Edge-isoperimetric inequalities for the symmetric product of graphs

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Abstract

The k-th symmetric product of a graph $G$ with vertex set $V$ with edge set $E$ is a graph $G^{[k]}$ with vertices as $k$-sets of $V$, where two $k$-sets are connected by an edge if and only if their symmetric difference is an edge in $E$. Using the isoperimetric properties of the vertex-induced subgraphs of $G$ and Sobolev inequalities on graphs, we obtain various edge-isoperimetric inequalities pertaining to the symmetric product of certain families of finite and infinite graphs.

1 Introduction

Given a graph $G = (V, E)$ with vertex set $V$ and edge set $E$, the edge boundary of the vertex subset $X \subseteq V$ with respect to $G$, commonly denoted as $\partial X$, is defined as the set of edges in $E$ with exactly one vertex in $X$ and one vertex in $V \setminus X$. In the well studied edge-isoperimetric problem [IIT64, AM85, Alo86, LPS88, Moh89, BL91, AC97, Til00, BE03], one seeks a lower bound on $|\partial X|$ for every $X$ with a fixed number of vertices. In this article, we say that a graph $G = (V, E)$ with a countable number of vertices has an isoperimetric dimension of $\delta$ with an isoperimetric number of $C$ if every $X \subseteq V$ has an edge boundary of size at least $C|X|^{1-1/\delta}$ whenever $|X|$ is at most half the cardinality of $V$.

The $k$-th symmetric product of a graph $G = (V, E)$ is a graph $G^{[k]}$ with vertices as the set of all $k$-sets of $V$, and two distinct $k$-sets $X$ and $Y$ are connected by an edge if and only if their symmetric difference $X \bigtriangleup Y = (X \setminus Y) \cup (Y \setminus X)$ is an edge in $E$ [Rud02, AGRR07]. The Johnson graph $J(n, k)$ is an example of a graph which can be expressed as a symmetric product of another graph. Namely, $J(n, k)$ is the $k$-th symmetric product of the complete graph $K_n$ on $n$ vertices. Since every eigenvalue of the combinatorial Laplacian of $J(n, k)$ is known, non-trivial edge-isoperimetric inequalities can be obtained using the second smallest eigenvalue. In this article, we study the edge-isoperimetric problem on the symmetric product graphs that need not be $K_n$.

When $G = (V, E)$ has a finite number of vertices, $G^{[k]} = G^{[|V|-k]}$ for every $k = 0, \ldots, |V|$. This is because complementing both the $k$-sets $X$ and $Y$ preserves their symmetric product. To see this, note that $(V \setminus X) \setminus (V \setminus Y) = (Y \setminus X)$, and $(V \setminus Y) \setminus (V \setminus X) = (X \setminus Y)$. Hence $(V \setminus X) \triangle (V \setminus Y) = (Y \setminus X) \triangle (Y \setminus Y) = X \triangle Y$. Moreover the graphs $G^{[0]}$ and $G^{[1]}$ are trivial; the graph $G^{[0]}$ has only one vertex and no edges, and the graph $G^{[1]}$ is equal to $G$. Hence we only study the properties of $G^{[k]}$ where $k = 2, \ldots, \lfloor |V|/2 \rfloor$.

Motivated by the graph isomorphism problem, several authors have studied the spectra properties of the symmetric square of graphs $G^{[2]}$ [AGRR07]. However, little is known about the higher symmetric powers of finite graphs apart from the fact that the $k$-th symmetric product of the complete graph $K_n$ is the Johnson graph $J(n, k)$.

Our edge-isoperimetric inequalities for symmetric products of the graph $G = (V, E)$ applies only when $V$ is a countable set. When $V$ is a finite set, we prove that if deleting any $k-1$ vertices from $G$ yields

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a vertex induced subgraph that has an isoperimetric number \(C\) and isoperimetric dimension \(\delta\), then \(G^{(k)}\) has isoperimetric dimension \(\delta\) with isoperimetric constant \(C|V|^{1/\delta}/(|V| - k + 1)\). When \(V\) is a countably infinite set, we prove that if deleting any \(k - 1\) vertices from \(G\) yields a vertex induced subgraph such that every connected component has an isoperimetric number \(C\) and isoperimetric dimension \(\delta\), then \(G^{(k)}\) has isoperimetric dimension \(\delta\) with isoperimetric constant \(Ck^{1-1/\delta}\). The proof of these results relies on analytical methods on the space of graphs.

