REPRESENTATIONS OF QUANTUM ALGEBRAS
AND q-SPECIAL FUNCTIONS

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Abstract

The connection between $q$-analogs of special functions and representations of quantum algebras has been developed recently. It has led to advances in the theory of $q$-special functions that we here review.

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## 1. Introduction

Most special functions of mathematical physics obey second-order differential equations for which Lie algebra theory is well known to provide a natural and useful setting.$^{1-3}$ Many of them have $q$-analogs$^4$ that satisfy $q$-difference equations. The algebraic interpretation of these $q$-special functions was initiated by Miller more than twenty years ago.$^5$ A lot of interest in this subject developed recently when it was realized that $q$-special functions are connected to quantum algebras and quantum groups.$^6,7$ Advances in this direction$^8-12$ will be the subject of the present review. (See also Refs.$^{[13,14]}$.)

We shall present realizations of quantized universal enveloping algebras in terms of $q$-difference operators. In analogy with Lie theory, we shall use the $q$-exponentials of the quantum algebra generators. Various $q$-special functions will be identified as matrix elements of these operators and also as basis vectors of the representation spaces. This approach allows to derive many generating relations and addition formulas that will be given.

In Section 2, we collect elements of $q$-analysis that will be used throughout.

In Section 3, we discuss the two-dimensional Euclidean quantum algebra $E_q(2)$. We establish its relation with $q$-Bessel functions using a realization in one complex variable. A two-variable realization of $E_q(2)$ is also given from which $q$-analogs of the Lommel and Graf addition formulas for the $q$-Bessel functions are obtained.
Section 4 is devoted to \( q \)-oscillator representations. We first present realizations of the oscillator quantum algebra on functions of one complex variable. Irreducible modules on which the spectrum of the number operator \( N \) is unbounded, bounded from below or above are examined. They provide an interpretation of the \( q \)-Laguerre functions and polynomials. An additional one-variable realization (with \( N \) bounded below) is also given, where the \( q \)-Hermite polynomials appear in the basis vectors. Finally, the \( q \)-oscillators are used to construct the metaplectic representation of \( su_q(1,1) \) which gives an algebraic interpretation of a certain \( q \)-generalization of the Gegenbauer polynomials.

In Section 5, we focus on \( sl_q(2) \). We start by considering first-order \( q \)-difference operators in one complex variable and discuss realizations corresponding to representations that are unbounded, bounded from either below or above and finite-dimensional. Basic hypergeometric functions of the type \( \phi_{1}^{2} \) are associated to the infinite-dimensional modules, while little \( q \)-Jacobi polynomials arise in the finite-dimensional case. We conclude by giving another realization of \( sl_q(2) \), this time in terms of second-order \( q \)-difference operators, which allows to obtain different generating relations for the basic hypergeometric function \( \phi_{1}^{2} \).

2. REVIEW OF \( q \)-ANALYSIS

We here collect results and formulas of \( q \)-analysis that will prove useful.\(^4\) For \( a \) and \( \alpha \) arbitrary complex numbers, we denote by \( (a; q)_\alpha \) the \( q \)-shifted factorial:

\[
(a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty},
\]

where

\[
(a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k), \quad |q| < 1.
\]

Note that for \( \alpha \) a positive integer \( n \),

\[
(a; q)_n = (1 - a)(1 - aq)\ldots(1 - aq^{n-1}).
\]

These products satisfy various identities like for instance

\[
q^{\frac{n(n-1)}{2}} (a^{-1}q^{1-n}; q)_n = (-a^{-1})^n (a; q)_n.
\]

The \( q \)-binomial symbol is defined by

\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_q = \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}.
\]
Note that \((q;q)_n/(1-q)^n \to n!\) as \(q \to 1^-\), so that (2.5) reduces to the usual binomial symbol in that limit. Of fundamental importance is Heine’s \(q\)-binomial theorem which states that

\[
\sum_{n=0}^{\infty} \left( \frac{a}{q} \right)_n z^n = \frac{(a z; q)_\infty}{(z; q)_\infty}, \quad |z| < 1, \quad |q| < 1.
\] (2.6)

Two \(q\)-exponential functions are obtained from the above formula. On the one hand, upon setting \(a = 0\), one gets

\[
e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} z^n = \frac{1}{(z; q)_\infty}, \quad |z| < 1,
\] (2.7)

while on the other, upon replacing \(z\) by \(-z/a\) in (2.6), using (2.4) and letting \(a \to \infty\), one finds

\[
E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q; q)_n} z^n = (-z; q)_\infty.
\] (2.8)

It is easy to see that \(e_q(z) E_q(-z) = 1\) and that

\[
\lim_{q \to 1^-} e_q(z(1-q)) = \lim_{q \to 1^-} E_q(z(1-q)) = e^z.
\] (2.9)

Let \(T_q\) be the \(q\)-dilatation operator acting as follows on functions of the complex variable \(z\)

\[
T_q f(z) = f(qz).
\] (2.10)

The \(q\)-difference operators \(D^+_z\) and \(D^-_z\) are given by

\[
D^+_z = z^{-1}(1 - T_q),
\] (2.11a)

\[
D^-_z = z^{-1}(1 - T^{-1}_q).
\] (2.11b)

Observe that \(1/(1-q) D^+_z \to d/dz\) and \(1/(1-q^{-1}) D^-_z \to d/dz\) as \(q \to 1\) and that the \(q\)-exponentials obey

\[
D^+_z e_q(z) = e_q(z),
\] (2.12a)

\[
D^-_z E_q(z) = -q^{-1} E_q(z).
\] (2.12b)

The basic hypergeometric series \(r \phi_s\) is defined by

\[
_r \phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q; z)
= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \left[ (-1)^n q^{n(n-1)/2} \right]^{1+s-r} z^n,
\] (2.13)

with \(q \neq 0\) when \(r > s + 1\). Since \((q^{-m}; q)_n = 0\), for \(n = m + 1, m + 2, \ldots\), the series \(r \phi_s\) terminates if one of the numerator parameters \(\{a_i\}\) is of the form \(q^{-m}\) with \(m = 0, 1, 2, \ldots\), and \(q \neq 0\).
When \( b \to q^{-m} \), with \( m \) a nonnegative integer, the functions \( _0\phi_1(b; q; z) \), \( _1\phi_1(a; b; q; z) \) and \( _2\phi_1(a_1, a_2; b; q; z) \) satisfy the following limit relations:

\[
_0\phi_1(q^{-m}; q; z) (q^{-m}; q)_{m+1} = z^{m+1} _0\phi_1(q^{m+2}; q; z q^{2(m+1)}) \frac{q^{m(m+1)}}{(q; q)_{m+1}}, \quad (2.14)
\]

\[
_1\phi_1(a; q^{-m}; z) (q^{-m}; q)_{m+1} = z^{m+1} _1\phi_1(a q^{m+1}; q^{m+2}; z q^{m+1}) (-1)^{m+1} q^{2m(m+1)} \frac{(a; q)_{m+1}}{(q; q)_{m+1}}, \quad (2.15)
\]

\[
_2\phi_1(a, b, q^{-m}; q; z) (q^{-m}; q)_{m+1} = z^{m+1} _2\phi_1(a q^{m+1}, b q^{m+1}; q^{m+2}; z) \frac{(a; q)_{m+1} (b; q)_{m+1}}{(q; q)_{m+1}}. \quad (2.16)
\]

