HECKE-CLIFFORD SUPERALGEBRAS AND CRYSTALS OF TYPE $D^{(2)}_l$

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Dedicated to Professor Tetsuji Miwa on the occasion of his sixtieth birthday

Abstract. In [BK], Brundan and Kleshchev showed that some parts of the representation theory of the affine Hecke-Clifford superalgebras and its finite-dimensional “cyclotomic” quotients are controlled by the Lie theory of type $A^{(2)}_{2l}$ when the quantum parameter $q$ is a primitive $(2l + 1)$-th root of unity. We show in this paper that similar theorems hold when $q$ is a primitive $4l$-th root of unity by replacing the Lie theory of type $A^{(2)}_{2l}$ with that of type $D^{(2)}_l$.

1. Introduction

It is known that we can sometimes describe the representation theory of “Hecke algebra” by “Lie theory”. In this paper, we use the terminology “Lie theory” as a general term for objects related to or arising from Lie algebra, such as highest weight representations, quantum groups, Kashiwara’s crystals, etc.

A famous example is Lascoux-Leclerc-Thibon’s interpretation [LLT] of Kleshchev’s modular branching rule [KL1]. It asserts that the modular branching graph of the symmetric groups in characteristic $p$ coincides with Kashiwara’s crystal associated with the level 1 integrable highest weight representation of the quantum group $U_q(g(A^{(1)}_{p-1}))$. Brundan’s modular branching rule for the Iwahori-Hecke algebras of type $A$ at the quantum parameter $q = \sqrt{p}$ over $\mathbb{C}$ is a similar result and can be regarded as a $q$-analogue of the above example [Br1].

Another beautiful example is Ariki’s theorem [Ari] generalizing Lascoux-Leclerc-Thibon’s conjecture for the Iwahori-Hecke algebras of type $A$ [LLT]. It relates the decomposition numbers of the Ariki-Koike algebras at $q = \sqrt{p}$ over $\mathbb{C}$ and Kashiwara-Lusztig’s canonical basis of a suitable integrable highest weight representation of $U_q(g(A^{(1)}_{p-1}))$. Varagnolo-Vasserot’s generalization of Ariki’s theorem to $q$-Schur algebras [VV] and Yvonne’s conjectural generalization for cyclotomic $q$-Schur algebras [Yvo] are also examples of connections between Hecke algebras and Lie theory.

However, all the Lie theory involved so far is only that of type $A^{(1)}_n$. Subsequently, based on the work of Grojnowski [Gro] and Grojnowski-Vazirani [GV], Brundan and Kleshchev showed that some parts of the representation theory of the affine Hecke-Clifford superalgebras introduced by Jones and Nazarov [JN] and its finite-dimensional “cyclotomic” quotients1 introduced by Brundan and Kleshchev [BK, §3.4-b] are controlled by the Lie theory of type $A^{(2)}_{2l}$ when the quantum parameter $q$ is a primitive $(2l + 1)$-th root of unity. Let $\mathcal{H}_n$ be the affine Hecke-Clifford

1As a special case they include the Hecke-Clifford superalgebras introduced by Olshanski [Ols].
superalgebra (see Definition 3.1) over an algebraically closed field \( F \) of characteristic different from 2 and let \( q \) be a \((2l + 1)\)-th primitive root of unity for \( l \geq 1 \). Their main results are as follows.

(1) The direct sum of the Grothendieck groups \( K(\infty) = \bigoplus_{n \geq 0} K_0(\text{Rep} \, \mathcal{H}_n) \) of the category \( \text{Rep} \, \mathcal{H}_n \) of integral \( \mathcal{H}_n \)-supermodules has a natural structure of a commutative graded Hopf \( \mathbb{Z} \)-algebra by induction and restriction [BK, Theorem 7.1] and the restricted dual \( K(\infty)^* \) is isomorphic to the positive part of the Kostant \( \mathbb{Z} \)-form of the universal enveloping algebra of \( g(A^{(2)}_{2l}) \) [BK, Theorem 7.17].

(2) The disjoint union \( B(\infty) = \bigsqcup_{n \geq 0} \text{irr}(\text{Rep} \, \mathcal{H}_n) \) of the isomorphism classes of irreducible integral \( \mathcal{H}_n \)-supermodules has a natural crystal structure in the sense of Kashiwara and it is isomorphic to Kashiwara’s crystal associated with \( U_-^q(g(A^{(2)}_{2l})) \) [BK, Theorem 8.10].

(3) For each positive integral weight \( \lambda \) of \( A^{(2)}_{2l} \), one can define a finite-dimensional quotient superalgebra \( \mathcal{H}_n^\lambda \) of \( \mathcal{H}_n \), called the cyclotomic Hecke-Clifford superalgebra [BK, §3, §4-b].

(4) Consider the direct sums of the Grothendieck groups \( K(\lambda) = \bigoplus_{n \geq 0} K_0(\mathcal{H}_n^\lambda - \text{smod}) \) of the category of finite-dimensional \( \mathcal{H}_n^\lambda \)-supermodules and \( K(\lambda)^* = \bigoplus_{n \geq 0} K_0(\text{Proj} \, \mathcal{H}_n^\lambda) \) of the category \( \text{Proj} \, \mathcal{H}_n^\lambda \) of finite-dimensional projective \( \mathcal{H}_n^\lambda \)-supermodules. Then \( K(\lambda)_Q = \mathbb{Q} \otimes_{\mathbb{Z}} K(\lambda) \) is naturally identified\(^2\) with the integrable highest weight \( U_Q \)-module of highest weight \( \lambda \) where \( U_Q \) stands for the \( \mathbb{Q} \)-form of the universal enveloping algebra of \( g(A^{(2)}_{2l}) \) [BK, Theorem 7.16.(i)]. Moreover, the Cartan map \( K(\lambda)^* \to K(\lambda) \) is injective [BK, Theorem 7.10] and \( K(\lambda)^* \subseteq K(\lambda) \) are dual lattices in \( K(\lambda)_Q \) under the Shapovalov form [BK, Theorem 7.16.(iii)].

(5) The disjoint union \( B(\lambda) = \bigsqcup_{n \geq 0} \text{irr}(\mathcal{H}_n^\lambda - \text{smod}) \) is isomorphic to Kashiwara’s crystal associated with the integrable \( U_-^q(g(A^{(2)}_{2l})) \)-module of highest weight \( \lambda \) [BK, Theorem 8.11].

Analogous results for the degenerate affine Sergeev superalgebras of Nazarov [Naz] and its cyclotomic quotients [BK, §4-i] over an algebraically closed field \( F \) of char \( F = 2l + 1 \) are also established in [BK] parallel to those for the affine Hecke-Clifford superalgebras and its cyclotomic quotients at \( q = 2^{l+1} \sqrt{1} \) over an algebraically closed field \( F \) of char \( F \neq 2 \). As a very special corollary of the results for the degenerate superalgebras, they beautifully obtain a modular branching rule of the spin symmetric groups \( \tilde{S}_n \). However, it may be a reason why they deal only with the case \( q = 2^{l+1} \sqrt{1} \) for the affine Hecke-Clifford superalgebras in [BK].

Note that exactly the same results as above hold when \( q \) is a primitive \((2l+1)\)-th root of unity for \( l \geq 1 \). This follows from the fact that \(-q\) is a primitive \((2l+1)\)-th root of unity and the superalgebra isomorphism between the affine Hecke-Clifford superalgebras (see Definition 3.1) \( \mathcal{H}_n(q) \) and \( \mathcal{H}_n(-q) \) given by

\[
\mathcal{H}_n(q) \xrightarrow{\sim} \mathcal{H}_n(-q), \quad X_i \mapsto X_i, \quad C_i \mapsto C_i, \quad T_j \mapsto -T_j
\]

\(^2\)It is not proved so far but expected that the weight space decomposition of \( K(\lambda)_Q \) coincides with the block decomposition of \( \{\mathcal{H}_n^\lambda\}_{n \geq 0} \) under this identification. In fact, it is settled in the following analogous situation, when \( \mathcal{H}_n^\lambda \) is replaced by Ariki-Koike algebra [LM], degenerate Ariki-Koike algebra [Br2] and odd level cyclotomic quotient of the degenerate affine Sergeev superalgebra [Ruf] respectively. See also [BK, §2].
for $1 \leq i \leq n$ and $1 \leq j < n$. However, the case when the multiplicative order of $q$ is divisible by 4 is yet untouched.

The purpose of this paper is to show that Brundan-Kleshchev’s method is still applicable to the case when $q$ is a primitive $4l$-th root of unity for any $l \geq 2$. In this case we have very similar results by replacing $A_{2l}^{(2)}$ with $D_l^{(2)}$ in the above summary. Roughly speaking, we prove the following four statements (for the precise statements, see Corollary 6.11, Corollary 6.12, Theorem 6.13 and Theorem 6.14).

**Theorem 1.1.** Let $F$ be an algebraically closed field of characteristic different from 2 and let $q$ be a primitive $4l$-th root of unity for $l \geq 2$. For each positive integral weight $\lambda$ of $D_l^{(2)}$, we can define a finite-dimensional quotient superalgebra $\mathcal{H}_n^\lambda$ of $\mathcal{H}_n$ (see Definition 4.1) so that the followings hold.

(i) the graded dual of $K(\infty) = \bigoplus_{n \geq 0} K_0(\text{Rep} \mathcal{H}_n)$ is isomorphic to $U_Z^+$ as graded $\mathbb{Z}$-Hopf algebra (see Theorem 6.14).

(ii) $K(\lambda) = \bigoplus_{n \geq 0} \mathbb{Q} \otimes K_0(\mathcal{H}_n^\lambda - \text{smod})$ has a left $U_Q$-module structure which is isomorphic to the integrable highest weight $U_Q$-module of highest weight $\lambda$ (see Theorem 6.13 for details).

(iii) $B(\infty) = \bigsqcup_{n \geq 0} \text{Irr}(\text{Rep} \mathcal{H}_n)$ is isomorphic to Kashiwara’s crystal associated with $U_v^{-}(\mathfrak{g}(D_l^{(2)}))$ (see Corollary 6.11).

(iv) $B(\lambda) = \bigsqcup_{n \geq 0} \text{Irr}(\mathcal{H}_n^\lambda - \text{smod})$ is isomorphic to Kashiwara’s crystal associated with the integrable $U_v(\mathfrak{g}(D_l^{(2)}))$-module of highest weight $\lambda$ (see Corollary 6.12).

Here $U_Z^+$ is the positive part of the Kostant $\mathbb{Z}$-form of the universal enveloping algebra of $\mathfrak{g}(D_l^{(2)})$ and $U_Q$ is the $\mathbb{Q}$-subalgebra of the universal enveloping algebra of $\mathfrak{g}(D_l^{(2)})$ generated by the Chevalley generators (see §2.2).

A difference between our paper and [BK] is a behavior of representations of low rank affine Hecke-Clifford superalgebras which are treated at length in §5.

Finally, let us explain a reason behind our searching the “missing” connection between Hecke algebra and Lie theory of type $D_{n+1}^{(2)}$. It is well known that the level 1 crystal $B(\Lambda_0)$ associated with $U_v(\mathfrak{A}_{n+1}^{(2)})$ or $U_v(A_{2n}^{(2)})$ is described by partitions [MM, Kan]. It is interesting that some of the combinatorics appearing in their descriptions had been already studied in the representation theory of the (spin) symmetric groups [Jam, Mor, MY], and such combinatorics controls modular branching of the (spin) symmetric groups [KI1, KI2, BK]. Thus, it is natural to ask which level 1 crystal has such a combinatorial realization, i.e., its underlying set is a subset of the set of partitions.

This problem is related to Kyoto path model [KMN12, KMN22] or its combinatorial counterpart, Kang’s Young wall [Kan]. The key tool underlying their realizations is a notion of perfect crystal [KMN22, Definition 1.1.1] which is introduced in [KMN12] to compute one-point functions of vertex models in statistical mechanics. As seen in [Kan], in order to realize $B(\Lambda_0)$ as a subset of the set of partitions, we need a perfect crystal of level 1 which has no branching point\(^3\). As shown in [KMN22],

\(^{3}\text{Let } G = (V, E) \text{ be a directed graph meaning that } V \text{ is the set of vertices and } E \subseteq V \times V \text{ is the adjacent relations meaning that } (v, w) \in E \text{ if and only if there exists a directed arrow from } v \text{ to } w. \text{ We say that a vertex } w \text{ is a branching point of } G \text{ if there exist } u \text{ and } v \text{ such that } u \neq v, u \neq w, v \neq w, (v, u) \in E \text{ and } (w, v) \in E.\)
such a perfect crystal of level 1 exists in types $A_n^{(1)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. Conversely, we can show that a pair of affine type and its perfect crystal of level 1 which has no branching point is one of the followings\footnote{$\langle A_1^{(1)} (B^{1,1}, (B^{1,1})^2), (A_1^{(1)}, B^{1,1} (n \geq 2)\),
(A_n^{(1)}, B^{n-1}) (n \geq 2), (A_{2n}^{(2)}, B^{1,1} (n \geq 1), (D_{n+1}^{(2)}, B^{1,1}) (n \geq 2)$}

\[
(A_1^{(1)}, B^{1,1}), \quad (A_1^{(1)}, (B^{1,1})^2), \quad (A_n^{(1)}, B^{1,1}) (n \geq 2),
(A_n^{(1)}, B^{n-1}) (n \geq 2), \quad (A_{2n}^{(2)}, B^{1,1}) (n \geq 1), \quad (D_{n+1}^{(2)}, B^{1,1}) (n \geq 2)
\]

if we assume the conjecture that any perfect crystal is a finite number of tensor product of Kirillov-Reshetikhin perfect crystals $B^{r,s}$ as stated in the first paragraph of the introduction of [KNO] and also assume the conjectural properties [HKOTY, Conjecture 2.1] [HKOTT, Conjecture 2.1] of Kirillov-Reshetikhin modules $W_s^{(r)}$.

This crystal-theoretic fact distinguishes types $A_n^{(1)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ from the other affine types and it is a reason behind our searching the “missing” connection between Hecke algebra and Lie theory of type $D_{n+1}^{(2)}$.

**Organization of the paper** The paper is organized as follows. In §2, we recall our conventions and necessary facts for superalgebras, supermodules and Kashiwara’s crystal theory. In §3 (resp. §4), we define the affine Hecke-Clifford superalgebras (resp. the cyclotomic Hecke-Clifford superalgebras) and review fundamental theorems for them along with [BK]. In §5, we give some preparatory character calculations concerning behavior of representations of low rank affine Hecke-Clifford superalgebras $H_2$, $H_3$ and $H_4$ which are responsible for the appearance of Lie theory of type $D_4^{(2)}$. Finally, in §6 we prove Theorem 1.1.

**Acknowledgements** The author would like to express his profound gratitude to Professor Jonathan Brundan and Professor Alexander Kleshchev since he owes a lot to their paper [BK]. Actually, many parts of the arguments in this paper are essentially the same as theirs. He also would like to thank Professor Masaki Kashiwara and Professor Susumu Ariki for their valuable comments when he read [K12] in their seminar and Professor Masato Okado for kindly answering questions on perfect crystals. Especially, he is grateful to Professor Masaki Kashiwara for reading the manuscript and giving him many useful comments and to Professor Jonathan Brundan for answering questions on [BK] and updated [BK’]. He is supported by JSPS fellowships for Young Scientists (No. 20-5306).

## 2. Preliminaries

**2.1. Superalgebras and supermodules.** We briefly recall our conventions and notations for superalgebras and supermodules following [BK, §2-b] (see also the references therein). In the rest of the paper, we always assume that our field $F$ is algebraically closed with char $F \neq 2$.

By a vector superspace, we mean a $Z/2Z$-graded vector space $V = V_\pi \oplus V_\pi$ over $F$ and denote the parity of a homogeneous vector $v \in V$ by $\pi \in Z/2Z$. Given two vector superspaces $V$ and $W$, an $F$-linear map $f : V \to W$ is called homogeneous if there exists $p \in Z/2Z$ such that $f(V_i) \subseteq W_{p+i}$ for $i \in Z/2Z$. In this case we call $p$ the parity of $f$ and denote it by $\overline{p}$.

A superalgebra $A$ is a vector superspace which is an unital associative $F$-algebra such that $A_iA_j \subseteq A_{i+j}$ for $i, j \in Z/2Z$. By an $A$-supermodule, we mean a vector
superspace $M$ which is a left $A$-module such that $A_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. In the rest of the paper, we only deal with finite-dimensional $A$-superalgebras. Given two $A$-superalgebras $V$ and $W$, an $A$-homomorphism $f : V \to W$ is an $F$-linear map such that

$$f(av) = (-1)^{\overline{a} \overline{f}}af(v)$$

for $a \in A$ and $v \in V$. We denote the set of $A$-homomorphisms from $V$ to $W$ by $\text{Hom}_A(V, W)$. By this, we can form a superadditive category $\text{A-smod}$ whose hom-set is a vector superspace in a way that is compatible with composition. However, we adapt a slightly different definition of isomorphisms from the categorical one\(^5\). Two $A$-superalgebras $V$ and $W$ are called evenly isomorphic (and denoted by $V \sim W$) if there exists an even $A$-homomorphism $f : V \to W$ which is an $F$-vector space isomorphism. They are called isomorphic (and denoted by $V \cong W$) if $V \sim W$ or $V \cong W$. Here for an $A$-superalgebra $M$, $\Pi M$ is an $A$-superalgebra defined by the same but the opposite grading underlying vector superspace $(\Pi M)_i = M_{i+1}$ for $i \in \mathbb{Z}/2\mathbb{Z}$ and a new action given as follows from the old one

$$a \cdot_{\text{new}} m = (-1)^{\overline{a} \overline{m}}a \cdot_{\text{old}} m.$$

We denote the isomorphism class of an $A$-superalgebra $M$ by $[M]$ and denote the set of isomorphism classes of irreducible $A$-superalgebras by $\text{Irr}(A\text{-smod})$.

Given two superalgebras $A$ and $B$, $A \otimes B$ with multiplication defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\overline{b_1} \overline{a_2}}(a_1a_2) \otimes (b_1b_2)$$

for $a_i \in A, b_j \in B$ is again a superalgebra. Let $V$ be an $A$-superalgebra and let $W$ be a $B$-superalgebra. Their tensor product $V \otimes W$ is an $A \otimes B$-superalgebra by the action given by

$$(a \otimes b)(v \otimes w) = (-1)^{\overline{a} \overline{v}}(av) \otimes (bw)$$

for $a \in A, b \in B, v \in V, w \in W$. Let us assume that $V$ and $W$ are both irreducible. We say that $V$ is type $Q$ if $V \cong \Pi V$ otherwise type $M$. If $V$ and $W$ are both of type $Q$, then there exists a unique (up to odd isomorphism) irreducible $A \otimes B$-superalgebra $X$ of type $M$ such that

$$V \otimes W \cong X \oplus \Pi X$$

as $A \otimes B$-superalgebras. We denote $X$ by $V \oplus W$. Otherwise $V \otimes W$ is irreducible but we also write it as $V \oplus W$. Note that $V \otimes W$ is defined only up to isomorphism in general and $V \otimes W$ is of type $M$ if and only if $V$ and $W$ are of the same type.

We extend the operation $\otimes$ as follows. Let $A$ and $B$ be superalgebras and let $V$ be an $A$-superalgebra and let $W$ be a $B$-superalgebra. Consider a pair $(V, \theta_V)$ where $\theta_V$ is either an odd involution of $V$ or $\theta_V = \text{id}_V$, and also consider a similar pair $(W, \theta_W)$. If $\theta_V = \text{id}_V$ or $\theta_W = \text{id}_W$, then we define $(V, \theta_V) \otimes (W, \theta_W) = V \otimes W$. If $\theta_V$ and $\theta_W$ are both odd involutions, then

$$\theta_V \otimes \theta_W : V \otimes W \to V \otimes W, \quad v \otimes w \mapsto (-1)^{\overline{\theta_V(v)} \overline{\theta_W(w)}} \theta_V(v) \otimes \theta_W(w)$$

\(^5\)Note that for irreducible $A$-superalgebras $V$ and $W$, the following statements are equivalent.

(i) there exist $f \in \text{Hom}_A(V, W)$ and $g \in \text{Hom}_A(W, V)$ such that $fg = \text{id}_W$ and $gf = \text{id}_V$.

(ii) there exist $f \in \text{Hom}_A(V, W)$ and $g \in \text{Hom}_A(W, V)$ which are both homogeneous and satisfy $f \circ g = \text{id}_W, g \circ f = \text{id}_V$. 
is an even $A \otimes B$-supermodule homomorphism such that $(\theta_V \otimes \theta_W)^2 = -\text{id}_{V \otimes W}$. Thus, $V \otimes W$ decomposes into $\pm \sqrt{-1}$-eigenspaces $X_{\pm \sqrt{-1}}$. Note that $X_{+ \sqrt{-1}}$ and $X_{- \sqrt{-1}}$ are oddly isomorphic since we have
\[
(\theta_V \otimes \text{id}_W)(X_{\pm \sqrt{-1}}) = (\text{id}_V \otimes \theta_W)(X_{\pm \sqrt{-1}}) = X_{\mp \sqrt{-1}}.
\]
Now we define $(V, \theta_V) \oplus (W, \theta_W) = X_{\pm \sqrt{-1}}$. Of course, we can pick the other summand, but such specification makes arguments simpler when we consider functoriality.

We also introduce Hom version of this operation. Assume further that $B$ is a subsuperalgebra of $A$. If $\theta_V = \text{id}_V$ or $\theta_W = \text{id}_W$, then we define $\text{Hom}_B((W, \theta_W), (V, \theta_V)) = \text{Hom}_B(W, V)$ which can be regarded as a supermodule over $C(A, B) \overset{\text{def}}{=} \{a \in A \mid ab = (-1)^{|a||b|}ba \text{ for all } b \in B\}$ by the action $(cf)(v) = c(f(v))$ for $c \in C(A, B)$ and $f \in \text{Hom}_B(W, V)$. If $\theta_V$ and $\theta_W$ are both odd involutions, then
\[
\Theta : \text{Hom}_B(W, V) \rightarrow \text{Hom}_B(W, V), \quad f \mapsto (\Theta(f))(v) = (-1)^{\overline{f} \overline{V}} \theta_V(f(\theta_W(v)))
\]
is an even $C(A, B)$-supermodule homomorphism such that $\Theta^2 = \text{id}_{\text{Hom}_B(W, V)}$. Thus, $\text{Hom}_B(W, V)$ decomposes into $\pm 1$-eigenspaces $X_{\pm 1}$. Similarly, we see that $X_{\pm 1} \cong \Pi X_{\mp 1}$, and we define $\text{Hom}_B((W, \theta_W), (V, \theta_V)) = X_{\mp 1}$.

