Almost existence from the feral perspective and some questions

JOEL W. FISH† and HELMUT H. W. HOFER‡

† Department of Mathematics, University of Massachusetts Boston, Boston, MA, USA
(e-mail: joel.fish@umb.edu)
‡ School of Mathematics, Institute for Advanced Study, Princeton, NJ, USA
(e-mail: hofer@math.ias.edu)

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Abstract. We use feral pseudoholomorphic curves and adiabatic degeneration to prove an extended version of the so-called ‘almost existence result’ for regular compact Hamiltonian energy surfaces. That is, that for a variety of symplectic manifolds equipped with a Hamiltonian, almost every (non-empty) compact energy level has a periodic orbit.

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1. Introduction and historical background
This paper is concerned with the almost existence phenomenon for periodic orbits of Hamiltonian dynamical systems. We shall describe some of the background of this
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phenomenon, and we relate it to the new feral curve theory, [8], which was recently initiated by the authors. The new approach to the almost existence phenomenon suggests a larger context which also features questions around the $C^\infty$-closing lemma in the following sense. Consider a compact symplectic manifold $(W, \Omega)$ equipped with a smooth Hamiltonian $H: W \to \mathbb{R}$. Given a regular energy surface $H^{-1}(E)$, one can ask the question: is it the case that after a small smooth perturbation of $H$ the new $H'^{-1}(E)$ has the property that the set of periodic orbits is dense? This is a classical question, and the $C^1$-closing lemma shows that the above assertion is true for a $C^2$-small perturbation of the Hamiltonian, that is, for a $C^1$-close Hamiltonian vector field. Whereas the $C^1$-closing lemma holds in general, it is known that the $C^\infty$-closing lemma does not; see [11]. We believe that the analysis below contributes to a growing body of evidence which suggests that the validity of the Hamiltonian $C^\infty$-closing lemma is intimately connected to the existence of a sufficiently rich Gromov–Witten theory of the ambient space. We touch on this speculation in §1.4.

The main goal of this paper is to set up a body of results to utilize the feral curve theory to study questions around almost existence and the closing lemma. Our results show that, given a compact pile of Hamiltonian energy surfaces, a sufficient supply of pseudoholomorphic curves associated to this pile implies ‘almost existence’; see Theorem 1.9. This is always attainable provided the pile of energy surfaces can be viewed as lying in a compact symplectic manifold with a sufficient supply of pseudoholomorphic curves. As an exercise the reader might enjoy using our more local results to re-prove the almost existence result for regular compact energy surfaces in $\mathbb{R}^{2n}$ by using the large supply of pseudoholomorphic curves obtained as a deformation of holomorphic curves in $\mathbb{CP}^n$. Building on previous work in [8], we also show that under suitable topological constraints, the almost existence result can be improved to the following: every energy level contains a non-trivial closed invariant subset, and for almost all of these energy levels this set is a periodic orbit; see Theorem 1.9. For the convenience of the reader we recall the necessary background.

1.1. The Weinstein conjecture. As a starting point for our discussion, we consider two seminal papers by P. Rabinowitz, namely [21, 22], which are concerned with the existence of a periodic orbit on a given compact regular energy surface

$$M = H^{-1}(E)$$

for an autonomous Hamiltonian system defined on the standard phase space $\mathbb{R}^{2n}$. Rabinowitz showed the existence of periodic orbits whenever suitable geometric conditions are met. We refer the reader to the introduction of [1] for the interesting broader historical perspective of Rabinowitz’s results.

A. Weinstein analyzed these results, particularly [22], and proposed the far-reaching Weinstein conjecture in [27], which we shall describe momentarily. First, though, we provide some definitions. Consider an odd-dimensional smooth manifold $M^{2n+1}$, with a 1-form $\lambda$ for which $\lambda \wedge (d\lambda)^n$ is a volume form. In this case, we call $\lambda$ a contact form for $M$, which then uniquely determines a vector field $X$ by the following equations:

$$\lambda(X) \equiv 1 \quad \text{and} \quad i_X d\lambda \equiv 0.$$
In this case, we call $X$ the Reeb vector field associated to the contact form $\lambda$. In a modified form, the Weinstein conjecture is stated as follows.

**Conjecture.** (Weinstein, 1978) Let $M$ be a smooth closed odd-dimensional manifold equipped with a contact form and an associated Reeb vector field $X$. Then the dynamical system given by

$$\dot{x} = X(x)$$

has a non-trivial periodic orbit.

The first breakthrough concerning the Weinstein conjecture was Viterbo’s celebrated result in [26], which showed that a regular compact contact-type energy surface of a Hamiltonian system in $\mathbb{R}^{2n}$ carries a non-trivial periodic orbit. In 1993 Hofer, in [13], showed that for a contact form $\lambda$ on a closed 3-manifold $M$ the Weinstein conjecture is true provided at least one of the following holds: either $M = S^3$, or $\pi_2(M) \neq 0$, or $\lambda$ is overtwisted. In 2007 Taubes proved in [25] that the Weinstein conjecture in dimension 3 is true for all $(M, \lambda)$.

A natural question then becomes whether or not a generalization of this result holds for more general compact energy surfaces in $\mathbb{R}^{2n}$. However, for $n \geq 3$, results in [9, 12] show that periodic orbits might not exist on a given energy surface. There is also a result for $n = 2$ (see Ginzburg and Gürel [10]), and thus it becomes interesting to study, in some sense, how often and how the generalization fails. This is the content of §1.2.

1.2. Almost existence. By analyzing Viterbo’s paper, the second author and E. Zehnder established in [15] that many compact Hamiltonian energy levels contain periodic orbits. Indeed, after some refinement by Rabinowitz in [23] and Struwe in [24], this phenomenon became referred to as ‘almost existence’. Working in a context in which the almost existence phenomenon holds, the actual existence question for periodic orbits of Hamiltonian systems quite often can be phrased in terms of whether ‘a priori estimates imply existence’; see [4]. Finally, the phenomenon was explained in [17] in terms of differentiability properties of the so-called Hofer–Zehnder capacity; see also [16].

For our explicit purpose of connecting this topic to feral curve theory, we approach the subject in a particular way. In a first definition we give an abstraction of a regular, smooth and compact Hamiltonian energy surface in a symplectic manifold, which we call a framed Hamiltonian manifold. By forgetting some of the information carried by a framed Hamiltonian manifold we obtain what is called an odd-symplectic manifold.

**Definition 1.1.** (Framed Hamiltonian manifold and odd-symplectic manifold) A framed Hamiltonian manifold $(M, \lambda, \omega)$ consists of a smooth closed odd-dimensional manifold $M = M^{2n+1}$, a 2-form $\omega$ and a 1-form $\lambda$ such that $\lambda \wedge \omega^n$ is a volume form. When such a $\lambda$ exists but we only specify $(M, \omega)$, we call the pair an odd-symplectic manifold.

A framed Hamiltonian manifold $(M, \lambda, \omega)$ defines a dynamical system. Namely, there exists a non-singular vector field $X$ on $M$ uniquely characterized by the equations

$$i_X \lambda \equiv 1 \quad \text{and} \quad i_X \omega \equiv 0.$$
As in the more special contact case we shall refer to $X$ as the Reeb vector field (associated to $(M, \lambda, \omega)$).

Assume that we are given a symplectic manifold $(W, \Omega)$ without boundary and consider a compact, smooth, regular and co-oriented hypersurface $M$ in $W$. Denoting by $i : M \to W$ the inclusion, we abbreviate $\omega = i^* \Omega$ and obtain the odd-symplectic manifold $(M, \omega)$. If $H : W \to \mathbb{R}$ is a smooth Hamiltonian and $H^{-1}(E) = M$ for some number $E \in \mathbb{R}$ and $dH(m) \neq 0$ for $m \in M$ we can take a 1-form $\lambda$ on $M$ such that $\lambda(X_H(m)) = 1$ for $m \in M$. Then $\lambda \wedge \omega^n$ is a volume form on $M$. Hence we obtain a framed Hamiltonian manifold $(M, \lambda, \omega)$. One easily verifies that the Reeb vector field $X$ satisfies

$$X_H(m) = X(m) \quad \text{for all } m \in M,$$

where $X_H$ is the Hamiltonian vector field associated to $H$ and defined by $i_{X_H}\Omega = -dH$.

In order to study the almost existence phenomenon we also need to consider neighborhoods of a smooth, regular and compact energy surface. Given a co-orientable, compact, smooth and regular hypersurface $M$ contained in $W$, where $(W, \tilde{\Omega})$ is a symplectic manifold, we obtain an odd-symplectic manifold $(M, \omega)$ as previously described. Namely, we take the inclusion $i : M \to W$ and define $\omega := i^* \tilde{\Omega}$.

We fix a 1-form $\lambda$ such that $\lambda \wedge \omega^n$ is a volume form on $M$. We can define on $\mathbb{R} \times M$ with coordinates $(t, m)$ a 2-form $\Omega$ by

$$\Omega = d(t\lambda) + \omega.$$

It is a trivial exercise that there exists an open neighborhood $U$ of $M = \{0\} \times M$ such that $\Omega|U$ is a symplectic form. Moreover, if we take $U$ small enough we find an embedding $\psi : U \to W$ onto an open neighborhood of $M \subset W$ such that $\psi^* \tilde{\Omega} = \Omega|U$ and, in addition, $\psi(0, m) = m$ for all $m \in M$. Note that this also implies that, given $\lambda_1$ and $\lambda_2$ so that $\lambda_1 \wedge \omega^n$ are volume forms, we find for the corresponding $\Omega_i$ defined by

$$\Omega_i = d(t\lambda_i) + \omega,$$

open neighborhoods $U_1$ and $U_2$ of $M \subset \mathbb{R} \times M$ such that $\Omega_i|U_i$ are symplectic and there exists a symplectic diffeomorphism $\psi : U_1 \to U_2$, which is the identity on $M = \{0\} \times M$. Depending on the case, whether or not $\lambda_1 \wedge \omega^n$ or $\lambda_2 \wedge \omega^n$ define the same orientation on $M$, we must have that $\psi$ maps $(s, m)$ for some $s > 0$ to some $(s', m')$ with $\pm s' > 0$.

From the previous discussion it follows that instead of working with a suitable open neighborhood $U$ of $M$ we may assume that $U = \mathbb{R} \times M$ is equipped with a symplectic form $\Omega$ such that $\omega = i^* \Omega$, where $i(m) = (0, m)$ for $m \in M$.

**Definition 1.2.** (Compatible $\Omega$) Given an odd-symplectic $(M, \omega)$, we call a symplectic form $\Omega$ on $\mathbb{R} \times M$ compatible provided $i^* \Omega = \omega$, where $i : M \to \mathbb{R} \times M$ is defined by $i(m) = (0, m)$ for $m \in M$.

The notion of almost existence will be associated to the behavior of small neighborhoods of $M = \{0\} \times M$ in $\mathbb{R} \times M$ for a symplectic form which restricts to $\omega$.

**Definition 1.3.** (Almost existence property—odd symplectic) Consider a smooth, compact odd-symplectic manifold $(M, \omega)$, denote by $\Omega$ a compatible symplectic form on $\mathbb{R} \times M$
and identify \( M \equiv \{0\} \times M \). We say that \((M, \omega)\) has the almost existence property provided there exists an open neighborhood \( U \) of \( M \) with the following property. Given any proper, smooth and surjective Hamiltonian \( H : V \to (-\varepsilon, \varepsilon) \) for some \( \varepsilon > 0 \), where \( V \) is an open neighborhood of \( M \) contained in \( U \) such that \( H^{-1}(0) = \iota(M) \) and \( dH(s, m) \neq 0 \) for \( m \in M \) and \( s \in (-\varepsilon, \varepsilon) \), define the set \( \Sigma_H \) by

\[
\Sigma_H = \{ E \in (-\varepsilon, \varepsilon) \mid \dot{x} = X_H(x) \text{ has a periodic orbit with } H(x) = E \}.
\]

We say that \((M, \omega)\) has the almost existence property provided for suitable \( U \) it holds for all \( H \) as described above that \( \text{measure}(\Sigma_H) = 2 \cdot \varepsilon \).

Finally, we can introduce a special class of symplectic manifolds.

**Definition 1.4. (Almost existence property—symplectic manifold)** A symplectic manifold \((W, \Omega_1)\) without boundary has the almost existence property provided that for every regular, compact and co-oriented hypersurface \( M \) the pair \((M, \iota^*\Omega)\) has the almost existence property, where \( \iota : M \to W \) is the inclusion.

We know that the standard symplectic vector space \( \mathbb{R}^{2n} \) has the almost existence property; see [16]. We also know that \( T^4 \) can be equipped with a symplectic form so that \((T^4, \Omega)\) does not have the almost existence property; see also [16]. In other words, the almost existence property is non-trivial. We state the following theorem for the convenience of the reader. It is based on some known facts which we identify as a local property.

**Theorem 1.5. (Local almost existence property)** Every symplectic manifold \((W, \Omega)\) without boundary has the following property. Given a point \( w \in W \), there exists an open neighborhood \( U(w) \) so that for every closed regular hypersurface \( M \subset U \) the pair \((M, \iota^*\Omega)\) has the almost existence property.

In other words, every symplectic manifold without boundary has the ‘local almost existence property’. It then becomes an interesting question what kind of more global compact, regular hypersurfaces in \((W, \Omega)\) have the almost existence property. We formulate a more precise question next.

**Question 1.6.** Assume that \((W, \Omega)\) is a symplectic manifold without boundary. Suppose that \( M \subset W \) is a smooth, compact, regular hypersurface without boundary so that the inclusion \( \iota : M \to W \) is isotopic to a small hypersurface, that is, contained in some \( U(w) \); see Theorem 1.5. Does \( M \) have the almost existence property? If this is not always true, then for which class of symplectic manifolds (other than \( \mathbb{R}^{2n} \)) is it true?

We note that the literature suggests that compact symplectic manifolds with a sufficiently rich Gromov–Witten theory have the almost existence property. The first paper, pre-dating Gromov–Witten theory, where such an idea is used is [14]. It shows that having a suitable moduli space of pseudoholomorphic curves associated to a symplectic manifold implies that the Weinstein conjecture holds for suitable energy surfaces. The method in [14] was then used in [19] and combined with Gromov–Witten theory for a more convenient packaging of properties of moduli spaces. This Gromov–Witten style approach
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has also been used to prove almost existence results in certain cases; see, for example, [20, Theorem 1.10].

1.3. Statement of the main result. Here we state the main result of the paper. The terms used below are standard; however, they are provided explicitly in §2 below. For example, the notion of an Ω-tame almost complex structure is provided in Definition 2.1; the notion of a pseudoholomorphic map is provided in Definition 2.2; and the notion of the genus of such a map is provided in Definition 2.3.

THEOREM 1.7. (Main result) Let \((W, \Omega)\) be a symplectic manifold without boundary, and let \(H : W \to \mathbb{R}\) be a smooth proper† Hamiltonian. Fix \(E_-, E_+ \in H(W) \subset \mathbb{R}\) with \(E_- < E_+\), as well as positive constants \(C_g > 0\) and \(C\Omega > 0\). Suppose that for each \(\Omega\)-tame almost complex structure \(J\) on \(W\) there exists a proper pseudoholomorphic map

\[
u : (S, j) \to \{p \in W : E_- < H(p) < E_+\}
\]

without boundary, which also satisfies the following conditions:

1. (genus and area bounds) The following inequalities hold:

\[
\text{Genus}(S) \leq C_g \quad \text{and} \quad \int_S \nu^*\Omega \leq C\Omega.
\]

2. (energy surjectivity) The map \(H \circ \nu : S \to (E_-, E_+)\) is surjective.

Then there is a periodic Hamiltonian orbit on almost every energy level in the range \((E_-, E_+)\). That is, if we let \(I \subset (E_-, E_+)\) denote the energy levels of \(H\) which contain a Hamiltonian periodic orbit, then \(I\) has full measure:

\[
\mu(I) = \mu((E_-, E_+)) = E_+ - E_-.
\]

At this point there are a few points worth making. The first is that the pseudoholomorphic maps in question here need not be compact—in fact, a careful inspection of the requirements reveals that they cannot be compact.

A second point is that the pseudoholomorphic maps allowed by the above hypotheses may have domains \((S, j)\) which are diffeomorphic to an open annulus, but sometimes the domains will be much worse. For example, our assumptions allow for the possibility that a domain of a pseudoholomorphic map may be an infinitely punctured open disk with infinitely many closed disks removed. Or worse: the open disk with the Cantor set removed. Or any closed Riemann surface with any closed set removed. To be clear, such assumptions are highly unusual in the standard theory of pseudoholomorphic curves; however, they are standard in feral curve theory. This is because feral curves in general are much wilder. We illustrate this with a plausible example.

Suppose \((W, \Omega)\) is a closed symplectic manifold, with an \(\Omega\)-compatible almost complex structure \(J\), a smooth Hamiltonian \(H : W \to \mathbb{R}\) for which 0 is a regular value, and a

† By ‘proper’ here, we mean that for each compact set \(K \subset \mathbb{R}\), the set \(H^{-1}(K)\) is compact.
pseudoholomorphic map $\tilde{u} : \mathcal{S} \to W$ where $\mathcal{S}$ is a closed Riemann surface such as a sphere or torus. For $\epsilon > 0$, what structure does the set have?

$$S := \{ z \in \mathcal{S} : -\epsilon < H(\tilde{u}(z)) < \epsilon \}$$  \hfill (1)

As it need not be the case that $\pm \epsilon$ are regular values of $H \circ u$, there is no reason that $S$ should necessarily admit a smooth compactification to a compact Riemann surface with smooth boundary. Indeed, all one can really say is that it has the structure of a closed Riemann surface with some closed set removed, which is exactly the sort of domain surface that Theorem 1.7 allows. Conceptually, it may be easier to think of the maps we allow as arising from restrictions like $u := \tilde{u}|_S$, with $S$ as in equation (1); however, it is worth noting that the existence of such an extension (or lack thereof) plays no role in our proof.

With such unusual freedom allowed for the domains of our pseudoholomorphic maps, the attentive reader may be concerned about the precise notion of genus. This is made rigorous in Definition 2.3 below, although it amounts to exhausting $S$ by compact two-dimensional manifolds with boundary and taking the limit of the associated genera.

