Analytically computable tangle for three-qubit mixed states

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Abstract. We present a new tripartite entanglement measure for three-qubit mixed states. The new measure \( t_r(\rho) \), which we refer to as the \( r \)-tangle, is given as a kind of the tangle, but has a feature which the tangle does not have; if we can derive an analytical form of \( t_r(\rho) \) for a three-qubit mixed state \( \rho \), we can also derive \( t_r(\rho') \) analytically for any states \( \rho' \) which are SLOCC-equivalent to the state \( \rho \). The concurrence of two-qubit states also satisfies the feature, but the tangle does not. These facts imply that the \( r \)-tangle \( t_r \) is the appropriate three-partite counterpart of the concurrence. We also derive an analytical form of the \( r \)-tangle \( t_r \) for mixtures of a generalized GHZ state and a generalized W state, and hence for all states which are SLOCC-equivalent to them.

1. Introduction

Quantum tasks beyond the classical tasks, such as quantum computing, teleportation, superdense coding, etc., utilize the entanglement as an important resource [1,2,3,4]. On one hand, with the development of the quantum information processing, manipulating many particles entangled to each other has become possible [5,6]. On the other hand, however, the quantification of the entanglement is still a fundamental problem in the field of quantum information. Vigorous effort has been made, and the problem has been solved for two-qubit pure and mixed states as well as for three-qubit pure states. The concurrence [7,8,9] and the negativity [10] make it possible for us to quantify the entanglement analytically for two-qubit pure and mixed states. The stochastic LOCC classification of three-qubit pure states revealed [11] that there exist two types of three-partite entanglement, namely the GHZ-type and the W-type. The tangle [12] and \( J_5 \) [13] enabled us to quantify the entanglements of these two types. With using the concurrence, the tangle, the parameter \( J_5 \) and the parameter \( Q_e \) introduced in Ref. [14], a necessary and sufficient condition of the possibility of deterministic LOCC transformations is given for arbitrary three-qubit pure states [14].

We thereby understood the features of two-qubit pure and mixed states as well as three-qubit pure states. Apart from the above, however, our comprehension is not
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Although there have been many researches on the tangle for three-qubit mixed states, its analytical form has been derived only in restricted regions [15, 16, 17, 18, 19]. The approach for deterministic LOCC used in Ref. [14] cannot be applied to three-qubit mixed states directly, because an important feature which holds for the tangle of pure states does not hold for the tangle of mixed states; when we perform a measurement \( \{M(i)\} \) on the qubit \( A \) of a three-qubit pure state \( |\psi\rangle \), the tangle \( \tau \) of the \( i \)th result

\[
|\psi(i)\rangle = M(i)|\psi\rangle / \sqrt{p(i)}
\]

and the tangle \( \tau \) of \( |\psi\rangle \) satisfy the following equation:

\[
\tau(|\psi(i)\rangle) = \alpha^2(i) \tau(|\psi\rangle), \quad \alpha(i) \equiv \frac{\sqrt{\det M^\dagger(i)M(i)}}{p(i)}.
\]  

(1.1)

This feature does not generally hold for the tangle of mixed states; we give an example that \( \tau(\rho(i)) \neq \alpha^2(i) \tau(\rho) \) in Appendix A.

In the present paper, we introduce a new tripartite entanglement measure for three-qubit mixed states, which we refer to as the r-tangle. The r-tangle \( t_r \) can be interpreted as a kind of the tangle; when the state is pure, the square of the r-tangle is equal to the tangle. The r-tangle also satisfies the following equation:

\[
t_r(\rho(i)) = \alpha(i)t_r(\rho), \quad \rho(i) \equiv \frac{M(i)\rho M^\dagger(i)}{p(i)},
\]  

