Robust, Expressive, and Quantitative Linear Temporal Logics

Daniel Neider\textsuperscript{1}, Alexander Weinert\textsuperscript{2} and Martin Zimmermann\textsuperscript{2}

\textsuperscript{1} Max Planck Institute for Software Systems, 67663 Kaiserslautern, Germany
neider@mpi-sws.org
\textsuperscript{2} Reactive Systems Group, Saarland University, 66123 Saarbrücken, Germany
\{weinert,zimmermann\}@react.uni-saarland.de

Abstract. Linear Temporal Logic (LTL) is the standard specification language for reactive systems and successfully applied in industrial settings. However, many shortcomings of LTL have been identified in the literature, among them the limited expressiveness, the lack of quantitative features, and the inability to express robustness. Typically, each one of these deficiencies is addressed in isolation. This is insufficient for applications, where all shortcomings manifest themselves simultaneously. Here, we tackle this issue by introducing logics that address more than one shortcoming at a time. To this end, we combine the logics robust LTL, Prompt-LTL, and Linear Dynamic Logic, each addressing one aspect, to new logics. For all combinations of two aspects, the resulting logic has the same desirable algorithmic properties as plain LTL. In particular, the highly efficient algorithmic backends that have been developed for LTL are also applicable to these new logics. Finally, we introduce a logic addressing all three aspects.

1 Introduction

Linear Temporal Logic (LTL) \cite{Pnueli77} is amongst the most prominent and most important specification languages for reactive systems, e.g., non-terminating controllers interacting with an antagonistic environment. Verification of such systems against LTL specifications is routinely applied in industrial settings nowadays \cite{Pradl19, Sipma18}. Underlying this success story is the exponential compilation property \cite{Vardi98}: every LTL formula can be effectively translated into an equivalent Büchi automaton of exponential size (and it turns out that this upper bound is tight). In fact, almost all verification algorithms for LTL are based on this property, which is in particular true for the popular polynomial space model checking algorithm and the doubly-exponential time synthesis algorithms. Other desirable properties of LTL include its compact and variable-free syntax and its intuitive semantics.

Despite the success of LTL, a plethora of extensions of LTL have been studied, all addressing individual and specific shortcomings of LTL. One such shortcoming is the well-known fact that LTL is strictly weaker than Büchi automata, i.e., it does not harness the full expressive power of the backends. Thus, increasing the expressiveness of LTL has generated much attention \cite{Barthe16a, Pnueli77, Vardi98}. As an example, consider the specification “p holds at every even time point, but may or may not hold at odd time points”. It is well-known that this property is not expressible in LTL, as LTL, intuitively, is unable to count modulo a fixed number. However, the specification is easily expressible in LDL as \( [r] \Diamond p \), where \( r \) is the regular expression \( (\texttt{tt} \cdot \texttt{tt})^* \), requiring \( p \) to be satisfied at every position \( j \) such that the prefix up to position \( j \) matches the regular expression \( r \) (which implies that \( j \) is even), i.e., \( \texttt{tt} \) is an atomic regular expression that matches every letter.

Another shortcoming of both LTL and LDL is their inability to adequately express timing bounds. For example, consider the specification “every request \( q \) is eventually answered by a response \( p \)”, which is expressed in LTL as \( [q \rightarrow \Diamond p] \). It is satisfied, even if the waiting time between requests and responses diverges to infinity, although such a behavior is typically undesired. Again, a long line of research has addressed this second shortcoming of LTL \cite{Pradl19, Sipma18}. The most basic one is Prompt-LTL \cite{Balzer92}, which adds the parameterized eventually operator \( \Diamond[p] \) to LTL. The semantics is now defined with an additional parameter \( k \), which bounds the scope of \( \Diamond[p] \): \( [q \rightarrow \Diamond[p] p] \) requires every request \( q \) to be answered within

\* Supported by the Saarbrücken Graduate School of Computer Science.
$k$ steps, when evaluated with respect to $k$. The resulting logic is a quantitative one: either one quantifies the parameter $k$ existentially and obtains a boundedness problem, e.g., “is there a bound $k$ such that every request can be answered within $k$ steps”, or one even aims to determine the optimal bound $k$. Again, Prompt-LTL retains the desirable properties of LTL, i.e., the exponential compilation property as well as intuitive syntax and semantics.

Finally, a third extension of LTL is concerned with the concept of robustness, which is much harder to formalize. This is reflected by a multitude of incomparable notions of robustness in verification [4,6,9,10,12,19,20,25,26]. Here, we are interested in robust LTL [26], which equips LTL with a five-valued semantics that captures different degrees of violations of universal specifications. As an example, consider the specification “if property $\varphi$ always holds true, then property $\psi$ also always holds true”, which is expressed in LTL as $\Box \varphi \rightarrow \Box \psi$ and is typical for systems that have to interact with an antagonistic environment. In classical semantics, the whole formula is satisfied as soon as the assumption $\varphi$ is violated once, even if the guarantee $\psi$ is violated as well. By contrast, the semantics of robust LTL ensures that the degree of the violation of $\Box \psi$ is always proportional to the degree of the violation of $\Box \varphi$. To this end, the degree of a violation of a property $\Box \varphi$ is expressed by five different truth values: either $\varphi$ always holds, or $\varphi$ is violated only finitely often, infinitely often, almost always, or always. Again, robust LTL has the exponential compilation property and an intuitive syntax (though its semantics is more intricate).

Commonly, extensions of LTL as described above are only studied in isolation, e.g., the logics are either more expressive, or quantitative, or robust. One notable exception is Parametric LDL (PLDL) [13], which adds quantitative operators to LDL while maintaining the exponential compilation property and intuitive syntax and semantics. In practical settings, however, it does not suffice to address one shortcoming of LTL while ignoring the others. Instead, one needs a logic that combines multiple extensions while still maintaining the desirable properties of LTL. The overall goal of this paper is, hence, to bridge this gap, thereby enabling expressive, quantitative, and robust verification and synthesis. Towards this goal, we proceed in three steps (see also Figure 1).

In Section 3 we “robustify” LDL: we introduce a novel logic, named rLDL, by lifting the five-valued semantics of robust LTL to LDL. We show that it has the exponential compilation property by generalizing the translation of LDL to alternating automata to rLDL.

In Section 4 we “robustify” Prompt-LTL. More precisely, we introduce a novel logic, named rPrompt-LTL, by extending the five-valued semantics from robust LTL to Prompt-LTL. Due to the lack of appropriate quantitative automata models for parameterized linear temporal logics, we develop a different approach to proving the exponential compilation property: we show that every rPrompt-LTL formula can be rewritten as a sequence of Prompt-LTL formulas that captures the original formula. As this translation only linearly increases the size of the formula and as Prompt-LTL has the exponential compilation property, we obtain the desired result for rPrompt-LTL.

Finally, in Section 5 we combine all three aspects and introduce a five-valued robust semantics for Prompt-LDL, obtaining the novel logic rPrompt-LDL. As our techniques developed for rLDL and rPrompt-LTL cannot easily be combined, we identify a fragment of rPrompt-LDL, which restricts the use of regular expressions. We demonstrate that this fragment indeed allows expressing interesting properties and show how its formulas can be translated into Prompt-LDL. However, we leave the question of whether rPrompt-LDL has the exponential compilation property for future work.

These results show that one can indeed combine any two of the three extensions of LTL while still preserving the desirable algorithmic properties of LTL. In particular, let us stress again that all highly sophisticated algorithmic backends developed for LTL are applicable to these novel logics as well, e.g., we show that the verification problem and the synthesis problem for each of these logics is solvable without an (asymptotic) increase in complexity.
2 Preliminaries

We denote the non-negative integers by \( \mathbb{N} \), the set \( \{0, 1\} \) of Boolean truth value by \( \mathbb{B} \), and the power set of \( S \) by \( 2^S \). By convention, we have \( \min \emptyset = 1 \) and \( \max \emptyset = 0 \) when the operators range over subsets of \( \mathbb{B} \). Throughout this work, we fix a finite non-empty set \( P \) of atomic propositions. For a set \( A \subseteq P \) and a propositional formula \( \phi \) over \( P \), we write \( A \models \phi \) if the variable valuation mapping elements in \( A \) to 1 and elements in \( P \setminus A \) to 0 satisfies \( \phi \).

The set of truth values considered for robust semantics is \( \mathbb{B}_4 = \{0000 < 0001 < 0011 < 0111 < 1111\} \). We write \( \preceq \) for the non-strict variant of \( \prec \) and define \( \min \emptyset = 1111 \) and \( \max \emptyset = 0000 \) when the operators range over subsets of \( \mathbb{B}_4 \).

A trace (over \( P \)) is an infinite sequence \( w \in (2^P)^\omega \). Given a trace \( w = w(0)w(1)w(2)\cdots \) and a position \( n \in \mathbb{N} \), we define \( w[0, n) = w(0) \cdots w(n - 1) \) and \( w[n, \infty) = w(n)w(n + 1)w(n + 2)\cdots \), i.e., \( w[0, n) \) is the prefix of length \( n \) of \( w \) and \( w[n, \infty] \) the remaining suffix.

Our work is based on three logics, which we introduce in the appendix: Robust Linear Temporal Logic \([26]\) in Section A.1, Linear Dynamic Logic \([28]\) in Section A.2, and Prompt Linear Temporal Logic \([17]\) in Section A.3. Further, we introduce Prompt Linear Dynamic Logic \([18]\) in Section A.4. We denote the semantics for all of these logics by evaluation functions \( V \) mapping a trace and a formula to a truth value. This is prudent for robust semantics, hence we also use this approach also for the other logics, which are typically defined via satisfaction relations. Our definitions here are equivalent to the original definitions.

As we have to define several evaluation functions, we distinguish them by decorating them with an \( r \) to denote robust semantics, an \( n \) to denote prompt, i.e., quantitative, semantics, and a \( d \) to denote that the logic extends LDL. Also, we use combinations of these letters to denote evaluation functions for logics that combine these aspects.

3 Robust Linear Dynamic Logic

In this section, we “robustify” LDL by generalizing the ideas underlying robust LTL to LDL, obtaining the logic rLDL (following Tabuada and Neider \([26]\) we only consider the fragment rLTL(□, ◦) without next, until and release). Following the precedent of robust LTL, we equip robust operators with dots to distinguish them from non-robust ones. Then, we study the expressiveness of the new logic and solve the canonical verification problems.

The formulas of rLDL are given by the grammar

\[
\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \mid (r\cdot) \varphi \mid [r\cdot] \varphi
\]

\[
r ::= \phi \mid \varphi ? \mid r + r \mid r ; r \mid r^* \,
\]

where \( p \) ranges over the atomic propositions in \( P \) and \( \phi \) over propositional formulas over \( P \). We refer to formulas of the form \( (r) \varphi \) and \( [r] \varphi \) as diamond formulas and box formulas, respectively. In both cases, we call \( r \) the guard of the operator.

We denote the set of subformulas of \( \varphi \) by \( \text{cl}(\varphi) \). Regular expressions are not subformulas, but the formulas appearing in the tests are, e.g., we have \( \text{cl}(p'\cdot q) = \{p, p', (p'\cdot q) p'\} \). The size \( |\varphi| \) of \( \varphi \) is the sum of \( |\text{cl}(\varphi)| \) and the sum of the lengths of the regular expressions appearing in \( \varphi \) (counted with multiplicity and measured in the number of operators).

Before we introduce the semantics for rLDL we first recall the cases underlying the semantics of the robust always operator \( [r] \varphi \) in robust LTL: either all positions are \( \varphi \)-satisfying (□), almost all positions are \( \varphi \)-satisfying (\( \Diamond \)), infinitely many positions are \( \varphi \)-satisfying (\( \Box \)), some position is \( \varphi \)-satisfying (\( \Diamond \Box \)), or no position is \( \varphi \)-satisfying.

A similar approach for a robust box formula \([r\cdot] \varphi \) would be to consider the following possibilities, where a position \( n \) is an \( r \)-match if the prefix up to and including position \( n - 1 \) is in the language of \( r \), and is \( \varphi \)-satisfying if the suffix starting at position \( n \) satisfies \( \varphi \): all \( r \)-matches are \( \varphi \)-satisfying, almost all \( r \)-matches are \( \varphi \)-satisfying, infinitely many \( r \)-matches are \( \varphi \)-satisfying, some \( r \)-match is \( \varphi \)-satisfying, or no \( r \)-match is \( \varphi \)-satisfying. If a trace \( w \) has infinitely many \( r \)-matches, then this is the natural generalization of the robust semantics for LTL. The trace may, however, only contain finitely many, or even no \( r \)-match. In the former case, there are not infinitely many \( \varphi \)-satisfying \( r \)-matches, but all \( r \)-matches could satisfy
Thus, the monotonicity of the cases is violated. We overcome this by interpreting “almost all” as “all” and “infinitely many” as “some”, if there are only finitely many r-matches.

Also, the guard r may contain tests, which have to be evaluated to determine whether a position is an r-match. For this, we have to use the appropriate semantics for the robust box operator. For example, if we interpret [r] ϕ to mean “almost all r-matches satisfy ϕ”, then the robust box operators in tests of r are evaluated with this interpretation as well. This may, however, violate monotonicity (see Example 2), which we therefore hardcode in the definition of the semantics.

We now formalize the informal description above and subsequently show that this formalization satisfies all desired properties. Formally, we define an evaluation function V^{RD} mapping a trace w and a formula ϕ to a truth value in 2. To simplify our notation, we write V^{RD}(w, ϕ) for i ∈ {1, 2, 3, 4} to denote the i-th bit of V^{RD}(w, ϕ), i.e., V^{RD}(w, ϕ) = V^{RD}(w, ϕ)_1V^{RD}(w, ϕ)_2V^{RD}(w, ϕ)_3V^{RD}(w, ϕ)_4V^{RD}(w, ϕ).

