ZETA FUNCTIONS OF NONDEGENERATE HYPERSURFACES IN TORIC VARIETIES VIA CONTROLLED REDUCTION IN $p$-ADIC COHOMOLOGY

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ABSTRACT. We give an interim report on some improvements and generalizations of the Abbott–Kedlaya–Roe method to compute the zeta function of a nondegenerate ample hypersurface in a projectively normal toric variety over $\mathbb{F}_p$ in linear time in $p$. These are illustrated with a number of examples including K3 surfaces, Calabi–Yau threefolds, and a cubic fourfold. The latter example is a non-special cubic fourfold appearing in the Ranestad–Voisin coplanar divisor on moduli space; this verifies that the coplanar divisor is not a Noether–Lefschetz divisor in the sense of Hassett.

1. Introduction

We consider the problem of computing the zeta function $Z(\mathcal{X}, t)$ of an explicitly specified variety $\mathcal{X}$ over a finite field $\mathbb{F}_q$ of characteristic $p$. For curves and abelian varieties, Schoof’s method and variants [Sch85, Pil90, AH96, GH00, GS04, GKS11, GS12] can compute $Z(\mathcal{X}, t)$ in time and space polynomial in $\log q$ and exponential in the genus/dimension; these have only been implemented for genus/dimension at most 2. Such methods may be characterized as $\ell$-adic, as they access the $\ell$-adic cohomology (for $\ell \neq p$ prime) of the variety via torsion points; there also exist $p$-adic methods which compute approximations of the Frobenius action on $p$-adic cohomology (Monsky–Washnitzer cohomology), and which have proven to be more viable in practice for large genus. Early examples include Kedlaya’s algorithm [Ked01] for hyperelliptic curves, in which the time/space dependence is polynomial in the genus and quasi-linear in $p$, and Harvey’s algorithm [Har07] which improves the dependence on $p$ to $p^{1/2+\epsilon}$. These methods have been subsequently generalized [GG01, DV06a, DV06b, Har12], notably by Tuitman’s algorithm [Tui16, Tui17] which applies to (almost) all curves while keeping the quasi-linear dependence on $p$. In another direction, Harvey [Har14] has shown that when computing the zeta functions of reductions of a fixed hyperelliptic curve over a number field, $p$-adic methods can achieve average polynomial time in $\log p$ and the genus; this has been implemented in small genus [HS14, HS16].

One advantage of $p$-adic methods over $\ell$-adic ones is that they scale much better to higher-dimensional varieties. For example, there are several $p$-adic constructions that apply to arbitrary varieties with reasonable asymptotic complexity [LW08, Har15], although we are not aware of any practical implementations. Various algorithms, and some implementations, have been given using Lauder’s

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deformation method of computing the Frobenius action on the Gauss–Manin connection of a pencil [Lau04a, Lau04b, Ger07, Hub08, Hub11, Ked13, PT15, Tui18].

In this paper, we build on an algorithm of Abbott–Kedlaya–Roe [AKR10] which adapts the original approach of [Ked01] to smooth projective hypersurfaces. Here, we add two key improvements.

- We use controlled reduction in de Rham cohomology, as described in some lectures of Harvey [Har10a, Har10b, Har10c], to preserve sparsity of certain polynomials, thus reducing the time (respectively, space) dependence on $p$ from polynomial to quasi-linear (respectively, $O(\log p)$). The resulting controlled AKR method was implemented, with further improvements, in Costa’s Ph.D. thesis [Cos15], with examples of generic surfaces and threefolds over $\mathbb{F}_p$ for $p \sim 10^6$ [Cos15, Section 1.6]; by contrast, the largest $p$ used in [AKR10] is 29. Costa and Harvey are currently preparing a paper on this method; meanwhile, Costa’s GPL-licensed code is available on GitHub [Cos], and is slated to be integrated into SageMath [Sag].

- We also generalize to toric hypersurfaces, subject to a standard genericity condition called nondegeneracy. This greatly increases the applicability of the method while preserving much of its efficiency. Some previous attempts have been made to compute zeta functions in this setting, such as work of Castryck–Denef–Vercauteren [CDV06] for curves and Sperber–Voight [SV13] in general; it is the combination with controlled reduction that makes our approach the most practical to date.

It may be possible to improve the dependence on $p$ to square-root (as in [Har07]) or average polynomial time (as in [Har14]), but we do not attempt to do so here.

For reasons of space, we give only a summary of the algorithm, with further details to appear elsewhere. In lieu of these details, we present a number of worked examples in dimensions 2–4 that demonstrate the practicality of this algorithm in a wide range of cases. The results are based on an implementation in C++, using NTL [Sho] for the underlying arithmetic operations. Our examples in dimensions 2 and 3 were computed on one core of a desktop machine with an Intel(R) Core(TM) i5–4590 CPU @ 3.30GHz; our sole example in dimension 4 was computed on one core of a server with an AMD Opteron Processor 6378 @ 1.6GHz. (We have not yet optimized our vector-matrix multiplications in any way; as a consequence, we observe a serious performance hit whenever the working moduli exceeds $2^{62}$.)

In dimensions 2 and 3, our examples are Calabi–Yau varieties, i.e., smooth, proper, simply connected varieties with trivial canonical bundle. In dimension 1, these are simply elliptic curves. In dimension 2, they are K3 surfaces, whose zeta functions are of computational interest for various reasons. For instance, these zeta functions can (potentially) be used to establish the infinitude of rational curves on a K3 surface (see the introduction to [CT14] for discussion); there has also been recent work on analogues of the Honda–Tate theorem, establishing conditions under which particular zeta functions are realized by K3 surfaces [Tae16, Ito16].

As for Calabi–Yau threefolds, much of the interest in their zeta functions can be traced back to mirror symmetry in mathematical physics. An early example is the work of Candelas–de la Ossa–Rodriguez Villegas [CdlORV03] on the Dwork pencil; a more recent example is [DKS+16], in which (using $p$-adic cohomology) certain mirror families of Calabi–Yau threefolds are shown to have related zeta functions.
Our four-dimensional example is a cubic projective fourfold. Such varieties occupy a boundary region between rational and irrational varieties; it is expected that a cubic fourfold is rational if and only if it is special in the sense of having a primitive cycle class in codimension 2. The geometry of special cubic fourfolds is in turn closely linked to that of K3 surfaces; in many cases, the Hodge structure of a K3 surface occurs (up to a twist) inside the Hodge structure of a special cubic fourfold, and (modulo standard conjectures) this implies a similar relationship between zeta functions. See [Has16] for further discussion.

The specific example we consider is related to the geometry of the moduli space of cubic fourfolds over $\mathbb{C}$. On this space, one can construct various divisors consisting entirely of special cubic fourfolds; Hassett calls these Noether–Lefschetz divisors. Recently, Ranestad–Voisin [RV17] exhibited four divisors which they believed not to be Noether–Lefschetz, but only checked this in one case. Addington–Auel [AA18] checked two more cases by finding in these divisors some cubic fourfolds over $\mathbb{Q}$ with good reduction at 2, such that the zeta functions over $\mathbb{F}_2$ show no primitive Tate classes in codimension 2. By replacing the brute-force point counts of Addington–Auel with $p$-adic methods, we are able to work modulo a larger prime to find an example showing that the fourth Ranestad–Voisin divisor is not Noether–Lefschetz.

To sum up, the overall goal of this project is to vastly enlarge the collection of varieties for which computing the zeta function is practical. It is our hope that doing so will lead to a rash of new insights, conjectures, and theorems of interest to a broad range of number theorists and algebraic geometers.

2. Toric Hypersurfaces

We begin by reviewing the construction of a projective toric variety from a lattice polytope. For more details we recommend [CLS11].

