Remarks on the principles of statistical fluid mechanics

Koji Ohkitani

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan

This is an idiosyncratic survey of statistical fluid mechanics centering on the Hopf functional differential equation. Using the Burgers equation for illustration, we review several functional integration approaches to the theory of turbulence. We note in particular that some important contributions have been brought about by researchers working on wave propagation in random media, among which Uriel Frisch is not an exception. We also discuss a particular finite-dimensional approximation for the Burgers equation.

This article is part of the theme issue “Scaling the turbulence edifice (part 1)’.

1. Introduction

In field theory, one important task is the following. Given an action integral $S[\phi]$, e.g. that of the $\phi^4$-model [1]

$$S[\phi] = \int \int dxdt \left\{ \frac{1}{2} (\phi_t^2 - |\nabla \phi|^2) - \frac{m^2}{2} \phi^2 - \lambda \phi^4 + J \phi \right\},$$

written with standard notations, we consider the partition function

$$Z(J) = N \int \exp(iS[\phi]) D[\phi].$$

The objectives are to figure out how to make sense of what the symbol means mathematically and to extract physically meaningful information from it. In view of a correspondence between field theory and statistical mechanics, such a view can be carried over to fluid mechanics. Hence in principle statistical theory of turbulence should be based on that footing.

We will revisit the formulation of the Hopf functional differential equation (hereafter FDE) [2,3] which governs statistics of fluid turbulence. It is not our intention to give an exhaustive survey of the subject matter. Rather we will restrict our attention to one of its facets, that is, functional
integration approaches to the theory of turbulence. In §2, we compare some FDEs for statistical fluid mechanics with historical notes. We discuss decaying turbulence in §3 and forced turbulence in §4. In §5, we consider the Hopf equation for a discrete model. Section 6 is a brief summary. This is basically a survey article, except for §4(b) and 5.

2. The birth of the Hopf equation

The primary objective of so-called ergodic theory is to show that the ensemble average equals the time average for dynamical systems of physical interest. In the 1930s, there was substantial progress in the field, including the proofs of the mean ergodic theorem (strong in $L^2$) [4] and the individual ergodic theorem (strong in $L^1$) [5]. Hopf extended von Neumann’s work to weak convergence and published a monograph [6]. For its brief history, e.g. [7].

In the development of $L^2$-theory, an innocent-looking but far-reaching idea was put forward in [8], that is, the introduction of a kind of composition operators [9]. That enabled: (i) linearization by considering observables and (ii) spectral analysis due to unitarity of operators. The price we have to pay is that we need to handle an infinite-dimensional space. For an observable $f(\cdot)$ and a phase point $P$, the Koopman operator

$$U_tf(P) = f(\phi_tP),$$

where $\phi_t$ denotes a phase flow. In the language of probability theory, the Koopman operator is the adjoint group to the Perron–Frobenius operator and the process of lifting of a dynamical system to a group of operators is called Koopmanism [12]. Hopf did not mention explicitly the notion of the Koopman operator in [2], but his formulation of statistical fluid mechanics clearly rests on it, as the Hopf–Foias approach can be regarded as such a manifestation.

Statistical study of the Navier–Stokes equations was initiated on the basis of a functional differential equation that now bears his name [2,3]. There the functional equation was heuristically derived for the characteristic functional of the velocity field. On the other hand, mathematical study of the Navier–Stokes equations was initiated in [13,14] on the basis of the Liouville equation, a functional equation for the probability measure. Further developments in this direction were summarized in [15], including the so-called first integral method. As a rule of thumb, table 1 shows a rough classification of several representations in mechanics. A related but different topic which we will not discuss here is the so-called Koopman mode analysis. There has been significant progress recently therein and interested readers ought to consult e.g. [16–18].

For the sake of simplicity, to illustrate the main ideas in this article, we will make use of the Burgers equation in $\mathbb{R}^1$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = v \frac{\partial^2 u}{\partial x^2},$$

$$u(x,0) = u_0(x),$$

where $u_0(x)$ is a smooth initial condition. They are transposable at least formally to the Navier–Stokes equations. Thus, when we say turbulence in this paper, we mean a flow of the 1D model.

Hopf considered the characteristic functional of the velocity field

$$\Phi[\theta(x),t] = \left\langle \exp \int u(x,t)\delta\theta(x)\,dx \right\rangle,$$

where $\langle \cdot \rangle$ denotes an ensemble average with respect to $u_0$. The governing equation he derived in [2] reads

$$\frac{\partial \Phi}{\partial t} = i \int \theta(x) \frac{\partial}{\partial x} \delta\theta(x) \frac{\partial^2 \Phi}{\partial x^2} \,dx + v \int \theta(x) \frac{\partial}{\partial x} \delta\theta(x) \frac{\partial \Phi}{\partial x} \,dx.$$

The same concept was introduced independently by A. Weil in connection with [10]. See also [11], p. 32. Paying attention to an invariant measure in Hamiltonian systems, Weil realized that such a system defines a one-parameter group of unitary transformations in an $L^2$-space with respect to the measure.
This is a functional differential equation (FDE). We recall that when the functional varies as
\[ \delta \Phi[\theta(x)] \approx \int A(x) \delta \theta(x) \, dx \]
for a small variation \( \delta \theta(x) \), the functional derivative \( \delta \Phi / \delta \theta(x) \) dx is defined by \( A(x) \).

