No-Regret Dynamics in the Fenchel Game: A Unified Framework for Algorithmic Convex Optimization

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Abstract

We develop an algorithmic framework for solving convex optimization problems using no-regret game dynamics. By converting the problem of minimizing a convex function into an auxiliary problem of solving a min-max game in a sequential fashion, we can consider a range of strategies for each of the two-players who must select their actions one after the other. A common choice for these strategies are so-called no-regret learning algorithms, and we describe a number of such and prove bounds on their regret. We then show that many classical first-order methods for convex optimization—including average-iterate gradient descent, the Frank-Wolfe algorithm, Nesterov’s acceleration methods, the accelerated proximal method—can be interpreted as special cases of our framework as long as each player makes the correct choice of no-regret strategy. Proving convergence rates in this framework becomes very straightforward, as they follow from plugging in the appropriate known regret bounds. Our framework also gives rise to a number of new first-order methods for special cases of convex optimization that were not previously known.

Keywords: Online learning, No-regret learning, Zero-sum game, Convex optimization, Frank-Wolfe method, Nesterov’s accelerated gradient methods, Momentum methods
1 Introduction

The main goal of this work is to develop a framework for solving convex optimization problems using iterative methods. Given a convex function \( f : \mathbb{R}^d \to \mathbb{R} \), domain \( K \subseteq \mathbb{R}^d \), and some tolerance \( \epsilon > 0 \), we want to find an approximate minimizer \( w \in K \) so that \( f(w) - \min_{w' \in K} f(w') \leq \epsilon \), using a sequence of oracle calls to \( f \) and its derivatives. This foundational problem has received attention for decades, and researchers have designed numerous methods for this problem under a range of oracle query models and structural assumptions on \( f(\cdot) \). What we aim to show in this paper is that a surprisingly large number of these methods—including those of Nesterov [1–5], Frank and Wolfe [6], and Beck and Teboulle [8]—can all be described and analyzed through a single unified algorithmic framework, which we call the Fenchel game no-regret dynamics (FGNRD). We show that several novel methods, with fast rates, emerge from FGNRD as well.

Let us give a short overview before laying out the FGNRD framework more precisely. A family of tools, largely developed by researchers in theoretical machine learning, consider the problem of sequential prediction and decision making in non-stochastic environments, often called adversarial online learning. This online learning setting has found numerous applications in several fields beyond machine learning—finance, for example, as well as statistics—but it has also emerged as a surprisingly useful tool in game theory. What we call no-regret online learning algorithms are particularly well-suited for computing equilibria in two-player zero-sum games, as well as solving saddle point problems more broadly. If each agent employs a no-regret online learning algorithm to choose their action at each of a sequence of rounds, it can be shown that the agents’ choices will converge to a saddle point, and at a rate that depends on their choice of learning algorithm. Thus, if we are able to simulate the two agents’ sequential strategies, where each aims to minimize the “regret” of their chosen actions, then what emerges from the resulting no-regret dynamics (NRD) can be implemented explicitly as an algorithm for solving min-max problems.

Our goal is to demonstrate how NRD helps develop and analyze methods for minimizing a closed convex function \( f \), i.e. solving \( \min_{w \in K} f(w) \), by solving a particular saddle-point problem whose two-input “payoff” function \( g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) is defined by

\[
g(x, y) := \langle x, y \rangle - f^*(y),
\]

where \( f^*(\cdot) \) is the conjugate of \( f(\cdot) \). We view this as a game in the sense that if one player selects an action \( x \) and a second player selects action \( y \), then \( g(x, y) \) is the former’s “cost” and the latter’s “gain” associated to their decisions. If the two players continue to update their decisions sequentially, first choosing \( x_1 \) and \( y_1 \) then \( x_2 \) and \( y_2 \), etc., and each player relies on some no-regret algorithm for this purpose, then one can show that the time-averaged iterates \( \bar{x}, \bar{y} \) form an approximate equilibrium of the Fenchel game—that is, \( g(\bar{x}, y') - \epsilon \leq g(\bar{x}, \bar{y}) \leq g(x', \bar{y}) + \epsilon \) for any alternative \( x', y' \). But indeed, this approximate equilibrium brings us right back to where we started, since using
the construction of the Fenchel game, it is easy to show that \( w \leftarrow \bar{x} \) satisfies \( f(w) - \min_{w' \in \calK} f(w') \leq \epsilon \). The approximation factor \( \epsilon \) is important, and we will see that it depends upon the number of iterations of the dynamic and the players’ strategies.

What FGNRD gives us is a recipe book for constructing and analyzing iterative algorithms for convex optimization. To simulate a dynamic we still need to make particular choices as for both players’ strategies and analyze their performance. We begin in Section 3 by giving a brief overview of tools from adversarial online learning, and we introduce a handful of simple online learning algorithms, including variants of FollowTheLeader and OnlineMirrorDescent, and prove bounds on the weighted regret—we generalize slightly the notion of regret by introducing weights \( \alpha_t > 0 \) for each round. We prove a key result that relates the error \( \epsilon \) of the approximate equilibrium pair \( \bar{x}, \bar{y} \), which are the weighted-average of the iterates of the two players, to the weighted regret of the players’ strategies. In Section 4 we show how several algorithms, including Frank-Wolfe’s method [6], several variants of Nesterov Accelerated Gradient Descent [1–5], and the accelerated proximal method [8], are all special cases of the FGNRD framework, all with special choices of the learning algorithms for the \( x \)- and \( y \)-player, and the weights \( \alpha_t \); see Table 1 for a summary of these recipes. In addition we provide several new algorithms using FGNRD in Section 5, summarized in Table 2.

The work presented in this manuscript combines results from our previous publications, [9–11], with a significantly cleaner and more general analysis. But we also include a number of new algorithms and convergence guarantees, described in Algorithm 8, Algorithm 9, Algorithm 10, Algorithm 20, Algorithm 11, and Algorithm 21, and Theorem 6, Theorem 7, Theorem 8, Theorem 9, Theorem 23, Theorem 10, Theorem 24. Moreover, for a family of online learning strategies listed in Algorithm 4, we develop a meta lemma in this work, Lemma 3, such that the guarantee of each online algorithm is obtained by an easy application of the meta lemma.

2 Preliminaries

2.1 Convex Analysis

We summarize some results in convex analysis that will be used in this paper. We also refer the readers to some excellent textbooks (e.g. [5, 12–16]).

Smoothness and strong convexity

A function \( f(\cdot) \) on \( \mathbb{R}^d \) is \( L \)-smooth with respect to a norm \( \| \cdot \| \) if \( f(\cdot) \) is everywhere differentiable and it has Lipschitz continuous gradient \( \| \nabla f(x) - \nabla f(z) \|_* \leq L \| x - z \| \), where \( \| \cdot \|_* \) denotes the dual norm. A function \( f(\cdot) \) is \( \mu \)-strongly convex w.r.t. a norm \( \| \cdot \| \) if the domain of \( f(\cdot) \) is convex and that \( f(\theta x + (1-\theta)z) \leq \theta f(x) + (1-\theta)f(z) - \frac{\mu}{2}(1-\theta)\| x - z \|^2 \) for all \( x, z \in \text{dom}(f) \) and \( \theta \in [0,1] \). If a function is \( \mu \)-strongly convex, then \( f(z) \geq f(x) + f_x^*(z-x) + \frac{\mu}{2}\| z - x \|^2 \) for all \( x, z \in \text{dom}(f) \), where \( f_x \) denotes a
Table 1 Summary of recovering existing optimization algorithms from *Fenchel Game.* Here $T$ denotes the total number of iterations, $\alpha_t$ are the weights which set the emphasis on iteration $t$, the last two columns on the table indicate the specific strategies of the players in the FGNRD.

| Algorithm                      | rate      | weights | $y$-player | $x$-player |
|--------------------------------|-----------|---------|------------|------------|
| Gradient Descent with averaging | Thm. 7    | $\alpha_t = 1$ | Sec. 3.3.1 | Sec. 3.3.9 |
| Cumulative Gradient Descent    | Thm. 10   | $\alpha_t = 1$ | Sec. 3.3.4 | Sec. 3.3.9 |

$L$-smooth convex optimization: $\min_{w \in \mathcal{K}} f(w)$ as a game $g(x,y) := (x,y) - f^*(y)$.

| Algorithm                      | rate      | weights | $y$-player | $x$-player |
|--------------------------------|-----------|---------|------------|------------|
| Frank-Wolfe method [6]         | Prop. 3 and 4 | $\alpha_t = t$ | Sec. 3.3.3 | BestResp$^+$\* |
| Linear rate FW [17]            | Thm. 3    | $\alpha_t = \frac{1}{\ell_t(x_t)^2}$ | Sec. 3.3.3 | BestResp$^+$\* |
| Gradient Descent with averaging | Thm. 8    | $\alpha_t = 1$ | Sec. 3.3.1 | Sec. 3.3.9 |
| Single-call extra-gradient with averaging | Thm. 9 | $\alpha_t = 1$ | Sec. 3.3.1 | BestResp$^+$\* | OptimisticMD |
| Nesterov's 1-memory method [4] | Thm. 14   | $\alpha_t = t$ | Sec. 3.3.7 | OptimiaticFTL |
| Nesterov's $\infty$-memory method [3] | Thm. 15   | $\alpha_t = t$ | Sec. 3.3.7 | FTRL$^+$ |
| Nesterov's first acceleration method [2] | Thm. 14   | $\alpha_t = t$ | Sec. 3.3.7 | OMD$^+$ |

Composite optimization: $\min_{w \in \mathbb{R}^d} f(w) + \psi(w)$, where $\psi(\cdot)$ is possibly non-differentiable, as a game $g(x,y) := (x,y) - f^*(y) + \psi(x)$.

| Algorithm                      | rate      | weights | $y$-player | $x$-player |
|--------------------------------|-----------|---------|------------|------------|
| Accelerated proximal method [8] | Thm. 16   | $\alpha_t = t$ | Sec. 3.3.7 | OMD$^+$ |

$L$-smooth and $\mu$ strongly convex optimization: $\min_{w \in \mathcal{K}} f(w)$ as a game $g(x,y) := (x,y) - f^*(y) + \frac{\mu}{2}\|\cdot\|^2$, where $\tilde{f}(\cdot) := f(\cdot) - \frac{\mu}{2}\|\cdot\|^2$.

| Algorithm                      | rate      | $\alpha_t$ | $y$-player | $x$-player |
|--------------------------------|-----------|------------|------------|------------|
| Nesterov’s method [5]          | Thm. 18   | $\alpha_t \exp(t)$ | Sec. 3.3.7 | FTRL$^+$ |

$L$-smooth convex finite-sum optimization: $\min_{w \in \mathcal{K}} f(w) := \sum_{i=1}^n f_i(w)$ as a game $g(x,y) := (x,y) - f^*(y)$.

| Algorithm                      | rate      | weights | $y$-player’s alg. | $x$-player’s alg. |
|--------------------------------|-----------|---------|-------------------|-------------------|
| Incremental FW [18]            | Thm. 6    | $\alpha_t = t$ | Appendix. D | Sec. 3.3.4 |

subgradient of $f$ at $x$. The subdifferential $\partial f(x)$ is the set of all subgradients of $f$ at $x$, i.e., $\partial f(x) = \{f_x : f(z) \geq f(x) + f_x^T(z-x), \forall z\}$. When $f$ is differentiable, we have $\partial f(x) = \{\nabla f(x)\}$. 
Non-smooth convex optimization: \( \min_{w \in K} f(w) \), where \( K \) is a \( \lambda \)-strongly convex set as a game \( g(x, y) := \langle x, y \rangle - f^*(y) \).

Assume that the norm of cumulative gradient does not vanish, \( \| \frac{1}{T} \sum_{s=1}^{T} \partial f(x_s) \| \geq \rho \).

| Algorithm | rate | weights | y-player’s alg. | x-player’s alg. |
|-----------|------|---------|----------------|----------------|
| Boundary  | Thm. 20 | \( \alpha_t = 1 \) | Sec. 3.3.3 | Sec. 3.3.1 |
| FW       | \( O \left( \frac{1}{\sqrt{TT}} \right) \) | | FTL | BestResp+ |

\( L \)-smooth convex optimization: \( \min_{w \in K} f(w) \),

where \( K \) is a \( \lambda \)-strongly convex set that is centrally symmetric and contains the origin, as a game \( g(x, y) := \langle x, y \rangle - f^*(y) \).

| Algorithm | rate | weights | y-player’s alg. | x-player’s alg. |
|-----------|------|---------|----------------|----------------|
| Gauge     | Thm. 22 | \( \alpha_t = t \) | Sec. 3.3.7 | FTRL+ with gauge function |
| FW       | \( O \left( \frac{1}{\sqrt{T^2T}} \right) \) | | OptimisticFTL | |

\( L \)-smooth and \( \mu \)-strongly convex optimization: \( \min_{w \in K} f(w) \),

where \( K \) is a \( \lambda \)-strongly convex set that is centrally symmetric and contains the origin, as a game \( g(x, y) := \langle x, y \rangle - \tilde{f}^*(y) + \frac{\mu}{\lambda} \gamma_{K}(x) \), where \( \tilde{f}(\cdot) := f(\cdot) - \frac{\mu}{\lambda} \gamma_{K}(x) \).

| Algorithm | rate | weights | y-player’s alg. | x-player’s alg. |
|-----------|------|---------|----------------|----------------|
| Gauge     | Thm. 23 | \( \alpha_t = \exp(t) \) | Sec. 3.3.7 | FTRL+ with gauge function |
| FW       | \( O \left( \exp \left( - \sqrt{\frac{\mu}{\lambda} T} \right) \right) \) | | OptimisticFTL | |

\( L \)-smooth convex optimization: \( \min_{w \in K} f(w) \),

as a game \( g(x, y) := \langle x, y \rangle - f^*(y) \).

| Algorithm | rate | weights | y-player’s alg. | x-player’s alg. |
|-----------|------|---------|----------------|----------------|
| Optimistic Mirror Descent with averaging | Thm. 24 | \( \alpha_t = t \) | Sec. 3.3.4 | Sec. 3.3.11 |
| | \( O \left( \frac{1}{\sqrt{T}} \right) \) | | FTL+ | OptimisticMD |

In Section 4 and Section 5, we will cover several optimization algorithms. For those that are labeled for minimizing non-smooth functions, their guarantees will not need differentiability of \( f(\cdot) \); otherwise, they are labeled for minimizing smooth functions, and they need differentiability of \( f(\cdot) \).

**Convex function and conjugate**

For any convex function \( f(\cdot) \), its Fenchel conjugate is

\[
    f^*(y) := \sup_{x \in \text{dom}(f)} \langle x, y \rangle - f(x).
\]

(1)

It is noted that the conjugate function \( f^*(\cdot) \) is convex, as it is a supremum over a linear function. Furthermore, if the function \( f(\cdot) \) is closed and convex, the following are equivalent: (I) \( y \in \partial f(x) \), (II) \( x \in \partial f^*(y) \), and (III)

\[
    \langle x, y \rangle = f(x) + f^*(y),
\]

(2)

which also implies that the bi-conjugate is equal to the original function, i.e. \( f^{**}(\cdot) = f(\cdot) \). Moreover, when the function \( f(\cdot) \) is differentiable, we have
\[ \nabla f(x) = \arg \sup_{y \in \text{dom}(f^*)} \langle x, y \rangle - f^*(y); \text{ on the other hand, when } f(\cdot) \text{ is not differentiable, we have } f_x = \arg \sup_{y \in \text{dom}(f^*)} \langle x, y \rangle - f^*(y), \text{ where } f_x \in \partial f(x) \text{ is a subgradient at } x. \] We refer to the readers to [19, 20] for more details of Fenchel conjugate. Throughout this paper, we assume that the underlying convex function is closed.

An important property of a closed and convex function is that \( f(\cdot) \) is \( L \)-smooth w.r.t. some norm \( \| \cdot \| \) if and only if its conjugate \( f^*(\cdot) \) is \( 1/L \)-strongly convex w.r.t. the dual norm \( \| \cdot \|^* \) (e.g. Theorem 6 in [20]).

**Bregman Divergence.**

We will denote the Bregman divergence \( D_{\phi}^z(\cdot) \) centered at a point \( z \) with respect to a \( \beta \)-strongly convex distance generating function \( \phi(\cdot) \) as

\[ D_{\phi}^z(x) := \phi(x) - \langle \nabla \phi(z), x - z \rangle - \phi(z). \quad (3) \]

**Strongly convex sets.**

A convex set \( K \subseteq \mathbb{R}^m \) is an \( \lambda \)-strongly convex set w.r.t. a norm \( \| \cdot \| \) if for any \( x, z \in K \), any \( \theta \in [0, 1] \), the \( \| \cdot \| \) ball centered at \( \theta x + (1 - \theta)z \) with radius \( \theta(1-\theta)\frac{\lambda}{2}\|x-z\|^2 \) is included in \( K \) [21]. Examples of strongly convex sets include \( \ell_p \) balls: \( \| x \|_p \leq r, \forall p \in (1, 2] \), Schatten \( p \) balls: \( \| \sigma(X) \|_p \leq r \) for \( p \in (1, 2] \), where \( \sigma(X) \) is the vector consisting of singular values of the matrix \( X \), and Group \( (s,p) \) balls: \( \| X \|_{s,p} = \| (\| X_1 \|_s, \| X_2 \|_s, \ldots, \| X_m \|_s) \|_p \leq r, \forall s, p \in (1, 2] \).

In Appendix G, we discuss more about strongly-convex sets.

### 2.2 Min-max Problems and the Fenchel Game

Throughout the rest of the paper we will be focusing on the following problem. For some natural number \( d > 0 \), have some convex and closed set \( K \subseteq \mathbb{R}^d \) and a closed and convex function \( f : K \to \mathbb{R} \). Our goal is to solve the minimization problem \( \min_{x \in K} f(x) \). While we consider both bounded and unbounded sets \( K \) (including \( K = \mathbb{R}^d \)), we assume throughout that a minimizer of \( f \) exists in some bounded region of \( K \).

**Min-max problems and (approximate) Nash equilibrium**

Given a zero-sum game with payoff function \( g(x, y) \) which is convex in \( x \) and concave in \( y \), define \( V^* = \inf_{x \in X} \sup_{y \in Y} g(x, y) \). An \( \epsilon \)-equilibrium of \( g(\cdot, \cdot) \) is a pair \( \hat{x}, \hat{y} \) such that

\[ V^* - \epsilon \leq \inf_{x \in X} g(x, \hat{y}) \leq V^* \leq \sup_{y \in Y} g(\hat{x}, y) \leq V^* + \epsilon, \quad (4) \]

where \( X \) and \( Y \) are convex decision spaces of the \( x \)-player and the \( y \)-player respectively.
The Fenchel Game.

One of the core tools of this paper is as follows. In order to solve the problem \( \min_{x \in K} f(x) \), we instead construct a saddle-point problem which we call the Fenchel Game. We define \( g : K \times Y \) as follows:

\[
g(x, y) := \langle x, y \rangle - f^*(y),
\]

where \( Y \) is the gradient space of \( f \) that will be precisely defined later in Section 3.2. This payoff function is useful for solving the original optimization problem, since an equilibrium of this game provides us with a solution to \( \min_{x \in K} f(x) \). Let \( \hat{x} \in K \) and \( \hat{y} \in Y \) be any equilibrium pair of \( g \), which exists even when the convex sets \( K \) and \( Y \) can be unbounded, thanks to the assumption that a minimizer of a closed convex function \( f(\cdot) \) is attained in a bounded region. We have \( V^* = \sup_{y \in Y} g(\hat{x}, y) \). Furthermore,

\[
\inf_{x \in K} f(x) = \inf_{x \in K} \sup_{y \in Y} \{\langle x, y \rangle - f^*(y)\} = \sup_{y \in Y} g(\hat{x}, y) = \sup_{y \in Y} \{\langle \hat{x}, y \rangle - f^*(y)\} = f(\hat{x}).
\]

In other words, given an equilibrium pair \( \hat{x}, \hat{y} \) of \( g(\cdot, \cdot) \), we immediately have a minimizer of \( f(\cdot) \). This simple observation can be extended to approximate equilibria as well.

Lemma 1 If \( (\hat{x}, \hat{y}) \) is an \( \epsilon \)-equilibrium of the Fenchel Game (5), then \( f(\hat{x}) - \min_x f(x) \leq \epsilon \).

Lemma 1 sets us up for the remainder of the paper. The framework, which we lay out precisely in Section 3.2, will consider two players sequentially playing the Fenchel game, where the \( y \)-player sequentially outputs iterates \( y_1, y_2, \ldots \), while alongside the \( x \)-player returns iterates \( x_1, x_2, \ldots \). Each player may use the previous sequence of actions of their opponent in order to choose their next point \( x_t \) or \( y_t \), and we will rely heavily on the use of no-regret online learning algorithms described in Section 3. In addition, we need to select a sequence of weights \( \alpha_1, \alpha_2, \ldots > 0 \) which determine the “strength” of each round, and can affect the players’ update rules. What we will be able to show is that the \( \alpha \)-weighted average iterate pair, defined as

\[
(\hat{x}, \hat{y}) := \left( \frac{\alpha_1 x_1 + \cdots + \alpha_T x_T}{\alpha_1 + \cdots + \alpha_T}, \frac{\alpha_1 y_1 + \cdots + \alpha_T y_T}{\alpha_1 + \cdots + \alpha_T} \right),
\]

is indeed an \( \epsilon \)-equilibrium of \( g(\cdot, \cdot) \), and thus via Lemma 1 we have that \( \hat{x} \) approximately minimizes \( f \). To get a precise estimate of \( \epsilon \) requires us to prove a family of regret bounds, which is the focus of the following section.
3 No-regret learning algorithms

An algorithmic framework, often referred to as no-regret learning or online convex optimization, which has been developed mostly within the machine learning research community, has grown quite popular as it can be used in a broad class of sequential decision problems. As we explain in Section 3.1, one imagines an algorithm making repeated decisions by selecting a vector of parameters in a convex set, and on each round is charged according to a varying convex loss function. The algorithm’s goal is to minimize an objective known as regret. In Section 3.2 we describe how online convex optimization algorithms with vanishing regret can be implemented in a two-player protocol which sequentially computes an approximate equilibria for a convex-concave payoff function. This is the core tool that allows us to describe a range of known and novel algorithms for convex optimization, by modularly combining pairs of OCO strategies. In Section 3.3 we provide several such OCO algorithms, most of which have been proposed and analyzed over the past 10-20 years.

### Protocol 1 Weighted Online Convex Optimization

1: **Input:** convex decision set \( Z \subseteq \mathbb{R}^d \)

2: **Input:** number of rounds \( T \)

3: **Input:** weights \( \alpha_1, \alpha_2, \ldots, \alpha_T > 0 \)

4: **Input:** algorithm OAlg

5: **for** \( t = 1, 2, \ldots, \) **do**

6: **Return:** \( z_t \leftarrow \text{OAlg} \)

7: **Receive:** \( \alpha_t, \ell_t(\cdot) \rightarrow \text{OAlg} \)

8: **Evaluate:** \( \text{Loss} \leftarrow \text{Loss} + \alpha_t \ell_t(z_t) \)

9: **end for**

3.1 Online Convex Optimization and Regret

Here we describe the framework, given precisely in Protocol 1, for online convex optimization. We assume we have some learning algorithm known as OAlg that is tasked with selecting “actions” from a convex decision set \( Z \subseteq \mathbb{R}^d \). On each round \( t = 1, \ldots, T \), OAlg returns a point \( z_t \in Z \), and is then presented with the pair \( \alpha_t, \ell_t \), where \( \alpha_t > 0 \) is a weight for the current round and \( \ell_t : Z \rightarrow \mathbb{R} \) is a convex loss function that evaluates the choice \( z_t \). While OAlg is essentially forced to “pay” the cost \( \alpha_t \ell_t(z_t) \), it can then update its state to provide better choices in future rounds.

