A new theory of the coefficient of earth pressure at rest

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To my boys, Edor and Geron…may you blossom in wisdom!

Abstract

One of the well-known and widely applied parameters in soil mechanics is the so-called coefficient of earth pressure at rest (denoted by a $K_0$). Field and laboratory investigations show that this parameter is correlated with the friction angle and the overconsolidation ratio of the soil deposit. For normally consolidated clays, the semi-emirical expression developed by Jaky is considered to hold good. The extension of Jaky’s equation for overconsolidated soils is built entirely on empirical data; and there is lack of a theoretical framework that satisfactorily explains the relationships. In this paper, these relationships are investigated from the nature of energy dissipation of soil aggregates. For the same, the cyclic stress-dilatancy relationship proposed by the author is employed, here for a plane strain condition. Then, the relationship between the stress ratio and the mobilized friction angle that maximizes the plastic dissipation is obtained; the value of the stress ratio at this state is shown to be reasonably close to the relationship proposed by Jaky. An exponential relationship between $K_0$ and OCR is derived, and for a special case, the exponent is shown to be the sine of the critical state friction angle. Finally, some generalizations are given and aspects which need further investigation are highlighted.

Key words (phrases): Soil plasticity, soil mechanics, earth pressure coefficient at rest, $K_0$, lateral earth pressure

1 Introduction

A well-known parameter related to an in-situ stress condition is the so-called coefficient of earth pressure at rest, $K_0$ [1]. Bishop [2] defined the coefficient of earth pressure at rest as the ratio of the lateral to the vertical effective stresses in a soil consolidated under the condition of no lateral deformation, the stresses being principal stresses with no shear stress applied on planes which these stresses act, i.e.,

$$K_0 = \frac{\sigma_{0h}}{\sigma_{0v}},$$

where $\sigma_{0h}$ is the at-rest lateral effective stress assumed equal in all lateral directions and $\sigma_{0v}$ is the at-rest vertical effective stress. Note that all stresses in this paper are effective without distinguishing them with a prime.

There are several methods, both intrusive and non-intrusive, that are used and calibrated for the determination of the $K_0$-in-situ [4] [5], Table 1. Amongst the intrusive tests that are used to estimate the in-situ stress states are, for example, self-boring pressure meter tests (SBPM), dilatometer tests (DMT).

1 According to Jefferies et al., there exists another definition based on incremental stresses as $K_0 = \frac{\Delta \sigma_h}{\Delta \sigma_v}$ for the condition that the radial strain is zero. They noted that this equation is interchangeably used with Equation (1) in the literature which may have led to confused interpretation and documentation of data.
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and cone penetration tests (CPT). Nonintrusive/partially nonintrusive geophysical field explorations, such as cross hole seismic tests (CHST) and downhole seismic tests (DHST), e.g., [6], have also been applied for the determination of the $K_0$. These explorations involve transmitting shear waves to a soil medium and measuring the shear wave velocity that travels across the medium. The measured shear wave velocities are then correlated to effective stresses, e.g. [7]. An obvious advantage of such nonintrusive tests over the intrusive ones are that disturbances are limited only around the boreholes. Measurements and interpretations from these tests indicated that there are several factors that the coefficient of earth pressure at rest a function of. Some of the factors that are suggested in the literature that the $K_0$ depends on are summarized as[8]

$$K_0 = f\{\varphi - fric. ang., OCR \mid history, age, soil properties, age, fabric, density, saturation, mineralogy, cementation, etc.\},$$

where OCR is overconsolidation ratio given by the ratio of the highest overconsolidation stress experienced to the current stress. Often very simple semi-empirical and empirical correlation equations are used to estimate the $K_0$, e.g. [1, 9-12]. The most popular relationship that is often used for describing the coefficient of earth pressure at rest for sands and normally consolidated clays is Jaky's [9]. Jaky proposed a relationship between internal friction angle $\varphi$ and the $K_{0,NC}$ stress ratio as

$$K_{0,NC} = (1 - \sin \varphi) \frac{1 + \frac{\varphi}{2} \sin \varphi}{1 + \sin \varphi},$$

which was later simplified to

$$K_{0,NC} = 1 - \sin \varphi.$$

Figure 1: (a) slip lines in a wage-shaped sand prism [16] and (b) heap of sand and angle of repose.

\[\text{Figure 1: (a) slip lines in a wage-shaped sand prism [16] and (b) heap of sand and angle of repose.}\]

