CLOSURES OF $B$-CONJUGACY CLASSES OF 2-NILPOTENT MATRICES HAVE RATIONAL SINGULARITIES

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Abstract. We prove that closures of Borel conjugacy classes of 2-nilpotent matrices have a rational resolution of singularities.

INTRODUCTION

Let $k$ be an algebraically closed field and consider the action of $G = \text{GL}_n(k)$ on $\mathfrak{g} = \text{gl}_n(k)$ given by conjugation. A $G$-orbit $G.x$ is called nilpotent if $x$ is a nilpotent linear map. Nilpotent orbits are indexed by partitions of $n$, and the closure order on nilpotent $G$-orbits corresponds to the dominance order on partitions. Due to a result of S. Donkin \cite{Donkin} the closure of a nilpotent $G$-conjugacy class is normal, for char$(k)$ arbitrary. For char$(k) = 0$ this was already proved by H. Kraft and C. Procesi \cite{Kraft-Procesi}.

Denoting by $B \subseteq G$ a Borel subgroup, it is natural to ask what can be said about the singularities in closures of $B$-conjugacy classes of nilpotent matrices. It is known that in general there are infinitely many such $B$-orbits; a nilpotent $G$-orbit consisting of finitely many $B$-orbits if and only if the matrices are of nilpotency order at most 2. (cf. D. I. Panyushev’s article \cite{Panyushev}, where a sphericity criterion for an orbit under the adjoint action of an arbitrary semisimple algebraic group is given).

The purpose of this article is to prove the following.

Theorem 0.0.1. Closures of Borel conjugacy classes of 2-nilpotent matrices have a rational resolution of singularities. In particular, they are normal and Cohen-Macaulay.

Our strategy is as follows: We start by proving that $N_2$, the scheme of nilpotent matrices of order 2, is a spherical variety of minimal rank, i.e. the ranks of all $B$-orbits in a given $G$-orbit coincide. This will be seen by presenting, the (unique) 2-nilpotent conjugacy class of rank $r$, as a parabolic induction on the spherical ($\text{GL}_r(k) \times \text{GL}_r(k)$)-variety $\text{GL}_r(k)$. An almost direct consequence of this will be that a $B$-orbit closure in $N_2$ has a Bott-Samelson-Demazure-Hansen resolution of singularities. Further, these resolutions have connected fibers.

Due to results of M. Brion \cite{Brion} the rationality of singularities would follow if the variety $\mathfrak{g}$ admits a $B$-canonical Frobenius splitting compatibly splitting all $B$-orbit closures. We prove this by parabolic induction using the analogous statement for the $\text{GL}_r(k)$-embedding given by the variety of $(r \times r)$-matrices. The Frobenius splitting statement for matrices was proved by X. He and J.F. Thomsen in \cite{He-Thomsen}.

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The last section of this article is devoted to more explicit computations. Using a Lemma from [2] we specify a set of representatives for the $B$-orbits in $N_2$. It is noteworthy that these matrices coincide with those obtained by M. Boos and M. Reineke in [1]; this article contains a description of the closure order on $B\setminus N_2$. We describe the closure order on the set of $B$-orbits in a fixed $G$-conjugacy class in terms of the Bruhat-Chevalley order on the symmetric group $S_n$. Finally, $B$-conjugacy class closures are described as sets of linear maps and we provide a simple example of a non-Gorenstein $B$-conjugacy class closure. This contrasts with the situation of $G$-conjugacy class closures, which are all Gorenstein.

1. $G$-orbits

1.1. Notation. Let $(e_i)_{i\in[1,n]}$ be the canonical basis of $k^n$ and for $m \leq n$, let $V_m = \langle e_i \mid i \in [1,m] \rangle$ be the span of the first $m$ basis vectors. We write $G$ for $GL_n(k)$, $g$ for its Lie algebra $gl_n(k)$ and $N_2$ for the set of nilpotent elements of order at most 2. This is a closed $G$-stable subscheme in $g$ and its $G$-orbit closures are indexed by the rank. Note that the rank $r$ of a nilpotent element of order 2 in $g$ satisfies $2r \leq n$. Denoting by $O_r$ the subscheme of nilpotent elements of order 2 and rank $r$ in $g$ we have

