BIHOMOGENEITY AND MENER MANIFOLDS

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ABSTRACT. It is shown that for every triple of integers \((\alpha, \beta, \gamma)\) such that \(\alpha \geq 1\), \(\beta \geq 1\), and \(\gamma \geq 2\), there is a homogeneous, non-bihomogeneous continuum whose every point has a neighborhood homeomorphic the Cartesian product of Menger compacta \(\mu^\alpha \times \mu^\beta \times \mu^\gamma\). In particular, there is a homogeneous, non-bihomogeneous, Peano continuum of covering dimension four.

INTRODUCTION

A space is \(n\)-homogeneous if for any pair of \(n\)-point sets the space admits a homeomorphism sending one of the sets onto the other. A homogeneous space is a 1-homogeneous space. A space \(X\) is bihomogeneous if for any pair of points \(p\) and \(q\) in \(X\), there is a homeomorphism \(h : X \to X\) such that \(h(p) = q\) and \(h(q) = p\). A space \(X\) is strongly locally homogeneous if for every \(p \in X\) and every neighborhood \(U_p\) there is a neighborhood \(V_p\) such that for every \(q \in V_p\) there is a homeomorphism \(h : X \to X\) with \(h(p) = q\) and \(h(x) = x\) for \(x \notin U_p\). A continuum is a compact, connected, metric space containing more than one point.

Around 1921, B. Knaster asked whether homogeneity implies bihomogeneity, and C. Kuratowski (Kazimierz Kuratowski) [Kur] gave an example of a 1-dimensional, non-locally compact, homogeneous, non-bihomogeneous subset of the plane. An example similar to that of Kuratowski can be easily described as follows: Let \(p\) and \(q\) be distinct points in the same composant of a nontrivial solenoid \(\Sigma\). The composant of \(\Sigma - \{p\}\) containing \(q\) is homogeneous but it is not bihomogeneous; one end of the composant is dense in it whereas the other end is not, making swapping points impossible. It is also easy to obtain nonmetric examples. In 1986, H. Cook [Cook] described a locally compact, 2-dimensional, homogeneous, non-bihomogeneous metric space. It is still not known if there is a 1-dimensional [locally] compact metric example.

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In 1930, D. van Dantzig [Dan] restated Knaster’s question for continua, which was answered in [KuK] by an example of a locally connected, homogeneous, non-bihomogeneous continuum. The construction of the example starts with a space that is bihomogeneous but has a property in some sense contrary to bihomogeneity. This space consists of compatibly oriented circular fibers such that any homeomorphism maps a fiber onto a fiber, and to swap certain fibers the homeomorphism must reverse the orientation of the fibers. The next step is to replace each fiber with a homogeneous space containing it as a retract and not admitting a homeomorphism reversing the orientation of the original fiber. The fibers are held together by a rigid grid that is locally homeomorphic to the Cartesian product of two Menger universal curves. Each circle is then replaced by a larger fiber, a manifold, which contains \( S^1 \) as its retract, but admits no homeomorphism changing the sign of the generator of the first homology group represented by this \( S^1 \). The dimension of the example equals the dimension of the manifold plus two, giving a 7-dimensional continuum. The seemingly unrelated property of local connectedness is important for the notion of 2-homogeneity: G. S. Ungar [Un] proved that 2-homogeneous continua are locally connected. However, homogeneity does not imply 2-homogeneity for Peano continua [KKT1, Ke1, Ke2, KKT2, Gar] and, as the above example shows, it does not imply bihomogeneity.

A substantially simpler, although not locally connected, example of a homogeneous, non-bihomogeneous continuum was given by P. Minc [Minc]. The “model” space of Minc’s example is a solenoid, whose each arc component is replaced by a sequence of “glued together” mapping cylinders of a degree \( m \geq 2 \) map of \( S^1 \) onto \( S^1 \). For most pairs of composants of a solenoid, a homeomorphism swapping the composants must be orientation reversing (see [Minc]). Hence in the above continuum, not every two arc components can be swapped. To achieve homogeneity, Minc takes the Cartesian product of this continuum and the Hilbert cube. To get a finite dimensional example, a manifold of the same homotopy type as the above mapping cylinder can be used to replace the solenoid composants. Recently, K. Kawamura [Kaw] noticed that using Menger manifolds and an “\( n \)-homotopy mapping cylinders” (see [CKT and CKS]), the dimension of Minc’s example can be lowered to 2.

