Approximate controllability of nonlocal problem for non-autonomous stochastic evolution equations

Pengyu Chen* and Xuping Zhang
Department of Mathematics, Northwest Normal University
Lanzhou 730070, China

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Abstract. In this paper, we are considered with approximate controllability for a class of non-autonomous stochastic evolution equations of parabolic type with discrete nonlocal initial conditions. Some new results about existence of mild solutions as well as approximate controllability are established under more natural conditions on nonlinear functions and control operator by introducing a new Green function and using the theory of evolution family, Schauder fixed point theorem and the resolvent operator condition. At last, as a sample of application, these results are applied to a class of non-autonomous stochastic partial differential equation of parabolic type with discrete nonlocal initial conditions. The results obtained in this paper is a supplement to the existing literature and essentially extends some existing results in this area.

1. Introduction. The theory of nonlocal evolution equations has become an important area of investigation in recent years because they can application to various problems arising in physics, biology, aerospace and medicine. It is demonstrated that the nonlocal initial condition can be applied in physics with better effect than the classical initial condition $u(0) = u_0$. In fact, for constants $a > 0$, $0 < t_1 < t_2 < \cdots < t_m < a$, $m \in \mathbb{N}$ and $c_k \neq 0$, $k = 1, 2, \cdots, m$, the following discrete nonlocal initial condition

$$u(0) = \sum_{k=1}^{m} c_k u(t_k) \quad (1)$$

has stronger application background and evolution equations with this kind of nonlocal initial condition have been studied extensively. For example, as early as in 1993, Deng [24] used the nonlocal condition (1) to describe the diffusion phenomenon of a small amount of gas in a transparent tube, in this case, nonlocal condition (1) allows the additional measurements at $t_k$, $k = 1, 2, \cdots, m$, which is more precise than the measurement just at $t = 0$. In 1999, Byszewski [6] pointed out that if $c_k \neq 0$,
$k = 1, 2, \cdots, m$, then the results obtained in [6] can be applied to kinematics to determine the location evolution $t \rightarrow u(t)$ of a physical object for which we do not know the positions $u(0), u(t_1), \cdots, u(t_m)$, but we know that the nonlocal condition (1) holds. Consequently, the nonlocal condition can be more useful than the standard initial condition $u(0) = u_0$ to describe some physical phenomena. The importance of nonlocal conditions have also been discussed in [8, 9, 10, 16, 25, 26, 35, 45].

On the other hand, in recent years, the stochastic differential equations have attracted great interest because of their practical applications in many areas such as physics, chemistry, economics, social sciences, finance and other areas of science and engineering. For more details about stochastic differential equations we refer to the books by Da Prato and Zabczyk [23], Grecksch and Tudor [32], Liu [37], Mao [40] and Sobczyk [43]. One of the branches of stochastic differential equations is the theory of stochastic evolution equations especially stochastic evolution equations with nonlocal initial conditions. Since semilinear stochastic evolution equations with nonlocal initial conditions are abstract formulations for many problems arising in the domain of engineering technology, biology and economic system etc., stochastic evolution equations with nonlocal initial conditions have attracted increasing attention in recent years and the existence, uniqueness and asymptotic behavior of mild solutions to stochastic evolution equations with nonlocal initial conditions have been considered by many authors, see Chen, Abdelmonem and Li [7], Chen and Li [11], Chen, Zhang and Li [14], Cui, Yan and Wu [21], Farahi and Guendouzi [27] and Zhang et al. [49, 50] for more comments and citations.

The concept of controllability, when it was first introduced by Kalman [34] in 1963, has become an active area of investigation due to its great applications in the field of physics. There are various works on approximate controllability of systems represented by differential equations, integro-differential equations, differential inclusions, neutral functional differential equations, and impulsive differential equations in Banach spaces. In 1983, Zhou [51] obtained sufficient condition for the approximate controllability to a class of control systems governed by the semilinear abstract equation, which is suitable not only to the infinite-dimensional case but also to the finite-dimensional case. Latter, Mahmudov [39] investigated the approximate controllability for abstract semilinear deterministic and stochastic control systems under the natural assumption that the associated linear control system is approximately controllable by using new properties of symmetric operators, compact semi-groups, the Schauder fixed point theorem and the contraction mapping principle in 2003. In 2015, Liu and Li [38] investigated the control systems governed by fractional evolution differential equations involving Riemann–Liouville fractional derivatives in Banach spaces under some suitable assumptions. In 2017, by constructing a control function involving Gramian controllability operator, Wang, Fečkan and Zhou [46] studied the approximate controllability for a class of Sobolev-type fractional evolution systems with nonlocal conditions in Hilbert spaces using Schauder fixed point theorem. By constructing a control function involving Gramian controllability operator and using Schauder’s fixed-point theorem as well as the theory of evolution family, Chen, Zhang and Li [15] invested existence of mild solutions as well as approximate controllability for a class of non-autonomous evolution system of parabolic type with nonlocal conditions in Banach spaces in 2019.

In particular, the approximate controllability of stochastic evolution equations especially stochastic evolution equations with nonlocal initial conditions are also
studied in recent years. In 2005, Balasubramaniam and Dauer [4] give some sufficient conditions for controllability to a class of semilinear stochastic evolution equations with nonlocal conditions by using a new fixed point analysis approach. Later, by using Sadovskii’s fixed point theorem and stochastic analysis theory, Farahi and Guendouzi [27] derive a new set of sufficient conditions for the approximate controllability of semilinear fractional neutral stochastic evolution equations with nonlocal conditions under the assumption that the corresponding linear system is approximately controllable in 2014. In 2016, Sakthivel, Ren, Debbouche and Mahmudov [42] obtain a new set of sufficient conditions for the approximate controllability of nonlinear fractional stochastic differential inclusions under the assumption that the corresponding linear system is approximately controllable by utilizing the fixed-point theorem for multivalued operators and fractional calculus.

However, we find that among the previous researches, most of researchers focus on the case that the differential operators in the main parts are independent of time $t$, which means that the problems under considerations are autonomous. In fact, when treating some parabolic evolution equations, it is usually assumed that the partial differential operators depend on time $t$ on account of this class of operators appears frequently in the applications, for the details please see Acquistapace [1], Acquistapace and Terreni [2], Amann [3], Chen, Zhang and Li [16, 17, 18, 19, 20], Fu [29], Fu and Huang [30], Fu and Zhang [31], Liang, Liu and Xiao [36], Tanabe [44], Wang, Ezzinbi and Zhu [47] and Wang and Zhu [48]. Therefore, it is interesting and significant to investigate non-autonomous stochastic evolution equations with nonlocal initial conditions, i.e., the differential operators in the main parts of the considered problems are dependent of time $t$. Base on all the above mentioned aspects, we investigate the existence of mild solutions as well as approximate controllability for the following non-autonomous stochastic evolution equations of parabolic type (NP)

$$du(t) - A(t)u(t)dt = [Bu(t) + f(t, u(t))]dt + g(t, u(t))dW(t), \quad t \in (0, a]$$

with nonlocal initial conditions (1) in this paper, where $A(t)$ is a family of (possibly unbounded) linear operators depending on time and having the domains $\mathcal{D}(A(t))$ for every $t \in [0, a]$, the state $u(\cdot)$ takes values in the real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|_\mathbb{H}$; $U$ is another separable Hilbert space with inner product $(\cdot, \cdot)_U$ and norm $\| \cdot \|_U$, $\{W(t) : t \geq 0\}$ is a cylindrical $U$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$: The same notation $\| \cdot \|$ is used to present the norm of $\mathcal{L}(U, \mathbb{H})$, which denotes the space of all bounded linear operators from $U$ into $\mathbb{H}$, $\mathcal{L}(\mathbb{H}) = \mathcal{L}(\mathbb{H}, \mathbb{H})$; The control function $v(t)$ takes values in $L^2_\mathbb{F}([0, a], U)$ of admissible control functions, $B$ is a bounded linear operator from $U$ into $\mathbb{H}$; $f : [0, a] \times \mathbb{H} \to \mathbb{H}$ and $g : [0, a] \times \mathbb{H} \to \mathcal{L}(U, \mathbb{H})$ are two continuous nonlinear functions.

