Gaussian Process Classification with Privileged Information by Soft-to-Hard Labeling Transfer

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Abstract

Learning using privileged information is an attractive problem setting that helps many learning scenarios in the real world. A state-of-the-art method of Gaussian process classification (GPC) with privileged information is GPC+, which incorporates privileged information into a noise term of the likelihood. A drawback of GPC+ is that it requires numerical quadrature to calculate the posterior distribution of the latent function, which is extremely time-consuming. To overcome this limitation, we propose a novel classification method with privileged information based on Gaussian processes, called “soft-label-transferred Gaussian process (SLT-GP).” Our basic idea is that we construct another learning task of predicting soft labels (continuous values) obtained from privileged information and we perform transfer learning from this task to the target task of predicting hard labels. We derive a PAC-Bayesian bound of our proposed method, which justifies optimizing hyperparameters by the empirical Bayes method. We also experimentally show the usefulness of our proposed method compared with GPC and GPC+

1 Introduction

Learning using privileged information is an attractive problem setting that helps many learning scenarios in the real world. The first method that can solve the problem of learning using privileged information is the support vector machine plus (SVM+) [18]. SVM+ uses privileged information to predict the optimal slack variables in SVM. It has been proven that the convergence of the generalization error of SVM+ is faster than that of standard SVM. However,
Table 1: Comparison between three classification methods using Gaussian processes. The likelihood function is modified in GPC+ so as to incorporate privileged information while the likelihood function is the same as GPC in the case of SLT-GP. In SLT-GP, the prior can be regarded to be modified by soft labels.

|                | GPC  | GPC+          | SLT-GP (This work) |
|----------------|------|---------------|--------------------|
| **Prior**      | \( f \sim \mathcal{GP}(0, k) \) | \( f \sim \mathcal{GP}(0, k) \)  | \( f_T \sim \mathcal{GP}(m_{x,s}, k_X) \) |
| **Likelihood** | \( p(y|f, x) = \Phi(tf(x)) \) | \( p(y|f, g, x, x^*) = \Phi \left( \frac{yf(x)}{\sqrt{\exp(g(x^*)})} \right) \) | \( p(y|f_T, x) = \Phi(yf_T(x)) \) |
| **Update in EP** | analytic | with numerical quadrature | analytic |
| **PAC-Bayes bound** | \( \sqrt{\text{[15]}} \) | n/a | \( \sqrt{\text{(Section [4])}} \) |

The accuracy of SVM+ has been empirically reported not to be so much better than that of standard SVM [16, 7].

Gaussian process classification (GPC) [19], which enables us to model the uncertainty of prediction, has been extended to GPC+ [7] so as to use privileged information. GPC+ incorporates privileged information into a noise term of the likelihood. However, the formulation of GPC+ requires numerical quadrature to calculate the posterior distribution. Since the number of numerical quadratures performed in GPC+ is proportional to the number of training examples, computation of GPC+ takes a long time. Moreover, it is hard to theoretically analyze the expected classification risk in a PAC-Bayes way [12] because the negative log likelihood loss function cannot be defined from only the input and the output in the case of GPC+.

To overcome the above limitations of GPC+, we propose a novel classification method with privileged information based on Gaussian processes. We formulate the classification problem with privileged information as a soft-label-transferred Gaussian processes (SLT-GP). We assume that there is a relationship between two tasks: (a) predicting hard labels and (b) predicting soft labels. Hard labels are binary values provided in a training dataset, while soft labels are continuous values extracted from privileged information similarly to the framework of generalized distillation [11]. We then formulate the classification problem with privileged information as transfer learning from the soft-labeling task to the hard-labeling task. In our formulation, we can derive an efficient expectation propagation algorithm to approximate the posterior distribution. Also, we can analyze the expected risk of our proposed method by using the PAC-Bayesian theorem [2] because it can be regarded that the prior distribution is modified from that of GPC in our formulation while the likelihood function is modified from that of GPC in the formulation of GPC+. Our theoretical analysis justifies the optimization of hyperparameters by the empirical Bayes method. Table 1 summarizes the relationship between standard GPC, GPC+, and our SLT-GP method proposed in this paper. Finally, we experimentally...
demonstrate the usefulness of the SLT-GP method.

2 Proposed formulation

In this section, we provide background information and the formulation for the model of the proposed method. The learning algorithm for the proposed method is described in the next section.

2.1 Preliminary

We prepare several notations for datasets. A training dataset consists of training input data \( x_i \in \mathcal{X} \), target labels \( y_i \in \{+1, -1\} \), and privileged information \( x^*_i \in \mathcal{X}^* \), where \( \mathcal{X} \) is the input space and \( \mathcal{X}^* \) is the space of privileged information. Note that since our method is based on Gaussian processes and it works with only covariance functions (and mean functions), \( \mathcal{X} \) and \( \mathcal{X}^* \) can be any spaces as long as covariance functions are defined on those spaces.

The overall dataset with \( n \) training examples is denoted by \( D = \{ (x_i, y_i, x^*_i) \}_{i=1}^n \). For brevity, we introduce some additional notations for the training dataset, \( X = (x_1, \ldots, x_n) \in \mathcal{X}^n \), \( X^* = (x^*_1, \ldots, x^*_n) \in (\mathcal{X}^*)^n \), and \( y = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n \), where \( \top \) denotes the transpose of a vector and a matrix.

