ON A WEAK TOPOLOGY FOR HADAMARD SPACES

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ABSTRACT. We investigate whether existing notions of weak sequential convergence on Hadamard spaces can be topologized, that is whether there exist corresponding notions of weak topologies. We provide an affirmative answer on what we call weakly proper Hadamard spaces. A notion of dual space is proposed and it is shown that our weak topology and dual space coincide with the standard ones in the case of a Hilbert space. We extend several results from classical functional analysis to the setting of Hadamard spaces, and we compare our topology to existing notions of weak topologies.

1. Introduction

A metric space \((X,d)\) is called a CAT(0) space if it is geodesically connected, and if every geodesic triangle in \(X\) is at least as thin as its corresponding comparison triangle in the Euclidean plane. CAT(0) spaces generalize the concept of spaces with non-positive sectional curvature in the sense of Alexandrov [1]. For an extensive treatment of these spaces, see, e.g., Bridson and Haefliger [4], D. Burago et al. [5], or Alexander et al. [19].

A metrically complete CAT(0) space is often called a Hadamard space, which we denote by \((H,d)\). Hilbert spaces constitute perhaps the simplest examples of Hadamard spaces. It is therefore natural to ask for notions of weak convergence and weak topologies that extend these notions from Hilbert spaces to general Hadamard spaces. The concept of \(\Delta\)-convergence (reviewed below) offers such an extension in terms of weak sequential convergence. Considering a slightly different notion of weak convergence by restricting to what we call weakly proper Hadamard spaces, we propose a weak topology such that the resulting notion of sequential convergence is indeed equivalent to that of weak convergence. We begin by reviewing \(\Delta\)-convergence.

The concept of \(\Delta\)-convergence. Lim [14] introduced the concept of \(\Delta\)-convergence on a general metric space \((X,d)\). A sequence \((x_n)\) in \(X\) is said to \(\Delta\)-converge to \(x\), written as \(x_n \xrightarrow{\Delta} x\), if

\[
\limsup_k d(x_{n_k}, x) \leq \limsup_k d(x_{n_k}, y)
\]

for every subsequence \((x_{n_k})\) of \((x_n)\) and for every \(y \in X\). As pointed out by Lim, this is equivalent to \(x\) being an asymptotic center of every subsequence of \((x_n)\). This result can be strengthened for Hadamard spaces in terms of uniqueness of asymptotic centers. Indeed, let \((x_n) \subset H\) be a bounded sequence. For \(y \in H\) define

\[
r(y, (x_n)) := \limsup_{n \to \infty} d(x_n, y) \quad \text{and let} \quad r((x_n)) := \inf_{y \in H} r((x_n), y)
\]

denote the asymptotic radius of \((x_n)\). Then the set \(\{y \in H \mid r(y, (x_n)) = r((x_n))\}\) consists of a single point, called the asymptotic center of \((x_n)\). Accordingly, a bounded sequence \((x_n) \subset H\) is said to be \(\Delta\)-convergent to a point \(x \in H\) whenever \(x\) is the asymptotic center of the sequence \((x_n)\).

It is worth noting that the notion of \(\Delta\)-convergence in CAT(0) spaces shares many properties with the usual notion of weak convergence in Banach spaces. For example,
Figure 1. Convergence of projections $P_\gamma x_n$ to $x$ along a geodesic $\gamma$ starting at $x$. The blue region is part of the elementary set $U_x(\varepsilon; \gamma)$, which might extend infinitely far to the left of $x$.

$\Delta$-convergence in CAT(0) satisfies the Kadec–Klee property (see [13, Kirk and Panyanak]) and a Banach–Saks-type property (see [6, Bącak]).

The concept of $\Delta$-convergence on Hadamard spaces is sometimes referred to as weak convergence. However, we reserve this term for a slightly different notion, which we introduce below. In particular, we drop the requirement for boundedness in our notion of weak convergence. Notice that the question of boundedness highlights one of the differences between Hadamard spaces and Banach spaces, since for the latter the Banach–Steinhaus theorem implies that every weakly convergent sequence must indeed be bounded.

**Weak convergence.** We base our notion of weak convergence on a notion proposed by Jost [9]. (See also Sosov [20] and Espíñola and Fernández-León [8].) We say that a sequence $(x_n) \subseteq H$ weakly converges to $x \in H$ if and only if $P_\gamma x_n \to x$ as $n \to +\infty$ along any geodesic $\gamma : [0, 1] \to H$ starting at $x$, where $P_\gamma$ denotes the projection to $\gamma$, see Figure 1.

Analogous definition can be stated in terms of nets, see §2.4. We denote weak convergence by $x_n \overset{w}{\to} x$. Notice that there is no requirement on boundedness in this definition, but if $(x_n)$ is bounded, then $x_n \overset{\Delta}{\to} x$ if and only if $x_n \overset{w}{\to} x$, see, e.g., [6, Proposition 3.1.3]. Therefore, our notion of weak convergence agrees with the notion of $\Delta$-convergence for bounded sequences in Hadamard spaces. Moreover, for Hilbert spaces, our notion of weak convergence agrees with the usual one.

**A weak topology inducing weak convergence.** An open question has been the identification of a topology $\tau_\Delta$ on $H$ such that for a given bounded sequence $(x_n) \subset H$ and $x \in H$ we have $x_n \overset{\Delta}{\to} x$ if and only if $(x_n)$ converges sequentially to $x$ with respect to $\tau_\Delta$. This was solved recently by [15, Lytchak–Petrunin] who introduce such a topology. However the question of constructing a topology that also characterizes weak convergence of unbounded sequences remained open. Inspired by a question of Bącak [7], the author had offered in his thesis [2] such a weak topology $\tau_w$ on what we call weakly proper Hadamard spaces. Although initially we did not know whether or not all Hadamard spaces were weakly proper, [15, §4] provides a counterexample. Nevertheless a wide range of Hadamard spaces are in fact weakly proper among which Hilbert spaces, locally compact Hadamard spaces, the Hilbert ball, and the infinite dimensional hyperbolic space are. More specifically we show:

1. On any weakly proper Hadamard space, weak sequential convergence and weak convergence coincide.
2. Weakly proper Hadamard spaces are always Hausdorff in $\tau_w$.
3. On a Hilbert space, our weak topology coincides with the usual weak topology.
4. On a locally compact Hadamard space, weak and strong topology coincide.

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$^1$This example was originally noted by M. Wardetzky.
Moreover, motivated by problems in optimization, we show that Hadamard spaces \((H,d)\) have some useful properties that resemble non-linear counterparts of the respective properties from the linear setting of Banach spaces:

1. A convex set is closed if and only if it is weakly closed.
2. The analogue of Mazur’s lemma holds.
3. If \((H,d)\) is separable and Hausdorff in \(\tau_w\), then weak compactness agrees with weak sequential compactness on bounded sets.
4. If \((H,d)\) is separable, then any bounded closed convex set is weakly compact.

Finally, we offer a comparison of our notion to previous notions of weak topologies in Hadamard spaces. In particular, we compare our notion to that of Monod [17], which offers a notion of a weak topology that is weaker than ours, but which does not yield convergence along geodesics and is thus too weak in terms of \(\Delta\)-convergence. Likewise, we compare our notion to that of Kakavandi [10], who offers a topology that is stronger than ours and for which it is unknown whether compactness results hold in general. We close in §4.4 with a brief discussion about the weak topology introduced by [15, Lytchak–Petrunin].