Several researchers define $\varphi$ differently. Originally, Jaky [8] assumed that the angle of internal friction and the angle of repose, Figure 1b, to be equal. Some use the term effective angle of shearing resistance probably to mean the same thing. According to Talesnick [15], $K_0$ and thus $\varphi$ depends on density, test apparatus, specimen preparation and compaction methods. Mesri and Vardhanabhuti [16] reported that maximum $K_0$ value and the minimum void ratio are reached almost simultaneously which implies that $K_0$ is density dependent. There is thus no similar take among researchers regarding which friction angle should be used in Jaky's formula.
Michalowski [16], found it surprising that Equation (4) could give a good representation of the “true stress ratio” in soils at rest due to the following two reasons: First, “the relationship was derived from a stress field distribution in a wedge-shaped prism that has less to do with the one dimensional straining process which $K_0$ is often associated with”. Second, “the formula results in unrealistic distribution of the normal stress at the base of the wage,” Figure 1.

For sands and normally consolidated clays, Jaky’s formula is used as it is or with some minor modifications [1]. For overconsolidated clays, some modifications were proposed, e.g., [1, 10, 11]. The most popular is perhaps the one proposed by Schmidt [11] which goes

$$K_{0,OC} = K_{0,NC} \cdot \text{OCR}^x,$$

where $K_{0,NC}$ is given by Equation (4). Schmertmann [27] suggested that the value of $x$ varies from 0.4 - 0.5. Numerous data have been presented by Mayne and Kulhawy [10] to support the relationship between $K_0$ and OCR; they also arrived at the conclusion that $x = \sin \varphi$. L’Heureux et al. [28], based on

| Method | References | Type | Description |
|--------|------------|------|-------------|
| SBPM   | [17], [18, 19], [20, 21] | Intrusive | The SBMP is penetrated to a desired depth. Pore pressure is measured by two piezometers until the excess pore pressure becomes less than 5 kPa. The effective lateral stress is then calculated by subtracting the hydrostatic pressure from the total stress applied to the probe. In theory, the SBPM drills into the ground with little to no disturbance [18, 19]. For cohesive soils, the SBPM seems a promising method [20, 21]. However, in the case of non-cohesive soils readings may be compromised due to disturbances. |
| DMT    | [22], [23] | Intrusive | The DMT contains a flat blade and a flexible stainless steel membrane located at one of the faces of the blade [22]. The membrane is usually inflated using pressurized gas. The blade is connected to a control unit on the surface. Full details of the standard test procedures are given in ASTM 1986 [23]. In general, the measured values are the external pressure which must be applied to the membrane in free air to keep it in contact with its seating and the internal pressure, which in free air, lifts the membrane center 1.0 mm from its seating. These measurements are used as a basis for calculation of, for example, indices which correlate to lateral stresses among some other properties. |
| CPT    | [24]       | Intrusive | The cone penetration test (CPT) has also been used for estimating the $K_0$ in-situ [24]. Correlations of $K_0$ with CPT data can also be obtained in Ku and Mayne [7]. However, the disturbance incurred in the penetration process of CPT would make it questionable whether a reasonable estimation of the in-situ $K_0$ value could be obtained. Often, a varying degree of disturbance is incurred in the boring process. The degree of disturbance depends on the type of soil as well. |
| CHST   | [6]        | Non-intrusive | CHST involves a minimum of two vertically drilled boreholes, one borehole where the signal source is placed, a depth containing the desired layer, and the other where the receiver is. Horizontal wave propagation velocity is then calculated for the first arriving signal. The recorded velocity is correlated to various geotechnical variables of the layer including the effective stress state of the medium. See ASTM D4428/D4428M-07 [6] for testing procedure and assumptions related to the test. |
| DHST   | [25], [26] | Non-intrusive | DHST requires only one borehole. The signal source is placed at the surface and receivers are placed along the borehole length at desired intervals. While the background mechanics in the DHST is similar to that of the CHST, targeting a specific layer is often difficult and noises due to refraction may have a significant effect on the readings of DHST [25]. See ASTM D7400-08 [26]. |
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several field and laboratory investigations arrived at the \( K_0 \) – OCR relationship of the form in Equation (5) for Norwegian clays.

