Galois covers of \( \mathbb{P}^1 \) and number fields with large class groups\(^*\)

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Abstract

For each finite subgroup \( G \) of \( \text{PGL}_2(\mathbb{Q}) \), and for each integer \( n \) coprime to 6, we construct explicitly infinitely many Galois extensions of \( \mathbb{Q} \) with group \( G \) and whose ideal class group has \( n \)-rank at least \( \#G - 1 \). This gives new \( n \)-rank records for class groups of number fields.

1 Introduction

If \( M \) is a finite abelian group, and if \( m > 1 \) is an integer, we define the \( m \)-rank of \( M \) to be the maximal integer \( r \) such that \( (\mathbb{Z}/m\mathbb{Z})^r \) is a subgroup of \( M \); we denote it by \( \text{rk}_m(M) \). If \( K \) is a number field, we denote by \( \text{Cl}(K) \) the ideal class group of \( K \).

Our motivation for the present paper is the following conjecture on class groups of number fields, which belongs to folklore, and is a consequence of the Cohen-Lenstra heuristics.

Conjecture 1.1. Let \( d > 1 \) and \( n > 1 \) be two integers. Then \( \text{rk}_n(\text{Cl}(K)) \) is unbounded when \( K \) runs through the number fields of degree \( [K : \mathbb{Q}] = d \).

When \( n = d \), and more generally when \( n \) divides \( d \), this conjecture follows easily from class field theory \[ \text{Br}65, \text{RZ}69. \] On the other hand, when \( n \) and \( d \) are coprime, there is not a single case where Conjecture 1.1 is known to hold. For a survey of known results, see \[ \text{GL}20. \]

It was proved by Nakano \[ \text{Nak}84, \text{Nak}85 \] that, given \( n > 1 \) and \( (r_1, r_2) \in \mathbb{N}^2 \), there exist infinitely many number fields \( K \) with \( r_1 \) real places and \( r_2 \) complex places such that

\[ \text{rk}_n(\text{Cl}(K)) \geq r_2 + 1. \]

This is currently the best known result for general \( (r_1, r_2) \) and \( n \). To our knowledge, the only improvements to this general bound are for \( (r_1, r_2) = (3, 0) \) \[ \text{Nak}86 \], or for specific values of \( n \); in particular, a better bound was obtained by Levin \[ \text{Lev}07 \] in the case when \( d \geq n^2 \).

In the present paper, we improve on Nakano’s inequality in the case when

\[ (r_1, r_2) = (4, 0), (6, 0), (0, 3), (0, 4) \text{ and } (0, 6) \]

by constructing fields \( K \) which are in addition Galois extensions of \( \mathbb{Q} \). Our strategy is closely related to the “geometric” techniques developed in \[ \text{GL}12 \text{ and } \text{BG}18, \] an overview of whose is given in \[ \text{GL}20. \] The main new ingredient of our work is the construction of specific Galois covers of \( \mathbb{P}^1_\mathbb{Q} \) from finite subgroups of \( \text{PGL}_2(\mathbb{Q}) \).

Our main result is as follows:

\[^*\text{MSC classes: 11R29 (Primary) 11R16 (Secondary)}\]

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**Theorem 1.2.** Let $G \leq \text{PGL}_2(\mathbb{Q})$ be a finite subgroup. Let $n$ be an integer coprime to 6. Then there exist infinitely many isomorphism classes of number fields $K$ such that

1. $K/\mathbb{Q}$ is Galois with group $G$;
2. $\text{rk}_n \text{Cl}(K) \geq \#G - 1$.

The list of finite subgroups of $\text{PGL}_2(\mathbb{Q})$ is well known [Bea10, Prop. 1.1]:

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, D_2 = (\mathbb{Z}/2\mathbb{Z})^2, D_3 = \mathfrak{S}_3, D_4, D_6,$$  \hspace{1cm} (1)

where $D_r$ denotes the dihedral group of order $2r$.

The condition that $n$ is coprime to 6 in Theorem 1.2 can be relaxed depending on $G$. We sum up our results for each group $G$ in Table 1, with relevant references in the case of quadratic and cubic extensions, which were already known.

The proof of Theorem 1.2 follows from a general strategy, and ends up with a case-by-case analysis. For each finite subgroup $G \leq \text{PGL}_2(\mathbb{Q})$, we give an explicit family of fields satisfying the conclusions of Theorem 1.2, and we count isomorphism classes of these fields, ordered by discriminant.

Table 1: Table of $G$ and $n$ for which there exist infinitely many Galois extensions $K/\mathbb{Q}$ with group $G$ such that $\text{rk}_n \text{Cl}(K) \geq \#G - 1$. All these are current $n$-rank records, except for imaginary quadratic (resp. biquadratic) extensions, where the $n$-rank 2 (resp. 4) can be achieved, as was shown by Yamamoto [Yam70].

| $G$ | $n$     | signature | author(s)               |
|-----|---------|-----------|-------------------------|
| $\mathbb{Z}/2\mathbb{Z}$ | any     | imaginary | Ankeny-Chowla 1955 [AC55] |
|      | any     | real      | Yamamoto 1970 [Yam70]  |
|      |         |           | Weinberger 1973 [Wei73] |
| $\mathbb{Z}/3\mathbb{Z}$ | any     | real      | Nakano 1986 [Nak86]     |
| $\mathbb{Z}/4\mathbb{Z}$ | odd     | real      |                         |
| $\mathbb{Z}/6\mathbb{Z}$ | coprime to 6 | real |                         |
| $D_2 = (\mathbb{Z}/2\mathbb{Z})^2$ | any     | imaginary | Yamamoto 1970 [Yam70]  |
|      | any     | real      |                         |
| $D_3 = \mathfrak{S}_3$ | odd     | imaginary |                         |
| $D_4$ | coprime to 6 | imaginary |                         |
| $D_6$ | coprime to 6 | imaginary |                         |

**Example 1.3.** The splitting field of the polynomial

$$x^6 + 3x^5 + 24829767x^4 + 49659529x^3 + 24829767x^2 + 3x + 1$$

(obtained by specializing our polynomial $D_3 P$ in §4.4 at $y^n = 19^7$) is a totally imaginary Galois extension of $\mathbb{Q}$ with group $\mathfrak{S}_3$. One computes with Pari/GP [The19] that its ideal class group has 42-rank exactly 5.
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Finally, this work would not have seen the light of day without the use of the Pari/GP software [The19], which allowed us to carry out various numerical experiments.

2 Galois covers of $\mathbb{P}^1$ whose Picard group has large $n$-rank

Throughout this paper, by “curve” we mean a smooth projective geometrically irreducible curve defined over some field $k$.

Let $G$ be a finite group. A Galois cover of curves with group $G$ is a finite surjective morphism $\phi : C_1 \to C_2$ of curves, together with an action of $G$ on $C_1$ under which $\phi$ is invariant, such that $C_1 \to C_2$ is generically an étale $G$-torsor (i.e. $k(C_1)/k(C_2)$ is a Galois extension with group $G$).

Lemma 2.1. Let $k$ be a field. Let $G \leq \text{PGL}_2(k)$ be a finite subgroup. Let $a, b \in \mathbb{P}^1(k)$ be two points which do not lie in the same orbit under the action of $G$. Then the map

$$h : \mathbb{P}^1 \to \mathbb{P}^1; \quad x \mapsto \prod_{\sigma \in G} \frac{x - \sigma(a)}{x - \sigma(b)}$$

is a (geometrically connected) Galois cover with group $G$.

In the statement above, and in the rest of the paper, we fix an embedding $\mathbb{A}^1 \to \mathbb{P}^1$ and identify rational functions on $\mathbb{A}^1$ and on $\mathbb{P}^1$. In particular, given $x_0 \in k$, the function $x - x_0$ has a unique zero at $x_0$ and a unique pole at $\infty$. We extend this property by declaring that the expression $x - \infty$ is the constant function equal to 1. Thus, the rational function $h$ above is well-defined even if the orbit of $a$ or that of $b$ contains $\infty$.

If $G \leq \text{PGL}_2(k)$ is a finite subgroup and $x_0 \in \mathbb{P}^1(k)$ is a point, we denote by $\text{orb}(x_0)$ its orbit under the action of $G$, a slight abuse of notation.

Proof. We note that $h$ is a rational map whose divisor is

$$\text{div}(h) = \sum_{\sigma \in G} \sigma(a) - \sum_{\sigma \in G} \sigma(b)$$

Now, for any $\tau \in G$, $h \circ \tau$ is a rational map, and

$$\text{div}(h \circ \tau) = \tau^{-1}(\text{div}(h)) = \text{div}(h)$$

because $\text{div}(h)$ is invariant under the action of $\tau^{-1}$, which is an element of $G$. We know that there exists, up to a multiplicative constant, a unique rational map $\mathbb{P}^1 \to \mathbb{P}^1$ with given divisor. Therefore, there exist $\mu_\tau \in k^\times$ such that $h \circ \tau = \mu_\tau h$. But we know that $\tau$ has at least one fixed point $x_0$ in $\mathbb{P}^1(k)$. If this point $x_0$ is neither a zero nor a pole of $h$, then by evaluating $h \circ \tau$ at $x_0$ we see that $\mu_\tau = 1$. If $x_0$ is a zero or a pole of $h$, then we let $r := \text{ord}_{x_0}(h)$ and the same conclusion holds by evaluating at $x_0$ the equality

$$(x - x_0)^{-r} h(\tau(x)) = \mu_\tau(x - x_0)^{-r} h(x).$$

This proves that $h : \mathbb{P}^1 \to \mathbb{P}^1$ is invariant under the action of $G$. Moreover, $h$ is finite of degree $\#G$ by construction. Therefore, $h : \mathbb{P}^1 \to \mathbb{P}^1$ induces an isomorphism $\mathbb{P}^1 \simeq \mathbb{P}^1/G$ (quotient with
respect to the fppf topology). Moreover, \( G \) being finite, it acts freely on \( \mathbb{P}^1 \) outside finitely many points, hence \( h \) is generically a \( G \)-torsor. As \( G \) is constant, \( G \)-torsors for the étale and the fppf topology coincide. We conclude that \( h \) is a Galois cover with group \( G \).

