DIVISORS ON MATROIDS AND THEIR VOLUMES

CHRISTOPHER EUR

ABSTRACT. The classical volume polynomial in algebraic geometry measures the degrees of ample (and nef) divisors on a smooth projective variety. We introduce an analogous volume polynomial for matroids, and give a complete combinatorial formula. For a realizable matroid, we thus obtain an explicit formula for the classical volume polynomial of the associated wonderful compactification. We then introduce a new invariant called the volume of a matroid as a particular specialization of its volume polynomial, and discuss its algebro-geometric and combinatorial properties in connection to graded linear series on blow-ups of projective spaces.

1. Introduction

The volume polynomial in classical algebraic geometry measures the self-intersection number of a nef divisor, or equivalently the volume of its Newton-Okounkov body. Here we define analogously the volume polynomial \( VP_M \) for a matroid \( M \). Let \( M \) be a matroid on a ground set \( E \) with lattice of flats \( \mathcal{L}_M \), and denote \( \overline{\mathcal{L}}_M : = \mathcal{L}_M \setminus \{ \emptyset, E \} \). Recall the definition of the Chow ring of a matroid:

**Definition 1.1.** The Chow ring of a simple matroid \( M \) is the graded ring

\[
A^\bullet(M) := \langle x_F x_{F'} : F, F' \text{ incomparable} \rangle + \langle \sum_{i,j} x_F - \sum_{G \supset j} x_G | i, j \in \overline{\mathcal{L}}_M \rangle
\]

In analogy to Chow rings in algebraic geometry, we call elements of \( A^1(M) \) divisors on a matroid \( M \). Recall that \( A^\bullet(M) \) satisfies Poincaré duality with the degree map

\[
\deg_M : A^{rkM-1}(M) \sim \mathbb{R}, \quad \text{where } \deg_M(x_{F_1} x_{F_2} \cdots x_{F_d}) = 1
\]

for every maximal chain \( F_1 \subset F_2 \subset \cdots \subset F_d \) in \( \overline{\mathcal{L}}_M \).

In other words, \( A^\bullet(M) \) is an Artinian Gorenstein \( \mathbb{R} \)-algebra, so that Macaulay’s inverse system gives a well-defined cogenerator \( VP_M \) of \( A^\bullet(M) \).

**Definition 3.1.** Let \( M \) be a matroid of rank \( r = d + 1 \). The volume polynomial \( VP_M(t) \in \mathbb{R}[t_F : F \in \overline{\mathcal{L}}_M] \) is the cogenerator of \( A^\bullet(M) \), where \( VP_M(t) \) is normalized so that the coefficient of any monomial \( t_{F_1} t_{F_2} \cdots t_{F_d} \) corresponding to a maximal chain of flats in \( \overline{\mathcal{L}}_M \) is \( d! \).

When \( M \) is realizable, \( VP_M \) agrees with the classical volume polynomial of the wonderful compactification \( Y_M \) of the complement of the associated hyperplane arrangement \( \mathcal{A}_M \) of \( M \). While \( VP_M \) is initially defined purely algebraically, we prove a completely combinatorial formula for \( VP_M \), which follows from our first main theorem on the intersection numbers of divisors on a matroid.

**Theorem 3.2.** Let \( M = (E, \mathcal{B}) \) be a matroid, \( \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E \) a chain of flats in \( \mathcal{L}_M \) of ranks \( r_i := \text{rk} F_i \), and \( d_1, \ldots, d_k \) be positive integers such that \( \sum_i d_i = d := \text{rk} M - 1 \). Denote by \( \vec{d}_i := \sum_{j=0}^{i} d_j \) (where \( d_0 := 0 \)). Then

\[
\deg(x_{F_1}^{d_1} \cdots x_{F_k}^{d_k}) = (-1)^{d-k} \prod_{i=1}^{k} \left( \frac{d_i - 1}{d_i - r_i} \right) \mu^{\vec{d}_i - r_i}(M|F_{i+1}/F_i)
\]

where \( \mu^i(N) \) denotes the \( i \)-th unsigned coefficient of the reduced characteristic polynomial \( \chi_N(t) = \mu^0(N) t^{rkN-1} - \mu^1(N) t^{rkN-2} + \cdots + \mu^{rkN-1}(N) \) of a matroid \( N \).
Corollary 3.3. Let the notations be as above. The coefficient of \( t_1^{d_1} \cdots t_k^{d_k} \) in \( VP_M(t) \) is

\[
(-1)^{d-k} \left( \frac{d}{d_1, \ldots, d_k} \right) \prod_{i=1}^{k} \left( \frac{d_i - 1}{d_i - r_i} \right) \mu^{d_i - r_i}(M[F_{i+1}/F_i]).
\]

As a first application, we give an explicit formula for the volumes of generalized permutohedra, adding to the ones given by Postnikov in [Pos09].

Proposition 4.4. Let \( z_\cdot : [n] \supset I \mapsto z_I \in \mathbb{R} \) be a submodular function on the boolean lattice of subsets of \([n] := \{1, \ldots, n\}\). Then the volume of the generalized permutohedron \( P_\cdot(z) = \{ (x) \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = z|_n, \sum_{i \in I} x_i \leq z_I \forall I \subset [n] \} \) is

\[
(n-1)! \cdot Vol(P(z)) = \sum_{I \star, \mathcal{d}} (-1)^{d-k} \left( \frac{d}{d_1, \ldots, d_k} \right) \prod_{i=1}^{k} \left( \frac{d_i - 1}{d_i - |I_i|} \right) \left( \frac{|I_{i+1}| - |I_i| - 1}{\tilde{d}_i - |I_i|} \right) z_I
\]

where the summation is over chains \( \emptyset \subset I_1 \subset \cdots \subset I_k \subset I_{k+1} = [n] \) and \( \mathcal{d} = (d_1, \ldots, d_k) \) such that \( \sum d_i = n-1 \) and \( \tilde{d}_j := \sum_{i=1}^{j} d_i \).

While the formula given here seems to be a “cancellation-free” version of the formula given in [Pos09, Corollary 9.4], it was not clear to the author how the two formulas are related.

Question 4.4. Can one derive the formula in Proposition directly from [Pos09, Corollary 9.4], or vice versa?

While the volume polynomial \( VP_M \) has the same information as the Chow ring \( A(M) \) of a matroid, it lends itself more naturally as an invariant of a matroid than the Chow ring. As an illustration, by considering \( VP_M(t) \) as a polynomial in \( \mathbb{R}[t_S : S \in 2^E] \), we show that the map \( M \mapsto VP_M \) is a valuation under matroid polytope subdivisions.

Proposition 4.6. \( M \mapsto VP_M \) is a (non-additive) matroid valuation in the sense of [AFR10].

That the volume polynomial of a matroid behaves well with respect to the matroid polytope and that the matroid-minor Hopf monoid structure arises in the proof of the proposition above suggest there may be a generalization of Hodge theory of matroids to arbitrary Lie type.

Question 4.8. Is there a Hodge theory of Coxeter matroids, generalizing the Hodge theory of matroids as described in [AIHK18]؟ (For Coxeter matroids, see [BGW03]).

Moreover, any natural specialization of \( VP_M \) defines an invariant of a matroid. In our case, we consider specializing \( VP_M \) via the rank function of a matroid to obtain a new invariant.

Definition 5.1. For a matroid \( M \), define its distinguished nef divisor \( D_M \) to be

\[
D_M := \sum_{F \in \mathcal{F}_M} (rk F)x_F,
\]

and define the volume of a matroid \( M \) to be the volume of its distinguished nef divisor:

\[
Vol(M) := VP_M(t_F := rk F) = \deg \left( \sum_{F \in \mathcal{F}_M} (rk F)x_F \right)^{rk M - 1}.
\]

The volume of a matroid seems to be a genuinely new invariant, as it is unrelated to classical invariants such as the Tutte polynomial or the volume of the matroid polytope; see Remark 5.3. For realizable matroids, the volume of a matroid measures how close the matroid is to the uniform matroid.

Theorem 5.5. Let \( M \) be a realizable matroid of rank \( r \) on \( n \) elements. Then

\[
Vol(M) \leq Vol(U_{r,n}) = n^{r-1} \quad \text{with equality iff } M = U_{r,n}.
\]
The primary tool for the proof of Theorem 5.5 is algebro-geometric in nature, relying on the realizability of the matroid. Computational evidence seems to show that the theorem might hold for general matroids.

**Conjecture 5.8.** The maximum volume among matroids of rank $r$ on $n$ elements is achieved uniquely by the volume of the uniform matroid $\text{Vol}(U_{r,n}) = n^{r-1}$.

Trying to apply similar method as in the proof of Theorem 5.5 for the non-realizable matroids naturally leads to the following question.

**Question 5.9.** Is there a naturally associated convex body for which this volume polynomial is measuring the volume of? In other words, is there a theory Newton-Okounkov body for general matroids (a.k.a. linear tropical varieties)?

**Remark 1.2.** Recent works on the Chow ring of matroids have led to the resolution of the longstanding conjecture of Rota on the log-concavity of the coefficients of chromatic polynomials, first proven for realizable matroids in [Huh12] and [HK12], and for general matroids in [AHK18]. Among the key tools in [Huh12] is the Teissier-Khovanskii inequality for intersection numbers of nef divisors, which can be understood as a phenomenon of convexity: The Newton-Okounkov body $\Delta(D)$ of a nef divisor $D$ is a convex body whose volume is the self-intersection number of $D$, and its existence reduces the Teissier-Khovanskii inequality to the Brunn-Minkowski inequality for volumes of convex bodies.

For general matroids, a combinatorial version of Teissier-Khovanskii inequality is proven in [AHK18] by establishing Hodge theory analogues for the Chow ring of a matroid without explicit use of convex bodies. Noting that matroids can be considered as tropical linear varieties ([AK06]), the results of [AHK18] suggest an existence for an analogue of Newton-Okounkov bodies for tropical linear varieties, and perhaps tropical varieties in general. Our results here can be seen as a first step towards such a direction.