Despite numerous data that are backing the relationship between \( K_0 \) and OCR, there is still some skepticism towards the relationship. The unique dependence of the \( K_0 \) value on OCR has, for instance, been criticized by Jefferies et al. [3] who stressed that \( K_0 \) and OCR are unrelated if the definition in Equation (1) is considered. Accordingly, they suggested that \( K_0 \) should be regarded an independent variable and measured \textit{in-situ}. Furthermore, Jefferies and Been [29] emphasized that the dominant tendency to rely on empirical correlations to determine \textit{in-situ} properties is undesirable. The author of this paper agrees to this suggestion. However, the repeatedly similar outcome across diverse tests and types of clay may suggest some underlying principle that governs the relationship. With this spirit, in this paper, the author wishes to present a theoretical framework for understanding the relationships between \( K_0 \), critical state friction angle and OCR which hitherto were empirical/semiempirical.

2 Plastic dissipation and stress-dilatancy relationships for loading and unloading in a plane strain condition

Assuming coaxiality between eigen directions of stresses and plastic strain increments, the plastic dissipation in a plane strain condition may be written as

\[
D^p = (\sigma_i + a)\dot{\varepsilon}_i^p + (\sigma_3 + a)\dot{\varepsilon}_3^p, \tag{6}
\]

wherein \( \sigma_i \) and \( \dot{\varepsilon}_i^p \) are principal stress and plastic strain rates respectively (\( i = 1 \) for major, and \( i = 3 \) for minor) and \( a \) is attraction [30]. Note that strain rates defined in this paper refer generally to an artificial time increment and can likewise be considered infinitesimal strain increments.

Considering Coulomb's shear strength theory in a Mohr-circle, the relationship between principal stress components is written as

\[
\sigma_i + a = N_\psi (\sigma_3 + a), \tag{7}
\]

where \( N_\psi \) is the stress ratio. We also assume that orthogonal plastic strain rates are related as

\[
\dot{\varepsilon}_3^p = -N_\psi \dot{\varepsilon}_1^p, \tag{8}
\]

where \( N_\psi \) is termed here as dilatancy ratio.

The plastic rate of work per unit volume can now be written as

\[
D^p = (\sigma_1 + a)\dot{\varepsilon}_1^p d_N, \tag{9}
\]

where

\[
d_N = 1 - \frac{N_\psi}{N_\varphi}. \tag{10}
\]

Considering the hypothesis of complementarity of stress-dilatancy conjugates proposed by the author [8],

\[
\delta d_N = -N_\psi \delta (N_\psi) + N_\psi \delta N_\psi = 0 \tag{11}
\]
yields

\[ C_N N_\psi = N_\psi, \quad (12) \]

where \( C_N \) is a ‘constant’ which may have different values for different modes of shearing. Note that, although phrased in a more advantageous form, the variation in Equation (11) is equivalent to Rowe’s minimum energy ratio or least work hypothesis. Equation (12) describes a stress-dilatancy relationship, as commonly known, and \( N_\tau \) and \( N_\psi \) are referred to as stress-dilatancy conjugates [8].

From Equations (6) and (12), the plastic dissipation is obtained as

\[ D_N^p = (\sigma_1 + a) \dot{\varepsilon}_1 \left( \frac{C_N - 1}{C_N} \right) \geq 0. \quad (13) \]

Assuming nonnegative plastic dissipation, the inequality

\[ C_N = \langle -s \rangle C_N^U + \langle s \rangle C_N^L, \quad s = \text{sgn} \left( \dot{\varepsilon}_1 \right), \quad 0 < C_N^U = 1/C_N^L \leq 1, \quad (14) \]

was proposed [8, 31] where \( \langle \rangle \) is the Macaulay bracket, the superscripts \( L \) and \( U \) respectively indicate loading and unloading. Note that \( C_N^U \) does not have to be the inverse of the \( C_N^L \) but its value must be less than unity for making sure that the plastic dissipation is nonnegative during unloading. The inverse relationship is just one of the possibilities.

