On the Skitovich–Darmois Theorem for the Group of $p$-Adic Numbers

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Abstract Let $\Omega_p$ be the group of $p$-adic numbers, and let $\xi_1$ and $\xi_2$ be independent random variables with values in $\Omega_p$ and distributions $\mu_1$ and $\mu_2$. Let $\alpha_j, \beta_j$ be topological automorphisms of $\Omega_p$. Assuming that the linear forms $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$ are independent, we describe possible distributions $\mu_1$ and $\mu_2$ depending on the automorphisms $\alpha_j, \beta_j$. This theorem is an analogue for the group $\Omega_p$ of the well-known Skitovich–Darmois theorem, where a Gaussian distribution on the real line is characterized by the independence of two linear forms.

Keywords Group of $p$-adic numbers · Characterization theorem

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1 Introduction

The classical characterization theorems of mathematical statistics were extended to different algebraic structures such as locally compact Abelian groups, Lie groups, quantum groups, and symmetric spaces (see e.g., [1–3,6–13], and also [4], where one can find necessary references). In particular, much attention has been devoted to the study of the Skitovich–Darmois theorem, where a Gaussian distribution is characterized by the independence of two linear forms, for some classes of locally compact Abelian groups, and the Heyde theorem, where a Gaussian distribution is characterized by the symmetry of the conditional distribution of one linear form given another. In these cases, coefficients of linear forms are topological automorphisms of a group.
The article is devoted to the Skitovich–Darmois theorem for the group of $p$-adic numbers $\Omega_p$. To the best of our knowledge, the characterization problems for the group $\Omega_p$ have not been studied earlier.

We recall that according to the classical Skitovich–Darmois theorem, if $\xi_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, are independent random variables, $\alpha_j$, $\beta_j$ are nonzero constants, and the linear forms $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ and $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$ are independent, then all random variables $\xi_j$ are Gaussian. This theorem was generalized by Ghurye and Olkin to the case when $\xi_j$ are independent vectors in the space $\mathbb{R}^m$ and the coefficients $\alpha_j$, $\beta_j$ are nonsingular matrices. They proved that the independence of $L_1$ and $L_2$ implies that all random vectors $\xi_j$ are Gaussian ([15, Ch. 3]).

Let $X$ be a second countable locally compact Abelian group, $\operatorname{Aut}(X)$ be the group of topological automorphisms of $X$, $\xi_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables with values in $X$ and distributions $\mu_j$. Consider the linear forms $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ and $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$, where $\alpha_j$, $\beta_j \in \operatorname{Aut}(X)$. In the earlier papers, the main attention was paid to the following problem: For which groups $X$ the independence of $L_1$ and $L_2$ implies that all $\mu_j$ are either Gaussian distributions, or belong to a class of distributions, which we can consider as a natural analogue of the class of Gaussian distributions. This problem was studied for different classes of locally compact Abelian groups ([4, Ch. V]). It turned out that in contrast to the classical situation, the cases of $n = 2$ and an arbitrary $n$ are essentially different. For $n = 2$, this problem was solved for the class of finite Abelian groups in [1], for the class of compact totally disconnected Abelian groups in [8], and for the class of discrete Abelian groups in [9]. We also note that group analogues of the Skitovich–Darmois theorem for $n = 2$ are closely connected with the positive definite functions of product type introduced by Schmidt (see [5, 17]).

In the article, we continue these investigations. On the one hand, we prove that the Skitovich–Darmois theorem, generally speaking, fails for the group of $p$-adic numbers $\Omega_p$. On the other hand, we give the complete descriptions of all automorphisms $\alpha_j$, $\beta_j \in \operatorname{Aut}(\Omega_p)$ such that the independence of the linear forms $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$ implies that $\mu_1$ and $\mu_2$ are idempotent distributions, i.e., shifts of the Haar distributions of compact subgroups of $\Omega_p$. We note that since $\Omega_p$ is a totally disconnected group, the Gaussian distributions on $\Omega_p$ are degenerated ([16, Ch. 4]).

