Analysis of Segregated Boundary-Domain Integral Equations for Variable-Coefficient Dirichlet and Neumann Problems with General Data

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Abstract

Segregated direct boundary-domain integral equations (BDIEs) based on a parametrix and associated with the Dirichlet and Neumann boundary value problems for the linear stationary diffusion partial differential equation with a variable coefficient are formulated. The PDE right hand sides belong to the Sobolev space $H^{-1}(\Omega)$ or $\tilde{H}^{-1}(\Omega)$, when neither classical nor canonical co-normal derivatives are well defined. Equivalence of the BDIEs to the original BVP, BDIE solvability, solution uniqueness/non-uniqueness, and as well as Fredholm property and invertibility of the BDIE operators are analysed in Sobolev (Bessel potential) spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

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1 Introduction

Many applications in science and engineering can be modeled by boundary-value problems (BVPs) for partial differential equations with variable coefficients. Reduction of the BVPs with arbitrarily variable coefficients to explicit boundary integral equations is usually not possible, since the fundamental solution necessary for such reduction is generally not available in an analytical form (except for some special dependence of the
coefficients on coordinates). Using a parametrix (Levi function) introduced in [21], [18] as a substitute of a fundamental solution, it is possible however to reduce such a BVP to a system of boundary-domain integral equations, BDIEs, (see e.g. [34, Sect. 18], [36, 35], where the Dirichlet, Neumann and Robin problems for some PDEs were reduced to indirect BDIEs). However, many questions about their equivalence to the original BVP, solvability, solution uniqueness and invertibility of corresponding integral operator remained open for rather long time.

In [3, 5, 27, 6, 8], the 3D mixed (Dirichlet-Neumann) boundary value problem (BVP) for the variable-coefficient stationary diffusion PDE with a square integrable right hand side was considered. Such equations appear e.g. in electrostatics, stationary heat transfer and other diffusion problems for inhomogeneous media. The BVP has been reduced to either segregated or united direct Boundary-Domain Integral or Integro-Differential Equations, some of the which are associated with those formulated in [26].

For a function from the Sobolev space $H^1(\Omega)$, a classical co-normal derivative in the sense of traces may not exist. However, when this function satisfies a second order partial differential equation with a right-hand side from $H^{-1}(\Omega)$, the generalised co-normal derivative can be defined in the weak sense, associated with the first Green identity and an extension of the PDE right hand side to $\tilde{H}^{-1}(\Omega)$ (see [23, Lemma 4.3], [28, Definition 3.1]). Since the extension is non-unique, the co-normal derivative appears to be a non-unique operator, which is also non-linear in $u$ unless a linear relation between $u$ and the PDE right hand side extension is enforced. This creates some difficulties in formulating the boundary-domain integral equations.

These difficulties are addressed in this paper presenting formulation and analysis of direct segregated BDIE systems equivalent to the Dirichlet and Neumann boundary value problems for the divergent-type PDE with a variable scalar coefficient and a general right hand side from $H^{-1}(\Omega)$ extended when necessary to $\tilde{H}^{-1}(\Omega)$. This needed a non-trivial generalisation of the third Green identity and its co-normal derivative for such functions, which essentially extends the approach implemented in [3, 5, 27, 6, 8] for the right hand side from $L_2(\Omega)$. Equivalence of the BDIEs to the original BVP, BDIE solvability, solution uniqueness/non-uniqueness, as well as Fredholm property and invertibility of the BDIE operators are analysed in Sobolev (Bessel potential) spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

Note that our analysis is aimed not at the boundary-value problems, which properties are well-known nowadays, but rather at the BDIE systems per se. The analysis is interesting not only in its own rights but is also to be used further on for analysis of convergence and stability of BDIE-based numerical methods for PDEs, see e.g. [15, 26, 31, 30, 37, 38, 40, 43, 44].
2 Co-normal derivatives and boundary value problems

Let $\Omega$ be a bounded open three-dimensional region of $\mathbb{R}^3$. For simplicity, we assume that the boundary $\partial \Omega$ is a simply connected, closed, infinitely smooth surface. Let $a \in C^\infty(\Omega)$, $a(x) > 0$ for $x \in \Omega$. Let also $\partial x_j := \partial/\partial x_j$ ($j = 1, 2, 3$), $\partial x := (\partial x_1, \partial x_2, \partial x_3)$.

We consider the scalar elliptic differential equation, which for sufficiently smooth $u$ has the following strong form,

$$ Au(x) := A(x, \partial x) u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( a(x) \frac{\partial u(x)}{\partial x_i} \right) = f(x), \quad x \in \Omega, $$

(2.1)

where $u$ is an unknown function and $f$ is a given function in $\Omega$.

In what follows $\mathcal{D}(\Omega) = C^\infty_{\text{comp}}(\Omega)$, $H^s(\Omega) = H^s_2(\Omega)$, $H^s(\partial \Omega) = H^s_2(\partial \Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number (see, e.g., [22, 23]). We recall that $H^s$ coincide with the Sobolev–Slobodetski spaces $W^s_2$ for any non-negative $s$. We denote by $\tilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^3)$,

$$ \tilde{H}^s(\Omega) := \{ g : g \in H^s(\mathbb{R}^3), \supp g \subset \overline{\Omega} \}, $$

while $H^s(\Omega)$ denotes the space of restrictions on $\Omega$ of distributions from $H^s(\mathbb{R}^3)$,

$$ H^s(\Omega) := \{ r_\Omega g : g \in H^s(\mathbb{R}^3) \}, $$

where $r_\Omega$ denotes the restriction operator on $\Omega$. We will also use notation $g|_\Omega := r_\Omega g$. We denote by $H^s_{\partial \Omega}$ the following subspace of $H^s(\mathbb{R}^3)$ (and $\tilde{H}^s(\Omega)$),

$$ H^s_{\partial \Omega} := \{ g : g \in H^s(\mathbb{R}^3), \supp g \subset \partial \Omega \}. $$

(2.2)

From the trace theorem (see e.g. [22, 12, 23]) for $u \in H^1(\Omega)$, it follows that $\gamma^+ u \in H^{1/2}(\partial \Omega)$, where $\gamma^+ = \gamma^+_{\partial \Omega}$ are the trace operators on $\partial \Omega$ from $\Omega$. Let also $\gamma^{-1} : H^{1/2}(\partial \Omega) \rightarrow H^1(\Omega)$ denote a (non-unique) continuous right inverse to the trace operators $\gamma^+$, i.e., $\gamma^+_{\partial \Omega} \gamma^{-1} w = w$ for any $w \in H^{1/2}(\partial \Omega)$, and $(\gamma^{-1})^* : \tilde{H}^{-1}(\Omega) \rightarrow H^{-1/2}(\partial \Omega)$ is the continuous operator dual to $\gamma^{-1} : H^{1/2}(\partial \Omega) \rightarrow H^1(\Omega)$, i.e.,

$$ \langle (\gamma^{-1})^* \tilde{f}, w \rangle_{\Omega} := \langle \tilde{f}, \gamma^{-1} w \rangle_{\Omega} $$

for any $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ and $w \in H^{1/2}(\partial \Omega)$.

For $u \in H^2(\Omega)$ we can denote by $T^+$ the corresponding classical (strong) co-normal derivative operator on $\partial \Omega$ in the sense of traces,

$$ (T^+ u)(x) := \sum_{i=1}^{3} a(x) n_i(x) \gamma^+ \frac{\partial u(x)}{\partial x_i} = a(x) \gamma^+ \frac{\partial u(x)}{\partial n(x)}, $$

(2.3)

where $n^+(x)$ is the outward (to $\Omega$) unit normal vectors at the point $x \in \partial \Omega$. However the classical co-normal derivative operator is, generally, not well defined if $u \in H^1(\Omega)$ (cf. an example in Section A in Appendix).

For $u \in H^1(\Omega)$, the partial differential operator $A$ is understood in the sense of distributions,

$$ \langle Au, v \rangle_{\Omega} := - \mathcal{E}(u, v) \quad \forall v \in \mathcal{D}(\Omega), $$

(2.4)
where
\[ E(u, v) := \int_{\Omega} a(x) \nabla u(x) \cdot \nabla v(x) dx, \]
and the duality brackets \( \langle g, \cdot \rangle_{\Omega} \) denote value of a linear functional (distribution) \( g \), extending the usual \( L_2 \) dual product.

Since the set \( \mathcal{D}(\Omega) \) is dense in \( \tilde{H}^1(\Omega) \), the above formula defines a continuous operator \( A : H^1(\Omega) \rightarrow H^{-1}(\Omega) = [\tilde{H}^1(\Omega)]^* \),
\[ \langle Au, v \rangle_{\Omega} = -E(u, v), \quad \forall \ u \in H^1(\Omega), \ v \in \tilde{H}^1(\Omega). \] (5.5)
Let us consider also the different operator, \( \tilde{A} : H^1(\Omega) \rightarrow \tilde{H}^{-1}(\Omega) = [H^1(\Omega)]^* \),
\[ \langle \tilde{A}u, v \rangle_{\Omega} = -E(u, v) = -\int_{\Omega} a(x) \nabla u(x) \cdot \nabla v(x) dx = -\int_{\mathbb{R}^3} \tilde{E}[a \nabla u](x) \cdot \nabla V(x) dx \]
\[ = \langle \nabla \cdot \tilde{E}[a \nabla u], V \rangle_{\mathbb{R}^3} = \langle \nabla \cdot \tilde{E}[a \nabla u], v \rangle_{\Omega}, \quad \forall \ u \in H^1(\Omega), \ v \in H^1(\Omega), \] (5.6)
which is evidently continuous and can be written as
\[ \tilde{A}u := \nabla \cdot \tilde{E}[a \nabla u]. \] (5.7)

Here \( V \in H^1(\mathbb{R}^3) \) is such that \( r_{\Omega} V = v \) and \( \tilde{E} \) denotes the operator of extension of the functions, defined in \( \Omega \), by zero outside \( \Omega \) in \( \mathbb{R}^3 \). For any \( u \in H^1(\Omega) \), the functional \( \tilde{A}u \) belongs to \( \tilde{H}^{-1}(\Omega) \) and is an extension of the functional \( Au \in H^{-1}(\Omega) \), which domain is thus extended from \( \tilde{H}^1(\Omega) \) to the domain \( H^1(\Omega) \) for \( \tilde{A}u \).

Inspired by the first Green identity for smooth functions, we can define the generalised co-normal derivative (cf., for example, [23 Lemma 4.3], [28 Definition 3.1], [20 Lemma 2.2]).

**DEFINITION 2.1.** Let \( u \in H^1(\Omega) \) and \( Au = r_{\Omega} \tilde{f} \) in \( \Omega \) for some \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \). Then the generalised co-normal derivative \( T^+(\tilde{f}, u) \in H^{-\frac{1}{2}}(\partial \Omega) \) is defined as
\[ \langle T^+(\tilde{f}, u), w \rangle_{\partial \Omega} := \langle \tilde{f} - \tilde{A}u, \gamma^{-1} w \rangle_{\Omega} + E(u, \gamma^{-1} w) = \langle \tilde{f} - \tilde{A}u, \gamma^{-1} w \rangle_{\Omega}, \quad \forall \ w \in H^{1/2}(\partial \Omega), \] (2.8)
that is, \( T^+(\tilde{f}, u) := (\gamma^{-1})^*(\tilde{f} - \tilde{A}u) \).

By [23 Lemma 4.3], [28 Theorem 5.3], we have the estimate
\[ \|T^+(\tilde{f}, u)\|_{H^{-1/2}(\partial \Omega)} \leq C_1\|u\|_{H^1(\Omega)} + C_2\|\tilde{f}\|_{\tilde{H}^{-1}(\Omega)}, \] (2.9)
and the first Green identity holds in the following form for \( u \in H^1(\Omega) \) such that \( Au = r_{\Omega} \tilde{f} \) in \( \Omega \) for some \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \),
\[ \langle T^+(\tilde{f}, u), \gamma^+ v \rangle_{\partial \Omega} = \langle \tilde{f}, v \rangle_{\Omega} + E(u, v) = \langle \tilde{f} - \tilde{A}u, v \rangle_{\Omega} \quad \forall \ v \in H^1(\Omega). \] (2.10)

As follows from Definition 2.1, the generalized co-normal derivative is nonlinear with respect to \( u \) for a fixed \( \tilde{f} \), but still linear with respect to the couple \( (\tilde{f}, u) \), i.e.,
\[ \alpha_1 T^+(\tilde{f}_1, u_1) + \alpha_2 T^+(\tilde{f}_2, u_2) = T^+(\alpha_1 \tilde{f}_1, \alpha_1 u_1) + T^+(\alpha_2 \tilde{f}_2, \alpha_2 u_2) = T^+(\alpha_1 \tilde{f}_1 + \alpha_2 \tilde{f}_2, \alpha_1 u_1 + \alpha_2 u_2) \] (2.11)
for any complex numbers $\alpha_1, \alpha_2$.

Let us also define some subspaces of $H^s(\Omega)$, cf. [11, 23, 29, 29].

