Dynamics of Simplest Chiral Gauge Theories

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Arguably, the simplest chiral gauge theories are SO(10) with \( N_f \) fermion fields in the spinor representation 16. We study their dynamics using their supersymmetric limits perturbed by an infinitesimal anomaly-mediated supersymmetry breaking as a guide. We predict the theory is gapped for \( N_f = 1, 2 \), while the SU\((N_f)\) global symmetry is broken to SO\((N_f)\) for moderately large \( N_f \geq 3 \).

INTRODUCTION

The fact that the weak interactions distinguish left from right came as a big surprise to physicists, as established by the \(^{60}\)Co \( \beta \)-decay experiment led by Chien-Shiumg Wu. This idea of parity violation became the essential foundation of the Standard Model of particle physics, namely that it is based on chiral gauge theories. Left-handed and right-handed particles have different quantum numbers, and actually they are fundamentally different particles unless the Higgs boson condensate connects them.

Despite the essential importance of chiral gauge theories in describing nature, its definition has not been clear. Even though perturbation theory can be developed with somewhat ad hoc ways of regularizing divergences, a non-perturbative definition on the lattice has been difficult due to the fermion doubling problem. There has been a lot of progress in this area (see, e.g., for reviews and references therein), yet numerical simulation has not been possible for four-dimensional spacetime due to the sign problem. Therefore, it is of paramount importance to come up with other approaches in defining and working out chiral gauge theories.

Recently, one of the authors (HM) proposed the methodology to use a supersymmetric (SUSY) version of gauge theories and perturb them by an infinitesimal supersymmetry breaking in the form called anomaly mediation in [19]. In many cases, supersymmetry allows for exact understanding of non-perturbative effects in gauge theories. In addition, anomaly mediation has the property called “UV insensitivity” such that its impact can be worked out whether the fields are elementary or composites. Therefore, the non-perturbative dynamics can be understood even in the presence of supersymmetry breaking. This methodology has already been applied to some chiral gauge theories [11, 12] with rather surprising results which did not agree with previous suggestions.

In the end, we will need computer simulations to understand which suggestions correctly describe non-perturbative dynamics of chiral gauge theories. Kikukawa suggested that SO(10) gauge theories with fermions in spinor (16) representations are likely the first of such theories to be simulated on computers [13].

SUPERSYMMETRIC SO(10) WITH 16’S

We can obtain non-perturbative superpotentials for the SO(10) theories with \( N_f \) 16s by first introducing additional \( 2(4 - N_f) + 1 \) fields in the vector representation. Then the theories “s-confine” and the superpotentials are given already in [19]. By adding mass terms to the extra fields and integrating them out, we obtain the desired superpotential of the original theories. Yet, the expressions in [19] are somewhat sketchy without precise numerical coefficients and inadequate for our purpose. Below we rederive the same results from the microscopic dynamics with precise expressions.

We consider the chiral superfields \( \psi_\alpha \) (\( \alpha = 1, \cdots, N_f \)) that are in the 16 representation of SO(10) (or Spin(10) to be more precise) and the fundamental representation of the SU\((N_f)\) global symmetry. They have gauge-invariant polynomials \( S_{\alpha \beta \gamma \delta} \) constructed as

\[
S_{\alpha \beta \gamma \delta} = \frac{1}{3!} A^{IJK}_{\alpha \beta} A^I_{\gamma \delta} = S_{\gamma \delta \alpha \beta}, \tag{1}
\]

\[
A^{IJK}_{\alpha \beta} = \psi^T_\alpha C_{I10} \Gamma^{IJK} \psi_\beta = -A^{IJK}_{\beta \alpha}. \tag{2}
\]

Here, \( I, J, K = 1, \cdots, 10 \) are SO(10) vector indices, and \( C_{10} \) is the “charge conjugation” matrix of SO(10), see the Appendix. Therefore, \( S_{\alpha \beta \gamma \delta} \) correspond to the following Young tableau of the SU\((N_f)\) flavor symmetry,

\[
S_{\alpha \beta \gamma \delta} = \begin{array}{c|c}
\alpha & \gamma \\
\hline
\beta & \delta
\end{array}
\]

(3)
For our convention of SO(10) gamma matrices (see the Appendix).