On the flip side of maximal volumes, we have the following conjecture on the minimal values.

**Conjecture 5.10.** The minimum volume among simple matroids of rank $r$ on $n$ is achieved uniquely by the matroid $U_{r-2,r-2} \oplus U_{2,n-r+2}$, and its volume is $r^{r-2}((n-r+1)(r-1)+1)$.

**Structure of the paper.** In section §2, we review relevant definitions and results about Chow rings of matroids and volume polynomials in algebraic geometry. In section §3, we define the volume polynomial of a matroid and give a combinatorial formula. First applications of the volume polynomial is discussed in section §4, where we give another formula for the volumes of generalized permutohedra and show that the volume polynomial is a matroid valuation. In section §5, we define the volume of a matroid as a particular specialization of the volume polynomial, and analyze some of its algebro-geometric and combinatorial properties.

A Macaulay2 file volumeMatroid.m2, implementing many of notions here, and various worked out examples can be found at https://math.berkeley.edu/~ceur/research.html.

**Notations.** $|S|$ denotes the cardinality of a (finite) set $S$. We use $k$ for a field, which we always assume algebraically closed, and a variety is a reduced, irreducible, and separated scheme over an algebraically closed field $k$. A binomial coefficient $\binom{n}{m}$ is understood to be zero if $m < 0$ or $m > n$.

2. Preliminaries

In this section, we set up relevant notations and review previous results about Chow rings of matroids, wonderful compactifications, volume polynomials, and cogenerators in the classical setting.
Wonderful compactifications and Chow rings of matroids. See [Oxl11] for a general reference on matroids. For accounts tailored towards Chow ring of matroids, we recommend [Bak18] or [Kat16].

For a matroid \( M = (E, B) \) on a ground set \( E \) with basis \( B \), denote by \( \mathcal{L}_M \) the lattice of flats, \( r \) the rank of the matroid, and \( d := r - 1 \). An open interval in a lattice \( \mathcal{L} \) is denoted \( (\ell_1, \ell_2) = \{ \ell \in \mathcal{L} \mid \ell_1 < \ell < \ell_2 \} \), and denote by \( \overline{\mathcal{L}} := \mathcal{L} \setminus \{0, 1\} \). Because the invariants of a matroid we consider in this paper—Chow ring, volume polynomial, and volume—only depend on the lattice of flats, we often assume for simplicity that the matroid is simple \(^1\).

Let \( M \) be a simple matroid. Recall that the Bergman fan \( \Sigma_M \) of a matroid \( M \) is a compactification of the hyperplane arrangement complement \( \mathcal{L}_M \). For a matroid \( M \) realizable, say as vectors spanning a \( k \)-vector space \( V \), the Bergman fan \( \Sigma_M \) is comprised of cones
\[
\sigma_{\mathcal{F}} := \mathrm{Cone}(e_{F_1}, e_{F_2}, \ldots, e_{F_k}) + \mathbb{R}1 \subset (\mathbb{Z}^E/\mathbb{Z}1)_{\mathbb{R}}
\]
for each a chain of flats \( \mathcal{F} : F_1 \subset \cdots \subset F_k \) in \( \mathcal{L}_M \), where \( e_S \) for \( S \subset E \) denotes \( e_S := \sum_{i \in S} e_i \).

If the matroid \( M \) matroid is realizable, say as vectors spanning a \( k \)-vector space \( V \), denote by \( A_M \) the associated hyperplane arrangement in \( \mathbb{P}(V^*) \). The hyperplanes in \( A_M \subset \mathbb{P}(V^*) \) are given by \( H_i := \{ f \in \mathbb{P}(V^*) : f(v_i) = 0 \} \) when \( M \) is simple, and more generally a flat \( F \) of \( M \) correspond to \( c \)-codimensional planes \( H_F := \{ f \in \mathbb{P}(V^*) : f(v_i) = 0 \ \forall v_i \in F \} = \bigcap_{v_i \in F} H_i \).

The wonderful compactification \( Y_M \) of a realizable matroid \( M \) is then obtained as a blow-up of \( \mathbb{P}(V^*) \) in the following way: one first blows-up the points \( \{H_F\}_{rk(F)=r-1} \), then the strict transforms of the lines \( \{H_{F'}\}_{rk(F')=r-2} \), then the strict transforms of the planes, and so forth. \( Y_M \) is a compactification of the hyperplane arrangement complement \( C(A_M) = \mathbb{P}(V^*) \setminus A_M \) whose boundary \( Y_M \setminus C(A_M) \) consists of the exceptional divisors \( \widetilde{H}_F \), which have simple-normal-crossings. See [DCP95] for the original construction, or [Fei05] for a survey geared towards combinatorialists. The intersection theory of the boundary divisors is encoded in the matroid, which leads to the definition of the Chow ring of a matroid.

**Definition 2.1.** The Chow ring of a simple matroid \( M \) is the graded ring
\[
A^\bullet(M) := \mathbb{Z}[x_F : F \in \overline{\mathcal{L}_M}] / I
\]
where the ideal \( I \) is generated by the quadrics \( x_F x_{F'} \), for each \( F, F' \) incomparable in \( \mathcal{L}_M \) and by the linear relations \( \sum_{F \ni i} x_F = \sum_{G \ni j} x_G \) for each \( i, j \in E \). As with algebraic geometry, we call elements of \( A^1(M) \) *divisors* on \( M \).

Among the first places where \( A^\bullet(M) \) appears is [FY04] as a combinatorial abstraction of the Chow rings of wonderful compactifications. Recently in [AHK18], the ring \( A^\bullet(M)_\mathbb{R} := A^\bullet(M) \otimes \mathbb{R} \) has been shown to satisfy the whole Kähler package—Poincaré duality, hard Lefschetz property, and Hodge-Riemann relations—which led to the proof of Rota’s conjecture on the log-concavity of the unsigned coefficients of the characteristic polynomial of a matroid in the same paper. For our purposes, we only need the Poincaré duality.

**Proposition 2.2.** [AHK18, 5.10] The Chow ring \( A^\bullet(M)_{\mathbb{R}} \) of a matroid \( M \) of rank \( r = d + 1 \) is a finite graded \( \mathbb{R} \)-algebra satisfying:

(i) There exists a linear isomorphism \( \deg_M : A^d(M) \to \mathbb{R} \) uniquely determined by the property that \( \deg_M(x_{F_1} x_{F_2} \cdots x_{F_d}) = 1 \) for every maximal chain \( F_1 \subset \cdots \subset F_d \) in \( \overline{\mathcal{L}_M} \), and

(ii) The pairings \( A^i(M) \times A^{d-i}(M) \to A^d(M) \) of \( \deg \) are non-degenerate.

\(^1\)Most statements and proofs given here for simple matroids will work for general matroids just by changing the phrase “for \( i \in E \)” to “for \( i \) a rank 1 flat.” and letting \( |S| \) to mean the number of atoms in \( S \subset E \) rather than the cardinality of \( S \).
Remark 2.3. Noting that the Bergman fan $\Sigma_M$ of a realizable matroid $M$ is the tropicalization of a (very affine) linear variety ([AK06]), the wonderful compactification $Y_M$ can also be realized as a tropical compactification where we take the closure of $C(A_M)$ in the toric variety $X_{\Sigma_M}$ of the Bergman fan. The inclusion $Y_M \hookrightarrow X_{\Sigma_M}$ induces a Chow equivalence $A^*(Y_M) \simeq A^*(X_{\Sigma_M})$, and the Chow ring of $X_{\Sigma_M}$ following [Dan75] and [Bri96] satisfy $A^*(X_{\Sigma_M})_Q \simeq A^*(M)_Q$. For reference on tropical compactifications see [MS15] or [Den14]. This perspective will be useful for computing the combinatorial formula for the volume polynomial in section §3.

Notation. We will work with $A^*(M)_{\mathbb{R}}$ rather than $A^*(M)$ most of the time, and so by “Chow ring of a matroid” we often mean $A^*(M)_{\mathbb{R}}$.

Volumes of divisors and cogenerators. For general reference on intersection theory, see [EH16] and [Laz04]. Here we mostly follow the survey [ELM+05].

Let $X$ be a $d$-dimensional smooth projective variety over an algebraically closed field $k$ of dimension $n$, and let $A^*(X)$ be its Chow ring and denote by $\deg_X$ or $\int_X$ the degree map $A^d(X) \to \mathbb{Z}$ sending a class of a closed point to 1. For a Cartier divisor $D$ on $X$, the volume of $D$ is defined as

$$\text{vol}(D) := \lim_{t \to \infty} \frac{h^0(X, tD)}{t^d/d!}.$$ 

In other words, denoting by $R(D)_\bullet := \bigoplus_{t \geq 0} R^0(X, tD)$ the section ring of $D$, the volume measures the asymptotics of $\frac{\dim_k(R(D)_t)}{t^d/d!}$ as $t \to \infty$.

If $D$ is ample, then $\text{vol}(D) > 0$. By standard relation between Hilbert polynomials and intersection multiplicities, volume of an ample divisor can be geometrically interpreted as follows: If $m >> 0$ is such that $mD$ is very ample, then for general divisors $E_1, \ldots, E_d$ in the complete linear system $|mD|$ we have $\text{vol}(D) = \frac{1}{m^d} \deg_X [E_1 \cap E_2 \cap \cdots \cap E_d]$. In other words, $\text{vol}(D) = \int_X (c_1(D))^d$ if $D$ is ample.

