Edge-colouring and total-colouring chordless graphs

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May 14, 2013

Abstract

A graph $G$ is chordless if no cycle in $G$ has a chord. In the present work we investigate the chromatic index and total chromatic number of chordless graphs. We describe a known decomposition result for chordless graphs and use it to establish that every chordless graph of maximum degree $\Delta \geq 3$ has chromatic index $\Delta$ and total chromatic number $\Delta + 1$. The proofs are algorithmic in the sense that we actually output an optimal colouring of a graph instance in polynomial time.

Keywords: chordless graph, graph decomposition, edge-colouring, total-colouring.

1 Introduction

Let $G = (V, E)$ be a simple graph. The degree of a vertex $v$ in $G$ is denoted by $\text{deg}_G(v)$ while the maximum degree in $G$ is denoted by $\Delta(G)$.

Edge-colouring

An edge-colouring of $G$ is a function $\pi : E \to C$ such that no two adjacent edges receive the same colour $c \in C$. If $C = \{1, 2, ..., k\}$, we say that $\pi$ is a $k$-edge-colouring. The chromatic index of $G$ is the least $k$ for which $G$ has a
k-edge-colouring. The chromatic index of any graph $G$ is obviously at least $\Delta(G)$. Vizing’s theorem \cite{34} states that every graph $G$, is $(\Delta(G) + 1)$-edge-colourable.

It is NP-complete to determine whether a graph $G$ is $\Delta(G)$-edge-colourable \cite{17, 20}. The edge-colouring problem remains NP-complete for perfect graphs \cite{7}. Graph classes for which edge-colouring is polynomially solvable include bipartite graphs \cite{19}, split-indifference graphs \cite{28}, series-parallel graphs (hence outerplanar) \cite{19}, and $k$-outerplanar graphs, for $k \geq 1$ \cite{4}. The complexity of edge-colouring is unknown for several well-studied strongly structured graph classes, for which only partial results have been reported, such as cographs \cite{2}, join graphs \cite{13, 23}, planar graphs \cite{31}, chordal graphs, and several subclasses of chordal graphs such as indifference graphs \cite{12}, split graphs \cite{10} and interval graphs \cite{5}.

**Total-colouring**

An *element* of a graph is either a vertex or an edge. Two elements are *adjacent* if they are either adjacent vertices, adjacent edges or a vertex incident to an edge. A *total-colouring* of $G$ is a function $\pi : V \cup E \rightarrow C$ such that no two adjacent elements receive the same colour $c \in C$. If $C = \{1, 2, ..., k\}$, we say that $\pi$ is a $k$-total-colouring. The *total chromatic number* of $G$ is the least $k$ for which $G$ has a $k$-total-colouring. The total chromatic number of any graph $G$ is obviously at least $\Delta(G) + 1$. The *Total Colouring Conjecture (TCC)*, posed independently by Behzad \cite{3} and Vizing \cite{34}, states that every simple graph $G$ would be $(\Delta(G) + 2)$-total-colourable and it is a challenging open problem in Graph Theory.

It is NP-complete to determine whether a graph $G$ is $(\Delta(G) + 1)$-total-colourable \cite{30}. The total-colouring problem remains NP-complete when restricted to $r$-regular bipartite inputs \cite{27}, for each fixed $r \geq 3$. The total-colouring problem is known to be polynomial — and the TCC is valid — for few very restricted graph classes, such as cycles, complete graphs, complete bipartite graphs \cite{38}, grids \cite{8}, series-parallel graphs \cite{15, 36, 37}, and split-indifference graphs \cite{9}. The computational complexity of the total-colouring problem is unknown for several important and well-studied graph classes. The complexity of total-colouring planar graphs is unknown; in fact, even the TCC has not yet been settled for planar graphs \cite{35}. The complexity of total-colouring is open for the class of chordal graphs, and the partial results for the related classes of interval graphs \cite{5}, split graphs \cite{10} and dually chordal graphs \cite{11} expose the interest in the total-colouring problem restricted to chordal graphs. Another class for which the complexity of
total-colouring is unknown is the class of join graphs; the results found in
the literature consider very restricted subclasses of join graphs [16, 22].