This paper shows that by applying Kawamura’s idea to the construction of [KuK], a locally connected, homogeneous, non-bihomogeneous continuum of dimension 4 can be obtained. The factorwise rigidity of the Cartesian products of Menger compacta, immediately gives such examples in all dimensions greater or equal to four.

1. Factorwise rigidity

K. Menger [Men] defined \( n \)-dimensional universal compacta in terms of the intersection of a sequence of polyhedra in \( \mathbb{R}^{2n+1} \). R. D. Anderson [An1, An2] proved that the 1-dimensional universal compactum, the Menger universal curve \( \mu^1 \), is homogeneous, and strongly locally homogeneous. Not much was known about the higher dimensional Menger universal compacta until M. Bestvina [Bes] characterized the Menger universal
compactum $\mu^n$ as a space that is topologically defined as follows:

1. $\mu^n$ is a compact n-dimensional metric space,
2. $\mu^n$ is $LC^{n-1}$,
3. $\mu^n$ is $C^{n-1}$,
4. $\mu^n$ satisfies the Disjoint n-Disk Property, $DD^n P$.

By [Bes], the compacta $\mu^n$ are homogeneous and strongly locally homogeneous. An $n$-dimensional Menger manifold, i.e., $\mu^n$-manifold, is a metric space whose every point has a neighborhood homeomorphic to $\mu^n$.

**Definition.** The Cartesian product $X = \Pi_{\lambda \in \Lambda} X_\lambda$ is factorwise rigid if every homeomorphism $h : X \to X$ preserves the Cartesian factors: specifically, there is a permutation $\tau : \Lambda \to \Lambda$ and homeomorphisms $h_\lambda : X_{\tau(\lambda)} \to X_\lambda$ such that if $h(\langle x_\lambda \rangle) = \langle y_\lambda \rangle$, then $y_\lambda = h_\lambda(x_{\tau(\lambda)})$.

**Definition.** An $i$-fiber of $X = \Pi_{\lambda \in \Lambda} X_\lambda$ is a subset of $X$ of the form $\Pi_{\lambda \in \Lambda} A_\lambda$, where $A_i = X_i$ and the remaining factors are single points. An $i$-cofiber of $X = \Pi_{\lambda \in \Lambda} X_\lambda$ is a subset of $X$ of the form $\Pi_{\lambda \in \Lambda} A_\lambda$, where $A_i$ is a single point and $A_j = X_j$ for $j \neq i$.

**Definition.** Points $x, y \in X$ are homologically separated in dimension $n$, if they have respective neighborhoods $U_x$ and $U_y$ such that

$$i_*(\check{H}_n(U_x)) \cap j_*(\check{H}_n(U_y)) = 0,$$

where $i : U_x \hookrightarrow X$, $j : U_y \hookrightarrow X$ are the inclusions, and $\check{H}$ is the $n$-th Čech homology group.

The factorwise rigidity of the Cartesian product of two Menger universal curves was first determined in [KKT1] to show that the product is not 2-homogeneous. The factorwise rigidity of the Cartesian products of pseudo-arcs was showed in [BeKe] and [BeLy]. J. Kennedy Phelps [Ke1] proved that the Cartesian product of arbitrarily many copies of $\mu^1$ is factorwise rigid, and by an unpublished result of T. Yagasaki, Kennedy’s theorem extends to the Cartesian products of copies of $\mu^n$ (see [CKT], Section 3). D. J. Garity [Gar] used the Künneth and Eilenberg-Zilber formulas to show that finite products of at least two Menger universal compacta (of equal or different dimensions, but excluding the product with all factors $\mu^0$) are not 2-homogeneous. His proof is very close to imply factorwise rigidity. The notion of homology separation was introduced in [KKT2].