The cases for atomic propositions and Boolean connectives are as for rLTL: ⋄: 

- V^{RD}(w, p) = \begin{cases} 1111 & \text{if } p \in w(0), \\
0000 & \text{if } p \notin w(0), \end{cases} 
and V^{RD}(w, ¬ϕ) = \begin{cases} 0000 & \text{if } V^{RD}(w, ϕ) = 1111, \\
1111 & \text{if } V^{RD}(w, ϕ) \neq 1111, \end{cases} 

- V^{RD}(w, ϕ_0 ∧ ϕ_1) = \min\{V^{RD}(w, ϕ_0), V^{RD}(w, ϕ_1)\}, 
- V^{RD}(w, ϕ_0 ∨ ϕ_1) = \max\{V^{RD}(w, ϕ_0), V^{RD}(w, ϕ_1)\}, 
- V^{RD}(w, ϕ_0 → ϕ_1) = \begin{cases} 1111 & \text{if } V^{RD}(w, ϕ_0) = V^{RD}(w, ϕ_1), \\
0000 & \text{if } V^{RD}(w, ϕ_0) < V^{RD}(w, ϕ_1).
\end{cases} 

- V^{RD}(w, [r.] ϕ) = \min_{n ∈ \mathbb{N}} \max_{RD}(w, [r.] ϕ) V^{RD}(w[n, ∞), ϕ), 
- V^{RD}(w, [r.) ϕ) = \min_{n ∈ \mathbb{N}} \max_{RD}(w, [r.) ϕ) V^{RD}(w[n, ∞), ϕ), 
- V^{RD}(w, [r.) ϕ) = \min_{n ∈ \mathbb{N}} \max_{RD}(w, [r.) ϕ) V^{RD}(w[n, ∞), ϕ), 
- V^{RD}(w, [r.) ϕ) = \min_{n ∈ \mathbb{N}} \max_{RD}(w, [r.) ϕ) V^{RD}(w[n, ∞), ϕ), 

Here, the match set R_{i}^{RD} ⊆ \mathbb{N} for i ∈ {1, 2, 3, 4} contains all positions n such that w[0, n) matches r and is defined inductively as follows:

- R_{i}^{RD}(w, ϕ) = \{\} if w(0) = 0 and R_{i}^{RD}(ϕ, w) = 0 otherwise, for propositional ϕ.
- R_{i}^{RD}(w, ϕ?) = \{0\} if V^{RD}(w, ϕ) = 1 and R_{i}^{RD}(ϕ, w?) = 0 otherwise.
- R_{i}^{RD}(w, r_0 + r_1) = R_{i}^{RD}(w, r_0) ∪ R_{i}^{RD}(w, r_1).
- R_{i}^{RD}(w, r; n) = \{n | \exists n′ s.t. n′ ∈ R_{i}^{RD}(w, r_0) \text{ and } n - n′ ∈ R_{i}^{RD}(w[n′, ∞), r_1)\}.
- R_{i}^{RD}(w, r^+) = \{0\} ∪ \{n_k | \exists n_1, ... n_{k-1} s.t. n_j ∈ R_{i}^{RD}(w[n_1 + ... + n_{j-1}, ∞), r) \text{ for all } j ∈ \{1, ..., k\}\},

Due to tests, membership of some n in R_{i}^{RD}(w, r) does, in general, not only depend on the prefix w[0, n), but on the complete trace w. Also, the semantics of the propositional atom ϕ differ from the semantics of the test ϕ?: the former consumes an input letter, while tests do not. Hence, rLTL (as LDL) features both kinds of atoms.

To give an intuitive description of the semantics, let us first formalize the notion of r-matches and ϕ-satisfiability. We say that a position n of w is an r-match of degree β if n ∈ R_{i}^{RD}(w, r) for the unique i with β = 0^{i-1}1^{n-i}, which requires all tests in r to be evaluated w.r.t. V^{RD} (i.e., to some truth value at least to β). Similarly, we say that a position of w is ϕ-satisfying of degree β if V^{RD}(w, ϕ) ≥ β, or if, equivalently, V^{RD}(w, ϕ) = 1 for the unique i with β = 0^{i-1}1^{n-i}.
Now, we have \( b'_i = 1 \) if all \( r \)-matches of degree 1111 are \( \varphi \)-satisfying of degree 1111. This is in particular satisfied if there is no such match. Further, if there are infinitely (finitely) many \( r \)-matches of degree 0111, then \( b'_i = 1 \) if almost all (if all) those matches are \( \varphi \)-satisfying of degree 0111. Dually, if there are infinitely (finitely) many \( r \)-matches of degree 0011, then \( b'_i = 1 \) if infinitely many (at least one) of those matches are (is) \( \varphi \)-satisfying of degree 0011. Finally, if there is at least one \( r \)-match of degree 0001, then \( b'_i = 1 \) if at least one of those matches is \( \varphi \)-satisfying of degree 0001. The cases where there is no \( r \)-match of any degree are irrelevant due to monotonicity, so we just hardcode them to 1.

**Example 1.** Consider the formula \([r] q \rightarrow [r] p\) with \( r = (\tt; \tt)^*\), which expresses that the degree of violation of \( p \) at even positions should at most be the degree of violation of \( q \) at even positions. Such a property cannot be expressed in \( \text{LTL}(\Box, \Diamond)\), as even \([r] q\) is known to be inexpressible in \( \text{LTL} \) \([3]\).

First, we show that the semantics is well-defined. This is not obvious due to the case distinctions and the use of the matching sets \( R^i_n \) for different \( i \).

**Lemma 1.** We have \( V^{\text{rd}}(w, \varphi) \in \mathbb{B}_4 \) for every trace \( w \) and every formula \( \varphi \).

**Proof.** We proceed by induction over the structure of \( \varphi \). The cases of atomic propositions and Boolean connectives are trivial, as they return values from \( \mathbb{B}_4 \) by definition. Similarly, we have \( V^{\text{rd}}(w, [r] \varphi) \in \mathbb{B}_4 \), as the maximization “\( b_i = \max \{b'_1, \ldots, b'_r\}\)” in the definition enforces the desired monotonicity.

To conclude, consider a diamond formula \( [r] \varphi \). Applying the induction hypothesis to the tests of \( r \) and an induction over the construction of \( r \) shows

\[
R^1_n(w, r) \subseteq R^2_n(w, r) \subseteq R^3_n(w, r) \subseteq R^4_n(w, r)
\]

for every trace \( w \). Hence, an application of the induction hypothesis for \( \varphi \) yields

\[
\max_{n \in R^i_n(w, r)} V^{\text{rd}}_i(w[n, \infty), \varphi) \leq \max_{n \in R^i_n(w, r)} V^{\text{rd}}_2(w[n, \infty), \varphi) \leq \max_{n \in R^i_n(w, r)} V^{\text{rd}}_4(w[n, \infty), \varphi)
\]

for every trace \( w \). Hence, \( V^{\text{rd}}(w, [r] \varphi) \in \mathbb{B}_4 \).

To conclude, let us give an example witnessing that the maximization in the semantics of the box operator is indeed necessary to obtain monotonicity.

**Example 2.** Let \( \varphi = [r] \tt \tt \) with \( r = ([\tt] \tt)^* \)? and \( w = \emptyset \{p\}\). Then, we have \( V^{\text{rd}}(w, [\tt] \tt \{p\}) = 0111 \) and consequently \( R^1_n(w, r) = \emptyset \) and \( R^2_n(w, r) = \{0\} \). Hence, \( \min_{n \in R^1_n(w, r)} V^{\text{rd}}_1(w[n, \infty), \tt \tt) = \min \emptyset = 1 \), but \( \min_{n \in R^2_n(w, r)} V^{\text{rd}}_2(w[n, \infty), \tt \tt) = \min \{0\} = 0 \). The traces \( \emptyset \{p\} \) \( \tt \tt \) and \( \{p\} \emptyset \tt \tt \) witness the same result for the other \( V^{\text{rd}}_i \).

We show that \( r\text{LDL} \) has the exponential compilation property. This allows us to solve the model checking and the synthesis problem using well-known and efficient automata-based algorithms. Furthermore, we are able to show that the complexity of these algorithms is asymptotically the same as the complexity of the algorithms for plain LDL and LTL.

We follow the approach by Faymonville and Zimmermann who presented a bottom-up translation of parametric LDL, an extension of LDL with parameterized temporal operators, into alternating automata \([13]\). In our setting, in contrast, we do not have to deal with parameterized operators, but instead with the implications of the five-valued semantics.

As alternating parity automata can be effectively translated into equivalent Büchi automata of exponential size (see, e.g., \([3]\)), we thereby obtain that \( r\text{LDL} \) has the exponential compilation property. We refer to the textbook \([15]\) for more details on Büchi automata.

**Theorem 1.** Let \( \varphi \) be an \( r\text{LDL} \) formula, \( n = |\varphi| \), and \( \beta \in \mathbb{B}_4 \). There exists a non-deterministic Büchi automaton \( \mathfrak{B}_{\varphi, \beta} \) with \( 2^{O(n \log n)} \) states recognizing the language \( \{w \in (2^P)^\\omega \mid V^{\text{rd}}(w, \varphi) \geq \beta\} \).

We present the proof of Theorem 1 in Appendix \([3]\), as it is on the one hand slightly technical and requires the introduction of some machinery and on the other hand a generalization of the corresponding result for PLDL \([13]\).
Furthermore, as it is done for the similar construction for PLDL [39], one can show that the automata can indeed be constructed efficiently: the non-deterministic Büchi automaton $\mathfrak{B}_{\varphi,\beta}$ can be constructed on-the-fly in polynomial space.

Also, the non-deterministic Büchi automaton can, as usual, be determined into a deterministic parity automaton, involving an another exponential blowup.

Corollary 1. Let $\varphi$ be an rLDL formula, $n = |\varphi|$, and $\beta \in \mathbb{B}_4$. There is a deterministic parity automaton $\mathfrak{B}_{\varphi,\beta}$ with $2^{2^{O(n \log n)}}$ states and with $2^{O(n \log n)}$ colors recognizing the language $\{w \in (2^P)^\omega \mid V^{\text{rd}}(w, \varphi) \geq \beta\}$.

3.1 Expressiveness

In this section, we compare the expressiveness of rLDL to that of rLTL($\bigotimes\square$) and LDL. It turns out, as expected, that rLDL generalizes both and is in a sense equivalent to LDL.

Theorem 2. rLTL($\bigotimes\square$) and LDL are syntactic fragments of rLDL.

Proof. Let us first embed rLTL($\bigotimes\square$) into rLDL by showing that the syntactic embedding of LTL into LDL extends to robust semantics. Recall that rLTL($\bigotimes\square$) only has temporal operators $\Diamond$ and $\square$, which we replace by $(\tt^*)$ and $[\cdot \tt^*]$. Note that $R_1^i(w, \tt^*) = \mathbb{N}$ holds true for every $w$. Hence, a straightforward induction shows that the resulting rLDL formula is equivalent to the original rLTL($\bigotimes\square$) formula. In particular, only the first case in the case distinctions defining the semantics of the robust box operator of rLDL is used, which mimics the definition of the semantics of the always operator in rLTL($\bigotimes\square$).

Furthermore, embedding LDL into rLDL is trivial: we have $V^d(w, \varphi) = V^{\text{rd}}(w, \varphi')$ for every $w$ and every LDL formula $\varphi$, where $\varphi'$ is the rLDL formula obtained from $\varphi$ by replacing each $\langle r \rangle$ by $[r \cdot]$ and the sequence $[r \cdot]$, respectively. This is shown by a straightforward induction over the construction of $\varphi$.

As LTL is a syntactic fragment of LDL, we immediately obtain that LTL is a syntactic fragment of rLDL as well. On the other hand, we leave open the question whether full robust LTL with the until and release operator is a fragment of rLDL: while the until operator can easily be embedded syntactically, the robust semantics of the release operator are not compatible with the standard syntactic translation of the release operator to LDL.

Our next theorem states that LDL and rLDL are of equal expressiveness. The direction from LDL to rLDL was shown in Theorem 2, hence we focus on the other one. Following Tabuada and Neider [26], we construct for every rLDL formula $\varphi$ and every $i \in \{1, 2, 3, 4\}$ an LDL formula $\varphi_i$ such that $V^{\text{rd}}(w, \varphi) = V^i(w, \varphi_i)$.

Theorem 3. LDL and rLDL are equally expressive and the translations are effective.

Proof. As argued above, we only have to consider the direction from rLDL to LDL. Hence, fix an rLDL formula $\varphi$ and $i \in \{1, 2, 3, 4\}$. Due to Theorem 3, $\{w \in (2^P)^\omega \mid V^{\text{rd}}(w, \varphi) = 1\}$ is $\omega$-regular. Hence, due to LDL being equi-expressive to the $\omega$-regular languages [28], there is also an LDL formula $\varphi_i$ with $V^i(w, \varphi_i) = 1$ if and only if $V^{\text{rd}}(w, \varphi) = 1$. Hence, $\varphi_i$ has the desired properties.

Let us analyze the complexity of the translation in more detail. The non-deterministic Büchi automaton is in general of exponential size and has to be determined before it can be translated into LDL (say with max-parity acceptance), which incurs a second exponential blowup. The resulting deterministic automaton can then be translated into LDL with an exponential blowup. The resulting formula expresses that its unique run on the input satisfies the following property: there is an even color $c$ and a position $n$ such that after $n$ no larger color appears, $c$ appears at least once, and every time color $c$ appears, it is not the last occurrence of $c$ (see [30] for a similar construction). Both properties are easily expressed in LDL by constructing guards $r_{q,q'}$ that match infixes that take the parity automaton from $q$ to $q'$. The number of subformulas is polynomial, but the guards $r_{q,q'}$ have in general exponential size (both in the size of the doubly-exponential deterministic automaton). Hence, the full construction incurs a triply-exponential blowup. However, the resulting formula is test-free, i.e., it does not contain tests in its guards.
We leave the question of whether there are non-trivial lower bounds on the translation for future work. For the special case of translating rLTL(∃, ∅) into LTL, the translation only incurs a linear blowup.\footnote{The translation was presented by Tabuada and Neider \cite{tabuada13a}, but they only claimed an exponential upper bound. However, closer inspection shows that it is actually linear, as the size of formulas is measured in the number of distinct subformulas, not the length of the formula.}

### 3.2 Model Checking

Theorem \ref{thm:translation_upper_bound} immediately provides solutions for typical applications of rLDL, such as model checking and synthesis, by reducing the problem from the domain of rLDL to the one of LDL. However, the price to pay for this approach is a triply-exponential blow-up in the size of the resulting LDL formula, which is clearly prohibitive for any real-world application. For this reason, we now develop more efficient model checking and synthesis techniques that are based on our translation of rLDL into finite automata.