**Remark 2.2.** The ramification points of \( h \) are exactly the points of \( \mathbb{P}^1 \) with non-trivial stabilizer under the action of \( G \).

**Remark 2.3.** Another way to state the conclusion of Lemma 2.1 is the following: the splitting field over \( k(t) \) of the polynomial

\[
R_t := \prod_{\sigma \in G} (x - \sigma(a)) - t \prod_{\sigma \in G} (x - \sigma(b))
\]

is a regular Galois extension of \( k(t) \) with group \( G \).

While constructions of Galois extensions of \( k(t) \) from finite subgroups of \( \text{PGL}_2(k) \) are classical [Ser92, Chap. 1], the way we achieve this from the orbits of two rational points does not seem to appear in the literature. It follows from [Ser92, §1.1] that, when \( G \cong \mathbb{Z}/3\mathbb{Z} \) (more generally, any group of odd order), or \( G \) is the subgroup induced by the standard representation \( S_3 \to \text{GL}_2(k) \), the fields obtained are generic, whatever the value of \((a,b)\) is. When \( G = \mathbb{Z}/4\mathbb{Z} \) and \( \sqrt{-1} \notin k \), there is no generic equation over \( k(t) \) [Ser92, §1.2], so our construction for different values of \((a,b)\) may lead to distinct families of fields.

Combining Lemma 2.1 with Hilbert’s irreducibility theorem yields the following:

**Corollary 2.4.** Assume \( k \) is a number field, and let \( G, a \) and \( b \) satisfying the assumptions of Lemma 2.1. Then for all \( m \in k \) outside a thin set, the polynomial \( R_m \) obtained by specialization of \((2)\) is irreducible, and its splitting field is a Galois extension of \( k \) with group \( G \). Moreover, the action of \( G \) on its roots is induced by the natural action of \( G \) on \( \mathbb{P}^1(\overline{\mathbb{Q}}) \).

**Remark 2.5.** A Galois extension of \( \mathbb{Q} \) is either totally real or totally complex. Therefore, when \( k = \mathbb{Q} \), the splitting field of \( R_m \) is totally real if and only if \( R_m \) has at least one real root.

**Lemma 2.6.** Let \( G, a \) and \( b \) satisfying the assumptions of Lemma 2.1. Let \( n > 1 \) be an integer which is coprime to the orders of the stabilizers of \( a \) and \( b \), and let \( \lambda \in k^\times \).

Then the polynomial

\[
\prod_{\sigma \in G} (x - \sigma(a)) - \lambda y^n \prod_{\sigma \in G} (x - \sigma(b))
\]

defines a geometrically irreducible curve \( C \) over \( k \), such that:

1. the \( y \)-coordinate map \( C \to \mathbb{P}^1 \) is a Galois cover with group \( G \);
2. the Picard group of \( C \) contains a subgroup isomorphic to \((\mathbb{Z}/n\mathbb{Z})\#\text{orb}(a) + \#\text{orb}(b) - 2\).

**Proof.** (1) Let \( h = h_{a,b} \) be the map from Lemma 2.1. Then \( C \) is the fiber product

\[
\begin{array}{ccc}
C & \xrightarrow{x} & \mathbb{P}^1 \\
y & \downarrow & \downarrow \lambda^{-1}h_{a,b} \\
\mathbb{P}^1 & \xrightarrow{z} & \mathbb{P}^1 \\
\end{array}
\]

According to Lemma 2.1, \( \lambda^{-1}h_{a,b} \) is a Galois cover with group \( G \). It follows that the same holds for the \( y \)-coordinate map \( C \to \mathbb{P}^1 \), which is obtained by pulling-back \( \lambda^{-1}h_{a,b} \).
(2) Considering \(h_{a,b}\) as an element of \(k(x)\), the equation of \(C\) is equivalent to
\[
y^n = \lambda^{-1} h_{a,b}
\]  

The linear factors at the numerator (resp. denominator) of \(h_{a,b}\) appear with multiplicity equal to the order of the stabilizer of \(a\) (resp. \(b\)), which by assumption are coprime to \(n\). It follows that the map \(x : C \to \mathbb{P}^1\) has degree \(n\) and is totally ramified above the set \(\text{orb}(a) \cup \text{orb}(b)\). In particular, \(C\) is geometrically irreducible.

Each of the points \(c \in \text{orb}(a) \cup \text{orb}(b)\) has a unique preimage by the map \(x\), which we denote by \(T_c\). It follows that
\[
\text{div}(x - c) = nT_c - nT_b\]
for all \(c \in \text{orb}(a) \cup \text{orb}(b)\).

This yields \(#\text{orb}(a) + #\text{orb}(b) - 1\) nonzero divisors \(T_c - T_b\) whose classes are \(n\)-torsion. On the other hand, it follows from (3) that
\[
\text{div}(y) = \sum_{\sigma \in G} T_{\sigma(a)} - T_{\sigma(b)}
\]
and this is the only nontrivial relation between the divisor classes \(T_c - T_b\). Therefore, these classes generate a subgroup of \(\text{Pic}(C)\) isomorphic to \((\mathbb{Z}/n\mathbb{Z})^{#\text{orb}(a) + #\text{orb}(b) - 2}\).

**Remark 2.7.** Under the assumptions of Lemma 2.6, if the orbit of \(b\) contains \(\infty\), then for all \(c \in \text{orb}(a) \cup \text{orb}(b)\), we have \(\text{div}(x - c) = nT_c - nT_\infty\), where \(T_c\) and \(T_\infty\) are defined in the proof of Lemma 2.6.

**Example 2.8.** Let \(q\) be a power of some prime number. Then for any \(r \mid q - 1\), the group \(\mu_r(\mathbb{F}_q)\) acts on \(\mathbb{P}^1\) by multiplication, an action which stabilizes 0 and \(\infty\). If in addition \(r \neq q - 1\), then there exists two points \(a\) and \(b\) in \(\mathbb{F}_q^\times\) which have distinct orbits and trivial stabilizer, hence Lemma 2.6 gives an explicit cyclic degree \(r\) Galois cover \(C \to \mathbb{P}^1\) defined over \(\mathbb{F}_q\) such that
\[
\text{rk}_n \text{Pic}(C) \geq 2r - 2.
\]

### 3 Arithmetic specialization

If \(K\) is a number field, we define
\[
\text{Sel}^n(K) := \{ \gamma \in K^\times/(K^\times)^n; \forall v \text{ finite place of } K, v(\gamma) \equiv 0 \pmod{n} \}
\]
which is an analogue of the Selmer group for the multiplicative group over \(K\). Then we have an exact sequence
\[
1 \longrightarrow \mathcal{O}_K^\times/(\mathcal{O}_K^\times)^n \longrightarrow \text{Sel}^n(K) \longrightarrow \text{Cl}(K)[n] \longrightarrow 0.
\]

In fact, \(\text{Sel}^n(K)\) is none other than the flat cohomology group \(H^1_{\text{fp}}(\text{Spec}(\mathcal{O}_K), \mu_n)\), and the exact sequence above is the Kummer exact sequence in flat cohomology.

According to [GG19 Prop. 1.1], this exact sequence always splits. Therefore, we are able to deduce from [GL12 Lemma 2.6] that
\[
\text{rk}_n \text{Sel}^n(K) = \text{rk}_n \mathcal{O}_K^\times/(\mathcal{O}_K^\times)^n + \text{rk}_n \text{Cl}(K)[n].
\]

The following statement is a variant of [GL12 Theorem 2.4]. It is obtained by combining a quantitative version of Hilbert’s irreducibility theorem, due to Cohen [Coh81], with the equality (6) above. See also [GL20 Theorem 2.7].
Theorem 3.1. Let $C$ be a smooth projective geometrically irreducible curve over $\mathbb{Q}$, and let $n > 1$ be an integer. Let $D_1, \ldots, D_s$ be divisors on $C$ whose classes in $\text{Pic}(C)$ generate a subgroup isomorphic to $(\mathbb{Z}/n\mathbb{Z})^s$, and let $g_1, \ldots, g_s$ be rational functions on $C$ such that $\text{div}(g_i) = nD_i$ for all $i$. Assume that there exists a finite map $\phi : C \to \mathbb{P}^1$ of degree $d > 1$ such that, for all $t \in \mathbb{N}$, the point $P_t := \phi^{-1}(t)$ has the property that

$$g_1(P_t), \ldots, g_s(P_t) \text{ define classes in } \text{Sel}^n(\mathbb{Q}(P_t)), \quad (SC)$$

where $\text{Sel}^n$ is defined in [5]. Then for all but $O(\sqrt{N})$ values $t \in \{1, \ldots, N\}$, the field $\mathbb{Q}(P_t)$ satisfies $[\mathbb{Q}(P_t) : \mathbb{Q}] = d$ and

$$\text{rk}_n \text{Cl}(\mathbb{Q}(P_t)) \geq s - \text{rk}_Z O^x_{\mathbb{Q}(P_t)}.$$ 

Moreover, there are infinitely many isomorphism classes of such fields $\mathbb{Q}(P_t)$.

