A model of morphogen transport in the presence of glypicans II

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Abstract

A model of morphogen transport consisting of two evolutionary PDEs of reaction-diffusion type and three ODEs posed on a rectangular domain \((-L,L) \times (0,\epsilon H)\) is analysed. We prove that the problem is globally well-posed and that the corresponding solutions converge as \(\epsilon \to 0^+\) to the unique solution of the one dimensional (i.e. posed on \((-L,L)\)) system which was analysed in the first part of the paper [15]. Main difficulties in the analysis stem from the presence of a singular source term - a Dirac Delta combined with no smoothing effect in the ODE part of the system.

AMS classification 35B40, 35Q92.

Keywords dimension reduction, morphogen transport, reaction-diffusion equations.

1 Introduction

According to the French Flag Model, created in the late sixties by Lewis Wolpert (see [22]), morphogen are molecules which due to mechanism of positional signalling govern the fate of cells in living organisms. It has been observed that certain proteins (and other substances) after being secreted from a source, typically a group of cells, spread through the tissue and after a certain amount of time form a stable gradient of concentration. Next receptors located on the surfaces of the cells detect levels of morphogen concentration and transmit these information to the nucleus. This leads to the activation of appropriate genes, synthesis of proteins and finally differentiation of cells.

Although the role of morphogen in cell differentiation, as described above, is commonly accepted there is still discussion regarding the exact kinetic mechanism of the movement of morphogen molecules and the role of reactions of morphogen with receptors in forming the gradient of concentration (see [3], [8], [9]). To determine the mechanism of morphogen transport, several mathematical models consisting of systems of semilinear parabolic PDEs of reaction diffusion-type coupled with ODEs were recently proposed and analysed (see [6], [7], [20], [14], [19]).

In this series of papers we analyse model [HKCS] from [5] which describes the formation of the gradient of morphogen Wingless (Wg) in the imaginal wing disc of the Drosophila Melanogaster individual. Model [HKCS] has two counterparts - one and two dimensional, depending on the dimensionality of the domain representing the imaginal wing disc. We denote these models [HKCS].1D and [HKCS].2D respectively. In mathematical terms [HKCS].1D is a system of two semilinear parabolic PDEs of reaction diffusion type coupled with three nonlinear ODEs posed on the interval \(I^L = (−L,L)\), while [HKCS].2D consists of a linear parabolic PDE posed on rectangle \(\Omega^{L,H} = (−L,L) \times (0,H)\) which is coupled via nonlinear boundary condition on \(\partial_1 \Omega^{L,H} = (−L,L) \times \{0\}\) with a semilinear parabolic PDE and three ODEs.

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In [15] we have shown that [HKCS].1D is globally well posed and has a unique stationary solution. In this paper we turn our attention to the analysis of the [HKCS].2D model. Using analytic semigroup theory we prove its global well-posedness in appropriately chosen function setting and justify rigourously that [HKCS].1D can be derived from [HKCS].2D through "ironing of the wing disc" - i.e. dimension reduction of the domain in the direction perpendicular to the surface of the wing disc. The main analytic problem which we have to overcome stems from two factors: the lack of smoothing effect in the ODEs and the presence of a point source term (a Dirac Delta) in the boundary condition for the equation posed on \((-L, L) \times (0, H)\), which causes the solution to be unbounded for every \(t > 0\).

Stationary problem for the [HKCS].2D is analysed in [16].

1.1 The [HKCS].2D model.

In this section we present the model [HKCS].2D - a two dimensional counterpart of the [HKCS] model introduced in [5]. For the presentation and analysis of [HKCS].1D - a one dimensional counterpart we refer to [15].

For \(L, H > 0\), \(\infty \geq T > 0\), denote

\[
I^L = (-L, L), \quad I^1 = (-1, 1)
\]

\[
\Omega^{L,H} = (-L, L) \times (0, H), \quad \partial \Omega^{L,H} = \partial_0 \Omega^{L,H} \cup \partial_1 \Omega^{L,H}, \quad \partial_1 \Omega^{L,H} = (-L, L) \times \{0\},
\]

\[
\Omega^L_T = (0, T) \times \Omega^{L,H}, \quad (\partial \Omega^{L,H})_T = (0, T) \times \partial \Omega^{L,H}, \quad \Omega = \Omega^{1,1}.
\]

The domain \(\Omega^{L,H}\) represents the imaginal wing disc of the Drosophila Melanogaster individual and the \(x_2\) direction corresponds to the thickness of the disc, so that in practice \(H \ll L\). Let \(\nu\) denote a unit outer normal vector to \(\partial \Omega^{L,H}\) and let \(\delta\) be a one dimensional Dirac Delta located at \(x = 0\) (that is \(\delta(\phi) = \phi(0)\) for any \(\phi \in C([-L, L])\)).

![Graph of the domain \(\Omega^{L,H}\).

Figure 1: Graph of the domain \(\Omega^{L,H}\). The arrow pointing towards the rectangle represents a point source of the morphogen (a Dirac Delta) on the boundary.]

[HKCS].2D is a system which consists of one evolutionary PDE posed on \(\Omega^{L,H}\), one evolutionary PDE and 3 ODE’s posed on \(\partial_1 \Omega^{L,H}\):

\[
\begin{align*}
\partial_t W - D \Delta W &= -\gamma W, & (t, x) \in \Omega_T^{L,H} \\
\partial_t W^* - D^* \partial_2^2 W^* &= -\gamma^* W^* + [k GW - k' W^*] - [k_{Rg} RW^* - k_{Rg} R^*_g], & (t, x) \in (\partial \Omega^{L,H})_T \\
\partial_t R &= -[k_R RW - k'_R R^*] - [k_{Rg} RW^* - k'_{Rg} R^*_g] - \alpha R + \Gamma, & (t, x) \in (\partial \Omega^{L,H})_T \\
\partial_t R^*_e &= [k_R RW - k'_R R^*_e] - \alpha^* R^*, & (t, x) \in (\partial \Omega^{L,H})_T \\
\partial_t R^*_g &= [k_{Rg} RW^* - k'_{Rg} R^*_g] - \alpha^* R^*_g, & (t, x) \in (\partial \Omega^{L,H})_T
\end{align*}
\]
supplemented by the boundary conditions:

\[
D \nabla W \nu = 0, \quad (t, x) \in (\partial_0 \Omega^{L,H})_T \tag{2a}
\]

\[
D \nabla W \nu = -[kGW - k'W^s] - [k_R RW - k'_R R^s] + s \delta, \quad (t, x) \in (\partial_1 \Omega^{L,H})_T \tag{2b}
\]

\[
\partial_{x_1} W^s = 0, \quad (t, x) \in (\partial \Omega^{L,H})_T \tag{2c}
\]

and initial conditions:

\[
W(0) = W_0, \quad x \in \Omega^{L,H} \tag{3a}
\]

\[
W^s(0) = W^s_0, \quad R(0) = R_0, \quad R^s(0) = R^s_0, \quad x \in \partial_1 \Omega^{L,H} \tag{3b}
\]

In [1],[2],[3] \( W, G, R \) denote concentrations of free morphogens \( W_g \), free glypicans \( D_l \) and free receptors, \( W^*, R^* \) denote concentrations of morphogen-glypican and morphogen-receptor complexes, \( R_g^s \) denotes concentration of morphogen-glypican-receptor complexes. It is assumed that \( W \) is located on \( \Omega^{L,H} \), while other substances are present only on \( \partial_1 \Omega^{L,H} \). The model takes into account association-dissociation mechanism of

- \( W \) and \( G \) with rates \( k, k' (kGW - k'W^s) \)
- \( W \) and \( R \) with rates \( k_R, k'_R (k_RW - k'_RR^s) \)
- \( W^* \) and \( R \) with rates \( k_{Rg}, k'_{Rg} (k_{Rg}RW^* - k'_{Rg}R_g^s) \)

Other terms of the system account for

- Diffusion of \( W \) in \( \Omega^{L,H} \) (resp. \( W^* \) on \( \partial_1 \Omega^{L,H} \)) with rate \( D \) (resp \( D^* \)): \(-D \Delta W \) (resp. \(-D^* \partial_{x_1}^2 W^s\)).
- Degradation of \( W \) in \( \Omega^{L,H} \) (resp. \( W^* \) on \( \partial_1 \Omega^{L,H} \)) with rate \( \gamma \) (resp. \( \gamma^* \)): \(-\gamma W \) (resp. \(-\gamma^* W^s\)).
- Internalisation (endocytosis) of \( R \) (resp. \( R^*, R_g^s \)) with rate \( \alpha \) (resp. \( \alpha^*, \alpha^*_g \)): \(-\alpha R \) (resp. \(-\alpha^* R^*, -\alpha^*_g R_g^s\)).
- Secretion of \( W \) with rate \( s \) from the source localised at the boundary point \( x = 0 \in \partial_1 \Omega^{L,H} \): \(s \delta\).
- Production of \( R \): \( \Gamma \).

For simplicity we assume that \( G \) and \( \Gamma \) are given and constant (in time and space).

In order to analyse the reduction of the dimension of the domain we introduce for \( \epsilon > 0 \) the [HKCS].(2D,\( \epsilon \)) model, which is obtained from [HKCS].2D by changing \( \Omega^{L,H} \) into \( \Omega^{L,\epsilon H} \) and rescaling the source term for \( W \) in the boundary conditions [2]:

[HKCS].(2D,\( \epsilon \))

\[
\partial_t W^\epsilon - D \Delta W^\epsilon = -\gamma W^\epsilon, \quad (t, x) \in \Omega^{L,\epsilon H}_T \tag{5a}
\]

\[
\partial_t W^s,\epsilon - D^* \partial_{x_1}^2 W^s,\epsilon = -\gamma^* W^{s,\epsilon} + [kGW^\epsilon - k'W^{s,\epsilon}] - [k_{Rg}R^sW^{s,\epsilon} - k'_{Rg}R_g^s,\epsilon], \quad (t, x) \in (\partial_1 \Omega^{L,\epsilon H})_T \tag{5b}
\]

\[
\partial_t R^\epsilon = -[k_R R^\epsilon W^\epsilon - k'_R R^s,\epsilon] - [k_{Rg}R^sW^{s,\epsilon} - k'_{Rg}R_g^s,\epsilon] - \alpha R^\epsilon + \Gamma, \quad (t, x) \in (\partial_1 \Omega^{L,\epsilon H})_T \tag{5c}
\]

\[
\partial_t R^s,\epsilon = [k_{Rg}R^sW^{s,\epsilon} - k'_{Rg}R_g^s,\epsilon] - \alpha^* R^{s,\epsilon}, \quad (t, x) \in (\partial_1 \Omega^{L,\epsilon H})_T
\]

\[
\partial_t R_g^s,\epsilon = [k_{Rg}R^sW^{s,\epsilon} - k'_{Rg}R_g^s,\epsilon] - \alpha^*_g R_g^{s,\epsilon}, \quad (t, x) \in (\partial_1 \Omega^{L,\epsilon H})_T
\]

with boundary conditions

\[
\epsilon^{-1} D \nabla W^\epsilon \nu = 0, \quad (t, x) \in (\partial_0 \Omega^{L,\epsilon H})_T \tag{5a}
\]

\[
\epsilon^{-1} D \nabla W^s,\epsilon = -[kGW^\epsilon - k'W^{s,\epsilon}] - [k_{Rg}R^sW^\epsilon - k'_{Rg}R_g^s,\epsilon] + s \delta, \quad (t, x) \in (\partial_0 \Omega^{L,\epsilon H})_T \tag{5b}
\]

\[
\partial_{x_1} W^s,\epsilon = 0, \quad (t, x) \in (\partial \Omega^{L,\epsilon H})_T \tag{5c}.
\]
and initial conditions
\[
W^*(0) = W_0^*, \\
W^{*,*}(0) = W_0^*, \\
R^*(0) = R_0^*, \\
R^*_{g*}(0) = R_g^*_0, \\
x \in \Omega^{L,\epsilon,H}
\]
\[
W_0^*(x_1, x_2) = W_0(x_1, x_2/\epsilon).
\]

Observe that [HKCS].(2D,1) = [HKCS].2D. Roughly speaking besides the well-posedness of [HKCS].2D, our main result is that
\[
\lim_{\epsilon \to 0^+} [HKCS].(2D,\epsilon) = [HKCS].1D,
\]
where [HKCS].1D was analysed in [15]. The precise meaning of the limit will be given later.

### 1.2 Nondimensionalisation

Introduce the following nondimensional parameters:
\[
T = L^2/D, \quad K_1 = k_R T, \quad K_2 = k_R T/H, \quad h = \epsilon H/L, \quad d = D^*/D,
\]
\[
b = (b_1, b_2, b_3, b_4, b_5) = (T\gamma, T\gamma^*, T\alpha, T\alpha^*, T\alpha^*),
\]
\[
c = (c_1, c_2, c_3, c_4, c_5) = (Tk_G/H, Tk', Hk_Rg/k_R, Tk_R'Tk_R', Tk_R'),
\]
\[
p = (p_1, p_2, p_3, p_4, p_5) = (K_2 T_s, 0, K_2 T_T, 0, 0).
\]

For \((t, x) = (t, x_1, x_2) \in \Omega_T = (0, T) \times (-1, 1) \times (0, 1)\) we define functions
\[
u^h(t, x_1, x_2) = K_1 W^*(Tt, Lx_1, \epsilon Hx_2),
\]
\[
u^h(t, x_1) = K_2 W^{*,*}(Tt, Lx_1),
\]
\[
u^h(t, x_1) = K_2 R^{*,*}(Tt, Lx_1),
\]
\[
u^h = (u_1^h, u_2^h, u_3^h, u_4^h, u_5^h),
\]
\[
u_{01}(x_1, x_2) = K_1 W_0^*(Lx_1, \epsilon Hx_2) = K_1 W_0(Lx_1, Hx_2),
\]
\[
u_02(x_1) = K_2 W_0^*(Lx_1),
\]
\[
u_03(x_1) = K_2 R_0(Lx_1),
\]
\[
u_04(x_1) = K_2 R_0^*(Lx_1),
\]
\[
u_0 = (u_0, u_{02}, u_{03}, u_{04}, u_{05}),
\]

then system [HKCS].(2D,\epsilon) rewritten in the nondimensional form reads
\[
\partial_t u_1^h + div(J_h(u_1^h)) = -b_1 u_1^h,
\]
\[
\partial_t u_2^h - d\partial_{x_1}^2 u_2^h = c_1 u_1^h - (b_2 + c_2 + c_3 u_2^h) u_2^h + c_5 u_3^h,
\]
\[
\partial_t u_3^h = -(b_3 + u_1^h + c_3 u_2^h) u_3^h + c_4 u_4^h + c_5 u_5^h + p_3,
\]
\[
\partial_t u_4^h = u_1^h u_3^h - (b_4 + c_1) u_4^h,
\]
\[
\partial_t u_5^h = c_3 u_2^h u_3^h - (b_5 + c_3) u_5^h,
\]

with boundary and initial conditions
\[
-J_h(u_1^h) \nu = 0,
\]
\[
-J_h(u_1^h) \nu = -(c_1 + u_2^h) u_1^h + c_2 u_2^h + c_4 u_4^h + p_1 \delta,
\]
\[
\partial_{x_1} u_2^h = 0,
\]
\[
u^h(0, \cdot) = u_0
\]
where
\[ J_h(u) = -[\partial_{x_1} u, h^{-2} \partial_{x_2} u] \] denotes the flux of \( u_1^h \),
\[ \nu \] denotes the outer normal unit vector to \( \partial \Omega \),
\[ \delta \] denotes a one dimensional Dirac Delta \( \delta(\phi) = \phi(0) \) for any \( \phi \in C([-1, 1]) \).

From now on we impose the following natural assumptions on the signs of the constant parameters and (possibly nonconstant) initial conditions

\[ d, b > 0, \ c, p, u_0 \geq 0. \quad (8) \]

### 1.3 Overview

The paper is divided into four sections as follows

1. Introduction
   (a) The [HKCS].2D model
   (b) Nondimensionalisation
   (c) Overview
2. Notation
3. Analitical Tools
   (a) Inequalities
   (b) Existence result for system of abstract ODEs
   (c) Operators, semigroups, estimates
   (d) Auxiliary functions
4. The case of regular source
5. The case of singular source
   (a) Definition of M-mild solution
   (b) The main results - well-posedness and dimension reduction
   (c) Proof of Theorem 2 and Theorem 3

### 2. Notation

If \( X \) is a linear space and \( Y \) is an arbitrary subset of \( X \) we denote by \( \text{lin}(Y) \) the linear space which consists of all linear combinations of elements of \( Y \). If \( X \) is a topological space and \( U \) is an arbitrary subset of \( X \) we denote by \( \text{cl}_X(U) \) the closure of \( U \) in \( X \). If \( X \) is a normed vector space we denote by \( \| \cdot \|_X \) its norm and by \( X^* \) its topological dual. If \( x \in X \) and \( x^* \in X^* \) we denote by \( \langle x^*, x \rangle_{(X^*, X)} = x^*(x) \) a natural pairing between \( X \) and its dual. If \( X \) is a Hilbert space we denote by \( (\cdot, \cdot)_X \) its scalar product. In particular \( (x|y)_{\mathbb{R}^n} = \sum_{i=1}^n x_i y_i \) and \( (f|g)_{L^2(U)} = \int_U f g \). To get more familiar with the notation observe that, due to Riesz theorem, for any \( x^* \in X^* \) there exists a unique \( x \in H \) such that \( \langle x^*, y \rangle_{(X^*, X)} = (x|y)_X \) for all \( y \in X \). If \( X_1 \) and \( X_2 \) are Hilbert spaces and \( u \in X_1 \), \( v \in X_2 \) we denote by \( X_1 \otimes X_2 \) and \( u \otimes v \) tensor products. For a comprehensive treatment on normed, Hilbert and Banach spaces we refer to [17], Chap. I-III.

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If $X, Y$ are normed spaces we denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators between $X$ and $Y$, topologized by the operator norm. If $A \in \mathcal{L}(X, Y)$ then we denote by $A' \in \mathcal{L}(Y^*, X^*)$ the transpose of $A$. If $D(A)$ is a linear (not necessarily dense) subspace of a Banach space $X$ we write $A : X \supset D(A) \to X$ when $A$ is an unbounded linear operator with domain $D(A)$. We denote by $G(A)$ the graph of $A$ (i.e. the set $\{(u, Au) : u \in D(A)\}$). For a closed, unbounded operator $A : X \supset D(A) \to X$ we denote by $\rho(A)$ the resolvent set of $A$, by $\sigma(A) = \mathbb{C} \setminus \rho(A)$ its spectrum and by $\sigma_p(A)$ its pure point spectrum (i.e. set of eigenvalues). The resolvent operator is denoted by $R(\lambda, A)$ or $(\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. If $X, Y$ are Hilbert spaces and $A : X \supset D(A) \to Y$ is a densely defined unbounded operator we denote by $A^* : Y \supset D(A^*) \to X$ the adjoint of $A$. Let us point out that if $X$ is a real vector space than by the spectrum of operator $A$ we understand the spectrum of its complexification. For a comprehensive treatment on bounded and unbounded operators as well as their spectral theory we refer to [17, Chap. VI-VIII].

