Leaky-box approximation to the fractional diffusion model

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Abstract. Two models based on fractional differential equations for galactic cosmic ray diffusion are applied to the leaky-box approximation. One of them (Lagutin-Uchaikin, 2000) assumes a finite mean free path of cosmic ray particles, another one (Lagutin-Tyumentsev, 2004) uses distribution with infinite mean distance between collision with magnetic clouds, when the trajectories have form close to ballistic. Calculations demonstrate that involving boundary conditions is incompatible with spatial distributions given by the second model.

1. Introduction

We continue investigation of problems arising in the fractional model of cosmic rays propagation in the Galaxy. Recall the situation. The first version of this model was proposed in [1]. It was based on the three-dimensional Lévy-Feldheim flight process in an infinite homogeneous medium obeying the diffusion equation with a fractional power of Laplacian

\[ \frac{\partial}{\partial t} + D(-\Delta)^{\alpha/2} \] G(r, t) = \delta(r)\delta(t),

containing the “diffusion coefficient” \( D \), which depends on the energy \( E \) and remains constant in the process of motion. The fractional Laplacian is given by its Fourier transform

\[ \int e^{ikr}(-\Delta)^{\alpha/2} f(r) dr = |k|^\alpha \int e^{ikr} f(r) dr \equiv |k|^\alpha \hat{f}(k), \quad 0 < \alpha \leq 2. \]

This equation and its solution expressed through the isotropic Lévy-Feldheim distribution \( \Psi^{(\alpha)}(r) \)

\[ G(r, t) = (Dt)^{-3/\alpha} \Psi^{(\alpha)}(r(Dt)^{-1/\alpha}) \]

were known to that time (see [2]). The fractional power of the Laplacian has been explained by a fractal (friable) large scale structure of the Galaxy which causes an enhanced kind of diffusion (superdiffusion). The form of such a diffusion packet is described by the isotropic Levy-Feldheim distribution, and its width grows with time proportionally to \( t^{1/\alpha} \). When \( \alpha = 2 \), the medium becomes homogeneous and anomalous diffusion reduces to the normal one with Gaussian profile and the width \( \propto t^{1/2} \). As a result, we saw that this solution on the assumption of power type of energy dependence of diffusivity \( (D \propto E^\delta) \) and source spectrum \( (S \propto E^{-p}) \) reveals an effect similar to the observed “knee” in primary spectra. As noted at the end of the cited work, the best fit of experimental data for H, He, CNO, Ne-Si and Fe-group were observed at \( \alpha = 5/3 \approx 1.67, \delta = 0.25 \) and \( p = 2.9 \). We will refer to this model as the LU-model.
2. Lagutin-Tyumentsev model

Later, the model was modified by inserting a fractional time-derivative instead of the first-order one [3] and lowering the orders \( \alpha \) of the fractional Laplacian to 0.3 [4]:

\[
\left[ t^\alpha D_t^\beta + D(-\Delta)^{\alpha/2} \right] G(\mathbf{r}, t) = \delta(\mathbf{r})\delta(t), \quad \delta(t) = t^{-\beta}/\Gamma(1-\beta), \quad \alpha = 0.3, \quad \beta = 0.8.
\]

For the sake of convenience, we will refer to this modification as the LT-model.

The following case for such a choice was given. The value \( \beta = 0.8 \) was taken from the work [5] devoted to investigation how photospheric convective motions transport magnetic flux elements. Experimental data exhibit subdiffusion behaviour of solar magnetic elements. Observations of solar magnetic bright points analyzed in the cited work led to conclusion that the waiting time distribution density follows \( t^{-\beta-1} \), \( \beta = 0.61 \pm 0.09 \), during interval 0.3-22 minutes and then rapidly damps. Thus, we do not see here any reasons for application of the power law with \( \beta = 0.8 \) to description of propagation of cosmic rays through interstellar medium: these phenomena are quite different by space-time scales and even by nature. Moreover, the power-law behavior is observed at small times only and disappears in the long-time region. This corresponds to the value \( \beta = 1 \).

