Renormalization group and diffusion equation

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Abstract

We study relationship between renormalization group and diffusion equation. We consider the exact renormalization group equation for a scalar field that includes an arbitrary cutoff function and an arbitrary quadratic seed action. As a generalization of the result obtained by Sonoda and Suzuki, we find that the correlation functions of diffused fields with respect to the bare action agree with those of bare fields with respect to the effective action, where the diffused field obeys a generalized diffusion equation determined by the cutoff function and the seed action and agrees with the bare field at the initial time.

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1 Introduction

It is recognized that the diffusion equation has to do with renormalization group, since diffusion can be regarded as a continuum analog of block-spin transformation or coarse-graining. Indeed, the solution to the diffusion equation in $d$ dimensions

$$\frac{\partial \varphi(\tau, x)}{\partial \tau} = \partial_i^2 \varphi(\tau, x) \quad (1.1)$$

with an initial condition $\varphi(0, x) = \phi(x)$ is given by

$$\varphi(\tau, x) = \int d^d x' K(x, x', \tau) \phi(x') \quad , (1.2)$$

where $K(x, x', \tau)$ is the heat kernel

$$K(x, x', \tau) = \frac{1}{(4\pi \tau)^{d/2}} \exp \left[ -\frac{(x - x')^2}{4\tau} \right] \quad . \quad (1.3)$$

Here $\varphi(\tau, x)$ can be viewed as a coarse-grained field obtained by smearing $\phi$ in a ball centered at $x$ with the radius $\sqrt{\tau}$. On the other hand, the gradient flow equation in gauge theories, which is regarded as a generalized diffusion equation respecting gauge symmetry, has been recently used to renormalize composite operators and so on [1–3].

Thus it is natural to expect that there is relationship between (generalized) diffusion equations or gradient flow equations and renormalization group equations. Indeed, the relationship has been studied in [4–11]. In particular, it was shown in [11] that the correlation functions of the diffused fields (1.2) with respect to the bare action agree with those of bare fields with respect to the effective action, where the effective action obeys the exact renormalization group (ERG) equation with a cutoff function and a seed action. This implies that correlation functions of the composite operators consisting of the diffused fields (1.2) are finite so that the diffused fields can be used to renormalize the composite operators.

In this paper, we study generalization of the result in [11]. We consider the ERG equation for a scalar field with an arbitrary cutoff function and an arbitrary quadratic seed action. We show that the correlation functions of diffused fields with respect to the bare action agree with those of bare fields with respect to the effective action obeying the above ERG equation, where the diffused field obeys a generalized diffusion equation determined by the cutoff function and the seed action and agrees with the bare field at the initial $\tau$. 

1
We also perform the \( \epsilon \) expansion using the derivative expansion as a check of validity of the ERG equation. We reproduce the well-known scaling dimensions of operators around the Wilson-Fisher fixed point.

Throughout this paper except section 5, we work in the momentum space, and introduce a notation
\[
\int_p \equiv \int \frac{d^d p}{(2\pi)^d}. \tag{1.4}
\]
Note that the Fourier transforms of (1.1), (1.2) and (1.3) are
\[
(\partial_\tau + p^2)\varphi(\tau, p) = 0, \tag{1.5}
\]
\[
\varphi(\tau, p) = K(p, \tau)\phi(p) \tag{1.6}
\]
and
\[
K(p, \tau) = e^{-\tau p^2}, \tag{1.7}
\]
respectively.

This paper is organized as follows. In section 2, we give our statement on relationship between the ERG equation and a generalized diffusion equation. In section 3, we prove our statement using functional differential equations for generating functionals of the correlation functions. In section 4, we provide another proof of our statement by solving the ERG equation using a functional integration kernel. In section 5, we perform the \( \epsilon \) expansion using the derivative expansion. We reproduce the well-known scaling dimensions of operators around the Wilson-Fisher fixed point. Section 6 is devoted to conclusion and discussion. We clarify the reason why we need to restrict ourselves to seed actions that are quadratic in \( \phi \).

## 2 Relation between renormalization group and diffusion equation

The ERG equation is a functional differential equation that describes nonperturbatively how the effective action \( S_\Lambda \) at the energy scale \( \Lambda \) changes when \( \Lambda \) is decreased to \( \Lambda - d\Lambda \). The one
for a scalar field in $d$ dimensions is specified by a cutoff function $\hat{C}_\Lambda(p^2)$ and a seed action $\hat{S}_\Lambda$, and takes the form \[12\textendash}15\]

\[-\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda[\phi]} = -\frac{1}{2} \int_p \hat{C}_\Lambda(p^2) \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} e^{-S_\Lambda[\phi]} - \int_p \hat{C}_\Lambda(p^2) \frac{\delta}{\delta \phi(p)} \left( \frac{\delta \hat{S}_\Lambda[\phi]}{\delta \phi(-p)} e^{-S_\Lambda[\phi]} \right).\]

(2.1)

$\hat{C}_\Lambda$ is quasi-local and incorporates UV regularization, while the seed action $\hat{S}_\Lambda$ is a functional of $\phi$ that has derivative expansion. Here we consider an arbitrary cutoff function $\hat{C}(p^2)$ and an arbitrary quadratic seed action that satisfy the above conditions. The coarse graining procedure is fixed by the cutoff function and the seed action. We parametrize the seed action as

$$\hat{S}_\Lambda[\phi] = -\frac{1}{2} \int_p \hat{C}^{-1}(p^2) \chi_\Lambda(p^2) \phi(p) \phi(-p),$$

(2.2)

where $\chi_\Lambda$ is an arbitrary regular function. The initial condition for (2.1) is given at a bare cutoff scale $\Lambda_0$, and $S_{\Lambda_0} = S_{\Lambda=\Lambda_0}$ is a bare action.

