Correlation Robustly Optimal Auctions

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Abstract

We study the design of auction within the correlation-robust framework in which the auctioneer is assumed to have information only about marginal distributions, but does not know the correlation structure of the joint distribution. The performance of a mechanism is evaluated in the worst-case over the uncertainty of joint distributions that are consistent with the marginal distributions. For the two-bidder case, we characterize the Second Price Auction with Uniformly Distributed Reserves as a maxmin auction among dominant strategy incentive compatible (DSIC) and ex-post individually rational (EPIR) mechanisms under the robust-version regularity conditions. For the $N$-bidder ($N \geq 3$) case, we characterize the Second Price Auction with Beta($\frac{1}{N-1}$, 1) Distributed Reserves as a maxmin auction among exclusive (a bidder whose bid is not the highest will never be allocated) DSIC and EPIR mechanisms under the general robust-version regularity conditions (I).

Keywords: Auction design, revenue maximization, correlated private values, maxmin, worst-case, dominant strategy incentive compatible, ex-post individual rational, exclusive, randomization, duality.

JEL Codes: C72, D82, D83.

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1 Introduction

In this paper, we conduct a robustness analysis for a single indivisible good. We consider a correlated private value environment in which the valuation profile is drawn from some joint general (correlated) distribution. It is assumed that the auctioneer knows the marginal distributions of every individual bidder, but does not have any knowledge about possible correlation across different bidders. Any mechanism is evaluated according to the auctioneer’s expected profit derived in the worst case, over all possible joint distributions consistent with the given set of marginal distributions. The objective of the auctioneer is to design a mechanism that maximizes worst-case revenue among some general class of mechanisms.\(^1\)

This framework is in the same spirit as the robust mechanism literature in that assuming away detailed knowledge of the auctioneer. It is motivated by the observation that the correlation structure is a much higher-dimensional object than the marginal distribution of a generic bidder. Therefore it is more difficult to estimate the correlation structure. Practically, it fits the situations where bidder pool changes constantly and then there is no data for estimating the correlation structure. Another situation where the auctioneer may only know the marginal distribution for each bidder is the one where identities of the participating bidders cannot be observed.\(^2\) He and Li (2020) characterize optimal robust reserve prices for the second price auction using the same framework. Bei et al. (2019) compare the performance of several DSIC mechanisms used in practice, including sequential posted price mechanisms. Building on their works, we take a step further to characterize the maxmin mechanism among some general class of mechanisms. That is, we will not restrict attention to some particular format of mechanisms.

We take the saddle point approach for our main results. Specifically, we reformulate the designer’s problem into a zero-sum game between the designer and Nature, who chooses a feasible joint distribution consistent with the known marginal distributions to minimize expected revenue. Finding an optimal mechanism is equivalent to finding a saddle point of the zero-sum game.

Theorem 1 states that the Second Price Auction with Uniformly Distributed Reserves and the Adversarial Correlation Structure form a saddle point under the robust-version regularity

\(^1\)The framework is originally proposed by Carroll (2017) for the multi-dimensional screening problem. His solution is simple and conveys a clear and intuitive message: if you do not know how to bundle, then do not. It is natural to adapt his framework to an environment with multiple bidders whose private valuations may be correlated.

\(^2\)We restrict attention to the set of DSIC and EPIR mechanisms for the two-bidder case, and the set of exclusive DSIC and EPIR mechanisms for the N-bidder \((N \geq 3)\) case.

\(^3\)See He and Li (2020) for a detailed discussion.
conditions for the two-bidder case. The intuition for the Adversarial Correlation Structure can be summarized as follows. We find the revenue of an optimal auction is equivalent to the expected sum of the allocation times the conditional virtual value of each bidder, which will be defined in the main context. Then the adversarial Nature’s problem is to reduce this expected sum as much as possible. We find a class of feasible operations by which Nature could reduce this expected sum through reducing the conditional virtual values for some value profiles while maintaining the consistency with the marginal distribution. This motivates the Adversarial Correlation Structure, which exhibits the property that the high bidder’s conditional virtual value is 0 for all values but the upper bound of the support. We find this property pins down an ordinary differential equation, from which we derive the Adversarial Correlation Structure. The key property of the Second Price Auction with Uniformly Distributed Reserves is that it fully insures the auctioneer against Nature: the expected revenues are the same across all possible correlation structures consistent with the marginal distributions. Finally, the robust-version regularity conditions play a dual role: they (i) guarantee that the Adversarial Correlation Structure is feasible; (ii) they guarantee that the Adversarial Correlation Structure exhibits inter-bidder monotonicity, i.e., the conditional virtual value of the high bidder is weakly greater than that of the low bidder. Indeed, (ii) guarantees that the Second Price Auction with Uniformly Distributed Reserves is a best response among DSIC and EPIR mechanisms to the Adversarial Correlation Structure. In addition, we find the robust-version regularity conditions are essentially necessary for the Second Price Auction with Uniformly Distributed Reserves to be a maxmin auction. The key observation behind is that the Second Price Auction with Uniformly Distributed Reserves is strictly monotone with respect to the bid when the bidder is likely to get the object. When the robust-version regularity conditions fail, this auction can not be a part of Nash equilibrium since Myerson’s ironing procedure implies the allocation rule in equilibrium should exhibit “flatness” across a range of bids.

For tractability, we restrict attention to a smaller class of mechanisms: exclusive DSIC and EPIR mechanisms, for the \( N \)-bidder \( (N \geq 3) \) case. Theorem 2 states that the Second Price Auction with \( Beta(\frac{1}{N-1}, 1) \) Distributed Reserves and the General Adversarial Correlation Structure form a saddle point under the general robust-version regularity conditions (I) hold for the \( N \)-bidder case. The intuition for the General Adversarial Correlation Structure can be summarized as follows. Given the restriction on exclusive mechanisms, only the highest bidders are possible to generate positive revenue to the auctioneer. Thus, the other bidders’ values except for the highest one are “wasted”. Note the adversarial Nature tries to reduce the expected revenue as much as possible. To do so,

\footnote{In an almost-sure sense.}
the Nature maximizes the waste by promoting the other bidders’ values as much as possible until all the other bidders’ values are the same. This motivates the General Adversarial Correlation Structure, which exhibits the first property that the support lies in the set of joint valuations in which either all bidders have the same value or there is a unique highest bidder and the other bidders have the same value. The second property of the General Adversarial Correlation Structure is that the highest bidder's conditional virtual value is 0 for all values but the upper bound of the support. This is a generalization of the property of the Adversarial Correlation Structure and the same intuition applies. We find the two properties pin down an ordinary differential equation, from which we derive the General Adversarial Correlation Structure. Then we guess the format of the maxmin mechanism is a second price auction with some random reserve, and we use duality theory to derive the distribution of the random reserve. To do so, we derive the dual program and use the complementarity slackness condition to pin down an ordinary differential equation regarding the distribution of the random reserves, to which we show the solution is \( \text{Beta}(\frac{1}{N-1}, 1) \) distribution. Finally, the general robust-version regularity conditions (I) guarantee that the General Adversarial Correlation Structure is feasible, which is sufficient for the Second Price Auction with \( \text{Beta}(\frac{1}{N-1}, 1) \) Distributed Reserves to be a best response among exclusive DSIC and EPIR mechanisms to the General Adversarial Correlation Structure. The intuition for the essential necessity of the general robust-version regularity conditions is similar with the that for the two-bidder case.

In addition to the main results, we propose a family of second price auctions with \( \text{Beta} \) distributed reserves of different upper bounds. We identify a non-trivial lower bound of the revenue guarantee for each auction in this family. From this we are able to find an auction from this family that achieves strictly better revenue guarantee than both the posted price mechanism and the second price auction with the optimal deterministic reserve, regardless of marginal distributions (Theorem 3 and Theorem 4).

The remaining of the paper proceeds as follows. Section 2 discusses related literature. Section 3 presents the model. Section 4 characterizes our main results. Section 5 extends the idea to construct a family of auctions and established its robust dominance. Section 6 is a discussion. Section 7 concludes. All proofs are in the Appendix A and B. Appendix C contains the result for the essential necessity of the robust-version regularity conditions.

2 Related Literature

The closestly related paper is He and Li (2020), who consider exactly the same setting, but restrict attention to second price auctions. Specifically, they find the worst-case
correlation structure for the second price auction without a reserve price, characterize the optimal deterministic reserve for the second price auction, and characterize the optimal random reserve for the second price auction under the sufficient condition that $xf(x)$ is non-decreasing.\footnote{$f(x)$ is the density function of the marginal distribution. They assume continuous distributions and does not allow mass points.} The major distinction is that we search for maxmin mechanisms among some general class of mechanisms. In addition, we find the (essentially) necessary and sufficient conditions for our proposed mechanism to be a maxmin mechanism. In contrast to their conditions, our conditions contain a probability mass condition for the upper bound. Methodologically, both papers use duality theory to proceed the analysis. The novelty of this paper is that we use a property of the virtual value to obtain an ordinary differential equation, from which we construct the worst-case correlation structure.

The paper is also closely related to Che (2020) and Koçyiğit et al. (2020), both of which consider a model of auction design in which the auctioneer only knows the expectation of each bidder’s value. Specifically, Che (2020) characterize the optimal random reserve prices for the second price auction and further shows that it is also a maxmin mechanism within a class of competitive mechanisms; Koçyiğit et al. (2020) characterize the maxmin mechanism among highest-bidder lotteries\footnote{This is the same as exclusive DSIC and EPIR mechanisms.} for the case where the known expectations are the same across bidders. Similarly, this paper also considers some general class of mechanisms. The main difference is that we assume the auctioneer knows exactly the marginal distributions.\footnote{That is, we assume the auctioneer knows more and therefore the revenue guarantee in our setting is an upper bound of theirs.}

Broadly, this paper joins the robust mechanism design literature (Bergemann and Morris (2005)). There are other papers searching optimal solutions in the worst-case over the space of parameters (e.g., Carroll and Meng (2016), Garrett (2014), Bergemann and Schlag (2011), Carroll (2017), Giannakopoulos et al. (2020), Chen et al. (2019)). Bergemann et al. (2016), Du (2018) and Brooks and Du (2020) consider a model of auction design with common values. They assume the bidders’ values for the item are drawn from a commonly known prior, but they may have arbitrary information (high-order beliefs) about the prior distribution unknown to the seller. An auction’s performance is measured by the worst expected revenue across a class of incomplete information correlated equilibria termed Bayes correlated equilibria (BCE) in Bergemann and Morris (2013). In this paper, we completely ignore the beliefs of the buyers by focusing on the DSIC mechanisms. The DSIC assumption may be more appropriate for situations in which we can not say much about the bidders’ beliefs.
3 Preliminaries

We introduce the following technical notations. All spaces considered are polish spaces; we endow them with their Borel $\sigma$–algebra. Second, product spaces are endowed with product $\sigma$–algebra. Third, we use $\Delta(X)$ to denote the set of all probability measures over $X$.

We consider an environment where a single indivisible good is sold to $N \geq 2$ risk-neutral bidders. We denote by $I = \{1, 2, ..., N\}$ the set of bidders. Each bidder $i$ has private information about her valuation for the object, which is modeled as a random variable $v_i$ with cumulative distribution function $F_i$. Throughout the paper, we focus on symmetric environment, i.e., $F_i = F_j := F$ for any $i, j \in I$. We use $f_i(v_i)$ to denote the density of $v_i$ in the distribution $F$ when $F$ is differentiable at $v_i$; We use $Pr(v_i)$ to denote the probability of $v_i$ in the distribution $F$ when $F$ has a probability mass at $v_i$. We denote $V_i$ as the support of $F$. We assume each $V_i$ is bounded. Without loss of generality, we assume $V_i = [0, 1]$ as a normalization. The joint support of all $F$ is denoted as $V := \times_{i=1}^{N} V_i = [0, 1]^N$ with a typical value profile $v$. The joint distribution is denoted as $P$. We denote bidder $i$’s opponent value profiles as $v_{-i}$, i.e., $v_{-i} \in V_{-i} := \times_{j \neq i} V_j$.