2. A weak topology on Hadamard spaces

2.1. Weakly open sets. Consider a Hadamard space \((H,d)\). A constant speed geodesic \(\gamma : [0,1] \to H\) is a curve that satisfies \(d(\gamma(s),\gamma(t)) = |s-t|d(\gamma(0),\gamma(1))\) for all \(s,t \in [0,1]\). For a given \(x \in H\) we define \(\Gamma_x(H)\) to be the set of all constant speed geodesics \(\gamma : [0,1] \to H\) such that \(\gamma(0) = x\). A set \(U \subset H\) is weakly open if for every \(x \in U\) there exists some \(\varepsilon > 0\) and a finite family of geodesics \(\gamma_1,\gamma_2,\ldots,\gamma_n \in \Gamma_x(H)\) such that the set

\[
U_x(\varepsilon;\gamma_1,\ldots,\gamma_n) := \{y \in H : d(x,P_{\gamma_i}y) < \varepsilon \ \forall i = 1,2,\ldots,n\}
\]

is contained in \(U\). Here \(P_{\gamma_i}y\) denotes the projection of \(y\) to the geodesic \(\gamma_i\). For a given geodesic segment \(\gamma \in \Gamma_x(H)\) we call \(U_x(\varepsilon;\gamma)\) an elementary set; see Figure [1].

Notice that the above collection of weakly open sets is non-empty (e.g., \(H\) is clearly weakly open). Moreover, this collection indeed defines a topology, which we call the weak topology \(\tau_w\). However, for this weak topology to be useful, we need to guarantee that it is not too coarse. Therefore, we introduce the following notion:

**Definition 1.** We call a Hadamard space weakly proper if every \(x \in H\) is an interior point for every elementary set \(U_x(\varepsilon;\gamma)\) in the weak topology \(\tau_w\), i.e., if for every \(x \in H\) and every \(\gamma \in \Gamma_x(H)\) there exists \(V \in \tau_w\) such that \(x \in V \subseteq U_x(\varepsilon;\gamma)\).

Notice that weak properness does neither imply that elementary sets are weakly open nor that their intersections are weakly open, but that, conversely, \(H\) is weakly proper if all elementary sets are weakly open.

It is straightforward to check that Hilbert spaces are weakly proper and that the resulting weak topology agrees with the usual one. Indeed, the latter is due to the fact that geodesics in Hilbert spaces are straight lines and projections to (oriented) straight lines can be identified with inner products and therefore with bounded linear functionals, see. An important non-trivial example of a weakly proper Hadamard space is the infinite dimensional hyperbolic space. We also show in Corollary [7] that locally compact Hadamard spaces are weakly proper. In fact, weak and strong topology coincide in this case. In general the product of weakly proper spaces is again weakly proper and in particular the product of any non-Euclidean locally compact Hadamard space with a Hilbert space. These form a large class of non-Hilbertian Hadamard spaces that are weakly proper and at the same time non-locally compact.
Basic properties. We follow standard notions in topology, see e.g. [12][16][18]. We start discussing a few basic properties of the weak topology \( \tau_w \). First notice that \( \tau_w \) is indeed weaker than the metric topology:

**Lemma 1.** Let \((H,d)\) be a Hadamard space. Then \( U_x(\varepsilon;\gamma) \) is open in the metric topology.

*Proof.* In a Hadamard space, projections \( P_C \) to closed convex sets \( C \) are nonexpansive operators, see, e.g., [4]. In particular, \( P_\gamma \) is nonexpansive since every geodesic segment \( \gamma \) is a closed convex set, i.e.,

\[
d(P_\gamma x, P_\gamma y) \leq d(x, y) \quad \forall x, y \in H.
\]

Now let \( y \in U_x(\varepsilon;\gamma) \). Then \( s := d(x, P_\gamma y) \) satisfies \( s < \varepsilon \). Therefore, the open geodesic ball \( B(y, \varepsilon - s) := \{ z \in H : d(x, y) < \varepsilon - s \} \) is contained in \( U_x(\varepsilon;\gamma) \). Indeed, let \( z \in B(y, \varepsilon - s) \). Then

\[
d(x, P_\gamma z) \leq d(x, P_\gamma y) + d(P_\gamma y, P_\gamma z) \leq d(x, P_\gamma y) + d(y, z) < s + (\varepsilon - s) = \varepsilon,
\]

where the penultimate inequality follows from (2). Therefore, \( z \in U_x(\varepsilon;\gamma) \). \( \Box \)

From nonexpansiveness of projections to geodesics (which are closed convex sets), we immediately obtain that strong convergence implies weak convergence:

**Lemma 2.** If \( x_n \to x \) then \( x_n \overset{w}{\to} x \).

We say that a sequence \((x_n) \subseteq H\) sequentially converges in \( \tau_w \) to an element \( x \in H \), and we denote this by \( x_n \overset{\tau_w}{\to} x \), if and only if for every weakly open set \( U \) containing \( x \) all but finitely many elements of the sequence \((x_n)_{n \in \mathbb{N}}\) are in \( U \).

**Theorem 1.** Let \((x_n)_{n \in \mathbb{N}} \subseteq H \) and let \( x \in H \). If \( x_n \overset{\tau_w}{\to} x \) then \( x_n \overset{w}{\to} x \). Moreover, if the Hadamard space is weakly proper then \( x_n \overset{\tau_w}{\to} x \) implies \( x_n \overset{w}{\to} x \).

*Proof.* Let \( x_n \overset{w}{\to} x \). Then for every \( \gamma \in \Gamma_x(H) \) we have that \( \lim_{n \to \infty} d(x, P_\gamma x_n) = 0 \), or equivalently \( x_n \in U_x(\varepsilon;\gamma) \) for all sufficiently large \( n \). Let \( U \in \tau_w \) contain \( x \). Then there exist \( \gamma_1, \ldots, \gamma_n \in \Gamma_x(H) \) and \( \varepsilon > 0 \) such that \( U_x(\varepsilon;\gamma_1, \ldots, \gamma_n) \subseteq U \). Since \( x_n \in U_x(\varepsilon;\gamma_1, \ldots, \gamma_n) \) for all sufficiently large \( n \), we obtain that \( x_n \in U \) for all sufficiently large \( n \).

Let \( H \) be weakly proper. Suppose that \( x_n \overset{\tau_w}{\to} x \) but \( x_n \overset{w}{\not\to} x \). Then there exists \( \gamma \in \Gamma_x(H) \) and \( \varepsilon > 0 \) such that \( x_n \notin U_x(\varepsilon;\gamma) \) for infinitely many \( n \). Weakly properness implies that there is an open set \( V \subseteq \tau_w \) containing \( x \) such that \( V \subseteq U_x(\varepsilon;\gamma) \). Therefore, \( x_n \notin V \) for infinitely many \( n \). However, this contradicts \( x_n \overset{\tau_w}{\to} x \). \( \Box \)

**Corollary 1.** If \( x_n \to x \) then \( x_n \overset{\tau_w}{\to} x \).

**Remark 1.** Weak limits are unique. Indeed if \( x_n \overset{w}{\to} x \) and \( x_n \overset{w}{\to} y \) then the elementary inequality \( d(x, y) \leq d(x, P_\gamma x_n) + d(P_\gamma x_n, y) = d(x, P_\gamma x_n) + d(P_\gamma x_n, y) \to 0 \) implies \( x = y \). Here \( \gamma : [0,1] \to H \) are geodesics such that \( \gamma(0) = x \), \( \gamma(1) = y \) and \( \tilde{\gamma}(0) = y \), \( \tilde{\gamma}(1) = x \). The same argument holds if sequence is replaced by a net.

