The further analytical discussions on the $U(1)$ gauged Q-balls with $N$-power potential

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Abstract
We discuss the $U(1)$ gauged Q-balls with $N$-power potential to examine their properties analytically. More numerical descriptions and some analytical consideration have been contributed to the models governed by four-power potential. We also demonstrate strictly some new limitations that the stable $U(1)$ gauged Q-balls should accept instead of estimating those with only some specific values of model variables numerically. Having derived the explicit expressions of radius, the Noether charge and energy of the gauged Q-balls, we find that these models under the potential of matter field with general power and the boundary conditions will exist instead of dispersing and decaying. The Noether charge of the large gauged Q-balls must be limited. The mass parameter of the model can not be tiny.

1. Introduction

There is considerable interest in the Q-balls as nontopological solitons. The Q-balls are some kinds of nontopological systems with defect structures possessing a conserved Noether charge because of a symmetry of the Lagrangian \([1–4]\). The systems appear in extended localized solution of models with certain self-interacting complex scalar field having the minimum energy configuration \([1–4]\). It was deduced that the Q-balls could form as a part of solitogenesis during a phase transition in the early universe and can survive now \([5–11]\). There are a lot of global Q-balls showing that the matter or energy distributes within a finite region. The spinning Q-balls are the flat spacetime limit of rotating boson stars. A limited frequency range can allow the spherically symmetric Q-balls and boson stars to exist \([12, 13]\). In those cases the coupling to gravity leads a spiral-like frequency relating to the mass and charge of boson stars \([12, 13]\). The further discussion on the interacting boson stars and Q-balls indicated that this kind of system can not be stable with sufficiently large interaction \([14]\). The signum-Gordon typed Q-balls put forward by Arodz et al are compact \([15, 16]\) and this kind of models in electrodynamics were also studied \([17]\). The analytical discussions were presented on the Q-balls with conical-shaped potential in the higher-dimensional spacetime \([18]\). Linear Q-balls with the description of a single complex scalar field were constructed \([19]\). Q-balls can become the candidates of dark matter because of their lifetime or cross section \([20, 21]\). The Q-balls were also be used to explain the origin of the baryon asymmetry \([22, 23]\) and so do the other kinds of the compact objects \([24]\). In the case of gauge-mediated supersymmetric Q-balls with large enough charge, the energy per unit charge is less than the rest mass of one particle, so this kind of Q-balls are stable instead of dispersing \([6, 20]\). Further this kind of Q-balls were explored in the inflation scenario to explain the baryon asymmetry and dark matter of the universe \([25]\). During the decay of the gravity-mediated supersymmetric Q-balls, the lightest supersymmetric particles can generate to become candidates of dark matter and so do some other stable gauge-mediated-type Q-balls \([26–29]\). The details of evolution of Q-ball were considered. In the case of gauge-mediated SUSY breaking, the appearance of unstable Q-balls is necessary for AD baryogenesis in GMSB in view of the astrophysical constraints from the stability of neutron stars. The Q-ball decay leads the gravitino dark matter \([30]\). Some efforts were devoted to the Q-balls under the thermal logarithmic potential showing the influence from the expansion of the universe and this kind of Q-balls can also be used to describe the baryon asymmetry and dark matter \([31–34]\).
In addition to the global Q-balls mentioned above, the models of gauged Q-balls also attract more attention. The non-topological solitons with gauge $U(1)$ symmetry were studied [35]. Further the so-called $U(1)$ gauged Q-balls were investigated [36]. A lot of theoretical considerations were paid to the gauged Q-balls and the valuable numerical estimations were obtained [17, 37–42]. Recently, Gulamov et al discussed the $U(1)$ gauged Q-balls with the potential involving the highest power term as $(\Phi^+\Phi)^2$ and compare their properties with those of Q-balls in the nongauged case while the two kinds of Q-balls have the scalar field potential in the same form to show the considerable difference between the two types [43]. It was shown that the Q-balls consisting of two kinds of complex scalar fields and $U(1)$ gauge field survive in the universe, one complex scalar field with positive electric charge having baryon charge, and the other negative-electric-charged field carrying lepton charge [44]. The charged boson stars made of massive complex scalar fields connecting the $U(1)$ gauge field and gravity governed by $V$-shaped scalar potential were considered in the de Sitter or anti de Sitter spacetime [45].