A third point is that the hypothesis in Theorem 1.7 regarding the existence of a suitable pseudoholomorphic map for each $\Omega$-tame almost complex structure seems very restrictive; however, in practice this is not the case. For example, Gromov–Witten invariants are invariants of a closed symplectic manifold obtained by algebraically counting pseudoholomorphic maps of specified genus, homology class, and incidence conditions. In particular, Gromov–Witten invariants are independent of the choice of almost complex structure (provided that it is $\Omega$-tame), and thus a sufficiently rich Gromov–Witten theory for a closed symplectic manifold $(W, \Omega)$ is sufficient to guarantee that the hypotheses of Theorem 1.7 are satisfied in many cases.

We will momentarily contrast the feral curve techniques used below with the methods used in [19, 20]; here we point out that the proof of Theorem 1.7 will show that its conclusion holds under weaker assumptions regarding the pseudoholomorphic curves. Specifically, one only needs the existence of pseudoholomorphic curves for a particular sequence of adiabatically degenerating almost complex structures. The notion of this adiabatic degeneration is rather technical and is made precise in Definition 2.8; however, the idea is to degenerate the almost complex structure so as to geometrically ‘stretch the neck’ along a continuum of energy levels. With such an concept internalized, we direct the reader’s attention to Theorem 3.1 in §3 below, which is a localized version of Theorem 1.7 with the assumptions stripped to the absolute essentials.

With the exception of Struwe’s results in [24], which pre-date modern symplectic methods, all proofs of almost existence follow a similar pattern: prove that the Hofer–Zehnder capacity of a domain containing the energy surface is finite, and the desired result follows from [17]. For example, in [20] Lu takes cues from [19] to use the Gromov–Witten invariants to define a pseudo-capacity which is finite and bounds the Hofer–Zehnder capacity; the almost existence result is then immediate. In contrast, the proof of Theorem 1.7 makes no use of capacities at all, and only makes use of pseudoholomorphic curves—specifically feral curve theory. The idea is to stretch the neck along a continuum of energy levels, and analyze some basic properties in the limit. For those familiar with methods from symplec-
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In field theory, it is worth pointing out that the key obstacle to overcome is that there are no Hofer energy bounds in this case, and hence infinite energy pseudoholomorphic curves (that is, feral curves) can appear in the limit ‘building’. The picture that emerges from this analysis is rather interesting. It seems that as one takes this adiabatic degeneration, curves which cross the region of degeneration are inexorably drawn to collapse onto families of periodic orbits. Or more precisely, such a collapse to families of orbits occurs almost everywhere, and on the complementary measure-zero set the curves are allowed to jump between such families, or even jump to a more general closed invariant subset. Further analysis of such limiting curves seems well poised to illuminate additional dynamical features.

We close this introductory section with an application which appears to be inaccessible to methods relying on the finiteness of the Hofer–Zehnder capacity. We state this as Theorem 1.9 below, but first provide a definition.

**Definition 1.8.** (Positive contact type) Let \((W, \Omega)\) be a compact symplectic manifold with boundary. Let \(M^+\) be a union of connected components of \(\partial W\). We say \(M^+ \subset \partial W\) is of positive contact type provided there exists an outward-pointing nowhere vanishing vector field \(Y\) defined in a neighborhood of \(M^+\) in \(W\) for which \(L_Y \Omega = \Omega\); here \(L\) denotes the Lie derivative. In this case \(\lambda := i_Y \Omega|_{M^+}\) is a contact form on \(M^+\).

**Theorem 1.9.** (Intertwining existence and almost existence) Let \((W, \Omega)\) be a four-dimensional compact connected exact symplectic manifold with boundary \(\partial W = M^+ \cup M^-\). Suppose \(M^+\) is positive contact type in the sense of Definition 1.8, and suppose that one of the following three conditions holds:

1. \(M^+\) has a connected component diffeomorphic to \(S^3\);
2. there exists an embedded \(S^2 \subset M^+\) which is homotopically non-trivial in \(W\);
3. \((M^+, \lambda)\) has a connected component which is overtwisted.

Then, for each Hamiltonian \(H \in C^\infty(W, \mathbb{R})\) for which \(H^{-1}(\pm 1) = M^\pm\), the following assertion is true. For each \(s \in [-1, 1]\) the energy level \(H^{-1}(s)\) contains a closed non-empty set other than the energy level \(H^{-1}(s)\) itself which is invariant under the Hamiltonian flow of \(X_H\); moreover, for almost every \(s \in [-1, 1]\), this closed invariant subset is a periodic orbit.

Note that the image of \(H\) is not required to lie in \([-1, 1]\).

The proof of Theorem 1.9 is provided in §3 below.

1.4. Speculation on the \(C^\infty\)-closing property. Before moving on to the proofs of Theorems 1.7 and 1.9, we wish to draw some connections to the \(C^\infty\)-closing lemma. We also aim to pose some speculative questions which we believe the feral curve techniques below seem well poised to eventually answer.

We begin with an odd-symplectic manifold \((M, \omega)\), and take a compatible symplectic form \(\Omega\) on \(\mathbb{R} \times M\). We consider the Fréchet space \(C^\infty(M, \mathbb{R})\) and observe that every element \(f\) defines a hypersurface in \(\mathbb{R} \times M\) by setting

\[
M_f = \{(f(m), m) \in \mathbb{R} \times M : m \in M\}.
\]
Considering the hypersurface \( M_f \subset (\mathbb{R} \times M, \Omega) \), we obtain the distinguished line bundle \( \mathcal{L}_f \to M_f \), given by
\[
\mathcal{L}_f := \ker(\omega_f) \subset TM_f,
\]
where \( \omega_f \) is the pull-back of \( \Omega \) by the inclusion \( M_f \to \mathbb{R} \times M \). Since \( \mathcal{L}_f \subset TM_f \) is a dimension-1 sub-bundle, it is an integrable distribution and we are interested in the closed leaves. We denote by \( C_f \) the union of all points in \( M_f \) which lie on a closed leaf. We say that the periodic orbits are dense on \( M_f \) provided \( \text{cl}(C_f) = M_f \).

**Definition 1.10.** (\( C^\infty \)-closing property) We say that the odd-symplectic manifold \((M, \omega)\) has the \( C^\infty \)-closing property provided that there exists a compatible \( \Omega \) on \( \mathbb{R} \times M \) so that, for a Baire subset \( \Sigma \) of \( C^\infty(M, \mathbb{R}) \),
\[
\text{cl}(C_f) = M_f \quad \text{for all } f \in \Sigma.
\]

Again one can use the closing property to define a particular class of symplectic manifolds.

**Definition 1.11.** (\( C^\infty \)-closing property—symplectic manifolds) We say a symplectic manifold \((W, \Omega)\) has the \( C^\infty \)-closing property provided that for every regular compact co-oriented hypersurface \( M \) in \( W \) the induced \((M, \omega)\) has the \( C^\infty \)-closing property.

One can play around with the above definition by allowing only hypersurfaces isotopic to small ones or those carrying a suitable topology. Alternatively, one might choose to only allow contact-type hypersurfaces. We leave it to the reader to explore these ideas and we only mention the following conjecture.

**Conjecture 1.12.** (Local \( C^\infty \)-closing property) The standard symplectic vector space \((\mathbb{R}^{2n}, \Omega_{\text{standard}}), n \geq 2\), has the \( C^\infty \)-closing property. In particular, all symplectic manifolds without boundary have the local \( C^\infty \)-closing property.

The conjecture is open for all \( n \geq 2 \) and basically nothing is known for \( n \geq 3 \). In the case of \( n = 2 \) one knows a partial result, namely that compact, regular hypersurfaces in \( \mathbb{R}^4 \) of contact type have the \( C^\infty \)-closing property. However, nothing is known for general compact regular hypersurfaces in \( \mathbb{R}^4 \). Indeed, by a result of Irie [18], every \((M, \lambda, d\lambda)\), where \( M \) is a closed 3-manifold equipped with a contact form \( \lambda \), has the \( C^\infty \)-closing property. In particular, every compact regular hypersurface of contact type in \( \mathbb{R}^4 \) has the \( C^\infty \)-closing property. In the background of Irie’s result and a follow-up result [2] is the important volume formula by Cristofaro-Gardiner, Hutchings, and Ramos [5]. At present the proof of [18] based on [5] only works in the three-dimensional case. Due to the use of Seiberg–Witten–Floer theory it will need some new ideas to attack the higher-dimensional cases—perhaps feral curves.

The germ of an idea goes as follows. The key upshot of feral curves is that one can stretch the neck along any hypersurface. The downside is that one may find limit sets which are much more complicated than a finite set of periodic orbits. However, Theorem 1.7 strongly suggests that generically (in the right sense) one can stretch the neck while following a single curve for each almost complex structure, and pass to the limit to find a
periodic orbit. That is, at least one periodic orbit is (generically) found by tracking just one curve. What if there are many curves to track? High-dimensional families of curves, for example. Here the proposed richness of the Gromov–Witten invariants comes into play. For example, consider \( \mathbb{R}^{2n} \simeq \mathbb{C}^n = \mathbb{C}P^n \setminus \mathbb{C}P^{n-1} \), and suppose we consider stretching the neck along some generic hypersurface \( M \subset \mathbb{R}^{2n} \). By considering curves of high degree, one obtains high-dimensional families of curves which stretch and break along periodic orbits. This raises a question: which orbits can be found by stretching the neck and tracking curves of any fixed (but arbitrarily large) degree? All orbits, or just a subset? A dense subset? If it turns out that neck stretching can find (nearly) every orbit then feral curves seem well poised to recover the \( \mathbb{C}^{\infty} \)-closing lemma for arbitrary regular, compact hypersurfaces in \( \mathbb{R}^4 \). Currently the known results about the contact-type case in \( \mathbb{R}^4 \) depend on Seiberg–Witten theory. The feral curve theory should be important in removing the contact-type hypothesis. In order to prove the results in higher dimensions one way to succeed seems to be the development of suitable techniques to use higher-dimensional moduli spaces.

2. Background and geometric framework
Here we recall some standard background material, and then we provide a geometric framework very specific to the problem of study. We begin with the notions of an almost complex manifold and pseudoholomorphic maps.

Definition 2.1. (Almost complex structures; compatible and tame) Let \( W \) be a smooth manifold, not necessarily closed, possibly with boundary, and let \( J \in \Gamma(\text{End}(TW)) \) be a smooth section for which \( J \circ J = -1 \). We call \( J \) an almost complex structure for \( W \), and the pair \( (W, J) \) an almost complex manifold. In the case that \( \Omega \) is a symplectic form on \( W \), we say \( J \) is \( \Omega \)-compatible provided that \( g(v, w) := \Omega(v, Jw) \) is a Riemannian metric. Under the weaker assumption that \( \Omega(v, Jv) \geq 0 \) for all \( v \in TW \), and with equality if and only if \( v = 0 \), we say instead that \( J \) is only \( \Omega \)-tame.

Definition 2.2. (Pseudoholomorphic map) Let \( (S, j) \) and \( (W, J) \) be smooth almost complex manifolds with \( \text{dim}(S) = 2 \), each possibly with boundary. A \( C^{\infty} \)-smooth map \( u : S \to W \) is said to be pseudoholomorphic provided \( J \cdot Tu = Tu \cdot j \). That is, the tangent map of \( u \) intertwines the almost complex structures on domain and target. Unless otherwise specified, we allow \( S \) to be disconnected. We say such a map is proper provided the pre-image of any compact set is compact.

It is worth noting that constant maps are always pseudoholomorphic. We also note that we will allow the domains of our pseudoholomorphic maps to be disconnected, which in conjunction with the fact that constant maps are always pseudoholomorphic allows for the possibility that any given map may have many constant components—perhaps infinitely many. At present we allow this, while noting that second countability of the domain Riemann surfaces forces any given map to have at most countably many constant components.

Because the domains of our pseudoholomorphic maps are Riemann surfaces which need not be compact, we make the notion of genus precise with the following definition.
Definition 2.3. (Genus) Let $S$ be a two-dimensional oriented manifold, possibly with boundary, with at most countably many connected components, and with the property that each connected component of $\partial S$ is compact.

1. If $S$ is closed and connected, then define $\text{Genus}(S) := g$ where $\chi(S) = 2 - 2g$ is the Euler characteristic of $S$.
2. If $S$ is compact and connected with $n$ boundary components, define $\tilde{S} = (S \sqcup (\bigsqcup_{k=1}^n D^2))/\sim$ to be the closed surface capped off by $n$ disks, and define $\text{Genus}(S) := \text{Genus}(\tilde{S})$.
3. If $S$ is compact (possibly with boundary), then $\text{Genus}(S)$ is defined to be the sum of the genera of each connected component.
4. If $S$ is not compact, then $\text{Genus}(S)$ is defined by taking any nested sequence $S_1 \subset S_2 \subset S_3 \subset \cdots$ of compact surfaces (possibly with boundary) such that $S_k \subset S$ for all $k \in \mathbb{N}$ and such that $S = \bigcup_k S_k$; then we define $\text{Genus}(S) := \lim_{k \to \infty} \text{Genus}(S_k)$.

We now turn our attention to geometric structures more specific to the proof of Theorem 1.7. The first key idea is what we call a framed Hamiltonian energy pile (see Definition 2.4) which is essentially the neighborhood of a compact energy level with enough structure to regard it as something like a family of framed Hamiltonian manifolds. This relationship is made precise with Lemma 2.6. Also important is Lemma 2.5, which is a means to identify the neighborhood of a compact energy level in a general symplectic manifold, with the structure of a framed Hamiltonian energy pile, thereby localizing the almost existence problem.

Definition 2.4. (Framed Hamiltonian energy pile) Let $I_{\epsilon}$ be the interval $(-\epsilon, \epsilon)$ equipped with the coordinate $s$, and let $M$ be a closed odd-dimensional manifold. Assume $I_{\epsilon} \times M$ is equipped with the symplectic form $\Omega$. Let $H : I_{\epsilon} \times M \to \mathbb{R}$ be the smooth Hamiltonian $H(s, p) = s$, and let $X_H$ be the associated Hamiltonian vector field determined by $i_{X_H} \Omega = -dH$ and let $\hat{\lambda}$ be a 1-form on $I_{\epsilon} \times M$ which satisfies the following three conditions:

(H1) $\hat{\lambda}(\partial_s) = 0$;
(H2) $\mathcal{L}_{\partial_s} \hat{\lambda} = 0$ where $\mathcal{L}$ is the Lie derivative;
(H3) $\hat{\lambda}(X_H) > 0$.

We call the triple $(I_{\epsilon} \times M, \Omega, \hat{\lambda})$ a framed Hamiltonian energy pile.

If we consider local coordinates $x_1, \ldots, x_N$ on $M$, $N = 2n + 1$, and the coordinate $s$ on $I_{\epsilon}$ then $\hat{\lambda}$ can be written at the point $(s, x)$ as

$$\hat{\lambda}(s, x) = \sum_{i=1}^{N} a_i(x) \, dx_i$$  \hspace{1cm} (2)

due to the imposed conditions (H1) and (H2). We shall show in Lemma 2.5 how framed Hamiltonian energy piles arise near a compact regular energy surface. We begin, however, by deriving a few geometric structures which arise as an immediate consequence of having a framed Hamiltonian energy pile. Define the two-plane distribution $\hat{\rho}$ on $I_{\epsilon} \times M$ by

$$\hat{\rho} = \text{Span}(\partial_s, X_H).$$  \hspace{1cm} (3)
and define the codimension-2 plane distribution $\xi$ on $\mathcal{I}_\varepsilon \times M$ by
\[ \xi = \ker (ds \wedge \hat{\lambda}) = (\ker ds) \cap (\ker \hat{\lambda}). \] (4)

We observe that the vector bundle $\xi \to \mathcal{I}_\varepsilon \times M$ is $\mathbb{R}$-invariant in the following sense. Given $s_0, s_1 \in \mathcal{I}_\varepsilon$, the map $(h + s_0, m) \to (h + s_1, m)$, which is defined for small $|h|$, pushes forward (the obvious restrictions of) $\xi$ to $\xi$. Moreover, we have the splitting
\[ T(\mathcal{I}_\varepsilon \times M) = \hat{\rho} \oplus \xi, \]
and the associated projections
\[ \pi^{\hat{\rho}} : \hat{\rho} \oplus \xi \to \hat{\rho} \quad \text{and} \quad \pi^\xi : \hat{\rho} \oplus \xi \to \xi. \]

Define the 2-form $\hat{\omega}$ by
\[ \hat{\omega} = \Omega \circ (\pi^\xi \times \pi^\xi). \] (5)

Here and throughout, we will also employ the notation
\[ \hat{X} := \frac{X_H}{\hat{\lambda}(X_H)}. \]

**Lemma 2.5.** (Localization to a framed Hamiltonian energy pile) Let $(\tilde{W}, \tilde{\Omega})$ be a symplectic manifold without boundary, and let $\tilde{H} : \tilde{W} \to \mathbb{R}$ be a $C^\infty$-smooth proper Hamiltonian. Suppose further that zero is a regular value of $\tilde{H}$. Then there exist an $\varepsilon > 0$, a framed Hamiltonian energy pile $(\mathcal{I}_\varepsilon \times M, \Omega, \hat{\lambda})$, and a $C^\infty$-smooth diffeomorphism
\[ \Phi : \mathcal{I}_\varepsilon \times M \to \{|\tilde{H}| < \varepsilon\} \subset \tilde{W} \]
for which
\[ (\tilde{H} \circ \Phi)(s, p) = s \quad \text{and} \quad \Phi^*\tilde{\Omega} = \Omega. \]

Additionally, the framed Hamiltonian energy pile can be found such that along the energy level $\{0\} \times M$, the following assertions are true.

1. $\hat{\lambda}(X_H) = 1$.
2. The vector bundles $\hat{\rho}$ and $\xi$ are symplectic complements over $\{0\} \times M$. That is, for each $q \in \{0\} \times M$, each $v_q \in \hat{\rho}_q$, and each $w_q \in \xi_q$, we have $\Omega(v_q, w_q) = 0$.

