AVERAGE SIZE OF A SELF-CONJUGATE (s, t)-CORE PARTITION

WILLIAM Y. C. CHEN, HARRY H. Y. HUANG, AND LARRY X. W. WANG

(Communicated by Patricia Hersh)

Abstract. Armstrong, Hanusa and Jones conjectured that if s, t are coprime integers, then the average size of an (s, t)-core partition and the average size of a self-conjugate (s, t)-core partition are both equal to \((s+t+1)(s-1)(t-1)\) \(\frac{24}{s+t}\). Stanley and Zanello showed that the average size of an \((s, s+1)\)-core partition equals \((s+1)^2/2\). Based on a bijection of Ford, Mai and Sze between self-conjugate \((s, t)\)-core partitions and lattice paths in an \(\lfloor s^2 \rfloor \times \lfloor t^2 \rfloor\) rectangle, we obtain the average size of a self-conjugate \((s, t)\)-core partition as conjectured by Armstrong, Hanusa and Jones.

1. Introduction

In this paper, employing a bijection of Ford, Mai and Sze between self-conjugate \((s, t)\)-core partitions and lattice paths, we prove a conjecture of Armstrong, Hanusa and Jones on the average size of a self-conjugate \((s, t)\)-core partition.

A partition is called a \(t\)-core partition, or simply a \(t\)-core, if its Ferrers diagram contains no cells with hook length \(t\). A partition is called an \((s, t)\)-core, or simply an \((s, t)\)-core, if it is simultaneously an \(s\)-core and a \(t\)-core. Let \(r = \gcd(s, t)\). If \(r > 1\), then each \(r\)-core is an \((s, t)\)-core, and so there are infinitely many \((s, t)\)-cores. When \(s\) and \(t\) are coprime, Anderson [1] showed that the number of \((s, t)\)-core partitions equals

\[
1 + \binom{s + t}{2} - \frac{s + t}{s + t + 1}.
\]

For coprime integers \(s\) and \(t\), Ford, Mai and Sze [4] characterized the set of hook lengths of diagonal cells in self-conjugate \((s, t)\)-core partitions, and they showed that the number of self-conjugate \((s, t)\)-core partitions equals

\[
\left( \left\lfloor \frac{s^2}{2} \right\rfloor + \left\lfloor \frac{t^2}{2} \right\rfloor \right).
\]  

(1.1)

A partition is of size \(n\) if it is a partition of \(n\). Aukerman, Kane and Sze [3] conjectured that the largest size of an \((s, t)\)-core partition for coprime numbers \(s\) and \(t\) is \((s^2-1)(t^2-1)/24\). Olsson and Stanton [6] proved this conjecture and obtained the following uniqueness property.
Theorem 1.1. If \( s \) and \( t \) are coprime, then there is a unique \((s, t)\)-core partition with the largest size \( \frac{(s^2-1)(t^2-1)}{24} \). Moreover, such an \((s, t)\)-core with the largest size is self-conjugate.

A short proof for the conjecture of Aukerman, Kane and Sze was given by Tripathi [8]. Vandehey [9] proved the following containment property of the unique \((s, t)\)-core partition with the largest size.

Theorem 1.2. There exists an \((s, t)\)-core partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) such that for each \((s, t)\)-core \( \mu = (\mu_1, \mu_2, \ldots) \), we have \( \lambda_i \geq \mu_i \) for all \( i \).

Clearly, the \((s, t)\)-core in the above theorem is the unique \((s, t)\)-core with the largest size.

Armstrong, Hanusa and Jones [2] proposed the following conjecture concerning the average size of an \((s, t)\)-core and the average size of a self-conjugate \((s, t)\)-core.