$$N_2 = \{0\} \cup \bigsqcup_{2r \leq n} O_r$$

and this is the $G$-orbit decomposition of $N_2$. For $2r \leq n$, let $P^r$ be the parabolic subgroup of $G$ stabilising the flag $V_r \subset V_{n-r}$. We have a natural morphism

$$p : O_r \to G/P^r$$

defined by $p(x) = (\text{Im}(x), \text{Ker}(x))$. This morphism extends to a birational transform of the closure of $O_r$ as follows. Let $X_r$ be the variety of pairs

$$X_r = \{(x, (I, K)) \in g \times G/P^r \mid \text{Im}(x) \subset I \subset K \subset \text{Ker}(x)\}.$$ 

There is a natural $G$-equivariant morphism $p : X_r \to G/P^r$ given by the second projection as well as a $G$-equivariant morphism $\pi : X_r \to g$ given by the first projection. This last morphism is birational onto the closure of $O_r$:

$$\overline{O_r} = \{x \in g \mid x^2 = 0 \text{ and rk}(x) \leq r\}.$$ 

1.2. Induction. The above morphism $p$ can be seen as an induction over $G/P^r$. Let $A_r$ be the fiber of $p$ over the flag $V_r \subset V_{n-r}$ i.e.

$$A_r = \{x \in g \mid \text{Im}(x) \subset V_r \subset V_{n-r} \subset \text{Ker}(x)\}.$$ 

For the adjoint action by conjugation, the parabolic subgroup $P^r$ stabilises $A_r$. We may therefore define the contracted product $G \times P^r A_r$ and an easy check gives an isomorphism $G \times P^r A_r \simeq X_r$. If we identify these two varieties and write $[g, x]$ for the class of an element $(g, x) \in G \times A_r$ in the contracted product, we can describe the morphisms $p$ and $\pi$ as follows:

$$p([g, x]) = g \cdot P_r \text{ and } \pi([g, x]) = gx = gxg^{-1}.$$
1.3. Spherical varieties. The above description of \( X_r \) as being induced from \( A_r \) gives an easy way for proving that \( X_r \) is spherical.

First note that \( A_r \) identifies with the space of square matrices of size \( r \) and that the action of \( P_r \) on it is given as follows:

\[
\begin{pmatrix}
A & B & C \\
0 & D & E \\
0 & 0 & F
\end{pmatrix} \cdot M = AMF^{-1}.
\]

In particular, \( A_r \) is a spherical variety for the Levi subgroup of \( P_r \) (it is a \( GL_r(k) \times GL_r(k) \)-equivariant compactification of \( GL_r(k) \) — note that \( GL_r(k) \times GL_r(k) \) is a quotient of the Levi subgroup of \( P_r \)).

Lemma 1.3.1. The varieties \( X_r \) and \( O_r \) are \( G \)-spherical.

Proof. This follows from the fact that \( X_r \) is induced from \( A_r \), a spherical variety, and the result [2, Lemma 6]. The result for \( O_r \) follows from this, as \( \pi \) maps the open \( B \)-orbit of \( X_r \) to an open \( B \)-orbit of \( O_r \).

\[\square\]

2. Orbits of Borel subgroups

Let \( B \) be the Borel subgroup of upper triangular matrices in \( G \).

2.1. Weak order. Recall the following general results on orbits of Borel subgroups in a spherical variety \( Z \) (see for example [12, Section 4.4]).

Let \( B(Z) \) be the set of \( B \)-orbits in \( Z \). It is a finite set. We define the weak Bruhat order on \( B(Z) \) as follows. Let \( Y \in B(Z) \). If \( P \) is a minimal parabolic subgroup of \( G \) containing \( B \) such that \( Y \subseteq PY \), we set \( Y' \) for the dense \( B \)-orbit in \( PY \) and say that \( P \) raises \( Y \) to \( Y' \). We write \( Y \prec Y' \). These relations are the covering relations of the weak Bruhat order. For \( Y \preceq Y' \) raised by \( P \), consider the proper morphism \( q : P \times^B Y \to PY \). The following result is proved in [15] for symmetric varieties and extends readily to the general case.