2.2 Soft labels

Our method uses soft labels extracted from GPC with privileged information. Soft labels are more informative than hard labels because soft labels have continuous values while hard labels have only sign information. Soft labels represents the degree of confidence of belonging to each class. To extract soft labels from the privileged information, we use standard GPC. The Gaussian process over the space of privileged information is defined by \( f^* \sim \mathcal{GP}(0, k^*) \), where \( \mathcal{GP}(m, k) \) denotes the Gaussian process with mean function \( m \) and covariance function \( k \), \( 0 \) is the zero function, and \( k^* : \mathcal{X}^* \times \mathcal{X}^* \rightarrow \mathbb{R} \) is a covariance function over the space of privileged information \( \mathcal{X}^* \). With an appropriate likelihood for classification, e.g. the probit model \( p(y = 1 | f(x)) = \Phi(f(x)) \), the posterior distribution \( p(f^*|X^*, y) \) can be approximated, where \( \Phi(z) \) is the cumulative distribution function of the standard normal distribution. Especially, the posterior distribution of \( f^* = (f^*(x^*_1), \ldots, f^*(x^*_n))^\top \) can be approximated as a multivariate normal distribution,

\[
p(f^*|X^*, y) \simeq q(f^*) = \mathcal{N}(f^*|\mu^*, \Sigma^*),
\]

(1)

where \( \mu^* \) is the approximated posterior mean vector and \( \Sigma \) is the posterior covariance matrix. The posterior mean is treated as soft labels, \( s_i = \mu^*_i \), and equivalently denoted by \( s = (s_1, \ldots, s_n)^\top = \mu^* \).
2.3 Soft-to-hard labeling transfer

Once the soft labels are obtained, we can consider two tasks: (a) the task predicting the hard label $y_i$ from the input $x_i$ and (b) the task predicting the soft label $s_i$ from the input $x_i$. By using the soft labels, we do not need to care about the privileged information itself since it is expected that the soft labels have enough information to predict the target labels as well as the privileged information. These two tasks are highly related when the privileged information $x_i^*$ has useful information about the input $x_i$ to predict the target label $y_i$, that is, the soft label $s_i$ represents the appropriate confidence of being positive or negative for the input $x_i$. Furthermore, it might be easier to predict the target label thorough predicting soft labels than predicting hard labels directly since soft labels have richer information than hard labels. This is similar to the idea of SVM+, which conceptually predicts the slack variables in the SVM using the privileged information [17]. However, there is no guarantee that the privileged information has the appropriate information about the data in real-world datasets. In such cases, making a prediction only from the soft labels causes severe performance deterioration.

To cope with this problem, we propose using transfer learning with Gaussian processes. The target task is predicting binary hard labels and the source task is predicting soft labels. Thus, the source task is a regression problem and the target task is a classification problem. We should note that the domains of these tasks are the same, $\mathcal{X}$, and the training input data $X$ are also the same for both tasks. A similar formulation is explored in multi-task setting [3], in transfer setting where both source and target tasks are regression problems [4], and in multi-output setting [2].

In our proposed method, both the source and target tasks have the same Gaussian process as a prior distribution over latent functions. Both Gaussian process regression and classification assume that a Gaussian process prior has the covariance function over the input space, $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$. The prior distributions over the latent functions of the source task $f^{(S)}$ and the target task $f^{(T)}$ are common: $f^{(S)}, f^{(T)} \sim \mathcal{GP}(0, k)$. To define a Gaussian process over multiple tasks, the inter-task covariance should be defined. The covariance function between the source task and the target task is defined as follows:

$$E[(f^{(S)}(x), f^{(T)}(x'))] = \rho k(x, x'),$$

(2)

where $\rho \in [0, 1]$ is a task similarity between the source and target tasks. This task similarity represents the relationship between hard labels and soft labels extracted from the privileged information. If the task similarity is close to 1, the latent functions of both tasks have close outcomes. Thus the information from soft labels are fully incorporated in the target task. If the task similarity is 0, there is no similarity between the outcomes of the source and target latent functions. Therefore, the latent functions $f^{(S)}$ and $f^{(T)}$ are independent and the soft labels are ignored. The task similarity parameter can be estimated by the empirical Bayes method as explained later.

The difference between the source task and the target task is reduced to
the difference of their likelihood functions. On the one hand, the source task, predicting soft labels, is essentially a regression problem. Thus, the likelihood for the source task is modeled by a Gaussian likelihood function, $p(s_i | f_i^{(S)}) = \mathcal{N}(s_i | f_i^{(S)}, 1)$, where $f_i^{(S)} = f^{(S)}(x_i)$. On the other hand, the target task is a classification problem. Thus, the likelihood for the target task is modeled by a Bernoulli likelihood function, $p(y_i = 1 | f_i^{(T)}) = \Phi(f_i^{(T)})$, where $f_i^{(T)} = f^{(T)}(x_i)$.

Since the likelihood terms in the source and target domains are different, we need to devise a learning algorithm for this transfer formulation.

### 3 Learning algorithm

Since the likelihood of the target tasks is non-Gaussian, the posterior distribution of $f^{(T)}$ is analytically intractable. Therefore, we need to approximate the posterior distribution. We use expectation propagation (EP) [14] to approximate the posterior Gaussian process. The difference between EP of the standard GPC and that of our method is the likelihood term. The likelihood term for GPC is the Bernoulli distribution while the likelihood term for our method includes both the Bernoulli and Gaussian likelihoods. We propose an EP method for a mixture of multiple likelihoods to estimate the posterior distribution in our method.