On each round \( t \), the learner must select a point \( z_t \in Z \), and is then “charged” a loss of \( \alpha_t \ell_t(z_t) \) for this choice. Typically it is assumed that, when the learner selects \( z_t \) on round \( t \), she has observed all loss functions \( \alpha_1 \ell_1(\cdot), \ldots, \alpha_{t-1} \ell_{t-1}(\cdot) \) up to, but not including, time \( t \). However, we will also consider learners that are prescient, i.e. that can choose \( z_t \) with knowledge of the loss functions up to and including time \( t \). The objective of interest in most
of the online learning literature is the learner’s regret, defined as

$$\alpha\text{-Reg}^z(z^*) := \sum_{t=1}^{T} \alpha_t \ell_t(z_t) - \sum_{t=1}^{T} \alpha_t \ell_t(z^*),$$

where $z^* \in Z$ is a comparator that the online learner is compared to. Often-times we will want to refer to the average regret, or the regret normalized by the time weight $A_T := \sum_{t=1}^{T} \alpha_t$, which we will denote $\alpha\text{-Reg}^z(z^*) := \frac{\alpha\text{-Reg}^z(z^*)}{A_T}$. Note that in online learning literature, what has become a cornerstone of online learning research has been the existence of no-regret algorithms, i.e. learning strategies that guarantee $\alpha\text{-Reg}^z(z^*) \to 0$ as $A_T \to \infty$.

### 3.2 Framework: optimization as Fenchel Game

In this paper, we consider Fenchel game (5) with weighted losses depicted in Protocol 2. In this game, one of the players play first, and the other player would see what its opponent plays before choosing its action. What Protocol 2 describes is the case when the $y$-player plays before the $x$-player plays, but we emphasize that we can swap the order and let the $x$-player plays first, which will turn out to be helpful when designing and analyzing certain algorithms. The $y$-player receives loss functions $\alpha_t \ell_t(\cdot)$ in round $t$, in which $\ell_t(y) := f^*(y) - \langle x_t, y \rangle$, while the $x$-player receives its loss functions $\alpha_t h_t(\cdot)$ in round $t$, in which $h_t(x) := \langle x, y_t \rangle - f^*(y_t)$. Consequently, we can define the weighted regret of the $x$ and $y$ players as

$$\alpha\text{-Reg}^y := \sum_{t=1}^{T} \alpha_t \ell_t(y_t) - \min_{y \in Y} \sum_{t=1}^{T} \alpha_t \ell_t(y),$$

$$\alpha\text{-Reg}^x := \sum_{t=1}^{T} \alpha_t h_t(x_t) - \min_{x \in X} \sum_{t=1}^{T} \alpha_t h_t(x),$$

where the decision space of the $x$-player $X$ is that of the underlying optimization problem $\min_{x \in K} f(x)$, i.e. $\mathcal{X} = K \subseteq \mathbb{R}^d$, while the decision space of the $y$-player is the gradient space, i.e. $Y := \bigcup_{x \in X} \partial f(x)$, which is the union of subdifferentials over the domain $\mathcal{X}$. One can check that the closure of the set $\mathcal{Y}$ is a convex set. Appendix A describes the proof.

**Theorem 1** The closure of the gradient space $\bigcup_{x \in \mathcal{X}} \partial f(x)$ is a convex set.

At times when we want to refer to the regret on another sequence $y'_1, \ldots, y'_T$ we may refer to this as $\alpha\text{-Reg}(y'_1, \ldots, y'_T)$. We also denote $A_t$ as the cumulative sum of the weights $A_t := \sum_{s=1}^{t} \alpha_s$ and the weighted average regret $\overline{\alpha\text{-Reg}} := \frac{\alpha\text{-Reg}}{A_T}$. 
Fenchel Game No-Regret Dynamics

Protocol 2 Fenchel Game No-Regret Dynamics
1: \textbf{Input:} number of rounds $T$
2: \textbf{Input:} Convex decision sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^d$
3: \textbf{Input:} Convex-concave payoff function $g : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$
4: \textbf{Input:} weights $\alpha_1, \alpha_2, \ldots, \alpha_T > 0$ \hspace{1cm} // Weights determined in advance
5: \textbf{Input:} algorithms $\text{OAlg}^\mathcal{Y}, \text{OAlg}^\mathcal{X}$ \hspace{1cm} // Learning algorithms for both players
6: for $t = 1, 2, \ldots, T$ do
7: \hspace{1cm} Return: $y_t \leftarrow \text{OAlg}^\mathcal{Y}$ \hspace{1cm} // y-player returns a point $y_t$
8: \hspace{1cm} Update: $\alpha_t, h_t(\cdot) \to \text{OAlg}^\mathcal{X}$ \hspace{1cm} // x-player updates with $\alpha_t$ and loss $g(\cdot, y_t)$
9: \hspace{1.5cm} where $h_t(\cdot) := g(\cdot, y_t)$
10: \hspace{1cm} Return: $x_t \leftarrow \text{OAlg}^\mathcal{X}$ \hspace{1cm} // x-player returns a point $x_t$
11: \hspace{1cm} Update: $\alpha_t, \ell_t(\cdot) \to \text{OAlg}^\mathcal{Y}$ \hspace{1cm} // y-player updates with $\alpha_t$ and loss $-g(x_t, \cdot)$
12: \hspace{1.5cm} where $\ell_t(\cdot) := -g(x_t, \cdot)$
13: end for
14: Output $(\bar{x}_T, \bar{y}_T) := \left( \frac{\sum_{T=1}^{T} \alpha_t x_t}{\sum_{T=1}^{T} \alpha_t}, \frac{\sum_{T=1}^{T} \alpha_t y_t}{\sum_{T=1}^{T} \alpha_t} \right)$.

Theorem 2 Assume a $T$-length sequence $\alpha$ are given. Suppose in Protocol 2 the online learning algorithms $\text{OAlg}^\mathcal{X}$ and $\text{OAlg}^\mathcal{Y}$ have the $\alpha$-weighted average regret $\overline{\alpha-\text{Reg}}^\mathcal{X}$ and $\overline{\alpha-\text{Reg}}^\mathcal{Y}$ respectively. Then the output $(\bar{x}_T, \bar{y}_T)$ is an $\epsilon$-equilibrium for $g(\cdot, \cdot)$, with $\epsilon = \overline{\alpha-\text{Reg}}^\mathcal{X} + \overline{\alpha-\text{Reg}}^\mathcal{Y}$. Moreover, if $f(x) := \sup_{y \in \mathcal{Y}} g(x, y)$, then it follows that

$$f(\bar{x}_T) - \min_{x \in \mathcal{X}} f(x) \leq \overline{\alpha-\text{Reg}}^\mathcal{X} + \overline{\alpha-\text{Reg}}^\mathcal{Y}.$$ 

Proof Suppose that the loss function of the $x$-player in round $t$ is $\alpha_t h_t(\cdot) : \mathcal{X} \to \mathbb{R}$, where $h_t(\cdot) := g(\cdot, y_t)$. The $y$-player, on the other hand, observes her own sequence of loss functions $\alpha_t \ell_t(\cdot) : \mathcal{Y} \to \mathbb{R}$, where $\ell_t(\cdot) := -g(x_t, \cdot)$.

$$\frac{1}{\sum_{s=1}^{T} \alpha_s} \sum_{t=1}^{T} \alpha_t g(x_t, y_t) = \frac{1}{\sum_{s=1}^{T} \alpha_s} \sum_{t=1}^{T} -\alpha_t \ell_t(y_t)$$
$$\geq -\frac{1}{\sum_{s=1}^{T} \alpha_s} \inf_{y \in \mathcal{Y}} \left( \sum_{t=1}^{T} \alpha_t \ell_t(y) \right) - \frac{\overline{\alpha-\text{Reg}}^\mathcal{Y}}{\sum_{s=1}^{T} \alpha_s}$$
$$= \sup_{y \in \mathcal{Y}} \left( \frac{1}{\sum_{s=1}^{T} \alpha_s} \sum_{t=1}^{T} \alpha_t g(x_t, y) \right) - \overline{\alpha-\text{Reg}}^\mathcal{Y}$$
$$\text{(Jensen)} \geq \sup_{y \in \mathcal{Y}} \left( \frac{1}{\sum_{s=1}^{T} \alpha_s} \sum_{t=1}^{T} \alpha_t x_t, y \right) - \overline{\alpha-\text{Reg}}^\mathcal{Y}$$
$$= \sup_{y \in \mathcal{Y}} \left( \bar{x}_T, y \right) - \overline{\alpha-\text{Reg}}^\mathcal{Y}$$
$$\geq \inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} \left( x, y \right) - \overline{\alpha-\text{Reg}}^\mathcal{Y}. \hspace{1cm} (9)$$

$$\geq \inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} \left( x, y \right) - \overline{\alpha-\text{Reg}}^\mathcal{Y}. \hspace{1cm} (10)$$
Let us now apply the same argument on the right hand side, where we use the $x$-player’s regret guarantee.

\[
\frac{1}{\sum_{s=1}^{T} \alpha_s} \sum_{t=1}^{T} \alpha_t g(x_t, y_t) = \frac{1}{\sum_{s=1}^{T} \alpha_s} \sum_{t=1}^{T} \alpha_t h_t(x_t)
\]

\[
\leq \inf_{x \in \mathcal{X}} \left( \sum_{t=1}^{T} \frac{1}{\sum_{s=1}^{T} \alpha_s} \alpha_t g(x, y_t) \right) + \frac{\alpha_{-\text{Reg}}^x}{\sum_{s=1}^{T} \alpha_s}
\]

\[
= \inf_{x \in \mathcal{X}} \left( \sum_{t=1}^{T} \frac{1}{\sum_{s=1}^{T} \alpha_s} \alpha_t g(x, y_t) \right) + \alpha_{-\text{Reg}}^x
\]

(Jensen) \leq \inf_{x \in \mathcal{X}} g \left( x, \sum_{t=1}^{T} \frac{1}{\sum_{s=1}^{T} \alpha_s} \alpha_t y_t \right) + \alpha_{-\text{Reg}}^x
\]

\[
\leq \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} g(x, y) + \alpha_{-\text{Reg}}^x. \tag{11}
\]

By (12) and (9), we have $\sup_{y \in \mathcal{Y}} g(\bar{x}_T, y) \leq V^* + \alpha_{-\text{Reg}}^x + \alpha_{-\text{Reg}}^y$. By (11) and (10), we have $V^* - \alpha_{-\text{Reg}}^x - \alpha_{-\text{Reg}}^y \leq \inf_{x \in \mathcal{X}} g(x, \bar{y}_T)$. Thus $(\bar{x}_T, \bar{y}_T)$ is an $\epsilon = \alpha_{-\text{Reg}}^x + \alpha_{-\text{Reg}}^y$ equilibrium.

We also have

\[
\frac{1}{\sum_{s=1}^{T} \alpha_s} \sum_{t=1}^{T} \alpha_t g(x_t, y_t) = \frac{1}{\sum_{s=1}^{T} \alpha_s} \sum_{t=1}^{T} \alpha_t h_t(x_t)
\]

\[
\leq \left( \sum_{t=1}^{T} \frac{1}{\sum_{s=1}^{T} \alpha_s} \alpha_t h_t(x^*) \right) + \frac{\alpha_{-\text{Reg}}^x}{\sum_{s=1}^{T} \alpha_s} \tag{13}
\]

\[
= \left( \sum_{t=1}^{T} \frac{1}{\sum_{s=1}^{T} \alpha_s} \alpha_t g(x^*, y_t) \right) + \alpha_{-\text{Reg}}^x
\]

(Jensen) \leq g \left( x^*, \sum_{t=1}^{T} \frac{1}{\sum_{s=1}^{T} \alpha_s} \alpha_t y_t \right) + \alpha_{-\text{Reg}}^x
\]

\[
\leq \sup_{y \in \mathcal{Y}} g(x^*, y) + \alpha_{-\text{Reg}}^x, \tag{14}
\]

where we denote $x^* \leftarrow \arg\min_{x \in \mathcal{K}} f(x)$ and (13) is because $\sum_{t=1}^{T} \alpha_t h_t(x_t) - \sum_{t=1}^{T} \alpha_t h_t(x^*) \leq \sum_{t=1}^{T} \alpha_t h_t(x_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} \alpha_t h_t(x) := \alpha_{-\text{Reg}}^x$. Combining (9) and (14), we see that $f(\bar{x}_T) - \min_{x \in \mathcal{X}} f(x) \leq \alpha_{-\text{Reg}}^x + \alpha_{-\text{Reg}}^y$. \hfill \Box

In order to utilize minimax duality, we have to define decision sets for two players, and we must produce a convex-concave payoff function. First we will assume, for convenience, that $f(x) := \infty$ for any $x \notin \mathcal{X}$. That is, it takes the value $\infty$ outside $\mathcal{X}$, which ensures that $f(\cdot)$ is lower semi-continuous and convex. We will also assume that the solution(s) to the underlying problem $x^* \leftarrow \arg\min_{x \in \mathcal{K}} f(x)$ has a finite size. As we will see in Theorem 14 and Theorem 16, the convergence rate of certain algorithms depend on the initial
distance to $x^*$, and hence this assumption guarantees that the initial distance is finite.

**Faster rates for min-max problems.**

We conclude this section by noting that many of the accelerated rates that we prove in this work emerge from a new set of tools that have been developed within the past decade. The use of no-regret learning to solve zero-sum games, as described in Theorem 2, was popularized by Freund and Schapire [22] in 1996, although the idea goes back to Blackwell [23] and Hannan [24] in the 1950s. While this basic tool has been used in many settings and applications, a key trick was discovered by Rakhlin and Sridharan [25] in 2013, showing that the rate for solving a simple matrix game could be improved from $O(1/\sqrt{T})$ to $O(1/T)$ via a more sophisticated type of algorithm which we herein refer to as optimistic; this will be explained precisely in Section 3.3. The first known presentation of such an algorithm was given by [26], who showed that a better regret bound is achievable if the sequence of loss functions changes slowly. As explained in [25], this is very useful when used in Protocol 2. This new trick is a core piece of many of the tools presented in this paper, and we emphasize that all of the accelerated optimization algorithms we describe through the FGNRD framework rely on one of the players using an optimistic online learning algorithm.

On the other hand, classic optimization algorithms like mirror prox of Nemirovski [27] and dual extrapolation of Nesterov [28] were invented earlier than the aforementioned no-regret learning approaches for solving min-max problems [25, 26, 29]. Both mirror prox and dual extrapolation can be applied to convex-concave problems and obtain a $O(1/T)$ fast rate, compared to a $O(1/\sqrt{T})$ slow rate [30]. Furthermore, both can be more broadly applied to solve variational inequalities in monotone operators [31–36], where finding a saddle point of a convex-concave function is a special case, see also [37–39] for related treatments. We note that there are some recent advancements based on mirror prox and dual extrapolation, e.g., [40], and there are also some progress on deriving lower bounds for solving certain classes of min-max problems, e.g., [41, 42].

**Min-max formulations of a primal problem**

The idea of casting a primal problem as a saddle point problem via Fenchel conjugate is not new. For example, Chambolle and Pock [43] consider solving a primal problem of the form, $\min_{x \in \mathcal{K}} f(Mx) + \psi(x)$, via solving a saddle point problem $\min_{x \in \mathcal{K}} \max_{y \in \Theta} \langle Mx, y \rangle + \psi(x) - f^*(y)$, where $M : \mathcal{K} \to \Theta$ is a continuous linear operator. When $Mx = x$ and $\psi(\cdot) = 0$, it reduces to our Fenchel Game formulation. Other examples include [44–49], to name just a few. However, the algorithmic approaches and the focuses in these related works are quite different from ours. For example, the algorithms of Chambolle and Pock [43] assume that both $f(\cdot)$ and $\psi(\cdot)$ are simple in the sense that the corresponding resolvent operators can be efficient computed, i.e., problems of
the form $\arg\min_{x \in K} f(x) + \frac{\|x - z\|^2}{2\eta}$ can be efficiently solved, while our work does not need this assumption. Moreover, we focus on demonstrating how no-regret learning together with the reformulation technique help design and analyze convex optimization algorithms systematically, enable to capture many existing algorithms and results, as well as to give rise to new ones, which is very different from existing works of primal-dual approaches. It is also interesting to note that in our approach, the $y$-player always outputs a gradient of the underlying function, which is not necessarily the case in other works, e.g., [43–49].

Other relevant works include Gutman and Peña [50, 51]. In [50], they consider solving convex optimization problems of the form, $\min_{x \in K} f(Mx) + \psi(x)$, where $M : K \rightarrow \text{dom}(f)$ is a linear operator. The design and analysis of their work focuses on minimizing the corresponding Fenchel duality gap,

$$\Delta = f(Mx) + \psi(x) + f^*(z) + \psi^*(-M^Tz),$$

where $z$ is the dual variable. Their algorithmic framework also considers outputs a weighted average of points and allows different weights on each iteration. From their framework, they derive generalized Frank-Wolfe, proximal gradient method and its variants. Our work differs from [50] because we consider applying no-regret learning and the corresponding analysis, which is not present in [50]. In [51], they provide a proof of $O(1/T^2)$ rate of accelerated proximal gradient method [8], where the analysis is done by weighting a certain duality gap. On the other hand, we will show a simpler proof of the accelerated proximal method by plugging in and summing the regret bounds of a pair of online learning strategies, see Theorem 16. Interestingly, our proof of the accelerated proximal method and that of Nesterov’s 1-memory method (Theorem 14) is the same from our Fenchel Game interpretation.

**The use of the weighted regret**

We note that the idea of the weighted regret can also be found in the works of Ho-Nguyen and Kılınç-Karzan [52, 53]. They consider applying no-regret learning algorithms to the associated Lagrangian of a constrained optimization problem, while ours is for the Fenchel Game. Specifically, [53] consider solving a convex problem with functional constraints,

$$\min_{x \in K} f(x), \quad \text{s.t. } h_i(x) \leq 0, \forall i \in [m],$$

via solving the min-max problem, $\min_{x \in K} \max_{y_i \geq 0, \forall i \in [m]} f(x) + y_i h_i(x)$. Their algorithm obtains an $O(1/T)$ rate and one of the techniques is emphasizing the per-round regret on the later rounds. In [52], they consider optimization under uncertainty sets: $\min_{x \in K} f(x), \quad \text{s.t. } \sup_{u_i \in U_i} h_i(x, u_i), \forall i \in [m]$, where each $U^i$ is a fixed uncertainty set. Their algorithm achieves a fast $O(1/T)$ rate when the weight is $\alpha_t = t$, while other works of online saddle-point optimization have a slower $O(1/\sqrt{T})$ rate, e.g., [54, 55].
We also note that a monograph [56] has a subsection regarding an analysis of the weighted average regret of Online Mirror Descent.

### 3.3 Online Convex Optimization: An Algorithmic Menu

In this section, we introduce and analyze several core online learning algorithms. We also refer the reader to some tutorial for more online learning algorithms (see e.g. [57–60]). Later in Sections 4 & 5, we will show how composing different online learning algorithm within the Fenchel Game No-Regret Dynamics (Protocol 2) enables to easily recover known results and methods for convex optimization (Section 4), as well as to design new algorithm with novel guarantees (Section 5).

We present two families of online learning strategies: the “batch-style” family of algorithms, and the “update” family, although we note that one can relate these two. Within each family, we have algorithms that require a regularization function $R(\cdot)$, or a “link” or “distance-generating” function $\phi(\cdot)$. In some cases, the algorithm may peek at the current loss function $\ell_t$ before selecting $z_t$ (these are denoted with a superscript +), or they may only have access to $\ell_1, \ldots, \ell_{t-1}$. We also present “optimistic” variants, where the algorithm can not see $\ell_t$ but can try to “guess” this function with some $m_t$, and use this guess in it’s optimization to compute $z_t$. We note, importantly, that we have given different names to these algorithms, keeping up with historical precedents in the online learning community, but we observe that with the appropriate parameters every one of the batch-style algorithms is in fact a special case of OptimisticFTRL, which is in some sense the Master Algorithm. We utilize this fact when proving regret bounds, as a Master Bound is provided in Lemma 3 of Subsection 3.3.2, and nearly all of the remaining bounds of the “batch-style” family of algorithms in Algorithm 4 follow as an easy corollary. Therefore, in the following, we describe the online learning strategies with their regret bounds first, and we defer the proofs in Subsection 3.3.8. It is noted that the online learning algorithms that will be introduced in this subsection do not need the loss function $\ell_t(\cdot)$ to be differentiable.

Before diving into the two families, we begin with one simple algorithm $\text{BESTRESP}^+$ that stands as something of an outlier.

#### Algorithm 3 $\text{BESTRESP}^+$ - Best Response

- **Parameters:** convex set $Z$
- **Receives:** $\alpha_1, \ldots, \alpha_T > 0, \ell_1, \ldots, \ell_T : Z \to \mathbb{R}$
- **Action:** $z_t \leftarrow \arg\min_{z \in Z} \{\ell_t(z)\}$
Algorithm 4 A family of batch-style online learning strategies

| Parameters: | convex set $\mathcal{Z}$, initial point $z_{\text{init}} \in \mathcal{Z}$ |
| Receives:   | $\alpha_1, \ldots, \alpha_T > 0, \ell_1, \ldots, \ell_T : \mathcal{Z} \to \mathbb{R}$ |
| $\text{FTL}[z_{\text{init}}]$ | $z_t \leftarrow z_{\text{init}}$ if $t = 1$ otherwise |

$$z_t \leftarrow \argmin_{z \in \mathcal{Z}} \left( \sum_{s=1}^{t-1} \alpha_s \ell_s(z) \right)$$

$$\text{FTL}^+ : \quad z_t \leftarrow \argmin_{z \in \mathcal{Z}} \left( \sum_{s=1}^{t} \alpha_s \ell_s(z) \right)$$

$\text{FTRL}[\ell(\cdot), \eta]$ : $z_t \leftarrow \argmin_{z \in \mathcal{Z}} \left( \sum_{s=1}^{t-1} \alpha_s \ell_s(z) + \frac{1}{\eta} R(z) \right)$

$\text{FTRL}^+[\ell(\cdot), \eta]$ : $z_t \leftarrow \argmin_{z \in \mathcal{Z}} \left( \sum_{s=1}^{t} \alpha_s \ell_s(z) + \frac{1}{\eta} R(z) \right)$

$\text{OPTIMISTICFTL} : \quad z_t \leftarrow \argmin_{z \in \mathcal{Z}} \left( \alpha_t m_t(z) + \sum_{s=1}^{t-1} \alpha_s \ell_s(z) \right)$

Requires “guesses” $m_1, \ldots, m_T : \mathcal{Z} \to \mathbb{R}$

$\text{OPTIMISTICFTRL}[\ell(\cdot), \eta]$ : $z_t \leftarrow \argmin_{z \in \mathcal{Z}} \left( \alpha_t m_t(z) + \sum_{s=1}^{t-1} \alpha_s \ell_s(z) + \frac{R(z)}{\eta} \right)$

Requires “guesses” $m_1, \ldots, m_T : \mathcal{Z} \to \mathbb{R}$

Algorithm 5 A family of update-style online learning strategies

| Parameters: | convex set $\mathcal{Z}$, initial point $z_0 = z_{-\frac{1}{2}} \in \mathcal{Z}$ |
| Receives:   | $\alpha_1, \ldots, \alpha_T > 0, \ell_1, \ldots, \ell_T : \mathcal{Z} \to \mathbb{R}$ |
| Set:        | $\alpha_0 := 0, \ell_0(\cdot) := 0$ |

$\text{OMD}[\phi(\cdot), z_0, \gamma] : \quad z_t \leftarrow \argmin_{z \in \mathcal{Z}} \left( \alpha_{t-1} \ell_{t-1}(z) + \frac{1}{\gamma} D_{z_{t-1}}^\phi(z) \right)$

$\text{OMD}^+[\phi(\cdot), z_0, \gamma] : \quad z_t \leftarrow \argmin_{z \in \mathcal{Z}} \left( \alpha_t \ell_t(z) + \frac{1}{\gamma} D_{z_{t-1}}^\phi(z) \right)$

$\text{OPTIMISTICMD}[\phi(\cdot), z_0, \gamma] : \quad z_t \leftarrow \argmin_{z \in \mathcal{Z}} \left( \alpha_t \langle z, \nabla \ell_t(z_t) \rangle + \frac{1}{\gamma} D_{z_{t-1}}^\phi(z) \right)$

$z_{t+\frac{1}{2}} \leftarrow \argmin_{z \in \mathcal{Z}} \left( \alpha_t \langle z, \nabla \ell_t(z_t) \rangle + \frac{1}{\gamma} D_{z_{t-1}}^\phi(z) \right)$

Requires “guesses” $m_1, \ldots, m_T : \mathcal{Z} \to \mathbb{R}$
3.3.1 Best Response – \textsc{BestResp}⁺

Perhaps the most trivial strategy for a prescient learner is to ignore the history of the $\ell_s$’s, and simply play the best choice of $z_t$ on the current round. We call this algorithm \textsc{BestResp}⁺. The ability to see the cost of your decisions in advance guarantees, quite naturally, that you will not suffer positive regret. We note that algorithms with a + superscript may peek” at the loss $\ell_t$ before choosing $z_t$.

