TWISTED INTERNAL COHOM OBJECTS IN THE CATEGORY OF QUANTUM LINEAR SPACES

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ABSTRACT. Adapting the idea of twisted tensor products [1] to the category of finitely generated algebras, we define on its opposite, the category QLS of quantum linear spaces [10][11], a family of objects $\text{hom}^\tau [A,B]^\text{op}$ in QLS, one for each pair $A^\text{op}, B^\text{op} \in \text{QLS}$, with analogous properties to $\text{hom} [A,B]^\text{op} = \text{Hom} [B^\text{op}, A^\text{op}]$ (its internal Hom objects), but representing spaces of transformations whose coordinate rings $\text{hom}^\tau [A,B]$ and the ones of their respective domains $B^\text{op}$ do not commute among themselves. (The work is mainly developed in a full subcategory of QLS whose opposite objects are given by finitely generated graded algebras.) They give rise to a QLS-based category different from the one defined by the function $(A,B) \mapsto \text{hom} [A,B]$. The mentioned non commutativity is controlled by a collection of twisting maps $\tau_{A,B}$ which define product spaces $[\text{hom}^\tau [A,B] \circ_\tau B]^\text{op}$ satisfying the inclusions $\text{hom}^\tau [A,B] \circ_\tau B \subset \text{hom}^\tau [A,B] \otimes B$. We show that the (bi)algebras $\text{end}^\tau [A] = \text{hom}^\tau [A,A]$, under certain circumstances, are 2-cocycle twisting of the quantum semigroups $\text{end} [A]$. This fact generalizes the twist equivalence (at a semigroup level) between, for instance, the quantum groups $GL_q(n)$ and their multiparametric versions $GL_{q,\phi}(n)$.

INTRODUCTION

Quantum linear spaces, or simply quantum spaces, were defined by Y. Manin as opposite (or dual) objects to finitely generated algebras (FGA). He described the latter by pairs $(A_1,A)$ where $A$ is a unital algebra generated by some finite linear subspace $A_1 \subset A$; arrows between them are unital algebra homomorphisms $A \rightarrow B$ which restricted to $A_1 \subset A$ give linear morphisms $A_1 \rightarrow B_1 (\subset B)$. Every pair $A = (A_1,A)$ is interpreted as the (generically) noncommutative coordinate ring of the quantum space $A^\text{op}$. In other words, if we call QLS the category of quantum spaces and FGA the category of their coordinate rings (the pairs), then QLS $\cong$ FGA$^\text{op}$. Since the duality between QLS and FGA, for quantum spaces we understand the objects of each one of these categories.

Initially, Manin [10] defined quantum spaces in terms of quadratic algebras, which constitute a full subcategory QA of FGA. Later [11], he extended the concept to arbitrary finitely generated algebras. Then we refer to the objects of QA and FGA as quadratic and general quantum spaces, respectively.

FGA is a monoidal category (FGA, $\circ, I$). Thus QLS has, by duality, an associated direct product of quantum spaces (of course, the term ‘direct’ is not used in the sense of ‘cartesian’).

Manin’s work is based on the existence of internal coHom objects in FGA, which define the quantum matrix spaces $\text{hom} [B,A]$ with related coevaluation morphisms $A \rightarrow \text{hom} [B,A] \circ B$; or in their dual version, the internal Hom objects

$$\text{Hom} [B^\text{op}, A^\text{op}] = \text{hom} [B,A]^\text{op}$$

of the monoidal category (QLS, $\times, \text{T}^\text{op}$), giving rise to the (noncommutative) algebraic geometric version of quantum groups. More precisely, he defined quantum groups as the universal Hopf envelope $\text{aut} [A] = GL [A]$ of the bialgebras $\text{end} [A] = \text{hom} [A,A]$. Restricted to QA, this approach is close to the FRT construction of quantum matrix bialgebras [10][11].

However the product $\circ$ is in essence the usual tensor product of algebras. Following the work of Čap, Schichl and Vanžura [1], we may ask why such a product is used in the construction above. Using it, one is assuming that the coordinates of the space $\text{Hom} [B^\text{op}, A^\text{op}]$ commute with the ones of $B^\text{op}$, although the coordinates on the individual factors do not commute among themselves. The main aim of this work is to carry out a twisted version of the quantum matrix spaces, by means of replacing on the coevaluation arrows the product $\circ$ by a twisted tensor product, which we shall indicate $\circ_\tau$. It is worth remarking that we are not going to change the monoidal structure on the category FGA, which would be to make the Manin construction in a Yang-Baxter or braided category [1]. We just change the products between certain objects. In other words, we study certain subclasses of maps $A \rightarrow H \circ_\tau B$ for fixed $A$ and $B$ (each subclass representing a particular kind of non commutativity between factors $H^\text{op}$ and $B^\text{op}$ of a quantum space product $(H \circ_\tau B)^\text{op}$) from which we construct universal objects, the twisted internal coHom objects $\text{hom}^\tau [B,A]$, with analogous properties to

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the proper coHom ones. In this way we extend, and endow with an universal character, the results given in a previous work [13]. We shall see that, in some cases, the associated twisted coEnd objects, given by bialgebras $end^T[A]$, correspond to twisting by counital 2-cocycles of the bialgebras $end[A]$ in the sense of Drinfeld twisting process [3]. That is to say, $end[A]$ and $end^T[A]$ are twist equivalent as bialgebras.

The paper is organized as follows. In §1 we analyze an interesting full subcategory of FGA (formed out by finitely generated graded algebras) on which our main work is based. This class of objects will be called conic quantum spaces, and they can be regarded as an intermediate version between general and quadratic ones. Before that, we review some well-known properties of the latter classes such as monoidal structures and related internal (co)Hom objects. We point out in §2 a related semigroupoid structure associated to certain comma categories of which the internal coHom objects are initial objects. This structure is the main guide towards our construction. We show how to arrive at the notions of (co)evaluation, (co)identity morphisms and (co)composition of morphisms (in a compatible way) using the universal character of the objects $hom[B, A]$ and the above mentioned semigroupoid structure, and then relate those notions to the concept of based categories. At this point we introduce the necessary ingredients to pose in precise algebraic terms the plan of our work and the main results.

In §3 we define a twisted tensor product $\circ_s$ between objects of (FGA, $\circ$) and morphisms between the resulting objects. They are such that, as in the algebra case, their isomorphism classes are in bijection to linear transformations with the same properties as twisting maps [π]. We introduce this concept in the algebraic geometric terminology used by Manin, which allows us to associate the (trivial or) non twisted product isomorphism class (i.e. the class related to the standard product $\circ$, or to the flipping map) to commuting points of factor spaces. Conversely, we are connecting the idea of (generically) non commuting points with the concepts of (non trivial) twisted tensor products and twisting maps.

Finally, we build up in §4 the twisted quantum matrix spaces $hom^T[B, A]$. They give rise to a QLS or FGA-op-based category different from the one defined by the proper Hom objects, and with an evaluation notion coming from the arrow (dual to) $A \to hom^T[B, A] \circ_B B$. We also show the twist equivalence $end^T[A] \simeq end[A]$, as bialgebras.

We often adopt definitions and notation extracted from Mac Lane’s book [3]. By $k$ we indicate some of the numerics fields, $\mathbb{R}$ or $\mathbb{C}$. The usual tensor product on $k$–Alg and $Vct_k$ (the categories of unital associative $k$-algebras and of $k$-vector spaces, respectively) is denoted by $\otimes$.

1. QUANTUM LINEAR SPACES

From its very definition, any pair $(A_1, A)$ in FGA has associated a canonical epimorphism of unital algebras $\Pi : A_1^\otimes \to A$, where $A_1^\otimes = \bigoplus_{n \in \mathbb{N}_0} A_1^{\otimes n}$ is the tensor algebra of $A_1$ ($\mathbb{N}_0$ denoting the non negative integers), such that restricted to $A_1$ gives the inclusion $A_1 \hookrightarrow A$, and it defines a canonical isomorphism $A_1^\otimes / \ker \Pi \simeq A$.

Moreover, the gradation in $A_1^\otimes$ induces a filtration in $A$ and $\ker \Pi$, with

$$A = \bigcup_{n \in \mathbb{N}_0} F_n; \quad F_n = \Pi \left( \bigoplus_{i=0}^{n} A_1^{\otimes i} \right)$$

and $\ker \Pi = \bigcup_{n \in \mathbb{N}_0} f_n$; $f_n \subset \bigoplus_{i=0}^{n} A_1^{\otimes i}$; $f_{0,1} = \{0\}$. On the other hand, for any pair of quantum spaces $(A_1, A)$ and $(B_1, B)$, the arrows $(A_1, A) \to (B_1, B)$ are given by algebra homomorphisms $\alpha : A \to B$ completely defined by the linear maps $\alpha|_{A_1} : A_1 \to B_1$. In fact, they can be characterized by arrows $\alpha_1 : A_1 \to B_1$ in $Vct_k$ such that

$$\alpha_1(\ker \Pi_A) \subset \ker \Pi_B, \text{ being } \alpha_1^\otimes = \bigoplus_{n \in \mathbb{N}_0} A_1^{\otimes n} : A_1^\otimes \to B_1^\otimes$$

the unique extension of $\alpha_1$ to $A_1^\otimes$ as a morphism of algebras. We will say that $\alpha$ is the quotient map of $\alpha_1^\otimes$.

In these terms, the category QA of quadratic algebras, which constitute a full subcategory of FGA, is formed by pairs $(A_1, A)$ such that the kernel of $\Pi : A_1^\otimes \to A$ is a (bilateral ideal) algebraically generated by a subspace $I \subset A_1 \otimes A_1$, i.e.

$$\ker \Pi = \bigoplus_{n \geq 2} I_n; \quad I_n = \sum_{k=0}^{n-2} A_1^{\otimes k} \otimes I \otimes A_1^{\otimes n-k-2}.$$ 

These objects are called quadratic quantum spaces. Its arrows can be characterized by linear maps $\alpha_1$ such that $\alpha_1^\otimes(I) \subset J$, being $J \subset B_1^{\otimes 2}$ the generator subspace of $\ker \Pi_B$.

Now let us present an intermediate version between quadratic and general quantum spaces. We shall name conic algebras or conic quantum spaces those pairs $(A_1, A)$ of FGA such that $A$ is a graded algebra

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$$A = \bigoplus_{n \in \mathbb{N}_0} A_n; \quad A_n = \Pi(A_1^{\otimes n}),$$
or equivalently, ker Π = \( \bigoplus_{n \in \mathbb{N}_0} I_n \), \( I_n \subset A_1^\otimes n \) and \( I_n \otimes I_m \subset I_{n+m} \). The arrows between conic quantum spaces are morphisms \( \alpha : A_1 \to B_1 \) in \( \text{Vct}_2 \) such that \( \alpha \otimes 1 \) \((I_n) \subset J_n \). If we name CA the category of conic quantum spaces, the inclusion of full subcategories \( \mathcal{Q}A \subset \mathcal{C}A \subset \mathcal{F}GA \) follows.

Examples of conic quantum spaces are, beside quadratics ones, those with related ideals of the form \( \ker \Pi = A_1^\otimes \otimes I \otimes A_1^\otimes m \), with \( I \subset A_1^\otimes m \) for some \( m \in \mathbb{N} \). We call them \( m \)-th quantum spaces, and denote by \( CA^m \) the full subcategory of \( \mathcal{F}GA \) which has these pairs as objects. Thus \( QA = CA^2 \).

Because our main constructions will be made on \( CA \), we are going to study its associated internal coHom objects and some functorial structures in terms of which they can be written. To this end, we first refresh the monoidal (and other relevant) structures on \( \mathcal{F}GA \) and \( QA \).

1.1. Monoidal structure on \( \mathcal{F}GA \) and its internal coHom objects. On \( \mathcal{F}GA \) a bifunctor \( \circ : \mathcal{F}GA \times \mathcal{F}GA \to \mathcal{F}GA \) can be defined, such that to every pair of quantum spaces \( A = (A_1, A) \) and \( B = (B_1, B) \) it assigns the quantum space

\[
A \circ B = (A_1 \otimes B_1, A \circ B),
\]

where \( A \circ B \) is the subalgebra of \( A \otimes B \) generated by \( A_1 \otimes B_1 \). That means, from the (canonical) isomorphism

\[
[A_1 \otimes B_1]_\otimes \cong \left( [A_1 \otimes B_1]_\otimes \cong \bigoplus_{n \in \mathbb{N}_0} A_1^\otimes n \otimes B_1^\otimes n \right)
\]

where \( [A_1 \otimes B_1]_\otimes \) is the subalgebra of \( A_1^\otimes \otimes B_1^\otimes \) generated by \( A_1 \otimes B_1 \), the kernel of the canonical projection \( [A_1 \otimes B_1]_\otimes \to A \circ B \) is isomorphic to

\[
(A_1^\otimes \otimes \ker \Pi_B + \ker \Pi_A \otimes B_1^\otimes) \cap [A_1 \otimes B_1]_\otimes.
\]

Note that \( A_1^\otimes \otimes B_1^\otimes = [A_1 \otimes B_1]_\otimes \). We frequently identify the latter algebra and \( [A_1 \otimes B_1]_\otimes \). Returning to the functor \( \circ \), on morphisms \( \alpha : A \to A' \) and \( \beta : B \to B' \), this functor gives

\[
\alpha \circ \beta \overset{\text{def}}{=} \alpha \otimes \beta |_{A \circ B} : A \circ B \to A' \circ B',
\]

the restriction of \( \alpha \otimes \beta \) to the subalgebra \( A \circ B \subset A \otimes B \). It is easy to see that \( \circ \) supplies \( \mathcal{F}GA \) with a structure of symmetric monoidal category, with unit object \( \mathcal{I} = (k, k) \). The functor \( \circ \) also defines a monoid on the \( m \)-th’s and conic quantum spaces, but the unit object in these cases is \( \mathcal{K} = (k, k^\otimes) = (k, k^{[e]}) \), i.e. the free algebra generated by the indeterminate \( e \).

For every couple of objects \( A \) and \( B \) in \( \mathcal{F}GA, \circ, \mathcal{I} \) (c.f. \( \mathcal{I} \)) we have a (left) internal coHom object \( \text{hom}(B, A) \), i.e. an initial object of the comma category \( (\mathcal{A} \downarrow \mathcal{F}GA \circ \mathcal{B}) \). The objects of each \( (\mathcal{A} \downarrow \mathcal{F}GA \circ \mathcal{B}) \), diagrams in the Manin terminology, are pairs

\[
\langle \varphi, \mathcal{H} \rangle_{A,B} = \langle \varphi, \mathcal{H} \rangle, \text{ with } \mathcal{H} \in \mathcal{F}GA,
\]

and \( \varphi \) a morphism \( A \to \mathcal{H} \circ B \); and its arrows \( \langle \varphi, \mathcal{H} \rangle \to \langle \varphi', \mathcal{H}' \rangle \) are morphisms \( \alpha : \mathcal{H} \to \mathcal{H}' \) satisfying

\[
\varphi' = (\alpha \circ I_B) \varphi.
\]

For every such comma category there exist an embedding

\[
\mathfrak{P} : (\mathcal{A} \downarrow \mathcal{F}GA \circ \mathcal{B}) \hookrightarrow \mathcal{F}GA / \mathfrak{P} \langle \varphi, \mathcal{H} \rangle = \mathcal{H}.
\]

If \( A \) and \( B \) are generated by \( A_1, B_1 \in \text{Vct}_2 \) with dim \( A_1 = n \) and dim \( B_1 = m \), the initial object of \( (\mathcal{A} \downarrow \mathcal{F}GA \circ \mathcal{B}) \) is the pair \( \langle \delta_{A,B}, \text{hom}(B, A) \rangle \) where \( \text{hom}(B, A) \) is given by an algebra \( E_1^\otimes / K \) being

\[
E_1 = \text{span} \left[ z_i^j \right]_{i,j=1}^{n,m},
\]

and \( K \) a bilateral ideal depending on \( \ker \Pi_A \) and \( \ker \Pi_B \), and \( \delta_{A,B} = \delta \) is the coevaluation map

\[
\delta : A \to \text{hom}(B, A) \circ B / a_i \mapsto z_i^j \otimes b_j,
\]

with \( \{a_i\} \) and \( \{b_j\} \) basis of \( A_1 \) and \( B_1 \). Generically each object \( \text{hom}(B, A) \) is equal to \( (E_1, E_1^\otimes / K) \), in the sense that symbols \( z_i^j \) are linearly independent.\(^2\) From formal properties of internal (co)Hom objects we have the cocomposition and coidentity maps

\[
\text{hom}(B, A) \to \text{hom}(C, A) \circ \text{hom}(B, C),
\]

\[
\text{end}[A] \overset{\text{def}}{=} \text{hom}(A, A) \to \mathcal{I},
\]

\(^1\)The above isomorphism is built up from permutations; for instance, restricted to \( [A_1 \otimes B_1]_\otimes^2 \) is given by the map \( S_{23} \) which acts as the flipping map on second and third factors, and trivially on the others.