We discuss cases $N_f = 1, 2, 3, 4$. For higher $N_f$, the theory has a magnetic description \[20\]. Unfortunately these do not have the SU($N_f$) global symmetry manifest in the description. Here we do not attempt to study higher $N_f \geq 5$.

**ANOMALY MEDIATION**

Anomaly mediation of supersymmetry breaking (AMSB) can be formulated with the Weyl compensator $\Phi = 1 + \theta^2 m$ \[21\] that appears in the supersymmetric Lagrangian as

$$\mathcal{L} = \int d^4 \Phi^* \Phi K + \int d^2 \Phi^* W + c.c. \quad (4)$$

Here, $K$ is the Kähler potential, $W$ is the superpotential of the theory, and $m$ is the size of supersymmetry breaking. When the theory is conformal, $\Phi$ can be removed from the theory by rescaling the fields $\phi_i \rightarrow \Phi^{-1} \phi_i$. On the other hand, violation of conformal invariance leads to supersymmetry breaking effects. Solving for auxiliary fields, the superpotential leads to the tree-level supersymmetry breaking terms

$$\mathcal{L}_{\text{tree}} = m \left( \phi_i \frac{\partial W}{\partial \phi_i} - 3W \right) + c.c. \quad (5)$$

In addition, conformal invariance is anomalously broken due to the running of coupling constants, and there are loop-level supersymmetry breaking effects in the scalar masses and gaugino masses,

$$m^2_i(\mu) = -\frac{1}{4} \gamma_i(\mu) m^2, \quad \text{(6)}$$

$$m_\lambda(\mu) = -\frac{\beta(g^2)}{2g^2} (\mu)m. \quad \text{(7)}$$

Here, $\gamma_i = d \ln Z_i(\mu)/d \ln \mu$, $\hat{\gamma} = d \gamma_i / d \ln \mu$, and $\beta(g^2) = dg^2/d \ln \mu$.

For asymptotically free gauge theories without a superpotential, $m^2$ is positive and the gaugino acquires a mass. Therefore the UV theory can correctly decouple scalars and gauginos. In the IR theory, we can apply the above formulae even for composite fields because they depend only on the particle content and interactions present at the respective energy scale. This is the property called “UV insensitivity” which allows us to study the IR behavior of the theory. In cases below, we rely on the tree-level AMSB Eq. \[5\] which is justified in the weakly-coupled limits.

$N_f = 1$ **CASE**

This case does not have a $D$-flat direction where we can analyze the theory with large field amplitudes and hence weak coupling. It was called “non-calculable” by Affleck, Dine, and Seiberg \[22\], who made a plausibility argument that U(1)$_R$ anomalies cannot be matched with a reasonable fermion content. On the other hand, the broken U(1)$_R$ implied dynamical supersymmetry breaking (later firmed up by \[23\]). Then the massless particle content must consist of one Nambu–Goldstone boson (NGB) of the broken U(1)$_R$ and one goldstino of the broken supersymmetry. Together with AMSB, the U(1)$_R$ symmetry as well as supersymmetry are explicitly broken. Therefore, both the goldstino and the U(1)$_R$ NGB are massive and the theory should be gapped.

To verify this plausibility argument explicitly, we use a trick proposed by HM \[24\] to include a chiral superfield $H(10)$ in the vector representation to make the theory “calculable.” In the end one introduces a mass term to $H$ as

$$W_{\text{tree}} = \frac{1}{2} MH^2 \quad \text{(8)}$$

to the superpotential. Raising $M$ beyond $\Lambda$ does not lead to a phase transition thanks to the holomorphy \[25\]. The $D$-flat direction is parameterized by

$$H = \frac{1}{\sqrt{2}}(0, 0, 0, 0, 0, 0, 0, 0, i(H^+ - H^-), (H^+ + H^-)), \quad \text{(9)}$$

$$\psi = \frac{1}{\sqrt{2}}(\uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \downarrow \otimes \downarrow \otimes \downarrow \otimes \uparrow) \chi. \quad \text{(10)}$$

