BOUNDARY POINTS, MINIMAL $L^2$ INTEGRALS AND CONCAVITY PROPERTY IV—FIBRATIONS OVER OPEN RIEMANN SURFACES

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Abstract. In this article, we consider the minimal $L^2$ integrals related to modules at boundary points on fibrations over open Riemann surfaces, and present a characterization for the concavity property of the minimal $L^2$ integrals degenerating to linearity.

1. Introduction

The strong openness property of multiplier ideal sheaves [20] (conjectured by Demailly [9]) is an important feature of multiplier ideal sheaves and has opened the door to new types of approximation technique (see e.g. [20, 34, 31, 13, 14, 10, 16, 28, 46, 47, 17, 32, 7]), where the multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the sheaf of germs of holomorphic functions $f$ such that $|f|^2 e^{-\varphi}$ is locally integrable (see e.g. [42, 35, 37, 11, 12, 9, 13, 33, 38, 39, 10, 29]), and $\varphi$ is a plurisubharmonic function on a complex manifold $M$ (see [8]).

Guan-Zhou [26] proved the strong openness property (the 2-dimensional case was proved by Jonsson-Mustată [30]). Using the strong openness property, Guan-Zhou [27] gave a proof of the following conjecture posed by Jonsson-Mustată (see [30]).

Conjecture J-M: If $e^F_0(\psi) < +\infty$, $\frac{1}{2} \mu\{\{e^F_0(\psi)\psi - \log |F| < \log r\}$ has a uniform positive lower bound independent of $r \in (0,1)$, where $\mu$ is the Lebesgue measure on $\mathbb{C}^n$, and $e^F_0(\psi) := \sup\{c \geq 0 : |F|^2 e^{-2c}\psi \text{ is } L^1 \text{ on a neighborhood of } o\}$ is the jumping number (see [30]).

Recall that Jonsson-Mustată [30] posed Conjecture J-M, and proved the 2-dimensional case, which deduced the 2-dimensional strong openness property. It is natural to ask: can one find a proof of Conjecture J-M independent of the strong openness property?

In [3], Bao-Guan-Yuan considered the minimal $L^2$ integrals related to modules at boundary points of the sublevel sets of plurisubharmonic functions on pseudoconvex domains, and established a concavity property of the minimal $L^2$ integrals, which deduced a proof of Conjecture J-M independent of the strong openness property. In [20], Guan-Mi-Yuan generalized the concavity property to weakly pseudoconvex Kähler manifolds.

Note that the linearity is a degenerate concavity. It is natural to ask:
Question 1.1. How to characterize the concavity property degenerating to linearity?

In [21], Guan-Mi-Yuan gave an answer to Question 1.1 for the case of open Riemann surfaces.

In this article, we give an answer to Question 1.1 for the case of fibrations over open Riemann surfaces.

1.1. Main result. Let $\Omega$ be an open Riemann surface, which admits a nontrivial Green function $G_\Omega$. Let $Y$ be an $n-1$ dimensional weakly pseudoconvex Kähler manifold, and let $K_Y$ be the canonical line bundle on $Y$. Let $M = \Omega \times Y$ be an $n-$dimensional complex manifold. Let $\pi_1$ and $\pi_2$ be the natural projections from $M$ to $\Omega$ and $Y$ respectively. Let $K_M$ be the canonical line bundle on $M$.

Let $\psi$ be a subharmonic function on $\Omega$. Let $\varphi_\Omega$ be a Lebesgue measurable function on $\Omega$ such that $\varphi_\Omega + \psi$ is subharmonic function on $\Omega$. Let $F$ be a holomorphic function on $\Omega$. Let $T \in [-\infty, +\infty)$. Denote that
\[
\tilde{\Psi} := \min\{\psi - 2\log|F|, -T\}.
\]

For any $z \in \Omega$ satisfying $F(z) = 0$, we set $\tilde{\Psi}(z) = -T$. Denote that $\Psi := \pi_1^*(\tilde{\Psi})$ on $M$. Let $\varphi_Y$ be a plurisubharmonic function on $Y$. Denote that $\varphi := \pi_2^*(\varphi_\Omega) + \pi_2^*(\varphi_Y)$.

Let $p \in M$ be a point. Denote that $\tilde{J}(\Psi)_p := \{f \in \mathcal{O}(\Psi < -t) \cap V : t \in \mathbb{R}$ and $V$ is a neighborhood of $p\}$, where $\mathcal{O}(\Psi < -t) \cap V$ is an $\mathcal{O}(\Psi < -t)$-module.

If $p \in \cap_{t>T}\{\Psi < -t\}$, then $J(\Psi)_p = \mathcal{O}_{M,p}$ (the stalk of the sheaf $\mathcal{O}_M$ at $p$), and $f_p$ is the germ $(f, p)$ of holomorphic function $f$. If $p \notin \cap_{t>T}\{\Psi < -T\}$, then $J(\Psi)_p$ is trivial.

Let $f_p, g_p \in J(\Psi)_p$ and $(h, p) \in \mathcal{O}_{M,p}$. We define $f_p + g_p := (f + g)_p$ and $(h, p) \cdot f_p := (hf)_p$. Note that $(f + g)_p$ and $(hf)_p \in J(\Psi)_p$ are independent of the choices of the representatives of $f, g$ and $h$. Hence $J(\Psi)_p$ is an $\mathcal{O}_{M,p}$-module.

For $f_p \in J(\Psi)_p$ and $a, b \geq 0$, we call $f_p \in I(a\Psi + b\varphi)$ if there exist $t \gg T$ and a neighborhood $V$ of $p$, such that $\int_{\{\Psi < -t\}\cap V} \left| f \right|^2 e^{-a\Psi - b\varphi} dV_M < +\infty$, where $dV_M$ is a continuous volume form on $M$. Note that $I(a\Psi + b\varphi)_p$ is an $\mathcal{O}_{M,p}$-submodule of $J(\Psi)_p$.

Let $Z_0 \subset M$ be a subset of $\cap_{t>T}\{\Psi < -t\}$ such that there exists a subset $\tilde{Z}_0$ of $\Omega$ such that $\tilde{Z}_0 = \tilde{Z}_0 \times Y$. Denote that $\tilde{Z}_1 := \{z \in \tilde{Z}_0 : v(dd^c(\psi), z) \geq 2ord_z(F)\}$ and $\tilde{Z}_2 := \{z \in \tilde{Z}_0 : v(dd^c(\psi), z) < 2ord_z(F)\}$, where $dd^c = \frac{\partial}{\partial \bar{\partial}} - 1$ and $v(dd^c(\psi), z)$ is the Lelong number of $dd^c(\psi)$ at $z$ (see [8]). Denote that $\tilde{Z}_4 := \{z \in \tilde{Z}_0 : v(dd^c(\psi), z) > 2ord_z(F)\}$. Note that $\{\Psi < -t\} \cup \tilde{Z}_3$ is an open Riemann surface for any $t \geq T$. Denote $Z_1 := \tilde{Z}_1 \times Y$, $Z_2 := \tilde{Z}_2 \times Y$ and $Z_3 := \tilde{Z}_3 \times Y$ respectively.

Let $c(t)$ be a positive function on $(T, +\infty)$ such that $c(t)e^{-t}$ is decreasing on $(T, +\infty)$, $c(t)e^{-t}$ is integrable near $+\infty$, and $c(-\Psi)e^{-\varphi}$ has a positive lower bound on $K \cap \{\Psi < -t\}$ for any compact subset $K \subset M \setminus \pi_1^{-1}(E)$, where $E$ is an analytic subset of $\Omega$ such that $E \subset \{\Psi = -\infty\}$. 
Let $f$ be a holomorphic $(n,0)$ form on $\{\Psi < -t_0\} \cap V$, where $V \supset Z_0$ is an open subset of $M$ and $t_0 > T$ is a real number. Denote

$$\inf \left\{ \int_{\{\Psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\Psi) : \tilde{f} \in H^0(\{\Psi < -t\}, \mathcal{O}(K_M)) \right\}$$

by $G(t;c,\Psi, \varphi, I(\varphi + \Psi), f)$, where $t \in [T, +\infty)$ and $|f|^2 := (\sqrt{-1})^n f \wedge \tilde{f}$ for any $(n,0)$ form $f$. Without misunderstanding, we denote $G(t;c,\Psi, \varphi, I(\varphi + \Psi), f)$ by $G(t)$ for simplicity.

Recall that $G(h^{-1}(r))$ is concave with respect to $r$ (see [21], see also Theorem 2.1.4), where $h(t) = \int_t^{+\infty} c(s)e^{-s}ds$ for any $t \geq T$. In this article, we give a characterization of the concavity degenerating to linearity.

Assume that $Z_3$ is finite, and denote that $Z_3 = \{z_1, z_2, \ldots, z_m\}$. Let $w_j$ be a local coordinate on a neighborhood $V_{z_j} \Subset \Omega$ of $z_j$ satisfying $w_j(z_j) = 0$ for any $j \in \{1, 2, \ldots, m\}$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any $j \neq k$. We give an answer to Question 1.1 for the case of fibrations over open Riemann surfaces as follows.

**Theorem 1.2.** For any $z \in Z_1$, assume that one of the following conditions holds:

(A) $\varphi$ is subharmonic near $z$ for some $\varphi \in [0, 1]$;

(B) $(\psi - 2q_z \log |w|)(z) > -\infty$, where $q_z = \frac{1}{2\pi} \int C \wedge \overline{C}$, $w$ is a local coordinate on a neighborhood of $z$ satisfying $w(z) = 0$.

If there exists $t_1 \geq T$ such that $G(t_1) \in (0, +\infty)$, then $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_T^{+\infty} c(s)e^{-s}ds)$ if and only if the following statements hold:

1. $f = \pi_1^\ast(a_jw_j^\ast dw_j) \wedge \pi_2^\ast(f_Y) + f_j$ on $(V_{z_j} \times Y) \cap \{\Psi < -t_0\} \cap V$ for any $j \in \{1, 2, \ldots, m\}$, where $a_j \in C \setminus \{0\}$, $k_j$ is a nonnegative integer, $f_Y$ is a holomorphic $(n - 1,0)$ form on $Y$ satisfying $\int_Y |f_Y|^2 e^{-\varphi} < \infty$, and $(f_j)_p \in \mathcal{O}(K_M)_p \otimes I(\varphi + \Psi)_p$ for any $p \in z_j \times Y$;

2. $\varphi \wedge \psi = 2\log |g| + 2\log |F|$, where $g$ is a holomorphic function on $\{\tilde{\Psi} < -T\} \cup Z_3 \subset \Omega$ such that $\text{ord}_{z_j}(g) = k_j + 1$ for any $j \in \{1, 2, \ldots, m\}$;

3. $Z_3 \neq \emptyset$ and $\psi = 2 \sum_{1 \leq j \leq m} (q_{z_j} - \text{ord}_{z_j}(F))G_{\Omega_t}(\cdot, z_j) + 2 \log |F| - t$ on $\Omega_t$ for any $j \in \{1, 2, \ldots, m\}$, where $\Omega_t = \{\tilde{\Psi} < -t\} \cup Z_3 \subset \Omega$ and $G_{\Omega_t}$ is the Green function on $\Omega_t$;

4. $\lim_{z \to z_j} \frac{dg}{a_jw_j^\ast dw_j} = c_0$ for any $j \in \{1, 2, \ldots, m\}$, where $c_0 \in C \setminus \{0\}$ is a constant independent of $j$.

When $F \equiv 1$, $\psi(z) = -\infty$ for any $z \in Z_0 = \tilde{Z}_3$ and condition (B) holds, Theorem 1.2 can be referred to [11] (see also Theorem 2.2.4 and Remark 2.3.4).

When $Z_3$ is an infinite analytic subset of $\Omega$, Proposition 3.2 gives a necessary condition of $G(h^{-1}(r))$ is linear.

### 2. Preparations

#### 2.1. Properties of products of Bergman spaces

In this section, we recall some results of products of Bergman spaces.

Let $U \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ be two open sets. Let $\varphi_1$ and $\varphi_2$ be two Lebesgue measurable functions on $U$ and $W$ respectively. Let $d\lambda_U$ and $d\lambda_W$ be the Lebesgue
measures on $U$ and $W$ respectively. Denote

$$A^2(U; e^{-\varphi_1}) := \{ f \in \mathcal{O}_U : \int_U |f|^2 e^{-\varphi_1} d\lambda_U < +\infty \},$$

$$A^2(W; e^{-\varphi_2}) := \{ g \in \mathcal{O}_W : \int_W |g|^2 e^{-\varphi_2} d\lambda_W < +\infty \}.$$ 

Denote $\|f\|_1 := (\int_U |f|^2 e^{-\varphi_1} d\lambda_U)^{\frac{1}{2}}$ for any $f \in A^2(U; e^{-\varphi_1})$ and $\|g\|_2 := (\int_W |g|^2 e^{-\varphi_2} d\lambda_W)^{\frac{1}{2}}$ for any $g \in A^2(W; e^{-\varphi_2})$. Let $M := U \times W$ and $\varphi := \varphi_1 + \varphi_2$. Denote

$$A^2(M; e^{-\varphi}) := \{ h \in \mathcal{O}_M : \int_M |h|^2 e^{-\varphi} d\lambda_M < +\infty \}$$

and $\|h\| := (\int_M |h|^2 e^{-\varphi} d\lambda_M)^{\frac{1}{2}}$ for any $h \in A^2(M; e^{-\varphi})$.

We firstly recall the following lemma.

**Lemma 2.1** (see [44]). Let $\mu$ be a real positive Lebesgue measurable function on open subset $D_1 \subset \mathbb{C}^n$. Assume that there exists a number $a > 0$ such that the function $\mu^{-a}$ is integrable on some open set $D \subset D_1$ with respect to Lebesgue measure $d\lambda$. Let $f$ be a holomorphic function on $D$. Assume that $\|f\|_\mu < +\infty$, where $\|f\|_\mu := (\int_D |f|^2 \mu d\lambda)^{\frac{1}{2}}$. Then for any compact set $K \subset D$, there exists a constant $C_K > 0$ such that

$$\sup_K |f(z)| \leq C_K \|f\|_\mu.$$ 

**Proof.** The following proof can be referred to [44]. Let $K$ be a compact subset of $D$. There exists a real number $r > 0$ (depends on $K$) such that $B(z, r) \subset D$ for any $z \in K$. Let $p := \frac{1}{1+a}$ and $q = 1 + a$. Note that $|f|^p$ is plurisubharmonic function on $D$. It follows from sub-mean value inequality that for any $z \in K$, we have

$$|f(z)|^\frac{p}{q} \leq \frac{1}{\text{Vol}(B(z, r))} \int_{B(z, r)} |f|^\frac{p}{q} d\lambda.$$ 

Then it follows from Hölder inequality that

$$\text{Vol}(B(z, r)) |f(z)|^\frac{p}{q} \leq \int_{B(z, r)} |f|^\frac{p}{q} \mu^\frac{1}{p} \mu^{-\frac{a}{p}} d\lambda$$

$$\leq \left( \int_{B(z, r)} |f|^2 \mu d\lambda \right)^\frac{p}{2} \left( \int_{B(z, r)} \mu^{-\frac{a}{p}} d\lambda \right)^\frac{1}{2}$$

$$\leq \|f\|_\mu^p \left( \int_{B(z, r)} \mu^{-a} d\lambda \right)^\frac{1}{2}. \quad (2.1)$$

Hence

$$|f(z)| \leq \left( \text{Vol}(B(z, r)) \right)^{\frac{p}{q}} \left( \int_{B(z, r)} \mu^{-a} d\lambda \right)^\frac{1}{2} \|f\|_\mu.$$ 

Denote $C_K := \left( \text{Vol}(B(z, r)) \right)^{\frac{p}{q}} \left( \int_{B(z, r)} \mu^{-a} d\lambda \right)^\frac{1}{2}$ and for any $z \in K$, we have

$$|f(z)| \leq C_K \|f\|_\mu.$$ 

**Lemma 2.1** has been proved. \qed
Remark 2.2. Let $\alpha \in \mathbb{Z}^n_{\geq 0}$ be a multi-index. Let $f$ be a holomorphic function on $D$. For any compact subset $K \subset D$, it follows from Lemma 2.1 and Cauchy integral formula that there exists a constant $C_{K,\alpha} > 0$ such that we have

$$
\sup_{K} |\partial^\alpha f(z)| \leq C_{K,\alpha} ||f||_\mu.
$$

In the following discussion, we assume that for any relatively compact set $U_1 \Subset U$ ($W_2 \Subset W$), there exists a real number $a_1 > 0$ ($a_2 > 0$) such that $e^{\alpha_1 \phi_1}$ ($e^{\alpha_2 \phi_2}$) is integrable on $U_1$ ($W_2$). Then we have the following proposition.

Proposition 2.3. $A^2(U; e^{-\phi_1})$, $A^2(W; e^{-\phi_2})$ and $A^2(M; e^{-\phi})$ are separable Hilbert spaces.

Proof. We prove that $A^2(U; e^{-\phi_1})$ is separable Hilbert space. The same proof also holds for $A^2(W; e^{-\phi_2})$ and $A^2(M; e^{-\phi})$.

It is clear that we only need to prove that $A^2(U; e^{-\phi_1})$ is complete and separable. Let $U_1 \Subset U$ be a relatively compact subset of $U$, then there exists $a_1 > 0$ such that $e^{\alpha_1 \phi_1}$ is integrable on $U_1$. It follows from Lemma 2.1 that we

$$
\sup_{U_1} |f(z)| \leq C_{U_1} ||f||_1.
$$

(2.2)

Let $\{f_n\}$ be a Cauchy sequence in $A^2(U; e^{-\phi_1})$. It follows from inequality (2.2) that $\{f_n\}$ is compactly convergent to a holomorphic function $F$ on $U$. Then

$$
\int_U |f_n - F|^2 e^{-\phi_1} d\lambda_U = \int_U \liminf_{k \to +\infty} |f_n - f_k|^2 e^{-\phi_1} d\lambda_U
\leq \liminf_{k \to +\infty} \int_U |f_n - f_k|^2 e^{-\phi_1} d\lambda_U
= \liminf_{k \to +\infty} ||f_n - f_k||_1.
$$

(2.3)

Hence when $n$ is large enough, we have $||F||_1 \leq ||f_n - F||_1 + ||f_n|| \leq \epsilon + ||f_n|| < +\infty$, i.e. $F \in A^2(U; e^{-\phi_1})$. It follows from $\{f_n\}$ is a Cauchy sequence in $A^2(U; e^{-\phi_1})$ and (2.3) that we know that when $n$ is large enough, we have $\liminf_{k \to +\infty} ||f_n - f_k||_1 < \epsilon$, where $\epsilon > 0$ is a small constant. Hence we have $\lim_{n \to +\infty} ||f_n - F||_1 = 0$. Hence the norm $|| \cdot ||_1$ is complete and $A^2(U; e^{-\phi_1})$ is a Hilbert space.

Let $L^2(U; e^{-\phi_1})$ be the space of the Lebesgue measurable function, which is square integrable with the weight $e^{-\phi_1}$ on $U$. Note that $L^2(U; e^{-\phi_1})$ is a separable Hilbert space (see [14]). $A^2(U; e^{-\phi_1})$ is a closed subspace of $L^2(U; e^{-\phi_1})$, hence $A^2(U; e^{-\phi_1})$ is a separable Hilbert space.

Next, we recall some results about products of Bergman spaces.

Lemma 2.4. Let $h \in A^2(M; e^{-\phi})$. Let $\alpha \in \mathbb{Z}^n_{\geq 0}$ be a multi-index (let $\beta \in \mathbb{Z}^m_{\geq 0}$ be a multi-index). For any $z_0 \in U$ ($w_0 \in W$), we have $\partial^\alpha h(z_0, w) \in A^2(W; e^{-\phi_2})$ ($\partial^\beta h(z, w_0) \in A^2(U; e^{-\phi_1})$).