For a superalgebra $A$, we define the Grothendieck group $K_0(A\text{-smod})$ to be the quotient of the $\mathbb{Z}$-module freely generated by all finite-dimensional $A$-supermodules by the $\mathbb{Z}$-submodule generated by
\begin{itemize}
  \item $V_1 - V_2 + V_3$ for every short exact sequence $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$ in $A\text{-smod}_\mathbb{Z}$
  \item $M - \Pi M$ for every $A$-supermodule $M$.
\end{itemize}
Here $A\text{-smod}_\mathbb{Z}$ is the abelian subcategory of $A\text{-smod}$ whose objects are the same but morphisms are consisting of even $A$-homomorphisms. Clearly, $K_0(A\text{-smod})$ is a free $\mathbb{Z}$-module with basis $\text{Irr}(A\text{-smod})$. The importance of the operation $\oplus$ lies in the fact that it gives an isomorphism
\[
(1) \quad K_0(A\text{-smod}) \otimes_\mathbb{Z} K_0(B\text{-smod}) \longrightarrow K_0(A \otimes B\text{-smod}), \quad [V] \otimes [W] \longmapsto [V \otimes W]
\]
for two superalgebras $A$ and $B$.

Finally, we make some remarks on projective supermodules. Let $A$ be a superalgebra. A projective $A$-supermodule is, by definition, a projective object in $A\text{-smod}$ and it is equivalent to saying that it is a projective object in $A\text{-smod}_\mathbb{Z}$ since there are canonical isomorphisms
\[
\text{Hom}_{A\text{-smod}}(V, W)_{\mathbb{Z}} \cong \text{Hom}_{A\text{-smod}_\mathbb{Z}}(V, W),
\]
\[
\text{Hom}_{A\text{-smod}}(V, W)_{\Pi} \cong \text{Hom}_{A\text{-smod}_\mathbb{Z}}(V, \Pi W)(\cong \text{Hom}_{A\text{-smod}_\mathbb{Z}}(\Pi V, W)).
\]
We denote by $\text{Proj} A$ the full subcategory of $A\text{-smod}$ consisting of all the projective $A$-supermodules.

Let us assume further that $A$ is finite-dimensional. Then, as in the usual finite-dimensional algebras, every $A$-supermodule $X$ has a (unique up to even isomorphism) projective cover $P_X$ in $A\text{-smod}_\mathbb{Z}$. If $X$ is irreducible, then it is (evenly) isomorphic to a principal indecomposable $A$-supermodule. From this, we easily see $M \cong N$ if and only if $P_M \cong P_N$ for $M, N \in \text{Irr}(A\text{-smod})$. Thus, $K_0(\text{Proj} A)$ is identified with $K_0(A\text{-smod})^* \overset{\text{def}}{=} \text{Hom}_Z(K_0(A\text{-smod}), Z)$ through the non-degenerate
canonical pairing

\[ \langle \cdot, \cdot \rangle_A : K_0(\text{Proj } A) \times K_0(A\text{-mod}) \to \mathbb{Z}, \]

\[ ([P_M], [N]) \mapsto \begin{cases} \dim \text{Hom}_A(P_M, N) & \text{if type } M = M, \\ \frac{1}{2} \dim \text{Hom}_A(P_M, N) & \text{if type } M = Q, \end{cases} \]

for all \( M \in \text{Irr}(A\text{-mod}) \) and \( N \in A\text{-mod} \). Note that the left hand side is nothing but the composition multiplicity \([N : M]\). We also reserve the symbol \( \omega_A : K_0(\text{Proj } A) \to K_0(A\text{-mod}) \) for the natural Cartan map.

2.2. Lie theory. We review necessary Lie theory for our purpose. Note that all the Lie-theoretic objects are considered over \( \mathbb{C} \) as usual although we are considering representations of “Hecke superalgebra” over \( F \).

Let \( A = (a_{ij})_{i,j \in I} \) be a symmetrizable generalized Cartan matrix and let \( \mathfrak{g} \) be the corresponding Kac-Moody Lie algebra. We denote the weight lattice by \( P \), the set of simple roots by \( \{ \alpha_i \mid i \in I \} \) and the set of simple coroots by \( \{ \alpha^*_i \mid i \in I \} \), etc. as usual. We denote by \( U_Q \) the \( \mathbb{Q} \)-subalgebra of the universal enveloping algebra of \( \mathfrak{g} \) generated by the Chevalley generators \( \{ e_i, f_i, h_i \mid i \in I \} \). In other words, \( U_Q \) is a \( \mathbb{Q} \)-subalgebra generated by \( \{ e_i, f_i, h_i \mid i \in I \} \) with the following relations

\[ [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \]

\[ [e_i, f_j] = \delta_{ij}h_i, \quad (\text{ad } e_i)^{1-a_{ik}}(e_k) = (\text{ad } f_i)^{1-a_{ik}}(f_k) = 0, \]

for all \( i, j, k \in I \) with \( i \neq k \). We also denote by \( U_Q^+ \) (resp. \( U_Q^- \)) the positive (resp. negative) part of the Kostant \( \mathbb{Z} \)-form of \( U_Q \), i.e., \( U_Q^\pm \) (resp. \( U_Q^- \)) is a subalgebra of \( U_Q \) generated by the divided powers \( \{ e_i^{(n)} \mid n \geq 1 \} \) (resp. \( \{ f_i^{(n)} \mid n \geq 1 \} \)).

Next, we recall the notion of Kashiwara’s crystal following [Kas].

**Definition 2.1.** A \( \mathfrak{g} \)-crystal is a 6-tuple \( (B, \text{wt}, \{ \varepsilon_i \}_{i \in I}, \{ \varphi_i \}_{i \in I}, \{ \bar{e}_i \}_{i \in I}, \{ \bar{f}_i \}_{i \in I}) \)

\[ \text{wt} : B \to P, \]

\[ \varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{ -\infty \}, \]

\[ \bar{e}_i, \bar{f}_i : B \cup \{ 0 \} \to B \cup \{ 0 \} \]

satisfies the following axioms.

(i) For all \( i \in I \), we have \( \bar{e}_i 0 = \bar{f}_i 0 = 0 \).

(ii) For all \( b \in B \) and \( i \in I \), we have \( \varphi_i(b) = \varepsilon_i(b) + \text{wt}(b)(h_i) \).

(iii) For all \( b \in B \) and \( i \in I \), \( \bar{e}_i b \neq 0 \) implies \( \varepsilon_i(\bar{e}_i b) = \varepsilon_i(b) - 1, \varphi_i(\bar{e}_i b) = \varphi_i(b) + 1 \) and \( \text{wt}(\bar{e}_i b) = \text{wt}(b) + \alpha_i \).

(iv) For all \( b \in B \) and \( i \in I \), \( \bar{f}_i b \neq 0 \) implies \( \varepsilon_i(\bar{f}_i b) = \varepsilon_i(b) + 1, \varphi_i(\bar{f}_i b) = \varphi_i(b) - 1 \) and \( \text{wt}(\bar{f}_i b) = \text{wt}(b) - \alpha_i \).

(v) For all \( b, b' \in B \) and \( i \in I \), \( \bar{e}_i b' \) is equivalent to \( b = \bar{e}_i b' \).

(vi) For all \( b \in B \) and \( i \in I \), \( \varphi_i(b) = -\infty \) implies \( \bar{e}_i b = \bar{f}_i b = 0 \).

**Definition 2.2.** Let \( B \) be a \( \mathfrak{g} \)-crystal. The crystal graph associated with \( B \) (and usually denoted by the same symbol \( B \)) is an \( I \)-colored directed graph whose vertices are the elements of \( B \) and there is an \( i \)-colored directed edge from \( b \) to \( b' \) if and only if \( b' = \bar{f}_i b \) for \( b, b' \in B \) and \( i \in I \).
Definition 2.3. Let $B$ and $B'$ be $\mathfrak{g}$-crystals. Their tensor product crystal $B \otimes B'$ is a $\mathfrak{g}$-crystal defined as follows.

$$B \otimes B' = B \times B',$$

$$\varepsilon_i(b \otimes b') = \max(\varepsilon_i(b), \varepsilon_i(b') - wt(b)(h_i)),$$

$$\varphi_i(b \otimes b') = \max(\varphi_i(b) + wt(b')(h_i), \varphi_i(b')),$$

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_ib \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b'), \\ b \otimes \tilde{e}_ib' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases}$$

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_ib \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_ib' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'), \end{cases}$$

$$wt(b \otimes b') = wt(b) + wt(b').$$

Here we regard $b \otimes 0$ and $0 \otimes b$ as 0.

Definition 2.4. Let $B$ and $B'$ be $\mathfrak{g}$-crystals. A crystal morphism $g : B \to B'$ is a map $g : B \sqcup \{0\} \to B' \sqcup \{0\}$ such that

(i) $g(0) = 0$.

(ii) If $b \in B$ and $g(b) \in B'$, then we have $wt(g(b)) = wt(b)$, $\varepsilon_i(g(b)) = \varepsilon_i(b)$ and $\varphi_i(g(b)) = \varphi_i(b)$ for all $i \in I$.

(iii) For $b \in B$ and $i \in I$, we have $g(\tilde{e}_ib) = \tilde{e}_i(g(b))$ if $g(b) \in B'$ and $g(\tilde{f}_ib) \in B'$.

(iv) For $b \in B$ and $i \in I$, we have $g(\tilde{f}_ib) = \tilde{f}_ig(b)$ if $g(b) \in B'$ and $g(\tilde{f}_ib) \in B'$.

If it commutes with all $\tilde{e}_i$ (resp. $\tilde{f}_i$), then we call it an $e$-strict (resp. $f$-strict) morphism. We call it a crystal embedding if it is injective, $e$-strict and $f$-strict.

Example 2.5. For each $\lambda \in P^+$, we denote by $T_\lambda = \{t_\lambda\}$ the $\mathfrak{g}$-crystal defined by

$$wt(t_\lambda) = \lambda, \quad \varphi_i(t_\lambda) = \varepsilon_i(t_\lambda) = -\infty, \quad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0.$$

Example 2.6. For each $i \in I$, we denote by $B_i = \{b_i(n) \mid n \in \mathbb{Z}\}$ the $\mathfrak{g}$-crystal defined by $wt(b_i(n)) = n\alpha_i$ and

$$\varepsilon_j(b_i(n)) = \begin{cases} -n & \text{if } j = i, \\ -\infty & \text{if } j \neq i, \end{cases} \quad \varphi_j(b_i(n)) = \begin{cases} n & \text{if } j = i, \\ -\infty & \text{if } j \neq i, \end{cases}$$

$$\tilde{e}_j(b_i(n)) = \begin{cases} b_i(n + 1) & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases} \quad \tilde{f}_j(b_i(n)) = \begin{cases} b_i(n - 1) & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

These pathological $\mathfrak{g}$-crystals are utilized in the following characterizations [KS, Proposition 3.2.3] [Sai, Proposition 2.3.1].

Proposition 2.7. We denote by $B(\infty)$ the associated $\mathfrak{g}$-crystal with the crystal base of $U^-_\nu(\mathfrak{g})$. Let $B$ be a $\mathfrak{g}$-crystal and $b_0$ an element of $B$ with $wt(b_0) = 0$. If the following conditions hold, then $B$ is isomorphic to $B(\infty)$.

(i) $wt(B) \subseteq \sum_{i \in I} \mathbb{Z}_{\leq 0}\alpha_i$.

(ii) $b_0$ is a unique element of $B$ such that $wt(b_0) = 0$.

(iii) $\varepsilon_i(b_0) = 0$ for every $i \in I$.

(iv) $\varphi_i(b) \in \mathbb{Z}$ for any $b \in B$ and $i \in I$.

(v) For every $i \in I$, there exists a crystal embedding $\Psi_i : B \to B \otimes B_i$ such that $\Psi_i(B) \subseteq B \times \{\tilde{f}_i^n b_i(0) \mid n \geq 0\}$.
Let us define our main ingredient $H$.

### Definition 3.1.

Let $H$ be the affine Hecke-Clifford superalgebra following [BK, 3]. Although Jones and Nazarov introduced it under the name of affine Sergeev algebra, a non-zero quantum parameter $q = 0$.

We denote by $U$ the associated $g$-crystal with the crystal base of the integrable highest $U_v(g)$-module of highest weight $\lambda \in P^+$. Let $B$ be a $g$-crystal and $b_\lambda$ an element of $B$ with $\text{wt}(b_\lambda) = \lambda$. If the following conditions hold, then $B$ is isomorphic to $\mathbb{B}(\lambda)$.

(i) $b_\lambda$ is a unique element of $B$ such that $\text{wt}(b_\lambda) = \lambda$.
(ii) There is an $f$-strict crystal morphism $\Phi : B(\infty) \otimes T_\lambda \rightarrow B$ such that $\Phi(b_0 \otimes t_\lambda) = b_\lambda$ and $\text{im} \Phi = B \sqcup \{0\}$. Here $b_0$ is the unique element of $B(\infty)$ with $\text{wt}(b_0) = 0$.
(iii) Consider the set $\{b \in B(\infty) \otimes T_\lambda \mid \Phi(b) \neq 0\}$. Then it is isomorphic to $B$ through $\Phi$ as a set.
(iv) For any $b \in B$ and $i \in I$, $\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k(b) \neq 0\}$ and $\varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k(b) \neq 0\}$.

### 3. Affine Hecke-Clifford superalgebras of Jones and Nazarov

#### 3.1. Definition and vector superspace structure.

From now on, we reserve a non-zero quantum parameter $q \in F^\times$ and set $\xi = q - q^{-1}$ for convenience. Let us define our main ingredient $H_n$, affine Hecke-Clifford superalgebra [JN, §3]. Although Jones and Nazarov introduced it under the name of affine Sergeev algebra, we call it affine Hecke-Clifford superalgebra following [BK, §2-d].

**Definition 3.1.** Let $n \geq 0$ be an integer. The affine Hecke-Clifford superalgebra $H_n$ is defined by even generators $X_1^{\pm 1}, \cdots, X_n^{\pm 1}, T_1, \cdots, T_{n-1}$ and odd generators $C_1, \cdots, C_n$ with the following relations.

(i) $X_i X_i^{-1} = X_i^{-1} X_i = 1, X_i X_j = X_j X_i$ for all $1 \leq i, j \leq n$.
(ii) $C_i^2 = 1, C_i C_j + C_j C_i = 0$ for all $1 \leq i \neq j \leq n$.
(iii) $T_i^2 = \xi T_i + 1, T_i T_j = T_j T_i, T_i T_{k+1} T_k = T_{k+1} T_k T_i$ for all $1 \leq k \leq n - 2$ and $1 \leq i, j \leq n - 1$ with $|i - j| \geq 2$.
(iv) $C_i X_i^{\pm 1} = X_i^{\mp 1} C_i, C_i X_j^{\pm 1} = X_j^{\mp 1} C_i$ for all $1 \leq i \neq j \leq n$.
(v) $T_i C_i = C_i T_i, (T_i + \xi C_i C_i) X_i T_i = X_{i+1}$ for all $1 \leq i \leq n - 1$.
(vi) $T_i C_i = C_i T_i, T_i X_i^{\pm 1} = X_i^{\mp 1} T_i$ for all $1 \leq i \leq n - 1$ and $1 \leq j \leq n$ with $j \neq i, i + 1$.

Note that the relations in Definition 3.1 implies the followings for $1 \leq i \leq n - 1$.

\begin{align*}
(3) & \quad T_i C_{i+1} = C_i T_i - \xi(C_i - C_{i+1}), \\
(4) & \quad T_i X_i = X_{i+1} T_i - \xi(X_{i+1} + C_i C_{i+1} X_i), \\
(5) & \quad T_i X_i^{-1} = X_{i+1}^{-1} T_i + \xi(X_i^{-1} + X_{i+1}^{-1} C_i C_{i+1}).
\end{align*}

We define the Clifford superalgebra $C_n$ by odd generators $C_1, \cdots, C_n$ with relation (ii) and also define the Iwahori-Hecke (super)algebra $H_n^{\text{IW}}$ of type $A$ by (even) generators $T_1, \cdots, T_{n-1}$ with relations (iii). By [BK, Theorem 2.2], natural superalgebra homomorphisms

\[ \alpha_A : F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \rightarrow H_n, \quad \alpha_B : C_n \rightarrow H_n, \quad \alpha_C : H_n^{\text{IW}} \rightarrow H_n \]
are all injective and we have the following isomorphism as vector superspaces.

\[ F[X^\pm_1, \cdots, X^\pm_n] \otimes C_n \otimes \mathcal{H}^N_n \cong \mathcal{H}_n, \quad x \otimes c \otimes t \mapsto \alpha_A(x)\alpha_B(c)\alpha_C(t). \]

In the sequel, we identify \( H \) this way.

Then we have the followings.

By (6), we easily see that the sequence of natural superalgebra homomorphisms

\[ \mathcal{H}_0 \twoheadrightarrow \mathcal{H}_1 \twoheadrightarrow \mathcal{H}_2 \twoheadrightarrow \cdots \]

are all injective and it forms a tower of superalgebras. We also see that for each \( \sigma \) of \( H \), there exists an automorphism \( \sigma \) of \( H \) defined by

\[ \begin{align*}
\sigma : T_i & \mapsto -T_{n-i} + \xi, \quad C_j \mapsto C_{j+1}, \quad X_j \mapsto X_{j+1}, \\
\tau : T_i & \mapsto T_i + \xi C_i C_{i+1}, \quad C_j \mapsto C_{j-1}, \quad X_j \mapsto X_j
\end{align*} \]

for \( 1 \leq i \leq n-1 \) and \( 1 \leq j \leq n \) [BK, §2-i].

Let \( M \) be an \( \mathcal{H}_n \)-supermodule. The dual space \( M^* \) has again an \( \mathcal{H}_n \)-supermodule structure by \( (hf)(m) = f(\tau(h)m) \) for \( f \in M^*, m \in M \) and \( h \in \mathcal{H}_n \). We denote this \( \mathcal{H}_n \)-supermodule by \( M^\sigma \). We also denote by \( M^\tau \) the \( \mathcal{H}_n \)-supermodule obtained by twisting the action of \( \mathcal{H}_n \) through \( \sigma \). Then we have the following [BK, Lemma 2.9, Theorem 2.14].

**Lemma 3.2.** Let \( M \) be an \( \mathcal{H}_m \)-supermodule and let \( N \) be an \( \mathcal{H}_n \)-supermodule. Then we have the followings.

1. \( (\text{Ind}_{\mathcal{H}_m}^{\mathcal{H}_n} M \otimes N)^\sigma \cong \text{Ind}_{\mathcal{H}_m}^{\mathcal{H}_n} N^\sigma \otimes M^\sigma. \)
2. \( (\text{Ind}_{\mathcal{H}_m}^{\mathcal{H}_n} M \otimes N)^\tau \cong \text{Ind}_{\mathcal{H}_m}^{\mathcal{H}_n} N^\tau \otimes M^\tau. \)

Moreover, if \( M \) and \( N \) are both irreducible, the same holds for \( \otimes \) in place of \( \otimes \).

3.3. **Cartan subsuperalgebra** \( A_n \). The subsuperalgebra

\[ A_n \overset{\text{def}}{=} (X^\pm_i, C_i)_{1 \leq i \leq n} (\subseteq \mathcal{H}_n) \]

plays a role of “Cartan subalgebra” in the rest of the paper.

**Definition 3.3.** For each integer \( i \in \mathbb{Z} \), we define

\[ q(i) = 2 \cdot \frac{q^{2i+1} + q^{-(2i+1)}}{q + q^{-1}}, \quad b_{\pm}(i) = \frac{q(i)}{2} \pm \sqrt{\frac{q(i)^2}{4} - 1} \]

and choose a subset \( I_q \subseteq \mathbb{Z} \) such that the map \( I_q \to \{ q(i) \mid i \in \mathbb{Z} \}, i \mapsto q(i) \) gives a bijection. An \( A_n \)-supermodule \( M \) is called integral if the set of eigenvalues of \( X_j + X_j^{-1} \) is a subset of \( \{ q(i) \mid i \in I_q \} \) for all \( 1 \leq j \leq n \). Let \( \mu \) be a composition of \( n \). An \( \mathcal{H}_\mu \)-supermodule \( M \) is called integral if \( \text{Res}^{\mathcal{H}_\mu}_{A_n} M \) is integral.

We denote the full subcategory of \( A_n \)-smod (resp. \( \mathcal{H}_\mu \)-smod) consisting of integral representations by \( \text{Rep} A_n \) (resp. \( \text{Rep} \mathcal{H}_\mu \)). We also denote by \( \text{ch}_\mu \) the induced \( \mathbb{Z} \)-linear homomorphism by the restriction functor \( \text{Res}^{\mathcal{H}_\mu}_{A_n} \)

\[ \text{ch}_\mu : K_0(\text{Rep} \mathcal{H}_\mu) \twoheadrightarrow K_0(\text{Rep} A_n) \]
between the Grothendieck groups. We always write \( ch \) instead of \( ch_n \) and call \( chM \) the formal character of \( \mathcal{H}_n \)-supermodule \( M \).

We recall a special case of covering modules [BK, §4-h].

**Definition 3.4.** Let \( m \geq 1 \) and let \( i \in I_q \). We define a \( 2m \)-dimensional \( \mathcal{H}_1 \)-supermodule \( L^\pm_m(i) \) with an even basis \( \{ w_1, \cdots, w_m \} \) and an odd basis \( \{ w'_1, \cdots, w'_m \} \) and the following matrix representations of actions of generators with respect to this basis.

\[
X_1 : \begin{pmatrix} J(b_+ (i); m) & O \\ O & J(b_- (i); m)^{-1} \end{pmatrix}, \quad C_1 : \begin{pmatrix} O & E_m \\ E_m & O \end{pmatrix}.
\]

Here \( J(\alpha; m) \) \( \overset{\text{def}}{=} (\delta_{i,j}\alpha + \delta_{i,j+1})_{1 \leq i,j \leq m} \) stands for the Jordan matrix of size \( m \).