The volume of a divisor depends only on its numerical equivalence class. Thus, letting $N^1(X)_{\mathbb{R}}$ be the group divisors modulo numerical equivalence generated by $\{\xi_1, \ldots, \xi_r\}$ and $\text{Nef}(X) \subset N^1(X)_{\mathbb{R}}$, the nef cone, the map $\text{vol} : \text{Nef}(X) \to \mathbb{R}$ defines the volume polynomial $VP_X \in \mathbb{R}[t_1, \ldots, t_r]$ where

$$VP_X(t_1, \ldots, t_r) = \text{vol}(t_1\xi_1 + \cdots + t_r\xi_r)$$

whenever $t_1\xi_1 + \cdots + t_r\xi_r \in \text{Nef}(X)$.

Macaulay’s inverse system and more generally the Matlis duality provide a purely algebraic approach to the notion of volume polynomial as the dual socle generator of an Artinian Gorenstein ring. We sketch the connection here and refer to [BH93] for details.

For a graded (or local complete) Noetherian ring $S$ with the residue field $k$ and its injective hull $E(k)$, the Matlis duality establishes a bijection

$$\{\text{Artinian ideals } I \subset S\} \leftrightarrow \{\text{Noetherian } S\text{-submodules of } E(k)\}$$

via $I \mapsto \text{Hom}_S(S/I, E(k)) = (0 :_E I)$. When $S/I$ is Artinian and Cohen-Macaulay, the bijection interchanges the type $r(S/I) := \dim_k \text{Hom}_S(k, S/I)$ and the minimal number of generators $\mu((0 :_E I))$. In particular, if $S/I$ is Artinian Gorenstein, then $(0 :_E I)$ is generated by a single element called the cogenerator or dual socle generator of $I$.

When $\text{char } k = 0$ and $S = k[x_0, \ldots, x_n]$ the standard graded polynomial ring, its injective hull is $E = k[\partial_0, \ldots, \partial_n]$ where $S$ acts on $E$ by $f \cdot \partial_i := \frac{\partial f}{\partial x_i}$. The following proposition then shows the equivalence of the cogenerator and the volume polynomial when the Chow ring $A(X)_Q$ is an Artinian Gorenstein ring.

Proposition 2.4. [CLS11, 13.4.7] Suppose a graded finite $k$-algebra $A = \bigoplus_{i=0}^d A_i$ satisfies the following:
(i) $A$ is generated in $A_1$, with $A_0 = k$,
(ii) there exists a $k$-linear isomorphism $\deg : A_d \to k$, and
(iii) $A_i \times A_{d-i} \to A_d \simeq k$ is a non-degenerate pairing.

Let $x_1, \ldots, x_n$ generate $A_1$, so that $A \simeq k[x]/I$ for some ideal $I$. Then there exists $P \in k[t_1, \ldots, t_n]$ such that

$$I = \{ f \in k[x] \mid f(\partial t_1, \ldots, \partial t_n) \cdot P = 0 \}.$$ 

Up to scaling by an element of $k$, this cogenerator is $\deg \left( (t_1 x_1 + \cdots + t_n x_n)^d \right)$ (where we extend $\deg : A_d \to k$ to $A_d[y_1, \ldots, y_n] \to k[t_1, \ldots, t_n]$).

**Remark 2.5.** In toric geometry, the volume of an ample divisor is realized as a volume of a rational convex polytope; for details see [CLS11, §9.13]. Moreover, recently this phenomenon of realizing the volume of a divisor as a volume of a convex body was extended to arbitrary smooth complete varieties where the Newton-Okounkov bodies take the place of the rational convex polytopes. See [KK12] for an approach using semigroups and with a view towards Alexandrov-Fenchel inequality and generalized Kushnirenko-Bernstein theorem. For an application with more representation theoretic flavor, see [Kav11] on volume polynomials and cohomology rings of spherical varieties.

### 3. The Volume Polynomial of a Matroid

As the Chow ring $A^*(M) \mathbb{R}$ of a matroid satisfies Poincaré duality, i.e. the conditions of Proposition 2.4, it is an Artinian Gorenstein algebra with a cogenerator, well-defined up to scaling by $k$.

**Definition 3.1.** Let $M$ be a matroid of rank $r = d+1$. The volume polynomial $VP_M \in \mathbb{R}[I_F : F \in \mathcal{L}_M]$ is the cogenerator of $A^*(M) \mathbb{R}$, where $VP_M$ is normalized so that the coefficient of any monomial $t_{F_1} t_{F_2} \cdots t_{F_d}$ corresponding to a maximal chain of flats in $\mathcal{L}_M$ is $d!$.

Equivalently, via Proposition 2.4 the volume polynomial is $VP_M = \deg_M \left( \left( \sum_{F \in \mathcal{L}_M} x_F t_F \right)^d \right)$ (where $\deg_M : A^d(M) \to \mathbb{R}$ is extended to $A^d[t_F's] \to \mathbb{R}[t_F's]$).

The coefficient of $t_{F_1}^{d_1} \cdots t_{F_k}^{d_k}$ for $d_1 + \cdots + d_k = d = \text{rk} M - 1$ in the volume polynomial $VP_M$ is $\deg_M(x_{F_1}^{d_1} \cdots x_{F_k}^{d_k})$. Thus, the knowing the volume polynomial amounts to knowing all the intersection numbers $\deg_M(x_{F_1}^{d_1} \cdots x_{F_k}^{d_k})$. The main theorem in this section is the combinatorial formula for all the intersection numbers.

**Theorem 3.2.** Let $M = (E, \mathcal{B})$ be a matroid, $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E$ a chain of flats in $\mathcal{L}_M$ of ranks $r_i := \text{rk} F_i$, and $d_1, \ldots, d_k$ be positive integers such that $\sum_i d_i = d := \text{rk} M - 1$. Denote by $d_i := \sum_{j=0}^i d_j$ (where $d_0 := 0$). Then

$$\deg_M(x_{F_1}^{d_1} \cdots x_{F_k}^{d_k}) = (-1)^{d-k} \prod_{i=1}^k \left( \frac{d_i - 1}{d_i - r_i} \right) \mu^{d_i-r_i}(M|F_{i+1}/F_i)$$

where $\mu^i(N)$ denotes the $i$-th unsigned coefficient of the reduced characteristic polynomial $\overline{\chi}_N(t) = \mu^0(N) t^{rkN-1} - \mu^1(N) t^{rkN-2} + \cdots \pm \mu^{rkN-1}(N)$ of a matroid $N$.

As an immediate corollary, we obtain a combinatorial formula for the volume polynomial.

**Corollary 3.3.** Let the notations be as above. The coefficient of $t^{d_1}_{F_1} \cdots t^{d_k}_{F_k}$ in $Vol_M(\emptyset)$ is

$$(-1)^{d-k} \left( \frac{d}{d_1, \ldots, d_k} \right) \prod_{i=1}^k \left( \frac{d_i - 1}{d_i - r_i} \right) \mu^{d_i-r_i}(M|F_{i+1}/F_i).$$
Remark 3.4 \((\mathcal{M}_{0,n})\). When \(M\) is realizable, \(VP_M\) agrees with the classical volume polynomial of the wonderful compactification \(Y_M\). In particular, when \(M = M(K_{n-1})\) the matroid of the complete graph on \(n - 1\) vertices, we have the following linear relation
\[
\langle e, |F_i \setminus F_i-1|e_{(F_{i+1} \setminus F_i)} - |F_{i+1} \setminus F_i|e_{(F_i \setminus F_{i-1})} \rangle = 0
\]
where \(e_S\) denotes \(\sum_{i \in S} e_i\). The dual elements \(m(\mathcal{F}, i)\) interact nicely with the chain \(\mathcal{F}\) in the following way.

Lemma 3.6. For \(F_j \in \mathcal{F}\), we have:
\[
\langle m(\mathcal{F}, i), u_{F_j} \rangle = \begin{cases} 0 & \text{if } j \neq i \\ -|F_i \setminus F_{i-1}| \cdot |F_{i+1} \setminus F_i| & \text{if } j = i \end{cases}
\]

Proof. Noting that \(e_S, u_{S'} = |S \cap S'|\) for any \(S, S' \subseteq E\), we have that if \(j < i\), then \(m(\mathcal{F}, i), e_{F_j}) = 0 - 0 = 0\); if \(j > i\), then \(m(\mathcal{F}, i), e_{F_j}) = |F_i \setminus F_{i-1}| \cdot |F_{i+1} \setminus F_i| - |F_{i+1} \setminus F_i| \cdot |F_i \setminus F_{i-1}| = 0\). Lastly, when \(i = j\), we get \(-|F_i \setminus F_{i-1}| \cdot |F_{i+1} \setminus F_i|\).

As a consequence, we obtain the following relation.

Corollary 3.7. For a chain \(\mathcal{F} \subseteq \mathcal{L}_M\) and \(F_i \in \mathcal{F}\), we have the following linear relation
\[
x_{F_i} = \sum_{F_j \in \mathcal{F}} \frac{|F \cap (F_{i+1} \setminus F_i)|}{|F_{i+1} \setminus F_i|} x_F - \sum_{F_j \in \mathcal{F}} \frac{|F \cap (F_i \setminus F_{i-1})|}{|F_i \setminus F_{i-1}|} x_F
\]
Proof. By the Lemma above, the linear relation defined by \( m(\mathcal{F}, i) \) is
\[
\sum_{F \in \mathcal{L}_M} \langle m(\mathcal{F}, i), u_F \rangle x_F = \sum_{F \notin \mathcal{F}} \langle m(\mathcal{F}, i), u_F \rangle x_F + (-|F_i \setminus F_{i-1}| : |F_{i+1} \setminus F_i|) x_{F_i} = 0,
\]
and dividing by \(|F_i \setminus F_{i-1}| : |F_{i+1} \setminus F_i|\) and rearranging gives the desired relation.

Hence, denoting by \( x_{\mathcal{F}} := x_{F_1} \cdots x_{F_k} \), the quadric relations in \( A^*(M) \) imply the following key lemma.