Conjecture 1.3. Assume that \( s \) and \( t \) are coprime. Then the average size of an \((s, t)\)-core and the average size of a self-conjugate \((s, t)\)-core are both equal to

\[
\frac{(s + t + 1)(s - 1)(t - 1)}{24}
\]

Stanley and Zanello [7] showed that the conjecture for the average size of an \((s, t)\)-core holds for \((s, s + 1)\)-cores. More precisely, they showed that the average size of an \((s, s + 1)\)-core equals \( \left( \frac{s+1}{3} \right)^2 \). In this paper, we prove the self-conjugate case of Conjecture 1.3.

2. THE AVERAGE SIZE OF A SELF-CONJUGATE \((s, t)\)-CORE

In this section we give a proof of Conjecture 1.3 for the case of self-conjugate \((s, t)\)-cores, which is stated as follows.

Theorem 2.1. Assume that \( s \) and \( t \) are coprime. Then the average size of a self-conjugate \((s, t)\)-core equals

\[
\frac{(s + t + 1)(s - 1)(t - 1)}{24}
\]

Before we present the proof, let us recall a characterization of self-conjugate \((s, t)\)-cores obtained by Ford, Mai and Sze [4]. They introduced an array \( A(s, t) = (A_{i,j})_{1 \leq i \leq \lfloor s/2 \rfloor, 1 \leq j \leq \lfloor t/2 \rfloor} \), where

\[
A_{i,j} = st - (2j - 1)s - (2i - 1)t.
\] (2.1)

Let \( \mathcal{P}(A(s, t)) \) be the set of lattice paths in \( A(s, t) \) from the lower-left corner to the upper-right corner. For example, Figure 2.1 gives an array \( A(s, t) \) for \( s = 8 \) and \( t = 11 \), where the solid lines represent a lattice path in \( \mathcal{P}(A(s, t)) \). For a lattice path \( P \) in \( \mathcal{P}(A(s, t)) \), let \( M_{A(s, t)}(P) \) denote the set of positive entries \( A_{i,j} \) below \( P \) along with the absolute values of negative entries above \( P \). The following theorem of Ford, Mai and Sze [4] establishes a connection between self-conjugate \((s, t)\)-cores and lattice paths in \( A(s, t) \).

Theorem 2.2. Assume that \( s \) and \( t \) are coprime. Let \( A(s, t) \) be the array as given in (2.1). Then there is a bijection \( \Phi \) between the set \( \mathcal{P}(A(s, t)) \) and the set of self-conjugate \((s, t)\)-core partitions such that for any \( P \in \mathcal{P}(A(s, t)) \), the set of main diagonal hook lengths of \( \Phi(P) \) is given by \( M_{A(s, t)}(P) \).
A lattice path \( P \) in the array \( A(8,11) \)

![Figure 2.1. A lattice path \( P \) in the array \( A(8,11) \)](image)

For example, in Figure 2.1, 5 is the only positive entry below \( P \), while \(-7\) and \(-13\) are the negative entries above \( P \). Thus \( M_{A(8,11)}(P) = \{5, 7, 13\} \). So we have \( \Phi(P) = (7, 5, 5, 3, 3, 1, 1) \), which is an \((8,11)\)-core partition.

The following lemma gives a formula for the size of a self-conjugate \((s,t)\)-core partition \( \lambda \) corresponding to a lattice path \( P \) in \( \mathcal{P}(A(s,t)) \).

**Lemma 2.3.** For any lattice path \( P \) in \( \mathcal{P}(A(s,t)) \), we have

\[
|\Phi(P)| = \frac{(s^2 - 1)(t^2 - 1)}{24} - \sum_{(i,j) \text{ is above } P} A_{i,j}.
\]

**Proof.** For a self-conjugate partition \( \lambda \), define

\[ MD(\lambda) = \{ h | h \text{ is the hook length of a cell on the main diagonal of } \lambda \} \]