The valuation profile $v$ is drawn from a joint distribution $P$, which may have arbitrary correlation structure. The auctioneer only knows marginal distributions $F$ for each bidder but does not know how these bidders’ valuations are correlated with each other. To the auctioneer, any joint distribution is plausible as long as the joint distribution is consistent with the marginals. We denote by

$$
\Pi_N(F) = \{\pi \in \Delta V : \forall i, A_i \subset V_i, \pi(A_i \times V_{-i}) = F(A_i)\}
$$

the collection of such joint distributions. We shall drop the subscript $N$ when there is no ambiguity.

The auctioneer seeks a dominant strategy incentive compatible (DSIC) and ex-post individually rational (EPIR) mechanism. A direct mechanism $(q, t)$ is defined as an allocation rule $q : V \rightarrow [0, 1]^N$ and a payment function $t : V \rightarrow \mathbb{R}^N$. With a little abuse of notations, each bidder submits a sealed bid $v_i \in V_i$ to the auctioneer. Upon receiving the bids profile $v = (v_1, v_2, \cdots, v_N)$, the allocation probabilities are $q(v) = (q_1(v), q_2(v), \cdots, q_N(v))$ and the payments are $t(v) = (t_1(v), t_2(v), \cdots, t_N(v))$ where $q(v) \geq 0$ and $\sum_i q_i(v) \leq 1$ for all $v$. Now we propose two definitions that will be useful for exposition.

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8We allow distributions to have probability mass. Furthermore, all results (with slight modifications) hold in discrete environments.

9It is without loss of generality to restrict attention to direct mechanisms. The revelation principle holds in our framework as we focus on dominant strategy mechanisms.
Definition 1. A mechanism \((q, t)\) is exclusive if for any \(v \in V\) and \(i \in I\) such that \(v_i < \max_{j \in I} v_j\), the allocation to the bidder \(i\) is \(q_i(v) = 0\).

Remark 1. Note that only the highest bidders are possible to be allocated in an exclusive auction.

Definition 2. \((q, t)\) is monotone if for any given \(i, v_{-i}\), we have \(q_i(v_i, v_{-i}) \leq q_i(v'_i, v_{-i})\) whenever \(v_i < v'_i\); \((q, t)\) is strictly monotone if for any pair of values \(v_i, v'_i\) in which \(q_i(v_i, v_{-i}) > 0, q_i(v'_i, v_{-i}) > 0\), we have \(q_i(v_i, v_{-i}) < q_i(v'_i, v_{-i})\) whenever \(v_i < v'_i\).

The set of all DSIC and EPIR mechanisms is denoted as \(\mathbb{M}_N\) when there are \(N\) bidders. We shall drop the dependency of \(\mathbb{M}_N\) on the number of bidders \(N\) when there is no confusion. The set of all exclusive DSIC and EPIR mechanisms is denoted as \(\mathbb{L}_N\) when there are \(N\) bidders. We shall drop the dependency of \(\mathbb{L}_N\) on the number of bidders \(N\) when there is no confusion.\(^{10}\)

We are interested in the auctioneer’s expected revenue in the dominant strategy equilibrium in which each bidder truthfully reports her valuation of the object. Then the expected revenue of a DSIC and EPIR mechanism \((q, t)\) when the correlation structure is \(\pi\) is \(U((q, t), \pi) \equiv \int_{v \in V} \pi(v) \sum_{i=1}^N t_i(v) dv\). The auctioneer evaluates each such mechanism \((q, t)\) by its worst-case expected revenue over correlation structures. The auctioneer’s goal is to find a mechanism among either \(\mathbb{M}\) (for the two-bidder case) or \(\mathbb{L}\) (for the \(N\)-bidder \((N \geq 3)\) case) with the maximal worst-case expected revenue for a given set of marginal distributions \(\{F_i\}_{i=1}^N\). Formally, the auctioneer tries to find a mechanism \((q^*, t^*)\) that solves the following problem:

\[
(q^*, t^*) \in \arg \max_{(q, t) \in \mathbb{M} (\text{or} \mathbb{L})} \min_{\pi \in \Pi(F)} \int_{v \in V} \pi(v) \sum_{i=1}^N t_i(v) dv
\]

s.t.

\[
v_i q_i(v) - t_i(v) \geq 0 \quad \forall i, v \quad (\text{EPIR})
\]

\[
v_i q_i(v) - t_i(v) \geq v_i q_i(v', v_{-i}) - t_i(v'_i, v_{-i}) \quad \forall i, v, v'_i \quad (\text{DSIC})
\]

4 Main Results

To facilitate the analysis, it will be useful to further simplify the problem. We will use the following proposition: its proof is standard but included in the Appendix A for completeness. And all formal proofs are deferred to the Appendix.

\(^{10}\)Our main results focus on DSIC and EPIR mechanisms for the two-bidder case and focus on exclusive DSIC and EPIR mechanisms for the \(N\)-bidder \((N \geq 3)\) case.
Proposition 1. Maxmin auctions have the following properties:
1. $q_i(\cdot, v_{-i})$ is nondecreasing in $v_i$ for all $v_{-i}$.
2. $t_i(v_i, v_{-i}) = v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(s, v_{-i}) ds$.

Proposition 1 is essentially a version of Myerson (1981). The payment rule of the maxmin auction can be characterized by the allocation rule.

In addition, we will define virtual values in our environment, which is useful to illustrate the results. We consider the problem that fixing any correlation structure $\pi$, the auctioneer designs an optimal auction $(q, t)$. We denote the density of value profile $v = (v_1, v_2, \cdots, v_N)$ as $\pi(v)$. We define $\pi(v_i) := \int_{[0,1]} N_{v_{-i}} \pi(v_i, v_{-i}) dv_{-i}$. We define $\pi(v_{-i}) := \int_0^1 \pi(v_i, v_{-i}) dv_i$. We denote the density of $v_i$ conditional on $v_{-i}$ as $\pi_i(v_i | v_{-i})$, the cumulative distribution function of $v_i$ conditional on $v_{-i}$ as $\Pi_i(v_i | v_{-i}) := \int_{s_i \leq v_i} \pi_i(s_i | v_{-i}) ds_i$. We define $\Pi_i(v_i, v_{-i}) \equiv \pi(v_{-i}) \Pi_i(v_i | v_{-i})$. An direct implication of Proposition 1 is that the expected revenue of $(q, t)$ under the correlation $\pi$ is

$$E[\sum_{i=1}^N t_i(v)] = E[\sum_{i=1}^N q_i(v) \phi_i(v)]$$

where

$$\phi_i(v) = \begin{cases} v_i - \frac{1 - \Pi_i(v_i | v_{-i})}{\pi_i(v_i | v_{-i})}, & \pi(v_i, v_{-i}) \neq 0 \\ 0, & \pi(v_i, v_{-i}) = 0 \end{cases}$$

Here $\phi_i(v)$ is defined as the “conditional virtual value” of bidder $i$ when the value profile is $v = (v_i, v_{-i})$. Thus the problem of designing an optimal auction given a correlation structure can be viewed as maximizing the expected sum of the allocation times the conditional virtual value\(^{11}\) given that the allocation rule is feasible and satisfies the monotonicity condition defined in Proposition 1.

We observe that the maxmin optimization problem can be interpreted as a two-player sequential zero-sum game. The two players are the auctioneer and Nature. The auctioneer first chooses a mechanism $(q, t) \in M$ (or $L$). After observing the auctioneer’s choice of the mechanism, Nature chooses a correlation structure $\pi \in \Pi(F)$. The auctioneer’s payoff is $U((q, t), \pi)$, and Nature’s payoff is $-U((q, t), \pi)$. Now instead of solving directly for such a subgame perfect equilibrium we can solve for a Nash equilibrium $((q^*, t^*), \pi^*)$ of the simultaneous move version of this zero-sum game, which corresponds to a saddle point of the payoff functional $U$, i.e.,

$$U((q^*, t^*), \pi) \geq U((q^*, t^*), \pi^*) \geq U((q, t), \pi^*)$$

for any $(q, t)$ and any $\pi$. The properties of a saddle point imply that the auctioneer’s

\(^{11}\)We also refer to this as the expected virtual surplus.
equilibrium strategy in the simultaneous move game, \((q^*, t^*)\), is also his maxmin strategy (i.e. his equilibrium strategy in the subgame perfect equilibrium of the sequential game).

### 4.1 Two Bidders

In this subsection, we characterize the maxmin auction among DSIC and EPIR mechanisms under certain regularity conditions for the two-bidder case. Our approach is to propose (guess) a pair \(((q, t), \pi)\) and then find the conditions under which our proposed pair is a saddle point. If we are able to do so, we then characterize the maxmin auction, at least for certain conditions.

#### 4.1.1 Second Price Auction with Uniformly Distributed Reserves

We propose the following pair in which the mechanism is “Second Price Auction with Uniformly Distributed Reserves” and the correlation structure is termed as “Adversarial Correlation Structure”. Formally, they are described as below.

**Second Price Auction with Uniformly Distributed Reserves**

Let \(v = (v_1, v_2)\) be the bid profile of the two bidders. If \(v_1 > v_2\), then \(q^*_1(v_1, v_2) = v_1, q^*_2(v_1, v_2) = 0\) and \(t^*_1(v_1, v_2) = \frac{v_1^2 + v_2^2}{2}, t^*_2(v_1, v_2) = 0\); if \(v_1 < v_2\), then \(q^*_1(v_1, v_2) = 0, q^*_2(v_1, v_2) = v_2\) and \(t^*_1(v_1, v_2) = 0, t^*_2(v_1, v_2) = \frac{v_2^2 + v_1^2}{2}\); if \(v_1 = v_2\) or \(v_1 = v_2 \equiv v\), then \(q^*_1(v_1, v_2) = q^*_2(v_1, v_2) = \frac{v}{2}, t^*_1(v_1, v_2) = t^*_2(v_1, v_2) = \frac{v^2}{2}\).\(^{12}\)

**Adversarial Correlation Structure\(^{13}\)**

Let \(\pi^*(v_1, v_2)\) denote the density of the value profile \((v_1, v_2)\) whenever the density exists. Let \(Pr^*(v_1, v_2)\) denote the probability mass of the value profile \((v_1, v_2)\) whenever there is some probability mass on \((v_1, v_2)\). The Adversarial Correlation Structure is defined by the follows:

\[
\pi^*(v_1, v_2) = \pi^*(v_2, v_1) = \begin{cases} 
  f(0) & v_1 = v_2 = 0 \\
  0 & v_1 > 0, v_2 = 0 \text{ or } v_1 = 0, v_2 > 0 \\
  \frac{1}{v_1} (v_2 f(v_2) - \int_0^{v_2} s^2 f(s) ds) & v_1 \geq v_2, 0 < v_1, v_2 < 1 \\
  v_2 f(v_2) - \frac{\int_0^{v_2} s^2 f(s)ds}{v_2} & v_1 = 1, 0 < v_2 < 1 
\end{cases}
\]

\(^{12}\)Note this mechanism is equivalent to the second price auction with \([0,1]\) uniformly distributed reserves. To see this, note that under the second price auction with \([0,1]\) uniformly distributed reserves, the high bidder with bid \(v\) will be allocated if and only if her bid is above the reserve, which occurs with probability \(v\).

