COMBINATORIAL CONSTRUCTIONS OF
THREE-DIMENSIONAL SMALL COVERS

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Abstract. In this paper we study two operations on 3-dimensional small
covers called a connected sum and a surgery. These operations correspond
to combinatorial operations on \((\mathbb{Z}_2)^3\)-colored simple convex polytopes. Then we
show that each 3-dimensional small cover can be constructed from \(T^3\), \(\mathbb{R}P^3\)
and \(S^1 \times \mathbb{R}P^2\) with two different \((\mathbb{Z}_2)^3\)-actions by using these operations. This
result is a generalization or an improvement of results in [3], [5], [8] and [12].

1. Introduction

A small cover was introduced by Davis and Januszkiewicz [1] as an
\(n\)-dimensional closed manifold \(M^n\) with a locally standard \((\mathbb{Z}_2)^n\)-action such that its orbit space
is a simple convex polytope \(P\) where \(\mathbb{Z}_2\) is the quotient additive group \(\mathbb{Z}/2\mathbb{Z}\). They
showed that there exists a one-to-one correspondence between small covers and
\((\mathbb{Z}_2)^n\)-colored polytopes (cf. [1, Proposition 1.8]). Here a pair \((P, \lambda)\)
is called a \((\mathbb{Z}_2)^n\)-colored polytope when \(P\) is an \(n\)-dimensional simple convex polytope with
the set of facets \(\mathcal{F}\) and a function \(\lambda: \mathcal{F} \to (\mathbb{Z}_2)^n\) satisfying the following condition:

\((\\ast)\) if \(F_1 \cap \cdots \cap F_n \neq \emptyset\) then \(\{\lambda(F_1), \cdots, \lambda(F_n)\}\) is linearly independent.

We say that two \((\mathbb{Z}_2)^n\)-colored polytopes \((P_i, \lambda_i) (i = 1, 2)\) are equivalent when
there exists a combinatorial equivalence of polytopes \(\phi: P_1 \to P_2\) such that \(\lambda_2 \phi = \theta \lambda_1\)
for some \(\theta \in \text{Aut}(\mathbb{Z}_2)^n\). The \(n\)-dimensional torus \(T^n\) and the real projective
space \(\mathbb{R}P^n\) with standard \((\mathbb{Z}_2)^n\)-actions are examples of small covers over the \(n\)-cube
\(I^n\) and the \(n\)-simplex \(\Delta^n\) respectively.

In this paper we are interested in constructions of 3-dimensional small covers \(M^3\)
from basic small covers by using some operations. In [3] Izmestiev studied a class
of 3-dimensional small covers which are called linear models and are correspondent
to 3-colored polytopes. He introduced two operations on linear models called a
connected sum \(\natural\) and a surgery \(\circ\) and proved the following theorem (cf. [3, Theorem
3]).

Theorem 1.1 (Izmestiev). Each linear model \(M^3\) can be constructed from \(T^3\) by
using three operations \(\natural, \circ\) and \(\circ^{-1}\) where \(\circ^{-1}\) is the inverse of \(\circ\).

In [12] we generalized Theorem 1.1 to orientable small covers \(M^3\) which are correspond-
to 4-colored polytopes. We introduced a new operation called the Dehn
surgery \(\circ^D\), and showed that each orientable small cover \(M^3\) can be constructed from
\(T^3\) and \(\mathbb{R}P^3\) by using four operations \(\natural, \circ, \circ^{-1}\) and \(\circ^D\) (cf. [12, Theorem 1.10]).
Later Lü and Yu [8] considered a construction of general small covers $M^3$. They introduced new operations $\sharp$, $\sharp^\text{ve}$, $\sharp^\Delta$ and $\sharp_i^\otimes$ ($i \geq 3$) and showed the following theorem (cf. [8] Theorem 1.2).

**Theorem 1.2** (Lü and Yu). Each small cover $M^3$ can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with a certain $(\mathbb{Z}_2)^3$-action by using seven operations $\sharp$, $\sharp^{-1}$, $\sharp^\text{ve}$, $\sharp^\Delta$, $\sharp_i^\otimes$ and $\sharp_i^\otimes$.

Operations appeared in Theorem 1.2 are all “non-decreasing” i.e. they do not decrease the number of faces of an orbit polytope, and therefore the use of the surgery $\sharp$ is prohibited unlike Theorem 1.1. In [5] Kuroki pointed out that the operations $\sharp^D$, $\sharp^e$ and $\sharp^\text{ve}$ can be obtained as compositions of $\sharp$ and $\sharp$ such as
$$
\sharp^D = \sharp \circ \sharp^p \mathbb{R}P^3, \quad \sharp^e = \sharp \circ \sharp^p \mathbb{R}P^3 \quad \text{and} \quad \sharp^\text{ve} = \sharp^2 \circ \sharp, \quad \text{respectively (cf. [5] Theorem 4.1)}.
$$

Therefore our result in [12] can be improved as follows: Each orientable small cover $M^3$ can be constructed from $\mathbb{R}P^3$ and $T^3$ by using three operations $\sharp$, $\sharp$ and $\sharp^{-1}$. (cf. [5] Corollary 4.4). Moreover Lü-Yu’s result can be rewritten by using $\sharp$ instead of $\sharp^e$ and $\sharp^\text{ve}$ as follows (cf. [5] Corollary 4.8): Each small cover $M^3$ can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with a certain $(\mathbb{Z}_2)^3$-action by using six operations $\sharp$, $\sharp$, $\sharp^{-1}$, $\sharp^\Delta$, $\sharp_i^\otimes$ and $\sharp_i^\otimes$. Then a problem arises (cf. [5] Problem 5.2).

**Problem 1.3**. What are basic small covers from which we can construct all 3-dimensional small covers using the three operations $\sharp$, $\sharp$ and $\sharp^{-1}$?

We give a solution to this problem. The following is our main result.

**Theorem 1.4**. Each small cover $M^3$ can be constructed from $T^3$, $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using two operations $\sharp$ and $\sharp$.

In the above theorem we do not use the inverse surgery $\sharp^{-1}$. As a corollary we obtain improvements of Theorem 1.1 and our previous result in [12].

**Corollary 1.5**. (1) Each linear model $M^3$ can be constructed from $T^3$ by using two operations $\sharp$ and $\sharp$.

(2) Each orientable small cover $M^3$ can be constructed from $T^3$ and $\mathbb{R}P^3$ by using two operations $\sharp$ and $\sharp$.

These results are equivariant analogues of a well-known result (cf. [4]): “Each closed 3-manifold can be constructed from the 3-sphere $S^3$ by using the Dehn surgeries”.

This paper is organized as follows. In section 2 we recall the definition and the basic facts about small covers briefly, and we introduce some basic 3-dimensional small covers. In section 3 we establish several operations on $(\mathbb{Z}_2)^3$-colored polytopes. In section 4 we discuss the constructions of $(\mathbb{Z}_2)^3$-colored polytopes, and prove Theorem 1.4. In section 5 we follow the standpoint of Lü and Yu, and discuss a non-decreasing construction of small covers by using the inverse surgery $\sharp^{-1}$ instead of the decreasing surgery $\sharp$. We shall point out that there is a gap in the proof of Theorem 1.2 in [8] (Remark 5.5) and improve their result as follows.