The spatial exponent \( \alpha \) was changed firstly from 1.67 to 1.00 in [6]. The authors wrote that “comparison of simulated characteristics with experiment indicates that the fractal structure of ISM with the parameter \( \alpha = 1 \) (Kraichnan spectrum of magnetic irregularities) gives local cosmic ray characteristics which are closest to the experiment”. However, as far back as in 1999 [7], see also [8], was found, that the fractal dimension \( d_F \) of a medium does not coincide with the exponent \( \alpha \) characterizing the free path distribution in this medium. This conclusion was supported by analytical calculations performed in [9]. They showed that \( \alpha \) depends not only on \( d_F \) but also on size of scattering objects (say, magnetic clouds). When it was recognized, Lagutin, Raikin and Tyumentsev [10] repeated the calculations and determined the spatial exponent \( \alpha = 0.3 \) from the linearized relation

\[
\alpha = 2 - d_F
\]

with \( d_F = 1.7 \) [10]. The latter number for the fractal dimension of the interstellar medium can be considered as a conventional value (see, for example, [11]). However, this formula could be used in the case if the interstellar magnetic field form small islands located at large distances of each other, but really it is not the case. Interstellar magnetic clouds affecting the cosmic ray transport have various sizes and may be close to each other. Thus, expecting shorter free paths, one should take essentially larger values of \( \alpha \). Looking at Fig.3 of the article [10], one can see an ambiguous \( \alpha(d_F) \) dependence: the \( \alpha \) is determined not only by the \( d_F \) but also by ratio size/distance for inhomogeneities. When this ratio grows, the fractal becomes less transparent and the free path pdf falls more rapidly. We refer to the work [12], where the numerical simulation with the Erlykin-Wolfendale model gave \( \alpha = 1.6 \pm 1.9 \) which, in the authors opinion, “is expected result, implying a fractal structure for interstellar medium”. The choice is also supported by recent article [13]. Using the known cosmic ray spectrum and radial gradient in the vicinity of the solar system to define an energy density and comparing with the modeling results shoved that the best fit for the value of \( \alpha \) is about 1.65 (possible fits range from 1.6-1.9, but not acceptable fit is found for \( \alpha = 2 \), which would correspond to conventional diffusion). Our initial value \( \alpha = 1.67 \) belongs to this region.

3. Bounded anomalous diffusion model

In our works [14, 15, 16] we show that random trajectories related to LT-model contains anomalously long rectilinear parts comparable with size of the Galaxy disc itself, and assumption
on instantaneous flights to such distances looks to say the least of it unphysical. The exit from this situation lies in using the fractional material derivative operator as it described in the above- 
sited our works: this operator takes into account that cosmic ray particles propagate through 
interstellar medium with a finite speed. the fractional material operator used in our bounded 
anomalous diffusion model [17], where the following equation

\[
A_\alpha \langle 0D_t^\alpha \rangle G(\mathbf{r}, t) = S_v(\mathbf{r}, t), \quad 0 < \alpha < 1;
\]

\[
\left[ \frac{1}{v} \partial_t + A_\alpha \langle 0D_t^\alpha \rangle \right] G(\mathbf{r}, t) = S_v(\mathbf{r}, t), \quad 1 < \alpha < 2;
\]

for cosmic ray propagation was represented. Here, \(G(\mathbf{r}, t)\) is the propagator, \(S_v(\mathbf{r}, t)\) is a source function, the angle brackets denote averaging over directions \(\Omega\) of propagation, the operator

\[
0D_t^\alpha G(\mathbf{r}, t) = \left( \frac{1}{v} \partial_t + \Omega \nabla \right)^\alpha G(\mathbf{r}, t)
\]

is the fractional generalization of the material derivative. When \(\alpha \in (1, 2)\), the equation reduces in the long time asymptotic region to the Levy-flight diffusion equation

