Weak approximation for homogeneous spaces over some two-dimensional geometric global fields

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Abstract

In this article, we study obstructions to weak approximation for connected linear groups and homogeneous spaces with connected or abelian stabilizers over finite extensions of \( \mathbb{C}((x, y)) \) or function fields of curves over \( \mathbb{C}((t)) \). We show that for connected linear groups, the usual Brauer-Manin obstruction works as in the case of tori. However, this Brauer-Manin obstruction is not enough for homogeneous spaces, as shown by the examples we give. We then construct an obstruction using torsors under quasi-trivial tori that explains the failure of weak approximation.

1 Introduction

Given an algebraic variety \( Z \) over a number field \( K \), we say that \( Z \) satisfies weak approximation if \( Z(K) \) is dense in \( \prod_{v \in \Omega} Z(K_v) \) with respect to the product of the \( v \)-adic topologies, where \( K_v \) denotes the \( v \)-adic completion and \( \Omega \) denotes the set of places of \( K \). Weak approximation is satisfied for simply connected groups, and when it is not satisfied, we want to characterize the closure \( \overline{Z(K)} \) of the set of rational points in \( \prod_{v \in \Omega} Z(K_v) \) as some subset of elements satisfying certain compatibility conditions, for example, as the Brauer-Manin set \( Z(K_\Omega)^{Br} \) (c.f. Section 8.2 of [Poo17]), and this gives an obstruction to weak approximation.

Going beyond number fields, recently there’s been an increasing interest in studying analogous questions over some two-dimensional geometric global fields. As in [IA21], we consider a field \( K \) of one of the following two types:

(a) the function field of a smooth projective curve \( C \) over \( \mathbb{C}((t)) \);
(b) the fraction field of a local, henselian, two-dimensional, excellent domain $A$ with
algebraically closed residue field of characteristic 0 (e.g. any finite extension of
$\mathbb{C}((x,y))$, the field of Laurent series in two variables over the field of complex num-
bers).

In case (a), one can consider the set $\Omega$ of valuations coming from the closed points of
the curve $C$. Colliot-Thélène/Harari (c.f. [CTH15]) proved the following exact sequence
describing the obstruction to weak approximation for the case of tori $T$:

$$1 \to T(K) \to \prod_{v \in \Omega} T(K_v) \to B_\omega(T)^D \to B(T)^D \to 1$$

(1.0.1)

where $B(Z)$ (resp. $B_\omega(Z)$) is defined to be the subgroup of $Br_1(Z)/Br(K)$ containing
elements vanishing in $Br_1(Z_v)/Br(K_v)$ for all places (resp. almost all places) $v \in C^{(1)}$, and $A^D$ denotes the group of continuous homomorphisms from $A$ to $\mathbb{Q}/\mathbb{Z}(-1)$.

In case (b), one can take $\Omega$ to be the set of valuations coming from prime ideals of
height one in $A$. Izquierdo (c.f. [Izq19]) proved the exact sequence 1.0.1 in such situa-
tions for tori $T$.

Using some dévissage arguments as in [CTH15], we can first generalize this result to
a connected linear group $G$ over $K$.

**Theorem 1.1.** (Theorem 4.1) We keep the notation as above. For a connected linear group
$G$ over $K$ of the type (a) or (b), there is an exact sequence

$$1 \to G(K) \to G(K_\Omega) \to B_\omega(G)^D \to B(G)^D \to 1.$$  

(1.1.1)

In particular, the obstruction to weak approximation is controlled by the Brauer set
$G(K_\Omega)^{B_\omega} := \text{Ker}(G(K_\Omega) \to B_\omega(G)^D)$. However, such an exact sequence doesn’t generalize
to homogeneous spaces, and we do find counter-examples:

**Theorem 1.2.** (Proposition 5.1) Let $Q$ be a flasque torus such that

$$H^1(K, Q) \to \bigoplus_{v \in \Omega} H^1(K_v, Q) \to \prod_{v \in \Omega} (\hat{Q})^D$$

is not exact (for example the one constructed in Corollary 9.16 of [CTH15]). Embed $Q$ into
some $\text{SL}_n$ and let $Z := \text{SL}_n / Q$. Then $Z(K) \nsubseteq Z(K_\Omega)^{B_\omega}$.