**DEFINITION 2.2.** Let $s \in \mathbb{R}$ and $A_s : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ be a linear operator. For $t \geq -\frac{1}{2}$, we introduce the space

$$H^{s,t}(\Omega; A_s) := \{ g : g \in H^s(\Omega), \ A_sg|_{\partial \Omega} = \tilde{f}_g|_{\partial \Omega}, \ \tilde{f}_g \in \tilde{H}^t(\Omega) \}$$

endowed with the norm $\| g \|_{H^{s,t}(\Omega; A_s)} := \left( \| g \|^2_{H^s(\Omega)} + \| \tilde{f}_g \|^2_{\tilde{H}^t(\Omega)} \right)^{1/2}$ and the inner product

$$(g, h)_{H^{s,t}(\Omega; A_s)} := (g, h)_{H^s(\Omega)} + (\tilde{f}_g, \tilde{f}_h)_{\tilde{H}^t(\Omega)}. \quad (2.12)$$

The distribution $\tilde{f}_g \in \tilde{H}^t(\Omega)$, $t \geq -\frac{1}{2}$, in the above definition is an extension of the distribution $A_sg|_{\partial \Omega} \in H^t(\Omega)$, and the extension is unique (if it does exist) since any distribution from the space $H^t(\mathbb{R}^3)$ with a support in $\partial \Omega$ is identical zero if $t \geq -1/2$ (see e.g. [23, Lemma 3.39], [28, Theorem 2.10]). We denote this extension as the operator $\tilde{A_s}$, i.e., $\tilde{A_s}g = \tilde{f}_g$. The uniqueness implies that the norm $\| g \|_{H^{s,t}(\Omega; A_s)}$ is well defined.

We will mostly use the operators $A$ or $\Delta$ as $A_s$ in the above definition. Note that since $Au - a\Delta u = \nabla a \cdot \nabla u \in L_2(\Omega)$ for $u \in H^1(\Omega)$, we have $H^{1,0}(\Omega; A) = H^{1,0}(\Omega; \Delta)$.

**DEFINITION 2.3.** For $u \in H^{1, -\frac{1}{2}}(\Omega; A)$, we define the canonical co-normal derivative $T^+u \in H^{\frac{1}{2}}(\partial \Omega)$ as

$$\langle T^+u, w \rangle_{\Omega} := \langle \tilde{A}_u, \gamma^{-1}w \rangle_{\Omega} + \mathcal{E}(u, \gamma^{-1}w) = \langle \tilde{A}_u - \tilde{A}u, v \rangle_{\Omega} \quad \forall \ w \in H^{\frac{1}{2}}(\partial \Omega), \quad (2.13)$$

that is, $T^+u := (\gamma^{-1})^*(\tilde{A}_u - \tilde{A}u)$.

The canonical co-normal derivatives $T^+u$ is independent of (non-unique) choice of the operator $\gamma^{-1}$, the operator $T^+ : H^{1, -\frac{1}{2}}(\Omega; A) \to H^{\frac{1}{2}}(\partial \Omega)$ is continuous, and the first Green identity holds in the following form,

$$\langle T^+u, \gamma^+v \rangle_{\partial \Omega} = \langle \tilde{A}_u, v \rangle_{\Omega} + \mathcal{E}(u, v) \quad \forall \ v \in H^1(\Omega). \quad (2.14)$$

The operator $T^+ : H^{1,t}(\Omega; A) \to H^{-\frac{1}{2}}(\partial \Omega)$ in Definition 2.3 is continuous for any $t \geq -\frac{1}{2}$. The canonical co-normal derivative is defined by the function $u$ and operator $A$ only and does not depend separately on the right hand side $\tilde{f}$ (i.e. its behaviour on the boundary), unlike the generalised co-normal derivative defined in (2.10), and the operator $T^+$ is linear. Note that the canonical co-normal derivative coincides with the classical co-normal derivative $T^+u = a \frac{\partial u}{\partial n}$ if the latter does exist in the trace sense, see [28, Corollary 3.14 and Theorem 3.16].

Let $u \in H^{1, -\frac{1}{2}}(\Omega; A)$. Then Definitions 2.1 and 2.3 imply that the generalised co-normal derivative for arbitrary extension $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ of the distribution $Au$ can be expressed as

$$\langle T^+(\tilde{f}, u), w \rangle_{\partial \Omega} = \langle T^+u, w \rangle_{\partial \Omega} + \langle \tilde{f} - \tilde{A}u, \gamma^{-1}w \rangle_{\Omega} \quad \forall \ w \in H^{\frac{1}{2}}(\Omega). \quad (2.15)$$
Let \( u \in H^1(\Omega) \) and \( v \in H^{1,0}(\Omega; A) \).

Swapping over the roles of \( u \) and \( v \) in (2.14), we obtain the first Green identity for \( v \),

\[
\mathcal{E}(u, v) + \int_{\Omega} u(x) Av(x) dx = \langle T^+ v, \gamma^+ u \rangle_{\partial \Omega}. \tag{2.16}
\]

If, in addition, \( Au = \tilde{f} \) in \( \Omega \), where \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \), then according to the definition of \( T^+(\tilde{f}, u) \) in (2.10), the second Green identity can be written as

\[
\langle \tilde{f}, v \rangle_{\Omega} - \int_{\Omega} u(x) Av(x) dx = \langle T^+(\tilde{f}, u), \gamma^+ v \rangle_{\partial \Omega} - \langle T^+ v, \gamma^+ u \rangle_{\partial \Omega}. \tag{2.17}
\]

If, moreover, \( u, v \in H^{1,0}(\Omega; A) \), then we arrive at the familiar form of the second Green identity for the canonical extension and canonical co-normal derivatives

\[
\int_{\Omega} [v(x) Au(x) - u(x) Av(x)] dx = \langle T^+ u, \gamma^+ v \rangle_{\partial \Omega} - \langle T^+ v, \gamma^+ u \rangle_{\partial \Omega}. \tag{2.18}
\]

### 3 Parametrix and potential type operators

We will say, a function \( P(x, y) \) of two variables \( x, y \in \Omega \) is a parametrix (the Levi function) for the operator \( A(x, \partial_x) \) in \( \mathbb{R}^3 \) if (see, e.g., \[21\] \[18\] \[34\] \[17\] \[36\] \[35\] \[26\])

\[
A(x, \partial_x) P(x, y) = \delta(x - y) + R(x, y), \tag{3.1}
\]

where \( \delta(\cdot) \) is the Dirac distribution and \( R(x, y) \) possesses a weak (integrable) singularity at \( x = y \), i.e.,

\[
R(x, y) = O(|x - y|^{-\kappa}) \quad \text{with} \quad \kappa < 3. \tag{3.2}
\]

It is easy to see that for the operator \( A(x, \partial_x) \) given by the left-hand side in (2.1), the function

\[
P(x, y) = \frac{1}{a(y)} P_{\Delta}(x, y) = \frac{-1}{4\pi a(y)|x - y|^3}, \quad x, y \in \mathbb{R}^3, \tag{3.3}
\]

is a parametrix, while the corresponding remainder function is

\[
R(x, y) = \nabla a(x) \cdot \nabla_x P(x, y) = -\frac{1}{a(y)} \nabla a(x) \cdot \nabla_y P_{\Delta}(x, y) = \frac{(x - y) \cdot \nabla a(x)}{4\pi a(y)|x - y|^3}, \quad x, y \in \mathbb{R}^3, \tag{3.4}
\]

and satisfies estimate (3.2) with \( \kappa = 2 \), due to the smoothness of the function \( a(x) \). Here

\[
P_{\Delta}(x, y) = \frac{-1}{4\pi |x - y|}, \quad x, y \in \mathbb{R}^n \tag{3.5}
\]

is the fundamental solution of the Laplace equation. Evidently, the parametrix \( P(x, y) \) given by (3.3) is related with the fundamental solution to the operator \( A(y, \partial_x) := a(y) \Delta(\partial_x) \) with "frozen" coefficient \( a(x) = a(y) \) and \( A(y, \partial_x) P(x, y) = \delta(x - y) \).
Let $a \in C^\infty(\mathbb{R}^3)$ and $a > 0$ a.e. \text{in} $\mathbb{R}^3$. For scalar functions $g$, for which the integrals have sense, the parametrix-based volume potential operator and the remainder potential operator, corresponding to parametrix (3.3) and to remainder (3.4) are defined as

$$P_g(y) := \int_{\mathbb{R}^3} P(x,y) g(x) \, dx, \quad y \in \mathbb{R}^3,$$

(3.6)

$$\mathcal{P}_g(y) := \int_{\Omega} P(x,y) g(x) \, dx, \quad y \in \Omega,$$

(3.7)

$$\mathcal{R}_g(y) := \int_{\Omega} R(x,y) g(x) \, dx, \quad y \in \Omega.$$

(3.8)

For $g \in H^s(\Omega)$, $s \in \mathbb{R}$, (3.6) is understood as $P_g = \frac{1}{a} P_\Delta g$, where the Newtonian potential operator $P_\Delta$ for the Laplace operator $\Delta$ is well defined in terms of the Fourier transform (i.e., as the pseudo-differential operator), on any space $H^s(\mathbb{R}^3)$. For $g \in \tilde{H}^s(\Omega)$, and any $s \in \mathbb{R}$, definitions (3.7) and (3.8) can be understood as

$$\mathcal{P}_g = \frac{1}{r_\Omega} P_\Delta g, \quad \mathcal{R}_g = -\frac{1}{r_\Omega} \nabla \cdot P_\Delta (g \nabla a),$$

(3.9)

while for $g \in H^s(\Omega)$, $-\frac{1}{2} < s < \frac{1}{2}$, as (3.9) with $g$ replaced by $\tilde{E}g$, where $\tilde{E} : H^s(\Omega) \to \tilde{H}^s(\Omega)$, $-\frac{1}{2} < s < \frac{1}{2}$, is the unique continuous extension operator related with the operator $\tilde{E}$ of extension by zero, cf. [28, Theorem 2.16].

The single and the double layer surface potential operators, are defined as

$$V_g(y) := -\int_{\partial\Omega} P(x,y) g(x) \, dS_x, \quad y \notin \partial\Omega,$$

(3.10)

$$W_g(y) := -\int_{\partial\Omega} \left[ T(x,n(x),\partial_x) P(x,y) \right] g(x) \, dS_x, \quad y \notin \partial\Omega,$$

(3.11)

where the integrals are understood in the distributional sense if $g$ is not integrable.

The corresponding boundary integral (pseudodifferential) operators of direct surface values of the single layer potential $V$ and of the double layer potential $W$, and the co-normal derivatives of the single layer potential $W'$ and of the double layer potential $L^+$ are

$$V_g(y) := -\int_{\partial\Omega} P(x,y) g(x) \, dS_x,$$

(3.12)

$$W_g(y) := -\int_{\partial\Omega} \left[ T_x^+ P(x,y) \right] g(x) \, dS_x,$$

(3.13)

$$W'(y) := -\int_{\partial\Omega} \left[ T_y^+ P(x,y) \right] g(x) \, dS_x,$$

(3.14)

$$L^+ g(y) := T^+ W_g(y),$$

(3.15)
where \( y \in \partial \Omega \).

From definitions (3.2), (3.10), (3.11) one can obtain representations of the parametrix-based potential operators in terms of their counterparts for \( a = 1 \) (i.e. associated with the Laplace operator \( \Delta \)), which we equip with the subscript \( \Delta \), cf. [3],

\[
P g = \frac{1}{a} P_\Delta g, \quad V g = \frac{1}{a} V_\Delta g, \quad W g = \frac{1}{a} W_\Delta (ag),
\]

(3.16)

\[
V g = \frac{1}{a} V_\Delta g, \quad W g = \frac{1}{a} W_\Delta (ag),
\]

(3.17)

\[
V g = \frac{1}{a} V_\Delta g, \quad W g = \frac{1}{a} W_\Delta (ag),
\]

(3.18)

\[
W' g = W' \Delta (ag) + \left[ a \frac{\partial}{\partial n} \left( \frac{1}{a} \right) \right] V g,
\]

(3.19)

\[
L^\pm g = L_\Delta (ag) + \left[ a \frac{\partial}{\partial n} \left( \frac{1}{a} \right) \right] W_\Delta^\pm (ag).
\]

(3.20)

Hence

\[
\Delta (aV g) = 0, \quad \Delta (aW g) = 0 \quad \text{in} \quad \Omega, \quad \forall g \in H^s(\partial \Omega) \quad \forall s \in \mathbb{R},
\]

(3.21)

\[
\Delta (aP g) = g \quad \text{in} \quad \Omega, \quad \forall g \in \tilde{H}^s(\Omega) \quad \forall s \in \mathbb{R},
\]

(3.22)

For \( g_1 \in H^{-\frac{1}{2}}(\partial \Omega) \), and \( g_2 \in H^{\frac{1}{2}}(\partial \Omega) \), there hold the following jump relations on \( \partial \Omega \)

\[
[V g_1(y)]^+ = V g_1(y)
\]

(3.23)

\[
[W g_2(y)]^+ = -\frac{1}{2} g_2(y) + W g_2(y),
\]

(3.24)

\[
[T(y,n(y),\partial_y)V g_1(y)]^+ = \frac{1}{2} g_1(y) + W' g_1(y),
\]

(3.25)

where \( y \in \partial \Omega \).

The jump relations as well as mapping properties of potentials and operators (3.10)-(3.8) are well known for the case \( a = \text{const} \). Employing (3.16)-(3.20), they were extended to the case of variable coefficient \( a(x) \) in [3, 5], and in addition to (3.23)-(3.25) some of them are presented in the Appendix for convenience.

4 The third Green identity and integral relations

We will apply in this section some limiting procedures (cf. [34], [17, S. 3.8]) to obtain the parametrix-based third Green identities.

