Stability and Hopf Bifurcation in an Hexagonal Governor System With a Spring

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Abstract

In this paper we study the Lyapunov stability and the Hopf bifurcation in a system coupling an hexagonal centrifugal governor with a steam engine. Here are given sufficient conditions for the stability of the equilibrium state and of the bifurcating periodic orbit. These conditions are expressed in terms of the physical parameters of the system, and hold for parameters outside a variety of codimension two.

Key-words: Hexagonal governor, Watt governor, Hopf bifurcation, stability, periodic orbit.

MSC: 70K50, 70K20.
1 Introduction

The centrifugal governor is a device that automatically controls the speed of an engine. The most important one, invented by James Watt in 1788 — Watt governor —, is regarded as the starting landmark for the theory of automatic control. The historical relevance of this device as well as its importance for present day theoretical and technological control developments — going from steam to diesel, gasoline engines and electronic governors — have been widely discussed by MacFarlane [6], Denny [2], Fasol [3] and Wellstead - Readman [12] among others.

The centrifugal governor design received several important modifications as well as other types of governors were also developed. From MacFarlane [6], p. 251, we quote:

“Several important advances in automatic control technology were made in the latter half of the 19th century. A key modification to the flyball governor was the introduction of a simple means of setting the desired running speed of the engine being controlled by balancing the centrifugal force of the flyballs against a spring, and using the preset spring tension to set the running speed of the engine”.

This paper is devoted to the study of the dynamic stability and simplest bifurcations of the system coupling the hexagonal centrifugal governor with a spring — called Hexagonal governor — and the steam engine. See Fig. 1 for an illustration. The system coupling the Hexagonal governor (resp. Watt governor, with no spring and with vanishing horizontal edges of the hexagon) and the steam engine will be called simply the Hexagonal Governor System (HGS) (resp. Watt Governor System (WGS)). The stability analysis of the stationary states and of small amplitude oscillations of this system will be pursued here.

The first mathematical analysis of the stability conditions in the WGS was due to Maxwell [7] and, in a user friendly style, likely to be better understood by engineers, by Vyshnegradskii [11]. A simplified version of the WGS local
stability based on the work of Vyshnegradskii is presented by Pontryagin [8].

The oscillatory, small amplitude, behavior in the WGS has been associated to a periodic orbit that appears from a Hopf bifurcation. This was established by Hassard et al. in [4], Al-Humadi and Kazarinoff in [1] and, in a more general context, by the authors in [9, 10].

In [10], restricting ourselves to Pontryagin’s system of differential equations for the WGS, we carried out a deeper investigation of the stability of the equilibrium along the critical Hopf bifurcations up to codimension 3, happening at a unique point at which the bifurcation diagram was established. A conclusion derived from the properties of the bifurcation diagram implied the existence of parameters where the WGS has an attracting periodic orbit coexisting with an attracting equilibrium.

The results of the present paper extend in a different direction the analysis in [9], as described below.

In Section 2 we introduce the differential equations that model the HGS illustrated in Fig. 1. The stability of the equilibrium point of this model is analyzed and a general version of the stability condition is obtained and presented in the terminology of Vyshnegradskii (Theorem 2.2 and Remark 2.3). The codimension 1 Hopf bifurcation for the HGS differential equations is studied in Section 3. An expression which determines the sign of the first Lyapunov coefficient is obtained (Theorem 3.1). Sufficient conditions for the stability of the bifurcating periodic orbit are given. Two pertinent particular cases (no spring and vanishing horizontal edge) are calculated and illustrated. See Theorem 3.6, Fig. 2 and Theorem 3.8, Fig. 3.

Concluding comments are presented in Section 4.
2 The Hexagonal governor system

2.1 Hexagonal governor differential equations

The HGS studied in this paper is shown in Fig. 1. There, \( \varphi \in (0, \frac{\pi}{2}) \) is the angle of deviation of the arms of the governor from its vertical direction axis \( S_1 \), \( \Omega \in [0, \infty) \) is the angular velocity of the rotation of the engine flywheel \( D \), \( \theta \) is the angular velocity of the rotation of \( S_1 \), \( l \) is the length of the arms, \( m \) is the mass of each ball, \( H \) is a sleeve which supports the arms and slides along \( S_1 \), \( T \) is a set of transmission gears and \( V \) is the valve that determines the supply of steam to the engine.

![Hexagonal centrifugal governor — steam engine system](image)

Figure 1: Hexagonal centrifugal governor — steam engine system.

The HGS differential equations can be found as follows. For simplicity, we neglect the mass of the sleeve and of the arms. There are four forces acting on the balls at all times. They are the tangential component of the gravity

\[ -mg \sin \varphi, \]

where \( g \) is the standard acceleration of gravity; the tangential component of
the centrifugal force
\[ m(L + l \sin \varphi)\theta^2 \cos \varphi, \]

\( 2L \geq 0 \) is the distance \( AA' \) in Fig. \[ \] the tangential component of the restoring force due to the spring
\[ -2kl(1 - \cos \varphi) \sin \varphi, \]

\( 2l \) is the natural length of the spring and \( k \geq 0 \) is the spring constant; and the force of friction
\[ -bl\dot{\varphi}, \]

\( b > 0 \) is the friction coefficient.

From Newton’s Second Law of Motion, using the transmission function \( \theta = c \Omega \), where \( c > 0 \), one has
\[ \ddot{\varphi} = c^2 \frac{L}{l} \Omega^2 \cos \varphi + \left( \frac{2k}{m} + c^2 \Omega^2 \right) \sin \varphi \cos \varphi - \frac{2kl + mg}{ml} \sin \varphi - \frac{b}{m} \dot{\varphi}. \] (1)

The torque acting on the flywheel \( D \) is
\[ I \dot{\Omega} = \mu \cos \varphi - F, \] (2)

where \( I \) is the moment of inertia of the flywheel, \( F \) is an equivalent torque of the load and \( \mu > 0 \) is a proportionality constant to represent the torque due to the steam which decreases with the angle \( \varphi \). See [8], p. 217, for more details.

