PERTURBATION DETERMINANTS, THE SPECTRAL SHIFT FUNCTION, TRACE IDENTITIES, AND ALL THAT

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Abstract. We discuss applications of the M. G. Krein theory of the spectral shift function to the multi-dimensional Schrödinger operator as well as specific properties of this function, for example, its high-energy asymptotics. Trace identities for the Schrödinger operator are derived.

1. Introduction

The spectral shift function (SSF) \( \xi(\lambda) = \xi(\lambda; H, H_0) \) is introduced by the relation

\[
\text{Tr} \left( f(H) - f(H_0) \right) = \int_{-\infty}^{\infty} \xi(\lambda)f'(\lambda)d\lambda, \tag{1.1}
\]
called the trace formula. The concept of the SSF in the perturbation theory appeared at the beginning of fifties in the physics literature in the paper [19] by I. M. Lifshitz. Its mathematical theory was shortly created by M. G. Krein who proved in [15] relation (1.1) for a pair of self-adjoint operators \( H_0, H \) with a trace-class difference \( V = H - H_0 \) and a wide class of functions \( f \). Then it was extended by him in [16] (see [17], for a more complete exposition) to operators \( H_0, H \) with a trace-class difference \( R(z) - R_0(z) \) of the resolvents. Moreover, a link (the Birman-Krein formula) with the scattering matrix (SM) \( S(\lambda) = S(\lambda; H, H_0) \) was found in [3] where it was shown that, for \( \lambda \) from a core of the spectrum of \( H_0 \),

\[
\text{Det } S(\lambda) = \exp \left( -2\pi i \xi(\lambda) \right). \tag{1.2}
\]

Later Krein’s theory was developed in various directions and applied to differential operators. A sufficiently detailed exposition of these developments can be found in survey [5] or in book [29] where however applications to differential operators were almost not discussed. In the present paper which can to a certain extent be considered as a continuation of [5], we try to partially fill in this gap. We study only the Schrödinger operator which looks as a sufficiently representative example.

In Section 2 we recall briefly basic results of Krein’s theory. In Section 3 we discuss its applications to the Schrödinger operator in the space \( \mathcal{H} = L_2(\mathbb{R}^d) \). Actually, our goal here is to extend Krein’s theory to the case of perturbations which are not of trace-class type. Interesting results in this direction can be found in [14] and [20]. The analogue of the SSF is called the regularized SSF. Our “direct” approach turns out to be fruitful even in the case of perturbations of trace-class type and allows us to reveal specific properties of the SSF which are not true for arbitrary self-adjoint operators. For example, we show that \( \xi(\lambda) \) is a continuous function of \( \lambda > 0 \) (in general, it belongs only to the class \( L_1^{(loc)} \)), but the proof of this result is curiously complicated for \( d > 3 \).

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Another goal of this paper is to obtain the high-energy asymptotic expansion of the SSF and of related objects which is the crucial ingredient in derivation of trace identities. Both these subjects were intensively studied in the literature; see [7, 9] for the one-dimensional case and [6, 8] for the case \( d = 3 \). We are mainly interested in the multi-dimensional case where rather stringent assumptions on the potential \( v(x) \) (it was required that \( v \) belongs to the Schwartz class) were imposed in [6, 8]. We note also papers [24, 25] where the high-energy asymptotic expansion of the SSF was obtained under natural assumptions on \( v(x) \) in the framework of the microlocal calculus.

In the present paper we suggest a new approach to construction of the asymptotic expansion (AE) of \( \xi(\lambda) \) as \( \lambda \to \infty \). Actually, two different problems arise here. The first of them is the proof of the existence of such expansion. The second problem consists in obtaining reasonably simple expressions for asymptotic coefficients. We solve the first problem in Section 6 using the high-energy AE of the SM \( S(\lambda) \) (see Section 5) and the Birman-Kreǐn formula. This entails also the AE of the function \( \text{Tr}(R^m(z) - R_0^m(z)) \), where \( m > d/2 - 1 \), as \( |z| \to \infty \) in the whole complex plane cut along \( \mathbb{R}_+ \). In Section 4 we use another approach to construction of this AE which works only away from the spectrum but gives reasonably simple expressions for asymptotic coefficients. This yields also an efficient expression for coefficients in the AE of the SSF. Another expression for these coefficients is given in terms of the so-called heat invariants. Finally, trace identities, both of integer and half-integer orders, are derived in Section 7.

2. Kreǐn’s theory

1. Below we use the following standard definitions (see, e.g., [10]): \( \mathcal{S}_p \), \( p \geq 1 \), is the symmetrically normed ideal (with the norm \( \| \cdot \|_p \) ) of the algebra of all bounded operators in a Hilbert space \( \mathcal{H} \); in particular, \( \mathcal{S}_1 \) is the ideal of trace operators and \( \mathcal{S}_2 \) is the ideal of Hilbert–Schmidt operators; the determinant \( \text{Det}(I + A) \) is well defined for \( A \in \mathcal{S}_1 \) and possesses standard properties which are basically the same as in the finite-dimensional case; more generally, the regularized determinant \( \text{Det}_p(I + A), p = 1, 2, \ldots \), is well defined for \( A \in \mathcal{S}_p \).

Let \( H_0, H \) be self-adjoint operators in a Hilbert space \( \mathcal{H} \), and let \( R_0(z) = (H_0 - z)^{-1}, R(z) = (H - z)^{-1} \) be their resolvents. The perturbation determinant (PD) for a pair \( H_0, H \) with the difference \( V = H - H_0 \in \mathcal{S}_1 \) is defined by the relation

\[
D(z) = D_{H/H_0}(z) = \text{Det}(I + VR_0(z))
\]

(2.1)
on the set \( \rho(H_0) \) of regular points of the operator \( H_0 \). The function \( D(z) \) is holomorphic on \( \rho(H_0) \),

\[
D(\bar{z}) = \overline{D(z)},
\]

(2.2)
and it has a zero \( z \) of order \( k \) if and only if \( z \) is an eigenvalue of multiplicity \( k \) of the operator \( H \). Moreover, \( D(z) \) satisfies the identity

\[
D^{-1}(z)D'(z) = \text{Tr} \left( R_0(z) - R(z) \right), \quad z \in \rho(H_0) \cap \rho(H).
\]

(2.3)
Since \( D(z) \to 1 \) as \( |\text{Im} \, z| \to \infty \), we can fix the branch of the function \( \ln D(z) \) in the half-planes \( \pm \text{Im} \, z > 0 \) by the condition \( \text{arg} \, D(z) \to 0 \) as \( |\text{Im} \, z| \to \infty \). These properties of PD extend to the case \( VR_0(z) \in \mathcal{S}_1 \).

The SSF is constructed in terms of the PD in the following theorem of M. G. Kreǐn.
Theorem 2.1. For \( V \in \mathcal{S}_1 \), there is the representation
\[
\ln D(z) = \int_{-\infty}^{\infty} \xi(\lambda)(\lambda - z)^{-1} d\lambda, \quad \text{Im } z \neq 0,
\]
where
\[
\xi(\lambda) = \pi^{-1} \lim_{\varepsilon \to 0^+} \arg D(\lambda + i\varepsilon).
\]
For almost every (a.e.) \( \lambda \in \mathbb{R} \) the limit in (2.5) exists and
\[
\int_{-\infty}^{\infty} |\xi(\lambda)| d\lambda \leq \|V\|_1, \quad \int_{-\infty}^{\infty} \xi(\lambda) d\lambda = \text{Tr } V.
\]
Moreover, \( \pm \xi(\lambda) \geq 0 \) if \( \pm V \geq 0 \).

In a gap of the continuous spectrum \( \xi(\lambda) \) depends on the shift of eigenvalues of the operator \( H \) relative to eigenvalues of \( H_0 \).

Proposition 2.2. On component intervals of the set of common regular points of the operators \( H_0 \) and \( H \) the SSF \( \xi(\lambda) \) assumes constant integral values. If \( \lambda \) is an isolated eigenvalue of finite multiplicity \( k_0 \) of the operator \( H_0 \) and \( k \) of the operator \( H \), then
\[
\xi(\lambda + 0) - \xi(\lambda - 0) = k_0 - k.
\]

The trace formula (1.1) was justified by M. G. Krein for a sufficiently wide class of functions.

Theorem 2.3. Suppose \( V \in \mathcal{S}_1 \) and the function \( f \) is continuously differentiable while its derivative admits the representation
\[
f'(\lambda) = \int_{-\infty}^{\infty} \exp(-it\lambda) dm(t), \quad |m|(\mathbb{R}) < \infty,
\]
with a finite (complex) measure \( m \). Then
\[
f(H) - f(H_0) \in \mathcal{S}_1
\]
and formula (1.1) holds.

Remark 2.4. Condition (2.6) can be somewhat modified and relaxed (see [4], [21]). For example, in the case of bounded operators \( H_0, H \) it suffices to require that \( f' \) is a Hölder continuous function.

2. Now we recall briefly basic notions of scattering theory (see, e.g., [29], for details). Let \( E(\cdot) \) be the spectral projection of the operator \( H \), let \( \mathcal{H}^{(a)} \) be its absolutely continuous subspace and let \( P \) be the orthogonal projection on \( \mathcal{H}^{(a)} \); the same objects for the operator \( H_0 \) are endowed with the index “0”. The wave operator for a pair of selfadjoint operators \( H_0 \) and \( H \) is defined by the relation
\[
W_\pm = W_\pm(H, H_0) = \lim_{t \to \pm \infty} \exp(iHt) \exp(-iH_0t) P_0(2.8)
\]
provided the corresponding strong limit exists. The wave operator is isometric on \( \mathcal{H}_0^{(a)} \) and enjoys the intertwining property \( W_\pm E_0(\Lambda) = E(\Lambda) W_\pm \) where \( \Lambda \subset \mathbb{R} \) is an arbitrary Borel set. In particular, its range \( \text{Ran } W_\pm \subset \mathcal{H}^{(a)} \). The operator \( W_\pm \) is said to be complete if \( \text{Ran } W_\pm = \mathcal{H}^{(a)} \). Thus, if the wave operators \( W_\pm \) exist and are complete, then the absolutely continuous parts of the operators \( H_0 \) and \( H \) are unitarily equivalent. In this case the scattering operator
\[
S = S(H, H_0) = W_+(H, H_0)W_-(H, H_0)
\]
commutes with $H_0$ and is unitary on the space $H_0^{(a)}$.