Let $n \geq 1$ be an integer. For any commutative ring $R$, let $R[x^\pm]$ denote the Laurent polynomial ring in $n$ variables $x_1, \ldots, x_n$ with coefficients in $R$. For $\alpha := (\alpha_i) \in \mathbb{Z}^n$, we write $x^\alpha$ for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We denote the $R$-torus by $T_R := \text{Spec}(R[x^\pm])$.

Let $\Delta \subset \mathbb{R}^n$ be the convex hull of a finite subset of $\mathbb{Z}^n$ that is not contained in any hyperplane, so that $\dim \Delta = n$. For $r \in \mathbb{R}$, let $r\Delta$ be the $r$-fold dilation of $\Delta$.

For an integer $d \geq 0$, let $P_d := \langle x^\alpha : \alpha \in d\Delta \cap \mathbb{Z}^n \rangle_R$ (resp. $P^\text{Int}_d := \langle x^\alpha : \alpha \in \text{Int}(d\Delta) \cap \mathbb{Z}^n \rangle_R$) be the free $R$-module on the set of monomials with exponents in $d\Delta \cap \mathbb{Z}^n$ (resp. $\text{Int}(d\Delta) \cap \mathbb{Z}^n$). Define the $R$-graded algebras

$$P_\Delta := \bigoplus_{d=0}^{+\infty} P_d \quad \text{and} \quad P^\text{Int}_\Delta := \bigoplus_{d=0}^{+\infty} P^\text{Int}_d,$$

with the usual multiplication in $R[x^\pm]$. We define the polarized toric variety associated to $\Delta$ as the pair $(\mathbb{P}_\Delta, \mathcal{O}_\Delta)$, where $\mathbb{P}_\Delta := \text{Proj} P_\Delta$ and $\mathcal{O}_\Delta$ is the ample line bundle on $\mathbb{P}_\Delta$ associated to the graded $P_\Delta$-module $P_\Delta(1)$. Note that $P_\Delta$ and $P^\text{Int}_\Delta$ admit $n$ commuting degree-preserving differential operators $\partial_i := x_i \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$.

In order to suppress some expository and algorithmic complexity, we make the simplifying assumption that $\Delta$ is a normal polytope; that is, the map

$$(\Delta \cap \mathbb{Z}^n)^d \to d\Delta \cap \mathbb{Z}^n : (x_1, \ldots, x_d) \mapsto x_1 + \cdots + x_d$$
is surjective for \( d \geq 1 \). This corresponds to the pair \((\mathbb{P}_\Delta, \mathcal{O}_\Delta)\) being projectively normal; this will be the case in our examples. As a consequence, we have that \( \mathcal{O}_\Delta \) is indeed very ample.

**Example 2.1.** Let \( \Delta \) be the regular \( n \)-simplex, the convex hull of 0, \( e_1, \ldots, e_n \). We may then identify \( P_d \) with the set of homogeneous polynomials of degree \( d \) in \( x_0, \ldots, x_n \), by identifying \( x^\alpha \in P_{\Delta,d} \) with the monomial \( x_0^{d-\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n} \); then \((\mathbb{P}_\Delta, \mathcal{O}_\Delta)\) is isomorphic to \((\mathbb{P}_n^d, \mathcal{O}(1))\).

We obtain the weighted projective space \( \mathbb{P}(w_0, \ldots, w_n) \) by taking

\[
\Delta = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n w_i x_i = w_0 \cdots w_n\}, \quad \text{see [Dol82, 1.2.5].}
\]

We obtain \( \mathbb{P}_R^1 \times_R \mathbb{P}_R^n \) by taking \( \Delta \) to be the Cartesian product of the regular \( k \)-simplex by the regular \( r \)-simplex [CLS11, §2.4].

We now turn our attention to toric hypersurfaces over \( R = \mathbb{F}_q \), the finite field with \( q = p^\alpha \) elements and characteristic \( p \). Let \( \mathcal{Y} \) be the hypersurface in \( \mathbb{T}_{\mathbb{F}_q}^n \) defined by a Laurent polynomial \( \mathcal{F} \in \mathbb{F}_q[x^\pm] \), \( \mathcal{Y} := V(\mathcal{F}) \subset \mathbb{T}_{\mathbb{F}_q}^n \). Let

\[
\text{supp} \mathcal{F} = \{\alpha \in \mathbb{Z}^n : \tau_\alpha \neq 0\}
\]

be the support of \( \mathcal{F} \) in \( \mathbb{R}^n \); the convex hull of \( \text{supp} \mathcal{F} \) is the Newton polytope of \( \mathcal{F} \), which we denote by \( \Delta \). We will work under the hypothesis that \( \mathcal{F} \) is \((\Delta-)\)-nondegenerate: \(^1\) for all faces \( \tau \subseteq \Delta \) (including \( \Delta \) itself), the system of equations

\[
\mathcal{F}|_\tau = 0 \quad \text{has no solution in} \quad \mathbb{F}_q^n,
\]

where \( \mathbb{F}_q \) denotes an algebraic closure of \( \mathbb{F}_q \). Furthermore, nondegeneracy implies quasi-smoothness, see [BC94, Definition 3.1 and Proposition 4.15]. For fixed normal \( \Delta \) over an infinite field, this condition holds for generic \( \mathcal{F} \). Others have given point-counting algorithms under this assumption [CDV06, SV13].

Let \( X := \text{Proj} P_{\Delta}(\mathcal{F}) \) denote the closure of \( \mathcal{Y} \) in \( \mathbb{P}_\Delta \) (placing \( \mathcal{F} \) in degree 1) and set \( \mathcal{U} := \mathbb{T}_{\mathbb{R}}^n \setminus \mathcal{Y} \). Let \( H^i_{\text{rig}} \) denote the \( i \)th rigid cohomology group in the sense of Berthelot [Ber97]. The Lefschetz hyperplane theorem, combined with Poincaré duality, show that the map

\[
H^i_{\text{rig}}(\mathbb{P}_\Delta) \to H^i_{\text{rig}}(X),
\]

induced by the inclusion \( X \hookrightarrow \mathbb{P}_\Delta \) is an isomorphism for \( i \neq n - 1 \) [BC94, 10.8]. This implies that the “interesting” part of the cohomology of \( X \) occurs in dimension \( n - 1 \) and consists of those classes that do not come from \( P_{\Delta} \). Denote by \( PH^i_{\text{rig}}(X) \) the primitive cohomology group of \( X \), defined by the (Frobenius-equivariant) exact sequence

\[
0 \to H^n_{\text{rig}}(\mathbb{P}_\Delta) \to H^{n-1}_{\text{rig}}(X) \to PH^n_{\text{rig}}(X) \to 0
\]

With this notation, we may write

\[
Z(\mathcal{X}, t) = Z(\mathbb{P}_\Delta, t)Q(t)^{(-1)^n}.
\]

where

\[
Q(t) := \det (1 - t \text{Frob}_q | PH^n_{\text{rig}}(X)).
\]

\(^1\)This condition was introduced by Dwork [Dwo62] without a name; the term nondegenerate first appears in [Kou76, Var76]. Synonyms include \( \Delta \)-regular [Bat90, §4] and schön [Tev07].
Thus given $T$, we would like to compute $Q(t)$.

The cohomology group $PH^{n-1}_{rig}(X)$ is closely related to $H^n_{rig}(P_\Delta \setminus X)$. For example, if $P_\Delta$ is a (weighted) projective space, as in [AKR10] and [Cos15], the two cohomology groups are isomorphic; see [BC94, 10.11].

3. de Rham cohomology of toric hypersurfaces

In preparation for our use of $p$-adic cohomology to compute $Q(t)$, we give an explicit description of the algebraic de Rham cohomology of a nondegenerate toric hypersurface in characteristic zero. We take $R$ to be the ring $\mathbb{Z}_p$, the ring of integers of $\mathbb{Q}_p$, the unramified extension of $\mathbb{Q}_p$.

