Decentralized Riemannian Gradient Descent on the Stiefel Manifold

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Abstract

We consider a distributed non-convex optimization where a network of agents aims at minimizing a global function over the Stiefel manifold. The global function is represented as a finite sum of smooth local functions, where each local function is associated with one agent and agents communicate with each other over an undirected connected graph. The problem is non-convex as local functions are possibly non-convex (but smooth) and the Steifel manifold is a non-convex set. We present a decentralized Riemannian stochastic gradient method (DRSGD) with the convergence rate of \(O(1/\sqrt{K})\) to a stationary point. To have exact convergence with constant stepsize, we also propose a decentralized Riemannian gradient tracking algorithm (DRGTA) with the convergence rate of \(O(1/K)\) to a stationary point. We use multi-step consensus to preserve the iteration in the local (consensus) region. DRGTA is the first decentralized algorithm with exact convergence for distributed optimization on Stiefel manifold.

1 Introduction

Distributed optimization has received significant attention in the past few years in machine learning, control and signal processing. There are mainly two scenarios where distributed algorithms are necessary: (i) the data is geographically distributed over networks and/or (ii) the computation on a single (centralized) server is too expensive (large-scale data setting). In this paper, we consider the following multi-agent optimization problem

\[
\min_{x} \frac{1}{n} \sum_{i=1}^{n} f_i(x_i) \\
\text{s.t.} \quad x_1 = x_2 = \ldots = x_n, \\
x_i \in \mathcal{M}, \quad \forall i = 1, \ldots, n,
\]

(1.1)

where \(f_i\) has \(L\)-Lipschitz continuous gradient in Euclidean space and \(\mathcal{M} := \text{St}(d, r) = \{x \in \mathbb{R}^{d \times r} : x^\top x = I_r\}\) is the Stiefel manifold. Unlike the Euclidean distributed setting, problem (1.1) is defined on the Stiefel manifold, which is a non-convex set. Many important applications can be written in the form (1.1), e.g., decentralized spectral analysis [15, 19, 20], dictionary learning [34], eigenvalue estimation of the covariance matrix [32] in wireless sensor networks, and deep neural networks with orthogonal constraint [4,18,41].

Problem (1.1) can generally represent a risk minimization. One approach to solving (1.1) is collecting all variables to a central server and running a centralized algorithm. However, when the dataset is massive (or the data dimension is large), this causes memory issues and computational burden on the central server. Then, it is more efficient to take a decentralized approach and use local computation based on a network topology. In this case, each local function \(f_i\) is associated with one agent in the network, and agents communicate with each other over an undirected connected graph. For example, for stochastic gradient descent (SGD), [23] show that the decentralized SGD can be faster than centralized SGD, especially when training neural networks. More importantly, a central server may not exist in practice.

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1.1 Our Contributions

In this paper, we focus on the decentralized setting and design efficient decentralized algorithms to solve (1.1) over any connected undirected network. Our contributions are as follows:

1. We show the convergence of the decentralized stochastic Riemannian gradient method (Algorithm 1) for solving (1.1). Specifically, the iteration complexity of obtaining an $\epsilon$-stationary point (see Definition 2.2) is $O(1/\epsilon^2)$ in expectation $^1$.

2. To achieve exact convergence with constant stepsizes, we propose a gradient tracking algorithm (DRGTA) (Algorithm 2) for solving (1.1). For DRGTA, the iteration complexity of obtaining an $\epsilon$-stationary point is $O(1/\epsilon)$ $^1$.

Importantly, both of the proposed algorithms are retraction-based and DRGTA is vector transport-free. These two features make the algorithms computationally cheap and conceptually simple. DRGTA is the first decentralized algorithm with exact convergence for distributed optimization on the Stiefel manifold.

1.2 Related works

Decentralized optimization has been well-studied in Euclidean space. The decentralized (sub)-gradient methods were studied in $^{[10,30,40,46]}$ and a distributed dual averaging subgradient method was proposed in $^{[12]}$. However, with a constant stepsize $\beta > 0$, these methods can only converge to a $O(\beta/\sigma_2)$-neighborhood of a stationary point, where $\sigma_2$ is a network parameter (see Assumption 1). To achieve exact convergence with a fixed stepsize, gradient tracking algorithms were proposed in $^{[11,29,33,37,44,47]}$, to name a few. The convergence analysis can be unified via a primal-dual framework $^{[3]}$. Another way to use the constant stepsize is decentralized ADMM and its variants $^{[6,8,27,38]}$. Also, decentralized stochastic gradient method for non-convex smooth problems were well-studied in $^{[5,23,43]}$, etc. We refer to the survey paper $^{[28]}$ for a complete review on the state-of-the-art algorithms and the role of network topology.

The problem (1.1) can be thought as a constrained decentralized problem in Euclidean space, but since the Stiefel manifold constraint is non-convex, none of the above works can solve the problem. On the other hand, we can also treat (1.1) as a smooth problem over the Stiefel manifold. However, the constraint $x_1 = x_2 = \ldots = x_n$ is difficult to handle due to the lack of linearity on $\mathcal{M}$. Since the Stiefel manifold is an embedded submanifold in Euclidean space, our viewpoint is to treat the problem in Euclidean space and develop new tools based on Riemannian manifold optimization $^{[1,7,13]}$. For the optimization problem (1.1), a decentralized Riemannian gradient tracking algorithm was presented in $^{[36]}$. The vector transport operation should be used in $^{[36]}$, which brings not only expensive computation but also analysis difficulties. Moreover, they need to use asymptotically infinite number of consensus steps. Other distributed algorithms were either specifically designed for the PCA problem $^{[15,32,34]}$ or in centralized topology $^{[14,19,42]}$. For these decentralized algorithms, diminishing stepsize or asymptotically infinite number of communication steps should be utilized to get exact solution. Different from all these works, DRGTA requires a finite number of communications using a constant step-size.

As a special case of problem (1.1), the Riemannian consensus problem is well-studied; see $^{[9,26,35,39]}$. Recently, it was shown in $^{[9]}$ that the multi-step consensus algorithm (DRCS) converges linearly to the global consensus in a local region.

**Definition 1.1 (Consensus).** Consensus is the configuration where $x_i = x_j \in \mathcal{M}$ for all $i, j \in [n]$. We define the consensus set as follows

$$\mathcal{X}^* := \{x \in \mathcal{M}^n : x_1 = x_2 = \ldots = x_n\}. \quad (1.2)$$

Specifically, DRCS iterates $\{x_k\}$ have the following convergence property in a neighborhood of $\mathcal{X}^*$

$$\text{dist}(x_{k+1}, \mathcal{X}^*) \leq \vartheta \cdot \text{dist}(x_k, \mathcal{X}^*), \quad \vartheta \in (0, 1), \quad (1.3)$$

where $\text{dist}^2(x, \mathcal{X}^*) := \min_{y \in \mathcal{M}} \frac{1}{n} \sum_{i=1}^n \|y - x_i\|^2_F$ and $x^\top = (x_1^\top \ x_2^\top \ \ldots \ x_n^\top)$. The linear rate of DRCS sheds some lights on designing the decentralized Riemannian gradient method on Stiefel manifold. More details will be provided in Section 3.

$^1$ We have omitted the dependence on network parameters here.
2 Preliminaries

Notation: The undirected connected graph $G = (\mathcal{V}, \mathcal{E})$ is composed of $|\mathcal{V}| = n$ agents. We use $x$ to denote a collection of all local variables $x_i$ by stacking them, i.e., $x^T = (x_1^T \; x_2^T \; \ldots \; x_n^T)$. For $\mathcal{M}$, the $n$-fold Cartesian product of $\mathcal{M}$ with itself is denoted as $\mathcal{M}^n = \mathcal{M} \times \ldots \times \mathcal{M}$. We use $[n] := \{1, 2, \ldots, n\}$. For $x \in \mathcal{M}^n$, we denote the $i$-th block by $[x]_i = x_i$. We denote the tangent space of $\mathcal{M}$ at point $x$ as $T_x\mathcal{M}$ and the normal space as $N_x\mathcal{M}$. The inner product on $T_x\mathcal{M}$ is induced from the Euclidean inner product $\langle x, y \rangle = \text{Tr}(x^T y)$. Denote $\|\cdot\|_F$ as the Frobenius norm and $\|\cdot\|_2$ as the operator norm. The Euclidean gradient of function $g(x)$ is $\nabla g(x)$ and the Riemannian gradient is $\text{grad}_g(x)$. Let $I_r$ and $0_r$ be the $r \times r$ identity matrix and zero matrix, respectively. And let $1_n$ denote the $n$ dimensional vector with all ones.

The network structure is modeled using a matrix, denoted by $W = W^T$; (ii) $W_{ij} \geq 0$ and $1 > W_{ii} > 0$ for all $i, j$; (iii) Eigenvalues of $W$ lie in $(-1, 1)$. The second largest singular value $\sigma_2$ of $W$ lies in $[0, 1]$.

We now introduce some preliminaries of Riemannian manifold and fundamental lemmas.

2.1 Induced Arithmetic Mean

Denote the Euclidean average point of $x_1, \ldots, x_n$ by

$$\hat{x} := \frac{1}{n} \sum_{i=1}^n x_i. \tag{2.1}$$

To measure the degree of consensus, the error $\|x_i - \hat{x}\|_F$ is typically used in the Euclidean decentralized algorithms. Instead, here we use the induced arithmetic mean(IAM) \cite{35} on $\text{St}(d, r)$, defined as follows

$$\bar{x} := \arg\min_{y \in \text{St}(d, r)} \sum_{i=1}^n \|y - x_i\|_F^2 = \mathcal{P}_{\text{St}}(\bar{x}), \tag{IAM}$$

where $\mathcal{P}_{\text{St}}(\cdot)$ is the orthogonal projection onto $\text{St}(d, r)$. Define

$$\bar{x} = 1_n \otimes \bar{x}. \tag{2.2}$$

Then the distance between $x$ and $\mathcal{X}^*$ is given by

$$\text{dist}^2(x, \mathcal{X}^*) = \min_{y \in \text{St}(d, r)} \frac{1}{n} \sum_{i=1}^n \|y - x_i\|_F^2 = \frac{1}{n} \|x - \bar{x}\|_F^2. \tag{l_{F,\infty}}$$

Furthermore, we define the $l_{F,\infty}$ distance between $x$ and $\bar{x}$ as

$$\|x - \bar{x}\|_{F,\infty} = \max_{i \in [n]} \|x_i - \bar{x}\|_F. \tag{l_{F,\infty}}$$

We will develop the analysis of decentralized Riemannian gradient descent by studying the error distance $\|x - \bar{x}\|_F$ and $\|x - \bar{x}\|_{F,\infty}$.

2.2 Optimality Condition

Next, we introduce the optimality condition on manifold $\mathcal{M}$. Consider the following centralized optimization problem over a matrix manifold $\mathcal{M}$

$$\min h(x) \quad \text{s.t.} \quad x \in \mathcal{M}. \tag{2.3}$$

Since we use the metric on tangent space $T_x\mathcal{M}$ induced from the Euclidean inner product $\langle \cdot, \cdot \rangle$, the Riemannian gradient $\text{grad}_g(x)$ on $\text{St}(d, r)$ is given by $\text{grad}_g(x) = \mathcal{P}_{T_x\mathcal{M}} \nabla h(x)$, where $\mathcal{P}_{T_x\mathcal{M}}$ is the orthogonal projection onto $T_x\mathcal{M}$. More specifically, we have

$$\mathcal{P}_{T_x\mathcal{M}} y = y - \frac{1}{2} x (x^T y + y^T x)$$
for any $y \in \mathbb{R}^{d \times r}$; see [1, 13]. The necessary first-order optimality condition of problem (2.3) is given as follows.

**Proposition 2.1.** [7, 45] Let $x \in \mathcal{M}$ be a local optimum for (2.3). If $h$ is differentiable at $x$, then $\text{grad} h(x) = 0$.

Therefore, $x$ is a first-order critical point (or critical point) if $\text{grad} h(x) = 0$. Let $\bar{x}$ be the IAM of $x$. We define the $\epsilon-$stationary point of problem (1.1) as follows.

**Definition 2.2 ($\epsilon$-Stationarity).** We say that $\mathbf{x}^T = (x_1^T \ x_2^T \ ... \ x_n^T)$ is an $\epsilon-$stationary point of problem (1.1) if the following holds:

$$\frac{1}{n} \sum_{i=1}^{n} \|x_i - \bar{x}\|_F^2 \leq \epsilon \quad \forall i, j \in [n]$$

and

$$\|\text{grad} f(\bar{x})\|_F^2 \leq \epsilon,$$

where we use the notation $f(\bar{x}) = \frac{1}{n} \sum_{i=1}^{n} f_i(\bar{x})$.

### 2.3 Basic Lemmas

Our goal is to develop the decentralized version of centralized Riemannian gradient descent on $\text{St}(d, r)$. Simply speaking, the centralized Riemannian gradient descent [1, 7] iterates as

$$x_{k+1} = R_{x_k}(-\alpha \text{grad} h(x_k)),$$

i.e., updating along a negative Riemannian gradient direction on the tangent space, then performing a operation called retraction $R_{x_k}$ to ensure feasibility. We use the definition of retraction in [7, Definition 1]. The retraction is the relaxation of exponential mapping, and more importantly, it is computationally cheaper. We also assume the second-order boundedness of retraction. It means that

$$R_x(\xi) = x + \xi + O(\|\xi\|_F^2).$$

That is, $R_x(\xi)$ is locally good approximation to $x + \xi$. Such kind of approximation is well enough to take the place of exponential map for the first-order algorithms.

**Lemma 2.3.** [7, 24] Let $\mathcal{R}$ be a second-order retraction over $\text{St}(d, r)$, we have

$$\|R_x(\xi) - (x + \xi)\|_F \leq M\|\xi\|_F^2,$$

$$\forall x \in \text{St}(d, r), \forall \xi \in T_x \mathcal{M}. \quad (P1)$$

Moreover, if the retraction is the polar decomposition. For all $\mathbf{x} \in \text{St}(d, r)$ and $\xi \in T_x \mathcal{M}$, the following inequality holds for any $y \in \text{St}(d, r)$ [22, Lemma 1]:

$$\|R_x(\xi) - y\|_F \leq \|x + \xi - y\|_F. \quad (2.4)$$

In the sequel, retraction refers to the polar retraction to present a simple analysis, unless otherwise noted. More details on the polar retraction is provided in appendix A. Throughout the paper, we assume that every $f_i(x)$ is Lipschitz smooth.

**Assumption 2.** Each $f_i(x)$ has $L-$Lipschitz continuous gradient, and let $D := \max_{x \in \text{St}(d, r)} \|\nabla f_i(x)\|_F$. Therefore, $\nabla f(x)$ is also $L-$Lipschitz continuous and $D = \max_{x \in \text{St}(d, r)} \|\nabla f(x)\|_F$.

We have two similar Lipschitz continuous inequalities on Stiefel manifold as the Euclidean-type ones [31]. We provide the proof in Appendix.
Lemma 2.4 (Lipschitz-type inequalities). For any \( x, y \in \text{St}(n,d) \) and \( \xi \in T_x\mathcal{M} \), if \( f(x) \) is \( L \)-Lipschitz smooth in Euclidean space, then there exists a constant \( L_y = L + L_n \) such that
\[
|f(y) - f(x) + \langle \nabla f(x), y - x \rangle| \leq \frac{L_y}{2} \|y - x\|^2_F, \tag{2.5}
\]
where \( L_n = \max_{x \in \text{St}(d,r)} \|\nabla f(x)\|_2 \). Moreover, define \( L_G = L + 2L_n \), one has
\[
\|\nabla f(x) - \nabla f(y)\|_F \leq L_G \|y - x\|_F. \tag{2.6}
\]

The difference between two Riemannian gradients is not well-defined on general manifold. However, since the Stiefel manifold is embedded in Euclidean space, we are free to do so. Another similar inequality as (2.5) holds in a local region defined as follows
\[
\mathcal{N} := \mathcal{N}_1 \cap \mathcal{N}_2,
\]
where
\[
\mathcal{N}_1 := \{ x : \|x - \bar{x}\|_2^2 \leq n\delta_1^2 \}, \tag{3.4}
\]
\[
\mathcal{N}_2 := \{ x : \|x - \bar{x}\|_{F,\infty} \leq \delta_2 \}, \tag{3.5}
\]
\( \delta_1, \delta_2 \) satisfy
\[
\delta_1 \leq \frac{1}{5\sqrt{r}} \delta_2 \quad \text{and} \quad \delta_2 \leq \frac{1}{6}. \tag{3.6}
\]

The following convergence result of DRCS can be found in [9, Theorem 2]. The formal statement is provided in Fact B.1 in Appendix.

