Fermion Zero Modes and Cosmological Constant

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Abstract

A general condition for the existence of fermion zero modes is derived for the M-5-brane, the M-2-brane and the $D = 4, N = 2$ Majumdar-Papapetrou 0-brane. The fermion zero modes of these $p$-branes do not exist if the supersymmetry spinor generator goes to a constant at the horizon and they exist only if it vanishes there. In particular it is shown that the fermion zero mode of the M-2-brane in $D = 11$ can be forbidden from existence if Rarita-Schwinger gamma tracelessness condition is imposed on the gravitino field. Non-existence of fermion zero mode is interpreted, in analogy to the three dimensional example of Becker et. al., as a world with zero cosmological constant without supersymmetric excited states. Also derived are the spin of the M-5-brane and its 3-form electric and magnetic dipole moments.

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1. Introduction

The cosmological constant problem in a supersymmetric theory is to reconcile the observed zero cosmological constant with the observed broken supersymmetry of the world. Witten \cite{1} has made a general observation that in three dimension the cosmological constant can be zero without the supersymmetric multiplet of physical states due to conically singular geometry of any massive three dimensional spacetimes. Three dimensional \( N = 2 \) supersymmetric abelian Higgs model coupled to supergravity has since been studied by Becker, Becker, and Strominger \cite{2} as an evidence of Witten’s claim. Their calculation involves proving that the vortex soliton solution of the model preserves some supersymmetries and yet fermion zero mode does not exist since the gravitino is not normalizable.

This paper will show that analogous results to that of Becker, Becker, and Strominger can be obtained in higher dimensional supergravities. In particular we will study the existence of the fermion zero modes of 2-brane and 5-brane solitons in \( D = 11 \) and show that for some functional choice of the spinor generator fermion zero modes can be forbidden from existence.

The first study of the supergravity fermion zero modes was done by Aichelburg and Embacher \cite{3} for the \( D = 4, N = 2 \) Majundar-Papapetrou 0-branes and the existence of the fermion zero modes was later related \cite{4} to the Rarita-Schwinger (RS) condition \( \Gamma^m \psi_m = 0 \). We will re-examine the existence of the fermion zero modes in this theory, and show that the RS condition is not a necessary condition but merely a sufficient \((D = 4)\) condition for the existence of the fermion zero modes.

More importantly we show that the fermion zero modes of \( D = 11 \) M-2-brane \cite{5} in fact do not exist if the RS condition is imposed i.e. it is not even a sufficient condition (for the M-2-brane.) This opens up the possibility that the RS condition needs not be imposed on the fermion zero modes in general. In fact it is known \cite{3} that the RS condition is not preserved at the second order in spinor parameter if imposed at the first order. In their original work \cite{6} the condition of Rarita and Schwinger is only one of a number of conditions imposed on \( \psi_m \) in order to make it behave as a spin \( \frac{3}{2} \) four dimensional particle. But in the context of constructing fermion zero modes described in this paper there seems to be absolutely no reason to impose this condition. We will argue that the RS condition should not be related to the existence of the fermion zero modes. Instead the desired existence will be shown to be related to the value of the supersymmetry generator at the horizon. The RS condition does fix that value but it is not the unique way.
The fermion zero modes exist if the gravitino is normalizable. The normalizability of the gravitino will be shown to be related to the behavior of $\epsilon$ at $r \to 0$ i.e. at the horizon. All fermion zero modes (if they exist) are equivalent if $\epsilon$ goes to the same constant as $r \to \infty$ but, as we will see, the gravitino is normalizable if and only if $\epsilon$ vanishes as $r \to 0$. In other words the supersymmetry must be completely broken at the horizon for the existence of the fermion zero mode! So long as $\epsilon \sim r^{-\delta}$ ($\delta \geq 0$) as $r \to 0$, the normalizability of $\psi_m$ is forbidden. This general result will be derived for the $D = 4$, $p = 0$ case and the M-2-brane as well as for the M-5-brane case. For the M-2-brane case the RS condition makes $\epsilon =$ constant and as a result the normalizability of $\psi_m$ is forbidden in this case.

