Reconstruction of the primordial fluctuation spectrum from the five-year WMAP data by the cosmic inversion method with band-power decorrelation analysis

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The primordial curvature fluctuation spectrum is reconstructed by the cosmic inversion method using the five-year Wilkinson Microwave Anisotropy Probe (WMAP) observation of the cosmic microwave background temperature anisotropy. We apply the covariance matrix analysis and decompose the reconstructed spectrum into statistically independent band-powers. The statistically significant deviation from a simple power-law spectrum suggested by the analysis of the first-year data is not found in the five-year data except possibly at one point near the border of the wavenumber domain where accurate reconstruction is possible.

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I. INTRODUCTION

Inflationary cosmology \([1, 2, 3]\) explains the origin of cosmic structure on various scales in an unified way. The expansion history during the inflationary stage of the early universe is recorded in the spectrum of seed structure which can be revealed by modern cosmological observations such as the Wilkinson Microwave Anisotropy Probe (WMAP) observation of the cosmic microwave background (CMB) \([4, 5, 6, 7, 8, 9, 10, 11, 12, 13]\). The WMAP mission evaluated every multipole moment of the CMB anisotropy spectrum in a wide range of scales which is potentially a record of the inflationary expansion history with high time resolution. It is a feasible challenge to probe the highly time-resolved behavior of the inflaton field (s) which drives inflation.

Since the initial release of the WMAP results \([4, 5, 6, 7, 8]\), it has been argued that the CMB temperature anisotropy spectrum has nontrivial features such as running of the spectral index, oscillatory behaviors on intermediate scales, and lack of power on large scales \([14, 15, 16, 17]\), which cannot be explained by a power-law primordial spectrum that is a generic prediction of simplest inflation models. These features may provide clues to unnoticed physics of inflation. Some of these anomalous structures disappeared on the three-year spectrum, however several anomalies are still existing \([10, 18]\). To explain these features, a number of inflation models have been proposed \([19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]\).

As an observational approach to these possible nontrivial features, which can be an alternative to model fitting, there have been several attempts to reconstruct the primordial spectrum using CMB anisotropy data without any prior assumptions about the shape of the primordial spectrum \([35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]\). One such attempt to reconstruct the primordial spectrum is a filtering method where the primordial spectrum is characterized by amplitudes on a few number of representative scales. While such methods have an advantage for reconstructing the global structure of the primordial spectrum, they may miss possible fine structures if their scale width is narrower than the filtering scale, which has been chosen rather arbitrarily so far.

On the other hand, there exist other methods which can reconstruct the primordial spectrum as a continuous function without any \textit{ad hoc} filtering scale to investigate detailed features such as the cosmic inversion method \([48, 49, 50, 51, 52]\), the Richardson-Lucy method \([53, 54, 55]\), or a nonparametric method \([56, 57]\). The cosmic inversion method has proved its ability of reconstructing the modulations of a power-law spectrum quite well by the analysis of mock anisotropy data. In the analysis of the first-year WMAP data, we pointed out the possibility of nontrivial structures in the primordial spectrum around the scales of \(2.4 \times 10^2\text{Mpc}\) and \(4.2 \times 10^2\text{Mpc}\).

Theoretically, according to the standard inflation paradigm, each wavenumber \((k\text{-})\) mode of the power spectrum is mutually independent. On the other hand, each \(k\)-mode of the power spectrum reconstructed from the observed CMB anisotropy has a strong correlation with neighboring modes because each multipole of CMB anisotropy, \(C_\ell\), depends on the \(k\)-modes in the wide range around \(k = \ell/d\) where \(d\) is the distance to the last scattering surface. In order to extract real features, therefore, it is important to decompose the reconstructed spectrum to mutually independent band-powers keeping resolution as high as possible.

The purpose of this work is to apply the cosmic inversion method to the five-year WMAP temperature anisotropy spectrum \([11]\), update the reconstructed primordial curvature spectrum, and perform the aforementioned band-power analysis by diagonalizing the covariance matrix. Then, we revisit the possibility of fine structures in the primordial spectrum. Because of the arbitrariness of the primordial spectrum, we inevitably incorporate infinite degree of freedom to our analysis, which results in degeneracy among spectral shape and cosmological parameters \([58, 59]\). In this paper, we consider the concordance adiabatic \(\Lambda\text{CDM}\) model, where the
cosmological parameters (except for the ones characterizing the primordial spectrum) are those of the WMAP team’s best-fit power-law model [12, 13], and instead focus on the detailed shape of the primordial spectrum. Note that, as shown in [51], different choices of cosmological parameters affect only the overall shape. The fine structures of the reconstructed power spectrum remain intact in the relatively small wavenumber range we probe.

