Factorisations of some partially ordered sets and small categories

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Received: date / Accepted: date

Abstract Orbits of automorphism groups of partially ordered sets are not necessarily congruence classes, i.e. images of an order homomorphism. Based on so-called orbit categories a framework of factorisations and unfoldings is developed that preserves the antisymmetry of the order Relation. Finally some suggestions are given, how the orbit categories can be represented by simple directed and annotated graphs and annotated binary relations. These relations are reflexive, and, in many cases, they can be chosen to be antisymmetric. From these constructions arise different suggestions for fundamental systems of partially ordered sets and reconstruction data which are illustrated by examples from mathematical music theory.

Keywords ordered set · factorisation · po-group · small category

Mathematics Subject Classification (2010) 06F15 · 18B35 · 00A65

1 Introduction

In general, the orbits of automorphism groups of partially ordered sets cannot be considered as equivalence classes of a convenient congruence relation of the corresponding partial orders. If the orbits are not convex with respect to the order relation, the factor relation of the partial order is not necessarily a partial order. However, when we consider a partial order as a directed graph, the direction of the arrows is preserved during the factorisation in many cases, while the factor graph of a simple graph is not necessarily simple. Even if the factor relation can be used to anchor unfolding information [1], this structure is usually not visible as a relation.

According to the common mathematical usage a fundamental system is a structure that generates another (larger structure) according to some given rules. In this article we will discuss several suggestions for such fundamental systems. For example, orbit categories and transversal categories can be considered as fundamental systems.

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However, ordered sets are considered as relational structures, which is not always possible for the above mentioned categories. When we map them into simple categories we often lose the direction information. This can sometimes be avoided when we throw away redundant information that can be regained during the reconstruction process.

For ordered sets local orders come into mind as possible fundamental systems. The usual definition defines a local order as a binary relation that is an order relation in every environment of some set covering. However, there is no natural covering for an ordered set. This means that any reflexive and antisymmetric relation can be considered as a local order. In the same way as order relations generalise to categories, local orders generalise to partial categories.

The current work is inspired by the objects of Mathematical Music Theory. Some of them (e.g. pitch and time intervals) have a natural notion of order \[2, 3\]. While pitch interval groups are usually commutative the time information is often equivalent to the affine group of the straight line \[4\]. In both cases some kind of periodicity plays an important role. The most prominent one for interval systems is the octave identification and the most obvious one in the time dimension is the metric periodicity. With the Shepard tones and the forward/backward directions we have aspects of the order relations that can still be discussed using the factor structures even though the usual factorisation of relations contradicts this fact \[5\].

It is a well known fact that the factor groups of po-groups with respect to convex normal subgroups are again po-groups. However, the above mentioned examples do not belong to this class.

Orbit factorisations of linearly ordered sets can be described using cyclic orders \[6\]. However, this fails in case of higher order dimension. This can be seen when we factor \((\mathbb{Z}, \leq) \times (\mathbb{Z}, \leq)\) by its subgroup \((4\mathbb{Z}, \leq) \times (3\mathbb{Z}, \leq)\). In that case we have the two paths

\[
(0, 0) \rightarrow (0, 1) \rightarrow (1, 1) \rightarrow (2, 1) \rightarrow (3, 1) \quad \text{and} \quad (0, 0) \rightarrow (1, 0) \rightarrow (2, 0) \rightarrow (3, 0) \rightarrow (3, 1) \rightarrow (0, 1).
\]

Consequently, the factored cyclic order relation contains the two triplets

\[
((0, 0), (0, 1), (3, 1)) \quad \text{and} \quad ((0, 0), (3, 1), (0, 1)).
\]

This violates the antisymmetry condition of cyclic orders and cyclically ordered groups.

In the previous case we still get a result when we consider the neighbourhood relation. Its factorisation leads again to a similar relation. So it is a good candidate when it comes to positive facts about a proper definition of a factor relation.

A drawback of this approach is that it does not work with orders containing infinite closed intervals like the product order \((\mathbb{Q}, \leq) \times (\mathbb{Q}, \leq)\).

In the approach of the current article during factorisation as much information is to be preserved as possible. Thus, the order relation is considered as a category. This also allows chaining of factorisations.

For non-commutative po-groups there is a difference between the group acting on itself via left and via right multiplication. For example David Lewin \[4\] distinguishes between both operations as “transpositions” and “interval preserving functions”, such that they get different interpretations. This suggests that either of these actions can be used independently of the other to describe the factorisations. Thus, it suffices to consider the factorisations of right po-groups. This simplifies the construction a lot as we can restrict ourselves to semi-regular group actions on ordered sets.
After a preliminary section (Section 2), in Section 3 we will introduce orbit categories that model factored po-groups. This will be enriched in Section 4 with additional information that is used to define and study basic properties of an unfolding operation based on group extensions. The actual reconstruction and its isomorphy to the original po-group is considered in Section 5. The following three sections describe modifications of the previous results which use the isomorphic category structure of the orbits. In Section 6 conditions are studied that allow to factor out the vertex monoids from the orbit categories. These so called “flat orbit categories” are used to define corresponding flat category representations and their unfolding in Section 7. In this section it is also proved that unfoldings of category representations and flat category representations are isomorphic. Finally, in Section 8 settings are discussed which allow to use partial categories as part of the representations. A list of additional properties of flat representations is introduced, including conditions under which the partial categories can be antisymmetric graphs. The article closes with some remarks about motivations and applications in mathematical music theory in Section 9 and some outlook in Section 10.

2 Preliminaries

Throughout this paper we consider an order relation to be a symmetric, reflexive and transitive relation. When it is not explicitly stated an order relation needs not to be linearly ordered. A partially ordered group (po-group) \((G, \cdot, ^-1, \leq)\) is a group with an order relation that for any \(x, y, a, b \in G\) fulfills the law \(a \leq b \iff axy \leq xby\). In the same way the wording “right po-group” is a group that fulfills the one sided relation \(a \leq b \iff ax \leq bx\). An element \(a \in G\) is called positive iff \(1 \leq a\) holds. The set of all positive elements \(G_+ := \{ a \in G \mid 1 \leq a \}\) is called positive cone. Likewise an element \(a \in G\) is called negative iff \(a \leq 1\) holds. The symbol \(\langle\) denotes a normal subgroup, while an ordinary subgroup is denoted by the order relation \(\leq\).

A left associative group action of a group \(G\) is a mapping \(G \times M \rightarrow M : (g, a) \mapsto g^a\) such that the equations \(1^a = a\) and \((a^g)^h = a^{(gh)}\) hold. The orbits are denoted by \(a^G := \{ a^g \mid g \in G \}\). In the same way a mapping \(G \times M \rightarrow M : (g, a) \mapsto a^g\) with \(1^a = a\) and \((b^a)^h = (b^h)^a\) is called right associative group action. The corresponding orbits are denoted by \(G^a\). In the same way a right associative group action is denoted by \(g^a\) and the right associative orbit with \(G^a\), respectively. The group acts semi-regular if for any set elements \(a\) and any group elements \(g\) the equation \(a = g^a\) implies \(g = 1\). It acts transitive \(a^G = M\). The action is called regular if it is semi-regular and transitive.

We will consider a category \(\mathcal{C}\) as a (not necessarily simple) directed graph with vertex set \(\mathcal{Ob}\ \mathcal{C}\) and edge set \(\mathcal{Mor}\ \mathcal{C}\). We refer to the edges as arrows. The set of arrows between two vertices \(x, y \in \mathcal{Ob}\ \mathcal{C}\) is denoted by \(\mathcal{Mor}_{\mathcal{C}}(x, y)\). The start and end vertex of an arrow \(a\) are denoted by \(\sigma(a)\) and \(\tau(a)\). For the concatenation \(*\) of a category \(\mathcal{C}\) a covariant notation has been chosen, leading to \(\sigma(a * b) = \sigma(a)\) and \(\tau(a * b) = \tau(b)\). The identity loop of a vertex \(x\) is denoted by \(id_x\).

Given a relation \(\rho \subseteq M \times N\), a mapping \(\phi : M \times N \rightarrow M' \times N'\) is called homomorphism into a relation \(\rho' \subseteq M' \times N'\), if for all elements \(x \in M\) and \(y \in N\) the equivalence \(x \rho y \Rightarrow \phi x \rho' \phi y\) holds. If furthermore \(\phi\) is bijective and \(x \rho y \Leftrightarrow \phi x \rho' \phi y\) holds, \(\phi\) is called an isomorphism. Considering two products sets \(M_1 \times M_2 \times \ldots \times M_n\) and \((\ldots (M_1 \times M_2) \times \ldots \).
\[ \cdots \times M_{n-1} \times M_n \) as equal we can extend these definitions to arbitrary relations. In this setting each \( f : M \to N \) is represented as a pair of relations \( \rho_f \) and \( \rho_f' \) with \( f(x) = y \iff x \rho_f y \) and \( f(x) \neq y \iff x \rho_f' y \). It is easy to see, that homomorphisms and isomorphisms of such pairs of relations fulfil the properties of the usual homomorphisms and isomorphisms of operations. Given two sets \( M, N \), a set \( R \) of relations on \( M \) and a set \( S \) of a relations on \( N \) together with an injective mapping \( \Psi : R \to S \) a mapping \( \varphi : M \to N \) is called a homomorphism (isomorphism) from \( (M, R) \) to \( (N, S) \), iff for each relation \( \rho \in R \) the mapping \( \varphi \) is a homomorphism (isomorphism) from \( \rho \) into \( \Psi(\rho) \). An automorphism is an isomorphism from \( (M, R) \) into itself. The set of automorphisms of a certain structure \( S \) is denoted by \( \text{Aut} S \).

A congruence relation is the kernel of a homomorphism. The kernel of a mapping \( f \) is denoted by \( \ker f \).

For a subset \( A \) of a structure \( B \) let \( \langle A \rangle_B \) denote the substructure generated by \( A \).

Let \( M \) be a set and \( \equiv \) an equivalence relation, then \( M_\equiv := \{ [x]_\equiv \mid x \in M \} \) is the partition into equivalence classes of \( M \). A set \( T \subseteq M \) is called transversal of the partition of \( \equiv \), if it contains exactly one element in each equivalence class, i.e. for all \( x \in M \) the equation \( |[x]_\equiv \cap T| = 1 \) holds. The factor relation of a relation \( \rho \in M^n \) is defined by the set

\[ \rho_\equiv := \{ ([x_1]_\equiv, \ldots, [x_n]_\equiv) \mid (x_1, \ldots, x_n) \in \rho \} \]

For a tuple \( t \) we denote the projection to the \( n \)-th place with the symbol \( \pi_n t \).

### 3 Factorisation

In this section we will define a structure that can be considered as a factorisation of po-groups. This work is inspired by binary relation orbifolds as discussed in [2][3][4]. However, these ideas are based on factor relations while we use a description here, that is more focused on the internal structure of the factor structure. For simplicity, we focus on semi-regular group actions, that generate orbits which are pairwise isomorphic with respect to the order relation. A well-known example for such group actions are po-groups.

In a right po-group \( (G, \leq) \) every group element \( x \in G \) acts as an automorphism on the order relation by simultaneous right multiplication to all group elements. Cancelability of the group tells us that \( G \) acts regular on \( (G, \leq) \). Thus every subgroup \( U \) of \( G \) acts semi-regular on \( G \). For po-groups the same is also true for the left multiplication. The elements of the group are either strictly negative, strictly positive, the neutral element or incomparable to the latter one, which means that for all elements \( a, b \in G \) and \( x \in U \) the inequality \( a \leq x a \) holds iff for all elements \( b \in G \) the same inequality \( b \leq x b \) holds. Furthermore, when we fix \( a \) and \( b \), the mapping \( \varphi : x \mapsto xa^{-1}b \) defines an isomorphism between the cosets \( U a \) and \( U b \), both with respect to the group action of \( U \) and with respect to the order relation. We will call this property translative, if the isomorphism commutes with the group operation in \( U \). We define it in the more general setting of small categories which allows us to express nested factorisations.

The factor relation of an order relation is not necessarily an order any more. Thus, we will focus on the orbits of the tuples of the relation. This may lead to a structure that has more than one arrow that may connect two objects. As a relation can always be considered as a simple directed graph we could generalise to non-simple graphs, in our case small categories, at least for intermediate results. For this view we need a notion that resembles the properties of right po-groups.
We call a category foldable by a permutation group iff its orbits can be considered again as a category in an obvious way.

**Definition 1** Let $\mathcal{K}$ be a category and $G$ be a group that acts on $\mathcal{K}$. Then $\mathcal{K}$ is called foldable by $G$ iff for any four arrows $a, b, c, d \in Mor_{\mathcal{K}}$ with $a^G = e^G$ and $b^G = d^G$ the following equation holds whenever $a \ast b$ and $c \ast d$ exist:

\[(a \ast b)^G = (c \ast d)^G.\] (1)

Actually, not every category is foldable with respect to every automorphism group as it can be seen in the following example.

**Example 1** Suppose the following category $\mathcal{K}$:

![Diagram](image)

Then, 

\[\varphi := (12)(ab)((a \ast c)(b \ast e))((a \ast d)(b \ast d))\] and 

\[\psi := (45)(cd)((a \ast c)(a \ast d))((b \ast c)(b \ast d))\]

are two automorphisms with $\varphi^2 = 1, \psi^2 = 1$ and $\varphi \psi = \psi \varphi$. The cyclic automorphism group $G = \langle \varphi, \psi \rangle$ has four orbits of arrows. These are $\{a, b\}, \{c, d\}, \{a \ast c, b \ast d\}$ and $\{a \ast d, b \ast e\}$.

Obviously, we get $c^G = d^G$, but $(a \ast c)^G \neq (a \ast d)^G$. Consequently $\mathcal{K}$ is not foldable with respect to $G$.

A different result gives the group $H = \langle \varphi, \psi \rangle$. It has three orbits of arrows: $\{a, b\}, \{c, d\}, \{a \ast c, b \ast d, a \ast d, b \ast e\}$. The category is obviously foldable by $H$.

So we can consider the set of orbits as a category:

**Definition 2** Let $\mathcal{K}$ be a category with the concatenation $\circ$ that is foldable by an automorphism group $G \leq Aut\mathcal{K}$. Then the category $\mathcal{K}/\!\!/G$ with

\[\mathcal{O}b(\mathcal{K}/\!\!/G) := (\mathcal{O}b\mathcal{K})/\!\!/G\] (2a)

\[Mor_{\mathcal{K}/\!\!/G}(x^G, y^G) := Mor_{\mathcal{K}}(x, y)^G\] (2b)

\[id_x^G := (id_x)^G\] (2c)

and the concatenation

\[a^G \ast b^G := (a \circ b)^G\] whenever $a \ast b$ exists, (2d)

is called orbit category of $\mathcal{K}$ by $G$.

**Lemma 1** For every category $\mathcal{K}$ and every semi-regular automorphism group $G \leq Aut\mathcal{K}$ there exists an orbit category $\mathcal{K}/\!\!/G$. 
Proof Let \( w^G \xrightarrow{a^G} x^G \xrightarrow{b^G} y^G \xrightarrow{c^G} z^G \) be a diagram of \( \mathcal{R} \parallel G \). Then, because of the semi-regularity of the group action there are unique vertices \( \hat{x}, \hat{y}, \hat{z} \) and arrows \( \hat{b} \in b^G \) and \( \hat{c} \in c^G \) such that \( w \xrightarrow{\hat{a}} \hat{x} \xrightarrow{\hat{b}} \hat{y} \xrightarrow{\hat{c}} \hat{z} \) is a diagram of \( \mathcal{R} \). So \( a^G \ast b^G \ast c^G = (a \circ b \circ c)^G \). In particular it shows
\[
(a^G \ast b^G) \ast c^G = ((a \circ b) \circ c)^G = (a \circ (b \circ c)^G) = a^G \ast (b^G \ast c^G).
\]
So the orbit category is indeed a category.

Subgroups of po-groups act semi-regular via left multiplication on their supergroups. So we can work with the orbit category.

The vertex monoids of an orbit category may not be isomorphic to each other. If they were isomorphic we could separate them from the category, which would allow an additional compression of the mathematical structure. As we will see later, all vertex monoids of po-groups are pairwise isomorphic. This information can be seen as a property of an automorphism group.

**Definition 3** Let \( \mathcal{R} \) a category and \( G \leq \text{Aut} \mathcal{R} \) an automorphism group. Then \( G \) acts transitively on \( \mathcal{R} \), iff \( G \) acts semi-regular on \( \mathcal{R} \) and for every two elements \( a, b \in \text{Ob} \mathcal{R} \) there exists a category isomorphism between the orbits \( \langle a^G \rangle \mathcal{R} \) and \( \langle b^G \rangle \mathcal{R} \) that commutes with the group action of \( G \) so that for all automorphisms \( g \in G \) the following diagram commutes:
\[
\begin{array}{ccc}
\langle a^G \rangle \mathcal{R} & \xrightarrow{g} & \langle a^G \rangle \mathcal{R} \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
\langle b^G \rangle \mathcal{R} & \xrightarrow{g} & \langle b^G \rangle \mathcal{R}
\end{array}
\]

In the following we will use this definition in order to separate the group action of a po-group on itself from the order relation:

**Lemma 2** Every subgroup \( S \trianglelefteq G \) of a po-group \( G \) acts transitively on \( G \) via right multiplication.

Proof Let \( U \) act left-associative on \( G \). Furthermore chose a transversal \( T \) of the orbits of \( U \) on \( G \). Then, each orbit can be represented in the form \( tU \) where \( t \in T \). For another element \( s \in T \) multiplication from the left by \( s \) forms an order isomorphism from \( tU \) to \( sU \). As \( U \) acts from right on itself, and the elements from \( T \) from the left, both commute with each other.