Let us mention quantitative versions of the last statement: it was proved by Dvornicich and Zannier [DZ94] that, for $N$ large enough, the number of isomorphism classes in the set

$$\{\mathbb{Q}(P_1), \ldots, \mathbb{Q}(P_N)\}$$

is $\gg N/\log N$, where the implicit constant depends on $\phi$ and $C$. Moreover, if the map $\phi$ has at least three distinct zeroes in $C(\overline{\mathbb{Q}})$, then by a result of Corvaja and Zannier [CZ03 Corollary 1], for $N$ large enough, the number of isomorphism classes is $\gg N$ (here, we use Vinogradov’s notation: $g \gg f$ means that $f = O(g)$ when $x \to +\infty$). For more details, we refer the reader to the introduction of [BL17].

3.1 Strategy of proof of Theorem 1.2

Our case-by-case strategy for proving Theorem 1.2 is as follows: for each group $G$ in the list [11], and for each suitable integer $n$, we shall explicitly give

1. one (or two) homographies which generate a subgroup isomorphic to $G$ in $\text{PGL}_2(\mathbb{Q})$;
2. two points $a, b \in \mathbb{P}^1(\mathbb{Q})$ satisfying the assumptions of Lemma 2.6 and such that the orbit of $b$ contains $\infty$;
3. a corresponding polynomial defining a curve $C$ as in Lemma 2.6 (i.e. a choice of $\lambda$);
4. a set of rational functions $g_1, \ldots, g_s$ ($s = \# \text{orb}(a) + \# \text{orb}(b) - 2$) on $C$ with $\text{div}(g_i) = nD_i$, such that the divisors $D_i$ generate a subgroup of $\text{Pic}(C)$ isomorphic to $(\mathbb{Z}/n\mathbb{Z})^s$.
5. a congruence condition on $y \in \mathbb{Z}$ such that the corresponding point $P_y$ on the curve $C$ satisfies the condition (SC).

Then one can construct from the congruence condition $y \equiv y_0 \pmod{N}$ a map $\phi = \frac{y - y_0}{N}$ which satisfies the hypotheses of Theorem 3.1. By Lemma 2.6 the map $y : C \to \mathbb{P}^1$ is a Galois cover with group $G$, hence the same holds for the map $\phi$.

It then follows from Theorem 3.1 that, for “most” values of $y$ satisfying the congruence condition, the field $\mathbb{Q}(P_y)$ is a Galois extension of $\mathbb{Q}$ with group $G$, and satisfies

$$\text{rk}_n \text{Cl}(\mathbb{Q}(P_y)) \geq s - \text{rk}_Z O^x_{\mathbb{Q}(P_y)}.$$ 

Finally, it suffices to determine the signature of the field $\mathbb{Q}(P_y)$ in order to deduce, by Dirichlet’s unit theorem, an explicit lower bound on its $n$-rank.
3.2 Detailed outline of the proof

Let us explain in more detail how each of the items in the general strategy is achieved in practice. We do not include the cases of \( \mathbb{Z}/2\mathbb{Z} \) and \( D_2 = (\mathbb{Z}/2\mathbb{Z})^2 \) in this general discussion, since these are slightly different. We shall treat them in the case-by-case analysis instead.

3.2.1 Choice of the homographies

We make, once and for all, the same choice as [Bea10, Prop. 1.1]. Let us fix \( r \geq 3 \), and let \( \zeta \) be a primitive \( r \)-th root of 1, then the homography

\[
f : z \mapsto \frac{(\zeta + \bar{\zeta} + 1)z - 1}{z + 1}
\]

is an element of order \( r \) in \( \text{PGL}_2(\mathbb{C}) \). Together with \( z \mapsto \frac{1}{z} \), it generates a subgroup of \( \text{PGL}_2(\mathbb{C}) \) isomorphic to \( D_r \). It is clear that, for \( r = 3, 4, 6 \), these homographies are defined over \( \mathbb{Q} \).

We note that \( f \) satisfies \( f(0) = -1 \) and \( f(-1) = \infty \).

**Lemma 3.2.** Let \( r \geq 3 \), and let \( G = \langle f \rangle \cong \mathbb{Z}/r\mathbb{Z} \) or \( G = \langle f, z \mapsto \frac{1}{z} \rangle \cong D_r \), where \( f \in \text{PGL}_2(\mathbb{C}) \) is defined above. Then for all \( b \in \text{orb}(\infty) \cap \mathbb{C} \) the rational map

\[
\varphi : z \mapsto \prod_{\sigma \in G} (\sigma(z) - b)
\]

is constant. In particular,

\[
\prod_{\sigma \in G} \sigma(z) = 1 \quad \text{and} \quad \prod_{\sigma \in G} (\sigma(z) + 1) = (\zeta + 1)^{\#G}.
\]

**Proof.** In fact, the first statement holds for any finite subgroup \( G \) of \( \text{PGL}_2(\mathbb{C}) \). Indeed, let \( b \in \text{orb}(\infty) \cap \mathbb{C} \) and let \( \varphi \) be the rational map defined above; then the divisor of \( \varphi \) is given by

\[
\text{div}(\varphi) = \sum_{\sigma \in G} (\sigma - b)^{-1}(0) - \sum_{\sigma \in G} (\sigma - b)^{-1}(\infty) = \sum_{\sigma \in G} \sigma^{-1}(b) - \sum_{\sigma \in G} \sigma^{-1}(\infty) = 0
\]

where the last equality holds because \( b \) and \( \infty \) have the same orbit. Thus, \( \varphi \) is a constant map. If we let \( b = 0 \) and \( b = -1 \) respectively, which belong to the orbit of \( \infty \), we obtain that the two quantities in \((\mathbb{S})\) are constants. In the case when \( G = \langle f \rangle \), these constants can be computed by observing that \( f(\zeta) = \zeta \), hence \( \zeta \) is a fixed point of \( G \). In the case when \( G = \langle f, z \mapsto 1/z \rangle \), we observe that

\[
G = \{ f^s, 1/f^s \mid s = 1, \ldots, r \}
\]

hence first equality in \((\mathbb{S})\) follows immediately from the previous case. The second one can be worked out explicitly:

\[
\prod_{\sigma \in G} (\sigma(\zeta) + 1) = (\zeta + 1)^r(\zeta^{-1} + 1)^r = (\zeta + 1)^r(\zeta + 1)^r \zeta^{-r} = (\zeta + 1)^{2r},
\]

hence the result. \( \square \)

**Remark 3.3.** A non-constant homography can be determined from its zero, its pole, and its value at 1. By applying this to the powers of our homography \( f \), which satisfies \( f^2(0) = \infty \), we deduce that for any integer \( s \) there exists a constant \( \mu_s \neq 0 \) such that

\[
f^s(z) = \mu_s \cdot \frac{z - f^{-s}(0)}{z - f^{2-s}(0)}.
\]
Remark 3.4. We note that (7) can be defined by a matrix of determinant \( \zeta + \bar{\zeta} + 2 \). Therefore, for any integer \( m \) coprime to
\[
\zeta + \bar{\zeta} + 2 = \begin{cases} 
1 & \text{if } r = 3 \\
2 & \text{if } r = 4 \\
3 & \text{if } r = 6 
\end{cases}
\]
the homography (7) reduces into a non-constant homography of the projective line over \( \mathbb{Z}/m\mathbb{Z} \). In other terms, it has good reduction outside \( \zeta + \bar{\zeta} + 2 \).

3.2.2 Choice of the points \( a \) and \( b \)

We put \( b = 0 \), whose orbit under \( G \) contains infinity, as noted above. We denote by \( \omega \) the order of the stabilizer of 0 under the action of \( G \), namely:
\[
\omega = \begin{cases} 
1 & \text{if } G \text{ is cyclic} \\
2 & \text{if } G \text{ is dihedral.}
\end{cases}
\]

In order to maximize the Picard group of \( C \), we choose \( a \) such that \( \# \text{orb}(a) = \#G \), in other terms its stabilizer under the action of \( G \) is trivial. Apart from that, the choice of \( a \) is far from being canonical, although we aim to choose the “smallest” integer which is not in the orbit of 0. This choice has an impact on the rest of the process, in particular the congruence conditions on \( y^n \) (hence the conditions on \( n \)) depend on it.

3.2.3 Choice of the curve

We note that the polynomial \( \prod_{\sigma \in G} (x - \sigma(a)) \) is monic of degree \( \#G \), with constant coefficient \( (-1)^{\#G} \) by (8), and the polynomial \( \prod_{\sigma \in G} (x - \sigma(0)) \) has degree \( \#G - \omega \), and has zero constant coefficient. Therefore, given \( \lambda \in \mathbb{Q}^\times \), the polynomial (in the variable \( x \))
\[
\prod_{\sigma \in G} (x - \sigma(a)) - \lambda y^n \prod_{\sigma \in G} (x - \sigma(0)) \quad (9)
\]
is monic, with constant coefficient \( (-1)^{\#G} \).

We choose \( \lambda \) in order to rescale the polynomials \( \prod_{\sigma \in G} (x - \sigma(a)) \) and \( \prod_{\sigma \in G} (x - \sigma(0)) \) into polynomials with integral coefficients, whose gcd is 1, and whose leading coefficient is positive (in other terms, polynomials with content 1). So, given \( a \), our choice of a curve \( C \) as in Lemma 2.6 is canonical.

More precisely, let us write as irreducible fractions \( \beta_i/\alpha_i \) the elements of \( \text{orb}(a) \), which by construction are rational numbers (we choose here \( \alpha_i > 0 \), hence the fraction is unique). Similarly we denote by \( \delta_j/\gamma_j \) the elements of \( \text{orb}(0) \setminus \{\infty\} \). We note that \( \prod_i \alpha_i = \prod_i \beta_i \) according to (8). Then we put
\[
\lambda := - \left( \prod_j \gamma_j \right)^{\omega} \times \left( \prod_i \alpha_i \right)^{-1}.
\]

Given our choice of \( \lambda \), when multiplying (9) by \( \prod_i \alpha_i \), we obtain the polynomial
\[
\prod_{i=1}^{\# \text{orb}(a)} (\alpha_i x - \beta_i) + y^n \prod_{j=1}^{\# \text{orb}(0) - 1} (\gamma_j x - \delta_j)^\omega \quad (10)
\]
We note that the second product has \( \# \text{orb}(0) - 1 \) factors because the point at infinity corresponds (by convention) to the factor 1, hence has been removed from the list.