If $U$ is a subset of $\mathbb{R}^n$ we denote by $\overline{U} = cl_{\mathbb{R}^n}(U)$ its closure and by $\partial U$ its boundary. For $1 \leq p \leq \infty$, $s \in \mathbb{R}$, we denote by $W^s_p(U)$ the fractional Sobolev (also known as Sobolev-Slobodecki) spaces. If $U$ is also bounded we denote by $C(\overline{U})$ the Banach space of continuous functions on $\overline{U}$ topologised by the supremum norm. By $\mathcal{M}(\overline{U}) = (C(\overline{U}))^*$ we denote the Banach space of finite, signed Radon measures. For $\theta \in [0, 1]$ we denote by $[\cdot, \cdot]_{\theta}$ the complex interpolation functor. For a comprehensive treatment on spaces $W^s_p$ and $[\cdot, \cdot]_\theta$ we refer to [21].

In the whole article $I_+ = (0, 1)$, $I = (-1, 1)$ and $\Omega = (-1, 1) \times (0, 1)$ are fixed domains. Moreover we denote by $\delta$ a one dimensional Dirac Delta: $\delta \in \mathcal{M}(\overline{I})$ and $\delta(\phi) = \int_I \phi \, d\delta = \phi(0)$ for any $\phi \in C(\overline{I})$. For $i, j \in \mathbb{N}$ we denote by $\delta_{ij}$ the Kronecker symbol i.e. $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

In estimates we will use a generic constant $C$ which may take different values even in the same paragraph. Constant $C$ may depend on various parameters, but it will never depend on $h$ nor any other parameter which could change due to a limiting process.

3 Analytical Tools

3.1 Inequalities

In Lemma 1 we collect three elementary estimates which are used in the following chapters. For completeness of the provision we provide short proofs.

**Lemma 1.** The following inequalities hold

$$\sup \{ t^\alpha e^{-rt} : t \geq t_0 \} \leq C (r^{-\alpha} + t_0^\alpha) e^{-rt_0}, \quad t_0 \geq 0, \alpha \geq 0, r > 0, \quad (9a)$$

$$\int_0^t \frac{d\tau}{\tau^\alpha (t - \tau)^\beta} \leq C t^{1-\alpha-\beta}, \quad t > 0, \alpha, \beta \geq 0, \alpha + \beta < 1, \quad (9b)$$

$$\int_0^t e^{-rt} \frac{d\tau}{\tau^\alpha (t - \tau)^\beta} \leq C \left( \frac{t^\alpha + \beta}{r^\beta} \right)^{\frac{1-(\alpha+\beta)}{1+\alpha+\beta}}, \quad t > 0, \alpha, \beta \geq 0, \alpha + \beta < 1, r > 0, \quad (9c)$$

where constant $C$ depends only on $\alpha$ and $\beta$.

**Proof.** To prove (9a) define for $t \geq 0$ function $f(t) = t^\alpha e^{-rt}$. Then $f'(t) = \alpha t^{\alpha-1} e^{-rt} - rt^\alpha e^{-rt} = t^{\alpha-1} e^{-rt}(\alpha - rt)$. Analysing the sign of $f'$ we obtain that function $f$ is increasing on $[0, \alpha/r]$ and decreasing on $[\alpha/r, \infty)$. It follows that

$$\sup \{ t^\alpha e^{-rt} : t \geq t_0 \} = \begin{cases} f(\alpha/r) & \text{if } t_0 \leq \alpha/r \\ f(t_0) & \text{if } t_0 > \alpha/r \end{cases} \leq f(\alpha/r) + f(t_0) \leq C(r^{-\alpha} + t_0^\alpha) e^{-rt_0},$$
where one can take $C = \max\{\alpha, 1\}$. To prove inequality (9a) we change variables $\tau = t\tau_1$. Then we have $$\int_0^t \frac{d\tau}{\tau^\alpha(t - \tau)^\beta} = t^{1 - \alpha - \beta} \int_0^1 \frac{d\tau_1}{\tau_1^\alpha(1 - \tau_1)^\beta} \leq C t^{1 - \alpha - \beta}.$$ Finally we prove (9c). Set $q = \frac{\alpha + \beta + 1}{2(\alpha + \beta)}$. It is easy to check that $1 < q$ and $(\alpha + \beta)q < 1$. Let $p = \frac{q}{q - 1} = \frac{1 + (\alpha + \beta)}{1 - (\alpha + \beta)}$ be $q$'s Hölder conjugate exponent. Using Hölder inequality we obtain

$$\int_0^t \frac{1}{\tau^\alpha(t - \tau)^\beta} d\tau = t^{1 - (\alpha + \beta)} \int_0^1 \frac{1}{\tau^q y^\alpha(1 - y)^\beta} dy \leq t^{1 - (\alpha + \beta)} \left( \int_0^1 \frac{1}{\tau^q (1 - \tau)^\beta} \right)^{1/q} \leq C t^{1 - (\alpha + \beta)} \left( t^{(\alpha + \beta)\frac{1 - (\alpha + \beta)}{1 + \alpha + \beta}} \right)^{1/p} = C \left( \frac{t^{\alpha + \beta}}{p} \right)^{\frac{1 - (\alpha + \beta)}{1 + \alpha + \beta}}.$$

Lemma 2 is an extension of the well known Gronwall inequality in integral form. Although several results of similar type can be found in the literature (for instance in [18]), we were not able to find a reference to the one which would cover the full range of parameters. Our method of proof is taken from [18].

Lemma 2. Let $0 \leq \alpha, \beta, \alpha + \beta < 1$, $0 \leq a, 0 < b, 0 < T < \infty$. Assume that $f \in L_\infty(0, T')$ for every $T' < T$ and that for a.e. $t \in (0, T)$ the following inequality holds

$$0 \leq f(t) \leq a + b \int_0^t \frac{f(\tau)}{\tau^\alpha(t - \tau)^\beta} d\tau,$$

then $f \in L_\infty(0, T)$ and

$$\|f\|_{L_\infty(0, T)} \leq a C \exp(b CT^{1 - (\alpha + \beta)}),$$

where $C$ depends only on $\alpha, \beta$. Moreover $C = 1$ when $\alpha = \beta = 0$.

Proof. When $\alpha = \beta = 0$ the result is the well known Gronwall inequality in integral form. Otherwise we proceed similarly as in the proof of inequality (9c). Fix $q > 1$ such that $q(\alpha + \beta) < 1$ and let $p = \frac{q}{q - 1}$ be $q$'s Hölder conjugate exponent. Using Hölder inequality we obtain

$$\int_0^t \frac{f(\tau)}{\tau^\alpha(t - \tau)^\beta} d\tau \leq \left( \int_0^t f(\tau)^p d\tau \right)^{1/p} \left( \int_0^t \frac{d\tau}{\tau^aq(t - \tau)^\beta} \right)^{1/q} = \left( \int_0^t f(\tau)^p d\tau \right)^{1/p} t^{1/q - (\alpha + \beta)} \left( \int_0^1 \frac{d\tau}{\tau^aq(1 - \tau)^\beta} \right)^{1/q} \leq CT^{1/q - (\alpha + \beta)} \left( \int_0^t f(\tau)^p d\tau \right)^{1/p},$$

thus

$$f(t)^p \leq \left( a + b CT^{1/q - (\alpha + \beta)} \left( \int_0^t f(\tau)^p d\tau \right)^{1/p} \right)^p \leq 2p - 1 a^p + 2p - 1 b^p C^p T^p q - p(\alpha + \beta) \int_0^t f(\tau)^p d\tau.$$

Using Lemma 2 with $\alpha = \beta = 0$ we obtain

$$f(t)^p \leq 2p - 1 a^p \exp \left( 2p - 1 b^p C^p T^p q - p(\alpha + \beta) t \right)$$

$$f(t) \leq 2^{1/q} a \exp \left( p - 1 \frac{1}{2p - 1 b^p C^p T^p q - p(\alpha + \beta)} t \right)$$

$$f(t) \leq 2^{1/q} a \exp \left( 2^{1/q} b CT^{1/q - (\alpha + \beta)} t^{1/p} \right) \leq 2^{1/q} a \exp \left( 2^{1/q} b CT^{1 - (\alpha + \beta)} \right),$$

from which our claim follows. \qed
3.2 Existence result for a system of abstract ODE’s

For \( i = 1, \ldots, n \) let \((X_i, X_i^1)\) be a densely injected Banach couple (i.e. \( X_i^1 \) is a dense subspace of \( X_i \) in the topology of \( X_i \)). For \( \alpha_i \in (0,1) \) denote \( X_i^{\alpha_i} = [X_i, X_i^1]_{\alpha_i} \) (where \([,]\)_{\alpha_i} is the complex interpolation functor). Finally note

\[
\alpha = (\alpha_1, \ldots, \alpha_n), \quad X = X_1 \times \ldots \times X_n, \quad X^1 = X_1 \times \ldots \times X_n^1, \quad X^{\alpha} = X_1^{\alpha_1} \times \ldots \times X_n^{\alpha_n}.
\]  

(10)

Lemma 3. Assume that for \( i = 1, \ldots, n \) the following three conditions are satisfied

1. Operator \( A_i : X_i \supset X_i^1 \rightarrow X_i \) generates an analytic strongly continuous semigroup \( e^{tA_i} \).
2. Map \( F_i : X^{\alpha} \rightarrow X_i \) is Lipschitz on bounded sets i.e.

\[
\forall R > 0 \exists C_R \left[ \| u \|_{X^{\alpha}}, \| w \|_{X^{\alpha}} \leq R \implies \| F_i(u) - F_i(w) \|_{X_i} \leq C_R \| u - w \|_{X^{\alpha}} \right]
\]

3. \( u_{0i} \in X_i^{\alpha_i} \).

Then the following system of abstract ODE’s

\[
\frac{d}{dt} u_i - A_i u_i = F_i(u), \quad t > 0
\]

(11)

\[
u_i(0) = u_{0i}
\]

(12)

has a unique maximal \( X^{\alpha} \) solution \( u = (u_1, \ldots, u_n) \) i.e. there exists a unique

\[
u \in C([0, T_{\text{max}}); X^{\alpha}) \cap C^1((0, T_{\text{max}}); X) \cap C((0, T_{\text{max}}); X^1),
\]

which satisfies system \([11]-[12]\) in the classical sense. For \( t \in (0, T_{\text{max}}) \) the following Duhamel formulas hold:

\[
u_i(t) = e^{tA_i} u_{0i} + \int_0^t e^{(t-\tau)A_i} F_i(u(\tau)) d\tau, \quad 1 \leq i \leq n.
\]

and \( T_{\text{max}} \) satisfies the blow-up condition:

\[
T_{\text{max}} < \infty \implies \limsup_{t \rightarrow T_{\text{max}}} \| u(t) \|_{X^{\alpha}} = \infty.
\]  

(13)

In particular if there exists \( C \) such that

\[
\sum_{i=1}^n \| F_i(u(t)) \|_{X_i} \leq C(\| u(t) \|_{X^{\alpha}} + 1), \quad t \in [0, T_{\text{max}})
\]

(14)

then \( T_{\text{max}} = \infty \).

Proof. If \( n = 1 \) the result is well-known and can be proved via the Banach fixed point argument (see for instance [12], Theorem 6.3.2). For \( n > 1 \) one can adapt the same method with obvious modifications. \( \square \)

3.3 Operators, semigroups, estimates

Let us recall that \( I_+ = (0,1), \quad I = (-1,1), \quad \Omega = I \times I_+ \).

For \( U \in \{I_+, I, \Omega\} \) we denote

\[
X(U) = L_2(U), \quad (\cdot, \cdot)_X(U) = (\cdot, \cdot)_{L_2(U)}.
\]

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For \( i, j \in \mathbb{N} \) we define functions \( u_i, v_i, w_{ij} \)
\[ u_i(x_1) = c_{1i} \cos(i\pi(x_1 + 1)/2), \; x_1 \in I, \quad v_i(x_2) = c_{2i} \cos(i\pi x_2), \; x_2 \in I_+ \]
\[ w_{ij}(x_1, x_2) = u_i(x_1)v_j(x_2), \; (x_1, x_2) \in \Omega, \] (15) (16)
where constants \( c_{1i}, c_{2i} \) are such that \( \|u_i\|_{X(I)} = \|v_i\|_{X(I_+)} = 1 \) i.e.
\[ c_{1i} = \begin{cases} 1/\sqrt{2} & \text{if } i = 0 \\ 1 & \text{if } i > 0 \end{cases}, \quad c_{2i} = \begin{cases} 1 & \text{if } i = 0 \\ \sqrt{2} & \text{if } i > 0 \end{cases}. \]

**Lemma 4.** The set \( \{v_i : i \in \mathbb{N}\} \) (resp. \( \{u_i : i \in \mathbb{N}\} \) and \( \{w_{ij} : i, j \in \mathbb{N}\}\) is a complete orthonormal system in \( X(I_+) \) (resp. \( X(I) \) and \( X(\Omega) \)).

**Proof.** The fact that \( \{v_i : i \in \mathbb{N}\} \) is a complete orthonormal system in \( X(I_+) \) is well known. Since \( u_i(x) = (c_{1i}/c_{2i})v_i((x + 1)/2) \) the thesis for the set \( \{u_i : i \in \mathbb{N}\} \) follows. Finally observe that since \( w_{ij} = u_i \otimes v_j \) and \( X(\Omega) = X(I) \otimes X(I_+) \) then the claim for the set \( \{w_{ij} : i, j \in \mathbb{N}\} \) follows from [17], Chap. II.4, Prop. 2.

Denote
\[ X_{fin}(I) = \text{lin}(\{u_i : i \in \mathbb{N}\}), \quad X_{fin}(\Omega) = \text{lin}(\{w_{ij} : i, j \in \mathbb{N}\}). \]

Define sequences
\[ \lambda_I^+ = (\lambda_i^+)_{i \in \mathbb{N}}, \quad \lambda_I^- = -(i\pi)^2, \; i \in \mathbb{N} \]
\[ \lambda_I^0 = (\lambda_i^0)_{i \in \mathbb{N}}, \quad \lambda_I^1 = -(i\pi/2)^2, \; i \in \mathbb{N} \]
\[ \lambda_{ij,h} = (\lambda_{ij,h}^0)_{i,j \in \mathbb{N}}, \quad \lambda_{ij,h}^0 = \lambda_I^1 + h^{-2}\lambda_I^0 = -(i\pi/2)^2 - (j\pi/h)^2, \; i, j \in \mathbb{N}, \; h \in (0, 1] \]
and denote \( \lambda_I^0, \lambda_{ij}^0 = \lambda_{ij,1}^0 \).

Define \( \tilde{A}_0 \) and \( \tilde{A}_h \) for \( h \in (0, 1] \) to be the unique unbounded linear operators such that
\[ \tilde{A}_0 : X(I) \supset X_{fin}(I) \rightarrow X(I), \quad \tilde{A}_0 u_i = \partial^2_{x_1 x_1} u_i = \lambda_{i}^1 u_i \]
\[ \tilde{A}_h : X(\Omega) \supset X_{fin}(\Omega) \rightarrow X(\Omega), \quad \tilde{A}_h w_{ij} = -\text{div} J_h(w_{ij}) = (\partial^2_{x_1 x_1} + h^{-2}\partial^2_{x_2 x_2})w_{ij} = \lambda_{ij,h}^0 w_{ij} \]
and denote \( \tilde{A} = \tilde{A}_1 \).

Define the unbounded linear operators \( A_0 \) and \( A_h \) for \( h \in (0, 1] \):
\[ A_0 : X(I) \supset D(A_0) \rightarrow X(I), \quad D(A_0) = \{u \in X(I) : \sum_{i \in \mathbb{N}} (1 - \lambda_i^1)^2 (u|u_i)^2_{X(I)} < \infty\}, \]
\[ A_0 u = \sum_{i \in \mathbb{N}} \lambda_{i}^1 (u|u_i)_{X(I)} u_i, \]
\[ A_h : X(\Omega) \supset D(A_h) \rightarrow X(\Omega), \quad D(A_h) = \{w \in X(\Omega) : \sum_{i,j \in \mathbb{N}} (1 - \lambda_{ij,h}^0)^2 (w|w_{ij})^2_{X(\Omega)} < \infty\}, \]
\[ A_h w = \sum_{i,j \in \mathbb{N}} \lambda_{ij,h}^0 (w|w_{ij})_{X(\Omega)} w_{ij} \]
and denote \( A = A_1 \). Observe that the domain \( D(A_h) \) does not depend on \( h \) since \( \lambda_{ij}^0 \leq \lambda_{ij,h}^0 \leq h^{-2}\lambda_{ij}^0 \), i.e.
\( D(A_h) = D(A) \).

**Lemma 5.**
1. Operator $A_0$ (resp. $A_h$) is the closure of operator $\tilde{A}_0$ (resp. $\tilde{A}_h$).

2. Operators $A_0$ and $A_h$ are self-adjoint and nonpositive.

3. The spectra of operators $A_0, A_h$ consist entirely of eigenvalues:
   \[
   \sigma(A_0) = \sigma_p(A_0) = \lambda^f, \quad \sigma(A_h) = \sigma_p(A_h) = \lambda^\Omega_h.
   \] (17)

4. Resolvent operators $R(\lambda, A_0)$ and $R(\lambda, A_h)$ satisfy
   \[
   R(\lambda, A_0)u = \sum_{i \in \mathbb{N}} (\lambda - \lambda_i^f)^{-1}(u|w_i)_X(\Omega)u_i, \quad \text{for } \lambda \in \rho(A_0), \ u \in X(I),
   \] (18)
   \[
   R(\lambda, A_h)w = \sum_{i,j \in \mathbb{N}} (\lambda - \lambda_{ij}^\Omega)^{-1}(w|w_{ij})_X(\Omega)w_{ij}, \quad \text{for } \lambda \in \rho(A_h), \ w \in X(\Omega).
   \] (19)

Proof. We give the proof only for $A_h$ (for $A_0$ it is similar). Moreover it is clear that it is enough to consider the case $h = 1$.