In order to better understand the obstruction to weak approximation for homogeneous spaces over fields of the type (a) or (b), we should somehow combine the Brauer-
Manin obstruction with the descent obstruction, another natural tool used in the study
of such questions, as done by Izquierdo and Lucchini Arteche in [IA21] for the study of
obstruction to rational points.
Theorem 1.3. (Proposition 5.5) For $G$ a connected linear group, we consider a homogeneous space $Z$ under $G$ with geometric stabilizer $\tilde{H}$ such that $G^{ss}$ is simply connected and $\tilde{H}^{torf}$ is abelian. Define

$$Z(K_{\Omega})^{qt, B_{\omega}} := \bigcap_{f : W \rightarrow Z, T \text{ quasi-trivial}} f(W(K_{\Omega})^{B_{\omega}}).$$

where $f$ runs over torsors $W \rightarrow Z$ under quasi-trivial tori $T$. Then $Z(K_{\Omega})^{qt, B_{\omega}} = \overline{Z(K)}$.

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3 Notation and preliminaries

The notation will be fixed in this section and used throughout this article. The setting is pretty much the same as in [IA21] where they treated the problem of Hasse principle for such varieties.

Cohomology The cohomology groups we consider are always in terms of Galois cohomology or étale cohomology.

Fields Throughout this article, we consider a field $K$ of one of the following two types:

(a) the function field of a smooth projective curve $C$ over $\mathbb{C}((t))$;

(b) the field of fractions of a two-dimensional, excellent, henselian, local domain $A$ with algebraically closed residue field of characteristic zero. An example of such a field is any finite extension of the field of fractions $\mathbb{C}((X, Y))$ of the formal power series ring $\mathbb{C}[[X, Y]]$.

Both these two types of fields share a number of properties that hold also for totally imaginary number fields:

(i) the cohomological dimension is two,
(ii) index and exponent of central simple algebras coincide,

(iii) for any semisimple simply connected group $G$ over $K$, we have $H^1(K, G) = 1$.

(iv) there is a natural set $\Omega$ of rank one discrete valuations (i.e. with values in $\mathbb{Z}$), with respect to which one can take completions. For type (a), let $\Omega$ be the set of valuations coming from $C^{(1)}$ the set of closed points of the curve $C$. For type (b), let $\Omega$ be the set of valuations coming from prime ideals of height one in $A$.

Weak approximation is satisfied for semisimple simply connected groups over such fields (c.f. Théorème 4.7 of [CTGP04] for type (b), and §10.1 of [CTH15] for type (a)).

**Sheaves and abelian groups** For $i > 0$, we denote by $\mu_n^{\otimes i}$ the $i$-time tensor product of the étale sheaf $\mu_n$ of $n$-th roots of unity with itself. We set $\mu_n^0 = \mathbb{Z}/n\mathbb{Z}$ and for $i < 0$, and we define $\mu_n^{\otimes i} = \text{Hom}(\mu_n^{\otimes (-i)}, \mathbb{Z}/n\mathbb{Z})$. Over the fields $K$ we consider, since $K$ contains an algebraically closed field and thus all the roots of unity, we have a (non-canonical) isomorphism $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$. Denote by $\mathbb{Q}/\mathbb{Z}(i)$ the direct limit of the sheaves $\mu_m^{\otimes i}$ for all $m > 0$. By choosing a compatible system of primitive $n$-th roots of unity for every $n$ (for example $\xi_n = \exp(2\pi i/n)$), we can identify $\mathbb{Q}/\mathbb{Z}(i)$ with $\mathbb{Q}/\mathbb{Z}$.

For an abelian group $A$ (that we always suppose to be equipped with the discrete topology if there isn’t any other topology defined on it), we denote by $A^D$ the group of continuous homomorphisms from $A$ to $\mathbb{Q}/\mathbb{Z}(-1)$. The functor $A \mapsto A^D$ is an anti-equivalence of categories between torsion abelian groups and profinite groups.

**Weak approximation and Brauer-Manin obstruction** Given a set $\Omega$ of places of $K$, let $Z(K_\Omega) := Z(K_\Omega)$ be the topological product where each $Z(K_v)$ is equipped with the $v$-adic topology, and $K_v$ denotes the completion at $v$. We say that a $K$-variety $Z$ satisfies weak approximation with respect to $\Omega$ if the set of rational points $Z(K)$, considered through the diagonal map as a subset of $Z(K_\Omega)$, is a dense subset. Weak approximation is a birational invariant, which is a consequence of the implicit function theorem for $K_v$ (c.f. Theorem 9.5.1 of [CTS21] and the argument in the Proposition 12.2.3 of [CTS21]).

There are varieties for which weak approximation fails, thus we try to introduce obstructions that give a more precise description of the closure of $Z(K)$ inside $Z(K_\Omega)$, thus explaining such failures.