(1.2)

where \( \alpha(i) \) is the same as in (1.1). The feature (1.2) has two merits. First, using the r-tangle, we may be able to derive a necessary and sufficient condition of the possibility of deterministic LOCC transformations for arbitrary three-qubit mixed states; because \( t^2_r(\rho(i)) = \alpha^2(i)t^2_r(\rho) \) holds, we may apply the approach in Ref. [14] to the mixed states by employing \( t^2_r(\rho) \) as a substitute for the tangle \( \tau(\rho) \). Second, we can derive the r-tangle analytically in broader regions than the tangle; if we can derive an analytical form of \( t_r(\rho) \) for a three-qubit mixed state \( \rho \), the equation (1.2) let us derive \( t_r(\rho') \) analytically for any states \( \rho' \) which are SLOCC-equivalent to the state \( \rho \).

Moreover, we also derive an analytical form of the r-tangle for mixtures of a generalized GHZ state and a generalized W state. For such states, the analytical form of the tangle also has been derived [17]. Using (1.2), we can derive the r-tangle not only for the mixtures but also for any states which are SLOCC-equivalent to the mixtures. Note again that we also cannot apply the approach to the tangle, because \( \tau(\rho(i)) = \alpha^2(i) \tau(\rho) \) does not hold generally.

2. Main Results

In the present section, we give two theorems for the r-tangle for three-qubit mixed states. First, we give the definition of the r-tangle:

\[
t_r(\rho) = \min_{\rho = \sum q_i|\psi_i\rangle\langle\psi_i|} \sum_i q_i \sqrt{\tau(|\psi_i\rangle)},
\]  

(2.1)
where $\sqrt{\tau(|\psi\rangle)}$ is written in terms of the coefficients $C_{ijk}$ as
\begin{equation}
|\psi\rangle = \sum_{p,q,r} C_{pqr} |pqr\rangle
\end{equation}

\begin{align}
\sqrt{\tau(|\psi\rangle)} &= 4|d_1 - 2d_2 + 4d_3|,
\end{align}
\begin{align}
d_1 &= C_{000}C_{111} + C_{001}C_{110} + C_{010}C_{101} + C_{100}C_{011},
d_2 &= C_{000}C_{111}C_{011}C_{110} + C_{000}C_{111}C_{010}C_{110} + C_{000}C_{111}C_{101}C_{001} + C_{011}C_{100}C_{101}C_{100} + C_{011}C_{100}C_{010}C_{100} + C_{101}C_{010}C_{110}C_{001},
d_3 &= C_{000}C_{110}C_{101}C_{011} + C_{111}C_{001}C_{010}C_{100}.
\end{align}

We refer to an ensemble \{\{q_i, |\psi_i\rangle\}\} of $\rho$ which minimizes the right-hand side of (2.1) as the optimal ensemble. We emphasize that $(t_r(\rho))^2 \neq \tau(\rho)$ because the mean of the square root is not equal to the square root of the mean. The equality $(t_r(\rho))^2 = \tau(\rho)$ is valid only when $\rho$ is pure.

Second, we give two theorems for the $r$-tangle. The first theorem below means that when we obtain the value of the $r$-tangle and the optimal ensemble of a state $\rho$, then we also obtain them of any states which are S-LOCC equivalent to $\rho$.

**Theorem 1** Suppose that a measurement \{\{M_{(j)}\}\} is performed on the qubit $A$ of an arbitrary three-qubit mixed state $\rho$ with the $r$-tangle $t_r(\rho)$ and the optimal ensemble \{\{q_i, |\psi_i\rangle\}\}. Suppose also that the state $\rho_{(j)} = M_{(j)}\rho M_{(j)}^\dagger/p_{(j)}$ is obtained as the $j$th result with the probability $p_{(j)}$. The $r$-tangle $t_r(\rho_{(j)})$ and the optimal ensemble \{\{r_{i,(j)}, |\varphi_{i,(j)}\rangle\}\} of $\rho_{(j)}$ are given by $t_r(\rho)$ and \{\{q_i, |\psi_i\rangle\}\} as follows:
\begin{align}
t_r(\rho_{(j)}) &= \alpha_{(j)} t_r(\rho), \quad \alpha_{(j)} \equiv \frac{\text{det} \sqrt{M_{(j)}^\dagger M_{(j)}}}}{p_{(j)}},
\end{align}
\begin{align}
r_{i,(j)} &= \frac{q_i}{p_{(j)}} \langle \psi_i | M_{(j)}^\dagger M_{(j)} | \psi_i \rangle,
\end{align}
\begin{align}
|\varphi_{i,(j)}\rangle &= \frac{M_{(j)} | \psi_i \rangle}{\sqrt{\langle \psi_i | M_{(j)}^\dagger M_{(j)} | \psi_i \rangle}}.
\end{align}