3 Review of consensus on Stiefel manifold

For the decentralized gradient-type algorithms [23, 29, 30, 37, 40, 46], they are based on the linear convergence of consensus iteration in Euclidean space.

The consensus problem over \( \text{St}(d,r) \) is to minimize the quadratic loss function on Stiefel manifold
\[
\min_{x} \varphi^A(x) := \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij} \|x_i - x_j\|_F^2 \tag{3.1}
\]

s.t. \( x_i \in \mathcal{M}, \forall i \in [n] \),

where the superscript \( t \geq 1 \) is an integer used to denote the \( t \)-th power of a doubly stochastic matrix \( W \). Note that \( t \) is introduced to provide flexibility for algorithm design and analysis, and computing \( W_{ij}^{t} \) corresponds to performing \( t \) steps of communication on the tangent space. The Riemannian gradient method DRCS proposed in [9] is given by for any \( i \in [n] \),
\[
x_{i,k+1} = R_{x_{i,k}}(\alpha P_{T_{x_{i,k}}} \mathcal{M}(\sum_{j=1}^{n} W_{ij}^{t} x_{j,k})). \tag{3.2}
\]

DRCS converges almost surely to consensus when \( r \leq \frac{d}{2} - 1 \) with random initialization [26]. However, to study the decentralized optimization algorithm to solve (1.1), the local Q-linear convergence of DRCS is more important for decentralized optimization. Due to the nonconvexity of \( \mathcal{M} \), the Q-linear rate of DRCS holds in a local region defined as follows
\[
\mathcal{N} := \mathcal{N}_1 \cap \mathcal{N}_2, \tag{3.3}
\]
where \( \delta_1, \delta_2 \) satisfy
\[
\delta_1 \leq \frac{1}{5\sqrt{r}} \delta_2 \quad \text{and} \quad \delta_2 \leq \frac{1}{6}. \tag{3.6}
\]

The following convergence result of DRCS can be found in [9, Theorem 2]. The formal statement is provided in Fact B.1 in Appendix.
Fact 3.1. (Informal) Under Assumption 1, for some \( \alpha \in (0, 1] \), if \( \alpha \leq \bar{\alpha} \) and \( t \geq \left\lfloor \log_2 \left( \frac{1}{2 \sqrt{n}} \right) \right\rfloor \), the sequence \( \{x_k\} \) of (3.2) achieves consensus linearly if the initialization satisfies \( x_0 \in \mathcal{N} \) defined by (3.6). That is, there exists \( \rho_i \in (0, 1) \) such that \( x_k \in \mathcal{N} \) for all \( k \geq 0 \) and
\[
\|x_{k+1} - x_k\| \leq \rho_i \|x_k - x_{k-1}\|.
\]

4 Decentralized Riemannian gradient descent

The results of consensus problem on Stiefel manifold lead us to combine the ideas of decentralized gradient method in Euclidean space with the Stiefel manifold optimization. In this section, we study a distributed Riemannian stochastic gradient method for solving problem (1.1), which is described in Algorithm 1. The algorithm is an extension of the decentralized subgradient descent [30].

Since we need to achieve consensus, the initial point \( x_0 \) should be in the consensus region \( \mathcal{N} \). One can simply initialize all agents from the same point. The step 5 in Algorithm 1 is first to perform a consensus step and then to update local variable using Riemannian stochastic gradient direction \( v_i,k \). The consensus step and computation of Riemannian gradient can be done in parallel\(^2\). The consensus stepsize \( \alpha \) satisfies \( \alpha \leq \bar{\alpha} \), which is the same as the consensus algorithm. The constant \( \bar{\alpha} \) is given in Fact B.1 in Appendix. Moreover, \( \alpha = 1 \) works in practice for any \( W \) satisfying Assumption 1. If \( x_1 = \ldots = x_n = z \), we denote
\[
f(z) := \frac{1}{n} \sum_{i=1}^{n} f_i(z).
\]

Moreover, we need the following assumptions on the stochastic Riemannian gradient \( v_i,k \) and the stepsize \( \beta_k \).

Algorithm 1 Decentralized Riemannian Stochastic Gradient Descent (DRSGD) for Solving (1.1)

1: Input: initial point \( x_0 \in \mathcal{N} \), an integer \( t \geq \left\lfloor \log_2 \left( \frac{1}{2 \sqrt{n}} \right) \right\rfloor \), \( 0 < \alpha \leq \bar{\alpha} \), where \( \bar{\alpha} \) is given in Fact 3.1.
2: for \( k = 0, \ldots \) do
3: Choose diminishing stepsize \( \beta_k = O(1/\sqrt{k}) \) \( \triangleright \) For each node \( i \in [n] \), in parallel
4: Compute stochastic Riemannian gradient \( v_i,k \) satisfying \( E\bar{v}_{i,k} = \text{grad} f_i(x_{i,k}) \)
5: Update \( x_{i,k+1} = R_{x_{i,k}}(\alpha \mathcal{P}_{X_{i,k}} M(\sum_{j=1}^{n} W_{ij}^T x_{j,k}) - \beta_k v_{i,k}) \)
6: end for

Assumption 3. 1. The stochastic gradient \( v_{i,k} \) is unbiased, i.e., \( E\bar{v}_{i,k} = \text{grad} f_i(x_{i,k}) \) for all \( i \in [n] \) and \( v_{i,k} \) is independent of \( v_{i,j} \) for any \( i \neq j \). Moreover, the variance is bounded: \( E \|v_{i,k} - \text{grad} f_i(x_{i,k})\|_F^2 \leq \Sigma^2 \).

2. We assume the uniform upper bound of \( \|v_i\|_F \) is \( D \), i.e., \( \max_{x \in S(d,r)} \|v_i\|_F \leq D \) for each \( i \in [n] \).

The Lipschitz smoothness of \( f_i(x) \) in Assumption 2 and unbiased estimation are quite standard in the literature. And Lemma 2.4 suggests that \( \text{grad} f_i \) is \( L_G \)-Lipschitz continuous. For the boundedness of \( \|v_i\|_F \), it is a weak assumption since the Stiefel manifold is compact. One example common is the finite-sum form: \( f_i = \frac{1}{m_i} \sum_{j=1}^{m_i} f_{ij} \), where \( f_{ij} \) is smooth. Then the stochastic gradient \( v_{i,k} \) is uniformly sampled from \( \text{grad} f_{ij}(x_{i,k}) \), \( i \in [m_i] \). We emphasize that the uniform boundedness of gradient is not needed for problems in Euclidean space, but Lipschitz continuity is necessary [17]. The step 5 can be seen as applying Riemannian gradient method to solve the following problem
\[
\min_{x \in \mathcal{M}} \beta_k f(x) + \alpha \varphi'(x).
\]

Similar as the analysis of DGD in Euclidean space, we need to ensure that \( \|x_k - \bar{x}_k\|_F \rightarrow 0 \). Hence, the effect of \( f \) should be diminishing. The following assumption on the stepsizes is also needed to get an \( \epsilon \)-solution.

\(^2\)One could also exchange the order of gradient step and communication step, i.e., \( x_{i,k+1} = R_{x_{i,k}}(-\beta_k v_{i,k}) \), \( x_{i,k+1} = R_{x_{i,k}}(\alpha \mathcal{P}_{X_{i,k}} M(\sum_{j=1}^{n} W_{ij}^T x_{j,k+1})) \). Our analysis can also apply to this kind of updates if \( x_0 \in \rho_i \mathcal{N} \), where \( \rho_i \mathcal{N} \) denotes region \( \mathcal{N} \) with shrunk radius \( \rho_i \beta_1, \rho_i \beta_2 \). For the Euclidean counterparts, when the graph is complete associated with equal weight matrix, the above updates are the same as centralized gradient step. However, they are different on Stiefel manifold.
Assumption 4 (Diminishing stepsize). The stepsize $\beta_k > 0$ is non-increasing and
\[
\sum_{k=0}^{\infty} \beta_k = \infty, \quad \lim_{k \to \infty} \beta_k = 0, \quad \lim_{k \to \infty} \frac{\beta_{k+1}}{\beta_k} = 1.
\]

The assumption $\lim_{k \to \infty} \frac{\beta_{k+1}}{\beta_k} = 1$ is additionally required to show the bound $\frac{1}{n} \|x_k - \bar{x}_k\|_F^2 = \mathcal{O}(\frac{\beta_k^2 D^2}{(1-\rho_k)^2})$, see Lemma D.3 in Appendix.

To proceed, we first need to guarantee that $x_k \in \mathcal{N}$, where $\mathcal{N}$ is the consensus contraction region defined in (3.3). Therefore, uniform bound $D$ and the multi-step consensus requirement $t \geq \lceil \log_{\sigma_2}(\frac{1}{\sqrt{\rho}}) \rceil$ are necessary in our convergence analysis. With appropriate stepsizes $\alpha$ and $\beta_k$, we get the following lemma using the consensus results in Fact 3.1. We provide the proof in Appendix.

Lemma 4.1. Under Assumptions 1 to 4, let the stepsize $\alpha$ satisfy $0 < \alpha \leq \bar{\alpha}$, $\beta_k$ satisfy $0 \leq \beta_k \leq \min\{\frac{\alpha_2}{D}, \frac{\alpha_1}{D}\}$, $\forall k \geq 0$, and $t \geq \lceil \log_{\sigma_2}(\frac{1}{\sqrt{\rho}}) \rceil$. If $x_0 \in \mathcal{N}$, it follows that $x_k \in \mathcal{N}$ for all $k \geq 0$ generated by Algorithm 1 and
\[
\|x_{k+1} - x_{k+1}\|_F \leq \rho_k^k \|x_0 - x_0\|_F + \sqrt{nD} \sum_{l=0}^{k} \rho_k^{k-l} \beta_l.
\]

We have $\beta_k = \mathcal{O}(\frac{1}{\sqrt{D}})$ when $\alpha = \mathcal{O}(1)$. Note that $t \geq \lceil \log_{\sigma_2}(\frac{1}{\sqrt{\rho}}) \rceil$ implies $\rho_k = \mathcal{O}(1)$; see appendix B. When $\beta_k$ is constant, Lemma 4.1 suggests that $x_k$ converges linearly to an $\mathcal{O}(\beta_k)$-neighborhood of $x_0$.

We present the convergence of Algorithm 1. The proof is based on the new Lipschitz inequalities for the Riemannian gradient in Lemma 2.4 and the properties of retraction in Lemma 2.3. We provide it in Appendix.

Theorem 4.2. Under Assumptions 1 to 4, suppose $x_k \in \mathcal{N}$, $t \geq \lceil \log_{\sigma_2}(\frac{1}{\sqrt{\rho}}) \rceil$, $0 < \alpha \leq \bar{\alpha}$. If
\[
\beta_k = \frac{1}{\sqrt{k+1}} \cdot \min\{\frac{\alpha_2}{5L_gD}, \frac{\alpha_1}{5D}, \frac{1-\rho_k}{D} \delta_1\}, \quad (4.1)
\]
it follows that
\[
\min_{k \leq K} \mathbb{E}\|\nabla f(\bar{x}_k)\|_F^2 \leq 4(f(\bar{x}_0) - f^*) + \frac{6L_gD^2}{\bar{\alpha}} \sum_{k=0}^{K} \beta_k^2
\]
\[
+ (2CD^2L_g^2 + 4T_nD^4) \sum_{k=0}^{K} \beta_k^2 + 4T_nL_gD^4 \sum_{k=0}^{K} \beta_k^2,
\]
where $C = \mathcal{O}(\frac{1}{(1-\rho_k)^2})$ is given in Lemma D.3 in Appendix. And $T_n = 2(4\sqrt{n} + 6\alpha)^2 C^2 + 8M^2$ and $T_2 = 20\alpha^2 C^2 + 9M^2$. Therefore, we have
\[
\min_{k \leq K} \mathbb{E}\|\nabla f(\bar{x}_k)\|_F^2 = \mathcal{O}\left(\frac{f(\bar{x}_0) - f^*}{\beta \sqrt{K+1}} + \frac{\Xi^2 \ln(K+1)}{n \sqrt{K+1}}\right)
\]
\[
+ \mathcal{O}\left(\frac{\max\{D^2, L_g^2\} \cdot (C + T_1 + T_2)}{\sqrt{K+1}}\right),
\]
where $\bar{\beta} = \min\{\frac{1}{L_g}, (1-\rho_k)/D\}$.

Theorem 4.2 together with Lemma D.3 implies that the iteration complexity of obtaining an $\epsilon$-stationary point defined in Definition 2.2 is $\mathcal{O}(1/\epsilon^2)$ in expectation. The communication round per iteration is $t \geq \lceil \log_{\sigma_2}(\frac{1}{\sqrt{\rho}}) \rceil$ since we need to ensure $x_k \in \mathcal{N}$. For sparse network, $t = \mathcal{O}(n^2 \log n)$ [9].

Following [23], if we use the constant stepsize $\beta_k = \frac{1}{2L_g + \sqrt{K+1}/n}$ where $K$ is sufficiently large, we can obtain the following result
\[
\min_{k=0,\ldots,K} \mathbb{E}\|\nabla f(\bar{x}_k)\|_F^2
\]
\[
\leq \frac{8L_g(f(\bar{x}_0) - f^*)}{K+1} + \frac{8(f(\bar{x}_0) - f^* + \frac{3L_g\alpha}{\rho_k})\Xi}{\sqrt{n(K+1)}}.
\]
More details are provided in Corollary D.5 in Appendix. Therefore, if $K$ is sufficiently large, the convergence rate is $O(1/\sqrt{nK})$. To obtain an $\epsilon$-stationary point, the computational complexity of single node is $O(1/\epsilon^2)$. However, the communication round $t \geq \lceil \log_{2\sqrt{n}}(nK) \rceil$ is too large. In practice, we find $t = 1$ performs almost the same as $t = \infty$, which is shown in Section 6. This may be because that when the stepsize is very small, DRSGD will not deviate from the consensus algorithm DRCS too much. We leave the further discussion as future work.

5 Gradient tracking on Stiefel manifold

In this section we study the decentralized gradient tracking method, which is based on the DIGing algorithm [29,33] for solving Euclidean problems. With an auxiliary gradient tracking sequence to estimate the full gradient, the constant stepsize can be used and faster convergence rate can be shown for the Euclidean algorithms [29,37]. We describe our algorithm in Algorithm 2, which is named as Decentralized Riemannian Gradient Tracking Algorithm (DRGTA).