It is necessary to investigate the $U(1)$ gauged Q-balls under the potential containing $(\Phi^+\Phi)^2$ term analytically with the help of virial theorem. The formation and characters of several kinds of gauged Q-balls have been evaluated [43–45]. According to the analytical discussions and numerical estimation, the important properties of $U(1)$ gauged Q-balls subject to four-power potential were demonstrated [43]. Gleiser et al put forward a generalized virial relation for Q-balls with general potential in the spacetime with arbitrary dimensionality to describe the Q-balls analytically instead of numerical estimations for several values of model parameters shown as a series of curves [46]. It is necessary to generalize the consideration to the case with term $(\Phi^+\Phi)^2$ in the potential [45] for the Q-balls with interactions and certainly their field equation is not easy to be discussed just in view of the numerical calculation. We think that the reliable and explicit relations among the model parameters consisting of the gauged Q-ball are difficult to be exhibited by performing the burden numerical calculation repeatedly because the field equations for this kind of model are nonlinear. It is fundamental to find the analytical expressions for Q-ball’s charge, radius and energy, the reliable and explicit relations among the model parameters and gauged interaction for the existence of Q-balls. The numerical results for a series of fixed values of system parameters can not reveal the model properties and the influence from interaction completely. The analytical description for gauged Q-balls is more powerful. We are going to make use of these results to scrutinize how the gauged interaction and the power $N$ affect the system feature. Here we plan to revisit the $U(1)$ gauged Q-balls with the $N$-power potential [43–45] analytically in virtue of the scheme from [46] and hope to find the necessary conditions that enable this kind of models to exist. First we estimate the energy per unit charge with the help of virial relation. We investigate the existence and stability in the cases of large ball and small ones respectively.

This paper is organized as follow. We find the virial relation for the $U(1)$ gauged Q-balls with $(\Phi^+\Phi)^2$ scalar field potential at first. Secondly we perform the analytical discussion of the radius and energy of these Q-balls to exhibit their properties in the cases of large radius or small ones respectively. During the research on this kind of Q-balls, we will find the upper limit to their Noether charge. The results are listed in the end.

2. The virial relation for $U(1)$ gauged Q-balls with $(\Phi^+\Phi)^2$ scalar field potential

In order to explore the profile of the energy of $U(1)$ gauged Q-balls, we employ the virial relation [46]. We choose the Lagrangian density of the system consisting of scalar field with $U(1)$ gauged symmetry as follows [43],

$$L = (D_\mu \Phi)^+ D^\mu \Phi - V(\Phi^+\Phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

(1)

where

$$D_\mu = \partial_\mu + ieA_\mu,$$

(2)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

(3)

and $\Phi = \Phi(x)$ is a complex scalar field, gauge fields $A^\mu = A^\mu(x)$. The index $\mu = 0, 1, 2, 3$ and the signature is $(+, -, -, -)$. The field equations are [43],

$$D_\mu D^\mu \Phi + \frac{\partial}{\partial \Phi^+} V(\Phi^+\Phi) = 0,$$

(4)

$$\partial_\mu F^{\mu\nu} = ie\Phi^+ D^\nu \Phi - ie(D^\nu \Phi)^+ \Phi.$$

(5)

The potential governing the system has a generalized form [43],

$$V(\Phi^+\Phi) = M^2(\Phi^+\Phi) - \lambda(\Phi^+\Phi)^2,$$

(6)

where $M$ is the mass of scalar field and the parameter $\lambda$ is positive. It is obvious that the potential (6) has a global minimum at $\Phi = 0$ and admits the formation of Q-balls. In order to become Q-balls, this kind of system with local $U(1)$ symmetry should carry a net particle number called Q which is conserved and should keep its energy
to be smaller than QM while the complex scalar fields and gauge field satisfy some boundary conditions to let the scalar fields distribute within a finite region. The associated conserved current density is defined as [2–4, 43–45],

\[ j^\mu = -i[\Phi^+ D^\mu \Phi - \Phi (D^\mu \Phi)^+] \tag{7} \]

and the corresponding conserved charge can be given by [2],

\[ Q = \int j^\mu d^4x. \tag{8} \]