**Lemma 2** (\textsc{BestResp}⁺) For any sequence of loss functions $\{\alpha_t \ell_t(\cdot)\}_{t=1}^T$, if $z_1, \ldots, z_T$ is chosen according to \textsc{BestResp}⁺ ensures,

$$\alpha\text{-Reg}^2(z^*) \leq \sum_{t=1}^T \alpha_t \ell_t(z_t) - \min_{z \in Z} \sum_{t=1}^T \alpha_t \ell_t(z) \leq 0,$$

for any comparator $z^*$.

**Proof** Since $z_t = \arg\min_{z \in Z} \ell_t(z)$, we have $\ell_t(z_t) \leq \ell_t(z)$ for any $z \in Z$. The result follows by summing the inequalities from $t = 1, \ldots, T$, and recalling that the $\alpha_t$’s are non-negative. \hfill \Box

3.3.2 A meta online learning algorithm: \textsc{OptimisticFTRL}

Here we describe \textsc{OptimisticFTRL}. As can be seen from Equation (20) in Algorithm 4, \textsc{OptimisticFTRL} employs a regularization term (similar to FTRL and FTRL⁺), and makes use of a hint sequence $m_t(\cdot)$ (similar to \textsc{OptimisticFTRL}). In Lemma 3 below, we state the regret guarantees of \textsc{OptimisticFTRL}. Later, we will show that all the other online learning strategies in Algorithm 4 — \textsc{FTL}, \textsc{FTL}⁺, FTRL, FTRL⁺, \textsc{OptimisticFTRL} are special cases of \textsc{OptimisticFTRL} and will demonstrate how their guarantees follow as corollaries of this lemma in Subsection 3.3.8.

**Lemma 3** (\textsc{OptimisticFTRL}[\textit{R(·, \eta)}]) Given $\eta > 0$ and a $\beta$-strongly convex $\textit{R} : Z \to \mathbb{R}$, assume we have $\{\alpha_t \ell_t(\cdot)\}_{t=1}^T$ a sequence of weights and loss functions such that each $\ell_t(\cdot)$ is $\mu_t$-strongly convex for some $\mu_t \geq 0$. Let $m_1, \ldots, m_T : Z \to \mathbb{R}$ be the sequence of hint functions given to \textsc{OptimisticFTRL}, where each $m_t(\cdot)$ is $\hat{\mu}_t$-strongly convex function over $Z$. Then \textsc{OptimisticFTRL}[\textit{R(·, \eta)}] satisfies

$$\alpha\text{-Reg}^2(z^*) \leq \sum_{t=1}^T \alpha_t (\ell_t(z_t) - \ell_t(w_{t+1}) - m_t(z_t) + m_t(w_{t+1})) \quad \text{(term (A))}$$

$$+ \frac{1}{\eta} (\textit{R}(z^*) - \textit{R}(w_1)) \quad \text{(term (B))}$$

$$- \frac{1}{2} \sum_{t=1}^T \left( \frac{\beta}{\eta} + \sum_{s=1}^{t-1} \alpha_s \mu_s \right) \|z_t - w_t\|^2 \quad \text{(term (C))}$$
\[
- \frac{1}{2} \sum_{t=1}^{T} \left( \frac{\beta}{\eta} + \alpha_t \hat{\mu}_t + \sum_{s=1}^{t-1} \alpha_s \mu_s \right) \|z_t - w_{t+1}\|^2, \quad \text{(term (D))}
\]

where \(w_1, \ldots, w_T\) is the alternative sequence chosen according to FTRL\([R(\cdot), \eta]\)
\(w_t := \arg \min_{z \in \mathcal{Z}} \sum_{s=1}^{t-1} \alpha_s \ell_s(z) + \frac{1}{\eta} R(z)\), and \(z^* \in \mathcal{Z}\) is arbitrary.

**Proof of Lemma 3** We can re-write the regret as
\[
\alpha \text{-Reg}^z(z^*) := \sum_{t=1}^{T} \alpha_t \ell_t(z_t) - \sum_{t=1}^{T} \alpha_t \ell_t(z^*)
\]
\[
= \sum_{t=1}^{T} \alpha_t (\ell_t(z_t) - \ell_t(w_{t+1})) - \alpha_t (m_t(z_t) - m_t(w_{t+1}))
\]
\[
+ \sum_{t=1}^{T} \alpha_t (m_t(z_t) - m_t(w_{t+1})) + \sum_{t=1}^{T} \alpha_t (\ell_t(w_{t+1}) - \ell_t(z^*)) \quad \text{(25)}
\]
In the following, we will denote
\[
D_T := \frac{1}{2} \left( \sum_{t=1}^{T} \left( \frac{\beta}{\eta} + \sum_{s=1}^{t-1} \alpha_s \mu_s \right) \|z_t - w_t\|^2 + \sum_{t=1}^{T} \left( \frac{\beta}{\eta} + \sum_{s=1}^{t-1} \alpha_s \mu_s + \alpha_t \hat{\mu}_t \right) \|z_t - w_{t+1}\|^2 \right)
\]
for brevity, and define \(D_0 := 0\). Let us first deal with the second term and the third term on (25). We will use induction to show that
\[
\sum_{t=1}^{T} \alpha_t (m_t(z_t) - m_t(w_{t+1})) + \sum_{t=1}^{T} \alpha_t (\ell_t(w_{t+1}) - \ell_t(z^*))
\]
\[
\leq \frac{1}{\eta} (R(z^*) - R(w_1)) - D_T \quad \text{(26)}
\]
for any point \(z^* \in \mathcal{Z}\). For the base case \(T = 0\), we have \(\sum_{t=1}^{0} \alpha_t (m_t(z_t) - m_t(w_{t+1})) + \sum_{t=1}^{0} \alpha_t (\ell_t(w_{t+1}) - \ell_t(z^*)) = 0 \leq \frac{1}{\eta} (R(z^*) - R(w_1)) - 0\), as \(w_1 := \arg \min_{z \in \mathcal{Z}} R(z)\). So the base case trivially holds.
Let us assume that the inequality (26) holds for \( t = 0, 1, \ldots, T - 1 \). Now consider round \( T \). We have

\[
\sum_{t=1}^{T} \alpha_t (m_t(z_t) - m_t(w_{t+1}) + \ell_t(w_{t+1}))
\]

\((a)\) we use the induction such that the inequality \((26)\) holds for any \( z^* \in \mathcal{Z} \) including \( z^* = w_T \), and \((b)\) is because

\[
\sum_{t=1}^{T-1} \alpha_t \ell_t(w_T) + \frac{1}{\eta} R(w_T) \leq \sum_{t=1}^{T-1} \alpha_t \ell_t(z_T) + \frac{1}{\eta} R(z_T) - \frac{1}{2} \left( \frac{\beta}{\eta} + \sum_{s=1}^{T-1} \alpha_s \mu_s \right) \| z_T - w_T \|^2,
\]

as \( w_T \) is the minimizer of a \( \frac{\beta}{\eta} + \sum_{t=1}^{T-1} \alpha_t \mu_t \) strongly convex function since \( w_T := \arg\min_{z \in Z} \sum_{s=1}^{T} \alpha_s \ell_s(z) + \frac{1}{\eta} R(z) \), and \((c)\) is because

\[
\sum_{t=1}^{T-1} \alpha_t \ell_t(z_T) + \alpha_T m_T(w_{T+1}) + \frac{1}{\eta} R(z_T) \leq \sum_{t=1}^{T-1} \alpha_t \ell_t(w_{T+1}) + \alpha_T m_T(w_{T+1}) + \frac{1}{\eta} R(w_{T+1}) - \frac{1}{2} \left( \frac{\beta}{\eta} + \sum_{s=1}^{T-1} \alpha_s \mu_s + \alpha_T \mu T \right) \| z_T - w_{T+1} \|^2,
\]

\((d)\) is because
as \( z_T \) is the minimizer of a \( \frac{\beta}{\eta} + \sum_{t=1}^{T-1} \alpha_t \mu_t + \alpha_T \bar{\mu} \) strongly convex function since 
\[ z_T := \arg\min_{z \in Z} \left( \sum_{s=1}^{T-1} \alpha_s \ell_s(z) \right) + \alpha_T m_T(z) + \frac{1}{\eta} R(z), \]
and (d) is due to \( w_{T+1} := \arg\min_{z \in Z} \sum_{t=1}^{T} \alpha_t \ell_t(z) + \frac{1}{\eta} R(z) \).

\[ \square \]

### 3.3.3 Follow The Leader – FTL

FTL, defined in (15), is perhaps the simplest strategy in online learning, it plays the best fixed action for the cumulative (weighted) loss seen in each previous round. It has been shown in various places (e.g. [57, 60]) that FTL exhibits logarithmic regret, at least when the loss functions are strongly convex. Our proof will ultimately be a consequence of a larger meta-result which we lay out in Section 3.3.2.

**Lemma 4 (FTL[\( z_{\text{init}} \)])** Let \( \alpha_1, \ldots, \alpha_T > 0 \) be arbitrary, and assume we have a sequence of \( \mu \)-strongly convex loss functions \( \ell_1(\cdot), \ldots, \ell_T(\cdot) \) for some \( \mu \geq 0 \). Then for any initial point \( z_{\text{init}} \in Z \) the OCO procedure FTL[\( z_{\text{init}} \)] satisfies

\[ \alpha\text{-Reg}^z(z^*) \leq \sum_{t=1}^{T} \frac{2\alpha_t^2}{\sum_{s=1}^{t} \alpha_s \mu} \| \delta_t \|_2^2, \]

where \( \delta_t \in \partial \ell_t(z_t) \) and \( z^* \in Z \) is arbitrary.

By Lemma 4, when we set the weights uniformly, i.e. \( \alpha_t = 1 \) \( \forall t \), and assume a bound on the gradient norms, i.e. \( \| \delta_t \|_2^2 \leq G \), the uniform regret is

\[ \text{Reg} := \sum_{t=1}^{T} \ell_t(z_t) - \sum_{t=1}^{T} \ell_t(z^*) \leq \frac{G \log(T + 1)}{2\mu}, \]

for any comparator \( z^* \in Z \).

It has been observed [61] that following the leader does not always lead to vanishing regret, for example when the loss function is linear, \( \ell_t(\cdot) := \langle \theta_t, \cdot \rangle \), the regret can grow linearly with \( T \) (see e.g. Example 2.2 [60]). On the other hand, if the constraint set \( Z \) is “suitably round”, then logarithmic regret is achievable even for linear losses.

**Lemma 5 (Theorem 3.3 in [62])** Let \{\( \ell_t(\cdot) := \langle \theta_t, \cdot \rangle \)\}_{t=1}^{T}\) be any sequence of linear loss functions. Denote \( G := \max_{t \leq T} \| \theta_t \| \) and assume that the support function \( \Phi(\cdot) := \max_{z \in Z} \langle z, \cdot \rangle \) has a unique maximizer for each cumulative loss vector \( L_t := \sum_{s=1}^{t} \theta_s \) at round \( t \). Define \( \nu_T := \min_{1 \leq t \leq T} \| L_t \| \). Let \( Z \subset \mathbb{R}^d \) be a \( \lambda \)-strongly convex set. Choose \( z_{\text{init}} \in \text{boundary}(Z) \). Then FTL[\( z_{\text{init}} \)] ensures,

\[ \text{Reg} := \sum_{t=1}^{T} \ell_t(z_t) - \sum_{t=1}^{T} \ell_t(z^*) \leq \frac{2G^2}{\lambda \nu_T} (1 + \log(T)), \]

for any comparator \( z^* \in Z \).
3.3.4 Be The Leader – FTL$^+$

FTL$^+$ (a.k.a. Be The Leader), defined in (16), is similar to FTL, except that the learner plays the best fixed action for the weighted cumulative loss seen thus far including the current round. Of course, in practical scenarios, we are unable to see the loss on the current round before we select our action, but it is nevertheless useful for analytical purposes to consider such “one-step lookahead” algorithms; in each such case we superscript the algorithm’s name with a $+$ symbol. In the FGNRD framework, we can design dynamics in where the player that acts second in the protocol can indeed observe the loss function prior to choosing its action, and thus it is quite natural to consider such one-step lookahead OCO algorithms.

Lemma 6 (FTL$^+$) Let $\alpha_1, \ldots, \alpha_T > 0$ be arbitrary, and assume we have a sequence of $\mu$-strongly convex loss functions $\ell_1(\cdot), \ldots, \ell_T(\cdot)$ for some $\mu \geq 0$. Then the regret of FTL$^+$ satisfies

$$\alpha\text{-Reg}^z(z^*) \leq -\sum_{t=1}^{T} \frac{\mu A_t - 1}{2} \|z_{t-1} - z_t\|^2 \leq 0,$$

for any comparator $z^* \in Z$.

3.3.5 Follow The Regularized Leader – FTRL

FTRL, also called dual averaging in optimization literature [63] is a classic algorithm in online learning (see e.g. [57, 59]). Looking at Equation (17) in Algorithm 4 one can notice that FTRL is similar to FTL with an additional regularization term $R(\cdot)$ scaled by parameter $1/\eta$. The regularization term induces stability into the decisions of the player, ensuring that consecutive decisions are close to each other, and this property is often crucial in order to ensure regret guarantees. For example, in the case of linear loss functions, FTRL, with appropriate choices of $\eta$ and $R(\cdot)$, can guarantee sublinear regret guarantees, while FTL cannot. In what follows we assume that $R(\cdot)$ is a $\beta$-strongly-convex function over $Z$.

Lemma 7 (FTRL$[R(\cdot), \eta]$) Given $\eta > 0$ and a $\beta$-strongly convex $R : Z \rightarrow \mathbb{R}$, assume we have $\{\alpha_t \ell_t(\cdot)\}_{t=1}^{T}$ be a sequence of loss functions such that each $\ell_t(\cdot)$ is $\mu$-strongly convex for some $\mu \geq 0$. Then FTRL$[R(\cdot), \eta]$, which generates the sequence $z_1, \ldots, z_T \in Z$, satisfies the following regret bound for any comparator $z^* \in Z$,

$$\alpha\text{-Reg}^z(z^*) \leq \sum_{t=1}^{T} \frac{2\alpha_t^2}{(\sum_{s=1}^{t} \alpha_s \mu) + \beta} \|\delta_t\|^2 + \frac{1}{\eta} (R(z^*) - R(z_1)),$$

where $\delta_t \in \partial \ell_t(z_t)$.

$^1$The term “Be The Leader” was coined by [61], who also showed it exhibits non-positive regret for linear loss functions.
3.3.6 Be The Regularized Leader – FTRL$^+$

FTRL$^+$ (((18))) is very similar to FTRL, with the difference that the former has an access to all past loss functions up to and including the current round. Recall that in our FGNRD template, one of the players is allowed to viewed the current loss prior to playing, and can therefore apply FTRL$^+$.

**Lemma 8** (FTRL$^+[R(\cdot),1/\eta]$) Given $\eta > 0$ and a $\beta$-strongly convex $R : \mathbb{Z} \to \mathbb{R}$, assume we have $\{\alpha_t \ell_t(\cdot)\}_{t=1}^T$ be a sequence of loss functions such that each $\ell_t(\cdot)$ is $\mu$-strongly convex for some $\mu \geq 0$. Then FTRL$^+[R(\cdot),\eta]$ satisfies the following regret bound for any comparator $z^* \in \mathbb{Z}$,

$$\alpha^\text{Reg}(z^*) \leq \frac{R(z^*) - R(z_0)}{\eta} - \sum_{t=1}^T \left( \frac{\mu A_t - 1}{2} + \frac{\beta}{2} \right) \|z_{t-1} - z_t\|^2,$$

where $z_0 = \min_{z \in \mathbb{Z}} R(z)$.

3.3.7 Optimistic Follow the Leader – OPTIMISTICFTL

In the previous subsection, we have seen that FTL$^+$ uses the knowledge of the loss function at rounds $t$ in order to ensure negative regret. While this knowledge is oftentimes unavailable, one can often access a “hint” or “guess” function $m_t(\cdot)$ that approximates $\ell_t(\cdot)$ prior to choosing an action $z_t$. As can be seen from Equation (19) in Algorithm 4 and Lemma 9, OPTIMISTICFTL makes use of the availability of such hints in order to provide better guarantees. The next statement shows that when we have “good” hints, in the sense that $m_t(\cdot) \approx \ell_t(\cdot)$, then OPTIMISTICFTL obtains improved guarantees compared to standard FTL.

**Lemma 9** (OPTIMISTICFTL) Let $\{\alpha_t \ell_t(\cdot)\}_{t=1}^T$ be a sequence of loss functions such that each $\ell_t(\cdot)$ is $\mu$-strongly convex, let $m_1(\cdot), \ldots, m_T(\cdot) : \mathbb{Z} \to \mathbb{R}$ be an arbitrary sequence of convex functions. Assume that we select our sequence of actions $z_1, \ldots, z_T$ according to OCO algorithm OPTIMISTICFTL, then for any $z^* \in \mathbb{Z}$, we have the following regret bound

$$\alpha^\text{Reg}(z^*) \leq \sum_{t=1}^T \alpha_t \left( \ell_t(z_t) - \ell_t(w_{t+1}) \right) - \alpha_t \left( m_t(z_t) - m_t(w_{t+1}) \right),$$

where we assume the sequence $w_2, \ldots, w_{T}$ satisfies $w_t \leftarrow \arg\min_{z \in \mathbb{Z}} \sum_{s=1}^{t-1} \ell_s(z)$.

3.3.8 Proof of FTL, FTL$^+$, FTRL, FTRL$^+$, OPTIMISTICFTL

Having described all the online learning algorithms in Algorithm 4, we are ready to show how their guarantees can be naturally obtained from that of OPTIMISTICFTL. The way that we derive them, i.e., showing the regret bound of OPTIMISTICFTL and then obtaining the rest of them as special cases, to our knowledge, is new, while we acknowledge that various versions of proofs of them can be found in the literature, e.g., [57, 59, 64].
Proof of Lemma 9 on OptimisticFTL. Observe that OptimisticFTL is actually OptimisticFTRL when \( R(\cdot) \) is a zero function. Therefore, let \( R(\cdot) = 0 \) in Lemma 3 and drop term (C) and (D) in (24) as they are non-positive, we obtain the result.

\( \square \)

Proof of Lemma 8 on FTRL\(^{+} \). Observe that the FTRL\(^{+} \) update is exactly equivalent to OptimisticFTRL with \( m_{t}(\cdot) = \ell_{t}(\cdot) \). Furthermore, \( w_{t+1} \) in Lemma 3 is actually \( z_{t} \) of FTRL\(^{+} \) shown on (18). So term (A) and term (D) on (24) in Lemma 3 is 0, Therefore, FTRL\(^{+} \) regret satisfies

\[
\alpha\text{-Reg}^{z}(z^{\ast}) \leq \frac{R(z^{\ast}) - R(z_{0})}{\eta} - \sum_{t=1}^{T} \left( \frac{\mu A_{t-1}}{2} + \frac{\beta}{2\eta} \right) \|z_{t-1} - z_{t}\|^2.
\]  

(34)

\( \square \)

Proof of Lemma 6 on FTL\(^{+} \). Observe that FTL\(^{+} \) is actually FTRL\(^{+} \) with \( R(\cdot) = 0 \). Therefore, let \( R(\cdot) = 0 \) and \( \beta = 0 \) in Equation (34), we obtain the regret of FTL\(^{+} \)

\[
\alpha\text{-Reg}^{z}(z^{\ast}) \leq - \sum_{t=1}^{T} \frac{\mu A_{t-1}}{2} \|z_{t-1} - z_{t}\|^2 \leq 0.
\]

(34)

\( \square \)

For the regret guarantee of FTRL, we will need the following supporting lemma.

**Lemma 10** (Lemma 5 in [65]) Let \( \psi_{1}(\cdot), \psi_{2}(\cdot) : \mathcal{Z} \to \mathbb{R} \) be two convex functions defined over a closed and convex domain. Denote \( u_{1} := \text{argmin}_{z \in \mathcal{Z}} \psi_{1}(z) \) and \( u_{2} := \text{argmin}_{z \in \mathcal{Z}} \psi_{2}(z) \). Assume that \( \psi_{2} \) is \( \sigma \)-strongly convex with respect to a norm \( \| \cdot \| \). Define \( \phi(\cdot) := \psi_{2}(\cdot) - \psi_{1}(\cdot) \). Then,

\[
\|u_{1} - u_{2}\| \leq \frac{2}{\sigma} \|\delta\|_{\ast}, \text{ where } \delta \in \partial \phi(u_{1}).
\]  

(35)

Furthermore, if \( \phi(\cdot) \) is convex, then,

\[
0 \leq \phi(u_{1}) - \phi(u_{2}) \leq \frac{2}{\sigma} \|\delta\|_{\ast}^{2}, \text{ where } \delta \in \partial \phi(u_{1}).
\]  

(36)

Proof of Lemma 7 on FTRL. Observe that FTRL is actually OptimisticFTRL where \( m_{t}(\cdot) = 0 \ \forall t \). Therefore, let \( m_{t}(\cdot) = 0 \) in Lemma 3, we obtain the regret of FTRL,

\[
\alpha\text{-Reg}^{z}(z^{\ast}) \leq \sum_{t=1}^{T} \alpha_{t} (\ell_{t}(z_{t}) - \ell_{t}(z_{t+1})) + \frac{1}{\eta} \left( R(z^{\ast}) - R(z_{1}) \right),
\]

(37)

where we have dropped term (C) and term (D) on (24) since they are non-positive, and we also note that \( u_{t} \) in Lemma 3 is the same as \( z_{t} \) here.

To continue, we use Lemma 10. Specifically, in Lemma 10, we let \( \psi_{1}(\cdot) \leftarrow \sum_{s=1}^{t-1} \alpha_{s} \ell_{s}(\cdot) + \frac{1}{\eta} R(\cdot) \) and \( \psi_{2}(\cdot) \leftarrow \sum_{s=1}^{t} \alpha_{s} \ell_{s}(\cdot) + \frac{1}{\eta} R(\cdot) \). Then, we have \( \phi(\cdot) = \sum_{s=1}^{t} \alpha_{s} \ell_{s}(\cdot) + \frac{1}{\eta} R(\cdot) \).
\(\alpha_t \ell_t(\cdot), u_1 = z_t, u_2 = z_{t+1}\) and that \(\sigma = \sum_{s=1}^{t} \alpha_s \mu + \beta\). So by Lemma 10 below, we have

\[
\alpha_t (\ell_t(z_t) - \ell_t(z_{t+1})) \leq \frac{2\alpha_t^2}{\left(\sum_{s=1}^{t} \alpha_s \mu\right) + \beta} \|\delta_t\|_2^2,
\]  

(38)

where \(\delta_t \in \partial \ell_t(z_t)\). Combining (37) and (38) leads to the result. \(\square\)

**Proof of Lemma 4 on FTL** Observe that FTL is actually FTRL with \(R(\cdot) = 0\). Therefore, let \(R(\cdot) = 0\) and \(\beta = 0\) in Lemma 10, we obtain the result. \(\square\)

### 3.3.9 Mirror Descent – OMD

OMD ((21) in Algorithm 5) is another popular algorithm in online learning. Compared to those in Algorithm 4, OMD updates the current iterate using only the latest loss function instead of the cumulative loss functions seen so far. Yet, it still has a nice regret guarantee.