If \( y \in \mathbb{Z} \) is chosen in such a way that the polynomial \( (\ref{eq:prime}) \) has integral coefficients (equivalently, the polynomial \( (\ref{eq:polynomial}) \) is zero modulo \( \prod_i \alpha_i \)), then its roots are algebraic units. This is our first assumption.

**Assumption (A1).** The integer \( y \in \mathbb{Z} \) has the property that the polynomial \( (\ref{eq:prime}) \) has integral coefficients.

**Lemma 3.5.** If \( G \) is cyclic, the condition

\[
y^n \equiv \lambda^{-1} \sum_{\sigma \in G} \sigma(a) \pmod{\prod \alpha_i}
\]

implies assumption (A1). Likewise, if \( G \) is dihedral, the condition

\[
y^n \equiv -\lambda^{-1} \sum_{(\sigma, \tau) \in G^2} \sigma(a)\tau(a) \pmod{\prod \alpha_i}
\]

implies assumption (A1).

**Proof.** Assume \( G = \langle f \rangle \) is cyclic. By construction, the equation of \( C \) can be written as

\[
y^n = -\frac{\prod_{i=1}^{\# \text{orb}(a)} (\alpha_i x - \beta_i)}{\prod_{j=1}^{\# \text{orb}(0) - 1} (\gamma_j x - \delta_j)}.
\]

Let \( \prod \alpha_i = \prod \beta_i = (\zeta + \bar{\zeta} + 2)^v m \), where \( m \) is coprime to \( (\zeta + \bar{\zeta} + 2) \), then by Remark 3.4 the reduction of \( f \) modulo \( m \) is an automorphism of the projective line over \( \mathbb{Z}/m\mathbb{Z} \), hence by reducing the equation above modulo \( m \) we obtain on the right-hand side a rational map whose zeroes and poles are \( G \)-invariant. But its numerator vanishes at zero, since its constant term is \( \pm \prod \beta_i \). It follows that this rational map is constant modulo \( m \), which is equal to \( \prod \alpha_i \) when \( r = 3 \).

On the other hand, given a homography \( \varphi \) and an irreducible fraction \( \beta/\alpha \), an elementary computation shows that \( \varphi(\beta/\alpha) \equiv \varphi(0) \pmod{\beta} \), including the case when \( \varphi(0) = \infty \). It follows immediately from this observation that the orbit of \( a \) and that of 0 coincide after reduction modulo any of the \( \beta_i \). When \( r = 4 \) we note that there is exactly one element of the orbit of \( a \) which reduces to 0 modulo \( \beta_i \), because the other elements in the orbit are \( \infty \) and \( \pm 1 \). It follows that the \( \beta_i \) are coprime, hence the Chinese remainder Theorem implies that our rational map is constant modulo \( \prod \beta_i \). When \( r = 6 \), the orbit of 0 is \( \{0, \infty, \pm 1, 2, 1/2\} \), so there are two points which reduce to zero modulo \( 2 = \zeta + \bar{\zeta} + 1 \), and 2 is the only prime for which this happens. But 2 is coprime to \( 3 = (\zeta + \bar{\zeta} + 2) \), hence 2 \( \mid m \) and the first part of our proof implies the result.

We have just proved that our map is constant modulo \( \prod \alpha_i \), hence assumption (A1) is satisfied whenever \( y^n \) is equal to this constant modulo \( \prod \alpha_i \). By construction, we have

\[
-\frac{\prod_{i=1}^{\# \text{orb}(a)} (\alpha_i x - \beta_i)}{\prod_{j=1}^{\# \text{orb}(0) - 1} (\gamma_j x - \delta_j)} = -\lambda^{-1} \left( \frac{x^r - (\sum_{\sigma \in G} \sigma(a))x^{r-1} + \ldots}{x^{r-1} + \ldots} \right)
\]

hence the constant is \( \lambda^{-1} \sum_{\sigma \in G} \sigma(a) \).

The dihedral case follows by a similar argument, with an additional grain of salt: the denominator of the rational map, corresponding to the orbit of 0, is the square of the previous one. Therefore, the condition obtained is on the coefficient of degree \( 2r - 2 \) of the numerator of the rational map. \( \square \)
Remark 3.6. Our proof includes a case-by-case analysis for the reader’s convenience, but it clearly holds for general values of \( r \), replacing \( \mathbb{Z} \) by the ring of integers of \( \mathbb{Q}(\zeta + \bar{\zeta}) \).

3.2.4 Choice of the functions

According to Remark 2.7, the functions \( x - c \), where \( c \) runs through \( \text{orb}(a) \cup \text{orb}(0) \), are natural building blocks for the maps \( g_i \) in the sense that any product or quotient of such functions has a divisor which is a multiple of \( n \), and the only nontrivial relation between the corresponding divisor classes comes from \( (\bar{\zeta} + 1) \).

Since \( b = 0 \), the function \( x \) is a natural candidate. In fact, under assumption (A1) from 3.2.3 the value of \( x \) at \( P_y \) is an algebraic unit, hence defines a class in the Selmer group. Clearly, the same holds for all other Galois conjugates of \( x \), which are:

\[
\begin{align*}
f(x) &= \frac{(\zeta + \bar{\zeta} + 1)x - 1}{x + 1} \\
f^2(x) &= \frac{(\zeta + \bar{\zeta})x - 1}{x} \\
f^3(x) &= \frac{(\zeta^2 + \bar{\zeta}^2 + \zeta + \bar{\zeta} + 1)x - (\zeta + \bar{\zeta} + 1)}{(\zeta + \bar{\zeta} + 1)x - 1} \\
f^4(x) &= \frac{(\zeta^2 + \bar{\zeta}^2 + 1)x - (\zeta + \bar{\zeta})}{(\zeta + \bar{\zeta})x - 1} \\
f^5(x) &= \frac{(\zeta^3 + \bar{\zeta}^3 + \zeta^2 + \bar{\zeta}^2 + \zeta + \bar{\zeta} + 1)x - (\zeta^2 + \bar{\zeta}^2 + \zeta + \bar{\zeta} + 1)}{(\zeta^2 + \bar{\zeta}^2 + \zeta + \bar{\zeta} + 1)x - (\zeta + \bar{\zeta} + 1)}
\end{align*}
\]

For \( r \leq 6 \), this list is complete (and it is redundant if \( r \leq 5 \)). We note that the numerators and denominators of these rational maps are of the form \( \gamma_j x - \delta_j \), where \( \delta_j/\gamma_j \) runs through \( \text{orb}(0) \) (notation from 3.2.3), as it was pointed in Remark 3.3.

It follows from the computation above that, when \( \delta_j/\gamma_j \) runs through \( \text{orb}(0) \), then either \( \gamma_j x - \delta_j \) or \( (\gamma_j x - \delta_j)/(x + 1) \) is amongst the functions

\[
x, \quad f(x), \quad xf^2(x), \quad f(x)f^3(x), \quad xf^2(x)f^4(x), \quad f(x)f^3(x)f^5(x).
\]

Under assumption (A1), the values of these functions at \( P_y \) are units. These functions will be the first elements in our list, which we denote by \( \varphi_j \) for \( j = 1, \ldots, \# \text{orb}(0) - 2 \). When \( r = 3 \), we let \( \varphi_1 = x \) and \( \varphi_2 = x + 1 \), which, according to the following lemma, both specialize into units under assumption (A1).

Lemma 3.7. Let \( x := x(P_y) \), then under assumption (A1) the algebraic integer \( x + 1 \) divides \( \zeta + \bar{\zeta} + 2 \).

Proof. Indeed, \( f(x) \) being a unit, \( x + 1 \) divides \( (\zeta + \bar{\zeta} + 1)x - 1 \), and it follows that \( x + 1 \) divides \( (\zeta + \bar{\zeta} + 1)(x + 1) - (\zeta + \bar{\zeta} + 1)x + 1 = \zeta + \bar{\zeta} + 2 \). \( \square \)

Lemma 3.8. Assume \( r = 4, 6 \). Then for all \( c \in \mathbb{Q} \), exactly half of the elements of \( \text{orb}(c) \) reduce to \(-1\) modulo \( \zeta + \bar{\zeta} + 2 \).

Proof. Under these assumptions, \( \zeta + \bar{\zeta} + 2 \) is a prime number. Since the map \( z \mapsto 1/z \) is well-defined modulo \( \zeta + \bar{\zeta} + 2 \) and fixes \(-1\), it is enough to consider the cyclic case. We first observe that

\[
f(z) = \frac{(\zeta + \bar{\zeta} + 2)z}{z + 1} - 1.
\]
Let $\beta/\alpha$ be an element of $\text{orb}(c)$, where $\alpha$ and $\beta$ are coprime integers. Then $\beta/\alpha$ reduces to $-1$ modulo $\zeta + \bar{\zeta} + 2$ if and only if $\zeta + \bar{\zeta} + 2$ divides $\alpha + \beta$. Now, according to the previous formula, we have

$$f(\alpha/\beta) = \frac{(\zeta + \bar{\zeta} + 2)\beta}{\alpha + \beta} - 1.$$  

Assume that $\beta/\alpha$ does not reduce to $-1$. Then the denominator $\alpha + \beta$ does not reduce to zero, hence $f(\alpha/\beta)$ reduces to $-1$. On the other hand, if $\beta/\alpha$ reduces to $-1$, then the denominator $\alpha + \beta$ reduces to zero, and $\beta$ does not (because $\beta \neq 0$ and $\gcd(\alpha, \beta) = 1$). So, the quantity $\frac{(\zeta + \bar{\zeta} + 2)\beta}{\alpha + \beta}$ does not reduce to zero, hence $f(\alpha/\beta)$ does not reduce to $-1$. This proves that exactly one out of two elements in the orbit of $c$ reduces to $-1$. 

