Spherically Symmetric Solutions on a Non-Trivial Frame in $f(T)$ Theories of Gravity

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A new solution with constant torsion is derived using the field equations of $f(T)$. Asymptotic forms of energy density, radial and transversal pressures are shown to meet the standard energy conditions, i.e., weak and null energy conditions according to some restrictions on $T_0$, $f(T_0)$ and $f(T)$. Other solutions are obtained for vanishing radial pressure and for specific choices of $f(T)$. The physics relevant to the resulting models is discussed.

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Recent observational data suggest that our universe is accelerating.\cite{1,2} This acceleration is explained in terms of so-called dark energy (DE). DE could also result from a cosmological constant, from an ideal fluid with a different equation of state shape and negative pressure,\cite{3} etc. It is unclear what type of DE can more seemingly explain the current era of the universe. A very attractive possibility that has already been mentioned is “modification of general relativity” (GR). Amendments to the Hilbert–Einstein action through the introduction of different functions of the Ricci scalar have been systematically explored by the so-called $f(R)$ gravity models, with reconstruction having been developed Ref.\cite{4-9}.

Recently, a new attractive modified gravity to account for the accelerating expansion of the universe, i.e., $f(T)$ theory, has been suggested by extending the action of teleparallel gravity\cite{8,10-13} similar to $f(R)$ theory, where $T$ is the torsion scalar. It has been demonstrated that $f(T)$ theory can not only explain the present cosmic acceleration with no need for dark energy,\cite{14} but can also provide an alternative to inflation without an inflation.$^{[15,16]}$ In addition, it is shown that $f(T)$ theories are not dynamically equivalent to teleparallel action plus a scalar field under conformal transformation.$^{[17]}$ It has therefore attracted some attention recently. In this regard, Linder\cite{18} proposed two new $f(T)$ models to explain the present accelerating expansion and found that $f(T)$ theory can unify a number of interesting extensions of geometry beyond GR.$^{[19]}$

The objective of this work is to find spherically symmetric solutions under the framework of $f(T)$ using anisotropic spacetime for a non-trivial frame. First, a brief review of $f(T)$ theory is presented. Then, non-trivial spherically symmetric spacetime is provided and applications to the field equation of $f(T)$ are presented. Various new spherically symmetric anisotropic solutions are derived, and several figures demonstrating the asymptotic behavior of energy density and transversal pressure are also given.

First, let us give a brief review of $f(T)$. In a spacetime with absolute parallelism parallel vector fields $h_\mu^a$ (Ref.\cite{20}), we identify the nonsymmetric connection

$$\Gamma^\lambda_{\mu\nu} \equiv h^a_\lambda h^\mu_{\nu a},$$

where $h_{\mu\nu a} = \delta_{\mu a} h_{\nu a}$. The metric tensor $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} \equiv \eta_{ab} h_\mu^a h_\nu^b$$

with $\eta_{ab} = (-1,+1,+1,+1)$ being the Minkowski spacetime. The torsion and the contorsion are defined as

$$T^\alpha_{\mu
u} \equiv \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = h^a_\alpha (\partial_\mu h^\nu_a - \partial_\nu h^\mu_a),$$

$$K_{\alpha\nu} \equiv -\frac{1}{2} (T^\alpha_{\mu\nu} - T^{\nu\nu}_{\alpha\mu} - T^{\mu\mu}_{\alpha\nu}) = \Gamma^\lambda_{\mu\nu} - \{\lambda_{\mu\nu}\}.$$  (3)

The tensor $S_{\alpha\nu}^\mu$ and the scalar tensor $T$ are defined as

$$S_{\alpha\nu}^\mu \equiv \frac{1}{2} (K_{\alpha\nu} + \delta_{\alpha\nu} T^\beta_{\mu\beta} - \delta_{\mu\nu} T^\beta_{\alpha\beta}),$$

$$T \equiv T^\alpha_{\alpha\nu} S_{\alpha\nu}^\mu.$$  (4)

Similar to the $f(R)$ theory, one can define the action of $f(T)$ theory as

$$\mathcal{L}(h_\mu^a, \Phi_A) = \int d^4x \left[ \frac{1}{16\pi} f(T) + \mathcal{L}_{\text{matter}}(\Phi_A) \right],$$

where $h = \sqrt{-g}$ and $\Phi_A$ represent the matter fields. Assume the action (5) as a functional of the fields $h_\mu^a$, $\Phi_A$. The vanishing of the variation with respect to the
field $h_{\alpha \mu}$ gives the equation of motion\cite{24} as follows:

$$
S_{\mu}^{\rho \nu} T_{\rho} f(T)_{TT} + \left[ - h^{-1} h_{\mu} \partial_{\rho} (h h_{\alpha} \partial_{\nu} h_{\alpha \rho}) - T_{\alpha \lambda \mu} S_{\lambda \nu} \right] f(T) - \frac{1}{4} \delta_{\mu}^{\nu} f(T) = 4 \pi T_{\mu}^{\nu},
$$