In this section, we focus on the rLDL model checking problem, which asks whether all executions of a given system satisfy a given specification expressed as an rLDL formula with truth value at least $\beta \in \mathbb{B}_4$. More formally, we assume the system under consideration to be modeled as a (labeled and initialized) transition system $\mathcal{S} = (S, s_I, E, \lambda)$ over $P$ consisting of a finite set $S$ of states containing the initial state $s_I$, a directed edge relation $E \subseteq S \times S$, and a state labeling $\lambda: S \to 2^P$ that maps each state to the set of atomic propositions that hold true in this state. A path through $\mathcal{S}$ is a sequence $\rho = s_0s_1s_2\cdots$ satisfying $s_0 = s_I$ and $(s_i, s_{i+1}) \in E$ for every $i \in \mathbb{N}$, and $\Pi_S$ denotes the set of all paths through $\mathcal{S}$. Finally, the trace of a path $\rho = s_0s_1s_2\cdots \in \Pi_S$ is the sequence $\lambda(\rho) = \lambda(s_0)\lambda(s_1)\lambda(s_2)\cdots$ of labels induced by $\rho$. Then, the rLDL model checking problem is defined as follows.

**Problem 1.** Let $\varphi$ be an rLDL formula, $\mathcal{S}$ a transition system, and $\beta \in \mathbb{B}_4$. Does $V^{rd}(\lambda(\rho), \varphi) \geq \beta$ hold true for all paths $\rho \in \Pi_S$?

The exponential compilation property (see Theorem \ref{thm:exponential_compilation}) and standard on-the-fly techniques for checking emptiness of exponentially-sized Büchi automata \cite{gilmore62} yields a PSPACE upper bound on the complexity of Problem \ref{prob:translation_upper_bound}. The matching lower bound follows from the subsumption of LDL shown above, as model checking LDL is PSPACE-complete.

**Theorem 4.** rLDL model checking is PSPACE-complete.

Using the translation of rLDL formulas to alternating parity automata and subsequently to non-deterministic Büchi automata, Problem \ref{prob:translation_upper_bound} can be solved as follows:

1. Translate the transition system $\mathcal{S}$ into a non-deterministic Büchi automaton $\mathfrak{B}_S$ with $L(\mathfrak{B}_S) = \{\lambda(\rho) \in (2^P)\omega \mid \rho \in \Pi_S\}$ in the usual way: $\mathfrak{B}_S$ has the same states as $\mathcal{S}$, the transition are $\{(s, \lambda(s), s') \mid (s, s') \in E\}$, and all states are accepting.
2. Complement $\mathfrak{A}_{\varphi, \beta}$ to obtain an alternating parity automaton $\overline{\mathfrak{A}_{\varphi, \beta}}$ accepting the language $\{w \in (2^P)\omega \mid V^{rd}(w, \varphi) \geq \beta\}$.
3. Complement $\overline{\mathfrak{A}_{\varphi, \beta}}$ to obtain an alternating parity automaton $\overline{\mathfrak{A}_{\varphi, \beta}}$ accepting the language $\{w \in (2^P)\omega \mid V^{rd}(w, \varphi) < \beta\}$.
4. Convert $\overline{\mathfrak{A}_{\varphi, \beta}}$ into an equivalent non-deterministic Büchi automaton $\mathfrak{B}_{\varphi, \beta}$ (see, e.g., \cite{tabuada13a}) and compute the product automaton $\mathfrak{B}$ with $L(\mathfrak{B}) = L(\mathfrak{B}_S) \cap L(\overline{\mathfrak{A}_{\varphi, \beta}})$ in the usual way.
5. Check whether $L(\mathfrak{B}) = \emptyset$ using a standard algorithm such as a nested depth-first search $\mathfrak{B}$. The answer to Problem \ref{prob:translation_upper_bound} is “yes” if and only if $L(\mathfrak{B}) = \emptyset$.

The number of states of the alternating parity automata in Step\ref{step:translation_upper_bound} and \ref{step:translation_upper_bound_complement} is both in $O(|\mathcal{S}|)$. Thus, the number of states of the non-deterministic Büchi automaton $\mathfrak{B}_{\varphi, \beta}$ constructed in Step\ref{step:translation_upper_bound_complement} is in $2^{O(|\mathcal{S}| \cdot \log |\varphi|)}$, and that of $\mathfrak{B}$ is in $|\mathcal{S}| \cdot 2^{O(|\varphi| \cdot \log |\varphi|)}$, where $|\mathcal{S}|$ denotes the number of states of the transition system $\mathcal{S}$ (cf. Theorem \ref{thm:exponential_compilation}). Finally, the time required for the emptiness check in Step\ref{step:translation_upper_bound_complement} is quadratic in the number of states of $\mathfrak{B}$ (linear in the number of $\mathfrak{B}$’s transitions). Consequently, the rLDL model checking problem can be solved in time $|\mathcal{S}|^2 \cdot 2^{O(|\varphi| \cdot \log |\varphi|)}$ and, hence, is in ExpTIME. The following proof shows that the rLDL model checking problem is in fact PSPACE-complete, matching exactly the complexity class of the model checking problem for LDL.
Proof (Proof of Theorem 4). As shown above, the rLDL model checking problem is in \( \text{Exptime} \). To show membership in \( \text{PSPACE} \), we use the observation that given two states of \( \mathcal{B} \), one can decide in polynomial space whether the second state is a successor of the first one (cf. Vardi and Wolper [29] and Faymonville and Zimmermann [13] for a similar construction). Moreover, one can represent states of \( \mathcal{B} \) in polynomial space. This allows running the classical model checking algorithm, which searches for a counterexample, in polynomial space by guessing an appropriate run. \( \text{PSPACE} \) hardness, on the other hand, follows immediately from the facts that (a) the LTL semantics is embedded in rLDL, via the embedding of LDL in rLDL (see Theorem 3), and (b) LTL model checking is \( \text{PSPACE} \)-hard [23]. Thus, rLDL model checking is \( \text{PSPACE} \)-hard as well.

Before we move on to reactive synthesis, let us briefly remark that the model checking problem for rLTL(□ □) is defined slightly differently. Instead of asking whether \( V^w(\lambda(\rho), \varphi) \geq \beta \), Tabuada and Neider [20] fix a set \( B \subseteq \mathbb{B}_4 \) and ask whether \( V^w(\lambda(\rho), \varphi) \in B \) for all paths \( \rho \in \mathcal{H}_S \). However, this slightly more general problem can easily be answered by a simple adaptation of Step 2 of the procedure above: given a (finite) set \( B \subseteq \mathbb{B}_4 \), we construct an alternating parity automaton accepting the language \( \{ w \in (2^P)^\omega \mid V^\text{rd}(w, \varphi) \in B \} \) using Boolean combinations of the automata \( \mathfrak{A}_{\varphi, \beta} \). Then, it is not hard to verify that this variant of the rLDL model checking problem is also \( \text{PSPACE} \)-complete.

### 3.3 Synthesis of Reactive Systems

Similar to model checking, the translation from rLDL formulas to automata provides us with an effective means to synthesize reactive controllers from rLDL specifications. In this context, we consider the classical reduction from reactive synthesis to infinite-duration two-player games over finite graphs. In particular, we show how to construct a finite-state winning strategy for games with rLDL winning conditions, which immediately correspond to implementations of reactive controllers. Throughout this section, we assume familiarity with games over finite graphs and follow the definitions and notations of [15, Chapter 2].

We consider rLDL games over \( P \), which are triples \( \mathcal{G} = (G, \varphi, \beta) \) consisting of a labeled game graph \( G \), an rLDL formula \( \varphi \), and a truth value \( \beta \in \mathbb{B}_4 \). A labeled game graph \( G = (V_0, V_1, E, \lambda) \) consists of a directed graph \( (V_0 \cup V_1, E) \), two finite, disjoint sets of vertices \( V_0 \) and \( V_1 \), and a function \( \lambda: V_0 \cup V_1 \to 2^P \) mapping each vertex to the set of atomic propositions that hold true in the vertex. We denote the set of all vertices by \( V = V_0 \cup V_1 \) and assume that game graphs do not have terminal vertices, i.e., \( \{v\} \times V \cap E \neq \emptyset \) for each vertex \( v \in V \).

As in the classical setting, rLDL games are played by two players, Player 0 and Player 1, who move a token along the edges of the game graph ad infinitum (if the token is currently placed on a vertex \( v \in V_i \), \( i \in \{0, 1\} \), then Player \( i \) decides the next move). The resulting infinite sequence \( \rho = v_0v_1v_2 \cdots \in (2^P)^\omega \) of vertices is called a play and induces a trace \( \lambda(\rho) = \lambda(v_0)\lambda(v_1)\lambda(v_2) \cdots \in (2^P)^\omega \). The valuation of \( \varphi \) on \( \lambda(\rho) \) determines the winner of a play: Player 0 wins if \( V^\text{rd}(\lambda(\rho), \varphi) \geq \beta \), whereas Player 1 wins if \( V^\text{rd}(\lambda(\rho), \varphi) < \beta \).

Let \( i \in \{0, 1\} \). A strategy of Player \( i \) is a mapping \( f: V^* V_i \to V \) that prescribes where to move the token depending on the finite play constructed so far. A play \( v_0v_1v_2 \cdots \) is played according to \( f \) if \( v_{n+1} = f(v_0 \cdots v_n) \) for every \( n \) with \( v_n \in V_i \). A strategy \( f \) of Player \( i \) is winning from a vertex \( v \in V \) if all plays that start in \( v \) and that are played according to \( f \) are winning for Player \( i \). Further, a (winning) strategy is a finite-state strategy if there exists a finite-state machine computing it in the usual sense (see [15, Chapter 2] for details).

We are interested in solving rLDL games, i.e., in solving the following problem.

**Problem 2.** Let \( \mathcal{G} \) be an rLDL game and \( v \) a vertex. Determine whether Player 0 has a winning strategy for \( \mathcal{G} \) from \( v \) and compute a finite-state winning strategy if so.

Theorem 4 provides a straightforward way to solve Problem 2 by reducing it to solving classical parity games (again, see [15, Chapter 2] for an introduction to parity games).

**Theorem 5.** Solving rLDL games is \( \text{2ExpTime-complete} \).

**Proof.** We proceed in several steps constructing a reduction to a parity game.

1. Construct the deterministic parity automaton \( \mathfrak{A}_{\rho, \beta} \) recognizing the language \( \{ w \in (2^P)^\omega \mid V^\text{rd}(w, \varphi) \geq \beta \} \) according to Corollary 1.
2. Construct the product of $\mathcal{P}_{x,\beta} = (Q, \Sigma, q_0, \delta, \Omega)$ and the labeled game graph $G = (V_0, V_1, E, \lambda)$. This product is a classical (non-labeled) parity game $G' = (G', \Omega')$ consisting of a game graph $G' = (V_0', V_1', E', \Omega')$ with $V_0' = V_0 \times Q$, $V_1' = V_1 \times Q$, and $E = \{(v, q), (v', \delta(q, \lambda(v))) \mid (v, v') \in E\}$ as well as a parity winning condition $\Omega'$ with $\Omega'(\langle v, q \rangle) = \Omega(q)$ for each $(v, q) \in V_0' \cup V_1'$. One obtains a play $\rho$ in the original game $G$ from a play $\rho'$ in the extended game $G'$ by projecting the vertices of $\rho'$ onto the first component. Thus, Player 0 wins a play $\rho'$ in $G'$ from a vertex $(v, q_1)$ if and only if the trace $\lambda(\rho)$ obtained from the corresponding play $\rho$ in $G$ satisfies $V^{\text{rp}}(\lambda(\rho), \varphi) \supseteq \beta$.

3. Solve the game $G'$ with standard algorithms for parity games, e.g., the recent quasi-polynomial time algorithm [5]. Finally, check and return whether Player 0 has a winning strategy from vertex $(v, q_1)$ and return a winning strategy if so.

The above reduction is a standard game reduction, whose correctness can be shown using standard techniques. In fact, it provides a 2ExpTime algorithm to solve Problem 3 due to Corollary 1, the deterministic parity automaton $\mathcal{P}_{x,\beta}$ constructed in Step 4 has $2^{2O(|\varphi| \cdot \log |\varphi|)}$ states and $2^{2O(|\varphi| \cdot \log |\varphi|)}$ colors; consequently, the parity game $G'$ of Step 2 has $|V| \cdot 2^{2O(|\varphi| \cdot \log |\varphi|)}$ vertices and $2^{2O(|\varphi| \cdot \log |\varphi|)}$ colors; thus, using the quasi-polynomial algorithm for solving parity games [5] results in a doubly-exponential algorithm for Problem 2.

On the other hand, the fact that rLDL subsumes LDL and, hence, LTL immediately implies that solving rLDL games is 2ExpTime-hard since solving LTL games is already 2ExpTime-hard [22].

4 Robust and Prompt Linear Temporal Logic

We now introduce robust semantics for Prompt-LTL, obtaining the new logic rPrompt-LTL. To solve the model checking and the synthesis problem we use reductions to Prompt-LTL.

The formulas of rPrompt-LTL are obtained by adding the prompt-eventually operator to rLTL($[\square, \diamond]$), while disallowing arbitrary negation and implications (as in Prompt-LTL), i.e., they are given by the grammar

$$\varphi ::= p \mid \neg p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \square \varphi \mid \diamond \varphi \mid \square p \mid \diamond p,$$

where $p$ ranges over the atomic propositions $P$. The size $|\varphi|$ of a formula $\varphi$ is the number of its distinct subformulas.