**Step 1** It is readily seen that operator $A$ is an extension of operator $\tilde{A}$. To show that operator $A$ is closed let us consider an arbitrary sequence $(w_n)_{n=1}^\infty \subset D(A)$ such that
   \[
   w_n \to w, \text{ in } X(\Omega), \quad Aw_n \to v, \text{ in } X(\Omega),
   \]
for certain $w, v \in X(\Omega)$. It follows that $(w_n)_{n=1}^\infty$ is a Cauchy sequence in the Hilbert space $X^1(\Omega) = (D(A), \langle \cdot | \cdot \rangle_{X^1(\Omega)})$, where
   \[
   (w|w')_{X^1(\Omega)} = \sum_{i,j \in \mathbb{N}} (1 - \lambda_{ij}^f)^2(w|w_{ij})_X(\Omega)(w'|w_{ij})_X(\Omega), \quad \text{for } w, w' \in D(A).
   \]
Thus $w \in D(A)$ and $Aw = v$ which proves that $A$ is closed. It is left to prove that $G(A) \subset cl_{X(\Omega) \times X(\Omega)}G(\tilde{A})$. To achieve this goal choose an arbitrary $(w, Aw) \in G(A)$. Then sequence $(w_n)_{n=1}^\infty$ defined by $w_n = \sum_{i+j \leq n} (w|w_{ij})_X(\Omega)w_{ij}$ satisfies
   \[
   w_n \in D(\tilde{A}), \text{ for } n \geq 1,
   \]
   \[
   (w_n, Aw_n) \to (w, Aw), \text{ in } X(\Omega) \times X(\Omega),
   \]
which completes the proof of 1.

**Step 2** To prove that operator $A$ is symmetric and nonpositive let us observe that for $w, w' \in D(A)$ we have
   \[
   (Aw|w')_X(\Omega) = \sum_{i,j \in \mathbb{N}} (Aw|w_{ij})_X(\Omega)(w'|w_{ij})_X(\Omega) = \sum_{i,j \in \mathbb{N}} \lambda_{ij}^f(w|w_{ij})_X(\Omega)(w'|w_{ij})_X(\Omega) =
   \]
   \[
   = \sum_{i,j \in \mathbb{N}} (w|w_{ij})_X(\Omega)(Aw'|w_{ij})_X(\Omega) = (w|Aw')_X(\Omega),
   \]
   \[
   (Aw|w)_X(\Omega) = \sum_{i,j \in \mathbb{N}} \lambda_{ij}^f(w|w_{ij})_X^2(\Omega) \leq 0.
   \]
Moreover $A$ is densely defined since $X_{fin}(\Omega) \subset D(A)$ and $X_{fin}(\Omega)$ is dense in $X(\Omega)$ by Lemma (cite), thus it is possible to define the adjoint operator $A^*$. To prove that $A$ is self-adjoint it is left to prove that $D(A^*) = D(A)$, which is equivalent to $D(A^*) \subset D(A)$ since the opposite inclusion always holds. Choose arbitrary $w' \in D(A^*)$. By definition of $D(A^*)$ there exists unique $v \in X(\Omega)$ such that for every $w \in D(A)$ one has $(w'|Aw)_X(\Omega) = (v|w)_X(\Omega)$. Testing by $w = w_{ij}$ we obtain $\lambda_{ij}^f(w'|w_{ij})_X(\Omega) = (v|w_{ij})_X(\Omega)$. Finally $w' \in D(A)$ since $\sum_{ij}(1 - \lambda_{ij}^f)^2(w'|w_{ij})^2_X(\Omega) \leq 2 \sum_{ij}(1 + |\lambda_{ij}^f|^2)(w'|w_{ij})^2_X(\Omega) = 2(\|w'\|^2_X(\Omega) + \|v\|^2_X(\Omega)) < \infty$.

**Step 3** Since $Aw_{ij} = \lambda_{ij}^f w_{ij}$ we obtain that $\lambda^\Omega \subset \sigma_p(A)$. For $\lambda \notin \lambda^\Omega$ define operator
   \[
   B(\lambda, A)w = \sum_{i,j}(\lambda - \lambda_{ij}^\Omega)^{-1}(w|w_{ij})_X(\Omega)w_{ij}.
   \]
One checks easily that $B(\lambda, A) \in \mathcal{L}(X(\Omega))$, $B(\lambda, A)w \in D(A)$ for $w \in X(\Omega)$. Moreover $B(\lambda, A)(\lambda - A)w = w$ for $w \in D(A)$ and $(\lambda - A)B(\lambda, A)w = w$ for $w \in X(\Omega)$ which is easily seen for $w \in X_{fin}(\Omega)$ and by the density argument can be extended to $D(A)$ and $X(\Omega)$. Thus $\rho(A) = \mathbb{C} \setminus \lambda^\Omega$, $R(\lambda, A) = B(\lambda, A)$ for $\lambda \in \rho(A)$ and $\sigma(A) = \sigma_p(A) = \lambda^\Omega$. □

Since operators $A_0, A_h$ are self-adjoint and nonpositive they generate strongly continuous analytic semigroups $e^{tA_0}$ and $e^{tA_h}$:

$$e^{tA_0}u = \sum_{i \in \mathbb{N}} e^{t\lambda_i^0}(u|u_i)^X(I)u_i, \text{ for } u \in X(I),$$

$$e^{tA_h}w = \sum_{i,j \in \mathbb{N}} e^{t\lambda_{ij}^h}(w|w_{ij})^X(\Omega)w_{ij}, \text{ for } w \in X(\Omega).$$

Since operators $I - A_0$ and $I - A$ are self-adjoint and positive one can define their fractional powers $(I - A_0)^s$ and $(I - A)^s$ for $s \geq 0$. Their domains $D((I - A_0)^s)$ and $D((I - A)^s)$ become Hilbert spaces (which we denote $X^s(I)$ and $X^s(\Omega)$) when equipped with appropriate scalar products. For $s \geq 0$ spaces $X^s(I)$ and $X^s(\Omega)$ are defined as follows

$$X^s(I) = \{ u \in X(I) : \sum_{i \in \mathbb{N}} (1 - \lambda_i^0)^{2s}(u|u_i)^2_X(I) < \infty \},$$

$$(u|u')^X(I) = \sum_{i \in \mathbb{N}} (1 - \lambda_i^0)^{2s}(u|u_i)^X(I)(u'|u_i)^X(I),$$

$$X^s(\Omega) = \{ w \in X(\Omega) : \sum_{i,j \in \mathbb{N}} (1 - \lambda_{ij}^\Omega)^{2s}(w|w_{ij})^2_X(\Omega) < \infty \},$$

$$(w|w')^X(\Omega) = \sum_{i,j \in \mathbb{N}} (1 - \lambda_{ij}^\Omega)^{2s}(w|w_{ij})^X(\Omega)(w'|w_{ij})^X(\Omega).$$

Lemma 6.

1. For $s_1 > s_2 \geq 0$ the following equalities hold

$$(u|u_i)^{X^{s_1}}(I) = (1 - \lambda_i^0)^{2(s_1 - s_2)}(u|u_i)^{X^{s_2}}(I), \text{ for } u \in X^{s_1}(I)$$

$$(w|w_{ij})^{X^{s_1}}(\Omega) = (1 - \lambda_{ij}^\Omega)^{2(s_1 - s_2)}(w|w_{ij})^X(\Omega), \text{ for } w \in X^{s_1}(\Omega).$$

2. The set $\{ u_i : i \in \mathbb{N} \}$ (resp. $\{ w_{ij} : i, j \in \mathbb{N} \}$) is a complete orthogonal system in $X^s(I)$ (resp. $X^s(\Omega)$) for any $s \geq 0$. In particular if $s_1 > s_2 \geq 0$ then $X^{s_1}(I)$ (resp. $X^{s_1}(\Omega)$) is a dense subspace of $X^{s_2}(I)$ (resp. $X^{s_2}(\Omega)$).

Proof.

Step 1

For any $s \geq 0$ we have

$$(u|u_i)^{X^s}(I) = \sum_{k \in \mathbb{N}} (1 - \lambda_k^s)^2(u|u_k)^X(I)(u_i|u_k)^X(I) = (1 - \lambda_i^0)^{2s}(u|u_i)^X(I),$$

from which (22) follows. The proof of (23) is similar.

Step 2

Orthogonality in $X^s$ follows from (22), (23) and Lemma 4. For the proof of completeness of $\{ u_i : i \in \mathbb{N} \}$ notice that if $u \in X^s(I)$ then denoting $u_n = \sum_{i=0}^n (u|u_i)^X(I)u_i$ one has $\lim_{n \to \infty} \| u - u_n \|_{X^s(I)} = \lim_{n \to \infty} \sum_{k \geq n+1} (1 - \lambda_k^s)^2(u|u_k)^2_X(I) = 0$. Similarly one proves completeness of $\{ w_{ij} : i, j \in \mathbb{N} \}$. □
Next we extend the scale of Hilbert spaces $X^s(I), X^s(\Omega)$ to $s \in [-1,0)$ by duality. More precisely for any $s \in [-1,0)$ we define $X^s(I) = (X^{-s}(I))^*$, $X^s(\Omega) = (X^{-s}(\Omega))^*$. Then for $s \in [-1,0)$ Banach spaces $X^s$ become Hilbert spaces when equipped with the following scalar products

$$
(u|u')_{X^s(I)} = \sum_{i \in \mathbb{N}} (1 - \lambda_i^s)^{2s} \left( u_i, u_i \right)_{(X^s(I),X^{-s}(I))},
$$

$$
(w|w')_{X^s(\Omega)} = \sum_{i,j \in \mathbb{N}} (1 - \lambda_{ij}^s)^{2s} \left( w_i, w_j \right)_{(X^s(\Omega),X^{-s}(\Omega))}.
$$

Observe that assertions of Lemma 3 are still valid without assuming that $s, s_1, s_2$ are nonnegative.

**Lemma 7.** $[X^{s_1}(U), X^{s_2}(U)]_\theta = X^{s_1(1-\theta)+s_2\theta}(U)$ for $s_1, s_2 \geq -1$, $\theta \in [0,1]$, $U \in \{I, \Omega\}$.

**Proof.** We provide the proof for $U = \Omega$ as the one for $U = I$ can be carried out similarly. For $s \geq -1$ define Hilbert spaces $Z^s_{ij} = (\mathbb{R},(a|a')_{Z^s_{ij}})$, where $(a|a')_{Z^s_{ij}} = (1 - \lambda_{ij}^s)^{2s}a a'$ and $l_2(Z^s_{ij}) = \{ (a, (a_{ij}))_{ij \in \mathbb{N}} : a_{ij} \in \mathbb{R}, \sum_{i,j \in \mathbb{N}} |a_{ij}|^2 Z^s_{ij} < \infty \}$. Define map

$$
\Phi(w) = \left( \left( w_i, w_{ij} \right)_{(X^{-1}(\Omega), X^1(\Omega))} \right)_{i,j \in \mathbb{N}},
$$

for $w \in X^{-1}(\Omega)$. Observe that $\Phi$ is an isometric isomorphism between $X^s(\Omega)$ and $l_2(Z^s_{ij})$ for any $s \geq -1$. This fact allows as to justify the first and the fourth equality in

$$
[X^{s_1}(\Omega), X^{s_2}(\Omega)]_\theta = [l_2(Z^{s_1}_{ij}), l_2(Z^{s_2}_{ij})]_\theta = l_2([Z^{s_1}_{ij}, Z^{s_2}_{ij}]_\theta) = l_2(Z^{s_1(1-\theta)+s_2\theta}_{ij}),(\mathbb{R}), X^1(\Omega)) = X^{s_1(1-\theta)+s_2\theta}(\Omega),
$$

while the second equality follows from [21, Chap. 1.18.1, Theorem].

**Lemma 8.** For $s \in [0,1]$, $U \in \{I, \Omega\}$ we have the following characterisation of the spaces $X^s(U)$:

$$
X^s(U) = \begin{cases} W_2^s(U) & \text{if } 0 \leq s < 3/4 \\
W_{2,N}^s(U) & \text{if } 3/4 \leq s \leq 1 \end{cases}, X^{3/4}(U) \subset W_2^{3/4}(U). \tag{24}
$$

**Proof.** The case when $U$ is an open bounded domain of $\mathbb{R}^n$ with a smooth boudary (in particular $U = I$) or $U$ is a half space - $U = \mathbb{R}^+ \times \mathbb{R}^{n-1}$ was treated in [3, Theorem 2]. The case when $U = \Omega$ we divide in several steps.

**Step 1** We will show that

$$
X^1(\Omega) = W_{2,N}^2(\Omega). \tag{25}
$$

Denote

$$
\eta_i(x_1) = c_{i1} \sin(i \pi (x_1 + 1)/2), \quad x_1 \in I, \quad \eta_i(x_2) = c_{i2} \sin(i \pi x_2), \quad x_2 \in I_+.
$$

Reasoning similarly as in the proof of Lemma 4 we obtain that the set $\{\eta_i : i \in \mathbb{N}_+\}$ (resp. $\{\bar{\eta}_i : i \in \mathbb{N}_+\}$, $\{\eta_i \otimes \bar{\eta}_j : i, j \in \mathbb{N}_+\}$, $\{\eta_i \otimes v_j : i \in \mathbb{N}_+, j \in \mathbb{N}\}$, $\{\eta_i \otimes v_j : i \in \mathbb{N}_+, j \in \mathbb{N}\}$) is a complete orthonormal system in $X(I)$ (resp. $X(I_+, X(\Omega), X(\Omega))$). Compute

$$
\partial_{x_1} w_{ij} = -(i \pi/2) \eta_i \otimes v_j = -\sqrt{|\lambda_i^j|} \eta_i \otimes v_j, \quad \partial_{x_2} w_{ij} = -(j \pi) u_i \otimes \bar{\eta}_j = -\sqrt{|\lambda_i^j|} u_i \otimes \bar{\eta}_j
$$

$$
\partial_{x_1}^{2} w_{ij} = -(i \pi/2)^2 w_{ij} = \lambda_i^j w_{ij}, \quad \partial_{x_2}^{2} w_{ij} = -(j \pi)^2 w_{ij} = \lambda_i^j w_{ij}
$$

$$
\partial_{x_1 x_2} w_{ij} = \partial_{x_2 x_1} w_{ij} = (i \pi/2)(j \pi) \eta_i \otimes \eta_j = \sqrt{|\lambda_i^j|} \lambda_i^j |\eta_i \otimes \eta_j|
$$

Observe that $X_{fin}(\Omega) \subset W_2^{2,N}(\Omega)$. Let $w \in X_{fin}(\Omega)$. Using the triangle inequality and $(a+b)^2 \leq 2(a^2 + b^2)$ we estimate

$$
\|w\|^2_{X^1(\Omega)} = \|(I - \Delta) w\|^2_{L_2(\Omega)} \leq 2 \|w\|^2_{L_2(\Omega)} + 2(\|\partial_{x_1 x_1} w\|_{L_2(\Omega)} + \|\partial_{x_2 x_2} w\|_{L_2(\Omega)}) \leq 4 \|w\|^2_{W_2^2(\Omega)}.
$$
On the other hand

\[
\|w\|_{W^2_{2}(\Omega)}^2 = \|w\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \|\partial_{x_i} w\|_{L^2(\Omega)}^2 + \sum_{i,j=1}^{2} \|\partial_{x_i x_j} w\|_{L^2(\Omega)}^2 \\
= \sum_{i,j} \left(1 + |\lambda_i^j| + |\lambda_j^i| + |\lambda_i^j|^2 + |\lambda_j^i|^2 + 2|\lambda_i^j||\lambda_j^i|\right)(w|w_{ij})_{X(\Omega)}^2 \\
\leq 2 \sum_{i,j} \left(1 - \lambda_i^j - \lambda_j^i\right)^2(w|w_{ij})_{X(\Omega)}^2 \\
= 2 \sum_{i,j} \left(1 - \lambda_{ij}^2\right)^2(w|w_{ij})_{X(\Omega)}^2 = 2\|w\|_{X^1(\Omega)}^2.
\]

Thus norms \(\|\cdot\|_{W^2_{2}(\Omega)}\) and \(\|\cdot\|_{X^1(\Omega)}\) are equivalent on \(X_{fin}(\Omega)\).

In particular \(X^1(\Omega) = cl_{X^1(\Omega)}(X_{fin}(\Omega)) \subset W^2_{2,N}(\Omega)\). It is left to prove that \(W^2_{2,N}(\Omega) \subset X^1(\Omega)\). Choose arbitrary \(u \in W^2_{2,N}(\Omega)\) and let \(f = u - \Delta u\). Then \(f \in X(\Omega)\). Let \(w = R(1,A)f\). Since \(X^1(\Omega) \subset W^2_{2,N}(\Omega)\) thus \(w \in W^2_{2,N}(\Omega)\) and \(f = w - \Delta w\). We have

\[
0 = \int_{\Omega} (w - u)(f - f) = \int_{\Omega} (w - u)^2 - \int_{\Omega} (w - u)\Delta(w - u) = \int_{\Omega} (w - u)^2 + \int_{\Omega} |\nabla (w - u)|^2,
\]

since \(\nabla (w - u) \cdot \nu = 0\) on \(\partial\Omega\). Finally we obtain that \(u = w \in X^1(\Omega)\).

**Step 2** We will show that for \(U \in \{(\mathbb{R}^+)^2, \Omega\}:\n
\[
[L_2(U), W^2_{2,N}(U)]_s = \begin{cases} W^{2s}_{2}(U) \text{ if } 0 \leq s < 3/4 \\
W^{2s}_{2,N}(U) \text{ if } 3/4 < s \leq 1. \end{cases}
\]

To prove (26) for \(U = (\mathbb{R}^+)^2\) we proceed as in the proof of [3, Theorem 2] for the case \(U = \mathbb{R}_+ \times \mathbb{R}\) substituting functions \(\pi\) and \(\nu\) from that proof by

\[
\nu' : L_2((\mathbb{R}^+)^2) \to L_2(\mathbb{R}^2), \quad \nu'(x_1, x_2) = u(|x_1|, |x_2|),
\]

\[
\pi' : L_2(\mathbb{R}^2) \to L_2((\mathbb{R}^+)^2), \quad \pi'v(x_1, x_2) = \frac{1}{4} \sum_{\epsilon_1, \epsilon_2 \in \{-1, 1\}} v(\epsilon_1 x_1, \epsilon_2 x_2).
\]

Observe that the only nonsmooth points of rectangle \(\Omega\) are the corners. We choose the covering of \(\Omega\) by four open subsets \(\{\Omega_i\}\) such that each of them contains exactly one corner. Then a standard argument involving partition of unity inscribed in the covering \(\{\Omega_i\}\) allows us to adapt (26) from \(U = (\mathbb{R}^+)^2\) to \(U = \Omega\).

