On the empirical spectral distribution for certain models related to sample covariance matrices with different correlations

Alicja Dembczak-Kołodziejczyk, Anna Lytova

Abstract

Given \( n, m \in \mathbb{N} \), we study two classes of large random matrices of the form

\[
\mathcal{L}_n = \sum_{\alpha=1}^{m} \xi_{\alpha} y_{\alpha} y_{\alpha}^{T} \quad \text{and} \quad \mathcal{A}_n = \sum_{\alpha=1}^{m} \xi_{\alpha} (y_{\alpha} x_{\alpha}^{T} + x_{\alpha} y_{\alpha}^{T}),
\]

where for every \( n \), \( (\xi_{\alpha})_{\alpha} \subset \mathbb{R} \) are iid random variables independent of \((x_{\alpha}, y_{\alpha})_{\alpha} \), and \((x_{\alpha})_{\alpha}, (y_{\alpha})_{\alpha} \subset \mathbb{R}^n \) are two (not necessarily independent) sets of independent random vectors having different covariance matrices and generating well concentrated bilinear forms. We consider two main asymptotic regimes as \( n, m(n) \to \infty \): a standard one, where \( m/n \to c \), and a slightly modified one, where \( m/n \to \infty \) and \( \mathbb{E} \xi \to 0 \) while \( m \mathbb{E}\xi/n \to c \) for some \( c \geq 0 \). Assuming that vectors \((x_{\alpha})_{\alpha} \) and \((y_{\alpha})_{\alpha} \) are normalized and isotropic “in average”, we prove the convergence in probability of the empirical spectral distributions of \( \mathcal{L}_n \) and \( \mathcal{A}_n \) to a version of the Marchenko-Pastur law and so called effective medium spectral distribution, correspondingly. In particular, choosing normalized Rademacher random variables as \((\xi_{\alpha})_{\alpha} \), in the modified regime one can get a shifted semicircle and semicircle laws. We also apply our results to the certain classes of matrices having block structures, which were studied in \([9, 21]\).

1 Introduction

In \([9, 21]\), the authors studied the empirical spectral distributions of the following two related models of sparse block matrices. Given \( r, d \in \mathbb{N} \), let \((\mathbf{v}^{k\ell})_{1 \leq k < \ell \leq r} \) be independent copies of a random vector \( \mathbf{v} \) uniformly distributed on the unit sphere in \( \mathbb{R}^d \), and let \((\mathbf{e}_{k\ell})_{1 \leq k < \ell \leq r} \) be independent copies of a \( 0/1 \) random variable \( \xi = \xi_r \) such that \( \mathbb{E} \xi = p_r \) for some \( p_r \in (0, 1] \). Define \( \mathcal{A}_{rd} \) and \( \mathcal{L}_{rd} \) as \( rd \times rd \) block matrices of the form

\[
\mathcal{A}_{rd} = \left( (1 - \delta_{kl}) \mathbf{B}_{k\ell} \right)^{r}_{k,\ell=1}, \quad \mathcal{L}_{rd} = \left( \delta_{kl} \sum_{j \neq k} \mathbf{B}_{j\ell} \right)^{r}_{k,\ell=1} - \mathcal{A}_{rd}, \quad (1.1)
\]

where for \( 1 \leq k < l \leq r \) blocks

\[
\mathbf{B}_{k\ell} = \xi_{k\ell} \mathbf{v}^{k\ell} \mathbf{v}^{k\ell T} \quad (1.2)
\]

are \( d \times d \) rank-one matrices with probability \( p_r \) (and 0 otherwise). These models were introduced in \([9]\) while studying the elastic vibrational modes of amorphous solids. Roughly speaking they describe a system of \( r \) \( d \)-dimensional points connected by springs (see also \([15, 27]\) and references therein for the application of these models in the study of certain disordered systems). Evidently, for \( d = 1 \) matrices \( \mathcal{A}_{rd} \) and \( \mathcal{L}_{rd} \) reduce to the adjacency matrix and Laplacian of the Erdős–Rényi graph.

Recall that given a Hermitian or symmetric \( n \times n \) matrix \( \mathcal{M}_n \) with eigenvalues \((\lambda_i)_i \), the empirical spectral distribution \( N_{\mathcal{M}_n} \) of \( \mathcal{M}_n \) and its Stieltjes transform \( s_{\mathcal{M}_n} \) are defined by the formulas

\[
N_{\mathcal{M}_n}(\Delta) = \frac{|\{i : \lambda_i \in \Delta\}|}{n}, \quad \forall \Delta \subset \mathbb{R}, \quad \text{and} \quad s_{\mathcal{M}_n}(z) = \int \frac{N_{\mathcal{M}_n}(d\lambda)}{\lambda - z} = \frac{1}{n} \text{Tr}(\mathcal{M}_n - z)^{-1}, \quad \forall z \neq 0.
\]
In [9, 21], the authors studied the empirical spectral distributions $N_{A_r}$ and $N_{L_r}$ as $r \to \infty$ in different asymptotic regimes, depending on $d$ and $p_r$. In the case when $d$ and $rp_r$ are some fixed numbers (sparse matrices), the first several moments of the limiting distributions were computed. In the “dense” regime when
\[ d \to \infty, \quad p_r = O(1), \quad \text{and} \quad p_r r/d \to c > 0, \quad \text{as} \quad r \to \infty, \]
the convergence in mean of $N_{L_r}$ and $N_{A_r}$ to the Marchenko-Pastur law and to the so called effective medium spectral distribution, correspondingly, was proved. (Here we follow terminology from [9, see also 23].) It was shown that the limits of the corresponding Stieltjes transforms, $f_L := \lim_{r \to \infty} E_s L_r$ and $f_A := \lim_{r \to \infty} E s_{A_r}$, satisfy equations
\[ 2z f^2_L + (z + 2 - c) f_L + 1 = 0 \quad \text{and} \quad zf^2_A + (1 - c)f^2_A - zf_A - 1 = 0. \tag{1.3} \]
Also the first several moments of the limiting distributions were computed. In the “dense” regime, the first several moments of the empirical spectral distributions $N_{L_r}$ and $N_{A_r}$ coincide with the moments of the semicircle and shifted semicircle distributions and, in particular,
\[ 2f^2_{L_r} + (z - c_r) f_{L_r} + 1 = 0 \quad \text{and} \quad c_r f^2_{A_r} + z f_{A_r} + 1 = 0, \tag{1.4} \]
where $c_r := rp_r/d \to \infty$, as $r \to \infty$.

In our work we suppose that $d$ grows to infinity with $r$. We modify the dilute regime as follows: $p_r \to 0$ and $r/d \to \infty$ while $c_r = rp_r/d \to c \geq 0$ as $r, d \to \infty$. This guarantees that the corresponding sequences of empirical spectral distributions $N_{L_r}$ and $N_{A_r}$ are tight and also allows to unify two regimes as follows:
\[ d \to \infty \quad \text{and} \quad rp_r/d \to c \geq 0 \quad \text{as} \quad r \to \infty. \tag{1.5} \]

We consider models (1.1) corresponding to normalized isotropic vectors $(v^{k\ell})_{k<\ell}$ which generate well-concentrated bilinear forms (see Assumption 1), and applying the Stieltjes transform method give a straightforward proof of the convergence in probability of $N_{L_r}$ and $N_{A_r}$ to the Marchenko-Pastur law and the effective medium spectral distribution. We show that the limits are the same for both regimes (in contrast to (1.3) and (1.4)), and that to get (1.4) with $c_r = c$ in the new dilute regime, $(\xi^{k\ell})_{k<\ell}$ needs to take both negative and positive values with non-zero probability. In particular, one can get (1.4) if $(\xi^{k\ell})_{k<\ell}$ are properly normalized Rademacher random variables (see Example 3 and Remark 1.6).

Note that we can rewrite matrices $A_r$ and $L_r$ in the form
\[ L_r = \sum_{1 \leq k < \ell \leq r} \xi_{k\ell} Y^{k\ell} Y^{k\ell T} \quad \text{and} \quad A_r = \sum_{1 \leq k \neq \ell \leq r} \xi_{k\ell} X^{k\ell} X^{k\ell T}, \tag{1.6} \]
where $\xi_{k\ell} = \xi_{\ell k}$, $(X^{k\ell})_{k \neq \ell}$ and $(Y^{k\ell})_{k < \ell}$ are sparse block vectors in $\mathbb{R}^{rd}$ given by
\[ Y^{k\ell} = (\delta_{jk} - \delta_{j\ell}) v^{k\ell j}_{r} \quad \text{and} \quad X^{k\ell} = (\delta_{jk} v^{k\ell j}_{r})_{j=1}. \]

This naturally leads to the study of more general classes of random matrices of the form
\[ L_n = \sum_{\alpha} \xi_{\alpha} Y_\alpha Y_\alpha^T \quad \text{and} \quad A_n = \sum_{\alpha} \xi_{\alpha} (y_\alpha x_\alpha^T + x_\alpha y_\alpha^T), \tag{1.7} \]
where $\xi_{\alpha} \in \mathbb{R}$ and $y_\alpha, x_\alpha \in \mathbb{R}^n$, $\alpha \leq m$, are some random variables and vectors. We mainly concentrate on model $L_n$, which is closely related to the sample covariance matrices, only that here we allow vectors $(y_\alpha)_\alpha$ to have different covariance matrices $Q_\alpha := E y_\alpha y_\alpha^T$, $\alpha \leq m$ (note that here $Q_\alpha$ are not necessarily centered). We suppose that these vectors are normalized and isotropic "in average", $m^{-1} \sum_{\alpha} Q_\alpha \approx n^{-1} I_n$, which allows to show that the empirical spectral distributions still converge to the Marchenko-Pastur
law. A similar model was considered in [25] and [16], where the convergence of spectral distributions was studied, in particular, for matrices of the form $\sum_{\alpha} y_{\alpha} y_{\alpha}^T$, corresponding to vectors with essentially different covariance matrices (not isotropic in average). In these papers the limiting distribution is given implicitly (in terms of asymptotic closeness to the solution of a certain system of equations), and our result does not follow directly from [25, 16]. Certain closely related models were also studied in [8], where the authors proved the convergence to the Marchenko-Pastur law of empirical spectral measures corresponding to the certain block-independent models and tensor models (in [8], see also a review of known results on convergence to the Marchenko-Pastur law with relaxed independence requirements including [18, 26, 14, 11, 22, 13, 24]).

