Noise induced transition from an absorbing phase to a regime of stochastic
spatiotemporal intermittency

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We introduce a stochastic partial differential equation capable of reproducing the main features of spatiotemporal intermittency (STI). Additionally the model displays a noise induced transition from laminarity to the STI regime. We show by numerical simulations and a mean-field analysis that for high noise intensities the system globally evolves to a uniform absorbing phase, while for noise intensities below a critical value spatiotemporal intermittency dominates. A quantitative computation of the loci of this transition in the relevant parameter space is presented.

Spatiotemporal chaos (STC) is a complex behavior, common to many spatially extended nonlinear dynamical systems [1]–[3]. This behavior is characterized by a combination of chaotic time evolution and spatial incoherence made evident by correlations decaying both in space and time. In spite of considerable theoretical and experimental effort devoted to give a precise definition of STC and its different regimes, the present status is still unsatisfactory. A possible strategy to make progress in the understanding of STC is to investigate scenarios based on simple models, with few controlled ingredients, that reproduce the spatio-temporal structures under study. Such models are instrumental in searching for generic mechanisms leading to such complex behavior. Among the available scenarios, few of them consider the framework of stochastic partial differential equations (SPDE) [3]–[5] to describe STC. A successful example is, however, the mapping of the Kuramoto-Shivashinsky equation (describing a STC regime named phase turbulence) to the stochastic model of surface growth known as Kardar-Parisi-Zhang equation [6].

A particular instance of STC is a regime called spatiotemporal intermittency (STI) which is present in a large variety of systems [1]–[3]. Generally speaking, this regime is a chaotic spatiotemporal evolution (the turbulent phase) irregularly and continuously interrupted by the spontaneous formation of domains with a wide range of sizes and lifetimes, where the behavior is ordered (laminar). The borders of the laminar regions propagate as fronts and eventually cause the collapse of the corresponding region into the turbulent background. There are strong indications that the STI regime has many features in common with phenomena of probabilistic nature. For example, it appears in some systems that critical exponents at the onset of STI are in the universality class of directed percolation [7]. STI has also been related to nucleation [8], another process associated with stochastic fluctuations. However, no description of STI in terms of SPDEs has been put forward so far.

The purpose of this Letter is to introduce a model, entirely based on a simple SPDE, that describes the main features of STI, and report the existence of a noise induced transition from laminarity to STI. The laminar phase is associated to an equilibrium state called absorbing in the SPDE parlance. The role of the turbulent phase is played by a strongly fluctuating state driven by noise. For either large values of the noise intensity or small values of the diffusion constant, the system is globally attracted to the absorbing state. However, for a fixed diffusion rate, STI sets in below a critical noise intensity. We investigate the nature of this transition both analytically using a mean-field scheme, and numerically by means of indicators such as the change of the mean velocity of fronts between absorbing and turbulent regions, the one-site probability distribution function and the order parameter characterizing the average system fraction in the laminar phase.

We consider the following one-dimensional Itô-SPDE [4] for the evolution of a real field \( u(x, t) \) [12]:

\[
\dot{u}_t = -\frac{\partial \phi}{\partial u} + D u_{xx} + \sqrt{\epsilon} g(u) \xi
\]

(1)

The r.h.s. of (1) is composed of a gradient term derived from a potential \( \phi(u) \), a diffusive spatial coupling with diffusion constant \( D \), and a multiplicative Gaussian white noise \( \xi(x, t) \) of zero mean and correlations \( \langle \xi(x, t) \xi(x', t') \rangle = 2 \delta(x-x') \delta(t-t') \). For the results described below, the essential aspects of our model are (i) a bistable potential \( \phi(u) \) and (ii) a multiplicative noise function \( g(u) \) which vanishes at the metastable state. Different forms of the noise amplitude \( g(u) \) give rise to different universality classes [3].

We make the simplest choice for the function \( g(u) = u - u - u + u + u^6 \) so that the sign of \( u(x, t) \) is preserved during the evolution of our system, allowing us to restrict our attention to the case \( u(x, t) \geq 0 \), by picking positive defined initial conditions. With this phase-space restriction, we choose the potential \( \phi(u) = a u^2 - u^4 + h u^6 \) (with \( h > 0 \)), which for \( 0 < a < a_M = 1/4h \) has a relative minimum at \( u = 0 \) and an absolute minimum at \( u = u_+ \) separated by a max-
minimum at $u = u_\pm$. At $a = a_M$ we are at the “Maxwell point” where $\phi(0) = \phi(u_\pm)$, while at $a = 0$ the $u = 0$ state switches from metastable to unstable. In the absence of noise a front between a region of $u = 0$ and a region of $u = u_+$ moves at finite speed towards the former, due to diffusive coupling. Consequently, any finite region of $u = 0$ surrounded by $u_+$ will eventually shrink letting the system evolve into the uniform $u_+$ attractor. This is the case except for initial configurations laying completely on the relatively small basin of attraction of $u = 0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Space-time evolution of $u(x,t)$ in a situation where the STI phase persists for all times (black: $u = 0$, grey: $u > 0$). Space ranges in $x \in (0,400)$ (with periodic boundary conditions) and time in $t \in (0,90)$. The numerical method uses a finite difference approximation for the Laplacian, and an Euler algorithm for the time integrator which respects the positivity of the field. The initial condition is chosen randomly in the interval $u_0(x) \in (0,2.4)$. Other values of the parameters are $\epsilon = 0.95$, $a = 0.5$, $D = 2.0$, $h = 0.22$, $\Delta x = 1$, $\Delta t = 0.001$.}
\end{figure}