For a variety $Z$, the cohomological Brauer group $\text{Br}(Z)$ is defined to be $H^2(Z, \mathbb{G}_m)$. Define also

- $\text{Br}_0(Z) := \text{Im}(\text{Br}(K) \rightarrow \text{Br}(Z))$,
\[ \begin{align*}
\text{Br}_1(Z) & := \ker(\text{Br}(Z) \to Br(Z_K)), \\
\text{Br}_a(Z) & := \text{Br}_1(Z)/\text{Br}_0(Z), \\
\mathcal{B}(Z) & := \ker(\text{Br}_a(Z) \to \prod_{v \in \Omega} \text{Br}_a(Z_{K_v})), \\
\mathcal{B}_S(Z) & := \ker(\text{Br}_a(Z) \to \prod_{v \in S} \text{Br}_a(Z_{K_v})) \text{ where } S \text{ is a finite subset of } \Omega, \\
\mathcal{B}_\omega(Z) & : \text{ subgroup of } \text{Br}_a(Z) \text{ containing the elements vanishing in } \text{Br}_a(Z_{K_v}) \text{ for almost all places } v \in \Omega.
\end{align*} \]

Similar to the exact sequence from Class Field Theory for number fields, there is also an exact sequence (c.f. Prop. 2.1.(v) of [CTH15] and Thm. 1.6 of [Izq19])

\[ \text{Br}(K) \to \bigoplus_{v \in \Omega} \text{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0 \]

and a Brauer-Manin pairing

\[ Z(K_\Omega) \times \mathcal{B}_\omega \to \mathbb{Q}/\mathbb{Z} \]

\[ ((P_v), \alpha) \mapsto \sum_{v \in \Omega} < P_v, \alpha > \]

with the following property: \( Z(K) \) is contained in the subset \( Z(K_\Omega)^{\mathcal{B}_\omega} \) of \( Z(K_\Omega) \) defined as \( \{(P_v) \in Z(K_\Omega) : ((P_v), \alpha) = 0 \text{ for all } \alpha \in \mathcal{B}_\omega(Z)\} \): the set of points that are orthogonal to \( \mathcal{B}_\omega(Z) \). When \( Z(K) \) is non-empty and \( \overline{Z(K)} = Z(K_\Omega)^{\mathcal{B}_\omega} \), we say that the Brauer-Manin obstruction with respect to \( \mathcal{B}_\omega \) is the only one to weak approximation.

**Tate-Shafarevich groups** For a Galois module \( M \) over the field \( K \) and an integer \( i \geq 0 \), we define the following Tate-Shafarevich groups:

\[ \begin{align*}
\mathfrak{III}^i(K, M) & := \ker(H^i(K, M) \to \prod_{v \in \Omega} H^i(K_v, M)), \\
\mathfrak{III}_S(K, M) & := \ker(H^i(K, M) \to \prod_{v \in S} H^i(K_v, M)) \text{ where } S \text{ is a finite subset of } \Omega, \\
\mathfrak{III}_\omega^i(K, M) & : \text{ subgroup of } H^i(K, M) \text{ containing the elements vanishing in } H^i(K_v, M) \text{ for almost all places } v \in \Omega.
\end{align*} \]
Algebraic groups and homogeneous spaces  For a linear algebraic $K$-group $G$, the following notation will be used:

- $D(G)$: the derived subgroup of $G$,
- $G^0$: the neutral connected component of $G$,
- $G^d := G/G^0$ the group of connected components of $G$, which is a finite group,
- $G^u$: the unipotent radical of $G^0$,
- $G^{\text{red}} := G^0/G^u$ which is a reductive group,
- $G^{\text{ss}} := D(G^{\text{red}})$ which is a semisimple group,
- $G^{\text{tor}} := G^{\text{red}}/G^{\text{ss}}$ which is a torus,
- $G^{\text{ssu}} := \ker(G^0 \to G^{\text{tor}})$ which is an extension of $G^{\text{ss}}$ by $G^u$,
- $G^{\text{torf}} := G/G^{\text{ssu}}$ which is an extension of $G^d$ by $G^{\text{tor}}$,
- $\hat{G}$: the Galois module of the geometric characters of $G$.

A unipotent group over $K$ a field of characteristic 0 is isomorphic to an affine space $A^a_K$, thus $K$-rational and satisfies weak approximation.

A torus $T$ is said to be quasi-trivial if $\hat{T}$ is an induced $\text{Gal}(\bar{K}/K)$-module.

