Comment on Two Dimensional $O(N)$ and $Sp(N)$ Yang Mills Theories as String Theories

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We write down all orders large $N$ expansions for the dimensions of irreducible representations of $O(N)$ and $Sp(N)$. We interpret all the terms in these expansions as symmetry factors for singular worldsheet configurations, involving collapsed crossovers and tubes. We use it to complete the interpretation of two dimensional Yang Mills Theories with these gauge groups, on arbitrary two dimensional manifolds, in terms of a String Theory of maps of the type considered by Gross and Taylor. We point out some intriguing similarities to the case of $U(N)$ and discuss their implications.
1. Introduction

Gross and Taylor [1], and Minahan [2] have established that the partition function of two-dimensional $U(N)$ or $SU(N)$ Yang Mills theory can be expanded in $1/N$ to give terms that all have a geometrical interpretation in terms of maps of a string worldsheet to the target space. The partition function for an orientable closed manifold of genus $G$ is

$$Z_M = \int [DA] e^{i \frac{1}{4 g^2} \int_M d^2 x \sqrt{g} Tr F_{\mu \nu} F^{\mu \nu}} = \sum_R (\dim R)^{2-2G} e^{-\frac{\lambda}{2N} C_2(R)},$$

(1.1)

where $\lambda = g^2 N$.

Our purpose is to investigate if a similar picture holds for $O(N)$ and $Sp(N)$ Yang Mills with the hope of learning more about the class of string theories related to two dimensional Yang Mills theories. The investigation of the stringy interpretation of $O(N)$ and $Sp(N)$ Yang Mills theories has been carried out in [3], for closed Riemann surfaces (orientable or nonorientable), for the terms coming from the casimir and the leading two terms of the $1/N$ expansion of $\dim R$, which suffices for a complete description of the theory on closed manifolds of Euler character zero. In this paper we write the full $1/N$ expansion for the dimension and identify the $\Omega$ point for $O(N)$ and $Sp(N)$, interpreting all the coefficients that appear in the $\Omega$ point in terms of a localised set of singularities. We also identify the appropriate observables which allow a stringy interpretation of the theory on manifolds with boundary. We will point out some very intriguing similarities between the $\Omega$ points of $O(N)$, $Sp(N)$ and the omega point for ‘composite’ representations of $U(N)$. We will refer frequently to the ‘chiral’ omega point ($\Omega_n$) and the ‘coupled’ omega point ($\Omega_{n,\tilde{n}}$) of [1]. They are defined in terms of the dimensions of $U(N)$ reps as

$$\dim R = \frac{N^n}{n!} \chi_R(\Omega_n),$$

(1.2)

where $R$ is the representation associated to a Young tableau with $n$ boxes, and

$$\dim(R\bar{S}) = \frac{N^{n+\tilde{n}}}{n!\tilde{n}!} \chi_{R\bar{S}}(\Omega_{n\tilde{n}})$$

(1.3)

where $R$ has $n$ boxes and $S$ has $\tilde{n}$ boxes. $(R\bar{S})$ is the irreducible representation of largest dimension in the tensor product of $R$ with the complex conjugate of $S$.

We will discuss in detail in the sections 2-4 the case of $O(N)$ and in section 5 we describe the small modification for $Sp(N)$. In section 2 we quote a character formula for
irreducible tensor representations of $O(N)$ and convert it to a form appropriate for our use. We specialise it to give the expansion of the dimension. We will not consider the spinor representations in detail in this section. They have casimirs of order $N^2$ and do not contribute to the large $N$ large $A$ asymptotics but may have nonperturbative effects. In section 3, we discuss the geometrical interpretation of the expansion. Some of the terms have a familiar interpretation similar to the case of the chiral expansion for $U(N)$. In addition there is a set of permutations which we identify as describing multiply branched sheets equipped with a collapsed crosscap or with an infinitesimal tube connecting two cycles. To prove that the coefficients in the expansion are really the symmetry factors for the singular maps described above, we derive, in section 4, a simple formula for a certain sum of characters of $S_n$, called $X_{\sigma_1}$, which appears in the character formula for $O(N)$. We comment on the formula for the dimensions of the spinor representations. In section 5, we write the large $N$ expansion for $Sp(N)$ and show that the geometrical interpretation carries over with one modification, the ‘collapsed crosscaps on branch points’ do not come with a minus sign as in the case of $O(N)$.

2. Formula for characters and dimensions of $O(N)$ representations

The following formula gives the large $N$ expansion for the dimension of a tensor representation of $O(N)$ associated with a Young tableau with $n$ boxes.