The following three aspects should be noted to distinguishing the present paper from earlier works on stochastic evolution equations with nonlocal initial conditions. On the one hand, to the best of the authors’ knowledge, most of the existing articles are devoted to the study of the approximate controllability of autonomous evolution equations, only a few papers, for example Chen, Zhang and Li [15], Fu [29], Fu and Huang [30], Fu and Zhang [31], investigated the the approximate controllability of deterministic non-autonomous evolution equations, i.e., the differential operators in the main parts of the considered problems are dependent of time $t$. On
the other hand, although the approximate controllability for semilinear autonomous stochastic evolution equations with nonlocal initial conditions have been studied by Balasubramaniam and Dauer [4], Farahi and Guendouzi [27] and Sakthivel, Ren, Debbouche and Mahmudov [42], the strong assumptions on nonlocal function $g$ are required. How to delete the strong restriction condition on nonlocal function $g$ is an interesting problem. At last, we have not see the relevant article to study the approximate controllability for non-autonomous stochastic evolution equations with nonlocal initial conditions. Therefore, in this paper we will extend the approximate controllability for semilinear autonomous stochastic evolution equations with nonlocal initial conditions which studied in [4], [27] and [42] to non-autonomous case, and investigate the approximate controllability for the non-autonomous stochastic evolution equations of parabolic type with nonlocal initial conditions (2)-(1) don’t require any condition on nonlocal term.

2. Notations and preliminaries. We begin with this section by giving some notations. Let $\mathbb{H}$ and $\mathbb{U}$ be two real separable Hilbert spaces, we denote by $(\cdot, \cdot)$ and $(\cdot, \cdot)_U$ their inner products, and by $\| \cdot \|$ and $\| \cdot \|_U$ their vector norms, respectively. We denote by $\mathcal{L}(\mathbb{H})$ the Banach space of all linear and bounded operators on $\mathbb{H}$ endowed with the topology defined by operator norm. In this paper, we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a complete filtered probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. Let $\{e_k, k \in \mathbb{N}\}$ be a complete orthonormal basis of $\mathbb{U}$. Suppose that $\{\mathbb{W}(t) : t \geq 0\}$ is a cylindrical $\mathbb{U}$-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a finite trace nuclear covariance operator $Q \geq 0$, denote $\text{Tr}(Q)=\sum_{k=1}^{\infty}\lambda_k = \lambda < \infty$, which satisfies that $Qe_k = \lambda_ke_k$, $k \in \mathbb{N}$. Let $\mathcal{W}(k), k \in \mathbb{N}$ be a sequence of one-dimensional standard Wiener processes mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ such that

$$\mathbb{W}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\mathbb{W}_k(t)e_k.$$ 

We further assume that $\mathcal{F}_t = \sigma\{\mathbb{W}(s), 0 \leq s \leq t\}$ is the $\sigma$-algebra generated by $\mathbb{W}$ and $\mathcal{F}_\infty = \mathcal{F}$. For $\varphi, \psi \in \mathcal{L}(\mathbb{U}, \mathbb{H})$, we define $(\varphi, \psi) = \text{Tr}(\varphi Q^* \psi^*)$, where $\psi^*$ is the adjoint of the operator $\psi$. Clearly, for any bounded operator $\psi \in \mathcal{L}(\mathbb{U}, \mathbb{H})$,

$$\|\psi\|_Q^2 = \text{Tr}(\psi Q^* \psi^*) = \sum_{k=1}^{\infty} \|\sqrt{\lambda_k}\psi e_k\|.$$ 

If $\|\psi\|_Q^2 < \infty$, then $\psi$ is called a $Q$-Hilbert-Schmidt operator.

The collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, denoted $L^2(\Omega, \mathbb{H})$, which is a Banach space equipped with the norm $\|u(\cdot)\|_{L^2} = (\mathbb{E}\|u(\cdot, \mathbb{W})\|^2)^{\frac{1}{2}}$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}u = \int_{\Omega} u(\mathbb{W})d\mathbb{P}$. An important subspace of $L^2(\Omega, \mathbb{H})$ is given by

$$L^2_0(\Omega, \mathbb{H}) = \{u \in L^2(\Omega, \mathbb{H}) | \text{ $u$ is $\mathcal{F}_0$-measurable}\}.$$ 

We denote by $C([0, a], L^2(\Omega, \mathbb{H}))$ the space of all continuous $\mathcal{F}_t$-adapted measurable processes from $[0, a]$ to $L^2(\Omega, \mathbb{H})$ satisfying $\sup_{t \in [0,a]}\mathbb{E}\|u(t)\|^2 < \infty$. Then it is easy to see that $C([0, a], L^2(\Omega, \mathbb{H}))$ is a Banach space endowed with the supnorm

$$\|u\|_C^2 = \left(\sup_{t \in [0,a]}\mathbb{E}\|u(t, \mathbb{W})\|^2\right)^{\frac{1}{2}}.$$
Let $L^2_\mathbb{F}([0, a], \mathbb{U})$ be the Banach space of all $\mathbb{U}$-value Bochner square integrable functions defined on $[0, a]$ with the norm

$$\|u\|_2 = \left( \int_0^a \mathbb{E}\|u(t, \mathbb{W})\|^2 dt \right)^{\frac{1}{2}}, \quad u \in L^2_\mathbb{F}([0, a], \mathbb{U}).$$

By [22, Proposition 2.8], we have the following result which will be used throughout this paper.

**Lemma 2.1.** If $g : [0, a] \times \mathbb{H} \to \mathcal{L}(\mathbb{U}, \mathbb{H})$ is continuous and $u \in C([0, a], L^2(\Omega, \mathbb{H}))$, then

$$\mathbb{E}\left\| \int_0^t g(t, u(t))d\mathbb{W}(t) \right\|^2 \leq \text{Tr}(Q) \int_0^t \mathbb{E}\|g(t, u(t))\|^2 dt.$$ 

Throughout the paper, we assume that $\{A(t) : 0 \leq t \leq a\}$ is a family of closed and densely defined operator on Hilbert space $\mathbb{H}$, which satisfies the known conditions of Acquistapace and Terreni:

(\text{AT}_1) For each $t \in [0, a]$, $A(t)$ is a closed linear operator on $\mathbb{H}$ and there exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $M_1 \geq 0$ such that $\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0)$ and for all $\lambda \in \Sigma_\theta$ and $0 \leq s \leq t \leq a$,

$$\|R(\lambda, A(t) - \lambda_0)\|_{\mathcal{L}(\mathbb{H})} \leq \frac{M_1}{1 + |\lambda|};$$

(\text{AT}_2) There exist constants $M_2 > 0$ and $\vartheta, \beta \in (0, 1]$ with $\vartheta + \beta > 1$ such that for all $\lambda \in \Sigma_\theta$ and $0 \leq s \leq t \leq a$,

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq \frac{M_2|t - s|^{\vartheta}}{|\lambda|^{\beta}},$$

where

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\lambda| \leq \theta\}.$$ 

Conditions (\text{AT}_1) and (\text{AT}_2), which are initiated by Acquistapace and Terreni in [2] and Acquistapace in [1] for $\lambda_0 = 0$, are well understood and widely used in the literature. Under the above conditions (\text{AT}_1) and (\text{AT}_2), the family $\{A(t) : 0 \leq t \leq a\}$ generates a unique linear evolution system, or called linear evolution family, $\{U(t, s) : 0 \leq s \leq t \leq a\}$. Furthermore, by an obvious rescaling from [1, Theorem 2.3] and [2, Theorem 2.1] combined with the Acquistapace and Terreni conditions (\text{AT}_1) and (\text{AT}_2) one gets the following properties for the family of linear operator $\{U(t, s) : 0 \leq s \leq t \leq a\}$:

**Lemma 2.2.** The family of the linear operator $\{U(t, s) : 0 \leq s \leq t \leq a\}$ satisfies the following properties:

(i) $U(t, r)U(r, s) = U(t, s)$, $U(t, t) = I$ for $0 \leq s \leq r \leq t \leq a$;

(ii) The map $(t, s) \mapsto U(t, s)x$ is continuous for all $x \in \mathbb{H}$ and $0 \leq s \leq t \leq a$;

(iii) $U(t, s) \in C^1((s, \infty), \mathcal{L}(\mathbb{H}))$, $\frac{\partial U(t, s)}{\partial t} = A(t)U(t, s)$ for $t > s$, and $\|A^k(t)U(t, s)\| \leq M(t - s)^{-k}$ for $0 < t - s \leq 1$ and $k = 0, 1$;

(iv) $\frac{\partial U(t, s)}{\partial s} = -U(t, s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in D(A(s))$.

From the property (iii) we know that

$$\|U(t, s)\|_{\mathcal{L}(\mathbb{H})} \leq M \quad \text{for} \quad 0 \leq s \leq t \leq a.$$  

(3)

In (3) and property (iii), $M > 0$ is a constant.
**Definition 2.3.** An evolution family \( \{U(t,s) : 0 \leq s \leq t \leq a\} \) is said to be compact if for all \( 0 \leq s < t \leq a \), \( U(t,s) \) is continuous and maps bounded subsets of \( \mathbb{H} \) into precompact subsets of \( E \).

**Lemma 2.4.** ([28]) For each \( t \in [0,a] \) and some \( \lambda \in \rho(A(t)) \), if the resolvent \( R(\lambda, A(t)) \) is a compact operator, then \( U(t,s) \) is a compact operator whenever \( 0 \leq s < t \leq a \).

**Lemma 2.5.** ([41]) Let \( \{U(t,s) : 0 \leq s \leq t \leq a\} \) be a compact evolution family on \( \mathbb{H} \). Then for each \( s \in [0,a] \), the function \( t \mapsto U(t,s) \) is continuous by operator norm for \( t \in (s,a) \).

In this paper, we always assume the following condition

\[
(H_C) \sum_{k=1}^{m} |c_k| < \frac{1}{\lambda}
\]
is satisfied. By the condition \((H_C)\) and \((3)\), we have

\[
\left\| \sum_{k=1}^{m} c_k U(t_k,0) \right\| \leq M \sum_{k=1}^{m} |c_k| < 1.
\]

By \((4)\) and operator spectrum theorem, we know that

\[
B := \left( I - \sum_{k=1}^{m} c_k U(t_k,0) \right)^{-1}
\]
exists, bounded and \( D(B) = \mathbb{H} \). Furthermore, by Neumann expression, \( B \) can be expressed by

\[
B = \sum_{n=0}^{\infty} \left( \sum_{k=1}^{m} c_k U(t_k,0) \right)^n.
\]

Therefore

\[
\left\| B \right\| \leq \sum_{n=0}^{\infty} \left\| \sum_{k=1}^{m} c_k U(t_k,0) \right\|^n = \frac{1}{1 - \left\| \sum_{k=1}^{m} c_k U(t_k,0) \right\|} \leq \frac{1}{1 - M \sum_{k=1}^{m} |c_k|}.
\]

For convenience, we introduce the Green’s function \( G(t,s) \) as follows

\[
G(t,s) = \sum_{k=1}^{m} \chi_{t_k}(s) U(t,0) B U(t_k,s) + \chi(t) U(t,s), \quad t, s \in [0,a],
\]
where

\[
\chi_{t_k}(s) = \begin{cases} c_k, & s \in [0,t_k), \\ 0, & s \in [t_k,a], \end{cases} \quad \chi(t) = \begin{cases} 1, & s \in [0,t), \\ 0, & s \in [t,a]. \end{cases}
\]

Therefore, by the above discuss and the proof of [8, Lemma 2.2] combined with the aid of the Green’s function \( G(t,s) \) defined by \((6)\), we can give the definition of mild solutions for NP \((2)-(1)\) as follows.

**Definition 2.6.** An \( \mathcal{F}_t \)-adapted stochastic process \( u : [0,a] \to \mathbb{H} \) is called a mild solution of NP \((2)-(1)\) if \( u(t) \in \mathbb{H} \) has càdlàg paths on \( t \in [0,a] \) almost surely and for each \( t \in [0,a] \), \( u(t) \) \( \mathbb{P} \)-almost surely satisfies the integral equation

\[
u(t) = \int_{0}^{a} G(t,s) [Bv(s) + f(s,u(s))] ds + \int_{0}^{a} G(t,s) g(s,u(s)) d\mathbb{W}(s),
\]
where \( G(t,s) \) is the Green’s function defined by \((6)\).
Definition 2.7. Let $u$ be a mild solution of NP (2)-(1) corresponding to the control $v \in L^2_{F}[\{0, a\}, U]$. NP (2)-(1) is said to be approximately controllable on the interval $[0, a]$ if $K_a(v) = \mathbb{H}$, where the set

$$K_a(v) = \{ u(a) \in U : v \in L^2_{F}[\{0, a\}, U]\}$$

is called the reachable set of NP (2)-(1).

3. Existence of mild solutions. In this section, we will investigate the existence of mild solutions for NP (2)-(1) by using the Schauder fixed-point theorem. For this purpose, we impose the following restrictions on nonlinear functions $f$, $g$ and linear operator $B$:

$$(H_f)$$ The function $f : [0, a] \times \mathbb{H} \to \mathbb{H}$ is continuous and there exists a function $\varphi_f \in L([-\infty, a], \mathbb{R}^+)$ such that $\mathbb{E}\|f(t, u)\|^2 \leq \varphi_f(t)$ for all $u \in \mathbb{H}$ and $t \in [0, a]$.

$$(H_g)$$ The function $g : [0, a] \times \mathbb{H} \to \mathcal{L}(U, \mathbb{H})$ is continuous and there exists a function $\varphi_g \in L([-\infty, a], \mathbb{R}^+)$ such that $\mathbb{E}\|g(t, u)\|^2 \leq \varphi_g(t)$ for all $u \in \mathbb{H}$ and $t \in [0, a]$.

$$(H_B)$$ There exists a function $\varphi_B \in L([-\infty, a], \mathbb{R}^+)$ such that $\mathbb{E}\|B(t)\|^2 \leq \varphi_B(t)$ for all $v \in L^2_{F}[\{0, a\}, U]$.

Theorem 3.1. Assume that the evolution family $\{U(t, s) : 0 \leq s \leq t \leq a\}$ generated by $\{A(t) : 0 \leq t \leq a\}$ is compact. If the conditions $(H_C)$, $(H_f)$, $(H_g)$ and $(H_B)$ are satisfied, then NP (2)-(1) has at least one mild solution on $[0, a]$.

Proof. Consider the operator $Q : C([-\infty, a], L^2(\Omega, \mathbb{H})) \to C([0, a], L^2(\Omega, \mathbb{H}))$ defined by

$$(Qu)(t) = \int_0^a G(t, s)Bv(s) + f(s, u(s))ds + \int_0^a G(t, s)g(s, u(s))d\mathbb{W}(s), \quad t \in [0, a],$$

where $G(t, s)$ is the Green’s function defined by (6). By direct calculation, we know that the operator $Q$ is well defined in $C([-\infty, a], L^2(\Omega, \mathbb{H}))$. From Definition 2.6, it is easy to see that the mild solution of NP (2)-(1) on $[0, a]$ is equivalent to the fixed point of the operator $Q$ defined by (8). In what follows, we will prove that the operator $Q$ has at least one fixed point by applying the famous Schauder Fixed Point Theorem.