Clearly, the main diagonal cells have distinct hook lengths and the size of a self-conjugate partition equals the sum of elements in \( MD(\lambda) \). Let \( P \) be a lattice path in \( \mathcal{P}(A(s,t)) \). By Theorem 2.2, we find that

\[
|\Phi(P)| = \sum_{h \in MD(\Phi(P))} h
\]

\[
= \sum_{(i,j) \text{ is below } P, A_{i,j} > 0} A_{i,j} - \sum_{(i,j) \text{ is above } P, A_{i,j} < 0} A_{i,j}
\]

\[
= \sum_{A_{i,j} > 0} A_{i,j} - \sum_{A_{i,j} < 0} A_{i,j}.
\]

To show that

\[
\sum_{A_{i,j} > 0} A_{i,j} = \frac{(s^2 - 1)(t^2 - 1)}{24},
\]

let \( Q \) be the lattice path along the left and upper borders of \( A(s,t) \). Note that \( M_{A(s,t)}(Q) \) consists of positive entries of \( A(s,t) \). Let \( \lambda = \Phi(Q) \). By Theorem 2.2, the set of main diagonal hook lengths of \( \lambda \) equals \( M_{A(s,t)}(Q) \). It follows that

\[
|\lambda| = \sum_{A_{i,j} > 0} A_{i,j}.
\]
We now proceed to show that
\begin{equation}
|\lambda| = \frac{(s^2 - 1)(t^2 - 1)}{24}.
\end{equation}

By Theorem 1.1, there is a unique \((s,t)\)-core \(\mu\) with the largest size \(\frac{(s^2 - 1)(t^2 - 1)}{24}\).

To prove \((2.4)\), it suffices to show that \(\mu = \lambda\). Let \(l(\lambda)\) and \(l(\mu)\) denote the lengths of \(\lambda\) and \(\mu\), respectively, where the length of a partition is meant to be the number of positive parts. By Theorem 2.2, there is a lattice path \(R \in P(A(s,t))\) such that \(\mu = \Phi(R)\). Using Theorem 1.2, we find that
\begin{equation}
l(\mu) \geq l(\lambda)
\end{equation}
and
\begin{equation}
\mu_i \geq \lambda_i
\end{equation}
for all \(i\). Combining \((2.5)\) and \((2.6)\), we obtain that
\begin{equation}
\mu_1 + l(\mu) - 1 \geq \lambda_1 + l(\lambda) - 1.
\end{equation}

Next we show that
\begin{equation}
\lambda_1 + l(\lambda) - 1 \geq \mu_1 + l(\mu) - 1.
\end{equation}

Notice that the largest main diagonal hook length of \(\lambda\) is \(\lambda_1 + l(\lambda) - 1\), that is,
\begin{equation}
\max MD(\lambda) = \lambda_1 + l(\lambda) - 1.
\end{equation}

Since \(\lambda = \Phi(Q)\), by Theorem 2.2, we deduce that
\begin{equation}
MD(\lambda) = M_{A(s,t)}(Q) = \{A_{i,j} | A_{i,j} > 0, 1 \leq i \leq \lceil s/2 \rceil, 1 \leq j \leq \lceil t/2 \rceil\}.
\end{equation}

Clearly, \(A_{1,1}\) is largest among all positive entries in \(A(s,t)\). It follows from \((2.4)\)
and \((2.10)\) that
\begin{equation}
A_{1,1} = \lambda_1 + l(\lambda) - 1.
\end{equation}

On the other hand, since \(\mu_1 + l(\mu) - 1\) is the hook length of the cell in the upper-left corner of \(\mu\), Theorem 2.2 ensures the existence of an entry \(A_{i,j}\) of \(M_{A(s,t)}(R)\) such that
\begin{equation}
|A_{i,j}| = \mu_1 + l(\mu) - 1.
\end{equation}

We claim that
\begin{equation}
A_{1,1} \geq |A_{i,j}|
\end{equation}
for any entry \(A_{i,j}\). Observe that
\begin{equation}
A_{1,1} > |A_{[s/2],[t/2]}|,
\end{equation}
since
\[A_{1,1} + A_{[s/2],[t/2]} = (st - s - t) + (st + s + t - 2t \lceil s/2 \rceil - 2s \lceil t/2 \rceil) > 0.\]

Notice that \(A_{1,1}\) is the largest entry in \(A(s,t)\) and \(A_{[s/2],[t/2]}\) is the smallest entry in \(A(s,t)\). Thus \((2.13)\) implies \((2.14)\). This proves \((2.8)\).