Chordless graphs

A cycle $C$ in a graph $G$ is a sequence of vertices $v_1v_2 \ldots v_nv_1$, that are distinct except for the first and the last vertex, such that for $i = 1, \ldots, n-1$, $v_iv_{i+1}$ is
an edge and $v_nv_1$ is an edge — we call these edges the edges of $C$. An edge of
$G$ with both endvertices in a cycle $C$ is called a chord of $C$ if it is not an edge
of $C$. A chordless graph is a graph whose cycles are all chordless. Chordless
graphs were first studied independently by Dirac [14] and Plummer [29]. In particular, they give several characterizations of these graphs, prove that they are 3-vertex-colourable and that they are 2-connected if and only if they are minimally 2-connected (meaning that the removal of any edge yields a non 2-connected graph). Lévêque, Maffray and Trotignon [21] studied the
class of chordless graphs independently. The motivation was to compute the chromatic number of line graphs with no induced generalized wheels,
where a generalized wheel is a graph made of a cycle together with a vertex
that has at least 3 neighbours on the cycle. They observed that these line
graphs are the line graphs of chordless graphs of maximum degree at most 3, and they prove that chordless graphs of maximum degree at most 3 are
3-edge-colourable. On the way to this result, they prove a decomposition
theorem for chordless graphs that is seemingly independent of the results in [14, 29].

There is a second motivation for the study of the chromatic index of
chordless graphs: they are a subclass of the class of unichord-free graphs
which are graphs that do not contain, as induced subgraph, a cycle with
unique chord. Trotignon and Vušković [33] proved a decomposition theo-
rem for these graphs, that in fact contains implicitly the decomposition of
chordless graphs from [21]. Both edge-colouring and total-colouring prob-
lems are NP-complete problems when restricted to unichord-free graphs, as
proved by Machado, de Figueiredo and Vušković [24] and by Machado and de
Figueiredo [25]; hence, it is of interest to determine subclasses of unichord-
free graphs for which edge-colouring and total-colouring are polynomial.

Our main result is the following.

**Theorem 1.** Let $G$ be a chordless graph of maximum degree at least 3.
Then $G$ is $\Delta(G)$-edge-colourable and $(\Delta(G) + 1)$-total-colourable. Moreover, these colourings may be obtained in time $O(|V(G)|^3|E(G)|)$.  

Note that the edge-colouring of chordless graphs with maximum degree 3
was already established in [21]. Theorem 1 relies on the decomposition theorem from [21]. We emphasize, however, that there are differences between the proof of [21] and ours. Most remarkably, since [21] deals only with the case $\Delta = 3$, any cutset of two non-adjacent vertices actually determines a cutset of two non-adjacent edges. Such an edge-cutset is used to construct a natural induction on the decomposition blocks. Our proof uses a different strategy based on the existence of an extreme decomposition tree, in which one of the decomposition blocks is 2-sparse. This leads to our third and main motivation for this work: to understand how such kind of decomposition results, which are classically applied to the design of vertex colouring algorithms, can be useful in the development of algorithms for other colouring problems, in particular edge-colouring [24] and total-colouring [25, 26] — the present work is successful in the sense that our chosen class of chordless graphs showed to be fruitful for the development of polynomial-time edge-colouring and total-colouring algorithms.

Section 2 reviews the decomposition result for chordless graphs established in [21]. Section 3 gives several results for a subclass of chordless graphs, the so-called 2-sparse graphs. Section 4 gives the proof of Theorem 1.

