BOUNDARY REGULARITY FOR MONGE-AMPÈRE EQUATIONS WITH UNBOUNDED RIGHT HAND SIDE

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Abstract. We consider Monge-Ampère equations with right hand side $f$ that degenerate to $\infty$ near the boundary of a convex domain $\Omega$, which are of the type

$$
det D^2u = f \quad \text{in } \Omega, \quad f \sim d_{\partial \Omega}^{-\alpha} \quad \text{near } \partial \Omega,
$$

where $d_{\partial \Omega}$ represents the distance to $\partial \Omega$ and $-\alpha$ is a negative power with $\alpha \in (0, 2)$. We study the boundary regularity of the solutions and establish a localization theorem for boundary sections.

1. Introduction

In this paper we consider degenerate Monge-Ampère equations of the type

$$
det D^2u = f \quad \text{in } \Omega, \quad f \sim d_{\partial \Omega}^{-\alpha} \quad \text{near } \partial \Omega,
$$

where $d_{\partial \Omega}$ represents the distance to the boundary of the domain $\Omega$ and $-\alpha$ is a negative power with $\alpha \in (0, 2)$.

Boundary estimates for the Monge-Ampère equation in the nondegenerate case $f \in C^2(\Omega)$, $f > 0$, were obtained by Ivočkina [8], Krylov [9], Caffarelli-Nirenberg-Spruck [3] (see also [1, 15]).

In [12], a localization theorem at boundary points was proved when the right hand side $f$ is only bounded away from $0$ and $\infty$. It states that under natural local assumptions on the domain and boundary data, the sections $S_h(x_0)$ with $x_0 \in \partial \Omega$ are “equivalent” to half-ellipsoids centered at $x_0$. This extends up to the boundary a result that is valid for sections compactly included in $\Omega$, which is a consequence of John’s lemma from convex geometry. These localization theorems are the key ingredients in establishing optimal $C^{2, \alpha}$ and $W^{2, p}$ estimates for solutions under further regularity properties of the right-hand side $f$ and boundary data (see [2, 12, 13]).

In [14], the first author studied degenerate Monge-Ampère equations of the type

$$
det D^2u = f \quad \text{in } \Omega, \quad f \sim d_{\partial \Omega}^{-\alpha} \quad \text{near } \partial \Omega,
$$

where $\alpha > 0$ is a positive power. A localization theorem and pointwise $C^2$ estimate were established in [14] and they were later used in [10] to prove the global smoothness for the eigenfunctions of the Monge-Ampère operator $(det D^2u)^{1/n}$.

In this paper, we consider the case of the Monge-Ampère equation with right hand side which degenerates to $\infty$ near the boundary of $\Omega$. This type of equations appear for example in the study of affine spheres in geometry [4, 5], the $p$-Minkowski problem [11], or in optimal transportation problems involving two densities with only one of them having compact support.

We study the case when $f$ is “comparable” with a negative power $d_{\partial \Omega}^{-\alpha}$ of the distance function to $\partial \Omega$. It can be checked from a simple 1D example that the Dirichlet problem for equation (1.1) is well posed only for $\alpha \in (0, 2)$. Moreover, when $\alpha \in (0, 1)$ solutions are expected to have bounded gradients, and when $\alpha \in [1, 2)$ the gradient should tend to $\infty$ as we approach the boundary. We study the geometry of boundary sections of solutions to (1.1) and prove two localization theorems Theorems 1.1 and 1.4 depending whether $\alpha$ is smaller or larger than 1.

We first give the localization theorem for the case $\alpha \in (0, 1)$. It states that under appropriate assumptions on the domain and boundary data, the sections

$$
S_h(x_0) := \{ x \in \bar{\Omega} \mid u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h \}
$$

with $x_0 \in \partial \Omega$ have the shape of half-ellipsoids centered at $x_0$.

Theorem 1.1. Assume $\Omega \subset \mathbb{R}^n$ is a bounded convex set, $\partial \Omega \in C^2$. Let $u : \bar{\Omega} \to \mathbb{R}$ be continuous, convex, satisfying

$$
det D^2u = f, \quad \lambda_0 d_{\partial \Omega}^{-\alpha} \leq f \leq \Lambda_0 d_{\partial \Omega}^{-\alpha} \quad \text{in } \Omega
$$

for some $\alpha \in (0, 1)$, and on $\partial \Omega$, $u$ separates quadratically from its tangent plane, namely

$$
\mu |x - x_0|^2 \leq u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0) \leq \mu^{-1} |x - x_0|^2, \quad \forall x, x_0 \in \partial \Omega,
$$

where $\lambda_0, \Lambda_0, \mu, \mu^{-1}$ are positive constants.
for some $\mu > 0$. Then there is a constant $c > 0$ depending only on $n, \lambda_0, \Lambda_0, \alpha, \mu, \text{diam}(\Omega)$ and $\|\partial\Omega\|_{C^2}$ such that for each $x_0 \in \partial\Omega$ and $h \leq c$ we have

$$E_h(x_0) \cap \overline{\Omega} \subset S_h(x_0) \subset E_{c-1}h(x_0),$$

where

$$E_h(x_0) := \{(x-x_0)_\tau^2 + |(x-x_0) \cdot \nu_{x_0}|^{2-\alpha} < h\}, \quad \forall h > 0,$$

$\nu_{x_0}$ denotes the unit inner normal to $\partial\Omega$ at $x_0$ and

$$(x-x_0)_\tau := (x-x_0) - [(x-x_0) \cdot \nu_{x_0}]\nu_{x_0}$$

is the projection of $x - x_0$ onto the tangent plane of $\partial\Omega$ at $x_0$.

Theorem 1.1 states that a boundary section $S_h$ is equivalent to an ellipsoid of axes $h^{\frac{\alpha}{\alpha-1}}$ in the tangential direction to $\partial\Omega$ and $h^{\frac{\alpha}{\alpha-1}}$ in the normal. As a corollary, it can be proved that the maximal interior sections have the same geometry as boundary sections. Namely, for any $y_0 \in \Omega$, let $S_h(y_0)$ denote the maximal interior section centered at $y_0$ which becomes tangent to $\partial\Omega$ at some point $x_0$. Then $S_h(y_0)$ is equivalent to an ellipsoid of axes $h^{\frac{\alpha}{\alpha-1}}$ in the tangential direction to $\partial\Omega$ at $x_0$ and $h^{\frac{\alpha}{\alpha-1}}$ in the normal $\nu_{x_0}$.

We remark that if $u_{|\partial\Omega} = \varphi$ and $\partial\Omega \in C^3, \varphi \in C^3(\partial\Omega)$, and $\Omega$ is uniformly convex, then the quadratic separation condition (1.4) is satisfied. The proof is given in [12, Proposition 3.2], where only the lower bound of $\det D^2u$ is used. Since in our degenerate case, $\det D^2u$ is also bounded below by a constant, the estimate still applies.

Theorem 1.1 implies global $W^{2,p}$ estimates of solutions if we assume further that $f = g d_{\partial\Omega}^\alpha$ for some function $g \in C(\overline{\Omega})$ which is strictly positive. In a subsequent work we will show that $u \in W^{2,p}(\Omega)$ for any $p < \frac{1}{\alpha}$.

For the case $\alpha \in (0, 1)$, we establish the following Liouville type theorem for global solutions to (1.1).

**Theorem 1.2.** Let $u \in C(\mathbb{R}^n_+)$ be a convex function that satisfies

$$c_0(|x'|^2 + x_n^{2-\alpha}) \leq u(x) \leq c_0^{-1}(|x'|^2 + x_n^{2-\alpha})$$

for some $c_0 > 0$ and

$$\det D^2u = x_n^{-\alpha}, \quad u(x', 0) = \frac{1}{2} |x'|^2.\tag{1.6}$$

Then

$$u(x) = \frac{1}{2} |x'|^2 + \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)},$$

Theorem 1.1 and the Liouville theorem imply a pointwise $C^2$ tangential estimate at the boundary.

**Theorem 1.3.** Assume that $\Omega \subset \{x_n > 0\}$ is a bounded convex set, $0 \in \partial\Omega$, $\partial\Omega \in C^2$ near the origin, and the principal curvatures of $\partial\Omega$ at $0$ are strictly positive. Let $u \in C(\overline{\Omega})$ be a convex solution to the equation

$$\det D^2u = f(x) d_{\partial\Omega}^\alpha \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega,$$

for some $\alpha \in (0, 1)$, where $f$ is a nonnegative function that is continuous at the origin and $f(0) > 0$, the boundary data $\varphi$ is $C^2$ at $0$, and it separates quadratically away from $0$. Assume further that

$$u(0) = 0, \quad \nabla u(0) = 0.$$

Then there exists a constant $a > 0$ such that

$$u(x) = Q(x') + ax_n^{2-\alpha} + o(|x'|^2 + x_n^{2-\alpha}),$$

where $Q$ represents the quadratic part of the boundary data $\varphi$ at the origin.

Next we give the localization theorem when $\alpha \in (1, 2)$. In this case we consider the maximal sections included in $\Omega$ which become tangent to $\partial\Omega$ at boundary points.

**Theorem 1.4.** Assume $\Omega \subset \mathbb{R}^n$ is uniformly convex, $\partial\Omega \in C^2$. Assume further that $0 \in \partial\Omega$ and the $x_n$ coordinate axis lies in the direction $\nu_0$ ($\nu_0$ is the unit inner normal to $\partial\Omega$ at $0$).

Let $u : \overline{\Omega} \to \mathbb{R}$ be continuous, convex, satisfying

$$\det D^2u = f, \quad \lambda_0 d_{\partial\Omega}^\alpha \leq f \leq \Lambda_0 d_{\partial\Omega}^\alpha \text{ in } \Omega,$$

for some $\alpha \in [1, 2)$, and assume $u_{|\partial\Omega} = \varphi \in C^2$. Suppose that $S_h(y_0)$ is the maximal section included in $\Omega$ which becomes tangent to $\partial\Omega$ at $0$. Then

$$\nabla x^\alpha u(y_0) = \nabla x^\alpha \varphi(0), \quad M = -u_n(y_0) \geq -C,$$

and the following hold:
i) If $\alpha \in (1, 2)$, denote $\beta := \frac{n+\alpha-1}{n}$, then we have
\[
ch \frac{\rho}{\beta} \leq \max\{M, 1\} \leq Ch \frac{\rho}{\beta}, \quad ch \frac{\rho}{\beta} \leq d_{\partial \Omega}(y_0) \leq Ch \frac{\rho}{\beta},
\]
\[
\{|x'|^2 + |x_n| \leq ch \frac{\rho}{\beta}\} \subset S_h(y_0) - y_0 \subset \{|x'|^2 + |x_n| \leq Ch \frac{\rho}{\beta}\}.
\]

ii) If $\alpha = 1$, denote $h_* := \min\{\bar{h}, 1\}$, then we have
\[
-c\log(ch) \leq |M|^\alpha \leq -C\log(ch), \quad ch^C \leq d_{\partial \Omega}(y_0) \leq Ch^c,
\]
\[
B_{ch^c} \subset \ S_h(y_0) - y_0 \subset B_{ch^c}.
\]

Here the constants $c, C$ depend only on $n, \lambda_0, \Lambda_0, \alpha, \text{diam}(\Omega)$, and $\varphi, \partial \Omega$ up to their second derivatives.

In the case $\alpha \in (1, 2)$, Theorem 1.4 states that for any $y_0 \in \Omega$, the maximal interior section $S_h(y_0)$ which becomes tangent to $\partial \Omega$ at some point $x_0$ is equivalent to an ellipsoid of axes $h^{\frac{1}{\alpha}-\frac{1}{\alpha^*}}$ in the tangential direction to $\partial \Omega$ at $x_0$ and $h^{\frac{1}{\alpha}}$ in the normal $\nu_{x_0}$. For the border line case $\alpha = 1$, it cannot be concluded from ii) that $S_h(y_0)$ is equivalent to an ellipsoid whose shape depends only on $h, y_0$ and $\Omega$. Probably more precise information is needed on the ratio between $f$ and $d_{\partial \Omega}^{-1}$ in order to reach a similar conclusion as in the case $\alpha \in (1, 2)$.

The proofs of Theorems 1.1 and 1.4 are quite different. Theorem 1.4 follows directly from comparison with explicit barriers. Theorem 1.1 is much more involved and most of the paper will be devoted towards its proof. We will follow similar ideas as in the nondegenerate case treated in [12].

The paper is organized as follows. In Section 2 we introduce some notation, then we reduce Theorem 1.1 to its local version Theorem 2.1. This is further reduced to Theorem 2.2, where the distance function is replaced by $x_n$. We also give a more precise quantitative version of Theorem 1.3 (see Theorem 2.3). Sections 3-4 are devoted to the proof of Theorem 2.2. In Section 5, the proof of Theorem 1.2 is given. In Section 6, we give the proof of Theorem 2.3 and then finish the proof of Theorem 1.3. In the last section, we give the proof of Theorem 1.4.