This paper is organized as follows: In Sec. II, the overview of our analysis is described. In Sec. III, we show the reconstructed primordial power spectrum from the five-year WMAP data and discuss its implication. Finally, Sec. IV is devoted to the conclusion.

II. INVERSION METHOD

A. Basic formulas

Before presenting the inversion method we first list basic formulas to be inverted. Although we only express formulas related to temperature anisotropy here, the same procedure can be repeated to polarization anisotropy as well [51], which will give us additional information in the future.

The temperature anisotropy of photons coming from direction $\hat{n}$ observed at $x$ is decomposed to Fourier modes and multipole moments as

$$
\frac{\delta T}{T}(\eta, x, \hat{n}) = \int \frac{d^3k}{(2\pi)^3} \Theta_k(\eta, \mu) e^{i k \cdot x} = \sum_{\ell, m} a_{\ell m}(\eta, x) Y_{\ell m}(\hat{n}),
$$

where $\mu = k \cdot \hat{n} / |k| \equiv \hat{k} \cdot \hat{n}$ and $\eta$ is the conformal time. In terms of multipole moment in the Fourier space, $\Theta_k(\eta)$, which is defined by

$$
\Theta_k(\eta, \mu) = \sum_\ell (-i)^\ell \Theta_{\ell k}(\eta) P_\ell(\mu),
$$

$a_{\ell m}$ is expressed as

$$
a_{\ell m}(\eta, x) = \int \frac{d^3k}{(2\pi)^3} \Theta_k(\eta, \mu) \frac{4\pi}{2\ell + 1} Y^*_{\ell m}(\hat{k}) e^{i k \cdot x}. \tag{3}
$$

Thanks to the assumption that there exist only adiabatic fluctuations, we can define the transfer function, $X_k(k)$, from the Fourier mode of primordial comoving curvature perturbation, $R_k(0)$, to $\Theta_{\ell k}(\eta_0)$ by

$$
\Theta_{\ell k}(\eta_0) \equiv X_k(k) R_k(0),
$$

where $\eta_0$ is the present conformal time. Then the angular power spectrum, $C_\ell \equiv \langle |a_{\ell m}|^2 \rangle$, of temperature anisotropy and the power spectrum of curvature perturbation, $P(k) \equiv \langle |R_k(0)|^2 \rangle$, are related by

$$
\frac{2\ell + 1}{4\pi} C_\ell = \int \frac{d^3k}{(2\pi)^3} \langle \Theta_{\ell k}(\eta_0)^2 \rangle \frac{2\ell + 1}{2\ell + 1} P(k). \tag{4}
$$

This is the master equation we wish to invert. Note that $X_k(k)$ depends on the cosmological parameters, too.

B. Cosmic inversion method

Let us introduce the cosmic inversion formula which relates the observational CMB anisotropy spectrum to the primordial curvature fluctuation spectrum by a first-order differential equation. Working in the longitudinal gauge,

$$
ds^2 = a^2(\eta) \left[-(1 + 2\Psi(x))dm^2 + (1 + 2\Phi(x))dx^2\right],
$$

the Boltzmann equation for $\Theta_k(\eta, \mu)$ can be transformed into the following integral form [60].

$$
\Theta_k(\eta_0, \mu) + \Psi_k(\eta_0) = \int_{\eta_0}^{\eta \approx \eta_{end}} d\eta \left\{ [\Theta_{0k} + \Psi_k - i\mu V_{0k}] \mathcal{V}(\eta) + \left[ \dot{\Psi}_k - \dot{\Phi}_k \right] e^{-\tau(\eta)} \right\} e^{ik\mu(\eta - \eta_0)}, \tag{6}
$$

where the overdot denotes the derivative with respect to the conformal time. Here, $\Psi_k$ and $\Phi_k$ are the gauge-invariant quantities representing the Fourier transform of the Newton potential and the spatial curvature perturbation in this gauge, respectively [61, 62], and

$$
\mathcal{V}(\eta) \equiv i e^{-\tau(\eta)}, \quad \tau(\eta) \equiv \int_{\eta_0}^{\eta} \tau d\eta, \tag{7}
$$

are the visibility function and the optical depth for Thomson scattering, respectively. In the limit that the thickness of the last scattering surface (LSS) is negligible, we find $\mathcal{V}(\eta) \approx \delta(\eta - \eta_s)$ and $e^{-\tau(\eta)} \approx \delta(\eta - \eta_s)$ where $\eta_s$ is the recombination time when the visibility function is maximal [60]. Taking the thickness of the LSS into account, we have a better approximation for Eq. (6) as

$$
\Theta_k(\eta_0, \mu) + \Psi_k(\eta_0) \approx \int_{\eta_{start}}^{\eta_{end}} d\eta \left\{ [\Theta_{0k} + \Psi_k - i\mu V_{0k}] \mathcal{V}(\eta) + \left[ \dot{\Psi}_k - \dot{\Phi}_k \right] e^{-\tau(\eta)} \right\} e^{ik\mu d}, \tag{8}
$$