The last two formulas are the \( q \)-analog's of limit relations that the confluent and ordinary hypergeometric functions, \( _1F_1 \) and \( _2F_1 \), respectively, verify.\(^{15,1}\)

### 3. \( q \)-Bessel Functions and the Two-Dimensional Euclidean Quantum Algebra

#### 3.1 One-variable realization and generating functions

The Bessel functions of the first kind \( J_\nu(z) \) have two \( q \)-analogs:\(^4\)

\[
J^{(1)}_\nu(z; q) = \frac{1}{(q; q)_\nu} \left( \frac{z}{2} \right)^\nu _2\phi_1 \left( 0, 0; q^{\nu+1}; q, -\frac{z^2}{4} \right), \quad (3.1a)
\]

\[
J^{(2)}_\nu(z; q) = \frac{1}{(q; q)_\nu} \left( \frac{z}{2} \right)^\nu _0\phi_1 \left( q^{\nu+1}; q, -\frac{z^2 q^{\nu+1}}{4} \right), \quad (3.1b)
\]

for \( 0 < q < 1 \). The function \( J^{(2)}_\nu(z; q) \) is an entire transcendental function and it is connected to \( J^{(1)}_\nu(z; q) \) through the formula:

\[
J^{(2)}_\nu(z; q) = (-z^2/4; q)_\infty J^{(1)}_\nu(z; q). \quad (3.2)
\]

One can check that in the limit \( q \to 1^- \) both functions (3.1) reduce to the ordinary Bessel functions,

\[
\lim_{q \to 1^-} J^{(k)}_\nu((1-q)z; q) = J_\nu(z), \quad k = 1, 2. \quad (3.3)
\]

The basic Bessel functions satisfy the following recursion relation:

\[
\frac{(1-q^\nu)}{z} J^{(k)}_\nu(z; q) = \frac{1}{2} \left( J^{(k)}_{\nu-1}(z; q) + q^\nu J^{(k)}_{\nu+1}(z; q) \right), \quad k = 1, 2. \quad (3.4)
\]
Acting with the difference operator (2.11a) on $J^{(1)}_{\nu}(z; q)$ one can check that

$$
\left(D_{z}^{+} - \frac{z}{4}\right) J^{(1)}_{\nu}(z; q) = \frac{1}{2} \left( J^{(1)}_{\nu-1}(z; q) - J^{(1)}_{\nu+1}(z; q) \right). \quad (3.5)
$$

An analogous formula holds for $J^{(1)}_{\nu}(z; q)$, where however the difference operator $D_{z}^{+}$ is replaced by $(-z^2/4; q)_{-1} D_{z}^{+} (-z^2/4; q)_{-1}$. The relations (3.4) and (3.5) can be combined to give

$$
\left[ \left(D_{z}^{+} - \frac{z}{4}\right) - \frac{(1-q^\nu)}{z}\right] J^{(1)}_{\nu}(z; q) = \frac{(1+q^\nu)}{2} J^{(1)}_{\nu-1}(z; q), \quad (3.6a)
$$

$$
\left[ -\left(D_{z}^{+} - \frac{z}{4}\right) + \frac{(1-q^\nu)}{z}\right] J^{(1)}_{\nu}(z; q) = \frac{(1+q^\nu)}{2} J^{(1)}_{\nu+1}(z; q). \quad (3.6b)
$$

When the index $\nu$ is an integer $n$, with the help of the limit formula (2.14) and of (3.2) one can check that:

$$
J^{(k)}_{-n}(z; q) = (-1)^{n} J^{(k)}_{n}(z; q), \quad k = 1, 2. \quad (3.7)
$$

The two-dimensional quantum algebra $E_{q}(2)$ has the same defining relations as its classical counterpart:

$$
[J, P_{\pm}] = \pm P_{\pm}, \quad [P_{\pm}, P_{-}] = 0. \quad (3.8)
$$

It is however endowed with a non-trivial Hopf structure by taking the following definitions of coproduct $\Delta$: $E_{q}(2) \to E_{q}(2) \otimes E_{q}(2)$, antipode $S$: $E_{q}(2) \to E_{q}(2)$ and counit $\varepsilon$: $E_{q}(2) \to \mathbb{C}$:

$$
\Delta(J) = J \otimes 1 + 1 \otimes J, \quad \Delta(P_{\pm}) = P_{\pm} \otimes q^{-J/2} + q^{J/2} \otimes P_{\pm},
$$

$$
S(J) = -J, \quad S(P_{\pm}) = -q^{\mp1/2} P_{\pm},
$$

$$
\varepsilon(J) = \varepsilon(P_{\pm}) = 0, \quad \varepsilon(1) = 1. \quad (3.9)
$$

As $q \to 1^{-}$ the trivial Hopf structure of $E(2)$ is recovered.

Take $J$ and $P_{\pm}$ to be the following operators acting on the space $\mathcal{H}$ of all finite linear combinations of the functions $z^n$, $z \in \mathbb{C}$, $n \in \mathbb{Z}$,

$$
J = m_0 + z \frac{d}{dz}, \quad P_{+} = \omega z, \quad P_{-} = \frac{\omega}{z}; \quad (3.10)
$$

$m_0$ and $\omega$ are complex parameters, such that $\omega \neq 0$ and $0 \leq \Re m_0 < 1$. It is easily seen to give a representation of the algebra (3.8), which we shall call $Q(\omega, m_0)$. Define the basis vectors $f_m$ of $\mathcal{H}$ by $f_m(z) = z^n$, where $m = m_0 + n$ and $n \in \mathbb{Z}$. Then,

$$
P_{\pm} f_m = \omega f_{m\pm 1}, \quad J f_m = m f_m. \quad (3.11)$$
In analogy with ordinary Lie theory, we introduce the operator

\[ U(\alpha, \beta, \gamma) = E_q(\alpha(1-q)P_+) E_q(\beta(1-q)P_-) E_q(\gamma(1-q)J) , \]  

(3.12)

which in the realization (3.10) becomes

\[ U(\alpha, \beta, \gamma) = E_q(\alpha \omega(1-q)z) E_q\left(\beta \omega(1-q)\frac{1}{z}\right) E_q\left(\gamma(1-q)\left(m_0 + z \frac{d}{dz}\right)\right) . \]  