In the static case, the components of gauge field can be chosen as [43],

\[ A^\mu(x) = (A_0(x), 0, 0, 0). \tag{9} \]

We take the ansatz for the fields configurations to be spherically symmetry like [43],

\[ \Phi(x) = f(r)e^{\pm it}. \tag{10} \]

\[ A_0(r) = A_0(r) \tag{11} \]

for the lowest energy. Here \( r = |r| \). There are boundary conditions imposed on the fields describing the gauged Q-balls [43],

\[ \lim_{r \to \infty} f(r) = 0, \tag{12} \]

\[ \lim_{r \to \infty} A_0(r) = 0. \tag{13} \]

According to Lagrangian (1), the total energy of the system is [43],

\[ E = \int \mathcal{H} d^4x = 4\pi \int_0^\infty (\omega^2 - e^2A_0^2)f^2 + \left( \frac{\partial f}{\partial r} \right)^2 + V(f) - \frac{1}{2} \left( \frac{\partial A_0}{\partial r} \right)^2 r^2 dr \tag{14} \]

Here we hire the ansatz (10), (11) and

\[ V(f) = M^2 f^2 - \lambda f^4. \tag{15} \]

The total Noether charge of the U(1) gauged Q-balls is,

\[ Q = \int j^\mu d^4x \]

\[ = 8\pi \int_0^\infty (\omega + eA_0)^2 f^2 r^2 dr. \tag{16} \]

According to [46], the virial relation as a generalization of Derrick’s theorem for Q-balls can be expressed as,

\[ 3 \langle V(f^2) \rangle = \frac{3}{4 \langle f^2 \rangle} [Q^2 - 4e^2(\langle A_0 f^2 \rangle)^2] + 3e^2 \langle A_0^2 f^2 \rangle - \left\{ \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{2} \left( \frac{\partial A_0}{\partial r} \right)^2 \right\} \tag{17} \]

leading to,

\[ \frac{E}{Q} = \omega \left\{ 1 + \left( 1 + \frac{2e}{\langle A_0 f^2 \rangle} \right) \left[ 1 + \frac{6(\langle V(f^2) \rangle - e^2 \langle A_0^2 f^2 \rangle)}{2\left( \left( \frac{\partial f}{\partial r} \right)^2 - \left( \frac{\partial A_0}{\partial r} \right)^2 \right)} \right]^{-

which keeps the Q-balls’ stability. Here \( \langle \cdots \rangle = \int \cdots d^4x \). If the ratio \( \frac{E}{Q} \) is small enough, the Q-balls will not decay into several kinds of particles such as scalars, fermions etc [20, 25, 29, 30]. It should be pointed out that the gauged Q-balls can radiate because of the interactions among the scalar fields.

3. The analytical discussion on the large U(1) gauged Q-balls with \((\Phi^+ \Phi)^2\) scalar field potential

We are going to describe the U(1) gauged Q-balls in the case of huge Noether charge and radius and probe the necessary conditions imposed on this kind of Q-balls as dark matter. As the first step we follow the Coleman’s procedure [1] to estimate the Q-balls. We select the scalar field composing the Q-balls to be step functions which are equal to be constants denoted as \( f_0 \) vanishing outside the balls. For the static charged Q-balls with uniform spherical charge distribution, the nonvanishing component of gauge field can be chosen as

\[ A_0(r) = \frac{\alpha Q}{4\pi r} \left( 3 - \frac{r^2}{R^2} \right) \text{ and } A_0(r) = \frac{\alpha Q}{4\pi r} \text{ within the Q-ball and outside ones respectively and } R \text{ is Q-ball’s radius [47]. This choice satisfies the equation (13) and conditions } \frac{dA_0(r)}{dr} \big|_{r=0} = 0 \text{ from [43]. The system energy is,} \]
where \( V(f^2) = M^2 f^2 - \lambda f^N \). We estimate the energy to find that the function can keep positive.