To define the SM $S(\lambda; H, H_0)$, we suppose for simplicity that the spectrum $\sigma_0 = \sigma(H_0)$ of the operator $H_0$ is absolutely continuous, consists of a finite (or locally finite) union of closed intervals and has constant multiplicity. Let $\mathfrak{h}$ be an auxiliary space whose dimension is equal to this multiplicity. Then there exists a unitary mapping $\mathcal{F} : \mathcal{H} \rightarrow L_2(\sigma_0; \mathfrak{h})$ such that the operator $\mathcal{F} H_0 \mathcal{F}^*$ acts as multiplication by the independent variable ($\lambda$) in the space $L_2(\sigma_0; \mathfrak{h})$. It follows from the relation $\mathcal{S} H_0 = H_0 \mathcal{S}$ that the operator $\mathcal{F} \mathcal{S} \mathcal{F}^*$ acts in the space $L_2(\sigma_0; \mathfrak{h})$ as multiplication by the unitary operator-valued function $S(\lambda) = S(\lambda; H, H_0) : \mathfrak{h} \rightarrow \mathfrak{h}$ defined for a.e. $\lambda \in \sigma_0$. Of course, this definition fixes the SM $S(\lambda)$ up to a unitary equivalence in the space $\mathfrak{h}$.

The Kato-Rosenblum theorem asserts that the wave operators $W_{\pm}(H, H_0)$ exist and are complete if $V \in \mathfrak{S}_1$. A link between the SSF and the SM was found in [3].

**Theorem 2.5.** If $V \in \mathfrak{S}_1$, then $S(\lambda) - I \in \mathfrak{S}_1$ and formula (1.2) holds for a.e. $\lambda \in \sigma_0$.

Relation (1.2) is often used for the definition (up to an integer number) of the SSF on the absolutely continuous part of the spectrum. Differentiating it formally, we obtain that

$$\text{Tr} \left( S^*(\lambda) S'(\lambda) \right) = -2\pi i \xi'(\lambda).$$

This relation is sometimes more convenient than (1.2) in applications since there exists a simple expression for the trace of an integral operator with smooth kernel.

3. In view of our applications to the Schrödinger operator, let us consider generalizations of Theorems 2.1, 2.3 and 2.5 specific for the semibounded case. With a shift by a constant, it may be achieved that the operators $H_0 + cI$ and $H + cI$ are positive definite. We assume that

$$R^m(z) - R_0^m(z) \in \mathfrak{S}_1$$

for $z = -c$ and some $m > 0$. It follows from Theorem 2.1 that the SSF for the pair $h_0 = (H_0 + cI)^{-m}, h = (H + cI)^{-m}$ exists and belongs to the space $L_1(\mathbb{R})$. Moreover, according to Proposition 2.2 it equals to zero on $\mathbb{R}^-$ and for sufficiently large $\lambda > 0$. For the initial pair $H_0, H$, the SSF is defined by the equalities

$$\xi(\lambda; H, H_0) = -\xi((\lambda + c)^{-m}; (H + cI)^{-m}, (H_0 + cI)^{-m})$$

for $\lambda > -c$ and $\xi(\lambda; H, H_0) = 0$ for $\lambda \leq -c$. The last requirement fixes the SSF uniquely. In terms of the PD $D_{h/h_0}(\zeta)$ for the pair $h_0, h$, the SSF can be constructed by the formula

$$\xi(\lambda; H, H_0) = \pi^{-1} \text{arg} \ D_{h/h_0}((\lambda + c + i0)^{-m}).$$

The class of admissible functions can be obtained from Theorem 2.3 (see also Remark 2.3) by the change of variables $\mu = (\lambda + c)^{-m}$. It follows from the Birman invariance principle that under assumption (2.10) the wave operators $W_{\pm}(H, H_0)$ exist and are complete. In this case the SM are related by the formula $S(\lambda; H, H_0) = S^*(\mu; h, h_0)$. Thus, the following result is a direct consequence of Theorems 2.1, 2.3 and 2.5.
Theorem 2.6. Let condition (2.10) hold. Define the SSF \( \xi(\lambda) = \xi(\lambda; H, H_0) \) by equalities (2.11) for \( \lambda > -c \) and \( \xi(\lambda) = 0 \) for \( \lambda \leq -c \). It satisfies the condition
\[
\int_{-\infty}^{\infty} |\xi(\lambda)|(1 + |\lambda|)^{-m-1}d\lambda < \infty
\] (2.13)
and is related to the SM by equality (1.2). Suppose that a function \( f \) has two locally bounded derivatives and
\[
(\lambda^{m+1} f'(\lambda))' = O(\lambda^{-1-\varepsilon}), \quad \varepsilon > 0, \quad \lambda \to \infty.
\] (2.14)
Then inclusion (2.7) and the trace formula (1.1) hold.

Corollary 2.7. The trace formula (1.1) holds for all functions \( f(\lambda) = (\lambda - z)^{-k} \) where \( k \geq m \) and \( z \in \rho(H_0) \cap \rho(H) \). In particular,
\[
\text{Tr}(R^m(z) - R_0^m(z)) = -m \int_{-\infty}^{\infty} \xi(\lambda)(\lambda - z)^{-m-1}d\lambda.
\] (2.15)

Remark 2.8. If condition (2.10) holds for \( m = 1 \), then the generalized PD
\[
\tilde{D}_{-c}(z) := \text{Det}(I + (z + c) R(-c) V R_0(z)) = D_{h/h_0}(z + c)^{-1}
\] (2.16)
is correctly defined and satisfies equation (2.3). Let us fix the continuous branch of \( \arg \tilde{D}_{-c}(z) \) by the condition \( \arg \tilde{D}_{-c}(-c) = 0 \). Then it follows from formula (2.12) that
\[
\xi(\lambda; H, H_0) = \pi^{-1} \arg \tilde{D}_{-c}(\lambda + i0).
\] (2.17)

The SSF has the definite sign for sign-definite perturbations of the operator \( H_0 \). We understand the sign of the perturbation in the sense of quadratic forms. The following result is obtained in [13] (see also Theorem 8.10.3 of [29]).

Theorem 2.9. Under assumption (2.10), \( \xi(\lambda) \geq 0 \) if \( H \geq H_0 \) and \( \xi(\lambda) \leq 0 \) if \( H \leq H_0 \).

4. In applications to differential operators it is convenient to use the concept of regularized determinant and to introduce the regularized PD
\[
D_p(z) = \text{Det}_p(I + V R_0(z)), \quad p = 2, 3, \ldots
\] (2.18)
This definition is good for \( V R_0(z) \in \mathfrak{S}_p \). In this case the function \( D_p(z) \) is holomorphic on the set \( \rho(H_0) \). Many properties of ordinary PD carry over to regularized PD. Thus, identity (2.2) remains true, and the generalization of (2.3) has the form
\[
D_p^{-1}(z) D_p'(z) = -\text{Tr} \left( R(z) - \sum_{k=0}^{p-1} (-1)^k R_0(z)(V R_0(z))^k \right).
\] (2.19)

3. THE REGULARIZED PD AND SSF FOR THE SCHRODINGER OPERATOR

1. Below we consider the pair of self-adjoint operators \( H_0 = -\Delta \),
\[
H = -\Delta + v(x), \quad v(x) = v(x),
\] (3.1)
in the space \( \mathcal{H} = L_2(\mathbb{R}^d) \). It is assumed that the potential \( v(x) \) decays sufficiently rapidly at infinity, that is
\[
|v(x)| \leq C(1 + |x|)^{-\rho},
\] (3.2)
where at least \( \rho > 1 \). Then the positive spectrum of the operator \( H \) is absolutely continuous, and its negative spectrum consists of eigenvalues \( \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \).
counted with their multiplicity (see, e.g., [23]). Wave operators (2.8) exist and are complete so that \( \text{Ran} W_\pm = E(\mathbb{R}_+)\mathcal{H} \) (see, e.g., [18]).

The canonical diagonalization of the operator \( H_0 \) is realized by the operator \( \mathcal{F} : \mathcal{H} \to L_2(\mathbb{R}_+; L_2(S^{d-1})) \) defined by the equation \((\mathcal{F} f)(\lambda; \omega) = (\Gamma_0(\lambda)f)(\omega)\) where
\[
(\Gamma_0(\lambda)f)(\omega) = 2^{-1/2} \lambda^{(d-2)/4} \tilde{f}(\lambda^{1/2} \omega), \quad \lambda > 0, \quad \omega \in S^{d-1},
\]
and \( \tilde{f} \) is the Fourier transform of \( f \). The SM \( S(\lambda) = S(\lambda; H, H_0) \) is correctly defined and is a unitary operator in the space \( L_2(S^{d-1}) \) for a.e. \( \lambda \in \mathbb{R}_+ \). Moreover, it admits the representation
\[
S(\lambda) = I - 2\pi i (A_0(\lambda) + \tilde{A}(\lambda))
\]
where
\[
A_0(\lambda) = \Gamma_0(\lambda) V \Gamma_0^*(\lambda), \quad \tilde{A}(\lambda) = -\Gamma_0(\lambda) V R(\lambda + i0) \Gamma_0^*(\lambda).
\]
These operators are well defined. Indeed, set \( \langle x \rangle^{-r} = (1 + |x|^2)^{-r/2} \) (we keep the same notation for the operator of multiplication by this function) and choose some \( r > 1/2 \). Then according to the Sobolev theorem the operator \( \Gamma_0(\lambda) \langle x \rangle^{-r} : \mathcal{H} \to L_2(S^{d-1}) \) is compact and depends continuously on \( \lambda > 0 \). The limiting absorption principle asserts that the operator-valued function \( \langle x \rangle^{-r} R_0(z) \langle x \rangle^{-r} \) is continuous in the complex plane up to the cut along \([0, \infty)\) with the point \( z = 0 \) possibly excluded. Thus, for \( \rho > 1 \), operators (3.4) can be factored in products of bounded operators depending continuously on \( \lambda > 0 \). It follows that the SM \( S(\lambda) \) is a continuous (in the operator topology) function of \( \lambda > 0 \).