**Step 3** Using Lemma 8 we obtain \(X^s(\Omega) = [X(\Omega), X^1(\Omega)]_s = [L_2(\Omega), W^2_{2,N}(\Omega)]_s\) from which (24) for \(U = \Omega\) follows.

For \(s \geq -1\) define operator \(A_{0,s}\) (resp. \(A_{h,s}\)) as \(X^s(I)\) (resp. \(X^s(\Omega)\)) realisation of operator \(A_0\) (resp. \(A_h\)) i.e.

\[
\begin{align*}
A_{0,s} & : X^s(I) \supset X^{s+1}(I) \to X^s(I), \quad A_{0,s}u = \sum_{i \in \mathbb{N}} \lambda_i^I(u|u_i)_{X(I)}u_i, \text{ for } u \in X^{s+1}(I), \\
A_{h,s} & : X^s(\Omega) \supset X^{s+1}(\Omega) \to X^s(\Omega), \quad A_{h,s}w = \sum_{i,j \in \mathbb{N}} \lambda_{ij,h}(w|w_{ij})_{X(\Omega)}w_{ij}, \text{ for } w \in X^{s+1}(\Omega).
\end{align*}
\]

Operators \(A_{0,s}, A_{h,s}\) are self-adjoint and nonpositive and thus generate strongly continuous, analytic semigroups of contractions \(e^{tA_{0,s}} \in \mathcal{L}(X^s(I)), \quad e^{tA_{h,s}} \in \mathcal{L}(X^s(\Omega))\).
If \( s_1 \geq s_2 \geq -1 \) then operators \( A_{0,s_1}, R(\lambda, A_{0,s_1}), e^{tA_{0,s_1}} \) are restrictions of operators \( A_{0,s_2}, R(\lambda, A_{0,s_2}), e^{tA_{0,s_2}} \) and operators \( A_{h,s_1}, R(\lambda, A_{h,s_1}), e^{tA_{h,s_1}} \) are restrictions of operators \( A_{h,s_2}, R(\lambda, A_{h,s_2}), e^{tA_{h,s_2}} \) i.e.
\[
A_{0,s_1} u = A_{0,s_2} u, \text{ for } u \in X^{s_1+1}(I) \\
R(\lambda, A_{0,s_1}) u = R(\lambda, A_{0,s_2}) u, e^{tA_{0,s_1}} u = e^{tA_{0,s_2}} u, \text{ for } u \in X^{s_1}(I) \\
A_{h,s_1} w = A_{h,s_2} w, \text{ for } w \in X^{s_1+1}(\Omega) \\
R(\lambda, A_{h,s_1}) w = R(\lambda, A_{h,s_2}) w, e^{tA_{h,s_1}} w = e^{tA_{h,s_2}} w, \text{ for } w \in X^{s_1}(\Omega)
\]

From now on we will lose \( s \)-dependence in notation and write \( A_0, A_h, R(\lambda, A_0), R(\lambda, A_h), e^{tA_0}, e^{tA_h} \) instead of \( A_{0,s}, A_{h,s}, R(\lambda, A_{0,s}), R(\lambda, A_{h,s}), e^{tA_{0,s}}, e^{tA_{h,s}} \).

**Lemma 9.** For \( h \in (0,1] \), \( \lambda > 0 \), \( t > 0 \) the following estimates hold
\[
\|R(\lambda, A_0)\|_{L(X^s(I), X^s(\Omega))} + \|R(\lambda, A_h)\|_{L(X^{s}(\Omega), X^{s'}(\Omega))} \leq C \frac{1}{\lambda^s} (1 + \lambda^{s'-s}), \quad -1 \leq s \leq s' \leq s + 1, \\
\|e^{tA_0}\|_{L(X^s(I), X^s(\Omega))} + \|e^{tA_h}\|_{L(X^{s}(\Omega), X^{s'}(\Omega))} \leq C \left( 1 + \frac{1}{\lambda^{s'-s}} \right), \quad -1 \leq s \leq s'.
\]
where \( C \) depends only on \( s, s' \).

**Proof.** The proof may be obtained with the use of spectral decomposition. For details we refer to proof of the Lemma 13 where we use the same technique.

Define operators
\[
E \in \mathcal{L}(X(I), X(\Omega)), \quad [Eu](x_1, x_2) = u(x_1), \text{ for } u \in X(I) \\
P \in \mathcal{L}(X(\Omega), X(I)), \quad [Pw](x_1) = \int_+ w(x_1, x_2)dx_2, \text{ for } w \in X(\Omega).
\]

**Lemma 10.** Operators \( E \) and \( P \) are mutually adjoint i.e. \( E^* = P \). Moreover
\[
E \in \mathcal{L}(X^s(I), X^s(\Omega)), \quad P \in \mathcal{L}(X^s(\Omega), X^s(I)), \text{ for } s \geq 0.
\]

**Proof.** To prove that \( E^* = P \) we need to show that
\[
(Eu|w)_{X(\Omega)} = (u|Pw)_{X(I)}, \text{ for } u \in X(I), w \in X(\Omega).
\]
Observe that for \( i, j, k \in \mathbb{N} \)
\[
Eu_k = w_{k0} \text{ and } Pw_{ij} = u_i \delta_{0j}.
\]
Thus
\[
(Eu_k|w_{ij})_{X(\Omega)} = (w_{k0}|w_{ij})_{X(\Omega)} = \delta_{ki} \delta_{0j} = (u_k|u_i \delta_{0j})_{X(I)} = (u_k|Pw_{ij})_{X(I)}.
\]
Owing to bilinearity of scalar products we obtain (30) for \( u \in X_{fin}(I), w \in X_{fin}(\Omega) \) and finally by density of \( X_{fin}(I) \) (resp. \( X_{fin}(\Omega) \)) in \( X(I) \) (resp. \( X(\Omega) \) ) and continuity of scalar products and operators \( E, P \) we obtain (30) for arbitrary \( u \in X(I), w \in X(\Omega) \).

For \( u \in X^s(I) \) we obtain
\[
\|Eu\|_{X^s(\Omega)}^2 = \sum_{i,j} (1 - \lambda_{ij}^s)^{2s} (Eu|w_{ij})_{X(\Omega)}^2 = \sum_{i,j} (1 - \lambda_{ij}^s)^{2s} (u|Pw_{ij})_{X(I)}^2 = \sum_{i,j} (1 - \lambda_{ij}^s)^{2s} (u|u_i)_{X(I)}^2 \delta_{0j}
\]
\[
= \sum_i (1 - \lambda_{ij}^s)^{2s} (u|u_i)_{X(I)}^2 = \|u\|_{X^s(I)}^2,
\]

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Using Cauchy-Schwarz inequality to estimate the inner sum we obtain further that

\[ \|P_{w}\|_{X^{s}(I)} = \sum_{k} (1 - \lambda_k^2)^{2s} |P_{w}|u_k^2 \leq \sum_{k} (1 - \lambda_k^2)^{2s} |Eu_k|^2 \]

\[ = \sum_{k} (1 - \lambda_k^2)^{2s} |w|w_{k0}^2 \leq \|w\|_{X^{s}(\Omega)}^2. \]

\[ \square \]

Using Lemma 10 we obtain that operators \( P_{-1} = E', E_{-1} = P' \) satisfy

\[ P_{-1} \in \mathcal{L}(X^{-s}(\Omega), X^{-s}(I)), \quad E_{-1} \in \mathcal{L}(X^{-s}(I), X^{-s}(\Omega)), \quad s \in [0, 1]. \]

Moreover for \( u \in X(I), \ w \in X(\Omega) \)

\[ P_{-1}w = E'w = E^*w = P_w, \quad E_{-1}u = P'u = P^*u = Eu. \]

From now on we will write \( E, P \) instead of \( E_{-1}, P_{-1} \).

For \( w \in X_{fin}(\Omega) \) denote by \( Tr \) the trace of \( w \) restricted to \( I \times \{0\} \) i.e.

\[ Tr(w)(x_1) = w(x_1,0), \quad \text{for} \ x_1 \in I. \]

**Lemma 11.** For any \( s > 1/4 \) there exists \( C \) depending only on \( s \) such that for any \( w \in X_{fin}(\Omega) \)

\[ \|Tr(w)\|_{X^{s-1/4}(I)} \leq C\|w\|_{X^{s}(\Omega)}. \] (32)

**Operator** \( Tr \) **can be uniquely extended to an operator** \( \tilde{Tr} \in \mathcal{L}(X^s(\Omega), X^{s-1/4}(I)) \).

**Proof.** For \( w = \sum_{i,j \geq 0} a_{ij} w_{ij} \) where only finitely many \( a_{ij} \) are nonzero we have \( Tr(w) = \sum_{i,j \geq 0} a_{ij} u_i v_j(0) = \sum_{i \geq 0} (\sum_{j \geq 0} a_{ij} c_{2j}) u_i \). Since sytem \( \{u_i\} \) is orthogonal in \( X^{s-1/4}(I), \ 0 < c_{2j} \leq \sqrt{2} \) and \( \|u_i\|_{X^{s-1/4}(I)}^2 = (1 - \lambda_i^2)^{2s-1/2} = (1 + (i \pi/2)^2)^{2s-1/2} \) we have

\[ \|Tr(w)\|_{X^{s-1/4}(I)}^2 = \sum_{i \geq 0} (\sum_{j \geq 0} a_{ij} c_{2j})^2 \|u_i\|_{X^{s-1/4}(I)}^2 \leq 2 \sum_{i \geq 0} (\sum_{j \geq 0} |a_{ij}|^2 (1 + (i \pi/2)^2)^{2s-1/2} \]

\[ \leq 2 \left( \frac{\pi}{2} \right)^{4s-1} \sum_{i \geq 0} (\sum_{j \geq 0} |a_{ij}|^2 (1 + i)^{4s-1}. \]

Using Cauchy-Schwarz inequality to estimate the inner sum we obtain further that

\[ \|Tr(w)\|_{X^{s-1/4}(I)}^2 \leq 2 \left( \frac{\pi}{2} \right)^{4s-1} \sum_{i \geq 0} (\sum_{j \geq 0} |a_{ij}|^2 (1 + i + j)^{4s} (\sum_{j \geq 0} \frac{1}{1 + i + j}^4s)(1 + i)^{4s-1} \]

\[ \leq 2 \left( \frac{\pi}{2} \right)^{4s-1} \frac{4s}{4s-1} \sum_{i \geq 0} (\sum_{j \geq 0} |a_{ij}|^2 (1 + i + j)^{4s}, \] (33)

where the last inequality is a consequence of the following estimate

\[ \sum_{j \geq 0} \frac{(1 + i)^{4s-1}}{(1 + i + j)^{4s}} = \frac{1}{1 + i} + \sum_{j \geq 1} \frac{(1 + i)^{4s-1}}{(1 + i + j)^{4s}} \leq 1 + (1 + i)^{4s-1} \int_{i+1}^{\infty} \frac{dt}{ti^{4s}} \]

\[ = 1 + (1 + i)^{4s-1} \frac{1}{4s-1} (1 + i)^{-4s} = \frac{4s}{4s-1}. \]

On the other hand since system \( \{w_{ij}\} \) is orthogonal in \( X^s(\Omega) \) and

\[ \|w_{ij}\|_{X^s(\Omega)}^2 = (1 - \lambda_{ij}^2)^{2s} = (1 + (i \pi/2)^2 + (j \pi)^2)^{2s} \geq 3^{-2s}(1 + i + j)^{4s} \]
we have
\[ \|w\|_{X^s(\Omega)}^2 = \sum_{i,j \geq 0} |a_{ij}|^2 \|w_{ij}\|_{X^s(\Omega)}^2 = \sum_{i \geq 0} \sum_{j \geq 0} |a_{ij}|^2 (1 + (i\pi/2)^2 + (j\pi/2)^2 s) \geq 3^{-2s} \sum_{i \geq 0} \sum_{j \geq 0} |a_{ij}|^2 (1 + i + j)^4s. \]
(34)

Combining (33) and (34) we obtain (32) with \( C^2 = 3^{2s}(\pi/2)^{4s} - 8s/(4s - 1) \). Since \( X_{fin}(\Omega) \) is dense in \( X^s(\Omega) \) (see part 2 of Lemma 5) the latter part of the Lemma 14 follows. \( \square \)

From now on we write \( Tr \) instead of \( Tr^\prime \).

**Lemma 12.** The following identities hold

\[ PT_r'u = Tr Eu = PE u = u, \quad u \in X^{-s}(I), \quad s > 0 \]
(35a)

\[ R(\lambda, A_h)E = ER(\lambda, A_0), \quad h > 0 \]
(35b)

\[ e^{tA_h} = E e^{tA_0}, \quad h > 0. \]
(35c)

**Proof.** Identities \( Tr Eu = PE u = u \) are obvious for \( u \in X(I) \) and can be extended to the case when \( u \in X^{-s}(I) \) by a density argument. Then

\[ PT_r'u = E' Tr'u = (Tr^\prime) u = u, \]

from which (35a) follows.

Since \( Eu_i = w_{i0} \) for any \( i \geq 0 \) hence

\[ R(\lambda, A_h)Eu_i = R(\lambda, A_h)w_{i0} = \frac{1}{\lambda - \lambda_{i0}^\Omega} w_{i0} = E \frac{1}{\lambda - \lambda_i^\Omega} u_i = ER(\lambda, A_0) u_i. \]

Since \( \{u_i\}_{i \geq 0} \) is a Schauder basis in every \( X^s(I) \) (see part 2 of Lemma 5) we obtain (35b). Similarly one proves (35c). \( \square \)

**Lemma 13.** For \( h \in (0, 1], \ s, s' \geq -1, \ t, \lambda > 0, \ w \in X^s(\Omega) \) the following estimates hold

\[ \|R(\lambda, A_h)(I - EP)w\|_{X^{s'}(\Omega)} \leq \frac{1}{\lambda - \lambda_{01,h}^\Omega} (1 + (\lambda - \lambda_{01,h}^\Omega)^{s'-s}) \| (I - EP)w\|_{X^s(\Omega)}, \quad 0 \leq s' - s \leq 1, \]
(36)

\[ \|e^{tA_h}(I - EP)w\|_{X^{s'}(\Omega)} \leq C \left( 1 + \frac{1}{1/t^{s'-s}} \right) e^{t\lambda_{01,h}^\Omega} \| (I - EP)w\|_{X^s(\Omega)}, \quad 0 \leq s' - s. \]
(37)

where \( C \) depends only on \( s, s' \).

**Proof.** Since \( X_{fin}(\Omega) \) is dense in \( X^s(\Omega) \) (see part 2 of Lemma 5) one can assume that \( w \in X_{fin}(\Omega) \) i.e. \( w = \sum_{i,j \geq 0} a_{ij} w_{ij} \) where only finitely many \( a_{ij} \) are nonzero. Since

\[ (I - EP)w_{ij} = w_{ij} - E(u_i \delta_{0j}) = w_{ij} - w_{i0} \delta_{0j} = w_{ij} (1 - \delta_{0j}), \]

system \( \{w_{ij}\} \) is orthogonal in \( X^{s'}(\Omega) \) and \( \|w_{ij}\|_{X^{s'}(\Omega)} = (1 - \lambda_{ij}^\Omega)^{s'-s} \|w_{ij}\|_{X^s(\Omega)} \) (part 1 of Lemma 5) we obtain

\[ \|R(\lambda, A_h)(I - EP)w\|_{X^{s'}(\Omega)}^2 = \sum_{i \geq 0, j \geq 1} \frac{1}{\lambda - \lambda_{ij,h}^\Omega} a_{ij} w_{ij}^2 \|w_{ij}\|_{X^s(\Omega)}^2 = \sum_{i \geq 0, j \geq 1} \frac{1}{(\lambda - \lambda_{ij,h}^\Omega)^2} a_{ij}^2 \|w_{ij}\|_{X^s(\Omega)}^2 \]
\[ = \sum_{i \geq 0, j \geq 1} \frac{1}{(\lambda - \lambda_{ij,h}^\Omega)^2} a_{ij}^2 (1 - \lambda_{ij}^\Omega)^{2(s'-s)} \|w_{ij}\|_{X^s(\Omega)}^2 \leq M_1^2 \sum_{i \geq 0, j \geq 1} a_{ij}^2 \|w_{ij}\|_{X^s(\Omega)}^2 \]
\[ = M_1^2 \| (I - EP)w \|_{X^s(\Omega)}^2, \]

where the last inequality is due to \( \{w_{ij}\} \) being an orthonormal basis of \( X^{s'}(\Omega) \).
where

\[ M_1 = \sup \left\{ \frac{(1 - \lambda_i^\Omega)^{s'-s}}{\lambda - \lambda_{ij}^\Omega} : i \geq 0, j \geq 1 \right\}. \]

To finish the proof of (36) it is left to show that

\[ M_1 \leq \frac{1}{\lambda - \lambda_{01}^\Omega}(1 + (\lambda - \lambda_{01}^\Omega)^{s'-s}). \] (38)

Using condition \( 0 \leq s' - s \leq 1 \) and the following inequality

\[ (1 + x)^\alpha \leq 1 + x^\alpha \text{ for } x > 0, \ 0 \leq \alpha \leq 1 \]

we estimate

\[
\frac{(1 - \lambda_i^\Omega)^{s'-s}}{\lambda - \lambda_{ij}^\Omega} \leq \frac{(1 - \lambda_i^\Omega)^{s'-s}}{\lambda - \lambda_{ij}^\Omega} \leq \frac{1 + (-\lambda_i^\Omega)^{s'-s}}{\lambda - \lambda_{ij}^\Omega} \leq \frac{1 + (\lambda - \lambda_{ij}^\Omega)^{s'-s}}{\lambda - \lambda_{ij}^\Omega} = \frac{1}{\lambda - \lambda_{ij}^\Omega}(1 + (\lambda - \lambda_{ij}^\Omega)^{s'-s})
\]

from which (38) and consequently (36) follows. We move to the proof of (37). Reasoning as in the proof of (36) we obtain that for \( w \in X^s(\Omega) \)

\[ \| e^{tA_\Omega}(w - EPw) \|_{X^s(\Omega)} \leq M_2 \| w - EPw \|_{X^s(\Omega)}, \]

where

\[ M_2 = \sup \{(1 - \lambda_i^\Omega)^{s'-s} \exp(t\lambda_i^\Omega) : i \geq 0, j \geq 1 \}. \]