Let us write down back to back the Liouville equation, which is also known as the Hopf–Foias equation
\[ \frac{d}{dt} \Psi[u] \, d\mu_t(u) = \int \left( \nu \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}, \frac{\delta \Psi}{\delta u(x)} \, dx \right) \, d\mu_t(u). \] (2.4)

This can be derived as follows [19]. Define a mapping \( T : u \rightarrow u + (\nu u_{xx} - uu_x) \, dt \) and consider an observable \( \Psi[u] \). By \( \mu_{t+dt} = \mu_t \circ T^{-1} \) (Hopf identity), we have \( \int \Psi[u] \, d\mu_{t+dt} = \int (\Psi \circ T) \, d\mu_t \). Now we have
\[ \text{LHS} \approx \int \Psi[u] \, d\mu_t + \frac{d}{dt} \left( \int \Psi[u] \, d\mu_t \right) \, dt, \]
whereas
\[ \text{RHS} \approx \int \Psi \left[ u + \left( \nu \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x} \right) \, dt \right] \, d\mu_t \]
\[ \approx \int \Psi[u] \, d\mu_t + \frac{d}{dt} \left( \int \Psi[u] \, d\mu_t \right) \, dt. \]

Hence the Liouville equation follows.

It should be noted that this equation is valid for any observables. In particular, to obtain the original form of the Hopf functional equation we may simply take \( \Psi[u] = \exp(\langle u, \theta \rangle) \), where \( \langle u, \theta \rangle \) denotes an inner-product, \( (u, \theta) = u \cdot \theta = \int u(x, t) \theta(x) \, dx \).

We note in passing that the equation for probability density function \( F = \Pi_k \delta(u_k - u_k(x, t)) \) was written down in [1, 20], where the symbolic product over \( k \) denotes the one over all \( x \). Adapting to the Burgers equation, it takes the following form:
\[ \frac{\partial F}{\partial t} = \int \frac{\delta}{\delta u(x)} \left( \nu \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x} \right) \, F \, dx. \] (2.5)

This is equivalent to the Hopf–Foias equation (2.4). Anyway, in the Liouville-type equations the underlying PDE appears explicitly. A complaint made by mathematical analysts about the Hopf equation is that they do not see PDEs in it. Actually PDEs do appear in the path integral representation of their formal solution of the latter.

3. Decaying turbulence

(a) Rosen’s action integral

Apparently the first path integral representation of this ilk is due to [21]. We begin considering freely decaying turbulence. A formal and symbolic solution was constructed to the Hopf equation
in [21]. See also [22]. Making use of the linearity of the Hopf equation we write the solution as

$$\Phi[\theta(x), t] = \int \Phi[\theta_0(x)]K[\theta_0(x)|\theta(x), t]D[\theta_0(x)],$$  

(3.1)

for a Green’s function \(K[\theta_0(x)|\theta(x), t]\). We will make use of the plane wave expansion of the Dirac mass

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) \, dk,$$  

(3.2)

or more precisely, in its functionally generalized form [23]

$$\delta[\psi] = \frac{1}{2\pi} \int \exp(i(\psi \cdot \phi))D[\phi].$$  

(3.3)

A symbolic expression of the Green’s function \(K[\theta_0(x)|\theta(x), t]\) was given as follows:

$$K[\theta_0(x)|\theta(x), t] = C \int_{\eta(x,0)=\theta(x)} \eta(x,t)=\theta(x) \exp \left\{ i \int_0^t dt \left( \frac{\partial \eta}{\partial \tau} + \eta Q[\xi] \right) dx \right\} D[\eta(x, \tau)]D[\xi(x, \tau)],$$  

(3.4)

where \(Q[\xi] = -\xi \cdot \xi_x + v \xi_{xx}\) and \(C\) is a formal normalization constant. In those expressions e.g. \(D[\xi] = \pi \cdot d\xi(x, t)\) denotes a fictitious (translation-invariant) measure.2

Actually the basic formula (3.3) is common to all the derivations we review in this article.

We recall how Rosen’s path integral representation is derived. Because of linearity it suffices to show that \(K\) satisfies the Hopf equation. For small \(\Delta t\), Green’s function is proportional to

$$\int \exp \left\{ \int (\eta - \eta') \cdot \xi + \Delta t Q[\xi] \right\} D[\xi]$$

$$= \int \exp \left\{ \int \eta \cdot (\eta - \eta') \right\} \exp \left\{ i\Delta t \int Q[\xi] \right\} D[\xi]$$

$$\approx \left( 1 + i\Delta t \int Q[\xi] \right) \int \exp \left\{ i\int (\eta - \eta') dx \right\} D[\xi]$$

$$= \left( 1 + i\Delta t \int dx \eta Q \left[ \frac{\delta}{i\partial\eta} \right] \right) \int \exp \left\{ i\int (\eta - \eta') dx \right\} D[\xi],$$

where we have written \(\eta = \eta(x, t + \Delta t), \eta' = \eta(x, t)\). Hence we have by (3.2)

$$K(t|t + \Delta t) = \left( 1 + i\Delta t \int dx \eta Q \left[ \frac{\delta}{i\partial\eta} \right] \right) \delta[\eta - \eta'],$$

that is,

$$\frac{\partial K}{\partial t} = i \int \eta Q \left[ \frac{\delta}{i\partial\eta} \right] K \, dx.$$  

Out of the double functional integrations, the \(D[\xi]\) integration can be carried out by the method of stationary phase [21], but the \(D[\eta]\) integration remains a challenge3 [25,26].