**Lemma 3.** Every weakly proper Hadamard space is Hausdorff with respect to the weak topology \( \tau_w \).

*Proof.* By weak properness it suffices to show that for any distinct elements \( x, y \in H \) it holds that \( U_x(\varepsilon;\gamma) \cap U_y(l_\gamma - \varepsilon; \tilde{\gamma}) = \emptyset \). There exist unique geodesics \( \gamma, \tilde{\gamma} : [0,1] \to H \) connecting \( x \) with \( y \) such that \( \gamma(0) = x \), \( \gamma(1) = y \) and \( \tilde{\gamma}(0) = y \), \( \tilde{\gamma}(1) = x \). Let \( l_\gamma := d(x, y) \), and let \( \varepsilon \in (0, l_\gamma) \). Clearly \( U_x(\varepsilon;\gamma) \cap U_y(l_\gamma - \varepsilon; \tilde{\gamma}) = \emptyset \). \( \Box \)
2.2. Convex sets in Hadamard spaces. We say that a set $S \subseteq H$ is weakly closed if it is closed with respect to $\tau_w$.

**Theorem 2.** Let $(H,d)$ be a Hadamard space. A convex set $C \subseteq H$ is strongly closed if and only if it is weakly closed.

**Proof.** Any weakly closed set is strongly closed. So let $C$ be a strongly closed convex set. We show that $C$ is weakly closed. It suffices to show that $H \setminus C$ is weakly open. Let $y \in H \setminus C$. Then $P_{Cy}$ exists and is unique. Let $\gamma : [0,1] \to H$ be the geodesic connecting $y$ with $P_{Cy}$ such that $\gamma(0) = y$ and $\gamma(1) = P_{Cy}$. For $\varepsilon \in (0,l(\gamma))$, where $l(\gamma) := d(y,P_{Cy})$ is the length of the geodesic $\gamma$, consider the elementary set $U_{\varepsilon}(\gamma)$. Since $(H,d)$ is weakly proper, it suffices to show that $U_{\varepsilon}(\gamma) \cap C = \emptyset$. Let $x \in C$, and let $z \in H$ be arbitrary. Since both $C$ and $\gamma$ are strongly closed and convex, we have the following quadratic inequalities (see, e.g., \cite[Theorem 1.12]{6}):

$$d(x,z)^2 \geq d(x,P_{Cy})^2 + d(P_{Cy},z)^2$$

and

$$d(x,P_{Cy})^2 \geq d(x,P_{x})^2 + d(P_{x},P_{Cy})^2,$$

where we have used that $x$ lies in the convex set $C$ for the first inequality and that $P_{Cy}$ lies in the convex set $\gamma$ for the second inequality. Now let $z = P_{x}$ be the projection of $x$ to $\gamma$. Since $z \in \gamma$, we have that $P_{Cy} = P_{Cy}$, see \cite[Theorem 2.1.12]{6}. Then the above two inequalities imply that $P_{Cy} = z = P_{x}$. From $d(y,P_{x}) = d(y,P_{Cy}) + \varepsilon > \varepsilon$ for all $x \in C$ it follows that $U_{\varepsilon}(\gamma) \cap C = \emptyset$. Since $y \in H \setminus C$ is arbitrary then $U_{\varepsilon}(\gamma) \subseteq H \setminus C$ yields the claim. \hfill \Box

For any set $S \subseteq H$ let $co(S)$ denote the smallest convex set containing $S$.

**Theorem 3.** (Mazur’s Lemma). Let $(x_n) \subseteq H$ be a sequence such that $x_n \xrightarrow{w} x$ for some $x \in H$. Then there exists a function $N : \mathbb{N} \to \mathbb{N}$ and a sequence $(y_n) \subseteq H$ such that $y_n \to x$ and $y_n \in co(\{x_1,x_2,\ldots,x_{N(n)}\})$ for all $n \in \mathbb{N}$.

**Proof.** Weak convergence $x_n \xrightarrow{w} x$ implies (by Theorem 1) that $x \in wcl(\{x_1,x_2,\ldots\})$, where $wcl(\{x_1,x_2,\ldots\})$ is the weak closure of $\{x_1,x_2,\ldots\}$. Moreover, $\{x_1,x_2,\ldots\} \subseteq co(\{x_1,x_2,\ldots\})$ implies that $wcl(\{x_1,x_2,\ldots\}) \subseteq wcl(co(\{x_1,x_2,\ldots\}))$. Hence $x \in wcl(co(\{x_1,x_2,\ldots\}))$. The strong closure $cl\ co(\{x_1,x_2,\ldots\})$ of the convex set $co(\{x_1,x_2,\ldots\})$ is a closed convex set. Indeed, if $C$ is (geodesically) convex, then so is $cl\ C$. In order to see this, let $u,v \in cl\ C$. Consider sequences $(u_n)$ and $(v_n)$ in $C$ such that $u_n \to u$ and $v_n \to v$. Let $\gamma$ be the geodesic connecting $u$ with $v$, and let $\gamma$ be the geodesics connecting $u_n$ with $v_n$. Then strong convexity of the squared distance function in Hadamard spaces implies that for each $t \in [0,1]$ we have that

$$d(\gamma(t),\gamma_n(t))^2 \leq (1-t)d(u,\gamma_n(t))^2 + td(v,\gamma_n(t))^2 - t(1-t)d(u,v)^2$$

$$\leq (1-t)((1-t)d(u,u_n)^2 + td(u,v_n)^2 - t(1-t)d(u,v_n)^2)$$

$$+ t((1-t)d(v,u_n)^2 + td(v,v_n)^2 - t(1-t)d(u,v_n)^2)$$

$$- t(1-t)d(u,v)^2.$$  

Taking the limit $n \to \infty$ yields $d(\gamma(t),\gamma_n(t)) \to 0$. Hence $cl\ C$ and therefore $cl\ co(\{x_1,x_2,\ldots\})$ are indeed convex. Therefore, Theorem 2 implies that $cl\ co(\{x_1,x_2,\ldots\})$ is weakly closed. It follows that

$$x \in wcl(co(\{x_1,x_2,\ldots\})) \subseteq cl(co(\{x_1,x_2,\ldots\}))$$

Then there exists some sequence $(y_n) \subseteq co(\{x_1,x_2,\ldots\})$ such that $y_n \to x$. Additionally, we have that $co(\{x_1,x_2,\ldots\}) = \bigcup_{k \in \mathbb{N}} co(\{x_1,x_2,\ldots,x_k\})$. Hence $y_n \in co(\{x_1,x_2,\ldots,x_{k(n)}\})$ for some $k(n)$. For each $n$ set $N(n) := k(n)$. \hfill \Box
2.3. **Weak compactness in Hadamard spaces.** A set $K \subseteq H$ is called *weakly sequentially compact* if every sequence in $K$ has a weakly convergent subsequence. Similarly, $K$ is called *$\tau_w$-sequentially compact* if every sequence in $K$ has a $\tau_w$-convergent subsequence. Weak sequential compactness implies $\tau_w$-sequential compactness, and these notions coincide on a weakly proper Hadamard space. Finally, a set $K$ is called *weakly compact* if for any open cover in $\tau_w$ of $K$ there is a finite subcover of $K$.

**Lemma 4.** ([6] Proposition 3.1.2) Every bounded sequence in a Hadamard space has a weakly convergent subsequence.

**Lemma 5** ([6] Lemma 3.2.1). Let $K \subseteq H$ be a closed convex set and $(x_n)_{n \in \mathbb{N}} \subset K$. If $x_n \xrightarrow{w} x$ then $x \in K$.