If estimate (3.2) is satisfied with \( \rho > d \), then inclusion (2.10) holds for all \( z \notin \sigma(H) \) and any positive integer \( m > d/2 - 1 \). This result is a straightforward consequence of the resolvent identity and of the inclusion \( \langle x \rangle^{-\rho/2} R_0(z) \langle x \rangle^{-\rho} \) is continuous if \( m = 1 \) but requires some tricks if \( m > 1 \) (see [23] or [24]). Thus, the assertion below follows from Theorems 2.6 and 2.9.

**Theorem 3.1.** Suppose that assumption (3.2) with \( \rho > d \) is satisfied. Let \( c > -\lambda_1 \), and let a positive integer \( m \) be such that \( 2(m + 1) > d \). Define the SSF by equality (2.11) for \( \lambda > -c \) and set \( \xi(\lambda; H, H_0) = 0 \) for \( \lambda \leq -c \). Then condition (2.13) is satisfied,
\[
\xi(\lambda) = 0 \quad \text{for} \quad \lambda < \lambda_1, \quad \xi(\lambda) = -n \quad \text{for} \quad \lambda \in (\lambda_n, \lambda_{n+1}),
\]
and the SSF is related to the SM by equality (2.12) for \( \lambda > 0 \). If a function \( f \) has two locally bounded derivatives and satisfies condition (2.14), then inclusion (2.7) and the trace formula (1.14) (in particular, (2.15) hold. Moreover, if \( v(x) \geq 0 \) \( (v(x) \leq 0) \), then \( \xi(\lambda) \geq 0 \) (respectively, \( \xi(\lambda) \leq 0 \)).

**Corollary 3.2.** Let \( d \leq 3 \). Then inclusion (2.10) and formula (2.15) hold for \( m = 1 \), and the SSF can be recovered via generalized PD (2.15) by relation (2.17).

2. Now we present a direct approach to construction of the SSF for the multi-dimensional Schrödinger operator (3.1). This allows us to study some specific properties of \( \xi(\lambda) \), such as its continuity for \( \lambda > 0 \) and behavior as \( \lambda \to \mp \infty \) and \( \lambda \to 0 \). We require here condition (3.2) for at least \( \rho > 1 \) but not necessarily for \( \rho > d \) as in the trace-class approach. Actually, \( \rho \) depends on the dimension \( d \) of the problem and is different in different assertions. For main results, we assume that \( \rho > 2 \). In this case the operator \( H \) might have only a finite number \( N \) of negative eigenvalues.
Proposition 3.3. Suppose that \( \rho > 1 \) if \( d = 1, 2 \), \( \rho > 3/2 \) if \( d = 3 \) and \( \rho \geq 2 \) if \( d \geq 4 \). Let \( p = 1 \) for \( d = 1 \), \( p = 2 \) for \( d = 2, 3 \) and \( p > d/2 \) for \( d \geq 4 \). Then the function \( D_p(z) \) is analytic in the complex plane cut along the positive half-axis \([0, \infty)\) and satisfies identities (2.2) and (2.19). For \( z \notin [0, \infty) \), this function may have zeros on the negative half-axis only, where they coincide with eigenvalues of the operator \( H \); the order of a zero of \( D_p(z) \) equals the multiplicity of the corresponding eigenvalue.

Let \( r = \rho/2 \), \( G = (x)^{-r} \) and let \( V \) be the multiplication operator by the bounded function \((x)^{\rho}v(x)\). Since non-zero eigenvalues of the operators \( VR_0(z) \) and \( GR_0(z)GV \) are the same, we have that

\[
D_p(z) = \text{Det}_p(I + GR_0(z)GV).
\]

This representation is more convenient than (2.19) because the operator-valued function \( GR_0(z)G \) is continuous up to the cut even in the classes \( \mathfrak{S}_p \) for sufficiently large \( p \). In the following assertion we take also into account that if \( D_p(\lambda \pm i0) = 0 \) for \( \lambda > 0 \), then \(-1\) is an eigenvalue of the compact operator \( GR_0(\lambda \pm i0)GV \) and hence \( \lambda \) is an eigenvalue of \( H \). However according to the Kato theorem (see, e.g., [23]) this operator does not have positive eigenvalues.

Proposition 3.4. Let \( \rho > 1 \) if \( d = 1, \rho > 3/2 \) if \( d = 2 \) and \( \rho > 2 \) if \( d \geq 3 \). Put \( p = 1 \) for \( d = 1 \), \( p = 2 \) for \( d = 2, 3 \) and \( p = d \) for \( d \geq 4 \). Then the function \( D_p(z) \) is continuous up to the cut along \([0, \infty)\), with the point \( z = 0 \) possibly excluded. The function \( D_p(\lambda \pm i0) \) does not have zeros for \( \lambda > 0 \).

As far as the high-energy behavior of the PD is concerned, we consider only the case \( d = 3 \). Let us use the identity

\[
\text{Det}_2(I + GR_0(z)GV) = \text{Det}_3(I + GR_0(z)GV) \exp(-2^{-1} \text{Tr}(VR_0(z))^2). \quad (3.6)
\]

Remark that

\[
||GR_0(z)G||^3 \leq ||GR_0(z)G||||GR_0(z)G||^2.
\]

The first factor in the right-hand side tends to zero as \( |z| \to \infty \), and it follows from the explicit formula for the integral kernel of the operator \( R_0(z) \) that the second factor is uniformly bounded. Therefore the first factor in the right-hand side of (3.6) tends to 1 as \( |z| \to \infty \). The same is true for the second factor because

\[
\text{Tr}(VR_0(z))^2 = (4\pi)^{-2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v(x)v(y)|x - y|^{-2e^{2i\sqrt{\lambda} |x-y|}}dxdy,
\]

and the integral tends to zero according to the Riemann-Lebesgue lemma. More generally, we have
Proposition 3.5. Under the assumptions of Proposition 3.4
\[
\lim_{|z| \to \infty} D_p(z) = 1 \quad (3.7)
\]
uniformly in \(\arg z \in [0, 2\pi]\).

The PD \(D_p(z)\) is singular at the point \(z = 0\) for \(d = 1\) and \(d = 2\) only.

Proposition 3.6. Let \(d \geq 3\), \(\rho > 2\), \(p = 2\) for \(d = 3\) and \(p = d\) for \(d \geq 4\). Then the PD \(D_p(0)\) is correctly defined and \(\|D_p(z) - D_p(0)\| \to 0\) as \(|z| \to 0\).

Clearly, \(D_p(0) = 0\) if and only if \(-1\) is an eigenvalue of the compact operator \(GR_0(0)GV\). In this case one says that the operator \(H\) has a zero-energy resonance (in particular, \(H\) might have zero eigenvalue).

We avoid a study of the PD \(D_p(z)\) as \(|z| \to 0\) imposing a mild a priori assumption
\[
\lim_{|z| \to 0} |z|^\alpha \ln D_p(z) = 0 \quad (3.8)
\]
for a suitable \(\alpha > 0\). Of course in the case \(d \geq 3\) condition (3.8) is satisfied for all \(\alpha > 0\) if \(D_p(0) \neq 0\). More generally, condition (3.8) can be deduced from the low-energy expansion (as \(z \to 0\)) of \(R(z)\) which requires however a somewhat stronger assumption than \(\rho > 2\).

3. We suppose that arg \(D_p(z)\) is a continuous function of \(z\) in the complex plane cut along \([\lambda_1, \infty)\) which is possible because \(D_p(z) \neq 0\) there. According to (3.7) we then fix its branch by the condition arg \(D_p(z) \to 0\) as \(|z| \to \infty\).

Now we are in a position to construct the regularized SSF \(\xi_p\).

Theorem 3.7. Let condition (3.2) where \(\rho > 2\) hold, and let \(p = 1\) for \(d = 1\), \(p = 2\) for \(d = 2, 3\) and \(p = d\) for \(d \geq 4\). Assume that (3.3) is true for \(\alpha = 1\). Define the regularized SSF \(\xi_p\) by equality (2.5) in terms of the regularized PD \(D_p(z)\). Then representation (2.4) for \(\ln D_p(z)\) holds with the function \(\xi_p(\lambda)\). The function \(\xi_p\) is determined by equalities (3.3) for \(\lambda < 0\). It is continuous for \(\lambda > 0\), \(\xi_p(\lambda) = o(1)\) as \(\lambda \to \infty\), \(\xi_p(\lambda) = o(\lambda^{-1})\) as \(\lambda \to 0\) and the integrals of \(\xi_p(\lambda)\lambda^{-1}\) are convergent (but not necessarily absolutely) at the points \(\lambda = \infty\) and \(\lambda = 0\), respectively.

