On certain products of algebraic groups over a finite field

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Abstract

Let \(G_1, \ldots, G_n\) be smooth connected and commutative algebraic groups over a finite field \(F\). We show that \(T(G_1, \ldots, G_n)(\text{Spec} F) = 0\) if \(n \geq 2\), where \(T(G_1, \ldots, G_n)\) is the reciprocity functor of Ivorra-Rülling associated with those algebraic groups as reciprocity functors. We also show that the finiteness of \((G_1 \otimes \cdots \otimes G_n)(\text{Spec} F)\) the tensor product of \(G_1, \ldots, G_n\) in the category of Mackey functors. We apply this to prove that, for a product of open curves, the finiteness of the relative Chow group and an abelian fundamental group which classifies abelian coverings with bounded ramification along the boundary.

1 Introduction

A Mackey functor over a perfect field \(F\) in the sense of \(\text{[5]}\) is a co- and contravariant functor from the category of étale schemes over \(F\) to the category of abelian groups. A smooth connected and commutative algebraic group \(G\) over the field \(F\) is regarded as a Mackey functor by the correspondence \(x \mapsto G(x)\). Such algebraic group \(G\) can be extended to a Nisnevich sheaf with transfers on the category of regular schemes over \(F\) with dimension \(\leq 1\). Furthermore, it satisfies the following condition which is the so-called reciprocity law (\(\text{[4]}, \text{Prop. 2.2.2}\)): For any open (=non-proper) regular connected curve \(G\) and a section \(a \in G(C)\) there exists an effective divisor \(D\) on the smooth compactification \(C\) of \(C\) with support in the boundary \(\overline{C} \setminus C\) such that

\[
\sum_{x \in C} v_x(f) \, \text{Tr}_{x/x} \, s_x(a) = 0
\]

for any \(f \neq 0\) in the function field \(F(C)\) of \(C\) such that \(f \equiv 1 \mod D\), that is, \(\text{div}(f - 1) \geq D\) as Weil divisors, where \(v_x\) is the valuation at \(x\), \(s_x : G(C) \to G(x)\) is the pull-back along the natural inclusion \(x \hookrightarrow C\) and \(\text{Tr}_{x/x} : G(x) \to G(x_C)\) is the push-forward along the finite map \(x \to x_C := \text{Spec} H^0(C, \mathcal{O}_C)\). F. Ivorra and K. Rülling [4] has introduced the notion of a reciprocity functor as a Nisnevich sheaf with transfers on the category of regular schemes over \(F\) with dimension \(\leq 1\) satisfying several axioms including the one like the reciprocity law as above. They have also introduced a “product” \(T(\mathcal{M}_1, \ldots, \mathcal{M}_n)\) associated to reciprocity functors \(\mathcal{M}_1, \ldots, \mathcal{M}_n\) in the quasi-abelian category of reciprocity functors (for the precise definition of the “product”, see [4] Def. 4.2.3). By the very construction of the product \(T\), as a Mackey functor \(T(\mathcal{M}_1, \ldots, \mathcal{M}_n)\) is a quotient of the tensor product \(\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n\) as Mackey functors (for the definition, see [2] in the next section). Hence we have a canonical surjection

\[
(\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n)(\text{Spec} F) \to T(\mathcal{M}_1, \ldots, \mathcal{M}_n)(\text{Spec} F).
\]

Although the tensor product \(\otimes\) gives a structure of a symmetric monoidal category in the abelian category of Mackey functors, it is not known that whether this product \(T\) satisfies the associativity and then gives a monoidal structure or not. However, this product coincides with the \(K\)-group of B. Kahn and T. Yamazaki [13] if we take homotopy invariant Nisnevich sheaves with transfers as reciprocity functors. In particular, we obtain an isomorphism

\[
T(G_1, \ldots, G_n)(\text{Spec} F) \simeq K(F; G_1, \ldots, G_n),
\]

for semi-abelian varieties \(G_1, \ldots, G_n\) over \(F\), where \(K(F; G_1, \ldots, G_n)\) is Somekawa’s \(K\)-group [13] which was limited on considering only semi-abelian varieties. For semi-abelian varieties \(G_1, \ldots, G_n\) over a finite
field $F$, B. Kahn in [4] showed that

$$(1) \quad K(F; G_1, \ldots, G_n) = (G_1 \otimes \cdots \otimes G_n)(\text{Spec } F) = 0$$

if $n > 1$. Because of the isomorphism ([13], Thm. 1.4)

$$K(F; G_m, \ldots, G_m) \cong K_n^M(F),$$

where $K_n^M(F)$ is the Milnor $K$-group of the field $F$, these results generalize the classical fact that $K_0^n(F) = 0$ if $F$ is a finite field $F$ and $n > 1$. For algebraic groups $G_1, G_2$ which may contain unipotent part, it is easy to see that $(G_1 \otimes G_2)(\text{Spec } F)$ may not be trivial. In this note, we shall show the following theorem.