#### 3.1 Expectation propagation

From Bayes’ theorem, the posterior distribution of the latent functions $f^{(T)}$ and $f^{(S)}$ are derived as follows:

$$
p(f^{(T)}, f^{(S)} | X, y, s) = p(y | f^{(T)}, X)p(s | f^{(S)}, X)p(f^{(T)}, f^{(S)})p(y, s | X),
$$

where $p(y, s | X)$ is the marginal likelihood integrating out $f^{(S)}$ and $f^{(T)}$. The same equation holds for the latent function values at the finite training points $f_T = (f_1^{(T)}, \ldots, f_n^{(T)})^\top$ and $f_S = (f_1^{(S)}, \ldots, f_n^{(S)})^\top$ as follows:

$$
p(f_T, f_S | X, y, s) = \frac{p(y | f_T)p(s | f_S)p(f_T, f_S)}{p(y, s | X)} = \prod_{i=1}^n [p(y_i | f_i^{(T)})p(s_i | f_i^{(S)})]p(f_T, f_S)
$$

$$
p(y, s | X),
$$

(4)
where

$$
p(\mathbf{f}_T, \mathbf{f}_S) = \mathcal{N} \left( \begin{bmatrix} \mathbf{f}_T \\ \mathbf{f}_S \end{bmatrix} | \mathbf{0}, K \right),
$$

(5)

$$
K = \left( \frac{1}{\rho} \begin{matrix} \rho \\ 1 \end{matrix} \right) \otimes K_X,
$$

(6)

$$
K_X = (k(x_i, x_j))_{i,j},
$$

(7)

and $\otimes$ denotes the Kronecker product. This is derived from the property of the Gaussian process we define.

While all the likelihood terms are approximated in GPC, only the likelihood for the target tasks is approximated in our method. To approximate the posterior distribution $p(f_T, f_S | X, y, s)$, the non-Gaussian likelihood $p(y|f_T)$ is approximated by an unnormalized Gaussian function of $f_T$ called “site approximation,”

$$
p(y_i|f_i^{(T)}) \simeq t_i^{(T)}(f_i^{(T)}) := \tilde{Z}_i \mathcal{N}(f_i^{(T)} | \tilde{\mu}_i, \tilde{\sigma}_i),
$$

(8)

where $\tilde{Z}_i$ is the normalization term. This is also denoted by natural parameters of the normal distribution. The moment parameters $\tilde{\mu}_i$ and $\tilde{\sigma}^2$ are converted into natural parameters $\tilde{\nu}_i$ and $\tilde{\tau}$ as follows:

$$
\tilde{\nu}_i = \frac{\tilde{\mu}_i}{\tilde{\sigma}^2_i}, \quad \tilde{\tau}_i = \frac{1}{\tilde{\sigma}^2_i}.
$$

(9)

For the Gaussian likelihood terms, no approximation is required,

$$
p(s_i|f_i^{(S)}) = t_i^{(S)}(f_i^{(S)}) := \mathcal{N}(f_i^{(S)} | s_i, 1).
$$

(10)

Once the likelihood terms are approximated by the site approximations $t_i^{(T)}(f_i^{(T)})$, the approximated posterior distribution $q(\mathbf{f}) \simeq p(\mathbf{f}|X, y, s)$ over both the source and target $\mathbf{f} = \begin{bmatrix} f^{(T)} \\ f^{(S)} \end{bmatrix}$ can be computed,

$$
q(\mathbf{f}) = \mathcal{N}(\mathbf{f} | \boldsymbol{\mu}, \Sigma),
$$

(11)

$$
\boldsymbol{\mu} = \Sigma \tilde{\Sigma}^{-1} \tilde{\mu} = \Sigma \tilde{\nu},
$$

(12)

$$
\Sigma = (K^{-1} + \tilde{\Sigma}^{-1})^{-1}
$$

$$
= K - K \tilde{T}^\frac{1}{2} B^{-1} \tilde{T}^\frac{1}{2} K,
$$

(13)

where

$$
\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_n, s_1, \ldots, s_n)^T \in \mathbb{R}^{2n},
$$

(14)

$$
\tilde{\Sigma} = \text{diag}(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2, 1, \ldots, 1) \in \mathbb{R}^{2n \times 2n},
$$

(15)

$$
\tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_n, s_1, \ldots, s_n)^T \in \mathbb{R}^{2n},
$$

(16)

$$
\tilde{T} = \text{diag}(\tilde{\tau}_1, \ldots, \tilde{\tau}_n, 1, \ldots, 1) \in \mathbb{R}^{2n \times 2n},
$$

(17)

$$
B = I + \tilde{T}^\frac{1}{2} K \tilde{T}^\frac{1}{2}.
$$

(18)
The site approximations $t_i^{(T)}(f_i^{(T)}) \ (i = 1, \ldots, n)$ are updated iteratively in the same way as the standard GPC.

