A simple solution to a continuous-time mean-variance portfolio selection via the mean-variance hedging

Naohiro Yoshida

1 Department of Industrial and Systems Engineering, Chuo University, 1-13-27 Kasuga, Bunkyo, Tokyo 112-8551, Japan
E-mail kelly-opcalc.05t@chuo-u.ac.jp

Received February 03, 2018, Accepted February 12, 2019

Abstract
In this paper, an explicit solution to a continuous-time mean-variance portfolio selection problem in a continuous semimartingale model is provided through the Lagrange multiplier method and results of a mean-variance hedging problem. Without reformulation of the problem which is usually employed in the literature, we get a more straightforward method of solution than earlier studies.

Keywords mean-variance portfolio selection, continuous-time, mean-variance hedging, continuous semimartingale model, the Lagrange multiplier method

Research Activity Group Mathematical Finance

1. Introduction

In this paper, a continuous-time mean-variance portfolio selection problem in a continuous semimartingale model is solved using results of the mean-variance hedging problem.

Mean-variance portfolio selection is a problem of the allocation of wealth among various securities so as to attain the optimal trade-off between the expected return of the portfolio and its risk measured by the variance of the portfolio. This problem was first proposed and solved in the single-period setting by [1]. Markowitz formulated the problem of minimizing a portfolio’s variance subject to the constraint that its expected return equals a constant level. This analysis has long been recognized as the basis of modern portfolio theory.

Being widely used in both academia and industry, this mean-variance paradigm has also inspired the development of the multiperiod mean-variance portfolio selections. A typical formulation of this problem is

$$\inf_{\theta \in \Theta} Var(V_T(\theta))$$

subject to $$E[V_T(\theta)] \geq A$$

where $$\Theta$$ is some set of trading strategies, $$V_T(\theta)$$ denotes the value of a portfolio corresponding to a trading strategy $$\theta$$ at the terminal time $$T$$ and $$A$$ is a sufficiently large constant. The precise description of this problem will be provided in Section 3 below. In discrete time, among others, [2] has solved a mean-variance portfolio selection in a general discrete-time model where the growth rates of the security prices at each period are assumed to be independent random variables. They used the framework of multiobjective optimization and introduced an embedding technique which embeds the original problem in quadratic utility optimization problems

$$\inf_{\theta \in \Theta} E[\mu V_T^2(\theta) - \lambda V_T(\theta)]$$

for some parameters $$\mu$$ and $$\lambda$$ (see, for example, [3, Theorem 3.1] about this reformulation) so that dynamic programming can be used to obtain explicit solutions.

Continuous-time mean-variance portfolio selections have been studied by various approaches. As examples of those studies, we have [3–7]. The embedding technique has also been employed by [3] to solve a continuous-time mean-variance portfolio selection using the stochastic linear-quadratic (LQ) control theory in a diffusion model with deterministic coefficients. [4] has also dealt with the problem by the stochastic LQ control in a diffusion model when the coefficients are random. [5] treated a continuous-time mean-variance portfolio selection in a jump-diffusion model as an application of their main result about the stochastic maximum principle in the model. [6] tackled the problem in a continuous-time Markovian model driven by two Brownian motions directly applying dynamic programming without the embedding technique and they derived the time-consistent solution to the problem. [7] has studied a continuous-time mean-variance portfolio selection with a condition which prohibits the portfolio taking negative value so that the theory can be more suitable for the investment situation in the real world.

The multiperiod mean-variance criteria can be applied to practical investment problems because of their solvability and explicit results. For example, both [8] and [9] solved optimal investment problems for insurance companies which are formulated as continuous-time mean-variance portfolio selections. Moreover [6] stated that the multiperiod mean-variance criteria can be used as a benchmark for evaluation of investment. These articles motivate us to struggle to obtain more simple and elementary solution to multiperiod mean-variance portfolio selections. In this paper, we will solve the continuous-time mean-variance portfolio selection in a continuous semimartingale model through the Lagrange multiplier
method and a solution of a mean-variance hedging without reformulating the problem.

The rest of the paper is organized as follows. In Section 2, results of a mean-variance hedging in a continuous semimartingale model are recalled. A continuous-time mean-variance portfolio selection is solved in Section 3 using the solution of the mean-variance hedging in continuous time. Section 4 concludes the paper.

2. A mean-variance hedging in continuous time

In this section, we recall results of the mean-variance hedging in continuous time.