2 Sobolev inequalities on graphs

Given a graph \(G = (V, E)\) and a function \(f : V \to \mathbb{R}\) on the vertex set, the discrete Sobolev seminorm of \(f\) corresponding to the edge set \(E\) is defined by

\[
\|f\|_E = \sum_{\{u,v\} \in E} |f(u) - f(v)|.
\]

The discrete Sobolev seminorm on graphs [Ost05] is related to the size of the edge boundaries of subsets of vertices. We call any inequality which involves the seminorm \(\|\cdot\|_E\) a discrete Sobolev inequality. Denote \(1_X : V \to \{0, 1\}\) as the indicator function on \(X\) for all \(X \subseteq V\), where \(1_X(x) = 1\) if \(x \in X\) and \(1_X(x) = 0\) if \(x \in V \setminus X\). The discrete Sobolev seminorm of \(1_X\) is the size of the edge boundary of \(X\). This identity is

\[
|\partial X| = \|1_X\|_E.
\]

Let \(\Phi_V\) denote the function space on \(V\), which is the set all functions \(f : V \to \mathbb{R}\). Lower bounds on \(\|1_X\|_E\) via functionals on \(\Phi_V\) have been studied for example by Chung and Yau [CY95], Tillich [Til00] and Ostrovskii [Ost05]. In this article, we restrict our attention to three families of functionals. The functionals are \(\|\cdot\|_p, g_p\) and \(\rho_p\) for \(p \geq 1\) where \(\|f\|_p = (\sum_{x \in V} |f(x)|^p)^{1/p}\),

\[
g_p(f) = \left(\frac{1}{|V|} \sum_{x,y \in V} |f(x) - f(y)|^p\right)^{1/p},
\]

and

\[
\rho_p(f) = \left(\sum_{x \in V} |f(x) - \mathbb{E}(f)|^p\right)^{1/p},
\]

where \(\mathbb{E}(f) = \frac{1}{|V|} \sum_{v \in V} f(v)\) denotes the expectation value of \(f\). Given \(C > 0\) and a functional \(\mathcal{F} : \Phi_V \to \mathbb{R}\), we say that \(G\) is \((C, \mathcal{F})\)-isoperimetric if for every \(X \subseteq V\),

\[
\|1_X\|_E \geq C \mathcal{F}(1_X). \tag{2.1}
\]

Evaluating the functionals \(\|\cdot\|_p, g_p,\) and \(\rho_p\) on the indicator function \(1_X\), we get \(\|1_X\|_p = |X|^{1/p}, g_p(1_X) = \left(\frac{2|X||V \setminus X|}{|V|}\right)^{1/p}\) and \(\rho_p(1_X) = \left(\sum_{x \in V} |1_X(x) - \frac{|X|}{|V|}|^p\right)^{1/p}\). The discrete Sobolev inequality (2.1) is closely related to the isoperimetric number and isoperimetric dimension of a graph as given in the following proposition.

**Proposition 2.1.** Let \(G = (V, E)\) be graph and \(C > 0\) and \(\delta > 1\). Then the following are true.

1. If \(V\) is countably infinite, \(G\) is \((C, \|\cdot\|_{\delta/(\delta - 1)})\)-isoperimetric if and only if \(G\) has an isoperimetric dimension of \(\delta\) with isoperimetric number \(C\).
2. If $V$ is finite and $G$ is $(C, g_{\delta/(\delta-1)})$-isoperimetric, then $G$ has an isoperimetric dimension of $\delta$ with isoperimetric number $C$.

3. If $V$ is finite and $G$ has an isoperimetric dimension of $\delta$ with isoperimetric number $C$, then $G$ is $(2^{-\delta/(\delta-1)}C, g_{\delta/(\delta-1)})$-isoperimetric.

Hence we can address graphs of finite isoperimetric dimension with the functionals $\rho_p$ using the two-sided bounds on $\rho_p(1_X)$ in terms of $g_p(1_X)$ as given in the following lemma.

