ON DUOIDAL ∞-CATEGORIES

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Abstract. A duoidal category is a category equipped with two monoidal structures in which one is (op)lax monoidal with respect to the other. In this paper we introduce duoidal ∞-categories which are counterparts of duoidal categories in the setting of ∞-categories. There are three kinds of functors between duoidal ∞-categories, which are called bilax, double lax, and double oplax monoidal functors. We make three formulations of ∞-categories of duoidal ∞-categories according to which functors we take. Furthermore, corresponding to the three kinds of functors, we define bimonoids, double monoids, and double comonoids in duoidal ∞-categories.

1. Introduction

The goal of this paper is to introduce duoidal categories in the setting of ∞-categories. A duoidal category is a category equipped with two monoidal products in which they are compatible in the following sense: the functors and natural transformations defining one monoidal structure are (op)lax monoidal with respect to the other monoidal structure.

The notion of duoidal categories have been introduced in [1] under the name of 2-monoidal categories. Aguiar and Mahajan [1] have developed a basic theory of duoidal categories and three kinds of functors between them. The name of duoidal categories has appeared in [20]. Inspired by 3-manifold invariants, Street [20] has developed a theory of duoidal categories and centers of monoids in monoidal categories. After that, duoidal categories have been used in many papers. In [2] Batanin and Markl have considered a theory of centers and homotopy centers of monoids in monoidal categories which are enriched in duoidal categories, and in [3] they have proved duoidal Deligne conjecture using their previous work. Booker and Street have further developed a general theory of duoidal categories in [7]. We can consider bimonoids in a duoidal category, which is a generalization of bialgebras in a symmetric monoidal category. In order to generalize the notion of Hopf algebras, Böhm, Chen, and Zhang have studied Hopf monoids and the fundamental theorem of Hopf modules in duoidal categories in [5]. Furthermore, Böhm and Lack have studied a notion of antipode on bimonoids in duoidal categories under some circumstances in [6].

There are many approaches to the theory of higher categories, and we have many models of (∞, 1)-categories, where (∞, 1)-categories are ∞-categories in which all k-morphisms are invertible for k > 1. Quasi-categories are one of these models, which were originally introduced by Boardman-Vogt [4] in their study of homotopy invariant algebraic structures. After that, quasi-categories are extensively developed by Joyal [12, 13], Lurie [14, 15], and many others. Quasi-categories and other models of (∞, 1)-categories have many applications in many areas of mathematics. According to Lurie [14], we use the term ∞-category to refer to quasi-categories and other models of (∞, 1)-categories.

Comodules over Hopf algebroids of co-operations for generalized (co)homology theories play fundamental roles in chromatic homotopy theory [17, 18, 16]. We can regard Hopf algebroids as...
examples of bimonoids in the duoidal category of bimodules over graded commutative rings. In [21] we have studied $\infty$-categories of comodule spectra over coalgebras of co-operations for generalized (co)homology theories. Although we did not use multiplicative structures on $\infty$-categories of comodule spectra in [21], we realized that we need to develop a theory of duoidal categories in the setting of $\infty$-categories in order to discuss multiplicative structures on these $\infty$-categories of comodule spectra.

In this paper we will introduce duoidal categories in the setting of $\infty$-categories, which we call duoidal $\infty$-categories. We have three kinds of functors between duoidal categories, which are called bilax, double lax, and double oplax monoidal functors. Corresponding to these functors, we define three kinds of functors between duoidal $\infty$-categories. We construct three $\infty$-categories

$$Duo_{\infty}^{\text{bilax}}, \quad Duo_{\infty}^{\text{dlax}}, \quad Duo_{\infty}^{\text{doplax}}$$

of duoidal $\infty$-categories with bilax, double lax, and double oplax monoidal functors, respectively.

There are notions of bimonoids, double monoids, and double comonoids in duoidal categories. They are generalizations of bialgebras, commutative algebras, counitive coalgebras in braided monoidal categories, respectively. We will introduce counterparts in duoidal $\infty$-categories, and construct three $\infty$-categories

$$b\text{Mon}(X), \quad d\text{Mon}(X), \quad dc\text{Mon}(X)$$

of bimonoids, double monoids, and double comonoids in a duoidal $\infty$-category $X$. These constructions determine three functors

$$b\text{Mon} : Duo_{\infty}^{\text{bilax}} \to \text{Cat}_{\infty},$$

$$d\text{Mon} : Duo_{\infty}^{\text{dlax}} \to \text{Cat}_{\infty},$$

$$dc\text{Mon} : Duo_{\infty}^{\text{doplax}} \to \text{Cat}_{\infty}.$$

For a duoidal $\infty$-category $X$ with two monoidal structures $\boxtimes$ and $\otimes$, where $\otimes$ is lax monoidal with respect to $\boxtimes$, we will show that the $\infty$-category $c\text{Alg}^{\otimes}(X)$ of coalgebra objects of the monoidal $\infty$-category $(X, \otimes)$ supports a monoidal structure induced by $\boxtimes$ (Theorem 6.8). This construction determines a functor $c\text{Alg}^{\boxtimes} : Duo_{\infty}^{\text{bilax}} \to \text{Mon}^{\text{lax}}(\text{Cat}_{\infty})$, where $\text{Mon}^{\text{lax}}(\text{Cat}_{\infty})$ is the $\infty$-category of monoidal $\infty$-categories and lax monoidal functors. Similarly, we will show that there is a functor $\text{Alg}^{\boxtimes} : Duo_{\infty}^{\text{bilax}} \to \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty})$, which associates to a duoidal $\infty$-category $X$ the $\infty$-category of algebra objects of the monoidal $\infty$-category $(X, \boxtimes)$, where $\text{Mon}^{\text{oplax}}(\text{Cat}_{\infty})$ is the $\infty$-category of monoidal $\infty$-categories and oplax monoidal functors (Remark 6.11).

One of our main theorems is as follows.

**Theorem 1.1** (Theorem 6.11, Corollary 6.12, and Remark 6.13). For a duoidal $\infty$-category $X$, there are natural equivalences

$$b\text{Mon}(X) \simeq \text{Alg}(c\text{Alg}^{\otimes}(X)) \simeq c\text{Alg}(\text{Alg}^{\boxtimes}(X))$$

of $\infty$-categories. As a result, there are equivalences

$$b\text{Mon} \simeq \text{Alg} \circ c\text{Alg}^{\otimes} \simeq c\text{Alg} \circ \text{Alg}^{\boxtimes}$$

of functors from $Duo_{\infty}^{\text{bilax}}$ to $\text{Cat}_{\infty}$.

We will prove similar results on the functors $d\text{Mon} : Duo_{\infty}^{\text{dlax}} \to \text{Cat}_{\infty}$ and $dc\text{Mon} : Duo_{\infty}^{\text{doplax}} \to \text{Cat}_{\infty}$ (Theorems 6.15, 6.17, and 6.19).
Based on the results in this paper, we will generalize duoidal $\infty$-categories to higher monoidal $\infty$-categories in \[22\]. In \[23\] we will give an example of duoidal $\infty$-categories of operadic modules. In \[24\] we will give another example of duoidal $\infty$-categories obtained from map monoidales in monoidal $\infty$-bicategories.

The organization of this paper is as follows: In §2 we recall the notion of duoidal categories and three kinds of functors between them in the classical setting. In §3 we review monoidal $\infty$-categories and functors between them. We recall lax and oplax monoidal functors between monoidal $\infty$-categories. In §4 we introduce mixed fibrations and study their properties. In §4.1 we study monoid objects of the $\infty$-category of a slice $\infty$-category $\text{Cat}_{\infty/T}$. We consider lax and oplax monoidal functors between monoid objects of $\text{Cat}_{\infty/T}$. Motivated by the results in §4.1 we introduce mixed fibrations in §4.2. Although mixed fibrations have an obvious duality by definition, we also give another asymmetric description of mixed fibrations. We also study mixed fibrations over marked simplicial sets. In §5 we introduce duoidal $\infty$-categories as mixed fibrations which satisfy the Segal conditions. As in the case of classical setting, there are three kinds of functors between duoidal $\infty$-categories. According to which kind of functors of duoidal $\infty$-categories we take, we will give three formulations of $\infty$-categories of duoidal $\infty$-categories. In §6.1 we introduce notions of bimonoids, double monoids, and double comonoids in duoidal $\infty$-categories. In §6.2 we define bimonoids in duoidal $\infty$-categories and prove Theorem 1.1. In §6.3 we define double monoids and double comonoids in duoidal $\infty$-categories and prove similar results on them to Theorem 1.1.

**Remark 1.2.** In this paper we think ordinary categories are a special kind of $\infty$-categories. Thus, we do not distinguish notationally between ordinary categories and their nerves. In particular, we identify the simplicial indexing category $\Delta$ with its nerve $N(\Delta)$.

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2. DUOIDAL CATEGORIES IN THE CLASSICAL SETTING

The notion of duoidal categories has been introduced in \[1\] by the name of 2-monoidal categories, and the name of duoidal categories has appeared in \[20\]. In this section we recall duoidal categories in the classical setting and three kinds of functors between them.

**Definition 2.1** (cf. \[1, Definition 6.1\]). A duoidal category is a quintuple $(D, \boxtimes, 1_{\boxtimes}, \otimes, 1_{\otimes})$, where $(D, \boxtimes, 1_{\boxtimes})$ and $(D, \otimes, 1_{\otimes})$ are monoidal categories, along with a natural transformation

$$\zeta_{A,B,C,D} : (A \otimes B) \boxtimes (C \otimes D) \to (A \boxtimes C) \otimes (B \otimes D),$$

and three morphisms

$$\Delta : 1_{\boxtimes} \to 1_{\boxtimes} \otimes 1_{\boxtimes}, \quad \mu : 1_{\otimes} \boxtimes 1_{\otimes} \to 1_{\otimes}, \quad \iota = \epsilon : 1_{\boxtimes} \to 1_{\otimes}$$

satisfying the following conditions:

**Associativity.** The following diagrams commute

\[
\begin{array}{c}
\begin{array}{c}
((A \otimes B) \boxtimes (C \otimes D)) \boxtimes (E \otimes F) \\
\zeta \boxtimes \iota
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\alpha \circ \zeta \\
\id \boxtimes \zeta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
((A \boxtimes C) \otimes (B \otimes D)) \boxtimes (E \otimes F) \\
\zeta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(A \otimes B) \boxtimes ((C \otimes D) \otimes (E \otimes F)) \\
\zeta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(A \boxtimes (C \otimes E)) \otimes (B \otimes (D \otimes F)) \\
\alpha \otimes \alpha
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
((A \boxtimes C) \otimes E) \otimes ((B \otimes D) \otimes F) \\
\zeta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
((A \otimes B) \boxtimes ((C \otimes E) \otimes D \otimes F)) \\
\id \boxtimes \zeta
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(A \boxtimes (C \otimes E)) \otimes (B \boxtimes (D \otimes F)) \\
\zeta
\end{array}
\end{array}\]
we obtain 2-categories \( \text{Mon} \) is a monoidal 2-category under Cartesian product. Applying \( \text{Mon} \) domonoids, and 2-cells are morphisms between oplax morphisms.

the 2-category whose 0-cells are pseudomonoids in \( M \) monoids, and 2-cells are morphisms between lax morphisms. We also denote by \( M \) the 2-category whose 0-cells are pseudomonoids in \( \text{Cat} \).

Proposition 2.2 (cf. [1, Proposition 6.73]). A duoidal category is identified with a pseudomonoid in \( \text{Mon}^{\text{lax}}(\text{Cat}) \). A duoidal category is also identified with a pseudomonoid in \( \text{Mon}^{\text{oplax}}(\text{Cat}) \).
Next, we recall three kinds of functors between duoidal categories. Since \( \text{Mon}^{\text{lax}}(\text{Cat}) \) is a monoidal 2-category, we can consider a 2-category
\[
\text{Mon}^{\text{lax}}(\text{Mon}^{\text{lax}}(\text{Cat})).
\]
By Proposition 2.2, 0-cells of \( \text{Mon}^{\text{lax}}(\text{Mon}^{\text{lax}}(\text{Cat})) \) are identified with duoidal categories.

**Definition 2.3.** A bilax monoidal functor is a 1-cell of \( \text{Mon}^{\text{lax}}(\text{Mon}^{\text{oplax}}(\text{Cat})) \).

In other words, a bilax monoidal functor between duoidal categories \( (C, \boxtimes, 1_\Box, \otimes, 1_\bowtie) \) and \( (D, \boxtimes, 1_\Box, \otimes, 1_\bowtie) \) is a triple \( (F, \varphi, \psi) \), where

1. \( F : C \to D \) is a functor,
2. \( (F, \varphi) : (C, \boxtimes, 1_\Box) \to (D, \boxtimes, 1_\Box) \) is lax monoidal, and
3. \( (F, \psi) : (C, \otimes, 1_\bowtie) \to (D, \otimes, 1_\bowtie) \) is oplax monoidal,

satisfying appropriate compatibility conditions (see [1, Definition 6.50] for details).

Note that there is an equivalence
\[
\text{Mon}^{\text{lax}}(\text{Mon}^{\text{oplax}}(\text{Cat})) \simeq \text{Mon}^{\text{oplax}}(\text{Mon}^{\text{lax}}(\text{Cat}))
\]
of 2-categories by [1, Proposition 6.75]. Hence we can identify bilax monoidal functors with 1-cells of the 2-category \( \text{Mon}^{\text{oplax}}(\text{Mon}^{\text{lax}}(\text{Cat})) \).

We can also consider a 2-category
\[
\text{Mon}^{\text{lax}}(\text{Mon}^{\text{lax}}(\text{Cat})).
\]

**Definition 2.4.** A double lax monoidal functor is a 1-cell of \( \text{Mon}^{\text{lax}}(\text{Mon}^{\text{lax}}(\text{Cat})) \).

As in the case of bilax monoidal functors, a double lax monoidal functor between \( (C, \boxtimes, 1_\Box, \otimes, 1_\bowtie) \) and \( (D, \boxtimes, 1_\Box, \otimes, 1_\bowtie) \) is a triple \( (F, \varphi, \psi) \), where

1. \( F : C \to D \) is a functor,
2. \( (F, \varphi) : (C, \boxtimes, 1_\Box) \to (D, \boxtimes, 1_\Box) \) is lax monoidal, and
3. \( (F, \psi) : (C, \otimes, 1_\bowtie) \to (D, \otimes, 1_\bowtie) \) is lax monoidal,

satisfying appropriate compatibility conditions (see [1, Definition 6.54] for details).

Similarly, we can consider a 2-category
\[
\text{Mon}^{\text{oplax}}(\text{Mon}^{\text{oplax}}(\text{Cat}_\infty)).
\]

**Definition 2.5.** A double oplax monoidal functor is a 1-cell of \( \text{Mon}^{\text{oplax}}(\text{Mon}^{\text{oplax}}(\text{Cat}_\infty)) \).

By duality, a double oplax monoidal functor between duoidal categories \( (C, \boxtimes, 1_\Box, \otimes, 1_\bowtie) \) and \( (D, \boxtimes, 1_\Box, \otimes, 1_\bowtie) \) is a triple \( (F, \varphi, \psi) \) such that \( (F^{\text{op}}, \varphi^{\text{op}}, \psi^{\text{op}}) \) is a double lax monoidal functor from \( (C^{\text{op}}, \otimes, 1_\bowtie, \boxtimes, 1_\Box) \) to \( (D^{\text{op}}, \otimes, 1_\bowtie, \boxtimes, 1_\Box) \) (see [1, Definition 6.55] for details).