\(^{13}\)The marginal distributions that this result covers have probability mass on the upper bound 1. \(Pr(1)\) denote the probability mass on the value of 1. In the Adversarial Correlation Structure, there is (non-negative) probability mass on the point \((1,1)\). In addition, it can be either a density or a probability mass on 0. If it is a probability mass, \(f(0)\) should be replaced with \(Pr(0)\).
\[ Pr^*(1, 1) = Pr(1) - \int_{s \in (0, 1)} s^2 f(s) ds \]

We next propose conditions about the marginal distribution which we term “robust-version regularity conditions”.

**Robust-version regularity conditions**: \( x^2 f(x) \) is non-decreasing for \( x \in (0, 1) \) and \( Pr(1) \geq \int_{x \in [0,1]} x^2 f(x) dx \).

**Remark 2.** If we assume the support of the marginal distribution is \([0, B]\) for general positive \( B \). The result can be derived similarly. And the robust-version regularity conditions will be \( x^2 f(x) \) is non-decreasing for \( x \in (0, B) \) and \( Pr(B) \geq \int_{x \in [0,B]} x^2 f(x) dx \). Note the point mass condition will vanish as \( B \to \infty \) if \( \int_{x \in [0,B]} x^2 f(x) dx \) is of order \( B^\alpha \) with \( \alpha < 2 \).

**Theorem 1.** The Second Price Auction with Uniformly Distributed Reserves and the Adversarial Correlation Structure form a Nash equilibrium under the robust-version regularity conditions. The revenue guarantee is \( E[X^2] \).

Now we illustrate the intuition behind the Adversarial Correlation Structure. Consider the minimax problem of Nature. Since the auctioneer is maximizing the expected virtual surplus under feasible and monotone allocations, the adversarial Nature would wish to reduce the expected virtual surplus as much as possible given that the correlation structure is consistent with the marginal distributions. For ease of exposition, we fix a symmetric correlation structure \( \pi \). Consider a square of value profiles \( \{(i, i), (i, j), (j, i), (j, j)\} \) where \( 0 < i = j - \delta \) for some small and positive \( \delta \). We conjecture that in the optimal auction to this correlation structure, the allocation is exclusive, strictly monotone and positive for the high bidder whose value is positive. Furthermore, we conjecture that \( j(q_1(j, i) - q_1(j, j)) - iq_1(i, i) - \int_j^i q_1(s, i) ds \geq 0 \).

Now consider the following operation: we add \( \epsilon \) (which is some positive but small number) to the probabilities of the value profiles \( (i, i) \) and \( (j, j) \), and subtract \( \epsilon \) from the probabilities of value profiles \( (i, j) \) and \( (j, i) \) (other parts of the correlation structure remain the same). First note this operation is “legal” in the sense that the new correlation structure is still consistent with the marginal distributions. Furthermore, this operation reduces the expected revenue. To see this, first note it suffices to consider the change in the expected revenue from bidder 1 since the correlation structure is symmetric. Then, note given the conjectured allocation, bidder 1’s allocation is positive only.
when she is the high bidder. Thus the modified correlation structure impacts the expected revenue from bidder 1 only through the value profiles in which $i \leq v_1 \leq j$, $v_2 = i$ and in which $(v_1, v_2) = (j, j)$. Specifically, the virtual surplus from bidder 1 is increased by $i q_1(i, i)\epsilon$ at the value profile $(i, i)$, increased by $jq_1(j, j)\epsilon$ at the value profile $(j, j)$, increased by $\int_{i}^{j} q_1(s, i)\epsilon ds$ at the value profiles $(s, i)$ in which $i \leq s \leq j$, and reduced by $jq_1(j, i)\epsilon$ at the value profile $(j, i)$. By some algebra, the expected revenues from bidder 1 is reduced by

$$\epsilon j(q_1(j, i) - q_1(j, j)) - \epsilon i q_1(i, i) - \epsilon \int_{i}^{j} q_1(s, i)\epsilon ds$$

Furthermore, note the “limit” of such modification is when the virtual value of the high bidder (bidder 1) at the value profile $(j, i)$ is zero, since if it is negative, then reducing the allocation $q_1(j, i)$ would yield a higher expected revenue to the auctioneer\(^{18}\), contradicting our conjectured optimal allocation. The above argument shows that given certain allocation rule, there may be “operations” Nature can adopt to reduce the expected revenue. The proposed operation suggests that increasing the probability of the value profiles in the diagonal and decreasing the probability of the value profiles in the remaining parts could reduce the expected revenue under some allocation rule. Furthermore, the worst-case correlation structure should exhibit many zero virtual values. This motivates the Adversarial Correlation Structure, whose salient feature is\(^{19}\)

$$\phi_1(v_1, v_2) = 0 \quad \text{for} \quad 1 > v_1 \geq v_2 \geq 0 \quad (1)$$

$$\phi_1(v_1, v_2) \geq 0 \quad \text{for} \quad v_1 = 1 \quad (2)$$

That is to say, for each bidder, her virtual value is zero when her value is weakly greater than her opponent and smaller than the upper bound of the support; her virtual values are possible to be positive only when her valuation is the upper bound of the support.

How do we obtain the Adversarial Correlation Structure? It can be shown that the above properties are equivalent to an ordinary differential equation, which is the essential step for the construction. Details are left to the Appendix A.

Suppose, furthermore, that the bidder 2’s virtual values are non-positive when bidder 1’s valuation exceeds that of bidder 2. Then we are able to design optimal auctions to the Adversarial Correlation Structure. Indeed, any exclusive, monotone and feasible auction that fully allocates the good to the bidder(s) with the highest value 1 is optimal. Formally,

\(^{18}\)Specifically, we would also adjust the allocations $q_1(s, i)$ for $i \leq s < j$ to iron out strict monotonicity.

\(^{19}\)Since the Adversarial Correlation Structure is symmetric, we describe the property for bidder 1 only and omit the description for bidder 2.
as we focus on symmetric mechanisms, any auction satisfying the following constraints is optimal.

\[ q_1(1, 1) = q_2(1, 1) = \frac{1}{2} \tag{3} \]
\[ q_1(1, v_2) = q_2(v_1, 1) = 1 \quad \text{for} \quad 0 \leq v_1, v_2 < 1 \tag{4} \]
\[ q \quad \text{is exclusive, monotone and feasible.} \tag{5} \]

Thus, there are many mechanisms optimal to the Adversarial Correlation Structure. Then which mechanism together with the Adversarial Correlation Structure will form a Nash equilibrium? To wit, among optimal mechanisms against the Adversarial Correlation Structure, which one will make the Adversarial Correlation Structure a best response for Nature? Based on insights from robust mechanism literature, the maxmin solutions should exhibit indifference across many correlation structures. We thus propose a conjecture that there is a maxmin solution exhibiting full-insurance property, i.e., its expected revenue is independent of correlation structures. Thus, the adversarial correlation structure will be a best response, as any correlation structure. Therefore, the problem boils down to finding a full-insurance mechanism that satisfies conditions (3),(4) and (5). Now we will construct such a mechanism. First note that given the constructed Adversarial Correlation Structure, we can compute the expected revenue of any optimal mechanism, which, by some algebra, is \( \int_{[0,1]} v^2 dF(v) \). Look at the integrand: \( v \cdot v \cdot f(v) \). Note the mechanism should generate the same revenue across all correlations, thus including the maximally positively correlated one. Now the first \( v \) can be interpreted as the value of the bidder, the second \( v \) can be interpreted as the allocation when the value profile is \( (v, v) \) and \( f(v) \) can be interpreted as the probability of \( (v, v) \) in the maximally positively correlated structure. This gives us the allocation rule when the two bidders have the same valuations. In order for the mechanism to exhibit full-insurance property, it can be shown that the sum of the payment from value profiles \( (i, i) \) and \( (j, j) \) equals that from value profiles \( (i, j) \) and \( (j, i) \) for any \( i, j \). From this, we derive the allocation rule when the two bidders have the different valuations. Finally, we observe the mechanism we derived is equivalent to the Second Price Auction with Uniformly Distributed Reserves.

Now we illustrate the robust-version regularity conditions. These conditions \( (i) \) guarantee that the construction of the Adversarial Correlation Structure is possible and \( (ii) \) guarantee that the Adversarial Correlation Structure is inter-bidder “monotone”:

\[ \phi_1(v_1, v_2) \geq \phi_2(v_1, v_2) \quad \text{for} \quad v_1 \geq v_2 \tag{6} \]

so that the Second Price Auction with Uniformly Distributed Reserves is a best response to
the Adversarial Correlation Structure. Therefore, the robust-version regularity conditions are sufficient for our proposed pair to be a Nash equilibrium. Note both sets of conditions imply that the marginal probability should not decrease (if it decreases) too fast, which resembles the well known Myerson’s regularity condition.

### 4.1.2 A Special Case: Equal Revenue Distribution

It is easy to check that a special marginal distribution, which is termed as the equal revenue distribution\(^{20}\) in the literature, satisfies the (continuous) robust-version regularity conditions. We can apply our construction to find the adversarial correlation structure, which turns out to be the independent joint distribution. Thus, we have the following corollary.

**Corollary 1.** When the two bidders have the equal revenue distribution \((\alpha \in (0, 1))\)

\[
F(v) = \begin{cases} 
1 - \frac{\alpha}{v} & \alpha \leq v < 1 \\
1 & v = 1
\end{cases}
\]

the maxmin auction is the Second Price Auction with Uniformly Distributed Reserves. The minimax correlation structure is the independent joint distribution.

Note the independent joint distribution exhibits zero virtual values for all but value profiles in which some bidder has the highest valuation 1. Then by verifying that the Second Price Auction with Uniformly Distributed Reserves is a best response to the independent joint distribution, we establish the corollary\(^{21}\).

### 4.1.3 Examples

There are other distributions satisfying the robust-version regulation conditions. We now provide some examples.

**Example 1.** Any (truncated) Pareto distribution with \(\alpha \in (0, 1), \beta \in (0, 1]\):

\[
F(v) = \begin{cases} 
1 - \frac{\alpha^\beta}{v^\beta} & \alpha \leq v < 1 \\
1 & v = 1
\end{cases}
\]

To see this, note \(v^2 f(v) = \alpha^\beta \beta v^{1-\beta}\) is non-decreasing when \(\beta \in (0, 1]\). And \(Pr(1) = \alpha^\beta \geq \alpha^\beta \frac{\beta}{2-\beta} (1 - \alpha^{2-\beta}) = \int_0^1 v^2 f(v)dv\) when \(\beta \in (0, 1]\).

\(^{20}\)The reason behind the name is that in the monopoly problem, the revenue from setting any price in the support is the same.

\(^{21}\)Alternatively, we can just derived the independent correlation structure from the general formula of Adversarial Correlation Structure.
Example 2. Uniformly distributed on \([0, 1)\) with a probability mass \(Pr(1) \geq \frac{1}{4}\).

To see this, note the first part of the conditions are trivial since it is uniformly distributed on \([0, 1)\). For the probability mass condition, note \(Pr(1) \geq \int_0^1 v^2(1 - Pr(1))dv = \int_0^1 v^2 f(v)dv\) when \(Pr(1) \geq \frac{1}{4}\).

4.2 \(N\) Bidders

In this section, we extend our analysis to general number of bidders. We still take the saddle point approach, but restrict attention to exclusive DSIC and EPIR mechanisms, i.e., \((q, t) \in L\). We present a candidate strategy profile and show it is a Nash equilibrium under certain conditions.

4.2.1 Second Price Auction with \(Beta\left(\frac{1}{N-1}, 1\right)\) Distributed Reserves

We propose the following pair in which the mechanism is “Second Price Auction with \(Beta\left(\frac{1}{N-1}, 1\right)\) Distributed Reserves” and the correlation structure is termed as “General Adversarial Correlation Structure”. Formally, they are described as below.