### 3.2 Inference

After estimating the site approximation $t_i^{(T)}(f_i^{(T)})$, we can infer the predictive distribution for an unknown input $\hat{x}$. Since the likelihood terms are approximated by unnormalized Gaussian functions, the predictive distribution of the latent function value can be calculated in the same way as the regression case:

$$p(f(\hat{x})|\hat{x}, X, y) \simeq \mathcal{N}(f(\hat{x})|\hat{\mu}, \hat{\sigma}^2),\quad (19)$$

$$\hat{\mu} = \frac{1}{\rho} \otimes \left( \begin{array}{c} k(\hat{x}, x_1) \\ \vdots \\ k(\hat{x}, x_n) \end{array} \right), \quad (22)$$

$$\hat{\sigma}^2 = k(\hat{x}, \hat{x}) - \frac{1}{\rho} \otimes \left( \begin{array}{c} k(\hat{x}, x_1) \\ \vdots \\ k(\hat{x}, x_n) \end{array} \right).$$

The predictive distribution for the target labels can also be calculated, using the predictive distribution of the latent function value as follows:

$$p(\hat{y} = 1|\hat{x}, X, y, s) = \Phi \left( \frac{\hat{\mu}}{\sqrt{1 + \hat{\sigma}^2}} \right). \quad (23)$$

### 3.3 Marginal likelihood

Usually, a covariance function $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ has several hyperparameters. For example, the radial basis function kernel $k_{\text{RBF}}(x, x')$ has a length scale parameter $l^2$:

$$k_{\text{RBF}}(x, x') = \exp \left( -\frac{||x - x'||^2}{2l^2} \right). \quad (24)$$

Additionally, there is a task similarity parameter $\rho$ that controls the importance of soft labels. The hyperparameters including the kernel hyperparameters and the task similarity are typically estimated by the empirical Bayes method, which maximizes the marginal likelihood of the model.

The marginal likelihood of both target and source labels can be approximated by using the site approximation. Since our problem is a transfer learning problem, our objective is to maximize the marginal likelihood of target labels conditioned on source labels. Although what we can compute in EP is the
joint marginal likelihood over target and source labels, the conditional marginal likelihood can be computed using Bayes’ theorem as follows:

\[
Z_{X,Y|s} := \log p(y|s, X) = \log \frac{p(y,s|X)}{p(s|X)} \\
= \log p(y|s,X) - \log p(s|X).
\]  

The latter term \( \log p(s|X) \) is the log marginal likelihood over soft labels. Since the likelihood for soft labels is Gaussian, the marginal likelihood can be calculated analytically.

4 PAC-Bayes bound

In this section, we provide a theoretical analysis of our method based on the probably approximately correct (PAC) Bayesian theorem [13]. The PAC-Bayesian theorem gives an upper bound of the generalization error for randomized estimators. We focus on the expected risk with the negative log likelihood loss

\[
\mathcal{L}^\text{all}_{D}(f_T) = \mathbb{E}_{(x,y) \sim D} \ell_{\text{all}}(f_T, x, y),
\]

where \( D \) is a data generating distribution over the pair of input \( X \) and output \( Y = \{+1, -1\} \) and \( \ell_{\text{all}} \) is the negative log likelihood loss of the target task defined by

\[
\ell_{\text{all}}(f_T, x, y) = -\log p(y|f_T(x)) = -\log \Phi(yf_T(x)).
\]

The expected risk for our proposed method can be bounded by the log marginal likelihood as in Theorem 1.

**Theorem 1.** If the data distribution \( D \) over \( X \times Y \) and the prior distribution \( P \) over a set of hypothesis \( \mathcal{F} \subset \{f : X \rightarrow \mathbb{R}\} \) have the following sub-Gaussian property with a variance factor \( \sigma_0^2 < \frac{1}{2} \):

\[
\forall \lambda \in \mathbb{R} : \log \mathbb{E}_{f_T \sim P} \mathbb{E}_{(x',y') \sim D} \exp (\lambda y'f_T(x')) \leq \frac{\lambda^2 \sigma_0^2}{2},
\]

then the expectation of the expected risk over the posterior distribution \( \hat{Q}(f_T) = p(f_T|X,Y,s) \) is bounded with probability at least \( 1 - \delta \) as follows:

\[
\mathbb{E}_{f_T \sim \hat{Q}} \mathcal{L}^\text{all}_{D}(f_T) \leq -\frac{1}{n} \log(\delta Z_{X,Y|s}) + b_{\sigma_0^2},
\]

where

\[
b_{\sigma_0^2} = \inf_{a > r_{\sigma_0}} \left[ \frac{1}{2} \log(2\pi(a + 4)) \\
- \frac{a}{2(a + 4)} + 4\sigma_0^4 \left( \frac{a + 5}{a + 4} \right)^2 \right],
\]

\[
c_{\sigma_0^2} = \frac{10\sigma_0^2 - 4}{1 - 2\sigma_0^2}.
\]
Theorem 1 suggests that the negative log marginal likelihood $-\log Z_{X,Y}$ becomes an upper bound of the expected risk with additional constants when $\sigma_0^2$ is fixed or bounded. The value of $\sigma_0^2$ itself is generally hard to calculate analytically. However, by choosing the covariance function in the Gaussian process prior so that it has a small enough value, the sub-Gaussian condition is expected to be satisfied.

**Proof of Theorem 1.** We start with the previous bound [1, 6]. The PAC-Bayes bound gives the relationship between the expected risk $L_\ell_D(f) = \mathbb{E}_{(x,y) \sim D} \ell(f, x, y)$ and the empirical risk $\hat{L}_\ell^{\ell}(X, y)(f) = \frac{1}{n} \sum_{i=1}^{n} \ell(f, x_i, y_i)$, where $\ell : \mathcal{F} \times \mathcal{X} \times \mathcal{Y}$ is a loss function over a hypothesis set $\mathcal{F} \subset \{ f : \mathcal{X} \to \mathbb{R} \}$, input space $\mathcal{X}$ and output space $\mathcal{Y}$, and $(X, y) \sim D^n$. The relationship between the expected and empirical risk is represented by an inequality with the Kullback-Leibler divergence from the posterior distribution $Q$ to the prior distribution $P$ over $\mathcal{F}$, $\text{KL}[Q||P] = \mathbb{E}_{f \sim Q} \left[ \log \frac{Q(f)}{P(f)} \right]$. The statement of the theorem is described as follows.

