The Minimal Position of a Stable Branching Random Walk

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Abstract In this paper, a branching random walk \((V(x))\) in the boundary case is studied, where the associated one dimensional random walk is in the domain of attraction of an \(\alpha\)-stable law with \(1 < \alpha < 2\). Let \(M_n\) be the minimal position of \((V(x))\) at generation \(n\). We established an integral test to describe the lower limit of \(M_n - \frac{1}{\alpha} \log n\) and a law of iterated logarithm for the upper limit of \(M_n - (1 + \frac{1}{\alpha}) \log n\).

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1. INTRODUCTION

We consider a discrete-time one-dimensional branching random walk. It starts with an initial ancestor particle located at the origin. At time 1, the particle dies, producing a certain number of new particles. These new particles are positioned according to the distribution of the point process \(\Theta\). At time 2, these particles die, each giving birth to new particles positioned (with respect to the birth place) according to the law of \(\Theta\). And the process goes on with the same mechanism. We assume the particles produce new particles independently of each other. This system can be seen as a branching tree \(T\) with the origin as the root. For each vertex \(x\) on \(T\), we denote its position by \(V(x)\). The family of the random variables \((V(x))\) is usually referred as a branching random walk (Biggins [4]). The generation of \(x\) is denoted by \(|x|\).

We assume throughout the remainder of the paper, including in the statements of theorems and lemmas, that

\[
E\left(\sum_{|x|=1} 1\right) > 1, \quad E\left(\sum_{|x|=1} e^{-V(x)}\right) = 1, \quad E\left(\sum_{|x|=1} V(x)e^{-V(x)}\right) = 0.
\]

Condition (1.1) means that the branching random walk \((V(x))\) is supercritical and in boundary case (see for example, Biggins and Kyprianou [7]). Every branching random walk satisfying certain mild integrability assumptions can be reduced to this case by some renormalization; see Jaffuel [14] for more details.

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Denote $M_n = \min_{|x|=n} V(x)$, i.e., the minimal position at generation $n$. We introduce the conditional probability $P^*(\cdot) := P(\cdot|\text{non-extinction})$. Under (1.1), $M_n \to \infty, P^*-a.s.$ (See for example, [5] and [17]). The asymptotic behaviors of $M_n$ have been extensively studied in [1], [2], [8], [13], etc. In particular, under (1.1) and certain exponential integrability conditions, Hu and Shi [13] obtained the following:

\[
\limsup_{n \to \infty} \frac{M_n}{\log n} = \frac{3}{2}, \quad P^*-a.s.
\]

\[
\liminf_{n \to \infty} \frac{M_n}{\log n} = \frac{1}{2}, \quad P^*-a.s.
\]

It showed that there is a phenomena of fluctuation at the logarithmic scale. Aidekon [2] proved the convergence in law of $M_n - \frac{3}{2} \log n$ when (1.1) and the following two conditions hold:

\[
(1.2) \quad E\left( \sum_{|x|=1} V^2(x)e^{-V(x)} \right) < \infty,
\]

\[
(1.3) \quad E(X (\log_+ X)^2 + \tilde{X} (\log_+ \tilde{X})) < \infty,
\]

where $X := \sum_{|x|=1} e^{-V(x)}$, $\tilde{X} := \sum_{|x|=1} V(x)+e^{-V(x)}$, and $V(x) := \max\{V(x), 0\}$. Later, Aidekon and Shi [3] proved that under (1.1)-(1.3),

\[
\liminf_{n \to \infty} (M_n - \frac{1}{2} \log n) = -\infty, \quad P^*-a.s.
\]

Based on this result, Hu [11] established the second order limit under the same assumptions (1.1)-(1.3):

\[
\liminf_{n \to \infty} \frac{M_n - \frac{1}{2} \log n}{\log \log n} = -1, \quad P^*-a.s.
\]

When (1.1), (1.3) and a higher order integrability condition for $V(x)$ hold (i.e. $E\left( \sum_{|x|=1} (V(x)_+)^3 e^{-V(x)} \right) < \infty$), the upper limit was obtained by Hu [12]:

\[
\limsup_{n \to \infty} \frac{M_n - \frac{3}{2} \log n}{\log \log n} = 1.
\]

Throughout the following, $c, c', c_1, c_2, \cdots$ will denote some positive constants whose value may change from place to place. $f(x) \sim g(x)$ as $x \to \infty$ means that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$; $f(x) = O(g(x))$ as $x \to \infty$ means that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$.

In this paper, we shall consider the random walk by assuming that

\[
(1.4) \quad E\left( \sum_{|x|=1} 1\{V(x) \leq -y\} e^{-V(x)} \right) = O(y^{-\alpha-\varepsilon}), \quad y \to \infty;
\]

\[
(1.5) \quad E\left( \sum_{|x|=1} 1\{V(x) \geq y\} e^{-V(x)} \right) \sim \frac{c}{y^\alpha}, \quad y \to \infty;
\]

\[
(1.6) \quad E(X(\log_+ X)^\alpha + \tilde{X}(\log_+ \tilde{X})^{\alpha-1}) < \infty,
\]

where $\alpha \in (1, 2), \varepsilon > 0, c > 0$. Under (1.4) and (1.5), in Section 2, we shall see that there is one-dimensional random walk $\{S_n\}$ corresponding to $(V(x))$, where $S_1$ belongs to the domain of attraction of a spectrally positive stable law. We call $(V(x))$ a stable random
walk, we shall study the asymptotic behavior of $M_n$ for the stable random walk $(V(x))$ under the conditions (1.1), (1.4)--(1.6). Our main results are the following Theorems 1.1--1.5.