From Eqs. (1) and (2) the differential equations of our model are given by
\[
\begin{align*}
\frac{d \varphi}{d\tau} &= \psi \\
\frac{d \psi}{d\tau} &= c^2 \frac{L}{l} \Omega^2 \cos \varphi + \left( \frac{2k}{m} + c^2 \Omega^2 \right) \sin \varphi \cos \varphi - \frac{2kl + mg}{ml} \sin \varphi - \frac{b}{m} \psi \\
\frac{d \Omega}{d\tau} &= \frac{1}{I} (\mu \cos \varphi - F)
\end{align*}
\] (3)
where $\tau$ is the time.

The standard Watt governor differential equations as presented in Pontryagin [8], p. 217, are obtained from (3) by taking $L = 0$ and $k = 0$,

$$
\begin{align*}
\frac{d \varphi}{d\tau} &= \psi \\
\frac{d \psi}{d\tau} &= e^2 \Omega^2 \sin \varphi \cos \varphi - \frac{g}{l} \sin \varphi - \frac{b}{m} \psi \\
\frac{d \Omega}{d\tau} &= \frac{1}{I} (\mu \cos \varphi - F)
\end{align*}
$$

(4)

Performing the following changes in the coordinates, parameters and time

$$
\begin{align*}
x &= \varphi, \quad y = \left( \frac{ml}{2kl + mg} \right)^{1/2} \psi, \quad z = c \left( \frac{ml}{2kl + mg} \right)^{1/2} \Omega, \\
t &= \left( \frac{2kl + mg}{ml} \right)^{1/2} \tau, \quad \rho = \frac{L}{l}, \quad \kappa = \frac{2kl}{2kl + mg}, \\
\varepsilon &= \frac{b}{m} \left( \frac{ml}{2kl + mg} \right)^{1/2}, \quad \alpha = \frac{c \mu}{I} \left( \frac{ml}{2kl + mg} \right), \quad \beta = \frac{F}{\mu},
\end{align*}
$$

(5)

where $\rho \geq 0$, $0 \leq \kappa < 1$, $\varepsilon > 0$, $\alpha > 0$ and $0 < \beta < 1$, the differential equations (3) can be written as

$$
\begin{align*}
x' &= \frac{dx}{dt} = y \\
y' &= \frac{dy}{dt} = \rho \; z^2 \cos x + (z^2 + \kappa) \sin x \cos x - \sin x - \varepsilon \; y \\
z' &= \frac{dz}{dt} = \alpha \; (\cos x - \beta)
\end{align*}
$$

(6)

or equivalently by

$$
\mathbf{x}' = f(\mathbf{x}, \zeta),
$$

(7)

where

$$
f(\mathbf{x}, \zeta) = (y, \rho \; z^2 \cos x + (z^2 + \kappa) \sin x \cos x - \sin x - \varepsilon \; y, \alpha \; (\cos x - \beta)),
$$

$$
\mathbf{x} = (x, y, z) \in \left( 0, \frac{\pi}{2} \right) \times \mathbb{R} \times [0, \infty)
$$

and

$$
\zeta = (\beta, \alpha, \varepsilon, \rho, \kappa) \in (0, 1) \times (0, \infty) \times (0, \infty) \times [0, \infty) \times [0, 1).
$$
2.2 Stability analysis at the equilibrium point

The HGS differential equations (6) have only one admissible equilibrium point

\[ P_0 = (x_0, y_0, z_0) = \left( \arccos \beta, 0, \frac{(1 - \kappa \beta)^{1/2}(1 - \rho^2)^{1/4}}{\beta^{1/2}(\rho + (1 - \beta^2)^{1/2})^{1/2}} \right). \]  \hfill (8)

The Jacobian matrix of \( f \) at \( P_0 \) has the form

\[
Df \left( P_0 \right) = \begin{pmatrix}
0 & 1 & 0 \\
-\omega_0^2 & -\varepsilon & \xi \\
-\alpha (1 - \beta^2)^{1/2} & 0 & 0
\end{pmatrix},
\]  \hfill (9)

where

\[
\omega_0 = \sqrt{\frac{(1 - \beta^2)^{3/2} + \rho(1 - \kappa \beta^3)}{\beta(\rho + (1 - \beta^2)^{1/2})}} \]  \hfill (10)

and

\[ \xi = 2\beta^{1/2}(1 - \beta^2)^{1/4}(1 - \kappa \beta)^{1/2}(\rho + (1 - \beta^2))^{1/2}. \]

For the sake of completeness we state the following lemma whose proof can be found in \[8\], p. 58.

**Lemma 2.1** The polynomial \( L(\lambda) = p_0 \lambda^3 + p_1 \lambda^2 + p_2 \lambda + p_3, \) \( p_0 > 0 \), with real coefficients has all roots with negative real parts if and only if the numbers \( p_1, p_2, p_3 \) are positive and the inequality \( p_1 p_2 > p_0 p_3 \) is satisfied.