Lemma 3.8.
\[
x_{\mathcal{F}} \cdot x_{F_i} = x_{\mathcal{F}} \cdot (-1) \left( \sum_{F \in (F_i, F_{i+1})} \frac{|F_i \setminus F|}{|F_{i+1} \setminus F|} x_F + \sum_{F \in (F_i, F_{i+1})} \frac{|F \setminus F_{i-1}|}{|F_{i+1} \setminus F_i|} x_F \right).
\]

Proof. Using Corollary 3.7 above to substitute for \( x_{F_i} \), and noting the quadric relations, we first have that the \( F \)'s appearing on the RHS of 3.7 must be either in \((\emptyset, F_{i-1})\), \((F_{i-1}, F_{i+1})\), \((F_{i+1}, E)\). If \( F \in (\emptyset, F_{i-1}) \) or \( F \in (F_{i-1}, E) \), then the coefficients of \( x_F \) in the two terms in RHS of 3.7 are equal (both 0 or both 1). If \( F \in (F_i, F_{i+1}) \), then RHS of 3.7 has in its sum
\[
\frac{|F \setminus F_{i-1}|}{|F_{i+1} \setminus F_{i}|} x_F = -1 - \frac{|F \setminus F_i|}{|F_{i+1} \setminus F_i|} x_F = \frac{|F \setminus F_{i-1}|}{|F_{i+1} \setminus F_i|} x_F.
\]
If \( F \in (F_{i-1}, F_i) \), then \( F \cap (F_{i+1} \setminus F_i) = \emptyset \) so that we get a contribution of \(-\frac{|F \setminus F_{i-1}|}{|F_{i+1} \setminus F_i|} x_F\).

We now use these observations to expand \( x_{F_1}^{d_1} \cdots x_{F_k}^{d_k} \) into square-free monomials. For convenience, denote \( x_{F_1}^{d_1} \cdots x_{F_k}^{d_k} \) by \( x_{\mathcal{F}}^{\mathcal{d}} \), and let \( r_i := \text{rk}\ F_i \) and \( d_i := \sum_{j=0}^i d_j \) (where \( d_0 := 0 \)). If \( F_1, \ldots, F_k \) do not form a chain, then the quadric relation on \( A(M) \) immediately makes \( x_{\mathcal{F}}^{\mathcal{d}} = 0 \). So we assume that \( \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E \) is a chain \( \mathcal{F} \) in \( \mathcal{L}_M \).

Observation 3.9. The key sequence of Lemma 3.8 is that \( x_{F_1}^{d_1} \cdots x_{F_i}^{d_i} \cdots x_{F_k}^{d_k} \) is a linear combination of monomials of the form \( x_G \cdot x_{F_1}^{d_1} \cdots x_{F_i}^{d_i-1} \cdots x_{F_k}^{d_k} \) where \( G \in (F_{i-1}, F_i) \) or \( G \in (F_i, F_{i+1}) \) (and zero if no such \( G \in \mathcal{L}_M \) exists). More precisely, we have
\[
x_{F_1}^{d_1} \cdots x_{F_i}^{d_i} \cdots x_{F_k}^{d_k} = \sum_{G \in (F_{i-1}, F_i)} -\frac{|G \setminus F_{i-1}|}{|F_i \setminus F_{i-1}|} x_G \cdot x_{F_1}^{d_1} \cdots x_{F_i}^{d_i-1} \cdots x_{F_k}^{d_k} + \sum_{G \in (F_i, F_{i+1})} -\frac{|F_{i+1} \setminus G|}{|F_{i+1} \setminus F_i|} x_G \cdot x_{F_1}^{d_1} \cdots x_{F_i}^{d_i-1} \cdots x_{F_k}^{d_k}.
\]
Call the terms in the RHS of the above where \( G \in (F_{i-1}, F_i) \) (resp. \( G \in (F_i, F_{i+1}) \)) the downward (resp. upward) descendants of \( x_{\mathcal{F}}^{\mathcal{d}} \) at \( F_i \) of 1st iteration, and by of 2nd iteration the immediate downward (upward) descendants of the downward descendants of 1st iteration at \( F_i \).

Consider the downward descendants of \( x_{\mathcal{F}}^{\mathcal{d}} \) at \( F_1 \) of \((\tau_1 - 1)\)th iteration assuming that \( \tau_1 \leq d_1 \). These are of the form (ignoring the coefficient) \( x_{G_1} \cdots x_{G_{\tau-1}} x_{F_1} x_{F_2}^{d_2} \cdots x_{F_k}^{d_k} \) where \( \emptyset \subsetneq G_1 \subsetneq \cdots \subsetneq G_{\tau-1} \subsetneq x_{F_1} \) form a maximal chain in \( [\emptyset, F_1] \subset \mathcal{L}_M \). Noting the restriction in the flats of the new variable when taking downward descendants given by Lemma 3.8, these can only arise in the following way: In the 1st iteration, we only consider the terms where \( \text{rk} G = 1 \), then in the 2nd iteration, we only consider terms whose flat of the new (added) variable is of rank 2, and so forth. One makes similar observation for upward descendants.

The first result from the above observation is the following.
Definition 3.12. For \( \leq 0 \) define
\[
\sum_{i=1}^{\gamma} \gamma \equiv (-1)^{\gamma+1}. \tag{3.3}
\]
Moreover, the sign of \( \deg(x_d^G) \) is \((-1)^{\gamma} \gamma = (-1)^{d-k} \).

Proof. After expanding everything so that there are no powers, we are looking at monomials \( x_G \) where \( \mathcal{F}' \) is a maximal chain in \( \mathcal{L}_F \) extending \( \mathcal{F} \). Given the restriction on the rank of the flats that get added at each step, one can think of \( x_G \) expanding to fill up the flats with ranks \( r \) in the range \( \tilde{d}_{i-1} < r \leq \tilde{d}_i \). This proves the first claim. The second is immediate from 3.8. \( \square \)

Proposition 3.11. If \( \tilde{d}_i = r_i \forall i \), then \( \deg(x_d^G) = (-1)^{d-k} \). In other words, there are “no multiplicities added” when “expanding downwards.”

Proof. With Observation 3.9, one can reduce to showing \( \deg x_d^F = (-1)^{d-1} \) for \( \rk F = d \). Reverse inducting on the number of downward iterations, the statement reduces to the following easy fact: For a loopless matroid \( M' = (E', \mathcal{I}) \) of rank 2, we have \( \sum_{F \in \mathcal{I}} |F| = 1 \). (Note that \( M|F_i/F_{i-1} \) is always loopless). \( \square \)

Thus, interesting multiplicities only come when taking upward descendants. These multiplicities are a priori do not even seem to be integers. We first analyze these multiplicities.

Definition 3.12. For \( M \) a loopless matroid on a ground set \( E \) of rank \( d+1 \) and \( -1 \leq i \leq d+1 \), define \( \gamma(M,i) \) as follows. For \( i = -1, d+1 \), define \( \gamma(M,-1) = -1 \) and \( \gamma(M,d+1) = 0 \); for \( 0 \leq i \leq d \), define
\[
\gamma(M,i) := \sum_{\mathcal{G} \in \mathcal{L}^M_{\leq i}} (-1) \left( -\frac{|G_1 \setminus G_0|}{|G_1|} \right) \left( -\frac{|G_2 \setminus G_1|}{|G_2|} \right) \cdots \left( -\frac{|G_{i+1} \setminus G_i|}{|G_{i+1}|} \right)
\]
where \( \mathcal{L}^M_{\leq i} \) consists of chains \( \mathcal{G} : \emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_i \subsetneq G_{i+1} = E \) such that \( \rk G_j = j \) for \( j = 0, \ldots, i \).

Proposition 3.13. Let \( \overline{x}_M(t) \) be the reduced characteristic polynomial of \( M \). Then \( \overline{x}_M(t) = \sum_{i=0}^{d} \gamma(M,i) t^{d-i} \).

Proof. Weisner’s theorem [Sta12, §3.9] states that for \( G \in \mathcal{L}_M \) a flat and \( a \in G \) we have
\[
\mu(\emptyset, G) = -\sum_{a \not\in F \subsetneq G} \mu(a, F)
\]
where \( F \subsetneq G \) means \( G \) covers \( F \) in the lattice \( \mathcal{L}_M \). Thus, we have that
\[
|G| \mu(\emptyset, G) = -\sum_{a \in G} \sum_{a \not\in F \subsetneq G} \mu(\emptyset, a, F) = -\sum_{F \subsetneq G} |G \setminus F| \mu(\emptyset, F).
\]
In other words, we have \( \mu(\emptyset, G) = \sum_{F \subsetneq G} |G \setminus F| \mu(\emptyset, F) \). Repeatedly applying this identity gives
\[
\mu(\emptyset, G) = -\gamma(G, \rk G - 1).
\]
Now, since one can define \( \gamma(M,i) \) recursively as
\[
\gamma(M,i) := \begin{cases} 
-1 & \text{for } i = -1 \\
\sum_{\rk F = i} -\frac{|E \setminus F|}{|E|} \gamma(M|F, i-1) & \text{for } i \geq 0 
\end{cases}
\]
we have
\[
\gamma(M,i) = \sum_{\rk F = i} \frac{|E \setminus F|}{|E|} \mu(\emptyset, F)
\]
for $0 \leq i \leq d + 1$. Now, to show the desired equality, we need show that $\gamma(M, 0) = 1$ and $\gamma(M, i + 1) - \gamma(M, i) = \sum \mu_G \mu(\emptyset, G)$ for $0 \leq i \leq d$. Well, $\gamma(M, 0) = 1$ is immediate, and
\[
\gamma(M, i + 1) - \sum \mu_G \mu(\emptyset, G) = \sum \frac{|E \setminus G| - 1}{|E|} \mu(\emptyset, G)
\]
\[
= \sum \frac{|G|}{|E|} \mu(\emptyset, G)
\]
\[
= \sum \sum \frac{|G \setminus E|}{|E|} \mu(\emptyset, F)
\]
\[
= \sum \frac{|E \setminus F|}{|E|} \mu(\emptyset, F) = \gamma(M, i)
\]
where the last equality follows from the cover-partition property of the flats-axioms for matroids. 

