A Note on the Borel-Cantelli Lemma

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Abstract

In this short note, we discuss the Barndorff-Nielsen lemma, which is a generalization of well-known Borel-Cantelli lemma. Although the result stated in the Barndorff-Nielsen lemma is correct, it does not follow from the argument proposed in the corresponding proof. In this note, we show this and offer an alternative proof of this lemma. We also propose a new generalization of Borel-Cantelli lemma.

Keywords and Phrases: Borel-Cantelli lemma; Barndorff-Nielsen lemma; limit laws.

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1 Introduction

Suppose \( A_1, A_2, \ldots \) is a sequence of events on a common probability space and that \( A_i^c \) denotes the complement of event \( A_i \). The Borel-Cantelli lemma, presented below as Lemma 1.1, is used for producing strong limit results.

Lemma 1.1. 1. If, for any sequence \( A_1, A_2, \ldots \) of events,

\[
\sum_{n=1}^{\infty} P(A_n) < \infty,
\]  

then \( P(A_n \text{ i.o.}) = 0 \);

2. If \( A_1, A_2, \ldots \) is a sequence of independent events and if \( \sum_{n=1}^{\infty} P(A_n) = \infty \), then \( P(A_n \text{ i.o.}) = 1 \).

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The independence condition in the second part of Lemma 1.1 has been weakened by a number of authors, including Chung and Erdos (1952), Erdos and Renyi (1959), Lamperti (1963), Kochen and Stone (1964), Spitzer (1964), Chandra (1999, 2008), Petrov (2002, 2004), Frolov (2012) and others.

The first part of Borel-Cantelli lemma has been generalized in Barndorff-Nielsen (1961), Balakrishnan and Stepanov (2010) and Frolov (2014). For a review on the Borel-Cantelli lemma, one may refer to the book of Chandra (2012). The result of Barndorff-Nielsen is presented below as Lemma 1.2.

Lemma 1.2. Let $A_n (n \geq 1)$ be a sequence of events such that $P(A_n) \to 0$. If

$$\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty,$$  

then $P(A_n \text{ i.o.}) = 0$.

Observe that condition (1.2) in Lemma 1.2 is weaker than condition (1.1) in the Borel-Cantelli lemma.

In this note, we show that the result stated in the Barndorff-Nielsen lemma does not follow from the argument proposed in the proof. That way, although the result is correct, the proof presented in Barndorff-Nielsen (1961) is incomplete and not rigorous.

In connection with the above, we propose an alternative proof of this lemma based on our earlier result, presented as Lemma 2.1. Lemma 2.1, in its turn, follows from a more general result, given here as Lemma 2.2 and proved in Balakrishnan and Stepanov (2010). In the end of this short note, we also propose a new generalization of Borel-Cantelli lemma.

## 2 On the Proof of Barndorff-Nielsen Lemma

First, we cite the proof of Lemma 1.2 from Barndorff-Nielsen (1961) and discuss it.

"Since

$$P(A_n^c \text{ i.o.}) = \lim_{n \to \infty} P(\cup_{i=n}^{\infty} A_i^c) \geq \lim_{n \to \infty} P(A_n^c) = 1$$

we have, in consequence of (1.2) and the Borel-Cantelli lemma,

$$P(A_n \text{ i.o.}) = P(A_n \cap A_{n+1}^c \text{ i.o.}) = 0."$$

Observe that (2.1) implies only that

$$P(A_n \text{ i.o.}) = P(A_n \text{ i.o.} \cap A_{n+1}^c \text{ i.o.}).$$
On p. 151 of Shiryaev (1989), one can find that for some sequence $s$ of events $B_n, C_n \ (n \geq 1)$

$$P(B_nC_n \ i.o.) < P(B_n \ i.o. \cap C_n \ i.o.).$$

The last inequality shows us that Lemma 1.2 does not follow directly from the argument proposed in its proof. We then present an alternative rigorous proof of Lemma 1.2.