Algorithm 2 Decentralized Riemannian Gradient Tracking over Stiefel manifold (DRGTA) for Solving (1.1)

1: Input: initial point $x_0 \in \mathcal{N}$, an integer $t \geq \log_{2\sqrt{n}}(1/\epsilon^2)$, $0 < \alpha \leq \alpha^*$ and stepsize $\beta$ according to (5.2).
2: Let $y_{i,0} = \text{grad} f_i(x_{i,0})$ on each node $i \in [n]$.
3: for $k = 0, \ldots$ do  
   4: Projection onto tangent space: $v_{i,k} = \mathcal{P}_{T_{x_{i,k}}M}y_{i,k}$.
   5: Update $x_{i,k+1} = \mathcal{R}_{x_{i,k}}(\alpha \mathcal{P}_{T_{x_{i,k}}M}(\sum_{j=1}^n W_{ij}x_{j,k}) - \beta v_{i,k})$.
   6: Riemannian gradient tracking: $y_{i,k+1} = \sum_{j=1}^n W_{ij}y_{j,k} + \text{grad} f_i(x_{i,k+1}) - \text{grad} f_i(x_{i,k})$.
7: end for

In Algorithm 2, the step 4 is to project the direction $y_{i,k}$ onto the tangent space $T_{x_{i,k}}M$, which follows a retraction update. The sequence $\{y_{i,k}\}$ is to approximate the Riemannian gradient $\text{grad} f_i(x_{i,k})$. More specifically, the sequence $\{y_k\}$ tracks the average Riemannian gradient $\frac{1}{n} \sum_{i=1}^n \text{grad} f_i(x_{i,k})$. Although, it is not mathematically sound to do addition operation between different tangent space in differential geometry, we can view $\text{grad} f_i(x_{i,k})$ as the projected Euclidean gradient. Note that $y_{i,k}$ is not necessarily on the tangent space $T_{x_{i,k}}M$. Therefore, it is important to define $v_{i,k} = \mathcal{P}_{T_{x_{i,k}}M}y_{i,k}$ so that we can use the properties of retraction in Lemma 2.3. Such a projection onto tangent space step, followed by the retraction operation, distinguishes the algorithm from the Euclidean space gradient tracking algorithms. Multi-step consensus of gradient is also required in step 5 and step 6. The consensus stepsize $\alpha$ satisfies the same condition as that of Algorithm 1.

5.1 Convergence of Riemannian gradient tracking

We first briefly revisit the idea of gradient tracking (GT) algorithm DIGing in Euclidean space. Note that if we consider the decentralized optimization problem (1.1) without the Stiefel manifold constraint, then Algorithm 2 is exactly the same as the DIGing. Since the Riemannian gradient $\text{grad} f_i$ becomes simply the Euclidean gradient $\nabla f_i$ and projection onto the tangent space and retraction are not needed. The main advantage of Euclidean gradient tracking algorithm is that one can use constant stepsize $\beta > 0$, which is due to following observation: for all $k \geq 0$, it follows that

$$\frac{1}{n} \sum_{i=1}^n y_{i,k} = \frac{1}{n} \sum_{i=1}^n \nabla f_i(x_{i,k}).$$
That is, the average of sequence $y_{i,k}$ is the same as that of $\nabla f_i(x_{i,k})$. It can be shown that if the following inexact gradient sequence, then it converges to a stationary point [29]

$$
x_{i,k+1} = \frac{1}{n} \sum_{i=1}^{n} W_{ij} x_{j,k} - \frac{\beta}{n} \sum_{i=1}^{n} \nabla f_i(x_{i,k}).
$$

However, the average of gradient information is unavailable in the decentralized setting. Therefore, GT uses $\frac{1}{n} \sum_{i=1}^{n} y_{i,k}$ to approximate $\frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_{i,k})$. Inspired by this, $y_{i,k}$ is used to approximate the Riemannian gradient, i.e., if

$$
y_{i,k+1} = \frac{1}{n} \sum_{j=1}^{n} W_{ij} y_{j,k} + \text{grad}_i(x_{i,k+1}) - \text{grad}_i(x_{i,k}),
$$

then it follows that

$$
\frac{1}{n} \sum_{i=1}^{n} y_{i,k} = \frac{1}{n} \sum_{i=1}^{n} \text{grad}_i(x_{i,k}) \quad \text{i.e.} \quad \hat{y}_k = \hat{y}_k.
$$

Therefore, $\{y_k\}$ tracks the average of Riemannian gradient, and if $\|\hat{y}_k\|_F \rightarrow 0$ and the sequence $\{x_k\}$ achieves consensus, then $x_k$ also converges to the critical point. This is because

$$
\|\text{grad}(\bar{x}_k)\|^2_F \leq 2\|\hat{y}_k\|^2_F + 2\|\text{grad}(\bar{x}_k) - \hat{y}_k\|^2_F
$$

$$
\leq 2\|\hat{y}_k\|^2_F + \frac{2L^2}{\beta} \|x_k - \bar{x}_k\|^2_F.
$$

To achieve consensus, we still need multi-step consensus in DRGTA as DRSGD. The multi-step consensus also helps us to show the uniform boundedness of $y_{i,k}$ and $v_{i,k}$, $i \in [n]$ for all $k \geq 0$, which is important to guarantee $x_k \in \mathcal{N}$. We get that the sequence stays in consensus region $\mathcal{N}$ in Lemma 5.1. We provide the proof in Appendix.

**Lemma 5.1 (Uniform bound of $y_{i}$ and stay in $\mathcal{N}$).** Under Assumptions 1 and 2, let $x_0 \in \mathcal{N}$, $t \geq \log_{\sigma_2}(\frac{1}{2\sqrt{n}})$, $\alpha$ satisfy $0 < \alpha \leq \bar{\alpha}$, $\beta$ satisfy $0 \leq \beta \leq \bar{\beta} := \min\{\frac{1}{4L_G G_3}, \frac{\alpha \delta_1}{\sqrt{2}(L_G + 2D)}\}$, then $\|y_{i,k}\|_F \leq L_G + 2D$ for all $i \in [n]$ and $x_k \in \mathcal{N}$ for all $k \geq 0$. Moreover, we have

$$
\frac{1}{n} \|x_k - \bar{x}_k\|^2_F \leq C_1 (L_G + 2D)^2 \beta^2, k \geq 0
$$

(5.1)

for some $C_1 = O(\frac{1}{\sqrt{n}})$ and $C_1$ is independent of $L_G, D$.

We present the $O(1/\epsilon)$ iteration complexity to obtain the $\epsilon$--stationary point of (1.1) as follows. The proof of DIGing can be unified by the primal-dual framework [3]. However, DRGTA cannot be rewritten in the primal-dual form. The proof is mainly established with the help of Lemma 2.4 and the properties of IAM. We provide it in Appendix.

**Theorem 5.2.** Under Assumptions 1 and 2, let $x_0 \in \mathcal{N}$, $t \geq \log_{\sigma_2}(\frac{1}{2\sqrt{n}})$, $0 < \alpha \leq \bar{\alpha}$, and

$$
0 < \beta \leq \min\{\bar{\beta}, \frac{1}{8L_G G_4}, \frac{1}{4L_G G_3} + (8\bar{C}_0 + \frac{1}{2}\bar{C}_2)\alpha \delta_1\},
$$

(5.2)

where $\bar{\beta}$ is given in Lemma 5.1. Then it follows that for the sequences generated by Algorithm 2

$$
\min_{k=0,\ldots,K} \frac{1}{n} \|y_k\|^2_F \leq \frac{8(f(\bar{x}_0) - f^* + \hat{C}_4 + G_4 L_G)}{\beta \cdot K},
$$

(5.3)

$$
\min_{k=0,\ldots,K} \frac{1}{n} \|x_k - \bar{x}_k\|^2_F \leq \frac{8\beta(f(\bar{x}_0) - f^* + \hat{C}_4 + G_4 L_G)\hat{C}_0 + \hat{C}_1}{K},
$$

(5.4)
\[
\min_{k \leq K} \|\nabla f(\bar{x}_k)\|_F^2 
\leq \frac{(16 + \alpha^2 \delta_1^2 \tilde{C}_0)(f(\bar{x}_0) - f^* + \tilde{C}_4 + \mathcal{G}_4 L_G) + \tilde{C}_1 L_G}{\beta \cdot K},
\]  
(5.5)

where the constants above are given by

\[ \mathcal{G}_3 = \mathcal{G}_1 \tilde{C}_0 + \mathcal{G}_0 \tilde{C}_0 + \mathcal{G}_2, \]
\[ \mathcal{G}_4 = \frac{\mathcal{G}_0 \alpha \delta_1^2}{25} + \tilde{C}_1 (\mathcal{G}_1 + 4r \mathcal{C}_1), \]
\[ \tilde{C}_0 = \frac{2}{(1 - \rho_t)^2}, \quad \tilde{C}_1 = \frac{2}{1 - \rho_t} \cdot \frac{1}{n} \|x_0 - \bar{x}_0\|_F^2, \]
\[ \tilde{C}_2 = \frac{2}{(1 - \sigma_1^2)^2}, \quad \tilde{C}_3 = \frac{2}{1 - \sigma_1^2} \cdot \frac{1}{n} \|y_0 - \bar{y}_0\|_F^2, \]
\[ \tilde{C}_4 = (8\alpha^2 \tilde{C}_1 \tilde{C}_2 L_G^2 + \tilde{C}_3) \cdot \frac{\beta}{2} = O\left(\frac{L_G}{(1 - \sigma_1^2)^2}\right). \]

The constants \( \mathcal{G}_0 = \mathcal{O}(r^2 \mathcal{C}_1), \mathcal{G}_1 = \mathcal{O}(r^2 \mathcal{C}_1) \) and \( \mathcal{G}_2 = \mathcal{O}(M) \) are given in Lemma E.2 in the appendix. We have \( \mathcal{G}_3 = \mathcal{O}\left(\frac{r^2 \mathcal{C}_1}{(1 - \rho_t)} + M\right) \) and \( \mathcal{G}_4 = \mathcal{O}\left(\frac{r^2 \mathcal{C}_1 \delta_1^2}{1 - \rho_t}\right). \) Recall that \( \beta \leq \tilde{\beta} \) is required to guarantee that the sequence \( \{x_k\} \) always stays in the consensus region \( \mathcal{N} \). And note that \( \rho_t \) is the linear rate of Riemannian consensus, which is greater than \( \sigma_1^2 \). The stepsize \( \beta \) follows

\[ \beta = \mathcal{O}\left(\min\left\{ \frac{1 - \rho_t}{L_G + 2D}, \frac{(1 - \rho_t)^2}{L_G}, \frac{1}{r^2 C_1 + M(1 - \rho_t)^2} \right\}\right). \]

This matches the bound of DIGing \([29, 33]\). Then Theorem 5.2 suggests that the consensus error rate is \( \mathcal{O}\left(\frac{1}{r^2 C_1 + M L_G} f(\bar{x}_0) - f^* + \frac{\|x_0 - \bar{x}_0\|_F^2}{n(1 - \rho_t) K}\right) \) and the convergence rate of \( \min_{k=0, \ldots, K} \|\nabla f(\bar{x}_k)\|_F^2 \) is given by \( \mathcal{O}\left(\frac{r^2 C_1 + M(1 - \rho t)^2}{K(1 - \rho_t)^2} (f(\bar{x}_0) - f^*) + \frac{\|x_0 - \bar{x}_0\|_F^2}{n(1 - \rho_t)^2} + \frac{r^2 C_1 \delta_1^2 L_G}{K(1 - \rho_t)^2}\right) \). Moreover, if the initial points satisfy \( x_{1,0} = x_{2,0} = \ldots = x_{n,0} \), we have \( \tilde{C}_1 = \tilde{C}_3 = \tilde{C}_4 = 0 \).

6 Numerical experiment

We solve the following decentralized eigenvector problem:

\[ \min_{x \in \mathbb{M}^n} -\frac{1}{2n} \sum_{i=1}^{n} x_i^T A_i^T A_i x_i, \quad \text{s.t.} \quad x_1 = \ldots = x_n, \]  
(6.1)

where \( A_i \in \mathbb{R}^{m_i \times d}, i \in [n] \) is the local data matrix in local agent and \( m_i \) is the sample size. Denote the global data matrix by \( A := [A_1^T A_2^T \ldots A_n^T]^T \). It is known that the global minimizer of (6.1) is given by the first \( r \) leading eigenvectors of \( A^T A = \sum_{i=1}^{n} A_i^T A_i \), denoted by \( x^* \). DRSGD and DRGTA are only proved to converge to the critical points, but we find they always converge to \( x^* \) in our experiments. Denote the column space of a matrix \( x \) by \( [x] \). To measure the quality of the solution, the distance between column space \( [x] \) and \( [y] \) can be defined via the canonical correlations between \( x \in \mathbb{R}^{d \times r} \) and \( y \in \mathbb{R}^{d \times r} \) \([16]\). One can define it by

\[ d_s(x, y) := \min_{Q \in O(r)} \|uQ - v\|_F, \]

where \( O(r) \) is the orthogonal group, \( u \) and \( v \) are the orthogonal basis of \( [x] \) and \( [y] \), respectively. In the sequel, we fix \( \alpha = 1 \) and generate the initial points uniformly randomly satisfying \( x_{1,0} = \ldots = x_{n,0} \in \mathcal{M} \). If full batch gradient is used in Algorithm 1, we call it DRDG, otherwise one stochastic gradient is uniformly sampled without replacement in DRSGD. In DRSGD, one epoch represents the number of complete passes through the dataset, while one iteration is used in the deterministic algorithms. For DRSGD, we set the
maximum epoch to 200 and early stop it if $d_s(\bar{x}_k, x^*) \leq 10^{-5}$. For DRGTA and DRDGD, we set the maximum iteration number to $10^4$ and the termination condition is $d_s(\bar{x}_k, x^*) \leq 10^{-8}$ or $\|\text{grad}f(\bar{x}_k)\|_F \leq 10^{-8}$. We set $
abla_k = \frac{\hat{\beta}_k}{\sum_{i=1}^{n} m_i}$ for DRGTA and DRDGD where $\hat{\beta}$ will be specified later. For DRSGD, we set $\beta = \frac{\hat{\beta}}{\sqrt{200}}$. We select the weight matrix $W$ to be the Metroplis constant weight [37].

6.1 Synthetic data

We report the convergence results of DRSGD, DRDGD and DRGTA with different $t$ and $\hat{\beta}$ on synthetic data. We fix $m_1 = \ldots = m_n = 1000$, $d = 100$ and $r = 5$ and generate $m_1 \times n$ i.i.d samples following standard multi-variate Gaussian distribution to obtain $A$. Let $A = U S \tilde{V}$ be the truncated SVD. Given an eigengap $\Delta \in (0, 1)$, we modify the singular values of $A$ to be a geometric sequence, i.e. $S_{i,j} = S_{0,0} \times \Delta^i, j \in [d]$. Typically, larger $\Delta$ results in more difficult problem. In Figure 1, we show the results of DRSGD, DRDGD and DRGTA on the data with $n = 32$ and $\Delta = 0.8$. The y-axis is the log-scale distance $d_s(\bar{x}_k, x^*)$. The first four lines in each testing case are for the ring graph, and the last one is on a complete graph with equally weighted matrix, which aims to show the case of $t \to \infty$. In Figure 1(a), when fixing $\hat{\beta}$, it is shown that that smaller $\hat{\beta}$ produces higher accuracy, which indicates the Theorem 4.2. We also see DRSGD performs almost the same with different $t \in \{1, 10, \infty\}$. For the two deterministic algorithms DRDGD and DRGTA, we see that DRDGD can use larger $\hat{\beta}$ if more communication rounds $t$ is used in Figure 1(b), (c). DRDGD cannot achieve exact convergence with the constant stepsize, while DRGTA successfully solves the problem using $t \in \{1, 10, \infty\}$, $\hat{\beta} = 0.05$.

Next, we report the numerical results on different networks and data size.

Figure 2 shows the results on the same data set as that of Figure 1. However, the network is an Erdős-Rényi model $\text{ER}(n, p)$, which means the probability of each edge is included in the graph with probability $p$. The Metropolis constant matrix is associated with the graph. Since the ER(32, 0.3) is more well-connected than the ring graph, we see that the results for different $t \in \{1, 10, \infty\}$ are almost the same except for DRDGD with $\hat{\beta} = 0.05$. Moreover, the solutions accuracy and convergence rate of DRDGD and DRGTA are better than those shown in Figure 1.

In Figure 3, we show the results when the initial point does not satisfy $x_0 \in \mathcal{N}$. Specifically, we randomly generate $x_{1,0}, \ldots, x_{n,0}$ on $\mathcal{M}$, and the other settings are the same as Figure 1. Surprisingly, we find that the proposed algorithms still converge. As suggested by [9, 26], the consensus algorithm can achieve global consensus with random initialization when $r \leq \frac{2}{3} d - 1$. The iteration in DRSGD and DRGTA is a perturbation of the consensus iteration. It will be interesting to study it further.