$$\dim R = \sum_{\sigma \in S_n} \frac{\chi_R(\sigma)}{n!} \sum_{n_1 \geq 0} \sum_{\substack{T_{\sigma_1}, T_{\sigma_2} = T_{\sigma} \in S_{n_1} \times S_{n_2} \mid n_1 + n_2 = n \mid T_{\sigma_1} \mid T_{\sigma_2}}} N^{K_{\sigma_2}} X_{\sigma_1} \frac{n!}{n_1! n_2!} \frac{|T_{\sigma_1}| |T_{\sigma_2}|}{|T_{\sigma}|}$$

$$= \frac{N^n}{n!} \chi_R(\Omega_n). \quad (2.1)$$

For each $n_1$ we are summing over conjugacy classes $T_{\sigma_1}$ of $S_{n_1}$, with $|T_{\sigma_1}|$ being the order of the conjugacy class and $\sigma_1$ a representative of the conjugacy class. The condition $T_{\sigma_1} T_{\sigma_2} = T_{\sigma}$ means that the cycles of $\sigma$ can be separated into two sets one of which characterises a conjugacy class in $S_{n_1}$ and the other a conjugacy class in $S_{n_2}$ ($n_2 = n - n_1$). $X_{\sigma_1}$ is defined by

$$X_{\sigma_1} = (-1)^{n_1/2} \sum_{R \in Y_{n_1}} \frac{\chi_R(\sigma_1)}{n_1!}, \quad (2.2)$$
where $Y_{n}^{*}$ is a subset of the Young tableaux with $n$ (even) boxes, which is described in the appendix. The expression for $\Omega_n$ is

$$
\Omega_n = \sum_{\sigma \in S_n} \sum_{n_1 \geq 0} \sum_{T_{\sigma_1}, T_{\sigma_2} = T_{\sigma}} N^{-n_1} X_{\sigma_1} N^{K_{\sigma_2} - n_2} \frac{n!}{n_2!} \left| \frac{T_{\sigma_1}}{T_{\sigma}} \right| \left( -1 \right)^{n_2} n_1 / 2 
= (1 + O(1/N)).
$$

The leading term in (2.1) is obtained, when $n_1$ is zero, and $\sigma_2$ is the identity element and the next term when it is an element belonging to the conjugacy class characterised by 1 cycle of length 2 and all other cycles of length 1 (the class $T_2$ in the notation of [1]). These are easily seen to agree with the terms as computed in [5] from a hook length type formula.

We now prove equation (2.1). Littlewood [6] (page 240) gives the following formula for the characters of $O(N)$

$$
\chi_R(U) = \{ R \} + \sum_{n_1 \geq 2} \sum_{R_1 \in Y_{n_1}^*} \sum_{R_2 \in Y_{n_2}} (-1)^{n_1/2} g(R_1, R_2; R) \{ R_2 \}. 
$$

(2.4)

$R$ is a Young tableau having $n$ boxes. $R_1$ runs over a certain subset of the Young tableaux with $n_1$ (even) boxes we will call $Y_{n_1}^*$ and describe in detail in Appendix A. $g(R_1, R_2; R)$ are the Littlewood-Richardson coefficients. $R_2$ runs over all tableaux with $n_2$ boxes. $\{ R \}$ and $\{ R_2 \}$ are symmetric functions in the $N$ eigenvalues of the $O(N)$ matrix $U$. The symmetric function $\{ R \}$ can be written in terms of characters of $S_n$:

$$
\{ R \} = \sum_{\sigma \in S_n} \frac{\chi_R(\sigma)}{n!} P_{\sigma}. 
$$

(2.5)

$P_{\sigma}$ is a power sum symmetric polynomial,

$$
P_{\sigma}(x_1, x_2, \cdots) = \prod_i (x_{1_i}^{r_i} + x_{2_i}^{r_i} + \cdots)^{n_i},
$$

(2.6)

where $i$ runs over the cycle lengths $(r_i)$, which have multiplicity $n_i$, in the cycle decomposition of $\sigma$. If there are $N$ arguments $x_1, x_2, \cdots, x_N$ which are the $N$ eigenvalues of the matrix $U$, this is the $Y_{\sigma}(U)$ of Gross-Taylor [1].

We will rewrite the character formula in a form appropriate for a geometric interpretation a la Gross-Taylor. Let us find $X_{\sigma_1}$ such that

$$
\sum_{\sigma \in S_n} X_{\sigma} \chi_R(\sigma) = \sum_{R' \in Y_{n_1}^*} \delta(R, R') (-1)^{n_2/2},
$$

(2.7)
the delta function is 1 if \( R \) and \( R' \) are the same and zero otherwise. From the orthogonality relation,

\[
\sum_{R \in Y_n} \chi_R(\sigma) \chi_R(\tau) = \delta_{T_\sigma, T_\tau} \frac{n!}{T_\sigma},
\]

we get

\[
X_{\sigma_1} = (-1)^{n/2} \sum_{R \in Y_n} \chi_R(\sigma_1) \frac{n!}{n!}.
\]

Now if \( \sigma \) is a permutation of \( n \) elements and it can be written as \( \sigma_1 \sigma_2 \) in cycle notation, where \( \sigma_1 \in S_{n_1}, \sigma_2 \in S_{n_2}, n_2 = (n - n_1) \) i.e \( \sigma_1 \sigma_2 \) also lives in the subgroup \( S_{n_1} \times S_{n_2} \) of \( S_n \), then we can write

\[
\chi_R(\sigma) = \sum_{R_1 \in Y_{n_1}, R_2 \in Y_{n_2}} g(R_1, R_2; R) \chi_{R_1}(\sigma_1) \chi_{R_2}(\sigma_2)
\]

(2.10)

The existence of such an expansion follows from the fact that the left hand side is a class function for the subgroups \( S_{n_1} \) and \( S_{n_2} \). The fact that the coefficients are precisely the Littlewood-Richardson coefficients follows from applications of the Frobenius reciprocity theorem [7].