We shall now give the key ingredient of our proof: the equation of our curve $C$ being equivalent to the vanishing of (10), the following relation holds in the function field of $C$

$$\frac{\# \text{orb}(a)}{\# \text{orb}(0)} \prod_{i=1}^{\# \text{orb}(a)} (\alpha_i x - \beta_i) = -y^n \prod_{j=1}^{\# \text{orb}(0) - 2} (\gamma_j x - \delta_j)$$  

where the $\beta_i/\alpha_i$ run through $\text{orb}(a)$ (notation from §3.2.3). We observe that $(x + 1)$ is amongst the factors on the right-hand side, since $-1$ belongs to the orbit of $0$.

If $r = 3$ we keep this relation unchanged. If $r = 4, 6$, we let

$$\varepsilon_i := \begin{cases} 1 & \text{if } \zeta + \bar{\zeta} + 2 \text{ divides } \alpha_i + \beta_i \\ 0 & \text{otherwise} \end{cases}$$

We claim that exactly half of the $\varepsilon_i$ are equal to one. Indeed, $\zeta + \bar{\zeta} + 2$ divides $\alpha_i + \beta_i$ if and only if $\beta_i/\alpha_i$ reduces to $-1$ modulo $\zeta + \bar{\zeta} + 2$, and, according to Lemma 3.8, exactly half of the elements of the orbit of $a$ have this property.

Dividing both sides of (11) by $(x + 1)^{\#G/2}$, we obtain the equality

$$\frac{\# \text{orb}(a)}{\# \text{orb}(0)} \prod_{i=1}^{\# \text{orb}(a)} (\alpha_i x - \beta_i) = -y^n \prod_{j=1}^{\# \text{orb}(0) - 2} \phi_j$$  

We denote by $\psi_i$ the functions

$$\psi_i := \frac{(\alpha_i x - \beta_i)}{(x + 1)^{\varepsilon_i}} \text{ for } i = 1, \ldots, \# \text{orb}(a).$$

Then the values of these functions at $P_y$ are algebraic integers: if $\varepsilon_i = 0$ this is trivial; if $\varepsilon_i = 1$ one can write

$$\frac{(\alpha_i x - \beta_i)}{(x + 1)} = \alpha_i - \frac{(\alpha_i + \beta_i)}{(x + 1)}$$

and the right-hand side is an algebraic integer because $x + 1$ divides $\zeta + \bar{\zeta} + 2$ which itself divides $\alpha_i + \beta_i$ under our assumptions.

The functions $\psi_i$ being defined for $r = 4, 6$, the equation (12) becomes

$$\prod_{i=1}^{\# \text{orb}(a)} \psi_i = -y^n \prod_{j=1}^{\# \text{orb}(0) - 2} \phi_j$$  

11
By construction, the $\varphi_j(P_y)$ are units and the $\psi_i(P_y)$ are algebraic integers. We shall prove that, under some additional condition on $y$, the $\psi_i(P_y)$ are coprime. If this holds then, by unique factorization into prime ideals in the ring of integers of $\mathbb{Q}(P_y)$, since the valuations of the right-hand side are multiples of $n$, the same holds for each of the factors $\psi_i(P_y)$ of the left-hand side, in other terms, the Selmer property is satisfied by the $\psi_i$.

If $r = 3$, we let $\psi_i = \alpha_i x - \beta_i$ and consider the equation (11) instead, which reads

$$\prod_{i=1}^{\#\text{orb}(a)} \psi_i = -y^n \varphi_1^s \varphi_2^s$$

(14)

The same reasoning as above applies to this situation as well.

**Lemma 3.9.** Let $(\alpha, \beta)$ and $(\gamma, \delta)$ be two distinct pairs of coprime integers in $\mathbb{N}^* \times \mathbb{Z}$. Then for any algebraic integer $x$, the ideal $(\alpha x - \beta, \gamma x - \delta)$ divides

$$\frac{\alpha \delta - \beta \gamma}{\gcd(\alpha, \gamma)}.$$

**Proof.** The identity

$$\frac{\gamma}{\gcd(\alpha, \gamma)}(\alpha x - \beta) - \frac{\alpha}{\gcd(\alpha, \gamma)}(\gamma x - \delta) = \frac{\alpha \delta - \beta \gamma}{\gcd(\alpha, \gamma)}$$

proves that the desired quantity belongs to the ideal $(\alpha x - \beta, \gamma x - \delta)$, hence the result. \hfill \Box

**Remark 3.10.** The above Lemma is generically optimal in the sense that, under its assumptions, the ideal $(\alpha X - \beta, \gamma X - \delta)$ of the polynomial ring $\mathbb{Z}[X]$ satisfies

$$(\alpha X - \beta, \gamma X - \delta) \cap \mathbb{Z} = \left( \frac{\alpha \delta - \beta \gamma}{\gcd(\alpha, \gamma)} \right).$$

So, we require $y$ to be coprime to all the quantities obtained from the above lemma:

**Assumption (A2).** The integer $y$ is coprime to

$$\prod_{i \neq j} \frac{\alpha_i \beta_j - \beta_i \alpha_j}{\gcd(\alpha_i, \alpha_j)}$$

(15)

where $\beta_i/\alpha_i$ run through $\text{orb}(a)$.

At the end of the day, we obtain a list of rational functions $\varphi_j, \psi_i$, of the form

$$\frac{\alpha x - \beta}{x+1}$$

where $\alpha, \beta$ are coprime integers such that $\beta/\alpha$ belongs to $\text{orb}(a) \cup \text{orb}(0)$. If $r = 4, 6$, the number of these functions is $\# \text{orb}(a) + \# \text{orb}(0) - 2$, and if $r = 3$, the number of functions is $\# \text{orb}(a) + 2 = \# \text{orb}(a) + \# \text{orb}(0) - 1$.

In both cases, these functions are not multiplicatively independent, because they satisfy the non-trivial relation (13) (resp. (14)), so in order to find an independent set we remove one of these functions from the list. If $r = 3$, we obtain $\# \text{orb}(a) + \# \text{orb}(0) - 2$ independent functions, which is the required number (it coincides with the $n$-rank of the Picard group). If $r = 4, 6$, in order to complete the list we add a last function

$$\frac{(x+1)^2}{\zeta + \zeta + 2}$$

(16)

which is multiplicatively independent from the previous ones.
Lemma 3.11. For \( r = 4, 6 \), the value of \((16)\) at \( P_y \) in an algebraic number of norm \( \pm 1 \).

Proof. Let \( Q \) be the polynomial \((9)\). Then the minimal polynomial of \( x + 1 \) is \( Q(T - 1) \), and its constant coefficient \( Q(-1) = (-1)^{\#G} \prod_{\sigma \in G} (\sigma(a) + 1) \) is equal (up to sign) to the norm of \( x + 1 \). According to \((8)\), this quantity is \((\zeta + 1)^{\#G} \). Assume \( G \) is cyclic, then we have

\[
(\zeta + 1)^r = ((\zeta + 1)^2)^{r/2} = \zeta^{r/2}(\zeta + \bar{\zeta} + 2)^{r/2} = -(\zeta + \bar{\zeta} + 2)^{r/2},
\]

from which one deduces that \((x + 1)^2/(\zeta + \bar{\zeta} + 2)\) has norm \( \pm 1 \), since \( Q \) has degree \( r \). The dihedral case can be proved along the same lines.

Under assumption (A1) the value of this last function \((16)\) at \( P_y \) need not be a unit, although its minimal polynomial has coefficients in \( \mathbb{Z}[\zeta + \bar{\zeta} + 2] \). We shall numerically check that, under an additional congruence condition on \( y^n \), this polynomial has integral coefficients, so the value of \((16)\) at \( P_y \) is a unit, hence defines a class in the Selmer group.

### 3.2.5 Determination of the congruence condition

In order to complete our proof, we shall explicitly determine, in each case, conditions on the integer \( y \) of the form:

(C1) \( y^n \equiv t_0 \pmod{m} \), where \( m = (\zeta + \bar{\zeta} + 2)^v \prod_i \alpha_i \) for some \( v \geq 0 \).

(C2) \( y \) is coprime to \( p_1 \cdots p_l \), where the \( p_i \) are the prime factors of \((15)\) which do not divide \( m \).

These conditions are chosen in such a way that (C1) and (C2) imply both (A1) and (A2), and additionally that the minimal polynomial of \((16)\) has integral coefficients.

In practice, the integers \( m \) and \( t_0 \) in condition (C1) are determined by a case-by-case computation, with the help of Pari/GP: we start from the congruence condition modulo \( \prod_i \alpha_i \) given by Lemma 3.11, which implies (A1). Then we compute the minimal polynomial of \((16)\) and require it to have integral coefficients. This gives us a slightly stronger condition modulo \((\zeta + \bar{\zeta} + 2)^v \prod_i \alpha_i \) for some \( v \geq 0 \).

Next, we observe that \( t_0 \) is coprime to \( m \) and to \( p_1 \cdots p_l \), in particular (C1) and (C2) imply that \( y^n \) is coprime to \((15)\), so that assumption (A2) is satisfied.

It follows from Bézout’s identity that the equation \( y^n \equiv t_0 \pmod{m} \) has a solution if and only if the order of \( t_0 \) in \((\mathbb{Z}/m\mathbb{Z})^\times\) is coprime to \( n \). So, a first step is to assume that \( n \) satisfies this condition. In all our examples, we tried to find the weakest possible condition on \( n \), like \( n \) is coprime to 2, to 3, or to 6 depending on the cases. There does not seem to exist a direct relation between the order of \( G \) and the condition on \( n \).