6.2 Real-world data

We compare our algorithms with a recently proposed algorithm decentralized Sanger’s algorithm (DSA) [15], which is a Euclidean-type algorithm. To solve the eigenvector problem (6.1), DSA is shown to converge
linearly to a neighborhood of the optimal solution. The computation of DSA iteration is cheaper than DRDGD since there is no retraction step. For simplicity, we fix $t = 1$ and $r = 5$ in this section.

We provide some numerical results on the MNIST dataset [21]. The graph is still the ring and $W$ is the Metropolis constant weight matrix. For MNIST, there are 60000 samples and the dimension is given by $d = 784$. We normalize the data matrix by dividing 255 such that the elements are in $[0, 1]$. The data set is evenly partitioned into $n$ subsets. The stepsizes of DRDGD and DRGTA are set to $\hat{\beta} = \frac{1}{60000}$.

The results for MNIST data set with $n = 20, 40$ are shown in Figure 4. We see that the convergence rate of DSA and DRDGD are almost the same and DRGTA with $\hat{\beta} = 0.1$ can achieve the most accurate solution. When $n$ becomes larger, the convergence rate of all algorithms is slower. Although the computation of DSA is cheaper than DRDGD, we find that when $\hat{\beta} = 0.5, n = 20$, DSA does not converge, which is not shown in the Figure 4 (a). This is probably because DSA is not a feasible method and needs carefully tuned stepsize.

Finally, we demonstrate the linear speedup of DRSGD for different $n$. The experiments are evaluated in a HPC cluster, where each computation node is associated with an Intel Xeon E5-2670 v2 CPU. The computation nodes are connected by FDR10 Infiniband. We use 10 CPU cores each computation node in the HPC cluster. And we treat one CPU core as one network node in our problem. The codes are implemented in python with mpi4py.

We set the maximum epoch as 300 in all experiments. The stepsize is set to $\beta = \frac{\sqrt{n}}{10000\sqrt{300}} \hat{\beta}$, where $\hat{\beta}$ is tuned for the best performance. The results in Figure 5 are $\log d_s(\bar{x}_k, x^*)$ v.s. epoch and $\log d_s(\bar{x}_k, x^*)$ v.s. CPU time, respectively. As we see in Figure 5(a), the solutions accuracy of $n = 16, 32, 60$ are almost the same, while the CPU time in Figure 5(b) can be accelerated by nearly linear ratio.
7 Conclusions

We discuss the decentralized optimization problem over Stiefel manifold and propose the two decentralized Riemannian gradient method and establish their convergence rate. Future topics could be cast into the following several folds: Firstly, for the eigenvector problem (6.1), it will be interesting to establish the linear convergence of DRGTA. Secondly, the analysis is based on the local convergence of Riemannian consensus, which results in multi-step consensus. It would be interesting to design algorithms based on Euclidean consensus.

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A About the polar retraction

Given the polar decomposition of $x + \xi = QH$, where $Q \in \mathbb{R}^{d \times r}$ is orthogonal and $H \in \mathbb{R}^{r \times r}$ is positive definite. The polar retraction is the polar factor

$$\mathcal{R}_x(\xi) = Q(x + \xi)(I_r + \xi^T \xi)^{-1/2}, \tag{A.1}$$

which is also the orthogonal projection of $x + \xi$ onto $\text{St}(d, r)$. The computation complexity is $\mathcal{O}(dr^2)$. [24, Append. E] showed that if $\|\xi\|_F \leq 1$ then $M = 1$ for polar retraction. The boundedness of $\xi$ can be verified in our convergence analysis. Therefore, we have $M = 1$ in this paper.

B More details on linear rate of consensus

The following results were provided in [9].

If there exists an integer $t \geq 0$ such that

$$\max_{i \in [n]} \| \sum_{j=1}^n (W_{ij} - 1/n)(x_j - x_\bar{\xi}) \|_F \leq \max_{i \in [n]} \| \sum_{j=1}^n (W_{ij} - 1/n) \|_F \leq \frac{1}{\alpha} \|x - x_\bar{\xi}\|_{F,\infty}, \tag{B.1}$$

then it suffices to show the sequence $\{x_k\}$ of DRCS satisfying $x_k \in \mathcal{N}$ with $t \geq \lceil \log_{\|\sigma_2(L)\|} \right \}$ steps of communication.

Denote the smallest eigenvalue of $W^t$ by $\lambda_\ell(W^t)$, the constant $L_\ell$ is given by

$$L_\ell = 1 - \lambda_\ell(W^t). \tag{B.2}$$

It is the Lipschitz constant of $\nabla \varphi^t(x)$. Since $L_\ell \in (0, 2]$, if $\lambda_\ell(W^t)$ is unknown, one can use $L_\ell = 2$. Define the second largest eigenvalue of $W^t$ by $\lambda_{\ell^*}(W^t)$

$$\mu_\ell = 1 - \lambda_{\ell^*}(W^t).$$

The formal statement of Fact 3.1 is given as follows.

Fact B.1. [9] Under Assumption 1, let the stepsize $\alpha$ satisfy $0 < \alpha \leq \bar{\alpha} := \min\{\nu \Phi_{m, 1}, 1, 1/M\}$ and $t \geq \lceil \log_{\|\sigma_2(L)\|} \right \}$, where $\nu \in [0, 1]$, $\Phi = 2 - \delta_2^2$ and $M$ is given in Lemma 2.3. The sequence $\{x_k\}$ of (3.2) achieves consensus linearly if the initialization satisfies $x_0 \in \mathcal{N}$ defined by (3.6). That is, we have $x_k \in \mathcal{N}$ for all $k \geq 0$ and

$$\|x_{k+1} - x_{k+1}\|_F \leq \|x_k - \alpha \nabla \varphi^t(x_k) - x_k\|_F \leq \sqrt{1 - 2(1 - \nu)\alpha \gamma_t \|x_k - x_k\|_F, \tag{B.3}$$

where $\gamma_t = (1 - 4r\delta_t^2)(1 - \frac{\delta_t^2}{2})\mu_\ell \geq \frac{\alpha}{\nu} \geq \frac{1 - \sigma_2^2}{2}$.

If $\nu = 1/2$, we have $\alpha \leq \bar{\alpha} := \min\{\frac{\Phi}{2M}, 1, 1/M\}$ and

$$\rho_t = \sqrt{1 - \gamma_t \alpha}.$$ 

Recall that $M$ is the constant given in Lemma 2.3. We also have $M = \mathcal{O}(1)$ which is discussed in appendix A.

If $\alpha = 1$ is admissible, then the rate is $\rho_t = \sqrt{1 + \sigma_2^2}$ which is worse that the Euclidean rate $\sigma_2^2$. Moreover, it was shown in [9] in a smaller region, i.e., $\varphi^t(x) = \mathcal{O}(\sigma_2^2)$ and $\|x - x\|_F^2 = \mathcal{O}(1)$, it follows asymptotically $\rho_t = \sigma_2^2$ with $\alpha = 1$. For simplicity, we will only discuss the convergence of our proposed algorithms using (B.3) with $\nu = 1/2$. Note that this may imply $\bar{\alpha} < 1$, but we find that $\alpha = 1$ always works for our proposed algorithms.
C Proofs for Section 2

Denote $\mathcal{P}_{N_x M}$ as the orthogonal projection onto the normal space $N_x M$. One can rewrite the projection $\mathcal{P}_{T_x M}(y - x), \forall y \in \text{St}(d, r)$ \cite{9} as follows

$$\mathcal{P}_{T_x M}(y - x) = y - x - \mathcal{P}_{N_x M}(y - x) = y - x + \frac{1}{2} x(x - y)^\top(x - y).$$

This implies that

$$\mathcal{P}_{T_x M}(y - x) = y - x + \mathcal{O}(\|y - x\|_F^2).$$

The relationship (P2) helps us to prove Lemma 2.4.

**Proof of Lemma 2.4.** Firstly, since $\nabla f(x)$ is $L$–Lipschitz in Euclidean space, one has

$$|f(y) - [f(x) + \langle \nabla f(x), y - x \rangle]| \leq \frac{L}{2} \|y - x\|_F^2. \quad (C.1)$$

Since $\text{grad} f(x) = \mathcal{P}_{T_x M} \nabla f(x)$, we have

$$\langle \text{grad} f(x), y - x \rangle = \langle \nabla f(x), \mathcal{P}_{T_x M}(y - x) \rangle = \langle \nabla f(x), y - x \rangle + \left\langle \nabla f(x), \frac{1}{2} x(x - y)^\top(y - x) \right\rangle. \quad (P2)$$

Using

$$\left\langle \nabla f(x), \frac{1}{2} x(y - x)^\top(y - x) \right\rangle \leq \|\nabla f(x)\|_2 \cdot \|x\|_2 \cdot \frac{1}{2} \|x - y\|_F^2 \leq \frac{1}{2} \|\nabla f(x)\|_2 \cdot \|y - x\|_F^2$$

implies

$$|\langle \text{grad} f(x), y - x \rangle - \langle \nabla f(x), y - x \rangle| \leq \frac{1}{2} \max_{x \in \text{St}(d, r)} \|\nabla f(x)\|_2 \cdot \|y - x\|_F^2, \quad (C.2)$$

where $\|\nabla f(x)\|_2$ represents the operator norm of $\nabla f(x)$. Since $\text{St}(d, r)$ is a compact set and $\nabla f(x)$ is continuous, we denote $L_n = \max_{x \in \text{St}(d, r)} \|\nabla f(x)\|_2$. Let $L_y = L_n + L$. Combining (C.1) with (C.2) yields

$$|f(y) - [f(x) + \langle \text{grad} f(x), y - x \rangle]| \leq \frac{L_y}{2} \|y - x\|_F^2. \quad (C.3)$$

Secondly, using $\text{grad} f(x) = \nabla f(x) - \mathcal{P}_{N_x M} \nabla f(x)$ and $\text{grad} f(y) = \nabla f(y) - \mathcal{P}_{N_y M} \nabla f(y)$ implies

$$\|\text{grad} f(x) - \text{grad} f(y)\|_F \
\leq \|\nabla f(x) - \nabla f(y)\|_F + \|\mathcal{P}_{N_x M} \nabla f(y) - \mathcal{P}_{N_y M} \nabla f(y)\|_F \
= \|\nabla f(x) - \nabla f(y)\|_F + \frac{1}{2} \|x(x^\top \nabla f(y) + \nabla f(y)^\top x) - y(y^\top \nabla f(y) + \nabla f(y)^\top y)\|_F \
\leq \|\nabla f(x) - \nabla f(y)\|_F + 2L_n \|x - y\|_F \
\leq (L + 2L_n) \|x - y\|_F. \quad (C.4)$$

In (C.4) we used

$$\|x(x^\top \nabla f(y) + \nabla f(y)^\top x) - y(y^\top \nabla f(y) + \nabla f(y)^\top y)\|_F \
\leq \|x(x - y)^\top \nabla f(y) + \nabla f(y)^\top (x - y)\|_F + \|(x - y)(y^\top \nabla f(y) + \nabla f(y)^\top y)\|_F \
\leq 4L_n \|x - y\|_F.$$

The proof is completed.
C.1  Comparison on different Lipschitz-type inequalities

Using Taylor’s Theorem [1, Lemma 7.4.7], $L'_g$ corresponds to the leading eigenvalue of Riemannian Hessian. According to [2], it follows for any $\eta \in T_xM$ that

$$\text{Hess}(f)[\eta] = P_{T_xM} (D\text{grad}(x)[\eta])$$

$$= P_{T_xM} \nabla^2 f(x) \eta - \eta^T P_{N_x} \nabla f(x) - \frac{1}{2} \left( \eta^T P_{N_x} \nabla f(x_i) + (P_{N_x} \nabla f(x))^T \eta \right),$$  \hfill (C.5)

where $P_{N_x}$ is the orthogonal projection onto the normal space $N_xM$. Since $x \frac{1}{2} \left( \eta^T P_{N_x} \nabla f(x_i) + (P_{N_x} \nabla f(x))^T \eta \right) \in N_xM$, we have

$$\langle \eta, \text{Hess}(f)[\eta] \rangle = \langle \eta, \nabla^2 f(x) \eta \rangle - \langle \eta, \eta^T P_{N_x} \nabla f(x) \rangle = \langle \eta, \nabla^2 f(x) \eta \rangle - \left\langle \eta, \eta \frac{1}{2} (x^T \nabla f(x) + \nabla f(x)^T x) \right\rangle,$$  \hfill (C.6)

where we use $P_{T_xM} \nabla^2 f(x) \eta = \nabla^2 f(x) \eta - P_{N_x} \nabla^2 f(x) \eta$. Therefore, we get

$$L'_g \leq \lambda_{\max}(\nabla^2 f(x)) + \max_{x \in \text{St}(d,r)} \|\nabla f(x)\|_2 = L + L_n.$$  \hfill (C.7)

The restricted inequality proposed in [7] is related to the pull back function $g(\xi) := f(R_\xi(\xi))$, whose Lipschitz constant $\tilde{L}_g$ relies on the retraction. Specifically, $\tilde{L}_g = M_0^2 L + 2ML_n$, where $M_0$ is a constant related to the retraction, $M$ and $L_n$ are the same constants in Lemma 2.3.

C.2  Technical lemmas

**Lemma C.1.** [9] For any $x \in \text{St}(d,r)^n$, let $\hat{x} = \frac{1}{n} \sum_{i=1}^n x_i$ be the Euclidean mean and denote $\bar{x} = 1_n \otimes \hat{x}$. Similarly, let $\bar{x} = 1_n \otimes \bar{x}$, where $\bar{x}$ is the IAM defined in (IAM). Moreover, if $\|x - \bar{x}\|_F^2 \leq n/2$, one has

$$\|\bar{x} - \hat{x}\|_F^2 \leq \frac{2\sqrt{n}}{n} \|x - \bar{x}\|_F^2.$$  \hfill (P1)

The following lemma will be useful to bound the Euclidean distance between two average points $\bar{x}_k$ and $\bar{x}_{k+1}$.

**Lemma C.2.** [9] Suppose $x, y \in N_1$, where $N_1$ is defined in (3.4). Then we have

$$\|\bar{x} - \bar{y}\|_F \leq \frac{1}{1 - 2\delta_i} \|\hat{x} - \hat{y}\|_F,$$

where $\bar{x}$ and $\bar{y}$ are the IAM of $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, respectively.

We also need the following bounds for grad$\varphi^t(x)$.