Using inequality (9a) from Lemma 1 we estimate for \( i \geq 0, j \geq 1 \)

\[
(1 - \lambda_i^\Omega)^{s'-s} \exp(t\lambda_i^\Omega) = (1 + (i\pi/2)^2 + (j\pi)^2)^{s'-s} \exp(-t((i\pi/2)^2 + (j\pi/h)^2))
\]

\[
= (1 + (i\pi/2)^2 + (j\pi)^2)^{s'-s} \exp(-\frac{t}{h^2}(1 + (i\pi/2)^2 + (j\pi)^2)) \exp(-t(i\pi/2)^2) \exp\left(\frac{t}{h^2}(1 + (i\pi/2)^2)\right)
\]

\[
\leq \sup \left\{ x^{s'-s} \exp\left(\frac{t}{h^2}x \right) : x \geq 1 + (i\pi/2)^2 + \pi^2 \right\} \exp(-t(i\pi/2)^2) \exp\left(\frac{t}{h^2}(1 + (i\pi/2)^2)\right)
\]

\[
\leq C\left(\frac{h^2}{t}\right)^{s'-s} + (1 + (i\pi/2)^2 + \pi^2)^{s'-s} \exp\left(-\frac{t}{h^2}(1 + (i\pi/2)^2 + \pi^2)\right) \exp(-t(i\pi/2)^2) \exp\left(\frac{t}{h^2}(1 + (i\pi/2)^2)\right)
\]

\[
= C\left(\frac{h^2}{t}\right)^{s'-s} + (1 + (i\pi/2)^2 + \pi^2)^{s'-s} \exp(-t(i\pi/2)^2) \exp\left(-\frac{t\pi^2}{h^2}\right)
\]

\[
\leq C\left(\frac{1}{t^{s'-s}} + \frac{(t(i\pi/2)^2)^{s'-s}}{t^{s'-s}}\right) \exp(-t(i\pi/2)^2) \exp\left(-\frac{t\pi^2}{h^2}\right)
\]

\[
\leq C + \frac{1}{t^{s'-s}} \sup \left\{ x^{s'-s} \exp(-x) : x \geq 0 \right\} \exp\left(-\frac{t\pi^2}{h^2}\right) \leq C(1 + \frac{1}{t^{s'-s}}) \exp\left(-\frac{t\pi^2}{h^2}\right).
\]

\( \square \)

For \( 1 \leq p < \infty \) and \( 0 \geq f \in L_p(I) \) we define multiplication operator \( M_f \)

\[ M_f : L^\infty(I) \supset D(M_f) \to L^\infty(I), \ M_f u = fu, \] (39)

where \( D(M_f) = \{ u \in L^\infty(I) : fu \in L^\infty(I) \} \). Observe that if \( u \in L^\infty(I) \) and \( \text{Re}(\lambda) > 0 \) then \( R(\lambda, M_f)u = \frac{u}{\lambda} \in L^\infty(I) \) and \( \| R(\lambda, M_f)u \|_{L^\infty(I)} \leq 1/|\lambda| \), which proves that \( M_f \) is sectorial and thus generates an analytic semigroup \( e^{tM_f} \):

\[ e^{tM_f}u = e^{tf}u, \ u \in L^\infty(I). \]
Lemma 14. Assume that $0 \geq f, f_1, f_2 \in L_p(I)$. Then for $t, t' \geq 0$

\begin{align}
  \|e^{tMf}\|_{L(L_\infty(I))} & \leq 1, \quad (40) \\
  \|e^{tMf} - e^{t'Mf}\|_{L(L_\infty(I), L_p(I))} & \leq |t' - t|\|f\|_{L_p(I)}, \quad (41) \\
  \|e^{tMf_1} - e^{tMf_2}\|_{L(L_\infty(I), L_p(I))} & \leq t\|f_1 - f_2\|_{L_p(I)}. \quad (42)
\end{align}

Proof. Using the following inequalities

$$
0 < e^x \leq 1, \quad |e^x - e^y| \leq |x - y|, \quad x, y < 0,
$$

we get for $u \in L_\infty(I)$

\begin{align}
  \|e^{tMf}u\|_{L_\infty(I)} & = \|e^{tf}u\|_{L_\infty(I)} \leq \|e^{tf}\|_{L(L_\infty(I))}\|u\|_{L_\infty(I)} \leq \|u\|_{L_\infty(I)} \\
  \|(e^{tMf} - e^{t'Mf})u\|_{L_p(I)} & \leq \|(e^{tf} - e^{t'f})u\|_{L_p(I)} \leq \|e^{tf} - e^{t'f}\|_{L_p(I)}\|u\|_{L_\infty(I)} \leq |t' - t|\|f\|_{L_p(I)}\|u\|_{L_\infty(I)} \\
  \|(e^{tMf_1} - e^{tMf_2})u\|_{L_p(I)} & \leq \|(e^{tf_1} - e^{tf_2})u\|_{L_p(I)} \leq \|e^{tf_1} - e^{tf_2}\|_{L_p(I)}\|u\|_{L_\infty(I)} \leq t\|f_1 - f_2\|_{L_p(I)}\|u\|_{L_\infty(I)}
\end{align}

from which (40), (41) and (42) follow. \qed

3.4 Auxiliary functions

Let us introduce the standard one dimensional mollifier

$$
\eta(x_1) = \begin{cases} 
  C \exp \left( \frac{1}{|x_1| - 1} \right), & |x_1| < 1 \\
  0, & |x_1| \geq 1
\end{cases}, \quad \eta^\epsilon(x_1) = \eta(x_1/\epsilon)/\epsilon, \quad \epsilon > 0
$$

where $C$ is such that $\int_\mathbb{R} \eta = 1$.

Lemma 15. For any $0 < s$ the following convergence holds

$$
\lim_{\epsilon \to 0^+} \|\eta^\epsilon - \delta\|_{X^{-1/4-s}(I)} = 0. \quad (43)
$$

Proof. Without loss of generality assume that $s < 1/8$. It is easy to show that every sequence $(\epsilon_n)_{n=1}^\infty$ of positive numbers which converges to 0 has a subsequence $(\epsilon_{nk})_{k=1}^\infty$ such that

$$
\eta^{\epsilon_{nk}} \to \delta \text{ in } X^{-1/4-s}(I). \quad (44)
$$

Fix any sequence $(\epsilon_n)_{n=1}^\infty$ of positive numbers which converges to 0. Since $(\eta^{\epsilon_n})_{n=1}^\infty$ is a bounded sequence in $\mathcal{M}(\overline{T})$ and the latter space imbeds compactly into $X^{-1/4-s}(I)$ ($\mathcal{M}(\overline{T}) = C(\overline{T})^*$, $X^{-1/4-s}(I) = X^{1/4+s}(I)^*$ = $W^{1/2+2s}(I)^*$ and $W^{1/2+2s}(I)$ imbeds compactly and densely into $C(\overline{T})$), one can choose a subsequence $(\epsilon_{nk})_{k=1}^\infty$ such that

$$
\eta^{\epsilon_{nk}} \to u \text{ in } X^{-1/4-s}(I),
$$

for certain $u \in X^{-1/4-s}(I)$. Finally observe that for any $v \in X^{1/4+s}(I)$ one has

$$
\langle u, v \rangle_{(X^{-1/4-s}(I), X^{1/4+s}(I))} = \lim_{k \to \infty} \langle \eta^{\epsilon_{nk}}, v \rangle_{(X^{-1/4-s}(I), X^{1/4+s}(I))} = \lim_{k \to \infty} \int_I \eta^{\epsilon_{nk}} v = v(0),
$$

where the first equality is a consequence of the fact that strong convergence in $X^{-1/4-s}(I)$ implies convergence in the weak star topology of $X^{-1/4-s}(I)$ while the third equality follows from a well known fact that $\eta^\epsilon$ converges to $\delta$ in the weak star topology of $\mathcal{M}(\overline{T})$. Thus $u = \delta$ and (44) follows. \qed
Lemma 16. The following estimates hold

\begin{align}
\|m^\mu\|_{X^{1/2-s}(\Omega)} &\leq C, \quad 0 < s \quad \text{ (48)} \\
\|m^\mu - m^{\mu_0}\|_{X^{1/2-s}(\Omega)} &\leq C\|\eta^f - \delta\|_{X^{-1/4-s}(I)}, \quad 0 < s \quad \text{ (49)} \\
\|m^{\mu_0} - Em^0\|_{X^{1/2-s}(\Omega)} &\leq C\frac{1}{|\lambda^1_{01,h}|^{s/2}}, \quad 0 < s \leq 2, \quad \text{ (50)}
\end{align}

where C does not depend on μ. Moreover \( m^\mu, m^0 \geq 0 \).

Proof. Using (27), (32), (43) we estimate

\[ \|m^\mu\|_{X^{1/2-s}(\Omega)} \leq p_1\|R(b_1, A_h)\|_{\mathcal{L}(X^{-1/2-s}(\Omega), X^{1/2-s}(\Omega))}\|Tr^f\|_{\mathcal{L}(X^{-1/4-s}(I), X^{-1/2-s}(\Omega))}\|\eta^f\|_{X^{-1/4-s}(I)} \leq C, \]

from which (48) follows. To prove (49) observe that:

\[ \|m^\mu - m^{\mu_0}\|_{X^{1/2-s}(\Omega)} \leq p_1\|R(b_1, A_h)\|_{\mathcal{L}(X^{-1/2-s}(\Omega), X^{1/2-s}(\Omega))}\|Tr^f\|_{\mathcal{L}(X^{-1/4-s}(I), X^{-1/2-s}(\Omega))}\|\eta^f - \delta\|_{X^{-1/4-s}(I)}. \]

Using (35a) we get that \( \delta = PT^r\delta \) hence using (35b) and (36) we obtain:

\[ \|m^{\mu_0} - Em^0\|_{X^{1/2-s}(\Omega)} = p_1\|R(b_1, A_h)(I - EP)Tr^\delta\|_{X^{1/2-s}(\Omega)} \leq C\left(\frac{1}{(b_1 - \lambda^\mu_{01,h})^{s/2}} + \frac{1}{b_1 - \lambda^\mu_{01,h}}\right)\|I - EP\|_{\mathcal{L}(X^{-1/2-s/2}(\Omega), X^{1/2-s/2}(\Omega))}\|Tr^f\|_{\mathcal{L}(X^{-1/4-s/2}(I), X^{-1/2-s/2}(\Omega))}\|\delta\|_{X^{-1/4-s/2}(I)} \leq C. \]

Moreover using (32), (43) we have

\[ \|I - EP\|_{\mathcal{L}(X^{-1/2-s/2}(\Omega), X^{1/2-s/2}(\Omega))}\|Tr^f\|_{\mathcal{L}(X^{-1/4-s/2}(I), X^{-1/2-s/2}(\Omega))}\|\delta\|_{X^{-1/4-s/2}(I)} \leq C. \]

Finally to finish the proof of (50) observe that

\[ \frac{1}{(b_1 - \lambda^\mu_{01,h})^{s/2}} + \frac{1}{b_1 - \lambda^\mu_{01,h}} \leq C\frac{1}{|\lambda^\mu_{01,h}|^{s/2}}. \]

Using maximum principle for elliptic boundary value problem (47) we get that \( m^\mu \geq 0 \) for \( \epsilon > 0 \). Then (49) implies that \( m^{\mu_0} \geq 0 \) while \( m^0 \geq 0 \) follows from (50).

Lemma 17. \( m^\mu \in W^1_p(\Omega) \) for any \( 1 \leq p < 2 \).

Proof. The claim follows from [16, Lemma 1].
In this section we study system (7) with $\delta$ with boundary and initial conditions.

Define operators $\alpha$.

Define spaces $X$.

Observe that due to Lemma 7 we have $u$ and for $\mathbf{u} \in X^\alpha$ set

$$F_i(\mathbf{u}) = \frac{\partial u_i}{\partial t}(t,x) + \mathbf{b}_i \cdot \nabla u_i + \mathbf{c}_i \cdot u_i = 0, \quad (t,x) \in \Omega_T$$

$$\mathbf{A}_i u = A_i - b_i u_i, \quad u \in X^\alpha_i, \quad i = 3, 4, 5$$

$$\mathbf{F}_i(\mathbf{u}) = f_i(Tr(u_i), u_2, \ldots, u_5), \quad i = 2, 3, 4, 5.$$
**Theorem 1.** Assume \([52]\). Then for every \(0 \leq u_0 \in X^\alpha\):

1. System \([51]\) has a unique, maximal \(X^\alpha\) solution \(u\).

2. \(u(t)\) is nonnegative for \(t \in [0, T_{\text{max}}]\).

3. There exists \(C\) depending only on \(\|u_0\|_\infty + \|u_40\|_\infty + \|u_50\|_\infty, b_3, b_4, b_5, p_3\) such that for \(t \in [0, T_{\text{max}}]\)

\[
\|u_3(t)\|_\infty + \|u_4(t)\|_\infty + \|u_5(t)\|_\infty \leq C.
\]  

(53)

4. \(T_{\text{max}} = \infty\).

**Proof.**

**Step 1 - local existence.**

Observe that due to well known Sobolev imbeddings we have \(X^\alpha = X^4(\Omega) \times X^{1/2}(I) \times (L_\infty(I))^3 \subset C(\overline{\Omega}) \times (L_\infty(I))^4\), in particular for \(u_1 \in X^{\alpha_1}_1\) one has

\[
\|Tr(u_1)\|_{L_\infty(I)} \leq C\|u_1\|_{X^{\alpha_1}_1}
\]

(54)

From \([54]\) we deduce that for \(u, w \in X^\alpha\) the following estimates hold

\[
\sum_{i=1}^{5} \|F_i(u)\|_{X^\alpha_i} \leq C\left\{ (1 + \sum_{i=1}^{2} \|u_i\|_{X^{\alpha_i}_i}) (1 + \|u_3\|_{X^{\alpha_3}_i}) + \sum_{i=4}^{5} \|u_i\|_{X^{\alpha_i}_i} + \|\omega\|_{X(I)} \right\}
\]

(55)

\[
\sum_{i=1}^{5} \|F_i(u) - F_i(u')\|_{X^{\alpha_i}_i} \leq C\left\{ \sum_{i=1}^{2} \|u_i - u'_i\|_{X^{\alpha_i}_i} (1 + \|u_3\|_{X^{\alpha_3}_i}) + \|u_3\|_{X^{\alpha_3}_i} \right\}
\]

(56)

Using above estimates we conclude that assumptions of Lemma \([3]\) are satisfied which results in the existence of unique maximally defined \(X^\alpha\) solution to \([51]\).

**Step 2 - nonnegativity of solutions.**

Reasoning as in step 1 we obtain that system

\[
\begin{align*}
\partial_t v_1 + \text{div}(J_h(v_1)) + b_1 v_1 &= 0, & (t, x) &\in \Omega_T \\
\partial_t v_2 - d \partial^2_{x_1} v_2 &= f_{2+}(v), & (t, x) &\in (\partial_1 \Omega)_T \\
\partial_t v_3 &= f_{3+}(v), & (t, x) &\in (\partial_1 \Omega)_T \\
\partial_t v_4 &= f_{4+}(v), & (t, x) &\in (\partial_1 \Omega)_T \\
\partial_t v_5 &= f_{5+}(v), & (t, x) &\in (\partial_1 \Omega)_T \\
\end{align*}
\]

(57a) - (57e)

with boundary and initial conditions

\[
\begin{align*}
-J_h(v_1)\nu &= 0, & (t, x) &\in (\partial_0 \Omega)_T \\
-J_h(v_1)\nu &= f_{1+}(v) + \omega, & (t, x) &\in (\partial_1 \Omega)_T \\
\partial_{x_1} v_2 &= 0, & (t, x) &\in (\partial_1 \Omega)_T \\
v(0, \cdot) &= u_0 \\
\end{align*}
\]

where for \(i = 1, \ldots, 5\) and \(v \in \mathbb{R}^5\)

\[
f_{i+}(v) = f_i((v_1)_+, \ldots, (v_5)_+)
\]
has a unique maximal $X^\alpha$ solution $v(t)$. Note by $T'_{\text{max}}$ its time of existence.

Testing (57a), (57b) by $(v_1), \ldots, (v_5)$ we obtain

$$-\frac{1}{2} \frac{d}{dt} \|(v_1)-\|_{X(\Omega)}^2 - \|\partial_x (v_1)-\|_{X(\Omega)}^2 - h^{-2} \|\partial x (v_1)-\|_{X(\Omega)}^2 - b_1 \|v_1\|_X^2 =$$

$$\int_I (f_1+(v_1(x,t),0,v_2(x),\ldots, v_5(x))) + (v_1(x,t),0)_{x_1} dx_1$$

$$-\frac{1}{2} \frac{d}{dt} \|(v_2)-\|_{X(I)}^2 - d \|\partial_x (v_2)-\|_{X(I)}^2 = \int_I f_2+(v_1(x,t),0,v_2(x),\ldots, v_5(x))) (v_2(x))_{x_1} dx_1$$

$$-\frac{1}{2} \frac{d}{dt} \|(v_3)-\|_{X(I)}^2 = \int_I f_3+(v_1(x,t),0,v_2(x),\ldots, v_5(x))) (v_3(x))_{x_1} dx_1, \ i = 3, 4, 5.$$ 

Since right hand sides of above equalities are nonnegative we obtain that

$$\frac{d}{dt} \|(v_1)-\|_{X(\Omega)}^2 + 5 \sum_{i=2}^5 \|v_i\|_{X(I)}^2 \leq 0$$

$$\|(v_1(t))-\|_{X(\Omega)}^2 + 5 \sum_{i=2}^5 \|v_i(t)\|_{X(I)}^2 \leq \|(v_0)-\|_{X(\Omega)}^2 + 5 \sum_{i=2}^5 \|v_0\|_{X(I)}^2 = 0.$$ 

Which proves that the only solution of system (57) is nonnegative. Since for $v \geq 0$ there is $f_i+(v) = f_i(v)$ we see that $T'_{\text{max}} \geq T'_{\text{max}}$ and $u(t) = v(t)$ for $t \in [0,T'_{\text{max}})$. Finally observe that if $T'_{\text{max}} < \infty$ then owing to the blow-up condition (13)

$$\lim_{t \to T_{\text{max}}^-} \|u(t)\|_{X^\alpha} = \lim_{t \to T'_{\text{max}}^-} \|v(t)\|_{X^\alpha} = \infty$$

whence $T_{\text{max}} = T'_{\text{max}}$ and finally $u(t) \geq 0$ for $t \in [0,T_{\text{max}})$.