**Lemma 2.2.** Let $G = (V, E)$ be a graph, $X \subseteq V$ and $p \geq 1$. Then

$$\frac{1}{2^{1-1/p}} g_p(1_X) \leq \rho_p(1_X) \leq g_p(1_X).$$

**Proof.** By definition, $\rho_p(1_X) = \left( \sum_{x \in V} \left| 1_X(x) - \frac{|X|}{|V|} \right|^p \right)^{1/p}$. Splitting the summation over $V$ into the disjoint subsets $X$ and $V \setminus X$ yields

$$\rho_p(1_X) = \left( |X| \left( 1 - \frac{|X|}{|V|} \right)^p + (|V| - |X|) \left( \frac{|X|}{|V|} \right)^p \right)^{1/p}. \tag{2.2}$$

Since $\left( 1 - \frac{|X|}{|V|} \right)^p \leq \left( 1 - \frac{|X|}{|V|} \right)$ and $\left( \frac{|X|}{|V|} \right)^p \leq \left( \frac{|X|}{|V|} \right)$ for $p \geq 1$, we get $\rho_p(1_X) \leq g_p(1_X)$. Since both $|X| \left( 1 - \frac{|X|}{|V|} \right)^p$ and $|V| - |X| \left( \frac{|X|}{|V|} \right)^p$ are at least $\left( \frac{|X||V|}{|V|} \right) \left( \frac{1}{2} \right)^{p-1}$, we get $\rho_p(1_X) \geq g_p(1_X)(1/2^{p-1})^{1/p}$. \hfill $\square$

Lemma 2.2 implies the following for $C > 0$ and $\delta > 1$.

1. If a graph is $(C, \rho_{\delta/(\delta-1)})$-isoperimetric, the graph also has an isoperimetric dimension of $\delta$ with an isoperimetric number of $2^{-\delta}C$.

2. If a graph has an isoperimetric dimension of $\delta$ with an isoperimetric number of $C$, the graph is also $(2^{-\delta/(\delta-1)}C, g_{\delta/(\delta-1)})$-isoperimetric.

The analytic inequalities of Tillich [Til00] establish the equivalence between edge-isoperimetric inequalities and discrete Sobolev inequalities on functionals that are either seminorms or quasilinear. We state Tillich’s result on seminorms in the following theorem.

**Theorem 2.3** (Tillich [Til00]). Let $G = (V, E)$ be a graph, $C > 0$, and $\rho$ be a seminorm on $\Phi_V$. Then $G$ is $(C, \rho)$-isoperimetric if and only if $\|f\|_E \geq C \rho(f)$ for every function $f : V \to \mathbb{R}$.

Tillich used the co-area formula to prove Theorem 2.3. The co-area formula is well known, and its proof can be found in [Til00] for instance.

**Lemma 2.4.** (Co-area formula) Let $G = (V, E)$ be a graph and $f : V \to \mathbb{R}$ be a non-negative function. Let $\Omega_t = \{x \in V : f(x) > t\}$. Then

$$\|f\|_E = \int_0^\infty |\partial \Omega_t| dt = \int_0^\infty \|1_{\Omega_t}\|_E dt.$$

We prove Theorem 2.3 for completeness because we could not find the complete proof in print.
Proof of Theorem 2.3. If \(\|f\|_E \geq C \rho(f)\), substituting \(f = 1_X\) for any subset \(X \subseteq V\) implies that \(\|1_X\|_E \geq C \rho(1_X)\) and hence \(G\) is \((C, \rho)\)-isoperimetric.

To prove the converse, Tillich used Lemma 2.4 to prove that

\[
\|f\|_E \geq C \left\{ \int_0^\infty \rho(1_{\Omega^+_i})dt + \int_0^\infty \rho(1_{\Omega^-_i})dt \right\},
\]

where \(\Omega^+_i = \{x \in V : \max(f(x), 0) \geq t\}\) and \(\Omega^-_i = \{x \in V : \max(-f(x), 0) \geq t\}\). Linearity of the integral and the seminorm properties of \(\rho\) then gives

\[
\int_0^\infty \rho(1_{\Omega^+_i})dt + \int_0^\infty \rho(1_{\Omega^-_i})dt = \int_0^\infty \rho(1_{\Omega^+_i})dt + \int_0^\infty \rho(-1_{\Omega^-_i})dt \geq \int_0^\infty \rho(1_{\Omega^+_i} - 1_{\Omega^-_i})dt = \rho(\int_0^\infty (1_{\Omega^+_i} - 1_{\Omega^-_i})dt).
\]

Noting that \(\int_0^\infty (1_{\Omega^+_i}(x) - 1_{\Omega^-_i}(x))dt = f(x)\) for all \(x \in V\) completes the proof. \(\square\)

In this article, we use Theorem 2.3 where \(\rho\) is either \(\rho_p\) or \(\|\cdot\|_p\) for \(p \geq 1\). In the discrete isoperimetric inequalities of Chung and Yau [CY95] and Ostrovskii [Ost05], functionals similar to \(\rho_p\) are used where the expectation value is replaced by the median, and evaluated on a different measure. We do not however pursue this direction of enquiry.