where $d \equiv \eta_0 - \eta_s$ is the conformal distance from the present to the LSS and $\eta_{start}$ and $\eta_{end}$ are the time when the recombination starts and ends, respectively. Here, we introduce the transfer functions, $f(k)$ and $g(k)$, defined by

$$
f(k) R_k(0) \equiv \int_{\eta_{start}}^{\eta_{end}} d\eta \left\{ [\Theta_{0k}(\eta) + \Psi_k(\eta)] \mathcal{V}(\eta) + \left[ \dot{\Psi}_k(\eta) - \dot{\Phi}_k(\eta) \right] e^{-\tau(\eta)} \right\}, \tag{9}
$$

$$
g(k) R_k(0) \equiv \int_{\eta_{start}}^{\eta_{end}} d\eta \Theta_{1k}(\eta) \mathcal{V}(\eta). \tag{10}
$$

We can calculate $f(k)$ and $g(k)$ numerically, which depend only on the cosmological parameters, for we are assuming that only adiabatic fluctuations are present.
Then, we find the approximated multipole moments as
\[
\Theta^{\text{app}}_\ell(0) = f(k)j_\ell(kd) + g(k)j'_\ell(kd) \, R_\ell(0),
\]
and the approximated angular correlation function as
\[
C^{\text{app}}(r) = \sum_{\ell=\ell_{\min}}^{\ell_{\max}} \frac{2\ell+1}{4\pi} C^{\text{app}}_\ell P_\ell \left(1 - \frac{r^2}{2d^2}\right),
\]
where \(C^{\text{app}}_\ell\) is obtained by putting Eq. (11) into Eq. (11), \(P^{\text{app}}_\ell\) is defined as \(r = 2d\sin(\theta/2)\) on the LSS, and \(\ell_{\min}\) and \(\ell_{\max}\) are lower and upper bounds on \(\ell\) due to the limitation of the observation. In the small scale limit \(r \ll d\), using the Fourier sine formula, we obtain a first-order differential equation for the primordial spectrum of the curvature perturbation,
\[
-k^2 f^2(k)P'(k) + \left[ -2k^2 f(k)f'(k) + kg^2(k) \right] P(k) = 4\pi \int_0^\infty dr \frac{1}{r} \frac{\partial}{\partial r}\left( r^3 C^{\text{app}}(r) \right) \sin kr \equiv \Xi(k).
\]
(13)

Since \(f(k)\) and \(g(k)\) are oscillatory functions around zero, we can find values of \(P(k)\) at the zero-points of \(f(k)\) as
\[
P(k_s) = \frac{\Xi(k_s)}{k_s g^2(k_s)} \quad \text{for} \quad f(k_s) = 0,
\]
(14)
assuming that \(P'(k)\) is finite at the singularities, \(k = k_s\). If the cosmological parameters and the angular power spectrum are given, we can solve Eq. (13) as a boundary value problem between singularities.

However, because Eq. (13) is derived by adopting the approximation (3), \(C^{\text{app}}_\ell\) is different from the exact angular spectrum \(C^{\text{ex}}_\ell\) for the same initial spectrum. The errors caused by the approximation, or the relative differences between \(C^{\text{app}}_\ell\) and \(C^{\text{ex}}_\ell\) are as large as about 30%. Thus, we should not use the observed power spectrum \(C^{\text{obs}}_\ell\) directly in Eq. (11). Instead, we must find \(C^{\text{app}}_\ell\) that would be obtained for the real \(P(k)\). Although this is impossible in the rigorous sense, we have found an empirical remedy to find \(C^{\text{app}}_\ell\) corresponding to \(C^{\text{obs}}_\ell\) in the following way. The crucial observation is that the ratio,
\[
b_\ell = \frac{C^{\text{ex}}_\ell}{C^{\text{app}}_\ell},
\]
(15)
is found to be almost independent of \(P(k)\) (19). Using this fact, we first calculate the ratio, \(b^{(0)}_\ell = C^{\text{ex}(0)}_\ell / C^{\text{app}(0)}_\ell\), for a known fiducial initial spectrum \(P^{(0)}(k)\) such as the WMAP team’s best-fit power-law spectrum. Then, inserting \(C^{\text{obs}}_\ell / b^{(0)}_\ell\), which is much closer to the actual \(C^{\text{app}}_\ell\), into the source term of Eq. (13), we may solve for \(P(k)\) with good accuracy. We may continue this procedure iteratively.

