A NON-RELATIVISTIC MODEL OF PLASMA PHYSICS
CONTAINING A RADIATION REACTION TERM

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Abstract. While a fully relativistic collisionless plasma is modeled by the Vlasov-Maxwell system a good approximation in the non-relativistic limit is given by the Vlasov-Poisson system. We modify the Vlasov-Poisson system so that damping due to the relativistic effect of radiation reaction is included. We prove existence and uniqueness as well as higher regularity of local classical solutions. Our results also include the higher regularity of classical solutions of the Vlasov-Poisson system depending on the regularity of the initial datum.

1. Introduction. In studying interactions of charged matter and electromagnetic fields it is a quite common situation that the velocity of matter is small if compared to the velocity of light. In such situations an asymptotic expansion of the equations in the small parameter $\frac{|v|}{c}$ can help to reduce the complexity of the original problem, e.g. for numerical computations. Since in many relativistic settings the zeroth order contribution of this expansion gives the Newtonian limit such expansions are usually called post-Newtonian expansions.

In numerous applications of asymptotic analysis it is quite straightforward to establish the equations of the expansion and it is the main task to prove a relation between solutions of the asymptotic expansion and solutions of the original equations. In contrast to that in interactions of charged matter and electrodynamic fields also the formulation of the approximation scheme becomes delicate if effects due to radiation damping need to be included. It is well-known that accelerated charged matter loses energy by radiation; an electromagnetic radiation field is generated which transports energy to null-infinity. This gives a damping effect on the motion of the matter known as radiation damping. There is a large amount of literature for single particle models concerning effective equations which incorporate radiation damping without giving a fully relativistic description, see [19] and the literature cited therein.

On the other hand the Vlasov-Maxwell system is a well-known fully relativistic description of a large ensemble of collisionless particles which interacts by means of the electromagnetic fields collectively generated by these particles. In a non-relativistic setting this situation is governed by the Vlasov-Poisson system. While
the asymptotic limit relation between the Vlasov-Poisson and the Vlasov-Maxwell system is well understood, see [18, 14], and also higher order relations between the so called Vlasov-Darwin and the Vlasov-Maxwell model are established, see [3], all these approximation models do not inhibit radiation damping.

It is the aim of this paper to establish and study a model in between the fully relativistic description and the non-relativistic description where effects due to radiation damping are included. In Section 2 we repeat some heuristics from [11, 12] leading to two candidates for such a model, see equations (6) and (7). While the first of these systems is already studied in [11, 12] we give an additional heuristic leading to (7). In the third section we shall reformulate (7) establishing the topic of this paper, the 'reduced Vlasov-Poisson system with radiation damping', see (rVPRD.)

In this situation the main criterion to decide wether or not a model is suitable, is the approximation property. In a second paper [2] it is proved that solutions of (rVPRD.) together with some lower order terms approximate solutions of the Vlasov-Maxwell system to a higher order in \( v/c \) than solutions of the Vlasov-Poisson or the Vlasov-Darwin model, see also [4] for an overview.

In Section 4 we present a proof of local existence, uniqueness and smoothness of solutions of (rVPRD.). This proof is an adaption of the usual proof of local existence of classical solutions, essentially going back to [1]. In this contribution we follow the presentation in [17]. With regard to the issue of higher regularity we have to use an additional induction loop.

Motivated by a preprint version of this paper the existence of global solutions for small initial data has already been proved in [6].

2. Heuristic derivations of models including radiation damping.

2.1. The Vlasov-Poisson system with radiation damping. How could such a model look like? In [5, Thm. 1.4] it is shown that for suitable solutions of the Vlasov-Maxwell system which are isolated from incoming radiation in the non-relativistic limit \( c \to \infty \), \( c \) the velocity of light, the total rate of radiated energy is given by

\[
\frac{2}{3c^3} |\vec{D}(t)|^2
\]

where \( \vec{D} \) is the dipole moment of the non-relativistic charge distribution, i.e. of the dipole moment of the solution of the Vlasov-Poisson system. This result yields a mathematical formulation and rigorous proof of Larmor’s formula, see [9, (14.22)], in the case of Vlasov matter.