3 The symmetric product of infinite graphs

In this section, we let \(G = (V, E)\) be a graph with a countably infinite number of vertices. To address the edge-isoperimetric problem on the graph \(G^{(k)}\) for a fixed positive integer \(k\), we rely on the edge-isoperimetric properties of the vertex-induced subgraphs of \(G\).

More precisely, for any vertex subset \(W\) of \(V\), we denote \(G[V \setminus W]\) as the vertex-induced subgraph obtained by deleting the vertices in \(W\) from the graph \(G\). Denote the subgraphs \(G_i[V \setminus W] = (V_i[V \setminus W], E_i[V \setminus W])\) as the connected components of \(G[V \setminus W]\). We now state our main result on edge-isoperimetric inequalities for the symmetric product of infinite graphs.

Theorem 3.1. Let \(G = (V, E)\) be an infinite graph, and let \(p \geq 1\) and \(C > 0\). Suppose that for every \(W \subseteq V\) such that \(|W| = k - 1\) where \(k \geq 2\), every connected component \(G_i[V \setminus W]\) of every vertex-induced subgraph \(G[V \setminus W]\) is \((C, \|\cdot\|_p)\)-isoperimetric. Then \(G^{(k)}\) is \((Ck^{1/p}, \|\cdot\|_p)\)-isoperimetric.

Before we begin to prove Theorem 3.1, we require the following equivalence between the cardinalities of two different sets that establishes the equivalence of expressions summed over these sets.

Proposition 3.2. Let \(V\) be a countable set and \(k\) be a integer such that \(k = 1, \ldots, |V|\). Then the sets \(\mathcal{A} = \{(W, x) : W \subseteq V, |W| = k - 1, x \in V \setminus W\}\) and \(\mathcal{A}' = \{(X, x) : W \subseteq V, |X| = k, x \in X\}\) have the same cardinality.

Proof. Let \(f : \mathcal{A} \to \mathcal{A}'\) where \(f \mapsto (W, x) = (W \cup \{x\}, x)\) for all \(W \subseteq V\) and \(x \in V \setminus W\). The map \(f\) is invertible, and is therefore a bijection from \(\mathcal{A}\) to \(\mathcal{A}'\). Hence \(\mathcal{A}\) and \(\mathcal{A}'\) have the same cardinality. \(\square\)

The proof of Theorem 3.1 relies mainly on Theorem 2.3 and uses the seminorm \(\ell_p\)-norm \(\|\cdot\|_p\).
Proof of Theorem 3.1. For all $\Omega \subseteq V^{(k)}$, note that $|\partial \Omega| = \|1_{\Omega}\|_{E^{(k)}}$. Two $k$-sets $X$ and $Y$ in $\Omega$ are adjacent in the graph $G^{(k)}$ if and only if the symmetric difference of $X$ and $Y$ is an edge in $E$. Hence

$$|\partial \Omega| = \sum_{W \subseteq V} \sum_{|W| = k-1} \sum_{x \in V \setminus W} |1_{\Omega}(W \cup \{x\}) - 1_{\Omega}(W \cup \{x\})|.$$  \hspace{1cm} (3.1)

Applying Theorem 2.3 using the seminorm $\| \cdot \|_p$ on every connected component $G_i[V \setminus W]$ of each induced subgraph $G[V \setminus W]$ for every $(k-1)$-set $W$ with respect to the function $1_{\Omega}(W \cup \{\cdot\})$, we get

$$|\partial \Omega| \geq \sum_{W \subseteq V} \sum_{|W| = k-1} C \left( \sum_{x \in V \setminus W} |1_{\Omega}(W \cup \{x\})|^p \right)^{1/p}.$$ \hspace{1cm} (3.2)

By subadditivity of the function $(\cdot)^{1/p}$ for all $p \geq 1$, the inequality (3.2) becomes

$$|\partial \Omega| \geq C \left( \sum_{W \subseteq V} \sum_{|W| = k-1} 1_{\Omega}(W \cup \{x\}) \right)^{1/p}.$$ \hspace{1cm} (3.3)