In order to describe a mean-variance hedging which we use and its solution, we have some preliminaries. We set a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0 \leq t \leq T})$ with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual condition. For simplicity, we assume that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$. Let $X$ be a continuous $\mathbb{R}^d$-valued semimartingale which represents the price of $d$ risky assets. In this paper, we assume that the growth rate of the risk-free asset is zero.

We define the whole set of trading strategies by

$$\Theta := \left\{ \theta = (\theta_s)_{0 \leq s \leq T} \mid \theta \text{ is an } \mathbb{R}^d \text{-valued predictable process such that } \int_0^T \theta_s^\top dX_s \text{ is a square-integrable semimartingale} \right\},$$

where $\top$ represents the transpose of vectors. In this paper, we assume that $G_T(\Theta) := \{\int_0^T \theta_s^\top dX_s \mid \theta \in \Theta\}$ is closed in $L^2(P)$.

Next, we define the variance-optimal martingale measure in this model. The following three definitions are taken from [10].

**Definition 1** Let $W$ denotes the linear subspace of $L^\infty(\Omega, \mathcal{F}, P)$ spanned by the simple stochastic integrals of the form $Y = h^\top (X_{T_1} - X_{T_1})$ where $T_1 \leq T_2 \leq T$ are stopping times such that $X_{T_1}$ is bounded and $h$ is any bounded $\mathbb{R}^d$-valued $\mathcal{F}_{T_1}$-measurable random variable.

**Definition 2** We define a set $M^s(P)$ as the space of all signed measures $Q \ll P$ with $Q(\Omega) = 1$ and

$$E \left[ \frac{dQ}{dP} Y \right] = 0, \forall Y \in W.$$

Moreover we define the following set of densities:

$$D^s := \left\{ \frac{dQ}{dP} \mid Q \in M^s(P) \right\}.$$

We assume that $D^s \cap L^2(P) \neq \emptyset$ hereafter.

**Definition 3** The variance-optimal martingale measure $\tilde{P}$ is the element of $M^s(P)$ such that $\tilde{D} = d\tilde{P}/dP$ is in $L^2(P)$ and minimizes $E[D^2]$ over all $D \in D^s \cap L^2(P)$.

The following lemma from [11] shows that $\tilde{D}$ can be represented by a stochastic integral with respect to $X$.

Lemma 4 $\tilde{P}$ is given by

$$\tilde{D} = E[\tilde{D}^2] + \int_0^T \tilde{\zeta}_s^\top dX_s$$

for some $\tilde{\zeta} \in \Theta$.

**Proof** See [11, Lemma 1 (b)].

(QED)

Here, we can describe a solution of a mean-variance hedging in continuous time. Note that a mean-variance hedging is a problem which minimizes the square mean of the hedging error of a derivative. In this paper, we only use the result of the mean-variance hedging where the payoff of the derivative is a constant. Its solution is given by [10] as follows.

**Lemma 5** Assume that $G_T(\Theta)$ is closed in $L^2(P)$ and $D^s \cap L^2(P) \neq \emptyset$. Then the solution $\theta$ of a mean-variance hedging problem

$$\inf_{\theta \in D^s} E \left[ \left( 1 - \int_0^T \theta_s^\top dX_s \right)^2 \right]$$

is given by

$$\tilde{\theta}_s = - \frac{\tilde{\zeta}_s}{E[\tilde{D}^2]}$$

for $0 \leq s \leq T$ where $\tilde{D}$ is the density of the variance-optimal martingale measure $\tilde{P}$ and $\tilde{\zeta}$ is the integrand in (1).

**Proof** See [10, Lemma 4].

(QED)

3. A mean-variance portfolio selection in continuous time

In this section, a continuous-time mean-variance portfolio selection in continuous semimartingale model is solved explicitly.

Let $c > 0$ be the initial endowment. Then the value of the self-financing portfolio corresponding to a strategy $\theta$ at the terminal time $T$ can be expressed as $c + \int_0^T \theta_s^\top dX_s$.

We consider the following problem:

$$\inf_{\theta \in \Theta} \text{Var} \left[ c + \int_0^T \theta_s^\top dX_s \right]$$

subject to $E \left[ c + \int_0^T \theta_s^\top dX_s \right] \geq A$ (2)

where $A$ is a sufficiently large constant such that $A > c$. Hereafter, we suppose that $E[\tilde{D}^2] > 1$. Otherwise, the original probability measure $P$ and the variance-optimal martingale measure $\tilde{P}$ almost surely coincide and it results in $E[c + \int_0^T \theta_s^\top dX_s] = c$ for any $\theta \in \Theta$. Then, in the current situation where $A > c$, it is obvious that the mean-variance portfolio selection (2) does not admit any optimal solution.