The semantics is again given by an evaluation function $V^{\text{rp}}$ mapping a trace $w$, a bound $k$ for the prompt-eventualities, and a formula $\varphi$ to a truth value from $\{\text{false}, \text{true}\}$. This function coincides with the evaluation function for rLTL($[\square, \diamond]$) but for the prompt-eventually:

$$V^{\text{rp}}(w, k, p) = \begin{cases} 1111 & \text{if } p \in w(0), \\ 0000 & \text{if } p \notin w(0), \end{cases} \text{ and } V^{\text{rp}}(w, k, \neg p) = \begin{cases} 1111 & \text{if } p \notin w(0), \\ 0000 & \text{if } p \in w(0), \end{cases}$$

$$V^{\text{rp}}(w, k, \varphi_0 \land \varphi_1) = \min\{V^{\text{rp}}(w, k, \varphi_0), V^{\text{rp}}(w, k, \varphi_1)\},$$

$$V^{\text{rp}}(w, k, \varphi_0 \lor \varphi_1) = \max\{V^{\text{rp}}(w, k, \varphi_0), V^{\text{rp}}(w, k, \varphi_1)\},$$

$$V^{\text{rp}}(w, k, \square \varphi) = b_1 b_2 b_3 b_4 \text{ where } b_i = \text{max}_{0 \leq n \leq k} V^{\text{rp}}(w[n, \infty), k, \varphi).$$

Thus, as for Prompt-LTL, the semantics of the prompt-eventually is obtained by bounding the scope to the next $k$ positions.

rLTL($[\square, \diamond]$) is a syntactic fragment of rPrompt-LTL, but Prompt-LTL is not, as we disallow until and release operators. We discuss this issue in Section 6.

Example 3. Consider the formula $[\square(\neg q \lor \diamond p)]$ and fix a bound $k$ to evaluate the prompt eventualities. We interpret occurrences of $q$ as requests and occurrences of $p$ as responses.

Then, the different degrees of satisfaction of the formula express the following possibilities: (i) every request is answered within $k$ steps, (ii) if there are infinitely many requests, then almost all of them are answered within $k$ steps, (iii) if there are infinitely many requests, then infinitely many are answered within $k$ steps, (iv) some request is answered within $k$ steps, and (v) no request is answered within $k$ steps.
4.1 Model Checking and Synthesis

In this section, we solve the model checking and the synthesis problem for rPrompt-LTL by a reduction to Prompt-LTL. To this end, we translate every rPrompt-LTL formula into a sequence of five Prompt-LTL formulas that capture the five degrees of satisfaction by making the semantics of the robust always operator explicit.

**Lemma 2.** For every rPrompt-LTL formula \( \varphi \) and every \( \beta \in \mathbb{B}_4 \), there is a Prompt-LTL formula \( \varphi_\beta \) of size \( O(|\varphi|) \) such that \( V^\text{rp}(w,k,\varphi) \geq \beta \) if and only if \( V^\text{r}(w,k,\varphi_\beta) = 1 \).

**Proof.** If \( \beta = 0000 \), then we can pick \( \varphi_\beta = \text{tt} \), independently of \( \varphi \). Otherwise, we obtain the result by induction over the construction of \( \varphi \):

- \( p_\beta = p \) and \( \neg p_\beta = \neg p \) for all atomic propositions \( p \in P \) and all \( \beta \gg 0000 \).
- \( (\varphi_0 \land \varphi_1)_\beta = (\varphi_0)_\beta \land (\varphi_1)_\beta \) for all \( \beta \gg 0000 \).
- \( (\varphi_0 \lor \varphi_1)_\beta = (\varphi_0)_\beta \lor (\varphi_1)_\beta \) for all \( \beta \gg 0000 \).
- \( (\lozenge \varphi)_\beta = \lozenge(\varphi_\beta) \) for all \( \beta \gg 0000 \).
- \( (\square \varphi)_1111 = \square(\varphi_1111) \),
- \( (\varphi_0111 = \lozenge\square(\varphi_0111) \),
- \( (\lozenge \varphi)_0011 = \lozenge\lozenge(\varphi_0011) \),
- \( (\square \varphi)_0001 = \lozenge(\varphi_0001) \), and
- \( (\lozenge p \varphi)_\beta = \lozenge p(\varphi_\beta) \) for all \( \beta \gg 0000 \).

A straightforward induction shows that the resulting formula has the desired properties.

Solutions to the model checking and synthesis problem for rPrompt-LTL are now simple consequences of Lemma 2 as the transformation is effective and techniques for Prompt-LTL can then be applied straightforwardly [17]. We start with the model checking problem where the bound \( k \) is existentially quantified as usual.

**Problem 3.** Let \( \varphi \) be an rPrompt-LTL formula, \( S \) a transition system, and \( \beta \in \mathbb{B}_4 \). Is there a \( k \in \mathbb{N} \) such that \( V^\text{rp}(\lambda(\rho),k,\varphi) \geq \beta \) holds true for all paths \( \rho \in I_S \)?

**Theorem 6.** rPrompt-LTL model checking is in \( \text{PSPACE} \).

**Proof.** By Lemma 2 there exists a \( k \in \mathbb{N} \) such that \( V^\text{rp}(\lambda(\rho),k,\varphi) \geq \beta \) holds true for all paths \( \rho \in I_S \) if and only if there exists a \( k \in \mathbb{N} \) such that \( V^\text{r}(\lambda(\rho),k,\varphi_\beta) = 1 \) for all paths \( \rho \in I_S \). The latter is an instance of the Prompt-LTL model checking problem, which is in \( \text{PSPACE} \) [17].

We do not claim \( \text{PSPACE} \)-hardness, since model checking the fragment of LTL with disjunction, conjunction, diamond, and box operator (and classical semantics) is \( \text{NP} \)-complete [2]. As this fragment can be embedded into rPrompt-LTL, we obtain at least \( \text{NP} \)-hardness for Problem 3. As we have no until and release operators (by our own volition), we cannot easily claim \( \text{PSPACE} \)-hardness. In contrast, the solution of the Prompt-LTL model checking problem consists of a reduction to LTL model checking that introduces until operators. Hence, we leave the fragment mentioned above, for which \( \text{NP} \) membership is known.

Finally, let us consider synthesis. An rPrompt-LTL game \((G,\varphi,\beta)\) is defined as an rLDL game but with \( \varphi \) being an rPrompt-LTL formula. Here, we say that a strategy \( f \) for Player 0 is a winning strategy from a vertex \( v \) if there exists a \( k \in \mathbb{N} \) such that \( V^\text{rp}(\lambda(\rho),k,\varphi) \geq \beta \) for every play \( \rho \) that starts in \( v \) and is played according to \( f \).

**Problem 4.** Let \( G \) be an rPrompt-LTL game and \( v \) a vertex. Determine whether Player 0 has a winning strategy for \( G \) from \( v \) and compute a finite-state winning strategy if so.

Our solution to this problem relies on Lemma 2 and techniques for Prompt-LTL [17].

**Theorem 7.** Solving rPrompt-LTL games is 2\text{ExpTime}-complete.

**Proof.** The lower bound follows from the special case of LTL [22]. On the other hand, the upper bound is again proven by a reduction to Prompt-LTL: Player 0 having a winning strategy for \((G,\varphi,\beta)\) from \( v \) is equivalent to her having a winning strategy for the Prompt-LTL game \((G,\varphi,\beta)\) from \( v \). The latter problem can be solved in doubly-exponential time and a finite-state strategy can effectively be computed [32].
5 Robust and Prompt Linear Dynamic Logic

Finally, we combine the capabilities of all logics considered so far in order to obtain a robust version of Prompt-LDL, which we call rPrompt-LDL. As this logic is an extension of Prompt-LTL, it features negations only at the level of atomic propositions, and does not allow implications. The formulas of rPrompt-LDL are given by the grammar

\[
  \phi ::= p \mid \phi \land \phi \mid \phi \lor \phi \mid \langle r \cdot \rangle \phi \mid \langle r \cdot \rangle_p \phi
\]

\[
r ::= \phi \mid \phi^* \mid r + r \mid r; r
\]

where \( p \) again ranges over the atomic propositions in \( P \) and \( \phi \) over propositional formulas over \( P \). Furthermore, the size of a formula is defined as for LDL and rLDL. The size of a formula is measured as for rLDL.

The semantics are again defined by an evaluation function \( V^\text{rpd} \) that maps a trace, a bound for the parameterized diamond operators, and a formula to a truth value:

\[
  V^\text{rpd}(w, k, p) = \begin{cases} 
  1111 & \text{if } p \in w(0), \\
  0000 & \text{if } p \notin w(0), \\
\end{cases}
\]

\[
  V^\text{rpd}(w, k, \varphi_0 \land \varphi_1) = \min \{ V^\text{rpd}(w, k, \varphi_0), V^\text{rpd}(w, k, \varphi_1) \},
\]

\[
  V^\text{rpd}(w, k, \varphi_0 \lor \varphi_1) = \max \{ V^\text{rpd}(w, k, \varphi_0), V^\text{rpd}(w, k, \varphi_1) \},
\]

\[
  V^\text{rpd}(w, k, \langle r \cdot \rangle \varphi) = b_1 b_2 b_3 b_4 \text{ where } b_1 = \max_{n \in \mathbb{R}_2^{(w, k, r)}} V^\text{rpd}_2(w[n, \infty), k, \varphi),
\]

\[
  b_2 = \begin{cases} 
  \max_{0 \leq m} \min_{n \in \mathbb{R}_2^{(w, k, r)}} V^\text{rpd}_2(w[n, \infty), k, \varphi) & \text{if } |R^\text{rpd}_2(w, k, r)| = \infty, \\
  1 & \text{if } 0 < |R^\text{rpd}_2(w, k, r)| < \infty,
\end{cases}
\]

\[
  b_3 = \begin{cases} 
  \min_{0 \leq m} \max_{n \in \mathbb{R}_2^{(w, k, r)}} V^\text{rpd}_3(w[n, \infty), k, \varphi) & \text{if } 0 < |R^\text{rpd}_3(w, k, r)| < \infty, \\
  1 & \text{if } |R^\text{rpd}_3(w, k, r)| = \infty,
\end{cases}
\]

\[
  b_4 = \begin{cases} 
  \max_{n \in \mathbb{R}_2^{(w, k, r)}} V^\text{rpd}_4(w[n, \infty), k, \varphi) & \text{if } |R^\text{rpd}_4(w, k, r)| > 0, \\
  1 & \text{if } |R^\text{rpd}_4(w, k, r)| = 0,
\end{cases}
\]

\[
  V^\text{rpd}(w, k, \langle r \cdot \rangle_p \varphi) = b_1 b_2 b_3 b_4 \text{ where } b_i = \max_{n \in \mathbb{R}_2^{(w, k, r)}} \min_{[0, n]} V^\text{rpd}_2(w[n, \infty), k, \varphi).
\]

Here, we adapt the definition of \( \mathbb{R}_2^{(w, k, r)} \) to account for the parameter \( k \): \( \mathbb{R}_2^{(w, k, r)}(w[k, r]) = \{0\} \) if \( |R^\text{rpd}_4(w, k, \varphi)| = 1 \) and \( \mathbb{R}_2^{(w, k, r)}(w[k, \varphi^?]) = 0 \) otherwise. All other cases are defined as before, but propagate the parameter \( k \).

Example 4. Consider the formula \( \langle r \cdot \rangle \langle \neg q \lor \langle r \cdot \rangle_p \rangle \) with \( r = (tt; tt)^* \). It is similar to the formula of Example 3 but only considers requests and responses at even positions.

rLDL (and thus also LDL) and Prompt-LDL are syntactic fragments of rPrompt-LDL, but Prompt-LTL is not, as the robust semantics of the release operator are not compatible with the semantics of rPrompt-LDL.

5.1 Model Checking and Synthesis for the Test-free Fragment

It turns out that our model checking and synthesis techniques for rLDL and rPrompt-LTL cannot easily be combined (we discuss the reasons why below). Thus, we identify a fragment of rPrompt-LDL that restricts the use of regular expression as guards of always and eventually operators: we only allow so-called test-free and limit-matching expression. For this fragment, we show that both the model checking and synthesis is decidable.

We say that a guard \( r \) is test-free if it does not contain tests as atoms, but only propositional formulas over the atomic propositions. A formula is test-free if each of its guards is test-free. In the remainder, we only consider test-free formulas. As the adaptations made to define \( \mathbb{R}_2^{(w, k, r)} \) are only concerned with tests, they can be ignored when reasoning about test-free formulas.
Remark 1. Let $r$ be a test-free guard. Then, $R_i^{rp}(w, k, r)$ is independent of $i$ and $k$ for every trace $w$.

We say that a test-free guard $r$ is limit-matching if we have $|R_i^{rp}(w, k, r)| = \infty$ for every trace $w$. This is well-defined due to the previous remark. Again, a test-free formula is limit-matching if each of its guards is limit-matching.

Lemma 3. The problem “Given a test-free formula $\varphi$, is $\varphi$ limit-matching?” is in PSPACE.

Proof. The problem is in PSPACE if one can decide in polynomial space whether a single test-free guard is limit-matching. Hence, let $r$ be such a guard, which is limit-matching if and only if infinitely many prefixes of each trace $w$ match $r$. An application of König’s Lemma yields that the latter condition is equivalent to each $w$ being Büchi-accepted by $\mathcal{G}_r$, which is an $\varepsilon$-NFA due to test-freeness. Hence, $r$ is limit-matching if and only if $\mathcal{G}_r$ is universal, which can be decided in polynomial space \text{[24]} (after eliminating $\varepsilon$-transitions). The automaton being of the same size as the guard and being efficiently constructible concludes the proof.

In the following, we consider the model checking and the synthesis problem for test-free limit-matching formulas. To this end, we proceed as in the case of rPrompt-LTL: we reduce these problems to those for Prompt-LDL by transforming formulas to “remove” robustness. Due to only considering limit-matching formulas, we do not have to deal with the cases of having only finitely many matches of a guard. On the other hand, we have to “split” guards to capture the semantics of the robust diamond operator. Here, we exploit the formula under consideration being test-free. We discuss both restrictions in Section 6. Let us just note here that for LDL, the test-free fragment is of equal expressive power as full LDL, albeit potentially less succinct. The former claim follows, e.g., from Theorem \text{[3]}

Example 5. Consider the formula $\varphi = [\langle tt; tt \rangle^*]p \lor [\langle tt; tt \cdot tt \rangle^*]q$. It specifies that either at every even position $p$ holds, or that at every third position $q$ holds, i.e., $\varphi$ implements modulo counting. Moreover, $\varphi$ is test-free and limit-matching.

In general, test-free and limit-matching rPrompt-LDL formulas can make use of arbitrary modulo counting, a significant advance in expressiveness over classical LTL, thus witnessing the usefulness of the fragment.

The main technical result on rPrompt-LDL states that the logic can be derobustified, i.e., translated into Prompt-LDL.