We can use Theorem 1 as follows; when we obtain the $r$-tangle for a mixed state $\rho$, we can also obtain it for any states in the same SLOCC class as $\rho$. Similarly, when we obtain the optimal ensemble for a mixed state $\rho$, we can also obtain it for any states in the same SLOCC class as $\rho$. Theorem 1 does not hold for the tangle $\tau(\rho)$; note again that $\tau(\rho) \neq (t_r(\rho))^2$. We show an explicit example of the case $\alpha_{(j)}^2 \tau(\rho_{(j)}) \neq \tau(\rho)$ in Appendix A.

The second theorem below gives $t_r(\rho)$ analytically when $\rho$ is a mixture of generalized GHZ and generalized W states.
Theorem 2  We have

\[ t_r(\rho(p)) = \begin{cases} 
0 & (0 \leq p \leq p_0) \\
2|ab| \frac{p-p_0}{1-p_0} & (p_0 \leq p \leq 1)
\end{cases} \tag{2.10} \]

where

\[ p_0 = \frac{s^{2/3}}{1 + s^{2/3}}, \tag{2.11} \]

\[ s = \frac{4cdf}{a^2b} > 0, \tag{2.12} \]

for the family of three-qubit mixed states

\[ \rho(p) = p \mid gGHZ_{a,b} \rangle \langle gGHZ_{a,b} \mid + (1-p) \mid gW_{c,d,f} \rangle \langle gW_{c,d,f} \mid, \tag{2.13} \]

which consists of a generalized GHZ state

\[ \mid gGHZ_{a,b} \rangle = a \mid 000 \rangle + b \mid 000 \rangle, \quad |a|^2 + |b|^2 = 1 \tag{2.14} \]

and a generalized W state

\[ \mid gW_{c,d,f} \rangle = c \mid 001 \rangle + d \mid 010 \rangle + f \mid 010 \rangle, \quad |c|^2 + |d|^2 + |f|^2 = 1. \tag{2.15} \]

Note that the analytical form of \( t_r(\rho(p)) \) is simpler than that of \( \tau(\rho(p)) \) in Ref. [17]. The r-tangle (2.10) consists of two straight lines as a function of \( p \), whereas the function \( \tau(\rho(p)) \) in Ref. [17] consists of two straight lines and a curve.

3. Proofs of Theorems

Proof of Theorem 1: We first consider the case in which \( \rho \) is pure. In this case, we have the equality \( t_r(\rho) = \sqrt{\tau(\rho)} \), and therefore Theorem 1 is included in Lemma 1 of Ref. [13].