4 Weak approximation for connected linear groups

The aim of this section is to prove the following theorem which concerns the Brauer-Manin obstruction to weak approximation for connected linear groups:

**Theorem 4.1.** Let $K$ be a field of the type (a) or (b) and let $G$ be a connected linear group over $K$. Then there is an exact sequence

$$1 \to G(K) \to G(K_\Omega) \to B_\omega(G)^D \to B(G)^D \to 1.$$  

We complete the proof by a series of lemmas.

**Lemma 4.2.** Suppose that Theorem 4.1 holds for $G^n$, then it also holds for $G$.

**Proof.** It follows from the fact that all of the operations in the exact sequence above take products to products. (For example, the closure of a product is the product of closures.)
An exact sequence

$$1 \to F \to H \times_K P \to G \to 1$$

where $H$ is semi-simple simply connected, $P$ is a quasitrivial $K$-torus and $F$ is finite and central is called a special covering of the reductive $K$-group $G$. For any reductive $K$-group $G$, there exists an integer $n > 0$ such that $G^n$ admit a special covering. By the lemma above, we can suppose without loss of generality that $G$ admits a special covering itself.

**Lemma 4.3.** The following sequence is exact

$$1 \to G(K) \to \prod_{v \in \Omega} G(K_v) \to B_\omega(G)^D$$

where $G$ is a reductive connected linear group.

**Proof.** The proof is exactly the same as in Lemma 9.6 of [CTH15], using the special covering and the fact that $H$ and $P$ both satisfy weak approximation (so does their product) and $H^1(K, H) = H^1(K, P) = H^1(K, H \times_K P) = 0$. Let $S$ be a finite set of places in $\Omega$. We have a commutative diagram where the rows are exact and the columns are complexes.

\[
\begin{array}{cccccc}
H \times_K P(K) & \to & G(K) & \to & H^1(K, F) & \to 1 \\
\downarrow & & \downarrow & & \downarrow & \\
\prod_{v \in S} H \times_K P(K_v) & \to & \prod_{v \in S} G(K_v) & \to & \prod_{v \in S} H^1(K_v, F) & \to 1 \\
\downarrow & & \downarrow & & \downarrow & \\
B_S(H \times P)^D & \to & B_S(G)^D & \to & \text{III}^2_S(\hat{F})^D & \to 1 \\
\end{array}
\]

Recall that we have an isomorphism $B_S(H \times P) \simeq \text{III}^2_S(\hat{P})$. The exactness of the last row comes from the fact that $H^1(K, \hat{F}) \simeq \text{Ker}(\text{Br}_a(G) \to \text{Br}_a(H \times_K P))$ and $\text{Br}_a(H \times_K P) \simeq H^2(K, \hat{P})$ (Cor 7.4 and Lemme 6.9 of [San81]). The commutativity of the right-bottom square can be deduced from Lemme 8.11 of [San81].

Since $G$ also admits a flasque resolution, with the same proof as in Corollary 9.9 of [CTH15], we have the following result taking the limit on $S$:

$$1 \to G(K) \to G(K_\Omega) \to B_\omega(G)^D$$

\[\square\]

**Lemma 4.4.** The following sequence is exact

$$1 \to G(K) \to \prod_{v \in \Omega} G(K_v) \to B_\omega(G)^D$$

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where $G$ is a connected linear group over $K$.

**Proof.** We use the resolution $1 \to G^u \to G \to G^{\text{red}} \to 1$ and the induced diagram

\[
\begin{array}{c}
G^u(K) \to G(K) \to G^{\text{red}}(K) \to 1 \\
\downarrow \quad \downarrow \quad \downarrow \\
G^u(K_\Omega) \to G(K_\Omega) \to G^{\text{red}}(K_\Omega) \\
\downarrow \quad \downarrow \\
\mathcal{B}_\omega(G)^D \to \mathcal{B}_\omega(G^{\text{red}})^D.
\end{array}
\]

taking into account the vanishing of $H^1(K, G^u)$. The unipotent radical $G^u$ satisfies weak approximation (see Section 3). The exactness of the rightmost column is known from the previous lemma, then a diagram chasing gives the result. \qed

**Lemma 4.5.** The sequence $G(K_\Omega) \to \mathcal{B}_\omega(G)^D \to \mathcal{B}(G)^D$ is exact, where $G$ is a connected linear group over $K$.

