, \quad \text{for} \quad i = 1, 2.
\]

\[
\frac{\partial}{\partial \tau} y_2 (x, t) - \Delta y_2 (x, t) + y_2 - y_1 = f_2.
\]

“tested” against \( \phi_1 (x), \phi_2 (x) \) respectively.

Let us multiply both sides in (4.14), (4.15) by \( \phi_1 (x), \phi_2 (x) \) respectively and applying Green’s formula (Lemma 3.6), we have

\[
\int_Q \left[ \frac{\partial}{\partial \tau} y_1 - \Delta y_1 (t) + y_1 + y_2 \phi_1 (x) \right] dxd\tau = \frac{f_1 \phi_1 (x) dxd\tau}{\nu}, \quad \text{for} \quad \phi_1 (x) \in H^1_0(\Omega),
\]

\[
\int_Q \left[ \frac{\partial}{\partial \tau} y_2 - \Delta y_2 (t) + y_2 - y_1 \phi_2 (x) \right] dxd\tau = \frac{f_2 \phi_2 (x) dxd\tau}{\nu}, \quad \text{for} \quad \phi_2 (x) \in H^1_0(\Omega)
\]

applying Green’s formula (Corollary 3.6), we have

\[
- \frac{B(\alpha)}{1 - \alpha} \int_Q \int_0^1 y_1 (x, 0) E_{\alpha, \alpha} \left[ - \gamma t^\alpha \right] \phi_1 (x, t) dxd\tau + \frac{B(\alpha)}{1 - \alpha} \int_Q \int_0^1 y_2 (x, 0) E_{\alpha, \alpha} \left[ - \gamma t^\alpha \right] \phi_2 (x, t) dxd\tau
\]

\[
+ \int_Q \int_0^T \frac{\partial \phi_1}{\partial \tau} \phi_1 (x, t) dxd\tau - \int_Q \int_0^T \frac{\partial \phi_2}{\partial \tau} \phi_2 (x, t) dxd\tau = \int_Q \int_0^T \int_\Omega y_1 (x, t) \left( - \int_0^1 \frac{\partial \phi_1}{\partial \tau} T_0 \alpha (\phi_1 (x, t)) \right) d\sigma d\tau dxd\tau
\]

whence comparing with (4.11), (4.12), we get

\[
- \int_0^T \int_\Omega \frac{\partial \phi_1}{\partial \tau} \phi_1 (x, t) dxd\tau + \frac{B(\alpha)}{1 - \alpha} \int_Q \int_0^T \int_\Omega y_1 E_{\alpha, \alpha} \left[ - \gamma t^\alpha \right] \phi_1 (x, t) dxd\tau
\]

\[
= \frac{B(\alpha)}{1 - \alpha} \int_Q \int_0^T \int_\Omega y_2 E_{\alpha, \alpha} \left[ - \gamma t^\alpha \right] \phi_2 (x, t) dxd\tau,
\]

\[
- \int_0^T \int_\Omega \frac{\partial \phi_2}{\partial \tau} \phi_2 (x, t) dxd\tau
\]

From this we deduce the initial conditions

\[
y_1 (x, 0) = y_{0,1}, \quad x \in \Omega,
\]

\[
y_2 (x, 0) = y_{0,2}, \quad x \in \Omega,
\]

which completes the proof.
5. Optimization theorem and the fractional control problem

For a control \( u = (u_1, u_2) \in (L^2(Q))^2 \), the state \( y(u) = (y_1(u), y_2(u)) \) of the system is given by the fractional variation coupled systems:

\[
\begin{align*}
\frac{ABC}{0} D_t^\alpha y_1(u) - \Delta y_1(u) + y_1(u) - y_2(u) &= f_1(t) + u_1, \quad \text{in } Q, \quad \text{a.e. } t \in ]0, T[, f_1 \in L^2(Q), \quad \text{(5.1)} \\
\frac{ABC}{0} D_t^\alpha y_2(u) - \Delta y_2(u) + y_2(u) + y_1(u) &= f_2(t) + u_2, \quad \text{in } Q, \quad \text{a.e. } t \in ]0, T[, f_2 \in L^2(Q), \quad \text{(5.2)}
\end{align*}
\]

\[
\begin{align*}
y_1(x, 0; u) &= y_{01}(x) \in L^2(\Omega), \quad x \in \Omega, \quad \text{(5.3)} \\
y_2(x, 0; u) &= y_{02}(x) \in L^2(\Omega), \quad x \in \Omega, \quad \text{(5.4)} \\
y_1(x, t) &= 0, \quad y_2(x, t) = 0, x \in \Gamma, t \in (0, T). \quad \text{(5.5)}
\end{align*}
\]