\(^2\)Nevertheless, suppose a relation \( c^2a_i + \lambda 1 \) (\( \lambda \neq 0 \)) is present in a quantum space \( A \), such that \( c^k = 0 \), for some \( k \). Then \( \text{end}[A] \) is generated by elements \( z_i^j \) related, among other possible constraints, by the linear form \( c^k z_i^j \).
which on each coEnd objects \( \text{end}[A] \in \text{FGA} \) define the (injective) comultiplications
\[
\Delta : \text{end}[A] \rightarrow \text{end}[A] \circ \text{end}[A] \quad / \quad z_i^j \mapsto z_i^k \otimes z_k^j
\]
and (in our case surjective) counits
\[
\varepsilon : \text{end}[A] \rightarrow I \quad / \quad z_i^j \mapsto \delta_i^j \in k,
\]
giving \( \text{end}[A] \) a bialgebra structure \((\Delta, \varepsilon)\). In addition, the (injective) coevaluation \( A \rightarrow \text{end}[A] \circ \text{A} \) defines a coaction which supplies \( A \) with a structure of left \( \text{end}[A]-\)comodule algebra.

The functor \( \mathfrak{f} \): The following observation will be crucial to construct our twisted quantum matrix spaces. For any \( \langle \varphi, \mathcal{H} \rangle \) in \( (A \downarrow \text{FGA} \circ B) \), with \( \mathcal{H} = (H_1, H) \), there exist an associated linear space
\[
H_1^\varphi = L \left\langle h_{i,j} \right\rangle_{i,j=1}^{n,m} \subset H_1
\]
defined by the restriction of \( \varphi \) to \( A_1 \), i.e. by a linear map \( a_i \mapsto h_i^j \otimes b_j \). In particular, there is a linear surjection
\[
\pi^\varphi : B_1^* \otimes A_1 \rightarrow H_1^\varphi : b_i^j \otimes a_i \mapsto h_i^j,
\]
being \( \{b_i^j\} \subset B_1^* \) the dual basis of \( \{b_i\} \). In what follows we identify the linear spaces \( \text{Lin}[B_1, A_1] \) and \( B_1^* \otimes A_1 \).

Moreover, \( \langle \varphi, \mathcal{H} \rangle \) can be related to a subalgebra \( H_1^\varphi \) of \( H_1 \), and the corresponding quantum space \( \mathcal{H}^\varphi = (H_1^\varphi, \mathcal{H}) \). For the initial object, we have generically that \( E_1 = E_1^\varphi \) and \( \pi^\varphi \) gives an isomorphism \( \text{Lin}[B_1, A_1] \cong E_1 \).

In this way, for every couple \( A, B \in \text{FGA} \), the map
\[
\langle \varphi, \mathcal{H} \rangle \in (A \downarrow \text{FGA} \circ B) \rightarrow \mathcal{H}^\varphi \in \text{FGA},
\]
extended to any arrow \( \langle \varphi, \mathcal{H} \rangle \rightarrow \langle \psi, \mathcal{G} \rangle \) as \( \alpha \mapsto \alpha|_{\mathcal{H}^\varphi} \), provides a collection of functors
\[
\mathfrak{f} : (A \downarrow \text{FGA} \circ B) \rightarrow \text{FGA}.
\]
For any \( \alpha : \langle \varphi, \mathcal{H} \rangle \rightarrow \langle \psi, \mathcal{G} \rangle \), \( \mathfrak{f} \alpha = \alpha|_{\mathcal{H}^\varphi} : \mathcal{H}^\varphi \rightarrow \mathcal{G}^\varphi \) defines the epi arrow
\[
(1.4) \quad \mathfrak{f} \alpha : \mathfrak{f} \langle \varphi, \mathcal{H} \rangle \rightarrow \mathfrak{f} \langle \psi, \mathcal{G} \rangle.
\]
Obviously, \( \mathfrak{f} = \mathfrak{P} \) for all \( \langle \varphi, \mathcal{H} \rangle \) such that \( \mathcal{H}^\varphi = \mathcal{H} \); in particular, \( \mathfrak{f} = \mathfrak{P} \) for generic initial objects.

The geometric role of initial objects: We will make a brief comment on the algebraic geometric interpretation of the relationship between diagrams \( \langle \varphi, \mathcal{H} \rangle \) and internal coHom objects in FGA.

The initial property of any \( \text{hom}[B, A] \) means that for a given object \( \mathcal{H} \) and a morphism \( \varphi : A \rightarrow \mathcal{H} \circ B \), there exist a unique morphism of quantum spaces \( \alpha : \text{hom}[B, A] \rightarrow \mathcal{H} \) making commutative the diagram
\[
A \quad \overset{\delta_{A,B}}{\xrightarrow{\varphi}} \quad \overset{\mathfrak{f}}{\xrightarrow{\alpha \circ I_B}} \quad \mathcal{H} \circ B
\]
From Eq. (1.4), we have the epi (or epimorphism, for the underlying algebras)
\[
(1.5) \quad \text{hom}[B, A] \rightarrow \mathfrak{f} \langle \varphi, \mathcal{H} \rangle
\]
for every \( \langle \varphi, \mathcal{H} \rangle \in (A \downarrow \text{FGA} \circ B) \). Thus, the opposite of any \( \mathfrak{f} \langle \varphi, \mathcal{H} \rangle \) can be regarded as a subspace
\[
\mathfrak{f} \langle \varphi, \mathcal{H} \rangle^{\text{op}} \hookrightarrow \text{hom}[B, A]^{\text{op}} = \text{Hom}[B^{\text{op}}, A^{\text{op}}]
\]
of the space of quantum linear maps from \( B \) to \( A \). More precisely, the monic \( \mathfrak{f} \langle \varphi, \mathcal{H} \rangle^{\text{op}} \hookrightarrow \text{hom}[B, A]^{\text{op}} \) gives the representative of an equivalence class of monics defining a subobject of \( \text{hom}[B, A]^{\text{op}} \).

We could say \( \text{hom}[B, A] \) is a ‘generic point’ of the ‘noncommutative algebraic variety’ \( \text{Hom}[B^{\text{op}}, A^{\text{op}}] \) (or that the quantum space \( \text{Hom}[B^{\text{op}}, A^{\text{op}}] \) is the ‘locus of \( \text{hom}[B, A] \))”, and regard the object \( \mathfrak{f} \langle \varphi, \mathcal{H} \rangle = \mathcal{H}^\varphi \) as a ‘specialization’ of it.
1.2. Quadratic quantum spaces. There is another symmetric monoidal structure on QA given by the bifunctor \( \bullet : QA \times QA \rightarrow QA \) such that to every pair of quadratic quantum spaces \( A = (A_1, A) \) and \( B = (B_1, B) \), it assigns the quantum space

\[
A \bullet B \doteq (A_1 \otimes B_1, A \bullet B) ; \quad A \bullet B \doteq [A_1 \otimes B_1]^\hat{\otimes} / I [I \otimes J],
\]

where \( I [X] \) means the bilateral ideal generated by the set \( X \). For the morphisms \( A \rightarrow A' \) and \( B \rightarrow B' \), the morphism \( \alpha \circ \beta : A \bullet B \rightarrow A' \bullet B' \) is defined as the quotient map of

\[
\alpha_1^\otimes \otimes \beta_1^\otimes |_{[A_1 \otimes B_1]^\hat{\otimes}} = \{\alpha_1 \otimes \beta_1\}^\otimes = \bigoplus_{n \in \mathbb{N}_0} \alpha_1^n \otimes \beta_1^n,
\]

which is well-defined because \( \alpha_1^\otimes \otimes \beta_1^\otimes \) is a morphism of \( I \otimes J' \). The unit object is \( U = (k, U) \), being \( U = k[e] / I [e^2] \).

Now, consider the covariant functor \( ! : QA^{op} \rightarrow QA \) such that

\[
A' = (A_1', A), \quad A' \doteq A_1^\otimes / I [I^\perp];
\]

where \( I^\perp \doteq \{r \in A_1^\otimes : \langle r, q \rangle = 0, \forall q \in I\} \) is the annihilator of \( I \) in relation with the standard pairing. For a morphism \( \alpha : A \rightarrow B \) it assigns \( \alpha^! : B^! \rightarrow A' \), being \( \alpha^! \) the quotient map of \( \alpha_1^\otimes : B_1^\otimes \rightarrow A_1^\otimes \). The main relationships among the functors \( \circ, \bullet \) and \( ! \) can be summarized in the following equations

\[
\text{(1.7)} \quad A'^! \subset A, \quad (A \circ B)^! \subset A' \bullet B', \quad (A \bullet B)^! \subset A' \circ B', \quad K^! \subset U.
\]

Manin showed that \((QA, \circ, K)\) has internal coHom objects given by \( \text{hom} [B, A] = B^! \bullet A \), with (co)evaluations, compositions and (co)identities (replacing \( I \) by \( K \)) given by \( [1.2] \) and \( [1.3] \). QA also has internal coHom objects in relation with the monoidal structure \( \bullet \), that is, \((QA, \bullet, U)\). They can be defined as

\[
\text{Hom} [B, A] = B^! \circ A \doteq \text{hom} [B^!, A]^!.
\]

We shall be mainly concerned with coHom objects, because they define on \( QLS = FGA^{op} \) the spaces of quantum linear maps

\[
\text{Hom} [B^{op}, A^{op}] = \text{hom} [B, A]^{op} \in QLS,
\]

or the noncommutative algebraic varieties given by the locus of \( \text{hom} [B, A] \).

The structures described in this section can be defined also on \( m \)-th quantum spaces (we just have to change 2 by \( m \) in all definitions). For instance, a unit element for \((CA^m, \bullet)\) would be \( U_m = (k, U_m) \), being \( U_m = k[e] / I [e^m] \). Note that \( U_2 = U \), and that we can define \( U_\infty \doteq K \).

1.3. The conic case. From now on, we understand by quantum spaces the conic ones, unless we say the contrary. If \((A_1, A)\) and \((B_1, B)\) are objects of this class, then

\[
\ker \Pi_A = \bigoplus_{n \in \mathbb{N}_0} I_n, \quad \ker \Pi_B = \bigoplus_{n \in \mathbb{N}_0} J_n,
\]

Define the functors \( \circ \) and \( ! \) as

\[
A \circ B \doteq [A_1 \otimes B_1]^\hat{\otimes} / \bigoplus_{n \in \mathbb{N}_0} I_n \otimes J_n
\]

and \( A' \doteq A_1^\otimes / I \left[ \bigoplus_{n \geq 2} I_n \right] \), where now

\[
I_n^\perp = \{x \in A_1^\otimes : \langle x, y \rangle = 0, \forall y \in I_n\}, \quad n \geq 2, \quad I_{0,1}^\perp = \{0\}.
\]

A unit element for \((CA, \circ)\) is \( U \) as for QA, however the restriction of \( \circ \) to every \( CA^m \) (in particular QA) does not coincide with the analogous functor \( \bullet \) defined on these subcategories. On the other hand, the functor \( ! \) does coincide with the corresponding to the \( m \)-th cases because

\[
\text{(1.8)} \quad (A_1^\otimes \circ I \otimes A_1^\otimes)^! = A_1^\otimes \circ I^\perp \otimes A_1^\otimes, \quad \forall r, s \in \mathbb{N}_0,
\]

where \( I \subset A_1^\otimes \) is the generator of \( \ker \Pi \); but \( !^2 \not= \text{id}_{CA} \). The preserved properties are (w.r.t. Eq. \( [1.7] \)),

\[
(A \circ B)^! \subset A' \circ B', \quad K^! \subset U, \quad K \subset U^!.
\]

while \((A \circ B)^! \not= A'_! \otimes B'_! \) and \( A'^! \not= A \).
Although the functors $*$ and $!$ do not have the nice properties of $\bullet$ and $!$, they lead us to define on $CA$ some kind of covariant mixing of them, namely

$$\triangleright : CA^{op} \times CA \rightarrow CA, \quad \triangleleft : CA \times CA^{op} \rightarrow CA,$$

$$\odot : CA^{op} \times CA^{op} \rightarrow CA,$$

given on objects by (identifying each object with its opposite)$^5$

$$A \triangleright B \equiv [\{a_1 \otimes b_1\}]^{\triangleleft} / I \left[ \bigoplus_{n \in J_0} I_n \otimes J_n \right],$$

$$A \triangleleft B \equiv [\{a_1 \otimes b_1\}]^{\circ} / I \left[ \bigoplus_{n \in J_0} I_n \otimes J_n \right],$$

$$A \odot B \equiv [\{a_1 \otimes b_1\}]^{\circ} / I \left[ \bigoplus_{n \in J_0} I_n \otimes J_n \right].$$

Using (1.8), it can be shown that restricted to every $CA^m$,

$$\triangleright = \bullet (! \times id), \quad \triangleleft = \bullet (id \times !), \quad \odot = \bullet (! \times !).$$

Associated to $\triangleright$ and $\triangleleft$ we have $K$ as left and as a right unit, respectively, in the sense that $K \triangleright A \cong A$ and $A \triangleleft K \cong A$ for any $A$ in $CA$ (while $A \triangleright U \cong A', U \triangleleft A \cong A'$ and $K \odot A \cong A \odot K \cong A'$).

**Theorem 1.** The monoidal category $(CA, \odot, K)$ has internal coHom objects given by

$$\underline{hom}[B, A] = B \triangleright A.$$  

In other words, the functor $\triangleright$ is an internal coHom functor. $\blacksquare$

Before passing to the proof, we introduce some notation and make some observations useful in next sections. Consider $A = (A_1, A)$ and $B = (B_1, B)$, with associated graded ideals linearly generated by the relations

$$(1.10) \quad \left\{ R_{\lambda_n}^{k_1, ..., k_n} a_{k_1} ... a_{k_n} \right\}_{\lambda_n \in \Lambda_n} \subset I_n, \quad \left\{ S_{\mu_n}^{k_1, ..., k_n} b_{k_1} ... b_{k_n} \right\}_{\mu_n \in \Phi_n} \subset J_n,$$

(sum over repeated indices are assumed) being $\{a_i\}$ and $\{b_i\}$ basis of $A_1$ and $B_1$. For each $J_n$, consider some complement $J_n^c$, such that $J_n \oplus J_n^c = B_1^{\otimes n}$, and indicate by $\left\{ J_{\omega_n}^c \right\}_{\omega_n \in \Omega_n}$ a basis for $J_n^c$. Then we can write

$$(1.11) \quad b_{k_1} ... b_{k_n} = (S_{(\omega_n)}^{k_1, ..., k_n} J_{(\omega_n)}^c + J_{(k_1, ..., k_n)}), \quad J_{(k_1, ..., k_n)} \in J_n.$$

It is important to note that each $J_n \subset B_1^{\otimes n}$ is precisely the linear space spanned by

$$\left\{ b_{k_1} ... b_{k_n} \right\}_{\omega_n \in \Omega_n},$$

being $\{b^i\} \subset B_1^*$ the dual basis of $\{b_i\}$. Those subspaces define the ideal associated to $B^i$.

**Proof.** An internal coHom object $\underline{hom}[B, A]$ of $CA$, if there exists, is an initial object of the comma category $(A \downarrow CA \circ B)$. So, let us show that there exists an initial diagram $(\delta, \mathcal{E})$ in every such a category.

Let $E_i$ be as in Eq. (1.11), and consider the map $\delta_1 : a_i \mapsto z_i^j \otimes b_j$. Then, $\delta_1$ evaluated on an element

$$R_{\lambda_n}^{k_1, ..., k_n} a_{k_1} ... a_{k_n} \in I_n$$

defines $E \equiv E_1^{\odot} / K$, with $K = \bigoplus K_n$, the ideal algebraically generated by

$$(1.12) \quad \left\{ k_{\lambda_n}^{k_1, ..., k_n} z_{j_1}^{k_1} ... z_{j_n}^{k_n} (S_{(j_1, ..., j_n)}^{(\omega_n)} J_{(\omega_n)}^c + J_{(k_1, ..., k_n)}^c) \right\}_{\lambda_n \in \Lambda_n} \subset K_n.$$

$\delta_1$ can be extended to an homomorphism $\delta : A \rightarrow E \otimes B$, resulting the latter a quotient map of $\delta_1^{\odot}$. Thus, the pair $E = (E_1, E)$ defines the diagram $(\delta, \mathcal{E})$. Its initial character is obvious. Moreover, from the comment leading to Eq. (1.12), it is immediate that the linear map $z_i^j \mapsto b^i \otimes a_i$ extends to an algebra isomorphism $E \cong B \triangleright A$. Then, we can define $\underline{hom}[B, A] = B \triangleright A$ $(\cong \mathcal{E})$. $\blacksquare$

Nevertheless, the existence of internal Hom objects for $(CA, \odot, U)$ can not be established, as we have made for each $(CA^m, \bullet, U_m)$, with $m \geq 2$.