Now we can use equations (2.4), (2.7) and (2.10), to write the \( O(N) \) character formula as follows:

\[
\chi_R(U) = \{R\} + \sum_{n_1 \geq 2} \sum_{n_2 \geq 0} \sum_{\sigma_1 \in S_{n_1}, \sigma_2 \in S_{n_2}} X_{\sigma_1} \frac{Y_{\sigma_2}(U)}{(n_2)!} \chi_R(\sigma_1 \sigma_2).
\]

(2.11)

A more compact way of writing the formula absorbs the first term by allowing \( n_1 \) to be 0 as well, defining \( X_{\sigma_1} = 1 \) for \( n_1 = 0 \), to give

\[
\chi_R(U) = \sum_{n_1 \geq 0} \sum_{n_2 \geq 0} \sum_{\sigma_1 \in S_{n_1}, \sigma_2 \in S_{n_2}} X_{\sigma_1} \frac{Y_{\sigma_2}(U)}{(n_2)!} \chi_R(\sigma_1 \sigma_2)
\]

\[
= \sum_{\sigma \in S_n} \chi_R(\sigma) \sum_{n_1 \geq 0} \sum_{T_{\sigma_1} T_{\sigma_2} = T_{\sigma}} X_{\sigma_1} \frac{Y_{\sigma_2}(U)}{(n_2)!} \frac{|T_{\sigma_1}| |T_{\sigma_2}|}{|T_{\sigma}|}
\]

\[
= \sum_{\sigma \in S_n} \chi_R(\sigma) \tilde{Y}_\sigma(U).
\]

(2.12)

In the second line the sum is over all ways of separating the cycles of each \( \sigma \in S_n \) into 2 sets of cycles \( \tilde{Y}_\sigma(U) \). The first set of cycles corresponds to a conjugacy class in \( S_{n_1}, \) a representative

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1 I am grateful to S. G. Naculich, H. A. Riggs, and H. J. Schnitzer for a correspondence pointing out a subtlety in going from the first to the second line of (2.12), which was missed in the first version of this paper.
of which is $\sigma_1$ and the second set to a conjugacy class in $S_{n-n_1}$, a representative of which is $\sigma_2$. The last line is a definition of $\bar{Y}_\sigma(U)$. By putting $U = 1$ we get the large $N$ expansion for the dimension of the $O(N)$ representation (2.1).

3. Geometrical interpretation

Having a formula for the $\Omega$ point in the form above, the same steps of Gross-Taylor can be used to write the 2D Yang-Mills partition function on an arbitrary manifold $\Sigma$ as a sum over homomorphisms from $\Pi_1(\Sigma \setminus \{\text{punctures}\})$ into the symmetric group of permutations on the sheets of the cover, where there are punctures for the branch points and the $|2 - 2G|$ omega points. The same argument as in [1] can be used to show that inverse powers of the dimension are written in terms of the inverse powers of $\Omega$. The argument relies on the fact that the element $\Omega_n$ of the group algebra of $S_n$ commutes with $S_n$, which is also true here because $\Omega_n$ is a sum of elements of $S_n$ with coefficients that only depend on the cycle structure (conjugacy class) of the element, as is clear from inspection of (2.1). Negative powers of the omega point can be dealt with in a $1/N$ expansion as in [1]. For arbitrary closed non-orientable surfaces, the reality of the reps of $S_n$ [5] guarantees that the full partition function can be expressed as a sum of homomorphisms to $S_n$.

We recall from [1] how the $\Omega$ point can be inserted in the partition function of the theory. Take the case of sphere,

$$Z = \sum_R (\dim R)^2 e^{-\frac{2\lambda}{N} C_2(R)}$$

$$= \sum_{n=0}^{\infty} \sum_{R \in Y_n} \frac{N^{2n}}{(n!)^2} \chi(\Omega_n)^2 e^{-\frac{2\lambda}{N}(nN + \frac{n(n-1)\chi_R(T_2)}{2N} - n)}$$

$$= \sum_{n=0}^{\infty} \sum_{R \in Y_n} \sum_{i \geq 0} \sum_{p_1 \ldots p_i \in T_2} \frac{N^{2n-i}}{(n!)^2} e^{-\frac{n\lambda A}{2N}} e^{\frac{n\lambda A}{2N}} d_R X_R(p_1 \ldots p_i \Omega_n^2)(-1)^i \frac{(\lambda A)^i}{i!}$$

$$= \sum_{n=0}^{\infty} \sum_{i \geq 0} \sum_{p_1 \ldots p_i \in T_2} N^{2n-i} e^{-\frac{n\lambda A}{2N}} e^{\frac{n\lambda A}{2N}} (-1)^i \frac{(\lambda A)^i}{i!} [1_{n!} \delta(p_1 \ldots p_i \Omega_n^2)]. \quad (3.1)$$

In combining the product of the characters into a single character, we use the fact that $\Omega_n$ commutes with $S_n$. The extra factor $e^{n\lambda A/2N}$ compared to $U(N)$ is accounted by infinitesimal crosscaps being mapped to a point [4]. We notice also that for either $O(N)$ or $SO(N)$ (whose lie algebras are isomorphic) the casimir does not give rise to collapsed handles and tubes as for $SU(N)$. Finally as observed in [3] the above expansion, which
is the analog of chiral expansion in the case of unitary groups, suffices to give non trivial answers on both orientable and non orientable surfaces, unlike the $SU(N)$ case. This is related to the reality of the reps of $O(N)$.