It has been shown by Gibbons [7] for the Majumdar-Papapetrou 0-brane and more recently by Gibbons and Townsend [8] for the M-p-branes that all p-branes of interest here interpolate between two maximally supersymmetric vacua: $\mathcal{M}_D$ at $r \to \infty$ and $AdS_{p+2} \times S^{D-p-2}$ at $r \to 0$. The vacuum at infinity has zero cosmological constant and only if the supersymmetry is completely broken at the horizon fermion zero modes exist! The generalization of the observation by Gibbons and Townsend is then, for $\epsilon \to 0$ case, these p-branes interpolate between $\mathcal{M}_D$ with supersymmetric excited states and $AdS_{p+2} \times S^{D-p-2}$ with zero supersymmetry generator. For the alternative behavior of $\epsilon \to 0$ constant the p-branes interpolate between $\mathcal{M}_D$ without fermion zero mode and $AdS_{p+2} \times S^{D-p-2}$ with nonzero supersymmetry generator. There is then a sort of duality between zero cosmological constant universe with nonsupersymmetric states and a negative cosmological constant world with nonzero supersymmetric generator and vice versa.

It should be emphasized that the present paper does not claim to solve the cosmological constant problem [9] in all its gory details. The models presented here, consistent as they are, may or may not correspond to the reality. It also raises a question of why the amount of supersymmetry at the horizon is unbroken in such a way that the observed nonsupersymmetric universe has zero cosmological constant. But the question is no different from question such as why the observed four dimensional universe is compactified, in some very special way, from a higher dimensional manifold. Anthropic considerations may or may not have the answer.

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1 We will use the isotropic coordinates so that $r \to 0$ is the horizon limit. $r$ is the radial distance transverse to the world volume.

2 This is a classical result but quantum Casimir energies cancel between equal contributions of boson and fermion loops as explained in [9] and first derived in [10]. Such arguments are still valid because these p-branes preserve some supersymmetry.
The main results of this paper are the connection between the supersymmetry breaking at the horizon and the existence (or nonexistence) of the fermion zero modes and extending the results of reference [2] to higher dimensional theories. In section 2 we illustrate the generation of fermion zero modes using the M-5-brane as an example. We will see that the result is a spinning M-5-brane, called fivebrane superpartner, with two dipole moments of 3-form gauge field. We then derive the conditions for the existence of the fermion zero modes for three \( p \)-branes.

2. Fivebrane Superpartner

We adopt the eleven dimensional convention of [11] and split eleven coordinates \((x^m = x^0, x^+, x^-, x^8, x^7, x^6, x^1, \ldots, x^5)\) into \( p + 1 \) coordinates \( x^a \) tangent to the world volume of the M-\( p \)-brane and \( D - p - 1 \) coordinates \( x^\alpha \) transverse to the world volume. For the case of \( p = 5 \) we will take \((x^a = x^0, x^+, x^-, x^8, x^7, x^6)\) and \((x^\alpha = x^1, \ldots, x^5)\). For the \( p = 2 \) case to be discussed later it will be understood that there are three \( x^a \) and eight \( x^\alpha \) but it will not be necessary to say which of the spatial \( x^m \) belong to \( x^a \). The bosonic 5-brane solution of Güven [12], described by the following fields

\[
d s^2 = \frac{\eta_{ab}}{f} d x^a d x^b + f^2 \delta_{\alpha\beta} d x^\alpha d x^\beta, \quad F_{\alpha_1 \ldots \alpha_4} = - s \varepsilon_{\alpha_1 \ldots \alpha_4} \partial_\alpha (f^3), \quad \psi_m = 0 \quad (1)
\]

is invariant under the following eleven dimensional supersymmetry transformation

\[
\delta \psi_m = \left[ \partial_m + \frac{1}{4} \omega^\hat{\mathfrak{m}}_m \hat{\mathfrak{m}} \sigma_{\hat{\mathfrak{m}}n} \right] \Gamma_{np} + \frac{1}{288} (\Gamma_m^{npqr} - 8 \delta_m^{n} \Gamma^{pqrs}) F_{npqr} \right] \epsilon \quad (3)
\]