Combining \((2.7)\) and \((2.8)\), we deduce that
\begin{equation}
\lambda_1 + l(\lambda) - 1 = \mu_1 + l(\mu) - 1.
\end{equation}

In view of \((2.11)\) and \((2.15)\), we see that
\[A_{1,1} = \mu_1 + l(\mu) - 1.\]
Thus \( A_{1,1} \) lies in \( MD(\mu) \). By Theorem 2.2, \( A_{1,1} \) belongs to \( M_{A(s,t)}(R) \). Since \( A_{1,1} > 0 \), \( R \) is the lattice path along the left and upper borders, namely, \( Q = R \) and \( \lambda = \mu \). So we conclude that \( \lambda \) is the largest \((s,t)\)-core. This completes the proof. \( \square \)

As to the case of Conjecture 1.3 for self-conjugate cores, we need some identities on the number of lattice paths in a rectangular region. Let \( m \) and \( n \) be positive integers, and \( B_{mn} \) be an \( m \times n \) diagram. The positions of the cells of the first row are \((1,1),(1,2), \ldots, (1,n)\), and so on. The set of lattice paths from the lower-left corner to the upper-right corner of \( B_{mn} \) is denoted by \( \mathcal{P}(B_{mn}) \). Let \( f(i,j) \) be the number of lattice paths in \( \mathcal{P}(B_{mn}) \) that lie below the cell \((i,j)\), possibly containing the right or lower border of the cell \((i,j)\).

**Lemma 2.4.** For positive integers \( m \) and \( n \), we have

\[
(2.16) \quad \sum_{1 \leq i \leq m, 1 \leq j \leq n} f(i,j) = \frac{mn}{2} \binom{m+n}{m}.
\]

**Proof.** It is clear that the number of lattice paths in \( \mathcal{P}(B_{mn}) \) below the cell \((i,j)\) equals the number of lattice paths above the cell \((m-i+1,n-j+1)\). Hence we have

\[
f(i,j) + f(m-i+1,n-j+1) = |\mathcal{P}(B_{mn})|.
\]

Note that the total number of lattice paths in \( \mathcal{P}(B_{mn}) \) equals \( \binom{m+n}{m} \). So we get

\[
(2.17) \quad f(i,j) + f(m-i+1,n-j+1) = \binom{m+n}{m}.
\]

Summing (2.17) over \( i \) and \( j \), we obtain (2.16). \( \square \)

**Lemma 2.5.** For positive integers \( m \) and \( n \), we have

\[
(2.18) \quad \sum_{1 \leq i \leq m, 1 \leq j \leq n} if(i,j) = \binom{m+2}{3} \binom{m+n}{m+1}
\]

and

\[
(2.19) \quad \sum_{1 \leq i \leq m, 1 \leq j \leq n} jf(i,j) = \binom{n+2}{3} \binom{m+n}{n+1}.
\]

**Proof.** Let

\[
G(m,n) = \sum_{1 \leq i \leq m, 1 \leq j \leq n} if(i,j).
\]

To prove (2.18), we claim that for \( m,n \geq 2 \),

\[
(2.20) \quad G(m,n) = G(m-1,n) + G(m,n-1) + \binom{m+1}{2} \binom{m+n-1}{m}.
\]

To prove (2.20), let \( T \) be the set of triples \((P,C_1,C_2)\), where \( P \) is a path in \( \mathcal{P}(B_{mn}) \), \( C_1 \) and \( C_2 \) are two cells above \( P \) such that they are in the same column and \( C_1 \) is at least as high as \( C_2 \). Notice that \( C_1 \) and \( C_2 \) are allowed to be the same cell.