**Theorem 1.1** (Thm. [2, 2, 2]). Let $G_1, \ldots, G_n$ be smooth commutative and connected algebraic groups over a finite field $F$ with characteristic $\neq 2$ for $n > 1$. Then we have

$$T(G_1, \ldots, G_n)(\text{Spec } F) = 0, \quad (G_1 \otimes \cdots \otimes G_n)(\text{Spec } F) \text{ is finite.}$$

As an application of [1], the class field theory of a product of projective smooth curves over a finite field, a special case of the higher dimensional class field theory of S. Bloch, K. Kato and S. Saito (e.g., [8]) is reduced from the classical (unramified) class field theory (= class field theory of curves over a finite field) and Lang’s theorem; the reciprocity map on a normal variety over a finite field has dense image. In Section 3 we will pursue related results on the (ramified) class field theory of a product of open (=non-proper) curves as a byproduct of the above theorem. In particular, we obtain a finiteness of the relative Chow group $\text{CH}_0(X, D)$ for a product of smooth curves $X = X_1 \times \cdots \times X_n$ over a finite field and an effective divisor $D$ on the smooth compactification $\overline{X}$ of $X$ with support in $\overline{X} \setminus X$ (Thm. 3.1).

Throughout this note, we mean by an **algebraic group** a smooth connected and commutative group scheme over a field. For a field $F$, we denote by $\text{char}(F)$ the characteristic of $F$.

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## 2 Finiteness

First we recall the definition and some properties of the product $\otimes$ in the category of Mackey functors and those of the product of reciprocity functors following [1].

Let $F$ be a perfect field. We call a morphism $x \to \text{Spec } F$ a **finite point** if $x = \text{Spec } E$ for some finite field extension of $F$. A Mackey functor $M$ over $F$ in the sense of [5] is determined by its value $M(x)$ on finite points $x \to \text{Spec } F$. For Mackey functors $M_1, \ldots, M_n$, the product $M_1 \otimes \cdots \otimes M_n$ called the **Mackey product** is defined as follows. For any finite point $x \to \text{Spec } F$,

$$(2) \quad (M_1 \otimes \cdots \otimes M_n)(x) := \left( \bigoplus_{y \to x: \text{finite}} \left( M_1(y) \otimes \cdots \otimes M_n(y) \right) \right) / \text{R}(x)$$

where $y \to x$ runs all finite points over $x$, and $\text{R}(x)$ is the subgroup generated by elements of the following form: For any morphism $j : y' \to y$ of finite points over $x$, and if $a_{i_0}' \in M_{i_0}(y')$ and $a_i \in M_i(y)$ for $i \neq i_0$, then

$$(3) \quad j^*(a_1) \otimes \cdots \otimes a_{i_0}' \otimes \cdots \otimes j^*(a_n) = a_1 \otimes \cdots \otimes j_* (a_{i_0}') \otimes \cdots \otimes a_n \in \text{R}(x),$$

where $j^*$ and $j_*$ are the pull-back and the push-forward along $j$ respectively. We write $\{a_1, \ldots, a_n\}_{y/x}$ for the image of $a_1 \otimes \cdots \otimes a_n \in M_1(y) \otimes \cdots \otimes M_n(y)$ in the product $(M_1 \otimes \cdots \otimes M_n)(x)$. Using this symbol, the above relation (3) defining $\text{R}(x)$ above gives the following equation which is often called the **projection formula**:

$$(4) \quad \{j^*(a_1), \ldots, a_{i_0}', \ldots, j^*(a_n)\}_{y'/x} = \{a_1, \ldots, j_*(a_{i_0}'), \ldots, a_n\}_{y/x}.$$
The Mackey product $M \otimes$ gives a tensor product in the abelian category of the Mackey functors. Its unit is the constant Mackey functor $\mathbb{Z}$. In particular, the product commutes with the direct sum $\oplus$ and satisfies the associativity: $M_1 \otimes M_2 \otimes M_3 \simeq (M_1 \otimes M_2) \otimes M_3$.