**Proof of Lemma 1.2** In order to prove Lemma 1.2 we adduce Lemma 2.1.

**Lemma 2.1.** Let $A_n, \ (n \geq 1)$ be a sequence of events such that $P(A_n) \to 0$. If

$$\sum_{n=1}^{\infty} P(A_nA_{n+1}) < \infty, \quad (2.3)$$

then $P(A_n \ i.o.) = 0$.

Now, the result of Lemma 1.2 readily follows from Lemma 2.1 and the identity

$$\sum_{n=1}^{\infty} P(A_nA_{n+1}^c) = P(A_1) + \sum_{n=1}^{\infty} P(A_nA_{n+1}).$$

Lemma 2.1, in its turn, follows from a more general result, Lemma 2.2 obtained in Balakrishnan and Stepanov (2010). For the convenience of reader, the last lemma and its proof are also presented here.

**Lemma 2.2.** Let $A_1, A_2, \ldots$ be a sequence of events such that $P(A_n) \to 0$. If, for some $m \geq 0$,

$$\sum_{n=1}^{\infty} P(A_n^cA_{n+1}^c \ldots A_{n+m-1}^cA_{n+m}^c) < \infty,$$

then $P(A_n \ i.o.) = 0$.

**Proof of Lemma 2.2** We first note that

$$P(A_n \ i.o.) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = \lim_{n \to \infty} P\left(\bigcup_{k=n}^{\infty} A_k\right).$$

However,

$$P\left(\bigcup_{k=n}^{\infty} A_k\right) = P(A_n) + P(A_n^cA_{n+1}) + P(A_n^cA_{n+1}^cA_{n+2}) + \ldots$$

$$\leq P(A_n) + P(A_n^cA_{n+1}) + \ldots + P(A_n^c \ldots A_{n+m-2}^cA_{n+m-1}^c)$$

$$+ \sum_{k=n}^{\infty} P(A_k^c \ldots A_{k+m-1}^cA_{k+m}) \to 0$$

as $n \to \infty$. Hence, the result follows. \(\square\)

Observe that Lemma 2.1 follows from Lemma 2.2 if we choose $m = 1$. 
3 New Result

In this section, we present a new theoretical result, which generalizes the second part of Borel-Cantelli lemma. We first introduce a new notion.

**Definition 3.1.** Let $A$ and $B$ be some events. We say that $\alpha \geq 0$ is the power-$A$ coefficient of dependence between $A$ and $B$ if $P(AB) = (P(A))^\alpha P(B)$, provided that $\alpha = 1$ if $P(B) = 0$.

Obviously, if $A$ and $B$ are independent, then $\alpha = 1$.

Let now $\bar{A}_n^c = A_n^cA_{n+1}^c \ldots$ and $\alpha_n$ be the power-$A_n^c$ coefficient of dependence between $A_n^c$ and $\bar{A}_{n+1}^c$. In the following lemma we present sufficient conditions for $P(A_n i.o.) = 1$.

**Lemma 3.1.** Let $A_1, A_2, \ldots$ be a sequence of events. If

$$\sum_{n=1}^{\infty} \alpha_n P(A_n) = \infty, \quad (3.1)$$

then

$$P(A_n i.o.) = 1, \quad (3.2)$$

**Proof of Lemma 3.1** Indeed,

$$P(\bar{A}_n^c) = (P(A_n^c))^\alpha P(\bar{A}_{n+1}^c)$$

$$= \ldots = (P(A_n^c))^\alpha \ldots (P(A_{n+k-1}^c))^\alpha_{n+k-1} P(\bar{A}_{n+k}^c) \quad (n, k \geq 1).$$

By the inequality $\log(1-x) \leq -x \ (0 \leq x < 1)$, we get that

$$P(\bar{A}_n^c) \leq e^{-\sum_{i=n+k-1}^{n+k-1} \alpha_i P(A_i)} P(\bar{A}_{n+k}^c) \quad (n, k \geq 1).$$