2. Statement of main results

We introduce some notation. We denote points in $\mathbb{R}^n$ as
\[
x = (x_1, \ldots, x_n) = (x', x_n), \quad x' \in \mathbb{R}^{n-1}.
\]

Let $u$ be a convex function defined on a convex set $\overline{\Omega}$, we denote by $S_h(x_0)$ the section centered at $x_0$ and at height $h > 0$,
\[
S_h(x_0) := \{x \in \overline{\Omega} : u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h\}.
\]

When $x_0 \in \partial \Omega$, the term $\nabla u(x_0)$ is understood in the sense that
\[
x_{n+1} = u(x_0) + \nabla u(x_0) \cdot (x - x_0)
\]
is a supporting hyperplane for the graph of $u$ at $x_0$ but for any $\epsilon > 0$,
\[
x_{n+1} = u(x_0) + (\nabla u(x_0) + \epsilon \nu_{x_0}) \cdot (x - x_0)
\]
is not a supporting hyperplane, where $\nu_{x_0}$ denotes the unit inner normal to $\partial \Omega$ at $x_0$. We denote for simplicity $S_h = S_h(0)$, and sometimes when we specify the dependence on the function $u$ we use the notation $S_h(u) = S_h$.

For a set $E \subset \mathbb{R}^n$, we always denote $\pi(E)$ the projection of $E$ into $\mathbb{R}^{n-1}$, i.e.,
\[
\pi(E) := \{x' \in \mathbb{R}^{n-1} : \exists \ t \in \mathbb{R} \ s.t. \ (x', t) \in E\}.
\]

In the case $\alpha \in (0, 1)$, for any $h > 0$ we often use the particular sets
\[
\mathcal{E}_h := \{|x'|^2 + x_n^{2-\alpha} < h\}, \quad \mathcal{E}^+_h := \mathcal{E}_h \cap \{x_n > 0\},
\]
and the diagonal matrix
\[
F_h := \text{diag}(h^{\frac{1}{\alpha}}, h^{\frac{1}{\alpha}}, \ldots, h^{\frac{1}{\alpha}}, h^{\frac{1}{\alpha^*}})
\]
in our estimates.

Next we give a local version of Theorem 1.1. Our assumptions are the following.

Let $\Omega \subset \mathbb{R}^n$ be a open convex set. Assume that for some fixed small $\rho > 0$,
\[
B_\rho(\rho e_n) \subset \Omega \subset \{x_n > 0\} \cap B_{\frac{1}{\rho}},
\]
and
\[
\Omega \text{ contains an interior ball of radius } \rho \text{ tangent to } \partial \Omega \text{ at each point on } \partial \Omega \cap \{x_n \leq \rho\}.
\]
The part $\partial \Omega \cap \{x_n \leq \rho\}$ is then given by $x_n = g(x')$ for some convex function $g$, where

$$g \in C^2 \left( \pi(\partial \Omega \cap \{x_n < \rho\}) \right), \quad g(0) = 0, \quad \nabla g(0) = 0. \tag{2.3}$$

Let $u : \overline{\Omega} \to \mathbb{R}$ be a convex solution to

$$\det D^2 u = f, \quad 0 < \lambda(x_n - g)^{-\alpha} \leq f \leq \Lambda(x_n - g)^{-\alpha} \quad \text{in } \Omega \cap \{x_n < \rho/2\} \tag{2.4}$$

for some $\alpha \in (0, 1)$. Moreover, $x_{n+1} = 0$ is the tangent plane to $u$ at $0$,

that is,

$$x_{n+1} = 0, \quad u(0) = 0, \quad \nabla u(0) = 0 \quad \text{in the sense that } x_{n+1} = \epsilon x_n \text{ is not a supporting plane for the graph of } u \text{ at } 0 \text{ for any } \epsilon > 0.$$

We also assume that $u$ separates quadratically on $\partial \Omega$ (in a neighborhood of $\{x_n = 0\}$) from the tangent plane at $0$, i.e.,

$$\mu |x|^2 \leq u(x) \leq \mu^{-1} |x|^2 \quad \text{on } \partial \Omega \cap \{x_n \leq \rho\}. \tag{2.6}$$

**Theorem 2.1.** Assume $\Omega$ and $u$ satisfy (2.1)-(2.6). Then there is a constant $c > 0$ depending only on $n, \lambda, \Lambda, \alpha, \mu$ and $\rho$ such that for each $h \leq c$ we have

$$E_{tr} \cap \overline{\Omega} \subset S_h \subset E_{c^{-1}h}. \tag{2.7}$$

Assume $\Omega$ and $u$ satisfy the hypotheses in Theorem 1.1. Fix a point $x_0 \in \partial \Omega$, by a translation and a rotation of coordinates we can assume that $x_0 = 0$, and the $x_n$ coordinate axis lies in the direction of $\nu_{x_0}$. Since $\partial \Omega \in C^2$, there exists $\rho > 0$ such that (2.1)-(2.3) hold, and after subtracting a linear function we have (2.5) and (2.6). By (2.1)-(2.3), it is easy to see that

$$\|D^2 g\|_{C(\pi(\partial \Omega \cap \{x_n \leq \rho/2\}))} \leq C(n, \rho) \tag{2.8}$$

and therefore

$$d_{\partial \Omega}(x) \leq x_n - g(x') \leq C'(n, \rho)d_{\partial \Omega}(x),$$

where $C(n, \rho)$ and $C'(n, \rho)$ are constants depending only on $n$ and $\rho$. It follows that $u$ satisfies (2.4) with $\lambda := \lambda_0, \Lambda := C'(n, \rho)\Lambda_0$. Therefore we reduce the proof of Theorem 1.1 to that of Theorem 2.1 above.

Let $\Omega$ and $u$ satisfy the hypotheses in Theorem 2.1. By constructing some lower barrier for $u$, we will prove in Section 3 that in some domain $\Omega_0 \subset \Omega$ we have $x_n - g \sim x_n$, and $u$ still satisfies the quadratic separation (2.6) on $\partial \Omega_0$ in a neighborhood of $\{x_n = 0\}$. Therefore we reduce the proof of Theorem 2.1 to that of Theorem 2.2 below.

We assume (2.1), (2.5), (2.6) hold while replacing the equation (2.4) by

$$\det D^2 u = f, \quad 0 < \lambda x_n^{-\alpha} \leq f \leq \Lambda x_n^{-\alpha} \quad \text{in } \Omega \cap \{x_n < \rho\}. \tag{2.9}$$

Note that we do not assume (2.2) and (2.3) hold here.

**Theorem 2.2.** Assume $\Omega$ and $u$ satisfy (2.1), (2.5), (2.6) and (2.9). Then there is a constant $c > 0$ depending only on $n, \lambda, \Lambda, \alpha, \mu$ and $\rho$ such that for each $h \leq c$ we have

$$E_{tr} \cap \overline{\Omega} \subset S_h \subset E_{c^{-1}h}. \tag{2.10}$$

We prove Theorem 2.2 using the compactness methods in [12]. We first obtain some preliminary estimates about $u$. Next we consider the rescaling $v$ of $u$. Then we reduce the proof of the theorem to that of a statement about $v$. We reduce this to the proof of a statement (Proposition 4.2) about the limiting function (still denoted by $u$) of such $v$. Different from the case that $\alpha = 0$ (in this case the estimate of the volume of $S_t(v)$ is $|S_t(v)|^2 \sim t^n$), the estimate of the volume of $S_t(v)$ becomes

$$|x^*_t(v) \cdot e_n|^{-\alpha} |S_t(v)|^2 \sim t^n, \tag{2.11}$$

where $x^*_t(v)$ is the center of mass of $S_t(v)$. The limiting function $u$ also satisfies this estimate. To prove Proposition 4.2, we construct some lower barrier for the limiting function $u$ and use (2.9). Since we do not have the estimate of $|S_t(u)|$, we also use the convexity of the original solution to estimate the quantity $x^*_t(v) \cdot e_n$ from below. The estimate (2.9) brings another difficulty when we prove Proposition 4.2. We use John’s lemma and find an ellipsoid $E_h$ equivalent to the section $S_h(u)$ of the limiting solution $u$. In the case $\alpha = 0$, we use the estimate $|E_h|^2 \sim h^n$ to estimate the shape of $S_h(u)$, but in our degenerate case, we do not have the estimate of the volume of $E_h$. For this, we use the estimate $\left|x^*_t(u) \cdot e_n\right|^{-\alpha} |E_h|^2 \sim h^n$ to obtain an estimate of the shape of $S_h(u)$ in terms of the quantity $x^*_t(u) \cdot e_n$. Using this estimate, we rescale $u$ and reduce Proposition 4.2 to the lower-dimensional case. Again, since we do not have the estimate of $|E_h|$, we perform a different rescaling (which corresponds to our estimate (2.9)) from the $\alpha = 0$ case.

At the end of this section we give a more precise quantitative version of Theorem 1.3 as follows.
Theorem 2.3. For any $\eta > 0$ there exists $\epsilon_0 > 0$ depending only on $\eta, n, \alpha$ such that if (2.1)-(2.5) hold with $\lambda = 1 - \epsilon_0, \Lambda = 1 + \epsilon_0$ and

\[
\frac{1}{2} - \epsilon_0 \right |x'|^2 \leq u(x) \leq \left( \frac{1}{2} + \epsilon_0 \right ) |x'|^2 \quad \text{on } \partial \Omega \cap \{ x_n \leq \rho \},
\]
then for all $h \leq c$, we have

\[
(1 - \eta) S_h(U_0) \cap \Pi \subset S_h(u) \subset (1 + \eta) S_h(U_0),
\]

where

\[
U_0(x) := \frac{1}{2} |x'|^2 + \frac{x_n^2 - \alpha}{(2 - \alpha)(1 - \alpha)}, \quad S_h(U_0) := \{ x \in \mathbb{R}^n : U_0(x) < h \},
\]

and the constant $c > 0$ depends only on $\eta, n, \alpha, \rho$.

3. Proof of Theorem 2.2 (I)

As mentioned in Section 2, we first show that we can reduce the proof of Theorem 2.1 to that of Theorem 2.2.

Proposition 3.1. Theorem 2.2 implies Theorem 2.1.

Proof. In this proof we always denote by $c, C, c_i, C_i$ $(i = 0, 1, 2, \ldots)$ constants depending only on $n, \lambda, \Lambda, \mu, \alpha$ and $\rho$. For simplicity of notation, their values may change from line to line whenever there is no possibility of confusion.

Let

\[
v_0 := \mu |x'|^2 + \frac{\Lambda}{(2 - \alpha)(1 - \alpha) \mu^{n-1}} (x_n - g)^{2-\alpha}.
\]

Then by straightforward computation and using (2.7), we obtain that

\[
\det D^2 v_0 = \frac{\Lambda}{\mu^{n-1}} (x_n - g)^{-\alpha} \det \left( 2 \mu I_{n-1} - \frac{\Lambda}{(1 - \alpha) \mu^{n-1}} (x_n - g)^{1-\alpha} D^2 g \right)
\]

\[
\geq \Lambda (x_n - g)^{-\alpha} \quad \text{in } \Omega \cap \{ x_n < c_* \},
\]

where $c_* \leq \rho/2$ is small depending only on $n, \lambda, \mu, \alpha$ and $\rho$.

Denote $D := \pi (\Omega \cap \{ x_n = c_* \})$. For $x' \in D$, define

\[
g^*(x') := \sup \left \{ l(x') : l \leq g \text{ in } D, l \text{ is affine, and } |\nabla l| \leq \frac{c_* \rho}{2} \right \}.
\]

Then $g^*$ is convex in $D$ since it is the supremum of a family of convex functions.

We claim that for any $x \in \Omega \cap \{ x_n = c_* \}$, we have

\[
x_n - g^*(x') \geq \frac{c_* \rho}{2}.
\]

Indeed, if $l$ is affine, $l \leq g$ in $D$ and $|\nabla l| \leq \frac{c_* \rho}{2}$, then

\[
0 = g(0) \geq l(0) = l(x') - \nabla l \cdot x',
\]

it follows that

\[
l(x') \leq \nabla l \cdot x' \leq \frac{c_* \rho}{2} \cdot \frac{1}{\rho} = \frac{c_*}{2},
\]

where we use the fact that $\Omega \subset B_{1/\rho}$. Thus the claim follows.

We also claim that

\[
\pi (\Omega \cap \{ x_n \leq c_0 \rho \}) \subset D \cap \left \{ |\nabla g| \leq \frac{c_0 \rho}{2} \right \} \subset \{ g^* = g \}
\]

for some small constant $c_0$.