Second Price Auction with \(Beta\left(\frac{1}{N-1}, 1\right)\) Distributed Reserves

Let \(v = (v_i, v_{-i})\) be the bid profile of the \(N\) bidders. Denote the highest bid in this bid profile as \(v^{(1)}\), the second highest bid in this bid profile as \(v^{(2)}\). If \(\#\{k : v_k = v^{(1)}\} = 1\), then

\[
q^*_i(v) = v^{(1)}, \quad q^*_j(v) = 0 \quad \text{and} \quad t^*_i(v) = \frac{1}{N} v^{(1)} + \frac{N-1}{N} v^{(2)}, \quad t^*_j(v) = 0 \quad \text{for} \quad i \in \{k : v_k = v^{(1)}\}
\]

and \(j \notin \{k : v_k = v^{(1)}\}\); if \(\#\{j : v_j = v^{(1)}\} = K \geq 2\), then

\[
q^*_i(v) = \frac{v^{(1)}}{K}, \quad q^*_j(v) = 0 \quad \text{and} \quad t^*_i(v) = \frac{1}{K} v^{(1)}, \quad t^*_j(v) = 0 \quad \text{for} \quad i \in \{k : v_k = v^{(1)}\} \quad \text{and} \quad j \notin \{k : v_k = v^{(1)}\}.
\]

General Adversarial Correlation Structure\(^{22}\)

Let \(\pi^*(v)\) denote the density of the value profile \(v\) whenever the density exists. Let \(Pr^*(v)\) denote the probability mass of the value profile \(v\) whenever there is some probability mass on \(v\). We define \(V^+ := \{v|\exists i \text{ s.t. } v_i = v^{(1)} > v_j = v^{(2)} \forall j \neq i \text{ or } v_i = v^{(1)} \forall i\}\). The General

\(^{22}\)The marginal distributions that this result covers have probability mass on the upper bound 1. \(Pr(1)\) denote the probability mass on the value of 1. In the Adversarial Correlation Structure, there is (non-negative) probability mass on the point \((1, \cdots, 1)\). In addition, it can be either a density or a probability mass on 0. If it is a probability mass, \(f(0)\) should be replaced with \(Pr(0)\).
Adversarial Correlation Structure is symmetric, and is defined as follows:

\[
\pi^*(v_i, v_{-i}) = \begin{cases} 
  f(0) & v = (0, \cdots, 0) \\
  0 & 0 = v_j < v_i, \forall j \neq i; v \notin V^+
  \\
  \frac{1}{(N-1)v_{(1)}} (v(2)f(v(2)) - \frac{v_{(2)}}{N-1} \int_{0}^{v(2)} s^{\frac{N}{N-1}} f(s) ds) & v \in V^+, 0 < v_j \leq v_i = v(1) < 1, \forall j \neq i \\
  \frac{1}{N-1} (v(2)f(v(2)) - \frac{v_{(2)}}{N-1} \int_{0}^{v(2)} s^{\frac{N}{N-1}} f(s) ds) & v \in V^+, 0 < v_j < v_i = 1, \forall j \neq i
\end{cases}
\]

We next propose two sets of conditions about the marginal distribution which we term “general robust-version regularity conditions (I)” and “general robust-version regularity conditions (II)”.

**General robust-version regularity conditions (I):** \( f(x) \geq \frac{x^{-1} - \frac{N}{N-1}}{N-1} \int_{0}^{x} s^{\frac{N}{N-1}} f(s) ds \) for \( x \in (0, 1) \) and \( Pr(1) \geq \frac{1}{N-1} \int_{0}^{1} s^{\frac{N}{N-1}} f(s) ds \).

**General robust-version regularity conditions (II):** \( x^2 f(x) \) is non-decreasing for \( x \in (0, 1) \) and \( Pr(1) \geq \frac{1}{N-1} \int_{0}^{1} s^{\frac{N}{N-1}} f(s) ds \).

**Remark 3.** Note the probability mass condition will vanish as the number of the bidders goes to infinity. To see this, note \( \frac{1}{N-1} \int_{0}^{1} s^{\frac{N}{N-1}} f(s) ds < \frac{1}{N-1} \int_{0}^{1} 1 \cdot f(s) ds < \frac{1}{N-1} \rightarrow 0 \) as \( N \rightarrow \infty \).

**Theorem 2.** Second Price Auction with Beta(\( \frac{1}{N-1}, 1 \)) Distributed Reserves and General Adversarial Correlation Structure form a Nash equilibrium under the general robust-version regularity conditions (I). The revenue guarantee is \( E[X^{\frac{N}{N-1}}] \). In addition, the general robust-version regularity conditions (II) imply the general robust-version regularity conditions (I).

**Remark 4.** Note the Second Price Auction with Beta(\( \frac{1}{N-1}, 1 \)) Distributed Reserves is asymptotically optimal, regardless of the marginal distributions. To see this, first note \( E[X] \) is an upper bound of the revenue guarantee for any mechanism, since the adversarial Nature can always choose the maximally positively correlated distribution. But by the Dominated Convergence Theorem, we have

\[
E[X^{\frac{N}{N-1}}] \rightarrow E[X]
\]
as \( N \rightarrow \infty \).
highest bidder(s). Formally, for this correlation structure,

\[ \phi_i(v_i, v_{-i}) = 0 \quad \text{for} \quad v \in V^+, 1 > v_i \geq \max_{j \neq i} v_j \]  

(7)

\[ \phi_i(v_i, v_{-i}) \geq 0 \quad \text{for} \quad v \in V^+, v_i = 1 \]  

(8)

That is to say, for each value profile in the support, the highest bidder(s) have zero virtual value(s) provided that the highest value is smaller than the upper bound of the support; the virtual value are possible to be positive only when the value is the upper bound of the support. The construction of the General Adversarial Correlation Structure also follows a similar procedure. Details are left to the Appendix A.

Now we are able to design optimal auctions among exclusive DSIC and EPIR mechanisms to the General Adversarial Correlation Structure. Indeed, any exclusive, monotone and feasible auction that fully allocates the good to the bidder(s) with the highest value of 1 is optimal. Formally, as we focus on symmetric mechanisms, the following constraints hold for any optimal exclusive auction:

\[ q_i(v) = \frac{1}{K} \quad \text{for} \quad v_i = 1, \#\{k : v_k = 1\} = K \geq 1 \]  

(9)

\[ q \quad \text{is exclusive, monotone and feasible} \]  

(10)

Therefore, there are many exclusive auctions optimal to the General Adversarial Correlation Structure. To find an auction among them such that the General Adversarial Correlation Structure is a best response for the Nature, we will use duality theory.

Consider the Nature’s problem of finding a worst correlation structure \( \pi \) to any mechanism \((q, t)\). Note this is an infinite dimensional linear program. We derive its dual maximization program. By weak duality, the value of the dual maximization problem is weakly lower than the value of the primal minimization problem\(^{23}\). Our idea is to propose a mechanism and then construct a set of feasible dual variables such that the value of the dual program equals the value of the primal program. We conjecture that the mechanism is the second price auction with some (not known yet) random reserves. To characterize the distribution of the random reserves, we observe that the complementarity slackness conditions have to hold. This yields an ordinary differential equation, to which the solution is the \( \text{Beta}(\frac{1}{N-1}, 1) \) distribution. Then we verify the dual constraints hold for all value profiles under the constructed dual variables. Finally, we verify that the values of the two programs are the same. This establishes that the General Adversarial Correlation Structure

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\(^{23}\)See, for example, He and Li (2020).
is a best response to the Second Price Auction with \( Beta(\frac{1}{N-1}, 1) \) Distributed Reserves. Also note the Second Price Auction with \( Beta(\frac{1}{N-1}, 1) \) Distributed Reserves satisfies (9) and (10). Thus we have found a Nash equilibrium.

Now we illustrate the general robust-version regularity conditions (I). These conditions guarantee that the construction of the General Adversarial Correlation Structure is possible so that the Second Price Auction with \( Beta(\frac{1}{N-1}, 1) \) Distributed Reserves is a best response to the Adversarial Correlation Structure among exclusive DSIC and EPIR mechanisms. In addition, we find the general robust-version regularity conditions (II) imply the general robust-version regularity conditions (I). That is, we find simpler sufficient conditions for the Second Price Auction with \( Beta(\frac{1}{N-1}, 1) \) Distributed Reserves to be a maxmin auction among exclusive DSIC and EPIR mechanisms. However, in contrast to the two-bidder case, the General Adversarial Correlation Structure is not inter-bidder monotone, i.e., the virtual value of the second highest bidder (with any value above 0) is strictly higher than that of the highest bidder (with any value below 1). To see this, note there is no density (or probability mass) on the value profiles in which there is a unique second highest bidder. This implies that the Second Price Auction with \( Beta(\frac{1}{N-1}, 1) \) Distributed Reserves is not a maximin auction among DSIC and EPIR mechanisms for any given marginal distributions. We summarize this observation as follows.

**Corollary 2.** The Second Price Auction with \( Beta(\frac{1}{N-1}, 1) \) Distributed Reserves is not a maxmin auction among DSIC and EPIR mechanisms when there are more than two bidders \((N \geq 3)\).

### 4.2.2 A Special Case: Equal Revenue Distribution

It is easy to check that the equal revenue distribution also satisfies the general robust-version regularity conditions. We can apply our construction to find the worst-case correlation structure, which, in contrast to the two-bidder case, is not the independent joint distribution.

**Corollary 3.** When the \( N \) bidders have the equal-revenue distribution \((\alpha \in (0, 1))\)

\[
F(v) = \begin{cases} 
1 - \frac{\alpha}{v} & \alpha \leq v < 1 \\
1 & v = 1 
\end{cases}
\]

the maxmin auction among exclusive DSIC and EPIR mechanisms is the Second Price Auction with \( Beta(\frac{1}{N-1}, 1) \) Distributed Reserves. The minimax correlation structure is as
follows.

\[
\pi(v_i, v_{-i}) = \begin{cases} 
\frac{(v(2)) - N}{(N-1)v^{(1)} (N-1)} & \alpha \leq v_i = v(1) < 1, \alpha \leq v_j = v(2) < 1, \forall j \neq i \\
\frac{v(2)}{v^{(1)}} N & v_i = 1, \alpha \leq v_j = v(2) < 1, \forall j \neq i \\
0 & \text{others}
\end{cases}
\]

In addition, the revenue guarantee is \(N\alpha - (N - 1)\alpha^{N-1}\).

4.2.3 Performance of The Second Price Auction with \(Beta(\frac{1}{N-1}, 1)\) Distributed Reserves

We have shown that the Second Price Auction with \(Beta(\frac{1}{N-1}, 1)\) Distributed Reserves is a maxmin solution under the general robust-version regularity conditions (I). Then what if those conditions do not hold? How does the Second Price Auction with \(Beta(\frac{1}{N-1}, 1)\) Distributed Reserves perform? In this section, we compare the performance of the Second Price Auction with \(Beta(\frac{1}{N-1}, 1)\) Distributed Reserves with that of some common dominant strategy mechanisms used in practice when the general robust-version regularity conditions (I) may not hold.

**Definition 3.** A mechanism \(M_1\) robustly dominates a mechanism \(M_2\) for a set \(\mathcal{H}\) of marginal distributions if the revenue guarantee of \(M_1\) is weakly greater than that of \(M_2\) for any marginal distribution in \(\mathcal{H}\), and the inequality is strict for some marginal distribution in \(\mathcal{H}\).

**Definition 4.** A mechanism \(M_1\) robustly dominates a mechanism \(M_2\) if the revenue guarantee of \(M_1\) is strictly greater than that of \(M_2\) for any marginal distribution.