Next, we consider the case in which \( \rho \) is mixed. Let us refer to the optimal ensembles of \( \rho \) and \( \rho_{(j)} \) as \( \{ q_i, \mid \psi_i \rangle \} \) and \( \{ r_{kj}, \mid \varphi_{kj} \rangle \} \), respectively. We will prove that

\[ t_r(\rho_{(j)}) = \alpha_{(j)} t_r(\rho) \quad \text{and} \quad \{ r_{kj}, \mid \varphi_{kj} \rangle \} = \{ r_{i(j)}, \mid \varphi_{i(j)} \rangle \}. \]

First, we consider the case in which \( \sqrt{\det(M_{(j)}^\dagger M_{(j)})} = 0 \) holds. In the present case, the qubit \( A \) becomes separable after the measurement, and thus the equation \( t_r(\rho_{(j)}) = 0 \) also holds. Thus, (2.7) is valid. We can also prove that the ensemble \( \{ r_{i(j)}, \mid \varphi_{i(j)} \rangle \} \) is optimal, because the states \( \mid \varphi_{i(j)} \rangle \) are separable or biseparable states: the qubits \( A \) of the states \( \mid \varphi_{i(j)} \rangle \) are separable. Thus Theorems 1 is valid when \( \sqrt{\det(M_{(j)}^\dagger M_{(j)})} = 0 \) holds.

Second, let us consider the case in which \( \sqrt{\det(M_{(j)}^\dagger M_{(j)})} \neq 0 \). First, we show that if we can prove the following two equations, we can also prove Theorem 1:

\[ \alpha_{(j)} t_r(\rho) \leq \sum_{kj} r_{kj} t_i(\mid \varphi_{kj} \rangle), \tag{3.1} \]

\[ \sum_i r_{i(j)} t_i(\mid \varphi_{i(j)} \rangle) \leq \alpha_{(j)} \sum_i q_i t_r(\mid \psi_i \rangle). \tag{3.2} \]
Because \( \{ r_{kj}, |\varphi_{kj}\rangle \} \) is the optimal ensemble of \( \rho_{(j)} \) and because \( \{ r_{i,(j)}, |\varphi_{i,(j)}\rangle \} \) is an ensemble of \( \rho_{(j)} \),
\[
t_t(\rho_{(j)}) = \sum_{k_j} r_{kj} t_t(\langle \varphi_{kj} |) \leq \sum_i r_{i(j)} t_t(\langle \varphi_{i(j)} |)
\]  
(3.3)
is valid. Note that \( \{ q_i, |\psi_i\rangle \} \) is the optimal ensemble of \( \rho \), and thus \( t_t(\rho) = \sum_i q_i t_t(\langle \psi_i |) \).
Thus, if (3.1) and (3.2) hold,
\[
\alpha_{\langle j} t_t(\rho_{(j)}) \leq \sum_i r_{i(j)} t_t(\langle \varphi_{i(j)} |) \leq \alpha_{\langle (j) t_t(\rho)
\]  
(3.4)
also holds. We can reduce (3.4) to
\[
\alpha_{\langle j} t_t(\rho_{(j)}) = t_t(\rho_{(j)}) = \sum_i r_{i(j)} t_t(\langle \varphi_{i(j)} |) = \alpha_{\langle (j) t_t(\rho),
\]  
(3.5)
and thus if we can prove Eqs. (3.1) and (3.2), we can also prove Theorem 1.

Let us prove (3.1) and (3.2). First, we prove (3.1). We prove (3.1) by introducing an ensemble \( \{ r_{kj}/L_{kj}^2, |\tilde{\varphi}_{kj}\rangle \} \) of \( \rho \) which satisfies
\[
\alpha_{\langle j} \sum_k r_{kj} t_t(\langle \tilde{\varphi}_{kj} |) = \sum_k r_{kj} t_t(\langle \tilde{\varphi}_{kj} |)
\]  
(3.6)
If we can introduce such ensemble of \( \rho \), we can prove (3.1) from (3.6); note that because \( \{ r_{kj}/L_{kj}^2, |\tilde{\varphi}_{kj}\rangle \} \) is an ensemble of \( \rho \),
\[
t_t(\rho) \leq \sum_k r_{kj}/L_{kj} t_t(\langle \tilde{\varphi}_{kj} |)
\]  
(3.7)
is valid.