**Proof.** We begin by fixing an auxiliary $\tilde{\Omega}$-compatible almost complex structure on $\tilde{W}$; denote it by $\tilde{J}$. This gives rise to the Riemannian metric $g_{\tilde{J}} = \tilde{\Omega} \circ (\text{Id} \times \tilde{J})$. Using this metric to compute the gradient of $\tilde{H}$, we let $\varphi^t$ be the time-$t$ flow associated to the vector field $\nabla \tilde{H} \|_{g_{\tilde{J}}}^{-2} \nabla \tilde{H}$ in a neighborhood $\{|\tilde{H}| < \varepsilon\}$ for some small $\varepsilon > 0$ yet to be determined. Defining $M := \tilde{H}^{-1}(0)$, it follows that for all sufficiently small $\varepsilon > 0$,
\[ \Phi : \mathcal{I}_\varepsilon \times M \to \{|\tilde{H}| < \varepsilon\} \subset \tilde{W} \]
\[ \Phi(s, p) = \varphi^s(p) \]
is a diffeomorphism, and $H := \tilde{H} \circ \Phi$ satisfies $H(s, p) = s$. 

By construction, the vector field \( \tilde{H} \) is tangent to level sets of \( \tilde{H} \), and thus along \( \{ \tilde{H} = 0 \} \) we can define \( \tilde{\lambda} \) to be the 1-form uniquely determined by the conditions

\[
\tilde{\lambda}(X_{\tilde{H}}) = 1 \quad \text{and} \quad \ker \tilde{\lambda} = \mathbb{R} \nabla \tilde{H} \oplus (\text{Span}(X_{\tilde{H}}, \nabla \tilde{H}))^\perp
\]

where \( \perp \) denotes the \( \tilde{\Omega} \)-symplectic complement. We then define

\[
\hat{\lambda} = \text{pr}_2^* \tilde{\lambda}
\]

where \( \text{pr}_2 : \mathcal{I}_\epsilon \times M \to M \) is the canonical projection. It is straightforward to verify that

\[
\hat{\lambda}(\partial_s) = 0, \quad \text{and} \quad L_{\partial_s} \hat{\lambda} = 0.
\]

Because \( \hat{\lambda}(X_{H}) \mid_{\{0\} \times M} = 1 \), it then follows that \( \hat{\lambda}(X_{H}) > 0 \) for all sufficiently small \( \epsilon > 0 \). By construction then, \((W, \Omega, \hat{\lambda})\) is a framed Hamiltonian energy pile provided that \( \epsilon > 0 \) is sufficiently small. Moreover, by construction, along \( \{0\} \times M \) it is the case both that \( \hat{\lambda}(X_{H}) = 1 \) and that \( \hat{\rho} \) and \( \hat{\xi} \) are \( \Omega \)-symplectic complements. This completes the proof of Lemma 2.5.

We make the following important observation.

**Lemma 2.6.** (Energy levels are framed Hamiltonian manifolds) Let \((\mathcal{I}_\epsilon \times M, \Omega, \hat{\lambda})\) be a framed Hamiltonian energy pile, and let \( \hat{\rho}, \hat{\xi} \) and \( \hat{\omega} \) be the associated structures defined above; see (3), (4) and (5). Then for each \( s_0 \in \mathcal{I}_\epsilon \), the restriction of \( \hat{\lambda} \) and \( \hat{\omega} \) to the energy level \( \{s_0\} \times M \) is a framed Hamiltonian structure for this energy level.

**Proof.** We begin by recalling that a framed Hamiltonian structure \( \eta = (\lambda, \omega) \) for an odd-dimensional manifold \( M \) is a 1-form \( \lambda \) and a closed 2-form \( \omega \) for which \( \lambda \wedge \omega \wedge \cdots \wedge \omega = \text{vol}_M \) is a volume form. To show this latter non-degeneracy condition is satisfied on our energy levels, first note that on \( \mathcal{I}_\epsilon \times M \) we have that \( T(\mathcal{I}_\epsilon \times M) = \hat{\rho} \oplus \hat{\xi} \) is a splitting and \( \Omega \) is non-degenerate on each of \( \hat{\rho} \) and \( \hat{\xi} \). Moreover, by construction

\[
\hat{\xi} = \ker (ds \wedge \hat{\lambda}) \quad \text{and} \quad \hat{\rho} = \ker (\hat{\omega}) = \text{Span}(\partial_s, X_H),
\]

and \( \hat{\omega} = \Omega \circ (\pi^\xi \times \pi^\xi) \) so that

\[
ds \wedge \hat{\lambda} \wedge \hat{\omega}_n > 0 \quad \text{on} \quad \mathcal{I}_\epsilon \times M;
\]

here \( \dim(M) = 2n + 1 \). It immediately follows that

\[
\hat{\lambda} \wedge \hat{\omega}_n > 0 \quad \text{on} \quad \{s_0\} \times M \quad \text{for each} \quad s_0 \in \mathcal{I}_\epsilon.
\]

This establishes the non-degeneracy condition.

Next, we establish that the restriction of \( \hat{\omega} \) to each energy level \( \{s_0\} \times M \) is closed. To that end, let \( v, w \in T(\{s_0\} \times M) \). Then there exist \( a, b \in \mathbb{R} \) and \( v^\xi, w^\xi \in \xi \) such that

\[
v = aX_H + v^\xi \quad \text{and} \quad w = bX_H + v^\xi,
\]

and then

\[
\Omega(v, w) = \Omega(aX_H + v^\xi, bX_H + w^\xi)
\]

\[
= \Omega(a, b^\xi)
\]

\[
= \Omega(\pi^\xi(v), \pi^\xi(w))
\]

\[
= \hat{\omega}(v, w)
\]
which shows that for the inclusion \( \iota : \{s_0\} \times M \hookrightarrow \mathcal{I}_e \times M \) we have \( \hat{\omega}|_{\{s_0\} \times M} = \iota^* \Omega \), and thus the restriction of \( \hat{\omega} \) to \( \{s_0\} \times M \) is closed. This completes the proof of Lemma \( \square \)

For the following considerations of adapted almost complex structures we summarize the salient points of the previous discussion. By definition a framed Hamiltonian manifold \( (\hat{M}, \hat{\lambda}, \hat{\omega}) \) has an associated Reeb vector field \( \hat{X} \) uniquely determined by the equations

\[
\hat{\lambda}(\hat{X}) = 1 \quad \text{and} \quad i_{\hat{X}} \hat{\omega} = 0.
\]

Given a framed Hamiltonian energy pile \( (\mathcal{I}_e \times M, \Omega, \hat{\lambda}) \), we can consider the Hamiltonian \( H \) given by \( H(s, m) = s \), which has Hamiltonian vector field \( X_H \) defined by \( i_{X_H} \Omega = -dH \). We normalize it by setting

\[
\hat{X} = X_H / \hat{\lambda}(X_H).
\]

Then we observe that \( \hat{X} \) satisfies

\[
\hat{\lambda}(\hat{X}) = 1, \quad ds(\hat{X}) = 0 \quad \text{and} \quad i_{\hat{X}} \hat{\omega} = 0.
\]

Consequently, on each energy level \( \{s_0\} \times M \) the Reeb vector field associated to the framed Hamiltonian structure \( (\hat{\lambda}, \hat{\omega})|_{\{s_0\} \times M} \) is the restriction of \( \hat{X} \).

We now turn to equipping framed Hamiltonian energy piles with a certain class of almost complex structures, which we now define.

**Definition 2.7.** (Weakly adapted almost complex structure) Let \( (\mathcal{I}_e \times M, \Omega, \hat{\lambda}) \) be a framed Hamiltonian energy pile, and let \( \hat{\rho}, \xi \) and \( \hat{\omega} \) be the associated structures defined above. Let \( J \) be an almost complex structure on \( \mathcal{I}_e \times M \) which satisfies the following conditions.

1. **(J1)** It preserves the splitting \( \hat{\rho} \oplus \xi \); that is,

\[
J : \hat{\rho} \to \hat{\rho} \quad \text{and} \quad J : \xi \to \xi.
\]

2. **(J2)** There exists a smooth function of the form

\[
\phi : \mathcal{I}_e \times M \to (0, 1]
\]

\[
\phi(s, p) = \phi(s)
\]

with the property that

\[
\phi \cdot J \partial_s = \hat{X} \quad \text{where} \quad \hat{X} = \frac{X_H}{\hat{\lambda}(X_H)}.
\]

3. **(J3)** The following is a Riemannian metric:

\[
g_J(v, w) = (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jw).
\]

In this case we say \( J \) is an almost complex structure weakly adapted to the framed Hamiltonian energy pile \( (\mathcal{I}_e \times M, \Omega, \hat{\lambda}) \).

We pause for a moment to discuss the manner in which these almost complex structures are weakly adapted. Consider a fixed framed Hamiltonian energy pile \( (\mathcal{I}_e \times M, \Omega, \hat{\lambda}) \).
Associated to this data we have the plane field bundles
\[ \xi = \ker (ds \wedge \hat{\lambda}) \quad \text{and} \quad \hat{\rho} = \ker (\hat{\omega}) = \text{Span}(\partial_s, X_H) = \text{Span}(\partial_s, \hat{X}), \]
with the 2-form \( \hat{\omega} = \Omega \circ (\pi^\xi \times \pi^\xi) \).

Here we recall that
\[ T(I_\varepsilon \times M) = \hat{\rho} \oplus \xi. \]

Following motivation from symplectic field theory as proposed in [6], there is a somewhat natural choice of class of almost complex structures here, namely those of the form
\[ J \partial_s = \hat{X} \quad \text{and} \quad J : \xi \to \xi \]
where \( \hat{\omega} \circ (\text{Id} \times J)|_\xi \) is symmetric and positive definite. There are two issues of note, which make such a choice somewhat different from the weakly adapted almost complex structures defined above. The first such difference is that in general \( \hat{\omega} \) is not translation invariant. Put another way, this means that \( \hat{X} \) has \( s \) dependency, or rather that \( \hat{X}(s, p) \neq \hat{X}(s', p) \) in general for \( s \neq s' \). Because \( \hat{X} \) is not translation invariant, and in fact in general the line bundle \( \mathbb{R} \hat{X} \subset \hat{\rho} \) will fail to be translation invariant, it follows that any almost complex structure which preserves \( \hat{\rho} \) will also fail to be translation invariant.

It is worth recalling that the framework of symplectic field theory typically requires a translation-invariant almost complex structure (in cylindrical homogeneous regions of symplectizations) or else is only required to be symplectically tame (in the inhomogeneous or cobordant regions). In contrast, Definition 2.7 requires the almost complex structure to be carefully ‘adapted’ in an inhomogeneous region. Perhaps a more important feature of Definition 2.7 is the ‘weakness’ condition, which allows the almost complex structure to have the form \( \phi \cdot J \partial_s = \hat{X} \). This is best understood as a partial degeneration of the almost complex structure which can be undone by a finite amount of ‘stretching the neck’ along the first factor of \( I_\varepsilon \times M \). More precisely, we define the embedding
\[ \Psi : I_\varepsilon \times M \to \mathbb{R} \times M \]
where
\[ \psi(s) = \int_0^s \frac{1}{\phi(t)} \, dt. \]

To see the utility of the map \( \Psi \), we consider an example, which already explains its key features with respect to the other relevant data. Specifically, we consider the case that \( \phi(s) = \delta \) where \( \delta \) is some very small positive number, and we assume that \( J_\xi := J|_\xi : \xi \to \xi \) is translation invariant in the \( s \) direction; that is, \( \mathcal{L}_{\partial_s} J_\xi = 0 \). For \( v^\xi \in \xi \) we then have
\[ J(a \partial_s + b \hat{X} + v^\xi) = -\delta b \partial_s + \frac{a}{\delta} \hat{X} + J_\xi v^\xi. \]

We also see that in this case
\[ \Psi : (-\varepsilon, \varepsilon) \times M \to \left( -\frac{\varepsilon}{\delta}, \frac{\varepsilon}{\delta} \right) \times M \quad \text{(diffeomorphism)} \]
\[ \Psi(s, p) = (s\delta^{-1}, p) \]
and thus
\[
(\Psi_\ast J)(a\partial \hat{s} + b(\Psi_\ast \hat{X}) + v^\xi) = -b\partial \hat{s} + a(\Psi_\ast \hat{X}) + J_\xi v^\xi,
\]
where \(\hat{s}\) is the coordinate on \((-\epsilon \delta^{-1}, \epsilon \delta^{-1})\). Note that if we abuse notation by identifying \(\{s_0\} \times M\) with \(M\), then
\[
(\Psi_\ast \hat{X})(\hat{s}, p) = \hat{X}(\delta \hat{s}, p).
\]
Similarly, if \(J_\xi = J|_\xi\) has \(s\) dependence, then by abusing notation again, we have
\[
(\Psi_\ast J_\xi)(\hat{s}, p) = J_\xi(\delta \hat{s}, p).
\]
Put another way, because the tangent bundle splits as \(\hat{\rho} \oplus \xi\), and because the almost complex structures preserve this splitting, and the stretching direction is contained in \(\hat{\rho}\), it follows that this ‘neck stretching’ does not degenerate the restriction \(J_\xi\), and it is chosen to stretch the compressed \(J\) into something better behaved. For example, \((\Psi_\ast J)\partial \hat{s} = \Psi_\ast \hat{X}\). It is left to the reader to observe that this qualitative behavior is preserved when one changes from the example case of \(\phi(s) = \delta\) to the more general case that \(\phi: \mathcal{I}_\epsilon \times M \to (0, 1]\) with \(\phi(s, p) = \phi(s)\).

We will elaborate further on this stretching construction below; however, it will be helpful to put it in the context of ‘adiabatic degeneration’, which amounts to fully neck-stretching along a continuum of energy levels simultaneously. We make this precise with the following definition.

**Definition 2.8.** (Adiabatically degenerating almost complex structures) Let \((\mathcal{I}_\epsilon \times M, \Omega, \hat{\lambda})\) be a framed Hamiltonian energy pile, let \(\hat{\rho}, \xi\) and \(\hat{\omega}\) be the associated structures defined above, and let \(\hat{X} = X_H / \hat{\lambda}(X_H)\) as above. Let \(\mathcal{J} = \{J_k\}_{k \in \mathbb{N}}\) be a sequence of weakly adapted almost complex structures in the sense of Definition 2.7, with \(\phi_k \cdot J_k \partial s = \hat{X}\), which also satisfies the following conditions.

\(\mathcal{J}1\) (Adiabatic degeneration) We require that \(\phi_k \to 0\) in \(C^\infty\) as \(k \to \infty\).

\(\mathcal{J}2\) (Symplectic area controls metric area) For each \(k \in \{1, 2, 3, \ldots\}\) and each \(v \in T(\mathcal{I}_\epsilon \times M)\), we require that
\[
(ds \wedge \hat{\lambda} + \hat{\omega})(v, J_k v) \leq 2\Omega(v, J_k v).
\]

\(\mathcal{J}3\) (Geometrically bounded) There exist a sequence \(\{C_n\}_{n=0}^\infty\) of positive constants and an auxiliary translation-invariant metric \(\tilde{g}\) on \(\mathbb{R} \times M\) for which
\[
\sup_{k \in \mathbb{N}} \| (\Psi_k)_* J_k \|_{C^n} \leq C_n
\]
for each \(n \in \mathbb{N}\); here \(\| \cdot \|_{C^n}\) is the \(C^n\)-norm on \(\mathbb{R} \times M\) with respect to the auxiliary metric \(\tilde{g}\), and the \(\Psi_k\) are the embeddings as in equation (7).

We then say that \(\mathcal{J} = \{J_1, J_2, \ldots\}\) is a sequence of adiabatically degenerating almost complex structures adapted to the framed Hamiltonian energy pile given by \((\mathcal{I}_\epsilon \times M, \Omega, \hat{\lambda})\).

We note that, given a framed Hamiltonian energy pile, it is easy to construct a candidate sequence of adiabatically degenerating almost complex structures. For example, one might
fix a translation-invariant $J_\xi = J|_\xi$, and then define

$$J_k \partial_s = k \hat{X} \quad \text{and} \quad J_k|_\xi = J_\xi.$$  

In such a candidate, one can easily see that $\phi_k = k^{-1} \to 0$ as $k \to \infty$, and in fact $(\Psi_k)^* J_k$ is independent of $k$. Consequently $(\mathcal{J}1)$ and $(\mathcal{J}3)$ are immediately satisfied. However, $(\mathcal{J}2)$ is less obvious. That is, it is unclear if the following estimate holds for all $J_k$:

$$(ds \wedge \hat{\lambda} + \hat{\omega})(v, J_kv) \leq 2\Omega(v, J_kv).$$

Geometrically, the concern is that while the almost complex structures $J_k$ are all $\Omega_1$-tame, they are not $\Omega_1$-compatible. Put another way, although the $J_k$ preserve the splitting $T(\mathcal{I}_\epsilon \times M) = \hat{\rho} \oplus \xi$, and $\hat{\rho}$ and $\xi$ are each symplectic subspaces, these subspaces are not symplectic complements; that is, there exist $v \in \hat{\rho}$ and $w \in \xi$ such that $\Omega(v, w) \neq 0$. That the cross terms might cause an issue is then compounded by the fact that we are degenerating adiabatically so that $\phi_k \to 0$ as $k \to \infty$.

We resolve this issue by making use of the fact that Lemma 2.5 allows us to guarantee that along $\{0\} \times M$ the sub-bundles $\hat{\rho}$ and $\xi$ are $\Omega_1$-symplectic complements. As it will turn out, condition $(\mathcal{J}2)$ can then be achieved by first fixing $\epsilon > 0$ sufficiently small. We will couple this with a convenient means of obtaining a sequence of adiabatically degenerating almost complex structures from some easily verified bounds and some $C^\infty$ converging functions $\phi_k$. This is achieved via the following proposition.