Lemmas 4 and 5 immediately imply the following result:

**Theorem 4.** Any bounded closed convex set $K$ in a Hadamard space is weakly sequentially compact and therefore $\tau_w$-sequentially compact.

**Proposition 2.** Let $(H, d)$ be separable. Then any weakly sequentially compact set $K \subset H$ is weakly compact.

*Proof.* The proof proceeds by contradiction. Suppose that $K \subseteq H$ is weakly sequentially compact but not weakly compact. Then there exists some open cover $\{U_i\}_{i \in I}$ of $K$ in $\tau_w$ that has no finite subcover. By assumption $(H, d)$ is separable. Hence $(H, d)$ is a Lindelöf space, i.e., every open cover (in the strong topology) has a countable subcover; see, e.g., [16, Theorem 6.7]. Since $\{U_i\}_{i \in I}$ is an open cover in the usual metric topology, there exists a countable subcover $\{U_j\}_{j \in J}$. Let $V_n := \bigcup_{j=1}^n U_j$. Then $W_n := H \setminus V_n$ is weakly closed for all $n$. Moreover, the family of sets $W_n$ satisfies $W_{n+1} \subseteq W_n$. Because $V_n$ cannot cover $K$, we have that $W_n \cap K$ is nonempty for every $n \in \mathbb{N}$. Let $x_n \in W_n \cap K$. Since $K$ is weakly sequentially compact, and consequently $\tau_w$-sequentially compact, the sequence $(x_n)$ has a subsequence $(x_{n_k})$ that converges in $\tau_w$ to some element $x^* \in K$. Let $U_w(x^*)$ denote the collection of weakly open sets containing $x^*$. Then for each $U \in U_w(x^*)$ and for each $n \in \mathbb{N}$ there exists $m \geq n$ such that $U \cap W_m \neq \emptyset$, and in particular $U \cap W_n \neq \emptyset$, implying that $x^* \in \text{wcl} W_n = W_n$. Since $n$ is arbitrary, we have that $x^* \in \bigcap_n W_n$. Hence $x^* \in \bigcap_n W_n \cap K$. Therefore, $\bigcap_{n \in \mathbb{N}} W_n \cap K \neq \emptyset$, which together with $\bigcap_{n \in \mathbb{N}} W_n \cap K \subseteq K$, yields that $K \supseteq K \setminus (\bigcap_{n \in \mathbb{N}} W_n \cap K) = K \setminus \bigcap_{n \in \mathbb{N}} (K \setminus V_n) = \bigcup_{n \in \mathbb{N}} (K \setminus V_n) = K$. □

**Remark 2.** Notice that the previous proof also applies to the more general setting of a topology that is weaker than the metric topology in any separable metric space.

**Proposition 3.** Let $(H, d)$ be a Hadamard space that is Hausdorff with respect to $\tau_w$ and let $K \subseteq H$ be bounded. If $K$ weakly compact then $K$ is weakly sequentially compact.

*Proof.* Let $(x_n)_{n \in \mathbb{N}} \subseteq K$, then $(x_n)$ is bounded. By Lemma 4 there is a subsequence $(x_{n_k}) \xrightarrow{w} x$ for some $x \in H$. By Theorem 4 we have $x_{n_k} \xrightarrow{\tau_w} x$. Assumption $\tau_w$ is Hausdorff implies that $K$ is weakly closed, consequently $\tau_w$-sequentially closed, hence $x \in K$. □

As an immediate consequence of Proposition 2 and Proposition 3 we obtain:

**Corollary 4.** Let $(H, d)$ be separable and Hausdorff with respect to $\tau_w$. If $K \subseteq H$ is bounded then $K$ is weakly compact if and only if $K$ is weakly sequentially compact.

**Corollary 5.** Closed bounded and convex sets in a separable Hadamard space are weakly compact.

**Question 1.** Is a weakly compact unbounded set $K \subseteq H$ also weakly sequentially compact?
This question is motivated by the example of the simplicial tree given by Monod [17]. This tree consists of countably many rays of finite but ever increasing length, all meeting at one vertex. If \( \{V_i\}_{i \in I} \) is some open cover in \( \tau_w \) for the simplicial tree then there exists some \( i \in I \) such that \( V_i \) contains an elementary set that has the common vertex. By construction this elementary set covers most of the space except for a certain ray which can be covered by at most a finite number of elementary sets. Therefore we can always find a finite subcover from our original cover \( \{V_i\}_{i \in I} \) i.e. the simplicial tree is weakly compact.

2.4. Locally compact spaces. We now turn to locally compact Hadamard spaces. Recall that a topological space \( (X, \tau) \) is said to be locally compact if for every \( x \in X \) there exists an open set \( U \in \tau \) and a compact set \( K \) such that \( x \in U \subseteq K \). Let \((\mathcal{A}, \geq)\) be a directed set and \((x_\alpha)_{\alpha \in \mathcal{A}} \) a net in \( H \) then \( x_\alpha \xrightarrow{w} x \) if and only if for every \( \gamma \in \Gamma_x(H) \) and every \( \varepsilon > 0 \) there is \( \alpha_\varepsilon \in \mathcal{A} \) such that \( d(x, P_\gamma x_\alpha) < \varepsilon \) for all \( \alpha \geq \alpha_\varepsilon \) i.e. \( x_\alpha \) is eventually in \( U_x(\varepsilon; \gamma) \) for every \( \gamma \in \Gamma_x(H) \) and every \( \varepsilon > 0 \).

**Theorem 5.** [10, Proposition 4.3] The followings statements are equivalent for a Hadamard space \((H, d)\):

- \((H, d)\) is locally compact.
- Every closed and bounded subset of \((H, d)\) is compact.
- Every bounded sequence in \((H, d)\) has a convergent subsequence.

**Lemma 6.** Let \((H, d)\) be locally compact Hadamard space and \((x_\alpha)_{\alpha \in \mathcal{A}} \) a net in \( H \). If \( x_\alpha \xrightarrow{w} x \), then \((x_\alpha)\) is bounded.

**Proof.** Let \( x_\alpha \xrightarrow{w} x \), but suppose that \((x_\alpha)\) is unbounded. Then we can w.l.o.g. assume that \( d(x_\alpha, x) \geq R > 0 \) for all \( \alpha \in \mathcal{A} \). Consider the closed geodesic ball \( B(x, R) \). Denote by \( P_C x_\alpha \) the projection of \( x_\alpha \) onto \( C \) for every \( \alpha \in \mathcal{A} \). By assumption \((H, d)\) is locally compact. Theorem 5 implies that \( C \) is compact, and in particular the boundary \( \partial C \) is compact. Hence, \( (P_C x_\alpha) \) is a net in the compact set \( \partial C \). By [12, §5, Theorem 2] there exits a subnet \((P_C x_\beta)_{\beta \in \mathcal{B}} \) converging to some element \( z \in \partial C \). Let \( \gamma : [0, 1] \to H \) denote the geodesic segment connecting \( x \) to \( z \). Evidently, \( \gamma \subset C \). Denote by \( \gamma_\beta : [0, 1] \to H \) the geodesic segment connecting \( x \) with \( P_C x_\beta \) for each \( \beta \in \mathcal{B} \). Let \( P_\gamma x_\beta \) denote the projection of \( x_\beta \) onto the geodesic segment \( \gamma \). From the triangle inequality we obtain

\[
d(P_\gamma x_\beta, z) \geq d(x_\beta, P_C x_\beta) - d(x_\beta, z),
\]

which implies that \( \lim_\beta |d(x_\beta, P_C x_\beta) - d(x_\beta, z)| = 0 \). Since both \( C \) and \( \gamma \) are strongly closed and convex, we have the following quadratic inequalities (see, e.g., [6, Theorem 2.1.12]):

\[
d(x_\beta, P_C x_\beta)^2 + d(P_C x_\beta, P_\gamma x_\beta)^2 \leq d(x_\beta, P_\gamma x_\beta)^2,
\]

implying that \( d(x_\beta, P_C x_\beta) \leq d(x_\beta, P_\gamma x_\beta) \leq d(x_\beta, z) \). Therefore, we have that

\[
\lim_\beta |d(x_\beta, P_\gamma x_\beta) - d(x_\beta, P_C x_\beta)| = 0.
\]