We define the vev with respect to the effective action $S_\Lambda$ as

$$\langle \cdots \rangle_\Lambda = \frac{1}{Z_\Lambda} \int D\phi \cdots e^{-S_\Lambda[\phi]}$$

(2.3)

with

$$Z_\Lambda = \int D\phi e^{-S_\Lambda[\phi]}.$$  

(2.4)

We show the following relation on the correlation functions

$$\langle \prod_{a=1}^n \phi(p_a) \rangle_\Lambda^c = \langle \prod_{a=1}^n \varphi(\tau, p_a) \rangle_{\Lambda_0}^c + \delta_{n,2} (2\pi)^d \delta^d(p_1 + p_2) \ r(\Lambda, p_1^2),$$

(2.5)

where $c$ stands for the connected part, and

$$\tau = \frac{1}{\Lambda^2} - \frac{1}{\Lambda_0^2}.$$  

(2.6)

$\varphi(\tau, p)$ is a solution to the differential equation

$$\partial_{\tau} \varphi(\tau, p) = \frac{1}{2} \Lambda^2 \hat{C}_\Lambda(p^2) \frac{\delta \hat{S}_\Lambda[\phi]}{\delta \varphi(\tau, -p)} = -\frac{1}{2} \Lambda^2 \chi_\Lambda(p^2) \varphi(\tau, p),$$

(2.7)
or equivalently
\[-\Lambda \frac{\partial}{\partial \Lambda} \varphi(\tau, p) = \dot{C}_\Lambda(p^2) \frac{\delta \hat{S}_\Lambda[\varphi]}{\delta \varphi(\tau, -p)} = -\chi_\Lambda(p^2) \varphi(\tau, p) \] (2.8)

with the initial condition
\[\varphi(0, p) = \phi(p) . \] (2.9)

\(r(\Lambda, p^2)\) obeys a differential equation
\[\Lambda \frac{\partial r(\Lambda, p^2)}{\partial \Lambda} = \dot{C}_\Lambda(p^2) + 2\chi_\Lambda(p^2) r(\Lambda, p^2) \] (2.10)

and satisfies an initial condition
\[r(\Lambda_0, p^2) = 0 . \] (2.11)

(2.7) or (2.8) can be viewed as a generalized diffusion equation. (2.5) gives a relation between the ERG equation and the generalized diffusion equation.

For later convenience, we solve (2.10) and (2.8). The solution to (2.10) with the initial condition (2.11) is given by
\[r(\Lambda, p^2) = A_{\Lambda, \Lambda_0}^{-1}(p^2) B_{\Lambda, \Lambda_0}^{-2}(p^2) \] (2.12)

with
\[A_{\Lambda, \Lambda_0}(p^2) = \left[ -\int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \dot{C}_{\Lambda'}(p^2) B_{\Lambda', \Lambda_0}^{2}(p^2) \right]^{-1} , \] (2.13)
\[B_{\Lambda, \Lambda_0}(p^2) = \exp \left[ \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \chi_{\Lambda'}(p^2) \right] . \] (2.14)

The solution to (2.8) with the initial condition (2.9) is given by
\[\varphi(\tau, p^2) = B_{\Lambda, \Lambda_0}^{-1}(p^2) \phi(p) . \] (2.15)

We will give a proof of (2.5) in the next section. Before closing this section, we give an example of \(\dot{C}\) and \(\chi_\Lambda(p^2)\):
\[\dot{C}_\Lambda(p^2) = -\frac{2}{\Lambda^2} K_\Lambda(p^2) + \frac{\zeta}{p^2} K_\Lambda(p^2)(1 - K_\Lambda(p^2)) \] (2.16)
with

\[ K_\Lambda(p^2) = e^{-p^2/\Lambda^2}, \tag{2.17} \]

and

\[ \chi_\Lambda(p^2) = 2 \frac{p^2}{\Lambda^2} - \frac{\eta}{2}. \tag{2.18} \]

Setting \( \zeta = \eta \) in (2.16) yields an ERG equation studied in \([11, 16]\). We rescale \( \varphi \) as

\[ \varphi'(\tau, p) = \left( \frac{\Lambda}{\Lambda_0} \right)^{\frac{\eta}{2}} \varphi(\tau, p). \tag{2.19} \]

Then, (2.5) reduces to

\[ \langle \prod_{a=1}^{n} \varphi(p_a) \rangle_\Lambda^c = \left( \frac{\Lambda_0}{\Lambda} \right)^{\frac{n\eta}{2}} \langle \prod_{a=1}^{n} \varphi'(\tau, p_a) \rangle_{\Lambda_0}^c \]

\[ + \delta_{n,2} (2\pi)^d \delta^d(p_1 + p_2) K_\Lambda(p^2) \int_{D} e^{-S_{\Lambda_0}[\varphi]} \left\{ 1 - K_\Lambda(p^2) \right\} \left\{ 1 - \frac{K_\Lambda(p^2)}{K_\Lambda(p^2)} \right\} \right]. \tag{2.20} \]

We see from (2.7) that \( \varphi' \) satisfies (1.5) and is given by (1.6). (2.20) is nothing but the relation obtained in \([11]\).