At first, we prove that the operator $Q : C([-\infty, a], L^2(\Omega, \mathbb{H})) \to C([0, a], L^2(\Omega, \mathbb{H}))$ is continuous. To this end, let the sequence $\{u_n\}_{n=1}^{\infty} \subset C([-\infty, a], L^2(\Omega, \mathbb{H}))$ such that $\lim_{n \to +\infty} u_n = u$ in $C([-\infty, a], L^2(\Omega, \mathbb{H}))$. By the continuity of the nonlinear functions $f$ and $g$, we have

$$\lim_{n \to +\infty} f(s, u_n(s)) = f(s, u(s)), \quad a.e. \ s \in J$$

and

$$\lim_{n \to +\infty} g(s, u_n(s)) = g(s, u(s)), \quad a.e. \ s \in J$$

From the conditions $(H_f)$ and $(H_g)$, we get that for a.e. $s \in J$,

$$\mathbb{E}\|f(s, u_n(s)) - f(s, u(s))\|^2 \leq 2\mathbb{E}\|f(s, u_n(s))\|^2 + 2\mathbb{E}\|f(s, u(s))\|^2 \leq 4\varphi_f(s)$$

and

$$\mathbb{E}\|g(s, u_n(s)) - g(s, u(s))\|^2 \leq 2\mathbb{E}\|g(s, u_n(s))\|^2 + 2\mathbb{E}\|g(s, u(s))\|^2 \leq 4\varphi_g(s).$$

Using the fact that the functions $s \to 4\varphi_f(s)$ and $s \to 4\varphi_g(s)$ are Lebesgue integrable for a.e. $s \in [0, t]$ and every $t \in [0, a]$, combined with Lemma 2.1, (3), (5),
Therefore, \((8)-(10)\) and the Lebesgue dominated convergence theorem, we know that
\[
E\|((Q_{u_n})(t) - (Q u)(t))\|^2 \\
\leq 2E\left\| \int_0^a G(t, s)[f(s, u_n(s)) - f(s, u(s))]ds \right\|^2 \\
+ 2E\left\| \int_0^a G(t, s)[g(s, u_n(s)) - g(s, u(s))]dW(s) \right\|^2 \\
\leq 4M^2\|\|B\|^2 \int_0^a \sum_{k=1}^m |\chi_{\{t_k\}}(s)|^2 \cdot ||U(t, s)||^2 \cdot E\|f(s, u_n(s)) - f(s, u(s))\|^2ds \\
+ 4\int_0^a |\chi_{\{t\}}(s)|^2 \cdot ||U(t, s)||^2 \cdot E\|f(s, u_n(s)) - f(s, u(s))\|^2ds \\
+ 4\text{Tr}(Q)M^2\|\|B\|^2 \int_0^a \sum_{k=1}^m |\chi_{\{t_k\}}(s)|^2 \cdot ||U(t, s)||^2 \cdot E\|g(s, u_n(s)) - g(s, u(s))\|^2ds \\
+ 4\text{Tr}(Q)\int_0^a |\chi_{\{s\}}(s)|^2 \cdot ||U(t, s)||^2 \cdot E\|g(s, u_n(s)) - g(s, u(s))\|^2ds \\
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
where
\[
\Lambda = \left[ \frac{M \sum_{k=1}^m |c_k|}{1 - M \sum_{k=1}^m |c_k|} \right]^2.
\]
(11)

The above inequality implies that that
\[
\|((Q_{u_n}) - (Q u))\|_C = \left( \sup_{t \in [0, a]} E\|((Q_{u_n})(t) - (Q u)(t))\|^2 \right)^{1/2} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Therefore, \(Q : C([0, a], L^2(\Omega, \mathbb{H})) \rightarrow C([0, a], L^2(\Omega, \mathbb{H}))\) is a continuous operator.

In what follows, we prove that there exists a positive constant \(R\) such that the operator \(Q\) defined by (10) maps the bounded closed convex set \(B_R \subset C([0, a], L^2(\Omega, \mathbb{H}))\) defined as
\[
B_R = \{ u \in C([0, a], L^2(\Omega, \mathbb{H})) : \|u\|_{L^2} \leq R \}
\]
into \(B_R\). In fact, if we choosing
\[
R \geq 4M^2(\Lambda + 1)\|2(\Lambda f + \Lambda_B) + \text{Tr}(Q)\Lambda_g\|,
\]
where
\[
\Lambda_f = \|\varphi f\|_{L([0, a], \mathbb{R}^+)} , \quad \Lambda_g = \|\varphi g\|_{L([0, a], \mathbb{R}^+)} , \quad \Lambda_B = \|\varphi_B\|_{L([0, a], \mathbb{R}^+)}. \quad (13)
\]
Then for every \( u \in B_R \), it follows from Lemma 2.1, (3), (5), (6), (8), (11)-(13) and the conditions \((H_C), (H_f), (H_g)\) and \((H_B)\) that

\[
\mathbb{E}||Qu(t)||^2 \leq 2\mathbb{E}\left|\int_0^a G(t, s)[Bv(s) + f(s, u(s))]ds\right|^2 + 2\mathbb{E}\left|\int_0^a G(t, s)g(s, u(s))d\mathbb{W}(s)\right|^2
\]

\[
\leq 4M^2||\|\int_0^a \sum_{k=1}^m |\chi_{t_k}(s)|^2 \cdot ||U(t_k, s)||^2 \cdot \mathbb{E}||Bv(s) + f(s, u(s))||^2 ds
\]

\[
+ 4\int_0^a |\chi(t)|^2 \cdot ||U(t, s)||^2 \cdot \mathbb{E}||Bv(s) + f(s, u(s))||^2 ds
\]

\[
+ 4\text{Tr}(Q)M^2||\|\int_0^a \sum_{k=1}^m |\chi_{t_k}(s)|^2 \cdot ||U(t_k, s)||^2 \cdot \mathbb{E}||g(s, u(s))||^2 ds
\]

\[
+ 4\text{Tr}(Q)\int_0^a |\chi(t)|^2 \cdot ||U(t, s)||^2 \cdot \mathbb{E}||g(s, u(s))||^2 ds
\]

\[
\leq 8M^2A\int_0^{t_k} [\varphi_f(s) + \varphi_B(s)]ds + 8M^2\int_0^{t_k} [\varphi_f(s) + \varphi_B(s)]ds
\]

\[
+ 4\text{Tr}(Q)M^2A\int_0^{t_k} \varphi_g(s)ds + 4\text{Tr}(Q)M^2\int_0^{t_k} \varphi_g(s)ds
\]

\[
\leq 4M^2(A + 1)[2(\Lambda_f + \Lambda_B) + \text{Tr}(Q)\Lambda_B] \leq R.
\]

Therefore, we have proved that \( Q : B_R \to B_R \) is a continuous operator.