### 3. Monoidal \( \infty \)-Categories

In this section we recall the definition of monoidal \( \infty \)-categories and strong monoidal functors as monoid objects and morphisms between them in the \( \infty \)-category of (small) \( \infty \)-categories. Furthermore, we recall the definition of lax monoidal functors between monoidal \( \infty \)-categories by regarding them as morphisms of nonsymmetric \( \infty \)-operads. By reformulating monoidal \( \infty \)-categories in terms of Cartesian fibrations, we introduce oplax monoidal functors. We think that results in this section are well known and do not claim any originality on them.
3.1. **Monoidal ∞-categories and strong monoidal functors.** In this subsection we recall the definitions of monoidal ∞-categories and strong monoidal functors as monoid objects and morphisms between them in the ∞-category Cat∞ of (small) ∞-categories (cf. [13] Definition 2.4.2.1 and [8] Definition 3.5.1).

Let ∆ be the simplicial indexing category. A morphism \( f : [n] \to [m] \) in ∆ is said to be inert if \( f \) is an injection given by \([n] = \{0,1,\ldots,n\} \cong \{i,i+1,\ldots,i+n\} \hookrightarrow \{0,1,\ldots,m\} = [m] \) for some \( 0 \leq i \leq m-n \). We denote by \( \rho^i : [1] \to [n] \) the inert morphism in ∆ given by \([1] = \{0,1\} \cong \{i-1,i\} \hookrightarrow \{0,1,\ldots,n\} = [n] \) for \( 1 \leq i \leq n \).

We recall the definition of monoid objects in an ∞-category with finite products (cf. [13] Definition 2.4.2.1 and [8] Definition 3.5.1]).

**Definition 3.1.** Let \( \mathcal{X} \) be an ∞-category which admits finite products, and let \( M : \Delta^\text{op} \to \mathcal{X} \) be a simplicial object in \( \mathcal{X} \). The inert morphism \( \rho^i \) induces a morphism \( M(\rho^i) : M([n]) \to M([1]) \) for \( 1 \leq i \leq n \). Taking a product of \( M(\rho^i) \), we obtain a morphism

\[
\prod_{1 \leq i \leq n} M(\rho^i) : M([n]) \to M([1]) \times \cdots \times M([1]).
\]

We call this morphism a Segal morphism. A monoid object in \( \mathcal{X} \) is a simplicial object in \( \Delta^\text{op} \) such that the Segal morphisms are equivalences in \( \mathcal{X} \) for all \([n] \in \Delta^\text{op} \). A morphism between monoid objects is a morphism of simplicial objects in \( \mathcal{X} \). We denote by

\[
\text{Mon}(\mathcal{X})
\]

the ∞-category of monoid objects in \( \mathcal{X} \) and morphisms between them. Note that \( \text{Mon}(\mathcal{X}) \) is a full subcategory of the ∞-category \( \text{Fun}(\Delta^\text{op}, \mathcal{X}) \) of simplicial objects in \( \mathcal{X} \).

Let \( \text{Cat}_\infty \) be the ∞-category of (small) ∞-categories. Since \( \text{Cat}_\infty \) has finite products, we can consider monoid objects in the ∞-category \( \text{Cat}_\infty \).

**Definition 3.2.** A monoidal ∞-category is a monoid object in \( \text{Cat}_\infty \), and a strong monoidal functor between monoidal ∞-categories is a morphism of monoid objects in \( \text{Cat}_\infty \). The ∞-category

\[
\text{Mon}(\text{Cat}_\infty)
\]

is the ∞-category of monoidal ∞-categories and strong monoidal functors.

3.2. **Monoidal ∞-categories as coCartesian fibrations.** In this subsection we recall another formulation of monoidal ∞-categories as coCartesian fibrations of nonsymmetric ∞-operads (cf. [13] Example 2.4.2.4 and [9] Definition 3.5.1).

First, we recall the straightening and unstraightening equivalence by Lurie [14] Theorem 3.2.0.1]. Let \( S \) be an ∞-category, and let \( \text{Cat}_\infty/S \) be the slice category of \( \text{Cat}_\infty \) over \( S \). We denote by \( \text{coCart}/s^t \) the subcategory of \( \text{Cat}_\infty/S \) consisting of coCartesian fibrations over \( S \) and functors over \( S \) which preserve coCartesian morphisms. By Lurie’s straightening and unstraightening functors [14] §3.2, there is an equivalence

\[
\text{Fun}(S, \text{Cat}_\infty) \simeq \text{coCart}/s^t
\]

of ∞-categories.

Thus, monoid objects of \( \text{Cat}_\infty \) corresponds to coCartesian fibrations over \( \Delta^\text{op} \) with some properties under the equivalence between \( \text{Fun}(\Delta^\text{op}, \text{Cat}_\infty) \) and \( \text{coCart}/(\Delta^\text{op}) \). We shall describe coCartesian fibrations over \( \Delta^\text{op} \) corresponding to monoidal ∞-categories.

Let \( p : X \to \Delta^\text{op} \) be a coCartesian fibration. We denote by \( X_{[n]} \) the fiber of \( p \) at \([n] \in \Delta^\text{op} \). The inert morphism \( \rho^i \) of \( \Delta \) induces a functor \( (\rho^i)_! : X_{[n]} \to X_{[1]} \) of ∞-categories. Taking a product of
(\rho^i)_{1 \leq i \leq n}$; for $1 \leq i \leq n$, we obtain a functor
\[
\prod_{1 \leq i \leq n} (\rho^i)! : X[n] \rightarrow X[1] \times \cdots \times X[1],
\]
which is also called a Segal morphism. A coCartesian fibration $p : X \rightarrow \Delta^{op}$ corresponds to a monoidal $\infty$-category if and only if the Segal morphisms are equivalences for all $[n] \in \Delta^{op}$. We also call such a coCartesian fibration a monoidal $\infty$-category.

Next, we shall describe strong monoidal functors between monoidal $\infty$-categories in terms of coCartesian fibrations. Let $p : X \rightarrow \Delta^{op}$ and $q : Y \rightarrow \Delta^{op}$ be coCartesian fibrations which are monoidal $\infty$-categories. We consider a functor $f : X \rightarrow Y$ of $\infty$-categories over $\Delta^{op}$. The functor $f$ corresponds to a strong monoidal functor if and only if $f$ carries $p$-coCartesian morphisms to $q$-coCartesian morphisms.

We denote by \( \text{Mon}(\text{Cat}_\infty) \) the full subcategory of $\text{Cart}/(\Delta^{op})^\sharp$ spanned by monoidal $\infty$-categories. By the above argument, there is an equivalence of $\infty$-categories
\[
\text{Mon}(\text{Cat}_\infty) \simeq \text{Mon}(\text{Cat}_\infty).
\]

**Remark 3.3.** Let $\text{Cart}/\Delta^t$ be the subcategory of $\text{Cat}_\infty/\Delta$ consisting of Cartesian fibrations over $\Delta$ and functors over $\Delta$ which preserve Cartesian morphisms. By Lurie's straightening and unstraightening functors [14 §3.2], there is an equivalence $\text{Fun}(\Delta^{op}, \text{Cat}_\infty) \simeq \text{Cart}/\Delta^t$ of $\infty$-categories. Thus, we can also describe monoidal $\infty$-categories in terms of Cartesian fibrations. We say that a Cartesian fibration $p : X \rightarrow \Delta$ is a monoidal $\infty$-category if the Segal morphisms $\prod_{1 \leq i \leq n} (\rho^i)! : X[n] \rightarrow X[1] \times \cdots \times X[1]$ are equivalences for all $[n] \in \Delta$. We define
\[
\text{Mon}'(\text{Cat}_\infty)
\]
to be the full subcategory of $\text{Cart}/\Delta^t$ spanned by monoidal $\infty$-categories.

Let $(-)^{op} : \text{Cat}_\infty \rightarrow \text{Cat}_\infty$ be the functor which assigns to an $\infty$-category $C$ its opposite $\infty$-category $C^{op}$. The functor $(-)^{op}$ induces an equivalence between $\text{Cart}/\Delta^t$ and $\text{coCart}/(\Delta^{op})^t$. Furthermore, this equivalence respects the Segal conditions. Hence $(-)^{op}$ induces an equivalence of $\infty$-categories
\[
(-)^{op} : \text{Mon}'(\text{Cat}_\infty) \simeq \text{Mon}(\text{Cat}_\infty).
\]

### 3.3. Lax and oplax monoidal functors

In this subsection we recall the definition of lax monoidal functors between monoidal $\infty$-categories as morphisms of coCartesian fibrations of non-symmetric $\infty$-operads (cf. [15 Definition 2.1.2.7] and [9 Definition 4.23]). By duality, we introduce oplax monoidal functors by using the description of monoidal $\infty$-categories as Cartesian fibrations.

First, we recall the definition of lax monoidal functors between monoidal $\infty$-categories. Let $p : X \rightarrow \Delta^{op}$ and $q : Y \rightarrow \Delta^{op}$ be objects of $\text{Mon}(\text{Cat}_\infty)$. We consider a functor $f : X \rightarrow Y$ of $\infty$-categories over $\Delta^{op}$.

**Definition 3.4** (cf. [15 Definition 2.1.2.7] and [9 Definition 4.23]). We say that $f$ is a lax monoidal functor if $f$ carries $p$-coCartesian morphisms over inert morphisms of $\Delta^{op}$ to $q$-coCartesian morphisms. We define an $\infty$-category
\[
\text{Mon}^{\text{lax}}(\text{Cat}_\infty)
\]
to be the subcategory of $\text{Cat}_\infty/\Delta^{op}$ consisting of monoidal $\infty$-categories and lax monoidal functors.
For a monoidal ∞-category \( p : X \to \Delta^{\text{op}} \), we call the fiber \( X_{[1]} \) at \([1]\) \( \in \Delta^{\text{op}} \) the underlying ∞-category of the monoidal ∞-category. Assigning to a monoidal ∞-category its underlying ∞-category, we obtain a functor
\[
ev_1 : \text{Mon}^{\text{op lax}}(\text{Cat}_\infty) \to \text{Cat}_\infty.
\]

**Remark 3.5.** A monoidal ∞-category can be identified with a non-symmetric ∞-operad \( p : X \to \Delta^{\text{op}} \) such that \( p \) is a coCartesian fibration. A lax monoidal functor between monoidal ∞-categories is a map of non-symmetric ∞-operads. Thus, the ∞-category \( \text{Mon}^{\text{op lax}}(\text{Cat}_\infty) \) is a full subcategory of \( \text{Op}^\text{NS}_\infty \) spanned by monoidal ∞-categories, where \( \text{Op}^\text{NS}_\infty \) is the ∞-category of non-symmetric ∞-operads.

Next, we shall define an oplax monoidal functor between monoidal ∞-categories. For this purpose, it is convenient for us to regard monoidal ∞-categories as Cartesian fibrations. Let \( p : X \to \Delta \) and \( q : Y \to \Delta \) be objects of \( \text{Mon}^{\text{op}}(\text{Cat}_\infty) \). We consider a functor \( f : X \to Y \) of ∞-categories over \( \Delta \).

**Definition 3.6.** We say that \( f \) is an oplax monoidal functor if \( f \) carries \( p \)-Cartesian morphisms over inert morphisms of \( \Delta \) to \( q \)-Cartesian morphisms. We define an ∞-category
\[
\text{Mon}^{\text{oplax}}(\text{Cat}_\infty)
\]
to be the subcategory of \( \text{Cat}_\infty/\Delta \) consisting of monoidal ∞-categories and oplax monoidal functors.

For an object \( q : Y \to \Delta \) of \( \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \), we say that the fiber \( Y_{[1]} \) at \([1]\) \( \in \Delta \) is the underlying ∞-category of the monoidal ∞-category. Assigning the underlying ∞-category to a monoidal ∞-category, we obtain a functor
\[
ev_1 : \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \to \text{Cat}_\infty.
\]

Finally, we shall describe a relationship between \( \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \) and \( \text{Mon}^{\text{lax}}(\text{Cat}_\infty) \) under the functor \( (-)^{\text{op}} \). Recall that \( (-)^{\text{op}} : \text{Cat}_\infty \to \text{Cat}_\infty \) is the functor which assigns to an ∞-category \( C \) its opposite ∞-category \( C^{\text{op}} \), and that \( (-)^{\text{op}} \) induces an equivalence between \( \text{Mon}^{\text{op}}(\text{Cat}_\infty) \) and \( \text{Mon}(\text{Cat}_\infty) \).

Let \( f : X \to Y \) be a morphism in \( \text{Cat}_\infty/\Delta \) between monoidal ∞-categories. We observe that \( f \) is an oplax monoidal functor if and only if \( f^{\text{op}} \) is a lax monoidal functor. Hence we obtain a functor
\[
(-)^{\text{op}} : \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \to \text{Mon}^{\text{lax}}(\text{Cat}_\infty)
\]
by assigning \( p^{\text{op}} : X^{\text{op}} \to \Delta^{\text{op}} \) to a monoidal ∞-category \( p : X \to \Delta \). We note that the functor \( (-)^{\text{op}} : \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \to \text{Mon}^{\text{lax}}(\text{Cat}_\infty) \) is an equivalence of ∞-categories, and that there is a pullback diagram
\[
\begin{array}{ccc}
\text{Mon}^{\text{oplax}}(\text{Cat}_\infty) & \xrightarrow{(-)^{\text{op}}} & \text{Mon}^{\text{lax}}(\text{Cat}_\infty) \\
\downarrow \text{ev}_1 & & \downarrow \text{ev}_1 \\
\text{Cat}_\infty & \xrightarrow{(-)^{\text{op}}} & \text{Cat}_\infty
\end{array}
\]
in the ∞-category \( \text{Cat}_\infty \) of (large) ∞-categories.

4. **Mixed fibrations**

In this section we introduce mixed fibrations and study their properties. We remark that mixed fibrations are also studied in \([10, 11]\) by name of curved orthofibration, and in \([19]\) by lax two-sided fibration. In \([4, 42]\) we study monoid objects of the ∞-category of a slice category \( \text{Cat}_\infty/T \). We consider (op)lax monoidal functors between monoid objects of \( \text{Cat}_\infty/T \). In \([4, 42]\) motivated
by the results in §4.1 we introduce mixed fibrations. Although mixed fibrations have an obvious
duality by definition, we also give another asymmetric description of mixed fibrations. We consider
mixed fibrations over marked simplicial sets and call them marked mixed fibrations. By considering
morphisms between marked mixed fibrations which preserve (co)Cartesian morphisms over marked
edges, we introduce an ∞-category of marked mixed fibrations.

4.1. Monoid objects of Cat_{∞/T} and (op)lax monoidal functors. Let T be an ∞-category. The
∞-category Cat_{∞/T} of (small) ∞-categories over T has finite products. We consider a sub-
category C of Cat_{∞/T} which is closed under finite products and equivalences in Cat_{∞/T}. In this
subsection we shall introduce an ∞-category Mon_{lax}(C) of monoid objects in C and lax monoidal
functors. By duality, we also introduce an ∞-category Mon_{oplax}(C) of monoid objects in C and
oplax monoidal functors.

First, we consider a description of the ∞-category Fun(S, Cat_{∞/T}) of functors from an ∞-
category S to Cat_{∞/T} in terms of fibrations over S × T. Let P : S → Cat_{∞/T} be a functor
of ∞-categories. The composite of P with the projection Cat_{∞/T} → Cat_{∞} gives rise to a co-
Cartesian fibration p\_S : X → S under the straightening and unstraightening equivalence [14,
Theorem 3.2.0.1]. Furthermore, we obtain a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p} & S \times T \\
\downarrow{p_\mathcal{S}} & & \downarrow{\pi_\mathcal{S}} \\
S & & \\
\end{array}
\]

where \(\pi_\mathcal{S}\) is the projection and \(p\) carries \(p_\mathcal{S}\)-coCartesian morphisms to \(\pi_\mathcal{S}\)-coCartesian morphisms.
We may assume that \(p\) is a categorical fibration by decomposing \(p\) into a composite of a categorical
equivalence and a categorical fibration.