Our main result which provides a solution to this problem can be written as follows.

**Theorem 6** For the mean-variance portfolio selection...
problem (2), an optimal strategy \( \theta^* \) is given by
\[
\theta^*_s = -\frac{A - c}{E[\hat{D}^2] - 1} \tilde{\xi}_s
\]
for \( 0 \leq s \leq T \) where \( \tilde{\xi} \) is the integrand in (1) and the optimal solution is obtained by
\[
\text{Var} \left( c + \int_0^T \theta_s^* dX_s \right) = \frac{(A - c)^2}{E[\hat{D}^2] - 1}. \tag{3}
\]

**Proof**

We begin with solving the following problem:
\[
\inf_{\theta \in \Theta} \text{Var} \left( c + \int_0^T \theta_s^* dX_s \right)
\]
subject to \( E \left[ c + \int_0^T \theta_s^* dX_s \right] = B \tag{4} \)
where \( B \) is a constant such that \( B \geq A \). It is obvious that the solution of the original problem (2) can be obtained by minimizing the solution of the above problem (4) in terms of \( B \). The Lagrangian corresponding to this problem (4) is obtained by
\[
\mathcal{L}(\theta, \lambda) = E \left[ \left( c + \int_0^T \theta_s^* dX_s - B \right)^2 + \lambda \left( B - E \left[ c + \int_0^T \theta_s^* dX_s \right] \right) \right]
\]
where \( \lambda \in \mathbb{R} \) is a Lagrange multiplier. By [12, Theorem 2 in Section 8.4], the optimal solution \( \theta^B \in \Theta \) and \( \lambda^B \in \mathbb{R} \) to the problem (4) is given by a saddle point of \( \mathcal{L} \), i.e., \( \theta^B \) and \( \lambda^B \) which satisfy
\[
\mathcal{L}(\theta^B, \lambda) \leq \mathcal{L}(\theta^B, \lambda^B) \leq \mathcal{L}(\theta, \lambda^B)
\]
for all \( \theta \in \Theta \) and \( \lambda \in \mathbb{R} \). Here, we minimize \( \mathcal{L}(\theta, \lambda) \) in terms of \( \theta \) for given \( \lambda \in \mathbb{R} \). As the example described in [13, Section 3.3], the Lagrangian can be rewritten as
\[
\mathcal{L}(\theta, \lambda) = E \left[ \left( c + \int_0^T \theta_s^* dX_s - B + \frac{1}{2} \lambda \right)^2 \right] - \frac{1}{4} \lambda^2. \tag{5}
\]

The problem of minimizing this with respect to \( \theta \) can be regarded as a mean-variance hedging when the payoff of a derivative is \( B - c + \lambda/2 \). Therefore, by Lemma 5, the strategy \( \theta^B \) which minimizes (5) is given by
\[
\theta^B_s = -\frac{B - c + \lambda/2}{E[\hat{D}^2]} \tilde{\xi}_s \tag{6}
\]
for \( 0 \leq s \leq T \). Next, we maximize \( \mathcal{L}(\theta^B, \lambda) \) in terms of \( \lambda \). From (1), we have
\[
\mathcal{L}(\theta^B, \lambda) = E \left[ \left( -\frac{B - c + \lambda/2}{E[\hat{D}^2]} \int_0^T \tilde{\xi}_s dX_s - \left( B + \frac{1}{2} \lambda \right)^2 \right) \right] - \frac{1}{4} \lambda^2
\]
\[
= E \left[ c - \frac{B - c + \lambda/2}{E[\hat{D}^2]} \left( \tilde{\xi}_s - B + E[\hat{D}^2] \right) \right]
\]
This is maximized when \( \lambda \) equals
\[
\lambda^B = \frac{2(B - c)}{E[\hat{D}^2] - 1}.
\]
Substituting \( \lambda = \lambda^B \) into (6), \( \theta^B \) is revealed to be
\[
\theta^B_s = -\frac{B - c}{E[\hat{D}^2]} \tilde{\xi}_s \tag{7}
\]
for \( 0 \leq s \leq T \). Then the optimal solution of (4) is obtained by
\[
\text{Var} \left( c + \int_0^T \theta_s^{B^*} dX_s \right) \tag{8}
\]
\[
= E \left[ \left( c - \frac{B - c}{E[\hat{D}^2]} \int_0^T \tilde{\xi}_s dX_s - B \right)^2 \right]
\]
\[
= (B - c)^2 E \left[ \left( 1 + \frac{\tilde{D} - E[\hat{D}^2]}{E[\hat{D}^2] - 1} \right)^2 \right]
\]
\[
= (B - c)^2 \frac{E((\tilde{D} - 1)^2)}{(E[\hat{D}^2] - 1)^2}
\]
\[
= \frac{(B - c)^2}{E[\hat{D}^2] - 1} \tag{9}
\]
for each \( B \geq A \). Since this is minimized when \( B = A \), the solution of the original problem (2) can be obtained by substituting \( B = A \) into (7) and (9) and this concludes the proof.