Now we follow the procedure from [46] to introduce the following field profiles to demonstrate the true large \( U(1) \) gauged Q-balls,

\[
    f(r) = \begin{cases} 
    f_c & r < R \\
    f_c e^{\alpha(r-R)} & r \geq R 
    \end{cases}
\]

where \( \alpha \) is positive variational parameters and \( R \) represents a region where the fields of Q-balls distribute. Within the region this kind of field configuration keeps constant instead of diminishing quickly, which means that the scalar field of Q-balls can extend a little widely. According to the large-Q-ball ansatzs (20), the conserved charge is,

\[
    E = E[f, A_0] \approx \frac{\alpha_Q^2}{R^3} + \beta_Q^2 + a_1 r^2 + b_1 r^3,
\]

where

\[
    \alpha_Q^2 = \frac{3Q^2}{16\pi f_c^2},
\]

\[
    \beta_Q^2 = \frac{3Q^2 e^{\frac{M^2 f_c^2}{25\pi}}}{25\pi},
\]

\[
    a_1 = 2\pi \alpha f_c^2 + \frac{2\pi}{\alpha} (M^2 f_c^2 - \frac{2\lambda f^N}{N}),
\]

\[
    b_1 = \frac{4\pi}{3} (M^2 f_c^2 - \lambda f^N).
\]

The above approximation is performed by leaving several dominant terms in the expression of the energy and this approximation is acceptable for large Q-balls.

In order to investigate the stability of this kind of Q-balls and the constrains on them, we extremize the total energy with respect to \( R \) to determine the minimum energy. We proceed the performance to find the equation that the critical radius \( R_{cl} \) of Q-balls refers to,

\[
    3b_1 R_{cl}^4 + 2a_1 R_{cl}^3 + \beta_Q^2 R_{cl} - 3\alpha_Q^2 = 0
\]

The approximate solution to equation (27) is,

\[
    R_{cl} = \left( \frac{3}{32\pi f_c^2 b_1} \right)^{\frac{1}{2}} \xi Q^2 - \frac{a_1}{9b_1 + \frac{6e^{\frac{M^2 f_c^2}{25\pi}}}{25\pi} \left( \frac{32\pi f_c^2 b_1}{3} \right)^{\frac{1}{2}}} Q^2,
\]

where

\[
    \xi = \left[ 1 - \frac{2\beta_Q^2}{729\alpha_Q^2 b_1^2} \right] + \left[ 1 - \frac{4\beta_Q^2}{729\alpha_Q^2 b_1^2} \right]^{\frac{3}{2}} + \left[ 1 - \frac{2\beta_Q^2}{729\alpha_Q^2 b_1^2} \right] - \left[ 1 - \frac{4\beta_Q^2}{729\alpha_Q^2 b_1^2} \right]^{\frac{3}{2}} - \frac{2\beta_Q^2}{9(\alpha_Q b_1)^2}
\]

We find that the minus term involving mass parameter will become smaller with larger \( M \). In order to keep the critical radius of the Q-ball real, the root should not be negative,

\[
    1 - \frac{4\beta_Q^2}{729\alpha_Q^2 b_1^2} \geq 0
\]
We find the upper limit on the Noether charge of the gauged Q-ball, which is also consistent with the conclusion drawn with numerical estimation in the case of four-power field potential \([43]\). Our results involving the power \(N\) and the gauge coupling indicate that the Noether charge must be limited or the Q-ball radius will not be real. The expression \((31)\) declares that the upper limit will approach the infinity if the gauge coupling \(e\) vanishes, which means that no restriction on the charge \(Q\) in the global case.

Further we impose the condition \(\frac{\partial E}{\partial \alpha_{\gamma}} = 0\) into equation \((28)\) to obtain,

\[
\alpha_{c}^{2} = M^{2} - \frac{2\lambda f_{c}^{N-2}}{N} \tag{32}
\]

which requires that \(M > \left(\frac{2\lambda f_{c}^{N-2}}{N}\right)^{\frac{1}{2}}\) similar to that of \([43]\). The energy of the U(1) gauged Q-ball with the critical radius and critical parameter is,

\[
\frac{EF_{\alpha_{\gamma}=\alpha_{c}}}{Q} = 4b_{1}\left(\frac{3}{32\pi f_{c}^{2}b_{1}}\right)^{\frac{1}{2}} \left(\frac{2}{\xi_{c}^{4}} + \xi_{c}^{4}\right) + 4\pi f_{c}^{2} \alpha_{c} \left(\frac{3}{32\pi f_{c}^{2}b_{1}}\right)^{\frac{1}{2}} \xi_{c}^{4} Q^{-\frac{1}{2}}. \tag{33}
\]