**Proof.** As in the previous lemmas, we can first treat the case where $G$ is reductive. Suppose now that $G$ is reductive, we use a coflasque resolution

\[1 \to P \to G' \to G \to 1\]

where $P$ is a quasitrivial torus and $G'$ fits into the exact sequence

\[1 \to G^{\text{sc}} \to G' \to T \to 1\]

where $T$ is a coflasque torus and $G^{\text{sc}}$ is a semisimple simply connected group. Suppose that the exactness of the sequence in question is known for $G'$ (replacing $G$). Then we can prove the exactness for $G$ by chasing in the following diagram

\[
\begin{array}{c}
\prod_{v \in S} G'(K_v) \to \prod_{v \in S} G(K_v) \to 0 \\
\downarrow \quad \downarrow \\
\mathcal{B}_\omega(P)^D \to \mathcal{B}_\omega(G')^D \to \mathcal{B}_\omega(G)^D \to 0 \\
\downarrow \quad \downarrow \\
\mathcal{B}(P)^D \to \mathcal{B}(G')^D \to \mathcal{B}(G)^D \to 0.
\end{array}
\]

The leftmost arrow is an isomorphism by Proposition 2.6 of [CTH15]. The last two rows are exact, and this can be proved by chasing the following diagram, as done in the proof of Lemma 4.4 of [Bor96].
(The exactness of the rows can be deduced from Proposition 6.10 of [San81] and the fact that \( \text{Pic}(P) = H^1(K, \widehat{P}) = 0 \) since \( \widehat{P} \) is a permutation module.)

The exactness of the sequence in question is known for \( T \) (Corollary 9.9 of [CTH15] for \( K \) of type (a), and Theorem 4.9 of [Izq19] for \( K \) of type (b)), by a similar diagram chasing, we can prove the result for \( G' \), thus completing the proof. The essential condition we need in this diagram chasing is that we should have an injection \( \mathbb{B}_\omega(G') \hookrightarrow \mathbb{B}_\omega(T) \), or equivalently a surjection \( \mathbb{B}_\omega(T) \twoheadrightarrow \mathbb{B}_\omega(G') \). This is indeed true because we have an exact sequence \( \mathbb{B}_a(T) \rightarrow \mathbb{B}_a(G') \rightarrow \mathbb{B}_a(G'^{sc}) \) and \( \mathbb{B}(G'^{sc}) = \mathbb{B}(K) \) since \( G'^{sc} \) a simply connected group over a field of characteristic 0 (c.f. Corollary in §0 of [Gil09]).

Now for a connected linear group \( G \) (not necessarily reductive), we use again the resolution \( 1 \rightarrow G^u \rightarrow G \rightarrow G^{\text{red}} \rightarrow 1 \) and the induced commutative diagram

\[
\begin{array}{ccccccccc}
\prod_{v \in S} G(K_v) & \longrightarrow & \prod_{v \in S} G^{\text{red}}(K_v) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{B}_\omega(G)^D & \longrightarrow & \mathbb{B}_\omega(G^{\text{red}})^D \\
\downarrow & & \downarrow & & \\
\mathbb{B}(G)^D & \longrightarrow & \mathbb{B}(G^{\text{red}})^D & & \\
\end{array}
\]

taking into account the vanishing of \( H^1(K_v, G^u) \). The injectivity of the middle line follows from the fact that \( \mathbb{B}_a(G^u) = \mathbb{B}_a(A^n_K) = (\mathbb{B}_a(A^1_K))^n = 0 \) (c.f. Lemme 6.6 of [San81], Theorem 4.5.1(viii) of [CTS21]). Then a diagram chasing gives the desired result.

\[\Box\]

5 Weak approximation for homogeneous spaces

Now we look at weak approximation for homogeneous spaces over such fields \( K \). The usual Brauer-Manin obstruction we used in last section is not enough in this case, as
shown by the following construction of examples:

**Proposition 5.1.** Let \( Q \) be a flasque torus such that
\[
H^1(K, Q) \to \bigoplus_{v \in \Omega} H^1(K_v, Q) \to \Pi^1_{\omega}(\hat{Q})^D
\]
is not exact. Such a torus exists (see Corollary 9.16 of [CTH15] for a construction.) Embed \( Q \) into some \( SL_n \) and let \( Z := SL_n / Q \). Then \( \overline{Z(K)} \subset Z(K)_{B^n} \).