In (1.6), we choose $(y_{\alpha})_{\alpha}$ and $(\xi_{\alpha})_{\alpha}$ from the following classes.

**Assumption 1.** We suppose that for every $n \in \mathbb{N}$, $y_{\alpha} = y_{\alpha,n} \in \mathbb{R}^n$, $\alpha \leq m$, are mutually independent random vectors such that for all deterministic matrices $D = D_n$ with $\|D\|_{op} = 1$ we have

$$\sup_{\alpha} \text{Var}(Dy_{\alpha}, y_{\alpha}) = o(1), \quad n \to \infty. \tag{1.8}$$

Here and in what follows, given a matrix $D$ we use notations $\|D\|_{op}$ and $\|D\|_{HS}$ for its operator and Hilbert-Schmidt norms.

**Assumption 2.** For every $n \in \mathbb{N}$, let $\xi_{\alpha} = \xi_{\alpha,n} \in \mathbb{R}$, $\alpha \leq m$, be mutually independent copies of a random variable $\xi_n$ with a cumulative distribution function $\sigma_n$. To treat simultaneously both cases, $m/n = O(1)$ and $m/n \to \infty$, $\frac{m}{n}E\xi_n = O(1)$, we introduce a signed measure $\tilde{\sigma}_n$, which controls $\frac{n}{m}E\xi_n$. Let $\tilde{\sigma}_n$ be defined as follows: for every finite $\Delta \subset \mathbb{R}$

$$\tilde{\sigma}_n(\Delta) = \frac{m}{n} \int_{\Delta} \xi d\sigma_n(\xi).$$

We suppose that as $n \to \infty$, $\tilde{\sigma}_n$ converges weakly to a signed measure $\tilde{\sigma}$ such that $|\tilde{\sigma}(\mathbb{R})| < \infty$, and

$$\sup_n \int |\xi^p d\tilde{\sigma}_n(\xi)| = \sup_n \frac{m}{n}E|\xi_n|^{p+1} < \infty, \quad p = 1, 2, 3.$$

We use notation $c_1 := \tilde{\sigma}(\mathbb{R}) \in \mathbb{R}$, $|c_1| < \infty$.

**Remark 1.1.** Note that if $m/n \to c > 0$, then $\tilde{\sigma}_n = c\sigma_n + o(1)$ as $n \to \infty$. Also, a bit more delicate but quite standard nowadays argument based on a truncation procedure for $\xi_n$ (see, for example, [19]) allows to show that the results of Theorem 1.2 below remain valid without any moment conditions on $\sigma_n$ in the case $m/n = O(1)$ and with the only moment condition $\frac{m}{n}E\xi_n = O(1)$ in the case $m/n \to \infty$, $m/n^2 \to 0$.

Our main result concerns convergence of the empirical spectral distributions of $(\mathcal{L}_n)_n$, it can be considered as a generalization of Theorem 3.3 of [19] (for $H^{(0)} = 0$) on the case of "samples" with entries having different covariance matrices.

**Theorem 1.2.** Given $m, n \in \mathbb{N}$, consider $n \times n$ matrices

$$\mathcal{L}_n = \sum_{\alpha} \xi_{\alpha}y_{\alpha}y_{\alpha}^T,$$

where $\xi_{\alpha}$ and $y_{\alpha} \in \mathbb{R}^n$, $\alpha \leq m$, are mutually independent random variables satisfying Assumptions 1, 2. Let $Q_{\alpha} := Ey_{\alpha}y_{\alpha}^T$ be such that

$$\sup_{\alpha} \|Q_{\alpha}\|_{op} = O(n^{-1}), \quad \sup_{\alpha} |\text{Tr} Q_{\alpha} - 1| = o(1), \quad \text{and} \tag{1.9}$$

$$Q_{(n)} := \frac{1}{m} \sum_{\alpha} Q_{\alpha} = \frac{1}{n} I_n + B_n, \quad \text{where} \quad \|B_n\|_{HS} = o(n^{-1/2}), \quad n \to \infty. \tag{1.10}$$
Then as \( n \to \infty \) the empirical spectral distributions \( N_{\xi_n} \) converge in probability to a non-random probability measure \( N_{\xi} \) which Stieltjes transform \( f \) is uniquely determined by the equation

\[
z f(z) = -1 + f(z) \int \frac{d\sigma(\xi)}{1 + \xi f(z)}
\]  

(1.11)
in the class of Stieltjes transforms of non-negative measures.

**Remark 1.3.** A simple renormalization allows to show that if in (1.9) and (1.10) we have

\[
\sup_\alpha |\text{Tr} Q_\alpha - a| = o(1) \quad \text{and} \quad Q_{(n)} = \frac{a}{n} I_n + B_n \quad \text{for some} \quad a > 0,
\]

then \( z f = -1 + a f (1 + a \xi f)^{-1} d\sigma(\xi) \). Also in the case \( d\sigma(\xi) = c \xi d\sigma(\xi) \) we restore the Marchenko-Pastur distribution.

Some additional information about the moments of \( \xi_\alpha \) allows to solve (1.11) exactly, here are several simple examples:

**Examples.** In the following three examples we use notations \( \delta \) and \( \delta' \) for the Dirac delta function and its generalized derivative, and \( \rho, \nu, \nu' \) for the densities of \( N, \sigma, \sigma' \), correspondingly.

1. Let \( (\xi_n)_n \) be not random, and \( \xi_\alpha, n = b_n \to b, m b_n / n \to c_1 \) as \( n \to \infty \). Then

\[
\nu_n = \frac{m}{n} \delta(\xi - b_n) \to \nu = c_1 \delta(\xi - b),
\]

and by (1.11) \( f \) satisfies \( b z f^2 + f(z + b - c_1) + 1 = 0 \), so that

\[
\rho(\lambda) = \begin{cases} 
\frac{1}{2 \pi b \lambda} \sqrt{(c_1 - \lambda)(\lambda - c_2)} & \text{if} \quad b \neq 0,
\delta(\lambda - c_1) & \text{if} \quad b = 0,
\end{cases}
\]  

(1.12)

where \( x_+ = x \) if \( x \geq 0 \) and \( 0 \) otherwise and \( c^\pm = (\sqrt{b} \pm \sqrt{c_1})^2 \).

2. Let for every \( n \), \( (\xi_n)_n \) are \( 0/1 \) random variables such that \( P(\xi_n = 1) m / n \to c_1 \) as \( n \to \infty \). Then again \( \nu = c_1 \delta(\xi - 1) \), and we get (1.12) with \( b = 1 \).

3. Suppose that all moments of \( \sigma \) are finite,

\[
c_j := \int \xi^{j-1} d\sigma(\xi) = \lim_{n \to \infty} \frac{m}{n} E_{\xi_n}^{j} < \infty, \quad \forall j \geq 1,
\]

and for some \( k_0 \geq 1 \) we have \( c_j = 0 \) \( \forall j > k_0 \). Note that this is possible only if \( k_0 \leq 2 \), and moreover, for \( k_0 = 2 \) the condition \( 0 < c_2 < \infty \) while \( c_3 = 0 \) is not fulfilled for pure non-negative (or pure non-positive) random variables. Indeed, if \( \xi_n \geq 0 \) a.s. then by the Schwartz inequality we would have

\[
0 < c_2 = \lim \frac{m}{n} E_{\xi_n}^{2} \leq \lim \frac{m}{n} (E_{\xi_n}^{3} E_{\xi_n}^{1})^{1/2} = c_1 c_3 = 0.
\]

So let \( \xi_n \) take both negative and positive values with positive probability, and \( c_j = 0 \) \( \forall j \geq 3 \). Then expanding \( (1 + \xi f(z))^{-1} \) into the Taylor’s series we get from (1.11) \( c_2 f^2 + (c_1 - z) f + 1 = 0 \), thus in this case the limiting density is given by the shifted semicircle law,

\[
\rho(\lambda) = \frac{1}{2 \pi c_2} \sqrt{(4 c_2 - (\lambda - c_1)^2)}_+.
\]

For example, if \( (\xi_n)_n \) take values \( \pm \sqrt{n / m} \) with probability \( 1/2 \), than \( \nu_n \to \nu = \delta' \), \( c_2 = 1, c_j = 0, j \neq 2 \), and \( \rho(\lambda) = \frac{1}{2 \pi} \sqrt{(4 - \lambda^2)}_+ \).
Remark 1.6.\hspace{1em} Theorem 1.5.\hspace{1em} transforms of non-negative measures. \hspace{1em} Let\hspace{1em} A_{rd} \hspace{1em} be defined in (1.1) - (1.2), where for every \( r \in \mathbb{N} \), \((\xi_{kl})_{1 \leq k < l \leq r}\) are iid copies of a 0/1 random variable \( \xi = \xi_r \) with \( \mathbb{P}(\xi = 1) = p_r \), and \((v^{kl})_{1 \leq k < l \leq r}\) are mutually independent normalized isotropic random vectors, \( \mathbb{E}v^{kl}v^{k'l'} = d^{-1}I_d \), satisfying Assumption 1 and having norms uniformly bounded in \( r \). Then in regime (1.3), \( N_{\mathcal{L}_{rd}} \) converge in probability to a non-random probability measure \( N_{\mathcal{L}} \) with the density
\[
\rho(\lambda) = \frac{1}{4\pi\lambda} \sqrt{((c_+ - \lambda)(\lambda - c_-))_+}, \quad c_\pm = (\sqrt{2} \pm \sqrt{c})^2.
\]
As to adjacency matrices \( A_{rd} \), in Section 4 we first treat matrices having a more general structure and prove an analog of Theorem 1.2 for matrix \( A_n \) defined in (1.7) (see Theorem 1.5). Then using essentially the same scheme we get the following result for \( A_{rd} \):

Theorem 1.5. Let \( A_{rd} \) be defined in (1.1) - (1.2), where for every \( r \in \mathbb{N} \), \((\xi_{kl})_{1 \leq k < l \leq r}\) are iid copies of a 0/1 random variable \( \xi = \xi_r \) with \( \mathbb{P}(\xi = 1) = p_r \), and \((v^{kl})_{1 \leq k < l \leq r}\) are mutually independent normalized isotropic random vectors, \( \mathbb{E}v^{kl}v^{k'l'} = d^{-1}I_d \), satisfying Assumption 1 and having norms uniformly bounded in \( r \). Then in regime (1.3), \( N_{\mathcal{L}_{rd}} \) converge in probability to a non-random probability measure \( N_{\mathcal{L}} \) which Stieltjes transform \( f_{\mathcal{L}} \) is uniquely determined by the second equation in (1.3) in the class of Stieltjes transforms of non-negative measures.