Therefore the goal is to implement an additional term into the Vlasov-Poisson system causing this loss of energy. As already suggested in [11, 12] we modify the Vlasov equation by incorporating a small additional force term

\[
\partial_t f^\pm + p \cdot \nabla_x f^\pm = \left( E + \frac{2}{3c^3} \vec{D} \right) \cdot \nabla_p f^\pm = 0.
\]  

(1)

Here \( f^+ \) and \( f^- \) give the distribution density of two species of charged particles where the mass of the particles is set to unity and the charge of the particles is +1 and −1, respectively. These densities are functions of time \( t \in \mathbb{R} \) and phase space variables \( (x,p) \in \mathbb{R}^3 \times \mathbb{R}^3 \). The resulting charge density is given by \( \varrho \),

\[
\varrho^\pm(t,x) = \int f^\pm(t,x,p) \, dp \quad \text{and} \quad \varrho = \varrho^+ - \varrho^-,
\]  

(2)
\[ E(t, x) = -\nabla \int \frac{\varrho(t, y)}{|x - y|} \, dy, \]
and finally \( D \) is the dipole moment of the charge distribution,
\[ D(t) = \int x \varrho(t, x) \, dx. \] (3)

We will refer to this set of equations (1)–(3) as the Vlasov-Poisson system with radiation damping. The additional term in (1) is a generalization of the radiation reaction force term used in particle models, see [9, (16.8)]. We also note that for this system the quantity
\[ E_S = \frac{1}{2} \int \int \rho^2 (f^+ + f^-)(t, x, p) \, dp \, dx + \frac{1}{8\pi} \int |E(t, x)|^2 \, dx - \frac{2}{3c^3} \dot{D} \cdot \ddot{D} \]
is decreasing, more precisely one obtains
\[ \frac{d}{dt} E_s(t) = -\frac{2}{3c^3} |\ddot{D}(t)|^2, \]
the subscript \( S \) referring to the name ‘Schott’-energy under which this energy can be found in the literature. This decreasing of energy is usually attributed to the effect of radiation damping.

2.2. Problems with the Cauchy problem. In order to formulate the Cauchy problem of this system we of course have to give initial values for the particle densities \( f^\pm \), but we also have to fix data for the dipole moment. Let us for a moment assume that we are equipped with a smooth solution of (1)–(3) such that the spatial support of \( f^\pm \) is compact on every time slice with constant time. We define the bare mass by
\[ M(t) = \int \int (f^+ + f^-)(t, x, p) \, dp \, dx. \] (4)
Then (1) yields mass conservation \( \partial_t M = 0 \) as well as charge conservation \( \partial_t \varrho^\pm + \nabla \cdot j^\pm = 0 \) for both species, where
\[ j^\pm = \int p f^\pm \, dp \] (5)
are the current densities. Using equation (1) we compute the first three time derivatives of \( D \) and find \( \ddot{D}(t) = \int j^+ - j^- \, dx \) as well as
\[ \dot{D}(t) = D^{[2]}(t) + \frac{2}{3c^3} M \ddot{D}(t) \]
where \( D^{[2]} \) is defined by
\[ D^{[2]}(t) = \int E(t, x) (\varrho^+ + \varrho^-)(t, x) \, dx. \]
Introducing the new variable
\[ y(t) := \ddot{D}(t) \]
and abbreviating \( \varepsilon := \frac{2}{3c^3} \)
we can recast (1) into
\[ \partial_t f^\pm = -p \cdot \nabla_x f^\pm \mp \left( E + \frac{y - D^{[2]}}{M} \right) \cdot \nabla_p f^\pm, \] (SGPP_\varepsilon)
\[ \varepsilon y = \frac{y - D^{[2]}}{M}, \] (SGPP_\varepsilon)\]

where we have to supply initial data \( f^{\circ, \pm} \) and \( y_0 \in \mathbb{R}^3 \). While the physical setting only delivers initial data for the particle distribution the question arises how to specify \( y_0 \) in order to get a solution which is physically reasonable. We shall discuss now several approaches to deal with this problem.