The product $- \otimes M$ is right exact for any Mackey functor $M$.

For any finite point $j : x' \to x$, the push-forward $j_* : (M_1 \otimes \cdots \otimes M_n)(x') \to (M_1 \otimes \cdots \otimes M_n)(x)$ along $j$ is given by $j_* \{a_1, \ldots, a_n\}_{x'/x} = \{a_1, \ldots, a_n\}_{x/x}$ on symbols.

Lemma 2.1. Let $G$ be a unipotent (smooth and commutative) algebraic group over $F$ and $A$ a semi-abelian variety over $F$. If $F$ is a perfect field of $\text{char}(F) = p > 0$, we have $G^M \otimes A = 0$.

Proof. The unipotent group $G$ has a composition series:

$$0 = G^r \subset \cdots \subset G^1 \subset G,$$

each $G^i/G^{i+1}$ being isomorphic to $\mathbb{G}_a$. By the right exactness (M2), it is enough to show $(G^M \otimes A) = 0$. By (M3) above, the assertion is reduced to showing $\{a, b\}_{x/x} = 0$ for any $a \in G_a(x), b \in A(x)$. There exists a finite point $j : x' \to x$ such that $j^*(b) = pb'$ for some $b' \in A(x')$. Since the trace map ($= \text{the push-forward map on } G_a$) $j_* = \text{Tr}_{x'/x} : G_a(x') \to G_a(x)$ is surjective, we obtain

$$\{a, b\}_{x/x} = \{\text{Tr}_{x'/x}(a'), b\}_{x/x} \quad \text{for some } a' \in G_a(x'),$$

$$\{a', j^*b\}_{x'/x} \quad \text{by the projection formula},$$

$$\{a', pb'\}_{x/x}$$

$$= 0.$$

The assertion follows from this. \hfill \Box

As noted before, a reciprocity functor is a Nisnevich sheaf with transfers on the category of regular schemes with dimension $\leq 1$ satisfying several axioms ([3], Def. 1.5.1). As examples, a constant Nisnevich sheaf, algebraic groups, the Milnor $K$-theory $K^n_0$, the higher Chow group $\mathcal{C}H_0(X, n)$ for some scheme $X$, Suslin’s singular homology group $h_0(X)$ and the sheaf of the absolute Kähler differentials $\Omega^n$ given by $X \to \Omega^n_{X/\mathbb{Z}}$ are reciprocity functors ([3], Sect. 2). Note that any reciprocity functor gives a Mackey functor by restricting to the category of points over $F$. It is known that the category of reciprocity functors forms a quasi-abelian category, especially, an exact category. Hence the notion of an admissible exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ and a right exact functor are defined in the category of reciprocity functors. The “product” $T(\mathcal{M}_1, \ldots, \mathcal{M}_n)$ for reciprocity functors $\mathcal{M}_1, \ldots, \mathcal{M}_n$ is a reciprocity functor and satisfies some functorial properties.

(R1) There are functorial isomorphisms

$$T(\mathcal{M}_1, \ldots, \mathcal{M}_i, \ldots, \mathcal{M}_j, \ldots, \mathcal{M}_n) \simeq T(\mathcal{M}_1, \ldots, \mathcal{M}_j, \ldots, \mathcal{M}_i, \ldots, \mathcal{M}_n)$$

and

$$T(\mathcal{M}_1, \ldots, \mathcal{M}_i \oplus \mathcal{M}'_i, \ldots, \mathcal{M}_n) \simeq T(\mathcal{M}_1, \ldots, \mathcal{M}_i, \ldots, \mathcal{M}_n) \oplus T(\mathcal{M}_1, \ldots, \mathcal{M}'_i, \ldots, \mathcal{M}_n).$$

There is an admissible epimorphism

$$T(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) \to T(T(\mathcal{M}_1, \mathcal{M}_2), \mathcal{M}_3).$$

However, it is not known whether this map becomes an isomorphism.

(R2) The functor $T(\mathcal{M}_1, \ldots, \mathcal{M}_{n-1}, -)$ is right exact ([3], Cor. 4.2.9).