3 Proof of the relation (2.5)

In this section, we give a proof of the relation (2.5). For this purpose, we define two functionals of \( J(p) \):

\[ U[J(\cdot), \Lambda] = \ln \int \mathcal{D}\varphi e^{i \int_{\mathbb{R}} J(p)\varphi(-p)} e^{-S_\Lambda[\varphi]}, \tag{3.1} \]

\[ V[J(\cdot), \Lambda] = V'[J(\cdot), \Lambda] + R[J(\cdot), \Lambda] \tag{3.2} \]

with

\[ V'[J(\cdot), \Lambda] = \ln \int \mathcal{D}\varphi e^{i \int_{\mathbb{R}} J(p)\varphi(-p)} e^{-S_{\Lambda_0}[\varphi]}, \tag{3.3} \]

\[ R[J(\cdot), \Lambda] = \frac{i^2}{2} \int_{\mathbb{R}} r(\Lambda, p^2) J(p) J(-p). \tag{3.4} \]
$U$ is the generating functional for the connected correlation functions of $\phi$ with respect to $S_\Lambda$, while $V'$ is the one for the connected correlation functions of $\varphi$ with respect to $S_{\Lambda_0}$. $U$ and $V$ agree at $\Lambda = \Lambda_0$ because $\varphi$ and $r(\Lambda, p^2)$ satisfy (2.9) and (2.11), respectively:

$$U[J(\cdot), \Lambda_0] = V[J(\cdot), \Lambda_0]. \quad (3.5)$$

In the following, we will show that $U$ and $V$ satisfy the same functional differential equation, which is the first order in the $\Lambda$ derivative. First, we calculate

$$-\Lambda \frac{\partial}{\partial \Lambda} U[J(\cdot), \Lambda] \quad (3.6)$$

as follows:

$$= \frac{1}{e^U \int D\phi \; e^{i \int_p J(p)\phi(-p)}} \left( -\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda[\phi]} \right)$$

$$= \frac{1}{e^U \int D\phi \; e^{i \int_p J(p)\phi(-p)}} \left( -\frac{1}{2} \int_p \hat{C}_\Lambda(p^2) \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} e^{-S_\Lambda[\phi]} - \int_p \hat{C}_\Lambda(p^2) \frac{\delta}{\delta \phi(p)} \left( \frac{\delta \hat{S}_\Lambda[\phi]}{\delta \phi(-p)} e^{-S_\Lambda[\phi]} \right) \right)$$

$$= -\frac{i^2}{2} \int_p \hat{C}_\Lambda(p^2) J(p) J(-p) - \frac{i}{e^U} \int D\phi \; e^{i \int_p J(p)\phi(-p)} \int_p \hat{C}_\Lambda(p^2) J(p) \frac{\delta \hat{S}_\Lambda[\phi]}{\delta \phi(p)} e^{-S_\Lambda[\phi]}$$

$$= -\frac{i^2}{2} \int_p \hat{C}_\Lambda(p^2) J(p) J(-p) - \frac{i}{e^U} \int D\phi \; e^{i \int_p J(p)\phi(-p)} \int_p \hat{C}_\Lambda(p^2) J(p) \frac{\delta \hat{S}_\Lambda[\phi]}{\delta \phi(p)} \left( \frac{1}{i \delta J(\cdot)} \right) e^U$$

$$= -\frac{i^2}{2} \int_p \hat{C}_\Lambda(p^2) J(p) J(-p) - \int_p \chi_\Lambda(p^2) J(p) \frac{\delta U}{\delta J(p)}. \quad (3.7)$$

Here we have used (2.1) in the second equality and performed partial path-integrations in the third equality. In the fourth equality, $\frac{\delta \hat{S}_\Lambda[\phi]}{\delta \phi(p)} \left[ \frac{1}{i \delta J(\cdot)} \right]$ is obtained by replacing $\phi(p')$ in a functional $\frac{\delta \hat{S}_\Lambda[\phi]}{\delta \phi(p)}$ with $\frac{1}{i \delta J(-p')}$. In the fifth equality, we have used the explicit form of $\hat{S}_\Lambda$ (2.2).