Indeed, the function \(\ln D_p(z)\) is analytic in the complex plane cut along \([\lambda_1, \infty)\) and is continuous up to the cut with exception of the points \(\lambda_1, \ldots, \lambda_N\) and, possibly, zero. Let us consider in the complex plane the closed contour \(\Gamma_{R,\varepsilon}\) which consists of the intervals \((\lambda_1, R + i0)\) and \((R - i0, \lambda_1)\) lying on the upper and lower edges of the cut and of the circle \(C_R\) of radius \(R\) passed in the counterclockwise direction. Moreover, we bypass every point \(\lambda_j, j = 1, \ldots, N\), and the point 0 by semicircles \(C_\varepsilon^+(\lambda_j)\) and \(C_\varepsilon^-(0)\) of radius \(\varepsilon\). By virtue of the Cauchy theorem, for an arbitrary complex \(z\), a sufficiently small \(\varepsilon\) and a sufficiently large \(R\)
\[
\ln D_p(z) = (2\pi i)^{-1} \int_{\Gamma_{R,\varepsilon}} \ln D_p(\zeta)(\zeta - z)^{-1} d\zeta. \quad (3.9)
\]
According to (3.7), the integral over \(C_R\) tends to zero as \(R \to \infty\). Since the function \(D(z)\) has only zeros of finite order at the points \(\lambda_1, \ldots, \lambda_N\), the integrals over \(C_\varepsilon^+(\lambda_j)\) tend to zero as \(\varepsilon \to 0\). The integrals over \(C_\varepsilon^-(0)\) tend to zero as \(\varepsilon \to 0\) according to condition (3.8). Hence representation (2.4) for \(D_p(z)\) with function \(\xi_p(\lambda)\) defined by (2.5) follows from equations (2.2) and (3.9).

Let us now obtain the trace formula.
Theorem 3.8. Let \( f \) be a bounded rational function with non-real poles. Then under the assumptions of Theorem 3.7

\[
\text{Tr} \left( f(H) - \sum_{k=0}^{p-1} \frac{1}{k!} \frac{d^k f(H_0 + \varepsilon V)}{d\varepsilon^k} \right) \bigg|_{\varepsilon=0} = \int_{-\infty}^{\infty} \xi_p(\lambda) f'(\lambda) d\lambda \tag{3.10}
\]

and, in particular,

\[
\text{Tr} \left( R(z) - \sum_{k=0}^{p-1} (-1)^k R_0(z) (VR_0(z))^k \right) = -\int_{-\infty}^{\infty} \xi_p(\lambda)(\lambda - z)^{-2} d\lambda. \tag{3.11}
\]

Indeed, differentiating representation (2.4) for \( D_p(z) \) and taking into account formula (2.19), we obtain first (3.11) which yields (3.10) for \( f(\lambda) = (\lambda - z)^{-1} \), \( \text{Im} z \neq 0 \). Then differentiating (3.11), we extend formula (3.10) to functions \( f(\lambda) = (\lambda - z)^{-m} \) where \( m = 2, 3, \ldots \). Clearly, formula (3.10) remains true for linear combinations of such functions.

The trace formula (1.1) makes of course no sense if \( \rho \leq d \), and representation (3.10) can be regarded as its regularization. If \( d = 1 \), then \( \xi = \xi_1 \) and Theorem 3.8 is contained in Theorem 2.6.

Applying the argument principle to the function \( D_p(z) \), we obtain the trace identity of zero order.

Theorem 3.9. Under the assumptions of Proposition 3.6 suppose that \( D_p(0) \neq 0 \). Then

\[
\arg D_p(\infty) - \arg D_p(0) = \pi N, \tag{3.12}
\]

where \( N \) is the total number of negative eigenvalues of the operator \( H \).

We emphasize that according to definition (2.3) under the assumption \( D_p(0) \neq 0 \) the limit \( \xi_p(+0) \) exists. Formula (3.12) (known as the Levinson formula) means that the regularized SSF \( \xi_p(\lambda) \) is continuous at the point \( \lambda = 0 \). In the cases \( d = 1 \) and \( d = 2 \) the singularity of the PD \( D_p(z) \) at the point \( z = 0 \) should be taken into account.

For positive \( \lambda \), the regularized SSF is related to the SM \( S(\lambda) \). For simplicity we consider the case \( p = 2 \) only. Let the operator \( A_0(\lambda) \) (known as the first Born approximation to the SM) be defined by formula (3.4). It is self-adjoint, does not depend on the resolvent of \( H \) and is constructed directly in terms of the Fourier transform of the potential \( v \). We set

\[
\nu(\lambda) = \text{Tr} \left( S(\lambda) - I + 2\pi i A_0(\lambda) \right) = -2\pi i \text{Tr} \tilde{A}(\lambda).
\]

The next result can be considered as a modification of the Birman-Krein formula (1.2). Its proof relies on the identity

\[
\det_2 \left( I - (V - VR(z)V)(R_0(z) - R_0(\bar{z})) \right)
= D_2(z)D_2(\bar{z})^{-1} \exp\left( -\text{Tr}(VR(z)V(R_0(z) - R_0(\bar{z})) \right)
\]

where we pass to the limit \( z \rightarrow \lambda + i0 \).

Theorem 3.10. Let condition (5.22) be satisfied for \( \rho > 3/2 \) if \( d = 2 \) and for \( \rho > 2 \) if \( d = 3 \). Then the function \( \xi_2(\lambda) \) is related to the SM \( S(\lambda) \) by the formula

\[
\det_2 S(\lambda) = e^{-2\pi i \xi_2(\lambda) - \nu(\lambda)}, \quad \lambda > 0.
\]
4. Suppose now that \( v \) satisfies condition (3.2) with \( \rho > d \). Then inclusion (2.10) holds for \( 2(m + 1) > d \), and the usual SSF \( \xi(\lambda) \) is defined by relation (2.11). We will find a relation between \( \xi \) and the regularized SSF \( \xi_p \) which will allow us to obtain a new information on the SSF \( \xi \) supplementing the results of subs. 1.

Let first \( d \leq 3 \) so that \( m = 1 \). The relation between the generalized PD \( \hat{D}(z) = \hat{D}_{-c}(z), \) \( -c < \lambda_1 \), defined by equality (2.16) and the regularized PD \( D_2(z) \) can easily be obtained by comparison of equations (2.13) for \( \hat{D}(z) \) and (2.19) for \( p = 2 \).

**Proposition 3.11.** If \( d = 2 \) or \( d = 3 \) and condition (3.2) is satisfied with \( \rho > d \), then

\[
\hat{D}(z) = D_{z-1}^0(-c)D_2(z) \exp\left((d\pi)^{-1}\int_{\mathbb{R}^d} v(x)dx b_d(z)\right),
\]

where \( b_2(z) = -\ln(-z/c), \ b_3(z) = c^{1/2} - (-z)^{1/2} \) and \( \arg(-z) \in (-\pi, \pi) \).

Taking the arguments of both sides of (3.13) and passing to the limit \( z \to \lambda + i0 \), we obtain the relation between the SSF \( \xi(\lambda) \) which can be defined by formula (2.17) and \( \xi_2(\lambda) \). Then we use Theorems 3.7 and 3.9.

**Theorem 3.12.** Let the assumptions of Proposition 3.11 be satisfied. Then for \( \lambda > 0 \) the SSF \( \xi(\lambda) \) admits the representation

\[
\xi(\lambda) = \xi_2(\lambda) + 2^{-1}(2\pi)^{-d|\mathbb{S}^{d-1}|}\int_{\mathbb{R}^d} v(x)dx \lambda^{(d-2)/2}.
\]

The function \( \xi(\lambda) \) is continuous for \( \lambda > 0 \) and

\[
\xi(\lambda) = 2^{-1}(2\pi)^{-d|\mathbb{S}^{d-1}|}\int_{\mathbb{R}^d} v(x)dx \lambda^{(d-2)/2} + o(1), \quad \lambda \to \infty.
\]

If \( d = 3 \) and the operator \( H \) does not have the zero-energy resonance, then there exists the limit \( \xi(\lambda) = -\pi N \) so that the SSF is continuous at the point \( \lambda = 0 \).

If \( d = 1 \), then \( D_2(z) \) is related to the usual PD \( D(z) \) by a formula similar to (3.13). This implies that the SSF \( \xi(\lambda) \) is continuous for \( \lambda > 0 \) and satisfies relation (3.14) where the remainder is \( o(\lambda^{-1/2}) \).

The case \( d \geq 4 \) is considerably more difficult because the SSF \( \xi(\lambda) \) is expressed only in terms of the PD for the pair \( (H_0 + cI)^{-m}, (H + cI)^{-m} \) which is not directly related to \( D_p(z) \) if \( m > d/2 - 1 \geq 1 \). To check that \( \xi(\lambda) \) is a continuous function of \( \lambda > 0 \), we compare formulas (2.15) and (3.11) and use that the function \( Tr(d^{m-2}(R_0(z))(VR_0(z))^k)/dz^{m-2}, k \geq 2, \) is continuous up to the cut along \((0, \infty)\). This implies that the integral of \( (\xi(\lambda) - \xi_p(\lambda))(\lambda - z)^{-m} \) over \( \lambda \) is also a continuous function of \( z \) up to this cut. Then standard results on the Cauchy type integrals show that the function \( \xi(\lambda) - \xi_p(\lambda) \) belongs to the class \( C^{m-1} \) for \( \lambda > 0 \). Now the continuity of \( \xi(\lambda) \) follows from Theorem 3.7. Let us formulate this result.

**Theorem 3.13.** Let condition (3.2) with \( \rho > d \) be satisfied. Then the SSF \( \xi(\lambda) \) is continuous for \( \lambda > 0 \).

Using representation (3.3) it is easy to show that under the assumption \( \rho > d \) the SM \( S(\lambda) \) depends continuously on \( \lambda > 0 \) in the trace-class topology and hence \( \text{det} S(\lambda) \) is also a continuous function of \( \lambda > 0 \). Nevertheless the continuity of the SSF \( \xi(\lambda) \) does not follow from formula (2.12) because its integer jumps are not a priori excluded.
The leading term of the high-energy asymptotics of the SSF is given by the first term in the right-hand side of (4.14) for all $d \geq 4$. This result is proven in Section 6, together with a complete AE of $\xi(\lambda)$ as $\lambda \to \infty$, where more stringent assumptions on $v(x)$ are imposed.