THEOREM 4.1. (i) If \( u \in H^1(\Omega) \), then following third Green identity holds,

\[
u + R u + W^+ u = P \tilde{A} u \quad \text{in} \quad \Omega,
\]

(4.1)

where the operator \( \tilde{A} \) is defined in (2.7), and for \( u \in C^1(\overline{\Omega}) \),

\[
P \tilde{A} u := (\tilde{A} u, P(\cdot, y))_\Omega = -E(u, P(\cdot, y)) = -\int_\Omega a(x) \nabla u(x) \cdot \nabla_x P(x, y) \, dx.
\]

(4.2)
(ii) If \( Au = \tilde{f}|_\Omega \) in \( \Omega \), where \( \tilde{f} \in \widetilde{H}^{-1}(\Omega) \), then the generalised third Green identity takes form,

\[
 u + R u - V T^+(\tilde{f}, u) + W \gamma^+ u = P \tilde{f} \quad \text{in } \Omega. \tag{4.3}
\]

Proof. (i) Let first \( u \in D(\overline{\Omega}) \). Let \( y \in \Omega \), \( B_\epsilon(y) \subset \Omega \) be a ball centered in \( y \) with sufficiently small radius \( \epsilon \), and \( \Omega_\epsilon := \Omega \setminus \overline{B_\epsilon(y)} \). For the fixed \( y \), evidently, \( P(\cdot, y) \in D(\overline{\Omega_\epsilon}) \subset H^{1,0}(A; \Omega_\epsilon) \) and has the coinciding classical and canonical conormal derivatives on \( \partial \Omega_\epsilon \). Then from (4.1) and the first Green identity (2.1) employed for \( \Omega_\epsilon \) with \( v = P(\cdot, y) \) we obtain

\[
 - \int_{\partial B_\epsilon(y)} T^+_x P(x, y) \gamma^+ u(x) dS(x) - \int_{\partial \Omega} T^+_x P(x, y) \gamma^+ u(x) dS(x) + \int_{\Omega_\epsilon} u(x) R(x, y) dx = - \int_{\Omega_\epsilon} a(x) \nabla u(x) \cdot \nabla_x P(x, y) dx. \tag{4.4}
\]

Taking limits as \( \epsilon \to 0 \), equation (4.4) reduces to the third Green identity (4.1)+(4.2) for any \( u \in D(\overline{\Omega}) \). Taking into account the density of \( D(\overline{\Omega}) \) in \( H^1(\Omega) \), and the mapping properties of the integral potentials, see Appendix, we obtain that (4.1) holds true also for any \( u \in H^1(\Omega) \).

(ii) Let \( \{ \tilde{f}_k \} \in D(\Omega) \) be a sequence converging to \( \tilde{f} \) in \( \widetilde{H}^{-1}(\Omega) \) as \( k \to \infty \). Then, according to Theorem [3.1] there exists a sequence \( \{ u_k \} \in D(\overline{\Omega}) \) converging to \( u \) in \( H^1(\Omega) \) such that \( Au_k = r_\Omega \tilde{f}_k \) and \( T^+(u_k) = T^+(\tilde{f}_k, u_k) \) converges to \( T^+(\tilde{f}, u) \) in \( H^{-\frac{1}{2}}(\partial \Omega) \). For such \( u_k \) we have by (4.2) and (2.10),

\[
 P A u_k(y) = \frac{1}{a(y)} \nabla_y \cdot \int_\Omega a(x) P_\Delta(x, y) \nabla u_k(x) dx
\]

\[
 = - \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} a(x) \nabla u_k(x) \cdot \nabla_x P(x, y) dx = - \lim_{\epsilon \to 0} E_{\Omega_\epsilon}(u_k, P(\cdot, y))
\]

\[
 = \lim_{\epsilon \to 0} \left[ \int_{\Omega_\epsilon} \tilde{f}_k P(x, y) dx - \int_{\partial B_\epsilon(y)} P(x, y) T^+ u_k(x) dS(x) - \int_{\partial \Omega} P(x, y) T^+ u_k(x) dS(x) \right] = P \tilde{f}_k + V T^+ u_k(y).
\]

Taking limits as \( k \to \infty \), we obtain \( PAu(y) = P \tilde{f} + V T^+(\tilde{f}, u) \), which substitution to (4.1) gives (4.3). \( \Box \)

For some functions \( \tilde{f}, \Psi, \Phi \), let us consider a more general "indirect" integral relation, associated with (4.3),

\[
 u + R u - V \Psi + W \Phi = P \tilde{f} \quad \text{in } \Omega. \tag{4.5}
\]

The following statement extends Lemma 4.1 from [3], where the corresponding assertion was proved for \( \tilde{f} \in L_2(\Omega) \).

**Lemma 4.2.** Let \( u \in H^1(\Omega), \Psi \in H^{-\frac{3}{2}}(\partial \Omega), \Phi \in H^{\frac{3}{2}}(\partial \Omega) \), and \( \tilde{f} \in \widetilde{H}^{-1}(\Omega) \) satisfy (4.5). Then

\[
 Au = r_\Omega \tilde{f} \quad \text{in } \Omega, \tag{4.6}
\]

\[
 r_\Omega V(\Psi - T^+(\tilde{f}, u)) - r_\Omega W(\Phi - \gamma^+ u) = 0 \quad \text{in } \Omega, \tag{4.7}
\]

\[
 \gamma^+ u + \gamma^+ R u - V \Psi - \frac{1}{2} \Phi + \mathcal{W} \Phi = \gamma^+ P \tilde{f} \quad \text{on } \partial \Omega, \tag{4.8}
\]
\[ T^+(\tilde{f}, u) + T^+R u - \frac{1}{2} \Psi - W'\Psi + L^+\Phi = T^+(\tilde{f} + \tilde{E} R_s \tilde{f}, \mathcal{P} \tilde{f}) \] on \( \partial \Omega, \quad (4.9) \]

where \( R_s \tilde{f} \in L_2(\Omega) \) is defined as
\[ R_s \tilde{f} := -\sum_{j=1}^{3} \partial_j[(\partial_j a)\mathcal{P} \tilde{f}]. \quad (4.10) \]

**Proof.** Subtracting (4.5) from identity (4.1), we obtain
\[ V\Psi(y) - W(\Phi - \gamma^+ u)(y) = \mathcal{P}[\tilde{A}u - \tilde{f}](y), \quad y \in \Omega. \quad (4.11) \]

Multiplying equality (4.11) by \( a(y) \), applying the Laplace operator \( \Delta \) and taking into account (3.21), (3.22), we get \( r_\Omega \tilde{f} = r_\Omega \tilde{A}u = Au \) in \( \Omega \). This means \( \tilde{f} \) is an extension of the distribution \( Au \in H^{-1}(\Omega) \) to \( \tilde{H}^{-1}(\Omega) \), and \( u \) satisfies (4.6). Then (2.10) implies
\[ \mathcal{P}[Au - \tilde{f}](y) = \langle Au - \tilde{f}, P(\cdot, y)\rangle_\Omega = -\langle T^+(\tilde{f}, u), P(\cdot, y)\rangle_{\partial \Omega} = V T^+(\tilde{f}, u), \quad y \in \Omega. \quad (4.12) \]

Substituting (4.12) into (4.11) leads to (4.7).

Equation (4.8) is implied by (4.5), (3.22) and (3.24).

To prove (4.9), let us first remark that
\[ A\mathcal{P} \tilde{f} = \tilde{f} + R_s \tilde{f} \quad \text{in} \quad \Omega, \quad (4.13) \]
which implies, due to (4.6), \( A(\mathcal{P} \tilde{f} - u) = R_s \tilde{f} \quad \text{in} \quad \Omega \), where \( R_s \) is defined by (4.10) and thus \( R_s \tilde{f} \in L_2(\Omega) \). Then \( A(\mathcal{P} \tilde{f} - u) \) can be canonically extended (by zero) to \( \tilde{A}(\mathcal{P} \tilde{f} - u) = \tilde{E} R_s \tilde{f} \in \tilde{H}^0(\Omega) \subset \tilde{H}^{-1}(\Omega) \). This implies that there exists a canonical co-normal derivative of \( (\mathcal{P} \tilde{f} - u) \), for which, due to (2.13) and (2.6), we have
\[ \langle T^+(\mathcal{P} \tilde{f} - u), w\rangle_{\partial \Omega} = \langle \tilde{A}(\mathcal{P} \tilde{f} - u) - \tilde{A} \mathcal{P} \tilde{f} + Au, \gamma^-w\rangle_\Omega = \langle \tilde{E} R_s \tilde{f} - \tilde{A} \mathcal{P} \tilde{f} + Au, \gamma^-w\rangle_\Omega = \langle \tilde{f} + \tilde{E} R_s \tilde{f} - \tilde{A} \mathcal{P} \tilde{f} + Au, \gamma^-w\rangle_\Omega = \langle \tilde{T}^+(\tilde{f}, u), w\rangle_{\partial \Omega} \quad \forall w \in H^{\frac{1}{2}}(\partial \Omega), \]
where \( \tilde{f} + \tilde{E} R_s \tilde{f} \in \tilde{H}^{-1}(\Omega) \) is an extension of \( A\mathcal{P} \tilde{f} \) associated with (4.13). That is,
\[ T^+(\mathcal{P} \tilde{f} - u) = T^+(\tilde{f} + \tilde{E} R_s \tilde{f}, \mathcal{P} \tilde{f}) - T^+(\tilde{f}, u) \quad \text{on} \quad \partial \Omega. \quad (4.14) \]

From (4.5) we have \( \mathcal{P} \tilde{f} - u = Ru - V\Psi + W\Phi \) in \( \Omega \). Substituting this in the left hand side of (4.14) and taking into account jump relation (3.25), we arrive at (4.9). \( \square \)

**Remark 4.3.** If \( \tilde{f} \in \tilde{H}^{-1/2}(\Omega) \subset \tilde{H}^{-1}(\Omega) \), then \( \tilde{f} + \tilde{E} R_s \tilde{f} \in \tilde{H}^{-1/2}(\Omega) \) as well, which implies \( \tilde{f} + \tilde{E} R_s \tilde{f} = \tilde{A}\mathcal{P} \tilde{f} \) and
\[ T^+(\tilde{f} + \tilde{E} R_s \tilde{f}, \mathcal{P} \tilde{f}) = T^+(\tilde{A}\mathcal{P} \tilde{f}, \mathcal{P} \tilde{f}) = T^+\mathcal{P} \tilde{f}. \quad (4.15) \]
Furthermore, if the hypotheses of Lemma 4.2 are satisfied, then (4.6) implies $u \in H^{1, -1/2}(\Omega, A)$ and $T^+(\tilde{f}, u) = T^+(\tilde{A}u, u) = T^+ u$. Henceforth, (4.9) takes the familiar form, cf. [3, equation (4.5)],

$$T^+ u + T^+ Ru - \frac{1}{2} \Psi - W' \Psi + L^+ \Phi = T^+ P \tilde{f} \quad \text{on} \quad \partial \Omega.$$

**REMARK 4.4.** Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ and a sequence $\{\phi_i\} \in \tilde{H}^{-1/2}(\Omega)$ converge to $\tilde{f}$ in $\tilde{H}^{-1}(\Omega)$. By the continuity of operators (C.1) and (C.3) in the Appendix, estimate (2.9) and relation (4.15) for $\phi_i$, we obtain that

$$T^+(\tilde{f} + \tilde{E} R_\ast \tilde{f}, P \tilde{f}) = \lim_{i \to \infty} T^+(\phi_i + \tilde{E} R_\ast \phi_i, P \phi_i) = \lim_{i \to \infty} T^+ P \phi_i$$

in $H^{-1/2}(\partial \Omega)$, cf. also Theorem B.1.

**COROLLARY 4.5.** If $u \in H^1(\Omega)$ and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ are such that $Au = r_\Omega \tilde{f}$ in $\Omega$, then

$$\frac{1}{2} \gamma^+ u + \gamma^+ Ru - VT^+(\tilde{f}, u) + W' \gamma^+ u = \gamma^+ P \tilde{f} \quad \text{on} \quad \partial \Omega,$$

$$\frac{1}{2} \gamma^+ u + \gamma^+ Ru - W'T^+(\tilde{f}, u) + L^+ \gamma^+ u = T^+(\tilde{f} + \tilde{E} R_\ast \tilde{f}, P \tilde{f}) \quad \text{on} \quad \partial \Omega. \tag{4.17}$$

The following statement is well known, see e.g. Lemma 4.2 in [3] and references therein.

**LEMMA 4.6.**

(i) If $\Psi^* \in H^{-\frac{1}{2}}(\partial \Omega)$ and $r_\Omega V \Psi^* = 0$ in $\Omega$, then $\Psi^* = 0$.

(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial \Omega)$ and $r_\Omega W \Phi^* = 0$ in $\Omega$, then $\Phi^* = 0$.

**THEOREM 4.7.** Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$. A function $u \in H^1(\Omega)$ is a solution of PDE $Au = \tilde{f}|_\Omega$ in $\Omega$ if and only if it is a solution of BDIDE (4.3).

**Proof.** If $u \in H^1(\Omega)$ solves PDE $Au = \tilde{f}|_\Omega$ in $\Omega$, then it satisfies (4.3). On the other hand, if $u$ solves BDIDE (4.3), then using Lemma 4.2 for $\Psi = T^+(\tilde{f}, u)$, $\Phi = \gamma^+ u$ completes the proof.

---

5 Segregated BDIE systems for the Dirichlet problem

Let us consider the **Dirichlet Problem:** Find a function $u \in H^1(\Omega)$ satisfying equations

$$Au = f \quad \text{in} \quad \Omega, \tag{5.1}$$

$$\gamma^+ u = \varphi_0 \quad \text{on} \quad \partial \Omega, \tag{5.2}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial \Omega)$, $f \in H^{-1}(\Omega)$.

Equation (5.1) is understood in the distributional sense (2.4) and the Dirichlet boundary condition (5.2) in the trace sense. The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.
THEOREM 5.1. The Dirichlet problem (5.1)-(5.2) is uniquely solvable in $H^1(\Omega)$. The solution is $u = (A^D)^{-1}(f, \varphi_0)^\top$, where the inverse operator, $(A^D)^{-1} : H^{1/2}(\partial \Omega) \times H^{-1}(\Omega) \to H^1(\Omega)$, to the left hand side operator, $A^D : H^1(\Omega) \to H^{1/2}(\partial \Omega) \times H^{-1}(\Omega)$, of the Dirichlet problem (5.1)-(5.2), is continuous.