In the presence of multiplicative noise, the state $u(x) \equiv 0$ becomes an absorbing state, i.e. a state where the system can be driven to by fluctuations but not removed from by neither these nor by the deterministic dynamics. This is because the selected noise amplitude function $g(u)$ vanishes at the fixed point $u = 0$. On the other hand, when the field takes values near $u_+$, the absolute minimum of $\phi(u)$, fluctuations are always active. However, without diffusion ($D = 0$) the fate of the system is to asymptotically settle in the absorbing state, since for any $\epsilon \neq 0$ a fluctuation large enough to push the system to the basin of attraction of $u = 0$ will eventually occur.

From this analysis, we infer that noise and diffusion play opposite roles on the asymptotic space-time evolution of the field $u$. While noise nurtures the development of regions dominated by the absorbing state, diffusion favors the dominance of $u$ values fluctuating around the global minimum of $\phi(u)$. We associate the quiescent uniform absorbing state of this system with laminarity and the disordered fluctuations around the global potential minimum with turbulence. In Fig. [1] we show a space-time plot of a numerical solution of Eq. (1). Notice that for these values of $\epsilon$ and $D$ there are occasional nucleations of regions of the laminar or absorbing state that eventually collapse under the progress of the fluctuating or turbulent state. The wide range of sizes and lifetimes of the laminar areas, characteristic of STI, is evident in the picture.

A similar mechanism, also involving the nucleation of a laminar state, has been invoked to be responsible of STI in a deterministic reaction-diffusion system [11]. In this case, the role of noise is played by a chaotic dynamics generated via a Hopf bifurcation.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Space-time plot of $u(x,t)$ starting from a random initial condition $u_0(x)$ of mean $u_+$ for $x \in (0,400)$ except in the interval $x \in (150,250)$ where $u_0(x) = 0$. The noise intensity $\epsilon$ is changed during the evolution: $\epsilon = 0.005$ for $t \in (0,30)$, $\epsilon = 0.145$ for $t \in (30,105)$ and $\epsilon = 0.250$ for $t \in (105,180)$. Other parameters are $a = 1.0$, $D = 2.0$.

In Fig. [2] for fixed $a$ and $D$, we compare the evolution of fronts separating absorbing and active regions for three different noise levels. For the lowest noise intensity $\epsilon$ these fronts invade the absorbing phase, as in the deterministic system, and the system asymptotically tends to be globally in the active phase $u_+$. On the other extreme, for the highest $\epsilon$, these fronts invade on average the active phase which consequently tends to disappear. The evolution for an intermediate value of $\epsilon$ shows a situation where the fronts are at rest on average [13]. This suggests a possible mechanism for a noise induced transition from the STI phase to the absorbing phase.

In Fig. [3] we give a quantitative evidence of this transition. There the average fraction $R$ of the system in the absorbing phase is plotted as a function of the noise intensity $\epsilon$. This figure shows a transition from $R \approx 0$ (STI) to $R = 1$ (absorbing phase) at roughly the same critical noise intensity as for the front velocity reversal. Another useful measure to characterize this transition is the one-site stationary probability density function (pdf) $P(u)$. In Fig. [3] we plot $P(u)$ for different values of the noise intensity. We observe a transition from a pdf which has a hump around the active phase $u_+$, to a pdf which is highly peaked at the absorbing state $u = 0$. Notice that this transition occurs at the value of $\epsilon$ for which
$R$ also changes abruptly.

\begin{equation}
P(u; \bar{u}) = \frac{1}{Z \ g(\bar{u})^2} \exp \int_0^u \frac{f(v) - 2D_0(v - \bar{u})}{g(v)^2} \ dv \quad (3)
\end{equation}

with $Z$ the normalization constant and where the dependence on the mean value $\bar{u}$ has been explicit. The value of $\bar{u}$ arises from the consistency relation:

\begin{equation}
\bar{u} = \int_0^\infty v \ P(u; \bar{u}) \ dv \quad (4)
\end{equation}