Considering the stress-dilatancy relation in Equation (12) and further assuming \( d\sigma_1 \dot{\varepsilon}_1 + d\sigma_3 \dot{\varepsilon}_3 = 0 \), i.e., associated plasticity, we have [32]

\[ C_N \frac{d\sigma_1}{d\sigma_3} = N_\sigma, \quad N_\psi = \frac{d\sigma_1}{d\sigma_3}. \quad (15) \]

The solution to this differential equation is:

\[ \sigma_1 + a = C \left( \sigma_3 + a \right)^{\frac{1}{C_N}}. \quad (16) \]

The constant of integration \( C \) may be established by considering a boundary condition along the curves described by Equation (16). A boundary condition considered here is where along the curve defined by Equation (16) the stress state is isotropic, i.e., \( \sigma_1 = \sigma_3 \). We call this stress apparent pre-consolidation stress and denote it with \( p_c \). At this point

\[ C = \left( p_c + a \right)^{\frac{C_N-1}{C_N}}. \quad (17) \]

Combining these Equations (16) and (17), we have:

\[ \frac{\sigma_1 + a}{p_c + a} = \left( \frac{\sigma_3 + a}{p_c + a} \right)^{\frac{1}{C_N}}. \quad (18) \]

The stress ratio, \( N_\phi \), can now be given as:
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\[ N_\psi = \frac{\sigma_3 + a}{\sigma_3 + a} = \left( \frac{\sigma_3 + a}{p_e + a} \right)^{1-C_N}. \]  \hfill (19)

Or, the plastic potential/yield function can be written as

\[ f = \frac{\sigma_3 + a}{p_e + a} \left( \frac{\sigma_3 + a}{p_e + a} \right)^{1-C_N} = 0. \]  \hfill (20)

This was previously derived and generalized by the author [32] and is, then, called an Associated Cyclic Stress Dilatancy plastic potential, abbreviated ACStD.

The mobilized dilatancy angle is defined as:

\[ \sin \psi_m := \frac{1 - N_\psi}{1 + N_\psi} = \frac{\left( \frac{\sigma_3 + a}{p_e + a} \right)^{1-C_N} - C_N \left( p_e + a \right)^{1-C_N}}{\left( \frac{\sigma_3 + a}{p_e + a} \right)^{1-C_N} + C_N \left( p_e + a \right)^{1-C_N}}. \]  \hfill (21)

The apparent pre-consolidation stress can also be related to the dilatancy angle as

\[ p_e + a = \left( \frac{1}{C_N} \frac{1}{1 + \sin \psi_m} \right) \left( \frac{C_N}{1 + \sin \psi_m} \right) \left( \sigma_3 + a \right). \]  \hfill (22)

The mobilized dilatancy angle may also be written in terms of the mobilized friction angle and the critical state friction angle as follows. Let us consider a Mohr-Coulomb (MC) material where the mobilized stress ratio and the critical state stress ratio are defined as

\[ N_{MC}^{MC} = \frac{1 + \sin \phi_m}{1 - \sin \phi_m} \quad \text{and} \quad C_N^{MC} = N_c = \frac{1 + \sin \phi_c}{1 - \sin \phi_c} \]  \hfill (23)

in which \( \phi_m \) is the mobilized friction angle, \( \phi_c \) is the critical state friction angle. The mobilized dilatancy angle can then be written as [8]

\[ -\sin \psi_m = \frac{\sin \phi_m - s \sin \phi_c}{1 - s \sin \phi_m \sin \phi_c}, \]  \hfill (24)

where \( s = 1 \) during loading in shear (shear mobilization away from isotropy) and \( s = -1 \) during unloading in shear (shear mobilization towards isotropy) – sign convention of soil mechanics applies, thus contraction is positive. When \( s = 1 \), the relationship is just the well-known Rowe's [33] stress-dilatancy relationship.
The coefficient of earth pressure from maximum plastic dissipation

The plastic dissipation for plane strain condition may conveniently be written as

\[ D^p_M = (p + a) \gamma^p \left( \sin \varphi_m + \sin \psi_m \right), \]  

where \( p = \frac{(\sigma_1 + \sigma_3)}{2}, \gamma^p = \varepsilon_1^P - \varepsilon_3^P. \)

Let us investigate at what friction and dilatancy angles the maximum dissipation occurs. This can be investigated by considering the plastic dissipation in Equation (25) and the variation

\[ \frac{\partial D^p_M}{\partial \sin \varphi_m} = 0, \quad D^p_M = \frac{D^p_M}{p \gamma^p}, \]  

which yields [8] (compression positive),

\[ \sin \varphi_m = \sin \psi_m = \frac{1 - \sqrt{1 - \sin^2 \varphi_c}}{\sin \varphi_c}. \]  