2 Structure of chordless graphs

The goal of the present section is to review the structure theorem from [21]. A graph is 2-sparse if every edge is incident to at least one vertex of degree at most 2. A 2-sparse graph is chordless because any chord of a cycle is an edge between two vertices of degree at least three.

A proper 2-cutset of a connected graph $G = (V, E)$ is a pair of non-adjacent vertices $a, b$ such that $V$ can be partitioned into non-empty sets $X$, $Y$ and $\{a, b\}$ so that: there is no edge between $X$ and $Y$; and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an $ab$-path and neither of $G[X \cup \{a, b\}]$ nor $G[Y \cup \{a, b\}]$ is a chordless path. We say that $(X, Y, a, b)$ is a split of this proper 2-cutset.

The decomposition result stated in Theorem 2 is implicit in [33], explicit in [21], and for the sake of completeness, we include a short proof from [1].

**Theorem 2** (Léveque, Maffray and Trotignon [21]). If $G$ is a 2-connected chordless graph, then either $G$ is 2-sparse or $G$ admits a proper 2-cutset.

**Proof** — If $G$ is not 2-sparse, then it has an edge $e = uv$ such that $u$ and $v$ have both degree at least 3. If $G \setminus e$ is 2-connected, then it contains a cycle
through $u$ and $v$, hence $uv$ is a chord in $G$, a contradiction. It follows that $G \setminus e$ is disconnected or has a cutvertex.

If $G \setminus e$ is disconnected, then $u$ and $v$ are cutvertices of $G$, a contradiction. So $G \setminus e$ is connected and has a cutvertex $w$. Since $G$ is 2-connected, the graph $(G \setminus e) \setminus w$ has exactly two connected components $G_u$ and $G_v$, containing $u$ and $v$ respectively, and $V = V(G_u) \cup V(G_v) \cup \{w\}$. Let $u' \notin \{v, w\}$ be a neighbour of $u$ ($u'$ exists since $u$ has degree at least 3). So, $u' \in V(G_u)$.

In $G$, vertex $u$ is not a cutvertex, so there is a path $P_u$ from $u'$ to $w$ whose interior is in $G_u$. Together with a path $P_v$ from $v$ to $w$ with interior in $G_v$, $P_u, uu'$ and $e$ form a cycle, so $uw \notin E(G)$ for otherwise $uw$ would be a chord of this cycle. Also, since $u$ is of degree at least 3 and $G$ is chordless, $u$ has a neighbour in $G_u \setminus P_u$. Hence $(V(G_u) \setminus \{u\}, V(G_v), u, w)$ is a split of a proper 2-cutset of $G$.

The blocks $G_X$ and $G_Y$ of a graph $G$ with respect to a proper 2-cutset with split $(X,Y,a,b)$ are defined as follows. Block $G_X$ (resp. $G_Y$) is the graph obtained by taking $G[X \cup \{a,b\}]$ (resp. $G[Y \cup \{a,b\}]$) and adding a new vertex $w$ adjacent to $a, b$. Vertex $w$ is a called the marker vertex of the block $G_X$ (resp. $G_Y$). Clearly, blocks $G_X$ and $G_Y$ of a 2-connected chordless graph are 2-connected chordless graphs as well.

Theorem 3 states that a not 2-sparse chordless graph has an extremal decomposition, that is, a decomposition in which (at least) one of the blocks is 2-sparse.

**Theorem 3.** Let $G$ be a 2-connected not 2-sparse chordless graph. Let $(X,Y,a,b)$ be a split of a proper 2-cutset of $G$ such that $|X|$ is minimum among all possible such splits. Then $a$ and $b$ both have at least two neighbours in $X$, and $G_X$ is 2-sparse.