$^5$The maps on morphisms are immediate; for instance, we have for $\triangleright$ that $\alpha^{op} \times \beta$ is sent to the quotient map of $[\alpha_i^1 \otimes \beta_1]^{\circ}$. 
2. Semigroupoids and (co)based categories

The main consequences related to the existence of internal (co)Hom objects in a given category are the notions of (co)evaluation or (co)action, (co)composition or (co)multiplication, and (co)identity or (co)unit, which enable us to think of them as the ‘(co)spaces of morphisms’ between the objects in the corresponding category. We will first see how such notions arise in FGA. Then, we will encode them in the framework of (co)based categories, since in this scheme our twisted version of quantum matrices is developed.

2.1. A semigroupoid structure on the comma categories. Let $FGA^\circ$ the disjoint union of the family $\{(A \downarrow FGA \circ B)\}_{A,B \in FGA}$,

$$FGA^\circ = \bigvee_{A,B \in FGA} (A \downarrow FGA \circ B).$$

The objects of $FGA^\circ$ are the disjoint union of objects of $(A \downarrow FGA \circ B)$, for all pairs $A,B \in FGA$, and its morphisms are the union of morphisms of each comma category (in particular, the set of morphisms from objects of $(A \downarrow FGA \circ B)$ to the ones of $(C \downarrow FGA \circ D)$ are empty, unless $A = C$ and $B = D$). The functors $\Psi$ of §1.1 extend to an obvious embedding $FGA^\circ \hookrightarrow FGA$ which we shall also call $\Psi$.

From the formal properties of the monoidal product in $(FGA, \circ, I)$, $FGA^\circ$ inherits a semigroupoid structure given by a partial product functor

$$\circ : \bigvee_{A,B \in FGA} (A \downarrow FGA \circ B) \times (B \downarrow FGA \circ C) \to FGA^\circ,$$

(2.1)

such that

$$\langle \varphi, H \rangle_{A,B} \circ \langle \chi, G \rangle_{B,C} = \langle (id \circ \chi) \varphi, H \circ G \rangle_{A,C}, \quad f \circ g = f \circ g.$$

If $f$ and $g$ are arrows $\langle \varphi, H \rangle_{A,B} \to \langle \varphi', H' \rangle_{A,B}$ and $\langle \chi, G \rangle_{B,C} \to \langle \chi', G' \rangle_{B,C}$, then $f \circ g$ will be an arrow $H \circ G \to H' \circ G'$ in FGA satisfying

$$(id \circ \chi') \varphi' = (f \circ g \circ id) (id \circ \chi) \varphi,$$

because equations $\varphi' = (f \circ id) \varphi$ and $\chi' = (g \circ id) \chi$ hold. Hence $f \circ g$ is a morphism

$$\langle (id \circ \chi) \varphi, H \circ G \rangle_{A,C} \to \langle (id \circ \chi') \varphi', H' \circ G' \rangle_{A,C}.$$

The associativity of $\circ$ follows from that of $\circ$, and it is immediate that the objects $\langle \ell_A, I \rangle \in (A \downarrow FGA \circ A)$ are units for $\circ$. In particular, $\circ$ supplies $(A \downarrow FGA \circ A)$ with a monoidal structure, for any $A \in FGA$. Moreover, $\Psi \circ = \circ (\Psi \times \Psi)$ and $\Psi (\ell_A, I) = I$, so $\Psi : FGA^\circ \hookrightarrow FGA$ is a functor of categories with unital associative partial products.

This groupoid structure on $FGA^\circ$ is independent of the existence of initial objects on each $(A \downarrow FGA \circ B)$, nevertheless, every object $(C \downarrow FGA \circ I)$, $C \in FGA$ has an initial object given by the pair $\langle r_C, C \rangle$, being $r_C$ the functorial isomorphism $C \simeq C \circ I$. (That is true for all monoidal categories.)

Since

$$\langle (id \otimes \delta_{B,C}) \delta_{B,C}, hom [C,A] \circ hom [B,C] \rangle \in (A \downarrow FGA \circ B)$$

and $\langle \ell_A, I \rangle \in (A \downarrow FGA \circ A)$, there exist unique arrows, cocomposition and coidentity (in $FGA^\circ$)

$$\Delta_{B,C,A} : hom [B,A] \rightarrow hom [C,A] \circ hom [B,C],$$

$$\varepsilon_A : hom [A,A] = end [A] \rightarrow I,$$

(2.2)

for every $A,B,C \in FGA$. There is also the notion of (left) codual object through $A^* = hom [A,I]$, with coevaluation $I \rightarrow A^* \circ A$. As remarked above, $hom [C,I] \cong C$, $\forall C \in FGA$; hence any coevaluation $A \rightarrow hom [B,A] \circ B$ can be regarded as the cocomposition

$$hom [I, A] \rightarrow hom [B, A] \circ hom [I, B],$$

(2.3)

i.e. $\delta_{A,B} = \Delta_{I,B,A}$. In particular, for duality we have

$$hom [I, I] \rightarrow hom [A, I] \circ hom [I, A].$$

(2.4)
Due to associativity and existence of unit element for \( \cdot \), the following diagrams (associative composition and identity morphisms) are commutative:\(^6\)

\[
\begin{align*}
\text{hom}[C, A] \circ \text{hom}[B, C] & \longrightarrow \text{hom}[D, A] \circ \text{hom}[C, D] \circ \text{hom}[B, C] \\
\text{hom}[B, A] & \longrightarrow \text{hom}[D, A] \circ \text{hom}[B, D] \\
I \circ \text{hom}[B, A] & \longrightarrow \text{end}[A] \circ \text{hom}[B, A]
\end{align*}
\]

\[(2.5)\]

These diagrams together with Eq. \((2.6)\) reflects the compatibility between the coevaluation and the notions of cocomposition and coidentity. In dual terms (and for concrete categories) they give equations like

\[
[f \circ g](v) = f(g(v)) \quad \text{and} \quad \text{id}(v) = v.
\]

On the other hand, diagram \((2.4)\) states that, for every \( A, B \) in FGA, arrows \( \text{hom}[B, A] \to \text{end}[A] \circ \text{hom}[B, A] \) are left invertible.

In particular, there exist injections \( \text{end}[A] \hookrightarrow \text{end}[A] \circ \text{end}[A] \), \( A \hookrightarrow \text{end}[A] \circ A \) as it was mentioned in §1.1.

### 2.2. The (co)based categories.

Given a monoidal category \((C, \circ, I)\), a C-based category is (c.f. \(\mathbb{S}\)):

a) A set of objects \( a, b, c, \ldots \);

b) A function which assigns to each ordered pair of objects \( (a, b) \) an object \( \text{Hom}[a, b] \in C \), called the Hom objects (to resemble the Hom sets that define a proper category);

c) For each ordered triple of objects \( (a, b, c) \) a morphism

\[
\text{Hom}[c, b] \circ \text{Hom}[a, c] \to \text{Hom}[a, b];
\]

d) For each object a morphism \( I \to \text{Hom}[a, a] = \text{End}[a] \).

These morphisms must satisfy obvious associativity and unit constraints (dual to diagrams \((2.5)\) and \((2.6)\)).\(^7\)

In a (co)based category there is not, \textit{a priori}, a notion of (co)evaluation.

It is clear that every category \((C, \circ, I)\) with internal Hom objects gives rise to a C-based category with objects in \(C.\)\(^8\) In the dual case, of internal coHom objects, there is a related C\(^{op}\)-based category, which we call C-cobased category. The function \((A, B) \mapsto \text{hom}[B, A]\), together with arrows \((2.2)\) and diagrams \((2.5)\) and \((2.6)\), form an FGA-cobased category. Dually, the function

\[
\text{hom}[B, A] \to \text{hom}[\text{hom}[B, A], A] = \text{hom}[\text{hom}[B, A], A]^{op},
\]

with reversed arrows and diagrams, defines a QLS-based category with objects in QLS.

This can be settled in more general terms: suppose a family of categories \( \{F^{A,B}, \mathcal{A}, \mathcal{B} \in \text{FGA}\} \) (or equivalently, a function \((A, B) \mapsto F^{A,B}\) to categories) with initial objects is given together a related family of functors \( \mathcal{P}^{F} : F^{A,B} \to \text{FGA} \) and a semigroupoid structure \( \circ_{F} \) on its disjoint union \( F^{-}\). Suppose in addition that

\[
\mathcal{P}^{F} \circ_{F} \sigma_{F} = \circ \left(\mathcal{P}^{F} \times \mathcal{P}^{F}\right)
\]

and the respective units are preserved. Then, the function \((A, B) \mapsto \text{hom}_{F}[B, A], \) being \( \text{hom}^{F}[B, A] \) the image under \( \mathcal{P}^{F} \) of the initial object of \( F^{A,B}\), defines an FGA-cobased category (for the dual case we just must change initial by terminal objects). That is a direct consequence of the discussion we have made in the previous subsection.

**Example.** Consider the category \( \text{CA}_{\bullet} \) of conic quantum spaces equipped with a marked basis (and with the same morphisms of \( \text{CA}\)). For each \( A_{\bullet} = (A, g), \mathcal{B}_{\bullet} = (\mathcal{B}, f) \in \text{CA}_{\bullet}, \) where \( g = \{a_{i}\} \subset A_{\bullet} \) and \( f = \{b_{j}\} \subset B_{\bullet} \) are the marked bases of \( A \) and \( \mathcal{B} \), we define the categories \( F^{\bullet} \) as the ones given by triples \( (H_{\bullet}, \varphi, \varphi^{\dagger}) \), being \( \varphi : A \to H \circ B \) and \( \varphi^{\dagger} : A^{\dagger} \to H \circ B^{\dagger} \) arrows in \( \text{CA} \) such that

\[
\langle \varphi^{\dagger}(a^{\dagger}), b_{j} \rangle = \langle b^{j}, \varphi(a_{i}) \rangle,
\]

\[(2.8)\]

\(^6\)Of course, we have a similar commutative diagram where \( I \) is on the right.

\(^7\)Any (small) category can be regarded as a Set-based category.

\(^8\)In the latter case and for concrete categories (in which the objects \( \text{Hom}[a, b] \) define Hom-sets), it says that \( C \) has been \textit{enriched} by the objects \( \text{Hom}[a, b] \in C. \) This is the case of \((\mathcal{Q}, \circ, \bullet)\), which is enriched by its internal Hom objects \( B^{\dagger} \circ A. \)
where \( (h \otimes x, y) = (x, y \otimes h) \) \( h \in H \), and \( \{a^i\}, \{b^i\} \) the corresponding duals to \( g \) and \( f \). Arrows
\[
(\mathcal{H}_*, \varphi^\dagger) \to (\mathcal{G}_*, \psi, \psi^\dagger)
\]
between triples are morphisms \( \mathcal{H} \to \mathcal{G} \in \text{CA} \) such that \( \psi = (\alpha \circ I) \varphi \) (i.e. they are morphisms in \((A \downarrow \text{CA} \circ B)\)).

Of course, the functors \( \text{hom}^T \) are the embedding \((\mathcal{H}_*, \varphi, \varphi^\dagger) \to \mathcal{H} \). The initial objects are quotients of the corresponding to \((A \downarrow \text{CA} \circ B)\). In particular, for each \( f^{-1} \), we can define \text{end}^f \{A_\bullet\} \cong \text{e} \{A, g\}, the one defined by Manin (for QA) in [10] to add the so-called \text{missing relations}.

The semigroupoid structure on \( f^{-1} \) is given by
\[
(\mathcal{H}_*, \varphi, \varphi^\dagger) \times (\mathcal{G}_*, \psi, \psi^\dagger) \to [\mathcal{H} \circ \mathcal{G}_*], (I_H \circ \psi), (I_H \circ \psi^\dagger) \varphi^\dagger
\]
and \( \alpha \times \beta \mapsto \alpha \circ \beta \), where the marked basis of \( \mathcal{H} \circ \mathcal{G} \) is the direct product between the ones of \( \mathcal{H} \) and \( \mathcal{G} \). ■

In §4 we construct a family of categories \( \{ \mathcal{Y}^{A,B} : A, B \in \text{FGA} \} \) whose objects are essentially pairs
\((A \to \mathcal{H} \circ_r B, \mathcal{H}), \mathcal{H} \in \text{FGA}, \)

with \( \circ_r \) some given twisted tensor product between \( \mathcal{H} \) and \( B \) (as we shall define in the next chapter). Each category \( \mathcal{Y}^{A,B} \) has an initial object, \( \text{hom}^T \{B, A\} \), and a map \( A \to \text{hom}^T \{B, A\} \circ_r B \). They also have a related faithful functor \( \text{hom}^T : \mathcal{Y}^{A,B} \to \text{FGA} \). Moreover, on the subfamily \( \{ \mathcal{Y}^{A,B} : A, B \in \text{CA} \} \) corresponding to the conic case, a semigroupoid structure \( \circ_T \) can be defined in such a way that
\[
\text{hom}^T \circ_T = \circ (\text{hom}^T \times \text{hom}^T).
\]

As a consequence, the function \((A, B) \to \text{hom}^T \{B, A\} \) defines an CA-cobased category and, in addition, the maps \( A \to \text{hom}^T \{B, A\} \circ_r B \) provide a coevaluation notion. (We must mention that the last notion is not compatible with cocomposition and counit -see the comments before Eq. [17].)

3. Twisted tensor product of quantum spaces

In this chapter we define a twisted tensor product structure on quantum spaces in a way intimately related to the corresponding algebra case, following the developments accomplished in ref. [1]. In doing so, we firstly redefine the twisted tensor products on the category \( k \text{-Alg} \) (with respect to its usual monoid \( \otimes \)) in geometric terms.

3.1. The twisted tensor products of algebras and the geometric language.

Given an algebra \((A, m, \eta)\), \( m \) denotes its associative product and \( \eta : k \to A \) its unit map. The unit element of any algebra (including \( k \)) is denoted by “1”.

Given \( A, B \in k \text{-Alg} \), the set \( \text{Hom}_{k \text{-Alg}} \{A, B\} \) of algebra homomorphisms is also called the set of \textit{B-points} of \( A \). This terminology is adequate when we are thinking about \( A \) and \( B \) as the coordinate (or function) rings of certain -noncommutative- spaces. It says that a \textit{B-point of } \textit{A} is \textit{generic} if it is defined by a monomorphism \( A \hookrightarrow B \); and two \textit{B-points} \( \alpha : X \to B \) of \( X \) and \( \beta : Y \to B \) of \( Y \) are \textit{commuting} if
\[
\alpha(x) \cdot \beta(y) = \beta(y) \cdot \alpha(x), \forall x \in X, \forall y \in Y,
\]
or
\[
[\alpha, \beta]_B = (m_B - m_B^\eta) (\alpha \otimes \beta) = 0.
\]

Let us express in these terms the definition of twisted tensor products of algebras given in [1].

Definition 1. Consider a pair of algebras \( A \) and \( B \). The \textbf{twisted tensor product (TTP) of } \textit{A and B} is a triple \((C, i_A, i_B)\) where \( C \) is an algebra, \( i_A \) is a generic \textbf{C-point} of \( A \) and \( i_B \) is a generic \textbf{C-point} of \( B \), such that the linear map
\[
\varphi \equiv m_C \{i_A \otimes i_B\} : A \otimes B \to C
\]
is an isomorphism. A morphism of TTP’s \((C, i_A, i_B) \to (C', i'_A, i'_B)\) is given by an homomorphism of algebras \( \rho : C \to C' \), with \( \rho i_A = i'_A \) and \( \rho i_B = i'_B \). ■

Given algebras \((A, m_A, \eta_A)\) and \((B, m_B, \eta_B)\), examples of these objects are algebras \( A \otimes_r B \) built over the vector space \( A \otimes B \) with product and unit map
\[
m_{A \otimes B} = (m_A \otimes m_B) (I_A \otimes \tau \otimes I_B); \quad \eta_{A \otimes B} = \eta_A \otimes \eta_B,
\]
being \( \tau : B \otimes A \to A \otimes B \) a linear map satisfying
\[
\tau (m_B \otimes I_A) = (I_A \otimes m_B) (\tau \otimes I_B) (I_B \otimes \tau),
\]
\[
\tau (I_B \otimes m_A) = (m_A \otimes I_B) (I_A \otimes \tau) (\tau \otimes I_A),
\]

\footnote{Note that \( \psi = (\alpha \circ I) \varphi \) and Eq. [23] implies \( \psi^\dagger = (\alpha \circ I) \varphi^\dagger \).