The maximum plastic dissipation for a plane strain deformation may then be written as

\[ D^p_M = \frac{2p \gamma^p}{\sin \varphi_c} \left(1 - \sqrt{1 - \sin^2 \varphi_c}\right) = 2p \gamma^p (1 - \cos \varphi_c) / \sin \varphi_c, \]

\[ = \frac{2p \gamma^p}{\sin \varphi_c} \left(\frac{1}{2} \sin \varphi_c + \frac{\sin^2 \varphi_c}{8} + \frac{\sin^3 \varphi_c}{16} + \ldots\right), \]

\[ = p \gamma^p \left(1 + \frac{\sin \varphi_c}{4} + \frac{\sin^2 \varphi_c}{8} + \ldots\right). \]  

Considering the mobilized friction angle given in Equation (27), the corresponding stress ratio can now be readily obtained as

\[ \frac{\sigma_3 + a}{\sigma_1 + a} = \frac{1}{\sqrt{N_c}} = \frac{1 - \sin \varphi_c}{1 + \sin \varphi_c}. \]  

We can also perform Taylor expansion and find that

\[ \sqrt{\frac{1 - \sin \varphi_c}{1 + \sin \varphi_c}} = 1 - \sin \varphi_c + \frac{\sin^2 \varphi_c}{2} - \frac{\sin^3 \varphi_c}{2} + \frac{3 \sin^4 \varphi_c}{8} + \ldots, \]  

which implies that \( 1 - \sin \varphi_c \) is the first order approximation of \( \sqrt{(1 - \sin \varphi_c)/(1 + \sin \varphi_c)}. \) This stress ratio is reasonably close to Jaky's formula in Equation (4), especially at lower friction angles. At higher friction angles, the higher order terms seem to gain some weight and hence the value of \( \sqrt{(1 - \sin \varphi_c)/(1 + \sin \varphi_c)} \) tends to deviate from Jaky's formula. Visual comparison is given in Figure 2. The tendency to maximize plastic dissipation may therefore be the reason why the stress ratio at rest for normally consolidated soil deposits tends to a \( K_0 \) value that is fairly captured by Jaky's formula (4).
Considering the schematics in Figure 3, for the stress state at the maximum dissipation, the following relations hold:

\[
\frac{p_{\sigma_0} + a}{\sigma_3 + a} = C_N \frac{C_N}{2(C_N - 1)}, \quad \frac{p_{\sigma_0} + a}{\sigma_3 + a} = C_N \frac{1}{2(C_N - 1)}
\]

(31)

Therefore,

\[
N_\nu = \frac{\sigma_1 + a}{\sigma_3 + a} = \left(\frac{\sigma_3 + a}{p_{\sigma_0} + a}\right)^{1-C_N} = \sqrt{C_N},
\]

(32)

and at this stress state the friction angle is

\[
\sin \phi_{m} = \frac{\sqrt{C_N} - 1}{\sqrt{C_N} + 1}
\]

(33)

The plastic strain rates, \( \tilde{\varepsilon}^P_2 \) and \( \tilde{\varepsilon}^P_3 \) are normal to the plastic potential function,

\[
g = \sigma_1 - \sqrt{1/C_N} \sigma_3 + k,
\]

(34)

which is normal to the line of the maximum plastic dissipation itself and, therefore, the plastic strain increments are aligned with the direction of the respective stress components; \( k \) is just a constant. It can also be seen that the plastic strain increment along the line of the maximum plastic dissipation is always contractive. With the dilatancy angle given, we can also show that, when slightly perturbed, the ratio of the plastic strain increments and the stress increments are in the proportion:

\[
N_\nu = \frac{d\sigma_1}{d\sigma_3} = \frac{1}{\sqrt{C_N}}.
\]

(35)

**Figure 2:** Comparison of Equation (29) with Jaky's \( K_0 \) formulations.
Figure 3: Schematics of yield functions during loading and unloading, critical state and line of maximum dissipation in $s - t$ space.
4 Relationship between $K_0$ and unloading induced OCR

We have now shown that a stress ratio at the maximum plastic dissipation for a plane strain condition is reasonably close to the coefficient of earth pressure defined by Jaky’s equation (4). We will now proceed further with our exposition and show how $K_0$ created by unloading is related to the OCR. For the same, we will consider the schematics in Figure 4 which is a plot of the yield function presented in Section 3 for varying values of apparent pre-consolidation stress. Note that the yield curves are infinitely many and only a few are shown for illustration. The material pays in terms of strains while mobilizing across several yield curves from one stress state to another neighbouring stress state. The yield curve that contains A is assumed for a virgin loading and the material is assumed not to have experienced a stress beyond this curve. For this virgin state, it is assumed that the stress ratio would preferably be at point A for maximizing its plastic dissipation. During unloading from A to B, if not forced, the stress state will mobilize closely following the yield function for unloading, indicated by the green path. This postulate is only an approximation. The stress changes during unloading may mobilize the stress state to a neighbouring yield curve that is described by a different $p_{\alpha}$ (defined below). This mobilization is assumed to be rather limited while moving towards isotropy, but it may gain significance after it crosses the line of isotropy.