Let $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$ be a lift of $f$ to characteristic zero with the same support as $f$ (it will also be nondegenerate). Consider $Y := V(f) \subset T := \mathbb{T}_q$, and $X$, the closure of $Y$ in $P_\Delta$. Write $U := T \setminus Y$, and $V := P_\Delta \setminus X \simeq \text{Spec}(A)$, where $A$ is the coordinate ring of $V$; explicitly,

$$A \simeq \bigcup_{d=0}^{+\infty} f^{-d}P_d.$$

Let $I_f$ be the ideal in $P_\Delta$ generated by $f, \partial_1 f, \ldots, \partial_n f$. We call $I_f$ the toric Jacobian ideal and the quotient ring $J_f := P_\Delta/I_f$ the toric Jacobian ring. Since $f$ is nondegenerate, the ideal $I_f$ is irrelevant in $P_\Delta$ and $\text{rank}_{\mathbb{Q}_q} J_f = n! \text{Vol}(\Delta)$; furthermore, $(J_f)_d = 0$ for $d > n$ [Bat93, §4]. If $O_\Delta$ is not very ample, then $I_f$ might not be generated in degree 1 and we might have $(J_f)_d = 0$ only for $d > n$.

Let $\Omega^*$ denote the logarithmic de Rham complex of $V$ with poles along $P_\Delta \setminus T$. Let $H^\bullet$ be the cohomology groups of $\Omega^*$; these are naturally isomorphic to $H^\bullet_{dR}(V \cap T = T \setminus Y = U)$ and $H^\bullet_{dR}(T_q \setminus \mathcal{V} = T_q \\
$ [Kat89].

We now provide an explicit description of the group $H^n$, as in [Bat93, §6 and 7], in which we will compute $Q(t)$. Set

$$\omega := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \in \Omega^n,$$

and define the ascending filtration in $\Omega^n$ by

$$\text{Fil}^d \Omega^n := \{ g f^{-d} \omega : g \in P_d \}.$$

The associated graded ring

$$\Omega^n := \bigoplus_{d=0}^{+\infty} \text{Gr}^d \Omega^n, \quad \text{Gr}^d \Omega^n := \text{Fil}^d \Omega^n / \text{Fil}^{d-1} \Omega^n$$

is then isomorphic to $P_\Delta/(f)$ (again placing $f$ in degree 1).

Equip $H^n$ with the filtration induced from $\Omega^n$, and view $H^n$ as the quotient of $\Omega^n$ by the $\mathbb{Q}_q$-submodule generated by the relations

$$\frac{g}{f^d} \omega - \frac{gf}{f^{d+1}} \omega \quad \text{and} \quad \frac{\partial_i(g)}{f^d} \omega - \frac{dg \partial_i(f)}{f^{d+1}} \omega$$

for each $i = 1, \ldots, n$, each nonnegative integer $d$, and each $g \in P_d$. From these relations, we see that

$$\text{Gr}^1 H^n \simeq P_1/(f) \quad \text{and} \quad \text{Gr}^d H^n \simeq (J_f)_d \quad (d > 1).$$
This gives a way to compute explicitly in $H^n$: for any $h \in (J_f)_{d+1}$ with $d > n$, we can find a relation of the form
\begin{equation}
\frac{d}{f^{d+1}} h = \frac{d g_0 f + \sum_{i=1}^{n} g_i \partial_i f}{f^{d+1}} \omega = \frac{dg_0 + \sum_{i=1}^{n} \partial_i g_i}{f^{d}} \omega.
\end{equation}

because $P_d \subset (I_f)_d$, so in $H^n$ we can reduce the pole order of any form to at most $n$. This process was introduced for smooth projective hypersurfaces in [Gri69] and attributed to Dwork; it is commonly known as Griffiths–Dwork reduction.

With the above representation of $H^n$, we may also identify $PH^{n-1}_d(X)$ with $(P^{\text{int}} + I_f)/I_f \subset H^n$, where the filtration by pole order is the Hodge filtration; see [Bat93, BC94, §9, §11].

We now introduce a variation of Griffiths–Dwork reduction, called \textit{controlled reduction}. This will be crucial for our application to $p$-adic cohomology, as careless application of Griffiths–Dwork reduction to a sparse form will easily lead to a dense form.

For $d = 1, \ldots, n + 1$, choose a $\mathbb{Z}_q$-linear splitting $P_d \approx (J_{f_j})_d \oplus C_d$, where $(J_{f_j})_d$ is a lift of $(J_f)_d$ into $P_d$. Let $\rho_d : P_d \to (J_{f_j})_d$ and $\pi_{d,0}, \ldots, \pi_{d,n} : P_d \to P_{d-1}$ be $\mathbb{Z}_q$-linear maps such that
\begin{equation}
g = \rho_d(g) + \pi_{d,0}(g) \cdot f + \sum_{i=1}^{n} \pi_{d,i}(g) \cdot \partial_i f; \quad g \in P_d.
\end{equation}

These maps may be constructed one monomial at a time.

\textbf{Proposition 3.1} (Controlled reduction). Let $x^\mu \in P_1$ and $x^\nu \in P_d$ be two monomials and define the following $\mathbb{Z}_q$-linear maps:
\begin{align*}
R_{\mu, \nu}(g) &:= (d + n)\pi_{n+1,0}(x^\nu g) + \sum_{i=1}^{n} (\partial_i + \mu_i)(\pi_{n+1,i}(x^\nu g)) \\
S_{\nu}(g) &:= \pi_{n+1,0}(x^\nu g) + \sum_{i=1}^{n} \nu_i \pi_{n+1,i}(x^\nu g)
\end{align*}

Then for any $g \in P_n$ and any nonnegative integer $j$, in $H^n$ we have
\begin{equation}
g^{(d+1)\nu + \mu}_{f^{d+n+j+1}} \omega \equiv (d + n + j)^{-1}(R_{\mu, \nu}(g) + jS_{\nu}(g)) \frac{x^{\nu \mu + \mu}}{f^{d+n+j}} \omega.
\end{equation}

\textbf{Proof.} This is straightforward from (3.1) and (3.2). \qed

Note that Proposition 3.1 enables us to reduce the pole order of a differential form from $d + n + j + 1$ to $d + n + j$ without increasing its total number of monomials; we can thus reduce the pole order of a sparse form without making it dense.

\textbf{Corollary 3.2.} With notation as in Proposition 3.1, let $k$ be a positive integer. Then for any $g \in P_n$,
\begin{equation}
g^{x^{\mu + k\nu}}_{f^{d+n+k}} \omega \equiv \prod_{j=0}^{k-1}(R_{\mu, \nu} + jS_{\nu})(g) \frac{x^{\mu}}{f^{d+n+j}} \omega,
\end{equation}

forming the composition product from left to right.

Using Proposition 3.1 amounts to performing linear algebra on matrices of size $\#(n\Delta \cap \mathbb{Z}^n) \sim n^n \text{Vol}(\Delta)$. One can reduce this by a factor of $n^n/n! \sim e^n$ at the expense of making the expression for the reduction matrix more convoluted; compare [Cos15, Proposition 1.17 and 1.18].
4. Monsky–Washnitzer Cohomology

We now indicate how Monsky–Washnitzer cohomology, as introduced in [MW68, Mon68, Mon71], provides a crucial link between algebraic de Rham cohomology and \( p \)-adic rigid cohomology, by transferring to the former the canonical Frobenius action on the latter; see also [vdP86]. To simplify, we assume \( p > \max\{n, 2\} \).

Let \( A^i \) denote the weak \( p \)-adic completion of \( A \), the ring consisting of formal sums \( \sum_{d=0}^{\infty} a_d f^{-d} \) such that for some \( a, b > 0 \), \( a_d \in p^{\max\{0,\lfloor ad-b\rfloor\}} P_d \) for all \( d \geq 0 \).