(3.13)

We define the matrix elements \( U_{kn}(\alpha, \beta, \gamma) \) through

\[ U(\alpha, \beta, \gamma) f_{m_0+n} = \sum_{k=-\infty}^{\infty} U_{kn}(\alpha, \beta, \gamma) f_{m_0+k} . \]  

(3.14)

Using (3.1b) and (2.14) the matrix elements \( U_{kn}(\alpha, \beta, \gamma) \) are shown to have in general the following expression:

\[ U_{kn}(\alpha, \beta, \gamma) = E_q(\gamma (1-q)(m_0 + n))q^{\frac{1}{2}(k-n)^2} \left(\frac{-\alpha}{\beta}\right)^{(k-n)/2} \times J^{(2)}_{k-n}\left(2 \omega (1-q) \left(-\frac{\alpha \beta}{q}\right)^{1/2} ; q\right) ; \]  

(3.15)

in the particular cases \( \alpha = \gamma = 0 \) and \( \beta = \gamma = 0 \), the non-zero entries take the simpler form:

\[ U_{kn}(0, \beta, 0) = q^{\frac{1}{2}(n-k)(n-k-1)} \frac{(\omega \beta (1-q))^{n-k}}{(q; q)_{n-k}} , \quad k \leq n , \]  

(3.16a)

\[ U_{kn}(\alpha, 0, 0) = q^{\frac{1}{2}(k-n)(k-n-1)} \frac{(\omega \alpha (1-q))^{k-n}}{(q; q)_{k-n}} , \quad k \geq n . \]  

(3.16b)

A generating function for the \( q \)-Bessel functions is now straightforwardly obtained. Setting \( \alpha = 1, \beta = -q, \gamma = 0, n = 0 \) and \( 2 \omega (1-q) = x \), in (3.14) gives

\[ E_q\left(\frac{x z}{2}\right) E_q\left(-\frac{q x}{2z}\right) = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}k(k-1)} J^{(2)}_k(x; q) z^k . \]  

(3.17)

Similarly, using the \( q \)-exponential function \( e_q \) defined in (2.7) instead of \( E_q \), one obtains the following generating function for \( J^{(1)}_k \)

\[ e_q\left(\frac{x z}{2}\right) e_q\left(-\frac{x}{2z}\right) = \sum_{k=-\infty}^{\infty} J^{(1)}_k(x; q) z^k . \]  

(3.18)
These are $q$-analogs of the generating relation for the ordinary Bessel functions $J_k(x)$,

$$e^{x(z-z^{-1})/2} = \sum_{k=-\infty}^{\infty} J_k(x) z^k,$$

(3.19)

to which they reduce in the limit $q \to 1^-$, when $x$ is replaced by $(1-q)x$.

3.2 Two-variable realization and addition formulas

We will now discuss realizations of $E_q(2)$ on a space of functions of two complex variables, $x$ and $y$. We shall look for a realization of $P_\pm$ and $J$ in terms of $q$-difference operators acting on the space generated by basis vectors of the form $f_m(x,y) = y^m F_m(x)$, $m = m_0 + n$, $n \in \mathbb{Z}$, such that (3.8) are satisfied. The constant $\omega$ is nonessential, since it can be changed by rescaling the generators $P_+$ and $P_-; for simplicity, we shall henceforth set $\omega = 1$. With the help of the relations (3.6), it is easy to check that such a realization is provided by the operators

$$P_+ = 2y \left[ -(D_x^+ - \frac{x}{4}) + \frac{y}{x} D_y^+ \right] (1 + T_y)^{-1},$$

(3.20a)

$$P_- = 2y^{-1} \left[ (D_x^+ - \frac{x}{4}) T_y + \frac{y}{x} D_y^+ \right] (1 + T_y)^{-1},$$

(3.20b)

$$J = y \frac{\partial}{\partial y},$$

(3.20c)

and the basis functions

$$f_m(x,y) = y^m J_m^{(1)}(x; q),$$

(3.21)

with $m = m_0 + n$, $n \in \mathbb{Z}$. Note that the equation $P_+ P_- f_m = \omega^2 f_m$ is now a second-order difference equation for $J_m^{(1)}(x; q)$:

$$\left\{ \left[ -(D_x^+ - \frac{x}{4}) + \frac{(1-q^{m-1})}{x} \right] \left[ (D_x^+ - \frac{x}{4}) q^m + \frac{(1-q^m)}{x} \right] \right.$$  

$$- \frac{(1+q^m)(1+q^{m-1})}{4} \right\} J_m^{(1)}(x; q) = 0.$$  

(3.22)

This is the $q$-analog of the Bessel differential equation, to which it reduces in the limit $q \to 1^-$, provided $x$ is replaced by $(1-q)x$.

Since we have a realization of the representation $Q(1, m_0)$, one gets from (3.14), with $m = m_0 + n$,

$$U(\alpha, \beta, \gamma) f_m(x,y) = \sum_{k=-\infty}^{\infty} U_{km-m_0}(\alpha, \beta, \gamma) f_{m_0+k}(x, y),$$

(3.23)
where the model independent matrix elements $U_{kn}(\alpha, \beta, \gamma)$ are still given by the formulas (3.15) or (3.16), with $\omega = 1$. To get addition formulas for $q$-Bessel functions, one now needs to evaluate explicitly the l.h.s. of (3.23), i.e. to compute directly the action of $U(\alpha, \beta, \gamma)$ on the basis functions (3.21), when $P_{\pm}$ and $J$ are realized as in (3.20). This can be done thanks to the $q$-binomial theorem (2.6) and various identities between $q$-shifted factorials. In the end, one arrives at the following summation formulas:\(^{12}\)

\[
\left(\frac{x}{2}\right)^m \frac{(-2\beta/x; q)_m}{(q; q)_m} 2 \phi_1 \left(0, -\frac{2\beta}{x} q^m, q^{m+1}, q, -\frac{x^2}{4}\right) = \sum_{l=0}^{\infty} q^{l(l-1)/2} \frac{\beta^l}{(q; q)_l} J^{(1)}_{m-l}(x; q),
\]

when $\alpha = \gamma = 0$,

\[
\left(\frac{x}{2}\right)^m \frac{1}{(q; q)_m} 2 \phi_1 \left(\frac{2\alpha}{x}, 0; q^{m+1}, q, -\frac{x^2}{4}\right) = \sum_{l=0}^{\infty} q^{l(l-1)/2} \frac{\alpha^l}{(q; q)_l} J^{(1)}_{m+l}(x; q), \tag{3.25}
\]

when $\beta = \gamma = 0$, and for $\alpha \neq 0$ and $\beta = -q\alpha$

\[
\left(\frac{x}{2}\right)^m \frac{zq/xy; q)_m}{(q; q)_m} 2 \phi_1 \left(y, z, q^m, q^{m+1}, q, -\frac{x^2}{4}\right) = \sum_{l=0}^{\infty} q^{l(l-1)/2} y^l J^{(2)}_l(z; q) J^{(1)}_{m+l}(x; q), \tag{3.26}
\]

where we have set $2\alpha = z$.