Remark 1.6. 1. One can find the explicit forms of the solution of the cubic equation (1.3) and the density of \( N_{\mathcal{L}} \) in [23] and [9].

2. It can be shown that if \((\xi_{kl})_{k \neq l}\) take values \( \pm \sqrt{d/r} \) with probability 1/2, then \( f_{\mathcal{L}} \) solves equation \( f_{\mathcal{L}}^2 + zf_{\mathcal{L}} + 1 = 0 \) (cf (1.4)), so that the limiting density is given by the semicircle law \( \rho(\lambda) = \frac{1}{2\pi \sqrt{4 - \lambda^2}} \) (see also Remark 1.2)

The structure of the remaining part of the paper is very simple: in Sections 2, 3 and 4 we give the proofs of Theorems 1.1, 1.4 and 1.5 correspondingly. The proof of Theorem 1.2 (based on [19]) is more detailed, while in the rest of the proofs we mostly discuss places which should be modified.

Acknowledgments A.L. was supported by grant nr 2018/31/B/ST1/03937 National Science Centre, Poland. A.L. also would like to thank the organizers of XV Brunel – Bielefeld Workshop on Random Matrix Theory and Applications for the excellent conditions and Prof. Cicuta for the introducing to the problem during this workshop.

2 Proof of Theorem 1.2

The proof is based on the standard nowadays method of Stieltjes transform which goes back to [18] (see [2, 3, 5, 20] for the details of the method and main properties of the Stieltjes transform), and which is used in a huge number of results on convergence of empirical spectral distributions of random matrices. This method is based on the fact that there is a one-to-one continuous correspondence between non-negative measures and their Stieltjes transforms, so that to find a weak limit in probability of random probability measures \( N_{\mathcal{L}_n} \) it is enough to show that for every \( z \in \mathbb{C} \setminus \mathbb{R} \) the Stieltjes transforms \( s_n := s_{\mathcal{L}_n} \) of \( N_{\mathcal{L}_n} \) converge in probability to a deterministic limit \( f \) satisfying \( \lim_{n \to \infty} \eta |f(\eta)| = 1 \). Then \( f \) is the Stieltjes
transform of a probability measure \( N \) such that \( N_{\mathcal{L}_n} \) converge weakly in probability to \( N \) and for every \( \Delta \subset \mathbb{R} \)

\[ N(\Delta) = \frac{1}{\pi} \lim_{\eta \to +0} \int_\Delta f(\lambda + i\eta)d\lambda. \]

Our scheme of the proof is as follows: in Lemma 2.2 we show that \( \text{Var}_{s_n}(z) = o(1) \) as \( n \to \infty \), that reduces the problem to finding the limit of the expectations \( E_{s_n} := f_n \), then in the main body of the proof (Lemma 2.3) we show that for every convergent subsequence of \( (f_n) \), its limit satisfies \( (1.11) \), and finally, the unique solvability of \( (1.11) \) in the class of the Stieltjes transforms of probability measures follows from Lemma 2.4 below.

**Lemma 2.1. (Solvability and uniqueness).** Let \( \bar{\sigma} \) be a signed measure defined in Assumption 1. Then there is a unique solution \( f \) of \( (1.11) \) in the class of Stieltjes transforms of the non-negative measures. Moreover, \( \lim_{\eta \to \infty} \eta|f(i\eta)| = 1 \), so that the corresponding to \( f \) measure \( N \) is a probability measure, \( N(\mathbb{R}) = 1 \).

**Proof.** We show first that if \( f \) is the Stieltjes transform of a non-negative measure \( N \) then for any \( \xi \in \mathbb{R} \)

\[ |1 + \xi f(z)|^{-1} \leq \max\{2, 4|\xi|/|\Im z|\}. \]  

(2.1)

(Note also that for \( \xi > 0 \) we have a simpler bound \( |1 + \xi f(z)|^{-1} \leq |z|/|\Im z| \), which follows from the inequality \( \Im z \Im f(z) \geq 0 \).) To this end given \( \xi \in \mathbb{R} \) define

\[ E_\xi := \{ z : |\xi||f(z)| = |\xi| \int (\lambda - z)^{-1}dN(\lambda) < 1/2 \}. \]

If \( z \in E_\xi \), we have \( |1 + \xi f(z)| > 1/2 \). If \( z \notin E_\xi \), by the Schwartz inequality \( \int |\lambda - z|^{-2}dN \geq 1/(4|\xi|)^2 \)

so that \( |1 + \xi f(z)| \geq |\xi|/|\Im z| \int |\lambda - z|^{-2}dN \geq |\Im z|/(4|\xi|) \), and (2.1) follows.

By the conditions of the lemma we have

\[ \int |d\bar{\sigma}(\xi)| = c_1 < \infty \quad \text{and} \quad \int |\xi^p d\bar{\sigma}(\xi)| < \infty, \quad p = 1, 2, 3. \]  

(2.2)

Note that \( \int |d\bar{\sigma}(\xi)| \) is not necessarily finite, that is why it is better to rewrite (1.11) in the form

\[ zf(z) = -1 + c_1 f(z) - f(z)^2 \int \frac{\xi d\bar{\sigma}(\xi)}{1 + \xi f(z)}, \]

where now by (2.1) – (2.2), \( \int |\xi(1 + \xi f(z))^{-1}d\bar{\sigma}(\xi)| \) is uniformly bounded in

\[ z \in C_{\eta_0} := \{ z \in \mathbb{C} : \eta = \Im z \geq \eta_0 \} \]  

(2.3)

for some \( \eta_0 > 0 \). In particular this allows to show that \( \lim_{\eta \to \infty} \eta|f(i\eta)| = 1 \). Next, if there are two solutions \( f_1, f_2 \) of this equation, than

\[ z(f_1 - f_2) = (f_1 - f_2)(c_1 - \int \frac{\xi(f_1 + f_2 + \xi f_1 f_2)d\bar{\sigma}(\xi)}{1 + \xi f_1(1 + \xi f_2)}), \]

where as it follows from (2.1) – (2.2), if \( f_1 \neq f_2 \) then the r.h.s. is uniformly bounded and the l.h.s. tends to infinity as \( z \to \infty \). Hence \( f_1 = f_2 \). The solvability of (1.11) in the class of Stieltjes transforms of the non-negative measures follows from the Banach fixed-point theorem.

Let \( G(z) := (\mathcal{L}_n - zI_n)^{-1}, z \in \mathbb{C} \setminus \mathbb{R} \), be the resolvent of \( \mathcal{L}_n \), so that \( s_n = n^{-1} \text{Tr} \, G \). Lemma 2.2 below shows that the variance of \( s_n \) tends to zero as \( n \to \infty \), hence, by Chebyshev’s inequality the convergence of \( (s_n)_n \) in probability follows from the convergence in means.
Lemma 2.2. (Self-averaging properties.) Under conditions of Theorem 1.2 we have uniformly in \( z \in \mathbb{C}_{\eta_0} \) for big enough \( \eta_0 \)

\[
(i) \quad \text{Var} n^{-1} \text{Tr} G(z) = O(n^{-1}), \quad \text{and} \\
(ii) \quad \sup_\beta \text{Var} \text{Tr} Q_\beta G(z) = O(n^{-1}), \quad n \to \infty.
\]  

(2.4) (2.5)

Proof. Our proof is based on a standard martingale technique introduced in random matrix theory by Girko (see [10] and [20]). For every \( 1 \leq \alpha \leq m \), introduce

\[
\mathcal{L}_n^\alpha = \mathcal{L}_n - \xi_\alpha y_\alpha y_\alpha^T \quad \text{and} \quad G_\alpha^\alpha(z) = (\mathcal{L}_n^\alpha - z I_n)^{-1},
\]

so that \( \mathcal{L}_n^\alpha, G_\alpha^\alpha \) do not depend on \( \xi_\alpha \) and \( y_\alpha \). Applying the result of [10], one can get

\[
\text{Var} n^{-1} \text{Tr} G(z) \leq \frac{1}{n^2} \sum_\alpha \mathbf{E} \left| \text{Tr}(G - E_\alpha G) \right|^2 \leq \frac{4}{n^2} \sum_\alpha \mathbf{E} \left| \text{Tr}(G - G^\alpha) \right|^2,
\]

(see also Lemma 3.2 [17]), where by the resolvent identity

\[
\mathbf{E} \left| \text{Tr}(G - G^\alpha) \right|^2 = \mathbf{E} \left| \xi_\alpha (G^\alpha G y_\alpha, y_\alpha) \right|^2 \leq \mathbf{E} |\xi_\alpha|^2 \mathbf{E} \|y_\alpha\|_2^4 / \eta_0^4,
\]

and (2.4) follows. Here we also used that as it follows from Assumptions 1, 2, \( \mathbf{E} |\xi_\alpha|^2 \) and \( \mathbf{E} \|y_\alpha\|_2^4 \) are bounded. Similarly we have

\[
\text{Var} \text{Tr} Q_\beta G(z) \leq 4 \sum_\alpha \mathbf{E} \left| \text{Tr} Q_\beta (G - G^\alpha) \right|^2 \leq 4 \sum_\alpha \mathbf{E} |\xi_\alpha|^2 \mathbf{E} \|y_\alpha\|_2^4 \|Q_\beta\|_\infty^2 / \eta_0^4,
\]

(2.6)

and by (1.9) we get (2.5). \( \square \)

As it follows from Lemma 2.2 (i), it remains to show that for every \( z \in \mathbb{C} \setminus \mathbb{R} \) the expectations of \( s_n \) converge to \( f \) which solves (1.11). Then since \( f \) is the Stieltjes transform of a non-negative measure (which is in fact a probability measure due to the tightness of \( \mathbf{E} \mathcal{L}_n \)), Lemma 2.1 finishes the proof of Theorem 1.2.