4 Representation and Unfolding

In the preceding section far we have made a category from a larger one. In a similar way to group extensions \([9, 10]\) we may define a representation of the larger category using the orbit category and some additional mathematical magic similar to \([7, 11, 1]\).

Throughout this paper we will use the name *annotation* for a contravariant homomorphism from a category into a group. Consequently whenever there is a (covariant) homomorphism \( F \) from one category \( \mathcal{R} \) into another one \( \mathcal{L} \) and an annotation \( A : \mathcal{L} \to G \), then also the concatenation \( F \circ A \) is an annotation.
Definition 4 Let \( \mathcal{R} \) be a category, \( G \) a be group and \( A : \mathcal{R} \rightarrow G \) be an annotation. The triplet \( (\mathcal{R}, A, G) \) is called a representation (of \( \mathcal{R} \)).

If \( A \) is faithful, \( (\mathcal{R}, A, G) \) is said to be faithful.

In our case the representation represents a category. First we introduce the concatenation, and the we define the unfolding as the object the representation refers to.

Lemma 3 Let \( (\mathcal{R}, A, G) \) be a representation and \( R = \text{Mor}_{\mathcal{R}} \times G \). Then the partial operation defined by

\[
\circ : R \times R \rightarrow R : \left( (a, g), (b, A(a)g) \right) \mapsto (a \ast b, g), \text{ whenever } a \ast b \text{ exists} \tag{3}
\]

is associative.

Proof Let \( a, b, c \in \text{Mor}_{\mathcal{R}} \) such that \( a \ast b \) and \( b \ast c \) exist. Then

\[
(\circ (a, g) \circ (b, A(a)g)) \circ (c, A(b)A(a)g) = (a \ast b, g) \circ (c, A(a \ast b)g) = (a \ast b \ast c, g) = (a, g) \circ (b \ast c, A(a)g) = (a, g) \circ ((b, A(a)g) \circ (c, A(b)A(a)g))
\]

\( \square \)

Definition 5 Let \( (\mathcal{R}, A, G) \) be a representation.

The category \( \mathcal{U} \) with vertices \( \text{Ob}\mathcal{U} := \text{Ob}\mathcal{R} \times G \) and arrows \( \text{Mor}\mathcal{U} := \text{Mor}\mathcal{R} \times G \) and start and end of the arrows according to

\[
\sigma (a, g) := (\sigma a, g), \tag{4a}
\]

\[
\tau (a, g) := (\tau a, A(a)g) \tag{4b}
\]

and the concatenation for \( \tau a = \sigma b \)

\[
(\sigma a, g) \circ (\tau b, A(b)g) := (a \ast b, g) \tag{4c}
\]

is called the unfolding \( \mathcal{R}\uparrow A \) of \( (\mathcal{R}, A, G) \). In this case, we call \( (\mathcal{R}, A, G) \) a representation of \( \mathcal{U} \).

The group of a representation has an induced automorphism action on its unfolding.

Lemma 4 Let \( (\mathcal{R}, A, G) \) be a representation. Then, the pair of mappings \( (\Phi, \Psi) \) with

\[
\Phi : G \times (\text{Ob}(\mathcal{R}\uparrow A G)) \rightarrow \text{Ob}(\mathcal{R}\uparrow A G) : (g, (x, h)) \mapsto (x, hg). \tag{5a}
\]

\[
\Psi : G \times (\text{Mor}(\mathcal{R}\uparrow A G)) \rightarrow \text{Mor}(\mathcal{R}\uparrow A G) : (g, (x, h)) \mapsto (x, hg). \tag{5b}
\]

defines a right associative automorphism action on \( \mathcal{R}\uparrow A G \).
Proof  Let

\[(a, h) \in \text{Mor}_{\mathcal{A}}(b, b') \in \text{Mor}_{\mathcal{A}}(b, b') \]

be two arrows from \(\mathcal{A} \rightarrow G\). Then:

\[A(a, h) = b'h^{-1}, \quad A(b, b') = b'h^{-1} \]

\[\sigma(a, h) = (x, h), \quad \sigma(b, b') = (y, h') \]

\[\tau(a, h) = (y, h'), \quad \tau(b, b') = (z, h'') \]

Applying \(\Phi\) and \(\Psi\) together with a group element \(g \in G\), we get

\[\Phi(g, \sigma(a, h)) = (x, hg) \quad \Phi(g, \sigma(b, b')) = (y, h'g) \]

\[\Phi(g, \tau(a, h)) = (y, h'g) \quad \Phi(g, \tau(b, b')) = (z, h''g) \]

For start and end of an arrow \((a, h)\) we get the equation \(h'g \cdot (hg)^{-1} = h'g^{-1}h^{-1} = h'h^{-1} = A(a)\). Thus, \((a, hg)\) is an arrow in the set \(\text{Mor}_{\mathcal{A}}(b, b')\), and the equations \(\sigma(\Psi(a, h)) = \Phi(\sigma(a, h))\) and \(\tau(\Psi(a, h)) = \Phi(\tau(a, h))\) hold.

Applying another group element \(g' \in G\) to these formulas substitutes \(g\) with \(gg'\) since \(g\) is used only as factor in the right multiplication. For \(h = g^{-1}\) we get the identity automorphism.

Consequently, the two mappings are right associative group actions.

In combination with Definition [2] this leads to the equations

\[\Phi(g, \sigma(a)) = \sigma(\Psi(g, a)), \quad \Phi(g, \tau(a)) = \tau(\Psi(g, a)) \]

\[\Psi(g, a \ast b) = \Psi(g, a) + \Psi(g, b)\]

Thus, \(\Phi\) and \(\Psi\) together form a right associative automorphism. \(\square\)

In the following we call the automorphism action of the previous lemma induced automorphism action of \(G\) on \(\mathcal{A} \rightarrow G\).

The next two lemmas show, that orbit category and unfolding correspond to each other.

**Lemma 5** Let \((\mathcal{A}, \mathfrak{A}, G)\) denote a representation. Then

\[\pi : \text{Ob} \mathfrak{A} \rightarrow \text{Ob} \mathcal{A} : (x, g) \mapsto x\]

\[\text{Mor}_{\mathfrak{A}} \rightarrow \text{Mor}_{\mathcal{A}} : (a, g) \mapsto a\]

is a full faithful category homomorphism.

**Proof** Let \((a, g), (b, A(a)g) \in \text{Mor}_{\mathcal{A}}\). Then the following equations hold:

\[\sigma(\pi(a, g)) = \sigma a = \pi(\sigma(a), g) = \pi(\sigma(a, g))\]

\[\tau(\pi(a, g)) = \tau a = \pi(\tau(a), A(a)g) = \pi(\tau(a, g))\]

for the concatenation we get

\[\pi(a, g) \ast \pi(b, A(a)g) = a \ast b = \pi(a \ast b, g)\]
Thus $\pi$ is a homomorphism. Two arrows are mapped to the same image iff they coincide in their first component, which means that they are either identical or have different second components. In the latter case they start at different vertices.

For every arrow $a$ of the category $\mathcal{A}$ the set $\{a\} \times \mathbb{G}$ is the subset of the arrows of the unfolding, which maps to the set $\{a\}$. So the homomorphism is full, too. $\square$

**Lemma 6** Under the conditions of Lemma 5 we get the following equation

$$\left(\mathcal{A} \times \mathbb{G}\right) /_{\ker \pi} \sim \left(\mathcal{A} \times \mathbb{G}\right) / / \mathbb{G}$$  \(6\)

**Proof** Let $(a, g), (b, h) \in \text{Mor}_{\mathcal{A} \times \mathbb{G}}$ two arrows. If the concatenation $(a, g) \ast (b, h)$ exists, the following equations hold:

$$[([a, g])_{\ker \pi} \ast ([b, h])_{\ker \pi}] = [(a, g) \ast (b, h)]_{\ker \pi} = ([a \ast b, g])_{\ker \pi} = \{(c, \xi) \mid c = a \ast b, \xi \in \mathbb{G}\} = \{a \ast b\} \times \mathbb{G}$$

in particular,

$$[([a, g])_{\ker \pi} \ast ([\text{id}_\mathcal{A}, h])_{\ker \pi}] = (a, g)^G$$  \(7\) and $$[([\text{id}_\mathcal{A}, g])_{\ker \pi} \ast ([\text{id}_\mathcal{A}, h])_{\ker \pi}] = (\text{id}_\mathcal{A}, g)^G,$$

so (6) holds. $\square$

**Corollary 1** The orbit category $(\mathcal{A} \times \mathbb{G}) / / \mathbb{G}$ of an unfolding $\mathcal{A} \times \mathbb{G}$ of a category $\mathcal{A}$ with respect to the same group $\mathbb{G}$ and the group action from Lemma 4 is isomorphic to the original category $\mathcal{A}$.

**Proof** The projection $\pi$ can be divided into an epimorphism $\varphi : \mathcal{A} \times \mathbb{G} \to \ker \pi = (\mathcal{A} \times \mathbb{G}) / / \mathbb{G}$ and an isomorphism between $\ker \pi$ and the image of $\pi$. Since $\pi$ is full, the image of $\pi$ is the whole category $\mathcal{A}$. $\square$

Now, as we can re-fold an unfolded representation into the corresponding category we can hope that we can find a similar equivalence for unfolding folded categories.

### 5 Reconstruction

As we have seen, there is a relationship between orbit categories, the unfolding and their representations. So far we did not answer the following question: Which categories can be folded into a representation such that the unfolding is isomorphic to the original category? This shall be addressed in the current section.

One part of the unfolding is given by the orbit category. It is sufficient to find an annotation which unfolds into a category annotation that is natural to the original category.

One such candidate is hidden in the group action. If $(\mathbb{G}, \leq)$ is a po-group. Then the mapping

$$A : \mathbb{G} \times \mathbb{G} \to \mathbb{G} : (g, h) \mapsto hg^{-1}$$

is an annotation of $(\mathbb{G}, \leq)$. In $\mathbb{S} \leq \mathbb{G}$ is a subgroup of $\mathbb{G}$. The orbits of the right associative group action of $\mathbb{S}$ results in the coset partition $\{g \mathbb{S} \mid g \in \mathbb{G}\}$. By fixing one Element $g_0$ we can transfer the group structure of $\mathbb{S}$ to $g_0 \mathbb{S}$, where $g_0$ is the neutral element and $g_0 \mathbb{S} \mathbb{S}^{-1} = (g_0 \mathbb{S})g_0^{-1} = g_0 S$. In that way we get a similar construction for every orbit. We formalise this idea with the help of a transversal set.
For a category $\mathcal{K}$ and a translative automorphism group $G \leq \text{Aut}_\mathcal{K}$ a \textit{transversal (set)} $T \subseteq \text{Ob}\mathcal{K}$ is a set such that $T^G = \text{Ob}\mathcal{K}$ and $\forall g \in G : |g^G \cap T| = 1$. Obviously, a transversal always exists. As the group acts semi-regular, for every vertex $x \in \text{Ob}\mathcal{K}$ there exists a unique automorphism $g_T(x)$ such that $x^{g_T(x)^{-1}} \in T$ holds. This automorphism is called \textit{canonical}, in the same way as the pair $(x^{g_T(x)^{-1}}, g_T(x))$ is denoted by the name \textit{canonical vertex annotation}. Consequently, for every vertex, every translative automorphism group and every transversal there exists a unique vertex annotation. In the same way we call the pair $(a^G, g_T(\sigma a))$ \textit{canonical arrow annotation}. The canonical arrow annotation is unique for every arrow.

In the following the mapping $A_T : \text{Mor}_\mathcal{K} \to G : a \mapsto g_T(\tau a)(g_T(\sigma a))^{-1}$ will be called \textit{natural annotation}. Obviously the natural annotation is constant on every orbit of $\text{Mor}_\mathcal{K}$ under the action of $G$.

With these ingredients we construct a representation, now. First we rebuild the orbit category from the transversal and all arrows which start in the transversal. Let $\mathcal{C}$ be a category such that

\[ \text{Ob}\mathcal{C} := T \]
\[ \text{Mor}_\mathcal{C}(x, y) := \bigcup_{z \in y^G} \text{Mor}_\mathcal{R}(x, z) \]
\[ a *_C b := a * b^{A_T(a)}. \]

Then we get

\[ \sigma_C a = \sigma_R a \in T \]

and – by abuse of notation –

\[ \tau_C a = \tau_R a^G \cap T \]

As the natural annotation is constant on the orbits of the arrows of $\mathcal{R}$, we can define

\[ A(a) := A_T(a). \]

Finally, the tuplet $(\mathcal{C}, A, G)$ describes a representation. For the unfolding of this representation we can prove the following:

\textbf{Lemma 7} The unfolding $\mathcal{C}^\mathcal{R} \sim_G \mathcal{R}$ is isomorphic to $\mathcal{R}$.

\textit{Proof} Let $\mathcal{R}^\prime := \mathcal{C}^\mathcal{R} \mathcal{R}$ denote the unfolding of $(\mathcal{C}, A, G)$. Then:

\[ \text{Ob}\mathcal{R}^\prime = \{(x, g) \mid g \in G, x \in T\} \]
\[ \text{Mor}_\mathcal{R}^\prime((x, g), (y, h)) = \{(a, g) \mid a \in \text{Mor}_\mathcal{C}(x, y), A_T(a) = h^{-1}\} \]

with the concatenation

\[ (a, g) \circ (b, h) := (a * g, b, g) \]
We observe that for any arrow \((a,g)\) ∈ \(\text{Mor}_K\) the following equations hold:

\[
\sigma_K(a,g) = (\sigma a, g)
\]

\[
\tau_K(a,g) = (\sigma a, A_T(a)g)
\]

It is easy to see that the two mappings

\[
\Phi: \text{Ob}_K' \to \text{Ob}_K: (x,g) \mapsto x^g,
\]

\[
\Psi: \text{Mor}_K' \to \text{Mor}_K: (a,g) \mapsto a^g
\]

form an isomorphism between \(K\) and \(K'\) with the inverse mappings

\[
\Phi^{-1}: \text{Ob}_K / / G \to \text{Ob}_K': x \mapsto (x^{(\sigma T(x))^{-1}}, g_T(x))
\]

\[
\Psi^{-1}: \text{Mor}_K / / G \to \text{Mor}_{K'}: a \mapsto (a^{(g_T(\sigma a))^{-1}}, g_T(\sigma a))
\]

Each vertex can be uniquely described as a pair of a transversal element and an automorphism, since the automorphism group acts semi-regular on \(\mathcal{R}\). This means that for each fixed transversal element the group action is a bijection between the automorphism group and the orbit of that transversal element.

The same is true for the orbits of the arrows while we use the starting vertex of every arrow as index into the arrow orbits. So also \(\Psi\) is bijective. Let \((a,g)\) ∈ \(\text{Mor}_{K'}\) and \((b,h)\) ∈ \(\text{Mor}_{K'}\) two arrows. Then \((a,g) \circ (b,h)\) is defined iff \(a * b\) exists and \(h = A_T(a)g\) holds. Then, we get

\[
\Psi((a,g) \circ (b,h)) = \Psi(a \sigma e b, g) = \Psi(a \star b^{\sigma(a)}, g) = (a \star b^{\sigma(a)})^g
\]

\[
= a^g \star (b^{\sigma(a)g}) = a^g \star b^g = \Psi(a,g) \star \Psi(b,h)
\]

Thus, the pair \((\Phi, \Psi)\) is an isomorphism between the categories \(\mathcal{R}\) and \(\mathcal{R}'\).

So we have some way to reconstruct the category from a transversal representation. This means that the category \(\mathcal{C}\) above can be considered as a fundamental system of \(\mathcal{R}\). Now, we replace the transversal category by the orbit category.

**Lemma 8** The categories \(\mathcal{R} / / G\) and \(\mathcal{C}\) are isomorphic.

**Proof** Consider the mappings

\[
\Phi: \mathcal{D}b\mathcal{C} \to \mathcal{D}b\mathcal{R} / / G: x \mapsto x^G
\]

\[
\Psi: \text{Mor}_{\mathcal{C}} \to \text{Mor}_{\mathcal{R} / / G}: a \mapsto a^G.
\]

Both are obviously bijective with the inverse mappings (when we identify singleton sets with their elements)

\[
\Phi^{-1}: \mathcal{D}b\mathcal{R} / / G \to \mathcal{D}b\mathcal{C}: x \mapsto x \cap T
\]

\[
\Psi^{-1}: \text{Mor}_{\mathcal{R} / / G} \to \text{Mor}_{\mathcal{C}}: a \mapsto \hat{a} \in a : \sigma \hat{a} \in T.
\]

As all arrows in one orbit differ pairwise in their starting vertex, the inverse mappings are well defined.
Obviously the mappings act as a homomorphism on the starting points of the arrows. For the end points we get
\[ \Phi(\tau a) = (\tau a)^G = \tau(a^G) = \tau\Psi(a). \]
And for the concatenation \( a * c b \) of two arrows \( a, b \in \mathcal{C} \) we get:
\[ \Psi(a * c b) = (\sigma^G b)(\alpha^G a) = \Psi(a * c b) \]
since \( G \) acts semi-regular both on \( b^G \) and on \( (\sigma^G b)(\alpha^G a) \), we get
\[ \Psi(a * c b) = a^G \cdot g \]
As \( A_T \) is constant on the orbits, we get for every arrow \( a \in \mathbb{M} \) the equation \( A_T[a] = A(\Phi(a)) \).

Consequently we can prove the unfolding.