Figure 4: Schematics of a set of loading and unloading yield functions, the path of maximum energy dissipation for the plastic work in Eq. (25) and critical state lines.
The yield function, the stress state obeys during unloading can be written as

\[ f_U = \frac{\sigma_1 + a}{p_{cu} + a} - \left( \frac{\sigma_3 + a}{p_{cu} + a} \right)^{N_c} = 0 \tag{36} \]

in which \( p_{cu} \) is where the path intersects the isotropic axis. As we have shown earlier, the stress state at the maximum dissipation is described by

\[ (\sigma_1 + a)_{\lambda} = \sqrt{N_c} (\sigma_3 + a)_{\lambda}, \tag{37} \]

Combining Equations (31) and (37) leads to

\[ N_c^{\frac{N_c}{2(1-N_c)}} (p_{c0} + a) = (\sigma_3 + a)_{\lambda} \tag{38} \]

and

\[ N_c^{\frac{N_c}{2(N_c-1)}} = \frac{p_{c0} + a}{p_{cu} + a}, \quad N_c^{\frac{N_c}{2(N_c-1)}} (p_{c0} + a) = p_{cu} + a \tag{39} \]

Substituting the relations in Equation (39) into Equation (36) one is led to

\[ f_U = \frac{\sigma_1 + a}{p_{c0} + a} - N_c^{\frac{N_c}{4(1-N_c)}} (\frac{\sigma_3 + a}{p_{c0} + a})^{N_c} = 0 \tag{40} \]

Considering \( p_{c0} + a = \text{OCR}(p_c + a) \) into Equation (40) and further rearranging one is led to:

\[ \text{OCR}^{N_c-1} (\sigma_1 + a) N_c^{\frac{N_c}{2(N_c-1)}} = (p_c + a)^{1-N_c} (\sigma_3 + a)^{N_c} \tag{41} \]

Rearranging Equation (20) for the current stress state one obtains

\[ \frac{(\sigma_1 + a)^{N_c}}{\left( \sigma_3 + a \right)^{N_c}} = p_{c} + a \tag{42} \]

Substituting Equation (42) into Equation (41)

\[ \text{OCR}^{N_c-1} (\sigma_1 + a)^{N_c+1} N_c^{\frac{N_c}{2(N_c-1)}} = (\sigma_3 + a)^{N_c+1} \tag{43} \]

which gives us

\[ \sigma_3 + a = \text{OCR}^{\frac{N_c-1}{N_c+1}} N_c^{\frac{1}{2}} (\sigma_1 + a) \tag{44} \]

Note that \( \frac{N_c-1}{N_c+1} = \sin \phi_c \), and therefore, the \( K_0 \) stress state may be stated as:

\[ \sigma_v + a = K_{0,NC} \text{OCR}^{\sin \phi_c} (\sigma_v + a); \quad K_{0,NC} = N_c^{-\frac{1}{2}} \tag{45} \]

where \( \sigma_h \) is the horizontal effective stress and \( \sigma_v \) is the vertical stress.
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Equation (45) rhymes with the empirical formula in Equation (5). Albeit some minor differences in the expression for the $K_{0,NC}$ and the additional attraction term, with the exponent $x = \sin \varphi_c$, we have, rather theoretically, arrived to the same expression that Mayne and Kulhawy [10] arrived at after investigating the stress state of a significant number of overconsolidated samples. If we had a priori taken Jaky’s formula to hold good and disregarded attraction, we would be led to exactly the same equation as in Mayne and Kulhawy [10].

5 Some points for further consideration

The following are pointed out for further consideration and possible extension of the theoretical framework presented above.