Indeed, the second inclusion in (3.3) follows easily from the convexity of $g$ and the definition of $g^*$. Therefore we only need to prove the first inclusion. Let $c_0 > 0$ be a small constant to be chosen. For any $x_0 \in \partial \Omega \cap \{ x_n \leq c_0 \rho \}$, we have $B_{c_0 \rho}(y_0) \subset \Omega \subset \{ x_n \geq 0 \}$ by (2.2), where $y_0 := x_0 + \rho v_{x_n}$. Let

\[
l = \inf_{x \in B_{c_0 \rho}(y_0)} x_n,
\]

then $(y'_0, l) \in \partial B_{c_0 \rho}(y_0)$ and

\[
\rho v_{x_n} \cdot e_n = (y_0 \cdot e_n - l) - (x_0 \cdot e_n - l) = \rho - (x_0 \cdot e_n - l) \geq (1 - c_0) \rho,
\]

which gives

\[
\frac{1}{\sqrt{1 + |\nabla g(y'_0)|^2}} \geq v_{x_n} \cdot e_n \geq 1 - c_0.
\]
Hence,

$$|\nabla g(x_0)| \leq \sqrt{\left(\frac{1}{1-c_0}\right)^2} - 1 \leq \frac{c_0\rho}{2}$$

if $c_0$ is small. The desired conclusion (3.3) follows.

Let

$$v^* := \mu|x|^2 + \frac{\Lambda}{(2 - \alpha)(1 - \alpha)^{\mu\alpha - 1}}(x_n - g)^{2 - \alpha} - C^*(x_n - g^*(x')).$$ 

Then $v^*$ is a lower barrier for $u$ in $\Omega \cap \{x_n \leq c_\ast\}$ if $C^*$ is large depending only on $n, \Lambda, \mu, \alpha$ and $\rho$.

Indeed, since $g^*$ is convex, we find from (3.1) that $u^*$ is a subsolution of the equation

$$\det D^2 w = \Lambda(x_n - g)^{-\alpha}.$$ 

On $\partial\Omega \cap \{x_n \leq c_\ast\}$, we have $x_n - g^* = g - g^* \geq 0$, which implies

$$v^* \leq \mu|x|^2 \leq u.$$ 

On $\Omega \cap \{x_n = c_\ast\}$, we obtain from (3.2) that

$$v^* \leq \frac{\mu}{\rho^2} + \frac{\Lambda}{(2 - \alpha)(1 - \alpha)^{\mu\alpha - 1}}c_\ast^{2 - \alpha} - C\ast \frac{c_\ast}{2} \leq 0 \leq u$$

if $C^*$ is large.

Thus,

$$v^* \leq u \quad \text{in} \quad \Omega \cap \{x_n \leq c_\ast\}.$$ 

This together with (3.3) implies that

$$u \geq \mu|x|^2 - C^*(x_n - g(x')) \quad \text{in} \quad \Omega \cap \{x_n \leq c_0\rho\}.$$ 

Therefore, if $\delta$ is small, we have

$$u \geq \frac{\mu|x|^2}{2} \quad \text{in} \quad \Omega \cap \{x_n \leq c_0\rho\} \cap \{x_n \leq g(x') + \delta|x'|^2\}.$$ 

On the other hand, the convexity of $u$ and the quadratic separation of $u$ on $\partial\Omega \cap \{x_n \leq \rho\}$ (see (2.6)) implies that

$$u \leq C|x|^2 \quad \text{in} \quad \Omega \cap \{x_n \leq c_0\rho\} \cap \{x_n \leq g(x') + \delta|x'|^2\}.$$ 

In particular, if we denote $\Omega_0 := \Omega \cap \{x_n < c_0\rho\} \cap \{x_n > g(x') + \delta|x'|^2\}$, then the above two estimates hold on $\partial\Omega \cap \{x_n \leq c_0\rho\}$.

We have

$$\lambda x_n^{-\alpha} \leq \det D^2 u \leq Cx_n^{-\alpha} \quad \text{in} \quad \Omega_0.$$ 

We apply Theorem 2.2 to $u$ in $\Omega_0$ and obtain that

$$E_{ch} \cap \overline{\Omega_0} \subset S_h \cap \overline{\Omega_0} \subset E_{Ch}, \quad \forall h \leq c.$$ 

We claim that the last estimate also holds for $S_h$ (instead of $S_h \cap \overline{\Omega_0}$). Indeed, we have by (3.7)

$$(\overline{\Omega} \setminus \Omega_0) \cap E_{ch} \subset S_h$$

and therefore

$$E_{ch} \cap \overline{\Omega} \subset S_h.$$ 

On the other hand, we obtain from (3.6) that

$$(S_h \cap \{x_n \leq c_0\rho\}) \setminus \Omega_0 \subset \{|x'| \leq Ch^{\frac{1}{2}}\}.$$ 

Since

$$\overline{\Omega} \setminus \Omega_0 \cap \{x_n \leq c_0\rho\} \subset \{x_n \leq g(x') + \delta|x'|^2\},$$

we obtain

$$S_h \setminus \Omega_0 \subset \{|x'| \leq Ch^{\frac{1}{2}}, x_n \leq Ch\} \subset E_{Ch}.$$ 

$\square$

In the following we give the first part of the proof of Theorem 2.2. In the remaining part of this section we denote by $c, C, c_i, C_i (i = 0, 1, 2, \ldots)$ positive constants depending on $n, \lambda, \Lambda, \mu$ and $\alpha$. The dependence of various constants also on $\rho$ will be denoted by $c(\rho), C(\rho), c_i(\rho), C_i(\rho) (i = 0, 1, 2, \ldots)$. 
Proposition 3.2. Assume that $\Omega$ and $u$ satisfy the hypotheses of Theorem 2.2. Then, for each $h \leq c(\rho)$ there exists a linear transformation (sliding along $x_n = 0$)
\[ A_h x = x - \nu n, \quad \nu_n = 0, \quad |\nu| \leq C(\rho)h^{-\frac{n}{2(\alpha+1-n)}}, \]
such that the rescaled function
\[ \tilde{u}(A_h x) = u(x) \]
satisfies in
\[ \tilde{S}_h := A_h S_h = \{ \tilde{u} < h \} \]
the following
(i) the center of mass $\tilde{x}_h^* \text{ of } \tilde{S}_h \text{ lies on the x}_n \text{ axis, i.e. } \tilde{x}_h^* = d_h e_n.$
(ii) $c_0h \leq |\tilde{S}_h|^2 d_h^{-\alpha} \leq C_0h^n.$
And after a rotation of the $x_1, \ldots, x_{n-1}$ variables we have
\[ \tilde{x}_h^* + cD_B B_1 \subset \tilde{S}_h \subset CD_B B_1, \]
where $D_B := \text{diag}(d_1, d_2, \ldots, d_{n-1}, d_n)$ is a diagonal matrix that satisfies
\[ (3.8) \quad \left( \prod_{i=1}^{n-1} d_i^{2\alpha} \right) \frac{d_{n-1}^{\alpha}}{h^n} = h^n \]
and
\[ \rho \geq \frac{\rho}{\alpha} \leq \frac{\alpha}{\rho} \leq \frac{\rho}{\alpha}. \]
(iii) Denote $\tilde{\Omega}_h := A_h \Omega$ and $\tilde{G}_h := \partial \tilde{S}_h \cap \{ \tilde{u} < h \}$, then $\tilde{G}_h$ is a graph i.e.
\[ \tilde{G}_h = (x', \tilde{g}_h(x')) \quad \text{with} \quad \tilde{g}_h(x') \leq \frac{2}{\rho} \left| x' \right|^2 \]
and the function $\tilde{u}$ satisfies on $\tilde{G}_h$
\[ \frac{\mu}{2} \left| x' \right|^2 \leq \tilde{u}(x) \leq 2\mu^{-1}\left| x' \right|^2. \]

Proof. Let
\[ v := \mu \left| x' \right|^2 + \frac{\Lambda}{(2-\alpha)(1-\alpha)\mu^{n-1}} x_n^{2\alpha} - C(\rho) x_n, \]
where $C(\rho)$ is large such that
\[ \frac{\Lambda}{(2-\alpha)(1-\alpha)\mu^{n-1}} x_n^{2\alpha} - C(\rho) x_n \leq 0 \quad \text{in } \Omega \cap \{ x_n \leq \rho \}, \]
then it is straightforward to check that $v$ is a lower barrier for $u$ in $\Omega \cap \{ x_n \leq \rho \}$. It follows that
\[ (3.9) \quad \tilde{S}_h \cap \{ x_n \leq \rho \} \subset \{ v < h \} \subset \{ x_n > c(\rho)(\mu \left| x' \right|^2 - h) \}. \]
Let $x_h^*$ be the center of mass of $S_h$ and $d_h := x_h^* \cdot e_n$. We claim that
\[ (3.10) \quad d_h \geq c(\rho)h^{\frac{\alpha}{n-1}} \]
for some small $c(\rho) > 0$.
Indeed, if
\[ d_h \geq c(n)\rho \]
with $c(n)$ depending only on $n$, then (3.10) holds clearly. On the other hand, if
\[ d_h \leq c(n)\rho, \]
then by John’s lemma, for some constant $C(n)$ depending only on $n$ we have
\[ \tilde{S}_h \subset \left\{ x_n \leq C(n) d_h \leq \frac{\rho}{2} \right\} \]
if $c(n)$ is small. If (3.10) does not hold, then from the last estimate, (3.9) and John’s lemma that
\[ \tilde{S}_h \subset \left\{ x_n \leq C(n) c(\rho) h^{\frac{\alpha}{n-1}} \right\} \cap \left\{ \left| x' \right| \leq C_1(\rho) h^{\frac{n-1}{n-\alpha}} \right\}. \]
Define
\[ w = \epsilon x_n + \frac{h}{2} \left( \frac{\left| x' \right|}{C(\rho) h^{\alpha(n-1)}} \right)^2 + \frac{\Lambda |C(\rho)|^{2(n-1)} h}{\left( \frac{\left| x' \right|}{\left( \frac{\left| x' \right|}{\left| x' \right|} \right)^{2-\alpha}} \right)^{\frac{n-1}{n-\alpha}}}. \]
Then we have in $S_h$,
\[ w \leq \epsilon + \frac{h}{2} + \frac{\Lambda |C(\rho)|^{2(n-1)} h}{\left( \frac{\left| x' \right|}{\left( \frac{\left| x' \right|}{\left| x' \right|} \right)^{2-\alpha}} \right)^{\frac{n-1}{n-\alpha}}} \leq h \]
if \( c_0(\rho) \) is small. On \( S_h \cap \partial \Omega \),
\[
w \leq \frac{\epsilon}{\rho} |x'|^2 + \frac{h^{\frac{1-n}{2}}}{2C_1(\rho)^2} |x'|^2 + \frac{\Lambda |C_1(\rho)|^{2(n-1)} h^{\frac{1-n}{2}}}{(2-\alpha)(1-\alpha)} \cdot \frac{|x'|^2}{\rho} \leq \mu |x'|^2
\]
if \( h \leq c(\rho) \). In conclusion,
\[
w \leq u \quad \text{in } S_h,
\]
which contradicts that \( \nabla u(0) = 0 \). Thus (3.10) holds.

Now we prove that for all small \( h \) we have
\[
d_h \leq C_0 h^{-\frac{\alpha}{n}}
\]
for some large constant \( C_0 \).