Proposition 2.1.1. One of the following occurs.

\begin{itemize}
    \item Type \( U \): \( PY = Y \cup Y' \) and \( q \) is birational.
    \item Type \( N \): \( PY = Y \cup Y' \) and \( q \) has degree 2.
    \item Type \( T \): \( PY = Y \cup Y' \cup Y'' \) with \( \dim Y'' = \dim Y \) and \( q \) is birational.
\end{itemize}

Remark 2.1.2. The following is involved in the proof of Proposition 2.1.1 and explains the above notation: Choose \( y \) in the open \( B \)-orbit of \( Y \), with isotropy group \( P_y \subseteq P \). This group acts on \( P/B \). Then, the image of \( P_y \) in \( \text{Aut}(P/B) \cong \text{Aut}(\mathbb{P}^{r-1}) = \text{PGL}_2(k) \) is either a torus (Type T), the normalizer of a torus (Type N), or contains a non-trivial unipotent subgroup (Type U).

Definition 2.1.3. A \( G \)-spherical variety is called of minimal rank if the covering relations are only of type \( U \).

The variety \( A_r \) being a group compactification of \( GL_r(k) \) it is a \( GL_r(k) \times GL_r(k) \)-spherical variety of minimal rank (see for example [14]) and by [2, Lemma 6] (see also [14]) we get

Lemma 2.1.4. The induced variety \( X_r \) is of minimal rank.

Corollary 2.1.5. The variety \( N_2 \) is of minimal rank.
Proof. A $G$-orbit in $N_2$ is given by $\mathcal{O}_s$, for some $s$ with $0 \leq 2s \leq n$. Further, $\mathcal{O}_s$ is isomorphic to the open $G$-orbit of $X_s$. \hfill \Box

Corollary 2.1.6. Any $G$-orbit in $X_r$ or $N_2$ contains a unique minimal $B$-orbit for the weak order.

Proof. This holds for any spherical variety of minimal rank (see [14]). \hfill \Box

Lemma 2.1.7. Let $Z$ be a $G$-spherical variety of minimal rank and $Y = \overline{B.y} \subseteq Z$ a $B$-orbit closure. Further, denote by $Y_0 \subseteq Z$ the closure of the minimal $B$-orbit in $G.y$. Then there exists a sequence of minimal parabolics $P_1, \ldots, P_m$ and a morphism

$$f : P_1 \times B \ldots \times B P_m \times B Y_0 \rightarrow Y, \quad [p_1, \ldots, p_m, x] \mapsto p_1 \ldots p_m x,$$

which is surjective, projective, birational, with connected fibers. In particular, $f$ is a resolution of singularities if and only if $Y_0$ is a smooth variety.

Proof. Since the dense $B$-orbits of $Y$ and $Y_0$ are in the same $G$-orbit $G.y$ and since the dense $B$-orbit in $Y_0$ is the only minimal orbit for the weak order in $G.y$, there exists a sequence of minimal parabolic subgroups $P_1, \ldots, P_m$ raising $Y_0$ to $Y$. Since case (N) does not occur, we see that the surjective and projective map

$$f : P_1 \times B \ldots \times B P_m \times B Y_0 \rightarrow Y$$

is birational. We now prove that $f$ has connected fibers, by induction on $m \in \mathbb{N}$. For $m = 1$, it suffices to prove that

$$f^{-1}(x) \simeq \{p \in P_1 \mid p^{-1} x \in Y_0\} / B \subseteq P_1 / B$$

is connected for all $x \in Y_0$, because $f$ is $P_1$-equivariant. If $P_1.x \subseteq Y_0$, then $f^{-1}(x) = P_1 / B$ is connected. If $P_1.x \not\subseteq Y_0$, we use that $P_1 \times B x \to P_1 x$ is not of type (N). Using Remark 2.1.2, we may conclude that the stabilizer of $x$, $(P_1)_x$, is contained in $B$, hence

$$f^{-1}(x) \simeq \{p \in P_1 \mid p^{-1} x \in B.x\} / B = (P_1)_x B / B = \{1 \cdot B\}$$

is connected. For $m > 1$, we factor $f$ into

$$P_1 \times B P_2 \times B \ldots \times B P_m \times B Y_0 \xrightarrow{f_1} P_1 \times B P_2 \ldots P_m Y_0 \xrightarrow{f_2} Y.$$ 