Lemma 2.3. (Convergence in mean.) Let \( f_n := \mathbf{E}s_n = n^{-1} \mathbf{E} \text{Tr} G \). Then for every \( z \in \mathbb{C} \setminus \mathbb{R} \) there exists \( \lim_{n \to \infty} f_n(z) =: f(z) \), and \( f \) satisfies (1.11).

Proof. Since \( |f_n(z)| \leq |3z|^{-1} \), there is a subsequence \( (f_{n_j})_{n_j} \) and an analytic function \( f(z), z \in \mathbb{C} \setminus \mathbb{R} \), such that \( (f_{n_j})_{n_j} \) converges to \( f \) uniformly on every compact set in \( \mathbb{C} \setminus \mathbb{R} \). Due to the uniqueness property of analytic functions it suffices to consider domain \( \mathbb{C}_{\eta_0} \) for some fixed \( \eta_0 > 0 \) which will be chosen later, and to show that every convergent subsequence converges in \( \mathbb{C}_{\eta_0} \) to a solution \( f \) of (1.11).

Saving notation \( (f_n)_n \) for a convergent subsequence, applying the resolvent identity, \( zG = -1 + GL_n \), and a rank-one perturbation formula

\[
G - G^\alpha = -\frac{\xi_\alpha G^\alpha y_\alpha y_\alpha^T G^\alpha}{1 + \xi_\alpha (G^\alpha y_\alpha, y_\alpha)},
\]

(2.7)

we get

\[
z f_n + 1 = \frac{1}{n} \sum_\alpha \mathbf{E} \xi_\alpha (G y_\alpha, y_\alpha) = \frac{1}{n} \sum_\alpha \mathbf{E} \frac{\xi_\alpha (G^\alpha y_\alpha, y_\alpha)}{1 + \xi_\alpha (G^\alpha y_\alpha, y_\alpha)}
\]

\[
= \int \frac{1}{m} \sum_\alpha \mathbf{E} \frac{(G^\alpha y_\alpha, y_\alpha)}{1 + \xi (G^\alpha y_\alpha, y_\alpha)} d\sigma_n(\xi).
\]

(2.8)
By the conditions of the theorem we have
\[
\mathbf{E}(G^\alpha y_\alpha, y_\alpha) = \mathbf{E} \text{Tr} Q_\alpha G^\alpha, \\
\text{Var}_\alpha(G^\alpha y_\alpha, y_\alpha) = o(1), \quad n \to \infty, 
\]
where in the first equality we can replace \(G^\alpha\) with \(G\). Indeed, by the resolvent identity and (1.9) we have
\[
|\mathbf{E} \text{Tr} Q_\alpha(G^\alpha - G)| = |\mathbf{E} \xi_\alpha(GQ_\alpha G^\alpha y_\alpha, y_\alpha)| \leq \|Q_\alpha\|_{op} |\xi| \text{ Tr} Q_\alpha/\eta_0^2 = O(n^{-1}),
\]
(more precisely, \(\int |\mathbf{E} \text{Tr} Q_\alpha(G^\alpha - G)d\tilde{\sigma}_n(\xi)| = o(1)\)), and we also used that \(\mathbf{E}\|y_\alpha\|^2 = \text{Tr} Q_\alpha\). Hence, introducing
\[
f_{n,\alpha} := \mathbf{E} \text{Tr} Q_\alpha,
\]
and applying (1.10) we get
\[
\mathbf{E}(G^\alpha y_\alpha, y_\alpha) = f_{n,\alpha} + O(n^{-1}) \quad \text{and} \quad f_n = \frac{1}{m} \sum_{\alpha} f_{n,\alpha} + o(1).
\]
Using the above equalities we get
\[
\frac{1}{1 + \xi(G^\alpha y_\alpha, y_\alpha)} = \frac{1}{1 + \xi f_n} \left(1 + \frac{\xi(f_n - f_{n,\alpha})}{1 + \xi(G^\alpha y_\alpha, y_\alpha)} - \frac{\xi(G^\alpha y_\alpha, y_\alpha)^o}{1 + \xi(G^\alpha y_\alpha, y_\alpha)}\right) + o(1),
\]
where \(x^o = x - \mathbf{E}x\). This and (2.8) yield
\[
z f_n + 1 = \int \frac{f_n}{1 + \xi f_n} d\tilde{\sigma}_n(\xi) + R_n + R_n' + o(1),
\]
\[
R_n = \int \frac{1}{1 + \xi f_n} \frac{1}{m} \sum_{\alpha} \mathbf{E} \frac{G^\alpha y_\alpha, y_\alpha)^o}{1 + \xi(G^\alpha y_\alpha, y_\alpha)} d\tilde{\sigma}_n(\xi),
\]
\[
R_n' = \int \frac{1}{1 + \xi f_n} \frac{1}{m} \sum_{\alpha} \mathbf{E} \frac{f_{n,\alpha} - f_n}{1 + \xi(G^\alpha y_\alpha, y_\alpha)} d\tilde{\sigma}_n(\xi).
\]
By the Schwartz inequality \(\mathbf{E}|(G^\alpha y_\alpha, y_\alpha)^o| \leq (\text{Var}(G^\alpha y_\alpha, y_\alpha))^{1/2}\), where by (2.5) and (2.12)
\[
\text{Var}(G^\alpha y_\alpha, y_\alpha) = E\text{Var}_\alpha(G^\alpha y_\alpha, y_\alpha) + \text{Var} \text{Tr} G^\alpha Q_\alpha = o(1).
\]
Note that by (2.1),
\[
|1 + \xi f_n(z)|^{-1} \leq \max \{2, 4|\xi|/|3z|\}.
\]
Also it follows from (2.7) that \((1 + \xi_\alpha(G^\alpha y_\alpha, y_\alpha))^{-1} = 1 - \xi_\alpha(Gy_\alpha, y_\alpha)\), hence,
\[
\frac{1}{|1 + \xi(G^\alpha y_\alpha, y_\alpha)|} \leq 1 + |\xi|\|y_\alpha\|_2^2/\eta_0.
\]
This and Assumption 2 allow to get
\[
|R_n| = o(1) \quad \text{and} \quad |R_n'| \leq \frac{C}{n} \sum_{\alpha} |f_{n,\alpha} - f_n| \leq C \Delta_n(z), \quad \text{where} \quad \Delta_n(z) = \max_{\alpha} |f_{n,\alpha}(z) - f_n(z)|
\]
(2.13)
and \( C > 0 \) depends only on \( \eta_0 \). To finish the proof it remains to show that \( \Delta_n = o(1) \). Repeating all the steps leading to (2.11) - (2.13) one can get

\[
z f_{n,\alpha} + \text{Tr} Q_\alpha = \sum_{\alpha} E \frac{\xi_\alpha (Q_\alpha G^{\alpha} y_\alpha, y_\alpha)}{1 + \xi_\alpha (G^{\alpha} y_\alpha, y_\alpha)} = \int \frac{n}{m} \sum_{\alpha} E \frac{(Q_\alpha G^{\alpha} y_\alpha, y_\alpha)}{1 + \xi(G^{\alpha} y_\alpha, y_\alpha)} d\sigma_n(\xi) = \int \frac{f_{n,\alpha}}{1 + \xi f_n} d\sigma_n(\xi) + R_{n,\alpha} + R'_{n,\alpha} + o(1),
\]

where

\[
R_{n,\alpha} = -\int \frac{1}{1 + \xi f_n} \frac{n}{m} \sum_{\beta} E \frac{\xi (Q_\alpha G^{\beta} y_\beta, y_\beta)(G^{\beta} y_\beta, y_\beta)}{1 + \xi (G^{\beta} y_\beta, y_\beta)} d\sigma_n(\xi) = o(1),
\]

\[
R'_{n,\alpha} = -\int \frac{1}{1 + \xi f_n} \frac{n}{m} \sum_{\beta} E \frac{\xi (Q_\alpha G^{\beta} y_\beta, y_\beta)((f_{n,\beta} - f_n)\beta)}{1 + \xi (G^{\beta} y_\beta, y_\beta)} d\sigma_n(\xi), \quad |R'_{n,\alpha}| \leq C \Delta_n(z),
\]

and we used additionally that by (1.9) \( \|Q_\alpha\|_{op} = O(n^{-1}) \). It follows from (2.11) and (2.14) that

\[
z (f_{n,\alpha} - f_n) + (1 - \text{Tr} Q_\alpha) = (f_{n,\alpha} - f_n) \int \frac{d\sigma_n(\xi)}{1 + \xi f_n} + R'_{n,\alpha} - R'_{n,\alpha} + o(1),
\]

hence, using bounds for \( R'_{n,\alpha} \) and \( R'_{n,\alpha} \) and also (1.9) we get

\[
|z - \int \frac{d\sigma_n(\xi)}{1 + \xi f_n} | f_{n,\alpha} - f_n | \leq C \Delta_n(z) + o(1),
\]

where \( C > 0 \) is uniformly bounded in \( \eta_0 \). Choosing \( \eta_0 \) big enough one can get

\[
|z - \int \frac{d\sigma_n(\xi)}{1 + \xi f_n} | > 2C,
\]

which implies \( 2C |f_{n,\alpha} - f_n| \leq C \Delta_n(z) + o(1) \). Taking the maximum over \( \alpha \leq m \) we get

\[
\Delta_n(z) = o(1), \quad n \to \infty.
\]