5.1 BDIE formulations and equivalence to the Dirichlet problem

Let us consider reduction the Dirichlet problem (5.1)-(5.2) with $f \in H^{-1}(\Omega)$, for $u \in H^1(\Omega)$, to two different segregated Boundary-Domain Integral Equation (BDIE) systems. Corresponding formulations for the mixed problem for $u \in H^{1,0}(\Omega; \Delta)$ with $f \in L^2(\Omega)$ were introduced and analysed in [3, 5, 27].

Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ be an extension of $f \in H^{-1}(\Omega)$ (i.e., $f = r_\Omega \tilde{f}$), which always exists, see [28, Lemma 2.15 and Theorem 2.16]. Let us represent in (4.3), (4.16) and (4.17) the generalised co-normal derivative and the trace of the function $u$ as

$$T^+(\tilde{f}, u) = \psi, \quad \gamma^+ u = \varphi_0,$$

and will regard the new unknown function $\psi \in H^{-1/2}(\partial \Omega)$ as formally segregated of $u$. Thus we will look for the couple $(u, \psi) \in H^1(\Omega) \times H^{-1/2}(\partial \Omega)$.

BDIE system (D1) To reduce the Dirichlet BVP (5.1)-(5.2) to the BDIE system (D1), we will use equation (4.3) in $\Omega$ and equation (4.16) on $\partial \Omega$. Then we arrive at the following system, (D1), of the boundary-domain integral equations,

\begin{align*}
u + \mathcal{R} u - V \psi &= \mathcal{F}^{D1}_1 \quad \text{in } \Omega, \quad (5.3) \\
\gamma^+ \mathcal{R} u - \mathcal{V} \psi &= \mathcal{F}^{D1}_2 \quad \text{on } \partial \Omega, \quad (5.4)
\end{align*}

where

$$\mathcal{F}^{D1} = \begin{bmatrix} \mathcal{F}^{D1}_1 \\ \mathcal{F}^{D1}_2 \end{bmatrix} = \begin{bmatrix} F^D_1 \\ \gamma^+ F^D_0 - \varphi_0 \end{bmatrix} \quad \text{and } F^D_0 := \mathcal{P} \tilde{f} - W \varphi_0 \quad \text{in } \Omega. \quad (5.5)$$

Note that for $\varphi_0 \in H^{1/2}(\partial \Omega)$, we have the inclusion $F^D_0 \in H^1(\Omega)$ if $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ due to the mapping properties of the Newtonian (volume) and layer potentials, cf. (C.4), (C.19).

BDIE system (D2) To obtain a segregated BDIE system of the second kind, (D2), we will use equation (4.3) in $\Omega$ and equation (4.17) on $\partial \Omega$. Then we arrive at the following BDIE system (D2),

\begin{align*}
u + \mathcal{R} u - V \psi &= \mathcal{F}^{D2}_1 \quad \text{in } \Omega, \quad (5.6) \\
\frac{1}{2} \psi + T^+ \mathcal{R} u - \mathcal{W}^+ \psi &= \mathcal{F}^{D2}_2 \quad \text{on } \partial \Omega, \quad (5.7)
\end{align*}

where

$$\mathcal{F}^{D2} = \begin{bmatrix} \mathcal{F}^{D2}_1 \\ \mathcal{F}^{D2}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{P} \tilde{f} - W \varphi_0 \\ T^+(\tilde{f} + \mathcal{E} \mathcal{R} \tilde{f}, \mathcal{P} \tilde{f}) - \mathcal{L}^+ \varphi_0 \end{bmatrix}. \quad (5.8)$$
Due to the mapping properties of the operators involved in (5.8) we have $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

Let us prove that BVP (5.1)–(5.2) in $\Omega$ is equivalent to both systems of BDIEs, (D1) and (D2).

**THEOREM 5.2.** Let $\varphi_0 \in H^\frac{1}{2}(\partial\Omega)$, $f \in H^{-1}(\Omega)$, and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ is such that $r_\Omega \tilde{f} = f$.

(i) If a function $u \in H^1(\Omega)$ solves the Dirichlet BVP (5.1)–(5.2), then the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, where

$$
\psi = T^+ (\tilde{f}, u) \quad \text{on} \quad \partial\Omega,
$$

solves the BDIE systems (D1) and (D2).

(ii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves one of the BDIE systems, (D1) or (D2), then this solution is unique and solves the other system, while $u$ solves the Dirichlet BVP, and $\psi$ satisfies (5.9).

**Proof.** (i) Let $u \in H^1(\Omega)$ be a solution to BVP (5.1)–(5.2). It is unique due to Theorem 5.1. Setting $\psi$ by (5.9) evidently implies $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$. Then it immediately follows from Theorem 4.7 and relations (4.16) and (4.17) that the couple $(u, \psi)$ solves systems (D1) and (D2) with the right hand sides (5.5) and (5.8), respectively, which completes the proof of item (i).

(ii) Let now a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (5.3)-(5.4). Taking trace of equation (5.3) on $\partial\Omega$ and subtracting equation (5.4) from it, we obtain,

$$
\gamma^+ u(y) = \varphi_0(y), \quad y \in \partial\Omega,
$$

i.e. $u$ satisfies the Dirichlet condition (5.2).

Equation (5.3) and Lemma 4.2 with $\Psi = \psi$, $\Phi = \varphi_0$ imply that $u$ is a solution of PDE (5.1) and

$$
V \Psi^*(y) - W \Phi^*(y) = 0, \quad y \in \Omega,
$$

where $\Psi^* = \psi - T^+ (\tilde{f}, u)$ and $\Phi^* = \varphi_0 - \gamma^+ u$. Due to equation (5.10), $\Phi^* = 0$. Then Lemma 4.2(i) implies $\Psi^* = 0$, which completes the proof of condition (5.9). Thus $u$ obtained from solution of BDIE system (D1) solves the Dirichlet problem and hence, by item (i) of the theorem, $(u, \psi)$ solve also BDIE system (D2).

Due to (5.5), the BDIE system (5.3)-(5.4) with zero right hand side can be considered as obtained for $\tilde{f} = 0$, $\varphi_0 = 0$, implying that its solution is given by a solution of the homogeneous BVP (5.1)–(5.2), which is zero by Theorem 5.1. This implies uniqueness of solution of the inhomogeneous BDIE system (5.3)-(5.4).

Let now a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (5.6)-(5.7). Lemma 4.2 for equation (5.6) implies that $u$ is a solution of equation (2.1), and equations (4.7) and (4.9) hold for $\Psi = \psi$ and $\Phi = \varphi_0$. Subtracting (4.9) from equation (5.7) gives

$$
\Psi^* := \psi - T^+ (\tilde{f}, u) = 0 \quad \text{on} \quad \partial\Omega,
$$

that is, equation (5.9) is proved.
Equations (4.7) and (5.11) give $W \Phi^* = 0$ in $\Omega$, where $\Phi^* = \varphi_0 - \gamma^+ u$. Then Lemma 4.6(ii) implies $\Phi^* = 0$ on $\partial \Omega$. This means that $u$ satisfies the Dirichlet condition (5.2). Thus $u$ obtained from solution of BDIE system (D2) solves the Dirichlet problem and hence, by item (i) of the theorem, $(u, \psi)$ solve also BDIE system (D1).

Due to (5.8), the BDIE system (5.6)-(5.7) with zero right hand side can be considered as obtained for $\tilde{f} = 0$, $\varphi_0 = 0$, implying that its solution is given by a solution of the homogeneous BVP (5.1)-(5.2), which is zero by Theorem 5.1. This implies uniqueness of solution of the inhomogeneous BDIE system (5.6)-(5.7).

**REMARK 5.3.** For a given function $f \in H^{-1}(\Omega)$, its extension $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ is not unique. Nevertheless, since solution of the Dirichlet BVP (5.1)-(5.2) does not depend on this extension, equivalence Theorem 5.2(ii) implies that $u$ in the solution of BDIE systems (D1) and (D2) does not depend on the particular choice of extension $\tilde{f}$. However, $\psi$ does obviously depends on the choice of $\tilde{f}$, see (5.9).

### 5.2 BDIE system operators invertibility, for the Dirichlet problem

BDIE systems (D1) and (D2) can be written as

$$\mathcal{D}^1 \mathcal{U} = \mathcal{F}^{D1} \quad \text{and} \quad \mathcal{D}^2 \mathcal{U} = \mathcal{F}^{D2},$$

respectively. Here $\mathcal{U}^D := (u, \psi)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$,

$$\mathcal{D}^1 := \begin{bmatrix} I - \mathcal{R} & -V \\ \gamma^+ \mathcal{R} & -V \end{bmatrix}, \quad \mathcal{D}^2 := \begin{bmatrix} I + \mathcal{R} & -V \\ T^+ \mathcal{R} & \frac{1}{2} I - \mathcal{W}' \end{bmatrix}, \quad (5.12)$$

while $\mathcal{F}^{D1}$ and $\mathcal{F}^{D2}$ are given by (5.5) and (5.8).

Due to the mapping properties of the operators participating in definitions of the operators $\mathcal{D}^1$ and $\mathcal{D}^2$ as well as the right hand sides $\mathcal{F}^{D1}$ and $\mathcal{F}^{D2}$ (see [3, 27] and the Appendix), we have $\mathcal{F}^{D1} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$, $\mathcal{F}^{D2} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$, while the operators

$$\mathcal{D}^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega), \quad (5.13)$$

$$\mathcal{D}^2 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \quad (5.14)$$

are continuous. Due to Theorem 5.2(ii), operator (5.13) and (5.14) are injective.

**THEOREM 5.4.** Operators (5.13) and (5.14) are continuous and continuously invertible.

*Proof.* The continuity is proved above. To prove the invertibility of operator (5.13), let us consider the operator

$$\mathcal{D}_0^1 := \begin{bmatrix} I & -V \\ 0 & -V \end{bmatrix}. $$
As a result of compactness properties of the operators $\mathcal{R}$ and $\gamma^+\mathcal{R}$ (see Corollary C.7 in the Appendix), the operator $\mathfrak{D}_0^1$ is a compact perturbation of operator (5.13). The operator $\mathfrak{D}_0^1$ is an upper triangular matrix operator with the following scalar diagonal invertible operators

$$I : H^1(\Omega) \to H^1(\Omega),$$
$$\mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega),$$

cf. [12, Ch. XI, Part B, §2, Theorem 3] for $\mathcal{V}$. This implies that

$$\mathfrak{D}_0^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is an invertible operator. Thus (5.13) is a Fredholm operator with zero index. The injectivity of operator (5.13), which is already proved, completes the theorem proof for operator (5.13).

The operator

$$\mathfrak{D}_0^2 := \begin{bmatrix} I & -\mathcal{V} \\ 0 & \frac{1}{2} I \end{bmatrix}$$

is a compact perturbation of operator (5.14) due to compactness properties of the operators $\mathcal{R}$ and $\mathcal{W}$, see [3] [5] [27] and Corollary C.7 from the Appendix. The invertibility of operator (5.14) then follows by the arguments similar to those for operator (5.13).

6 Segregated BDIE systems for the Neumann Problem

Let us consider the Neumann Problem: Find a function $u \in H^1(\Omega)$ satisfying equations

$$Au = r_\Omega \tilde{f} \quad \text{in} \quad \Omega, \quad (6.1)$$
$$T^+(\tilde{f}, u) = \psi_0 \quad \text{on} \quad \partial\Omega, \quad (6.2)$$

where $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$, $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$.

Equation (6.1) is understood in the distributional sense (2.4) and Neumann boundary condition (6.2) in the weak sense (2.10). The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.

**THEOREM 6.1.**

(i) The Neumann homogeneous problem, associated with (6.1)-(6.2), admits only one linearly independent solution $u^0 = 1$ in $H^1(\Omega)$.

(ii) The non-homogeneous Neumann problem (6.1)-(6.2) is solvable if only if the following solvability condition is satisfied

$$\langle \tilde{f}, u^0 \rangle_\Omega - \langle \psi_0, \gamma^+ u^0 \rangle_{\partial\Omega} = 0. \quad (6.3)$$
6.1 BDIE formulations and equivalence to the Neumann problem

We will explore different possibilities of reducing the Neumann problem \((6.1)-(6.2)\) to a BDIE system. Let us represent in \((4.3)\), \((4.16)\) and \((4.17)\) the generalised co-normal derivative and the trace of the function \(u\) as

\[
T^+(\tilde{f},u) = \psi_0, \quad \gamma^+ u = \varphi,
\]

and will regard the new unknown function \(\varphi \in H^{\frac{1}{2}}(\partial\Omega)\) as formally segregated of \(u\). Thus we will look for the couple \((u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)\).

**BDIE system (N1)** First, using equation \((4.3)\) in \(\Omega\) and equation \((4.17)\) on \(\partial\Omega\), we arrive at the following BDIE system (N1) of two equations for the couple of unknowns, \((u, \varphi)\),

\[
\begin{align*}
  &u + \mathcal{R}u + W\varphi = \mathcal{F}^{N1} \text{ in } \Omega, \\
  &T^+\mathcal{R}u + \mathcal{L}^+ \varphi = \mathcal{F}^{N1}_2 \text{ on } \partial\Omega,
\end{align*}
\]

where

\[
\mathcal{F}^{N1} = \begin{bmatrix}
\mathcal{F}^{N1}_1 \\
\mathcal{F}^{N1}_2
\end{bmatrix} = \begin{bmatrix}
\mathcal{P}\tilde{f} + V\psi_0 \\
T^+(\tilde{f} + \tilde{E}\mathcal{R}_+\tilde{f}, \mathcal{P}\tilde{f}) - \frac{1}{2} \psi_0 + W'\psi_0
\end{bmatrix}.
\]

Due to the mapping properties of the operators involved in \((6.6)\) we have \(\mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)\).