**Proposition 2.9.** (Adiabatically degenerating constructions) Let $(\mathcal{I}_\epsilon \times M, \Omega, \hat{\lambda})$ be a framed Hamiltonian energy pile, and let $\hat{\rho}, \xi$ and $\hat{\omega}$ be the associated structures defined above. Assume, as in the conclusions of Lemma 2.5, that along $\{0\} \times M$ the sub-bundles $\hat{\rho}$ and $\xi$ are symplectic complements and $\hat{\lambda}(X_H) = 1$. Let $\hat{X} = X_H/\hat{\lambda}(X_H)$ as above. Let $\{\hat{J}_k\}_{k \in \mathbb{N}}$ be a sequence of almost complex structures on $\mathcal{I}_\epsilon \times M$ which satisfy the following conditions.

\begin{enumerate}[label=(D\arabic*)]
  \item $\hat{J}_k : \hat{\rho} \to \hat{\rho}$ and $\hat{J}_k : \xi \to \xi$ for each $k \in \mathbb{N}$.
  \item $(ds \wedge \hat{\lambda} + \hat{\omega}) \circ (\text{Id} \times \hat{J}_k)$ is a Riemannian metric for each $k \in \mathbb{N}$.
  \item There exist constants $\{C'_n\}_{n \in \mathbb{N}}$ such that $\sup_{k \in \mathbb{N}} \|\hat{J}_k\|_{C^n} \leq C'_n$.
\end{enumerate}

Then there exists an $\epsilon > 0$ with the following significance. For any sequence of functions $\phi_k : (-\epsilon, \epsilon) \to (0, 1]$ which converge in $C^\infty$ (though not necessarily to zero), the weakly adapted almost complex structures defined by

$$\phi_k : J_k \partial_s = \hat{X} \quad \text{and} \quad J_k|_\xi := \hat{J}_k|_\xi$$

satisfy the following properties.

\begin{enumerate}[label=(E\arabic*)]
  \item For each $v \in T(\mathcal{I}_\epsilon \times M)$ we have
    $$(ds \wedge \hat{\lambda} + \hat{\omega})(v, J_kv) \leq 2\Omega(v, J_kv).$$
\end{enumerate}
Almost existence from the feral perspective and some questions

There exist a sequence \( \{K_n\}_{n=0}^\infty \) of positive constants and an auxiliary translation-invariant metric \( \tilde{g} \) on \( \mathbb{R} \times M \) for which

\[
\sup_{k \in \mathbb{N}} \| (\Psi_k)_* J_k \|_{C^n} \leq K_n
\]

for each \( n \in \mathbb{N} \); here \( \| \cdot \|_{C^n} \) is the \( C^n \) norm on \( \mathbb{R} \times M \) with respect to the auxiliary translation-invariant metric \( \tilde{g} \), and the \( \Psi_k \) are the embeddings as in equation (7).

In particular, if \( \epsilon' \in (0, \epsilon) \) and \( \phi_k|_{(-\epsilon',\epsilon')} \to 0 \), then the almost complex structures \( J_k \) are adiabatically degenerating on \( I_{\epsilon'} \times M \) in the sense of Definition 2.8, and on \( I_{\epsilon} \times M \) these almost complex structures are all tame.

The proof of Proposition 2.9 is elementary but somewhat lengthy, so we relegate it to Appendix A.1; here we confine ourselves to indicating its utility. We start by noting that a natural way to perform an adiabatic degeneration is to start with some fixed almost complex structure \( J \) which preserves the splitting \( \hat{\rho} \oplus \xi \) and for which \( (ds \wedge \hat{\lambda} + \hat{\omega}) \circ (\text{Id} \times J) \) is a Riemannian metric. This almost complex structure will then be \( \Omega \)-compatible along \( \{0\} \times M \), and hence \( \Omega \)-tame in some neighborhood of \( \{0\} \times M \). Consequently, \( J \) can be adjusted away from \( \{0\} \times M \) so that it is \( \Omega \)-tame everywhere and still satisfies our compatibility conditions in a neighborhood of \( \{0\} \times M \). Proposition 2.9 then guarantees that we can find \( \phi_k \) which tend to zero in a neighborhood of zero so that the sequence of almost complex structures given by

\[
\phi_k \cdot J_k \partial_s = \hat{\chi} \quad \text{and} \quad J_k|_{\xi} = J|_{\xi}
\]

are everywhere tame, while also adiabatically degenerating a neighborhood of \( \{0\} \times M \). Among other things, this means

\[
(ds \wedge \hat{\lambda} + \hat{\omega})(v, J_k v) \leq 2\Omega(v, J_k v)
\]

for all \( k \in \mathbb{N} \). The latter condition is important, since by assumption the pseudoholomorphic curves we will study will have uniform bounds on \( \Omega \)-energy, which (as a consequence of the above inequality) will give us bounds on the area associated to the metric \( (ds \wedge \hat{\lambda} + \hat{\omega}) \circ (\text{Id} \times J) \).

With Proposition 2.9 stated and its use outlined, we now turn our attention to proving our main result. This is the topic of the next section.

3. Proof of the main result

In this section we prove Theorem 1.7, which will be accomplished as follows. In light of Lemma 2.5 and Proposition 2.9, we see that the general problem can essentially be reduced to a localized problem involving framed Hamiltonian energy piles and adiabatically degenerating almost complex structures. As such, our first task is to prove Theorem 3.1 below, which is essentially a localized version of Theorem 1.7. The proof of Theorem 3.1 is somewhat technical and will take the bulk of this section. Once this is established, Theorem 1.7 will follow rather quickly. We now proceed with our first main technical argument.

**Theorem 3.1.** (Localized almost existence) Let \( (I_\epsilon \times M, \Omega, \hat{\lambda}) \) be a framed Hamiltonian energy pile, and let \( \hat{\rho}, \xi \) and \( \hat{\omega} \) be the associated structures. Let \( \{J_k\}_{k \in \mathbb{N}} \) be a sequence
of adiabatically degenerating almost complex structures in the sense of Definition 2.8. Suppose that for each \( k \in \mathbb{N} \) there exists a proper pseudoholomorphic map \( u_k : (S_k, j_k) \rightarrow (\mathcal{I}_e \times M, J_k) \) without boundary such that \( s \circ u_k(S_k) = \mathcal{I}_e \), with the property that there exist no connected components of \( S_k \) on which \( u_k \) is the constant map. Suppose further that there exist positive constants \( C_g \) and \( C/\Omega_1 \) for which

\[
\text{Genus}(S_k) \leq C_g \quad \text{and} \quad \int_{S_k} u_k^* \Omega \leq C/\Omega_1.
\]

Then for almost every point \( s \in \mathcal{I}_e \), there exists a periodic orbit of the Hamiltonian vector field \( X_H \) on the energy level \( \{s\} \times M \). That is, the set \( \mathcal{I}_e' \subset \mathcal{I}_e \) of energy levels \( H(s, p) = s \) which contain a Hamiltonian periodic orbit has full measure:

\[
\mu(\mathcal{I}_e') = \mu(\mathcal{I}_e) = 2\epsilon.
\]

In order to begin, we will first need to recall a version of the co-area formula as follows.

**Proposition 3.2.** (The co-area formula) Let \((S, g)\) be an oriented \( C^1 \)-Riemannian manifold of dimension 2; we allow that \( S \) need not be complete\(^\dagger\). Suppose that \( \beta : S \rightarrow [a, b] \subset \mathbb{R} \) is a \( C^1 \) function without critical points. Let \( f : S \rightarrow [0, \infty) \) be a measurable function with respect to \( d\mu_g \). Then

\[
\int_S f \|\nabla \beta\|_g \ d\mu_g^2 = \int_a^b \left( \int_{\beta^{-1}(t)} f \ d\mu_g^1 \right) dt \tag{10}
\]

where \( \nabla \beta \) is the gradient of \( \beta \) computed with respect to the metric \( g \).

**Proof.** This is a well-known result; however, the details of this specific version are provided in [8, Appendix A.2]. \( \square \)

The co-area formula above will be used in a rather particular way, namely as a means of expressing \( \int_S u^*(ds \wedge \hat{\lambda}) \) as a double integral. More precisely, we have the following lemma.

**Lemma 3.3.** (Co-area application) Let \((\mathcal{I}_e \times M, \Omega, \hat{\lambda})\) be a framed Hamiltonian energy pile, and let \( J \) be a weakly adapted almost complex structure in the sense of Definition 2.7. Let \((u, S, j)\) be a \( J \)-holomorphic curve in \( \mathcal{I}_e \times M \) for which \( \partial S = \emptyset \). Then

\[
\int_S u^*(ds \wedge \hat{\lambda}) = \int_{\mathcal{I}_e} \left( \int_{(s \circ u)^{-1}(t) \setminus \mathcal{X}} u^* \hat{\lambda} \right) dt, \tag{11}
\]

where \( \mathcal{X} := \{ \xi \in S : d(s \circ u)\xi = 0 \} \); that is, \( \mathcal{X} \) is the set of critical points of the function \( s \circ u : S \rightarrow \mathcal{I}_e \subset \mathbb{R} \).

**Proof.** Define \( \widetilde{S} := S \setminus \mathcal{X} \), which is a manifold since \( \widetilde{S} \subset S \) is open. Observe that by definition of \( \mathcal{X} \) it follows that \( u^*(ds \wedge \hat{\lambda})|_{\mathcal{X}} \equiv 0 \), so

\[
\int_S u^*(ds \wedge \hat{\lambda}) = \int_{\widetilde{S}} u^*(ds \wedge \hat{\lambda}).
\]

\(^\dagger\) That is, there may exist Cauchy sequences, with respect to \( g \), which do not converge in \( S \).
Since \( u : \tilde{S} \to \mathcal{I}_s \times M \) is an immersion, we may define the metric \( \gamma = u^*g_J \) where \( g_J \) is the Riemannian metric as in equation (6) in Definition 2.7; note that \( J \) is a \( g_J \)-isometry. The almost complex structure \( j \) on \( S \) induces an orientation on \( \tilde{S} \), and hence we have

\[
\int_{\tilde{S}} u^*(ds \wedge \hat{\lambda}) = \int_{\tilde{S}} u^*(ds \wedge \hat{\lambda})(v, \tau) \, d\mu^2_\gamma, \tag{12}
\]

where \((v, \tau)\) is a positively oriented \( \gamma \)-orthonormal frame field, and \( d\mu^2_\gamma \) is the volume form on \( \tilde{S} \) associated to the metric \( \gamma \). This observation is elementary; however, details are provided in [8, Appendix A.2]. Note that equation (12) holds for arbitrary orthonormal frame field \((v, \tau)\). However, we shall henceforth make use of the following particular frame:

\[
v := \frac{\nabla (s \circ u)}{||\nabla (s \circ u)||_\gamma} \quad \text{and} \quad \tau := jv. \tag{13}
\]

Because \( u : S \to \mathcal{I}_s \) is a \( J \)-holomorphic map and \( J \) is a \( g_J \)-isometry, it follows that \( j \) is a \( \gamma \)-isometry. Also note that for each \( v^\xi \in \xi \) and \( a, b \in \mathbb{R} \), we have

\[
J(a\partial_s + b\hat{X} + v^\xi) = \frac{a}{\phi} \hat{X} - \phi b\partial_s + Jv^\xi
\]

with \( Jv^\xi \in \xi \), and hence

\[
ds \circ J = -\phi \hat{\lambda} \quad \text{and} \quad \hat{\lambda} \circ J = \frac{1}{\phi} ds.
\]

With \( v \) and \( \tau \) as in equation (13), it is then straightforward to verify the following:

\[
0 = (u^*\lambda)(v) = u^*ds(\tau);
\]

\[
0 < u^*\left(\frac{1}{\phi} ds\right)(v) = (u^*\hat{\lambda})(\tau);
\]

\[
1 = ||\tau||^2_\gamma = ||v||^2_\gamma.
\]

Also,

\[
||\nabla (s \circ u)||_\gamma = \sup_{x \in \mathcal{T}_s S \atop ||x||_\gamma = 1} d(s \circ u)(x) = \sup_{x \in \mathcal{T}_s S \atop ||x||_\gamma = 1} ds(T u \cdot x) = u^*ds(v), \tag{14}
\]

and

\[
(\phi \circ u) \, ||u^*\hat{\lambda}||_\gamma = u^*(\phi \hat{\lambda})(\tau) = -u^*ds(jjv) = ||\nabla (s \circ u)||_\gamma. \tag{15}
\]

With \((v, \tau)\) defined as such, we have

\[
\int_{\tilde{S}} u^*(ds \wedge \hat{\lambda})(v, \tau) \, d\mu^2_\gamma = \int_{\tilde{S}} ds(T u \cdot v)\hat{\lambda}(T u \cdot \tau) \, d\mu^2_\gamma
\]
\[
= \int_{\tilde{S}} \frac{1}{\phi \circ u} ||\nabla (s \circ u)||^2_\gamma \, d\mu^2_\gamma.
\]
We employ Lemma 3.3 on $\tilde{S}$ with $\beta = s \circ u$, and $f = 1/\phi \circ u \| \nabla (s \circ u) \|_{\gamma}$ to obtain
\[
\int_{\tilde{S}} \frac{1}{\phi \circ u} \| \nabla (s \circ u) \|^2_{\gamma} \, d\mu_{\gamma} = \int_{I_{c}} \left( \int_{(s \circ u)^{-1}(t) \setminus \mathcal{X}} \frac{1}{\phi \circ u} \| \nabla (s \circ u) \|_{\gamma} \, d\mu_{\gamma} \right) \, dt
\]
\[
= \int_{I_{c}} \left( \int_{(s \circ u)^{-1}(t) \setminus \mathcal{X}} \left( u^* \lambda \right) \, d\mu_{\gamma} \right) \, dt
\]
and hence by combining equalities we have
\[
\int_{\tilde{S}} u^* (ds \wedge \lambda) = \int_{I_{c}} \left( \int_{(s \circ u)^{-1}(t) \setminus \mathcal{X}} u^* \lambda \right) \, dt,
\]
which establishes equation (11).

With these preliminaries established, our next main task is to carefully pass to a certain subsequence of our almost complex structures and pseudoholomorphic curves. To that end, we first recall that $\phi_k \cdot J_k \partial_s = \hat{X} = X_H / \hat{\lambda}(X_H)$ where $\{\phi_k\}_{k \in \mathbb{N}}$ is a sequence of positive functions converging to zero in $C^\infty$. Next we define the following intervals. For each $t \in I_{\epsilon}$, and for each sufficiently large $k \in \mathbb{N}$, define the open interval $I(t,k) = (t_0, t_1) = \{a \in \mathbb{R} : t_0 < a < t_1\}$

where
\[
\int_{t}^{t_1} \frac{1}{\phi_k(s)} \, ds = 1 = \int_{0}^{t} \frac{1}{\phi_k(s)} \, ds.
\]
Introduce the map
\[
Sh_c : \mathbb{R} \times M \rightarrow \mathbb{R} \times M
\]
\[
Sh_c(s, p) = (s - c, p).
\]

**Lemma 3.4. (The $\Psi^t_k$ are diffeomorphisms)** We introduce the maps $\Psi^t_k$ defined by
\[
\Psi^t_k = Sh_{\psi_k(t)} \circ \psi_k
\]
where $\psi_k$ and $\psi_k$ are respectively the maps given in equations (7) and (8). Then $\Psi^t_k$ induces a diffeomorphism
\[
\Psi^t_k : I(t, k) \times M \rightarrow (-1, 1) \times M.
\]
of the form $(s, p) \rightarrow (g_k(s), p)$ with $\Psi^t_k(t, p) = (0, p)$.

**Proof.** By definition
\[
\Psi_k(s, p) = \left( \int_{0}^{s} \frac{1}{\phi_k(\tau)} \, d\tau, p \right) = \left( \int_{0}^{t} \frac{1}{\phi_k(\tau)} \, d\tau + \int_{t}^{s} \frac{1}{\phi_k(\tau)} \, d\tau, p \right).
\]
Hence
\[
\Psi_k(s, p) = \left( \psi_k(t) + \int_{t}^{s} \frac{1}{\phi_k(\tau)} \, d\tau, p \right).
\]
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It follows immediately that $\Psi_1^t_k$ has image $(-1, 1) \times M$ and is a diffeomorphism.

As we have seen, when $k$ is large, $\phi_k$ is close to zero, and $\Psi_1^t_k$ then ‘stretches the neck’ to undo the partial degeneration done by $\phi_k$ (via the condition that $\phi_k \cdot J_k \partial_s = \hat{X}$); the shift function, $S_h$, then $\mathbb{R}$-shifts in the target to put the image of $t \in I(t, k)$ at $0 \in (-1, 1)$. More concisely, $I(t, k)$ is an open neighborhood of $t \in I$, which when neck-stretched and shifted becomes the standard interval $(-1, 1)$.

Under the hypotheses of Theorem 3.1 we are given a sequence of pseudoholomorphic curves

$$u_k : (S_k, j_k) \to (I \times M, J_k)$$

where we have a genus bound $C_g$ and a symplectic bound

$$g(S_k) \leq C_g \quad \text{and} \quad \int_{S_k} u_k^* \omega \leq C_\Omega. \tag{17}$$

Note that we obtain from this, because each of the 2-forms $ds \wedge \hat{\lambda}$ and $\hat{\omega}$ evaluates non-negatively on $J_k$-complex lines, the following estimate:

$$\int_{S_k} u_k^*(ds \wedge \hat{\lambda}) + \int_{S_k} u_k^* \hat{\omega} = \int_{S_k} u_k^*(ds \wedge \hat{\lambda} + \hat{\omega}) \leq 2 \int_{S_k} u_k^* \Omega \leq 2C_\Omega. \tag{18}$$

Note that the first inequality follows from the assumption that the $J_k$ are adiabatically degenerating; see Definition 2.8 and, more specifically, condition $J2$. Define $\tilde{C}_\Omega := 2 \cdot C_\Omega$ so that

$$\int_{S_k} u_k^* \hat{\omega} \leq \tilde{C}_\Omega \quad \text{and} \quad \int_{S_k} u_k^*(ds \wedge \hat{\lambda}) \leq \tilde{C}_\Omega. \tag{19}$$

Our goal is the rather careful construction of subsequences with a certain number of good properties. To that end, it will be helpful to recall that if $\{x_k\}_{k \in \mathbb{N}}$ is a sequence of points, then a subsequence can be specified using a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ by writing $\{x_f(k)\}_{k \in \mathbb{N}}$. A further subsequence can be defined using a strictly increasing function $h : \mathbb{N} \to f(\mathbb{N})$ giving $\{x_h(k)\}_{k \in \mathbb{N}}$. The following result is at the heart of our further constructions. We shall write $\mathbb{N}_0$ for the union of $\mathbb{N}$ and $\{0\}$.