The final observation equations are given by

\[
\begin{align*}
z_i(u) &= y_i(u, T), \quad \text{for each } i = 1, 2. \quad \text{(5.6)}
\end{align*}
\]

The cost function \( J(v) \) for \( v = \{v_1, v_2\} \) is given by

\[
\begin{align*}
J(v) &= \int_Q \left[ (y_1(v, T) - z_{d1})^2 + (y_2(v, T) - z_{d2})^2 \right] \, dx \, dt \\
&\quad + (Nu, v - u)_U \quad \text{(5.7)}
\end{align*}
\]

where \( z_d = [z_{d1}, z_{d2}] \) is a given element in \( (L^2(Q))^2 \) and \( N = [N_1, N_2] \in L(L^2(Q), L^2(Q)) \) is Hermitian positive definite operator:

\[
(Nu, u) \geq c ||u||^2_{L^2(Q)}, \quad c > 0, \quad \text{for each } i = 1, 2. \quad \text{(5.8)}
\]

Control constraints: We define \( U_{ad}(\) set of admissible controls) is closed, convex subset of \( U = L^2(Q) \times L^2(Q) \).

Control problem: We want to minimize \( J(v) \) over \( U_{ad} \), i.e.

\[
\begin{align*}
J(u) &= \inf_{v \in \{v_1, v_2\} \in U_{ad}} J(v). \quad \text{(5.9)}
\end{align*}
\]

Under the given considerations, we have the following theorem.

**Theorem 5.1:** The problem (5.9) admits a unique solution given by (5.1)–(5.5) and the optimality condition

\[
\begin{align*}
\int_Q \left[ p_1(v_1 - u_1) + p_2(v_2 - u_2) \right] \, dx \, dt + (Nu, v - u)_U \\
&\geq 0, \quad \forall v \in U_{ad}, \quad u \in U_{ad}, \quad \text{(5.10)}
\end{align*}
\]

where \( p(u) = [p_1(u), p_2(u)] \) is the adjoint state.

**Proof:** Since the control \( u \in U_{ad} \) is optimal if and only if

\[
J'(u)(v - u) \geq 0 \quad \text{for all } v \in U_{ad}. \quad \text{(5.11)}
\]

The above condition, when explicitly calculated for this case, gives

\[
\begin{align*}
(y_1(u) - z_{d1}, y_1(v) - y_1(u))_{L^2(Q)} + (y_2(u) - z_{d2}, y_2(v) - y_2(u))_{L^2(Q)} + (Nu, v - u)_U \geq 0, \quad \text{(5.12)}
\end{align*}
\]

i.e.

\[
\begin{align*}
\int_Q [y_1(u) - z_{d1}, y_1(v) - y_1(u) + (y_2(u) - z_{d2})(y_2(v) - y_2(u))] \, dx \, dt + (Nu, v - u)_{L^2(Q)^2} \geq 0. \quad \text{(5.13)}
\end{align*}
\]

For the control \( u \in (L^2(Q))^2 \), the adjoint state \( p(u) = [p_1(u), p_2(u)] \in (L^2(Q))^2 \) is defined by

\[
\begin{align*}
- \frac{ABC}{0} D_t^\alpha p_1(u) + \Delta p_1(u) + p_1(u) + p_2(u) &= y_1(u) - z_{d1}, \quad \text{in } Q, \quad \text{(5.14)} \\
- \frac{ABC}{0} D_t^\alpha p_2(u) + \Delta p_2(u) + p_2(u) - p_1(u) &= y_2(u) - z_{d2}, \quad \text{in } Q, \quad \text{(5.15)} \\
p_1(u) &= 0, \quad p_2(u) = 0 \quad \text{on } \Sigma, \quad \text{(5.16)} \\
p_1(x, T; u) &= 0, \quad p_2(x, T; u) = 0 \quad \text{in } \Omega. \quad \text{(5.17)}
\end{align*}
\]