**BDIE system (N2)** If we use equation \((4.3)\) in \(\Omega\) and equation \((4.16)\) on \(\partial\Omega\), we arrive for the couple \((u, \varphi)\) at the following BDIE system (N2) of two equations of the second kind,

\[
\begin{align*}
  &u + \mathcal{R}u + W\varphi = \mathcal{F}^{N2}_1 \text{ in } \Omega, \\
  &\frac{1}{2} \varphi + \gamma^+ \mathcal{R}u + W\varphi = \mathcal{F}^{N2}_2 \text{ on } \partial\Omega,
\end{align*}
\]

where

\[
\mathcal{F}^{N2} = \begin{bmatrix}
\mathcal{F}^{N2}_1 \\
\mathcal{F}^{N2}_2
\end{bmatrix} = \begin{bmatrix}
F^N_0 \\
\gamma^+ F^N_0
\end{bmatrix}, \quad F^N_0 := \mathcal{P}\tilde{f} + V\psi_0 \text{ in } \Omega.
\]

Due to the mapping properties of the operators involved in \((6.9)\), we have \(\mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)\).

**THEOREM 6.2.** Let \(\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)\) and \(\tilde{f} \in \tilde{H}^{-1}(\Omega)\).

(i) If a function \(u \in H^1(\Omega)\) solves the Neumann problem \((6.1)-(6.2)\) then the couple \((u, \varphi)\) with \(\varphi = \gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)\) solves BDIE systems (N2) and (N1).

(ii) Vice versa, if a couple \((u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)\) solves one of the BDIE systems, (N1) or (N2), then the couple solves the other one BDE system and \(u\) solves the Neumann problem \((6.1)-(6.2)\) and \(\gamma^+ u = \varphi\).
(iii) The homogeneous BDIE systems (N1) and (N2) have unique linear independent solution \( U^0 = (u^0, \varphi^0)^\top = (1, 1)^\top \) in \( H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \). Condition (6.3) is necessary and sufficient for solvability of the nonhomogeneous BDIE systems (N1) and (N2) in \( H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \).

Proof. (i) Let \( u \in H^1(\Omega) \) be a solution of the Neumann problem (6.1)-(6.2). Then from Theorem 4.7 and relations (4.16) and (4.17) we see that the couple \((u, \varphi)\) with \( \varphi = \gamma^+ u \) solves LBDIE systems (N1) and (N2), which proves item (i).

(ii) Let a couple \((u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)\) solve LBDIE system (N1). Lemma 4.2 for equation (6.4) implies that \( u \) is a solution of equation (2.1), and equations (4.7)-(4.9) hold for \( \Psi = \psi_0 \) and \( \Phi = \varphi \). Subtracting (4.9) from equation (6.5) gives \( T^+ (\tilde{f}, u) = \psi_0 \) on \( \partial \Omega \). Further, from (4.7) we derive \( W(\gamma^+ u - \varphi) = 0 \) in \( \Omega^+ \), whence \( \gamma^+ u = \varphi \) on \( \partial \Omega \) by Lemma 4.6, completing item (ii) for LBDIE system (N1).

Let now a couple \((u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)\) solve the LBDIE system (N2). Further, taking the trace of (6.7) on \( \partial \Omega \) and comparing the result with (6.8), we easily derive that \( \gamma^+ u = \varphi \) on \( \partial \Omega \). Lemma 4.2 for equation (6.7) implies that \( u \) is a solution of equation (2.1), while equations (4.7)-(4.9) hold for \( \Psi = \psi_0 \) and \( \Phi = \varphi \). Further, from (4.7) we derive

\[
V(\psi_0 - T^+ (\tilde{f}, u)) = 0 \quad \text{in} \quad \Omega^+,
\]

whence \( T^+ u = \psi_0 \) on \( \partial \Omega \) by Lemma 4.6 i.e., \( u \) solves the Neumann problem (6.1)-(6.2), which completes the proof of item (ii) for LBDIE system (N2).

(iii) Theorem 6.1 along with items (i) and (ii) imply the claims of item (iii) for LBDIE systems (N2) and (N1).

\[\square\]

6.2 Properties of BDIE system operators for the Neumann problem

BDIE systems (N1) and (N2) can be written, respectively, as

\[
\mathcal{A}^1 U^N = \mathcal{F}^{N1}, \quad \mathcal{A}^2 U^N = \mathcal{F}^{N2},
\]

where \( U^N = (u, \varphi)^\top \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \),

\[
\mathcal{A}^1 := \begin{bmatrix} I + R & W \\ T^+ R & \mathcal{L}^+ \end{bmatrix}, \quad \mathcal{A}^2 := \begin{bmatrix} I + R & W \\ \gamma^+ R & \frac{1}{2} I + W \end{bmatrix}.
\]

Due to the mapping properties of the potentials, \( \mathcal{F}^{N1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \), \( \mathcal{F}^{N2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \).

**THEOREM 6.3.** The operators

\[
\mathcal{A}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega), \quad (6.10)
\]

\[
\mathcal{A}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega). \quad (6.11)
\]
are continuous Fredholm operators with zero index. They have one-dimensional null-spaces, \( \ker \mathfrak{N}^1 = \ker \mathfrak{N}^2 \), in \( H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \), spanned over the element \((u^0, \varphi^0) = (1, 1)\).

**Proof.** The mapping properties of the potentials, see Appendix, imply continuity of operators (6.11) and (6.10).

First consider operator (6.10). Let us denote \( L^\perp_0 g := L^\perp_\Delta (ag) \). Hence the operator \( L^\perp_0 : H^{\frac{1}{2}}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega) \) is a Fredholm operator with zero index (cf. e.g. [11, Theorem 2], [12, Ch. XI, Part B, §3,]). Therefore the operator

\[
\mathcal{A}^N_{01} := \begin{bmatrix} I & W \\ 0 & L^\perp_0 \end{bmatrix} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega).
\]

(6.12)
is also Fredholm with zero index. Operator (6.10) is a compact perturbation of \( \mathcal{A}^N_{01} \) since the operators

\[
\mathcal{R} : H^1(\Omega) \to H^1(\Omega)
\]

\[
L^\perp + L^\perp_\Delta : H^{\frac{1}{2}}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega),
\]

\[
T^+ \mathcal{R} : H^1(\Omega) \to H^{-\frac{1}{2}}(\partial \Omega)
\]

are compact, due to relation (3.20) and Theorem C.4. Thus operator (6.10) is Fredholm with zero index as well. The claims that \( \ker \mathfrak{N}^1 \) is one-dimensional and the couple \((u^0, \varphi^0) = (1, 1)\) belongs to \( \ker \mathfrak{N}^1 \) directly follow from Theorem 6.2(iii).

The proof for operator (6.11) is similar. \( \square \)

To describe in more details the ranges of operators (6.10) and (6.11), i.e., to give more information about the co-kernels of these operators, we will need several auxiliary assertions. First of all, let us remark that for any \( v \in H^{s-\frac{1}{2}}(\partial \Omega) \), \( s < \frac{3}{2} \), the single layer potential can be defined as

\[
V_v(y) := -\langle \gamma P(\cdot, y), v \rangle_{\partial \Omega} = -\langle P(\cdot, y), \gamma^* v \rangle_{\mathbb{R}^3} = -\mathbf{P} \gamma^* v(y), \quad y \in \mathbb{R}^3 \setminus \partial \Omega.
\]

(6.13)

where \( \gamma^* : H^{s-\frac{1}{2}}(\partial \Omega) \to H^{\frac{s-1}{2}}_{\partial \Omega} \), \( s < \frac{3}{2} \), is the operator adjoined to the trace operator \( \gamma : H^{2-s}(\mathbb{R}^3) \to H^{\frac{s-1}{2}}_{\partial \Omega} \), and the space \( H^{\frac{s-1}{2}}_{\partial \Omega} \) is defined by (2.22).

**LEMMA 6.4.** Let \( \tilde{f} \in \tilde{H}^{s-2}(\Omega) \), \( s > \frac{1}{2} \). If

\[
\rho_\Omega \mathbf{P} \tilde{f} = 0 \quad \text{in} \quad \Omega,
\]

(6.14)

then \( \tilde{f} = 0 \) in \( \mathbb{R}^3 \).

**Proof.** Multiplying (6.14) by \( a \), taking into account (6.16) and applying the Laplace operator, we obtain \( \rho_\Omega \tilde{f} = 0 \), which means \( \tilde{f} \in H^{s-2}_{\partial \Omega} \). If \( s \geq \frac{3}{2} \), then \( \tilde{f} = 0 \) by Theorem 2.10 from [28]. If \( \frac{1}{2} < s < \frac{3}{2} \), then by the same theorem there exists \( v \in H^{s-\frac{1}{2}}(\partial \Omega) \) such that \( \tilde{f} = \gamma^* v \). This gives \( \mathbf{P} \tilde{f} = \mathbf{P} \gamma^* v = -V v \) in \( \mathbb{R}^3 \). Then (6.14) reduces to \( V v = 0 \) in \( \Omega \), which implies \( v = 0 \) on \( \partial \Omega \) (see e.g. Lemma 4.6 for \( s = 1 \), which can be easily generalized to \( \frac{1}{2} < s < \frac{3}{2} \)) and thus \( \tilde{f} = 0 \) in \( \mathbb{R}^3 \). \( \square \)
THEOREM 6.5. Let $\frac{1}{2} < s < \frac{3}{2}$. The operator

$$P : \tilde{H}^{s-2}(\Omega) \to H^s(\Omega)$$

(6.15)

and its inverse

$$(P)^{-1} : H^s(\Omega) \to \tilde{H}^{s-2}(\Omega)$$

(6.16)

are continuous and

$$(P)^{-1}g = [\Delta \tilde{E}(I - V_{\Delta}V_{\Delta}^{-1}\gamma^+) - \gamma^*V_{\Delta}^{-1}\gamma^+]\gamma g \in \mathbb{R}^3, \forall g \in H^s(\Omega).$$

(6.17)

Proof. The continuity of (6.15) is well known, cf. [3, Theorem 3.8]. By Lemma 6.4, operator (6.15) is injective. Let us prove its surjectivity. To this end, for arbitrary $g \in H^s(\Omega)$ let us consider the following equation with respect to $\tilde{f} \in \tilde{H}^{s-2}(\Omega)$,

$$P_\Delta \tilde{f} = g \quad \text{in} \quad \Omega.$$  

(6.18)

Let $g_1 \in H^s(\Omega)$ be the (unique) solution of the following Dirichlet problem: $\Delta g_1 = 0$ in $\Omega$, $\gamma^+ g_1 = \gamma^+ g$, which can be particularly presented as $g_1 = V_{\Delta}V_{\Delta}^{-1}\gamma^+ g$, see e.g. [11] or proof of Lemma 2.6 in [28]. Let $g_0 := g - g_1$. Then $g_0 \in H^s(\Omega)$ and $\gamma^+ g_0 = 0$ and thus $g_0$ can be uniquely extended to $\tilde{E}g_0 \in \tilde{H}^s(\Omega)$, where $\tilde{E}$ is the operator of extension by 0 outside $\Omega$. Thus by (6.13), equation (6.18) takes form

$$r_\Omega P_\Delta [\tilde{f} + \gamma^*V_{\Delta}^{-1}\gamma^+]g = g_0 \quad \text{in} \quad \Omega.$$  

(6.19)

Any solution $\tilde{f} \in \tilde{H}^{s-2}(\Omega)$ of the corresponding equation on $\mathbb{R}^3$,

$$P_\Delta [\tilde{f} + \gamma^*V_{\Delta}^{-1}\gamma^+]g = \tilde{E}g_0 \quad \text{in} \quad \mathbb{R}^3,$$

(6.20)

will evidently solve (6.19). If $\tilde{f}$ solves (6.20), then acting with the Laplace operator on (6.20), we obtain

$$\tilde{f} = \tilde{Q}g := \Delta \tilde{E}g_0 - \gamma^*V_{\Delta}^{-1}\gamma^+ g = \Delta \tilde{E}(g - r_\Omega V_{\Delta}V_{\Delta}^{-1}\gamma^+ g) - \gamma^*V_{\Delta}^{-1}\gamma^+ g \quad \text{in} \quad \mathbb{R}^3.$$  

(6.21)

On the other hand, substituting $\tilde{f}$ given by (6.21) to (6.20) and taking into account that $P_\Delta P_\Delta \tilde{h} = \tilde{h}$ for any $\tilde{h} \in \tilde{H}^s(\Omega)$, $s \in \mathbb{R}$, we obtain that $\tilde{Q}g$ is indeed a solution of equation (6.20) and thus (6.19). By Lemma 6.4 the solution of (6.19) is unique, which means that the operator $\tilde{Q}$ is inverse to operator (6.15), i.e., $\tilde{Q} = (r_\Omega P)^{-1}$. Since $\Delta$ is a continuous operator from $\tilde{H}^s(\Omega)$ to $\tilde{H}^{s-2}(\Omega)$, equation (6.21) implies that the operator $(r_\Omega P)^{-1} = \tilde{Q} : H^s(\Omega) \to \tilde{H}^{s-2}(\Omega)$ is continuous. The relations $P = \frac{1}{a}P_\Delta$ and $a(x) > c > 0$ then imply invertibility of operator (6.15) and ansatz (6.17). \(\square\)

LEMMA 6.6. For any couple $(\mathcal{F}_1, \mathcal{F}_2) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$, there exists a unique couple $(\bar{f}_**, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^\frac{1}{2}(\partial \Omega)$ such that

$$\mathcal{F}_1 = P\bar{f}_* - W\Phi_* \quad \text{in} \quad \Omega,$$

(6.22)
Moreover, \((\tilde{f}_{**}, \Phi_*) = C_{**}(F_1, F_2)\) and \(C_{**}: H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \to \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)\) is a linear continuous operator given by

\[
\tilde{f}_{**} = \Delta(aF_1) + \gamma^*(F_2 + (\gamma^*F_1)\partial_n a),
\]

\[
\Phi_* = \frac{1}{a} \left( -\frac{1}{2} I + W_\Delta \right)^{-1} \gamma^* \{ -aF_1 + P_\Delta[\Delta(aF_1) + \gamma^*(F_2 + (\gamma^*F_1)\partial_n a)] \},
\]

where \(\Delta(aF_1) = \nabla \cdot \dot{E} \nabla (aF_1)\).