**Lemma C.3.** [9] For any $x \in \text{St}(d,r)^n$, it follows that

$$\|\sum_{i=1}^n \text{grad}\varphi^t(x_i)\|_F \leq L_t \|x - \bar{x}\|_F^2$$  \hfill (C.8)

and

$$\|\text{grad}\varphi^t(x)\|_F \leq L_t \|x - \bar{x}\|_F,$$  \hfill (C.9)

where $L_t$ is the constant given in (B.2). Moreover, suppose $x \in N_2$, where $N_2$ is defined by (3.5). We then have

$$\max_{i \in [n]} \|\text{grad}\varphi^t(x)\|_F \leq 2\delta_2.$$  \hfill (C.10)

Applying Lemma C.2 to the update rule of our algorithms gives the following lemma.
Lemma C.4. If \( x_k \in \mathcal{N}, x_{k+1} \in \mathcal{N} \) and \( x_{i,k+1} = R_{x_{i,k},
abla}(\nabla x_{i,k} + \beta v_{i,k}), \) where \( u_{i,k} \in T_{x_{i,k}} \mathcal{M}, \) \( 0 \leq \alpha \leq \frac{1}{M}, \) \( 0 \leq \beta. \) Let \( u^{\top}_k = (u^{\top}_{1,k}, \ldots, u^{\top}_{n,k}) \) and \( \hat{u}_k = \frac{1}{n} \sum_{i=1}^n u_{i,k}. \) It follows that
\[
\|x_k - x_{k+1}\|_F \leq \frac{1}{1 - 2\delta_1^2} \left( \frac{2L_2^2\alpha + L_1\alpha}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{2M\beta^2}{n} \|u_k\|_F^2 \right).
\]

Proof. From Lemma 2.3 and Lemma C.3, we have
\[
\|\hat{x}_k - \hat{x}_{k+1}\|_F
\leq \|\hat{x}_k + \frac{1}{n} \sum_{i=1}^n (-\nabla \varphi_i(x_k) + \beta v_{i,k}) - \hat{x}_{k+1}\|_F + \frac{1}{n} \sum_{i=1}^n (-\nabla \varphi_i(x_k) + \beta v_{i,k})\|_F
\leq \frac{M}{n} \sum_{i=1}^n \|\nabla \varphi_i(x_k) + \beta v_{i,k}\|_F + \frac{1}{n} \sum_{i=1}^n \|\nabla \varphi_i(x_k)\|_F + \beta \|\hat{u}_k\|_F
\leq \frac{2M\alpha^2}{n} \|\nabla \varphi(x_k)\|_F^2 + \frac{2M\beta^2}{n} \|u_k\|_F^2 + \frac{1}{n} \sum_{i=1}^n \|\nabla \varphi_i(x_k)\|_F + \beta \|\hat{u}_k\|_F
\leq \frac{2L_2^2M\alpha^2 + L_1\alpha}{n} \|x_k - x_{k+1}\|_F^2 + \frac{2M\beta^2}{n} \|u_k\|_F^2 + \beta \|\hat{u}_k\|_F.
\]
Therefore, it follows from Lemma C.2 that
\[
\|x_k - x_{k+1}\|_F \leq \frac{1}{1 - 2\delta_1^2} \|x_k - \bar{x}_k\|_F \leq \frac{1}{1 - 2\delta_1^2} \left( \frac{2L_2^2\alpha + L_1\alpha}{n} \|x_k - \bar{x}_k\|_F^2 + \beta \|\hat{u}_k\|_F + \frac{2M\beta^2}{n} \|u_k\|_F^2 \right),
\]
where we use the fact that \( \alpha \leq \frac{1}{M}. \)

\section{D Proofs for Section 4}

We use the notations
\[
v_k = [v_{1,k}^\top \ldots v_{n,k}^\top]^\top, \quad \hat{v}_k = \frac{1}{n} \sum_{i=1}^n v_{i,k},
\]
\[
g_{i,k} = \nabla f_i(x_{i,k}) \quad \text{and} \quad \hat{g}_k = \frac{1}{n} \sum_{i=1}^n g_{i,k}.
\]
The following lemma is useful to show \( x_k \in \mathcal{N} \) for all \( k. \)

Lemma D.1. \[9, Lemma 11\] Given any \( x \in \mathcal{N}_2, \) where \( \mathcal{N}_2 \) is defined in (3.5), if \( t \geq \lfloor \log_{\sigma_2}(\frac{1}{2\sqrt{n}}) \rfloor, \) we have
\[
\max_{i \in [n]} \| \sum_{j=1}^n (W_{ij}^t - 1/n)x_j \|_F \leq \frac{\delta_2}{2}.
\]

Lemma D.2. Under the same conditions of Fact B.1, if \( x_k \in \mathcal{N} \) and
\[
x_{i,k+1} = R_{x_{i,k},
abla}(\nabla x_{i,k} + \beta v_{i,k}), \quad \forall i \in [n],
\]
where \( v_{i,k} \in T_{x_{i,k}} \mathcal{M}, \) the following holds
\[
\|x_{k+1} - x_k\|_F \leq \rho_1 \|x_k - \bar{x}_k\|_F + \beta \|\hat{v}_k\|_F.
\]

Proof. By the definition of IAM, we have
\[
\|x_{k+1} - \bar{x}_{k+1}\|_F^2 \leq \|x_{k+1} - \bar{x_k}\|_F^2
\]
\[
= \sum_{i=1}^n \|R_{x_{i,k}}(\nabla x_{i,k} + \beta v_{i,k}) - \bar{x}_k\|_F^2
\leq \sum_{i=1}^n \|x_{i,k} - \nabla \varphi_i(x_k) + \beta v_{i,k} - \bar{x}_k\|_F^2.
\]
Let \( v_k = [v_{1,k}^\top \ldots v_{n,k}^\top]^\top \). Then, we get
\[
\|x_{k+1} - z_{k+1}\|_F \leq \|x_k - \nabla \phi(x_k) - \beta_k v_k - z_k\|_F \\
\leq \|x_k - \nabla \phi(x_k) - \tilde{z}_k\|_F + \beta_k \|v_k\|_F.
\] (D.3)
By combining inequality (B.3) of Fact B.1, we get
\[
\|x_{k+1} - \tilde{z}_{k+1}\|_F \leq \rho_t \|x_k - \tilde{z}_k\|_F + \beta_k \|v_k\|_F.
\] (D.4)
The proof is completed.

**Proof of Lemma 4.1.** We prove that \( x_k \in \mathcal{N} \) for all \( k \geq 0 \) by induction. Suppose \( x_k \in \mathcal{N} \), let us show \( x_{k+1} \in \mathcal{N} \). Note \( \|v_k\|_F \leq \sqrt{nD} \). Using Lemma D.2 yields
\[
\|x_{k+1} - \tilde{z}_{k+1}\|_F \leq \rho_t \|x_k - \tilde{z}_k\|_F + \beta_k \sqrt{nD} \\
\leq \rho_t \sqrt{n\delta_1} + \beta_k \sqrt{nD} \\
\leq \sqrt{n\delta_1},
\] where the last inequality follows from \( \beta_k \leq \frac{1-\rho_t}{D} \delta_1 \). Hence \( x_{k+1} \in \mathcal{N}_1 \). Secondly, let us verify \( x_{k+1} \in \mathcal{N}_2 \). It follows from \( \beta_k \leq \frac{a_0}{nD} \leq \frac{\alpha}{2D} \) and \( \alpha \leq 1 \) that
\[
\|x_{k+1} - x_k\|_{F,\infty} \leq \max_{i \in [n]} \|\nabla \phi_i(x_{k,i})\|_F + \beta_k D \leq 2\alpha \delta_2 + \frac{\alpha}{2} \leq 1 - \delta_1^2.
\] Using Lemma C.4 yields
\[
\|\bar{x}_k - \bar{z}_{k+1}\|_F \leq \frac{1}{1 - 2\delta_1^2} \left( \frac{2L_t^2 \alpha + L_t \alpha}{n} \|x_k - \tilde{z}_k\|_F \right) + \beta_k \|v_k\|_F \\
\leq \frac{1}{1 - 2\delta_1^2} \left[ \frac{2L_t^2 \alpha + L_t \alpha}{n} \delta_1^2 + \beta_k D + 2M \beta_k^2 D^2 \right].
\] Furthermore, since \( L_t \leq 2, \beta_k \leq \frac{a_0}{nD}, \alpha \leq 1/M, \) we get
\[
\|\bar{x}_k - \bar{z}_{k+1}\|_F \leq \frac{1}{1 - 2\delta_1^2} \left( \frac{252}{25} \delta_1^2 + \frac{a_0}{5} \right) \leq \frac{1}{1 - 2\delta_1^2} \left( \frac{252}{625} \delta_2^2 + \frac{1}{5G} \delta_2 \right),
\] (D.6)
where the last inequality follows from \( \delta_1 \leq \frac{1}{\sqrt{G}} \delta_2 \). Then, one has
\[
\|x_{i,k+1} - \tilde{x}_{k+1}\|_F \\
\leq \|x_{i,k+1} - \tilde{x}_k\|_F + \|\bar{x}_k - \bar{z}_{k+1}\|_F \\
\leq \|x_{i,k} - \nabla \phi_i(x_k) - \beta_k v_{i,k} - \bar{x}_k\|_F + \|\bar{x}_k - \bar{z}_{k+1}\|_F + \|\bar{x}_k - \bar{z}_{k+1}\|_F.
\] (D.7)
Now, we proceed by using the same lines in the proof of [9, Lemma 13] as follows
\[
\nabla \phi_i(x) = x_i - \sum_{j=1}^n W_{ij} x_j - \frac{1}{2} x_i \sum_{j=1}^n W_{ij} (x_i - x_j)^\top (x_i - x_j),
\] (D.8)
Combining this with (D.7) implies
\[ (D.8) \quad \| (1 - \alpha)(x_{i,k} - \bar{x}_k) + \alpha(\bar{x}_k - \bar{x}_k) + \alpha \sum_{j=1}^{n} W_{ij}^t (x_{j,k} - \bar{x}_k) + \frac{\alpha}{2} x_{i,k} \sum_{j=1}^{n} W_{ij}^t (x_{i,k} - x_{j,k}) \|^F \]
\[ \leq (1-\alpha)\delta_2 + \alpha\| \bar{x}_k - \bar{x}_k \|^F + \alpha \| \sum_{j=1}^{n} (W_{ij}^t - \frac{1}{n}n_{x,j,k}) \|^F + \frac{1}{2} \| \alpha \sum_{j=1}^{n} W_{ij}^t (x_{i,k} - x_{j,k}) \|^F \] (D.9)
\[ \leq (1-\alpha)\delta_2 + 2\alpha\delta_1^2 \sqrt{\tau} + \alpha \| \sum_{j=1}^{n} (W_{ij}^t - \frac{1}{n}n_{x,j,k}) \|^F + 2\alpha\delta_2^2 \] (D.10)
\[ \leq (1-\alpha)\delta_2 + 2\alpha\delta_1^2 \sqrt{\tau} + 2\alpha\delta_2^2, \] (D.11)

where (D.9) follows from \( \alpha \in [0, 1] \), (D.10) holds by Lemma C.1 and (D.11) follows from Lemma D.1. Combining this with (D.7) implies
\[ \| x_{i,k+1} - \bar{x}_{k+1} \|^F \leq (1-\alpha)\delta_2 + 2\alpha\delta_1^2 \sqrt{\tau} + 2\alpha\delta_2^2 + \frac{1}{5} \alpha \delta_1 + \| \bar{x}_k - \bar{x}_{k+1} \|^F \]
\[ \leq (1-\alpha)\delta_2 + 2\alpha\delta_1^2 \sqrt{\tau} + 2\alpha\delta_2^2 + \frac{1}{5} \alpha \delta_1 + \frac{1}{1-2\delta_2^2} \left( \frac{252}{625} \alpha \delta_2^2 + \frac{1}{25\sqrt{\tau}} \alpha \delta_2 \right). \] (D.12)

Therefore, substituting the conditions (3.6) on \( \delta_1, \delta_2 \) into (D.12) yields
\[ \| x_{i,k+1} - \bar{x}_{k+1} \|^F \leq \delta_2. \]

The proof of the first statement is completed. Finally, it follows from (D.5) that
\[ \| x_{k+1} - \bar{x}_{k+1} \|^F \leq \rho_t \| x_k - \bar{x}_k \|^F + \beta_k \sqrt{nD} \]
\[ \leq \rho_t \| x_0 - \bar{x}_0 \|^F + \sqrt{nD} \sum_{l=0}^{k} \rho_t^{k-l} \beta_l. \] (D.13)

An immediate result of Lemma 4.1 is that the rate of consensus \( \| x_k - \bar{x}_k \|^2_F = O(\beta_k^2) \) if \( \beta_k = O(\frac{1}{\sqrt{n}}) \). The proof is similar as [25, Proposition 8], we provide it for completeness.

**Lemma D.3.** Under Assumptions 1 to 4, for Algorithm 1, if \( x_0 \in \mathcal{N}, 0 < \alpha \leq \min\{\frac{\Phi}{2\tau^2}, 1, \frac{1}{\tau^2}\}, t \geq \lceil \log_2(\frac{1}{\sqrt{n}\beta_k}) \rceil \) and
\[ \beta_k = \min\left\{ \frac{\alpha \delta_1}{5D}, \frac{1}{(k+1)^{\delta_1}}, \frac{1-\rho_t \delta_1}{D} \right\}, \quad p \in (0, 1], \] (D.14)
then there exists a constant \( C > 0 \) such that \( \frac{1}{n} \| x_k - \bar{x}_k \|_F^2 \leq CD^2 \beta_k^2 \) for any \( k \geq 0 \), where \( C \) is independent of \( D \) and \( n \).

**Proof of Lemma D.3.** The proof relies on Lemma 4.1. Let \( a_k := \frac{\| x_k - \bar{x}_k \|^F}{\sqrt{n}D} \).

It follows from (D.13) that
\[ a_{k+1} \leq \rho_t a_k + D \cdot \frac{\beta_k}{\beta_{k+1}} \leq \rho_t \beta_k^{k+1-K} a_K + D \sum_{l=K}^{k} \rho_t^{k-l} \beta_l \beta_{l+1}. \] (D.15)

Recall that \( \beta_k = O(1/D) \) and \( \frac{1}{n} \| x_0 - \bar{x}_0 \|_F^2 \leq \delta_2^2 \), it follows that \( a_0 \leq \delta_1 / \beta_0 = O(D) \). Since \( \lim_{k \to \infty} \beta_{k+1} / \beta_k = 1 \), there exists sufficiently large \( K \) such that
\[ \frac{\beta_k}{\beta_{k+1}} \leq 2, \quad \forall k \geq K. \]
For $0 \leq k \leq K$, there exists some $C' > 0$ such that
\[ a_k^2 \leq C'D^2, \]
where $C'$ is independent of $D$ and $n$. For $k \geq K$, using (D.15) gives $a_k^2 \leq CD^2$, where $C = 2C' + \frac{8}{(1-\rho^2)^2}$. Hence, we get $\|x_k - \bar{x}_k\|/n \leq CD^2\beta_k^2$ for all $k \geq 0$, where $C = O(\frac{1}{(1-\rho^2)^2})$.