**Proof.** We consider the following commutative diagram with exact rows:
\[
\begin{array}{c}
Z(K) \xrightarrow{f_1} H^1(K, Q) \xrightarrow{f_3} 1 \\
\downarrow{f_2} \hspace{1cm} \downarrow{f_4} \hspace{1cm} \downarrow{f_6} \\
Z(K_\Omega) \xrightarrow{f_4} \prod_{v \in \Omega} H^1(K_v, Q) \xrightarrow{f_6} 1 \\
\downarrow{f_5} \hspace{1cm} \\
1 \xrightarrow{f_7} B_\omega(Z) \xrightarrow{f_7} \Pi^1_{\omega}(\hat{Q})^D \xrightarrow{f_7} 1.
\end{array}
\]
The vanishing of \( Br_a(SL_n) \) on the bottom-left corner comes from the Corollary of [Gil09] (section 0). We prove the result by diagram chasing. Since the right column is not exact, there exists \( a \in \prod_{v \in \Omega} H^1(K_v, Q) \) such that \( f_6(a) \) vanishes but \( a \notin f_3(H^1(K, Q)) \). Since \( f_4 \) is surjective, we can find \( b \in Z(K_\Omega) \) such that \( f_4(b) = a \). By the commutativity of the bottom square, \( f_5(b) \) vanishes. We prove that \( b \notin \overline{Z(K)} \) by contradiction. We suppose \( b \in \overline{Z(K)} \). The torus \( Q \) being flasque implies that \( \prod_{v \in \Omega} H^1(K_v, Q) \) is finite (c.f. Proposition 9.1 of [CTH15]), thus the preimage \( f_4^{-1}(a) \) is open (\( f_4 \) is continuous, c.f. [Čes15]), containing \( b \in \overline{Z(K)} \), so we should be able to find \( c \in Z(K) \) lying in \( f_4^{-1}(a) \). Then \( f_3(f_1(c)) = a \) by the commutativity of the top square, contradicting \( a \notin \text{Im } f_3 \). Therefore, we found \( b \in Z(K_\Omega)_{B^n} \setminus \overline{Z(K)} \).

Therefore, we need to find other obstructions. In the rest of this section, we consider homogeneous spaces \( Z \) under a connected linear group \( G \) with geometric stabilizer \( \bar{H} \) such that \( G^{ss} \) is simply connected and \( \bar{H}^{torf} \) is abelian. This is the assumption already used in [IA21] and [Bor96], and is satisfied by every homogeneous space under a connected linear \( K \)-group with connected stabilizers (c.f. Lemma 5.2 in [Bor96]).

We can find \( Z \leftrightarrow W \rightarrow W' \) such that

- \( W \) is a \( K \)-homogeneous space under \( G \times T \),
- \( T \) is a quasitrivial torus into which \( H^{torf} \) injects, where \( H^{torf} \) is the canonical \( K \)-form of \( \bar{H}^{torf} \) associated to \( Z \),
• $W \to Z$ is a $T$-torsor,

• $W'$ is the quotient variety $Z/G^{ss}$, which is also a homogeneous space of $G^{tor} \times T$ with geometric stabilizer $\overline{H}^{tor}$ and the fibers of $W \to W'$ are homogeneous spaces of $G^{ssu}$ with geometric stabilizers $\overline{H}^{ssu}$.

Indeed, we can embed $H^{tor}$ in a quasi-trivial torus $T$ and consider the diagonal morphism $H \to G \times T$ induced by the inclusion $H \hookrightarrow G$ and the composition $H \to H^{tor} \to T$. Then we define $W = (G \times T)/H$, and $W \to Z$ is induced by the projection to the first coordinate. Define $W'$ to be the quotient variety $W/G^{ssu}$ and we get what we want.

**Proposition 5.2.** The fiber $W_P$ above a $K$-point $P \in W'(K)$ satisfies weak approximation.

**Proof.** Since $H^{ss}$ is semi-simple, we consider its simply connected covering $1 \to F \to H^{sc} \to H^{ss} \to 1$ where $F$ is finite and $H^{sc}$ is simply connected.

We first prove that this covering induces isomorphisms $H^1(K, H^{ss}) \simeq H^2(K, F)$ and $H^1(K_v, H^{ss}) \simeq H^2(K_v, F)$ for all $v \in \Omega$. For $K$ and $K_v$, the two conditions in Theorem 2.1 of [CTGP04] are satisfied, and thus we have a bijection $H^1(K, H^{ad}) \to H^2(K, \mu)$ coming from the central isogeny

$$1 \to \mu \to H^{sc} \to H^{ad} \to 1$$

associated to the center $\mu$ of $H^{sc}$. Since $F$ is contained in the center $\mu$, we have the exact sequence

$$1 \to \mu/F \to H^{ss} \to H^{ad} \to 1$$

and the commutative diagram with exact rows

$$
\begin{array}{ccccccc}
H^1(K, \mu/F) & \to & H^1(K, H^{ss}) & \to & H^1(K, H^{ad}) & \to & H^2(K, \mu/F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(K, \mu/F) & \to & H^2(K, F) & \to & H^2(K, \mu) & \to & H^2(K, \mu/F). \\
\end{array}
$$

Then the four-lemma gives the surjectivity of $H^1(K, H^{ss}) \to H^2(K, F)$. The injectivity comes from the vanishing of $H^1(K, H^{sc})$. The proof is the same when $K$ is replaced by $K_v$.