Next, we calculate

$$-\Lambda \frac{\partial}{\partial \Lambda} V[J(\cdot), \Lambda] \quad (3.8)$$

as follows:
\[
= -\frac{i^2}{2} \int_p \Lambda \frac{\partial r(\Lambda, p^2)}{\partial \Lambda} J(p)J(-p) + \frac{i}{e^{\nu R}} \int \mathcal{D}\phi \ e^{i \int_p J(p) \varphi(\tau, -p)} \int_p J(p) \left( -\Lambda \frac{\partial}{\partial \Lambda} \varphi(\tau, -p) \right) e^{-S_{\lambda_0}[\varphi]}
\]
\[
= -\frac{i^2}{2} \int_p (\dot{C}_\Lambda(p^2) + 2\chi_\Lambda(p^2)r(\Lambda, p^2)) J(p)J(-p)
\]
\[
+ \frac{i}{e^{\nu R}} \int \mathcal{D}\phi \ e^{i \int_p J(p) \varphi(\tau, -p)} \int_p J(p) \dot{C}_\Lambda(p^2) \frac{\delta S_\Lambda[\varphi]}{\delta \varphi(\tau, p)} e^{-S_{\lambda_0}[\varphi]}
\]
\[
= -\frac{i^2}{2} \int_p (\dot{C}_\Lambda(p^2) + 2\chi_\Lambda(p^2)r(\Lambda, p^2)) J(p)J(-p) + \frac{i}{e^{\nu R}} \int \dot{C}_\Lambda(p^2) J(p) \frac{\delta S_\Lambda[\varphi]}{\delta \varphi(p)} \left[ \frac{1}{i} \frac{\delta}{\delta J(p)} \right] e^{\nu R}
\]
\[
= -\frac{i^2}{2} \int_p (\dot{C}_\Lambda(p^2) + 2\chi_\Lambda(p^2)r(\Lambda, p^2)) J(p)J(-p) - \int_p \chi_\Lambda(p^2) J(p) \frac{\delta V}{\delta J(p)} (V - R)
\]
\[
= -\frac{i^2}{2} \int_p \dot{C}_\Lambda(p^2) J(p)J(-p) - \int_p \chi_\Lambda(p^2) J(p) \frac{\delta V}{\delta J(p)} \] \quad (3.9)

Here we have used \((2.8)\) and \((2.10)\) in the second equality, \((2.2)\) in the fourth equality and \((3.4)\) in the last equality.

We see from \((3.7)\) and \((3.9)\) that \(U\) and \(V\) satisfy the same functional differential equation, which is the first order in the \(\Lambda\) derivative. We therefore conclude with \((3.5)\) that
\[
U[J(\cdot), \Lambda] = V[J(\cdot), \Lambda].
\] \quad (3.10)

Finally, we calculate the LHS of the relations \((2.5)\):
\[
\left\langle \prod_{a=1}^n \phi(p_a) \right\rangle_{\Lambda_0} = \frac{1}{i^n} \frac{\delta^n}{\delta J(-p_1) \cdots \delta J(-p_n)} U[J(\cdot), \Lambda] \bigg|_{J=0}
\]
\[
= \frac{1}{i^n} \frac{\delta^n}{\delta J(-p_1) \cdots \delta J(-p_n)} V[J(\cdot), \Lambda] \bigg|_{J=0}
\]
\[
= \frac{1}{i^n} \frac{\delta^n}{\delta J(-p_1) \cdots \delta J(-p_n)} V'[J(\cdot), \Lambda] \bigg|_{J=0} + \delta_{n,2}(2\pi)^d \delta^d(p_1 + p_2)r(\Lambda, p_1^2)
\]
\[
= \left\langle \prod_{a=1}^n \varphi(\tau, p_1) \right\rangle_{\Lambda_0} + \delta_{n,2}(2\pi)^d \delta^d(p_1 + p_2)r(\Lambda, p_1^2). \] \quad (3.11)

Thus, we have completed the proof of \((2.5)\).

4 Functional integration kernel

In this section, we show \((2.5)\) by solving \((2.1)\) in terms of a functional integration kernel, which is a functional analog of the heat kernel \((1.3)\). We seek for a functional integra-
tion kernel $\mathcal{K}[\phi, \phi', \Lambda, \Lambda_0]$ that is a solution to the ERG equation (2.1) satisfying an initial condition

$$\mathcal{K}[\phi, \phi', \Lambda_0, \Lambda_0] = \Delta[\phi - \phi'],$$  \hspace{1cm} (4.1)

where $\Delta$ is the delta functional. It is easy to show that $\mathcal{K}$ is given by

$$\mathcal{K}[\phi, \phi', \Lambda, \Lambda_0] = \det[A^2_{\Lambda, \Lambda_0} B_{\Lambda, \Lambda_0}] \exp \left[ -\frac{1}{2} \int_p A_{\Lambda, \Lambda_0}(p^2)(B_{\Lambda, \Lambda_0}(p^2)\phi(p) - \phi'(p))(B_{\Lambda, \Lambda_0}(p^2)\phi(-p) - \phi'(-p)) \right],$$  \hspace{1cm} (4.2)

where $A_{\Lambda, \Lambda_0}$ and $B_{\Lambda, \Lambda_0}$ are defined in (2.13) and (2.14), respectively, and satisfy the initial conditions

$$A_{\Lambda_0, \Lambda_0} = \infty, \quad B_{\Lambda_0, \Lambda_0} = 1.$$  \hspace{1cm} (4.3)

Then, (2.1) is solved in terms of $\mathcal{K}$ as

$$e^{-S_{\Lambda}[\phi]} = \int \mathcal{D}\phi' \mathcal{K}[\phi, \phi', \Lambda, \Lambda_0] e^{-S_{\Lambda_0}[\phi']}.$$  \hspace{1cm} (4.4)

By substituting (4.4) into the definition of $U$ (3.1), performing $\phi$ integration and using (2.12) and (2.15), we can show (3.10). Thus, we give another proof of the relation (2.5).

5 $\epsilon$ expansion

In this section, as a check of validity of the ERG equation (2.1), we perform the $\epsilon$ expansion in $d = 4 - \epsilon$ dimensions using the derivative expansion. We restrict ourselves to the case in which $\hat{C}_\Lambda(p^2)$ and $\chi_\Lambda(p^2)$ are given by (2.16) and (2.18), respectively. We will see that the scaling dimensions of operators around the Wilson-Fisher fixed point are reproduced for arbitrary $\eta$ and $\zeta$. The $\epsilon$ expansion using the derivative expansion was studied for the Polchinski equation [18], which can be viewed as an ERG equation, in [17] and for another specific ERG equation in [9]. Here we perform procedure done in [9].