**Theorem 1**
\[ (\mathcal{R}/G)_{\eta} \cong \mathcal{R}. \]

**Proof** The unfolding \( C \) of \( \mathcal{R}/G \) is isomorphic to the category \( \mathcal{R} \). As \( \mathcal{R}/G \) is isomorphic to \( \mathcal{C} \) and this isomorphism preserves the annotation, we can replace the elements (vertices and arrows) of \( \mathcal{C} \) in \( C \) by their isomorphic image and get \( (\mathcal{R}/G)_{\eta} \cong \mathcal{R}. \)

## 6 Vertex categories

The idea behind a fundamental system is to reduce the information that is managed by a certain structure. If an orbit category has cycles that don’t occur in the original category, a possibly infinite number of arrows is preserved between any two vertices of the cycle. In some case the information about their structure can be stored more efficiently in the preimage of the vertex monoids. These are the generated subcategories of vertex orbits. We call them vertex categories.

According to the definition, for a translative automorphism group \( G \in \text{Aut}\mathcal{R} \) of a category \( \mathcal{R} \), any two orbits \( x^G, y^G \in \mathcal{R}^G \) are isomorphic to each other. As the group action is semi-regular, for every arbitrary, but fixed vertex \( x \in \mathbb{D} \) there is a bijection
\[ \phi_x : G \to x^G : g \mapsto x^g \]
between the group \( G \) and the orbit \( x^G \). So we can define a category \( \mathcal{G} \) in the following way:

**Definition 6** Let \( \mathcal{R} \) be a category and \( G \in \text{Aut}\mathcal{R} \) a translative automorphism group. Then, for every vertex \( x \in \mathbb{D} \) the right groupal category \( \mathcal{G}_x := \langle x^G \rangle \mathcal{R} \) with the additional group action \( \phi_x : \mathcal{S}_x \times \mathbb{D} \to \mathcal{S}_x \) defined via
\begin{enumerate}
  \item \( \forall g, h \in G : x^g \cdot \phi_x h = x^{g \cdot h} \)
  \item \( \forall a \in \mathbb{M}, h \in G : a \cdot \phi_x h = a^h \)
\end{enumerate}
is called *vertex category*. 
If the category $\mathcal{R}$ is an ordered set, the vertex category is a right po-group. If it is a po-group, which is factorised by one of its normal subgroups, the vertex categories are po-groups that are isomorphic to the normal subgroup with respect to the group operation and the order relation.

Every po-group can be considered as a category whose objects act on itself via a left- and a right-associative automorphism actions. Both actions commute due to the associativity of the group operation. For simple categories this actions also define a binary operator on the arrows. In general such a category with such a binary operator is called “groupal category”. Here, we consider only strict associativity.

**Definition 7** Let $\mathcal{R}$ be a category. Furthermore let $\cdot : \mathcal{R} \times \mathcal{Ob} \mathcal{R} \to \mathcal{R}$ a binary operation that acts as a group operation on the objects $\mathcal{Ob} \mathcal{R}$ and as a left-associative automorphism action on $\mathcal{R}$. Then $(\mathcal{R}, \cdot)$ is called a right-groupal category. Dually, the term left-groupal category denotes such a category if the group action is right-associative.

The category is called a groupal category, if it is both left groupal and right groupal, such that both group actions commute with each other and the following equation holds for all arrows $a, b \in \mathcal{R}$:

$$a^\sigma b \ast a^\tau b = a^\sigma a^b \ast a^\tau b$$  \hspace{1cm} (8)

**Corollary 2** Every vertex category is a right-groupal category.

In the following we will denote the neutral element with respect to the group operation of a right groupal category by the symbol “1”.

It is a well-known fact that every right po-group can be considered as a right-groupal category and every po-group can be considered as a groupal category.

In the same way as with right po-groups, right-groupal categories are completely defined by the set of arrows that start or end in the neutral element 1. For that to prove we use the two operators for a category $\mathcal{R}$:

$$\downarrow_{\mathcal{R}} x := \bigcup_{y \in \mathcal{Ob} \mathcal{R}} \mathcal{Mor}_{\mathcal{R}}(y, x) \text{ and } \uparrow_{\mathcal{R}} x := \bigcup_{y \in \mathcal{Ob} \mathcal{R}} \mathcal{Mor}_{\mathcal{R}}(x, y),$$  \hspace{1cm} (9)

which can be interpreted as the objects of the slice category and the coslice category of a given object. Note that both operators correspond to each other by the duality principle. Both $\downarrow_{\mathcal{R}} x$ and $\uparrow_{\mathcal{R}} x$ contain the vertex monoid $\mathcal{Mor}_{\mathcal{R}}(x, x)$ as a subset. Then, $\mathcal{Mor}_{\mathcal{R}}$ is completely defined by $\mathcal{Ob} \mathcal{R}$ and $\downarrow_{\mathcal{R}} 1$. The concatenation is defined by the concatenation on $\downarrow_{\mathcal{R}} 1 \cup \uparrow_{\mathcal{R}} 1$.

**Lemma 9** Let $(\mathcal{R}, \cdot)$ be a right-groupal category. Then,

$$\mathcal{Mor}_{\mathcal{R}}(x, y) = \mathcal{Mor}_{\mathcal{R}}(1, y \cdot x \cdot x^{-1}) \cdot_{\mathcal{R}} x = \mathcal{Mor}_{\mathcal{R}}(x \cdot y \cdot x^{-1}, 1) \cdot_{\mathcal{R}} y,$$  \hspace{1cm} (10)

and for all arrows $a, b \in \mathcal{Mor}_{\mathcal{R}}$ with $\tau a = \sigma b$ there exist arrows $\hat{a} \in \downarrow_{\mathcal{R}} 1$ and $\hat{b} \in \uparrow_{\mathcal{R}} 1$ such that $a \ast b = (\hat{a} \ast \hat{b}) \cdot \tau a$.

**Proof** Let $x, y \in \mathcal{Ob} \mathcal{R}$ then (10) follows directly from the definition. Now, we consider the diagram $x \xrightarrow{a} y \xrightarrow{b} z$. This is the same as $(x \cdot y^{-1} \cdot a \cdot y^{-1}, 1 \cdot b \cdot y^{-1} \cdot z \cdot y^{-1}) \cdot y$. Consequently $a \ast b = (a \cdot y^{-1} \ast b \cdot y^{-1}) \cdot y$. As $a \cdot y^{-1} \in \downarrow_{\mathcal{R}} 1$ and $b \cdot y^{-1} \in \uparrow_{\mathcal{R}} 1$ hold, the lemma is proved. \hfill $\Box$
As with groups, also the group action of a right groupal category can be used to define a
group operation that mainly shifts the neutral element.

**Lemma 10** Let $\mathcal{K}$ be a right groupal category and $a \in \text{Ob}\mathcal{K}$ a vertex. Then the binary
operator

\[ x \cdot_a y : \text{Ob}\mathcal{K} \times \text{Ob}\mathcal{K} \to \text{Ob}\mathcal{K} : (x, y) \mapsto xa^{-1}y \]  

(11)

is a group operation on $\text{Ob}\mathcal{K}$ such that $\varphi : \mathcal{K} \to \mathcal{K} : x \mapsto xa$ is a right groupal category
isomorphism which maps the group operations to each other.

**Proof** First we show that $\varphi$ is a group isomorphism: It is bijective as the group operation is
bijective in each argument. Let $a, b \in \text{Ob}\mathcal{K}$. Then

\[ \varphi(xy) = xya = xaa^{-1}ya = \varphi(x) \cdot_a \varphi(y). \]

Thus $\varphi$ is a group isomorphism.

Let further $\tau, \eta \in \text{Mor}\mathcal{K}$ be two arrows such that $\tau \ast \eta \in \mathcal{K}$ exists. Then both

\[ (\tau \ast \eta)^a = \tau^a \ast \eta^a \in \text{Mor}\mathcal{K}, \]
\[ (\tau \ast \eta)^{-1} = \tau^{-1} \ast \eta^{-1} \in \text{Mor}\mathcal{K} \quad \text{and} \quad ((\tau \ast \eta)^a)^{-1} = \tau \ast \eta \in \text{Mor}\mathcal{K} \]

hold. Thus, the bijection $\varphi$ is also with respect to the category structure an isomorphism. □

The group operations of two vertex categories of the same category with the same translative
automorphism group can be chosen in a way that is compatible with the translative group
action. So, we can consider the vertex categories of all vertices as isomorphic right groupal
categories.

**Lemma 11** Let $\mathcal{K}$ and $\mathcal{L}$ be right groupal categories and $\varphi : \mathcal{K} \to \mathcal{L}$ a category isomorphism
between them with the following properties:

\[ \varphi(xy) = \varphi(x)(\varphi(1))^{-1}\varphi(y). \]  

(12)

Then, the mapping $\psi : \mathcal{K} \to \mathcal{L} : x \mapsto \varphi(1)$ is an isomorphism of right groupal categories.

**Proof** The equation (12) can be rewritten in the form:

\[ \varphi(xy) = \varphi(x) \cdot_{\varphi(1)} \varphi(y) \]

That means that $\varphi$ is also a group isomorphism with respect to the group operation $\cdot_{\varphi(1)}$ on
$\text{Ob}\mathcal{L}$. Thus $\varphi$ is a right groupal category isomorphism between $\mathcal{K}$ with the standard group
operation and $\mathcal{L}$ with the group operation $\cdot_{\varphi(1)}$.

Let $\psi : \mathcal{L} \to \mathcal{L} : x \mapsto x \varphi(1)$ the right groupal isomorphism on $\mathcal{L}$ that makes $\varphi(1)$ the
neutral element. Then $\varphi \circ \psi^{-1}$ is a right groupal isomorphism between $\mathcal{K}$ and $\mathcal{L}$ both with
their standard group operation. □

For each orbit we can find a representative, so that the group operation created above coincides with the normal group operation of a vertex category.

**Lemma 12** Let $\mathcal{K}$ be a category and $G \leq \text{Aut}\mathcal{K}$ a translative automorphism group. Then
for any two vertices $a, b \in \text{Ob}\mathcal{K}$ a third vertex $c \in G^2$ exists such that there exists a right
groupal category isomorphism between the vertex categories $\mathcal{G}_a$ and $\mathcal{G}_c$ that commutes with the automorphisms from $G$. 
Proof Since $G$ is translative, there exists a category isomorphism $\phi : \mathcal{G}_b \to \mathcal{G}_a$. It induces a group structure on $\mathcal{G}_b$, whose neutral element will be denoted by $c$ and the group operation by $\cdot$. It remains to show that the induced group structure is the same as in $\mathcal{G}_c$. Let $x, y \in \mathcal{Ob}_{\mathcal{G}_c}$ two vertices in $\mathcal{G}_c$. Then, there exist two elements $g, h \in \mathcal{G}$ with $c^g = x$ and $c^h = y$, such that

\[
x \cdot \mathcal{G}_c y = c^g \cdot \mathcal{G}_c c^h = c^g c^h = c^{g \cdot h} = \phi^{-1}(\phi(c^{g \cdot h})) = \phi^{-1}(\phi(c^g \cdot c^h)) = \phi^{-1}(\phi(c^g) \cdot \phi(c^h)) = \phi^{-1}(\phi(c^g) \cdot \phi(c^h)) = x \cdot y.
\]

\[\phi(a^g a^h) = \phi(a)^{g \cdot h}.
\]

Let \(b^f = c := \phi(a)\):

\[
\phi(a^g a^h) = b^f h = c^e c^{-1} h = \phi(a^e) \phi(a)^{-1} \phi(a^h).
\]

As $a$ is the neutral element in $\mathcal{G}_a$, the previous lemmas \[\text{[10]}\] and \[\text{[11]}\] ensure the existence of a right groupal category isomorphism between the vertex categories $\mathcal{G}_a$ and $\mathcal{G}_c$. Now, we look at the group action of $h \in \mathcal{G}$. Let $x = a^g \in a^\mathcal{G}$, then

\[
\phi((a^g)^h) = \phi(a^{gh}) = \phi(a)^{gh} = (c^e)^h = c^e c^{-1} h.
\]

As we can see in the last proof the role of the orbit vertices with respect to the automorphism action of the group depends on the choice of the neutral element. However, the action of the automorphism group does not depend on that choice.

7 Flat orbit categories

In some cases the orbit categories contain lots of redundant information. For example for po-groups we are often more interested in the local structure than in far relationships. As the latter are mainly a result of transitivity, we may omit them and reconstruct them from transitivity.

As all vertex monoids of the orbit category are isomorphic, every orbit category is the homomorphic image of the product of a vertex monoid with some other category. Since a translative group acts semi-regular on vertices, we can encode the information of the vertex monoids in the automorphism group.

In analogy to normal sub(semi)groups we can also consider vertex monoids as normal, if we find a way to exchange the operands of the concatenation.
Definition 8 Let $\mathcal{R}$ denote a category and $G \leq \text{Aut}\mathcal{R}$ be a translative automorphism group. Then, $G$ is called right-normal on $\mathcal{R}$ (in symbols: $G \leq_{r\text{-normal}} \mathcal{R}$), iff for every two arrows $a \in \text{Mor}\mathcal{R}$ and $x \in \text{Mor}\mathcal{R}_\sigma a$ there exists an arrow $C(a^G, x^G) \in \text{Mor}\mathcal{R}_\sigma a / G$ such that in the orbit category $\mathcal{R}/G$ the following equations hold:

$$\begin{align*}
\tau^G \ast a^G & = a^G \ast C(a^G, x^G), \quad (13a) \\
\text{id}_{\tau a} & = C(a^G, \text{id}_{\tau a}). \quad (13b)
\end{align*}$$

We can express right normal group actions in terms of the underlying category.

Lemma 13 Let $\mathcal{R}$ denote a category and $G \leq \text{Aut}\mathcal{R}$ be a translative automorphism group. Then, $G$ is right-normal, iff for every two arrows $a \in \text{Mor}\mathcal{R}$ and $x \in \text{Mor}\mathcal{R}_\sigma a$ with $\tau a = \sigma a$ there exists an automorphism $g \in G$ and an arrow $C'(a, x) \in \text{Mor}\mathcal{R}_\sigma a$ such that

$$\begin{align*}
\tau \ast a & = a^G \ast C'(a, x), \quad (14a) \\
\text{id}_{\tau a} & = C'(a, \text{id}_{\tau a}). \quad (14b)
\end{align*}$$

Proof Let first $\tau a = \sigma a$ hold. As a semi-regular group action is regular on its orbits there exist unique homomorphisms $h, g \in G$ such that the concatenation $\tau a = \sigma a^G$ holds. Let $\eta \in C(a^G, x^G)$. Then, an automorphism $f \in G$ exists such that the concatenation $\sigma a^G \ast \eta f$ exists. As $G$ acts translative, the orbit $a^G \ast \eta f$ has exactly one arrow starting in $\sigma a$, which is by construction the arrow $a^G \ast \eta f$. So we can define $C'(a, x) := \eta f$, which fulfills the condition $\tau a = \sigma a$.

Conversely suppose Equation $\tau a = \sigma a$ holds under the mentioned conditions. As $G$ is translative there exists a choice function $c : \text{Mor}\mathcal{R}/G \rightarrow \text{Mor}\mathcal{R}_\sigma a / G$ such that for $(\hat{a}, \hat{x}) = c(a^G, x^G)$ with $\hat{a}^G \in \text{Mor}\mathcal{R}_\sigma a / G$ the equation $\tau \hat{a} = \sigma \hat{a}$ holds.

For two arrows $a^G, \hat{a}^G \in \text{Mor}\mathcal{R}/G$ and $x^G \in \text{Mor}\mathcal{R}_\sigma a / G(\sigma a^G, \sigma \hat{a}^G)$ we chose a pair of arrows $(\hat{a}, \hat{x}) = c(a^G, x^G)$ Then,

$$\tau^G \ast a^G = (\hat{\tau} \ast \hat{a})^G = (\hat{\sigma} \ast \hat{x} \ast C'(a, x))^G = \hat{a}^G \ast C'((\hat{a}, \hat{x}))^G = a^G \ast C(c(a^G, x^G))^G.$$  

So we can define $C(a^G, x^G) := C'(c(a^G, x))^G$, which fulfills Equation $\tau a = \sigma a$.

The equations $(14b)$ and $(13b)$ can be proved in an analogous way. $\square$

Corollary 3 If a partial mapping $C''$ fulfills Equation $\tau a = \sigma a$, then the partial mapping

$$C'(a, x) := \begin{cases} 
\text{id}_{\tau a}, & \tau = \text{id}_{\sigma a} \\
C''(a, x), & \text{else}
\end{cases}$$

fulfills Equations $(14a)$ and $(14b)$.

Lemma 14 Every subgroup of a groupal category is right-normal on its category structure.

Proof Let $\mathcal{R}$ a groupal category, $G \leq \text{Ob}\mathcal{R}$ and $a \in \text{Mor}\mathcal{R}$ and $x \in \text{Mor}\mathcal{R}_\sigma a$. Then,

$$\begin{align*}
\sigma \tau a & = a^G \ast \tau a \\
\tau \ast a^{-1} \tau & = \sigma a^{-1} \sigma \tau a \\
\tau \ast b & = b \ast \tau b^{-1}
\end{align*}$$

Substituting $b := \sigma a^{-1} \ast \tau$ leads to $\sigma a^{-1} = b \ast \tau^{-1}$.

$$\tau \ast b = b \ast \tau b^{-1} \ast \tau = b \ast \tau b^{-1} \ast \tau = \sigma \tau a^{-1} \ast \tau \ast b^{-1} \ast \tau$$
Corollary 4. Every subgroup of a po-group acts right-normal on its category structure.

In many cases normal group actions define isomorphisms between their vertex categories. As an example we prove it for simple categories.