**Theorem 1.6**. (1) Each linear model $M^3$ can be constructed from $T^3$ by using three operations $\sharp$, $\sharp^e$ and $\sharp^{-1}$.

(2) Each orientable small cover $M^3$ can be constructed from $T^3$ and $\mathbb{R}P^3$ by using three operations $\sharp$, $\sharp^e$ and $\sharp^{-1}$.

(3) Each small cover $M^3$ can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using four operations $\sharp$, $\sharp^e$, $\sharp^{-1}$ and $\sharp_i^\otimes$. 

In section 6 we shall make a remark on a 2-torus manifold which is an object of a little wider class than small covers. If the object is expanded to this class, the argument becomes easier. We prove the following theorem.

**Theorem 1.7.** (1) Each linear model of a locally standard 2-torus manifold over $D^3$ can be constructed from $S^3$ by using inverse surgery $\natural^{-1}$.

(2) Each orientable locally standard 2-torus manifold over $D^3$ can be constructed from $S^3$ by using two surgeries $\natural^{-1}$, $\natural^D$ and the blow up $\mathbb{R}P^3$.

(3) Each locally standard 2-torus manifold over $D^3$ can be constructed from $S^3$ by using the inverse surgery $\natural^{-1}$ and connecting $\mathbb{R}P^3$, $S^1 \times_{\mathbb{Z}_2} S^2$, $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions by operations $\natural$ and $\natural^e$.

2. Basics of small covers

In this section we recall the definitions and basic facts on small covers (see [1] for detail). Let $P$ be an $n$-dimensional simple convex polytope with facets (i.e., codimension-one faces) $\mathcal{F} = \{F_1, \cdots, F_m\}$. A small cover $M$ over $P$ is an $n$-dimensional closed manifold with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is $P$. For a facet $F$ of $P$, we define $\lambda(F)$ to be the generator of the isotropy subgroup at $x \in \pi^{-1}(\text{int} F)$ where $\pi : M \to P$ is the orbit projection. Then a function $\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$ is called a characteristic function of $M$ which satisfies the following condition.

\[
(*) \text{ if } F_1 \cap \cdots \cap F_n \neq \emptyset \text{ then } \{\lambda(F_1), \cdots, \lambda(F_n)\} \text{ is linearly independent.}
\]

Therefore $\lambda$ is a kind of face-coloring of $P$. Then we call a function satisfying $(*)$ a $(\mathbb{Z}_2)^n$-coloring of $P$. We say that two $(\mathbb{Z}_2)^n$-colored polytopes $(P_1, \lambda_1)$ and $(P_2, \lambda_2)$ are equivalent when there exists a combinatorial equivalence of polytopes $\phi : P_1 \to P_2$ such that $\lambda_2 \phi = \theta \lambda_1$ for some $\theta \in \text{Aut}(\mathbb{Z}_2)^n$. Conversely, given a simple convex polytope $P$ and a $(\mathbb{Z}_2)^n$-coloring $\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$ satisfying $(*)$, we can construct a small cover $M$ such that its characteristic function is the given $\lambda$ as follows:

\[
M(P, \lambda) := P \times (\mathbb{Z}_2)^n / \sim,
\]

where $(x, t) \sim (y, s)$ is defined as $x = y \in P$ and $s - t$ is contained in the subgroup generated by $\lambda(F_1), \cdots, \lambda(F_k)$ such that $x \in \text{int}(F_1 \cap \cdots \cap F_k)$. We say that two small covers $M_i$ over $P_i$ ($i = 1, 2$) are $\text{GL}(n, \mathbb{Z}_2)$-equivalent on a combinatorial equivalence of polytopes $\phi : P_1 \to P_2$ when there exists a $\theta$-equivariant homeomorphism $f : M_1 \to M_2$ such that $\pi_2 \circ f = \phi \circ \pi_1$ i.e. $f(g \cdot x) = \theta(g) \cdot f(x) \ (g \in (\mathbb{Z}_2)^n, \ x \in M_1)$ for some $\theta \in \text{Aut}(\mathbb{Z}_2)^n$. Moreover we say that two small covers are equivalent when they are $\text{GL}(n, \mathbb{Z}_2)$-equivalent on some equivalence $\phi : P_1 \to P_2$. In [7] this equivalence and a $\text{GL}(n, \mathbb{Z}_2)$-equivalence on the identity are called a weakly equivariant homeomorphism and a $D$-$J$ equivalence, respectively. Davis and Januszkiewicz proved that a small cover $M$ over $P$ with a characteristic function $\lambda$ is $\text{GL}(n, \mathbb{Z}_2)$-equivalent on the identity to $M(P, \lambda)$ when we fix a polytope $P$ (cf. [1] Proposition 1.8). Therefore we can identify the equivalence class of a small cover $M(P, \lambda)$ with the equivalence class of a $(\mathbb{Z}_2)^n$-colored polytope $(P, \lambda)$.

**Example 2.1.** The real projective space $\mathbb{R}P^n$ and the $n$-dimensional torus $T^n$ with standard $(\mathbb{Z}_2)^n$-actions are examples of small covers over the $n$-simplex $\Delta^n$ and the $n$-cube $I^n$ respectively. Figure [1] shows their characteristic functions on the polytopes in the case $n = 3$, where $\{\alpha, \beta, \gamma\}$ is a basis of $(\mathbb{Z}_2)^3$. We notice that a
\((\mathbb{Z}_2)^n\)-coloring on \(\Delta^n\) is unique up to equivalence. Therefore we denote the colored simplex by \(\Delta^n\) by omitting coloring.

**Figure 1.** Characteristic functions of \(\mathbb{R}P^3\) and \(T^3\).

A small cover over \(P\) with \(n\)-coloring (i.e. \(\lambda(F)\) is a basis of \((\mathbb{Z}_2)^n\)) is called a linear model. An example of a linear model is the torus \(T^n\) shown in Example\[2.1\]

In this case the \(n\)-coloring of \(P\) (i.e. the linear model) is unique up to equivalence. In case \(n = 3\), it is well-known that a simple convex polytope is 3-colorable if and only if each face contains an even number of edges.

In [11, Theorem 1.7], we gave a criterion of when a small cover is orientable. We recall the criterion in the case \(n = 3\).

**Theorem 2.2.** A 3-dimensional small cover \(M(P, \lambda)\) is orientable if and only if \(\lambda(F)\) is contained in \(\{\alpha, \beta, \gamma, \alpha + \beta + \gamma\}\) for a suitable basis \(\{\alpha, \beta, \gamma\}\) of \((\mathbb{Z}_2)^3\).

From the above theorem the small covers \(\mathbb{R}P^3\) and \(T^3\) given in Figure 1 are both orientable. We call a \((\mathbb{Z}_2)^3\)-coloring satisfying the orientability condition in the above theorem an orientable coloring of \(P\). Since each triple of \(\{\alpha, \beta, \gamma, \alpha + \beta + \gamma\}\) is linearly independent, the orientable coloring is just an ordinary 4-coloring.