Next, we show that \( Q : B_R \to B_R \) is a compact operator. To prove this, we first show that \( \{(Qu(t)) : u \in B_R\} \) is relatively compact in \( \mathbb{H} \) for every \( t \in [0, a] \). It is easy to see from (6)-(7), (8) and Lemma 2.2 that for every \( u \in B_R \),

\[
(Qu)(0) = \int_0^a G(0, s)[Bv(s) + f(s, u(s))]ds + \int_0^a G(0, s)g(s, u(s))d\mathbb{W}(s)
\]

\[
= \int_0^a \sum_{k=1}^m \chi_{t_k}U(0, 0)\mathbb{E}U(t_k, s)[Bv(s) + f(s, u(s))]ds
\]

\[
+ \int_0^a \sum_{k=1}^m \chi_{t_k}U(0, 0)\mathbb{E}U(t_k, s)g(s, u(s))d\mathbb{W}(s)
\]

\[
= \sum_{k=1}^m c_k\mathbb{E} \int_0^{t_k} U(t_k, s)[Bv(s) + f(s, u(s))]ds
\]

\[
+ \sum_{k=1}^m c_k\mathbb{E} \int_0^{t_k} U(t_k, s)g(s, u(s))d\mathbb{W}(s).
\]  

(14)

For any \( 0 < \epsilon < t_k \) \((k = 1, 2, \cdots, m)\) and \( u \in B_R \), we define the set \( \{(Qu)(0) : u \in B_R\} \) by

\[
(Qu)(0) = U(t_k, t_k - \frac{\epsilon}{2})U(t_k - \frac{\epsilon}{2}, t_k - \epsilon) \sum_{k=1}^m c_k\mathbb{E} \int_0^{t_k - \epsilon} U(t_k - \epsilon, s)
\]
For every $u_1 \in 10 PENGYU CHEN AND XUPING ZHANG$

Since $u_1 \in \{Q^e u \in Q \cup \epsilon \}$ is compact in $H$, the set $\{Q^e u(0) : u \in B_R \}$ is relatively compact in $H$ for every $\epsilon \in (0, t_k) (k = 1, 2, \cdots, m)$. Moreover, for every $u \in B_R$, by Lemma 2.1, (3), (5)-(7), (11), (14), (15) and the conditions $(H_f)$, $(H_g)$ and $(H_B)$, we get that

$$E\|Q^e u(0) - (Q u)(0)\|^2$$

$$\leq 4E\|U(t_k, t_k - \epsilon)U(t_k - \epsilon, t_k - \epsilon) \sum_{k=1}^m c_k B \int_0^{t_k - \epsilon} U(t_k, s) [Bv(s) + f(s, u(s))] ds$$

$$- \sum_{k=1}^m c_k B \int_0^{t_k - \epsilon} U(t_k, s) [Bv(s) + f(s, u(s))] ds \|^2$$

$$+ 4E\| \sum_{k=1}^m c_k B \int_{t_k - \epsilon}^{t_k} U(t_k, s) ds \|^2$$

$$+ 4E\| U(t_k, t_k - \epsilon)U(t_k - \epsilon, t_k - \epsilon) \sum_{k=1}^m c_k B \int_0^{t_k - \epsilon} U(t_k, s) g(s, u(s)) dW(s)$$

$$- \sum_{k=1}^m c_k B \int_0^{t_k - \epsilon} U(t_k, s) g(s, u(s)) dW(s) \|^2$$

$$+ 4E\| \sum_{k=1}^m c_k B \int_{t_k - \epsilon}^{t_k} U(t_k, s) g(s, u(s)) dW(s) \|^2$$

$$\leq 4\Lambda \int_{t_k - \epsilon}^{t_k} \varphi_f(s) ds + 4\Lambda \int_{t_k - \epsilon}^{t_k} \varphi_B(s) ds + 4\Lambda \int_{t_k - \epsilon}^{t_k} \psi_g(s) ds$$

$$\to 0 \quad \text{as} \quad \epsilon \to 0.$$ 

Hence, we have proved that there are relatively compact set $\{Q^e u(0) : u \in B_R \}$ arbitrarily close to the set $\{(Q u)(0) : u \in B_R \}$, this means that the set $\{Q^e u(0) : u \in B_R \}$ is relatively compact in $H$. For $0 < t \leq a$ be given, $0 < \epsilon < t$ and $u \in B_R$, we define the set $\{Q^e u(t) : u \in B_R \}$ by

$$(Q^e u)(t) = \sum_{k=1}^m c_k U(t, 0) B \int_0^{t_k} U(t_k, s) [Bv(s) + f(s, u(s))] ds$$

$$+ U(t, t - \epsilon)U(t - \epsilon, t - \epsilon) \int_0^{-\epsilon} U(t - \epsilon, s) [Bv(s) + f(s, u(s))] ds$$

$$+ \sum_{k=1}^m c_k U(t, 0) B \int_0^{t_k} U(t_k, s) g(s, u(s)) dW(s)$$

$$+ U(t, t - \epsilon)U(t - \epsilon, t - \epsilon) \int_0^{-\epsilon} U(t - \epsilon, s) g(s, u(s)) dW(s).$$
Since $U(t, t - \frac{\alpha}{2}) \in \mathcal{L}(\mathbb{H})$, $U(t - \frac{\alpha}{2}, t)$ and $U(t, 0)$ are compact in $\mathbb{H}$, the set \{$(\mathcal{Q}^\alpha u)(t) : u \in B_R$\} is relatively compact in $\mathbb{H}$ for every $\epsilon \in (0, t)$. By applying a similar method which used in (16), we can prove that there are relatively compact set \{$(\mathcal{Q}^\alpha u)(t) : u \in B_R$\} arbitrarily close to the set \{$(\mathcal{Q}u)(t) : u \in B_R$\} in $\mathbb{H}$ for $0 < t \leq a$. Therefore, the set \{$(\mathcal{Q}u)(t) : u \in B_R$\} is also relatively compact in $\mathbb{H}$ for $0 < t \leq a$. Hence, combined with this fact that the set \{$(\mathcal{Q}u)(0) : u \in B_R$\} is relatively compact in $\mathbb{H}$, we get that the set \{$(\mathcal{Q}u)(t) : u \in B_R$\} is relatively compact in $\mathbb{H}$ for $0 \leq t \leq a$.

At last, we demonstrate that \{$\mathcal{Q}u : u \in B_R$\} is a family of equicontinuous functions in $C([0, a], L^2(\Omega, \mathbb{H}))$. For any $u \in B_R$ and $0 \leq t' < t'' \leq a$, by Lemma 2.1, (5), (10) and the conditions $(H_f)$, $(H_g)$ and $(H_B)$, we have

$$
\mathbb{E}\|(\mathcal{Q}u)(t'') - (\mathcal{Q}u)(t')\|^2 = \mathbb{E}\|\int_0^{t'} G(t'', s) - G(t', s) \|[Bv(s) + f(s, u(s))]ds
$$

$$
+ \int_0^{t'} [G(t'', s) - G(t', s)]g(s, u(s))d\mathbb{W}(s)\|^2
$$

$$
\leq 6\mathbb{E}\||U(t'', 0) - U(t', 0)\| \int_0^a \sum_{k=1}^m \chi_{t_k}(s)\mathbb{E}|U(t_k, s)[Bv(s) + f(s, u(s))]ds\|^2
$$

$$
+ 6\mathbb{E}\|\int_0^a \chi_{t'}(s)|U(t'', s) - U(t', s)|[Bv(s) + f(s, u(s))]ds\|^2
$$

$$
+ 6\mathbb{E}\|\int_0^a [\chi_{t''}(s) - \chi_{t'}(s)]U(t'', s)[Bv(s) + f(s, u(s))]ds\|^2
$$

$$
+ 6\mathbb{E}\||U(t'', 0) - U(t', 0)\| \int_0^a \sum_{k=1}^m \chi_{t_k}(s)\mathbb{E}|U(t_k, s)g(s, u(s))d\mathbb{W}(s)\|^2
$$

$$
+ 6\mathbb{E}\|\int_0^a \chi_{t'}(s)|U(t'', s) - U(t', s)|g(s, u(s))d\mathbb{W}(s)\|^2
$$