We also define for \( m \geq 1 \) an \( \mathcal{H}_1 \)-homomorphisms \( g_m^\pm : L^\pm_{m+1}(i) \rightarrow L^\pm_m(i) \) by

\[
w_k \mapsto \begin{cases} w_k & \text{if } 1 \leq k \leq m, \\ 0 & \text{if } k = m + 1, \end{cases} \quad w'_k \mapsto \begin{cases} w'_k & \text{if } 1 \leq k \leq m, \\ 0 & \text{if } k = m + 1. \end{cases}
\]

Here \( w_k \) and \( w'_k \) in the left hand side are those of \( L^\pm_{m+1}(i) \) whereas \( w_k \) and \( w'_k \) in the right hand side are those of \( L^\pm_m(i) \). Note that there is an odd isomorphism \( g_m^\circ : L^+_m(i) \overset{\sim}{\rightarrow} L^-_m(i) \) since \( J(b_+(i); m) \) and \( J(b_-(i); m)^{-1} \) are similar. For convenience, we abbreviate \( L^+_m(i) \) (resp. \( L^-_m(i) \)) to \( L_m(i) \) (resp. \( L(i) \)) and \( g_m^\circ \) to \( g_m \).

**Definition 3.5.** For \( i \in I_q \) we define an \( \mathcal{H}_1 \)-supermodule \( R_m(i) = \mathcal{H}_1 / N(i) \) where \( N(i) \) is a two-sided ideal generated by

\[
f(i) = \begin{cases} (X_1 + X_1^{-1} - q(i))^m & \text{if } q(i) \neq \pm 2, \\ (X_1 - b_+(i))^m (= (X_1 - b_-(i))^m) & \text{if } q(i) = \pm 2. \end{cases}
\]

As in [BK, §4-h] (or by elementary linear algebra), we have the following.

**Lemma 3.6.** Let \( i \in I_q \).

(i) If \( q(i) \neq \pm 2 \), then there exists an even isomorphism \( R_m(i) \simeq L^+_m(i) \oplus L^-_m(i) \) for \( m \geq 1 \) which commutes with the obvious surjection \( R_m(i) \rightarrow R_{m+1}(i) \).

\[
(7) \quad \begin{array}{cccc}
R_1(i) & R_2(i) & R_3(i) & \cdots \\
\downarrow & \downarrow & \downarrow & \\
L_1(i) \oplus \Pi L_1(i) & L_2(i) \oplus \Pi L_2(i) & L_3(i) \oplus \Pi L_3(i) & \cdots
\end{array}
\]

(ii) If \( q(i) = \pm 2 \), then we have \( R_m(i) \simeq L^+_m(i) = L^-_m(i) \) and there exist odd involutions \( g_k^\circ \) for \( k \geq 1 \) make the following diagram commutes.

\[
(8) \quad \begin{array}{cccc}
R_1(i) & R_2(i) & R_3(i) & \cdots \\
\downarrow & \downarrow & \downarrow & \\
L_1(i) & L_2(i) & L_3(i) & \cdots
\end{array}
\]

In virtue of \( A_n \cong A_1^{\otimes n} \) and (1), we have the following (see [BK, Lemma 4.8]).
Lemma 3.7. We have $\text{Irr}(\text{Rep}\ A_n) = \{L(i_1) \oplus \cdots \oplus L(i_n) \mid (i_1, \cdots, i_n) \in I_q^n\}$. Note that for $(i_1, \cdots, i_n) \in I_q^n$, $L(i_1) \oplus \cdots \oplus L(i_n)$ is of type $Q$ if and only if $\# \{1 \leq k \leq n \mid q(i_k) = \pm 2\}$ is odd.

3.4. Block decomposition. The (super)center $Z(H_n)$ of $H_n$ is naturally identified with the algebra of symmetric polynomials of $X_1 + X_1^{-1}, \cdots, X_n + X_n^{-1}$ [JN, Proposition 3.2(b)], [BK, Theorem 2.3] via

$$F[X_1 + X_1^{-1}, \cdots, X_n + X_n^{-1}] \cong Z(H_n), \quad f \mapsto f.$$ 

Thus, for any $M \in \text{Rep} H_n$, we have a decomposition $M = \bigoplus_{\gamma \in I_q^n / S_n} M[\gamma]$ with

$$M[\gamma] = \{m \in M \mid \forall f \in Z(H_n), \exists N \in \mathbb{Z}_{>0}, (f - \gamma(f))^N m = 0\}$$

in $\text{Rep} H_n$. Here $\chi_\gamma$ is a central character attached for $\gamma = [(\gamma_1, \cdots, \gamma_n)]$ by

$$\chi_\gamma : Z(H_n) \rightarrow F, \quad f(X_1 + X_1^{-1}, \cdots, X_n + X_n^{-1}) \mapsto f(q(\gamma_1), \cdots, q(\gamma_n)).$$

Note that if $\gamma_1 \neq \gamma_2$ in $I_q^n / S_n$, then $\chi_{\gamma_1} \neq \chi_{\gamma_2}$.

Definition 3.8. Let $M \in \text{Irr}(\text{Rep} H_n)$. Then there exists a unique $\gamma \in I_q^n / S_n$ such that $M = M[\gamma]$. In this case, we say that $M$ belongs to the block $\gamma$.

We remark that this terminology coincides with the usual notion of block since the set $\{\chi_\gamma \mid \gamma \in I_q^n / S_n\}$ exhausts the possible central characters arising from $\text{Rep} H_n$. In fact, for any $\gamma = [(\gamma_1, \cdots, \gamma_n)] \in I_q^n / S_n$, all the composition factors of $L(\gamma_1) \oplus \cdots \oplus L(\gamma_n)$ belongs to $\gamma$ since we have

$$\text{ch} L(\gamma_1) \oplus \cdots \oplus L(\gamma_n) = \sum_{w \in S_n} [L(i_w(1)) \oplus \cdots \oplus L(i_w(n))].$$

This identity [BK, Lemma 4.10] follows from the Mackey theorem [BK, Theorem 2.8].

3.5. Kashiwara operators. Recall the Kato supermodule $L(i^n) \overset{\text{def}}{=} \text{Ind}^{H_n}_{A_n} L(i)^{\otimes n}$ [BK, §4-g]. Using them, we can introduce Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ that send an irreducible supermodule to another one (if defined). We first recall a fundamental property of Kato’s modules [BK, Theorem 4.16.(i)].

Theorem 3.9. For $i \in I_q$ and $n \geq 1$, $L(i^n)$ is irreducible of the same type as $L(i)^{\otimes n}$ and it is the only irreducible supermodule in its block of $\text{Rep} H_n$.

Definition 3.10. For $i \in I_q$, $0 \leq m \leq n$ and $M \in \text{Rep} H_n$, we denote by $\Delta_{i,m} M$ the simultaneous generalized $q(i)$-eigenspace of the commuting operators $X_k + X_k^{-1}$ for all $n - m < k \leq n$. Note that $\Delta_{i,m} M$ is an $H_{n-m,m}$-supermodule. We also define $\varepsilon_i(M) = \max \{m \geq 0 \mid \Delta_{i,m} M \neq 0\}$.

By [BK, §5-a], we have the followings [BK, Lemma 5.5, Theorem 5.6, Corollary 5.8].

Theorem 3.11. Let $i \in I_q$, $m \geq 0$ and $M \in \text{Irr}(\text{Rep} H_n)$.

(i) $N \overset{\text{def}}{=} \text{Cosoc} \text{Ind}^{H_{n+m}}_{H_{n,m}} M \oplus L(i^n)$ is irreducible with $\varepsilon_i(N) = \varepsilon_i(M) + m$ and any other irreducible composition factor $L$ of $\text{Ind}^{H_{n+m}}_{H_{n,m}} M \oplus L(i^n)$ satisfies $\varepsilon_i(L) < \varepsilon_i(M) + n$. 

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(ii) Assume that $0 \leq m \leq \varepsilon_i(M)$. There exists (up to isomorphism) an irreducible $\mathcal{H}_{n-m}$-supermodule $L$ such that $\text{type } L = \text{type } M, \varepsilon_i(L) = \varepsilon_i(M) - m$ and $\text{Soc } \Delta_{m}^{\mathcal{H}_{n-1}} M \cong L \oplus L(i^m)$.

(iii) Assume that $\varepsilon_i(M) > 0$. Then we have

$$\text{Soc } \text{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n-1}} \Delta_i(M) \cong \begin{cases} L \oplus \Pi L & \text{if type } M = Q \text{ or } q(i) \neq \pm 2, \\ L & \text{if type } M = M \text{ and } q(i) = \pm 2, \end{cases}$$

for some irreducible $\mathcal{H}_{n-1}$-module $L$ of the same type as $M$ if $q(i) \neq \pm 2$ and of the opposite type to $M$ if $q(i) = \pm 2$.

**Definition 3.12.** Let us write $B(\infty) \overset{def}{=} \bigsqcup_{n \geq 0} \text{Irr}(\text{Rep } \mathcal{H}_n)$. For $i \in I_q$, we define maps $\overline{e}_i, \overline{f}_i : B(\infty) \sqcup \{0\} \rightarrow B(\infty) \sqcup \{0\}$ as follows.

- $\overline{e}_0 = 0 = \overline{f}_0$.
- For $M \in \text{Irr}(\text{Rep } \mathcal{H}_n)$, we set $\overline{f}_i M = \text{Cosoc Ind}_{\mathcal{H}_{n+1}}^{\mathcal{H}_{n}} M \oplus L(i)$.
- For $M \in \text{Irr}(\text{Rep } \mathcal{H}_n)$, we set $\overline{e}_i M = 0$ if $\varepsilon_i(M) = 0$ otherwise $\overline{e}_i M = L$ for a unique $L \in \text{Irr}(\text{Rep } \mathcal{H}_{n-1})$ with $\text{Soc } \Delta_{i} M \cong L \oplus L(i)$.

Note that we have $\varepsilon_i(M) = \max\{m \geq 0 \mid (\overline{e}_i)^m M \neq 0\}$ by Theorem 3.11 (ii).

By [BK, Lemma 5.10], $\overline{e}_i$ and $\overline{f}_i$ satisfy one of the axioms of Kashiwara’s crystal (see Definition 2.1 (v)), i.e.,

**Lemma 3.13.** For $M, N \in B(\infty)$ and $i \in I_q$, $\overline{f}_i M = N$ is equivalent to $\overline{e}_i N = M$.

**Definition 3.14.** For $i = (i_1, \ldots, i_n) \in I_q^n$, we define $L(i) = \overline{f}_{i_n} \overline{f}_{i_{n-1}} \cdots \overline{f}_{i_2} \overline{f}_{i_1} 1$. Here $1$ is the trivial representation of $\mathcal{H}_0 = F$.

Note that $L(i)$ applied for $i = (i, \ldots, i)$ coincides with the Kato supermodule $L(i^n)$ by Theorem 3.9. By an inductive use of Lemma 3.13, we have the following [BK, §5-d, Lemma 5.15].

**Corollary 3.15.** For any $L \in \text{Irr}(\text{Rep } \mathcal{H}_n)$ there exists $i \in I_q^n$ such that $L \cong L(i)$. $\text{Res}_{\mathcal{H}_1}^{\mathcal{H}_n} L(i)$ has a submodule isomorphic to $L(i_1) \oplus \cdots \oplus L(i_n)$.

Also a repeated use of Theorem 3.11 (ii) implies the following [BK, Lemma 5.14].

**Corollary 3.16.** Let $M \in \text{Irr}(\text{Rep } \mathcal{H}_n)$ and let $\mu$ be a composition of $n$. For any irreducible composition factor $N$ of $\text{Res}_{\mathcal{H}_1}^{\mathcal{H}_n} M$, we have $\text{type } M = \text{type } N$.

### 3.6. Root operators

We shall define root operators $e_i$ as a direct summand of $\text{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n-1}} \Delta_i$. Note that for any $M \in \text{Rep } \mathcal{H}_n$ and $i \in I_q$, we have a natural identification

$$\text{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n-1}} \Delta_i M \cong \lim_{\rightarrow m} \text{Hom}_{\mathcal{H}_1}(R_m(i), M).$$

Here $\mathcal{H}_1$ stands for a subsuperalgebra in $\mathcal{H}_n$ generated by $\{X^\pm_1, C_n\}$ isomorphic to $\mathcal{H}_1$. Considering (7) or (8), we can chose a summand of $\text{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n-1}} \Delta_i M$ appropriately as follows.

**Definition 3.17.** For $M \in \text{Irr}(\text{Rep } \mathcal{H}_n)$ and $i \in I_q$, we define

$$e_i M = \lim_{\rightarrow m} \text{Hom}_{\mathcal{H}_1}(L_m(i), \theta_m^n), (M, \theta_M))(\in \text{Rep } \mathcal{H}_{n-1}).$$

Here the $\theta$’s are defined as follows.
Theorem 3.19. Let $\theta_m^o = \text{id}_{M_m}$ if $q(i) \neq \pm 2$, and $\theta_m^o = g_m^o$ otherwise. 

$\theta_M = \text{id}_M$ if type $M = M$, and $\theta_M$ is an odd involution of $M$ otherwise.

Thus, by Theorem 3.11 (iii), we have

$$\text{Res}_{\mathcal{H}_{n-1}}\Delta_i(M) \simeq \begin{cases} e_iM & \text{if type } M = M \text{ and } q(i) \pm 2, \\ e_iM \oplus \Pi e_iM & \text{if type } M = Q \text{ or } q(i) \neq \pm 2. \end{cases}$$

By the commutativity of $\text{Res}_{\mathcal{H}_{n-1}}$ and $\tau$-duality, we see the following [BK, Lemma 6.6.(i)].

Corollary 3.18. Let $M \in \text{Irr}(\text{Rep} \mathcal{H}_n)$ and $i \in I_q$. Then $e_iM$ is non-zero if and only if $\tilde{c}_iM$ is non-zero, in which case $e_iM$ is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $\tilde{c}_iM$.

Also, as seen in [BK, §6-d], we have the following [BK, Theorem 6.11].

Theorem 3.19. Let $M \in \text{Irr}(\text{Rep} \mathcal{H}_n)$ and $i \in I_q$.

(i) In $K_0(\text{Rep} \mathcal{H}_n)$, we have $[e_iM] = \varepsilon_i(M)[\tilde{c}_iM] + \sum c_a[N_a]$ where $N_a$ are irreducibles with $\varepsilon_i(N_a) < \varepsilon_i(M) - 1$.

(ii) If $q(i) \neq \pm 2$, then $\varepsilon_i(M)$ is the maximal size of a Jordan block of $X_n + X_n^{-1}$ on $M$ with eigenvalue $q(i)$.

(iii) If $q(i) = \pm 2$, then $\varepsilon_i(M)$ is the maximal size of a Jordan block of $X_n$ on $M$ with eigenvalue $b_+(i) = b_-(i)$.

(iv) $\text{End}_{\mathcal{H}_{n-1}}(e_iM) \simeq \text{End}_{\mathcal{H}_{n-1}}(\tilde{c}_iM)^{\oplus \varepsilon_i(M)}$ as vector superspaces.

3.7. Kashiwara’s crystal structure. In this subsection, let $A = (a_{ij})_{i,j \in I_q}$ be an arbitrary symmetrizable generalized Cartan matrix indexed by $I_q$. We identify $I^n_q/\mathcal{G}$ and $\Gamma_n \overset{\text{def}}{=} \{\sum_{i \in I_q} k_i \alpha_i \in \sum_{i \in I_q} \mathbb{Z}_{\geq 0} \alpha_i \mid \sum_{i \in I_q} k_i = n\}$ by

$$b_A : I^n_q/\mathcal{G} \overset{\sim}{\rightarrow} \Gamma_n, \quad [(\gamma_1, \cdots, \gamma_n)] \mapsto \sum_{k=1}^n \alpha_{\gamma_k}.$$ 

For $M \in \text{Irr}(\text{Rep} \mathcal{H}_n)$ belonging to a block $\gamma \in I^n_q/\mathcal{G}_n$ and $i \in I_q$, we define $\text{wt}(M) = -b_A(\gamma), \quad \varphi_i(M) = \varepsilon_i(M) + \langle \nu_i, \text{wt}(M) \rangle$.

By Theorem 3.11 and Lemma 3.13, we can check the following [BK, Lemma 8.5].

Lemma 3.20. The 6-tuple $(B(\infty), \text{wt}, \{\varepsilon_i\}_{i \in I_q}, \{\varphi_i\}_{i \in I_q}, \{\tilde{c}_i\}_{i \in I_q}, \{\tilde{f}_i\}_{i \in I_q})$ is a $g(A)$-crystal.

Finally, we introduce $\sigma$-version of the above operations for $M \in B(\infty) \text{ and } i \in I_q$.

$$\tilde{c}_i^\sigma M = (\tilde{c}_i(M^\sigma))^\sigma, \quad \tilde{f}_i^\sigma M = (\tilde{f}_i(M^\sigma))^\sigma, \quad \varepsilon_i^\sigma(M) = \varepsilon_i(M^\sigma).$$

Of course, we have $\varepsilon_i^\sigma(M) = \max\{k \geq 0 \mid (\tilde{c}_i)^k M \neq 0\}$. However $\varepsilon_i^\sigma(M)$ has another description as follows by Theorem 3.19 (ii) and Theorem 3.19 (iii).

Lemma 3.21. Let $i \in I_q$ and $M \in \text{Irr}(\text{Rep} \mathcal{H}_n)$.

- If $q(i) \neq \pm 2$, then $\varepsilon_i^\sigma(M)$ is the maximal size of a Jordan block of $X_1 + X_1^{-1}$ on $M$ with eigenvalue $q(i)$.
- If $q(i) = \pm 2$, then $\varepsilon_i^\sigma(M)$ is the maximal size of a Jordan block of $X_1$ on $M$ with eigenvalue $b_+(i) = b_-(i)$.
We also quote results concerning the commutativity of $\bar{e}_i$ and $\bar{f}_j$ [BK, Lemma 8.1, Lemma 8.2, Lemma 8.4].

**Lemma 3.22.** Let $M \in \text{Irr}(\text{Rep} \mathcal{H}_n)$ and $i, j \in I_q$.

(i) $\varepsilon_i(\bar{f}_j^* M) = \varepsilon_i(M)$ or $\varepsilon_i(\bar{f}_j^* M) = \varepsilon_i(M) + 1$.

(ii) If $i \neq j$, then $\varepsilon_i(\bar{f}_j^* M) = \varepsilon_i(M)$.

(iii) If $\varepsilon_i(\bar{f}_j^* M) = \varepsilon_i(M)$ (denoted by $\varepsilon$), then $\bar{e}_i^* \bar{f}_j^* M \cong \bar{f}_j^* \bar{e}_i^* M$.

(iv) If $\varepsilon_i(\bar{f}_j^* M) = \varepsilon_i(M) + 1$, then $\bar{e}_i^* \bar{f}_j^* M \cong M$.

3.8. **Hopf algebra structure.** Consider the graded $\mathbb{Z}$-free module

$$K(\infty) = \bigoplus_{n \geq 0} K_0(\text{Rep} \mathcal{H}_n)$$

with natural basis $B(\infty)$ and define $\mathbb{Z}$-linear maps

$$\phi_{m,n} : K_0(\text{Rep} \mathcal{H}_m) \otimes K_0(\text{Rep} \mathcal{H}_n) \xrightarrow{\sim} K_0(\text{Rep} \mathcal{H}_{m+n}) \xrightarrow{\text{Ind}_{m+n}^{m,n}} K_0(\text{Rep} \mathcal{H}_m),$$

$$\Delta_{m,n} : K_0(\text{Rep} \mathcal{H}_{m+n}) \xrightarrow{\text{Res}_{m+n}^{m,n}} K_0(\text{Rep} \mathcal{H}_{m,n}) \xrightarrow{\sim} K_0(\text{Rep} \mathcal{H}_m) \otimes K_0(\text{Rep} \mathcal{H}_n),$$

$$\phi = \sum_{m,n \geq 0} \phi_{m,n} : K(\infty) \otimes K(\infty) \xrightarrow{\sim} K(\infty), \quad \iota : \mathbb{Z} \xrightarrow{\sim} K_0(\text{Rep} \mathcal{H}_0) \xrightarrow{\text{proj}} K_0(\infty)$$

$$\Delta = \sum_{m,n \geq 0} \Delta_{m,n} : K(\infty) \xrightarrow{\sim} K(\infty) \otimes K(\infty), \quad \varepsilon : K_0(\infty) \xrightarrow{\text{proj}} K_0(\text{Rep} \mathcal{H}_0) \xrightarrow{\sim} \mathbb{Z}.$$ 

Note that $\phi_{m,n}$ is well-defined since for any $M \in \text{Rep} \mathcal{H}_{m,n}$ we have $\text{Ind}_{m+n}^{m,n} M \in \text{Rep} \mathcal{H}_{m+n}$ by [BK, Lemma 4.6].

Transivity of induction and restriction makes $(K_0(\infty), \phi, \iota)$ a graded $\mathbb{Z}$-coalgebra. Injectivity of the formal character map $\text{ch} : K_0(\text{Rep} \mathcal{H}_{\infty}) \hookrightarrow K_0(\text{Rep} \mathcal{A}_{\infty})$ [BK, Theorem 5.12] implies $L \cong L^\tau$ for all $L \in B(\infty)$ [BK, Corollary 5.13]. Combine it with Lemma 3.2 (ii), we see that the multiplication of $(K_0(\infty), \phi, \iota)$ is commutative. By Mackey theorem [BK, Theorem 2.8], we see that $(K(\infty), \phi, \Delta, \iota, \varepsilon)$ is a graded $\mathbb{Z}$-bialgebra$^6$. Since a connected (non-negatively) graded bialgebra is a Hopf algebra [Swe, pp.238], we get the following [BK, Theorem 7.1].