The formulas (3.24) and (3.25) are the $q$-generalizations of the Lommel summation theorems for ordinary Bessel functions\(^{1,17}\)

\[
\left(1 + \frac{2\beta}{x}\right)^{m/2} J_m \left(x \left(1 + \frac{2\beta}{x}\right)^{1/2}\right) = \sum_{l=0}^{\infty} \frac{\beta^l}{l!} J_{m-l}(x), \quad \left|\frac{2\beta}{x}\right| < 1, \tag{3.27a}
\]

\[
\left(1 - \frac{2\alpha}{x}\right)^{-m/2} J_m \left(x \left(1 - \frac{2\alpha}{x}\right)^{1/2}\right) = \sum_{l=0}^{\infty} \frac{\alpha^l}{l!} J_{m+l}(x), \quad \left|\frac{2\alpha}{x}\right| < 1, \tag{3.27b}
\]

while (3.26) is a $q$-analog of the Graf summation theorem,\(^{1,17}\)

\[
\left(1 - \frac{z}{xy}\right)^{m/2} \left(1 - \frac{yz}{x}\right)^{-m/2} J_m \left(x \left(1 - \frac{z}{xy}\right)^{1/2} \left(1 - \frac{yz}{x}\right)^{1/2}\right) = \sum_{l=-\infty}^{\infty} y^l J_l(z) J_{m+l}(x), \quad \left|\frac{z}{xy}\right| < 1, \left|\frac{yz}{x}\right| < 1. \tag{3.28}
\]
4. q-OSCILLATORS AND BASIC SPECIAL FUNCTIONS

4.1 One-variable model and q-Laguerre functions and polynomials

The $q$-oscillator algebra is generated by three elements $A$, $A^\dagger$ and $N$ satisfying the defining relations

\[ [N, A] = -A \quad [N, A^\dagger] = A^\dagger, \]
\[ AA^\dagger - q A^\dagger A = 1. \]  \hspace{1cm} (4.1a, 4.1b)

By introducing the redefined generators

\[ a = q^{-\frac{N}{2}} A \quad a^\dagger = q^{\frac{1-N}{2}} A^\dagger, \] \hspace{1cm} (4.2)

the algebra (4.1) becomes

\[ [N, a] = -a \quad [N, a^\dagger] = a^\dagger \]
\[ a a^\dagger - q^{\frac{1}{2}} a^\dagger a = q^{-\frac{N}{2}} \quad a a^\dagger - q^{-\frac{1}{2}} a^\dagger a = q^{\frac{N}{2}}. \] \hspace{1cm} (4.3)

This is the form in which the defining relations of the $q$-oscillator algebra are more often presented.\textsuperscript{18–20}

In the limit $q \to 1^-$, (4.1) (and (4.3)) reduce to the canonical commutation relations of the harmonic oscillator annihilation, creation and number operators. This algebra is known to have representations in which the number operator is unbounded, or bounded from either below or above.\textsuperscript{1} In this Section we shall construct $q$-deformations of these representations;\textsuperscript{11} they will be denoted $R_q(\omega, m_0)$, $R_q^\uparrow(\omega)$ and $R_q^\downarrow(\omega)$.

We shall start with the representation $R_q(\omega, m_0)$ and take the following realization of the generators $A$, $A^\dagger$ and $N$, in the space $\mathcal{H}$ of all finite linear combinations of the monomials $z^n$, $z \in \mathbb{C}$, $n \in \mathbb{Z}$,

\[ A = \frac{1}{1-q} D^+_z + \frac{\omega + m_0}{z} T_q, \quad A^\dagger = z, \quad N = m_0 + z \frac{d}{dz}; \] \hspace{1cm} (4.4)

$\omega$ and $m_0$ are complex constants, such that $0 \leq \Re m_0 < 1$, and $\omega + m_0$ is not an integer. On the space $\mathcal{H}$ we shall again take as basis vectors the functions $f_m = z^n$, with $m = m_0 + n$ for all $n \in \mathbb{Z}$. For convenience, we set

\[ m_0 + \omega = \frac{1-q^\rho}{1-q}; \]

the action of the generators on the basis vectors is then easily obtained,

\[ Af_m = \frac{1-q^{m-m_0+\rho}}{1-q} f_{m-1}, \]
\[ A^\dagger f_m = f_{m+1}, \]
\[ N f_m = m f_m. \] \hspace{1cm} (4.5)
As in the previous section we introduce the operator

\[ U(\alpha, \beta, \gamma) = E_q \left( (1 - q)A^\dagger \right) E_q \left( (1 - q)A \right) E_q \left( (1 - q)N \right) . \]  

The matrix elements \( U_{kn}(\alpha, \beta, \gamma) \) of the operator \((4.6)\) are defined as in \((3.14)\). In the representation \( R_q(\omega, m_0) \) they are found to take the following form

\[ U_{kn}(\alpha, \beta, \gamma) = E_q \left( \gamma(1 - q)(m_0 + n) \right) q^{\frac{1}{2}n-k} \beta^{n-k} L_{\rho+k}^{(n-k)} \left( -\frac{\alpha \beta}{q} ; q \right) . \]  

where \( L_{\nu}^{(\lambda)}(x; q) \) stands for the \( q \)-Laguerre functions

\[ L_{\nu}^{(\lambda)}(x; q) = \frac{(q^{\lambda+1}; q)_\nu}{(q; q)_\nu} \left( q^{-\nu}; q^{\lambda+1}; q^{-1} - (1 - q) q^{\lambda+\nu+1} x \right) . \]  

A generating relation for these functions can now be obtained. Direct evaluation shows that

\[ U(\alpha, \beta, 0) z^n = E_q \left( \alpha (1 - q) z \right) z^n \left( -\frac{\beta}{z} ; q \right)_{\rho+n} . \]  

Inserting \((4.9)\) and \((4.7)\) in \((3.14)\), and taking \( n = 0, \beta = -q, \gamma = 0 \) and \( t = -1/z \), one obtains

\[ E_q \left( -(1 - q) \alpha/t \right) (-qt; q)_\rho = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}k(k+1)} t^{k} L_{\rho-k}^{(k)}(\alpha; q) . \]  

This is the \( q \)-analog of the relation^{15,1}

\[ e^{-\alpha/t} (1 + t)^\rho = \sum_{k=-\infty}^{\infty} t^{k} L_{\rho-k}^{(k)}(\alpha) , \]

for the ordinary Laguerre functions, to which it reduces in the limit \( q \to 1^- \).