Now, multiplying Equations (5.14) and (5.15) by \( y_1(v) - y_1(u) \), \( y_2(v) - y_2(u) \) respectively and applying Green’s formula, we obtain

\[
\begin{align*}
\int_Q \left[ (y_1(u) - z_{d1}, y_1(v) - y_1(u)) \right] \, dx \, dt \\
= \int_Q \left[ \frac{-ABC}{0} D_t^\alpha p_1(u) + \Delta p_1(u) + p_1(u) \right] \, dx \, dt \\
&\quad + p_2(u)(y_1(v) - y_1(u)) \, dx \\
= \frac{B(\alpha)}{1 - \alpha} \int_0^T \int_\Omega \left[ \frac{\partial y_1(u)}{\partial v} - \frac{\partial y_1(u)}{\partial v_A} \right] \, d\Sigma \\
&\quad + \int_\Sigma \frac{\partial p_1(u)}{\partial v}(y_1(v) - y_1(u)) \, d\Sigma \\
&\quad + \int_Q \left[ p_1(u) \left( \frac{ABC}{0} D_t^\alpha + \Delta + 1 \right) \right] + p_2(u)(y_1(v) - y_1(u)) \, dx \, dt, \quad \text{(5.18)}
\end{align*}
\]

\[
\begin{align*}
\int_Q \left[ (y_2(u) - z_{d2}, y_2(v) - y_2(u)) \right] \, dx \, dt \\
= \int_Q \left[ \frac{-ABC}{0} D_t^\alpha p_2(u) + \Delta p_2(u) + p_2(u) \right] \, dx \, dt \\
&\quad - p_1(u)(y_2(v) - y_2(u)) \, dx \\
= \frac{B(\alpha)}{1 - \alpha} \int_0^T \int_\Omega \left[ \frac{\partial y_2(u)}{\partial v} - \frac{\partial y_2(u)}{\partial v_A} \right] \, d\Sigma \\
&\quad - \int_\Sigma \frac{\partial p_2(u)}{\partial v}(y_2(v) - y_2(u)) \, d\Sigma \\
&\quad + \int_Q \left[ p_2(u) \left( \frac{ABC}{0} D_t^\alpha + \Delta + 1 \right) \right] + p_2(u)(y_2(v) - y_2(u)) \, dx \, dt,
\end{align*}
\]

\[
\begin{align*}
\int_Q \left[ (y_1(v) - y_1(u)) \right] \, dx \, dt \\
= \int_Q \left[ \frac{-ABC}{0} D_t^\alpha y_1(v) + \Delta y_1(v) + y_1(v) \right] \, dx \, dt \\
&\quad + y_1(u)(y_1(v) - y_1(u)) \, dx \\
= \frac{B(\alpha)}{1 - \alpha} \int_0^T \int_\Omega \left[ \frac{\partial y_1(v)}{\partial v} - \frac{\partial y_1(v)}{\partial v_A} \right] \, d\Sigma \\
&\quad - \int_\Sigma \frac{\partial y_1(v)}{\partial v}(y_1(v) - y_1(u)) \, d\Sigma \\
&\quad + \int_Q \left[ y_1(v) \left( \frac{ABC}{0} D_t^\alpha + \Delta + 1 \right) \right] + y_1(v)(y_1(v) - y_1(u)) \, dx \, dt.
\end{align*}
\]
\[-y_2(u, v, 0) \, dt \, dx + \int_{\Sigma} \sigma_0 (u) \left( \frac{\partial y_2(v)}{\partial \nu} - \frac{\partial y_2(u)}{\partial \nu} \right) \, d\Sigma \]
\[\int_{\partial \Omega} \left( y_2(v) - y_2(u) \right) \, d\nu \]

which can be written as

\[\int_{\Omega} \left( \frac{\partial z_1(v)}{\partial \nu} \right) \, d\Omega \]
\[\int_{\partial \Omega} \left( y_2(v) - y_2(u) \right) \, d\nu \]

and hence (5.13) is equivalent to

\[\int_{\Omega} p_1(u) (v_1 - u_1) \, dx + \int_{\Omega} p_2(u) (v_2 - u_2) \, dx + (N u, v - u) (L^2(Q))^2 \geq 0, \]

which can be written as

\[\int_{\Omega} \left[ (p_1(u) + N_1 u_1)(v_1 - u_1) \]
\[+ (p_2(u) + N_2 u_2)(v_2 - u_2) \right] \, dx \geq 0, \]

which completes the proof. \[\blacksquare\]

6. Mathematical examples and applications

This section is devoted to study some mathematical examples and applications to illustrate our results in this paper.