For dimensional reasons, any two points in $\mu^n$ are homologically separated in dimension $n$. At every $x \in \mu^n$, there are arbitrarily small spheres $S^n$ embedded in $\mu^n$ as retracts. Let $X = \mu^{n_1} \times \cdots \times \mu^{n_k}$. The $m$-cycles, where $m = n_1 + \cdots + n_k = \dim X$, carried by two disjoint tori of form $S^{n_1} \times \cdots \times S^{n_k}$ are not homologous. Therefore, if $h_1, h_2 : X \to X$ are isotopic homeomorphisms, then $h_1 = h_2$. The lemma below has analogs in the above mentioned papers, but treats factorwise rigidity as a local property.
Lemma 1. Let $X = X_1 \times \cdots \times X_k$, where $X_i$ is homeomorphic to $\mu^{\alpha_i}$, $1 \leq \alpha_1 \leq \cdots \leq \alpha_k$. Let $U = U_1 \times \cdots \times U_k$, $U_i \subset X_i$, be an open connected subset of $X$, and let $h: U \to X$ be an open embedding. Then $h(x_1, \ldots, x_k) = (h_1(x_{\tau(1)}), \ldots, h_k(x_{\tau(k)}))$, where $\tau$ is a permutation and $h_i : U_{\tau(i)} \to X_i$ is an embedding.

Proof. Let $x = (x_1, \ldots, x_k)$ and $\bar{x} = (x_1, \bar{x}_2, \ldots, \bar{x}_k)$ be two points in the same 1-cofiber of $U$. There is a sequence of $\alpha_1$-dimensional spheres $C_n$ in $U_1$ containing $x_1$ with $\operatorname{diam}(C_n) \to 0$. Let $K_n = \{(y^n, x_2, \ldots, x_k) | y^n \in C_n\}$ and $\bar{K}_n = \{(y^n, \bar{x}_2, \ldots, \bar{x}_k) | y^n \in C_n\}$. The spheres $K_n$ are retracts of $X$, and the spheres $h(K_n)$ are retracts of $h(U)$. Every point of $h(U)$ has small neighborhoods $V_1 \times \cdots \times V_k$ in $h(U)$, which are retracts of $X$. Hence sufficiently small spheres $h(K_n)$ are retracts of $X$. For some $i$ and infinitely many $n$’s, $\pi_i \circ h : K_n \to X$ is essential. Note that $\alpha_1 = \alpha_i$, so by the classical Hurewicz theorem, an essential map $S^{\alpha_1} \to \mu^{\alpha_i}$ is homologically nontrivial. If the $i$-coordinates of $h(x)$ and $h(\bar{x})$ are different, then for sufficiently large $n$, the images $\pi_i(h(K_n))$ and $\pi_i(h(\bar{K}_n))$, are disjoint. Since the 1-cofibers of $U$ are arcwise connected, there is a tube $S^{\alpha_i} \times [0, 1]$ in $U$ joining $K_n$ and $\bar{K}_n$, which implies that the nontrivial cycles represented by $\pi_i(h(K_n))$ and $\pi_i(h(\bar{K}_n))$ are homologous, contradicting the fact that distinct points in $\mu^{\alpha_i}$ are homologically separated in dimension $\alpha_i$. Therefore the points $h(x)$ and $h(\bar{x})$ are in the same 1-cofiber of $X$. Hence every 1-cofiber in $U$ is mapped into a cofiber in $X$. By continuity, 1-cofibers close to a 1-cofiber mapped into an $i$-cofiber are mapped into $i$-cofibers for the same $i$. The compositions of the natural inclusions of $U_2 \times \cdots \times U_k$ into 1-cofibers, $h$ restricted to these 1-cofibers, and the projection $\pi_i$ are isotopic, so they are identical. The proof is completed by induction. \qed

Corollary. [See Problem 2, KKT2.] Finite Cartesian products of Menger universal compacta (also Menger manifolds) are factorwise rigid.

2. Grids

A homeomorphism $h : X \to X$ is periodic with period $k \geq 1$ if $h^k(p) = p$ for every $p \in X$, but for every $1 \leq i < k$ and $p \in X$, $h^i(p) \neq p$. A closed subset $A$ of a compact metric space $X$ is a $Z$-set if for every $\epsilon > 0$, there is a map $f : X \to X$ $\epsilon$-close to the identity with $f(X) \cap A = \emptyset$.