**Theorem 2 ([1]).** Given a distribution $D$ over $\mathcal{X} \times \mathcal{Y}$, a hypothesis set $\mathcal{F}$, a loss function $\ell : \mathcal{F} \times \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, a prior distribution $P$ over $\mathcal{F}$ and real numbers $\delta \in (0, 1]$ and $\lambda > 0$, the following inequality holds with probability at least $1 - \delta$ over the choice of $(X, y) \sim D^n$:

$$\forall Q \text{ on } \mathcal{F} : \frac{1}{n} \mathbb{E}_{f \sim Q} L_\ell_D(f) \leq \frac{1}{n} \mathbb{E}_{f \sim Q} \hat{L}_\ell^{\ell}(X, y)(f) + \frac{1}{\lambda} \text{KL}[Q||P] + \frac{1}{\delta} \Psi_{\ell, P, D} (\lambda, n),$$

(32)

where

$$\Psi_{\ell, P, D} (\lambda, n) = \log \mathbb{E}_{f \sim P(X', y') \sim D^n} \mathbb{E} \exp \left[ \lambda \left( L_\ell_D(f) - \hat{L}_\ell^{\ell}(X', y')(f) \right) \right].$$

(33)

From Theorem 2 and the discussion by [6], we can relate the PAC-Bayesian theorem to the empirical Bayes method. We consider the negative log likelihood loss $\ell_{\text{all}}(f, x, y) = -\log p(y|f(x))$ and the optimal Gibbs posterior

$$\tilde{Q}(f) = \frac{P(f) \exp(-n\hat{L}_{\text{all}}^{\ell}(f))}{Z_{X,Y}},$$

(34)

where

$$Z_{X,Y} = \int P(f) \exp(-n\hat{L}_{\text{all}}^{\ell}(f))df$$

(35)

is the marginal likelihood of the training data $(X, Y)$. By substituting $n$ for $\lambda$, the expectation of the expected risk is rewritten as follows:

$$\mathbb{E}_{f \sim \tilde{Q}} L_\ell^{\ell}_{\text{all}}(f) \leq -\frac{1}{n} \log(\delta Z_{X,Y}) + \frac{1}{n} \Psi_{\ell_{\text{all}}, P, D}(n, n).$$

(36)
Table 2: The data-generating processes for synthetic datasets where \( d = 50, d^* = 3, c = 2, c^* = 2, J = \{1, 2, 3\}, k_0(x, x') = 10 \exp \left(-\frac{\|x-x'\|^2}{2}\right), k_0^*(x, x') = 10 \exp \left(-\frac{\|x-x'\|^2}{2}\right)\) and \(J = \{A \subset \{1, \ldots, d\} | |A| = d^*\}. \alpha \in \mathbb{R}^d\) and \(\alpha^* \in \mathbb{R}^{d^*}\) are sampled from the standard normal distribution for each dataset. \(\mathcal{U}(\mathbb{Z})\) denotes the uniform distribution over the set \(\mathbb{Z}\). 200 samples were generated for training and 1000 samples were generated for testing.

| Clean soft label                      | Clean feature                      | Relevant feature                      |
|--------------------------------------|------------------------------------|--------------------------------------|
| \(x_i \sim \mathcal{N}(0, I_d)\)    | \(x_i^* \sim \mathcal{N}(0, I_d)\) | \(x_i \sim \mathcal{N}(0, I_d)\)    |
| \(\epsilon_i \sim \mathcal{N}(0, 1)\) | \(y_i = \text{sign}(x_i^* + \epsilon_i)\) | \(y_i \sim \text{sign}(\alpha^*^\top x_i^*)\) |

| Independent feature | Latent GP | Noise variance |
|---------------------|-----------|----------------|
| \(x_i \sim \mathcal{N}(0, I_d)\) | \(x_i \sim \mathcal{U}([0, 10[^c]\) | \(x_i \sim \mathcal{U}([0, 10[^c]\) |
| \(J_i \sim \mathcal{U}(J)\) | \(K_0 = (k_0(x_i, x_j))_{i,j}\) | \(K_0 = (k_0(x_i, x_j))_{i,j}\) |
| \(x_i^* \sim \mathcal{N}(0, 1)\) | \(\epsilon_i \sim \mathcal{N}(0, 1)\) | \(\epsilon_i \sim \mathcal{N}(0, 1)\) |
| \(y_i \sim \text{sign}(\alpha^*^\top x_i^*)\) | \(y_i \sim \text{sign}(\alpha^*^\top x_i^*)\) | \(y_i \sim \text{sign}(\alpha^*^\top x_i^*)\) |

In our method, the marginal likelihood of the target hard labels are conditioned on the soft labels \(p(y | X, s)\). This modification can be understood as modification of the prior distribution, i.e.,

\[
P(f_T) = p(f_T | X, s) = \mathcal{GP}(f_T | m_{X,s}, k_X),
\]

where

\[
m_{X,s}(x) = \rho k_X(x) K_X^{-1} s,
\]

\[
k_X(x, x') = k(x, x') - \rho^2 k_X(x) K_X^{-1} k_X(x'),
\]

\[
k_X(x) = (k(x, x_1), \ldots, k(x, x_n))^\top.
\]

The marginal likelihood for our method is defined as follows:

\[
Z_{X,y | s} = \int p(f_T | X, s) \prod_{i=1}^n p(y_i | f_T(x_i)) df_T.
\]

From the following lemma, the last term in (36) can be bounded by a constant.