(R3) Assume $\text{char}(F) \neq 2$. We have $T(\mathcal{M}_3, \ldots, \mathcal{M}_n) = 0$ if two of $\mathcal{M}_i$’s are unipotent algebraic groups over $F$ ([3], Thm. 5.5.1).
By the very construction, for any finite point \( x \) over \( F \), the product \( T(\mathcal{M}_1, \ldots, \mathcal{M}_n)(x) \) evaluated at \( x \) is a quotient of the Mackey product \( (\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n)(x) \). More precisely, the product \( T(\mathcal{M}_1, \ldots, \mathcal{M}_n) \) is defined to be the Nisnevich sheafification \( \mathcal{L}_\infty^\ast \) of a quotient \( \mathcal{L} \) of the product \( \mathcal{L} := \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n \) whose underlying Mackey functor is the Mackey product \( \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n \). However, an isomorphism \( \mathcal{L}_\infty^\ast (x) \xrightarrow{\sim} \mathcal{L}_N^\ast (x) \) exists since any Nisnevich covering of \( \text{Spec} F \) refines a trivial covering.

Next we show the following theorem a part of the main theorems:

**Theorem 2.2.** Let \( F \) be a finite field with \( \text{char}(F) \neq 2 \) and \( \mathcal{M}_1, \ldots, \mathcal{M}_n \) reciprocity functors over \( F \). If two of \( \mathcal{M}_i \)'s are (non-trivial) algebraic groups, then the underlying Mackey functor of \( T(\mathcal{M}_1, \ldots, \mathcal{M}_n) \) is trivial. In other words, \( T(\mathcal{M}_1, \ldots, \mathcal{M}_n)(x) = 0 \) for any finite point \( x \to \text{Spec} F \).

**Proof.** Here we show \( T(G_1, G_2, \mathcal{M}) = 0 \) for any algebraic groups \( G_1, G_2 \) and a reciprocity functor \( \mathcal{M} \). The proof of the assertion \( T(G_1, G_2, \mathcal{M}_1, \ldots, \mathcal{M}_n) = 0 \) for \( n > 1 \) is exactly same as in the case of \( n = 1 \). (Note also that since the product of the reciprocity functors does not satisfy the associativity \( (R1) \), we cannot reduce the assertion to \( n = 2 \)). The algebraic group \( G_i \) has a decomposition

\[
0 \to U_i \to G_i \to A_i \to 0,
\]

where \( U_i \) is a unipotent algebraic group and \( A_i \) is a semi-abelian variety. The short exact sequence gives an admissible exact sequence in the category of reciprocity functors ([4], Lem. 3.2.12). From the right exactness \( (R2) \) (and \( (R1) \)), we obtain the following admissible exact sequences

\[
\begin{align*}
T(U_1, U_2, \mathcal{M}) &\longrightarrow T(A_1, U_2, \mathcal{M}) \\
T(U_1, G_2, \mathcal{M}) &\longrightarrow T(G_1, G_2, \mathcal{M}) \\
T(U_1, A_2, \mathcal{M}) &\longrightarrow T(A_1, A_2, \mathcal{M}) \\
0 &\longrightarrow 0
\end{align*}
\]

By \( (R3) \), we obtain \( T(U_1, U_2, \mathcal{M}) = 0 \) as reciprocity functors. Since there is the surjection \( (A_1 \otimes A_2 \otimes \mathcal{M}) \to T(A_1, A_2, \mathcal{M}) \) of Mackey functors, and the canonical isomorphism \( A_1 \otimes A_2 \otimes \mathcal{M} \simeq (A_1 \otimes A_2) \otimes \mathcal{M} \), we obtain \( T(A_1, A_2, \mathcal{M}) = 0 \) as a Mackey functor by Kahn’s theorem ([1]). Similarly, Lemma 2.1 implies \( (U_1 \otimes A_2 \otimes \mathcal{M}) = T(U_1, A_2, \mathcal{M}) = 0 \) and \( (A_1 \otimes U_2 \otimes \mathcal{M}) = T(A_1, U_2, \mathcal{M}) = 0 \). The assertion \( T(G_1, G_2, \mathcal{M}) = 0 \) follows from the above diagram.

**Corollary 2.3.** Let \( \mathcal{M}_1, \ldots, \mathcal{M}_m \) be reciprocity functors over a finite field \( F \).

(i) \( T(\mathcal{K}_n^M, \mathcal{M}_1, \ldots, \mathcal{M}_m) = 0 \) as a Mackey functor if \( n \geq 2 \).