**Theorem 3.23.** $(K(\infty), \phi, \Delta, \iota, \varepsilon)$ is a commutative graded Hopf algebra over $\mathbb{Z}$. Thus, $K(\infty)^*$ is a cocommutative graded Hopf algebra over $\mathbb{Z}$. 

Here $K(\infty)^*$ is a graded dual of $K(\infty)$, i.e., $K(\infty)^* = \bigoplus_{n \geq 0} \text{Hom}_\mathbb{Z}(K_0(\text{Rep} \mathcal{H}_n), \mathbb{Z})$. $K(\infty)^*$ has a natural $\mathbb{Z}$-free basis $\{ \delta_M \mid M \in B(\infty) \}$ defined by $\delta_M([M]) = 1$ and $\delta_M([N]) = 0$ for all $[N] \in B(\infty)$ with $N \not\cong M$.

6In checking the details, we need the commutativity of the following diagrams for $m \geq k$ and $n \geq l$ and it follows from Corollary 3.16.

\[
\begin{array}{cccc}
K_0(\text{Rep} \mathcal{H}_{m,n}) & \xrightarrow{\sim} & K_0(\text{Rep} \mathcal{H}_{m}) \otimes K_0(\text{Rep} \mathcal{H}_n) & \xrightarrow{\sim} K_0(\text{Rep} \mathcal{H}_{m}) \otimes K_0(\text{Rep} \mathcal{H}_n) \\
\text{Res}_{m,n}^{m,m} & \text{Res}_{m,n}^{m,n} \otimes \text{Res}_{m,n}^{n,n} & \text{Ind}_{m,n}^{m,m} & \text{Ind}_{m,n}^{m,n} \otimes \text{Ind}_{m,n}^{n,n} \\
K_0(\text{Rep} \mathcal{H}_{k,l}) & \xrightarrow{\sim} & K_0(\text{Rep} \mathcal{H}_{k}) \otimes K_0(\text{Rep} \mathcal{H}_l) & \xrightarrow{\sim} K_0(\text{Rep} \mathcal{H}_{k}) \otimes K_0(\text{Rep} \mathcal{H}_l). \end{array}
\]
3.9. **Left $K(\infty)^*$-module structure on $K(\infty)$.** By [Swe, Proposition 2.1.1], for a coalgebra $C$ and a right $C$-comodule $\omega : M \to M \otimes C$, $M$ is turned into a left $C^*$-module by

$$C^* \otimes M \xrightarrow{\text{id}_{C^*} \otimes \omega} C^* \otimes M \otimes C \xrightarrow{\text{swap} \otimes \text{id}_C} M \otimes C^* \otimes C \xrightarrow{\text{id}_M \otimes (\cdot)} M \otimes Z \xrightarrow{\sim} M.$$ 

It implies that each coalgebra $C$ is naturally regarded as a left $C^*$-module. It is easily seen that if $C$ is connected (non-negatively) graded coalgebra then the left action of $C^*$ is faithful. Thus, $K(\infty)$ has a natural faithful left $K(\infty)^*$-module structure and it coincides with the root operators $e_i$ in the following sense [BK, Lemma 7.2, Lemma 7.4].

**Lemma 3.24.** For $i \in I_q, r \geq 1$ and $M \in K(\infty)$, we have $\delta_{L^{(r)}} \cdot M = e_i^{(r)} M$.

Note that $e_i^{(r)}$ is a priori an operator on $K(\infty)_0 \overset{\text{def}}{=} \mathbb{Q} \otimes K(\infty)$, however as seen in Lemma 3.24 it is a well-defined operator on $K(\infty)$. We can prove it directly by defining a divided power root operators $e_i^{(r)}$ in a module-theoretic way [BK, §6-c].

4. **Cyclotomic Hecke-Clifford superalgebra**

4.1. **Definition and vector superspace structure.**

**Definition 4.1.** Let $n \geq 1$ and assume that $R = a_d X_1^d + \cdots + a_0 \in F[X_1](\subseteq \mathcal{H}_n)$ satisfies $C_1 R = a_0 X_1^{-d} RC_1$ (equivalently saying, the coefficients $\{a_i\}_{i=0}^d$ of $R$ satisfies $a_d = 1$ and $a_i = a_0 a_d^{-1}$ for all $0 \leq i \leq d$). We define the cyclotomic Hecke-Clifford superalgebra $\mathcal{H}^R_n = \mathcal{H}_n/(R)$ for $n \geq 1$ and define $\mathcal{H}^R_0 = F$.

Note that the antiautomorphism $\tau$ of $\mathcal{H}_n$ induces an anti-automorphism of $\mathcal{H}^R_n$ also written by $\tau$. As in the affine case, for an $\mathcal{H}^R_n$-supermodule $M$ we write $M^\tau$ the dual space $M^*$ with $\mathcal{H}^R_n$-supersupersmodule structure obtained by $\tau$.

By [BK, Theorem 3.6], $\mathcal{H}^R_n$ is a finite-dimensional superalgebra whose basis is a canonical images of the elements

$$\{X_1^{\alpha_1} \cdots X_n^{\alpha_n} C_{\beta_1} \cdots C_{\beta_n} T_w | 0 \leq \alpha_k < d, \beta_k \in \mathbb{Z}/2\mathbb{Z}, w \in S_n\}.$$ 

Thus, we have the following commutativity between towers of superalgebras.

$$\mathcal{H}_0 \xrightarrow{\tau} \mathcal{H}_1 \xrightarrow{\tau} \mathcal{H}_2 \xrightarrow{\tau} \cdots$$

$$\mathcal{H}^R_0 \xrightarrow{\tau} \mathcal{H}^R_1 \xrightarrow{\tau} \mathcal{H}^R_2 \xrightarrow{\tau} \cdots$$

It makes us possible to define inductions and restrictions for $\{\mathcal{H}^R_n\}_{n \geq 0}$ as well as $M^\tau$ and we have the following [BK, Theorem 3.9, Corollary 3.15].

**Theorem 4.2.** Let $M$ be an $\mathcal{H}^R_n$-supermodule.

(i) There is a natural isomorphism of $\mathcal{H}^R_n$-modules.

$$\text{Res}^{\mathcal{H}^R_{n+1}}_{\mathcal{H}^R_n} \text{Ind}^{\mathcal{H}^R_{n+1}}_{\mathcal{H}^R_n} M \simeq (M \oplus \Pi M)^d \oplus \text{Ind}^{\mathcal{H}^R_{n-1}}_{\mathcal{H}^R_{n-1}} \text{Res}^{\mathcal{H}^R_{n-1}}_{\mathcal{H}^R_n} M.$$ 

(ii) The functors $\text{Res}^{\mathcal{H}^R_{n+1}}_{\mathcal{H}^R_n}$ and $\text{Ind}^{\mathcal{H}^R_{n+1}}_{\mathcal{H}^R_n}$ are left and right adjoint to each other.

(iii) There is a natural isomorphism as $\mathcal{H}^R_{n+1}$-modules $\text{Ind}^{\mathcal{H}^R_{n+1}}_{\mathcal{H}^R_n} (M^\tau) \simeq (\text{Ind}^{\mathcal{H}^R_{n+1}}_{\mathcal{H}^R_n} M)^\tau$. 

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We also define two natural functors. Note that $pr^R$ is a left adjoint to $infl^R$.

$$pr^R : H_n\text{-smod} \rightarrow H_n^R\text{-smod}, \quad M \mapsto M/\langle R \rangle M,$$

$$infl^R : H_n^R\text{-smod} \rightarrow H_n\text{-smod}, \quad M \mapsto Res_{H_n}^R M.$$ 

4.2. Kashiwara operators. Kashiwara operators for cyclotomic superalgebras are defined using those defined for affine superalgebras as follows. By Lemma 3.13, $\tilde{e}_i^R$ and $\tilde{f}_i^R$ clearly satisfy Definition 2.1 (v).

**Definition 4.3.** Let us write $B(R) \triangleq \bigsqcup_{n \geq 0} \text{Irr}(H_n^{R}\text{-smod})$. For $i \in I_q$, we define maps $\tilde{e}_i^R, \tilde{f}_i^R : B(R) \sqcup \{0\} \rightarrow B(R) \sqcup \{0\}$ as follows.

- For $M \in \text{Irr}(H_n^{R}\text{-smod})$, we set $\tilde{e}_i^R M = (pr^R \circ \tilde{e}_i \circ \text{infl}^R) M$ and $\tilde{f}_i^R M = (pr^R \circ \tilde{f}_i \circ \text{infl}^R) M$.

We also define for $M \in B(R)$ and $i \in I_q$,

$$\varepsilon_i^R(M) = \max \{k \geq 0 \mid (\tilde{e}_i^R)^k(M) \neq 0\} = \varepsilon_i(\text{infl}^R M),$$

$$\varphi_i^R(M) = \max \{k \geq 0 \mid (\tilde{f}_i^R)^k(M) \neq 0\} = \varphi_i(\text{infl}^R M).$$

Note that although $\varphi_i^R(M)$ may take the value $+\infty$, it always takes a finite value as seen in Lemma 4.9 (ii) below.

4.3. Root operators.

**Definition 4.4.** For $M \in H_n^{R}\text{-smod}$ such that $\text{infl}^R M$ belongs to a block $\gamma \in I_q^n/\mathfrak{S}_n$ with $-b_A(\gamma) = \sum_{\alpha_i} k_i \alpha_i$, we define

$$\text{Res}_i^R M = \begin{cases} 
pr^R((\text{infl}^R \text{Res}_{H_n}^R M)[b_A^{-1}(-(\gamma - \alpha_i))]) & \text{if } k_i > 0, \\
0 & \text{if } k_i = 0,
\end{cases}$$

$$\text{Ind}_i^R M = pr^R((\text{infl}^R \text{Ind}_{H_n}^R M)[b_A^{-1}(-(\gamma + \alpha_i))]).$$

In general, for $M \in H_n^{R}\text{-smod}$ we define $\text{Res}_i^R M$ (resp. $\text{Ind}_i^R M$) by applying $\text{Res}_i^R$ (resp. $\text{Ind}_i^R$) for each summand of $M = \bigoplus_{\gamma \in I_q^n/\mathfrak{S}_n} pr^R((\text{infl}^R M)[\gamma]).$

By Theorem 4.2 and central character consideration, we get the following [BK, Lemma 6.1].

**Corollary 4.5.** Let $i \in I_q$.

(i) $\text{Res}_i^R$ and $\text{Ind}_i^R$ are left and right adjoint to each other.

(ii) For each $M \in H_n^{R}\text{-smod}$ there are natural isomorphisms

$$\text{Ind}_i^R (M^\gamma) \simeq (\text{Ind}_i^R M)^\gamma, \quad \text{Res}_i^R (M^\gamma) \simeq (\text{Res}_i^R M)^\gamma.$$ 

Note that $\text{Res}_i^R$ is nothing but $pr^R \circ \text{Res}_{H_n}^{R_{n-1}} \circ \Delta_i \circ \text{infl}^R$ and it can be described as follows (see also (9)). Replacing each operator with its left adjoint and checking the well-definedness, we have the following [BK, Lemma 6.2].
Lemma 4.6. Let $M \in \mathcal{H}_n^R$-smod and $i \in I_q$. There are natural isomorphisms
\[
\text{Res}_i^R M \simeq \lim_{\longrightarrow} \text{pr}^R \text{Hom}_{\mathcal{H}_n^R}(R_m(i), \text{infl}^R M),
\]
\[
\text{Ind}_i^R M \simeq \lim_{\longleftarrow} \text{pr}^R \text{Ind}_{\mathcal{H}_n^R \otimes \mathcal{H}_1^R}^R((\text{infl}^R M) \otimes R_m(i)).
\]
Here both limits are stabilized after finitely many terms.

As in the affine case, we can choose a suitable summand of $\text{Res}_i^R M$ and $\text{Ind}_i^R M$ using (7) or (8).

Definition 4.7. Let $M \in \text{Irr}(\mathcal{H}_n^R$-smod). We define
\[
e_i^R X = \lim_{\longrightarrow} \text{pr}^R \text{Hom}_{\mathcal{H}_n^R}( (L_m(i), \theta_m^e), (\text{infl}^R X, \text{infl}^R \theta_X)),
\]
\[
f_i^R X = \lim_{\longleftarrow} \text{pr}^R \text{Ind}_{\mathcal{H}_n^R \otimes \mathcal{H}_1^R}^R((\text{infl}^R X, \text{infl}^R \theta_X) \otimes (L_m(i), \theta_m^o))
\]
for each $X = M$ or $X = P \overset{def}{=} P_M$ and $i \in I_q$. Here $\theta$’s are defined as follows.

- $\theta_m^e = \text{id}_{L_m(i)}$ if $q(i) \neq \pm 2$, and $\theta_m^o = g_m^e$ otherwise.
- $\theta_M = \text{id}_M$ if type $M = M$, and $\theta_M$ is an odd involution of $M$ otherwise.
- $\theta_P = \text{id}_P$ if type $M = M$, and $\theta_P$ is an odd involution of $P$ whose existence is guaranteed by [Kl2, Lemma 12.2.16] elsewise.

Note that for a principal indecomposable $P$ and $i \in I_q$, $e_i^R P$ and $f_i^R P$ are again projectives since they are summands of $\text{Res}_i^R P$ and $\text{Ind}_i^R P$ respectively (see also Corollary 4.5). Thus, we define operators $e_i^R$ and $f_i^R$ on $K(R) \overset{def}{=} \bigoplus_{n \geq 0} K_0(\mathcal{H}_n^R$-smod) and $K(R)^* \overset{def}{=} \bigoplus_{n \geq 0} K_0(\text{Proj} \mathcal{H}_n^R)$.

Lemma 4.8. For any principal indecomposable $\mathcal{H}_n^R$-supermodule $P$ and $i \in I_q$, we have in $K_0(\mathcal{H}_n^R$-smod) and $K_0(\mathcal{H}_{n+1}^R$-smod) respectively
\[
e_i^R(\omega_{\mathcal{H}_n^R}(P)) = \omega_{\mathcal{H}_{n-1}^R}([e_i^R P]), \quad f_i^R(\omega_{\mathcal{H}_n^R}(P)) = \omega_{\mathcal{H}_{n+1}^R}([f_i^R P]).
\]

Proof. Let $A$ and $B$ be superalgebras and consider an (even) exact functor $X : A$-smod $\to B$-smod which sends every projective to a projective. Then for any principal indecomposable projective $A$-supermodule $P$, we easily see $X(\omega_A(P)) = \omega_B([XP])$ in $K_0(\text{B-smod})$. By Corollary 4.5 (i), it implies that
\[
\text{Res}_i^R(\omega_{\mathcal{H}_n^R}(P)) = \omega_{\mathcal{H}_{n-1}^R}([\text{Res}_i^R P]), \quad \text{Ind}_i^R(\omega_{\mathcal{H}_n^R}(P)) = \omega_{\mathcal{H}_{n+1}^R}([\text{Ind}_i^R P]).
\]
We shall only show $e_i^R(\omega_{\mathcal{H}_n^R}(P)) = \omega_{\mathcal{H}_{n-1}^R}([e_i^R P])$ in $K_0(\mathcal{H}_{n+1}^R$-smod) because the other is similar. By (7), (8), Lemma 4.6 and Definition 4.7, we have
\[
[e_i^R P] = \begin{cases} [\text{Res}_i^R P] & \text{if } q(i) = \pm 2 \text{ and type Cosoc } P = M, \\ \frac{1}{2} [\text{Res}_i^R P] & \text{if otherwise}. \end{cases}
\]
in $\mathbb{K}_0(\text{Proj } \mathcal{H}_{n-1}^R)$. Similarly, for $M \in \text{Irr}(\mathcal{H}_{n-1}^R)$-mod we have
\[
[e_i^R M] = \begin{cases} 
\text{Res}_i^R M & \text{if } q(i) = \pm 2 \text{ and type } M = M, \\
\frac{1}{2}\text{Res}_i^R M & \text{if otherwise}
\end{cases}
\]
in $\mathbb{K}_0(\mathcal{H}_{n-1}^R)$-mod. Thus, it is enough to show that for each irreducible factor $N$ of $P$ we have type $N = \text{type Cosoc } P$. Take a unique $\gamma \in I_q^R / S_n$ such that $P = P[\gamma]$. It is clear that $N$ also belongs to the block $\gamma$. By Corollary 3.16, type $N$ is determined by its central character.

Since $e_i^R = pr^R \circ e_i \circ \text{infl}^R$ and $\varepsilon_i^R = pr^R \circ e_i \circ \text{infl}^R$, Corollary 3.18 and Theorem 3.19 hold for $M \in \text{Rep } \mathcal{H}_n^R$ and $i \in I_q$ by replacing $e_i$, $\varepsilon_i$ and $e_i$ appearing there with $e_i^R$, $\varepsilon_i^R$ and $\varepsilon_i^R$ respectively. We quote the corresponding properties of $f_i^R$, $\widetilde{f}_i^R$ and $\varphi_i^R$ [BK, Lemma 6.6(ii), Lemma 6.18, Corollary 6.24].

**Lemma 4.9.** Let $M \in \text{Irr}(\mathcal{H}_{n}^R \text{-mod})$ and $i \in I_q$.

(i) $f_i^R M$ is non-zero if and only if $\widetilde{f}_i^R M$ is non-zero, in which case it is a selfdual indecomposable module with irreducible socle and cosocle isomorphic to $f_i^R M$.

(ii) $\varphi_i^R(M)$ is the smallest $m \geq 1$ (thus, takes a finite value by Lemma 4.6) such that $f_i^R = pr^R \text{Ind}_{H_{n+1}^R}^H_i (\text{infl}^R M, \text{infl}^R \theta_M) \oplus (L_m(i), \theta_{m}^\iota)$ if $f_i^R M \neq 0$.

If $f_i^R M = 0$ then $\varphi_i^R(M) = 0$.

(iii) In $\mathbb{K}_0(\text{Rep } \mathcal{H}_n)$, we have $[f_i^R M] = \varphi_i^R(M)[\tilde{f}_i^R M] + \sum c_n[N_n]$ where $N_n$ are irreducibles with $\varepsilon_i^R(N_n) < \varepsilon_i^R(M) + 1$.

(iv) $\text{End}_{\mathcal{H}_{n-1}^R}(f_i^R M) \simeq \text{End}_{\mathcal{H}_{n-1}^R}(\tilde{f}_i^R M) \oplus \varphi_i^R(M)$ as vector superspaces.

**Corollary 4.10.** For any $M \in \text{Irr}(\mathcal{H}_{n}^R \text{-mod})$ and $i \in I_q$, we have $(e_i^R + \varepsilon_i^R)(M) + 1 [M] = (f_i^R + \varphi_i^R(M) + 1)[M] = 0$ in $K(R)$.

**Proof.** $(e_i^R + \varepsilon_i^R)(M) + 1 [M] = 0$ follows from Theorem 3.19(i). To prove $(f_i^R + \varphi_i^R(M) + 1)[M] = 0$, it is enough to show that $(f_i^R)^m[M] \neq 0$ implies $(\tilde{f}_i^R)^m[M] \neq 0$ for any $m \geq 0$.

By the definition, $(f_i^R)^m[M] \neq 0$ is equivalent to $([\text{Ind}^R_i]^m M) \neq 0$. By Corollary 4.5 (i), we have
\[
\text{Hom}_{\mathcal{H}_{n+m}^R}([\text{Ind}^R_i]^m M, N) \cong \text{Hom}_{\mathcal{H}_{n}^R}(M, (\text{Res}^R_{n+m})^\iota N)
\]
\[
= \text{Hom}_{\mathcal{H}_{n}^R}(\text{infl}^R M, (\text{Res}^R_{n+m})^\iota \Delta^{n+m} \text{infl}^R N)
\]
for any $N \in \mathcal{H}_{n+m}^R$-mod. Since $(\text{Ind}^R_i)^m M \neq 0$, there exists an $N \in \text{Irr}(\mathcal{H}_{n+m}^R)$ such that (10) is non-zero. Take any irreducible sub $\mathcal{H}_n$-supermodule $X \cong \text{infl}^R M$ of $\text{Res}^R_{n+m} \Delta^{n+m} \text{infl}^R N$ and consider $\mathcal{H}_n$-supermodule $X' \overset{\text{def}}{=} \mathcal{H}_n X$ where $\mathcal{H}_n$ stands for a subsuperalgebra in $\mathcal{H}_{n+m}$ generated by $\{X_{\pm l}, C_k, T_l \mid n < k \leq n+m, n < l < n+m\}$ isomorphic to $\mathcal{H}_n$. Then $\text{ch}_{(n,m)} X' = c[X \oplus L(i^m)]$ for some $c \in \mathbb{Z}_{\geq 1}$ by Theorem 3.9. Comparing with $\text{Soc} \Delta^{n+m} \text{infl}^R N \cong (\varepsilon_i^R \text{infl}^R N) \oplus L(i^m)$ by Theorem 3.11(ii) (see also [BK, Lemma 5.9.(i)]), we see $\text{infl}^R M \cong X \cong \varepsilon_i^R \text{infl}^R N$ which implies $(\tilde{f}_i^R)^m M \cong N \neq 0$.

As proved in [BK, Lemma 7.14], $[\text{Res}^R_i \text{Ind}^R_i M] - [\text{Ind}^R_i \text{Res}^R_i M]$ is a multiple of $[M]$ for any $M \in \text{Irr}(\mathcal{H}_{n}^R \text{-mod})$. By Theorem 3.19(i) and Lemma 4.9(iii), it implies the following.
Corollary 4.11. For any $M \in \text{Irr}(\mathcal{H}_{n}^{R} \text{-smod})$ and $i, j \in I_{q}$, we have $\varepsilon_{i}^{R}(f_{j}^{R}[M]) - f_{j}^{R}(\varepsilon_{i}^{R}[M]) = \delta_{i,j}(\varepsilon_{i}^{R}(M) - \varepsilon_{i}^{R}(M)) \cdot [M]$ in $K(R)$.

By Schur’s lemma, Theorem 4.2 (i), Theorem 3.19 (iv), Lemma 4.9 (ii) and Lemma 4.9 (iv), we have the following. See also [BK, Lemma 6.20].