Example 6.1 (Neumann problem): We consider an example of a diffusion equation which is analogous to that considered in Section 2 but with Neumann boundary condition and boundary control.

In this example, we consider the space

\[\mathcal{V}(0, T) := \{ y : y \in L^2(0, T; H^1(\Omega)), \Delta \} \]

in which a solution of a fractional differential systems is contained. Let \(y(u) = \{ y_1(u), y_2(u) \} \in \mathcal{V}(0, T)\) be the state of the system which is given by

\[\frac{\partial y_1(x, t)}{\partial \nu} = f_1(t), \text{ in } Q, \quad \text{a.e. } t \in [0, T], f_1 \in L^2(Q), \]
\[\frac{\partial y_2(x, t)}{\partial \nu} = f_2(t), \text{ in } Q, \quad \text{a.e. } t \in [0, T], f_2 \in L^2(Q), \]

\[y_1(x, 0, u) = y_0, x \in \Omega, \]
\[y_2(x, 0, u) = y_0, x \in \Omega, \]

\[\frac{\partial y_1(x, t)}{\partial \nu} \bigg|_{\Sigma} = u_1, \quad \frac{\partial y_2(x, t)}{\partial \nu} \bigg|_{\Sigma} = u_2, x \in \Gamma, t \in (0, T). \]

The control \(u = \{u_1, u_2\}\) is taken in \(L^2(\Sigma) \times L^2(\Sigma)\):

\[u = \{u_1, u_2\} \in U = L^2(\Sigma) \times L^2(\Sigma). \]

Problem (6.2)–(6.6) admits a unique solution. To see this we apply Theorem (1.2) [15], with

\[V = H^1(\Omega) \times H^1(\Omega), \quad \phi = \{\phi_1, \phi_2\} \in V, \]

\[\pi(t; y, \phi) = \pi(y, \phi) = \int_{\Omega} \Delta y_1(x) \phi_1(x) \, dx \]
\[+ \int_{\Omega} \Delta y_2(x) \phi_2(x) \, dx \]
\[+ \int_{\Omega} \left[ y_1 \phi_1 + y_2 \phi_2 - y_1 \phi_1 + y_1 \phi_2 \right] \, dx, \]

\[L(\phi) = f(t, \phi) = \int_{\Omega} [f_1(x, t) \phi_1(x) + f_2(x, t) \phi_2(x)] \, dx \]
\[+ \int_{\Gamma} \left[ u_1(t) \phi_1(x) + u_2(t) \phi_2(x) \right] \, d\Gamma. \]

Let us consider the case where we have partial observation of the final state

\[z(v) = y_1(x, T; v), \]

(6.11)
and the cost function $J(v)$ for $v = (v_1, v_2)$ is given by

$$J(v) = \int_\Omega (y_1(x, T; v) - z_d)^2 \, dx$$

$$+ \langle Nu, v \rangle_{(L^2(\sum), L^2(\sum))}, \quad z_d \in L^2(\Omega),$$

where $N = \{N_1, N_2\} \in C(L^2(\sum), L^2(\sum))$ is Hermitian positive definite operator:

$$\langle Nu, u \rangle \geq c\|u\|^2_{L^2(\sum)}, \quad c > 0.$$  

Control constraints: We define $U_{ad}$ (set of admissible controls) is closed, convex subset of $U = L^2(\sum) \times L^2(\sum)$.

Control problem: We want to minimize $J$ over $U_{ad}$, i.e. find $u = (u_1, u_2)$ such that

$$J(u) = \inf_{v = (v_1, v_2) \in U_{ad}} J(v).$$

The adjoint state is given by

$$-t \ ABC Df^T_0 p_1(u) + \Delta p_1(u) + p_1(u) + p_2(u) = 0, \quad \text{in } Q,$$

$$-t \ ABC Df^T_0 p_2(u) + \Delta p_2(u) + p_2(u) - p_1(u) = 0, \quad \text{in } Q,$$

$$\frac{\partial p_1(u)}{\partial \nu} = 0, \quad \frac{\partial p_2(u)}{\partial \nu} = 0, \quad \text{on } \Sigma,$$

$$p_1(x, T; u) = y_1(u) - z_d, \quad \text{in } \Omega,$$

$$p_2(x, T; u) = 0, \quad \text{in } \Omega.$$ 

The optimality condition is

$$\int_\sum \left[ p_1(u)(v_1 - u_1) + p_2(u)(v_2 - u_2) \right] \, d\Sigma$$

$$+ \langle Nu, v \rangle_{(L^2(\sum), L^2(\sum))} \geq 0, \quad \forall v \in U_{ad}, \ u \in U_{ad}.$$ 