**Proposition 3.5.** (The key inductive construction) **Given a sequence of pseudoholomorphic curves**

$$u_k : (S_k, j_k) \to (I \times M, J_k)$$

**satisfying the bound** (17), **there exist a sequence** $(f_m)_{m \in \mathbb{N}_0}$ **of strictly increasing maps** $f_m : \mathbb{N} \to \mathbb{N}$ **and a sequence** $(L_m)_{m \in \mathbb{N}_0}$ **of finite sets** $L_m \subset \text{cl}(I)$ **with the following properties.**

1. $L_0 = \emptyset$ and $f_0(k) = k$ for all $k \in \mathbb{N}$.
2. $L_{m-1} \subset L_m$ for $m \in \mathbb{N}$.
3. $f_m : \mathbb{N} \to f_{m-1}(\mathbb{N})$ and $L_{m-1} \subset L_m$ for $m \in \mathbb{N}$.
(4) For \( m \in \mathbb{N}_0 \), given \( t \in L_m \cap \mathcal{I}_\varepsilon \), there exists a sequence \( (\tau_k) \subset \mathcal{I}_\varepsilon \) converging to \( t \) such that the following limit exists and satisfies the given inequality:
\[
\lim_{k \to \infty} \int_{(s \circ f_m(k))^{-1}(\mathcal{I}(\tau_k, f_m(k)))} u^*_m(k) \hat{\omega} > \frac{\bar{C}_\Omega}{2^m}.
\]

(5) For \( m \in \mathbb{N}_0 \) and given \( t \in \mathcal{I}_\varepsilon \setminus L_m \), there exists no sequence \( \tau_k \to t \) such that
\[
\limsup_{k \to \infty} \int_{(s \circ f_m(k))^{-1}(\mathcal{I}(\tau_k, f_m(k)))} u^*_m(k) \hat{\omega} > \frac{\bar{C}_\Omega}{2^m}.
\]

Proof. We now begin an inductive process of constructing a sequence of nested subsequences. The start of the inductive construction is obvious. We define \( L_0 := \emptyset \) and \( f_0 : \mathbb{N} \to \mathbb{N} \) by \( f_0(k) = k \) for \( k \in \mathbb{N} \). Then item (1) holds and for \( m = 0 \) also item (4) is trivially satisfied since \( L_0 \cap \mathcal{I}_\varepsilon = \emptyset \). Again for \( m = 0 \) we see that \( \mathcal{I}_\varepsilon \setminus L_0 = \mathcal{I}_\varepsilon \). If we take an element \( t \in \mathcal{I}_\varepsilon \), we see that the symplectic bound (19) implies that there is no sequence \( \tau_k \to t \) with the property (5). The statements of items (2) and (3) are only relevant for \( m \geq 1 \). Hence, with the choices we have made all relevant statements hold for \( m = 0 \).

Let us assume that for some \( m \in \mathbb{N} \) we have carried out the constructions of \( L_i \) and \( f_i \) for \( i = 0, \ldots, m-1 \) so that our (relevant) statements in items (1)–(5) hold for \( i = 1, \ldots, m-1 \). We shall now construct \( f_m \) and \( L_m \). Specifically, we note that the construction will be made so that
\[
f_m : \mathbb{N} \to f_{m-1}(\mathbb{N}) \quad \text{and} \quad L_{m-1} \subset L_m.
\]

Since \( \bar{C}_\Omega/2^m < \bar{C}_\Omega/2^{m-1} \) it follows from item (4) (for the case \( m-1 \)) that for every \( t \in L_{m-1} \cap \mathcal{I}_\varepsilon \) there exists a sequence \( \tau_k \to t \) such that the following limit exists and satisfies the given inequality:
\[
\lim_{k \to \infty} \int_{(s \circ f_{m-1}(k))^{-1}(\mathcal{I}(\tau_k, f_{m-1}(k)))} u^*_{m-1}(k) \hat{\omega} > \frac{\bar{C}_\Omega}{2^{m-1}}.
\]

Of course, this remains true if we pass to a subsequence. By the inductive construction we know that for a given \( t \in \mathcal{I}_\varepsilon \setminus L_{m-1} \) there does not exist a sequence \( \tau_k \to t \) with the property
\[
\limsup_{k \to \infty} \int_{(s \circ f_{m-1}(k))^{-1}(\mathcal{I}(\tau_k, f_{m-1}(k)))} u^*_{m-1}(k) \hat{\omega} > \frac{\bar{C}_\Omega}{2^{m-1}}.
\]

However, there might be a sequence for such a \( t \) if we replace the right-hand side by the smaller number \( \bar{C}_\Omega/2^m \). Thus we consider the possible cases. Assume first that for every \( t \in \mathcal{I}_\varepsilon \setminus L_{m-1} \) there is no sequence for which
\[
\limsup_{k \to \infty} \int_{(s \circ f_{m-1}(k))^{-1}(\mathcal{I}(\tau_k, f_{m-1}(k)))} u^*_{m-1}(k) \hat{\omega} > \frac{\bar{C}_\Omega}{2^m}.
\]

holds. In this case we define \( L_m := L_{m-1} \) and \( f_m(k) = f_{m-1}(k) \). Then \( f_m : \mathbb{N} \to f_{m-1}(\mathbb{N}) \) and \( L_{m-1} \subset L_m \). Moreover, items (2)–(5) hold for \( m \) and the construction for the \( m \)-case is complete.
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Assume next we find a \( t_{m,1} \in I \backslash L_{m-1} \) for which we have a sequence \( \tau_k \rightarrow t_{m,1} \) for which (21) holds. Then we take a subsequence \( h_{m,1} : \mathbb{N} \rightarrow f_{m-1}(\mathbb{N}) \) for which

\[
\lim_{k \rightarrow \infty} \int_{(s \circ u_{h_{m,1}(k)})^{-1}(\mathcal{I}(\tau_{m,1}(k), h_{m,1}(k)))} u_{h_{m,1}(k)}^* \hat{\omega} > \frac{\bar{C}_\Omega}{2^m}.
\]

(22)

Next we can ask if we find a \( t_{m,2} \in I \backslash (L_{m-1} \cup \{t_{m,1}\}) \) for which we have a sequence \( \tau_k \rightarrow t_{m,2} \) such that

\[
\limsup_{k \rightarrow \infty} \int_{(s \circ u_{h_{m,2}(k)})^{-1}(\mathcal{I}(\tau_k, h_{m,1}(k)))} u_{h_{m,2}(k)}^* \hat{\omega} > \frac{\bar{C}_\Omega}{2^m}.
\]

If that is not the case we define \( f_m(k) = h_{m,1}(k) \) and \( L_m = L_{m-1} \cup \{t_{m,1}\} \). One easily verifies items (2)–(5) and the construction for the \( m \)-case is complete.

Otherwise we find a \( t_{m,2} \in I \backslash (L_{m-1} \cup \{t_{m,1}\}) \) with the above-mentioned property and we can take \( h_{m,2} : \mathbb{N} \rightarrow h_{m,1}(\mathbb{N}) \) so that the left-hand limit exists and is greater than \( \frac{\bar{C}_\Omega}{2^m} \). Again we ask if we find a third point \( t_{m,3} \in L_{m-1} \cup \{t_{m,1}, t_{m,2}\} \) with the same property. If that is not the case we define \( L_m = L_{m-1} \cup \{t_{m,1}, t_{m,2}\} \) and \( f_m(k) = h_{m,2}(k) \). Again one verifies items (2)–(5). Otherwise we obtain a sequence \( \tau_k \rightarrow t_{m,3} \) and \( h_{m,3} : \mathbb{N} \rightarrow h_{m,2}(\mathbb{N}) \) such that

\[
\limsup_{k \rightarrow \infty} \int_{(s \circ u_{h_{m,3}(k)})^{-1}(\mathcal{I}(\tau_k, h_{m,3}(k)))} u_{h_{m,3}(k)}^* \hat{\omega} > \frac{\bar{C}_\Omega}{2^m}.
\]

(23)

In the outline above we consider several cases and in some of them the procedure terminates after a finite number of steps. The only possible way for the procedure not to stop is that we find more and more points \( t_{m,n}, n \in \mathbb{N} \), with the previously described properties. However, we claim that this procedure terminates after a finite number of steps \( n_m \) so that we can define \( L_m = L_{m-1} \cup \{t_{m,1}, \ldots, t_{m,n_m}\} \) and \( f_m(k) = h_{m,n_m}(k) \) which satisfies by construction items (2)–(5). To see that the procedure terminates, assume otherwise. Pick a positive integer \( N \) such that

\[
N \cdot 2^{-m} > 1
\]

and consider the \( N \) different points \( t_{m,1}, \ldots, t_{m,N} \in I \backslash L_{m-1} \). By construction we have the maps

\[
\begin{align*}
    h_{m,1} : \mathbb{N} &\rightarrow f_{m-1}(\mathbb{N}), \\
    h_{m,2} : \mathbb{N} &\rightarrow h_{m,1}(\mathbb{N}), \\
    & \vdots \\
    h_{m,N} : \mathbb{N} &\rightarrow h_{m,N-1}(\mathbb{N}),
\end{align*}
\]

with the properties that for \( t_{m,n}, n \in \{1, \ldots, N\} \), there exists a sequence \( \tau_{m,n}^k \rightarrow t_{m,n} \) such that

\[
\lim_{k \rightarrow \infty} \int_{(s \circ u_{h_{m,n}(k)})^{-1}(\mathcal{I}(\tau_{m,n}^k, h_{m,n}(k)))} u_{h_{m,n}^*(k)} \hat{\omega} > \frac{\bar{C}_\Omega}{2^m}.
\]

(23)
Using (23), we obtain with \(H(k) := h_{m,N}(k)\) and the sequences \(\tau^n_k \to t_n\) for \(n = 1, \ldots, N\),

\[
\tilde{C}_\Omega \geq \limsup_{k \to \infty} \int_{S_k} u^*_H(k) \hat{\omega} \\
\geq \sum_{n=1}^N \lim_{k \to \infty} \int_{(s \circ u_H(k))^{-1}(I(\tau^n_k, H(k)))} u^*_H(k) \hat{\omega} \\
\geq N \cdot 2^{-m} \cdot \bar{C}_\Omega > \bar{C}_\Omega.
\]

Observe that we use that the sets \(I(\tau^n_k, H(k))\) are mutually disjoint for large \(k\). Indeed, since \(\tau^n_k \to t_{m,n}\) as \(k \to \infty\) and the points \(t_{m,n}\) are mutually disjoint, this follows from the fact that the diameter of the intervals \(I(\tau^n_k, H(k))\) is shrinking to 0. Thus we have shown that for each fixed \(m\), the procedure which generates the set \(\{t_{m,1}, t_{m,2}, \ldots\}\) terminates after a finite number of iterations, and hence this set is finite. This completes the proof of Proposition 3.5.

With these subsequences established, we now pass to a diagonal subsequence by defining

\[
k_m := f_m(m) \quad \text{for } m \in \mathbb{N}.
\]

We also define the countable subset

\[
L := \bigcup_{m \in \mathbb{N}} L_m.
\]

The following result is then an immediate consequence of the above construction.

**Lemma 3.6. (Vanishing horizontal area)** Let the \((I_\epsilon \times M, \Omega, \hat{\lambda})\), \(\{J_k\}_{k \in \mathbb{N}}\) and \(u_k : (S_k, j_k) \to (I_\epsilon \times M, J_k)\) be as above, and let \(\{k_m\}_{m \in \mathbb{N}}\) be the subsequence given in equation (24). Then for each \(t_0 \in I_\epsilon \setminus L\), we have

\[
\lim_{m \to \infty} \int_{(s \circ u_{k_m})^{-1}(I(t_0, k_m))} u^*_{k_m} \hat{\omega} = 0.
\]

**Proof.** Since \(t_0 \in I_\epsilon \setminus L\), it follows that \(t_0 \notin L_m\) for every \(m \in \mathbb{N}_0\). Hence there does not exist a sequence \(\tau_k \to t_0\) with

\[
\limsup_{k \to \infty} \int_{(s \circ u_{f_m(k)})^{-1}(I(\tau_k, f_m(k)))} u^*_{f_m(k)} \hat{\omega} \geq \frac{\tilde{C}_\Omega}{2^m},
\]

and specifically we must have

\[
\limsup_{k \to \infty} \int_{(s \circ u_{f_m(k)})^{-1}(I(t_0, f_m(k)))} u^*_{f_m(k)} \hat{\omega} \leq \frac{\tilde{C}_\Omega}{2^m}.
\]
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Since \( k_m := f_m(m) \) is the diagonal sequence we deduce from (26) that for every \( m \in \mathbb{N}_0 \),
\[
\limsup_{i \to \infty} \int_{(s \circ u_{k_i})^{-1}(\mathcal{I}(t_0, k_i))} u_{k_i}^* \hat{\omega} \leq \frac{\tilde{C} \Omega}{2^m}.
\] (27)
This implies the assertion
\[
\lim_{m \to \infty} \int_{(s \circ u_{k_m})^{-1}(\mathcal{I}(t_0, k_m))} u_{k_m}^* \hat{\omega} = 0.
\]

We will next make use of this subsequence, and with it we will be interested in the \( \hat{\lambda} \) integrals of our curves along various levels \( \{t\} \times M \). This is made precise in equation (29) below; however, for the moment we note that we will be interested in various properties of these functions, (e.g. that they are measurable and integrable). Indeed, studying properties and limits of such functions will ultimately yield the desired result regarding existence of periodic orbits on almost every energy level. We now proceed with this argument.

For each \( m \in \mathbb{N} \) we define \( \mathcal{Y}_m \) to be the set of critical values of the functions \( s \circ u_{k_m} : S_{k_m} \to \mathcal{I}_\epsilon \subset \mathbb{R} \). By Sard’s theorem, each of these sets has measure zero; that is, \( \mu(\mathcal{Y}_m) = 0 \) for each \( m \in \mathbb{N} \). By countable sub-additivity, we then also have
\[
\mu(\mathcal{Y}) = 0 \quad \text{where} \quad \mathcal{Y} := \bigcup_{m \in \mathbb{N}} \mathcal{Y}_m.
\] (28)

For each \( m \in \mathbb{N} \) we then define the functions
\[
F_m : \mathcal{I}_\epsilon \to [0, \infty) \subset \mathbb{R}
\] (29)

\[
F_m(t) = \begin{cases} \int_{(s \circ u_{k_m})^{-1}(t)} u_{k_m}^* \hat{\lambda} & \text{if } t \in \mathcal{I}_\epsilon \setminus \mathcal{Y}, \\ 0 & \text{otherwise}. \end{cases}
\]

We now make the following claim.

**Lemma 3.7.** \((F_m \text{ is measurable})\) For each \( m \in \mathbb{N} \), the function \( F_m \) defined above is a measurable function. Moreover, for each \( m \in \mathbb{N} \), the function \( F_m \) agrees with the function
\[
t \mapsto \int_{(s \circ u_{k_m})^{-1}(t) \setminus \chi_m} u_{k_m}^* \hat{\lambda},
\] (30)
augmented everywhere; here \( \chi_m = \{ \xi \in S : d(s \circ u_{k_m})\xi = 0 \} \) as in Lemma 3.3.

**Proof.** First fix \( m \in \mathbb{N} \) and define
\[
\tilde{F}_m : \mathcal{I}_\epsilon \to [0, \infty) \subset \mathbb{R}
\]

\[
\tilde{F}_m(t) = \begin{cases} \int_{(s \circ u_{k_m})^{-1}(t)} u_{k_m}^* \hat{\lambda} & \text{if } t \in \mathcal{I} \setminus \mathcal{Y}_m, \\ 0 & \text{otherwise}. \end{cases}
\]

Observe that \( F_m \) and \( \tilde{F}_m \) agree almost everywhere, and consequently if \( \tilde{F}_m \) is measurable so is \( F_m \). Because \( s \circ u_{k_m}(\chi_m) = \mathcal{Y}_m \) which has measure zero, it follows that \( \tilde{F}_m \) is almost everywhere equal to the function defined in equation (30). This establishes the second part of the lemma.
To establish the first part of the lemma it is sufficient to show that $\tilde{F}_m$ is measurable. That is, it is sufficient to show that for each $r \in [0, \infty)$, the set $\tilde{F}_m^{-1}([0, r))$ is measurable. To that end, note that by assumption in Theorem 3.1 we have $s \circ u_{km}(S_{km}) = \mathcal{I}_e$, and consequently for each $s_0 \in \mathcal{I}_e \setminus \mathcal{Y}_m$ we have

$$\tilde{F}_m(s_0) = \int_{(s \circ u_{km})^{-1}(s_0)} u_{km}^* \hat{\lambda} > 0.$$ 

It also follows that $\tilde{F}_m^{-1}(0) = \mathcal{Y}_m$. Note that $\mathcal{Y}_m$ is closed in $\mathcal{I}_e$. Also note that $\tilde{F}_m|_{\mathcal{I}_e \setminus \mathcal{Y}_m}$ is differentiable, and hence continuous, and thus

$$A_r := (\tilde{F}_m|_{\mathcal{I}_e \setminus \mathcal{Y}_m})^{-1}([0, r)) \text{ is open in } \mathcal{I}_e \setminus \mathcal{Y}_m.$$ 

That is, there exists an open set $O \subset \mathcal{I}_e$ such that $A_r = O \cap (\mathcal{I}_e \setminus \mathcal{Y}_m)$. However, $\mathcal{I}_e \setminus \mathcal{Y}_m$ is open in $\mathcal{I}_e$ and hence $A_r$ is open in $\mathcal{I}_e$. Consequently, $A_r$ is measurable. However, we then have

$$\tilde{F}_m^{-1}([0, r)) = (\tilde{F}_m|_{\mathcal{I}_e \setminus \mathcal{Y}_m})^{-1}([0, r)) \cup (\tilde{F}_m|_{\mathcal{Y}_m})^{-1}([0, r))$$

$$= A_r \cup \mathcal{Y}_m,$$

with $A_r$ open (and hence measurable) and $\mathcal{Y}_m$ having measure zero (and hence being measurable). We conclude that $\tilde{F}_m^{-1}([0, r))$ is measurable, which completes the proof of Lemma 3.7. \hfill \Box

For the following discussion we introduce the map $F$ which is defined as follows:

$$F : \mathcal{I}_e \to \mathbb{R}^+ \cup \{\infty\}$$

$$F(s) = \lim_{m \to \infty} \inf F_m(s).$$

(31)

Then by standard measure theory results $F$ is an extended measurable function; see [3]. With this definition in place the next guiding observation (made rigorous below) is that two important results hold. The first one is given in the next proposition.