Assume by contradiction that \( d_h \geq C_0 h^{-\frac{\alpha}{n}} \). Then (3.9) implies that
\[
|\langle x^*_h \rangle'| \leq C(\rho) d_h^{\frac{\alpha}{n}}.
\]
From (2.1) and (2.6) we know that if \( h \leq c(\rho) \), then \( S_h \) contains the set \( \partial \Omega \cap \{ x_n \leq \rho \} \cap \{ x : |x'| \leq (ch)^{\frac{1}{2}} \} \)
for some small \( c \) depending only on \( \mu \). Therefore \( S_h \) contains the convex set generated by \( \partial \Omega \cap \{ x_n \leq \rho \} \cap \{ x : |x'| \leq (ch)^{\frac{1}{2}} \} \) and the point \( x^*_h \). Let \( x_n = b \) be a hyperplane in \( \mathbb{R}^n \), where \( b \leq \rho \) is chosen such that
\[
ch + (b - \rho)^2 = \rho^2.
\]
For each \( x_0 \in \partial \Omega \cap \{ x_n \leq \rho \} \cap \{ x : |x'| = (ch)^{\frac{1}{2}} \} \), let \( y_0 \) be the intersection of the segment \( x_0 x^*_h \) (which is the segment joining \( x_0 \) and \( x^*_h \)) and the hyperplane \( x_n = b \). We can write
\[
y_0 = (1 - \theta)x_0 + \theta x^*_h
\]
for some \( \theta = \theta(x_0) \in (0,1) \). Since
\[
(1 - \theta)x_0 \cdot e_n + \theta d_h = y_0 \cdot e_n = b \leq \frac{ch}{\rho},
\]
we obtain
\[
\theta \leq \frac{ch}{\rho d_h}.
\]
Recall that \( d_h \geq C_0 h^{-\frac{\alpha}{n}} \), then by (3.12) we obtain that for all small \( h \)
\[
|y_0| = |(1 - \theta)x_0' + \theta(x^*_h)| \geq |x_0'| - \theta (\langle |x^*_h|' \rangle + |x_0'|) \geq (ch)^{\frac{1}{2}} - \frac{ch}{\rho d_h} \left( C(\rho) d_h^{\frac{\alpha}{n}} + (ch)^{\frac{1}{2}} \right) \geq \frac{(ch)^{\frac{1}{2}}}{2}.
\]
Since \( S_h \) contains the convex set generated by all such \( y_0 \) and \( x^*_h \), this means that \( S_h \) contains a convex set of measure \( c(n) \left( \frac{(ch)^{\frac{1}{2}}}{2} \right)^{n-1} d_h \), and therefore
\[
|S_h| \geq c(n) \left( \frac{(ch)^{\frac{1}{2}}}{2} \right)^{n-1} d_h.
\]
Let \( v \) solves
\[
\det D^2 v = \lambda(C(n) d_h)^{-\alpha} \leq \det D^2 u \quad \text{in } S_h, \quad v = h \quad \text{on } \partial S_h.
\]
Then
\[
v \geq u \geq 0 \quad \text{in } S_h,
\]
It follows
\[
h^n \geq |h - \min_{S_h} v|^n \geq c(n, \alpha) \lambda d_h^{-\alpha} |S_h|^2.
\]
Namely,
\[
d_h^{-\alpha} |S_h|^2 \leq C(n, \lambda, \alpha) h^n.
\]
It follows from (3.13) and (3.14) that
\[
d_h \leq Ch^{\frac{1}{1-\alpha}}.
\]
We reach a contradiction if \( C_0 \) is sufficiently large, hence (3.11) is proved.
Define
\[
A_h x = x - \nu x_n, \quad \nu = \frac{(x^*_h)' \rho}{d_h}
\]
and \[ \tilde{u}(A_h x) = u(x). \]

Then the center of mass of \( \tilde{S}_h = A_h S_h \) is
\[ \tilde{x}_h^* = A_h x_h^* \]
and lies on the \( x_n \)-axis from the definition of \( A_h \). We obtain from (3.9) and (3.10) that
\[ (3.15) \quad |\nu| = \frac{|(x_n^* )'|}{d_h} \leq C(\rho) d_h^{-\frac{1}{2}} \leq C(\rho) h^{-\frac{1}{2(n-1)}}. \]

Part (i) of Proposition 3.2 follows.

Let \( \tilde{\Omega}_h := A_h \Omega \) and \( \tilde{G}_h := \tilde{\partial S}_h \cap \partial \tilde{\Omega}_h = \tilde{\partial S}_h \cap \{ \tilde{u} < h \} \).

On \( \partial \tilde{\Omega} \cap \{ x_n \leq \rho \} \cap \{|x'| \leq (\mu h)^\frac{1}{2} \} \), we have
\[ |A_h x - x| = |\nu| x_n \leq C(\rho) h^{-\frac{1}{2(n-1)}} |x'|^2 \leq C(\rho) h^{-\frac{1}{2(n-1)}} |x'|. \]

Note that
\[ \partial \tilde{S}_h \cap \partial \Omega \subset \{ x_n \leq \rho \} \cap \{|x'| \leq (\mu h)^\frac{1}{2} \}, \]
thus on \( \tilde{G}_h = \tilde{\partial S}_h \cap \partial \tilde{\Omega}_h \),
\[ x_n \leq \frac{1}{\rho} |(A_h^{-1} x')'| \leq \frac{2}{\rho} |x'|^2 \]
and
\[ \frac{\mu}{2} |x'|^2 \leq \tilde{u}(x) = u(A_h^{-1} x) \leq \mu^{-1} |(A_h^{-1} x)|^2 \leq 2\mu^{-1} |x'|^2. \]

It remains to prove (ii). After a rotation of \( x_1, \ldots, x_{n-1} \) variables, we can assume that \( \tilde{S}_h \cap \{ x_n = d_h \} \) is equivalent to an ellipsoid of axes \( d_1 \leq d_2 \leq \cdots \leq d_{n-1} \) i.e.
\[ \left\{ \sum_{i=1}^{n-1} \frac{x_i}{d_i}^2 \leq 1 \right\} \cap \{ x_n = d_h \} \subset \tilde{S}_h \cap \{ x_n = d_h \} \subset \left\{ \sum_{i=1}^{n-1} \left( \frac{x_i}{d_i} \right)^2 \leq C(\rho) \right\}. \]

Thus,
\[ \tilde{S}_h \subset \left\{ \sum_{i=1}^{n-1} \frac{x_i}{d_i}^2 \leq C(n) \right\} \cap \{ 0 \leq x_n \leq C(n) d_h \}. \]

Since \( \tilde{u} \leq 2\mu^{-1} |x'|^2 \) on \( \tilde{G}_h \), we see that the domain of definition of \( \tilde{G}_h \) contains a ball in \( \mathbb{R}^{n-1} \) of radius \( (\mu h/2)^\frac{1}{2} \). This implies that
\[ (3.16) \quad d_i \geq c_1 h^{\frac{1}{2}}, \quad i = 1, \ldots, n-1. \]

Now we prove that
\[ (3.17) \quad d_h^{2-\alpha} \prod_{i=1}^{n-1} d_i^2 \geq c_2 h^\alpha. \]

Indeed, if the last estimate does not hold, then we construct
\[ w := c x_n + \left[ \sum_{i=1}^{n-1} \left( \frac{x_i}{d_i} \right)^2 + \left( \frac{x_n}{d_h} \right)^{2-\alpha} \right] \cdot c h. \]

If \( c_2 \) is small, then we have
\[ \det D^2 w \geq \frac{c^2 \prod_{i=1}^{n-1} d_i^2}{c_2} \geq \frac{c^2}{c_2} d_h^{n-\alpha} \geq \Lambda x_n^{-\alpha}. \]

On \( \partial \tilde{S}_h \setminus \tilde{G}_h \),
\[ w \leq c + C(n, \alpha) c h \leq h, \]
and on \( \tilde{G}_h \), we use (3.16) and (3.10) to obtain
\[ w \leq \frac{2 c}{\rho} |x'|^2 + c h \sum_{i=1}^{n-1} \left( \frac{x_i}{d_i} \right)^2 + c h C(n)^{1-\alpha} \frac{2}{\rho d_h} \leq \frac{\mu}{2} |x'|^2 \]
if \( c \) is small. We conclude that \( w \leq \tilde{u} \) in \( \tilde{S}_h \). This contradicts \( \nabla \tilde{u}(0) = 0 \) and therefore (3.17) holds.

Since \( \tilde{S}_h \) contains the convex set generated by \( \left\{ \sum_{i=1}^{n-1} \left( \frac{x_i}{d_i} \right)^2 \leq 1 \right\} \cap \{ x_n = d_h \} \) and the point 0, we have
\[ |\tilde{S}_h| \geq c(n) \left( \prod_{i=1}^{n-1} d_i \right) \cdot d_h. \]
This together with (3.17), (3.14) implies that

\[(3.18) \quad Ch^n \geq \delta_h^{-\alpha} |\tilde{S}_h|^2 \geq c(n)\delta_h^{2-\alpha} \prod_{i=1}^{n-1} d_i \geq ch^n.\]

Define \(d_n\) from \(d_1, \ldots, d_{n-1}\) by (3.8), and (3.18) gives

\[cd_h \leq d_n \leq Cd_h.\]

This proves \((ii)\). \(\square\)

Theorem 2.2 follows from Proposition 3.2 and the following result.

**Lemma 3.1.** Assume that \(\Omega\) and \(u\) satisfy the hypotheses of Theorem 2.2. Then for any \(h \leq c(\rho)\), we have

\[(3.19) \quad d_n \geq ch^{-\varpi}.\]

Lemma 3.1 implies Theorem 2.2.

From Lemma 3.1 and Proposition 3.2 we obtain

\[ch^{\frac{\varpi}{\mu}} \leq d_i \leq Ch^{\frac{\varpi}{\mu}}, \quad i = 1, \ldots, n-1, \quad ch^{\frac{\varpi}{\mu}} \leq d_n \leq Ch^{\frac{\varpi}{\mu}}.\]

It follows that

\[(3.20) \quad \tilde{x}_h + cF_hB_1 \subset A_hS_h \subset CF_hB_1,\]

where we recall from Section 2 that

\[F_h x = (h^{\frac{1}{2}} x', h^{\frac{\varpi}{\alpha}} x_n).\]

Since the domain of definition of \(\tilde{G}_h\) contains a ball of radius \((\mu h/2)^\frac{1}{2}\), we have

\[(3.21) \quad cF_hB_1 \cap \Omega \subset A_hS_h \subset CF_hB_1.\]

It follows that

\[(3.22) \quad c\mathcal{E}_h \cap \Omega \subset A_hS_h \subset C\mathcal{E}_h.\]

Denote \(A_h x = x - \nu_h x_n\). Using in (3.20) that \(S_{h/2} \subset S_h\) we find

\[|\nu_{h/2} - \nu_h| \leq Ch^{\frac{\varpi}{\mu} - \frac{\varpi}{\mu}}, \quad \forall h \leq c(\rho),\]

which gives

\[(3.23) \quad |\nu_h| \leq C(\rho)h^{\frac{\varpi}{\mu} - \frac{\varpi}{\mu}}, \quad \forall h \leq c(\rho).\]

This easily implies that

\[(3.24) \quad \mathcal{E}_{c_1(\rho)h} \subset A_h^{-1} \mathcal{E}_h \subset \mathcal{E}_{c_1(\rho)h}\]

for some constants \(c_1(\rho), C_1(\rho) > 0\).

The conclusion of Theorem 2.2 follows from (3.22) and (3.24).

In order to prove Lemma 3.1, we modify the definition of the quantity \(b_u(h)\) in [12].

Fix \(\alpha \in (0, 1)\). Given a convex function \(u\) we define

\[(3.25) \quad b_u(h) = h^{-\frac{1}{\alpha}} \sup_{S_h} x_n.\]

Whenever there is no possibility of confusion we drop the subindex \(u\) and write for simplicity \(b(h)\).

\(b(h)\) satisfies the following properties which are slightly different from those in [12].

1) If \(h_1 \leq h_2\), then

\[\left( h_1 \right)^{\frac{1}{\alpha}} \leq b(h_1) \leq \left( h_2 \right)^{\frac{1}{\alpha}}.\]

2) If \(A\) is a linear transformation which leaves the \(x_n\)-coordinate invariant and

\[\tilde{u}(Ax) = u(x),\]

then

\[b_{\tilde{u}}(h) = b_u(h).\]

3) If \(A\) is a linear transformation which leaves the plane \(\{x_n = 0\}\) invariant, then

\[\frac{b_{\tilde{u}}(h_1)}{b_{\tilde{u}}(h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.\]
4) If 

$$\tilde{u}(x) = \beta u(x)$$

with $\beta$ a positive constant, then

$$b_\beta(\beta h) = \beta \frac{b_\beta(h)}{b_\beta(h_1)}$$

and therefore

$$\frac{b_\beta(\beta h_1)}{b_\beta(\beta h_2)} = \frac{b_\beta(h_1)}{b_\beta(h_2)}$$

From part (ii) of Proposition 3.2 we know that

$$cd_n \leq d_h = x_1^* \cdot e_n \leq Cd_n,$$

and it follows that

$$cd_n \leq b_\beta(h) \frac{1}{h} = \sup_{\partial S_n} x_n \leq Cd_n.$$ 

Thus Lemma 3.1 will follow if we show that $b_\beta(h)$ is bounded below. This will follow from property 1) above and the following lemma.

**Lemma 3.2.** If $h \leq c(\rho)$ and $b_\beta(h) \leq c_0$, then

$$\frac{b_\beta(h)}{b_\beta(h)} > 2$$

for some $t \in [c_0, 1]$.