**Theorem 1.1.** Assume (1.1), (1.4)--(1.6). For any nondecreasing function $f$ satisfying 
$$\lim_{x \to \infty} f(x) = \infty,$$ 
we have

\begin{equation}
\mathbb{P}^\star(M_n - \frac{1}{\alpha} \log n < -f(n), \text{ i.o.}) = \begin{cases} 0 & \text{if } \int_0^\infty \frac{1}{t e^{f(t)}}dt < \infty, \\ 1 & \text{if } \int_0^\infty \frac{1}{t e^{f(t)}}dt = \infty. \end{cases}
\end{equation}

The behavior of the minimal position $M_n$ is closely related to the so-called additive martingale $(W_n)_{n \geq 0}$:

$$W_n := \sum_{|u|=n} e^{-V(u)}, n \geq 0.$$ 

By [6] and [17], $W_n \to 0$ almost surely as $n \to \infty$. A similar integral test for the upper limits of $W_n$ can be described as follows:

**Theorem 1.2.** Assume (1.1), (1.4)--(1.6). For any nondecreasing function $f$ satisfying 
$$\lim_{x \to \infty} f(x) = \infty,$$ 
we have 

\begin{equation}
\limsup_{n \to \infty} n^{\frac{1}{\alpha}} W_n f(n) = \begin{cases} 0 & \text{if } \int_0^\infty \frac{1}{t f(t)}dt < \infty, \\ \infty & \text{if } \int_0^\infty \frac{1}{t f(t)}dt = \infty. \end{cases}
\end{equation}

**Theorem 1.3.** Assume (1.1), (1.4)--(1.6). We have

\begin{equation}
\liminf_{n \to \infty} \frac{M_n - \frac{1}{\alpha} \log n}{\log \log n} = -1, \mathbb{P}^\star-\text{a.s.}
\end{equation}

**Remark 1.4.** For the random walk $(V(x))$ satisfying (1.1)--(1.3), where the one-dimensional random walk associated with $(V(x))$ has finite variance, Hu [11] Theorem 1.1, 1.2, Proposition 1.3] established the corresponding theorems for $M_n$ and $W_n$. Theorem 1.1, 1.2, 1.3 are extensions of them for the stable random walk under (1.1), (1.4)--(1.6). Now the one-dimensional random walk \{$S_n$\} associated with $(V(x))$ has no finite variance (see Section 2 for details).

**Theorem 1.5.** Assume (1.1), (1.4)--(1.6). We have

$$\limsup_{n \to \infty} \frac{M_n - (1 + \frac{1}{\alpha}) \log n}{\log \log \log n} \geq 1, \mathbb{P}^\star-\text{a.s.}$$

**Remark 1.6.** The upper limit for $M_n$ is established in Hu [12] under (1.1)--(1.3) and finite third order moment 

$$\mathbb{E} \left( \sum_{|x|=1} (V(x)_+)^3 e^{-V(x)} \right) < \infty.$$ 

While in this paper, under (1.1), (1.4)--(1.6), for $k \geq \alpha$, $(V(x))$ no longer satisfies the integrability condition 

$$\mathbb{E} \left( \sum_{|x|=1} (V(x)_+)^k e^{-V(x)} \right) < \infty.$$ 

In this case, we have not got the condition for the upper limit of $\frac{M_n - (1+\frac{1}{\alpha}) \log n}{\log \log \log n}$. In Theorem 1.5, only a lower bound is obtained.
2. Stable random walk

In this section, we first introduce an one-dimensional random walk associated with the branching random walk.

For \( a \in \mathbb{R} \), we denote by \( P_a \) the probability distribution associated to the branching random walk \((V(x))\) starting from \( a \), and \( E_a \) the corresponding expectation. For any vertex \( x \) on the tree \( T \), we denote the shortest path from the root \( \emptyset \) to \( x \) by \((\emptyset, x) := \{x_0, x_1, x_2, \ldots, x_\|$\rangle\). Here \( x_i \) is the ancestor of \( x \) at the \( i \)-th generation. For any \( \mu, \nu \in T \), we use the partial order \( \mu < \nu \) if \( \mu \) is an ancestor of \( \nu \). Under \((1.1)\), there exists a sequence of independently and identically distributed real-valued random variables \( S_1, S_2 - S_1, S_3 - S_2, \ldots \), such that for any \( n \geq 1, a \in \mathbb{R} \) and any measurable function \( g : \mathbb{R}^n \rightarrow [0, \infty) \),

\[
E_a \left( \sum_{|x|=n} g(V(x_1), \ldots, V(x_n)) \right) = E_a \left( e^{S_n-a}g(S_1, \ldots, S_n) \right),
\]

where, under \( P_a \), we have \( S_0 = a \) almost surely. \((2.1)\) is called the \textit{many-to-one formula}. We will write \( P \) and \( E \) instead of \( P_0 \) and \( E_0 \). Since \( E(\sum_{|x|=1} V(x)e^{-V(x)}) = 0 \), we have \( E(S_1) = 0 \). By \((1.4)\) and \((1.5)\), it is not difficult to see that \( ES_k^n = \infty \) for \( k \geq \alpha \). Under conditions \((1.4)\) and \((1.5)\), \( S_1 \) belongs to the domain of attraction of a spectrally positive stable law with characteristic function

\[
G(t) := \exp \left\{ -c_0|t|^\alpha \left( 1 - i \frac{t}{|t|} \tan \frac{\pi \alpha}{2} \right) \right\}, \quad c_0 > 0.
\]

The following are some estimates on \((S_n)\), which are key in the proofs of the main theorems.

**Lemma 2.1.** Let \( 0 < \lambda < 1 \). There exist positive constants \( c_1, c_2, \ldots, c_5 \) such that for any \( a \geq 0, b \geq -a, 0 \leq u \leq v \) and \( n \geq 1 \),

\[
\begin{align*}
(2.2) \quad & P(S_n \geq -a) \leq c_1 \frac{(1 + a)^{n\alpha}}{n^{2\alpha}}, \\
(2.3) \quad & P(-S_n \geq -a) \leq c_2 \frac{(1 + a)^{n\alpha - 1}}{n^{1 - \frac{\alpha}{2}}}, \\
(2.4) \quad & P(S_n \leq b, S_n \geq -a) \leq c_3 \frac{(1 + a)(1 + a + b)^{\alpha}}{n^{1 + \frac{\alpha}{2}}}, \\
(2.5) \quad & P(S_n \geq -a, \min_{i \in [\lambda n, n]} S_i \geq b, S_n \in [b + u, b + v]) \leq c_4 \frac{(1 + v)^{\alpha - 1}(1 + v - u)(1 + a)}{n^{1 + \frac{\alpha}{2}}}, \\
(2.6) \quad & P(S_n \geq -a, \min_{\lambda n \leq i < n} S_i > b, S_n \leq b) \leq c_5 (1 + a)n^{-1 - \frac{\alpha}{2}},
\end{align*}
\]

where \( S_n := \min_{0 \leq i \leq n} S_i \) and \(-S_n := \min_{0 \leq i \leq n} (-S_i)\).