1. As stated earlier, the condition in Equation (14), that the constant for unloading is the inverse of the constant for loading is not a necessary requirement (but it seems natural to assume so.) The framework presented above can therefore be considered a special condition. We may generalize it as follows. Suppose $C_l = \frac{1}{C^l_N}$ and $C_u = \frac{1}{C^u_N}$, where $C^l_N$ and $C^u_N$ are as defined in Equation (14), with the condition that $0 < C_l = \frac{1}{N_c} \leq 1$; $C_u \geq 1$ but not necessarily $C^u_N = \frac{1}{C^l_N}$, one obtains:

$$\sigma_v + a = \text{OCR} \left( \sigma_v + a \right),$$

(46)

The $K_{0,NC}$ will then be given by

$$K_{0,NC} = C_i \frac{C_i(C_i-1)}{2(C_i+C_i)} \left( \sigma_v + a \right),$$

(47)

Suppose $C_u = fN_c \geq 1$, $C_l = \frac{1}{N_c}$, and the exponent over OCR is denoted by $x$, one can write

$$K_{0,NC} = N_c \frac{1}{2} \left( f \right) \frac{fN_c(1-N_c)}{(N_c - 1)},$$

(48)

$$x = \frac{(fN_c - 1)(N_c - 1)}{fN^2_c - 1}.$$  

(49)

A special value of $f$ we may be interested to establish is for the condition that the $K_{0,NC}$ is described by Jaky’s equation, i.e., $K_{0,NC} = 1 - \sin \varphi_c$. This requires finding the solution of $f$ for the condition

$$\left( f \right) \frac{fN_c(1-N_c)}{(N_c - 1)} = \cos \varphi_c.$$  

(50)

The closed form solution for $f$ is

$$f = \exp \left[ \frac{-2N_c \ln (\cos \varphi_c)}{N_c - 1} \right] + W \left[ \frac{2 \ln (\cos \varphi_c)}{N_c (N_c - 1)} \right] \exp \left[ \frac{2N_c \ln (\cos \varphi_c)}{N_c - 1} \right],$$

(51)

where $W$ is Lamberts W-function. Equation (51) may be approximated by $f \approx 16.586 \exp \left[ -2.792 \cos \varphi_c \right]$. When $f$ is specified by Equation (51), the exponent, $x$, shows a slight increase over $\sin \varphi_c$. The plots in Figure 5 illustrate how the value of $f$ changes with the cosine of the critical state friction angle and, accordingly, how the exponent, $x$, changes with the critical state friction angle. Note that the exponent, $x$, cannot be larger than unity for any value of $f$ in Equation (51). For realistic values of the critical state friction angle, the value of
$x$ will be 0.4 - 0.7. For this range, the value of $x$ described by Equations (49) and (51) is only about 11% higher than $\sin \phi_c$.

**Figure 5:** Plots of the value of $f$ with the cosine of the critical state friction angle (left) and the exponent $x$ with critical state friction angle (right); shaded regions are for relastic ranges of the critical state friction angle.

2. Note also that the $K_0$ - OCR relationship we so far derived hinges on the assumption that the virgin stress state the soil has ever experienced is created along the $K_{0,NC} = N_c^{-\frac{1}{2}}$ stress path (which may be a naturally preferred condition for maximizing plastic dissipation), otherwise, if it is created by any other random stress path, say $\sigma_h + a = K_{NC}(\sigma_v + a)$, then the corresponding stress proportion that is created by unloading is still exponentially related to the corresponding OCR and is given by

$$\sigma_h + a = K_{0,NC}^{OCR} \left( \sigma_v + a \right).$$

If one wishes to write the stress ratio using an OCR created along a stress path of proportion $K_{NC} > K_{0,NC} = N_c^{-\frac{1}{2}}$, using the $K_{0,NC} = N_c^{-\frac{1}{2}}$, one can still do that but must use some other virtual OCR, say OCR$_v$, such that

$$\sigma_h + a = K_{0,NC}^{OCR_v} \left( \sigma_v + a \right).$$

The OCR$_v$ may also multiplicatively split as OCR$_v = OCR_p OCR_i$, where OCR$_p$ is the real OCR created by any random $K_{NC}$ consolidation path and OCR$_i$ is the additional OCR that would have been obtained if the mass was normally consolidated along the $K_{0,NC} = N_c^{-\frac{1}{2}}$. Suppose $K_{NC} = \beta K_{0,NC}$, then

$$\sigma_h + a = K_{0,NC}^{OCR_p} \beta OCR_i \left( \sigma_v + a \right); \beta = \frac{K_{NC}}{K_{0,NC}}.$$  

Stress states created by unloading from other $K_{NC}$ paths than the $K_{0,NC}$ may be the reason that some researchers found higher values for the exponent over the OCR. The comment by Jefferies et al. [3], that there is no unique relation between OCR and $K_0$, may be true for stress states created by unloading along a constrained path and the soil cannot react in its innate tendencies in one or more of its stress components while some changes occur on one or more of them. The
writer believes that such stresses states are less probable for a natural soil deposit; and therefore, the exponential relationship is applicable for practical purposes.