**Step 3 - global solvability:** $T_{\text{max}} = \infty$.

Adding equations (51c), (51d), (51e) and using nonnegativity of $u$ we get

$$\partial_t (u_3 + u_4 + u_5) + \min \{b_3, b_4, b_5\} (u_3 + u_4 + u_5) \leq p_3$$

from which we conclude that there exists $C$ depending only on $\|u_{30}\|_{\infty} + \|u_{40}\|_{\infty} + \|u_{50}\|_{\infty}, b_3, b_4, b_5, p_3$ such that

$$\|u_3(t)\|_{\infty} + \|u_4(t)\|_{\infty} + \|u_5(t)\|_{\infty} \leq C, \ t \in [0,T_{\text{max}}).$$

Using (58) and (55) we get that condition (14) is satisfied which gives $T_{\text{max}} = \infty$.

$$\square$$

5 The case of a singular source

In this section we study the well-posedness of system (7). Moreover we analyse limit $h \to 0^+$. Recall that $\mu = (h, \epsilon) \in (0, 1] \times [0, 1]$. Substituting in (7) singular source $\delta$ by its mollification $\eta^\nu$ we get

$$\partial_t u_1^\nu + \text{div} (J_3(u_1^\nu)) + b_1 u_1^\nu = 0, \quad (t, x) \in \Omega_T$$

$$\partial_t u_2^\nu - d \partial_x^2 u_2^\nu = f_2(u^\nu), \quad (t, x) \in (\partial_1 \Omega)_T$$

$$\partial_t u_3^\nu = f_3(u^\nu), \quad (t, x) \in (\partial_1 \Omega)_T$$

$$\partial_t u_4^\nu = f_4(u^\nu), \quad (t, x) \in (\partial_1 \Omega)_T$$

$$\partial_t u_5^\nu = f_5(u^\nu), \quad (t, x) \in (\partial_1 \Omega)_T$$
with boundary and initial conditions

\[-J_h(u^\mu_1)\nu = 0, \quad (t, x) \in (\partial_0 \Omega)_T\]
\[-J_h(u^\mu_1)\nu = f_1(u^\mu) + p_1\eta^\mu, \quad (t, x) \in (\partial_1 \Omega)_T\]
\[\partial_{x_1} u^\mu_1 = 0, \quad (t, x) \in (\partial_1 \Omega)_T\]
\[u^\mu(0, \cdot) = u_0,\]

where

\[u^\mu = (u^\mu_1, u^\mu_2, u^\mu_3, u^\mu_4, u^\mu_5).\]

Using Theorem 1 we obtain that for \(\epsilon \in (0, 1]\) system (59) has a unique globally defined \(X^\alpha\) solution. Unfortunately for \(\epsilon = 0\) presence of a singular source term \(\eta^0 = \delta\) in (60) causes \(u^\mu_1\) to be an unbounded function of \(x\) for any positive time which prevents us from proving the existence of a \(X^\alpha\) solution. This motivates us to generalize the notion of solution. We rewrite our problem in the new variables so that system (59) has a unique globally defined \(X^\alpha\) solution. Unfortunately for \(\epsilon = 0\) presence of a singular source term \(\eta^0 = \delta\) in (60) causes \(u^\mu_1\) to be an unbounded function of \(x\) for any positive time which prevents us from proving the existence of a \(X^\alpha\) solution. This motivates us to generalize the notion of solution. We rewrite our problem in the new variables so that system (59) with singular source term is transformed into system (63) with regular sources and low regularity initial data.

Observe that putting

\[z^\mu = (z^\mu_1, z^\mu_2, z^\mu_3, z^\mu_4, z^\mu_5) = M(u^\mu_1 - m^\mu, u^\mu_2, u^\mu_3, u^\mu_4, u^\mu_5),\]
\[z^\mu_0 = (z^\mu_{01}, z_{02}, z_{03}, z_{04}, z_{05}) = M(u_{01} - m^\mu, u_{02}, u_{03}, u_{04}, u_{05}),\]

where \(M\) denotes the following matrix

\[M = \begin{bmatrix}
1, 0, 0, 0, 0 \\
0, 1, 0, 0, 0 \\
0, 0, 1, 0, 0 \\
0, 0, 1, 1, 0 \\
0, 0, 1, 1, 1
\end{bmatrix},\]

system (59) can be rewritten as

\[\partial_t z^\mu_1 + \text{div}(J_h(z^\mu_1)) + b_1 z^\mu_1 = 0, \quad (t, x) \in \Omega_T \quad (63a)\]
\[\partial_t z^\mu_2 - \partial_x^2 z^\mu_2 = g_2^\mu(z^\mu), \quad (t, x_1) \in (\partial_1 \Omega)_T \quad (63b)\]
\[\partial_t z^\mu_3 + Tr(m^\mu)z^\mu_3 = g_3(z^\mu), \quad (t, x_1) \in (\partial_1 \Omega)_T \quad (63c)\]
\[\partial_t z^\mu_4 = g_4(z^\mu), \quad (t, x_1) \in (\partial_1 \Omega)_T \quad (63d)\]
\[\partial_t z^\mu_5 = g_5(z^\mu), \quad (t, x_1) \in (\partial_1 \Omega)_T \quad (63e)\]

with boundary and initial conditions

\[-J_h(z^\mu_1)\nu = 0, \quad (t, x) \in (\partial_0 \Omega)_T\]
\[-J_h(z^\mu_1)\nu = g_1^\mu(z^\mu), \quad (t, x_1) \in (\partial_1 \Omega)_T\]
\[\partial_{x_1} z^\mu_1 = 0, \quad (t, x_1) \in (\partial_1 \Omega)_T\]
\[z^\mu(0, \cdot) = z^\mu_0,\]

where

\[g_1^\mu, g_2^\mu : I \times \mathbb{R}^5 \to \mathbb{R}, \quad g_3, g_4, g_5 : \mathbb{R}^5 \to \mathbb{R}\]
\[g_1^\mu(z) = -c_1 z_1 + c_2 z_2 - z_1 z_3 + c_4(z_4 - z_3) - (c_1 + z_3)Tr(m^\mu)\]
\[g_2^\mu(z) = -b_2 z_2 + c_1 z_1 - c_2 z_2 - c_3 z_2 z_3 + c_5(z_5 - z_4) + c_1 Tr(m^\mu)\]
\[g_3(z) = -b_3 z_3 - z_1 z_3 - c_3 z_2 z_3 + c_4(z_4 - z_3) + c_5(z_5 - z_4) + p_3\]
\[g_4(z) = -b_3 z_3 - b_4(z_4 - z_3) - c_3 z_2 z_3 + c_5(z_5 - z_4) + p_3\]
\[g_5(z) = -b_3 z_3 - b_4(z_4 - z_3) - b_5(z_5 - z_4) + p_3\]
5.1 Definition of M-mild solution to (59)

Assume that:

\[ d, b > 0, \ c, p \geq 0, \quad 2 < p < \infty, \ 0 < \theta < \min \left\{ \frac{1}{16}, \frac{1}{2p} \right\}, \tag{64a} \]

\[ 0 \leq u_0 = (u_{01}, \ldots, u_{05}) \in X^{1/2+\theta}(\Omega) \times X^{1/2}(I) \times \{L_\infty(I)\}^3. \tag{64c} \]

Define Banach spaces

\[ Z = Z_1 \times Z_2 \times Z_3 \times Z_4 \times Z_5 = X^{-1/4-\theta}(\Omega) \times X(I) \times L_p(I) \times L_p(I) \times L_p(I) \]

\[ Z_+ = Z_1+ \times Z_2+ \times Z_3+ \times Z_4+ \times Z_5+ = X^{1/2+\theta}(\Omega) \times X^{1/2}(I) \times L_\infty(I) \times L_\infty(I) \times L_\infty(I) \]

and for \( z \in Z_+ \) put

\[ G_1^\mu(z) = Tr^*(g_1^\mu(Tr(z_1), z_2, z_3, z_4, z_5)), \]

\[ G_2^\mu(z) = g_2^\mu(Tr(z_1), z_2, z_3, z_4, z_5), \]

\[ G_i(z) = g_i(Tr(z_1), z_2, z_3, z_4, z_5), \quad i \in \{3, 4, 5\}. \]

**Definition 1.** For \( \mu = (h, \epsilon) \in (0, 1] \times [0, 1] \) we say that \( u^\mu \) is a **M-mild** solution of system (59) on \([0, T]\) if

1. conditions (64) are satisfied,

2. for every \( T' < T \)

\[ z_1^\mu \in C([0, T], Z_1), \quad t^{2\theta}z_1^\mu \in L_\infty(0, T'; Z_{1+}) \tag{65a} \]

\[ z_2^\mu \in C([0, T], Z_2) \tag{65b} \]

\[ z_3^\mu \in C([0, T], Z_3) \cap L_\infty(0, T'; Z_{3+}) \tag{65c} \]

\[ z_i^\mu \in C([0, T], Z_{i+}), \quad i \in \{4, 5\} \tag{65d} \]

3. for every \( t \in [0, T] \) the following Duhamel formulas hold

\[ z_1^\mu(t) = e^{t(A_{h-b_1})}z_{01}^\mu + \int_0^t e^{(t-\tau)(A_{h-b_1})}G_1^\mu(z^\mu(\tau))d\tau \tag{66a} \]

\[ z_2^\mu(t) = e^{tA_0}z_{02} + \int_0^t e^{(t-\tau)dA_0}G_2^\mu(z^\mu(\tau))d\tau \tag{66b} \]

\[ z_3^\mu(t) = e^{-tTr(m_\mu)}z_{03} + \int_0^t e^{-(t-\tau)Tr(m_\mu)}G_3(z^\mu(\tau))d\tau \tag{66c} \]

\[ z_i^\mu(t) = z_{0i} + \int_0^t G_i(z^\mu(\tau))d\tau, \quad i \in \{4, 5\}. \tag{66d} \]

where \( z^\mu, z_i^\mu \) are related with \( u^\mu, u_0 \) by (61) and (62).

Using (65a), Lemma 17 and imbedding \( Z_{1+} \subset W_p^1(\Omega) \) for \( p < 2 \) we obtain

**Remark 1.** If \( u^\mu \) is a M-mild solution of system (59), then \( t^{2\theta}u_1^\mu = t^{2\theta}(z_1^\mu + m^\mu) \in L_\infty(0, T'; W_p^1(\Omega)), \ p < 2. \)
5.2 The main results

We will prove the following two theorems

Theorem 2. Assume \((64)\). Then

1. For every \(\mu = (h, \epsilon) \in (0, 1) \times (0, 1)\), \(0 < T \leq \infty\) system \((59)\) has a unique M-mild solution \(u^\mu\) defined on \([0, T)\). This solution is nonnegative and is also \(X^\alpha\) solution.

2. For every \(h \in (0, 1), 0 < T \leq \infty\) system \((59)\) has a unique M-mild solution \(u^{\mu_0}\) defined on \([0, T)\). The solution is nonnegative. Moreover if \(T = \infty\) then for every \(0 < T' < \infty\) the following convergence holds

\[
\lim_{\epsilon \to 0^+} \left\{ \sum_{i=1}^{5} \| u^\mu_i - u^{\mu_0}_i \|_{L_\infty(0,T';Z_i)} \right\} = 0.
\]  

(67)

Theorem 3. Assume \((64)\) and let \(u^{\mu_0}\) be for \(h \in (0, 1)\) the unique M-mild solution of system \((59)\) (with \(\epsilon = 0\)) on \([0, \infty)\). Then for every \(0 < T < \infty\)

\[
\lim_{h \to 0^+} \left\{ \| z^{\mu_0}_1 - z^0_1 \|_{L_\infty(0,T;Z_{1+})} + \sum_{i=2}^{5} \| u^{\mu_0}_i - u^0_i \|_{L_\infty(0,T;Z_i)} \right\} = 0,
\]  

(68)

where \(z^{\mu_0}_1 = u^{\mu_0}_1 - m^{\mu_0}, z^0_1 = E(u^0_1 - m^0)\) and \(u^0 = (u^0_1, \ldots, u^0_5)\) is the unique classical solution of

\[
\begin{align*}
\partial_t u_1 - \partial_{x_1}^2 u_1 + b_1 u_1 &= f_1(u) + p_1 \delta & (t, x_1) \in I_T \\
\partial_t u_2 - d_2 \partial_{x_1}^2 u_2 &= f_2(u), & (t, x_1) \in I_T \\
\partial_t u_3 &= f_3(u), & (t, x_1) \in I_T \\
\partial_t u_4 &= f_4(u), & (t, x_1) \in I_T \\
\partial_t u_5 &= f_5(u), & (t, x_1) \in I_T 
\end{align*}
\]

(69)

with boundary and initial conditions

\[
\begin{align*}
\partial_{x_1} u_1 &= \partial_{x_2} u_2 = 0, & (t, x_1) \in (\partial I)_T \\
u(0, .) &= u^0 = [Pu_{01}, u_{02}, u_{03}, u_{04}, u_{05}].
\end{align*}
\]

Remark 2. For global well-posedness of system \((69)\) we refer to \([13]\).

5.3 Proof of Theorem \((2)\)

Step 1 - estimates for \(G_i\)’s.

Lemma 18. For \(z, z' \in \mathbb{Z}_+, \mu \in \mathbb{Z}_+ \times [0, 1]\) the following estimates hold

\[
\begin{align*}
\sum_{i=1}^{2} \| G_i^\mu(z) \|_{Z_{i-}} + \sum_{i=3}^{5} \| G_i(z) \|_{Z_{i+}} &\leq C \left( (1 + \sum_{i=1}^{2} \| z_i \|_{Z_{i+}})(1 + \| z_3 \|_{Z_{3+}}) + \sum_{i=4}^{5} \| z_i \|_{Z_{i+}} \right), \\
\sum_{i=1}^{2} \| G_i^\mu(z) - G_i^\mu(z') \|_{Z_{i-}} + \sum_{i=3}^{5} \| G_i(z) - G_i(z') \|_{Z_{i+}} &\leq C \left( (1 + \| z_3 \|_{Z_{3+}} + \| z_3' \|_{Z_{3+}}) \sum_{i=1}^{2} \| z_i - z_i' \|_{Z_i} + (1 + \sum_{i=1}^{2} (\| z_i \|_{Z_{i+}} + \| z_i' \|_{Z_{i+}})) \| z_3 - z_3' \|_{Z_3} + \sum_{i=4}^{5} \| z_i - z_i' \|_{Z_i} \right),
\end{align*}
\]

25
\[
\sum_{i=1}^{2} \|G_i^n(z) - G_i^n(z')\|_{Z_{i-}} + \sum_{i=3}^{5} \|G_i(z) - G_i(z')\|_{Z_{i+}} \\
\leq C\left(1 + \|z_3\|_{Z_{3+}} + \|z_3'\|_{Z_{3+}}\right) + \left(1 + \sum_{i=1}^{2} \left(\|z_i\|_{Z_{i+}} + \|z_i'\|_{Z_{i+}}\right)\right)\|z_3 - z_3'\|_{Z_{3+}} + \sum_{i=4}^{5} \|z_i - z_i'\|_{Z_{i+}}),
\]
\[
\sum_{i=1}^{2} \|G_i^n(z) - G_i^{m_0}(z)\|_{Z_{i-}} \leq C(1 + \|z_3\|_{Z_{3+}})\|\eta' - \delta\|_{X^{-1/4-\theta}(I)},
\]
where \(C\) does not depend on \(\mu\).

Proof. We will prove inequalities involving \(G_1^n\) and \(G_3\). Inequalities involving \(G_2^n, G_4\) and \(G_5\) can be derived analogously. Using condition \((64b)\) we get that \(X^{1/4-\theta}(I) = W^{1/2-2\theta}(I) \subset L_p(I)\) from which \(Tr \in L(Z_{1}, L_p(I))\). Using above observation and Hölder inequality we estimate
\[
\|G_1^n(z)\|_{Z_{1-}} = \|Tr^n g_1^n(Tr(z_1), z_2, z_3, z_4, z_5)\|_{X^{-1/4-\theta}(I)} \leq C\|g_1^n(Tr(z_1), z_2, z_3, z_4, z_5)\|_{L_2(I)}
\]
\[
\leq C\left(\|Tr(z_1)\|_{L_2(I)} + \|z_2\|_{L_2(I)} + \|z_1\|_{L_2(I)}\|z_3\|_{L_\infty(I)} + \|z_4\|_{L_2(I)} + \|z_3\|_{L_2(I)} + (1 + \|z_3\|_{L_\infty(I)})\|Tr(m^n)\|_{L_2(I)}\right)
\]
\[
\leq C\left(1 + \sum_{i=1}^{2} \|z_i\|_{Z_{i+}}(1 + \|z_3\|_{Z_{3+}}) + \sum_{i=4}^{5} \|z_i\|_{Z_{i+}}\right)
\]
\[
\|G_3(z)\|_{Z_{3+}} \leq C\left(\|z_3\|_{L_\infty(I)} + \|Tr(z_1)\|_{L_\infty(I)}\|z_3\|_{L_\infty(I)} + \|z_2\|_{L_\infty(I)}\|z_3\|_{L_\infty(I)} + \|z_4\|_{L_\infty(I)}\|z_3\|_{L_\infty(I)} + \sum_{i=3}^{5} \|z_i\|_{L_\infty(I)} + 1\right)
\]
\[
\leq C\left(1 + \sum_{i=1}^{2} \|z_i\|_{Z_{i+}}(1 + \|z_3\|_{Z_{3+}}) + \sum_{i=4}^{5} \|z_i\|_{Z_{i+}}\right)
\]
\[
\|G_1^n(z) - G_1^{m_0}(z')\|_{Z_{1-}} \leq C\|g_1^n(Tr(z_1), z_2, z_3, z_4, z_5) - g_1^{m_0}(Tr(z_1'), z_2', z_3', z_4', z_5')\|_{L_2(I)}
\]
\[
\leq C\left(\|Tr(z_1 - z_1')\|_{L_2(I)} + \|z_2 - z_2'\|_{L_2(I)} + \|z_1 - z_1'\|_{L_2(I)}\|z_3\|_{L_\infty(I)} + \|z_4 - z_4'\|_{L_2(I)} + \|z_3 - z_3'\|_{L_2(I)}\|Tr(z_1')\|_{L_\infty(I)} + \|z_2 - z_2'\|_{L_2(I)} + \|Tr(m^n)\|_{L_\infty(I)}\|z_3 - z_3'\|_{L_\infty(I)}\right)
\]
\[
\leq C\left(1 + \|z_3\|_{Z_{3+}} + \|z_3'\|_{Z_{3+}}\right)\sum_{i=1}^{2} \|z_i - z_i'\|_{Z_{i}}
\]
\[
+ (1 + \sum_{i=1}^{2} (\|z_i\|_{Z_{i+}} + \|z_i'\|_{Z_{i+}})\|z_3 - z_3'\|_{Z_{3+}} + \sum_{i=4}^{5} \|z_i - z_i'\|_{Z_{i+}}\right)
\]
\[
\|G_3(z) - G_3(z')\|_{Z_{3}} \leq C\left(\|z_3 - z_3'\|_{L_\infty(I)} + \|Tr(z_1 - z_1')\|_{L_\infty(I)}\|z_3\|_{L_\infty(I)} + \|z_3 - z_3'\|_{L_\infty(I)}\|Tr(z_1')\|_{L_\infty(I)}\right)
\]
\[
+ \|z_2 - z_2'\|_{L_\infty(I)}\|z_3\|_{L_\infty(I)} + \|z_3 - z_3'\|_{L_\infty(I)}\|z_2^I\|_{L_\infty(I)} + \sum_{i=4}^{5} \|z_i - z_i'\|_{L_\infty(I)}\right)
\]
\[
\leq C\left(1 + \|z_3\|_{Z_{3+}} + \|z_3'\|_{Z_{3+}}\right)\sum_{i=1}^{2} \|z_i - z_i'\|_{Z_{i+}} + (1 + \sum_{i=1}^{2} (\|z_i\|_{Z_{i+}} + \|z_i'\|_{Z_{i+}}))\|z_3 - z_3'\|_{Z_{3+}} + \sum_{i=4}^{5} \|z_i - z_i'\|_{Z_{i+}}\right)
\]
\[
\|G_1^n(z) - G_1^{m_0}(z)\|_{Z_{1-}} \leq C\|(c_1 + z_3)Tr(m^n - m^{m_0})\|_{L_2(I)} \leq C(1 + \|z_3\|_{L_\infty(I)})\|Tr(m^n - m^{m_0})\|_{L_2(I)}
\]
\[
\leq C(1 + \|z_3\|_{Z_{3+}})\|\eta' - \delta\|_{X^{-1/4-\theta}(I)}
\]
Step 2 - uniqueness of M-mild solutions.