For a manifold with $G$ handles and $b$ boundaries the appropriate observables for a stringy interpretation are

$$Z(G, \lambda A, N; \{\sigma_i\}) = \prod_i \tilde{Y}_{\sigma_i}(U_i) >$$

$$= \int dU_1 dU_2 \ldots dU_b Z(G, \lambda A; U_1, \ldots , U_b) \prod_i \tilde{Y}_{\sigma_i}(U_i). \quad (3.2)$$

The orthogonality property of the $\tilde{Y}_\sigma$ will follow from that of $O(N)$ and $S_n$ characters, as shown for the analogous functions in the unitary case [1]. This orthogonality will guarantee [1] that the partition function defined above will count maps to the target space with permutations in the conjugacy class of $\sigma_i$ on the sheets covering the boundaries.

We now have to find a geometrical interpretation for all the coefficients appearing in $\Omega_n$ of equation (2.3). For terms for which $n_1 = 0$, $n_2 = n$, the coefficient of $\sigma_2 = \sigma \in S_n$ is $N^{K_\sigma - n}$. This has a simple interpretation in terms of multiple branch points, and the power of $N$ correctly gives the Euler character of a surface branched in the way prescribed by $\sigma$. The product of $\frac{T_{\sigma}}{(n)!}$ (which is equal to $\frac{1}{n!}$ times the number of distinct homomorphisms describing the same configuration of unlabelled sheets) with the coefficient of $N^{K_\sigma - n}\sigma$ in the omega point gives $\frac{1}{|S_{\nu}|}$ where $|S_{\nu}|$ is the symmetry factor for the branched covering described by $\sigma$. This kind of singularity and symmetry factor are familiar from the chiral omega point of $U(N)$ or $SU(N)$ [1]. When $n_1$ is not zero, we have a sum over decompositions of the cycles of $\sigma$ into 2 sets, such that the first set can be associated with a permutation $\sigma_1$ in the class $T_{\sigma_1}$ of $S_{n_1}$ and the second with a permutation $\sigma_2$ in the class $T_{\sigma_2}$ of $S_{n_2}$. Now $\frac{1}{(n)!}T_{\sigma}$ (which is again $\frac{1}{n!}$ times the number of distinct homomorphisms into $S_n$ describing the same configuration of unlabelled sheets) multiplied by the coefficient of a given decomposition of the cycles of $\sigma$ into two sets, separates into $|T_{\sigma_1}|X_{\sigma_1}N^{-n_1}$ times $\frac{T_{\sigma_2}N^{K_{\sigma_2} - n_2}}{n_2!}$. The latter term again correctly describes the genus and symmetry factors for multiple branch points described by $\sigma_2$. The coefficient of $\sigma_1$ has a power of $N$ which is smaller than the one expected from the branch points responsible for the permutation. A similar situation is met in the $U(N)$ case for the coupled $\Omega$ point where there are collapsed tubes connecting cycles of the same length. So this is a useful ingredient, and we call the $n_1$ sheets the singular sector which we will now study.
We will need one more singular object. Let us look at a configuration that can arise from the $O(N)$ omega point, for concreteness, say on a target space which has the topology of a disc, and consider terms where there is no power of the area so that there are no branch points coming from the casimir. Let us further specialise to a term where $n = n_2$ and $\sigma_2$ is made of one cycle of length $n_2$. This term contributes to $\dim R$ a factor $N^{K_{\sigma_2}} = N^1 = N^{2g-b}$, where $g$ and $b$ are the number of handles and boundaries, respectively, of the worldsheet. If we specialised instead to a term where $n = n_1$ and $\sigma_1$ is made of one cycle only, then the associated power of $N$ is zero. This can be understood if the branch point comes with a tiny crosscap as well. We also observe that $n_1$ is even, so odd total number of sheets do not appear in the singular sector (which we recall are the sheets on which $\sigma_1$ are acting). This suggests that these crosscaps can only live on even cycles. The power of $N$ associated with $\sigma_1$ is always correctly accounted for, if all the cycles either have a crosscap or share a tube with another cycle. We will consider the implications of this simplest set of singularities needed to account for the powers of $N$, for example we do not invoke pants connecting more than two cycles or such higher contact terms. And we will show that the few singularities can correctly account for the $X_{\sigma_1}$. Now what kind of tubes are allowed? Do they only connect cycles of the same length as in the coupled $U(N)$ case? We will look at some simple examples of the coefficient $X_{\sigma_1}$ to understand the symmetry factors associated with each singularity and then use the geometrical picture that emerges to guess a general formula for $X_{\sigma_1}$. We will prove this formula in the next section.