In order to prove Lemma 3.2, we recall the function $\tilde{u}$, the section $S_{th}$ and the matrix $D_h$ in Proposition 3.2. Define

$$v(x) = \frac{1}{h} \tilde{u}(D_h x) = \frac{1}{h} u(A_h^{-1} D_h x).$$

The section $S_1(v) = \{ v < 1 \} = D_h^{-1} A_h S_{th}$ satisfies

$$x^* + cB_1 \subset S_1(v) \subset C B_1$$

with $x^*$ the center of mass of $S_1(v)$. The function $v$ satisfies in $S_1(v)$

$$\lambda x_n^{-\alpha} \leq \det D^2 v(x) = d_n^\alpha \det D^2 u(A_h^{-1} D_h x) \leq \Lambda x_n^{-\alpha}$$

and

$$v(0) = 0, \quad 0 \leq v \leq 1.$$ 

Moreover, let $0 < t \leq 1$, $x^*_t(v)$ and $x^*_{th}$ be the centers of mass of $S_t(v)$ and $S_{th}(u)$ respectively, and $d_{th} = x_{th} \cdot e_n$. Then

$$(x^*_t(v) \cdot e_n)^{-\alpha} |S_t(v)|^2 = d_{th}^\alpha |S_{th}(u)|^2.$$

Since Proposition 3.2 implies that $c(t h)^n \leq d_{th}^\alpha |S_{th}(u)|^2 \leq C(t h)^n$, we obtain

$$ct^n \leq (x^*_t(v) \cdot e_n)^{-\alpha} |S_t(v)|^2 \leq Ct^n.$$ 

From the convexity of $u$ we have

$$x^*_t(v) \cdot e_n = \frac{d_{th}}{d_n} \geq \frac{c d_{th}}{d_n} \geq \frac{c \sup_{S_{th}(u)} x_n}{\sup_{S_n} x_n} \geq ct.$$

Let $G_v := \partial S_1(v) \cap \{ v < 1 \}$. We claim that

$$G_v \subset \{ x_n \leq \sigma \}, \quad \sigma = C(\rho) h^{\frac{1}{\alpha + 1}}.$$ 

Indeed, for $x \in G_v = D_h^{-1} \tilde{G}_v$,

$$d_n x_n \leq \frac{2}{\rho} |D_h x'|^2 \leq C(\rho) h,$$

which gives

$$x_n \leq C(\rho) h^{1 - \frac{1}{\alpha + 1}} = \sigma$$

by (3.10). Thus the claim follows.

We also have

$$v = 1 \quad \text{on} \partial S_1(v) \setminus G_v.$$ 

On $G_v$,

$$\mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq v(x) \leq \mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2,$$
where
\[ a_i = \frac{d_i}{h^2} \geq c_1, \quad i = 1, \ldots, n - 1 \]
by (3.16).

In order to prove Lemma 3.2, we only need to show that there exists constants \( c(\rho), c_0 \) small and \( C \) sufficiently large such that if \( h \leq \varepsilon(\rho) \) and \( \max_{1 \leq i \leq n - 1} a_i \geq C \), then the rescaled function \( v \) satisfies
\[ b_v(t) \geq 2h(1) \]
for some \( t \in [c_0, 1] \).

4. Proof of Theorem 2.2 (II)

We consider the class of solutions \( v \) that satisfy the properties above. After relabeling the constants \( \mu \) and \( a_i \), and by abuse of notation writing \( u \) instead of \( v \), we may assume we are in the following case.

Fix \( \mu, \lambda, \Lambda \) and \( \alpha \in (0, 1) \). For an increasing sequence
\[ a_1 \leq a_2 \leq \cdots \leq a_{n-1} \]
we consider the family of solutions
\[ u \in \mathcal{D}_\sigma^\mu(a_1, a_2, \ldots, a_{n-1}) \]
of convex functions \( u : \Omega \to \mathbb{R} \) that satisfy
\[
(4.1) \quad \lambda x_n^{-\alpha} \leq \det D^2 u \leq \Lambda x_n^{-\alpha}, \quad 0 \leq v \leq 1 \quad \text{in } \Omega;
\]
\[
(4.2) \quad 0 \in \partial \Omega, \quad B_{\mu}(x_0) \subset \Omega \subset B_{1/\mu}^+;
\]
\[
(4.3) \quad \mu h^\alpha \leq (x_k^* \cdot e_n)^{-\alpha}|S_k|^2 \leq \mu h^{-\alpha}, \quad x_k^* \cdot e_n \geq \mu h
\]
with \( x_k^* \) the center of mass of \( S_k \).

Moreover, there exists a closed set \( G \subset \partial \Omega \) such that
\[
(4.4) \quad G \subset \partial \Omega \cap \{ x_n \leq \sigma \},
\]
and \( G \) is a graph in the \( e_n \) direction with projection \( \pi(G) \) along \( e_n \).

The boundary values of \( u = \varphi \) on \( \partial \Omega \) satisfy
\[
(4.5) \quad \{ \mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq 1 \} \subset \pi(G) \subset \{ \mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq 1 \}.
\]

The boundary values of \( u = \varphi \) on \( \partial \Omega \) satisfy
\[
(4.6) \quad \varphi = 1 \quad \text{on } \partial \Omega \setminus G,
\]
and
\[
(4.7) \quad \mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq \varphi(x) \leq \min \{ 1, \mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2 \} \quad \text{on } G.
\]

As explained in [12] (see Page 79 there), Property (3.26) is a corollary of the following proposition.

**Proposition 4.1.** For any \( M > 0 \) there exists \( C_* \) depending only on \( M, n, \mu, \lambda, \Lambda \) and \( \alpha \) such that if \( u \in \mathcal{D}_\sigma^\mu(a_1, a_2, \ldots, a_{n-1}) \) with
\[
a_{n-1} \geq C_* \quad \text{and} \quad \sigma \leq C_*^{-1}
\]
then
\[
b(h) = (\sup_{S_n} x_n)h^{-\frac{1}{\alpha - \mu}} \geq M
\]
for some \( h \in [C_*^{-1}, 1] \).

We prove Proposition 4.1 by compactness as in [12]. We introduce the limiting solutions from the class
\[ \mathcal{D}_\sigma^\mu(a_1, \ldots, a_{n-1}) \]
when \( a_{k+1} \to \infty \) and \( \sigma \to 0 \).

For an increasing sequence
\[ a_1 \leq a_2 \leq \cdots \leq a_k \]
with
\[ a_1 \geq \mu, \]
we denote by
\[ \mathcal{D}_\sigma^\mu(a_1, \ldots, a_k, \infty, \ldots, \infty), \quad 0 \leq k \leq n - 2, \]
the class of functions \( u \) that satisfy
\[
\lambda x_n^{-\alpha} \leq \det D^2 u \leq \Lambda x_n^{-\alpha}, \quad 0 \leq u \leq 1 \quad \text{in } \Omega;
\]
\[
0 \in \partial \Omega, \quad B_{\mu}(x_0) \subset \Omega \subset B_{1/\mu}^+;
\]
\[
\mu h^\alpha \leq (x_h^* \cdot e_n)^{-\alpha}|S_h|^2 \leq \mu^{-1} h^\alpha, \quad x_h^* \cdot e_n \geq \mu h,
\]
where \( x_h^* \) is the center of mass of \( S_h \). There exists a closed set \( G \) such that
\[
G \subset \partial \Omega \cap \{ x_i = 0, i > k \}.
\]
If we restrict to the space generated by the first \( k \) coordinates, then
\[
\{ \mu^{-1} \sum_{i=1}^k a_i^2 x_i^2 \leq 1 \} \subset G \subset \{ \mu \sum_{i=1}^k a_i^2 x_i^2 \leq 1 \}.
\]
The boundary values of \( u = \varphi \) on \( \partial \Omega \) satisfy
\[
\varphi = 1 \quad \text{on } \partial \Omega \setminus G,
\]
and
\[
\mu \sum_{i=1}^k a_i^2 x_i^2 \leq \varphi(x) \leq \min\{1, \mu^{-1} \sum_{i=1}^k a_i^2 x_i^2 \} \quad \text{on } G.
\]

As in [12], Proposition 4.1 will follow from the proposition below.

**Proposition 4.2.** For any \( M > 0 \) and \( 0 \leq k \leq n - 2 \) there exists \( c_k \) depending only on \( M, k, n, \mu, \lambda, \Lambda \) and \( \alpha \) such that if
\[
u \in D_0^0(a_1, \ldots, a_k, \infty, \ldots, \infty),
\]
then
\[
b(h) = (\sup_{S_h} x_n) h^{-\frac{1}{\alpha}} \geq M
\]
for some \( h \in [c_k, 1] \).

To prove the above proposition, we use the notation introduced in [12]. Denote
\[
x = (y, z, x_n), \quad y = (x_1, \ldots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \ldots, x_{n-1}) \in \mathbb{R}^{n-1-k}.
\]
A sliding along the \( y \) direction is defined as follows:
\[
T_x := x + \nu_1 z_1 + \nu_2 z_2 + \cdots + \nu_{n-k-1} z_{n-k-1} + \nu_{n-k} x_n
\]
with
\[
\nu_1, \nu_2, \ldots, \nu_{n-k} \in \text{span}\{ e_1, \ldots, e_k \}.
\]

**Lemma 4.1.** Assume that
\[
u \geq p(\|z| - q x_n)
\]
for some \( p, q > 0, q \leq q_0 \) and assume that for each section \( S_h \) of \( u, h \in (0, 1) \), there exists \( T_h \), a sliding along the \( y \) direction, such that
\[
T_h S_h \subset C_0 F_h B_1^+\n\]
forsome constant \( C_0 \). Then
\[
u \notin D_0^0(1, \ldots, 1, \infty, \ldots, \infty).
\]

**Proof.** Assume by contradiction that \( u \in D_0^0(1, \ldots, 1, \infty, \ldots, \infty) \). We will show that
\[
u \geq p'(|z| - q' x_n), \quad q' = q - \eta,
\]
where \( \eta > 0 \) depends only on \( q_0, C_0, \lambda, \mu, n, \alpha \) and \( 0 < p' \ll p \).

Apply this result a finite number of times we obtain
\[
u \geq \epsilon(|z| + x_n)
\]
for some \( \epsilon > 0 \) small. Thus we obtain \( S_h \subset \{ x_n \leq \epsilon^{-1} h \} \) and it follows that
\[
T_h S_h \subset \{ x_n \leq \epsilon^{-1} h \}.
\]
This together with the hypothesis of the lemma and (4.10) in the definition of the class \( D_0^0 \) implies that
\[
\mu h^\alpha \leq (x_h^* \cdot e_n)^{-\alpha}|S_h|^2 \leq \mu^{-1} h^\alpha |T_h S_h|^2 \leq C h^{n+1-\alpha},
\]
where \( C \) is a constant depending only on \( \epsilon, C_0, n, \mu \) and \( \alpha \). This is a contradiction as \( h \to 0 \).

It remains to prove (4.15). Since \( u \in D_0^0(1, \ldots, 1, \infty, \ldots, \infty) \), there is a closed set
\[
G_h \subset \partial S_h \cap \{ (z, x_n) = 0 \}
\]
such that when we restrict to the subspace \( \{(z, x_n) = 0\} \),
\[
\{\mu^{-1}|y|^2 \leq h\} \subset G_h \subset \{\mu|y|^2 \leq h\},
\]
and the boundary values \( \varphi_h \) of \( u \) on \( \partial S_h \) satisfy
\[
\varphi_h = h \quad \text{on } \partial S_h \setminus G_h;
\]
\[
\mu|y|^2 \leq \varphi_h \leq \min\{h, \mu^{-1}|y|^2\} \quad \text{on } G_h.
\]
Define
\[
w(x) = \frac{1}{h} u(T_h^{-1} F_h x).
\]
Then
\[
S_1(w) = F_h^{-1} T_h S_h \subset C_0 B^+_1,
\]
and
\[
\lambda x_n^{-\alpha} \leq \det D^2 w \leq \Lambda x_n^{-\alpha} \quad \text{in } S_1(w).
\]
Also,
\[
w(x) \geq \frac{1}{h^n} p(|h^\frac{1}{2} x|) - q h^{\frac{1}{2n} - \alpha} x_n = \frac{p}{h^\frac{1}{2}} (|z| - q h^{\frac{1}{2n} - \alpha} x_n).
\]
Moreover, the boundary values \( \varphi_w \) of \( w \) on \( \partial S_1(w) \) satisfy
\[
\varphi_w = 1 \quad \text{on } \partial S_1(w) \setminus G_w;
\]
\[
\mu|y|^2 \leq \varphi_w \leq \min\{1, \mu^{-1}|y|^2\} \quad \text{on } G_w = F_h^{-1} G_h.
\]
Define
\[
v := \delta \left( |x'|^2 + \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)} \right) + \frac{\Lambda}{\delta^{n-1}} \left( z_1 - q h^{\frac{1}{2n} - \alpha} x_n \right)^2 + N \left( z_1 - q h^{\frac{1}{2n} - \alpha} x_n \right) + \delta h^{\frac{1}{2n} - \alpha} x_n,
\]
where \( \delta \) is small depending only on \( \mu, C_0, \alpha \) and \( N \) is large such that
\[
\frac{\Lambda}{\delta^{n-1}} t^2 + N t
\]
is increasing in the interval \(|t| \leq (1 + q_0)C_0\).

By straightforward computation and similar arguments to the proof of [12, Lemma 5.4], we find that \( v \) is a lower barrier for \( w \) in \( S_1(w) \), which implies
\[
w \geq N \left( z_1 - q h^{\frac{1}{2n} - \alpha} x_n \right) + \delta h^{\frac{1}{2n} - \alpha} x_n \quad \text{in } S_1(w).
\]
Since this inequality holds for all directions in the \( z \)-plane, we obtain
\[
w \geq N \left[ |z| - \left( q - \frac{\delta}{N} \right) h^{\frac{1}{2n} - \alpha} x_n \right] .
\]
Back to \( u \) we have
\[
\frac{u(x) - h w(F_h^{-1} T_h x)}{h} \geq h^\frac{1}{2} N \left[ |z| - \left( q - \frac{\delta}{N} \right) x_n \right] \quad \text{in } S_h.
\]
From the convexity of \( u \) and \( u(0) = 0 \), we know that this inequality holds in \( \Omega \) and therefore (4.15) is proved.
\[
\square
\]
Now we give the proof of Proposition 4.2.