For positive integers $\alpha$, $\beta$, and $k \geq 2$, choose $M$, $f_M$, $N$, $A$, $B$, $f_N$, $F$, and $Q$ as follows.

1. $f_M : M \to M$, where $M$ is a $\mu^\alpha$-manifold, is a periodic homeomorphism of period $k$,
2. $N = \mu^\beta$,
3. $A$ and $B$ are disjoint, nonempty, homeomorphic $Z$-sets in $N$, and $f_N : A \to B$ is a homeomorphism, such that the quotient space of $N$ obtained by identifying each point $n \in A$ with its image $f_N(n) \in B$, is a $\mu^\beta$-manifold,
4. $F : M \times A \to M \times B$ is given by $F(m, n) = (f_M(m), f_N(n))$,
5. $Q = (M \times N)/F$ is the quotient space obtained by identifying each point $(m, n)$ with the point $F(m, n)$.
We refer to the continuum $Q$ as the $k$-grid. By a slight abuse of notation, points in the quotient space are denoted in the same way as the corresponding points in $M \times (N - A)$.

For $p = (m_p, n_p) \in Q$, let

$$M_p = \{(m, n) \in Q \mid n = n_p\},$$

$$N_p = \{(m, n) \in Q \mid m = f^i_M(m_p), i = 0, \ldots, k - 1\},$$

$$O_p = M_p \cap N_p.$$

Call the sets $M_p$ and $N_p$ horizontal and vertical fibers respectively. The intersection of a horizontal fiber and a vertical fiber is a necklace and its elements are beads. Note that the number of beads on each necklace $O_p$ is $k$.

**Lemma 2.** A homeomorphism $h : Q \rightarrow Q$ takes each horizontal and vertical fiber onto a horizontal or vertical fiber, and a necklace onto a necklace.

**Proof.** Using Lemma 1, the proof is identical to the proof of Lemmas 5 and 6 in [KuK], where this is shown for the case $\alpha = \beta = 1$ and a specific $k$. □

There is a cyclic order of a given set of beads on a necklace, which cannot be arbitrarily disturbed by a homeomorphism of $Q$. Let $\phi : Q \rightarrow Q$ be the homeomorphism given by $\phi(m, n) = (f_M(m), n)$. For a point $p_0$ in $Q$, denote by $p_i$ the point $\phi^i(p_0)$. Thus the necklace $O_{p_0}$ is the set $\{p_0, \ldots, p_{k-1}\}$.

**Lemma 3.** [Compare with Lemma 7 in KuK] Suppose that $h : Q \rightarrow Q$ is a homeomorphism and $h(p_0) = p_0$. Then there is an $s$ such that $h(p_i) = p_{(si) \mod k}$ for $i = 0, \ldots, k - 1$.

**Proof.** There is an arc $L_0$ joining $p_0$ and $p_1$ such that distinct $\phi^i(L_0)$ and $\phi^j(L_0)$ do not intersect except for a possible common end point. Denote $\phi^i(L_0)$ by $L_i$, and by $K$ the simple closed curve $\bigcup_{i=0}^{k-1} L_i$. Note that $K$ is the union of necklaces. Hence $h(K)$ is the union of necklaces; if $p \in h(L_0)$ then $O_p \subset h(K)$. We have

$$h(K) = \bigcup_{i=0}^{k-1} h(L_i) = \bigcup_{i=0}^{k-1} \phi^i h(L_0).$$

The points $h(p_i)$ are ordered on the simple closed curve $h(K)$ in such a way that the difference modulo $k$ in the indices between $h(p_i)$ and $h(p_{i+1})$ is a constant. Therefore if $\phi(p_1) = p_s$, then $h(p_i) = p_{(si) \mod k}$. □

**Lemma 4.** If $h : Q \rightarrow Q$ is a homeomorphism and $h(p_0) \in O_{p_0}$, then there are integers $r$ and $s$ such that $h(p_i) = p_{(r+si) \mod k}$ for $i = 0, \ldots, k - 1$.