**Theorem 2.2** If

\[ \varepsilon > \varepsilon_c = \frac{2\alpha \beta^{3/2}(1 - \beta^2)^{3/4}(1 - \kappa \beta)^{1/2}(\rho + (1 - \beta^2)^{1/2})^{3/2}}{(1 - \beta^2)^{3/2} + \rho(1 - \kappa \beta^3)}, \]

then the HGS differential equations (6) have an asymptotically stable equilibrium point at \( P_0 \). If

\[ 0 < \varepsilon < \varepsilon_c \]

then \( P_0 \) is unstable.
Proof. The characteristic polynomial of $Df(P_0)$ is given by $p(\lambda)$, where

$$-p(\lambda) = \lambda^3 + p_1 \lambda^2 + p_2 \lambda + p_3,$$

$p_1 = \varepsilon$, $p_2 = \frac{(1 - \beta^2)^{3/2} + \rho(1 - \kappa \beta^3)}{\beta(\rho + (1 - \beta^2)^{1/2})}$

and

$$p_3 = \frac{2\alpha \beta^{3/2}(1 - \beta^2)^{3/4}(1 - \kappa \beta)^{1/2}(\rho + (1 - \beta^2)^{1/2})^{3/2}}{\beta(\rho + (1 - \beta^2)^{1/2})}.$$

The coefficients of $-p(\lambda)$ are positive. Thus a necessary and sufficient condition for the asymptotic stability of the equilibrium point $P_0$, as provided by the condition for one real negative root and a pair of complex conjugate roots with negative real part, is given by (11), according to Lemma 2.1.

Remark 2.3 In terms of the HGS physical parameters, condition (11) is equivalent to

$$\frac{bI}{m} \eta > 1,$$

where

$$\eta = \frac{|d\Omega_d|}{dF} = \frac{(1 - \beta^2)^{3/2} + \rho - \beta^3 \kappa \rho}{2 \beta^{3/2}(1 - \beta^2)^{3/4}(1 - \kappa \beta)^{1/2}((1 - \beta^2)^{1/2} + \rho)^{3/2}}$$

is the non-uniformity of the performance of the engine which quantifies the change in the engine speed with respect to the load (see [8], p. 219, for more details). Eq. (13) can be written in terms of the original parameters of the HGS, but this expression is too long to be put in print.

The rules formulated by Vyshnegradskii to enhance the stability of the system follow directly from (12). In particular, the interpretation of (12) is that a sufficient amount of damping $-b$— must be present relative to the other physical parameters for the system to be stable at the desired operating speed. Condition (12) is equivalent to the original condition given by Vyshnegradskii for the WGS (see [8], p. 219).
3 Hopf bifurcation analysis

In this section we study the stability of $P_0$ under the condition

$$\varepsilon = \varepsilon_c,$$  \hspace{1cm} (14)

that is, on the Hopf hypersurface which is complementary to the range of validity of Theorem 2.2.

3.1 Generalities on Hopf bifurcations

The study outlined below is based on the approach found in the book of Kuznetsov [5], pp 177-181.

Consider the differential equations

$$x' = f(x, \mu),$$  \hspace{1cm} (15)

where $x \in \mathbb{R}^3$ and $\mu \in \mathbb{R}^m$ is a vector of control parameters. Suppose (15) has an equilibrium point $x = x_0$ at $\mu = \mu_0$ and represent

$$F(x) = f(x, \mu_0)$$  \hspace{1cm} (16)

as

$$F(x) = Ax + \frac{1}{2} B(x, x) + \frac{1}{6} C(x, x, x) + O(||x||^4),$$  \hspace{1cm} (17)

where $A = f_x(0, \mu_0)$ and

$$B_i(x, y) = \sum_{j,k=1}^3 \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k} \bigg|_{\xi=0} x_j y_k,$$  \hspace{1cm} (18)

$$C_i(x, y, z) = \sum_{j,k,l=1}^3 \frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} \bigg|_{\xi=0} x_j y_k z_l,$$  \hspace{1cm} (19)

for $i = 1, 2, 3$. Here the variable $x - x_0$ is also denoted by $x$.

Suppose $(x_0, \mu_0)$ is an equilibrium point of (15) where the Jacobian matrix $A$ has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i \omega_0$, $\omega_0 > 0$, and no other critical (i.e., on the imaginary axis) eigenvalues.
The two dimensional center manifold can be parametrized by \( w \in \mathbb{R}^2 = \mathbb{C} \), by means of \( x = H(w, \bar{w}) \), which is written as

\[
H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2 \leq j+k \leq 3} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^4),
\]

with \( h_{jk} \in \mathbb{C}^3 \), \( h_{jk} = \bar{h}_{kj} \).

Substituting these expressions into (15) and (17) one has

\[
H_w(w, \bar{w})w' + H_{\bar{w}}(w, \bar{w})\bar{w}' = F(H(w, \bar{w})).
\]  

(20)

Let \( p, q \in \mathbb{C}^3 \) be vectors such that

\[
Aq = i\omega_0 q, \quad A^\top p = -i\omega_0 p, \quad \langle p, q \rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1.
\]  

(21)

The complex vectors \( h_{ij} \) are to be determined so that equation (20) writes as follows

\[
w' = i\omega_0 w + \frac{1}{2} G_{21} w|w|^2 + O(|w|^4),
\]  

(22)

with \( G_{21} \in \mathbb{C} \).

Solving the linear system obtained by expanding (20), the coefficients of the quadratic terms of (16) lead to

\[
h_{11} = -A^{-1}B(q, \bar{q}),
\]  

(23)

\[
h_{20} = (2i\omega_0 I_3 - A)^{-1}B(q, q),
\]  

(24)

where \( I_3 \) is the unit \( 3 \times 3 \) matrix.