**Proposition 3.8.** (F is almost everywhere finite) With $F$ as just defined in (31), we have that

$$\text{measure}(\{s \in \mathcal{I}_e : F(s) = \infty\}) = 0.$$

**Proof.** In view of (11), we have the formula

$$\int_{S_{km}} u_{km}^* (ds \wedge \hat{\lambda}) = \int_{\mathcal{I}_e} \left( \int_{(s \circ u_{km})^{-1}(s)} u_{km}^* \hat{\lambda} \right) ds.$$

Using (18) and (19), we infer that

$$2 \cdot C_\Omega = \tilde{C}_\Omega \geq \int_{S_{km}} u_{km}^* (ds \wedge \hat{\lambda}) = \int_{\mathcal{I}_e} F_m(s) ds.$$
In view of Fatou’s lemma, recalling that \( F = \lim \inf F_m \), we obtain

\[
\bar{C}_\Omega \geq \lim \inf_{m \to \infty} \int_{\mathcal{I}_c} F_m(s) \, ds \geq \int_{\mathcal{I}_c} F(s) \, ds.
\]

This shows that \( \{ s \in \mathcal{I}_c : F(s) = \infty \} \) has vanishing measure. 

The second result, which is more substantial, studies the points \( s \) satisfying \( F(s) < \infty \). We will establish that there is a periodic orbit on \( \{ s \} \times M \) provided \( F(s) < \infty \) and \( s \in \mathcal{I}_c \setminus (L \cup Y) \). The proof of Theorem 3.1 then follows immediately since we have just established that \( \{ s \in \mathcal{I}_c : F(s) < \infty \} \) has (full measure) \( 2\varepsilon \) and \( L \cup Y \) has measure zero.

**Proposition 3.9.** (Bounded \( F \) yields periodic orbit) Let \( (\mathcal{I}_c \times M, \Omega, \hat{\lambda}), \{ J_k \}_{k \in \mathbb{N}}, \) and \( u_k : (\mathcal{I}_c, j_k) \to (\mathcal{I}_c \times M, J_k) \) be as in the hypotheses of Theorem 3.1. Let \( L \) be defined as in equation (25), \( Y \) be as defined in equation (28), and let \( F_m \) be the functions defined in equation (29). Also, let \( s_0 \in \mathcal{I}_c \setminus (L \cup Y) \), and suppose

\[
F(s_0) = \lim \inf_{m \to \infty} F_m(s_0) = \lim_{N \to \infty} \inf_{m \geq N} F_m(s_0) < \infty.
\]

Then there exists a periodic orbit on the energy level \( \{ s_0 \} \times M \).

**Proof.** We begin by passing to a subsequence (still denoted with subscripts \( m \)) so that

\[
\lim_{N \to \infty} \inf_{m \geq N} F_m(s_0) = \lim_{m \to \infty} F_m(s_0) = \lim_{m \to \infty} \int_{(\sigma \circ u_{km})^{-1}(s_0)} u_{km}^* \hat{\lambda} =: C\hat{\lambda} < \infty.
\]

Let \( \Psi_{km}^{s_0} : \mathcal{I}_c \times M \to \mathbb{R} \times M \) be the embedding provided in equation (16), whose restriction \( \tilde{\Psi}_m \) to \( \mathcal{I}(s_0, k_m) \times M \) given by

\[
\tilde{\Psi}_m : \mathcal{I}(s_0, k_m) \times M \to (-1, 1) \times M
\]

is a diffeomorphism. The key feature is that \( \tilde{\Psi}_m \) maps \( s_0 \) to 0 and stretches out the shrinking intervals to length 2. Define

\[
\tilde{S}_m := (s \circ u_{km})^{-1}(\mathcal{I}(s_0, k_m)) \quad \text{and} \quad \tilde{j}_m := j_{km} |_{\tilde{S}_m},
\]

and

\[
\tilde{u}_m : \tilde{S}_m \to (-1, 1) \times M
\]

\[
\tilde{u}_m = \tilde{\Psi}_m \circ u_{km}.
\]

Also, with \( a \) the coordinate on \( (-1, 1) \), define the following structures on \( (-1, 1) \times M \):

\[
\tilde{j}_m := (\tilde{\Psi}_m)_* J_{km};
\]

\[
\tilde{\lambda} := (\tilde{\Psi}_m)_* \hat{\lambda} \quad \text{(note that there is no } m \text{-dependence on the left-hand side!)};
\]

\[
\tilde{\omega}_m := (\tilde{\Psi}_m)_* \tilde{\omega};
\]

\[
g_m := (da \wedge \tilde{\lambda} + \tilde{\omega}_m) \circ (\text{Id} \times \tilde{j}_m) = da^2 + \tilde{\lambda}^2 + \tilde{\omega}_m \circ (\text{Id} \times \tilde{j}_m).
\]

We now make several observations which follow immediately from our construction.
The maps $\tilde{u}_m : (\tilde{S}_m, \tilde{j}_m) \to (\mathbb{R} \times (1, 1) \times M)$ are proper pseudoholomorphic maps without boundary and lack constant components.

(2) Genus($\tilde{S}_m$) $\leq C_g$.

(3) $u^{-1} \circ \tilde{u}_m (0) \neq \emptyset$.

(4) $\int_{(a \circ \tilde{u}_m)^{-1}(0)} \tilde{u}_m^* \hat{\lambda} \to C_{\hat{\lambda}} < \infty$.

(5) $(-1, 1) \times M$, equipped with any of the $(\hat{\lambda}, \tilde{\omega}_m)$, is a realized Hamiltonian homotopy in the sense of [8, Definition 2.9], with adapted almost Hermitian structures $(\tilde{J}_m, \tilde{g}_m)$, recalled in Definition A.3 in the Appendix.

We then pass to a subsequence (still denoted with subscripts $m$) of all sufficiently large $m$ to pass to a further subsequence (still denoted with subscripts $m$) to a (nodal) limit pseudoholomorphic map $\tilde{u}_m (\tilde{S}_m \setminus \tilde{\Sigma}_m) \subset (\tilde{\Sigma}_m, \tilde{j}_m)$ with smooth boundary for which

$$\tilde{u}_m (\tilde{S}_m \setminus \tilde{\Sigma}_m) \subset ((-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)) \times M$$

for all $m \in \mathbb{N}$,

(32)

and for which the sequence $\tilde{u}_m : \tilde{\Sigma}_m \to (-1, 1) \times M$ converges in a Gromov sense to a (nodal) limit pseudoholomorphic map $\tilde{u}_\infty : (\tilde{\Sigma}_\infty, \tilde{j}_\infty) \to ((-1, 1) \times M, \tilde{J}_\infty)$; here $(\tilde{\Sigma}_\infty, \tilde{j}_\infty)$ is compact and may have smooth boundary. Note that because $(a \circ \tilde{u}_m)^{-1}(0) \neq \emptyset$ for all $m \in \mathbb{N}$, it follows from equation (32) that $\tilde{\Sigma}_\infty \neq \emptyset$, and if $\partial \tilde{\Sigma}_\infty \neq \emptyset$, then $\tilde{u}_\infty (\partial \tilde{\Sigma}_\infty) \subset ((-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)) \times M$. Because $s_0 \in (\mathbb{R} \setminus (L \cup \mathcal{X}))$ (specifically because $s_0 \notin L$), we must have

$$\int_{\tilde{\Sigma}_m} \tilde{u}_m^* \tilde{\omega}_m \to 0 \quad \text{and} \quad \int_{\tilde{\Sigma}_m} \tilde{u}_m^* \tilde{\omega}_m \to \int_{\tilde{\Sigma}_\infty} \tilde{u}_\infty^* \tilde{\omega}_\infty$$

so

$$\int_{\tilde{\Sigma}_\infty} \tilde{u}_\infty^* \tilde{\omega}_\infty = 0.$$ 

We now make the following claim.

**LEMMA 3.10.** (Non-trivial limit component with boundary) There exists a connected component $\tilde{\Sigma}'$ of $\tilde{\Sigma}_\infty$ for which $\tilde{\Sigma}' \cap (a \circ \tilde{u}_\infty)^{-1}(0) \neq \emptyset$ and $\partial \tilde{\Sigma}' \neq \emptyset$. □

**Proof.** Suppose not. For notational clarity, we let $S_0$ denote the set of connected components of $\tilde{\Sigma}_\infty$ which have non-empty intersection with $(a \circ \tilde{u}_\infty)^{-1}(0)$. Then there are three possibilities.
Case I. $S_0 = \emptyset$.
In this case, $\tilde{\Sigma}_{\infty}$ can be written as the disjoint union $\tilde{\Sigma}_{\infty} = \tilde{\Sigma}_+^\infty \cup \tilde{\Sigma}_-^\infty$ where

$$a \circ \tilde{u}_\infty(\tilde{\Sigma}_1^\infty) \subset (0, 1) \quad \text{and} \quad a \circ \tilde{u}_\infty(\tilde{\Sigma}_-^\infty) \subset (-1, 0).$$

If either of $\tilde{\Sigma}_\pm^\infty$ are empty, then by Gromov convergence there must be some large $m \in \mathbb{N}$ and some number $a_0 \in (-\frac{1}{2}, \frac{1}{2})$ for which $(a \circ \tilde{u}_m)^{-1}(a_0) = \emptyset$, and hence there exists a $s_0 \in \mathcal{I}_\epsilon$ (specifically $s_0 = \psi_{k_m}^{-1}(a_0)$) for which $(s \circ u_{k_m})^{-1}(s_0) = \emptyset$. However, this violates the assumption in Theorem 3.1 which states that $s \circ u_k(S_k) = \mathcal{I}_\epsilon$ for all $k$. Consequently, $\tilde{\Sigma}_+^\infty \neq \emptyset$ and $\tilde{\Sigma}_-^\infty \neq \emptyset$.

Next note that because $\tilde{\Sigma}_{\infty}$ is compact, it follows that each of $\tilde{\Sigma}_\pm^\infty$ is compact and non-empty. However, we then have

$$\sup_{z \in \tilde{\Sigma}_{\infty}} a \circ \tilde{u}_\infty(z) = a_- < 0 < a_+ = \inf_{z \in \tilde{\Sigma}_{\infty}} a \circ \tilde{u}_\infty(z).$$

But then again, by Gromov convergence, this will violate the assumption that $s \circ u_k(S_k) = \mathcal{I}_\epsilon$ for all $k$. Thus case I is impossible.

Case II. For each $\tilde{\Sigma}' \in S_0$, the restriction $\tilde{u}_\infty|_{\tilde{\Sigma}'}$ is the constant map.
In this case, we can write $\tilde{\Sigma}_{\infty}$ as the disjoint union $\tilde{\Sigma}_{\infty} = \tilde{\Sigma}_+^\infty \cup \tilde{\Sigma}_0^\infty \cup \tilde{\Sigma}_-^\infty$ where

$$a \circ \tilde{u}_\infty(\tilde{\Sigma}_1^\infty) \subset (0, 1), \quad a \circ \tilde{u}_\infty(\tilde{\Sigma}_0^\infty) = \{0\}, \quad \text{and} \quad a \circ \tilde{u}_\infty(\tilde{\Sigma}_-^\infty) \subset (-1, 0).$$

The argument then proceeds as in case I, which shows that case II is also impossible.

Case III. There exists $\tilde{\Sigma}' \in S_0$, such that the restriction $\tilde{u}_\infty|_{\tilde{\Sigma}'}$ is not constant.
Note that by the contradiction hypothesis we must have $\partial \tilde{\Sigma}' = \emptyset$, and by Gromov convergence $\tilde{\Sigma}'$ is compact. That is, $(\tilde{\Sigma}', \tilde{f}_\infty)$ is a closed Riemann surface. Next, we make use of the fact that

$$\int_{\tilde{\Sigma}_{\infty}} \tilde{u}_\infty^* \tilde{\omega}_{s_0} = 0$$

together with the fact that $\tilde{\omega}_{s_0}$ evaluates non-negatively on $\tilde{f}_\infty$-complex lines to conclude that for each $z \in \tilde{\Sigma}'$ we must have

$$\text{Image}(T_z \tilde{u}_\infty) \subset \ker(\tilde{\omega}_{s_0}) \tilde{u}_\infty(z) = \text{Span}(\partial_u, X(\tilde{u}_\infty(z))),$$

where $X$ is the Hamiltonian vector field $X(a, p) = \tilde{X}(s_0, p)$. Consequently, there exists a Hamiltonian trajectory $\gamma : \mathbb{R} \to \{0\} \times M$, solving $\gamma'(t) = X(\gamma(t))$ for all $t$ for which

$$\tilde{u}_\infty(\tilde{\Sigma}') \subset (-1, 1) \times \gamma(\mathbb{R}).$$

If $\gamma$ is not periodic, then the map

$$\Phi : (-1, 1) \times \mathbb{R} \to \mathbb{R} \times M \quad \Phi(s, t) = (s, \gamma(t))$$
is an injective pseudoholomorphic immersion, and hence the map
\[ \Phi^{-1} \circ \tilde{u} \infty : \tilde{\Sigma}' \to (-1, 1) \times \mathbb{R} \subset \mathbb{C} \]
is a holomorphic map from a closed Riemann surface into \( \mathbb{C} \). By the maximum principle, and the fact that \( \Phi \) is an immersion, it follows that \( \tilde{u} \infty : \tilde{\Sigma}' \to (-1, 1) \times M \) is a constant map, but this contradicts the assumption defining case III.

The case in which \( \gamma \) is a periodic orbit is treated similarly by holomorphically parameterizing \((-1, 1) \times \gamma(S^1)\) by an annulus in \( \mathbb{C} \). Again the maximum principle applies and we conclude that \( \tilde{u} \infty : \tilde{\Sigma}' \to (-1, 1) \times M \) is a constant map, which is impossible. We conclude that case III is impossible.

All cases are impossible, and hence this completes the proof by contradiction of Lemma 3.10.

With Lemma 3.10 established, we now observe that there exists a connected component \( \tilde{\Sigma}' \subset \tilde{\Sigma} \) for which \( \tilde{u} \infty : \tilde{\Sigma}' \to (-1, 1) \times M \) is a non-constant holomorphic map from a connected compact Riemann surface \( \tilde{\Sigma}' \) with non-empty boundary into \( \mathbb{C} \). Moreover, this holomorphic map satisfies the following two conditions:

1. \( \tilde{u} \infty : \tilde{\Sigma}' \to (-1, 1) \times \mathbb{R} \subset \mathbb{C} \)
2. \( \tilde{u}^{-1}([0] \times \mathbb{R}) \neq \emptyset \).

However, by the maximum principle, this is impossible. We conclude that \( \gamma \) must be a periodic trajectory of the Hamiltonian vector field \( X \) which satisfies
\[ \hat{\lambda}(X) = 1 \quad \text{and} \quad i_X \omega^{\infty} = 0. \]

Or in other words, for the symplectic manifold \((\mathcal{I}_e \times M, \Omega)\), and Hamiltonian function \( H(s, p) = s \), there exists a periodic Hamiltonian orbit on energy level \( \{s_0\} \times M \). This completes the proof of Proposition 3.9.

Let us summarize the already established facts involving the map \( F : \mathcal{I}_e \to [0, +\infty] \).
Recall that \( F \) has been given as \( F(s) := \lim \inf_{m \to \infty} F_m \), where the \( F_m \) have been
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previously defined in (30) by

\[
F_m : \mathcal{I}_{\varepsilon} \to [0, \infty) \subset \mathbb{R}
\]

\[
F_m(t) = \begin{cases} 
\int_{(s_{out})^{-1}(t)} u_{k_m}^* \dot{\lambda} & \text{if } t \in \mathcal{I}_{\varepsilon} \setminus \mathcal{Y}, \\
0 & \text{otherwise.}
\end{cases}
\]

The \( \mathcal{Y} \) has been defined in (28) and we have shown the following.

- The set \( \{ s \in \mathcal{I}_{\varepsilon} : F(s) = \infty \} \) has measure zero.
- The set \( L \) is countable and the set \( \mathcal{Y} \) has measure zero.
- If \( s \in \{ \mathcal{I}_{\varepsilon} : F(s) < \infty \} \setminus (L \cup \mathcal{Y}) \) then \( \{ s \} \times M \) contains a periodic orbit.

We are now in a position to prove the following theorem.

**THEOREM 3.11. (Main Result)** Let \((W, \Omega)\) be a symplectic manifold without boundary, and let \(H : W \to \mathbb{R}\) be a smooth proper Hamiltonian. Fix \(E_- < E_+ \in H(W) \subset \mathbb{R}\) with \(E_- < E_+\), as well as positive constants \(C_g > 0\) and \(C_\Omega > 0\). Suppose that for each \(\Omega\)-tame almost complex structure \(J\) on \(W\) there exists a proper pseudoholomorphic map

\[
u : (S, j) \to \{ p \in W : E_- < H(p) < E_+ \}
\]

without boundary, which also satisfies the following conditions.

1. **(genus and area bounds)** The following inequalities hold:

\[
\text{Genus}(S) \leq C_g \quad \text{and} \quad \int_S u^* \Omega \leq C_\Omega.
\]

2. **(energy surjectivity)** The map \(H \circ u : S \to (E_-, E_+)\) is surjective.

Then there is a periodic Hamiltonian orbit on almost every energy level in the range \((E_-, E_+)\). That is, if we let \(\mathcal{I} \subset (E_-, E_+)\) denote the energy levels of \(H\) which contain a Hamiltonian periodic orbit, then \(\mathcal{I}\) has full measure:

\[
\mu(\mathcal{I}) = \mu((E_-, E_+)) = E_+ - E_-.
\]

**Proof.** In order to prove this result, we will make use of our localization results, but first we need to properly reframe the problem. For notational convenience we begin by defining

\[
\tilde{W} = \{ p \in W : E_- < H(p) < E_+ \}.
\]

Next, for each \(c \in \mathbb{R}\), we define the function

\[
H^c : \tilde{W} \to \mathbb{R}
\]

\[
H^c(q) = H(q) - c
\]

and observe that \(H\) and \(H^c\) generate identical Hamiltonian vector fields on \(\tilde{W}\). Consequently, \(\gamma : \mathbb{R} \to \tilde{W}\) is a Hamiltonian periodic orbit of \(H\) if and only if it is a Hamiltonian periodic orbit of \(H^c\). Next we make the following claim.

\[\dagger\] By ‘proper’ here, we mean that for each compact set \(\mathcal{K} \subset \mathbb{R}\), the set \(H^{-1}(\mathcal{K})\) is compact.
Claim. To prove Theorem 1.7, it is sufficient to show that for each regular value \( c \in (E_-, E_+) \) of \( H \), there exists a \( \delta = \delta(c) > 0 \) such that the set of energy levels \( \{|H^c| < \delta\} \) containing a Hamiltonian periodic orbit has measure \( 2\delta \).