We can easily solve (4) in the no-coupling limit $D_0 = 0$, where $P(u) \sim u^{-2(1+a)}$ which is non-normalizable (remember that $a > 0$) so in fact we have $P(u) = \delta(u)$, and hence $\bar{u} = 0$, which describes the absorbing phase. On the other hand, the limit $D_0 = \infty$ can be treated using a saddle-point expansion in $D_0$ which results in the equation $f(\bar{u}) = 0$ [20]. This equation coincides with the steady state result of the deterministic analysis for a spatially averaged field. Such analysis predicts a transition from the absorbing state $u = 0$ to the state $u = u_+$ at the Maxwell point $a = a_M$. These limiting results show that the transition from STI to absorbing appears in our parameter regime $0 < a < a_M$ as a joint effect of fluctu- tions and spatial coupling. We note that other transition to an absorbing phase, recently studied in the context of SPDE’s, already exists for $D_0 = 0$ [1].

**FIG. 3.** (a) Fraction portion of absorbing phase $R$ as a function of $\epsilon$ and two different values of $a$. The data are the result of averaging over 40 realizations, for a time $t = 900$. Other simulation parameters as in Fig. [1]. (b) One-site probability distribution $P(u)$ for different values of $\epsilon$. The ordinate axis of the pdf corresponding to $\epsilon = 1.025$ is scaled down 10 times.

The transition from the active phase to the absorbing phase that we have numerically characterized above, can be described within a Weiss-like mean field theory specially devised to study the effect of fluctuations [19]. We first notice that, by a suitable rescaling of the space variable $x$, (1) can be rewriting as: $u_i = f(u) + D_0 u_{xx} + g(u) \xi$ with $f(u) = -\frac{2D_0}{u}$ and $D_0 \equiv D/\epsilon^2$. We next consider a spatial discretization of the field where $u_i = u(x_i, t)$. One can write the multi-variate Fokker–Planck equation for the set of variables $\{u_1, u_2 \ldots \}$ which, after integration of all the $u_i$ variables except one, yields the equation for the stationary one-site pdf $P(u)$:

\begin{equation}
\partial_t [(–f(u) + 2D_0(u - E(u)))P] + \partial_u^2(g^2(u)P) = 0 \quad (2)
\end{equation}

where $E(u) = \int u' \ P(u'|u) \ du'$ is the steady-state conditional average of the field at a nearest neighbor site of a site in which the field takes the value $u$. The above mentioned mean–field approximation takes $E(u) \approx \bar{u}$, the yet unknown mean field value of $u$. This is analogous to the traditional Weiss mean–field approach in the theory of critical phenomena. We integrate (2) to obtain the steady state pdf:
for non-zero velocity of the fronts, in analogy to what is observed in deterministic models \[11\].

[Image: FIG. 5. Numeric integration of (1) with periodic boundary conditions and spatially correlated noise with \(\langle x(x,t)\rangle = \exp(-\frac{1}{2} x^2)\delta(t-t')\) for two noise intensities: \(\epsilon = 0.35\) for \(t \in (0,150)\) and \(\epsilon = 0.50\) for \(t \in (150,250)\); \(x \in (0,400)\) and \(\gamma = 8.0, D = 2.0\).

As a conclusion, we have shown that STI can be modeled by means of a simple SPDE. Our model provides a new insight into the complex behavior of the STI phase as a stochastic nucleation of an absorbing metastable state. One of our main results is the occurrence of a noise induced transition where STI disappears in favor of an absorbing phase at sufficiently high noise intensity or low enough spatial coupling. Notice that both, transitions to absorbing states \[17\] and noise induced phase transition \[22\] are known in the SPDE context. The former phenomenon does not require spatial coupling. The latter has been so far associated with transitions between statistically homogeneous stationary states. The noise induced transition shown here is a genuine consequence of stochasticity in extended systems. It describes a transition between an absorbing phase and a dynamically active and structured phase which is a form of STC. Specific features of this active phase are described by considering space-time configurations associated with the individual realizations of the stochastic dynamics. In summary we believe the SPDE approach to be general to explore new generic mechanisms of spatiotemporal chaos.

We acknowledge helpful comments from J. M. Sancho and M. A. Muñoz, and financial support from DGES (Spain) projects PB94-0167, PB97-0141-C02-01. M.G.Z. is supported from a post-doctoral grant of the MEC (Spain).

†\[http://www.ime.dea.uib.es/PhysDept\]

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[16] This choice make our SPDE model in the multiplicative noise universality class \[7\].
[17] For the chosen potential \(\Phi(u) = \sqrt{(1 + 3/2 a u/3)}\).
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[21] For each value of \(D_0\) the consistency equation \[1\] for \(0 < a < a_M\) gives 3 solutions which continuously match the solutions given by \(f(\tilde{a}) = 0\) as \(D_0 \to \infty\). The relative stability can be analyzed by means of a Maxwell–like construction, which predicts a transition at the solid line in Fig. \[1\].
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