\( k = 0 \): Assume Proposition 4.2 is not true, then by compactness, there exist \( M > 0 \) and \( u \in D^\mu_0(\infty, \ldots, \infty) \) such that \( b(h) \leq M \) for any \( 0 < h \leq 1 \). Let
\[
v := \delta \left( |x'| + \frac{1}{2} |x'|^2 \right) + \frac{\Lambda}{\delta^{n-1}(2-\alpha)(1-\alpha)} x_n^{2-\alpha} - N x_n,
\]
where \( \delta \) is small depending only on \( \mu \) and \( N \) is large such that
\[
\frac{\Lambda}{\delta^{n-1}(2-\alpha)(1-\alpha)} x_n^{2-\alpha} - N x_n \leq 0
\]
in \( B^+_1 \). It is easily seen that
\[
v \leq u \quad \text{in } \Omega.
\]
It follows that
\[
u \geq \delta |x'| - N x_n
\]
and then
\[
S_h \subset \{|x'| \leq \delta^{-1}(N x_n + h)\}.
\]
Since \( b(h) \leq M \) implies that \( x_n \leq M h^{\frac{1}{\alpha}} \), we obtain
\[
S_h \subset \{ |x'| \leq Ch^{\frac{1}{2}}, x_n \leq M h^{\frac{1}{\alpha}} \},
\]
where \( C \) is a constant depending only on \( M, \mu, \Lambda \) and \( \alpha \). This contradicts Lemma 4.1 and therefore Proposition 4.2 is true for \( k = 0 \).

**Assume Proposition 4.2 holds for \( 0, 1, \ldots, k - 1, 1 \leq k \leq n - 2 \), and now we prove it for \( k \).**

By the induction hypothesis, it suffices to consider the case \( a_k \leq C_k \), where \( C_k \) is a constant depending only on \( M, k, n, \mu, \lambda, \Lambda \) and \( \alpha \). Assume in contradiction that no \( c_k \) exists, then we can find a limiting solution \( u \) such that
\[
u \in D^0_{\mu}(1, \ldots, 1, \infty, \ldots, \infty)
\]
with
\[
b(h) \leq M, \quad \forall h > 0,
\]
where \( \bar{\mu} \) depends only on \( \mu \) and \( C_k \).

Denote as before
\[
x = (y, z, x_n), \quad y = (x_1, \ldots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \ldots, x_{n-1}) \in \mathbb{R}^{n-1-k}.
\]

Similar to the case \( k = 0 \), the function
\[
v := \delta \left( |z| + 1 \right) + \frac{\Lambda}{\delta^n (2 - \alpha)(1 - \alpha)} x_n^{-\alpha} - N x_n
\]
is a lower barrier for \( u \), where \( \delta \) is small depending only on \( \bar{\mu} \) and \( N \) is large. Therefore,
\[
u \geq \delta |z| - N x_n.
\]
This together with (4.18) implies that
\[
S_h \subset \{ |z| \leq \delta^{-1} (N x_n + h) \} \cap \{ x_n \leq M h^{\frac{1}{\alpha}} \}.
\]

From John’s lemma, there is an ellipsoid \( E_h \) such that
\[
E_h \subset \tilde{S}_h - x_h^\star \subset C(n) E_h
\]
with \( x_h^\star \) the center of mass of \( \tilde{S}_h \). By a fact in linear algebra (see the arguments in [12, Page 83]), there is \( T_h \), a sliding along the \( y \) direction, such that
\[
T_h E_h = |E_h|^{\frac{n}{2}} AB_1,
\]
where, after rotating coordinates in the \((y, 0, 0)\) and \((0, z, 0)\) subspaces, the matrix \( A \) satisfies
\[
A(y, z, x_n) = (A_1 y, A_2 (z, x_n)),
\]
\[
A_1 = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_k \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \gamma_{k+1} & 0 & \cdots & 0 & \theta_{k+1} \\ 0 & \gamma_{k+2} & \cdots & 0 & \theta_{k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{n-1} & \theta_{n-1} \\ 0 & 0 & \cdots & 0 & \theta_n \end{pmatrix}
\]
with
\[
0 < \beta_1 \leq \cdots \leq \beta_k, \quad \gamma_j > 0, \quad \theta_n > 0, \quad \left( \prod_{i=1}^{k} \beta_i \right) \left( \prod_{j=1}^{n-1} \gamma_j \right) \theta_n = 1.
\]

Let
\[
\tilde{u}(x) = u(T_h^{-1} x), \quad \tilde{S}_h = T_h S_h,
\]
then (4.21) implies that
\[
\tilde{x}_h^\star + |E_h|^{\frac{1}{2}} AB_1 \subset \tilde{S}_h \subset C(n) |E_h|^{\frac{1}{2}} AB_1,
\]
where \( \tilde{x}_h^\star \) is the center of mass of \( \tilde{S}_h \).

Since \( u \in D^0_{\mu}(1, \ldots, 1, \infty, \ldots, \infty) \), there exists \( \tilde{G}_h = G_h \),
\[
\tilde{G}_h \subset \{ (z, x_n) = 0 \} \cap \partial \tilde{S}_h
\]
such that on the subspace \( \{ (z, x_n) = 0 \} \),
\[
\{ \tilde{\mu}^{-1} |y|^2 \leq h \} \subset \tilde{G}_h \subset \{ \tilde{\mu} |y|^2 \leq h \},
\]
and the boundary values \( \tilde{\phi}_h \) of \( \tilde{u} \) on \( \partial \tilde{S}_h \) satisfy
\[
\tilde{\phi}_h = h \quad \text{on} \quad \partial \tilde{S}_h \setminus \tilde{G}_h; \quad \text{on} \quad \partial \tilde{S}_h \setminus \tilde{G}_h;
\]
\[ \bar{\mu}|y|^2 \leq \bar{\varphi}_h \leq \min\{h, \bar{\mu}^{-1}|y|^2\} \quad \text{on } \bar{G}_h. \]

For any \( h > 0 \), denote \( d_h := \bar{x}_h^* \cdot e_n \), then
\[ \bar{\mu}^n \leq (\bar{x}_h^* \cdot e_n)^{-\alpha}|\bar{S}_h|^2 \leq d_h^{-\alpha}|S_h|^2 \leq \bar{\mu}^n, \quad \bar{x}_h^* \cdot e_n = d_h \geq \bar{\mu}h. \]

We will show that
\[ |E_h|^2 AB_1 \subset C\text{diag}(h^{\frac{1}{2}}, \ldots, h^{\frac{1}{2}}, h^{\frac{1}{2}} d_h^{-\frac{1}{2}}) B_1, \]
where \( C \) is a constant depending only on \( \mu, M, k, \lambda, \Lambda, n \) and \( \alpha \).

This together with (4.23), (4.18) gives
\[ T_h S_h \subset C\text{diag}(h^{\frac{1}{2}}, \ldots, h^{\frac{1}{2}}, h^{\frac{1}{2}} \bar{\mu}^{-\frac{1}{2}}) B_1. \]

Now we prove (4.25). Let
\[ A = |E_h|^2 h^{-\frac{1}{2}} \text{diag}(h^{-\frac{1}{2}}, \ldots, h^{-\frac{1}{2}}, h^{-\frac{1}{2}} d_h^{-\frac{1}{2}}) A \]
\[ = |E_h|^2 h^{-\frac{1}{2}} \left( \begin{array}{cc} A_1 & \vdots \\ \vdots & A_2 \end{array} \right) = \left( \begin{array}{c} A_1 \\ \bar{A}_2 \end{array} \right). \]

Since \( \bar{G}_h \subset \partial \bar{S}_h \cap \{(z, x_n) \neq 0\} \) contains a ball in \( \mathbb{R}^h \) of radius \( (\bar{\mu}h)^{1/2} \), then from the second inclusion in (4.23) we obtain
\[ \bar{\beta}_i := h^{-\frac{1}{2}}|E_h|^2 \beta_i \geq c, \quad i = 1, \ldots, k, \]
where \( c \) is a constant depending only on \( n \) and \( \bar{\mu} \).

From (4.18) we know that for any \( x = (y, z, x_n) \in \bar{S}_h \) we have
\[ x_n \leq C(n)d_h \leq C(n)(Mh^{-\frac{1}{2}}) d_h^{\frac{1}{2}} \leq C(n, M, \alpha) h^{\frac{1}{2}} d_h^{\frac{1}{2}}, \]
combining this, (4.24) and (4.20) we obtain that
\[ \bar{S}_h \subset \{(z, x_n) \leq Ch^{\frac{1}{2}} d_h^{\frac{1}{2}}\}. \]

This together with the first inclusion in (4.23) implies that \( ||A_2|| \leq C \) and if follows that
\[ \gamma_j := h^{-\frac{1}{2}}|E_h|^2 \gamma_j \leq C, \quad h^{-\frac{1}{2}}|E_h|^2 |\theta_n\bar{\mu}| \leq C, \quad \bar{\theta}_n := h^{-\frac{1}{2}}|E_h|^2 d_h^{-\frac{1}{2}} \theta_n \leq C, \]
where \( C \) is a constant depending only on \( n, \bar{\mu}, \Lambda, \alpha \) and \( M \).

Also, we have by (4.23)
\[ |E_h|^2 \theta_n \leq \bar{x}_h^* \cdot e_n = d_h \leq C(n)|E_h|^2 \theta_n. \]

We define
\[ w(x) := \frac{1}{h}(E_h|^2 A)x, \]
then from (4.23) we know that
\[ B_1(x_0) \subset S_1(w) = |E_h|^2 A^{-1}\bar{S}_h \subset C(n)B_1 \]
for some \( x_0 \), and from (4.24) and (4.29) we find that
\[ \bar{\lambda} x_n^{-\alpha} \leq \det D^2 w \leq \bar{\lambda} x_n^{-\alpha} \]
with \( \bar{\lambda}, \bar{\Lambda} \) depending only on \( \lambda, \Lambda, n, \alpha, \tilde{\mu} \).

Moreover, for \( t > 0 \) let \( x_t^*(w) \) be the center of mass of the section \( S_t(w) \), then
\[ S_t(w) = |E_h|^2 A^{-1} T_h S_{th}(w), \]
and we have by (4.29)
\[ \frac{d_{th}}{d_h} \leq x_t^*(w) \cdot e_n = |E_h|^2 \theta_n^{-1} d_{th} \leq C(n) \frac{d_{th}}{d_h}. \]

Then (4.24) implies that
\[ C t^n \leq (x_t^*(w) \cdot e_n)^{-\alpha} |S_t(w)|^2 \leq C t^n \]
for some constants \( c, C \) depending only on \( n, \alpha \) and \( \bar{\mu} \).

Let \( G_w = \partial S_1(w) \cap \{ w < 1 \} = |E_h|^2 A^{-1} \bar{G}_h \), then the boundary values \( \varphi_w \) of \( w \) satisfy
\[ \varphi_w = 1 \quad \text{on } \partial S_1(w) \setminus G_w. \]
and from the definition of $\beta_i$ we find that

$$\tilde{\mu} \sum_{i=1}^{k} \beta_i^2 y_i^2 \leq \varphi_{\omega} \leq \tilde{\mu}^{-1} \sum_{i=1}^{k} \beta_i^2 y_i^2.$$  

This implies that

$$w \in D_0^s(\tilde{\beta}_1, \ldots, \tilde{\beta}_k, \infty, \ldots, \infty)$$  

for some $\tilde{\mu}$ depending only on $\mu, M, k, \lambda, \Lambda, n$ and $\alpha$.

Note that (4.24) implies that

$$c \leq \left( \prod_{i=1}^{k} \beta_i \right) \left( \prod_{j=k+1}^{n-1} \xi_j \right) \tilde{\theta}_n = h^{-1} \left| E_{h} \right| d_{h}^{-1} \leq C$$

with $c, C$ depending only on $n$ and $\tilde{\mu}$.

We claim

$$\tilde{\theta}_n \geq c_\ast$$

for some small $c_\ast$ to be chosen.

Indeed, if we $c_\ast$ is small, then (4.28) and (4.30) imply that

$$\beta_k \geq C_k(\tilde{\mu}, M, \tilde{\lambda}, \Lambda, n, \alpha)$$

with $M := 2\tilde{\mu}^{-1}$. Then by the induction hypothesis,

$$b_w(\tilde{h}) \geq M \geq 2b_w(1)$$

for some $h > C_k^{-1}$. It follows that

$$\frac{b_w(\tilde{h})}{b_w(h)} = \frac{b_w(\tilde{h})}{b_w(1)} \geq 2,$$

which implies $b_w(\tilde{h}) \geq 2b_w(h)$ for any $h > 0$. This contradicts (4.18) and therefore the claim holds.