**Proof.** Let us first assume that there exist \((\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)\) satisfying equations \(6.22\), \(6.23\) and find their expressions in terms of \(F_1\) and \(F_2\). Let us re-write \(6.22\) as

\[
F_1 - P\tilde{f}_{**} = -W\Phi_* \quad \text{in} \ \Omega, \quad (6.26)
\]

Multiplying \(6.26\) by \(a\) and applying the Laplace operator to it, we obtain,

\[
\Delta(aF_1) - P\Delta \tilde{f}_{**} = \Delta(aF_1) - \tilde{f}_{**} = -\Delta W_\Delta(a\Phi_*) = 0 \quad \text{in} \ \Omega, \quad (6.27)
\]

which means

\[
\Delta(aF_1) = r_\Omega \tilde{f}_{**} \quad \text{in} \ \Omega \quad (6.28)
\]

and \(aF_1 - P\Delta \tilde{f}_{**} \in H^{1,0}(\Omega; \Delta)\) and hence \(F_1 - P\tilde{f}_{**} \in H^{1,0}(\Omega; A)\). The latter implies that the canonical conormal derivative \(T^+(F_1 - P\tilde{f}_{**})\) is well defined. It can be also written in terms of the generalised conormal derivatives,

\[
T^+(F_1 - P\tilde{f}_{**}) = T^+(\tilde{A}(F_1 - P\tilde{f}_{**}), F_1 - P\tilde{f}_{**}) = T^+(\dot{E}A(F_1 - P\tilde{f}_{**}), F_1 - P\tilde{f}_{**})
\]

\[
= T^+(\dot{E} \nabla \cdot (a \nabla (F_1 - P\tilde{f}_{**})), F_1 - P\tilde{f}_{**})
\]

\[
= T^+(\dot{E} \Delta(aF_1 - P\Delta \tilde{f}_{**}) - \dot{E} \nabla \cdot (F_1 - P\tilde{f}_{**})\nabla a), F_1 - P\tilde{f}_{**})
\]

\[
= T^+(\dot{E} \nabla \cdot (F_1 \nabla a) - \dot{E} R_\ast \tilde{f}_{**}, F_1 - P\tilde{f}_{**}), \quad (6.29)
\]

where \(11.10\) and \(11.10\) were taken into account. Applying the canonical conormal derivative operator \(T^+\) to the both sides of equation \(6.26\) and substituting there \(6.29\), we obtain

\[
T^+(\tilde{f}_{**} - \dot{E} \nabla \cdot (F_1 \nabla a), F_1) - T^+(\tilde{f}_{**} + \dot{E} R_\ast \tilde{f}_{**}, P\tilde{f}_{**}) = -\mathcal{L}^+ \Phi_* \quad \text{on} \ \partial \Omega. \quad (6.30)
\]

Subtracting this from \(6.23\), we obtain

\[
F_2 = T^+(\tilde{f}_{**} - \dot{E} \nabla \cdot (F_1 \nabla a), F_1) \quad \text{on} \ \partial \Omega. \quad (6.31)
\]
Due to (6.23), we can represent

\[ \tilde{f}_{ss} = \tilde{\Delta}(aF_1) + \tilde{f}_{1s} = \nabla \cdot \tilde{E} \nabla (aF_1) - \gamma^* \Psi_{ss}, \]  

(6.32)

where \( \tilde{f}_{1s} \in H^{-1}_0(\Omega) \) and hence, due to e.g. [28, Theorem 2.10] can be in turn represented as \( \tilde{f}_{1s} = -\gamma^* \Psi_{ss} \), with some \( \Psi_{ss} \in H^{-\frac{1}{2}}(\partial \Omega) \). Then (6.38) is satisfied and

\[ T^+(\tilde{f}_{ss} - \tilde{E} \nabla \cdot (F_1 \nabla a), F_1) = (\gamma^{-1})^* [\tilde{f}_{ss} - \tilde{E} \nabla \cdot (F_1 \nabla a) - \tilde{A} F_1] \]

\[ = (\gamma^{-1})^* [\nabla \cdot \tilde{E} \nabla (F_1 \nabla a) - \gamma^* \Psi_{ss} - \tilde{E} \nabla \cdot (F_1 \nabla a) - \nabla \cdot \tilde{E} (a \nabla F_1)] \]

\[ = (\gamma^{-1})^* [\nabla \cdot \tilde{E} (F_1 \nabla a) - \gamma^* \Psi_{ss} - \tilde{E} \nabla \cdot (F_1 \nabla a)] = -\Psi_{ss} - (\gamma^+ F_1) \partial_n a \]  

(6.33)

because

\[ \left< (\gamma^{-1})^* [\nabla \cdot \tilde{E} (F_1 \nabla a) - \gamma^* \Psi_{ss} - \tilde{E} \nabla \cdot (F_1 \nabla a)], w \right>_{\partial \Omega} = \left< \nabla \cdot \tilde{E} (F_1 \nabla a) - \gamma^* \Psi_{ss} - \tilde{E} \nabla \cdot (F_1 \nabla a), \gamma^{-1} w \right>_{\Omega} \]

\[ = \left< \nabla \cdot \tilde{E} (F_1 \nabla a), \gamma^{-1} w \right>_{\mathbb{R}^3} - \Psi_{ss} - \left< \tilde{E} \nabla \cdot (F_1 \nabla a), \gamma^{-1} w \right>_{\Omega} \]

\[ = -\left< \tilde{E} (F_1 \nabla a), \nabla \gamma^{-1} w \right>_{\mathbb{R}^3} - \Psi_{ss} + \left< F_1 \nabla a, \nabla \gamma^{-1} w \right>_{\Omega} - \left< n \cdot \gamma^+ (F_1 \nabla a), \gamma^+ \gamma^{-1} w \right>_{\Omega} = -\left< (\gamma^+ F_1) \partial_n a, w \right>_{\partial \Omega} - \Psi_{ss}. \]

Hence (6.31) reduces to

\[ \Psi_{ss} = -F_2 - (\gamma^+ F_1) \partial_n a. \]  

(6.34)

and (6.32) to (6.24).

Now (6.26) can be written in the form

\[ W_\Delta(a \Phi_*) = F_\Delta \quad \text{in} \quad \Omega, \]  

(6.35)

where

\[ F_\Delta := -a F_1 + P_\Delta \tilde{f}_{ss} = -a F_1 + P_\Delta [\tilde{\Delta}(a F_1) + \gamma^* (F_2 + (\gamma^+ F_1) \partial_n a)], \]  

(6.36)

is a harmonic function in \( \Omega \) due to (6.27). The trace of equation (6.35) gives

\[ -\frac{1}{2} a \Phi_* + W_\Delta(a \Phi_*) = \gamma^+ F_\Delta \quad \text{on} \quad \partial \Omega. \]  

(6.37)

Since the operator \( -\frac{1}{2} I + W_\Delta : H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega) \) is an isomorphism (see e.g. [12, Ch. XI, Part B, §2, Remark 8]), this implies

\[ \Phi_* = \frac{1}{a} \left( -\frac{1}{2} I + W_\Delta \right)^{-1} \gamma^+ F_\Delta = \frac{1}{a} \left( -\frac{1}{2} I + W_\Delta \right)^{-1} \gamma^+ \{ -a F_1 + P_\Delta [\tilde{\Delta}(a F_1) + \gamma^* (F_2 + (\gamma^+ F_1) \partial_n a)] \}. \]  

(6.38)
Proof. Let us consider the equation
\[ (\tilde{f}_s, \Phi_s) = C_{ss}(\mathcal{F}_1, \mathcal{F}_2), \]
where \( C_{ss} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \) is a linear continuous operator, as requested. We still have to check that the functions \( \tilde{f}_{ss} \) and \( \Phi_s \), given by (6.24) and (6.38), satisfy equations (6.22) and (6.23). Indeed, \( \Phi_s \) given by (6.38) satisfies equation (6.37) and thus \( \gamma^+ W_{\Delta}(a\Phi_s) = \gamma^+ \mathcal{F}_\Delta \). Since both \( W_{\Delta}(a\Phi_s) \) and \( \mathcal{F}_\Delta \) are harmonic functions, this implies (6.35)-(6.36) and by (6.24) also (6.22). Finally, (6.24) implies by (6.33) that (6.31) is satisfied, and adding (6.30) to it leads to (6.23).

Let us now prove that the operator \( C_{ss} \) is unique. Indeed, let a couple \((\tilde{f}_s, \Phi_s) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \) be a solution of linear system (6.22)-(6.23) with \( \mathcal{F}_1 = 0 \) and \( \mathcal{F}_2 = 0 \). Then (6.28) implies that \( r_\Omega \tilde{f}_{ss} = 0 \) in \( \Omega \), i.e., \( \tilde{f}_{ss} \in H^{-1}_{\partial\Omega} \subset \tilde{H}^{-1}(\Omega) \). Hence, (6.31) reduces to
\[
0 = T^+(\tilde{f}_{ss}, 0) \quad \text{on} \quad \partial\Omega.
\]
By the first Green identity (2.10), this gives,
\[
0 = \left( T^+(\tilde{f}_{ss}, 0), \gamma^+ v \right)_{\partial\Omega} = (\tilde{f}_{ss}, v)_\Omega \quad \forall \ v \in H^1(\Omega),
\]
which implies \( \tilde{f}_{ss} = \gamma^* \Psi_s \). Finally, (6.33) gives \( \Phi_s = 0 \). Hence, any solution of non-homogeneous linear system (6.22)-(6.23) has only one solution, which implies uniqueness of the operator \( C_{ss} \).

\[ \square \]

**Theorem 6.7.** The cokernel of operator (6.10) is spanned over the functional
\[
g^* := ((\gamma^+)^* \partial_n a, 1)^T
\]
in \( \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \), i.e., \( g^*(\mathcal{F}_1, \mathcal{F}_2) = ((\gamma^+ \mathcal{F}_1) \partial_n a + \mathcal{F}_2, \gamma^+ u^0)_{\partial\Omega} \), where \( u^0 = 1 \).

**Proof.** Let us consider the equation \( \mathfrak{H}^1 \mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^T \), i.e., the BDIE system (N1),
\[
u + \mathcal{R} u + W\varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega,
\]
\[
T^+ \mathcal{R} u + \mathcal{L}^+ \varphi = \mathcal{F}_2 \quad \text{on} \quad \partial\Omega,
\]
with arbitrary \((\mathcal{F}_1, \mathcal{F}_2) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \), for \((u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \). By Lemma 6.6 the right hand side of the system has form (6.22)-(6.23), i.e., system (6.32)-(6.33) reduces to
\[
u + \mathcal{R} u + W(\varphi + \Phi_s) = \mathcal{P} \tilde{f}_{ss} \quad \text{in} \quad \Omega,
\]
\[
T^+ \mathcal{R} u + \mathcal{L}^+(\varphi + \Phi_s) = T^+(\tilde{f}_{ss} + \tilde{E} \mathcal{R}_s \tilde{f}_{ss}, \mathcal{P} \tilde{f}_{ss}) \quad \text{on} \quad \partial\Omega,
\]
where the couple \((\tilde{f}_{ss}, \Phi_s) \in \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \) is given by (6.24), (6.25). Up to the notations, system (6.44)-(6.45) is the same as in (6.6) with \( \psi_0 = 0 \). Then Theorems 6.2 iii) and 6.5 imply that BDIE system (6.44)-(6.45) and hence (6.32)-(6.33) is solvable if and only if
Let us recall that \( P \) is spanned over the functional \( g^{*1} \) generated by (6.41). Therefore, we can always represent \( u_0(x) = 1 \).

**THEOREM 6.8.** The cokernel of operator (6.11) is spanned over the functional

\[
g^{*2} := \begin{pmatrix}
-a\gamma^{*2} \left( \frac{1}{2} + \mathcal{W}_x \right) \mathcal{V}_x^{-1} \gamma^{+} u^0 \\
-a \left( \frac{1}{2} - \mathcal{W}_x \right) \mathcal{V}_x^{-1} \gamma^{+} u^0
\end{pmatrix}
\]

in \( \tilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \), i.e.,

\[
g^{*2}(F_1, F_2) = \left\langle \begin{array}{c}
-a\gamma^{*2} \left( \frac{1}{2} + \mathcal{W}_x \right) \mathcal{V}_x^{-1} \gamma^{+} u^0, F_1 \\
-a \left( \frac{1}{2} - \mathcal{W}_x \right) \mathcal{V}_x^{-1} \gamma^{+} u^0, F_2
\end{array} \right\rangle_{\partial\Omega},
\]

where \( u^0(x) = 1 \).

**Proof.** Let us consider the equation \( \mathcal{R}u = (F_1, F_2)^{\top} \), i.e., the BDIE system (N2),

\[
\begin{align*}
\frac{1}{2} \varphi + \gamma^{+} \mathcal{R} u + \mathcal{W} \varphi &= F_1 \quad \text{in } \Omega, \\
\frac{1}{2} \varphi + \gamma^{+} \mathcal{R} u + \mathcal{W} \varphi &= F_2 \quad \text{on } \partial\Omega,
\end{align*}
\]

with arbitrary \((F_1, F_2) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)\) for \((u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)\).