We find the Second Price Auction with \(Beta(\frac{1}{N-1}, 1)\) Distributed Reserves achieves relatively better performance than the posted price auction for a wide range of distribution. Note for the posted price mechanism, the worst-case correlation structure is the maximally positively correlated distribution, the revenue guarantee thus is

\[
\max_x x(1 - F(x))
\]

which is the monopoly profit when there is only one bidder.
Proposition 2. For any $N \geq 2$, the Second Price Auction with Beta($\frac{1}{N-1}, 1$) Distributed Reserves robustly dominates the posted price mechanism when the revenue function $x \cdot (1 - F(x))$ is concave\textsuperscript{24}.

For the second price auction, He and Li (2020) have shown that the revenue guarantee for the second price auction is

$$\frac{N}{N-1} \cdot \int_{0}^{F^{-1}(\frac{N-1}{N})} x dF(x)$$

We find the revenue guarantee of the Second Price Auction with Beta($\frac{1}{N-1}, 1$) Distributed Reserves is (weakly) greater than that of the second price auction for some distributions, including uniform distributions\textsuperscript{25}.

He and Li (2020) also characterize the robust deterministic reserve for the second price auction. For the uniform distribution, the robust reserve is $\frac{1}{N+1}$ and the revenue guarantee is $\frac{N-1}{2(N+1)}$, which is strictly lower than the guarantee of the Second Price Auction with Beta($\frac{1}{N-1}, 1$) Distributed Reserves when $N \geq 3$.\textsuperscript{26}

5 Robust Dominance

In this section, motivated by the main idea embedded in the construction, we propose a family of second price auctions with Beta distributed reserves of different upper bounds. Formally, the distribution function for the random reserve is

$$F(x) = \left(\frac{x}{r}\right)^{\frac{1}{N-1}}$$

for $x \in [0, r]$ where $r \in (0, 1)$. That is, when the value of the highest bidder is above some threshold $r$, the good will be fully allocated to her. For each $r$, we are able to identify a non-trivial lower bound of the revenue guarantee by constructing a set of feasible dual variables.

Lemma 1. The revenue guarantee of the second price auction with random reserves distributed as $F(x) = \left(\frac{x}{r}\right)^{\frac{1}{N-1}}$ is at least

$$\int_{0}^{r} \frac{x^{N-1}}{r^{N-1}} dF(x) + r(1 - F(r))$$

\textsuperscript{24}Note the marginal distribution can be fully continuous.

\textsuperscript{25}For the uniform distribution, the revenue guarantee is $\frac{N-1}{2N}$ under the Second Price Auction with Beta($\frac{1}{N-1}, 1$) Distributed Reserves, while $\frac{N-1}{2N}$ under the second price auction.

\textsuperscript{26}When $N = 2$, the revenue guarantees are the same, which is $\frac{1}{3}$. 19
We are interested in the performance of the proposed family of auctions. The above lemma implies a potential criterion under which the auctioneer selects an auction from this family. Although the revenue guarantees may depend on the details of the marginal distribution and thus are hard to be identified, the auctioneer can at least select one whose lower bound of the revenue guarantee is the highest among this family. Formally, the auctioneer can find

\[ r^* \in \arg \max_r \int_0^r x \frac{F \left( \frac{x}{r^{N-1}} \right)}{r^{N-1}} dF(x) + r(1 - F(r)) \]

Now we compare its performance with some standard dominant strategy auctions used in practice.

**Theorem 3.** For any \( N \geq 2 \), the second price auction with random reserves distributed as \( F(x) = \left( \frac{x}{r^*} \right)^{\frac{1}{N-1}} \) robustly dominates the posted price mechanism.

**Theorem 4.** For any \( N \geq 2 \), the second price auction with random reserves distributed as \( F(x) = \left( \frac{x}{r^*} \right)^{\frac{1}{N-1}} \) robustly dominates the second price auction with the optimal deterministic reserve.

Note surprisingly, we do not require any distributional assumption for these two theorems to hold. Thus, the second price auction with random reserves distributed as \( F(x) = \left( \frac{x}{r^*} \right)^{\frac{1}{N-1}} \) is more desirable than both the post price mechanism and the second price auction with the optimal deterministic reserve in terms of correlation-robustness considerations for any number of bidders. Alternatively, Theorem 3 can be interpreted as that competition effect dominates adversarial correlation effect. To see this, note Theorem 3 implies that we find one auction for the two-bidder that generates strictly higher worst-case revenue than the monopoly revenue from one agent, regardless of the marginal distribution. Thus, even if the Nature picks the most adversarial correlation structure, it is always strictly more desirable for the auctioneer to have just one more bidder.

6 Discussion

6.1 Maxmin Auctions

The Second Price Auction with \( Beta(\frac{1}{N-1}, 1) \) Distributed Reserves is interesting in its own right. First, this auction fully insures the auctioneer against all possible correlation structures consistent with the marginal distribution for the two-bidder case. A rough intuition behind is that the competition is weak and Nature is relatively more powerful for two-bidder case. Therefore this auction that fully insures the auctioneer turns out to have better performance
for many environments. Second, this auction is strictly monotone and only bidder(s) with the value of the upper bound of the support can get full allocation. Third, this auction does not require the information of the marginal distribution except for the support\footnote{The auctioneer need the entire information of the marginal distribution to check whether the Second Price Auction with $Beta\left(\frac{1}{N-1}, 1\right)$ Distributed Reserves is a maxmin auction, but the description of the auction only requires information about the support.}. Thus, the auction is easier for the auctioneer to apply compared with auctions with a reserve price, e.g., Myerson’s auction, which often requires the full information of the marginal distribution to calculate the optimal reserve.

### 6.2 Probability Mass Condition

Due to the nature of the construction of the General Adversarial Correlation Structure, we require the probability mass condition on the upper bound of the support. In contrast, traditional mechanism design literature often assumes the distribution of values is fully continuous. However, distributions with a probability mass arise endogenously in many robust design environments. Condorelli and Szentes (2020) shows that a particular equal revenue distribution is the buyer-optimal distribution in a version of the hold-up problem. Che (2020) considers a robust auction design problem and finds that the worst-case marginal distribution has a point mass on both the upper bound and an interior point. Zhang (2021) considers a robust public good mechanism design problem and finds that the worst-case marginal distribution has a point mass on the upper bound. In addition, as Remark 3 suggests, provided that the number of bidders is large, the general robust-version regularity conditions (II) allow any distribution with a point mass on the upper bound as long as $x^2 f(x)$ is non-decreasing for $x$ below the upper bound. In this sense the probability mass condition can be regarded as non-restrictive when the number of bidders is large.

### 7 Concluding Remarks

To my knowledge, this paper is the first to characterize the maxmin auctions among some general class of mechanisms (DISC and EPIR mechanisms for the two-bidder case and exclusive DISC and EPIR mechanisms for the $N$-bidder ($N \geq 3$) case) in the correlation-robust framework. It remains an open question what the maxmin auctions among DSIC and EPIR mechanisms are for general number of bidders. The constructive method may shed light on other robust design problems and general robust optimization problems.
8 Appendix

A Proofs for Section 4

A.1 Proof of Proposition 1

(i) \( q_i(\cdot, v_{-i}) \) increasing:

Dominant strategy incentive compatibility for a type \( v \) requires that for any \( v' \neq v \):

\[
v q_i(v, v_{-i}) - t_i(v, v_{-i}) \geq v q_i(v', v_{-i}) - t_i(v', v_{-i})
\]

DSIC also requires that:

\[
v' q_i(v', v_{-i}) - t_i(v', v_{-i}) \geq v' q_i(v, v_{-i}) - t_i(v, v_{-i}).
\]

Adding the two inequalities, we have that:

\[
(v - v')(q_i(v, v_{-i}) - q_i(v', v_{-i})) \geq 0
\]

It follows that \( q_i(v, v_{-i}) \geq q_i(v', v_{-i}) \) whenever \( v > v' \).

(ii) \( t_i(v, v_{-i}) = v q_i(v, v_{-i}) - \int_0^{v_i} q_i(s, v_{-i}) ds \):

Fix \( v_{-i} \), Define

\[
U_i(v) = v q_i(v, v_{-i}) - t_i(v, v_{-i})
\]

By the two inequalities in (i), we get

\[
(v' - v) q_i(v, v_{-i}) \leq U_i(v') - U_i(v) \leq (v' - v) q_i(v', v_{-i})
\]

Dividing throughout by \( v' - v \):

\[
q_i(v, v_{-i}) \leq \frac{U_i(v') - U_i(v)}{(v' - v)} \leq q_i(v', v_{-i})
\]

As \( v \uparrow v' \), we get:

\[
\frac{dU_i(v)}{dv} = q_i(v, v_{-i})
\]

Then we get

\[
t_i(v, v_{-i}) \leq v q_i(v, v_{-i}) - \int_{s<v} q_i(s, v_{-i}) - U_i(0)
\]
Note $U_i(0) \geq 0$ by the ex post IR constraint. If $U_i(0) > 0$, then we can reduce it to 0 so that we can increase the revenue from all value profiles and the value of the problem will be strictly greater. Thus, for any maxmin auction, $U_i(0) = 0$ and $t_i(v_i, v_{-i}) = v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(s, v_{-i})$.

A.2 Proof of Theorem 1

From a high level, our approach is reverse engineering in the sense we first formed a educated guess of the Nash equilibrium, and then identify the conditions validating the guess.

As is mentioned in the section of Main Results, the idea for the worst-case correlation structure is to exhibit indifference across many mechanisms. Specifically, we guess that it exhibits the properties (1) and (2). Now we show that Adversarial Correlation Structure exhibits (1) and (2) by constructing a symmetric correlation structure exhibiting (1) and (2), which is summarized by the lemma below.

Lemma 2. The Adversarial Correlation Structure exhibits (1) and (2).

Proof. First, note by allocating all marginal density $f(0)$ (or marginal probability mass $Pr(0)$) to value profile $(0, 0)$, we have $\phi_i(v_i, 0) = \phi_j(0, v_j) = 0$ for any $v_i, v_j$. Thus, (1) and (2) trivially hold for these value profiles. Now let $A_{kj} := \{v|k \leq v_1 \leq j, v_2 = k\}$, define $c(0) := f(0)$ and $c(k) := \int_{A_{k1}} d\pi^*$ for $k > 0$. Consider the value profile $(v_1, v_2)$ where $0 < v_2 \leq v_1 < 1$. In order for the virtual values to satisfy (1), we must have

$$\phi_1(v_1, v_2) = v_1 - \frac{c(v_2) - \int_{v_2}^{v_1} \pi^*(s, v_2) ds}{\pi^*(v_1, v_2)} = 0 \quad \forall 0 < v_2 \leq v_1 < 1$$

These equations are essentially a system of ordinary differential equations, whose solution is well known:

$$\pi^*(v_1, v_2) = \frac{1}{v_1} v_2 c(v_2) \quad 0 < v_2 \leq v_1 < 1 \quad (11)$$

$$\pi^*(1, v_2) = v_2 c(v_2) \quad 0 < v_2 < 1 \quad (12)$$

By symmetry, we also obtain $\pi^*(v_2, v_1) = \pi^*(v_1, v_2)$ for $0 < v_2 \leq v_1 < 1$ and $\pi^*(v_2, 1) = \pi^*(1, v_2)$ for $0 < v_2 < 1$. Finally,

$$Pr^*(1, 1) = Pr(1) - \int_{j\in[0,1]} j c(j) dj$$

Now we solve for $c(k)$ and therefore for $\pi^*$. Note since the marginal distribution is the same

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28The solution is reminiscent of the equal revenue distribution.
across two bidders, given the above derivation, $c(k)$ must satisfy the following condition:

$$c(k) = f(k) - \frac{\int_0^k jc(j) \, dj}{k^2} \quad \forall 0 < k < 1 \quad (14)$$

To see this, note

$$f(k) = \int_{\{0 \leq v_1 \leq 1, v_2 = k\}} d\pi^* = \int_{A_k^1 \cup \{0 \leq v_1 < k, v_2 = k\}} d\pi^* = \int_{A_k^1} d\pi^* + \int_{\{0 \leq v_1 < k, v_2 = k\}} d\pi^* + \int_{\{v_1 = k, 0 \leq v_2 < k\}} d\pi^*$$

where the last equality follows from symmetry. Multiplying $k$ by both sides of (14), we obtain

$$kc(k) = kf(k) - \frac{\int_0^k jc(j) \, dj}{k} \quad \forall 0 < k < 1$$

Define $g(k) := \int_0^k jc(j) \, dj$ for $0 < k < 1$. Then we have

$$g'(k) = kf(k) - \frac{g(k)}{k} \quad \forall 0 < k < 1$$

Note this is an ordinary differential equation, and we solve for $g(k)$:

$$g(k) = \frac{1}{k} \int_0^k j^2 f(j) \, dj \quad \forall 0 < k < 1 \quad (15)$$

From this we compute $c(k)$ for $0 < k < 1$,

$$c(k) = f(k) - \frac{\int_0^k j^2 f(j) \, dj}{k^3} \quad \forall 0 < k < 1 \quad \forall 0 < k < 1 \quad (16)$$

Plugging (16) to (11), (12) and (13), we obtain the Adversarial Correlation Structure as stated in Theorem 1. And by construction, it satisfies (1) and (2).