The coefficients of the cubic terms are also uniquely calculated, except for the term \( w^2\bar{w} \), whose coefficient satisfies a singular system for \( h_{21} \)

\[
(i\omega_0 I_3 - A)h_{21} = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q,
\]  

(25)

which has a solution if and only if

\[
\langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q \rangle = 0.
\]
Therefore

\[ G_{21} = \langle p, C(q, q, \bar{q}) + B(\bar{q}, (2i\omega_0I_3 - A)^{-1}B(q, q)) - 2B(q, A^{-1}B(q, \bar{q}))) \rangle. \]  

(26)

The first Lyapunov coefficient \( l_1 \) is defined by

\[ l_1 = \frac{1}{2\omega_0} \text{Re} \ G_{21}. \]  

(27)

From (22) its sign decides the stability, when negative, or instability, when positive, of the equilibrium.

A Hopf point \((x_0, \mu_0)\) is an equilibrium point of (15) where the Jacobian matrix \( A \) has a pair of purely imaginary eigenvalues \( \lambda_{2,3} = \pm i\omega_0, \omega_0 > 0 \), and no other critical eigenvalues. At a Hopf point, a two dimensional center manifold is well-defined, which is invariant under the flow generated by (15) and can be smoothly continued to nearby parameter values.

A Hopf point is called transversal if the curves of complex eigenvalues cross the imaginary axis with non-zero derivative.

In a neighborhood of a transversal Hopf point with \( l_1 \neq 0 \) the dynamic behavior of the system (15), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to the complex normal form

\[ w' = (\gamma + i\omega)w + l_1|w|^2, \]  

(28)

\( w \in \mathbb{C}, \gamma, \omega \) and \( l_1 \) are smooth continuations of 0, \( \omega_0 \) and the first Lyapunov coefficient at the Hopf point [5]. When \( l_1 < 0 \) \((l_1 > 0)\) a family of stable (unstable) periodic orbits can be found on this family of center manifolds, shrinking to the equilibrium point at the Hopf point.

### 3.2 Hopf bifurcation in the HGS

From [7] write the Taylor expansion (17) of \( f(x) \). Define

\[ \omega_1 = \sqrt{\frac{1 - \beta^2}{\beta}} \]  

(29)
\[ \sigma = \sqrt{\frac{1 - \kappa \beta}{\rho + \omega_1 \beta^{1/2}}} \]  

(30)

Thus, with \( \omega_0 \) given in Eq. (10),

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
-\omega_0^2 & -\varepsilon_c & \frac{\varepsilon_c \omega_0^2}{\alpha \beta^{1/2} \omega_1} \\
-\alpha \beta^{1/2} \omega_1 & 0 & 0
\end{pmatrix},
\]  

(31)

and, with the notation in (17) we have

\[
F(x) - Ax = (0, F_2(x) + O(\|x\|^4), F_3(x) + O(\|x\|^4)) ,
\]  

(32)

where

\[
F_2(x) = -\frac{3}{2} \beta^{1/2} \omega_1 (1 - \rho \sigma^2) x^2 + 2\sigma (\beta^{1/2} \omega_1)^{1/2} (2 \beta^2 - 1 - \beta^{1/2} \rho \omega_1) x z + \\
\beta (\rho + \beta^{1/2} \omega_1) z^2 + \frac{1 + (3 - 7 \beta^2) (1 - \rho \sigma^2)}{6 \beta} x^3 - \\
\beta^{1/2} \sigma (\beta^{1/2} \omega_1)^{1/2} (\rho + 4 \beta^{1/2} \omega_1) x^2 z + (2 \beta^2 - 1 - \beta^{1/2} \rho \omega_1) x z^2,
\]

and

\[
F_3(x) = -\frac{1}{2} \alpha \beta x^2 + \frac{1}{6} \alpha \beta^{1/2} \omega_1 x^3.
\]

From (31) the eigenvalues of \( A \) are

\[
\lambda_1 = -\varepsilon_c, \quad \lambda_2 = i \omega_0, \quad \lambda_3 = -i \omega_0.
\]

(33)

The eigenvectors \( q \) and \( p \) satisfying (21) are respectively

\[
q = \left( -i, \omega_0, \frac{\alpha \beta^{1/2} \omega_1}{\omega_0} \right)
\]  

(34)

and

\[
p = \left( -\frac{i}{2} \omega_0 - i \varepsilon_c, \omega_0 (\varepsilon_c + i \omega_0), -\frac{i}{2} \varepsilon_c \omega_0 (\varepsilon_c + i \omega_0), 2 \alpha \beta^{1/2} \omega_1 (\omega_0^2 + \varepsilon_c^2) \right).
\]

(35)

The main result of this section can be formulated now.
Theorem 3.1 Consider the family of differential equations \([6]\). The first Lyapunov coefficient at the point \((8)\) for parameter values satisfying \((14)\) is given by

\[
l_1(\beta, \alpha, \rho, \kappa) = -\frac{R(\beta, \alpha, \rho, \kappa)}{4\beta \varepsilon \omega_0^2 (\varepsilon_1^2 + 5\varepsilon_0^2 + 4\omega_1^4)}, \tag{36}
\]