Then, we prove that

$$H^2(K, F) \to \bigoplus_{v \in S} H^2(K_v, F)$$

is surjective, where $S$ is a finite set of places. This is proved by chasing the following commutative diagram.
The row in the middle is exact (c.f. [Izq16] Theorem 2.7 applying $d = 0$. The surjectivity of $\oplus_{v \in S} H^2(K_v, F) \to H^0(K, \hat{F})^D$ comes from the injectivity taking the duals $H^0(K, \hat{F}) \leftrightarrow \oplus_{v \in S} H^0(K_v, \hat{F})$. This yields the surjectivity of $H^2(K, F) \to \prod_{v \in S} H^2(K_v, F)$, which is $H^1(K, H^{ss}) \to \oplus_{v \in S} H^1(K_v, H^{ss})$.

Finally, by Lemme 1.13 of [San81], we have $H^1(K, H^{ss}) = H^1(K, H^{ssu})$ and $H^1(K_v, H^{ss}) = H^1(K_v, H^{ssu})$. With the same argument as in Proposition 3.2 of [San81], we can prove that $G^{ssu}$ satisfies weak approximation (c.f. Proposition 4.1 of [Bor66] for the vanishing of $H^1(K, H^u)$ and $H^1(K_v, H^u)$. The unipotent radical $H^u$ satisfies weak approximation). Therefore, we consider the commutative diagram with exact rows (the set $W_P(K)$ is non-empty by Proposition 3.1 of [IA21]):

\[
\begin{array}{cccccc}
G^{ssu}(K) & \longrightarrow & W_P(K) & \longrightarrow & H^1(K, H^{ssu}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod_{v \in S} G^{ssu}(K_v) & \longrightarrow & \prod_{v \in S} W_P(K_v) & \longrightarrow & \prod_{v \in S} H^1(K_v, H^{ssu}) & \longrightarrow & 1.
\end{array}
\]

A diagram chasing then gives the desired result. 

**Proposition 5.3.** We have $W(K_\Omega)^{B_{w}(W)} = \overline{W(K)}$.

**Proof.** Consider $(P_v) \in W(K_\Omega)^{B_{w}(W)}$, and any open subset $U$ containing $(P_v)$, we want to find $Q \in U \cap W(K)$ using fibration methods.

Denote by $(P'_v)$ the image of $(P_v)$ under the induced map $g : W(K_\Omega) \to W'(K_\Omega)$. By the functorality of the Brauer-Manin pairing, $(P'_v)$ lies in $W'(K_\Omega)^{B_{w}(W')}$. Since $W \to W'$ is smooth, and the $K_v$ are henselian, we have open maps $W(K_v) \to W'(K_v)$, and thus $g$ is open. In particular, $V = g(U) \subseteq W'(K_\Omega)$ is also open. Since $W'$ is a torus (c.f. Theorem 3.2 of [IA21]), we have $W'(K_\Omega)^{B_{w}(W')} = \overline{W'(K)}$. Therefore, there exists $P' \in V \cap W'(K)$. Let $(Q_v) \in g^{-1}(P') \cap U \subseteq W(K_\Omega)$, and it can be seen as in $W_{P'}(K_\Omega)$ too, as shown in the following diagram.
Since for all $Z$, the functorality of the Brauer-Manin pairing, we have

$$
\prod_{\nu \in \Omega} \text{Spec } K_{\nu}.
$$

This diagram also shows that $W_{p^r}(K_{\Omega}) \subseteq W(K_{\Omega})$, and $U_{|_{W_{p^r}(K_{\Omega})}} := U \cap W_{p^r}(K_{\Omega})$ is an open neighborhood of $(Q_{v})$ in $W_{p^r}(K_{\Omega})$. Since $W_{p^r}$ satisfies weak approximation by proposition 5.2, there exists $Q \in W_{p^r}(K) \cap U_{|_{W_{p^r}(K_{\Omega})}}$, and thus in $W(K) \cap U$, which proves that $W(K_{\Omega})^{B_{\omega}(W)} \subseteq \overline{W(K)}$.

As in [IA21], we are naturally led to consider the following definition.