C. Numerical calculation

Given an initial condition and cosmological parameters, we can calculate the transfer functions \(f(k)\) and \(g(k)\) numerically. Then, with the angular correlation function (or equivalently anisotropy spectrum), Eq. (13) is solved as a boundary value problem between the neighboring singularities, and hence \(P(k)\) is reconstructed. Hereafter, we treat \(A(k) \equiv k^2 P(k)\) instead of \(P(k)\) itself for the comprehensive display purpose and consistency with the common normalization of fluctuation amplitude.

We adopt the adiabatic initial condition and fiducial cosmological parameter set which is the WMAP team’s best-fit power-law model (12) to calculate the transfer functions. For the reconstruction from the five-year WMAP data, the cosmological parameters are \(h = 0.724, \Omega_b = 0.0432, \Omega_\Lambda = 0.751, \Omega_m = 0.249\), and \(\tau = 0.089\). In this case, the positions of the singularities given by Eq. (13) are \(k_d \simeq 70, 430, 690, \ldots\), where \(d \simeq 1.42 \times 10^3\) Mpc. Around the singularities, the reconstructed spectrum has large numerical errors that are amplified by the observational errors. Using various model power spectra to investigate the accuracy of our inversion formula, we found that we can achieve the reconstruction with good accuracy in the limited range \(120 \lesssim k d \lesssim 380\) or \(8.5 \times 10^{-3}\) Mpc\(^{-1} \lesssim k \lesssim 2.7 \times 10^{-2}\) Mpc\(^{-1}\). In this region, the errors due to our inversion method turn out to be much smaller than those due to observational errors including the cosmic variance.

In practice, we cannot take the upper bound of the integration in the right-hand side of Eq. (13) to be infinite. The integrand in Eq. (13) is oscillating with its amplitude increasing with \(r\). To evaluate the right-hand side of Eq. (13) as finite, we convolve an exponentially decreasing function with a cutoff scale \(r_{\text{cut}}\) into the integration of the Fourier sine transform. As the cutoff scale is made larger, the rapid oscillations of the integrand with increasing amplitude become numerically uncontrollable. On the other hand, if the cutoff scale is made smaller, the resolution in the \(k\)-space becomes worse as \(\Delta k \sim \pi / r_{\text{cut}}\). For both numerical stability and resolution in \(k\)-space, we adopt the optimized cutoff scale of \(r_{\text{cut}} \simeq 0.8d\).

For implementing the inversion scheme, we employ the routines of CMBFAST code (63) to calculate the transfer functions, but we have modified it to adopt much finer resolution than the original one in both \(k\) and \(\ell\) so that we can compute the fine structure of angular power spectra accurately.

D. Monte-Carlo simulation

In order to incorporate observational errors and the cosmic variance and to obtain mutually independent band-powers, we employ Monte-Carlo method to calculate the covariance matrix of the reconstructed power spectrum. Producing 50000 realizations of a temperature anisotropy spectrum based on the WMAP team’s
best-fit power-law model whose statistics obey the likelihood function provided by WMAP team with good precision, we obtain 50000 realizations of a reconstructed primordial spectrum. (The prescription of simulating anisotropy spectra is described in Appendix.) About 1000 samples are sufficient for convergence of the covariance matrix introduced below, which assures the robustness of our conclusion.

The covariance matrix of the reconstructed power spectrum is defined by

$$
K_{ij} \equiv \frac{1}{N} \sum_{\alpha=1}^{N} A_{\alpha}(k_i) A_{\alpha}(k_j) - \frac{1}{N} \sum_{\alpha=1}^{N} A_{\alpha}(k_i) \frac{1}{N} \sum_{\beta=1}^{N} A_{\beta}(k_j)
\equiv \langle A_{\alpha}(k_i) A_{\alpha}(k_j) \rangle_\alpha - \langle A_{\alpha}(k_i) \rangle_\alpha \langle A_{\beta}(k_j) \rangle_\beta,
$$

where $A_{\alpha}(k_i)$ represents the value of the reconstructed power spectrum at $k = k_i$ in the $\alpha$-th realization, and $N = 50000$ as mentioned above. The resultant $K_{ij}$ is a square matrix with its dimension equal to the number of sampling points $k_i$, $N$. When we solve the cosmic inversion equation, we discretize the relevant range of $k$ to more than 2400 points. We calculate $K_{ij}$ at each point to estimate the error of $A(k)$ there. In practice, however, the neighboring $k$-modes are strongly correlated with each other, as mentioned above, and so the number of independent modes are much smaller. Hence we do not need to, and in fact, should not take so many points to calculate $K_{ij}$.