**Lemma 3.**

\[
\Psi_{\ell_{	ext{uni}}, c, \mathcal{D}}(n, n) \leq nb_{\tilde{g}}.
\]

The proof of Lemma 3 is provided in Appendix A.
Table 3: Results of synthetic datasets. Each experiment was repeated 100 times. The mean and standard deviation (enclosed in parentheses) of accuracy are displayed. The best accuracy for each dataset except the rightmost column is highlighted in **boldface**. The rightmost column is the test accuracy of GPC where privileged information is given as input data for both training and test dataset.

| Datasets          | GPC          | GPC+         | SLT-GP       | GPC (reference) |
|-------------------|--------------|--------------|--------------|-----------------|
| Train             | X            | $(X, X^*)$   | $(X, X^*)$   | $X^*$           |
| Test              | X            | X            | X            | $X^*$           |
| Clean soft label  | 87.89 (2.39) | 88.21 (1.57) | **95.27 (0.89)** | 95.41 (0.82)   |
| Clean feature     | 65.40 (2.31) | 68.89 (2.01) | **69.99 (1.65)** | 89.72 (1.38)   |
| Relevant feature  | 89.85 (1.82) | 89.88 (1.85) | **98.92 (0.57)** | 99.09 (0.58)   |
| Independent feature | 50.68 (1.68) | 50.81 (1.80) | **50.95 (1.78)** | 99.01 (0.58)   |
| Latent GP         | 82.20 (2.83) | 80.05 (9.14) | **86.75 (2.35)** | 89.97 (1.77)   |
| Noise variance    | 77.36 (6.05) | **78.85 (6.54)** | 77.90 (5.97) | 55.34 (5.09)   |

### 5 Experiments

In this section, we present the results of experiments on synthetic and real-world datasets. In the experiments on synthetic datasets in Section 5.1–5.6, our proposed method (Soft-Label Transferred Gaussian Process classification, SLT-GP) was compared with three methods—Gaussian process classification (GPC) with only privileged information (treated as input data), standard GPC without privileged information, and GPC+. The performance of GPC with only privileged information is calculated by privileged information of test data, which is not actually available in our problem setting but its results are useful to understand the improvement of the algorithm. The experimental settings of Section 5.1–5.4 were originally presented in the previous work [11]. The data-generating processes for the synthetic datasets in Section 5.1–5.6 are summarized in Table 2 and their results are given in Table 3. In Section 5.7, the performance of our proposed SLT-GP method and GPC+ is compared by using real-world datasets. Finally, we investigate the change in the task similarity $\rho$ estimated by empirical Bayes method compared with the optimal one in Section 5.8. The hyperparameters in the covariance function of each method and the task similarity are estimated by the empirical Bayes method for all experiments.

#### 5.1 Clean soft labels as privileged information

In this setting, the true distance (including the sign corresponding to the target class) from the decision boundary is given as privileged information. The data-generating process is presented in the Table 2 as “clean soft label.” We used the linear kernel, $k_{\text{linear}}(x, x') = \sigma^2 x^\top x'$, as the covariance function of the latent function over input for GPC, SLT-GP and GPC+. The Gaussian radial basis function $k_{\text{RBF}}(x, x') = \exp\left(-\frac{\|x-x\|^2}{2\sigma^2}\right)$ was used as the covariance function of
the latent function over privileged information for GPC+ because it is assumed that the relationship between the privileged information and the latent function over the privileged information is not linear. The row “clean soft label” in Table 3 shows the result of this experiment. This result shows that our method outperforms the baselines because the privileged information and the soft labels generated from the privileged information have enough information to classify each sample.

5.2 Clean features as privileged information
In this setting, each input feature is contaminated and the feature before contamination is given as privileged information. The data-generating process is presented in Table 2 as “clean feature.” The covariance functions used in each method were the same as the experiment in Section 5.1. The results of this experiment are shown in Table 3 as the dataset “clean feature.” In this dataset, the relationship between the input data and the privileged information is weak because the input is contaminated by noise. Therefore, the privileged information is not so useful and there are few differences between our method and the baselines.

5.3 Relevant features as privileged information
In this setting, only a subset of all the features is relevant to the output label. The relevant features were given as privileged information. The data-generating process is presented in Table 2 as “relevant feature.” We used the same covariance functions as the ones described in Section 5.1 for each method. The results of this experiment are shown in Table 3 as the dataset “relevant feature.” Since only the subset of all the features were given as privileged information, unnecessary information is eliminated from the input data. Therefore, privileged information improved the performance of our method. However, the improvement for GPC+ was negligible.

5.4 Independent relevant features as privileged information
In this setting, only the subset of all the features is relevant as the previous dataset “relevant feature.” But relevant features are independent for each data example. The data-generating process is presented in Table 2 as “independent feature.” We used the same covariance functions as the ones described in Section 5.1 for each method. The results of this experiment are shown in Table 3 as the dataset “independent feature.” In this dataset, though only the relevant features were given as privileged information, the privileged information did not have the information about which features are relevant to each output. Thus, the privileged information rarely improved the performance over the standard GPC in both SLT-GP and GPC+.
Figure 1: Computation time versus accuracy. Dataset size was changed from 20 to 200, repeating 100 times for each dataset size. The horizontal axis represents computation time in seconds and the vertical axis represents the accuracy.