We now show that for permutations $\sigma_1$, made of one or two cycles, the quantity $|T_{\sigma_1}|X_{\sigma_1} = TX_{\sigma_1}$ ($|T_{\sigma_1}|$ is the number of elements in the conjugacy class of $\sigma_1$) is the symmetry factor for some singular worldsheet configurations involving multiple branch points, crosscaps, and tiny tubes connecting two cycles. We can use the sum of characters given in (2.9) when $\sigma_1$ is a permutation made of one cycle only, or two cycles, appendix A describes some details. For one cycle of even length $2m$, $T X_{\sigma_1} = \frac{1}{2m}$. So each crosscap in an $\Omega$ point comes with a minus sign. Note that the ‘crosscap with branch point’ in the $\Omega$ point comes with the same symmetry factor as a pure branch point in the $\Omega$ point, but has an extra minus sign. For two odd cycles of different lengths the $TX$ vanishes, which is consistent with the idea that odd cycles with crosscaps do not occur and that tubes only connect cycles of equal length. For two even cycles of unequal length it is just the product of the symmetry factors for each, suggesting again that cycles of different lengths do not interact. For two cycles of equal odd length $r$, $TX = \frac{1}{2r}$. For two cycles of
equal even length \( r \), \( TX = \frac{1}{2r} + \frac{1}{2r^2} \). The first term in each case comes from a collapsed
tube connecting the 2 cycles. Except for the factor of 2, the symmetry factor and the
minus sign are exactly the same as for a tube connecting two sheets of opposite orientation
in the coupled \( \Omega \) point of \( U(N) \). The extra symmetry is expected because the sheets
being connected are not equipped with different orientations as in the case of \( U(N) \). The
additional term in the case of two even cycles comes from configurations where the two
even cycles are each carrying a crosscap. Note that we do not have separate factors for
infinitesimal Klein bottles and tori because in the singular limit they are the same.

We now summarise the complete set of singularities in the \( O(N) \) omega point. In
the singular sector (described by \( \sigma_1 \)) then, an even cycle has to carry a crosscap or be
connected to another cycle of the same length by a collapsed tube. An odd cycle must be
connected to another cycle of the same length. Each crosscap or collapsed tube comes with
a minus sign. These singular configurations come with the natural symmetry factors. The
sheets not in the singular sector (described by \( \sigma_2 \) ) come with multiple branch points. The
only new singularity compared to the coupled \( U(N) \) case is the ‘crosscap on even cycles’.
These singular configurations are shown in figure 1.

This predicts that the \( TX_\sigma \) should factorise into separate factors for each cycle length.
And the contribution is zero from an odd number of odd cycles. For an even number \( 2m \)
of odd cycles of length \( r \), it should be

\[
TX_{[r^{2m}]} = (-1)^m \frac{1}{m!2^m r^m}.
\]

(3.3)

The minus signs come from the \( m \) tubes, \( 1/(2r) \) is the symmetry factor of a pair of cycles
connected by a tiny tube, and \( m! \) comes from the symmetry operation of permuting the \( m \)
identical pairs of cycles. For any number \( n \) of even cycles (order \( r \)), we sum over \( i \) collapsed
handles (going from zero to the integer part of \( n/2 \)), with \( (n - 2i) \) crosscaps, to get

\[
TX_{[r^n]} = \sum_{i=0}^{\text{int}(n/2)} \frac{(-1)^{n-i}}{i!(n-2i)!2^i r^{n-i}}
\]

\[
= \sum_{i=0}^{\text{int}(n/2)} \frac{(-1)^{n-2i}}{(n-2i)!r^{n-2i}} \times \frac{(-1)^i}{i! r^i 2^i},
\]

(3.4)

where, in the last line, we have separated out the factors for crosscaps and tubes.
4. Derivation of Formulae for the Sum of Characters $X_{\sigma}$

Directly summing the characters to get (3.3) and (3.4) in the case of arbitrary permutations $\sigma_1$ appears hard. The following identity between symmetric functions [6] will be very useful:

\[
\prod_{i<j}(1-\alpha_i\alpha_j)\prod_i(1-\alpha_i^2) = 1 + \sum_{R\in Y^*}(-1)^{n/2}\sum_{\sigma\in S_n}\chi_R(\sigma)\frac{P_\sigma}{n!}P_\sigma(\alpha_1, \alpha_2 \cdots)
\]

\[
= 1 + \sum_{n \geq 2} \sum_{\sigma \in S_n} X_{\sigma} P_\sigma.
\]

We first determine $X_{\sigma}$ for $\sigma_1$ made of $n_1$ cycles all of length $r_1$. To start we consider the case $r_1 = 1$. We only need to keep $n_1$ distinct $\alpha$'s in the identity (1.1), and we need to find the coefficient of the power sum symmetric polynomial $(\alpha_1 + \alpha_2 + \cdots + \alpha_{n_1})^{n_1}$. This polynomial is the only one containing the monomial $\alpha_1 \alpha_2 \cdots \alpha_{n_1}$. On LHS the monomial can be built by taking pairs of $\alpha$'s from the different factors. The number of ways of doing that is equal to the number of ways of pairing up all the elements from $n_1$ objects which is zero if $n_1$ is odd, and if $n_1 = 2m_1$ we get the identity.