Let $z \in Y$. By induction hypothesis $f_1$ and $f_2$ have connected fibers. The closedness of $f_1$ then ensures that the preimage of a connected subscheme under $f_1$ is again connected. Therefore

$$f^{-1}(z) = f_1^{-1}(f_2^{-1}(z))$$

is connected.

Now, $f$ being a resolution of singularities is equivalent to the smoothness of $P_1 \times B \ldots \times B P_m \times B Y_0$. This variety is smooth if and only if $Y_0$ is smooth. \hfill \Box

Corollary 2.1.8. Any $B$-orbit closure in $X_r$ or $N_2$ has a resolution of singularities as in Lemma 2.1.7.

Proof. As both varieties are $G$-spherical of minimal rank, it remains to show that closures of minimal $B$-orbits in $X_r$ or $N_2$ are smooth.

The closure in $N_2$ of the minimal $B$-orbit in the $G$-orbit $\mathcal{O}_r$ is isomorphic to the closure of the minimal $B_r \times B_r$-orbit in the dense $GL_r(k) \times GL_r(k)$-orbit in $A_r$ (here $B_r$ is the Borel subgroup of upper triangular matrices of $GL_r(k)$). This orbit
is simply $B_r \subset \mathbb{A}_r$ thus its closure is the set of upper triangular matrices, which is smooth.

A $G$-orbit of $G \times^{P'} \mathbb{A}_r$ is of the form $G.[1,u] = G \times^{P'} L^{r,u}$, where $L^{r,u}$ is the Levi subgroup of $P'$. Further, $[1,v] \in G.[1,u]$ if and only if $rk(v) = rk(u)$. Hence $G$-orbits of $X_r$ are indexed by the non-negative number $l := rk(u) \leq r$. For such an integer we define

$$x_l(e_i) = \begin{cases} 
0 & \text{if } i = 1, \ldots, n - l \\
e_{i-(n-l)} & \text{if } i = n - l + 1, \ldots, n \in \mathbb{A}_r
\end{cases}$$

Then $B.[1,x_l]$ is the minimal $B$-orbit in the $G$-orbit $G.[1,y_l]$. Using the isomorphism

$$G \times^{P'} \mathbb{A}_r \to X_r, \quad [g,x] \mapsto (g,x,g(k^r),g(k^{n-r}))$$

we see that $B.[1,x_l]$ identifies with the minimal $B$-orbit of $\mathcal{O}_l$. Hence the closure of the minimal $B$-orbit in a $G$-orbit of $X_r$ is smooth, as well. \hfill \Box

### 3. Frobenius splitting

In this section we assume $\text{char}(k) = p > 0$.

**3.1. Generalities on Frobenius splitting.** Our basic reference for Frobenius splitting is [3]. For a variety $X$ defined over $k$, we say that $X$ is Frobenius split if the absolute Frobenius morphism $F: X \to X$ has a left inverse on the level of structural sheaves: the $\mathcal{O}_X$-module homomorphism $\mathcal{O}_X \to F_\ast \mathcal{O}_X$ has a left inverse $\varphi : F_\ast \mathcal{O}_X \to \mathcal{O}_X$. A subvariety $Y \subset X$ is compatibly split for $\varphi$ if its sheaf of ideals $\mathcal{I}_Y$ satisfies $\varphi(F_\ast \mathcal{I}_Y) \subset \mathcal{I}_Y$. If the variety has a $B$-action, then $\text{Hom}(F_\ast \mathcal{O}_X, \mathcal{O}_X)$ is a $B$-module and a splitting $\varphi$ is $B$-canonical if it is fixed by the maximal torus and killed by the divided power operators $e_\alpha^{(n)}$ for $n \geq p$ and $\alpha$ a simple root of $G$ (see [3] Lemma 4.1.6).