Theorem 8. For every test-free limit-matching rPrompt-LDL formula $\varphi$ and every $\beta \in \mathcal{B}_3$, there is a Prompt-LDL formula $\varphi_\beta$ such that $V^r_{\varphi_\beta}(w, k, \varphi) \geq \beta$ if and only if $V^r_p(w, k, \varphi_\beta) = 1$.

Proof. Before we present the translation, we need to explain how to “split” guards, which is necessary to implement the semantics of the robust box operator, i.e., we have to check that almost all $r$-matches are $\psi$ satisfying for some guard $r$ and some subformula $\psi$. In LTL, “almost all” is expressed by $\square\square$. We will use the analogous LDL operators, i.e., a formula of the form $\langle \cdot \rangle [\cdot]$. But now we need guards $r_0$ and $r_1$ for the diamond and the box operator so that the concatenation $r_0r_1$ is equivalent to $r$. To this end, we transform $r$ into a deterministic automaton and then have such a pair of guards for every intermediate state that can be reached by the automaton. Ultimately, we then end up with a disjunction of formulas of the form $\langle \cdot \rangle [\cdot]$.

Let $r$ be a test-free guard. Applying Lemma \text{[3]} to $r$ yields an $\varepsilon$-NFA $\mathcal{G}_r$ without marked states. Hence, eliminating $\varepsilon$-transitions and determining the resulting automaton yields a deterministic finite automaton $\mathcal{D}_r$ such that $w[0, n]$ is accepted by $\mathcal{D}_r$ if and only if $w \in \mathcal{R}(w, r)$. Furthermore, due to test-freeness, acceptance of $w[0, n]$ by $\mathcal{D}_r$ only depends on the prefix $w[0, n]$ of $w$, but not on the corresponding suffix $w[n, \infty)$. Let $Q$ be the set of states of $\mathcal{D}_r$, $q_I$ the initial state, and $F$ the set of final states. Then, one can efficiently construct regular expressions (i.e., guards) $r_{q_I,q}$ and $r_{q,F}$ such that $w \in (2^n)^*$ is in the language of $r_{q_I,q}$ (of $r_{q,F}$) if the unique run of $\mathcal{D}_r$ starting in $q_I$ (in $q$) ends in $q$ (in $F$).

Now, we are ready to construct $\varphi_\beta$. Again, the case $\beta = 0000$ is trivial. Hence, we assume $\beta > 0000$ in the following. We proceed by induction over the construction of the formula:

- $p_\beta = p$ and $\neg p_\beta = \neg p$ for all atomic propositions $p \in P$ and all $\beta > 0000$.
- $(\varphi_0 \land \varphi_1)_\beta = (\varphi_0)_\beta \land (\varphi_1)_\beta$ for all $\beta > 0000$.
- $(\varphi_0 \lor \varphi_1)_\beta = (\varphi_0)_\beta \lor (\varphi_1)_\beta$ for all $\beta > 0000$. 

12
We addressed the problems of verification and synthesis with robust, expressive, and quantitative linear temporal logics. As is, the robust semantics of the release operator (see [26]) is not expressible as a generalization of the translation proposed in Section 5.1. However, due to the blowup when splitting the guards, it is unlikely that a naive implementation of this approach yields the elusive exponential compilation property for rPrompt-LDL. One alternative to this latter challenge is to study the semantics for the robust box operator proposed in Footnote 3 on Page 4. This is expressible as a generalization of the translation proposed in Section 5.1. Furthermore, decidability for the full logic is left open. We show the problems to be decidable for an important fragment, but due to a blowup of size and we still have to translate the formula $\varphi$ containing these guards into (deterministic) automata to solve the problems.

A straightforward induction over the construction of $\varphi$, relying on the fact that $\varphi$ is limit-matching, yields the correctness of the translation.

Note that our approach for the fragment, which relies on a translation to Prompt-LDL, cannot easily be extended to formulas with tests and to formulas with non-limit-matching guards. The existence of tests complicates the construction of the deterministic automaton required to “split” the guards. For example, consider the guard $(\varphi_0 ?; a ; \varphi'_0) + (\varphi_1 ?; a ; \varphi'_1)$: after processing an $a$, depending on which tests hold true before the $a$, the automaton still has to distinguish whether $\varphi_0$ or $\varphi_1$ has to hold after processing the $a$. Complicating the situation even further, the lack of negations in parameterized logics does not allow to “disambiguate” the guard. Similarly, allowing non-limit-matching guards requires us to implement the full case distinction in the definition of the semantics of the robust box operator. However, implementing the case distinction in Prompt-LDL is again complicated by the lack of negations.

Now, the model checking and the synthesis problem for rPrompt-LDL, which are defined as expected, can be solved by reducing them to their analogues for Prompt-LDL (cf. Section 4.1). We obtain the following results.

**Corollary 2.** The rPrompt-LDL model checking and synthesis problem are decidable.

We refrain from specifying the exact complexity of the algorithms, as we conjecture them to be several exponents away from optimal algorithms: The guards $r_{q,F}$ and $r_{q,F}$ are already of doubly-exponential size and we still have to translate the formula $\varphi$ containing these guards into (deterministic) automata to solve the problems.

## 6 Conclusion

We addressed the problems of verification and synthesis with robust, expressive, and quantitative linear temporal specifications. Inspired by robust Linear Temporal Logic, we have first developed robust extensions of the logics LDL and Prompt-LTL, named rLDL and rPrompt-LTL, respectively. Then, we combined rLDL and rPrompt-LTL into a third logic, named rPrompt-LDL, which has the expressiveness of $\omega$-regular languages and allows reasoning about timing bounds and robustness at the same time.

For rLDL and rPrompt-LTL, we have shown how to solve the model checking and synthesis problem relying on the exponential compilation property. Hence, all these problems are not harder than those for plain LTL. The situation for the combination of all three basic logics, i.e., for rPrompt-LDL, is less encouraging. We show the problems to be decidable for an important fragment, but due to a blowup of the formulas during the reduction, we (most likely) do not obtain asymptotically optimal algorithms. Furthermore, decidability for the full logic is left open.

In future work, we aim to determine the exact complexity of the model checking and synthesis problem for (full) rPrompt-LDL. One promising approach is to generalize the translation of rLDL into alternating parity automata. However, this requires a suitable quantitative alternating automata model with strong closure properties that can be transformed into equivalent non-deterministic and deterministic automata.

One alternative to this latter challenge is to study the semantics for the robust box operator proposed in Footnote 3 on Page 4. This is expressible as a generalization of the translation proposed in Section 5.1. However, due to the blowup when splitting the guards, it is unlikely that a naive implementation of this approach yields the elusive exponential compilation property for rPrompt-LDL.

Finally, we leave open whether full robust LTL, i.e., with until and release, can be embedded into the logics we have introduced here. As is, the robust semantics of the release operator (see [26]) is not compatible with our robust semantics for rLDL. In future work, we plan to study generalizations of full robust LTL.
References

1. Alur, R., Etessami, K., Torre, S.L., Peled, D.: Parametric temporal logic for “model measuring”. ACM Trans. Comput. Log. 2(3), 388–407 (2001)
2. Alur, R., Torre, S.L.: Deterministic generators and games for LTL fragments. ACM Trans. Comput. Log. 5(1), 1–25 (2004)
3. Baier, C., Katoen, J.P.: Principles of Model Checking. The MIT Press (2008)
4. Bloem, R., Chatterjee, K., Greimel, K., Henzinger, T.A., Jobstmann, B.: Robustness in the presence of liveness. In: CAV 2010. LNCS, vol. 6174, pp. 410–424. Springer (2010)
5. Calude, C.S., Jain, S., Koussainov, B., Li, W., Stephan, F.: Deciding parity games in quasipolynomial time. In: STOC 2017, pp. 252–263. ACM (2017)
6. Dalal, E., Neider, D., Tabuada, P.: Synthesis of safety controllers robust to unmodeled intermittent disturbances. In: CDC 2016, pp. 7425–7430 (2016)
7. De Giacomo, G., Vardi, M.Y.: Linear temporal logic and linear dynamic logic on finite traces. In: Rossi, F. (ed.) IJCAI, IJCAI/AAAI (2013)
8. Demri, S., Goranko, V., Lange, M.: Temporal Logics in Computer Science: Finite-State Systems. Cambridge Tracts in Theoretical Computer Science, Cambridge University Press (2016)
9. Douzé, A., Maler, O.: Robust satisfaction of temporal logic over real-valued signals. In: FORMATS 2010. LNCS, vol. 6246, pp. 92–106. Springer (2010)
10. Doyen, L., Henzinger, T.A., Legay, A., Nickovic, D.: Robustness of sequential circuits. In: ACSD 2010, pp. 77–84. IEEE Computer Society (2010)
11. Eisner, C., Fisman, D.: A Practical Introduction to PSL. Integrated Circuits and Systems, Springer (2006)
12. Faïnekos, G.E., Pappas, G.J.: Robustness of temporal logic specifications for continuous-time signals. Theor. Comput. Sci. 410(42), 4262–4291 (2009)
13. Faymonville, P., Zimmermann, M.: Parametric linear dynamic logic. Inf. Comput. 253, 237–256 (2017)
14. Fix, L.: Fifteen years of formal property verification in intel. In: Grumberg, O., Veith, H. (eds.) 25 Years of Model Checking - History, Achievements, Perspectives. LNCS, vol. 5000, pp. 139–144. Springer (2008)
15. Grädel, E., Thomas, W., Wilke, T. (eds.): Automata, Logics, and Infinite Games: A Guide to Current Research, LNCS, vol. 2500. Springer (2002)
16. Koymans, R.: Specifying real-time properties with metric temporal logic. Real-Time Systems 2, 255–299 (1990)
17. Kupferman, O., Piterman, N., Vardi, M.Y.: From liveness to promptness. Formal Methods in System Design 34(2), 83–103 (2009)
18. Leucker, M., Sánchez, C.: Regular linear temporal logic. In: Jones, C.B., Liu, Z., Woodcock, J. (eds.) ICTAC 2007. LNCS, vol. 4711, pp. 291–305. Springer (2007)
19. Majumdar, R., Saha, I.: Symbolic robustness analysis. In: RTSS 2009, pp. 355–363. IEEE Computer Society (2009)
20. Neider, D., Weinert, A., Zimmermann, M.: Synthesizing optimally resilient controllers. In: Ghika, D., Jung, A. (eds.) CSL 2018. Schloss Dagstuhl - LZI (2018), to appear
21. Pnueli, A.: The temporal logic of programs. In: FOCS 1977, pp. 46–57. IEEE (Oct 1977)
22. Pnueli, A., Rosner, R.: On the synthesis of an asynchronous reactive module. In: Ausiello, G., Dezani-Ciancaglini, M., Rocca, S.R.D. (eds.) ICALP 1989. LNCS, vol. 372, pp. 652–671. Springer (1989)
23. Sistla, A.P., Clarke, E.M.: The complexity of propositional linear temporal logics. J. ACM 32(3), 733–749 (1985)
24. Sistla, A.P., Vardi, M.Y., Wolper, P.: The complementation problem for Büchi automata with applications to temporal logic (extended abstract). In: Brauer, W. (ed.) ICALP 1985. LNCS, vol. 194, pp. 465–474. Springer (1985)
25. Tabuada, P., Caliskan, S.Y., Rungger, M., Majumdar, R.: Towards robustness for cyber-physical systems. IEEE Trans. Automat. Contr. 59(12), 3151–3163 (2014)
26. Tabuada, P., Neider, D.: Robust linear temporal logic. In: Talbot, J., Regnier, L. (eds.) CSL 2016. LIPIcs, vol. 62, pp. 10:1–10:21. Schloss Dagstuhl - LZI (2016)
27. Thompson, K.: Programming techniques: Regular expression search algorithm. Commun. ACM 11(6), 419–422 (Jun 1968)
28. Vardi, M.Y.: The rise and fall of LTL. In: D’Agostino, G., Torre, S.L. (eds.) GandALF 2011. EPTCS, vol. 54 (2011)
29. Vardi, M.Y., Wolper, P.: Reasoning about infinite computations. Inf. Comput. 115(1), 1–37 (1994)
30. Weinert, A., Zimmermann, M.: Visibly linear dynamic logic. In: Lal, A., Akshay, S., Saurabh, S., Sen, S. (eds.) FSTTCS 2016. LIPIcs, vol. 65, pp. 28:1–28:14. Schloss Dagstuhl - LZI (2016)
31. Wolper, P.: Temporal logic can be more expressive. Information and Control 56(1/2), 72–99 (1983)
32. Zimmermann, M.: Optimal bounds in parametric LTL games. Theor. Comput. Sci. 493, 30–45 (2013)
33. Zimmermann, M.: Parameterized linear temporal logics meet costs: still not costlier than LTL. Acta Inf. 55(2), 129–152 (2018)
A A Zoo of Temporal Logics

In this section, we introduce the logics we generalize in this work. See Table 1 for an overview.

| Logic          | Operators | Complexity          | Games               |
|----------------|-----------|---------------------|---------------------|
| rLTL(□ ◊)     | ¬, ∧, ∨, →, □ ◊ | NP-hard/in PSPACE in 2ExpTime |
| LDL           | ¬, ∧, ∨, →, (r), [r] | PSPACE-compl.       |
| Prompt-LTL    | ∧, ∨, U, R, ◊p | PSPACE-compl.       |
| Prompt-LDL    | ∧, ∨, (r), [r], (r)p | PSPACE-compl.       |

Table 1. The logics are work is based on.

A.1 Robust Linear Temporal Logic

When Tabuada and Neider introduced robust LTL [26], they first considered the fragment rLTL(□ ◊) without the untill and release operator, which already captures the most interesting problems arising from adding robustness. Then, they added the until and release operator and studied the full logic. Here, we follow their approach and consider generalizations of the fragment rLTL(□ ◊) that only contains the temporal operators □ and ◊.

Formally, the formulas of rLTL(□ ◊) are given by the grammar

\[ \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \mid \varphi \diamond \varphi \mid \square \varphi, \]

where \( p \) ranges over the atomic propositions in \( P \). Following the convention introduced in the original paper, we denote the temporal operators of robust logics with dots as to make the distinction to the original logics more visible. Moreover, note that the syntax of rLTL(□ ◊) explicitly contains implication and conjunction; due to the many-valued semantics of rLTL(□ ◊) introduced below, these two operators cannot be recovered from disjunction and negation. We define the size \( |\varphi| \) of a formula as the number of distinct subformulas of \( \varphi \).