The $D$-flatness for the (9,10) generator requires

$$|H^+|^2 - |H^-|^2 - \frac{1}{2} |\chi|^2 = 0. \quad \text{(11)}$$

Along this flat direction, SO(10) is broken to SO(7). It is obvious that $H$ above breaks SO(10) to SO(8), where $16$ decomposes as $8_s \oplus 8_a$ as the spinor and anti-spinor. Using the triality of SO(8), the anti-spinor can be regarded as a vector. Its vacuum expectation value (VEV) breaks SO(8) further to SO(7) with a non-trivial embedding. Indeed, the Higgs mechanism in supersymmetry gives the counting $45 - 2 \times 16 = 21 - 2$ where 2 classical moduli fields can be identified with $\psi \psi H$ and $H^2$. The gaugino condensate of SO(7) generates the dynamical superpotential

$$W_{\text{dyn}} = 5 \left( \frac{\Lambda^{21}}{(\psi \psi H)^2} \right)^{1/5}, \psi \psi H = \sqrt{2} H^+ \chi^2. \quad \text{(12)}$$

It is easy to obtain a minimum that breaks supersymmetry with a massless goldstino and a massless U(1)$_R$ NGB. The Witten index vanishes because of the dynamically broken supersymmetry. Given that the Witten index is topologically invariant, it stays vanishing as $M \rightarrow \infty$, establishing the dynamical supersymmetry breaking in the SO(10) theory with a single $16$ \[24\].
Together with AMSB, both supersymmetry and U(1)_R are explicitly broken with no global symmetry. Correspondingly, we confirmed that both the goldstino and the U(1)_R NGB acquire mass in the limit m ≪ M ≪ Λ where a perturbative analysis is justified. We verified that the spectrum is indeed gapped, namely, there is no massless field.

\[ N_f = 2 \text{ CASE} \]

The D-flat direction can be understood using the hierarchical symmetry breaking. Under the SO(8) subgroup, we have two pairs of (8_s ⊕ 8_a). Using the triality of SO(8), we can regard them as two (8_s ⊕ 8_s). While two 8_s breaks SO(10) to SO(8), one 8_s breaks it to SO(7) and then the other to G_2. Recall that a spinor VEV of SO(7) breaks it to G_2.

Explicit D-flat direction for the classical moduli space is described by \( S_{\alpha \beta \gamma \delta} = S_{\epsilon \alpha \beta \gamma \delta} \) which is SU(2) singlet,

\[ \psi_1 = \frac{v}{2} (\zeta_4 + \zeta_7), \quad \psi_2 = \frac{v}{2} (\zeta_1 + \zeta_6), \quad (13) \]

giving \( S = 3v^4/2 \). See Appendix for the definitions of \( \zeta_i \).

In fact, 45 - 14 = 16 × 2 - 1 and hence all \( \psi_{1,2} \) are eaten except for the D-flat direction \( S \).

The gaugino condensate of \( G_2 \) generates the superpotential

\[ W_{\text{dyn}} = 4 \left( \frac{\Lambda^{20}}{S^2} \right)^{1/4}. \quad (14) \]

With AMSB, the potential using the canonical Kähler potential along \( v \) is

\[ V = \frac{128 \Lambda^{10}}{3v^6} - m \frac{80 \Lambda^5}{\sqrt{6} v^2}, \quad (15) \]

which settles to a minimum with no SU(2) breaking,

\[ v = \left( \frac{384 \Lambda^{10}}{25 m^2} \right)^{1/8}. \quad (16) \]

When \( m \ll \Lambda \), the minimum is \( v \gg \Lambda \) and hence the gauge coupling is weak, justifying the analysis using the canonical Kähler potential. The light spectrum consists of scalar of mass squared \( 100m^2/3 \), pseudo-scalar \( 50m^2/3 \), and Majorana fermion \( 25m^2 \), satisfying the vanishing supertrace. The spectrum is gapped.

\[ N_f = 3 \text{ CASE} \]

The general D-flat direction breaks SO(10) to SU(2). It was non-trivial to come up with explicit parameterization of the general D-flat direction as is often the case for chiral gauge theories. It is helpful to observe that the unbroken SU(2) can be identified with SU(2)_R of the maximal Pati–Salam subgroup SO(10) ⊃ SO(6) × SO(4) ≃ SU(4) × SU(2)_L × SU(2)_R where the decomposition is

\[ 16 = (4^*, 1, 2) + (4, 1, 2). \quad (17) \]