2 Definitions and Notation

We will use some results of the duality theory for the locally compact Abelian groups (see [14]). Before we formulate the main theorem, we recall some definitions and agree on notation. For an arbitrary locally compact Abelian group $X$ let $Y = X^*$ be its character group, and $(x, y)$ be the value of a character $y \in Y$ at an element $x \in X$. If $K$ is a closed subgroup of $X$, we denote by $A(Y, K) = \{ y \in Y : (x, y) = 1 \text{ for all } x \in K \}$ its annihilator. If $\delta : X \mapsto X$ is a continuous homomorphism, then the adjoint homomorphism $\overline{\delta} : Y \mapsto Y$ is defined by the formula $(x, \overline{\delta} y) = (\delta x, y)$ for all $x \in X$, $y \in Y$. We note that $\delta \in \operatorname{Aut}(X)$ if and only if $\overline{\delta} \in \operatorname{Aut}(Y)$. Denote by $I$ the identity automorphism of a group.
Let $M^1(X)$ be the convolution semigroup of probability distributions on $X$. For a distribution $\mu \in M^1(X)$ denote by

$$
\hat{\mu}(y) = \int_X (x, y) d\mu(x)
$$

its characteristic function (Fourier transform), and by $\sigma(\mu)$ the support of $\mu$. For $\mu \in M^1(X)$, we define the distribution $\check{\mu} \in M^1(X)$ by the formula $\check{\mu}(E) = \mu(-E)$ for any Borel set $E \subset X$. Observe that $\hat{\mu}(y) = \check{\mu}(y)$. Let $K$ be a compact subgroup of $X$. Denote by $m_K$ the Haar distribution on $K$. We note that the characteristic function of $m_K$ is of the form

$$
\hat{m}_K(y) = \begin{cases} 
1, & y \in A(Y, K), \\
0, & y \notin A(Y, K).
\end{cases}
$$

Denote by $I(X)$ the set of all idempotent distributions on $X$, i.e., the set of shifts of the Haar distributions $m_K$ of the compact subgroups $K$ of $X$. Let $x \in X$. Denote by $E_x$ the degenerate distribution concentrated at the point $x$.

3 The Main Theorem

Let $p$ be a prime number. We need some properties of the group of $p$-adic numbers $\Omega_p$ (see [14, §10]). As a set $\Omega_p$ coincides with the set of sequences of integers of the form $x = (\ldots, x_{-n}, x_{-n+1}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, x_{n+1}, \ldots)$, where $x_n \in \{0, 1, \ldots, p-1\}$, such that $x_n = 0$ for $n < n_0$, where the number $n_0$ depends on $x$. We correspond to each element $x \in \Omega_p$ the series $\sum_{k=-\infty}^{\infty} x_k p^k$. Addition and multiplication of the series are defined in a natural way, and they define the operations of addition and multiplication in $\Omega_p$. With respect to these operations, $\Omega_p$ is a field. Denote by $\Lambda_k$ a subgroup of $\Omega_p$ consisting of $x \in \Omega_p$ such that $x_n = 0$ for $n < k$. The subgroup $\Lambda_0$ is called the group of $p$-adic integers and is denoted by $\Delta_p$. We note that $\Lambda_k = p^k \Delta_p$. The family of the subgroups $\{\Lambda_k\}_{k=-\infty}^{\infty}$ forms an open basis at zero of the group $\Omega_p$ and defines a topology on $\Omega_p$. With respect to this topology the group $\Omega_p$ is locally compact, noncompact, and totally disconnected. We note that the group $\Omega_p$ is represented as a union $\Omega_p = \bigcup_{j=-\infty}^{\infty} p^j \Delta_p$. The character group $\Omega_p^*$ of the group $\Omega_p$ is topologically isomorphic to $\Omega_p$, and the value of a character $y \in \Omega_p^*$ at an element $x \in \Omega_p$ is defined by the formula

$$
(x, y) = \exp\left[2\pi i \left(\sum_{n=-\infty}^{\infty} x_n \left(\sum_{s=n}^{\infty} y_{-s} p^{-s+n-1}\right)\right)\right],
$$

where for given $x$ and $y$ the sums in (2) actually are finite. Each automorphism $\alpha \in \text{Aut}(\Omega_p)$ is of the form $\alpha g = x_{\alpha} g$, $g \in \Omega_p$, where $x_{\alpha} \in \Omega_p$, $x_{\alpha} \neq 0$. For $\alpha \in \text{Aut}(\Omega_p)$, we identify the automorphism $\alpha \in \text{Aut}(\Omega_p)$ with the corresponding element $x_{\alpha} \in \Omega_p$, i.e., when we write $\alpha g$, we suppose that $\alpha \in \Omega_p$. We note that $\bar{\alpha} = \alpha$. Denote
by $\Delta^0$ the subset of $\Omega_p$ consisting of all invertible in $\Delta_p$ elements, $\Delta^0_p = \{ x \in \Omega_p : x_n = 0$ for $n < 0, x_0 \neq 0 \}$. We note that each element $g \in \Omega_p$ is represented in the form $g = p^k c$, where $k$ is an integer, and $c \in \Delta^0_p$. Hence, multiplication on $c$ is a topological automorphism of the group $\Delta_p$.