Proof. It follows from Remark 2.2 that there exists a constant $C_0 > 0$ ($C_0$ is independent of $w$) such that for any $w \in W$,

$$
|\partial^\alpha h(z_0, w)| \leq C_0 ||h(z, w)||_1.
$$
Then it follows from $h \in A^2(M, e^{-\varphi})$ and Fubini’s theorem that we have
\[
|\partial_z^\alpha h(z_0, w)|^2 := \int_W |\partial_z^\alpha h(z_0, w)|^2 e^{-\varphi_2} d\lambda_W
\]
\[
\leq C_0 \int_W |h(z, w)|^2 e^{-\varphi_2} d\lambda_W
\]
\[
= C_0 \int_W \int_U |h(z, w)|^2 e^{-\varphi_1 - \varphi_2} d\lambda_M
\]
\[
= C_0 ||h||^2 < +\infty.
\]
Hence we have $\partial_z^\alpha h(z_0, w) \in A^2(W, e^{-\varphi_2})$.

The same proof as above shows that $\partial_w^\alpha h(z, w_0) \in A^2(U, e^{-\varphi_1})$.

Lemma 2.4 has been proved. \(\square\)

The following lemma will be used in the proof of Lemma 2.6

**Lemma 2.5.** Let $h \in A^2(M, e^{-\varphi})$ and $T \in A^2(U, e^{-\varphi_1})$. For any $w \in W$, denote
\[
H(w) := \int_U h(z, w) T(z) e^{-\varphi_1(z)} d\lambda_U.
\]
Then $H(w)$ is a holomorphic function on $W$ and $H(w) \in A^2(W, e^{-\varphi_2})$.

**Proof.** When $w \in W$ is fixed, it follows from Lemma 2.4 that $h(z, w)$ belongs to $A^2(U, e^{-\varphi_1})$. It follows from Cauchy-Schwarz inequality and both $T$ and $h(z, w)$ belong to $A^2(U, e^{-\varphi_1})$ that
\[
|H(w)| \leq \left( \int_U |h(z, w)|^2 e^{-\varphi_1(z)} d\lambda_U \right)^\frac{1}{2} \left( \int_U |T(z)|^2 e^{-\varphi_1(z)} d\lambda_U \right)^\frac{1}{2} < +\infty,
\]
for any $w \in W$.

Let $w = (w_1, w_2, \ldots, w_m) \in W$ be given. Let $W_0 \subseteq W$ be an open convex neighborhood of $w$ in $W$. Let $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_m) \in W_0$. It follows from Remark 2.2 that we have
\[
|h(z, \tilde{w}) - h(z, w) - \sum_{j=1}^m (w_j - \tilde{w}_j) \partial_{w_j} h(z, w)|^2
\]
\[
= \left| \int_0^1 dh(z, w + t(\tilde{w} - w)) \frac{dt}{dt} \right|^2 - \sum_{j=1}^m (w_j - \tilde{w}_j) \partial_{w_j} h(z, w)|^2
\]
\[
= \left| \sum_{j=1}^m (w_j - \tilde{w}_j) \int_0^1 \partial_{w_j} h(z, w + t(\tilde{w} - w)) dt - \sum_{j=1}^m (w_j - \tilde{w}_j) \partial_{w_j} h(z, w)|^2
\]
\[
= \sum_{j=1}^m \sum_{k=1}^m (w_j - \tilde{w}_j)(w_k - \tilde{w}_k) \int_0^1 \int_0^1 t_1 (\partial_{w_j} \partial_{w_k} h(z, w + t_1 t_2(\tilde{w} - w)) dt_1 dt_2 dt_1|^2
\]
\[
\leq |\tilde{w} - w|^4 \frac{1}{4} \sum_{j=1}^m \sum_{k=1}^m \sup_{\tilde{w} \in W_0} |\partial_{w_j} \partial_{w_k} h(z, \tilde{w})|^2
\]
\[
\leq C_1 |\tilde{w} - w|^4 ||h(z, \cdot)||^2_2,
\]
where \( C_1 > 0 \) is a constant independent of \( z \). Then we have

\[
|H(\tilde{w}) - H(w) - \sum_{j=1}^{m} (w_j - \tilde{w}_j) \int_{U} (\partial_{w_j} h(z, w)) T(z) e^{-\varphi_1(z)} d\lambda_U|^2
\]

\[
= \left| \int_{U} \left( h(z, \tilde{w}) - h(z, w) - \sum_{j=1}^{m} (w_j - \tilde{w}_j) \partial_{w_j} h(z, w) \right) T(z) e^{-\varphi_1(z)} d\lambda_U \right|^2
\]

\[
\leq \|T\|^2 \int_{U} |h(z, \tilde{w}) - h(z, w) - \sum_{j=1}^{m} (w_j - \tilde{w}_j) \partial_{w_j} h(z, w)|^2 e^{-\varphi_1(z)} d\lambda_U
\]

\[
\leq \|T\|^2 \|\tilde{w} - w\|^4 C_1 \int_{U} \|h(z, \cdot)\|^2 e^{-\varphi_1(z)} d\lambda_U
\]

\[
= \|T\|^2 \|\tilde{w} - w\|^4 C_1 \int_{W} \int_{U} |h(z, \tilde{w})|^2 e^{-\varphi_2(\tilde{w})} e^{-\varphi_1(z)} d\lambda_W d\lambda_U
\]

\[
= C_1 \|T\|^2 \|\tilde{w} - w\|^4 \|h\|^2
\]

It follows from inequality \( 2.5 \) that \( H(w) \) is a holomorphic function on \( W \). Note that

\[
\|H(w)\|_2^2 \leq \int_{W} |H(w)|^2 e^{-\varphi_2} d\lambda_W
\]

\[
\leq \left( \int_{U} |T(z)|^2 e^{-\varphi_1(z)} d\lambda_U \right) \int_{W} \int_{U} |h(z, w)|^2 e^{-\varphi_1(z)} d\lambda_U d\lambda_W
\]

\[
= \|T\|^2 \cdot \|h\|^2 < +\infty.
\]

(2.6)

Hence we have \( H(w) \in A^2(W, e^{-\varphi_2}) \). \( \square \)

The following lemma implies that the product of bases of \( A^2(U; e^{-\varphi_1}) \) and \( A^2(W; e^{-\varphi_2}) \) make a basis of \( A^2(M; e^{-\varphi}) \).

**Lemma 2.6.** Let \( \{f_i(z)\}_{i \in \mathbb{Z}_{>0}} \) and \( \{g_j(w)\}_{j \in \mathbb{Z}_{>0}} \) be the complete orthonormal bases of \( A^2(U, e^{-\varphi_1}) \) and \( A^2(W, e^{-\varphi_2}) \) respectively. Then \( \{f_i(z)g_j(w)\}_{i,j \in \mathbb{Z}_{>0}} \) is a complete orthonormal basis of \( A^2(M, e^{-\varphi}) \).

**Proof.** It follows from Fubini’s theorem that \( \{f_i(z)g_j(w)\}_{i,j \in \mathbb{Z}_{>0}} \) is orthonormal basis of \( A^2(M, e^{-\varphi}) \). Now we prove \( \{f_i(z)g_j(w)\}_{i,j \in \mathbb{Z}_{>0}} \) is complete. Let \( h(z, w) \in A^2(M, e^{-\varphi}) \) such that \( \int_{M} h(z, w) f_i(z) g_j(w) e^{-\varphi} d\lambda_M = 0 \) for any \( i, j \in \mathbb{Z}_{>0} \).

For any \( i_0 \in \mathbb{Z}_{>0} \), denote

\[
H_{i_0}(w) := \int_{U} h(z, w) f_{i_0}(z) e^{-\varphi_1} d\lambda_U,
\]
where \( w \in W \). It follows from Lemma 2.5 that \( H_{i_0}(w) \in A^2(W, e^{-\varphi_2}) \). It follows from Fubini’s theorem that for any \( j_0 \in \mathbb{Z}_{\geq 0} \), we have

\[
0 = \int_M h(z, w) f_{i_0}(z) g_{j_0}(w) e^{-\varphi} d\lambda_M
= \int_W \left( \int_U h(z, w) f_{i_0}(z) e^{-\varphi_1} d\lambda_U \right) g_{j_0}(w) e^{-\varphi_2} d\lambda_W
= \int_W H_{i_0}(w) g_{j_0}(w) e^{-\varphi_2} d\lambda_W. \tag{2.7}
\]

As \( j_0 \) is arbitrarily chosen and \( \{g_j(w)\}_{j \in \mathbb{Z}_{\geq 0}} \) is the complete orthonormal basis of \( A^2(W, e^{-\varphi_2}) \), we know that \( H_{i_0}(w) = 0 \) for any \( i_0 \in \mathbb{Z}_{\geq 0} \).

For any \( w \in W \), it follows from \( H_i(w) = 0 \) for any \( i \in \mathbb{Z}_{\geq 0} \) and \( \{f_i(z)\}_{i \in \mathbb{Z}_{\geq 0}} \) is the complete orthonormal basis of \( A^2(U, e^{-\varphi_1}) \), we know that \( h(z, w) = 0 \), for any \( z \in U \). Since \( w \) is arbitrarily chosen, we know that \( h(z, w) \equiv 0 \) on \( M \), for any \( z \in U \). This shows that \( \{f_i(z)g_j(w)\}_{i,j \in \mathbb{Z}_{\geq 0}} \) is complete.

Lemma 2.6 has been proved. \( \square \)

Let \( \Delta \subset \mathbb{C} \) be the unit disk, and let \( Y \) be an \((n-1)\)-dimensional complex manifold, and let \( N = \Delta \times Y \). Let \( \pi_1 \) and \( \pi_2 \) be the natural projections from \( N \) to \( \Delta \) and \( Y \) respectively. Let \( \rho_1 \) be a nonnegative Lebesgue measurable function on \( \Delta \) satisfying that \( \rho_1(w) = \rho_1(|w|) \) for any \( w \) and the Lebesgue measure of \( \{w \in \Delta : \rho_1(w) > 0\} \) is positive. Let \( \rho_2 \) be a nonnegative Lebesgue measurable function on \( Y \), and denote that \( \rho = \pi_1^*\rho_1 \times \pi_2^*\rho_2 \) on \( N \). We recall the following lemma.

**Lemma 2.7 (see [2]).** For any holomorphic \((n, 0)\) form \( F \) on \( N \), there exists a unique sequence of holomorphic \((n-1, 0)\) forms \( \{F_j\}_{j \in \mathbb{Z}_{\geq 0}} \) on \( Y \) such that

\[
F = \sum_{j \in \mathbb{Z}_{\geq 0}} \pi_1^*(w) dw_j \wedge \pi_2^*(F_j),
\]

where the right term of the above equality is uniformly convergent on any compact subset of \( N \). Moreover, if \( \int_N |F|^2 \rho < +\infty \), we have

\[
\int_Y |F_j|^2 \rho_2 < +\infty
\]

for any \( j \in \mathbb{Z}_{\geq 0} \).

In the following discussion, we assume that \( U \subset \mathbb{C} \) is an open subset containing origin \( 0 \) in \( \mathbb{C} \).

**Lemma 2.8.** There exists a countable complete orthonormal basis \( \{f_i(z)\}_{i \in \mathbb{Z}_{\geq 0}} \) of \( A^2(U, e^{-\varphi_1}) \) such that \( k_i := ord_0(f_i) \) is strictly increasing with respect to \( i \).

**Proof.** For any \( k \in \mathbb{Z}_{\geq 0} \), denote

\[
A^2_k(U, e^{-\varphi_1}) := \{ f \in A^2(U, e^{-\varphi_1}) : (\partial^k_0 f)(0) = 1 \& (\partial^k_j f)(0) = 0 \text{ for } j < k \}. \]

If \( A^2_k(U, e^{-\varphi_1}) \neq \emptyset \), denote \( S_k := \inf \{ ||f||_1 : f \in A^2_k(U, e^{-\varphi_1}) \} \). We prove that there exists a \( F_k \in A^2_k(U, e^{-\varphi_1}) \) such that \( S_k = ||F_k||_1 \).

Let \( \{F_j\}_{j \in \mathbb{Z}_{\geq 0}} \) be a sequence of holomorphic functions in \( A^2_k(U, e^{-\varphi_1}) \) such that \( ||F_j||_1 \to S_k \) as \( j \to +\infty \). It follows from Lemma 2.4 that for any relative compact subset \( U_1 \subset U \), we have \( \sup_{U_1} |F_j| \leq C_1 ||F_j||_1 \) for some \( C_1 > 0 \). As \( ||F_j||_1 \to S_k \),
when $j \to +\infty$, we know that $\sup_j \sup_{U,e} |F_j| < C_2$ for some $C_2 > 0$. Hence there exists a subsequence of $\{F_j\}_{j \in \mathbb{Z}_{>0}}$ (also denote by $\{F_j\}_{j \in \mathbb{Z}_{>0}}$) compactly convergent to a holomorphic function $F_k$ on $U$. As $\{F_j\}_{j \in \mathbb{Z}_{>0}}$ is compactly convergent to $F_k$ and $F_j \in A^2_k(U, e^{-\varphi_1})$, we know that $(\partial^k_z F_k)(0) = 1$ and $(\partial^k_z F_k)(0) = 0$ for $j < k$.

It follows from Fatou’s Lemma that we know
\[
\int_U |F_k|^2 e^{-\varphi_1} d\lambda_U \leq \liminf_{j \to +\infty} \int_U |F_j|^2 e^{-\varphi_1} d\lambda_U = S_k,
\]
which implies that $F_k \in A^2_k(U, e^{-\varphi_1})$ and $\|F_k\|_1 \leq S_k$. By definition of $S_k$, we know that $\|F_k\|_1 = S_k$.

Denote $\mathbb{K} := \{k \in \mathbb{Z}_{>0} : A^2_k(U, e^{-\varphi_1}) \neq \emptyset\}$. Note that $\mathbb{K}$ is a countable infinite subset of $\mathbb{Z}_{>0}$. For any $k \in \mathbb{K}$, we take $f_k := \frac{F_k}{\|F_k\|_1}$.

Now we prove $\{f_k(z)\}_{k \in \mathbb{K}}$ is a complete orthonormal basis of $A^2(U, e^{-\varphi_1})$.

Firstly, we prove that $\{f_k(z)\}_{k \in \mathbb{K}}$ is orthonormal. Let $k \in \mathbb{K}$. We prove that for any $g \in A^2(U, e^{-\varphi_1})$ satisfying $(\partial^k_z g)(0) = 0$ for any $j \leq k$, we have $(f_k, g)_1 = 0$, where $(\cdot, \cdot)_1$ is the inner product on $A^2(U, e^{-\varphi_1})$. Note that for any $\alpha \in \mathbb{C}$, we have $f_k + \alpha g \in A^2_k(U, e^{-\varphi_1})$. Hence for any $\alpha \in \mathbb{C}$, $\|f_k + \alpha g\|_1 \geq S_k$. However let $\alpha = -\frac{(f_k, g)_1}{\|g\|_1^2}$, then by direct computation we have
\[
\|f_k + \alpha g\|_1^2 = \|f_k\|^2 - \frac{(f_k, g)_1^2}{\|g\|_1^2}.
\]
If $(f_k, g)_1 \neq 0$, we will have $\|f_k + \alpha g\|_1 < S_k$, which is a contradiction. Hence we have $(f_k, g)_1 = 0$. By the above discussion, we also know that $f_k$ is the unique holomorphic function in $A^2_k(U, e^{-\varphi_1})$ such that $\|f_k\|_1 = S_k$. For any $k_1 < k_2 \in \mathbb{K}$, following above discussion, we have $(f_{k_1}, f_{k_2})_1 = 0$. By definition, we also have $\|f_k\|_1 = 1$. Hence $\{f_k(z)\}_{k \in \mathbb{K}}$ is orthonormal basis.

Secondly, we prove that $\{f_k(z)\}_{k \in \mathbb{K}}$ is complete. If $k \in \mathbb{Z}_{>0} \setminus \mathbb{K}$, then we set $f_k(z) \equiv 0$. Let $h \in A^2(U, e^{-\varphi_1})$ be a holomorphic function. Let $\{a_k\}_{k \in \mathbb{Z}_{>0}}$ be a sequence of complex numbers which will be determined later. Let
\[
h_k(z) = \sum_{j \leq k} a_j f_j(z).
\]
Now we will choose $\{a_k\}_{k \in \mathbb{Z}_{>0}}$ such that for any $j \leq k$,
\[
(\partial^j_z h_k)(0) = (\partial^j_z h)(0). \tag{2.8}
\]
If $0 \in \mathbb{Z}_{>0} \setminus \mathbb{K}$, we set $a_0 = 0$. If $0 \notin \mathbb{Z}_{>0} \setminus \mathbb{K}$, we set $a_0 = h(0)||F_0||_1$. By the construction of $\{f_j(z)\}_{j \in \mathbb{Z}_{>0}}$, we know that $h_k(0) = h(0)$. Assume that for any $j \leq k - 1$, the sequence $\{a_j\}_{0 \leq j \leq k - 1}$ has been chosen. Now we choose the number $a_k$. If $k \in \mathbb{Z}_{>0} \setminus \mathbb{K}$, then $A^2_k(U, e^{-\varphi_1}) = \emptyset$. We also note that $h - h_{k-1}$ satisfies
\[
(\partial^j_z(h - h_{k-1}))(0) = 0, \text{ for any } j \leq k - 1.
\]
Then we have $\left(\partial^k_z(h - h_{k-1})\right)(0) = 0$, otherwise $\frac{(h(z) - h_{k-1}(z))}{(\partial^k_z(h - h_{k-1}))(0)} \in A^2_k(U, e^{-\varphi_1})$ which is a contradiction. Since $\left(\partial^k_z(h - h_{k-1})\right)(0) = 0$, we set $a_k = 0$. If $k \in \mathbb{K}$, it follows from $(\partial^k_z h_k)(0) = (\partial^k_z h)(0)$ that we have the following equation
\[
a_0(\partial^k_z f_0)(0) + a_1(\partial^k_z f_1)(0) + \cdots + a_k(\partial^k_z f_k)(0) = (\partial^k_z h)(0). \tag{2.9}
\]
Note that $(\partial^k f_k)(0) = \frac{1}{||f_k||} \neq 0$ and $\{a_j\}_{0 \leq j \leq k-1}$ has been chosen. The equation \([2.3]\) can be solved, i.e. we can find $a_k$ such that $(\partial^k h_k)(0) = (\partial^k h)(0)$ holds. Hence we can choose $\{a_k\}_{k \in \mathbb{Z}_{\geq 0}}$ by induction such that for any $j \leq k$,

$$(\partial^j h_k)(0) = (\partial^j h)(0).$$

Note that for any $j \leq k$, we have $(\partial^j (h_k - h))(0) = 0$. Hence we know that

$$\langle h_k - h, f_k \rangle_1 = 0,$$

By Bessel inequality, we have $\sum_{j=0}^{+\infty} |a_j|^2 \leq ||h||_2^2$. Denote $H(z) := \sum_{j=0}^{+\infty} a_j f_j(z)$, where the right hand-side is uniformly convergent to $H(z)$ on any compact subset of $U$ and we have $H(z) \in A^2(U, e^{-\nu^2})$. Hence for any $k \in \mathbb{Z}_{\geq 0}$, we have

$$(\partial^k H)(0) = \lim_{j \to +\infty} (\partial^k h_k)(0) = (\partial^k h)(0).$$

Hence $h \equiv H$, which implies that $\{f_k(z)\}_{k \in \mathbb{K}}$ is complete.