**Definition 5.4.** For an arbitrary $K$-variety $Z$, we define

$$Z(K_{\Omega})^{qt,B_{\omega}} := \bigcap_{f: W \rightarrow Z, T \text{ quasi-trivial}} f(W(K_{\Omega})^{B_{\omega}}).$$

Using the torsor $W \rightarrow Z$ defined above, we have $Z(K_{\Omega})^{qt,B_{\omega}} \subseteq \overline{Z(K)}$. In fact, this is an equality.

**Proposition 5.5.** Let the notation be as above. Then $Z(K_{\Omega})^{qt,B_{\omega}} = \overline{Z(K)}$.

**Proof.** Since for all $f$, we have

$$Z(K) \subseteq f(W(K)) \subseteq f(W(K_{\Omega})^{B_{\omega}}),$$

so $Z(K) \subseteq Z(K_{\Omega})^{qt,B_{\omega}}$. To prove Proposition 5.5, it is equivalent to prove that $Z(K_{\Omega})^{qt,B_{\omega}}$ is closed. It suffices to prove that $f(W(K_{\Omega})^{B_{\omega}})$ is closed for every torus $f : W \rightarrow Z$ with $T$ quasi-trivial. Since $f$ is smooth and the $K_{\nu}$ are henselian, the induced map $f : W(K_{\Omega}) \rightarrow Z(K_{\Omega})$ is open. Now we’ll prove that $f(W(K_{\Omega})^{B_{\omega}})$ doesn’t meet $f(W(K_{\Omega}) \setminus W(K_{\Omega})^{B_{\omega}})$, and combined with the surjectivity of $f$, we’ll get $f(W(K_{\Omega})^{B_{\omega}})$ closed.

Fixing $w \in W(K)$ which is a $K$-point of $W$, we have the canonical isomorphism $\text{Br}_{a}(W) \simeq \text{Br}_{1,w}(W)$. Define the map $i_{w} : T \rightarrow W$ as $t \mapsto t.w$, and we have an induced map $i^{*}_{w} : \text{Br}_{1,w}(W) \rightarrow \text{Br}_{1,e}(T)$. Let $\alpha$ be an element in $\text{B}_{\omega}(W)$ and we still denote by $\alpha$ its image in $\text{Br}_{1,w}(W)$. Let $(x_{v}, y_{v}) \in W(K_{\Omega})$ above $(P_{v})$. We’ll prove that

$$\sum_{\nu \in \Omega} < x_{\nu}, \alpha > = \sum_{\nu \in \Omega} < y_{\nu}, \alpha >.$$  

Let $m : T \times W \rightarrow W$ denotes the action of $T$ on $W$. Since $W_{F_{\nu}}$ is a $K_{\nu}$-torsor under $T$, there exists $t_{v} \in T(K_{\nu})$ such that $x_{v} = t_{v}.y_{v}$. By the functorality of the Brauer-Manin paring, we have $< x_{v}, \alpha > = < (t_{v}, y_{v}), m^{*}\alpha >$. By
Lemme 6.6 of [San81], we have $\text{Br}_{1,(w,e)}(W \times T) = \text{Br}_{1,w}(W) \times \text{Br}_{1,e}(T)$, and the formula (24) in [BD13] gives $m^*\alpha = p_T^*i^*_w(\alpha) + p_W^*\alpha$ where $p_T$ and $p_W$ are the two natural projections. Then $< x_v, \alpha >= < t_v, i^*_w(\alpha) > + < y_v, \alpha >$. But $P$ is a quasi-trivial torus, we have $B_\omega(P) \simeq B(P)$ (c.f. Prop. 2.6 and §8.1 of [CTH15]). Therefore, by the exact sequence in Theorem 4.1, the map $P(K_\Omega) \rightarrow B_\omega(P)^D$ is 0, i.e. $\sum_{\nu \in \Omega} < t_v, i^*_w(\alpha) >= 0$ and we get what we want.

Remark 5.6. Actually the above proof also shows that

$$Z(K_\Omega)^{qt,B_\omega} = \bigcap_{f:W \rightarrow Z,T \text{ quasi-trivial}} \bigcap_{\alpha \in B_\omega(W)} f(W(K_\Omega)^{\alpha}).$$

With this description, we can prove as done in Theorem 6.4 of [IA21] that

$$Z(K_\Omega)^{tor} \subseteq Z(K_\Omega)^{qt,B_\omega}. \quad (5.6.1)$$

But this doesn't necessarily give an obstruction to weak approximation since we don't know if $Z(K_\Omega)^{tor}$ is closed. We don't know if (5.6.1) is an equality. One could wonder whether a “purely descent” description of $Z(K_\Omega)^{qt,B_\omega}$ exists.

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