**Example 2.3.** We consider small covers on the 3-sided prism \(P^3(3) = I \times \Delta^2\). There exist three types of \((\mathbb{Z}_2)^3\)-coloring on \(P^3(3)\) shown in Figure\[2\] up to equivalence. The first example \(M(P^3(3), \lambda_1)\) is non-equivariantly homeomorphic to \(S^1 \times \mathbb{R}P^2\). The second example \(M(P^3(3), \lambda_2)\) is not equivariantly homeomorphic to \(M(P^3(3), \lambda_1)\) but non-equivariantly homeomorphic to \(S^1 \times \mathbb{R}P^2\) (cf. [8, Lemmas 4.2 and 4.3]). The last example \(M(P^3(3), \lambda_3)\) is orientable and homeomorphic to \(\mathbb{R}P^3 \sharp \mathbb{R}P^3\) where \(\sharp\) is the connected sum (see the following section).

**Figure 2.** Basic three types of \((\mathbb{Z}_2)^3\)-coloring on 3-sided prism \(P^3(3) = I \times \Delta^2\); \(\lambda_1, \lambda_2\) and \(\lambda_3\) respectively.

**Example 2.4.** It is easily verified that there exist four types of \((\mathbb{Z}_2)^3\)-coloring on the 3-cube \(I^3 = P^3(4)\). One of them is the 3-colored cube which is already seen in Figure\[1\] and is denoted by \((I^3, \lambda_0)\). The other three types are shown in Figure\[3\] The associated small covers are homeomorphic to \(S^1 \times K\), a twisted \(K\)-bundle over \(S^1\) and a twisted \(T^2\)-bundle over \(S^1\) according to \(\lambda_1, \lambda_2\) and \(\lambda_3\) respectively, where \(K = \mathbb{R}P^2 \sharp \mathbb{R}P^2\) is the Klein’s bottle (more precisely see [8, Lemmas 5.3 and 5.4]).
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Figure 3. Three types of \((\mathbb{Z}_2)^3\)-coloring on 3-cube \(I^3\): \(\lambda_1\), \(\lambda_2\) and \(\lambda_3\) respectively (except the 3-colored cube in Figure 1).

Remark 2.5. In [8] the \(GL(3,\mathbb{Z}_2)\)-equivalence on the identity (D-J equivalence) is adopted as an equivalence relation of \((\mathbb{Z}_2)^3\)-colored polytopes, i.e. \((P,\lambda) \sim (P,\theta\lambda)\) for \(\theta \in \text{Aut}(\mathbb{Z}_2)^3\). Therefore it is written that there exist five (resp. seven) types of \((\mathbb{Z}_2)^3\)-coloring on \(P^3(3)\) (resp. \(I^3\)) in [8]. Discussing D-J equivalence classes only when the orbit polytope \(P\) is fixed has the meaning. However, the orbit polytopes will be not fixed in the following sections. Then we adopt our equivalence (the weakly equivariantly homeomorphism) instead of the D-J equivalence. In this paper we shall rewrite results in [8] to our standpoint by our equivalence. The difference between the D-J equivalence and our equivalence is not essential in the discussion of the following sections.

3. OPERATIONS ON SMALL COVERS

Henceforth we assume that \(n = 3\) and \((P,\lambda)\) is a pair of a 3-dimensional simple convex polytope \(P\) with a \((\mathbb{Z}_2)^3\)-coloring \(\lambda\), and \(\{\alpha, \beta, \gamma\}\) is a basis of \((\mathbb{Z}_2)^3\). We call a 3-dimensional simple convex polytope a 3-polytope for simplicity. From the Steinitz’s theorem (see [2] etc.) combinatorially equivalence classes bijectively correspond to 3-connected 3-valent simple planar graphs i.e. 1-skeleton of \(P\). Here a graph \(\Gamma\) is called \(k\)-connected, \(l\)-valent and simple if \(\Gamma\) is connected after cutting any \((k - 1)\) edges, the degree of each vertex is \(l\), and there is no loop and no multi-edge, respectively. In this section we recall some operations on \((\mathbb{Z}_2)^3\)-colored polytopes (or small covers), which were introduced in [3], [8] and [12].

Definition 3.1 (the connected sum \(\#\)). The operation \(\#\) in Figure 4 (from left to right) is called the connected sum (at vertices) and its inverse (from right to left) is denoted by \(\#^{-1}\). These operations also can be defined for non-colored polytopes. Remark that \(P_1\# P_2\) is also a 3-polytope for any 3-polytopes \(P_i\) \((i = 1, 2)\) from the Steinitz’s theorem. The operation \(\#\) corresponds to the connected sum \(M(P_1, \lambda_1)\sharp M(P_2, \lambda_2)\) around fixed points of them (cf. [11] 1.11 or [8] Definition 3). We say that \((P,\lambda)\) is decomposable (as a \((\mathbb{Z}_2)^3\)-colored polytope) when there exist two \((\mathbb{Z}_2)^3\)-colored polytopes \((P_i, \lambda_i)\) \((i = 1, 2)\) such that \((P,\lambda) = (P_1,\lambda_1)\sharp(P_2,\lambda_2)\). Similarly we say that \(P\) is decomposable as a non-colored polytope when \(P = P_1\sharp P_2\) as non-colored polytopes for some \(P_i\) \((i = 1, 2)\).

Specifically the connected sum with \(\Delta^3\) on polytopes, denoted by \(\sharp\Delta^3\) (and often called a cutting vertex or bistellar 0-move), corresponds to the operation called a blow up on small covers (Figure 5). Its inverse \(\#^{-1}\Delta^3\) (often called a bistellar 2-move) is called a blow down.

Definition 3.2 (the surgery \(\natural\)). The operation \(\natural\) in Figure 6 (from left to right) is called the surgery along an edge \(e\) and its inverse \(\natural^{-1}\) (from right to left) is called the inverse surgery along a pair of edges \(e_1\) and \(e_3\). The operations \(\natural\) and \(\natural^{-1}\) both
Figure 4. The connected sum $\#$ and its inverse $\#^{-1}$.

Figure 5. The blow up $\# \Delta^3$ and the blow down $\#^{-1} \Delta^3$.

correspond to the ordinaly surgeries on small covers (cf. [3]). In the previous papers [3], [5], [8] and [12], surgeries $\#$ and $\#^{-1}$ were not distinguished and they both were denoted by the same symbol $\#$.

Figure 6. The surgery $\#$ and its inverse $\#^{-1}$.

We do not allow the surgeries $\#$ and $\#^{-1}$ when the 3-connectedness of the 1-skeleton of $P$ is destroyed after doing it, i.e. the following cases respectively:

**in case $\#$:** if and only if $F_2$ and $F_4$ are adjacent to a same face except $F_1$ and $F_3$ (involve the case when $F_1$ or $F_3$ is a quadrilateral),

**in case $\#^{-1}$:** if and only if $F'_1$ is adjacent to $F'_3$.

**Definition 3.3** (the connected sum along edges $\#^e$). The operation $\#^e$ in Figure 7 (from left to right) is called the connected sum along edges and its inverse is denoted by $(\#^e)^{-1}$. We notice that the operation $\#^e$ is obtained as the composition $\#^e = \# \circ \#$ as shown in the same figure (cf. [5] Theorem 4.1(2)). The operation $\#^e$ corresponds to the connected sum along the circle $\pi^{-1}(e)$ on a small cover $M$ where $\pi : M \to P$ is the projection (cf. [5]).