**Proof** — First, we show that $a$ and $b$ both have at least two neighbours in $X$. Suppose that one of $a, b$, say $a$, has a unique neighbour $a' \in X$. We claim that $a'$ is not adjacent to $b$. For otherwise, since $G[X \cup \{a, b\}]$ does not induce a path and $G$ is 2-connected, there is a path in $G[X \cup \{b\}] \setminus a'b$ from $b$ to $a'$, that together with the edge $a'a$ and a path from $a$ to $b$ with interior in $Y$ forms a cycle with a chord: $a'b$. So, $a'$ is not adjacent to $b$. Hence, by replacing $a$ by $a'$, we obtain a proper 2-cutset that contradicts the minimality of $X$.

Let $w$ be the marker vertex of $G_X$. Suppose that 2-connected chordless graph $G_X$ is not 2-sparse. According to Theorem 2 $G_X$ has a proper 2-cutset with split $(X_1, X_2, u, v)$. Choose it so that $u$ and $v$ both have degree at least 3 (this is possible as explained at the beginning of the proof). Note
that \( w \notin \{u, v\} \). W.l.o.g. \( w \in X_1 \). But then \( \{a, b\} \subseteq X_1 \cup \{u, v\} \) and hence \( (X_2, Y \cup X_1 \setminus \{w\}, u, v) \) is a proper 2-cutset of \( G \), contradicting the minimality of \(|X|\). Therefore, \( G_X \) is 2-sparse.

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3 2-Sparse graphs

A 2-sparse graph is easily shown to be \( \Delta(G) \)-edge-colourable and \( (\Delta(G) + 1) \)-total-colourable (see below). But we need slightly more for the induction in our main proof.

**Theorem 4** (König). Any bipartite graph \( G \) is \( \Delta(G) \)-edge-colourable.

**Theorem 5** (Borodin, Kostochka, and Woodall [6]). Let \( G = (V, E) \) be a bipartite graph and suppose that a list \( L_{uv} \) of colours is associated to each edge \( uv \in E \). If for each edge \( uv \in E \), \( |L_{uv}| \geq \max\{\deg_G(u), \deg_G(v)\} \), then there is an edge-colouring \( \pi \) of \( G \) such that, for each edge \( uv \in E \), \( \pi(uv) \in L_{uv} \).

The proof of the preceding theorem in [6] is quite involved. It relies on counting arguments in such a way that it does not imply clearly an algorithm outputting the colouring whose existence is proved. We propose the following simple weakening stated in Lemma 6 which is enough for our purpose and whose proof is clearly algorithmic.

Let \( V_{\geq 3}(G) \) be the set of vertices with degree at least 3 in \( G \). Note that if \( G \) is 2-sparse then \( V_{\geq 3}(G) \) is a stable (i.e., independent, possibly empty) set. If \( S \) is a stable set in \( V(G) \), let \( B(G, S) \) denote the bipartite subgraph of \( G \) whose vertices are the vertices of \( S \) and their neighbours, and whose edges are the edges of \( G \) that have one end in \( S \).

**Lemma 6.** Let \( G = (V, E) \) be a bipartite graph with a bipartition \( (X, Y) \) such that \( V_{\geq 3}(G) \subseteq X \). Suppose that a list \( L_{uv} \) of colours is associated to each edge \( uv \in E \) such that for all \( u \in X \), all edges incident to \( u \) receive the same list. If for each edge \( uv \in E \), \( |L_{uv}| \geq \max\{\deg_G(u), \deg_G(v)\} \), then there is an edge-colouring \( \pi \) of \( G \) such that, for each edge \( uv \in E \), \( \pi(uv) \in L_{uv} \).

**Proof** — We prove the theorem by induction on the number of edges. We may assume that \( G \) is connected for otherwise we work on components separately. If all edges receive the same list (in particular when there are no edges at all) then the result follows from Theorem 4. So, let \( xy \) and
x'y be adjacent edges with different lists. Note that from our assumptions, 
\( x, x' \in X \) and \( y \in Y \). W.l.o.g. colour 1 is available for \( xy \) and not for \( x'y \). 
Let \( P = v_1, \ldots, v_k \) be a sequence of vertices such that \( v_1 = y, v_2 = x, \) vertices \( v_1, \ldots, v_{k-1} \) are distinct, and for all \( 1 \leq i < k \), \( v_iv_{i+1} \in E(G) \) and \( \deg_G(v_i) = 2 \). Suppose that \( P \) is maximal w.r.t. these properties. Note that \( v_k \) has degree 2 if and only if \( G \) is a cycle.