Lemma 15. Let \( \mathcal{K} \) be a category and \( G \leq \text{Aut} \mathcal{K} \) a translative automorphism group, and \( \mathcal{C} \) defined as in Lemma 13. Then for all arrows \( a \in \text{Mor} \mathcal{K} \) and \( \tau, \eta \in \mathcal{C}_a \), there exist automorphisms \( g, h \in G \) such that the following equations hold:

\[
\begin{align*}
  a \ast \mathcal{C}(a, \tau \ast \eta) &= a \ast \mathcal{C}(a, \tau) \ast \mathcal{C}(a^h, \eta) & (16a) \\
  a &= a \ast \mathcal{C}(a, \text{id}_{\mathcal{C}a}) & (16b) \\
  \tau &= \mathcal{C}(\text{id}_{\mathcal{C}a}, \tau) & (16c)
\end{align*}
\]

Proof

\[
\begin{align*}
  a \ast \mathcal{C}(a, \tau \ast \eta) &= \tau \ast \eta \ast a^h = \tau \ast a^h \ast \mathcal{C}(a^h, \eta) = a \ast \mathcal{C}(a, \tau) \ast \mathcal{C}(a^h, \eta) \\
  a &= \text{id}_{\mathcal{C}a} \ast a = a \ast \mathcal{C}(a, \text{id}_{\mathcal{C}a}) \\
  \tau &= \tau \ast \text{id}_{\mathcal{C}a} = \text{id}_{\mathcal{C}a} \ast \mathcal{C}(\text{id}_{\mathcal{C}a}, \tau) = \mathcal{C}(\text{id}_{\mathcal{C}a}, \tau)
\end{align*}
\]

Lemma 16. Let \( \mathcal{K} \) be a category, \( G \leq \text{Aut} \mathcal{K} \) a translative automorphism group, \( a \in \text{Mor} \mathcal{K} \) and \( \mathcal{C} \) defined as in Lemma 13. For every arrow \( \tau \in \text{Mor} \mathcal{C}_a \) let \( h_\tau(\tau) := \tau \ast \sigma \ast \tau^{-1} \) denote the automorphism that shifts the arrow \( a \) such that \( a^{h_\tau(\tau)} \ast \tau \) exists. If the mapping

\[
\phi_a : \text{Mor} \mathcal{C}_a \to \text{Mor} \mathcal{C}_a : \tau \mapsto \mathcal{C}(a^{h_\tau(\tau)}, \tau).
\]

is injective, Eq. (14a) defines a category homomorphism

\[
\phi_a : \text{Mor} \mathcal{C}_a \to \text{Mor} \mathcal{C}_a : \tau \mapsto \mathcal{C}(a^{h_\tau(\tau)}, \tau).
\]

Proof Let \( \tau_1, \tau_2 \in \text{Mor} \mathcal{C}_a \) be two arrows such that \( \tau_1 \ast \tau_2 \) exists. Consequently an automorphism \( g \in G \) exists such that with \( 16a \) the following equation holds:

\[
a^{g} \ast \mathcal{C}(a^{h_\tau_1(\tau_1)}, \tau_1 \ast \tau_2) = (\tau_1 \ast \tau_2) \ast a = a \ast \mathcal{C}(a^{h_\tau_1(\tau_1)}, \tau_1) \ast \mathcal{C}(a^{h_\tau_2(\tau_2)}, \tau_2).
\]

As \( s_a \) is injective, this implies \( \phi_a(\tau_1 \ast \tau_2) = \phi_a(\tau_1) \ast \phi_a(\tau_2) \).

Corollary 5. If \( a \) is an epimorphism, \( \phi_a \) is a homomorphism.

In general, the inverse implication is not true, as we are considering only a subcategory, here.

Lemma 17. In Lemma 16 \( \phi_a \) commutes with every automorphism in \( G \).
Proof Let \( g \in G \) be an automorphism. We use that the group action is semi-regular, so that distinct arrows of an orbit have distinct starting points. Following the notation of Lemma \( \text{16} \), two automorphisms \( \hat{g}, \hat{g} \in G \) exist such that the application of equation \( \text{(14a)} \) leads to
\[
\alpha^{h_{a}(x)^{g}} * \phi'(a, x)^{g} = \alpha^{h_{a}(x)^{g}} * \phi'(a, x)^{g} = \alpha^{h_{a}(x)^{g}} * \phi'(a, x)^{g},
\]
since \( \sigma \alpha^{h_{a}(x)^{g}} = \sigma \alpha^{h_{a}(x)^{g}} \). This implies \( \phi_{a}(x)^{g} = \phi_{a}(x)^{g} \). \( \square \)

For groupal categories we may use the left multiplication for the construction of \( \phi \) in Eq. \( \text{(14a)} \) and \( \text{(14b)} \), which is shown by the following two lemmas.

**Lemma 18** Let \( \mathcal{R} \) denote a groupal category and \( G \leq \mathcal{R} \mathcal{b} \mathcal{R} \) be a translative right-associative automorphism group, then for any two vertices \( x, y \in \mathcal{R} \mathcal{b} \mathcal{R} \) the mapping
\[
\phi_{x,y}(x) : \mathcal{G}_{x} \to \mathcal{G}_{y} : x \mapsto x^{y^{x^{-1}}}
\]
is an isomorphism that commutes with the right group action of \( G \).

Proof As \( \mathcal{R} \) is groupal, the mapping \( \phi_{x,y} \) can be extended to an automorphism on \( \mathcal{R} \) so it is injective and invertible if it is well-defined. It remains to show that it is well-defined and surjective.

The mapping is well-defined. Let \( z \in x^{G} \) then \( x^{y^{x^{-1}}} z \in y^{x^{-1}} x^{G} = y^{G} \), which follows from the group properties of \( \mathcal{R} \mathcal{b} \mathcal{R} \).

In the same way the mapping \( \phi_{x,y} \) is well-defined, injective and invertible. Furthermore for \( \eta \in \mathcal{M}_{\sigma \phi} \),
\[
\phi_{x,y}(\phi_{x,y}(\eta)) = \phi_{x,y}(\eta y^{-1}) = x^{y^{-1} y^{-1}} \eta = \eta.
\]
So \( \phi_{x,y} \) is surjective and \( \phi_{x,y} \) is its inverse.

As \( \phi_{x,y} \) is defined by applying a left group action of the objects, it commutes by definition with the right group action. \( \square \)

**Lemma 19** In Lemma \( \text{18} \) for any arrow \( x \in \mathcal{G}_{x} \), any automorphism \( g \in G \) and any arrow \( a \in \mathcal{M}_{\sigma \phi}(x, y) \) the following equation holds:
\[
x^{g} * a^{h_{a}(x)^{g}} = (a * \phi_{x,y}(x))^{g}
\]

Proof As we consider the categories \( \mathcal{G}_{x} \) and \( \mathcal{G}_{y} \) the vertices \( x \) and \( y \) play the role of the neutral element with respect to the group operation. Furthermore \( \sigma x = x = l_{\mathcal{G}_{x}} \). So we get
\[
(x * a^{\sigma a^{-1}})^{g} = a^{\sigma a^{-1}} * (a * a^{-1})^{g} = a^{\sigma a^{-1}} * \phi_{\sigma a, a}(x)^{g} = a^{\sigma a^{-1}} * \phi_{x,y}(x)^{g}.
\]

\( \square \)

**Lemma 20** Let \( \mathcal{R} \) be a category, \( S \leq G \leq \text{Aut} \mathcal{R} \) two translative automorphism groups with \( G \triangleleft \mathcal{R} \). Then, \( S \triangleleft \mathcal{R} \).
Proof Since $G \preceq K$, by Lemma 13 for every arrow $a \in \mathcal{M}_{\sigma G}$ and $x \in S_{\sigma a}$ there exist automorphisms $g \in S, h \in G$ and an arrow $\mathcal{C}'(a^h, x) \in G_{\sigma a}$ such that

$$\tau \ast a^h = a^h \ast \mathcal{C}'(a^h, x). \quad (*)$$

Since $\sigma \tau = \sigma a^h$ and $\tau \sigma = \sigma a$ holds, we derive the equation $h = g(\tau \sigma)^{-1} \sigma \tau \in S$. For start and end of $\mathcal{C}'(a^h, x)$ we get the equations $\sigma \mathcal{C}'(a^h, x) = \tau a^h = \tau a^h(\tau \sigma)^{-1} \sigma \tau$ and $\mathcal{C}'(a^h, x) = \tau a^h$. So $\sigma \mathcal{C}'(a^h, x) \in \mathcal{C}'(a^h, x)$, which means $\mathcal{C}'(a^h, x) \in S_{\tau \sigma}$. So we can apply Lemma 13 with respect to the group $S$ to show that there exists a partial mapping $\mathcal{C}$ such that

$$\tau^S \ast a^S = a^S \ast \mathcal{C}(a^S, x).$$

With the same lemma, we may shift Equation (13b) to (14b) and back to prove

$$\text{id}_{\tau a}^S = \mathcal{C}(a, \text{id}_{\sigma a}).$$

So we have shown $S \preceq K$. □

As we have seen, we can replace loops on the left hand side of a concatenation in the orbit category by loops concatenated from the right. Now, we introduce a possibility to use this knowledge to transfer the vertex monoids to the unfolding group. We must identify those arrows that have to be preserved in our simplified orbit category.

Definition 9 Let $\mathcal{K}$ be a category and $x, y \in \mathcal{O}b\mathcal{K}$ vertices of $\mathcal{K}$. We say an arrow $a \in \mathcal{M}_{\sigma G}(x, y)$ is reducible iff arrows $b \in \mathcal{M}_{\sigma G}(x, y)$ and $\eta \in \mathcal{M}_{\sigma G}(y, y) \setminus \{\text{id}_y\}$ exist such that

$$a = b \ast \eta. \quad (21)$$

Otherwise $a$ is called irreducible.

If every non-loop arrow of $\mathcal{K}$ is the concatenation of an irreducible arrow and a loop, the category is called representable by irreducible arrows. It is called uniquely representable by irreducible arrows, iff for each arrow $a \in \mathcal{M}_{\sigma G}$ there exists exactly one representation in the form (21).

Corollary 6 All non-identity loops are reducible.

Corollary 7 Identity loops are irreducible.

Lemma 21 Let $(G, \leq)$ be an $\ell$-group with a lattice ordered normal subgroup $N \preceq G$. Then for each arrow orbit $a$ in the preceding definition there exists at most one irreducible arrow $b$ such that (21) holds. In that case arrow $\eta$ in that equation is unique. iff $b$ exists.

Proof Suppose we have two descriptions $a^G = b^G \ast \eta^G = c^G \ast e^G$, with arrows $\eta^G, e^G \in \mathcal{M}_{\sigma G}(\tau a^G, \tau a^G)$. W.l.o.g. the arrows can be chosen such that $\sigma a = \sigma b = \sigma c$. Then, the relations $\sigma a \leq \tau b$ and $\sigma a \leq \tau c$ hold. Consequently an arrow $\delta$ exists with $\sigma a = \sigma \delta$ and $\tau \delta := \tau b \land \tau c \geq \sigma a$.

On the other hand there exists an element $n \in N$ such that $\tau b^n = \tau c$ so that we get

$$\tau \delta = \tau b \land \tau c = \tau b \land \tau b^n = \tau b(1 \land n) \geq \sigma a.$$

As $(N, \leq |_N)$ is a sublattice of $(G, \leq)$, $1 \land n \in N$. With $1 \land n \leq l$, it follows that $1 \lor n^{-1} = (1 \land n)^{-1} \geq 1$ and $b = \delta \ast \{\tau b(1 \land n), \tau b\}$, where $(\tau b(1 \land n), \tau b) \in \mathcal{E}_{\tau b}$. So $\delta = b$ or $b$ is not irreducible. The same holds for $c$. So, if $b$ and $c$ are irreducible, $b = c$ holds. As there is only one arrow between two vertices, the arrow $(\tau b(1 \land n), \tau a)$ is uniquely defined. □
This lemma covers an important class of po-groups. However, it depends on the uniqueness of certain greatest lower bounds. So we can easily rephrase this lemma in terms of category theory and finite products.

The statement of Lemma 21 is not true in general for normal subcategories. Take for example the po-group \( \mathbb{Z} \times \mathbb{Z} \) with the product order and the normal subgroup \( N \) that is generated by the set \( \{(1, 1), (1, -1)\} \). Then the arrow \( ((0, 0), (2, 1))^N \) can be reduced to both \( ((0, 0), (0, 1))^N \) and \( ((0, 0), (1, 0))^N \) but none can be reduced to the other since \( (0, 1) \) and \( (1, 0) \) are incomparable.

The uniqueness of irreducible arrows suggests that we can use for a further compression of the orbit categories in some cases. For lattice po-groups such a representation is unique. Before we can do that we need some partial operations:

**Definition 10** Let \( \mathcal{A} \) be a category, \( G \leq \text{Aut} \mathcal{A} \) a translative automorphism group such that \( \mathcal{A}/G \) is uniquely representable by irreducible arrows. Then we denote by

\[
\tau : \text{Mor}_{\mathcal{A}/G} \to \text{Mor}_{\mathcal{A}/G} \tag{22}
\]

the partial mapping that maps each arrow into the set of irreducible arrows, and

\[
n : \text{Mor}_{\mathcal{A}/G} \to \text{Mor}_{\mathcal{A}/G} \tag{23}
\]

the complementary partial mapping such that for every reducible \( \tau \in \text{Mor}_{\mathcal{A}/G} \) arrow the following equation holds:

\[
\tau = \tau(\tau) \ast n(\tau). \tag{24}
\]

**Corollary 8** Let \( (G, \leq) \) denote an \( \ell \)-group with a normal lattice ordered subgroup \( N \triangleleft G \). Then, for each arrow \( a \) and every arrow \( n \in \text{Mor}_{(G, \leq)/N} \) the following equation holds:

\[
\tau(a) = \tau(a \ast n) \tag{25}
\]

if \( a \ast n \) is defined and \( \tau(a) \) and \( \tau(a \ast n) \) exist.

**Corollary 9** For the mappings \( \tau \) and \( n \) from Definition 10 the following holds:

\[
\begin{align*}
\tau(\text{id}_x) &= \text{id}_x & n(\text{id}_x) &= \text{id}_x \tag{26a} \\
\tau(a + id \cdot a) &= \tau(a) & n(a + id \cdot a) &= n(a) \tag{26b} \\
\tau(id + a) &= \tau(a) & n(id + a) &= n(a) \tag{26c} \\
n(\tau(a)) &= \text{id}_{\tau \cdot a} \tag{26d}
\end{align*}
\]

As a next step we will construct a concatenation based on a category \( \mathcal{A} \) that is uniquely representable by irreducible arrows, so that \( \mathcal{A}/G \) equipped with this concatenation is a category again.

**Lemma 22** Under the conditions of Lemma 16, if \( \mathcal{A}/G \) is uniquely representable by irreducible arrows then for each arrow \( a \in \text{Mor}_{\mathcal{A}/G} \) the homomorphism \( \phi_a \) from (15) is injective.

**Proof** Let \( \tau, \eta \in \text{Mor}_{\mathcal{A}/G} \). If \( \tau \ast \tau = \tau \ast \eta \), then \( \tau \ast \tau \) is not uniquely representable or \( \tau = \eta \). \( \square \)
Lemma 23  Let $\mathfrak{R}$ be a category, $G \leq \text{Aut} \mathfrak{R}$ a transitive automorphism group that acts normal on $\mathfrak{R}$ such that $\mathfrak{R}//G$ is uniquely representable by irreducible arrows. Then the partial mapping

$$\bullet : \text{Mor}_{\mathfrak{R}//G} \times \text{Mor}_{\mathfrak{R}//G} \to \text{Mor}_{\mathfrak{R}//G} : (\tau, \eta) \mapsto \tau(\tau \ast \eta)$$  \hfill (27)

is a category concatenation.

Proof  Using Corollary 8 and Definition 8 we know: $\eta \in \text{Mor}_{\mathfrak{R}//G}(\tau \mathfrak{R}, \tau \mathfrak{R})$ that $\tau \bullet \eta = \tau(\tau)$. In general,

$$\tau \bullet \eta = \tau(\tau \ast \eta \ast \tau \ast \eta)$$
$$= \tau(\tau \ast \eta \ast \tau \ast \eta)$$
$$= \tau(\tau \ast \eta)$$
$$= \tau \bullet \eta$$

Consequently we can prove the associativity:

$$(\tau \bullet \eta) \bullet \zeta = \tau(\tau(\tau \ast \eta) \ast \tau(\tau \ast \eta) \ast \tau(\tau \ast \eta) \ast \tau(\tau \ast \eta) \ast \tau(\tau \ast \eta))$$
$$= \tau(\tau(\tau \ast \eta) \ast \tau(\tau \ast \eta))$$
$$= \tau(\tau(\tau \ast \eta) \ast \tau(\tau \ast \eta))$$
$$= \tau(\tau(\tau \ast \eta) \ast \tau(\tau \ast \eta))$$
$$= \tau(\tau(\tau \ast \eta) \ast \tau(\tau \ast \eta))$$
$$= \tau \bullet (\eta \bullet \zeta).$$

$\Box$

Corollary 10  The mapping $\tau$ is a category homomorphism from $\mathfrak{R}$ with the concatenation $\ast$ to $\tau[\mathfrak{R}]$ with the concatenation $\bullet$.

Definition 11  Let $\mathfrak{R}$ be a category and $G \leq \text{Aut} \mathfrak{R}$ a transitive automorphism group that acts normal on $\mathfrak{R}$. Let further $\mathfrak{R}//G$ representable by irreducible arrows. Then we call the category $\tau[\mathfrak{R}//G]$ with the concatenation from Lemma 23 flat orbit category of $\mathfrak{R}$.