The asymptotic behavior of the energy per unit charge of large charged Q-balls with critical radius \(R_{\alpha_{\gamma}}\) critical parameter \(\alpha_{c}\) and huge charge \(Q\) is,

\[
\lim_{Q \to \infty} \frac{E[F]}{Q} = 4b_{1}\left(\frac{3}{32\pi f_{c}^{2}b_{1}}\right)^{\frac{1}{2}} \left(\frac{2}{\xi_{c}^{4}} + \xi_{c}^{4}\right). \tag{34}
\]

It is obvious that the above explicit expression is finite and its result can be regulated to be lower than the kinetic energy. The energy per unit charge lower than the kinetic energy per unit charge can keep this kind of U(1) gauged Q-balls to survive instead of dispersing. The figure 1 shows the dependence of the minimum energy per unit charge of \(U(1)\) gauged Q-balls with \(N\)-power potential on charge \(Q\) for power \(N = 4, 6, 8\) respectively.

4. The analytical discussion on the small \(U(1)\) gauged Q-balls with \((\phi^{+}\phi)^{2}\) scalar field potential

Here we focus on the small \(U(1)\) gauged Q-balls with the help of scheme from \([46]\) and discuss the possibility that this kind of small balls could be dark matter. The small \(U(1)\) gauged Q-balls that we will consider is also
controlled by the potential (15) from equation (6) [43]. We bring about the Gaussian ansatz like [46],

\[ f(r) = f_c e^{-\frac{r^2}{2r^2}}, \]

where \( f_c \) is constant. This kind of function lets the field decrease directly and rapidly and certainly the size of the Q-ball will be small. Substituting the ansatz (35) into the expression (16), we write the conserved charge belonging to the small gauged Q-balls as follows,

\[ Q = \sqrt{\frac{\pi^3}{2}} \omega f_c^2 R^3 + e^{2Qf_c^2} \left[ \frac{3}{16e^2} + \frac{9}{32} \sqrt{\frac{\pi}{2}} \text{erf}(\sqrt{2}) \right], \]

where \( \text{erf}(z) \) is the error function [47]. The total energy in the case of small Q-balls is,

\[ E = E[f_c] \approx \frac{a_Q}{R^3} + b_c R + c_s R^3 + \frac{d_s}{R}, \]

where

\[ a_Q = \frac{Q^2}{\sqrt{2} \pi^2 f_c^2}, \]

\[ b_c = \frac{e^{2Qf_c^2}}{\sqrt{2} \pi^2} \left[ \frac{3}{16e^2} + \frac{9}{32} \sqrt{\frac{\pi}{2}} \text{erf}(\sqrt{2}) \right] \]

\[ \left[ \frac{27}{64e^2} - \frac{169}{128 \sqrt{\frac{\pi}{2}}} \text{erf}(\sqrt{2}) \right] + 3 \left( \frac{\pi}{2} \right) \frac{1}{f_c^2}, \]

\[ c_s = \left( \frac{\pi}{2} \right) \frac{1}{f_c^2} \left[ M f_c^2 - \left( \frac{2}{N} \right) \frac{1}{2} \right], \]

\[ d_s = \left\{ \frac{\sqrt{2}}{\pi^2} \left[ \frac{3}{16e^2} + \frac{9}{32} \sqrt{\frac{\pi}{2}} \text{erf}(\sqrt{2}) \right] - \frac{3}{20\pi} \right\} e^{2Qf_c^2}. \]