At this point, although the Adversarial Correlation Structure exhibits properties (1) and (2), we cannot say much about the optimal auctions to the Adversarial Correlation Structure since we haven’t said anything about the virtual value of the low bidder. If the virtual value of the low bidder were higher than that of the high bidder, it would be possible that low bidder gets strictly positive allocation under the optimal auction. We thus require an inter-bidder monotonicity condition (6). Next, we show the dual role of the robust-version regularity conditions.

**Lemma 3.** The robust-version regularity conditions guarantee that it is possible to construct the Adversarial Correlation Structure and guarantee that the Adversarial Correlation Structure will exhibits (6).
Proof. To guarantee that it is possible to construct the Adversarial Correlation Structure, it has to be a legal joint distribution in the sense the density (or probability mass) has to be non-negative for all value profiles, i.e., \( \pi^*(v_1, v_2) \geq 0 \) for \( 0 \leq v_1, v_2 < 1 \) and \( Pr^*(1, 1) \geq 0 \). Therefore, we have,

\[
f(k) - \frac{\int_0^k j^2 f(j) dj}{k^3} \geq 0 \quad \forall 0 < k < 1 \tag{17}
\]

\[
Pr(1) \geq \int_{s \in [0,1)} s^2 f(s) ds \tag{18}
\]

Slightly rewriting (17):

\[
k^3 f(k) - \int_0^k j^2 f(j) dj \geq 0 \quad \forall 0 < k < 1 \tag{19}
\]

Now we show the first part of the robust-version regularity conditions implies (17). To see this, we first define \( H(k) := k^3 f(k) - \int_0^k j^2 f(j) dj \). Note \( H(0) = 0 \). In addition, for any \( k \geq k' \geq 0 \), we have\(^{29}\)

\[
H(k) - H(k') = k^3 f(k) - (k')^3 f(k') - \int_{k'}^k j^2 f(j) dj
\geq k^3 f(k) - k' k^2 f(k) - \int_{k'}^k j^2 f(j) dj
\geq k^2 f(k)(k - k') - (k - k') k^2 f(k)
= 0
\]

where the first inequality follows from \( k^2 f(k) \geq (k')^2 f(k') \) for \( k \geq k' \) and the second inequality follows \( k^2 f(k) \geq j^2 f(j) \) for any \( j \leq k \). Therefore \( H(k) \geq 0 \) for any \( 0 < k < 1 \), i.e., (17) holds. Thus, the robust-regularity conditions guarantee that it is possible to construct the Adversarial Correlation Structure.

Now given that the construction is possible, we argue that the robust-version regularity conditions guarantee that the virtual valuations are inter-bidder monotone, i.e., (6) holds. Given the Adversarial Correlation Structure, it suffices to show

\[\phi_2(v_1, v_2) \leq 0\]

\(^{29}\)I thank Songzi Du for the proof without relying on differentiability of \( f(x) \).
if $0 < v_2 \leq v_1 < 1$. We now calculate $\phi_2(v_1, v_2)$ for $0 < v_2 \leq v_1 < 1$:

$$\phi_2(v_1, v_2) = v_2 - \frac{f(v_1) - \int_0^{v_2} \pi^*(v_1, t) dt}{\pi^*(v_1, v_2)} =$$

$$v_2 - \frac{f(v_1) - \int_0^{v_2} c(t) \frac{t}{v_1} dt}{c(v_2) \frac{v_2}{v_1}} =$$

$$v_2 - \frac{f(v_1) - \frac{1}{v_1 v_2} \int_0^{v_2} t f(t) dt}{(f(v_2) - \frac{\int_0^{v_2} t^2 f(t) dt}{v_2}) v_2}$$

Now it is easy to see that

$$\phi_2(v_1, v_2) \leq 0 \iff f(v_1) \geq \frac{v_2^2}{v_1^2} f(v_2)$$

for any $0 < v_2 \leq v_1 < 1$, which is exactly the first part of the robust-version regularity conditions.

The following virtual value matrix holds for the Adversarial Correlation Structure under the robust-version regularity conditions.

\[
\begin{pmatrix}
(0, 0),_0 & (0, 0),_0 > 0 & \cdots & \cdots & (0, 0),_0 < 1 & (0, 0),_0 \leq 1 \\
(0, 0),_0 > 0 & (0, 0),_0 = v_2 > 0 & \cdots & \cdots & \cdots & (0, 0),_1 < 1 \leq 1 \\
\vdots & (0, 0),_1 = v_1 > v_2 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots \\
(0, 0),_1 < 0 & (0, 0),_1 < 1 > 0 & \cdots & \cdots & (0, 0),_1 = v_2 < 1 & (0, 0),_1 < 1 \leq 1 \\
(0, 0),_1 < 1 > 0 & (0, 0),_1 < 1 < 0 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

Here “0” means zero virtual value, “−” means non-positive virtual values, “+” means non-negative virtual values, “<” means the virtual value is the bidder is weaker smaller than that of her opponent. The subscript denotes the corresponding value profile. Now given the robust-version regularity conditions, it is easy to design optimal auctions to the Adversarial Correlation Structure. Indeed, any monotone, exclusive and feasible auction that fully allocates the good to the bidder(s) with the highest possible value of 1 is optimal.

**Lemma 4.** The Second Price Auction with Uniformly Distributed Reserves is a best response to the Adversarial Correlation Structure under the robust-version regularity conditions.

**Proof.** Note the Second Price Auction with Uniformly Distributed Reserves is monotone,
exclusive and feasible. In addition, the bidder(s) with the value of 1 is allocated with probability 1. Thus, this lemma holds by Lemma 2 and Lemma 3.

A general insight from the robust mechanism design literature is that the maxmin auction will exhibit a lot of indifference across correlation structures. With some preliminary calculation, we find the Second Price Auction with Uniformly distributed Reserves generates the same expected revenue for many different correlation structures, which make it a good candidate for maxmin solution. Indeed, we are able to show that it exhibits the full-insurance property; it generates the same expected revenue across all plausible correlation structures. Then, trivially, the Adversarial Correlation Structure is a best response for Nature.

**Lemma 5.** The Second Price Auction with Uniformly Distributed Reserves exhibits the full-insurance property. Thus, the Adversarial Correlation Structure is a best response to the Second Price Auction with Uniformly Distributed Reserves.

**Proof.** Let \( t(v_1, v_2) := t_1(v_1, v_2) + t_1(v_1, v_2) \).

Note for any \( \{(i, k), (j, k), (i, l), (j, l)\} \), we have

\[
t(i, k) + t(j, l) = t(i, l) + t(j, k) = \frac{i^2 + k^2 + j^2 + l^2}{2} \quad (FI)
\]

Indeed (FI) guarantees that the mechanism exhibits the full-insurance property. To see this, starting from any correlation structure, if we wish to have a different one, then necessarily it will involve operations in which for some \( \{(i, k), (j, k), (i, l), (j, l)\} \), the density of \((i, k)\) and the density of \((j, l)\) increase (decrease) by some positive \( \epsilon \), and simultaneously, the density of \((j, k)\) and the density of \((i, l)\) decrease (increase) by some positive \( \epsilon \). Therefore, (FI) guarantees that the mechanism full insures the auctioneer, which implies that any plausible correlation structure is a best response.

Formally, for any joint distribution \( \pi \) consistent with the given marginal distributions, the expected revenue under the Second Price Auction with Uniformly Distributed Reserves is

\[
\int_{[0,1]^2} t(v_1, v_2) d\pi(v_1, v_2) = \int_{[0,1]^2} \frac{v_1^2 + v_2^2}{2} d\pi(v_1, v_2) \\
= \int_{[0,1]^2} \frac{v_1^2}{2} d\pi(v_1, v_2) + \int_{[0,1]^2} \frac{v_2^2}{2} d\pi(v_1, v_2) \\
= \int_{[0,1]} \frac{v_1^2}{2} dF(v_1) + \int_{[0,1]} \frac{v_2^2}{2} dF(v_2) \\
= E[X^2]
\]
Theorem 1 is implied by Lemma 4 and Lemma 5.

A.3 Proof of Corollary 1

This is directly implied by Theorem 1 by checking that the equal-revenue distribution satisfies the robust-version regularity conditions. Alternatively, we observe that the independent joint distribution is a best response since this auction is a full-insurance auction. For the independent joint distribution, since only value profiles in which at least one of the two bidders has the value of 1 render positive virtual values, any mechanism that fully allocate the good to the high bidder(s) when at least one of the bidder has the value of 1 is optimal. Thus the Second Price Auction with Uniformly distributed Reserves. Therefore we conclude they form a Nash equilibrium. In addition, the revenue guarantee is $1 - (1 - \alpha)^2 = 2\alpha - \alpha^2$.

A.4 Proof of Theorem 2

The proof strategy is similar to that for Theorem 1.

Lemma 6. The General Adversarial Correlation Structure exhibits (7) and (8).