We set $A_0 = \infty$ so that (2.6) reduces to $\tau = 1/\Lambda^2$. We expand $\hat{C}_\Lambda$ in terms of $\tau$ as

\[ \text{Here we keep a factor } e^{-\tau p^2} \text{ as a regularization of the delta function, which includes ambiguity. Indeed, we have checked that the conclusion is unchanged if } e^{-\tau p^2} \text{ is replaced by } be^{-\tau p^2} \text{ with } b \text{ being a constant. A similar ambiguity in the regularization of the delta function is also seen in (5.13).} \]
\[ \dot{C}_\Lambda(p^2) = -e^{-\tau p^2} \left\{ 2\tau - \frac{\zeta}{p^2} \left( 1 - e^{-\tau p^2} \right) \right\} \]
\[ = -2\tau e^{-\tau p^2} \left( 1 - \frac{\zeta}{2} + O(\tau) \right). \]  

(5.1)

Then, we rewrite (2.1) in the coordinate space as
\[ \partial_\tau S_\Lambda[\phi] = \int_x \left( -\partial^2 \phi(x) - \frac{\eta}{4\pi} \phi(x) \frac{\delta S_\Lambda[\phi]}{\delta \phi(x)} \right) \]
\[ - \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) \int_{x,y} K(x, y, \tau) \left\{ \frac{\delta S_\Lambda[\phi]}{\delta \phi(x)} \frac{\delta S_\Lambda[\phi]}{\delta \phi(y)} - \frac{\delta^2 S_\Lambda[\phi]}{\delta \phi(x) \delta \phi(y)} \right\}, \]  

(5.2)

where \( \int_x = \int d^d x \) and \( K(x, y, \tau) \) is defined in (1.3).

In the following, we make the derivative expansion keeping up to two derivatives. The effective action is expanded as
\[ S_\Lambda[\phi] = \int_x \left[ V_\tau(\phi(x)) + \frac{1}{2} W_\tau(\phi(x)) (\partial \phi(x))^2 \right]. \]  

(5.3)

By using a formula
\[ \int_{x,y} K(x, y, \tau) f(\phi(x)) g(\phi(y)) \]
\[ = \int_x \left[ f(\phi(x)) g(\phi(x)) - \tau (\partial \phi(x))^2 f'(\phi(x)) g'(\phi(x)) + O(\tau^2) \right], \]  

(5.4)

we obtain from (5.2)
\[ \partial_\tau S_\Lambda[\phi] = \int_x \left[ -\frac{\eta}{4\pi} \phi V'_\tau - \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) V''_\tau + \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) (4\pi\tau)^{-d/2} \left( V''_\tau + \frac{d}{2\tau} W_\tau \right) \right] \]
\[ + \left( \partial \phi \right)^2 \left\{ V''_\tau - \frac{\eta}{4\pi} \left( \frac{1}{2} \phi W'_\tau + W_\tau \right) - \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) \left( V''_\tau W'_\tau + 2V''_\tau W_\tau - \tau V'''_\tau \right) \right\} \]
\[ + \frac{1}{4} \left( 1 - \frac{\zeta}{2} \right) (4\pi\tau)^{-d/2} W''_\tau \]  

(5.5)

On the other hand, we obtain from (5.3)
\[ \partial_\tau S_\Lambda[\phi] = \int_x \left[ \partial_\tau V_\tau(\phi) + \frac{1}{2} \left( \partial \phi \right)^2 \partial_\tau W_\tau(\phi) \right]. \]  

(5.6)

Comparing (5.5) and (5.6) gives rise to
\[ \partial_\tau V_\tau(\phi) = -\frac{\eta}{4\pi} \phi V'_{\tau} - \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) \phi V''_{\tau} + \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) (4\pi\tau)^{-d/2} (V''_{\tau} + \frac{d}{2\tau} W_\tau), \]  

(5.7)
\[ \partial_{\tau} W_\tau(\phi) = 2V'' - \frac{n}{2\tau} \left( \frac{1}{2} \phi W'_\tau + W_\tau \right) - \left( 1 - \frac{\zeta}{2} \right) \left( V'_\tau W'_\tau + 2V''_\tau W_\tau - \tau V''_\tau \right) \]
\[ + \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) (4\pi \tau)^{-d/2} W''_\tau . \] (5.8)

We normalize the kinetic term in (5.3) as \( W_\tau(\phi) = 1 \), so that
\[ W'_\tau(\phi) = 0, \quad W''_\tau(\phi) = 0 . \] (5.9)

Then, (5.7) and (5.8) reduce to
\[ \partial_{\tau} V_\tau(\phi) = -\frac{n}{4\tau} \phi V'_\tau - \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) V'^2_\tau + \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) (4\pi \tau)^{-d/2} \left( V'_\tau + \frac{d}{2\tau} \right) , \] (5.10)
\[ \partial_{\tau} W_\tau(\phi) = -\frac{n}{2\tau} + \zeta V''_\tau + \left( 1 - \frac{\zeta}{2} \right) \tau V''_\tau . \] (5.11)