(6)

where $T_{\rho} = \frac{\partial f}{\partial T}$, $f(T)_{TT} = \frac{\partial f(T)}{\partial T}$, and $T_{\mu}^{\nu}$ is the energy momentum tensor. In this study we consider the matter content to have an anisotropic form, i.e., given by

$$
T_{\mu \nu} = \text{diag}(\rho, -p_r, -p_t, -p_t),
$$

(7)

where $\rho$, $p_r$, and $p_t$ are the energy density, the radial and tangential pressures, respectively. Next we apply the field Eq. (6) to a spherically symmetric spacetime and try to find new solutions.

Assume that the non-trivial manifold possesses stationary and the spherical symmetry has the form\cite{21}

$$
(h_{\alpha \mu}) = \begin{bmatrix}
  e^{A(r)} & 0 & 0 & 0 \\
  0 & e^{B(r)} \sin \theta \cos \phi & r \cos \phi \cos \phi & -r \sin \theta \sin \phi \\
  0 & e^{B(r)} \sin \sin \phi & r \cos \sin \phi & r \sin \sin \phi \\
  0 & e^{B(r)} \cos \theta & -r \sin \theta & 0
\end{bmatrix},
$$

(8)

where $A(r)$ and $B(r)$ are two unknown functions of $r$. The metric associated with Eq. (8) takes the form

$$
ds^2 = -e^{A(r)} dt^2 + e^{B(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
$$

It is important to note that for the same metric and the same coordinate basis, different frames result in different forms of equations of motion.\cite{22} Using Eq. (8), one can obtain $h = \det(h_{\alpha \mu}) = e^{(A+2B)/2} r^2 \sin \theta$. With the use of Eqs. (3) and (4), one can obtain the torsion scalar and its derivatives in terms of $r$ in the form

$$
T(r) = \frac{2(1-2e^{-B/2} - r A' e^{-B/2} [1 - e^{-B/2}] + e^{-B})}{r^2},
$$

$$
A' = \frac{\partial A(r)}{\partial r},
$$

$$
T'(r) = \frac{e^{-B/2}}{r^2} (B' r^2 A'(2e^{-B/2} - 1) + 2r(e^{-B/2} - 1) + 2r [r A'' - A'] [1 - e^{-B/2}] - 4[2 - e^{-B/2} - e^{-B/2}]).
$$

(9)

The field equations (6) for an anisotropic fluid have the form

$$
4 \pi \rho = -\frac{e^{-3B/2} f TT}{r^4} (4[r A' - r^2 A'' + r B' + 3] + 3r^2 [r A' B' + 4 e^{-B} + e^{-B/2} (r^2 [2 A'' - A' B']) - 2r [B' + A'] - 12 + e^{-B/2} (2 r^2 A'' - 2 r A' - 2 r B' - 2 r^2 A' B' - 4)])
$$

$$
- e^{-B/2} f_T (2 - r A' + e^{-B/2} [r A' + r B' - 2]) + \frac{f}{4},
$$

4 \pi p_r = \frac{e^{-3B/2} f TT}{8 r^2} (8 - 4e^{-B/2} + e^{-B/2} [2 r A' + 2 r^2 A' - r^2 A'' - 2 r A' - 4] + \frac{f}{4});
$$

(10)

It is of interest to note that Eq. (8) was used in the literature.\cite{23} In this study we will find new solutions to Eq. (10) without any assumption on the unknown function $B(r)$. Next we try to find several solutions to Eq. (10) assuming some conditions on $f(T)$.

First assumption: $T = \text{const} = T_0$. From Eq. (9), it can be shown that $A(r)$, which satisfies $T = T_0$ and $T' = 0$, has the form

$$
A(r) = \frac{1}{2} \int \frac{2 - 4 e^{-B/2} + 2 e^{-B/2} + 2 e^{-B/2} (1 + e^{-B/2})}{r (1 - e^{-B/2})} dr + c_1,
$$

(11)

where $c_1$ is a constant of integration. The weak and null energy conditions have the form

$$
\rho \geq 0, \quad \rho + p_r \geq 0, \quad \rho + p_t \geq 0.
$$

(12)

Figure 1 shows the asymptotic form of energy density, radial and transversal pressures when $T = T_0$. One can find that the relevant physics meets the standard energy conditions given by Eq. (12). We put restrictions on the values of $T_0$, $f(T_0)$, $f_T(T_0)$ and $B(r)$ such that Eq. (12) is satisfied.