(QED)

**Remark 7**

We can check that \( E[c + \int_0^T \theta_s^* dX_s] = A \) as follows:
\[
E \left[ c - \frac{A - c}{E[\hat{D}^2]} \int_0^T \tilde{\xi}_s dX_s \right]
\]
\[
= c - \frac{A - c}{E[\hat{D}^2]} E[\tilde{D} - E[\hat{D}^2]]
\]
\[
= c - \frac{A - c}{E[\hat{D}^2] - 1} (1 - E[\hat{D}^2])
\]
\[
= A.
\]

**Remark 8**

Note that our result (3) is consistent with [3, Eq. (6.9)] if our \( E[\hat{D}^2] \), which is the square mean of the density of the variance-optimal martingale measure is equal to \( \exp \left\{ \int_0^T r(t) dt \right\} \) in [3] and the risk-free rate is zero.

**Remark 9**

Our approach may be valid for continuous-time mean-variance portfolio selections in general semi-martingale models if we employ the results given by [14] to handle the mean-variance hedging. To our knowledge, there is no study which solved such a problem.
4. Conclusion

In this paper, an explicit solution to a continuous-time mean-variance portfolio selection problem in a continuous semimartingale model is provided. Without reformulating the problem, we get a simpler method of solution than earlier studies applying results of a mean-variance hedging problem. Our approach may be valid for continuous-time mean-variance portfolio selections in general semimartingale models if we employ the results given by [14] to handle the mean-variance hedgings. However, we have always assumed that the growth rate of the risk-free asset is zero in this paper and a way to remove this limitation is not obvious. We have also not concerned the condition which prohibits the terminal value of the portfolio becoming negative as [7]. It is left for the future to overcome these shortcomings.

Acknowledgments

The author was supported by a fellowship of Japan Society for the Promotion of Science (JSPS). This research was also supported by JSPS KAKENHI Grant Number JP 16J02354.

References

[1] H. Markowitz, Portfolio selection, J. Finance, 7 (1952), 77–91.
[2] D. Li and W. L. Ng, Optimal dynamic portfolio selection: multiperiod mean-variance formulation, Math. Finance, 10 (2000), 387–406.
[3] X. Y. Zhou and D. Li, Continuous-time mean-variance portfolio selection: A stochastic LQ framework, Appl. Math. Optim., 42 (2000), 19–33.
[4] A. E. B. Lim, Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market, Math. Oper. Res., 29 (2004), 132–161.
[5] N. C. Framstad, B. Øksendal and A. Sulem, Sufficient stochastic maximum principle for the optimal control of jump diffusions and applications to finance, J. Optim. Theory Appl., 121 (2004), 77–98.
[6] S. Basak and G. Chabakauri, Dynamic mean-variance asset allocation, Rev. Financ. Stud., 23 (2010), 2970–3016.
[7] T. R. Bielecki, H. Jin, S. R. Pliska and X. Y. Zhou, Continuous-time mean-variance portfolio selection with bankruptcy prohibition, Math. Finance, 15 (2005), 213–244.
[8] L. Delong and R. Gerrard, Mean-variance portfolio selection for a non-life insurance company, Math. Methods Oper. Res., 66 (2007), 339–367.
[9] Z. Wang, J. Xia and L. Zhang, Optimal investment for an insurer: The martingale approach, Insurance Math. Econom., 40 (2007), 322–334.
[10] T. Rheinländer and M. Schweizer, On $L^2$-projections on a space of stochastic integrals, Ann. Probab., 25 (1997), 1810–1831.
[11] M. Schweizer, Approximation pricing and the variance-optimal martingale measure, Ann. Probab., 24 (1996), 206–236.
[12] D. G. Luenberger, Optimization by Vector Space Methods, John Willy & Sons, New York, 1969.
[13] B. K. Øksendal and A. Sulem, Applied Stochastic Control of Jump Diffusions, Springer, Berlin, 2005.
[14] M. Jeanblanc, M. Mania, M. Santacroce and M. Schweizer, Mean-variance hedging via stochastic control and BSDEs for general semimartingales, Ann. Appl. Probab., 22 (2012), 2388–2428.