In a similar way as the orbit categories, also flat orbit categories can be unfolded, and both unfoldings lead to isomorphic categories. This isomorphism is presented here, while more information about the resulting representations is given in the next section.

Lemma 24  Let $\mathfrak{R}$ be a category, $G \leq \text{Aut} \mathfrak{R}$ a transitive automorphism group that acts normal on $\mathfrak{R}$ such that $\mathfrak{R}//G$ is uniquely representable by irreducible arrows, $(\mathfrak{R}//G, A, G)$ a representation of $\mathfrak{R}$, and $C$ the mapping from Definition 8. Under the conditions of Lemma 23 let

$$\hat{n} : \text{Mor}_{\mathfrak{R}} \times \text{Mor}_{\mathfrak{R}} \to \text{Mor}_{\mathfrak{R}} : (a, b) \mapsto n(a \ast b).$$  \hfill (28)

Then, the mapping

$$\varphi : \tau[\mathfrak{R}//G] \times \langle G \rangle_{\mathfrak{R}//G} \to \mathfrak{R}//G : (\tau, \eta) \mapsto \tau \ast \eta$$  \hfill (29)

from the orbit category $\mathfrak{R}//G$ into the product category $\tau[\mathfrak{R}//G] \times \langle G \rangle_{\mathfrak{R}//G}$ with the concatenation

$$(\tau \ast \eta) \ast (\tau \ast \eta) = (\tau(\tau \ast \eta) \ast \tau(\tau \ast \eta))$$

is an isomorphism.
Proof. Let \((ra,x), (rb,η) \in r\langle R/G \rangle \times (G)_R/G\). Then,
\[
\varphi(ra,x) \ast \varphi(rb,η) = (ra \ast x) \ast (rb \ast η) = ra \ast rb \ast C(rb,x) \ast η \\
= \varphi(ra \ast rb), n(ra \ast rb) \ast C(rb,x) \ast η \\
= \varphi((ra,x) \ast (rb,η)). 
\]

According to Lemma 16 and Lemma 22, a category homomorphism \(ψ_{r_2} : Mor(σ_{r_2})_{\langle R/G \rangle} \to Mor(\langle r_2 \rangle)_{\langle G/G \rangle}\) exists such that \(η_1 \ast r_2 = r_2 \ast η_1\). Lemma 21 assures that this arrow is unique as the arrow \(r_2 \ast η_2 η_1\) has a unique representation in the form (21). For the same reason the representation \(r_1 \ast r_2 \ast η_2 \ast η_1\) is unique. □

Corollary 11. Let \(A : \langle R/G \rangle \to G\) be an annotation. Then
\[
A(rb)A(ra) = A(ra \ast rb) = A(\hat{n}(ra \ast rb))A(r(\tau ra rb)) \tag{31}
\]

Corollary 12. The mapping \(\hat{n}\) from (28) fulfills the following equation:
\[
\hat{n}(a, id_{ra}) = id_{ra} \tag{32a}
\]
\[
\hat{n}(id_{ra}, a) = id_{ra} \tag{32b}
\]

Recall, that we aim at a description of orbit categories of po-groups. As they form groups with binary relations on them, it is an interesting question, in which case the orbit category can be described by a binary relation. So, in which case is the orbit category or the flat orbit category a binary relation aka simple category? The following lemma should help.

Lemma 25. Let \(R\) denote a category and \(G \leq Aut R\), a translative automorphism group that acts normal on \(R\). Let further \(\langle R/G \rangle\) be representable by irreducible arrows. If \(R\) is a simple category and for some vertex \(x \in \mathfrak{Ob} R\) for any two vertices \(x', x'' \in \mathfrak{Ob} G\), there exists a vertex \(y \in \tau[σ, x'] \cap τ[σ, x'']\), then the flat orbit category \(\tau[\langle R/G \rangle]\) is a simple category.

Proof. Suppose \(\tau[\langle R/G \rangle]\) is not a simple category. Then there are two vertices \(x, y \in \mathfrak{Ob} \tau[\langle R/G \rangle]\) and two different arrows \(a, b \in Mor(\tau[\langle R/G \rangle] (x, y))\). Additionally, we can find vertices \(x' \in x\) and \(y', y'' \in y\) and arrows \(a' \in Mor_R(x', y') \cap a\) and \(b' \in Mor_R(x', y'') \cap b\). As \(R\) is a simple category from \(a \neq b\) follows \(y' \neq y''\). Then we know the following facts:

- There is no direct arrow from \(y'\) to \(y''\). Otherwise, the identity \(a = b\) holds, as \(\langle R/G \rangle\) is representable by irreducible arrows. And,

- If there exists a vertex \(z \in y\) with arrows \(y' \xrightarrow{z_1} \ x \xrightarrow{z_2} y''\), then the equation \(a' \ast z_1 = b' \ast z_2\) holds as \(R\) is a simple category. Since \(\tau[\langle R/G \rangle]\) is uniquely representable by irreducible arrows at least one of \(a'\) or \(b'\) must be reducible. Thus, \(a = b\).

As there cannot be two distinct arrows in the same direction between the same two vertices, the category \(\tau[\langle R/G \rangle]\) is a simple category. □

Corollary 13. If \(R\) is a directed po-group, the flat orbit category is a simple category.

Also circles simplify the structure of flat orbit categories a lot.

Lemma 26. Let \(R\) a category that is representable by irreducible arrows. Then each subcategory of \(\tau[\mathfrak{R}]\) that is generated by a finite circle, is simple.
Proof In a category every circle can be reduced to two antiparallel arrows between two arbitrary vertices of the circle. If \( r[K] \) contains a circle with two parallel arrows, then we find vertices \( x, y \in \text{Ob} r[K] \) and arrows \( a, b \in \text{Mor}_{r[K]}(x, y) \) and \( c \in \text{Mor}_{r[K]}(y, x) \). Then in \( r \) the arrows \( a * r c \) and \( c * r b \) are loops. This leads to the equation
\[
a = \tau(a * r c) = \tau((a * r c) * r b) = \tau(b * r c(a * r c)) = b
\]
Consequently, there can be only one arrow in each direction between two vertices of the circle in \( r[K] \).

It is easy to see that the flat orbit category of an \( \ell \)-group is isomorphic to the factor relation that arises from the underlying simple category of \( r \) via factorisation by the orbit partition of a normal subgroup \( \mathfrak{H} \) acting on \( r \).

At the first glance it also suffers from the fact that the direction information might get lost. However, any annotation on the orbit category is still available as a mapping from the flat orbit category into the group. It is not necessarily a homomorphism anymore. However, it can be used together with Lemma 24 to recreate the annotation on the product category. If \( A \) is an annotation of \( \mathfrak{H} / \mathfrak{H} \mathfrak{M} \) and – consequently – \( \mathfrak{H} / \mathfrak{H} \mathfrak{M} \) is isomorphic to the factor relation \( \mathfrak{H} / \mathfrak{H} \mathfrak{M} \), and if there exist three arrows \( a, b, \tau \in \mathfrak{H} / \mathfrak{H} \mathfrak{M} \) of the flat orbit category with \( a * b = c \), then \( c \) is redundant if there exists an arrow \( \tau \in \text{Mor}_{\mathfrak{H} / \mathfrak{H} \mathfrak{M}}(\tau b, \tau b) \) such that \( c = \tau a \). This is the case when four arrows \( \tau a, b, \tau c \in \mathfrak{H} \mathfrak{M} \) exist such that \( \tau c = \tau a \). We will explore these facts in the following sections.

8 Flat category representations

As we have seen, we can express the category part of a representation by a category, a groupal category and two mappings. This should be enough in order to define representations that use the idea of a flat category.

In analogy to group extensions we can define also category extensions. The isomorphism \( (\mathfrak{A}, \mathfrak{B}) \) is a good candidate. Actually, we have to resemble some results from [10, 9] with respect to flat orbit categories.

Definition 12 Let \( r \) denote a category with concatenation \( \bullet \) and \( \mathfrak{G} \) another category with the concatenation \( * \) and \( \text{Ob} \mathfrak{G} = \{ 1 \} \), and let
\[
\sigma_{\mathfrak{A}} : \text{Mor}_{r} \times \text{Mor}_{\mathfrak{G}} \to \text{Ob} \mathfrak{A} : (a, g) \mapsto \sigma a
\]
\[
\tau_{\mathfrak{A}} : \text{Mor}_{r} \times \text{Mor}_{\mathfrak{G}} \to \text{Ob} r : (a, g) \mapsto \tau a
\]
\[
\mathcal{E} : \text{Mor}_{r} \times \text{Mor}_{\mathfrak{G}} \to \text{Mor}_{\mathfrak{G}},
\]
\[
\text{ mappings, where } \mathcal{E} \text{ fulfils } (33a) \text{ and } (13b), \text{ and}
\]
\[
n : \text{Mor}_{r} \times \text{Mor}_{r} \to \text{Mor}_{\mathfrak{G}},
\]
\[
*_{\mathfrak{G}} : (\text{Mor}_{r} \times \mathfrak{G})^2 \to \text{Mor}_{r} \times \mathfrak{G}
\]
\[
: ((a, \tau), (b, \eta)) \mapsto (a \bullet b, n(a, b) * \mathcal{E}(b, \tau) * \eta),
\]
two partial mappings such that the equations
\[
\mathcal{E}(a, id_{1}) = id_{1}
\]
\[
n(id_{1}, id_{1}) = n(a, id_{1}) = n(id_{\mathfrak{A}}, a) = id_{1}
\]
hold and \( \mathcal{L} := (\text{Ob} r, \text{Mor}_{r} \times \text{Mor}_{\mathfrak{G}}, \sigma_{\mathfrak{A}}, \tau_{\mathfrak{A}}, *_{\mathfrak{G}} \) is a small category. Then, the category \( \mathcal{L} \) is called a singleton category extension of \( \mathfrak{A} \) by \( \mathfrak{G} \). This is denoted by \( \mathfrak{E}(\mathfrak{A}, n, \mathcal{E}, \mathfrak{G}) := \mathcal{L} \).
For convenience we identify the category $\mathcal{L}$ with the singleton category extension that generates it. This can be used to define flat category representations as another kind of an unfoldable factor structure.

**Definition 13** Let $\mathcal{L}$ denote a category, $\mathcal{G}$ a right grupal category, such that the singleton category extension $L := \mathcal{L}(\mathcal{R}, n, \mathcal{L}, \mathcal{G} \bowtie \mathcal{G})$ exists, and let

\[
A : \text{Mor}_R \rightarrow \mathcal{G},
\]

\[
A' : \text{Mor}_R \times \text{Mor}_R \bowtie \mathcal{G} \rightarrow \mathcal{G}
\]

(33a)

\[
: (a, x^\mathcal{G}) \mapsto \tau_{x}(\sigma_{x})^{-1}A(a)
\]

(34b)

two mappings. Then, the tuple $(\mathcal{R}, A, n, \mathcal{G}, \mathcal{G})$ is called a *flat category representation* of $\mathcal{L}$ via $(\mathcal{L}, A', \mathcal{G})$, iff $(\mathcal{L}, A', \mathcal{G})$ is a representation.

**Corollary 14** In equation (34b) $A'$ is well-defined i.e. for every vertex $g \in \mathcal{G}$ the following equation holds:

\[
A'(a, x^\mathcal{G}) = A'(a, (x^\mathcal{G})^\mathcal{G})
\]

(35)

*Proof* $\tau_{x}^{-1}(\sigma_{x})^{-1} = \tau_{x}^{-1}(\sigma_{x})^{-1}$

Additionally to the indirect definitions above, we can find also axiomatic descriptions of singleton category extensions and flat category representations.

**Theorem 2** Let $\mathcal{R}$ denote a category with concatenation $\bullet$ and $\mathcal{G}$ another category with $\mathcal{G} = \{1\}$ and concatenation $\ast$, and let $\mathcal{E}, \sigma_{\mathcal{E}}, \tau_{\mathcal{E}}$ mappings and $n$ and $\ast_{\mathcal{E}}$ partial mappings according to the properties (35a) to (35g) from Definition 7. Then the structure $\mathcal{L} := (\mathcal{R}, \text{Mor}_R \times \text{Mor}_R, \sigma_{\mathcal{E}}, \tau_{\mathcal{E}}, \ast_{\mathcal{E}})$ is a singleton category extension of $\mathcal{R}$ by $\mathcal{G}$ iff the following equations hold:

\[
(id_{a, x}) \ast_{\mathcal{E}} (a, id_{1}) = (a, \mathcal{E}(a, x))
\]

(36a)

\[
n(a, b \bullet c) \ast_{\mathcal{E}} n(b, c) = n(a \bullet b, c) \ast_{\mathcal{E}} (c, n(a, b))
\]

(36b)

\[C(a, x \ast_{\mathcal{E}} y) = C(a, x) \ast_{\mathcal{E}} (C(a, \eta))
\]

(36c)

\[C(a \bullet b, x) \ast_{\mathcal{E}} n(a, b) = n(a, b) \ast_{\mathcal{E}} (C(b, C(a, x))
\]

(36d)

*Proof* Let us first assume that $\mathcal{L}$ is a singleton category extension of $\mathcal{R}$. Then we can prove the Equations (36a), (36b), (36c) and (36d). Equation (36a) is a direct consequence from the definition:

\[
(id_{a, x}) \ast_{\mathcal{E}} (a, id_{1}) = (id_{a, x} \bullet a, n(id_{a, x}, a) \ast_{\mathcal{E}} (a, x) \ast_{\mathcal{E}} id_{1}) = (a, \mathcal{E}(a, x)) 
\]

The transitivity of the category $\mathcal{L}$ follows

\[
(a \bullet b \bullet c, n(a, b \bullet c) \ast n(b, c)) = (a, id_{1}) \ast_{\mathcal{E}} (b \bullet c, n(b, c))
\]

\[
= (a, id_{1}) \ast_{\mathcal{E}} (b, id_{1}) \ast_{\mathcal{E}} (c, id_{1})
\]

\[
= (a \bullet b, n(a, b)) \ast_{\mathcal{E}} (c, id_{1})
\]

\[
= (a \bullet b \bullet c, n(a \bullet b, c) \ast_{\mathcal{E}} C(c, n(a, b)))
\]
Equation (36c) follows from
\[ (a, \mathcal{C}(a, x \ast y)) = (\text{id}_{\sigma a}, x \ast y) \ast \mathcal{L} (a, \text{id}_1) = (\text{id}_{\sigma a}, x) \ast \mathcal{L} (\text{id}_{\sigma a}, y) \ast \mathcal{L} (a, \text{id}_1) \]
\[ = (\text{id}_{\sigma a}, x) \ast \mathcal{L} (a, \mathcal{C}(a, y)) = (a, \mathcal{C}(a, x \ast y)) \]

The same scheme can be used to prove Equation (36d):
\[ (a \bullet b, \mathcal{C}(a \bullet b, x) \ast n(a, b)) = (\text{id}_{\sigma a}, x) \ast \mathcal{L} (a \bullet b, n(a, b)) = (\text{id}_{\sigma a}, x) \ast \mathcal{L} (a, \text{id}_1) \ast \mathcal{L} (b, \text{id}_1) = (a, \mathcal{C}(a, x)) \ast \mathcal{L} (b, \text{id}_1) \]
\[ = (a \bullet b, n(a, b) \ast \mathcal{L} (b, \mathcal{C}(a, x))) \]

Now, suppose that the Equations (36a), (36b), (36c) and (36d) hold. We prove that \( \mathcal{L} \) is a category. Start and end of the arrows and the identities inherit their category structure from \( \mathcal{R} \) and (33a). It remains to show that the partial mapping \( \ast \mathcal{L} \) is a category concatenation, which means that we have to prove that it is associative:
\[ ((a, x) \ast \mathcal{L} (b, y)) \ast \mathcal{L} (c, z) = (a \bullet b, n(a, b) \ast \mathcal{C}(b, x) \ast y) \ast \mathcal{L} (c, z) \]
\[ = \left( (a \bullet b \bullet c, n(a \bullet b, c) \ast \mathcal{C}(n(a, b) \ast \mathcal{C}(b, x) \ast y)) \ast \mathcal{L} (c, z) \right) \]
with (36c) we can rewrite this term as
\[ = \left( (a \bullet b \bullet c, n(a \bullet b, c) \ast \mathcal{C}(c, n(a, b))) \ast \mathcal{C}(c, \mathcal{C}(b, x))) \ast \mathcal{L} (c, \mathcal{C}(y)) \right) \ast \mathcal{L} (c, \mathcal{C}(z)) \]
(36b) leads to
\[ = \left( a \bullet b \bullet c, n(a, b \bullet c) \ast \mathcal{C}(c, \mathcal{C}(b, x))) \ast \mathcal{L} (c, \mathcal{C}(y)) \right) \ast \mathcal{L} (c, \mathcal{C}(z)) \]
with (36d) this is equivalent to
\[ = \left( a \bullet b \bullet c, n(a, b \bullet c) \ast \mathcal{C}(b \bullet c, x) \ast n(b, c) \ast \mathcal{C}(c, \mathcal{C}(y)) \right) \ast \mathcal{L} (c, \mathcal{C}(z)) \]
\[ = \left( a, x \ast \mathcal{L} (b \bullet c, n(b, c) \ast \mathcal{C}(c, \mathcal{C}(y))) \right) \ast \mathcal{L} (c, \mathcal{C}(z)) \]
\[ = \left( a, x \ast \mathcal{L} (b, \mathcal{C}(y) \ast \mathcal{L} (c, z)) \right) \]
So the partial mapping \( \ast \mathcal{L} \) is a category concatenation. Now, it is obvious that \( \mathcal{L} \) is a category.