We define the associated logarithmic de Rham complex \( \Omega^{\cdot \cdot} \) action on the latter; see also [vdP86]. To simplify, we assume \( p > p_{\text{Mon68, Mon71}} \), provides a crucial link between algebraic de Rham cohomology and \( \text{Mad Monsky–Washnitzer cohomology groups} \ H^{\cdot \cdot} \) of \( X \) that receive an action of the Frobenius automorphism, which we can make explicit by constructing a lift \( \hat{\sigma} \) of the Witt vector Frobenius on \( Q \). We finally extend \( \hat{\sigma} \) to \( \Omega^{\cdot \cdot} \) by \( \Omega^{\cdot \cdot} \to \Omega^{\cdot \cdot} \otimes_{Z_q} Q_q \) is a quasi-isomorphism [Mon70, vdP86, Kat89]; that is, the induced maps \( H^i \otimes_{Z_q} Q_q \to H^{i, \cdot} \otimes_{Z_q} Q_q \) are isomorphisms. We can thus identify the algebraic de Rham cohomology of \( U \) with the Monsky–Washnitzer cohomology of \( U \).

On the other hand, we also have \( H^{1, \cdot} \otimes_{Z_q} Q_q \simeq H^{\text{rig}}_X(U) \) and the latter object is functorial with respect to geometry in characteristic \( p \) [Ber97]. In this way, \( H^{1, i} \) receives an action of the Frobenius automorphism, which we can make explicit by constructing a lift \( \sigma \) of the \( p \)-th power Frobenius on \( F_q \) to \( A^1 \). To do so, we take the Witt vector Frobenius on \( Z_q \) and set \( \sigma(\mu) = \mu^p \) for any monomial \( \mu \in P_\Delta \). We then extend \( \sigma \) to \( A^1 \) by the formula

\[
\sigma \left( \frac{g}{f^d} \right) := \sigma(g)\sigma(f)^{-d} = \sigma(g) \sum_{i \geq 0} \left( \begin{array}{c} -d \\ i \end{array} \right) \frac{(\sigma(f) - \mu^p)^i}{f^{p(d+i)}}.
\]

The above series converges (because \( p \) divides \( \sigma(f) - \mu^p \)) and the definitions ensure that \( \sigma \) is a semilinear (with respect to the Witt vector Frobenius) endomorphism of \( A^1 \). We finally extend \( \sigma \) to \( \Omega^{1, \cdot} \) by \( \sigma(gh) := \sigma(g) \sigma(h) \).

5. Sketch of the algorithm

We now indicate briefly how to use controlled reduction to compute the Frobenius action on the cohomology of nondegenerate toric hypersurfaces. We start as in [Har07, Proposition 4.1], by rewriting the Frobenius action in a sparser form.

Lemma 5.1. For any positive integers \( d, N \) and \( g \in P_d \), in \( A^1 \) we have

\[
\sigma \left( \frac{g}{f^d} \right) \equiv \sum_{j=0}^{N-1} \left( \begin{array}{c} -d \\ j \end{array} \right) \left( \frac{d + N - 1}{d + j} \right) \sigma(gf^j)f^{-p(d+j)} \pmod{p^N}.
\]

Proof. This follows from (4.1) by truncating the sum and then rewriting formally; see [Cos15, Lemma 1.10].

In order to compute a \( p \)-adic approximation of the Frobenius action on \( PH^{n-1}(X) \), we must first fix a basis of the latter; we do this by constructing a monomial basis for \( PH^{n-1}_d(X) \) via explicit linear algebra. We then apply Frobenius to each basis element in the sparse truncated form given by Lemma 5.1; recursively reduce the pole order using Corollary 3.2 (using \( k = p \) as much as possible); and project to the chosen monomial basis. The dominant step is controlled reduction, which amounts to \( O(pm^N \text{Vol}(\Delta)) \) matrix multiplications of size \( n! \text{Vol}(\Delta) \) per basis element.

We will not address precision estimates in this report, except to note that the machinery of [AKR10, §3.4] applies. In general, if we want \( N \) digits of \( p \)-adic accuracy, we must apply Lemma 5.1 with \( N \) replaced by \( N' = N + O(n + \log N) \).
and work modulo $p^{O(N')}$. Hence, with respect to $p$ alone, we expect our algorithm to run in quasi-linear time in $p$ and use $O(\log p)$ space.

6. K3 surfaces

We now turn our attention to examples, starting with K3 surfaces. For $X$ a K3 surface, $\dim H^2(X) = 22$ and the Hodge numbers are $(1, 20, 1)$. A common example of a K3 surface is a smooth quartic surface in $\mathbb{P}^3$; however, they also occur in other ways, such as hypersurfaces in weighted projective spaces. Using a criterion of Miles Reid [Rei80], Yonemura [Yon90] found the complete list of (polarized) weighted projective spaces in which a generic hypersurface is a K3 surface; there are 95 of these. For toric varieties, the corresponding classification is that of reflexive 3-dimensional polytopes, of which there are 4,319 in all [KS98].

In the following examples, we worked modulo $p^4$ in order to obtain $Q(t)$ with $2p$-adic significant digits. As a result, we observe a performance hit for $p > 2^{16}$.

**Example 6.1.** Consider the projective quartic surface $X \subset \mathbb{P}^3_{\mathbb{F}_p}$ defined by

$$x^4 + y^4 + z^4 + w^4 + \lambda xyzw = 0;$$

it is a member of the Dwork pencil. For $p = 2^{20} - 3$ and $\lambda = 1$, using the controlled AKR algorithm in 22h7m we compute that

$$Z(X, t)^{-1} = (1 - t)(1 - pt)^{16}(1 + pt)^3(1 - p^2 t)Q(t),$$

where the “interesting” factor is

$$Q(t) = (1 + pt)(1 - 1688538 t + p^2 t^2).$$

For this family, the remaining factors, apart from $Q(t)$, could have also been deduced by a $p$-adic formula of de la Ossa-Kadir [Kad04, Chapter 6]. In this context, the Hodge numbers of $PH^2(X)$ are $(1, 19, 1)$.

A similar runtime would be expected if we used our current implementation to compute $Z(X, t)$ with $\Delta$ being the 3-simplex (tetrahedron), as indicated by the outer polytope at right. Instead, we observe that the monomials defining $X$ generate a sublattice of index $4^2$ in $\mathbb{Z}^3$; hence, we can instead run our algorithm with a polytope of significantly smaller volume ($32/3 \approx 10.66$ versus $2/3 \approx 0.66$), as indicated by the inner polytope at right. This leads to a dramatic speedup: with our current implementation, we computed $Q(t)$ in 1m33s.

We present the running times for other $p$ in Table 1, and the memory usage is about 16MB.

In the new framework, $X$ is given by the closure (in $\mathbb{P}_\Delta$) of the affine surface defined by the Laurent polynomial

$$x^4 y z^{-1} + \lambda x + y + z + 1,$$

and the Hodge numbers of $PH^2(X)$ are $(1, 1, 1)$, which explains why $\deg Q(t) = 3$.