**Proposition 3.1.1.** Let $X = G \times^P Z$ be induced from $Z$ over $G/P$. If $Z$ admits a $B$-canonical splitting compatibly splitting a subvariety $Z'$, then $X$ has a $B$-canonical splitting compatibly splitting $G \times^P Z'$ and all the varieties $BwP \times^P Z$ for $w \in W$ the Weyl group of $G$.

**Proof.** This result is proved in [3] Theorem 4.1.17 and Exercise 4.1.E.(4)] for $P = B$. One deduces the result from this case and [3] Lemma 1.1.8] since the map $f : G \times^P Z \to G \times^P Z$ satisfies $f_\ast \mathcal{O}_{G \times^P Z} = \mathcal{O}_{G \times^P Z}$.

**3.2. Frobenius splitting for $X_r$.** Recall the notation $B_r$ for the Borel subgroup of upper triangular matrices in $GL_r(k)$. The following result is proved in [3] Proposition 7.1].

**Proposition 3.2.1.** The $GL_r(k)$-embedding $\mathbb{A}_r$ has a $B_r \times B_r$-canonical splitting compatibly splitting all the $B_r \times B_r$-orbit closures.

**Remark 3.2.2.** The compatibly split subvarieties for the above splittings are described in [13]. The $B_r \times B_r$-orbit closures in $\mathbb{A}_r$ are matrix Schubert varieties. For such a variety $Z$, there exists a Schubert variety $Z'$ in $GL_{2r}(k)$ and a smooth and surjective morphism $Z' \to Z$ (see [6]). In particular, $Z$ has rational singularities, as this holds for Schubert varieties.

**Corollary 3.2.3.** The variety $X_r$ has a $B$-canonical splitting compatibly splitting all the $B$-orbit closures.
Proof. We only need to check that the above $B_r \times B_r$-canonical splitting of $\mathbb{A}_r$ is also a $B$-canonical splitting. The invariance for the torus is clear. We have an action of $P^r$ on $\mathbb{A}_r$, thus a $B$-action. Furthermore, this action factors through the quotient $GL_r(k) \times GL_r(k)$ of the Levi subgroup of $P^r$. In particular the $B$-action factors through the $B_r \times B_r$-action therefore the divided difference operators coincide and the result follows from the fact that the splitting is $B_r \times B_r$-canonical. □

Corollary 3.2.4. The variety $\mathcal{O}_r$ admits a $B$-canonical splitting compatibly splitting all the $B$-orbit closures.

Proof. Since $\mathcal{O}_r$ is normal (see [5]), we have $\pi_* \mathcal{O}_X = \mathcal{O}_{\mathcal{O}_r}$ due to Zariski’s main theorem. The result follows from [4, Lemma 1.18]. □

4. Regularity results

We prove that the $B$-orbit closures of nilpotent elements of order 2 are normal and admit a rational resolution.

Theorem 4.0.1. Any $B$-orbit closure in $X_r$ or in $\mathcal{O}_r$ has a rational resolution.

Proof. We prove that the $B$-orbit closures are normal. Once this result holds, the general arguments of [3, Section 3, Remark 2] imply that the map described in Corollary 2.1.8 is a rational resolution.

We first assume char$(k) = p > 0$. Since any $B$-orbit closure is Frobenius split, it is quasi-normal. By Corollary 2.1.8 there exists a resolution of that orbit closure with connected fibers proving the result. For char$(k) = 0$, the result follows from the positive characteristic case and [7, Théorème 12.2.4.(4)]. □

5. Explicit description of $B$-conjugacy class closures

5.1. Closure order on $\mathcal{O}_r$. In this section we use the results of [2, Lemma 6] to give a complete description of the $B$-orbits in $\mathcal{O}_r$.