For the flat category representation we need some axioms that properly define the annotation.

**Theorem 3** Let \( \mathcal{R} \) be a category with concatenation \( \bullet, \mathcal{G} \) a right groupoidal category, such that the singleton category extension \( \mathcal{L} := \mathcal{C}(\mathcal{R}, n, \mathcal{C}, \mathcal{G} \setminus \mathcal{D} \mathcal{B} \mathcal{G}) \) exists, and let
\[ A : \text{Mor}_\mathcal{R} \to \mathcal{D} \mathcal{B} \mathcal{G}, \]
\[ A' : \text{Mor}_\mathcal{R} \times \text{Mor}_{\mathcal{G} \setminus \mathcal{D} \mathcal{B} \mathcal{G}} \to \mathcal{D} \mathcal{B} \mathcal{G} \]
\[ : (a, f) \mapsto \tau f (\sigma f)^{-1} A(a) \]
(37a)
\[ (37b) \]
two mappings. Then, the tuple \( (\mathcal{R}, A, n, \mathcal{C}, \mathcal{G}) \) is a flat category representation of \( \mathcal{L} \setminus \mathcal{D} \mathcal{B} \mathcal{G} \) via \( (\mathcal{L}, A', \mathcal{D} \mathcal{B} \mathcal{G}) \), if the following equations hold:
\[ A'(a, x) = A'(\text{id}_\mathcal{R}, x) A'(a, \text{id}_1) \]
(38a)
\[ \tau n(a, b)(\sigma n(a, b))^{-1} = A(b) A(a) (A(a \bullet b))^{-1} \]
(38b)
\[ A(b) \tau f (\sigma f)^{-1} = \tau C(b, x) (\sigma C(b, x))^{-1} A(b) \]
(38c)
Proof If \((\mathcal{R}, A, n, \mathcal{C}, \mathcal{G})\) is a flat representation of \(\mathcal{L} \dashv \mathcal{L} \bowtie \mathfrak{G}\), then Equation \((38a)\) follows from Corollary \([12]\) and Equation \((38b)\) is a direct consequence of Corollary \([11]\). Equation \((38c)\) can also be easily proved:

\[
A(b) \tau_x(\sigma) = 1 A(b) \tau_x(\sigma)^{-1} A(\text{id}_b)
\]

\[
= A'(b, \text{id}_b) A'(\text{id}_b, x)
\]

\[
= A'(\text{id}_b, x) \ast \mathcal{C}(b, x)
\]

\[
= A'(b, \mathcal{C}(b, x))
\]

\[
= \tau \mathcal{C}(b, x)(\sigma \mathcal{C}(b, x))^{-1} A(b)
\]

Now, Suppose Equations \((38a)\), \((38b)\) and \((38c)\) hold. We have to prove that \(A'\) is an annotation.

\[
A'(a, x) = A'(a * b, n(a, b) * \mathcal{C}(b, x) * \eta)
\]

\[
= \tau \eta(\sigma)^{-1} \tau \mathcal{C}(b, x)(\sigma \mathcal{C}(b, x))^{-1} \tau n(a, b)(\sigma n(a, b))^{-1} A(a * b)
\]

\[
= \tau \eta(\sigma)^{-1} \tau \mathcal{C}(b, x)(\sigma \mathcal{C}(b, x))^{-1} A(b) A(a)
\]

\[
= \tau \eta(\sigma)^{-1} A(b) \tau x(\sigma)^{-1} A(a)
\]

\[
= A'(b, \eta) A'(a, x)
\]

So \(A'\) is an annotation from \(\mathcal{L}\) into the group \(\mathcal{L} \bowtie \mathfrak{G}\) and the tuple \((\mathcal{L}, A', \mathcal{L} \bowtie \mathfrak{G})\) is a representation, which proves the theorem. \(\square\)

Now, we can prove that small categories that are uniquely representable by irreducible arrows give rise to flat category representations.

**Lemma 27** Let \(\mathcal{R}\) be a category, \(G \leq \text{Aut}\mathcal{R}\) a translative automorphism group that acts normal on \(\mathcal{R}\) such that \(\mathcal{R} \bowtie G\) is uniquely representable by irreducible arrows and \((\mathcal{R} \bowtie G, A, \mathcal{G})\) a representation of \(\mathcal{R}\). Under the conditions of Lemma \([23]\) and Lemma \([24]\) for every \(x \in \mathcal{L} \bowtie \mathcal{G}\), the tuple \((\mathcal{R} \bowtie G, A, \mathcal{G})\) is a flat category representation via \((\mathcal{R} \bowtie G, A, \mathcal{G})\).

**Proof** As \(A\) is determined by a natural annotation of \(((\mathcal{R} \bowtie G, A, \mathcal{G})) / G\), it is sufficient to prove the lemma for the case where \(A\) is a natural annotation.

Let us first assume that \(A\) is a natural annotation. Lemma \([24]\) tells us that the category \(\mathcal{L}\) with \(\mathcal{L} \bowtie G = \mathcal{L} \bowtie (\mathcal{R} \bowtie G)\), \(\mathcal{M}(\mathcal{G})(x, y) = \mathcal{M}(\mathcal{G})(x, y) \times \mathcal{M}(\mathcal{G})(x, y)\) and the concatenation from Lemma \([24]\) is a category. So we have to prove that \(A'\) from equation \((34b)\) is an annotation. With the definition

\[
A'^{\prime} : \mathcal{M}(\mathcal{G}) \times \mathcal{M}(\mathcal{G}) \to \mathcal{L} \bowtie \mathfrak{G} : (\mathcal{G}, \mathcal{G}) \to A'(\mathcal{G}, \mathcal{G})
\]

we know that

\[
A'^{\prime}(\mathcal{G}, \mathcal{G})A'^{\prime}(\mathcal{G}, \mathcal{G}) = \tau \eta(\sigma)^{-1} A(\mathcal{G}, \mathcal{G}) \tau x(\sigma)^{-1} A(\mathcal{G}, \mathcal{G})
\]

As \(A\) is a natural annotation of \(\mathcal{R}\), \(A|_{\mathcal{G}}\) is also a natural annotation which coincides on \(\mathcal{M}(\mathcal{G})\) with the natural annotation \(A|_{\mathcal{G}}\) that is defined on \(\mathcal{G}\). So we can rewrite this:

\[
A'(\mathcal{G}, \mathcal{G}) = A(\mathcal{G}, \mathcal{G}) \mathcal{A}(\mathcal{G}, \mathcal{G}) A(\mathcal{G}, \mathcal{G}) = A(\mathcal{G}, \mathcal{G}) \mathcal{G}(\mathcal{G}, \mathcal{G}) A(\mathcal{G}, \mathcal{G})
\]

\[
A'^{\prime}(\mathcal{G}, \mathcal{G}) * (\mathcal{G}, \mathcal{G}) = A'^{\prime}((\mathcal{G}, \mathcal{G}) * \mathcal{G}(\mathcal{G}, \mathcal{G}) * (\mathcal{G}, \mathcal{G}) = A'^{\prime}((\mathcal{G}, \mathcal{G}) * \mathcal{G}(\mathcal{G}, \mathcal{G}) * (\mathcal{G}, \mathcal{G})
\]

\[
= A'^{\prime}((\mathcal{G}, \mathcal{G}) * \mathcal{G}(\mathcal{G}, \mathcal{G}) * (\mathcal{G}, \mathcal{G})
\]

\[
= A'^{\prime}((\mathcal{G}, \mathcal{G}) * \mathcal{G}(\mathcal{G}, \mathcal{G}) * (\mathcal{G}, \mathcal{G})
\]
Then there exist arrows \( n', c', \eta' \in Mor_R \) with \( \eta'^G = \hat{n}(ra^G, rb^G), \ c'^G = C(rb^G, x^G) \) and \( \eta'^G = \eta \) such that

\[
= \tau \eta' (\sigma' n')^{-1} \tau c' (\sigma c')^{-1} \tau n'(\sigma' n')^{-1} A(\tau(ra^G * rb^G)) = A(ra^G * rb^G)
\]

As \( A \) is a natural annotation, we can rewrite this:

\[
= A(\eta^G) A(c^G) A(ra^G * rb^G)
\]
\[
= A(\eta^G) A(C(rb^G, x^G)) A(rb^G) A(ra^G)
\]

And with \( rb^G \ast C(rb^G, x^G) = x^G * rb^G \) we get

\[
= A(\eta^G) A(rb^G) A(x^G) A(ra^G)
\]

So \( A'' \) is an annotation. Furthermore \( A'' = A \). □

**Corollary 15** Let \( (R, A, G) \) denote a representation where \( R \) is uniquely representable by irreducible arrows. Then there exists a flat representation \( (\tau[R/G], A[\hat{\tau}[r][\tau[G]], \hat{n}, \hat{C}, \hat{\Theta}_x) \) of \( R \tau[G]. \)

Now, that we know that we can unfold a flat representation via a representation, it would be handy to have a direct description of the unfolding of a flat representation. First we define it, afterwards we prove that the definition is correct.

**Definition 14** Let \( (R, A, n, C, \Theta) \) be a flat category representation. Then, the structure \( R \tau_{A,n,C} \Theta \) with

\[
\text{Ob} R \tau_{A,n,C} \Theta := \text{Ob}(R \times \Theta), \quad \text{(39a)}
\]
\[
\text{Mor}_{R \tau_{A,n,C} \Theta} ((x, g), (y, h)) := \left\{ (a, g) \in \text{Mor}_{R \times \Theta} : \tau g = h, \sigma g = A(a) g \right\} \quad \text{and} \quad \text{(39b)}
\]
\[
(a, g) \ast_{R \tau_{A,n,C} \Theta} (b, h) := (a \ast b, n(a, b) \ast C(b, g) \ast h) \quad \text{(39c)}
\]

whenever the concatenations are defined in \( R \) and \( \Theta \), is called **unfolding** of \( (R, A, n, C, \Theta) \).

**Lemma 28** Let \( (R, A, n, C, \Theta) \) be a flat category representation together with a mapping

\[
A' : \text{Mor}_R \times \text{Mor}_{\Theta} / \text{Ob} \Theta : (a, g) \mapsto A(a).
\]

Then, \( A' \) is a category annotation of \( E(R, n, \Theta, / \Theta) \), and the pair of mappings

\[
\Phi : \text{Ob} R \times \text{Ob} \Theta \rightarrow \text{Ob} R \times \text{Ob} \Theta : (a, g) \mapsto (a, g) \quad \text{(40b)}
\]
\[
\Psi : \text{Mor}_R \times \text{Mor}_{\Theta} \rightarrow \text{Mor}_R \times \text{Mor}_{\Theta} / \text{Ob} \Theta \times \text{Mor}_{\Theta}
\]
\[
: (a, g) \mapsto \left( (a, g), \left( A(a) \right)^{-1} \sigma g \right) \quad \text{(40c)}
\]

form an isomorphism between the unfoldings \( R \tau_{A,n,C} \Theta \) and \( E(R, n, \Theta, / \Theta) \).
Proof For convenience we write $\mathcal{L} := \mathcal{X} \mathbb{A} \mathbb{C}$ and $\mathcal{M} := \mathbb{L} \mathbb{R} \mathbb{C} / \mathcal{S} \mathcal{D} \mathcal{B} \mathcal{R}$. With Theorem 3 the mapping $A'$ is an annotation of $\mathcal{M}$.
Furthermore,

$$\Phi(\sigma_{\mathcal{L}}(a, g)) = \Phi\left(\sigma_{\mathcal{H}} a, (A(a))^{-1} \sigma_{\mathcal{G}} g\right)$$

$$= \left(\sigma_{\mathcal{H}} a, (A(a))^{-1} \sigma_{\mathcal{G}} g\right)$$

$$= \sigma_{\mathcal{H}} \left((a, g^Dg^\mathcal{G}), (A(a))^{-1} \sigma_{\mathcal{G}} g\right)$$

$$= \sigma_{\mathcal{H}} \Phi(a, g)$$

$$\Phi(\tau_{\mathcal{L}}(a, g)) = \Phi(\tau_{\mathcal{H}} a, \tau_{\mathcal{G}} g)$$

$$= \left(\tau_{\mathcal{H}} a, \tau_{\mathcal{G}} g(\sigma_{\mathcal{G}} g^{-1}A(a)(A(a))^{-1} \sigma_{\mathcal{G}} g\right)$$

$$= \tau_{\mathcal{H}} \left((a, g^Dg^\mathcal{G}), (A(a))^{-1} \sigma_{\mathcal{G}} g\right)$$

$$= \tau_{\mathcal{H}} \Phi(a, g)$$

$\Psi((a, g) \ast_{\mathcal{L}} (b, h)) = \Psi(a \ast_{\mathcal{H}} b, n(a, b) \ast_{\mathcal{G}} \mathbb{L}(b, g) \ast_{\mathcal{G}} h)$

$$= \left(\left(a \ast_{\mathcal{H}} b, \left(n(a, b) \ast_{\mathcal{G}} \mathbb{L}(b, g) \ast_{\mathcal{G}} h\right)^Dg^\mathcal{G}\right),
\left(A(a \ast_{\mathcal{H}} b)\right)^{-1} \sigma(n(a, b) \ast_{\mathcal{G}} \mathbb{L}(b, g) \ast_{\mathcal{G}} h)\right)$$

$$= \left(\left(a \ast_{\mathcal{H}} b, \left(n(a, b) \ast_{\mathcal{G}} \mathbb{L}(b, g) \ast_{\mathcal{G}} h\right)^Dg^\mathcal{G}\right),
\left(A(a \ast_{\mathcal{H}} b)\right)^{-1} \sigma(n(a, b)) \left(\mathcal{L}(n(a, b))\right)^{-1} \sigma(\mathbb{L}(b, g) \ast_{\mathcal{G}} h)\right)$$

Since $\mathcal{D} \mathcal{B} \mathcal{R}$ acts translatively on $\mathcal{S}$, this can be rewritten into

$$= \left(\left(a \ast_{\mathcal{H}} b, \left(n(a, b)^Dg^\mathcal{G}/\mathcal{S} \mathcal{D} \mathcal{B} \mathcal{R} \mathbb{L}(b, g)^Dg^\mathcal{G}/\mathcal{S} \mathcal{D} \mathcal{B} \mathcal{R} b^Dg^\mathcal{G}\right),
\left(A(a)\right)^{-1} \left(A(b)\right)^{-1} \sigma(\mathbb{L}(b, g)) \left(\mathcal{L}(b, g)\right)^{-1} \sigma(h)\right)$$

$$= \left(\left(a \ast_{\mathcal{H}} b, \left(n(a, b) \ast_{\mathcal{G}} \mathbb{L}(b, g) \ast_{\mathcal{G}} h\right)^Dg^\mathcal{G}\right),
\left(A(a)\right)^{-1} \sigma(\mathbb{L}(b, g)) \left(\mathcal{L}(b, g)\right)^{-1} \sigma(h)\right)$$

We can apply Eq. (14a) in order to prove the equivalence under the brace. Then, we get

$$= \left(\left(a, g^Dg^\mathcal{G}\right)^* \mathcal{E}(\mathcal{H}, n, \mathbb{E}, \mathcal{S}/\mathcal{D} \mathcal{B} \mathcal{R}) \left(b, h^Dg^\mathcal{G}\right),
\left(A(a)\right)^{-1} \sigma(\mathbb{L}(b, g)) \left(\mathcal{L}(b, g)\right)^{-1} \sigma(h)\right)$$
The brace can be omitted as \( \tau_L(a, g) = \sigma_L(b, h) \), and thus \( \tau g = (A(b))^{-1} \sigma h \).

\[
\begin{align*}
&= \left( (a, g^{D'b\rho}) \ast_{\sigma (\sigma (\sigma \tau_{b\rho} = b, h, \rho) \circ D'b\rho)} (b, h^{D'b\rho}), (A(a))^{-1} \sigma g \right) \\
&= \left( (a, g), (A(a))^{-1} \sigma g \right) \ast_{\sigma R} \left( (b, h), (A(b))^{-1} \sigma h \right) \\
&= \Psi(a, g) \ast_{\sigma R} \Psi(b, h)
\end{align*}
\]

So it remains to show the bijectivity of \( \Psi \) from which follows the bijectivity of \( \Phi \). The injectivity follows directly from Eq. \((*)\) below. Consider the mapping

\[
\Psi' : Mor_{\rho} \times Mor_{\rho} \rightarrow Mor_{\rho} \times Mor_{\rho}
\]

\[
:\left( (a, g^{D'b\rho}), h \right) \mapsto (a, g') \text{ where } g' \in \rho^{D'b\rho} \text{ with } \Psi g' = (A(a))h.
\]

This mapping is well-defined since \( D'b\rho \) acts translatively on \( \rho \). Obviously, the equation \( \Psi \left( (a, g^{D'b\rho}), h \right) = \left( (a, g^{D'b\rho}), h \right) \) holds for all arrows of \( \rho \). So \( \Psi' \) is the inverse of \( \Psi \) which proves the bijectivity.

\( \square \)

Now, we can consider the base structure of our representation as a reflexive and transitive binary relation with some additional decorations. If we know that the unfolded structure is a transitive relation, it may be more helpful to preserve antisymmetry than transitivity.

Before we discuss this in detail, let us have some philosophical remarks: In \([9,10]\), Two mappings are considered: \( A^B \) which corresponds to \( \rho(B,A) \) and \( A_B^B' \) which corresponds to \( \rho(B,B') \), here. Both group extensions as well as unfoldings are reconstructions of factored structures. And both constructions consider the factor structure as transversal of the unfolded structure: They describe, how the elements of the transversal are related to each other and how the equivalence classes of the kernel of the canonical homomorphism are mapped to each other.