To see that this claim is true, we first consider the case that \( H \) has no critical points in \( \tilde{W} \). In this case, every \( c \in (E_-, E_+) \) is a regular energy value, and thus for each such \( c \) we define \( \delta_c = \delta(c) \) so that the set of energy levels \( \{|H^c| < \delta_c\} \) containing a Hamiltonian periodic orbit has measure \( 2\delta_c \). It follows that \( \{(c - \delta_c, c + \delta_c)\}_{E_- < c < E_+} \) is an open cover of \( (E_-, E_+) \). Using the fact that this is an open cover, together with the fact that the open interval \( (E_-, E_+) \) can be written as the countable union of compact intervals, for example

\[
(E_-, E_) = \bigcup_{\mathcal{I} \in \mathcal{E}} \mathcal{I} \quad \text{where} \quad \mathcal{E} = \left\{ \left[ E_- + \frac{1}{n} L, E_+ - \frac{1}{n} L \right] \right\}_{n \in \mathbb{N}} \quad \text{and} \quad L = \frac{E_+ - E_-}{4},
\]

it follows that there exists a countable set \( \{c_i\}_{i \in \mathbb{N}} \) such that

\[
(E_-, E_) = \bigcup_{i \in \mathbb{N}} (c_i - \delta_{c_i}, c_i + \delta_{c_i}).
\]

That the set of energy levels in \( (E_-, E_+) \) has measure \( E_+ - E_- \) then follows essentially from countable additivity. More specifically, if \( \Xi \) denotes those energy levels in \( (E_-, E_+) \) which have a periodic orbit, and \( A_i = (c_i - \delta_{c_i}, c_i + \delta_{c_i}) \), and \( B_n = B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i \), then

\[
\mu(\Xi) = \mu\left( \Xi \cap \bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} \mu(\Xi \cap B_n) = \sum_{n=1}^{\infty} \mu(B_n) = \mu\left( \bigcup_{n=1}^{\infty} B_n \right) = E_+ - E_-
\]

The case that \( H : \tilde{W} \to \mathbb{R} \) has critical points is treated similarly, by first observing that the set of critical values of \( H \) is closed and has measure zero. Thus the set of regular values of \( H \) has full measure and can be written as the disjoint union of open intervals. The claim is then established by another application of countable additivity.

With the claim established, we can now apply Lemma 2.5 to the Hamiltonian \( H^c : \tilde{W} \to \mathbb{R} \) for each regular value \( c \) of \( H \). For each such \( c \), this establishes a framed Hamiltonian energy pile \( (\mathcal{I}^c \times M, \Omega^c, \hat{\lambda}^c) \), and a diffeomorphism

\[
\Phi^c : \mathcal{I}^c \times M \to \{|H^c| < \epsilon^c\}
\]

for which \( H^c \circ \Phi^c(s, p) = s \) and \( (\Phi^c)^* \Omega = \Omega^c \). Moreover, we obtain associated structures \( \hat{\rho}^c, \xi^c \) and \( \hat{\omega}^c \) on \( \mathcal{I}^c \times M \), and Lemma 2.5 also guarantees that along \( \{0\} \times M \) we have \( \hat{\lambda}^c(X_{H^c}) = 1 \) and also along \( \{0\} \times M \) the sub-bundles \( \hat{\rho}^c \) and \( \xi^c \) are symplectic complements. We then observe that in light of the above claim we have just established, it follows that in order to complete the proof of Theorem 1.7 it is sufficient to show that for each such \( c \), there exists a \( \delta^c > 0 \) such that almost every energy level of the framed Hamiltonian energy pile \( (\mathcal{I}^c \times M, \Omega^c, \hat{\lambda}^c) \) has a Hamiltonian periodic orbit.

To find such a \( \delta^c > 0 \), the aim will be to apply Proposition 2.9. However, to do that we must first have at our disposal a sequence of suitable almost complex structures. To construct these, we start by defining an almost complex \( J^c \) on \( \mathcal{I}^c \times M \) by requiring that \( J^c \partial_s = \tilde{X}^c \) and that \( J^c : \xi^c \to \xi^c \) have the property that \( J^c \xi^c := J^c|_{\xi^c} \) is translation
invariant, and that \( \hat{\omega}^{c} \circ (\text{Id} \times J_{c^{\pm}}) \) is symmetric and positive definite. We then treat \( J^{c} \) as a constant sequence and apply Proposition 2.9 which guarantees the existence of an \( \ell^{c} \) (stated in the proposition as \( \epsilon \)) with the following significance. Let \( \phi_{k}^{c} : \mathcal{I}_{\ell^{c}} \rightarrow (0, 1] \) be a sequence of functions which converge in \( C^{\infty} \) to a limit function \( \phi_{\infty}^{c} \) which satisfies

\[
\phi_{\infty}^{c}(s) = \begin{cases} 
1 & \text{if } |s| \geq \frac{1}{2} \ell^{c}, \\
0 & \text{if } |s| \leq \frac{1}{4} \ell^{c}.
\end{cases}
\]

Then a consequence of Proposition 2.9 is that the almost complex structures defined by

\[ \phi_{k}^{c} \cdot J_{k} \partial_{s} = \tilde{X}^{c} \text{ and } J_{k} |_{\xi} = J_{\xi}^{c} \]

are each \( \Omega^{c} \)-tame on \( \mathcal{I}_{\ell^{c}} \times M \), and they are adiabatically degenerating on \( \mathcal{I}_{\ell^{c}/4} \times M \). We define \( \delta^{c} := \frac{1}{4} \ell^{c} \). We then observe that because these almost complex structures are all tame, on \( \mathcal{I}_{\ell^{c}} \times M \) they can be seen as arising as \( J_{k} = (\phi_{k}^{c})^{*} \tilde{J}_{k} \) for some \( \Omega \)-tame almost complex structures \( \tilde{J}_{k} \) on \( \tilde{W} \). The hypotheses of Theorem 1.7 then guarantee the existence of a sequence of pseudoholomorphic curves with bounded symplectic area and genus, and which span the energy levels in \( \mathcal{I}_{\delta^{c}} \times M \), while the almost complex structures are adiabatically degenerating on \( \mathcal{I}_{\delta^{c}} \times M \). Recall that second countability of the domains of these pseudoholomorphic curves guarantees that the set of connected components on which each is a constant map is countable, and hence for any such curve the set of energy levels containing a constant component is countable. Making use of the fact that our pseudoholomorphic maps are continuous and proper, it follows that one may remove the constant connected components while still guaranteeing `energy surjectivity'. Thus after removing constant components, we may apply Theorem 3.1, which guarantees that the set of energy levels with periodic orbits has full measure in \((−\delta^{c}, \delta^{c})\). Because we have reduced the proof of Theorem 1.7 to establishing just this result, we see that we have completed the proof of Theorem 1.7.

With Theorem 1.7 established, we now prove Theorem 1.9.

**Theorem 3.12.** (Intertwining existence and almost existence) Let \( (W, \Omega) \) be a four-dimensional compact connected exact symplectic manifold with boundary \( \partial W = M^{+} \cup M^{-} \). Suppose \( M^{+} \) is positive contact type in the sense of Definition 1.8, and suppose that one of the following three conditions holds:

1. \( M^{+} \) has a connected component diffeomorphic to \( S^{3} \);
2. there exists an embedded \( S^{2} \subset M^{+} \) which is homotopically non-trivial in \( W \);
3. \( (M^{+}, \lambda) \) has a connected component which is overtwisted.

Then for each Hamiltonian \( H \in C^{\infty}(W) \) for which \( H^{-1}(\pm 1) = M^{\pm} \), the following is true. For each \( s \in [−1, 1] \) the energy level \( H^{-1}(s) \) contains a closed non-empty set other than the energy level \( H^{-1}(s) \) itself which is invariant under the Hamiltonian flow of \( X_{H} \). Moreover, for almost every \( s \in [−1, 1] \) this closed invariant subset is a periodic orbit.
Proof. First observe that if \( s_0 \in [-1, 1] \) is a critical value of \( H \), then there exists \( p \in H^{-1}(s_0) \) such that \( dH(p) = 0 \), and hence the constant trajectory
\[
\gamma: \mathbb{R} \rightarrow H^{-1}(s_0) \\
\gamma(t) = p
\]
is a periodic orbit. Also note that because \( M^+ = H^{-1}(1) \) is contact type, and is either \( S^3 \), overtwisted, or contains a homotopically non-trivial \( S^2 \), it follows from [13] that \( M^+ \) has a periodic orbit. Then for any regular value \( s_0 \in [-1, 1] \), it follows from [8] (specifically Theorem 2) that the flow of \( X_H \) on \( H^{-1}(s_0) \) is not minimal. This establishes that each energy level \( H^{-1}(s) \) with \( s \in [-1, 1] \) is not minimal. To complete the proof of Theorem 1.9, it remains to show that almost every energy level \( H^{-1}(s) \) for \( s \in (-1, 1) \) has a periodic orbit. This follows from Theorem 1.7 above, provided we can guarantee the existence of the required pseudoholomorphic curves for arbitrary tame almost complex structure. However, this is fairly standard and the relevant details are provided in [8] (specifically the proof of Theorem 2); however, we recall the key points here.

Choose an \( \Omega \)-tame almost complex structure \( J \) on \( W \), and fix regular values \( E_- \) and \( E_+ \) of \( H \) so that
\[
-1 < E_- < E_+ < 1.
\]
Consider the case that tight \( S^3 \) is a connected component of \( M^+ \). Then along this spherical component, \( W \) can be symplectically capped off by \( \mathbb{C}P^2 \setminus O \) where \( O \) is diffeomorphic to the 4-ball, and the resulting symplectic manifold we denote by \((\tilde{W}, \tilde{\Omega})\). Note that in order to guarantee that \( \tilde{\Omega} \) is indeed symplectic, one may need to require that its restriction to \( \mathbb{C}P^2 \setminus O \subset \tilde{W} \) be a large constant multiple of the Fubini–Study metric. By adjusting \( J \) in a neighborhood of \( S^3 \subset M^+ \), the almost complex structure \( J \) can be extended to an \( \tilde{\Omega} \)-tame almost complex structure \( \tilde{J} \) on \( \tilde{W} \), which is also standard in a neighborhood of \( \mathbb{C}P^1 \subset \mathbb{C}P^2 \setminus O \). Define the constant \( C_{\tilde{\Omega}} := \int_{\mathbb{C}P^1} \tilde{\Omega} \) to be the \( \tilde{\Omega} \)-area of this sphere at infinity. One then considers the connected component of the moduli space of \( \tilde{J} \)-pseudoholomorphic curves which contain this \( \mathbb{C}P^1 \subset \mathbb{C}P^2 \setminus O \subset \tilde{W} \). As detailed in [8], automatic transversality guarantees that this four-real-dimensional moduli space (of unparameterized curves) is cut out transversely, each distinct pair of curves intersects at exactly one point, and by requiring \( \Omega \) to be exact there cannot be any ‘bubbles’ that arise in the compactification. Homotopy invariance of intersection numbers guarantees that each curve in this moduli space intersects \( \mathbb{C}P^1 \subset \mathbb{C}P^2 \setminus O \subset \tilde{W} \). It remains to show that this family of curves satisfies the energy surjectivity condition; that is, that this family of curves extends from \( M^+ = H^{-1}(1) \) to every energy level \( H^{-1}(s) \) for \( s \in (-1, 1) \). This too is detailed in [8], and follows from a mixture of automatic transversality and compactness of the family of curves; the latter is guaranteed by the fact that \( \tilde{J} \) is \( \tilde{\Omega} \)-tame. This then guarantees energy surjectivity. That is, for each \( \Omega \)-tame almost complex structure \( J \), there exists a pseudoholomorphic map \( \tilde{u}: (\tilde{S}, j) \rightarrow (\tilde{W}, \tilde{J}) \) with the property that the restricted map
\[
u := \tilde{u}|_S \quad \text{where} \quad S := \{z \in \tilde{S} : H \circ \tilde{u}(z) \in (E_-, E_+)\}
\]
satisfies \( H \circ \nu(S) = (E_-, E_+) \). It is also easily checked that Genus\((S) = 0 \) and \( \int_S \nu^*\tilde{\Omega} \leq C_{\tilde{\Omega}} \). Consequently, Theorem 1.7 applies, and hence almost every energy level in
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(E−, E+) contains a periodic orbit. By letting E− → −1 and E+ → 1, the desired result is immediate.

This covers the tight S3 case; the overtwisted case and the homotopically non-trivial S3 ⊂ M+ case are very similar, although the mechanism to generate the curves is different. The reader is directed to [8] for the details.

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A. Appendix. Miscellaneous support

This section is mostly devoted to providing a few support definitions and results which are important but are otherwise a bit of a distraction from more important arguments.

A.1. Supporting proofs. Our primary goal of this section is to prove Proposition 2.9; however, to do so we must first establish a few important supporting results. The first of these is the following.

**Lemma A.1.** (Cross term control) Let (Iℓ × M, Ω, ˆλ) be a framed Hamiltonian energy pile, and let ˆρ and ξ be the associated structures defined in §2. Assume as in the conclusions of Lemma 2.5 that along {0} × M, the sub-bundles ˆρ and ξ are Ω-symplectic complements. Let J0 be an Ω-compatible almost complex structure on Iℓ × M, with associated Riemannian metric g0 = Ω ◦ (Id × J0). Then for each δ > 0, there exists an ε = ε(δ, Ω, ˆλ, ˆω, J0) ∈ (0, ℓ), with the property that for each v = v^ ˆρ + v^ξ ∈ T(Iℓ × M) with v^ ˆρ ∈ ˆρ and v^ξ ∈ ξ we have

$$|Ω(v^ ˆρ, v^ξ)| ≤ δ ∥v^ ˆρ∥_g0 ∥v^ξ∥_g0.$$  

**Proof.** Given s ∈ Iℓ, we consider the embedding

$$M → Iℓ × M : m → (s, m)$$

and the pull-backs of ˆρ → Iℓ × M and ∈ Ω, ˆλ, ˆω, J0) ⊂ (0, ℓ), with the property that for each v = v^ ˆρ + v^ξ ∈ T(Iℓ × M) with v^ ˆρ ∈ ˆρ and v^ξ ∈ ξ we have

$$|Ω(v^ ˆρ, v^ξ)| ≤ δ ∥v^ ˆρ∥_g0 ∥v^ξ∥_g0.$$  

It is straightforward to show that Θ is continuous and that Θ(0) = 0. By compactness

$$Θ : [0, ℓ) → \mathbb{R},$$

$$s → \max_{s ∈ [−s, s]} Θ(s)$$

is also continuous and ˜Θ(0) = 0. The desired result is then immediate.
In order to proceed further, we will need to define a certain metric bound on the geometry of a weakly adapted almost complex structure. We denote this quantity by \(|J\langle|\), and establish it as follows. Given a framed Hamiltonian energy pile

\[(I_\ell \times M, \Omega, \hat{\lambda}),\]

we first fix a background metric \(g_0\) on \(I_\ell \times M\) associated to an auxiliary \(\Omega\)-compatible almost complex structure \(J_0\) by the usual formula \(g_0 := \Omega \circ (\text{Id} \times J_0)\). Of course, near the ends of \(I_\ell \times M\) the metric \(g_0\) might not behave well. Our Hamiltonian energy pile comes with the structures \(\hat{\omega}\) and \(\xi\). Given any other weakly adapted almost complex structure \(J\), we define the following quantity, where \(\ell' \in (0, \ell)\):

\[
|J\langle(\mathcal{I}_{\ell' \times M}, \Omega, \lambda, \hat{\omega}, J_0) := \sup_{q \in \mathcal{I}_{\ell' \times M} \setminus \{0\}} \max_{v_\lambda \in \mathcal{I}_{\ell' \times M}} \left( \frac{\|v_\xi^J\|_{g_J}}{\|v_\xi^J\|_{g_0}}, \frac{\|v_\xi^J\|_{g_J}}{\|v_\xi^J\|_{g_0}}, \frac{\|v_\xi^J\|_{g_J}}{\|v_\xi^J\|_{g_0}}, \frac{\|v_\xi^J\|_{g_J}}{\|v_\xi^J\|_{g_0}} \right); \tag{34}
\]

here \(g_J\) is the metric as defined in equation (6) in Definition 2.7.

An immediate benefit of such a definition is that whenever it is the case that \(|J\langle| \leq C_1\), it is also the case that the following hold for each \(v_\xi^J \in \xi\) lying above a point in \(T_{\ell' \times M}\):

\[
\|v_\xi^J\|_{g_0} \leq C_1 \|v_\xi^J\|_{g_J}; \tag{35}
\]

\[
\|v_\xi^J\|_{g_J} \leq C_1 \|v_\xi^J\|_{g_0}; \tag{36}
\]

\[
\|Jv_\xi^J\|_{g_0} \leq C_1 \|Jv_\xi^J\|_{g_J}; \tag{37}
\]

\[
\|v_\xi^J\|_{g_0} \leq C_1 \|Jv_\xi^J\|_{g_J}. \tag{38}
\]

It is also worth noting that if \(J_1\) and \(J_2\) are weakly adapted almost complex structures which agree on \(\xi\), then \(|J_1\langle| = \langle|J_2\langle|\). Indeed, the quantity \(|J\langle|\) depends only on \(J\langle^\xi\). With this definition established, we can now proceed with an important application.

**Proposition A.2.** (Metric area controlled by symplectic area) Let \((I_\ell \times M, \Omega, \hat{\lambda})\) be a framed Hamiltonian energy pile, and let \(\hat{\rho}, \xi\) and \(\hat{\omega}\) be the associated structures defined in equations (2), (3), (4). Assume, as in the conclusions of Lemma 2.5, that along \(0 \times M\) the sub-bundles \(\hat{\rho}\) and \(\xi\) are symplectic complements and \(\hat{\lambda}(X_H) = 1\). Let \(J_0\) be an auxiliary \(\Omega\)-compatible almost complex structure on \(I_\ell \times M\), with associated Riemannian metric \(g_0 = \Omega \circ (\text{Id} \times J_0)\). Fix a large positive constant \(C_1 > 1\). Then there exists a number \(\varepsilon = \varepsilon(\Omega, \hat{\lambda}, \hat{\omega}, J_0) \in (0, \ell)\) with the property that after trimming \(I_\ell \times M\) to \(I_\varepsilon \times M\), the following holds. If \(J\) is a weakly adapted almost complex structure for which

\[
|J\langle(\mathcal{I}_\varepsilon \times M, \Omega, \hat{\lambda}, \hat{\omega}, J_0) \leq C_1
\]

in the sense of equation (34), then

\[
(ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv) \leq 2\Omega(v, Jv)
\]

for each \(v \in T(I_\varepsilon \times M)\). Here \(J\) only needs to be defined over \(J_\varepsilon \times M\).