\[
\delta A_{mn} = - 6 \bar{\epsilon} \Gamma_{[mn]} \psi_p, \quad \delta e^\hat{m}_m = 2 \bar{\epsilon} \tilde{\Gamma}^m \psi_m \quad (2)
\]

where \( \epsilon \) is an anticommuting Majorana spinor, \( s = \pm 1 \) the sign of the fivebrane magnetic charge, \( \varepsilon_{12345} = \varepsilon_{12345}^{12345} = 1, \{\tilde{\Gamma}_m, \tilde{\Gamma}_n\} = 2 \eta^mn, \) and \( \Gamma_m = e^\hat{m}_m \hat{\mathfrak{m}} \Gamma_n \). The hatted quantities are flat space ones and \( \Gamma \) with multiple indices product of dirac matrices with all indices different. The supertorsionless equation consistent with equation (3) is \( de^\hat{m}_m = e^\tilde{\mathfrak{m}} \wedge \omega^\mathfrak{m}_n \hat{\mathfrak{m}} + \) terms involving \( \psi \) (which we do not need here.) The function \( f = f(x^\alpha) \) satisfies \( \delta_{\alpha\beta} \partial_\alpha \partial_\beta f^3 = 0 \) which solution of interest is

\[
f = \left( 1 + \frac{6M}{|\vec{r}|^3} \right)^{\frac{1}{3}} \quad (4)
\]

where \( \vec{r} = (x^1, \ldots, x^5) \). Invariance of the solution (11) under (2) is easy to see. But the solution (1) is invariant [12] under (3) only if \( \epsilon = f^{-\frac{1}{4}} \lambda \) with constant \( \lambda \) satisfying \( (1 + \tilde{\Gamma} \lambda) = 0 \) where \( \tilde{\Gamma} \equiv \tilde{\Gamma}^{12345} \). Half of the independent components of \( \lambda \) are zero because
\( \bar{\Gamma} \) squares to unity and \( \text{tr}\bar{\Gamma} = 0 \). The fivebrane solution therefore preserves half of the supersymmetry.

The fermion zero mode of the fivebrane (1) is obtained by acting on the solution with a spinor \( \epsilon = \mathcal{E}\lambda \) with the property

\[
(1 - \bar{\Gamma}s)\lambda = 0
\]

(5)

where we have placed all spacetime dependence of \( \epsilon \) in the function \( \mathcal{E} = \mathcal{E}(x^m) \). From equation (3) we will then generate nonzero \( \psi_m \) which in turn can be plugged into equation (2) to generate the fermionic corrections to the bosonic fields. Equations of motions of the supergravity are invariant under equations (3) and (2) with \( \epsilon \) any function of spacetime coordinates but the resultant fermion zero mode will have the desirable physical properties only if we take \( \mathcal{E} \) to be a function of the transverse spatial coordinates with the property \( \mathcal{E} \to \infty \) a nonzero constant. This property is necessary in order to obtain the asymptotically flat spacetime and amounts to restricting local supersymmetry. By rescaling \( \lambda \) we may set the asymptotic value of \( \mathcal{E} \) to 1. These properties of \( \mathcal{E} \) are obtained by taking \( \mathcal{E} = f^{-\delta_5} \).

In this section we will assume that the value of \( \delta_5 \) is such that the resultant gravitino is normalizable. For the most part in this section we will not need to know the precise value of \( \delta_5 \) since we will mainly be interested in \( r \to \infty \) behavior of various quantities. Also, as explained in [5], the interactions between two superpartners are well defined only at distance far away from the horizon.