**Proof.** The proofs of \((2.2)-(2.5)\) have been given in \([10]\) Lemmas 2.1–2.4]. Here we only prove \((2.6)\). Let \( f(x) := P(S_1 \leq -x) \). Denote the event in \((2.6)\) by \( \mathbb{E}_{\text{ext}} \). Applying the
Markov property of \((S_i)\) at \(n - 1\), and using (2.5) we have that
\[
P(E_{2.6}) \leq \sum_{j=0}^{\infty} f(j) P\left( S_{n-1} \geq -a, \min_{\lambda_n \leq i < n} S_i > b, b + j < S_{n-1} \leq b + j + 1 \right)
\]
\[
\leq c (1 + a)n^{-1-\frac{\alpha}{2}} \sum_{j=0}^{\infty} f(j)(2 + j)^{\alpha-1}.
\]
(2.7)

By (1.4),
\[
\sum_{j=0}^{\infty} f(j)(2 + j)^{\alpha-1} \leq cE((-S_1)^{\alpha}1_{\{S_1 \leq 0\}}) < \infty.
\]

Then the proof is completed. \(\square\)

### 3. Proofs of Theorems 1.1–1.3

In this section, first we prove Theorem 1.1 and Theorem 1.2. Noticing that \(W_n \geq e^{-M_n}\), we only need to prove the the convergence part in Theorem 1.2, i.e.,
\[
\int_0^{\infty} dt \frac{t}{f(t)} < \infty \Rightarrow \limsup_{n \to \infty} \frac{n^\frac{1}{\alpha}W_n}{f(n)} = 0, \quad P^*-\text{a.s.}
\]
and the divergence part in Theorem 1.1, i.e.,
\[
\int_0^{\infty} dt \frac{t}{e^{f(t)}} = \infty \Rightarrow P^*(M_n - \frac{1}{\alpha} \log n < -f(n), \quad \text{i.o.}) = 1.
\]

We define the set containing brothers of vertex \(x\) by \(\Omega(x)\), i.e., \(\Omega(x) = \{y : y|_{|y|-1} = x|_{|x|-1}, y \neq x\}\). For \(\beta \geq 0\), define
\[
W^\beta_n := \sum_{|x|=n} e^{-V(x)}1_{\{V(x) \geq -\beta\}},
\]
where \(V(x) := \min_{0 \leq i \leq |x|} V(x_i)\).

To prove (3.1), we need the following lemma.

**Lemma 3.1.** Assume (1.1), (1.4)–(1.6). For any \(\beta \geq 0\), there exists a constant \(c\) such that for any \(1 < n \leq m\) and \(\lambda > 0\), we have
\[
P\left( \max_{n \leq k \leq m} k^\frac{1}{\alpha}W^\beta_k > \lambda \right) \leq c \left( \frac{\log n}{n^\frac{1}{\alpha}} + \frac{1}{\lambda} \right) + \frac{1}{\lambda}.
\]

**Proof.** We introduce another martingale related to \(W^\beta_k\):
\[
W_k^{(\beta,n)} := \sum_{|x|=k} e^{-V(x)}1_{\{V(x_n) \geq -\beta\}}, \quad n \leq k \leq m + 1,
\]
where \(V(x_n) := \min_{1 \leq i \leq n} V(x_i)\). Therefore,
\[
P\left( \max_{n \leq k \leq m} k^\frac{1}{\alpha}W^\beta_k > \lambda \right) \leq P\left( \max_{n \leq k \leq m} k^\frac{1}{\alpha}W^{(\beta,n)}_k > \lambda \right) + P\left( \min_{n \leq k \leq m} \min_{|x|=k} V(x) < -\beta \right).
\]
By the branching property, for \(n \leq k \leq m\), we have that
\[
E(W^{(\beta,n)}_k | F_k) = W^{(\beta,n)}_k.
\]
Hence, by Doob’s maximal inequality,

\[ P( \max_{n \leq k \leq m} k^{1/\alpha} W_k^{(\beta,n)} \geq \lambda) \leq \frac{m^{\frac{1}{\alpha}}}{\lambda} \mathbb{E}(W_n^{(\beta,n)}) = \frac{m^{\frac{1}{\alpha}}}{\lambda} \mathbb{E}(W_n^{\beta}). \]

From (2.1) and (2.2), it follows that

\[ P( \max_{n \leq k \leq m} k^{1/\alpha} W_k^{(\beta,n)} \geq \lambda) \leq c_1 \frac{1}{\lambda} \binom{m}{n}^{\frac{1}{\alpha}}. \]

On the other hand, By Aidekon [2, P. 1403] we know

\[ P( \inf_{x \in T} V(x) < -x) \leq e^{-x} \text{ for } x \geq 0. \]

Hence

\[ P( \min_{n \leq k \leq m} \min_{|x| = k} V(x) < -\beta) \leq \frac{1}{n} + \sum_{k=n}^{m} \mathbb{E}\left( \sum_{|x| = k} 1 \{ V(x) < -\beta, V(x_n) \geq -\beta, \ldots, V(x_{k-1}) \geq -\beta, V(x) \geq -\log n \} \right). \]

By (2.1) and (2.2),

\[ P( \min_{n \leq k \leq m} \min_{|x| = k} V(x) < -\beta) \leq \frac{1}{n} + \sum_{k=n}^{m} \mathbb{E}(S_k) \leq c \frac{\log n}{n^{\frac{1}{\alpha}}}, \]

which together with (3.5), completes the proof.

\[ \blacksquare \]

**Proof of (3.1).** Let \( n_j = 2^j \). According to Lemma 3.1, for all large \( j \) we have

\[ P( \max_{n_j \leq k \leq n_{j+1}} k^{1/\alpha} W_k^{\beta} > f(n_j)) \leq c \left( \frac{\log n_j}{n_j^{\frac{1}{\alpha}}} + \frac{2^{\frac{1}{\alpha}}}{f(n_j)} \right). \]