By assumption \( x_\alpha \xrightarrow{w} x \), and in particular, \( x_\beta \xrightarrow{w} x \). Therefore, it follows that \( P_\gamma x_\beta \to x \). Consider the geodesic segment \( \eta_\beta : [0, 1] \to H \) connecting \( x \) with \( x_\beta \). Then there exists \( z_\beta \in \eta_\beta \) such that \( z_\beta \in \partial C \) for every \( \beta \in \mathcal{B} \). Since \( z_\beta \in \partial C \), we obtain that \( d(x_\beta, P_C x_\beta) \leq d(x_\beta, z_\beta) \) and thus

\[
d(x_\beta, x) = d(x_\beta, z_\beta) + d(z_\beta, x) \geq d(x_\beta, P_C x_\beta) + R, \quad \forall \beta \in \mathcal{B}
\]

which in turn implies that \( |d(x_\beta, x) - d(x_\beta, P_C x_\beta)| \geq R > 0 \). By the triangle inequality we again have

\[
d(P_\gamma x_\beta, x) \geq |d(x_\beta, x) - d(x_\beta, P_\gamma x_\beta)|, \quad \forall \beta \in \mathcal{B}.
\]
Therefore, \( \lim_{\beta} P_{\gamma} x_\beta = x \) implies that \( \lim_{\beta} |d(x_\beta, x) - d(x_\beta, P_{\gamma} x_\beta)| = 0 \). Moreover we have

\[
0 < R \leq |d(x_\beta, x) - d(x_\beta, P_C x_\beta)| \leq |d(x_\beta, x) - d(x_\beta, P_{\gamma} x_\beta)| + |d(x_\beta, P_{\gamma} x_\beta) - d(x_\beta, P_C x_\beta)|,
\]

which together with (3) yields a contradiction since the right side vanishes in the limit. \( \square \)

**Corollary 6.** Let \((H, d)\) be locally compact Hadamard space and \((x_n)_{n \in \mathbb{N}}\) a sequence in \(H\). If \(x_n \xrightarrow{w} x\), then \((x_n)\) is bounded.

**Theorem 6.** In a locally compact Hadamard space \((H, d)\) weak topology and strong topology coincide.

**Proof.** Any weakly open set is open. For the converse direction let \(U\) be an open set, and suppose that \(U\) is not weakly open. Then there is some \(x \in U\) such that for any finite collection of geodesic segments \(\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma_x(H)\) and any \(\varepsilon > 0\) the set \(\bigcap_{n=1}^{\infty} U_x(\varepsilon; \gamma_i)\) is not entirely contained in \(U\). Let \(U\) be the collection of all finite subsets from \(\Gamma_x(H)\). Two sets \(F_1, F_2 \in \mathcal{F}\) are said to be equivalent whenever \(|F_1| = |F_2|\). Let \([F]\) denote an equivalence class and \(|[F]| = n_F\) its cardinality. Consider the following sets

\[
V_F = \{ y \in C : d(x, P_y y) < 1/n_F, \gamma \in F \}.
\]

By the above observation then for every \(F \in \mathcal{F}\) there exists \(x_F \in V_F \setminus U\). If we set \(F_1 \supseteq F_2\) whenever \(F_1 \subseteq F_2\) then \((\mathcal{F}, \supseteq)\) is a directed set. In particular \((x_F)_{F \in \mathcal{F}}\) is a net and by construction \((x_F)_{F \in \mathcal{F}}\) weakly converges to \(x\). From Lemma 3 the net \((x_F)_{F \in \mathcal{F}}\) is bounded and consequently by Theorem 3 the (strong) closure \(\overline{\{(x_F)_{F \in \mathcal{F}}\}}\) is compact in \(H\). Then by virtue of [12, §5, Theorem 2] there is a subnet \((x_{F'})_{F' \in \mathcal{F}'}\) of the net \((x_F)_{F \in \mathcal{F}}\) converging to a certain element \(y \in \overline{\{(x_F)_{F \in \mathcal{F}}\}}\). In particular \(x_{F'} \xrightarrow{w} y\). By Remark 1 we have that \(y = x\). This is impossible since by construction \(x_{F'} \notin U\) for all \(F' \in \mathcal{F}'\). \( \square \)

**Corollary 7.** Every locally compact Hadamard space is weakly proper.

**Proof.** It follows from Theorem 6 that the elementary sets \(U_x(\varepsilon; \gamma)\) are weakly open for every \(x \in H, \gamma \in \Gamma_x(H)\) and \(\varepsilon > 0\). In particular \(x\) is an interior point in \(\tau_w\) of any elementary set \(U_x(\varepsilon; \gamma)\). \( \square \)

**Corollary 8.** In a locally compact Hadamard space weak and strong convergence coincide.

3. The dual space

3.1. Construction of the dual and weak-* topology. The weak topology can also be characterized by considering functions \(\phi_\gamma(x; \cdot) : H \to \mathbb{R}_+\) given by

\[
\phi_\gamma(x; y) := d(x, P_\gamma y) \quad \forall y \in H, \gamma \in \Gamma_x(H).
\]

Indeed, let \(\gamma \in \Gamma_x(H)\), then \(U_x(\varepsilon; \gamma) = \phi_\gamma^{-1}(x; \cdot)[0, \varepsilon)\). Notice that nonexpansiveness of projections to convex sets implies that \(\phi_\gamma\) is Lipschitz continuous with Lipschitz constant equal to 1. Let

\[
\|\phi_\gamma - \phi_\eta\|_\infty := \sup_{y \in H \setminus \{x\}} \frac{|\phi_\gamma(x; y) - \phi_\eta(x; y)|}{d(x, y)}.
\]

Let \(H^*_x := \{ \phi_\gamma(x; \cdot) : \gamma \in \Gamma_x(H) \}\), and let \(d_*(\phi_\gamma, \phi_\eta) := \|\phi_\gamma - \phi_\eta\|_\infty\). Notice that due to the fact that \(\phi_\gamma(x; y) \leq d(x, y)\), the value of \(d_*(\phi_\gamma, \phi_\eta)\) is finite. We claim that \((\Phi_x(H), d_*)\) is a metric space. In order to show definiteness, we require the notions of comparison triangles and Alexandrov angles. A geodesic triangle \(\Delta(pqr)\) determined by three points \(p, q, r \in H\) consists of three geodesic segments \([p, q], [q, r], [r, p]\). To each geodesic triangle \(\Delta(pqr)\) there corresponds a comparison triangle \(\Delta(\overline{pqr})\) in the Euclidean plane \(\mathbb{E}^2\) with vertices \(\overline{p}, \overline{q}, \overline{r}\) such that \(d(p, q) = \|\overline{p} - \overline{q}\|, d(q, r) = \|\overline{q} - \overline{r}\|\) and \(d(p, r) = \|\overline{p} - \overline{r}\|\).
interior angle of $\Delta(pqr)$ at $p$ is called the comparison angle between $q$ and $r$ at $p$ and it is denoted by $\overline{Z}_p(q, r)$. Now let $\gamma$ and $\eta$ be two geodesic segments emanating from the same point $p$, i.e., $\gamma(0) = \eta(0) = p$. The Alexandrov angle between $\gamma$ and $\eta$ at $p$ is the number $\angle_p(\gamma, \eta) \in [0, \pi]$ defined as

$$
\angle_p(\gamma, \eta) := \limsup_{t,t' \to 0} \overline{Z}_p(\gamma(t), \eta(t'))
$$

In a Riemannian manifold, the Alexandrov angle coincides with the usual angle.