Since $K$ is a real symmetric matrix, it can be diagonalized by a real unitary matrix $U$, to yield

$$
UKU^\dagger = \text{diag} (\lambda_1, \lambda_2, ..., \lambda_N) \equiv \Lambda,
$$

where $\lambda_i$’s are the eigenvalues of $K$. We find they are positive definite as they should be, provided that we take $N$ small enough so that neighboring modes are not degenerate with each other. In the present case, we find that if we take $N \lesssim 50$, the covariance matrix is well behaved in the sense that the following procedure is possible with positive definite $\lambda_i$ and the well-behaved window matrix defined below. In terms of

$$
\Lambda^{-1/2} \equiv \text{diag} \left( \lambda_1^{-1/2}, \lambda_2^{-1/2}, ..., \lambda_N^{-1/2} \right),
$$

we define inverse square root of $K$ as $K^{-1/2} \equiv U^\dagger \Lambda^{-1/2} U$.

We also define a window matrix $W$ by

$$
W_{ij} = \frac{(K^{-1/2})_{ij}}{\sum_{m=1}^{N} (K^{-1/2})_{im}},
$$

which satisfies the normalization condition $\sum_{j=1}^{N} W_{ij} = 1$. Convolving $P_{\alpha}(k_i)$ with this window function, we define a new statistical variable $S_{\alpha}(k_i)$ as

$$
S_{\alpha}(k_i) \equiv \sum_{j=1}^{N} W_{ij} A_{\alpha}(k_j),
$$

whose correlation matrix is diagonal and reads

$$
\langle S_{\alpha}(k_i) S_{\alpha}(k_j) \rangle_\alpha - \langle S_{\alpha}(k_i) \rangle_\alpha \langle S_{\alpha}(k_j) \rangle_\beta = (WK^t W)_{ij} = \left[ \sum_{m=1}^{N} (K^{-1/2})_{im} \right]^{-2} \delta_{ij},
$$

where $W$ denotes a transposed matrix.

Note that in the previous band-power analysis of the power spectrum in the literature, decomposition into band-powers or wavelets is done by hand without calculating the covariance matrix; hence they result in either undersampling to lack traceability of fine structures or oversampling which generates unwanted correlations between different modes.

III. RESULTS OF RECONSTRUCTION

A. Reconstructed primordial spectrum

Figure 1 shows the primordial spectrum $A(k)$ reconstructed from the five-year WMAP data by the cosmic inversion method. In this figure, the solid wavy curve depicts the result of reconstruction and the solid straight line is the best-fit power-law $\Lambda$CDM model obtained by the five-year WMAP observations, namely, the power-law spectrum with $A = 2.39 \times 10^{-9}$ and $n_s = 0.961$, where $A$ is the amplitude of curvature perturbation at $k_0 = 0.002\text{Mpc}^{-1}$. The dotted curves around the power-law are associated 1$\sigma$ standard errors which correspond to the diagonal elements of the error covariance matrix (Fig. 3) calculated by Monte Carlo simulation.

The modulations of the reconstructed spectrum roughly fit inside the 1$\sigma$ borders; therefore this oscillatory deviation from the power-law spectrum is attributed to the fact that we are analyzing only one random realization of an ensemble of a simple power-law spectrum. It is not necessarily required that the inflation model responsible for creation of our Universe should predict such a highly nontrivial spectrum, for our Universe is merely one of the realizations of quantum ensemble that is accompanied by significant fluctuations.

Once the possibility of prominent dips around $kd \simeq 180, 220$ and 350 was pointed out in the analysis of first-year WMAP data [50]. As seen in Fig. 1, the modulations on such scales are fairly degraded in the reconstructed spectrum from the five-year data. Since many of the glitches and bits seen in the first-year WMAP anisotropy spectrum have disappeared in the five-year WMAP spectrum, the updated primordial spectrum is much more smoothed.

B. Restored anisotropy spectrum

Figure 2 illustrates the CMB temperature anisotropy spectrum which we restored from the updated primordial spectrum by adopting the cosmological parameters
of the best-fit power-law model and grafted the best-fit power-law spectrum outside of the investigated range. The effective \(\chi^2\) value in the range of \(120 \leq \ell \leq 380\) for this restored anisotropy spectrum, which we calculate using the likelihood tool provided by the WMAP team [64], is 246, while that for the best-fit power-law model is 273. Although the degree of fit is improved significantly, it does not have the original statistical meaning because we have incorporated a functional degree of freedom.

\section{Band-power analysis}

Despite the fact that our reconstructed power spectrum restores the fine structures of the observed angular power spectrum of CMB anisotropy well, the oscillations observed in the raw reconstructed spectrum may not have true statistical meaning. As mentioned in the previous section and we can see in the covariance matrix Fig. 3 explicitly, neighboring \(k\)-modes of the reconstructed power spectrum are correlated with each other up to the range \(\Delta kd \sim 10\). The origin of the correlation is the convolution with \(X_\ell(k)\) between \(A(k)\) and \(C_\ell\). In particular, mapping the primordial spectrum to \(C_\ell\) erases the information that is responsible for the fine structure whose characteristic scale is much below the correlation width of \(\Delta kd \sim 10\).