$W = S_n$ is the Weyl group of $G$, considered as a subgroup of $G$. We denote by $W^P_r$ the set of minimal length representatives in $W$ of the quotient $W/W^P_r$ and by $W_r$ the Weyl group of $GL_r(k)$. Let $B_r$ be the Borel subgroup of upper triangular matrices of $GL_r(k)$. Define $x_r \in \mathbb{A}_r$ by

$$x_r(e_i) = \begin{cases} 0 & \text{if } i = 1, \ldots, n - r \\ e_{i-(n-r)} & \text{if } i = n - r + 1, \ldots, n \end{cases}$$

Denote the stabilizer of $x_r$ in $G$ by $C_r$. A direct computation shows that $C_r$ is the subgroup of $P^r$ consisting of the matrices whose upper-left and lower-right block coincide.

Denote by $\prec$ the Bruhat-Chevalley order on $W$. By $s_i \in W$ we mean the transposition switching $i$ with $i + 1$. The corresponding minimal parabolic subgroup containing $B$ is denoted by $P_i$. Further, we set $W(C_r) := W \cap C_r$.

Lemma 5.1.1. The following holds:

1. $\mathcal{O}_r = \bigsqcup_{(\sigma, w) \in W^P_r \times W_r} B\sigma w.x_r$.
2. $B\sigma' w'.x_r \subseteq B\sigma w.x_r \subseteq \mathcal{O}_r$ if and only if there exists $\tau \prec \sigma w$ with $\sigma'w'W(C_r) = \tau W(C_r)$.
3. $\dim B\sigma w.x_r = l(\sigma) + l(w) + \binom{r+1}{2}$.
**Proof.** With \( A_r^0 := \{ x \in A_r \mid \text{rk}(x) = r \} \) we have an \( B \)-equivariant isomorphism
\[
G \times P^r A_r^0 \rightarrow \mathcal{O}_r, \quad [g, x] \mapsto g.x.
\]
The identification \( A_r^0 \cong \text{GL}_r(k) \) provides a Bruhat decomposition
\[
A_r^0 = \bigsqcup_{w \in W_r} Bw.x_r.
\]
The first claim and the minimality of the \( B \)-orbit \( B.x_r \) follow from [2, Lemma 6].

Now, choose a reduced expression \( \sigma w = s_{i_1} \ldots s_{i_r} \). Using that all covering relations are of type (U) (again by [2, Lemma 6]) we see that the closure of \( B\sigma w.x_r \) in \( \mathcal{O}_r \) is given by
\[
P_{i_1} \ldots P_{i_r}.x_r = \bigcup_{\tau \prec \sigma w} B\tau.x_r.
\]
The second claim follows, because \( \tau'.x_r = \tau.x_r \) if and only if \( \tau'W(C_r) = \tau W(C_r) \).

For the third claim, note that \( l(\sigma w) = l(\sigma) + l(w) \) since \( w \in W_r \subseteq W_{P^r} \).

Therewith
\[
\dim B\sigma w.x_r = \dim P_{i_1} \ldots P_{i_r}.x_r = l(\sigma w) + \dim(B.x_r)
\]
\[
= l(\sigma w) + \dim(B_r) = l(\sigma) + l(w) + \binom{r+1}{2}
\]
and the last claim follows. \( \square \)

**Remark 5.1.2.** The results of [2, Lemma 6] give a description of the \( B \)-orbits in \( X = G \times P^r A_r \) as well: They are indexed by \( W_{P^r} \times P(r) \), where \( P(r) \) is the set of partial permutation matrices in \( M_r(k) \).

**Remark 5.1.3.** As mentioned in the introduction, \( B \)-orbits in \( N_2 \) have been investigated before in [1]. In order to compare our results with these previous results, denote for \( (i, j) \in [n]^2 \) by \( E_{i,j} \in g \) the corresponding elementary matrix. Then
\[
\sigma w.x_r = \sigma w x_r (\sigma w)^{-1} = \sum_{j=1}^r E_{\sigma w(j), \sigma(n-r+j)}.
\]
This is the 2-nilpotent matrix associated to the oriented link pattern on \( r \) arcs
\[
(\sigma(n-r+1), \sigma w(1)), \ldots, (\sigma(n), \sigma w(r))
\]
as in [1]. It was there shown that oriented link patterns parametrize \( B \)-orbits in \( \mathcal{O}_r \) by using representation theory of quivers. The closure order on \( B \backslash \mathcal{O}_r \) was determined in [1] as well. The term oriented link pattern refers to an extension of A. Melnikov’s [10] link patterns which parametrize \( B \)-orbits of 2-nilpotent upper-triangular matrices.