Denote $\hat{f}_1 := f_{k_1}$, where $k_1$ satisfies $\text{ord}_{0} f_{k_1}$ is minimal among $k \in \mathbb{K}$. Denote $\hat{f}_2 := f_{k_2}$, where $k_2$ satisfies $\text{ord}_{0} f_{k_2}$ is minimal among $k \in \mathbb{K}\setminus\{k_1\}$. Denote $\hat{f}_i := f_{k_i}$, where $k_i$ satisfies $\text{ord}_{0} f_{k_i}$ is minimal among $k \in \mathbb{K}\setminus\{k_1, k_2, \ldots, k_{i-1}\}$.

Now $\{\text{ord}_{0} \hat{f}_i\}_{1 \leq i \leq k}$ is strictly increasing with respect to $i$. Note that $\{\hat{f}_i(z)\}_{i \geq 0}$ is a rearrangement of $\{f_k(z)\}_{k \in \mathbb{K}}$ and $\{f_k(z)\}_{k \in \mathbb{K}}$ is a complete orthonormal basis of $A^2(U, e^{-\nu^2})$. We know that $\{\hat{f}_i(z)\}_{i \geq 0}$ satisfies the requirement of Lemma \[2.8\].

We are done.

\[\square\]

In the following discussion, let $U = \Delta \subset \mathbb{C}$ be the unit disk. It follows from Lemma \[2.8\] that there exists a complete orthonormal basis $\{\hat{f}_i(z)\}_{i \geq 0}$ of $A^2(U, e^{-\nu^2})$ which satisfies $\text{ord}_{0} \hat{f}_i)$ is strictly increasing with respect to $i$. Let $W \subset \mathbb{C}^m$ be an open set. Let $\{g_j(w)\}_{j \in \mathbb{Z}_{\geq 0}}$ be the complete orthonormal basis of $A^2(W, e^{-\nu^2})$. Let $F$ be a holomorphic function on $M := U \times W$ satisfying that $F \in A^2(M, e^{-\nu^2})$. It follows from Lemma \[2.7\] that $F = \sum_{l \geq k} z^l F_l(w)$, where $\{F_l\}_{l \geq k}$ is a sequence of holomorphic functions on $W$ and $F_{k}(w) \neq 0$. Denote $k_i := \text{ord}_{0} \hat{f}_i)$ for any $i \in \mathbb{Z}_{\geq 0}$ and $k := \inf_{i \in \mathbb{Z}_{\geq 0}}\{k_i\} = k_1$.

**Lemma 2.9.** We have $\hat{k} \geq k$ and for any $l \geq \hat{k}$, $F_l \in A^2(W, e^{-\nu^2})$.

**Proof.** It follows from Lemma \[2.8\] that $\{\hat{f}_i(z)g_j(w)\}_{i, j \in \mathbb{Z}_{\geq 0}}$ is a complete orthonormal basis of $A^2(M, e^{-\nu^2})$.

As $F \in A^2(M, e^{-\nu^2})$, we know that

$$F(z, w) = \sum_{i, j \in \mathbb{Z}_{\geq 0}} a_{ij} \hat{f}_i(z)g_j(w) = \sum_{i \in \mathbb{Z}_{\geq 0}} \hat{f}_i(z) \sum_{j \in \mathbb{Z}_{\geq 0}} a_{ij}g_j(w),$$

for some $a_{ij} \in \mathbb{C}$, where the right-hand side is uniformly convergent on any compact subset of $M$. It follows from $F \in A^2(M, e^{-\nu^2})$ that we know $\sum_{j \in \mathbb{Z}_{\geq 0}} a_{ij}g_j(w) \in A^2(W, e^{-\nu^2})$ for any $i \in \mathbb{Z}_{\geq 0}$.

As $U \subset \mathbb{C}$ is unit disc, we assume that for each $i \in \mathbb{Z}_{\geq 0}$, $\hat{f}_i(z) = \sum_{l \geq 0} b_{il} z^l$ on $U$ for some $b_{il} \in \mathbb{C}$, where the right-hand side is uniformly convergent on any compact subset of $U$. Then we know that $F(z, w) = \sum_{l \geq k} z^l \left( \sum_{i \in \mathbb{Z}_{\geq 0}} b_{il}(\sum_{j \in \mathbb{Z}_{\geq 0}} a_{ij}g_j(w)) \right)$,
where the index set \( I_l := \{ i \in \mathbb{Z}_{\geq 0} : k_i \leq l \} \) and the right-hand side is uniformly convergent on any compact subset of \( M \). For fixed \( l \geq k \), as \( k_i \in \mathbb{Z}_{\geq 0} \) is strictly increasing with respect to \( i \), we know that \( I_l \) is a finite set for any \( l \geq k \).

Recall that it follows from Lemma 2.7 that \( F = \sum_{l \geq k} z^l F_l(w) \) and \( \{ F_l \}_{l \geq k} \) is unique. The uniqueness of \( \{ F_l \}_{l \geq k} \) implies that we have \( \sum_{i \in I_l} b_i (\sum_{j \in \mathbb{Z}_{\geq 0}} a_{ij} g_j(w)) = 0 \) for \( k \leq l < \tilde{k} \) and \( F_l(w) = \sum_{i \in I_l} b_i (\sum_{j \in \mathbb{Z}_{\geq 0}} a_{ij} g_j(w)) \) for any \( l \geq k \). It follows from \( I_l \) is a finite set for any \( l \geq k \) and \( \sum_{j \in \mathbb{Z}_{\geq 0}} a_{ij} g_j(w) \in A^2(W, e^{-\varphi_z}) \) for any \( i \in \mathbb{Z}_{\geq 0} \). that we have \( F_l \in A^2(W, e^{-\varphi_z}) \) for any \( l \geq \tilde{k} \).

**Remark 2.10.** By the definition of \( k \), we know that for any \( l \geq k \), \(|z|^2 e^{-\varphi_z} \) is \( L^1 \) integrable near \( 0 \in \mathbb{C} \). As \( k \geq k \), we have \(|z|^2 e^{-\varphi_z} \) is \( L^1 \) integrable near \( 0 \in \mathbb{C} \) for any \( l \geq k \).

### 2.2. Concavity property on weakly pseudoconvex Kähler manifolds.

In this section, we recall some results about the concavity property on weakly pseudoconvex Kähler manifolds [21] (see also [20]). Let \( M \) be a complex manifold. Let \( X \) and \( Z \) be closed subsets of \( M \). We call that a triple \((M, X, Z)\) satisfies condition \((A)\), if the following two statements hold:

I. \( X \) is a closed subset of \( M \) and \( X \) is locally negligible with respect to \( L^2 \) holomorphic functions; i.e., for any local coordinated neighborhood \( U \subseteq M \) and for any \( L^2 \) holomorphic function \( f \) on \( U \setminus X \), there exists an \( L^2 \) holomorphic function \( f \) on \( U \) such that \( f|_{U \setminus X} = f \) with the same \( L^2 \) norm;

II. \( Z \) is an analytic subset of \( M \) and \( M \setminus (X \cup Z) \) is a weakly pseudoconvex Kähler manifold.

Let \( M \) be an \( n \)-dimensional complex manifold. Assume that \((M, X, Z)\) satisfies condition \((A)\). Let \( K_M \) be the canonical line bundle on \( M \). Let \( dV_M \) be a continuous volume form on \( M \). Let \( F \) be a holomorphic function on \( M \). Assume that \( F \) is not identically zero. Let \( \psi \) be a plurisubharmonic function on \( M \). Let \( \varphi \) be a Lebesgue measurable function on \( M \) such that \( \varphi + \psi \) is a plurisubharmonic function on \( M \).

Let \( T \in (-\infty, +\infty) \).

Denote that \( \Psi := \min\{\psi - 2 \log |F|, -T\} \).

For any \( z \in M \) satisfying \( F(z) = 0 \), we set \( \Psi(z) = -T \).

**Definition 2.11 (21).** We call that a positive measurable function \( c \) on \((T, +\infty)\) is in class \( P_{T, M, \Psi} \) if the following two statements hold:

1. \( c(t)e^{-t} \) is decreasing with respect to \( t \);

2. For any \( t_0 > T \), there exists a closed subset \( E_0 \) of \( M \) such that \( E_0 \subseteq Z \cap \{ \Psi(z) = -\infty \} \) and for any compact subset \( K \subseteq M \setminus E_0 \), \( e^{-\varphi} c(-\Psi) \) has a positive lower bound on \( K \cap \{ \Psi < -t_0 \} \).

For \( f_{z_0} \in J(\Psi)_{z_0} \) and \( a, b \geq 0 \), we call \( f_{z_0} \in I(a\Psi + b\varphi)_{z_0} \) if there exist \( t \gg T \) and a neighborhood \( V \) of \( z_0 \) such that \( \int_{\Psi < -t} |f|^2 e^{-a\Psi - b\varphi} dV_M < +\infty \). Note that \( I(a\Psi + b\varphi)_{z_0} \) is an \( \mathcal{O}_{M, z_0} \)-submodule of \( J(\Psi)_{z_0} \).

Let \( Z_0 \) be a subset of \( \cap_{t \geq T} \{ \Psi < -t \} \). Let \( f \) be a holomorphic \((n, 0)\) form on \( \{ \Psi < -t_0 \} \cap V \), where \( V \) contains \( Z_0 \) as an open subset of \( M \) and \( t_0 \geq T \) is a real number. Let \( J_{z_0} \) be an \( \mathcal{O}_{M, z_0} \)-submodule of \( J(\Psi)_{z_0} \) such that \( I(\Psi + \varphi)_{z_0} \subset J_{z_0} \), where \( z_0 \in Z_0 \). Let \( J \) be the \( \mathcal{O}_{M, z_0} \)-module sheaf with stalks \( J_{z_0} \), where \( z_0 \in Z_0 \).
Denote the minimal $L^2$ integral related to $J$

\[
\inf \left\{ \int_{\{ \Psi < -t \}} |\tilde{f}|^2 e^{-\varphi} c(-\Psi) : \tilde{f} \in H^0(\{ \Psi < -t \}, \mathcal{O}(K_M)) \right\}
\]

& (\tilde{f} - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}, \text{ for any } z_0 \in Z_0 \right\}
\]

(2.10)

by $G(t; c, \Psi, \varphi, J, f)$, where $t \in [T, +\infty)$ and $|f|^2 := |f|^2$ for any $(n, 0)$ form $f$. Without misunderstanding, we denote $G(t; c, \Psi, \varphi, J, f)$ by $G(t)$ for simplicity. For various $c(t)$, we denote $G(t; c, \Psi, \varphi, J, f)$ by $G(t; c)$ respectively for simplicity.

We recall the concavity property for $G(t)$.

**Theorem 2.12** ([21]). Let $c \in \hat{P}_{T,M,\Psi}$ satisfying that $\int_{T_1}^{+\infty} c(s) e^{-s} dz < +\infty$, where $T_1 > T$. If there exists $t \in [T, +\infty)$ satisfying that $G(t) < +\infty$, then $G(h^{-1}(r))$ is concave with respect to $r \in (0, \int_{T_1}^{+\infty} c(t) e^{-t} dt)$, $\lim_{t \to T+0} G(t) = G(T)$ and $\lim_{t \to +\infty} G(t) = 0$, where $h(t) = \int_t^{+\infty} c(s) e^{-s} ds$.

When $F = 1$ and $\psi(z) = -\infty$ for any $z \in Z_0$, $J_z$ is an ideal of $\mathcal{O}_{M,z}$ for any $z \in Z_0$ and Theorem 2.12 degenerates to the concavity property with respect to the ideals at inner points ([21], see also [19]).

Let $c(t)$ be a nonnegative measurable function on $(T, +\infty)$. Denote that

\[
\mathcal{H}^2(t; c) := \left\{ \tilde{f} : \int_{\{ \Psi < -t \}} |\tilde{f}|^2 e^{-\varphi} c(-\Psi) < +\infty, \tilde{f} \in H^0(\{ \Psi < -t \}, \mathcal{O}(K_M)) \right\}
\]

\& (\tilde{f} - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}, \text{ for any } z_0 \in Z_0 \right\},
\]

where $t \in [T, +\infty)$.

The following corollary gives a necessary condition for the concavity degenerating to linearity.

**Corollary 2.13** ([21]). Let $c \in \hat{P}_{T,M,\Psi}$ satisfying that $\int_{T_1}^{+\infty} c(s) e^{-s} ds < +\infty$, where $T_1 > T$. Assume that $G(t) \in (0, +\infty)$ for some $t \geq T$, and $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_{T_1}^{+\infty} c(s) e^{-s} ds)$, where $h(t) = \int_t^{+\infty} c(s) e^{-s} ds$.

Then there exists a unique holomorphic $(n, 0)$ form $\tilde{F}$ on $\{ \Psi < -T \}$ such that $(\tilde{F} - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ holds for any $z_0 \in Z_0$, and $G(t) = \int_{\{ \Psi < -t \}} |\tilde{F}|^2 e^{-\varphi} c(-\Psi)$ holds for any $t \geq T$.

Furthermore

\[
\int_{\{ \Psi < -t_1 \}} |\tilde{F}|^2 e^{-\varphi} a(-\Psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(t) e^{-t} dt} \int_{t_1}^{t_2} a(t) e^{-t} dt
\]

holds for any nonnegative measurable function $a$ on $(T, +\infty)$, where $T \leq t_2 < t_1 \leq +\infty$ and $T_1 \in (T, +\infty)$.
Remark 2.14. If $\mathcal{H}^2(t_0; \tilde{c}) \subset \mathcal{H}^2(t_0; c)$ for some $t_0 \geq T$, we have
\[ G(t_0; \tilde{c}) = \int_{\{\Psi < -t_0\}} |F|^2 e^{-\tilde{c} \Psi} d\Psi = \frac{G(T_1; c)}{\int_{T_1}^\infty c(t)e^{-t} dt} \int_{T_1}^{\infty} \tilde{c}(s)e^{-s} ds, \]
where $\tilde{c}$ is a nonnegative measurable function on $(T, +\infty)$ and $T_1 \in (T, +\infty)$. Thus, if $\mathcal{H}^2(t; \tilde{c}) \subset \mathcal{H}^2(t; c)$ for any $t > T$, then $G(h^{-1}(r); \tilde{c})$ is linear with respect to $r \in [0, \int_{T}^{\infty} c(s)e^{-s} ds]$.

We recall a characterization of $G(t) = 0$, where $t \geq T$.

Lemma 2.15. Let $c \in \tilde{P}_{T,M,\Psi}$ satisfying that $\int_{T_1}^{\infty} c(s)e^{-s} ds < +\infty$, where $T_1 > T$. Let $t_0 \geq T$. The following two statements are equivalent:
\begin{enumerate}
  \item $G(t_0) = 0$;
  \item $f_{t_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$.
\end{enumerate}

2.3. Properties of $\mathcal{O}_{M,p}$-module $J_p$. In this section, we present some properties of $\mathcal{O}_{M,p}$-module $J_p$. The notation used in this section can be referred to Section 1.6.

Let $T \in [-\infty, +\infty)$. Denote
\[ \Psi := \min\{\pi_1^* (\psi - 2 \log |F|), -T\}. \]
If $F(z) = 0$ for some $z \in \Omega$, we set $\Psi(z, w) = -T$ for any $w \in Y$. Let $T_1 > T$ be any real number.

Denote
\[ \varphi_1 := 2 \max\{\pi_1^* (\psi) + T_1, \pi_1^* (2 \log |F|)\}, \]
and
\[ \Psi_1 := \min\{\Psi, -T_1\}. \]

Denote $\varphi := \pi_1^* (\varphi_1) + \pi_2^* (\varphi_\Omega)$. By definition we have $I(\Psi_1 + \varphi)_p = I(\Psi + \varphi)_p$, for any $p \in M$. Recall that $\varphi_\Omega + \psi$ and $\psi$ are subharmonic functions on $\Omega$, hence for any relatively compact subset $U \subseteq \Omega$, there exists a real number $a_1 > 0$ such that $e^{a_1 \psi_\Omega}$ is integrable on $U$. Note that $\varphi_\Omega$ is a plurisubharmonic function on $Y$ and then for any relatively compact set $W \subseteq Y$, there exists a real number $a_2 > 0$ such that $e^{a_2 \varphi_\Omega}$ is integrable on $W$.

Let $c(t)$ be a positive measurable function on $(T, +\infty)$ such that $c(t) \in \tilde{P}_{T,M,\Psi}$. Let $dV_M$ be a continuous volume form on $M$. Let $p \in M$ be a point, denote that $H_p := \{f_p \in I(\Psi)_p : f_p |_{\{\Psi < -t_1\} \cap V_0} |F|^2 e^{-\varphi_1 c(-\Psi)} dV_M < +\infty$ for some $t > T$ and $V_0$ is an open neighborhood of $p\}$ and $H_p := \{f(p) \in \mathcal{O}_{M,p} : \int_{V_0} |F|^2 e^{-\varphi_1 c(-\Psi_1)} dV_M < +\infty$ for some open neighborhood $V_0$ of $p\}$.

As $c(t) \in \tilde{P}_{T,M,\Psi}$, hence $c(t) e^{-t}$ is decreasing with respect to $t$ and we have $I(\Psi_1 + \varphi)_p = I(\Psi + \varphi)_p \subset H_p$. We also note that $H_p$ is an ideal of $\mathcal{O}_{M,p}$.

In [21], we proved the following proposition of $H_p/I(\varphi + \Psi_1)_p$.

Proposition 2.16. There exists an $\mathcal{O}_{M,p}$-module isomorphism $P : H_p/I(\varphi + \Psi_1)_p \rightarrow H_p/I(\varphi + \varphi_1 + \Psi_1)_p$.

We recall the following closedness property of submodule of $\mathcal{O}_{C^\infty,\alpha}$.

Lemma 2.17 (see [18]). Let $N$ be a submodule of $\mathcal{O}_{C^\infty,\alpha}$, $1 \leq q < +\infty$, let $f_j \in \mathcal{O}_{C^\infty(U)^q}$ be a sequence of $q$-tuples holomorphic in an open neighborhood $U$ of the origin $o$. Assume that the $f_j$ converge uniformly in $U$ towards a $q$-tuples $f \in \mathcal{O}_{C^\infty(U)^q}$, assume furthermore that all germs $(f_j, o)$ belong to $N$. Then $(f, o) \in N$. 

Lemma 2.19. Let \( u \) be a plurisubharmonic function on \( \Delta^n \subset \mathbb{C}^n \). If \( v(dd^c u, o) < 1 \), then \( e^{-2u} \) is \( L^1 \) on a neighborhood of \( o \), where \( o \in \Delta^n \) is the origin.

Following from Lemma 2.18 and Siu’s Decomposition Theorem, we can obtain the following well-known result (see [21]).

Lemma 2.20. Let \( \phi \) be a subharmonic function on the unit disc \( \Delta \subset \mathbb{C} \). Then we have \( I(\phi)_o = (z^k)_o \) if and only if \( v(dd^c(\phi), o) \in [2k, 2k + 2] \).

Proof. It follows from Proposition 2.10 that \( H_p = I(\phi + \Psi_1)_p \) if and only if \( H_p = I(\phi + \Psi_1 + \varphi_1)_p \), where \( \mathcal{H}_p := \{(h, p) \in \mathcal{O}_{M, p} : |h|^2e^{-\phi - \varphi_1}c(-\Psi_1) \) is integrable near \( p \}. \) Now, we prove \( H_p = I(\phi + \Psi_1 + \varphi_1)_p \).