We define \( \Lambda_{mn...} \equiv \bar{\Lambda}_{mn...}\lambda \) and raise and lower indices of \( \Lambda \) with the Minkowski metric. Carrying out the procedure described in the previous paragraph we get the \( \lambda^2 \) quantities to be

\[
\psi = \left\{ \frac{1}{2}f^{-\frac{5}{2}}f_\alpha \bar{\Gamma}_{a}^\alpha dx^a - \frac{1}{f} \left[ f_\beta \bar{\Gamma}_{\beta}^\alpha + f_\alpha (\delta_5 + \frac{1}{4}) \right] \right\} \epsilon
\]

(6)

\[
A_{a_1a_2a_3} \sim \frac{3Qx^a \varepsilon_{a_1a_2a_3} b_1b_2b_3 \Lambda_{b_1b_2b_3}}{2r^5}
\]

(8)

\[
A_{c\alpha_1\alpha_2} \sim \frac{9Qx^\gamma \alpha_1\alpha_2 \gamma_{\beta_1\beta_2} \Lambda_{\beta_1\beta_2}}{2r^5}
\]

(9)

\[
e^a \sim \frac{6Mx^a \Lambda^{a\alpha}}{r^5} \tag{7}
\]

\[
e^a \sim \frac{3Mx^a \Lambda^{a\beta}}{r^5} \tag{7}
\]

\[
g_{\alpha c} \sim \frac{9Mx^\gamma \Lambda_{\beta\alpha c}}{r^5}
\]

(10)

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3 We discuss a more general form of \( \mathcal{E} \) in section 4.
where \( f_\alpha = \partial_\alpha f \), \( Q \equiv sM \) the magnetic charge, and \( \sim \) means \( r \to \infty \) limit. We have also used the conventions \( \varepsilon_{0+876} = \varepsilon^{0+867} = 1 \), and \( \hat{\Gamma}^{0+87612345} = 1 \). Other \( \lambda^2 \) components of the bosonic fields are zero on account of the Majorana property of \( \lambda \).

The “generalized” spin \( J \) of the M-5-brane can be read off from the off-diagonal components of the metric (10) as follows

\[
g_{ac} \sim \frac{x^\beta J_{ac}}{r^5} \quad \Rightarrow \quad J_c^{\alpha\beta} = 9M \Lambda_c^{\alpha\beta}.
\]  

(11)

We will call the components \( A_{a_1a_2a_3} \) electric and the electric dipole moments \( P \) can be read off by defining

\[
A_{a_1a_2a_3} \sim \frac{x^\gamma \varepsilon_{a_1a_2a_3} b_1b_2b_3 P_{\beta b_1b_2b_3}}{6r^5} \quad \Rightarrow \quad P_{abc}^{\beta} = 9Q \Lambda^{\beta}_{abc}.
\]

(12)

As in 3+1 dimensional electrodynamics, the electric dipole moments transform as vectors in the transverse space if the tangent space indices are considered mere labels of the vectors. There are thus 20 electric dipole moments. We will call the remaining components of the gauge field magnetic and the magnetic dipole moments \( \mu \) can be read off by defining

\[
A_{c\alpha_1\alpha_2} \sim \frac{x^\gamma \varepsilon_{\alpha_1\alpha_2\gamma} \beta_1\beta_2 \mu_{c\beta_1\beta_2}}{2r^5} \quad \Rightarrow \quad \mu_c^{\alpha\beta} = 9Q \Lambda_c^{\alpha\beta}.
\]

(13)

As in 3+1 dimensional electrodynamics, the magnetic dipole moments transform as antisymmetric tensors in the transverse space.\(^4\) Six tangent space indices can be considered as labels of these tensors.

Because \( \mu \) and \( J \) have the same index structure it is possible to calculate the gyromagnetic ratio from the naive formula

\[
\mu = g \frac{Q}{2M} J
\]

to get \( g=2 \). But this value will depend sensitively on the definition (13). It should be compared with the usual \( g \) in \( D = 4, p = 0 \) case only after careful comparison of two theories which we have not done here.

It is surprising that a spinning purely magnetic object can have dipole moments other than the electric ones. Some physicists \([13]\) have already observed an analogous phenomena in four dimensions where they interpret a similar result as spinning electric charge with both electric and magnetic dipole moments. We remark that the recent result \([14]\) of M-5-brane superalgebra including both 2-form and 5-form charges has been interpreted in \([15]\) as M-5-brane being dyonic. Perhaps these objects are composite like the garden variety neutron which has zero net electric charge as well as highly nontrivial magnetic dipole moment.