(ii) \( T(\Omega^n, \mathcal{M}_1, \ldots, \mathcal{M}_m) = 0 \) as a Mackey functor if \( n \geq 1 \).

(iii) Let \( X \) be a quasi-projective smooth and geometrically connected variety over \( F \). Assume that there is a smooth projective and connected variety \( \overline{X} \) over the algebraically closed field \( F \) containing \( X \) as an open subscheme. Then we have \( T(h_0(X)^0, \mathcal{M}_1, \ldots, \mathcal{M}_m) = 0 \) as a Mackey functor if one of \( \mathcal{M}_i \)'s is an algebraic group, where \( h_0(X)^0 \) is the kernel of the degree map \( h_0(X) \to h_0(\text{Spec} F) \simeq \mathbb{Z} \) in the category of reciprocity functors.

**Proof.** By the computations of the products ([1], Sect. 5) we have \( T(\mathbb{G}_m^\times \times \mathbb{G}_m^\times) \simeq \mathcal{K}_n^M \) as Mackey functors and \( T(\mathbb{G}_a, \mathbb{G}_m^\times \times \mathbb{G}_m^\times) \simeq \Omega^n \) as reciprocity functors. Here we used the notation \( T(\mathcal{M}, \mathcal{N}^\times \times ...) := T(\mathcal{M}, \mathcal{N}, \ldots) \) for some reciprocity functors \( \mathcal{M} \) and \( \mathcal{N} \). For any finite point \( x \), there are surjections

\[
((\mathbb{G}_m)^M \otimes \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_m)(x) \to (T(\mathbb{G}_m^\times \times \mathbb{G}_m^\times) \otimes \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_m)(x) \to T(\mathcal{K}_n^M, \mathcal{M}_1, \ldots, \mathcal{M}_m)(x).
\]

The assertion (i) follows from ([1]). The proof of (ii) is same as (i) (by using Lemma 2.1 instead of ([1])). For (iii), we show only \( T(h_0(X)^0, G)(x) = 0 \) for \( x = \text{Spec} F \) and an algebraic group \( G \) over \( F \). The general case
can be proved by the same way. The proof is essentially same as in the one of Proposition 3.7 in [1]. To show the assertion, it is enough to show that the image \( \{a, b\}_{x/x} \) of \( \{a, b\}_{x/x} \) in \( T(h_0(X)^0), G(x) \) is trivial. The element \( a \in h_0(X)(x) = h_0(X) \) is represented by a 0-cycle \( \sum_{i=1}^{n} n_i x_i \) (\( n_i \neq 0 \)) on \( X \). By the Bertini-type theorem of Poonen, there exists a smooth projective curve \( C \subset X \) such that \( C \) contains \( x_i \) for all \( i \). Denoting \( C := C \cap X \), \( \{a, b\}_{x/x} \) is in the image of the canonical map \( T(h_0(C)^0), G(x) \to T(h_0(X)^0), G(x) \) induced by the inclusion \( C \to X \). Since \( h_0(C)^0 \) is isomorphic to the generalized Jacobian variety \( J_C \) of Rosenlicht which is a semi-abelian variety, we obtain \( \{a, b\}_{x/x} = 0 \) from Theorem 2.2. (The field \( F \) may have char(\( F \)) = 2, since we do not need (R3) to show the assertion.)

Finally, we consider the Mackey product of algebraic groups over a finite field.

**Theorem 2.4.** Let \( G_1, \ldots, G_n \) be algebraic groups over a finite field \( F \). Then the group \( (G_1 \otimes \cdots \otimes G_n)(x) \) is finite for any finite point \( x \) over \( F \).

**Proof.** The algebraic group \( G_i \) has a decomposition

\[
0 \to U_i \to G_i \to A_i \to 0
\]

with unipotent part \( U_i \) and a semi-abelian variety \( A_i \) over \( F \). Replacing the product of reciprocity functors by the Mackey products in (6), Kahn’s theorem (1) and Lemma 2.1 gives a surjection \( (U_1 \otimes \cdots \otimes U_n)(x) \to (G_1 \otimes \cdots \otimes G_n)(x) \). Thus it is enough to show the assertion for \( G_i = U_i \) for all \( i \). A composition series of the unipotent group \( G := G_i \):

\[
0 = G^r \subset \cdots \subset G^3 \subset G,
\]