Corollary 4.12. For any $M \in \text{Irr}(\mathcal{H}_{n}^{R} \text{-smod})$, we have

$$
\sum_{i \in I_{q}} \langle 2 - \delta_{b_{+}(i), b_{-}(i)}(\varepsilon_{i}^{R}(M) - \varepsilon_{i}^{R}(M)) = d.
$$

4.4. Left $K(\infty)^{\ast}$-module structure on $K(R)$. Clearly, $\text{inf}^{R}$ induces an injection $K(R) \hookrightarrow K(\infty)$ and a map $\Delta^{R} : K(R) \rightarrow K(R) \otimes K(\infty)$ with the following commutative diagram

$$
\begin{array}{ccc}
K(\infty) & \xrightarrow{\Delta^{R}} & K(\infty) \otimes K(\infty) \\
\text{inf}^{R} \downarrow & & \downarrow \text{inf}^{R} \otimes \text{id}_{K(\infty)} \\
K(R) & \xrightarrow{\Delta^{R}} & K(R) \otimes K(\infty).
\end{array}
$$

Thus, $K(R)$ is a subcomodule of the right regular $K(\infty)$-comodule. It implies that $K(R)$ is a $K(\infty)^{\ast}$-submodule of a left $K(\infty)^{\ast}$-module $K(\infty)$ in §3.9 where an operator $(e_{i}^{R})^{(r)}$ acts as $\delta_{L,(r)}^{i}$ by Lemma 3.24 for $i \in I_{q}$ and $r \geq 1$.

4.5. Injectivity of the Cartan map. The purpose of this subsection is to show the injectivity of the Cartan map $\omega_{T}^{R}$ of $\mathcal{H}_{n}^{R}$ [BK, Theorem 7.10]. It is essentially the same as [BK, §7-c] but arguments are slightly different because we don’t define divided power operators $e_{i}^{R}, (e_{i}^{R})^{(r)}$ and $(f_{i}^{R})^{(r)}$ in a module-theoretic way as [BK, §6-c].

We first recall the following formula [BK, Lemma 7.6] which follows from the definitions that $e_{i}^{R}$ and $f_{i}^{R}$ are suitable summands of $\text{Res}^{R}_{i}$ and $\text{Ind}^{R}_{i}$ respectively.

Lemma 4.13. For any $x \in K_{0}(\text{Proj} \mathcal{H}_{n}^{R})$ and $y_{\pm} \in K_{0}(\mathcal{H}_{n \pm 1}^{R} \text{-smod})$, we have

$$
\langle e_{i}^{R}x, y_{-} \rangle_{\mathcal{H}_{n-1}^{R}} = \langle x, f_{i}^{R}y_{-} \rangle_{\mathcal{H}_{n}^{R}}, \quad \langle f_{i}^{R}x, y_{+} \rangle_{\mathcal{H}_{n+1}^{R}} = \langle x, e_{i}^{R}y_{+} \rangle_{\mathcal{H}_{n}^{R}}.
$$

Since $(e_{i}^{R})^{(r)}$ is a well-defined operator on $K(R)$, we have the following. See also [BK, Corollary 7.7].

Corollary 4.14. $(f_{i}^{R})^{(r)}$ is a well-defined operator on $K(R)^{\ast}$ for any $i \in I_{q}$ and $r \geq 1$. More precisely, if

$$
(e_{i}^{R})^{(r)}[M] = \sum_{N \in \text{Irr}(\mathcal{H}_{n-1}^{R} \text{-smod})} a_{M,N}[N], \quad (f_{i}^{R})^{(r)}[M] = \sum_{N \in \text{Irr}(\mathcal{H}_{n+1}^{R} \text{-smod})} b_{M,N}[N]
$$

in $K_{0}(\mathcal{H}_{n-1}^{R} \text{-smod})$ and $\mathbb{Q} \otimes K_{0}(\mathcal{H}_{n+1}^{R} \text{-smod})$ respectively, then we have

$$
(f_{i}^{R})^{(r)}[P_{N}] = \sum_{M \in \text{Irr}(\mathcal{H}_{n-1}^{R} \text{-smod})} a_{M,N}[P_{M}], \quad (e_{i}^{R})^{(r)}[P_{N}] = \sum_{M \in \text{Irr}(\mathcal{H}_{n+1}^{R} \text{-smod})} b_{M,N}[P_{M}]
$$

in $K_{0}(\text{Proj} \mathcal{H}_{n-1}^{R})$ and $\mathbb{Q} \otimes K_{0}(\text{Proj} \mathcal{H}_{n+1}^{R})$ respectively.
Lemma 4.15. Let $M \in \text{Irr}(\mathcal{H}_n^R - \text{smod})$ and $i \in I_q$. For $m \leq \varepsilon \overset{\text{def}}{=} \varepsilon^R_i(M)$, we have

\[(\varepsilon^R_i)^m[P_M] = \sum_{L \in \text{Irr}(\mathcal{H}_{n-m}^R - \text{smod})} b_L[P_L] \]

in $K_0(\text{Proj} \mathcal{H}_n^R)$. Moreover, in case $m = \varepsilon$, we have

\[(\varepsilon^R_i)^\varepsilon[P_M] = \varepsilon!(\varepsilon + \varphi^R_i(M)) [P_{(\varepsilon^R_i)^\varepsilon M}] + \sum_{L \in \text{Irr}(\mathcal{H}_{n-\varepsilon}^R - \text{smod}) \atop \varepsilon^R_i(L) > 0} b_L[P_L].\]

Proof. By Corollary 4.14, $b_L$ is the coefficient of $[M]$ in $(f^R_i)^m[L]$ in $K_0(\mathcal{H}_n^R - \text{smod})$. Note by Lemma 4.9 (iii), we have

$(f^R_i)^m[L] \in \sum_{N \in \text{Irr}(\mathcal{H}_{n-m}^R - \text{smod})} \varepsilon^R_i(N) \leq m + \varepsilon^R_i(L)$

This implies $\varepsilon \leq m + \varepsilon^R_i(L)$ if $b_L \neq 0$ and completes the proof of (11).

Suppose $b_L \neq 0$ and $\varepsilon^R_i(L) = 0$. Again, by Lemma 4.9 (iii), we have $(f^R_i)^\varepsilon L \cong M$ and $b_L = \varepsilon!(\varepsilon^R_i(L))$. Thus, we have $L \cong (f^R_i)^\varepsilon M$ and $b_L = \varepsilon!(\varepsilon^R_i(M))$. \qed

Theorem 4.16. $\omega_{\mathcal{H}_n^R} : K_0(\text{Proj} \mathcal{H}_n^R) \to K_0(\mathcal{H}_n^R - \text{smod})$ is injective for all $n \geq 0$.

Proof. We prove by induction on $n$. The case $n = 0$ is clear.

Suppose $n > 0$ and $\omega_{\mathcal{H}_n^R}$ is injective for all smaller $n' < n$. We show that if

\[(\varepsilon^R_i)^\varepsilon\bigg(\sum_{M \in \text{Irr}(\mathcal{H}_n^R - \text{smod})} a_M[P_M]\bigg) = 0\]

for $a_M \in \mathbb{Z}$, then we have $a_M = 0$ for all $M \in \text{Irr}(\mathcal{H}_n^R - \text{smod})$. To prove it, it is enough to show that for each $i \in I_q$, we have $a_M = 0$ for all $M \in \text{Irr}(\mathcal{H}_n^R - \text{smod})$ with $\varepsilon^R_i(M) > 0$. This is because there exists some $i \in I_q$ such that $\varepsilon^R_i(M) > 0$ for any $M \in \text{Irr}(\mathcal{H}_n^R - \text{smod})$ if $n > 0$.

For each $i \in I_q$, we prove it by induction on $\varepsilon^R_i(M) > 0$. Suppose that for a given $M$ with $\varepsilon \overset{\text{def}}{=} \varepsilon^R_i(M) > 0$ we have $a_N = 0$ for all $N$ with $0 < \varepsilon^R_i(N) < \varepsilon$. Apply $(\varepsilon^R_i)^\varepsilon$ to (12), we have

\[0 = \sum_{L \in \text{Irr}(\mathcal{H}_{n-\varepsilon}^R - \text{smod}) \atop \varepsilon^R_i(L) = \varepsilon} \varepsilon!(\varepsilon + \varphi^R_i(L)) a_L \omega_{\mathcal{H}_{n-\varepsilon}^R}([P_{(\varepsilon^R_i)^\varepsilon L}]) + \omega_{\mathcal{H}_{n-\varepsilon}^R}(X)\]

where $X \in \sum_{L' \in \text{Irr}(\mathcal{H}_{n-\varepsilon}^R - \text{smod})} \varepsilon^R_{i(L')} > 0 \mathbb{Z}[P_{L'}]$ by Lemma 4.8 and Lemma 4.15. By induction hypothesis, we have $a_M = 0$. \qed

4.6. Symmetric non-degenerate bilinear form on $K(R)_Q$. By Theorem 4.16, $\bigoplus_{n \geq 0} K_0(\text{Proj} \mathcal{H}_n^R) \cong K(R)^* \subseteq K(R)$ are two integral lattices of $K(R)_Q$ $\overset{\text{def}}{=} \mathbb{Q} \otimes K(R)$. Thus, by tensoring $\mathbb{Q}$, $\bigoplus_{n \geq 0} \overline{\langle \cdot, \cdot \rangle}_n : K(R)^* \times K(R) \to \mathbb{Z}$ induces a non-degenerate bilinear form on $K(R)_Q$ which we denote by $\langle \cdot, \cdot \rangle_R$. 

\[\mathbb{Q} \otimes K(R)\]
Lemma 4.17. Let $M \in \text{Irr}(\mathcal{H}^R_n)$ and $i \in I_q$. We have

$$[P_M] = (f^R_i) \cdot [P_{\epsilon^R_i} \cdot M] - \sum_{L \in \text{Irr}(\mathcal{H}^R_n \text{-mod})} a_L[P_L]$$

for $\varepsilon = \epsilon^R_i(M)$ in $K_0(\text{Proj} \mathcal{H}^R_n)$.

Proof. Write $f^R_i \cdot [P_{\epsilon^R_i} \cdot M] = \sum_{L \in \text{Irr}(\mathcal{H}^R_n \text{-mod})} b_L[P_L]$ in $K_0(\text{Proj} \mathcal{H}^R_n)$. By Corollary 4.14, $b_L$ is the coefficient of $[(\epsilon^R_i)^t M]$ of $(\epsilon^R_i)^t[L]$ in $K_0(\mathcal{H}^R_{n-\varepsilon} \text{-mod})$. Thus, $b_L \neq 0$ implies $\epsilon^R_i(L) \geq \varepsilon$. Finally, suppose $b_L \neq 0$ and $\epsilon^R_i(L) = \varepsilon$. By Theorem 3.19 (i), we have $b_L = 1$ and $(\epsilon^R_i)^t[L] \cong (\epsilon^R_i)^t M$, i.e., $L \cong M$. \hfill \Box

A repeated use of Lemma 4.17 implies the following [BK, Theorem 7.9].

Theorem 4.18. We have $\bigoplus_{n \geq 0} K_0(\text{Proj} \mathcal{H}^R_n) = U^-_Z[1_R]$ where $1_R$ is the trivial supermodule of $\mathcal{H}^R_0$.

Proof. We prove $[P_M] \in U^-_Z[1_R]$ for all $M \in B(R)$. Suppose for a contradiction an existence of $M \in \text{Irr}(\mathcal{H}^R_n \text{-mod})$ such that $[P_M] \notin U^-_Z[1_R]$. We take such an $M$ with minimum $n$. Since $n > 0$, there exists an $i \in I_q$ with $\varepsilon^R_i(M) > 0$. We take $N$ with maximum $\varepsilon^R_i(N) \geq \varepsilon^R_i(M) > 0$ in $\{N \in \text{Irr}(\mathcal{H}^R_n \text{-mod}) \mid [P_N] \notin U^-_Z[1_R] \} \neq \emptyset$. However, $[P_N] \in U^-_Z[1_R]$ by a choice of $N$ and Lemma 4.17, a contradiction. \hfill \Box

Using Lemma 4.13 inductively along with $K_0(\mathcal{H}^R_{n+1} \text{-mod})_Q = \sum_{i \in I_q} f^R_i K_0(\mathcal{H}^R_n \text{-mod})_Q$ by Theorem 4.18, we get the following result [BK, Theorem 7.11].

Corollary 4.19. The non-degenerate bilinear form $(,)_R$ on $K(R)_Q$ is symmetric.

5. Character calculations

The purpose of this section is to give preparatory character calculations concerning behavior of representations of low rank affine Hecke-Clifford superalgebras $\mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ for §6.2. Since they are responsible for the appearance of Lie theory of type $D_1(2)$ and omitted in [BK], we give detailed and self-contained calculations.

5.1. Preparations. We note that if a given $M \in \text{Irr}(\text{Rep} \mathcal{H}_n)$ has a formal character of the form $\text{ch} M = c \cdot [L(i_1) \otimes \cdots \otimes L(i_m)]$ for some $c \in \mathbb{Z}_{\geq 1}$ then $M \cong L(i_1, \ldots, i_n)$ by Corollary 3.15. We also recall the Shuffle lemma [BK, Lemma 4.11] to compute the formal characters.

Lemma 5.1. For $M \in \text{Irr}(\text{Rep} \mathcal{H}_m)$ and $N \in \text{Irr}(\text{Rep} \mathcal{H}_n)$ with $\text{ch} M = \sum_{i \in I_q} a_i [L(i_1) \otimes \cdots \otimes L(i_m)]$ and $\text{ch} N = \sum_{j \in I_q} b_j [L(j_1) \otimes \cdots \otimes L(j_n)]$, we have

$$\text{ch} \text{Ind}^R_{\mathcal{H}_{m+n}} M \otimes N = \sum_{i \in I_q^m} a_i b_j \left( \sum_{k \in I_q^{m+n}} [L(k_1) \otimes \cdots \otimes L(k_{m+n})]. \right)$$

Here $k \in I_q^{m+n}$ runs satisfying the following condition: there exist $1 \leq v_1 < \cdots < v_n \leq m + n$ such that $(k_{u_1}, \ldots, k_{u_m}) = (i_1, \ldots, i_m), (k_{v_1}, \ldots, k_{v_n}) = (j_1, \ldots, j_n)$ and $\{u_1, \ldots, u_m\} \cup \{v_1, \ldots, v_n\} = \{1, \ldots, m+n\}$.

We also need the following [BK, Lemma 4.3] which follows by direct calculation.

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Lemma 5.2. Suppose we are given \( a, b \in F^\times \) with \( a + a^{-1} = q(i) \) and \( b + b^{-1} = q(j) \) for some \( i, j \in I_q \). If \(|i - j| \leq 1\), then the following vanishes.

\[
a^{-2}(ab - 1)^2(ab - 1)^2(a^{-2}(ab - 1)^2(ab - 1)^2 - \xi^2 a^{-1}b^{-1}(ab - 1)^2 - \xi^2 a^{-1}b(ab - 1)^2).
\]

Corollary 5.3. For any \( i, j \in \mathbb{Z} \) with \(|i - j| = 1\) and \( q(j) \neq q(i) \), we have

\[
\frac{\xi^2}{(q(j) - q(i))^2}(q(i)q(j) - 4) = 1.
\]

Proof. We take \( a \) and \( b \) to satisfy \( a + a^{-1} = q(i) \) and \( b + b^{-1} = q(j) \). We have

\[
a^{-2}(ab - 1)^2(ab - 1)^2 - \xi^2 a^{-1}b^{-1}(ab - 1)^2 - \xi^2 a^{-1}b(ab - 1)^2 = 0.
\]

by Lemma 5.2 and \( q(i) \neq q(j) \). Direct calculation shows that the left hand side is equal to \( (q(i) - q(j))^2 - \xi^2(q(i)q(j) - 4) \).

In the rest of this section, for each \( i \in I_q \) we write the basis elements \( w_1 \) and \( w_1' \) of \( L(i) = L_1(i) \) in Definition 3.4 as \( v_1^+ \) and \( v_1^- \) respectively. Recall that the irreducible \( H_2 \)-supermodule \( L(i) = Fv_1^+ \sqcup Fv_1^- \) is given by the grading \( L(i)_j = Fv_j \) for \( j \in \mathbb{Z}/2\mathbb{Z} \) and the following action.

\[
X_1^+ v_1^+ = b_+(i)v_0^+, \quad X_1^- v_1^- = b_-(i)v_0^-, \quad C_1 v_0^+ = v_0^+, \quad C_1 v_0^- = v_0^-.
\]

5.2. On the block \([i, j]\) with \(|i - j| = 1\).

Lemma 5.4. For any \( i, j \in \mathbb{Z} \) such that

\[
|i - j| = 1, \quad q(j) \neq q(i), \quad \text{(type } L(i), \text{type } L(j)) \neq (Q, Q),
\]

we define \( H_2 \)-supermodule \( M \) and \( A_2 \)-supermodule \( N \) as follows.

\[
M \overset{\text{def}}{=} \text{Ind}_{H_{i,1}}^{H_2} L(j) \otimes L(i), \quad N \overset{\text{def}}{=} (X_2 + X_2^{-1} - q(i))M \subseteq \text{Res}_{H_{i,1}}^{H_2} M.
\]

Then the following two statements hold.

(i) \( N \) is \( T_1 \)-invariant, i.e., \( N \) is an \( H_2 \)-supermodule.

(ii) \( \text{ch } N = [L(i) \otimes L(j)] \).

Proof. Note that we have \( \text{ch}_{1,1} N = [L(i) \otimes L(j)] \) because \( 0 \subseteq N \subseteq M \) and \( \text{ch } M = [L(i) \otimes L(j)] + [L(j) \otimes L(i)] \) by Lemma 5.1 and \( \text{ch } \text{Cosoc}(M) = \text{ch } L(j) \) contains a term \([L(j) \otimes L(i)]\) by Corollary 3.15. Thus, it is enough to show that \( T_1 N \subseteq N \).

By (3) and (4), we have

\[
(X_2 + X_2^{-1} - q(i))T_1 = T_1(X_1 + X_1^{-1} - q(i)) + \xi(X_2 + C_1 C_2 X_1 - X_1^{-1} - X_2^{-1} C_1 C_2).
\]

From this, we see that the following \( X \) and \( Y \) form a basis of \( N_{\overline{0}} \).

\[
X \overset{\text{def}}{=} (X_2 + X_2^{-1} - q(i))T_1 \otimes v_0^+ \otimes v_0^- = (q(j) - q(i))T_1 \otimes v_0^+ \otimes v_0^- + \xi((b_+(i) - b_-(j))1 \otimes v_0^+ \otimes v_0^- - \xi(b_+(i) - b_+(j))1 \otimes v_0^+ \otimes v_0^-),
\]

\[
Y \overset{\text{def}}{=} (X_2 + X_2^{-1} - q(i))T_1 \otimes v_0^+ \otimes v_0^- = (q(j) - q(i))T_1 \otimes v_0^+ \otimes v_0^- + \xi((b_-(i) - b_-(j))1 \otimes v_0^+ \otimes v_0^- + (b_-(i) - b_+(j))1 \otimes v_0^+ \otimes v_0^-). \]
To show $T_1 N \subseteq N$, it is enough to show $T_1 N_\mathfrak{P} \subseteq N_\mathfrak{P}$. For this purpose, it is enough to show the following equalities which follows from Corollary 5.3.

$$T_1 X = \xi(1 + \frac{b_+(i) - b_-(j)}{q(j) - q(i)}) X - \xi \frac{b_+(i) - b_+(j)}{q(j) - q(i)} Y,$$

$$T_1 Y = \xi \frac{b_-(i) - b_-(j)}{q(j) - q(i)} X + \xi(1 + \frac{b_-(i) - b_+(j)}{q(j) - q(i)}) Y.$$

\[ \square \]

**Corollary 5.5.** For any $i, j \in \mathbb{Z}$ such that $|i - j| = 1, \ q(j) \neq q(i), \ (\text{type } L(i), \text{type } L(j)) \neq (Q, Q)$.

We have the following descriptions of $L(ij)$.

(i) $\text{ch } L(ij) = [L(i) \otimes L(j)]$.

(ii) There exists a basis $\{X, Y\}$ of $L(ij)_{\mathfrak{P}}$ such that the matrix representations of $L(ij)$ with respect to the basis $\{X, Y, C_1 X, C_1 Y\}$ is as follows.

$$X_1^{\pm 1} : \begin{pmatrix} b_+(i) & 0 & 0 & 0 \\ 0 & b_+(i) & 0 & 0 \\ 0 & 0 & b_+(i) & 0 \\ 0 & 0 & 0 & b_+(i) \end{pmatrix}, \quad X_2^{\pm 1} : \begin{pmatrix} b_+(j) & 0 & 0 & 0 \\ 0 & b_+(j) & 0 & 0 \\ 0 & 0 & b_+(j) & 0 \\ 0 & 0 & 0 & b_+(j) \end{pmatrix},$$

$$C_1 : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C_2 : \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$T_1 : \begin{pmatrix} 1 & b_+(j) - b_-(i) & b_-(i) - b_-(j) & 0 & 0 \\ b_+(j) - b_+(i) & b_-(j) - b_+(i) & 0 & 0 \\ 0 & 0 & b_+(j) - b_+(i) & b_-(j) - b_+(i) \\ 0 & 0 & b_-(i) - b_+(j) & b_-(i) - b_-(j) \end{pmatrix}.$$

5.3. **On the block** $[(i, i, j)]$ with $|i - j| = 1$.

**Lemma 5.6.** For any $i, j \in \mathbb{Z}$ such that $|i - j| = 1, \ q(j) \neq q(i), \ (\text{type } L(i), \text{type } L(j)) = (M, M)$.

We define $\mathcal{H}_3$-supermodule $M$ and $\mathcal{H}_{2,1}$-supermodule $N$ as follows.

$$M \overset{\text{def}}{=} \text{Ind}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} L(ij) \otimes L(i), \quad N \overset{\text{def}}{=} (X_3 + X_3^{-1} - q(i))M \subseteq \text{Res}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} M.$$

If $q(i)q(j) + q(j)^2 - 8 \neq 0$, then we have $T_2 N \nsubseteq N$ and $M$ is irreducible.