where

\[
R(\beta, \alpha, \rho, \kappa) = \varepsilon_0^2 \omega_0^2 (3\rho \sigma^2 - 4)(\varepsilon_1^2 + 4\omega_0^2) + 8\alpha^2 \beta^{11/4} \varepsilon \sigma \omega_1^4 \omega_1^2 (\varepsilon_1^2 + 4\omega_0^2) + 8\alpha^2 \beta^{3/2} \varepsilon \sigma^2 \omega_0^2 \omega_1^2 - 4\alpha^3 \beta^{13/4} \varepsilon \sigma \omega_1^2 (\varepsilon_1^2 + 10\omega_0^2) + 8\alpha^3 \beta^{21/4} \varepsilon \sigma \omega_1^2 (\varepsilon_1^2 + 10\omega_0^2) - 4\alpha^5 \beta^4 \varepsilon \sigma \omega_1^2 (\varepsilon_1^2 + 10\omega_0^2) + 4\alpha^5 \beta^6 \varepsilon \sigma \omega_1^2 (\varepsilon_1^2 + 8\omega_0^2) - 2\alpha \beta^7 \varepsilon \sigma \omega_1^2 \omega_1^2 (14\omega_0^4 + 3\varepsilon \omega_1^2 (\rho \sigma^2 - 1) + 30\omega_0^2 \omega_1^2 (\rho \sigma^2 - 1)) + 8\beta^{11/2} (4\alpha^2 \varepsilon_0^2 \sigma^2 \omega_1^2 \omega_1^2 + \alpha^4 \rho \omega_1^2 (\varepsilon_1^2 + 8\omega_0^2) - 4\alpha \beta^{11/4} \varepsilon \sigma \omega_1^2 (\varepsilon_1^2 + 10\omega_0^2) + 30\omega_0^2 \omega_1^2 (\rho \sigma^2 - 1)) + 8\beta^{13/2} (4\alpha^2 \varepsilon \sigma \omega_0^2 \omega_1^2 \omega_1^2 + \alpha^4 \rho \omega_1^2 (\varepsilon_1^2 + 8\omega_0^2) - 4\alpha \beta^{13/4} \varepsilon \sigma \omega_1^2 (\varepsilon_1^2 + 10\omega_0^2) + 30\omega_0^2 \omega_1^2 (\rho \sigma^2 - 1)) + 8\beta^{15/2} (4\alpha^2 \varepsilon_0^2 \sigma^2 \omega_1^2 \omega_1^2 + \alpha^4 \rho \omega_1^2 (\varepsilon_1^2 + 8\omega_0^2) - 4\alpha \beta^{15/4} \varepsilon \sigma \omega_1^2 (\varepsilon_1^2 + 10\omega_0^2) + 30\omega_0^2 \omega_1^2 (\rho \sigma^2 - 1)).
\]

Proof. The proof depends on preliminary calculations presented below. From \([17], [18], [19]\) and \([32]\) one has

\[
B(\mathbf{x}, \mathbf{y}) = (0, B_2(\mathbf{x}, \mathbf{y}), -\alpha \beta x_1 y_1), \tag{37}
\]

where

\[
B_2(\mathbf{x}, \mathbf{y}) = -3 \beta^{1/2} \omega_1 (1 - \rho \sigma^2) x_1 y_1 + 2\beta (\rho + \omega_1 \beta^{1/2}) x_3 y_3 + \frac{2\sigma (\beta^{1/2} \omega_1)^{1/2} (2\beta^2 - 1 - \rho \omega_1 \beta^{1/2})}{\beta^{1/2}} (x_1 y_3 + x_3 y_1),
\]

\[13\]
The first Lyapunov coefficient is given by (27). From (35) and (41) one has

\[ C(x, y, z) = \left(0, C_2(x, y, z), \alpha \beta^{1/2} \omega_1 x_1 y_1 z_1 \right), \quad (38) \]

where

\[ C_2(x, y, z) = \frac{1 + (1 - \rho \sigma^2) (3 - 7 \beta^2)}{\beta} x_1 y_1 z_1 + 2 (2 \beta^2 - 1 - \rho \beta^{1/2} \omega_1) \]

\[ (x_1 y_3 z_3 + x_3 y_1 z_3 + x_3 y_3 z_1) - 2 \beta^{1/2} \sigma (\beta^{1/2} \omega_1)^{1/2} (\rho + 4 \beta^{1/2} \omega_1) \]

\[ (x_1 y_1 z_3 + x_1 y_3 z_1 + x_3 y_1 z_1). \]

Referring to the notation in (37), (38) and (34) one has

\[ B(q, \bar{q}) = (0, B_2(q, q), \alpha \beta), \quad (39) \]

where

\[ B_2(q, q) = \frac{\beta \omega_x^2}{\omega_0^2 (\beta^{1/2} \omega_1)^{3/2}} \left[ 2 \alpha^2 \beta (\beta^{1/2} \omega_1)^{3/2} (\rho + \beta^{1/2} \omega_1) + 3 (1 - \rho \sigma^2) \omega_0^2 (\beta^{1/2} \omega_1)^{1/2} + i 4 \alpha \sigma \omega_0 \omega_1 (1 - 2 \beta^2 + \beta^{1/2} \rho \omega_1) \right], \]

\[ B(q, \bar{q}) = (0, B_2(q, \bar{q}), -\alpha \beta), \quad (40) \]

where

\[ B_2(q, \bar{q}) = \frac{\beta^{1/2} \omega_1 (3 \omega_0^2 (\rho \sigma^2 - 1) + 2 \alpha^2 \beta^{3/2} \omega_1 (\rho + \beta^{1/2} \omega_1))}{\omega_0^2}, \]

\[ C(q, q, \bar{q}) = \left(0, C_2(q, q, \bar{q}), -i \alpha \beta^{1/2} \omega_1 \right), \quad (41) \]

where

\[ C_2(q, q, \bar{q}) = \frac{-i}{\beta \omega_0} \left[ \omega_0^2 (4 - 3 \rho \sigma^2 + 7 \beta^2 (\rho \sigma^2 - 1)) + 2 \alpha^2 \beta^2 \omega_1^3 \
\[ (2 \beta^2 - 1 - \beta^{1/2} \rho \omega_1) - i 2 \alpha \beta^2 \omega_0 \omega_1 (\beta^{1/2} \omega_1)^{1/2} (\rho + 4 \beta^{1/2} \omega_1) \right]. \]