each \( G^i/G^{i+1} \cong \mathbb{G}_a \). By the right exactness (M2), we may assume \( G_i = \mathbb{G}_a \) for all \( i \) and \( x = \text{Spec } F \) without loss of generality. We show the finiteness of \( (\mathbb{G}_a)^{\otimes n}(x) \) by induction on \( n \). The case of \( n = 1 \), there is nothing to show. For \( n > 1 \), we assume that \( (\mathbb{G}_a)^{\otimes (n-1)}(x) \) is finite. The group \( (\mathbb{G}_a)^{\otimes n}(x) \) has a structure of an \( F \)-vector space given by \( a(a_1, \ldots, a_n)_{x/x} := \{j^*(a)_{a_1, \ldots, a_n}_{x'_{/x}}\} \) for \( a \in F \) and a symbol \( \{a_1, \ldots, a_n\}_{x/x} \) on a finite point \( j : x' \to x = \text{Spec } F \). Consider a subspace \( I(x) \) of \( (\mathbb{G}_a)^{\otimes n}(x) \) generated by the elements of the form

\[
\{1, a_2, \ldots, a_n\}_{x'/x} - \{a_2 \cdots a_n, 1, \ldots, 1\}_{x'/x}.
\]

By identifying the canonical isomorphism \( (\mathbb{G}_a)^{\otimes n} \cong \mathbb{G}_a \otimes (\mathbb{G}_a)^{\otimes (n-1)} \) by (M1) and \( j^*(1) = 1 \in \mathbb{G}_a(x) \), we have

\[
\{1, a_2, \ldots, a_n\}_{x'/x} = \{j^*(1), \{a_2, \ldots, a_n\}_{x'/x}\}_{x'/x} = \{1, j^*(a_2, \ldots, a_n)_{x'/x}\}_{x/x} \quad \text{by the projection formula}
\]

\[
= \{1, a_2, \ldots, a_n\}_{x'/x} \quad \text{by (M3)}.
\]

By the induction hypothesis, the elements of this form are finite. On the other hand, the projection formula implies

\[
\{a_2 \cdots a_n, 1, \ldots, 1\}_{x'/x} = \{j_*(a_2 \cdots a_n), 1, \ldots, 1\}_{x/x}.
\]

Thus the subspace \( I(x) \) is finite. Next we consider a subspace \( S(x) \) of the quotient \( Q(x) := (\mathbb{G}_a)^{\otimes n}(x)/I(x) \) generated by symbols of the form \( \{a_1, \ldots, a_n\}_{x/x} \). Here we denote by \( \{a_1, \ldots, a_n\}_{x'/x} \) the image of \( \{a_1, \ldots, a_n\}_{x'/x} \) in the quotient \( Q(x) \). It is easy to see that the subspace \( S(x) \) is finite. For any symbol \( \{a_1, \ldots, a_n\}_{x/x} \) in \( Q(x) \) on a finite point \( j : x' \to x \), we have

\[
\{a_1, \ldots, a_n\}_{x/x} = j_*(\{a_1, \ldots, a_n\}_{x'/x}) \quad \text{by (M3)}
\]

\[
= j_*(a_1, a_2, \ldots, a_n)_{x'/x'} \quad \text{because of } \{a_1, \ldots, a_n\}_{x'/x} \in Q(x')
\]

\[
= \{a_1 \cdots a_n, 1, \ldots, 1\}_{x/x} \quad \text{by (M3)}
\]

\[
= \{j_*(a_1 \cdots a_n), 1, \ldots, 1\}_{x/x} \quad \text{by the projection formula}.
\]

Thus we obtain \( Q(x) = S(x) \) and the assertion follows from it.

\[\square\]
3 Applications

Let $X$ be a smooth (and connected) variety over a finite field $F$. Assume that there is a smooth compactification $\overline{X}$ of $X$, that is, a projective smooth variety which contains $X$ as an open subscheme. Let $D$ be an effective divisor on $\overline{X}$ with support in $\overline{X} \setminus X$. To consider the ramification along the divisor on $D$ here we recall the relative Chow group $\text{CH}_0(X,D)$ of the pair $(X,D)$ (cf. [2], Sect. 8.1, see also [10], Sect. 3.4 and 3.5). Define

$$\text{CH}_0(X,D) := \text{Coker} \left( \text{div} : \bigoplus_{\phi : C \to X} P_C(\phi^*D) \to Z_0(X) \right),$$

where the direct sum runs over the normalization $\phi : C \to X$ of a curve in $X$, $Z_0(X)$ is the group of 0-cycles on $X$, $\overline{C} : \to \overline{X}$ is the extension of the map $\phi$ to the smooth compactification $\overline{C}$ of $C$, the map div is given by the divisor map on each curve $C$ and

$$P_C(\phi^*D) := \{ f \in F(C)^\times \mid f \equiv 1 \mod \phi^*D + (\overline{C} \setminus C)_{\text{red}} \}.$$  