**Lemma 7.** $(\Phi_x(H), d_x)$ is a metric space.

**Proof.** Symmetry and the triangle inequality are evident. In order to show definiteness, suppose that $d_x(\phi_\gamma, \phi_\eta) = 0$, then this means $\phi_\gamma(x; y) = \phi_\eta(x; y)$ for all $y \in H$. In particular, we obtain that $\phi_\eta(x; \gamma(1)) = d(x, \gamma(1))$ and $\phi_\gamma(x; \eta(1)) = d(x, \eta(1))$. On the other hand, we have the equation $d(x, \gamma(1)) = d(x, P_\gamma \eta(1)) + d(P_\gamma \eta(1), \gamma(1))$ and analogously the equation $d(x, \eta(1)) = d(x, P_\eta \gamma(1)) + d(P_\eta \gamma(1), \eta(1))$. These two equations imply that $d(P_\gamma \eta(1), \gamma(1)) = d(P_\eta \gamma(1), \eta(1)) = 0$. We claim that $\gamma(1) = \eta(1)$. If not then the Alexandrov angle $\angle_p(\gamma(1), [\gamma(1), \eta(1)])$ implies $\angle_p(\gamma(1), [\gamma(1), \eta(1)]) \geq \pi/2$ and analogously $\angle_p(\gamma(1), [\gamma(1), \eta(1)]) \geq \pi/2$ (see Theorem 2.1.12). On the other hand by the cosine formula for Euclidean triangles, the comparison angles $\overline{Z}_{\gamma(1)}(x, \eta(1))$ and $\overline{Z}_{\eta(1)}(x, \gamma(1))$ satisfy $\overline{Z}_{\gamma(1)}(x, \eta(1)) \geq \angle_p(\gamma(1), [\gamma(1), \eta(1)])$ and $\overline{Z}_{\eta(1)}(x, \gamma(1)) \geq \angle_p(\gamma(1), [\gamma(1), \eta(1)])$. This in turn would imply $\overline{Z}_{\gamma(1)}(x, \eta(1)) \geq \pi/2$ and $\overline{Z}_{\eta(1)}(x, \gamma(1)) \geq \pi/2$, which is impossible. □

We call $(H^*, d_x)$ the dual of $H$ at $x$. We introduce a topology that is weaker than the one induced by $d_x(\cdot, \cdot)$. The concept of weak-* convergence is defined as follows: A sequence $(\phi_{\gamma_n}) \subseteq H^*_x$ is said to weak-* converge to some $\phi_\gamma \in H^*_x$ if and only if $\lim_{n \to \infty} \phi_{\gamma_n}(x; y) = \phi_\gamma(x; y)$ for all $y \in H$. It is obvious that strong convergence implies weak-* convergence. Weak-* convergence gives rise to a topology which we call the weak-* topology on $H^*_x$. A basis for this topology is determined by the sets

$$
U_\gamma(\varepsilon; y_1, y_2, \ldots, y_n) := \{\phi_\eta \in H^*_x : |\phi_\gamma(x; y_i) - \phi_\eta(x; y_i)| < \varepsilon, \forall i = 1, 2, \ldots, n\}, \quad n \in \mathbb{N}.
$$

i.e., any open set in the weak-* topology is a union of sets of the form $(\ref{5})$. We denote this topology by $\tau_{w^*}$.

**Theorem 7.** The following properties hold:

1. A sequence $(\phi_{\gamma_n})$ weak-* converges to $\phi_\gamma$ if and only if $\phi_{\gamma_n} \xrightarrow{\tau_{w^*}} \phi_\gamma$.
2. $(H^*_x, \tau_{w^*})$ is a Hausdorff space.
3. A weak-* closed set in $H^*_x$ is closed.

**Proof.** The first property follows from the definition of weak-* topology and that of open sets given in $(\ref{5})$. For the second property it suffices to show that for any two distinct elements $\phi_\gamma$ and $\phi_\eta$ there is $\varepsilon > 0$ such that the open sets $U_\gamma(\varepsilon; y)$ and $U_\eta(\varepsilon; y)$ have empty intersection for some $y \in H$. Let $\varepsilon := |\phi_\gamma(x; y) - \phi_\eta(x; y)|/2$ and suppose there is $\phi_\mu \in U_\gamma(\varepsilon; y) \cap U_\eta(\varepsilon; y)$ then

$$
|\phi_\gamma(x; y) - \phi_\eta(x; y)| \leq |\phi_\gamma(x; y) - \phi_\mu(x; y)| + |\phi_\mu(x; y) - \phi_\eta(x; y)| < \frac{1}{2}|\phi_\gamma(x; y) - \phi_\eta(x; y)| + \frac{1}{2}|\phi_\gamma(x; y) - \phi_\eta(x; y)| = |\phi_\gamma(x; y) - \phi_\eta(x; y)|
$$

which is impossible. In order to show the third property, let $S \subseteq H^*_x$ be a weak-* closed set, and let $\phi_{\gamma_n}$ be a sequence in $S$. Suppose that $\phi_{\gamma_n} \to \phi_\gamma$ in the strong topology, then $\phi_{\gamma_n} \xrightarrow{\tau_{w^*}} \phi_\gamma$, which implies that $\phi_\gamma \in S$. Therefore $S$ is (strongly) closed. □
3.2. The case of a Hilbert space. Let $H$ be a Hilbert space $(\mathcal{H}, \| \cdot \|)$ equipped with its canonical norm. Note that for every $x \in \mathcal{H}$ each geodesic segment $\gamma \in \Gamma_x(\mathcal{H})$ corresponds to a unique line $l$ passing through $x$. Given $\gamma, \eta \in \Gamma_x(\mathcal{H})$ we say $\gamma$ is equivalent to $\eta$, and write $\gamma \sim \eta$, if and only if $\gamma, \eta$ belong to the same line $l$. Let $[l]$ denote the equivalence class of all geodesic segments $\gamma \in \Gamma_x(\mathcal{H})$ sharing the same line $l$.

Our aim is to show that our notion of weak convergence along geodesic segments coincides with the usual notion of weak convergence in a Hilbert space. Moreover we prove that a Hilbert space is weakly proper. Our aim is to show that our dual $\mathcal{H}^*_x$ coincides with the usual notion of the dual of a Hilbert space. We require the following lemma.

**Lemma 8.** A sequence $(x_n) \subset \mathcal{H}$ weakly converges (in the sense of Section 3) to $x \in \mathcal{H}$ if and only if $P_l x_n \to x$ as $n \uparrow + \infty$ for all lines $l$ containing $x$.