\(^4\) Of course in the elementary exposition of 3+1 electrodynamics magnetic dipole is a (pseudo-)vector which in 3+1 dimension is an antisymmetric tensor in disguise.
3. The Normalizability of $\psi_m$

The norm of the gravitino is by definition

$$\|\psi\|^2 = \int \psi_m^\dagger \psi_n g^{mn} \tag{14}$$

where the integral is over all of the transverse space from horizon to infinity. We will always work in spherical coordinates. Above $g^{mn}$ is the background metric.

2.1. M-5-brane

From equation (6) we get

$$\|\psi\|^2 = \Omega_4 \int_0^\infty r^4 f^5 \psi_m^\dagger \psi_n g^{mn} \, dr = 36 [5 + (\delta_5 + \frac{1}{4})^2] M^2 \Omega_4 \lambda^\dagger \lambda \int_0^\infty f^{-(2\delta_5+3)} \frac{dr}{r^4}$$

$$= 12 [5 + (\delta_5 + \frac{1}{4})^2] \Omega_4 \lambda^\dagger \lambda \left\{ -(1/4\delta_5)(1 + 6\infty)^{-2\delta_5}, \quad \text{if } \delta_5 < 0; \right.$$  

$$+ (1/6) \ln(1 + 6\infty), \quad \text{if } \delta_5 = 0; \right.$$  

$$+ (M/4\delta_5), \quad \text{if } \delta_5 > 0 \right. \tag{15}$$

where $\Omega_z = 2\pi^{z/2}/(\frac{z}{2} - 1)!$.

Imposing the RS condition we have

$$\Gamma^m \psi_m = \left[ \frac{1}{2} f^{-2} f_\alpha \hat{\Gamma}^\alpha \hat{\alpha}_a + f^{-2} f_\beta \hat{\Gamma}^\alpha \hat{\beta}_a - f^{-2} (\delta_5 + \frac{1}{4}) \hat{\Gamma}^\alpha f_\alpha \right] \epsilon$$

$$= f^{-2} f_\alpha \hat{\Gamma}^\alpha (-3 + 4 - \delta_5 - \frac{1}{4}) \epsilon = 0 \tag{15}$$

$$\Longrightarrow \delta_5 = \frac{3}{4}$$

where we have used an obvious identity $\hat{\Gamma}_\alpha^\beta = \hat{\Gamma}_\alpha^\beta - \delta_\alpha^\beta$. Therefore the RS condition is consistent with the finiteness (and positivity) of the gravitino norm.

2.2. M-2-brane

Recall that in this subsection the $a$ type indices take on three values whereas the $\alpha$ type indices take on eight spatial values. The gravitino field derived in [5] can be easily generalized for $\epsilon = f^{-\delta_2} \lambda$ with constant $\lambda$ satisfying an analogous condition to equation (5). The result is

$$\psi = \left\{ f^{-\frac{5}{2}} f_\alpha \hat{\Gamma}^\alpha_a dx^a - \frac{1}{2f} \left[ \hat{\Gamma}^\beta_\alpha f_\beta + (2\delta_2 + 1) f_\alpha \right] dx^\alpha \right\} \epsilon \tag{16}$$
where the background metric and the function \( f = f(x^\alpha) \) are given by \[16\]

\[
\frac{\eta_{ab} dx^a dx^b + f \delta_{\alpha \beta} dx^\alpha dx^\beta}{f^2} = \left(1 + \frac{3M}{r^6}\right)^{\frac{1}{3}} , \quad r = \sqrt{\delta_{\alpha \beta} x^\alpha x^\beta} .
\]

We have

\[
\parallel \psi \parallel^2 = \Omega^7 \int_0^\infty r^7 f^4 \psi_m^\dagger \psi_n g^{mn} dr = 9 \lambda^\dagger \lambda M^2 [11 + (2\delta_2 + 1)^2] \Omega^7 \int_0^\infty f^{-(2\delta_2 + 3)} \frac{dr}{r^7}
\]

\[
= \frac{3}{2} \lambda^\dagger \lambda M \Omega^7 [11 + (2\delta_2 + 1)^2] \int_0^\infty (1 + 3t)^{-\frac{2\delta_2}{3} + 1} dt
\]

\[
= \frac{3}{2} \lambda^\dagger \lambda \Omega^7 [11 + (2\delta_2 + 1)^2] \begin{cases} -(1/2\delta_2)(1 + 3\infty)^{-\frac{2\delta_2}{3}} , & \text{if } \delta_2 < 0; \\ (1/3) \ln(1 + 3\infty) , & \text{if } \delta_2 = 0; \\ (M/2\delta_2) , & \text{if } \delta_2 > 0. \end{cases}
\]