For the particular case of the heat equation, at least we know

$$C \int_{\eta(x,0)=\theta(x)} \exp \left\{ i \int_0^t dt \left( \frac{\partial \eta}{\partial \tau} + v \frac{\partial^2 \eta}{\partial x^2} \right) dx \right\} D[\eta(x, \tau)]D[\xi(x, \tau)] = \delta[\theta_0 - \exp(v\Delta t)\theta],$$

so that we recover \(\Phi[\theta, t] = \Phi_0[\exp(v\Delta t)\theta]\). This is a solution which represents the final period of decay in turbulence [2,3].

2We may call it ‘abstract Feynman measure’. It may be compared with the classical Wiener measure in \(\mathbb{R}^n\) and the abstract Wiener measure in \(\mathbb{R}^\infty\), both of which are well understood. Abstract Feynman measure does not exit because Feynman measure does not, see appendix A.

3If it were possible to carry out the \(D[\eta]\) integration, the resultant functional would be termed the Onsager–Machlup action in analogy with similar results for statistics of linear dynamical systems. e.g. Section VI.9 of [24].
(b) Alternative derivation for Rosen’s path integral

Rosen’s original derivation of the characteristic functional based on the Hopf equation is mostly straightforward. We recall that Rosen’s symbolic expression can be obtained \textit{without} using the Hopf equation, which is allegedly based on the idea of Novikov, §29.5 of [25]. See also [27], p. 252. The rationale for including this is that the second derivation stands valid when we have external forcing which is not necessarily white-in-time Gaussian. For simplicity, we illustrate it here using the unforced equation.

We formally rewrite \(\partial u/\partial t = Q[u]\) as

\[
u(x, t) = u_0(x) + \int_0^t Q[u(x, \tau)] d\tau.
\]

In fact, this virtually trivial recasting leads to a non-trivial result. Consider

\[
\exp (i\vartheta(x) \cdot u(x, t)) = \exp \left\{ i\vartheta(x) \cdot \left( u_0(x) + \int_0^t Q[u(x, \tau)] d\tau \right) \right\} \\
= \int \exp \left\{ i\vartheta(x) \cdot \left( u_0(x) + \int_0^t Q[v(x, \tau)] d\tau \right) \right\} \exp \left\{ i \int_0^t (u(x, \tau) - v(x, \tau)) w(x, \tau) d\tau \right\} D[v] D[w],
\]

by virtue of (3.3). Plugging \(u(x, \tau) = u_0(x) + \int_0^\tau Q[v(x, \tau')] d\tau'\) into the above equation, we have

\[
\exp (i\vartheta(x) \cdot u(x, t)) = \int \exp \left\{ i\vartheta(x) \cdot \left( u_0(x) + \int_0^t Q[v(x, \tau)] d\tau \right) \right\} \\
\times \exp \left\{ i \int_0^t \left( u_0(x) + \int_0^\tau Q[v(x, \tau')] d\tau' - v(x, \tau) \right) w(x, \tau) d\tau \right\} D[v] D[w].
\]

Integrating by parts we find

\[
\int_0^t d\tau \int_0^\tau \left[ \int_0^\tau Q[v(x, \tau')] d\tau' \right] w(x, \tau) = \left[ \int_0^\tau Q[v(x, \tau')] d\tau' \int_0^\tau w(x, \tau') d\tau' \right]_{\tau=0}^t - \int_0^t d\tau \int_0^\tau w(x, \tau') d\tau' Q[v(x, \tau)] \\
= + \int_0^t d\tau \int_\tau^t w(x, \tau') d\tau' Q[v(x, \tau)],
\]

and hence deduce

\[
e^{i(u, \vartheta)} = \int \exp \left\{ iu_0(x) \cdot \left( \vartheta(x) + \int_0^t w(x, \tau') d\tau' \right) \right\} \exp \left\{ i \int_0^t d\tau Q[v] \cdot \left( \vartheta(x) + \int_\tau^t w(x, \tau') d\tau' \right) \right\} \times \exp \left\{ -i \int_0^t v(x, \tau) \cdot w(x, \tau) d\tau \right\} D[v] D[w].
\]

Averaging over \(u_0\), we find

\[
\Phi[\theta(x), t] = \int \Phi_0 \left[ \left\{ \theta(x) + \int_0^t w(x, \tau') d\tau' \right\} \right] \times \exp \left\{ i \int_0^t d\tau \left( Q[v] \cdot \left( \theta(x) + \int_\tau^t w(x, \tau') d\tau' \right) - v(x, \tau) \cdot w(x, \tau) \right) \right\} D[v] D[w].
\]

We rename \(v(x, \tau) = \zeta(x, \tau)\) and \(\theta(x) + \int_0^t w(x, \tau') d\tau' = \eta(x, \tau)\) with \(\eta(x, t) = \theta(x), \quad \eta(x, 0) = \theta(x) + \int_0^t w d\tau'\) and \(d\eta(x, \tau) = -w(x, \tau) d\tau\). Replacing \(D[v] = D[\eta]\) and \(D[w] \propto D[\eta]\) and subsuming extra factors into the prefactor, we recover (3.1), (3.4) with the lower limit of integration unspecified. \(\blacksquare\)
4. Stationary turbulence

(a) Generalized Hopf equation

We now turn our attention to statistically steady turbulent flows driven by an external forcing. The Burgers equation with the forcing reads

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f(x, t). \]

For simplicity, let us consider a white-in-time Gaussian random force \( f(x, t) \) [20], which satisfies

\[ \langle f(x, t)f(y, s) \rangle = F(x - y)\delta(t - s). \]

Here, \( F(r) \) denotes the correlation function and \( \langle \cdot \rangle \) a statistical average. That way we can decouple the joint probability distributions of the velocity and the forcing.