3. There is some mobilization of the yield function during unloading and, therefore, the assumption that the stress path follows a single yield curve during unloading is not entirely true. However, for shear mobilization towards isotropy, the plastic mobilization of the yield function is assumed to be negligible. This may be studied by employing the theory in an elastoplastic constitutive framework.

4. In the current treatment of the subject, a plane strain deformation mode is considered. However, axisymmetric deformation modes can also be treated in the same way – considering the plastic dissipation for the respective deformation modes. In that case, the mobilized friction angle that maximizes the plastic dissipation for the respective modes of shear will be different; it will be lesser for the triaxial extension and higher for the triaxial compression.

5. Whether the friction angle at the phase transformation fits better instead of the critical state friction angle remains to be investigated. If it happens that the phase transformation angle is better fitting, then the \( K_0 \) will depend on the density of the soil sample.

6. In the theory presented in this paper, the OCR is defined based on unloading. Creep theories suggest that the OCR for clay soils increases with time. Then, it may follow from this that time has some effect on \( K_0 \). But, recent advancements in this area seem to suggest such may not be true, at least not in the same sense as that of the OCR created by unloading [34]. In the outset, unloading involves a direct change on the stress components while volumetric creep seems to remove the apparent overconsolidation ratio further by creep compaction; and has therefore, little to do with the actual change of the stress components. The question would then be how can, during observation, the OCR that is induced by unloading be differentiated from that which is induced by time. This aspect needs further investigation. We may, for the sake of brevity and rather less rigorously, suggest a simple approach. The total OCR may be multiplicatively split into that which is caused by unloading, say OCR\(_u\), and that which caused by time, say OCR\(_t\), i.e., OCR = OCR\(_u\) x OCR\(_t\). Such a split relies on the assumption that one of the components may be obtained in some way. For instance, the OCR\(_t\) may be determined from other assumptions such as the age of the soil deposit. The \( K_0 \) may then be determined using only part of the OCR that comes from unloading (i.e., OCR\(_u\)). However, there is some fundamental issue that needs to be settled regarding the OCR-age relationship suggested in creep theories. Most creep theories seem to suggest that older deposits would tend to have higher OCR. However, measurements and interpretations of the OCR indicate that the general trend is such that overconsolidation ratio tends towards unity (i.e., towards normally consolidated) with increasing depth. That means, if we follow those theoretical frameworks for creep, then we will be led to the absurd conclusion that deeper soil layers are at a younger age than overlying soil layers.

7. The undrained shear strength, \( c_u \), of clay soils may also be approximated employing the theoretical framework laid out here. Assuming the constant for unloading is the inverse of the constant for loading, the author arrived at

\[
c_u = \frac{1}{2} \left( N_s - 1 \right) N_v \left( \frac{1}{N_s - 1} \right) \frac{N_s \sin \varphi}{N_s - 1} \left( \sigma_s + a \right).
\]

This is in the same format as that of the SHANSEP. The author intends to treat the full exposition of this and related aspect of the theory in a separate paper.
6 Conclusion

A theoretical framework is established for describing the relationships between $K_0$, critical state friction angle and overconsolidation ratio. The cyclic stress-dilatancy relationship developed by the author was instrumental in these derivations. The coefficient of earth pressure at rest for normally consolidated soils is derived based on the assumption that the vertical and the horizontal stress components in a soil deposit tend to be in a proportion that maximizes plastic dissipation unless they are subjected to stress changes that would force the stress components change their proportion into some other, for instance due to unloading or loading in constrained stress paths. This consideration, for a plane strain condition gave a stress ratio close to the one proposed by Jaky. For soils that are overconsolidated by unloading, a Schmidt type $K_0$-OCR relationship is derived. Furthermore, extensions and generalizations are put forward. The current treatment is limited to a plane strain condition. The author wishes to follow this up with generalizations for other modes of deformation and with investigation of the possible influence time may have on the $K_0$. 
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