4. The Green function for large values of the spectral parameter

In subs. 1 we construct an AE of the integral kernel $R(x,x';z)$ of the resolvent (of the Green function) as $|z| \to \infty$ provided $z \in \Pi_\theta$ where $\arg z \in (\theta, 2\pi - \theta)$. This method works only for $\theta > 0$ but gives explicit expressions for the coefficients of this expansion and does not require any decay of $v(x)$ at infinity. Under some decay assumptions this method yields also an AE of $\Tr(R^m(z) - R^m_0(z))$. In subs. 2 we obtain a local AE of the parabolic Green function (heat kernel) $G(x,x';t)$ as $t \to 0$ which requires almost no assumptions on $v(x)$ at infinity. The Laplace transform relates these expansions with the AE of $R(x,x';z)$ as $|z| \to \infty$. In subs. 3 these results are used to enhance the results on the local AE of $R(x,x';z)$. On the contrary, the AE of $\Tr(R^m(z) - R^m_0(z))$ is used to derive an AE of $\Tr(e^{-Ht} - e^{-H_0t})$ as $t \to 0$.

1. We proceed from a modification of the iterated resolvent identity which is a special case of the non-commutative Taylor formula of [12]. We formulate it for the Schrödinger operator, but actually this identity has abstract nature.

Proposition 4.1. Suppose that $v \in C^\infty(\mathbb{R}^d)$ and that $v$ as well as all its derivatives are bounded functions. Define the operators $X_n$ by the recurrent relations

$$X_0 = I \quad \text{and} \quad X_{n+1} = X_n H_0 - H X_n$$

so that

$$X_n = \sum_{k=0}^n (-1)^k \binom{n}{k} H^k H_0^{n-k}. $$

Then, for all $N \geq 0$,

$$R(z) = \sum_{n=0}^N X_n R_0^{n+1}(z) + R(z) X_{N+1} R_0^{N+1}(z). \quad (4.2)$$

The operators $X_n$ can easily be computed. For example, $X_1 = -v$, $X_2 = -2((\nabla v), \nabla) - (\Delta v) + v^2$ and

$$X_3 = -4(\text{Hess} v \nabla, \nabla) + 6v((\nabla v), \nabla) - (\Delta^2 v) + 2|\nabla v|^2 + 3v(\Delta v) - v^3.$$ 

Proposition 4.2. The $X_n$ is a differential operator of order $n - 1$ so that

$$X_n = \sum_{|\alpha| \leq n-1} p_{\alpha,n} \partial^\alpha, \quad p_{\alpha,n}(x) = \overline{p_{\alpha,n}(x)}, \quad \alpha = \{\alpha_1, \ldots, \alpha_d\}. \quad (4.3)$$

Under the assumptions of Proposition 4.1 all coefficients $p_{\alpha,n}$ as well as all its derivatives are bounded functions. Moreover, if $v$ satisfies estimates

$$|\partial^\kappa v(x)| \leq C_\kappa (1 + |x|)^{-\rho-|\kappa|}, \quad \rho > 0, \quad (4.4)$$

for all multi-indices $\kappa$, then $(\partial^\kappa p_{\alpha,n})(x) = O(|x|^{-\rho-|\kappa|-\epsilon(n-1)\epsilon})$, $\epsilon = \min\{\rho, 1\}$, for all $n$, $\alpha$ and $\kappa$ as $|x| \to \infty$. 

Both Propositions 4.1 and 4.2 can be verified by induction in \( n \) quite straightforwardly.

Let us use identity (4.2) to obtain an AE of the integral kernel \( R(x, x'; z) \) of \( R(z) \) as \( |z| \to \infty \), \( z \in \Pi_\theta \). According to (4.3) the operator \( T_n(z) = X_n R_{0}^{n+1}(z) \) for \( n \geq 1 \) has integral kernel
\[
T_n(x, x'; z) = (2\pi)^{-d} \sum_{|\alpha| \leq n-1} i^{|\alpha|} p_{\alpha, n}(x) \int_{\mathbb{R}^d} e^{i(x-x'; \xi)} \xi^\alpha (|\xi|^2 - z)^{-n-1} \, d\xi. \tag{4.5}
\]
It follows that
\[|T_n(x, x'; z)| \leq C \sum_{|\alpha| \leq n-1} |p_{\alpha, n}(x)||z|^{-(n+3-d)/2} \]
if \( n + 3 > d \). Of course formula (4.5) can be rewritten as
\[
T_n(x, x'; z) = n!^{-1} \sum_{|\alpha| \leq n-1} p_{\alpha, n}(x) \partial_\xi^n R_{0}^{n}(x - x'; z) \tag{4.6}
\]
where \( R_{0}^{n}(x - x'; z) = \partial_\xi^n R_{0}(x - x'; z) \) and \( R_{0}(x - x'; z) \) is integral kernel of the resolvent \( R_{0}(z) \). The estimate of the remainder \( R(z)X_{N+1}R_{0}^{N+1}(z) \) in (4.2) relies only on the trivial bound \( \|R(z)\| = O(|z|^{-1}) \) and explicit formula (4.5). Thus, identity (4.2) yields the following result.

**Theorem 4.3.** Suppose that \( \nu \in C^\infty(\mathbb{R}^d) \) and that \( \nu \) as well as all its derivatives are bounded functions. Then, for all sufficiently large \( N \) \( (N \geq d-3) \), the asymptotic relation
\[
R(x, x'; z) = \sum_{n=0}^{N} T_n(x, x'; z) + O(|z|^{-(N-d)/2-2}) \tag{4.7}
\]
is valid as \( |z| \to \infty \), \( z \in \Pi_\theta \). The estimate of the remainder here is uniform with respect to \( x, x' \in \mathbb{R}^d \). Moreover, relation (4.7) can be infinitely differentiated in \( x, x' \) (then \( N \) increases) and \( z \).

It follows from (4.3) that the functions \( T_n(x, x'; z) \) decay exponentially as \( |x - x'| |z|^{1/2} \to \infty \), \( z \in \Pi_\theta \). Therefore, although valid for all \( x, x' \in \mathbb{R}^d \), AE (4.7) is of interest in the region \( |x - x'| = O(|z|^{-1/2}) \) only. Observe that the functions \( T_n(x, x'; z) \) are singular on the diagonal \( x = x' \) but are getting smoother as \( n \) increases. In particular, these functions are continuous if \( n > d-3 \) and \( n \geq 1 \) \( (T_0(x, x'; z) \) is continuous for \( d = 1 \) only). Thus, (4.7) yields the expansion of the Green function both for large \( |z| \) and in smoothness. Note also that diagonal singularities of the functions \( T_n(x, x'; z) \) are getting weaker after differentiations with respect to \( z \).

Let us discuss expansion (4.7) for the case \( x = x' \). If \( d = 1 \), then we can directly set \( x = x' \) in (4.7). In the case \( d \geq 2 \) we previously differentiate (4.5) and (4.7) \( (m-1) \)-times with respect to \( z \). If \( 2(m+1) > d \), then integral kernel of the operator \( R^{(m-1)}(z) - R_{0}^{(m-1)}(z) \) is continuous. Therefore setting \( x = x' \) and collecting together terms of the same power of \( z \), we obtain
Corollary 4.4. Let $2(m + 1) > d$. Set $c_{\alpha,k} = \int_{\mathbb{R}^d} \xi^{2\alpha}(|\xi|^2 + 1)^{-k} d\xi$ where $k > |\alpha| + d/2$. Then

\[
\left( R^{(m-1)}(x,x';z) - R_0^{(m-1)}(x,x';z) \right) \bigg|_{x=x'} = (m-1)! \sum_{n=0}^{\infty} r_n^{(m)}(x)(-z)^{d/2-m-n-1}
\]

where (the definition below makes sense for $n + m + 1 > d/2$)

\[
r_n^{(m)}(x) = (2\pi)^{-d} \sum_{|\alpha| \leq n} (-1)^{|\alpha|} \binom{|\alpha| + n + m}{m - 1} c_{\alpha,n+m+1} x_0^{n+1} \phi_{\alpha}^{(n+1)}(x). \tag{4.9}
\]

For potentials decaying at infinity, AE (4.8) can be integrated over $x \in \mathbb{R}^d$ because the derivative of order $m - 1$ of the remainder in (4.2) can be estimated in the trace-class norm. This yields the following result.

Theorem 4.5. Let assumption (4.3) where $\rho > d$ hold, and let $2(m + 1) > d$. Then the AE as $|z| \to \infty$

\[
\text{Tr}(R^m(z) - R_0^m(z)) = \sum_{n=0}^{\infty} r_n^{(m)}(-z)^{d/2-m-n-1}. \tag{4.10}
\]

is valid for $z \in \Pi_\theta$. The real coefficients $r_n^{(m)}$ are defined by the formula

\[
r_n^{(m)} = \int_{\mathbb{R}^d} r_n^{(m)}(x) dx. \tag{4.11}
\]

Remark 4.6. Instead of taking derivatives, we can remove from $R(z)$ several (instead of one as in (4.10)) terms of its expansion (4.2). This allows us to relax also the condition $\rho > d$. For example, if $d < 6$ and (4.3) is satisfied for $\rho > d/2$, then

\[
\text{Tr}(R(z) - R_0(z) + R_0(z)VR(z)) = \sum_{n=1}^{\infty} r_n^{(1)}(-z)^{d/2-n-2}. \tag{4.12}
\]

2. The parabolic Green function $G(x,x';t)$ is integral kernel of the operator $\exp(-Ht)$, $t > 0$. It satisfies the parabolic equation

\[
\partial G(x,x';t)/\partial t = \Delta G(x,x';t) - v(x)G(x,x';t), \quad \Delta = \Delta_x. \tag{4.13}
\]

We seek (cf. [2]) its approximate solution in the form

\[
G_N(x,x';t) = G_0(x,x';t) \sum_{n=0}^{N} a_n(x,x') t^n. \tag{4.14}
\]

where $G_0(x,x';t) = G_0(x-x';t)$ is integral kernel of the operator $\exp(-H_0 t)$ and $g_0(x,x') = 1$. Let us plug expression (4.14) into (4.13), use the equation $\partial G_0/\partial t = \Delta G_0$ and divide by the common factor $G_0$. Requiring then that the coefficients at $t^n$, $n = 0, 1, \ldots, N - 1$, vanish, we find recurrent equations for the functions $g_{n+1}(x,x')$. For a fixed $x'$, this yields an ordinary differential equation $(n+1)g_{n+1} + \rho \partial g_{n+1}/\partial r = \Delta g_n - v g_n$ in the variable $r = |x-x'|$. Solving it under the assumption that $g_{n+1}(x,x)$ is finite, we obtain the formula

\[
g_{n+1}(x,x') = \int_0^1 \sigma^n (\Delta g_n - v g_n)(x' + \sigma(x-x'), x') d\sigma. \tag{4.15}
\]
The construction above does not require any assumptions on \( v(x) \) at infinity. Our justification of the local AE of \( G(x, x'; t) \) as \( t \to 0 \) requires only very mild assumptions. Using representation (4.15), one easily proves the following assertion by induction.