2.3. **Methods of reduction.** The Vlasov-Poisson system with radiation damping serves as an approximation of a relativistic system valid up to order \( \varepsilon^{-3} \) or \( \varepsilon \). On account of that it seems to be acceptable to modify this system by terms which are formally of higher order in \( \varepsilon \). In \([11]\) there are several ways of reduction discussed, each of it consisting of two basic steps in certain combinations: on the one hand a time derivative will be replaced modulo higher order terms via the Vlasov equation, e.g. \( \bar{D} = D^{[2]} + O(\varepsilon) \) and on the other hand a time derivative will be cancelled by a substitution in the characteristic equation \( \lambda^\pm = \mathcal{P}^\pm, \mathcal{P}^\pm = \pm(E + \varepsilon \bar{D}) \). The so-called 2+1 reduction \( \lambda^\pm = \lambda^\pm, \bar{P}^\pm = \mathcal{P}^\pm \pm \varepsilon \bar{D} \) and \( \bar{D} = D^{[2]} \) leads to the Vlasov equation

\[ \partial_t f^\pm + (p \pm \varepsilon D^{[2]}) \cdot f^\pm \nabla_x \pm E \cdot \nabla_p f^\pm = 0 \] (6)

which is thoroughly studied in \([12]\). Using a 3+0 reduction scheme, replacing \( \bar{D} \) by \( D^{[2]} \), we find the Vlasov-equation

\[ \partial_t f^\pm + p \cdot \nabla_x f^\pm \pm (E + \varepsilon D^{[2]}) \cdot \nabla_p f^\pm = 0. \] (7)

2.4. **A reduction via geometric perturbation arguments.** Here we want to give an alternative formal derivation of equation (7) using an analogy to single particle models. First note that in case of a single particle the problem of the physical reasonable initial value is discussed since the beginning of the last century, see e.g. \([9, \text{chapter } 16]\). In \([13]\) it has been observed that in particle models this problem has the structure of a geometric singular perturbation problem, and the “physical” dynamics are obtained on a center-like manifold of the full dynamics. For sake of discussion let us assume for a moment that the first equation in (SGPP_\varepsilon) is an ODE instead of a PDE and to simplify notations in this subsection \( f \) denotes the couple \( (f^+, f^-) \). The same convention will be used for \( f^\circ = (f^{\circ, +}, f^{\circ, -}) \).

Then theorems from singular geometric perturbation theory, see \([10]\), support us with invariant manifolds. More precisely, the manifold \( \mathcal{M}_0 = \{ y = h_0(f) = D^{[2]} \} \) being invariant under the dynamics of (SGPP_\varepsilon=0) persists for small \( \varepsilon > 0 \): For sufficiently small \( \varepsilon > 0 \) there is a manifold \( \mathcal{M}_\varepsilon = \{ y = h_\varepsilon(f) \} \) being invariant under the dynamics of (SGPP_\varepsilon) which remains close to \( \mathcal{M}_0 \) in a suitable sense and carries those trajectories which are bounded in \( y \).

Unfortunately, this theory does not apply in our case because we are dealing with a phase-space of infinite dimension; thus the proof of the existence of an invariant manifold is hard. For sake of discussion we shall take the existence of a smooth invariant manifold for granted and assume that it is given by means of a smooth function \( h_\varepsilon = h_\varepsilon(f^\circ) \), acting on \( C^\infty_0(\mathbb{R}^3 \times \mathbb{R}^3) \times C^\infty_0(\mathbb{R}^3 \times \mathbb{R}^3) \) and taking values in \( \mathbb{R}^3 \). The manifold \( \mathcal{M}_\varepsilon = \{(f^\circ, h_\varepsilon(f^\circ))\} \) is invariant under the flow of (SGPP_\varepsilon) if the solution of (SGPP_\varepsilon) subject to the initial conditions \( (f^\circ, y(0)) = (f^\circ, h_\varepsilon(f^\circ)) \) satisfies

\[ y(t) = h_\varepsilon(f(t, \cdot, \cdot)) \] (8)
for all times the solution exits. In order to find an appropriate approximation of 
\((\text{SGPP}_\varepsilon)\) let us assume that we are furnished with smooth maps \(h_\varepsilon\) defining the manifolds \(\mathcal{M}_\varepsilon\) and that we are able to expand in \(\varepsilon\),

\[ h_\varepsilon = h_0 + \varepsilon h_1 + \mathcal{O}(\varepsilon^2). \] (9)