$$
+ 6\mathbb{E}\|\int_0^a [\chi_{t''}(s) - \chi_{t'}(s)]U(t'', s)g(s, u(s))d\mathbb{W}(s)\|^2
$$

$$
\leq 6\mathbb{E}\||U(t'', 0) - U(t', 0)\| \int_0^a \sum_{k=1}^m \chi_{t_k}(s)\mathbb{E}|U(t_k, s)[Bv(s) + f(s, u(s))]ds\|^2
$$

$$
+ 12 \int_0^{t'} ||U(t'', s) - U(t', s)||^2 \varphi_f(s)ds + 12 \int_0^{t'} ||U(t'', s) - U(t', s)||^2 \varphi_B(s)ds
$$

$$
+ 12M^2 \varphi_f(s)ds + 12M^2 \varphi_B(s)ds
$$

$$
+ 6\mathbb{E}\||U(t'', 0) - U(t', 0)\| \int_0^a \sum_{k=1}^m \chi_{t_k}(s)\mathbb{E}|U(t_k, s)g(s, u(s))d\mathbb{W}(s)\|^2
$$

$$
+ 6\mathbb{E}\int_0^{t'} ||U(t'', s) - U(t', s)||^2 \varphi_f(s)ds + 6M^2 \varphi_f(s)ds
$$

$$
:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8,
$$
where
\[
I_1 = 6\mathbb{E}\left\| U(t'',0) - U(t',0) \right\|_2^2 \int_0^a \sum_{k=1}^m \chi_{t_k}(s) B U(t_k, s)[B v(s) + f(s,u(s))] ds,
\]
\[
I_2 = 12 \int_0^{t'} \left\| U(t'',s) - U(t',s) \right\|^2 \varphi_f(s) ds,
\]
\[
I_3 = 12 \int_0^{t'} \left\| U(t'',s) - U(t',s) \right\|^2 \varphi_B(s) ds,
\]
\[
I_4 = 12M^2 \int_{t'}^{t''} \varphi_f(s) ds,
\]
\[
I_5 = 12M^2 \int_{t'}^{t''} \varphi_B(s) ds,
\]
\[
I_6 = 6\mathbb{E}\left\| U(t'',0) - U(t',0) \right\|_2^2 \int_0^a \sum_{k=1}^m \chi_{t_k}(s) B U(t_k, s) g(s,u(s)) d\varpi(s),
\]
\[
I_7 = 6\text{Tr}(Q) \int_0^{t'} \left\| U(t'',s) - U(t',s) \right\|^2 \varphi_g(s) ds,
\]
\[
I_8 = 6M^2\text{Tr}(Q) \int_{t'}^{t''} \varphi_g(s) ds.
\]

In order to prove that \( \mathbb{E}\left\| (Qu)(t'') - (Qu)(t') \right\|^2 \to 0 \) independently of \( u \in B_R \) as \( t'' - t' \to 0 \), we only need to check \( I_1 \to 0 \) when \( t'' - t' \to 0 \) for \( i = 1, 2, \ldots, 8 \).

For \( I_1 \) and \( I_6 \), by Lemma 2.1, (5), (9), the conditions \( (H_f), (H_g) \) and \( (H_B) \), we get that
\[
\mathbb{E}\left\| \int_0^a \sum_{k=1}^m \chi_{t_k}(s) B U(t_k, s)[B v(s) + f(s,u(s))] ds \right\|^2 \\
\leq 2\Lambda \int_0^{t_k} \varphi_f(s) ds + 2\Lambda \int_0^{t_k} \varphi_B(s) ds \\
\leq 2\Lambda(\Lambda_f + \Lambda_B)
\]
and
\[
\mathbb{E}\left\| \int_0^a \sum_{k=1}^m \chi_{t_k}(s) B U(t_k, s) g(s,u(s)) d\varpi(s) \right\|^2 \\
\leq \Lambda\text{Tr}(Q) \int_0^{t_k} \varphi_g(s) ds \\
\leq \Lambda\text{Tr}(Q) \Lambda_g.
\]

Therefore, by Lemma 2.2, (17) and (18) one can easily get that \( I_1 \to 0 \) and \( I_6 \to 0 \) as \( t'' - t' \to 0 \).

For \( t' = 0, 0 < t'' \leq a \), it is easy to see that \( I_2 = I_3 = I_7 = 0 \). For \( 0 < t' < a \) and arbitrary \( 0 < \delta < t' \), by Lemma 2.1, Lemma 2.5, the conditions \( (H_f), (H_g) \) and \( (H_B), (3) \) and the arbitrariness of \( \delta \), we get that
\[
I_2 \leq 12 \int_0^{t'-\delta} \left\| U(t'',s) - U(t',s) \right\|^2 \varphi_f(s) ds \\
+ 12 \int_{t'-\delta}^{t'} \left\| U(t'',s) - U(t',s) \right\|^2 \varphi_f(s) ds \\
\leq \sup_{s \in [0,t'-\delta]} \left\| U(t'',s) - U(t',s) \right\|^2_{L^2([0,1])} \cdot 12 \int_0^{t'-\delta} \varphi_f(s) ds + 24M^2 \int_{t'-\delta}^{t'} \varphi_f(s) ds \\
\to 0 \quad \text{as} \quad t'' - t' \to 0 \quad \text{and} \quad \delta \to 0,
\]
\[ I_3 \leq 12 \int_{0}^{t' - \delta} \| U(t'', s) - U(t', s) \|^2 \varphi_B(s) ds + 12 \int_{t' - \delta}^{t'} \| U(t'', s) - U(t', s) \|^2 \varphi_B(s) ds \]
\[ \leq \sup_{s \in [0, t' - \delta]} \| U(t'', s) - U(t', s) \|^2 \int_{0}^{t' - \delta} \varphi_B(s) ds + 12 M^2 \int_{t' - \delta}^{t'} \varphi_B(s) ds \]
\[ \to 0 \quad \text{as} \quad t'' - t' \to 0 \quad \text{and} \quad \delta \to 0, \]

and
\[ I_7 \leq 6 \text{Tr}(Q) \int_{0}^{t' - \delta} \| U(t'', s) - U(t', s) \|^2 \varphi_g(s) ds + 6 \text{Tr}(Q) \int_{t' - \delta}^{t'} \| U(t'', s) - U(t', s) \|^2 \varphi_g(s) ds \]
\[ \leq \sup_{s \in [0, t' - \delta]} \| U(t'', s) - U(t', s) \|^2 \int_{0}^{t' - \delta} \varphi_g(s) ds + 12 M^2 \text{Tr}(Q) \int_{t' - \delta}^{t'} \varphi_g(s) ds \]
\[ \to 0 \quad \text{as} \quad t'' - t' \to 0 \quad \text{and} \quad \delta \to 0. \]

For \( I_4, I_5 \) and \( I_8 \), by the conditions (\( H_f \)), (\( H_g \)) and (\( H_B \)), we get that
\[ I_4 = 12 M^2 \int_{t'}^{t''} \varphi_f(s) ds \to 0 \quad \text{as} \quad t'' - t' \to 0, \]
\[ I_5 = 12 M^2 \int_{t'}^{t''} \varphi_B(s) ds \to 0 \quad \text{as} \quad t'' - t' \to 0 \]
and
\[ I_8 = 6 M^2 \text{Tr}(Q) \int_{t'}^{t''} \varphi_g(s) ds \to 0 \quad \text{as} \quad t'' - t' \to 0. \]

As a result, we have proved that \( \mathbb{E} \| (Qu)(t'') - (Qu)(t') \|^2 \to 0 \) independently of \( u \in B_R \) as \( t'' - t' \to 0 \), which means that the operator \( Q : B_R \to B_R \) is equicontinuous. Hence, combined with the Arzela-Ascoli theorem one gets that \( Q : \Omega_R \to \Omega_R \) is a compact operator. Therefore, by Schauder Fixed Point Theorem we know that \( Q \) has at least one fixed point \( u \in B_R \), which is in turn a mild solution of NP (2)-(1) on \([0, a] \). This completes the proof of Theorem 3.1.