Since the Dwork pencil is a “small” deformation of the Fermat quartic, we may also use the Pancratz–Tuitman implementation of the deformation method [PT15] to compute every factor of $Z(X, t)$. We did this and verified that our results agree; we compare running times in Table 1. To interpret these results fairly, note that the Pancratz–Tuitman implementation works in $\mathbb{P}_\Delta$ instead of $\mathbb{P}^3$, and so computes
the whole numerator of $Z(\mathcal{X}, t)$ rather than just $Q(t)$. (We also observe that the newer algorithm of [Tui18] has a square-root dependence on $p$, as in [Har07].)

| $p$ | CHK time | PT time | $p$ | CHK time |
|-----|----------|---------|-----|----------|
| $2^5 - 5$ | 0.03s | 1.66s | $2^{17} - 1$ | 11.9s |
| $2^9 - 3$ | 0.04s | 3.64s | $2^{18} - 5$ | 23.4s |
| $2^{10} - 3$ | 0.04s | 7.39s | $2^{19} - 1$ | 46.9s |
| $2^{11} - 9$ | 0.06s | 14.65s | $2^{20} - 3$ | 1m33s |
| $2^{12} - 3$ | 0.08s | 34.80s | $2^{21} - 9$ | 3m6s |
| $2^{13} - 1$ | 0.13s | 34.80s | $2^{22} - 3$ | 6m15s |
| $2^{14} - 3$ | 0.22s | 6m43s | $2^{15} - 19$ | 14m14s |
| $2^{16} - 15$ | 0.72s | 23m12s | $2^{17} - 19$ | 2m33s |
| $2^{18} - 19$ | 0.41s | 2m33s | $2^{19} - 3$ | 3m6s |
| $2^{20} - 9$ | 0.22s | 6m43s | $2^{21} - 9$ | 6m15s |
| $2^{22} - 15$ | 0.72s | 23m12s | $2^{23} - 3$ | 6m15s |

Table 1. The second and fifth columns use our current implementation to compute $Q(t)$. The third column uses the Pancratz–Tuitman implementation [PT15] to compute $Z(\mathcal{X}, t)$.

**Example 6.2.** Consider the projective quartic surface $\mathcal{X} \subset \mathbb{P}_p^3$ defined by

$$x^3y + y^4 + z^4 + w^4 - 12xyzw;$$

it contains a hypergeometric motive (see [DKS+16, Section 5]). For $p = 2^{15} - 19$, using the controlled AKR algorithm in 27m12s we compute that

$$Z(\mathcal{X}, t)^{-1} = (1 - t)(1 - pt)^2(1 + pt)^2(1 - pt + p^2t^2)^2(1 - p^2t^2 + p^4t^4)^2(1 - p^2t)Q(t),$$

where the “interesting” factor is (up to rescaling)

$$pQ(t/p) = p + 20508t^1 - 18468t^2 - 26378t^3 - 18468t^4 + 20508t^5 + pt^6.$$

As in the previous example, the Newton polytope has volume 8, but the defining monomials generate a sublattice of index 4 in $\mathbb{Z}^3$; we may thus work instead with a polytope of volume 2 (depicted at right) and observe a significant speedup. In this setting, the Hodge numbers of $\text{PH}^2(\mathcal{X})$ are $(1, 4, 1)$. With our current implementation we computed $Q(t)$ in 4s. We present the running times for other $p$ in Table 2, where the memory footprint was about 52MB.

Alternatively, one could try to use Magma [BCP97] to confirm $Q(t)$. Unfortunately, Magma is only able to confirm the linear coefficient:

```plaintext
> C2F2 := HypergeometricData([6,12], [1,1,1,2,3]);
> EulerFactor(C2F2, 2^10 * 3^6, 2^15 - 19: Degree:=1);
1 + 20508*$1 + O($1^2)
```

**Example 6.3.** Consider the closure $\mathcal{X}$ in $\mathbb{P}_\Delta$ (which in this case is not a weighted projective space) of the affine surface defined by the Laurent polynomial

$$3x + y + z + x^{-2}y^2z + x^3y^{-6}z^{-2} + 3x^{-2}y^{-1}z^{-2} - 2 - x^{-1}y - y^{-1}z^{-1} - x^2y^{-4}z^{-1} - xy^{-3}z^{-1};$$

it is a K3 surface of geometric Picard rank 6, and the Hodge numbers of $PH^2(X)$ are $(1, 14, 1)$. For $p = 2^{15} - 19$, using our current implementation, in 6m20s we obtain the “interesting” factor of $Z(X, t)$:

$$pQ(t/p) = (1 - t) \cdot (1 + t) \cdot (p + 330571 + 15641^2 - 14296t^3 - 1185t^4 + 5107t^5 + 27955t^6 + 25963t^7 + 27955t^8 + 5107t^9 - 1185t^10 - 14296t^{11} + 15641t^{12} + 33057t^{13} + pt^{14})$$.

We present the running times for other $p$ in Table 3, where the peak memory usage was about 144MB.

The vertices of the associated polytope correspond to the first six terms displayed; the remaining terms are interior points. We depict this polytope of volume 8 at right.

We know of no previous algorithm that can compute $Z(X, t)$ for $p$ in this range. The defining polynomial is “dense” from the point of the Sperber–Voight algorithm [SV13], which is based on Dwork cohomology and scales with the number of monomials away from the vertices of the Newton polytope.

| $p$  | time | $p$  | time | $p$  | time |
|------|------|------|------|------|------|
| $2^7 - 1$ | 6.46s | $2^{10} - 3$ | 18.93s | $2^{13} - 1$ | 1m46s |
| $2^9 - 5$ | 9.50s | $2^{11} - 9$ | 31.34s | $2^{14} - 3$ | 3m24s |
| $2^9 - 3$ | 12.64s | $2^{12} - 3$ | 56.23s | $2^{15} - 19$ | 6m20s |

Table 3. Running times for Example 6.3.

**Example 6.4.** Let $X$ be the smooth projective surface in $\mathbb{P}^3$ defined by the fully dense, randomly chosen quartic polynomial

$$-9x^4 - 10x^3y - 9x^2y^2 + 2xy^3 - 7y^4 + 6x^3z + 9x^2yz - 2xy^2z + 3y^3z + 8x^2z^2 + 6y^2z^2 + 2xz^3 + 7yz^3 + 9z^4 + 8x^3w + x^2yw - 8xy^2w - 7y^3w + 9x^2zw - 9xyzw + 3y^2zw - xz^2w - 3yz^2w + z^3w - x^2w^2 - 4xyw^2 - 3xzw^2 + 8yzw^2 - 6z^2w^2 + 4xw^3 + 3yw^3 + 4zw^3 - 5w^4;$$

then $\Delta$ is the 3-simplex (tetrahedron) of volume $32/3 \approx 10.66$. For this example, we have $PH^2(X) \simeq H^3(\mathbb{P}^3 \setminus X)$, the Hodge numbers are $(1, 19, 1)$, and

$$Z(X, t)^{-1} = (1 - t)(1 - pt)(1 - p^2t)Q(t)$$.
where \( \deg Q(t) = 21 \). For \( p = 2^{15} - 19 \), we obtain
\[
pQ(t/p) = (1 + t)(p - 53159t^4 + 10023t^2 - 3204t^3 + 49736t^4 - 56338t^5
+ 43086t^6 - 48180t^7 + 44512t^8 - 4204t^9 + 47794t^{10}
- 42681t^{11} + 44512t^{12} - 48180t^{13} + 43086t^{14} - 56338t^{15}
+ 49736t^{16} - 3204t^{17} + 10023t^{18} - 53159t^{19} + pt^{20})
\]
using the controlled AKR algorithm in 38m27s; our current implementation takes roughly the same time. We present the running times for other \( p \) in Table 4. The memory footprint was about 230MB.

Unfortunately, the deformation method is not suitable for dense quartics with \( p \) in this range. For example, for \( p = 31 \) the running time was 2h8m and its memory footprint was around 7GB, and both time and space should scale linearly with \( p \).

| \( p \)  | time     | \( p \)  | time     | \( p \)  | time     |
|-------|----------|-------|----------|-------|----------|
| \( 2^7 - 1 \) | 25.41s   | \( 2^{10} - 3 \) | 1m30s   | \( 2^{13} - 1 \) | 9m26s   |
| \( 2^8 - 17 \) | 37.73s   | \( 2^{11} - 9 \) | 2m37s   | \( 2^{14} - 3 \) | 18m42s  |
| \( 2^9 - 3 \) | 55.82s   | \( 2^{12} - 3 \) | 4m50s   | \( 2^{15} - 19 \) | 36m29s  |

Table 4. Running times for Example 6.4.