Lemma D.4. Under Assumptions 1 to 4, suppose $x_k \in \mathcal{N}$, $t \geq \log_2(\frac{1}{2\alpha})$, $0 < \alpha \leq \min\{\frac{\Phi}{2\pi}, 1, \frac{1}{\pi}\}$. If $x_{i,k+1} = R_{x_i,k}(-\nabla f(x_{i,k}) - \beta_k v_{i,k})$, $0 < \beta_k \leq \min\{\frac{1}{\sqrt{L_g}}, \frac{K}{2D}\}$ and $\beta_k \geq \beta_{k+1}$, where $v_{i,k}$ satisfies Assumption 3 and $L_g$ is given in Lemma 2.4. It follows that
\begin{align*}
\mathbb{E}_k f(\bar{x}_{k+1}) &\leq f(\bar{x}_k) + \langle \nabla f(\bar{x}_k), \mathbb{E}_k(\bar{x}_{k+1} - \bar{x}_k) \rangle + \frac{L_g}{2} \mathbb{E}_k \|\bar{x}_{k+1} - \bar{x}_k\|_F^2 \\
&= f(\bar{x}_k) - \langle \nabla f(\bar{x}_k), \beta_k \hat{g}_k \rangle + \langle \nabla f(\bar{x}_k), \mathbb{E}_k[\bar{x}_{k+1} - \bar{x}_k + \beta_k \hat{v}_k] \rangle + \frac{L_g}{2} \mathbb{E}_k \|\bar{x}_{k+1} - \bar{x}_k\|_F^2 \\
&= f(\bar{x}_k) - \beta_k \|\nabla f(\bar{x}_k)\|_F^2 + \beta_k \|\nabla f(\bar{x}_k) - \hat{g}_k\|_F^2 + \frac{L_g}{2} \mathbb{E}_k \|\bar{x}_{k+1} - \bar{x}_k\|_F^2 \\
&+ \langle \nabla f(\bar{x}_k), \mathbb{E}_k(\bar{x}_{k+1} - \bar{x}_k + \beta_k \hat{g}_k) \rangle + \frac{L_g}{2} \mathbb{E}_k \|\bar{x}_{k+1} - \bar{x}_k\|_F^2,
\end{align*}
where $\hat{v}_k = \frac{1}{n} \sum_{i=1}^{n} v_{i,k}$ and we use $\mathbb{E}_k \hat{v}_k = \hat{g}_k$ in the first equation. Note that for $\beta_k > 0$, we have
\[ \langle \nabla f(\bar{x}_k), \mathbb{E}_k(\bar{x}_{k+1} - \bar{x}_k + \beta_k \hat{g}_k) \rangle \leq \frac{\beta_k}{4} \|\nabla f(\bar{x}_k)\|_F^2 + \frac{1}{\beta_k} \|\mathbb{E}_k(\bar{x}_{k+1} - \bar{x}_k + \beta_k \hat{g}_k)\|_F^2. \]
Plugging this into (D.17) yields
\begin{align*}
\mathbb{E}_k f(\bar{x}_{k+1}) \\
\leq f(\bar{x}_k) - \beta_k \|\hat{g}_k\|_F^2 + \frac{\beta_k}{4} \|\nabla f(\bar{x}_k)\|_F^2 + \frac{\beta_k}{2} \|\nabla f(\bar{x}_k) - \hat{g}_k\|_F^2 + \frac{1}{\beta_k} \|\mathbb{E}_k(\bar{x}_{k+1} - \bar{x}_k + \beta_k \hat{g}_k)\|_F^2 \\
+ \frac{L_g}{2} \mathbb{E}_k \|\bar{x}_{k+1} - \bar{x}_k\|_F^2.
\end{align*}
Using Lemma 2.4 implies
\[ a_1 \leq \frac{1}{n} \sum_{i=1}^{n} \|\nabla f(x_{i,k}) - \nabla f(\bar{x}_k)\|_F^2 \overset{(2.6)}{=} \frac{L_g^2}{n} \|x_k - \bar{x}_k\|_F^2, \]
\[ \overset{(2.6)}{=} \frac{L_g^2}{n} \|x_k - \bar{x}_k\|_F^2. \]
Secondly, we use the following inequality to derive the upper bound of \(a_2\). From Lemma 4.1, we have \(x_{k+1} \in \mathcal{N}\). One has

\[
\|x_{k+1} - \bar{x}_k + \beta_k \hat{v}_k\|_F \\
\leq \|x_k - \bar{x}_k\|_F + \|x_{k+1} - \bar{x}_{k+1}\|_F + \|\hat{x}_k - \beta_k \hat{v}_k - \bar{x}_{k+1}\|_F \\
\overset{(P1)}{\leq} \frac{2\sqrt{n}}{n} \left(\|x_k - \bar{x}_k\|_F^2 + \|x_{k+1} - \bar{x}_{k+1}\|_F^2\right) + \|\hat{x}_k - \beta_k \hat{v}_k - \bar{x}_{k+1}\|_F \\
\overset{(C.8)}{\leq} \frac{4\sqrt{n}}{n} \|x_k - \bar{x}_k\|_F^2 + \|\hat{x}_k - \beta_k \hat{v}_k - \bar{x}_{k+1}\|_F, \tag{D.19}
\]

where we use \(\|x_k - \bar{x}_k\|_F^2 \geq \|x_{k+1} - \bar{x}_{k+1}\|_F^2\) in the last inequality.

For the second term, since \(v_{i,k} \in T_{x_{i,k}} \mathcal{M}\) we have

\[
\|x_k - x_{k+1}\|_F \leq \frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \alpha \nabla f_i(x_k) - \beta_k v_{i,k} - x_{i,k+1}\|_F + \frac{\alpha}{n} \sum_{i=1}^{n} \nabla f_i(x_k) \|
\overset{(P1)}{\leq} \frac{M}{n} \sum_{i=1}^{n} \|\alpha \nabla f_i(x_k) + \beta_k v_{i,k}\|_F^2 + \frac{\alpha}{n} \sum_{i=1}^{n} \nabla f_i(x_k) \|
\overset{(C.8)}{\leq} \frac{2M\alpha^2}{n} \|\nabla f_i(x_k)\|_F^2 + \frac{2M\beta_k^2}{n} \|v_k\|_F^2 + \frac{L_t \alpha}{n} \|x_k - \bar{x}_k\|_F^2 \tag{D.20}
\]

\[
\overset{(C.9)}{\leq} \frac{2L_t^2 M\alpha^2 + L_t \alpha}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{2M\beta_k^2}{n} \|v_k\|_F^2 \\
\leq \frac{10\alpha}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{2M\beta_k^2}{n} \|v_k\|_F^2,
\]

where we use \(\alpha \leq \frac{1}{M}\) and \(L_t \leq 2\) in the last inequality. Plugging (D.20) into (D.19) yields

\[
\|x_{k+1} - \bar{x}_k + \beta_k \hat{v}_k\|_F^2 \leq 2\left(\frac{4\sqrt{n} + 10\alpha}{n}\right)^2 \|x_k - \bar{x}_k\|_F^2 + 2\left(\frac{2M\beta_k^2}{n}\right)^2 \|v_k\|_F^2. \tag{D.21}
\]

Then, using Jensen’s inequality and \(\|v_k\|_F^2 \leq nD^2\) implies

\[
a_2 \leq E_k[\|x_{k+1} - \bar{x}_k + \beta_k \hat{v}_k\|_F^2] \leq 2\left(\frac{4\sqrt{n} + 10\alpha}{n}\right)^2 \|x_k - \bar{x}_k\|_F^2 + 8M^2\beta_k^4 D^4.
\]

Thirdly, invoking Lemma C.4 yields

\[
\|x_k - \bar{x}_{k+1}\|_F \leq \frac{1}{1 - 2\delta^2} \left(\frac{10\alpha}{n} \|x_k - \bar{x}_k\|_F^2 + 2M\beta_k^2 D^2 + \beta_k \|\hat{v}_k\|_F\right).
\]

Hence, it follows that

\[
a_3 \leq \frac{2}{(1 - 2\delta^2)^2} \left(\frac{10\alpha}{n} \|x_k - \bar{x}_k\|_F^2 + 2M\beta_k^2 D^2\right)^2 + \frac{2}{(1 - 2\delta^2)^2} \beta_k^2 E_k \|\hat{v}_k\|_F^2 \\
\overset{=}{=} \frac{2}{(1 - 2\delta^2)^2} \left[\frac{10\alpha}{n} \|x_k - \bar{x}_k\|_F^2 + 2M\beta_k^2 D^2\right]^2 + \frac{2}{(1 - 2\delta^2)^2} \beta_k^2 E_k \|\hat{v}_k - \bar{g}_k\|_F^2 + \frac{2}{(1 - 2\delta^2)^2} \beta_k^2 \|\bar{g}_k\|_F^2 \\
\overset{\overset{(i)}{\leq}}{=} \frac{2}{(1 - 2\delta^2)^2} \left[\frac{10\alpha}{n} \|x_k - \bar{x}_k\|_F^2 + 2M\beta_k^2 D^2\right]^2 + \frac{2}{(1 - 2\delta^2)^2} \beta_k^2 \|\sum_{i=1}^{n} E_k v_{i,k} - g_k\|_F^2 + \frac{2}{(1 - 2\delta^2)^2} \beta_k^2 \|\bar{g}_k\|_F^2 \\
\overset{\overset{(ii)}{\leq}}{=} \frac{4}{(1 - 2\delta^2)^2} \left[\frac{100\alpha^2}{n^2} \|x_k - \bar{x}_k\|_F^2 + 4M^2\beta_k^4 D^4\right] + \frac{2}{(1 - 2\delta^2)^2} \beta_k^2 \|\bar{g}_k\|_F^2 + \frac{2}{(1 - 2\delta^2)^2} \beta_k^2 \|\bar{g}_k\|_F^2,
\]

where (i) and (ii) hold by the independence of \(v_{i,k}\) and bounded variance of Assumption 3, respectively.
Therefore, by combining $a_1, a_2, a_3$ with (D.18) implies that
\[
\mathbb{E}_k f(\bar{x}_{k+1}) \leq f(\bar{x}_k) - \frac{\beta_k}{2} \frac{1}{2} \left\| \nabla f(\bar{x}_k) \right\|^2_F + \frac{\beta_k}{2} \frac{1}{2} \left\| \nabla f(\bar{x}_k) \right\|^2_F + \frac{\beta_k}{2} \frac{1}{2} \left\| \nabla f(\bar{x}_k) \right\|^2_F + \frac{1}{\beta_k} a_1 + \frac{1}{\beta_k} a_2 + \frac{L_g}{2} a_3 \\
\leq f(\bar{x}_k) - \left( \frac{\beta_k}{2} - \frac{L_g \beta_k^2}{(1-2\delta_1^2)^2} \right) \frac{1}{2} \left\| \nabla f(\bar{x}_k) \right\|^2_F + \frac{\beta_k}{2} \frac{1}{2} \left\| \nabla f(\bar{x}_k) \right\|^2_F + \frac{\beta_k L_g^2}{2n} \left\| x_k - \bar{x}_k \right\|^2_F + \frac{2}{\beta_k} \left( \frac{4\sqrt{r} + 10\alpha^2}{n} \right)^2 \left\| x_k - \bar{x}_k \right\|^2_F \\
+ 8M^2 \beta_k^3 D^4 + \frac{2L_g}{(1-2\delta_1^2)^2} \left[ \frac{100\alpha^2}{n^2} \left\| x_k - \bar{x}_k \right\|^2_F + 4M^2 \beta_k^4 D^4 \right] + \frac{L_g}{(1-2\delta_1^2)^2} \beta_k^2 \Xi^2.
\]

By Lemma D.3, we have $\left\| x_k - \bar{x}_k \right\|^2_F \leq nCD^2 \beta_k^2$. It follows that
\[
\mathbb{E}_k f(\bar{x}_{k+1}) \\
\leq f(\bar{x}_k) - \left( \frac{\beta_k}{2} - \frac{L_g \beta_k^2}{(1-2\delta_1^2)^2} \right) \frac{1}{2} \left\| \nabla f(\bar{x}_k) \right\|^2_F + \frac{\beta_k}{2} \frac{1}{2} \left\| \nabla f(\bar{x}_k) \right\|^2_F + \frac{\beta_k L_g^2}{2n} \left\| x_k - \bar{x}_k \right\|^2_F + \frac{2}{\beta_k} \left( \frac{4\sqrt{r} + 10\alpha^2}{n} \right)^2 \left\| x_k - \bar{x}_k \right\|^2_F \\
+ 8M^2 \beta_k^3 D^4 + \frac{2L_g}{(1-2\delta_1^2)^2} \left[ \frac{100\alpha^2}{n^2} \left\| x_k - \bar{x}_k \right\|^2_F + 4M^2 \beta_k^4 D^4 \right] + \frac{L_g}{(1-2\delta_1^2)^2} \beta_k^2 \Xi^2.
\]

where we use $\frac{1}{(1-2\delta_1^2)^2} \leq 1.002$ and $\beta_k \leq \frac{1}{3L_g}$ in the last inequality. The proof is completed.

**Proof of Theorem 4.2.** Using (D.16) implies
\[
\mathbb{E}_k f(\bar{x}_{k+1}) \\
\leq f(\bar{x}_k) - \frac{\beta_k}{4} \left\| \nabla f(\bar{x}_k) \right\|^2_F + \frac{3L_g \Xi^2}{2n} \beta_k^2 + \frac{CD^4 G^2}{2} \beta_k^4 + (T_1 D^4) \beta_k^4 + T_2 L_g D^4 \beta_k^4,
\]
Taking the expectation on all $k$ and telescoping the right hand side give us for any $K > 0$
\[
\sum_{k=0}^K \frac{\beta_k}{4} \mathbb{E} \left\| \nabla f(\bar{x}_k) \right\|^2_F \leq f(\bar{x}_0) - f^* + \frac{3L_g \Xi^2}{2n} \sum_{k=0}^K \beta_k^2 + \frac{CD^4 G^2}{2} \sum_{k=0}^K \beta_k^4 + (T_1 D^4) \sum_{k=0}^K \beta_k^4 + T_2 L_g D^4 \sum_{k=0}^K \beta_k^4,
\]
where $f^* = \min_{x \in \text{St}(d, r)} f(x)$. Dividing both sides by $\sum_{k=0}^K \frac{\beta_k}{4}$ yields
\[
\min_{k=0, \ldots, K} \mathbb{E} \left\| \nabla f(\bar{x}_k) \right\|^2_F \leq f(\bar{x}_0) - f^* + \frac{3L_g \Xi^2}{2n} \sum_{k=0}^K \frac{\beta_k^2}{4} + \frac{CD^4 G^2}{2} \sum_{k=0}^K \beta_k^4 + (T_1 D^4) \sum_{k=0}^K \beta_k^4 + T_2 L_g D^4 \sum_{k=0}^K \beta_k^4.
\]
Let $\beta = \min\{1, L_g, \frac{1}{\alpha_d} \}$. Noticing that $\beta_k = \mathcal{O}(\min\{\frac{1}{\alpha_d}, \frac{1}{L_g} \} \cdot \frac{1}{k})$, $\sum_{k=0}^K \frac{\beta_k^2}{4} = \mathcal{O}(\frac{\ln(K+1)}{\sqrt{K+1}})$, $\sum_{k=0}^K \beta_k^4 = \mathcal{O}(\frac{\beta^4}{\sqrt{K+1}})$ and $\sum_{k=0}^K \frac{\beta_k^4}{4} = \mathcal{O}(\frac{\beta^4}{\sqrt{K+1}})$. The proof is completed.

The following corollary follows [23], in which the convergence results of constant stepsize $\beta_k$ is given.

**Corollary D.5.** Under Assumptions 1 to 4, suppose $x_k \in N$, $t \geq \left\lceil \log_{\alpha_d} \left( \frac{\alpha_d}{\alpha_d - 1} \right) \right\rceil$, $0 < \alpha \leq 1$. If constant stepsize $\beta_k \equiv \beta = \frac{1}{2L_G + \sqrt{\ln(K+1)/n}}$, where
\[
K + 1 \geq \max\left\{ \frac{n}{\Xi^2} \left( \max\{3L_G, \frac{5D}{\alpha_d^3}, \frac{D_d}{1-\rho_t} \} \right)^2, \frac{n^3}{\Xi^2} \left( \frac{CD^4 G^2 + (2T_1 + T_2) D^4}{2(f(\bar{x}_0) - f^*) + 3L_G} \right)^2 \right\},
\]

25
if follows that

$$
\min_{k=0,\ldots,K} \mathbb{E}\|\nabla f(\bar{x}_k)\|_F^2 \leq \frac{8L_G(f(\bar{x}_0) - f^*)}{K+1} + \frac{8(f(\bar{x}_0) - f^* + 3\frac{L_G}{2})\Xi}{\sqrt{n(K+1)}}.
$$

Proof. Since $K + 1 \geq \frac{1}{2L_G} (\max\{3L_G, \frac{5D}{\alpha\delta_1}, \frac{D \delta_1}{1-\rho_1}\})^2$, we have

$$
\beta_k \leq \min\left\{ \frac{1}{5L_G}, \frac{\alpha\delta_1}{5D}, 1 - \frac{\rho_1}{D} \delta_1 \right\}
$$

for all $k = 0, 1, \ldots, K$. Therefore, it follows that $x_k \in \mathcal{N}$ for $k = 0, 1, \ldots, K$. Using Theorem 4.2, we have

$$
\min_{k=0,\ldots,K} \mathbb{E}\|\nabla f(\bar{x}_k)\|_F^2 \\
\leq \frac{4(f(\bar{x}_0) - f^*)}{(K+1)\beta} + \frac{6L_g\Xi^2}{n} + (2CD^2L_g^2 + 4\mathcal{T}_1D^4)\beta^2 + 4\mathcal{T}_2L_gD^4
$$

$$
\leq \frac{8L_G(f(\bar{x}_0) - f^*)}{K+1} + \frac{4(f(\bar{x}_0) - f^*)\Xi}{\sqrt{n(K+1)}} + \frac{6L_g\Xi^2}{2nL_G + \Xi\sqrt{n(K+1)}} + \frac{2CD^2L_g^2 + (4\mathcal{T}_1 + 2\mathcal{T}_2)D^4}{(2L_G + \Xi\sqrt{(K+1)/n})^2},
$$

(D.23)

$$
\leq \frac{8L_G(f(\bar{x}_0) - f^*)}{K+1} + \frac{4(f(\bar{x}_0) - f^* + 3\frac{L_G}{2})\Xi}{\sqrt{n(K+1)}} + \frac{2nCD^2L_g^2 + (4\mathcal{T}_1 + 2\mathcal{T}_2)nD^4}{\Xi^2(K+1)},
$$

(D.24)

where we use $\beta \leq \frac{1}{2L_G} \leq \frac{1}{2L_g}$ in (D.23).