**Remark 1.** Theorem 3.1 in this paper extends the studying of solvability for non-autonomous parabolic evolution equations with nonlocal initial conditions in [15], [18], [19], [20], [36], [47] and [48] to the case of stochastic non-autonomous evolution equations with nonlocal initial conditions.
exists a continuous function $E$ of mild solutions for NP (2)-(1) in this section, with the aid of Green’s function for NP (2)-(1) obtained in Section 3, we investigate the approximate controllability respectively, $B$ and $U(t,s)$, respectively,

$$G^*(a, s) = \sum_{k=1}^{m} \chi_{t_k}(s)U^*(a, 0)B^*U^*(t_k, s) + \chi_a(s)U^*(a, s), \quad s \in [0, a].$$

Let $u$ be the mild solution of NP (2)-(1) corresponding to the control $v \in L^2_{\mu}([0, a], U)$. Then NP (2)-(1) is said to be approximately controllable on interval $[0, a]$ if for every desired final state $u_a \in L^2(\Omega, H)$ and $\epsilon > 0$, there exists a control $v \in L^2_{\mu}([0, a], U)$ such that $u$ satisfies $\|u(a) - u_a\|^2 < \epsilon$.

In the following, we will show that for every $\mu > 0$ and $u_a \in L^2(\Omega, H)$ there exists a continuous function $u \in C([0, a], L^2(\Omega, H))$ such that

$$u(t) = \int_0^a G(t, s)[f(s, u(s)) + Bv_{\mu}(s)]ds + \int_0^a G(t, s)g(s, u(s))d\mathbb{W}(s),$$

where the function $v_{\mu}$ is the control function defined by

$$v_{\mu}(t) = B^*G^*(a, t)R(\mu, \Gamma_0)p(u(\cdot))$$

with

$$p(u(\cdot)) = u_a - \int_0^a G(a, s)f(s, u(s))ds - \int_0^a G(a, s)g(s, u(s))d\mathbb{W}(s).$$

The following natural condition is needed in our discussions:

(H_R) $\mu R(\mu, \Gamma_0) \to 0$ as $\mu \to 0^+$ in the strong operator topology.

**Remark 2.** The condition (H_R) hold if and only the following linear evolution equation

$$u'(t) = A(t)u(t) + Bv(t), \quad t \in [0, a]$$

with discrete nonlocal initial condition (1) is approximately controllable.

**Theorem 4.1.** Assume that the evolution family $\{U(t, s) : 0 \leq s \leq t \leq a\}$ generated by $\{A(t) : 0 \leq t \leq a\}$ is compact. If the conditions (H_C), (H_I), (H_2), (H_B) and (H_R) are satisfied, then NP (2)-(1) is approximately controllable on $[0, a]$.

**Proof.** By Theorem 3.1 we know that NP (2)-(1) has at least one mild solution $u_{\mu} \in B_R$, which means that for every $t \in [0, a],$

$$u_{\mu}(t) = \int_0^a G(t, s)[f(s, u_{\mu}(s)) + Bv_{\mu}(s)]ds + \int_0^a G(t, s)g(s, u_{\mu}(s))d\mathbb{W}(s) \quad (19)$$

with

$$v_{\mu}(t) = B^*G^*(a, t)R(\mu, \Gamma_0)p(u_{\mu}(\cdot)) \quad (20)$$

and

$$p(u_{\mu}(\cdot)) = u_a - \int_0^a G(a, s)f(s, u_{\mu}(s))ds - \int_0^a G(a, s)g(s, u_{\mu}(s))d\mathbb{W}(s). \quad (21)$$
Hence, combined (6), (7) and (19)-(21) with an easy computation one gets that
\[
u_\mu(a) = \int_0^a G(a,s)[f(s,u_\mu(s)) + Bu_\mu(s)]ds + \int_0^a G(a,s)\varphi(s)ds
\]
\[
= u_a - p(u_\mu(\cdot)) + \int_0^a G(a,s)BB^*G^*(a,s)\varphi(s)ds
\]
\[
= u_a - p(u_\mu(\cdot)) + \int_0^a G(a,s)\varphi(s)ds
\]
\[
= u_a - p(u_\mu(\cdot)) + \int_0^a \varphi(s)ds
\]
\[
= u_a - p(u_\mu(\cdot)) + \int_0^a \varphi(s)ds
\]
\[
= u_a - \mu R(\mu, \Gamma_0) p(u_\mu(\cdot)).
\]
(22)
In addition, from the conditions \((H_f)\) and \((H_g)\) we know that
\[
\left( \int_0^a \mathbb{E}(\|f(s,u_\mu(s))\|^2)ds \right)^{\frac{1}{2}} \leq \left( \int_0^a \varphi_f(s)ds \right)^{\frac{1}{2}} = \sqrt{\Lambda_f} < \infty
\]
(23)
and
\[
\left( \int_0^a \mathbb{E}(\|g(s,u_\mu(s))\|^2)ds \right)^{\frac{1}{2}} \leq \left( \int_0^a \varphi_g(s)ds \right)^{\frac{1}{2}} = \sqrt{\Lambda_g} < \infty,
\]
(24)
where \(\Lambda_f\) and \(\Lambda_g\) are the constants defined in (13). (23) implies that the sequence \( \{f(\cdot,u_\mu(\cdot)) \mid \mu > 0\} \) is bounded in Hilbert space \(L^2_f([0,a], \mathbb{U})\), and therefore, there exists a subsequence of \( \{f(\cdot,u_\mu(\cdot)) \mid \mu > 0\} \), still denoted by \( \{f(\cdot,u_\mu(\cdot)) \mid \mu > 0\} \), which converges weakly to some point \(F(\cdot) \in L^2_f([0,a], \mathbb{U}).\) (24) implies that the sequence \( \{g(\cdot,u_\mu(\cdot)) \mid \mu > 0\} \) is bounded in Hilbert space \(L^2_g([0,a], \mathbb{U})\), and therefore, there exists a subsequence of \( \{g(\cdot,u_\mu(\cdot)) \mid \mu > 0\} \), still denoted by \( \{g(\cdot,u_\mu(\cdot)) \mid \mu > 0\} \), which converges weakly to some point \(G(\cdot) \in L^2_g([0,a], \mathbb{U}).\)

Now, we write
\[
\varpi := u_a - \int_0^a G(a,s)F(s)ds - \int_0^a G(a,s)G(s)d\nu_\mu(s).
\]
(25)
Hence, from (21) and (25) one gets that
\[
\mathbb{E}(\|p(u_\mu) - \varpi\|^2) \leq 2\mathbb{E}\left( \int_0^a G(a,s)[f(s,u_\mu(s)) - F(s)]ds \right)^2
\]
\[
+ 2\mathbb{E}\left( \int_0^a G(a,s)[g(s,u_\mu(s)) - G(s)]d\nu(s) \right)^2.
\]
(26)
From the fact that the evolution family \(U(t,s)\) is compact operators for \(0 \leq s < t \leq a\) combined with (6) one gets that the Green’s function \(G(t,s)\) is compact for \(t > s \geq 0\). This means that the mappings
\[
f(t) \rightarrow \int_0^t G(t,s)f(s)ds\quad \text{and}\quad g(t) \rightarrow \int_0^t G(t,s)g(s)d\nu(s)
\]
are compact for \(t \in [0,a]\), which imply that
\[
\int_0^a G(a,s)[f(s,u_\mu(s)) - F(s)]ds \rightarrow 0 \quad \text{as} \quad \mu \rightarrow 0^+\quad \text{(27)}
\]
and
\[
\int_0^a G(a,s)[g(s,u_\mu(s)) - G(s)]d\nu(s) \rightarrow 0 \quad \text{as} \quad \mu \rightarrow 0^+.
\]
(28)
Therefore, from (26)-(28) we know that
\[ \mathbb{E} \| p(u_{\mu}) - \varpi \|^2 \to 0 \text{ as } \mu \to 0^+. \] (29)
Therefore, (22), (29) and the assumption \((H_R)\) imply that
\[ \mathbb{E} \| u_{\mu}(a) - u_a \|^2 \leq 2 \mathbb{E} \| \mu R(\mu, \Gamma^a_0) p(u_{\mu}) \|^2 + 2 \mathbb{E} \| \mu R(\mu, \Gamma^a_0) \|^2 \cdot \mathbb{E} \| p(u_{\mu}) - \varpi \|^2 \to 0 \text{ as } \mu \to 0^+. \]
Hence, NP (2)-(1) is approximately controllable on \([0, a]\). This completes the proof of Theorem 4.1.