The Hopf equation was generalized to accommodate random forcing in [28]. This is based on the following Furutsu–Novikov–Donsker theorem [28–31], which states for any functional \( R \)

\[ \langle f(x)R[f] \rangle = \int F(x - x') \left( \frac{\delta R[f]}{\delta x'} \right) dx'. \]  

(4.1)

The generalized Hopf equation turned out to be

\[ \frac{\partial \Phi}{\partial t} = \frac{i}{2} \int \theta(x) \frac{\partial}{\partial x} \frac{\delta^2 \Phi}{\delta \theta(x) \delta x^2} dx + \nu \int \theta(x) \frac{\partial^2}{\partial x^2} \frac{\delta \Phi}{\delta \theta(x)} dx - \frac{1}{2} \int \theta(x) F(x - y) \theta(y) dx dy \Phi. \]  

(4.2)

For the Hopf equation subject to more general forcing, see [32].

Just like the original Hopf equation, it seems that this equation has not been put to practical use. Here let us see how we may modify Rosen’s functional expression to accommodate the forcing and verify that it satisfies the generalized Hopf equation. To that end the following paragraph by Uriel Frisch would be instructive, which we quote here verbatim from [33].

We consider first the randomly perturbed heat equation

\[ \frac{\partial}{\partial t} \psi(t, r) = \Delta \psi + \mu(r) \psi, \quad \psi(0, r) = \delta(r), \]  

(3.44)

where \( \mu(r) \) is a centred random Gaussian function with covariance

\[ \Gamma(r, r') = \mathbb{E}[\mu(r)\mu(r')]. \]  

(3.45)

For any realization of \( \mu(r) \), equation (3.44) can be solved by means of Kac’s formula [cf. Appendix Eq. (A 12)]

\[ \psi(t, r) = \mathbb{E}_W \left\{ \delta(\rho(t) - r) \exp \left\{ \int_0^t \mu(\rho(\tau)) d\tau \right\} \right\}. \]  

(3.46)

To find \( \mathbb{E}_\mu[\psi(t, r)] \), we must average again over \( \mu \). Assuming that the \( \mu \) and \( W \) averages can be interchanged, we obtain

\[ \mathbb{E}_\mu[\psi(t, r)] = \mathbb{E}_W \left\{ \delta(\rho(t) - r) \mathbb{E}_\mu \left\{ \exp \left\{ \int_0^t \mu(\rho(\tau)) d\tau \right\} \right\} \right\}. \]  

(3.47)

For a fixed Brownian path \( \rho(\tau), \ 0 \leq \tau \leq t \), the random variable

\[ \phi = \int_0^t \mu(\rho(\tau)) d\tau, \]  

(3.48)

is a linear functional of the Gaussian random function \( \mu \); hence

\[ \mathbb{E}_\mu[\exp \phi] = \exp \left\{ \frac{1}{2} \mathbb{E}_\mu[\phi^2] \right\} = \exp \left\{ \frac{1}{2} \int_0^t \int_0^t \Gamma(\rho(\tau), \rho(\tau')) d\tau d\tau' \right\}. \]  

(3.49)
Equation (3.47) can now be rewritten as

\[
\mathcal{E}_\mu(\Psi(t, \tau)) = \mathcal{E}_W \left\{ \delta(\mathbf{r}(t) - \mathbf{r}) \exp \left\{ \frac{1}{2} \int_0^t \int_0^t \Gamma(\mathbf{r}(\tau_1), \mathbf{r}(\tau_2)) \, d\tau_1 \, d\tau_2 \right\} \right\}.
\]

(b) Linking Rosen’s action integral to Wyld’s

Note that the action integral for turbulence driven by Gaussian white forcing is called Wyld functional [34] and appears in the so-called Martin–Siggi–Rose formalism (hereafter, MSR) [35]. The formalism builds a field theory associated with statistical mechanics of a given dynamical system. That appears in a now standard field-theoretic treatment of turbulence, particularly instanton theory. Such an action integral is sometimes called after the names of Wyld [4] and Rosen e.g. [37]. The link is well expected in the folklore, but best described here explicitly with some justifications.

Before that it is suitable to refer some cornerstone papers on the theory of turbulence. The space–time characteristic functional was introduced in [38], the path integral representation for the MSR theory was derived in [39,40] and a connection to the Kolmogorov theory was sought in [41]. See also §9.5 of [42] for a succinct survey.

We will see that (i) the symbolic expression associated with Rosen’s action integral subject to random forcing yields with the MSR action and (ii) it is indeed a solution of Novikov’s generalized Hopf equation. To include forcing, we replace RHS of the Burgers equation \( Q \) with the MSR action and (ii) it is indeed a solution of Novikov’s generalized instanton theory. Such an action integral is sometimes called after the names of Wyld [4] and Rosen e.g. [37]. The link is well expected in the folklore, but best described here explicitly with some justifications.