The semantics of rLTL(□ ◊) is given by an evaluation function \( V^R \) mapping a trace \( w \) and a formula \( \varphi \) to a truth value from \( \mathbb{B}_4 \) that is defined as follows:

\[
V^R(w, p) = \begin{cases} 
1111 & \text{if } p \in w(0), \\
0000 & \text{if } p \notin w(0), 
\end{cases}
\]

\[
V^R(w, \neg \varphi) = \begin{cases} 
1111 & \text{if } V^R(w, \varphi) \neq 1111, \\
0000 & \text{if } V^R(w, \varphi) = 1111, 
\end{cases}
\]

\[
V^R(w, \varphi_0 \land \varphi_1) = \min\{V^R(w, \varphi_0), V^R(w, \varphi_1)\},
\]

\[
V^R(w, \varphi_0 \lor \varphi_1) = \max\{V^R(w, \varphi_0), V^R(w, \varphi_1)\},
\]

\[
V^R(w, \varphi_0 \rightarrow \varphi_1) = \begin{cases} 
1111 & \text{if } V^R(w, \varphi_0) \preceq V^R(w, \varphi_1), \\
V^R(w, \varphi_1) & \text{if } V^R(w, \varphi_0) \succ V^R(w, \varphi_1), 
\end{cases}
\]

\[
V^R(w, \diamond \varphi) = b_1b_2b_3b_4 \text{ with } b_i = \max_{n \geq 0} V^R_n(w[n, \infty], \varphi), \text{ and}
\]

\[
V^R(w, \square \varphi) = b_1b_2b_3b_4 \text{ with }
\]

- \( b_1 = \min_{n \geq 0} V^R_1(w[n, \infty], \varphi), \)
- \( b_2 = \max_{m \geq 0} \min_{n \geq m} V^R_2(w[n, \infty], \varphi), \)
- \( b_3 = \min_{m \geq 0} \max_{n \geq m} V^R_3(w[n, \infty], \varphi), \) and
- \( b_4 = \max_{m \geq 0} V^R_4(w[n, \infty], \varphi). \)

Here, \( V^R_i(w, \varphi) \) denotes the projection of \( V^R(w, \varphi) \) to its \( i \)-th component, i.e., we have \( V^R(w, \varphi) = V^R_4(w, \varphi)V^R_3(w, \varphi)V^R_2(w, \varphi)V^R_1(w, \varphi). \)

The first bit of the semantics captures the classical semantics of LTL, i.e., we have \( V^R_1(w, \varphi) = 1 \) if and only if \( w \) satisfies \( \varphi \) classically. Intuitively, the next three bits are obtained by weakening the
semantics of the subformulas of the form [□ϕ]: instead of (classically) requiring every position to satisfy ϕ, the second bit is one if almost all positions satisfy ϕ (i.e., [□□ϕ] holds), the third bit is one of infinitely many positions satisfy ϕ (i.e., [□□□ϕ] holds), and the fourth bit is one if at least one position satisfies ϕ (i.e., [□ϕ] holds). Note that negation and implication are also non-classical and break this intuition for subformulas of the form [□ϕ] in the scope of a negation or implication. For a full motivation and explanation of the semantics, we refer to the original work introducing rLTL([□, □]) [26].

Verification problems with rLTL([□, □]) specifications have an additional input, a threshold β ∈ ℜ4, and ask every trace to evaluate to at least β. Tabuada and Neider showed that the model checking problem with rLTL([□, □]) specifications can be solved in polynomial space and that infinite games with rLTL([□, □]) specifications can be solved in doubly-exponential time [26]. There are no matching lower bounds, as we only consider the fragment rLTL([□, □]) without the until and release operators (see also the discussion below Theorem 5).

### A.2 Linear Dynamic Logic

The formulas of LDL are given by the grammar

\[ \varphi ::= p \mid \lnot \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \Rightarrow \varphi \mid \langle r \rangle \varphi \mid [r] \varphi \]

\[ r ::= \phi \mid \varphi ? \mid r + r \mid r ; r \mid r^* \]

where \( p \) ranges over the atomic propositions in \( P \) and where \( \phi \) ranges over arbitrary propositional formulas over \( P \). We use the abbreviations \( \top = p \lor \lnot p \) and \( \bot = p \land \lnot p \) for some atomic proposition \( p \).

The regular expressions have two types of atoms: propositional formulas \( \phi \) over the atomic propositions and tests \( \varphi ? \), where \( \varphi \) is again an LDL formula. As we will see later, the semantics of these two kinds of atoms differ significantly. We refer to formulas of the form \( \langle r \rangle \varphi \) and \( [r] \varphi \) as diamond formulas and box formulas, respectively. In both cases, we call \( r \) the guard of the operator.

We denote the set of subformulas of \( \varphi \) by \( cl(\varphi) \). Guards are not subformulas, but the formulas appearing in the tests are, e.g., we have \( cl(\langle p? ; q \rangle p') = \{ p, p', \langle p? ; q \rangle p' \} \). The size \( |\varphi| \) of \( \varphi \) is the sum of \( |cl(\varphi)| \) and the sum of the lengths of the guards appearing in \( \varphi \) (counted with multiplicity and measured in the number of operators).

Analogously to the definition for rLTL([□, □]), and slightly non-standard, we define the semantics of LDL by specifying an evaluation function \( V^\varphi \) mapping a trace \( w \) and a formula \( \varphi \) to a truth value from \( \mathbb{B} \) denoting whether \( w \) satisfies \( \varphi \) or not. Also, our presentation of the semantics here is slightly cumbersome, in particular the definition for the implication, again to align with the definition for rLTL([□, □]). Nevertheless, our definition below is equivalent to the classical semantics of LDL (cf. [7,13]) via a satisfaction relation \( \models \) in the following sense: we have \( V^\varphi(w, \varphi) = 1 \) if and only if \( w \) satisfies \( \varphi \) in the classical sense.

\[
- V^\varphi(w, p) = \begin{cases} 1 & \text{if } p \in w(0), \\ 0 & \text{if } p \notin w(0), \end{cases}
- V^\varphi(w, \lnot \varphi) = \begin{cases} 1 & \text{if } V^\varphi(w, \varphi) = 0, \\ 0 & \text{if } V^\varphi(w, \varphi) = 1, \end{cases}
- V^\varphi(w, \varphi_0 \land \varphi_1) = \min\{V^\varphi(w, \varphi_0), V^\varphi(w, \varphi_1)\},
- V^\varphi(w, \varphi_0 \lor \varphi_1) = \max\{V^\varphi(w, \varphi_0), V^\varphi(w, \varphi_1)\},
- V^\varphi(w, \varphi_0 \Rightarrow \varphi_1) = \begin{cases} 1 & \text{if } V^\varphi(w, \varphi_0) \leq V^\varphi(w, \varphi_1), \\ V^\varphi(w, \varphi_0) & \text{if } V^\varphi(w, \varphi_0) > V^\varphi(w, \varphi_1), \end{cases}
- V^\varphi(w, \langle r \rangle \varphi) = \max_{n \in R(w, r)} V^\varphi(w[n, \infty), \varphi), \text{ and}
- V^\varphi(w, [r] \varphi) = \min_{n \in R(w, r)} V^\varphi(w[n, \infty), \varphi).
\]

Here, the match set \( R(w, r) \subseteq \mathbb{N} \) contains all positions \( n \) such that \( w[0, n] \) matches \( r \) and is defined inductively as follows:

\[
- R(w, \phi) = \{1\} \text{ if } w(0) \models \phi \text{ and } R(\phi, w) = \emptyset \text{ otherwise, for propositional } \phi.
\]

Tabuada and Neider only showed that their algorithm runs in exponential time, but using standard on-the-fly techniques [29] it can also be implemented in polynomial space.
- \( R(w, \varphi?) = \{0\} \) if \( V^\varphi(w, \varphi) = 1 \) and \( R(w, \varphi?) = \emptyset \) otherwise.
- \( R(w, r_0 + r_1) = R(w, r_0) \cup R(w, r_1) \).
- \( R(w, n') = \{n \mid \exists n'' \text{s.t. } n'' \in R(w, r_0) \text{ and } n - n' \in R(w[n', \infty), r_1)\} \).
- \( R(w, r^*) = \{0\} \cup \{n_k \mid \exists n_1, \ldots, n_k \text{s.t. } n_j \in R(w[n_1 + \cdots + n_{j-1}, \infty), r) \text{ for all } j \in \{1, \ldots, k\} \} \),

where we use \( n_1 + \cdots + n_0 = 0 \).

Due to tests, membership of some \( n \in R(w, r) \) does, in general, not only depend on the prefix \( w[0, n) \), but on the complete trace \( w \). Also, the semantics of the propositional atom \( \phi \) differ from the semantics of the test \( \phi? \); the former consumes an input letter, while tests do not. Hence, LDL features both kinds of atoms.

Fix some trace \( w \), a formula \( \varphi \), and a guard \( r \). We say that a position \( n \) of \( w \) is an \( r \)-match if \( n \in R(w, r) \). Further, \( n \) is a \( \varphi \)-satisfying position of \( w \) if \( V^\varphi(w[n, \infty), \varphi) = 1 \). Thus, the formula \( \langle r \rangle \varphi \) requires some \( \varphi \)-satisfying \( r \)-match to exist. Dually, \( [r] \varphi \) requires every \( r \)-match of \( w \) to be \( \varphi \)-satisfying (in particular, this is the case if there is no \( r \)-match in \( w \)). Thus, the diamond operator generalizes the eventually operator and the box operator generalizes the always operator, which are the respective special cases for a trivial guard that matches every position, e.g., \( TT^\omega \). Similarly, the next-, until-, and release operator of LTL can be expressed in LDL (the latter two use tests in the guards). Thus, LTL is a fragment of LDL. Furthermore, it is known that LDL captures the \( \omega \)-regular languages [28].

Model checking against LDL specifications is PSPACE-complete and solving LDL games is 2ExpTime-complete [13,28].

A.3 Prompt Linear Temporal Logic

The formulas of Prompt-LTL are given by the grammar

\[
\varphi ::= p \mid \neg p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \mathbin{\nu} \varphi \mid \varphi \mathbin{\langle} \varphi \mid \varphi \mathbin{\square} \varphi
\]

where \( p \) ranges over the atomic propositions in \( P \). The size \( |\varphi| \) of a formula \( \varphi \) is defined as the number of its distinct subformulas.

In Prompt-LTL, formulas are in negation normal form and implication is disallowed. Both requirements are necessary to preserve monotonicity of the prompt-eventually \( \mathbin{\square} \) with respect to the parameter \( k \) bounding it (Alur et al. [1] provide a detailed discussion). Again, let \( TT = p \lor \neg p \) and \( FF = p \land \neg p \) for some atomic proposition \( p \), which we use to define the shorthands \( \mathbin{\diamond} \varphi = TT \varphi \) and \( \mathbin{\Box} \varphi = FF \varphi \).

Again, we define the semantics by an evaluation function \( V^\varphi \) mapping a trace \( w \), a bound \( k \in \mathbb{N} \) for the parameterized operators, and a formula \( \varphi \) to a truth value in \( \mathbb{B} \) (which is again equivalent to the standard definition):

- \( V^\varphi(w, k, p) = \begin{cases} 1 & \text{if } p \in w(0), \\ 0 & \text{if } p \notin w(0), \end{cases} \)
- \( V^\varphi(w, k, \neg p) = \begin{cases} 1 & \text{if } p \notin w(0), \\ 0 & \text{if } p \in w(0), \end{cases} \)
- \( V^\varphi(w, k, \varphi_0 \land \varphi_1) = \min \{V^\varphi(w, k, \varphi_0), V^\varphi(w, k, \varphi_1)\} \),
- \( V^\varphi(w, k, \varphi_0 \lor \varphi_1) = \max \{V^\varphi(w, k, \varphi_0), V^\varphi(w, k, \varphi_1)\} \),
- \( V^\varphi(w, k, \varphi_0 \mathbin{\nu} \varphi_1) = \max_{0 \leq n} \min \{V^\varphi(w[n, \infty), k, \varphi_1), \min_{0 \leq m < n} V^\varphi(w[m, \infty)), k, \varphi_0\} \),
- \( V^\varphi(w, k, \varphi_0 \mathbin{\langle} \varphi_1) = \min_{0 \leq n} \max \{V^\varphi(w[n, \infty), k, \varphi_1), \max_{0 \leq m \leq n} V^\varphi(w[m, \infty)), k, \varphi_0\} \),
- \( V^\varphi(w, k, \varphi_0 \mathbin{\square} \varphi_1) = \max_{0 \leq n < k} V^\varphi(w[n, \infty), k, \varphi) \).

In verification problems for Prompt-LTL, the bound \( k \) on the prompt-eventually is existentially quantified. Kupferman et al. proved that Prompt-LTL model checking is PSPACE-complete and that solving games with Prompt-LTL winning conditions is 2ExpTime-complete [17].

---

\[6\] Instead of games, they actually considered the related framework of realizability, an abstract type of game without underlying graph. However, realizability and graph-based games are interreducible (also, see [32]).
A.4 Prompt Linear Dynamic Logic

In our proofs, we also use Prompt-LDL, which can be seen as a combination of LDL and Prompt-LTL. This logic has been studied by Faymonville and Zimmermann as a fragment of PLDL (although it has never been explicitly named).

The formulas of Prompt-LDL are given by the grammar

\[ \varphi ::= p | \neg p | \varphi \land \varphi | \varphi \lor \varphi | \langle r \rangle \varphi | [r] \varphi | \langle r \rangle p \varphi \]

\[ r ::= \phi | \varphi ? | r + r | r ; r | r^* \]

where \( p \) again ranges over the atomic propositions in \( P \) and \( \phi \) ranges over propositional formulas over \( P \). As in Prompt-LTL, we have to disallow arbitrary negations and implications. The size of a formula is defined as for LDL.