Similarly, we obtain that

$$\tilde{\xi}_j \geq \tilde{c}_\ast$$

for some small $\tilde{c}_\ast$.

We obtain from (4.30), (4.31), (4.32) that

$$\beta_i \leq C, \quad i = 1, \ldots, k,$$

where $C$ is a constant depending only on $\mu, M, k, \lambda, \Lambda, n$ and $\alpha$. This implies that $\|A_i\| \leq C$ and therefore $\|A\| \leq C$.

Thus, the estimate (4.25) holds. Then the proof is finished because (4.19), (4.26) and (4.17) contradict Lemma 4.1.

5. PROOF OF THEOREM 1.2

In this section, we always denote by $c, C, c_i, C_i, i \in \mathbb{N}$ constants depending only on $n, c_0$ and $\alpha$ ($c_0$ is the constant in (1.5)). Their values may change from line to line whenever there is no possibility of confusion.

Lemma 5.1. Assume the hypotheses in Theorem 1.2 hold, then for $i = 1, \ldots, n - 1$ we have $u_i \in C(\mathbb{R}^n_t)$.

Proof. We first claim that for some constant $c_1$ small, we have

$$|\nabla u| \leq c_1^{-1}, \quad \text{in} \, B^+_c.$$

Indeed, we note that (1.5) implies that

$$B^+_c \subset S_i(u) \subset B^+_{c_1},$$

for some constant $k$ depending only on $c_0$ and $\alpha$. We can use the convexity of $u$ and obtain an upper bound for $u_n$ and all $|u_i|, 1 \leq i \leq n - 1$, in $B^+_{c_1}$. On the other hand, for any $x_0 \in B^+_c$, the function

$$w(x_0, 0, x) := \frac{1}{2} |x_0|^2 + x_0 \cdot (x' - x_0) + \delta|x'-x_0|^2 + \frac{\delta^{1-n}}{(2-\alpha)(1-\alpha)} (x_0^{2-\alpha} - k^{-1}x_n)$$

is a lower barrier $u$ in $S_i(u)$, where $\delta$ is small depending only on $n, c_0$ and $\alpha$. This together with the convexity of $u$ gives a lower bound for $u_n(x_0)$.

Next we prove that for any $1 \leq i \leq n - 1$, $u_i$ is continuous at any point $x_0 \in \{|x'| \leq c_1/2, x_n = 0\}$. Indeed, fix $1 \leq i \leq n - 1$ and $x_0 \in \{|x'| \leq c_1/2, x_n = 0\}$, define

$$u_{x_0}(x) = u(x_0 + x) - u(x_0) - \nabla u(x_0) \cdot x.$$
We only need to prove that $\partial_i u_{x_0}$ is continuous at 0. Assume there is a sequence $x^{(m)} \to 0, m \to \infty$ with
\[
\partial_i u_{x_0}(x^{(m)}) \geq \epsilon
\]
for some $\epsilon > 0$. We have
\[
u_{x_0} \geq u_{x_0}(x^{(m)}) + \nabla u_{x_0}(x^{(m)}) \cdot (x - x^{(m)}).
\]
Note that $|\nabla u_{x_0}(x^{(m)})|$ is bounded by (5.1). Let $m \to \infty$, we obtain
\[
u_{x_0} \geq a \cdot x
\]
for some $a = (a', a_n) \in \mathbb{R}^n$ with $a_i \geq \epsilon$. From the value of $u_{x_0}$ on the boundary $\{x_n = 0\}$ we find that $a' = 0$. This is a contradiction.

For any $\lambda > 0$, we define
\[
u_{\lambda}(y) := \frac{1}{\lambda} u(F_{\lambda} y),
\]
then $u_{\lambda}$ satisfies (1.5) and (1.6). The results above show that for any $1 \leq i \leq n - 1$, $\partial_i u_{\lambda}$ is continuous on $\{||x|| \leq c_i/2, x_n = 0\}$. Therefore, $u_i$ is continuous on $F_{\lambda}\{||x|| \leq c_i/2, x_n = 0\}$. Let $\lambda \to \infty$ and we conclude that $u_i$ is continuous on $\{x_n = 0\}$. $\Box$

**Proof of Theorem 1.2**: As before we have
\[
(5.2) \quad B_k^+ \subset S_i(u) \subset B_k^{i-1},
\]
and
\[
(5.3) \quad |u_i| \leq C \quad \text{in } B_k^{i/4}, \quad i = 1, \ldots, n - 1,
\]
where $k$ is a constant depending only on $c_0$ and $\alpha$.
Let
\[
L\varphi := \text{tr}[(D^2 u)^{-1} D^2 \varphi]
\]
be the linearized Monge-Ampère operator for $u$. Then for $i = 1, \ldots, n - 1$ we have
\[
Lu = n, \quad u_i = x_i \quad \text{on } \{x_n = 0\},
\]
and if we define $P(x) = \delta|x'|^2 + \delta^{1-n}-(2-a)(2-a)|1-\alpha|$ with $\delta > 0$ a small constant to be chosen, then
\[
LP = \text{tr}[(D^2 u)^{-1} D^2 P] \geq n[\det(D^2 u)^{-1} \det D^2 P]^{1/2} > n.
\]
Let $\gamma_1, \gamma_2$ be large constants to be chosen and define
\[
v^+(x) := x_i + \gamma_1 \left[\delta|x|^2 + \delta^{1-n}\left(\frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)} - \gamma_2 x_n\right) - u(x)\right].
\]
We have
\[
Lv^- = \gamma_1 [LP - Lu] > 0.
\]
On $\partial B_{k/4}^+ \cap \{x_n = 0\}$, we choose $\delta \leq 1/2$ and obtain
\[
v^- = x_i + \gamma_1 \left[\delta|x|^2 - \frac{1}{2} |x|^2\right] \leq x_i.
\]
We choose $\gamma_2$ large such that
\[
\frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)} - \gamma_2 x_n \leq 0 \quad \text{in } B_{k/4}^+.
\]
Then on $\partial B_{k/4}^+ \cap \{x_n > 0\}$, we use (1.5) and obtain
\[
v^- \leq x_i + \gamma_1 \left[\delta|x|^2 - c_0|x|^2 - c_0 x_n^{2-\alpha}\right] \leq x_i - \frac{\gamma_1 c_0}{2}(|x|^2 + x_n^{2-\alpha}) \leq -C,
\]
where $C$ is the constant in (5.3), and we choose $\delta \leq c_0/2$ and $\gamma_1$ large.

By Lemma 5.1, $u_i \in C(B_{k/4}^+)$ and therefore the maximum principle for linear elliptic equations implies that
\[
v^- \leq u_i \quad \text{in } B_{k/4}^+.
\]
Similarly,
\[
v^+ \geq u_i \quad \text{in } B_{k/4}^+.
\]
Therefore,
\[
(5.4) \quad |u_i - x_i| \leq \gamma_1 [\delta^{1-n} \gamma_2 x_n + u] \quad \text{in } B_{k/4}^+.
\]
For any $\lambda > 0$, we define
\[ u_\lambda(y) := \frac{1}{\lambda} u(F_\lambda y), \]
then $u_\lambda$ satisfies (1.5) and (1.6).

Apply (5.4) with $u \sim u_\lambda$ and we obtain
\[ |\partial_i u_\lambda(y) - y_i| \leq \gamma_1 |\delta^{1-n} \gamma_2 \rho_n + u_\lambda(y)| \quad \text{in } B^+_{k/4}. \]

Back to $u$ we have
\[ |u_i(x) - x_i| \leq \gamma_1 \left[ \delta^{1-n} \gamma_2 \frac{1}{c} x_n + \lambda^{\frac{1}{c}} u(x) \right] \quad \text{in } F_\lambda B^+_{k/4}. \]

Let $\lambda \to \infty$, we obtain
\[ u_i = x_i, \quad \forall x \in \mathbb{R}^n_+. \]

For any $x = (x', x_n) \in \mathbb{R}^n_+$,
\[ u(x', x_n) = u(0, x_n) + \int_0^1 \nabla_{x'} u(\theta x', x_n) \cdot x'd\theta = u(0, x_n) + \frac{1}{2} |x'|^2. \]

Thus,
\[ \det D^2u = u_{nn}(0, x_n) = x_n^{-\alpha}, \]

it follows that
\[ u(0, x_n) = \frac{x_n^{-\alpha}}{(2-\alpha)(1-\alpha)} + Ax_n + B \]
for some constants $A, B \in \mathbb{R}$.

Since $u \in C(\mathbb{R}^n_+)$ and satisfies (1.5), we obtain $A = B = 0$. The conclusion of the theorem follows from this and (5.7).

6. Proof of Theorem 1.3

**Proof of Theorem 2.3**: By the localization theorem (Theorem 2.1),
\[ cU_0(x) \leq u(x) \leq c^{-1} U_0(x) \quad \text{in } \Omega \cap S_\eta(U_0), \]
where $c$ is a constant depending only on $n, \alpha$ and $\rho$. Let $\Omega_h = F^{-1}_h \Omega$, then $\Omega_h \cap S_1(U_0)$ can be denoted by $x_n = g_h(x')$, where
\[ g_h(x') = h^{-\frac{1}{n-\alpha}} g(h^{\frac{1}{2}} x') \leq Ch^\frac{\alpha}{2-n} |x'|^2 \]
for some constant $C = C(n, \rho)$. Define
\[ u_h(x) = \frac{1}{h} u(F_h x), \quad x \in \Omega_h. \]

Then we have
\[ cU_0(x) \leq u_h(x) \leq c^{-1} U_0(x) \quad \text{in } \Omega_h \cap S_1(U_0). \]

The assumptions of Theorem 2.3 imply that
\[ (1 - \epsilon_0)(x_n - g_h(x'))^{-\alpha} \leq \det D^2 u_h(x) \leq (1 + \epsilon_0)(x_n - g_h(x'))^{-\alpha} \quad \text{in } \Omega_h \cap S_1(U_0) \]
and
\[ \left( \frac{1}{2} - \epsilon_0 \right) |x'|^2 \leq u_h(x) \leq \left( \frac{1}{2} + \epsilon_0 \right) |x'|^2 \quad \text{on } \partial \Omega_h \cap S_1(U_0). \]

Assume by contradiction that Theorem 2.3 does not hold. Then there is a constant $\eta > 0$ such that for any $m \in \mathbb{N}, m \geq 1$, there exist $\Omega^m, g^m, u^m$ that satisfy the hypotheses of Theorem 2.3 with $\epsilon_0 \to 1/m$, and some $0 < h_m \leq \min\{1/m, c\}$ such that if we denote $\Omega^m_{h_m} = F^{-1}_h \Omega^m$ and
\[ u^m_{h_m}(x) := \frac{1}{h_m} u^m(F_{h_m} x), \]
then the part $\partial \Omega^m_{h_m} \cap S_1(U_0)$ is given by $x_n = g^m_{h_m}(x')$ for some convex function $g^m_{h_m}$ satisfying (6.1) with $h \to h_m$, and the function $u^m_{h_m}$ satisfies (6.2)-(6.4) with $\epsilon_0 \to 1/m$, while the inclusion in Theorem 2.3 does not hold for $\eta$ and $S_{h_m}(u^m)$.

Let $m \to \infty$, we can extract a subsequence $u^m_{h_m}$ that converges uniformly on compact sets to a global solution $u_0$ defined in $\mathbb{R}^n_+$ such that
\[ cU_0(x) \leq u_0(x) \leq c^{-1} U_0(x) \quad \text{in } \mathbb{R}^n_+. \]
and
\[(6.6)\quad \det D^2u_0(x) = x_n^{-\alpha} \text{ in } \mathbb{R}^n, \quad u_0(x', 0) = \frac{1}{2}|x'|^2.\]

Theorem 2 implies that
\[
u_0 = U_0 = \frac{1}{2}|x'|^2 + \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)}.
\]

We reach a contradiction.

**Proof of Theorem 1.3:** Assume the hypotheses in Theorem 1.3 hold, then we can assume that
\[
\varphi = \frac{1}{2}(Mx', x') + o(|x'|^2),
\]
for some positive definite matrix \(M \in \mathbb{R}^{(n-1) \times (n-1)}\).