We make a change of variable \( \phi \to \varphi \) such that \( W_{\tau+\delta \tau} = 1 \):
\[ d\varphi(x) = \sqrt{W_{\tau+\delta \tau}(\phi)} \, d\phi(x) . \] (5.12)

Then, the Jacobian for the change of variable is
\[ \left| \frac{\delta \phi}{\delta \varphi} \right| = 1 - \delta \tau \int_{x,y} aK(x, y, \tau) \left\{ -\frac{n}{4\tau} + \frac{\zeta}{2} V''_\tau + \left( 1 - \frac{\zeta}{2} \right) \tau V''_\tau \right\} \delta^{(d)}(x - y) \]
\[ = \exp \left[ -\delta \tau \int_{x} a(4\pi \tau)^{-d/2} \left\{ -\frac{n}{4\tau} + \frac{\zeta}{2} V''_\tau + \left( 1 - \frac{\zeta}{2} \right) \tau V''_\tau \right\} \right] , \] (5.13)
where we have regularized the delta function \( \delta^{(d)}(x - y) \) by \( aK(x, y, \tau) \) with \( a \) being a constant.

We further make quantities dimensionless as
\[ x \to \tau^{1/2} x, \quad \varphi(x) \to \tau^{-(d-2)/4} \varphi(x), \quad V_\tau \to \tau^{-d/2} V_\tau . \] (5.14)

Then, from (5.7), (5.13) and (5.14), we obtain
\[ \tau \partial_{\tau} V_\tau = \frac{d}{2} V'_\tau - \frac{d - 2}{4} \varphi V'_\tau - \frac{1}{2} \left( 1 + \frac{\zeta}{2} \right) V'^2_\tau + \frac{B}{2} \left\{ 1 + \left( a - \frac{1}{2} \right) \zeta \right\} V''_\tau + \frac{B}{2} \left( \frac{a - \zeta}{2} \right) V''_\tau \]
\[ + \frac{B}{4} \left( d - a\eta - \frac{d}{2} \zeta \right) - \frac{1}{2} \left( 1 - \frac{\zeta}{2} \right) V'_\tau \int_{0}^{\varphi} d\varphi' V''_\tau , \] (5.15)
where \( B = (4\pi)^{-d/2} \).

We expand \( V_\tau \) in terms of \( \phi \) as
\[ V_\tau = v_0 + \frac{1}{2!} v_2 \varphi^2 + \frac{1}{4!} v_4 \varphi^4 + \frac{1}{6!} v_6 \varphi^6 + \frac{1}{8!} v_8 \varphi^8 + \frac{1}{10!} v_{10} \varphi^{10} + \frac{1}{12!} v_{12} \varphi^{12} + \frac{1}{14!} v_{14} \varphi^{14} + \cdots . \] (5.16)
By substituting (5.16) into (5.15), we obtain

\[ \tau \partial_{\tau} v_2 = v_2 - \left(1 + \frac{\zeta}{2}\right)v_2^2 - \left(1 - \frac{\zeta}{2}\right)v_2^3 + \frac{B}{2} \left\{ 1 + \left( a - \frac{1}{2}\right) \zeta \right\} v_4 + aB \left(1 - \frac{\zeta}{2}\right)v_2v_4 \]

\[ \tau \partial_{\tau} v_4 = \frac{\epsilon}{2} v_4 - 4 \left(1 + \frac{\zeta}{2}\right)v_2v_4 - 6 \left(1 - \frac{\zeta}{2}\right)v_2^2 v_4 + \frac{B}{2} \left\{ 1 + \left( a - \frac{1}{2}\right) \zeta \right\} v_6 + aB \left(1 - \frac{\zeta}{2}\right) (3v_2^2 + v_2v_6) \]  

\[ \tau \partial_{\tau} v_6 = (-1 + \epsilon)v_6 - \left(1 + \frac{\zeta}{2}\right)(10v_4^2 + 6v_2v_6) + \frac{B}{2} \left\{ 1 + \left( a - \frac{1}{2}\right) \zeta \right\} v_8 + aB \left(1 - \frac{\zeta}{2}\right) (15v_4v_6 + v_2v_8) - \left(1 - \frac{\zeta}{2}\right)(38v_2 v_4^2 + 9v_2^2 v_6) \]

\[ \tau \partial_{\tau} v_8 = \left(-2 + \frac{3}{2} \epsilon\right)v_8 - \left(1 + \frac{\zeta}{2}\right)(56v_4v_6 + 8v_2v_8) + \frac{B}{2} \left\{ 1 + \left( a - \frac{1}{2}\right) \zeta \right\} v_{10} + aB \left(1 - \frac{\zeta}{2}\right) (35v_6^2 + 28v_2v_8 + v_2v_{10}) \]

\[ \tau \partial_{\tau} v_{10} = (-3 + 2\epsilon)v_{10} - \left(1 + \frac{\zeta}{2}\right)(126v_6^2 + 120v_4v_8 + 10v_2v_{10}) + \frac{B}{2} \left\{ 1 + \left( a - \frac{1}{2}\right) \zeta \right\} v_{12} + aB \left(1 - \frac{\zeta}{2}\right) (210v_6 v_8 + 45v_4v_{10} + v_2v_{12}) - \left(1 - \frac{\zeta}{2}\right)(602v_2 v_6^2 + 520v_2v_4v_8 + 15v_2^2 v_10 + 2556v_4^2 v_6) \]

\[ \tau \partial_{\tau} v_{12} = \left(-4 + \frac{5}{2} \epsilon\right)v_{12} - \left(1 + \frac{\zeta}{2}\right)(792v_6 v_8 + 220v_4v_{10} + 12v_2v_{12}) + \frac{B}{2} \left\{ 1 + \left( a - \frac{1}{2}\right) \zeta \right\} v_{14} + aB \left(1 - \frac{\zeta}{2}\right) (462v_8^2 + 495v_6v_{10} + 66v_4v_{12} + v_2v_{14}) - \left(1 - \frac{\zeta}{2}\right)(4104v_2 v_6 v_8 + 980v_2v_4v_{10} + 18v_2^2 v_{12} + 19580v_4v_6^2 + 8536v_4^2 v_8) \]

where \( \epsilon = 4 - d \).