In other words, following [AHK18] and denoting by $\mu'(M)$ the (unsigned) coefficient of $t^{rk M - 1 - i}$ in the reduced characteristic polynomial of $M$, we have $|\gamma(M, i)| = \mu'(M)$. We are now finally ready to prove the main theorem of this section.

Proof of Theorem 3.2. Note that $(\bar{d}_i - 1)$ is nonzero iff $\bar{d}_{i-1} < r_i \leq \bar{d}_i$ since $(r_i - \bar{d}_{i-1} - 1) + (\bar{d}_i - r_i) = d_i - 1$, so the condition for the above expression to be nonzero agrees with Proposition 3.10. Proposition 3.11 implies that we need not worry about multiplicities from iterative downward descendants of $x^d_{F_i}$'s. For going up the rank, we get the multiplicities of $(\bar{d}_i - r_i)$'s from the fact that we can expand downwards then upwards, or vice versa. Lastly, using Lemma 3.8 for expanding upwards, and noting that $\frac{|E \setminus F|}{|E'|}$ can be expressed as $\frac{|E \setminus F|}{|E'|}$ where $E'$ is the ground set of $M|F_{i+1}/F_i$ and $F$ is the flat considered as a flat in $M|F_{i+1}/F_i$, we have
\[
\deg(x^d_{F_1} \cdots x^d_{F_k}) = (-1)^{d-k} \cdot \prod_{i=1}^k \left( \frac{d_i - 1}{d_i - r_i} \right) |\gamma(M|F_{i+1}/F_i, \bar{d}_i - r_i)|.
\]
Applying Proposition 3.13 then gives us the desired formula.

4. First applications of the volume polynomial

We give some first applications of the volume polynomial of a matroid. In this section, we always set the ground set of a matroid to be $[n] := \{1, \ldots, n\}$ for some $n$.

Volumes of generalized permutohedra. As an immediate application, we compute the volume of a generalized permutohedron using the volume polynomial of a boolean matroid. The explicit formula given here can be viewed as a “cancellation-free” version of the one given in [Pos09, Corollary 9.4]; we review the results briefly here and point to [Pos09] and [AA17] as a general reference.

A generalized permutohedron $P$ is obtained by sliding the facets of the permutohedron. More precisely, for a submodular function $z(x) : 2^n \rightarrow \mathbb{R}$ (where $[n] \supset I \mapsto z_I$) on the boolean lattice $\mathcal{L}_{2^n}$, we define
\[
P(z) := \{(x) \in \mathbb{R}^n \mid \sum_{i \in \emptyset} x_i = z_{[n]}, \sum_{i \in I} x_i \leq z_I \forall I \subset \emptyset\}
\]
which is a polytope of dimension at most $n - 1$ in $\mathbb{R}^n$.

For nonnegative set of numbers $\{y_I \mid I \subset [n]\}$, one can consider a Minkowski sum $P(y) = \sum_{I \subset \emptyset} y_I \Delta_I$ where $\Delta_I = \text{Conv}(e_i : i \in I) \subset \mathbb{R}^n$. By setting $z_I = \sum_{J \subset I} y_J$, one has $P(z) =$
$P(y)$ ([Pos09, Proposition 6.3]). Note that not every generalized permutohedra is realized as $P(y)$ however ([Pos09, Remark 6.4]). The volume of $P(y)$ was computed by Postnikov:

**Theorem 4.1.** [Pos09, Corollary 9.4] The volume of $P(y)$ is

$$\text{Vol} \ P(y) = \frac{1}{(n-1)!} \sum_{(I_1, \ldots, I_{n-1})} y_{I_1} \cdots y_{I_{n-1}}$$

where the summation is over ordered subsets $(I_1, \ldots, I_{n-1})$ of $[n]$ such that for any $\{i_1, \ldots, i_k\} \subset [n-1]$ we have $|I_{i_1} \cup \cdots \cup I_{i_k}| \geq k + 1$.

To convert the above formula into one in terms of $z_I$'s, one uses Möbius inversion formula to get

$$z_I = \sum_{I \subseteq J} y_I \iff y_I = \sum_{J \subseteq I} z_I \mu(J,I) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} z_J.$$

Here we give another formula in terms of the $z_I$'s. The Bergman fan $\Sigma_n$ of $U_{n,n}$ is the normal fan of the permutohedron, whose rays correspond to subsets of $[n]$. As nef torus invariant divisors on the toric variety $X_{\Sigma_n}$ exactly correspond to submodular functions, well-known results from toric geometry on volumes of torus invariant divisors imply that

$$\text{Vol} \ P(z) = \frac{1}{(n-1)!} V P_{U_{n,n}}(z).$$

Note that the flats of $U_{n,n}$ are just subsets of $[n]$. As $U_{n,n}|I/J \simeq U_{m,m}$ where $m = |I \setminus J|$, we have via Corollary 3.3 that

**Proposition 4.2.** Let $z(\cdot) : I \mapsto z_I \in \mathbb{R}$ be a submodular function, then the volume of the generalized permutohedron $P(z)$ is

$$(n-1)! \text{Vol} \ P(z) = \sum_{d, d_i \geq 0} (-1)^{d-k} \binom{d}{d_1, \ldots, d_k} \prod_{i=1}^{k} \left( \frac{d_i - 1}{d_i - |I_i|} \right) \left( |I_{i+1}| - |I_i| - 1 \right) z_{I_i}$$

where the summation is over chains $\emptyset \subseteq I_1 \subseteq \cdots \subseteq I_k \subseteq I_{k+1} = [n]$ and $d = (d_1, \ldots, d_k)$ such that $\sum d_i = n - 1$ and $d_j = \sum_{i=1}^{j} d_i$.

**Remark 4.3.** Setting $z_I = \frac{(n-|I|)!|I|}{2}$, one recovers that the volume of the permutohedron $P_n := \text{Conv}(\{\sigma(n), \sigma(n-1), \ldots, \sigma(1)\}) \in \mathbb{R}^n | \sigma \in S_n)$ is $n^{n-2}$.

It was not clear to the author whether one could derive the formula in the above Proposition directly from [Pos09, Corollary 9.4].

**Question 4.4.** Can one derive the formula in Proposition directly from [Pos09, Corollary 9.4], or vice versa?

**Valuativensness of the volume polynomial.** While the volume polynomial $VP_M$ has the same information as the Chow ring $A(M)$, it lends itself more naturally as a valuation on a matroid when viewed as a map $M \mapsto VP_M \in \mathbb{R}[t_S : S \in 2^{[n]}]$. In this subsection, we illustrate this by showing that $M \mapsto VP_M$ is a valuation under matroid polytope subdivisions, a statement that would not make sense for $M \mapsto A(M)$.

We first give a brief sketch on matroid polytopes and matroid valuations; for more on matroid valuations, we point to [AFR10] and [DF10]. Given a matroid $M$ on $[n]$ of rank $r$ with bases $B$, its **matroid polytope** is defined as

$$\Delta(M) := \text{Conv}(e_B \ | \ B \in B) \subset \mathbb{R}^n$$
where \( e_S := \sum_{i \in S} e_i \) for \( S \subset [n] \) and \( e_i \)'s are the standard basis of \( \mathbb{R}^n \). Its vertices are the indicator vectors for the bases of \( M \), and it follows from a theorem of Gelfand, Goresky, MacPherson, and Serganova that the faces of \( \Delta(M) \) are also matroid polytopes ([GGMS87]). A matroid subdivision \( S \) of a matroid polytope \( \Delta \) is a polyhedral subdivision \( S : \Delta = \bigsqcup \Delta(M_i) \) such that each \( \Delta(M_i) \) is a matroid polytope of some matroid \( M_i \) (necessarily of rank \( r \) on with ground set \([n]\)).

Denote by \( \text{Int}(S) \) the faces of \( \Delta(M_i) \)'s that is not on the boundary of \( \Delta(M) \). It is often of interest to see whether a valuation on matroids behave well via inclusion-exclusion with respect to matroid subdivisions:

**Definition 4.5.** Let \( R \) be an abelian group, and let \( \mathcal{M} := \bigcup_{n \geq 0} \{ \text{matroids on ground set } [n] \} \) be the set of all matroids. A map \( \varphi : \mathcal{M} \to R \) is a matroid valuation (or is valuative) if for any \( M \in \mathcal{M} \) and a matroid subdivision \( S : \Delta(M) = \bigsqcup_i \Delta(M_i) \) one has

\[
\varphi(M) = \sum_{Q \in \text{Int}(S)} (-1)^{\text{dim} \Delta(M) - \text{dim} Q} \varphi(M_i).
\]

Many interesting functions on matroids are matroid valuations, for example the Tutte polynomial ([AFR10, Corollary 5.7]) and the quasi-symmetric functions \( \psi^{\text{BJR}} \) introduced by Billera, Jia, and Reiner in [BJR09]. It was also shown recently in [LdMRS17] that matroid analogues of Chern-Schwartz-MacPherson (CSM) classes are matroid valuations. Here we show that the volume polynomial of a matroid is also a matroid valuation.

**Proposition 4.6.** \( M \to VP_M(t) \in \mathbb{R}[t_S : S \in 2^{[n]}] \) is a matroid valuation.

We state a useful lemma before the proof.

**Lemma 4.7.** Let \( f, g : \mathcal{M} \to \mathbb{Q} \) be matroid valuations, and let \( A \subset [n] \). Then \( f * A g \) defined by \( (f * A g)(M) = f(M|A)g(M/A) \) is also a matroid valuation.