5.5 Latent Gaussian process as privileged information

In this setting, the data was generated from a latent Gaussian process. The output was the sign of the value of the latent function, while the privileged information was the value of the latent function itself. This setting is a non-linear version of Section 5.1. The data-generating process is presented in Table 2 as “latent GP.” We used the Gaussian radial basis function $k_{\text{RBF}}$ as the covariance function in all methods. The results of this experiment are shown in Table 3 as the dataset “latent GP.” This result shows that our method outperforms other baselines even with the non-linear dataset.

5.6 Noise variance as privileged information

In this setting, the noise variance of the latent function depends on the privileged information. This dataset simulates the assumption of the data generation process of GPC+, which is presented in Table 2 as “noise variance.” We used the Gaussian radial basis function $k_{\text{RBF}}$ as the covariance function in all methods. The results of this experiment are shown in Table 3 as the dataset “noise variance.” In this dataset, the accuracy of GPC+ is slightly higher than our proposed method. However, the improvement is marginal.

5.7 Real-world dataset (FlowCAP)

We consider analyzing real-world cell data measured by a flow cytometer. As a possible case, we can consider computer-aided diagnoses based on flow cytometry.
etry in the following situation. In large hospitals, there are highly functional flow cytometers, while many small hospitals have only limited functional flow cytometers. In such cases, diagnoses in small hospitals might be improved by using the datasets obtained by highly functional flow cytometers as privileged information even though limited input features are available for diagnoses.

In this experiment, we used the FlowCAP dataset [5]. The dataset has three types of cells, one for a class of normal cells and two for abnormal cells. Each cell has a six-dimensional feature vector: two light scatter characteristics and four fluorescence intensity values. In our preliminary experiment, two of the fluorescence intensity values were good at separating the normal and abnormal classes. Therefore, we used these two features as privileged information and remaining four features as input data. 200 examples were sampled for training and 1000 examples for testing. The numbers of positive examples (abnormal cells) and negative examples (normal cells) are the same in the training data and test data, respectively. The covariance functions used for all four methods were the Gaussian radial basis function kernels $k_{\text{RBF}}$.

The classification accuracy and computation time are plotted in Figure 1, where the number of training data is increased from 20 to 200. As the figure shows, our proposed method achieves almost the same accuracy as GPC+ with much lower computation time.
5.8 On estimation of task similarity

Finally, we numerically investigate the change in the task similarity $\rho$ in the case where the privileged information is contaminated. Also, the relationship between the true expected risk and its upper bound (used in the empirical Bayes method) is explained by estimating $\rho$.

We synthetically generated data as follows:

$$
\begin{align*}
 x_i &\sim \mathcal{U}([0, 10]^2) \quad (i = 1, \ldots, n), \\
 (f_1^*, \ldots, f_n^*)^\top &\sim \mathcal{N}(\mathbf{0}, 10K_0), \\
 g_i^* &\sim \mathcal{N}(0, 10) \quad (i = 1, \ldots, n), \\
 \epsilon_i &\sim \mathcal{N}(0, 1) \quad (i = 1, \ldots, n), \\
 x_i^* &\leftarrow (1-r)f_i^* + rg_i^* \quad (i = 1, \ldots, n), \\
 y_i &\leftarrow \text{sign}(f_i^* + \epsilon_i) \quad (i = 1, \ldots, n),
\end{align*}
$$

where $r \in [0, 1]$ is a parameter controlling the noise in privileged information,

$$
K_0 = (k_0(x_i, x_j))_{i,j}, \quad (43)$$

$$
k_0(x, x') = \exp \left( -\frac{\|x - x'\|^2}{2} \right), \quad (44)
$$

We generated 200 samples as training data and observed the changes in $\rho$ chosen by two objectives, which was repeated 100 times for different value of $r$. We compared $\rho$ calculated by minimizing our upper bound of the expected risk (equivalent to the empirical Bayes method) with $\rho$ calculated by minimizing the true expected risk calculated with 1000 test data. We used the Gaussian radial basis function with the fixed variance amplitude $k(x, x') = \frac{1}{2} \exp \left( -\frac{\|x - x'\|^2}{2l^2} \right)$, where $l^2$ is a length-scale parameter optimized by the empirical Bayes method.

The result is shown in Figure 2. The changes in $\rho$ have the same tendency between the two objectives and $\rho$ becomes smaller as the noise rate becomes larger. However, hyperparameter tuning based on the upper bound systematically yielded slightly larger values of $\rho$ than the optimal ones in terms of the expected risk. The difference in the value of $\rho$ might have been resulted from the constant term in our bound. The difference would have been reduced if we can derive a computable tighter upper bound, which is left as future work.

6 Conclusion

Our contributions are summarized as follows. (A) We proposed a novel classification method with privileged information (SLT-GP). (B) We analyzed our proposed method by the PAC-Bayesian theorem. (C) We empirically compared our proposed method and the previous methods based on Gaussian processes.

Our proposed method incorporates privileged information as soft labels that contain the information about the confidence belonging to each class and trans-
fers the information to the target task predicting the hard labels. This formulation includes a hyperparameter corresponding to “task similarity,” which controls how much information should be transferred from the soft-labeling task to the hard-labeling task. The task similarity parameter can be determined by the empirical Bayes method and becomes an interpretable index for the effect of the privileged information.