Example 6.2 (No constraints problem): In the case of no constraint on the control ($U = U_{ad}$) and $N = \{N_1, N_2\}$ is a diagonal matrix of operators. Then (6.19) reduces to

$$p_1 + N_1 u_1 = 0, \quad \text{on } \Sigma, \quad p_2 + N_2 u_2 = 0, \quad \text{on } \Sigma,$$

equivalent to

$$u_1 = -N_1^{-1}(p_1(u)|_\sum), \quad u_2 = -N_2^{-1}(p_2(u)|_\sum).$$

The fractional optimal control is obtained by solving (6.2)–(6.6) and (6.15)–(6.18) simultaneous, (where we eliminate $u_1, u_2$ with the aid of (6.21)) and then utilizing (6.21).

Example 6.3: If we take

$$U_{ad} = \{u|u_i \in L^2(\sum), u_i \geq 0 \}, \quad \text{almost everywhere on } \Sigma, i = 1, 2,$$

and $N = v \times \text{Identity}$, (6.19) gives

$$u_1 \geq 0, \quad p_1(u) + v_1 u_1 \geq 0, \quad u_1(p_1(u) + v_1 u_1) = 0, \quad \text{on } \Sigma,$$

$$u_2 \geq 0, \quad p_2(u) + v_2 u_2 \geq 0, \quad u_2(p_2(u) + v_2 u_2) = 0, \quad \text{on } \Sigma.$$

The fractional optimal control is obtained by the solution of the fractional problem

$$\frac{\partial y_1(x, t)}{\partial \nu} = f_1(t), \quad \text{in } Q, \quad \text{a.e. } t \in [0,T], \ f_1 \in L^2(Q),$$

$$\frac{\partial y_2(x, t)}{\partial \nu} = f_2(t), \quad \text{in } Q, \quad \text{a.e. } t \in [0,T], \ f_2 \in L^2(Q),$$

$$y_1(x, 0; u) = y_0,1(x) \in L^2(\Omega), \quad x \in \Omega,$$

$$y_2(x, 0; u) = y_0,2(x) \in L^2(\Omega), \quad x \in \Omega,$$

$$p_1(x, T; u) = y_1(u) - z_d, \quad \text{in } \Omega,$$

$$p_2(x, T; u) = 0, \quad \text{in } \Omega.$$ 

$$\frac{\partial y_1(x, t)}{\partial \nu}|_\Sigma = 0, \quad x \in \Gamma, \ t \in (0,T),$$

$$\frac{\partial y_2(x, t)}{\partial \nu}|_\Sigma = 0, \quad x \in \Gamma, \ t \in (0,T),$$

$$\frac{\partial y_1(x, t)}{\partial \nu}|_\Sigma = 0, \quad x \in \Gamma, \ t \in (0,T),$$

$$\frac{\partial p_1(u)}{\partial \nu} = 0, \quad \text{on } \Sigma,$$

$$\frac{\partial p_2(u)}{\partial \nu} = 0, \quad \text{on } \Sigma,$$

$$p_1 + \frac{\partial y_1}{\partial \nu} \geq 0, \quad \frac{\partial y_1}{\partial \nu} \left[ p_1 + v_1 \frac{\partial y_1}{\partial \nu} \right] = 0, \quad \text{on } \Sigma,$$

$$p_2 + \frac{\partial y_2}{\partial \nu} \geq 0, \quad \frac{\partial y_2}{\partial \nu} \left[ p_2 + v_2 \frac{\partial y_2}{\partial \nu} \right] = 0, \quad \text{on } \Sigma,$$

$$u_1 = \frac{\partial y_1}{\partial \nu}|_\Sigma,$$

$$u_2 = \frac{\partial y_2}{\partial \nu}|_\Sigma.$$ 

Example 6.4 (Riemann–Liouville sense): We consider an example analogous to that considered in
Example (6.1), but the fractional time derivative is considered in a Riemann–Liouville sense. The state equations are given by