In order to keep SU(2)_R unbroken, the VEVs reside only in \((4, 2, 1)\). Out of 45 generators of SO(10), 15 + 3 + 3 = 21 generators preserve the Pati–Salam group, while the remaining 24 generators are \((6, 2, 2)\) that map \((4, 2, 1)\) to \((4^*, 1, 2)\) and vice versa. As long as \((4^*, 1, 2)\) all vanish, \( D \)-flatness under SU(4)×SU(2)_L is sufficient to guarantee the \( D \)-flatness under the whole SO(10). Explicit D-flat direction for the classical moduli space is described by

\[ S_{\alpha \beta \gamma \delta} = \epsilon_{\alpha \beta \eta \xi} \bar{\psi}_\eta \bar{\psi}_\xi \quad (18) \]
in the $6^*$ representation of the global SU(3) symmetry. We find the $D$-flat direction
\[
\psi_1 = v \begin{pmatrix} \cos \theta & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, 
\psi_2 = v \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \cos \theta \end{pmatrix}, 
\psi_3 = v \begin{pmatrix} \sin \theta \sin \phi & -\sin \theta \cos \phi \\ 0 & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi \end{pmatrix},
\]
which gives
\[
\mathcal{S} = v^4 \begin{pmatrix} 0 & -12 \sin^2 \theta & 6 \sin 2\theta \sin \phi \\ -12 \sin^2 \theta & 6 \sin 2\theta \sin \phi \\ 6 \sin 2\theta \sin \phi & 6 \sin 2\theta \sin \phi & -24 \cos^2 \theta \end{pmatrix}.
\]
We verified that all possible eigenvalues can be generated by appropriate choice of $v, \theta, \phi$.

Along the $D$-flat directions, SO(10) is generically broken to SU(2)$_R$. Out of $16 \times 3 = 48$ chiral superfields, $45 - 3 = 42$ are eaten, leaving $48 - 42 = 6$ components described by $\mathcal{S}^{\frac{1}{2}}$. The remaining SU(2)$_R$ gauge group develops a gaugino condensate which leads to the runaway behavior of $\det S$,
\[
W_{\text{dyn}} = 2 \left( \frac{\Lambda^{18}}{\det S} \right)^{1/2}.
\]
To work out the potential, it is important that all chiral superfields participate in its derivation even if they do not have an expectation value. We find
\[
V_{\text{SUSY}} = \frac{\csc^6(\theta) \sec^2(\theta) \sec^2(\phi) \Lambda^{18}}{13824 v^{14}} 
\times \left( (\cos(2\theta) + 3) \sec^2(\theta) \sec^2(\phi) - 9 \cos(2\theta) + 13 \right),
\]
and
\[
V_{\text{AMS}} = -\sqrt{2} \frac{m \Lambda^{18}}{4 v^{14}} \csc^2(\theta) \sec(\theta) \sec(\phi).
\]
With AMSB, it settles to a minimum with $\theta = \arctan \sqrt{2}$ and $\phi = 0$, where three eigenvalues of $\mathcal{S}$ are the same up to a unitary transformation. Therefore, the theory breaks SU(3) to SO(3). The minimum is at
\[
v = \left( \frac{49 \Lambda^{18}}{18432 m^2} \right)^{1/16}
\]
and hence $v \gg \Lambda$ as long as $m \ll \Lambda$ which justifies the weakly coupled analysis using canonical Kähler potential. The light spectrum consists of multiplets $6^* = 1 + 5$ under SO(3) with mass-squared
\[
\text{scalars: } \frac{648}{7} m^2, \frac{162}{49} m^2 \times 5,
\]
\[
\text{pseudo-scalars: } \frac{486}{7} m^2, 0 \times 5,
\]
\[
\text{fermions: } 81 m^2, \frac{81}{49} m^2 \times 5.
\]

Figure 3: Top: Contour plot of the potential on the $(\theta, \phi)$ plane with $\Lambda = 1$, $m = 0.01$, and $v$ at its minimum Eq. (24). It shows the minimum at $\phi = 0$ and $\theta = \arctan \sqrt{2} \approx 0.955$. The other minimum is equivalent under $(\theta, \phi) \simeq (-\theta, -\phi)$. Bottom: The behavior of the potential along the $v$ direction with (red) and without (blue) the anomaly mediation.

Five massless pseudo-scalars are the NGBs of SU(3)/SO(3), and they together satisfy the vanishing supertrace.