**Example 5.1.4.** For \( (n, r) = (4, 2) \) we have \( W_r = \langle s_1 \rangle, \ W(C_r) = \langle s_1, s_3 \rangle \) and
\[
W^{P^r} = \{ \sigma \in W \mid \sigma(1) < \sigma(2), \sigma(3) < \sigma(4) \} = \{ \text{id}, s_2, s_1 s_2, s_3 s_2, s_1 s_3 s_2, s_2 s_1 s_3 s_2 \}.
\]
Using Lemma 5.1.1 we obtain
5.2. \( B \)-orbit closures in \( N_2 \) as sets of linear maps. Recall that \( V_i \subseteq k^n \) denotes the coordinate subspace generated by \( e_1, \ldots, e_i \). For \( 0 \leq 2r \leq n \) and \((\sigma, w) \in W^{Pr} \times W_r\) we consider \( Z(\sigma, w) = B\sigma w.x_r \subseteq N_2 \). All \( B \)-conjugacy class closures in \( N_2 \) are of this form. In order to describe \( Z(\sigma, w) \) explicitly, we define

\[
\rho(i, j, x) := \dim (x(V_i) + V_j), \text{ for } (i, j) \in [n] \times [n] \cup \{0\},
\]

where we have set \( V_0 := \{0\} \).

**Lemma 5.2.1.** As a set of linear maps,

\[
Z(\sigma, w) = \{ x \in N_2 \mid \rho(i, j, x) \leq \rho(i, j, \sigma w.x_r), \text{ for all } (i, j) \in [n] \times [n] \cup \{0\} \}.
\]

**Proof.** This result appears in [16]. We give a shorter proof. The space \( g = M_n(k) \) is the disjoint union of \((B, B)\)-double cosets of partial permutation matrices. Further, for \( x, y \in g \) one has \( BxB = ByB \) if and only if

\[
\rho(i, j, x) = \rho(i, j, y), \text{ for all } (i, j) \in [n] \times [n] \cup \{0\}.
\]

From Lemma 5.1.1 we derive that \( B\sigma w.x_r = \mathcal{O}_{r} \cap B(\sigma w.x_r)B \). Consequently,

\[
B\sigma w.x_r = \{ x \in \mathcal{O}_{r} \mid \rho(i, j, x) = \rho(i, j, \sigma w.x_r), \text{ for all } (i, j) \in [n] \times [n] \cup \{0\} \}.
\]

Taking the closure in \( N_2 \) we obtain the desired description of \( Z(\sigma, w) \). \( \square \)

**Example 5.2.2.** Let \( (n, r) = (3, 1) \). Then \( W_r = \{1\} \) and \( W^{Pr} = W \). Take \((\sigma, w) = (s_1s_2, 1)\). Then \( Z(s_1s_2, 1) \) is 3-dimensional. Using Lemma 5.2.1 we see it is isomorphic to

\[
Z = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \mid (A, v) \in M_2(k) \times k^2, \ A^2 = 0 \text{ and } Av = 0 \right\}.
\]
This is a toric variety: The open torus is
\[ T_Z := Z \cap \left( M_2(k^*) \times (k^*)^2 \right) = \left\{ \left( \begin{pmatrix} t_1 & -t_2 \\ t_2 & -t_1 \end{pmatrix}, \begin{pmatrix} t_3 \\ t_1 t_3 \\ t_2 \end{pmatrix} \right) \mid t_1, t_2, t_3 \in k^* \right\} . \]

By considering 1-parameter subgroups of \( T_Z \) we see that \( Z \) is the toric variety associated with the cone
\[ \mathbb{Q}_{\geq 0} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mathbb{Q}_{\geq 0} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{Q}_{\geq 0} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mathbb{Q}_{\geq 0} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} . \]

We conclude that \( Z \) is not Gorenstein, as the generators of this cone are not contained in an affine hyperplane in \( \mathbb{Q}^3 \).

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