Denote by $\mathbb{Z}(p^\infty)$ the set of rational numbers of the form $\{ k/p^n : k = 0, 1, \ldots, p^n - 1, n = 0, 1, \ldots \}$. If we define the operation in $\mathbb{Z}(p^\infty)$ as addition modulo 1, then $\mathbb{Z}(p^\infty)$ is transformed into an Abelian group which we consider in the discrete topology. Obviously, this group is topologically isomorphic to the multiplicative group of all $p^n$th roots of unity, where $n$ goes through the set of nonnegative integers, considering in the discrete topology. For a fixed $n$ denote by $\mathbb{Z}(p^n)$ a subgroup of $\mathbb{Z}(p^\infty)$ consisting of all elements of the form $\{ k/p^n : k = 0, 1, \ldots, p^n - 1 \}$. Note that the group $\mathbb{Z}(p^n)$ is topologically isomorphic to the multiplicative group of all $p^n$th roots of unity, considering in the discrete topology. Observe that the groups $\mathbb{Z}(p^\infty)$ and $\Delta_p$ are the character groups of one another.

Now we will prove the main result of the paper. We will do this for the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$, where $\alpha \in \operatorname{Aut}(\Omega_p)$, and then will show how the general case is reduced to this one.

**Theorem 1** Let $X = \Omega_p$, $\alpha \in \operatorname{Aut}(X)$, $\alpha = p^k c$, $c \in \Delta^0_p$. Then the following statements hold.

1. Assume that either $k = 0$ or $|k| = 1$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. Assume that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$ are independent. Then
   1(i) If $k = 0$, then $\mu_1, \mu_2 \in I(X)$; moreover if $c = (0, 0, \ldots, 0, 1, c_1, \ldots)$, then $\mu_1$ and $\mu_2$ are degenerate distributions;
   1(ii) If $|k| = 1$, then either $\mu_1 \in I(X)$ or $\mu_2 \in I(X)$.
2. If $|k| \geq 2$, then there exist independent random variables $\xi_1$ and $\xi_2$ with values in $X$ and distributions $\mu_1$ and $\mu_2$ such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$ are independent whereas $\mu_1, \mu_2 \notin I(X)$.

To prove Theorem 1, we need some lemmas. Let $\xi$ be a random variable with values in a second countable locally compact Abelian group $X$ and distribution $\mu$. Taking into account that the characteristic function of the distribution $\mu$ is the expectation $E[(\xi, y)]$, exactly as in the classical case, we may prove the following statement.

**Lemma 1** Let $X$ be a second countable locally compact Abelian group. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. Then the independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$, where $\alpha \in \operatorname{Aut}(X)$, is equivalent to the fact that the characteristic functions $\widehat{\mu}_1(y)$ and $\widehat{\mu}_2(y)$ satisfy the equation

$$
\widehat{\mu}_1(u+v)\widehat{\mu}_2(u+\alpha v) = \widehat{\mu}_1(u)\widehat{\mu}_2(u)\widehat{\mu}_1(v)\widehat{\mu}_2(\alpha v), \quad u, v \in Y.
$$

**Lemma 2** Let $X = \Omega_p$ and $\alpha \in \operatorname{Aut}(X)$, $\alpha = p^k c$, $c \in \Delta^0_p$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$ such that $\mu_j(y) \geq 0$, $j = 1, 2$. Assume that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$
are independent. Then there exists a subgroup $B = p^j \Delta_p$ in $Y$ such that $\hat{\mu}_j(y) = 1$ for $y \in B, \ j = 1, 2$.

**Proof** We use the fact that the family of the subgroups $\{ p^j \Delta_p \}_{j=-\infty}^\infty$ forms an open basis at zero of the group $Y$. Since $\hat{\mu}_1(0) = \hat{\mu}_2(0) = 1$, we can choose $m$ in such a way that $\hat{\mu}_j(y) > 0$ for $y \in L = p^m \Delta_p, \ j = 1, 2$. Put $M = L$ if $k \geq 0$, and $M = p^{-k}L$ if $k < 0$. Then $M$ is a subgroup of $L$ and $\alpha(M) \subset L$. Put $\psi_j(y) = -\log \hat{\mu}_j(y), \ y \in L, \ j = 1, 2$.