\[
\frac{\partial G}{\partial t} = -D_v(-\Delta)^{\alpha/2}G(\mathbf{r}, t), \quad G(\mathbf{r}, 0) = \delta(\mathbf{r}),
\]

with diffusivity [18]

\[
D_v = \frac{D_\infty}{1 + w/v},
\]

where \(w\) stands for the mean path covered by Lévy-jumps per unit time and \(v\) is the free motion velocity. The solution of Eq.(2) for an unbounded fractal medium is expressed through the isotropic Lévy-Feldheim distribution

\[
G(\mathbf{r}, t) = (D_v t)^{-\alpha/3} \Psi(\alpha)((D_v t)^{-1/\alpha} \mathbf{r}), \quad 1 < \alpha < 2.
\]

4. Fractional Laplacian in a bounded domain
In this work, we consider another aspect of cosmic ray propagation, provoked by the long-
distant parts, namely the influence of boundaries on the fractional Laplacian. Indeed, the true 
Laplacian is a local operator, having the same form independently of presence or absence of 
boundaries. However, the fractional Laplacian is a non-local operator and for this reason it has 
a form depending on boundaries. In particular, the definition based on the Fourier transform 
can not be applied to the fractional Laplacian acting in a bounded medium.

The statement of such a problem should be accompanied with a specification of the desired 
function values throughout an outer region. So, we have to return to the integral representation 
of the operator. The random flight interpretation can help in specifying the conditions but some 
subtle points such as distinction between first-passage and first-arrival times or between free and 
reflecting boundary conditions appear [19]. In [19] have investigated the matrix representation 
of the one-dimensional fractional Laplacian and solved numerically in connection to the first-
passage problem (the Lévy-flights under absorbing boundary conditions) and to the long-ranged 
interfaces with no constraints at the ends (the free boundary conditions).

Krepysheva et al ([20]) analyze the symmetric Lévy flights restricted to a semi-infinite domain 
by a reflective barrier. They show that the introduction of the boundary condition induces a 
modification in the kernel of the nonlocal operator:

\[
-(\Delta)^{\alpha/2} f(x, t) = -\frac{1}{2 \cos(\alpha \pi/2) \Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_0^\infty |x - \xi|^{1-\alpha} f(\xi, t) d\xi, \quad 1 < \alpha < 2,
\]
\[ \to -(-\triangle)^{\alpha/2} f(x, t) = -\frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_0^\infty \left[ |x - \xi|^{1-\alpha} + (x + \xi)^{1-\alpha} \right] f(\xi, t) d\xi. \]

The operators \(-(-\triangle)^{\alpha/2}\) and \(-(-\triangle)^{\alpha/2}\) differ in the kernels, but the difference becomes small when \(x + \xi\) is large. Nevertheless, omitting the term \((x + \xi)^{1-\alpha}\) we would get a decreasing integral with respect to \(x\), whereas the total amount of the diffusing matter should be preserved.

Rafeiro and Samko ([21]) have introduced a version of the fractional Laplacian for a bounded domain as a generalization of the Marchaud formula for one-dimensional fractional derivatives on an interval \((a, b), -\infty < a < b \leq \infty\), to the multidimensional case of functions defined on a region \(G \subset \mathbb{R}^d\):

\[ D^\alpha_G f(x) = C(\alpha) \left[ a_G(x) f(x) + \int_G \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} dy \right], \quad x \in G \subset \mathbb{R}^d, \]

where \(\alpha \in (0, 1)\),

\[ C(\alpha) = \frac{\alpha 2^{\alpha-1} \Gamma((d + \alpha)/2)}{\pi^{d/2} \Gamma(1 - \alpha/2)} \]

and

\[ a_G(x) = \int_{\mathbb{R}^d \setminus G} \frac{dy}{|x - y|^{d+\alpha}}. \]

In other words, this is the Riesz fractional derivative of the zero continuation of \(f(x)\) from \(G\) to the whole space \(\mathbb{R}^d\).

