A SHORT PROOF OF A CONJECTURE ON THE HIGHER CONNECTIVITY OF GRAPH COLORING COMPLEXES

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Abstract. The Hom–complexes were introduced by Lovász to study topological obstructions to graph colorings. It was conjectured by Babson and Kozlov, and proved by Čukić and Kozlov, that $\text{Hom}(G, K_n)$ is $(n - d - 2)$–connected, where $d$ is the maximal degree of a vertex of $G$. We give a short proof of the conjecture.

Introduction

It was conjectured by Babson and Kozlov [1], and proved by Čukić and Kozlov [4], that $\text{Hom}(G, K_n)$ is $(n - d - 2)$–connected, where $d$ is the maximal degree of a vertex of $G$. We give a shorter proof of this, by generalizing the proof of that $\text{Hom}(K_m, K_n)$ is $(n - m - 1)$–connected in Babson and Kozlov [1].

For definitions and basic theorems on Hom–complexes used in this text, see the papers mentioned above, or the survey by Kozlov [6].

1. An analogue of the chromatic number

An independent subset of vertices of a graph is a set, such that no vertices of it are adjacent. The minimal number of sets needed to partition the vertex set of a graph $G$ into independent sets is the chromatic number $\chi(G)$.

Definition 1.1. A covering $I_1, I_2, \ldots, I_k$ of $G$ is a sequence of independent subsets of $V(G)$ such that they partition $V(G)$, and $I_i$ is a maximal independent set in the induced subgraph of $G$ with vertex set $I_i \cup I_{i+1} \cup \ldots \cup I_k$, for all $i$, where $1 \leq i \leq k$.

A partition of $G$ into $\chi(G)$ independent sets can always be transformed to a covering by ordering the independent sets and if needed enlarging them. But a covering can use more than $\chi(G)$ sets. Define $\hat{\chi}(G)$ to be the maximal number of sets in a covering of $G$. Clearly $\hat{\chi}(G) \geq \chi(G)$.

Lemma 1.2. If $d$ is the maximal degree of a vertex of $G$, then $\hat{\chi}(G) \leq d + 1$.

Proof. Let $I_1, I_2, \ldots, I_{\hat{\chi}(G)}$ be a covering of $G$, and $v \in I_{\hat{\chi}(G)}$. For each $i$, where $1 \leq i < \hat{\chi}(G)$, there is a $w \in I_i$ adjacent to $v$, because otherwise $I_i$ would not be a maximal independent set. Hence the degree of $v$ is at least $\hat{\chi}(G) - 1$. The degree of $v$ is at most $d$, thus $\chi(G) \leq d + 1$. $\square$

Lemma 1.3. If $H$ is an induced subgraph of $G$, then $\hat{\chi}(H) \leq \hat{\chi}(G)$.

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Proof. It suffices to prove this when $H$ and $G$ only differ by a vertex $v$ of $G$. Let $I_1, I_2, \ldots, I_{\check{\chi}(H)}$ be a covering of $H$. If $v$ is adjacent to a vertex in each of the sets $I_i$, then $\{v\}, I_1, I_2, \ldots, I_{\check{\chi}(H)}$ is a covering of $G$ and $\check{\chi}(H) + 1 \leq \check{\chi}(G)$. Otherwise, let $I_j$ be the first set in the covering such that $v$ is not adjacent to any vertex of $I_j$. Then $I_1, I_2, \ldots, I_j \cup \{v\}, \ldots, I_{\check{\chi}(H)}$ is a covering of $G$, and $\check{\chi}(H) \leq \check{\chi}(G)$. \qed

Lemma 1.4. If $I$ is a maximal independent set of $G$, then $\check{\chi}(G) > \check{\chi}(G \setminus I)$.

Proof. Let $I_1, I_2, \ldots, I_{\check{\chi}(G \setminus I)}$ be a covering of $G \setminus I$. Then $I, I_1, I_2, \ldots, I_{\check{\chi}(G \setminus I)}$ is a covering of $G$ with $1 + \check{\chi}(G \setminus I)$ sets. \qed

2. Higher connectivity of $\text{Hom}(G, K_n)$

Lemma 2.1. If $I$ is an independent set of $G$, and $I' \subset I$, then $\Delta = \{\eta \in \text{Hom}(G, K_n)|n \in \eta(i) \Rightarrow i \in I\}$ collapses onto $\Delta' = \{\eta \in \text{Hom}(G \setminus (I \setminus I'), K_n)|n \in \eta(i) \Rightarrow i \in I'\}$.