We plug (8) and (9) into the second line of \((\text{SGPP}_\varepsilon)\). Assuming that we are allowed to exchange the order of differentiation with respect to \(t\) and expansion in \(\varepsilon\) we compute

\[ \varepsilon \dot{h}_0(f) + \mathcal{O}(\varepsilon^2) = h_0(f) - D[2] + \varepsilon h_1(f) + \mathcal{O}(\varepsilon^2). \]

Equating powers of \(\varepsilon\) we obtain

\[ h_0(f) = D[2] = \int E(\varrho^+ + \varrho^-) \, dx \quad \text{and} \quad h_1(f) = D[2] = \frac{d}{dt} \int E(\varrho^+ + \varrho^-) \, dx. \]

Note that \(D[2]\) is a function of \(E\) and \(\varrho^\pm\). But because \(E\) itself is defined by means of the Poisson integral with source \(\varrho\) it is a function of \(f^\pm\), too. Therefore \(D[2] = D[2](f)\). Neglecting higher order terms we also end with the Vlasov-equation (7), which shall be discussed in the following.\(^1\)

3. The reduced Vlasov-Poisson system with radiation damping, main results.

3.1. Reformulation using principal values. First we have to remove an obstacle. On the one hand we need \(\partial_t f(t)\) and \(f(t)\) to define \(h_1\); therefore \(h_1\) is not defined on \(C^0_0(\mathbb{R}^3 \times \mathbb{R}^3) \times C^{\infty}_0(\mathbb{R}^3 \times \mathbb{R}^3)\). On the other hand in [11, p. 3579] it is argued that (7) is a borderline case for the usual proof of local existence of classical solutions: in order to obtain a local existence theorem the orders of differentiability of the different unknowns should fit together properly. If it is supposed that the potential \(u\) of \(E\) is \(k\)-times differentiable, then \(f\) and \(\varrho\) are \(k-2\) times differentiable and thus by solving the Poisson equation \(\Delta U = 4\pi \varrho\) two orders of differentiability should be gained which is not the case in spaces of pointwise differentiable functions.

We shall get rid of both of these problems by using the Vlasov-equation (7) and integration by parts one more time, but now we are forced to use principal values. To this end we additionally introduce the notation \(E^{\pm} = -\nabla \int \varrho^\pm(y) \frac{dy}{|x-y|}\); then some elementary calculations, using the identity \(\nabla_x |y-x|^{-1} = -\nabla_y |y-x|^{-1}\), lead to

\[
\begin{align*}
 h_1(f) &= -\frac{d}{dt} \int \nabla_x (|x-y|^{-1}) (\varrho^+ (y) - \varrho^- (y)) (\varrho^+(x) + \varrho^-(x)) \, dy \, dx \\
 &= 2 \int E^+ \partial_t \varrho^- - E^- \partial_t \varrho^+ \, dx.
\end{align*}
\]

Using charge conservation \(\partial_t \varrho^\pm + \nabla \cdot j^\pm = 0\) we e.g. compute

\[
\begin{align*}
\int E^+ \partial_t \varrho^- \, dx &= - \int E^+ \nabla \cdot \dot{j}^- \, dx \\
&= - \int \int \frac{x-y}{|x-y|^3} \varrho^+(y) \nabla_x \cdot j^-(x) \, dy \, dx \\
&= - \int \varrho^+(y) \int \frac{z}{|x|^3} \nabla_z \cdot j^-(y+z) \, dz \, dy \\
&= - \int \varrho^+(y) \left( \lim_{\eta \to 0} \int_{|z| > \eta} \frac{z}{|x|^3} \nabla_z \cdot j^-(y+z) \, dz \right) \, dy.
\end{align*}
\]