\[ 0^\alpha D_t^\alpha y_1(u) - \Delta y_1(u) + y_1(u) - y_2(u) = f_1(t), \text{ in } Q, \quad a.e. t \in [0, T], \quad f_1 \in L^2(Q), \]  
\[ 0^\alpha D_t^\alpha y_2(u) - \Delta y_2(u) + y_2(u) + y_1(u) = f_2(t), \text{ in } Q, \quad a.e. t \in [0, T], \quad f_2 \in L^2(Q), \]  
\[ L^{1-\alpha}y_1(x, 0^+; u) = y_{01}(x) \in L^2(\Omega), \quad x \in \Omega, \]  
\[ L^{1-\alpha}y_2(x, 0^+; u) = y_{02}(x) \in L^2(\Omega), \quad x \in \Omega, \]  
\[ \frac{\partial y_1(x, t)}{\partial \nu} |_{\Sigma} = y_1, \quad \frac{\partial y_2(x, t)}{\partial \nu} |_{\Sigma} = y_2, \quad x \in \Gamma, t \in (0, T), \]  

where the fractional integral \( L^{1-\alpha} \) and the derivative \( 0^\alpha D_t^\alpha \) are understood here in the Riemann–Liouville sense, \( L^{1-\alpha}y(x, 0^+; u) = \lim_{t \to 0^+} L^1 \alpha y(x, t; u) \).

The adjoint state is given by

\[ 0^\alpha D_t^\alpha y_1(u) - \Delta y_1(u) + y_1(u) - y_2(u) = f_1(t), \text{ in } Q, \]  
\[ 0^\alpha D_t^\alpha y_2(u) - \Delta y_2(u) + y_2(u) + y_1(u) = f_2(t), \text{ in } Q, \]  
\[ L^{1-\alpha}y_1(x, 0^+; u) = y_{01}(x) \in L^2(\Omega), \quad x \in \Omega, \]  
\[ L^{1-\alpha}y_2(x, 0^+; u) = y_{02}(x) \in L^2(\Omega), \quad x \in \Omega, \]  
\[ \frac{\partial y_1(x, t)}{\partial \nu} |_{\Sigma} = y_1, \quad \frac{\partial y_2(x, t)}{\partial \nu} |_{\Sigma} = y_2, \quad x \in \Gamma, t \in (0, T), \]  

and the cost function \( J(v) \) for \( v = [v_1, v_2] \) is given by

\[ J(v) = \int \Omega (y_1(x, T; v) - z_d)^2 \, dx \]  
\[ + (Nu, v - u)_{L^2(\Sigma)}^2 \geq 0, \quad \forall v \in U_{ad}, \quad u \in U_{ad}. \]  

The control \( u = [u_1, u_2, \ldots, u_n] \) is taken in \( (L^2(\Sigma))^n \):

\[ u = [u_1, u_2, \ldots, u_n] \in U = (L^2(\Sigma))^n. \]  

Problem (6.41)–(6.44) admits a unique solution. To see this, we use the method developed in [15]:

\[ V = (H^1(\Omega))^n, \quad \phi = [\phi_1, \phi_2, \ldots, \phi_n] \in V, \]  

\[ \pi(t, y, \phi) = \pi(y, \phi) = \int \Omega \sum_{i=1}^n \frac{\partial}{\partial \nu} y_i(x) \frac{\partial}{\partial x_i} \phi_i(x) \, dx \]  
\[ + \int \sum_{i=1}^n b_i y_i(x) \phi_i(x) \, dx, \]  
\[ L(\phi) = (f, \phi) = \int \Omega \sum_{i=1}^n f_i(x, t) \phi_i(x) \, dx \]  
\[ + \int \sum_{i=1}^n u_i(t) \phi_i(x) \, d\Gamma. \]  

Let us consider the case where we have partial observation of the final state

\[ z(v) = y_1(x, T; v), \]  

and the cost function \( J(v) \) for \( v = [v_1, v_2] \) is given by

\[ J(v) = \int \Omega (y_1(x, T; v) - z_d)^2 \, dx \]  
\[ + (Nu, v - u)_{L^2(\Sigma)}^2 \geq 0, \quad \forall v \in U_{ad}, \quad u \in U_{ad}. \]  

Control constraints: We define \( U_{ad} \) (set of admissible controls) is closed, convex subset of \( U = (L^2(\Sigma))^n \).