The representation \( R_q^+(\omega) \) can be realized on the space \( \mathcal{H}^{(+)} \) of all finite linear combinations of the functions \( z^n; z \in C, n \in Z^+ \), by taking

\[ A = z \ , \quad A^\dagger = \frac{1}{1-q} D_z^- \ , \quad N = -\omega - z \frac{d}{dz} , \]

with \( \omega \) a complex parameter. As basis vectors in \( \mathcal{H}^{(+)} \) we now choose the functions \( f_m = z^n \), with \( m = -\omega - n, n \geq 0 \). The matrix elements of the operator \( U(\alpha, \beta, \gamma) \), defined now by

\[ U(\alpha, \beta, \gamma) z^n = \sum_{k=0}^{\infty} U_{kn}(\alpha, \beta, \gamma) z^k , \]

are again given in terms of the \( q \)-Laguerre functions. Note that the \( q \)-Laguerre polynomials \( L_{n}^{(\lambda)}(x; q) \), with \( n \) integer, do not occur as matrix elements of \( R_q(\omega, m_0) \) nor \( R_q^+(\omega) \). They arise instead in the matrix elements of the representation \( R_q^+(\omega) \), to which we now come.
The representation $R^q\uparrow_q(\omega)$ can be formally obtained from the representation $R_q(\omega, m_0)$ by letting $m_0 = -\omega$ (i.e. $\rho = 0$). The generators of the $q$-oscillator algebra are now realized on the space $H^{(+)}$ as,

$$A = \frac{1}{1-q} D^+_z , \quad A^\dagger = z , \quad N = -\omega + z \frac{d}{dz} . \quad (4.14)$$

The basis vectors $f_m$ in $H^{(+)}$ are defined by $f_m = z^n$, with $m = -\omega + n$. The action of the generators (4.14) on them is as in (4.5), with $\rho - m_0$ replaced by $\omega$.

The matrix elements of the operator $U(\alpha, \beta, \gamma)$, defined as in (4.13), are found to be expressible in terms of the $q$-Laguerre polynomials,

$$L_k^{(\lambda)}(x; q) = \frac{(q^{\lambda+1}; q)_k}{(q; q)_k} \sum_{l=0}^{k} \frac{(q^{-k}; q)_l q^{(\lambda+l-1)l} (1-q)^l (q^{k+\lambda+1}x)^l}{(q^{\lambda+1}; q)_l (q; q)_l} , \quad (4.15)$$

as follows

$$U_{kn}(\alpha, \beta, \gamma) = E_q\left(\gamma(1-q)(n-\omega)\right) q^\frac{n}{2} (n-k)(n-k-1) \beta^{n-k} L_k^{(n-k)}\left(-\frac{\alpha\beta}{q}\right). \quad (4.16)$$

Inserting this expression back into (4.13) and using (4.9) with $\rho = 0$, one gets the following generating function for the $q$-Laguerre polynomials:

$$E_q\left(-\alpha(1-q)z\right) \left(-\frac{q}{z}; q\right)_n z^n = \sum_{k=0}^{\infty} q^\frac{n}{2} (n-k)(n-k+1) L_k^{(n-k)}(\alpha; q) z^k . \quad (4.17)$$

This is the $q$-analog of the relation

$$e^{-\alpha z} (1 + z)^n = \sum_{k=0}^{\infty} L_k^{(n-k)}(\alpha) z^k , \quad (4.18)$$

for ordinary Laguerre polynomials, to which (4.17) reduces in the limit $q \to 1^-$.

### 4.2 q-Oscillators and q-Hermite polynomials

There is another realization of the representation $R^q\uparrow_q(0)$ on functions of the complex variable $w$ which allows to relate the $q$-oscillator algebra to $q$-Hermite polynomials. It is defined by taking

$$A = \frac{1}{1-q} \frac{1}{w} \left(1 - \sqrt{w} T_w\right) , \quad A^\dagger = w \left(1 - \sqrt{\frac{q}{w} T_w}\right) , \quad (4.19a)$$

$$N = \frac{\ln[1 - (1-q)A^\dagger A]}{\ln q} . \quad (4.19b)$$
The basis functions \( f_n(w) \) are here expressed in terms of the \( q \)-Hermite polynomials\(^{22-24}\)

\[
H_n(w; q) = (w + T_w)^n \cdot 1 = \sum_{k=0}^{n} \binom{n}{k}_q w^k ,
\]

as follows:

\[
f_n(w) = (−\sqrt{q})^n f_0(w) \ H_n\left(-\frac{w}{\sqrt{q}}; q\right) ,
\]

with

\[
f_0(w) = \left[ \sum_{k=-\infty}^{\infty} q^{k(k-1)/2} w^k \right]^\frac{1}{2} .
\]

The transformation properties of the basis vectors under \( U(\alpha, \beta, \gamma) \) are model independent and now provide a relation between the \( q \)-Laguerre and the \( q \)-Hermite polynomials. After direct evaluation of the l.h.s. of (4.13) in the realization (4.19) and some simplifications, one obtains\(^8\)

\[
(-w)^n E_q\left(\left(1 - q\right)\alpha (w + T_w)\right) \cdot 1
\]

\[
\equiv (-w)^n \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \left(\frac{1 - q}{q; q}\right)_k \alpha^k H_k(w; q)
\]

\[
= \sum_{k=0}^{\infty} (-1)^k q^\frac{1}{2} k(k-1)-n(n+k) \ L_k^{(n-k)}(q^{(n+1)}; q) H_k(q^n w; q) .
\]

It should be noted that

\[
\lim_{q \to 1^-} H_n(w; q) = (w + 1)^n .
\]

Substituting \( z \) for \( -(w + 1) \), it easily seen that (4.23) goes into (4.18) in the limit \( q \to 1^- \).

### 4.3 The metaplectic representation of \( su_q(1,1) \) and the \( q \)-Gegenbauer polynomials

The metaplectic representation of \( su_q(1,1) \) is defined from the representation (4.14) of the \( q \)-oscillator algebra by taking the generators \( K_\pm \) and \( K_0 \) of \( su_q(1,1) \) to be given by\(^9\)

\[
K_+ = \frac{1}{[2]^q^{1/2}} (A^\dagger)^2 \quad K_- = \frac{1}{[2]^q^{1/2}} A^2 \quad K_0 = \frac{1}{2} (N + \frac{1}{2}) .
\]

The defining relations

\[
K_- K_+ - q^2 K_+ K_- = q^{2K_0} [2K_0]_q
\]

\[
[K_0, K_\pm] = \pm K_\pm ,
\]

(4.26)
are thus realized. The following redefinition of the generators $\tilde{J}_\pm = \pm q^{-K_0+\frac{x}{2} \pm \frac{1}{2}} K_\pm$, $\tilde{J}_3 = K_0$ casts the defining relations in the standard form\(^6,7\)

$$
[\tilde{J}_+, \tilde{J}_-] = \frac{q^{2\tilde{J}_3} - q^{-2\tilde{J}_3}}{q - q^{-1}}
$$

$$
[\tilde{J}_3, \tilde{J}_\pm] = \pm \tilde{J}_\pm .
$$

This representation decomposes into two irreducible components, with the invariant subspaces $\mathcal{H}^{(e)}$ and $\mathcal{H}^{(o)}$ formed out of $N$-eigenstates with even and odd eigenvalues, respectively.