Let \( K[\theta_0(x)|\theta(x), \mathbf{r}] = C \int_{\eta(x,0)=\theta_0(x)}^{\eta(x,t)=\theta(x)} \exp \left\{ i \int_0^t \int_0^t \left( \frac{\partial \eta}{\partial \tau} \xi + \eta \xi \right) \, d\tau \, dx \right\} \times \exp \left\{ i \int_0^t \int_0^t \eta(x)f(x) \, dx \right\} \mathcal{D}[\eta] \right\}.

Randomize \( f(x, t) \) and average over \( \mathcal{D}[f \mathbf{r}, t] \) with the weight

\[
\exp \left\{ -\frac{1}{2} \int_0^t \int_0^t \int_0^t \eta(x)f(x) \, d\mathbf{r} \right\},
\]

where \( \kappa(x, y) = \kappa(x - y) \) is an inverse of \( F(x, y) = \kappa(x - y) \). We find

\[
K = C \int_{\eta(x,0)=\theta_0(x)}^{\eta(x,t)=\theta(x)} \exp \left\{ i \int_0^t \int_0^t \left( \frac{\partial \eta}{\partial \tau} \xi + \eta \xi \right) \, d\tau \, dx \right\} \times \exp \left\{ i \int_0^t \int_0^t \eta(x)f(x) \, dx \right\} \mathcal{D}[\eta] \right\}.
\]

It can be seen to be equivalent to the Wyld functional in the MSR scheme. First, integrating by parts under the constraints \( \xi(x,0) = \xi(x, t) = 0 \), we have

\[
K = C \int_{\eta(x,0)=\theta_0(x)}^{\eta(x,t)=\theta(x)} \exp \left\{ -i \int_0^t \int_0^t \eta(x) \left( \frac{\partial \xi}{\partial \tau} - Q[\xi] \right) \, d\tau \, dx \right\} \times \exp \left\{ i \int_0^t \int_0^t \eta(x)f(x) \, dx \right\} \mathcal{D}[\eta] \right\}.
\]

Second, to verify that the expression defined with the above kernel satisfies the generalized Hopf equation we note

\[
\frac{\partial}{\partial t} \exp \left\{ -\frac{1}{2} \int_0^t \int_0^t \eta(x, \tau_1)F(x - y)\eta(y, \tau_2) \, d\tau_1 \, d\tau_2 \right\}
\]

\[
= -\frac{1}{2} \int_0^t \eta(x, t)F(x - y)\eta(y, t) \, d\tau \exp \left\{ -\frac{1}{2} \int_0^t \int_0^t \eta(x)F(x - y)\eta(y) \, dx \, dy \right\}.
\]

The explicit functional form, however, does not appear explicitly in [36].
The additional contribution to the RHS of the Hopf equation is

\[ -\frac{C}{2} \int_{\eta(x,0) = \theta_0(x)}^{\eta(x,t) = \theta(x)} \eta(x,t)F(x-y)\eta(y,t) \, dx \, dy |_{\eta(x,t) = \theta(t)} \times \exp \left\{ -i \int_0^t \int \left( \frac{\partial \xi}{\partial \tau} - Q[\xi] \right) - \frac{1}{2} \int_0^t \int \eta(x)F(x-y)\eta(y) \, dx \, dy \right\} \mathcal{D}[\eta] \mathcal{D}[\xi]. \]

\[ = -\frac{1}{2} \int_{\eta(x,0) = \theta_0(x)}^{\theta(x)} \eta(x)F(x-y)\theta(y) \, dx \, dy \times \exp \left\{ -i \int_0^t \int \left( \frac{\partial \xi}{\partial \tau} - Q[\xi] \right) \right\} \]

\[ = -\frac{1}{2} \int_0^t \int \eta(x)F(x-y)\eta(y) \, dx \, dy \times \mathcal{D}[\eta] \mathcal{D}[\xi], \]

which indeed yields the final term on the RHS of Novikov’s equation (4.2).

(c) **Instanton theory**

The functional integrations remain merely symbolic. However, the action integral that appears in the exponent is just \((n + 1)\)-dimensional integral and well-defined \((n = 1, 2, 3)\). Hence, it is sensible to consider stationary conditions by writing down its Euler–Lagrange equations. For Rosen’s action, a pair of equations consists of the Burgers equation and its adjoint (dual)

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} = -v \frac{\partial^2 v}{\partial x^2}, \]

which is linear and runs backward in time. Statistical fluid mechanics has inherently control-theoretic aspects built-in [43].

Here, we will have a look at standard manipulations in instanton theory, following [44,45]. See also [43,46] for the numerical implementations, including the Navier–Stokes equations.