It is important to decompose the power spectrum into mutually independent modes for evaluating the statistical significance of the oscillations in the reconstructed spectrum. Here we construct band-powers using the window matrix \(W\) defined in Sec. II D which diagonalizes the covariance matrix and gives mutually independent errors. We take the dimension of the window matrix to \(N = 40\) and decompose \(A(k)\) in the region \(100 \leq kd \leq 400\) to 40 band-powers. Figure 4 shows the window functions for each mode.

Figure 5 is the result of band-power analysis of the five-year WMAP data. In this graph, \(i\)-th data point indicates the value of

\[
S(k_i) = \sum_{j=1}^{N} W_{ij} A(k_j),
\]

and the vertical error bar represents the variance

\[
\left[\langle \langle S_\alpha^2(k_i) \rangle \rangle - \langle \langle S_\alpha(k_i) \rangle \rangle^2 \right]^{1/2} = \left[ \sum_{m=1}^{N} (K^{-1/2})_{im} \right]^{-1}.
\]

Here \(k_i\) is the location of the peak of the \(i\)-th line of the window matrix \(W_{ij}\). The horizontal bar, on the other hand, indicates the width of the window matrix, where the dispersion of the fitted Gaussian is shown (see Fig. 4). We find their typical full width is \(\Delta kd \approx 10\) and the neighboring horizontal bars barely overlap with each other. As is seen clearly in Fig. 5, our band-power reconstruction of the five-year power spectrum basically agrees with the best-fit power-law as a whole.

The \(i\)-th band-power depends on the multipole moment on the relevant angular scale of \(\ell \sim k_i d\) and also on the surrounding multipoles of \(\ell \sim k_i d \pm 5\). Indeed, we found that the statistics of each band-power subject to Gaussian distribution due to the superposition of several multipole moments even though the distribution of \(C_\ell\) is non-Gaussian (see Appendix). By virtue of our band-power analysis, we can also estimate the statistical significance of the reconstructed spectrum itself by evaluating the deviation from the best-fit power-law spectrum at every band simultaneously. We estimated the whole statistical significance of the central 34 bands which correspond to the scales of \(120 \leq kd \leq 380\) and found that, in terms of reduced \(\chi^2\) value which is 0.64, the reconstructed

\[
10^9 A(k)
\]

\[
kd
\]

\[
T_{\text{CMB}}^2 [\ell+1/2, \ell/\pi, [\mu^2]_l]
\]

\[
l
\]
FIG. 3: The error covariance matrix of the reconstructed primordial spectrum, $10^9 A(k)$, calculated by Monte Carlo simulation based on the cosmic inversion method. The central strip is indicating positive correlation and the surrounding strip is indicating negative correlation. Error enhancement on the largest scales of our analysis comes from the numerical instability due to the singularity of transfer function while error regain on the small scale end is due to the domination of measurement error.

spectrum from the five-year WMAP data is $(-1.5\sigma$ realization. It is quite consistent with the best-fit power-law spectrum.

Here the absence of the large modulations around $kd \simeq 180$, 220 and 350 is confirmed again. In the band-power analysis, we find dips around $kd \simeq 180$, 220 and 350 with the statistical significance about $1\sigma$. The nontrivial features reported previously have practically disappeared in the reconstructed spectrum from the five-year data.

Note, however, that in Fig. 5 we find a prominent deviation around $kd \simeq 120$ which would be a true signal of deviation from a simple power-law spectrum if our reconstruction method could be trusted there. While it is possible that this peak is indeed a real signal associated with the feature observed around $\ell \simeq 120$ in the angular power spectrum of the temperature anisotropy, it may simply be an artifact because $kd \simeq 120$ is so close to the singularity $k_s \simeq 70$ that we can count on our method only marginally there.

IV. CONCLUSION

Using the cosmic inversion method, we have reconstructed the primordial power spectrum of curvature perturbations assuming the absence of isocurvature modes and the best-fit values of cosmological parameters for the power-law $\Lambda$CDM model. While the range of accurate reconstruction is rather narrow, $120 \lesssim kd \lesssim 380$, we can reproduce the fine structures of modulations off a simple power-law with which we can recover the highly oscillatory features observed in $C_\ell$.

The statistical significance of the oscillatory structures in the reconstructed power spectrum is difficult to quantify due to the strong correlation among the neighboring $k$-modes. We have therefore performed the covariance matrix analysis to calculate the window matrix, $W$, which diagonalizes the covariance matrix into statistically independent modes. We have chosen the large enough number of band-powers, $N = 40$, to probe the feature of the primordial spectrum as precisely as possible while keeping the overlap of the window functions $W_{ij}$ small enough.
As a result we have found all the independent modes are consistent with the best-fit power-law $\Lambda$CDM model for the five-year WMAP data except possibly for a point around $kd \approx 120$. Whereas, the possible deviation from a simple power-law around $kd \approx 180$, 220 and 350 reported in the analysis of the first-year WMAP data has disappeared. This difference is due to the fact that the observed $C_{\ell}$ in the five-year WMAP data has become much smoother than the first-year counterpart around the first peak.