By Proposition 3.2 we can reorder the summation in (3.3) to get

$$|\partial \Omega| \geq C \left( \sum_{X \in V^{(k)}} \sum_{x \in X} 1_{\Omega}(X) \right)^{1/p} = C \left( k \sum_{X \in V^{(k)}} 1_{\Omega}(X) \right)^{1/p} = C(k|\Omega|)^{1/p}.$$ \hspace{1cm} (3.4)

\[\Box\]

4 The symmetric product of finite graphs

Given a graph $G = (V,E)$ with finite number of vertices, we address the edge-isoperimetric problem on the graph $G^{(k)}$ for a fixed positive integer $k = 2, \ldots, \lfloor |V|/2 \rfloor$. Again we rely on the edge-isoperimetric properties of the vertex-induced subgraphs of a graph $G$. Here, we use the seminorm $\rho_p$ with Theorem 4.1 instead of $\| \cdot \|_p$ for $p \geq 1$.

Our main result is a lower bound on $|\partial \Omega|$, which is the size of the edge boundary of any vertex subset $\Omega$ in $G^{(k)}$. Our lower bound on $|\partial \Omega|$ is provided in terms of $|\partial_J \Omega|$, which is the size of the edge boundary of $\Omega$ in the Johnson graph $J(n,k)$.

Theorem 4.1. Let $G = (V,E)$ be a graph with $n$ vertices, and let $p \geq 1$ and $C > 0$. Suppose that every vertex-induced subgraph of $G$ with $n - k + 1$ vertices is $(C, \rho_p)$-isoperimetric. Then for every $\Omega \subseteq V^{(k)}$,

$$|\partial \Omega| \geq \frac{C}{n - k + 1} (2|\partial_J \Omega|)^{1/p}.$$  \hspace{1cm} (4.1)

Proof of Theorem 4.1. For all $\Omega \subseteq V^{(k)}$, note that $|\partial \Omega| = \|1_{\Omega}\|_{E^{(k)}}$. Two $k$-sets $X$ and $Y$ in $\Omega$ are adjacent in the graph $G^{(k)}$ if and only if the symmetric difference of $X$ and $Y$ is an edge in $E$. Hence

$$|\partial \Omega| = \sum_{W \subseteq V} \sum_{|W| = k-1} |1_{\Omega}(W \cup \{u\}) - 1_{\Omega}(W \cup \{v\})|.$$ \hspace{1cm} (4.1)

\[\Box\]
Applying Theorem 2.3 with seminorm $\rho_p$ on each induced subgraph $G[V \setminus W]$ for every $(k-1)$-set $W$ with respect to the function $\mathbf{1}_{\Omega}(W \cup \{\cdot\})$, we get

$$|\partial \Omega| \geq \sum_{W \subset V} C \left( \sum_{x \in V \setminus W} \mathbf{1}_{\Omega}(W \cup \{x\}) - \sum_{y \in V \setminus W} \frac{\mathbf{1}_{\Omega}(W \cup \{y\})}{n-k+1} \right)^{p}/p. \quad (4.2)$$

By subadditivity of the function $(\cdot)^{1/p}$ for all $p \geq 1$, the inequality (4.2) becomes

$$|\partial \Omega| \geq C \left( \sum_{W \subset V} \sum_{x \in V \setminus W} \mathbf{1}_{\Omega}(W \cup \{x\}) - \sum_{y \in V \setminus W} \frac{\mathbf{1}_{\Omega}(W \cup \{y\})}{n-k+1} \right)^{1/p}. \quad (4.3)$$

By Proposition 3.2 we can reorder the summation in (4.3) to get

$$|\partial \Omega| \geq C \left( \sum_{X \in \Omega} \sum_{x \in X} \left( \mathbf{1}_{\Omega}(X) - \sum_{y \in V \setminus (X \setminus \{x\})} \frac{\mathbf{1}_{\Omega}(X \triangle \{x,y\})}{n-k+1} \right)^{1/p}. \quad (4.4)$$

Each $k$-set $X$ appearing in the inequality (4.4) either belongs to $\Omega$ or not. Applying simple arithmetic on the right hand side of (4.4) above then yields

$$C \left( \sum_{x \in \Omega} \sum_{y \in X \setminus \{x\}} \frac{1 - \mathbf{1}_{\Omega}(X \triangle \{x,y\})}{n-k+1} \right)^{p} + C \sum_{X \notin \Omega} \sum_{y \in V \setminus (X \setminus \{x\})} \frac{\mathbf{1}_{\Omega}(X \triangle \{x,y\})}{n-k+1}^{p}/p. \quad (4.5)$$