7. Calabi–Yau threefolds

We next consider Calabi–Yau threefolds. Unlike for K3 surfaces, the middle Betti numbers of Calabi–Yau threefolds are not \( a \) \( priori \) bounded; the largest value of which we are aware is 984 (found in [KS00]).

A common example is a smooth quintic surface in \( \mathbb{P}^4 \). Again, additional constructions arise from generic hypersurfaces in weighted projective spaces, of which there are 7,555 in all, or more generally from toric varieties corresponding to reflexive 4-dimensional polytopes, of which there are 473,800,776 in all [KS00]. In all of the following examples, we worked modulo \( p^6 \) in order to obtain \( Q(t) \) and our memory footprint ranged between 100MB and 270MB.

Example 7.1. Consider the projective quintic threefold \( X \subset \mathbb{P}^3_{\mathbb{P}^4} \) defined by
\[
x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_0x_1x_2x_3x_4 = 0;
\]
it is a member of the Dwork pencil. We have
\[
Z(X; t) = \frac{R_1(pt)^{20}R_2(pt)^{30}Q(t)}{(1 - t)(1 - pt)(1 - p^2t)(1 - p^3t)}
\]
where \( R_1 \) and \( R_2 \) are the numerators of the zeta functions of certain curves given by a formula of Candelas–de la Ossa–Rodriguez Villegas [CdlORV03].

As it is presented, we would work with \( \mathbb{P}_{\Delta} = \mathbb{P}^4 \) where \( \Delta \) is the 4-simplex of volume 625/24. As in Example 6.1, the monomials of the equation generate a sublattice of index 5^4 in \( \mathbb{Z}^4 \), so we may instead work with a polytope whose volume is smaller by a factor of 5^3. For \( p = 2^{20} - 3 \), we compute the “interesting” factor
\[
Q(t) = 1 - 1576492860t^3 + 2672053179370pt^2 - 1576492860p^3t^3 + p^6t^4
\]
Example 7.2. Let $X$ be the threefold defined by
\[ x_0^8 + x_1^3 x_2 + x_0^2 x_1 x_2 x_3 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_0 x_1 x_2 x_3 x_4 + x_2 x_3 x_4^2 \]
in the weighted projective space $\mathbb{P}(1,14,18,20,25)$. The Newton polytope has volume $11/3 \approx 3.67$; by changing the lattice we may instead work with a polytope of volume $1/3 \approx 0.33$. In this setting, the Hodge numbers of $PH^3(X)$ are $(1,1,1,1)$.

For $p = 2^{20} - 3$, we compute the “interesting” factor of $Z(X,t)$
\[ 1 - 618297672t^1 + 390956360946pt^2 - 618297672p^3t^3 + p^6t^4 \]
in $32m33s$. We present the running times for other $p$ in Table 6.

| $p$  | time  | $p$  | time  | $p$  | time  |
|------|-------|------|-------|------|-------|
| $2^8 - 5$ | 1.90s | $2^{13} - 1$ | 18.2s | $2^{18} - 5$ | 8m0s |
| $2^9 - 3$ | 1.96s | $2^{14} - 3$ | 32.9s | $2^{19} - 1$ | 16m8s |
| $2^{10} - 3$ | 2.06s | $2^{15} - 19$ | 1m6s | $2^{20} - 3$ | 32m33s |
| $2^{11} - 9$ | 7.48s | $2^{16} - 15$ | 2m4s | $2^{21} - 9$ | 1h5m |
| $2^{12} - 3$ | 10.9s | $2^{17} - 1$ | 4m3s | $2^{22} - 3$ | 2h23m |

Table 6. Running times for Example 7.2.

Example 7.3. Let $X$ be the threefold defined by
\[ x_1^7 + x_0^5 x_1 x_2 + x_0^2 x_1^2 x_2 x_3 + x_0 x_1 x_2^2 x_4 + x_0 x_1^2 x_3 + x_0^2 x_1 x_2 x_3 x_4 + x_2 x_3 x_4^2 \]
in the weighted projective space $\mathbb{P}(10,11,16,19,21)$. Again, by choosing the right lattice, we reduce the volume of the Newton polytope from $55/12 \approx 4.58$ to $11/24 \approx 0.46$, and the Hodge numbers of $PH^3(X)$ are $(1,2,2,1)$. For $p = 2^{20} - 3$, we computed the “interesting” factor of $Z(X,t)$
\[ 1 - 2068001468t^1 + 3449674041773pt^2 - 3772715295733197p^2t^3 + 3449674041773p^4t^4 - 2068001468p^6t^5 + p^9t^6 \]
in $2h10m$. We present the running times for other $p$ in Table 7.

| $p$  | time  | $p$  | time  | $p$  | time  |
|------|-------|------|-------|------|-------|
| $2^8 - 5$ | 0.73s | $2^{13} - 1$ | 6.41s | $2^{18} - 5$ | 2m50s |
| $2^9 - 3$ | 0.77s | $2^{14} - 3$ | 11.61s | $2^{19} - 1$ | 5m38s |
| $2^{10} - 3$ | 0.80s | $2^{15} - 19$ | 21.98s | $2^{20} - 3$ | 11m18s |
| $2^{11} - 9$ | 2.54s | $2^{16} - 15$ | 43.07s | $2^{21} - 9$ | 22m41s |
| $2^{12} - 3$ | 3.80s | $2^{17} - 1$ | 1m25s | $2^{22} - 3$ | 52m37s |

Table 5. Running times for Example 7.1.
Example 7.4. Let $\mathcal{X}$ be the closure in $\mathbb{P}_\Delta$ (which is not a weighted projective space) of the threefold defined by the Laurent polynomial

$$xy_z^2w^3 + x + y + z - 1 + y^{-1}z^{-1} + x^{-2}y^{-1}z^{-2}w^{-3} = 0.$$ 

Choosing the correct lattice reduces the volume of the Newton polytope from $9/8 \approx 1.12$ to $3/8 \approx 0.38$, and the Hodge numbers of $PH^3(\mathcal{X})$ are $(1, 2, 2, 1)$. For $p = 2^{20} - 3$, we computed the “interesting” factor of $Z(\mathcal{X}, t)$

$$(1 + 718pt + p^3t^2)(1 + 1188466826t^4 + 1915150034310pt^2 + 1188466826p^3t^3 + p^6t^4)$$

in $1h15m$. We present the running times for other $p$ in Table 8.

| $p$  | time    | $p$  | time    | $p$  | time    |
|------|---------|------|---------|------|---------|
| $2^5 - 5$ | 2.74s   | $2^{13} - 1$ | 39.28s | $2^{18} - 5$ | 18m34s |
| $2^9 - 3$ | 2.80s   | $2^{14} - 3$ | 1m3s   | $2^{19} - 1$ | 38m8s  |
| $2^{10} - 3$ | 3.00s  | $2^{15} - 19$ | 1m21s | $2^{20} - 3$ | 1h15m |
| $2^{11} - 9$ | 14.86s  | $2^{16} - 15$ | 4m45s | $2^{21} - 9$ | 2h32m |
| $2^{12} - 3$ | 22.32s  | $2^{17} - 1$ | 9m12s | $2^{22} - 3$ | 5h39m |

Table 8. Running times for Example 7.4.

8. Cubic fourfolds

For our final example, we consider a cubic fourfold. For $X$ a smooth cubic fourfold in $\mathbb{P}^5$, $\dim H^4(X) = 23$ and the Hodge numbers are $(0, 1, 21, 1, 0)$.