Proof. It suffices to prove this when $I \setminus I' = \{v\}$. Let $\eta_1, \eta_2, \ldots, \eta_k$ be an ordering of $\{\eta \in \Delta|n \notin \eta(v)\}$ such that if $\eta(w) \supseteq \eta'(w)$ for all $w \in V(G)$ then $\eta$ is not after $\eta'$. Define $\eta^*_i$ as $\eta^*_i(w) = \eta_i(w)$ for $w \neq v$, and $\eta^*_i(v) = \eta_i(v) \cup \{n\}$. Each successive removal of $\eta^*_i$ together with $\eta_i$ from $\Delta$ for $i = 1, 2, \ldots, k$ is a collapse step. The cells left are $\Delta'' = \{\eta \in \Delta|\eta(v) = \{n\}\}$. Finally, there is a bijection between the face posets of $\Delta'$ and $\Delta''$ by extending each $\eta \in \Delta'$ with $\eta(v) = \{n\}$. \qed

The main use of lemma 2.1 is when $I' = \emptyset$. Then $n \notin \eta(w)$ for all $\eta \in \Delta'$ and $w \in V(G) \setminus I$, so $\Delta' = \text{Hom}(G \setminus I, K_{n-1})$. Another way to prove the lemma is to use discrete Morse theory [5].

We will use a variation of a Nerve Lemma, (Björner 10.6(ii) [6], Björner et.al. [7]). A regular cell complex $\Delta$ is $m$-connected if there is a family of subcomplexes $\{\Delta_i\}$ such that $\Delta = \cup \Delta_i$, all of the subcomplexes $\Delta_i$ are $m$-connected, and all of the intersection of several $\Delta_i$’s are $(m-1)$-connected.

Theorem 2.2. $\text{Hom}(G, K_n)$ is $(n - \check{\chi}(G) - 1)$-connected.

Proof. We use induction on $\check{\chi}(G)$ and on $n - \check{\chi}(G)$. When $\check{\chi}(G) = 1$, $G$ have no edges, so $\text{Hom}(G, K_n)$ is contractible, and in particular $(n - \check{\chi}(G) - 1)$-connected.

If $n - \check{\chi}(G) = 0$ then $n \geq \check{\chi}(G)$ so $\text{Hom}(G, K_n)$ is non-empty, and $(n - \check{\chi}(G) - 1)$-connected.

For all $I \in \mathcal{I}$, let $\Delta_I = \{\eta \in \text{Hom}(G, K_n)|n \in \eta(i) \Rightarrow i \in I\}$, where $\mathcal{I}$ is the family of maximal independent subsets of $G$. Clearly $\text{Hom}(G, K_n) = \cup_{I \in \mathcal{I}} \Delta_I$. By lemma 2.1, the complex $\Delta_I$ is homotopy equivalent to $\text{Hom}(G \setminus I, K_{n-1})$, which is $((n-1) - (\check{\chi}(G) - 1) - 1)$-connected by lemma 1.3 and induction. If $\mathcal{I} \supseteq \mathcal{I}' \neq \emptyset$ then $\cap_{I \in \mathcal{I}' \setminus \mathcal{I}} \Delta_I = \{\eta \in \text{Hom}(G, K_n)|n \in \eta(i) \Rightarrow i \in \cap_{I \in \mathcal{I}'} I\}$ is homotopy equivalent to $\text{Hom}(G \setminus (\cap_{I \in \mathcal{I}'} I), K_{n-1})$ by lemma 2.1 and $((n-1) - \check{\chi}(G) - 1)$-connected by lemma 1.3 and induction. By the Nerve Lemma we are done. \qed

Corollary 2.3. $\text{Hom}(G, K_n)$ is $(n - d - 2)$-connected.

Proof. Lemma 1.2 states that $\check{\chi}(G) \leq d + 1$. \qed
REFERENCES

1. E. Babson, D.N. Kozlov, *Complexes of graph homomorphisms*. \texttt{arXiv:math.CO/0310056} 23 pages, to appear in Israel J. Math.

2. A. Björner, Topological Methods, in: “Handbook of Combinatorics” (eds. R. Graham, M. Grötschel, and L. Lovász), North-Holland, 1995, pp. 1819–1872.

3. A. Björner, L. Lovász, S.T. Vrećica, R.T. Živaljević, *Chessboard complexes and matching complexes*, J. London Math. Soc. (2) 49 (1994), 25–49.

4. S. Ćukić, D.N. Kozlov, *Higher connectivity of graph coloring complexes*. \texttt{arXiv:math.CO/0410335} 16 pages, to appear in Int. Math. Res. Notices.

5. R. Forman, *Morse theory for cell complexes*, Adv. Math. 134, no. 1, (1998), 90-145.

6. D.N. Kozlov, *Chromatic numbers, morphism complexes, and Stiefel-Whitney characteristic classes*, invited contribution to *Geometric Combinatorics*, IAS/Park City Mathematics Series 14, American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, N.J.

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