Specifically the operations $\#^e P^3(3)$ (along a vertical edge in Figure 2) and $\#^e \Delta^3$ are often called the cutting edge and the bistellar 1-move, respectively (Figure 8). The former (left diagram) corresponds to a blow up along the circle $\pi^{-1}(e)$ on a small cover. In this diagram we can choose not only $\beta + \gamma$ but also $\alpha + \beta + \gamma$ as a color of the center square when $* = 0$. The latter operation $\#^e \Delta^3 = \# \circ \# \Delta^3$ corresponds to the Dehn surgery of type $\Delta^3$ on a small cover (cf. [12] or [5] 3.5). This operation is denoted by $\#^D$ and is called the Dehn surgery. This operation can
Figure 7. The connected sum along the edges \( \#^e \) and its inverse \((\#^e)^{-1}\). The figure also shows that \( \#^e = \natural \circ \sharp \).

be done along an edge \( e \) which satisfies the following condition:

\[
\sum_{e} \lambda(F) := \sum_{\{F \in \mathcal{F} | e \cap F \neq \emptyset\}} \lambda(F) = 0.
\]

We call such an edge 0-sum edge (or 4-colored edge in orientable case). We notice that the Dehn surgery \( \natural^D \) does not change the number of faces, and is invertible because \((\natural^D)^{-1} = \natural^D\).

Figure 8. The cutting edge \( \#^e P^3(3) \) and the Dehn surgery \( \natural^D = \#^e \Delta^3 \).

From the Steinitz's theorem, a 3-polytope \( P \) is decomposable as a non-colored polytope if and only if there exist three edges such that they are not adjacent to each other and the 1-skeleton of \( P \) becomes disconnected after cutting them. Obviously if an orientable (4-)colored polytope \( P \) is decomposable as a non-colored polytope then \((P, \lambda)\) is also decomposable (as a \((\mathbb{Z}_2)^3\)-colored polytope). However we need a little attention for non-orientable colored polytopes. We say that \((P, \lambda)\) is quasi-decomposable when there exist two \((\mathbb{Z}_2)^3\)-colored polytopes \((P_1, \lambda_1)\) and \((P_2, \lambda_2)\) such that either \((P, \lambda) = (P_1, \lambda_1)\#^e(P_2, \lambda_1)\) or \((P, \lambda) = (P_1, \lambda_1)\#^e(P_2, \lambda_2)\), except \( P = P_1\#^e \Delta^3(= \#^D P_1)\).

Remark 3.4. Notice that if a 1-skeleton of \( P \) becomes disconnected after cutting three edges \( \{e', e'', e'''\} \) then these three edges are not adjacent to each other or meet at a vertex. In fact if a pair \( \{e', e''\} \) of these three edges is adjacent to each
other and the other edge $e''$ is not adjacent to $e' \cap e''$ then the 1-skeleton of $P$ becomes disconnected after cutting the edge $e''$ and the edge which is adjacent to $e' \cap e''$ and different from $e'$ and $e''$. This contradicts the 3-connectedness of the 1-skeleton of $P$.

**Proposition 3.5.** Let $(P, \lambda)$ be a $(\mathbb{Z}_2)^3$-colored polytope, but not $P^3(3)$. If $P$ is decomposable as a non-colored polytope then $(P, \lambda)$ is quasi-decomposable.

**Proof.** It is sufficient to treat the case that $P$ is indecomposable as a $(\mathbb{Z}_2)^3$-colored polytope. Since $P$ is decomposable as a non-colored polytope, there exist three non-adjacent edges such that $P$ becomes disconnected after cutting them out, and colors of the three faces adjacent to these edges are not linearly independent as shown in Figure 9.

![Figure 9](image)

**Figure 9.** The decomposition of a polytope along a 3-cycle of 2-independent faces.

Since $P \neq P^3(3)$, $P$ has at least six faces so we may assume that there are at least two faces under the pillar ($F_i$'s) in the first diagram. We first assume that $F'_3 = F'_2$ (equivalently $e_{21} = e_{31}$ because if it is not so, the 1-skeleton of $P$ becomes disconnected after cutting these two edges). Then the 1-skeleton of $P$ becomes disconnected after cutting three edges $e_1$, $e_{23}$, and $e_{32}$. Since $P$ is indecomposable, these three edges actually meet at a vertex $F_1' \cap F_2' \cap F_3'$ (see Remark 3.4). It should be $F_1' = F_2' = F_3'$ and it is a triangle. This contradicts the assumption that there are at least two faces under the pillar. Therefore the assumption $F'_2 = F'_3$ is denied, and by a similar discussion we can reach the conclusion that $F'_i (i = 1, 2, 3)$ are different faces each other. We notice that if $F_3' \cap F'_2 = \emptyset$ then it is clear that $F_1' \cap F'_2 = F_2' \cap F'_1 = \emptyset$. Therefore we can assume that $F'_3 \cap F'_2 = \emptyset$ by changing the role of $F_i$'s if necessary.

Now we can do the surgery $\natural^{-1}$ for edges $e_1$ and $e_{32}$, and decompose $P$ into two $(\mathbb{Z}_2)^3$-colored polytopes $P_1$ and $P_2$ by cutting three non-adjacent edges $e_{1}'$, $e_2$ and $e_{31}$ (second and third diagrams). Then we have $\natural^{-1}P = P_1 \natural P_2$ or equivalently $P = P_1 \natural P_2$. □

Notice that the surgery $\natural$ and the Dehn surgery $\natural^D$ are not allowed along an edge of a quadrilateral and a triangle, respectively, and the inverse surgery $\natural^{-1}$ is not allowed along a pair of adjacent edges. The following is a key lemma to relate the surgery to the connected sum.

**Lemma 3.6.** Let $(P, \lambda)$ be a $(\mathbb{Z}_2)^3$-colored polytope. Suppose that the 3-connectedness of the 1-skeleton of $P$ is destroyed after doing surgeries $\natural^{-1}$ or $\natural^D$, but not the above trivial prohibited cases. Then $(P, \lambda)$ is quasi-decomposable. In particular when $(P, \lambda)$ is (orientable) 4-colored, $(P, \lambda)$ is decomposable as a $(\mathbb{Z}_2)^3$-colored polytope.
Proof. In consequence of Proposition 3.6 it is sufficient to prove that \((P, \lambda)\) is decomposable as a non-colored polytope.

(1) **in case** \(\tau^{-1}\): When the inverse surgery \(\tau^{-1}\) is not allowed in the right diagram of Figure 6, \(F'_2\) is adjacent to \(F'_3\). Then cutting the three non-adjacent edges \(e_1, e_3\) and \(F'_1 \cap F'_3\) makes the 1-skeleton of \(P\) disconnected. That is \(P\) is decomposable as a non-colored polytope.

(2) **in case** \(\tau^D\): Since \(\tau^D = (\tau^{-1}\Delta^3) \circ \tau^{-1}\) and there is no obstacle for the blow down \(\tau^{-1}\Delta^3\), the allowance of \(\tau^D\) depends only on that of \(\tau^{-1}\).

\(\square\)

4. Constructions of Small Covers

In this section we discuss constructions of \((\mathbb{Z}_2)^3\)-colored polytopes (i.e. small covers) by using two operations \(\sharp\) and \(\star\). Henceforth polytopes are considered as \((\mathbb{Z}_2)^3\)-colored polytopes. In [3], Izmestiev proved the following theorem which is a combinatorial translation of Theorem 1.1.