In this example, we worked modulo $p^6$ in order to obtain $Q(t)$.

Example 8.1. Let $\mathcal{X}$ be the smooth projective cubic fourfold in $\mathbb{P}^5_{\mathbb{F}_p}$ defined by

$$x_0^3 + x_1^3 + x_2^3 + (x_0 + x_1 + 2x_2)^3 + x_3^3 + x_4^3 + x_5^3 + 2(x_0 + x_3)^3 + 3(x_1 + x_4)^3 + (x_2 + x_5)^3;$$

it is nondegenerate in $\mathbb{P}^5$. For $p = 31$, in $21h31m$ we computed

$$Z(\mathcal{X}, t)^{-1} = (1 - t)(1 - pt)(1 - p^2t)(1 - p^3t)(1 - p^4t)Q(t)$$

where the “interesting” factor is an irreducible Weil polynomial given by

$$pQ(t/p^2) = p - 7t + 21t^2 - 52t^3 - 8t^4 - 28t^5 + 21t^6 + 35t^7 + 39t^8 + 62t^{10} + 23t^{11} + 62t^{12} + 39t^{13} + 35t^{15} + 21t^{16} - 28t^{17} - 8t^{18} - 52t^{19} + 21t^{20} - 7t^{21} + pt^{22};$$

the coefficient of $t^1$ may be confirmed independently by counting $\mathcal{X}(\mathbb{F}_p)$ using the Sage function `count_points`. For $p = 127$ the running time was $23h15m$ and for $p = 499$ it was $24h55m$. In both cases, we also observed that the “interesting”
factor is an irreducible Weil polynomial. In these three computations, the memory footprint was around 36.5GB.

In this high-dimensional setting, the bottleneck seems to be the linear algebra required to set up controlled reduction. In this example, for $p = 31$ more than half of the running time (15h32m) is spent solving a linear problem of size $15,504 \times 37,128$ modulo $p^6$. With a more careful implementation of this step (for example, avoiding Hensel lifts) we would expect a significant speedup.

Note that the defining equation for $X$ is quite sparse. To assess the effect of this sparsity, as well as to cross-check the answer, we recomputed $Z(X, t)$ after applying a random linear change of variables to obtain a dense defining equation. For $p = 31$, in 27h55m and using about 41GB we obtained the same value for $Z(X, t)$ as above.

As described in the introduction, Example 8.1 has an implication for the moduli of cubic fourfolds. A cubic fourfold is coplanar if it is defined by an expression of the form $\sum_{i=1}^{10} c_i^3$, in which each $c_i$ is a linear form and some four of the $c_i$ are linearly dependent. Ranestad–Voisin [RV17] show that the Zariski closure of the coplanar locus on the moduli space of cubic fourfolds is a divisor, denoted $D_{\text{copl}}$. Example 8.1 is a coplanar cubic fourfold over $\mathbb{Q}$ which is non-special: the existence of a primitive cycle class in codimension 2 would imply that $p\mathbb{Q}(t/p^2)$ is divisible by some cyclotomic polynomial. This shows (modulo a detailed description and validation of the algorithm) that $D_{\text{copl}}$ is not a Noether–Lefschetz divisor.

References

[AA18] Nicolas Addington and Asher Auel. Some non-special cubic fourfolds. preprint, 2018. arXiv:1703.05923.

[AH96] Leonard M. Adleman and Ming-Deh A. Huang. Counting rational points on curves and abelian varieties over finite fields. In Algorithmic number theory (Talence, 1996), volume 1122 of Lecture Notes in Comput. Sci., pages 1–16. Springer, Berlin, 1996.

[AKR10] Timothy G. Abbott, Kiran S. Kedlaya, and David Roe. Bounding Picard numbers of surfaces using $p$-adic cohomology. In Arithmetic, geometry, and coding theory (AGCT 2005), volume 21 of Sémin. Congr., pages 125–159. Soc. Math. France, Paris, 2010.

[Bat93] Victor V. Batyrev. Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori. Duke Math. J., 69(2):349–409, 1993.

[BC94] Victor V. Batyrev and David A. Cox. On the Hodge structure of projective hypersurfaces in toric varieties. Duke Math. J., 75(2):293–338, 1994.

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. J. Symbolic Comput., 24(3-4):235–305, 1997. Computational algebra and number theory (London, 1993).

[Ber97] Pierre Berthelot. Finitude et pureté cohomologique en cohomologie rigide. Invent. Math., 128(2):329–377, 1997. With an appendix in English by Aise Johan de Jong.

[CdIORV03] Philip Candelas, Xenia de la Ossa, and Fernando Rodriguez-Villegas. Calabi-Yau manifolds over finite fields. II. In Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001), volume 38 of Fields Inst. Commun., pages 121–157. Amer. Math. Soc., Providence, RI, 2003.

[CDV06] W. Castryck, J. Denef, and F. Vercauteren. Computing zeta functions of nondegenerate curves. IMRP Int. Math. Res. Pap., pages Art. ID 72017, 57, 2006.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. Toric varieties, volume 124 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.

[Cos] Edgar Costa. controlledreduction: C++ implementation of the controlled reduction method to compute Hasse–Weil zeta functions of smooth projective hypersurfaces over finite fields. https://github.com/edgarcosta/controlledreduction.
[Cos15] Edgar Costa. *Effective computations of Hasse–Weil zeta functions*. PhD thesis, New York University, 2015.

[CT14] Edgar Costa and Yuri Tschinkel. Variation of Néron-Severi ranks of reductions of K3 surfaces. *Exp. Math.*, 23(4):475–481, 2014.

[DKS+16] Charles F. Doran, Tyler L. Kelly, Adriana Salerno, Steven Sperber, John Voight, and Ursula Whitcher. Zeta functions of alternate mirror Calabi-Yau families. *preprint*, 2016. *arXiv:1612.09249*.

[Dol82] Igor Dolgachev. Weighted projective varieties. In *Group actions and vector fields* (Vancouver, B.C., 1981), volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

[DV06a] Jan Denef and Frederik Vercauteren. Counting points on $C_{ab}$ curves using Monsky-Washnitzer cohomology. *Finite Fields Appl.*, 12(1):78–102, 2006.

[DV06b] Jan Denef and Frederik Vercauteren. An extension of Kedlaya’s algorithm to hyperelliptic curves in characteristic 2. *J. Cryptology*, 19(1):1–25, 2006.

[Dwo62] Bernard Dwork. On the zeta function of a hypersurface. *Inst. Hautes Études Sci. Publ. Math.*, (12):5–68, 1962.

[Ger07] Ralf Gerkmann. Relative rigid cohomology and deformation of hypersurfaces. *Int. Math. Res. Pap. IMRP*, (1):Art. ID rpm003, 67, 2007.

[GG01] Pierrick Gaudry and Nicolas Gürel. An extension of Kedlaya’s point-counting algorithm to superelliptic curves. In *Advances in cryptology—ASIACRYPT 2001* (Gold Coast), volume 2248 of *Lecture Notes in Comput. Sci.*, pages 480–494. Springer, Berlin, 2001.

[GH00] Pierrick Gaudry and Robert Harley. Counting points on hyperelliptic curves over finite fields. In *Algorithmic number theory* (Leiden, 2000), volume 1838 of *Lecture Notes in Comput. Sci.*, pages 313–332. Springer, Berlin, 2000.

[GS04] Pierrick Gaudry and Éric Schost. Construction of secure random curves of genus 2 over prime fields. In *Advances in cryptology—EUROCRYPT 2004*, volume 3027 of *Lecture Notes in Comput. Sci.*, pages 239–256. Springer, Berlin, 2004.

[GKS11] Pierrick Gaudry, David Kohel, and Benjamin Smith. Counting points on genus 2 curves with real multiplication. In *Advances in cryptology—ASIACRYPT 2011*, volume 7073 of *Lecture Notes in Comput. Sci.*, pages 504–519. Springer, Heidelberg, 2011.