A morphism \(F : P \to Q\) in Fun(S, Cat_{∞/T}) gives rise to a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
p & & q \\
\downarrow{p_\mathcal{S}} & & \downarrow{q_\mathcal{S}} \\
S \times T & & \\
\end{array}
\]

where \(p\) and \(q\) are categorical fibrations corresponding to \(P\) and \(Q\), respectively, and \(f\) sends
\(p_\mathcal{S}\)-coCartesian morphisms to \(q_\mathcal{S}\)-coCartesian morphisms.

Next, we consider an ∞-category Fun(S, C), where C is a subcategory of Cat_{∞/T} which is closed
under equivalences in Cat_{∞/T}. We shall give a similar description of Fun(S, C) in terms of fibrations
over S × T as above.

For an object \(P \in \text{Fun}(S, C)\), the composite of \(P\) with the inclusion functor \(C \hookrightarrow \text{Cat}_{∞/T}\)
duces a categorical fibration \(p : X \to S \times T\) such that \(p_\mathcal{S} = \pi_\mathcal{S} \circ p\) is a coCartesian fibration and
\(p\) preserves coCartesian morphisms as above. We denote by \(X_s\) the fiber of \(p_\mathcal{S} : X \to S\) at \(s \in S\).
By restriction of \(p\), we obtain a functor \(p_\mathcal{s} : X_s \to \{s\} \times T \simeq T\), which is equivalent to \(P(s)\) in \(C\)
for any \(s \in S\). Furthermore, a morphism \(e : s \to s'\) in \(S\) induces a functor \(p_\mathcal{e} : X_s \to X_{s'}\). Since \(p\)
preserves coCartesian morphisms, we obtain a commutative diagram

\[
\begin{array}{ccc}
X_s & \xrightarrow{p_\mathcal{e}} & X_{s'} \\
p_\mathcal{s} & & p_\mathcal{s'} \\
\downarrow{p_\mathcal{s}} & \downarrow{p_\mathcal{s'}} & \\
T & & \\
\end{array}
\]

which is equivalent to the morphism \(P(e) : P(s) \to P(s')\) in \(C\).
Let $F : P \rightarrow Q$ be a morphism in $\text{Fun}(S, C)$. We have categorical fibrations $p : X \rightarrow S \times T$ and $q : Y \rightarrow S \times T$ which correspond to $P$ and $Q$, respectively. Regarding $F$ as a morphism in $\text{Fun}(S, \text{Cat}_{\infty}^\text{op}/T)$ by using the inclusion functor $C \hookrightarrow \text{Cat}_{\infty}^\text{op}/T$, we obtain a functor $f : X \rightarrow Y$ over $S \times T$ such that $f$ carries $p_S$-coCartesian morphisms to $q_S$-coCartesian morphisms. By restriction of $f$, we obtain the following commutative diagram

$$
\begin{array}{ccc}
X_s & \xrightarrow{f_s} & Y_s \\
p_s & \downarrow & q_s \\
T_s & \xrightarrow{\pi_s} & S,
\end{array}
$$

which is equivalent to $F(s) : P(s) \rightarrow Q(s)$ in $C$ for any $s \in S$.

Motivated by the description of $\text{Fun}(S, C)$ in terms of fibrations over $S \times T$ as above, we introduce an $\infty$-category $\text{Fun}(S, C)$ which is equivalent to $\text{Fun}(S, C)$.

**Definition 4.1.** Let $S$ and $T$ be $\infty$-categories, and let $C$ be a subcategory of $\text{Cat}_{\infty}^\text{op}/T$ which is closed under equivalences in $\text{Cat}_{\infty}^\text{op}/T$. We define a subcategory $\text{Fun}(S, C)$ of $\text{Cat}_{\infty}/S \times T$ as follows. The objects of $\text{Fun}(S, C)$ are categorical fibrations $p : X \rightarrow S \times T$ such that $p_S = \pi_S \circ p : X \rightarrow S$ is a coCartesian fibration, $p$ carries $p_S$-coCartesian morphisms to $\pi_S$-coCartesian morphisms, the restriction $p_s : X_s \rightarrow T$ is an object of $C$ for any $s \in S$, and the induced functor $p_e : X_s \rightarrow X_{s'}$ over $T$ is a morphism in $C$ for any morphism $e : s \rightarrow s'$ in $S$. The morphisms of $\text{Fun}(S, C)$ between $p : X \rightarrow S \times T$ and $q : Y \rightarrow S \times T$ are functors $f : X \rightarrow Y$ over $S \times T$ such that $f$ carries $p_S$-coCartesian morphisms to $q_S$-coCartesian morphisms and the induced functor $f_s : X_s \rightarrow Y_s$ over $T$ is a morphism in $C$ for any $s \in S$.

**Proposition 4.2.** There is an equivalence of $\infty$-categories

$$\text{Fun}(S, C) \simeq \text{Fun}(S, C).$$

**Remark 4.3.** Let $S$ and $T$ be $\infty$-categories, and let $C$ be a subcategory of $\text{Cat}_{\infty}^\text{op}/T$ which is closed under equivalences in $\text{Cat}_{\infty}^\text{op}/T$. By duality, we define a subcategory $\text{Fun}^\text{op}(S^\text{op}, C)$ of $\text{Cat}_{\infty}^\text{op}/S \times T$ as follows. The objects of $\text{Fun}^\text{op}(S^\text{op}, C)$ are categorical fibrations $p : X \rightarrow S \times T$ such that $p_S = \pi_S \circ p : X \rightarrow S$ is a Cartesian fibration, $p$ carries $p_S$-Cartesian morphisms to $\pi_S$-Cartesian morphisms, the restriction $p_s : X_s \rightarrow T$ is an object of $C$ for any $s \in S$, and the induced functor $p_e : X_s \rightarrow X_{s'}$ over $T$ is a morphism in $C$ for any morphism $e : s' \rightarrow s$ in $S$. The morphisms of $\text{Fun}^\text{op}(S^\text{op}, C)$ between $p : X \rightarrow S \times T$ and $q : Y \rightarrow S \times T$ are functors $f : X \rightarrow Y$ over $S \times T$ such that $f$ carries $p_S$-Cartesian morphisms to $q_S$-Cartesian morphisms and the induced functor $f_s : X_s \rightarrow Y_s$ over $T$ is a morphism in $C$ for any $s \in S$. By the same argument as above, we see that there is an equivalence of $\infty$-categories

$$\text{Fun}^\text{op}(S^\text{op}, C) \simeq \text{Fun}(S^\text{op}, C).$$

Next, we shall consider monoid objects in a subcategory $C$ of $\text{Cat}_{\infty}^\text{op}/T$. We assume that $C$ is closed under finite products and equivalences. Since $C$ is closed under finite products, we can consider the $\infty$-category $\text{Mon}(C)$ of monoid objects in $C$. The $\infty$-category $\text{Mon}(C)$ is a full subcategory of $\text{Fun}(\Delta^\text{op}, C)$ and we have an equivalence between $\text{Fun}(\Delta^\text{op}, C)$ and $\text{Fun}(\Delta^\text{op}, C)$ by Proposition 4.2. This motivates us to introduce an $\infty$-category $\text{Mon}(C)$ as follows.

**Definition 4.4.** We define an $\infty$-category

$$\text{Mon}(C)$$
to be a full subcategory of $\text{Fun}(\Delta^{\text{op}}, C)$ spanned by those objects $p : X \to \Delta^{\text{op}} \times T$ of $\text{Fun}(\Delta^{\text{op}}, C)$ such that the induced morphisms

$$X_{[n]} \to X_{[1]} \times T \cdots \times T X_{[1]}$$

are equivalences in $C$ for all $[n] \in \Delta^{\text{op}}$.

**Remark 4.5.** By duality, we define an $\infty$-category $\text{Mon}'(C)$ to be a full subcategory of $\text{Fun}'(\Delta, C)$ spanned by those objects $p : X \to \Delta \times T$ of $\text{Fun}'(\Delta, C)$ such that the induced morphisms $X_{[n]} \to X_{[1]} \times T \cdots \times T X_{[1]}$ are equivalences in $C$ for all $[n] \in \Delta^{\text{op}}$.

Using Proposition 4.12 we can verify that a monoid object of $C$ corresponds to an object of $\text{Mon}(C)$ and a morphism between monoid objects in $C$ corresponds to a morphism of $\text{Mon}(C)$. Thus, we obtain the following proposition.

**Proposition 4.6.** There is an equivalence of $\infty$-categories

$$\text{Mon}(C) \simeq \text{Mon}(C).$$

**Remark 4.7.** By duality, there is an equivalence $\text{Mon}'(C) \simeq \text{Mon}(C)$ of $\infty$-categories.

**Remark 4.8.** Let $p : X \to \Delta^{\text{op}} \times T$ be an object of $\text{Mon}(C)$. The restriction $p_t : X_t \to \Delta^{\text{op}}$ is a monoidal $\infty$-category for each $t \in T$. Let $f : X \to Y$ be a morphism of $\text{Mon}(C)$. Then $f$ induces a strong monoidal functor $f_t : X_t \to Y_t$ for each $t \in T$.

We shall consider a lax monoidal functor between monoid objects of $C$.

**Definition 4.9.** Let $f : X \to Y$ be a functor over $\Delta^{\text{op}} \times T$, where $p : X \to \Delta^{\text{op}} \times T$ and $q : Y \to \Delta^{\text{op}} \times T$ are objects of $\text{Mon}(C)$. We say that $f$ is a lax monoidal functor if $f_{[n]} : X_{[n]} \to Y_{[n]}$ is a morphism of $C$ for each $[n] \in \Delta^{\text{op}}$, and if $f$ carries $p_{\Delta^{\text{op}}}$-coCartesian morphisms over inert morphisms of $\Delta^{\text{op}}$ to $q_{\Delta^{\text{op}}}$-coCartesian morphisms. We denote by

$$\text{Mon}^{\text{lax}}(C)$$

the subcategory of $\text{Cat}_{\infty/\Delta^{\text{op}} \times T}$ consisting of objects of $\text{Mon}(C)$ and lax monoidal functors between them.

**Remark 4.10.** Let $f : X \to Y$ be a functor of $\text{Mon}^{\text{lax}}(C)$. Then $f$ induces a lax monoidal functor $f_t : X_t \to Y_t$ for each $t \in T$.

By duality, we shall define an $\infty$-category $\text{Mon}^{\text{oplax}}(C)$ of monoid objects in $C$ and oplax monoidal functors.

**Definition 4.11.** Let $f : X \to Y$ be a functor over $\Delta \times T$, where $p : X \to \Delta \times T$ and $q : Y \to \Delta \times T$ are objects of $\text{Mon}(C)$. We say that $f$ is an oplax monoidal functor if $f_{[n]} : X_{[n]} \to Y_{[n]}$ is a morphism of $C$ for each $[n] \in \Delta$, and if $f$ carries $p_{\Delta}$-Cartesian morphisms over inert morphisms of $\Delta$ to $q_{\Delta}$-Cartesian morphisms. We denote by

$$\text{Mon}^{\text{oplax}}(C)$$

the subcategory of $\text{Cat}_{\infty/\Delta \times T}$ consisting of objects of $\text{Mon}'(C)$ and oplax monoidal functors between them.

**Remark 4.12.** Let $f : X \to Y$ be a functor of $\text{Mon}^{\text{oplax}}(C)$. Then $f$ induces an oplax monoidal functor $f_t : X_t \to Y_t$ for each $t \in T$. 
Mixed fibrations. In this subsection, motivated by \cite{10} we introduce mixed fibrations and study their properties. We remark that mixed fibrations are also studied in \cite{10,11} by name of curved orthofibration, and in \cite{19} by lax two-sided fibration.

Let \( p : X \to S \times T \) be a categorical fibration, where \( S \) and \( T \) are \( \infty \)-categories. We set \( p_S = \pi_S \circ p \) and \( p_T = \pi_T \circ p \), where \( \pi_S : S \times T \to S \) and \( \pi_T : S \times T \to T \) are the projections. We consider the following conditions on \( p \):

**Condition 4.13.** The map \( p_S \) is a coCartesian fibration and the functor \( p \) carries \( p_S \)-coCartesian morphisms to \( p_T \)-Cartesian morphisms.

**Condition 4.14.** The map \( p_T \) is a Cartesian fibration and the functor \( p \) carries \( p_T \)-Cartesian morphisms to \( p_S \)-coCartesian morphisms.

**Definition 4.15** (cf. \cite{10} Definition 2.3.1 and \cite{19} Definition 2.1.1). Let \( S \) and \( T \) be \( \infty \)-categories. We say that a categorical fibration \( p : X \to S \times T \) is a mixed fibration over \((S,T)\) if it satisfies Conditions 4.13 and 4.14. We define an \( \infty \)-category

\[ \text{Mfib}/(S,T) \]

to be the full subcategory of \( \text{Cat}_{\infty}/S \times T \) spanned by mixed fibrations over \((S,T)\), and call it the \( \infty \)-category of mixed fibrations over \((S,T)\).

**Remark 4.16.** The functor \((-)^{op}\) induces an equivalence of \( \infty \)-categories between \( \text{Cat}_{\infty}/S \times T \) and \( \text{Cat}_{\infty}/T^{op} \times S^{op} \). By symmetry of the definition of mixed fibrations, \((-)^{op}\) induces an equivalence of \( \infty \)-categories

\[ \text{Mfib}/(S,T) \simeq \text{Mfib}/(T^{op},S^{op}). \]

We consider characterizations of mixed fibrations. We denote by \( X_s \) the fiber of the map \( p_S : X \to S \) at \( s \in S \), and by \( X_t \) the fiber of the map \( p_T : X \to T \) at \( t \in T \). By restriction, there are maps \( p_s : X_s \to \{s\} \times T \simeq T \) and \( p_t : X_t \to S \times \{t\} \simeq S \).

**Lemma 4.17.** Let \( p : X \to S \times T \) be a categorical fibration and let \( e : x \to x' \) be a morphism in \( X_t \) for \( t \in T \). If \( e \) is a \( p_S \)-coCartesian morphism, then it is a \( p_t \)-coCartesian morphism. Conversely, if \( e \) is a \( p_t \)-coCartesian morphism and if \( p : X \to S \times T \) satisfies Condition 4.13, then \( e \) is a \( p_S \)-coCartesian morphism.

**Proof.** If \( e \) is \( p_S \)-coCartesian, then \( e \) is \( p \)-coCartesian by \cite{10} Proposition 2.2.1]. Since \( p_t : X_t \to S \) is a pullback of \( p : X \to S \times T \) along \( S \simeq S \times \{t\} \to S \times T \), we see that \( e \) is \( p_t \)-coCartesian.

Now, assume that \( e \) is \( p_t \)-coCartesian and that \( p \) satisfies Condition 4.13. By \cite{10} Corollary 2.2.2, there is a \( p_S \)-coCartesian morphism \( e' : x \to x'' \) in \( X_t \) over \( p_S(e) \). By the first part of the proof, \( e' \) is \( p_t \)-coCartesian. This implies that \( e \) and \( e' \) are equivalent in \( X_t \), and hence \( e \) is \( p_S \)-coCartesian.

**Remark 4.18.** Let \( p : X \to S \times T \) be a categorical fibration and let \( e : x \to x' \) be a morphism in \( X_s \) for \( s \in S \). By duality, if \( e \) is a \( p_T \)-Cartesian morphism, then \( e \) is a \( p_s \)-Cartesian morphism. Conversely, if \( e \) is a \( p_s \)-Cartesian morphism and if \( p \) satisfies Condition 4.14, then \( e \) is a \( p_T \)-Cartesian morphism.