In case of group extensions, the kernel of the canonical homomorphism is defined by the partition of right cosets of a normal subgroup. These are mapped to each other by internal homomorphisms \( (A^B) \) where the neutral elements of the group operations of the cosets are the elements of a transversal. The relationship between the transversal elements is encoded by the elements \( A_B^B' \). These elements also encode how larger cycles are factored into smaller cycles.

When we build representations with orbit categories, the category structure is mainly encoded in the orbit category. The relationship between the transversal elements and the location of particular arrows are encoded in the annotation. So, the offset correction behaves differently in unfoldings than \( A_B^B' \) for Schreier’s group extensions. The relationship of arrows between the elements of the same vertex congruence class is encoded in the corresponding vertex monoid and the annotation.

During the transition to a flat representation the category structure of the vertex monoids is transferred to the unfolding group, resulting in a right-groupal category. So the connection between the individual loops and the arrows between different vertices gets destroyed. This must be reconstructed using the two operators \( \rho \) and \( n \). So \( \rho \) plays the role of an homomorphism and \( n \) encodes how the annotations match each other.

As we will see later, the natural annotations of the images of \( n \) change by inner automorphisms of the annotation group when the underlying transversal changes. In contrast the corresponding mapping for Schreier’s group extensions may result in arbitrary group elements when the underlying transversal changes.
In order to describe this, we introduce isomorphisms between flat category representations.

**Definition 15** Under the conditions of Lemma 27 two flat category representations \((K, A, \hat{n}, C, \mathcal{G})\) and \((L, B, \hat{m}, D, \mathcal{H})\) are called **isomorphic** if there exist a category isomorphism \(\phi: L \rightarrow K\), a groupal category isomorphism \(\psi: \mathcal{H} \rightarrow \mathcal{G}\), and a mapping \(h: \text{Ob}L \rightarrow \text{Aut}_{\text{CAT}}\mathcal{G}\) from the vertices of category \(L\) into the set of category automorphisms on \(\mathcal{G}\) such that for all \(a, b \in L\) and \(x \in \mathcal{H}\) and all vertices \(x' \in \text{Ob}\mathcal{H}\) and \(f, g \in \text{Ob}\mathcal{G}\) the following equations hold:

\[
\begin{align*}
    h(x)(a)^{\phi(x)} &= h(x)(a^\phi) \\
    B(a) &= h(\tau a)^{-1}(1)A(\sigma(a))h(\sigma a)(1) \\
    m(a, b) &= h(\tau b)^{-1}(m(\phi a, \phi b)) \\
    D(a, x) &= h(\tau a)^{-1}\left(C(\phi a, h(\sigma a)(x))\right)
\end{align*}
\]

In this case the triplet \((\phi, \psi, h)\) is called **isomorphism**.

The flat category representations form a category together with the isomorphisms. This is shown next.

**Lemma 29** The concatenation of two isomorphisms of flat categories is also an isomorphism.

**Proof** Let \((K, A, \hat{n}, C, \mathcal{G})\), \((L, B, \hat{m}, D, \mathcal{H})\), \((M, C, \hat{l}, E, \mathcal{I})\) flat category representations.

Let further the triplets \(\phi: L \rightarrow K\), \(\psi: \mathcal{H} \rightarrow \mathcal{G}\), \(h: \text{Ob}L \rightarrow \text{Aut}_{\text{CAT}}\mathcal{G}\) and \(\phi': M \rightarrow L\), \(\psi': \mathcal{I} \rightarrow \mathcal{H}\), \(h': \text{Ob}M \rightarrow \text{Aut}_{\text{CAT}}\mathcal{H}\) isomorphisms.

Then for the mappings

\[
\begin{align*}
    \phi: M &\rightarrow K: a \mapsto \phi'(a) \\
    \psi: \mathcal{I} &\rightarrow \mathcal{H}: a \mapsto \psi'(a) \\
    h: \text{Ob}M &\rightarrow \text{Aut}_{\text{CAT}}\mathcal{H}: x \mapsto h'(x) \circ h(\phi(x))
\end{align*}
\]
The following equations hold:

\[
\hat{h}(x)(b^\ell) = h(\varphi(x))(h'(x)(b^\ell)) = h(\varphi(x))(h'(x)(b)^{\Phi(x)}) \\
= h(\varphi(x))(h'(x)(b))^\varphi = h(\varphi(x))(h'(x)(b))^{\Phi(x)} \\
= \hat{h}(x)(b^\ell)^{\Phi(x)}
\]

\[
C(a) = h'(\tau a)^{-1}(1)B(\varphi'(a))h'(\sigma a)(1) \\
= h'(\tau a)^{-1}(1)h(\tau \varphi'(a))^{-1}(1)A(\varphi(\varphi'(a)))h(\sigma \varphi'(a))(1)h'(\sigma a)(1) \\
= h'(\tau a)^{-1}(1)A(\varphi(\varphi'(a))(h'(\sigma a))(1)) \\
= (\hat{h}(\tau a)^{-1}(1)A(\hat{\varphi}(a))\hat{h}(\sigma a)(1) \\
\hat{1}(a, b) = h'(\tau b)^{-1}(\hat{h}(\varphi'(a), \varphi'(b))) \\
= h'(\tau b)^{-1}(\hat{h}(\varphi'(a), \varphi'(b))) \\
= \hat{h}(\tau b)^{-1}(\hat{h}(\varphi'(a), \varphi'(b))) \\
E(a, \tau) = h'(\tau a)^{-1}(\hat{D}(\varphi'(a), \varphi'(a)(\tau))) \\
= h'(\tau a)^{-1}(\hat{E}(\varphi'(a), \varphi'(a)\varphi'(a))) \\
= (\hat{h}(\tau a)^{-1}(\hat{E}(\varphi(a), \varphi(a)(\tau)))) \\
= (\hat{h}(\tau a)^{-1}(\hat{E}(\varphi(a), \varphi(a)(\tau))))
\]

Obviously \(\hat{\varphi}\) and \(\hat{\phi}\) are isomorphisms, which fulfil Equations 41a to 41d. So \((\hat{R}, A, \hat{n}, E, \Phi)\), is isomorphic to \((\hat{M}, C, I, E, \Phi)\).

In fact, the isomorphisms are invertible. We leave the proof to the interested reader. We use the term “isomorphism”, here, because two isomorphic flat category representations unfold into isomorphic categories. So they are isomorphic in the sense that they represent essentially the same thing.

**Lemma 30** The unfoldings of two isomorphic flat category representations are isomorphic categories.

**Proof** We use the same notations as in Lemma 13. At first we consider the case where \(\varphi\) and \(\psi\) are trivial. Then Equations 41a to 41d are reduced to

\[
h(x)(g^\ell) = h(x)(g^\ell) = h(x)(g^\ell) = h(x)(g^\ell) \\
B(a) = h'(\tau a)^{-1}(1)A(\varphi(\varphi'(a))h(\sigma a)(1) \\
\hat{n}(a, b) = h'(\tau b)^{-1}(\hat{h}(a, b)) \\
\hat{D}(a, \tau) = h'(\tau a)^{-1}(\hat{D}(a, \varphi(a)(\varphi'(\tau))))
\]
Then, the unfolding $R^{-}_{b,h,D} \Theta$ is defined by

$$
\mathcal{Ob} R^{-}_{b,h,D} \Theta := \mathcal{Ob}(R \times \Theta),
$$

$$
\forall g \in \mathcal{Ob} R^{-}_{b,h,D} \Theta \ s.t \ \tau g = h, \sigma g = B(a)g \ \ and
\ (a,g) *_{R^{-}_{b,h,D} \Theta} (b,h) := (a*b,*_{\mathcal{Ob}(R \times \Theta)} (b,g) *_{\mathcal{Ob}(R \times \Theta)} b)
$$

Now, we substitute $B$, $m$ and $D$ according to the Equations (11a) to (11d):

$$
\forall g \in \mathcal{Ob} R^{-}_{b,h,D} \Theta \ s.t \ \tau g = h, \sigma g = A(a)g \ \ and
\ (a,g) *_{R^{-}_{b,h,D} \Theta} (b,h) := (a*b,h(\tau b) \ (a,b) + h(\tau b)^{-1}(\sigma)(b,h(\sigma b)(g)) *_{\mathcal{Ob}(R \times \Theta)} b)
$$

So, obviously, the mapping

$$
\Phi : \mathcal{Ob} R^{-}_{b,h,D} \Theta \rightarrow \mathcal{Ob} R^{-}_{b,h,A} \Theta \ s.t \ (a,g) \rightarrow (a,h(\tau a)(g))
$$

is a bijection and compatible with the category concatenation. Thus, it raises a category isomorphism between $R^{-}_{b,h,D} \Theta$ and $R^{-}_{b,h,A} \Theta$.

As the isomorphisms $\varphi$ and $\psi$ are compatible with the category structure and $\psi$ is also an isomorphism with respect to the group operation, we can substitute them in the corresponding equations without changing their syntactic applicability. Thus

$$
\Psi : \mathcal{Ob} R^{-}_{b,h,D} \Theta \rightarrow \mathcal{Ob} R^{-}_{b,h,A} \Theta \ s.t \ (a,g) \rightarrow (\varphi(a),h(\tau a)(\psi g))
$$

raises a category isomorphism between $R^{-}_{b,h,D} \Theta$ and $R^{-}_{b,h,A} \Theta$. □

The isomorphism $\Phi$ in the previous proof corresponds to a change of the transversal for a natural annotation. Equation (11b) tells us that the natural annotations of the unfoldings of any image of the mapping $\hat{n}$ are related by inner automorphisms of the group $\mathcal{Ob} \Theta$: The equations

$$
B(b)B(a) = \tau m(a,b)(\sigma m(a,b))^{-1}B(a*b)
$$

$$
A(\varphi b)A(\varphi a) = \tau n(\varphi a,\varphi b)(\varphi a \ (\sigma(\varphi a,\varphi b))^{-1}A(\varphi a \ b)
$$

lead to the equations

$$
\Psi(\tau m(a,b))\Psi(\sigma m(a,b))^{-1}h(\tau(a*b))^{-1}(1)A(\psi(a*b))h(\sigma(a*b))(1) = h(\tau b)^{-1}(1)A(\psi(b))h(\sigma b(1))\tau(1)A(\psi(a))(1)\ \ and
\ \tau n(\varphi a,\varphi b)(\sigma n(\varphi a,\varphi b))^{-1}A(\varphi a \ b) = A(\varphi(b))A(\varphi(a))
These can be combined into a single equation connecting $n$ and $m$.

$$\psi(\tau m(a,b))\psi(\sigma m(a,b))^{-1} = h(\tau b)^{-1}(1)\tau n(\phi a, \phi b)(\sigma n(\phi a, \phi b))^{-1}h(\tau b)(1)$$

This is especially important for those arrows $a, b$ where $n(a,b)$ is a loop. In this case being a loop does not depend on the choice of the transversal elements of a natural annotation. So, we can consider the pair of arrows $(a, b)$ to be consistent. On the other hand, we would call a pair of arrows $a, b$ a long pair if $n(a,b) \neq 1$. In case of a po-group a long pair is essentially a pair of arrows such that for all convex transversals at least one of the two or their concatenation comes from or points to a vertex outside of the convex transversal.

9 Flat representations

Being a long pair does not necessarily mean that one of the arrows is redundant. Examples of po-groups with uniquely representable orbit categories can easily be constructed. On the other hand the orbit category $(\mathbb{Q} \times \mathbb{Q}, \leq)/(12\mathbb{Z} \times 12\mathbb{Z})$ has a system of pairwise consistent arrows that – in combination with a vertex category – represent the whole category itself.

Now, we will construct a structure that represents the antisymmetry of the flat orbit category in the flat category representation. This means, that we must restrict ourselves to partial categories, where the concatenation is not always defined. In our case, each partial category is constructed with a category in mind.

**Definition 16** Let $\mathcal{R}$ denote a category with concatenation $\ast$. Any subgraph $\Psi$ of $\mathcal{R}$ together with a partial binary operator $\cdot$ called concatenation is called partial subcategory of $\mathcal{R}$, if the following conditions are met:

1. Every vertex has an identity loop.
2. For any two arrows $a, b \in \text{Mor}_P$ the concatenation $a \cdot b = a \ast b$ exists iff the concatenation $a \ast b \in \text{Mor}_P$ exists in $\Psi$.

A partial subcategory $\Psi$ of $\mathcal{R}$ is called full if there exists a homomorphism from the path category $\text{Pfad}_P$ onto $\mathcal{R}$ that is compatible with the concatenation in $\Psi$.

The partial subcategory $\Psi$ is called fully defining if for the congruence relation $\equiv$ defined on the path category $\text{Pfad}_P$ and generated by formula

$$a \ast b \equiv a \cdot b$$

the factor category $\text{Pfad}_P/\equiv$ is isomorphic to $\mathcal{R}$.

It is called flat defining if for the congruence relation $\bowtie$ generated by the formula

$$\forall x \in \mathcal{Ob} \Psi, \tau \in \text{Mor}_P(x,x) : \tau \bowtie \text{id}_x$$

the category $\text{Pfad}_P(\equiv \lor \bowtie)$ is isomorphic to $\mathcal{R}$.

With Lemma 25 and 26 we can tell, when uniquely representable orbit categories of po-groups give rise to simple partial categories.

Our goal is now, that we want to express flat category representations in the sense of partial categories. This means we somehow need a way to reconstruct the flat category representation. If a flat category representation $(\mathcal{R}, A, n, C, G)$ has a simple vertex category $\mathcal{G}$, there is at most one arrow between any two vertices in $\mathcal{G}$ and thus, the mapping $n$ can be
reconstructed according to Equation \([36d]\) from the annotation \(A\) and the category concatenation. Then, \(n(a, b)\) is the arrow between \(\tau b\) and \(\tau b^\circ (a)\). Applying this rule recursively, we can extend this construction to the concatenation of any paths in \(\mathcal{P}\). This implies that the partial subcategory must include all concatenated arrows \(a\) for which the annotation \(\mathcal{A}(b \circ c)\) differs from the product \(\mathcal{A}(a)\) of all possible combinations \(a = b \circ c\).

In the same way we can reconstruct the mapping \(\mathcal{C}\) from \(\mathcal{E}|_{\mathcal{P} \times \mathcal{P} \times \mathcal{P}}\) by iteratively applying \(\mathcal{C}|_{\mathcal{P} \times \mathcal{P} \times \mathcal{P}}\) on all arrows of the paths according to Equation \([36d]\). This is well-defined if \(\mathcal{E}\) is a simple category.

This means that under certain conditions (e.g. the above mentioned ones) we can define a flat representation based on a partial subcategory.

Given a category \(\mathcal{R}\) and an annotation \(A : \mathcal{R} \to \mathcal{G}\) into a group then for each partial subcategory \(\mathcal{P} \leq \mathcal{R}\) we call \(A|_{\mathcal{P}}\) annotation of \(\mathcal{P}\).

**Definition 17** Let \((\mathcal{R}, A, n, \mathcal{G}, \mathcal{E})\) be a flat category representation and \(\mathcal{P}\) be a partial category of \(\mathcal{R}\). Then we call the tuple \((\mathcal{P}, A, n, \mathcal{E}, \mathcal{G})\) flat representation. It is called simple, iff \(\mathcal{P}\) is a simple graph, and it is called faithful iff \(A\) is faithful.

If \(\Omega \leq \mathcal{P}\) is a partial subcategory of \(\mathcal{R}\) and a subgraph of \(\mathcal{P}\), the tuple \((\Omega, A|_{\Omega}, n, \mathcal{E}, \mathcal{G})\) is called flat subrepresentation of \((\mathcal{P}, A, n, \mathcal{E}, \mathcal{G})\).

We say a flat category representation \((\mathcal{R}, \hat{\mathcal{A}}, \hat{n}, \hat{\mathcal{E}}, \mathcal{G})\) is called a completion of a flat representation \((\mathcal{P}, A, n, \mathcal{E}, \mathcal{G})\), if there exists an embedding \(\varphi : \mathcal{P} \to \mathcal{R}\) such that \(\mathcal{P}\) is a fully defining partial subcategory of \(\mathcal{R}\) and \(A = \varphi \circ \hat{A}|_{\varphi(\mathcal{P}) \times \varphi(\mathcal{P})}\). \(n = (\varphi \times \varphi) \circ \hat{n}|_{\varphi(\mathcal{P}) \times \varphi(\mathcal{P})}\), and \(\mathcal{E} = (\varphi \times \varphi) \circ \hat{\mathcal{E}}|_{\varphi(\mathcal{P}) \times \varphi(\mathcal{P})}\). So that for the partial mappings \(n, \hat{n}, \mathcal{E}, \hat{\mathcal{E}}\) and the annotations \(A\) and \(\hat{A}\) the following diagrams commute.

\[
\begin{array}{ccc}
\text{Mor}_\mathcal{P} \times \text{Mor}_\mathcal{P} & \xrightarrow{\varphi \times \varphi} & \text{Mor}_\mathcal{R} \times \text{Mor}_\mathcal{R} \\
\text{Mor}_\mathcal{P} \times \text{Mor}_\mathcal{\hat{P}} & \xrightarrow{\varphi \times \text{id}} & \text{Mor}_\mathcal{R} \times \text{Mor}_\mathcal{\hat{P}} \\
\langle n \rangle & \xrightarrow{\varphi} & \langle \hat{n} \rangle \\
\mathcal{P} & \xrightarrow{A} & \hat{\mathcal{A}} \\
\mathcal{Db} & \xrightarrow{\hat{A}} & \mathcal{G}
\end{array}
\]

Obviously a flat category representation of a simple category is faithful. Consequently its flat representations are also faithful.