$N_f = 4$ CASE

In this case, the SO(10) gauge group is generically completely broken on the moduli space. Out of $16 \times 4 = 64$ chiral superfields, 45 are eaten, leaving $64 - 45 = 19$ moduli fields. They are described by 20 of SU(4), or equivalently traceless symmetric tensor $S = S^T$, $\text{Tr} S = 0$ of SO(6) with one constraint. Unfortunately, the expression given in [19] is too sketchy for our purpose. We find that the classical constraint is $4 \text{Tr} S^4 - (\text{Tr} S^2)^2 = 0$ by explicitly looking at many $D$-flat solutions. This factor of four is crucial because it tells us which symmetry breaking pattern is possible within the constraint. Quantum mechanically, the constraint is modified and we have the superpotential
\[
W_{\text{dyn}} = X \left( 4 \text{Tr} S^4 - (\text{Tr} S^2)^2 - \Lambda^{16} \right).
\]

The minimum turns out to be near $\Lambda$ and is not weakly coupled. Yet the fact that anomaly matching conditions
are satisfied using $S$ as the dynamical degree of freedom implies that Kähler potential is non-singular for $S$. We use canonical Kähler potential for the $S$ field as well as the Lagrange multiplier field $X$ for the analysis.

We studied following possible ground states,

\begin{align}
\text{SO}(4) \times \text{SO}(2) : & \quad S \propto \text{diag}(1, 1, 1, 1, -2, -2), \quad (29) \\
\text{SO}(2)^3 : & \quad S \propto \text{diag}(a, a, b, b, -a - b, -a - b), \quad (30) \\
\text{SO}(3) \times \text{SO}(3) : & \quad S \propto \text{diag}(1, 1, 1, 1, -1, -1), \quad (31) \\
\text{SO}(5) : & \quad S \propto \text{diag}(1, 1, 1, 1, -5), \quad (32) \\
\text{SO}(4) : & \quad S \propto \text{diag}(a, a, a, b, -a - b). \quad (33) 
\end{align}

The first two satisfy the classical constraint but not the quantum modified constraint, and hence are not on the moduli space quantum mechanically. The last one is minimized for $b = -5a$ which is actually the SO(5) configuration. Its vacuum energy is $V_0 = -0.000052A^4$ for $m = 0.01A$, but has tachyonic pseudo-scalar in the traceless symmetric tensor direction of SO(5) and it is hence unstable. The SO(3) \times \text{SO}(3) configuration has the vacuum energy $V_0 = -0.00017A^4$ and is the deepest. Note also that there is no run-away direction given that $m_b^2 > 0$ in the UV.

We consider \text{SO}(6)/(\text{SO}(3) \times \text{SO}(3)) = \text{SU}(4)/\text{SO}(4) as the main candidate for the ground state. However this analysis is not rigorous, given that the theory is in the strongly coupled regime and we do not have control over the Kähler potential. Therefore we leave the possibility of \text{SO}(6)/\text{SO}(5) = \text{SU}(4)/\text{Sp}(4) open to be conservative.

**NON-SUPERSYMMETRIC LIMITS**

As $m$ is increased and approaches $\Lambda$, we cannot solve the theory any more. There may be a phase transition around $m \sim \Lambda$. If there isn’t, the theory remains in the same universality class, namely that it has the same massless particle content and symmetry breaking pattern. We hope to learn non-SUSY limits this way. At the same time, it is important to discuss what we may expect in the non-SUSY limit to see if it is plausible that they are connected continuously. In this discussion, we abuse our notation to refer to Weyl fermions using the same symbol $\psi_{\alpha}$ as the chiral superfields above.

There is no fermionic composite operator, given that it would require an odd power of $\psi$, while each $\psi$ is odd under the $\mathbb{Z}_2$ center of Spin(10), and hence an odd power of $\psi$ cannot be gauge invariant. Therefore, there cannot be a massless composite fermion for the anomaly matching. The SU($N_f$) global symmetry must be dynamically broken to an anomaly-free subgroup. The candidate subgroups are SO($N_f$) or Sp($2[N_f/2]$). Using the Hilbert series techniques developed for the Standard Model Effective Field Theory (SMEFT) in [26], one finds 1, 6, 21, 56 of them for $N_f = 1, 2, 3, 4$, respectively [31]. They correspond to the following four-fermion quadrilinear operators with a possible condensate,

\begin{align}
Z_{\alpha \beta \gamma \delta} = & \, (X^{I}_{\alpha \beta}, X^{I}_{\gamma \delta}), \quad (34) \\
X^{I}_{\alpha \beta} = & \, X^{I}_{\alpha \beta} \equiv \psi_{\alpha}^{T}CC_{10}^{I}\psi_{\beta}, \quad (35)
\end{align}