We consider the following conditions:

**Condition 4.19.** The maps \( p_t : X_t \to S \) are coCartesian fibrations for all \( t \in T \).

**Condition 4.20.** The maps \( p_s : X_s \to T \) are Cartesian fibrations for all \( s \in S \).

We have the following characterizations of mixed fibrations by \cite{10} Proposition 2.3.3).

**Proposition 4.21** (\cite{10} Proposition 2.3.3). The following conditions are equivalent for \( p : X \to S \times T \):
The map $p$ is a mixed fibration.

The map $p$ satisfies Conditions 4.13 and 4.20.

The map $p$ satisfies Conditions 4.14 and 4.19.

Now, we study properties of maps between mixed fibrations. For this purpose, we introduce a notion of marked mixed fibrations.

First, we recall the definition of marked simplicial sets (cf. [14, Definition 3.1.0.1]). A marked simplicial set is a pair $(S, E)$, where $S$ is a simplicial set and $E$ is a set of edges of $S$ which contains all degenerate edges. When $E$ is the set $S_1$ of all edges of $S$, we write $S^e = (S, S_1)$. When $E$ is the set $s_0(S_0)$ of all degenerate edges of $S$, we write $S^\flat = (S, s_0(S_0))$.

Let $(S, E)$ be a marked simplicial set, where $S$ is an $\infty$-category. We define an $\infty$-category $\text{coCart}/(S, E)$ to be a subcategory of $\text{Cat}_\infty/S$ as follows. The objects of $\text{coCart}/(S, E)$ are coCartesian fibrations over $S$. The morphisms between coCartesian fibrations $p : X \to S$ and $q : Y \to S$ are functors $f : X \to Y$ over $S$ such that $f$ preserves coCartesian morphisms over $E$.

Remark 4.22. Let $\mathcal{P} = (E, T, \emptyset)$ be a categorical pattern on $S$ in the sense of [15, Definition B.0.19], where $T$ is the set of all 2-simplices of $S$. By [15, Theorem B.0.20], there exists a left proper combinatorial simplicial model structure on $(\text{Set}_\Delta^+)_/\mathcal{P}$. We can regard the $\infty$-category $\text{coCart}/(S, E)$ as a full subcategory of the underlying $\infty$-category of $(\text{Set}_\Delta^+)_/\mathcal{P}$.

Similarly, we define an $\infty$-category $\text{Cart}/(S, E)$ to be a subcategory of $\text{Cart}_\infty/S$ consisting of Cartesian fibrations over $S$ and functors over $S$ which preserve Cartesian morphisms over $E$.

Let $S$ and $T$ be $\infty$-categories, and let $p : X \to S \times T$ be a mixed fibration over $(S, T)$. For any morphism $e : s \to s'$ of $S$, we have a commutative diagram

$$
\begin{array}{ccc}
X_s & \xrightarrow{e_1} & X_{s'} \\
\downarrow{p_s} & & \downarrow{p_{s'}} \\
T & & \\
\end{array}
$$

Similarly, for any morphism $f : t' \to t$ of $T$, we have a commutative diagram

$$
\begin{array}{ccc}
X_t & \xrightarrow{f'} & X_{t'} \\
\downarrow{p_t} & & \downarrow{p_{t'}} \\
S & & \\
\end{array}
$$

Suppose that we have sets $E$ and $F$ of morphisms of $S$ and $T$, respectively, which contain all degenerate ones. We consider the following conditions on $p$:

**Condition 4.23.** For any $e : s \to s'$ in $E$, the functor $e_1$ carries $p_s$-Cartesian morphisms over $F$ to $p_{s'}$-Cartesian morphisms.

**Condition 4.24.** For any $f : t' \to t$ in $F$, the functor $f'$ carries $p_t$-coCartesian morphisms over $E$ to $p_{t'}$-coCartesian morphisms.

**Proposition 4.25.** Let $p : X \to S \times T$ be a mixed fibration over $(S, T)$, and let $E$ and $F$ be subsets of morphisms of $S$ and $T$, respectively, which contain all degenerate ones. The mixed fibration $p$ satisfies Condition 4.23 if and only if $p$ satisfies Condition 4.24.
We define a marked mixed fibration over \( ((S, E), (T, F)) \) to be a mixed fibration \( p : X \to S \times T \) over \( (S, T) \) which satisfies the equivalent Conditions 4.13 and 4.14. Let \( p : X \to S \times T \) and \( q : Y \to S \times T \) be marked mixed fibrations over \( ((S, E), (T, F)) \). A morphism \( f \) of marked mixed fibrations over \( ((S, E), (T, F)) \) between \( p \) and \( q \) is a morphism of marked fibrations over \( (S, T) \) such that \( f \) carries \( p_s \)\{-\}coCartesian morphisms over \( E \) to \( q_s \)\{-\}coCartesian morphisms and that \( p_t \)\{-\}Cartesian morphisms over \( F \) to \( q_t \)\{-\}Cartesian morphisms. We define an \( \infty \)-category

\[
\text{Mfib}((S, E), (T, F))
\]

to be the \( \infty \)-category of marked mixed fibrations over \( ((S, E), (T, F)) \) and morphisms between them.

**Remark 4.27.** The notion of marked mixed fibration over \( (S^t, T^t) \) is equivalent to that of orthofibration over \( T \times S \) in [10] and of two-sided fibration over \( S \times T \) in [19].

Next, we consider a description of the \( \infty \)-category \( \text{Fun}(S, \text{Cart}/(T, F)) \) in terms of marked mixed fibrations.

**Proposition 4.28.** There is an equivalence of \( \infty \)-categories

\[
\text{Fun}(S, \text{Cart}/(T, F)) \simeq \text{Mfib}/((S^t, (T, F)).
\]

**Proof.** By Proposition 4.2, \( \text{Fun}(S, \text{Cart}/(T, F)) \) is equivalent to \( \text{Fun}(S, \text{Cart}/(T, F)) \). By Definition 4.1, an object of \( \text{Fun}(S, \text{Cart}/(T, F)) \) is a categorical fibration \( p : X \to S \times T \) such that \( p \) satisfies Conditions 4.13 \( p_s : X_s \to T \) is a Cartesian fibration for any \( s \in S \), and \( p_e : X_s \to X_{s'} \) over \( T \) carries \( p_s \)\{-\}Cartesian morphisms over \( F \) to \( p_{s'} \)\{-\}Cartesian morphisms for any morphism \( e : s \to s' \) in \( S \). By Proposition 4.21, we see that \( p \) is a mixed fibration over \( (S, T) \). Thus, the objects of \( \text{Fun}(S, \text{Cart}/(T, F)) \) are the marked mixed fibrations \( p : X \to S \times T \) over \( (S^t, (T, F)) \).

Let \( p : X \to S \times T \) and \( q : Y \to S \times T \) be objects of \( \text{Fun}(S, \text{Cart}/(T, F)) \). A morphism between \( p \) and \( q \) in \( \text{Fun}(S, \text{Cart}/(T, F)) \) is a functor \( f : X \to Y \) over \( S \times T \) such that \( f \) carries \( p_s \)\{-\}coCartesian morphisms over \( q_s \)\{-\}coCartesian morphisms and the functor \( f_s : X_s \to Y_s \) over \( T \) carries \( p_s \)\{-\}Cartesian morphisms over \( F \) to \( q_s \)\{-\}Cartesian morphisms for any \( s \in S \). By Remark 4.18, we see that \( f \) carries \( p_t \)\{-\}Cartesian morphisms over \( F \) to \( q_t \)\{-\}Cartesian morphisms. Thus, the morphisms between \( p \) and \( q \) in \( \text{Fun}(S, \text{Cart}/(T, F)) \) are the morphisms of mixed fibrations over \( (S^t, (T, F)) \).

**Remark 4.29.** By duality, there is an equivalence of \( \infty \)-categories

\[
\text{Fun}(T^{op}, \text{coCart}/(S, E)) \simeq \text{Mfib}((S^t, E), T^t).
\]

5. Duoidal \( \infty \)-Categories

In [22] we recalled the notion of duoidal category in the classical setting. In this section we introduce a duoidal category in the setting of \( \infty \)-categories and call it a duoidal \( \infty \)-category. In [22] we
also recalled that there are three kinds of functors between duoidal categories. We introduce corresponding functors between duoidal ∞-categories. According to which kind of functors of duoidal ∞-categories we take, we will give three formulations of ∞-category of duoidal ∞-categories.

5.1. Duoidal ∞-categories. In this subsection we introduce duoidal ∞-categories which are analogues of duoidal categories in the setting of ∞-categories.

We recall that $\text{Mon}_{\text{oplax}}(\text{Cat}_\infty)$ is the ∞-category of monoidal ∞-categories and oplax monoidal functors. The objects of $\text{Mon}_{\text{oplax}}(\text{Cat}_\infty)$ are Cartesian fibrations $p : X \to \Delta$ such that $p^{\op} : X^{\op} \to \Delta^{\op}$ is a monoidal ∞-category. The morphisms between objects $X \to \Delta$ and $Y \to \Delta$ are functors $f : X \to Y$ over $\Delta$ such that $f$ preserves Cartesian morphisms over inert morphisms of $\Delta$. Let $\Delta_{\text{int}}$ be the set of all inert morphisms of $\Delta$. We set $\Delta^2 = (\Delta, \Delta_{\text{int}})$. Then $\text{Mon}_{\text{oplax}}(\text{Cat}_\infty)$ is a full subcategory of $\text{Cart}/\Delta^2$ spanned by monoidal ∞-categories.

We can verify that $\text{Mon}_{\text{oplax}}(\text{Cat}_\infty)$ has finite products as follows. For objects $p : X \to \Delta$ and $q : Y \to \Delta$ of $\text{Mon}_{\text{oplax}}(\text{Cat}_\infty)$, the fiber product $p \times_{\Delta} q : X \times_{\Delta} Y \to \Delta$ is a product of $p$ and $q$, and the identity functor $\Delta \to \Delta$ is a final object in $\text{Mon}_{\text{oplax}}(\text{Cat}_\infty)$. Thus, we can consider monoid objects in $\text{Mon}_{\text{oplax}}(\text{Cat}_\infty)$.

**Definition 5.1.** A duoidal ∞-category is a monoid object in $\text{Mon}_{\text{oplax}}(\text{Cat}_\infty)$.

**Example 5.2.** Braided monoidal categories can be regarded as duoidal categories. We consider a generalization of this fact in ∞-categories. Let $E_{\boxtimes}$ be the little 2-cubes operad. We will show that an $E_2$-monoidal ∞-category determines a duoidal ∞-category. Suppose that $m : E_{\boxtimes}^2 \to \text{Cat}_\infty$ is an $E_2$-monoid object of $\text{Cat}_\infty$. Using the functor $\text{Cut} : \Delta^{\op} \to \text{Assoc}$ of [15, Construction 4.1.2.9], we obtain a functor $d : \Delta^{\op} \times \Delta^{\op} \to \text{Assoc}_{\boxtimes} \times \text{Assoc}_{\boxtimes} \simeq E_{\boxtimes}^{\otimes} \times E_{\boxtimes}^{\otimes} \to E_{\boxtimes}^{\otimes}$. By composing $m$ with $d$, we obtain a functor $d \circ m : \Delta^{\op} \times \Delta^{\op} \to \text{Cat}_\infty$, which determines a functor $\Delta^{\op} \to \text{Mon}(\text{Cat}_\infty) \subset \text{Fun}(\Delta^{\op}, \text{Cat}_\infty)$. By regarding it as a functor $\Delta^{\op} \to \text{Mon}(\text{Cat}_\infty) \simeq \text{Mon}(\text{Cat}_\infty) \subset \text{Mon}_{\text{oplax}}(\text{Cat}_\infty)$, we obtain a duoidal ∞-category.

We give another description of duoidal ∞-categories in terms of mixed fibrations. Let $S$ be an ∞-category. First, we consider the ∞-category $\text{Fun}(S, \text{Mon}_{\text{oplax}}(\text{Cat}_\infty))$. By Proposition 4.22 we have an equivalence $\text{Fun}(S, \text{Mon}_{\text{oplax}}(\text{Cat}_\infty)) \simeq \text{Fun}(S, \text{Mon}_{\text{oplax}}(\text{Cat}_\infty))$. The inclusion functor $\text{Mon}_{\text{oplax}}(\text{Cat}_\infty) \hookrightarrow \text{Cart}/\Delta^2$ induces a fully faithful functor $\text{Fun}(S, \text{Mon}_{\text{oplax}}(\text{Cat}_\infty)) \to \text{Fun}(S, \text{Cart}/\Delta^2)$, and we have an equivalence $\text{Fun}(S, \text{Cart}/\Delta^2) \simeq \text{Mfib}/(S^2, \Delta^2)$ by Proposition 4.28. Hence there is a fully faithful functor

$$\text{Fun}(S, \text{Mon}_{\text{oplax}}(\text{Cat}_\infty)) \to \text{Mfib}/(S^2, \Delta^2).$$

We determine the essential image of this functor. Let $p : X \to S \times \Delta$ be a mixed fibration over $(S, \Delta)$. Any morphism $\phi : [m'] \to [m]$ of $\Delta$ induces a functor $\phi^*: X_{[m']} \to X_{[m]}$ over $S$. In particular, we obtain a morphism

$$X_{[m]} \to X_{[1]} \times_S \cdots \times_S X_{[1]}$$

in $\text{coCart}/S^2$ for any $[m] \in \Delta$. We also call it a Segal morphism. We consider the following condition on $p$:

**Condition 5.3.** The Segal morphisms $X_{[m]} \to X_{[1]} \times_S \cdots \times_S X_{[1]}$ are equivalences in the ∞-category $\text{coCart}/S^2$ for all $[m] \in \Delta$.

**Lemma 5.4.** Let $p : X \to S \times \Delta$ be a mixed fibration over $(S, \Delta)$. If $p$ satisfies Condition 5.3, then $p$ is a marked mixed fibration over $(S^2, \Delta^2)$. 
Proof. If \( p \) satisfies Condition 5.3, then the functor \((\rho^i)^\dagger : X_{[m]} \to X_{[1]}\) over \( S \) preserves coCartesian morphisms for any \( [m] \in \Delta \) and \( 1 \leq i \leq m \). For an inert morphism \( \phi : [m'] \to [m] \) of \( \Delta \), there is a commutative diagram

\[
\begin{array}{c}
X_{[m]} \xrightarrow{\phi'} X_{[1]} \times_S \cdots \times_S X_{[1]} \\
\downarrow \quad \quad \quad \downarrow \\
X_{[m']} \xrightarrow{\phi^\dagger} X_{[1]} \times_S \cdots \times_S X_{[1]},
\end{array}
\]

where the right vertical arrow is a projection. This implies that \( \phi' : X_{[m]} \to X_{[m']} \) preserves coCartesian morphisms. Hence \( p \) is a marked mixed fibration over \((S^\dagger, \Delta^\dagger)\).

\[\square\]

**Proposition 5.5.** A mixed fibration \( p : X \to S \times \Delta \) over \((S, \Delta)\) satisfies Condition 5.3 if and only if \( p \) is an object of \( \text{Fun}(S, \text{Mon}^{\text{op lax}}(\text{Cat}_{\infty})) \).