\footnote{Such relations turns \text{end} \{A\} into a commutative algebra \text{e} \{A\} whenever \( A \) is commutative.}
and

\[ \tau(I_B \otimes \eta_A) = \eta_A \otimes I_B; \quad \tau(\eta_B \otimes I_A) = I_A \otimes \eta_B. \]

The generic \( A \otimes_{\tau} B \)-points correspond to the usual inclusion of algebras

\[ i_A \triangleq i_A : a \mapsto a \otimes 1, \quad i_B \triangleq i_B : b \mapsto 1 \otimes b. \]

The maps \( \tau \) (satisfying (3.1) and (3.2)) are called **twisting maps**. Essentially, these are all of TTP’s, namely: any triple \((C, i_A, i_B)\) is isomorphic to a unique triple of the form \((A \otimes_{\tau} B, i_A, i_B)\), i.e. there exist a unique twisting map \( \tau : B \otimes A \to A \otimes B \) such that

\[ (A \otimes_{\tau} B, i_A, i_B) \simeq (C, i_A, i_B) \]

(for a proof see \([1]\)). The isomorphism is given by \( \varphi : A \otimes B \simeq C \) and \( \tau \) by the equation

\[ \tau(b \otimes a) = \varphi^{-1}(i_B(b) \cdot i_A(a)), \quad a \in A, \ b \in B. \]

In other words, the equivalence classes of TTP’s are in one to one correspondence to twisting maps.

We often refer to a TTP of \( A \) and \( B \) as an algebra \( A \otimes_{\tau} B \) (omitting the canonical inclusion maps), being \( \tau \) some twisting map.

Any triple \((C, i_k, i_A)\) or \((C, i_A, i_k)\) (remember that \( k \) is a unit object for the monoidal category \((k-\text{Alg}, \otimes)\)), because of the Eq. (3.2), is isomorphic to the algebra \( A \), i.e. \( A \otimes_k k = k \otimes A = A \) for all twisting map \( \tau \).

We shall see that the canonical **flipping map** \( \tau_o \), \( \tau_o(b \otimes a) = a \otimes b \), is related to the TTP’s with commuting generic points, as it could be expect and is implicit in \([1]\). This class of TTP’s belongs to the class of the usual tensor product of algebras \( A \otimes B \).

**Proposition 1.** Given a pair of algebras \( A \) and \( B \), the set of twisted tensor products \((C, i_A, i_B)\) such that \([i_A, i_B]_C = 0\), i.e. \( i_A \) and \( i_B \) are commuting points, form an equivalence class whose associated twisting map is the flipping map.

**Proof.** Let \((C, i_A, i_B)\) be a TTP of \( A \) and \( B \). By the above theorem, there exist a unique twisting map \( \tau(b \otimes a) = \varphi^{-1}(i_B(b) \cdot i_A(a)) \), such that \( \varphi \) defines the algebra isomorphism \( A \otimes_{\tau} B \simeq C \), satisfying

\[ \varphi i_A = i_A \quad \text{and} \quad \varphi i_B = i_B. \]

Therefore, if \([i_A, i_B]_C = 0\),

\[ \tau(b \otimes a) = \varphi^{-1}(i_B(b) \cdot i_A(a)) = \varphi^{-1}(i_A(a) \cdot i_B(b)) = \varphi^{-1}(i_A(a)) \cdot \tau \varphi^{-1}(i_B(b)) = i_A(a) \cdot \tau i_B(b) = (a \otimes 1) \cdot \tau (1 \otimes b) = a \otimes b \]

where we are denoting “\( \cdot \)” for the product in \( C \) and “\( \cdot \tau \)” for the one in \( A \otimes B \). The last equation means that \( \tau \) is the canonical flipping map. 

Therefore, it can be stated that those twisting \( \tau \) different from flipping map give rise to a (direct) product space where the points of each factor do not commute among themselves.

For latter convenience, we will characterize the classes of TTP’s with bijective twisting maps.

**Definition 2.** Let \((C, i_A, i_B)\) be a TTP for \( A \) and \( B \). It will be called **symmetric** twisted tensor product (STTP) provided the linear map

\[ \varphi^{op} \overset{\sim}{=} m_C(i_B \otimes i_A) : B \otimes A \to C \]

is a bijection. 

**Proposition 2.** The class of a STTP corresponds to a bijective twisting map, and viceversa.

**Proof.** The twisting map related to \((C, i_A, i_B)\) is \( \tau = \varphi^{-1} \varphi^{op} \), as it can see from (3.2). Then \( \tau \) is bijective iff \( \varphi^{op} \) is a bijection.

In these cases \( \varphi^{op} \) defines the algebra isomorphism \( B \otimes_{\tau^{op}} A \simeq C \), with

\[ \tau^{op} = (\varphi^{op})^{-1} \varphi = (\varphi^{-1} \varphi^{op})^{-1} = \tau^{-1}. \]

In particular we have \( B \otimes_{\tau^{-1}} A \simeq A \otimes_{\tau} B \).

Now we shall rephrase the above definitions for quantum spaces.
3.2. Twisted monoid on FGA’s objects. The monoid \( \circ \) stems from the usual tensor product \( \otimes \), being \( A \circ B \) a subalgebra of \( A \otimes B \). This enables us to introduce a natural definition of \( \text{TTP} \) on quantum spaces as follows:

**Definition 3.** Let \( A = (A_1, A) \) and \( B = (B_1, B) \) be two objects of FGA. A **TTP of \( A \) and \( B \)** is a quadruple

\[
(C, C, i_A, i_B), \quad C = (C_1, C),
\]

with \( C \) a subalgebra of \( C \), and \( i_A \) and \( i_B \) are generic \( C \)-points of \( A \) and \( B \), resp., such that the linear map \( \varphi = m_C (i_A \otimes i_B) \) is an isomorphism, with the additional condition that its restriction to \( A_1 \otimes B_1 \) gives \( \varphi : A_1 \otimes B_1 \rightarrow C_1 \). A morphism of TTP’s is a morphism of algebras \( \rho : C \rightarrow C' \), with \( \rho i_A = i_A' \), \( \rho i_B = i_B' \), and \( \rho (C_1) \subset C'_1 \), i.e. \( \rho \) restricted to \( C \) defines a morphism \( C \rightarrow C' \).

**Proposition 3.** Any quadruple \( (C, C, i_A, i_B) \) is isomorphic to a unique one

\[
(A \circ \tau, B \otimes \tau, i_A, i_B),
\]

where \( \tau \) is a twisting map, \( A \circ \tau, B \otimes \tau = (A_1 \otimes B_1, A \circ \tau B), \) and \( A \circ \tau B \) is the subalgebra of \( A \otimes \tau B \) generated by \( A_1 \otimes B_1 \). The isomorphism is given by \( \varphi \), the associated isomorphism of quantum spaces is

\[
\varphi|_{A \circ \tau B} : A \circ \tau B \cong C,
\]

and again \( \tau (b \otimes a) = \varphi^{-1} (i_B (b) \cdot i_A (a)) \), \( a \in A, b \in B \).

**Proof.** It follows from the (general) algebra case.

Since this is a one to one correspondence, we often refer to TTP’s of \( A \) and \( B \) as a quantum space \( A \circ \tau B \), being \( \tau \) some twisting map.

**Remark 1.** Analogously to the previous section, any quadruple \( (C, C, i_A, i_B) \) is isomorphic to \( A \), because \( A \circ \tau \mathcal{I} = \mathcal{I} \circ \tau A = A \) for any twisting map (remember that \( \mathcal{I} \) is a unit object for (FGA, \( \circ \))). The same is not true for the unit element \( \mathcal{K} \) of \( (CA, \circ) \) (and every \((CA^n, \circ)\)), because the underlying algebra of \( \mathcal{K} = (k, k[e]) \) is not just generated by the unit element \( e \) (as the object \( \mathcal{I} = (k, k) \)), but for the set \( \{1, e^n\}_{n \geq 1} \).

Respect to the relationship between commuting points and the monoid \( \circ \) of quantum spaces, we have:

**Proposition 4.** Given \( A, B \in \text{FGA} \), the set of TTP’s \( (C, C, i_A, i_B) \) such that \( [i_A, i_B]_C = 0 \), form an equivalence class with associated twisting map equal to the flipping map. In other words, the above set is the equivalence class of \( A \circ B \).

It is worth mentioning that the underlying vector spaces of \( A \circ B \) and \( A \circ B \) (subspaces of \( A \otimes B \)) do not coincide in general (while for the algebra case we have the equality of vector spaces \( A \otimes B = A \otimes B \)). Nevertheless, let us consider the symmetric analogous of TTP of algebras:

**Definition 4.** A **symmetric** twisted tensor product of \( A \) and \( B \), is a twisted tensor product \( (C, C, i_A, i_B) \) such that

\[
\varphi^{op} = m_C (i_B \otimes i_A) : B \otimes A \rightarrow C
\]

is a linear isomorphism which restricted to \( B_1 \otimes A_1 \) gives \( B_1 \otimes A_1 \subset C_1 \).

As in the algebra case we have \( A \circ \tau B \cong B \circ_{\tau^{-1}} A \).

A direct consequence of the above definition is the following.

**Proposition 5.** The class of an STTP corresponds to a bijective twisting map such that

\[
\tau (B_1 \otimes A_1) \subset A_1 \otimes B_1,
\]

and vice versa.

**Proof.** As in the previous section, the twisting map related to \( (C, C, i_A, i_B) \) can be written \( \tau = \varphi^{-1} \varphi^{op} \), from which we see that \( \tau \) is a bijection iff \( \varphi^{op} \) is a bijection. On the other hand,

\[
\varphi\tau (B_1 \otimes A_1) = \varphi^{op} (B_1 \otimes A_1) \subset C_1
\]

iff Eq. \( 3.2 \) holds.

---

\(^{11}\)It is enough to take, for \( A = A^\otimes \) and \( B = B_1^\otimes \), a twisting map \( \tau : B \otimes A \rightarrow A \otimes B \) satisfying \( 3.2 \), and in some complement \( \mathcal{I}' \) of \( 1 = B \otimes k + k \otimes A \) (\( k = A_0, B_0 \)), to put \( \tau|_{\mathcal{I}'} = 0 \). Hence, we will have \( A \circ \tau B = k \), clearly distinct from \( A \circ B \).
We call symmetric twisting maps those \( \tau \)'s related to STTP's. They are completely defined by isomorphisms 
\[
\tau|_{B_1 \otimes A_1} : B_1 \otimes A_1 \simeq A_1 \otimes B_1,
\]
since properties (21) and (22). For convenience, we often write the above maps as 
\[
\hat{\tau} : B_1 \otimes A_1 \simeq B_1 \otimes A_1,
\]
using the flipping map \( \tau_o \) to reorder the factors, i.e. \( \hat{\tau} = \tau_o \tau \). Therefore, given a pair of basis \( \{a_i\} \) and \( \{b_j\} \) of \( A_1 \) and \( B_1 \), resp., we have \( \tau \) defined by 
\[
\hat{\tau}(b_i \otimes a_j) = \hat{\tau}_{ij} b_i \otimes a_k,
\]
where we are using the sum over repeated indices convention.

**Example.** Consider a pair of quadratic quantum spaces \( A \) and \( B \), such that, in terms of basis \( \{a_i\}_{i=1}^n \) and \( \{b_j\}_{j=1}^m \) of \( A_1 \) and \( B_1 \), resp., the kernel of the associated canonical maps \( A_1 \rightarrow A \) and \( B_1 \rightarrow B \) are generated by subspaces 
\[
\text{span} \left[ A_{ij} a_k \otimes a_l \right]_{i,j=1}^n \subset A_1^{\otimes 2} \quad \text{and} \quad \text{span} \left[ B_{ij} b_k \otimes b_l \right]_{i,j=1}^m \subset B_1^{\otimes 2}.
\]
Taking an invertible matrix \( \hat{\tau} \) such that there exist \( \Lambda \) and \( \Omega \) satisfying 
\[
A^{ik}_{ij} \hat{\tau}_{si} \hat{\tau}_{ts} = \Lambda^{ik}_{ij} A^{pq}_{kl} \quad \text{and} \quad B^{ik}_{ij} \hat{\tau}_{si} \hat{\tau}_{ts} = \Omega^{ik}_{ij} B^{pq}_{kl},
\]
we get a STTP \( A \circ B \). (Such matrices \( \Lambda \) and \( \Omega \) insure \( \hat{\tau} \) preserve the ideals.) For example, calling \( A \) and \( B \) the endomorphisms \( A_1^{\otimes 2} \rightarrow A_1^{\otimes 2} \) and \( B_1^{\otimes 2} \rightarrow B_1^{\otimes 2} \), defined by the coefficients \( A^{ik}_{ij} \) and \( B^{ik}_{ij} \) (on above basis), resp., any pair of linear isomorphisms \( \alpha : A_1 \simeq A_1 \) and \( \beta : B_1 \simeq B_1 \) such that \( [\alpha \cdot \alpha] = [\beta \cdot \beta] = 0 \), give rise to a symmetric twisting map \( \tau \) with \( \hat{\tau} = \beta \circ \alpha \). In fact, for such a case we can take 
\[
\Lambda^{ik}_{ij} = \tau_{ri}^{sk} \tau_{ts}^{kl} \quad \text{and} \quad \Omega^{ik}_{ij} = \tau_{is}^{kl} \tau_{jr}^{kl}.
\]

Now, the main properties of STTP's.

**Proposition 6.** Given a symmetric twisting map \( \tau \) on \( A \) and \( B \), the underlying vector spaces to \( A \circ B \) and \( A \circ B \) are equal. Moreover, the related gradations of \( A \circ B \) and \( A \circ B \) coincide.

**Proof.** As in the example above, let us indicate by \( \{a_i\} \) and \( \{b_j\} \) a pair of basis for \( A_1 \) and \( B_1 \), respectively. The underlying vector space of \( A \circ B \subset A \otimes B \) and \( A \circ B \subset A \otimes B \) are generated by the words written with elements \( \{a_i \otimes b_j\} \) under \( m_{A \otimes B} \) and \( m_{A \otimes B} \). If we write \( a_i \otimes b_j = ab \), \( m_{A \otimes B} = \cdot \), and \( m_{A \otimes B} = \cdot \), those words are related by 
\[
(3.5) \quad ab \cdot a \cdot b \cdot \ldots = \Gamma \quad ab \cdot a \cdot b \cdot \ldots
\]
where \( \Gamma \) is a matrix constructed from \( \hat{\tau} \) by using successively Eq. (31) (in any order, because both products are associative). Because \( \hat{\tau} \) is invertible, \( \Gamma \) is also invertible. Then, we can write \( ab \cdot a \cdot b = \Gamma^{-1} ab \cdot a \cdot b \). Hence, any element of \( A \circ B \) is an element of \( A \circ B \), and vice versa.

The proof of the last statement is immediate from Eq. (36). □

**Proposition 7.** Given \( A \) and \( B \) in \( CA \), a STTP \( A \circ B \) is in \( CA \). Moreover, the related gradations of \( A \circ B \) and \( A \circ B \) are equals.

**Proof.** The symmetric property of a twisting map \( \tau : B \otimes A \rightarrow A \otimes B \), where \( A \) and \( B \) are graded algebras, insures that \( \tau(B_n \otimes A_m) \subset A_m \otimes B_n \) (which can be seen by direct calculation), and so the product \( m_{A \otimes B} \) satisfies 
\[
m_{A \otimes B}^{\tau} (A_n \otimes B_m) = (m_{A \otimes B} (A_n \otimes \tau(B_n \otimes A_m) \otimes B_m) 
\subset A_{m+n} \otimes B_{n+m}.
\]

Thus, as a vector space \( A \circ B = \bigoplus_n A_n \otimes B_n \) and 
\[
(A_n \otimes B_m) \cdot (A_m \otimes B_n) \subset A_{m+n} \otimes B_{n+m},
\]
as we wanted to show. □

To conclude this section, let us observe that, in view of last proposition, to define a STTP of conic quantum spaces \( A \) and \( B \) is the same as fixing a particular algebra structure on the graded vector space \( A \circ B \), such that \( A_1 \circ B_1 \) is the generating subspace. That means, given a symmetric twisting map \( \tau \), the conic quantum space \( A \circ B \) (and in particular \( A \circ B \)) is an object of GrVctk (the category of graded vector spaces and homogeneous linear maps) with an additional structure. Hence, defining on GrVctk the monoidal product 
\[
V \otimes W = \bigoplus_{n \in \mathbb{N}_0} V_n \otimes W_n,
\]
for $V = \bigoplus_{n \in \mathbb{N}} V_n$ and $W = \bigoplus_{n \in \mathbb{N}} W_n$, the forgetful functor $\mathfrak{f} : CA \rightarrow \text{GrVct}_k$ turns into a monoidal one, and the equalities

$$\mathfrak{f}(A \circ B) = \mathfrak{f}(A) \circ \mathfrak{f}(B) = \mathfrak{f}(A) \circ \mathfrak{f}(B) = A \circ B$$

hold for all twisting maps $\tau$.