[Gri69] Phillip A. Griffiths. On the periods of certain rational integrals. I, II. *Ann. of Math.* (2) 90 (1969), 460-495; *ibid.* (2), 90:496–541, 1969.

[GS04] Pierrick Gaudry and Éric Schost. Construction of secure random curves of genus 2 over prime fields. In *Advances in cryptology—EUROCRYPT 2004*, volume 3027 of *Lecture Notes in Comput. Sci.*, pages 239–256. Springer, Berlin, 2004.

[Har07] David Harvey. Kedlaya’s algorithm in larger characteristic. *Int. Math. Res. Not. IMRN*, (22):Art. ID rnm095, 29, 2007.

[Har10a] David Harvey. Computing zeta functions of certain varieties in larger characteristic. *http://web.maths.unsw.edu.au/~davidharvey/talks/zetasqrfp-talk.pdf*, 2010. [Online; accessed 15-January-2018].

[Har10b] David Harvey. Computing zeta functions of projective surfaces in large characteristic. *http://web.maths.unsw.edu.au/~davidharvey/talks/zetasqrfp-talk3.pdf*, 2010. [Online; accessed 15-January-2018].

[Har12] Michael C. Harrison. An extension of Kedlaya’s algorithm for hyperelliptic curves. *J. Symbolic Comput.*, 47(1):89–101, 2012.

[Har14] David Harvey. Counting points on hyperelliptic curves in average polynomial time. *Ann. of Math.* (2), 179(2):783–803, 2014.

[Har15] David Harvey. Computing zeta functions of arithmetic schemes. *Proc. Lond. Math. Soc.* (3), 111(6):1379–1401, 2015.

[Has16] Brendan Hassett. Cubic fourfolds, K3 surfaces, and rationality questions. In *Rationality problems in algebraic geometry*, volume 2172 of *Lecture Notes in Math.*, pages 29–66. Springer, Cham, 2016.
[HS14] David Harvey and Andrew V. Sutherland. Computing Hasse-Witt matrices of hyperelliptic curves in average polynomial time. *LMS J. Comput. Math.*, 17(suppl. A):257–273, 2014.

[HS16] David Harvey and Andrew V. Sutherland. Computing Hasse-Witt matrices of hyperelliptic curves in average polynomial time, II. In *Frobenius distributions: Lang-Trotter and Sato-Tate conjectures*, volume 663 of *Contemp. Math.*, pages 127–147. Amer. Math. Soc., Providence, RI, 2016.

[Hub08] Hendrik Hubrechts. Point counting in families of hyperelliptic curves. *Found. Comp. Math.*, 8:137–169, 2008.

[Hub11] Hendrik Hubrechts. Memory efficient hyperelliptic curve point counting. *Int. J. Num. Theory*, 7(1):203–214, 2011.

[Ito16] Kazuhiro Ito. Unconditional construction of K3 surfaces over finite fields with given L-function in large characteristic. *preprint*, 2016. arXiv:1612.05382.

[Kad04] Shabnam N. Kadir. *The arithmetic of Calabi–Yau manifolds and mirror symmetry*, PhD thesis, Univ. of Oxford, 2004.

[Kat89] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Ked01] Kiran S. Kedlaya. Counting points on hyperelliptic curves using Monsky-Washnitzer cohomology. *J. Ramanujan Math. Soc.*, 16(4):323–338, 2001.

[Ked13] Kiran S. Kedlaya. Effective p-adic cohomology for cyclic cubic threefolds. In *Computational algebraic and analytic geometry*, volume 572 of *Contemp. Math.*, pages 127–171. Amer. Math. Soc., Providence, RI, 2013.

[Kou76] A. G. Kouchnirenko. Polyèdres de Newton et nombres de Milnor. *Invent. Math.*, 32(1):1–31, 1976.

[KS00] Maximilian Kreuzer and Harald Skarke. Complete classification of reflexive polyhedra in four dimensions. *Adv. Theor. Math. Phys.*, 4(6):1209–1230, 2000.

[Lau04a] Alan G. B. Lauder. Counting solutions to equations in many variables over finite fields. *Found. Comput. Math.*, 4(3):221–267, 2004.

[Lau04b] Alan G. B. Lauder. Deformation theory and the computation of zeta functions. *Proc. London Math. Soc.*, 3:565–602, 2004.

[LW08] Alan G. B. Lauder and Daqing Wan. Counting points on varieties over finite fields of small characteristic. In *Algorithmic number theory: lattices, number fields, curves and cryptography*, volume 44 of *Math. Sci. Res. Inst. Publ.*, pages 579–612. Cambridge Univ. Press, Cambridge, 2008.

[Monsky analysis and zeta functions*, volume 4 of *Lectures in Mathematics, Department of Mathematics, Kyoto University*. Kinokuniya Book-Store Co., Ltd., Tokyo, 1970.

[Mon71] Paul Monsky. Formal cohomology. III. Fixed point theorems. *Ann. of Math.* (2), 93:315–343, 1971.

[MW68] P. Monsky and G. Washnitzer. Formal cohomology. I. *Ann. of Math.* (2), 88:181–217, 1968.

[Pila90] J. Pila. Frobenius maps of abelian varieties and finding roots of unity in finite fields. *Math. Comp.*, 55(192):745–763, 1990.

[PT15] Sebastian Pancratz and Jan Tuitman. Improvements to the deformation method for counting points on smooth projective hypersurfaces. *Found. Comput. Math.*, 15(6):1413–1404, 2015.

[Reid80] Miles Reid. Canonical 3-folds. In *Journées de Géométrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[RV17] Kristian Ranestad and Claire Voisin. Variety of power sums and divisors in the moduli space of cubic fourfolds. *Doc. Math.*, 22:455–504, 2017.

[Sag] The Sage Developers. *SageMath, the Sage Mathematics Software System*. http://www.sagemath.org.
ZETA FUNCTIONS OF TORIC HYPERSURFACES

[Sch85] René Schoof. Elliptic curves over finite fields and the computation of square roots mod p. *Math. Comp.*, 44(170):483–494, 1985.

[Sho] Victor Shoup. NTL: Number Theory Library. http://www.shoup.net/ntl/.

[SV13] Steven Sperber and John Voight. Computing zeta functions of nondegenerate hypersurfaces with few monomials. *LMS J. Comput. Math.*, 16:9–44, 2013.

[Tae16] Lenny Taelman. K3 surfaces over finite fields with given $L$-function. *Algebra Number Theory*, 10(5):1133–1146, 2016.

[Tev07] Jenia Tevelev. Compactifications of subvarieties of tori. *Amer. J. Math.*, 129(4):1087–1104, 2007.

[Tui16] Jan Tuitman. Counting points on curves using a map to $\mathbb{P}^1$. *Math. Comp.*, 85(298):961–981, 2016.

[Tui17] Jan Tuitman. Counting points on curves using a map to $\mathbb{P}^1$, II. *Finite Fields Appl.*, 45:301–322, 2017.

[Tui18] Jan Tuitman. Computing zeta functions of generic projective hypersurfaces in larger characteristic. *Math. Comp.*, 2018.

[Var76] A. N. Varchenko. Zeta-function of monodromy and Newton’s diagram. *Invent. Math.*, 37(3):253–262, 1976.

[vdP86] Marius van der Put. The cohomology of Monsky and Washnitzer. *Mém. Soc. Math. France (N.S.)*, (23):4, 33–59, 1986. Introductions aux cohomologies $p$-adiques (Luminy, 1984).

[Yon90] Takashi Yonemura. Hypersurface simple $K3$ singularities. *Tohoku Math. J. (2)*, 42(3):351–380, 1990.

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