Consider a formal normalization \( \mathcal{N} = \int \delta(u_t + uu_x - vu_{xx} - f) \mathcal{D}[u] \) and the partition function

\[ Z = \int \exp \left( -\frac{1}{2} \int \int f(x,t)\kappa(x-y)f(y,t) \, dx \, dy dt \right) \mathcal{D}[f] \]

\[ = \frac{1}{\mathcal{N}} \int \delta(u_t + uu_x - vu_{xx} - f) \exp \left( -\frac{1}{2} \int \int f(x,t)\kappa(x-y)f(y,t) \, dx \, dy dt \right) \mathcal{D}[f] \mathcal{D}[u]. \]

We rewrite this by (3.3), redefining the normalization constant

\[ Z = \frac{1}{\mathcal{N}} \int \exp \left( i \int v(u_t + uu_x - vu_{xx} - f) \, dx dt \right) \]

\[ - \frac{1}{2} \int \int f(x,t)\kappa(x-y)f(y,t) \, dx \, dy dt \mathcal{D}[f] \mathcal{D}[u] \mathcal{D}[v] \]

\[ = \frac{1}{\mathcal{N}} \int \exp \left( i \int v(u_t + uu_x - vu_{xx}) \, dx dt \right) \]

\[ \times \exp \left( -i \int vf \, dx dt - \frac{1}{2} \int \int f(x,t)\kappa(x-y)f(y,t) \, dx \, dy dt \right) \mathcal{D}[f] \mathcal{D}[u] \mathcal{D}[v] \]

\[ = \frac{1}{\mathcal{N}} \int \exp \left( i \int v(u_t + uu_x - vu_{xx}) \, dx dt \right) \]

\[ \times \exp \left( -\frac{1}{2} \int \int v(x,t)F(x-y)v(y,t) \, dx \, dy dt \right) \mathcal{D}[u] \mathcal{D}[v], \]

where a functional Fourier transform of the Gaussian function has been taken in the final line.

(d) **Yet another form of path integral**

A simple conversion of the Burgers equation into an integral equation was considered in 3(b). By virtue of Duhamel principle, we can also convert the Burgers equation into another integral
equation of the following form:

\[ u - G : uu = \dot{u}, \]

where \( \dot{u} = g * u_0 \) is a heat flow defined with the heat kernel \( g(x, t) = (4\pi vt)^{-1/2} \exp(-x^2/4vt) \).

We have also denoted

\[ G : uu = \int_0^t ds \int_{-\infty}^{\infty} dy G(x - y, t - s)u(y, s)^2, \]

where \( G(x, t) \equiv -\frac{1}{2} \beta g(x, t) \). On this basis, the following representation can be derived as in [47,48]

\[ \Phi[\theta] = \int \exp \left( i \int ((\theta - v) \cdot u + v \cdot G : uu) \, dx - A[u] \right) \hat{\Phi}[v] D[v] / D[u], \tag{4.3} \]

where \( A[u] \) denotes an unknown functional to account for possible breakdown of classical solutions and \( J = \det(\delta \dot{u}/\delta u) \).

In fact, we can confirm that \( J \equiv 1 \) as in the 3D incompressible case. We also have \( A[u] \equiv 0 \) because the Burgers equation is globally well-posed. Hence we find

\[ \Phi[\theta] = \int \exp \left( i \int ((\theta - v) \cdot u + v \cdot G : uu) \, dx \right) \hat{\Phi}[v] D[v] / D[u]. \tag{4.4} \]

This formula connects the characteristic functional \( \hat{\Phi}[v] \) of the heat equation with that of the Burgers equation \( \Phi[\theta] \). If we discard the second term of the exponent, it is clear that we recover \( \Phi[\theta] = \hat{\Phi}[v] \) by the property of the Dirac measure (3.3). This form of path integral is valid only for a viscous fluid, in contrast to the previous one which can cover an inviscid fluid as well. It is of interest to study whether and how the above expression is simplified further when we make use of the Cole–Hopf transform which linearizes the Burgers equation.

5. Hopf equation for a discrete Burgers equation

As seen above the Hopf functional equation faces a daunting mathematical difficulty because of the infinite-dimensional character of the functional differential equation. To alleviate the difficulty at least partially, we consider a discrete version of the Burgers equation in this section so that its ‘Hopf equation’ is a PDE rather than an FDE.

There are some works in the similar spirit [49–51]. The master equation was presented for a discrete Burgers equation, but that does not respect Cole–Hopf transform. See also [52]. Here we take up a discretization of Burgers equation which respects the Cole–Hopf transformation and present asymptotic analysis using a series expansion in time.

(a) Discrete Burgers equation

The following version of discretization was proposed in [53] for the Burgers equation:

\[
\frac{du(x, t)}{dt} = -\frac{2\nu^2}{\epsilon^2} \Delta_x \left\{ \exp \left( -\frac{\epsilon}{2\nu} u(x, t) \right) + \exp \left( \frac{\epsilon}{2\nu} u(x - \epsilon, t) \right) - 2 \right\} \\
= -\frac{2\nu^2}{\epsilon^3} \left\{ \exp \left( -\frac{\epsilon}{2\nu} u(x + \epsilon, t) \right) - \exp \left( -\frac{\epsilon}{2\nu} u(x, t) \right) + \exp \left( \frac{\epsilon}{2\nu} u(x, t) \right) - \exp \left( \frac{\epsilon}{2\nu} u(x - \epsilon, t) \right) \right\}, \tag{5.1} 
\]

where \( \epsilon \) denotes spatial spacing and \( \Delta_x f(x) = (f(x + \epsilon) - f(x))/\epsilon \). Writing \( u_n(t) = u(x_n, t) \) with \( x_n = n\epsilon, \ (N = 1, \ldots, N) \), it can also be written

\[
\frac{du_n}{dt} = -\frac{2\nu^2}{\epsilon^3} \Delta_n \left\{ \exp \left( -\frac{\epsilon}{2\nu} u_n \right) + \exp \left( \frac{\epsilon}{2\nu} u_{n-1} \right) - 2 \right\}, \tag{5.2} 
\]