We colour \( v_1v_2 \) with Colour 1 and colour greedily the edges \( v_2v_3, \ldots, v_{k-1}v_k \) with some available colour, that is for each edge a colour in the list of the edge, not used for the preceding edge. Note that this gives a colouring of the edges of \( P \), even if \( v_{k-1}v_k = x'y \) (this happens when \( G \) is a cycle), because in this case, the colour used for \( v_1v_2 = xy \) is not in the list of \( v_{k-1}v_k = x'y \). We delete all the edges of \( P \) and, if \( v_k \) has degree at least 3, we remove from all lists of edges incident to it, the colour used for \( v_{k-1}v_k \). We edge-colour what remains by the induction hypothesis.

Lemma 6 extends to 2-sparse graphs in the following way.

**Lemma 7.** Let \( G = (V,E) \) be a 2-sparse graph and suppose that a list \( L_{uv} \) of colours is associated to each edge \( uv \in E \). Let \( S \) be a stable set of \( G \) such that \( V \geq 3(G) \subset S \). Suppose that for every vertex \( u \in S \), all edges incident to \( u \) receive the same list. If for each edge \( uv \in E \), \( |L_{uv}| \geq \max\{\deg_G(u),\deg_G(v)\} \) and for each edge \( uv \in E \) with no end in \( S \), \( |L_{uv}| \geq 3 \), then there is an edge-colouring \( \pi \) of \( G \) such that, for each edge \( uv \in E \), \( \pi(uv) \in L_{uv} \).

**Proof —** Let \( G' = B(G,S) \). Note that \( G' \subset G \), so each edge of \( G' \) has a list of colours. We edge-colour \( G' \) by Lemma 6. It remains to colour the edges in \( E(G) \setminus E(G') \). Each of them has a list of colours of size 3 and is adjacent to at most two edges of \( G \). So, they can be coloured greedily.

**Lemma 8.** A 2-sparse graph \( G \) of maximum degree \( \Delta \geq 3 \) is \( \Delta(G) \)-edge-colourable.

**Proof —** We associate to each edge the list \( \{1,\ldots,\Delta+1\} \) of colours and apply Lemma 7.

From here on, we study total colouring of 2-sparse graphs.

**Lemma 9.** Let \( G = (V,E) \) be a 2-sparse graph of maximum degree \( \Delta \geq 4 \) and let \( S \) be a stable set of \( G \) such that \( V_{\geq 3}(G) \subset S \). Suppose that each vertex of \( S \) is precoloured with some colour from \( \{1,\ldots,\Delta+1\} \). Suppose that a list \( L_{uv} \) of colours from \( \{1,2,\ldots,\Delta+1\} \) is associated to every edge
uv with one end in S, and for every vertex \( u \in S \), all edges incident to \( u \) receive the same list. Suppose that for each edge uv with one end \( u \in S \), \(|L_{uv}| \geq \max\{\deg_G(u), \deg_G(v)\} \) and \( u \) is not precoloured with a colour from \( L_{uv} \).

Then there is a total-colouring \( \pi \) of \( G \) with colours in \{1, \ldots, \Delta + 1\} that extends the precolouring of the vertices of \( S \) and such that \( \pi(uv) \in L_{uv} \) for each edge \( uv \) with one end in \( S \).

**Proof** — Let \( G' = B(G, S) \subset G \). We edge-colour \( G' \) by Lemma 6. It remains to colour edges in \( E(G) \setminus E(G') \) and the vertices of degree at most 2 of \( G \). Each of them is an element of \( G \) incident to at most four elements of \( G \) and at least five colours are available. So, they can be coloured greedily.