On the other hand, there still remain some nontrivial deviations from the expected $C_{\ell}$ of a simple power-law spectrum around $\ell \approx kd \approx 40$ even in the five-year data. Unfortunately, the cosmic inversion method cannot probe the primordial spectrum in this region. So we need to develop a different method which can probe the power spectrum for smaller wavenumber region $kd \lesssim 120$ precisely.

From the covariance matrix analysis, we have shown that statistically independent bands of the primordial spectrum have an effective width $\Delta kd \approx 10$. This means that it is difficult to probe the possible fine structures on $A(k)$ predicted by, say, trans-Planckian processes, if the scale width of their characteristic modulation is narrower than $\Delta kd \approx 10$. In other words, even if our result is consistent with a simple power-law spectrum, this does not rule out such possibility of narrow modulation of $\Delta kd \ll 10$.

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APPENDIX: PRESCRIPTION OF GENERATING MOCK ANISOTROPY SPECTRA

In order to perform a proper error analysis of the reconstructed power spectrum, we must prepare a number of realizations of $C_{\ell}$’s which obey the correct statistical distribution function. In the ideal situation with full-sky, uncontaminated observation, each multipole coefficient $a_{\ell m}$ is Gaussian distributed and mutually independent if primordial perturbation is random Gaussian. Consequently each angular multipole $C_{\ell}$ is $\chi^2$-distributed with $2\ell + 1$ degrees of freedom, and different multipoles are uncorrelated. In practice, however, reliable observation can be made in only a finite fraction, $f_{\text{sky}}$, of the sky and different $\ell$–modes are somewhat correlated. In this situation, the following form of likelihood function has been proposed and used in the statistical analysis of WMAP data [11].

$$-2\ln L = \frac{1}{3} \sum_{\ell, \ell' = \ell_{\text{min}}}^{\ell_{\text{max}}} (C_{\ell}^{\ell_{\text{max}}} - C_{\ell}^{th}) F_{\ell \ell'} (C_{\ell'}^{\ell_{\text{max}}} - C_{\ell'}^{th})$$

$$+ \frac{2}{3} \sum_{\ell, \ell' = \ell_{\text{min}}}^{\ell_{\text{max}}} (Z_{\ell}^{\ell_{\text{max}}} - Z_{\ell}^{th}) F_{\ell \ell'} (Z_{\ell'}^{\ell_{\text{max}}} - Z_{\ell'}^{th}),$$

where $C_{\ell} \equiv \ell(\ell + 1)C_{\ell}/2\pi$ and $Z_{\ell} \equiv \ln(C_{\ell} + N_{\ell})$ with $N_{\ell} \equiv \ell(\ell + 1)N_{\ell}/2\pi$. Here $F_{\ell \ell'}$ is the Fisher matrix of $C_{\ell}$ and $F_{\ell \ell'}^{th}$ is related with $F_{\ell \ell'}$ as

$$F_{\ell \ell'}^{th} \equiv (C_{\ell}^{th} + N_{\ell}) F_{\ell \ell'} (C_{\ell}^{th} + N_{\ell}).$$

Superscripts $D$ and $th$ denote data and the theoretical model, respectively.

The value of each component of Fisher matrix $F_{\ell \ell'}$ is given in [22] where we find diagonal components are larger than neighboring off-diagonal elements typically by a factor of $\sim 10^2 - 10^3$. When we depict $C_{\ell}$’s, these diagonal components of the Fisher matrix are used to indicate error bars associated with the respective multipole. However, since different multipoles are correlated we would obtain an erroneous result if we created random samples based on these errors only. We should first find a basis consisting of linear combinations of $C_{\ell}$’s which diagonalize the Fisher matrix to obtain statistically independent quantities.

Here we describe the prescription to find an appropriate basis to diagonalize the Fisher matrix. First we note that, since the Fisher matrix is a real symmetric matrix, it can be diagonalized by a real unitary matrix which we denote by $Q$, namely,

$$Q^\dagger F Q = \text{diag}(r_{\ell_{\text{min}}}, r_{\ell_{\text{min}}+1}, \ldots, r_{\ell_{\text{max}}}) \equiv R,$$

where $r_{\ell_i}$’s are eigenvalues of $F_{\ell \ell'}$ and they are all positive. In terms of

$$R^{1/2} \equiv \text{diag}(r_{\ell_{\text{min}}}^{1/2}, r_{\ell_{\text{min}}+1}^{1/2}, \ldots, r_{\ell_{\text{max}}}^{1/2}),$$