\[
(-1)^{m_1} \frac{(2m_1)!}{2^{m_1}(m_1)!} = (2m_1)!TX_{[(1^{2m_1})]}
\]

\[
\Rightarrow TX_{[(1^{2m_1})]} = \frac{(-1)^{m_1}}{2^{m_1}m_1!}.
\]

This agrees with (3.3). Figure 2 illustrates the case $m_1 = 5$ and compares with a similar configuration in the $U(N)$ omega point. For general $r_1$ we choose the $r_1n_1$ indeterminates in the following way

\[
\alpha_{i+kn_1} = \alpha_i e^{2\pi ik/r_1}
\]

where $i$ runs from 1 to $n_1$; and $k$ runs from 0 to $r_1 - 1$. This guarantees that $(\alpha_1^K + \alpha_2^K + \cdots + \alpha_{r_1n_1}^K)$ vanishes unless $K$ is a multiple of $r_1$. This means that the only power sum symmetric polynomial which contains the monomial $\alpha_1^{r_1} \alpha_2^{r_1} \cdots \alpha_{n_1}^{r_1}$ is $(r_1\alpha_1^{r_1} + r_1\alpha_2^{r_1} + \cdots + r_1\alpha_{n_1}^{r_1})^{n_1}$ whose coefficient is $TX_{[(r_1)^{n_1}]}$. With our choice of variables, the LHS can be written, for odd $r_1$, as

\[
\prod_{1 \leq i < j \leq n_1} (1-\alpha_i^{r_1}\alpha_j^{r_1})^{r_1} \prod_{1 \leq i \leq n_1} (1-\alpha_i^{2r_1})^{(r_1+1)/2},
\]

\[
\Rightarrow TX_{[(r_1)^{n_1}]} = \frac{(-1)^{m_1}}{2^{m_1}m_1!}.
\]
and, for even $r_1$, as :

$$\prod_{1 \leq i < j \leq n_1} (1 - \alpha_i^{r_1} \alpha_j^{r_1}^{r_1}) \prod_{1 \leq i \leq n_1} (1 - \alpha_i^{2r_1})^{r_1/2}(1 - \alpha_i^{r_1}). \quad (4.5)$$

Comparing coefficients of the monomial gives the answer in (3.3) and (3.4).

Now we consider the proof of factorisation. Consider for a start the case of $n_1$ cycles of length $r_1$ and $n_2$ cycles of length $r_2$. Choose the variables

$$\alpha_1 \cdots \alpha_{r_1n_1+r_2n_2}$$

as follows. Let

$$\alpha_i + k_1n_1 = \tilde{\alpha}_i e^{\frac{2\pi ik_1}{r_1}}, \quad 1 \leq i \leq n_1, \quad 0 \leq k_1 \leq (r_1 - 1), \quad (4.6a)$$

$$\alpha_i + r_1n_1 + k_2n_2 = \tilde{\beta}_i e^{\frac{2\pi ik_2}{r_2}}, \quad 1 \leq i \leq n_2, \quad 0 \leq k_2 \leq (r_2 - 1) \quad (4.6b).$$

With this choice there is only one term on the RHS containing the monomial

$$\tilde{\alpha}_1^{r_1} \tilde{\alpha}_2^{r_1} \cdots \tilde{\alpha}_{n_1}^{r_1} \tilde{\beta}_1^{r_2} \tilde{\beta}_2^{r_2} \cdots \tilde{\beta}_{n_2}^{r_2}. \quad (4.7)$$

And it is the polynomial

$$(\alpha_1^{r_1} + \alpha_2^{r_1} + \cdots + \alpha_{r_1n_1+r_2n_2}^{r_1})^{n_1}(\alpha_1^{r_2} + \alpha_2^{r_2} + \cdots + \alpha_{r_1n_1+r_2n_2}^{r_2})^{n_2},$$

whose coefficient is $TX[(r_1^{n_1}, r_2^{n_2})]$. Now suppose without loss of generality that $r_2 > r_1$. Then powers of $\tilde{\beta}$ in the monomial we are looking for cannot come from the first $n_1$ factors, so must be chosen from the last $n_2$ factors only. So the powers of $\tilde{\alpha}$ have to come from the first $n_1$. The combinatoric factor on the RHS is thus the product of those that determined the $TX[(r_1^{n_1})]$ and $TX[(r_2^{n_2})]$. Now on the LHS of (4.1) we can separate out the product into terms containing $\tilde{\alpha}$ only and terms containing $\tilde{\beta}$ only, and mixed terms. The mixed terms can be computed to be

$$\prod_{1 \leq i \leq n_1, 1 \leq j \leq n_2} (1 - \tilde{\alpha}_i^{r_1} \tilde{\beta}_j^{r_2} \tilde{\alpha}_i^{r_2} \tilde{\beta}_j^{r_1})^r,$$
where \( r \) is the greatest common divisor of \( r_1 \) and \( r_2 \). This does not contribute to the monomial (4.7). This argument for factorisation clearly generalises to the the case of an arbitrary number of cycle lengths. We have proved then, that there are no infinitesimal tubes connecting cycles of different lengths.