The following shows how a total precolouring of the ends of a path can be extended to the path when only four colours are available.

**Lemma 10.** Let \( k \geq 3 \) be an integer and \( P = p_1 \ldots p_k \) a path. Suppose that \( p_1, p_1p_2, p_{k-1}p_k \) and \( p_k \) are total-precoloured in such a way that \( p_1 \) and \( p_k \) receive the same colour. This can be extended to a 4-total-colouring of \( P \).

**Proof** — W.l.o.g. we suppose that \( p_1 \) and \( p_k \) are precoloured with Colour 1, \( p_1p_2 \) is precoloured with Colour 2 and \( p_{k-1}p_k \) is precoloured with Colour 2 or 3. We proceed by induction on \( k \). W.l.o.g. we suppose that \( p_1, p_1p_2, p_{k-1}p_k \) and \( p_k \) are precoloured with colours 1, 2, 1 or 1, 2, 3, 1. If \( k = 3 \) then \( p_2 \) is the only uncoloured element; give it Colour 4. If \( k = 4 \) then colour \( p_2, p_2p_3 \) and \( p_3 \) with colours 3, 1, 4. If \( k \geq 5 \) then colour \( p_2, p_2p_3, p_{k-2}p_k-1 \) and \( p_k-1 \) with colours 4, 3, 1, 4; this can be extended to a total-colouring of \( P \) by the induction hypothesis applied to path \( p_2 \ldots p_{k-1} \).

The following shows how to extend a total precolouring in a 2-sparse graph of maximum degree 3.

**Lemma 11.** Let \( G \) be a 2-connected 2-sparse graph of maximum degree 3. Then \( G \) is 4-total-colourable. Moreover suppose that \( u \) is a vertex of degree 2 that has two neighbours \( a, b \) of degree 3 and suppose that \( a, b, au, ub \) receive respectively Colours 1, 1, 2, 3. This can be extended to a total-colouring of \( G \) using 4 colours.

**Proof** — Let \( G' = B(G, S) \subset G \), where \( S = V_{\geq 3}(G) \). We first total-colour \( G' \). We give Colour 1 to all vertices of \( S \). Then by Theorem 4 we edge-colour \( G' \) with Colours 2, 3, 4 (up to a relabeling, it is possible to give Colour 2, 3 to
au, ub respectively). We extend this to a total-colouring of $G$ by considering one by one the paths of $G$ whose interior vertices have degree 2 and whose ends have both degree 3. Note that since $G$ is 2-connected with maximum degree 3, these paths edge-wise partition $G$ and vertex-wise cover $G$. Let $P = p_1 \ldots p_k$ ($k \geq 3$ since $G$ is 2-sparse) be such a path. The following elements are precoloured: $p_1, p_1 p_2, p_{k-1} p_k, p_k$. The precolouring satisfies the requirement of Lemma 10 so we can extend it to $P$.

**Lemma 12.** A 2-sparse graph $G$ of maximum degree $\Delta \geq 3$ is $(\Delta + 1)$-total-colourable.

**Proof** — We may assume that $G$ is connected, since otherwise it suffices to total-colour its components.

When $\Delta(G) = 3$, we prove by induction a formally stronger statement: if $\Delta(G) \leq 3$ then $G$ is 4-total-colourable. If $G$ has at most two vertices then the result clearly holds. If $G$ is 2-connected and $\Delta(G) = 2$ then $G$ is a cycle and a 4-total-colouring exists. If $\Delta(G) = 3$ and $G$ is 2-connected, then we may rely on Lemma 11. So we may assume that $G$ is not 2-connected and has a cutvertex $v$. So, let $X$ and $Y$ be two disjoint non-empty sets that partition $V \setminus \{v\}$, with no edges between them. The graphs $G[X \cup \{v\}]$ and $G[Y \cup \{v\}]$ are 4-total-colourable by induction, and we assume up to a relabeling that $v$ has the same colour in both total-colourings and that the edges incident to $v$ are coloured in both with (at most 3) different colours. So, we obtain a 4-total-colouring of $G$ by giving to each element the colour it has in one of the blocks.