When

$$
K + 1 \geq \frac{n^3}{\Xi^6} \left( \frac{CD^2L_g^2 + (2\mathcal{T}_1 + \mathcal{T}_2)D^4}{2(f(\bar{x}_0) - f^*) + 3L_G} \right)^2,
$$

the second term in (D.24) is greater than the third term, we get

$$
\min_{k=0,\ldots,K} \mathbb{E}\|\nabla f(\bar{x}_k)\|_F^2 \\
\leq \frac{8L_G(f(\bar{x}_0) - f^*)}{K+1} + \frac{8(f(\bar{x}_0) - f^* + 3\frac{L_G}{2})\Xi}{\sqrt{n(K+1)}},
$$

which completes the proof. \hfill \Box

## E Proofs for Section 5

In this section, we use the following notations

$$
G_k := \begin{bmatrix} \nabla f_1(x_{1,k}) \\ \vdots \\ \nabla f_n(x_{n,k}) \end{bmatrix}, \quad y_k := \begin{bmatrix} y_{1,k} \\ \vdots \\ y_{n,k} \end{bmatrix}, \quad \hat{y}_k := \frac{1}{n} \sum_{i=1}^{n} y_{i,k},
$$

$$
\hat{y}_k := \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_{i,k}), \quad \hat{G}_k := (I_n \otimes I_n)\hat{y}_k.
$$

**Proof of Lemma 5.1.** We prove it by induction. Let $\hat{y}_{-1} = \hat{y}_0$, one has $\|y_{i,0}\|_F \leq D$ and

$$
\|y_{i,0} - \hat{y}_{-1}\|_F \leq \|y_{i,0}\|_F + \|\hat{y}_{-1}\|_F \leq D + \frac{1}{n} \sum_{j=1}^{n} \|y_{j,0}\|_F \leq 2D
$$

for all $i \in [n]$ by Assumption 2. Suppose for some $k \geq 0$, it follows that $\|y_{i,k}\|_F \leq 2D + L_G$ and $\|y_{i,k} - \hat{y}_{k-1}\|_F \leq 2D + L_G$. 

We note that the bound of $v_i$ becomes $2D + L_G$ here since $\|v_{i,k}\|_F = \|P_{T_{i,k}} y_{i,k}\|_F \leq \|y_{i,k}\|_F$. Following the same argument in the proof of Lemma 4.1, we get $x_{k+1} \in \mathcal{N}$ since $0 < \alpha \leq \min\{\frac{\beta}{2L_G}, 1, \frac{1}{M}\}$ and $0 \leq \beta \leq \min\{\frac{1 - \rho_i}{L_G + 2D}, \frac{\alpha\delta_i}{5(L_G + 2D)}\}$.

Then, we have

$$\|y_{i,k+1} - \hat{g}_k\|_F = \|\sum_{j=1}^n W_{i,j}^t y_{j,k} - \hat{g}_k + \text{grad}\, f(x_{i,k+1}) - \text{grad}\, f(x_{i,k})\|_F$$

$$= \|\sum_{j=1}^n (W_{i,j}^t - \frac{1}{n})(y_{j,k} - \hat{g}_{k-1}) + \text{grad}\, f(x_{i,k+1}) - \text{grad}\, f(x_{i,k})\|_F$$

$$\leq \sigma^2 2\sqrt{n}\|y_{j,k} - \hat{g}_{k-1}\|_F + L_G\|x_{i,k+1} - x_{i,k}\|_F$$

$$\leq \sigma^2 2\sqrt{n}\|y_{j,k} - \hat{g}_{k-1}\|_F + L_G\|\text{grad}\, f(x_{i,k})\|_F + \beta\|y_{i,k}\|_F$$

$$\leq \frac{1}{2}(2D + L_G) + 2\delta_2 L_G + \frac{L_G}{5}\delta_1\alpha$$

$$\leq D + L_G.$$ 

Hence, $\|y_{i,k+1}\|_F \leq \|y_{i,k+1} - \hat{g}_k\|_F + \|\hat{g}_k\|_F \leq L_G + 2D$, where we use $\|\hat{g}_k\|_F \leq D$. Therefore, we get $\|y_{i,k}\|_F \leq L_G + 2D$ for all $i, k$ and $x_k \in \mathcal{N}$.

Using the same argument of Lemma D.3, there exists some $C_1 = \mathcal{O}(\frac{1}{(1 - \rho_i)^2})$ that is independent of $L_G$ and $D$ such that

$$\frac{1}{n}\|x_k - \bar{x}_k\|^2_F \leq C_1(L_G + 2D)^2\beta^2, k \geq 0.$$ 

(E.1)

The proof is completed.

Next, we present the relations between the consensus error and the gradient tracking error.

**Lemma E.1.** Under the same conditions of Lemma 5.1, one has the following error bounds for any $k \geq 0$:

1. **Successive gradient error:**

$$\|G_{k+1} - G_k\|_F \leq 2\alpha L_G\|x_k - \bar{x}_k\|_F + \beta L_G\|y_k\|_F.$$ 

(E.2)

2. **Successive tracking error:**

$$\|y_{k+1} - \hat{G}_{k+1}\|_F \leq \sigma^2\|y_k - \hat{G}_k\|_F + \|G_{k+1} - G_k\|_F.$$ 

(E.3)

3. **Successive consensus error:** for $\rho_t = \sqrt{1 - \gamma_t\alpha} \in (0, 1)$,

$$\|x_{k+1} - \bar{x}_{k+1}\|_F \leq \rho_t\|x_k - \bar{x}_k\|_F + \beta\|y_k\|_F.$$ 

(E.4)

4. **Associating $y_k, \hat{G}_k$ with above items:**

$$\|y_k\|_F \leq \|y_k - \hat{G}_k\|_F + \|\hat{G}_k\|_F.$$ 

(E.5)

**Proof of Lemma E.1.** By Lemma 5.1, we know $x_k \in \mathcal{N}$ for all $k \geq 0$.

1. Using Lemma 2.4 yields

$$\|G_{k+1} - G_k\|_F \leq L_G\|x_{k+1} - x_k\|_F.$$ 

By Lemma 2.3, it follows that

$$\|x_k - x_{k+1}\|_F \leq \alpha\|\text{grad}\, f(x_k)\|_F + \beta\|v_k\|_F \leq 2\alpha\|x_k - \bar{x}_k\|_F + \beta\|y_k\|_F,$$

where we use $\|v_k\|_F \leq \|y_k\|_F$. Hence, the inequality (E.2) is proved.
2. Denote $J = \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^\top$. Note that
\[
y_{k+1} - \hat{G}_{k+1} = \((I_n - J) \otimes I_n\)y_{k+1} = \((I_n - J) \otimes I_n\)((W \otimes I_n)y_k + G_{k+1} - G_k) = \((I_n - J) \otimes I_n\)y_k + ((I_n - J) \otimes I_n)(G_{k+1} - G_k)
\]
where we use \((I_n - J) \otimes I_n\)(\(W \otimes I_n\)) = \((I_n - J) \otimes I_n\). It follows that
\[
\|y_{k+1} - \hat{G}_{k+1}\|_F \leq \sigma_k^2 \|y_k - \hat{G}_k\|_F + \|G_{k+1} - G_k\|_F
\]
3. Note that \(\|v_k\|_F \leq \|y_k\|_F\). Then the desired result follows the same line as that of Lemma D.2.
4. This follows from the triangle inequality.

To show Theorem 5.2, we firstly show a descent lemma. Note that an extra \(\|\hat{G}_k\|_F^2 = n\|\hat{g}_k\|_F^2\) appears in (E.5), what is we aim at bounding in the optimization problem (1.1). By combining with the following lemmas, we can quickly obtain the final convergence result.

**Lemma E.2.** Under the same conditions of Lemma 5.1, it follows that
\[
f(\bar{x}_{k+1}) \leq f(\bar{x}_k) - (\beta - 4L_G\beta^2)\|\hat{g}_k\|_F^2 + \frac{L_G}{n} \|\bar{x}_{k+1} - \bar{x}_k\|_F^2 + G_0 \frac{L_G}{n} \|\bar{x}_k - \bar{x}_{k-1}\|_F^2 + G_1 \frac{L_G}{n} \|\bar{x}_k - \bar{x}_{k-1}\|_F^2 + G_2 \frac{L_G}{n} \beta^2 \|y_k\|_F^2,
\]
where \(G_0 = \frac{4r(L_G + 2D)^2 C_1}{L_G^2}, G_1 = 1 + G_0 + \frac{2D_r + 8MD\alpha^2}{L_G} + 13C_1\delta^2 \alpha^4, G_2 = \frac{2MD}{L_G} + \frac{\delta^2}{2} + 5\) and \(C_1\) is given in Lemma 5.1.

Since \(D = \max_{x \in \mathbb{S}(d,r)} \|\nabla f(x)\|_F \leq \sqrt{T} \max_{x \in \mathbb{S}(d,r)} \|\nabla f(x)\|_2 = \sqrt{T}L_d\). By the choice of \(\alpha\), the constants in Lemma E.2 are given by \(G_0 = O(r^2C_1), G_1 = O(r^2C_1)\) and \(G_2 = O(M)\).

**Proof of Lemma E.2.** It follows from Lemma 2.4 that
\[
\|\hat{g}_k - \text{grad} f(\bar{x}_k)\|_F^2 \leq \frac{1}{n} \sum_{i=1}^n \|\text{grad} f_i(x_{i,k}) - \text{grad} f(\bar{x}_k)\|_F^2 \leq \frac{L_G^2}{n} \|\bar{x}_k - x_{k-1}\|_F^2. \tag{E.7}
\]
By invoking Lemma 2.4 and noting \(L_g \leq L_G\), we also have
\[
f(\bar{x}_{k+1}) \leq f(\bar{x}_k) + \langle \text{grad} f(\bar{x}_k), \bar{x}_{k+1} - \bar{x}_k \rangle + \frac{L_G}{2} \|\bar{x}_{k+1} - \bar{x}_k\|_F^2 \\
\leq f(\bar{x}_k) + \langle \hat{g}_k, \bar{x}_{k+1} - \bar{x}_k \rangle + \langle \text{grad} f(\bar{x}_k) - \hat{g}_k, \bar{x}_{k+1} - \bar{x}_k \rangle + \frac{L_G}{2} \|\bar{x}_{k+1} - \bar{x}_k\|_F^2 \\
\leq f(\bar{x}_k) + \langle \hat{g}_k, \bar{x}_{k+1} - \bar{x}_k \rangle + \frac{L_G}{n} \|\bar{x}_k - \bar{x}_{k-1}\|_F^2 + \frac{3L_G}{4} \|\bar{x}_{k+1} - \bar{x}_k\|_F^2 \\
\leq f(\bar{x}_k) + \langle \hat{g}_k, \bar{x}_{k+1} - \bar{x}_k \rangle + \frac{L_G}{n} \|\bar{x}_k - \bar{x}_{k-1}\|_F^2 + \frac{3L_G}{4} \|\bar{x}_{k+1} - \bar{x}_k\|_F^2 \\
= f(\bar{x}_k) + \langle \hat{g}_k, \bar{x}_{k+1} - \bar{x}_k \rangle + \langle \hat{g}_k, \bar{x}_{k+1} - \bar{x}_k \rangle + \frac{L_G}{n} \|\bar{x}_k - \bar{x}_{k-1}\|_F^2 + \frac{3L_G}{4} \|\bar{x}_{k+1} - \bar{x}_k\|_F^2. \tag{E.8}
\]
Note that for \(\beta > 0\), we have
\[
\langle \hat{g}_k, \bar{x}_{k+1} - \bar{x}_k \rangle \leq \frac{\beta^2 L_G}{2} \|\hat{g}_k\|_F^2 + \frac{1}{\beta^2 L_G} \|\bar{x}_k - \bar{x}_{k-1}\|_F^2 + \frac{1}{\beta^2 L_G} \|\bar{x}_{k+1} - \bar{x}_k\|_F^2.
\]
Plugging this into (E.8) yields
\[
f(\tilde{x}_{k+1}) \leq f(\bar{x}_k) + (\tilde{g}_k, \tilde{x}_{k+1} - \tilde{x}_k) + \frac{\beta^2 L_G}{2} \|\tilde{g}_k\|_F^2 + \frac{1}{\beta^2 L_G} (\|\bar{x}_k - \bar{x}_k\|_F^2 + \|\bar{x}_{k+1} - \bar{x}_{k+1}\|_F^2) \\
+ \frac{L_G}{n} \|\bar{x}_k - \bar{x}_k\|_F^2 + \frac{3L_G}{4} \|\bar{x}_{k+1} - \bar{x}_k\|_F^2. \tag{E.9}
\]

Firstly, we have
\[
b_1 = (\tilde{g}_k, \tilde{x}_{k+1} - \tilde{x}_k - \beta \tilde{g}_k + \beta \bar{g}_k)
= -\beta \|\tilde{g}_k\|_F^2 + \left(\tilde{g}_k, \frac{1}{n} \sum_{i=1}^n [x_{i,k+1} - (x_{i,k} - \beta v_{i,k} - \alpha g_i\varphi_i(x_k))]\right)
+ \left(\tilde{g}_k, \frac{1}{n} \sum_{i=1}^n [\beta (y_{i,k} - v_{i,k}) - \alpha g_i\varphi_i(x_k)]\right). \tag{E.10}
\]

Since \(y_{i,k} - v_{i,k} \in N_{\bar{x}_{i,k}} M\), it follows that
\[
\left\langle \tilde{g}_k, \frac{\beta}{n} \sum_{i=1}^n (y_{i,k} - v_{i,k}) - \alpha g_i\varphi_i(x_k) \right\rangle \\
\leq \frac{\beta}{n} \sum_{i=1}^n (\tilde{g}_k - g_i f_i(x_{i,k}, y_{i,k} - v_{i,k})) + \frac{2\alpha}{n} \|\tilde{g}_k\|_F \cdot \|x_k - \bar{x}_k\|_F^2 \\
\leq \frac{1}{4nL_G} \sum_{i=1}^n \|\tilde{g}_k - g_i f_i(x_{i,k})\|_F^2 + \frac{\beta^2 L_G}{n} \sum_{i=1}^n \|P_{N_{x_{i,k}} y_{i,k}}\|_F^2 + \frac{2\alpha D}{n} \|x_k - \bar{x}_k\|_F^2 \\
\leq \frac{1}{4n^2L_G} \sum_{i=1}^n \sum_{j=1}^n \|\tilde{g}_k - g_i f_i(x_{i,k})\|_F^2 + \frac{\beta^2 L_G}{n} \|y_k\|_F^2 + \frac{2\alpha D}{n} \|x_k - \bar{x}_k\|_F^2 \\
\leq \frac{L_G + 2\alpha D}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{\beta^2 L_G}{n} \|y_k\|_F^2,
\]
where we use Lemma 2.4 in the last inequality. This, together with (E.10) and (C.8) implies
\[
b_1 \leq -\beta \|\tilde{g}_k\|_F^2 + \frac{D}{n} \sum_{i=1}^n \|x_{i,k} - \alpha g_i\varphi_i(x_k) - \beta v_{i,k} - x_{i,k+1}\|_F + \frac{L_G + 2\alpha D}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{\beta^2 L_G}{n} \|y_k\|_F^2 \tag{P1}
\]
\[
\leq -\beta \|\tilde{g}_k\|_F^2 + \frac{MD\alpha^2}{n} \|\alpha g_i\varphi_i(x_k) + \beta v_{i,k} - x_{i,k+1}\|_F + \frac{L_G + 2\alpha D}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{\beta^2 L_G}{n} \|y_k\|_F^2 \tag{C.8}
\]
\[
\leq -\beta \|\tilde{g}_k\|_F^2 + \frac{2MD\alpha^2}{n} \|\alpha g_i\varphi_i(x_k)\|_F^2 + \frac{L_G + 2\alpha D}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{\beta^2 L_G}{n} \|y_k\|_F^2 \tag{C.9}
\]
where we use \(\|\tilde{g}_k\|_F \leq D\).