We proceed to compute \(|T|\) in two ways. It is easily seen that \( if(i,j) \) is the number of triples \((P,C_1,C_2)\) in \( T \) with \( C_1 = (i,j) \). For \( m,n \geq 1 \), we have \(|T| = G(m,n)|.

For a given lattice path \( P \) in \( \mathcal{P}(B_{mn}) \), the cells above \( P \) form the Ferrers diagram of a partition, denoted by \( \mu \). Let \( \mu' \) be the conjugate of \( \mu \). In the \( j \)-th column of
Clearly, proof. \( \Box \)

We are now ready to prove Theorem 2.1.
Proof of Theorem 2.1 Let \( SC(s, t) \) denote the set of self-conjugate \((s, t)\)-cores. We aim to show that

\[
(2.27) \quad \sum_{\lambda \in SC(s, t)} |\lambda| = \frac{(s + t + 1)(s - 1)(t - 1)}{24} \left( \left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right).
\]

By Theorem 2.2, we find that

\[
(2.28) \quad \sum_{\lambda \in SC(s, t)} |\lambda| = \sum_{P \in P(A(s, t))} |\Phi(P)|.
\]

Using Lemma 2.3, we get

\[
(2.29) \quad \sum_{P \in P(A(s, t))} |\Phi(P)| = \frac{(s^2 - 1)(t^2 - 1)}{24} \left( \left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right)
\]

\[
- \sum_{P \in P(A(s, t))} \sum_{(i, j) \text{ is above } P} A_{i,j}.
\]

Combining (2.28) and (2.29), we obtain that

\[
(2.30) \quad \sum_{\lambda \in SC(s, t)} |\lambda| = \frac{(s^2 - 1)(t^2 - 1)}{24} \left( \left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right)
\]

\[
- \sum_{P \in P(A(s, t))} \sum_{(i, j) \text{ is above } P} A_{i,j}.
\]

By the definition (2.1) of the array \( A(s, t) \), we deduce that

\[
\sum_{P \in P(A(s, t))} \sum_{(i, j) \text{ is above } P} A_{i,j} = \sum_{P \in P(A(s, t))} \sum_{(i, j) \text{ is above } P} (st + s + t - 2sj - 2ti)
\]

\[
= (st + s + t) \sum_{1 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor, 1 \leq j \leq \left\lfloor \frac{t}{2} \right\rfloor} f(i, j)
\]

\[
- 2s \sum_{1 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor, 1 \leq j \leq \left\lfloor \frac{t}{2} \right\rfloor} jf(i, j)
\]

\[
- 2t \sum_{1 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor, 1 \leq j \leq \left\lfloor \frac{t}{2} \right\rfloor} if(i, j).
\]

Using Lemma 2.4 and Lemma 2.5 with \( m = \left\lfloor \frac{s}{2} \right\rfloor \) and \( n = \left\lfloor \frac{t}{2} \right\rfloor \), the above relation becomes

\[
(2.31) \quad \sum_{P \in P(A(s, t))} \sum_{(i, j) \text{ is above } P} A_{i,j} = (st + s + t) \left( \left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right) \left( \left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right)
\]

\[
- 2s \left( \left\lfloor \frac{s}{2} \right\rfloor + 2 \right) \left( \left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right)
\]