**Proof.** By [DF10, Corollary 5.5, 5.6], we reduce to the case when \( f = \sum_{i=0}^{\infty} s_X(i) q(i) \) and \( g = \sum_{i=0}^{\infty} s_Y(i) q(i) \), where \( X(i) \) is a flag \( \emptyset = X_0(i) \subseteq X_1(i) \subseteq \cdots \subseteq X_k(i) = [i] \) and \( q(i) \) is an increasing sequence \( 0 = q_0 \leq q_1 < q_2 < \cdots < q_k \) (likewise for \( Y \) and \( z \)), and \( s_X(i) q(i) \) is a function on the set of matroids with ground set \([i]\) defined by

\[
s_X(i) q(i)(M) = \begin{cases} 1 & \text{if } M \text{ has ground set } [i] \text{ and } \text{rk}_M(X_j(i)) = q_j(i) \text{ for all } j = 0, \ldots, i \ , \\ 0 & \text{otherwise} \end{cases}.
\]

Let \( |A| = \ell \). Then, \( f * A g \) (where the ground set \( A \subset [n] \) of \( M|A \) is relabelled to be \([\ell]\) by increasing order and likewise for \([n] \setminus A \) for \( M/A \)) is equal to \( s_Z p \), where \( Z \) is the concatenation of \( X(i) \) and \( \psi^{(n-\ell)} \cup A \), and \( p \) is the concatenation of \( q(i) \) and \( X(n-\ell) + q_k \) (by union or plus we mean adding to each element in the sequence). Hence, \( f * A g \) is also a matroid valuation. \( \square \)

**Proof of Proposition 4.6.** We show that the maps \( M \mapsto (\text{coefficient of } t_{S_1}^{d_1} \cdots t_{S_k}^{d_k} \text{ in } VP_M) \) are matroid valuations. As the Tutte polynomial is valuative, so are the coefficients of the reduced characteristic polynomial. Moreover, note that if \( \varphi : \mathcal{M} \to R \) is a matroid valuation, then so is \( \tilde{\varphi}(n, r) \) for any fixed \( n \) and \( r \), where \( \tilde{\varphi}(n, r) \) is defined by

\[
\tilde{\varphi}(n, r)(M) := \begin{cases} \varphi(M) & \text{if } M \text{ is a matroid of rank } r \text{ on } n \text{ elements} \\ 0 & \text{otherwise} \end{cases}.
\]

Thus, for \( s, t, u, v \in \mathbb{Z}_{\geq 0} \) such that \( s \geq t \) and \( u \geq v \), we define

\[
\tilde{\mu}(s, t, u, v) = \begin{cases} 0 & \text{if } \text{rk}(M) \neq u - v \\ \frac{1}{n(t-1)} \mu^{s-v}(M) & \text{if } \text{rk}(M) = u - v \end{cases}.
\]
Then for a flag $\emptyset \subset S_1 \subset \ldots \subset S_k \subset S_{k+1} = [n]$, an increasing sequence $0 = \tilde{d}_0 < \tilde{d}_1 < \ldots < \tilde{d}_k = d$, and an increasing sequence $0 = r_0 < r_1 < \cdots < r_k < r_{k+1} = d + 1$, we have that (letting $d_i := \tilde{d}_i - \tilde{d}_{i-1}$)

$$(1)^{d-k} \mu(\tilde{d}_0, d_0, r_1, r_0) * S_1 \tilde{\mu}(\tilde{d}_1, d_1, r_2, r_1) * S_2 \cdots * S_k \tilde{\mu}(\tilde{d}_k, d_k, r_{k+1}, r_k)$$

is a matroid valuation that evaluates a matroid $M$ to $(1)^{d-k}(d_1, \ldots, d_k) \prod_{i=1}^{k} (\frac{d_i - 1}{d_i - r_i}) \mu(\tilde{d}_i - r_i) (M|S_i+1/S_i)$ if $M$ is of rank $d + 1$ with $S_i$ being a chain of flats with $rk_M(S_i) = r_i$, and 0 otherwise. Thus, summing over all sequences $r_i$ of the above function, we have that taking the coefficient of $t_i^{d_1} \cdots t_i^{d_k}$ in the $VP_M$ is a matroid valuation.

That the matroid volume polynomial behaves well with respect to matroid polytope subdivisions and the appearance of matroid-minor Hopf monoid structure appearing in its expression as in the proof above suggests that there may be a generalization of Chow ring of matroids to Coxeter matroids of arbitrary Lie type (where matroids are the type $A$ case). For Coxeter matroids see [BGW03].

**Question 4.8.** Is there a Hodge theory of Coxeter matroids, generalizing the Hodge theory of matroids as described in [AHK18]?

### 5. The volume of a matroid

Let $M = (E, B)$ be a matroid. Following [AHK18], a (strictly) submodular function $c(\cdot) : 2^E \to \mathbb{R}$ gives a combinatorially nef (ample) divisor $D = \sum_{F \in X_M} c_F x_F \in A^1(M)_{\mathbb{R}}$. For realizable matroids, if the divisor $D \in A^1(M)_{\mathbb{R}}$ is combinatorially nef (ample) then as an element of $A^1(Y_M)_{\mathbb{R}}$ the divisor $D$ is nef (ample) in the classical sense. As the rank function is a distinguished submodular function of a matroid, we define the following notions.

**Definition 5.1.** For a matroid $M$, define its distinguished nef divisor $D_M$ to be

$$D_M := \sum_{F \in X_M} (rk F) x_F,$$

and define the volume of a matroid $M$ to be the volume of its distinguished nef divisor:

$$Vol(M) := \deg \left( \sum_{F \in X_M} (rk F) x_F \right)^{rk_M - 1}.$$

It follows immediately from Proposition 4.6 that the volume of a matroid is a valuative invariant.

**Corollary 5.2.** The map $M \mapsto Vol(M)$ is a matroid valuation.

We do not know of a purely combinatorial meaning of the volume of a matroid. In fact, it seems to be a genuinely new invariant of a matroid.

**Remark 5.3** (Relation to other invariants, or lack thereof). We point to `volumeMatroid.m2` for computations supporting the statements below.

- Volume $Vol(M)$ is not a complete invariant.
- Same Tutte polynomial does not imply same volume, and vice versa. The two graphs in Figure 2 of [CLP15] have the same Tutte polynomial but their matroids are not isomorphic; their volumes are 1533457 and 1534702. There are many examples of matroids with same volume but with different Tutte polynomials.
- Same volume of the matroid polytope does not imply same volume, and vice versa.

**Remark 5.4.** Since the two graphs in the Figure 2 of [CLP15] have the same Tutte polynomial but different matroid volumes, we thus obtain an example of two simple matroids with same Tutte polynomials but with different $\mathcal{G}$-invariant (see [Der09] for $\mathcal{G}$-invariant of a matroid).
For realizable matroids however, the volume measures how general the associated hyperplane arrangement is. More precisely,

**Theorem 5.5.** Let $M$ be a realizable matroid of rank $r$ on $n$ elements. Then

$$\text{Vol}(M) \leq \text{Vol}(U_{r,n}) = n^{r-1} \quad \text{with equality iff } M = U_{r,n}. $$

The proof is algebro-geometric in nature, and follows from the following.

**Proposition 5.6.** Let $M = (E, B)$ be a simple realizable matroid of rank $r = d + 1$, and let $\pi : Y_M \to \mathbb{P}_k^d$ be the blow-down map from the wonderful compactification. Then $|D_M|$ is a linear subseries of $|\pi^* (\mathcal{O}_{\mathbb{P}_k^d}(n))|$, with equality iff $M = U_{r,n}$.

More precisely, $|D_M|$ is isomorphic to the linear subseries $L \subset |\mathcal{O}_{\mathbb{P}_k^d}(n)|$ consisting of homogeneous degree $n$ polynomials on $\mathbb{P}_k^d$ vanishing on $H_F$ with order at least $|F| - \text{rk } F$.

**Proof.** Let $A_M = \{ H_F \}_{F \in \mathcal{X}_M}$ be the hyperplane arrangement as noted in §2.1, and denote $h := c_1(\mathcal{O}_{\mathbb{P}_k^d}(1))$ the hyperplane class. Then for any $i \in E$, we have

$$\pi^* h = \pi^*[H_i] = \sum_{i \in F} x_F$$

by the construction of $Y_M$ as consecutive blow-ups. Thus, we have

$$D_M = \sum_F (\text{rk } F)x_F = \sum_{i \in E} x_i + \sum_{\text{rk } F > 1} (\text{rk } F)x_F$$

$$= \sum_{i \in E} \left( \pi^* h - \sum_{\subseteq F} x_F \right) + \sum_{\text{rk } F > 1} (\text{rk } F)x_F$$

$$= n\pi^* h + \sum_{\text{rk } F > 1} (|F| - \text{rk } F)x_F$$

Hence, noting that the rank function on $M$ satisfies $|S| \geq \text{rk } S$ for any subset $S \subseteq E$, we see that the divisors in $|D_M|$ are exactly the divisors in $|\pi^* \mathcal{O}_{\mathbb{P}_k^d}(n)|$ that vanish on $H_F$ with order at least $|F| - \text{rk } F \geq 0$. Moreover, since $|F| - \text{rk } F \geq |G| - \text{rk } G$ for any flats $F \supset G$, the divisors in $|D_M|$ are in fact elements of $|\mathcal{O}_{\mathbb{P}_k^d}(n)|$ vanishing on $H_F$ with order at least $|F| = \text{rk } F$. Lastly, the only simple matroids with the property $|F| = \text{rk } F$ for all flats $F \in \mathcal{L}_M$ are the uniform matroids. (Given an $r$-subset $S$, its closure $\overline{S}$ is a flat not equal to $E$ iff $S$ is not a basis). \[\square\]

Two immediate consequences follow.