Let

$$
U(\alpha, \beta, \gamma) = E_{q^2} (\alpha(1-q^2)[2]_{q^{1/2}K_+}) E_{q^2} (\beta(1-q^2)[2]_{q^{1/2}K_-}) E_{q^2} (\gamma(1-q^2)2K_0) ;
$$

its matrix elements in the spaces $\mathcal{H}^{(e)}$ and $\mathcal{H}^{(o)}$ are defined by

$$
U(\alpha, \beta, \gamma) z^{2n} = \sum_{k=0}^{\infty} U_{kn}^{(e)} (\alpha, \beta, \gamma) z^{2k} \tag{4.29a}
$$

$$
U(\alpha, \beta, \gamma) z^{2n+1} = \sum_{k=0}^{\infty} U_{kn}^{(o)} (\alpha, \beta, \gamma) z^{2k+1} . \tag{4.29b}
$$

They are evaluated to be

$$
U_{kn}^{(e)} (\alpha, \beta, \gamma) = E_{q^2} \left( \gamma(1-q^2)(2n + \frac{1}{2}) \right) q^{(k+n)^2+k-n} \frac{(q; q)_{2n}}{(q; q^2)_{n+k}} \alpha^k \beta^n
\times (1 + q)^{k+n} (1 - q)^{k-n} C_{2k}^{(-n-k)} \left( q; q \right) (1 + q) \sqrt{\alpha \beta} , \tag{4.30}
$$

$$
U_{kn}^{(o)} (\alpha, \beta, \gamma) = -E_{q^2} \left( \gamma(1-q^2)(2n + \frac{3}{2}) \right) q^{(k+n+1)^2+k-n} \frac{(q; q)_{2n+1}}{(q; q^2)_{n+k+1}} \alpha^k \beta^n
\times \sqrt{\alpha \beta} (1 + q)^{k+n+1} (1 - q)^{k-n} C_{2k+1}^{(-n-k-1)} \left( q; q \right) (1 + q) \sqrt{\alpha \beta} , \tag{4.31}
$$

where $C_k^{(\lambda)}$ are the following $q$-generalizations of the Gegenbauer polynomials

$$
C_k^{(\lambda)} (q; z) \equiv C_k (q; q^{2\lambda}; z) = \sum_{l=0}^{[k/2]} (-1)^l q^{l(l-1)} (q^{2\lambda}; q^2)_{k-l} z^{k-2l} , \tag{4.32}
$$

with $[x]$ the integer part of $x$. They obey the three-term recurrence relation

$$
(1-q^{k+1}) C_{k+1}^{(\lambda)} (q; z) - (1-q^{2(\lambda+k)}) z C_k^{(\lambda)} (q; z) + q^{k-1} (1-q^{2\lambda+k-1}) C_{k-1}^{(\lambda)} (q; z) = 0 , \tag{4.33}
$$

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and thus form an orthogonal set. Introducing the discrete $q$-Hermite polynomials

$$h_k(q; z) = (q; q)_k C_k(q; 0; z)$$

$$= \sum_{l=0}^{[k/2]} \frac{(-1)^l q^{l(l-1)} (q; q)_k}{(q^2; q^2)_l (q; q)_{k-2l}} z^{k-2l},$$

(4.34)

and proceeding as before, one gets the following generating functions for the $q$-Gegenbauer polynomials:

$$E_q^2 \left( -\frac{w^2}{q^2} \right) h_{2n} \left( q; \frac{w x}{q^2} \right)$$

$$= \sum_{k=0}^{\infty} (-1)^{n+k} q^{(n+k)(n+k-1)} \frac{(q; q)_{2n}}{(q^2; q^2)_{n+k}} C_{2k}^{(-n-k)}(q; x) w^{2k},$$

(4.35a)

$$E_q^2 \left( -\frac{w^2}{q^2} \right) h_{2n+1} \left( q; \frac{w x}{q^2} \right)$$

$$= \sum_{k=0}^{\infty} (-1)^{n+k+1} q^{(n+k+1)(n+k)} \frac{(q; q)_{2n+1}}{(q^2; q^2)_{n+k+1}} C_{2k+1}^{(-n-k-1)}(q; x) w^{2k+1}.$$  

(4.35b)

In the limit $q \to 1^-$, $C_k^{(\lambda)}(q; z)$ become the ordinary Gegenbauer polynomials $C_k^{(\lambda)}(z)$ and $(1 + q)^{k/2} (1 - q)^{-k/2} h_k(q; \sqrt{1 - q^2} z)$ the usual Hermite polynomials $H_k(z)$; in this limit the equations (4.35) tend to

$$e^{-w^2} H_{2n}(x) = \sum_{k=0}^{\infty} (-1)^{n+k} \frac{(2n)!}{(n+k)!} C_{2k}^{(-n-k)}(x) w^{2k},$$

(4.36a)

$$e^{-w^2} H_{2n+1}(x) = \sum_{k=0}^{\infty} (-1)^{n+k+1} \frac{(2n+1)!}{(n+k+1)!} C_{2k+1}^{(-n-k-1)}(x) w^{2k+1}.$$  

(4.36b)

These relations express the usual Hermite polynomials in terms of the usual Gegenbauer polynomials.
5. THE QUANTUM ALGEBRA $sl_q(2)$ AND q-HYPERGEOMETRIC FUNCTIONS

5.1 Realizations in terms of first-order q-difference operators

The quantum algebra $sl_q(2)$ is generated by three elements $J_+, J_-$ and $J_3$ satisfying the defining relations

\[
J_+ J_- - q^{-1} J_- J_+ = \frac{1 - q^2 J_3}{1 - q},
\]

\[
[J_3, J_\pm] = \pm J_\pm.
\]

(5.1)

In the limit $q \to 1^-$, the relations (5.1) reduce to the Lie brackets of the ordinary algebra $sl(2)$. If we redefine the generators according to $\tilde{J}_\pm = q^{-\frac{1}{2}(J_3 + \frac{1}{2})} J_\pm, \tilde{J}_3 = J_3$, the algebra (5.1) takes the form given in (4.27) with $q$ replaced by $q^{1/2}$.