**Theorem 4.1** (Izmestiev). Each 3-colored polytope \((P^3, \lambda)\) can be constructed from \((F^3, \lambda_0)\) by using three operations \(\sharp, \star\) and \(\tau^{-1}\).

We start from linear models and consider constructions of orientable small covers (i.e. 4-colored polytopes). Let \(P\) be an \(l\)-gonal face of \(P\). We say that \(F\) is \(j\)-independent \((j = 2, 3)\) when the rank of \(\{\lambda(F_1), \ldots, \lambda(F_1)\}\) is \(j\) where \(F_1, \ldots, F_l\) are faces adjacent to \(F\). In the case of orientable small covers, a \(j\)-independent face is a face such that the number of colors of adjacent faces is \(j\) \((j = 2, 3)\). Similarly we say that an edge is \(j\)-colored \((j = 3\) or \(4)\) when the number of the four faces adjacent to the edge is \(j\).

**Proposition 4.2.** Each 4-colored polytope \((P^3, \lambda)\) can be constructed from 3-colored polytopes and \(\Delta^3\) by using two operations \(\sharp\) and \(\tau^D\).

**Proof.** By induction on the number of faces of \(P\), it is sufficient to prove the following

\((*)\) Each 4-colored polytope \(P \neq \Delta^3\) can be decomposed into two polytopes after doing the Dehn surgery \(\tau^D \circ (\tau^D)^{-1}\) finitely many times.

Assume that \(P\) is 4-colored and not \(\Delta^3\). Then there exists a 3-independent face. Let \(F\) be a 3-independent face such that the number of its edges is minimum among 3-independent faces of \(P\), and \(k\) be this number. We prove the above \((*)\) by induction on \(k\). If \(k = 3\) (i.e., \(F\) is a triangle) then we get a colored decomposition \(P = P^0\sharp \Delta^3\) immediately. We assume \(k \geq 4\). Since \(F\) is a 3-independent face, there exists a 4-colored edge \(e\) of \(F\) (see Figure 10).

We notice that there exist no triangular face of \(P\) because \(k \geq 4\). If the Dehn surgery \(\tau^D\) is not allowed along an edge then \(P\) decomposes into two polytopes from Lemma 3.6. Therefore we may assume that the Dehn surgery \(\tau^D\) is allowed along every 4-colored edge of \(F\). If the 3-independence of \(F\) is preserved under the Dehn surgery \(\tau^D\) along some edge, then we can reduce \(P\) to \(\tau^D P\) which has a \((k - 1)\)-gonal 3-independent face, and the proof ends by induction on \(k\). Therefore it is sufficient to show the existence of such an edge.

In Figure 10 we assume that \(F\) becomes 2-independent after doing \(\tau^D\) along the edge \(e\). Then an adjacent face of \(F\) which is painted as \(\beta\) must be unique, and the other faces are painted by \(\alpha\) and \(\gamma\) alternatively such as \(* = \gamma, \ldots, * = \alpha\). In particular when \(k = 4\) (or even), the contradiction arises because \(* = *\).
Figure 10. A 4-colored edge $e$ of a 3-independent face $F$.

$k \geq 5$ and this situation arises, we can do the Dehn surgery $\natural D$ along the edge $e'$ (or $e''$) preserving the 3-independence of $F$.

**Remark 4.3.** In the proof of Proposition 4.2 when we ignore the coloring of $P$, the Dehn surgery $\natural D$ can be continued until a triangle appears for all faces, and then leads to a well-known fact that “Each 3-polytope is bistellarly equivalent to each other” or equivalently “the PL-homeomorphism class of $S^2$ is unique” (cf. [10]).

Combining the above proposition and Theorem 4.1 and noting the relation $\natural D = \natural \circ (\sharp \Delta^3)$, we have the following corollary immediately (cf. [12, Theorem 1.10] and [5, Corollary 4.4]).

**Corollary 4.4.** Each 4-colored polytope $(P^3, \lambda)$ can be constructed from $(I^3, \lambda_0)$ and $\Delta^3$ by using three operations $\sharp$, $\natural$ and $\natural^{-1}$.

Next we consider a construction of all $(\mathbb{Z}_2)^3$-colored polytope. We recall the basic fact that each 3-polytope has a face which has edges less than six (cf. [2] etc.). Such a face is called a small face. If each small face can be compressed so that the number of faces of $P$ decreases then we can reduce all $(\mathbb{Z}_2)^3$-colored polytopes to some basic polytopes by induction on the number of faces. At first we compress 3-independent small faces.

**Proposition 4.5.** Let $P$ be a $(\mathbb{Z}_2)^3$-colored polytope except $\Delta^3$ and $P^3(3)$ as a non-colored polytope. If there exists a 3-independent small face of $P$, then either $P$ or $\natural D P$ is quasi-decomposable.

**Proof.** If there exists a triangular face of $P$ except $\Delta^3$ and $P^3(3)$ then $P$ is decomposable as a non-colored polytope and so $(P, \lambda)$ is quasi-decomposable from Proposition 3.5. Therefore we can assume that $P$ has no triangular face. Let $F$ be a 3-independent small face of $P$.

(1) When $F$ is a quadrilateral, the situation around $F$ is shown as left of Figure 11 where $a_i, b_j \in \mathbb{Z}_2$ with $b_2a_3 = 0$ and at least one of $a_1$ and $b_1$ is nonzero. By a symmetry we may assume that $a_1 = 1$. Since an adjacent triangle does not exist, and we can always blow down $(\sharp^e)^{-1} P^3(3)$ for $F$ along the horizontal edges (if $a_3b_1 = 0$) or the vertical edges (if $b_1 = 1, b_2 = 0$), as shown in Figure 8. That is $P = P^e \sharp^e P^3(3)$.

(2) When $F$ is a pentagon, the situation around $F$ is shown as right of Figure 11 where $a_i, b_j, c_k \in \mathbb{Z}_2$ with $a_2b_3 + b_2 + b_1c_3 + b_3 = 1$ and at least one of $a_1$, $b_1$ and $c_1$ is nonzero. We prove that there exists a 0-sum edge $e$ of $F$ such that $F$ is transformed by $\natural D P$ into a 3-independent quadrilateral. Then $\natural D P$ is quasi-decomposable from the case (1). Here if the Dehn surgery $\natural D$ is not allowed then $P$ is quasi-decomposable from Lemma 3.6.

i) The case $a_1 = 1$ (the case $c_1 = 1$ can be treated similarly).
a) When \(a_2 = 1\), \(e_2\) is a 0-sum edge. If \(c_1 = 0\) or \(b_1 + b_2 = 1\) then the Dehn surgery \(\hat{z}^D\) along the edge \(e_2\) preserves the 3-independence of \(F\) because the rank of \(\{\lambda(F_1), \lambda(F_3), \lambda(F_4)\}\) is three. If \(c_1 = 1\) and \(b_1 = b_2 = 0\) then we have \(b_3 = 1\) and \(e_3\) is a 0-sum edge and \(\{\lambda(F_1), \lambda(F_2), \lambda(F_3)\}\) is linearly independent. If \(c_1 = 1\) and \(b_1 = b_2 = 1\) then we have \(c_3 = 1\) and \(e_1\) is a 0-sum edge and \(\{\lambda(F_2), \lambda(F_3), \lambda(F_4)\}\) is linearly independent. In all cases the Dehn surgery \(\hat{z}^D\) along a certain 0-sum edge preserves the 3-independence of \(F\).