Secondly, we use the following inequality to derive the upper bound of \(b_2\). From Lemma 5.1, we have \(x_{k+1} \in N\). One has
\[
\|\tilde{x}_k - \tilde{x}_k\|_F^2 + \|\tilde{x}_{k+1} - \tilde{x}_{k+1}\|_F^2 \tag{P1}
\]
\[
\leq \frac{4\alpha}{n^2} (\|x_k - \bar{x}_k\|_F^2 + \|x_{k+1} - \bar{x}_{k+1}\|_F^2). \tag{E.12}
\]
We then obtain

\[ b_2 \leq \frac{4r}{n^2\beta_2^2 L_G} (\|x_k - \bar{x}_k\|_F^2 + \|x_{k+1} - \bar{x}_{k+1}\|_F^2). \]

Thirdly, invoking Lemma C.4 and \( \alpha \leq 1/M \) yields

\[
\|\bar{x}_k - \bar{x}_{k+1}\|_F \leq \frac{1}{1 - 2\delta_1^2} \left[ \frac{10\alpha}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{2M\beta^2}{n} \|y_k\|_F^2 + \|\gamma_k\|_F \right].
\]

Then, it follows from \( \beta \|y_k\|_F \leq \frac{\alpha\delta_1}{\alpha} \) that

\[
b_3 \leq \frac{3L_G}{4} \left( \frac{2}{1 - 2\delta_1^2} \left[ \frac{10\alpha}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{2M\beta^2}{n} \|y_k\|_F^2 \right]^2 + \frac{2}{(1 - 2\delta_1^2)^2} \beta^2 \|\gamma_k\|_F^2 \right) \leq \frac{3L_G}{1 - 2\delta_1^2} \left[ \frac{100\alpha^2}{n^2} \|x_k - \bar{x}_k\|_F^2 + \frac{(M\alpha\delta_1\beta)^2}{10n} \|y_k\|_F^2 \right] + \frac{3L_G}{1 - 2\delta_1^2} \beta^2 (\|\gamma_k\|_F^2 + \|\gamma_k - \hat{y}_k\|_F^2) \leq \frac{3L_G}{1 - 2\delta_1^2} \left[ \frac{100\alpha^2}{n^2} \|x_k - \bar{x}_k\|_F^2 + \frac{(M\alpha\delta_1\beta)^2}{10n} \|y_k\|_F^2 \right] + \frac{3L_G}{1 - 2\delta_1^2} \beta^2 (\|\gamma_k\|_F^2 + \frac{1}{n} \|y_k\|_F^2),
\]

where we use \( \hat{y}_k = \gamma_k \) and \( \|\gamma_k - \hat{y}_k\|_F^2 \leq \frac{1}{n} \|P_{N_{\delta_1}, k} y_i, k\|_F^2 \leq \frac{1}{n} \|y_k\|_F^2 \). It follows from (5.1) that

\[
\|x_k - \bar{x}_k\|_F^2 \leq C_1 (L_G + 2D)^2 \beta^2 \leq \frac{C_1 \alpha^2 \delta_1^2}{25},
\]

where we use \( \beta \leq \frac{\alpha\delta_1}{\alpha \alpha \delta_1 + 2D} \). Therefore, we get

\[
b_2 \leq \frac{4r(L_G + 2D)^2 C_1}{nL_G} (\|x_k - \bar{x}_k\|_F^2 + \|x_{k+1} - \bar{x}_{k+1}\|_F^2), \tag{E.13}
\]

and

\[
b_3 \leq \frac{3L_G}{1 - 2\delta_1^2} \left[ \frac{4C_1 \alpha^2 \delta_1^2}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{\delta_1^2}{10n} \beta^2 \|y_k\|_F^2 \right] + \frac{3L_G}{1 - 2\delta_1^2} \beta^2 (\|\gamma_k\|_F^2 + \frac{1}{n} \|y_k\|_F^2) \leq \frac{13L_G C_1 \alpha^2 \delta_1^2}{n} \|x_k - \bar{x}_k\|_F^2 + \frac{7}{2} L_G \beta^2 \|\gamma_k\|_F^2 + \frac{\delta_1^2}{n} + \frac{4}{n} L_G \beta^2 \|y_k\|_F^2, \tag{E.14}
\]

where we use \( \alpha \leq \frac{1}{M} \) and \( \frac{1}{(1 - 2\delta_1^2)^2} \leq 1.002 \). Therefore, by combining the upper bound of \( b_1, b_2, b_3 \) with (E.9) implies

\[
\begin{aligned}
f(\bar{x}_{k+1}) &\leq f(\bar{x}_k) + b_1 + \frac{\beta^2 L_G}{2} \|\hat{y}_k\|_F^2 + b_2 + \frac{L_G}{n} \|x_k - \bar{x}_k\|_F^2 + b_3 \\
&\leq f(\bar{x}_k) - (\beta - 4L_G\beta^2) \|\hat{y}_k\|_F^2 + \frac{L_G + 4r(L_G + 2D)^2 C_1}{nL_G} + 2D\alpha + 8MD\alpha^2 + 13L_G C_1 \alpha^2 \delta_1^2 \|x_k - \bar{x}_k\|_F^2 \\
&\quad + \frac{4r(L_G + 2D)^2 C_1}{nL_G} \|x_{k+1} - \bar{x}_{k+1}\|_F^2 + \frac{2MD + (\delta_1^2 + 5)L_G}{2} \beta^2 \|y_k\|_F^2.
\end{aligned}
\]

The proof is completed.

\[ \square \]

To proceed, we need the following recursive lemma, which is helpful to combine Lemma E.1 and Lemma E.2. It is a little different from the original one in [44]. We only change \( \sqrt{\sum_{t=0}^k u_t^2} \) and \( \sqrt{\sum_{t=0}^k w_t^2} \) to be \( \sum_{t=0}^k u_t^2 \) and \( \sum_{t=0}^k w_t^2 \).
Lemma E.3. [44, Lemma 2] Let \( \{u_k\}_{k \geq 0} \) and \( \{w_k\}_{k \geq 0} \) be two positive scalar sequences such that for all \( k \geq 0 \)

\[
u_{k+1} \leq \eta \nu_k + w_k,
\]

where \( \eta \in (0, 1) \) is the decaying factor. Let \( \Gamma(k) = \sum_{i=0}^k u_i^2 \) and \( \Omega(k) = \sum_{i=0}^k w_i^2 \). Then we have

\[
\Gamma(k) \leq c_0 \Omega(k) + c_1,
\]

where \( c_0 = \frac{2}{(1-\eta^2)} \) and \( c_1 = \frac{2}{1-\eta} \).  

Proof of Theorem 5.2. Applying Lemma E.3 to (E.4) yields

\[
\frac{1}{n} \sum_{k=0}^K \|x_k - \bar{x}_k\|_F^2 \leq \tilde{C}_0 \cdot \frac{\beta^2}{n} \sum_{k=0}^K \|y_k\|_F^2 + \tilde{C}_1, \tag{E.15}
\]

where \( \tilde{C}_0 = \frac{2}{(1-\rho^2)} \) and \( \tilde{C}_1 = \frac{2}{1-\rho^2} \frac{1}{n} \|x_0 - \bar{x}_0\|_F^2 \).

It follows from Lemma E.2 that

\[
f(\tilde{x}_{K+1}) \leq f(\tilde{x}_0) - (\beta - 4L_G \beta^2) \sum_{k=0}^K \|\tilde{g}_k\|_F^2 + \frac{G_1 L_G}{n} \sum_{k=0}^K \|x_k - \bar{x}_k\|_F^2 + \frac{G_0 L_G}{n} \sum_{k=1}^{K+1} \|x_k - \bar{x}_k\|_F^2 + \frac{G_2 L_G}{n} \beta^2 \sum_{k=0}^K \|y_k\|_F^2 \tag{E.15}
\]

\[
\leq f(\tilde{x}_0) - (\beta - 4L_G \beta^2) \sum_{k=0}^K \|\tilde{g}_k\|_F^2 + G_3 \frac{L_G \beta^2}{n} \sum_{k=0}^K \|y_k\|_F^2 + G_4 L_G, \tag{E.16}
\]

where we use \( \beta \leq \min\left\{ \frac{1}{\sqrt{\lambda_{G}}, \frac{\alpha \delta}{(L_G + 2D)^2}} \right\}, \beta^2 \|y_{K+1}\|_F^2 \leq n(L_G + 2D)^2 \beta^2 \leq \frac{4 \alpha^2 n}{25} \) and \( G_3 := G_1 \tilde{C}_0 + G_0 \tilde{C}_0 + G_2 \) and \( G_4 := \frac{\tilde{C}_0 \beta^2}{25} \tilde{C}_1 \).  

We are going to associate \( \|\tilde{g}_k\|_F^2 \) with \( \|y_k\|_F^2 \). By (E.5), we get

\[
- \frac{1}{n} \sum_{k=0}^K \|\tilde{g}_k\|_F^2 = - \frac{1}{n} \sum_{k=0}^K \|G_k\|_F^2 \leq \frac{1}{n} \sum_{k=0}^K \|y_k - \bar{G}_k\|_F^2 - \frac{1}{2n} \sum_{k=0}^K \|y_k\|_F^2 \tag{E.17}
\]

Again, applying Lemma E.3 to (E.3) yields

\[
\frac{1}{n} \sum_{k=0}^K \|y_k - \hat{G}_k\|_F^2 \leq \tilde{C}_2 \frac{1}{n} \sum_{k=0}^K \|G_{k+1} - \hat{G}_k\|_F^2 + \tilde{C}_3 \tag{E.2}
\]

\[
\leq \tilde{C}_2 \frac{1}{n} \sum_{k=0}^K (8\alpha^2 L_G^2 \|x_k - \bar{x}_k\|_F^2 + 2\beta^2 L_G^2 \|y_k\|_F^2) + \tilde{C}_3 \tag{E.15}
\]

\[
\leq \tilde{C}_2 \frac{1}{n} \sum_{k=0}^K (8\alpha^2 \tilde{C}_0 \tilde{C}_2 + 2\tilde{C}_2) L_G^2 \beta^2 \frac{1}{n} \sum_{k=0}^K \|y_k\|_F^2 + 8\alpha^2 \tilde{C}_1 \tilde{C}_2 L_G^2 + \tilde{C}_3 \tag{E.15}
\]

\[
\leq (8\tilde{C}_0 + \frac{1}{2} \tilde{C}_2) \alpha \delta_1 L_G \beta \frac{1}{n} \sum_{k=0}^K \|y_k\|_F^2 + 8\alpha^2 \tilde{C}_1 \tilde{C}_2 L_G^2 + \tilde{C}_3, \tag{E.15}
\]

where \( \tilde{C}_2 = \frac{2}{(1-\eta^2)} \) and \( \tilde{C}_3 = \frac{2}{1-\eta} \frac{1}{n} \|y_0 - \tilde{G}_0\|_F^2 \). The last line is due to \( \beta \leq \frac{\alpha \delta_1}{5L_G} \) and \( \alpha^2 \tilde{C}_2 \leq \tilde{C}_2 \leq \frac{2}{(1-\eta^2)} \leq 5 \). Plugging this into (E.17) implies

\[
- \frac{1}{n} \sum_{k=0}^K \|\tilde{g}_k\|_F^2 \leq \left( \frac{8\tilde{C}_0 + \frac{1}{2} \tilde{C}_2}{\tilde{C}_1} \alpha \delta_1 L_G \beta - \frac{1}{2} \right) \frac{1}{n} \sum_{k=0}^K \|y_k\|_F^2 + 8\alpha^2 \tilde{C}_1 \tilde{C}_2 L_G^2 + \tilde{C}_3. \tag{E.18}
\]
Hence, it follows from equation (E.16) that

$$f(\tilde{x}_{K+1})$$

(E.18)

$$\leq f(\bar{x}) - \frac{\beta}{2} \left( \frac{1}{2} - \left[ 2G_3 + (8\tilde{C}_0 + \frac{1}{2}\tilde{C}_2)\alpha\delta_1 \right] L_G\beta \right) \frac{1}{n} \sum_{k=0}^{K} \|y_k\|_F^2 + \frac{\beta}{2} \left( 8\alpha^2\tilde{C}_1\tilde{C}_2L_G^2 + \tilde{C}_3 \right) + G_4L_G$$

\[\leq f(\bar{x}) - \frac{\beta}{8} \frac{1}{n} \sum_{k=0}^{K} \|y_k\|_F^2 + \frac{\beta}{2} \left( 8\alpha^2\tilde{C}_1\tilde{C}_2L_G^2 + \tilde{C}_3 \right) + G_4L_G \]  

(E.19)

where the last inequality is due to $\beta \leq \frac{1}{4L_G(2G_3 + (8\tilde{C}_0 + \frac{1}{2}\tilde{C}_2)\alpha\delta_1)}$.

Then, we get

$$\frac{\beta}{8} \sum_{k=0}^{K} \|\hat{g}_k\|_F^2 \leq \frac{\beta}{8} \cdot \frac{1}{n} \sum_{k=0}^{K} \|y_k\|_F^2 \leq f(\bar{x}) - f^* + \tilde{C}_4 + G_4L_G,$$

(E.20)

where $\tilde{C}_4 = (8\alpha^2\tilde{C}_1\tilde{C}_2L_G^2 + \tilde{C}_3)\frac{\beta}{2} = O(\frac{r\delta^2}{(1-\sigma^2)^2})$ and $f^* = \min_{x \in \text{St}(d,r)} f(x)$. This implies

$$\min_{k=0,\ldots,K} \|\hat{g}_k\|_F^2 = \min_{k=0,\ldots,K} \|\hat{y}_k\|_F^2 \leq \min_{k=0,\ldots,K} \frac{1}{n} \|y_k\|_F^2 \leq \frac{8(f(\bar{x}) - f^* + \tilde{C}_4 + G_4L_G)}{\beta \cdot K}.$$  

(E.21)

It then follows from (E.15) that

$$\min_{k=0,\ldots,K} \frac{1}{n} \|x_k - \bar{x}_k\|_F^2 \leq \frac{8\beta(f(\bar{x}) - f^* + \tilde{C}_4 + G_4L_G)\tilde{C}_0 + \tilde{C}_1}{K}.$$  

Finally, noticing $\beta \leq \frac{\alpha\delta_1}{L_G}$ and

$$\|\nabla f(\bar{x}_k)\|_F^2 \leq 2\|\hat{g}_k\|_F^2 + 2\|\nabla f(\bar{x}_k) - \hat{g}_k\|_F^2 \leq 2\|\hat{g}_k\|_F^2 + \frac{2L_G^2}{n} \|x_k - \bar{x}_k\|_F^2.$$  

We finally have

$$\min_{k=0,\ldots,K} \|\nabla f(\bar{x}_k)\|_F^2 \leq \frac{(16 + \alpha^2\delta^2\tilde{C}_0)(f(\bar{x}) - f^* + \tilde{C}_4 + G_4L_G) + \tilde{C}_1L_G}{\beta \cdot K}.$$  

The proof is completed. \qed