When $\Delta(G) \geq 4$, we give Colour 1 to each vertex of degree at least 3. We associate to each edge $uv \in E(G)$ such that $\deg_G(u) \geq 3$ or $\deg_G(v) \geq 3$ the list $L_{uv} = \{2, \ldots, \Delta(G) + 1\}$. Then, we apply Lemma 9 to find the colouring.

## 4 Proof of Theorem 1

**Proof** — Let $G = (V, E)$ be a chordless graph of maximum degree $\Delta \geq 3$. We shall prove that $G$ is $\Delta$-edge-colourable and $(\Delta + 1)$-total-colourable by induction on $|V|$. By Lemmas 8 and 12 this holds for 2-sparse graphs (in particular for the claw, the smallest chordless graph of maximum degree at least 3).

If $G$ is not 2-connected, an edge- (resp. total-) colouring of $G$ can be recovered from an edge (resp. total-) colouring of its blocks. So, we may
assume that $G$ is 2-connected but not 2-sparse. According to Theorem 2, $G$ has a proper 2-cutset with split $(X, Y, a, b)$ and we choose such a 2-cutset with minimum $|X|$. By Theorem 3, the block $G_X$ is 2-sparse.

Let $S$ be the set of all vertices of degree at least 3 in the block $G_X$. Note that by Theorem 3, $S$ contains $a$ and $b$, and since $G_X$ is 2-sparse, $S$ is a stable set of $G_X$, and therefore of $G$. In what follows, we consider three cases: edge-colouring, total-colouring with $\Delta \geq 4$, and total-colouring with $\Delta = 3$.

**Edge-colouring**

By induction, we know that $G[Y \cup \{a, b\}]$ (this is not $G_Y$, we do not use the marker vertex) is $\Delta$-edge-colourable. Let $C_a$ and $C_b$ be the sets of the colours given to the edges incident to $a$ and $b$ respectively in such a colouring. We show how the elements of $G[Y \cup \{a, b\}]$ coloured so far give the required list conditions in Lemma 7 in order to edge-colour the 2-sparse $G[X \cup \{a, b\}]$. To do so, we associate to each edge incident to $a$ and $b$ the list $\{1, \ldots, \Delta\} \setminus C_a$ and $\{1, \ldots, \Delta\} \setminus C_b$ of colours respectively. To the other edges, we associate the list $\{1, \ldots, \Delta\}$. We apply Lemma 7 to $G[X \cup \{a, b\}]$ and $S$ to colour the edges of $G[X \cup \{a, b\}]$. We obtain an edge-colouring of $G$ with colours $\{1, \ldots, \Delta\}$.

**Total-colouring, $\Delta \geq 4$**

By induction, we know that $G[Y \cup \{a, b\}]$ is $(\Delta + 1)$-total-colourable. Let $C_a$ (resp. $C_b$) be the sets of the colours given to $a$ (resp. $b$) and to the edges incident to $a$ (resp. $b$). We show how the elements of $G[Y \cup \{a, b\}]$ coloured so far give the required list conditions in Lemma 9 in order to total-colour the 2-sparse $G[X \cup \{a, b\}]$. To do so, we colour $a, b$ with the colours they have in $G[Y \cup \{a, b\}]$ and all other vertices from $S$ with Colour 1. We associate to each edge incident to $a$ and $b$ the list $\{1, \ldots, \Delta + 1\} \setminus C_a$ of colours and $\{1, \ldots, \Delta + 1\} \setminus C_b$ respectively. To all the other edges incident to a vertex $u$ of degree at least 3 (so, $u \notin \{a, b\}$), we associate the list $\{2, \ldots, \Delta + 1\}$. By Lemma 9 applied to $G[X \cup \{a, b\}]$ and $S$, we extend this to a total-colouring of $G$ with colours in $\{1, \ldots, \Delta + 1\}$.