We look for the Wilson-Fisher fixed point by assuming that

\[ v_2^* = O(\epsilon), \quad v_4^* = O(\epsilon), \quad v_6^* = O(\epsilon^2), \quad v_n^* = O(\epsilon^3) \quad \text{for} \quad n \geq 8 . \]  

We set \( \tau \partial_{\tau} v_n^* = 0 \) in (5.17)-(5.19), and expand the RHS of (5.17) up to the first order in \( \epsilon \) and those of (5.18) and (5.19) up to the second order in \( \epsilon \). Then, we obtain equations determining the fixed point as follows:

\[ 0 = v_2^* + \frac{B}{2} \left\{ 1 + \left( a - \frac{1}{2}\right) \zeta \right\} v_4^* . \]
We take a

We solve these equations as

up to the first order in $\epsilon$

By substituting $v = (4\pi)^{-d/2} = B_0 + O(\epsilon) = (4\pi)^{-2} + O(\epsilon)$.

We solve these equations as

\begin{align}
v^*_2 &= -\frac{\epsilon}{6A}\left\{1 + \left(a - \frac{1}{2}\right)\zeta\right\} + O(\epsilon^2), \\
v^*_4 &= \frac{\epsilon}{3B_0A} + O(\epsilon^2), \\
v^*_6 &= -\frac{10\epsilon^2}{(3B_0A)^2}\left(1 + \frac{\zeta}{2}\right) + O(\epsilon^3),
\end{align}

where

\[ A \equiv \left(a - \frac{1}{2}\right)\zeta^2 + 3a\zeta - 2a + 2. \]

We take $a$ such that $A \neq 0$. Note that

\[ B = (4\pi)^{-d/2} = B_0 + O(\epsilon) = (4\pi)^{-2} + O(\epsilon). \]

By substituting $v_n = v^*_n + \delta v_n$ into (5.17)-(5.22) and expanding the RHS of (5.17)-(5.22) up to the first order in $\epsilon$ and $\delta v_n$, we obtain

\begin{align}
\tau \partial_\tau \delta v_2 &= \left\{1 - 2\left(1 + \frac{\zeta}{2}\right)v^*_2 + aB\left(1 - \frac{\zeta}{2}\right)v^*_4\right\}\delta v_2 \\
&\quad + \frac{B}{2}\left\{1 + \left(a - \frac{1}{2}\right)\zeta + 2a\left(1 - \frac{\zeta}{2}\right)v^*_2\right\}\delta v_4, \\
\tau \partial_\tau \delta v_4 &= -4\left(1 + \frac{\zeta}{2}\right)v^*_4\delta v_2 + \left\{\frac{\zeta}{2} - 4\left(1 + \frac{\zeta}{2}\right)v^*_2 + 6aB\left(1 - \frac{\zeta}{2}\right)v^*_4\right\}\delta v_4 \\
&\quad + \frac{B}{2}\left\{1 + \left(a - \frac{1}{2}\right)\zeta + 2a\left(1 - \frac{\zeta}{2}\right)v^*_2\right\}\delta v_6, \\
\tau \partial_\tau \delta v_6 &= -20\left(1 + \frac{\zeta}{2}\right)v^*_6\delta v_4 + \left\{-1 + \epsilon - 6\left(1 + \frac{\zeta}{2}\right)v^*_2 + 15aB\left(1 - \frac{\zeta}{2}\right)v^*_4\right\}\delta v_6 \\
&\quad + \frac{B}{2}\left\{1 + \left(a - \frac{1}{2}\right)\zeta + 2a\left(1 - \frac{\zeta}{2}\right)v^*_2\right\}\delta v_8, \\
\tau \partial_\tau \delta v_8 &= -56\left(1 + \frac{\zeta}{2}\right)v^*_8\delta v_6 + \left\{-2 + \frac{3}{2}\epsilon - 8\left(1 + \frac{\zeta}{2}\right)v^*_2 + 28aB\left(1 - \frac{\zeta}{2}\right)v^*_4\right\}\delta v_8 \\
&\quad + \frac{B}{2}\left\{1 + \left(a - \frac{1}{2}\right)\zeta + 2a\left(1 - \frac{\zeta}{2}\right)v^*_2\right\}\delta v_{10}, \\
\tau \partial_\tau \delta v_{10} &= -120\left(1 + \frac{\zeta}{2}\right)v^*_4\delta v_8 + \left\{-3 + 2\epsilon - 10\left(1 + \frac{\zeta}{2}\right)v^*_2 + 45aB\left(1 - \frac{\zeta}{2}\right)v^*_4\right\}\delta v_{10} \\
&\quad + \frac{B}{2}\left\{1 + \left(a - \frac{1}{2}\right)\zeta + 2a\left(1 - \frac{\zeta}{2}\right)v^*_2\right\}\delta v_{12},
\end{align}
τ∂τv_{12} = -220 \left(1 + \frac{ζ}{2}\right)v_4^* v_{10} + \left\{-4 + \frac{5}{2} \epsilon - 12 \left(1 + \frac{ζ}{2}\right)v_2^* + 66aB \left(1 - \frac{ζ}{2}\right)v_4^*\right\} \delta v_{12} \\
+ \frac{B}{2} \left(1 + \left(a - \frac{1}{2}\right)ζ + 2a \left(1 - \frac{ζ}{2}\right)v_2^*\right) \delta v_{14} . \tag{5.37}