The first Lyapunov coefficient is given by (27). From (35) and (41) one has

\[ \text{Re}(p, C(q, q, \bar{q})) = \frac{-1}{2 \beta \omega_0^2 (\epsilon_x^2 + \omega_0^2)} \left[ 2 \alpha \beta^{9/4} \sigma \omega_0^2 \omega_1^{3/2} (\rho + 4 \beta^{1/2} \omega_1) + \omega_0^2 (3 \rho \sigma^2 - 4 + 7 \beta^2 (1 - \rho \sigma^2)) + \beta \omega_1^4 \right] + \epsilon_c \left( \omega_0^2 (3 \rho \sigma^2 - 4 + 7 \beta^2 (1 - \rho \sigma^2)) + \beta \omega_1^4 + 2 \alpha^2 \beta^2 \omega_1^2 (1 - 2 \beta^2 + \beta^{1/2} \rho \omega_1) \right). \]
From (35), (37), (34) and (24) one has

\[
\begin{align*}
\text{Re}(p, 2B(q, h_{12})) &= \frac{-\beta^{3/4}}{\varepsilon_c \omega_0^4 \omega_1^1 (\varepsilon_e^2 + \omega_0^2)} \bigg[ 4\alpha^3 \varepsilon_c \rho \sigma \omega_1^{11/2} (2\beta^{7/2} - \beta^{3/2}) + \\
4\alpha^4 \beta^{13/4} \rho^2 \omega_1^5 + 8\alpha^3 \beta^3 \varepsilon_c \sigma \omega_1^{13/2} + 8\alpha^4 \beta^{15/4} \rho \omega_1^7 - 4\alpha^3 \beta^{5/2} \varepsilon_c \rho \sigma \omega_1^{15/2} + \\
4\alpha^4 \beta^{17/4} \omega_1^8 + 8\alpha^3 \beta^3 \varepsilon_c \omega_0^4 \omega_1^1 (\omega_0^2 + \omega_1^2 (\rho \sigma^2 - 1)) + \\
2\alpha^2 \beta^{7/4} \rho \omega_0^2 \omega_1^3 (\omega_0^2 + \omega_1^2 (\rho \sigma^2 - 1)) + \\
2\alpha^2 \beta^{9/4} \omega_0^2 \omega_1^4 (\omega_0^2 + \omega_1^2 (\rho \sigma^2 - 1)) - \\
2\alpha \varepsilon_c \sigma \omega_0^2 \omega_1^{5/2} (2\omega_0^2 + \omega_1^2 (\rho \sigma^2 - 1)) - \\
2\alpha \beta^{1/2} \varepsilon_c \rho \sigma \omega_0^2 \omega_1^{7/2} (2\omega_0^2 + \omega_1^2 (\rho \sigma^2 - 1)) - \\
4\alpha \beta^2 \varepsilon_c \sigma \omega_1^{5/2} (\alpha^2 \omega_1^5 (1 + \rho^2) - 3 \omega_0^2 \omega_1^2 (\rho \sigma^2 - 1) - 2 \omega_1^4) \bigg].
\end{align*}
\]

From (35), (37), (34) and (24) one has

\[
\text{Re}(p, B(\bar{q}, h_{20})) = \frac{\vartheta(\beta, \alpha, \rho, \kappa)}{2 \omega_0^4 \omega_1^1 (\varepsilon_e^4 + 5 \varepsilon_e^2 \omega_0^2 + 4 \omega_0^4)},
\]
Substituting (42), (43) and (44) into (26) and (27), the theorem is proved.

Remark 3.2 The denominator of the first Lyapunov coefficient given by Eq. (36) is positive. Thus the sign of $l_1$ is determined by the sign of the function $R$, the numerator of $l_1$.

The expression for $l_1$ depends only the parameters $\alpha, \beta, \rho$ and $\kappa$, although in the expression in (36) appear also $\omega_0, \omega_1, \sigma$ and $\varepsilon_c$. This is due to the fact that these last parameters are functions of the previous ones as shown in (10), (29) and (30).

Proposition 3.3 Consider the family of differential equations (6) regarded as dependent on the parameter $\varepsilon$. The real part, $\gamma$, of the pair of complex eigenvalues verifies

$$\gamma'(\varepsilon_c) = -\frac{\omega_0^2}{2(\omega_0^2 + \varepsilon_c^2)} < 0.$$  (45)

Therefore, the transversality condition holds at the Hopf point.

Proof. Let $\lambda(\varepsilon) = \lambda_{2,3}(\varepsilon) = \gamma(\varepsilon) \pm i\omega(\varepsilon)$ be eigenvalues of $A(\varepsilon)$ such that $\gamma(\varepsilon_c) = 0$ and $\omega(\varepsilon_c) = \omega_0$, according to (33). Taking the inner product of $p$ with the derivative of $A(\varepsilon)q(\varepsilon) = \lambda(\varepsilon)q(\varepsilon)$ at $\varepsilon = \varepsilon_c$ one has

$$\left\langle p, \frac{dA}{d\varepsilon} \bigg|_{\varepsilon=\varepsilon_c} q \right\rangle = \gamma'(\varepsilon_c) \pm \omega'(\varepsilon_c).$$

Thus the transversality condition is given by

$$\gamma'(\varepsilon_c) = \Re \left\langle p, \frac{dA}{d\varepsilon} \bigg|_{\varepsilon=\varepsilon_c} q \right\rangle.$$  (46)

As

$$\frac{dA}{d\varepsilon} \bigg|_{\varepsilon=\varepsilon_c} q = (0, -\omega_0, 0),$$

the proposition follows from (35) and a simple calculation.
The full expression of \( l_1 \) in terms of the parameters \( \alpha, \beta, \rho, \kappa \) seems too long to be of use in qualitative arguments. Two special cases are considered below for the sake of illustration.