**Lemma 11** (OMD \([\phi(\cdot), z_0, \gamma]\)) Assume that \(\phi(\cdot)\) is \(\beta\)-strongly convex w.r.t \(\|\cdot\|\). For any sequence of proper lower semi-continuous convex loss functions \(\{\alpha_t \ell_t(\cdot)\}_{t=1}^{T}\), OMD \([\phi(\cdot), z_0, \gamma]\) satisfies the following regret bound for any comparator \(z^* \in \mathcal{Z}\),

\[
\alpha\text{-Reg}^z(z^*) \leq \frac{1}{\gamma} D^\phi_{z_0}(z^*) + \frac{\gamma}{2\beta} \sum_{t=1}^{T} \|\alpha_t \delta_t\|_2^2,
\]  

(39)

where \(\delta_t \in \partial \ell_t(z_t)\).

The result in Lemma 11 is quite well-known, see e.g. [57, 59], nevertheless, we replicate the proof in Appendix B for completeness.

### 3.3.10 Prescient Mirror Descent – OMD+

OMD+ ((22) in Algorithm 5) assumes that the online player is prescient, i.e., it knows the loss functions \(\ell_t\) prior to choosing \(z_t\). Recall that in FGNRD (protocol 2) one of the players is allowed to view the current loss prior to playing, and can therefore apply OMD+.

**Lemma 12** (OMD+ \([\phi(\cdot), z_0, \gamma]\)) Assume that \(\phi(\cdot)\) is \(\beta\)-strongly convex. For any sequence of proper lower semi-continuous convex loss functions \(\{\alpha_t \ell_t(\cdot)\}_{t=1}^{T}\), OMD+ \([\phi(\cdot), z_0, \gamma]\) satisfies the following regret bound for any comparator \(z^* \in \mathcal{Z}\),

\[
\alpha\text{-Reg}^z(z^*) \leq \frac{1}{\gamma} D^\phi_{z_0}(z^*) - \sum_{t=1}^{T} \frac{\beta}{2\gamma} \|z_{t-1} - z_t\|_2^2
\]

We will need the following supporting lemma for proving Lemma 12 below.
**Lemma 13** (Property 1 in [66]) For any proper lower semi-continuous convex function \( \theta(z) \), let \( z^+ = \arg\min_{z \in \mathcal{Z}} \theta(z) + D^{\phi}_c(z) \). Then, it satisfies that for any \( z^* \in \mathcal{Z} \),

\[
\theta(z^+) - \theta(z^*) \leq D^{\phi}_c(z^*) - D^{\phi}_{z^+}(z^*) - D^{\phi}_c(z^+).
\]

**Proof of Lemma 12** The key inequality we need is Lemma 13; using the lemma with \( \theta(z) = \gamma \alpha_t \ell_t(z), z^+ = z_t \) and \( c = z_{t-1} \) we have

\[
\gamma \alpha_t \ell_t(z_t) - \gamma \alpha_t \ell_t(z^*) = \theta(z_t) - \theta(z^*) \leq D^{\phi}_c(z^*) - D^{\phi}_{z^+}(z^*) - D^{\phi}_c(z^+). \quad (41)
\]

Therefore, the regret with respect to any comparator \( z^* \in \mathcal{Z} \) can be bounded as

\[
\alpha\text{-REG}^z(z^*) := \sum_{t=1}^{T} \alpha_t \ell_t(z_t) - \sum_{t=1}^{T} \alpha_t \ell_t(z^*) \leq \frac{1}{\gamma} \sum_{t=1}^{T} D^{\phi}_{z^+}(z^*) - D^{\phi}_{z_t}(z^*) - D^{\phi}_c(z^+),
\]

where the last inequality uses the strong convexity of \( \phi \), which grants \( D^{\phi}_{z^+}(z^*) \geq \frac{\beta}{2} \|z_t - z_{t-1}\|^2 \).

**3.3.11 Optimistic Mirror Descent – OPTIMISTICMD**

When a hint of the loss function at the beginning of each round is available, the online learner can exploit the hint in the hope to get a better regret. Denote \( m_t \) a vector that represents the hint before the learner outputs a point at \( t \). **OPTIMISTICMD** in Algorithm 5 has the following regret guarantee.

**Lemma 14** (OPTIMISTICMD \( [\phi(\cdot), z_0, \gamma] \)) Assume that \( \phi(\cdot) \) is \( \beta \)-strongly convex w.r.t \( \| \cdot \| \). For any sequence of proper lower semi-continuous convex loss functions \( \{\alpha_t \ell_t(\cdot)\}_{t=1}^{T} \), **OPTIMISTICMD** \( [\phi(\cdot), z_0, \gamma] \) satisfies the following regret bound for any comparator \( z^* \in \mathcal{Z} \),

\[
\alpha\text{-REG}^z(z^*) \leq \frac{1}{\gamma} D^{\phi}_{z^+}(z^*) + \frac{\gamma}{2\beta} \sum_{t=1}^{T} \alpha_t^2 \|\delta_t - m_t\|^2,
\]

where \( \delta_t \in \partial \ell_t(z_t) \).

**OPTIMISTICMD** were proposed by [26, 67], to our knowledge. We replicate the proof of Lemma 14 in Appendix B for completeness.
3.3.12 Discussion about the decision space

Before closing this section, we would like to discuss whether the decision space $Z$ of the aforementioned online learning algorithms can indeed be unconstrained without incurring a potentially vacuous regret. For those algorithms that are prescient (BESTRESP$^+$, FTL$^+$, FTRL$^+$, OMD$^+$), their decision spaces can be unconstrained, i.e. $Z = \mathbb{R}^d$, while enjoying either a non-positive regret or a constant regret that is independent of the horizon $T$. On the other hand, for those that are unable to see their loss functions before outputting an action, e.g., FTL, FTRL, or OMD, their decision spaces might need to be a compact convex set to avoid a vacuous regret bound. Take Lemma 11 of OMD as an example, its regret bound consists of a sum of square gradient norms, and hence one might require the decision space $Z$ to be a compact convex set instead of the unconstrained one $\mathbb{R}^d$ to further bound the gradient norms.

4 Recovery of existing algorithms

What we are now able to establish, using the tools developed above, is that several iterative first order methods to minimize a convex function can be cast as simple instantiations of the Fenchel game no-regret dynamics. But more importantly, using this framework and the various regret bounds stated above, we able to establish a convergence rate for each via a unified analysis.

For every one of the optimization methods we explore below we provide the following:

1. We state the update method described in its standard iterative form, alongside an equivalent formulation given as a no-regret dynamic. To provide the FGNRD form, we must specify the payoff function $g(\cdot, \cdot)$—typically the Fenchel game, with some variants—as well as the sequence of weights $\alpha_t$, and the no-regret algorithms $OAlg^Y, OAlg^X$ for the two players.

2. We provide a proof of this equivalence, showing that the FGNRD formulation does indeed produce the same sequence of iterates as the iterative form; this is often deferred to the appendix.

3. Leaning on Theorem 2, we prove a convergence rate for the method.

4.1 Frank-Wolfe method and its variants

The Frank-Wolfe method (FW) [6], also known as conditional gradient, is known for solving constrained optimization problems. FW is entirely first-order, while requiring access to a linear optimization oracle. Specifically, given a compact and convex constraint set $K \subset \mathbb{R}^d$, FW relies on the ability to (quickly) answer queries of the form $\arg\min_{x \in K} \langle x, v \rangle$, for any vector $v \in \mathbb{R}^d$. In many cases this linear optimization problem is much faster for well-behaved constraint sets; e.g. simple convex polytopes, the PSD cone, and various balls defined by vector and matrix norms [68, 69]. When the constraint set is the
nuclear norm ball, which arises in matrix completion problems, then the linear optimization oracle corresponds to computing a top singular vector, which requires time roughly linear in the size of the matrix [69].

Algorithm 6 Frank-Wolfe [6]

| Given: $L$-smooth $f(\cdot)$, convex domain $K \subset \mathbb{R}^d$, arbitrary $w_0$, iterations $T$. |
|---|
| $\gamma_t \leftarrow \frac{2}{t+1}$ |
| $v_t \leftarrow \arg\min_{v \in K} \langle v, \nabla f(w_{t-1}) \rangle$ |
| $w_t \leftarrow (1 - \gamma_t)w_{t-1} + \gamma_tv_t$ |
| $g(x, y) := \langle x, y \rangle - f^*(y)$ |
| $\alpha_t \leftarrow t$ |
| $\text{OAlg}^Y := \text{FTL}[\nabla f(w_0)]$ |
| $\text{OAlg}^X := \text{BESTRESP}^+$ |

Output: $w_T = \bar{x}_T$

We describe the Frank-Wolfe method precisely in Algorithm 6, in both its iterative form and its FGNRD interpretation. We begin by showing that these two representations are equivalent.

**Proposition 3** The two interpretations of Frank-Wolfe, as described in Algorithm 6, are equivalent. That is, for every $t$, the iterate $w_t$ computed iteratively on the left hand side is identically the weighted-average point $\bar{x}_t$ produced by the dynamic on the right hand side.

**Proof** We show, via induction, that the following three equalities are maintained for every $t$. Note that three objects on the left correspond to the iterative description given in Algorithm 6 whereas the three on the right correspond to the FGNRD description.

\[
\begin{align*}
\nabla f(w_{t-1}) &= y_t \quad (43) \\
v_t &= x_t \quad (44) \\
w_t &= \bar{x}_t. \quad (45)
\end{align*}
\]

To start, we observe that since the $\text{OAlg}^Y$ is set as $\text{FTL}[\nabla f(w_0)]$, we have the base case for $(43)$, $y_1 = \nabla f(w_0)$, holds by definition. Furthermore, we observe that for any $t$ we have $(43) \implies (44)$. This is because, if $y_t = \nabla f(w_{t-1})$, the definition of $\text{BESTRESP}^+$ implies that

\[
\alpha_t \leftarrow t
\]

Next, we can show that $(44) \implies (45)$ for any $t$ as well using induction. Assuming that $w_{t-1} = \frac{\sum_{s=1}^{t-1} \alpha_s x_s}{\sum_{s=1}^{t-1} \alpha_s} = \frac{\sum_{s=1}^{t-1} sv_s}{\sum_{s=1}^{t-1} s}$, a bit of algebra verifies

\[
\begin{align*}
w_t &:= (1 - \gamma_t)w_{t-1} + \gamma_tv_t = \left(\frac{t-1}{t+1}\right) \frac{\sum_{s=1}^{t-1} sv_s}{\sum_{s=1}^{t-1} s} + \left(\frac{2}{t+1}\right) v_t \\
&= \frac{\sum_{s=1}^{t} sv_s}{\sum_{s=1}^{t} s} = \frac{\sum_{s=1}^{t} \alpha_s x_s}{\sum_{s=1}^{t} \alpha_s} =: \bar{x}_t
\end{align*}
\]

Finally, we show that (43) holds for $t > 1$ via induction. Recall that $y_t$ is selected via FTL against the sequence of loss functions $\alpha_t \ell_t(\cdot) := -\alpha_t g(x_t, \cdot)$ Precisely this means that, for $t > 1$,

$$y_t := \arg\min_{y \in \mathcal{Y}} \left\{ \frac{1}{A_{t-1}} \sum_{s=1}^{t-1} \alpha_s \ell_s(y) \right\}$$

$$= \arg\min_{y \in \mathcal{Y}} \left\{ \frac{1}{A_{t-1}} \sum_{s=1}^{t-1} (\alpha_s \langle -x_s, y \rangle + f^*(y)) \right\}$$

$$= \arg\max_{y \in \mathcal{Y}} \{ \langle \bar{x}_{t-1}, y \rangle - f^*(y) \} = \nabla f(\bar{x}_{t-1}).$$

The final line follows as a result of the Legendre transform [15]. Finally, by induction, we have $\bar{x}_{t-1} = w_{t-1}$, and hence we have established (43). This completes the proof. □

Now that we have established Frank-Wolfe as an instance of Protocol 2, we can now prove a bound on convergence using the tools established in Section 3.

**Theorem 4** Let $w_T$ be the output of Algorithm 6. Let $f$ be $L$-smooth and let $\mathcal{K}$ have square $\ell_2$ diameter no more than $D$. Then we have

$$f(w_T) - \min_{w \in \mathcal{K}} f(w) \leq \frac{8LD}{T+1}.$$

**Proof** By Proposition 3 and Theorem 2, we obtain

$$f(w_T) - \min_{w \in \mathcal{K}} f(w) \leq \underbar{\alpha-Reg}^x[\text{BestResp}^+] + \underbar{\alpha-Reg}^y[\text{FTL}].$$

Recall that we have $\underbar{\alpha-Reg}^x[\text{BestResp}^+] \leq 0$ by Lemma 2. Let us then turn our attention to the regret of $\text{OAlg}^Y$.

First note that, since $f(\cdot)$ is $L$-smooth, its conjugate $f^*(\cdot)$ is $\frac{1}{L}$-strongly convex, and thus the function $-g(x, \cdot)$ is also $\frac{1}{L}$-strongly convex in its second argument. Next, if we define $\ell_t(\cdot) := -g(x_t, \cdot)$, then we can bound the norm of the gradient as

$$\|\nabla \ell_t(y_t)\|^2 = \|x_t - \nabla f^*(y_t)\|^2 = \|x_t - \bar{x}_{t-1}\|^2 \leq D.$$

Combining with Lemma 4 we see that

$$\underbar{\alpha-Reg}^y[\text{FTL}] \leq \frac{1}{AT} \sum_{t=1}^{T} 2\alpha_t^2 \|\nabla \ell_t(y_t)\|^2 = \frac{8L}{T(T+1)} \sum_{t=1}^{T} \frac{t^2D}{t(t+1)} \leq \frac{8LD}{T+1}.$$

This completes the proof. □

**4.1.1 Variant 1: a linear rate Frank-Wolfe over strongly convex set**

[17, 70, 71] show that Frank-Wolfe for smooth convex function (not necessarily a strongly convex function) for strongly convex sets has linear rate under certain conditions. Algorithm 7 shows that a similar result can be derived
from the game framework, in which the $y$-player uses a variant of FTL called AFTL (defined in Algorithm 22 of Appendix C).

**Algorithm 7 Adaptive Frank-Wolfe**

| Given: $L$-smooth convex $f(\cdot)$, arbitrary $\bar{x}_0 = x_0 = w_0 \in \mathcal{K} \subset \mathbb{R}^d$, iterations $T$, $A_0 = 0$. |
|---|
| $x_t \leftarrow \arg\min_{x \in \mathcal{K}} \langle x, \nabla f(w_{t-1}) \rangle$ |
| $\alpha_t \leftarrow \frac{1}{\|x_t - x_{t-1}\|}$ |
| $A_t \leftarrow A_{t-1} + \alpha_t$ |
| $w_t \leftarrow \frac{1}{A_t} \sum_{s=1}^{t} \alpha_s x_s$ |
| $g(x, y) := \langle x, y \rangle - f^*(y)$ |
| $\text{OAlg}^Y := \text{AFTL}[\nabla f(x_0)]$ |
| $\text{OAlg}^X := \text{BestResp}^+$ |
| Iterative Description | FGNRD Equivalence |
| Output: $w_T = \bar{x}_T$ |

**Theorem 5** Suppose that $f(\cdot)$ is $L$-smooth convex, and that $\mathcal{K}$ is a $\lambda$-strongly convex set. Also assume that the gradients of the $f(\cdot)$ in $\mathcal{K}$ are bounded away from 0, i.e., $\max_{w \in \mathcal{K}} \|\nabla f(w)\| \geq B$. Then, Algorithm 7 has

$$f(w_T) - \min_{w \in \mathcal{K}} f(w) = O\left(\exp\left(-\frac{\lambda B}{L} T\right)\right).$$

Note that the weights $\alpha_t$ are not predefined but rather depend on the queries of the algorithm. The proof of Theorem 5 is described in full detail in Appendix C.

### 4.1.2 Variant 2: an incremental Frank-Wolfe

Recently, [18] and [72] propose stochastic Frank-Wolfe algorithms for optimizing smooth convex finite-sum functions, i.e. $\min_{x \in \mathcal{K}} f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x)$, where each $f_i(x) := \phi(x^\top z_i)$ represents a loss function $\phi(\cdot)$ associated with sample $z_i$. In each iteration the algorithms only require a gradient computation of a single component. [18] show that their algorithm has $O\left(\frac{c_\kappa}{T}\right)$ expected convergence rate, where $c_\kappa$ is a number that depends on the underlying data matrix $z$ and in worst case is bounded by the number of components $n$. Algorithm 8 shows a similar algorithm which picks a sample in each iteration by cycling through the data points. It is obtained when the $y$-player uses a variant of FTL called LazyFTL defined in Appendix D. We have the following theorem that shows an $\tilde{O}\left(\frac{n}{T}\right)$ deterministic convergence rate. Its proof is in Appendix D.
Algorithm 8 Incremental Frank-Wolfe

Given: $L$-smooth $f(\cdot)$, convex domain $K$, arbitrary $w_0 \in K \subset \mathbb{R}^d$, iterations $T$.
Init: For each sample $i$, compute $g_{i,0} := \frac{1}{n} \nabla f_i(w_0) \in \mathbb{R}^d$ so that $\nabla f(w_0) = \sum_{i=1}^n g_{i,0}$.

Select $i_t$ via by cycling through $[n]$ 
$g_{i_t,t} \leftarrow \frac{1}{n} \nabla f_{i_t}(w_t)$ and $g_{j,t} \leftarrow g_{j,t-1}$, for $j \neq i_t$
$g_t \leftarrow \sum_{i=1}^n g_{i,t}$
$v_t \leftarrow \arg\min_{x \in K} \langle x, g_t \rangle$
$w_t \leftarrow (1 - \gamma_t) w_{t-1} + \gamma_t v_t$

$g(x,y) := \langle x, y \rangle - f^*(y)$
$\alpha_t \leftarrow 1$
$\text{OAlg}^Y := \text{LazyFTL}[\nabla f(w_0)]$
$\text{OAlg}^X := \text{BestResp}^+$

Output: $w_T = \bar{x}_T$

Iterative Description FGNRD Equivalence

Theorem 6 Assume that $f(\cdot)$ is $L$-smooth convex and that its conjugate is $L_0$-Lipschitz. Algorithm 8 has

$$f(w_T) - \min_{w \in K} f(w) = O\left( \frac{\max\{LR, L(L_0 + r)n\} \log T}{T} \right),$$

where $r$ is a bound of the length of any point $x$ in the constraint set $K$, i.e. $\max_{x \in K} ||x|| \leq r$, and $R$ is the square of the diameter of $K$.

4.1.3 More related works

There has been growing interest in Frank-Wolfe in recent years, e.g. [73, 74]. For the constraint set that belongs to a certain class of convex polytopes, there are FW-like algorithms achieving linear convergence rate, see e.g. [68, 75–79]. There are also works that study Frank-Wolfe on various aspects, e.g. minimizing some structural norms [80], reducing the number of gradient evaluations [81]. Bach [82] shows that for certain types of objectives, subgradient descent applied to the primal domain is equivalent to FW applied to the dual domain.

4.2 Vanilla gradient descent, with averaging

In Algorithm 9 we describe a variant of gradient descent. It outputs a weighted average of iterates of the gradient descent steps. The algorithm is obtained by swapping the ordering of the players. The $x$-player plays first according to OMD and then the $y$-player plays BestResp$^+$. Theorem 7 below shows that the algorithm has a guarantee even for non-smooth convex functions. It is noted that Algorithm 9 with the convergence rate guarantee shown in Theorem 7 can also be obtained from the classical online-to-batch conversion (e.g., [83] or Theorem 9.5 in [59]). Nevertheless, we provide an analysis based on our framework.

Theorem 7 Denote the constant $G = \max_{y \in \partial f(w), w \in K} ||y||^2$, and let $R$ be an upper bound on $||w_0 - w^*||$, where $w^* := \arg\min_{w \in K} f(w)$. The output $\tilde{w}_T = \bar{x}_T$ of
Algorithm 9 Vanilla gradient descent, with averaging

Given: Convex \( f(\cdot) \) and iterations \( T \).

Init: \( w_0 = x_0 \in \mathcal{K} \subseteq \mathbb{R}^d \). Set: \( \eta = \gamma = \begin{cases} \frac{R}{G\sqrt{T}}, & \text{if } f(\cdot) \text{ is non-smooth}, \\ \frac{1}{2L}, & \text{if } f(\cdot) \text{ is } L\text{-smooth}. \end{cases} \)

Iterative Description

\[
\begin{align*}
  w_t &\leftarrow w_{t-1} - \eta \delta_{t-1}, \quad \text{where } \delta_{t-1} \in \partial f(w_{t-1}) \\
  \bar{w}_t &\leftarrow \frac{1}{t} \sum_{s=1}^{t} w_s
\end{align*}
\]

Output: \( \bar{w}_T = \bar{x}_T \)

Algorithm 9 satisfies

\[
f(\bar{w}_T) - \min_{w \in \mathcal{K}} f(w) = O\left( \frac{GR}{\sqrt{T}} \right).
\]

Proof The equivalence \( w_t = x_t \) between the two displays can be easily shown by noting that the update of OMD is \( x_t = x_{t-1} - \alpha_t \gamma y_{t-1} \), where \( \gamma = \frac{R}{G\sqrt{T}} \) and \( \alpha_t = 1 \), and the update of BestResp is \( y_t = \arg\min_{y \in Y} f^*(y) - \langle x_t, y \rangle \in \partial f(x_t) \).

By Lemma 11, the regret of the \( x \)-player is \( \alpha\text{-Reg}^x[\text{OMD}] \leq \frac{1}{\gamma A_t} \left( \frac{1}{\gamma} D_{x_0}^\phi(x^*) + \frac{T}{2} \sum_{t=1}^{T} \|\alpha_t y_t\|^2 \right) \). On the other hand, by Lemma 2, the regret of the \( y \)-player is 0. By setting \( \alpha_t = 1 \), adding average regrets of both players, and plugging in \( \gamma = \frac{R}{G\sqrt{T}} \),

\[
\alpha\text{-Reg}^x[\text{OMD}] + \alpha\text{-Reg}^y[\text{BestResp}^+] \leq \frac{1}{T} \left( \frac{R^2}{\gamma} + \frac{T G^2}{2} \right) = O\left( \frac{GR}{\sqrt{T}} \right). \tag{47}
\]

We should note that BestResp\(^+\) is not necessarily a well-defined algorithm, when the required argmin doesn’t have a unique solution. This is relevant in the case when \( f(\cdot) \) is not a differentiable function, which is why we have set \( y_t = \delta_t \) for any \( \delta_t \in \partial f(x_t) \). On the other hand, the convergence in Theorem 7 holds for any choice of subgradients.

We can also recover the known \( O\left( \frac{1}{T} \right) \) convergence rate of GD when the function \( f(\cdot) \) is smooth. To achieve this, we will use the following known result to upper-bound the r.h.s. of (47).

Lemma 15 (see e.g., Lemma A.2 in [84]) Assume \( f(\cdot) \) is \( L \)-smooth. Denote \( x^* = \arg\min_{x \in \mathbb{R}^d} f(x) \). Then,

\[
\|\nabla f(x)\|^2 \leq 2L \left( f(x) - f(x^*) \right), \forall x \in \mathbb{R}^d.
\]
Theorem 8 Assume \( f(\cdot) \) is \( L \)-smooth. Then, the output \( \bar{w}_T = \bar{x}_T \) of Algorithm 9 satisfies
\[
f(\bar{w}_T) - \min_{w \in \mathbb{R}^d} f(w) \leq \frac{2L\|w_0 - w^*\|^2}{T}.
\]

Proof Let \( \alpha_t = 1, \forall t \) and hence \( A_t = T \). We start by upper-bounding the last term on the r.h.s. of (47). We have
\[
\begin{align*}
\alpha\text{-Reg}^x[\text{OMD}] + \alpha\text{-Reg}^y[\text{BestResp}^+] &\leq \frac{1}{A_t} \left( \frac{1}{\gamma} D^{\phi}_{x_0}(x^*) + \sum_{t=1}^{T} \frac{\gamma}{2} \|\alpha_t y_t\|^2 \right) \\
&\leq \frac{1}{T} D^{\phi}_{x_0}(x^*) + \gamma L \sum_{t=1}^{T} (f(x_t) - f(x^*)) \\
&\leq \frac{1}{T} D^{\phi}_{x_0}(x^*) + \gamma L \sum_{t=1}^{T} \langle \nabla f(x_t), x_t - x^* \rangle,
\end{align*}
\]
where (a) uses Lemma 15 and that the update of \( \text{BestResp}^+ \) is \( y_t = \arg \min_{y \in Y} f^*(y) - \langle x_t, y \rangle = \nabla f(x_t) \), and (b) uses convexity.