The ordinary algebra $sl(2)$ is known to have representations that are unbounded, bounded from either below or above and finite dimensional. In the following we shall consider the $q$-analogs of these representations, and call them $D_q(u, m_0), D_q^\uparrow(j), D_q^\downarrow(j)$ and $D_q(2j)$, respectively.

The representation $D_q(u, m_0)$ is characterized by two complex constants $u$ and $m_0$ such that neither $m_0 + u$ nor $m_0 - u$ is an integer, and $0 \leq \Re m_0 < 1$. On the space $\mathcal{H}$ of all finite linear combinations of the functions $z^n, z \in \mathbb{C}, n \in \mathbb{Z}$, the generators $J_\pm$ and $J_3$ are realized as (see also Ref.[25]),

\[
J_+ = q^{\frac{1}{2}(m_0 - u + 1)} \left[ \frac{z^2}{1 - q} D_z^+ - \frac{1 - q^{u-m_0}}{1 - q} z \right],
\]

\[
J_- = -q^{\frac{1}{2}(m_0 - u + 1)} \left[ \frac{1}{1 - q} D_z^+ + \frac{1 - q^{u+m_0}}{1 - q} \frac{1}{z} T_q \right],
\]

\[
J_3 = m_0 + z \frac{d}{dz}.
\]

(5.2a)

(5.2b)

(5.2c)

The basis vectors $f_m$ in $\mathcal{H}$ are still defined by $f_m = z^n$ for $m = m_0 + n$ and all integers $n$. Thus,

\[
J_+ f_m = q^{\frac{1}{2}(u-m_0+1)} \frac{1 - q^{m-u}}{1 - q} f_{m+1},
\]

\[
J_- f_m = -q^{\frac{1}{2}(m_0-u+1)} \frac{1 - q^{m+u}}{1 - q} f_{m-1},
\]

\[
J_3 f_m = m f_m.
\]

(5.3)

As in the previous sections, we introduce the operator

\[
U(\alpha, \beta, \gamma) = E_q (\alpha(1-q)J_+) E_q (\beta(1-q)J_-) E_q (\gamma(1-q)J_3),
\]

acting on the space $\mathcal{H}$ through the realization (5.2), and define its matrix elements again as in (3.14). The elements $U_{kn}(\alpha, \beta, \gamma)$ are expressed in terms of the basic hypergeometric
function $2\phi_1$:

\[
U_{kn}(\alpha, \beta, \gamma) = E_q \left( \gamma (1 - q) \left( n + \frac{1}{2} (s - t) \right) \right) \left( q^{s+n+\frac{1}{2}(1-t)} \beta \right)^{n-k} \\
\times \frac{(q^{-s-n}; q)_{n-k}}{(q; q)_{n-k}} 2\phi_1 \left( q^{-s-k}, q^{t-k+1}, q^{n-k+1}; q; -q^{s-t+n+k} \alpha \beta \right), \quad \text{if } k \leq n ,
\]

\[
(5.5a)
\]

\[
U_{kn}(\alpha, \beta, \gamma) = E_q \left( \gamma (1 - q) \left( n + \frac{1}{2} (s - t) \right) \right) \left( -q^{k-1+\frac{1}{2}(1-t)} \alpha \right)^{k-n} \\
\times \frac{(q^{t-k+1}; q)_{k-n}}{(q; q)_{k-n}} 2\phi_1 \left( q^{-s-n}, q^{t-n+1}, q^{k-n+1}; q; -q^{s-t+n+k} \alpha \beta \right), \quad \text{if } k \geq n ,
\]

\[
(5.5b)
\]

where $s = u + m_0$ and $t = u - m_0$. Note that neither $s$ nor $t$ is an integer. Thanks to the limiting relation (2.16), either one of the two expressions for $U_{kn}(\alpha, \beta, \gamma)$ remain valid over the whole range of the indices. Using these results the following identity can be obtained from (3.14)

\[
\sum_{r=0}^{\infty} \frac{(q^{-s}; q)_r}{(q; q)_r} \beta^r \phi_1 \left( q^{r-t}; -; q; q^{t+1} \frac{x}{\beta} \right) = \sum_{k=-\infty}^{\infty} \frac{(q^{-s}; q)_k}{(q; q)_k} \beta^k \phi_1 \left( q^{k-s}, q^{t+k+1}; q^{k+1}; q; q^{-k} x \right),
\]

\[
(5.6)
\]

which can be considered as a generating function for $2\phi_1$.

The representation $D_q^\dagger(j)$ can be obtained from the one just discussed by setting $m_0 = -u \equiv -j$, with $2j$ not a non-negative integer. The generators $J_+$ and $J_3$ act now on the space $\mathcal{H}^{(+)}$ of all finite linear combinations of the functions $z^n$, $z \in \mathbb{C}$, $n \in \mathbb{Z}^+$. The analysis of this case leads to the following generating function for the terminating series $2\phi_1$ as a finite combination of functions $1\phi_1$

\[
\sum_{m=0}^{n} (-1)^m q^{\frac{1}{2}(m(m+1)-n(n+1))} \left[ \frac{n}{m} \right]_q (-z)^l \phi_1 \left( q^{m-2j}, -; q; q^{2j+1} x z \right) = \sum_{k=0}^{\infty} \frac{(q^{-n}; q)^{n-k}}{(q; q)_{n-k}} z^k \phi_1 \left( q^{-k}, q^{2j-k+1}; q^{n-k+1}; q; q^k x \right).
\]

\[
(5.7)
\]

The representation $D_q^\dagger(j)$ of $sl_2(2)$ can be formally obtained from $D_q^\dagger(j)$ by letting $J_+ \rightarrow q^{-(J_3+1/2)} J_+$, $J_- \rightarrow q^{-(J_3-1/2)} J_+$ and $J_3 \rightarrow -J_3$.