Without loss of generality, we can assume that \( M = \Delta \times \Delta^{n-1}, z = 0 \in \Delta \) and \( p = o \in M \) (the origin of \( M \)). As \( c(t)e^{-t} \) is decreasing, we have \( I(\phi + \Psi_1 + \varphi_1)_o \subset \mathcal{H}_o \). For any \( (h, o) \in \mathcal{H}_o \), there exists \( r_1 > 0 \) such that \( \int_{\Delta_{r_1} \times \Delta_{r_1}^{n-1}} |h|^2e^{-\phi - \varphi_1}c(-\Psi_1) < +\infty \), which implies that

\[
\int_{\Delta_{r_1} \times \Delta_{r_1}^{n-1}} |h|^2e^{-\phi - \varphi_1} < +\infty,
\]

(2.11)

where \( \Delta_{r_1} = \{ z \in \mathbb{C} : |z| < r_1 \} \). Hence we know that \( h \in A^2(\Delta_{r_1} \times \Delta_{r_1}^{n-1}, e^{-\phi - \varphi_1}) \). It follows from Lemma 2.9 and Remark 2.10 that we know that \( h(z, w) = \sum_{l \geq k} z^l F_l(w) \), where \( \{F_l\}_{l \geq k} \) is a sequence of holomorphic functions on \( W \) satisfying \( F_k(w) \neq 0 \), \( F_l \in A^2(\Delta_{r_1}^{n-1}, e^{-\varphi_1}) \) and \( |z^2|e^{-\varphi_1 - (\pi_1)_1, (\varphi_1)} - \varphi_1 \) is \( L^1 \) integrable near \( 0 \in \Delta \) for any \( l \geq k \).

Denote that \( x_1 := v(dd^c(\phi), 0) \) and \( x_2 := v(dd^c(\varphi_1 + \psi), 0) \). It follows from Siu’s Decomposition Theorem that

\[
\psi = x_1 \log |z| + \tilde{\psi},
\]

(2.12)

where \( \tilde{\psi} \) is a subharmonic function on \( \Delta \) satisfying that \( v(dd^c(\tilde{\psi}), 0) = 0 \). As \( v(dd^c(\psi), 0) < 2ord_0(F) \), we have

\[
\psi \leq \frac{1}{2}(\pi_1)_1, (\varphi_1) = \max\{\psi + T_1, 2\log |F|\} \leq x_1 \log |z| + C_1
\]

(2.13)

near \( 0 \), where \( C_1 \) is a constant, which implies that \( v(dd^c(\frac{1}{2}(\pi_1)_1, (\varphi_1)), 0) = x_1 \). As \( x_2 = v(dd^c(\varphi_1 + \psi), 0) \), we have

\[
\varphi_1 + \psi \leq x_2 \log |z| + C_2
\]

(2.14)

near \( 0 \), where \( C_2 \) is a constant. Combining inequality (2.11), equality (2.12), inequality (2.13) and inequality (2.14), we get that there exists \( r_2 \in (0, r_1) \) such
that
\[
\int_{\Delta_{r_2}} |z|^{2k - x_1 - x_2} e^{\hat{\psi}} \\
\leq C_3 \int_{\Delta_{r_2}} |z|^{2k - \frac{1}{2}(\pi_1)_* (\phi_1) - \phi_0 - \psi + \tilde{\psi}} \\
= C_3 \int_{\Delta_{r_2}} |z|^{2k} e^{-\frac{1}{2}(\pi_1)_* (\phi_1) - \phi_0 - x_1 \log |w|} \\
\leq C_3 e^{C_1} \int_{\Delta_{r_2}} |z|^{2k} e^{-\phi_0 - (\pi_1)_* (\phi_1)} \\
< +\infty.
\]  
(2.15)

For any \( p > 1 \), as \( v(dd^c (\hat{\psi}), 0) = 0 \), it follows from Lemma 2.18 that there exists \( r_3 \in (0, r_2) \) such that \( \int_{\Delta_{r_3}} e^{-\frac{1}{2}\tilde{\psi}} < +\infty \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). It follows from inequality (2.16) and Hölder inequality that
\[
\int_{\Delta_{r_3}} |z|^{2k - x_1 - x_2} \\
\leq \left( \int_{\Delta_{r_3}} |z|^{2k - x_1 - x_2} e^{\hat{\psi}} \right)^{\frac{p}{p'}} \left( \int_{\Delta_{r_3}} e^{-\frac{1}{2}\tilde{\psi}} \right)^{\frac{1}{q'}} \\
< +\infty,
\]  
which shows that \( |z|^{2k - x_1 - x_2} \) is integrable near \( o \) for any \( p > 1 \). As \( x_1 + x_2 = v(dd^c (\psi), 0) + v(dd^c (\phi_0 + \psi), 0) \notin \mathbb{Z} \), we have \( |z|^{2k - x_1 - x_2} \) is integrable near \( o \). Note that
\[
v(dd^c (\phi_0 + \phi_1)_* (\psi_1 + \psi_1)), 0) = v(dd^c (\phi_0 + \psi + \frac{1}{2}(\pi_1)_* (\phi_1)), 0) = x_1 + x_2.
\]
It follows from Lemma 2.19 that \((z^k, 0) \in \mathcal{I}(\phi_0 + \phi_1)_* (\psi_1 + \psi_1)\).

It follows from \((z^l, 0) \in \mathcal{I}(\phi_0 + \phi_1)_* (\psi_1 + \psi_1)\) and \( F_l \in A^2(\Delta_{r_1}^{|z|^1}, e^{-\psi}) \) for any \( l \geq k \) that we have \( z^l F_l(w) \in \mathcal{I}(\phi + \psi + \phi_1)_o \) for any \( l \geq k \). It follows from Lemma 2.17 and \( h(z, w) = \sum_{l \geq k} z^l F_l(w) \) that we have \((h, o) \in \mathcal{I}(\phi + \psi + \phi_1)_o\). Hence we obtain that \( \mathcal{H}_o = \mathcal{I}(\phi + \psi + \phi_1)_o \).

Lemma 2.20 is proved.

Lemma 2.21. For any \( z \in \tilde{Z}_1 \), assume that one of the following two conditions holds:

(\( A \)) \( \phi + a \psi \) is subharmonic near \( z \) for some \( a \in [0, 1) \);

(\( B \)) \( (\psi - 2p_z \log |w|)(z) > -\infty \), where \( p_z = \frac{1}{2} v(dd^c (\psi), z) \) and \( w \) is a local coordinate on a neighborhood of \( z \) satisfying that \( w(z) = 0 \).

Let \( c(t) = \frac{e^t}{t^2} \). Then for any \( p = (z, w) \in Z_1 \setminus Z_3 \), we have \( H_p = I(\phi + \psi)_p \).

Proof. It follows from Proposition 2.10 that \( H_p = I(\phi + \psi)_p \) if and only if \( \mathcal{H}_p = \mathcal{I}(\phi + \psi + \phi_1)_p \), where \( \mathcal{H}_p := \{ (h, p) \in \mathcal{O}_{M, p} : |h|^2 e^{-\phi - \phi_1} c(\psi_1) \in \mathcal{I}(\phi + \psi + \phi_1)_o \} \) is integrable near \( p \). Now, we prove \( \mathcal{H}_p = \mathcal{I}(\phi + \psi + \phi_1)_p \).

Without loss of generality, we can assume that \( M = \Delta \times \Delta_{v-1}^n \), \( z = 0 \in \Delta \) and \( p = o \in M \) (the origin of \( M \)). As \( c(t)e^{-t} \) is decreasing, we have \( \mathcal{I}(\phi + \psi + \phi_1)_o \subset \mathcal{H}_o \). For any \( (h, o) \in \mathcal{H}_o \), there exists \( r_1 > 0 \) such that \( \int_{\Delta_{r_1} \times \Delta_{v-1}^n} |h|^2 e^{-\phi - \phi_1} c(\psi_1) < +\infty \). Denote that \( x_1 := v(dd^c (\psi), 0) \) and \( x_2 := v(dd^c (\phi_0 + \psi), 0) \).

Firstly, we prove \( \mathcal{H}_o \subset \mathcal{I}(\phi + \psi + \phi_1)_o \) under condition \((A) \) \( \phi + a \psi \) is subharmonic near \( 0 \) for some \( a \in [0, 1) \). As \( c(t) = \frac{e^t}{t^2} \), we can assume that \( c(t) \geq e^{at} \).
when \( t \geq T_1 \). Then we have
\[
\int_{\Delta_{r_1} \times \Delta_{r_1}^{n-1}} |h|^2 e^{-\varphi - a\Psi_1 - \varphi_1} \leq \int_{\Delta_{r_1} \times \Delta_{r_1}^{n-1}} |h|^2 e^{-\varphi - \varphi_1} c(-\Psi_1) < +\infty.
\]

Hence we know that \( h \in A^2(M, e^{-\varphi - a\Psi_1 - \varphi_1}) \). It follows from Lemma 2.10 and Remark 2.11 that we know that \( h(z, w) = \sum_{l \geq k} z^l F_l(w) \), where \( \{F_l\}_{l \geq k} \) is a sequence of holomorphic functions on \( W \) satisfying \( F_k(w) \neq 0 \), \( F_l \in A^2(\Delta^{n-1}, e^{-\varphi y}) \) and \( |z|^2 e^{-\varphi y - (\pi_1)(a\Psi_1 + \varphi_1)} \) is \( L^1 \) integrable near \( 0 \in \Delta \) for any \( l \geq k \).

Note that \( \varphi_\Omega + (\pi_1)_*(a\Psi_1 + \varphi_1) = \varphi_\Omega + a\psi + (2 - a) \max \{\psi, 2 \log |F|\} \) and \( v(dd^c(\varphi_\Omega + a\psi), 0) = v(dd^c(\varphi_\Omega + \psi), 0) - (1 - a)v(dd^c(\psi), 0) = x_2 - (1 - a)x_1 \).

As \( v(dd^c(\psi), 0) = 2\partial a\partial d(F) \), we have \( v(dd^c(\varphi_\Omega + (\pi_1)_*(a\Psi_1 + \varphi_1)), 0) = x_2 - (1 - a)x_1 + (2 - a)x_2 = x_2 + x_1 \). Note that \( v(dd^c(\varphi_\Omega + (\pi_1)_*(a\Psi_1 + \varphi_1)), 0) = x_1 + x_2 \). It follows from \( |z|^2 e^{-\varphi y - (\pi_1)(a\Psi_1 + \varphi_1)} \) is \( L^1 \) integrable near \( 0 \in \Delta \) for any \( l \geq k \) and Lemma 2.19 that \( (z', 0) \in I(\varphi_\Omega + (\pi_1)_*(a\Psi_1 + \varphi_1))_0 \) for any \( l \geq k \).

It follows from (2.16) and \( h(z, w) = \sum_{l \geq k} z^l F_l(w) \) that we have \( h(o) \in I(\varphi + \Psi_1 + \varphi_1)_o \).

Hence we obtain that \( \mathcal{H}_o = I(\varphi + \Psi_1 + \varphi_1)_o \) under condition (A).

Now, we prove \( \mathcal{H}_o \subset I(\varphi + \Psi_1 + \varphi_1)_o \) under condition \( (B) \), \( (\psi - x_1 \log |w|)(a) > -\infty \). For any \( (h, o) \in \mathcal{H}_o \), there exists \( r_2 > 0 \) such that \( \int_{\Delta_{r_2} \times \Delta_{r_2}^{n-1}} |h|^2 e^{-\varphi - \varphi_1} c(-\Psi_1) < +\infty \), which implies that
\[
\int_{\Delta_{r_2} \times \Delta_{r_2}^{n-1}} |h|^2 e^{-\varphi - \varphi_1} < +\infty.
\]

Hence we know that \( h \in A^2(\Delta_{r_2} \times \Delta_{r_2}^{n-1}, e^{-\varphi - \varphi_1}) \). It follows from Lemma 2.10 and Remark 2.11 that we know that \( h(z, w) = \sum_{l \geq k} z^l F_l(w) \), where \( F_k(w) \neq 0 \), \( \{F_l\}_{l \geq k} \) is a sequence of holomorphic functions on \( W \) satisfying \( F_l \in A^2(\Delta^{n-1}, e^{-\varphi y}) \) and \( |z|^2 e^{-\varphi y - (\pi_1)(\varphi_1)} \) is \( L^1 \) integrable near \( 0 \in \Delta \) for any \( l \geq k \).

It follows from Siu’s Decomposition Theorem that
\[
\psi = x_1 \log |z| + \tilde{\psi},
\]
where \( \tilde{\psi} \) is a subharmonic function on \( \Delta \) satisfying that \( v(dd^c(\tilde{\psi}), 0) = 0 \). Note that \( \tilde{\psi}(0) > -\infty \) and \( \varphi_\Omega + (\pi_1)_*(\varphi_1) \leq \varphi_\Omega + 2x_1 \log |z| + C_1 = \varphi_\Omega + \psi - \tilde{\psi} + x_1 \log |z| \leq (x_1 + x_2) \log |z| - \tilde{\psi} + C_2 \) near \( 0 \), where \( C_1 \) and \( C_2 \) are constants. Note that \( e^{\tilde{\psi}} \) is subharmonic, then it follows from \( |z|^2 e^{-\varphi y - (\pi_1)(\varphi_1)} \) is \( L^1 \) integrable near \( 0 \in \Delta \) for any \( l \geq k \) and the sub-mean value inequality that there exists \( r_3 \in (0, r_2) \) such that for any \( l \geq k \), we have
such that
\[
\{ \phi \exists \text{ a sequence } \{ G \}\text{ Riemann surface, which admits a nontrivial Green function (see [36], see also [43])}
\]
Lemma 2.24

Lemma 2.23

property of plurisubharmonic functions on Stein manifolds.

(see [22])

is harmonic near bound near \( z \) negative subharmonic function on \( \Omega \)

Lemma 2.22

lemmas.

2.4. Some required lemmas. In this section, we recall and present some required lemmas.

Lemma 2.22 (see [24]). Let \( c(t) \) be a positive measurable function on \((0, +\infty)\), and let \( a \in \mathbb{R} \). Assume that \( \int_{t}^{+\infty} c(s)e^{-as}ds \in (0, +\infty) \) when \( t \) near \( +\infty \). Then we have

1. \( \lim_{t \to +\infty} \int_{t}^{+\infty} c(s)e^{-as}ds = 1 \) if and only if \( a = 1 \);
2. \( \lim_{t \to +\infty} \int_{t}^{+\infty} c(s)e^{-as}ds = 0 \) if and only if \( a > 1 \);
3. \( \lim_{t \to +\infty} \int_{t}^{+\infty} c(s)e^{-as}ds = +\infty \) if and only if \( a < 1 \).

The following Lemma belongs to Fornæss and Narasimhan on approximation property of plurisubharmonic functions on Stein manifolds.

Lemma 2.23 ([15]). Let \( X \) be a Stein manifold and \( \varphi \in \text{PSH}(X) \). Then there exists a sequence \( \{ \varphi_n \}_{n=1}^{\infty} \) of smooth strongly plurisubharmonic functions such that \( \varphi_n \searrow \varphi \).

Now, we recall some basic properties of the Green functions. Let \( \Omega \) be an open Riemann surface, which admits a nontrivial Green function \( G_{\Omega} \), and let \( z_0 \in \Omega \).

Lemma 2.24 (see [33], see also [13]). Let \( w \) be a local coordinate on a neighborhood of \( z_0 \) satisfying \( v(z_0) = 0 \). \( G_{\Omega}(z, z_0) = \sup_{v \in \Delta_{\Omega}(z_0)} v(z) \), where \( \Delta_{\Omega}(z_0) \) is the set of negative subharmonic function on \( \Omega \) such that \( v - \log |w| \) has a locally finite upper bound near \( z_0 \). Moreover, \( G_{\Omega}(\cdot, z_0) \) is harmonic on \( \Omega \setminus \{ z_0 \} \) and \( G_{\Omega}(\cdot, z_0) - \log |w| \) is harmonic near \( z_0 \).

Lemma 2.25 (see [22]). For any open neighborhood \( U \) of \( z_0 \), there exists \( t > 0 \) such that \( \{ z \in \Omega : G_{\Omega}(z, z_0) < -t \} \) is a relatively compact subset of \( U \).
Let $S := \{z_j : j \in \mathbb{Z}_{\geq 1} \& j < \gamma\}$ be a discrete subset of the open Riemann surface $\Omega$, where $\gamma \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. We recall the following two lemmas about $\sum_{1 \leq j < \gamma} q_j G_\Omega(\cdot, z_j)$.

**Lemma 2.26** (see [23]). Let $\psi$ be a negative subharmonic function on $\Omega$ such that $\frac{1}{2}v(\partial^c \psi, z_j) \geq q_j$ for any $j$, where $q_j > 0$ is a constant. Then $2\sum_{1 \leq j < \gamma} q_j G_\Omega(\cdot, z_j)$ is a subharmonic function on $\Omega$ satisfying that $2\sum_{1 \leq j < \gamma} q_j G_\Omega(\cdot, z_j) \geq \psi$ and $2\sum_{1 \leq j < \gamma} q_j G_\Omega(\cdot, z_j)$ is harmonic on $\Omega \setminus S$.

The following lemma will be used in the proofs of Proposition 2.22 and Proposition 2.34.

**Lemma 2.27** (see [23]). Let $\psi$ be a negative plurisubharmonic function on $\Omega$ satisfying $\frac{1}{2}v(\partial^c \psi, z_0) \geq q_j > 0$ for any $j$, where $q_j$ is a constant. Assume that $\psi \neq 2\sum_{1 \leq k < \gamma} q_j G_\Omega(\cdot, z_j)$. Let $l(t)$ be a positive Lebesgue measurable function on $(0, +\infty)$ satisfying $l$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} l(t) dt < +\infty$. Then there exists a Lebesgue measurable subset $V$ of $M$ such that $l(-\psi(z)) < l(-2\sum_{1 \leq k < \gamma} q_j G_\Omega(z, z_j))$ for any $z \in V$ and $\mu(V) > 0$, where $\mu$ is the Lebesgue measure on $\Omega$.

Let $Y$ be an $n - 1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and $K_M$ be the canonical line bundle on $M$. Let $\pi_1, \pi_2$ be the natural projections from $M$ to $\Omega$ and $Y$. Let $\psi_1$ be a subharmonic function on $\Omega$ such that $q_j = \frac{1}{2}v(\partial^c \psi_1, z_j) > 0$ for any $z_j \in S = \{z_j : 1 \leq j < \gamma\}$, and let $\varphi_\Omega$ be a Lebesgue measurable function on $\Omega$ such that $\psi_1 + \varphi_\Omega$ is subharmonic on $\Omega$. Let $\varphi_Y$ be a plurisubharmonic function on $Y$. Denote that $\psi := \pi_1^*(\psi_1)$, $\varphi := \pi_1^*(\varphi_\Omega) + \pi_2^*(\varphi_Y)$. Using the Weierstrass Theorem on open Riemann surfaces (see [1]) and Siu's Decomposition Theorem, we have

$$\varphi_\Omega + \psi_1 = 2 \log |g_0| + 2u_0,$$

where $g_0$ is a holomorphic function on $\Omega$ and $u_0$ is a subharmonic function on $\Omega$ such that $v(\partial^c u_0, z) \in (0, 1)$ for any $z \in \Omega$.

Let $w_j$ be a local coordinate on a neighborhood $V_{z_j} \subset \Omega$ of $z_j$ satisfying $w_j(z_j) = 0$ for $z_j \in \tilde{Z}_0$, where $V_j \cap V_{z_k} = \emptyset$ for any $j \neq k$. Without loss of generality, assume that $w_j(V_{z_j}) = \{w \in \mathbb{C} : |w| < s_j\}$, where $s_j > 0$. Denote that

$$V_0 := \bigcup_{1 \leq j < \gamma} V_{z_j}.$$ Assume that $g_0 = d_j w_j^k h_j$ on $V_{z_j}$, where $d_j$ is a constant, $k_j$ is a nonnegative integer, and $h_j$ is a holomorphic function on $V_{z_j}$ such that $h_j(z_j) = 1$ for any $1 \leq j < \gamma$.

Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing and $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$. We recall the following optimal $L^2$ extension theorem.

**Theorem 2.28** (see [1]). Let $F$ be a holomorphic $(n, 0)$ form on $V_0 \times Y$ such that $F = \pi_1^*(w_j^k f_j dw_j) \wedge \pi_2^*(\tilde{F}_j)$ on $V_{z_j} \times Y$, where $\tilde{k}_j$ is a nonnegative integer, $f_j$ is a holomorphic form on $V_{z_j}$ such that $f_j(z_j) = a_j \in \mathbb{C} \setminus \{0\}$, and $\tilde{F}_j$ is a holomorphic $(n - 1, 0)$ form on $Y$ for any $1 \leq j < \gamma$.

Denote that $I_F := \{j : 1 \leq j < \gamma \& \tilde{k}_j + 1 - k_j \leq 0\}$. Assume that $\tilde{k}_j + 1 - k_j = 0$ and $u_0(z_j) > -\infty$ for $j \in I_F$. If

$$\int_Y |\tilde{F}_j|^2 e^{-\varphi_Y} < +\infty$$

(2.19)
for any $1 \leq j < \gamma$, and
\[
\sum_{j \in I_F} \frac{2\pi |a_j|^2 e^{-2u_0(z_j)}}{q_{z_j}|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\tilde{\varphi}_V} < +\infty, \tag{2.20}
\]
then there exists a holomorphic $(n,0)$ form $\tilde{F}$ on $M$, such that $(\tilde{F} - F_i(z_j,y)) \in \mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $1 \leq j < \gamma$ and $y \in Y$, and
\[
\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}_V} c(-\psi) \leq \left( \int_0^{+\infty} c(s)e^{-\tilde{\varphi}_V} ds \right) \sum_{j \in I_F} \frac{2\pi |a_j|^2 e^{-2u_0(z_j)}}{q_{z_j}|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\tilde{\varphi}_V}. \tag{2.21}
\]

The following lemma will be used in the proofs of Proposition 2.32 and Proposition 2.34.

Lemma 2.29. For any $1 \leq j < \gamma$, assume that one of the following conditions holds:

(A) $\varphi + a\psi$ is subharmonic near $z_j$ for some $a \in [0,1]$;

(B) $(\psi - 2q_0, \log |w_j|)(z_j) > -\infty$, where $q_{z_j} = \frac{1}{t} \psi(df'(\psi), z_j)$.

Let $F$ be a holomorphic $(n,0)$ form on $M$ such that $F = \sum_{l \geq k_j} \pi_1^*(w_j^j dw_j) \wedge \pi_2^*(F_{j,l})$ on $V_j \times Y$ for any $j$ according to Lemma 2.7, where $k_j$ is a nonnegative integer, $F_{j,l}$ is a holomorphic $(n-1,0)$ form on $Y$ satisfying that $\tilde{F}_j := F_{j,k_j} \neq 0$. Denote that
\[
I_F := \{ j : 1 \leq j < \gamma \& \hat{k}_j + 1 - k_j \leq 0 \}.
\]

Assume that there exists a constant $C > 0$ such that
\[
\int_{\{ \psi < -t \}} |F|^2 e^{-\tilde{\varphi}_V} \tilde{c}(-\psi) \leq C \int_t^{+\infty} \tilde{c}(s)e^{-\tilde{\varphi}_V} ds
\]
for any $t \geq 0$ and any nonnegative Lebesgue measurable function $\tilde{c}(t)$ on $(0, +\infty)$. Then $k_j + 1 - k_j = 0$ for any $j \in I_F$, $\int_Y |F_{j,l}|^2 e^{-\tilde{\varphi}_V} < +\infty$ for any $j, l$, and
\[
\sum_{j \in I_F} \frac{2\pi |a_j|^2 e^{-2u_0(z_j)}}{q_{z_j}|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\tilde{\varphi}_V} \leq C.
\]

Proof. Firstly, we prove that $\int_Y |F_{j,l}|^2 e^{-\tilde{\varphi}_V} < +\infty$ for any $j, l$. Fixed $z_j$, there exists $r_j > 0$ such that $\{|w_j| < r_j\} \subset V_j$ and
\[
\int_{\{|w_j| < r_j\} \times Y} |F|^2 e^{-\tilde{\varphi}_V} \tilde{c}(-\psi) < +\infty \tag{2.22}
\]
for any nonnegative Lebesgue measurable function $\tilde{c}(t)$ on $(0, +\infty)$ satisfying $\int_0^{+\infty} \tilde{c}(s)e^{-\tilde{\varphi}_V} ds < +\infty$.

If there exists $a \in [0,1]$ such that $\varphi + a\psi$ near $z_j$, note that there exists a positive Lebesgue measurable function $\tilde{c}(t)$ on $(0, +\infty)$ such that $\tilde{c}(t)e^{-at}$ (denoted by $b(t)$) is increasing near $+\infty$ and $\int_0^{+\infty} \tilde{c}(s)e^{-\tilde{\varphi}_V} ds < +\infty$, then there exist $\tilde{r}_j \in (0, r_j)$ and $C_1 < 0$ such that $b(t)$ is increasing on $(-C_1, +\infty)$
\[
\varphi + a\psi \leq C_1, \\
\psi \leq C_1
\]

on \( \{|w_j| < \tilde{r}_j\} \). Inequality (2.22) shows that
\[
\int_{\{|w_j| < \tilde{r}_j\} \times Y} |F|^2 e^{-\pi_1^j(\varphi Y)}\,d\mu = C_1
\leq \int_{\{|w_j| < \tilde{r}_j\} \times Y} |F|^2 e^{-\pi_2^j(\varphi Y)}\,d\mu \left(-\pi_1^j(\psi_1)\right)
\leq +\infty. \tag{2.23}
\]

It follows from Lemma 2.24 and inequality (2.23) that \( \int_Y |F_j|^2 e^{-\varphi} < +\infty \) for any \( l \geq \tilde{k}_j \).

If \((\psi_1 - 2q_{z_j} \log |w_j|)(z_j) > -\infty\), then \( \tilde{\psi}_1 = \psi_1 - 2q_{z_j} \log |w_j| \) is a subharmonic function on \( V_{z_j} \) satisfying \( \tilde{\psi}_1(z_j) > -\infty \). Choosing \( \tilde{c} \equiv 1 \), it follows from Fubini’s Theorem and sub-mean value inequality of subharmonic functions that
\[
\int_{\{|w_j| < \tilde{r}_j\} \times Y} |F|^2 e^{-\varphi} \geq C_2 \int_{\{|w_j| < \tilde{r}_j\} \times Y} |F|^2 e^{-\varphi} \,d\mu = C_2 \int_{\{|w_j| < \tilde{r}_j\} \times Y} \left|\pi_1^j(w_j^j \,dw_j) \wedge \pi_2^j(F_j^j, d\mu)\right|^2 e^{-\varphi} \tag{2.24}
\]
where \( C_2 > 0 \) is a constant. Combining inequality (2.22) and inequality (2.24), we get that \( \int_Y |\tilde{F}_j|^2 e^{-\varphi} < +\infty \). It follows from Lemma 2.24 and Remark 2.10 that \(|w_j|^2 e^{-c_2l} \) is integrable near \( z_j \) for any \( l \geq \tilde{k}_j \), hence there exists \( r_j' \in (0, r_j) \) such that
\[
\int_{\{|w_j| < r_j'\} \times Y} \left|\pi_1^j(w_j^j \,dw_j) \wedge \pi_2^j(F_j^j, d\mu)\right|^2 e^{-\varphi} \leq 2 \int_{\{|w_j| < r_j'\} \times Y} \left|\pi_1^j(w_j^j \,dw_j) \wedge \pi_2^j(F_j^j, d\mu)\right|^2 e^{-\varphi} + 2 \int_{\{|w_j| < r_j'\} \times Y} |F|^2 e^{-\varphi} \tag{2.25}
\]
\(< +\infty. \)

Combining Fubini’s Theorem and sub-mean value inequality of subharmonic functions, inequality (2.25) shows that \( \int_Y |\tilde{F}_{jk+1}|^2 e^{-\varphi} < +\infty \). Hence, we know that \( \int_Y |F_{jk}|^2 e^{-\varphi} < +\infty \) for any \( l \geq \tilde{k}_j \) by induction.

Now, we prove that \( \tilde{k}_j + 1 - k_j = 0 \) for any \( j \in I_F \) and
\[
\sum_{j \in I_F} \frac{2e^{-2q_{z_j}}(z_j)}{q_{z_j} d_j} \int_Y |\tilde{F}_j|^2 e^{-\varphi} \leq C. \tag{2.26}
\]

According to Siu’s Decomposition Theorem, Lemma 2.24 and 2.26 we can assume that
\[
\psi_1 = 2 \sum_{1 \leq j < \gamma} q_{z_j} G_\Omega(\cdot, z_j) + \psi_0, \tag{2.26}
\]
where \( \psi_0 \) is a negative subharmonic function on \( \Omega \) such that \( v(d\varphi(\psi_0), z_j) = 0 \) for any \( 1 \leq j < \gamma \). Lemma 2.24 shows that there exist smooth subharmonic functions \( \{u_1\}_{l \in \mathbb{Z}_{>1}} \) and \( \{\tilde{\psi}_1\}_{l \in \mathbb{Z}_{>1}} \) on \( \Omega \) such that \( u_1 \) and \( \tilde{\psi}_1 \) are decreasingly convergent to
$u_0$ and $\psi_0$ with respect to $l$, respectively. For any positive integer $m$ satisfying $2 \leq m \leq \gamma$, denote that $I_m := \{ j \in I_F : 1 \leq j < m \}$. Denote that
\[
G := 2 \sum_{1 \leq j < \gamma} q_{z_j} G_{\Omega}(\cdot, z_j).
\]

Fixing a positive integer $m$, there exists $a \in [0, 1)$ such that $\varphi_\Omega + a\psi_1$ is subharmonic near $z_j$ for any $1 \leq j < m$ satisfying $(\psi_1 - 2q_{z_j} \log |w_j|)(z_j) = -\infty$. Note that there exists a positive Lebesgue measurable function $\tilde{c}(t)$ on $(0, +\infty)$ such that $\tilde{c}(t)e^{-at}$ (denoted by $b(t)$) is increasing near $+\infty$ and $\int_0^{+\infty} \tilde{c}(s)e^{-s}ds < +\infty$. Note that
\[
\int_{\{\psi < t\}} |F|^2 e^{-\varphi} \tilde{c}(-\psi) \geq \sum_{1 \leq j < m} \int_{\{\psi < t\} \cap (V_{z_j} \times Y)} |F|^2 e^{-\varphi} \tilde{c}(-\psi)
\geq \sum_{j \in I_m} \int_{\{\psi < t\} \cap (V_{z_j} \times Y)} |F|^2 e^{-\varphi} \tilde{c}(-\psi). \tag{2.27}
\]

Fixing $j \in I_m$, we will prove that $\tilde{k}_j + 1 = k_j$ and
\[
\liminf_{t \to +\infty} \int_{\{\psi < t\} \cap (V_{z_j} \times Y)} |F|^2 e^{-\varphi} \tilde{c}(-\psi) \geq \frac{2\pi e^{-2u_0(z_j)}}{q_{z_j}|d_j|^2} \int_Y |\tilde{F}|^2 e^{-\varphi}Y. \tag{2.28}
\]

If $(\psi_1 - 2q_{z_j} \log |w_j|)(z_j) > -\infty$, we have $\psi_0(z_j) > -\infty$. For any $\epsilon > 0$, there exists $r, j > 0$ such that $U_j := \{ |w_j(z)| < r, z \in V_{z_j} \} \in V_{z_j}$,
\[
\sup_{z \in U_j} 2|u_1(z) - \psi_0(z_j)| < \epsilon,
\]
and
\[
\sup_{z \in U_j} |H_j(z) - H_j(z_j)| < \epsilon,
\]
where $H_j = G - 2q_{z_j} \log |w_j| + \psi_1 + \epsilon$ is a smooth function on $V_{z_j}$. There exists $t_1 > 0$ such that $(2q_{z_j} \log |w_j| + H_j(z_j) < -t_1) \Subset U_j$ and $\tilde{c}$ is increasing on $[t_1, +\infty)$, then we get that
\[
\int_{\{\psi < t\} \cap (V_{z_j} \times Y)} |F|^2 e^{-\varphi} \tilde{c}(-\psi)
\geq \int_{\{\pi_1^{*}(G + \tilde{\psi}) < t\} \cap (V_{z_j} \times Y)} |F|^2 e^{-\varphi} \tilde{c}(-\psi) \tag{2.29}
\geq \int_{\{2q_{z_j} \log |w_j| + H_j(z_j) < -t\} \times Y} |F|^2 e^{\pi_1^{*}(-2\log |g_0| - 2u_0(z_j) - \epsilon + G + \psi_0) + \pi_1^{*}(-\varphi)}
\times \tilde{c}(\pi_1^{*}(-2q_{z_j} \log |w_j| - H_j(z_j)))
\]
for $t \geq t_1$. Note that there exists a holomorphic function $\tilde{g}_j$ on $U_j$ such that $|\tilde{g}_j|^2 = e^{\tilde{c}}$. Note that $\text{ord}_z(\tilde{g}_j) = 1$, and let $\tilde{d}_j = \lim_{z \to z_j} \frac{\tilde{g}_j(z)}{w_j(z)} \in \mathbb{C} \setminus \{0\}$. Note that $\lim_{z \to z_j} \frac{g_0(z)}{w_j(z)} = \tilde{d}_j \in \mathbb{C} \setminus \{0\}$. Without loss of generality, assume that $\{g_0(z) = 0 : z \in U_j\} = \{z_j\}$. It follows from Fubini’s Theorem, sub-mean value
inequality of subharmonic functions and inequality \((2.29)\) that

\[
\int_{\{\psi < -t\} \cap (V_{z_j} \times Y)} |F|^2 e^{-c(\psi)} c(-\psi)
\geq \int_{(2q_{j_1} \log |w_j| + H_j(z_j) < -t) \times Y} |F|^2 |\pi_1^*(\tilde{g}_j)|^{2q_{j_1}} e^{\pi_1^*(-2\log |g_0| - 2u_j(z_j) - \epsilon + \psi_0 + \pi_1^*(-\varphi))}
\times c(\pi_1^*(-2q_{j_1} \log |w_j| - H_j(z_j)))
\geq 4\pi |\tilde{d}_j|^{2q_{j_1}} e^{-2u_j(z_j) - \epsilon + \psi_0(z_j)} \int_0^{e^{-\frac{\pi}{q_{j_1}}}} r^{2\tilde{k}_j + 2q_{j_1} - \tilde{k}_j + 1} c(-2q_{j_1} \log |r| - H_j(z_j)) dr
\times \int_Y |\tilde{F}_j|^2 e^{-\varphi(y)}
\geq \frac{2\pi |\tilde{d}_j|^{2q_{j_1}} e^{-2u_j(z_j) - \epsilon + \psi_0(z_j)}}{q_{j_1}|d_j|^2} e^{\left(-\frac{\tilde{k}_j - k_j + 1}{q_{j_1}}\right) H_j(z_j) \int_t^{+\infty} e^{-\left(\frac{1}{q_{j_1}}\right) s} c(s) ds}
\times \int_Y |\tilde{F}_j|^2 e^{-\varphi(y)}
\]  

(2.30)

for \(t \geq t_1\). As \(\lim_{t \to +\infty} \int_{\{\psi < -t\} \cap (V_{z_j} \times Y)} |F|^2 e^{-c(\psi)} c(-\psi) < +\infty\) and \(\tilde{F}_j \neq 0\), it follows from Lemma \((2.22)\) and \(\tilde{k}_j - k_j + 1 \leq 0\) that \(\tilde{k}_j = k_j + 1 = 0\). Hence letting \(\epsilon \to 0 + 0\), inequality \((2.30)\) implies that

\[
\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\} \cap (V_{z_j} \times Y)} |F|^2 e^{-c(\psi)} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds}
\geq \frac{2\pi |\tilde{d}_j|^{2q_{j_1}} e^{-2u_j(z_j) + \psi_0(z_j) - (G - 2q_{j_1} \log |w_j|)(z_j) - \psi_1(z_j)}}{q_{j_1}|d_j|^2} \times \int_Y |\tilde{F}_j|^2 e^{-\varphi(y)}.
\]

(2.31)

Note that \(\tilde{d}_j|^{2q_{j_1}} e^{-(G - 2q_{j_1} \log |w_j|)(z_j)} = 1\), \(\lim_{t \to +\infty} u_j(z_j) = u_0(z_j)\) and \(\lim_{t \to +\infty} \tilde{\psi}_l(z_j) = \psi_0(z_j)\). Letting \(t \to +\infty\), it follows from inequality \((2.31)\) that

\[
\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\} \cap (V_{z_j} \times Y)} |F|^2 e^{-c(\psi)} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds}
\geq \frac{2\pi e^{-2u_0(z_j)}}{q_{j_1}|d_j|^2} \times \int_Y |\tilde{F}_j|^2 e^{-\varphi(y)}.
\]