Furthermore, the semantics of Prompt-LDL are obtained by combining the one of LDL and the one of Prompt-LTL: Again, we define an evaluation function \( V^{RD} \) mapping a trace \( w \), a bound \( k \), and a formula \( \varphi \) to a truth value.

\[ V^{RD}(w, k, p) = \begin{cases} 1 & \text{if } p \in w(0), \\ 0 & \text{if } p \notin w(0), \end{cases} \]

\[ V^{RD}(w, k, \neg p) = \begin{cases} 1 & \text{if } p \notin w(0), \\ 0 & \text{if } p \in w(0), \end{cases} \]

\[ V^{RD}(w, k, \varphi_0 \land \varphi_1) = \min \{ V^{RD}(w, k, \varphi_0), V^{RD}(w, k, \varphi_1) \}, \]

\[ V^{RD}(w, k, \varphi_0 \lor \varphi_1) = \max \{ V^{RD}(w, k, \varphi_0), V^{RD}(w, k, \varphi_1) \}, \]

\[ V^{RD}(w, k, [r] \varphi) = \max_{n \in \mathbb{N}} V^{RD}(w|n, \infty), k, \varphi, \]

\[ V^{RD}(w, k, \langle r \rangle \varphi) = \min_{n \in \mathbb{N}} V^{RD}(w|n, \infty), k, \varphi, \]

\[ V^{RD}(w, k, \langle r \rangle p \varphi) = \max_{n \in \mathbb{N}} V^{RD}(w|n, \infty), k, \varphi, \]

Here, \( \mathcal{R}(w, k, r) \) is defined as \( \mathcal{R}(w, r) \), but propagates the bound \( k \) to evaluate tests. Hence, we define \( \mathcal{R}(w, k, \varphi?) = \{0\} \) if \( V^{RD}(w, k, \varphi) = 1 \) and \( \mathcal{R}(w, k, \varphi?) = \emptyset \) otherwise. All other cases are defined as for LDL.

Prompt-LDL as defined here is a syntactic fragment of Parametric LDL \([13]\), for which the model checking problem is \( \text{PSPACE}\)-complete and the synthesis problem is \( 2\text{ExpTime}\)-complete. Hence, the same results apply to Prompt-LDL as well. Here, the bound \( k \) is again uniformly existentially quantified.

B Translating Robust LDL into Automata

In this appendix, we prove Theorem 1. To this end, in Section 1 we first recall how to translate guards \( r \) into finite non-deterministic automata with special features to account for tests. Then, in Section 2 we present the translation of rLDL into alternating parity automata of linear size, which can then be further transformed into non-deterministic Büchi automata of exponential size and deterministic parity automata of doubly-exponential size. Such automata are needed for solving the model checking problem and the synthesis problem, respectively.

B.1 Translating Guards into Automata

Recall that \( P \) is the (finite) set of atomic propositions. An automaton with tests \( \mathfrak{G} = (Q, 2^P, q_I, \delta, F, t) \) consists of a finite set \( Q \) of states, the alphabet \( 2^P \), an initial state \( q_I \in Q \), a transition function \( \delta : Q \times (\Sigma \cup \{\varepsilon\}) \to 2^P \), a set \( F \) of final states, and a partial function \( t \), which assigns to some states \( q \in Q \) an rLDL formula \( \tau(q) \) (which should be thought of as the analogue of tests). We write \( q \xrightarrow{a} q' \) if \( q' \in \delta(q, a) \) for \( a \in \Sigma \cup \{\varepsilon\} \). An \( \varepsilon \)-path \( \pi \) from \( q \) to \( q' \) in \( \mathfrak{G} \) is a sequence \( \pi = q_1 \cdots q_k \) of \( k \geq 1 \) states with \( q = q_1 \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} q_k = q' \). The set of all \( \varepsilon \)-paths from \( q \) to \( q' \) is denoted by \( \Pi(q, q') \) and \( t(\pi) = \{ t(q_i) \mid 1 \leq i \leq k \} \) is the set of tests visited by \( \pi \).

A run of \( \mathfrak{G} \) on \( w(0) \cdots w(n-1) \in \Sigma^* \) is a sequence \( q_0 q_1 \cdots q_n \) of states such that \( q_0 = q_I \) and for every \( i \) in the range \( 0 \leq i \leq n-1 \) there is a state \( q_i' \) reachable from \( q_i \) via an \( \varepsilon \)-path \( \pi_i \) and such that \( q_{i+1} \in \delta(q_i', w(i)) \). The run is accepting if there is a \( q_n' \in F \) reachable from \( q_n \) via an \( \varepsilon \)-path \( \pi_n \). This
slightly unusual definition of runs (but equivalent to the standard one) simplifies our reasoning below. Also, the definition is oblivious to the tests assigned by \( t \). To take them into account, we define for \( i \in \{1, 2, 3, 4\} \):

\[
\mathcal{R}_i^r(w, \Sigma) = \{ n \mid \Sigma \text{ has an accepting run on } w[0, n) \text{ with } \varepsilon\text{-paths } \pi_0, \ldots, \pi_n \text{ s.t.} \\
V_i^{\text{rd}}(w[n', \infty), t(\pi_n)) = 1 \text{ for every } n' \text{ in the range } 0 \leq n' \leq n \}.
\]

Every guard (which is just a regular expression with tests) can be turned into an equivalent automaton with tests via a straightforward generalization of the classical Thompson construction \[27\] turning classical regular expressions into \( \varepsilon \)-NFA.

**Lemma 4** (\[13\]). Every guard \( r \) can be translated into an automaton with tests \( \Sigma_r \), such that \( \mathcal{R}_i^r(w, r) = \mathcal{R}_i^r(w, \Sigma_r) \) for every \( i \in \{1, 2, 3, 4\} \) and with \(|\Sigma_r| \in \mathcal{O}(|r|)\). Furthermore, all final states of \( \Sigma_r \) are terminal, i.e., they have no outgoing transitions.

Note that the automaton \( \Sigma_r \) is independent of \( i \), as this value only determines how tests are evaluated. These are handled “externally” in the definition of the semantics. Having thus demonstrated how to turn guards into automata, we now demonstrate how do the same for whole rLDL formulas.

### B.2 Translating rLDL into Alternating Automata

An alternating parity automaton \( \mathfrak{A} = (Q, \Sigma, q_I, \delta, \Omega) \) consists of a finite set \( Q \) of states, an alphabet \( \Sigma \), an initial state \( q_I \in Q \), a transition function \( \delta: Q \times \Sigma \rightarrow B^+(Q) \), and a coloring \( \Omega: Q \rightarrow \mathbb{N} \) of the states. Here, \( B^+(Q) \) denotes the set of positive Boolean combinations over \( Q \), which contains in particular the formulas \( \top \) (true) and \( \bot \) (false).

A run of \( \mathfrak{A} \) on input string \( w = w(0)w(1)w(2) \cdots \in \Sigma^\omega \) is a directed graph \( \rho = (V, E) \) where \( V \subseteq Q \times \mathbb{N} \) and \( ((q, n), (q', n')) \in E \) implies \( n' = n + 1 \) such that \( (q_I, 0) \in V \) and such that for all \( (q, n) \in V \) we have \( \text{Succ}_0(q, n) = \delta(q, w(n)) \). Here \( \text{Succ}_0(q, n) \) denotes the set of successors of \( (q, n) \) in \( \rho \) projected to \( Q \). A run \( \rho \) is accepting if all infinite paths (projected to \( Q \)) through \( \rho \) satisfy the (max) parity condition, i.e., the maximal color occurring infinitely often on the path is even. The language \( L(\mathfrak{A}) \) contains all \( w \in \Sigma^\omega \) that have an accepting run of \( \mathfrak{A} \).

Alternating parity automata are easily seen to be closed under all Boolean operations. Fix automata \( \mathfrak{A}_0 = (Q_0, \Sigma, q_0^I, \delta_0, \Omega_0) \) and \( \mathfrak{A}_1 = (Q_1, \Sigma, q_1^I, \delta_1, \Omega_1) \).

- \((Q_0, \Sigma, q_0^I, \overline{\Pi}, \overline{\Omega})\) recognizes \( \Sigma^\omega \setminus L(\mathfrak{A}_0) \), where \( \overline{\Pi}(q) = \Omega(q) + 1 \) and where \( \overline{\Omega}_0 \) is the dual of \( \Omega_0 \), i.e., \( \overline{\delta}_0(q, A) \) is obtained from \( \delta_0(q, A) \) by replacing each disjunction by a conjunction, each conjunction by a disjunction, each \( \top \) by \( \bot \), and each \( \bot \) by \( \top \).
- The disjoint union of \( \mathfrak{A}_0 \) and \( \mathfrak{A}_1 \) with a fresh initial state \( q_I \) and \( \delta(q_I, A) = \delta_0(q_I^0, A) \land \delta_1(q_I^1, A) \) recognizes \( L(\mathfrak{A}_0) \cap L(\mathfrak{A}_1) \).
- The disjoint union of \( \mathfrak{A}_0 \) and \( \mathfrak{A}_1 \) with a fresh initial state \( q_I \) and \( \delta(q_I, A) = \delta_0(q_I^0, A) \lor \delta_1(q_I^1, A) \) recognizes \( L(\mathfrak{A}_0) \cup L(\mathfrak{A}_1) \).

The latter two constructions can obviously be generalized to unions and intersections of arbitrary arity while still only requiring a single fresh state.

We prove the following Lemma, which implies Theorem \[4\] as alternating parity automata can be translated into non-deterministic Büchi automata of exponential size (see, e.g., \[8\]).

**Lemma 5.** For every rLDL formula \( \varphi \) and every \( \beta \in \mathbb{B}_4 \), there is an alternating parity automaton \( \mathfrak{A}_{\varphi, \beta} \) with \( \mathcal{O}(|\varphi|) \) states recognizing the language \( \{ w \in (2^\mathbb{N})^\omega \mid V^{\text{rd}}(w, \varphi) \geq \beta \} \).

**Proof.** We first construct the desired automaton by structural induction over the construction of \( \varphi \). Then, we estimate its size. We begin by noting that \( \mathfrak{A}_{\varphi, 0000} \) is trivial for every formula \( \varphi \), as it has to accept every input. Hence, we only consider \( \beta \gg 0000 \) in the remainder of the proof.

For an atomic proposition \( p \in P \), \( \mathfrak{A}_{p, \beta} \) for \( \beta \gg 0000 \) is an automaton that accepts exactly those \( w \) with \( p \in w(0) \). Such an automaton can easily be constructed.

Now, consider a negation \( \varphi = \neg \varphi' \): by definition, we have \( V^{\text{rd}}(w, \varphi) = 0000 \) if \( V^{\text{rd}}(w, \varphi') = 1111 \), and \( V^{\text{rd}}(w, \varphi') = 1111 \) if \( V^{\text{rd}}(w, \varphi') \neq 1111 \). Thus, \( \mathfrak{A}_{p, \beta} \) for \( \beta \gg 0000 \) has to accept the language \( \{ w \mid V^{\text{rd}}(w, \varphi') \neq 1111 \} \), which is the complement of the language of \( \mathfrak{A}_{\varphi', 1111} \).

19
Next, consider a conjunction $\varphi = \varphi_0 \land \varphi_1$. To this end, recall that $V^\text{rd}(w, \varphi) = \min\{V^\text{rd}(w, \varphi_0), V^\text{rd}(w, \varphi_1)\}$. Hence, $\mathcal{A}_{\varphi, \beta}$ has to recognize the language

$$L(\mathcal{A}_{\varphi_0, \beta_0}) \cap L(\mathcal{A}_{\varphi_1, \beta_1}).$$

Such an automaton can be constructed using the closure operations described above.\footnote{The description of the language (and thus the automaton) can be simplified by exploiting the fact that $\beta \leq \beta'$ implies $\mathcal{A}_{\varphi, \beta} \subseteq \mathcal{A}_{\varphi, \beta'}$ for every $\varphi$. The same is true for disjunction and implication.}

The construction for a disjunction $\varphi = \varphi_0 \lor \varphi_1$ is dual to the conjunction: we have $V^\text{rd}(w, \varphi) = \max\{V^\text{rd}(w, \varphi_0), V^\text{rd}(w, \varphi_1)\}$ and thus construct $\mathcal{A}_{\varphi, \beta}$ such that it recognizes the language

$$L(\mathcal{A}_{\varphi_0, \beta_0}) \cap L(\mathcal{A}_{\varphi_1, \beta_1}).$$

For an implication $\varphi = \varphi_0 \rightarrow \varphi_1$, we again implement the semantics via Boolean combinations of automata. Recall that $V^\text{rd}(w, \varphi_0 \rightarrow \varphi_1)$ is equal to 111 if $V^\text{rd}(w, \varphi_0) \preceq V^\text{rd}(w, \varphi_1)$. Otherwise, it is equal to $V^\text{rd}(w, \varphi_1)$. Hence, we construct $\mathcal{A}_{\varphi, \beta}$ so that it recognizes the language

$$\left( \bigcup_{\beta_0, \beta_1 \in \mathbb{R}_4} L(\mathcal{A}_{\varphi_0, \beta_0}) \cap L(\mathcal{A}_{\varphi_1, \beta_1}) \right) \cup \left( \bigcup_{\beta_0, \beta_1 \in \mathbb{R}_4} L(\mathcal{A}_{\varphi_0, \beta_0}) \cap L(\mathcal{A}_{\varphi_1, \beta_1}) \right).$$

The left part covers all cases in which the implication evaluates to 111. Due to 111 $\succeq \beta$ for every $\beta$, this part is equal for all automata. The right part covers all other cases, which depend on $\beta$.

Now, we turn to the constructions for the guarded temporal operators, which are more involved as we have to combine automata for guards, for the tests occurring in them, and for formulas. We follow the general construction presented in \cite{13}, but generalize it to deal with the richer truth values underlying the robust semantics.

First, we consider a diamond formula $\varphi = (\langle \cdot \rangle \varphi')$ with tests $\theta_1, \ldots, \theta_n$ in $r$. Recall that we have $V^\text{rd}(w, \varphi) = b_1 b_2 b_3 b_4$ where $b_i = \max_{n \in \mathbb{N}} V^\text{rd}_i(w[n, \infty), \varphi')$ for all $i \in \{1, 2, 3, 4\}$. Thus, $\mathcal{A}_{\varphi, \beta}$ has to accept $w$ if and only if $w$ has an $r$-match of degree $\beta$ that is $\varphi'$-satisfying of degree $\beta$.