## 4. Twisted Quantum Matrix Spaces

In this section we finally study the consequences of replacing $\circ$ by $\circ_{\tau}$ in maps $A \rightarrow H \circ B$ (the diagrams of $(A \downarrow CA \circ B)$), obtaining in this way the categories $\Upsilon^{A,B}$ mentioned in §2.2. Because $CA \circ B$ is no longer a functor on $CA$, this change cannot be performed in the framework of comma categories. However, one is able to define arrows $A \rightarrow H \circ B$ (which essentially give the objects defining $\Upsilon^{A,B}$) in terms of the morphisms $\mathfrak{f}_A \rightarrow \mathfrak{f}H \circ \mathfrak{f}B$ in $\text{GrVct}_k$. Hence, the comma categories$^{12}$ $\Upsilon(CA \circ B)$, where $\mathfrak{f}(CA \circ B)$ is the composition of the functors $CA \circ B$ and $\mathfrak{f}$, will be the cornerstone in the following construction, since we will define the categories $\Upsilon^{A,B}$ as full subcategories of them.

### 4.1. The categories $\Upsilon$.

As we made at the end of §1.1, for every $\langle \varphi, H \rangle$ in $\Upsilon(CA \circ B)$ it can be defined the surjection $\pi^\varphi : B_1^1 \otimes A_1 \rightarrow H_1^\varphi$, such that $b^j \otimes a_i \mapsto h^i_j \in H_1$, and a related functor $\tilde{\mathfrak{f}} : (\mathfrak{f}_A \downarrow \mathfrak{f}(CA \circ B)) \rightarrow CA$ such that

$$\tilde{\mathfrak{f}}(\varphi, H) = H^\varphi = (H_1^\varphi, H^\varphi) ; \quad \alpha \mapsto \alpha|_{H^\varphi},$$

where $H^\varphi$ is the subalgebra of $H$ generated by $H_1^\varphi$. (Note that in general, $H^\varphi$ has no relation with the image of $\varphi$.)

Now, consider a linear bijection

$$\hat{\tau} : B_1 \otimes \text{Lin}[B_1, A_1] \simeq B_1 \otimes \text{Lin}[B_1, A_1],$$

associated to $A$ and $B$. Eventually, it could be defined on $B_1 \otimes H_1^\varphi$ through $\pi^\varphi : \text{Lin}[B_1, A_1] \rightarrow H_1^\varphi$ in such a way that the diagram

$$
\begin{array}{ccc}
B_1 \otimes \text{Lin}[B_1, A_1] & \rightarrow & B_1 \otimes H_1^\varphi \\
\downarrow & & \downarrow \\
B_1 \otimes \text{Lin}[B_1, A_1] & \rightarrow & B_1 \otimes H_1^\varphi
\end{array}
$$

be commutative, and extend it to all of $B \otimes H^\varphi$ as a symmetric twisting map using the properties (3.1) and (3.2).

### Definition 5.

**For every pair $A, B \in CA$ and every linear bijection $\hat{\tau}$ (as in Eq. (4.2)), we define $\Upsilon^{A,B}$ as the full subcategory of $(\mathfrak{f}_A \downarrow \mathfrak{f}(CA \circ B))$, associated to $\hat{\tau}$, formed out by diagrams $\langle \varphi, H \rangle$ such that $\hat{\tau}$ defines a symmetric twisting map for $\tilde{\mathfrak{f}}(\varphi, H) = H^\varphi$ and $B$, and the homogeneous linear map $\varphi$ is a morphism of quantum spaces $A \rightarrow H^\varphi \circ B$.**

Given now a collection $\{\hat{\tau}_{A,B}\}_{A,B \in CA}$ of linear bijections, we name $\Upsilon$ the disjoint union of the categories $\Upsilon^{A,B}$ defined above. Clearly, $CA^\varphi$ is a category $\Upsilon^\varphi$ with associated collection

$$\hat{\tau}_{A,B} = I_{B_1 \otimes \text{Lin}[B_1, A_1]}, \quad \forall A, B \in CA.$$ 

Moreover, calling $\mathfrak{f}CA^\varphi$ the disjoint union of $(\mathfrak{f}_A \downarrow \mathfrak{f}(CA \circ B))$, it follows that any $\Upsilon^\varphi$ (in particular $CA^\varphi$) is a full subcategory of $\mathfrak{f}CA^\varphi$. This sets $CA^\varphi$ and a generic $\Upsilon^\varphi$ on an equal footing.

### Theorem 2.

**Each category $\Upsilon^{A,B} \subset \Upsilon^\varphi$ has initial object.**

**Proof.** Given $A$ and $B$ generated by linear spaces $A_1$ and $B_1$ with dim $A_1 = n$ and dim $B_1 = m$, the initial object of $\Upsilon^{A,B}$ is a pair

$$\left< \delta_{A,B}, \text{hom}^\Upsilon[B, A] \right> ; \quad \text{hom}^\Upsilon[B, A] = (D_1, D) ; \quad D_1 = \text{span} [z_{ij}]_{i,j=1}^{n,m}.$$ 

Also, $\delta_{A,B} = \delta$ is defined by the injective linear map $\delta_1 : a_i \mapsto z_{ij} \otimes b_j$. Let us show it.

Because of $\text{Lin}[B_1, A_1] \simeq D_1$, $\hat{\tau}_{A,B} = \hat{\tau}$ can be extended to all of $B_1^\otimes \otimes D_1^\otimes$ as a symmetric twisting map $\pi^\varphi : B_1^\otimes \otimes D_1^\otimes \rightarrow D_1^\otimes \otimes B_1^\otimes$, and $\delta_1$ extended to an algebra homomorphism $\delta^\otimes_1 : A_1^\otimes \rightarrow D_1^\otimes \otimes \otimes B_1^\otimes$. We are

12Its objects are pairs $\langle \varphi, H \rangle$ where $H \in \text{FGA}_A$, and $\varphi$ is an arrow in $\text{GrVct}_k$.

$$\varphi : \mathfrak{f}_A \rightarrow \mathfrak{f}(H \circ B) = \mathfrak{f}H \circ \mathfrak{f}B.$$
going to build up an algebra $D$ as a quotient of $D^\infty$, such that the linear map $\tau : B \otimes D \to D \otimes B$ and the algebra homomorphism $A \to D \circ_\tau B$, given as the quotient maps of $\tau^\infty$ and $\delta^\infty$, respectively, are well-defined.

For $J = \ker [B^\infty \to B]$ consider some complement $J^c$, such that $J \oplus J^c = B^\infty$. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a set of linear generators for $I = \ker [A^\infty \to A]$ and indicate by $\{J^c_\omega\}_{\omega \in \Omega}$ a basis for $J^c$. Then, we can write
\[(4.3)\]
$$\delta^\infty (I_\lambda) = d^\infty_\lambda \otimes J^c_\omega + \ldots,$$
where $d^\infty_\lambda \in D^\infty$ are linear combinations of monomials of $z_i^\infty$ and "..." denotes terms contained in $D^\infty \otimes J$. On the other hand, given a set of generators $\{J_\theta\}_{\theta \in \Theta}$ for $J$, and basis $\{b_s\}_{s \in S}$ and $\{z_R\}_{R \in R}$ for $B^\infty$, respectively, (given the latter, for instance, by monomials on $B_1$ and $D_1$ basis elements, being $S$ and $R$ the obvious multi-indices), let us write
\[(4.4)\]
$$\tau^\infty (J_\theta \otimes z_R) = d^\infty_{\theta,R} \otimes J^c_\omega + \ldots; \quad \tau^\infty (b_s \otimes d^\infty_\lambda) = d^\infty_{S,\lambda} \otimes J^c_\omega + \ldots.$$ Evaluating $\tau^\infty$ on $b_{S_m} \otimes d^\infty_{\delta,\theta,R}$ and $b_{S_n} \otimes d^\infty_{\omega,\lambda}$, then evaluate the part of the result contained in $D^\infty \otimes J^c$, and so on, one arrives to (repeating the process $m$ times)
\[(4.5)\]
$$\tau^\infty \left( b_{S_m} \otimes d^\infty_{\delta,\theta,R} \right) = d^\infty_{S_m,\delta} \otimes J^c_\omega + \ldots$$
and
$$\tau^\infty \left( b_{S_m} \otimes d^\infty_{\omega,\lambda} \right) = d^\infty_{S_m,\lambda} \otimes J^c_\omega + \ldots.$$ Defining in $D^\infty$ the bilateral ideal $L$ algebraically generated by the set
\[\{d^\infty_{S_m,\delta} \} \otimes \bigcup_{\lambda \in \Lambda} \{d^\infty_{S_m,\lambda} \} \in \Omega \}
(\text{where for } m = 1 \text{ the subindexes } S \text{'s disappear}) we get for the algebra $D = D^\infty / L$ the symmetric twisting map $\tau : B \otimes D \to D \otimes B$ and the algebra homomorphism $\delta : A \to D \circ_\tau B$, given by the linear maps $\tau$ and $a_i \mapsto z_i^\infty \otimes b_j$, respectively.

From the gradation of $I$ and $J$, the ideal $L$ inherits a graded structure. Thus $D_{B,A} = D = (D_1, D)$ in $CA$ and $\delta$ defines the morphism of quantum spaces $A \to D \circ_\tau B$ (in particular, an homogeneous linear map). It is obvious that $D = \mathcal{F} \langle \delta, D \rangle = D^\delta$, so $\langle \delta, D \rangle$ is a diagram of $\mathcal{Y}^{A,B}$. The initial character of $\langle \delta, D \rangle$ is immediate.

Observe that above result does not involve the gradation of the algebras, so it can be extended to the general case. Consider the category of pairs $(A_1, A)$, with $A \in \text{Vct}_k$ and $A_1 \subset A$ a linear subspace of $A$, and the related forgetful functor $h : A \to (A_1, A)$. Let us construct the commutative diagrams $h(A) \mid (\text{FGA} \circ B)$ and their related disjoint union $h_{\text{FGA}}$. Its objects are diagrams $\langle \varphi, H \rangle$, with $H \in \text{FGA}$ and $\varphi$ a linear map $A \to H \circ B$ satisfying $\varphi(A_1) \subset H_1 \otimes B_1$. So, we can define a functor $f : h_{\text{FGA}} \to \text{FGA}$, as in Eq. (14), with $f(\varphi, H) = H^\varphi$, and define full subcategories $T \subset h_{\text{FGA}}$ associated to collections $\{\varphi\}$, whose objects are diagrams $\langle \varphi, H \rangle \in h_{\text{FGA}}$ such that $\varphi$ gives the arrow $A \to f(\varphi, H) \circ B$. It can be shown (directly from the proof of the previous theorem) that each $\mathcal{Y}^{A,B}$ has initial objects. However it is not clear for us how to endow this $T$ with a semigroupoid structure, as we will do for the conic case.

4.1.1. The initial objects of $\mathcal{Y}^{A,B}$ and the geometric interpretation. As at the end of §1.1, for every initial object $D_{B,A}$ there is an epimorphism $D_{B,A} \to \mathcal{F} \langle \varphi, H \rangle$, $\forall \langle \varphi, H \rangle \in \mathcal{Y}^{A,B}$. Thus, the opposite of any $\mathcal{F} \langle \varphi, H \rangle$ can be regarded as a non commutative subvariety $\mathcal{F} \langle \varphi, H \rangle^{\text{op}} \hookrightarrow D_{B,A}^{\text{op}}$. Following [10], we develop this idea in the terminology of $C$-points. Fix a couple of quantum spaces $A, B$ and consider the initial object $D_{B,A} = D$ of $\mathcal{Y}^{A,B}$. Suppose an algebra $C$ and a pair of $C$-points $i_B$ and $i_D$, of $B$ and $D$, are given, in such a way that
\[(4.6)\]
$$i_B (b_i) \cdot i_D (z_k^m) = \tilde{\tau}^\infty_{\text{ink}} i_D (z_k^m) \cdot i_B (b_i).$$
We are denoting by "⋅" the product of $C$. If we write $i_B (b_i) = b_i$ and $i_D (z_k^m) = z_k^m$, then the equation above says
$$b_i \cdot z_k^m = \tilde{\tau}^\infty_{\text{ink}} z_k^m \cdot b_i.$$ That means $i_B$ and $i_D$ are non commuting $C$-points, and the non commutativity is given by a symmetric twisting map defining $\mathcal{Y}^{A,B}$. Since the last theorem the map $i_A \equiv (i_D \cdot i_B) \delta$, given by
$$i_A (a_j) = a_j \equiv z_j^k \cdot b_k \in C,$$
defines a $C$-point of $A$. Moreover,
Theorem 3. Consider the quantum spaces \((A_1, A)\) and \((H_1, H)\), with
\[
H_1 = \text{span}\left[ h_j^{k_i} \right]_{i,j=1}^n, \quad h_j^k \in H,
\]
an algebra \(C\), and \(C\)-points \(i_B : b_j \mapsto b_j^i\) and \(i_B^* : h_j^k \mapsto h_j^{k^i}\) satisfying Eq. (4.7) (changing \(z\) by \(h\)). Suppose in addition that \(i_B\) is a generic point. Then, the map \(a_j \in A_1 \mapsto h_j^k \cdot b_k \in C\) can be extended to a \(C\)-point of \(A\) iff the assignment \(z_j^k \mapsto h_j^{k^i}\) defines a \(C\)-point of \(D\).

Proof. Suppose \(a_j \mapsto h_j^k \cdot b_k\) can be extended to a \(C\)-point of \(A\), and call \(i_A\) such a map. Let us come back to the proof of previous theorem. There, for instance, we denote by \(I_\lambda\) an element of the ideal \(I\) related to \(A\). Here, we will identify \(I_\lambda\) with its projection (by \(\Pi\)) over \(A\), so \(I_\lambda = 0\). Applying an analogous criterium to the other symbols, we have from Eq. (4.3) that

\[
0 = i_A (I_\lambda) = i_H (d_\lambda^i) \cdot i_B (J_\omega^c).
\]

Now, the symbols \(d_\lambda^i \in H\) represent linear combinations of monomials in \(h_j^k\). They are formally identical to the ones in \(D\) (we just have to replace \(z\) by \(h\)). Since \(i_B\) is a generic \(C\)-point, the elements \(i_B (J_\omega^c)\) are linearly independent, and consequently each \(i_H (d_\lambda^i)\) must be the null element. Similarly, from Eq. (4.3) follows that

\[
0 = i_B (J_\omega) \cdot i_H (h_R) = i_H \left( \frac{d_\omega}{d_{\omega,R}} \right) \cdot i_B (J_\omega^c),
\]

and for the same reason, \(i_H \left( \frac{d_\omega}{d_{\omega,R}} \right) = 0\). The equation above can be seen as the result of passing \(i_H (h_R)\) through \(i_B (J_\omega)\) using Eq. (4.6) (replacing \(z\) by \(h\)). Following that process, we finally arrive at the equations (see Eq. 4.5)

\[
i_H \left( \frac{d_\omega}{d_{\omega,R}} \right) = 0 \quad \text{and} \quad i_H \left( \frac{d_\omega}{d_{\omega,R}} \right) = 0.
\]

That means, the elements \(i_H (h^k) = h_j^k \in C\) must satisfy unless the same relations that have to satisfy the elements \(z_j^k \in D\). Thus, \(z_j^k \mapsto h_j^k\) extends to an algebra homomorphism.

Reciprocally, if \(z_j^k \mapsto h_j^k\) extends to an algebra homomorphism, then the elements \(h_j^k\) must satisfy unless the relations (4.7), what implies that \(a_j \mapsto h_j^k \cdot b_k\) is a well defined \(C\)-point. \(\blacksquare\)

Thus, this theorem is simply another way to express the initial property of \(D\) in the category \(Y^{A,B}\). Note that again, the gradation has not been used.

4.1.2. Factorizable bijections. We can see from Equation (4.6) or (4.7) that the algebras \(D\) could be very small. Things change by considering special classes of twisting maps.

We say that a collection \(\{\tilde{\tau}_{A,B}\}_{A \in CA}\) is factorizable if there exists another collection

\[
\{\sigma_A\}_{A \in CA}, \quad \sigma_A \in \text{Aut} [A_1],
\]

such that

\[
\tilde{\tau}_{A,B} = \text{id} \otimes \sigma_B^{-1} \otimes \sigma_A : B_1 \otimes B_1^\otimes A_1 \to B_1 \otimes B_1^\otimes A_1.
\]

In that case we say \(\tilde{\tau}_{A,B}\) is a factorizable bijection.