This leads to \( R'_{n} = o(1) \) as \( n \to \infty \) and finishes the proofs of the lemma and of the theorem. \( \square \)

### 3 Proof of Theorem 1.4

Given \( r, d \in \mathbb{N} \), let \( \mathcal{L}_{rd} \) be defined in (1.1) - (1.2) and (1.6):

\[
\mathcal{L}_{rd} = \sum_{1 \leq k < \ell \leq r} \xi_{k\ell} Y_{k\ell} Y_{k\ell}^T, \quad Y_{k\ell} = (Y_j)_{j=1}^{r} = ((\delta_{jk} - \delta_{jt})v_{k\ell})_{j=1}^{r} \in \mathbb{R}^{r \times d},
\]

where \( (\xi_{k\ell})_{1 \leq k < \ell \leq r} \) are iid copies of a 0/1 random variable \( \xi = \xi_r \) with \( \mathbb{P}(\xi = 1) = p_r \) and \( (v_{k\ell})_{1 \leq k < \ell \leq r} \) are mutually independent normalized isotropic random vectors, \( E v_{k\ell} v_{k\ell}^T = d^{-1} \delta_{\alpha\beta} \), satisfying Assumption 1 and having norms uniformly bounded in \( r \),

\[
\sup_{k,\ell} \|v_{k\ell}\|_2^2 \leq C_0.
\]
for some \( C_0 > 0 \). Here for block vectors of the form \( X = (X_j)^{r}_{j=1} = (X_{\alpha_i})^{r,d}_{j=1} \) we use Latin indexes to count blocks and Greek indexes to count entries within a block. Let

\[
Q^{k\ell} = \left( Q^{k\ell}_{\gamma,j \beta} \right)^{r,d}_{i,j,\gamma,\beta = 1} = \mathbf{E} Y^{k\ell} Y^{k\ell T} = \left( \mathbf{E} Y_i^{k\ell} Y_j^{k\ell} \right)^{r}_{i,j=1}.
\]

By the definition of \( Y^{k\ell} \),

\[
Q^{k\ell} = \frac{1}{d} \left( (\delta_{jk} - \delta_{j\ell})(\delta_{ik} - \delta_{i\ell})I_d \right)^{r}_{i,j=1},
\]

so it has only four non-zero blocks (equal \( d^{-1}I_d \)). To check the conditions of Theorem 1.2 note first that now

\[
m = r(r - 1)/2, \quad n = rd, \quad \text{so that} \quad \lim_{n \to \infty} \frac{m}{n} = \lim_{r \to \infty} \frac{pr^r}{2d} = c/2 = c_1.
\]

For any \( rd \times rd \) block matrix \( D = (D_{ij})^{r}_{i,j=1} \) with \( d \times d \) blocks \( D_{ij} = (D_{\alpha_i, \beta_j})^{d}_{\alpha, \beta = 1} \) we have

\[
(DY^{k\ell}, Y^{k\ell}) = (\tilde{D}v^{k\ell}, v^{k\ell}), \quad \text{where} \quad \tilde{D} = D_{kk} + D_{\ell \ell} - D_{k\ell} - D_{\ell k},
\]

so that (2.8) for \( Y^{k\ell} \) follows from (1.8) for \( v^{k\ell} \). Also it is easy to check that \( \text{Tr} Q^{k\ell} = 2 \) and

\[
Q_{rd} := \frac{2}{r(r-1)} \sum_{1 \leq k < \ell \leq r} Q^{k\ell} = \frac{2}{r(r-1)} I_{rd} + B_{rd}, \quad \text{where} \quad B_{rd} = -\frac{2}{r(r-1)d} \left( 1 - \delta_{ij} \right) I_{i,j=1},
\]

and \( \| B_{rd} \|_{HS} = o((rd)^{-1/2}), \) \( r \to \infty, \) thus (1.10) is fulfilled. The only condition of Theorem 1.2 which is not fulfilled is the first part of (1.9), namely, we have \( \| Q^{k\ell} \|_{op} = O(d^{-1}) \) (instead of \( \| Q^{k\ell} \|_{op} = O((rd)^{-1}) \)).

On the other hand, matrix \( Q^{k\ell} \) is very sparse and has only four non-zero blocks.

Hence we need to go through the proof of Theorem 1.2 and check the places, where condition (1.9) was used. There are three such places: Lemma 2.2 (ii), (2.10), and (2.15) – (2.16). As to Lemma 2.2 (ii), we reprove it in Lemma 3.1 below. Now we recall the main steps of the proof of Lemma 2.3 and check (2.10) and (2.15) – (2.16).

Similar to (2.8), one can get

\[
z f_r + 1 = c_1 - \frac{c_1}{r(r-1)/2} \sum_{1 \leq k < \ell \leq r} \mathbf{E} \frac{1}{A_{k\ell}}, \quad \text{where} \quad A_{k\ell} = 1 + (G^{k\ell} Y^{k\ell}, Y^{k\ell}),
\]

\[
G^{k\ell} = (L_{rd} - \xi_{k\ell} Y^{k\ell} Y^{k\ell T} - z I_{rd})^{-1}. \quad \text{It is easy to show that}
\]

\[
|A_{k\ell}|^{-1}, \quad |\mathbf{E} A_{k\ell}|^{-1} \leq 1/(1 - 2C_0/\eta_0).
\]

We also have \( \mathbf{E}(G^{k\ell} Y^{k\ell}, Y^{k\ell}) = \mathbf{E} \text{Tr} G^{k\ell} G^{k\ell} \). Similar to (2.10), here we can replace \( G^{k\ell} \) with \( G \). Indeed, since by the definition of \( Q^{k\ell} \),

\[
(Q^{k\ell} X, Y) = \frac{1}{d} \sum_{\gamma} (X_{k \gamma} - X_{\ell \gamma})(Y_{k \gamma} - Y_{\ell \gamma}), \quad \forall X, Y \in \mathbb{R}^{rd},
\]

we have

\[
|\mathbf{E} \text{Tr} Q^{k\ell}(G^{k\ell} - G)| = |\mathbf{E} \xi_{k\ell}(Q^{k\ell} G^{k\ell} Y^{k\ell}, G Y^{k\ell})|
\]

\[
= |\mathbf{E} \xi_{k\ell} \frac{1}{d} \sum_{\gamma} ((G^{k\ell} Y^{k\ell})_{k \gamma} - (G^{k\ell} Y^{k\ell})_{\ell \gamma})((G Y^{k\ell})_{k \gamma} - (G Y^{k\ell})_{\ell \gamma})| \leq \frac{4}{d \eta_0} \mathbf{E} \| Y^{k\ell} \|_2^2 = O(d^{-1}).
\]
Thus
\[ E(G^{k\ell}Y^{k\ell}, Y^{k\ell}) = f_{r,k\ell} + O(d^{-1}), \quad f_{r,k\ell} = E \text{ Tr } Q^{k\ell} G, \]
and repeating the steps leading to (2.8) – (2.16), we get
\[ zf_r + 1 = \frac{2c_1 f_r}{1 + 2f_r} + R_r + o(1), \quad |R_r| \leq C \Delta_r(z), \quad \Delta_r = \max_{k,\ell} |f_{r,k\ell} - 2f_r|, \]
and
\[ zf_{r,k\ell} + 2 = \frac{2c_1 f_{r,k\ell}}{1 + 2f_r} + R_{r,k\ell} + o(1), \quad R_{r,k\ell} = \frac{2c_1}{1 + 2f_r} \sum_{1 \leq i < j \leq r} E \frac{(Q^{k\ell}G^{ij}Y^{ij}, Y^{ij}) A_{ij}^o}{A_{ij}} \]
\[ R'_{r,k\ell} = \frac{2c_1}{1 + 2f_r} \sum_{1 \leq i < j \leq r} E \frac{(Q^{k\ell}G^{ij}Y^{ij}, Y^{ij}) (f_{r,ij} - 2f_r)}{A_{ij}}. \]

It follows from (3.1) that for any \( X \in \mathbb{R}^{rd} \)
\[ (Q^{k\ell}X, Y^{ij}) = [\delta_{ik} + \delta_{i\ell} - \delta_{jk} - \delta_{j\ell}]^2 \]
\[ \sum_{\gamma} (X_{k\gamma} - X_{\ell\gamma}) v^{ij}_{k\gamma}. \]

Hence instead of the double sums over \( i, j \) in the expressions above we have single sums over \( i \) or over \( j \). This and the boundedness of \( v^{ij} \) and \( A_{ij} \) allows to treat \( R_{r,k\ell} \) and \( R'_{r,k\ell} \) similar to (2.15) – (2.16) and then to show that \( R_{n,\alpha}, R'_{n,\alpha} = o(1) \) and to get the equation for \( f_L = \lim f_r \) (see (1.3)).