Rearranging the above inequality and noting that \( \alpha\text{-Reg}^y[\text{BestResp}^+] = 0 \), we obtain
\[
(1 - \gamma L)\alpha\text{-Reg}^x[\text{OMD}] = \frac{D^{\phi}_{x_0}(x^*)}{\gamma T}.
\]
Setting \( \gamma = \frac{1}{2T} \) and recognizing that \( D^{\phi}_{x_0}(x^*) = \frac{1}{2} \|x_0 - x^*\|^2 = \frac{1}{2} \|w_0 - w^*\|^2 \), we obtain the result. \( \square \)

4.3 Single-gradient-call extra-gradient, with averaging
Extra-gradient method [85–87] and its single-gradient-call variants [43, 88–92] have drawn significant interest in recent years. In Algorithm 10 below, we show how a single-gradient-call variant can be obtained from the Fenchel game by pitting \text{OptimisticMD} against \text{BestResp}^+. More precisely, we obtain an \( O(1/T) \) convergence rate in the unconstrained setting. We remark that an \( O(1/T) \) convergence rate of single-call extra-gradient has been established in the literature, see e.g., [88], without restricting to the unconstrained setting \( K = \mathbb{R}^d \).

Theorem 9 The output \( \bar{w}_T = \bar{x}_T \) of Algorithm 10 satisfies
\[
f(\bar{w}_T) - \min_{w \in \mathbb{R}^d} f(w) = \frac{8LD^{\phi}_{w_0}(w^*) + \frac{1}{2T} \|\nabla f(w_0)\|^2}{T},
\]
where \( w^* := \arg \min_{w \in \mathbb{R}^d} f(w) \).
Algorithm 10 Single-gradient-call extra-gradient, with averaging

Given: $L$-smooth $f(\cdot)$, a $1$-strongly convex $\phi(\cdot)$, iterations $T$

Init: arbitrary $w_{-\frac{1}{T}} = w_0 = x_0 = x_{-\frac{1}{T}} \in \mathbb{R}^d$.

\[
\begin{align*}
\gamma & \leftarrow \frac{1}{T} \\
 w_t & \leftarrow \underset{w \in \mathbb{R}^d}{\text{argmin}} \left( \alpha_t \langle w, \nabla f(w_{t-1}) \rangle ight) \\
 & \quad + \frac{1}{T} D_{w_{t-\frac{1}{T}}}(w) \\
 w_{t+\frac{1}{T}} & \leftarrow \underset{w \in \mathbb{R}^d}{\text{argmin}} \left( \alpha_t \langle w, \nabla f(w_t) \rangle ight) \\
 & \quad + \frac{1}{T} D_{w_{t-\frac{1}{T}}}(w) \\

g(x, y) & := \langle x, y \rangle - f^*(y) \\
\alpha_t & := 1 \text{ for } t = 1, \ldots, T \\
oAlg^X & := \text{OPTIMISTICMD}[\phi(\cdot), x_0, \frac{1}{2L}] \\
oAlg^Y & := \text{BESTRESP}^+ \\
\text{Output: } w_T = x_T
\end{align*}
\]

Iterative Description FGNRD Equivalence

Proof of Theorem 9 The equivalence of the two displays, namely, for all $t$,

\[
w_t = x_t \\
w_{t-\frac{1}{T}} = x_{t-\frac{1}{T}} \\
\nabla f(w_t) = y_t
\]

can be trivially shown by induction. Specifically, the first two relations hold in the beginning by letting the initial point $w_0 = w_{-\frac{1}{T}} = x_0 = x_{-\frac{1}{T}} \in \mathbb{R}^d$. To show the last one, we apply the definition of $\text{BESTRESP}^+$,

\[
y_t \leftarrow \underset{y \in \mathcal{Y}}{\text{argmax}} \ell_t(y) = \underset{y \in \mathcal{Y}}{\text{argmax}} f^*(y) - \langle x_t, y \rangle = \nabla f(x_t).
\]

By induction, we have $w_t = x_t$; consequently, $y_t = \nabla f(w_t)$.

By summing the regret bound of each player, i.e. Lemma 14 and Lemma 6, we get

\[
\begin{align*}
\alpha\text{-REG}^T[\text{OPTIMISTICMD}] + \alpha\text{-REG}^T[\text{BESTRESP}^+] \\
\leq & \frac{1}{A_t} \left\{ \frac{1}{\gamma} D_{x_0}(x^*) + \frac{\gamma}{2} \sum_{t=1}^{T} \alpha_t^2 \|\nabla f(w_t) - \nabla f(w_{t-1})\|_2^2 \right\} \\
\leq & \frac{1}{A_t} \left\{ \frac{1}{\gamma} D_{x_0}(x^*) + \gamma \sum_{t=1}^{T} \alpha_t^2 \left( \|\nabla f(w_t)\|^2 + \|\nabla f(w_{t-1})\|^2 \right) \right\} \\
\leq & \frac{1}{A_t} \left\{ \frac{1}{\gamma} D_{x_0}(x^*) + \gamma \alpha_t^2 \|\nabla f(w_0)\|^2 + 4\gamma L \sum_{t=1}^{T} \alpha_t^2 (f(w_t) - f(w_*)) \right\} \\
\leq & \frac{1}{A_t} \left\{ \frac{1}{\gamma} D_{x_0}(x^*) + \gamma \alpha_t^2 \|\nabla f(w_0)\|^2 + 4\gamma L \sum_{t=1}^{T} \alpha_t^2 (\nabla f(w_t), w_t - w_*) \right\} \\
\leq & \frac{1}{T} \left\{ \frac{1}{\gamma} D_{x_0}(x^*) + \gamma \|\nabla f(w_0)\|^2 \right\} + 4\gamma L \frac{1}{T} \sum_{t=1}^{T} (\nabla f(w_t), w_t - w_*), \\
\overset{(d)}{=} & \alpha\text{-REG}^T[\text{OPTIMISTICMD}]
\end{align*}
\]
where in (a) we used the relation $m_t = y_{t-1} = \nabla f(w_{t-1})$ when we applied Lemma 14, (b) is by the triangle inequality, (c) is by Lemma 15, (d) uses the convexity, and (e) uses $\alpha_t = 1$.

Rearranging the above inequality and noting that $\alpha_{-\text{Reg}}[\text{BestResp}] = 0$, we obtain

$$
(1 - 4\gamma L)\frac{1}{\alpha_{-\text{Reg}}[\text{OMD}]} = \frac{1}{T}D_{x_0}^\phi(x^*) + \gamma\|\nabla f(w_0)\|_2^2.
$$

Setting $\gamma = \frac{1}{8L}$, we obtain the result.

We remark that we can get an $O(1/\sqrt{T})$ convergence rate when we replace $R_{\text{d}}$ with $K \subset \mathbb{R}^d$ in Algorithm 10. Specifically, we can simply bounding the square norm of the difference of the gradients on the r.h.s. of (48) by $G^2$, and then follow the proof of Theorem 7 to get a $O(1/\sqrt{T})$ rate by setting the parameters appropriately. Recovering the $O(1/T)$ rate for the constrained setting using the regret analysis is left open in this work.

### 4.4 Cumulative gradient descent

We describe yet another algorithm from the game for non-smooth optimization. Algorithm 11 below has the last-iteration guarantee, while subgradient descent method, i.e. $w_t = w_{t-1} - \eta f_x(w_{t-1})$, does not enjoy the last-iteration guarantee, see e.g. Chapter 2 of [93] or [94]. This highlights the benefit of the proposed algorithm.

**Algorithm 11** Cumulative gradient descent

Given: Convex $f(\cdot)$ and iterations $T$.

| Iterative Description | FGNRD Equivalence |
|-----------------------|-------------------|
| $w_t = (1 - \frac{1}{T})w_{t-1} - \frac{1}{T}\eta\sum_{s=1}^{t-1} \delta_s$, where $\delta_s \in \partial f(w_s)$ | $g(x,y) := \langle x,y \rangle - f^*(y)$ |
| $w_t = (1 - \frac{1}{T})w_{t-1} - \frac{1}{T}\eta \delta_s$ $- \frac{1}{1-T}\eta (w_{t-1} - (1 - \frac{1}{T})w_{t-2})$ | $\alpha_t := 1$ for $t = 1,\ldots,T$ |
| $\eta \leftarrow \frac{(R/G)}{\sqrt{T}}$ | OAlg$^X := \text{OMD}[\frac{1}{T} \cdot \|\cdot\|_2, x_0, \frac{R/G}{\sqrt{T}}]$ |
| Output: $w_T = \bar{x}_T$ | OAlg$^Y := \text{FTL}^+$ |

**Theorem 10** Denote the constant $G = \max_{\delta \in \partial f(w), w \in K} \|\delta\|^2$, and let $R$ be an upper bound on $\|w_0 - w^*\|$. The output $w_T = \bar{x}_T$ of Algorithm 11 satisfies

$$
f(w_T) - \min_{w \in K} f(w) = O\left(\frac{GR}{\sqrt{T}}\right),
$$

where $w^* := \arg\min_{w \in K} f(w)$.
Proof Let $\gamma = \frac{(R/G)}{\sqrt{T}}$ and $\alpha_t = 1$. The update of OMD is $x_t = x_{t-1} - \gamma y_{t-1}$, while the update of FTL is $y_t = \arg\max_{y \in Y} \sum_{s=1}^t \alpha_s (f^*_s(y) - \langle x_s, y \rangle) \in \partial f(x_t)$. Recursively expanding $x_t$ leads to:

$$\bar{x}_t = \left(1 - \frac{1}{t}\right) x_{t-1} + \frac{1}{t} x_t - \gamma y_{t-1}$$

We now see the equivalence of the two displays in Algorithm 11.

By Lemma 11, the regret of the $x$-player is, $\alpha$-Reg$_x \leq \frac{1}{\gamma} D_{x_0}(x^*) + \frac{2}{\gamma} \sum_{t=1}^T \|\alpha_t y_t\|^2$. On the other hand, by Lemma 2, the regret of the $y$-player is not greater than 0. Adding average regrets of both players, we get

$$\underline{\alpha}$-Reg$_x$[OMD] + $\underline{\alpha}$-Reg$_y$[FTL] $\leq \frac{1}{A_t} \left( \frac{1}{\gamma} D_{x_0}(x^*) + \sum_{t=1}^T \frac{2}{\gamma} \|\alpha_t y_t\|^2 \right) = O \left( \frac{GR}{\sqrt{T}} \right).$$

Let us comment that the same algorithm can be produced from Algorithm 1 in [95] with the online gradient descent strategy via a so called “anytime online-to-batch conversion”, though the algorithm is not explicitly stated in the paper. We also note that while Algorithm 11 depicts the equivalence when the problem is unconstrained, i.e., $K = \mathbb{R}^d$, we can also derive the projected version of the algorithm for its iterative description when the constraint set is a compact convex set. Specifically, the left column (iterative description) in Algorithm 11 would become

$$w_t = \left(1 - \frac{1}{t}\right) w_{t-1} + \frac{1}{t} x_t, \text{ where } x_t = \Pi_K [x_{t-1} - \gamma \delta_{t-1}], \delta_{t-1} \in \partial f(w_{t-1}).$$

### 4.5 Accelerated methods for smooth convex optimization

In this subsection, we are going to introduce several accelerated algorithms. An optimization procedure is generally referred to as accelerated if it achieves a faster convergence rate relative to a vanilla method, e.g. gradient descent. The convergence of gradient descent on smooth convex optimization problems was known to be $O(1/T)$, and it was believed by many up until the early 1980s that this was the fastest rate achievable despite the only known lower bound of $\Omega(1/T^2)$. It was shown by Nesterov [1] that indeed an upper bound of $O(1/T^2)$ is indeed possible with a slightly more complex update scheme.

We will discuss Nesterov Accelerated Gradient Descent, and its variants, in the context of our FGNRD framework. Nesterov’s first acceleration method [1, 2] (see also [96]) for the unconstrained setting is frequently described as follows. First initialize $z_0$ and $w_0$ arbitrarily, and then iterate as follows:
Algorithm 12 Unconstrained Nesterov Accelerated Gradient Descent

1: Input: $w_0, z_0$ arbitrary in $\mathbb{R}^d$.
2: for $t = 1, 2, \ldots, \text{ do}$
3: \hspace{1em} $\theta_t \leftarrow t \cdot \frac{L}{t+1}$, $\beta_t \leftarrow t-2 \cdot \frac{1}{t+1}$
4: \hspace{1em} $w_t \leftarrow z_{t-1} - \theta_t \nabla f(z_{t-1})$
5: \hspace{1em} $z_t \leftarrow w_t + \beta_t(w_t - w_{t-1})$
6: end for
7: return $w_T$

But to get the ball rolling let us first consider a related method of Polyak [7].

4.5.1 The Heavy Ball Method

In Algorithm 13 we describe the classical iterative version of the Heavy Ball algorithm in the box on the left. One observes that the update looks quite similar to vanilla gradient descent, but there’s the addition of a so-called momentum term $v_t$ which is the difference of the previous two iterates.

Algorithm 13 Heavy Ball

Given: $L$-smooth $f(\cdot)$, arbitrary $x_0 = w_0 = w_{-1} \in \mathbb{R}^d$, iterations $T$.

| $\eta_t \leftarrow \frac{t}{4(t+1)L}$, $\beta_t \leftarrow \frac{t-2}{t+1}$ | $g(x, y) := (x, y) - f^*(y)$
| $v_t \leftarrow w_{t-1} - w_{t-2}$ | $\alpha_t := t$ for $t = 1, \ldots, T$
| $w_t \leftarrow w_{t-1} - \eta_t \nabla f(w_{t-1}) + \beta_t v_t$ | $\text{OAlg}^Y := \text{FTL}[\nabla f(x_0)]$
| Iterative Description | $\text{OAlg}^X := \text{OMD}^+[\frac{1}{2}\| \cdot \|_2^2, x_0, \frac{1}{T}]$
| Output: $w_T = \bar{x}_T$ | FGNRD Equivalence

We provide a formulation of Heavy Ball in the right box of Algorithm 13, and let us begin by stating the equivalence. The proof is deferred to Appendix E.

Proposition 11 The two interpretations of Heavy Ball described in Algorithm 13 are equivalent.

Proof (sketch) Similar to the proof of Proposition 3, we need to show that the following three equalities are maintained throughout each iteration:

$$
\nabla f(w_{t-1}) = y_t
$$

$$
\bar{x}_t = \bar{x}_{t-1} - \frac{\gamma}{A_t} \nabla f(\bar{x}_{t-1}) + (\bar{x}_{t-1} - \bar{x}_{t-2}) \left( \frac{\alpha_t A_{t-1}}{A_t \alpha_{t-1}} \right),
$$

(49)

$$
w_t = \bar{x}_t
$$
The first and last lines follow along the same lines as the proof of Proposition 3. The expression (49) can be established when we see that the OMD+\([\frac{1}{2}, x_0, \frac{1}{4L}]\) update algorithm can be written as

\[ x_t = x_{t-1} - \alpha_t \gamma_t y_t = x_{t-1} - \frac{\alpha_t}{8L} y_t. \]

Plugging this expression into the definition of \(\bar{x}_t\) establishes the update (49). □

Denote \(\Pi_K[\cdot]\) the projection onto \(K\). We remark that for the case when \(K \subset \mathbb{R}^d\), the left column (iterative description) in Algorithm 13 would become

\[ w_t = (1 - \frac{2}{t+1}) w_{t-1} + \frac{2}{t+1} x_t, \text{ where } x_t = \Pi_K \left[ x_{t-1} - \frac{t}{8L} \nabla f(w_{t-1}) \right]. \]

**Theorem 12** The output \(\bar{x}_T\) of Algorithm 13 satisfies

\[ f(\bar{x}_T) - \min_{x \in K} f(x) = O \left( \frac{4L\|x_0 - x^*\|^2 + \sum_{t=1}^T 4L \left( \|\bar{x}_{t-1} - x_t\|^2 - \|x_{t-1} - x_t\|^2 \right)}{T^2} \right). \]

Furthermore, given a convex constraint set \(K \subset \mathbb{R}^d\) with diameter \(R\), we have \(f(\bar{x}_T) - \min_{w \in K} f(w) = O(\frac{LR}{T^2})\).

We note that the first bound in Theorem 12 does not require \(K\) to be a compact convex set. However, when \(K = \mathbb{R}^d\), unfortunately we cannot further upper-bound the trajectory-dependent term \(\|\bar{x}_{t-1} - x_t\|^2 - \|x_{t-1} - x_t\|^2\) by a constant. Whether one can use the regret analysis to show an \(O(1/T)\) rate of unconstrained Heavy Ball is left open.

### 4.5.2 Nesterov’s Methods

Starting from 1983, Nesterov has proposed various accelerated methods for smooth convex problems (i.e. [1–4]), all of which are often described as accelerated gradient descent. Our goal now will be to try to understand Nesterov’s various algorithms. We will show that all the methods can be generated from Fenchel game with some simple tweaks.

**Algorithm 14** Nesterov’s 1-memory method \([4, 66]\)

Given: \(L\)-smooth \(f(\cdot)\), convex domain \(K \subset \mathbb{R}^d\), arbitrary \(x_0 = v_0 \in K\), 1-strongly convex distance generating function \(\phi(\cdot)\), iterations \(T\).

\[
\begin{align*}
\beta_t &\leftarrow \frac{2}{t+1}, \gamma_t \leftarrow \frac{4}{4L} \\
z_t &\leftarrow (1 - \beta_t) w_{t-1} + \beta_t v_{t-1} \\
v_t &\leftarrow \arg\min_{x \in K} \gamma_t \langle \nabla f(z_t), x \rangle + D_{v_{t-1}}(x) \\
w_t &\leftarrow (1 - \beta_t) w_{t-1} + \beta_t v_t
\end{align*}
\]

Iterative Description

\[
\begin{align*}
g(x, y) := \langle x, y \rangle - f^*(y) \\
\alpha_t := t \text{ for } t = 1, \ldots, T \\
\text{OAlg}^Y := \text{OPTIMISTICFTL} \\
\text{OAlg}^X := \text{OMD}^+[\phi(\cdot), x_0, \frac{1}{4L}]
\end{align*}
\]

FGNRD Equivalence

Output: \(w_T = \bar{x}_T\).
Let us begin with the version known as the 1-memory method. To be precise, we adopt the presentation of Nesterov’s algorithm given in Algorithm 1 of [66].

**Proposition 13** The two interpretations of Nesterov’s 1-memory method in Algorithm 14 are equivalent. That is, the iteratively generated points \( w_t \) coincide with the sequence \( \bar{x}_t \) produced by the FGNRD dynamic.

**Proof** Allow us define some notations first.

\[
\bar{x}_t := \frac{1}{A_t} \sum_{s=1}^{t} \alpha_s x_s
\]

\[
\tilde{x}_t := \frac{1}{A_t} \left( \alpha_t x_{t-1} + \sum_{s=1}^{t-1} \alpha_s x_s \right), \tag{50}
\]

\[
\hat{y}_{t+1} := \arg\min_{y \in Y} \sum_{s=1}^{t} \alpha_s \left( f^*(y) - \langle x_s, y \rangle \right)
\]

Let us make a few observations about these sequences.

\[
\hat{y}_{t+1} = \arg\max_{y \in Y} \langle \bar{x}_t, y \rangle - f^*(y) = \nabla f(\bar{x}_t) \tag{51}
\]

\[
\tilde{x}_t - \bar{x}_t = \frac{\alpha_t}{A_t} (x_{t-1} - x_t). \tag{52}
\]

Equations (51) and (52) follow from simple arithmetic as well as elementary properties of Fenchel conjugation and the Legendre transform [14]. Equation (52) is by a simple algebraic calculation.

We show, via induction, that the following equalities are maintained for every \( t \). Note that three objects on the left correspond to the iterative description given in Algorithm 17 whereas the three on the right correspond to the FGNRD description.

\[
\nabla f(z_t) = y_t \tag{53}
\]

\[
v_t = x_t \tag{54}
\]

\[
w_t = \bar{x}_t. \tag{55}
\]

We first note that the initialization ensures that (53) holds for \( t = 1 \). Also, observe that we have \( \beta_t = \frac{A_t}{A_t} \). Therefore,

\[
w_t = (1 - \beta_t) w_{t-1} + \beta_t v_t = \left( 1 - \frac{A_t}{A_t} \right) w_{t-1} + \frac{\alpha_t}{A_t} v_t = \frac{A_{t-1}}{A_t} w_{t-1} + \frac{\alpha_t}{A_t} v_t \tag{56}
\]

From (56), we see that (54) and \( w_{t-1} = \bar{x}_{t-1} \) implies (55), as \( w_t \) is always an average of the updates \( v_t \). It remains to establish (53) and (54) via induction.

Let us first show (53). But before that, we show \( y_t = \nabla f(\bar{x}_t) \) as follows.

In Algorithm 14 the \( y \)-player is using the OptimisticFTL algorithm to choose the sequence of \( y_t \)'s, which is given by

\[
y_t = \arg\min_{y \in Y} \left\{ \alpha t \ell_{t-1}(\cdot) + \sum_{s=1}^{t-1} \alpha_s \ell_s(y) \right\}, \tag{57}
\]
where the guess $m_t(\cdot)$ to be $m_t(\cdot) \leftarrow \ell_{t-1}(\cdot)$, $\ell_s(y) := f^*(y) - \langle x_s, y \rangle$, and $\ell_0(x_0) := f^*(y) - \langle x_0, y \rangle$ so that $y_0 = \nabla f(x_0)$. Then, we have

$$y_t = \arg\min_{y \in Y} \{ A_T(f^*(y) - \langle \bar{x}_t, y \rangle) \} = \nabla f(\bar{x}_t). \; \; (58)$$

Given that $y_t = \nabla f(\bar{x}_t)$ shown in (58), it suffices to show that $\bar{x}_t = z_t$ for establishing (53). We have

$$z_t = (1 - \beta_t)w_{t-1} + \beta_tv_{t-1} = (1 - \frac{\alpha_t}{A_t})\bar{x}_{t-1} + \frac{\alpha_t}{A_t}x_{t-1}$$

$$= \frac{A_{t-1}}{A_t}\sum_{s=1}^{t-1} \alpha_s x_s + \frac{\alpha_t}{A_t}x_{t-1} = \bar{x}_t.$$  

Hence, (53) holds.

To show (54), observe that the update of $v_t$ is exactly equivalent to OMD$^+$ for which $\gamma = \frac{1}{A_T}$, $\alpha_t \leftarrow t$, and $\ell(t) \leftarrow \langle y_t, \cdot \rangle$. We thus have completed proof.  \qed

We remark that when the problem is unconstrained, i.e. $\mathcal{K} = \mathbb{R}^d$, Algorithm 14 with the distance generating function $\phi(\cdot) = \frac{1}{2}\|\cdot\|_2^2$ can be rewritten as Algorithm 12 — the Nesterov’s first acceleration method.