where $C = i\gamma_0\gamma_2$ is the usual charge conjugation matrix of the SO(3,1) Lorentz group. Since the fermions $\psi_{\alpha}$ ($\alpha = 1, \cdots N_f$) are in the fundamental representation of SU($N_f$), the possible condensate is in the representation

\begin{align}
\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\
\hline
N_f & + & 3 & 4 \\
\end{array}
\end{align}

or

\begin{align}
\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\
\hline
N_f^2 & 2(N_f^2 - 1) & 12 & N_f \\
\end{array}
\end{align}

In general, the quadrilinear condensate may be

\begin{align}
Z_{\alpha \beta \gamma \delta} \propto & \, \delta_{\alpha \beta}\delta_{\gamma \delta} + \delta_{\alpha \gamma}\delta_{\beta \delta} + \delta_{\alpha \delta}\delta_{\beta \gamma}, \quad (38)
\end{align}

for the former to leave SO($N_f$) ⊂ SU($N_f$) unbroken, or

\begin{align}
Z_{\alpha \beta \gamma \delta} \propto & \, \delta_{\alpha \beta}\delta_{\gamma \delta} - \delta_{\alpha \gamma}\delta_{\beta \delta} \quad \text{or} \quad J_{\alpha \gamma}J_{\beta \delta} + J_{\alpha \delta}J_{\beta \gamma}, \quad (39)
\end{align}

for the latter to leave SO($N_f$) or Sp($2[N_f/2]$) ⊂ SU($N_f$) unbroken. For the $N_f = 4$ case, there is another singlet operator,

\begin{align}
\epsilon^{\alpha \beta \gamma \delta}(\psi_{\alpha}^{T}CC_{10}^{IJK}\psi_{\beta})(\psi_{\gamma}^{T}CC_{10}^{IJK}\psi_{\delta}), \quad (40)
\end{align}

which is not relevant for symmetry breaking.

The analysis using supersymmetry above shows the preference for SO($N_f$) for $N_f = 3$ [38], while for Sp($2$) = SU($2$) for $N_f = 2$ [39]. For $N_f = 4$ our result prefers SO($4$) while it is inconclusive and may allow for Sp($4$).

One of the few methods proposed to analyze dynamics of chiral gauge theory is called the tumbling [27] (originally suggested in [28]). The tumbling hypothesis assumes a fermion bilinear condensate in the Most Attractive Channel (MAC). We could not find applications of the tumbling hypothesis to the SO(10) theory with 16s, and we attempt it here. Between two 161 and 162, there are $16^2 = 10_8 \oplus 120_4 \oplus 126_8$ channels. The one-gauge-boson-exchange potential is given by

\begin{align}
V = & \, g_2^2 \sum_a T^a_1T^a_2 = \frac{g_1^2}{r} \frac{1}{2}(C_{16}^2 - 2C_{16}), \quad (41)
\end{align}

with the quadratic Casimir operators $C_{R}$, and we find

\begin{align}
\sum_a T^a_1T^a_2 = \begin{cases}
\frac{-27}{\Delta_{16}(-10)} & 10 \\
\frac{1}{\Delta_{16}(+120)} & 120 \\
\frac{1}{\Delta_{16}(+126)} & 126
\end{cases}.
\end{align}

The MAC then suggests

\begin{align}
\langle \psi_{\alpha}^{T}CC_{10}^{I}\psi_{\beta} \rangle = \langle \psi_{\beta}^{T}CC_{10}^{I}\psi_{\alpha} \rangle \propto \Lambda^3\delta^{I,10}\delta_{\alpha\beta}, \quad (43)
\end{align}
which breaks SO(10) gauge group to SO(9), while the
global symmetry SU(N_f) to SO(N_f). The fermions
ψα (16) are real representations under the SO(9) gauge
group and hence the theory is vector-like. It allows for
the standard chiral symmetry breaking consistent with
the remaining SO(N_f) symmetry, resulting in the con-
 fined massive Majorana fermions. However, an order pa-
rameter such as (43) cannot be taken at face value be-
cause gauge-non-invariant operators cannot have VEVs
29. This argument can be taken only as suggestive at
best.