In the case of vanishing radial pressure, the second of Eqs. (10) reads

$$
\frac{f(T)}{2} = \frac{2 f e^{-B/2}}{r^2} \left(2 + r A' - 2 e^{-B/2} [r A' + 1] \right).
$$

(13)
We study various cases of Eq. (13) in the following. For the first form of \( f(T) \), let us assume \( f(T) \) to have the form,

\[
f(T) = a_0 + a_1 T + a_n T^n, \tag{14}
\]

where \( a_0, a_1 \) and \( a_n \) are constants. From Eqs. (13) and (14) one can, for the linear case of \( f(T) \), obtain

\[
A(r) = \frac{1}{2} \int_0^r \left[ \frac{e^{B(r)}(2a_1 - a_0 r^2) - 2a_1}{r}\right] dr + c_2, \tag{15}
\]

with \( c_2 \) being another constant of integration. The asymptote behavior of energy density and the transversal pressure, obtained from Eqs. (10) using Eq. (15), are shown in Fig. 2.

In the same way, one can obtain a solution for the nonlinear case, i.e., when \( n = 2 \), we have the unknown function \( A(r) \) which takes the form

\[
A(r) = \frac{1}{4} \int \frac{1}{r(3 + e^{B(r)}[1 - 4e^{-\frac{B(r)}{2}]})} \cdot \left( e^{2B(r)}[e^{B(r)} - 24a_2 e^{B(r)/2} - 12a_2(1 + e^{B(r)})] \right)
\]

\[
- 16a_2^2 e^{-3B(r)} + 16a_2 e^{-\frac{B(r)}{2}}[a_1 r^2 - 4a_2] - 2e^{-2B(r)}[12a_0 a_2 a_2 r^4 - 40a_1 a_2 a_2 r^2 + a_2^2 r^6 + 96a_2^2]
\]

\[
+ e^{-\frac{B(r)}{2}}[32a_1 a_2 r^2 - 16a_0 a_2 r^4 - 64a_2^2] + e^{-B(r)}[4a_2 a_2 r^4 - 8a_2 a_1 r^2] + 16a_2^2 e^{-\frac{B(r)}{2}} \right) ^{1/2} dr + c_3,
\]

with \( c_3 \) being a constant of integration.

From Fig. 3, it is clear that this model meets the standard energy condition when \( a_0 = a_1 = a_2 = c_3 = 1 \) and \( B(r) \sim 1/r \). Other options are not permitted due to the reasons discussed above.

Substituting Eq. (13) into Eq. (14) we obtain

\[
\frac{1}{r^2} \left[ r^2 \{a_0 + a_1 T + a_n T^n\} - e^{-\frac{B(r)}{2}} [4a_1
\]

\[
+ 2r a_1 A' + 2 n r a_n T^{n-1} A' + 4na_n T^{n-1}] + e^{-B(r)}[4na_n T^{n-1} + 4r a_1 A' + 4a_1 + 4nr a_n T^{n-1}] \right] = 0.
\]

According to the \( f(R) \) model, also used for the \( f(T) \) theory, as an alternative to the dark energy, \(^{15}\) by taking the function \( f(T) \) presented by Eq. (14) when \( n = -1 \), one can obtain a relation between the unknown function \( A(r) \) in terms of the unknown function \( B(r) \). The asymptote behavior of energy density and the tangential pressure are plotted in Fig. 4. Using the same procedure we obtain a physically acceptable model as shown in Fig. 4.

Now we will describe the main results and provide some discussion. All the solutions obtained in this work can be summarized as follows. (1) When the scalar torsion is taken to be constant, i.e., \( T = T_0 \), a quite general differential equation that governs the two unknown functions is obtained. By this relation, calculations of energy density, radial and transversal pressures are provided. The asymptote behavior
of these quantities are drawn in Fig. 1 for a specific asymptote of $B(r)$. From Fig. 1, it is clear that the asymptote of energy density, radial and transversal pressures depend on the constants as well as on the asymptote behavior of $B(r)$. (2) The condition of vanishing radial pressure is derived. A quite general assumption on the form of $f(T)$ is employed. Different cases have been studied using the assumptions: (i) when $n = 0$, following the procedure carried out in the case of constant scalar torsion, a differential equation that links the two unknown functions $A(r)$ and $B(r)$ is derived. The asymptote behaviors of the density and transversal pressure are given in Fig. 2. From this figure one can find that both energy density and transversal pressure are positive. (ii) When the condition $n = 2$, i.e., the nonlinear case is employed, a solution is obtained for a specific form of one of the two unknown functions. The density and tangential pressure are given in Fig. 3. From this figure one can conclude that the character of this model is physically acceptable since it satisfies the standard energy conditions. (iii) When the condition $n = -1$ is employed, this case is studied for the $f(T)$ theory as an alternative to the dark energy. By taking the function $f(T)$ as given by Eq. (14), a solution is obtained for a specific form of one of the two unknown functions. The density and tangential pressure are given in Fig. 4. From this figure one can conclude that this model is also physically acceptable for the reasons discussed above.

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