By our assumption for \( f \),

\[ \sum_{j \geq j_0} \frac{1}{f(n_j)} \leq \sum_{j \geq j_0} \frac{1}{\log 2} \int_{n_{j-1}}^{n_j} \frac{1}{f(x)} \, dx < \infty. \]

Hence

\[ \sum_{j \geq j_0} P( \max_{n_j \leq k \leq n_{j+1}} k^{1/\alpha} W_k^{\beta} > f(n_j)) < \infty. \]

By Borel-Cantelli Lemma, for all large \( k \),

\[ k^{1/\alpha} W_k^{\beta} \leq f(k), \quad \mathbb{P}\text{-a.s.} \]

Letting \( \beta \to \infty \), we have \( k^{1/\alpha} W_k \leq f(k) \), \( \mathbb{P}\text{-a.s.} \). As a consequence,

\[ \lim_{k \to \infty} \frac{k^{1/\alpha} W_k}{f(k)} \leq 1, \quad \mathbb{P}\text{-a.s.} \]

Replacing \( f \) by \( \varepsilon f \), and letting \( \varepsilon \to 0 \), we complete the proof.
Fix $K \geq 0$. Now we define for $n < k \leq \alpha n$,

$$A_{k}^{(n,\lambda)} := \{ x : |x| = k, V(x_i) \geq a_{i}^{(n,\lambda)}, 0 \leq i \leq k, V(x) \leq \frac{1}{\alpha} \log n - \lambda + K \},$$

$$B_{k}^{(n,\lambda)} := \{ x : |x| = k, \sum_{u \in \Omega(x_i+1)} (1 + (V(u) - a_{i}^{(n,\lambda)}))_+ e^{-(V(u) - a_{i}^{(n,\lambda)})} \leq c e^{-b_{i}^{(k,n)}}, 0 \leq i \leq k-1 \},$$

where $a_{i}^{(n,\lambda)} = 1_{\{\frac{\lambda}{4} n < i \leq k\}}(\frac{1}{\alpha} \log n - \lambda)$, $b_{i}^{(k,n)} = 1_{\{0 \leq i \leq \frac{\lambda}{4} n\}} i^\frac{1}{2} + 1_{\{\frac{\lambda}{4} n < i \leq \alpha n\}} (k - i)^\frac{1}{2}$, $K > 0$, $

\gamma = \frac{1}{\alpha(n+1)}$ and $c'$ is a positive constant chosen as in [10, Lemma 7.1].

Lemmas 3.2 and 3.3 are preparing works for the proof of (3.2).

**Lemma 3.2.** Assume (1.1), (1.4) - (1.6). There exist some positive constants $K$ and $c_{6}, c_{7}$ such that for all $n \geq 2$, $0 \leq \lambda \leq \frac{1}{2\alpha} \log n$,

$$c_{6} e^{-\lambda} \leq \mathbf{P} \left( \bigcup_{k=n+1}^{\alpha n} A_{k}^{(n,\lambda)} \cap B_{k}^{(n,\lambda)} \right) \leq c_{7} e^{-\lambda},$$

**Proof.** The proof of the lower bound goes in the same way in [10, Lemma 7.1] by replacing $\frac{1}{\alpha} \log n$ to $\frac{1}{\alpha} \log n - \lambda$. Let $s := \frac{1}{\alpha} \log n - \lambda$. Applying (2.1) and (2.5), we get

$$\mathbf{P} \left( \bigcup_{k=n+1}^{\alpha n} A_{k}^{(n,\lambda)} \right) \leq \sum_{k=n+1}^{\alpha n} \mathbf{E} \left( \sum_{|x| = k} 1_{\{V(x_i) \geq a_{i}^{(n,\lambda)}, i \leq k, V(x) \leq s + K\}} \right)$$

$$= \sum_{k=n+1}^{\alpha n} \mathbf{E} \left( e^{S_k} 1_{\{S_i \geq a_{i}^{(n,\lambda)}, i \leq k, S_k \leq s + K\}} \right)$$

$$\leq \sum_{k=n+1}^{\alpha n} e^{s + K} \mathbf{P} \left( S_i \geq a_{i}^{(n,\lambda)}, i \leq k, S_k \leq s + K \right)$$

$$\leq c \sum_{k=n+1}^{\alpha n} e^{s + K} \frac{1}{n^{1 + \frac{\lambda}{2\alpha}}}$$

$$\leq c e^{-\lambda},$$

completing the proof.

Denote the natural filtration of the branching random walk by $(\mathcal{F}_n, n \geq 0)$. Here we introduce the well-known change-of-probabilities setting in Lyons [17] and spinal decomposition. With the nonnegative martingale $W_n$, we can define a new probability measure $\mathbf{Q}$ such that for any $n \geq 1$,

$$(3.7) \quad \mathbf{Q}_{|\mathcal{F}_n} := W_n \cdot \mathbf{P} |_{\mathcal{F}_n},$$

where $\mathbf{Q}$ is defined on $\mathcal{F}_\infty := \vee_{n \geq 0} \mathcal{F}_n$. Similarly we denote by $\mathbf{Q}_a$ the probability distribution associated to the branching random walk starting from $a$, and $\mathbf{E}_Q$ the corresponding expectation related $\mathbf{Q}(:= \mathbf{Q}_0)$. Let us give a description of the branching random walk under $\mathbf{Q}$. We start from one single particle $\omega_0 := \varnothing$, located at $V(\omega_0) = 0$. At time $n + 1$, each particle $v$ in the $n$th generation dies and gives birth to a point process independently distributed as $(V(x), |x| = 1)$ under $\mathbf{P}_{V(v)}$ except one particle $\omega_n$, which dies and produces a point process distributed as $(V(x), |x| = 1)$ under $\mathbf{Q}_{V(\omega_n)}$. While $\omega_{n+1}$ is chosen to be $\mu$ among the children of $\omega_n$, proportionally to $e^{-V(\mu)}$. Next we state the following fact about the spinal decomposition.
Fact 7.1 (Lyons [17]). Assume (1.1).