If \((\psi_1 - 2q_{j_1} \log |w_j|)(z_j) = -\infty\), then \(\varphi_0 + a\psi_1\) is subharmonic on a neighborhood \(\tilde{V}_{z_j} \subset V_{z_j}\) of \(z_j\). Denote that

\[
\varphi_0 := \varphi_0 + a\psi_1 - 2s_1 \log |w_j|
\]

is a subharmonic function on \(\tilde{V}_{z_j}\), where \(s_1 = \frac{1}{2} v(\varphi_0 + a\psi_1, z_j)\). Following from Lemma \((2.28)\) there exist smooth subharmonic functions \(\{\varphi_l\}_{l \in Z_{z_j}}\) and \(\{\tilde{\psi}_l\}_{l \in Z_{z_j}}\) on \(\bar{\Omega}\) such that \(\varphi_l\) and \(\tilde{\psi}_l\) are decreasingly convergent to \(\varphi_0\) and \(\psi_0\) with respect to \(l\), respectively. For any \(\epsilon > 0\), there exists \(r_{j, 2} > 0\) such that \(U'_j := \{|w_j(z)| < r_{j, 2} : z \in V_{z_j}\} \subset \tilde{V}_{z_j}\),

\[
\sup_{z \in U'_j} |\varphi_l(z) - \varphi_0(z_j)| < \epsilon,
\]

for \(3 < l < j\).
and
\[
\sup_{z \in \tilde{V}_j} |H_j(z) - H_j(z_j)| < \epsilon,
\]
where \(H_j = G - 2q_{z_j} \log |w_j| + \tilde{\psi}_1 + \epsilon\) is a smooth function on \(\tilde{V}_j\). There exists \(t_2 > 0\) such that \(\{2q_{z_j} \log |w_j| + H_j(z_j) < -t_2\} \subseteq U_j\) and \(b(t) = \tilde{c}(t)e^{-at}\) is increasing on \([t_2, +\infty)\), then we get
\[
\int_{\{\psi < t\} \cap (V_j \times Y)} |F|^2 e^{-\psi\tilde{c}(-\psi)} \geq \int_{\{\psi < t\} \cap (V_j \times Y)} |F|^2 e^{-\pi_1(-\psi_1)(\varphi + a\psi_1)} b(-\pi_1'(\psi_1)) (2.32)
\]
\[
\geq \int_{\{2q_{z_j} \log |w_j| + H_j(z_j) < -t\} \cap (V_j \times Y)} |F|^2 e^{\pi_1(-2s_1 \log |w_j| - \varphi(z_j) - \epsilon) + \pi_2(-\varphi_1)} \times b(\pi_1'(-2q_{z_j} \log |w_j| - H_j(z_j)))
\]
for \(t \geq t_2\). It follows from Fubini’s Theorem, sub-mean value inequality of subharmonic functions and inequality \(\Box\) that
\[
\int_{\{\psi < t\} \cap (V_j \times Y)} |F|^2 e^{-\psi\tilde{c}(-\psi)} \geq 4\pi e^{-\psi_1(z_j) - \epsilon} \int_0^e e^{\frac{H_j(z_j)}{q_{z_j}}} t^{2q_{z_j} - 2s_1 + 1} b(-2q_{z_j} \log |r| - H_j(z_j)) dr \times \int_Y |\tilde{F}_j|^2 e^{-\varphi_1} \geq 2\pi e^{-\psi_1(z_j) - \epsilon} \int_0^e e^{\frac{\tilde{k}_j - s_1 + 1}{q_{z_j}}} H_j(z_j) \int_t^{+\infty} e^{-\frac{(k_j - s_1 + 1)}{q_{z_j}}} b(s) ds \times \int_Y |\tilde{F}_j|^2 e^{-\varphi_1} \geq 2\pi e^{-\psi_1(z_j) - \epsilon} \int_0^e e^{\frac{\tilde{k}_j - s_1 + 1}{q_{z_j}}} H_j(z_j) \int_t^{+\infty} e^{-\frac{(k_j - s_1 + 1)}{q_{z_j}} + a} \tilde{c}(s) ds \times \int_Y |\tilde{F}_j|^2 e^{-\varphi_1} (2.33)
\]
for \(t \geq t_1\). Note that
\[
s_1 + (1 - a)q_{z_j} = \frac{1}{2} (v(d^F(\varphi_1 + \psi_1), z_j) + (1 - a)(d^F(\tilde{\psi}_1), z_j)) = \frac{1}{2} (v(d^F(\varphi_1 + \psi_1), z_j)) = ord_{z_j}(g_0) = k_j,
\]
which implies that
\[
\frac{\tilde{k}_j - s_1 + 1}{q_{z_j}} + a = \frac{\tilde{k}_j - k_j + 1}{q_{z_j}} + 1.
\]
As \(\lim_{t_1 \rightarrow +\infty} \int_{\{\psi < t\} \cap (V_j \times Y)} |F|^2 e^{-\psi\tilde{c}(-\psi)} \int_t^{+\infty} \tilde{c}(s) e^{-ds} ds < +\infty\) and \(\tilde{F}_j \neq 0\), it follows from Lemma 2.22 and \(\tilde{k}_j - k_j + 1 \leq 0\) that \(\tilde{k}_j - k_j + 1 = 0\). Hence letting \(\epsilon \rightarrow 0 + 0\),
inequality (2.33) implies that
\[
\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\} \cap (V_\epsilon \times Y)} |F|^2 e^{-\varphi} \tilde{c}(-\psi)}{\int_t^{+\infty} \tilde{c}(s) e^{-s} ds} \geq 2\pi e^{-\varphi(\psi_j)} \frac{q_{z_j}}{\int_Y |\tilde{F}_j|^2 e^{-\varphi Y}}.
\] (2.34)

Note that
\[
\lim_{t \to +\infty} \varphi_1(z_j) + (1 - a) (G + \psi_1 - 2q_{z_j} \log |w_j|)(z_j)
\]

\[
= (1 - a) (G + \psi - 2q_{z_j} \log |w_j|)(z_j) + \varphi_0(z_j)
\]

\[
= (1 - a) (\psi_1 - 2q_{z_j} \log |w_j|)(z_j) + (\varphi_\Omega + a\psi_1 - 2s_1 \log |w_j|)(z_j)
\]

\[
= (\varphi_\Omega + \psi_1 - 2k_j \log |w_j|)(z_j)
\]

\[
= 2u_0(z_j) + 2 \log |d_j|,
\]

then it follows from inequality (2.34) that
\[
\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\} \cap (V_\epsilon \times Y)} |F|^2 e^{-\varphi} \tilde{c}(-\psi)}{\int_t^{+\infty} \tilde{c}(s) e^{-s} ds} \geq 2\pi e^{-2u_0(z_j)} \frac{q_{z_j}}{|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi Y}.
\]

Hence for any \( m \) satisfying \( 2 \leq m < \gamma \), we have \( \tilde{k}_j - k_j + 1 = 0 \) for any \( j \in I_m \), and inequality (2.27) implies that
\[
\sum_{j \in I_m} 2\pi e^{-2u_0(z_j)} \frac{q_{z_j}}{|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi Y} \leq \liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\} \cap (V_\epsilon \times Y)} |F|^2 e^{-\varphi} \tilde{c}(-\psi)}{\int_t^{+\infty} \tilde{c}(s) e^{-s} ds} \leq C.
\]

By the arbitrariness of \( m \), we have \( \tilde{k}_j + 1 - k_j = 0 \) for any \( j \in I_F \) and
\[
\sum_{j \in I_F} 2\pi e^{-2u_0(z_j)} \frac{q_{z_j}}{|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi Y} \leq C.
\]

\[\square\]

2.5. Linearity on fibrations over open Riemann surfaces at inner points.

In this section, we recall and present some results about the concavity degenerating to linearity on fibrations over open Riemann surfaces at inner points.

Let \( \Omega \) be an open Riemann surface, which admits a nontrivial Green function \( G_\Omega \), and let \( K_\Omega \) be the canonical (holomorphic) line bundle on \( \Omega \). Let \( Y \) be an \( n - 1 \) dimensional weakly pseudoconvex Kähler manifold. Let \( M = \Omega \times Y \) be a complex manifold, and \( K_M \) be the canonical line bundle on \( M \). Let \( \pi_1, \pi_2 \) be the natural projections from \( M \) to \( \Omega \) and \( Y \).

Let \( \tilde{Z}_0 \) be a (closed) analytic subset of \( \Omega \) and denote \( Z_0 := \pi_1^{-1}(\tilde{Z}_0) \) be an analytic subset of \( M \). Let \( \psi_1 \) be a negative subharmonic function on \( \Omega \) such that \( \psi_1(z) = -\infty \) for any \( z \in \tilde{Z}_0 \), and let \( \varphi_\Omega \) be a Lebesgue measurable function on \( \Omega \) such that \( \varphi_\Omega + \psi_1 \) is subharmonic on \( \Omega \). Let \( \varphi_Y \) be a plurisubharmonic function on \( Y \). Denote that \( \psi := \pi_1^{-1}(\psi_1), \varphi = \pi_1^{-1}(\varphi_\Omega) + \pi_2^{-1}(\varphi_Y) \). Let \( c \) be a positive function on \( (0, +\infty) \) such that \( c^{-1}(t)e^{-t} dt < +\infty \), \( c(t)e^{-t} \) is decreasing on \( (0, +\infty) \) and \( e^{-\varphi c(-\psi)} \) has a positive lower bound on any compact subset of \( M \setminus \pi_1^{-1}(E) \), where \( E \subset \{ \psi_1 = -\infty \} \) is a discrete subset of \( \Omega \).
Let $f$ be a holomorphic $(n,0)$ form on a neighborhood of $Z_0$. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \& (\tilde{f} - f, p) \in \mathcal{O}(K_M)_p \otimes \mathcal{I}(\varphi + \psi)_p \text{ for any } p \in Z_0 \right\}$$

by $G(t)$ for any $t \in [0, +\infty)$.

Recall some notations related to open Riemann surfaces (see [14], see also [22] [19]). Let $P : \Delta \to \Omega$ be the universal covering from the unite disc $\Delta$ to $\Omega$. The holomorphic function $\hat{f}$ on $\Delta$ is called a multiplicative function, if there is a character $\chi$, which is the representation of the fundamental group of $\Delta$ such that $g^* (\hat{f}) = \chi(g) \hat{f}$, where $|\chi| = 1$ and $g$ is an element of the fundamental group of $\Omega$. Denote the set of such $\hat{f}$ by $\mathcal{O}^\chi(\Omega)$.

It is known that for any harmonic function $u$ of $\Omega$, there exists a $\chi_u$ and a multiplicative function $f_u \in \mathcal{O}^\chi_u(\Omega)$ such that $|f_u| = P^*(e^u)$. If $u_1 - u_2 = \log |\tilde{f}|$, then $\chi_{u_1} = \chi_{u_2}$, where $u_1$ and $u_2$ are harmonic functions on $\Omega$ and $\tilde{f}$ is a holomorphic function on $\Omega$. Recall that for the Green function $G_\Omega(z, 0)$, there exists a $\chi_{z_0}$ and a multiplicative function $f_{z_0} \in \mathcal{O}^\chi_{z_0}(\Omega)$ such that $|f_{z_0}(z)| = P^*(e^{G_\Omega(z, z_0)})$ (see [11]).

Let $\tilde{Z}_0 = \{z_j : j \in \mathbb{Z}_{\geq 0} \& 1 \leq j \leq m\}$ be a finite subset of the open Riemann surface $\Omega$. Let $w_j$ be a local coordinate on a neighborhood $V_{z_j} \subset \Omega$ of $z_j$ satisfying $w_j(z_j) = 0$ for any $j \neq k$. Denote that $V_0 := \bigcup_{1 \leq j \leq m} V_{z_j}$. Without loss of generality, assume that $w_j(V_{z_j}) = \{w \in \mathbb{C} : |w| < r_j\}$, where $r_j > 0$. Let $f$ be a holomorphic $(n,0)$ form on $V_0 \times Y$. Theorem 2.12 implies that $G(h^{-1}(r))$ is concave with respect to $r$, where $h(t) = \int_0^t c(s)e^{-s}ds$. We recall a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the fibers over sets of finite points as follows.

Theorem 2.30 ($\Omega$). Assume that $e^{-\varphi} c(-\psi)$ has a positive lower bound on any compact subset of $M \setminus Z_0$, $G(0) \in (0, +\infty)$ and $(\psi_1 - 2q_{z_j} G\Omega(\cdot, z_j))(z_j) > -\infty$, where $q_{z_j} = \frac{1}{2} v(\mathcal{d}c(\psi_1), z_j) > 0$ for any $j \in \{1, 2, \ldots, m\}$. Then $G(h^{-1}(r))$ is linear with respect to $r$ in $(0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:

1. $\psi_1 = 2 \sum_{j=1}^m q_{z_j} G\Omega(\cdot, z_j)$;

2. for any $j \in \{1, 2, \ldots, m\}$, $f = \pi_1^1(a_jw_j w_{k_j}dw_{j_k}) \wedge \pi_2^2(f_Y) + f_j$ on $V_{z_j} \times Y$, where $a_j \in \mathbb{C} \setminus \{0\}$ is a constant, $k_j$ is a nonnegative integer, $f_Y$ is a holomorphic $(n-1,0)$ form on $Y$ such that $\int_Y |f_Y|^2 e^{-\varphi_Y} \in (0, +\infty)$, and $(f_j, (z_j, y)) \in \mathcal{O}(K_M)(z_j, y) \otimes \mathcal{I}(\varphi + \psi)(z_j, y)$ for any $j \in \{1, 2, \ldots, m\}$ and $y \in Y$;

3. $\varphi_0 + \psi_1 = 2 \log |g| + 2 \sum_{j=1}^m G\Omega(\cdot, z_j) + 2u$, where $g$ is a holomorphic function on $\Omega$ such that $ord_{z_j}(g) = k_j$ and $u$ is a harmonic function on $\Omega$;

4. $\prod_{j=1}^m \chi_{z_j} = \chi_{-u}$, where $\chi_{-u}$ and $\chi_{z_j}$ are the characters associated to the functions $-u$ and $G\Omega(\cdot, z_j)$ respectively;
(5) for any \( j \in \{1, 2, \ldots, m\},
\[
\lim_{z \to z_j} \frac{a_j w_j^{k_j} dw_j}{g P_\varepsilon \left( f_u \left( \prod_{l=1}^{m} f_{z_l} \right) \left( \sum_{l=1}^{m} p_l \frac{d f_{z_l}}{f_{z_l}} \right) \right)} = c_0,
\]
where \( c_0 \in \mathbb{C}\{0\} \) is a constant independent of \( j \).

**Remark 2.31** (see [21]). The statements (3), (4) and (5) hold if and only if the following statements hold:

1. \( \varphi_\Omega + \psi_1 = 2 \log |g_1| \), where \( g_1 \) is a holomorphic function on \( \Omega \) such that \( \operatorname{ord}_{z_j}(g_1) = k_j + 1 \) for any \( j \in \{1, 2, \ldots, m\}; \)
2. \( \operatorname{ord}_{z_j}(g_1) \) is a constant independent of \( j \).

We give a generalization of Theorem 2.30, which will be used in the proofs of Proposition 3.2 and Theorem 1.2.

**Proposition 2.32.** Let \( G(0) \in (0, +\infty) \) and \( q_{z_j} = \frac{1}{2} v(\partial \Omega, \partial \Omega (\psi)) > 0 \) for any \( j \in \{1, 2, \ldots, m\}. \) For any \( j \in \{1, 2, \ldots, m\}, \) assume that one of the following conditions holds:

(1) \( \psi_1 = 2 \sum_{j=1}^{m} q_{z_j} G(\psi), z_j) ; \)
(2) for any \( j \in \{1, 2, \ldots, m\}, \) \( f = \pi^1(a_j w_j^{k_j} dw_j) + \pi^2(f_Y) + f_j \) on \( V_{z_j} \times Y, \) where \( a_j \in \mathbb{C}\{0\} \) is a constant, \( k_j \) is a nonnegative integer, \( f_Y \) is a holomorphic \( (n-1,0) \) form on \( Y \) such that \( \int_Y |f_Y|^2 e^{-\varphi_Y} \in (0, +\infty), \) and \( (f_j, (z_j, y)) \in \mathcal{O}(K_M)(z_j, y) \otimes \mathcal{O}(\varphi + \psi)(z_j, y) \) for any \( j \in \{1, 2, \ldots, m\} \) and \( y \in Y; \)
(3) \( \varphi_\Omega + \psi_1 = 2 \log |g_1| \), where \( g_1 \) is a holomorphic function on \( \Omega \) such that \( \operatorname{ord}_{z_j}(g_1) = k_j + 1 \) for any \( j \in \{1, 2, \ldots, m\}; \)
(4) \( \operatorname{ord}_{z_j}(g_1) \) is a constant independent of \( j \).

**Proof.** The sufficiency follows from Theorem 2.30 and Remark 2.31. Thus, we just need to prove the necessity. It follows from Theorem 2.30 and Remark 2.31 that it suffices to prove \( \psi_1 = 2 \sum_{j=1}^{m} q_{z_j} G(\psi), z_j) \).

It follows from Corollary 2.13 that there exists a holomorphic \( (n,0) \) form \( F \) on \( M \) satisfying \( (F - f, p) \in \mathcal{O}(K_M) \otimes \mathcal{O}(\varphi + \psi)p \) for any \( p \in Z_0, \)
\[
G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)
\]
for any \( t \geq 0 \) and
\[
\int_{\mathcal{M}} |F|^2 e^{-\varphi} a(-\psi) = \int_{0}^{+\infty} G(0) \frac{G(0)}{t} c(s)e^{-s} ds \int_{0}^{+\infty} a(s)e^{-s} ds
\]
for any nonnegative measurable function \( a(t) \) on \((0, +\infty)\). It follows from Lemma 2.7 that \( F = \sum_{j \geq k_j} \pi_1^j(w_j^j dw_j) \wedge \pi_2^j(F_{j,l}) \) on \( V_j \times Y \) for any \( j \in \{1, 2, \ldots, m\} \), where \( k_j \) is a nonnegative integer and \( F_{j,l} \) is a holomorphic \((n-1, 0)\) form on \( Y \) satisfying that \( \tilde{F}_j := F_{j,k_j} \neq 0 \). Denote that

\[
I_F := \{ j : 1 \leq j \leq m & k_j + 1 - k_j \leq 0 \}.
\]

Using the Weierstrass Theorem on open Riemann surfaces (see [14]) and Siu’s Decomposition Theorem, we have

\[
\varphi_\Omega + \psi_1 = 2\log |g_0| + 2u_0,
\]

where \( g_0 \) is a holomorphic function on \( \Omega \) and \( u_0 \) is a subharmonic function on \( \Omega \) such that \( v(ddc^u_0, z) \in (0, 1) \) for any \( z \in \Omega \). There exists a nonnegative integer \( k_j \) such that \( d_j := \lim_{z \to z_j} \frac{g_0(z)}{|w_j^j(z)|} \in \mathbb{C} \setminus \{0\} \). It follows from Lemma 2.27 that \( \tilde{F}_j + 1 - k_j = 0 \) for any \( j \in I_F \), \( \int_Y |F_j|^{2e^{-\varphi_\psi}} < +\infty \) for any \( j, l \), and

\[
\sum_{j \in I_F} \frac{2\pi e^{-2u_0(z_j)}}{|d_j|^2} \int_Y |\tilde{F}_j|^{2e^{-\varphi_\psi}} \leq \frac{G(0)}{\int_0^{+\infty} c(s)e^{-s}ds}.
\]

Note that \( |w_j^j|^{2e^{-\varphi_\psi}} \) is integrable near \( z_j \) for any \( l \geq k_j \) and any \( j \in \{1, 2, \ldots, m\} \), hence it follows from Lemma 2.7 that \( (\sum_{j \geq k_j} \pi_1^j(w_j^j dw_j) \wedge \pi_2^j(F_{j,l}), p) \in (O(K_M) \otimes \mathcal{I}(\varphi + \psi)_p \otimes \mathcal{I}(\varphi + \psi))_p \) for any \( p \in Z_0 \). Denote that \( \tilde{\psi}_1 := 2 \sum_{j=1}^m q_j G_\Omega(\cdot, z_j) \) and \( \varphi_\Omega := \varphi_\Omega + \psi_1 - \tilde{\psi}_1 \). Note that \( \tilde{\psi}_1 \geq \psi_1, \varphi_\Omega + \tilde{\psi}_1 = \varphi_\Omega + \psi_1 \) and \( c(t)e^{-t} \) is decreasing on \((0, +\infty)\). Using Theorem 2.28, there exists a holomorphic \((n, 0)\) form \( \tilde{F} \) on \( M \), such that \( \tilde{F} - F, p \in (O(K_M) \otimes \mathcal{I}(\varphi_\Omega + \tilde{\psi}_1 + \pi_2^j(\varphi_Y))_p \otimes \mathcal{I}(\varphi + \psi))_p \) for any \( p \in Z_0 \) and

\[
G(0) \leq \int_M |\tilde{F}|^{2e^{-\varphi_\psi}} c(-\psi)
\]

\[
\leq \int_M |\tilde{F}|^{2e^{-\pi_1^j(\varphi_\Omega) - \pi_2^j(\varphi_Y)}} c(-\pi_1^j(\tilde{\psi}_1))
\]

\[
\leq \left( \int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j \in I_F} \frac{2\pi e^{-2u_0(z_j)}}{|d_j|^2} \int_Y |\tilde{F}_j|^{2e^{-\varphi_\psi}}.
\]

Combining equality 2.30, inequality 2.30 and inequality 2.27, we obtain that

\[
\int_M |\tilde{F}|^{2e^{-\varphi_\psi}} c(-\psi) = \int_M |\tilde{F}|^{2e^{-\pi_1^j(\varphi_\Omega) - \pi_2^j(\varphi_Y)}} c(-\pi_1^j(\tilde{\psi}_1)).
\]

As \( \tilde{F} \neq 0 \), it follows from Lemma 2.27 that \( \psi_1 = \tilde{\psi}_1 = 2 \sum_{j=1}^m q_j G_\Omega(\cdot, z_j) \).