The important fact is that \(\varepsilon\) does not depend on \(J\) and only on the numerical bound!
Proof. We first fix $\ell' \in (0, \ell)$. This is done since the background metric $g_0$ might not be well behaved near the ends of $I_\ell \times M$. We begin by defining the constants $C_2$ and $\delta$ by

$$C_2 := \sup_{q \in I_\ell' \times M} \|\partial_s\|_{g_0}$$

and

$$\delta := \frac{1}{8C_2C_1^2}. \quad (39)$$

Let $\epsilon = \epsilon(\delta, \Omega, \hat{\lambda}, \hat{\omega}, J_0)$ be the constant guaranteed by Lemma A.1, and we may assume without loss of generality that $0 < \epsilon \leq \ell'$. Recall that a consequence of Lemma A.1 is that for each $v = v^\hat{\rho} + v^\xi \in T(I_\epsilon \times M)$ with $v^\hat{\rho} \in \hat{\rho}$ and $v^\xi \in \xi$ we have

$$|\Omega(v^\hat{\rho}, v^\xi)| \leq \delta \|v^\hat{\rho}\|_{g_0} \|v^\xi\|_{g_0}. \quad (40)$$

Also recall that along $\{0\} \times M$ we have $\hat{\lambda}(X_H) = 1$, so that by shrinking $\epsilon > 0$ we can further guarantee that

$$\sup_{q \in I_\epsilon \times M} |\hat{\lambda}(X_H(q)) - 1| \leq \frac{1}{210}. \quad (41)$$

Next recall that

$$\hat{X} = \frac{X_H}{\hat{\lambda}(X_H)}, \quad J\partial_s = \frac{1}{\phi} \hat{X} \quad \text{and} \quad J\hat{X} = -\phi \partial_s.$$ 

Additionally, recall that we have the splitting $T(I_\epsilon \times M) = \hat{\rho} \oplus \xi$, and the associated projections $\pi^\hat{\rho}: \hat{\rho} \oplus \xi \to \hat{\rho}$ and $\pi^\xi: \hat{\rho} \oplus \xi \to \xi$. In general we will use the abbreviated notation $v^\hat{\rho} = \pi^\hat{\rho}(v)$ and $v^\xi = \pi^\xi(v)$. In the following we work with $\hat{\rho} \oplus \xi \rightarrow I_\epsilon \times M$, where $\epsilon > 0$ is the previously chosen number. We begin with the following claim.

$$(ds \wedge \hat{\lambda})(v^\hat{\rho}, Jv^\hat{\rho}) = \hat{\lambda}(X_H) \cdot \Omega(v^\hat{\rho}, Jv^\hat{\rho}) \quad \text{for each } v^\hat{\rho} \in \hat{\rho}. \quad (42)$$

To prove this, we first write $v^\hat{\rho} = a\partial_s + b\hat{X}$ for some $a, b \in \mathbb{R}$, and then compute as follows:

$$\Omega(v^\hat{\rho}, Jv^\hat{\rho}) = \Omega(a\partial_s + b\hat{X}, J(a\partial_s + b\hat{X}))$$

$$= \Omega\left( a\partial_s + b\hat{X}, \frac{1}{\phi} a\hat{X} - \phi b\partial_s \right)$$

$$= \left( \frac{a^2}{\phi} + \phi b^2 \right) \Omega(\partial_s, \hat{X})$$

$$= \left( \frac{a^2}{\phi} + \phi b^2 \right) \Omega(\partial_s, X_H) \hat{\lambda}(X_H)$$

$$= \left( \frac{a^2}{\phi} + \phi b^2 \right) \frac{dH(\partial_s)}{\hat{\lambda}(X_H)}$$

$$= \left( \frac{a^2}{\phi} + \phi b^2 \right) \frac{ds(\partial_s)}{\hat{\lambda}(X_H)}$$

$$= \left( \frac{a^2}{\phi} + \phi b^2 \right) \frac{1}{\hat{\lambda}(X_H)}.$$
Similarly, we compute the following (with the same \( \hat{\rho} \) as above):

\[
(ds \wedge \hat{\lambda})(v \hat{\rho}, Jv \hat{\rho}) = (ds \wedge \hat{\lambda})(a \partial s + b \hat{X}, J(a \partial s + b \hat{X}))
\]

\[
= (ds \wedge \hat{\lambda}) \left( a \partial s + b \hat{X}, \frac{1}{\phi} a \hat{X} - \phi b \partial s \right)
\]

\[
= \left( \frac{a^2}{\phi} + \phi b^2 \right)
\]

\[
= \hat{\lambda}(X_H) \Omega (v \hat{\rho}, Jv \hat{\rho}).
\]

This establishes equation (42). The case with \( \hat{\omega} \) is much easier. Indeed, recall that by definition we have

\[
\hat{\omega} = \Omega \circ (\pi^\xi \times \pi^\xi).
\]

It immediately follows that

\[
\hat{\omega}(v \xi, Jv \xi) = \Omega(v \xi, Jv \xi)
\]

for each \( v \xi \in \xi \). (43)

In just a moment we will be concerned with estimating cross terms; however, first we will need an elementary estimate. Starting with

\[
\| \partial s \|^2_{g_J} = (ds \wedge \hat{\lambda})(\partial s, J\partial s) = (ds \wedge \hat{\lambda}) \left( \partial s, \frac{1}{\phi} \hat{X} \right) = \frac{1}{\phi}
\]

and combining the above with the definition of \( C_2 \) in equation (39) yields

\[
\| \partial s \|^2_{g_0} \leq C_2^2 \| \partial s \|^2_{g_J} \leq C_2^2 \| \partial s \|^2_{g_J}.
\]

We can now estimate cross terms. To that end, we let \( v = v \hat{\rho} + v \xi \in T(\mathcal{I}_\epsilon \times M) \) with \( v \hat{\rho} = a \partial s + b \hat{X} \in \hat{\rho} \) and \( v \xi \in \xi \). Then

\[
|\Omega(v \hat{\rho}, Jv \xi)| = |\Omega(a \partial s + b \hat{X}, Jv \xi)|
\]

\[
\leq \delta \|a \partial s\|_{g_0} \|Jv \xi\|_{g_0}
\]

\[
\leq \delta C_1 \|a \partial s\|_{g_0} \|v \xi\|_{g_0}
\]

\[
\leq \delta C_1^2 \|a \partial s\|_{g_0} \|v \xi\|_{g_J}
\]

\[
\leq \delta C_1^2 C_2 \|a \partial s\|_{g_J} \|v \xi\|_{g_J} \quad \text{by equation (44)}
\]

\[
= \frac{1}{8} \|a \partial s\|_{g_J} \|v \xi\|_{g_J}
\]

\[
\leq \frac{1}{8} \left( \|a \partial s\|^2_{g_J} + \|v \xi\|^2_{g_J} \right)
\]

\[
\leq \frac{1}{8} \left( \|a \partial s\|^2_{g_J} + \|b \hat{X}\|^2_{g_J} + \|v \xi\|^2_{g_J} \right)
\]
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\[ \leq \frac{1}{8} (\|v^\hat{\rho}\|_{g_J}^2 + \|v^\xi\|_{g_J}^2) \]
\[ = \frac{1}{8} \|v\|_{g_J}^2 \]
\[ = \frac{1}{8} (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv). \]

The other cross term is estimated rather similarly. Again let \( v = v^\hat{\rho} + v^\xi \in T(I \times M) \) with \( v^\hat{\rho} = a \partial_s + b \hat{X} \in \hat{\rho} \) and let \( v^\xi \in \xi \). We estimate

\[ |\Omega(v^\xi, Jv^\hat{\rho})| = |\Omega(v^\xi, J(a \partial_s + b \hat{X}))| \]
\[ = |\Omega \left( v^\xi, \frac{a}{\phi} \hat{X} - b \phi \partial_s \right) | \]
\[ = |\Omega(v^\xi, b \phi \partial_s)| \]
\[ \leq \delta \|v^\xi\|_{g_0} \|b \phi \partial_s\|_{g_0} \]
\[ \leq \delta C_1 C_2 \phi \|v^\xi\|_{g_J} \|b \partial_s\|_{g_J} \]
\[ \leq \frac{1}{8} \|v^\xi\|_{g_J} \|b \partial_s\|_{g_J} \]
\[ \leq \frac{1}{8} (\|v^\xi\|_{g_J}^2 + \|b \partial_s\|_{g_J}^2) \]
\[ \leq \frac{1}{8} (\|v^\xi\|_{g_J}^2 + \|v^\hat{\rho}\|_{g_J}^2) \]
\[ \leq \frac{1}{8} \|v\|_{g_J}^2 \]
\[ = \frac{1}{8} (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv) \]

With these two estimates established, we can now use them to establish the following:

\[ |\Omega(v, Jv) - (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv)| \]
\[ = |\Omega(v^\hat{\rho} + v^\xi, J(v^\hat{\rho} + v^\xi)) - (ds \wedge \hat{\lambda} + \hat{\omega})(v^\hat{\rho} + v^\xi, J(v^\hat{\rho} + v^\xi))| \]
\[ = |\Omega(v^\hat{\rho} + v^\xi, J(v^\hat{\rho} + v^\xi)) - (ds \wedge \hat{\lambda})(v^\hat{\rho}, Jv^\hat{\rho}) - \hat{\omega}(v^\xi, Jv^\xi)| \]
\[ \leq |\Omega(v^\hat{\rho}, Jv^\hat{\rho}) - ds \wedge \hat{\lambda}(v^\hat{\rho}, Jv^\hat{\rho})| + |\Omega(v^\xi, Jv^\xi) - \hat{\omega}(v^\xi, Jv^\xi)| \]
\[ + |\Omega(v^\hat{\rho}, Jv^\xi)| + |\Omega(v^\xi, Jv^\hat{\rho})| \]
\[ \leq |\Omega(v^\hat{\rho}, Jv^\hat{\rho}) - (ds \wedge \hat{\lambda})(v^\hat{\rho}, Jv^\hat{\rho})| + |\Omega(v^\xi, Jv^\hat{\rho})| + |\Omega(v^\hat{\rho}, Jv^\xi)| \]
\[ \leq |\Omega(v^\hat{\rho}, Jv^\hat{\rho}) - (ds \wedge \hat{\lambda})(v^\hat{\rho}, Jv^\hat{\rho})| + \frac{1}{4} (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv) \]
\[ = \left| \frac{1}{\hat{\lambda}(X_H)} (ds \wedge \hat{\lambda})(v^\hat{\rho}, Jv^\hat{\rho}) - (ds \wedge \hat{\lambda})(v^\hat{\rho}, Jv^\hat{\rho}) \right| + \frac{1}{4} (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv) \]
\[ \leq \frac{1}{\hat{\lambda}(X_H)^{-1}} - 1 |(ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv) + \frac{1}{4} (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv) \]
\[ \leq \frac{1}{2} (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv) \]
In other words, we have shown that
\[ |\Omega(v, Jv) - (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv)| \leq \frac{1}{2} (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv) \]
and thus
\[ (ds \wedge \hat{\lambda} + \hat{\omega})(v, Jv) \leq 2\Omega(v, Jv) \]
for all \( v \in T(I \epsilon \times M) \).

With the above estimates established, we can now turn our attention to the main result of this section. Note that the metric \( \tilde{g} \) occurring below is a translation-invariant metric on \( \mathbb{R} \times M \) associated to a metric on \( M \). The norms \( \| \cdot \|_{C^n} \) use this metric.

**Proposition 2.9. (Adiabatically degenerating constructions)** Let \( (I \epsilon \times M, \Omega, \hat{\lambda}) \) be a framed Hamiltonian energy pile, and let \( \hat{\rho}, \xi, \) and \( \hat{\omega} \) be the associated structures defined in §2. Assume, as in the conclusions of Lemma 2.5, that along \( \{0\} \times M \) the sub-bundles \( \hat{\rho} \) and \( \xi \) are symplectic complements and \( \hat{\lambda}(X_H) = 1 \). Let \( \tilde{X} = X_H / \hat{\lambda}(X_H) \) as above and denote by \( \| \cdot \|_{C^n} \) the \( C^n \)-norm on \( \mathbb{R} \times M \) with respect to the auxiliary translation-invariant metric \( \tilde{g} \), and the \( \Psi_k \) are the embeddings as in equation (7). Let \( \{\tilde{J}_k\}_{k \in \mathbb{N}} \) be a sequence of almost complex structures on \( I \epsilon \times M \) which satisfy the following conditions.

(D1) \( \tilde{J}_k : \hat{\rho} \to \hat{\rho} \) and \( \tilde{J}_k : \xi \to \xi \) for each \( k \in \mathbb{N} \).

(D2) \( (ds \wedge \hat{\lambda} + \hat{\omega}) \circ (\text{Id} \times \tilde{J}_k) \) is a Riemannian metric for each \( k \in \mathbb{N} \).

(D3) there exist constants \( \{C'_n\}_{n \in \mathbb{N}} \) such that
\[ \sup_{k \in \mathbb{N}} \| \tilde{J}_k \|_{C^n} \leq C'_n. \]

Then there exists an \( \epsilon > 0 \) with the following significance. For any sequence of functions \( \phi_k : (-\epsilon, \epsilon) \to (0, 1] \) which converges in \( C^\infty \), the weakly adapted almost complex structures defined by
\[ \phi_k \cdot J_k \partial_s = \tilde{X} \quad \text{and} \quad J_k|\xi := \tilde{J}_k|\xi \]
(45)
satisfy the following properties.

(E1) For each \( v \in T(I \epsilon \times M) \) we have
\[ (ds \wedge \hat{\lambda} + \hat{\omega})(v, J_k v) \leq 2\Omega(v, J_k v). \]

(E2) There exists a sequence \( \{K_n\}_{n=0}^\infty \) of positive constants so that with respect to the auxiliary translation-invariant metric \( \tilde{g} \) on \( \mathbb{R} \times M \),
\[ \sup_{k \in \mathbb{N}} \| (\Psi_k)_* J_k \|_{C^n} \leq K_n \]
for each \( n \in \mathbb{N} \).

In particular, if \( \epsilon' \in (0, \epsilon) \) and \( \phi_k|_{(-\epsilon', \epsilon')} \to 0 \), then the almost complex structures \( J_k \) are adiabatically degenerating on \( I \epsilon' \times M \) in the sense of Definition 2.8, and on \( I \epsilon \times M \) these almost complex structures are all tame.
Proof. Fix an auxiliary $\Omega$-compatible almost complex structure $J_0$ on $\mathcal{I}_\ell \times M$, and let $g_0$ be the associated background Riemannian metric $g_0 = \Omega \circ (\text{Id} \times J_0)$. Pick an $\ell' \in (0, \ell)$ and consider the sequence $(\tilde{J_k})$. We define

$$C_1 := \sup_{k \in \mathbb{N}} |\tilde{J_k}|_{(\mathcal{I}_{\ell'} \times M, \Omega, \hat{\lambda}, \hat{\omega}, J_0)}$$

where $|J|$ is defined as in equation (34). We note that $C_1$ is finite because of hypothesis (D3). We then apply Proposition A.2 which guarantees the existence of an $\epsilon > 0$ for which (E1) holds for any weakly adapted almost complex structures $J_k$ for which $J_k|_{\xi} = \tilde{J_k}|_{\xi}$.

To establish (E2), assume that the $\phi_k$ have been fixed so they converge in $C^\infty$. Define the $J_k$ as in equation (45). To estimate the norms of the $(\Psi_k)_* J_k$ we first note that it is sufficient to establish $C^n$ bounds for

$$((\Psi_k)_* J_k)\partial \hat{\lambda} \quad \text{and} \quad ((\Psi_k)_* J_k)|_{\xi}.$$ 

However, each of these bounds is immediately obtained as a consequence of the fact that the $\phi_k$ are converging in $C^\infty$ together with hypothesis (D3). The remaining desired conclusions are immediate. This completes the proof of Proposition 2.9. □

A.2. Extra definitions. For the convenience of the reader, we provide a few definitions here which are used indirectly. The reader should consult [8], which contains further discussions of these terms.

**Definition A.3.** (Realized Hamiltonian homotopy) Let $M$ be a smooth (odd-dimensional) closed manifold, let $\mathcal{I} \subset \mathbb{R}$ be an interval equipped with the coordinate $t$, and let $\hat{\lambda}$ and $\hat{\omega}$ respectively be a 1-form and 2-form on $\mathcal{I} \times M$. We say $(\mathcal{I} \times M, (\hat{\lambda}, \hat{\omega}))$ is a realized Hamiltonian homotopy provided the following assertions hold.

1. $\hat{\lambda}(\partial_t) = 0$.
2. $i_{\partial_t} \hat{\omega} = 0$.
3. $d\hat{\omega}|_{t=\text{const}} = 0$
4. $dt \wedge \hat{\lambda} \wedge \hat{\omega} \wedge \ldots \wedge \hat{\omega} > 0$.
5. $\hat{\lambda}$ is invariant under the flow of $\partial_t$.
6. If $\mathcal{I}$ is unbounded, then there exists a neighborhood of $\{\pm \infty\} \times M$ on which $\hat{\omega}$ is invariant under the flow of $\partial_t$.

**Definition A.4.** (Adapted structures for a realized Hamiltonian homotopy) Let $(\mathcal{I} \times M, (\hat{\lambda}, \hat{\omega}))$ be a realized Hamiltonian homotopy. We say an almost Hermitian structure $(\hat{J}, \hat{g})$ on $\mathcal{I} \times M$ is adapted to this realized Hamiltonian homotopy provided the following hold.

1. $\hat{J}\partial_t = \hat{\lambda}$.
2. $\hat{J} : \hat{\xi} \rightarrow \hat{\xi}$.
3. $\hat{g} = (dt \wedge \hat{\lambda} + \hat{\omega})(\cdot, \hat{J} \cdot)$.
4. If $\mathcal{I}$ is unbounded, then there exists a neighborhood of $[\pm \infty] \times M$ on which the restriction $\hat{J}|_{\xi}$ is invariant under the flow of $\partial_t$. 