**Proof.** Since $\text{ch } \text{Cosoc } M = L(iij)$ contains a term $[L(i) \otimes L(j) \otimes L(i)]$ by Corollary 3.15 and $\text{ch } M = [L(i) \otimes L(j) \otimes L(i)] + 2[L(i) \otimes L(j)]$ by Lemma 5.1, if $M$ is reducible then $M$ has a unique irreducible submodule $M'$ with $\text{Res}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} M' \cong L(i^2) \otimes L(j)$ by Theorem 3.9. Thus, if $M$ is reducible then $\text{Res}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} M' = N$. It implies that if $T_2 N \nsubseteq N$ then $M$ is irreducible.

In the rest of the proof, we show that $T_2 N \nsubseteq N$ if $q(i)q(j) + q(j)^2 - 8 \neq 0$. We take a basis $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \overset{\text{def}}{=} (X, Y, C_1 X, C_1 Y)$ of $L(ij)$ in Corollary 5.5. Then a basis of $M$ is given by

$$\{X_{\beta,k,l} \overset{\text{def}}{=} \beta \otimes \alpha_k \otimes v_l^i \mid \beta \in \{1, T_2, T_1 T_2\}, k \in \{1, 2, 3, 4\}, l \in \mathbb{Z}/2\mathbb{Z}\}.$$
and a basis of $N_{\Upsilon}$ is given by \{$Y_k, Z_k \mid 1 \leq k \leq 4$\} where

$$Y_k \overset{\text{def}}{=} (X_3 + X_3^{-1} - q(i))X_{T_2,k,f(k)}, \quad Z_k \overset{\text{def}}{=} (X_3 + X_3^{-1} - q(i))X_{T_1,T_2,k,f(k)} (= T_1 Y_k)$$

for $k = 1, 2, 3, 4$ and $f(1) = f(2) = \Upsilon$ and $f(3) = f(4) = \Upsilon$. More explicitly,

$$Y_1 = (q(j) - q(i))T_2 \otimes \alpha_1 \otimes v_0^i + \xi((b_+ (i) - b_-(j))1 \otimes \alpha_1 \otimes v_0^i + (b_+ (i) - b_+(j))1 \otimes \alpha_4 \otimes v_1^T),$$

$$Y_2 = (q(j) - q(i))T_2 \otimes \alpha_2 \otimes v_0^i + \xi((b_+ (i) - b_-(j))1 \otimes \alpha_2 \otimes v_0^i + (b_+ (i) - b_+(j))1 \otimes \alpha_3 \otimes v_1^T),$$

$$Y_3 = (q(j) - q(i))T_2 \otimes \alpha_3 \otimes v_1^T + \xi((b_-(i) - b_+(j))1 \otimes \alpha_3 \otimes v_1^T + (b_-(i) - b_+(j))1 \otimes \alpha_2 \otimes v_0^1),$$

$$Y_4 = (q(j) - q(i))T_2 \otimes \alpha_4 \otimes v_1^T + \xi((b_-(i) - b_+(j))1 \otimes \alpha_4 \otimes v_1^T + (b_-(i) - b_+(i))1 \otimes \alpha_3 \otimes v_1^T),$$

$$Z_1 = (q(j) - q(i))T_1 T_2 \otimes \alpha_1 \otimes v_0^i + \frac{\xi^2}{q(j) - q(i)}((b_+ (i) - b_-(j))(b_+(j) - b_-(i))1 \otimes \alpha_1 \otimes v_0^i$$

$$+ (b_+(i) - b_-(j))(b_+(j) - b_-(i))1 \otimes \alpha_2 \otimes v_0^i + (b_+(i) - b_+(j))(b_-(j) - b_-(i))1 \otimes \alpha_3 \otimes v_1^T$$

$$+ (b_+(i) - b_+(j))(b_-(j) - b_-(i))1 \otimes \alpha_4 \otimes v_1^T),$$

$$Z_2 = (q(j) - q(i))T_1 T_2 \otimes \alpha_2 \otimes v_0^i + \frac{\xi^2}{q(j) - q(i)}((b_+(i) - b_-(j))(b_-(j) - b_-(i))1 \otimes \alpha_1 \otimes v_0^i$$

$$+ (b_+(i) - b_+(j))(b_-(j) - b_+(i))1 \otimes \alpha_2 \otimes v_0^i + (b_-(j) - b_-(i))(b_+(j) - b_+(i))1 \otimes \alpha_3 \otimes v_1^T$$

$$+ (b_-(j) - b_+(i))(b_-(j) - b_+(i))1 \otimes \alpha_4 \otimes v_1^T).$$

It is enough to show $T_2 Z_1 \notin N_{\Upsilon}$ to prove $T_2 N_{\Upsilon} \subset N_{\Upsilon}$. Note that we have

$$T_2 Z_1 = \xi((b_+(j) - b_-(i))T_1 T_2 \otimes \alpha_1 \otimes v_0^i + (b_+(j) - b_+(i))T_1 T_2 \otimes \alpha_2 \otimes v_0^i) + \Delta$$

for a suitable $\Delta \in \text{span}\{X_{T_2,k,l} \mid 1 \leq k \leq 4, l \in \mathbb{Z}/2\mathbb{Z}\}$. Thus, if $T_2 Z_1 \in N_{\Upsilon}$, then we must have

$$T_2 Z_1 = \frac{\xi}{q(j) - q(i)}(b_+(j) - b_-(i))Z_1 + \frac{\xi^2}{q(j) - q(i)}((b_+(i) - b_-(j))(b_+(j) - b_-(i))Y_1 + (b_+(i) - b_-(j))(b_+(j) - b_+(i))Y_2$$

$$+ (b_+(i) - b_+(j))(b_-(j) - b_+(i))Y_3 + (b_+(i) - b_+(j))(b_-(j) - b_-(i))Y_4).$$

Especially, the coefficient of $1 \otimes \alpha_1 \otimes v_0^i$ of the right hand side must be 0. It gives us

$$\frac{\xi^3}{(q(j) - q(i))^2}(b_+(i) - b_-(i))(q(i)q(j) + q(j)^2 - 8) = 0.$$
Proof. $q(i)q(j) + q(j)^2 - 8 = 0$ is equivalent to $q^{4i+3} + 3 = 1$ or $q^{4i+1} + 3 = 1$ by
\[
(q^{2i+1} + q^{-2(i+1)}) + (q^{2i+1} + q^{-(i+1)}) = 2(q + q^{-1})^2
\]
\[
= (q^{2(i+1)} + q^{-2(i+1)}) + (q^{2i+1} + q^{-(i+1)}) - 2(q + q^{-1})^2
\]
\[
= (q + q^{-1})(q^{2+1.5+1.5} - q^{-2(2i+1.5+1.5)})
\]
Since type $L(i) = M$, we have $l \geq 3$ and $1 \leq i \leq l - 2$. Thus we have $2 \leq 4i - 2 < 4i + 6 \leq 4l - 2$ and we see that $q^{4i+3} \neq 1$ and $q^{4i+1} \neq 1$.

By Lemma 5.6, $L(ij) \cong M = \text{Ind}_{H_{2,1}}^H L(i) \otimes L(i)$. Thus, $\text{ch} L(ij) = 2[L(i) \otimes L(j)]$ by Lemma 5.1. It implies $\Delta_j \neq 0$ and $\bar{e}_j M \cong L(i)$ by Theorem 3.9. Thus, we have $M \cong L(ij)$.

Finally, consider the irreducible supermodule $L(i"

Lemma 5.8. For any $i, j \in \mathbb{Z}$ such that

\[
|i - j| = 1, \quad q(j) \neq q(i), \quad \text{type } L(i), \text{type } L(j) = (Q, M),
\]
we define $\mathcal{H}_3$-supermodule $M$ and $\mathcal{H}_{2,1}$-supermodule $N$ as follows.

\[
M \defeq \text{Ind}_{H_{2,1}}^H L(i) \otimes L(i), \quad N \defeq (X_3 + X_3^{-1} - q(i))M \subseteq \text{Res}_{H_{2,1}}^H M.
\]

Then the following two statements hold.

(i) $N$ is $T_2$-invariant, i.e., $N$ is an $\mathcal{H}_3$-supermodule.

(ii) $\text{ch} N = 2[L(i) \otimes L(j)]$ and $\text{ch} M/N = [L(i) \otimes L(j) \otimes L(i)]$.

Proof. As in the first paragraph of the proof of Lemma 5.6, if $N$ is $T_2$-invariant then $\text{ch} N = 2[L(i) \otimes L(j)]$ and $\text{ch} M/N = [L(i) \otimes L(j) \otimes L(i)]$. Thus, it is enough to show that $N$ is $T_2$-invariant.

In the rest of the proof, we write $a$ instead of $b_+(i) = b_-(i)$ and take a basis \{X, Y, C_1 X, C_1 Y\} of $L(ij)$ in Corollary 5.5.

We can take a realization of $L(ij) \otimes L(i)$ as an $\mathcal{H}_{2,1}$-submodule $W$ of $L(ij) \otimes L(i)$ given as follows because direct calculation shows that $W$ is $\mathcal{H}_{2,1}$-invariant.

\[
W \defeq W_\pi + W_{\pi}, \quad W_\pi \defeq FX' + FY', \quad W_{\pi} \defeq F(C_1 X') \oplus F(C_1 Y'),
\]
\[
X' \defeq X \otimes v_\pi^a + \sqrt{-1}(C_1 X) \otimes v_\pi^a, \quad Y' \defeq Y \otimes v_\pi^a - \sqrt{-1}(C_1 Y) \otimes v_\pi^a.
\]

More precisely, we can check that the matrix representations with respect to the basis \{X', Y', C_1 X', C_1 Y'\} is given as follows.

\[
X_{i+1} \defeq \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad X_{i+1} \defeq \begin{pmatrix} b_+(j) & 0 & 0 & 0 \\ 0 & b_+(j) & 0 & 0 \\ 0 & 0 & b_+(j) & 0 \\ 0 & 0 & 0 & b_+(j) \end{pmatrix}, \quad X_{i+1} \defeq \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix},
\]
\[
C_1 \defeq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C_2 \defeq \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_3 \defeq \begin{pmatrix} 0 & 0 & \sqrt{-1} & 0 \\ 0 & 0 & 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 & 0 & 0 \\ 0 & \sqrt{-1} & 0 & 0 \end{pmatrix}.
\]
Hereafter, we put \((\alpha_1, \alpha_2, \alpha_3, \alpha_4) \equiv (X', Y', C_1X', C_1Y')\). Then a basis of \(M\) is given by \(\{X_{\beta, k} \equiv \beta \otimes \alpha_k \mid \beta \in \{1, T_2, T_1T_2\}, k \in \{1, 2, 3, 4\}\}\). It is enough to show that \(T_2N_{\overline{\pi}} \subseteq N_{\overline{\pi}}\). We can choose
\[
\{Y_k \equiv (X_3 + X_3^{-1} - q(i))X_{T_2, k}, Y_{k+2} \equiv (X_3 + X_3^{-1} - q(i))X_{T_1T_2, k} \mid 1 \leq k \leq 2\}
\]
as a basis of \(N_{\overline{\pi}}\). More explicitly, we have
\[
Y_1 = (q(j) - q(i))T_2 \otimes \alpha_1 + \xi(1 - \sqrt{1})(a - b_+(j))1 \otimes \alpha_1 + \sqrt{1} (a - b_+(j))1 \otimes \alpha_2),
\]
\[
Y_2 = (q(j) - q(i))T_2 \otimes \alpha_2 + \xi(\sqrt{1})(a - b_+(j))1 \otimes \alpha_1 + (a - b_+(j))1 \otimes \alpha_2),
\]
\[
Y_3 = (q(j) - q(i))T_1T_2 \otimes \alpha_1
\]
\[
+ \frac{\xi^2}{q(j) - q(i)}(b_+(j) - a)(a - b_-(j))((1 - \sqrt{1})1 \otimes \alpha_1 + (1 + \sqrt{1})1 \otimes \alpha_2),
\]
\[
Y_4 = (q(j) - q(i))T_1T_2 \otimes \alpha_2
\]
\[
+ \frac{\xi^2}{q(j) - q(i)}(b_+(j) - a)(a - b_-(j))((-1 + \sqrt{1})1 \otimes \alpha_1 + (1 + \sqrt{1})1 \otimes \alpha_2).
\]

Now we can check the following relations using Corollary 5.3.
\[
T_2Y_1 = \frac{\xi b_+(j) - a}{q(j) - q(i)} Y_1 + \frac{\xi (a - b_+(j))}{q(j) - q(i)} \sqrt{1} Y_2,
\]
\[
T_2Y_2 = \frac{\xi(a - b_-(j))}{q(j) - q(i)} Y_1 + \frac{\xi b_-(j) - a}{q(j) - q(i)} Y_2,
\]
\[
T_2Y_3 = \frac{\xi b_+(j) - a}{q(j) - q(i)} (Y_3 + Y_4) + \frac{\xi^2 (b_+(j) - a)(a - b_-(j))}{(q(j) - q(i))^2} ((1 - \sqrt{1})Y_1 + (1 + \sqrt{1})Y_2),
\]
\[
T_2Y_4 = \frac{\xi(a - b_-(j))}{q(j) - q(i)} (Y_3 - Y_4) + \frac{\xi^2 (b_+(j) - a)(a - b_-(j))}{(q(j) - q(i))^2} ((-1 + \sqrt{1})Y_1 + (1 + \sqrt{1})Y_2).
\]
$$T_2Y_4 = \frac{\xi(a - b_{-}(j))}{q(j) - q(i)}(Y_3 - Y_4) + \frac{\xi^2(b_{+}(j) - a)(a - b_{-}(j))}{(q(j) - q(i))^2}((1 + \sqrt{-1})Y_1 + (1 + \sqrt{-1})Y_2),$$

$$C_3Y_1 = -C_1Y_2, \quad C_3Y_2 = C_1Y_1,$$

$$C_3Y_3 = \sqrt{-1}(C_1Y_4) - \xi(1 + \sqrt{-1})(C_1Y_2),$$

$$C_3Y_4 = \sqrt{-1}(C_1Y_3) + \xi(1 - \sqrt{-1})(C_1Y_1).$$

**Proof.** It is enough to show the last 4 relations. Direct calculations using (3) gives us

$$C_1Y_1 = (q(j) - q(i))T_2 \otimes \alpha_3 + \xi((a - b_{-}(j))1 \otimes \alpha_3 + \sqrt{-1}(a - b_{+}(j))1 \otimes \alpha_4),$$

$$C_1Y_2 = (q(j) - q(i))T_2 \otimes \alpha_3 + \xi(\sqrt{-1}(a - b_{-}(j))1 \otimes \alpha_3 + (a - b_{+}(j))1 \otimes \alpha_4),$$

$$C_1Y_3 = -\sqrt{-1}(q(j) - q(i))T_1T_2 \otimes \alpha_3 + (q(j) - q(i))\xi(1 + \sqrt{-1})T_2 \otimes \alpha_3 + \Delta_1,$$

$$C_1Y_4 = \sqrt{-1}(q(j) - q(i))T_1T_2 \otimes \alpha_4 + (q(j) - q(i))\xi(1 - \sqrt{-1})T_2 \otimes \alpha_4 + \Delta_2,$$

$$C_3Y_2 = (q(j) - q(i))T_2 \otimes (-\alpha_4) + \Delta_3 = -C_1Y_2,$$

$$C_3Y_3 = -(q(j) - q(i))T_1T_2 \otimes \alpha_4 + \Delta_5 = \sqrt{-1}(C_1Y_4) - \sqrt{-1}\xi(1 - \sqrt{-1})(C_1Y_2),$$

$$C_3Y_4 = (q(j) - q(i))T_1T_2 \otimes \alpha_3 + \Delta_6 = \sqrt{-1}(C_1Y_3) - \sqrt{-1}\xi(1 + \sqrt{-1})(C_1Y_1).$$

Here $\Delta_1, \cdots, \Delta_6$ are suitable elements in $\text{span}\{1 \otimes \alpha_k \mid 1 \leq k \leq 4\}(\subseteq M)$. \hfill \Box

**5.4. On the block $[[i, i, j]]$ with $|i - j| = 1$ and (type $L(i), type L(j) = (Q, M)$.**

**Lemma 5.10.** For any $i, j \in \mathbb{Z}$ such that

$$|i - j| = 1, \quad q(j) \neq q(i), \quad (\text{type } L(i), \text{type } L(j)) = (Q, M),$$

we define $\mathcal{H}_4$-supermodule $M$ and $\mathcal{H}_{3,1}$-supermodule $N$ as follows.

$$M \overset{\text{def}}{=} \text{Ind}_{\mathcal{H}_{3,1}}^{\mathcal{H}_4} L((ij) \otimes L(i)), \quad N \overset{\text{def}}{=} (X_4 + X_4^{-1} - q(i))M \subseteq \text{Res}_{\mathcal{H}_{3,1}}^{\mathcal{H}_4} M.$$ 

If $q(j) + 2q(i) \neq 0$, then $T_3N \nsubseteq N$ and $M$ is irreducible.

**Proof.** By the same reasoning as the Lemma 5.6, if $T_3N \nsubseteq N$ then $M$ is irreducible.

In the rest of the proof, we show that if $q(j) + 2q(i) \neq 0$ then $T_3N \nsubseteq N$.

We write $a$ instead of $b_{+}(i) = b_{-}(i)$ as in the proof of Lemma 5.8 and we adapt a basis $\{Y_1, Y_2, Y_3, Y_4\}$ of $L((ij) \otimes L((ij))$ in Corollary 5.9. Thus, we can choose

$$\{Z_{\beta,k} \overset{\text{def}}{=} \beta \otimes Y_k \otimes v_\beta, W_{\beta,k} \overset{\text{def}}{=} \beta \otimes C_1Y_k \otimes v_\beta \mid \beta \in \{1, T_3T_2T_3T_1T_2T_3\}, k \in \{1, 2, 3, 4\}\}$$

as a basis of $M_\mathcal{P}$ and

$$\begin{align*}
\left\{Z_{\beta,k}' \overset{\text{def}}{=} & \epsilon(x_4^{-1} + x_4^{-1} - q(i))Z_{\beta,k}, \\
W_{\beta,k}' \overset{\text{def}}{=} & \epsilon(x_4^{-1} + x_4^{-1} - q(i))W_{\beta,k} \mid \beta \in \{T_3T_2T_3T_1T_2T_3\}, k \in \{1, 2, 3, 4\}\right\}
\end{align*}$$

as a basis of $N_\mathcal{P}$. More explicitly, we have

$$Z_{T_3,1}' = (q(j) - q(i))T_3 \otimes Y_1 \otimes v_\beta + \epsilon((a - b_{-}(j))1 \otimes Y_1 \otimes v_\beta + (a - b_{+}(j))1 \otimes C_1Y_2 \otimes v_\beta),$$

$$Z_{T_3,2}' = (q(j) - q(i))T_3 \otimes Y_2 \otimes v_\beta + \epsilon((a - b_{+}(j))1 \otimes Y_2 \otimes v_\beta + (b_{-}(j) - a)1 \otimes C_1Y_1 \otimes v_\beta),$$

$$Z_{T_3,3}' = (q(j) - q(i))T_3 \otimes Y_3 \otimes v_\beta + \epsilon((a - b_{-}(j))1 \otimes Y_3 \otimes v_\beta + \sqrt{-1}(b_{+}(j) - a)1 \otimes C_1Y_2 \otimes v_\beta),$$

$$Z_{T_3,4}' = (q(j) - q(i))T_3 \otimes Y_4 \otimes v_\beta + \epsilon((a - b_{+}(j))1 \otimes Y_4 \otimes v_\beta.$$
be the coefficient of 1, calculated as follows.

\[ W'_{T_{a,1}} = (q(j) - q(i))T_3 \otimes C_1 Y_3 \otimes v_{i,0} + \xi((a - b_-(j))1 \otimes C_1 Y_1 \otimes v_{i,0} + (a - b_+(j))1 \otimes Y_2 \otimes v_{i,0}), \]

\[ W'_{T_{a,2}} = (q(j) - q(i))T_3 \otimes C_1 Y_2 \otimes v_{i,0} + \xi((a - b_+(j))1 \otimes C_1 Y_2 \otimes v_{i,0} + (b_-(j) - a)1 \otimes Y_1 \otimes v_{i,0}), \]

\[ W'_{T_{a,3}} = (q(j) - q(i))T_3 \otimes C_1 Y_3 \otimes v_{i,0} + \xi((a - b_-(j))1 \otimes C_1 Y_3 \otimes v_{i,0} \]

\[ + \sqrt{-1}(b_+(j) - a)1 \otimes Y_4 \otimes v_{i,0} - \xi(1 - \sqrt{-1})(b_+(j) - a)1 \otimes Y_2 \otimes v_{i,0}), \]

\[ W'_{T_{a,4}} = (q(j) - q(i))T_3 \otimes C_1 Y_4 \otimes v_{i,0} + \xi((a - b_+(j))1 \otimes C_1 Y_4 \otimes v_{i,0} \]

\[ + \sqrt{-1}(b_-(j) - a)1 \otimes Y_3 \otimes v_{i,0} + \xi(1 - \sqrt{-1})(b_-(j) - a)1 \otimes Y_1 \otimes v_{i,0}), \]

\[ Z'_{T_{a,1},k} = T_2 Z'_{T_{a,1}} = (q(j) - q(i))T_2 T_3 \otimes Y_k \otimes v_{i,0} + \Delta_k \quad (1 \leq k \leq 4), \]

\[ W'_{T_{a,1},k} = T_2 W'_{T_{a,1}} = (q(j) - q(i))T_2 T_3 \otimes C_1 Y_k \otimes v_{i,0} + \Delta_{k+4} \quad (1 \leq k \leq 4). \]

Here each \( \Delta_m \) for \( 1 \leq m \leq 8 \) is a suitable element in \( \text{span}\{1 \otimes C_d^i Y_k \otimes v_e^i \mid k \in \{1, 2, 3, 4\}, d \in \{0, 1\}, e \in \mathbb{Z}/2\mathbb{Z}\} \subseteq M \) and we write \( \Delta_3 = \sum_{k=1}^{4} P_{k} Y_k \otimes v_{i,0} + \sum_{k=1}^{4} Q_{k} Y_k \otimes v_{i,0} \) with suitable coefficients. We define \( \Omega_m, \Omega_{Z,k} \) and \( \Omega_{W,k} \) to be the coefficient of \( 1 \otimes Y_1 \otimes v_{i,0} \) in \( \Delta_m, Z'_{T_{a,1},k} \) and \( W'_{T_{a,1},k} \) respectively. Now \( T_3 Z'_{T_2 T_3} \) is expanded as follows.