**Proof.** First, we suppose that \( p \) satisfies Condition 5.3. Then the Segal morphism \( X_{(s,[m])} \to X_{[s,1]} \times \cdots \times X_{[s,1]} \) is an equivalence of \( \infty \)-categories for any \( (s,[m]) \in S \times \Delta \). Furthermore, since \( p : X \to S \times \Delta \) is a marked mixed fibration over \((S^\dagger, \Delta^\dagger)\) by Lemma 5.4, the functor \( e : X_s \to X_{s'} \) over \( \Delta \) induced by any morphism \( e : s \to s' \) of \( S \) carries \( p_{s'} \)-Cartesian morphisms over inert morphisms of \( \Delta \) to \( p_{s'} \)-Cartesian morphisms. Thus, \( p \) is an object of \( \text{Fun}(S, \text{Mon}^{\text{op lax}}(\text{Cat}_{\infty})) \).

Conversely, we suppose that \( p \) is an object of \( \text{Fun}(S, \text{Mon}^{\text{op lax}}(\text{Cat}_{\infty})) \). Since \( p : X \to S \times \Delta \) is a marked mixed fibration over \((S^\dagger, \Delta^\dagger)\), the functor \( \phi : X_{[m]} \to X_{[m']} \) over \( S \) induced by any inert morphism \( \phi : [m'] \to [m] \) of \( \Delta \) preserves coCartesian morphisms. Hence the Segal morphism \( X_{[m]} \to X_{[1]} \times_S \cdots \times_S X_{[1]} \) preserves coCartesian morphisms over \( S \) for any \( [m] \in \Delta \). Furthermore, the maps \( X_{(s,[m])} \to X_{[s,1]} \times \cdots \times X_{[s,1]} \) on fibers are equivalences by the assumption for all \( s \in S \).

By the dual of \([11] \text{ Proposition 3.3.1.5}]\), we see that \( p \) satisfies Condition 5.3.

\[\square\]

**Corollary 5.6.** The \( \infty \)-category \( \text{Fun}(S, \text{Mon}^{\text{op lax}}(\text{Cat}_{\infty})) \) is equivalent to a full subcategory of \( \text{Mfib}/(S, \Delta^\dagger) \) spanned by marked mixed fibrations over \((S^\dagger, \Delta^\dagger) \) which satisfy Condition 5.3.

Next, we consider a description of the \( \infty \)-category \( \text{Mon}(\text{Mon}^{\text{op lax}}(\text{Cat}_{\infty})) \) in terms of mixed fibrations. By definition, \( \text{Mon}(\text{Mon}^{\text{op lax}}(\text{Cat}_{\infty})) \) is a full subcategory of \( \text{Fun}(\Delta^\text{op}, \text{Mon}^{\text{op lax}}(\text{Cat}_{\infty})) \), and hence there is a fully faithful functor

\[\text{Mon}(\text{Mon}^{\text{op lax}}(\text{Cat}_{\infty})) \to \text{Mfib}/((\Delta^\text{op})^\dagger, \Delta^\dagger)\]

by Corollary 5.6. We shall determine the essential image of this functor.

Let \( p : X \to \Delta^\text{op} \times \Delta \) be a mixed fibration over \((\Delta^\text{op} \times \Delta \), where \( \Delta^\text{op} = \Delta \). We denote by \( X_{[m],[n]}^\dagger \) the fiber of \( p \) at \(([m],[n]) \in \Delta^\text{op} \times \Delta \). Recall that we have coCartesian fibrations \( p_{[n]}^\dagger : X_{[n]}^\dagger \to \Delta^\text{op} \) for each \([n] \in \Delta \), and Cartesian fibrations \( p_{[m],[n]} : X_{[m],[n]}^\dagger \to \Delta \) for each \([m] \in \Delta^\text{op} \). We consider the following conditions on \( p \):

**Condition 5.7.** The Segal morphisms

\[
X_{[m]}^\dagger \to \underbrace{X_{[1]}^\dagger \times_{\Delta \otimes \Delta} X_{[1]}^\dagger \times_{\Delta \otimes \Delta} \cdots \times_{\Delta \otimes \Delta} X_{[1]}^\dagger}_{m\text{-fold}}
\]

are equivalences in the \( \infty \)-category \( \text{coCart}/(\Delta^\text{op})^\dagger \) for all \([n] \in \Delta \).

**Condition 5.8.** The Segal morphisms

\[
X_{[m]} \to \underbrace{X_{[1]} \times_{\Delta \otimes \Delta} X_{[1]} \times_{\Delta \otimes \Delta} \cdots \times_{\Delta \otimes \Delta} X_{[1]}}_{m\text{-fold}}
\]

are equivalences in the \( \infty \)-category \( \text{Cart}/(\Delta \otimes \Delta)^\dagger \) for all \([m] \in \Delta^\text{op} \).
Proposition 5.9. Let \( p : X \to \Delta^\text{op}_\otimes \times \Delta \) be a mixed fibration over \((\Delta^\text{op}_\otimes, \Delta)\). The map \( p \) is in the essential image of the functor \( \text{Mon}(\text{Mon}^{\text{oplax}}(\text{Cat}_\infty)) \to \text{Mfib}/((\Delta^\text{op}_2)^\otimes, (\Delta_\otimes)^\otimes) \) if and only if \( p \) satisfies Conditions 5.7 and 5.8.

Proof. First, we suppose that \( p \) is in the essential image of the functor \( \text{Fun}(\Delta^\text{op}_\otimes, \text{Mon}^{\text{oplax}}(\text{Cat}_\infty)) \to \text{Mfib}/((\Delta^\text{op}_2)^\otimes, (\Delta_\otimes)^\otimes) \). By Corollary 5.6, \( p \) satisfies Condition 5.7. By the assumption, the Segal morphisms \( X_{[m], \bullet} \to X_{[1], \bullet} \times_{\Delta_\otimes} \cdots \times_{\Delta_\otimes} X_{[1], \bullet} \) are equivalences in \( \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \) for all \( [m] \in \Delta^\text{op}_2 \). This implies that Condition 5.8 holds.

Conversely, we suppose that \( p \) satisfies Conditions 5.7 and 5.8. By Corollary 5.6, \( p \) is in the essential image of the functor \( \text{Fun}(\Delta^\text{op}_\otimes, \text{Mon}^{\text{oplax}}(\text{Cat}_\infty)) \to \text{Mfib}/((\Delta^\text{op}_2)^\otimes, (\Delta_\otimes)^\otimes) \). Since \( \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \) is a full subcategory of \( \text{Cart}/(\Delta_\otimes)^\otimes \), Condition 5.8 implies that the Segal morphisms \( X_{[m], \bullet} \to X_{[1], \bullet} \times_{\Delta_\otimes} \cdots \times_{\Delta_\otimes} X_{[1], \bullet} \) are equivalences in \( \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \) for all \( [m] \in \Delta^\text{op}_2 \). Hence \( p \) is in the essential image of the functor \( \text{Mon}(\Delta^\text{op}_2, \text{Mon}^{\text{oplax}}(\text{Cat}_\infty)) \to \text{Mfib}/((\Delta^\text{op}_2)^\otimes, (\Delta_\otimes)^\otimes) \). \( \square \)

Definition 5.10. We also say that a mixed fibration
\[
p : X \to \Delta^\text{op}_2 \times \Delta_\otimes
\]
over \((\Delta^\text{op}_2, \Delta_\otimes)\) is a duoidal \( \infty \)-category if it satisfies Conditions 5.7 and 5.8.

Example 5.11. If \( R \) is a commutative ring, then the category of \( R-R \)-bimodules has the structure of a duoidal category. We consider a generalization of this fact in \( \infty \)-categories. Let \( \mathcal{C} \) be a presentable \( E_2 \)-monoidal \( \infty \)-category, and let \( A \) be an \( \mathbb{E}_2 \)-algebra object of \( \mathcal{C} \). There is a duoidal \( \infty \)-category \( p : X \to \Delta^\text{op}_2 \times \Delta_\otimes \) in which \( X_{[1],[1]} \) is equivalent to the \( \infty \)-category \( A\text{BMod}_A(\mathcal{C}) \) of \( A-A \)-bimodules in \( \mathcal{C} \). For \( M, N \in A\text{BMod}_A(\mathcal{C}) \), we have \( M \otimes N \simeq M \otimes_A N \) and \( M \boxtimes N \simeq A \otimes_{A\otimes A} (M \otimes N) \otimes_{A\otimes A} A \). See [23] for more details and a generalization to operadic modules.

We shall show that the duoidal \( \infty \)-categories are closed under the functor \((-)^{\text{op}}\).

Proposition 5.12. Let \( p : X \to \Delta^\text{op}_2 \times \Delta_\otimes \) be a mixed fibration over \((\Delta^\text{op}_2, \Delta_\otimes)\). Then \( p \) is a duoidal \( \infty \)-category if and only if \( p^{\text{op}} : X^{\text{op}} \to \Delta^\text{op}_2 \times \Delta_\otimes \) is a duoidal \( \infty \)-category.

Proof. By Remark 1.10, if \( p \) is a mixed fibration over \((\Delta^\text{op}_2, \Delta_\otimes)\), then \( p^{\text{op}} \) is a mixed fibration over \((\Delta^\text{op}_2, \Delta_\otimes)\). By the symmetry of Conditions 5.7 and 5.8, we see that \( p \) is a duoidal \( \infty \)-category if and only if \( p^{\text{op}} \) is a duoidal \( \infty \)-category. \( \square \)

5.2. Bilax monoidal functors. Let \( p : X \to \Delta^\text{op}_2 \times \Delta_\otimes \) and \( q : Y \to \Delta^\text{op}_2 \times \Delta_\otimes \) be duoidal \( \infty \)-categories. Suppose we have a functor \( f : X \to Y \) over \( \Delta^\text{op}_2 \times \Delta_\otimes \) which is a morphism of \( \text{Mon}(\text{Mon}^{\text{oplax}}(\text{Cat}_\infty)) \). Then we obtain strong monoidal functors \( f_{\bullet,[n]} : X_{\bullet,[n]} \to Y_{\bullet,[n]} \) for each \([n] \in \Delta_\otimes\), and oplax monoidal functors \( f_{[m],\bullet} : X_{[m],\bullet} \to Y_{[m],\bullet} \) for each \([m] \in \Delta^\text{op}_2\). This is asymmetric under \((-)^{\text{op}}\). We shall introduce a bilax monoidal functor between duoidal \( \infty \)-categories, which is symmetric under \((-)^{\text{op}}\).

Definition 5.13. Let \( p : X \to \Delta^\text{op}_2 \times \Delta_\otimes \) and \( q : Y \to \Delta^\text{op}_2 \times \Delta_\otimes \) be duoidal \( \infty \)-categories, and let \( f : X \to Y \) be a functor over \( \Delta^\text{op}_2 \times \Delta_\otimes \). We say that \( f \) is a bilax monoidal functor between \( p \) and \( q \) if \( f \) is a morphism of \( \text{Mfib}/((\Delta^\text{op}_2)^\otimes, (\Delta_\otimes)^\otimes) \).

By definition, a bilax monoidal functor \( f : X \to Y \) between duoidal \( \infty \)-categories \( p : X \to \Delta^\text{op}_2 \times \Delta_\otimes \) and \( q : Y \to \Delta^\text{op}_2 \times \Delta_\otimes \) induces lax monoidal functors
\[
f_{\bullet,[n]} : X_{\bullet,[n]} \to Y_{\bullet,[n]}
\]
for each \([n] \in \Delta_\otimes\), and oplax monoidal functors
\[
f_{[m],\bullet} : X_{[m],\bullet} \to Y_{[m],\bullet}
\]
for each \([m] \in \Delta_{\infty}^\op\). By the symmetry of the definition of bilax monoidal functors, we easily obtain the following proposition.

**Proposition 5.14.** If \(f\) is a bilax monoidal functor between duoidal \(\infty\)-categories, then \(f^{\op}\) is also a bilax monoidal functor.

**Definition 5.15.** We define

\[
\text{Duo}_{\infty}^{\text{bilax}}
\]

to be the full subcategory of \(\text{Mfib}/((\Delta_{\infty}^\op)^I, (\Delta_\infty)^I)\) spanned by duoidal \(\infty\)-categories. We call \(\text{Duo}_{\infty}^{\text{bilax}}\) the \(\infty\)-category of duoidal \(\infty\)-categories and bilax monoidal functors.

**Remark 5.16.** The functor \((-)^{\op} : \text{Cat}_{\infty} \to \text{Cat}_{\infty}\) induces an equivalence

\[
(-)^{\op} : \text{Duo}_{\infty}^{\text{bilax}} \xrightarrow{\sim} \text{Duo}_{\infty}^{\text{bilax}}
\]
of \(\infty\)-categories.

For a duoidal \(\infty\)-category \(p : X \to \Delta_{\infty}^\op \times \Delta_\infty\), we call the fiber \(X_{[1],[1]}\) the underlying \(\infty\)-category. Assigning to a duoidal \(\infty\)-category its underlying \(\infty\)-category, we obtain a functor

\[
ev_{[1],[1]} : \text{Duo}_{\infty}^{\text{bilax}} \to \text{Cat}_{\infty}
\]
of \(\infty\)-categories.

We shall show that \(\text{Duo}_{\infty}^{\text{bilax}}\) is equivalent to the \(\infty\)-category \(\text{Mon}_{\infty}^{\text{oplax}}(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty}))\) of monoid objects and lax monoidal functors in \(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty})\).

**Theorem 5.17.** There is an equivalence

\[
\text{Duo}_{\infty}^{\text{bilax}} \simeq \text{Mon}_{\infty}^{\text{oplax}}(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty}))
\]
of \(\infty\)-categories.

**Proof.** The objects of \(\text{Duo}_{\infty}^{\text{bilax}}\) coincide with those of \(\text{Mon}_{\infty}^{\text{oplax}}(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty}))\). We shall show that the morphisms in \(\text{Duo}_{\infty}^{\text{bilax}}\) coincide with those in \(\text{Mon}_{\infty}^{\text{oplax}}(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty}))\).

Let \(p : X \to \Delta_{\infty}^\op \times \Delta_\infty\) and \(q : Y \to \Delta_{\infty}^\op \times \Delta_\infty\) be duoidal \(\infty\)-categories. The morphisms between \(p\) and \(q\) in \(\text{Duo}_{\infty}^{\text{bilax}}\) are functors \(f : X \to Y\) over \(\Delta_{\infty}^\op \times \Delta_\infty\) such that \(f\) carries \(p_{\Delta_{\infty}^\op}\)-coCartesian morphisms over inert morphisms of \(\Delta_{\infty}^\op\) to \(q_{\Delta_{\infty}^\op}\)-coCartesian morphisms, and that \(q_{\Delta_\infty}\)-Cartesian morphisms over inert morphisms of \(\Delta_\infty\) to \(q_{\Delta_\infty}\)-Cartesian morphisms. On the other hand, the morphisms between \(p\) and \(q\) in \(\text{Mon}_{\infty}^{\text{oplax}}(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty}))\) are functors \(f : X \to Y\) over \(\Delta_{\infty}^\op \times \Delta_\infty\) such that \(f\) carries \(p_{\Delta_{\infty}^\op}\)-coCartesian morphisms over inert morphisms of \(\Delta_{\infty}^\op\) to \(q_{\Delta_{\infty}^\op}\)-coCartesian morphisms, and that the morphism \(f_{[m]}, \bullet : X_{[m], \bullet} \to Y_{[m], \bullet}\) over \(\Delta_\infty\) is a morphism of \(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty})\) for each \([m] \in \Delta_{\infty}^\op\) by Definition 1.19. Recall that \(f_{[m], \bullet}\) is a morphism of \(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty})\) if and only if \(f_{[m], \bullet}\) sends \(p_{[m], \bullet}\)-Cartesian morphisms over inert morphisms of \(\Delta_\infty\) to \(q_{[m], \bullet}\)-Cartesian morphisms. By Remark 4.18, the morphisms between \(p\) and \(q\) in \(\text{Mon}_{\infty}^{\text{oplax}}(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty}))\) coincide with those in \(\text{Duo}_{\infty}^{\text{bilax}}\). This completes the proof.