It suffices to prove the theorem for the case \(f(0) = 1\) and \(M = I_{n-1}\). Indeed, let \(D' \in \mathbb{R}^{(n-1) \times (n-1)}\) be a positive definite matrix such that
\[
D'MD' = I_{n-1}.
\]

Let \(\lambda > 0\) be a constant to be chosen. Define
\[
D := \left( \begin{array}{c} D' \\ \lambda \end{array} \right).
\]

For any \(y \in \tilde{\Omega} = D^{-1} \Omega\), define
\[
\tilde{u}(y) = u(Dy)
\]
and \(\tilde{\varphi} := \tilde{u}|_{\partial \tilde{\Omega}}\). Then we have
\[
\det D^2\tilde{u}(y) = \tilde{f}(y)d_{\partial \tilde{\Omega}}^{-\alpha}(y), \quad \tilde{f}(y) := (\det M)^{-1}\lambda^2f(Dy)d_{\partial \tilde{\Omega}}^{-\alpha}(y).
\]

It is easy to see that
\[
\lim_{y \to 0} \frac{d_{\partial \Omega}(Dy)}{d_{\partial \tilde{\Omega}}(y)} = \lambda.
\]

Thus we can choose \(\lambda > 0\) such that
\[
\lim_{y \to 0} \tilde{f}(y) = (\det M)^{-1}\lambda^2f(0)\lambda^{-\alpha} = 1.
\]

Now we assume \(f(0) = 1\) and \(M = I_{n-1}\), and we will prove that
\[(6.7)\quad u(x) = \frac{1}{2}|x'|^2 + \frac{x_n^{2-\alpha}}{(2-\alpha)(1-\alpha)} + o(|x'|^2 + x_n^{2-\alpha}).\]

For any \(\epsilon_1 > 0\) small, we can choose \(R = R(\epsilon_1) > 0\) such that \(\partial \Omega \cap B_R\) is given by \(x_n = g(x')\) for some convex function \(g\), where
\[(6.8)\quad g \in C^2(\pi(\partial \Omega \cap B_R)), \quad g(0) = 0, \quad \nabla g(0) = 0, \quad D^2g(0) \geq k_0 I_{n-1} > 0,
\]
\[(6.9)\quad \det D^2u = f(x)d_{\partial \Omega}^{-\alpha}, \quad 1 - \epsilon_1 \leq f \leq 1 + \epsilon_1 \quad \text{in } \Omega \cap B_R,
\]
\[(6.10)\quad \left( \frac{1}{2} - \epsilon_1 \right)|x'|^2 \leq u(x) = \varphi(x') \leq \left( \frac{1}{2} + \epsilon_1 \right)|x'|^2 \quad \text{on } \partial \Omega \cap B_R,
\]

where \(k_0\) depends only on the principal curvatures of \(\partial \Omega\) at 0.

It is obvious that
\[
\lim_{x \to 0} \frac{x_n - g(x')}{d_{\partial \Omega}(x)} = 1.
\]

Therefore we can choose \(R = R(\epsilon_1)\) smaller such that
\[(6.11)\quad (1 - 4\epsilon_1)(x_n - g(x'))^{-\alpha} \leq \det D^2u \leq (1 + 4\epsilon_1)(x_n - g(x'))^{-\alpha} \quad \text{in } \Omega \cap B_R.
\]

For any \(\eta > 0\), let \(\epsilon_0\) be the constant given by Theorem 2.3 and \(\epsilon_1 := \epsilon_0/4\). Using (6.8), (6.10) and (6.11), we can choose \(\rho > 0\) such that the hypotheses of Theorem 2.3 hold. Then Theorem 2.3 implies that
\[
|u(x) - U_0(x)| \leq C\eta U_0(x) \quad \text{in } \Omega \cap S_\rho(U_0),
\]
where \(U_0\) is defined as in Theorem 2.3, \(c\) is a constant depending only on \(n, \alpha, \rho\) and \(C = C(n, \alpha)\) depends only on \(n, \alpha\). This proves (6.7) and therefore the proof of Theorem 1.3 is complete.
7. Proof of Theorem 1.4

In this section we always denote by \( c, C, c_i, C_i (i = 0, 1, \ldots) \) constants depending only on \( n, \lambda_0, \Lambda_0, \alpha, \text{diam}(\Omega) \), and \( \varphi, \partial \Omega \) up to their second derivatives. For any \( A, B \in \mathbb{R} \), we write \( A \sim B \) if

\[
eq \frac{A}{B} \leq C
\]

for some constants \( c, C \) depending only on \( n, \lambda_0, \Lambda_0, \alpha, \text{diam}(\Omega) \), and \( \varphi, \partial \Omega \) up to their second derivatives.

Suppose the assumptions of Theorem 1.4 hold. First we can use barriers to obtain that

\[
\|u\|_{C(\overline{\Omega})} \leq C.
\]

Now we restrict to a neighborhood of \( 0 \in \partial \Omega \). As in Section 2, we can assume that for some fixed small \( \rho > 0 \), the part \( \partial \Omega \cap \{x_n \leq \rho\} \) is given by \( x_n = g(x') \) for some convex function \( g \), where

\[
g \in C^2(\pi(\partial \Omega \cap \{x_n < \rho\})) , \quad g(0) = 0 , \quad \nabla g(0) = 0.
\]

The function \( u : \overline{\Omega} \to \mathbb{R} \) satisfies \( u = \varphi(x') \) on \( \partial \Omega \cap \{x_n \leq \rho\} \), and

\[
\det D^2u = f, \quad 0 < \lambda(x_n - g)^{-\alpha} \leq f \leq \Lambda(x_n - g)^{-\alpha} \quad \text{in} \ \Omega \cap \{x_n < \rho/2\},
\]

where \( \alpha \in [1, 2) \).

**Case 1 :** \( \alpha \in (1, 2) \).

Denote \( \beta := \frac{n(\alpha - 1)}{2} > 1 \). We claim that for any \( x \in \partial \Omega \cap \{x_n \leq \rho/2\} \),

\[
-\delta x_n - \delta^{-1}(x_n - g)^{2-\beta} \leq \tilde{u} \leq -\delta x_n - \delta(x_n - g)^{2-\beta},
\]

where \( \delta > 0 \) is a small constant, and

\[
\tilde{u} := u - u(0) - \nabla x' \varphi(0) \cdot x'.
\]

Indeed, let \( C_0, C_1 > 0 \) be two constants and define

\[
v^- := \varphi(0) + \nabla x' \varphi(0) \cdot x' - \frac{C_0}{(2-\beta)}(x_n - g)^{2-\beta} - C_1 x_n.
\]

Since \( \Omega \) is uniformly convex, \( \partial \Omega, \varphi \in C^2 \), and \( u \) is bounded below by \( (7.1) \), by straightforward computation we obtain that \( v^- \) is a lower barrier for \( u \) in \( \Omega \cap \{x_n \leq \rho/2\} \) if \( C_0, C_1 \) are sufficiently large. Similarly, the function

\[
v^+ := \varphi(0) + \nabla x' \varphi(0) \cdot x' - \frac{c_1}{(2-\beta)}(x_n - g)^{2-\beta} + C_1 x_n
\]

is an upper barrier for \( u \) in \( \Omega \cap \{x_n \leq \rho/2\} \) if \( c_1 \) is small and \( C \) is sufficiently large. Hence the claim follows.

The estimate \( (7.4) \) implies that

\[
|u - u(x_0)| \leq C|x - x_0|^{2-\beta} \quad \forall \ x_0 \in \partial \Omega, \ x \in \Omega.
\]

This together with the convexity of \( u \) implies that \( u \) is Hölder continuous in \( \Omega \) and

\[
\|u\|_{C^{0,2-\beta}(\overline{\Omega})} \leq C.
\]

Let \( y_0 \in \Omega \) and assume \( S_{\hat{h}}(y_0) \) is the maximal section included in \( \Omega \) which becomes tangent to \( \partial \Omega \) at \( 0 \) with \( \hat{h} \leq 0 \). Then we obtain that

\[
\nabla x' u(y_0) = \nabla x' \varphi(0), \quad S_{\hat{h}}(y_0) = \{ x \in \Omega : \tilde{u}(x) < u_n(y_0)x_n \}.
\]

Note that \( u_n(y_0) \) is bounded above since \( \varphi \in C^2 \) and \( \Omega \) is uniformly convex. Thus

\[
\hat{h} = -\tilde{u}(y_0) + u_n(y_0)y_0 \cdot e_n
\]

is bounded above.

Denote \( M := -u_n(y_0) \). We only need to consider two cases: \( -C < M < 2\delta^{-1} \) and \( M \geq 2\delta^{-1} \), where \( \delta \) is the constant in \( (7.4) \).

If \( -C < M < 2\delta^{-1} \), then at the point \( x = (0, c_0) \) with \( c_0 \) a small constant, we have by \( (7.4) \)

\[
\tilde{u} + Mx_n \leq 3\delta^{-1}x_n - \delta x_n^{2-\beta} \leq -\frac{\delta}{2}x_n^{2-\beta} \leq -\frac{\delta}{2}x_n^{2-\beta}.
\]

It follows that \( \tilde{h} \) is bounded below. Hence by \( (7.5) \),

\[
S_{\hat{h}}(y_0) \supset B_c(y_0).
\]
It remains to consider the case $M \geq 2\delta^{-1}$. For some $c_1$ small, the second inequality in (7.4) implies that the point $x = (0, c_1 M^{-1/\gamma}) \in S_h(y_0)$ and
\[
\tilde{u} + M x_n \leq -\frac{\delta}{2} x_n^{2-\beta} + M x_n = c_1 M^{2-\beta} \left[1 - \frac{\delta}{2} c_1^{1-\beta}\right] \leq -c_1 M^{2-\beta}.
\]
Therefore,
\[
(7.6)
\]
\[
\tilde{h} \geq c_1 M^{2-\beta}.
\]
On the other hand, the first inequality in (7.4) and the uniform convexity of $\Omega$ imply that
\[
(7.7)
\]
\[
S_h(y_0) \subset \{x_n \leq CM^{1/\gamma}, |x'| \leq CM^{1/\gamma}\}.
\]
Using this and the first inequality in (7.4) again, we obtain that for any $x \in S_h(y_0)$
\[
\tilde{u} + M x_n \geq \frac{M}{2} x_n - \delta^{-1} x_n^{2-\beta} \geq -\delta^{-1} x_n^{2-\beta} \geq -CM^{2-\beta}
\]
and therefore
\[
(7.8)
\]
\[
\tilde{h} \leq CM^{2-\beta}.
\]
By (7.4), (7.6) and (7.8), we have
\[
(7.9)
\]
\[
S_{h/2}(y_0) \subset \{x_n \geq cM^{1/\gamma}\}.
\]
Using the first inequality in (7.4), (7.7) and (7.9), we obtain that
\[
(7.10)
\]
\[
x_n - g \sim x_n \sim M^{1/\gamma} \quad \forall x \in S_{h/2}(y_0).
\]
The volume estimate for interior sections in [6, Corollary 3.2.4] and the definition of $\beta$ imply that
\[
(7.11)
\]
\[
|S_{h/2}(y_0)| \sim M^{\frac{n(2-\beta)+\alpha}{2(1-\beta)}} = M^{\frac{n+1}{2(1-\beta)}}.
\]
Define $T y := (M^{-1/\gamma} y', M^{-1/\gamma} y)\$, and
\[
v(y) := \frac{1}{M^{1/\gamma}}[\tilde{u}(Ty) + M(Ty) \cdot c_n + \tilde{h}/2] \quad \text{in } T^{-1}S_{h/2}(y_0).
\]
We have
\[
\det D^2v(y) = M^{\frac{\alpha}{2}} \det D^2u(Ty) \sim 1 \quad \text{in } T^{-1}S_{h/2}(y_0)
\]
and
\[
|T^{-1}S_{h/2}(y_0)| \sim 1.
\]
Since (7.7) implies that
\[
T^{-1}S_{h/2}(y_0) \subset B_C,
\]
hence the Aleksandrov's maximum principle [6, Theorem 1.4.2] implies that
\[
T^{-1}S_{h/2}(y_0) \supset B_C(T^{-1}y_0).
\]
Part i) of Theorem 1.4 is proved.

Case 2: $\alpha = 1$.

By straightforward computation, the functions
\[
v^- := \varphi(0) + \nabla_x \varphi(0) \cdot x' - C_0(x_n - g)(-\log(x_n - g))^{\frac{1}{\beta}} - C_1 x_n
\]
and
\[
v^+ := \varphi(0) + \nabla_x \varphi(0) \cdot x' - c_1(x_n - g)(-\log(x_n - g))^{\frac{1}{\beta}} + C x_n,
\]
are barriers for $u$ in $\Omega \cap \{x_n \leq \rho/2\}$ if $C_0, C_1, C$ are large constants and $c_1$ is small. Hence, we have in $\Omega \cap \{x_n \leq \rho/2\}$
\[
(7.12)
\]
\[
-\delta^{-1} x_n - \delta^{-1} (x_n - g)(-\log(x_n - g))^{\frac{1}{\beta}} \leq \tilde{u} \leq \delta^{-1} x_n - \delta (x_n - g)(-\log(x_n - g))^{\frac{1}{\beta}},
\]
where $\delta > 0$ is a small constant, and $\tilde{u}$ is defined as in the case $\alpha \in (1, 2)$.

Using (7.12) and similar arguments to the previous case, part ii) of the theorem is proved. The proof of Theorem 1.4 is complete.
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