We regard these as a linear transformation for \( δv_n \). Then, the eigenvalue equation reads

\begin{align*}
-λ^5 + λ^4 \left\{-5 + 5\epsilon - 15(2 + ζ)v_2^* + \frac{95}{2}aB(2 - ζ)v_4^*\right\} \\
+ λ^3 \left\{-5 + 15\epsilon - 50(2 + ζ)v_2^* + B(-100 + 270a - 235aζ + 25ζ^2 - 50aζ^2)v_4^*\right\} \\
+ λ^2 \left\{5 - 15(2 + ζ)v_2^* + B(-108 - 17a - \frac{199}{2}aζ + 27ζ^2 - 54aζ^2)v_4^*\right\} \\
+ λ \left\{6 - 17\epsilon + 44(2 + ζ)v_2^* + B(112 - 288a + 256aζ - 28ζ^2 + 56aζ^2)v_4^*\right\} \\
- 3\epsilon + 12(2 + ζ)v_2^* + B(48 - 36a + 66aζ - 12ζ^2 + 24aζ^2)v_4^* = 0 . \tag{5.38}
\end{align*}

Here we have ignored (5.37) and dropped the term proportional to \( δv_{12} \) in (5.36). By substituting (5.27) and (5.28) into this equation, we obtain

\begin{align*}
-λ^5 - λ^4 \left\{5 - \frac{ε}{6}(45 + 110D)\right\} - λ^3 \left\{5 - \frac{ε}{3}(20 + 110D)\right\} \\
+ λ^2 \left\{5 - \frac{ε}{6}(93 + 110D)\right\} + λ \left\{6 - \frac{ε}{3}(17 + 110D)\right\} + 3ε = 0 , \tag{5.39}
\end{align*}

where

\[ D \equiv \frac{a(2 - ζ)}{A} = \frac{a(2 - ζ)}{(a - 1/2)ζ^2 + 3aζ - 2a + 2} . \tag{5.40} \]

The solutions to (5.39) are

\[
\lambda_2 = 1 - \frac{1}{6}ε, \quad \lambda_4 = -\frac{1}{2}ε, \quad \lambda_6 = -1 - \frac{3}{2}ε, \quad \lambda_8 = -2 - \frac{19}{6}ε, \\
\lambda_{10} = -3 + \frac{ε}{6}(77 + 110D) . \tag{5.41}
\]

\( \lambda_2 \sim \lambda_8 \) are the well-known scaling dimensions of operators around the Wilson-Fisher fixed point. We have checked that \( \lambda_{10} \) is also obtained correctly if (5.37) is included.

6 Conclusion and discussion

In this paper, we studied relationship between renormalization group and diffusion equation. We considered the ERG equation for a scalar field that includes an arbitrary cutoff function and an arbitrary quadratic seed action. As a generalization of the result in ref. [11], we
found that the correlation functions of diffused fields with respect to the bare action agree
with those of bare fields with respect to the effective action, where the diffused field obeys
a generalized diffusion equation determined by the cutoff function and the seed action and
agrees with the bare field at the initial time. This result is reasonable in that diffusion is
associated with coarse graining and that the coarse graining procedure is fixed by the cutoff
function and the seed action. We performed the $\epsilon$ expansion using the derivative expansion as
a check of validity of the ERG equation. We reproduced the well-known scaling dimensions
of operators around the Wilson-Fisher fixed point.

We make a comment on a case in which the seed action includes terms that are higher
than the second order in $\phi$. In this case, $U$ satisfies a functional differential equation given
by the fourth equality in (3.7), while $V$ satisfies a functional differential equation given by
the third equality in (3.9). The latter equation includes terms that mix $\delta V/\delta J$ with $\delta R/\delta J$,
and those terms do not exist in the former equation. This implies that the relation (2.5)
does not hold for this case even if $r$ is modified. We need to generalize (2.5) in some way.

Finally, we note that (2.7) is a ‘gradient flow equation’

$$\partial_\tau \varphi(\tau, p) = -\frac{\delta \tilde{S}}{\delta \varphi(\tau, -p)}$$

with

$$\tilde{S} = \int_p \frac{\Lambda^2}{4} \chi(\Lambda^2) \varphi(\tau, p) \varphi(\tau, -p).$$

While the above gradient flow equation is at first sight conceptually different from the one
studied in the context of gauge theories, we hope that our findings in this paper will give
some insights into construction of an ERG equation with manifest gauge invariance.

Acknowledgements

A.T. was supported in part by Grant-in-Aid for Scientific Research (No. 18K03614) from
Japan Society for the Promotion of Science.

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