b) If \(a_2 = 0\) then we have \(b_2 = 1\) and \(b_3 + c_3 = 1\). Therefore we obtain \(\sum_{e_2} \lambda(F) = (b_1 + c_1)\alpha\) and \(\sum_{e_3} \lambda(F) = (b_1 + c_1 + 1)\alpha\), so either \(e_4\) or \(e_5\) is a 0-sum edge. Since \(\{\lambda(F_1), \lambda(F_2), \lambda(F_3)\}\) is linearly independent, the Dehn surgery \(\hat{z}^D\) along \(e_4\) or \(e_5\) preserves the 3-independence of \(F\).

ii) The case \(a_1 = c_1 = 0\) and \(b_1 = 1\). We have \(a_2 b_3 + b_2 = 1, b_2 c_3 + b_3 = 1\) and \(\{\lambda(F_1), \lambda(F_2), \lambda(F_4)\}\) is linearly independent. In this case since \(\sum_{e_3} \lambda(F) = (a_2 + b_2 + 1)\beta + (b_3 + 1)\gamma = a_2(1 + b_3)\beta + b_2 c_3 \gamma\) and \(\sum_{e_5} \lambda(F) = (b_2 + 1)\beta + (b_3 + c_3 + 1)\gamma = a_2 b_3 \beta + c_3(1 + b_2)\gamma\), either \(e_3\) or \(e_5\) is a 0-sum edge (if \(a_2 = c_3 = 1\) then \(b_2 + b_3 = 1\)). Then the Dehn surgery \(\hat{z}^D\) preserves the 3-independence of \(F\).

**Remark 4.6.** In the above proposition, except an irregular quasi-decomposition because of the prohibition of \(\hat{z}^D\), we have the fact that each 3-independent small face \(F\) is compressible: such as \(P = P^\rho \Delta^3\) when \(F\) is a triangle, \(P = P^\rho P^3(3)\) or \(P^\rho P^3(3)\) when \(F\) is a quadrilateral and \(P = \hat{z}^D(P^\rho P^3(3))\) or \(\hat{z}^D(P^\rho P^3(3))\) when \(F\) is a pentagon respectively. In all cases the number of faces decreases by this decomposition.

**Proposition 4.7.** Let \(P\) be a \((\mathbb{Z}_2)^3\)-colored polytope except \(\Delta^3, P^3(3)\) and \(I^3\) as a non-colored polytope. If there exists a 2-independent small face of \(P\) then either \(P\) or \(\hat{z}^{-1} P\) is quasi-decomposable.

**Proof.** If there exists a triangular face of \(P\) except \(\Delta^3\) and \(P^3(3)\) then \(P\) is decomposable as a non-colored polytope and so \((P, \lambda)\) is quasi-decomposable from Proposition 3.6. Therefore we can assume that \(P\) has no triangular face. Let \(F\) be a 2-independent small face of \(P\). We notice that the inverse surgery \(\hat{z}^{-1}\) is allowed in the category of \((\mathbb{Z}_2)^3\)-colored polytopes when \((P, \lambda)\) is not quasi-decomposable by Lemma 3.6.

1) When \(F\) is a quadrilateral, the number of quadrilaterals adjacent to \(F\) is at most two because \(P \neq I^3\) and the situation around a 2-independent quadrilateral \(F\) is shown as Figure 12 where \(\ast = \beta\) or 0. If \(F_1\) and \(F_2\) are quadrilateral (the third
Figure 12. The compression of a 2-independent quadrilateral.

Figure 13. The compression of a 2-independent pentagon.

Remark 4.8. When $F$ is a pentagon in the proof of the above proposition, although the compression of the triangle of $\sharp^{-1}P$ does not change the number of faces compared with the beginning, $F$ is transformed into a quadrilateral by this step (see the third diagram). Then we apply the argument (2) in the proof of Proposition 4.7 to the quadrilateral so that the number of faces in the resulting polytope is one less than the number of faces in $P$.

In consequence of Propositions 4.5 and 4.7 we can reduce any $(\mathbb{Z}_2)^3$-colored polytope to $\Delta^3$, $I^3$ and $P^3(3)$ with a certain coloring by using the surgeries $\sharp^{-1}$, $\sharp^D = (\sharp^{-1}\Delta^3) \circ \sharp^{-1}$ (without $\sharp$) and the inverses of connected sums $\sharp^\tau$, $\sharp^{\tau'} = \sharp \circ \sharp$. From Examples 2.3 and 2.4 the possible colorings on $P^3(3)$ (resp. $I^3$) are only three...
Corollary 4.10. We restrict the above theorem to 3- (resp. 4-) colored polytopes, and along vertical edges and \((\#)\) by using two operations \(\Delta^3\) and \(\Delta^3\). Since the surgeries \(\zeta\) and \(\zeta^{-1}\) preserves the number of colors of faces, and the connected sum \(\#\) increases the number of faces, it is clear that these four polytopes can not be constructed from others by using only \(\zeta\), \(\zeta\) and \(\zeta^{-1}\). Therefore we have,

**Theorem 4.9.** Each \((\mathbb{Z}_2)^3\)-colored polytope \((P^3, \lambda)\) can be constructed from \(\Delta^3\), \((I^3, \lambda_0)\), \((P^3(3), \lambda_1)\) and \((P^3, \lambda_2)\) by using two operations \(\zeta\) and \(\zeta\).

The topological translation of the above theorem is Theorem 4.3 shown in the introduction. We restrict the above theorem to 3- (resp. 4-) colored polytopes, and obtain improvements of Theorem 4.4 and Corollary 4.5 as follows.

**Corollary 4.10.** (1) Each 3-colored polytope \((P^3, \lambda)\) can be constructed from \((I^3, \lambda_0)\) by using two operations \(\zeta\) and \(\zeta\).

(2) Each 4-colored polytope \((P^3, \lambda)\) can be constructed from \(\Delta^3\) and \((I^3, \lambda_0)\) by using two operations \(\zeta\) and \(\zeta\).

5. **Non-decreasing constructions of small covers**

Since the operations \(\zeta\) and its inverse \(\zeta^{-1}\) both correspond to surgeries on small covers, we followed Izmestiev’s standpoint in [8] and used the surgery \(\zeta\) in the previous section. However in [8] Lü and Yu considered a “non-decreasing” construction by only operations that number of faces is not decreased, and therefore the use of \(\zeta\) is prohibited. To cancel some obstacles they produced new operations \(\zeta^{\text{even}}, \zeta^\Delta\) and \(\zeta^\ominus\), and showed the following theorem (cf. [8] Theorem 1.1).

**Theorem 5.1** (Lü and Yu). Each \((\mathbb{Z}_2)^3\)-colored polytope \((P^3, \lambda)\) can be constructed from \(\Delta^3\) and \((P^3(3), \lambda_2)\) by using seven operations \(\zeta, \zeta^\circ, \zeta^{\text{even}}, \zeta^{-1}, \zeta^\Delta, \zeta^\bigcirc\) and \(\zeta^\bigcirc\).