By Lemma 1, the characteristic functions $\hat{\mu}_j(y)$ satisfy Eq. (3). Taking into account that $\alpha = \alpha$, we get from (3) that the functions $\psi_j(y)$ satisfy the equation

$$\psi_1(u + v) + \psi_2(u + \alpha v) = \psi_1(u) + \psi_2(u) + \psi_1(v) + \psi_2(\alpha v), \ u \in L, \ v \in M.$$ (4)

Integrating Eq. (4) over the group $L$ with respect to the Haar distribution $dm_L(u)$ and using the fact that the Haar distribution $m_L$ is $L$-invariant, we obtain

$$\psi_1(v) + \psi_2(\alpha v) = 0, \ v \in M.$$

It follows from this that $\psi_1(v) = \psi_2(\alpha v) = 0$ for $v \in M$, and hence $\hat{\mu}_1(y) = \hat{\mu}_2(\alpha y) = 1, \ y \in M$. Put $B = M \cap \alpha(M)$. Then $B$ is the required subgroup. Lemma 2 is proved. □

**Lemma 3** ([4, §2]) Let $X$ be a second countable locally compact Abelian group, and $\mu \in M^1(X)$. Let $E = \{ y \in Y : \hat{\mu}(y) = 1 \}$. Then $E$ is a closed subgroup of $Y$, the characteristic function $\hat{\mu}(y)$ is $E$-invariant, i.e., $\hat{\mu}(y + h) = \hat{\mu}(y)$ for all $y \in Y, \ h \in E$, and $\sigma(\mu) \subset A(X, E)$.

An Abelian group $G$ is called $p$-prime if the order of every element of $G$ is a power of $p$. Denote by $\mathcal{P}$ the set of prime numbers. The following result follows from the proof of Theorem 1 in [8] (see also [4, §13]).

**Lemma 4** Let $X$ be a group of the form

$$\mathcal{P}_{p \in \mathcal{P}} (\Delta_p^{n_p} \times G_p),$$

where $n_p$ is a nonnegative integer, and $G_p$ is a finite $p$-prime group, may be $G_p = \{ 0 \}$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. If the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$, where $\alpha \in \text{Aut}(X)$, are independent, then $\mu_j = m_K \ast E_{x_j}$, where $K$ is a compact subgroup of $X$, and $x_j \in X, \ j = 1, 2$.

**Lemma 5** ([4, §13]) Let $X$ be a second countable locally compact Abelian group, $\xi_1$ and $\xi_2$ be independent identically distributed random variables with values in $X$ and distribution $m_K$, where $K$ is a compact subgroup of $X$. Let $\alpha \in \text{Aut}(X)$. Then the following statements are equivalent:
(i) the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$ are independent;
(ii) $(I - \alpha)(K) \supset K$.

Proof of Theorem 1 Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. Assume that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$ are independent. By Lemma 1, the characteristic functions of the distributions $\mu_j$ satisfy Eq. (3). It is obvious that the characteristic functions of the distributions $\tilde{\mu}_j$ also satisfy Eq. (3). This implies that the characteristic functions of the distributions $\nu_j = \mu_j \ast \tilde{\mu}_j$ satisfy Eq. (3) as well. We have $\tilde{\nu}_j(y) = |\tilde{\mu}_j(y)|^2 \geq 0$, $j = 1, 2$. Hence, when we prove Statements 1(i) and 1(ii), we may assume without loss of generality that $\mu_j(y) \geq 0$, $j = 1, 2$, because $\mu_j$ and $\nu_j$ are either degenerate distributions or idempotent distributions simultaneously. Moreover, if it is necessary, we may consider new independent random variables $\xi'_1 = \xi_1$ and $\xi'_2 = \alpha \xi_2$, and hence, we may assume that $k \geq 0$. Note also that the only nonzero proper closed subgroups of $\Omega_p$ are the subgroups $\Delta_k = p^k \Delta_p$, $k = 0, \pm 1, \ldots$ [14, (10.16)].