**Lemma 4.7.** Suppose that \( v \in C^{\infty}(\mathbb{R}^d) \) and \( (\partial^\alpha v)(x) = O(e^{\gamma |x|^2}) \) as \( |x| \to \infty \) for some \( \gamma \geq 0 \) and all \( \kappa \). Then \( |g_n(x, x')| \leq C_n e^{(n\gamma + \epsilon) |x-x'|^2} \) for all \( \epsilon > 0 \) uniformly with respect to \( x' \) from a compact subset of \( \mathbb{R}^d \). Moreover, these estimates can be infinitely differentiated in the variables \( x \) and \( x' \).

Let us set \( Q_N = G_0(-\Delta g_N + v g_N) t^N \). It follows from (4.15) and a similar equation for the function \( G_N \) that the difference \( F_N = G - G_N \) admits the representation

\[
F_N(x, x'; t) = -\int_0^t (e^{-(t-s)H} Q_N(\cdot, x'; s))(x) ds
\]  

where a point \( x' \) is fixed. Combining estimate of Lemma 4.7 for \( g_N \) with the obvious bound \( ||e^{-\tau H}|| \leq e^{-\lambda_1 \tau} \), one can estimate function (4.16) for small \( t \) by \( C_N t^{N-d/4+1} \) uniformly in \( x, x' \) from compact sets. This yields the following result.

**Theorem 4.8.** Let the assumptions of Lemma 4.7 hold. Assume that the operator \( H = -\Delta + v(x) \) is self-adjoint on a domain \( \mathcal{D}(H) \supset \mathcal{S}(\mathbb{R}^d) \) and that it is semi-bounded from below. Define the functions \( g_n \) by the recurrent relations \( g_0(x, x') = 1 \) and (4.15). Then

\[
G(x, x'; t) = (4\pi t)^{-d/2} \sum_{n=0}^{\infty} g_n(x, x') t^n
\]

as \( t \to 0 \) uniformly with respect to \( x \) and \( x' \) from compact subsets of \( \mathbb{R}^d \). Moreover, asymptotic expansion (4.17) can be infinitely differentiated in the variables \( x \), \( x' \) and \( t \). In particular, we have that

\[
G(x, x; t) = (4\pi t)^{-d/2} \sum_{n=0}^{\infty} g_n(x) t^n, \quad g_n(x) = g_n(x, x), \quad t \to 0.
\]

Of course expansion (4.17) is of interest for \( |x - x'| = O(t^{1/2}) \) only.

Proceeding from (4.15), one can give closed expressions for the functions \( g_n(x, x') \). In particular, it is not difficult to calculate explicitly the first functions \( g_n(x) \) (known as local heat invariants of the operator \( H \)):

\[
g_1 = -v, \quad g_2 = 2^{-1}v^2 - 6^{-1}\Delta v, \quad g_3 = -6^{-1}(v^3 - v\Delta v - 2^{-1}v|\nabla v|^2 + 10^{-1}\Delta^2 v),
\]

\[
g_4 = 24^{-1}v^4 + 30^{-1}\langle \nabla v, \nabla (\Delta v) \rangle + 60^{-1}v\Delta^2 v + 72^{-1}(\Delta v)^2 - 840^{-1}\Delta^3 v
\]

\[
-12^{-1}v^2\Delta v - 12^{-1}v|\nabla v|^2 + 90^{-1}\text{Tr}(\text{Hess } v)^2.
\]

For an arbitrary \( n \), the functions \( g_n(x) \) can be found (see [22]) and also [11] where the results of [1] were used) by the formula

\[
g_n(x) = (-1)^n\Gamma(n + d/2) \sum_{k=0}^{n-1} \frac{(-\Delta v + v(x))^{k+n}(|x-x'|^{2k})}{4^k k!(n + k)! (n - 1 - k)! \Gamma(k + d/2 + 1)}
\]

3. Since

\[
R(x, x'; z) = \int_0^{\infty} G(x, x'; t) e^{it} dt, \quad Re z < \inf \sigma(H) = \lambda_1,
\]
we can relate expansion \((4.17)\) as \(t \to 0\) with the asymptotic expansion of the resolvent kernel \(R(x, x'; z)\) as \(|z| \to \infty\) and thus to enhance the results of subs. 1. We emphasize that the boundedness of \(v(x)\) is not required now.

**Theorem 4.9.** Under the assumptions of Theorem 4.8 for sufficiently large \(N\),

\[
R(x, x'; z) = \sum_{n=0}^{N} g_n(x, x') R_0^{(n)}(x, x'; z) + O(|z|^{-N+\frac{d-2}{2}-2})
\]  

(4.21)

as \(|z| \to \infty\), \(\Re z < \inf \sigma(H)\), uniformly with respect to \(x, x'\) from a compact subset of \(\mathbb{R}^d\). Expansion (4.21) can be infinitely differentiated in \(x, x'\) and \(z\).

Theorems 4.3 and 4.9 give two different expansions of the resolvent kernel. Similarly to subs. 1, we can pass in (4.21) to the limit \(x' \to x\). Recall that kernel \(R_0(x, x'; z)\) is singular on the diagonal, but the derivatives \(R_0^{(n)}(x, x'; z)\) are getting smoother as \(n\) increases. In particular, the function \(R_0^{(n)}(x, x'; z)\) is continuous on the diagonal if \(2(n + 1) > d\) and

\[
R_0^{(n)}(x, x'; z) = (4\pi)^{-d/2} \Gamma(1 + n - d/2)(-z)^{d/2 - n - 1}.
\]

Therefore removing from (4.21) \(R_0(x, x'; z)\) and differentiating \((m - 1)\)-times with respect to \(z\), we can set \(x = x'\) if \(2(m + 1) > d\). This yields the AE

\[
(4\pi)^{-d/2} \sum_{n=0}^{\infty} \Gamma(n + m + 1 - d/2) g_{n+1}(x)(-z)^{d/2 - m - n - 1}
\]

for the left-hand side of (4.8). Comparing it with the right-hand side of (4.8), we find that

\[
r_n^{(m)}(x) = (4\pi)^{-d/2} (m - 1)!^{-1} \Gamma(n + m + 1 - d/2) g_{n+1}(x). \tag{4.22}
\]

Integrating (4.18) over \(x \in \mathbb{R}^d\), we formally obtain the AE

\[
\text{Tr}(e^{-Ht} - e^{-Ho}) = (4\pi i)^{-d/2} \sum_{n=1}^{\infty} g_n t^n, \quad \text{where} \quad g_n = \int_{\mathbb{R}^d} g_n(x) dx. \tag{4.23}
\]

The passage from (4.18) to (4.23) requires some estimates of \(F_N(x, x; t)\) at infinity which of course demand an appropriate decay of a potential as \(|x| \to \infty\). We avoid such estimates and deduce AE (4.23) from Theorem 4.5 using the formula (the inversion of the Laplace transform (4.20))

\[
\text{Tr}(e^{-Ht} - e^{-Ho}) = (2\pi i)^{-1} (m - 1)! t^{-m+1} \int_{\beta-i\infty}^{\beta+i\infty} \text{Tr}(R^m(x) - R_0^m(x)) e^{-tz} dz
\]

where \(2(m + 1) > d\) and \(\beta < \lambda_1\). Let us formulate the precise result.

**Theorem 4.10.** Suppose that \(v(x)\) satisfies estimates (4.4) where \(\rho > d\). Then AE (4.23) holds.

We note finally that according to (4.22)

\[
r_n^{(m)} = (4\pi)^{-d/2} (m - 1)!^{-1} \Gamma(n + m + 1 - d/2) g_{n+1}. \tag{4.24}
\]
5. High-energy asymptotics of the SM

1. Let assumption (4.4) be satisfied. Away from the diagonal $\omega = \omega'$, the
integral kernel $s(\omega, \omega'; \lambda)$ of the SM $S(\lambda)$ is $C^\infty$-function and decays faster than
any power of $\lambda^{-1}$ as $\lambda \to \infty$. On the contrary, it acquires diagonal singularities
which are determined by the fall-off of $v(x)$ at infinity. It turns out that these
singularities and the high-energy limit are described by the same formulas.