Putting $X_\phi := X \times_{\text{Spec } F} x$ and denoting by $D_x$ the pull-back of $D$ to $X_x := \overline{X} \times_{\text{Spec } F} x$ for any finite point $x \to \text{Spec } F$, the assignment

$$\mathcal{H}_0(X,D) : x \mapsto \text{CH}_0(X_\phi,D_x)$$

gives a Mackey functor $\mathcal{H}_0(X,D)$. As an application of the main theorem, we show the following finiteness theorem of the relative Chow group:

**Theorem 3.1.** Let $X_1, \ldots, X_n$ be smooth and connected curves over a finite field $F$ with $X_i(F) \neq \emptyset$ and put $X := X_1 \times \cdots \times X_n$. For an effective divisor $D$ on $X := \overline{X}_1 \times \cdots \times X_n$ with support in $\overline{X} \setminus X$, the kernel of the degree map $\text{CH}_0(X,D)^0 := \text{Ker}(\text{deg} : \text{CH}_0(X,D) \to \mathbb{Z})$ is finite.

**Proof.** We show the case $n = 2$ (the proof is same for $n > 2$). For sufficiently large divisors $D_i$ on $\overline{X}_i$ for $i = 1,2$, there is a surjection $\text{CH}_0(X,D_1 \times X_2 + X_1 \times D_2) \to \text{CH}_0(X,D)$. Thus we may assume that $D$ is a divisor of the form $D_1 \times X_2 + X_1 \times D_2$. Now we consider the map

$$\psi : (\mathcal{H}_0(X_1,D_1) \otimes \mathcal{H}_0(X_2,D_2))(\text{Spec } F) \to \text{CH}_0(X,D)$$

deфинищь by $\{a_1,a_2\} \times_{\text{Spec } F} \mapsto (j_x)_*(p_1^*a_1 \cap p_2^*a_2)$, where $p_i : \overline{X} = \overline{X}_1 \times \overline{X}_2 \to \overline{X}_i$ is the projection, $j_x : X_\phi \to X$ is given by the base change to $x$ and $\cap$ is the internal product defined similarly to the ordinal Chow group of 0-cycles. We show that the map $\psi$ is surjective. Take a cycle $[x]$ as a generator of $\text{CH}_0(X,D)$ which is represented by a closed point $x$ on $X$ and is a finite point $j_x : x \to \text{Spec } F$. By the definition of $\psi$, the push-forward map on the relative Chow group and the norm map on the Mackey product are compatible as in the following commutative diagram:

$$\begin{array}{ccc}
(\mathcal{H}_0(X_1,D_1) \otimes \mathcal{H}_0(X_2,D_2))(x) & \xrightarrow{\psi_x} & \text{CH}_0(X_\phi,D_x) \\
(j_x)_* \downarrow & & \downarrow (j_x)_* \\
(\mathcal{H}_0(X_1,D_1) \otimes \mathcal{H}_0(X_2,D_2))(\text{Spec } F) & \xrightarrow{\psi} & \text{CH}_0(X,D).
\end{array}$$

Thus to show the surjectivity of $\psi$ we may assume that $x$ is an $F$-rational point. The point $x$ is determined by maps $x_i \to X_i$. These points give $\psi([x_1],[x_2]) = [x]$ and thus $\psi$ is surjective.