Once the condition on \( n \) is found, we describe explicitly the solutions of the equation \( y^n \equiv t_0 \pmod{m} \), which yields congruence conditions of the form \( y \equiv y_0 \pmod{m} \), where \( y_0 \) is a suitable power of \( t_0 \), depending on the class of \( m \) modulo the multiplicative order of \( t_0 \) in \((\mathbb{Z}/m\mathbb{Z})^\times\).

Finally, since \( t_0 \) is coprime to \( m \) and to \( p_1 \cdots p_l \), so does \( y_0 \), and the condition

\[
y \equiv y_0 \pmod{mp_1 \cdots p_l}
\]

implies that both (C1) and (C2) above hold (it is in fact much stronger). As explained above, this yields a map \( \phi \) satisfying the hypotheses of Theorem 3.1. We shall not exhibit such a map, but its existence allows us to make use of Theorem 3.1.

Remark 3.12. Although we believe it should exist, we have not been able to prove the existence of conditions similar to (C1) and (C2) in full generality.
3.2.6 Counting the fields obtained

Given a curve as in Lemma 2.6, the zeroes of the map $y$ (hence those of the map $\phi$ whose existence is granted by the discussion at the end of §3.2.5) are in bijection with orb($a$). Therefore, if $\#\text{orb}(a) \geq 3$, then by the quantitative version of Theorem 3.1 the number of isomorphism classes of fields in \{\mathbb{Q}(P_1), \ldots, \mathbb{Q}(P_N)\} is $\gg N$. Since $\#\text{orb}(a) = \#G$, this is the case for all our groups, except $\mathbb{Z}/2\mathbb{Z}$, which for simplicity we exclude from this discussion.

Finally, we note that the discriminant of our polynomial (9) is $O(y^{ln})$ for some integer $l$. Needless to say, the same holds for the discriminant of the field $\mathbb{Q}(P_y)$. It follows that, for sufficiently large positive $X$, the number of isomorphism classes of fields $\mathbb{Q}(P_y)$ satisfying $rk_n\text{Cl}(\mathbb{Q}(P_y)) \geq \#G - 1$ and such that $|\text{Disc}(\mathbb{Q}(P_y)/\mathbb{Q})| \leq X$ is $\gg X^{1/ln}$.

4 Case-by-case proof of Theorem 1.2

4.1 $\mathbb{Z}/2\mathbb{Z}$

We shall recover the result of Yamamoto [Yam70] and Weinberger [Wei73] on real quadratic fields with $n$-rank at least one. We note that our family is not the same as theirs.

The homography $z \mapsto 1/z$ has order 2; choosing $a = 2$, $b = 0$ and $\lambda = -1/2$ as in §3.2.3 we obtain from Lemma 2.6 the polynomial

$$C_2P = \frac{1}{2}((x - 2)(2x - 1) + y^n x) = x^2 + \left(\frac{y^n - 5}{2}\right)x + 1.$$ 

Let $y \in \mathbb{Z}$, and let $P_y$ be the corresponding point on the curve defined by $C_2P$.

Congruence conditions

(i) $2 \nmid y$ and $3 \mid y$.

We claim that, under these conditions, the values at $P_y$ of the multiplicatively independent rational functions

$$x, \quad x - 2,$

define classes in $\text{Sel}^n(\mathbb{Q}(P_y))$. In other terms, the condition (SC) in Theorem 3.1 is satisfied by these rational maps and this choice of $P_y$.

Abusing notation, we simply denote by $x$ the value $x(P_y)$ in the proof below. The integer $y$ being odd, we deduce that $y^n$ is odd, hence the polynomial $C_2P$ has integral coefficients. It follows that $x$ is a unit, hence defines an element in the Selmer group.

On the other hand, we have by construction

$$(x - 2)(2x - 1) = -y^n x.$$ 

We know that $y$ is coprime to 3 and that $x$ is a unit, so the right-hand side is coprime to 3. It follows that the two terms on the left-hand side are also coprime to 3, hence are coprime because $(x - 2, 2x - 1)$ divides 3 by Lemma 3.9 Therefore, $x - 2$ and $2x - 1$ are both $n$-th powers of ideals, according to the identity above.
The discriminant of \( C_3P \) is \((y^n - 9)(y^n - 1)/4\), which is strictly positive unless \(1 \leq y^n \leq 9\). So, if \(y\) is large, the field \(\mathbb{Q}(P_y)\) is totally real, with discriminant \(O(y^{2n})\).

**Statement of the result**

According to Theorem 3.1, for all but \(O(\sqrt{N})\) values \(y \in \{1, \ldots, N\}\) satisfying the conditions above, \(\mathbb{Q}(P_y)\) is a real quadratic field, and

\[
\text{rk}_n \text{Cl}(\mathbb{Q}(P_y)) \geq 1.
\]

Moreover, for \(N\) large enough, the number of isomorphism classes of such \(\mathbb{Q}(P_y)\) when \(y\) runs through \(\{1, \ldots, N\}\) is \(\gg N/\log N\). It follows that, for \(X\) large enough, the number of such fields whose discriminant is bounded above by \(X\) is \(\gg X^{1/2n}/\log X\).

**4.2 \( \mathbb{Z}/3\mathbb{Z} \)**

We shall recover the result of Nakano [Nak86] on cyclic cubic fields with \(n\)-rank at least two. Our family is the same as his. In fact, it is constructed from the universal family of cyclic cubic fields given by Shanks [Sha74], known as the “simplest cubic fields”.

The homography

\[
z \mapsto -1/(z + 1)
\]

has order 3. The orbits of 0 and 1 are respectively given by

\[
\text{orb}(0) = \{0, -1, \infty\}; \quad \text{orb}(1) = \{1, -1/2, -2\}.
\]

The corresponding polynomial is

\[
C_3P = \frac{1}{2}((x - 1)(x + 2)(2x + 1) + y^nx(x + 1)) = x^3 + \left(\frac{y^n + 3}{2}\right)x^2 + \left(\frac{y^n - 3}{2}\right)x - 1.
\]

Let \(y \in \mathbb{Z}\), and let \(P_y\) be the corresponding point on the curve defined by \(C_3P\).

**Congruence conditions**

(i) 2 \(\nmid\) \(y\) and 3 \(\nmid\) \(y\).

One checks that, under these conditions, assumptions (A1) and (A2) are satisfied. It follows from the arguments in 3.2.4 that the values at \(P_y\) of the rational functions

\[
x, \quad x + 1, \quad x - 1, \quad x + 2
\]

define classes in Sel\(^n\)(\(\mathbb{Q}(P_y)\)).

**Signature**

The polynomial \(C_3P\) has degree three, hence has at least one real root. Therefore, according to Remark 2.5 the field \(\mathbb{Q}(P_y)\) is totally real, with discriminant \(O(y^{4n})\).
Statement of the result

According to Theorem 3.1, for all but \( O(\sqrt{N}) \) values \( y \in \{1, \ldots, N\} \) satisfying the conditions above, \( \mathbb{Q}(P_y) \) is a cyclic cubic field, and

\[
\text{rk}_n \text{Cl}(\mathbb{Q}(P_y)) \geq 2.
\]

Moreover, for \( N \) large enough, the number of isomorphism classes of such \( \mathbb{Q}(P_y) \) when \( y \) runs through \( \{1, \ldots, N\} \) is \( \gg N \). It follows that, for \( X \) large enough, the number of such fields whose discriminant is bounded above by \( X \) is \( \gg X^{1/4n} \).

4.3 \( \mathbb{Z}/4\mathbb{Z} \)

The results obtained in this section appear to be new. The family of cyclic quartic fields we consider is a subfamily of the one constructed by Gras [Gra78], also known as the “simplest quartic fields” by analogy with Shank’s terminology.

The homography

\[
z \mapsto (z - 1)/(z + 1)
\]

has order 4. The orbits of 0 and 2 are respectively given by

\[
\text{orb}(0) = \{0, -1, \infty, 1\}; \quad \text{orb}(2) = \{2, 1/3, -1/2, -3\}.
\]

The corresponding polynomial is

\[
C_4P = \frac{1}{6} ((x - 2)(2x + 1)(x + 3)(3x - 1) + y^n x(x - 1)(x + 1))
\]

\[
= x^4 + \left(\frac{y^n + 7}{6}\right) x^3 - 6x^2 - \left(\frac{y^n + 7}{6}\right) x + 1.
\]

Let \( y \in \mathbb{Z} \), and let \( P_y \) be the corresponding point on the curve defined by \( C_4P \).

Congruence conditions

(i) \( n \) is odd;

(ii) \( y \equiv 5 \) (mod 12);

(iii) \( 5 \nmid y \).

We claim that, under these conditions, the values at \( P_y \) of the rational functions

\[
x, \quad \frac{x - 1}{x + 1}, \quad \frac{x + 3}{x + 1}, \quad x - 2, \quad 2x + 1, \quad \frac{(x + 1)^2}{2},
\]

define classes in \( \text{Sel}^n(\mathbb{Q}(P_y)) \).

Condition (i) and (ii) imply that \( y^n \equiv 5 \) (mod 12), in particular the polynomial \( C_4P \) has integral coefficients, hence (A1) holds. Conditions (ii) and (iii) imply that \( y \) is coprime to 2 and 5, hence (A2) holds. It follows from the arguments in §3.2.4 that all the functions in the above list, except the last one, have the required property.