To describe $s(\omega, \omega'; \lambda)$ in a neighbourhood of the diagonal, we recall first a stan-
dard construction of approximate but explicit solutions $\psi_N$ of the Schrödinger equation
$-\Delta \psi + v(x)\psi = k^2 \psi$. This construction relies on a solution of the corresponding
transport equation by iterations. Let us set

$$\psi_N(x, \xi) = e^{i(x, \xi)}b_N(x, \xi), \quad \xi = k\omega \in \mathbb{R}^d, \quad k = \sqrt{\lambda},$$

where

$$b_N(x, \xi) = \sum_{n=0}^{N} (2ik)^{-n}b_n(x, \omega), \quad b_0(x, \omega) = 1. \quad (5.1)$$

Plugging these expressions into the Schrödinger equation and equating coefficients
at the same powers of $(2ik)^{-1}$, we obtain recurrent equations

$$\langle \omega, \nabla_x b_{n+1}(x, \omega) \rangle = -\Delta b_n(x, \omega) + v(x)b_n(x, \omega)$$

whence

$$b_{n+1}(x, \omega) = \int_{-\infty}^{0} (-\Delta b_n(x + tw, \omega) + v(x + tw)b_n(x + tw, \omega))dt. \quad (5.2)$$

It is easy to see that under assumption (4.4) for any $c < 1$ and $\varepsilon_0 = \min\{1, \rho - 1\}$

$$|\partial_x^n b_n(x, \omega)| \leq C_\alpha (1 + |x|)^{-\rho+1-\varepsilon_0(n-1)-|\alpha|}, \quad n \geq 1, \quad \text{if} \quad \langle x, \omega \rangle \leq c|x|. \quad (5.3)$$

Let us fix some point $\omega_0 \in \mathbb{S}^{d-1}$. Let $\Lambda(\omega_0)$ be the plane orthogonal to $\omega_0$
and $\Omega = \Omega(\omega_0, \delta) \subset \mathbb{S}^{d-1}$ be determined by the condition $\langle \omega, \omega_0 \rangle > \delta > 0$. Set
$x = \omega_0 z + y$, where $y \in \Lambda(\omega_0)$,

$$A_N(\omega, \omega', y; \lambda) = 2^{-1/2}\langle \omega + \omega', \omega_0 \rangle b_N(y, -k\omega)b_N(y, k\omega') + (2ik)^{-1}\left(b_N(y, -k\omega)\partial_2b_N(y, k\omega') - b_N(y, k\omega')\partial_2b_N(y, -k\omega)\right) \quad (5.4)$$

and

$$s_N(\omega, \omega'; \lambda) = (2\pi)^{-d+1}k^{d-1}\int_{\Lambda(\omega_0)} e^{ik(y, \omega' - \omega)}A_N(\omega, \omega', y; \lambda)dy \quad (5.5)$$

for $\omega, \omega' \in \Omega$. Since $|\langle y, \omega \rangle| \leq c|y|$ and $|\langle y, \omega' \rangle| \leq c|y|$ where $c < 1$ for $\omega, \omega' \in \Omega$, $y \in \Lambda(\omega_0)$, estimates (5.3) imply that oscillating integral (5.5) is well defined. As
shown in [30], the function $s_N(\omega, \omega'; \lambda)$ describes all singularities of $s(\omega, \omega'; \lambda)$ and approximates it with arbitrary accuracy as $\lambda \to \infty$.

**Theorem 5.1.** Let assumption (4.4) hold for $\rho > 1$. Let the function $b_N(x, \xi)$ be defined by formulas (5.1) and (5.2). Define for $\omega, \omega' \in \Omega$ the function $s_N$ by equalities (5.4), (5.5). Then, for an arbitrary number $p$ and a sufficiently large $N = N(p)$, the remainder $s(\omega, \omega'; \lambda) - s_N(\omega, \omega'; \lambda)$ belongs to the class $C^p(\Omega \times \Omega)$
and the $C^p$-norm of this function is $O(\lambda^{-p})$ as $\lambda \to \infty$. 
Another form of AE of the scattering amplitude \( s(\omega, \omega', \lambda) \) can be found in [27]. Theorem 5.1 extends to long-range potentials [31]. Formula (5.5) shows that we actually consider the SM \( S(\lambda) \) as a pseudo-differential operator (on the unit sphere) determined by its amplitude \( A(\omega, \omega', y; \lambda) \). Plugging expression (5.1) into the right-hand side of (5.4), we can expand this amplitude into asymptotic series in powers of \( (2ik)^{-1} \). Moreover, it is possible to reformulate Theorem 5.1 using the standard procedure (see, e.g., [26]) of passage from an operator of a pseudo-differential operator to its symbol. We note also that the operator-valued function \( S \) is a continuous function of \( \lambda \) as \( \lambda > 0 \) and the operator-valued function \( S' \) is a continuous function of \( \omega \). Standard results of the pseudo-differential calculus (see, e.g., [26]) show that, together with the operator-valued function \( S(\lambda) \) and \( S'(\lambda) \), the operator \( T(\lambda) \) is also a pseudo-differential operator on the unit sphere. Moreover, Theorem 5.1 entails the following result which we formulate in terms of its right symbol.

**Theorem 5.2.** Let assumptions (4.1) hold for \( \rho > 1 \). Then kernel of the time-delay operator \( T(\lambda) \) admits expansion into the asymptotic series

\[
t(\omega, \omega'; \lambda) = (2\pi)^{-d+1} \sum_{n=0}^{\infty} (2ik)^{-n} \int_{\Lambda_\omega} e^{ik(y, \omega'-\omega)} \tau_n(\omega', y) dy
\]

where \( \tau_n = \tau_n \) are smooth functions of \( \omega' \in \Omega(\omega, \delta), y \in \Lambda(\omega) \) and \( \partial_\omega^\epsilon \tau_n(\omega', y) = O(|y|^{-\rho+1-\epsilon n-|\alpha|}) \) as \( |y| \to \infty \). If \( \rho > d \), then \( t \) is a continuous function of \( \omega, \omega' \) and

\[
t(\omega, \omega'; \lambda) = (2\pi)^{-d+1} \lambda^{d/2-2} \sum_{n=0}^{\infty} (-4\lambda)^{-n} \tau_n(\omega),
\]

where \( \tau_n(\omega) \) are the integrals of \( \tau_{2n}(\omega, y) \) over \( y \in \Lambda(\omega) \). Moreover, \( T \in C^\infty(\mathbb{R}_+) \), and AE (5.6), (5.7) can be infinitely differentiated with respect to \( \lambda \).

6. **High-energy asymptotics of the SSF**

Our goal in this section is to find the complete AE of the SSF \( \xi(\lambda) \) as \( \lambda \to \infty \). At the same time we extend AE (4.10) to the whole complex plane cut along the positive half-axis. In subs. 1 we prove the existence of the complete AE of \( \xi(\lambda) \). However the method of this subsection gives complicated expressions for coefficients \( \xi_n \) of this expansion. This drawback is remedied in subs. 2 and 3 where a connection of the results of subs. 1 with the results of subs. 1 and 2, §4, is established. This yields expressions for \( \xi_n \) in terms of the coefficients \( r_n^{(m)} \) as well as of the heat invariants \( g_{n+1} \).

1. We proceed from equation (4.9) which follows from the Birman-Krein formula (1.2) because the SSF \( \xi \) is a continuous function of \( \lambda > 0 \) and the operator-valued function \( S(\lambda) \) is differentiable with respect to \( \lambda \) in the trace norm. Let us now use Theorem 5.2. Integrating (5.7) over \( \omega \in S^{d-1} \) and taking into account formula (2.9), we find the AE of the derivative \( \xi'(\lambda) \) of the SSF.
Theorem 6.1. Let assumption (1.1) where \( \rho > d \) hold. Then \( \xi \in C^\infty(\mathbb{R}_+) \) and the AE

\[
\xi' = \lambda^{d/2-2} \sum_{n=0}^{\infty} \eta_n \lambda^{-n}
\]

holds with the coefficients

\[
\eta_0 = 4^{-1}(d-2)(2\pi)^{-d}|S^{d-1}| \int_{\mathbb{R}^d} v(x) dx,
\eta_n = -(2\pi)^{-d}(-4)^{-n} \int_{S^{d-1}} \tau_n(\omega) d\omega.
\]

Expansion (6.1) can be infinitely differentiated.

Integrating (6.1) over \( \lambda \), we obtain also the following result.

Corollary 6.2. For a suitable constant \( \gamma \), the SSF \( \xi(\lambda) \) admits expansion into the asymptotic series

\[
\xi(\lambda) = \lambda^{d/2-1} \sum_{n=0, n \neq d/2-1}^{\infty} \xi_n \lambda^{-n} + \hat{\xi} \ln \lambda + \gamma,
\]

where \( \xi_n = (d/2 - n - 1)^{-1} \eta_n, \hat{\xi} = 0 \) for \( d \) odd and \( \hat{\xi} = \eta_{d/2-1} \) for \( d \) even.

Expressions (6.2) for the coefficients \( \eta_n, n \geq 1 \), are rather complicated. Below we shall find efficient expressions for \( \xi_n \). At the same time we shall show that \( \hat{\xi} = 0 \) for all \( d \) and find the constant \( \gamma \) in (6.3).

2. In view of representation (2.15), the following assertion can be deduced from Corollary 6.2 with a help of a version of the Privalov theorem.

Proposition 6.3. Let assumption (1.1) where \( \rho > d \) hold, and let \( 2(m+1) > d \).

Then, for some numbers \( \epsilon_n^{(m)} \), \( n \geq m + 1 \), the expansions into asymptotic series hold as \( |z| \to \infty, z \in \mathbb{C} \setminus \mathbb{R}_+ \):

\[
\text{Tr}(R^m(z) - R_{0}^m(z)) = \sum_{n=0}^{\infty} \epsilon_n^{(m)} z^{-n}
\]

if \( d \) is odd, and

\[
\text{Tr}(R^m(z) - R_{0}^m(z)) = \sum_{n=0}^{d/2-2} \epsilon_n^{(m)} (-z)^{d/2-m-n-1} - \gamma(-z)^{-m} + \sum_{n=m+1}^{\infty} \epsilon_n^{(m)} z^{-n}
\]

if \( d \) is even (the sum over \( n = 0, \ldots, d/2-2 \) is absent if \( d = 2 \)). Here \( \gamma \) is the same as in (6.3) and the asymptotic coefficients \( \epsilon_n^{(m)} \) are defined by the formula

\[
\epsilon_n^{(m)} = -mB(-n + d/2, m + n + 1 - d/2) \xi_n
\]

(\( B(p, q) \) is the beta-function) for all \( n \) if \( d \) is odd and for \( n = 0, \ldots, d/2-2 \) if \( d \) is even and \( \sigma_n^{(m)} = -(-1)^{n-d/2} m \left( \frac{n - d/2 + m}{m} \right) \xi_n \) for \( n = d/2, d/2 + 1, \ldots \).
We emphasize that the coefficients $c_n(m)$ are not determined by the AE of the SSF $\xi(\lambda)$ as $\lambda \to \infty$. Let us now compare Proposition 6.3 with Theorem 4.5. Expansion (4.10) should coincide with (6.4) or (6.5) in the angle $z \in \Pi_\theta$. Therefore the coefficients (4.11) and (6.6) are the same. Moreover, $\gamma = 0$, $c_n(m) = 0$ in (6.4) and $\gamma = - r_{d/2-1}, \sigma_n(m) = 0, c_n(m) = r_{n+d/2-1}$ in (6.5). Thus, Proposition 6.3 implies the following result.