Then

$$\lim_{k \to \infty} P(\bar{A}_n^c) = P(\bar{A}_n^c) \leq e^{-\sum_{i=n}^{\infty} \alpha_i P(A_i)} \ldots (1 - P(A_n i.o.)) \quad (n \geq 1). \quad (3.3)$$

Obviously, (3.1) and (3.3) imply (3.2). □

It should be noted that Lemma 3.1 is more of a theoretical value. Indeed, it is not easy to find the coefficients $\alpha_n$ in general case. In some situations, however, it is not difficult. Let us consider one of such cases. Let $I_A$ be the indicator-function of event $A$, i.e.

$$I_A = \begin{cases} 1, & \text{if } A \text{ happens,} \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 3.2.** We say that $A_n \ (n \geq 1)$ is a Markov sequence of events if the sequence of random variables $I_{A_n} \ (n \geq 1)$ is a Markov chain.
In respect to this definition, see also Stepanov (2014).

**Remark 3.1.** Let $\beta_n$ be the power-$A_n^c$ coefficient of dependence of $A_n^c$ and $A_{n+1}^c$. Then $\alpha_n = \beta_n$.

In the case when $A_n$ ($n \geq 1$) forms the Markov sequence of events, Remark 3.1 allows to find $\alpha_n$ easily. Indeed,

$$\alpha_n = \frac{\log P(A_n^c \mid A_{n+1})}{\log P(A_n^c)}.$$

**References**

Balakrishnan, N., Stepanov, A. (2010). Generalization of Borel-Cantelli lemma. *The Mathematical Scientist*, **35**, 61–62.

Barndorff-Nielsen, O. (1961). On the rate of growth of the partial maxima of a sequence of independent identically distributed random variables. *Math. Scand.*, 9, 383–394.

Chandra, T.K. (1999). A First Course in Asymptotic Theory of Statistics. Narosa Publishing House Pvt. Ltd., New Delhi.

Chandra, T.K. (2008). Borel-Cantelli lemma under dependence conditions. *Statist. Probab. Lett.*, **78**, 390-395.

Chandra, T.K., (2012). The Borel-Cantelli Lemma. Springer Briefs in Statistics.

Chung, K.L. and Erdos, P. (1952). On the application of the Borel-Cantelli lemma. *Trans. Amer. Math. Soc.*, 72, 179–186.

Erdos, P. and Renyi, A. (1959). On Cantor’s series with convergent $\sum 1/q_n$. *Ann. Univ. Sci. Budapest. Sect. Math.*, 2, 93–109.

Frolov, A.N. (2012). Bounds for probabilities of unions of events and the Borel-Cantelli lemma. *Statist. Probab. Lett.*, **82**, 2189–2197.

Frolov, A.N. (2014). On inequalities for probabilities of unions of events and the Borel-Cantelli lemma. *Vestnik St. Petersb. Univ. Math.*, 47, 68-75.

Kochen, S.B. and Stone, C.J. (1964). A note on the Borel-Cantelli lemma. *Illinois J. Math.*, 8, 248–251.

Lamperti, J. (1963). Wiener’s test and Markov chains. *J. Math. Anal. Appl.*, 6, 58–66.

Petrov, V.V. (2002). A note on the Borel-Cantelli lemma. *Statist. Probab. Lett.*, **58**, 283–286.
Petrov, V.V. (2004). A generalization of the Borel–Cantelli lemma. *Statist. Probab. Lett.*, 67, 233–239.

Spitzer, F. (1964). *Principles of Random Walk*. Van Nostrand, Princeton, New Jersey.

Shiryaev, A. (1989). *Probability*. Moscow, Nauka (in Russian).

Stepanov A. (2014). On the Use of the Borel-Cantelli Lemma in Markov Chains, *Statist. Probab. Lett.*, 90, 149–154.