(i) For any $|x| = n$, we have

$$Q(\omega_n = x|\mathcal{F}_n) = \frac{e^{-V(x)}}{W_n}.$$ 

(ii) The spine process $(V(\omega_n))_{n \geq 0}$ under $Q$ has the distribution of $(S_n)_{n \geq 0}$ (introduced in Section 2) under $P$.

(iii) Let $G_\infty := \sigma\{\omega_j, V(\omega_j), \Omega(\omega_j), (V(u))_{u \in \Omega(\omega_j)}, j \geq 1\}$ be the $\sigma$-algebra of the spine and its brothers. Denote by $\{\mu\nu, |\nu| \geq 0\}$ the subtree of $T$ rooted at $\mu$. For any $\mu \in \Omega(\omega_k)$, the induced branching random walk $(V(\mu\nu), |\nu| \geq 0)$ under $Q$ and conditioned on $G_\infty$ is distributed as $P_{V(\mu)}$.

For $n \geq 2$ and $0 \leq \lambda \leq \frac{1}{2\alpha} \log n$, we define

$$E(n, \lambda) := \bigcup_{k=n+1}^{\alpha m} (A_k^{(n,\lambda)} \cap B_k^{(n,\lambda)}).$$

Lemma 3.3. Assume (1.1), (1.4) – (1.6). There exists $c > 0$ such that for any $n \geq 2$, $0 \leq \lambda \leq \frac{1}{2\alpha} \log n$, $m \geq 4n$ and $0 \leq \mu \leq \frac{1}{2\alpha} \log m$,

$$P(E(n, \lambda) \cap F(m, \mu)) \leq c e^{-\lambda - \mu} + c e^{-\mu \log n \over n^\alpha}.$$ 

Proof. For convenience, we write $s := \frac{1}{\alpha} \log n - \lambda$, $t := \frac{1}{\alpha} \log m - \mu$.

$$P(E(n, \lambda) \cap F(m, \mu)) \leq E(1_{E(n, \lambda)} \sum_{k=m+1}^{\alpha m} \sum_{|x|=k} 1_{\{x \in A_k^{(m, \lambda)} \cap B_k^{(m, \mu)}\}})$$

$$= \sum_{k=m+1}^{\alpha m} E_Q \left(1_{E(n, \lambda)} e^{V(\omega_k)} 1_{\{\omega_k \in A_k^{(m, \lambda)} \cap B_k^{(m, \mu)}\}} \right)$$

$$\leq e^{t+K} \sum_{k=m+1}^{\alpha m} \sum_{l=n+1}^{\alpha m} E_Q \left( \sum_{|x|=l} 1_{\{x \in A_l^{(n, \lambda)} \cap B_l^{(n, \lambda)} \cap A_k^{(m, \mu)} \cap B_k^{(m, \mu)}\}} \right)$$

$$= e^{t+K} \sum_{k=m+1}^{\alpha m} \sum_{l=n+1}^{\alpha m} I(k, l).$$

Decomposing the sum on the brothers of the spine, we obtain

$$I(k, l) = Q(\omega_l \in A_l^{(n, \lambda)} \cap B_l^{(n, \lambda)}, \omega_k \in A_k^{(m, \mu)} \cap B_k^{(m, \mu)})$$

$$+ \sum_{p=1}^{l} E_Q \left(1_{\{\omega_k \in A_k^{(m, \mu)} \cap B_k^{(m, \mu)}\}} \sum_{x \in \Omega(\omega_p)} f_{k,l,p}(V(x)) \right)$$

$$= I_1(k, l) + \sum_{p=1}^{l} J(k, l, p),$$

(3.10)
where \( f_{k,l,p}(V(x)) := \mathbb{E}_q\left( \sum_{u \geq x, |u| = l} 1_{\{u \in A_i^{(m), \nu} \cap B_j^{(n), \lambda}\}} |G_\infty\right) \). Recalling Fact 7.1(iii), we obtain,

\[
\begin{align*}
 f_{k,l,p}(x) &\leq \mathbb{E}_x\left( \sum_{|\nu| = l-p} 1_{\{V(\nu) \geq a^{(m, \lambda)}_{i+p}, 0 \leq i \leq l-p, V(\nu) \leq s+K\}} \right) \\
 &\leq e^{-x+K} \mathbb{P}_x(S_i \geq a^{(m, \lambda)}_{i+p}, 0 \leq i \leq l-p, S_{l-p} \leq s + K),
\end{align*}
\]

(3.11)

where the last step is from (2.1). To estimate \( \sum_{p=1}^l J(k, l, p) \), we break the sum into two parts. Firstly consider the case \( 1 \leq p \leq \alpha n \). By (2.5), we have

\[
J(k, l, p) \leq ce^{-x+s+K} \frac{1 + x_+}{n^{1+\frac{1}{\alpha}}},
\]

Consequently,

\[
\sum_{p=1}^{\alpha n} J(k, l, p) \leq ce^s n^{-1+\frac{1}{\alpha}} \sum_{p=1}^{\alpha n} \mathbb{E}_q\left( 1_{\{\omega_k \in A_{k}^{(m, \mu)} \cap B_k^{(n, \nu)}\}} \sum_{x \in \Omega(\omega_p)} (1 + V(x)) e^{-V(x)} \right) \\
\leq ce^s n^{-1+\frac{1}{\alpha}} \sum_{p=1}^{\alpha n} \mathbb{E}_q\left( 1_{\{\omega_k \in A_{k}^{(m, \mu)} \cap B_k^{(n, \nu)}\}} e^{-(p-1)\frac{K}{1+\frac{1}{\alpha}}} \right),
\]

where the last inequality comes from the definition of \( B_k^{(m, \mu)} \). Note that by (2.5),

\[
\mathbb{Q}(\omega_k \in A_{k}^{(m, \mu)}) = \mathbb{P}(t \leq S_k \leq t + K, S_i \geq a^{(m, \mu)}_i, 0 \leq i \leq k) \leq cm^{-1-\frac{1}{\alpha}}
\]

for all \( m < k \leq \alpha n \). It follows that

\[
(3.12) \quad \sum_{p=1}^{\alpha n} J(k, l, p) \leq ce^s n^{-1+\frac{1}{\alpha}} \mathbb{Q}(\omega_k \in A_{k}^{(m, \mu)}) \leq ce^s n^{-1+\frac{1}{\alpha}} m^{-1+\frac{1}{\alpha}}.
\]