**Theorem 6.4.** Let assumption (4.4) with $\rho > d$ hold, and let $2(m+1) > d$. Then AE (4.10) as $|z| \to \infty$ is valid for all $z \in \mathbb{C} \setminus \mathbb{R}_+$. At the same time we have improved significantly Theorem 6.1 and Corollary 6.2 obtaining explicit expressions of coefficients $\xi_n$ in AE (6.3) in terms of $r_n(m)$.

**Theorem 6.5.** Let assumption (4.4) with $\rho > d$ hold. Then the SSF $\xi(\lambda)$ admits expansion into the asymptotic series

$$\xi(\lambda) = \lambda^{d/2-1} \sum_{n=0}^{\infty} \xi_n \lambda^{-n},$$

where the coefficients $\xi_n$ are determined by formulas (6.6) and (4.9), (4.11). In particular, $\xi_n = 0$ for all $n \geq d/2$ if $d$ is even.

3. Alternatively, AE (6.7) can be deduced from Theorem 4.10 (instead of Theorem 4.5). Since

$$\text{Tr}(e^{-Ht} - e^{-H_0t}) = -t \int_{-\infty}^{\infty} \xi(\lambda) e^{-t\lambda} d\lambda,$$

it follows (see, e.g., [8]) from AE (6.3) of the SSF that

$$\text{Tr}(e^{-Ht} - e^{-H_0t}) = -\sum_{n=0}^{\infty} \Gamma(d/2-n) \xi_n t^{n+1-d/2} - \pi \gamma + \sum_{n=1}^{\infty} (-1)^n c_n t^n$$

(6.8) if $d$ is odd, and

$$\text{Tr}(e^{-Ht} - e^{-H_0t}) = -\sum_{n=0}^{d/2-1} \Gamma(d/2-n) \xi_n t^{n+1-d/2} - \pi \gamma$$

$$-\sum_{n=d/2}^{\infty} (-1)^{n-d/2} (n-d/2)!^{-1} \xi_n t^{n+1-d/2} \ln t + \sum_{n=1}^{\infty} (-1)^n c_n t^n$$

(6.9) if $d$ is even. Here

$$c_n = (n-1)!^{-1} \left( \int_{-\infty}^{0} \xi(\lambda) \lambda^{n-1} d\lambda + \int_{0}^{\infty} \xi(\lambda) - \sum_{j<n+d/2-1} \xi_j \lambda^{d/2-j-1} \lambda^{n-1} d\lambda \right).$$

(6.10)

Comparing (6.8) and (6.9) with (4.23), we recover Theorem 6.5. Moreover, we obtain explicit expressions

$$\xi_n = -(4\pi)^{-d/2} \Gamma^{-1}(d/2-n) g_{n+1}, \quad n = 0, 1, \ldots,$$

(6.11)

for the asymptotic coefficients in (6.7) in terms of the heat invariants defined by formulas (4.19) and (4.23). Relation (6.11) can also be obtained by putting together formulas (4.23) and (6.6).
7. Trace identities

The contribution of the continuous spectrum to trace identities is, roughly speaking, given in terms of the function \( \text{Tr}(R(\lambda + i\lambda) - R_0(\lambda + i\lambda)) \). In the case of integer order the imaginary part and in the case of half-integer order the real part of this function are considered.

1. Let us start with formulas of integer order. The method used here does not require any specific study of the PD or the SSF in a neighborhood of the point \( \lambda = 0 \). In particular, in the case \( d = 1 \) Theorem 7.1 does not exclude that the operator \( H \) has infinite number of negative eigenvalues.

Let us compare the coefficient at \( t^n \), \( n = 1, 2, \ldots \), in expansion (4.23) with the same coefficient in expansions (6.8) or (6.9). According to (4.23) this coefficient is zero if \( d \) is odd and it equals \((4\pi)^{-d/2} g_{d/2+n} \) if \( d \) is even. On the other hand, according to (6.8) or (6.9) it equals \((-1)^n c_n \) and hence is determined by equality (6.10). Let us also take into account that, for \( \lambda < 0 \), the SSF is given by formula (3.3) which yields

\[
\int_{-\infty}^{\infty} \xi(\lambda) \lambda^{n-1} d\lambda = n^{-1} \sum_{j=1}^{N} \lambda_j^n.
\]

Thus, the equalities \( c_n = 0 \) for \( d \) odd and \( c_n = (-1)^n (4\pi)^{-d/2} g_{d/2+n} \) for \( d \) even give us identities known as the trace identities.

**Theorem 7.1.** Let assumption (4.4) where \( \rho > d/2 \) hold. Then for all \( n = 1, 2, \ldots \), we have the following identities. If \( d \) is odd, then

\[
\int_0^{\infty} \left( \xi(\lambda) - \sum_{j=0}^{(d-3)/2+n} \xi_j \lambda^{d/2-j-1} \right) \lambda^{n-1} d\lambda + n^{-1} \sum_{j=1}^{N} \lambda_j^n = 0.
\]

If \( d \) is even, then

\[
\int_0^{\infty} \left( \xi(\lambda) - \sum_{j=0}^{d/2-1} \xi_j \lambda^{d/2-j-1} \right) \lambda^{n-1} d\lambda + n^{-1} \sum_{j=1}^{N} \lambda_j^n = (-1)^n (4\pi)^{-d/2} (n-1)! g_{d/2+n}.
\]

2. Next we consider trace identities of half-integer order. Now we suppose that \( d \leq 3 \) which allows us to conveniently formulate results in terms of the regularized PD \( D_2(z) \) discussed in §3 (for \( d = 1 \) the results can be formulated in terms of \( D(z) \) - see [11]).

**Proposition 7.2.** Let \( d \leq 3 \), and let assumption (4.4) where \( \rho > d/2 \) hold. Then the AE

\[
\ln D_2(z) = \sum_{n=1}^{\infty} \delta_n (-z)^{d/2-1-n}
\]

as \( |z| \to \infty, z \in \mathbb{C} \setminus \mathbb{R}_+ \), is valid with the coefficients \( \delta_n = -(n + 1 - d/2)^{-1} r_n^{(1)} \) where the coefficients \( r_n^{(1)} \) are defined by (4.9) and (4.11). In particular, as \( \lambda \to \infty \), \( \ln |D_2(\lambda + i0)| = O(\lambda^{-\infty}) \) if \( d \) is odd, and

\[
\ln |D_2(\lambda + i0)| = (-1)^{d/2+1} \sum_{n=1}^{\infty} (-1)^n \delta_n \lambda^{d/2-1-n}
\]

if \( d \) is even.
Indeed, let us use Remark 4.6 to Theorem 4.5. Combining equation (2.19) for \( p = 2 \) with AE (4.12), we see that
\[
d\ln D_2(z)/dz = -\sum_{n=1}^{\infty} \langle n \rangle (-z)^{d/2-n}.
\]
It remains to integrate it and to take relation (3.7) into account.

By virtue of (4.24), we also have that
\[
\delta_n = -\left(4\pi\right)^{-d/2} \Gamma\left(n + 1 - d/2\right) \bar{g}_{n+1}.
\]

Theorem 7.3. Let \( d \leq 3 \), let assumption (4.4) where \( \rho > 2 \) be satisfied and let \( n = 0, 1, 2, \ldots \). Suppose also that condition (3.8) is satisfied for \( \alpha = n + 1/2 \). Let the numbers \( \delta_n \) be the same as in Proposition 7.2. Then, for \( d = 1, 3 \),
\[
\pi^{-1} (-1)^n \int_{0}^{\infty} \ln |D_2(\lambda + i0)| \lambda^{n-1/2} d\lambda - (n + 1/2)^{-1} \sum_{j=1}^{N} |\lambda_j|^{n+1/2} = \delta_n
\]
and, for \( d = 2 \) (the sum under the integral sign is absent for \( n = 0 \)),
\[
\pi^{-1} (-1)^n \int_{0}^{\infty} \left( \ln |D_2(\lambda + i0)| - \sum_{j=1}^{n} (-1)^j \delta_j \lambda^{d/2-1-j} \right) \lambda^{n-1/2} d\lambda
\]
\[
- (n + 1/2)^{-1} \sum_{j=1}^{N} |\lambda_j|^{n+1/2} = 0.
\]

The proof is similar to that of Theorem 3.7. By virtue of the Cauchy theorem, we have that
\[
\int_{\Gamma_{R,\varepsilon}} \ln D_2(z)(-z)^{n-1/2} dz = 0 \quad (7.2)
\]
where \( \Gamma_{R,\varepsilon} \) is the same contour as in (3.3). The integrals over \( C^{\pm}_{\varepsilon}(\lambda_j) \) and \( C^{\pm}_{\varepsilon}(0) \) tend again to zero as \( \varepsilon \to 0 \). Thus, it follows from (7.2) that
\[
2(-1)^n \int_{0}^{R} \ln |D_2(\lambda + i0)| \lambda^{n-1/2} d\lambda - 2\pi (n + 1/2)^{-1} \sum_{j=1}^{N} |\lambda_j|^{n+1/2}
\]
\[
= i \int_{C_R} \ln D(z)(-z)^{n-1/2} dz. \quad (7.3)
\]
The AE as \( R \to \infty \) of both integrals can easily be deduced from (7.1). Hereby the divergent terms cancel each other. It remains to pass to the limit \( R \to \infty \) in (7.3).

Trace identities of integer order can also be obtained by the method of proof of Theorem 7.3 (hereby \( n - 1/2 \) should be replaced by \( n - 1 \) in (7.2)) which however requires more stringent assumptions than Theorem 7.1.

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