Imposing the RS condition we have

\[
\Gamma^m \psi_m = \left[ f^{\frac{-3}{2}} f_\alpha \hat{\Gamma}^\alpha_{\ a} - f^{\frac{-3}{2}} f_\beta \hat{\Gamma}^\beta_{\ a} - f^{\frac{-3}{2}} f_\alpha \hat{\Gamma}^\alpha_{\ b} \right] \epsilon
\]

\[
= f^{-\frac{3}{2}} f_\alpha \hat{\Gamma}^\alpha_{\ a} (-3 + \frac{7}{2} - \delta_2 - \frac{1}{2}) \epsilon = 0
\]

\[
\implies \delta_2 = 0.
\]

We see that RS condition with \( \delta_2 = 0 \) forbids the existence of the fermion zero mode.

### 2.3. Majumdar-Papapetrou 0-brane

We will follow the metric signature and the supersymmetry transformation equations of \[3\] but three indices \( \alpha, \beta \) will denote the usual three dimensional spatial indices and \( m, n \) four spacetime coordinates. The background metric and the harmonic function are

\[
ds^2 = \frac{1}{f^2} dt^2 - f^2 dx^2 , \quad f = 1 + \frac{M}{r}
\]

Taking \( \epsilon = f^{-\delta_0} \lambda \) with \( \lambda \) satisfying an analogous relation to \( [\beta] \)) we can straightforwardly generalize the gravitino given in \( [\beta] \) (or \( \beta \)) to get

\[
\psi = \left\{-\frac{f_\beta}{f^3} \hat{\Gamma}^\beta dt - \frac{1}{f} \left[ \hat{\Gamma}^\beta \hat{\Gamma}_\alpha f_\beta + (\delta_0 - \frac{1}{2}) f_\alpha \right] dx^\alpha \right\} \epsilon.
\]

As shown in \[\beta\] the RS condition is equivalent to \( \delta_0 = \frac{1}{2} \). Bearing in mind that, in this subsection, the spatial dirac matrices are negative complex transpose of themselves we can derive

\[
\psi^\dagger = \epsilon^\dagger \left\{ \frac{f_\alpha}{f^3} \hat{\Gamma}^\alpha dt - \frac{1}{f} \left[ \hat{\Gamma}_\alpha \hat{\Gamma}^\alpha f_\beta + (\delta_0 - \frac{1}{2}) f_\alpha \right] dx^\alpha \right\}
\]
We next calculate $\psi_n^\dagger \psi^n$ and we get

$$\psi_m^\dagger \psi^m = \frac{1}{f^4} \epsilon^\dagger \left[ - f_\beta f_\alpha \widehat{\Gamma}^\alpha \widehat{\Gamma}^\beta - (\delta_0 - \frac{1}{2})^2 \delta^\alpha \delta^\beta f_\alpha f_\beta \\
+ f_\beta f_\gamma \widehat{\Gamma}_\alpha \widehat{\Gamma}^\gamma \widehat{\Gamma}^\alpha + 2(\delta_0 - \frac{1}{2}) \widehat{\Gamma}^\beta \widehat{\Gamma}^\alpha f_\alpha f_\beta \right] \epsilon
$$

$$= \frac{1}{f^4} \epsilon^\dagger \left\{ \frac{M^2}{r^4} \left[ 1 - (\delta_0 - \frac{1}{2})^2 - 2(\delta_0 - \frac{1}{2}) \right] - f_\gamma f_\beta \widehat{\Gamma}_\alpha \widehat{\Gamma}^\beta \left( \widehat{\Gamma}^\alpha \widehat{\Gamma}^\gamma + 2 \delta^\alpha \gamma \right) \right\} \epsilon
$$