\[
- 2t \left( \left\lfloor \frac{t}{2} \right\rfloor + 2 \right) \left( \left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right) - 1)\).
We claim that
\[
\frac{(s^2 - 1)(t^2 - 1)}{24}\left(\left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right)
= \frac{(s + t + 1)(s - 1)(t - 1)}{24}\left(\left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right)
+ (st + s + t) \frac{\left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor}{2}\left(\left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right)
- 2t \left(\frac{\left\lfloor \frac{s}{2} \right\rfloor + 2}{3}\left(\left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right) - 1\right)
- 2s \left(\frac{\left\lfloor \frac{s}{2} \right\rfloor + 2}{3}\left(\left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right) - 1\right),
\]
which simplifies to
\[
\frac{st(s - 1)(t - 1)}{24} = (st + s + t) \frac{\left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor}{2} - \frac{t}{3}\left(\left\lfloor \frac{s}{2} \right\rfloor + 2\right)\left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor
- \frac{s}{3}\left(\left\lfloor \frac{t}{2} \right\rfloor + 2\right)\left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor.
\]
When $s$ and $t$ are coprime, at least one of $s$ and $t$ is odd. Thus, we may assume, without loss of generality, that $s$ is odd. In this case, the above relation can be easily verified. So the claim holds. Combining (2.30), (2.31) and (2.32), we arrive at (2.27), and hence the proof is complete. 

\section*{Note added in proof}

Johnson \cite{5} has proved the conjecture of Armstrong, Hanusa and Jones concerning the average size of an $(s,t)$-core. He also gave an alternative derivation of Theorem 2.1.

\section*{References}

\bibitem{1} Jaclyn Anderson, \textit{Partitions which are simultaneously $t_1$- and $t_2$-core}, Discrete Math. \textbf{248} (2002), no. 1-3, 237–243, DOI 10.1016/S0012-365X(01)00343-0. MR1892698 (2002m:05021)

\bibitem{2} Drew Armstrong, Christopher R. H. Hanusa, and Brant C. Jones, \textit{Results and conjectures on simultaneous core partitions}, European J. Combin. \textbf{41} (2014), 205–220, DOI 10.1016/j.ejc.2014.04.007. MR3219261

\bibitem{3} David Aukerman, Ben Kane, and Lawrence Sze, \textit{On simultaneous $s$-cores/$t$-cores}, Discrete Math. \textbf{309} (2009), no. 9, 2712–2720, DOI 10.1016/j.disc.2008.06.024. MR2523778 (2011a:05026)

\bibitem{4} Ben Ford, Hoang Mai, and Lawrence Sze, \textit{Self-conjugate simultaneous $p$- and $q$-core partitions and blocks of $A_n$}, J. Number Theory \textbf{129} (2009), no. 4, 858–865, DOI 10.1016/j.jnt.2008.09.012. MR2499410 (2010j:05345)

\bibitem{5} Paul Johnson, \textit{Lattice points and simultaneous core partitions}, arXiv:1502.07934.

\bibitem{6} Jørn B. Olsson and Dennis Stanton, \textit{Block inclusions and cores of partitions}, Aequationes Math. \textbf{74} (2007), no. 1-2, 90–110, DOI 10.1007/s00010-006-2867-1. MR2355859 (2008j:05311)

\bibitem{7} Richard P. Stanley and Fabrizio Zanello, \textit{The Catalan Case of Armstrong’s Conjecture on Simultaneous Core Partitions}, SIAM J. Discrete Math. \textbf{29} (2015), no. 1, 658–666, DOI 10.1137/130950318. MR3327348

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
[8] Amitabha Tripathi, *On the largest size of a partition that is both $s$-core and $t$-core*, J. Number Theory 129 (2009), no. 7, 1805–1811, DOI 10.1016/j.jnt.2008.08.009. MR2524197 (2010f:05022)

[9] Joseph Vandehey, *Containment in $(s,t)$-core partitions*, arXiv:0809.2134.

Center for Applied Mathematics, Tianjin University, Tianjin 300072, People's Republic of China
*E-mail address*: chenyc@tju.edu.cn

Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, People's Republic of China
*E-mail address*: haoyangh@mail.nankai.edu.cn

Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, People's Republic of China
*E-mail address*: wswh2@nankai.edu.cn

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use