we define matrices

$$V \equiv Q R^{1/2} Q^\dagger = V^\dagger,$$

and

$$V_D \equiv \text{diag}(v_{\ell_{\text{min}}}^{1/2} r_{\ell_{\text{min}}}, \ldots, v_{\ell_{\text{max}}}^{1/2} r_{\ell_{\text{max}}}).$$

that is, $V$ is a “square root” of the Fisher matrix with $V^2 = F$, and $V_D$ is a diagonal matrix with their diagonal components identical to those of $V$. We find

$$(V^{-1} V_D)^\dagger F (V^{-1} V_D) = V_D^2 \equiv \text{diag}(v_{\ell_{\text{min}}}^2 r_{\ell_{\text{min}}}, \ldots, v_{\ell_{\text{max}}}^2 r_{\ell_{\text{max}}}),$$

that is, $F$ can also be diagonalized if it is sandwiched by $(V^{-1} V_D)^\dagger$ and $V^{-1} V_D$. Note, however, that $v_{\ell \ell'}^2$’s are not eigenvalues of $F$ because $V^{-1} V_D$ is not a unitary matrix. Nevertheless, defining
where we find the likelihood function is diagonalized as follows.

\[
-2 \ln \mathcal{L} = \sum_{\ell = \ell_{\text{min}}}^{\ell_{\text{max}}} \left( \frac{v_{\ell}^2}{3} B_{\ell}^2 + \frac{2v_{\ell}^2}{3} B_{\ell}^2 \right) .
\]  
\[
(A.10)
\]

Now \( B_{\ell} \) and \( B_{\ell}' \) are independent of each other. On the other hand, \( \tilde{B}_{\ell}'s \) are dependent on \( B_{\ell}'s \). Reflecting the properties of the Fisher matrix and thanks to the normalization by \( V_D^{-1} \), we find that each diagonal component of \( V_D^{-1} V \) is equal to unity. Among the off-diagonal components in each line, \( (V_D^{-1} V)_{\ell \ell' \pm 2} \) have the largest magnitudes and their magnitude is no larger than 0.01. The other off-diagonal components are even smaller. Therefore, with a good approximation, we may put \( B_{\ell} \approx C_{\ell}^P - C_{\ell}^{th} \) and \( \tilde{B}_{\ell} \approx (C_{\ell}^{th} + N_{\ell})(Z_{\ell}^P - Z_{\ell}^{th}) \). In fact, we do not adopt these approximations but assume a relationship

\[
\tilde{B}_{\ell} = (C_{\ell}^{th} + N_{\ell}) \ln \left( 1 + \frac{B_{\ell}}{C_{\ell}^{th} + N_{\ell}} \right) ,
\]

\[
(A.11)
\]

which is inferred from the above approximate relations, so that each multipole is completely isolated in the likelihood function \( (A.10) \). With this approximation we can generate random numbers for each \( \ell \) separately obeying a probability distribution function (PDF)

\[
\mathcal{P}[B_{\ell}]d \ln B_{\ell} \propto \exp \left[ -\frac{v_{\ell}^2}{6} B_{\ell}^2 \right. \\
-\frac{v_{\ell}^2}{3}(C_{\ell}^{th} + N_{\ell})^2 \ln^2 \left( 1 + \frac{B_{\ell}}{C_{\ell}^{th} + N_{\ell}} \right) \ln B_{\ell} .
\]

\[
(A.12)
\]

Without the approximation \( (A.11) \), it would be computationally formidable to generate many random samples of an angular power spectrum with the appropriate statistical distribution.

We generate random numbers for each \( B_{\ell} \) satisfying the above PDF \( (A.12) \) from which we constitute realizations of \( C_{\ell} \) according to

\[
\begin{pmatrix}
C_{\ell_{\text{min}}} \\
C_{\ell_{\text{min}} + 1} \\
\vdots \\
C_{\ell_{\text{max}}} 
\end{pmatrix} = (V_D^{-1} V)^{-1}
\begin{pmatrix}
B_{\ell_{\text{min}}} \\
B_{\ell_{\text{min}} + 1} \\
\vdots \\
B_{\ell_{\text{max}}} 
\end{pmatrix} +
\begin{pmatrix}
C_{\ell_{\text{min}}}^{th} \\
C_{\ell_{\text{min}} + 1}^{th} \\
\vdots \\
C_{\ell_{\text{max}}}^{th} 
\end{pmatrix} ,
\]

\[
(A.13)
\]

which have the desired correlation properties and PDFs.

We make 50000 random realizations based on the WMAP team’s best-fit power-law angular spectrum according to the above prescription and perform inversion using the cosmic inversion method to calculate the error covariance matrix of the reconstructed primordial spectrum.
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