By induction hypothesis, we have automata $\mathcal{A}_{\varphi', \beta}$ and $\mathcal{A}_{\theta_j, \beta}$ for every test $\theta_j$ in $r$. Also, we have an $\varepsilon$-NFA with tests $\mathcal{G}_r$ equivalent to $r$ due to Lemma \cite{13}. We combine these automata to the alternating automaton $\mathcal{A}_{\varphi, \beta}$ by non-deterministically guessing a (finite) run of $\mathcal{G}_r$. Whenever the run encounters a final state, the automaton may jump to the initial state of $\mathcal{A}_{\varphi', \beta}$ and then behave like that automaton. Furthermore, while simulating $\mathcal{G}_r$, $\mathcal{A}_{\varphi, \beta}$ also has to verify that the tests occurring along the guessed run of $\mathcal{G}_r$ hold true by universally spawning copies of $\mathcal{A}_{\theta_j, \beta}$ each time a transition state labeled with $\theta_j$ is traversed. Since we do not allow for $\varepsilon$-transitions in alternating automata, we have to eliminate the $\varepsilon$-transitions of $\mathcal{G}_r$, during the construction of $\mathcal{A}_{\varphi, \beta}$. Finally, in order to prevent $\mathcal{A}_{\varphi, \beta}$ from simulating $\mathcal{G}_r$, ad infinitum, the states copied from $\mathcal{G}_r$ are assigned an odd color, which forces the jump to $\mathcal{A}_{\varphi, \beta}$ to be executed eventually.

Formally, we define $\mathcal{A}_{\varphi, \beta} = (Q, 2^P, q_I, \delta, \Omega)$ where

$\begin{aligned}
- Q & \quad \text{is the disjoint union of the states of the automata } \mathcal{G}_r, \mathcal{A}_{\theta_j, \beta} \text{ for } j \in \{1, \ldots, n\}, \text{ and } \mathcal{A}_{\varphi', \beta}, \\
- q_I & \quad \text{is the initial state of } \mathcal{G}_r, \\
- \Omega & \quad \text{coincides with the colorings of the automata } \mathcal{A}_{\theta_j, \beta} \text{ and } \mathcal{A}_{\varphi', \beta} \text{ on their states and assigns color 1 to every state of } \mathcal{G}_r,
\end{aligned}$

and where $\delta$ is defined as follows: if $q$ is a state of $\mathcal{G}_r$, then

$$\delta(q, A) = \begin{cases} 
V_{q' \in Q} V_{p \in \mathcal{G}_r} (p \wedge \mathcal{A}_{\theta_j, \beta}(A) \wedge \mathcal{A}_{\varphi', \beta}(A)) \\
V_{q' \in P} V_{p \in \mathcal{G}_r} (\mathcal{A}_{\theta_j, \beta}(A) \wedge \mathcal{A}_{\varphi', \beta}(A))
\end{cases}$$
where \( q'_j \) and \( q'_r \) are the initial states of \( \mathfrak{A}_{\theta_j, \beta} \) and \( \mathfrak{A}_{\varphi', \beta} \), respectively, where \( Q^r \) (\( F^r \)) is the set of (final) states of \( \mathfrak{S}_r \), where \( \delta_r, \delta'_r \), and \( \delta_1 \) are the transition functions of \( \mathfrak{S}_r, \mathfrak{A}_{\varphi', \beta} \), and \( \mathfrak{A}_{\theta_j, \beta} \) respectively, and where the sets \( \Pi(q, q') \) of \( \varepsilon \)-paths are induced by \( \mathfrak{S}_r \). Furthermore, for states \( q \) of \( \mathfrak{A}_{\varphi', \beta} \), we define \( \delta(q, A) = \delta'(q, A) \) and for states \( q \) of \( \mathfrak{A}_{\theta_j, \beta} \) we define \( \delta(q, A) = \delta'(q, A) \). The resulting automaton accepts \( w \) if and only if \( w \) has at least one \( r \)-match of degree \( \beta \) that is \( \varphi' \)-satisfying of degree \( \beta \).

Finally, we consider the box operator, which requires the most involved construction due to the case distinction defining the \( b'_i \) and the subsequent maximization to obtain the \( b_i \). First, recall that the semantics of the box operator is not dual to the semantics of the diamond operator. Nevertheless, the dual construction of the one for the diamond operator is useful as a building block. We first present this construction before tackling the construction for the box operator.

In the dual construction, one interprets \( \mathfrak{S}_r \) as a universal automaton whose transitions are ignored if the test on the source of the transition fails. Furthermore, each visit to a final state spawns a copy of the automaton \( \mathfrak{A}_{\varphi', \beta} \), as every \( r \)-match has to be \( \varphi' \)-satisfying. Thus, the states of \( \mathfrak{S}_r \) are now accepting, as all \( r \)-matches have to be considered, and the automata for the tests are dualized in order to check for the failure of the test.

Formally, this approach yields the alternating parity automaton \( (Q, 2^P, q_1, \delta, \Omega) \) where \( Q \) and \( q_1 \) are as above, where

\[
\delta(q, A) = \left\{ \begin{array}{ll}
\land_{q' \in Q} \land_{p \in \Pi(q, q')} (p \lor \vee_{\theta \in \pi(q)} \delta'(q'_i, A)) & \text{for states } q \text{ of } \mathfrak{S}_r, \text{ where } q'_i \text{ and } q'_r \text{ are the initial states of } \mathfrak{A}_{\theta_j, \beta} \text{ and } \mathfrak{A}_{\varphi', \beta}, \text{ respectively, where } \delta(q, A) = \delta'(q, A) \text{ for states } q \text{ of } \mathfrak{A}_{\varphi', \beta} \text{, and where } \delta(q, A) = \delta'(q, A) \text{ for states } q \text{ of } \mathfrak{A}_{\theta_j, \beta}. \end{array} \right.
\]

for states \( q \) of \( \mathfrak{S}_r \), where \( q'_i \) and \( q'_r \) are the initial states of \( \mathfrak{A}_{\theta_j, \beta} \) and \( \mathfrak{A}_{\varphi', \beta} \), respectively, where \( \delta(q, A) = \delta'(q, A) \) for states \( q \) of \( \mathfrak{A}_{\varphi', \beta} \), and \( \delta(q, A) = \delta'(q, A) \) for states \( q \) of \( \mathfrak{A}_{\theta_j, \beta} \). Here, we use the fact that the final states of \( \mathfrak{S}_r \) have no outgoing transition, which implies that no match is missed by contracting an \( \varepsilon \)-path. Finally, states from \( \mathfrak{S}_r \) have color 0, states from \( \mathfrak{A}_{\varphi', \beta} \) keep their color, and the colors from the automata \( \mathfrak{A}_{\theta_j, \beta} \) are incremented by one. Recall that dualizing the transition relation and incrementing the colors of the automata \( \mathfrak{A}_{\theta_j, \beta} \) amounts to complementation. This allows to terminate runs of \( \mathfrak{S}_r \), if a test does not hold true. The resulting automaton accepts a trace if and only if every \( r \)-match of degree \( \beta \) is \( \varphi \)-satisfying of degree \( \beta \).

Now, we fix \( \varphi = [\cdot] \varphi' \). Recall that we have \( V^{\text{run}}(w, \varphi) = b_1 b_2 b_3 b_4 \) with \( b_i = \max\{b'_1, \ldots, b'_i\} \) for some bits \( b'_i \). The maximization is easily implemented using the Boolean closure properties of alternating automata provided we have automata checking that some bit \( b'_i \) is equal to one. Two cases are trivial: Indeed, we have \( b'_i = 1 \) if and only if every \( r \)-match of degree 1111 is \( \varphi \)-satisfying of degree 1111. This property is checked by the dual automaton constructed above. Furthermore, \( b'_i = 1 \) if and only if \( V^{\text{run}}(w, [\cdot] \varphi') \geq 0001 \) or if there is no \( r \)-match of degree 0001. The former language is recognized by \( \mathfrak{A}_{\langle \cdot \rangle \varphi', 0001} \), the latter one by an automaton we construct below. We then combine these two automata to obtain \( \mathfrak{A}_{\varphi', 0001} \).

Hence, it remains to consider \( b'_2 \) and \( b'_3 \), which are both defined by a case distinction over the number of \( r \)-matches of the trace. These case distinctions will be implemented using alternation. To this end, we first show how to test for the three cases, i.e., we argue that the following languages are recognizable by alternating parity automata, where \( i \in \{1, 2, 3, 4\} \):

1. \( L^i_w(r, i) = \{ w \in \{2^P\}_w \mid |\mathcal{R}^i_w(w, r)| = 0 \} \)
2. \( L^i_w(r, i) = \{ w \in \{2^P\}_w \mid 0 < |\mathcal{R}^i_w(w, r)| < \infty \} \)
3. \( L^i_w(r, i) = \{ w \in \{2^P\}_w \mid |\mathcal{R}^i_w(w, r)| = \infty \} \)

Let \( \theta_1, \ldots, \theta_n \) be the tests in \( r \). By induction hypothesis, we have alternating parity automata \( \mathfrak{A}_{\theta_j, \beta} \) for every \( \theta_j \) and every truth value \( \beta \).

The first case is already solved, as we have \( \mathcal{R}^i_w(w, r, \varepsilon) = 0 \) if and only if \( V^{\text{run}}(w, [\cdot] \varphi') \leq 0 \), which is in turn equivalent to \( V^{\text{run}}(w, [\cdot] \varphi') < 0^i \), i.e., the complement of the automaton \( \mathfrak{A}_{\langle \cdot \rangle \varphi', 0001} \) recognizes \( L^i_w(r, i) \).

Next, we construct an automaton for the language \( L^\infty_w(r, i) \). Then, the automaton for \( L^i_w(r, i) \) is obtained as the intersection of the complement automata for the other two languages (for the given \( r \) and \( i \)). Thus, we need to construct an automaton that accepts \( w \) if there are infinitely many \( r \)-matches of degree \( 0^{i-1}1^{5-1} \) or greater.
The construction of an automaton for $L_\infty^r(r,i)$ is more involved than the previous one, as the automaton $\mathfrak{G}_r$ checking for matches with $r$ is non-deterministic. Nevertheless, we show that standard arguments about such automata still yield the desired result. We say that an infinite sequence $q_0q_1q_2 \cdots$ of states is an (infinity) witness for $w$ if $q_0$ is the initial state of $\mathfrak{G}_r$, and if for every $n$, there is an accepting run of $\mathfrak{G}_r$ on some prefix of $w$ such that the tests on the associated $\varepsilon$-paths hold w.r.t. $V_{r,n}^{0,i}$, and such that $q_0 \cdots q_n$ is a prefix of this run. An application of König’s Lemma shows that $R_r^i(w,r)$ is infinite if and only if $\mathfrak{G}_r$ has a witness for $w$.

Thus, the automaton recognizing $L_\infty^r(r,i)$ has to find such a witness for $w$ while processing $w$. This is implemented as follows: we start with $\mathfrak{G}_r$, eliminate $\varepsilon$-transitions non-deterministically as above and spawn a copy of $A_{\theta_j,0 \rightarrow i \cdots \infty}$, when traversing a state with test $\theta_j$. Furthermore, every time a letter is processed, another copy of $\mathfrak{G}_r$ is spawned (with a disjoint set of states). The coloring of the original copy is chosen such that the automaton has to process infinitely many letters and such that the disjoint copies have to reach a final state of $\mathfrak{G}_r$. Hence, the resulting automaton accepts $w$ if and only if there is a witness for $w$. We leave the details to the industrious reader and just note that we have now constructed all automata we need to capture the cases if the case distinction.

Extending the construction just presented also allows us to construct an automaton that accepts a trace $w$ if and only if it has infinitely many $\varphi'$-satisfying $r$-matches (both of degree $0^{i-1}15^{-i}$). To this end, the copies spawned to check for matches do not terminate in an accepting sink, but instead spawn a copy of $A_{\varphi',0^{i-1}15^{-i}}$, to check for satisfaction of $\varphi'$. Similarly, we can construct an automaton that accepts a trace $w$ if and only if it has infinitely many $r$-matches of degree $0^{i-1}15^{-i}$ that are not $\varphi'$-satisfying of degree $0^{i-1}15^{-i}$. Again, we leave the details to the reader.

These automata also allow us to construct an automaton that accepts a trace $w$ if and only if $R_r^i(w,r)$ is infinite and almost all $r$-matches in $R_r^i(w,r)$ are $\varphi'$-satisfying (both of degree $0^{i-1}15^{-i}$). This automaton is obtained by taking the automaton checking for infinitely many $\varphi'$-satisfying $r$-matches (both of degree $\beta$) and intersecting it with the complement of the one checking for infinitely many $r$-matches that are not $\varphi'$-satisfying of degree $0^{i-1}15^{-i}$.

Combining the automata checking the cases of the case distinction with the automata checking for $\varphi'$-satisfiability yields the desired automata for $b_i^r$ and $b_i';$ A case distinction is easily implemented using the Boolean closure properties and all necessary auxiliary automata have been constructed above.

It remains to argue that $A_{\varphi,\beta}$ is of linear size in $|\varphi|$. To this end, we say that an alternating parity automaton $(Q', \Sigma, q'_1, \delta', \Omega')$ is a subautomaton of $(Q, \Sigma, q_1, \delta, \Omega)$ if $Q' \subseteq Q$, $\delta'(q, A) = \delta(q, A)$ for every $q \in Q'$ and every $A \in \Sigma$, and $\Omega'(q) = \Omega(q)$ for every $q \in Q'$.

Inspecting the construction above shows that an automaton $A_{\varphi,\beta}$ is built from automata for immediate subformulas (w.r.t. all truth values if necessary), a test automaton (if applicable), and a constant number of fresh states. Furthermore, if formulas share subformulas, then the construction can also share these subautomata. Hence, we obtain the desired linear upper bound on the size of $A_{\varphi,\beta}$.

It is not straightforward that the equivalent Büchi automata as in Theorem 1 can be constructed efficiently, as the definition of the alternating automaton involves $\varepsilon$-path of arbitrary length. However, these can be restricted to simple paths, which are of bounded length. Furthermore, as it is done for the similar construction for PLDL [13], one can show that the Büchi automata can be constructed on-the-fly in polynomial space. This is sufficient for our applications later on.