Let \( q := \frac{y^n - 5}{12} \), which is an integer under our conditions. Then one computes, using Pari/GP, that the minimal polynomial of \( (x(P_y) + 1)^2/2 \) is given by

\[
T^4 - 2(q^2 + q + 4)T^3 + (5q^2 + 10q + 19)T^2 - 2(q^2 + 3q + 4)T + 1
\]

which has integral coefficients. Therefore \( (x(P_y) + 1)^2/2 \) is a unit, hence the last function in the list satisfies the property.
Signature

The polynomial \( C_4P \) satisfies \( C_4P(0) = 1 \) and \( C_4P(1) = -4 \), hence has at least one real root. Therefore, according to Remark 2.5, the field \( \mathbb{Q}(P_y) \) is totally real, with discriminant \( O(y^{6n}) \).

Statement of the result

Assume \( n \) is odd. Then for all but \( O(\sqrt{N}) \) values \( y \in \{1, \ldots, N\} \) satisfying the conditions above, \( \mathbb{Q}(P_y) \) is a real cyclic quartic field, and
\[
\text{rk}_n \, \text{Cl}(\mathbb{Q}(P_y)) \geq 3.
\]

Moreover, for \( N \) large enough, the number of isomorphism classes of such \( \mathbb{Q}(P_y) \) when \( y \) runs through \( \{1, \ldots, N\} \) is \( \gg N \). It follows that, for \( X \) large enough, the number of such fields whose discriminant is bounded above by \( X \) is \( \gg X^{1/6n} \).

4.4 \( \mathbb{Z}/6\mathbb{Z} \)

The results obtained in this section appear to be new. Our family of cyclic sextic fields is similar to that constructed by Gras [Gra86], although the cyclic subgroup of \( \text{PGL}_2(\mathbb{Q}) \) on which her construction relies on is not the same than ours.

The homography
\[ z \mapsto (2z - 1)/(z + 1) \]
has order 6. The orbits of 0 and 3 are respectively given by
\[
\text{orb}(0) = \{0, -1, \infty, 1, 2, 1/2\}; \quad \text{orb}(3) = \{3, 5/4, 2/3, 1/5, -1/2, -4\}.
\]

The corresponding polynomial is
\[
C_6P = \frac{1}{120} ((x - 3)(x + 4)(2x + 1)(3x - 2)(4x - 5)(5x - 1) + y^n x(x - 1)(x + 1)(x - 2)(2x - 1))
= x^6 + \left(\frac{y^n - 37}{60}\right) x^5 - \left(\frac{y^n + 323}{24}\right) x^4 + 20x^3 + \left(\frac{y^n - 37}{60}\right) x^2 - \left(\frac{y^n + 323}{24}\right) x + 1.
\]

Let \( y \in \mathbb{Z} \), and let \( P_y \) be the corresponding point on the curve defined by \( C_6P \).

Congruence conditions

(i) \( n \) is coprime to 6;
(ii) \( y \equiv 397 \pmod{1080} \) if \( n = 12k + 1 \), \( y \equiv 37 \pmod{1080} \) if \( n = 12k + 5 \), \( y \equiv 613 \pmod{1080} \) if \( n = 12k + 7 \) and \( y \equiv 253 \pmod{1080} \) if \( n = 12k + 11 \);
(iii) \( 7 \nmid y \).

We claim that, under these conditions, the values at \( P_y \) of the rational functions
\[
x, \quad x - 1, \quad \frac{2x - 1}{x + 1}, \quad \frac{x - 2}{x + 1}, \quad x - 3, \quad \frac{x + 4}{x + 1}, \quad 2x + 1, \quad 3x - 2, \quad \frac{4x - 5}{x + 1}, \quad \frac{(x + 1)^2}{3}
\]
define classes in \( \text{Sel}^n(\mathbb{Q}(P_y)) \).
Condition (ii) implies that \( y^n \equiv 397 \pmod{1080} \), in particular the polynomial \( C_6 P \) has integral coefficients, hence (A1) holds. Condition (ii) and (iii) imply that \( y \) is coprime to 2, 3 and 7, hence (A2) holds. It follows from the arguments in §3.2.4 that all the functions in the above list, except the last one, have the required property.

Let \( q := \frac{y^n - 397}{1080} \), which is an integer under our conditions. Then one computes, using Pari/GP, that the minimal polynomial of \( (x(P_y) + 1)^2/3 \) is given by

\[
T^6 - 6(18q^2 + 15q + 5)T^5 + 15(39q^2 + 36q + 11)T^4 - 2(483q^2 + 483q + 151)T^3 \\
+ 15(39q^2 + 42q + 14)T^2 - 6(18q^2 + 21q + 8)T + 1
\]

which has integral coefficients. Therefore \( (x(P_y) + 1)^2/2 \) is a unit, hence the last function in the list satisfies the property.

**Signature**

The polynomial \( C_6 P \) satisfies \( C_6(P(0)) = 1 \) and \( C_6(P(-1)) = -27 \), hence has at least one real root. Therefore, according to Remark 2.5 the field \( \mathbb{Q}(P_y) \) is totally real, with discriminant \( O(y^{10n}) \).

**Statement of the result**

Assume \( n \) is coprime to 6. Then for all but \( O(\sqrt{N}) \) values \( y \in \{1, \ldots, N\} \) satisfying the conditions above, \( \mathbb{Q}(P_y) \) is a real cyclic sextic field, and

\[
\text{rk}_n \text{Cl}(\mathbb{Q}(P_y)) \geq 5.
\]

Moreover, for \( N \) large enough, the number of isomorphism classes of such \( \mathbb{Q}(P_y) \) when \( y \) runs through \( \{1, \ldots, N\} \) is \( \gg N \). It follows that, for \( X \) large enough, the number of such fields whose discriminant is bounded above by \( X \) is \( \gg X^{1/10n} \).

### 4.5 \( D_2 = (\mathbb{Z}/2\mathbb{Z})^2 \)

We shall construct, for any \( n > 1 \), real biquadratic fields whose class group has \( n \)-rank at least three. A detailed analysis reveals that each of the three quadratic subfields has at least one element of order \( n \) in its class group. While these examples appear to be new, it follows from Yamamoto’s result that one can achieve \( n \)-rank at least 4 in the imaginary case.

Let \( G \simeq (\mathbb{Z}/2\mathbb{Z})^2 \) be the subgroup of \( \text{PGL}_2(\mathbb{Q}) \) generated by the homographies

\[
z \mapsto (z + 1)/(z - 1) \quad \text{and} \quad z \mapsto -1/z
\]

Choosing \( a = 2, b = 0 \) and \( \lambda \) as in §3.2.3, we obtain from Lemma 2.6 the polynomial

\[
D_2 P = \frac{1}{6} ((x - 2)(2x + 1)(x - 3)(3x + 1) + y^n x(x - 1)(x + 1)) \\
= x^4 + \left( \frac{y^n - 25}{6} \right) x^3 + 2x^2 - \left( \frac{y^n - 25}{6} \right) x + 1.
\]

Let \( y \in \mathbb{Z} \), and let \( P_y \) be the corresponding point on the curve defined by \( D_2 P \).
Congruence conditions

(i) \( y \equiv 1 \pmod{12} \).

(ii) \( 5 \nmid y \) and \( 7 \nmid y \).

One checks that, under these conditions, the values at \( P_y \) of the rational functions

\[
\begin{align*}
x, \quad & \frac{x + 1}{x - 1}, \quad x - 2, \quad 2x + 1, \quad \frac{x - 3}{x - 1}, \quad \frac{(x - 1)^2}{2},
\end{align*}
\]

define classes in \( \text{Sel}^n(\mathbb{Q}(P_y)) \). In other terms, the condition [SC] in Theorem 3.1 is satisfied by these rational maps and this choice of \( P_y \).

Signature

The polynomial \( D_2P \) satisfies

\[
D_2^2P(0) = 1 \quad \text{and} \quad D_2^2P(1/2) = \frac{(-y^n + 50)}{16},
\]
hence has at least one real root when \( y^n > 50 \). Therefore, the field \( \mathbb{Q}(P_y) \) is totally real for \( y \) large enough.

Statement of the result

According to Theorem 3.1, for all but \( O(\sqrt{N}) \) values \( y \in \{1, \ldots, N\} \) satisfying the conditions above, \( \mathbb{Q}(P_y) \) is a real biquadratic field, and

\[
\text{rk}_n \text{Cl}(\mathbb{Q}(P_y)) \geq 3.
\]

Remark 4.1. Polynomials defining the three quadratic subfields are given by

\[
\begin{align*}
x^2 + \left( \frac{y^n - 25}{6} \right) x + 4 &= \frac{1}{6} ((2x - 3)(3x - 8) + y^n x); \\
x^2 + \left( \frac{y^n - 25}{6} \right) x + \left( \frac{y^n - 25}{6} \right) &= \frac{1}{6} ((x - 5)(6x + 5) + y^n(x + 1)); \\
x^2 + \left( \frac{y^n - 25}{6} \right) x - \left( \frac{y^n - 25}{6} \right) &= \frac{1}{6} ((2x - 5)(3x - 5) + y^n(x - 1));
\end{align*}
\]

and one checks that, under the above assumptions, each of these fields has at least one element of order \( n \) in its class group. When \( n \) is odd, this allows to recover the above result via Brauer’s class number relations [Wal79].

4.6 \( D_3 = \mathfrak{G}_3 \)

The subgroup generated by

\[
z \mapsto -1/(z + 1) \quad \text{and} \quad z \mapsto 1/z
\]

is isomorphic to \( D_3 = \mathfrak{G}_3 \). The orbits of 0 and 2 are respectively given by

\[
\text{orb}(0) = \{0, -1, \infty\}; \quad \text{orb}(2) = \{2, -1/3, -3/2, 1/2, -3, -2/3\}.
\]

The corresponding polynomial is

\[
D_3P = \frac{1}{36} ((x - 2)(2x - 1)(x + 3)(3x + 1)(2x + 3)(3x + 2) + y^n x^2(x + 1)^2)
\]

\[
= x^6 + 3x^5 + \left( \frac{y^n - 127}{36} \right) x^4 + \left( \frac{y^n - 217}{18} \right) x^3 + \left( \frac{y^n - 127}{36} \right) x^2 + 3x + 1.
\]

Let \( y \in \mathbb{Z} \), and let \( P_y \) be the corresponding point on the curve defined by \( D_3P \).
Congruence conditions

(i) $n$ is odd;
(ii) $y \equiv 19 \pmod{36}$;
(iii) $5 \nmid y$ and $7 \nmid y$.