Thus, Proposition 2.3 holds. \( \square \)

Let \( \tilde{Z}_0 = \{ z_j : j \in \mathbb{Z}_{\geq 1} \} \) be an infinite discrete subset of the open Riemann surface \( \Omega \). Let \( w_j \) be a local coordinate on a neighborhood \( V_j \subset \subset \Omega \) of \( z_j \) satisfying \( w_j(z_j) = 0 \) for any \( j \in \tilde{Z}_0 \), where \( V_j \cap V_k = \emptyset \) for any \( j \neq k \). Without loss of generality, assume that \( w_j(V_j) = \{ w \in \mathbb{C} : |w| < r_j \} \), where \( r_j > 0 \). Denote that \( V_0 := \bigcup_{j=1}^\infty V_j \). Let \( f \) be a holomorphic \((n, 0)\) form on \( V_0 \times Y \). We recall a
necessary condition such that $G(h^{-1}(r))$ is linear for the fibers over infinite analytic subsets as follows.

**Proposition 2.33 ([H]).** Assume that $e^{-r}c(-\psi)$ has a positive lower bound on any compact subset of $M \setminus \Omega$, $G(0) \in (0, +\infty)$ and $(\psi_1 - 2q_z G_\Omega(\cdot, z_j))(z_j) > -\infty$, where $q_j = \frac{1}{2}v(dd^c(\psi_1), z_j) > 0$ for any $j \in \mathbb{Z}_{\geq 1}$. Assume that $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t) e^{-t} dt]$, then the following statements hold:

1. $\psi_1 = 2 \sum_{j=1}^{\infty} q_j G_\Omega(\cdot, z_j);$
2. for any $j \in \mathbb{Z}_{\geq 1}$, $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y) + f_j$ on $V_{\Omega_j} \times Y$, where $a_j \in \mathbb{C} \setminus \{0\}$ is a constant, $k_j$ is a nonnegative integer, $f_Y$ is a holomorphic $(n-1, 0)$ form on $Y$ such that $\int_Y |f_Y|^2 e^{-\varphi_Y} \in (0, +\infty)$, and $(f_j, (z_j, y)) \in \mathcal{O}(K_M(z_j, y)) \otimes \mathcal{I}(\varphi + \psi)(z_j, y)$ for any $j \in \mathbb{Z}_{\geq 1}$ and $y \in Y$;
3. $\varphi_\Omega + \psi_1 = 2 \log |g|$, where $g$ is a holomorphic function on $\Omega$ such that ord$_{z_j}(g) = k_j + 1$ for any $j \in \mathbb{Z}_{\geq 1}$;
4. for any $j \in \mathbb{Z}_{\geq 1}$,
   \[
   \frac{d g}{\text{ord}_{z_j}(g) z \to z_j a_j w_j^{k_j} dw_j} = c_0,
   \]
where $c_0 \in \mathbb{C} \setminus \{0\}$ is a constant independent of $j$;
5. $\sum_{j \in \mathbb{Z}_{\geq 1}} q_j < +\infty$.

We give a generalization of Proposition 2.33 which will be used in the proof of Proposition 6.2.

**Proposition 2.34.** Let $G(0) \in (0, +\infty)$ and $q_j = \frac{1}{2}v(dd^c(\psi_1), z_j) > 0$ for any $j \in \mathbb{Z}_{\geq 1}$. For any $j \in \mathbb{Z}_{\geq 1}$, assume that one of the following conditions holds:

A. $\varphi_\Omega + a \psi_1$ is subharmonic near $z_j$ for some $a \in [0, 1]$;
B. $(\psi_1 - 2q_j G_\Omega(\cdot, z_j))(z_j) > -\infty$.

If $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t) e^{-t} dt]$, then the following statements hold:

1. $\psi_1 = 2 \sum_{j=1}^{\infty} q_j G_\Omega(\cdot, z_j);$
2. for any $j \in \mathbb{Z}_{\geq 1}$, $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y) + f_j$ on $V_{\Omega_j} \times Y$, where $a_j \in \mathbb{C} \setminus \{0\}$ is a constant, $k_j$ is a nonnegative integer, $f_Y$ is a holomorphic $(n-1, 0)$ form on $Y$ such that $\int_Y |f_Y|^2 e^{-\varphi_Y} \in (0, +\infty)$, and $(f_j, (z_j, y)) \in \mathcal{O}(K_M(z_j, y)) \otimes \mathcal{I}(\varphi + \psi)(z_j, y)$ for any $j \in \mathbb{Z}_{\geq 1}$ and $y \in Y$;
3. $\varphi_\Omega + \psi_1 = 2 \log |g|$, where $g$ is a holomorphic function on $\Omega$ such that ord$_{z_j}(g) = k_j + 1$ for any $j \in \mathbb{Z}_{\geq 1}$;
4. for any $j \in \mathbb{Z}_{\geq 1}$,
   \[
   \frac{d g}{\text{ord}_{z_j}(g) z \to z_j a_j w_j^{k_j} dw_j} = c_0,
   \]
where $c_0 \in \mathbb{C} \setminus \{0\}$ is a constant independent of $j$;
5. $\sum_{j \in \mathbb{Z}_{\geq 1}} q_j < +\infty$.

**Proof.** The proof of Proposition 2.34 is similar to Proposition 2.32.
It follows from Proposition 2.33 that it suffices to prove \( \psi_1 = 2 \sum_{j=1}^{+\infty} q_j G_\Omega(\cdot, z_j) \).

It follows from Corollary 2.14 that there exists a holomorphic \((n, 0)\) form \( F \) on \( M \) satisfying \((F - f, p) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_p \) for any \( p \in Z_0 \),

\[
G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \tag{2.38}
\]

for any \( t \geq 0 \) and

\[
\int_M |F|^2 e^{-\varphi} a(-\psi) = \frac{G(0)}{\int_0^{+\infty} c(s)e^{-s} ds} \int_0^{+\infty} a(s)e^{-s} ds
\]

for any nonnegative measurable function \( a(t) \) on \((0, +\infty)\). It follows from Lemma 2.24 that \( F = \sum_{j \geq k_j} \pi^*_i (w^j dw_j) \wedge \pi^*_j(F_j,i) \) on \( V_j \times Y \) for any \( j \in \mathbb{Z}_{\geq 1} \), where \( k_j \) is a nonnegative integer and \( F_j,i \) is a holomorphic \((n - 1, 0)\) form on \( Y \) satisfying that \( \tilde{F}_j := F_j,k_j \neq 0 \). Denote that

\[
I_F := \{ j : j \in \mathbb{Z}_{\geq 1} \text{ and } \tilde{k}_j + 1 - k_j \leq 0 \}.
\]

Using the Weierstrass Theorem on open Riemann surfaces (see [14]) and Siu’s Decomposition Theorem, we have

\[
\varphi_\Omega + \psi_1 = 2 \log |g_0| + 2u_0,
\]

where \( g_0 \) is a holomorphic function on \( \Omega \) and \( u_0 \) is a subharmonic function on \( \Omega \) such that \( v(\alpha^{\infty} u_0, z) \in [0, 1) \) for any \( z \in \Omega \). There exists a nonnegative integer \( k_j \) such that \( d_j := \lim_{z \to z_j} a_{w_j}(z) \wedge w_j(z) \in \mathbb{C} \setminus \{0\} \). It follows from Lemma 2.24 that \( \tilde{k}_j + 1 - k_j = 0 \) for any \( j \in I_F \), \( \int_Y |F_j,i|^2 e^{-\varphi} < +\infty \) for any \( j, l \), and

\[
\sum_{j \in I_F} \frac{2\pi e^{-2u_0(z_j)}}{q_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi} \leq \frac{G(0)}{\int_0^{+\infty} c(s)e^{-s} ds}. \tag{2.39}
\]

Note that \( |w^j_j|^2 e^{-\varphi} \) is integrable near \( z_j \) for any \( l \geq k_j \) and any \( j \in \mathbb{Z}_{\geq 1} \), hence it follows from Lemma 2.17 that \( \sum_{j \geq k_j} \pi^*_i (w^j dw_j) \wedge \pi^*_j(F_j,i) \) is integrable on \( Y \) for any \( p \in Z_0 \). Denote that \( \tilde{\psi}_1 := 2 \sum_{j=1}^{+\infty} q_j G_\Omega(\cdot, z_j) \) and \( \varphi_\Omega := \varphi_\Omega + \psi_1 - \tilde{\psi}_1 \). Note that \( \tilde{\psi}_1 \geq \psi_1, \varphi_\Omega + \tilde{\psi}_1 = \varphi_\Omega + \psi_1 \) and \( c(t)e^{-t} \) is decreasing on \((0, +\infty)\).

Using Theorem 2.38, there exists a holomorphic \((n, 0)\) form \( \tilde{F} \) on \( M \), such that \((\tilde{F} - F, p) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_p \) for any \( p \in Z_0 \) and

\[
G(0) \leq \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi)
\]

\[
\leq \int_M |\tilde{F}|^2 e^{-\pi^*_1(\tilde{\varphi}_\Omega) - \pi^*_j(\varphi_\Omega c(-\pi^*_1(\tilde{\psi}_1)))}
\]

\[
\leq \left( \int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi e^{-2u_0(z_j)}}{q_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi} \tag{2.40}
\]

Combining equality (2.38), inequality (2.39) and inequality (2.40), we obtain that

\[
\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) = \int_M |\tilde{F}|^2 e^{-\pi^*_1(\tilde{\varphi}_\Omega) - \pi^*_j(\varphi_\Omega c(-\pi^*_1(\tilde{\psi}_1)))}.
\]
As $\tilde{F} \neq 0$, it follows from Lemma 2.27 that $\psi_1 = \tilde{\psi}_1 = 2 \sum_{j=1}^{+\infty} q_j G_{\Omega}(\cdot, z_j)$.

Thus, Proposition 2.28 holds. \hfill \Box

3. A NECESSARY CONDITION FOR $G(h^{-1}(r))$ IS LINEARITY

Let $\Omega$ be an open Riemann surface, which admits a nontrivial Green function $G_{\Omega}$. Let $Y$ be an $n - 1$ dimensional weakly pseudoconvex Kähler manifold, and let $K_Y$ be the canonical line bundle on $Y$. Let $M = \Omega \times Y$ be an $n$-dimensional complex manifold. Let $\pi_1$ and $\pi_2$ be the natural projections from $M$ to $\Omega$ and $Y$ respectively. Let $K_M$ be the canonical line bundle on $M$.

Let $\psi$ be a subharmonic function on $\Omega$. Let $\varphi_{\Omega}$ be a Lebesgue measurable function on $\Omega$ such that $\varphi_{\Omega} + \psi$ is subharmonic function on $\Omega$. Let $F$ be a holomorphic function on $\Omega$. Let $T \in [-\infty, +\infty)$. Denote that

$$\tilde{\psi} := \min \{\psi - 2 \log|F|, -T\}.$$  

For any $z \in \Omega$ satisfying $F(z) = 0$, we set $\tilde{\Psi}(z) = -T$. Denote that $\Psi := \pi_1^*(\tilde{\Psi})$ on $M$. Let $\varphi_Y$ be a plurisubharmonic function on $Y$. Denote that $\varphi := \pi_1^*(\varphi_{\Omega}) + \pi_2^*(\varphi_Y)$.

Let $Z_0 \subset M$ be a subset of $\cap_{t > T}\{\Psi < -t\}$ such that there exists a subset $\tilde{Z}_0$ of $\Omega$ such that $Z_0 = \tilde{Z}_0 \times Y$. Denote that $\tilde{Z}_1 := \{z \in \tilde{Z}_0 : v(dd^\circ_1(\psi), z) \geq 2ord_1(F)\}$ and $\tilde{Z}_2 := \{z \in \tilde{Z}_0 : v(dd^\circ_1(\psi), z) < 2ord_1(F)\}$, where $dd^\circ_1 = \frac{\partial}{\partial \bar{z}}$ and $dd^\circ_1(\psi)$ is decreasing on $T$, $+\infty$, $c(t)e^{-t}$ is integrable near $+\infty$, and $c(-\psi)e^{-\psi}$ has a positive lower bound on $K \cap \{\psi < -T\}$ for any compact subset $K \subset M \setminus \{1\}^{-1}(E)$, where $E$ is an analytic subset of $\Omega$ such that $E \subset \{\tilde{\Psi} = -\infty\}$.

Let $f$ be a holomorphic $(n, 0)$ form on $\{\Psi < -t_0\} \cap V$, where $V \supset Z_0$ is an open subset of $M$ and $t_0 > T$ is a real number. Denote

$$\inf \left\{ \int_{\{\Psi < -t\}} |\tilde{f}|^2 e^{-\varphi}c(-\Psi) : \tilde{f} \in H^0(\{\Psi < -t\}, \mathcal{O}(K_M)) \right\}$$

$$\& (\tilde{f} - f)_p \in \mathcal{O}(K_M)_p \otimes I(\varphi + \Psi)_p \text{ for any } p \in Z_0$$

by $G(t; c, \Psi, \varphi, I(\varphi + \Psi), f)$, where $t \in [T, +\infty)$. Without misunderstanding, we denote $G(t; c, \Psi, \varphi, I(\varphi + \Psi), f)$ by $G(t)$ for simplicity.

Note that there exists a subharmonic function $\psi_1$ on $\Omega_1 := \{\tilde{\Psi} < -T\} \cup \tilde{Z}_3$ such that $\psi_1 + 2 \log|F| = \psi$. Denote that $M' := \Omega_1 \times Y \subset M$ and $\tilde{\psi}_1 := \pi_1^*(\psi_1)$.

For any $z \in \Omega_1$, if $F(z) = 0$, for any $y \in Y$, we have $\Psi((z, y)) \neq -\infty$, then we know that $e^{-\varphi(c(-\psi_1))} = e^{-\varphi(c(-\tilde{\psi}))}$ has a positive lower bound on $K \cap \{z \times Y\} \subset K \cap \{\psi < -T\}$, where $K \subset M$ is a neighborhood of $(z, y)$. Combining $e^{-\varphi(c(-\psi_1))} \leq C e^{-\psi_1} - \tilde{\psi} = C e^{-\tilde{\varphi}} + \tilde{\psi} - \tilde{\varphi} - 2 \log|F| - \tilde{\varphi}$ on $K$, we have $v(dd^\circ_1(\varphi_{\Omega} + \psi), z) \geq 2ord_1(F)$. Hence we have $\varphi_{\Omega} + \psi_1$ is a subharmonic function on $\Omega_1$.

For any $z \in \tilde{Z}_3$, if $F(z) \neq 0$, we know that $I(\varphi + \Psi)_{(z, y)} = I(\varphi + \tilde{\psi})_{(z, y)}$ is an ideal of $\mathcal{O}_{M, (z, y)}$ for any $y \in Y$. For any $z \in \tilde{Z}_3$, if $F(z) = 0$, we know that
For any $G$ with respect to $r$ satisfying that $\tilde{h}$ of $h$ near $(z, y)$ such that $(\tilde{h}, (z, y)) \in I(\phi + \psi)(z, y)$, where $y \in Y$ and $h(z, y) \in I(\phi + \psi)(z, y)$.

Let $f_1$ be a holomorphic $(n, 0)$ form on $V_1$, where $V_1 \supset Z_3$ is an open subset of $M'$. Denote

$$
\inf \left\{ \int_{\{\tilde{\psi} < -t\}} |\tilde{f}|^2 e^{-r}\omega(-\tilde{\psi}) : \tilde{f} \in H^0(\{\tilde{\psi} < -t\}, \mathcal{O}(K_M)) \right\}
$$

by $G_{f_1}(t)$, where $t \in [T, +\infty)$.

Note that $e^{-r}\omega(-\tilde{\psi})$ has a positive lower bound on any compact subset of $M' \setminus \{(\pi_1^{-1}(E) \cup \tilde{Z}_3)\}$ and $E \cup \tilde{Z}_3 \subset \{\psi_1 = -\infty\}$. Let $f_2$ be a holomorphic $(n, 0)$ form on $V \cap \{\psi < -t_0\} \cup \{\tilde{\psi} < -t_0\}$ such that $(f_2)_p \in \mathcal{O}(K_M)_p \otimes H_p$ for any $p \in Z_0$, where $H_p = \{h_p \in J(\Psi)_p : \int_{\{\psi < -t\} \cap U} |h|^2 e^{-r}\omega(-\Psi)dv_M < +\infty\}$ for some $t > T$ and some neighborhood $U$ of $p_1$. $dv_M$ is a continuous volume form on $M$, $V \supset Z_0$ is an open subset of $M$ and $t_0 > T$ is a real number. Theorem 212 shows that $G(h^{-1}(r); c, \Psi, \varphi, I(\phi + \Psi), f_2)$ and $G_{f_1}(h^{-1}(r))$ are concave with respect to $r$, where $h(t) = \int_1^{+\infty} c(x)e^{-x}ds$. We give a relationship between $G(h^{-1}(r); c, \Psi, \varphi, I(\phi + \Psi), f_2)$ and $G_{f_1}(h^{-1}(r))$, which will be used in the proof of Proposition 5.5.2.

**Lemma 3.1.** If $H_p = I(\phi + \Psi)_p$ for any $p \in Z_0 \setminus Z_3$, then $G(t; c, \Psi, \varphi, I(\phi + \Psi), f_2) = \tilde{G}_{f_2}(t)$ for any $t \geq T$.

**Proof.** For any $t \geq T$ and holomorphic $(n, 0)$ form $\tilde{f}$ on $\{\tilde{\psi} < -t\}$ satisfying $(\tilde{f} - f_2)_p \in \mathcal{O}(K_M)_p \otimes I(\phi + \Psi)_p$ for any $p \in Z_3$ and $\int_{\{\tilde{\psi} < -t\}} |\tilde{f}|^2 e^{-r}\omega(-\tilde{\psi}) < +\infty$, it follows from $(f_2)_p \in \mathcal{O}(K_M)_p \otimes H_p$ for any $p \in Z_0$ and $H_p = I(\phi + \Psi)_p$ for any $p \in Z_0 \setminus Z_3$ that $(\tilde{f} - f_2)_p \in \mathcal{O}(K_M)_p \otimes I(\phi + \Psi)_p$ for any $p \in Z_0$. As $\mu(Z_2) = 0$, where $\mu$ is the Lebesgue measure on $M$, the definitions of $G(t; c, \Psi, \varphi, I(\phi + \Psi), f_2)$ and $G_{f_2}(t)$ show that $G(t; c, \Psi, \varphi, I(\phi + \Psi), f_2) \leq \tilde{G}_{f_2}(t)$ for any $t \geq T$.

For any $t \geq T$, let $\tilde{f}$ be a holomorphic $(n, 0)$ form on $\{\Psi < -t\}$ satisfying $(\tilde{f} - f_2)_p \in \mathcal{O}(K_M)_p \otimes I(\phi + \Psi)_p$ for any $z \in Z_0$ and $\int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-r}\omega(-\Psi) < +\infty$. For any $p = (z, y) \in Z_3\{\Psi < -t\}$, we have $F(z) = 0$. Note that $\psi(d\omega(\psi), z) > 2\omega_d(F)$, then $e^{-r}\omega(-\Psi)$ has a positive lower bound on $K\setminus\{(z) \times Y \subset \{\Psi < -t\}$, where $K \subset M$ is a neighborhood of $(z, y)$. Following from $\int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-r}\omega(-\Psi) < +\infty$, we get that there exists a holomorphic $(n, 0)$ form $f_1$ on $\{\tilde{\psi} < -t\} = \{\Psi < -t\} \cup Z_3$ such that $f_1 = \tilde{f}$ on $\{\Psi < -t\}$, which implies that $(f_1 - f_2)_p \in \mathcal{O}(K_M)_p \otimes I(\phi + \Psi)_p$ for any $p \in Z_3$ and $\int_{\{\psi < -t\}} |f_1|^2 e^{-r}\omega(-\Psi) = \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-r}\omega(-\Psi)$. By the definitions of $G(t; c, \Psi, \varphi, I(\phi + \Psi), f_2)$ and $G_{f_2}(t)$, we have $G(t; c, \Psi, \varphi, I(\phi + \Psi), f_2) \geq \tilde{G}_{f_2}(t)$ for any $t \geq T$.