**Lemma 16** Consider the FGNRD implementation in Algorithm 14, where the function $f(\cdot)$ is convex and $L$-smooth with respect to the norm $\|\cdot\|$, whose dual norm is $\|\cdot\|_*$. Regardless of the sequence of points $x_1, \ldots, x_T$ generated from the protocol, we have

$$\alpha\text{-REG}^\|\text{OPTIMISTICFTL}\| \leq L \sum_{t=1}^{T} \frac{\alpha_t^2}{A_t} \|x_{t-1} - x_t\|^2. \; \; (59)$$

**Proof** By using Lemma 9 with the guess $m_t(\cdot) \leftarrow \ell_{t-1}(\cdot)$, $w_t \leftarrow \hat{y}_t$ of (51), and $z_t \leftarrow y_t$ of (58), and that $\alpha_t (\ell_t(y) - \ell_{t-1}(y)) = \alpha_t \langle x_{t-1} - x_t, y \rangle$ in Fenchel Game, we have

$$\sum_{t=1}^{T} \alpha_t \ell_t(y_t) - \alpha_t \ell_t(y^*) \leq \sum_{t=1}^{T} \alpha_t (\ell_t(y_t) - \ell_{t-1}(y_t) - (\ell_t(y_{t+1}) - \ell_{t-1}(y_{t+1})))$$

(Eqns. 51, 58)

$$\leq \sum_{t=1}^{T} \alpha_t \|x_{t-1} - x_t\| \|\nabla f(\bar{x}_t) - \nabla f(\bar{x}_t)\|_*$$

(Hölder’s Ineq.)

$$\leq L \sum_{t=1}^{T} \alpha_t \|x_{t-1} - x_t\| \|\nabla f(\bar{x}_t) - \nabla f(\bar{x}_t)\|_*$$

($L$-smoothness of $f$)

$$= L \sum_{t=1}^{T} \frac{\alpha_t^2}{A_t} \|x_{t-1} - x_t\| \|\bar{x}_t - \bar{x}_t\|$$

(Eqn. 52)

as desired, where the first inequality is because that $m_t(\cdot) = \ell_{t-1}(\cdot)$.  \qed
Theorem 14  Let us consider the output $\bar{x}_T \equiv w_T$ of Algorithm 14. Given that $f$ is $L$-smooth, $w_T$ satisfies
\[ f(w_T) - \min_{w \in \mathcal{K}} f(w) \leq \frac{8LDw_0(w^*)}{T^2}. \]

Proof  The equivalence $\bar{x}_T \equiv w_T$ was established in Proposition 13. We may now appeal to Theorem 2 to prove this result, which gives us that
\[ f(w_T) - \min_{w \in \mathcal{K}} f(w) \leq \frac{1}{A_T} (\alpha\text{-Reg}^y[\text{OPTIMISTICFTL}] + \alpha\text{-Reg}^x[\text{OMD}^+]). \]
We have a bound for $\alpha\text{-Reg}^x$ from Lemma 12, with parameters $\gamma = \frac{1}{4L}$ and $\beta = 1$, and a bound on $\alpha\text{-Reg}^y$ in Lemma 16. Combining these we obtain
\[ f(w_T) - \min_{w \in \mathcal{K}} f(w) \leq \frac{1}{A_T} \left( 4LDw_0(w^*) + \sum_{t=1}^{T} \left( \frac{\alpha_t^2}{A_t} L - 2L \right) \|x_{t-1} - x_t\|^2 \right). \]
Given the choice of weights $\alpha_t = t$, we have $A_t = \frac{t(t+1)}{2}$ and therefore $\frac{\alpha_t^2}{A_t} = \frac{2t^2}{t(t+1)} \leq 2$. With this in mind, the sum on the right hand side of (60) is non-positive. Noticing that $\frac{1}{A_T} \leq \frac{2}{T^2}$ completes the proof.

It is worth dwelling on exactly how we obtained the above result. A less refined analysis of the OMD$^+$ algorithm would have simply ignored the negative summation term in Lemma 12, and simply upper bounded this by 0. But the negative terms $\|x_t - x_{t-1}\|^2$ in this sum happen to correspond exactly to the positive terms one obtains in the regret bound for the $y$-player, but this is true only as a result of using the OPTIMISTICFTL algorithm. To obtain a cancellation of these terms, we need a $\gamma_t$ which is roughly constant, and hence we need to ensure that $\frac{\alpha_t^2}{A_t} = O(1)$. The final bound, of course, is determined by the inverse quantity $\frac{1}{A_T}$, and a quick inspection reveals that the best choice of $\alpha_t = \theta(t)$. This is not the only choice that could work, and we conjecture that there are scenarios in which better bounds are achievable for different $\alpha_t$ tuning. We show in Subsection 4.6 that a linear rate is achievable when $f(\cdot)$ is also strongly convex, and there we tune $\alpha_t$ to grow exponentially in $t$ rather than linearly.

Infinite memory method.  We finish this section by mentioning one additional algorithm. Nesterov has proposed another first-order method with $O(1/T^2)$ convergence, which requires maintaining a weighted average of gradients along the algorithm’s path. This method also has a natural and simple interpretation in our FGNRD framework, and we include it here for completeness.

We leave it to the reader to establish the equivalence described in Algorithm 15, and here we state the convergence rate whose proof is deferred to Appendix F.
Algorithm 15 Nesterov’s ∞-memory method [3, 66]

Given: $L$-smooth $f(\cdot)$, convex domain $K \subseteq \mathbb{R}^d$, arbitrary $v_0 = x_0 \in K$, 1-strongly convex regularizer $R(\cdot)$, iterations $T$.

Output: $w_T = \bar{x}_T$.

\[
\begin{align*}
\beta_t &\leftarrow \frac{2}{t+1}, \gamma_t \leftarrow \frac{t}{4L}, \\
z_t &\leftarrow (1 - \beta_t)w_{t-1} + \beta_tv_{t-1}, \\
v_t &\leftarrow \text{argmin}_{x \in K} \sum_{s=1}^{t} \gamma_s \langle \nabla f(z_s), x \rangle + R(x), \\
w_t &\leftarrow (1 - \beta_t)w_{t-1} + \beta_tv_t.
\end{align*}
\]

Iterative Description FGNRD Equivalence

Theorem 15 Assume that $f(\cdot)$ is $L$-smooth convex. Nesterov’s ∞-memory method (Algorithm 15) satisfies

\[
f(w_T) - \min_{w \in K} f(w) \leq O\left(\frac{L(R(w^*)) - \min_{x \in K} R(x)}{T^2}\right).
\]

4.5.3 Accelerated proximal methods

Algorithm 16 Accelerated proximal method.

Given: $L$-smooth $f(\cdot)$, arbitrary $v_0 = x_0 \in \mathbb{R}^d$, iterations $T$.

Output: $w_T = \bar{x}_T$.

\[
\begin{align*}
\beta_t &\leftarrow \frac{2}{t^2 + 1}, \gamma_t \leftarrow \frac{t}{4L}, \\
z_t &\leftarrow (1 - \beta_t)w_{t-1} + \beta_tv_{t-1}, \\
v_t &\leftarrow \text{prox}_{t^{-1}\gamma\psi}(x_{t-1} - t^{-1}\nabla f(z_t)), \\
w_t &\leftarrow (1 - \beta_t)w_{t-1} + \beta_tv_t.
\end{align*}
\]

Iterative Description FGNRD Equivalence

In this section, we consider solving composite optimization problems

\[
\min_{x \in \mathbb{R}^d} f(x) + \psi(x),
\]

where $f(\cdot)$ is smooth convex but $\psi(\cdot)$ is possibly non-differentiable convex. Examples of $\psi(\cdot)$ include $\ell_1$ norm ($\|\cdot\|_1$), $\ell_\infty$ norm ($\|\cdot\|_\infty$), and elastic net ($\|\cdot\|_1 + \gamma\|\cdot\|_2^2$ for $\gamma > 0$). We would like to show that the game analysis still applies to this problem. We just need to change the payoff function $g$ to account for $\psi(x)$. Specifically, we consider the following two-player zero-sum game,

\[
\min_{x \in \mathbb{R}^d} \max_{y \in Y} \{g(x,y) := \langle x, y \rangle - f^*(y) + \psi(x)\}.
\]

Notice that the minimax value of the game is $\min_{x \in \mathbb{R}^d} f(x) + \psi(x)$, which is exactly the optimum value of the composite optimization problem. Let us
denote the proximal operator \(^2\) as \(\text{prox}_{\lambda \psi}(v) := \arg\min_{x \in \mathbb{R}^d} (\psi(x) + \frac{1}{2\lambda} \|x - v\|^2)\).

We remark that the FGNRD interpretation in Algorithm 16 is essentially the same as Algorithm 14, except here the payoff function \(g\) is defined differently \((62)\). The weighting scheme \(\alpha_t\) and the players’ strategies remain the same, and we thus omit the equivalence between the two interpretations of Algorithm 16.

In this new game, the \(x\)-player plays OMD\(^+\) with the distance generating function \(\phi(x) = \frac{1}{2}\|x\|^2\) and receives the loss functions \(\alpha_t h_t(x) := \alpha_t \{\langle x, y_t \rangle + \psi(x)\}\), which leads to the following update,

\[
x_t = \arg\min_{x \in \mathbb{R}^d} \gamma(\alpha_t h_t(x)) + D_{x_{t-1}}(x) = \arg\min_{x \in \mathbb{R}^d} \gamma(\alpha_t \{\langle x, y_t \rangle + \psi(x)\}) + D_{x_{t-1}}(x)
\]

\[
= \arg\min_{x \in \mathbb{R}^d} \psi(x) + \frac{1}{2\alpha_t \gamma}(\|x\|^2 + 2\langle \alpha_t \gamma y_t - x_{t-1}, x \rangle)
\]

\[
= \text{prox}_{\alpha_t \gamma \psi}(x_{t-1} - \alpha_t \gamma \nabla f(\tilde{x}_t)),
\]

where the last equality follows that \(y_t = \nabla f(\tilde{x}_t)\) and the definition of the operator \(\text{prox}(\cdot)\). One can view Algorithm 16 as a variant of the so called “Accelerated Proximal Gradient” in \([8]\). Yet, the design and analysis of our algorithm is simpler than that of \([8]\).

**Theorem 16** The update \(w_T = \bar{x}_T\) in Algorithm 16 satisfies

\[
f(w_T) - \min_{w \in \mathbb{R}^d} f(w) \leq O\left(\frac{L\|w_0 - w^*\|^2}{T^2}\right).
\]

**Proof** Even though the payoff function \(g(\cdot, \cdot)\) is a bit different, the proof still essentially follows the same line as Theorem 14, as \(y\)-player plays OPTIMISTICFTL and the \(x\)-player plays OMD\(^+\). \(\square\)

**4.5.4 Related works**

In recent years, there are growing interest in giving new interpretations of Nesterov’s accelerated algorithms or proposing new variants. For example, \([66]\) gives a unified analysis for some Nesterov’s accelerated algorithms \([3–5]\), using the standard techniques and analysis in optimization literature. \([98, 99]\) connects the design of accelerated algorithms with dynamical systems and control theory. \([100, 101]\) gives a geometric interpretation of the Nesterov’s method for unconstrained optimization, inspired by the ellipsoid method. \([102]\) studies the Nesterov’s methods and the Heavy Ball method for quadratic non-strongly convex problems by analyzing the eigen-values of some linear dynamical systems. \([103]\) proposes a variant of accelerated algorithms by mixing the updates

\(^2\) It is known that for some \(\psi(\cdot)\), their corresponding proximal operations have closed-form solutions, see e.g. \([97]\) for details.
of gradient descent and mirror descent and showing the updates are complementary. [104, 105] propose a primal-dual view that recovers several first-order algorithms with careful discretizations of a continuous-time dynamic, which also leads to a new accelerated extra-gradient descent method. [40] show a simple acceleration proof of mirror prox [106] and dual extrapolation [107] based on solving the Fenchel game. [96, 108–113] analyze the acceleration algorithms via the lens of ordinary differential equations and the numerical analysis. See also a monograph [114] for various methods of acceleration, and a recent work [115] that shows acceleration of Heavy Ball compared to standard gradient descent for minimizing a class of Polyak-Łojasiewicz functions when the non-convexity is averaged-out. We also note an independent work [116, 117] provide a game interpretation of Nesterov’s accelerated method but does not have the idea of the regret analysis. In our work, we show a deeper connection between online learning and optimization, and propose a modular framework that is not limited to Nesterov’s method.

Finally, the accelerated algorithms in this work require the function to be smooth. For minimizing general non-smooth convex functions, the lower-bound of the convergence rate is $O(1/\sqrt{T})$ for any first-order gradient-based methods, see e.g., Subsection 3.2.1 in [5]. However, Nesterov [3] show that when a function has a specific structure, an $O(1/T)$ rate can be obtained via a smoothing technique. Specifically, the class of non-smooth functions is in the following form:

$$f(x) = \max_{z \in \Theta} \langle Mx, z \rangle - \psi(z),$$

where $\Theta$ is a compact convex set, $\psi(\cdot) : \Theta \to \mathbb{R}$ is convex, and $M : \mathcal{K} \to \Theta$ is a linear operator. The smoothing technique that Nesterov introduced is first by constructing a smooth approximation of $f(\cdot)$ in (63):

$$f_\nu(x) = \max_{z \in \Theta} \langle Mx, z \rangle - (\psi(z) + \nu d(z)),$$

where $d(\cdot)$ is 1-strongly convex on $\Theta$. Since $\psi(z) + \nu d(z)$ is a strongly convex function, one can therefore show that the approximated function $f_\nu(\cdot)$ is smooth. Then, one can apply those accelerated methods for smooth convex functions to $f_\nu(\cdot)$. By tuning the parameter $\nu$ to trade-off the smoothness constant of $f_\nu(\cdot)$ and the approximation error of $f_\nu(\cdot)$ to $f(\cdot)$ appropriately, one can get an $O(1/T)$ rate for minimizing the original function $f(\cdot)$. We refer the reader to the original paper [3] for the details. We also note that there are other smoothing techniques in the literature, e.g., Moreau envelope [118].

### 4.6 Proving Accelerated Linear Rates

Nesterov observed that for strongly convex smooth functions, one can achieve an accelerated linear rate (e.g. page 71-81 of [5]). It is natural to ask if the zero-sum game and regret analysis in the present work also recovers this faster rate in the same fashion. We answer this in the affirmative. Assume $f(\cdot)$ is
Algorithm 17 Accelerated Gradient with Linear Convergence

Given: $L$-smooth, $\mu$-strongly convex $f(\cdot)$, convex domain $\mathcal{K} \subseteq \mathbb{R}^d$, iterations $T$, and a distance generating function $\phi(\cdot)$ that is 1-strongly convex, $L_\phi$-smooth, and differentiable. Finally, let $R(x) = \phi(x)$.

Init: $w_0 = v_0 = x_0 = \arg\min_{x \in \mathcal{K}} \phi(x)$.

\[
\begin{align*}
\beta_t &\leftarrow \frac{1}{T} \sqrt{\frac{L}{L(1+L_\phi)}}, \\
\gamma_t &\leftarrow \alpha_t \\
z_t &\leftarrow (1 - \beta_t)w_{t-1} + \beta v_{t-1} \\
\Phi_t(x) &\leftarrow \sum_{s=1}^{t} \gamma_s (\langle \nabla f(z_s), x \rangle + \mu \phi(x)) \\
v_t &\leftarrow \arg\min_{x \in \mathcal{K}} R(x) + \Phi_t(x) \\
w_t &\leftarrow (1 - \beta_t)w_{t-1} + \beta v_t
\end{align*}
\]

Output: $w_T = \bar{x}_T$

\begin{tabular}{|c|c|}
\hline
\textbf{Algorithm 17} & \\
\hline
\text{Given:} & $L$-smooth, $\mu$-strongly convex $f(\cdot)$, convex domain $\mathcal{K} \subseteq \mathbb{R}^d$, iterations $T$, and a distance generating function $\phi(\cdot)$ that is 1-strongly convex, $L_\phi$-smooth, and differentiable. Finally, let $R(x) = \phi(x)$. \hline
\text{Init:} & $w_0 = v_0 = x_0 = \arg\min_{x \in \mathcal{K}} \phi(x)$. \hline
$\beta_t \leftarrow \frac{1}{T} \sqrt{\frac{L}{L(1+L_\phi)}}, \gamma_t \leftarrow \alpha_t$ & $\tilde{f}(x) := f(x) - \mu \phi(x)$ \hline
$z_t \leftarrow (1 - \beta_t)w_{t-1} + \beta v_{t-1}$ & $g(x, y) := \langle x, y \rangle - \tilde{f}^*(y) + \mu \phi(x)$ \hline
$\Phi_t(x) := \sum_{s=1}^{t} \gamma_s (\langle \nabla f(z_s), x \rangle + \mu \phi(x))$ & $\alpha_1 = \frac{1}{2L(1+L_\phi)}$ \hline
$v_t \leftarrow \arg\min_{x \in \mathcal{K}} R(x) + \Phi_t(x)$ & $\alpha_t = \frac{1}{L} \sqrt{\frac{\mu}{L(1+L_\phi)}}$ \hline
$w_t \leftarrow (1 - \beta_t)w_{t-1} + \beta v_t$ & for $t = 2, \ldots, T$ \hline
\hline
\end{tabular}

FGNRD Equivalence

Output: $w_T = \bar{x}_T$

\begin{tabular}{|c|c|}
\hline
\textbf{Proposition 17} & The two interpretations of Algorithm 17 are equivalent. That is, the iteratively generated points $w_t$ coincide with the sequence $\bar{x}_t$ produced by the FGNRD dynamic. \hline
\hline
\begin{proof}
The proof follows the same lines as that of Proposition 13 until the last line by setting $\beta_t = \beta = \frac{1}{T} \sqrt{\frac{L}{L(1+L_\phi)}} = \frac{\alpha_t}{\sqrt{T}}$. We only need to replace the last line of the proof. Specifically, the update of $v_t$ is exactly equivalent to FTRL$^+$ on the loss sequences $\alpha_t \ell_t(\cdot) \leftarrow \alpha_t \left( \langle \nabla f(z_t), \cdot \rangle + \mu \phi(\cdot) \right)$. Hence, the proof is completed.
\end{proof}

\end{tabular}
We now prove the accelerated linear convergence rate of Algorithm 17. First, we denote the condition number $\kappa := \frac{L}{\mu}$.

**Theorem 18** Assume we are given a norm $\| \cdot \|$, and that we have a distance-generating function $\phi(\cdot)$ that is differentiable, $L_\phi$-smooth, and $1$-strongly convex with respect to $\| \cdot \|$. Assume we also have a function $f(\cdot)$ that is $L$-smooth with respect to $\| \cdot \|$ and $\mu$-strongly convex in the sense of (65). Then the output of Algorithm 17 satisfies

$$f(w_T) - \min_{w \in \mathcal{K}} f(w) \leq c \exp \left( -\frac{1}{2 \sqrt{\kappa} \sqrt{1 + L_\phi}} T \right) (\phi(w^*) - \phi(w_0))$$

for some constant $c := 2L(1 + L_\phi) > 0$.

**Proof** As the proof of Lemma 16, we first bound the regret of the $y$-player as follows.\[\sum_{t=1}^{T} \alpha_t \ell_t(y_t) - \alpha_t \ell_t(y^*) \leq \sum_{t=1}^{T} \alpha_t \langle x_{t-1} - x_t, y_t - \tilde{y}_{t+1} \rangle\]

(Eqns. 51, 58) \[= \sum_{t=1}^{T} \alpha_t \langle x_{t-1} - x_t, \nabla \tilde{f}(\bar{x}_t) - \nabla \bar{f}^*(\bar{x}_t) \rangle\]

(Hölder’s Ineq.) \[\leq \sum_{t=1}^{T} \alpha_t \| x_{t-1} - x_t \| \| \nabla \tilde{f}(\bar{x}_t) - \nabla \bar{f}^*(\bar{x}_t) \|_*\]

= \sum_{t=1}^{T} \alpha_t \| x_{t-1} - x_t \| \| \nabla f(\bar{x}_t) - \mu \nabla \phi(\bar{x}_t) - \nabla f(\bar{x}_t) + \mu \nabla \phi(\bar{x}_t) \|_*\]

(Triangle Ineq. & Smooth $\phi$) \[\leq \sum_{t=1}^{T} \alpha_t \| x_{t-1} - x_t \| (\| \nabla f(\bar{x}_t) - \nabla f(\bar{x}_t) \|_* + \mu L_\phi \| \bar{x}_t - \tilde{x}_t \|)\]

(Smoothness of $f$ & $\mu \leq L$) \[\leq L(1 + L_\phi) \sum_{t=1}^{T} \alpha_t \| x_{t-1} - x_t \| \| \bar{x}_t - \tilde{x}_t \|\]

(Eqn. 52) \[= L(1 + L_\phi) \sum_{t=1}^{T} \frac{\alpha_t^2}{A_t} \| x_{t-1} - x_t \| \| x_{t-1} - x_t \|\]

So the regret satisfies\[\alpha\text{-Reg}^y \leq L(1 + L_\phi) \sum_{t=1}^{T} \frac{\alpha_t^2}{A_t} \| x_{t-1} - x_t \|^2\] (68)

For the $x$-player, according to Lemma 8, its regret is\[\alpha\text{-Reg}^x \leq R(x^*) - R(x_0) - \sum_{t=1}^{T} \left( \frac{\mu A_{t-1}}{2} + \frac{1}{2} \right) \| x_{t-1} - x_t \|^2\] (69)

where $x_0 = \text{argmin}_x R(x)$. Summing (68) and (69), we have\[\alpha\text{-Reg}^y[\text{OPTIMISTICFTL}] + \alpha\text{-Reg}^x[\text{FTRL}^+]\]

\[\leq R(x^*) - R(x_0) + \sum_{t=1}^{T} \left( \frac{L(1 + L_\phi) \alpha_t^2}{A_t} - \mu A_{t-1} + \frac{1}{2} \right) \| x_{t-1} - x_t \|^2.\]
By choosing the weight \( \{\alpha_t\} \) to satisfy \( \alpha_1 = \frac{1}{2L(1+L_\phi)} \) and that for \( t \geq 2, \frac{\alpha_t}{A_t} = \frac{1}{2} \sqrt{\frac{\mu}{L(1+L_\phi)}}, \) the coefficient of the distance terms will be non-positive for all \( t \), i.e.

\[
\left( \frac{L(1+L_\phi)\alpha_t^2}{A_t} - \frac{\mu A_{t-1}}{2} \right) \leq 0,
\]

which means that the distance terms will cancel out. Therefore, the optimization error \( \epsilon \) after \( T \) iterations satisfies that

\[
\epsilon \leq \sum_{t=1}^{T} \alpha_{t-1} \times \text{Reg}^y + \alpha_{t-1} \times \text{Reg}^x
\]

\[
\leq \frac{1}{A_1} \left( 1 - \frac{\alpha_2}{A_2} \right) \cdots \left( 1 - \frac{\alpha_T}{A_T} \right) (R(x^*) - R(x_0))
\]

\[
\leq \left( 1 - \frac{1}{2\sqrt{1+L_\phi\sqrt{\kappa}}} \right)^{T-1} \frac{R(x^*) - R(x_0)}{A_1} = O \left( \exp \left( -\frac{1}{2\sqrt{1+L_\phi\sqrt{\kappa}}} \right) \right).
\]

\[\square\]

5 New algorithms

In Section 4 we established that the FGNRD framework can be used to describe a set of known algorithms for convex optimization under various assumptions. But we have argued that the framework is indeed quite generic, given that we have flexibility to choose different no-regret update algorithms, to change the game’s payoff function, and to modify the protocol. In the present section we use FGNRD to describe two novel algorithms, and to provide strong convergence rates using the same methodology.

5.1 Boundary Frank-Wolfe

In the original formulation of the Fenchel Game No-Regret Dynamics, Protocol 2, we specified that the two players act in a particular order, first the \( y \)-player followed by the \( x \)-player. This ordering is important, as it allows us to consider scenarios in which \( x_t \) can be chosen in response to \( y_t \), as was the case in nearly all of the algorithms proposed in Section 4. But we can instead reverse the order of the two players, allowing the \( x \)-player to act before the \( y \)-player, while otherwise following the same protocol. This will give us a new algorithm for constrained convex optimization, Boundary Frank-Wolfe, which has the unusual property that it only computes gradients of \( f \) on the boundary of the constraint set \( \mathcal{K} \). We will show the algorithm has an \( O(\log T/T) \) convergence rate for any convex Lipschitz \( f \), without any smoothness nor strong-convexity requirement for \( f \), yet under two conditions: (1) the true minimizer of \( f \) in \( \mathcal{K} \) must exist on the boundary, and \( \mathcal{K} \) must be a strongly convex set. The result relies on recent work in online convex optimization under strongly convex constraints [62]. (Notably, there is prior work that develops a Frank-Wolfe-like algorithm that strongly convex decision sets [21] with \( O(1/T^2) \) convergence, but this result requires that \( f \) is both smooth and strongly convex as well.)
Algorithm 18 Boundary Frank-Wolfe

Given: possibly non-smooth convex $f(\cdot)$, convex domain $K \subset \mathbb{R}^d$, arbitrary $w_0 = z_0$, iterations $T$.

\[
\begin{align*}
  z_t &\leftarrow \arg\min_{z \in K} \frac{1}{t} \sum_{s=1}^{t-1} \langle z, \delta_s \rangle, \delta_s \in \partial f(z_s) \\
  w_t &\leftarrow \frac{(t-1)w_{t-1} + z_t}{t} \\
\end{align*}
\]

Iterative Description

Output: $w_T := \bar{x}_T$.