There are several possible directions to extend this work. In this work, we have only considered binary classification tasks. However, it would be interesting to extend our work to multi-class classification tasks because soft labels of multi-class classification tend to have much more information about the privileged information. Another area of research is devising a different hyperparameter optimization scheme based on the PAC-Bayesian bounds from the empirical Bayes method. In our numerical simulation, there was a gap between the expected risk and the upper bound. The gap and the expected risk would be reduced if we could derive a tighter upper bound and directly minimize it.

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A Proof of Lemma 3

In the discussion by [6], the following assumption is considered implicitly,

\[
E_{f \sim P(X',y')} \exp \left[ \sum_{i=1}^{n} V_i(f) \right] = E_{f_1,\ldots,f_n \sim P(X',y')} \exp \left[ \sum_{i=1}^{n} V_i(f_i) \right], \tag{45}
\]

where

\[ V_i(f) = \mathcal{L}_D^{\text{null}}(f) - \ell_{nll}(f, x'_i, y'_i). \tag{46} \]

By using the same assumption, the term \( \Psi_{\ell_{nll}, P, D}(n, n) \) can be modified as follows:

\[
\Psi_{\ell_{nll}, P, D}(n, n) = \log E_{f \sim P(X',y')} \exp \left[ n \left( \mathcal{L}_D^{\text{null}}(f) - \tilde{\mathcal{L}}_X^{\text{null}}(f) \right) \right] = \log E_{f \sim P(X',y')} \exp \left[ \sum_{i=1}^{n} V_i(f) \right] = \log \left. \sum_{i=1}^{n} \log E_{f \sim P(x',y')} \exp V_i(f) \right|_{f \sim P(x',y')} = \sum_{i=1}^{n} \log E_{f \sim P(x',y')} \exp V_i(f). \tag{47}
\]

We assume that \( y'f(x') \) over the prior distribution \( P \) is sub-Gaussian with the variance factor \( \sigma_0^2 \), i.e.

\[
\log E_{f \sim P(x',y')} \exp \lambda y'f(x') \leq \frac{\lambda^2 \sigma_0^2}{2} \quad (\lambda \in \mathbb{R}). \tag{49}
\]

From the fact about the cumulative distribution function of normal distribution [9],

\[
\Phi(z) > \frac{2\phi(z)}{\sqrt{z^2 + 4} - z} \quad (z \leq 0), \tag{50}
\]

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the inverse of cumulative distribution function can be evaluated for any $a > -4$,

$$
\Phi(z)^{-1} < \frac{\sqrt{z^2 + 4} - z}{2\phi(z)}
= \frac{\sqrt{2\pi} \left( \sqrt{z^2 + 4} - z \right)}{2} \exp\left( \frac{z^2}{2} \right)
\leq \sqrt{2\pi} \exp\left( \frac{1}{2} z^2 + \frac{1}{2} \log(z^2 + 4) \right)
\leq \sqrt{2\pi} \exp\left( \frac{1}{2} z^2 + \frac{1}{2} \left( \frac{z^2}{a + 4} + \log(a + 4) \right) \right)
= \sqrt{2\pi(a + 4)} \exp\left( \frac{a + 5}{2(a + 4)} z^2 - \frac{a}{2(a + 4)} \right). \quad (51)
$$

Note that the last inequality holds for any $z \in \mathbb{R}$. If $z$ is sub-Gaussian with a variance factor $\sigma_z^2$ such that \( \frac{a + 5}{2(a + 4)} \leq \frac{1}{4\sigma_z^2} \), the following inequality holds [8],

$$
\log \mathbb{E}_z \Phi(z)^{-1} \leq \log \mathbb{E}_z \left[ \sqrt{2\pi(a + 4)} \right.
\exp\left( \frac{a + 5}{2(a + 4)} z^2 - \frac{a}{2(a + 4)} \right)
= \frac{1}{2} \log(2\pi(a + 4)) - \frac{a}{2(a + 4)}
+ \log \mathbb{E}_z \exp\left( \frac{a + 5}{2(a + 4)} z^2 \right)
\leq \frac{1}{2} \log(2\pi(a + 4)) - \frac{a}{2(a + 4)}
+ 4 \left( \frac{a + 5}{a + 4} \right)^2 \sigma_z^2. \quad (52)
$$
Each term in (48) is evaluated as follows:

\[
\log \mathbb{E}_{f \sim P} \mathbb{E}_{(x', y') \sim D} \exp V_i(f) \\
\leq \log \mathbb{E}_{f \sim P} \exp L_{D}^{\text{true}}(f) \\
\leq \log \mathbb{E}_{f \sim P} \mathbb{E}_{(x', y') \sim D} \exp \ell_{\text{all}}(f, x', y') \\
= \log \mathbb{E}_{f \sim P} \mathbb{E}_{(x', y') \sim D} \exp(-\log \Phi(y'f(x'))) \\
= \log \mathbb{E}_{f \sim P} \mathbb{E}_{(x', y') \sim D} \Phi(y'f(x'))^{-1} \\
\leq \frac{1}{2} \log(2\pi(a + 4)) - \frac{a}{2(a + 4)} \\
+ 4 \left( \frac{a + 5}{a + 4} \right)^2 \sigma_0^4. \tag{53}
\]

Note that the last inequality (51) holds for any \( z \in \mathbb{R} \).

The condition that the above inequality holds is

\[
\frac{a + 5}{2(a + 4)} < \frac{1}{4\sigma_0^2} \iff a > \frac{10\sigma_0^4 - 4}{1 - 2\sigma_0^2}. \tag{54}
\]