Under these conditions, the values at $P_y$ of the rational functions

\[
x, \quad x + 1, \quad x - 2, \quad x + 3, \quad 2x - 1, \quad 2x + 3, \quad 3x + 1
\]

define classes in $\text{Sel}^n(\mathbb{Q}(P_y))$.

Signature

The polynomial $D_3P$ can be written as

\[
D_3P = P_1 + y^n P_2.
\]

We note that $P_1$ is monic of degree 6, hence has a global minimum over $\mathbb{R}$; that $P_2$ takes strictly positive values over $\mathbb{R}$, except at its roots 0 and −1; and that $P_1(0)$ and $P_1(−1)$ are strictly positive. By elementary calculus it follows from these observations that, for $y^n$ large enough, $D_3P$ takes strictly positive values over $\mathbb{R}$, hence the field $\mathbb{Q}(P_y)$ is totally imaginary. A detailed analysis reveals that, for $y^n > 100$, $D_3P$ takes strictly positive values.

Statement of the result

Assume $n$ is odd. Then for all but $O(\sqrt{N})$ values $y \in \{1, \ldots, N\}$ satisfying the conditions above, $\mathbb{Q}(P_y)$ is a totally imaginary Galois extension of $\mathbb{Q}$ with group $\mathfrak{S}_3$, and

\[
\text{rk}_n \text{Cl}(\mathbb{Q}(P_y)) \geq 5.
\]

Remark 4.2. The quadratic subfield is defined by the polynomial

\[
x^2 + 3x + \left(\frac{y^n - 19}{36}\right) = \frac{1}{36}((6x - 1)(6x + 19) + y^n).
\]

The three conjugate cubic subfields are defined by

\[
x^3 + 3x^2 + \left(\frac{y^n - 235}{36}\right)x + \left(\frac{y^n - 325}{18}\right) = \frac{1}{36}((2x - 5)(3x + 10)(6x + 13) + y^n(x + 2)).
\]

One checks that, under the above assumptions, the class group of the quadratic (resp. cubic) subfield has $n$-rank at least one (resp. two). When $n$ is coprime to 6, this allows to recover the above result via Brauer’s class number relations [Wal79].

Remark 4.3. The above factorizations are particular instances of a general phenomenon: let $f(z) = -1/(z + 1)$, and let $a \in \mathbb{Q}$ which does not belong to the orbit of 0, then the splitting field of the polynomial

\[
\prod_{i=1}^{3} (x - f^i(a))(x - \frac{1}{f^i(a)}) - tx^2(x + 1)^2
\]
is Galois over \( \mathbb{Q}(t) \) with group \( \mathfrak{S}_3 \), according to our construction. Its quadratic subfield is defined by the polynomial

\[
(x - (a + f(a) + f^2(a))) \left( x - \left( \frac{1}{a} + \frac{1}{f(a)} + \frac{1}{f^2(a)} \right) \right) - t,
\]

and the cubic subfields are defined by

\[
\prod_{i=1}^{3} \left( x - \left( f^i(a) + \frac{1}{f(a)} \right) \right) - t(x + 2).
\]

Needless to say, this can be generalized to arbitrary values of \( a, b \) and \( r \) (the order of \( f \)).

**Example 4.4.** Let \( n = 5 \) and \( y = 199 \), then our field is defined by the polynomial

\[
x^6 + 3x^5 + 8668877802x^4 + 17337755599x^3 + 8668877802x^2 + 3x + 1
\]

One checks using Pari/GP that its quadratic subfield has class group of 5-rank 2, and that its cubic subfields have 5-rank 2 each. It follows from Brauer’s class number relations that the class group of this field has 5-rank 6. In view of our numerical experiments, it seems that there should exist infinitely many such cases in our family.

### 4.7 \( D_4 \)

The subgroup generated by

\[
z \mapsto (z - 1)/(z + 1) \quad \text{and} \quad z \mapsto 1/z
\]

is isomorphic to \( D_4 \). The orbits of 0 and 2 are respectively given by

\[
\text{orb}(0) = \{0, -1, \infty, 1\}; \quad \text{orb}(2) = \{2, 1/3, -1/2, -3, 1/2, 3, -2, -1/3\}.
\]

The corresponding polynomial is

\[
D_4P = \frac{1}{36} \left( (x - 2)(2x - 1)(x + 2)(2x + 1)(x - 3)(3x - 1)(x + 3) + y^n x^2(x - 1)^2(x + 1)^2 \right)
\]

\[
= x^8 + \left( \frac{y^n - 481}{36} \right) x^6 - \left( \frac{y^n - 733}{18} \right) x^4 + \left( \frac{y^n - 481}{36} \right) x^2 + 1.
\]

Let \( y \in \mathbb{Z} \), and let \( P_y \) be the corresponding point on the curve defined by \( D_4P \).

**Congruence conditions**

(i) \( n \) is coprime to 6;

(ii) \( y \equiv 49 \pmod{144} \) if \( n = 3k + 1 \) and \( y \equiv 97 \pmod{144} \) if \( n = 3k + 2 \);

(iii) \( 5 \nmid y \) and \( 7 \nmid y \).

Under these conditions, the values at \( P_y \) of the rational functions

\[
x, \quad \frac{x - 1}{x + 1}, \quad x - 2, \quad x + 2, \quad \frac{x - 3}{x + 1}, \quad \frac{x + 3}{x + 1}, \quad 2x - 1, \quad 2x + 1, \quad \frac{3x - 1}{x + 1}, \quad \frac{(x + 1)^2}{2}
\]

define classes in \( \text{Sel}^n(\mathbb{Q}(P_y)) \).
A detailed analysis, similar to the $D_3P$ case, reveals that, when $y^n > 49$, the polynomial $D_4P$ takes strictly positive values over $\mathbb{R}$, hence the field $K$ is totally complex.

Statement of the result

Assume $n$ is coprime to 6. Then for all but $O(\sqrt{N})$ values $y \in \{1, \ldots, N\}$ satisfying the conditions above, $\mathbb{Q}(P_y)$ is a totally complex Galois extension of $\mathbb{Q}$ with group $D_4$, and
\[ \text{rk}_n \text{Cl}(\mathbb{Q}(P_y)) \geq 7. \]

Remark 4.5. Replacing $y^n$ by $y^{2n}$ in the polynomial $D_4P$, one can prove similar result when $n$ is only assumed to be coprime to 3. It suffices to generalize our constructions, starting from Lemma 2.6 in order to take into account the square factors in the orbit at infinity.

Remark 4.6. Choosing $a = 4$ instead, one finds another congruence condition on $y^n$, which has a solution for all $n$ coprime to 5. Therefore, the same result hold for $n$ coprime to 10 (in fact, 5 if one takes into account the previous remark).

4.8 $D_6$

The subgroup generated by
\[ z \mapsto (2z - 1)/(z + 1) \quad \text{and} \quad z \mapsto 1/z \]
is isomorphic to $D_6$. The orbits of 0 and $-2$ are respectively given by
\[ \text{orb}(0) = \{0, -1, \infty, 2, 1, 1/2\}; \quad \text{orb}(-2) = \{-2, 5, 3/2, 4/5, 1/3, -1/4, \text{and their inverses}\}. \]

The corresponding polynomial is
\[
D_6P = \frac{1}{120^2}((x+2)(2x+1)(x-3)(3x-1)(x+4)(4x+1)(x-5)(5x-1)(2x-3)(3x-2)(4x-5)(5x-4) + y^n x^2(x-1)^2(x+1)^2(x-2)^2(2x-1)^2)
\]

Let $y \in \mathbb{Z}$, and let $P_y$ be the corresponding point on the curve defined by $D_6P$.

Congruence conditions

(i) $n$ is coprime to 6;
(ii) $y^n \equiv 117649 \pmod{388800}$;
(iii) $7 \nmid y, 11 \nmid y$ and $13 \nmid y$.

Under these conditions, the values at $P_y$ of the rational functions
\[ x, \quad x - 1, \quad \frac{2x - 1}{x + 1}, \quad \frac{x - 2}{x + 1}, \quad x + 2, \quad \frac{x - 5}{x + 1}, \quad 2x - 3, \quad \frac{5x - 4}{x + 1}, \quad 3x - 1, \quad \frac{4x + 1}{x + 1}, \quad 2x + 1, \quad \frac{5x - 1}{x + 1}, \quad 3x - 2, \quad \frac{4x - 5}{x + 1}, \quad x - 3, \quad \frac{(x + 1)^2}{3} \]
define classes in $\text{Sel}^n(\mathbb{Q}(P_y))$. 

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Signature

As before, a detailed analysis reveals that, when \( y^n > 20449 \), the polynomial \( D_6 P \) takes strictly positive values over \( \mathbb{R} \), hence the field \( K \) is totally complex.

Statement of the result

Assume \( n \) is coprime to 6. Then for all but \( O(\sqrt{N}) \) values \( y \in \{1, \ldots, N\} \) satisfying the conditions above, \( \mathbb{Q}(P_y) \) is a totally complex Galois extension of \( \mathbb{Q} \) with group \( D_6 \), and

\[
\text{rk}_n \text{Cl}(\mathbb{Q}(P_y)) \geq 11.
\]

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