Control problem: We want to minimize \( J \) over \( U_{ad} \), i.e. find \( u = [u_1, u_2, \ldots, u_n] \) such that

\[ J(u) = \inf_{v=[v_1,v_2,\ldots,v_n]\in U_{ad}} J(v). \]  

The optimal state is given by

\[ -{\alpha \nu^\gamma D_t^\gamma p_1(u) + \Delta p_1(u) + \sum_{j=1}^n b_j p_j(u) = 0, \text{ in } Q, \]  
\[ \frac{\partial p_1(u)}{\partial \nu} = 0, \quad \text{on } \Sigma, \]  
\[ p_1(x, T; u) = y_1(u) - z_d, \quad \text{in } \Omega, \]  
\[ p_k(x, T; u) = 0, \quad k = 2, 3, \ldots, n, \quad \text{in } \Omega, \]  

where \( b_j \) are the transpose of \( b_j \). The optimality condition is

\[ \int \sum_{j=1}^n p_j(u)(v_j - u_j) \, d\Sigma + (Nu, v - u)_{L^2(\Sigma)}^2 \]  
\[ \geq 0, \quad \forall v \in U_{ad}, \quad u \in U_{ad}. \]
Remark 6.1: If we take $\alpha = 1$ in the previews sections, we obtain the classical results in the optimal control with integer derivatives.

7. Conclusion
In this paper, we considered optimal control problem for coupled diffusion systems with Atangana–Baleanu derivatives and final observation. The analytical results were given in terms of Euler–Lagrange equations for the fractional optimal control problems. The formulation presented and the resulting equations are very similar to those for classical optimal control problems for coupled parabolic systems. The optimization problem presented in this paper constitutes a generalization of the optimal control problem of diffusion equations with Dirichlet boundary conditions considered in [12, 13, 15, 22–24, 35, 39] to coupled systems with Atangana–Baleanu time derivatives.

Disclosure statement
No potential conflict of interest was reported by the authors.

Funding
This work was supported by Taibah University.

ORCID
G. M. Bahaa http://orcid.org/0000-0001-6670-3110
A. Hamiaz http://orcid.org/0000-0001-5605-8728