Remark 3. As the reader can see, Theorem 4.1 in this paper extends the approximately controllable results of autonomous stochastic evolution equations with nonlocal initial conditions obtained by Balasubramaniam and Dauer [4], Farahi and Guendouzi [27] and Sakthivel, Ren, Deboubec and Mahmudov [42] to non-autonomous stochastic evolution equations with nonlocal initial conditions, and completely deleted the strong restriction condition on nonlocal function \(g\) which required in [4], [27] and [42]. This distinguishes the present paper from earlier works on stochastic evolution equations with nonlocal initial conditions.

Remark 4. The Green’s function defined by (6) plays an important role in proof of Theorem 4.1 in this paper.

5. Example. In order to illustrate the applicability of our main results, we consider the following non-autonomous stochastic partial differential equation with discrete nonlocal initial conditions of the form
\[
\begin{cases}
\frac{\partial}{\partial t} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) + b(t)u(x, t) = \frac{\sin(x, t, u(x, t))}{1+t^{\frac{1}{3}}} + c(x)v(t) + t^2 \cos(\pi t) dW(t) + 1 + |u(x, t)| dt, & x \in [0, \pi], t \in [0, a], \\
u(0, t) = u_0(t), & t \in [0, a], \\
u(x, 0) = \sum_{k=1}^{m} c_k u(x, t_k), & x \in [0, \pi],
\end{cases}
\] (30)
where \(W(t)\) denotes a one-dimensional standard cylindrical Wiener process defined on a stochastic space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), \(a > 0\) is a constant, \(0 < t_1 < t_2 < \cdots < t_m < a\), \(c_k\) are real numbers, \(c_k \neq 0\), \(k = 1, 2, \cdots, m\), \(c \in L^2([0, \pi], \mathbb{R})\), \(v \in L^2([0, a], \mathbb{R})\), \(b : [0, a] \to \mathbb{R}\) is a continuously differentiable function and satisfies
\[ b_{\min} := \min_{t \in [0, a]} b(t) > -1. \] (31)

Let \(\mathbb{H} = \mathbb{U} = L^2([0, \pi], \mathbb{R})\) with the norm \(\| \cdot \|_2\) and inner product \((\cdot, \cdot)\). Consider the operator \(O\) on \(\mathbb{H}\) defined by
\[ Ou = \frac{\partial^2}{\partial x^2} u \]
with domain
\[ D(O) = \{ u \in L^2([0, \pi], \mathbb{R}) : u, u' \text{ are absolutely continuous, } u'' \in L^2([0, \pi], \mathbb{R}) \text{ and } u(0) = u(\pi) = 0 \}. \]
It is well known from Pazy [41] that $O$ has a discrete spectrum, and its eigenvalues are $-n^2$, $n \in \mathbb{N}^+$ with the corresponding normalized eigenvectors $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$.

Define the bounded linear operator $A(t)$ on $L^2([0, \pi], \mathbb{R})$ by

$$A(t)u = Ou - b(t)u$$

with domain

$$D(A(t)) = D(O), \quad t \in [0, a].$$

It follows from [41, Lemma 6.1 in Chapter 7] that there are constants $\theta \in \left(\frac{\pi}{2}, \pi\right]$ and $M_1 \geq 0$ such that $A(t)$ satisfy the condition $(AT_1)$. Furthermore, by again [41, Lemma 6.1 in Chapter 7] together with continuously differentiable of coefficient $b(t)$ one know that there exist constants $M_2 > 0$ and $\vartheta, \beta \in (0, 1]$ with $\vartheta + \beta > 1$ such that for all $\lambda \in \Sigma_\theta$ and $0 \leq s \leq t \leq a$, the condition $(AT_2)$ is satisfied. Therefore, the family $\{A(t) : 0 \leq t \leq a\}$ generates an strongly continuous evolution family $\{U(t, s) : 0 \leq s \leq t \leq a\}$ defined by

$$U(t, s)u = \sum_{n=1}^{\infty} e^{-\left(f \left(a(t)\right) + n^2(t-s)\right)} \langle u, e_n \rangle e_n, \quad 0 \leq s \leq t \leq a, \quad u \in H.$$

A direct calculation gives

$$\|U(t, s)\|_{\mathcal{L}(H)} \leq e^{-\left(1 + b_{\min}\right)}, \quad 0 \leq s \leq t \leq a. \quad (32)$$

(31) and (32) means that

$$M := \sup_{0 \leq s \leq t \leq a} \|U(t, s)\|_{\mathcal{L}(H)} = 1. \quad (33)$$

Note also from [33] that, for each $t, s \in [0, a]$ with $t > s$, the evolution family $U(t, s)$ is a nuclear operator, which implies the compactness of $U(t, s)$ for $t > s$.

Let

$$u(t) = u(\cdot, t), \quad f(t, u(t)) = \frac{\sin(\cdot, t, u(\cdot, t))}{1 + t^{1/3}}, \quad g(t, u(t)) = \frac{t^2 \cos(\pi t) dW(t)}{1 + |u(\cdot, t)|^2 dt}.$$

Define the bounded linear operator $B : U \to H$ by $Bv(t) = c(\cdot) v(t)$, then the non-autonomous stochastic partial differential equation with discrete nonlocal initial conditions (30) can be rewritten into the abstract form of NP (2)-(1) in $L^2([0, \pi], \mathbb{R})$.

**Theorem 5.1.** If $\sum_{k=1}^{m} |c_k| < 1$, then the non-autonomous stochastic partial differential equation with discrete nonlocal initial conditions (30) has at least one mild solution $u \in C([0, \pi] \times [0, a])$ and it is approximately controllable on $[0, a]$.

**Proof.** By the assumption $\sum_{k=1}^{m} |c_k| < 1$ and (33) it is easy to see that the condition $(H_c)$ hold. From the definition of nonlinear functions $f, g$ and linear operator $B$ we can easily to verify that the conditions $(H_f)$, $(H_g)$ and $(H_B)$ are satisfied with

$$\varphi_f(t) = \pi t^{-\frac{2}{3}}, \quad \varphi_g(t) = t^4 \cos^2(\pi t), \quad \varphi_B(t) = v^2(t) \int_{0}^{\pi} c^2(x) dx.$$ 

Furthermore, combined Remark 2 with the fact that the associated linear evolution equation is approximately controllable on $[0, a]$, we know that the condition $(H_B)$ is satisfied. Therefore, all the assumptions of Theorem 3.1 and Theorem 4.1 are satisfied. Hence, our conclusion follows from Theorem 3.1 and Theorem 4.1. This completes the proof of Theorem 5.1. \qed
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E-mail address: chpengyu123@163.com
E-mail address: lanyu9986@126.com