It remains to prove

**Lemma 3.1.** \( \text{Var } \text{Tr } Q^{k\ell} G = o(1), \ r \to \infty. \)

**Proof.** We have (see (2.6) and (3.1))
\[ \text{Var } \text{Tr } Q^{k\ell} G \leq 4 \sum_{i<j} E |\text{Tr } Q^{k\ell}(G^{ij} - G)|^2 = 4 \sum_{i<j} E |\xi_{ij}(Q^{k\ell}G^{ij}Y^{ij}, GY^{ij})|^2 \]
\[ = \frac{4}{d^2} \sum_{i<j} E \left| \sum_{\gamma} \xi_{ij}(G^{ij}Y^{ij})_{k\gamma} - (G^{ij}Y^{ij})_{\ell\gamma} \right|^2, \]
where
\[ \frac{1}{d^2} \sum_{i<j} E \left| \sum_{\gamma} \xi_{ij}(G^{ij}Y^{ij})_{k\gamma} (GY^{ij})_{k\gamma} \right|^2 \leq \frac{1}{d^2} \sum_{i<j} E \xi_{ij}^2 ||G^{ij}Y^{ij}||^2 \sum_{\gamma} |(GY^{ij})_{k\gamma}|^2 \]
\[ \leq \frac{C_0}{\eta_0 d^2} E \sum_{\gamma} \sum_{i<j} \xi_{ij} ||GY^{ij}||_{k\gamma}^2 \]
and by the definition of \( L_{rd} \) and the resolvent identity,
\[ \sum_{i<j} \xi_{ij} ||GY^{ij}||_{k\gamma}^2 = \sum_{i<j} \xi_{ij} (GY^{ij}Y^{ijT}G)_{k\gamma,k\gamma} = (G L_{rd} G)_{k\gamma,k\gamma} = (G(zG + I_{rd}))_{k\gamma,k\gamma} = O(1). \]

This finishes the proof of the lemma. \( \square \)
4 Adjacency matrices. Proof of Theorem 1.5

The scheme of the proof is essentially the same as in the case of Laplacian $L_{rd}$. The main difference is that here for every vector $X^{k\ell}$ in the definition of $A_{rd}$ (see (1.6)) there are two terms containing this vector, $X^{k\ell}X^{\ell kT}$ and $X^{k\ell}X^{\ell kT}$, so that in order to separate this vector from the rest we need to apply the rank one perturbation formula twice. Also it is convenient to consider first a more general model without block structure. We have

**Theorem 4.1.** Given $n, m \in \mathbb{N}$, consider an $n \times n$ matrix $A_n = \sum_\alpha \xi_\alpha(x_\alpha y^T_\alpha + y_\alpha x^T_\alpha)$, where

(i) $(\xi_\alpha)_\alpha$ are iid copies of a $0/1$ random variable $\xi = \xi_n$ with $\mathbb{P}(\xi = 1) = p_n$,

(ii) $\frac{m}{n}p_n \to c_1 > 0$ as $n \to \infty$ (without loss of generality we assume that $\frac{m}{n}p_n \equiv c_1$),

(iii) $(x_\alpha)_\alpha, (y_\alpha)_\alpha \subset \mathbb{R}^n$ are two sets of mutually independent random vectors such that $\|x_\alpha\|_2^2 \leq C_0$, $\|y_\alpha\|_2^2 \leq C_0$ for some $C_0 > 0$ and for all deterministic matrices $D = D_n$ with $\|D\|_{op} = 1$ we have (cf (1.8))

$$\sup_{{u, v} \in (x_\alpha, y_\alpha)_\alpha} \text{Var}(D u, v) = o(1), \quad n \to \infty.$$  

(iv) matrices $Q^{x_\alpha} := E x_\alpha x^T_\alpha$, $Q^{y_\alpha} := E y_\alpha y^T_\alpha$, and $Q^{xy_\alpha} := E x_\alpha y^T_\alpha = Q^{yx_\alpha T}$ have the operator norms of order $O(n^{-1})$ and

$$\sup_{{Q_\alpha} \in (Q^{x_\alpha}, Q^{y_\alpha})_\alpha} |\text{Tr} Q_\alpha - 1| = o(1),$$

(v) for every $n \times n$ matrices $K_1, K_2$ we have

$$\frac{1}{m} \sum_\alpha \text{Tr} Q^{x_\alpha} K_1 \text{Tr} Q^{y_\alpha} K_2 = \frac{1}{n} \text{Tr} K_1 \frac{1}{n} \text{Tr} K_2,$$

(vi) matrix $Q^{xy} := (\frac{1}{m} \sum_\alpha |Q^{xy}_i|)_{i,j}$ satisfies $\|Q^{xy}\|_{HS} = o(n^{-1/2}).$

Then as $n \to \infty$ the empirical spectral distributions $N_{A_n}$ converge in probability to a non-random probability measure $N_A$ which Stieltjes transform $f$ is uniquely determined by the equation

$$zf^3 + (1 - 2c_1)f^2 - zf - 1 = 0$$

in the class of Stieltjes transforms of non-negative measures.

**Remark 4.2.** A more general case corresponding to $\xi_\alpha$ satisfying Assumption 2 contains more pure technical details and we do not treat it here, but we strongly believe that following essentially the same scheme one can prove that in this case $f$ solves the equation

$$zf = -1 - 2f^2 \int \frac{\xi d\tilde{s}(\xi)}{1 - \xi^2 f^2}.$$  

**Proof.** Following the scheme of the proof of Theorem 1.2 note first that the proof of the analog of Lemma 2.1 is trivial in this case and the proof of the analog of Lemma 2.2 is essentially the same. Thus we only need to prove the convergence in mean (cf Lemma 2.3). To this end introduce

$$A_n^\alpha := A_n - \xi_\alpha(x_\alpha y^T_\alpha + y_\alpha x^T_\alpha) \quad \text{and} \quad G^\alpha(z) := (A_n^\alpha - zI_n)^{-1},$$

where $z \in \mathbb{C}_{\eta_0}$ for a big enough $\eta_0$. Given an $n \times n$ symmetric matrix $K$, applying twice (2.7) we get

$$(KGx_\alpha, y_\alpha) = \frac{(KG^\alpha x_\alpha, y_\alpha)(1 + \xi_\alpha(G^\alpha x_\alpha, y_\alpha)) - \xi^2_\alpha(KG^\alpha y_\alpha, y_\alpha)(G^\alpha x_\alpha, x_\alpha)}{(1 + \xi_\alpha(G^\alpha x_\alpha, y_\alpha))^2 - \xi^2_\alpha(G^\alpha y_\alpha, y_\alpha)(G^\alpha x_\alpha, x_\alpha)}.$$  

(4.2)
It follows from the resolvent identity, (ii) and (4.2) with $K = I$, that

$$zf_n(z) + 1 = \frac{2}{n} \sum_{\alpha} \mathbb{E} \xi_{\alpha}(Gx_{\alpha}, y_{\alpha})$$

$$= 2c_1 \frac{1}{m} \sum_{\alpha} \mathbb{E} \frac{(G^\alpha x_{\alpha}, y_{\alpha})(1 + (G^\alpha x_{\alpha}, y_{\alpha})) - (G^\alpha y_{\alpha}, y_{\alpha})(G^\alpha x_{\alpha}, x_{\alpha})}{(1 + (G^\alpha x_{\alpha}, y_{\alpha}))^2 - (G^\alpha y_{\alpha}, y_{\alpha})(G^\alpha x_{\alpha}, x_{\alpha})}$$

$$= 2c_1 - \frac{2c_1}{m} \sum_{\alpha} \mathbb{E} \frac{1 + (G^\alpha x_{\alpha}, y_{\alpha})}{A_{\alpha}}$$

$$= 2c_1 - \frac{2c_1}{m} \sum_{\alpha} \frac{1}{\mathbb{E} A_{\alpha}} \mathbb{E} (1 + (G^\alpha x_{\alpha}, y_{\alpha}))A_{\alpha}^\alpha + R_n, \quad (4.3)$$

where

$$A_{\alpha} = (1 + (G^\alpha x_{\alpha}, y_{\alpha}))^2 - (G^\alpha y_{\alpha}, y_{\alpha})(G^\alpha x_{\alpha}, x_{\alpha}),$$

$$R_n = \frac{2c_1}{m} \sum_{\alpha} \frac{1}{\mathbb{E} A_{\alpha}} \mathbb{E} (1 + (G^\alpha x_{\alpha}, y_{\alpha}))A_{\alpha}^\alpha.$$ 

Applying (iii) and an analog of Lemma 3.1 (ii), one can show that

$$|A_{\alpha}| \geq \frac{1}{2} - 3C_0/\eta_0^2 > 0,$$

$$\text{Var} A_{\alpha} = o(1), \quad \text{(see also (2.12)) and}$$

$$\mathbb{E} A_{\alpha} = (1 + \mathbb{E} \text{Tr} Q^{y_\alpha} G^\alpha)^2 - \mathbb{E} \text{Tr} Q^{y_\alpha} G^\alpha \mathbb{E} \text{Tr} Q^{y_\alpha} G^\alpha + o(1),$$

where with the help of (iv) $G^\alpha$ can be replaced with $G$ with an error term of order $O(n^{-1})$ (cf (2.10)) so that

$$\mathbb{E} A_{\alpha} = (1 + \mathbb{E} \text{Tr} Q^{y_\alpha} G)^2 - f_n^{x_\alpha} f_n^{y_\alpha} + o(1),$$

and we use notations

$$f_n^{x_\alpha} = \mathbb{E} \text{Tr} Q^{x_\alpha} G, \quad f_n^{y_\alpha} = \mathbb{E} \text{Tr} Q^{y_\alpha} G.$$ 

By (iv) and (vi), $|\text{Tr} Q^{y_\alpha} G| = O(1)$ and

$$\frac{1}{m} \sum_{\alpha} |\text{Tr} Q^{y_\alpha} G| \leq \sum_{i,j} Q_{ij}^{x_\alpha} |G_{ij}| \leq \|Q^{y_\alpha}\|_{HS} \|G\|_{HS} = o(1). \quad (4.4)$$