In order to establish the equation for the critical radius \( R_{c\alpha} \), we extremized the expression of the energy with respect to \( R \) like \( \frac{\partial E}{\partial R} |_{R=R_{c\alpha}} = 0 \), then

\[ 3c_s R_{c\alpha}^6 + b_c R_{c\alpha}^4 - h_c R_{c\alpha} - 3a_Q = 0 \]

showing the acceptable approximate solution as,

\[ R_{c\alpha} \approx R_0 \left[ 1 - \frac{1}{18c_s R_0^2} \left( b_c - \frac{h_c}{R_0^2} \right) \right], \]

where

\[ R_0 = \left[ \frac{2Q^2}{\pi^2 f_c^2 \left[ M f_c^2 - \left( \frac{2}{N} \right) \frac{1}{2} \right]} \right]^{\frac{1}{2}}, \]

where the magnitude must be real with \( M > \left( \frac{2}{N} \right) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \) similar to that from [43]. It is obvious that this magnitude is smaller than the previous requirement \( \left( \frac{2}{N} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right) \) in the case of larger balls. In a word the parameter \( M \) can not be too small. The minimum energy per unit charge for small \( U(1) \) gauged Q-balls can be expressed in terms of the critical radius as,

\[ \frac{E[f_c]|_{R=R_{c\alpha}}}{Q} = 2c_s \left[ \frac{1}{\sqrt{2} \pi^2 f_c^2 c_s} \right] \frac{1}{2} \left[ 1 + \frac{b_c}{2c_s} \left( \sqrt{2} \pi^2 f_c^2 c_s \right) \frac{1}{2} Q^{-\frac{1}{2}} \right] \]

\[ - \left( \frac{b_c^2}{36c_s^2} - \frac{d_s}{2c_s} \right) \left( \sqrt{2} \pi^2 f_c^2 c_s \right) \frac{1}{2} Q^{-\frac{1}{2}} + \frac{b_c d_s}{36c_s^2} \left( \sqrt{2} \pi^2 f_c^2 c_s \right) Q^{-\frac{1}{2}} \].

In fact the smaller Q-balls can also swallow more charge, so the minimum energy per unit charge is exhibited as,

\[ \lim_{Q \to \infty} \frac{E[f_c]}{Q} |_{R=R_{c\alpha}} = 2c_s \left( \frac{1}{\sqrt{2} \pi^2 f_c^2 c_s} \right) \frac{1}{2}. \]

The above magnitude is finite and can also be regulated to be small for a series of special values of model parameters to keep the small \( U(1) \) gauged Q-balls stability instead of dispersing and decaying. In figure 2, for
comparison we also select $f_c = 2$ without losing generality and show that the functions of minimum energy per unit charge of small $U(1)$ gauged Q-balls on charge $Q$ and power $N$ are similar to those of large ones. The smaller Q-balls can survive under the influence from $U(1)$ interaction.

5. Summary and conclusion

Here we study the $U(1)$ gauged Q-balls with N-power field potential strictly by means of variational estimation without solving the nonlinear field equations numerically with respect to several given values of model parameters. In particular, the analytical expressions for radius and energy of this kind of Q-ball are obtained. Our analytical discussions help us to declare that the energy per unit charge of Q-balls made of charged scalar field will not be divergent if the charge is large. The ratio of total energy and charge can be sufficiently small for the ranges of special values imposed on the Q-balls construction to keep their stability instead of dispersing or decaying, which are certainly consistent with results in the case of $N = 4$ [43]. Our analytical expressions are explicit and reliable. The radius and energy of Q-ball have the terms associated with the gauge coupling $e$ whose effect on the Q-balls’ properties is considerable and certainly the characters of gauged Q-balls are different from those of global ones. It should be indicated explicitly that the mass parameter can not be too small like $M > \left(\frac{a}{\xi^{N-2}}\right)^2$ based on our explicit expression for Q-ball radius. The Noether charge for the potential with $N = 4$ is within a region which was estimated with series of numerical consideration in [43]. With the help of scheme [46], we discover the radii of the $U(1)$ gauged Q-balls analytically and keep the balls radii to be real to obtain the explicit expression for charge’s upper limit associated with the model variable, the gauge coupling and the highest power $N$ when the gauged Q-balls under the general potential are rather large. Some further works have been proceeded.

Acknowledgments

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Figure 2. The solid, dot, dashed curves of the minimum energy per unit charge of small $U(1)$ gauged Q-balls in the $N$-power potential as functions of charge $Q$ for power $N = 4, 6, 8$ respectively.

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