As an example, consider \(A\) and \(B\) with \(A \cong A_1^\otimes / I\) and \(B \cong B_1^\otimes / J\), and ideals given by Eqs. (1.10) of §1.3. Let us take \(Y^{A,B}\) with associated factorizable bijection \(\tilde{\tau}_{A,B}\) as above. The algebra \(D\) will be

\[
D = \left[ \text{span} \left[ z_j^{n,m} \right]_{j=1}^{n,m} \right] / L,
\]

being \(L\) the ideal algebraically generated by the elements (compare to Eq. (4.13))

\[
R_{\lambda_0}^{k_1 \ldots k_n} \cdot z_j^{(m-1)j_1} \cdot z_k^{(m)j_2} \ldots z_{km}^{(m+n-2)j_n} \cdot (S_j^{\perp})^{\omega_n}_{j_1 \ldots j_n}, \quad z_u^{(r)} = [\sigma_A]_u^{p \ast} \cdot z_u^p \cdot [\sigma_B^{-1}]_u^{q}.
\]

\(\lambda_n \in \Lambda_n, \omega_n \in \Omega_n, m \in \mathbb{N}, n \in \mathbb{N}_0\). In addition, if \(\sigma_A\) and \(\sigma_B\) can be extended to \(A\) and \(B\) as algebras automorphisms, the ideal defined by Eq. (4.9) reduces to the one defined by the elements

\[
R_{\lambda_0}^{k_1 \ldots k_n} \cdot z_j^{j_1} \cdot z_k^{j_2} \cdot z_{k_3}^{j_3} \ldots z_{k_n}^{j_n} \cdot (S_j^{\perp})^{\omega_n}_{j_1 \ldots j_n},
\]

\(\lambda_n \in \Lambda_n, \omega_n \in \Omega_n, n \in \mathbb{N}_0\), as we will show in §4.2.2.
4.2. The semigroupoid structure of $\mathcal{Y}$. We begin this section with the following observation: $\mathcal{Y} \mathcal{A} \circ \mathcal{C}$ has a semigroupoid structure given by the partial product functor

\[(\varphi, \mathcal{H}) \times (\chi, \mathcal{G}) \mapsto (I_{\mathcal{H}} \circ \chi, \varphi, \mathcal{H} \circ \mathcal{G}); \quad \alpha \times \beta \mapsto \alpha \circ \beta,
\]

with domain

\[\bigvee_{A,B,C \in \mathcal{CA}} (\mathcal{Y} \mathcal{A} \downarrow \mathcal{Y} (\mathcal{A} \circ \mathcal{C})) \times (\mathcal{Y} \mathcal{C} \downarrow \mathcal{Y} (\mathcal{C} \circ \mathcal{B})),\]

codomain $\mathcal{Y} \mathcal{A} \circ \mathcal{C}$, and $\mathcal{CA} \subseteq \mathcal{Y} \mathcal{A} \circ \mathcal{C}$ is a sub-semigroupoid. Its associativity comes from that of $\circ$, and the unit elements are given by the diagrams $(\ell_A, K)$, where $\ell_A$ is the homogeneous isomorphism $A \cong k[e] \otimes A$, such that $a \mapsto e^n \otimes a$ if $a \in A_n$. Finally, it is immediate from Eq. (2.1) that $\mathcal{CA}$ is a sub-semigroupoid of $\mathcal{Y} \mathcal{A} \circ \mathcal{C}$.

Nevertheless, for a generic collection of linear bijections, $\mathcal{Y}$ fails to be a semigroupoid. To address this problem, we shall consider particular collections of twisting maps.

Given a pair of linear endomorphisms $\alpha$ and $\beta$ on vector spaces $\otimes_{i \in I} V_i$ and $\otimes_{i \in I} W_i$ ($I$ a set of indices), respectively, we shall say that $\alpha$ is strongly congruent to $\beta$, and denote $\alpha \sim \beta$, if there exist a family of isomorphisms $\zeta_i : V_i \cong W_i$, $\forall i \in I$, such that

\[\alpha = (\otimes_{i \in I} \zeta_i^{-1}) \beta (\otimes_{i \in I} \zeta_i).
\]

A collection $\{\widehat{\tau}_{A,B}\}_{A,B \in \mathcal{CA}}$ of linear bijections (4.2) will be called global collection when

\[\widehat{\tau}_{A,B} \sim \widehat{\tau}_{C,D} \iff A_1 \cong C_1 \text{ and } B_1 \cong D_1.
\]

Note that two $\widehat{\tau}$’s are strongly congruent iff there exists basis on the respective linear spaces such that the matrices defined by the maps $\widehat{\tau}$’s on these basis coincide. Hence, the equivalence classes can be represented by a collection $\{\tau_{n,m}\}_{n,m \in \mathbb{N}_0}$ of invertible matrices $\tau_{n,m} \in GL (m^2 \cdot n)$ or linear isomorphisms

\[\tau_{n,m} : k^m \otimes k^{m^2} \otimes k^n \cong k^m \otimes k^{m^2} \otimes k^n.
\]

In particular, for a factorizable $\tau_{n,m}$ we have $\tau_{n,m} = id_m \otimes \sigma_m^{-1} \otimes \sigma_n$, with $\sigma_n \in GL (n)$ or $\sigma_n : k^n \cong k^n$; in indices

\[[\tau_{n,m}]_{i\alpha k}^{j\beta} = \delta_i^j \left[\sigma_m^{-1}\right]_{\alpha}^{\beta} \left[\sigma_n\right]_{k}^{\alpha}.
\]

In these cases the equivalence classes are given by a collection $\{\sigma_n\}_{n \in \mathbb{N}_0}$.

In working with a global collection, it can be regarded as the replacement of the category FGA by another one formed out by pairs $(n, I)$, where $n \in \mathbb{N}_0$ and $I$ is a bilateral ideal of $[k^n] \otimes$, with arrows $(n, I) \rightarrow (m, J)$ given by linear maps $\alpha : k^n \rightarrow k^m$ defining an inclusion $\alpha_\ominus (I) \subseteq J$.

4.2.1. The $\mathcal{CA}$-cobased categories. Now, let us present main result of this section.

**Theorem 4.** If the category $\mathcal{Y}$ is associated to a global collection, the following statements are equivalent:

a) $\mathcal{Y} \subseteq \mathcal{Y} \mathcal{A} \circ \mathcal{C}$ is a sub-semigroupoid.

b) The collection that defines $\mathcal{Y}$ is factorizable.

**Proof.** a) $\Rightarrow$ b) Consider $\mathcal{A}$ with $dim A_1 = n$. If every $\mathcal{Y} \mathcal{A}$ is a submonoid of $(\mathcal{Y} \mathcal{A} \downarrow \mathcal{Y} (\mathcal{A} \circ \mathcal{C}))$, then $(I \circ \delta)$ defines the algebra morphism $\delta : \mathcal{A} \rightarrow (D \circ D) \otimes \mathcal{A}$

\[(I \circ \delta) \delta : \mathcal{A} \rightarrow (D \circ D) \otimes \mathcal{A}
\]

with $(I \circ \delta) \delta (a_m) = \mathcal{Z}_m \otimes \mathcal{Z}_l \otimes \mathcal{A}_l$. The twisting map $\tau$, given by a matrix $\tau_{n,m} = \tau_n$, is

\[\tau [a_i \otimes (\mathcal{Z}_m \otimes \mathcal{Z}_l)] = [\tau_n]^{k\alpha}_{ij\beta} [a_k \otimes (\mathcal{Z}_r \otimes \mathcal{Z}_l)].
\]

Note that $\tau$ is only defined on $\mathcal{A} \langle (I \circ \delta), D \circ D \rangle$, given the latter by the subalgebra of $D \circ D$ generated by

\[span \left[\sum_{j=1}^{n} \mathcal{Z}_j \otimes \mathcal{Z}_k \right] \subseteq D_1 \otimes D_1.
\]

Since Eq. 4.11 and 4.12, $(I \circ \delta)$ $\delta$ applied to $a_i \otimes a_j$ gives

\[\left[\tau_n\right]^{l\alpha}_{ij\beta} \left[\tau_n\right]^{r\beta\gamma}_{l\alpha\delta} - \delta_k^{\alpha\beta} \left[\tau_n\right]^{r\beta\gamma}_{l\alpha\delta}\left[\tau_n\right]^{s\alpha\delta}_{k\rho\beta} \left[\sigma_p\right]^{r\alpha\beta}_{j\gamma\delta} \left[\sigma_q\right]^{s\beta\gamma}_{j\rho\delta} = 0.
\]

Because this eq. must hold for all $\mathcal{A} \subseteq \mathcal{CA}$, in particular for those $\mathcal{A}$ defined by free algebras,

\[\left[\tau_n\right]^{l\alpha}_{ij\beta} \left[\tau_n\right]^{r\beta\gamma}_{l\alpha\delta} = \delta_k^{\alpha\beta} \left[\tau_n\right]^{r\beta\gamma}_{l\alpha\delta}, \quad \forall n \in \mathbb{N}_0,
\]

is necessary from the global character of the considered collection. On the other hand, due to $\mathcal{Y} \mathcal{A}$ is a submonoid, the unit $(\ell_A, K)$ of $(\mathcal{Y} \mathcal{A} \downarrow \mathcal{Y} (\mathcal{A} \circ \mathcal{A}))$, is also a unit for $\mathcal{Y} \mathcal{A}$. In particular, $\ell_A$ must define the isomorphisms $A \cong K \circ A$ and $A \cong K \circ A$ for all $\mathcal{A}$ with $dim A_1 = n$). Since the map $\widehat{\tau}_A$ for $(\ell_A, K)$ is given by an isomorphism $k \otimes A_1 \cong k \otimes A_1$, such that

\[a_i \otimes e \delta_j^\alpha \mapsto [\tau_n]^{k\alpha}_{ij\beta} [a_k \otimes e \delta_j^\alpha] = [\tau_n]^{k\alpha}_{ij\beta} a_k \otimes e,
\]
it is easy to see that $\tau_n^{kl\alpha} = \delta^k_{\beta}\tau^\alpha_i$ with $\hat{\tau}_A (a_i \otimes e) = \tau^a_i a_k \otimes e$. This immediately implies that $\ell_A$ defines the above isomorphisms iff

$$[\tau_n]^{kl\alpha}_{ij\beta} = \delta^k_\beta \delta^{\alpha}_\beta.$$  

The solutions to Eqs. (4.14) and (4.15) (see lemma at the end of this section) are

$$\tau_n = id_n \otimes \rho_n \otimes \sigma_n \text{ with } \sigma_n \in GL(n) \sslASH Z_{GL(n)},$$

or in indices $[\tau_n]^{kl\alpha}_{ij\beta} = \delta^k_\beta \delta^{\alpha}_\beta$, being $\sigma = \sigma^{-1}$.

We now address the general case. Let $B$ and $C$ be quantum spaces, and basis $\{b_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^p$ of $B_1$ and $C_1$. We shall note $\langle \delta, D \rangle$ and $\langle \delta', D' \rangle$ the corresponding initial objects of $\mathcal{Y}^{A,B}$ and $\mathcal{Y}^{B,C}$, respectively. (Again, the indices will run on the range that correspond to their associated dimensions.) Because of

$$\langle (I \circ \delta') \delta, D \circ D' \rangle \in \mathcal{Y}^{A,C},$$

the map $(I \circ \delta') \delta$ gives rise to the algebra morphism

$$A \to (D \circ D') \otimes C : a_m \mapsto z_m^r (z')^j_i \otimes c_i,$$

with twisting map $\tau$ given by (it is valid the same observation that leads to Eq. (4.13))

$$\hat{\tau}_{A,C} (c_i \otimes (z')^j_i \otimes (z')^j_i)) = [\tau_n]^{kl\alpha}_{ij\beta} c_k \otimes (z^\alpha_{i,j})^r \otimes (z^\alpha_{i,j})^r.$$  

To be $(I \circ \delta') \delta$ an algebra morphism defined by (4.16)

$$[\tau_n]^{slx}_{k\alphaij} [\tau_m]^{r\beta q}_{p\alpha pj} - \delta^q_\beta [\tau_n]^{r\beta x}_{p\alpha ij} [\tau_m]^{s\beta y}_{l\alpha pi} (z^k_{i,j} \cdot (z')^y_i \otimes z^\alpha_{i,j} \cdot (z')^p_i \otimes c_r \cdot c_i) = 0$$

is necessary. Asking that the above eq. holds for all $A, B, C \in CA$ with the associated dimensions $n, m, p$, we need

$$[\tau_n]^{slx}_{k\alpha ij} [\tau_m]^{r\beta y}_{p\alpha pj} = \delta^q_\beta [\tau_n]^{r\beta x}_{p\alpha ij} [\tau_m]^{s\beta y}_{l\alpha pi}, \forall n, m, p \in N_0.$$  

We can solve these equations in terms of the solution of the case $n = m = p$. Taking $n = p$, the last eq. can be written

$$[\tau_n]^{slx}_{k\alpha ij} [\tau_n]^{r\beta y}_{p\alpha pj} = \delta^q_\beta [\tau_n]^{r\beta x}_{p\alpha ij} [\tau_n]^{s\beta y}_{l\alpha pi} [\tau_n]^{\beta \gamma}_{p\alpha ij},$$

and contracting $x$ with $p$ and $\alpha$ with $s$, it reduces to

$$[\tau_n]^{slx}_{k\alpha ij} [\tau_n]^{r\beta y}_{p\alpha pj} = \delta^q_\beta [\tau_n]^{r\beta x}_{p\alpha ij} [\tau_n]^{s\beta y}_{l\alpha pi} [\tau_n]^{\beta \gamma}_{p\alpha ij} = 0$$

is fulfilled. Contracting $u$ with $\alpha$ and $p$ with $v$, and defining

$$[\beta_n]_{uv} = \frac{1}{m^2} [\tau_n]^{\beta \gamma}_{p\alpha ij} [\tau_n]^{\alpha \beta}_{p\alpha ij},$$

it sees that $[\tau_n]^{svx}_{k\alpha ij} = \delta^u_\beta [\beta_n]_{uv} [\sigma_n]^{\gamma}_{i,j}$; and changing $n$ by $m$,

$$[\tau_n]^{svx}_{k\alpha ij} = \delta^u_\beta [\sigma_n]^{\gamma}_{i,j} [\sigma_n]^{\gamma}_{i,j}.$$  

Now, substituting the last two expression on (4.15), it is straightforward to show that $\beta_n = \sigma_n$ and $\beta_m = \sigma_m$; therefore

$$\tau_{n,m} = id_m \otimes \sigma_m \otimes \sigma_n, \sigma_n \in GL(n), \sigma_m \in GL(m).$$  

- $b \Rightarrow a)$ Let us suppose that $\mathcal{Y}$ is defined by a factorizable collection of bijections. Consider again the quantum spaces $A, B$ and $C$, and diagrams $\langle \varphi, H \rangle \in \mathcal{Y}^{A,B}$ and $\langle \psi, G \rangle \in \mathcal{Y}^{B,C}$, with associated linear spaces (via the functor $\hat{\sigma}$)

$$H^\varphi = span \left[ h_{i,j}^m \right]_{i,j=1}^{n,m} \text{ and } G^\psi = span \left[ g_{i,j}^p \right]_{i,j=1}^{m,p}.$$  

Denoting by $h$ and $g$ the matrices with entries $\{h_{i,j}^m\} \subset H_1$ and $\{g_{i,j}^p\} \subset G_1$, respectively, and by $a, b$ and $c$ the vectors whose components are $\{a_i\} \subset A_1$, $\{b_i\} \subset B_1$ and $\{c_i\} \subset C_1$. Then, we can write

$$\hat{\tau}_{A,B} : b \otimes h \mapsto b \otimes \hat{h} \text{ and } \hat{\tau}_{B,C} : c \otimes g \mapsto c \otimes \hat{g},$$

being $\hat{h} = \sigma_n \cdot h \cdot \sigma_m$ and $\hat{g} = \sigma_m \cdot g \cdot \sigma_p$, where "\cdot" indicates matrix multiplication (respect to the chosen basis of $A_1, B_1$ and $C_1$). There is no contraction between the above matrices and vector, unless we include the symbols "$\otimes$" in expressions where they appear.
It must be shown that \( (I_H \circ \chi) \varphi, \mathcal{H} \circ \mathcal{G} \) is an object of \( \Upsilon^4 \), which is true if and only if \( (I_H \circ \chi) \varphi \) define the arrow \( \mathcal{A} \rightarrow (\mathcal{H} \circ \mathcal{G}) \circ \mathcal{C} \), with \( \tau \) given by \( \tilde{\tau}_{A,C} \), which in the basis above is defined by the assignment

\[
\tilde{\tau}_{A,C} : c \odot (\hat{h} \otimes g) \mapsto c \odot (\hat{h} \otimes \hat{g}).
\]

Let us see firstly that \( \tilde{\tau}_{A,C} \) can be effectively extended to a twisting map \( \tau \) from \( \mathcal{C} \otimes (\mathcal{H} \circ \mathcal{G}) \) to \( (\mathcal{H} \circ \mathcal{G}) \otimes \mathcal{C} \), and then that \( (I_H \circ \chi) \varphi \) defines the arrow above. Such an extension would be given by

\[
\tau : c_1 \ldots c_r \otimes h_1 \ldots h_s \otimes g_1 \ldots g_s \mapsto \hat{h}_1^{(r)} \ldots \hat{h}_s^{(r)} \otimes \hat{g}_1^{(r)} \ldots \hat{g}_s^{(r)} \otimes c_1 \ldots c_r
\]

where the \( h_k \)'s are copies of \( h \), the contractions are done between \( h_k \)'s and \( g_l \)'s with \( k = l \), and \( \hat{h}_k = (\sigma_\gamma)^L \cdot h \cdot (\sigma_\gamma)^L \) (idem \( g \)).