Statement 1(i) We can assume that $\alpha \neq 1$. In the opposite case, obviously, $\mu_1$ and $\mu_2$ are degenerate distributions. Since by the condition $k = 0$, we have $\alpha = c$, $c \in \Delta_p^0$, and hence, the restriction of the automorphism $\alpha \in \text{Aut}(X)$ to any subgroup $p^m \Delta_p$ is a topological automorphism of $p^m \Delta_p$. By Lemma 2, there exists a subgroup $B = p^l \Delta_p$ such that $\tilde{\mu}_j(y) = 1$, $j = 1, 2$, for $y \in B$. It follows from Lemma 3 that $\sigma(\mu_j) \subset A(X, B)$. Put $G = A(X, B)$. It is easy to see that $G = p^{\alpha l + 1} \Delta_p$. We have $G \cong \Delta_p$, and the restriction of $\alpha$ to the subgroup $G$ is a topological automorphism of $G$. Thus, we get that the independent random variables $\xi_1$ and $\xi_1$ take values in a group $G \cong \Delta_p$, they have distributions $\mu_1$ and $\mu_1$, and the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$, where $\alpha \in \text{Aut}(G)$, are independent. Applying Lemma 4, and taking into account that $\mu_j(y) \geq 0$, $j = 1, 2$, we obtain that $\mu_1 = \mu_2 = m_K$, where $K$ is a compact subgroup of $G$. Thus, we proved the first part of Statement 1(i). On the other hand, we have independent identically distributed random variables $\xi_1$ and $\xi_2$ with values in $X$ and distribution $m_K$ such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$ are independent. Hence, by Lemma 5 $(I - \alpha)(K) \supset K$. Suppose that $c = (0, 0, \ldots, 0, 1, c_1, \ldots)$, and $K \neq \{0\}$. It is obvious that in this case, $(I - \alpha)(K)$ is a proper subgroup of $K$. The obtained contradiction shows that $K = \{0\}$, i.e., $\mu_1$ and $\mu_2$ are degenerate distributions. We also proved the second part of Statement 1(i).

In particular, it follows from this reasoning that in the case, when $X = \Omega_2$, $\mu_1$ and $\mu_2$ are degenerate distributions, because if $c \in \Delta_2^0$, then $c_0 = 1$.

Statement 1(ii) Put $f(y) = \mu_1(y)$, $g(y) = \mu_2(y)$. Taking into account that $\alpha = \tilde{\alpha}$, we rewrite Eq. (3) in the form

$$f(u + v)g(u + \alpha v) = f(u)g(u)f(v)g(\alpha v), \quad u, v \in Y.$$  \hspace{1cm} (5)

Put

$$E = \{y \in Y : f(y) = g(y) = 1\}. \hspace{1cm} (6)$$

Obviously, we can assume that $\mu_j$ are nondegenerate distributions, and hence $E \neq \Omega_p$. By Lemma 2 $E \neq \{0\}$, and by Lemma 3, $E$ is a closed subgroup of $\Omega_p$. Thus, $E$,
as a nonzero proper closed subgroup of $\Omega_p$, is of the form $E = \Delta_0$. Since $k \geq 1$, we have $\alpha(E) \subseteq E$ and hence, $\alpha$ induces a continuous endomorphism $\hat{\alpha}$ on the factor group $L = Y/E$. Taking into account that by Lemma 3

$$f(y + h) = f(y), \quad g(y + h) = g(y),$$

for all $y \in Y$, $h \in E$, we can consider the functions $\hat{f}(y)$ and $\hat{g}(y)$ induced on $L$ by the functions $f(y)$ and $g(y)$. It follows from (6) that

$$\{y \in L : \hat{f}(y) = \hat{g}(y) = 1\} = \{0\}. \tag{7}$$

Passing from Eq. (5) on the group $Y$ to the induced equation on the factor group $L = Y/E$, we obtain

$$\hat{f}(u + v)\hat{g}(u + \hat{\alpha}v) = \hat{f}(u)\hat{g}(u)\hat{f}(v)\hat{g}(\hat{\alpha}v), \quad u, v \in L. \tag{8}$$