Assume that \(u, u'\) are two M-mild solutions of system (59) on \([0, T), 0 < T \leq \infty\), with the same initial condition. Let \(z, z'\) be related with \(u, u'\) by (61), (62). Fix \(T' < T\).

For \(t \in (0, T')\) denote \(f(t) = \sum_{i=1}^{5} \|z_i(t) - z'_i(t)\|_{Z_i}\). Put

\[
K_1(T') = \|t^{2\theta} z_1\|_{L_\infty(0, T'; Z_{1+})} + \|t^{2\theta} z'_1\|_{L_\infty(0, T'; Z_{1+})}, \quad K_i(T') = \|z_i\|_{L_\infty(0, T'; Z_{i+})} + \|z'_i\|_{L_\infty(0, T'; Z_{i+})}, \quad i = 2, 3
\]

Then \(\overline{K}(T') = \max\{K_1(T'), K_2(T'), K_3(T')\}\). Using Theorem 1 with \(z, z'\) we conclude that

\[
\sum_{i=1}^{5} \|G_i(z(t)) - G_i(z'(t))\|_{Z_i} + \sum_{i=3}^{5} \|G_i(z(t)) - G_i(z'(t))\|_{Z_i} \\
\leq C\left\{\left(1 + \|z_3(t)\|_{Z_{3+}} + \|z'_3(t)\|_{Z_{3+}}\right) \sum_{i=1}^{2} \|z_i(t) - z'_i(t)\|_{Z_i} + \left(1 + \sum_{i=1}^{2} (\|z_i(t)\|_{Z_{i+}} + \|z'_i(t)\|_{Z_{i+}}))\right) \|z_3(t) - z'_3(t)\|_{Z_3} \\
+ \sum_{i=4}^{5} \|z_i(t) - z'_i(t)\|_{Z_i} \leq C\left\{(1 + K_3(T')) \sum_{i=1}^{2} \|z_i(t) - z'_i(t)\|_{Z_i} + \left(1 + \frac{1}{t^{2\theta}} K_1(T') + K_2(T')\right) \|z_3(t) - z'_3(t)\|_{Z_3} \\
+ \sum_{i=4}^{5} \|z_i(t) - z'_i(t)\|_{Z_i} \leq C(1 + \overline{K}(T')) \left(1 + \frac{1}{t^{2\theta}}\right) f(t).
\]

Using Lemma 1 and owing to the fact that \(z, z'\) satisfy (66) we obtain for \(t \in (0, T')\)

\[
f(t) \leq \int_{0}^{t} \left\{\|e^{(t-\tau)A_0}\|_{L(Z_{1-}, Z_{1-})} \|G_i^\mu(z(\tau)) - G_i^\mu(z'(\tau))\|_{Z_{1-}} \\
+ \|e^{-(t-\tau)\tau}dA_0\|_{L(Z_{2-}, Z_{2-})} \|G_i^\mu(z(\tau)) - G_i^\mu(z'(\tau))\|_{Z_{2-}} + e^{-(t-\tau)T(\omega)}|z_3(\tau)| - |G_3(z(\tau)) - G_3(z'(\tau))|\|_{Z_{3+}} \\
+ \sum_{i=4}^{5} \|G_i(z(\tau)) - G_i(z'(\tau))\|_{Z_{i+}}\right\} d\tau \leq C(1 + \overline{K}(T')) \int_{0}^{t} \left(1 + \frac{1}{(t-\tau)^{3/4}} + \frac{1}{(t-\tau)^{1/2}}\right) \left(1 + \frac{1}{t^{2\theta}}\right) f(\tau) d\tau \\
\leq C(1 + \overline{K}(T'))(1 + (T')^{3/4 + 2\theta}) \int_{0}^{t} \frac{f(\tau)}{t^{2\theta}(t-\tau)^{3/4}} d\tau.
\]

Finally using Lemma 2 (see (64b)) we conclude that \(f \equiv 0\) on \((0, T')\) hence \(u \equiv u'\).

Step 3 - existence of global solutions for \(\epsilon > 0\) and \(\mu\)-independence of bounds.

Using Theorem 1 with \(s = 1/2 + \theta, s' = 1/2 + 2\theta, \omega = \eta\) we obtain that system (59) has for \(\mu \in (0, 1] \times (0, 1]\) a unique global \(X^\mu\) solution \(u^\mu\) which is nonnegative. Let \(z^\mu, z^\mu_0\) be related with \(u^\mu, u_0\) by (61) and (62). It is easy to see that \(z^\mu\) satisfies formulas (66) from which one concludes that \(u^\mu\) is also a M-mild solution of system (59). Using part 3 of Theorem 1 we get that

\[
M_3 = \sup_{\mu \in (0, 1] \times (0, 1]} \sum_{i=3}^{5} \|z^\mu_i\|_{L_\infty(0, \infty; Z_{i+})}
\]

is finite. Fix \(T < \infty\) and for 0 < \(t < T\) denote \(g(t) = 1 + t^{2\theta}\|z^\mu_1(t)\|_{Z_{1+}} + \|z^\mu_2(t)\|_{Z_{2+}}\). Owing to Lemma 18

\[
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\]
we obtain
\[
\sum_{i=1}^{2} \|G_i^\mu(z^\mu(t))\|_{L^\infty} \leq C \left( 1 + \|z_1^\mu(t)\|_{L^1} + \|z_2^\mu(t)\|_{L^2} \right) \left( 1 + \|z_3^\mu(t)\|_{L^1} + \|z_4^\mu(t)\|_{L^2} + \|z_5^\mu(t)\|_{L^2} \right)
\]
\[
\leq C(1 + M_3) \left( 1 + \|z_1^\mu(t)\|_{L^1} + \|z_2^\mu(t)\|_{L^2} \right) \leq C(1 + M_3) \left( 1 + \frac{1}{\tau_2} \right) g(t)
\]

Using (66) and Lemma 9 we estimate (recall that \(Z_2 = Z_{2+}\))
\[
g(t) \leq 1 + t^{2\theta} e^{tA_1} \|\mathcal{L}(Z_1, Z_{1+})\| z_{10}^1 \|Z_i + \|e^{tA_0} \|z_{02}^1\| z_2 + \\
\int_0^t \left( t^{2\theta} e^{(t-\tau)A_1} \|\mathcal{L}(Z_1, Z_{1+})\| G_1^\mu(z^\mu(\tau)) \|Z_1 + \|G_2^\mu(z^\mu(\tau)) \|z_{20}^1\| z_2 \right) d\tau
\]
\[
\leq C(1 + t^{2\theta}) \left( 1 + (1 + M_3) \int_0^t \left( 1 + \frac{1}{(t-\tau)^{3/4+2\theta}} + \frac{1}{(t-\tau)^{1/2}} \right) \left( 1 + \frac{1}{\tau_2} \right) g(\tau) d\tau \right)
\]
\[
\leq C(1 + T^{2\theta}) \left( 1 + (1 + M_3)(1 + T^{3/4+4\theta}) \int_0^t \frac{g(\tau)}{(t-\tau)^{3/4+2\theta} \tau_2} d\tau \right)
\]
\[
\leq C(1 + T^{2\theta}) + C(1 + M_3)(1 + T^{3/4+6\theta}) \int_0^t \frac{g(\tau)}{(t-\tau)^{3/4+2\theta} \tau_2} d\tau.
\]

Thus using Lemma 2 (see (64b)) we get that for every \(T > 0\)
\[
M_1(T) = \sup_{\mu \in (0,1] \times (0,1]} \|t^{2\theta} z_1^\mu\|_{L^\infty(0,T;Z_{1+})} \quad \text{and} \quad M_2(T) = \sup_{\mu \in (0,1] \times (0,1]} \|z_2^\mu\|_{L^\infty(0,T;Z_{2+})}
\]

are finite.

Step 4 - existence of local mild solutions for \(\epsilon = 0\).

For \(R, T > 0\) define
\[
Z_1 = \{ z_1 \in C([0, T], Z_1) : \|z_1\|_{L^\infty(0,T;Z_1)} + \|t^{2\theta} z_1\|_{L^\infty(0,T;Z_{1+})} \leq R \},
\]
\[
d_1(z_1, z_1') = \|z_1 - z_1'\|_{L^\infty(0,T;Z_1)} + \|t^{2\theta} (z_1 - z_1')\|_{L^\infty(0,T;Z_{1+})}
\]
\[
Z_2 = \{ z_2 \in C([0, T], Z_2) : \|z_2\|_{L^\infty(0,T;Z_2)} \leq R \}, \quad d_2(z_2, z_2') = \|z_2 - z_2'\|_{L^\infty(0,T;Z_2)}
\]
\[
Z_i = \{ z_i \in C([0, T], Z_i) : \|z_i\|_{L^\infty(0,T;Z_{i+})} \leq R \}, \quad d_i(z_i, z_i') = \|z_i - z_i'\|_{L^\infty(0,T;Z_{i+})}, \quad i = 3, 4, 5
\]
\[
Z = Z_1 \times \ldots \times Z_5, \quad d(Z, Z') = \sum_{i=1}^5 d_i(z_i, z_i')
\]

Observe that \(Z_i\) and \(Z\) are complete metric spaces.

For \(z \in Z, \ \mu = (h, \epsilon) \in (0, 1] \times (0, 1]\) define
\[
[\Phi_1^\mu(z)](t) = e^{t(A_h-b_1)} z_{01} + \int_0^t e^{(t-\tau)(A_h-b_1)} G_1^\mu(z(\tau)) d\tau
\]
\[
[\Phi_2^\mu(z)](t) = e^{tA_0} z_{02} + \int_0^t e^{(t-\tau)dA_0} G_2^\mu(z(\tau)) d\tau
\]
\[
[\Phi_3^\mu(z)](t) = e^{-T\tau(m_\tau)} z_{03} + \int_0^t e^{(t-\tau)T\tau(m_\tau)} G_3^\mu(z(\tau)) d\tau
\]
\[
[\Phi_i(z)](t) = z_{0i} + \int_0^t G_i(z(\tau)) d\tau, \quad i = 4, 5
\]
\[
\Phi^\mu = (\Phi_1^\mu, \Phi_2^\mu, \Phi_3^\mu, \Phi_4^\mu, \Phi_5^\mu)
\]
Lemma 19. There exist $R, T > 0$ such that for every $\mu \in (0, 1] \times [0, 1]$ the map $\Phi^\mu$ maps $\mathcal{Z}$ into itself and satisfies for every $z, z' \in \mathcal{Z}$ the following condition
\[
d_\mathcal{Z}(\Phi^\mu(z), \Phi^\mu(z')) \leq (1/2)d_\mathcal{Z}(z, z').
\] (72)

Proof. Fix $R \geq 1 \geq T > 0$. Using Lemma 18 we have for $t \in [0, T]$ and $z, z' \in \mathcal{Z}$
\[
\sum_{i=1}^{2} ||G_i^\mu(z(t))||_{Z_{+}} + \sum_{i=3}^{5} ||G_i(z(t))||_{Z_{+}} \leq CR^2 \left(1 + \frac{1}{t^{2\theta}} \right)
\] (73)
\[
\sum_{i=1}^{2} ||G_i^\mu(z(t)) - G_i^\mu(z'(t))||_{Z_{-}} + \sum_{i=3}^{5} ||G_i(z(t)) - G_i(z'(t))||_{Z_{-}} \leq CR \left(1 + \frac{1}{t^{2\theta}} \right)d_\mathcal{Z}(z, z').
\] (74)

Using (73) and Lemma 9 we estimate
\[
t^{2\theta} ||[\Phi^\mu(z)](t)||_{Z_{+}} + ||[\Phi^\mu(z)](t)||_{Z_{+}} + \sum_{i=1}^{5} ||[\Phi^\mu(z)](t)||_{Z_{+}} \leq (t^{2\theta} ||e^{tA_0}||_{\mathcal{L}(Z_{1}, Z_{1}+)} + ||e^{tA_0}||_{\mathcal{L}(Z_{1})}) ||z^0||_{Z_{1}}
\]
\[+ ||e^{tdA_0}||_{\mathcal{L}(Z_{2})} ||z_{02}||_{Z_{2}} + ||e^{-(t-Tr(m))}||_{Z_{+}} ||z_{03}||_{Z_{3}} + \sum_{i=4}^{5} ||z_{0i}||_{Z_{i}} + \int_{0}^{t} \left( (t^{2\theta} ||e^{((t-r)A_0}||_{\mathcal{L}(Z_{1}, Z_{1}+)}
\]
\[+ ||e^{(t-r)A_0}||_{\mathcal{L}(Z_{1}+Z_{2})} ||G_i(z(\tau))||_{Z_{+}} + ||e^{-(t-r)A_0}||_{\mathcal{L}(Z_{2}+Z_{3})} ||G_i(z(\tau))||_{Z_{+}} \right) d\tau \leq C(t^{2\theta} + 1) \left\{ ||z^0||_{Z_{1}} + ||z_{02}||_{Z_{2}} + \sum_{i=3}^{5} ||z_{0i}||_{Z_{i}} \right.
\]
\[+ R^2 \int_{0}^{t} \left( 1 + \frac{1}{(t-\tau)^{3/4}} + \frac{1}{(t-\tau)^{1/2}} \right) \left( 1 + \frac{1}{t^{2\theta}} \right) d\tau \right\} \leq C \left\{ ||z^0||_{Z_{1}} + ||z_{02}||_{Z_{2}} + \sum_{i=3}^{5} ||z_{0i}||_{Z_{i}} + CR^2 T^{1/4-\theta} \right\}
\]

Taking $R, T$ such that $R \geq \max\{1, 2C(||z^0||_{Z_{1}} + ||z_{02}||_{Z_{2}} + \sum_{i=3}^{5} ||z_{0i}||_{Z_{i}})\}$ and $T \leq \min\{1, (2CR)^{4/(16\theta-1)}\}$ we obtain
\[
t^{2\theta} ||[\Phi^\mu(z)](t)||_{Z_{+}} + ||[\Phi^\mu(z)](t)||_{Z_{+}} + \sum_{i=2}^{5} ||[\Phi^\mu(z)](t)||_{Z_{+}} \leq R/2 + R/2 = R
\]
which proves that $\Phi^\mu$ maps $\mathcal{Z}$ into itself. Using (74) we prove analogously that condition (72) holds after making $T$ smaller if needed.

We obtain from Lemma 19 that the map $\Phi^\mu : \mathcal{Z} \to \mathcal{Z}$ satisfies, for certain $R, T$ which are independent of $\mu$, the assumptions of Banach’s fixed point theorem. We conclude that system (59) has for $\epsilon = 0$ a unique maximally defined mild solution $u^{\mu_0}$ defined on $[0, T^h_{\text{max}}]$, where $T^* := \inf\{T^h_{\text{max}} : h \in (0, 1]\} > 0$.

Step 5 - For any fixed $h \in (0, 1]$: $u^\mu$ converges to $u^{\mu_0}$ as $\epsilon \to 0$. Moreover $T^h_{\text{max}} = \infty$.