where \( \Delta_n f_n = f_{n+1} - f_n \) and periodic boundary conditions \( f_N = f_0, f_{N+1} = f_1 \) are assumed.
Consider the characteristic function \( \Phi[\theta_n] = \left( \exp i \sum_{n=1}^{N} u_n(t) \theta_n \right) \), then its governing equation can be derived as

\[
\frac{\partial \Phi}{\partial t} = -\frac{2\nu^2 i}{\epsilon^2} \sum_{n=1}^{N} \theta_n \Delta_n \left( \Phi \left[ \theta_1, \ldots, \theta_n + \frac{\epsilon i}{2\nu}, \ldots, \theta_N \right] \right. \\
+ \left. \Phi \left[ \theta_1, \ldots, \theta_{n-1} - \frac{\epsilon i}{2\nu}, \ldots, \theta_N \right] - 2\Phi[\theta_1, \ldots, \theta_N] \right). \tag{5.3}
\]

It is readily checked that (5.3) reduces to (2.3) in the limit of small \( \epsilon \). Using exponential operators, we can recast it as

\[
\frac{\partial \Phi}{\partial t} = -\frac{2\nu^2 i}{\epsilon^2} \sum_{n=1}^{N} \theta_n \Delta_n \left\{ \exp \left( \frac{\epsilon i}{2\nu} \theta_n \right) + \exp \left( -\frac{\epsilon i}{2\nu} \theta_{n-1} \right) - 2 \right\} \Phi,
\]

\[
= \frac{2\nu^2 i}{\epsilon^2} \sum_{n=1}^{N} \left( \Delta_n \theta_{n-1} \right) \left\{ \exp \left( \frac{\epsilon i}{2\nu} \theta_n \right) + \exp \left( -\frac{\epsilon i}{2\nu} \theta_{n-1} \right) - 2 \right\} \Phi,
\]

where the final line follows from ‘summation by parts’ and \( \tilde{\Delta}_n \theta_n = (1/\epsilon) \Delta_n \theta_n, \theta_0 = \theta_N, \theta_{N+1} = \theta_1 \) (by periodicity). We can then write its formal solution

\[
\Phi = \exp \left[ \frac{2\nu^2 i t}{\epsilon^2} \sum_{n=1}^{N} (\Delta_n \theta_{n-1}) \left\{ \exp \left( \frac{\epsilon i}{2\nu} \theta_n \right) + \exp \left( -\frac{\epsilon i}{2\nu} \theta_{n-1} \right) - 2 \right\} \right] \Phi[\theta_1, \ldots, \theta_N]. \tag{5.4}
\]

Expanding in a power series in time, we have

\[
\Phi = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{2\nu^2 i t}{\epsilon^2} \right)^k \left[ \sum_{n=1}^{N} (\Delta_n \theta_{n-1}) \left\{ \exp \left( \frac{\epsilon i}{2\nu} \theta_n \right) + \exp \left( -\frac{\epsilon i}{2\nu} \theta_{n-1} \right) - 2 \right\} \right]^k \Phi[\theta_1, \ldots, \theta_N].
\]

These operators are non-commutative, e.g.

\[
\exp \left( \frac{\epsilon i}{2\nu} \theta_n \right) (\theta_n - \theta_{n-1}) = (\theta_n - \theta_{n-1}) \exp \left( \frac{\epsilon i}{2\nu} \theta_n \right) + \frac{\epsilon i}{2\nu},
\]

but the non-commutativity appears in higher order in \( \epsilon \). Hence, by multinomial expansion, we find to leading order

\[
[\ldots]^k \Phi_0 \approx \sum_{\sum_{i=1}^{N} k_i = k, k_i \geq 0} \frac{k!}{k_1! k_2! \ldots k_N!} \Pi_{j=1}^{N} (\Delta_j \theta_{j-1})^{k_j} \left\{ \exp \left( \frac{\epsilon i}{2\nu} \theta_j \right) + \exp \left( -\frac{\epsilon i}{2\nu} \theta_{j-1} \right) - 2 \right\}^{k_j} \Phi_0,
\]

Again, by multinomial expansion

\[
[\ldots]^k \Phi_0 = \sum_{\sum_{i=1}^{N} m_i = m, m_i \geq 0} \frac{k!}{m_1! m_2! m_3!} \exp \left( \frac{i\epsilon}{2\nu} m_1 \theta_i \right) \exp \left( -\frac{i\epsilon}{2\nu} m_2 \theta_{j-1} \right) (-2)^{m_3} \Phi_0[\theta_1, \ldots, \theta_N]
\]

\[
= \sum_{\sum_{i=1}^{N} m_i = m, m_i \geq 0} \frac{k!}{m_1! m_2! m_3!} \Phi_0 \left[ \theta_1, \ldots, \theta_{j-1} - \frac{i\epsilon}{2\nu} m_2, \theta_j + \frac{i\epsilon}{2\nu} m_1, m_3, \ldots, \theta_N \right].
\]

All together we obtain

\[
\Phi \approx \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{2\nu^2 i t}{\epsilon^2} \right)^k \sum_{\sum_{i=1}^{N} k_i = k, k_i \geq 0} \frac{k!}{k_1! k_2! \ldots k_N!} \Pi_{j=1}^{N} (\Delta_j \theta_{j-1})^{k_j} \times \sum_{\sum_{i=1}^{N} m_i = m, m_i \geq 0} \frac{k!}{m_1! m_2! m_3!} \Phi_0 \left[ \theta_1, \ldots, \theta_{j-1} - \frac{i\epsilon}{2\nu} m_2, \theta_j + \frac{i\epsilon}{2\nu} m_1, \ldots, \theta_N \right].
\]
Note that in the above expressions $j$-dependence of $m_1, m_2, m_3$ has been suppressed for simplicity.