Proof. First, note by allocating all marginal density $f(0)$ (or marginal probability mass $Pr(0)$) to the value profile $(0, \ldots, 0)$, we have $\phi_i(v) = 0$ for any $i$ and $v_i > 0, v_j = 0 \forall j \neq i$. Thus, (7) and (8) trivially hold for these value profiles. Now let $A_{kj} := \{v| k \leq v_1 \leq j, v_i = k \forall i \neq 1\}$, define (with slight abuse of notations) $c(0) := f(0)$ and $c(k) := \int_{A_{k1}} d\pi^*$ for $k > 0$. Consider the value profile $(v_1, v_2, \ldots, v_2)_{N-1}$ where $0 < v_2 \leq v_1 < 1$. In order for the virtual values of bidder 1 to satisfy (7), we must have

$$\phi_1(v_1, v_2, \ldots, v_2)_{N-1} = v_1 - \frac{c(v_2) - \int_{v_2}^{v_1} \pi^*(s, v_2, \ldots, v_2)ds}{\pi^*(v_1, v_2, \ldots, v_2)_{N-1}} = 0 \quad \forall 0 < v_2 \leq v_1 < 1$$

These equations are essentially a system of ordinary differential equations, whose solution is well known:

$$\pi^*(v_1, v_2, \ldots, v_2)_{N-1} = \frac{1}{v_1} v_2 c(v_2) \quad 0 < v_2 \leq v_1 < 1 \quad (20)$$

$$\pi^*(1, v_2, \ldots, v_2)_{N-1} = v_2 c(v_2) \quad 0 < v_2 < 1 \quad (21)$$

\[ \square \]
By symmetry, we also obtain $\pi^*(v) = \pi^*(v_1, v_2, \ldots, v_2)$ for $0 < v_j = v(2) \leq v_i = v(1) < 1, \forall j \neq i, \forall i$ and $\pi^*(v) = \pi^*(1, v_2, \ldots, v_2)$ for $0 < v_j = v(2) < v_i = 1, \forall j \neq i, \forall i$. Finally,

$$Pr^*(1, \ldots, 1) = Pr(1) - \int_{j[0,1)} j c(j) dj$$

(22)

Now we solve for $c(k)$ and therefore for $\pi^*$. Note since the marginal distribution is the same across all bidders, given the above derivation, $c(k)$ must satisfy the following condition:

$$f(k) = (N - 1)c(k) + \frac{\int_0^k j c(j) dj}{k^2} \quad \forall 0 < k < 1$$

(23)

To see this, suppose the bidder 1’s value is $k$. Then either $k$ is the highest value and other bidders all have value $j \in [0, k] \left( \frac{\int_0^k j c(j) dj}{k^2} \right)$ or $k$ is the second highest value and one of the other bidders has the highest value ($(N - 1)c(k)$). Multiplying $k$ by both sides of (23), we obtain

$$k f(k) = (N - 1) k c(k) + \frac{\int_0^k j c(j) dj}{k} \quad \forall 0 < k < 1$$

Define $g(k) := \int_0^k j c(j) dj$ for $0 < k < 1$. Then we have

$$k f(k) = (N - 1) g(k) + \frac{g(k)}{k} \quad \forall 0 < k < 1$$

Note this is an ordinary differential equation, and we solve for $g(k)$:

$$g(k) = \frac{1}{(N - 1) k^{N-1} k^2} \int_0^k j^{\frac{N}{N-1}} f(j) dj \quad \forall 0 < k < 1$$

(24)

From this we compute $c(k)$ for $0 < k < 1$,

$$c(k) = \frac{1}{N - 1} (f(k) - \int_0^k j^{\frac{N}{N-1}} f(j) dj) \quad \forall 0 < k < 1 \quad \forall 0 < k < 1$$

(25)

Plugging (25) to (20),(21) and (22), we obtain the General Adversarial Correlation Structure as stated in the main results. And by construction, it satisfies (7) and (8).

When is the General Adversarial Correlation Structure feasible?

**Lemma 7.** The general robust-version regularity conditions (I) guarantee that it is possible to
construct the General Correlation Structure. The general robust-version regularity conditions (II) imply the general robust-version regularity conditions (I).

**Proof.** The General Adversarial Correlation Structure has to be a legal joint distribution in the sense that the density (or probability mass) has to be non-negative for all value profiles. Therefore the first part of this lemma holds. To show the second part of this lemma, define (with abuse of notations)

\[ H(k) := (N - 1)k^{1 + \frac{N}{N-1}} f(k) - \int_0^k s^{N-1} f(s) ds. \]

Note \( G(0) = 0 \). In addition, for any \( k \geq k' \geq 0 \), we have

\[
H(k) - H(k') = (N - 1)k^{1 + \frac{N}{N-1}} f(k) - (N - 1)(k')^{1 + \frac{N}{N-1}} f(k') - \int_{k'}^k s^{N-1} f(s) ds \\
\geq (N - 1)k^{2} f(k)(k^{\frac{1}{N-1}} - (k')^{\frac{1}{N-1}}) - k^{2} f(k) \int_{k'}^k s^{\frac{1}{N-1}-1} ds \\
= (N - 1)k^{2} f(k)(k^{\frac{1}{N-1}} - (k')^{\frac{1}{N-1}}) - (N - 1)k^{2} f(k)(k^{\frac{1}{N-1}} - (k')^{\frac{1}{N-1}}) \\
= 0
\]

where the first inequality follows from \( k^{2} f(k) \geq (k')^{2} f(k') \) for \( k \geq k' \) and the second inequality follows \( k^{2} f(k) \geq s^{2} f(s) \) for any \( s \leq k \). Therefore \( H(k) \geq 0 \) for any \( 0 < k < 1 \). \( \square \)

Given that we now restrict attention to exclusive DSIC and EPIR mechanisms, the following lemma is immediate.

**Lemma 8.** The Second Price Auction with Beta\(\left(\frac{1}{N-1}, 1\right)\) Distributed Reserves is a best response among exclusive DSIC and EPIR mechanisms to the General Adversarial Correlation Structure under the general robust-version regularity conditions (I).

**Proof.** Note the Second Price Auction with Beta\(\left(\frac{1}{N-1}, 1\right)\) Distributed Reserves is monotone, exclusive and feasible. In addition, the bidder(s) with the value of 1 is allocated with probability 1. Thus, this lemma holds by Lemma 6 and Lemma 7. \( \square \)

Now we use duality theory to show the General Adversarial Correlation Structure is a best response to the Second Price Auction with Beta\(\left(\frac{1}{N-1}, 1\right)\) Distributed Reserves. We first write down the primal minimization problem for Nature given a mechanism \((q, t)\) and derive its dual maximization problem. Formally, let \( \{\lambda_i(v_i)\}_{i \in \{1,2,\ldots,N\}, v_i \in [0,1]} \) be dual variables.

\[
(P) \quad \min_{\pi \in \Pi(F)} \int_{v \in [0,1]^N} \sum_{i=1}^N t_i(v) d\pi(v)
\]
s.t.
\[ \int_0^1 \pi(v_i, v_{-i}) dv_{-i} = f(v_i) \quad \text{dual variables} \quad \lambda_i(v_i) \]
\[ \pi(v) \geq 0 \]

(\(D\)) \[ \max_{\{\lambda_i(v_i)\}} \sum_{i=1}^N \int_0^1 \lambda_i(v_i) dF(v_i) \]
s.t.
\[ \sum_{i=1}^N \lambda_i(v_i) \leq \sum_{i=1}^N t_i(v) \quad \forall v \in [0,1]^N \]
\[ \lambda_i(v_i) \in R \]

By the weak duality, this dual problem has weakly smaller value than the value of the primal problem. In order for the two programs to have the same value, the complementarity slackness conditions have to hold. Given the support of the General Adversarial Correlation Structure, we have

\[ \sum_{i=1}^N \lambda_i(v_i) = \sum_{i=1}^N t_i(v) \quad v_i = v(1) \forall i; \exists i, v_i = v(1) > v_j = v(2) > 0 \forall j \neq i \] (27)

We assume \( \lambda_i(v) = \lambda(v) \) for all \( i \), and assume the mechanism is the second price auction with random reserves distributed as \( G \), then (27) implies

\[ N \lambda(v_i) = v_i G(v_i) \quad \forall v_i \in [0,1] \] (28)
\[ \lambda(v(1)) + (N-1) \lambda(v(2)) = v(1) G(v(1)) - \int_{v(2)}^{v(1)} G(s) ds \quad \forall 1 \geq v(1) > v(2) > 0 \] (29)

Note by (28),
\[ \lambda(v_i) = \frac{v_i G(v_i)}{N} \] (30)

Plugging (30) to (29), we obtain for \( 1 \geq v(1) > v(2) > 0 \),
\[ \frac{v(1) G(v(1)) + (N-1) v(2) G(v(2))}{N} = v(1) G(v(1)) - \int_{v(2)}^{v(1)} G(s) ds \] (31)

Taking first order derivatives with respect to \( v(1) \) and \( v(2) \), we obtain the same ordinary differential equation:
\[ (N-1)v G'(v) = G(v) \] (32)
Given that $G(v)$ is a distribution, we have the solution

$$G(v) = v^{\frac{1}{N-1}}$$

This is the $Beta(\frac{1}{N-1}, 1)$ distribution.

**Lemma 9.** The General Adversarial Correlation Structure is a best response to the Second Price Auction with $Beta(\frac{1}{N-1}, 1)$ Distributed Reserves.

**Proof.** We construct the dual variables as follows. For all $i \in \{1, 2, \cdots, N\}$ and $v \in [0, 1]$

$$\lambda_i(v) = \frac{v^{\frac{N}{N-1}}}{N} \quad (33)$$

We argue they are feasible given the Second Price Auction with $Beta(\frac{1}{N-1}, 1)$ Distributed Reserves. To see this, we divide value profiles into two cases.

**Case 1:** $\# \{k : v_k = v(1)\} = 1$.

In this case, $t(v) = \frac{v^{\frac{N}{N-1}} + (N-1)v^{\frac{N}{N-1}}}{N}$. The LHS of (26) is maximized when all bidders except the highest bidder have the same value, and (26) holds with equality. Therefore (26) holds with strict inequality otherwise.

**Case 2:** $\# \{k : v_k = v(1)\} \geq 2$.

In this case, $t(v) = v^{\frac{N}{N-1}}$. The LHS of (26) is maximized when all bidders have the same value, and (26) holds with equality. Therefore (26) holds with strict inequality otherwise.

Now we argue the values of $(P)$ and $(D)$ are the same given the General Adversarial Correlation Structure, the Second Price Auction with $Beta(\frac{1}{N-1}, 1)$ Distributed Reserves, and the constructed dual variables. To see this, first note the value of $(P)$ is

$$\sum_{i=1}^{N} \int_{0}^{1} \lambda_i(v_i)dF(v_i) = \int_{0}^{1} v^{\frac{N}{N-1}}dF(v_i) \quad (34)$$

Note only the bidder(s) with the value of 1 will generate possible virtual value under the
General Adversarial Correlation Structure. The value of \((P)\) can be calculated as follows.

\[
Pr(\{v|\exists i, v_i = 1\}) \cdot 1 = Pr^*(1, \ldots, 1) + N \int_{\{v_i=1, 0 \leq v_j = v(2) \leq 1, \forall j \neq 1\}}^{} d\pi^* \\
= Pr(1) + \frac{1}{N-1} \int_{[0,1]} s^{N-1} f(s) ds + N \cdot \frac{1}{N-1} \int_{[0,1]} s^{N-1} f(s) ds \\
= Pr(1) + \int_{[0,1]} s^{N-1} f(s) ds \\
= \int_0^1 v_i^{N-1} dF(v_i)
\]

(35)

By (34) and (35), the values of \((P)\) and \((D)\) are the same. \(\Box\)

Theorem 2 is implied by Lemma 7, Lemma 8 and Lemma 9.

A.5 Proof of Corollary 2

We argue that the Second Price Auction with \(Beta(\frac{1}{N-1}, 1)\) Distributed Reserves will not be a best response to the General Adversarial Correlation Structure among DSIC and EPIR mechanisms. To see this, note for any value profile \(v\) in which \(#\{k : v_k = v(1)\} = 1\) and \(v(2) > 0, \phi_j(v) = v(2) > 0\) for any \(j\) such that \(v_j = v(2)\). Therefore, by allocating positive allocations to those second highest bidders, we have a feasible and profitable deviation.

A.6 Proof of Corollary 3

It is easy to check that the equal revenue distribution satisfies the general robust-version regularity conditions (II). Then we directly derive the worst-case correlation structure using the formula (in Theorem 2) for the General Adversarial Correlation Structure. The revenue guarantee is \(E[X^{N-1}] = \int_{\alpha}^1 x^{N-1} \cdot \frac{2}{x^2} dx + \alpha \cdot 1 = N\alpha - (N-1)\alpha^{\frac{N}{N-1}}\).