### 3.2.1 The case \( \rho = 0 \)

**Corollary 3.4** Consider the case where \( \rho = 0 \). Then the equilibrium point \( P_0 \) in (8) is given by

\[
P_0 = (x_0, y_0, z_0) = \left( \arccos \beta, 0, \left( \frac{1}{\beta} - \kappa \right)^{1/2} \right),
\]

the Hopf hypersurface (14) is given by

\[
\varepsilon_c = \varepsilon_c(\beta, \alpha, \kappa) = 2 \alpha \beta^{3/2} (1 - \kappa \beta)^{1/2}
\]

and the numerator of \( l_1 \) in (36) is given by

\[
G_1(\beta, \alpha, \kappa) = -3 + 5 \kappa \beta - (\alpha^2 - 5)\beta^2 + \kappa(\alpha^2 - 7)\beta^3 - 2\alpha^2 \kappa^2 \beta^4 - (\alpha^4 - 2\alpha^2 \kappa^2) \beta^6 + \alpha^4 \kappa^2 \beta^7.
\]

If \( G_1 \) is different from zero then the family of HGS differential equations (6) has a transversal Hopf point at \( P_0 \) for \( \varepsilon_c = 2 \alpha \beta^{3/2} (1 - \kappa \beta)^{1/2} \).

**Proof.** The proof is immediate by substituting \( \rho = 0 \) into Eqs. (8), (14) and (36). A sufficient condition for being a Hopf point is that the first Lyapunov coefficient \( l_1 \neq 0 \), since the transversality condition is satisfied by Proposition 3.3. But from (49) it is equivalent to \( G_1 \neq 0 \).

Remark 3.5 The expression (47) shows that the “running speed” of the system depends monotonically decreasing on \( \kappa \), which is monotonically increasing on \( k \), according to (5). This corroborates analytically the quotation of MacFarlane in the Introduction.
Equation (49) gives a simple expression to determine the sign of the first Lyapunov coefficient (36) for the case $\rho = 0$. The graph $G_1(\beta, \alpha, \kappa) = 0$ is illustrated in Fig. 2, where the signs of the first Lyapunov coefficient are also represented. The surface $l_1 = 0$ divides the hypersurface of critical parameters $\varepsilon_c = 2\alpha \beta^{3/2} (1 - \kappa \beta)^{1/2}$ into two connected components denoted by $S$ and $U$ where $l_1 < 0$ and $l_1 > 0$ respectively. In Fig. 2 the $\beta$ coordinates at the reference points $B_1$ and $B_2$ are 0.7746 and 0.5272, respectively.

![Figure 2: Signs of the first Lyapunov coefficient for $\rho = 0$.](image)

The following theorem summarizes the results in this subsection.

**Theorem 3.6** Consider the case where $\rho = 0$. If $(\beta, \alpha, \kappa) \in S \cup U$ then the family of differential equations (6) has a transversal Hopf point at $P_0$ for $\varepsilon = \varepsilon_c$. If $(\beta, \alpha, \kappa) \in S$ then the Hopf point at $P_0$ for $\varepsilon = \varepsilon_c$ is asymptotically stable and for each $\varepsilon < \varepsilon_c$, but close to $\varepsilon_c$, there exists a stable periodic orbit near the unstable equilibrium point $P_0$. If $(\beta, \alpha, \kappa) \in U$ then the Hopf point at $P_0$ for $\varepsilon = \varepsilon_c$ is unstable and for each $\varepsilon > \varepsilon_c$, but close to $\varepsilon_c$, there exists...
an unstable periodic orbit near the asymptotically stable equilibrium point $P_0$. See Fig 2.

3.2.2 The case $\kappa = 0$

Corollary 3.7 Consider the case where $\kappa = 0$. Then the equilibrium point $P_0$ in (8) is given by

$$P_0 = (x_0, y_0, z_0) = \left(\arccos \beta, 0, \frac{(1 - \beta^2)^{1/4}}{\beta^{1/2}(\rho + (1 - \beta^2)^{1/2})^{1/2}}\right),$$

(50)

the Hopf hypersurface (114) is given by

$$\varepsilon_c = \varepsilon_c(\beta, \alpha, \rho) = \frac{2\alpha \beta^{3/2} (1 - \beta^2)^{3/4}(\rho + (1 - \beta^2)^{1/2})^{3/2}}{\rho + (1 - \beta^2)^{3/2}}$$

(51)