Guan & Ma [22], investigating the reflected symmetric \(\alpha\)-stable processes, gave the name regional fractional Laplacian to the limit

\[ -(-\triangle)^{\alpha/2} f(x) = \lim_{\varepsilon \downarrow 0} C(\alpha) \int_{G, |x - y| > \varepsilon} \frac{f(x) - f(y)}{|y - x|^{d+\alpha}} dy, \]

provided it exists.

For more detail, the reader can be referred to the articles [23, 24, 25, 26, 27, 28]. Better understanding of the Laplacian in a bounded domain can be achieved on the base of the non-local operator theory [29, 31, 32].

This short review is done in order to underline that in contradistinction to classical case, the fractional Laplacian \(\triangle^{\alpha/2}\) change its form in a bounded domain and cannot be determined by its Fourier transform \(-|k|^\alpha\) anymore. Consequently, all results obtained by Lagutin et all in 2001-2011 years relate to infinite unbounded fractal medium. One should say, that referring to the normal model with the use of Gaussian distribution in a bounded model can not justify the similar use of the stable distributions because their long tails may easily get the boundary surfaces which are inaccessible for the normal Gaussian process.

5. Numerical simulation of anomalous diffusion in the Galactic disk

Taking into account the above-mentioned difficulties with statement of boundary conditions in analytic or numerical approach, we perform direct Monte Carlo simulations to investigate the CR propagation in the framework of a fractal Galaxy model. The first problem we consider here is the escape time distribution for the Galactic disk. As can be concluded from [33] (see also [34]), the leading contribution in this process belongs to plane boundaries so the escape through the cylindrical part of the boundary can in the first approximation be neglected. Thus, we will
simulate isotropic walk of particles in a fractal layer with two plane-parallel boundaries and the initial random point uniformly distributed between these boundaries (Fig. 1). Objects for study are escape time and escape path distributions in two models: LU ($\alpha = 1.67$, $\beta = 1$, $v = c$) and LT ($\alpha = 0.3$, $\beta = 0.8$, $v = \infty$). Results of Monte Carlo simulation are presented in Fig. 2. The coefficient of anomalous diffusion for both models $D = D_0 R^{0.27}$, $E = 10^6$ GeV, parameter $D_0 = 4 \cdot 10^{-6}$ pc/yr$^{0.3}$ in LT-model and $D_0 = 2.4 \cdot 10^{-5}$ pc$^{1.67}$/yr in LU-model.

Figure 1. The scheme of the model ($h = 150$ pc).

Figure 2. Numerically calculated pdf of first passage time (left panel) and path (right panel) in frameworks of GS-1963, LU-2000 and LT-2004 models.

In the LT-model, distributions of free path lengths and waiting times have the form of asymptotical power laws

$$P\{\xi > r\} \sim \frac{(c_\alpha r)^{-\alpha}}{\Gamma(1 - \alpha)}$$, $\alpha > 0$, $r \to \infty$; 

$$P\{\tau > t\} \sim \frac{(c_\beta t)^{-\beta}}{\Gamma(1 - \beta)}$$, $\beta > 0$, $t \to \infty$.

We take $c_\alpha = 270$ pc$^{-1}$ and $c_\beta = 10^{-2}$ yr$^{-1}$ (for $E = 10^6$ GeV) and simulate $\xi$ and $\tau$ as random variables with pdf in the form of fractional exponents:

$$\xi = \frac{|\ln U_1|^{1/\alpha}}{c_\alpha^{1/\alpha}} S(\alpha)$$, 

$$\tau = \frac{|\ln U_2|^{1/\beta}}{c_\beta^{1/\beta}} S(\beta).$$
Here $S(\alpha)$ and $S(\beta)$ are one-sided stable variables simulated according to Kanter’s algorithm

$$S(\alpha) = \frac{a \sin(\alpha \pi U_3)[\sin((1 - \alpha)\pi U_3)]^{1/\alpha - 1}}{[\sin(\pi U_3)]^{1/\alpha}[\ln U_4]^{1/\alpha - 1}},$$

where $U_1$, $U_2$, $U_3$ and $U_4$ are variables uniformly distributed in $(0, 1)$.