The equivalence between the two forms of Algorithm 18 can be proven in the usual fashion, and so we omit the proof of the following proposition.

**Proposition 19** The two interpretations of Algorithm 18 are equivalent. That is, the iteratively generated points $w_t$ coincide with the sequence $\bar{x}_t$ produced by the FGNRD dynamic.

**Theorem 20** Assume that the constraint set $K$ is $\lambda$-strongly convex, and that $\sum_{s=1}^{t} \delta_s$, where $\delta_s \in \partial f(x_s)$ has non-zero norm, then Boundary Frank-Wolfe (Algorithm 18) has

\[
f(w_T) - \min_{w \in K} f(w) = O\left(\frac{M \log T}{\lambda L_T T}\right),
\]

where $M := \sup_{\delta \in \partial f(x), x \in K} \|\delta\|$, $\Theta_t := \frac{1}{t} \sum_{s=1}^{t} \delta_s$, $\delta_s \in \partial f(x_s)$, and $L_T := \min_{1 \leq t \leq T} \|\Theta_t\|$.

**Proof** Since $y$-player plays $\text{BestResp}^+$, its regret $\leq 0$. For the $x$-player, we use Lemma 5, which grants $\alpha \text{-Reg}^\times \leq O\left(\frac{M \log T}{\lambda L_T T}\right)$. Using Theorem 2 we have

\[
f(\bar{x}_T) - \min_{x \in K} f(x) \leq \frac{1}{T}(\alpha \text{-Reg}^\times[\text{FTL}] + \alpha \text{-Reg}^y[\text{BestResp}^+])
\]

\[
= O\left(\frac{M \log T}{\lambda L_T T}\right)
\]

as desired. \qed

Note that the rate depends crucially on $L_T$, which is the smallest averaged-gradient norm computed during the optimization. According to Theorem 1, the closure of the gradient space is a convex set. This implies the following lemma that ensures $L_T$ is bounded away from 0.

**Lemma 17** If $0 \notin \text{closure} \{y : y \in \partial f(x), x \in K\}$, then the average of gradient norm $L_T$ satisfies $L_T > 0$. 

Proof By Theorem 1, we know $\Theta_t$, which is the average of some sub-gradients, is itself a subgradient. Moreover, $\Theta_t$ is not $0$ given that $0$ is not in the gradient space, hence we have $L_T > 0$.

Alternatively, as $0$ is not in the closure of the gradient space, which is a closed convex set by Theorem 1, the separation theorem implies that any point in the closure must be bounded away from $0$. Hence, $L_T > 0$. □

We remark that if all the subgradients at the points in the constraint set $\mathcal{K}$ have their sizes be lower-bounded by a constant $G > 0$, then we have that $L_T$ is lower-bounded by $G$, since the (closure) of the union of the subdifferentials is a convex set (Theorem 1) and therefore $\Theta_t$, which is the average of some sub-gradients, is itself a subgradient.

The proposed algorithm may only be useful when it is known, a priori, that the solution $w^*$ will occur not in the interior but on the boundary of $\mathcal{K}$. It is indeed an odd condition, but it does hold in many typical scenarios. One may add a perturbed vector to the gradient and show that with high probability, $L_T$ is a non-zero number. The downside of this approach is that it would generally grant a slower convergence rate.

5.2 Gauge Frank-Wolfe

Using the FGNRD framework further, we propose a new algorithm that is “FRANK-WOLFE-like” in the following sense: we can optimize a smooth objective function using only linear optimization oracle queries. This method requires the constraint set to be “suitably round” but achieves a rate of $O(1/T^2)$ without additional assumptions on $f$. The main trick is to utilize a regularization function that is adapted to the constraint set $\mathcal{K}$. With this in mind, define the “gauge function” of $\mathcal{K}$ [120, 121] as

$$
\gamma_\mathcal{K}(x) := \inf\{c \geq 0 : \frac{x}{c} \in \mathcal{K}\}.
$$

(70)

Notice that, for a closed convex $\mathcal{K}$ that contains the origin, one has $\mathcal{K} = \{x \in \mathbb{R}^d : \gamma_\mathcal{K}(x) \leq 1\}$. Furthermore, the boundary points on $\mathcal{K}$ satisfy $\gamma_\mathcal{K}(x) = 1$.

Definition: ($\lambda$-Gauge set) We call a $\lambda^2$-strongly convex set that contains the origin in its interior a $\lambda$-Gauge set.

Examples of the $\lambda$-Gauge set are given in Appendix G.

Theorem 21 (Theorem 4 in [122] and Theorem 2 in [123]) Let $\mathcal{K}$ be a $\lambda$-Gauge set. Then, its squared gauge function, i.e. $\gamma_\mathcal{K}^2(\cdot)$, is $\lambda$-strongly-convex.

Algorithmically, we will utilize FTRL$^+$ for the $x$-player with regularization $\gamma_\mathcal{K}^2(\cdot)$ in the Fenchel Game. Naturally, this requires us to solve problems of the form

$$
x = \arg\min_{x \in \mathcal{K}} \langle \zeta, x \rangle + \gamma_\mathcal{K}^2(x),
$$

(71)
for any arbitrary vector $\zeta$. While at first glance this does not appear to be a linear optimization problem, let us reparametrize the problem as follows. Denote $\text{bndry}(K)$ as the boundary of the constraint set $K$, and notice that (71) is equivalent to solving

$$\min_{\rho \in [0, 1], z \in \text{bndry}(K)} \langle \zeta, \rho z \rangle + \gamma^2_K(\rho z) = \min_{\rho \in [0, 1]} \left[ \left( \min_{z \in \text{bndry}(K)} \langle \zeta, z \rangle \right) \rho + \rho^2 \right],$$

(72)

where we are able to remove the dependence on the gauge function since it is homogeneous, i.e., $\gamma_K(\rho z) = \rho \gamma_K(z)$, and is identically 1 on the boundary of $K$. Notice that the inner minimization is now a linear problem, and the outer problem is a 1-parameter constrained quadratic problem which can be solved directly. In summary, we have

$$\arg\min_{x \in K} \langle \zeta, x \rangle + \gamma^2_K(x) = \rho^* z^*$$

where

$$\left\{ \begin{array}{l}
z^* = \arg\min_{z \in \text{bndry}(K)} \langle \zeta, z \rangle, \\
\rho^* = \max(0, \min(1, -\langle \zeta, z^* \rangle/2)).
\end{array} \right. $$

(73)

**Algorithm 19** Gauge Frank-Wolfe (smooth convex $f(\cdot)$)

Given: $L$-smooth convex $f(\cdot)$, arbitrary $v_0 = x_0 \in K \subset \mathbb{R}^d$, iteration $T$.

| Iterative Description | FGNRD Equivalence |
|-----------------------|------------------|
| $\beta_t \leftarrow \frac{2}{T+1}$, $\gamma_t \leftarrow \frac{1}{4\gamma}$ | $g(x, y) := \langle x, y \rangle - f^*(y)$ |
| $z_t \leftarrow (1 - \beta_t)w_{t-1} + \beta_tv_{t-1}$ | $\alpha_t := t$ for $t = 1, \ldots, T$ |
| $z^* \leftarrow \arg\min_{z \in \text{bndry}(K)} \langle \sum_{s=1}^t \gamma_s \nabla f(z_s), z \rangle$ | OAlg$^Y := \text{OPTIMISTICFTL}$ |
| $\gamma_t \leftarrow \frac{1}{2} \langle \sum_{s=1}^t \gamma_s \nabla f(z_s), z^* \rangle$ | OAlg$^X := \text{FTRL}^+ [\gamma^2_K(\rho x), \frac{1}{4\gamma}]$ |
| $\rho_t^* \leftarrow \max(0, \min(1, -\gamma_t))$ | |
| $v_t \leftarrow \rho_t^* z_t^*$ | |
| $w_t \leftarrow (1 - \beta_t)w_{t-1} + \beta_tv_t$ | |

Output: $w_T = \bar{x}_T$.

**Theorem 22** Suppose the constraint set $K$ is a $\lambda$-Gauge set. Assume that the function $f(\cdot)$ is $L$-smooth convex with respect to the induced gauge norm $\gamma_K(\cdot)$. Suppose that the step size $\eta$ satisfies $\frac{1}{CL} \leq \eta \leq \frac{1}{4L}$ for some constant $C \geq 4$. Then, the output $w_T$ of Algorithm 19 satisfies

$$f(w_T) - \min_{w \in K} f(w) \leq \frac{2CL}{\lambda T^2}.$$

**Proof** We have just shown that line 4-5 is due to that the $x$-player plays FTRL$^+$ with the square of the gauge function as the regularizer. So Algorithm 19 is an instance of Algorithm 15, and we can invoke Theorem 15 to obtain the convergence rate, noting that $\gamma^2_K(x^*) \leq 1$. \qed
We want to emphasize again that our analysis does not need the function $f(\cdot)$ to be strongly convex to show $O(1/T^2)$ rate. On the other hand, [21] shows the $O(1/T^2)$ rate under the additional assumption that the function is strongly convex. [21] raises an open question if an accelerated linear rate is possible under the assumption that the function is smooth and strongly convex. We partially answer the open question through Algorithm 20 and Theorem 23.

**Algorithm 20** Gauge Frank-Wolfe (smooth and strongly convex $f(\cdot)$)
Given: $L$-smooth and $\mu$-strongly convex $f(\cdot)$, arbitrary $w_0 = v_0 = x_0 \in \mathcal{K} \subset \mathbb{R}^d$, iterations $T$.

\[
\begin{align*}
\beta &\leftarrow \frac{1}{\bar{\lambda}} \sqrt{\frac{L(1+L_{\phi}/\lambda)}{\mu}}, \gamma_t \leftarrow \alpha_t \\
z_t &\leftarrow (1 - \beta)w_{t-1} + \beta v_{t-1} \\
z_t^* &\leftarrow \text{argmin}_{z \in \text{bdry}(\mathcal{K})} \langle \sum_{s=1}^{t} \gamma_s \nabla \hat{f}(z_s), z \rangle \\
\gamma_t &\leftarrow \frac{\langle \sum_{s=1}^{t} \gamma_s \nabla \hat{f}(z_s), z_t^* \rangle}{2L + \sum_{s=1}^{t} \gamma_s} \\
\rho_t^* &\leftarrow \max(0, \min(1, -\gamma_t)) \\
v_t &\leftarrow \rho_t^* z_t^* \\
w_t &\leftarrow (1 - \beta)w_{t-1} + \beta v_t
\end{align*}
\]

Output: $w_T = \bar{x}_T$, FGNRD Equivalence

**Theorem 23** Suppose the constraint set $\mathcal{K}$ is a $\lambda$-Gauge set and the induced square of the gauge function $\gamma^2_{\mathcal{K}}(\cdot)$ is differentiable and $L_{\phi}$-smooth. Assume that the function $f(\cdot)$ is $L$-smooth with respect to the induced gauge norm $\gamma_{\mathcal{K}}(\cdot)$ and $f(\cdot)$ is $\mu$-strongly convex satisfying (65) with $\frac{1}{2} \gamma^2_{\mathcal{K}}(\cdot)$ being the distance generating function. Then, the output $w_T$ of Algorithm 20 satisfies

\[
f(w_T) - \min_{w \in \mathcal{K}} f(w) = O\left(\exp\left(-\frac{T}{2\sqrt{1 + L_{\phi}/\lambda \sqrt{\lambda}}}\right)\right).
\]

Lemma 2.2 and Theorem A.2 in [124] implies that if a strongly convex set $\mathcal{K}$ which is centrally-symmetric at the origin satisfies the notion of a smooth set 3, then the induced square of the gauge function $\gamma^2_{\mathcal{K}}(\cdot)$ is $L_{\phi}$-smooth and differentiable for some constant $L_{\phi} > 0$; consequently, the condition of Theorem 23 holds. Examples include $l_p$ balls and Schatten $p$ balls for $1 < p \leq 2$ [124].

---

3For a compact convex set $\mathcal{K}$, it is smooth if and only if $\#(N_{\mathcal{K}}(z) \cap \partial \mathcal{K}^o) = 1$ for any point $z$ on the boundary of $\mathcal{K}$, i.e. $z \in \partial \mathcal{K}$, where $N_{\mathcal{K}}(z) := \{w \in \mathbb{R}^d : \langle z - x, w \rangle \geq 0, \forall x \in \mathcal{K}\}$ is the normal cone at $z$ and $\mathcal{K}^o := \{w \in \mathbb{R}^d : \langle w, x \rangle \leq 1, \forall x \in \mathcal{K}\}$ is the polar of $\mathcal{K}$, see e.g. Definition 2.1 in [124].
Proof The equivalence of the two displays in Algorithm 20 can be shown by following that of the proof of Proposition 17, where the update FTRL$^+$:

$$x_t \leftarrow \arg\min_{x \in \mathcal{K}} \sum_{s=1}^{t} \alpha_s h_s(x) + R(x) = \arg\min_{x \in \mathcal{K}} \sum_{s=1}^{t} \alpha_s \left( \langle y_s, x \rangle + \frac{\mu}{\lambda} \gamma_s^2(x) \right) + \frac{1}{\lambda} \gamma_t^2(x)$$

is equivalent to solving

$$(\rho_t^*, z_t^*) \leftarrow \arg\min_{\rho \in [0,1]} \left( \arg\min_{z \in \text{bndry} (\mathcal{K})} \left\{ \sum_{s=1}^{t} \alpha_s \nabla \tilde{f}(z_s), z \right\} \right) \rho + \left( \frac{1}{\mu} + \sum_{s=1}^{t} \alpha_s \right) \frac{\mu}{\lambda} \rho^2,$$

and setting $v_t = \rho_t^* z_t^*$. Specifically, Algorithm 20 is an instance of Algorithm 17 with the regularization of FTRL$^+$ being the square of the gauge function, i.e. $\phi(\cdot) = \frac{1}{\lambda} \gamma_s^2(\cdot)$. Hence, the convergence rate can be shown by following the proof of Theorem 18, where we bound the Lipschitzness of $\nabla \tilde{f}(\cdot)$ as $||\nabla \tilde{f}(\tilde{x}_t) - \nabla \tilde{f}(\bar{x}_t)|| \leq ||\nabla f(\bar{x}_t) - \nabla f(\tilde{x}_t)|| + \frac{\mu}{\lambda} \nabla \phi(\bar{x}_t)||_{\star} \leq ||\nabla f(\bar{x}_t) - \nabla f(\tilde{x}_t)||_{\star} + \mu \frac{L_{\phi}}{\lambda} ||\bar{x}_t - \tilde{x}_t|| \leq L \left( 1 + \frac{L_{\phi}}{\lambda} \right) ||\bar{x}_t - \tilde{x}_t||$. □

5.3 Optimistic mirror descent, with weighted averaging

We give another accelerated algorithm for smooth convex optimization in the following. Algorithm 21 is obtained when we swap the ordering of the player and let the $x$-player plays first using OPTIMISTICMD and then the $y$-player plays FTL$^+$. The reader might find some similarity of Algorithm 21 and some single-call variants of the extra-gradient methods, see e.g., [43, 88–92]. However, the gradient is taken at some weighted-average points in Algorithm 21 and the resulting converge rate is an accelerated rate $O(1/T^2)$, which seems to be different from the single-call extra-gradient methods in the literature, to the best of our knowledge. We note that the algorithm of Chambolle and Pock (Section 5 in [46]) has a $O(1/T^2)$ for a class of relevant problems with a single gradient call in each iteration, however, their algorithm needs the distance generating function of the Bregman divergence to be the square $l_2$ norm and its update is different from Algorithm 21.

Theorem 24 below shows an accelerated rate $O(1/T^2)$. Comparing Algorithm 10 and Algorithm 21 here, one can see that the difference is that the $y$-player plays FTL$^+$ in Algorithm 21 instead of BESTRESP$^+$ in Algorithm 10. This is crucial to get the accelerated rate, as the negative distance terms in the regret bound of FTL$^+$ cancel out that of OPTIMISTICMD. On the other hand, the regret bound of BESTRESP$^+$ cannot cancel out the distance terms from that of OPTIMISTICMD, which results in a slower $O(1/T)$ rate of Algorithm 10.

Theorem 24 The output $\tilde{w}_T = \tilde{x}_T$ of Algorithm 21 satisfies

$$f(\tilde{w}_T) - \min_{w \in \mathcal{K}} f(w) = \frac{2LD_{w_0}^\phi(w^*) + 2L||w_0 - w_1||^2}{T^2},$$

where $w^* := \arg\min_{w \in \mathcal{K}} f(w)$. 


The equivalence of the two displays, namely, for all $\alpha$ and Lemma 14 Appendix A Proof of Theorem can be trivially shown by induction. Specifically, the first two relations hold in the beginning by letting the initial point $w_0 = x_0 = x_{-\frac{1}{2}} \in K \subseteq \mathbb{R}^d$. 

\begin{algorithm}
\caption{Optimistic mirror descent, with weighted averaging}
Given: $L$-smooth $f(\cdot)$, a 1-strongly convex $\phi(\cdot)$, iterations $T$
Init: arbitrary $w_{-\frac{1}{2}} = w_0 = x_0 = x_{-\frac{1}{2}} \in K \subseteq \mathbb{R}^d$.
\begin{align*}
\gamma & \leftarrow \frac{1}{L} \\
w_t & \leftarrow \arg\min_{w \in K} \left( \alpha_t \langle w, \nabla f(w_{t-1}) \rangle + \frac{1}{\gamma} D_{w_{t-\frac{1}{2}}}^\phi(w) \right) \\
w_{t+\frac{1}{2}} & \leftarrow \arg\min_{w \in K} \left( \alpha_t \langle w, \nabla f(w_t) \rangle + \frac{1}{\gamma} D_{w_{t-\frac{1}{2}}}^\phi(w) \right)
\end{align*}
\end{algorithm}

Output: $\bar{w}_T = \bar{x}_T$

\textbf{Proof of Theorem 24} The equivalence of the two displays, namely, for all $t$,

\begin{align*}
w_t & = x_t \\
w_{t-\frac{1}{2}} & = x_{t-\frac{1}{2}} \\
\nabla f(\bar{w}_t) & = y_t
\end{align*}
can be trivially shown by induction. Specifically, the first two relations hold in the beginning by letting the initial point $w_0 = w_{-\frac{1}{2}} = x_0 = x_{-\frac{1}{2}} \in K$. To show the last one, we apply the definition of FTL$^+$,

\begin{align*}
y_t & \leftarrow \arg\max_{y \in Y} \sum_{s=1}^{t} \alpha_s \ell_s(y) = \arg\max_{y \in Y} \sum_{s=1}^{t} \left( \alpha_s f^*(y) - \alpha_s \langle x_s, y \rangle \right) = \nabla f(\bar{x}_t).
\end{align*}

By induction, we have $w_t = x_t$. So we have $\bar{w}_t = \bar{x}_t$ and consequently $y_t = \nabla f(\bar{w}_t)$.

By summing the regret bound of each player, i.e. Lemma 14 and Lemma 6, we get

\[\alpha\text{-REG}^\phi[\text{OPTIMISTICMD}] + \alpha\text{-REG}^\phi[\text{FTL}^+]\]

\begin{align*}
& \overset{(a)}{\leq} \frac{1}{A_t} \left\{ \frac{1}{\gamma} D_{x_0}^\phi(x^*) + \frac{\gamma}{2} \sum_{t=1}^{T} \alpha_t^2 \| \nabla f(\bar{w}_t) - \nabla f(\bar{w}_{t-1}) \|^2_\ast - \sum_{t=1}^{T} \frac{A_t-1}{L} \| \nabla f(\bar{w}_t) - \nabla f(\bar{w}_{t-1}) \|^2_\ast \right\} \\
& \overset{(b)}{\leq} \frac{LD^\phi_{x_0}(x^*)}{A_t} + L \| \bar{w}_0 - \bar{w}_1 \|^2 \leq \frac{2LD^\phi_{x_0}(x^*) + 2L \| w_0 - w_1 \|^2}{T^2},
\end{align*}

where in (a) we used the relation $m_t = y_{t-1} = \nabla f(\bar{w}_{t-1})$ when we applied Lemma 14 and the fact that the $y$-player’s loss function $\ell_t(y) := f^*(y) - \langle x, y_t \rangle$ is $\frac{1}{T}$-strongly convex w.r.t the dual norm $\| \cdot \|_\ast$ so that we have $\mu = \frac{1}{L}$ when we invoked Lemma 6, and in (b) we used that $\frac{\gamma \alpha_t^2}{2} \leq \frac{A_t-1}{L}$ for $\gamma \leq \frac{1}{2L}$ so that the distance terms cancel out except the one at $t = 1$.

\[\square\]

\section*{Appendix A Proof of Theorem 1}

\textbf{Proof} This is a result of the following lemmas.
Definition: [Definition 12.1 in [125]] A mapping $T : \mathcal{R}^n \to \mathcal{R}^n$ is called monotone if it has the property that
\[
\langle v_1 - v_0, x_1 - x_0 \rangle \geq 0 \text{ whenever } v_0 \in T(x_0), v_1 \in T(x_1).
\]
Moreover, $T$ is maximal monotone if there is no monotone operator that properly contains it.

Lemma 2: [Theorem 12.17 in [125]] For a proper, lower semi-continuous, convex function $f$, $\partial f$ is a maximum monotone operator.

Lemma 3: [Theorem 12.41 in [125]] For any maximal monotone mapping $T$, the set “domain of $T$” is nearly convex, in the sense that there is a convex set $C$ such that $C \subset \text{domain of } T \subset \text{cl}(C)$. The same applies to the range of $T$.

Therefore, the closure of the union of the sets $\bigcup_{x \in X} \partial f(x)$ is convex by the above lemmas. \hfill \qed

Appendix B  Proof of Lemma 11 and 14

Proof of Lemma 11 Let $\delta_t \in \partial \ell_t(z_t)$. The per-round regret with respect to any comparator $z^* \in Z$ can be bounded as
\[
\alpha_t \ell_t(z_t) - \alpha_t \ell_t(z^*) \leq \langle \alpha_t \delta_t, z_t - z^* \rangle
\]
\[
= \left( \frac{1}{\gamma} (\nabla \phi(z_{t+1}) - \nabla \phi(z_t)), z^* - z_{t+1} \right) + \langle \alpha_t \delta_t, z_t - z_{t+1} \rangle
\]
\[
+ \left( \frac{1}{\gamma} (\nabla \phi(z_t) - \nabla \phi(z_{t+1})), -\alpha_t \delta_t, z^* - z_{t+1} \right)
\]
\[
\leq \left( \frac{1}{\gamma} (\nabla \phi(z_{t+1}) - \nabla \phi(z_t)), z^* - z_{t+1} \right) + \langle \alpha_t \delta_t, z_t - z_{t+1} \rangle
\]
\[
= \frac{1}{\gamma} \left( D^\phi_{z_t}(z^*) - D^\phi_{z_{t+1}}(z^*) - D^\phi_{z_{t+1}}(z_t) \right) + \langle \alpha_t \delta_t, z_t - z_{t+1} \rangle
\]
\[
\leq \frac{1}{\gamma} \left( D^\phi_{z_t}(z^*) - D^\phi_{z_{t+1}}(z^*) \right) + \frac{\gamma}{2\beta} \|\alpha_t \delta_t\|^2_x,
\]
(B1)

where the second inequality uses the optimality of $z_{t+1}$ for $\min_{z \in Z} \alpha_t \ell_t(z) + \frac{1}{\gamma} D^\phi_{z_t}(z)$ and the last one uses that $\langle \alpha_t \nabla \ell_t(z_t), z_t - z_{t+1} \rangle \leq \frac{\gamma}{2\beta} \|\alpha_t \delta_t\|^2 + \frac{\beta}{2\gamma} \|z_t - z_{t+1}\|^2$ and that $\phi(\cdot)$ is $\beta$ strongly convex.