Thus, Lemma 5.5.1 holds.

Note that $\tilde{Z}_3$ is a discrete subset of $\Omega$. Let $\tilde{Z}_3 = \{z_j : 1 \leq j < \gamma\}$, where $\gamma \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. Let $w_j$ be a local coordinate on a neighborhood $V_{z_j} \in \Omega$ of $z_j$ satisfying that $w_j(z_j) = 0$ for any $z_j \in \tilde{Z}_3$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any $j \neq k$. We give a necessary condition for the concavity of $G(h^{-1}(r))$ degenerating to linearity.
Proposition 3.2. For any \( z \in \tilde{Z}_1 \), assume that one of the following conditions holds:

(A) \( \varphi_\Omega + a \psi \) is subharmonic near \( z \) for some \( a \in [0, 1) \);

(B) \( (\psi - 2q_z \log |w|)(z) > -\infty \), where \( q_z = \frac{1}{2} \psi(d\mathcal{F}(\psi), z) \) and \( w \) is a local coordinate on a neighborhood of \( z \) satisfying that \( w(z) = 0 \).

If there exists \( t_1 \geq T \) such that \( G(t_1) \in (0, +\infty) \) and \( G(h^{-1}(r)) \) is linear with respect to \( r \in (0, \int_0^\infty c(s)e^{e^{-d}s}ds) \), then the following statements hold:

1. \( f = \pi_1^1(a_j w_j^k dw_j) \land \pi_2^2(f_Y) + f_j \) on \( (V_{t_j} \times Y) \cap \{ \Psi < -t_0 \} \cap V \), where \( a_j \in C\{ \Psi < -t_0 \} \) \( k_j \) is a nonnegative integer, \( f_Y \) is a holomorphic \((n - 1, 0)\) form on \( Y \) satisfying \( \int_Y |f_Y|^2 e^{-\varphi_Y} \in (0, +\infty) \), and \( (f_j)_p \in O(K_\Omega)_p \otimes I(\varphi + \Psi)_p \) for any \( p \in \{ z_j \} \times Y \);

2. \( \varphi_\Omega + \psi = 2 \log |g| + 2 \log |F| \), where \( g \) is a holomorphic function on \( \{ \Psi < -T \} \cap Z_3 \subset \Omega \) such that \( \text{ord}_z(g) = k_j + 1 \) for any \( 1 \leq j < \gamma \);

3. \( \tilde{Z}_3 \neq \emptyset \) and \( \psi = 2 \sum_{1 \leq j < \gamma} (q_z - \text{ord}_z(F)) G_{\Omega_j}(\cdot, z_j) + 2 \log |F| - t \) on \( \Omega_t \) for any \( t \geq T \), where \( \Omega_t = \{ \Psi < -t \} \cup \tilde{Z}_3 \subset \Omega \) and \( G_{\Omega_t} \) is the Green function on \( \Omega_t \);

4. \( \lim_{z \to z_j} a_j w_j^{\text{ord}_z(F)} = c_0 \) for any \( 1 \leq j < \gamma \), where \( c_0 \in C\{ \emptyset \} \) is a constant independent of \( j \);

5. \( \sum_{1 \leq j < \gamma} (q_z - \text{ord}_z(F)) < +\infty \).

Proof. It follows from Remark 2.14 that we can assume that \( c(t) \geq \frac{t}{1+t} \) near \( +\infty \).

Lemma 2.21 shows that \( H_p = I(\varphi + \Psi)_p \) for any \( p \in Z_0 \cap Z_3 \).

For any \( z_0 \in \tilde{Z}_2 \), it follows from Lemma 2.21 and the following remark that \( H_p = I(\varphi + \Psi)_p \) for any \( p \in \{ z_0 \} \times Y \).

Remark 3.3. Let \( b \in (0, 1) \). Let \( \tilde{\varphi}_\Omega = \varphi_\Omega + b(\psi - 2 \log |F|) \) and \( \tilde{\psi} = (1 - b) \psi + 2b \log |F| \). Denote that \( \tilde{\Psi} := \min\{ \pi_1^1(\tilde{\psi} - 2 \log |F|), (1 - b)\Psi \} = (1 - b)\Psi \). Let \( \tilde{c}(t) = c \left( \frac{1}{1+b} \right) e^{-\frac{t}{1-b}} \) be a function on \((1 - b)T, +\infty)\), and we have \( \tilde{c}(t) \geq 1 \) near \( +\infty \). It is clear that \( \tilde{\psi} \) and \( \tilde{\varphi}_\Omega + \tilde{\psi} = \varphi_\Omega + \psi \) are subharmonic functions. Denote that \( \tilde{\varphi}_\Omega = \pi_1^1(\tilde{\varphi}_\Omega) + \pi_2^2(\varphi_\Omega) \).

Note that \( e^{-\tilde{\varphi}_\Omega \tilde{c}(\cdot - \tilde{\Psi})} = e^{-\pi_1^1(\varphi_\Omega + b(\psi - 2 \log |F|)) - \pi_2^2(\varphi_\Omega) c(-\Psi) e^{\tilde{\Psi}}} = e^{-\varphi c(-\Psi)} \) on \( \{ \Psi < -T \} \), \( \tilde{\varphi} = \tilde{\varphi}_\Omega + \tilde{\psi} = \tilde{\varphi}_\Omega + \tilde{\psi} \) on \( \{ \Psi < -t \} \) and \( \tilde{c}(t) e^{-t} = c \left( \frac{1}{1-b} \right) e^{\frac{t}{1-b}} \). As \( z_0 \in \tilde{Z}_2 = \{ z \in \tilde{Z}_2 : 2\text{ord}_z(F) > v(d\mathcal{F}(\psi), z) \} \), we can choose \( b \in (0, 1) \) such that \( v(d\mathcal{F}(\tilde{\varphi}_\Omega), z_0) + v(d\mathcal{F}(\tilde{\varphi}_\Omega + \tilde{\psi}), z_0) = v(d\mathcal{F}(\tilde{\psi}), z_0) + v(d\mathcal{F}(\tilde{\varphi}_\Omega + \tilde{\psi}), z_0) + b(2\text{ord}_z(F) - v(d\mathcal{F}(\tilde{\psi}), z_0)) \notin \mathbb{Z} \).

Thus, we have \( H_p = I(\varphi + \Psi)_p \) for any \( p \in Z_0 \cap Z_3 \).

It follows from Lemma 2.15 that there exists a holomorphic \((n, 0)\) form \( f_{t_1} \) on \( \{ \Psi < -t_1 \} \) such that \( (f_{t_1} - f)_p \in O(K_\Omega)_p \otimes I(\varphi + \Psi)_p \) for any \( p \in Z_0 \) and \( \int_{\{ \Psi < -t_1 \}} |f_{t_1}|^2 e^{-\varphi c(-\Psi)} < +\infty \), which implies that \( (f_{t_1})_p \in O(K_\Omega)_p \otimes H_p \) for any \( p \in Z_0 \).

Note that there exists a subharmonic function \( \psi_1 \) on \( \Omega_1 := \{ \tilde{\Psi} < -T \} \cup \tilde{Z}_3 \) such that \( \psi_1 + 2 \log |F| = \psi \). Denote that \( M' := \Omega_1 \times Y = \{ \tilde{\Psi} < -T \} \cup Z_3 \subset M \) and \( \psi := \pi_1^1(\psi_1) \). For any \( (z_0, y_0) \in Z_3 \cap \{ \Psi < -t_1 \} \), we have \( F(z_0) = 0 \). Note that \( v(d\mathcal{F}(\psi_1), z_0) > 2\text{ord}_z(F) \), then \( e^{-\varphi c(-\Psi)} \) has a positive lower bound on \( (V') \{ z_0 \} \times 0 \subset \{ \Psi < -t \} \), where \( V' \cap \Omega_1 \) is a neighborhood of \( z_0 \) and \( Y_0 \) is a neighborhood of \( y_0 \). Following from \( \int_{\{ \Psi < -t_1 \}} |f_{t_1}|^2 e^{-\varphi c(-\Psi)} < +\infty \), we get that
there exists a holomorphic $(n, 0)$ form $\tilde{f}_1$ on $\{\tilde{\psi} < -t_1\} = \{\tilde{\psi} < -t_1\} \cup Z_3$ such that $\tilde{f}_1 = f_1$ on $\{\Psi < -t_1\}$, which implies that $(\tilde{f}_1 - f)_p \in O(K_{1\Omega}) \otimes I(\varphi + \Psi)_p$ for any $p \in Z_0$. By the definition of $G(t; c, \Psi, \varphi, I(\varphi + \Psi), f)$, we have $G(t; c, \Psi, \varphi, I(\varphi + \Psi), f) = G(t; c, \Psi, \varphi, I(\varphi + \Psi), f_1)$ for any $t \geq T$. It follows from Lemma 3.1 and $H_p = I(\varphi + \Psi)_p$ for any $p \in Z_0 \setminus Z_3$ that $G_{\tilde{f}_1}(t) = G(t; c, \Psi, \varphi, I(\varphi + \Psi), f_1)$ for any $t \geq T$, which implies that $G_{\tilde{f}_1}(h^{-1}(r))$ is linear with respect to $r$. Denote that $Z_3' := \{z \in \tilde{Z}_3 : v(df^c(\varphi_1 + \psi_1), z) > 0\}$ and $Z_3' := \pi_1^{-1}(Z'_3)$. Denote

$$ \inf \left\{ \int_{\{\tilde{\psi} < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\tilde{\psi}) : \tilde{f} \in H^0(\{\tilde{\psi} < -t\}, O(K_M)) \right\}$$

by $\tilde{G}(t)$, where $t \in [T, +\infty)$. For any holomorphic $(n, 0)$ form $\tilde{f}$ on $\{\tilde{\psi} < -t\}$ satisfying $\int_{\{\tilde{\psi} < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\tilde{\psi}) < +\infty$ and any $p = (z_0, y_0) \in \tilde{Z}_3 \setminus \tilde{Z}'_3$, it follows from Lemma 2.7 and Lemma 2.9 that $\tilde{f} U_{z_0} \times Y = \sum_{j \geq 0} \pi_1^*(w^d w^j) \cap \pi_2^*(F_j)$, where $w$ is a local coordinate on a neighborhood $V_{z_0} \in \{\tilde{\psi} < -t\} \subset \Omega_1$ satisfying $w(z_0) = 0$ and $U_{z_0} = \{w(z) < 1\} \cap V_{z_0}$, $F_j$ is holomorphic $(n-1, 0)$ form on $Y$ satisfying $(F_j, y_0) \in (O(K_Y) \otimes I(\varphi_Y))_p$. Note that $\tilde{Z}_3 \setminus \tilde{Z}'_3 = \{z \in \tilde{Z}_3 : v(d\tilde{f}^c(\varphi_1 + \psi_1), z) = 0\}$, then it follows from Lemma 2.17 that $(\tilde{f}, p) \in O(K_M) \otimes I(\varphi + \tilde{\psi})_p$. Hence, we have $G(h^{-1}(r)) = G_{\tilde{f}_1}(h^{-1}(r))$ is linear with respect to $r$.

For any $z_0 \in \tilde{Z}_3'$, if $(\psi_1 - 2q_{z_0} \log |w|(z_0) = -\infty$, where $q_{z_0} = \frac{1}{2} v(df^c(\psi_1), z_0)$ and $w$ is a local coordinate on a neighborhood of $z_0$ satisfying $w(z_0) = 0$, we have $(\psi_1 - 2q_{z_0} \log |w|(z_0) = -\infty$ (where $q_{z_0} = \frac{1}{2} v(df^c(\psi_1), z_0)$), thus there exists $a \in [0, 1)$ such that $\varphi_1 + a \psi$ is subharmonic near $z_0$. As $v(df^c(\varphi_1 + \psi_1), z_0) = v(df^c(\varphi_1 + a \psi), z_0) - 2 a \log x(F_0) > 0$, there exists $a' \in [a, 1)$ such that $v(df^c(\varphi_1 + a' \psi_1) > 2 a' \log x(F_0) > 0$, which implies that $\varphi_1 + a' \psi$ is subharmonic near $z_0$. As $G(t_1; c, \Psi, \varphi, i(\varphi + \Psi), f_1) = G(t_1) \in (0, +\infty)$, we have $Z_3' \neq \emptyset$. It follows from Proposition 2.88 and Proposition 2.89 (replace $M, \psi, Z_0$ and $c(-t)$ respectively, where $t > T$) that the following statements hold:

(a) $\psi_1 + t = 2 \sum_{z \in \tilde{Z}_3} q_{z} \log |g|_z$ on $\Omega_1 = \{\tilde{\psi} < -t\}$ for any $t > T$;

(b) $f_{\tilde{f}_1} = \pi_1^*(a_j w^j \omega_{w^j} \psi_{w^j})$ on $\tilde{Z}_3'$, where $a_j \in C \setminus \{0\}$ is a constant, $k_j$ is a nonnegative integer, $f_{\tilde{f}}$ is a holomorphic $(n-1, 0)$ form on $Y$ such that $\int_{\tilde{Y}} |f_{\tilde{f}}| e^{-\varphi} c(\tilde{Y}) < \infty$, and $(f_{\tilde{f}}, (z_j, y)) \in O(K_M)(z_j, y) \otimes I(\varphi + \tilde{\psi})(z_j, y)$ for any $z_j \in \tilde{Z}_3'$ and $y \in Y$;

(c) $\varphi_1 + \psi_1 = 2 \log |g|$, where $g$ is a holomorphic function on $\Omega_1 = \{\Psi < -T\} \cup \tilde{Z}_3$ such that $ord_z(g) = k_j + 1$ for any $z \in \tilde{Z}_3'$;

(d) $\frac{q_{z}}{ord_z(g)} \lim_{z \to z_j} \frac{dq}{a_j \omega_{w^j} \omega_{w^j}} = c_0$ for any $z_j \in \tilde{Z}_3'$, where $c_0 \in C \setminus \{0\}$ is a constant independent of $z_j$;

(e) $\sum_{z \in \tilde{Z}_3} q_{z} \psi < +\infty$.

By definition of $\psi_1$, we know that $q_z = q_z - ord_z(F)$ for any $z \in \Omega_1$. It follows from (a), Lemma 2.9 and $Z_3 \in \{z \in \tilde{Z}_0 : v(df^c(\psi), z) > 2ord_z(F)\}$ that $\tilde{Z}_3 = \tilde{Z}_3' \neq \emptyset$ and $\psi = 2 \sum_{1 \leq j \leq s} (q_{z_j} - ord_z(F)) \log |g| - t$ on $\Omega_t$ for
any $t \geq T$. Note that $(f_1 - f)_p \in \Omega(K_M)_p \otimes I(\varphi + \Psi)_p$ for any $p \in Z_0$, and $(f_j, p) \in \Omega(K_M)_p \otimes I(\varphi + \psi)_p$ implies $(f_j)_p \in \Omega(K_M)_p \otimes I(\varphi + \Psi)_p$ for any $Z_3$. Let $f_j = f - \pi_1^j(a_jw_j^k dw_j) \wedge \pi_2^j(fy)$ on $(V_j \times Y) \cap \{\Psi < -t_0\} \cap V$, hence $(f_j)_p = (f - f_1 + f_j)_p \in \Omega(K_M)_p \otimes I(\varphi + \Psi)_p$ for any $p \in \{z_j\} \times Y$. It follows from $(c)$ and $\psi = \psi_1 + 2 \log |F|$ that $\varphi_0 + \psi = 2 \log |g| + 2 \log |F|$.

Thus, the five statements in Proposition 3.2 hold. \hfill \Box

4. Proof of Theorem 1.2

The necessity of Theorem 1.2 follows from Proposition 3.2. In the following, we prove the sufficiency.

As $\psi = 2 \sum_{1 \leq j \leq m} (g_{j_1} - ord_z(F))G_{\Omega_1}(\cdot, z_j) + 2 \log |F| - t$ on $\Omega_t$ and $\Psi = \min\{\pi_1^j(\psi - 2 \log |F|), T\}$, it follows from Lemma 3.14 that

$$Z_0 = Z_0 \cap (\cap_{s \geq T} \{\Psi < -s\}) = \{z_j : j \in \{1, 2, \ldots, m\}\} \times Y = Z_3.$$  

It follows from Lemma 2.13 that there exists a holomorphic $(n, 0)$ form $f_1$ on $\{\Psi < -t_1\}$ such that $(f_1 - f)_p \in \Omega(K_M)_p \otimes I(\varphi + \Psi)_p$ for any $p \in Z_0$ and $f_1(\Psi < -t_1)$ for any $p \in Z_0$, where $t_1 > 0$. Note that there exists a subharmonic function $\psi_1$ on $\Omega_t := \{\Psi < -T\} \cup Z_3$ such that $\psi_1 + 2 \log |F| = \psi$. Denote that $M' := \Omega_t \times Y = \{\Psi < -T\} \cup Z_3 \subset M$ and $\psi := \pi_1^j(\psi_1)$. For any $(z_0, y_0) \in Z_3 \{\Psi < -t_1\}$, we have $H(z_0) = 0$. Note that $v(\psi_1, z_0) > 2ord_{z_0}(F)$, then $e^{-\varphi}c(-\Psi)$ has a positive lower bound on $(V \setminus \{z_0\}) \times Y_0 \subset \{\Psi < -t\}$, where $V' \subset \Omega_t$ is a neighborhood of $z_0$ and $Y_0$ is a neighborhood of $y_0$. Following from $\int_{\{\Psi < -t_1\}} f_1 |e^{-\varphi}c(-\Psi)| < +\infty$, we get that there exists a holomorphic $(n, 0)$ form $f_2$, on $\{\Psi < -t_1\} \cup Z_3$ such that $f_1 = f_2$ on $\{\Psi < -t_1\}$, which implies that $(f_1 - f)_p \in \Omega(K_\Omega)_p \otimes I(\varphi + \Psi)_p$ for any $p \in Z_0$. Following from the definition of $G_{f_2}(t)$ in Section 3, it follows from Lemma 3.1 and $Z_0 = Z_3$ that $G_{f_2}(t) = G(t)$ for any $t \geq T$. Using Proposition 2.32 (replace $M$, $c(\cdot)$ and $\psi$ by $M'$, $c(\cdot + \hat{t})$ and $\pi_1^j(\psi_1) + \hat{t}$ respectively, where $\hat{t} > T$), we know that $G_{f_1}(h^{-1}(r) + \hat{t})$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s + \hat{t})e^{-s} ds)$ for any $\hat{t} > T$, where $h(t) = \int_0^{+\infty} c(s + \hat{t})e^{-s} ds = \int_0^{+\infty} c(s)e^{-s} ds = e^{\hat{t}}h(t + \hat{t})$. Note that $G(h^{-1}(r)) = G_{f_1}(h^{-1}(e^{\hat{t}}r + \hat{t}))$ for any $r \in (0, \int_0^{+\infty} c(s)e^{-s} ds)$. Hence we have $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s} ds)$.

Thus, Theorem 1.2 holds.

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