For spinor representations [6], the dimension can be written as \( 2^N \) times an expression of the form (2.1) with \( X_{\sigma_1} \) replaced by \( \hat{X}_{\sigma_1} \) whose generating function is the expression [6],

\[
\prod_i (1 - \alpha_i) \prod_{i<j} (1 - \alpha_i \alpha_j).
\] (4.8)

The same steps as above shows that similar expressions for \( \hat{X}_{\sigma_1} \) can be written as a sum over configurations. The only difference compared to the case of tensor representations is that the crosscaps live on odd cycles only.

5. About \( Sp(N) \)

Many similarities between \( O(N) \) and \( Sp(N) \) have been observed in the context of matrix models [9] and loop equations [10]. One interesting relation between the dimensions of \( O(N) \) and \( Sp(N) \) tensor representations will be useful here in giving a quick answer for the \( Sp(N) \) case using the result for \( O(N) \). From [11], [12], [13] we have

\[
\dim_{[Sp(N)]} R = (-1)^n \dim_{O(-N)} \tilde{R},
\] (5.1)

where \( \tilde{R} \) is the conjugate representation, related to \( R \) by exchanging rows with columns, and \( \dim_{O(-N)} \) is meant in the sense of analytic continuation. Using this equation and (2.1) we can write the following equation for the dimension of a representation of \( Sp(N) \)

\[
\dim R = \sum_{\sigma \in S_n} \frac{\chi_R(\sigma)}{n!} \sum_{n_1 \geq 0} \sum_{T_{\sigma_1}=T_\sigma} (-N)^{K_{\sigma_2}} X_{\sigma_1} n_1! n_2! (-1)^n \frac{|T_{\sigma_1}| |T_{\sigma_2}|}{|T_\sigma|}.
\] (5.2)

We used the fact [3] that \( \chi_R(\sigma) = \chi_R(\sigma)(-1)^p \) where \( p \) is 0 if the permutation is even and 1 if it is odd, and we wrote \((-1)^p = \chi_R(\sigma)(-1)^{n-K_\sigma} = (-1)^{n-K_\sigma_1-K_{\sigma_2}} \). This means that the geometrical interpretation of the \( \Omega \) point of \( Sp(N) \) is identical except for a minus sign \((-1)^{K_{\sigma_1}} \). From the geometrical interpretation of \( X_{\sigma_1} \), this is equivalent to saying that the
collapsed crosscaps associated with branch points do not come with a minus sign. The
natural generalisation of (5.2) to arbitrary characters is,

$$\chi_R(U) = \sum_{\sigma \in S_n} \frac{\chi_R(\sigma)}{n!} \sum_{n_1 \geq 0} \sum_{T_{\sigma_1} T_{\sigma_2} = T_{\sigma}} (-1)^{K_{\sigma_1}} X_{\sigma_1} Y_{\sigma_2}(U) \frac{n_1! |T_{\sigma_1}| |T_{\sigma_2}|}{n_2! |T_{\sigma}|},$$

(5.3)

where $U$ is now in $Sp(N)$. We need this equation for a geometrical interpretation of the
theory on a manifold with boundaries and it can be proved by using the same steps as in
the $O(N)$ case, as we now outline. It is pointed out in [13] that the analog for $Sp(N)$ of
(2.4) differs only in that the set $Y^*$ is replaced by its conjugate, which is the set that, in
Frobenius notation (see appendix A), consists of Young tableaux of the form

$$\left( \begin{array}{cccc}
  a & b & c & \ldots \\
  a + 1 & b + 1 & c + 1 & \ldots
\end{array} \right).$$

(5.4)

We can repeat the steps in section 2 to write a formula for $X_{\sigma_1}(Sp(N)) \equiv \tilde{X}_{\sigma_1}$ for which
the generating function replacing the one on the LHS of (1.1) is $\prod_{i<j} (1 - \alpha_i \alpha_j)$. By the
same choice of variables as in section 4 we can derive the analog of equation (4.4) which is

$$\prod_{1 \leq i \leq j \leq n_1} (1 - \alpha_i^{r_1} \alpha_j^{r_1})^{r_1} \prod_{1 \leq i \leq n_1} (1 - \alpha_i^{2r_1})^{(r_1-1)/2},$$

(5.5)

and the analog of (1.5) which is

$$\prod_{1 \leq i < j \leq n_1} (1 - \alpha_i^{r_1} \alpha_j^{r_1})^{r_1} \prod_{1 \leq i \leq n_1} (1 - \alpha_i^{2r_1})^{r_1 - 1} (1 + \alpha_i^{r_1}).$$

(5.6)

We observe that the terms which contribute to the monomial $\alpha_1^{r_1} \alpha_2^{r_1} \cdots \alpha_{n_1}^{r_1}$ are unchanged
except that the terms which correspond to crosscaps sitting on branch points have a plus
sign instead of a minus sign. The proof of factorisation is identical. Note that crosscaps
coming from the casimir are also of opposite sign for $O(N)$ and $Sp(N)$ [5].