and the numerator of $l_i$ in (27) is given by

$$G_2(\beta, \alpha, \rho) = -2\alpha^4 \beta^{22} + 2\alpha^4 \beta^{20}(8 + 7\rho((1 - \beta^2)^{1/2} + 3\rho)) + 3(-2 + \rho(-9(1 - \beta^2)^{1/2} + \rho(-15 - 10(1 - \beta^2)^{1/2}\rho + 3(1 - \beta^2)^{1/2}\rho^3 + \rho^4))) - 2\beta^{18}(-5 + \alpha^2(1 + \rho((1 - \beta^2)^{1/2} + 5\rho)) + \alpha^4(28 + \rho(50(1 - \beta^2)^{1/2} + 7\rho(22 + 5\rho((1 - \beta^2)^{1/2} + \rho)))) + \beta^{16}(-86 + \alpha^2(16 + \rho(27(1 - \beta^2)^{1/2} + 2\rho(52 + 9(1 - \beta^2)^{1/2} + 7\rho^2))) + 2\alpha^4(56 + \rho(153(1 - \beta^2)^{1/2} + 7\rho(69 + \rho(33(1 - \beta^2)^{1/2} + \rho(35 + 3(1 - \beta^2)^{1/2}\rho + \rho^2)))))) - \beta^{14}(-41(8 + (1 - \beta^2)^{1/2}\rho) + 2\alpha^2(28 + \rho(64(1 - \beta^2)^{1/2} + \rho(215 + 2\rho(40(1 - \beta^2)^{1/2} + \rho(34 + (1 - \beta^2)^{1/2}\rho)))) + \alpha^2(70 + \rho(260(1 - \beta^2)^{1/2} + \rho(630(1 - \beta^2)^{1/2} + 840 + \rho(700 + \rho(140(1 - \beta^2)^{1/2} + \rho(56 + (1 - \beta^2)^{1/2}\rho)))))))) + \beta^{12}(-13(56 + 21(1 - \beta^2)^{1/2}\rho + 4\rho^2) + \alpha^2(112 + \rho(319(1 - \beta^2)^{1/2} + \rho(976 + \rho(552(1 - \beta^2)^{1/2} + \rho(494 + 51(1 - \beta^2)^{1/2}\rho + 8\rho^2)))))) + 2\alpha^4(56 + \rho(265(1 - \beta^2)^{1/2} + \rho(875 + \rho(910(1 - \beta^2)^{1/2} + \rho(1050 + \rho(350(1 - \beta^2)^{1/2} + \rho(154 + 11(1 - \beta^2)^{1/2}\rho + \rho^2)))))))) - \beta^{10}(-259(4 + 3(1 - \beta^2)^{1/2}\rho) + \rho^2(-305 + 9(1 - \beta^2)^{1/2}\rho) + \alpha^2(140 + \rho(480(1 - \beta^2)^{1/2} + \rho(1376 + \rho(1011(1 - \beta^2)^{1/2} + 2\rho(458 + 93(1 - \beta^2)^{1/2}\rho + 23\rho^2)))))) + 2\alpha^4(28 + \rho(162(1 - \beta^2)^{1/2} + \rho(546 + \rho(735(1 - \beta^2)^{1/2} + \rho(875 + \rho(420(1 - \beta^2)^{1/2} + \rho(196 + \rho + \rho^2))))))$$

(52)
It is equivalent to $G$ since the transversality condition is satisfied by Proposition 3.3. But from Proposition 3.3, the condition for being a Hopf point is that the first Lyapunov coefficient $l_1 \neq 0$, since the transversality condition is satisfied by Proposition 3.3. But from Proposition 3.3, it is equivalent to $G_2 \neq 0$.

**Proof.** The proof is obtained by substituting $\kappa = 0$ into Eqs. (8), (14) and (36). The long expression above, being a challenge to hand calculation, has been performed with Computer Algebra. In the site [13] has been posted the main steps of the long calculations involved in this substitution. This has been done in the form of a **notebook** for MATHEMATICA 5 [13]. A sufficient condition for being a Hopf point is that the first Lyapunov coefficient $l_1 \neq 0$, since the transversality condition is satisfied by Proposition 3.3. But from Proposition 3.3, it is equivalent to $G_2 \neq 0$.

**Equation (52)** gives an expression to determine the sign of the first Lyapunov coefficient (36) for the case $\kappa = 0$. The graph $G_2(\beta, \alpha, \rho) = 0$ is illustrated in Fig. 3 where the signs of the first Lyapunov coefficient are also
represented. The surface $l_1 = 0$ divides the hypersurface of critical parameters $\varepsilon = \varepsilon_c$ into two connected components denoted by $S$ and $U$ where $l_1 < 0$ and $l_1 > 0$ respectively. At point $B_1$ the $\rho$ coordinate is 0.0478. See Fig. 3.

Figure 3: Signs of the first Lyapunov coefficient for $\kappa = 0$.

The following theorem summarizes the results in this subsection.

**Theorem 3.8** Consider the case $\kappa = 0$. If $(\beta, \alpha, \rho) \in S \cup U$ then the family of differential equations (6) has a transversal Hopf point at $P_0$ for $\varepsilon = \varepsilon_c$. If $(\beta, \alpha, \rho) \in S$ then the Hopf point at $P_0$ for $\varepsilon = \varepsilon_c$ is asymptotically stable and for each $\varepsilon < \varepsilon_c$, but close to $\varepsilon_c$, there exists a stable periodic orbit near the unstable equilibrium point $P_0$. If $(\beta, \alpha, \rho) \in U$ then the Hopf point at $P_0$ for $\varepsilon = \varepsilon_c$ is unstable and for each $\varepsilon > \varepsilon_c$, but close to $\varepsilon_c$, there exists an unstable periodic orbit near the asymptotically stable equilibrium point $P_0$. See Fig. 3.
4 Concluding comments

In this paper the original stability analysis due to Maxwell and Vyshnegradskii of the Watt Centrifugal Governor System —WGS— has been extended to the Hexagonal Governor System —HGS— where a more general force, due to the spring, acting on the sliding sleeve of the governor has been considered.

In Theorem 2.2 we have extended the stability results presented in Pontryagin [8] to include this more general system. See [9] for another possible extension.

Concerning the bifurcations of the HGS, this paper deals with the codimension one Hopf bifurcations in the Hexagonal governor differential equations. The general expression for the first Lyapunov coefficient at the Hopf point has been obtained in Theorem 3.1. More concrete consequences of this calculation have been synthesized in Theorems 3.6 and 3.8. These results give sufficient conditions for the stability of the points on the Hopf hypersurface and of the periodic orbit that bifurcates from the Hopf point for the Hexagonal governor differential equations (6) in two particular cases easier to visualize with the help of numerical plotting. See Figs. 2, 3 and the site [13].

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