\[ \xi(b_+(j) - a)(q(j) - q(i))((1 - \sqrt{-1})T_2 T_3 \otimes Y_3 \otimes v_{i,0} + (1 + \sqrt{-1})T_2 T_3 \otimes Y_4 \otimes v_{i,0}) \]

\[ + \sum_{k=1}^{4} P_{k} T_3 \otimes Y_k \otimes v_{i,0} + \sum_{k=1}^{4} Q_{k} T_3 \otimes C_1 Y_k \otimes v_{i,0}. \]

Thus, if \( T_3 Z'_{T_2 T_3} \in \mathbb{Q}_{\mathbb{Z}} \), then we must have

\[ T_3 Z'_{T_2 T_3} = \frac{\xi(b_+(j) - a)}{q(j) - q(i)}(Z'_{T_{a,1}} + Z'_{T_{a,2}}) + \sum_{k=1}^{4} \frac{P_{k} Z'_{T_{a,1},k} + Q_{k} W'_{T_{a,1},k}}{q(j) - q(i)} \]

\[ + \frac{\xi^2(b_+(j) - a)(a - b_-(j))}{q(j) - q(i)}((1 - \sqrt{-1})Z'_{T_{a,1}} + (1 + \sqrt{-1})Z'_{T_{a,2}}). \]

Especially, the coefficient of \( 1 \otimes \alpha_1 \otimes v_{i,0} \) of the right hand side must be 0, in other words

\[ S \triangleq \xi(b_+(j) - a)(q(j) - q(i))(\Omega_3 + \Omega_4) + \sum_{k=1}^{4} \frac{P_{k} \Omega_{Z,k} + Q_{k} \Omega_{W,k}}{q(j) - q(i)} \]

\[ + \frac{\xi^2(b_+(j) - a)(a - b_-(j))}{q(j) - q(i)}((1 - \sqrt{-1})\Omega_1 + (1 + \sqrt{-1})\Omega_2) = 0. \]

Note that \( \Omega_{Z,2} = \Omega_{Z,3} = \Omega_{Z,4} = \Omega_{W,1} = \Omega_{W,3} = 0 \) and necessary data are calculated as follows.

\[ \Omega_1 = \frac{\xi^2}{q(j) - q(i)}(a - b_-(j))(b_+(j) - a), \quad \Omega_2 = \frac{\xi^2 \sqrt{-1}}{q(j) - q(i)}(a - b_+(j))(a - b_-(j)), \]

\[ \Omega_3 = \frac{\xi^3(1 - \sqrt{-1})}{q(j) - q(i)}(a - b_-(j))^2(b_+(j) - a), \quad \Omega_4 = \frac{\xi^3(1 - \sqrt{-1})}{(q(j) - q(i))^2}((b_+(j) - a)^2(a - b_-(j)), \]

\[ \Omega_{Z,1} = \xi(a - b_-(j)), \quad \Omega_{W,2} = \xi(b_-(j) - a), \quad \Omega_{W,4} = \xi^2(1 - \sqrt{-1})(b_-(j) - a), \]
5.5. Since Corollary 5.13, we have

\[
P_1 = \Omega_3 = \frac{\xi^3(1 - \sqrt{-1})}{(q(j) - q(i))^2}(a - b_-(j))^2(b_+(j) - a), \quad Q_4 = \frac{\sqrt{-1} \xi^2}{q(j) - q(i)}(b_+(j) - a)(a - b_-(j)),
\]

\[
Q_2 = \frac{\xi^3(1 + \sqrt{-1})}{(q(j) - q(i))^2}(b_+(j) - a)^2(a - b_-(j)) + \frac{\xi^3(1 + \sqrt{-1})}{q(j) - q(i)}(b_+(j) - a)(a - b_-(j)).
\]

Using them, we have

\[
S = \frac{\xi^4(1 - \sqrt{-1})}{(q(j) - q(i))^3}(a - b_-(j))(b_+(j) - a)(4(a - b_-(j))(b_+(j) - a) + (b_+(j) - a)^2 + (a - b_-(j))^2).
\]

Note that \((a - b_-(j))(b_+(j) - a) = aq(j) - 2 \neq 0\) since \(q(j) \neq \pm 2\). Thus, we have

\[
4(a - b_-(j))(b_+(j) - a) + (b_+(j) - a)^2 + (a - b_-(j))^2 = (q(j) + 4a)(q(j) - 2a) = 0.
\]

Again, by \(q(j) \neq \pm 2\), we have \(q(j) + 4a = q(j) + 2q(i) = 0\) if \(T_3 \mathbb{Z} T_3 T_3, 3 \in N_7\). □

**Corollary 5.11.** Assume \(q\) be a primitive \(4l\)-th root of unity for \(l \geq 3\) and \(i, j \in \mathbb{Z}\) satisfy

\[
|i - j| = 1, \quad q(j) \neq q(i), \quad \text{(type } L(i), \text{ type } L(j)) = (\mathbb{Q}, \mathbb{M}).
\]

Then we have the following descriptions.

1. \(L(iij) \cong L(iijj) \cong \text{Ind}_{H_{3,1}} \otimes L(i)\).
2. \(\text{ch } L(iiji) = \text{ch } L(iiiij) = 6[L(i) \otimes L(j)] + 2[L(i) \otimes L(j)] \otimes L(i)\).
3. \(\text{ch } L(ijii) = 6[L(j) \otimes L(i)] + 2[L(i) \otimes L(j)] \otimes L(i)\).
4. \(\text{ch } L(ijii) = 2[L(i) \otimes L(j)] \otimes L(i) \otimes L(i)\).

**Proof.** We only need to consider the case \((i, j) = (0, 1), (l - 1, l - 2)\). In each case, we see that \(q(j) + 2q(i) = 0\) implies \(q^6 = 1\). Thus, we have \(L(iijj) \cong \text{Ind}_{H_{3,1}} \otimes L(i)\) by Lemma 5.10. By the same reasoning as Corollary 5.7, we have \(L(iijj) \cong L(iijj)\). Note that \(L(ijjj) \neq L(ijjj)\) since \(L(ji) \neq L(ij)\) by Corollary 5.5. Since \(q(ji) = 3\), we see that \(L(ijjj) \cong L(ijjj)\). Now it is easily seen that \(L(iiji) \cong \text{Ind}_{H_{3,1}} \otimes L(i)\). □

5.5. The case when \(q\) is a primitive \(8\)-th root of unity.

**Lemma 5.12.** Let \(q\) be a primitive \(8\)-th root of unity. We can take a basis \(B = \{w_1, w_2\}\) of \(L(01)\) such that \(w_1\) is even and \(w_2\) is odd and the matrix representations with respect to \(B\) is as follows.

\[
X_{i}^{\pm 1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_{2}^{\pm 1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & -q^2 \\ q^2 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} q & 0 \\ 0 & q^3 \end{pmatrix}.
\]

**Proof.** We can check by direct calculation that they satisfy the defining relations of \(H_2\). It is clearly irreducible and note that the whole space is a simultaneous \((2, -2) = (q(0), q(1))-\)eigenspace of \((X_1 + X_1^{-1}, X_2 + X_2^{-1})\). □

**Corollary 5.13.** We have \(\text{ch } L(01) = [L(0) \otimes L(1)]\) and \(\text{ch } L(10) = [L(1) \otimes L(0)]\).

**Lemma 5.14.** Let \(q\) be a primitive \(8\)-th root of unity. We can take a basis \(B = \{w_i, | 1 \leq i \leq 8\}\) of \(L(001)\) such that \(w_i\) is even and \(w_{i+4}\) is odd for \(1 \leq i \leq 4\) and the matrix representations with respect to \(B\) is as follows.

\[
X_i : \begin{pmatrix} M_{X} & O \\ O & M_{X} \end{pmatrix}, \quad X_{3}^{\pm 1} = -E_8, \quad X_{3}^{-1} = 2E_8 - X_1, \quad X_{2}^{-1} = 2E_8 - X_2.
\]
Corollary 5.16.}

\[
C_j : \begin{pmatrix} O & M_{C_j} \\ -M_{C_j} & O \end{pmatrix}, \quad T_1 : \frac{1}{1 + q^2} \begin{pmatrix} M_{T_1} & O \\ O & M_{T_1} \end{pmatrix}, \quad T_2 : \begin{pmatrix} M_{T_2} & O & O & O \\ O & M_{T_2} & O & O \\ O & O & M_{T_2} & O \\ O & O & O & M_{T_2} \end{pmatrix},
\]

for \(1 \leq i \leq 2\) and \(1 \leq j \leq 3\) where

\[
M_{X_1} = \begin{pmatrix} 1 & 0 & -2 & 2q \\ 0 & 2q^{-1} & 1 & 0 \\ 2q^{-1} & -2q^2 & 0 & 1 \end{pmatrix}, \quad M_{X_2} = \begin{pmatrix} -1 & -2q^{-1} & 0 & 0 \\ 2q & 0 & 0 & 0 \\ 0 & 0 & -1 & 2q \end{pmatrix},
\]

\[
M_{C_1} = \begin{pmatrix} q^2 & 0 & 2q^2 & 2q^{-1} \\ 0 & q^2 & 2q^{-1} & 0 \\ 2q^2 & 2q & -q^2 & 0 \\ 2q & 2 & 0 & -q^2 \end{pmatrix}, \quad M_{C_2} = \begin{pmatrix} 0 & 0 & q^2 & 0 \\ 0 & 0 & 2q^{-1} & 0 \\ q^2 & 0 & 0 & 0 \\ 2q & 1 & 0 & 0 \end{pmatrix},
\]

\[
M_{C_3} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & q^2 \\ 1 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \end{pmatrix}, \quad M_{T_1} = \begin{pmatrix} q^3 & q^2 & -q^3 & -1 \\ 0 & q^3 & 0 & q \\ q^3 & q^2 & q^3 & 1 \\ 0 & q & 0 & q^3 \end{pmatrix}, \quad M_{T_2} = \begin{pmatrix} q^3 + q & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Proof. We can check by direct calculation that they satisfy the defining relations of \(H_3\) and whole space is a simultaneous \((2, 2, -2) = (q(0), q(0), q(1))\)-eigenspace of \((X_1 + X_1^{-1}, X_2 + X_2^{-1}, X_3 + X_3^{-1})\). Since \(\dim L(0^2) \oplus L(1) = 8\) and Theorem 3.9, this submodule is irreducible. \(\Box\)

Corollary 5.15. Let \(q\) be a primitive 8-th root of unity. We have \(\text{ch} L(001) = 2[L(0) \otimes 2 \oplus L(1)], \text{ch} L(010) = [L(0) \oplus L(1) \oplus L(0)]\) and \(\text{ch} L(100) = 2[L(1) \oplus L(0) \otimes 2]\).

Proof. By \(\text{ch} L(001) = 2[L(0) \otimes 2 \oplus L(1)], \text{ch} L(010) = [L(0) \oplus L(1) \oplus L(0)]\) and \(\text{ch} L(100) = 2[L(1) \oplus L(0) \otimes 2]\), we have \(L(100) \cong L(010) \cong L(001)^\sigma\). Consider \(M = \text{Ind}_{H_{3,1}}^{H_3} L(01) \oplus L(0)\). By Corollary 5.13 and Lemma 5.1, we have \(\text{ch} M = [L(0) \oplus L(1) \oplus L(0)] + 2[L(0) \otimes 2 \oplus L(1)]\). Apply Theorem 3.11 (i), we see that \(L(010) \cong \text{Cococ} M\) with \(\text{ch} L(010) = [L(0) \oplus L(1) \oplus L(0)]\). \(\Box\)

Corollary 5.16. Let \(q\) be a primitive 8-th root of unity. Then \(M \overset{\text{def}}{=} \text{Ind}_{H_{3,1}}^{H_4} L(001) \oplus L(0)\) is an irreducible \(H_4\)-supermodule.

Proof. Take a basis \(\{w_i \mid 1 \leq i \leq 8\}\) in Lemma 5.14. Consider the following linear transformations with respect to this basis.

\[
X_4^{\pm 1} : E_8, \quad C_4 : \begin{pmatrix} O & -E_4 \\ -E_4 & O \end{pmatrix}.
\]

We can check that the matrix representations of \(\{X_4^{\pm 1}, C_4, T_j \mid 1 \leq i \leq 4, 1 \leq j \leq 3\}\) satisfy the defining relations of \(H_{3,1}\). Thus, they are also matrix representations of \(L(001) \oplus L(0)\).

To prove that \(M\) is irreducible, it is enough to show that \(H_{3,1}\)-supermodule \(N \overset{\text{def}}{=} (X_4 + X_4^{-1} - q(0))M\) is not \(T_3\)-invariant as in the proof of Lemma 5.10. Thus, it is enough to show that \(T_3 Z \neq (Z - W)/2\) where

\[
Z \overset{\text{def}}{=} (X_4 + X_4^{-1} - 2)T_3 \otimes w_1 = -4T_3 \otimes w_1 + 2\xi(w_1 + w_3),
\]
\[ W \overset{\text{def}}{=} (X_4 + X_4^{-1} - 2)T_3 \otimes w_3 = -4T_3 \otimes w_3 + 2\xi(w_3 - w_1), \]
\[ T_3Z = -2\xi(T_3 \otimes w_1 - T_3 \otimes w_3) - 4 \cdot 1 \otimes w_1. \]
This follows from \(2\xi \neq -4. \)

**Corollary 5.17.** Let \( q \) be a primitive 8-th root of unity. Then we have the following descriptions.

(i) \( L(0010) \cong L(0001) \cong \text{Ind}_{H_{2,1}}^{H_{3,1}} L(001) \otimes L(0). \)

(ii) \( \text{ch} L(0010) = \text{ch} L(0001) = 6[ L(0) \otimes L(1) + 2[ L(0) \otimes L(1) \oplus L(0) ]]. \)

(iii) \( \text{ch} L(1000) = 6[ L(1) \otimes L(0)^{\otimes 3} + 2[ L(0) \otimes L(1) \oplus L(0)^{\otimes 2}]. \)

(iv) \( \text{ch} L(0100) = 2[ L(0) \otimes L(1) \oplus L(0) \otimes L(0)^{\otimes 2}] + 2[ L(0)^{\otimes 2} \otimes L(1) \oplus L(0)]. \)

**Proof.** Same as the proof of Corollary 5.11. \( \square \)

6. Hecke-Clifford superalgebras and crystals of type \( D_l^{(2)} \)

Recall so far that \( F \) is an algebraically closed field of characteristic different from 2. From now on, we assume that \( q \) is a primitive 4l-th root of unity for \( l \geq 2 \) and choose \( \{ 0, 1, \ldots , l-1 \} \) as \( I_q \). Note that we have \( q(0) = 2 \) and \( q(l-1) = -2. \)

6.1. Lie theory of type \( D_l^{(2)} \). Consider the Dynkin diagram and the affine Cartan matrix indexed by \( I_q \) of type \( D_l^{(2)} \) as follows\(^8\).

\[
\begin{array}{cccccc}
  0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  l \geq 3 & 0 & 1 & 2 & l-3 & l-2 & l-1
\end{array}
\]

\[
\begin{pmatrix}
  2 & -2 & 0 & \cdots & 0 & 0 & 0 \\
  -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
  0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
  0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
  0 & 0 & 0 & \cdots & 0 & -2 & 2 \\
\end{pmatrix}
\]

In the rest of this section, let \( \mathfrak{g} \) be the corresponding Kac-Moody Lie algebra and apply definitions in §3.7 for \( A = D_l^{(2)} \).

6.2. Representations of low rank affine Hecke-Clifford superalgebras. The purpose of this subsection is to show that \( [BK, \text{Lemma 5.19}, \text{Lemma 5.20}] \) still hold in our setting, i.e., when \( q \) is a primitive 4l-th root of unity for \( l \geq 2 \). This fact is responsible for the appearance of the Lie theory of type \( D_l^{(2)} \).

**Lemma 6.1.** Let \( i, j \in I_q \) with \( |i - j| = 1 \). Then, for all \( a, b \geq 0 \) with \( a + b < -\langle h_i, \alpha_j \rangle \), there is a non-split short exact sequence

\[
0 \rightarrow L(i^{a+1} j^b) \rightarrow \text{Ind}_{H_{a+b+1,1}}^{H_{a+b+2,1}} L(i^{a} j^b) \oplus L(i) \rightarrow L(i^{a} j^b+1) \rightarrow 0.
\]

Moreover, for every \( a, b \geq 0 \) with \( a + b \leq -\langle h_i, \alpha_j \rangle \), we have

\[
\text{ch} L(i^{a} j^b) = a! b! [ L(i)^{\otimes a} \oplus L(j) \oplus L(i)^{\otimes b} ].
\]

**Proof.** (14) is established in Corollary 5.5, Corollary 5.9, Corollary 5.13 and Corollary 5.15. An existence of a non-split short exact sequence (13) follows from Lemma 5.1, Theorem 3.11 (i), Definition 3.14 and the injectivity of the formal character map \( \text{ch} : K_0(\text{Rep} H_n) \hookrightarrow K_0(\text{Rep} A_n) \) \( [BK, \text{Theorem 5.12}] \). \( \square \)

\(^8\)According to Kac’s notation \( [\text{Kac, TABLE Aff 1-3}] \), \( D_2^{(2)} \) should be regarded as \( A_1^{(1)} \).
Lemma 6.2. Let $i, j \in I_q$ with $|i - j| = 1$ and set $n = 1 - \langle h_i, \alpha_j \rangle$. Then $L(i^n j) \cong L(i^{n-1} j i)$. Moreover, for every $a, b \geq 0$ with $a + b = -\langle h_i, \alpha_j \rangle$, we have

$$L(i^n j i^{b+1}) \cong \text{Ind} \_{\mathcal{H}_{n+1}}^{\mathcal{H}_n} L(i^n j i^b) \oplus L(i) \cong \text{Ind} \_{\mathcal{H}_{n+1}}^{\mathcal{H}_n} L(i) \oplus L(i^n j i^b)$$

with character

$$a(l(b + 1)\lceil L(i) \oplus L(j) \oplus L(i)^{(b+1)} \rceil + (a + 1)!\lfloor L(i) \oplus (a + 1) \oplus L(j) \oplus (a + b)\rfloor.$$

Proof. Character formulas are established in Corollary 5.7, Corollary 5.11 and Corollary 5.16. The rest of reasoning is the same as the proof of Lemma 6.1. \hfill \square

Corollary 6.3. The operators $\{e_i : K(\infty) \to K(\infty) \mid i \in I_q\}$ satisfy the Serre relations, i.e.,

$$
\begin{align*}
& e_i e_j = e_j e_i \quad \text{if } |i - j| > 1, \\
& e_i^2 + e_i e_j e_i = 2e_i e_j e_i \quad \text{if } |i - j| = 1 \text{ and } i \neq 0 \text{ and } i \neq l - 1, \\
& e_i^3 e_j + 3e_i e_j e_i^2 = 3e_i^2 e_j e_i + e_j e_i^3 \quad \text{otherwise}.
\end{align*}
$$

Proof. By Lemma 3.24 and coassociativity of $\Delta$, it is enough to check the same relation on $K_0(\text{Rep } \mathcal{H}_2), K_0(\text{Rep } \mathcal{H}_3)$ and $K_0(\text{Rep } \mathcal{H}_4)$ respectively. It is achieved using the character formulas in Lemma 6.1 and Lemma 6.2. \hfill \square

The same arguments using Lemma 6.1 and Lemma 6.2 establishes the following [BK, Lemma 5.23].

Lemma 6.4. Take $M \in \text{Irr}(\text{Rep } \mathcal{H}_n)$ and $i, j \in I_q$ with $i \neq j$. Then the followings hold where $k = -\langle h_i, \alpha_j \rangle$ and $\varepsilon = \varepsilon_i(M)$.

(i) There exists a unique pair of non-negative integers $(a, b)$ with $a + b = k$ such that for every $m \geq 0$ we have $\varepsilon_i(f_i^m f_j M) = m + \varepsilon - a$.

(ii) $[\text{Cosoc Ind } f_i^m f_j M \oplus L(i^n j i^b) : f_i^m f_j M] > 0$ for $m \geq k$.

(iii) $[\text{Cosoc Ind } e_i^{3m-k} M \oplus L(i^n j i^b) : f_i^m f_j M] > 0$ for $0 \leq m < k \leq m + \varepsilon$.

Note that Lemma 6.4 (ii) and (iii) is equivalent to saying that we have

$$[\text{Cosoc Ind } (f_i^{3m-k} e_i^M) \oplus L(i^n j i^b) : f_i^m f_j M] > 0$$

for every $m \geq 0$ with $k \leq m + \varepsilon$.

Keep the setting in Lemma 6.4. Since there are surjections

$$\text{Ind } e_i^M \oplus L(i^{3m-k} e_i^M) \longrightarrow f_i^{3m-k} e_i^M, \quad \text{Ind } L(i^m) \oplus L(j^b) \longrightarrow L(i^n j i^b)$$

by Theorem 3.11 (i) and Lemma 6.1 respectively, we have

$$[\text{Cosoc Ind } (e_i^M \oplus L(i^{3m-k} e_i^M)) : f_i^m f_j M] > 0.$$

By Frobenius reciprocity there is a non-zero injective homomorphism

$$e_i^M \oplus L(i^{3m-k} e_i^M) \hookrightarrow \text{Res}_{n-1, n, m+b} L(i^b) \times f_i^m f_j M.$$

Thus, we also have a non-zero injective homomorphism

$$e_i^M \oplus L(i^{3m-k} e_i^M) \hookrightarrow \text{Res}_{n-1, n, m+b} f_i^m f_j M.$$

Again by Frobenius reciprocity, for every $m \geq 0$ with $k \leq m + \varepsilon$ we have

$$[\text{Res}_{n+b} f_i^m f_j M : f_i^m f_j M] > 0.$$
6.3. Cyclotomic Hecke-Clifford superalgebra.