**Corollary 5.7.** The volume of $U_{r,n}$ is $n^{r-1}$.

**Proof.** The map given by $\pi^* \mathcal{O}_{\mathbb{P}_k^d}(n)$ factors as $Y_M \xrightarrow{\pi} \mathbb{P}_k^d \xrightarrow{\nu_n} \mathbb{P}_k^{(d+n)-1}$ where $\nu_n$ is the $n$-tuple Veronese embedding, and the degree of the Veronese embedding is $n^d = n^{r-1}$. \[\square\]

**Proof of Theorem 5.5.** First assume $M$ is simple. Then Proposition 5.6 shows that $R(D_M) \subseteq R(\pi^* \mathcal{O}_{\mathbb{P}_k^d}(n))$, with equality iff $M = U_{r,n}$. As both divisors $D_M$ and $\pi^* \mathcal{O}_{\mathbb{P}_k^d}(n)$ are nef, the volume polynomial agrees with the volume of a divisor in the classical sense as in §2.2. If $M$ is not simple, then its volume is the same as the simple matroid $M'$ satifying $\mathcal{L}_M = \mathcal{L}_{M'}$, and $M'$ has at most $n - 1$ elements. \[\square\]

From computational experiments, we conjecture that Theorem 5.5 holds for matroids in general, not necessarily realizable.

**Conjecture 5.8.** The maximum volume among matroids of rank $r$ on $n$ elements is achieved uniquely by the volume of the uniform matroid $\text{Vol}(U_{r,n}) = n^{r-1}$. 

Trying to apply similar method as in the proof of Theorem 5.5 for the non-realizable matroids naturally leads to the following question.

**Question 5.9.** Is there a naturally associated convex body for which this volume polynomial is measuring the volume of? In other words, is there a theory Newton-Okounkov body for general matroids (a.k.a. linear tropical varieties)?

On the flip side of maximal volumes, we have the following conjecture on the minimal values.

**Conjecture 5.10.** The minimum volume among simple matroids of rank $r$ on $n$ is achieved uniquely by the matroid $U_{r-2,r-2} \oplus U_{2,n-r+2}$, and its volume is $r^{r-2}((n - r + 1)(r - 1) + 1)$. 
6. Examples

All the matroids in the examples are realizable to illustrate the algebro-geometric connections. Let’s first start with a sanity check:

Example 6.1. Let \( M := U_{2,3} \), the uniform matroid of rank 2 on 3 elements. The lattice of flats \( \mathcal{L}_M \) is

\[
\begin{array}{c}
0, 1, 2 \\
2 \\
1 \\
0
\end{array}
\]

and the Chow ring \( A(M) \) is

\[
A(M) = \frac{\mathbb{Z}[x_0, x_1, x_2]}{(x_2x_1, x_2x_0, x_1x_0, x_1 - x_0, x_2 - x_0)},
\]

which is isomorphic to \( \mathbb{Z}[x_0]/(x_0^2) \). As the complement of the hyperplane arrangement \( A_M \) is \( \mathbb{P}^1 \setminus \{p, q, r\} \) for \( p, q, r \) distinct points, the wonderful compactification is \( Y_M = \mathbb{P}^1 \), whose Chow ring is also \( \mathbb{Z}[h]/(h^2) \) where \( h \) is the class of a point on \( \mathbb{P}^1 \). The volume polynomial is

\[
VP_M = t_0 + t_1 + t_2.
\]

As expected, a divisor \( a_0x_0 + a_1x_1 + a_2x_2 = (a_0 + a_1 + a_2)h \) has volume \( a_0 + a_1 + a_2 \) (when the sum is \( \geq 0 \)). The distinguished divisor \( x_0 + x_1 + x_2 \) has volume 3 = 3^2 - 1.

The next two examples are of rank 3, whose wonderful compactifications are obtained from blowing-up points on \( \mathbb{P}^2 \). For these surfaces, we can compute the intersection numbers with classical algebraic geometry without much difficulty; see [Har77, §V.3] for reference on monoidal transformations. We check in these examples that the classical results and the combinatorial ones introduced in this paper indeed agree.

Example 6.2. Let \( M := U_{3,4} \). Its lattice of flats \( \mathcal{L}_M \) and the hyperplane arrangement \( A_M \subset \mathbb{P}^2 \) are

The wonderful compactification \( Y_M \) is given by blowing-up the six points \( H_{0,1}, \ldots, H_{2,3} \). The volume polynomial is

\[
VP_M(t) = -2t_1^2 - 2t_2^2 - 2t_0^2 - 2t_0t_2 + 2t_3t_2, 3 + 2t_2t_2, 3 - t_2^2, 3 + 2t_1t_1, 3 + 2t_3t_3, 3 - t_3^2, 3 + 2t_3t_0, 3 + 2t_0t_0, 3 - t_0^2, 3 + 2t_1t_0, 1 + 2t_0t_0, 1 - t_0^2.
\]
Notice that the coefficient of $t_i^2$ is $-2$ as expected from Corollary 3.3 since $M/\{i\} \simeq U_{2,3}$, so that $|\mu_1(M/\{i\})| = 2$, and the coefficients of $t_i^2$ is $-1$ in agreement with Proposition 3.11. The rest of the coefficients are maximal chains, so the coefficient is $\binom{2}{1,1} = 2$. The volume of $M$ is $(4)(-2)(1) + (12)(2)(2) + (6)(-4) = 16 = 4^2$, as expected from Theorem 5.5.

Let $\pi : Y_M \rightarrow \mathbb{P}^2$ be the blow-down map, $\bar{H} := \pi^*H$ the pullback of the hyperplane class $H \subset \mathbb{P}^2$, and $E_{ij}$’s the exceptional divisors from the blown-up points. Then Pic$Y_M = \mathbb{Z}\{\bar{H}, E_1, \ldots, E_6\}$, where intersection pairing of divisors are $E_i \cdot E_j = 0 \forall i \neq j$, $E_i \cdot \bar{H} = 0$, $\bar{H} \cdot \bar{H} = 1$, and $E_i \cdot E_i = -1$. Hence, $x_0 = \pi^*H_0 = \bar{H} + E_{01} + E_{02} + E_{03}$, so that $x_0^2 = 1 - 1 - 1 = -2$, as expected. Similarly, one computes that the volume of $M$ is $4^2 = 16$. Alternatively, note that the map $Y_M \rightarrow \mathbb{P}(H^0(4\bar{H}))$ given by $4\bar{H}$ factors birational through $\mathbb{P}^2$ as the 4-tuple Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^{14}$, whose degree is 16.

A side remark: the complete linear system $|3\bar{H} - E_{01} - \cdots - E_{23}|$ defines a birational map $Y_M \rightarrow \mathbb{P}^3$ whose image is the Cayley nodal cubic surface (as $(3\bar{H} - E_{01} - \cdots - E_{23})^2 = 3$). Indeed, as $3\bar{H} - E_{01} - \cdots - E_{23} = 3(x_0 + x_{0,1} + x_{0,2} + x_{0,3}) - (x_{0,1} + \cdots + x_{2,3}) = 3x_0 + 2x_{0,1} + 2x_{0,2} + 2x_{0,3} - x_{1,2} - x_{1,3} - x_{2,3}$, evaluating $V_{P_M}$ respectively gives $-2 \cdot 3^2 - 3 \cdot 2^2 - 3 \cdot 1^1 + 3 \cdot 2 \cdot 3 \cdot 2 = 3$.

Example 6.3. Let $M = U_{1,1} \oplus U_{2,3}$, another matroid of rank 3 on 4 elements, but not uniform as the above example. Its lattice of flats $\mathcal{L}_M$ and hyperplane arrangement $\mathcal{A}_M$ are

\begin{center}
\begin{tikzpicture}
\node at (0,0) {0};
\node at (1,0) {1};
\node at (2,0) {2};
\node at (3,0) {3};
\node at (1,1) {3};
\node at (1,-1) {2};
\node at (2,-1) {1};
\node at (1,2) {1};
\node at (1,3) {2};
\node at (2,2) {3};
\node at (2,3) {0,1};
\node at (3,2) {0,2};
\node at (3,3) {0,3};
\end{tikzpicture}
\end{center}

Its wonderful compactification $\pi : Y_M \rightarrow \mathbb{P}^2$ is the plane $\mathbb{P}^2$ blown-up at four points $H_{01}, H_{02}, H_{03}, H_{123}$. The volume polynomial is

$$V_{P_M}(t) = -t_0^2 - t_1^2 - t_2^2 + 2t_3t_0 + 2t_0t_3 + t_0^2 - 2t_3t_1 - 2t_0t_2 + 2t_3t_1 - 2t_2t_0 + 2t_3t_2 + 2t_1t_3.$$ 

Notice that the coefficient of $t_0^2$ is $-2$ while those of $t_i^2 (i \neq 0)$ are $-1$, since $\mathcal{L}_{M/0} \simeq \mathcal{L}_{U_{2,3}}$ whereas $\mathcal{L}_{M/1} \simeq \mathcal{L}_{U_{1,2}}$ (the reduced chromatic polynomials of $U_{2,3}$ and $U_{2,2}$ are $t - 2$ and $t - 1$). The volume of $M$ is $-5 + (4)(-2^2) + 9(2)(1 \cdot 2) = 15 < 16$.

Again, let $\bar{H} := \pi^*H$ the pullback of the hyperplane class $H$, and $E_{ij}$ the exceptional divisors of from the blown-up points. As in the proof of Theorem 5.5, the distinguished divisor $D_M$ is $4\bar{H} - E_{123}$, whose volume is $(4\bar{H} - E_{123})^2 = 16 - 1 = 15$. Alternatively, the map $Y_M \rightarrow \mathbb{P}(H^0(4\bar{H} - E_{123}))$ factors birational through $\mathbb{P}^2$ as a rational map $\mathbb{P}^2 \rightarrow \mathbb{P}^{13}$ given by a graded linear system $L \subset H^0(\mathcal{O}_{\mathbb{P}^2}(4))$ consisting of quartics through the point $H_{123}$. Its image is the blow-up of a point in $\mathbb{P}^2$ embedded in $\mathbb{P}^{13}$ with degree 15. In summary,

$$Y_M \xrightarrow{\pi} \text{Bl}_{H_{123}} \mathbb{P}^2 \xrightarrow{\text{Veronese}} \mathbb{P}(t_{14}^2 - 1) \xrightarrow{\pi} \mathbb{P}^{13}.$$
We feature one example of rank 4. Here, we can see that the computation of the volume polynomial and the intersection numbers already become somewhat nontrivial.