**Total-colouring, $\Delta = 3$**

By Theorem 3, vertices $a,b$ both have at least two neighbours in $X$. Hence, since $G$ has maximum degree 3, vertices $a,b$ both have a unique neighbour in $Y$, say $a', b'$ respectively. Notice that vertices $a'$ and $b'$ are distinct, for
otherwise \( Y = \{a'\} = \{b'\} \) and \( G[Y \cup \{a, b\}] \) is a path. Let \( G'_Y \) be the block obtained from \( G[Y \cup \{a, b\}] \) by contracting \( a \) and \( b \) into a vertex \( w_{ab} \). Block \( G_X \) is built as previously defined by adding to \( G[X \cup \{a, b\}] \) a vertex \( w \) adjacent to \( a, b \).

It is easy to check that \( G_Y \) is chordless. By induction we total-colour \( G_Y \) with 4 colours. W.l.o.g. the vertex \( w_{ab} \) receives Colour 1, edge \( w_{ab}a' \) Colour 2, edge \( w_{ab}b' \) Colour 3. In \( G_X \), precolour \( a \) and \( b \) with Colour 1, \( wa \) with Colour 2, and \( wb \) with Colour 3. Apply Lemma 11 to \( G_X \). This gives a total-colouring of \( G_X \) with 4 colours. A 4-total-colouring of \( G \) is obtained as follows: elements of \( G \) that are in one of \( G_X, G_Y \) receive the colour they have in this block. Edges \( aa' \) and \( bb' \) receive Colours 2 and 3 respectively.

Algorithm and complexity analysis

Here we sketch the description of an algorithm that computes the colourings whose existence is proved above. As usual, \( n \) denotes the number of vertices of the input graph, and \( m \) the number of its edges. All the proofs in Section 3 are clearly algorithmic and most arguments rely on very simple searches of the graph. The only exception, which is the slowest step, is when we use Theorem 4. The edge-colouring whose existence is claimed by Theorem 4 can be obtained in time \( O(nm) \), see Section 20.1 in [32]. Finding a proper 2-cutset satisfying the conditions of Theorem 3 can be performed in time \( O(n^2m) \) as follows. First observe that we may assume \( G \) is biconnected, for otherwise we can use Hopcroft and Tarjan [18] algorithm to compute its biconnected components, and reconstructing the colouring from the blocks is easy: just identify the colours of the vertices that are 1-cutsets. Now we have a biconnected graph. For each pair of vertices, (first) test whether such pair forms a proper 2-cutset and (second) choose, among all pairs, a proper 2-cutset with a block of minimum order. The time complexity of the first step (checking whether a pair \( \{a, b\} \) of vertices is a proper 2-cutset can be done in linear time \( O(n + m) \), which is the time for characterizing the connected components of \( G \setminus \{a, b\} \). Since there are \( O(n^2) \) pairs of vertices to be tested, we have shown the claimed complexity time of \( O(n^2m) \) to find a proper 2-cutset satisfying the conditions of Theorem 3.

Now, a colouring can be obtained as follows. Find a proper 2-cutset satisfying the conditions of Theorem 3 and if one is found build the two blocks with respect to it, one of which is 2-sparse. The not 2-sparse block (if any) is coloured recursively, and the colouring is extended to the 2-sparse block following the strategy in the proofs in Section 3. Since a block has fewer vertices than the original graph, there are at most \( n \) recursive calls, so, the global running time is
\(O(n^3m).\)

Acknowledgement

We are grateful to Pierre Aboulker and Marko Radovanović for pointing out a mistake in a preliminary version of this work.

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