Definition 6.5. For each positive integral weight $\lambda \in P^+$, we define a polynomial

$$f^\lambda = (X_1 - 1)^{\lambda(h_0)}(X_1 + 1)^{\lambda(h_{l-1})}\prod_{i=1}^{l-2}(X_1^2 - q(i)X_1 + 1)^{\lambda(h_i)}$$

Note that since the canonical central element is $c = h_0 + h_{l-1} + \sum_{i=1}^{l-2}2h_i$, the degree of $f^\lambda$ is $\lambda(c)$. We can easily check that $f^\lambda$ satisfies the assumption in Definition 4.1. From now on, we apply all the constructions in §4 for $R = f^\lambda$ and abbreviate $K(R)$, $e_i^R$, etc. to $K(\lambda)$, $e_i^\lambda$, etc. respectively.

As a Corollary of Lemma 3.21, we have the following characterization of $\text{Im}(\inf^\lambda : B(\lambda) \hookrightarrow B(\infty))$ [BK, Corollary 6.13].

Corollary 6.6. Let $\lambda \in P^+$ and $M \in B(\infty)$. We have $\text{pr}^\lambda M = M$ if and only if $\varepsilon_i^\lambda(M) \leq \lambda(h_i)$ for all $i \in I_q$.

Lemma 6.7. Let $i, j \in I_q$ with $i \neq j$ and $M \in \text{Irr}(H_n^\lambda)$ such that $\varphi_j^\lambda(M) > 0$. Then $\varphi_i^\lambda(f_j^mM) - \varepsilon_i^\lambda(f_j^mM) \leq \varphi_i^\lambda(M) - \varepsilon_i^\lambda(M) - a_{ij}$.

Proof. Put $\varepsilon = \varepsilon_i^\lambda(M) = \varepsilon_i(\inf^\lambda M)$. Apply Lemma 6.4 to $\inf^\lambda M$ and take a pair $(a, b)$ in Lemma 6.4 (i). Since $\varepsilon_i^\lambda(f_j^mM) = \varepsilon_i(\inf^\lambda M) = \varepsilon - a$, it is enough to show that $\varphi_i^\lambda(f_j^mM) \leq \varphi_i^\lambda(M) + b$. Note that $m > \varphi_i^\lambda(M) + b$ implies that $-a_{ij} \leq m + \varepsilon$ by $m + \varepsilon + a_{ij} > \varphi_i^\lambda(M) + (\varepsilon - a)$. Thus, we have

$$\varepsilon_i^\lambda(f_j^mM) \geq \varepsilon_i^\lambda(f_j^{m-b} \inf^\lambda M) > \lambda(h_i).$$

Here the 1st inequality follows from (16) and the 2nd inequality follows from Corollary 6.6 and $\sigma$-version of Lemma 3.22 (ii). Again by Corollary 6.6, we have $\text{pr}^\lambda f_i^m \inf^\lambda M = 0$ for each $m > \varphi_i^\lambda(M) + b$, i.e., $\varphi_i^\lambda(f_j^mM) \leq \varphi_i^\lambda(M) + b$. \qed

Theorem 6.8. For any $M \in \text{Irr}(H_n^\lambda)$ and $i \in I_q$, we have $\varphi_i^\lambda(M) - \varepsilon_i^\lambda(M) = \langle h_i, \lambda + \text{wt}(\inf^\lambda M) \rangle$.

Proof. By Corollary 6.6, we have $\varphi_i^\lambda(1_\lambda) = \lambda(h_i)$. Combined with the obvious $\varepsilon_i^\lambda(1_\lambda) = 0$ and Lemma 6.7 inductively, we have $\varphi_i^\lambda(M) - \varepsilon_i^\lambda(M) \leq \langle h_i, \lambda + \text{wt}(\inf^\lambda M) \rangle$. Thus, it is enough to show that

$$(\varphi_0^\lambda(M) - \varepsilon_0^\lambda(M)) + (\varphi_{l-1}^\lambda(M) - \varepsilon_{l-1}^\lambda(M)) + \sum_{i=1}^{l-2}2(\varphi_i^\lambda(M) - \varepsilon_i^\lambda(M)) = \lambda(h_i),$$

which is the same thing as Corollary 4.12. \qed

Corollary 6.9. The 6-tuple $(B(\lambda), \text{wt}^\lambda, \{\varepsilon_i^\lambda\}_{i \in I_q}, \{\varphi_i^\lambda\}_{i \in I_q}, \{\check{e}_i^\lambda\}_{i \in I_q}, \{\check{f}_i^\lambda\}_{i \in I_q})$ is a $\mathfrak{g}$-crystal by defining $\text{wt}^\lambda(M) = \lambda + \text{wt}(\inf^\lambda M)$ for $M \in B(\lambda)$.

6.4. Lie-theoretic descriptions of $B(\infty)$ and $B(\lambda)$.

Theorem 6.10. For each $i \in I_q$, the map

$$\Psi_i : B(\infty) \rightarrow B(\infty) \otimes B_i, \quad [M] \mapsto [(\check{e}_i^\lambda)^\ast(M)M] \otimes b_i(-\varepsilon_i^\lambda(M))$$

is a crystal embedding.
Proof. We prove $\Psi_i(\bar{f}_i M) = \bar{f}_i \Psi_i(M)$ for any $i, j \in I_q$ and $|M| \in B(\infty)$. In case $i \neq j$, this follows from $\sigma$-versions of Lemma 3.22 (ii) and (iii).

Let us assume $i = j$ and put $a = \varepsilon_i^*(M)$. By Definition 2.3,
\[
\bar{f}_i \Psi_i(M) = \begin{cases} 
[f_i(\tilde{c}_i^a) M] \otimes b_i(-a) & \text{if } \varepsilon_i(\tilde{c}_i^a M) + a + \langle h_i, \text{wt}(M) \rangle > 0, \\
[(\tilde{c}_i^a) M] \otimes b_i(-a - 1) & \text{if } \varepsilon_i(\tilde{c}_i^a M) + a + \langle h_i, \text{wt}(M) \rangle \leq 0.
\end{cases}
\]

Comparing with $\sigma$-versions of Lemma 3.22 (i), (iii) and (iv), it is enough to show the following.
\[
\varepsilon_i^*(\bar{f}_i M) = \begin{cases} 
a & \text{if } \varepsilon_i(\tilde{c}_i^a M) + a + \langle h_i, \text{wt}(M) \rangle > 0, \\
a + 1 & \text{if } \varepsilon_i(\tilde{c}_i^a M) + a + \langle h_i, \text{wt}(M) \rangle \leq 0.
\end{cases}
\]

Consider the case $\varepsilon_i(\tilde{c}_i^a M) + a + \langle h_i, \text{wt}(M) \rangle > 0$ and take $\lambda_1 \in P^+$ such that $\lambda_1(h_j)$ is big enough for any $j \neq i$ and $\lambda_1(h_i) = a$. Note that $M$ can be regarded as an element of $B(\lambda_1)$ by Corollary 6.6. By Theorem 6.8, we have
\[
\varphi_i^{\lambda_1}(pr^{\lambda_1} M) = \varepsilon_i(\tilde{c}_i^a M) + \langle h_i, \lambda_1 + \text{wt}(M) \rangle = \varepsilon_i(M) + a + \langle h_i, \text{wt}(M) \rangle \\
\geq \varepsilon_i(\tilde{c}_i^a M) + a + \langle h_i, \text{wt}(M) \rangle \geq 1.
\]

Thus, we have $\varepsilon_i^*(\bar{f}_i M) \leq \lambda_1(h_i) = a$ by Corollary 6.6. It implies $\varepsilon_i^*(\bar{f}_i M) = a$ by $\sigma$-version of Lemma 3.22 (i).

Finally, consider the case $\varepsilon_i(\tilde{c}_i^a M) + a + \langle h_i, \text{wt}(M) \rangle \leq 0$, i.e.,
\[
\varepsilon_i(\tilde{c}_i^a M^\sigma) + a + \langle h_i, \text{wt}(M^\sigma) \rangle = \varepsilon_i(\tilde{c}_i^a M^\sigma) - a + \langle h_i, \text{wt}(\tilde{c}_i^a M^\sigma) \rangle \leq 0.
\]

Take $\lambda_2 \in P^+$ such that $\lambda_2(h_j)$ is big enough for any $j \neq i$ and $\lambda_2(h_i) = r + \varepsilon_i^*(\tilde{c}_i^a M^\sigma)$ for $r = a - \varepsilon_i^*(\tilde{c}_i^a M^\sigma) - \langle h_i, \text{wt}(\tilde{c}_i^a M^\sigma) \rangle \geq 0$. Again $\tilde{c}_i^a M^\sigma$ can be regarded as an element of $B(\lambda_2)$ and we have
\[
\varphi_i^{\lambda_2}(pr^{\lambda_2}(\tilde{c}_i^a M^\sigma)) = \varepsilon_i(\tilde{c}_i^a M^\sigma) + \langle h_i, \lambda_2 + \text{wt}(\tilde{c}_i^a M^\sigma) \rangle \\
= \langle h_i, \lambda_2 + \text{wt}(\tilde{c}_i^a M^\sigma) \rangle = \langle h_i, \lambda_2 + \text{wt}(\tilde{c}_i^a M^\sigma) \rangle = \langle h_i, \lambda_2 + \text{wt}(\tilde{c}_i^a M^\sigma) \rangle
\]

by Theorem 6.8. Combined with Corollary 6.6, it implies
\[
\begin{cases} 
\varepsilon_i(M) = \varepsilon_i^*(M^\sigma) = \varepsilon_i^*(\bar{f}_i M^\sigma) \leq \lambda_2(h_i), \\
\varepsilon_i(\bar{f}_i M) = \varepsilon_i^*(\bar{f}_i M^\sigma) = \varepsilon_i^*(\bar{f}_i M^\sigma) \geq \lambda_2(h_i) + 1.
\end{cases}
\]

Thus, by Lemma 3.22 (i), we have
\[
\varepsilon_i(M) = \lambda_2(h_i) = a - \langle h_i, \text{wt}(\tilde{c}_i^a M^\sigma) \rangle = -a - \langle h_i, \text{wt}(M) \rangle.
\]

Take $\lambda_3 \in P^+$ such that $\lambda_3(h_j)$ is big enough for any $j \neq i$ and $\lambda_3(h_i) = a$. Again $M$ can be regarded as an element of $B(\lambda_3)$ and we have
\[
\varphi_i^{\lambda_3}(pr^{\lambda_3} M) = \varepsilon_i(\tilde{c}_i^a M) + \langle h_i, \lambda_3 + \text{wt}(M) \rangle = \varepsilon_i(M) + a + \langle h_i, \text{wt}(M) \rangle = 0
\]

by Theorem 6.8. Thus, we have $\varepsilon_i^*(\bar{f}_i M) > \lambda_3(h_i) = a$ by Corollary 6.6. It implies $\varepsilon_i^*(\bar{f}_i M) = a + 1$ by $\sigma$-version of Lemma 3.22 (i). \hfill \Box

Corollary 6.11. The $g$-crystal $B(\infty)$ is isomorphic to $B(\infty)$.

Proof. Apply Proposition 2.7 to $B = B(\infty)$ and $b_0 = [1]$.

Corollary 6.12. For each $\lambda \in P^+$, the $g$-crystal $B(\lambda)$ is isomorphic to $B(\lambda)$. 

Proof. Apply Proposition 2.8 to \( B = B(\lambda), b_\lambda = [1_\lambda] \) and a map
\[
\Phi : B(\infty) \otimes T_\lambda \longrightarrow B(\lambda), \quad [M] \otimes t_\lambda \longmapsto [pr^M M].
\]
The latter is an \( f \)-strict crystal morphism since \( f^\lambda_i = pr^\lambda \circ f_i \circ inf^\lambda \) by Definition 4.3 and \( f_i M \neq 0 \) for any \( M \in B(\infty) \) by Definition 3.12. \( \square \)

6.5. Lie-theoretic descriptions of \( K(\infty)_Q \) and \( K(\lambda)_Q \).

**Theorem 6.13.** For each \( \lambda \in P^+ \), we have the following.

(i) \( K(\lambda)_Q \) has a left \( U_Q(= (e_i, f_i, h_i \mid (2))_{i \in I_q}) \)-module structure by
\[
e_i [M] = [e_i^h M], \quad f_i [M] = [f_i^h M], \quad h_i [M] = \langle h_i, \text{wt}^h(M) \rangle [M],
\]
and it is isomorphic to the integrable highest weight \( U_Q \)-module of highest weight \( \lambda \) with highest weight vector \( [1_\lambda] \).

(ii) The symmetric non-degenerate bilinear form \( \langle , \rangle_\lambda \) on \( K(\lambda)_Q \) in §4.6 coincides with the usual Shapovalov form satisfying \( \langle [1_\lambda], [1_\lambda] \rangle_\lambda = 1 \) under the above identification.

(iii) \( \bigoplus_{n \geq 0} K_0(\text{Proj} \mathcal{H}_n^\lambda) \cong K(\lambda)^* \subseteq K(\lambda) \) are two integral lattices of \( K(\lambda)_Q \) containing \( [1_\lambda] \) with \( K(\lambda)^* = U_Q^- [1_\lambda] \) and \( K(\lambda) \) being its dual under the Shapovalov form.

**Proof.** By §4.4 and Corollary 6.3, the operators \( \{ e_i^h : K(\lambda) \to K(\lambda) | i \in I_q \} \) satisfy the Serre relations (15). It implies that the operators \( \{ f_i^h : (K(\lambda)^* \to K(\lambda)^*) | i \in I_q \} \) satisfy the Serre relations by Lemma 4.13. Thus, both operators satisfy the Serre relations on \( K(\lambda)_Q \) by Theorem 4.16. By Corollary 4.11 and Theorem 6.8, we have \( [e_i^h, f_i^h] = \delta_{i,j} h_i \) as operators on \( K(\lambda)_Q \). Since other relations of (2) are immediately deduced from the definition of the action of \( h_i \), \( K(\lambda)_Q \) has a left \( U_Q \)-module structure by the above actions. By Corollary 4.10, \( e_i^h \) and \( f_i^h \) are both nilpotent operators on \( K(\lambda)_Q \). Since the action of \( \{ h_i \mid i \in I_q \} \) is diagonalized with finite-dimensional weight spaces by the definition, \( K(\lambda)_Q \) is an integrable \( U_Q \)-module. By Theorem 4.18, \( K_0(\lambda)_Q \cong U_Q^- [1_\lambda] \) is a highest weight \( U_Q^- \)-module of highest weight \( \lambda \) with highest weight vector \( [1_\lambda] \). Now (ii) is a direct consequence of Lemma 4.13 and Corollary 4.19 and (iii) is a restatement of Theorem 4.16 and Corollary 4.18. \( \square \)

**Theorem 6.14.** There exists a graded \( \mathbb{Z} \)-Hopf algebra isomorphism \( U_\mathbb{Z}^+ \overset{\sim}{\longrightarrow} K(\infty)^* \) which takes \( e_i^{(r)} \) to \( \delta_{L(i),r} \) for each \( i \in I_q \) and \( r \geq 0 \).

**Proof.** By §3.9 and Corollary 6.3, there exists a graded \( \mathbb{Z} \)-algebra map \( \pi : U_\mathbb{Z}^+ \to K(\infty)^* \) which takes \( e_i^{(r)} \) to \( \delta_{L(i),r} \) for each \( i \in I_q \) and \( r \geq 0 \). It is easily checked that it is a graded \( \mathbb{Z} \)-coalgebra map since \( \delta_{L(i)} \) is mapped to \( \delta_{L(i)} \otimes 1 + 1 \otimes \delta_{L(i)} \) via the comultiplication of \( K(\infty)^* \). Thus, \( \pi \) is a graded \( \mathbb{Z} \)-Hopf algebra map by [Swe, Lemma 4.0.4].

It is enough to show that \( \pi \) is an isomorphism as graded \( \mathbb{Z} \)-modules. By Corollary 6.6, we have a natural isomorphism \( \lim_{\lambda \in P^+} K_0(\mathcal{H}_n^\lambda \text{-smod}) \cong K_0(\text{Rep} \mathcal{H}_n) = K_0(\text{Proj} \mathcal{H}_n) \). Combined with Theorem 4.18, it gives us
\[
\text{Hom}_\mathbb{Z}(K_0(\text{Rep} \mathcal{H}_n), \mathbb{Z}) \cong \lim_{\lambda \in P^+} \text{Hom}_\mathbb{Z}(K_0(\mathcal{H}_n^\lambda \text{-smod}), \mathbb{Z}) \cong \lim_{\lambda \in P^+} K_0(\text{Proj} \mathcal{H}_n^\lambda) \cong \lim_{\lambda \in P^+} (U_\mathbb{Z}^-)_n[1_\lambda] \overset{\sim}{\longrightarrow} (U_\mathbb{Z}^-)_n,
\]
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Theorem 6.13. By tracing this isomorphism, we see that the graded easily from the fact (U(Z)_n) is an isomorphism K(\lambda)_Q \cong U_Q / \sum_{i \in I} U_Q f_i^{(h_i)+1} as shown in Theorem 6.13. By tracing this isomorphism, we see that the graded \mathbb{Z}-module isomorphism K(\infty)^* \cong U(Z)_n is given by the composite

$$U(Z)_n \xrightarrow{\sim} U(Z)_n^+ \xrightarrow{\pi} K(\infty)^*$$

where U(Z)_n \xrightarrow{\sim} U(Z)_n^+ is the algebra anti-isomorphism given by f_i \mapsto e_i for all i \in I. See also the proof of [BK, Theorem 7.17] in [BK', §3].

References

[Ari] S. Ariki, On the decomposition numbers of the Hecke algebra of G(m, 1, n), J. Math. Kyoto Univ. 36 (1996), 789–808.
[BK] J. Brundan and A. Kleshchev, Hecke-Clifford superalgebras, crystals of type A_{2l} and modular branching rules for S_n, Represent. Theory 5 (2001), 317–403.
[BK'] J. Brundan and A. Kleshchev, Corrigenda to ‘Hecke-Clifford superalgebras, crystals of type A_{2l} and modular branching rules for S_n’, available in http://darkwing.oregon.edu/~brundan/papers/.
[Br1] J. Brundan, Modular branching rules and the Mullineux map for Hecke algebras of type A, Proc. London Math. Soc. 77 (1998), 551-581.
[Br2] J. Brundan, Centers of degenerate cyclotomic Hecke algebras and parabolic category O, Represent. Theory 12 (2008), 236–259.
[Gro] I. Grojnowski, Affine \mathfrak{sl}_n controls the modular representation theory of the symmetric group and related Hecke algebras, math.RT/9907129.
[GV] I. Grojnowski and M. Vazirani, Strong multiplicity one theorems for affine Hecke algebras of type A, Transform. Groups 6 (2001), 143–155.
[HKOTT] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Z. Tsuboi, Paths, crystals and fermionic formulae, in “MathPhys Odyssey 2001-Integrable Models and Beyond In Honor of Barry M.McCoy”, edited by M. Kashiwara and T. Miwa, Birkhäuser (2002) 205–272.
[HKOTY] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, Remarks on fermionic formula, Contemp. Math., 248 (1999), 243–291.
[Jam] G. James, The representation theory of the symmetric groups, Lecture Notes in Mathematics 682, Springer, 1978.
[JN] A. Jones and M. Nazarov, Affine Sergeev algebra and q-analogues of the Young symmetrizers for projective representations of the symmetric group, Proc. London Math. Soc. 78 (1999), 481–512.
[Kac] V. G. Kac. Infinite dimensional Lie algebras. Cambridge University Press, 1990.
[Kan] S.-J. Kang, Crystal bases for quantum affine algebras and combinatorics of Young walls, Proc. London Math. Soc. 86 (2003), 29–69.
[Kas] M. Kashiwara, Bases cristallines des groupes quantiques, Cours Spéc., vol. 9, Soc. Math. France, 2002.
[KI] A. Kleshchev, Branching rules for modular representations of symmetric groups II, J. Reine. Angew. Math. 459 (1995), 163–212.
[KI2] A. Kleshchev, Linear and Projective Representations of Symmetric Groups, Cambridge University Press, 2005.
[KMN1] S.-J. Kang, M. Kashiwara, K. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Affine crystals and vertex models, Internat. J. Modern Phys. A 7, Suppl. 1A (1992), 449–484.
[KMN2] S.-J. Kang, M. Kashiwara, K. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. 68 (1992), 499–607.
[KNO] M. Kashiwara, T. Nakashima and M. Okado, Affine geometric crystals and limit of perfect crystals, Trans. Amer. Math. Soc. 360 (2008), 3645–3686.
[KS] M. Kashiwara and Y. Saito, Geometric construction of crystal bases, Duke Math. J. 89 (1997), 9–36.
[LLT] A. Lascoux, B. Leclerc and J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Phys. 181 (1996), 205–263.
[LM] S. Lyle and A. Mathas, *Blocks of cyclotomic Hecke algebras*, Adv. Math. 216 (2007), 854–878.

[MM] K. Misra and T. Miwa, *Crystal base for the basic representation of $U_q(\mathfrak{sl}(n))*, Comm. Math. Phys. 134 (1990), 79–88.

[Mor] A.O. Morris, *The spin representation of the symmetric group*, Canad. J. Math. 17 (1965), 543–549.

[MY] A.O. Morris and A.K. Yaseen, *Some combinatorial results involving shifted Young diagrams*, Math. Proc. Cambridge Philos. Soc. 99 (1986), 23–31.

[Naz] M. Nazarov, *Young’s symmetrizers for projective representations of the symmetric group*, Adv. Math. 127 (1997), 190–257.

[Ols] G. Olshanski, *Quantized universal enveloping superalgebra of type $Q$ and a super-extension of the Hecke algebra*, Lett. Math. Phys. 24 (1992), 93–102.

[Ruf] O. Ruff, *Centers of cyclotomic Sergeev superalgebras*, arXiv:0811.3991.

[Sai] Y. Saito, *Crystal bases and quiver varieties*, Math. Ann. 324 (2002), 675–688.

[Swe] M. Sweedler, *Hopf algebras*, W. A. Benjamin, Inc., New York, 1969.

[VV] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. 100 (1999), 267–297.

[Yvo] X. Yvonne, *A conjecture for $q$-decomposition matrices of cyclotomic $\mathfrak{g}$-Schur algebras*, J. Algebra 304 (2006), 419–456.

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