**Example 6.4.** Let $M := U_{2,2} \oplus U_{2,3}$. Its lattice of flats $\mathcal{L}_M$ is as follows.

And its hyperplane arrangement $A_M$ can be illustrated as
The wonderful compactification $\pi : Y_M \to \mathbb{P}^3$ is obtained by blowing-up the four points, then the strict transforms of the 8 lines. The volume polynomial is

\[
V_{PM}(L) = t_1^3 + t_2^3 + 2t_1^3 - 2t_2^3 - 3t_1t_2^2 + 3t_3^2 - 2t_1t_3^2 - 4t_2t_3^2 - 3t_1t_2t_3 - 3t_1^2t_2t_3 + 3t_1t_2t_3^2 - 3t_1t_2t_3^2 - 3t_1t_2t_3^2 - 3t_1t_2t_3^2 - 3t_1t_2t_3^2 - 3t_1t_2t_3^2 - 3t_1t_2t_3^2.
\]

Its volume is 112, which is the smallest for simple matroids of rank 4 on 5 elements. Let $H = \pi^*H$ be the pullback of hyperplane $H \subset \mathbb{P}^3$ again, and let $E_F$’s be the exceptional divisors from the blow-ups. The distinguished divisor $D_M$, again following the proof of Theorem 5.5 is $5H - E_{0234} - E_{1234} - E_{234}$. Let’s call $P,Q$ the two points $E_{0234}, E_{1234}$, and $L$ the line $E_{234}$. Then the map given by $D_M$ on $Y_M$ through a rational map on $\mathbb{P}^3$ as follows. First, consider the rational map given by $L \subset H^0(\mathcal{O}_{\mathbb{P}^3}(5))$ consisting of divisors through the two points $P,Q$. Then, consider the map from $\text{Bl}_{P,Q} \mathbb{P}^3$ given by divisors in $H^0(\text{Bl}_{P,Q} \mathbb{P}^3)$ passing containing the strict transform of the line $L$. The image of the map is the blow-up of $L$ in $\mathbb{P}^3$ embedded in $\mathbb{P}^{49}$ with degree 112. In summary, we have

\[
\begin{align*}
Y_M &\xrightarrow{\pi} \text{Bl}_L(\text{Bl}_{P,Q} \mathbb{P}^3) \\
\text{Bl}_{P,Q} \mathbb{P}^3 &\xrightarrow{\pi} \mathbb{P}^{49} \\
\mathbb{P}^3 &\xrightarrow{\text{Veronese}} (\mathbb{P}^{49})^{*+3} \xrightarrow{-1} \mathbb{P}^{53} \\
\mathbb{P}^3 &\xrightarrow{\pi} \mathbb{P}^{49}
\end{align*}
\]

Acknowledgements. I would like to thank Bernd Sturmfels for suggesting this problem, Justin Chen for many helpful discussions and his Macaulay2 matroids package [Che15] which allowed for numerous computations that led to discoveries and conjectures, Alex Fink for generously providing the sketch of the proof for valuativeeness of the volume polynomial, and Federico Ardila for inspiring me to pursue this direction of research. I am also grateful for helpful conversations with David Eisenbud, June Huh, Vic Reiner, and Mengyuan Zhang.
References

[AA17] Federico Ardila and Marcelo Aguiar, Hopf monoids and generalized permutahedra, preprint (2017), available at arXiv:arXiv:1709.07504.

[AHK18] Karim Adiprasito, June Huh, and Eric Katz, Hodge Theory for combinatorial geometries, Annals of Mathematics 188 (2018), 381–452.

[AHK17] ———, Hodge theory of matroids, Notices Amer. Math. Soc. 64 (2017), no. 1, 26–30.

[AFR10] Federico Ardila, Alex Fink, and Felipe Rincón, Valuations for matroid polytope subdivisions, Canad. J. Math. 62 (2010), no. 6, 1228–1245.

[AK06] Federico Ardila and Caroline J. Klivans, The Bergman complex of a matroid and phylogenetic trees, J. Combin. Theory Ser. B 96 (2006), no. 1, 38–49.

[Bak18] Matthew Baker, Hodge theory in combinatorics, Bull. Amer. Math. Soc. (N.S.) 55 (2018), no. 1, 57–80.

[BJR09] Louis J. Billera, Ning Jia, and Victor Reiner, A quasisymmetric function for matroids, European J. Combin. 30 (2009), no. 8, 1727–1757.

[BGW03] Alexandre V. Borovik, I. M. Gelfand, and Neil White, Coxeter matroids, Progress in Mathematics, vol. 216, Birkhäuser Boston, Inc., Boston, MA, 2003.

[Bri96] Michel Brion, Piecewise polynomial functions, convex polytopes and enumerative geometry, Parameter spaces (Warsaw, 1994), Banach Center Publ., vol. 36, Polish Acad. Sci. Inst. Math., Warsaw, 1996, pp. 25–44.

[BH93] Winfried Bruns and Jürgen Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.

[Che15] Justin Chen, Matroids: a Macaulay2 package, preprint (2015), available at arXiv:1511.04618.

[CLP15] Julien Clancy, Timothy Leake, and Sam Payne, A note on Jacobians, Tutte polynomials, and two-variable zeta functions of graphs, Exp. Math. 24 (2015), no. 1, 1–7.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, Toric varieties, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[Dan78] V. I. Danilov, The geometry of toric varieties, Uspekhi Mat. Nauk 33 (1978), no. 2(200), 85–134, 247 (Russian).

[DCP95] C. De Concini and C. Procesi, Wonderful models of subspace arrangements, Selecta Math. (N.S.) 1 (1995), no. 3, 459–494.

[Den14] Graham Denham, Toric and tropical compactifications of hyperplane complements, Ann. Fac. Sci. Toulouse Math. (6) 23 (2014), no. 2, 297–333 (English, with English and French summaries).

[Der09] Harm Derksen, Symmetric and quasi-symmetric functions associated to polymatroids, J. Algebraic Combin. 30 (2009), no. 1, 43–86.

[DF10] Harm Derksen and Alex Fink, Valuative invariants for polymatroids, Adv. Math. 225 (2010), no. 4, 1840–1892.

[Eis95] David Eisenbud, Commutative algebra, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[EH16] David Eisenbud and Joe Harris, 3264 and all that—a second course in algebraic geometry, Cambridge University Press, Cambridge, 2016.

[ELM+05] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa, Asymptotic invariants of line bundles, Pure Appl. Math. Q. 1 (2005), no. 2, Special Issue: In memory of Armand Borel., 379–403.

[Fei05] Eva Maria Feichtner, De Concini-Procesi wonderful arrangement models: a discrete geometer’s point of view, Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., vol. 52, Cambridge Univ. Press, Cambridge, 2005, pp. 333–360.

[FY04] Eva Maria Feichtner and Sergey Yuzvinsky, Chow rings of toric varieties defined by atomic lattices, Invent. Math. 155 (2004), no. 3, 515–536.

[Ful93] William Fulton, Introduction to toric varieties, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[FS97] William Fulton and Bernd Sturmfels, Intersection theory on toric varieties, Topology 36 (1997), no. 2, 335–353.

[GGMS87] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova, Combinatorial geometries, convex polyhedra, and Schubert cells, Adv. in Math. 63 (1987), no. 3, 301–316.

[GK16] José Luis González and Kalle Karu, Some non-finitely generated Cox rings, Compos. Math. 152 (2016), no. 5, 984–996.

[Har77] Robin Hartshorne, Algebraic geometry, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Huh12] June Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, J. Amer. Math. Soc. 25 (2012), no. 3, 907–927.
[HK12] June Huh and Eric Katz, Log-concavity of characteristic polynomials and the Bergman fan of matroids, Math. Ann. 354 (2012), no. 3, 1103–1116.

[HW17] June Huh and Botong Wang, Enumeration of points, lines, planes, etc, Acta Math. 218 (2017), no. 2, 297–317.

[Kat16] Eric Katz, Matroid theory for algebraic geometers, Nonarchimedean and tropical geometry, Simons Symp., [Cham], 2016, pp. 435–517.

[Kee92] Sean Keel, Intersection theory of moduli space of stable n-pointed curves of genus zero, Trans. Amer. Math. Soc. 330 (1992), no. 2, 545–574.

[Kav11] Kiumars Kaveh, Note on cohomology rings of spherical varieties and volume polynomial, J. Lie Theory 21 (2011), no. 2, 263–283.

[KK12] Kiumars Kaveh and A. G. Khovanskii, Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory, Ann. of Math. (2) 176 (2012), no. 2, 925–978.

[Laz04] Robert Lazarsfeld, Positivity in algebraic geometry. I, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[LM09] Robert Lazarsfeld and Mircea Mustaţă, Convex bodies associated to linear series, Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 5, 783–835 (English, with English and French summaries).

[LdMRS17] Lucia Lopez de Medrano, Felipe Rincon, and Kristin Shaw, Chern-Schwartz-MacPherson classes of matroids, preprint (2017), available at arXiv:1707.07303.

[MS15] Diane Maclagan and Bernd Sturmfels, Introduction to tropical geometry, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015.

[OT92] Peter Orlik and Hiroaki Terao, Arrangements of hyperplanes, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992.

[Oxl11] James Oxley, Matroid theory, 2nd ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011.

[Pos09] Alexander Postnikov, Permutohedra, associahedra, and beyond, Int. Math. Res. Not. IMRN 6 (2009), 1026–1106.

[Sta12] Richard P. Stanley, Enumerative combinatorics. Volume I, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.