However there is a gap in the proof of their paper (we shall point it out later). In this section we also consider a non-decreasing construction of small covers in their standpoint. At first we start with 3-colored polytopes (i.e. linear models). In [8] Izmestiev claimed that each 3-colored polytope can be constructed from 3-colored prisms \(P^3(2I)\) by using \(\zeta\) and \(\zeta^{-1}\) in the proof of Theorem 5.1. From the relation \(P^3(2I) = P^3\zeta^e \cdots \zeta^e P^3\), we can obtain a construction of 3-colored polytopes as follows.

**Proposition 5.2.** Each 3-colored polytope \((P^3, \lambda)\) can be constructed from \((I^3, \lambda_0)\) by using three operations \(\zeta, \zeta^e\) and \(\zeta^{-1}\).

From the above examination we use the operation \(\zeta^e\) and \(\zeta^{-1}\) instead of \(\zeta\) below. Then we can also use the Dehn surgery \(\zeta^D\) and its inverse because of the relations \(\zeta^D = \zeta^e \Delta^3\) and \(\zeta^{D^{-1}} = \zeta^D\). Applying Proposition 5.2 to the above proposition, we have,

**Proposition 5.3.** Each 4-colored polytope \((P^3, \lambda)\) can be constructed from \(\Delta^3\) and \((I^3, \lambda_0)\) by using three operations \(\zeta, \zeta^e\) and \(\zeta^{-1}\).

On the other hand there exist some obstacles for the construction of general \((\mathbb{Z}_2)^3\)-colored polytopes. At first we must prove that Lemma 3.6 also holds for the surgery \(\zeta\).
Lemma 5.4. Let \((P, \lambda)\) be a \((\mathbb{Z}_2)^3\)-colored polytope and \(e\) be an edge of \(P\) but not an edge of a quadrilateral. Suppose that the 3-connectedness of the 1-skeleton of \(P\) is destroyed after doing surgery \(\natural\) along the edge \(e\). Then \((P, \lambda)\) is quasi-decomposable.

Proof. In the Figure\[1\] we assume that the surgery \(\natural\) destroys the 3-connectedness of the 1-skeleton of \(P\). Then there exists a face \(F\) such that \(F \cap F_2 \neq \emptyset\) and \(F \cap F_4 \neq \emptyset\) (see Figure\[1\]). Since neither \(F_1\) nor \(F_3\) is a quadrilateral, \(P \neq P^3(3)\) and we can assume that \(e_1\) is not adjacent to \(e_2\) (i.e., \(R \neq Q\)). When \(e'_1\) is adjacent to \(e_4\) (i.e., \(R' = Q'\)), the 1-skeleton of \(P\) becomes disconnected after cutting the three non-adjacent edges \(e_1, e_2, e'\). Therefore \(P\) is decomposable as a non-colored polytope, and so \(P\) is quasi-decomposable from Proposition\[1\]. We assume that \(e'_1\) is not adjacent to \(e_4\) (i.e., \(P' \neq Q'\)). We do the inverse surgery \(\natural^{-1}\) along the pair of edges \(\{e'_1, e_4\}\) where \(i = 3\) when \(\lambda(F)\) is either \(\alpha\) or \(\alpha + \beta\), and \(i = 1\) when it is not so. If the inverse surgery \(\natural^{-1}\) is not allowed then \((P, \lambda)\) is quasi-decomposable from Lemma\[3\]. Then the graph of \(\natural^{-1}P\) becomes disconnected after cutting the three non-adjacent edges \(e_2, e_i\) and the edge to which \(e'_1\) and \(e_4\) were glued by \(\natural^{-1}\), and \(\{\lambda(F), \lambda(F_2), \lambda(F_4)\}\) is linearly independent. Therefore \(\natural^{-1}P\) is decomposable as a \((\mathbb{Z}_2)^3\)-colored polytope such as \(\natural^{-1}P = P_1 \sharp P_2\), or equivalently \(P = P_1 \sharp e P_2\) i.e. \(P\) is quasi-decomposable. \(\square\)

![Figure 14. The obstacle of the surgery \(\natural\).

Remark 5.5. In [3] Izmestiev used the above lemma only when \(F_4\) in Figure\[1\] is a quadrilateral. In this case \(P\) is always decomposable as a non-colored polytope. In [8] Lü and Yu claimed that this argument can be generalized to every case under the hypothesis of Lemma\[5\] without a proof (cf. [8, Proposition 2.5]), and proved Theorem\[7\] using this claim when \(F_4\) is a pentagon, too. However their claim is incorrect (see Figure\[1\]). Although there is a gap in their proof of Theorem\[7\] the proof is complemented by using Lemma\[5\] instead of their key lemma [8, Proposition 2.5]. Furthermore the theorem is improved by replacing \(\natural\) with \(\natural^{\square}\) as follows: Each \((\mathbb{Z}_2)^3\)-colored polytope \((P^3, \lambda)\) can be constructed from \(\Delta^3\) and \((P^3(3), \lambda_2)\) by using seven operations \(\natural^3, \natural^e, \natural^{ee}, \natural^{-1}, \natural^c\) \((i = 3, 4, 5)\).

From the discussion of the previous section, we can reduce each \((\mathbb{Z}_2)^3\)-colored polytope \(P\) to polytopes which have less faces than \(P\) by using the inverses of \(\natural\) and \(\natural^c\) when \(P\) has a 3-independent small face, or a 2-independent triangle, or a pair of 2-independent quadrilaterals adjacent to each other. Moreover we point out that each 2-independent pentagon can be compressed by using the surgery \(\natural\) as shown in Figure\[1\].
Figure 15. A counter example of [8, Proposition 2.5].

Figure 16. Another compression of a 2-independent pentagon. In the first diagram we may assume that $F_3$ is not quadrilateral by replacing it by $F_2$ if necessary. Then we can do the surgery $\natural$ for the edge $e_3$ and transform $F$ into a triangle (second diagram). Here when the surgery $\natural$ is not allowed, $P$ is quasi-decomposable from Lemma 5.4. Then the triangle can be compressed by $(\sharp e)^{-1}P^3(3)$ and we have $P = \natural^{-1}(P'\sharp e^2P^3(3))$ (third diagram).

In general when colors of two faces on ends of an edge of big faces coincide, we can do the surgery $\natural$ along this edge and decrease the number of faces. Then we can reduce $P$ to $\tilde{P}$ which satisfies the following conditions:

1. $\tilde{P}$ is not quasi-decomposable,
2. each small face of $\tilde{P}$ is an isolated 2-independent quadrilateral,
3. two colors of faces on ends of every edge which is adjacent to big faces do not coincide.

Figure 17. Example of an irreducible polytope: truncated octahedron with a $(\mathbb{Z}_2)^3$-coloring (cf. [8, Example 2.1]).
There are many polytopes satisfying the above condition (see Figure 17). Obviously such a polytope is irreducible by using the inverses of only operations $\triangledown$, $\triangle$ and $\ast^{-1}$. Then we need a coloring change operation $\triangledown^c_i$ in [8].