It is easy to see that $L \cong \mathbb{Z}(p^\infty)$ and $\hat{\beta} = (I - \hat{\alpha}) \in \text{Aut}(L)$. Putting in (8) first $u = -\hat{\alpha}y$, $v = y$, and then $u = y$, $v = -y$, and taking into account that $\hat{f}(-y) = \hat{f}(y)$ and $\hat{g}(-y) = \hat{g}(y)$, we get

$$\hat{f}((I - \hat{\alpha})y) = \hat{f}(\hat{\alpha}y)\hat{g}^2(\hat{\alpha}y)\hat{f}(y), \quad y \in L, \tag{9}$$

$$\hat{g}((I - \hat{\alpha})y) = \hat{f}^2(y)\hat{g}(y)\hat{g}(\hat{\alpha}y), \quad y \in L. \tag{10}$$

Obviously, Eq. (9) implies that

$$\hat{f}(\hat{\beta}y) \leq \hat{f}(y), \quad y \in L. \tag{11}$$

We note now that any element of the group $L$ belongs to some subgroup $H$, $H \cong \mathbb{Z}(p^n)$, moreover, $\hat{\beta}(H) = H$. Since $H$ is a finite subgroup, $\hat{\beta}^n y = y$ for any $y \in H$, where $n$ depends generally on $y$. Then (11) implies that

$$\hat{f}(y) = \hat{f}(\hat{\beta}^n y) \leq \cdots \leq \hat{f}(\hat{\beta}y) \leq \hat{f}(y), \quad y \in L.$$

Thus, on each orbit $\{y, \hat{\beta}y, \ldots, \hat{\beta}^{n-1}y\}$, the function $\hat{f}(y)$ takes a constant value. The similar statement for the function $\hat{g}(y)$ follows from the equation induced by Eq. (10).

Assume that $\hat{f}(y_0) \neq 0$ at a point $y_0 \in L$, $y_0 \neq 0$. Then $\hat{f}(\hat{\beta}y_0) = \hat{f}(y_0) \neq 0$, and Eq. (9) implies that

$$\hat{f}(\hat{\alpha}y_0) = \hat{g}(\hat{\alpha}y_0) = 1. \tag{12}$$

It follows from (7) and (12) that $\hat{\alpha}y_0 = 0$. By the condition $\alpha = pc$, where $c \in \Delta_0^L$. This implies that $\hat{\alpha} = p\hat{c}$, where $\hat{c}$ is an automorphism of the group $L$, induced by the automorphism $c$. Hence, $y_0$ is an element of order $p$. Reasoning similarly we get from Eq. (10) that if $\hat{g}(y_1) \neq 0$, $y_1 \in L$, $y_1 \neq 0$, then $\hat{f}(y_1) = \hat{g}(\hat{\alpha}y_1) = 1$.

Let $w$ be an arbitrary element of $L$. Denote by $\langle w \rangle$ the subgroup of $L$ generated by $w$. It follows from $\hat{f}(y_1) = 1$ that $\hat{f}(y) = 1$ for all $y \in \langle y_1 \rangle$. Since $L \cong \mathbb{Z}(p^\infty)$ and
\( \langle y_1 \rangle \) is a subgroup of \( L \), we have \( \langle y_1 \rangle \cong \mathbb{Z}(p^m) \) for some \( m \), and hence \( \hat{\alpha}(\langle y_1 \rangle) \subset \langle y_1 \rangle \). Moreover, \( \hat{f}(\hat{\alpha}y_1) = 1 \). Thus the equalities

\[
\hat{f}(\hat{\alpha}y_1) = \hat{g}(\hat{\alpha}y_1) = 1
\]  

(13)

hold true. It follows from (7) and (13) that \( \hat{\alpha}y_1 = 0 \), and hence \( y_1 \) is also an element of order \( p \). Since \( L \cong \mathbb{Z}(p^\infty) \), the group \( L \) contains the only subgroup \( A \) topologically isomorphic to \( \mathbb{Z}(p) \). So, we proved that the functions \( \hat{f}(y) \) and \( \hat{g}(y) \) vanish for \( y \notin A \).

Consider the restriction of Eq. (8) to the subgroup \( A \). Taking into account that \( \hat{\alpha}y = 0 \) for all \( y \in A \), we obtain

\[
\hat{f}(u + v)\hat{g}(u) = \hat{f}(u)\hat{g}(u)\hat{f}(v), \quad u, v \in A.
\]  

(14)

If \( \hat{g}(u_0) \neq 0 \) at a point \( u_0 \in A \), \( u_0 \neq 0 \), then we conclude from (14) that

\[
\hat{f}(u_0 + v) = \hat{f}(u_0)\hat{f}(v), \quad v \in A.
\]