We now come to the finite-dimensional representation $D_q(2j)$, which is defined for $2j$ a nonnegative integer. The generators $J_+$ and $J_3$ are realized as in $D_q^\dagger(j)$, but they now act on the finite-dimensional space $\mathcal{H}^{(j)}$ of all linear combinations of the functions $z^n$, $z \in \mathbb{C}$, $n = 0, 1, \ldots, 2j$. For the basis vectors $f_m$, we take $f_m = z^n$, with $m = -j + n$, $0 \leq n \leq 2j.$
The matrix elements of the operator $U(\alpha, \beta, \gamma)$ are now defined by

$$U(\alpha, \beta, \gamma) \ z^n = \sum_{k=0}^{2j} U_{kn}(\alpha, \beta, \gamma) \ z^k , \quad (5.8)$$

and can be expressed in terms of little $q$-Jacobi polynomials,$^4$

$$p_n(x; a; b; q) = 2 \phi_1\left(q^{-n}, abq^{n+1}; aq; qx\right). \quad (5.9)$$

Indeed, one has

$$U_{kn}(\alpha, \beta, \gamma) = E_q(\gamma (1 - q)(n - j)) \left(\frac{q^{-n}; q}{q; q} n - k\right) \left(\frac{q^{-n+1/2}; q}{q; q} \right)^{n-k} \times p_k\left(-q^{n+k-2j-1} \alpha \beta; q^{n-k}; q^{2j-k-n}; q\right). \quad (5.10)$$

A straightforward computation shows that the action of $U(\alpha, \beta, 0)$ on $z^n$ can also be expressed in terms of the Stieljes-Wiegert polynomials,$^{26,27}$

$$s_n(x; q) = (-1)^n q^{(2n+1)/4} (q; q)_n^{-1/2} \sum_{l=0}^{n} \left[\begin{array}{c} n \\ l \end{array}\right]_q q^{l^2} (-q^{1/2} x)^l. \quad (5.11)$$

One finds,

$$U(\alpha, \beta, 0) \ z^n = \sum_{m=0}^{n} q^{\frac{1}{2}(n-n)(n-m-1)+2j+m+1/2} \left[\begin{array}{c} n \\ m \end{array}\right]_q \left((q; q)_{2j-m}\right)^{1/2} \left(-q^{n-m+1/2} \beta\right)^{n-m} (-z)^m s_{2j-m}(q^{m-j-1} \alpha z; q). \quad (5.12)$$

Inserting (5.12) and (5.10) in (5.8), after some redefinitions one gets

$$\sum_{m=0}^{n} q^{-\frac{1}{2}(n-n)(n-m-1)+2j+m+1/2} \left[\begin{array}{c} n \\ m \end{array}\right]_q \left((q; q)_{2j-m}\right)^{1/2} \left(-q^{n-m+1/2} \beta\right)^{n-m} (-z)^m s_{2j-m}(q^{m-j-1} \alpha z; q)$$

$$= \sum_{k=0}^{2j} q^{\frac{1}{2}(k(k-1)-n(n-1))} \left[\begin{array}{c} n \\ k \end{array}\right]_q z^k p_k\left(q^kw; q^{n-k}; q^{2j-k-n}; q\right). \quad (5.13)$$

This generating formula can also be equivalently rewritten in terms of the Wiegert-Szegő polynomials,$^{22-24}$

$$G_n(x; q) = \sum_{l=0}^{n} \left[\begin{array}{c} n \\ l \end{array}\right]_q q^{l(n-n)} x^l. \quad (5.14)$$
One easily proves that
\[
\sum_{m=0}^{n} q^{-\frac{1}{2}(n-m)(n+m-1)} \binom{n}{m}_q z^m G_{2j-m}(-q^{2j} wz) = \sum_{k=0}^{2j} q^{\frac{1}{2}(k(k-1)-n(n-1))} \binom{n}{k}_q z^k p_k \left( q^k w; q^{-n-k}; q^{2j-k-n}; q \right). \tag{5.15}
\]

5.2 A realization in terms of second-order q-difference operators

We can also represent the generators of the quantum algebra \( \text{sl}_q(2) \) on the space \( \mathcal{H} \) using quadratic expressions in q-difference operators. Here it is more convenient to use generators \( K_\pm \) and \( K_0 \) subjected to the defining relations (4.26). One takes\(^{10}\)

\[
K_- = \frac{1}{[2]_q^{1/2}} \left[ \frac{1}{(1-q^2)^2} D_z^2 + \frac{\omega}{z^2} T_q \right], \tag{5.16a}
\]
\[
K_+ = \frac{1}{[2]_q^{1/2}} z^2, \tag{5.16b}
\]
\[
K_0 = \frac{1}{2} \left( z \frac{d}{dz} + \frac{1}{2} \right), \tag{5.16c}
\]
where \( \omega \) is a complex parameter. This representation decomposes into two irreducible components, with the invariant subspaces \( \mathcal{H}^{(e)} \) and \( \mathcal{H}^{(o)} \) respectively spanned by the even and odd powers of \( z \).

The operator \( U(\alpha, \beta, \gamma) \) is as in (4.28) and its matrix elements in the spaces \( \mathcal{H}^{(e)} \) and \( \mathcal{H}^{(o)} \) are defined by

\[
U(\alpha, \beta, \gamma) z^{2n} = \sum_{k=-\infty}^{\infty} U^{(e)}_{kn}(\alpha, \beta, \gamma) z^{2k}, \tag{5.17a}
\]
\[
U(\alpha, \beta, \gamma) z^{2n+1} = \sum_{k=-\infty}^{\infty} U^{(o)}_{kn}(\alpha, \beta, \gamma) z^{2k+1}. \tag{5.17b}
\]

With
\[
\omega = \frac{1 + q^{-1} - q^\rho - q^{-(\rho+1)}}{1 - q^2},
\]
one obtains for these elements the following expressions

\[
U^{(e)}_{kn}(\alpha, \beta, \gamma) = E_{q^2} \left( \gamma(1-q^2)(2n+\frac{1}{2}) \right) \frac{q^{(k-n)(k-n-1)}}{(q^2; q^2)_{k-n}} ((1-q^2) \alpha)^{k-n}
\times 2\phi_1 \left( q^{-\rho-2n}, q^{\rho-2n+1}; q^{2k-2n+2}; q^2; \alpha \beta (1+q^2) q^{2n+2k-1} \right), \tag{5.18}
\]
\[ U_{kn}^{(o)}(\alpha, \beta, \gamma) = E_{q^2} \left( \gamma(1-q^2)(2n+\frac{3}{2}) \right) \frac{q^{(k-n)(k-n-1)}}{(q^2; q^2)_{k-n}} ((1-q^2)\alpha)^{k-n} \]
\[ \times \ _2\phi_1 \left( q^{n-2n}, q^{n-2n-1}; q^{2k-2n+2}, q^2; \alpha \beta (1+q)^2 q^{2n+2k+1} \right). \]

These formulas are defined for the whole range of \( k \) and \( n \) with the help of the limiting relation (2.16).

The generating function which is here obtained from (5.17a) reads as follows

\[ E_{q^2}(w) w^n \ _2\phi_0 \left( q^{-\rho-2n}, q^{\rho-2n+1}; q^2; -q^{2n} \frac{\alpha}{w} \right) \]
\[ = \sum_{k=-\infty}^{\infty} w^k \frac{q^{(k-n)(k-n-1)}}{(q^2; q^2)_{k-n}} \ _2\phi_1 \left( q^{-\rho-2n}, q^{\rho-2n+1}; q^{2k-2n+2}; q^2; \alpha q^{2k} \right). \]

An equivalent formula results from (5.17b).

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