Appendix

Let us see firstly that \( \mathcal{A} \rightarrow \mathcal{H} \otimes \mathcal{G} \otimes \mathcal{C} \) is a morphism \( \mathcal{A} \rightarrow (\mathcal{H} \circ \mathcal{G}) \circ \mathcal{C} \).

Finally, we observe that units \( \langle \ell_A \rangle \) are objects of \( \Upsilon^4 \). So, the theorem has been proved. □

**Remark 2.** We see from proof above that, for \( \Upsilon \) to be a sub-semigroupoid of \( \mathfrak{F} \mathcal{C} \mathcal{A} \), factorization property is a generic condition we must impose on the collection \( \{ \tau \} \). Thus, for global collections the mentioned generic condition is needed in order to have the inclusion \( \Upsilon \subset \mathfrak{F} \mathcal{C} \mathcal{A} \) of semigroupoids. □

Observe that, in proving b) ⇒ a) the global character of the associated collection that defines \( \Upsilon \) is not involved. So we can enunciate the following corollaries (without proof).

**Corollary 1.** Any category \( \Upsilon \) associated to a factorizable collection is a sub-semigroupoid of \( \mathfrak{F} \mathcal{C} \mathcal{A} \). □

We call \( \circ \Upsilon \) the partial product associated to such a semigroupoid. Because the embedding \( \mathfrak{R}^\Upsilon : \Upsilon \rightarrow \mathcal{A} \), such that \( (\varphi, \mathcal{H}) \mapsto \mathcal{H} \), satisfies \( \mathfrak{R}^\Upsilon \circ \Upsilon = \circ (\mathfrak{R}^\Upsilon \times \mathfrak{R}^\Upsilon) \) and \( \mathfrak{R}^\Upsilon \langle \ell_A \rangle = \mathcal{K} \) (see §2.2), it follows:

**Corollary 2.** For every factorizable collection, the function

\[
(\mathcal{A}, \mathcal{B}) \mapsto \text{hom}^\Upsilon [\mathcal{B}, \mathcal{A}] \subset \mathcal{D}_{\mathcal{B}, \mathcal{A}} \subset \mathcal{C} \mathcal{A}
\]

defines an \( \mathcal{A} \)-cobased or QLS-based category with arrows (opposite to)

\[
\text{hom}^\Upsilon [\mathcal{C}, \mathcal{A}] \rightarrow \text{hom}^\Upsilon [\mathcal{B}, \mathcal{A}] \circ \text{hom}^\Upsilon [\mathcal{C}, \mathcal{B}]
\]

the cocomposition, and for end\( ^\Upsilon [\mathcal{A}] \) the counit epimorphism

\[
\text{end}^\Upsilon [\mathcal{A}] \rightarrow \mathcal{K} / z^j_i \mapsto \delta^j_i e,
\]

and the monomorphic comultiplication

\[
\text{end}^\Upsilon [\mathcal{A}] \mapsto \text{end}^\Upsilon [\mathcal{A}] \circ \text{end}^\Upsilon [\mathcal{A}] / z^j_i \mapsto z^j_i \otimes z^j_i. \]

Now, the annotated lemma to conclude proof of above theorem.

**Lemma 1.** The solutions to Eqs. (4.2.1) and (4.2.5) are given by

\[
[\tau_n]_{ijkl} = \delta^l_k \sigma_{\alpha \beta} \sigma_{\gamma \delta} \sigma_{\epsilon \zeta}, \quad \sigma_{\alpha \beta} \in \text{GL}(n) / Z_{\text{GL}(n)}.
\]

**Proof.** We want to solve equations

\[
[\tau_n]_{ijkl} = \delta^l_k \tau_n^{\gamma \beta} \tau_n^{\alpha \rho} \tau_n^{\gamma \delta},
\]

and

\[
[\tau_n]_{ijab} = \delta^l_k \delta^j_i \delta^b_a.
\]

The solutions will be of the form

\[
[\tau_n]_{ijkl} = \delta^l_k \theta_{jy}.
\]

In fact, contracting \( p-q \) in (4.2.4) we have, using (4.2.1),

\[
[\tau_n]_{ijkl} \delta^p_a = \delta^l_k [\tau_n]_{ikab} \cdot
\]

Then, taking \( \alpha = r \), we arrive at the equality \( [\tau_n]_{ikab} = \delta^l_k [\tau_n]_{ikab} \). Hence, we can define \( \theta_{jy}^l = [\tau_n]_{ikab} \) and obtain (4.2.4).

Inserting (4.2.2) on (4.2.4) and (4.2.1), the latter reduce to

\[
\delta_{jy}^l \delta_{lq} = \delta_{jy}^{\beta q} \delta_{\beta q}^l \cdot \theta_{\beta q}^l = \delta_{\beta q}^l.
\]
If we define $\eta_{xy}^{ij} = \delta_{xy}^{ij}/n$, and contract $q$-$y$ we have $\eta_{xy}^{ij} \delta_{yp} = \eta_{jx}^{p} \eta_{xy}^{ip}$. Now, contracting $l$-$\beta$ in the latter, both equations reduce to $\eta_{\cdot \cdot} = \eta$ and $tr \eta = 1$, regarding $\eta$ as an $n^2 \times n^2$ matrix. The condition $\eta_{\cdot \cdot} = \eta$ says that $\eta$ is diagonalizable and has eigenvalues 1 and 0. If it has $n\lambda$ eigenvalues equals to $\lambda$, $\lambda = 0, 1$, then $tr \eta = n_0 \cdot 0 + n_1 \cdot 1 = n_1$. But $tr \eta = 1$, so $n_1 = 1$ and $n_0 = n^2 - 1$. In particular, $\eta$ can be diagonalized to a matrix $d = diag_{a\in \mathbb{N}}(1, 0, \ldots, 0)$ through some invertible matrix $x$; i.e.

$$\eta_{ij}^{kl} = x_{ij}^{kl} \delta_{vw}^{kl} (x^{-1})_{vw}^{kl} = x_{ij}^{kl} (x^{-1})_{ij}^{kl}.$$  

Coming back to $\theta$ we can write $\theta_{kl}^{ij} = n x_{ij}^{kl} (x^{-1})_{kl}^{ij}$, and defining $[\sigma_n]_i^j = n x_{ij}^{kl}$ and $[\sigma_n]_i^j = (x^{-1})_{ij}^{kl}$, we will have

$$\theta_{kl}^{ij} = [\sigma_n]_i^j [\sigma_n]_j^l.$$  

But $\theta_{kl}^{ij} = \delta_k^i$, so $[\sigma_n]_i^j = [\sigma_n]_j^i = [\sigma_n]_i^j = \delta_k^i$, i.e. $\sigma_n = \sigma_n^{-1}$. Thus, $\tau_n$ is a solution of (4.20) and (4.21) only if

$$\tau_{n}^{k} l_{xy} = \delta_k^x [\sigma_n]_y^l [\sigma_n]_x^l,$$

for some $\sigma_n \in GL(n)$. Reciprocally, if $\sigma_n \in GL(n)$, it is easy to see that last expression for $\tau_n$ gives a solution for (4.14) and (4.15).

### 4.2.2. Factorizable collections given by a family of automorphisms

We are going to study the case in which a given factorizable map

$$\tilde{\tau}_{A,B} = id \otimes \rho \otimes \phi,$$  

associated to $\Theta_{A,B}$, is such that $\sigma_A : A_1 \simeq A_1$ and $\sigma_B : B_1 \simeq B_1$ can be extended to quantum space automorphisms $A \simeq A$ and $B \simeq B$.

Consider $\mathcal{A}$ and $\mathcal{B}$ with related ideals $\mathfrak{I}$ and $\mathfrak{J}$ given in Eq. (4.10), and define another ideals, namely $\mathfrak{I}_\sigma$ and $\mathfrak{J}_\sigma$, linearly generated by

$$\left\{ \left\{ a R_{\lambda_n}^{k_1 \ldots k_n} a_{k_1 \ldots k_n} \right\}_{\lambda_n \in \Lambda_n} \right\}_{n \in \mathbb{N}},$$  

respectively, being

$$R_{\lambda_n}^{k_1 \ldots k_n} = R_{\lambda_n}^{k_1 \ldots k_n} \phi_{j_2}^{k_2} (\phi_{j_3}^{k_3} \ldots (\phi_{j_n}^{k_n})), $$

$$S_{\mu_n}^{k_1 \ldots k_n} = S_{\mu_n}^{k_1 \ldots k_n} (\rho_{j_2}^{-1})^{k_2} (\rho_{j_3}^{-1})^{k_3} \ldots (\rho_{j_n}^{-1})^{k_n}.$$  

Of course, $\mathfrak{I}_\sigma \simeq \mathfrak{I}$ and $\mathfrak{J}_\sigma \simeq \mathfrak{J}$. Therefore, $(\mathfrak{I}_\sigma)^{\perp} \simeq \mathfrak{I}_\sigma^\perp$ and $(\mathfrak{J}_\sigma)^{\perp} \simeq \mathfrak{J}_\sigma^\perp$. In particular, the spaces $(\mathfrak{J}_\sigma)^{\perp}$ will be generated by the set

$$\left\{ b_{j_2}^{k_2} \ldots b_{j_n}^{k_n} \rho_{j_2}^{k_2} (\rho_{j_3}^{-1})^{k_3} \ldots (\rho_{j_n}^{-1})^{k_n} (S^{\perp})_{j_1 \ldots j_n}^{k_1 \ldots k_n} \right\}_{\omega_n \in \Omega_n},$$  

so we can define

$$(S^{\perp})_{j_1 \ldots j_n}^{k_1 \ldots k_n} = \rho_{j_2}^{k_2} (\rho_{j_3}^{-1})^{k_3} \ldots (\rho_{j_n}^{-1})^{k_n} (S^{\perp})_{j_1 \ldots j_n}^{k_1 \ldots k_n}.$$  

### Proposition 8

If $\sigma_A : A_1 \simeq A_1$ and $\sigma_B : B_1 \simeq B_1$ can be extended to quantum space automorphisms $A \simeq A$ and $B \simeq B$, we can define

$$\hom^T [\mathcal{B}, \mathcal{A}] \simeq B^T \cdot A^T,$$  

$$A^T \simeq (A_1, A_1^\perp / \mathfrak{I}_\sigma),$$  

$$B^T \simeq (B_1, B_1^\perp / \mathfrak{J}_\sigma),$$  

and for $\mathcal{A}, \mathcal{B} \in QA$ (or any $\mathcal{C}^{\mathbb{N}}$, $m \in \mathbb{N}$), see Eq. (4.9).

In particular, $\hom^T[K, \mathcal{A}] = A^T$.

**Proof.** The maps $\sigma_A$ and $\sigma_B$ can be extended to algebra isomorphisms iff for every $n \in \mathbb{N}$ there exists numbers $C_{\lambda_n}^{\lambda_n}$’s and $D_{\mu_n}^{\mu_n}$’s (defining invertible matrices in $GL(\Lambda_n)$ and $GL(\Phi_n)$, resp.) such that

$$R_{\lambda_n}^{k_1 \ldots k_n} \phi_{j_1}^{k_1} \ldots \phi_{j_n}^{k_n} = C_{\lambda_n}^{\lambda_n} R_{\lambda_n}^{k_1 \ldots k_n},$$

$$S_{\mu_n}^{k_1 \ldots k_n} (\rho_{j_1}^{-1})^{k_1} \ldots (\rho_{j_n}^{-1})^{k_n} = D_{\mu_n}^{\mu_n} S_{\mu_n}^{k_1 \ldots k_n}.$$
In such a case, because \( \sigma_A^*: A_1^* \cong A_1 \) and \( \sigma_B^*: B_1^* \cong B_1 \) could be extended to algebra homomorphisms \( A^i \cong A^i \) and \( B^i \cong B^i \), we will have, in particular (for \( B \)), numbers \( E_{\omega}^{(n)} \)'s such that
\[
\rho_{j_2k_1}^{j_1k_2} \cdots \rho_{j_nk_n}^{j_1k_n} \left( S^\perp \right)^{\omega_n}_{j_1 \cdots j_n} = \left( S^\perp \right)^{\omega_n}_{j_1 \cdots j_n} \ E_{\omega}^{(n)}.
\]
Remember that for a factorizable map the underlying algebras of the initial objects of \( \mathcal{Y}^{A,B} \) are isomorphic to the one given by Eqs. (1.8) and (1.9). Hence, from (4.20) and the last eq., it is obvious that (as we affirm in §4.1.2) Eq. (4.20) reduces to
\[
(4.28) \quad \{ R^{k_1 \cdots k_n}_{\omega_n} \ z^{j_1}_{k_1} \ z^{j_2}_{k_2} \cdots z^{(n-1)j_n}_{k_n} \left( S^\perp \right)^{\omega_n}_{j_1 \cdots j_n} \} \chi_{n \in N_0},
\]
where we have to read \( z^i = b^i \otimes a_i \). This means that \( L \) is generated by
\[
\{ \sigma R^{k_1 \cdots k_n}_{\omega_n} \ z^{j_1}_{k_1} \ z^{j_2}_{k_2} \cdots z^{(n)j_n}_{k_n} \left( S^\perp \right)^{\omega_n}_{j_1 \cdots j_n} \} \chi_{n \in N_0},
\]
and immediately that \( D \cong \left( B^i_1 / J_r \right) \triangleright (A^i_1 / L_r) \).

To show the equality \( A^\mathcal{Y} = \text{hom}^\mathcal{Y}[\mathcal{K}, \mathcal{A}] \), it is enough to realize that \( \sigma_{\mathcal{K}}: \mathcal{K} \cong \mathcal{K} \) always can be extended to an automorphism \( \mathcal{K} \cong \mathcal{K} \) and always \( \mathcal{K}^\mathcal{Y} = \mathcal{K} \). So, \( \mathcal{K}^\mathcal{Y} \triangleright A^\mathcal{Y} = A^\mathcal{Y} \). In particular, \( \text{hom}^\mathcal{Y}[\mathcal{K}, \mathcal{K}] = \mathcal{K} \).

As an example, let us take \( A = B = A_R \), i.e. the quadratic quantum space generated by \( A_1 = \text{span} [a_i]_{i=1}^n \) and restricted to relations
\[
(4.29) \quad I_R = \text{span} \left( R^{k_1k_2}_{\omega} \ a_k \otimes a_l - a_j \otimes a_i \right)_{k,j},
\]
being \( R = R_{su(n)} \), the \( R \)-matrix related to the Lie algebra \( su(n) \) and the quantum groups \( GL_q(n) \) and \( SL_q(n) \).