**Remark 5.18.** By the symmetry of the definition of duoidal \(\infty\)-categories and bilax monoidal functors, there is an equivalence

\[
\text{Duo}_{\infty}^{\text{bilax}} \simeq \text{Mon}_{\infty}^{\text{oplax}}(\text{Mon}_{\infty}^{\text{oplax}}(\text{Cat}_{\infty}))
\]
of \(\infty\)-categories.
5.3. **Double lax and oplax monoidal functors.** In §5.2 we introduced bilax monoidal functors between duoidal ∞-categories. We have two other kinds of functors between duoidal ∞-categories. We will call these functors double lax monoidal functors and double oplax monoidal functors. In order to formulate these functors, we shall give other descriptions of duoidal ∞-categories. Using one of these descriptions, we will define an ∞-category of duoidal ∞-categories and double lax monoidal functors. By duality, we will also define an ∞-category of duoidal ∞-categories and double oplax monoidal functors.

By Definition 5.1 a duoidal ∞-category is a monoid object of $\text{Mon}^{\text{oplax}}(\text{Cat}_\infty)$. Since there is an equivalence $(\_)^{\text{op}} : \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \to \text{Mon}^{\text{lax}}(\text{Cat}_\infty)$ of ∞-categories, we can regard a duoidal ∞-category as a monoid object of $\text{Mon}^{\text{lax}}(\text{Cat}_\infty)$. Furthermore, by Proposition 4.6 we can identify a duoidal ∞-category with an object of $\text{Mon}(\text{Mon}^{\text{lax}}(\text{Cat}_\infty))$, which is a categorical fibration $p : X \to \Delta^{\text{op}}_\infty \times \Delta^{\text{op}}_\infty$ satisfying the following conditions:

**Condition 5.19.** Let $p : X \to \Delta^{\text{op}}_\infty \times \Delta^{\text{op}}_\infty$ be a categorical fibration.

1. The composite $\pi_\circ \circ p : X \to \Delta^{\text{op}}_\infty$ is a coCartesian fibration, and $p$ carries $\pi_\circ \circ p$-coCartesian morphisms to $\pi_\circ$-coCartesian morphisms.
2. For each $[n] \in \Delta^{\text{op}}_\infty$, the restriction $p[n] : X_{[n]} \to \Delta^{\text{op}}_\infty$ is a monoidal ∞-category,
3. For each morphism $[n] \to [n']$ in $\Delta^{\text{op}}_\infty$, the induced map $X_{[n]} \to X_{[n']}$, over $\Delta^{\text{op}}_\infty$ is a lax monoidal functor.
4. For each $[n] \in \Delta^{\text{op}}_\infty$, the Segal morphism

$$X_{[n]} \to X_{[1]} \times_{\Delta^{\text{op}}_\infty} \cdots \times_{\Delta^{\text{op}}_\infty} X_{[1]}$$

is an equivalence of coCartesian fibrations over $\Delta^{\text{op}}_\infty$.

**Definition 5.20.** We also say that a categorical fibration

$$p : X \to \Delta^{\text{op}}_\infty \times \Delta^{\text{op}}_\infty$$

is a duoidal ∞-category if it satisfies Condition 5.19.

**Definition 5.21.** Let $p : X \to \Delta^{\text{op}}_\infty \times \Delta^{\text{op}}_\infty$ and $q : Y \to \Delta^{\text{op}}_\infty \times \Delta^{\text{op}}_\infty$ be duoidal ∞-categories. Let $f : X \to Y$ be a functor over $\Delta^{\text{op}}_\infty \times \Delta^{\text{op}}_\infty$. We say that $f$ is a double lax monoidal functor if it carries $(\pi_\circ \circ p)$-coCartesian morphisms over inert morphisms of $\Delta^{\text{op}}_\infty$ to $(\pi_\circ \circ q)$-coCartesian morphisms and if the induced functor $f[n] : X_{[n]} \to Y_{[n]}$, over $\Delta^{\text{op}}_\infty$ is a lax monoidal functor for each $[n] \in \Delta^{\text{op}}_\infty$.

In other words, a double lax monoidal functor is a morphism in $\text{Mon}^{\text{lax}}(\text{Mon}^{\text{lax}}(\text{Cat}_\infty))$.

**Definition 5.22.** We define an ∞-category

$$\text{Duo}^{\text{dlax}}_\infty$$

to be the ∞-category $\text{Mon}^{\text{lax}}(\text{Mon}^{\text{lax}}(\text{Cat}_\infty))$. We call $\text{Duo}^{\text{dlax}}_\infty$ the ∞-category of duoidal ∞-categories and double lax monoidal functors.

Next, by duality, we introduce double oplax monoidal functors between duoidal ∞-categories.

**Definition 5.23.** We define an ∞-category

$$\text{Duo}^{\text{doplax}}_\infty$$

to be the ∞-category $\text{Mon}^{\text{oplax}}(\text{Mon}^{\text{oplax}}(\text{Cat}_\infty))$. We also say that an object of $\text{Duo}^{\text{doplax}}_\infty$ is a duoidal ∞-category. A morphism in $\text{Duo}^{\text{doplax}}_\infty$ is said to be a double oplax monoidal functor. We call $\text{Duo}^{\text{doplax}}_\infty$ the ∞-category of duoidal ∞-categories and double oplax monoidal functors.
Remark 5.24. By Definition 4.11 an object of Duo$^{oplax}$ is a categorical fibration $p : X \to \Delta^\otimes \times \Delta^{op}$ which satisfies the following conditions:

1. The composite $\pi_\otimes \circ p : X \to \Delta^\otimes$ is a Cartesian fibration, and $p$ carries $(\pi_\otimes \circ p)$-Cartesian morphisms to $\pi_\otimes$-Cartesian morphisms.
2. For each $[n] \in \Delta^\otimes$, the restriction $p_{[n],*} : X_{[n],*} \to \Delta^{op}$ is a monoidal $\infty$-category.
3. For each morphism $[n'] \to [n]$ in $\Delta^\otimes$, the induced map $X_{[n],*} \to X_{[n'],*}$ over $\Delta^{op}$ is an oplax monoidal functor.
4. For each $[n] \in \Delta^\otimes$, the Segal morphism

$$X_{[n],*} \to \overline{X_{[1],*} \times_{\Delta^\otimes} \cdots \times_{\Delta^{op}} X_{[1],*}}$$

is an equivalence of Cartesian fibrations over $\Delta^{op}$.

Let $p : X \to \Delta^\otimes \times \Delta^{op}$ and $q : Y \to \Delta^\otimes \times \Delta^{op}$ be duoidal $\infty$-categories. A double oplax monoidal functor from $p$ to $q$ is a functor $f : X \to Y$ over $\Delta^\otimes \times \Delta^{op}$ such that it carries $\pi_\otimes \circ p$-Cartesian morphisms over inert morphisms of $\Delta^\otimes$ to $\pi_\otimes \circ q$-Cartesian morphisms and that the induce functor $f_{[n],*} : X_{[n],*} \to Y_{[n],*}$ over $\Delta^{op}$ is an oplax monoidal functor for each $[n] \in \Delta^\otimes$.

Remark 5.25. For a duoidal $\infty$-category $p : X \to \Delta^{op}_\otimes \times \Delta^{op}_\otimes$, the map $p^{op} : X^{op} \to \Delta^\otimes \times \Delta^{op}$ is also a duoidal $\infty$-category. This induces an equivalence

$$(\text{id})^{op} : \text{Duo}^{oplax} \cong \text{Duo}^{op}$$

of $\infty$-categories.

6. Bimonoids, double monoids, and double comonoids in duoidal $\infty$-categories

We can consider bimonoids, double monoids, and double comonoids in duoidal categories. In this section we introduce notions of bimonoids, double monoids, and double comonoids in duoidal $\infty$-categories. For a duoidal $\infty$-category $p : X \to \Delta^{op}_\otimes \times \Delta^{op}_\otimes$, we construct a monoidal structure on the $\infty$-category $\text{Alg}^{\otimes}(X)$ of algebra objects of $(X, \otimes, 1)$ by using the monoidal structure $(\otimes, 1)$. Similarly, we construct a monoidal structure on the $\infty$-category $\text{cAlg}^{\otimes}(X)$ of coalgebra objects of $(X, \otimes, 1)$ by using the monoidal structure $(\otimes, 1)$. We show that bimonoid objects of $X$ are equivalent to algebra objects of $\text{Alg}^{\otimes}(X)$, that they are also equivalent to coalgebra objects of $\text{cAlg}^{\otimes}(X)$ by duality. We also prove similar results on double monoids and double comonoids in duoidal $\infty$-categories.

6.1. Algebra and coalgebra objects in monoidal $\infty$-categories

In this subsection we recall algebra and coalgebra objects in monoidal $\infty$-categories (cf. [15, 2.1.2] and [16, 4.3]).

First, we recall algebra objects in a monoidal $\infty$-category. Let $p : X \to \Delta^{op}$ be a monoidal $\infty$-category with underlying $\infty$-category $X_{[1]}$. An algebra object of the monoidal $\infty$-category $X_{[1]}$ is a section $s : \Delta^{op} \to X$ of $p$ such that $s$ takes inert morphisms of $\Delta^{op}$ to $p$-coCartesian morphisms of $X$.

We can describe algebra objects of $X_{[1]}$ in another way. Let $\text{id}_{\Delta^{op}} : \Delta^{op} \to \Delta^{op}$ be the identity functor of $\Delta^{op}$. Note that $\text{id}_{\Delta^{op}}$ is a final object of $\text{Mon}^{\text{lax}}(\text{Cat}_\infty)$. An algebra object of $X_{[1]}$ is a morphism from $\text{id}_{\Delta^{op}}$ to $p$ in $\text{Mon}^{\text{lax}}(\text{Cat}_\infty)$.

Let $\text{Fun}_{\Delta^{op}}(\Delta^{op}, X)$ be the $\infty$-category of sections of $p$. We denote by

$$\text{Alg}(X)$$

the full subcategory of $\text{Fun}_{\Delta^{op}}(\Delta^{op}, X)$ spanned by algebra objects of $X_{[1]}$. Let $f$ be a lax monoidal functor from $p : X \to \Delta^{op}$ to $q : Y \to \Delta^{op}$. Since $f$ carries $p$-coCartesian morphisms over inert
morphisms of $\Delta^{op}$ to $q$-coCartesian morphisms, it induces a functor
\[ \text{Alg}(f) : \text{Alg}(X) \to \text{Alg}(Y) \]
of $\infty$-categories. Assigning to a monoidal $\infty$-category $p : X \to \Delta^{op}$ the $\infty$-category $\text{Alg}(X)$ of its algebra objects, we obtain a functor
\[ \text{Alg} : \text{Mon}^{\text{lax}}(\text{Cat}_\infty) \to \text{Cat}_\infty \]
of $\infty$-categories.

Next, we consider coalgebra objects of a monoidal $\infty$-category. Let $p : X \to \Delta$ be a monoidal $\infty$-category, that is, $p$ is a Cartesian fibration such that the Segal morphisms are equivalences. A coalgebra object of the monoidal $\infty$-category $X$ is a section $s$ of $p$ such that $s$ takes inert morphisms of $\Delta$ to $p$-Cartesian morphisms. As in the case of algebra objects, we can regard a coalgebra object of $X$ as a morphism from $id_\Delta$ to $p$ in $\text{Mon}^{\text{op}}(\text{Cat}_\infty)$, where $id_\Delta : \Delta \to \Delta$ is a final object of $\text{Mon}^{\text{op}}(\text{Cat}_\infty)$.

Let $\text{Fun}_\Delta(\Delta, X)$ be the $\infty$-category of sections of $p$. We denote by
\[ \text{cAlg}(X) \]
the full subcategory of $\text{Fun}_\Delta(\Delta, X)$ spanned by coalgebra objects of $X$. Let $f$ be an oplax monoidal functor from $p : X \to \Delta$ to $q : Y \to \Delta$. Since $f$ carries $p$-Cartesian morphisms over inert morphisms of $\Delta$ to $q$-Cartesian morphisms, it induces a functor
\[ \text{cAlg}(f) : \text{cAlg}(X) \to \text{cAlg}(Y) \]
of $\infty$-categories. Assigning to a monoidal $\infty$-category $p : X \to \Delta$ the $\infty$-category $\text{cAlg}(X)$ of its coalgebra objects, we obtain a functor
\[ \text{cAlg} : \text{Mon}^{\text{op}}(\text{Cat}_\infty) \to \text{Cat}_\infty \]
of $\infty$-categories. Note that the functor $\text{cAlg}$ is equivalent to the composite of the functors
\[ \text{Mon}^{\text{op}}(\text{Cat}_\infty)(-)^{op} \to \text{Mon}(\text{Cat}_\infty) \to \text{Alg}(\text{Cat}_\infty) \to \text{Cat}_\infty. \]

### 6.2. Bimonoids in duoidal $\infty$-categories

In this subsection we introduce a notion of bimonoids in duoidal $\infty$-categories. For a duoidal $\infty$-category $p : X \to \Delta^{op} \times \Delta$, we construct an $\infty$-category $\text{bMon}(X)$ of bimonoids in $X$. This construction determines a functor $\text{bMon} : \text{Duo}^{\text{bilax}} \to \text{Cat}_\infty$ of $\infty$-categories. Furthermore, we show that the functor $\text{cAlg} : \text{Mon}^{\text{op}}(\text{Cat}_\infty) \to \text{Cat}_\infty$ can be upgraded to a functor $\text{cAlg}^{\oplus} : \text{Duo}^{\text{bilax}} \to \text{Mon}(\text{Cat}_\infty)$, and that the functor $\text{bMon}$ is equivalent to the composite $\text{Alg} \circ \text{cAlg}^{\oplus}$ of functors.

Recall that $\Delta = \Delta^{op} = \Delta$, and $\pi_0 : \Delta^{op} \times \Delta \to \Delta^{op}$ and $\pi_0 : \Delta^{op} \times \Delta \to \Delta$ are the projections. Note that the identity map $\text{id}_{\Delta^{op} \times \Delta} : \Delta^{op} \times \Delta \to \Delta^{op} \times \Delta$ is a final object of $\text{Duo}^{\text{bilax}}$.

**Definition 6.1.** Let $p : X \to \Delta^{op} \times \Delta$ be a duoidal $\infty$-category. We define a bimonoid in the duoidal $\infty$-category $X_{[1],[1]}$ to be a morphism $B : \text{id}_{\Delta^{op} \times \Delta} \to p$ in $\text{Duo}^{\text{bilax}}$. In other words, a bimonoid in $X_{[1],[1]}$ is a section $B$ of $p : X \to \Delta^{op} \times \Delta$: 

\[
\xymatrix{ \Delta^{op} \times \Delta \ar[r]^{\text{id}} & \Delta^{op} \times \Delta }
\]
such that $B$ sends $\pi_\otimes$-coCartesian morphisms over inert morphisms of $\Delta_{\otimes}^{op}$ to $(\pi_\otimes \circ p)$-coCartesian morphisms, and that $\pi_\otimes$-Cartesian morphisms over inert morphisms of $\Delta_{\otimes}$ to $(\pi_\otimes \circ p)$-Cartesian morphisms.