On the other hand, when \( \frac{\alpha n}{4} < p \leq l \), returning to (3.11),

\[
\begin{align*}
 f_{k,l,p}(x) &\leq e^{-x+s+K} \mathbb{P}_x(S_i \geq a^{(m, \lambda)}_{i+p}, 0 \leq i \leq l-p, S_{l-p} \leq s + K) \\
 &\leq ce^{-x+s+K} \frac{1 + (x-s)_+}{(1+\frac{1}{\alpha})},
\end{align*}
\]

which is from (2.4). Hence

\[
\sum_{\frac{\alpha n}{4} < p \leq l} J(k, l, p) \\
\leq ce^s + K \sum_{\frac{\alpha n}{4} < p \leq l} \frac{1}{(1+\frac{1}{\alpha})} \mathbb{E}_q\left( 1_{\{\omega_k \in A_{k}^{(m, \mu)} \cap B_k^{(n, \nu)}\}} \sum_{x \in \Omega(\omega_p)} e^{-V(x)} (1 + (V(x) - s)_+) \right) \\
\leq ce^s \sum_{\frac{\alpha n}{4} < p \leq l} \frac{e^{-(p-1)\frac{K}{1+\frac{1}{\alpha}}}}{(1+\frac{1}{\alpha})} \mathbb{Q}(\omega_k \in A_{k}^{(m, \mu)}).
\]

As a consequence,

\[
(3.13) \quad \sum_{\frac{\alpha n}{4} < p \leq l} J(k, l, p) \leq ce^s e^{-n\frac{K}{m^{1+\frac{1}{\alpha}}}}.
\]
It remains to estimate \( I_1(k, l) \) for \( n < l < \alpha n < \frac{\alpha m}{\alpha} < k \leq \alpha m \). Clearly,

\[
I_1(k, l) \leq Q(\omega_l \in A_l^{(n, \lambda)}, \omega_k \in A_k^{(m, \mu)})
\]

\[
= P(S_i \geq a_i^{(n, \lambda)}, 0 \leq i \leq l, S_i \leq s + K, S_j \geq a_j^{(m, \mu)}, 0 \leq j \leq k, S_k \leq t + K).
\]

We use the Markov property at \( l \) and (2.5) to arrive at

\[
I_1(k, l) \leq \frac{c}{(k-l)^{1+\frac{1}{\alpha}} E((1+S_l)1_{\{S_i \geq a_i^{(n,\lambda)}, 0 \leq i \leq l, S_i \leq s+K\}})}
\]

\[
\leq c(1+s+K)(k-l)^{-1-\frac{1}{\alpha} l^{-1-\frac{1}{\alpha}}},
\]

which together with (3.12) and (3.13) leads to

\[
I(k, l) \leq ce^s(n^{-1-\frac{1}{\alpha} m^{-1-\frac{1}{\alpha}}} + e^{-n^\alpha m^{-1-\frac{1}{\alpha}}}) + c(1+s+K)(k-l)^{-1-\frac{1}{\alpha} l^{-1-\frac{1}{\alpha}}}.
\]

Recalling (3.9), we have

(3.14)

\[
P(E(n, \lambda) \cap F(m, \mu)) \leq e^{t+K} \sum_{k=m+1}^{\alpha m} \sum_{l=\frac{n}{\alpha}}^{\alpha n} c\left(e^s(n^{-1-\frac{1}{\alpha} m^{-1-\frac{1}{\alpha}}} + (1+s+K)(k-l)^{-1-\frac{1}{\alpha} l^{-1-\frac{1}{\alpha}}})\right)
\]

\[
\leq c e^{-\lambda - \mu} + c e^{-\mu \log n/\alpha^\alpha}.
\]

\[\Box\]

**Proof of (3.2).** Let \( f \) be the nondecreasing function such that \( \int_0^\infty \frac{dt}{te^f(t)} = \infty \). By Erdős [9], we can assume that \( \frac{1}{2} \log(\log t) \leq f(t) \leq 2 \log(\log t) \) for all large \( t \) without any loss of generality. Let

\[
F_x := \{ M_n + x \leq \frac{1}{\alpha} \log n - f(n), \text{ i.o.}, x \in \mathbb{R} \}.
\]

We are going to prove that there exists \( c_8 > 0 \) such that for any \( x \),

(3.15)

\[
P(F_x) \geq c_8.
\]

Define \( n_i = 2^i, \lambda_i = f(n_{i+1}) + x, \) and \( E_i = E(n_i, \lambda_i) \). It is easy to see for any \( x \in \mathbb{R} \), we can choose \( i_0 = i_0(x) \) such that \( 0 \leq \lambda_i \leq \frac{1}{\alpha} \log n_i \) for \( i \geq i_0 \). According to Lemma 3.2 and Lemma 3.3, there exists \( c > 0 \) such that for any \( i \geq i_0, j \geq i + 2, \)

\[
\frac{1}{c} e^{-\lambda_i} \leq P(E_i) \leq c e^{-\lambda_i}, \quad i \geq i_0,
\]

\[
P(E_i \cap E_j) \leq c e^{-\lambda_i - \lambda_j + \lambda_j \log n_i/n_i^{\alpha}}.
\]

It follows that

\[
\sum_{i=i_0}^{k} P(E_i) \geq c \sum_{i=i_0}^{k} e^{-\lambda_i},
\]

(3.16)

\[
\sum_{i,j=i_0}^{k} P(E_i \cap E_j) \leq c \left( \sum_{i=i_0}^{k} e^{-\lambda_i} \right)^2 + c \left( \sum_{i=i_0}^{k} e^{-\lambda_i} \right) \left( \sum_{i=1}^{\infty} \log n_i/n_i^{\alpha} \right).
\]
Note that \( \sum_{i=0}^{n} e^{-\lambda_i} \geq c \sum_{i=0}^{n} e^{-f(n+1)} \geq c \sum_{i=0}^{n} e^{\int_{t_{i+1}}^{t_i} \frac{dt}{n+1}} = \infty. \) Thus, we can find a constant \( c_8 \) (notice that our choice of \( c_8 \) does not depend on \( x \)) such that

\[
\limsup_{k \to \infty} \frac{\sum_{i,j=1}^{k} P(E_i \cap E_j)}{\left( \sum_{i=1}^{k} P(E_i) \right)^2} \leq c_8.
\]