6. Conclusions and Speculations

In performing the large $N$ expansion of the partition function for the case of $U(N)$
gauge group, the naive expansion analogous to (3.1) did not correctly reproduce the large
$N$ asymptotics. One of its most obvious weaknesses [1] was that it gave a trivial answer for
nonorientable target spaces whereas there is no reason to expect the large $N$ asymptotics
of the formulae derived in [14] to be trivial. The coupled Ω point is necessary in order to get the correct large \( N \), large \( A \), approximation. For \( O(N) \) and \( Sp(N) \) the naive expansion does not give trivial answers on nonorientable target spaces. Moreover we have found close similarities between the structure of the \( O(N) \) and \( Sp(N) \) Ω points with the coupled \( U(N) \) Ω point which suggests that, the ‘naive’ expansion might give the correct large \( N \), large \( A \) asymptotics for the \( O(N) \) and \( Sp(N) \) gauge groups. The methods of [15], as applied to \( O(N) \) or \( Sp(N) \) could perhaps be used to see if the large \( N \) large \( A \) asymptotics for these gauge groups is indeed correctly given by the expansion in (3.1). For small \( A \) the spinor representations with the factor of \( 2^N \) in their dimension formula will probably have to be taken into account.

Our results on the Ω point together with those of [3], suggest that the string action of QCD2 must be of a form that is generalisable to describe \( O(N) \) and \( Sp(N) \) Yang Mills. Whereas many questions about 2D Yang Mills are most easily answered by starting with the original Yang Mills action, it seems that a satisfactory answer to the question of why there are no higher order contact terms than crosscaps and tubes connecting cycles of equal length can most naturally come from a string picture for all these theories.

One interesting fact about the infinitesimal tubes is that they are not of two distinct kinds, Klein bottle and torus type (although the theory does contain worldsheets with both topologies [3]). This is naturally understood if the Omega point is associated with singular objects on the worldsheet rather than just being a singularity of the map from worldsheet to target space. A degenerated torus is indistinguishable from a degenerated Klein bottle, whereas singular maps from a Klein bottle to a point and a torus to a point are distinct. This suggests that the ingredients that go into the omega point, infinitesimal crosscaps and infinitesimal tubes, are closely connected to the local worldsheet physics, and perhaps the local operator content of the worldsheet theory. Further tests of this picture should be made, perhaps in the framework set up by Kostov [16].

Identifying the appropriate observables for manifolds with boundary may be of help in developing a Das-Jevicki like picture for the \( O(N) \) and \( Sp(N) \) case, as done for \( U(N) \) in [7] [8]. It should also allow a computation of Wilson loops by the gluing method of [1], adapted to \( O(N) \) and \( Sp(N) \).

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Appendix A. Determination of $X_{[\sigma]}$ by Summing Characters

At this point we need to describe in more detail the set $Y^*$, which is done conveniently using the Frobenius notation. The Frobenius notation for Young tableaux describes them by an array of pairs of numbers. The number of pairs is equal to the number of boxes on the leading diagonal of the tableaux. The upper number of the pair is the number of boxes to the right of that box and the lower number is the number of boxes below that box. A tableau with row lengths $[6, 4, 3]$, for example, is described by

\[
\begin{pmatrix}
5 & 2 & 0 \\
2 & 1 & 0
\end{pmatrix}.
\]

The set $Y^*$ consists of tableaux which are of the form

\[
\begin{pmatrix}
a + 1 & b + 1 & c + 1 & \ldots \\
a & b & c & \ldots
\end{pmatrix},
\]

Clearly they can only have an even number of boxes. Using a formula for characters of a single cycle $[7]$, only one tableau in $Y^*$, with a single element along the diagonal, will contribute to $X_\sigma$ when $\sigma = (2m)$. We find that $X[(2m)] = \frac{-1}{(2m)!}$. The sum over all elements in this conjugacy class gives $\frac{-1}{(2m)!}$.

For conjugacy classes with two cycles $(n_1, n_2)$, $n_1 \neq n_2$ the sum over characters can still be done fairly easily, using for example the Murnaghan-Nakayama recursion formula for characters of $S_n$ $[7]$. Only tableaux with at most two boxes on the leading diagonal contribute. And we find

\[
TX[(n_1, n_2)] = \frac{1}{n_1 n_2}
\]

if they are both even, and zero if they are both odd.
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Figure 1a.
Ordinary branch point. Cyclic symmetry factor of 5 and weight \( \frac{1}{5} \).

Figure 1b.
2 cycles of equal length connected by a collapsed tube which carries a minus sign. Cyclic symmetry factor of 5, and symmetry of exchanging the 2 cycles gives weight of \( \frac{-1}{10} \).

Figure 1c.
An even branch point with a crosscap. Cyclic symmetry factor of 4 and weight \( \frac{-1}{4} \) in O(N) omega point.
Figure 2a
A configuration of 10 single sheets, with collapsed tubes (drawn as dotted lines) joining cycles of equal length (here 1), in the O(N) omega point. Symmetry factor is $2^5 \cdot 5!$ and each tube comes with a minus sign, giving a weight of $(-1)^5/(5! \cdot 2^5)$.

Figure 2b.
A configuration of 10 single sheets, with collapsed tubes connecting opposite orientations, in the coupled U(N) omega point. Symmetry factor is 5! and each tube comes with a minus sign, so the weight is $(-1)^5/(5!)$. No factor of $2^5$ because the two cycles being connected by a tube have opposite orientation.