Summing (B1) from $t = 1$ to $T$ leads to the bounds. \hfill \qed

Proof of Lemma 14 The proof in the following is based on the proof in [25]. We let $\delta_t \in \partial \ell_t(z_t)$ in the following.

By applying Lemma 13 to the update
\[
z_t \leftarrow \arg \min_{z \in Z} \left( \alpha_t \langle z, m_t \rangle + \frac{1}{\gamma} D^\phi_{z_{t-\frac{1}{2}}}(z) \right),
\]
we have
\[
\alpha_t\langle m_t, z_t - z_{t+\frac{1}{2}} \rangle \leq \frac{1}{\gamma} \left( D^\phi_{z_{t-\frac{1}{2}}}(z_{t+\frac{1}{2}}) - D^\phi_{z_t}(z_{t+\frac{1}{2}}) - D^\phi_{z_{t-\frac{1}{2}}}(z_t) \right). \tag{B2}
\]
Also, by applying Lemma 13 to \(z_{t+\frac{1}{2}}\), we get
\[
\alpha t \langle \nabla \ell_t(z_t), z_{t+\frac{1}{2}} - z^* \rangle \leq \frac{1}{\gamma} \left( D^\phi_{z_t-\frac{1}{2}}(z^*) - D^\phi_{z_{t+\frac{1}{2}}}(z^*) - D^\phi_{z_{t-\frac{1}{2}}}(z_{t+\frac{1}{2}}) \right). \tag{B3}
\]
Then, we can bound the per-round regret with respect to the comparator \(z^*\) as follows:
\[
\alpha t \ell_t(z_t) - \alpha t \ell_t(z^*) \leq \alpha t \langle \delta t, z_t - z^* \rangle
\]
\[
= \alpha t \langle \delta t, z_{t+\frac{1}{2}} - z^* \rangle + \alpha t \langle m t, z_t - z_{t+\frac{1}{2}} \rangle + \alpha t \langle \delta t - m t, z_t - z_{t+\frac{1}{2}} \rangle
\]
\[
\leq \frac{1}{\gamma} \left( D^\phi_{z_{t-\frac{1}{2}}}(z^*) - D^\phi_{z_{t+\frac{1}{2}}}(z^*) - D^\phi_{z_{t-\frac{1}{2}}}(z_t) \right) + \alpha t \| \delta t - m t \|_\ast || z_t - z_{t+\frac{1}{2}} \|
\]
\[
\leq \frac{1}{\gamma} \left( D^\phi_{z_{t-\frac{1}{2}}}(z^*) - D^\phi_{z_{t+\frac{1}{2}}}(z^*) - \frac{\beta}{2} || z_t - z_{t+\frac{1}{2}} ||^2 - \frac{\beta}{2} || z_t - z_{t-\frac{1}{2}} ||^2 \right)
\]
\[
+ \frac{\alpha t^2}{2\beta} || \delta t - m t ||_\ast^2 + \frac{\beta}{2} || z_t - z_{t+\frac{1}{2}} ||^2,
\]
where the second inequality is due to (B2) and (B3) and the last one is by the \(\beta\)-strong convexity of \(\phi(\cdot)\).

Summing the above from \(t = 1\) to \(T\) and noting that \(z_{-\frac{1}{2}} = z_0 \in \mathcal{K}\), we obtain the bound.

\[\square\]

**Appendix C  Proof of Theorem 5**

Note that in the algorithm that we describe below the weights \(\alpha_t\) are not pre-defined but rather depend on the queries of the algorithm. These adaptive weights are explicitly defined in Algorithm 22 which is used by the \(y\)-player. Note that Algorithm 22 is equivalent to performing FTL updates over the following loss sequence: \(\{\ell_t(y) := \alpha_t \ell_t(y)\}_{t=1}^T\). The \(x\)-player plays best response, which only involves the linear optimization oracle.

---

**Algorithm 22  Adaptive Follow-the-Leader (AFTL)**

1. for \(t = 1, 2, \ldots, T\) do
2.    Play \(y_t \in \mathcal{Y}\)
3.    Receive a strongly convex loss function \(\alpha_t \ell_t(\cdot)\) with \(\alpha_t = \frac{1}{\| \nabla \ell_t(y_t) \|^2}\).
4.    Update \(y_{t+1} = \min_{y \in \mathcal{Y}} \sum_{s=1}^t \alpha_y \ell_s(y)\)
5. end for

**Proof** We can easily show the equivalence of the two displays shown on Algorithm 7 by using the induction and noting that \(\nabla \ell_t(y_t) = x_t - \nabla f^\ast(y_t) = x_t - \bar{x}_{t-1}\), where in the last equality we used that \(y_t = \nabla f(\bar{x}_{t-1})\) if and only if \(\bar{x}_{t-1} = \nabla f^\ast(y_t)\).

Now we switch to show the convergence rate. Since the \(x\)-player plays best response, \(\alpha \text{-REG}^\ast [\text{BESTRESP}^\ast] = 0\), we only need to show that \(\alpha \text{-REG}^\ast [\text{SC-AFTL}] = O(\exp(-\frac{\Delta B}{L} T))\), which we do next.
We start by defining a function $s(y) := \max_{x \in \mathcal{X}} -x^Ty + f^*(y)$, which is a strongly convex function. We are going to show that $s(\cdot)$ is also smooth. We have

$$\|\nabla_w s(\cdot) - \nabla_z s(\cdot)\| = \|\text{argmax}_{x \in \mathcal{X}} \{-w^T \cdot + f^*(w)\} - \text{argmax}_{x \in \mathcal{X}} \{-z^T \cdot + f^*(z)\}\|
$$

$$= \|\text{argmax}_{x \in \mathcal{X}} \{-w^T x\} - \text{argmax}_{x \in \mathcal{X}} \{-z^T x\}\| \leq \frac{2\|w - z\|}{\lambda(\|w\| + \|z\|)} \leq \frac{\|w - z\|}{\lambda B},$$

where the second to last inequality uses Lemma 18 regarding $\lambda$-strongly convex sets, and the last inequality is by assuming the gradient of $\|\nabla f(\cdot)\| \geq B$ and the fact that $w, z \in \mathcal{Y}$ are gradients of $f$. This shows that $s(\cdot)$ is a smooth function with smoothness constant $L' := \frac{1}{\lambda B}$.

$$T = \sum_{t=1}^{T} \frac{\|\nabla \ell_t(y_t)\|^2}{\|\nabla \hat{\ell}_t(y_t)\|^2} \overset{Proposition 25}{=} 25 \sum_{t=1}^{T} \frac{\|\nabla s(y_t)\|^2}{\|\nabla \hat{\ell}_t(y_t)\|^2} \overset{Lemma 15}{\leq} \sum_{t=1}^{T} \frac{L'}{\lambda B} (s(y_t) - s(y^*))$$

$$\leq \sum_{t=1}^{T} \frac{L'}{\lambda B} (\ell_t(y_t) - \ell_t(y^*)),
$$

where we denote $y^* := \text{argmin}_{y \in \mathcal{Y}} s(y)$ and the last inequality follows from the fact that $s(y_t) := \ell_t(y_t)$ and $\ell_t(y) = -g(x_t, y) \leq -g(x_t, y) = s(y)$ for any $y$.

In the following, we will denote $c$ a constant such that $\|\nabla \ell_t(y_t)\| = \|x_t - \nabla f^*(y_t)\| = \|x_t - \bar{x}_t - 1\| \leq c$. We have

$$T \leq \sum_{t=1}^{T} \frac{L'}{\lambda B} (\ell_t(y_t) - \ell_t(y^*))$$

$$\overset{(a)}{=} \sum_{t=1}^{T} L'(\hat{\ell}_t(y_t) - \hat{\ell}_t(y^*))$$

$$\overset{(b)}{\leq} \frac{L \cdot L'}{2} \sum_{t=1}^{T} \frac{\|\nabla \ell_t(y_t)\|^2}{\|\nabla \hat{\ell}_t(y_t)\|^2}$$

$$\overset{(c)}{\leq} \frac{L \cdot L'}{2} \left(1 + \log \left(c^2 \sum_{t=1}^{T} \|\nabla \ell_t(y_t)\|^2\right)\right),$$

where (a) is by the definition of $\hat{\ell}_t(\cdot)$, and (b) is shown using Lemma 4 with strong convexity parameter of $\ell_t(\cdot)$ being $\frac{1}{L}$, and (c) is by Lemma 19 so that

$$\sum_{t=1}^{T} \frac{\|\ell_t(y_t)\|^2}{\|\ell_s(y_s)\|^2} \leq \sum_{t=1}^{T} \frac{c^2 \|\ell_t(y_t)\|^2}{\|\ell_s(y_s)\|^2} \leq 1 + \log(c^2 \sum_{t=1}^{T} \|\ell_t(y_t)\|^2).$$

Thus, we get

$$c^2 \sum_{t=1}^{T} \|\nabla \ell_t(y_t)\|^2 = O(e^{\frac{1}{L'} T}) = O(e^{\frac{L}{L'} T}). \quad (C4)$$

So

$$\frac{\alpha-\text{Reg}^y}{A_T} := \sum_{t=1}^{T} \alpha_t (\ell_t(y_t) - \ell_t(y^*)) \leq \frac{L}{2A_T} \sum_{t=1}^{T} \|\nabla \ell_t(y_t)\|^2$$

$$\leq \frac{Lc^2 \left(1 + \log \left(c^2 \sum_{t=1}^{T} \|\nabla \ell_t(y_t)\|^2\right)\right)}{2c^2 \sum_{t=1}^{T} \|\nabla \ell_t(y_t)\|^2} \overset{(a)}{\leq} O(e^{\frac{1}{L'} T}) \overset{(b)}{\leq} o\left(O\left(e^{\frac{\lambda B T}{2L T'}}\right)\right) = O\left(Lc^2 e^{-\frac{\lambda B T}{2L T'}}\right),$$
where (a) is by Lemma 19, (b) is by (C4) and the fact that $\frac{1+\log z}{z}$ is monotonically decreasing for $z \geq 1$. Thus, we have
\[
f(w_T) - \min_{w \in \mathcal{K}} f(w) \leq \alpha^{\text{REG}}[\text{BESTRESP}^+] + \alpha^{\text{REG}}[\text{AFTL}] = O \left( Lc^2 e^{-\frac{\lambda T}{2}} \right).
\]
This completes the proof. □

**Proposition 25** For arbitrary $y$, let $\ell(\cdot) := -g(x_y, \cdot)$. Then $-\nabla_y \ell(\cdot) \in \partial y_s(\cdot)$, where $x_y$ means that the $x$-player plays $x$ by BESTRESP $^+$ after observing the $y$-player plays $y$.

**Proof** Consider any point $w \in \mathcal{Y}$,
\[
s(w) - s(y) = g(x_y, y) - g(x_w, w)
\]
\[
= g(x_y, y) - g(x_y, w) + g(x_y, w) - g(x_w, w) \geq g(x_y, y) - g(x_y, w) + 0
\]
\[
\geq \langle g_y(x_y, y), w - y \rangle = \langle -\nabla_y \ell(y), w - y \rangle,
\]
where the first inequality is because that $x_w$ is the best response to $w$, the second inequality is due to the concavity of $g(x_y, \cdot)$. The overall statement implies that $-\nabla_y \ell(y)$ is a subgradient of $s$ at $y$. □

**Lemma 18** Denote $x_p = \arg\max_{x \in \mathcal{K}} \langle p, x \rangle$ and $x_q = \arg\max_{x \in \mathcal{K}} \langle q, x \rangle$, where $p, q \in \mathbb{R}^d$ are any nonzero vectors. If a compact set $\mathcal{K}$ is a $\lambda$-strongly convex set, then
\[
\|x_p - x_q\| \leq \frac{2\|p - q\|}{\lambda (\|p\| + \|q\|)}.
\]

**Proof** [122] show that a strongly convex set $\mathcal{K}$ can be written as intersection of some Euclidean balls. Namely,
\[
\mathcal{K} = \bigcap_{u: \|u\|_2 = 1} B_{\frac{1}{\lambda}} \left( x_u - \frac{u}{\|u\|} \right),
\]
where $x_u$ is defined as $x_u = \arg\max_{x \in \mathcal{K}} \langle \frac{u}{\|u\|}, x \rangle$.

Let $x_p = \arg\max_{x \in \mathcal{K}} \langle \frac{p}{\|p\|}, x \rangle$ and $x_q = \arg\max_{x \in \mathcal{K}} \langle \frac{q}{\|q\|}, x \rangle$. Based on the definition of strongly convex sets, we can see that $x_q \in B_{\frac{1}{\lambda}} (x_p - \frac{p}{\lambda \|p\|})$ and $x_p \in B_{\frac{1}{\lambda}} (x_q - \frac{q}{\lambda \|q\|})$. Therefore,
\[
\left\| x_q - x_p - \frac{p}{\lambda \|p\|} \right\|^2 \leq \frac{1}{\lambda^2},
\]
which leads to
\[
\|p\| \cdot \|x_p - x_q\|^2 \leq \frac{2}{\lambda} \langle x_p - x_q, p \rangle. \tag{C5}
\]
Similarly,
\[
\left\| x_p - x_q - \frac{q}{\lambda \|q\|} \right\|^2 \leq \frac{1}{\lambda^2}.
\]

[4][122] discuss the smoothness of the support function on strongly convex sets. Here, we state a more general result.
which results in
\[ \|q\| \cdot \|x_p - x_q\|^2 \leq \frac{2}{\lambda} \langle x_q - x_p, q \rangle. \] (C6)

Summing (C5) and (C6), one gets \( (\|p\| + \|q\|) \|x_p - x_q\|^2 \leq \frac{2}{\lambda} \langle x_p - x_q, p - q \rangle \). Applying the Cauchy-Schwarz inequality completes the proof. \(\square\)

Lemma 19 ([84]) For any non-negative real numbers \(a_1, \ldots, a_n \geq 1\),

\[ \sum_{i=1}^{n} \frac{a_i}{\sum_{j=1}^{i} a_j} \leq 1 + \log \left( \sum_{i=1}^{n} a_i \right). \]

Appendix D Proof of Theorem 6

Proof The equivalence of the updates follows the proof of Theorem 3. Specifically, we have that the objects on the left in the following equalities correspond to Alg. 2 and those on the right to Alg. 8.

\[ x_t = v_t \]
\[ \bar{x}_t = w_t. \]

To analyze the regret of the \(y\)-player, we define \(\tilde{y}_t\) as the points if the \(y\)-player would have played FTL, which is

\[ \tilde{y}_t := \arg\min_{y \in \mathcal{Y}} \frac{1}{t-1} \sum_{s=1}^{t-1} \ell_t(y) = \arg\max_{y \in \mathcal{Y}} \frac{1}{t-1} \sum_{s=1}^{t-1} \langle x_s, y \rangle - f^*(y) \]

\[ = \nabla f(\bar{x}_t) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{x}_t). \]

On the other hand, we call the actual strategy used by the \(y\)-player LAYYFTL as it only compute the gradient of a single component. We can show the average regret of
LAZYFTL as follows.

\[
\overline{\alpha_{\text{Reg}}}(\gamma) = \frac{1}{T} \sum_{t=1}^{T} (\ell_t(\tilde{y}_t) - \ell_t(y_*) + \frac{1}{T} \sum_{t=1}^{T} (\ell_t(y_t) - \ell_t(\tilde{y}_t))

\leq \frac{4LR \log T}{T} + \frac{1}{T} \sum_{t=1}^{T} (\ell_t(y_t) - \ell_t(\tilde{y}_t))

= \frac{4LR \log T}{T} + \frac{1}{T} \sum_{t=1}^{T} (f^*(y_t) - f^*(\tilde{y}_t) + \langle x_t, \tilde{y}_t - y_t \rangle)

\leq \frac{4LR \log T}{T} + \sum_{t=1}^{T} \frac{1}{T} (L_0 + r) \|y_t - \tilde{y}_t\|

= \frac{4LR \log T}{T} + \frac{1}{T} (L_0 + r) \sum_{t=1}^{T} \frac{1}{n} \sum_{i=1}^{n} \|g_{i,t} - \nabla f_i(\bar{x}_t)\|

= \frac{4LR \log T}{T} + \frac{1}{T} (L_0 + r) \sum_{t=1}^{T} \frac{1}{n} \sum_{i \neq i_t}^{n} \|\bar{x}_{\tau_t(i)} - \bar{x}_t\|

\leq \frac{4LR \log T}{T} + \frac{L(L_0 + r)}{Tn} \sum_{t=1}^{T} \sum_{i \neq i_t}^{n} 2nr \frac{2nr}{t}

= O \left( \frac{\max\{LR, L(L_0 + r)r\} \log T}{T} \right),
\]

where (a) is due to the regret of FTL (Lemma 4),

\[
\frac{1}{T} \left( \sum_{t=1}^{T} \ell_t(\tilde{y}_t) - \ell_t(y_*) \right) \leq \frac{1}{T} \sum_{t=1}^{T} 2\|\nabla \ell_t(\tilde{y}_t)\|^2 = \frac{4LR \log T}{T},
\]

and that \(\|\nabla \ell_t(\tilde{y}_t)\|^2 = \|x_t - \nabla f^*(\tilde{y}_t)\|^2 = \|x_t - \tilde{x}_{t-1}\|^2 \leq R\), (b) we assume that the conjugate is \(L_0\) Lipschitz and that \(\max_{x \in \mathcal{K}} \|x\| \leq r\), (c) we denote \(\tau_t(i) \in [T]\) as the last iteration that \(i_{th}\) sample’s gradient is computed at \(t\), and (d) is because that

\[
\|\bar{x}_{\tau_t(i)} - \bar{x}_t\| = \left\| \frac{\tau_t(i)}{t} \sum_{s=1}^{t} x_s - \frac{1}{t} \sum_{s=1}^{t} x_s \right\| \leq \left\| \sum_{s=1}^{t} x_s \left( \frac{1}{\tau_t(i)} - \frac{1}{t} \right) \right\| + \left\| \frac{1}{t} \sum_{s=\tau_t(i)+1}^{t} x_s \right\|

= \frac{t - \tau_t(i)}{t} \|\bar{x}_{\tau_t(i)}\| + \frac{1}{t} \sum_{s=\tau_t(i)+1}^{t} x_s \leq \frac{nr}{t} + \frac{1}{t} \sum_{s=\tau_t(i)+1}^{t} x_s = \frac{nr}{t} + \frac{t - \tau_t(i)}{t} \left\| \frac{1}{t - \tau_t(i)} \sum_{s=\tau_t(i)+1}^{t} x_s \right\|

\leq \frac{2nr}{t}.
\]
For the $x$-player, since it plays $\text{BestResp}^+$, the regret is non-positive, i.e.

$$f(w_T) - \min_{w \in K} f(w) \leq \alpha\text{-Reg}^{x}[\text{BestResp}^+] + \alpha\text{-Reg}^{y}[\text{LazyFTL}]$$

$$= O\left(\max\{LR, L(L_0 + r)n\} \log T\right).$$

Appendix E  Proof of Theorem 12

Proof First, we can bound the norm of the gradient as

$$\|\nabla \ell_t(y_t)\|^2 = \|x_t - \nabla f^*(y_t)\|^2 = \|x_t - \bar{x}_{t-1}\|^2.$$

Combining this with Lemma 4 we see that

$$\alpha\text{-Reg}^{y}[\text{FTL}] \leq \frac{1}{A_T} \sum_{t=1}^{T} 2a_t^2 \|\nabla \ell_t(y_t)\|^2 = \frac{1}{A_T} \sum_{t=1}^{T} 2a_t^2 \|x_t - \bar{x}_{t-1}\|^2.$$

On the other hand, the $x$-player plays OMD$^+$, according to Lemma 12, its regret satisfies

$$\alpha\text{-Reg}^{x} \leq \frac{1}{A_T} D^\phi_{x_0}(x^*) \leq \frac{1}{A_T} 4L \|x_{t-1} - x_t\|^2.$$

By adding the average regret of both players, we get

$$\alpha\text{-Reg}^{y}[\text{FTL}] + \alpha\text{-Reg}^{x}[\text{OMD}^+] \leq \frac{1}{A_T} \left(4L \|x_0 - x^*\|^2 + \sum_{t=1}^{T} 4L \left(\|\bar{x}_{t-1} - x_t\|^2 - \|x_{t-1} - x_t\|^2\right)\right),$$

where we used that $D^\phi_{x_0}(x^*) = \frac{1}{2} \|x_0 - x^*\|^2$. Since the distance terms may not cancel out, one can only bound the differences of the distance terms by a constant, which leads to the non-accelerated $O(1/T)$ rate.

Proof The proof is similar to that of Theorem 14. The only difference is that the $x$-player is FTRL$^+$ instead of OMD$^+$. By using a bound on $\alpha\text{-Reg}^y$ in Lemma 16 and the bound in Lemma 8 of FTRL$^+$ with $\mu = 0$ (as the $x$-player sees linear loss functions), we have

$$f(w_T) - \min_{w \in K} f(w) \leq \frac{1}{A_T} (\alpha\text{-Reg}^{x}[\text{OptimisticFTL}] + \alpha\text{-Reg}^{y}[\text{FTRL}^+])$$

$$\leq \frac{1}{A_T} \left(R(x^*) - \min_{x \in K} R(x)\right) + \sum_{t=1}^{T} \left(\frac{\alpha_t}{A_t} L - \frac{\beta}{2\eta}\right) \|x_{t-1} - x_t\|^2.$$

Choosing $\eta = \frac{1}{4L}$, $\beta = 1$, and the weight $\alpha_t = t$, we have $A_t := t(t+1)$ and therefore $\frac{\alpha_t}{A_t} = \frac{2t^2}{t(t+1)} \leq 2$. With this in mind, the sum on the right hand side of (F7) is non-positive. Noticing that $\frac{1}{A_T} \leq \frac{2}{T^2}$ completes the proof.
Appendix G  Examples of strongly convex sets

[21] lists three known classes of \( \lambda \)-strongly convex sets as follows, which are all gauge sets.

1. \( \ell_p \) balls: \( \| x \|_p \leq r, \forall p \in (1,2] \). The strong convexity of the set is \( \lambda = \frac{p-1}{r} \) and its square of gauge function is \( \frac{1}{r} \| x \|_p^2 \), which is a \( \frac{2(p-1)}{r^2} \)-strongly convex function with respect to norm \( \| \cdot \|_p \) by Lemma 4 in [21].

2. Schatten \( p \) balls: \( \| \sigma(X) \|_p \leq r \) for \( p \in (1,2] \), where \( \sigma(X) \) is the vector consisting of singular values of the matrix \( X \). The strong convexity of the set is \( \lambda = \frac{p-1}{r} \) and the square gauge function is \( \frac{1}{r^2} \| \sigma(X) \|_p^2 \), which is a \( \frac{2(p-1)}{r^2} \)-strongly convex function with respect to norm \( \| \sigma(\cdot) \|_p \) by Lemma 6 in [21].

3. Group \( (s,p) \) balls: \( \| X \|_{s,p} = \| (\| X_1 \|_s, \| X_2 \|_s, \ldots, \| X_m \|_s) \|_p \leq r \) where \( X \in \mathbb{R}^{m \times n} \), \( X_j \) represents the \( j \)-th row of \( X \), and \( s, p \in (1,2] \). The strong convexity of the set is \( \lambda = \frac{2(s-1)(p-1)}{r((s-1)+(p-1))} \) and its square gauge function is \( \frac{1}{r^2} \| X \|_{s,p}^2 \), which is a \( \frac{2(s-1)(p-1)}{r^2((s-1)+(p-1))} \)-strongly convex function with respect to norm \( \| \cdot \|_{s,p} \) by Lemma 8 in [21].

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