Fix $T < T^h_{\text{max}}$ and for $0 < t < T$ denote: $f^\mu(t) = \sum_{i=1}^{5} ||z^\mu_i(t) - z^{\mu_0}_0(t)||_{Z_i}$. Put
\[
K^h_i(T) = \sup_{\epsilon \in [0, 1]} ||t^{2\theta} z^\mu_i ||_{L_{\infty}(0,T;Z_{i+})}, \quad K^h_i(T) = \sup_{\epsilon \in [0, 1]} ||z^\mu_i ||_{L_{\infty}(0,T;Z_{i+})}, \quad i = 2, 3
\]

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Observe that due to (70), (71), $K^h(T)$ are finite. Denote $\overline{K}^h(T) = \max\{K_1^h(T), K_2^h(T), K_3^h(T)\}$. Using Lemma 18, we have for $0 < t < T$

\[
\sum_{i=1}^{2} \|G_i^u(z^u(t)) - G_i^u(z^{\mu_0}(t))\|_{Z_{-i}} + \sum_{i=3}^{5} \|G_i(z^u(t)) - G_i(z^{\mu_0}(t))\|_{Z_i} \leq C(1 + \overline{K}^h(T))(1 + \frac{1}{t^{2\theta}}) f^u(t)
\]

Thus owing to (66), we estimate

\[
\begin{align*}
\|G_3(z^{u_0}(t))\|_{Z_{3+}} & \leq C\left(1 + \frac{1}{t^{2\theta}}\right) \left(1 + (\overline{K}^h(T))^2\right) \\
\end{align*}
\]

Therefore, using Lemma 8 and Lemma (16), we obtain

\[
\begin{align*}
f^u(t) & \leq C \|m^u - m^{\mu_0}\|_{Z_1 + t}\|Tr(m^u - m^{\mu_0})\|_{Z_3}\|Z_3\|Z_{3+} \\
& + \int_{0}^{t} \left\{ \|e^{(t-\tau)A_0}L(Z_{1+},Z_1)\|Z_{1+} + \|e^{(t-\tau)dA_0}L(Z_{2+},Z_2)\|Z_{2+}\right\} d\tau \\
& \times \left( \sum_{i=1}^{2} \|G_i^u(z^u(\tau)) - G_i^u(z^{\mu_0}(\tau))\|_{Z_{-i}} + \sum_{i=3}^{5} \|G_i(z^u(\tau)) - G_i(z^{\mu_0}(\tau))\|_{Z_i} \right) \\
& + \int_{0}^{t} \left\{ \|e^{(t-\tau)A_0}L(Z_{1+},Z_1)\|Z_{1+} + \|e^{(t-\tau)dA_0}L(Z_{2+},Z_2)\|Z_{2+}\right\} \left( \sum_{i=1}^{2} \|G_i^u(z^{\mu_0}(\tau)) - G_i^u(z^{\mu_0}(\tau))\|_{Z_{-i}} \right) d\tau \\
& + \int_{0}^{t} \left\{ \|e^{(t-\tau)Tr(m^u - m^{\mu_0})}\|_{Z_3}\|G_3(z^{\mu_0}(\tau))\|_{Z_{3+}} \right\} d\tau \\
& \leq C(1 + t)\|\eta^u - \delta\|_{X_{-1/4-\theta}(I)} + C(1 + \overline{K}^h(T)) \int_{0}^{t} \left(1 + \frac{1}{(t-\tau)^{3/4} + \frac{1}{(t-\tau)^{1/2}}}\right) \left(1 + \frac{1}{t^{2\theta}}\right) f^u(\tau)d\tau \\
& + C(1 + \overline{K}^h(T))\|\eta^u - \delta\|_{X_{-1/4-\theta}(I)} \int_{0}^{t} \left(1 + \frac{1}{(t-\tau)^{3/4} + \frac{1}{(t-\tau)^{1/2}}}\right) d\tau \\
& + C\left(1 + (\overline{K}^h(T))^2\right)\|\eta^u - \delta\|_{X_{-1/4-\theta}(I)} \int_{0}^{t} (t-\tau)\left(1 + \frac{1}{t^{2\theta}}\right) d\tau \\
& \leq a_h(T)\|\eta^u - \delta\|_{X_{-1/4-\theta}(I)} + b_h(T) \int_{0}^{t} \frac{f^u(\tau)}{(t-\tau)^{3/4} \frac{1}{2\theta}} d\tau.
\end{align*}
\]

Using Lemma 2 (see (64B)), we get that

\[
\|f^u\|_{L^\infty(0,T)} \leq a_h(T)\|\eta^u - \delta\|_{X_{-1/4-\theta}(I)} C \exp\left(b_h(T)CT^{1-\left(\frac{3}{4} + \frac{1}{2\theta}\right)}\right).
\]
from which we conclude that \( \lim_{\epsilon \to 0^+} \|f^\mu\|_{L_\infty(0,T)} = 0 \) for every \( h \in (0,1] \), \( T < T_{\text{max}}^h \) by Lemma 15. In particular \( u^{\mu_0} \) is nonnegative on \([0,T_{\text{max}}^h]\) and for every \( T < T_{\text{max}}^h \)
\[
\|t^{2\theta} z_{1}^{\mu_0}\|_{L_\infty(0,T;Z_{1+})} \leq M_1(T) \\
\|z_{2}^{\mu_0}\|_{L_\infty(0,T;Z_{2+})} \leq M_2(T) \\
\sum_{i=3}^{5} \|z_{i}^{\mu_0}\|_{L_\infty(0,T;Z_{i+})} \leq M_3
\]

We observe thus that \( z^{\mu_0} \) does not blow-up in finite time in norm of the space \( Z_+ \). Using standard continuation argument we conclude that \( T_{\text{max}}^h = \infty \) for any \( h \in (0,1] \).

### 5.4 Proof of Theorem 3

Recall that \( m^0 \) is defined in (46). Denote
\[
\begin{align*}
    z_{01}^0 &= Pu_{01} - m^0 \\
    g_{1}^0, g_{2}^0 : I \times \mathbb{R}^5 &\to \mathbb{R} \\
    g_{1}^0(z) &= -c_1 z_1 + c_2 z_2 - z_1 z_3 + c_4(z_4 - z_3) - (c_1 + z_3)m^0 \\
    g_{2}^0(z) &= -b_2 z_2 + c_1 z_1 - c_2 z_2 - c_3 z_2 z_3 + c_5(z_5 - z_4) + c_1 m^0.
\end{align*}
\]

For \( z \in Z_+ \) define
\[
\begin{align*}
    G_{1}^0(z) &= Tr^*(g_{1}^0(Tr(z_1), z_2, z_3, z_4, z_5)) \\
    G_{2}^0(z) &= g_{2}^0(Tr(z_1), z_2, z_3, z_4, z_5)
\end{align*}
\]

Observe that since \( u^0 = (u_1^0, \ldots, u_5^0) \) solves (69a), \( z^0 = (z_1^0, \ldots, z_5^0) = M(E(u_1^0 - m^0), u_2^0, \ldots, u_5^0) \) satisfies the following Duhamel formulas:
\[
\begin{align*}
    z_1^0(t) &= E \left\{ e^{(A_0 - b_1)} z_{01}^0 + \int_0^t e^{(t-\tau)(A_0 - b_1)} P G_{1}^0(z^0(\tau)) d\tau \right\} \quad (75a) \\
    z_2^0(t) &= e^{t A_0} z_{02} + \int_0^t e^{(t-\tau)dA} G_{1}^0(z^0(\tau)) d\tau \quad (75b) \\
    z_3^0(t) &= e^{-tm^0} z_{03} + \int_0^t e^{-(t-\tau)m^0} G_{2}^0(z^0(\tau)) d\tau \quad (75c) \\
    z_i^0(t) &= z_{0i} + \int_0^t G_i(z^0(\tau)) d\tau, \ i \in \{4,5\} \quad (75d)
\end{align*}
\]

Fix \( T < \infty \). Denote
\[
\begin{align*}
    N(T) &= \sup_{h \in (0,1]} \left( \|t^{2\theta} z_{1}^{\mu_0}\|_{L_\infty(0,T;Z_{1+})} + \|t^{2\theta} z_{2}^{\mu_0}\|_{L_\infty(0,T;Z_{1+})} + \sum_{i=3}^{5} (\|z_{i}^{\mu_0}\|_{L_\infty(0,T;Z_{i+})} + \|z_{i}^{0}\|_{L_\infty(0,T;Z_{i+})}) \right) \\
    f^{\mu_0}(t) &= t^{2\theta} \|z_{1}^{\mu_0}(t) - z_{1}^0(t)\|_{Z_{1+}} + \sum_{i=2}^{5} \|z_{i}^{\mu_0}(t) - z_{i}^0(t)\|_{Z_{i}}
\end{align*}
\]
Observe that $N(T) \leq M_1(T) + M_2(T) + M_3 < \infty$ as was proved in Step 3. Owing to Lemma 18 and Lemma 16 for $0 < t < T < \infty$ we have

$$
\sum_{i=1}^{2} \|G_{1i}^{m_0}(t) - G_{1i}^{m_0}(0)\|_{Z_{i-}} + \sum_{i=3}^{5} \|G_{i}(z^{m_0}(t)) - G_{i}(z^{0}(t))\|_{Z_{i}} \leq C(1 + N(T)) \left(1 + \frac{1}{t^{2\theta}}\right) f^{m_0}(t)
$$

$$
\|G_{11}^{m_0}(0)(t)\|_{Z_{1-}} + \|G_{3}(z^{0}(t))\|_{Z_{3+}} \leq C(1 + (N(T))^2) \left(1 + \frac{1}{t^{2\theta}}\right)
$$

$$
\sum_{i=1}^{2} \|G_{1i}^{m_0}(0)(t) - G_{1i}^{m_0}(0)(t)\|_{Z_{i-}} \leq C(1 + N(T)) \frac{1}{|\lambda_{01,h}|^{\theta/2}}.
$$

Since $z^{m_0}$ (resp. $z^0$) satisfies (60) (resp. (75)) thus using (35) and Lemma 14 we obtain

$$
f^{m_0}(t) \leq t^{2\theta} \|e^{tA_{h}}(z^{m_0} - E z^{0})\|_{Z_{1+}} + \|e^{-tT_{m_0}} - e^{-tm_0}\|_{Z_{3+}}
$$

$$
+ \int_{0}^{t} \left\{ t^{2\theta} \|e^{(t-\tau)A_{h}}(G_{1i}^{m_0}(z^{m_0}(\tau)) - E P G_{1}^{i}(z^{0}(\tau)))\|_{Z_{i+}} \right\} d\tau + \int_{0}^{t} \left\{ \|e^{(t-\tau)dA_{h}}(G_{2i}^{m_0}(z^{m_0}(\tau)) - G_{2i}(z^{0}(\tau)))\|_{Z_{2}} \right\} d\tau
$$

$$
+ \int_{0}^{t} \left\{ \|e^{-(t-\tau)T_{m_0}}(G_{3i}^{m_0}(z^{m_0}(\tau))) - G_{3i}(z^{0}(\tau))\|_{Z_{3+}} + \|G_{1i}^{m_0}(z^{m_0}(\tau)) - G_{1i}(z^{0}(\tau))\|_{Z_{i-}} \right\} d\tau
$$

$$
+ \int_{0}^{t} \left\{ \|e^{-(t-\tau)T_{m_0}}(G_{2i}^{m_0}(z^{m_0}(\tau))) - G_{2i}(z^{0}(\tau))\|_{Z_{2-}} \right\} d\tau + \int_{0}^{t} \left\{ \|e^{-(t-\tau)T_{m_0}}(G_{3i}^{m_0}(z^{m_0}(\tau))) - G_{3i}(z^{0}(\tau))\|_{Z_{3}} \right\} d\tau
$$

Using Lemma 5, Lemma 37, Lemma 16, Lemma 13 and Lemma 1 we have

$$
f^{m_0}(t) \leq C t^{2\theta} e^{|\lambda_{01,h}|u_{01}} |Z_{1+} + C(1 + t^{2\theta}) \frac{1}{|\lambda_{01,h}|^{\theta/2}} + C t \frac{1}{|\lambda_{01,h}|^{\theta/2}} |z_{03}| |Z_{3+}|
$$

$$
+ C t^{2\theta} (1 + N(T)) \int_{0}^{t} \left( 1 + \frac{1}{(t-\tau)^{3/4+2\theta}} \right) \left( 1 + \frac{1}{t^{2\theta}} \right) f^{m_0}(\tau) + \frac{1}{|\lambda_{01,h}|^{\theta/2}} d\tau
$$

$$
+ C t^{2\theta} (1 + (N(T))^2) \int_{0}^{t} \left( 1 + \frac{1}{(t-\tau)^{3/4+2\theta}} \right) \left( 1 + \frac{1}{t^{2\theta}} \right) e^{-(t-\tau)\lambda_{01,h}} d\tau
$$

$$
+ C(1 + N(T)) \int_{0}^{t} \left( 1 + \frac{1}{(t-\tau)^{1/2}} \right) \left( 1 + \frac{1}{t^{2\theta}} \right) f^{m_0}(\tau) + \frac{1}{|\lambda_{01,h}|^{\theta/2}} d\tau
$$

$$
+ C(1 + (N(T))^2) \int_{0}^{t} \left( 1 + \frac{1}{t^{2\theta}} \right) f^{m_0}(\tau)\right\} d\tau
$$

$$
+ C(1 + (N(T))^2) \frac{1}{|\lambda_{01,h}|^{\theta/2}} \int_{0}^{t} (t-\tau) \left( 1 + \frac{1}{t^{2\theta}} \right) d\tau
$$

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\[ \leq C(1 + T)\left( \frac{1}{|\lambda_{01,h}|^{2\theta}} + \frac{1}{|\lambda_{01,h}|^{\theta/2}} \right) \]

\[ + C(1 + T^{2\theta})(1 + (N(T))^2)\left\{ \frac{1}{|\lambda_{01,h}|^{\theta/2}} \int_0^t \left( 1 + \frac{1}{(t - \tau)^{3/4 + 2\theta}} + \frac{1}{(t - \tau)^{1/2}} + (t - \tau) \left( 1 + \frac{1}{(t - \tau)^{2\theta}} \right) \right) d\tau \right\} \]

\[ + \int_0^t \left( 1 + \frac{1}{(t - \tau)^{3/4 + 2\theta}} \right) \left( 1 + \frac{1}{(t - \tau)^{1/2}} \right) e^{-\lambda_{01,h} t} d\tau + \int_0^t \left( 1 + \frac{1}{(t - \tau)^{3/4 + 2\theta}} + \frac{1}{(t - \tau)^{1/2}} \right) \left( 1 + \frac{1}{(t - \tau)^{2\theta}} \right) f_{\mu_0}(\tau) d\tau \]

\[ \leq a(T) \left( \frac{1}{|\lambda_{01,h}|^{\theta/2}} + \frac{1}{|\lambda_{01,h}|^{1/4 - 2\theta}} \right) C \exp \left( b(T) CT^{1 - (3/4 + 4\theta)} \right), \]

Using Lemma (2) (see (64b)) we get that

\[ \| f_{\mu_0} \|_{L^\infty(0,T)} \leq a(T) \left( \frac{1}{|\lambda_{01,h}|^{\theta/2}} + \frac{1}{|\lambda_{01,h}|^{1/4 - 2\theta}} \right) C \exp \left( b(T) CT^{1 - (3/4 + 4\theta)} \right), \]

from which we conclude that \( \lim_{h \to 0^+} \| f_{\mu_0} \|_{L^\infty(0,T)} = 0 \) since \( |\lambda_{01,h}| = (\pi/h)^2 \to \infty \) as \( h \to 0 \).

**Acknowledgement**

The author would like to express his gratitude towards his PhD supervisors Philippe Laurençot and Dariusz Wrzosek for their constant encouragement and countless helpful remarks and towards his numerous colleagues for stimulating discussions.

The author was supported by the International Ph.D. Projects Programme of Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007-2013 funded by European Regional Development Fund (Ph.D. Programme: Mathematical Methods in Natural Sciences).

The article is supported by NCN grant no 2012/05/N/ST1/03115.

This publication has been co-financed with the European Union funds by the European Social Fund. Part of this research was carried out during the author’s visit to the Institut de Mathématiques de Toulouse, Université Paul Sabatier, Toulouse III.

**References**

[1] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, Function Spaces, Differential Operators and Nonlinear Analysis (1993), pp. 9-126.

[2] T. Bollenbach, K. Kruse, P. Pantazis, M. González-Gaitán, F. Jülicher, *Morphogen transport in Epithelia*, Phys. Rev. E. 75, 011901 (2007).

[3] D. Fujiwara, *Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order*, Proc. Japan Acad., Vol. 43, (1967) pp. 82-86.

[4] J. B. Gurdon, P.-Y. Bourillot, *Morphogen gradient interpretation*, Nature, Vol. 413 (2001).

[5] L. Hufnagel, J. Kreuger, S. M. Cohen, B. I. Shraiman, *On the role of glypicans in the process of morphogen gradient formation*, Dev. Biol., Vol. 300 (2006), pp 512-522.

[6] P. Krzyżanowski, Ph. Laurençot, D. Wrzosek, *Well-posedness and convergence to the steady state for a model of morphogen transport*, SIMA, Vol. 40, Iss. 5 (2008), pp 1725-1749.

[7] P. Krzyżanowski, Ph. Laurençot, D. Wrzosek, *Mathematical models of receptor-mediated transport of morphogens*, M3AS, Vol. 20 (2010), pp 2021-2052.
A. Kicheva, P. Pantazis, T. Bollenbach, Y. Kalaidzidis, T. Bittig, F. Jülicher, M. González-Gaitán, *Kinetics of morphogen gradient formation*, Science, Vol. 315 (2007), pp 521-525.

M. Kerszberg, L. Wolpert, *Mechanisms for positional signalling by morphogen transport: a theoretical study*, J. Theor. Biol., Vol. 191 (1998), pp 103-114.

A. D. Lander, Q. Nie, Y. M. Wan, *Do morphogen gradients arise by Diffusion?*, Dev. Cell, Vol. 2 (2002), pp 785-796.

J. L. Lions, E. Magenes, *Problème aux limites non homogènes IV*, Ann. Sc. Norm. Sup. Pisa, 15 (1961) pp 311-326.

L. Lorenzi, A. Lunardi, G. Metafune, D. Pallara, *Analytic semigroups and reaction-diffusion problems*, Internet Seminar 2004-2005.

A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications, Vol. 16, Birkhauser (1995).

M. Malogrosz, *Well-posedness and asymptotic behavior of a multidimensional model of morphogen transport*, J. Evol. Eq., Vol. 12, Iss. 2 (2012), pp 353-366.

M. Malogrosz, *A model of morphogen transport in the presence of glypicans I*, Nonlinear Analysis: Theory, Methods & Applications, Vol. 83 (2013), pp 91-101.

M. Malogrosz, *A model of morphogen transport in the presence of glypicans III*, submitted.

M. Reed, B. Simon, *Methods of modern mathematical physics I: Functional analysis*, Academic Press, Inc. (1980)

M. Medved, *A new approach to an analysis of Henry type integral inequalities and their Bihari type versions*, Journal of Mathematical Analysis and Applications, 214 (1997), pp 349-366.

C. Stinner, J. I. Tello, M. Winkler, *Mathematical analysis of a model of chemotaxis arising from morphogenesis*, M2AS, Vol. 35 (2012), pp 445-465.

J. I. Tello, *Mathematical analysis of a model of morphogenesis*, Dis. Cont. Dyn. Syst., Vol. 25, Iss. 1 (2009), pp 343-361.

H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Mathematical Library, Vol. 18 (1978).

L. Wolpert *Positional information and the spatial pattern of cellular differentiation*. J. Theor. Biol., Vol. 25, Iss. 1 (1969), pp 1-47.