Introducing the new variable, \( \varphi' = \varphi - (F_2 - \gamma^{+} F_1) \), BDIE system (6.48)-(6.49) takes form

\[
\begin{align*}
\frac{1}{2} \varphi' + \gamma^{+} \mathcal{R} u + \mathcal{W} \varphi' &= \gamma^{+} F_1^{\prime} \quad \text{in } \Omega, \\
\frac{1}{2} \varphi' + \gamma^{+} \mathcal{R} u + \mathcal{W} \varphi' &= \gamma^{+} F_2^{\prime} \quad \text{on } \partial\Omega,
\end{align*}
\]

where

\( F_1^{\prime} = F_1 - W(F_2 - \gamma^{+} F_1) \in H^1(\Omega). \)

Let us recall that \( \mathcal{P} = r_\Omega \mathcal{P} : \tilde{H}^{s-2}(\Omega) \to H^s(\Omega) \) and then by Theorem 6.5, the operator \( \mathcal{P}^{-1} = (r_\Omega \mathcal{P})^{-1} : H^s(\Omega) \to \tilde{H}^{s-2}(\Omega) \) is continuous for \( \frac{1}{2} < s < \frac{3}{2} \). Hence, we can always represent \( F_1^{\prime} = \mathcal{P} \tilde{f}_s \), with

\[
\tilde{f}_s = [\Delta \tilde{E}(I - r_\Omega \mathcal{V}_\Delta^{-1} \gamma^{+}) - \gamma^{*} \mathcal{V}_\Delta^{-1} \gamma^{+}](aF_1^{\prime}) \in \tilde{H}^{-1}(\Omega).
\]

For \( F_1^{\prime} = \mathcal{P} \tilde{f}_s \), the right hand side of BDIE system (6.48)-(6.49) is the same as in (6.9) with \( \tilde{f} = \tilde{f}_s \) and \( \psi_0 = 0 \). Then Theorem 6.2(iii) implies that BDIE system (6.50)-(6.51) is solvable if and only if

\[
\langle \tilde{f}_s, u^0 \rangle = \langle [\Delta \tilde{E}(I - r_\Omega \mathcal{V}_\Delta^{-1} \gamma^{+}) - \gamma^{*} \mathcal{V}_\Delta^{-1} \gamma^{+}](aF_1^{\prime}), u^0 \rangle_{\mathbb{R}^3}
\]
\[
= \langle \hat{E}(I - r_\Omega V_\Delta V_\Delta^{-1} \gamma^+(aF_1'), \Delta u^0)_{\mathbb{R}^3} - \langle \gamma^+(aF_1'), V_\Delta^{-1} \gamma^+(aF_1'), u^0 \rangle_{\mathbb{R}^3},
= -\langle \gamma^+(aF_1'), V_\Delta^{-1} \gamma^+ u^0 \rangle_{\partial \Omega},
= -\left\langle \frac{1}{2}[\gamma^+(aF_1) + (aF_2)] - \Delta \omega_\Delta [a(F_2 - \gamma^+ F_1)], V_\Delta^{-1} \gamma^+ u^0 \right\rangle_{\partial \Omega},
\]
\[
= \left\langle -a \gamma^+ \left( \frac{1}{2} + \omega_\Delta \right) V_\Delta^{-1} \gamma^+ u^0, F_1 \right\rangle_{\Omega} + \left\langle -a \left( \frac{1}{2} - \omega_\Delta \right) V_\Delta^{-1} \gamma^+ u^0, F_2 \right\rangle_{\partial \Omega} = 0. \tag{6.52}
\]
Thus the functional \( g^{*2} \) defined by (6.47) generates the necessary and sufficient solvability condition of equation \( \mathfrak{N}^2 U = (F_1, F_2)^\top \). Hence \( g^{*2} \) is a basis of the cokernel of \( \mathfrak{N}^2 \).

\[\Box\]

6.3 Perturbed segregated BDIE systems for the Neumann problem

Theorem 6.2 implies, that even when the solvability condition (6.3) is satisfied, the solutions of both BDIE systems, (N1) and (N2), are not unique. By Theorem 6.3 in turn, the BDIE left hand side operators, \( \mathfrak{N}^1 \) and \( \mathfrak{N}^2 \), have non-zero kernels and thus are not invertible. To find a solution \( (u, \varphi) \) from uniquely solvable BDIE systems with continuously invertible left hand side operators, let us consider, following [25], some BDIE systems obtained from (N1) and (N2) by finite-dimensional operator perturbations. Note that other choices of the perturbing operators are also possible.

Below we use the notations \( \mathcal{U}^N = (u, \varphi)^\top \) and \( |\partial \Omega| := \int_{\partial \Omega} dS \).

6.3.1 Perturbation of BDIE system (N1)

Let us introduce the perturbed counterparts of the BDIE system (N1),

\[ \hat{\mathfrak{N}}^1 \mathcal{U}^N = F^N, \tag{6.53} \]

where

\[ \hat{\mathfrak{N}}^1 := \mathfrak{N}^1 + \hat{\mathfrak{N}}^1 \] and \( \hat{\mathfrak{N}}^1 \mathcal{U}^N(y) := g^0(\mathcal{U}^N) G^1(y) = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi(x) dS \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]

that is,

\[ g^0(\mathcal{U}^N) := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi(x) dS, \quad G^1(y) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

For the functional \( g^{*1} \) given by (6.41) in Theorem 6.7, \( g^{*1}(G^1) = |\partial \Omega|, \) while \( g^0(\mathcal{U}^0) = 1. \) Hence Theorem 6.7 in Appendix, extracted from [25], implies the following assertion.

THEOREM 6.9. (i) The operator \( \hat{\mathfrak{N}}^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \rightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \) is continuous and continuously invertible.

(ii) If condition \( g^{*1}(F^N) = 0 \) (or condition (6.3) for \( F^N \) in form (6.6)) is satisfied, then the unique solution of perturbed BDIDE system (6.53) gives a solution of original BDIE system (D2) such that

\[ g^0(\mathcal{U}^N) = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \gamma^+ u dS = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi dS = 0. \]
6.3.2 Perturbation of BDIE system (N2)

The perturbation operators chosen below for BDIE system (N2) are slightly different from those, used in [25] for the purely boundary integral equations, in [31, Section 3] for a united localised BDIE system and in [30, Section 2] for a united non-localised BDIE system.

Let us introduce the perturbed counterparts of the BDIE system (N1),

\[ \hat{N}^2 \mathcal{U}^N = \mathcal{F}^{N2}, \]  

where

\[ \hat{N}^2 := N^2 + \hat{N}^2 \quad \text{and} \quad \hat{N}^2 \mathcal{U}^N := g^0(\mathcal{U}^N) G^2 = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi(x) dS \begin{pmatrix} a^{-1}(y) \\ \gamma^+ a^{-1}(y) \end{pmatrix}, \]  

that is,

\[ g^0(\mathcal{U}^N) := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi(x) dS, \quad G^2 := \begin{pmatrix} a^{-1}(y) u^0(y) \\ \gamma^+ [a^{-1} u^0](y) \end{pmatrix}. \]

For the functional \( g^2 \) given by (6.47) in Theorem 6.8, since the operator \( V^{-1}_\Delta : H^\frac{1}{2}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega) \) is positive definite and \( u^0(x) = 1 \), there exists a positive constant \( C \) such that

\[
g^2(G^2) = \left\langle -a \gamma^+ \left( \frac{1}{2} + \mathcal{W}_\Delta \right) V^{-1}_\Delta \gamma^+ u^0, a^{-1} u^0 \right\rangle_\Omega + \left\langle -a \left( \frac{1}{2} - \mathcal{W}_\Delta \right) V^{-1}_\Delta \gamma^+ u^0, \gamma^+ (a^{-1} u^0) \right\rangle_{\partial \Omega}
\]

\[
= -\left\langle \left( \frac{1}{2} + \mathcal{W}_\Delta \right) V^{-1}_\Delta \gamma^+ u^0 + \left( \frac{1}{2} - \mathcal{W}_\Delta \right) V^{-1}_\Delta \gamma^+ u^0, \gamma^+ u^0 \right\rangle_{\partial \Omega}
\]

\[
\leq -C \| \gamma^+ u^0 \|_{H^\frac{1}{2}(\partial \Omega)}^2 \leq -C \| \gamma^+ u^0 \|_{L^2(\partial \Omega)}^2 = -C |\partial \Omega|^2 < 0. \quad (6.55)
\]

Due to (6.55) and \( g^0(\mathcal{U}^0) = 1 \), Theorem [D.1] in Appendix implies the following assertion.

**THEOREM 6.10.** (i) The operator \( \hat{N}^2 : H^1(\Omega) \times H^\frac{1}{2}(\partial \Omega) \rightarrow H^1(\Omega) \times H^\frac{1}{2}(\partial \Omega) \) is continuous and continuously invertible.

(ii) If condition \( g^2(\mathcal{F}^{N2}) = 0 \) (or condition (6.3) for \( \mathcal{F}^{N2} \) in form (6.6)) is satisfied, then the unique solution of perturbed BDIDE system (6.54) gives a solution of original BDIE system (N2) such that

\[ g^0(\mathcal{U}^N) = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \gamma^+ u dS = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi dS = 0. \]

**APPENDIX**

A Function from \( H^1(\Omega) \) with no classical or canonical conormal derivative

For functions from \( H^1(\Omega) \) the co–normal derivative \( a \partial_n u \) on \( \partial \Omega \) may not exist in the classical (trace) or even canonical sense. In this section we consider an example of such function.
Let $\Omega$ be a ball $B_{r_0} \subset \mathbb{R}^3$ of some radius $r_0 > 0$ with the centre at $x = 0$. Let $a = 1$ and hence $A$ be the Laplace operator $\Delta$. Let us consider the function

$$u(x) = (r_0^2 - |x|^2)^{3/4}, \quad x \in \Omega.$$ 

Evidently, this function is infinitely smooth in $\Omega$, vanishes on the boundary and its gradient

$$\nabla u(x) = -\frac{3}{2}x(r_0^2 - |x|^2)^{-1/4}$$

(A.1)

belongs to $L_p(\Omega)$, $0 < p < 4$ and hence to $L_2(\Omega)$. This implies that $u$ belongs to the Sobolev space $W^{1,p}_0(\Omega)$, $0 < p < 4$ and thus $u \in H^1(\Omega)$. For the classical conormal derivative we have,

$$T^+_c u(x) = n(x) \cdot \lim_{|x| \to r_0} \nabla u(x) = -\infty,$$

which evidently means that it does not belong to any Sobolev space on the boundary.

On the other hand, $u$ solves the Dirichlet problem

$$\Delta u = f \in H^{-1}(\Omega) \quad \text{in} \quad \Omega,$$
$$\gamma^+ u = 0 \quad \text{on} \quad \partial \Omega$$

(A.2)

(A.3)

with

$$f(x) = -\frac{9}{2}(r_0^2 - |x|^2)^{-1/4} + \frac{3}{4}|x|^2(r_0^2 - |x|^2)^{-5/4} \in H^{-1}(\Omega).$$

To define the canonical conormal derivative of $u$ according to Definitions 2.2 and 2.13, the function $f$ should at least belong to $H^{-\frac{1}{2}}(\Omega)$. Let us prove that this is not the case. Indeed, if $f \in H^{-\frac{1}{2}}(\Omega)$, then the dual form $\langle f, \tilde{g} \rangle_\Omega$ should be bounded for any test function $\tilde{g} \in \tilde{H}^{1/2}(\Omega)$. Let us take

$$\tilde{g}(x) = \begin{cases} (r_0^2 - |x|^2)^{1/4}, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}.$$ 

Estimating the Sobolev-Slobodetski norm of this function one can prove that $\tilde{g}$ belongs to the space $\tilde{H}^s(\Omega)$ for any $s < 2/3$ and particularly to $\tilde{H}^{1/2}(\Omega)$. However

$$f(x)\tilde{g}(x) = -\frac{9}{2} + \frac{3}{4}|x|^2(r_0^2 - |x|^2)^{-1} \quad \text{in} \quad \Omega$$

and hence $\langle f, \tilde{g} \rangle_\Omega = \int_\Omega f(x)\tilde{g}(x)dx$ is not bounded. This implies that $f \notin H^{-\frac{1}{2}}(\Omega)$ and the canonical conormal derivative is also not defined.

To calculate the generalised co-normal derivative, one has to extend the function $f \in H^{-1}(\Omega)$ to the function $\tilde{f} \in \tilde{H}^{-1}(\Omega)$. As remarked in [28, Lemma 2.15] this is always possible due to the Hahn-Banach theorem, and an explicit extension is suggested in [28, Theorem 2.16], although the extension is not unique. Particularly, one can assign $\tilde{f} = \tilde{A}u$, i.e., by (2.6),

$$\langle \tilde{f}, v \rangle_\Omega = -\int_\Omega \nabla u(x) \cdot \nabla v(x)dx = -\int_\Omega \nabla u(x) \cdot \nabla v(x)dx = \langle \nabla \cdot \hat{E}\nabla u, v \rangle_\Omega, \quad \forall v \in H^1(\Omega),$$

(A.4)
where $\nabla u$ is given by (A.1). Then (2.8) implies that the generalised conormal derivative, $T^+(\tilde{f}, u)$, is well defined on $\partial \Omega$ and is zero. Different extensions of $f$ to $\tilde{f}$ will lead to different conormal derivatives, and moreover, any distribution from $H^{-\frac{1}{2}}(\partial \Omega)$ can be nominated as conormal derivative by appropriate choice of extension $\tilde{f}$, cf. [1, Section 2.2, item 4], [28, Eq. (3.13)], [29, Eq. (5.10)].