Recall that \( GL_q(n) \) is a quadratic bialgebra generated by symbols \( T^R_i \) satisfying
\[
(4.30) \quad R^{k_1k_2}_{ij} T^R_i T^R_j - T^R_j T^R_i R^{k_1k_2}_{ij}.
\]
It coacts upon \( A_R \) through the map \( a_i \mapsto T^R_i \otimes a_k \), and consequently, \( GL_q(n) \) defines a diagram in the category \( (A \downarrow QA \circ A) \). It follows from the initiality of \( \text{end} [A_R] \) that there exists an epimorphism \( \text{end} [A_R] \twoheadrightarrow GL_q(n) \). Now, for the bijection \( \hat{\tau}_{A_R} = id \otimes \phi^{-1} \otimes \phi \), with \( \phi = \sigma_{A_R} : A_1 \rightarrow A_1 \) given by an invertible diagonal matrix, we have that \( [R, \phi \otimes \phi] = 0 \). Then, Eq. (4.20) holds, and
\[
\text{end}^\mathcal{Y} [A_R] = \left( (A_R)^\mathcal{Y} \right)^1 \cdot (A_R)^\mathcal{Y} = A^{\mathcal{Y}}_{A_R} \bullet A_{\mathcal{Y}}.
\]
Moreover, since straightforwardly \( A_{\mathcal{Y}} = A_{R_0} \), being
\[
R^\phi = R^\phi_{su(n)} \cdot (\phi^{-1} \otimes id) \ R_{su(n)} \cdot (id \otimes \phi),
\]
it follows that
\[
\text{end}^\mathcal{Y} [A_R] = A^{\mathcal{Y}}_{A_{R_0}} \bullet A_{\mathcal{Y}} = \text{end} [A_{R_0}] \cdot A_{\mathcal{Y}}.
\]

R^\phi defines precisely a (multiparametric) quantum group \( GL_{q,\phi}(n) \), so we have, as before, an epimorphic map \( \text{end}^\mathcal{Y} [A_R] \twoheadrightarrow GL_{q,\phi}(n) \). In particular, we can say that \( [GL_{q,\phi}(n)]_{op} \) is a quantum subspace of \( \text{End}^\mathcal{Y} [A_R^{op}] \).

4.3. **Twisted internal coEnd objects.** In this section, we shall connect \( \text{end}^\mathcal{Y} \)'s and \( \text{end}^\mathcal{Y} \)'s by 2-cocycle twisting of bialgebras, as \( GL_q(n) \) and its multiparametric versions. We actually use a dual approach with respect to Drinfeld one, following the book of Majid [9] and references therein.

Let us restrict ourself to collections \( \{ \sigma_A \}_{A \in \mathcal{C}} \) that define automorphisms \( \sigma_A : A \cong A \).

We shall see each object \( \text{end}^\mathcal{Y} [A] = A^\mathcal{Y} \triangleright A^\mathcal{Y} \), endowed with the bialgebra structure
\[
\Delta : \text{end}^\mathcal{Y} [A] \hookrightarrow \text{end}^\mathcal{Y} [A] \circ \text{end}^\mathcal{Y} [A]; \quad \varepsilon : \text{end}^\mathcal{Y} [A] \rightarrow \mathcal{K},
\]
which is a comutant 2-cocycle twisting of the proper coHom object \( \text{end} [A] \); and that the same twisting induces one on \( A \) by the coaction \( A \hookrightarrow \text{end} [A] \circ A \), giving the quantum space \( A^\mathcal{Y} \) (c.f. [9], page 54).

Consider a pair \( (A_1, A) \) with related automorphism \( \sigma_A \) given by \( a_i \mapsto \sigma^i_a a_i \). Let us write \( z^i = a^j \otimes a_i \) and define on \( \text{end} [A] = A \triangleright A \) the linear 2-form \( \chi : (A \triangleright A) \otimes (A \triangleright A) \rightarrow \mathbb{K} \),\footnote{It is well-defined because of Eq. (4.20) for \( \phi = \sigma \).}
\[
\chi (z_1 \cdots z_r \otimes z_{r+1} \cdots z_{r+s}) = (\sigma^{-r})_{r+1} \cdots (\sigma^{-r})_{r+s}.
\]

(4.31)
\[
\chi (1 \otimes x) = \chi (x \otimes 1) = \varepsilon (x) \in \mathbb{K}, \quad \forall x \in A \triangleright A.
\]
The first eq. must be understood as, for \( r = 2 \) and \( s = 3 \),
\[
\chi \left( z_{i_1} \cdot z_{i_2} \otimes z_{j_3} \cdot z_{j_4} \cdot z_{j_5} \right) = \left( \sigma^{-2} \right)^{j_3}_{i_3} \left( \sigma^{-2} \right)^{j_4}_{i_4} \left( \sigma^{-2} \right)^{j_5}_{i_5} \in \k.
\]

The last eq. says that \( \chi \) is counital. By \( * \) we indicate the convolution product of linear forms on a bialgebra.

**Proposition 9.** If \((\mu, \eta, \Delta, \varepsilon)\) is the bialgebra structure of \( A \triangleright A \), we can define another structure \((\mu_\chi, \eta, \Delta, \varepsilon)\), where \( \mu_\chi = \chi \ast \mu \ast \chi^{-1} \).

This new bialgebra is denoted \((A \triangleright A)_\chi\) and called the twisting of \( A \triangleright A \) by \( \chi \).

**Proof.** The counital property for \( \chi \) insures that \( \eta \) is a unit for \( \mu_\chi \). We just must prove \( \mu_\chi \) is associative. This follows from the fact that \( \chi \) is a 2-cocycle, i.e. \( \chi_{12} * (\mu \otimes I) = \chi_{23} * (I \otimes \mu) \); where, \( \chi_{12} = \varepsilon \otimes \chi \) and \( \chi_{23} = \varepsilon \otimes \chi \) and \( \chi_{13} = \chi_{12} (I \otimes \tau_0) \). In fact, acting with \( \chi_{12} * \chi \) (\( \mu \otimes I \)) upon the element
\[
z_1 \ldots z_r \otimes z_{r+1} \ldots z_{r+s} \otimes z_{r+s+1} \ldots z_{r+s+t} \in (A \triangleright A)^{\otimes 3}
\]
we obtain the matrix
\[
(\sigma^{-r})_{r+1} \ldots (\sigma^{-r})_{r+s}(\sigma^{-r-s})_{r+s+1} \ldots (\sigma^{-r-s})_{r+s+t} ;
\]
and acting with \( \chi_{23} * \chi \) (\( I \otimes \mu \)) we have
\[
(\sigma^{-s})_{r+s+1} \ldots (\sigma^{-s})_{r+s+t} \cdot (\sigma^{-r})_{r+1} \ldots (\sigma^{-r})_{r+s+t} =
\]
\[
= (\sigma^{-r})_{r+1} \ldots (\sigma^{-r})_{r+s}(\sigma^{-r-s})_{r+s+1} \ldots (\sigma^{-r-s})_{r+s+t} ,
\]
where \( \cdot \) indicates matrix multiplication. Thus \( \chi \) is a 2-cocycle. Therefore, \((\mu_\chi, \eta)\) define a unital algebra structure.

\( \Delta \) and \( \varepsilon \) are algebra morphisms for \((\mu_\chi, \eta)\) (see [9] for a proof), and consequently \((\mu_\chi, \eta, \Delta, \varepsilon)\) is a bialgebra structure on the linear space \( A \triangleright A \), as we wanted to show. \( \blacksquare \)

Now, we are going to show that \((A \triangleright A)_\chi\) is a quantum space, which is isomorphic to \( A^T \triangleright A^T \) as quantum spaces and as bialgebras. By definition, the underlying vector space of \((A \triangleright A)_\chi\) is \( A \triangleright A \), so is given by monomials in \( z_i^j = a^j \otimes a_i \), under the product \( \mu \), satisfying (see Eq. 12 of §1.3)
\[
R^k_{\chi_1 \ldots k_n} z^{j_1}_{k_1} \cdot z^{j_2}_{k_2} \ldots \cdot z^{j_n}_{k_n} \left( R^\perp \right)^{\omega_n}_{j_1 \ldots j_n} = 0 ,
\]
which can be written
\[
R^k_{\lambda_1 \ldots k_n} z^{j_1}_{k_1} \cdot z^{j_2}_{k_2} \ldots \cdot z^{j_n}_{k_n} \left( R^\perp \right)^{\omega_n}_{j_1 \ldots j_n} = 0
\]
or
\[
\sigma R^k_{\lambda_1 \ldots k_n} \cdot z^{j_1}_{k_1} \cdot z^{j_2}_{k_2} \ldots \cdot z^{j_n}_{k_n} \left( \sigma R^\perp \right)^{\omega_n}_{j_1 \ldots j_n} = 0 ,
\]
where \( \cdot \chi \equiv \mu_\chi \).\footnote{To see the equivalence of Eqs. 12 and 13 note that \( z_i^j \cdot z_i^{j_1} = z_i^{j_1} \cdot z_i^j \) (and use 13 for higher order monomials); and the equivalence of those equations with 13 follows from definition of \( \sigma R \) and \( \sigma R^\perp \) (see Eqs. 18-19 and 20-21).} Clearly the monomials in \( z_i^j \) under the product \( \mu_\chi \) also span \((A \triangleright A)_\chi\). Then, we have a quantum space
\[
(A \triangleright A)_\chi = \left( A^*_1 \otimes A_1, (A \triangleright A)_\chi \right)
\]
with algebra structure \((\mu_\chi, \eta)\) and related ideal generated by the elements given in Eq. 12, with \( \lambda_n \in \Lambda_n \), \( \omega_n \in \Omega_n \), \( n \in \mathbb{N} \). Because the elements that define the ideal of \( A^T \triangleright A^T \) are
\[
\sigma R^k_{\lambda_1 \ldots k_n} z^{j_1}_{k_1} \cdot z^{j_2}_{k_2} \ldots \cdot z^{j_n}_{k_n} \left( \sigma R^\perp \right)^{\omega_n}_{j_1 \ldots j_n} ,
\]
it is obvious that \( A^T \triangleright A^T \) and \((A \triangleright A)_\chi\) are isomorphic quantum spaces. On the other hand, the coalgebra structure are identical, hence they are isomorphic as bialgebras as well. The following theorem resumes these results.

**Theorem 5.** The twisting \((A \triangleright A)_\chi\) (of the bialgebra \( A \triangleright A \) by \( \chi \)) defines a pair
\[
\text{end} \left[ A \right]_\chi = (A \triangleright A)_\chi = (A^*_1 \otimes A_1, (A \triangleright A)_\chi)
\]
isomorphic to \( A^T \triangleright A^T = \text{end}^T \left[ A \right] \); in other terms, there exists a counital 2-cocycle \( \chi \) such that
\[
\text{end}^T \left[ A \right] \preceq \text{end} \left[ A \right]_\chi . \quad \blacksquare
\]
Finally, consider in $A$ the product
\[ m_\chi = (\chi \otimes m) (I \otimes \tau_\circ \otimes I) (\delta \otimes \delta), \]
where $\delta$ is the proper coevaluation $A \to \text{end}[A] \circ A$ and $\tau_\circ$ is the flipping map. In particular,
\[ a_i \cdot_\chi a_j = m_\chi (a_i \otimes a_j) = (\sigma^{-1})^k_{\lambda_n} a_i \cdot a_k. \]

Since $\chi$ is a counital 2-cocycle, $m_\chi$ is associative with the same unit as $m$, and $A_1$ generates $A$ under $m_\chi$. Then, the algebra $(A, m_\chi)$ (with the same unit as $(A, m)$) is a quantum space, namely $A_\chi$. Using $m_\chi$, the relations
\[ R^{k_1 \ldots k_n}_{\lambda_n} a_{k_1} \cdot \ldots \cdot a_{k_n} = 0 \]
can be expressed as
\[ (4.35) \]
\[ \sigma R^{k_1 \ldots k_n}_{\lambda_n} a_{k_1} \cdot _\chi \ldots \cdot a_{k_n} = 0, \]
then.

**Theorem 6.** The quantum space $A^\chi = \text{hom}^T [\mathcal{K}, A]$ is isomorphic to the twisting $A_\chi$ of $A$ induced by the coaction $A \to (A \triangleright \triangleleft) \circ A$ and the 2-form $\chi$ (given in [4,37]); in other words, $A^\chi \simeq A_\chi$. ■

**Corollary 3.** Since $\text{hom}^T [B, A] = B^\chi \triangleright A^\chi$, it follows that
\[ \text{hom}^T [B, A] \simeq B_\chi \triangleright A_\chi = \text{hom} [B_\chi, A_\chi]. \]

For coEnd objects we have in addition
\[ \text{end}^T [A] \simeq A_\chi \triangleright A_\chi = \text{end} [A_\chi] \simeq \text{end} [A]_\chi. \] ■

**Conclusions**

We carried out the construction of $\text{hom}^T [B, A]$ looking for the quantum spaces giving, in a universal way, a notion of ‘quantum space of transformations’ when one allows for some non commutativity among its points and the points of the space to be transformed. In doing so, we defined a twisted tensor product between objects of the category FGA of (general) quantum spaces (in an analogous way that the one defined by Čap and his collaborators for unital algebras). Making use of the Manin algebraic geometric terminology, we were able to connect the idea of twisted tensor product structures with the mentioned non commutativity. Then, among certain subclasses of maps $A \to \mathcal{H} \triangleright \triangleright B$, we find universal elements, defined by objects $\text{hom}^T [B, A] \in$ FGA, giving us for each subclass the notion of space of morphisms we are looking for. Moreover, on a subcategory $CA \subset$ FGA of quantum spaces, the opposite objects to $\text{hom}^T [B, A]$ define a QLS-based category.

We showed that under certain circumstances, the bialgebras $\text{end}^T [A]$ are 2-cocycle twisting of $\text{end} [A]$ (the proper internal coEnd objects of CA). In a forthcoming paper [1] we shall define twisting transformations of quantum spaces, partially controlled by a (multiplicative) cochain quasicomplex. In these terms, we will see that all objects $\text{hom}^T [B, A]$ (under similar circumstances) are twisting by 2-cocycles of the proper coHom ones. Then, we can say that both objects are twist or gauge equivalent, in an analogous sense to the gauge equivalence that Drinfeld defined on quasi-Hopf algebras. On the other hand, symmetric twisted tensor products $A \triangleright B \subset \tau B$ can be seen as particular 2-cocycle twisting of the quantum space $A\triangleright B$, which enable us to develope a generalization of the concept of twisted coHom objects in the setting of twisting of quantum spaces, as we shall discuss in [5].

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**Appendix**

Consider the general quantum spaces $\mathcal{A} = (A_1, A^\circ_1)$, $\mathcal{B} = (B_1, B^\circ_1)$ and $\mathcal{C} = (C_1, C)$, and the basis $\{a_i\} \subset A_1$, $\{b_i\} \subset B_1$ and $\{c_i\} \subset C_1$. The algebra of $C$ is given by $C = C^\circ_1 / I [D^k c_k + \lambda 1]$, with $\lambda \neq 0$. So, $C$ is not a graded algebra. Let $(\varphi, H)$ and $(\check{\varphi}, G)$ be initial diagrams in $\Upsilon^{A,B}$ and $\Upsilon^{B,C}$, respectively, where $\Upsilon$ is built up, just for simplicity, from a factorizable collection. Because $A$ and $\mathcal{B}$ are given by free algebras, we can take $H = (B^\circ_1 \otimes A_1, [B^\circ_1 \otimes A_1]^\circ_1)$, and $\varphi_1 = \varphi_{A_1} : a_i \mapsto b^i \otimes a_i \otimes b_j$. On the other hand, the algebra $\mathcal{G}$ of $\mathcal{G} = (G_1, G)$ can be taken as a quotient of $[C^\circ_1 \otimes B_1]^\circ_1$, where such a quotient is needed to make the corresponding $\tau$ of $\Upsilon^{B,C}$ well-defined. In particular, calling $g^j_i$ the generators of $G_1$, we must have
\[ \tilde{\tau}_{B,C} ((D^k c_k + \lambda 1) \otimes g^j_i) = D^k c_k \otimes g^j_i + \lambda 1 \otimes g^j_i = \lambda 1 \otimes (g^j_i - g^j_i) = 0, \]
what implies that \( \hat{g}_j^l - g_j^l = 0 \). Then, the symbols \( g_j^l \) are not linearly independent. Let \( \{ y_\mu \} \) be a basis for \( G_1 \), such that \( g_j^l = \alpha_{j}^{\mu} y_\mu \), and let us denote by \( h_n^l \) the basis of \( B_1^* \otimes A_1 \). If we want to define \( \hat{\tau}_{A,C} \) on \( H \circ G \otimes C \), we need that

\[
\hat{\tau}_{A,C} \left( (D^k c_k + \lambda 1) \otimes (h_n^l \otimes g_j^l) \right) = \lambda 1 \otimes \left( h_n^l \otimes g_j^l - \hat{h}_n^l \otimes \hat{g}_j^l \right)
\]

\[
= \lambda 1 \otimes \left( h_n^l - \hat{h}_n^l \right) \otimes g_j^l = \lambda 1 \otimes \left( h_n^l - \hat{h}_n^l \right) \alpha_{j}^{\mu} \otimes y_\mu = 0,
\]

which is true if and only if \( \left( h_n^l - \hat{h}_n^l \right) \alpha_{j}^{\mu} = 0 \). But the symbols \( h_n^l \) are linearly independent, hence, in general, \( \hat{\tau}_{A,C} \) is not well-defined on \( H \circ G \otimes C \).

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