**Proof.** Suppose that $\lim_n P_l x_n = x$ for all lines $l$ containing $x$. Let $\gamma \in \Gamma_x(\mathcal{H})$ such that $\gamma \subset l$. Then all but finitely many of the terms $P_l x_n$ are in the image of $\gamma$, i.e., $P_l x_n = P_{\gamma} x_n$ for all sufficiently large $n$. This means $\lim_n P_{\gamma} x_n = \lim_n P_l x_n = x$. Since this holds for any $l$ containing $x$ and for any $\gamma \in [l]$ then $\lim_n P_{\gamma} x_n = x$ for any geodesic segment $\gamma \in \Gamma_{x}(\mathcal{H})$. Now let $x_n \overset{w}{\to} x$. By definition $\lim_n P_{\gamma} x_n = x$ for all $\gamma \in \Gamma_{x}(\mathcal{H})$. Each $\gamma \in \Gamma_{x}(\mathcal{H})$ determines a unique line $l$ containing $x$. Then a similar argument shows that $\lim_n P_l x_n = x$ for all lines containing $x$.

To the collection $\{ \phi_{\gamma}(x; \cdot) \}_{\gamma \in [l]}$ we can associate a function $\phi_{[l]}(x; \cdot)$ defined as $\phi_{[l]}(x; y) := \| x - P_l y \|$, $\forall y \in \mathcal{H}$. Let $\mathcal{H}^*$ denote the dual of a Hilbert space $\mathcal{H}$, that is the space of all bounded linear continuous functionals on $\mathcal{H}$. A sequence $(x_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ is said to converge weakly (in the usual sense) to $x \in \mathcal{H}$ if and only if $\lim_n f(x_n) = f(x)$ for all $f \in \mathcal{H}^*$.

**Proposition 9.** In a Hilbert space weak convergence (in the sense of Section 3) coincides with the usual notion of weak convergence.

**Proof.** Let $x \in \mathcal{H}$ and denote by $\mathcal{H}^*_x := \{ \phi_{\gamma} \mid \gamma \in \Gamma_{x}(\mathcal{H}) \}$ where $\phi_{\gamma}(y) := \| x - P_{\gamma} y \|$ for every $y \in \mathcal{H}$. By Lemma 8 it follows that $\lim_n \phi_{\gamma}(x; x_n) = 0$ for all $\gamma \in [l]$ if and only if $\lim_n \phi_{[l]}(x; x_n) = 0$. Therefore it is sufficient to restrict to the family of functions $\{ \phi_{[l]}(x; \cdot) \}_L$ where $L$ is the set of all lines containing $x$. Clearly $\{ \phi_{[l]}(x; \cdot) \}_L = \mathcal{H}^*_x / \sim$. Since projection to lines in Hilbert spaces correspond to inner products with a unit vector along the respective line, it follows that the quotient space $\mathcal{H}^*_x / \sim$ together with its quotient metric coincides with the usual dual $\mathcal{H}^*$ of the Hilbert space $\mathcal{H}$. □

**Proposition 10.** A Hilbert space is weakly proper. In particular the usual weak topology coincides with $\tau_w$.

**Proof.** Let $x \in \mathcal{H}$ and consider the elementary set $U_x(\varepsilon; \gamma)$ for some $\varepsilon > 0$ and $\gamma \in \Gamma_{x}(\mathcal{H})$. Let $l$ be the unique line passing through $x$ corresponding to the geodesic segment $\gamma$. Let $y \in U_x(\varepsilon; \gamma)$ and define $l' := l + (y - x)$. Then $l'$ is parallel to $l$. Let $\alpha$ denote the place determined by $l$ and $l'$. Note that for any $z \in \mathcal{H}$ the projections $P_{\alpha} z, P_{l'} z$ and $P_{l''} z$ are collinear and moreover their common line $l''$ is perpendicular to both $l$ and $l'$. This argument together with $y \in U_x(\varepsilon; \gamma)$ imply that for some $\delta > 0$ and $\gamma' \in \Gamma_{y}(\mathcal{H})$ belonging to $[l']$ we have $U_y(\delta; \gamma') \subseteq U_x(\varepsilon; \gamma)$. Clearly weak topology coincides with $\tau_w$. □

**Corollary 11.** The Cartesian product of a locally compact Hadamard space and a Hilbert space is weakly proper.

**Proof.** Follows immediately from Corollary 7 and Proposition 10. □

For a given $y \in H$ and $z \in l$ let $\theta$ be the the angle between the vectors $y - x$ and $z - x$. Then from the cosine formula for inner product we get

$$
\langle y - x, z - x \rangle = \| y - x \| \| z - x \| \cos \theta
$$

(6)
Realizing that $\|y - x\| \cos \theta = \pm \|P_{\gamma} y - x\|$ then follows

$$\pm \phi_{|}(x; y) = \frac{1}{\|z_{l} - x\|} \langle y - x, z_{l} - x \rangle, \ \forall y \in \mathcal{H}$$

Using the linearity of the inner product one can rewrite (7) as

$$\phi_{|}(x; y) = \langle y - x, u_{l} \rangle, \ \forall y \in \mathcal{H}$$

where $u_{l} := \pm \frac{z_{l} - x}{\|z_{l} - x\|}$

From (8) it is evident that $H_{x}/\sim$ coincides with the dual $\mathcal{H}^{*}$ of the Hilbert space $\mathcal{H}$.

4. OTHER FORMS OF WEAK TOPOLOGY

There have been previous treatments of weak topologies for Hadamard spaces. These attempts have offered other perspectives on the notion of the weak convergence, which we compare to our notion.

4.1. Kakavandi’s weak topology. Kakavandi [10] proposed a notion of a weak topology, which is based on the following observation. In a Hilbert space $(\mathcal{H}, \| \cdot \|)$ equipped with its canonical norm $\| \cdot \|$ a sequence $(x_{n})$ converges weakly to an element $x \in \mathcal{H}$ iff $\lim_{n \to \infty} \langle x_{n} y \rangle = \langle x y \rangle$ for all $y \in \mathcal{H}$. This is equivalent to $\lim_{n \to \infty} \langle x_{n} - z, y - z \rangle = \langle x - z, y - z \rangle$ for all $y, z \in \mathcal{H}$. Then the identity

$$\langle x - z, y - w \rangle = \frac{1}{2}(\|x - y\|^{2} + \|z - w\|^{2} - \|x - w\|^{2} - \|z - y\|^{2})$$

gives rise to the possibility of extending the definition of weak convergence to a general metric space $(X, d)$ by expressing the right side of (9) in terms of the metric $d(\cdot, \cdot)$. Following Berg and Nikolaev [3] consider the Cartesian product $X \times X$ where $X$ is a general metric space. Each pair $(x, y) \in X \times X$ determines a so-called bound vector which is denoted by $\vec{x y}$. The point $x$ is called the tail of $\vec{x y}$ and $y$ is called the head. The zero bound vector is $0_{x} = \vec{x x}$. The length of a bound vector $\vec{x y}$ is defined as the metric distance $d(x, y)$.

Furthermore, if $\vec{x y} := \vec{x y}$, then $- \vec{u w} := \vec{y z}$. Let

$$\langle \vec{x z}, \vec{y w} \rangle := \frac{1}{2}(d(x, y)^{2} + d(z, w)^{2} - d(x, w)^{2} - d(z, y)^{2})$$

Kakavandi’s notion of weak convergence is defined in the following way. A sequence $(x_{n})$ in a Hadamard space $(H, d)$ converges weakly to an element $x \in H$ if and only if $\lim_{n \to \infty} \langle \vec{x x_{n}}, \vec{x y} \rangle = 0$ for all $y \in H$. This form of weak convergence coincides with the usual weak convergence in a Hilbert space. It turns out, however, that Kakavandi’s notion of convergence does not coincide with $\Delta$-convergence, see Example 4.7 in [10].