By Kochen and Stone’s version of Borel-Cantelli Lemma \([15]\), we have \( P(E_i, \ i.o.) \geq c_8 \), which implies (3.15). Let \( F_\infty := \bigcap_{x \in \mathbb{Z}} F_x \). We have \( P(F_\infty) \geq c_8 \) since \( F_x \) are non-increasing on \( x \). We then use the branching property to obtain

\[
P(F_\infty|F_k) = 1_{\{Z_k > 0\}}(1 - \prod_{|x| = k} (1 - P(V(x)(F_\infty)))) \leq 1_{\{Z_k > 0\}}(1 - (1 - c_8)Z_k).
\]

Letting \( k \to \infty \) in the above inequality, we conclude that

\[
1_{F_\infty} = 1_{\{\text{non-extinction}\}} P - a.s.
\]

The divergence part (3.2) is now proved.

\[\square\]

**Proofs of Theorems 1.1–1.2.** They are immediate by combining the above proofs of (3.1) and (3.2).

\[\square\]

**Proof of Theorem 1.3** In Theorem 1.2, taking \( f(n) = \log \log n \) and \((1 + \varepsilon) \log \log n \) for \( \varepsilon > 0 \), we obtain the desired result.

\[\square\]

### 4. Proof of Theorem 1.4

**Lemma 4.1.** Assume (1.1), (1.3)–(1.6). For any \( \lambda > 0 \), there is \( c_9 > 0 \) such that for each \( n \geq 1 \),

\[
P\left( M_n < (1 + \frac{1}{\alpha}) \log n - \lambda \right) \leq c_9(1 + \lambda)e^{-\lambda}.
\]

**Proof.** If we have proved that for any \( \lambda, \beta > 0 \), there exists \( c \) such that for any \( n \geq 1 \),

\[
P\left( M_n < (1 + \frac{1}{\alpha}) \log n - \lambda, \ \min_{|u| \leq n} V(u) \geq -\beta \right)
\]

\[
\leq c(1 + \beta)e^{-\lambda}\left(1 + \frac{\left(1 + (1 + \frac{1}{\alpha}) \log n - \lambda\right)_+}{n^{1/\alpha}}\right)^{2\alpha+1},
\]

then by the following fact

\[
P(\inf_{u \in T} V(u) < -\lambda) \leq e^{-\lambda},
\]

we can obtain the proof of Lemma 4.1.

Now we turn to prove (1.1). For brevity we write \( b = (1 + \frac{1}{\alpha}) \log n - \lambda - 1 \). Note that we can assume \( b > 1 > -\beta \), otherwise there is nothing to prove for (1.1). For \( |u| = n \) such that \( V(u) < b + 1 \), either \( \min_{\frac{n}{2} \leq j \leq n} V(u_j) > b \), or \( \min_{\frac{n}{2} \leq j \leq n} V(u_j) \leq b \). For the latter case, we shall consider the first \( j \in [\frac{n}{2}, n] \) such that \( V(u_j) \leq b \). Then

\[
P\left( M_n < (1 + \frac{1}{\alpha}) \log n - \lambda, \ \min_{|u| \leq n} V(u) \geq -\beta \right) \leq P(H_1) + P(H_2).
\]
with

\[ H_1 := \{ \exists |u| = n : V(u) < b + 1, V(u) \geq -\beta, \min_{\frac{n}{2} \leq j \leq n} V(u_j) > b \}, \]

\[ H_2 := \bigcup_{\frac{n}{2} \leq j \leq n} \{ \exists |u| = n : V(u) < b + 1, V(u) \geq -\beta, \min_{\frac{n}{2} \leq i < j} V(u_i) > b, V(u_j) \leq b \}. \]

By (2.1) and (2.5), we have

\[ \text{By (2.1) and (2.5), we have} \]

\[ \begin{align*}
\mathbb{P}(H_1) &\leq \mathbb{E} \left( \sum_{|u|=n} 1\{V(u) < b+1, V(u) \geq -\beta, \min_{\frac{n}{2} \leq j \leq n} V(u_j) > b\} \right) \\
&= \mathbb{E} \left( e^{S_n} 1\{S_n < b+1, S_n \geq -\beta, \min_{\frac{n}{2} \leq j \leq n} S_j > b\} \right) \\
&\leq c e^{b(1 + \beta)} n^{-1 - \frac{1}{\alpha}} \\
&\leq c (1 + \beta) e^{-\lambda}.
\end{align*} \]

(4.3)

To deal with \( \mathbb{P}(H_2) \), we consider \( v = u_j \) and use the notation \( |u|_v := |\mu| - |\nu| = n - j \) and \( V_v(u) := V(u) - V(v) \) for \( |u| = n \) and \( v < u \). Then by the Markov property,

\[ \begin{align*}
\mathbb{P}(H_2) &\leq \sum_{\frac{n}{2} \leq j \leq n} \mathbb{E} \left( \sum_{|v|=j} 1\{V(v) \geq -\beta, \min_{\frac{n}{2} \leq i < j} V(v_i) \geq b, V(v) \leq b\} \sum_{|u|=n-j} 1\{V_v(u) \leq b+1-V(v) \min_{\frac{n}{2} \leq i \leq n} V_v(u_i) \geq -\beta-V(v)\} \right) \\
&= \sum_{\frac{n}{2} \leq j \leq n} \mathbb{E} \left( \sum_{|v|=j} 1\{V(v) \geq -\beta, \min_{\frac{n}{2} \leq i < j} V(v_i) \geq b, V(v) \leq b\} \phi(V(v), n-j) \right) \\
&=: E_{(4.4)} + E'_{(4.4)}
\end{align*} \]

(4.4)

(4.5)

where \( E_{(4.4)} \) denotes the sum \( \sum_{\frac{n}{2} \leq j \leq n} \) and \( E'_{(4.4)} \) denotes the sum \( \sum_{\frac{n}{2} \leq j \leq n} \) in (4.5), and

\[ \phi(x, n-j) := \mathbb{E} \left( \sum_{|u|=n-j} 1\{V_v(u) \leq b+1-V(v) \min_{\frac{n}{2} \leq i \leq n} V_v(u_i) \geq -\beta-V(v)\} \left| V(v) = x \right. \right) \\
= \mathbb{E} \left( e^{S_{n-j}} 1\{S_{n-j} \leq b+1-x, S_{n-j} \geq -\beta-x\} \right). \]