**Proof.** Take $r$ and $s$ such that $h(p_0) = p_r$ and $\phi^{-r} \circ h(p_1) = p_s$. □
3. Circular fibers and fiber replacing

The next step is to construct a continuum built on the $k$-grid $Q$ obtained from $Q \times I$ by the identification $(p, 1) = (\phi^b(p), 0)$, where $k$ is the product of two positive integers $a$ and $b$. Each of the sets $O_p \times I$ transforms into $b$ circles called circular fibers; $X$ decomposes into pairwise disjoint copies of $S^1$. The number of beads of $O_p \times \{0\}$ on each circle $C$ is $a$. The order of the beads determines the orientation of $C$. By Lemma 3.1 of [KKT2], we have:

**Lemma 5.** A homeomorphism $h : X \to X$ takes each circular fiber onto a circular fiber.

Every point of $X$ has a neighborhood homeomorphic to the Cartesian product $M \times N \times I$. Orient the $I$-fiber of $Q \times I$ and transfer the orientation to the circular fibers of $X$. Similarly as in [KuK] (see Lemma 11), a homeomorphism of $X$ onto itself either preserves orientation on all circular fibers (it is then orientation preserving), or reverses orientation on all circular fibers (it is then orientation reversing).

As in the previous section let $O_{p_0} = \{p_0, \ldots, p_k\}$ be a necklace in $Q$. Denote the point $(p_i, 0) \in X$ by $q_i$.

**Lemma 6.** If $a \geq 3$, then every homeomorphism $h : X \to X$ such that $h(q_0) = q_1$, $h(q_1) = q_0$, and $h(Q \times \{0\}) = Q \times \{0\}$ is orientation reversing.

**Proof.** By Lemma 4, $h(q_i) = q_{(1-i) \mod ab}$. Suppose that $h$ is orientation preserving. Then $h(\{q_0, q_b, q_{2b}, \ldots\}) = \{q_1, q_{1+b}, q_{1+2b}, \ldots\}$ preserving order. Hence $h(q_b) = q_{1+b}$ and $1 + b = (1 - b) \mod ab$. So $2 \mod a = 0$, which is a contradiction. □

If one were to follow the procedure described in [KuK], each circular fiber $C$ of $X$ would be replaced by a manifold $E$ which contains $C$ as its retract, and such that every autohomeomorphism of $E$ takes the element of the first homology group represented by $C$ onto itself; in particular it does not change the sign of this element. (Note that in [KuK], $C$, $E$, and $X$ are denoted by different symbols.) The resulting continuum $D$ is the union of pairwise disjoint copies of the same manifold $E$, called manifold fibers. Since in [KuK] $a = b = 3$; each circular fiber in $X$ consists of three segments; each manifold fiber of $D$ consists of three identical pieces. $D$ contains a copy of $X$ as its retract. Although $X$ needs not be invariant under a homeomorphism $h : D \to D$, $h$ induces a homeomorphism of $X$ preserving the correspondence of the circular fibers to the manifold fibers given by the inclusion $X \hookrightarrow D$. It is shown that:

1. every homeomorphism $h : D \to D$ maps a manifold fiber onto a manifold fiber,
2. some manifold fibers cannot be swapped by a homeomorphism of $D$.

Kawamura’s idea [Kaw] to modify Minc’s example [Minc] can be also applied to modify $D$. Instead of the manifold $E$ take a Menger manifold $\Omega$ consisting of $a$ identical pieces homeomorphic to a $\mu^\gamma$-manifold $P$, where $\gamma \geq 2$. $P$ corresponds to the mapping
cylinder of a degree two map of $S^1$ onto $S^1$. The Menger manifold $\Omega$ is similar to the Menger manifold $L_n$ in [Kaw], Section 3, and has the following properties:

1. $\Omega = \bigcup_{i=0}^{a-1} \Omega_i$, and there are homeomorphisms $\tau_i : P \to \Omega_i$.
2. $\Omega_i \cap \Omega_j = \emptyset$ if $|(i - j) \mod a| \neq 1$.
3. There are two disjoint homeomorphic $Z$-sets in $P$, $P_0$ and $P_1$, such that $\Omega_i \cap \Omega_j = \tau_i(P_0) = \tau_j(P_1)$ if $(i - j) \mod a = 1$, and $\Omega_i \cap \Omega_j = \tau_i(P_1) = \tau_j(P_0)$ if $(j - i) \mod a = 1$.
4. There is a simple closed curve $K \subset \Omega$ intersecting each $\Omega_i$ in an arc such that $K$ is a retract of $\Omega$, every homeomorphism $\Omega \to \Omega$ takes the element of the first homology group represented by (oriented) $K$ onto itself without changing its sign.