We obtain \( \{ r_{kj}/L_{kj}^2, |\tilde{\varphi}_{kj}\rangle \} \) explicitly as follows. Now we consider the case in which \( \det(M_{(j)}) \neq 0 \) holds, and thus we can take \( M_{(j)}^{-1} \), which is the inverse of \( M_{(j)} \). We take the ensemble \( \{ r_{kj}/L_{kj}^2, |\tilde{\varphi}_{kj}\rangle \} \) as follows:
\[
|\tilde{\varphi}_{kj}\rangle \equiv L_{kj}\sqrt{p_{(j)}}M_{(j)}^{-1}|\varphi_{kj}\rangle,
\]  
(3.8)
where \( L_{kj} \) are normalization constants. We can prove that \( \{ r_{kj}/L_{kj}^2, |\tilde{\varphi}_{kj}\rangle \} \) satisfies (3.6), as follows;
\[
\sum_k r_{kj} t_t(\langle \varphi_{kj} |) = \sum_k r_{kj} t_t(\frac{M_{(j)}}{L_{kj}\sqrt{p_{(j)}}} |\tilde{\varphi}_{kj}\rangle) = \sum_k r_{kj} t_t(\langle \tilde{\varphi}_{kj} |) = \alpha_{\langle (j) \sum_k r_{kj}/L_{kj} t_t(\langle \tilde{\varphi}_{kj} |)
\]  
(3.9)
Finally, let us prove (3.2). Note that we can write \( \{ r_{i,(j)}, |\varphi_{i,(j)}\rangle \} \) as
\[
r_{i,(j)} = \frac{q_i}{N_{ij}^2}, \quad |\varphi_{i,(j)}\rangle = \frac{M_{(j)}}{\sqrt{p_{(j)}}}|\psi_i\rangle, \quad N_{ij} \equiv \frac{\sqrt{p_{(j)}}}{\sqrt{\langle \psi_i | M_{(j)}^{-1}\rangle M_{(j)} |\psi_i \rangle}}.
\]  
(3.10)
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Thus, we can derive (3.2) as follows:

$$\sum_i r_{i(j)} t_{i}(|\varphi_{i(j)}\rangle) = \sum_i \frac{q_i}{N^2_{ij}} t_{i} \left( N_{ij} \frac{M_{ij}}{\sqrt{p(j)}} |\psi_i\rangle \right)$$

$$= \sum_i \frac{q_i}{N^2_{ij}} N_{ij} \frac{\sqrt{\det(M_{ij}^\dagger M_{ij})}}{p(j)} t_{i} (|\psi_i\rangle) = \frac{\sqrt{\det(M_{ij}^\dagger M_{ij})}}{p(j)} \sum_i q_it_{i} (|\psi_i\rangle). \quad (3.11)$$

This completes the proof of Theorem 1. □

**Proof of Theorem 2:** We prove the present theorem by a method similar to the one used in Ref. [17]. First, we prove the following lemma.

**Lemma 1** If there is a function $f(p)$ which satisfies the following three conditions, it must be $t_{f}(\rho(p))$:

**Condition 1** The following inequality holds for any $p$ and $\varphi$:

$$f(p) \leq t_{f}(|p, \varphi\rangle), \quad (3.12)$$

where

$$|p, \varphi\rangle \equiv \sqrt{p} |gGHZ_{a,b}\rangle + \sqrt{1 - p} e^{i(\varphi - \tilde{\varphi}/3)} |gW_{c,d,f}\rangle, \quad (3.13)$$

$$\tilde{\varphi} \equiv \arg \left[ \frac{acdf}{a^2b} \right]. \quad (3.14)$$

**Condition 2** There exists an ensemble $\{p_i, |q_i, \varphi_i\rangle\}$ of $\rho(p)$ which satisfies the following equation:

$$f(p) = \sum_i p_i t_{f}(|q_i, \varphi_i\rangle). \quad (3.15)$$

**Condition 3** The function $f(p)$ is a convex function.

**Proof:** Because of Condition 2, we have $f(p) \geq t_{f}(\rho(p))$. We also prove $f(p) \leq t_{f}(\rho(p))$ as follows:

$$t_{f}(\rho(p)) = \sum_i \tilde{p}_i t_{f}(\tilde{q}_i, \tilde{\varphi}_i)$$

$$\geq \sum_i \tilde{p}_i f(\tilde{q}_i) \geq f(\sum_i \tilde{p}_i \tilde{q}_i) = f(p), \quad (3.16)$$

where $\{\tilde{p}_i, |\tilde{q}_i, \tilde{\varphi}_i\rangle\}$ is the optimal ensemble of $\rho(p)$. We have derived the first inequality from Condition 1 and the second inequality from Condition 3. □

Now we only have to prove that the right-hand side of (2.10), which we refer to as $g(p)$, satisfies Conditions 1–3. First, the function $g(p)$ is clearly convex, and thus Condition 3 holds. Second, we can take the ensemble of $\rho(p)$ which satisfies (3.15) as follows:

$$\rho(p) = \left\{ \begin{array}{ll}
\frac{p_0 - p}{p_0} |0, 0\rangle \langle 0, 0| + \frac{p}{3p_0} \sum_{n=0}^{2} |p_0, \frac{2n\pi}{3}\rangle \langle p_0, \frac{2n\pi}{3}| & (0 \leq p \leq p_0) \\
\frac{p - p_0}{1 - p_0} |1, 0\rangle \langle 1, 0| + \frac{1 - p}{3(1 - p_0)} \sum_{n=0}^{2} |p_0, \frac{2n\pi}{3}\rangle \langle p_0, \frac{2n\pi}{3}| & (p_0 \leq p \leq 1) 
\end{array} \right\}. \quad (3.17)$$
To prove that the above ensembles satisfy (3.15), we only have to notice that

\[ t_r(|p, \varphi \rangle) = 2|ab| \sqrt{|p^2 - \sqrt{p(1-p)}^3 e^{3i\varphi}|^{\frac{4cdf}{a^2b}}}, \]  

(3.18)

and especially

\[ t_r(|1, 0 \rangle) = 2|ab|, \]  

(3.19)

\[ t_r(|0, 0 \rangle) = t_r\left(|p_0, \frac{2n\pi}{3}\right) = 0. \]  

(3.20)

Thus, Condition 2 is valid.

Finally, let us prove Condition 1. Because \( t_r \) is non-negative, \( g(p) \) clearly satisfies Condition 1 for \( 0 \leq p \leq p_0 \). Note that for \( p_0 \leq p \leq 1 \), the function \( g(p) \) is a linear function of \( p \) and that the following three expressions hold:

\[ t_r(|p, 0 \rangle) \leq t_r(|p, \varphi \rangle), \]  

(3.21)

\[ t_r(|1, 0 \rangle) = g(1), \]  

(3.22)

\[ t_r(|0, 0 \rangle) = g(p_0). \]  

(3.23)

Thus, if we can prove \( t_r(|p, 0 \rangle) \) is concave for \( p_0 \leq p \leq 1 \), then we can also prove \( g(p) \leq t_r(|p, \varphi \rangle) \) for \( p_0 \leq p \leq 1 \). Let us prove the concaveness of \( t_r(|p, 0 \rangle) \). Only for simplicity, we refer to \( 4cdf/(a^2b) \) as \( s \). Then,

\[ \frac{d^2}{dp^2} t_r(|p, 0 \rangle) = 2|ab| \frac{d^2}{dp^2} \sqrt{p^2 - \sqrt{p(1-p)}^3 s} \]  

(3.24)

\[ = -\frac{1}{4(t_r(|p, 0 \rangle))^2} \left( \frac{12p^3 + 3p/2}{\sqrt{p(1-p)}} + s^2 - 4p^4 + 20p^3 - 3p^2 - 2p + 1 \right). \]  