The case for N_f = 2 is special where the supersymmetric
analysis suggests that the global SU(2) symmetry is un-
broken. It corresponds to the second attractive chan-
nel in 120. Namely the tumbling hypothesis and super-
symmetric analysis differ in the predictions, which would
be very interesting to be settled by future lattice simu-
lations. It may be that SU(N_f)/Sp(N_f) is the preferred
ground state when N_f is even.

It is worthwhile recalling the results for other chiral
gauge theories. For SU(N_c) with one anti-symmetric ten-
sor A and anti-fundamentals Fα (α = 1, · · · , N_c − 4),
there is SU(N_c − 4) × U(1) global symmetry. For N_c =
even, the global symmetry is broken to Sp(N_c − 4) and
there are no massless fermions. For N_c = odd, however,
the global symmetry is broken to Sp(N_c − 5) × U(1), and
there are N_c − 5 massless fermions that match anomali-
ies 11. For SU(N_c) with one symmetric tensor S and an-
ti-fundamentals Fα (α = 1, · · · , N_c + 4), there is
SU(N_c + 4) × U(1) global symmetry that is broken to
SO(N_c + 4) and there are no massless fermions 12. It
would be useful to study more examples to draw general
lessons on chiral gauge theories 30.

CONCLUSION

We studied dynamics of SO(10) gauge theory with N_f
fermions in the 16-representation. This is arguably the
simplest chiral gauge theory to be simulated on lattice
in the near future. We solved the theory exactly with
supersymmetry broken infinitesimally by anomaly medi-
ation. The theory is gapped for N_f = 1, 2, while ex-
hibits SU(3)/SO(3) symmetry breaking for N_f = 3. The
case N_f = 4 prefers SU(4)/SO(4) while SU(4)/Sp(4)
possibility remains. Even though we cannot exclude a
phase transition when supersymmetry breaking is in-
creased, we find it quite plausible that the general pat-
tern of symmetry breaking is SU(N_f)/SO(N_f) for the
non-supersymmetric limit.

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CONVENTIONS

We adopt the following conventions for the SO(10)
gamma matrices. Using the standard Pauli matrices
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

we define 32-by-32 gamma matrices that satisfy the Cliff-
ford algebra \{Γ^I, Γ^J\} = 2δ^IJ,
\[
\Gamma^1 = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0,
\Gamma^2 = \sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0,
\Gamma^3 = \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0,
\Gamma^4 = \sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0,
\Gamma^5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \sigma_0,
\Gamma^6 = \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \sigma_0,
\Gamma^7 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_0,
\Gamma^8 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_0,
\Gamma^9 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1,
\Gamma^{10} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2,
\Gamma^{11} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3.
\]
The 16 representation consists of the components Γ^{11} =
+1. The generators of SO(10) are given by
\[
\frac{1}{2} \Sigma^{IJ}, \quad \Sigma^{IJ} = \frac{i}{2} [\Gamma^I, \Gamma^J].
\]

We also use the rank-three matrices
\[
\Gamma^{IJK} = \frac{i}{3!} \Gamma^{IJ} \Gamma^K,
\]
where the indices I, J, K are totally anti-symmetric. The
“charge conjugation” matrix is defined by
\[
C_{10} = -\Gamma^2 \Gamma^4 \Gamma^6 \Gamma^8 \Gamma^{10} = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2,
\]
which satisfies C_{10} = -C_{10}^T = C_{10}^{-1}, and C_{10} Γ^I C_{10}^{-1} = -(Γ^I)^T.
Using this convention, the $D$-flat direction for the three flavor case Eq. (19) is in the $(4, 2, 1)$ components under the Pati–Salam subgroup $SU(4) \times SU(2)_L \times SU(2)_R$, and hence we focus on them,

$$\psi = v \begin{pmatrix} \zeta_1 & \zeta_5 \\ \zeta_2 & \zeta_6 \\ \zeta_3 & \zeta_7 \\ \zeta_4 & \zeta_8 \end{pmatrix},$$

where each of the components is given by

$$\zeta_1 = \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow,$$

$$\zeta_2 = \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow,$$

$$\zeta_3 = \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow,$$

$$\zeta_4 = \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow,$$

$$\zeta_5 = \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow,$$

$$\zeta_6 = \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow,$$

$$\zeta_7 = \downarrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow,$$

$$\zeta_8 = \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow.$$

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