Let $\operatorname{Fun}_{\Delta_{\otimes}^{op} \times \Delta_{\otimes}}(\Delta_{\otimes}^{op} \times \Delta_{\otimes}, X)$ be the $\infty$-category of sections of $p : X \to \Delta_{\otimes}^{op} \times \Delta_{\otimes}$. We define an $\infty$-category

$$\text{bMon}(X)$$

to be the full subcategory of $\operatorname{Fun}_{\Delta_{\otimes}^{op} \times \Delta_{\otimes}}(\Delta_{\otimes}^{op} \times \Delta_{\otimes}, X)$ spanned by bimonoids in $X_{[1],[1]}$, and call it the $\infty$-category of bimonoids of $X_{[1],[1]}$. Assigning to a duoidal $\infty$-category $p : X \to \Delta_{\otimes}^{op} \times \Delta_{\otimes}$ the $\infty$-category of bimonoids of $X_{[1],[1]}$, we obtain a functor

$$\text{bMon} : \text{Duo}_{\infty}^{\text{bilax}} \to \text{Cat}_{\infty}$$

of $\infty$-categories.

In the following of this subsection we shall define a functor

$$\text{cAlg}^{\otimes} : \text{Duo}_{\infty}^{\text{bilax}} \to \text{Mon}^{\text{lax}}(\text{Cat}_{\infty})$$

and show that the functor $\text{bMon} : \text{Duo}_{\infty}^{\text{bilax}} \to \text{Cat}_{\infty}$ is equivalent to the composite of the functors

$$\text{Duo}_{\infty}^{\text{bilax}} \xrightarrow{\text{cAlg}^{\otimes}} \text{Mon}^{\text{lax}}(\text{Cat}_{\infty}) \xrightarrow{\text{Alg}} \text{Cat}_{\infty}.$$  

By duality, we shall define a functor

$$\text{Alg}_{\infty}^{\otimes} : \text{Duo}_{\infty}^{\text{bilax}} \to \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty})$$

and show that $\text{bMon}$ is equivalent to the composite of the functors

$$\text{Duo}_{\infty}^{\text{bilax}} \xrightarrow{\text{Alg}_{\infty}^{\otimes}} \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty}) \xrightarrow{\text{cAlg}} \text{Cat}_{\infty}.$$  

Let $p : X \to \Delta_{\otimes}^{op} \times \Delta_{\otimes}$ be a duoidal $\infty$-category. We let $\text{cAlg}^{\otimes}(X)$ be a simplicial set over $\Delta_{\otimes}^{op}$ satisfying the following formula

$$\operatorname{Hom}_{\operatorname{Set}_{\Delta}/\Delta_{\otimes}}(K, \text{cAlg}^{\otimes}(X)) \cong \operatorname{Hom}_{\operatorname{Set}_{\Delta}/(\Delta_{\otimes}^{op} \times \Delta_{\otimes})}(K \times \Delta_{\otimes}, X)$$

for any simplicial set $K$ over $\Delta_{\otimes}^{op}$, where $\operatorname{Set}_{\Delta}$ is the category of simplicial sets. We denote by $\overline{\pi}$ the map $\text{cAlg}^{\otimes}(X) \to \Delta_{\otimes}^{op}$. Note that the fiber $\text{cAlg}^{\otimes}(X)_{[m]}$ of $\overline{\pi}$ at $[m] \in \Delta_{\otimes}^{op}$ is $\operatorname{Fun}_{\Delta_{\otimes}}(\Delta_{\otimes}, X_{[m]} \times).$

**Lemma 6.2.** The map $\overline{\pi} : \text{cAlg}^{\otimes}(X) \to \Delta_{\otimes}^{op}$ is a categorical fibration of simplicial sets. In particular, $\text{cAlg}^{\otimes}(X)$ is an $\infty$-category.

**Proof.** Since equivalences in $\Delta_{\otimes}^{op}$ are only identities, it suffices to show that $\overline{\pi} : \text{cAlg}^{\otimes}(X) \to \Delta_{\otimes}^{op}$ is an inner fibration by [14 Corollary 2.4.6.5]. Suppose we have a commutative diagram

$$\begin{array}{ccc}
\Lambda^n_i & \rightarrow & \text{cAlg}^{\otimes}(X) \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & \Delta_{\otimes}^{op}
\end{array}$$

of simplicial sets for $0 < i < n$. We would like to construct a map $\Delta^n \to \text{cAlg}^{\otimes}(X)$, which makes the whole diagram commutative.
By the definition of \( \mathsf{cAlg}^\otimes(X) \), the above diagram is equivalent to the following commutative diagram

\[
\begin{array}{ccc}
\Lambda^n \times \Delta & \longrightarrow & X \\
\downarrow & & \downarrow \\
\Delta^n \times \Delta & \longrightarrow & \Delta^{\text{op}} \times \Delta.
\end{array}
\]

The left vertical arrow is an inner anodyne map by [14, Corollary 2.3.2.4] and the right vertical arrow is an inner fibration by the assumption. Hence there is a map \( \Delta^n \times \Delta \to X \) which makes the whole diagram commutative. This induces a desired map \( \Delta^n \to \mathsf{cAlg}^\otimes(X) \).

**Lemma 6.3.** Suppose that we have a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & \xleftarrow{c} & \Delta^{\text{op}} \times \Delta
\end{array}
\]

of \( \infty \)-categories, where \( g \) and \( h \) are coCartesian fibrations, and \( f \) is a categorical fibration which carries \( h \)-coCartesian morphisms to \( g \)-coCartesian morphisms. Furthermore, suppose that we have a pullback diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & A \\
\downarrow{k} & & \downarrow{f} \\
Y & \xrightarrow{m} & B
\end{array}
\]

of \( \infty \)-categories. If the image of any morphism of \( Y \) under \( m \) is a \( g \)-coCartesian morphism, then \( k: X \to Y \) is a coCartesian fibration.

**Proof.** Note that \( k \) is a categorical fibration since it is a pullback of the categorical fibration \( f \) along \( m \). Let \( x \) be an object of \( X \) over \( y \in Y \) and let \( e : k(x) = y \to y' \) be a morphism of \( Y \). In order to prove that \( k \) is a coCartesian fibration, we shall show that there is a lifting \( \overline{e}: x \to x' \) in \( X \) of \( e \), which is a \( k \)-coCartesian morphism.

Since \( h \) is a coCartesian fibration, there is an \( h \)-coCartesian morphism \( \overline{e} : l(x) \to a \) over \( g(m(e)) \). From the fact that \( f \) carries \( h \)-coCartesian morphisms to \( g \)-coCartesian morphisms, \( f(\overline{e}) : f(l(x)) = m(y) \to f(a) \) is a \( g \)-coCartesian morphism over \( g(m(e)) \). By the assumption, \( m(e) : m(y) \to m(y') \) is also a \( g \)-coCartesian morphism over \( g(m(e)) \), and hence there is an equivalence \( e' : m(y) \to m(y') \) in \( B \) such that \( e' \circ f(\overline{e}) \simeq m(e) \). By [14, Corollary 2.4.6.5], we can take a lifting \( \overline{e}' : a \to a' \) in \( A \) of \( e' \), which is an equivalence, since \( f \) is a categorical fibration and \( B \) is an \( \infty \)-category. Using the fact that \( f \) is an inner fibration, we see that there is a \( h \)-coCartesian morphism \( \overline{c} : l(x) \to a' \) in \( A \), which is a lifting of \( m(e) \). By the dual of [14, Proposition 2.4.1.3(3)], \( \overline{c} \) is an \( f \)-coCartesian morphism. The lifting \( \overline{c} \) determines a lifting \( \overline{e} : x \to x' \) of \( e \), which is a \( k \)-coCartesian morphism by [14, Proposition 2.4.1.3(2)]. This completes the proof.

**Lemma 6.4.** The map \( \mathfrak{r} : \mathsf{cAlg}^\otimes(X) \to \Delta^{\text{op}}_{\Delta^\otimes} \) is a coCartesian fibration of \( \infty \)-categories.

**Proof.** We have a commutative triangle

\[
\begin{array}{ccc}
\text{Fun}(\Delta^\otimes, X) & \xrightarrow{f} & \text{Fun}(\Delta^\otimes, \Delta^{\text{op}}_{\Delta^\otimes}) \\
\downarrow{h} & & \downarrow{g} \\
\text{Fun}(\Delta^\otimes, \Delta^{\text{op}}_{\Delta^\otimes})
\end{array}
\]
of ∞-categories, where \( g \) and \( h \) are coCartesian fibrations, and the map \( f \) carries \( g \)-coCartesian morphisms to \( h \)-coCartesian morphisms by [14 Corollary 3.2.2.12]. We see that the map \( f \) is a categorical fibration by using [14 Corollary 2.2.5.4]. Furthermore, there is a pullback diagram

\[
\begin{array}{ccc}
c\text{Alg}^\otimes(X) & \xrightarrow{\pi} & \text{Fun}(\Delta_\otimes, X) \\
j \downarrow & & \downarrow f \\
\Delta_\otimes^\text{op} & \xrightarrow{e} & \text{Fun}(\Delta_\otimes, \Delta_\otimes^\text{op} \times \Delta_\otimes)
\end{array}
\]

of ∞-categories, where the bottom horizontal arrow \( j \) is an adjoint of the identity map of \( \Delta_\otimes^\text{op} \times \Delta_\otimes \). Since the image of any morphism of \( \Delta_\otimes^\text{op} \) under \( j \) is a \( g \)-coCartesian morphism in \( \text{Fun}(\Delta_\otimes, \Delta_\otimes^\text{op} \times \Delta_\otimes) \), the lemma follows from Lemma 6.3.

Remark 6.5. Let \( e \) be a morphism of \( c\text{Alg}^\otimes(X) \), which corresponds to a map \( \Delta^1 \times \Delta_\otimes \to X \). The morphism \( e \) is a \( \otimes \)-coCartesian morphism if and only if the composite \( \Delta^1 \simeq \Delta^1 \times \{[m]\} \to \Delta^1 \times \Delta_\otimes \to X \) is a \( p \)-coCartesian morphism for all \( [m] \in \Delta_\otimes \).

Definition 6.6. We define

\[
c\text{Alg}^\otimes(X)
\]

to be the full subcategory of \( c\text{Alg}^\otimes(X) \) spanned by coalgebra objects in \( X_{[m],[1]} \) for some \( [m] \in \Delta_\otimes^\text{op} \).

By restriction of \( \pi \), we obtain a map

\[
u : c\text{Alg}^\otimes(X) \to \Delta_\otimes^\text{op}
\]
of ∞-categories.

We shall show that \( u : c\text{Alg}^\otimes(X) \to \Delta_\otimes^\text{op} \) is a monoidal ∞-category.

Proposition 6.7. The map \( u : c\text{Alg}^\otimes(X) \to \Delta_\otimes^\text{op} \) is a coCartesian fibration of ∞-categories.

Proof. Let \( x \) be an object of \( c\text{Alg}^\otimes(X) \) over \( [m] \in \Delta_\otimes^\text{op} \) and let \( e : [m] \to [m'] \) be a morphism of \( \Delta_\otimes^\text{op} \). Since \( \pi : c\text{Alg}^\otimes(X) \to \Delta_\otimes^\text{op} \) is a coCartesian fibration by Lemma 6.4, there is a \( \otimes \)-coCartesian morphism \( x : x \to y \) over \( e \). In order to prove the proposition, it suffices to show that \( y \) is an object of \( c\text{Alg}^\otimes(X) \). We have a map \( e_1 : X_{[m]} \to X_{[m']}, \) over \( \Delta_\otimes \), which carries \( p_{[m]}, \) \( \otimes \)-Cartesian morphisms over inner morphisms of \( \Delta_\otimes \) to \( p_{[m']} \)-\( \otimes \)-Cartesian morphisms. If we regard \( x \in c\text{Alg}^\otimes(X) \) as a map \( \Delta_\otimes \to X_{[m]}, \) then the object \( y \) is identified with the composite \( e_1 \circ x : \Delta_\otimes \to X_{[m]} \). Since \( x \) carries inert morphisms of \( \Delta_\otimes \) to \( p_{[m']} \)-\( \otimes \)-Cartesian morphisms, \( e_1 \circ x \) carries inert morphisms of \( \Delta_\otimes \) to \( p_{[m']} \)-\( \otimes \)-Cartesian morphisms. Hence \( y \) is an object of \( c\text{Alg}^\otimes(X) \) over \( [m'] \in \Delta_\otimes^\text{op} \). This completes the proof.

Theorem 6.8. The map \( u : c\text{Alg}^\otimes(X) \to \Delta_\otimes^\text{op} \) is a monoidal ∞-category.

Proof. By Proposition 6.7, it suffices to show that the Segal morphism

\[
c\text{Alg}^\otimes(X)_{[m]} \to c\text{Alg}^\otimes(X)_{[1]} \times \cdots \times c\text{Alg}^\otimes(X)_{[1]}
\]

is an equivalence of ∞-categories for each \( [m] \in \Delta_\otimes^\text{op} \). Since \( p : X \to \Delta_\otimes^\text{op} \times \Delta_\otimes \) is a duoidal ∞-category, the Segal morphism \( X_{[m]} \to X_{[1]} \times \Delta_\otimes \cdots \times \Delta_\otimes X_{[1]} \times \Delta_\otimes \) is an equivalence in \( \text{Cart}/\Delta_\otimes \) for each \( [m] \in \Delta_\otimes^\text{op} \). This induces an equivalence

\[
c\text{Alg}^\otimes(X_{[m]}, \bullet) \to c\text{Alg}^\otimes(X_{[1]}, \bullet) \times \cdots \times c\text{Alg}^\otimes(X_{[1]}, \bullet)
\]
of ∞-categories. Since we can identify this map with (6.1), the map \( u : c\text{Alg}^\otimes(X) \to \Delta_\otimes^\text{op} \) is a monoidal ∞-category.
Next, we shall show that a bilax monoidal functor \( f : X \to Y \) of duoidal \( \infty \)-categories induces a lax monoidal functor between the monoidal \( \infty \)-categories \( \text{cAlg}^\otimes(X) \) and \( \text{cAlg}^\otimes(Y) \).

By the definition of \( \text{cAlg}^\otimes(\infty \text{-categories}) \), we have an evaluation map \( \text{ev} : \text{cAlg}^\otimes(X) \times \Delta^\otimes \to X \) over \( \Delta^\otimes_{\text{op}} \times \Delta^\otimes_\emptyset \). Let \( f : X \to Y \) be a bilax monoidal functor between duoidal \( \infty \)-categories. Composing \( \text{ev} \) with \( f \), we obtain a map \( \text{cAlg}^\otimes(X) \times \Delta^\otimes \to Y \) over \( \Delta^\otimes_{\text{op}} \times \Delta^\otimes_\emptyset \), which induces a map \( \text{cAlg}^\otimes(f) : \text{cAlg}^\otimes(X) \to \text{cAlg}^\otimes(Y) \) over \( \Delta^\otimes_{\text{op}} \).

Let \( s \) be an object of \( \text{cAlg}^\otimes(X)[m] \simeq \text{cAlg}(X[m], \bullet) \) for \( [m] \in \Delta^\otimes_{\text{op}} \). Since the bilax monoidal functor \( f \) induces an oplax monoidal functor \( f[m] : X[m], \bullet \to Y[m], \bullet \), we see that \( \text{cAlg}^\otimes(f)(s) \) lands in the full subcategory \( \text{cAlg}^\otimes(Y)[m] \simeq \text{cAlg}(Y[m], \bullet) \) of \( \text{cAlg}^\otimes(Y)[m] \). Thus, we obtain a functor

\[
\text{cAlg}^\otimes(f) : \text{cAlg}^\otimes(X) \to \text{cAlg}^\otimes(Y)
\]

of \( \infty \)-categories over \( \Delta^\otimes_{\text{op}} \).