References
[1] Atangana A. Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties. Phys A Stat Mech Appl. 2018;505:688–706.
[2] Atangana A, Baleanu D. New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model. Therm Sci. 2016;20(2):763–769.
[3] Atangana A, Gomez Aguila JF. Decolonisation of fractional calculus rules: breaking commutativity and associativity to capture more natural phenomena. Eur Phys J Plus. 2018;133(166):1–22. https://doi.org/10.1140/epjp/i2018-12021-3
[4] Baleanu D, Fernandez A. On some new properties of fractional derivatives with Mittag–Leffler kernel. Commun Nonlinear Sci Numer Simul. 2017;59:344–362.
[5] Baleanu D, Jajarmi A, Hajipour M. A new formulation of the fractional optimal control problems involving Mittag–Leffler nonsingular kernel. J Optim Theory Appl. 2017;175:718–737.
[6] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Prog Fract Differ Appl. 2015;1(2):73–85.
[7] Djida JD, Atangana A, Area I. Numerical computation of a fractional derivative with non-local and non-singular kernel. Math Model Nat Phenom. 2017;12(3):4–13.
[8] Djida JD, Mophou GM, Area I. Optimal control of diffusion equation with fractional time derivative with nonlocal and nonsingular Mittag–Leffler kernel. ArXiv preprint arXiv:1711.09070. 2017
[9] Gomez-Aguilar JF. Irving–Mullineux oscillator via fractional derivatives with Mittag–Leffler kernel. Chaos Soliton Fract. 2017;95(35):179–186.
[10] Gomez-Aguilar JF. Space–time fractional diffusion equation using a derivative with nonsingular and regular kernel. Phys A. 2017;465:562–572.
[11] Gomez-Aguilar JF, Atangana A, Morales-Delgado JF. Electrical circuits RC, LC, and RL described by Atangana–Baleanu fractional derivatives. Int J Circ Theor Appl. 2017;45(11):1514–1533. https://doi.org/10.1002/cta.2348.
[12] Agrawal OP. A general solution for the fourth-order fractional diffusion-wave equation. Fract Calc Appl Anal. 2003;6:31–12.
[13] Agrawal OP. Solution for a fractional diffusion-wave equation defined in a bounded domain. Nonlinear Dyn. 2002;29:145–155.
[14] Mophou GM, NGuerekata G. Optimal control of a fractional diffusion equation with state constraints. Comput Math Appl. 2011;62:1413–1426.
[15] Lions JL. Optimal Control of Systems Governed by Partial Differential Equations. Berlin Heidelberg: Springer-Verlag; 1971.
[16] Bahaa GM, Kotarski W. Time-optimal control of infinite order distributed parabolic systems involving multiple time-varying lags. Num Funct Anal Optim. 2016;37(9):1066–1088.
[17] Ahmad B, Ntouyas SK. Existence of solutions for fractional differential inclusions with four-point nonlocal Riemann–Liouville type integral boundary conditions. Filomat. 2013;27(6):1027–1036.
[18] Bahaa GM, Khidr S. Numerical solutions for optimal control problem governed by elliptic system on Lipschitz domains. JTSI https://doi.org/10.1080/16583655.2018.1522739
[19] Bahaa GM, El-Marouf SAA, Embaby OA. Pareto optimal control for mixed Neumann infinite-order parabolic system with state-control constraints. JTSI. 2015;9:264–273.
[20] Bahaa GM. Fractional optimal control problem for infinite order system with control constraints. Adv Diff Eq. 2016;250:1–16.
[21] Bahaa GM. Fractional optimal control problem for differential system with delay argument. Adv Diff Eq. 2017;69:1–19.
[22] Bahaa GM. Fractional optimal control problem for variable-order differential systems. Fract Calc Appl Anal. 2017;20(6):1447–1470.
[23] Mophou GM. Optimal control of fractional diffusion equation. Comput Math Appl. 2011;61:68–78.
[24] Mophou GM. Optimal control of fractional diffusion equation with state constraints. Comput Math Appl. 2011;62:1413–1426.
[25] Mophou GM, Tao S, Joseph C. Initial value/boundary value problem for composite fractional relaxation equation. Appl Math Comput. 2015;257:134–144.
[26] Debouche A, Nieto JJ. Sobolev type fractional abstract evolution equations with nonlocal conditions and optimal multi-controllers. Appl Math Comput. 2015;245:74–85.
[27] Debouche A, Nieto JJ. Relaxation in controlled systems described by fractional integro-differential equations with nonlocal control conditions. Elect J Diff Eq. 2015;89:1–18.
[28] Wang JR, Zhou Y. A class of fractional evolution equations and optimal controls. Nonlinear Anal Real World Appl. 2011;12:262–272.

[29] Tang Q, Ma QX. Variational formulation and optimal control of fractional diffusion equations with Caputo derivatives. Adv Diff Eq. 2015;283:1–15.

[30] Agrawal OP. Formulation of Euler–Lagrange equations for fractional variational problems. J Math Anal Appl. 2002;272:368–379.

[31] Agrawal OP. A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dyn. 2004;38:323–337.

[32] Bahaa GM. Fractional optimal control problem for variational inequalities with control constraints. IMA J Math Control Inform. 2018;35(1):107–122.

[33] Bahaa GM. Fractional optimal control problem for differential system with control constraints. Filomat. 2016;30(8):2177–2189.

[34] Agrawal OP, Defterli O, Baleanu D. Fractional optimal control problems with several state and control variables. J Vibr Cont. 2010;16(13):1967–1976.

[35] Mophou GM, Joseph C. Optimal control with final observation of a fractional diffusion wave equation. Dyn Contin Disc Impul Syst Ser A: Math Anal. 2016;23:341–364.

[36] Debbouche A, Torres DFM. Sobolev type fractional dynamic equations and optimal multi-integral controls with fractional nonlocal conditions. Fract Calc Appl Anal. 2015;18(1):95–121.

[37] Debbouche A, Nieto JJ, Torres DFM. Optimal solutions to relaxation in multiple control problems of Sobolev type with nonlocal nonlinear fractional differential equations. J Optim Theo Appl. 2017;174(1):7–31.

[38] Abdeljawad T, Baleanu D. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag–Leffler nonsingular kernel. J Nonlinear Sci Appl. 2017;10:1098–1107.

[39] Ahmed H.F. A numerical technique for solving multi-dimensional fractional optimal control problems. JTUS. 2018;12(5):494–505. doi:10.1080/16583655.2018.1491690.