**B Approximation of generalised conormal derivatives by classical ones**

**THEOREM B.1.** Let $u \in H^1(\Omega)$, $Au = r_\Omega \tilde{f}$ in $\Omega$ for some $\tilde{f} \in \tilde{H}^{-1}(\Omega)$, and $\{\tilde{f}_k\} \in \mathcal{D}(\Omega)$ be a sequence such that $\|\tilde{f} - \tilde{f}_k\|_{\tilde{H}^{-1}(\Omega)} \to 0$ as $k \to \infty$. Then there exists a sequence $\{u_k\} \in \mathcal{D}(\Omega)$ such that $Au_k = r_\Omega \tilde{f}_k$ and $\|u - u_k\|_{H^1(\Omega)} \to 0$ as $k \to \infty$. Moreover, $\|T^+(u_k) - T^+(\tilde{f}, u)\|_{H^{-\frac{1}{2}}(\partial \Omega)} \to 0$ as $k \to \infty$.

*Proof.* Let us consider the Dirichlet problem

\[ A u_k = \tilde{f}_k \text{ in } \Omega, \]  
\[ \gamma^+ u_k = \varphi_k \text{ on } \partial \Omega, \]

where $\{\varphi_k\} \in \mathcal{D}(\partial \Omega)$ is a sequence converging to $\gamma^+ u$ in $H^{\frac{1}{2}}(\partial \Omega)$. By Theorem 5.1 the unique solution of problem (B.1)-(B.2) in $H^1(\Omega)$ is $u_k = (A^D)^{-1}(\tilde{f}_k, \varphi_k) + \gamma$, where $(A^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \to H^1(\Omega)$ is a continuous operator. Hence the functions $u_k$ converge to $u$ in $H^1(\Omega)$ as $k \to \infty$. Due to infinite smoothness of the data $(\tilde{f}_k, \varphi_k)$ and the boundary $\partial \Omega$, the solution $u_k$ belongs to $\mathcal{D}(\Omega)$ implying that its classical conormal derivative $T^+ u_k$ is well defined. Since $\tilde{A} u_k = \tilde{f}_k \in \mathcal{D}(\Omega) \in L_2(\Omega)$, the canonical conormal derivative is also well defined and equals to the classical one. Then subtracting (2.13) for $u_k$ from (2.8), we obtain,

\[ \langle T^+(\tilde{f}, u) - T^+ u_k, w \rangle_{\partial \Omega} = (\tilde{f} - \tilde{f}_k, \gamma w)_{\Omega} + \mathcal{E}(u - u_k, \gamma w) \quad \forall w \in H^{\frac{1}{2}}(\partial \Omega). \]

Then

\[ \|T^+(\tilde{f}, u) - T^+ u_k\|_{H^{-\frac{1}{2}}(\partial \Omega)} \leq C \left( \|\tilde{f} - \tilde{f}_k\|_{\tilde{H}^{-1}(\Omega)} + \|\nabla(u - u_k)\|_{L^2(\Omega)} \right) \]

for some positive $C$. Since the right hand side of (B.3) tends to zero as $k \to \infty$, so does also the left hand side.

**C Properties of the surface and volume potentials**

The mapping and jump properties of the potentials of type (3.6)-(3.7), (3.10)-(3.11) and the corresponding boundary integral and pseudodifferential operators in the Hölder ($C^{k+\alpha}$), Bessel potential ($H^p_s$) and Besov ($B_{p,q}^s$) spaces are well studied nowadays for the constant coefficient, $a = const$, (see, e.g., a list of references in [3 [19]]). Employing relations (3.18)-(3.20), some of the properties were extended in [3 [5] to the case of variable positive coefficient $a \in C^\infty(\mathbb{R})$, and several of those results are provided here for convenience (without proofs).
THEOREM C.1. Let $\Omega$ be a bounded open three–dimensional region of $\mathbb{R}^3$ with a simply connected, closed, infinitely smooth boundary. The following operators are continuous

$$\mathcal{P} : \tilde{H}^s(\Omega) \to H^{s+2}(\Omega), \quad s \in \mathbb{R},$$  \hspace{1cm} (C.1)$$
$$\mathcal{R}, \mathcal{R}_* : \tilde{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R},$$  \hspace{1cm} (C.2)$$
$$\gamma^+\mathcal{P} : \tilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{1}{2};$$  \hspace{1cm} (C.3)$$
$$\gamma^+\mathcal{R} : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2};$$  \hspace{1cm} (C.4)$$
$$T^+\mathcal{P} : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2};$$  \hspace{1cm} (C.5)$$
$$T^+\mathcal{R} : \tilde{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2};$$  \hspace{1cm} (C.6)$$
$$\gamma^+\mathcal{R} : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2};$$  \hspace{1cm} (C.7)$$
$$\gamma^+\mathcal{P} : \tilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{1}{2};$$  \hspace{1cm} (C.8)$$
$$T^+\mathcal{P} : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2};$$  \hspace{1cm} (C.9)$$
$$T^+\mathcal{R} : \tilde{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2};$$  \hspace{1cm} (C.10)$$
$$\gamma^+\mathcal{R} : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2};$$  \hspace{1cm} (C.11)$$
$$\gamma^+\mathcal{P} : \tilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{1}{2};$$  \hspace{1cm} (C.12)

COROLLARY C.2. The following operators are continuous,

$$\mathcal{P} : \tilde{H}^s(\Omega) \to H^{s+2,\frac{1}{2}}(\Omega; L), \quad s \geq -\frac{1}{2},$$  \hspace{1cm} (C.13)$$
$$\mathcal{R}_* : \tilde{H}^s(\Omega) \to H^{s+2,\frac{1}{2}}(\Omega; L), \quad s > -\frac{1}{2};$$  \hspace{1cm} (C.14)$$
$$\mathcal{R} : \tilde{H}^s(\Omega) \to H^{s+1,\frac{1}{2}}(\Omega; L), \quad s > \frac{1}{2};$$  \hspace{1cm} (C.15)

Proof. Continuity of operators (C.1), (C.2) and (C.4) imply continuity of operator (C.13) for $s > -\frac{1}{2}$ as well as (C.14) and (C.15).

Let us prove (C.13) for $s = -\frac{1}{2}$. For $g \in \tilde{H}^{-\frac{1}{2}}(\Omega)$, we have, $\mathcal{P} g \in H^\frac{3}{2}(\Omega)$ due to (C.1), while

$$\Delta \mathcal{P} g = \Delta \left[ \frac{1}{a} \mathcal{P}_\Delta g \right] =$$

$$\frac{1}{a} g + 2 \sum_{j=1}^{3} \partial_j \left[ \frac{1}{a} \right] \partial_j [\mathcal{P}_\Delta g] + \left[ \Delta \frac{1}{a} \right] \mathcal{P}_\Delta g \quad \text{in} \quad \mathbb{R}^3,$$

where $\mathcal{P}_\Delta := \mathcal{P}|_{a=1}$, and we taken into account that $\Delta \mathcal{P}_\Delta g = g$. The first term in (C.16) belongs to $\tilde{H}^{-\frac{1}{2}}(\Omega)$, while, since $a \in C^\infty(\bar{\Omega})$, $a > 0$, the sum of the second and the third term belongs to $H^{\frac{3}{2}}(\Omega)$ and can be extended by zero to $\tilde{H}^0(\Omega) \subset \tilde{H}^{-\frac{1}{2}}(\Omega)$, which completes the proof of continuity for operator (C.13) for $s = -\frac{1}{2}$. \qed
THEOREM C.3. The following operators are continuous,

\begin{align*}
V : H^{s-3/2}(\partial \Omega) &\to H^s(\Omega), \quad s \in \mathbb{R}, \quad (C.17) \\
: H^{s-3/2}(\partial \Omega) &\to H^{s-3/2}(\Omega; L), \quad s > \frac{1}{2}; \quad (C.18) \\
W : H^{s-3/2}(\partial \Omega) &\to H^s(\Omega), \quad s \in \mathbb{R}, \quad (C.19) \\
: H^{s-3/2}(\partial \Omega) &\to H^{s-3/2}(\Omega; L), \quad s > \frac{1}{2}. \quad (C.20)
\end{align*}

THEOREM C.4. Let \( s \in \mathbb{R} \). The following pseudodifferential operators are continuous

\begin{align*}
\mathcal{V} : H^s(\partial \Omega) &\to H^{s+1}(\partial \Omega), \\
\mathcal{W}, \mathcal{W}' : H^s(\partial \Omega) &\to H^{s+1}(\partial \Omega), \\
\mathcal{L}^+ : H^s(\partial \Omega) &\to H^{s-1}(\partial \Omega).
\end{align*}

THEOREM C.5. Let \( s \in \mathbb{R} \). The operators

\begin{align*}
r_{s_2} \mathcal{V} : H^s(\partial \Omega) &\to H^s(\partial \Omega), \\
r_{s_2} \mathcal{W} : H^s(\partial \Omega) &\to H^s(\partial \Omega), \\
r_{s_2} \mathcal{W}' : H^s(\partial \Omega) &\to H^s(\partial \Omega)
\end{align*}

are compact.

THEOREM C.6. The operator

\( \mathcal{V} : H^{s-1}(\partial \Omega) \to H^s(\partial \Omega) \)

is continuously invertible for all \( s \in \mathbb{R} \).

COROLLARY C.7. The operators

\begin{align*}
\mathcal{R} : H^s(\Omega) &\to H^s(\Omega), \quad s > -\frac{1}{2}, \quad (C.21) \\
&\to H^s(\Omega) \to H^{s-\frac{1}{2}}(\Omega; L), \quad s > \frac{1}{2}; \quad (C.22) \\
\gamma^+ \mathcal{R} : H^s(\Omega) &\to H^{s-\frac{1}{2}}(\partial \Omega), \quad s > -\frac{1}{2}; \quad (C.23) \\
T^+ \mathcal{R} : H^s(\Omega) &\to H^{s-\frac{3}{2}}(\partial \Omega), \quad s > \frac{1}{2}. \quad (C.24)
\end{align*}

are compact for any infinitely smooth boundary curve \( \partial \Omega \).

Proof. Compactness of the operators \( (C.21), (C.23) \) and \( (C.24) \) follow from \( (C.4), (C.8) \), and \( (C.12) \), respectively, and the Rellich compact embedding theorem. Then \( (C.21) \) and \( (C.4) \) imply \( (C.22) \). \( \square \)
D Finite dimensional perturbation of operator equations

Theorem [D.1] below is implied by [25, Lemma 2] (see also [12, §21], [11, Section 21.4], where the particular case, \( h^*_i(x_j) = x^*_i(h_j) = \delta_{ij} \), has been considered). Another approach, although with hypotheses similar to the ones in Theorem [D.1], is presented in [16, Lemma 4.8.24].

**THEOREM D.1.** Let \( B_1 \) and \( B_2 \) be two Banach spaces. Let \( A : B_1 \to B_2 \) be a linear continuous Fredholm operator with zero index, \( A^* : B_2^* \to B_1^* \) be the operator adjoined to it, and \( \dim \ker A = \dim \ker A^* = n < \infty \), where \( \ker A = \text{span} \{ \delta_i \}_{i=1}^n \subset B_1 \), \( \ker A^* = \text{span} \{ h^*_i \}_{i=1}^n \subset B_2^* \). Let

\[
A_1 x := \sum_{i=1}^k h_i h^*_i(x),
\]

where \( h^*_i, h_i (i = 1, ..., n) \) are elements from \( B_1^* \) and \( B_2 \), respectively, such that

\[
\det[h^*_i(\delta_j)] \neq 0, \quad \det[\delta^*_i(h_j)] \neq 0 \quad i, j = 1, ..., n. \tag{D.1}
\]

Then:

(i) the operator \( A - A_1 : B_1 \to B_2 \) is continuous and continuously invertible;

(ii) if \( y \in B_2 \) satisfies the solvability conditions,

\[
\delta^*_i(y) = 0, \quad i = 1, ..., n, \tag{D.2}
\]

of equation

\[
A x = y, \tag{D.3}
\]

then the unique solution \( x \) of equation

\[
(A - A_1)x = y, \tag{D.4}
\]

is a solution of equation (D.3) such that

\[
h^*_i(x) = 0 \quad (i = 1, ..., k). \tag{D.5}
\]

(iii) Vice versa, if \( x \) is a solution of equation (D.4) satisfying conditions (D.3), then conditions (D.2) are satisfied for the right-hand side \( y \) of equation (D.4) and \( x \) is a solution of equation (D.3) with the same right-hand side \( y \).

Note that more results about finite-dimensional operator perturbations are available in [25].

Concluding remarks

The Dirichlet and Neumann problems for a variable-coefficient PDE with general right-hand side functions from \( H^{-1}(\Omega) \) and \( \tilde{H}^{-1}(\Omega) \), respectively, were equivalently reduced to two direct segregated boundary-domain
integral equation systems, for each of the BVPs. This involved systematic use of the generalised co-normal derivatives without assumption that they reduce to classical or canonical co-normal derivatives. The operators associated with the left-hand sides of all the BDIE systems were analysed in corresponding Sobolev spaces. It was shown that the operators of the BDIE systems for the Dirichlet problem are continuous and continuously invertible. For the Neumann problem the BDIE system operators are continuous but only Fredholm with zero index, their kernels and co-kernels were analysed, and appropriate finite-dimensional perturbations were constructed to make the perturbed operators invertible and provide a solution of the original BDIE systems and the Neumann problem. A further analysis of spectral properties of the two second kind equations obtained in the paper is needed to decide whether the resolvent theory and the Neumann series method (cf. \cite{24, 39} and references therein) are efficient for solving the equations.

The same approach can be used to extend, to the general PDE right hand sides, the BDIE systems for the mixed problems, unbounded domains, BDIEs of more general scalar PDEs and the systems of PDEs, as well as to the united and localised BDIEs, for which the analysis is now available for the right hand sides only from $L^2(\Omega)$, see \cite{3, 10, 27, 2, 13, 32, 33}. The conditions on smoothness of the variable coefficients and the boundary can be also essentially relaxed.

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