The largest category that can be generated by a graph is its path category. This implies that there is a homomorphism from the path category of \(\mathcal{P}\) into \(\mathcal{R}\). As the annotation of the concatenation of two arrows is the product of the annotations of both of them, it is uniquely defined. Thus, a faithful completion of a simple flat representation is the representation with the smallest possible category with respect to vertex-injective category homomorphisms that preserve the kernel of the annotation. Every category that is a non-injective image of the faithful completion must identify two arrows with different annotations.

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{id} & \mathcal{P}|_{\mathcal{A}=\mathcal{P}} & \xrightarrow{\varphi} & \mathcal{R} \\
\xrightarrow{A} & \xrightarrow{A} & \xrightarrow{A} & \xrightarrow{A} & \xrightarrow{A}
\end{array}
\]

There are other properties of representations, flat category representations and flat representations that might be useful for us. If \((\mathcal{R}, A, \mathcal{G})\) is a representation or \((\mathcal{R}, A, n, \mathcal{E}, \mathcal{G})\) is a flat representation, it is called
faithful if the annotation is faithful,
simple if the partial subcategory is simple
ordered if the category $\mathfrak{R}$ is an ordered set
$S$-symmetric if $S \leq \text{Aut} \, \mathfrak{R}$ is a transitive automorphism group of the partial
subcategory,
translatively $S$-symmetric if $S$ is a translatable automorphism group of the partial sub-
category and the representation is $S$-symmetric.
antisymmetric if $A(a)A(b) = 1$ implies $\tau a \neq \sigma b$ or $\sigma a \neq \tau b$ in $\mathfrak{R}$.
complete wrt. $\mathfrak{R}$ if there is a completion and an automorphism from the com-
pletion into the category $\mathfrak{R}$.
antisymmetrically $S$-complete If it is antisymmetric, $S$-symmetric and complete, and it is
not a proper flat subrepresentation of any antisymmetric, $S$-
symmetric and complete flat representation.

A po-group is a relational structure. Thus, we specialise on relational fundamental sys-
tems. This implies that we consider such flat representations whose category is a simple
graph: simple flat representations. In order to keep the direction information we should also
focus on antisymmetric flat representations. As in a po-group every group element acts as
an automorphism on the order relation we want this behaviour also for the factor group and
the corresponding relation.

The $S$-symmetry is of special interest for us, as this is the property which ensures that
the factorisation is structure-preserving for convex normal subgroups of po-groups and $\ell$-
groups. The factorisation of a po-group is symmetric with respect to its factor group. So
there is still some hope to preserve this property during “simplification” of the orbit category
to a simple partial subcategory.

Lemma 31 Let $\mathfrak{R} = (G, \leq)$ denote a po-group, $N < G$ a normal subgroup of $G$. Then,
every simple flat category representation $(r[\mathfrak{R}/N], A, n, \mathfrak{C}, N)$ of $\mathfrak{R}$ is translatively $G_N$-
symmetric.

Proof Recall, $G$ is translative, and so is $N$. Let $g \in G$ and $n, \hat{n} \in N$ denote three elements
of the group and its normal subgroup. All three act on $G$ as order automorphisms. For any
arrows $a \in \mathfrak{R} \sigma_{\mathfrak{R}}$ there exists a group element $\hat{n} \in N$ such that $(\sigma a)^{\hat{n}}(g^{\hat{n}}) = (\sigma a)^{(g^{\hat{n}})}$ holds.
The same is true if we exchange left and right. As either $n$ or $\hat{n}$, and – independently from
them – $\hat{n}$ can be chosen freely, this leads to the equation $(\sigma a)^{N((g^{\hat{n}}))} = (\sigma a)^{(g^{\hat{n}})}N = \sigma(a^{N(\hat{n})})$ from the corresponding set inclusions. In the same way we get the equations

$$(\tau a)^{N(g^{\hat{n}})} = \tau(a^{g^{\hat{n}}})$$

and

$$a^{N(g^{\hat{n}})} \ast b^{N(g^{\hat{n}})} = (a \ast b)^{g^{\hat{n}}}$$

if $a \ast b$ exists.

Consequently the induced action of $G_N$ is an automorphism action on $\mathfrak{R}/N$.

As $r[\mathfrak{R}/N]$ is a simple category, applying $r$ commutes with the action of $G_N$, which
shows that $G_N$ acts regular on $r[\mathfrak{R}/N]$. So this is a translative group action. \hfill \Box

This allows us to introduce the well-known factorisation results from classical theory of
po-groups into this framework. First we want to mention the fact, that for every po-group
$\mathfrak{R} = (G, \leq)$ and every of its convex normal subgroups $N < G$ the orbit category $\mathfrak{R}/N$
is isomorphic to the factor group $G_N$, when the latter is considered as a category. This
isomorphism is fully defined by its restriction to the objects.
**Theorem 4** The factorisation of a po-group \( R = (G, \leq) \) by a convex normal subgroup \( \mathbb{N} \) together with the natural annotation give rise to a faithful translative antisymmetrically \( G_{\mathbb{N}} \)-complete flat representation of \( R \).

**Proof** The factorisation and its natural annotation define the representation \( (R/\mathbb{N}, A_{\mathbb{N}}, \mathbb{N}) \).

It is complete, shown by the unfolding, it is antisymmetric, the underlying simple category of the partial subcategory is an ordered set, which we know from the theory of po-groups. Consequently the representation is anti-symmetrical and faithful. Finally it is translative \( G_{\mathbb{N}} \)-symmetric as shown in the previous lemma. \( \square \)

Given a category \( R \), its partial subcategories together with the canonical embeddings form an ordered set \( (P, \leq) \), actually a complete meet-semilattice. The partial categories which contain all vertices form a proper subset \( (P_V, \leq |P_V) \) of \( (P, \leq) \). As the discrete category is always a partial subcategory, there are simple partial categories among the elements of \( P_V \).

When we fix the annotation \( A \) and the two partial mappings \( n \) and \( c \), we can order the set of flat representations of a given representation according to the order relation \( \leq |P_V \) of the partial subcategory of the representations. We will call this order induced order in this context. In order to find the best candidates of flat representations, we can analyse this ordered set if we find some maximal or minimal flat representations.

**faithfulness** If a mapping is faithful, its restriction to a subset is also faithful. Thus the faithful flat representations form an order ideal. If a non-faithful flat representation has a faithful flat subrepresentation, this can happen only by removing conflicting arrows. Consequently the union of a chain of faithful representations is also faithful, which implies that there are maximal faithful flat representations.

**simplicity** Obviously the simple flat representations form an order ideal, too. The union of the partial categories of a chain of simple flat representations is simple. So also the ideal of simple flat representations has maximal elements.

**ordering** The intersection of a set of ordered sets is an ordered set. Thus, the ordered flat representations together with the maximal flat representation form a closure system.

**S-symmetric** The discrete category is also symmetric with respect to the symmetric group of its vertices. Given a fixed permutation group \( S \), \( S \)-symmetric partial categories below a given \( S \)-symmetric (partial subcategory) form a closure system: The intersection of a set of \( S \)-symmetric partial subcategories is always also \( S \)-symmetric. Consequently the simple \( S \)-symmetric partial categories together with the maximal flat representation form a closure system.

**translatively \( S \)-symmetric** As the action on the vertices is fixed the intersection of a set of translatively \( S \)-symmetric partial categories is still translatively \( S \)-symmetric. Thus the translatively \( S \)-symmetric flat representations form a closure system.

**antisymmetric** The antisymmetric partial subcategories form an order ideal. When we consider a chain in this ideal, its union is still antisymmetric.

**complete wrt. \( R \)** If a flat representation is not complete with respect to a category \( R \), then all its flat subrepresentations are also not complete. So the complete subrepresentations form an order filter.

**antisymmetrically \( S \)-complete** If there exist antisymmetrically \( S \)-complete flat representations they have maximal elements.

So the set of (simple) faithful antisymmetric translatively \( G_{\mathbb{N}} \)-complete flat representations of \( (G, \leq) \) is either empty or has maximal elements.
We call a maximal faithful, antisymmetric, translatively $\mathbb{G}_{\mathbb{N}}$-complete flat representation cyclically fundamental.

This catalogue of features has the advantage, that different structures can be chosen depending on the intended use. For completeness, we should also consider approaches as the abridged annotation introduced in [1], which considers the neighbourhood relation (also: Hasse relation) of a preordered set. However, this does not necessarily exist for every po-group.

10 Further results and applications to music theory

This article provides a fundamental construction for a certain type of factorisations of po-groups and similar small categories into orbits of certain automorphism groups. This is a starting point for further investigations. For example for the mathemusical applications it is helpful to describe group extensions by means of multiple annotations of (flat) representation. This is available as unpublished result by the author.

There are examples of factorisations of po-groups that do not permit simple flat representations. There exist ideas to characterise the existence of such representations by means of similarity or generalised neighbourhood relations.

The structures introduced in this article are inspired by ideas published in [7]. For simple small categories there exists a direct mapping between our structures and the ones developed by Monika Zickwolff.

The class of representations can be equipped with morphisms in a similar way as the isomorphisms are defined in [11, 1] and Definition 15, leading to the category of representations. Unfolding isomorphic representations leads to isomorphic groups.

Furthermore there are different applications of representations when it comes to fundamental systems of tone structures. The following examples are based on the notion of a tone system: A set of tones, equipped with a group of differences, called intervals. For details we refer to [12, 13, 14].

10.1 Chroma Systems

Chroma intervals can be considered a factorised interval group. They can be used to describe the cyclic notion of the intervals of a tone system in music theory. Examples include:

- The physical frequency or physical frequency space. The pitch of a tone can be roughly described as a frequency or the period time of a periodic vibration. Both lead to the po-group $(\mathbb{R}, \cdot, -1, \leq)$, though they are dually ordered. For many music theoretical considerations this po-groups can be factored by the octave relation, which leads to a representation that is isomorphic to $((\mathbb{R}, \cdot, \leq)/2\mathbb{Z}, o, 2\mathbb{Z})$ with $o(x) := 2^\lfloor \log_2 x \rfloor$, which is isomorphic via the logarithm to $((\mathbb{R}, \leq)/\mathbb{Z}, o', \mathbb{Z})$ with $o'(x) := \lfloor x \rfloor$.

- The Shepard tones [5] with their intervals and the relation that expresses that one tone $t_1$ is more likely perceived lower than another tone $t_2$ form an application of a flat representation $(\mathcal{S}, o', n', \mathcal{C}', (\mathbb{Z}, \leq))$ of $(\mathbb{R}/\mathbb{Z}, o', \mathbb{Z})$ with $\mathcal{O}b \mathcal{S} = \mathcal{O}b (\mathbb{R}, \leq)/\mathbb{Z}$ and $\mathcal{M}or \mathcal{S} (t_1, t_2) = \{ (x, y) \in \mathcal{M}or (\mathbb{R}, \leq)/\mathbb{Z} (x, y) \mid 0 \leq t_2 - t_1 < 1/2 \}$.

- The $n$-tone equal temperament ($n$-TET) is often represented as a flat representation of $(\mathbb{Z}, \leq)$ via the orbit category $(\mathbb{Z}, \leq)/n\mathbb{Z}$. In contrast to other temperaments it can often be modelled as a $\mathbb{Z}_{n'}$-symmetric flat representation.
Musical scales can be modelled as a (non-necessary symmetric) $m$-TET and a mapping into a $\mathbb{Z}_n$-symmetric $n$-TET. This mapping is usually not a homomorphism as the group structure is not preserved. However the category structure from the scale can be reconstructed from the category structure of the enclosing $n$-TET tone system. For example the diatonic scale is often be considered as a $\mathbb{Z}_7$-symmetric 7-TET that is embedded into a $\mathbb{Z}_{12}$-symmetric 12-TET.

- The torus of chromas is a 2-dimensional chroma system that describes the Tonnetz of major thirds and fifths. A natural model for it would be an antisymmetrically $\mathbb{Z}_1 \times \mathbb{Z}_{12} \times \mathbb{Z}_3$-complete flat representation

\[ \langle \mathcal{P}, A, n, \mathcal{E}, (\mathbb{Z}, \leq) \times 12(\mathbb{Z}, \leq) \times 2(\mathbb{Z}, \leq) \rangle \]

where $\mathcal{P}$ is a partial subcategory of

\[ \langle (\mathbb{Z}, \leq) \times (\mathbb{Z}, \leq) \times (\mathbb{Z}, \leq) \rangle / (\mathbb{Z} \times 12\mathbb{Z} \times 3\mathbb{Z}). \]

- Musical events have finite durations. When we describe them, we have the properties onset, duration and end. This allows to represent them as points in a 2-dimensional subspace of a 3-dimensional space. One possibility is to represent each event by onset and duration. The typical operations are translation of the onset and stretching a passage by a certain time factor. Both together form the group $AGL_1(\mathbb{R})$ of affine operations on the real line. As each event could be generated from $(1, 1)$ by one of the affine operations, we can model the space of musical events as $\langle AGL_1(\mathbb{R}), \leq \rangle$ where $\leq$ is a lexicographic product order, in which the duration is infinitesimal with respect to the onset. So the rhythmic space is an example where non-commutative po-groups play an important role in music theory. Typical rhythmic patters can come from the rhythm itself, or from an external structure like measures. The latter can be divided into parts or grouped into larger ensembles (typical groups consist of 3, 4, 6 or 8 measures). These generate subgroups of the translational subgroup of $AGL_1(\mathbb{R})$. Rhythmic patterns can be described as subsets in the orbit category of $AGL_1(\mathbb{R})$ by one of these subgroups.

Chroma systems describe the aspect of being a factor structure of a tone system. In this sense they cannot distinguish between different diatonic modes.

### 10.2 Fundamental Systems in the Narrow Sense

Given a partition on a set. A transversal is a set that contains exactly one element of each class of the partition. Given a tone system and a normal subgroup of its interval group that acts interval preserving on the tone system. In that case we can consider a transversal of the orbits of the tones together with a sufficient set of positive intervals as a scale. Such a fundamental system can be considered as a tile that can be repeated in all directions in order to generate the tone system.

Examples:

- Special cases of the diatonic scale as C-major or d-minor scales are often considered using partial categories which are not symmetric, and where all arrows start in one tone (the reference tone). This gives rise to the modern view on the so called „church modes“.
- Leitton in chroma systems
- Dominant sevenths chords in chroma systems
In Jazz music the notion “scale” is linked with the chroma supply that may be used in improvisation in a certain tonality. Jazz scales are constructed using two complementing orders: the order in the physical pitch space, which lead to a chroma system and harmonic orderings based on the tonality.

Fundamental systems tend to have a larger variety of intervals than chroma systems. For example, The diatonic C major mode in 12-tet has a minor 7th as interval in its fundamental system. This interval is reduced to a negative major second in the corresponding chroma system.

10.3 Extended Scales

Fundamental systems as discussed above contain information about intervals. In a tone system two tones are identical if they have the same interval to common third tone. This will not happen if we reconstruct a tone system from fundamental system as in section 10.2. But, why two tiles must not overlap? In the real world overlapping tiles can form a strong connection when glued together. Actually what we call gluing in mathematics can be better compared to welding in real live. Nevertheless, it is easy to define a binary relation $\sigma$ according to the rules

$$t_1 \sigma t_2 \Leftrightarrow \exists t_3 : \delta(t_1, t_3) = \delta(t_2, t_3).$$

It is easy to prove that $\sigma$ is an equivalence relation. Factorisation by $\sigma$ leads to the desired result.

In musical discussions sometimes scale steps beyond 7 occur. While the octave can still be considered as a representation of the prime or first scale step, the numbers 9 and above tones that are different from their equivalent scale steps below 8. However, these representations live in very small parts of musical compositions (often only one chord). Some short time later the same notes may be described by different numbers. This suggests that music theorists consider the tone system to be generated by substructures that cannot be described by transversals of the orbits. As they still generate the whole tone system the “fundamental” system must generate overlapping tiles which are glued together during composition or performance.

11 Further research topics

Some questions have already been raised in the previous sections. They are not repeated, here.

So far flat representations depend on categories that are uniquely representable by minimal arrows. In case of po-groups this implies that the order must be Archimedian. However, if the unfolding also reconstructs infinitesimal elements, this condition can be dropped. As the order relation of the positive cone is dually isomorphic to the one of the negative cone via the group inversion antiautomorphism, the relevant information is already encoded in the vertex category. It is an open question how this information can be made accessible for sufficiently simple algorithms.

In a po-group each principal ideal of an element is dually isomorphic to the principal filter of the same element and with respect to incoming and outgoing arrows. The order filters of a po-group form a monoid where the positive cone is the neutral element. The
elementwise product of two order filters is again an order filter. Together with the subset relation they form a monoidal category.

Actually the vertex annotation of a representation of an \( \ell \)-group can be understood as the defining elements of principal order filters in the vertex categories. This gives an order relation on the arrows between two vertices in the orbit category. It can be used to define flat representations based on valued categories on the category of order filters in the vertex category. So, flat representations of arbitrary po-groups can be considered. As long as we have no parallel arrows with the same annotation in the representation this setting can be easily extended to the case where the vertex categories are right groupal categories. If the restriction of parallel arrows with the same annotation can be lifted this type of annotation would be an analogy to the annotation as described in [7].

The order filters of a cancelable po-monoid together with the set inclusion morphisms and the block operation form a simple monoidal category \( OF \). Thus, flat representations could be modelled based on \( OF \)-valued categories.

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