In the LU-model, waiting times can be simulated according to the exponential distribution

$$P\{\tau > t\} = \exp(-\mu t).$$

We take $\mu = 10^{-2}$ yr$^{-1}$. Path length are simulated according to the Pareto distribution $P\{\xi > r\} = b r^{-\alpha}$ with $b = 8 \cdot 10^{-2}$ pc$^{-\alpha}$. These parameters are valid for $E = 10^6$ GeV.

Dotted line corresponds to the ballistic motion from a source. Distance between planes is equal to $2h = 300$ pc. Right panel of Fig. 2 shows the numerically calculated pdf of first passage time. Solid lines are for LU-model and dashed line is for LU model for the same parameters as in the left panel.

**Figure 3.** Escape time versus transparency. The instantaneous point source is situated on the middle plane.

The mean escape time can be calculating according to the following formula

$$\tau_{esc} = \varepsilon \tau_{SB} + \varepsilon(1-\varepsilon)[\tau_{SB} + \tau_{BB}] + \varepsilon(1-\varepsilon)^2[\tau_{SB} + 2\tau_{BB}] + \ldots = \tau_{SB} + \frac{1-\varepsilon}{\varepsilon}\tau_{BB},$$

where $\varepsilon$ is the transparency of boundaries, $\tau_{SB}$ is the mean passage time from source to boundary, and $\tau_{BB}$ is the mean passage time from boundary to boundary. In Fig. 3, the dependences of escape time on transparency in the models under consideration are shown. Corresponding values of $\tau_{SB}$ and $\tau_{BB}$ are indicated in the figure. In the LT-2004 model the mean escape time is infinite due to trapping times with asymptotically power law distributions.

**Conclusion**

As should be clear from the arguments presented above, equations with fractional derivatives possess a high potential for description of CR propagation. Taking into account quasi-fractal
Figure 4. Distribution density of the transverse coordinate $z$ of the cosmic rays in the region with specularly reflecting boundaries $z = \pm h$. The instantaneous point source is on the middle plane.

structure of ISM presented by index $\alpha$, and presence of magnetic plugs and traps allows to describe CR propagation in more details and more adequate than the ordinary diffusion. Kermani and Fatemi [13] state that the best fit for the value of $\alpha$ is about 1.65. They compare the results obtained in frames of anomalous diffusion model with cosmic ray spectrum and radial gradient in the vicinity of the solar system. They write that the data on radial gradient allows to use the fit range for $\alpha$ from 1.6 to 1.9.

In our work [16], it has been shown by Monte Carlo simulation that the LT-model provides large anisotropy for cosmic rays propagated in infinite space from a single source. From Fig. 2, we can see that even specularly reflecting boundaries can not change this situation. First passage time are distributed in very wide interval. Fig. 4 confirms this reasoning. It shows distribution density of the transverse coordinate $z$ for the case of random walk of a particle between two specularly reflecting boundaries with coordinates $z = \pm h$. Random walk starts from the middle plane $z = 0$. The densities are calculated for several times. One can see that even for $t = 10^6$ yr stationary distribution is not established. For the LU-model, the uniform distribution of transverse coordinate takes place at time $t \approx 5 \cdot 10^4$ yr for parameters indicated above.

6. Acknowledgments

This work is supported by Russian Foundation for basic Research (grants 11-01-00747, 12-01-00660, 13-01-00585, 12-01-33074) and Ministry of Education and Science of the Russian Federation (grant 2.1894.2001 and 14.B37.21.1296).

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