The Menger manifold $P$ can be used to replace the circular fiber of the continuum $X$ to obtain a non-bihomogeneous continuum $Y$. Namely, $Y$ is the quotient space obtained from $Q \times P$ by identifying each point $(p, x)$ with $(\phi^b(p), \tau_{(i+1)}^{-1} \circ \tau_i(x))$, where $Q$ is the $k$-grid considered in the beginning of this section, $p \in Q$, and $x \in P$. Note that $\tau_{(i+1)}^{-1} \circ \tau_i : P_1 \to P_0$.

**Lemma 7.** The continuum $Y$ is homogeneous.

**Proof.** The proof uses strong local homogeneity of the Menger compacta and is similar the proofs of Lemmas 3 and 4 in [KuK]. □

**Lemma 8.** If $a \geq 3$ and $b \geq 2$, then $Y$ is not bihomogeneous.

**Proof.** Since $Y$ is locally homeomorphic to $\mu^a \times \mu^b \times \mu^\gamma$, the local factorwise rigidity holds. Using the local Cartesian product structure, we may define the $\mu^a$, $\mu^b$, and $\mu^\gamma$-fibers, as well as the $\mu^a$, $\mu^b$, and $\mu^\gamma$-cofibers. The $\mu^\gamma$-fibers are homeomorphic to $P$, and the $\mu^\gamma$-cofibers are homeomorphic to $Q$. The necklaces in the $\mu^\gamma$-cofibers have $ab$ beads, whereas the necklaces in the $\mu^a$- and $\mu^b$-cofibers contain only $a$ beads. Therefore every homeomorphism $Y \to Y$ maps the $\mu^\gamma$-fibers onto $\mu^\gamma$-fibers and $\mu^\gamma$-cofibers onto $\mu^\gamma$-cofibers even if $\alpha = \beta = \gamma$. Since $b \geq 2$, there are at least two distinct $\mu^\gamma$-fibers passing through the same necklace of $Q \times \{x\}$, where $x$ is a point in $P - P_1$. By Lemma 6 and by the above consideration, there are two such fibers that cannot be swapped. □

The above lemma could be compared to Lemmas 13, 14, 15, and 16 in [KuK]. However, factorwise rigidity involving all three factors, $\mu^a$, $\mu^b$, and $\mu^\gamma$, makes the proof much simpler.

**Theorem.** For every triple of integers $(\alpha, \beta, \gamma)$ such that $\alpha \geq 1$, $\beta \geq 1$, and $\gamma \geq 2$, there is a homogeneous, non-bihomogeneous continuum whose every point has a neighborhood homeomorphic to $\mu^\alpha \times \mu^\beta \times \mu^\gamma$.

**Remark.** $Y$ is the quotient space of $\mu^a \times \mu^b \times \mu^\gamma$ with some identifications made along $Z$-sets. It is not known whether the same effect can be achieved on the product $\mu^1 \times \mu^1 \times \mu^1$. 


Question 1. Does there exist a homogeneous, non-bihomogeneous, Peano continuum of dimension lower than 4?

Question 2. Does there exist a homogeneous, non-bihomogeneous, Peano continuum whose every point has a neighborhood homeomorphic to $\mu^1 \times \mu^1 \times \mu^1$?

Remark. While the second question remains open, since the submission of this paper, the first question has been answered by G. Kuperberg [KuG]. For any pair of integers $\alpha, \beta$ such that $\alpha \geq 1$ and $\beta \geq 2$, he constructs a homogeneous, non-bihomogeneous Peano continuum with the local structure of $\mu^\alpha \times \mu^\beta$.

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