(3.25)

The term \( 12p^3 + 3p/2 \) is clearly positive. The term \( -4p^4 + 20p^3 - 3p^2 - 2p + 1 \) is also positive for \( 0 \leq p \leq 1 \) as shown in Fig1. Therefore, \( t_r(|p, 0 \rangle) \) is concave for \( p_0 \leq p \leq 1 \), and thus the function \( g(p) \) satisfies Conditions 1–3. Hence, because of Lemma 1, the equation \( t_r(\rho(p)) = g(p) \) is valid. □
4. Conclusion

In the present article, we introduced a new entanglement measure which we call the r-tangle. The r-tangle \( t_r(\rho_{ij}) = \alpha_{ij} t_r(\rho) \). Thanks to the feature, if we derive an analytical form of the r-tangle for a three-qubit mixed state, we can also derive the r-tangle analytically for any states which are SLOCC-equivalent to the state. Note that the concurrences also satisfy a similar feature \( C(\rho_{ij}) = \alpha_{ij} C(\rho) \), and that the tangle \( \tau(\rho) \) does not satisfy such a feature; we show an example that \( \tau(\rho_{ij}) \neq \alpha_{ij}^2 \tau(\rho) \) in Appendix A. These facts imply that we should consider the r-tangle instead of the tangle as the three-partite counterpart of the concurrence. Moreover, we derive the analytical form of the r-tangle for mixtures of generalized GHZ state and generalized W state. Although the tangle has been also derived for such states [17], the form of \( t_r(\rho_{ij}) \) is simpler than that of \( \tau(\rho(p)) \) as the function of \( p \). Using \( t_r(\rho_{ij}) = \alpha_{ij} t_r(\rho) \), we can derive the r-tangle not only for the mixture but also for any state which is SLOCC-equivalent to the mixtures. We cannot apply the approach to the tangle, because \( \tau(\rho_{ij}) = \alpha_{ij}^2 \tau(\rho) \) does not hold generally.

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Appendix A.

In the present appendix, we will show a counterexample of \( \tau(\rho_{ij}) = \alpha_{ij}^2 \tau(\rho) \). Let us consider the following three-qubit mixed state:

\[
\rho = \frac{4}{5} \left| g_{GHZ} \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \left\langle g_{GHZ} \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right| + \frac{1}{5} \left| g_{W} \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \left\langle g_{W} \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right|.
\] (A.1)

According to a result in Ref. [17], we have \( \tau(\rho) = (63 - \sqrt{465})/90 \). Let us perform the following measurement on the qubit A of \( \rho \):

\[
M_{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{pmatrix}, \quad M_{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{3}{\sqrt{10}} \end{pmatrix}.
\] (A.2)

The probability \( p_{(0)} \) that we obtain the result 0 is 29/50, for which the state becomes

\[
\rho_{(0)} = \frac{22}{29} \left| g_{GHZ} \frac{10}{11}, \frac{1}{\sqrt{11}} \right\rangle \left\langle g_{GHZ} \frac{10}{11}, \frac{1}{\sqrt{11}} \right| + \frac{7}{29} \left| g_{W} \frac{10}{21}, \frac{10}{21}, \frac{1}{\sqrt{21}} \right\rangle \left\langle g_{W} \frac{10}{21}, \frac{10}{21}, \frac{1}{\sqrt{21}} \right|.
\] (A.3)

According to the result in Ref. [17], \( \tau(\rho_{(0)}) = 160(9 - \sqrt{6})/7569 \). Thus,

\[
\alpha_{(0)}^2 = \frac{\det M_{(0)}^\dagger M_{(0)}}{p_{(0)}^2} = \frac{250}{841}
\]
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\[ \neq \frac{1600(9 - \sqrt{6})}{841(63 - \sqrt{465})} = \frac{\tau(\rho(0))}{\tau(\rho)}. \]  

(A.4)

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