$$= \frac{1}{f^4} \epsilon^\dagger \left\{ \frac{M^2}{r^4} \left[ 2 - (\delta_0 + \frac{1}{2})^2 \right] + f_\gamma f_\beta \left( \widehat{\Gamma}^\beta \widehat{\Gamma}^\gamma - 2 \widehat{\Gamma}^\gamma \widehat{\Gamma}^\gamma \right) \right\} \epsilon
$$

$$= f^{- (4 + 2\delta_0)} \lambda^\dagger \lambda \left[ 3 - (\delta_0 + \frac{1}{2})^2 \right] \frac{M^2}{r^4}$$

We finally get

$$||\psi||^2 = \Omega_3 \left[ 3 - (\delta_0 + \frac{1}{2})^2 \right] \lambda^\dagger \lambda M^2 \int_0^\infty f^{-(1 + 2\delta_0)} \frac{dr}{r^2}
$$

$$= \Omega_3 \left[ 3 - (\delta_0 + \frac{1}{2})^2 \right] \lambda^\dagger \lambda M \int_0^\infty (1 + t)^{-(1 + 2\delta_0)} \frac{dt}{t^2}
$$

$$= \Omega_3 \left[ 3 - (\delta_0 + \frac{1}{2})^2 \right] \lambda^\dagger \lambda \begin{cases} 
-(1/2\delta_0)(1 + \infty)^{-2\delta_0}, & \text{if } \delta_0 < 0; \\
\ln(1 + \infty), & \text{if } \delta_0 = 0; \\
(M/2\delta_0), & \text{if } \delta_0 > 0.
\end{cases}$$

We see that the RS condition with $\delta_0 = \frac{1}{2}$ is only a sufficient condition for the normalizability to the norm. In contrast to the eleven dimensional cases the positivity of the four dimensional norm sets an upper bound on $\delta_0$: $\delta_0 < \sqrt{3} - \frac{1}{2}$. For the ease of exposition we will have this bound in our mind whenever we speak of $\delta_0$ without mentioning the bound.

### 4. Summary

We have shown that the normalizability of the gravitino can be achieved if $\delta_p > 0$ for all three $p$’s. Because the dummy variable of integration is $t = r^{3 - p - D}$ the divergence of the integral is due to the behavior of $\epsilon$ at $r = 0$. From the fact that $\delta_p = 0$ gives rise to logarithmic divergences, it is easy to convince oneself that $\delta_p$ dependence of the normalizability will be unchanged if one has only demanded that $\epsilon \to 0$ $r$ $\delta_p$ regardless of the behavior of $\epsilon$ elsewhere.\footnote{Put it another way the norm exists if and only if $\epsilon$ vanishes at the horizon. We can take the vanishing of $\epsilon$ at the horizon as the complete breaking of supersymmetry there. On the other hand the norm diverges logarithmically if $\epsilon$ goes to a constant at the horizon and we can take constant $\epsilon$ at the horizon as some preserved...} Of course $\epsilon$ must not behave wildly elsewhere.

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supersymmetry there. Note that the logarithmic behavior of the divergence due to the infrared limit is the same as that observed in three dimensional example [2].

The existence of the norm is related to the existence of the quantum fermion states. Although there is no well defined notion of the classical fermion one can still ask in what sense the gravitino is well behaved classically. Because they depend on the coordinate system the values of $\psi_m$ have no meaning. The correct things to examine are $\psi_m$. In fact in the Hamiltonian formulation [17] $\psi_m$ are the fundamental quantities. It is trivial to check that $\psi_m$ in all three cases are nonsingular at the horizon so long as $\delta_p \geq 0$. Therefore the gravitino is well behaved in the classical sense for constant $\epsilon$ at the horizon despite the nonexistence of the quantum norm in this case.

For completeness we finally note that Deser and Teitelboim [18] had proposed the so called “natural” condition $\Gamma^{\alpha} \psi_\alpha = 0$ in the context of defining the supercharge. This “natural” condition turns out to be equivalent to $\delta_0 = \frac{3}{2}, \delta_2 = 3, \delta_5 = 3\frac{3}{4}$.

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