Putting here \( v = (p - 1)u_0 \), we get \( \hat{f}(u_0) = 1 \). Since \( p \) is a prime number, we have \( A = \langle u_0 \rangle \), and hence, \( \hat{f}(y) = 1 \) for \( y \in A \). If \( \hat{g}(y) = 0 \) for any \( y \in A \), \( y \neq 0 \), then, obviously, \( \hat{f}(y) \) may be an arbitrary positive definite function on \( A \). Thus, we proved that either

\[
\hat{f}(y) = \begin{cases} 
1, & y \in A, \\
0, & y \notin A,
\end{cases}
\]  

(15)

or

\[
\hat{g}(y) = \begin{cases} 
1, & y = 0, \\
0, & y \neq 0.
\end{cases}
\]  

(16)

Return from the induced functions \( \hat{f}(y) \) and \( \hat{g}(y) \) on \( L \) to the functions \( f(y) \) and \( g(y) \) on \( Y \). Taking into account (1) and the fact that a distribution is uniquely defined by its characteristic function, we obtain from (15) and (16) that either \( \mu_1 \in I(X) \), or \( \mu_2 \in I(X) \). Statement 1(ii) is proved.

**Statement 2** It is easy to see that without loss of generality, we can assume that \( k \geq 2 \). Consider on the group \( \Omega_\rho \) the distributions

\[
\mu_1 = am_{\Lambda_1} + (1 - a)m_{\Lambda_k + 2}, \quad \mu_2 = am_{\Lambda_k + 2} + (1 - a)m_{\Lambda_{k + 1}},
\]

where \( 0 < a < 1 \). As has been noted earlier, \( A(Y, \Lambda_m) = \Lambda_{m + 1} \). Therefore (1) implies that the characteristic functions \( f(y) = \hat{\mu}_1(y) \) and \( g(y) = \hat{\mu}_2(y) \) are of the form

\[
f(y) = \begin{cases} 
1, & y \in p^{k - 1}\Delta_p, \\
a, & y \in \Delta_p \setminus p^{k - 1}\Delta_p, \\
0, & y \notin \Delta_p,
\end{cases}
\]

\[
g(y) = \begin{cases} 
1, & y \in p^k\Delta_p, \\
a, & y \in p^{k - 1}\Delta_p \setminus p^k\Delta_p, \\
0, & y \notin p^{k - 1}\Delta_p.
\end{cases}
\]
Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $\Omega_p$ and distributions $\mu_1$ and $\mu_2$. It is obvious that $\mu_1, \mu_2 \notin I(X)$. We will check that the characteristic functions $f(y)$ and $g(y)$ satisfy Eq. (5). Then, by Lemma 1, the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$ are independent, and Statement 2 will be proved.

Consider 3 cases: 1. $u, v \in \Delta_p$; 2. $u \notin \Delta_p, v \in \Delta_p$; and 3. $v \notin \Delta_p$.

1. $u, v \in \Delta_p$. Note that since $k \geq 2$, we have $p^{-k-1} \Delta_p \subset \Delta_p$. Consider 3 subcases.

1a. $u \in p^{k-1} \Delta_p, v \in \Delta_p$. Since $u \in p^{k-1} \Delta_p$, we have $f(u) = 1$, and hence $f(u+v) = f(v)$. Since $\alpha v \in p^k \Delta_p$, we have $g(\alpha v) = 1$, and hence $g(u) = f(u) = g(u)$. Equation (5) takes the form $f(v)g(u) = f(v)g(u)$, and it is obviously true.

1b. $u \in \Delta_p \setminus p^{k-1} \Delta_p, v \in p^{k-1} \Delta_p$. Since $v \in p^{k-1} \Delta_p$, we have $f(v) = 1$, and hence $f(u) = f(v)$. Equation (5) takes the form $f(u)g(u) = f(u)g(u)$, and it is obviously true.

1c. $u \in \Delta_p \setminus p^{k-1} \Delta_p, v \in \Delta_p \setminus p^{k-1} \Delta_p$. Since $v \in \Delta_p$, we have $\alpha v \in p^k \Delta_p$. This implies that $g(\alpha v) = 1$, and hence $g(u) = g(u+\alpha v) = g(u)$. Since $u \notin p^{k-1} \Delta_p$, we have $g(u) = 0$. Thus, both sides of Eq. (5) vanish.

2. $u \notin \Delta_p, v \in \Delta_p$. This implies that $u + v \notin \Delta_p$, and hence $f(u) = 0$ and $f(u+v) = 0$. Thus, both sides of Eq. (5) vanish.