There is a natural topology associated to Kakavandi’s convergence generated by sets of the form

$$W(x, y; \varepsilon) := \{ z \in H \mid \langle \vec{x z}, \vec{y w} \rangle < \varepsilon \}, \ \text{for any} \ x, y \in H, \ \varepsilon > 0.$$ 

More precisely, the family of sets $\{W(x, y; \varepsilon) \mid x, y \in H, \varepsilon > 0\}$ forms a subbasis for Kakavandi’s topology $\tau_{K}$. One has $x_{n} \xrightarrow{\tau_{K}} x$ if and only if $x_{n} \xrightarrow{\tau_{K}} x$, see [10, Theorem 3.2]. Moreover, Kakavandi’s topology is Hausdorff.

Theorem 8. Let $(H, d)$ be a Hadamard space. Then the followings hold:

(i) $\tau_{w}$ is coarser than $\tau_{K}$.

(ii) Kakavandi convergence and weak convergence coincide in a locally compact space.
ON A WEAK TOPOLOGY FOR HADAMARD SPACES

Proposition 12. The following chain of inclusions holds: \( \tau_M \subseteq \tau_w \subseteq \tau_K \). All three topologies coincide with the usual weak topology whenever \((H,d)\) is a Hilbert space.

Proof. Let \((H,d)\) be a Hadamard space. Theorem 2 implies that a convex set is \(\tau_w\)-closed if and only if it is \(\tau_s\)-closed. Hence \(\tau_M \subseteq \tau_w\). The last statement of the theorem follows from the fact that Hilbert spaces are weakly proper together with the fact that \(\tau_M\) and \(\tau_K\) coincide with the usual weak topology on any Hilbert space, see [17, Example 18].

It is known that if \(K \subseteq H\) is compact then the restrictions of \(\tau_M\) and \(\tau_s\) to \(K\) coincide (see [17, Lemma 17]). Hence, in view of Proposition 12 the restrictions of all three weak topologies to a compact subset \(K\) of a Hadamard space \(H\) coincide with the strong topology. An important property of Monod’s topology is that any \(\tau_s\)-closed convex and bounded set is \(\tau_M\)-compact, see [17, Theorem 14]. However, it turns out that in general a Hadamard space is not Hausdorff with respect to \(\tau_M\). For example, the Euclidean cone of an infinite dimensional Hilbert space is not Hausdorff when equipped with \(\tau_M\) [11, Example 3.6].

4.3. Geodesically monotone operators. A continuous operator \(T: H \to H\) is said to be geodesically monotone if for all \(x_0, x_1 \in H\) the real-valued function \(\varphi : [0,1] \to \mathbb{R}_+\) defined by \(\varphi(\alpha;x_0,x_1) := d(Tx_0,Tx_\alpha)\) is monotone in \(\alpha\) where \(x_\alpha := (1-\alpha)x_0 + \alpha x_1\) is the convex combination along the geodesic from \(x_0\) to \(x_1\). The next theorem provides a sufficient condition for an arbitrary Hadamard space to be Hausdorff in Monod’s topology.

Theorem 9. If the projection \(P_\gamma\) is geodesically monotone for all geodesic segments \(\gamma\), then \((H,\tau_M)\) is Hausdorff.
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Figure 2. Example of a Hadamard space that arises by removing the solid wedge \((ABCD)\) from a solid cube. Both endpoints \(A\) and \(B\) of the geodesic segment \([AB]\) (orange) connecting \(A\) with \(B\) project to the point \(C\) on the geodesic \(\gamma\); however, the midpoint \(E\) of \([AB]\) projects to the point \(F\), which is not in the convex hull along \(\gamma\) of the projections of \(A\) and \(B\). This illustrates that projection to geodesics might not preserve monotonicity.

Proof. Let \(x, y \in H\) be two distinct points and \(\gamma : [0, 1] \to H\) a geodesic such that \(\gamma(0) = x, \gamma(1) = y\). Let \(l > 0\) denote the length of \(\gamma\). For some fixed number \(0 < \varepsilon < l\) let \(C(x, \varepsilon) := \{z \in H \mid d(x, P_\gamma z) \leq \varepsilon\}\). We claim that \(C(x, \varepsilon)\) is a closed convex set. Closedness follows immediately since \(P_\gamma\) is nonexpansive and therefore continuous. Let \(z_0, z_1 \in C(x, \varepsilon)\) be two distinct elements. Let \(z_\alpha := (1 - \alpha)z_0 \oplus \alpha z_1\) for some \(\alpha \in (0, 1)\). By assumption, \(P_\gamma\) is a geodesically monotone operator; thus, \(d(P_\gamma z_0, P_\gamma z_\alpha)\) is monotone in \(\alpha\), implying that \(z_\alpha \in C(x, \varepsilon)\). By definition of \(\tau_M\) it follows that \(C(x, \varepsilon)\) is \(\tau_M\)-closed. Hence \(H \setminus C(x, \varepsilon)\) is \(\tau_M\)-open. By construction \(y \in H \setminus C(x, \varepsilon)\). Using the same argument, it follows that \(C(y, \varepsilon) := \{z \in H \mid d(y, P_\gamma z) \leq l - \varepsilon\}\) is \(\tau_M\)-closed. Hence, \(H \setminus C(y, \varepsilon)\) is a \(\tau_M\)-open set containing \(x\). It is evident by construction that \((H \setminus C(x, \varepsilon)) \cap (H \setminus C(y, \varepsilon)) = \emptyset\). Therefore, \((H, \tau_M)\) is a Hausdorff space.

The contrapositive of this statement together with [11, Example 3.6] shows that the projection \(P_\gamma\) is not a geodesically monotone operator in a general Hadamard space. This is in contrast with projections to geodesic segments in Hilbert spaces, which are always geodesically monotone. Notice furthermore that the converse of Theorem 9 is not true. Indeed, Figure 2 provides an example.

Remark 3. Geodesic monotonicity of metric projections onto geodesics coincides with the so-called “(nice) N-property” introduced in [8].

4.4. Discussion. In relation to weak convergence of bounded sequences in CAT(0) spaces [15, Lytchak–Petrunin] recently introduced a topology \(\tau\) on a CAT(0) space \((X, d)\) in the following way: a set \(S \subseteq X\) is \(\tau\)-closed in \(X\) if any bounded sequence \((x_n)_{n \in \mathbb{N}}\) in \(S\) that weakly converges to \(x \in X\) implies \(x \in S\). By construction \(\tau\) is finer than \(\tau_w\). This topology differs from \(\tau_w\) in at least two ways. First \(\tau\) characterizes only bounded sequences (nets).
Second by construction $\tau$ is sequential, see [15, Theorem 1.2], whereas our weak topology $\tau_w$ is not necessarily so. This can be seen from the discussion in §3.2 where $\tau_w$ coincides with the usual weak topology on a Hilbert space and the latter topology is not sequential. Now concerning the example in [15, §4] it is proved that it is not Hausdorff in $\tau$, consequently it cannot be Hausdorff in $\tau_w$. By Lemma 3 it follows then that this example of Hadamard space cannot be weakly proper. Therefore not all Hadamard spaces are weakly proper.

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