Using the inequality $(\sum_i x_i)^p \geq \sum_i x_i^p$ for non-negative $x_i$, the expression (4.5) becomes

$$C \left( \sum_{x \in \Omega} \sum_{y \in V \setminus (X \setminus \{x\})} \frac{1 - \mathbf{1}_{\Omega}(X \triangle \{x,y\})}{(n-k+1)^p} + C \sum_{X \notin \Omega} \sum_{y \in V \setminus (X \setminus \{x\})} \frac{\mathbf{1}_{\Omega}(X \triangle \{x,y\})}{(n-k+1)^p} \right)^{1/p} = \frac{C}{n-k+1} \left( 2 \sum_{x \in \Omega, x \in X \setminus V \setminus (X \setminus \{x\})} \frac{\mathbf{1}_{\Omega}(X \triangle \{x,y\})}{(n-k+1)^p} \right)^{1/p}. \quad (4.6)$$

To complete the proof, note that

$$\sum_{x \notin \Omega} \sum_{y \in V \setminus (X \setminus \{x\})} \mathbf{1}_{\Omega}(X \triangle \{x,y\}) = |\partial \Omega|. \quad \square$$

The eigenvalues of the combinatorial Laplacian of the Johnson graph $J(n,k)$ for $k = 0, \ldots, \lfloor n/2 \rfloor$ are $j(n+1-j)$ with multiplicities $\binom{n}{j} - \binom{n-1}{j}$, where $j = 0, \ldots, k$ [BH11, Section 12.3.2]. If $\lambda$ is the second smallest eigenvalue of the combinatorial Laplacian of a graph, then that graph is $(\frac{1}{2}, g_1)$-isoperimetric [GR01, Lemma 13.7.1]. Since the second smallest eigenvalue of the combinatorial Laplacian of the Johnson graph $J(n,k)$ is always non-negative, $|\partial \Omega| \geq \frac{n}{2} g_1(\mathbf{1}_{\Omega})$ for every $\Omega \subseteq V^{(k)}$. Hence

$$(2|\partial \Omega|)^{1/p} \geq \left( \frac{n}{2} g_1(\mathbf{1}_{\Omega}) \right)^{1/p} = n^{1/p} g_p(\mathbf{1}_{\Omega}). \quad (4.6)$$

Using (4.6) with Theorem 4.1 together with Lemma 2.2 yields the following corollary.
Corollary 4.2. Let $G = (V, E)$ be a graph with $n$ vertices, and let $p \geq 1$ and $C > 0$. Suppose that every vertex-induced subgraph of $G$ with $n - k + 1$ vertices is $(C, \rho_p)$-isoperimetric. Then $G^{(k)}$ is $(\frac{Cn^{1/p}}{n-k+1}, \rho_p)$-isoperimetric and $(\frac{n^{1/p}}{n-k+1}, \rho_p)$-isoperimetric.

If a graph is $(C, \rho_1)$-isoperimetric, its vertex-induced subgraphs $G[V \setminus W]$ are $(C - |W|, \rho_1)$-isoperimetric for every $W \subseteq V$ [GR01, Theorem 13.5.1]. This together with (4.6) and the fact that $g_1(I_\Omega) = \rho_1(I_\Omega)$ from Lemma 2.2 yields the following corollary.

Corollary 4.3. Let $G = (V, E)$ be a graph with $n$ vertices that is $(C, \rho_1)$-isoperimetric, and let $k = 2, \ldots, \lfloor n/2 \rfloor$. Then $G^{(k)}$ is $(n^{1-k+1}, \rho_1)$-isoperimetric.

5 Additional remarks

1. The inequality in Theorem 4.1 is tight. To see this, consider the symmetric product of the complete graph.

2. The exact solution to the edge-isoperimetric problem for the Johnson graph, also known as the problem of Kleitman and West [Har91], remains an unsolved problem. Better edge-isoperimetric inequalities for the Johnson graph will improve the edge-isoperimetric inequalities of the symmetric product of finite graphs given in Corollaries 4.2 and 4.3.

3. It would be interesting to obtain edge-isoperimetric inequalities for vertex-induced subgraphs that have finite isoperimetric dimension. Such results can be combined with Theorem 4.1 to yield edge-isoperimetric inequalities for the symmetric product of graphs that are not expanders.

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