Hence $R_n = o(1)$ and

$$zf_n + 1 = 2c_1 - \frac{2c_1}{m} \sum_{\alpha} \frac{1}{1 - f_n^{x_\alpha} f_n^{y_\alpha}} + o(1).$$

It follows from (v) that $m^{-1} \sum_{\alpha} f_n^{x_\alpha} f_n^{y_\alpha} = f_n^2$, hence,

$$zf_n + 1 = 2c_1 - \frac{2c_1}{1 - f_n^2} + R'_n + o(1), \quad (4.5)$$

where

$$R'_n = - \frac{1}{1 - f_n^2} \frac{2c_1}{m} \sum_{\alpha} \frac{f_n^{x_\alpha} f_n^{y_\alpha} - f_n^2}{1 - f_n^{x_\alpha} f_n^{y_\alpha}}, \quad (4.6)$$

$$|R'_n| \leq \frac{2c_1}{\eta_0 (1 - \eta_0^{-2})^2} \Delta_n, \quad \Delta_n := \max_{\alpha} (|f_n^{x_\alpha} - f_n| + |f_n^{y_\alpha} - f_n|).$$
It remains to show that $\Delta_n = o(1)$. To this end we treat similarly $f_n^{x\alpha}$ (and $f_n^{y\alpha}$) and applying (4.2) with $K = Q^{x\alpha}$ we get
\[
z f_n^{x\alpha} + \text{Tr} Q^{x\alpha} = \frac{2c_1}{m} \sum_{\beta} E \left( \frac{(nQ^{x\alpha}G^\beta x_\beta, y_\beta)(1 + (G^\beta x_\beta, y_\beta)) - (nQ^{x\alpha}G^\beta y_\beta, y_\beta)(G^\beta x_\beta, x_\beta)}{A_{\beta}} \right).
\]
Note that $\|nQ^{x\alpha}\|_{op} = O(1)$, hence repeating steps leading to (1.5) – (1.6) and applying (v), one can get
\[
z f_n^{x\alpha} + \text{Tr} Q^{x\alpha} = -2c_1 \frac{I}{f_n} + R'_{n,\alpha} + o(1),
\]
where
\[
R'_{n,\alpha} = - \frac{1}{1 - f_n^2} \frac{2c_1}{m} \sum_{\beta} E \left( \frac{f_n^\beta E \text{Tr} Q^{x\alpha} Q^{y\beta} G \left( f_n^{x\alpha} f_n^{y\alpha} - f_n^2 \right)}{1 - f_n^\beta f_n^\beta} \right),
\]
\[
|R'_{n,\alpha}| \leq \frac{2c_1 C_0^2}{(1 - \eta_0^2)^2} \Delta_n.
\]
Now subtracting (4.7) from (4.5) and using (iv) one can show that $\Delta_n = o(1)$ (cf. (2.17)). Thus
\[
z f_n + 1 = - \frac{2c_1 f_n^2}{1 - f_n^2} + o(1),
\]
which leads to (1.1) and finishes the proof of Theorem 1.1.

**Proof of Theorem 1.5.** Now we have
\[
A_{rd} = \sum_{1 \leq k \neq \ell \leq r} \xi_{k\ell} X^{k\ell} X^{\ell k} T, \quad X^{k\ell} = (\delta_{jk} v^{k\ell})^n_{j=1},
\]
so that in terms of Theorem 1.4
\[
\sum_{\alpha} = \sum_{1 \leq k \neq \ell \leq r}, \quad m = r(r - 1)/2, \quad n = rd, \quad c_1 = c/2, \quad x_\alpha = X^{k\ell}, \quad y_\alpha = X^{\ell k}, \quad k < \ell,
\]
and the analogs of $Q^{x\alpha}$ and $Q^{y\alpha}$ are given by
\[
Q^{k\ell} := E X^{k\ell} X^{\ell k} T = d^{-1}(\delta_{ik} \delta_{jk} I_d)^r_{i=1}, \quad \text{Tr} Q^{k\ell} = 1, \quad \text{and}
\]
\[
Q^{k\ell} := E X^{k\ell} X^{\ell k} T = d^{-1}(\delta_{ik} \delta_{jk} I_d)^r_{i=1}.
\]
We suppose that $(v^{k\ell})_{k < \ell}$ have uniformly bounded norms, so let $C_0 > 0$ be such that $\|v^{k\ell}\|_2^2 \leq C_0$ for every $k < \ell$.

Checking the conditions of Theorem 1.1, note first that (iii) follows from the definition of $X^{k\ell}$ and conditions for $v^{k\ell}$. As to (v-vi), these conditions are not fulfilled with $\sum_{\alpha} = \sum_{k \neq \ell}$, but since in (4.8) we have $\sum_{k \neq \ell}$ by the definitions of $Q^{k\ell}$ we get the following analogs of (v-vi):
\[
(v') \quad \frac{1}{r} \sum_{k} \text{Tr} Q^{k\ell} K = \frac{1}{r d} \text{Tr} K \quad \text{and}
\]
\[
(v')' \quad \|\tilde{Q}^{xy}\|_{HS} = O((r\sqrt{d})^{-1}), \quad \text{where} \quad \tilde{Q}^{xy}_{i\gamma, j\beta} := \frac{1}{r^2} \sum_{k, \ell} |Q^{k\ell}_{i\gamma, j\beta}| = \frac{1}{db^2} \delta_{i\gamma, j\beta}.
\]

14
Thus again the only condition which is not fulfilled is the first part of (iv), because now $\|Q^{kk}\|_{op}, \|Q^{kk}\|_{op} = O(d^{-1})$ (instead of $O((rd)^{-1})$). On the other hand, matrices $(Q^{kk})_k$ are “orthogonal” up to normalisation:

$$Q^{kk}Q^{\ell\ell} = \frac{1}{d} \delta_{kp} Q^{k\ell}, \text{ and also } Q^{kk}X^{\ell\ell} = \frac{1}{d} \delta_{kp} X^{k\ell},$$

(4.9)

thus in the corresponding places of the proof we have single sums instead of double sums. This allows to repeat the proof of Theorem 4.1 with slight modifications and to get first

$$zf_r + 1 = \frac{1}{rd} \sum_{k \neq \ell} E \xi_{k\ell}(GX^{k\ell}, X^{\ell k}) = c - \frac{c}{r^2} \sum_{k \neq \ell} E\left[1 + \frac{\sum_{k \neq \ell} \delta_{kp}^2}{A_{k\ell}}\right],$$

(4.10)

where

$$A_{k\ell} = (1 + (G^{k\ell} X^{k\ell}, X^{\ell k}))^2 - (G^{kk} X^{kk}, X^{kk})(G^{k\ell} X^{k\ell}, X^{\ell k}).$$

(4.11)

Since for any matrix $B, \|Bv^{k\ell}|Bv^{k\ell}\| \leq C_0\|B\|_{op}$, we have

$$\text{Var} A_{k\ell} = C(\eta_0) \max_{k,\ell} \{\text{Var}(G^{k\ell} X^{k\ell}, X^{\ell k}), \text{Var}(G^{kk} X^{kk}, X^{kk})\},$$

where we use notation $C(\eta_0)$ for every positive function uniformly bounded in $\eta_0 \to \infty$,

$$C(\eta_0) = O(1), \quad \eta_0 \to \infty.$$ 

It follows from Assumption 1 and Lemma 4.3 below that

$$\text{Var}(G^{k\ell} X^{k\ell}, X^{\ell k}) = E\text{Var}_{k\ell}(G^{k\ell} X^{k\ell}, X^{\ell k}) + \text{Var} \text{ Tr } Q^{k\ell} G^{k\ell} = o(1), \quad \text{and}$$

$$\text{Var}(G^{kk} X^{kk}, X^{kk}) = E\text{Var}_{kk}(G^{kk} X^{kk}, X^{kk}) + \text{Var} \text{ Tr } Q^{kk} G^{kk} = o(1),$$

(4.12)

(cf. (4.11)), hence, $\text{Var} A_{k\ell} = o(1), r \to \infty$. Also similar to (4.4) one can show that the terms containing $E(G^{k\ell} X^{k\ell}, X^{\ell k}) = \text{Tr } Q^{k\ell} G^{k\ell}$ do not contribute to the limit. It follows from above that

$$zf_r + 1 = c - \frac{c}{r^2} \sum_{k,\ell} \frac{1}{1 - f_r^{kk} f_r^{\ell\ell}} + o(1)$$

(4.13)

$$= c - \frac{c}{1 - f_r^2} + R'_r + o(1),$$

where

$$f_r^{kk} = E \text{ Tr } Q^{kk} G = d^{-1} \sum_{\gamma} \text{E} G_{k\gamma, k\gamma}, \quad \frac{1}{r^2} \sum_k f_r^{kk} = f_r,$$

and

$$R'_r = -\frac{1}{1 - f_r^2} \sum_{k,\ell} \frac{f_r^{kk} f_r^{\ell\ell} - f_r^2}{1 - f_r^{kk} f_r^{\ell\ell}}, \quad |R'_r| \leq C(\eta_0) \Delta_r, \quad \Delta_r = \max_{\ell} |f_r - f_r^{\ell\ell}|.$$ 

Using (4.9) - (4.10), similar to (4.7) one can get for every $q \leq r$

$$zf_r^{qq} + 1 = \sum_{k \neq \ell} E \xi_{k\ell}(Q^{qq} GX^{k\ell}, X^{\ell k}) = \frac{1}{d} \sum_{k \neq q} E \xi_{kq}(GX^{kq}, X^{qk})$$

(4.14)

$$= c - \frac{c}{r} \sum_{k \neq q} \frac{1}{1 - f_r^{kk} f_r^{qq}} + o(1) = c - \frac{c}{1 - f_r^{qq} f_r} + R'_r + o(1),$$

15
Lemma 4.3. Let $V_r := \max_{1 \leq k, \ell \leq r} \text{Var} \text{Tr} Q^{k \ell} G$. Then $V_r = o(1)$ as $r \to \infty$.