A.7 Proof of Proposition 2

It is easy to see the worst-case correlation structure for posted price mechanisms is the maximally correlated one, thus the guarantee of which is \(\max x \cdot (1 - F(x))\). Since \(x^{\frac{N}{N-1}} \geq x^2\) for any \(x \in [0, 1]\) and \(N \geq 2\), it suffices to compare \(E[X^2]\) with \(\max x \cdot (1 - F(x))\). By integration by part, we can write

\[
E[X^2] = 2 \int_0^1 x(1 - F(x)) dx
\]
If the revenue function \( R(x) := x \cdot (1 - F(x)) \) is concave, let \( x^* \) denote the solution to \( \max_{x \in [0,1]} R(x) \). In addition, note \( R(0) = R(1) = 0 \). Then using graph it is easily seen that

\[
\int_0^1 R(x)dx \geq \frac{1}{2} \cdot 1 \cdot R(x^*)
\]

This is equivalent to

\[
E[X^2] = 2 \int_0^1 x(1 - F(x))dx \geq \max x \cdot (1 - F(x))
\]

Also note for some distribution whose revenue function is strictly concave, the inequality is strict, e.g., uniform distribution on [0,1]. Thus, we complete the proof.

B Proofs for Section 5

B.1 Proof of Lemma 1

For each \( r \), we construct the dual variables for the second price auction with random reserves distributed as \( F(x) = \left(\frac{x}{r}\right)^{\frac{1}{N-1}} \) as follows:

\[
\lambda_i(v) = \frac{v^N}{Nr^{N-1}} \quad \text{for} \quad 0 \leq v \leq r, \forall i
\]

\[
\lambda_i(v) = \frac{r}{N} \quad \text{for} \quad r < v \leq 1, \forall i
\]

Given the constructed dual variables above, the value of (D) is

\[
\int_0^r \frac{v^N}{r^{N-1}} dF(x) + r(1 - F(r))
\]

Then it suffices to show that the constructed dual variables are feasible. We divide the value profiles into three cases.

Case 1: \( v_{(1)} \leq r \).

(26) holds by similar argument with that in the proof of Lemma 9.

Case 2: \( v_{(1)} > r, \#\{k : v_k = v_{(1)}\} = 1 \).

When \( v_{(2)} > r, t(v) = v_{(2)} \). The LHS of (26) is maximized when \( v_i \geq r \) for all \( i \), yielding the value of \( N \cdot \frac{r}{N} = r < t(v) \);

When \( v_{(2)} \leq r, t(v) = v_{(1)} \cdot 1 - \int_r^{v_{(1)}} - \int_{v_{(2)}}^r \left(\frac{v}{r}\right)^{\frac{1}{N-1}} dx = \frac{r}{N} + \frac{(N-1)v_{(2)}}{N^{N-1}} \). The LHS of (26) is maximized when \( v_i = v_{(2)} \) for all \( i \notin \{k : v_k = v_{(1)}\} \), yielding the value of
\[
\frac{r}{N} + \frac{(N-1)v_{(2)}}{N-r/N-1} = t(v). \text{ Therefore (26) holds.}
\]

Case 3: \( v_{(1)} > r, \#\{k : v_k = v_{(1)}\} \geq 2. \)

Now \( t(v) = v_{(1)}. \) The LHS of (26) is maximized when \( v_i \geq r \) for all \( i, \) yielding the value of \( N \cdot \frac{r}{N} = r < t(v). \) Therefore (26) holds.

\[\text{B.1 Proof of Theorem 3}\]

By definition of \( r^* \) and Lemma 1, the revenue guarantee of the second price auction with random reserves distributed as \( F(x) = \left(\frac{x}{r^*}\right)^{N-1} \) is at least

\[
\max_r \int_0^r \frac{x^{N}}{r^{N-1}} dF(x) + r(1 - F(r))
\]

Note the revenue guarantee of the posted price mechanism is \( \max_{x \in [0,1]} x \cdot (1 - F(x)). \) Denote the solution as \( x^*. \) Since \( x^* > 0, \) then

\[
\max_{r \in [0,1]} \int_0^r \frac{x^{N}}{r^{N-1}} dF(x) + r(1 - F(r)) \geq \int_0^{x^*} \frac{x^{N}}{(x^*)^{N-1}} dF(x) + x^*(1 - F(x^*))
\]

\[
> x^*(1 - F(x^*))
\]

\[
= \max_{x \in [0,1]} x \cdot (1 - F(x))
\]

\[\text{B.1 Proof of Theorem 4}\]

By He and Li (2020), the revenue guarantee of the second price auction with the optimal deterministic reserve for \( N \)-bidder case is as follows:

\[
\frac{N}{N-1} \int_{r^*}^{c(r^*)} x dF(x)
\]

where \( r^* \) satisfies \( F(Nr^*) = F\left(\frac{N-1+F(r^*)}{N}\right) \) and \( c(r^*) = F^{-1}\left(\frac{N-1+F(r^*)}{N}\right). \) Note

\[
\max_{r \in [0,1]} \int_0^r \frac{x^{N}}{r^{N-1}} dF(x) + r(1 - F(r)) \geq \int_0^{c(r^*)} \frac{x^{N}}{c(r^*)^{N-1}} dF(x) + c(r^*)(1 - F(c(r^*)))
\]

\[
= \int_0^{c(r^*)} \frac{x^{N}}{c(r^*)^{N-1}} dF(x) + F^{-1}\left(\frac{N-1+F(r^*)}{N}\right)\frac{1 - F(r^*)}{N}
\]
Thus, it suffices to show
\[ \int_{0}^{\infty} \frac{x^{N}}{c(r^{*})^{N-1}} dF(x) + F^{-1}(\frac{N - 1 + F(r^{*})}{N}(1 - F(r^{*})) \geq \frac{N}{N-1} \int_{r^{*}}^{\infty} x dF(x) \]
for any \( F(x) \). Now define \( J(x) := \frac{N}{N-1} x - \frac{x^{N}}{c(r^{*})^{N-1}} \). Since \( J'(x) = \frac{N}{N-1} (1 - (\frac{x}{c(r^{*})})^{N-1}) \) and \( J''(x) = -\frac{Nc^{N-2}}{(N-1)^{2}} \leq 0 \), \( J(x) \) is maximized at \( x = c(r^{*}) \) and max \( J(x) = \frac{1}{N-1} c(r^{*}) \). Now, we have
\[
\frac{N}{N-1} \int_{r^{*}}^{\infty} x dF(x) - \int_{0}^{\infty} \frac{x^{N}}{c(r^{*})^{N-1}} dF(x) < \frac{N}{N-1} \int_{r^{*}}^{\infty} x dF(x) - \int_{r^{*}}^{\infty} \frac{x^{N}}{c(r^{*})^{N-1}} dF(x)
\]
\[
= \int_{r^{*}}^{\infty} (\frac{N}{N-1} x - \frac{x^{N}}{c(r^{*})^{N-1}}) dF(x)
\]
\[
\leq \int_{r^{*}}^{\infty} \frac{1}{N-1} c(r^{*}) dF(x)
\]
\[
= \frac{1}{N-1} c(r^{*})(F(c(r^{*})) - F(r^{*))
\]
\[
= \frac{1}{N-1} F^{-1}(\frac{N - 1 + F(r^{*})}{N}(1 - F(r^{*}))\geq \frac{N}{N-1} \int_{r^{*}}^{\infty} x dF(x)
\]
This finishes the proof.

C “Necessity” of the Conditions

**Proposition 3.** If the Second Price Auction with Uniformly Distributed Reserves is a maxmin auction among DSIC and EPIR mechanisms, then the robust-version regularity conditions hold almost surely.

**Proof.** The intuition behind this is the observation that given the Second Price Auction with Uniformly Distributed Reserves, the high bidder’s allocation is strictly monotone and strictly positive but less than 1 when the her value \( v \in (0, 1) \). Thus in a Nash equilibrium, the high bidder’s virtual value has to be 0 for \( v \in (0, 1) \) under the correlation structure, otherwise Myerson’s ironing argument implies that allocation rule in equilibrium should exhibit “flatness” across some range. Formally, we will establish Lemma 10 and Lemma 11 below.

**Lemma 10.** For the Second Price Auction with Uniformly distributed Reserves to be part of
a Nash equilibrium, the equilibrium correlation structure has to be the Adversarial Correlation Structure almost surely.

Proof. Let $\pi$ denote a best response to the Second Price Auction with Uniformly distributed Reserves. Suppose (1) does not hold for a set of $(v_1, v_2)$ where $1 > v_1 \geq v_2$ with some positive measure. If virtual values of bidder 1 for these value profiles are all positive, then consider a modified allocation exhibiting the property that the allocation to bidder 1 is one from the value profile in which the virtual value of bidder 1 becomes positive for the first time. Formally, let $\overline{v_1}(v_2) := \inf\{v_1 : \phi_1(v_1, v_2) > 0, v_1 \geq v_2\}$. Let $q(v_1, v_2) := 1$ for $v_1 > \overline{v_1}(v_2)$ and $q(v) := q^*(v)$ otherwise. Such modification is feasible since bidder 2 gets zero allocation for all these value profiles in the the Second Price Auction with Uniformly distributed Reserves. Thus we get a profitable and feasible deviation. If virtual values of bidder 1 for these value profiles are all negative, by similar argument, we rule out the possibility that Second Price Auction with Uniformly distributed Reserves is a best response to $\pi$. Now If virtual values of bidder 1 for these value profiles are not all positive and not all negative, we have to discuss two cases. The first case is where the virtual value is still (weakly) monotone. Then by similar argument, we can see that the Second Price Auction with Uniformly distributed Reserves can not be a best response to $\pi$. The second case is where the virtual value is not monotone, then a best response has to exhibit flatness across a range of value profiles, which can be done by Myerson’s ironing procedure. Note Second Price Auction with Uniformly distributed Reserves exhibits strict monotonicity for these value profiles. Thus, it cannot be a best response. Formally, suppose $\phi_1(v_1, v_2) > 0, \phi_1(v'_1, v_2) < 0$ but $v'_1 > v_1$, then we can modify the allocation of the Second Price Auction with Uniformly distributed Reserves by increasing the allocation by a small amount $\epsilon$ to bidder 1 when the value profile is $(v_1, v_2)$ and decreasing the allocation by a small amount $\epsilon$ to bidder 1 when the value profile is $(v'_1, v_2)$. Since this is a feasible and profitable deviation, we conclude that the Second Price Auction with Uniformly distributed Reserves can not be a best response.

Then by Lemma 10 and the proof of Lemma 2, the equilibrium correlation structure is the Adversarial Correlation Structure almost surely.

Lemma 11. For the Second Price Auction with Uniformly Distributed Reserves to be part of a Nash equilibrium, the Adversarial Correlation Structure exhibits (6) almost surely.

Proof. Suppose not. Then, there exists a set of $(v_1, v_2)$ where $v_1 > v_2$ but $\phi_2(v_1, v_2) > 0$ with some positive measure. Then by increasing the allocation to bidder 2 by a small amount $\epsilon$ when the value profile is $(v_1, v_2)$, we have a feasible and profitable deviation. Thus, the Second Price Auction with Uniformly distributed Reserves is not a best response.
Proposition 3 is implied by Lemma 3, Lemma 10 and Lemma 11.

Proposition 4. If the Second Price Auction with Beta($\frac{1}{N-1}, 1$) Distributed Reserves is a maxmin auction among exclusive DSIC and EPIR mechanisms, then the general robust-version regularity conditions (I) hold almost surely.

Proof. The key observation behind is that the highest bidder’s allocation is strictly monotone and strictly positive but less than 1 when the her value $v \in (0, 1)$ under the Second Price Auction with Beta($\frac{1}{N-1}, 1$) Distributed Reserves. Formally, we establish Lemma 12 below.

Lemma 12. For the Second Price Auction with Beta($\frac{1}{N-1}, 1$) Distributed Reserves to be part of a Nash equilibrium, the equilibrium correlation structure has to be the General Adversarial Correlation Structure almost surely.

Proof. As shown in Lemma 9, (26) holds with equality if and only if $v \in V^+$. This implies the equilibrium correlation structure has the support $V^+$. Then following the same argument as in the proof of Lemma 10, this lemma holds.

Proposition 4 is implied by Lemma 7 and Lemma 12.
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