3. $v \notin \Delta_p$. This implies that $f(v) = 0$ and hence, the right-hand side of Eq. (5) vanishes. If the left-hand side of Eq. (5) does not vanish, then the following inclusions

$$\begin{cases} u + v \in \Delta_p, \\ u + \alpha v \in p^{k-1} \Delta_p \end{cases} \quad (17)$$

hold true. On the one hand, since $k \geq 2$, it follows from (17) that $(I - \alpha) v \in \Delta_p$. On the other hand, since $k \geq 2$, we have $(I - \alpha) \in \text{Aut}(\Delta_p)$. Hence $v \in \Delta_p$. The obtained contradiction shows that the left-hand side of Eq. (5) vanishes as well.

We showed that the characteristic functions $f(y)$ and $g(y)$ satisfy Eq. (5). Thus, we proved Statement 2 and hence, Theorem 1 is completely proved.

Remark 1 As follows from the proof of Statement 1(i) if $k = 0$, then $\mu_j = m_K * E_{x_j}$, where $K$ is a compact subgroup of $\Omega_p, x_j \in \Omega_p, j = 1, 2$.

As a corollary from Theorem 1 and Remark 1, we derive the Kac-Bernstein theorem for the group $\Omega_p$ (see [4, §7]).

Corollary 1 Let $\xi_1$ and $\xi_2$ be independent random variables with values in $\Omega_p$ and distributions $\mu_1$ and $\mu_2$. Assume that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \beta \xi_2$ are independent. If $p = 2$, then $\mu_1$ and $\mu_2$ are degenerate distributions. If $p > 2$, then $\mu_j = m_K * E_{x_j}$, where $K$ is a compact subgroup of $\Omega_p, x_j \in \Omega_p, j = 1, 2$.

Remark 2 Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $\Omega_p$ and distributions $\mu_1$ and $\mu_2$. Assume that the linear forms $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$, where $\alpha_j, \beta_j \in \text{Aut}(\Omega_p)$ are independent. We can consider new independent random variables $\xi_1' = \alpha_1 \xi_1$ and $\xi_2' = \alpha_2 \xi_2$ and reduce the problem of
describing possible distributions $\mu_1$ and $\mu_2$ to the case, when $L_1 = \xi_1 + \xi_2$, $L_2 = \delta_1\xi_1 + \delta_2\xi_2$, where $\delta_1$, $\delta_2 \in \text{Aut}(\Omega_p)$. Since $L_1$ and $L_2$ are independent if and only if $L_1$ and $L'_2 = \delta_1^{-1}L_2$ are independent, the problem of describing possible distributions $\mu_1$ and $\mu_2$ is reduced to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha\xi_2$, where $\alpha \in \text{Aut}(\Omega_p)$, i.e., it is reduced to Theorem 1.

**Remark 3** Consider the group $\Omega_p$, where $p > 2$. Let $\xi_1$ and $\xi_2$ be independent identically distributed random variables with values in $\Omega_p$ and distribution $m_{\Delta_p}$. Let $\alpha = (0, 0, \ldots, 0, x_0, x_1, \ldots) \in \text{Aut}(\Omega_p)$, where $x_0 \neq 1$. It is easy to verify that the characteristic functions $\widehat{\mu}_1(y) = \widehat{\mu}_2(y) = \widehat{m}_{\Delta_p}(y)$ satisfy Eq. (3). This implies by Lemma 1 that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha\xi_2$ are independent. Thus, for the group $\Omega_p$, where $p > 2$, Statement 1(i) cannot be strengthened to the statement that both $\mu_1$ and $\mu_2$ are degenerate distributions.

**Remark 4** Statement 1(ii) cannot be strengthened to the statement that both $\mu_1$ and $\mu_2$ are idempotent distributions. Namely, if $k = 1$, then there exist independent random variables $\xi_1$ and $\xi_2$ with values in the group $X = \Omega_p$ and distributions $\mu_1$ and $\mu_2$ such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha\xi_2$ are independent, but one of the distributions $\mu_j \not\in I(X)$. We get the corresponding example if we put $\mu_1 = m_{\Lambda_1}$ and $\mu_2 = am_{\Lambda_1} + (1 - a)m_{\Lambda_0}$, where $0 < a < 1$. The proof is similar to the reasoning given in the proof of Statement 2.

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