Proof. Note that the simple trick based on the resolvent identity, which we have used in the last line of the proof of Lemma 3.1 to get rid of the double sum over $i, j$, does not work here. So we will go another way.

For every $q \leq r$, it follows from (4.14) that

$$z \text{Var} \text{Tr} Q^{qq} G = \text{Tr} Q^{qq} G (\text{Tr} Q^{qq} G) \circ \text{Var} \text{Tr} Q^{qq} G = \frac{1}{d} \sum_{k \neq q} \text{E} \xi_{k q} (G X^{k q}, X^{q k}) (\text{Tr} Q^{qq} G^{k q}) \circ + R_r,$$

where $x^o = x - E x$ and

$$R_r = \frac{1}{d} \sum_{k \neq q} \text{E} \xi_{k q} (G X^{k q}, X^{q k}) (\text{Tr} Q^{qq} (G - G^{k q})) \circ.$$

We have

$$|\text{Tr} Q^{qq} (G - G^{k q})| = |\xi_{k q} (Q^{qq} G^{k q} X^{k q}, G X^{q k}) + (Q^{qq} G^{k q} X^{q k}, G X^{k q})| \leq 2 ||Q^{qq}||_{op} ||\varphi^{k q}||^2 / \eta_0 = O(d^{-1}),$$

hence, $R_r = O(d^{-1})$. Also, we have

$$\text{Var} \text{Tr} Q^{qq} G = \text{Var} \text{Tr} Q^{qq} G^{k q} + O(d^{-1}).$$

Applying (4.2), one can continue (4.15) and get similar to (4.10)

$$z \text{Var} \text{Tr} Q^{qq} G = -\frac{c}{r} \sum_{k \neq q} \text{E} \frac{(1 + (G^{k q} X^{k q}, X^{q k})) (\text{Tr} Q^{qq} G^{k q}) \circ}{A_{k q}}$$

$$= -\frac{c}{r} \sum_{k \neq q} \text{E} (G^{k q} X^{k q}, X^{q k}) (\text{Tr} Q^{qq} G^{k q}) \circ \frac{A_{k q}}{E A_{k q}}$$

$$+ \frac{c}{r} \sum_{k \neq q} \frac{1}{E A_{k q}} \text{E} (1 + (G^{k q} X^{k q}, X^{q k})) (\text{Tr} Q^{qq} G^{k q}) \circ A_{k q}^o =: T_r^{(1)} + T_r^{(2)}.$$

Since by the Schwartz inequality

$$|\text{E} (G^{k q} X^{k q}, X^{q k}) (\text{Tr} Q^{qq} G^{k q}) \circ| = |\text{E} \text{Tr} Q^{k q} G^{k q} (\text{Tr} Q^{qq} G^{k q}) \circ|$$

$$\leq (\text{Var} \text{Tr} Q^{k q} G^{k q})^{1/2} (\text{Var} \text{Tr} Q^{qq} G^{k q})^{1/2} \leq V_r + O(d^{-1}),$$

we have $|T_r^{(1)}| \leq C(\eta_0) V_r + O(d^{-1})$. It follows from (4.11) – (4.12) that $\text{Var} A_{k q} \leq V_r + o(1)$. This and the Schwartz inequality allows to get

$$|T_r^{(2)}| \leq C(\eta_0) V_r^{1/2} \max_{k \neq q} \text{Var} A_{k q}^{1/2} \leq C(\eta_0) V_r + o(1).$$
Summarising we get from every $q \leq r$

$$\eta_0 \mathbb{V} \text{ar} \, \text{Tr} Q^{qq}G \leq C(\eta_0)V_r + o(1).$$

Similar, one can show that $\eta_0 \mathbb{V} \text{ar} \, \text{Tr} Q^{kq}G \leq C(\eta_0)V_r + o(1)$ for every $k, q$. Hence, taking maximum over $k, q$, we get $\eta_0 V_r \leq C(\eta_0)V_r + o(1)$, where $C(\eta_0)$ remains bounded as $\eta_0 \to \infty$. Thus choosing $\eta_0$ big enough we get $V_r = o(1)$ as $r \to \infty$. This finishes the proof of the lemma and the proof of Theorem 1.5.

References

[1] R. Adamczak, On the Marchenko-Pastur and circular laws for some classes of random matrices with dependent entries, Electron. J. Probab. 16 (2011), no. 37, 1068–1095. MR2820070.

[2] Akhiezer, N. I., Glazman, I. M. Theory of Linear Operators in Hilbert Space, Dover, New York, 1993.

[3] G. W. Anderson, A. Guionnet and O. Zeitouni, An introduction to random matrices, Cambridge Studies in Advanced Mathematics, 118, Cambridge University Press, Cambridge, 2010. MR2760897

[4] G. Aubrun, Random points in the unit ball of $l^p_n$, Positivity 10 (2006), no. 4, 755–759. MR2280648

[5] Z. Bai and J. W. Silverstein, Spectral analysis of large dimensional random matrices, second edition, Springer Series in Statistics, Springer, New York, 2010. MR2567175

[6] Bai, Z. D., Zhou, W. (2008). Large sample covariance matrices without independence structures in columns. Statistica Sinica, 18(2), 425.

[7] D. Banerjee and A. Bose, Bulk behaviour of some patterned block matrices, Indian J. Pure Appl. Math. 47 (2016), no. 2, 273–289. MR3517625

[8] Bryson, J., Vershynin, R. and Zhao, H., (2019). Marchenko-Pastur law with relaxed independence conditions. arXiv preprint [arXiv:1912.12724]

[9] G. M. Cicuta, J. Krausser, R. Milkus, A. Zaccone, Unifying model for random matrix theory in arbitrary space dimensions, Phys. Rev. E 97 (2018), no. 3, 032113, 8 pp. MR3789138

[10] Dharmadhikari, S. W., Fabian, V., and Jogdeo, K. (1968). Bounds on the moments of martingales, Ann. Math. Statist. 39, 1719–1723.

[11] J. S. Geronimo and T. P. Hill, Necessary and sufficient condition that the limit of Stieltjes transforms is a Stieltjes transform, J. Approx. Theory 121 (2003), no. 1, 54–60. MR1962995

[12] Girko, V. (2001). Theory of Stochastic Canonical Equations, vols.I, II Kluwer, Dordrecht.

[13] Göotze, F., Naumov, A.A., and Tikhomirov, A.N. (2014). Limit theorems for two classes of random matrices with dependent entries, Teor. Veroyatnost. i Primenen., 59(1), 61–80.

[14] F. Götze and A. N. Tikhomirov, Limit theorems for spectra of random matrices with martingale structure, in Stein’s method and applications, 181–193, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 2005, Singapore Univ. Press, Singapore. MR2205336

[15] A. Lemaitre and C. Maloney, Sum rules for the quasi-static and visco-elastic response of disordered solids at zero temperature, J. Stat. Phys. 123 (2006), no. 2, 415–453. MR2227089
[16] Louart, C. and Couillet, R., 2018. Concentration of measure and large random matrices with an application to sample covariance matrices. arXiv preprint arXiv:1805.08295.

[17] A. Lytova, Central limit theorem for linear eigenvalue statistics for a tensor product version of sample covariance matrices, J. Theoret. Probab. 31 (2018), no. 2, 1024–1057. MR3803923

[18] Marchenko, V., Pastur, L. (1967). The eigenvalue distribution in some ensembles of random matrices. Math. USSR Sbornik, 1, 457–483.

[19] Pajor, A. and Pastur, L. (2009). On the limiting empirical measure of eigenvalues of the sum of rank one matrices with log-concave distribution, Studia Math., 195(1), 11–29.

[20] L. Pastur and M. Shcherbina, Eigenvalue distribution of large random matrices, Mathematical Surveys and Monographs, 171, American Mathematical Society, Providence, RI, 2011. MR2808038

[21] M. Pernici and G. M. Cicuta, Proof of a conjecture on the infinite dimension limit of a unifying model for random matrix theory, J. Stat. Phys. 175 (2019), no. 2, 384–401. MR3968860

[22] S. O’Rourke, A note on the Marchenko-Pastur law for a class of random matrices with dependent entries, Electron. Commun. Probab. 17 (2012), no. 28, 13 pp. MR2955493

[23] G. Semerjian and L. F. Cugliandolo, Sparse random matrices: the eigenvalue spectrum revisited, J. Phys. A 35 (2002), no. 23, 4837–4851. MR1916090

[24] P. Yaskov, A short proof of the Marchenko-Pastur theorem, C. R. Math. Acad. Sci. Paris 354 (2016), no. 3, 319–322. MR3463031

[25] Yin, Y. (2018). On singular value distribution of large dimensional data matrices whose columns have different correlations. arXiv preprint arXiv:1802.01245.

[26] Y. Q. Yin and P. R. Krishnaiah, Limit theorems for the eigenvalues of product of large-dimensional random matrices when the underlying distribution is isotropic, Teor. Veroyatnost. i Primenen. 31 (1986), no. 2, 394–398. MR0851002

[27] Zaccone, A. and Scossa-Romano, E., 2011. Approximate analytical description of the nonaffine response of amorphous solids. Physical Review B, 83(18), p.184205.

Alicja Dembczak-Kołodziejczyk,
University of Opole,
48 Oleska, 45-052,
Opole, Poland.
e-mail: alicja.dembczak@uni.opole.pl

Anna Lytova,
University of Opole,
48 Oleska, 45-052,
Opole, Poland.
e-mail: alytova@uni.opole.pl