From the assumption $X_i(F) \neq \emptyset$, there exists a decomposition $\mathcal{H}_0(X_1,D_1) \simeq \mathbb{Z} \oplus J_{X_1,D_1}$ by the generalized Jacobian variety $J_{X_i,D_i}$ of the pair $(X_i,D_i)$ ([12]). According to this decomposition we obtain

$$\mathcal{H}_0(X_1,D_1) \otimes \mathcal{H}_0(X_2,D_2) \simeq \mathbb{Z} \oplus J_{X_1,D_1} \oplus J_{X_2,D_2} \oplus (J_{X_1,D_1} \otimes J_{X_2,D_2})$$

by (M1). From the product (7), there exists a surjection

$$J_{X_1,D_1}(F) \oplus J_{X_2,D_2}(F) \oplus (J_{X_1,D_1} \otimes J_{X_2,D_2})(\text{Spec } F) \twoheadrightarrow \text{CH}_0(X,D)^0.$$

The left is finite by Theorem [2] and so is $\text{CH}_0(X,D)^0$. \qed
Let $X$ be the product of curves over $F$, and $D$ as in the above theorem (Thm. 3.1). For each normalization $\phi : C \to X$ of a curve in $X$, we have a divisor $D_C := \phi (D) + (\overline{C} \smallsetminus C)_{\text{red}}$ on $C$. The category of étale coverings of $X$ with ramification bounded by the collection of divisors $(D_C)_{\phi : C \to X}$ forms a Galois category and gives a fundamental group $\pi_1(X, D)$ (8, Lem. 3.3). For each such $C$ and a point $x \in \overline{C} \smallsetminus C$ there is the canonical map $G_{C, x}^{ab} := \text{Gal}(F(C)_x^{ab}/F(C)_x) \to \pi_1(X)^{ab}$, where $F(C)_x^{ab}$ is the maximal abelian extension of the completion $F(C)_x$ at $x$. By the very definition of the coverings, we have

$$\text{Coker} \left( \bigoplus_{\phi : C \to X} G_{C, x}^{ab, m_x(D_C)} \to \pi_1(X)^{ab} \right) \to \pi_1(X, D)^{ab},$$

where $G_{C, x}^{ab, m}$ is the $m$-th (upper numbering) ramification subgroup of $G_{C, x}^{ab}$ (11, Chap. IV, Sect. 3) and $m_x(D_C)$ is the multiplication of the divisor $D_C$ at $x$. Using the idele theoretic description of the relative Chow group

$$\text{CH}_0(X, D) \simeq \text{Coker} \left( \bigoplus_{\phi : C \to X} F(C)^X \to Z_0(X) \oplus \bigoplus_{\phi : C \to X} F(C)_x^X/U^{ab, m_x(C, X)} \right),$$

where $U^{m_x(C, X)} = 1 + m_x^{D_C}$ is the higher unit group of $F(C)_x$, local class field theory (see e.g., [11], Chap. XV) induces the following commutative diagram:

$$\begin{array}{ccc}
Z_0(X)^0 & \longrightarrow & \text{CH}_0(X, D)^0 \\
\rho \downarrow & & \downarrow \rho_D \\
\pi_1(X)^{ab, 0} & \longrightarrow & \pi_1(X, D)^{ab, 0}.
\end{array}$$

Here $Z_0(X)^0$ is the kernel of the degree map $Z_0(X) \to \mathbb{Z}$, the left vertical map $\rho$ is the reciprocity map on $X$ and $\pi_1(X, D)^{ab, 0}$ is the geometric part of the abelian fundamental group (= the kernel of the canonical map $\pi_1(X, D)^{ab} \to \pi_1(\text{Spec}(F))^{ab}$). The image of the left reciprocity map $\rho : Z_0(X) \to \pi_1(X)^{ab}$ is known to be dense (due to Lang [9]) and the image of $\rho_D$ is finite by Theorem 3.1. Therefore, the map $\rho_D$ is surjective and we obtain the following finiteness result.

**Corollary 3.2.** Let $X$ and $D$ be as in Theorem 3.1. Then, the geometric part of the abelian fundamental group $\pi_1(X, D)^{ab, 0}$ is finite.

We conclude this note by referring to recent results in [2]. H. Esnault and M. Kerz showed the finiteness of the group $\text{CH}_0(X, D)^0$ for a smooth variety $X$ over a finite field and an effective Cartier divisor $D$ with support in the boundary $\overline{X} \smallsetminus X$ only assuming the existence of some normal compactification $\overline{X}$ of $X$ (2, Thm. 8.1). This is an application of Deligne’s finiteness theorem for $l$-adic Galois representations of function fields. Using this finiteness theorem instead of Theorem 3.1 in the above arguments, we also obtain the finiteness of $\pi_1(X, D)^{ab, 0}$ for the variety $X$.

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