6. Summary

After making a note on the Koopman (and Weil) operator, in this survey, we reviewed some functional integration methods for handling the Hopf equation for turbulence. We have seen a number of different methods of path integral representations for both decaying turbulence (with Rosen’s action) and forced turbulence (with Wyld’s action). We have also seen how those two actions are related via a standard technique known to researchers on wave propagation, as evidenced by Uriel’s paragraph. The technique is also useful in modern instanton theory. Common to all those derivations lies the plane wave expansion of the Dirac delta measure, which is essentially a tool in linear theory (Fourier analysis).

In the final part, we discuss a particular finite-dimensional approximation for the Burgers equation and construct approximate solutions to its Hopf characteristic function.

For recent references on statistical theory of turbulence from functional integration viewpoints, it may be in order to consult e.g. [27,54–57]. For the developments of turbulence theory in general, the following references are also of interest [58,59].

Data accessibility. This article has no additional data.

Competing interests. The author declares that he has no competing interests’.

Funding. No funding has been received for this article.

Appendix A. Non-existence of Feynman Measure

This counter-example appeared in the introduction section in [60]. The following exposition is based on [61]. See also Problem 64 of Ch. X in [62], [63], or Ch. 6 and appendix K in [64].

All we need to know is the convolution

$$p_1 * p_2(x) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp\left(-\frac{x^2}{2(\sigma_1^2 + \sigma_2^2)}\right)$$

of two Gaussian functions

$$p_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{x^2}{2\sigma_1^2}\right), p_2(x) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{x^2}{2\sigma_2^2}\right),$$

where $\sigma_1, \sigma_2 > 0$. For positive $\lambda$, we compute the $N$-fold convolution products of $(N + 1)$ functions by repeated use of it,

$$I = \left(\frac{\lambda}{2\pi\epsilon}\right)^{(N+1)/2} \int \ldots \int \exp\left(-\frac{\lambda}{2\epsilon} \sum_{l=0}^{N} (x_{l+1} - x_l)^2\right) \Pi_{l=1}^{N} dx_l$$

$$= \left(\frac{\lambda}{2\pi N\epsilon}\right)^{1/2} \exp\left(-\frac{\lambda}{2N\epsilon} (x_{N+1} - x_0)^2\right).$$

This also holds for complex-valued $\lambda$ provided that $\Re(\lambda) > 0$. If the above formula remains valid in the limit of $N \to \infty$ while $N\epsilon=\text{const}$, the path integral does make sense. Because absolute convergence implies the existence of the integral, we are led to estimate the majorant

$$|I| \leq \left(\frac{|\lambda|}{2\pi\epsilon}\right)^{(N+1)/2} \int \ldots \int \exp\left(-\frac{\Re(\lambda)}{2\epsilon} \sum_{l=0}^{N} (x_{l+1} - x_l)^2\right) \Pi_{l=1}^{N} dx_l.$$
We have
\[ J = \left( \frac{2\pi \epsilon}{\Im(\lambda)} \right)^{(N+1)/2} \left( \frac{\Im(\lambda)}{2\pi N \epsilon} \right)^{1/2} \exp \left( -\frac{\Im(\lambda)}{2N \epsilon} (x_{N+1} - x_0)^2 \right), \]
by an identity obtained by \( \lambda \to \Im(\lambda) \) in I above, and hence
\[
\left( \frac{\Im(\lambda)}{2\pi N \epsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left( -\frac{\Im(\lambda)}{2\epsilon} \sum_{l=0}^{N} (x_{l+1} - x_l)^2 \right) I_l^{N} \, dx_l
= \left( \frac{\Im(\lambda)}{2\pi N \epsilon} \right)^{1/2} \exp \left( -\frac{\Im(\lambda)}{2N \epsilon} (x_{N+1} - x_0)^2 \right).
\]
Calibrating the prefactors, we find
\[
|I| \leq \left( \frac{|\lambda|}{2\pi \epsilon} \right)^{(N+1)/2} \left( \frac{2\pi \epsilon}{\Im(\lambda)} \right)^{(N+1)/2} \left( \frac{\Im(\lambda)}{2\pi N \epsilon} \right)^{1/2} \exp \left( -\frac{\Im(\lambda)}{2N \epsilon} (x_{N+1} - x_0)^2 \right)
= \left( \frac{|\lambda|}{\Im(\lambda)} \right)^{N/2} \left( \frac{|\lambda|}{2\pi N \epsilon} \right)^{1/2} \exp \left( -\frac{\Im(\lambda)}{2N \epsilon} (x_{N+1} - x_0)^2 \right).
\]
When \( \Im(\lambda) \neq 0 \), we have \( \frac{|\lambda|}{\Im(\lambda)} > 1 \). Therefore, the final line diverges in the limit of \( N \to \infty \) with \( N \epsilon \) held fixed.

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