It follows from (2.4) that

\[ \phi(x, n-j) \leq c (1 + \beta + x) (2 + \beta + b)^{\alpha} (n-j+1)^{-1 - \frac{1}{\alpha}} e^{b-x}. \]

(4.6)
By (4.6), (2.1) and then (2.4), we obtain that
\[
E_{\mathbf{1}(\mathbf{4.6})} \leq c \sum_{\frac{n}{2} \leq j \leq \frac{3n}{4}} (2 + b + \beta)^{\alpha} n^{-1-\frac{1}{\alpha}} e^b E \left( (1 + \beta + S_j) 1_{\{S_j \geq -\beta, \min_{\frac{n}{4} \leq i < j} S_i > b, S_j \leq b\}} \right)
\]
\[
\leq c (2 + b + \beta)^{\alpha+1} e^{-\beta} \sum_{\frac{n}{2} \leq j \leq \frac{3n}{4}} P(\mathcal{S}_j \geq -\beta, S_j \leq b)
\]
\[
\leq c (1 + \beta) (2 + b + \beta)^{2\alpha+1} e^{-\lambda} \sum_{\frac{n}{2} \leq j \leq \frac{3n}{4}} j^{-1-\frac{1}{\alpha}}
\]
(4.7)
\[
\leq c (1 + \beta) \frac{(1 + \beta + (1 + \frac{1}{\alpha}) \log n - \lambda)^{2\alpha+1}}{n^\frac{1}{\alpha}} e^{-\lambda}.
\]
Meanwhile, by the estimate \( \phi(x, n - j) \leq e^{b+1-x} \), we get that
\[
E_{\mathbf{1}(\mathbf{4.6})} \leq \sum_{\frac{3n}{4} \leq j \leq n} E \left( \sum_{|v| = j} 1_{\{V(v) \geq -\beta, \min_{\frac{n}{4} \leq i < j} V(n_i) > b, V(v) \leq b\}} e^{b+1-V(v)} \right)
\]
\[
= e^{b+1} \sum_{\frac{3n}{4} \leq j \leq n} P \left( S_j \geq -\beta, \min_{\frac{n}{4} \leq i < j} S_i > b, S_j \leq b \right)
\]
\[
\leq c e^b (1 + \beta) n^{-1-\frac{1}{\alpha}}
\]
(4.8)
\[
\leq c (1 + \beta) e^{-\lambda}.
\]

Combing the estimates (4.2)–(4.8), we get (4.1), and then complete the proof. \( \square \)

**Proof of Theorem 1.5**
Consider large integer \( j \). Let \( n_j := 2^j \) and \( \lambda_j := a \log \log n_j \) with some constant \( 0 < a < 1 \). Put
\[
K_j := \{ M_{n_j} > (1 + \frac{1}{\alpha}) \log n_j + \lambda_j \}.
\]

Recall that if the system dies out at generation \( n_j \), then \( M_{n_j} = \infty \). Define \( M_{n_j}^{(u)} \) for the subtree \( \mathbb{T}_u \) just as \( M \) for \( \mathbb{T} \). Then
\[
K_j = \{|u| = n_{j-1}, M_{n_j-n_{j-1}}^{(u)} > (1 + \frac{1}{\alpha}) \log n_j + \lambda_j - V(u)\}.
\]

By the branching property at \( n_{j-1} \) we obtain
\[
P^*(K_j | F_{n_{j-1}}) = \prod_{|u|=n_{j-1}} P^* \left( M_{n_j-n_{j-1}} \geq (1 + \frac{1}{\alpha}) \log n_j + \lambda_j - x \right) \bigg|_{x=V(u)}.
\]

By Theorem 1.3 a.s. for all large \( j \), \( M_{n_{j-1}} \geq \frac{1}{2^a} \log n_{j-1} \sim c j \), hence \( x \equiv V(u) \gg \lambda_j \) since \( \lambda_j \sim a \log \log j \). By Lemma 4.1, on \( \{M_{n_{j-1}} \geq \frac{1}{2^a} \log n_{j-1}\} \), for some constant \( c > 0 \) and all \( |u| = n_{j-1}, \)
\[
P^* \left( M_{n_j-n_{j-1}} < (1 + \frac{1}{\alpha}) \log n_j + \lambda_j - x \right) \bigg|_{x=V(u)} \leq c V(u) e^{-(V(u) - \lambda_j)}.
\]
For sufficiently large $j$, it follows that
\[ P^*(K_j | F_{n_{j-1}}) \geq 1 \{ M_{n_{j-1}} \geq \frac{\log n_{j-1}}{\alpha} \} \prod_{|u| = n_{j-1}} \left( 1 - eV(u)e^{-(V(u) - \lambda_j)} \right) \]
\[ \geq 1 \{ M_{n_{j-1}} \geq \frac{\log n_{j-1}}{\alpha} \} \exp \left( -2c \sum_{|u| = n_{j-1}} V(u)e^{-(V(u) - \lambda_j)} \right) \]
\[ = 1 \{ M_{n_{j-1}} \geq \frac{\log n_{j-1}}{\alpha} \} \exp \left( -2ce^{\lambda_j} D_{n_{j-1}} \right). \]

By [10, Theorem 1.1], $D_{n_{j-1}} \to D_\infty$ a.s. Recalling $e^{\lambda_j} \sim (\log j)^a$ with $a < 1$, we get that
\[ \sum_j P^*(K_j | F_{n_{j-1}}) = \infty, \quad a.s. \]
which according to Lévy’s conditional form of Borel-Cantelli’s lemma ([16, Corollary 68]), implies that $P^*(K_i, i.o.) = 1$. Thus
\[ \limsup_{n \to \infty} \frac{M_n - (1 + \frac{1}{\alpha}) \log n}{\log \log \log n} \geq a, \quad P^* - a.s. \]
The proof is completed by letting $a \to 1$. 

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