**Definition 5.6** (The coloring change $\triangledown^c_i$). The operation in Figure 18 is called the coloring change $\triangledown^c_i$ for a 2-independent $i$-gon. This operation is defined as the connected sum along faces to the $i$-gonal prism $P^3(i)$ in particular $\triangledown^c_i = \triangledown^\Delta(P^3(3), \lambda_2)$ (see [8]). It is clear that $\triangledown^c_i$ is invertible because $(\triangledown^c_i)^{-1} = \triangledown^c_i$.

Figure 18. The coloring change $\triangledown^c_i$ for 2-independent $i$-gon.

By using the operation $\triangledown^c_i$, we can change a color of each 2-independent quadrilateral, and compress it by the surgery $\ast$. Moreover the 3-colored cube $(P^3, \lambda_0)$ is obtained by this operation from other basic polytopes such as $\triangledown^c_i(P^3, \lambda_i)$ ($i = 1$ or 3). Therefore we have an improvement of Theorem 5.1 as follows.

**Theorem 5.7.** Each $(\mathbb{Z}_2)^3$-colored polytope $(P^3, \lambda)$ can be constructed from $\Delta^3$, $(P^3(3), \lambda_1)$ and $(P^3(3), \lambda_2)$ by using four operations $\triangledown$, $\triangledown^c$, $\ast^{-1}$ and $\triangledown^c_i$.

The topological translations of Propositions 5.2, 5.3 and Theorem 5.7 are stated in Theorem 1.6.

6. **Locally standard 2-torus manifolds over $D^3$**

In this section we shall give a remark for 2-torus manifolds. A 2-torus manifold $M^n$ is an $n$-dimensional closed smooth manifold with an effective action of $(\mathbb{Z}_2)^n$ (see [6], [7] for detail). If the action is locally standard then the orbit space $Q$ is a nice manifold with corners. When $Q$ is a simple convex polytope, $M$ is a small cover.

We consider the case that $Q$ is a 3-dimensional disc $D^3$ with a simple cell decomposition of the boundary $\partial D^3$, i.e. a locally standard 2-torus manifold over $D^3$. This class is a little wider than 3-dimensional small covers. In fact the 1-skeleton of $Q$ is a 2-connected 3-valent planner graph. This graph is simple and 3-connected if and only if $Q$ is a simple convex polytope. In this category there is little obstacle of surgeries. Therefore it becomes easy to discuss in previous sections.

**Example 6.1.** In Figure 19 we show the characteristic functions of $S^3$ with a standard $(\mathbb{Z}_2)^3$-action and three different $(\mathbb{Z}_2)^3$-colorings of the 2-sided prism $P^3(2)$, respectively. Then the associated 2-torus manifolds $M(P^3(2), \lambda_i)$ are non-equivariantly homeomorphic to $S^1 \times S^2$, $S^2$-bundle over $S^1$ characterized by the conjugation $z \mapsto \bar{z}$ on $S^2 = CP^1$ and $S^1 \times S^2$ according to $i = 0, 1, 2$. We denote $M(P^3(2), \lambda_1)$ by $S^1 \times_{\mathbb{Z}_2} S^2$ where a $\mathbb{Z}_2$-action on $S^1 \times S^2$ is given as follows: $t \cdot (s, z) = (-s, \bar{z})$. 
Remark 6.2. We can easily verify the following relations:

1. $\# \circ$ is trivial and $\#^e \circ = \sharp$.
2. $\# P^3(2)$ (or $\#^e P^3(2)$ along the horizontal edge) is a blow up shown in Figure 20 and $\#^e P^3(2)$ (along the vertical edge) is trivial.
3. $\sharp I^3(1, \lambda_0) = (P^3(2), \lambda_0)$ and $\sharp (P^3(2), \lambda_0) = \circ$.
4. $\sharp D^3 = (P^3(2), \lambda_2)$.
5. $\sharp P^3(3, \lambda_1) = (P^3(2), \lambda_1)$.

We notice that if $Q$ is not 3-connected then $Q$ is decomposable as a $(\mathbb{Z}_2)^3$-colored cell decomposition of $D^3$. Therefore applying the above remark (3), (4) and (5) to Theorem 4.9 we obtain the following corollary immediately.

Corollary 6.3. Each $(\mathbb{Z}_2)^3$-colored cell decomposition of $D^3$ can be constructed from $\circ$, $(P^3(2), \lambda_0)$, $(P^3(2), \lambda_1)$ and $(P^3(2), \lambda_2)$ by using two operations $\sharp$ and $\sharp^e$.

In the category of 2-torus manifolds, there is little obstacle for surgeries and blow downs. Therefore we need not consider the case that surgeries are not allowed (e.g. Lemmas 3.6 and 5.4), and obtain the following theorem.

Theorem 6.4. (1) Each 3-colored cell decomposition of $D^3$ can be constructed from $\circ$ by using the inverse surgery $\sharp^{-1}$.

(2) Each 4-colored cell decomposition of $D^3$ can be constructed from $\circ$ by using the inverse surgery $\sharp^{-1}$, the Dehn surgery $\sharp D^3 (= \sharp^e D^3)$ and the blow up $\sharp^e D^3$.

(3) Each $(\mathbb{Z}_2)^3$-colored cell decomposition of $D^3$ can be constructed from $\circ$ by using the inverse surgery $\sharp^{-1}$ and connecting $D^3$, $(P^3(2), \lambda_1)$ and $(P^3(3), \lambda_2)$ by the operations $\sharp$ and $\sharp^e$.

Proof. Let $(Q, \lambda)$ be a $(\mathbb{Z}_2)^3$-colored cell decomposition of $D^3$ but not $\circ$. If a 2-gonal face appears in the following discussion then $(P^3(2), \lambda_1)$ is separated from $Q$ or we do the surgery $\sharp$ and a 2-gon is compressed immediately.
(1) Each 3-colored cell decomposition except $\varnothing$ can be done the surgery $\natural$ and decrease the number of faces.

(2) In the proof of Proposition 4.2 the Dehn surgery $\natural D$ can be continued until a triangle appears because there is no obstacle of $\natural D$. Therefore each 4-colored cell decomposition of $D^3$ can be reduced to a 3-colored cell decomposition by using $\natural D$ and the blow down $\natural - 1 \Delta_3$.

(3) In the proofs of Propositions 4.5 and 4.7 we need not consider the quasi-decomposition by prohibition of surgeries. When $Q$ has a 3-independent small face, $Q$ can be reduced by the blow downs $\natural - 1 P^3(3)$, $\natural - 1 P^3(2)$, $\natural$ and $\natural D$. When $Q$ has a 2-independent triangle, $Q$ can be reduced by the blow downs $\natural - 1 P^3(3)$ and $\natural - 1 P^3(2)$ (along the horizontal edge). Since each 2-independent quadrilateral (or pentagon) has a 3-colored edge, we can do the surgery $\natural$ along this edge in this category and decrease the number of faces. Therefore we can reduce $Q$ to the basic polytopes $\Delta_3$, $P^3(3)$ and $P^3(2)$ by using $\natural$ and inverses of $\natural$ and $\natural D$. From the relations (3), (4) and (5) in Remark 6.2, $\natural(P^3(3), \lambda_2) = \natural D \circ \natural \Delta_3$, $\natural(P^3(3), \lambda_1) = \natural - 1 \circ \natural(P^3(2), \lambda_1)$ and so on. Then the proof is complete.

The topological translation of the above theorem is stated in Theorem 1.7.

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