**Proposition 6.9.** A bilax monoidal functor \( f : X \to Y \) between duoidal \( \infty \)-categories induces a lax monoidal functor

\[
\text{cAlg}^\otimes(f) : \text{cAlg}^\otimes(X) \to \text{cAlg}^\otimes(Y)
\]

between monoidal \( \infty \)-categories.

**Proof.** We have to show that the induced map \( \text{cAlg}^\otimes(f) : \text{cAlg}^\otimes(X) \to \text{cAlg}^\otimes(Y) \) preserves coCartesian morphisms over inert morphisms of \( \Delta^\otimes_{\text{op}} \). Since \( f : X \to Y \) is a bilax monoidal functor, \( f \) preserves coCartesian morphisms over inert morphisms of \( \Delta^\otimes_{\text{op}} \). If \( e : [m] \to [m'] \) is an inert morphism of \( \Delta^\otimes_{\text{op}} \), we have a commutative diagram

\[
\begin{array}{ccc}
X[m], \bullet & \xrightarrow{f[m], \bullet} & Y[m], \bullet \\
e \downarrow & & \downarrow \sigma \\
X[m'], \bullet & \xrightarrow{f[m'], \bullet} & Y[m'], \bullet
\end{array}
\]

of \( \text{Cart}/(\Delta^\otimes)^{\text{op}} \). This induces the following commutative diagram

\[
\begin{array}{ccc}
\text{cAlg}(X[m], \bullet) & \xrightarrow{\text{cAlg}(f[m], \bullet)} & \text{cAlg}(Y[m], \bullet) \\
\downarrow e \downarrow & & \downarrow e \downarrow \\
\text{cAlg}(X[m'], \bullet) & \xrightarrow{\text{cAlg}(f[m'], \bullet)} & \text{cAlg}(Y[m'], \bullet)
\end{array}
\]

of \( \infty \)-categories. This means that the functor \( \text{cAlg}^\otimes(f) : \text{cAlg}^\otimes(X) \to \text{cAlg}^\otimes(Y) \) preserves co-Cartesian morphisms over inert morphisms of \( \Delta^\otimes_{\text{op}} \). This completes the proof. \( \square \)

By Theorem 6.8 and Proposition 6.9, we obtain a functor

\[
\text{cAlg}^\otimes : \text{Duo}^{\text{bilax}} \longrightarrow \text{Mon}^{\text{lax}}(\text{Cat}_{\infty})
\]

by assigning to a duoidal \( \infty \)-category \( X \to \Delta^\otimes_{\text{op}} \times \Delta^\otimes_\emptyset \) the monoidal \( \infty \)-category \( \text{cAlg}^\otimes(X) \to \Delta^\otimes_{\text{op}} \).

**Remark 6.10.** By duality, there is a functor

\[
\text{Alg}^\otimes : \text{Duo}^{\text{bilax}} \longrightarrow \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty})
\]
of $\infty$-categories. Note that we have a commutative diagram

\[
\begin{array}{ccc}
\text{Duo}_\infty^{\text{bilax}} & \xrightarrow{c\text{Alg}^\otimes} & \text{Mon}^{\text{lax}}(\text{Cat}_\infty) \\
(-)^{op} \downarrow & & \downarrow (-)^{op} \\
\text{Duo}_\infty^{\text{bilax}} & \xrightarrow{\text{Alg}^\otimes} & \text{Mon}^{\text{oplax}}(\text{Cat}_\infty)
\end{array}
\]

in $\hat{\text{Cat}}_\infty$.

By composing $c\text{Alg}^\otimes$ with the functor $\text{Alg} : \text{Mon}^{\text{lax}}(\text{Cat}_\infty) \to \text{Cat}_\infty$, we obtain a functor $\text{Alg} \circ c\text{Alg}^\otimes : \text{Duo}_\infty^{\text{bilax}} \to \text{Cat}_\infty$.

Unwinding the definitions, we obtain the following theorem.

**Theorem 6.11.** There is a natural isomorphism

\[
b\text{Mon}(X) \simeq (\text{Alg} \circ c\text{Alg}^\otimes)(X)
\]

of simplicial sets for any duoidal $\infty$-category $p : X \to \Delta_\otimes^{op} \times \Delta_\otimes$.

**Corollary 6.12.** The functor $b\text{Mon} : \text{Duo}_\infty^{\text{bilax}} \to \text{Cat}_\infty$ is equivalent to the composite $\text{Alg} \circ c\text{Alg}^\otimes$.

**Remark 6.13.** By duality, the functor $b\text{Mon}$ is equivalent to the composite $c\text{Alg} \circ \text{Alg}^\otimes$. Notice that there is a commutative diagram

\[
\begin{array}{ccc}
\text{Duo}_\infty^{\text{bilax}} & \xrightarrow{c\text{Alg}^\otimes} & \text{Mon}^{\text{lax}}(\text{Cat}_\infty) \\
(-)^{op} \downarrow & & \downarrow (-)^{op} \\
\text{Duo}_\infty^{\text{bilax}} & \xrightarrow{\text{Alg}^\otimes} & \text{Mon}^{\text{oplax}}(\text{Cat}_\infty) \\
\text{Cat}_\infty & \xrightarrow{c\text{Alg}} & \text{Cat}_\infty
\end{array}
\]

in $\hat{\text{Cat}}_\infty$.

**6.3. Double monoids and double comonoids in duoidal $\infty$-categories.** In this subsection we introduce notions of double monoids and double comonoids in duoidal $\infty$-categories. For a duoidal $\infty$-category $p : X \to \Delta_\otimes^{op} \times \Delta_\otimes$, we construct an $\infty$-category $d\text{Mon}(X)$ of double monoids in $X$.

This construction determines a functor $d\text{Mon} : \text{Duo}_\infty^{\text{dlax}} \to \text{Cat}_\infty$ of $\infty$-categories. Furthermore, we show that the functor $\text{Alg} : \text{Mon}^{\text{lax}}(\text{Cat}_\infty) \to \text{Cat}_\infty$ can be upgraded to a functor $\text{Alg}^\otimes : \text{Duo}_\infty^{\text{dlax}} \to \text{Mon}^{\text{lax}}(\text{Cat}_\infty)$, and that the functor $d\text{Mon}$ is equivalent to the composite $\text{Alg} \circ \text{Alg}^\otimes$. By duality, we also show the similar results on double comonoids.

First, we define double monoids in a duoidal $\infty$-category.

**Definition 6.14.** Let $p : X \to \Delta_\otimes^{op} \times \Delta_\otimes^{op}$ be a duoidal $\infty$-category, that is, it is a categorical fibration satisfying Condition [6.11.9]. We define a double monoid in the duoidal $\infty$-category $X_{[1],[1]}$ to be a morphism $D : \text{id}_{\Delta_\otimes^{op} \times \Delta_\otimes^{op}} \to p$ in $\text{Duo}_\infty^{\text{dlax}}$. In other words, a double monoid in $X_{[1],[1]}$ is a section $D$ of $p : X \to \Delta_\otimes^{op} \times \Delta_\otimes^{op}$.
such that $D$ sends $\pi_{\varnothing}$-coCartesian morphisms over inert morphisms of $\Delta_\varnothing^{op}$ to $(\pi_{\varnothing} \circ p)$-coCartesian morphisms, and that $\Delta_\varnothing^{op} \simeq \{[n]\} \times \Delta_\varnothing^{op} \to X_{[n],\bullet}$ is an algebra object of $X_{[1],[1]}$ for each $[n] \in \Delta_\varnothing^{op}$.

Let $\text{Fun}_{\Delta_\varnothing^{op} \times \Delta_\varnothing^{op}}(\Delta_\varnothing^{op} \times \Delta_\varnothing^{op}, X)$ be the $\infty$-category of sections of $p : X \to \Delta_\varnothing^{op} \times \Delta_\varnothing^{op}$. We define an $\infty$-category
d_{\text{Mon}}(X)

\text{to be the full subcategory of } \text{Fun}_{\Delta_\varnothing^{op} \times \Delta_\varnothing^{op}}(\Delta_\varnothing^{op} \times \Delta_\varnothing^{op}, X) \text{ spanned by double monoids in } X_{[1],[1]}, \text{ and call it the } \infty\text{-category of double monoids of } X_{[1],[1]}$. Assigning to a duoidal $\infty$-category $p : X \to \Delta_\varnothing^{op} \times \Delta_\varnothing^{op}$ the $\infty$-category of double monoids of $X_{[1],[1]}$, we obtain a functor
d_{\text{Mon}} : \text{Duo}^{dlax} \longrightarrow \text{Cat}_\infty

do\text{-categories.}

Next, in the same way as the construction of the functor $c\text{Alg}^{\otimes} : \text{Duo}^{dlax} \to \text{Mon}^{lax}(\text{Cat}_\infty)$ in §6.2, we construct a functor $\text{Alg}^{\otimes} : \text{Duo}^{dlax} \to \text{Mon}^{lax}(\text{Cat}_\infty)$, which is a lifting of the functor $\text{Alg} : \text{Mon}^{lax}(\text{Cat}_\infty) \to \text{Cat}_\infty$.

For a duoidal $\infty$-category $p : X \to \Delta_\varnothing^{op} \times \Delta_\varnothing^{op}$, we let $\overline{\text{Alg}}^{\otimes}(X)$ be a simplicial set over $\Delta_\varnothing^{op}$ satisfying the following formula

$$\text{Hom}_{\text{Set}}(K, \overline{\text{Alg}}^{\otimes}(X)) \cong \text{Hom}_{\text{Set}}(\Delta_\varnothing^{op} \times \Delta_\varnothing^{op} \times \Delta_\varnothing^{op}, X)$$

for any simplicial set $K$ over $\Delta_\varnothing^{op}$. We denote by $\overline{\pi}$ the map $\overline{\text{Alg}}^{\otimes}(X) \to \Delta_\varnothing^{op}$. Note that the fiber $\overline{\text{Alg}}^{\otimes}(X)_{[n]}$ of $\overline{\pi}$ at $[n] \in \Delta_\varnothing^{op}$ is $\text{Fun}_{\Delta_\varnothing^{op}}(\Delta_\varnothing^{op}, X_{[n],\bullet})$. We define

$$\text{Alg}^{\otimes}(X) \text{ to be the full subcategory of } \overline{\text{Alg}}^{\otimes}(X) \text{ spanned by algebra objects of the monoidal } \infty\text{-category } X_{[n],\bullet} \to \Delta_\varnothing^{op} \text{ for some } [n] \in \Delta_\varnothing^{op}. \text{ We define a map}$$

$$u : \text{Alg}^{\otimes}(X) \to \Delta_\varnothing^{op}$$

to be the restriction of $\overline{\pi}$.

By the same argument as in Theorem 6.8 and Proposition 6.9, we obtain the following theorem.

**Theorem 6.15.** The map $u : \text{Alg}^{\otimes}(X) \to \Delta_\varnothing^{op}$ is a monoidal $\infty$-category. A double lax monoidal functor $f : X \to Y$ of duoidal $\infty$-categories induces a lax monoidal functor

$$\text{Alg}^{\otimes}(f) : \text{Alg}^{\otimes}(X) \longrightarrow \text{Alg}^{\otimes}(Y).$$

**Remark 6.16.** By Remark 6.13, we have a monoidal $\infty$-category $\text{Alg}^{\otimes}(Y) \to \Delta_\varnothing$, which is a Cartesian fibration, for a duoidal $\infty$-category $q : Y \to \Delta_\varnothing^{op} \times \Delta_\varnothing^{op}$. Let $p : X \to \Delta_\varnothing^{op} \times \Delta_\varnothing^{op}$ be an equivalent duoidal $\infty$-category to $q$. By Theorem 6.15, we obtain a monoidal $\infty$-category

$$\text{Alg}^{\otimes}(X) \to \Delta_\varnothing^{op},$$

which is a coCartesian fibration. These two monoidal $\infty$-categories are equivalent.

By Theorem 6.15, we obtain a functor

$$\text{Alg}^{\otimes} : \text{Duo}^{dlax} \longrightarrow \text{Mon}^{lax}(\text{Cat}_\infty).$$

Unwinding the definitions, we obtain the following theorem.

**Theorem 6.17.** There is a natural isomorphism

$$d_{\text{Mon}}(X) \simeq (\text{Alg} \circ \text{Alg}^{\otimes})(X)$$

of simplicial sets for any duoidal $\infty$-category $p : X \to \Delta_\varnothing^{op} \times \Delta_\varnothing^{op}$. The functor $d_{\text{Mon}} : \text{Duo}^{dlax} \to \text{Cat}_\infty$ is equivalent to the composite

$$\text{Duo}^{dlax} \overset{\text{Alg}^{\otimes}}{\longrightarrow} \text{Mon}^{lax}(\text{Cat}_\infty) \overset{\text{Alg}}{\longrightarrow} \text{Cat}_\infty.$$
By duality, we define double comonoids in duoidal $\infty$-categories.

**Definition 6.18.** Let $p : X \to \Delta \otimes \Delta$ be a duoidal $\infty$-category, that is, it is a categorical fibration which satisfies the conditions in Remark 5.24. We define a double comonoid in the duoidal $\infty$-category $X_{[1],[1]}$ to be a morphism $C : \text{id}_{\Delta \otimes \Delta} \to p$ in $\text{Duo}_{\infty}$. In other words, a double comonoid in $X_{[1],[1]}$ is a section $C$ of $p : X \to \Delta \otimes \Delta$:

$$
\begin{array}{ccc}
\Delta \otimes \Delta & \xrightarrow{\text{id}} & \Delta \otimes \Delta \\
\downarrow & & \downarrow \\
X & \xrightarrow{p} & \text{Duo}_{\infty}
\end{array}
$$

such that $C$ sends $\pi_{\otimes}$-Cartesian morphisms over inert morphisms of $\Delta_{\otimes}$ to $(\pi_{\otimes} \circ p)$-Cartesian morphisms, and that $\Delta_{\otimes} \simeq \{[n]\} \times \Delta \to X_{[n],[1]}$ is a coalgebra object of $X_{[n],[1]}$ for each $[n] \in \Delta_{\otimes}$.

Let $\text{Fun}_{\Delta \otimes \Delta_{\otimes}}(\Delta \otimes \Delta_{\otimes}, X)$ be the $\infty$-category of sections of $p : X \to \Delta \otimes \Delta$. We define an $\infty$-category

$$
\text{dcMon}(X)
$$

of $\infty$-categories.

We define a functor

$$
\text{cAlg}^{\otimes} : \text{Duo}_{\infty} \longrightarrow \text{Cat}_{\infty}
$$

by making the following diagram commute

$$
\begin{array}{ccc}
\text{Duo}_{\infty} & \xrightarrow{\text{cAlg}^{\otimes}} & \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty}) \\
(-)^p \downarrow & & \downarrow (-)^p \\
\text{Duo}_{\infty} & \xrightarrow{\text{Alg}^{\otimes}} & \text{Mon}^{\text{lax}}(\text{Cat}_{\infty})
\end{array}
$$

in $\widehat{\text{Cat}_{\infty}}$.

By the dual of Theorem 6.17, we obtain the following theorem.

**Theorem 6.19.** There is a natural isomorphism

$$
\text{dcMon}(X) \simeq (\text{cAlg} \circ \text{cAlg})^{\otimes}(X)
$$

of simplicial sets for any duoidal $\infty$-category $p : X \to \Delta \otimes \Delta$. The functor $\text{dcMon} : \text{Duo}_{\infty} \longrightarrow \text{Cat}_{\infty}$ is equivalent to the composite

$$
\begin{array}{ccc}
\text{Duo}_{\infty} & \xrightarrow{\text{cAlg}^{\otimes}} & \text{Mon}^{\text{oplax}}(\text{Cat}_{\infty}) \\
& \xrightarrow{\text{cAlg}} & \text{Cat}_{\infty}
\end{array}
$$

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