\(^1\)On a very formal level \(h_\varepsilon(f)\) is given by \(\sum_{j=0}^\infty \varepsilon^j D[j]^2(f)\), where, using the Vlasov equation, all derivative w.r.t to time can be replaced by derivatives w.r.t to phase-space.
\[
= - \int \varrho^+(y) \lim_{\eta \to 0} \left( - \int_{|z| > \eta} |z|^{-3} \left(-3z \otimes z|z|^{-2} + 4\pi \right) j^- (y + z) \, dz \\
- \int_{|z| = \eta} z \otimes z|z|^{-4} j^- (y + z) \, dS(z) \right) \, dy \\
= \int \varrho^+ H(j^-) \, dy. 
\]

Here \( \text{id} \) denotes the identity mapping in \( \mathbb{R}^3 \) and for a suitable function \( j \) the operator \( H \) is defined by
\[
H(j)(y) := \int \frac{1}{|z|^3} \left(-3z \otimes z|z|^2 + 4\pi \right) j(y + z) \, dz + 4\pi \, \frac{3}{3} j(y). \tag{10}
\]

Note that the kernel \( K(z) = -3z \otimes z|z|^2 + 4\pi \) is bounded on \( \mathbb{R}^3 \setminus \{0\} \), is homogeneous of degree zero and satisfies \( \int_{|z|=1} K(z) \, dS(z) = 0. \) Thus, using the Calderón-Zygmund inequality, \( H \) is well defined for smooth functions with compact support and for \( 1 < p < \infty \) can be extended to a bounded linear operator mapping \( L^p(\mathbb{R}^3) \) to \( L^p(\mathbb{R}^3) \), see [20]. We call the following set of equations the ‘reduced Vlasov-Poisson system with radiation damping’
\[
\partial_t f^{\pm} + p \cdot \nabla_x f^{\pm} \pm (E + \varepsilon D^{[3]}(x)) \cdot \nabla_p f^{\pm} = 0, \\
E(t, x) = -\nabla_x \left( \int \frac{\varrho(t, y)}{|x - y|} \, dy \right), \\
D^{[3]}(t) = 2 \int \left( H(j^-) \varrho^+ - H(j^+) \varrho^- \right)(t, x) \, dx, 
\]
where \( \varrho^{\pm}, j^{\pm}, H(j^{\pm}) \) are defined according to (2), (5) and (10). To be precise we shall define what we mean by a classical solution. We adopt the formulation of the usual Vlasov-Poisson system given in [17].

**Definition 3.1.** The pair of functions \( f^{\pm} : J \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty) \) is a classical solution of the reduced Vlasov-Poisson system with radiation damping on the interval \( J \subset \mathbb{R} \) if the following holds:

(i) The functions \( f^{\pm} \) are continuously differentiable with respect to all its variables.

(ii) The induced spatial densities \( \varrho^{\pm} \) and \( j^{\pm} \), the induced electric field \( E \) and the induced damping term \( D^{[3]} \) exist on \( J \times \mathbb{R}^3 \) and \( J \), respectively, and are continuously differentiable.

(iii) For every compact subinterval \( I \subset J \) the electric field \( E \) is bounded on \( I \times \mathbb{R}^3 \).

(iv) The functions \( f^{\pm}, E \) and \( D^{[3]} \) satisfy the equations in (rVPRD) .

**3.2. Local solutions, uniqueness and higher regularity.** It is the main result of this paper to establish local existence and uniqueness of classical solutions of (rVPRD). Furthermore, we want to establish higher regularity of solutions depending on the smoothness of the initial data. To the best knowledge of the author also in the case \( \varepsilon = 0 \), which is the usual Vlasov-Poisson system, this question is only investigated in [15] and not published elsewhere. Surprisingly, in that contribution only smoothness in spatial directions is proved whereas the methods also allow for proving smoothness for all space-time directions. At last we want to establish certain bounds holding uniformly in \( 0 \leq \varepsilon \leq 1 \). These bounds are needed in [2] in order to prove the asymptotic approximation of solutions of the Vlasov-Maxwell
system by densities which are built from solutions of (rVPRD).\(^2\) We shall take our initial data from the following class of smooth functions with compact support. Fix an integer \(k\) and constants \(R_0\) and \(S_0\). With regard to the initial data we require that

\[
f^{\circ,\pm} \in C^\infty_0(\mathbb{R}^3 \times \mathbb{R}^3), \quad \|f^{\circ,\pm}\|_{W^{k,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)} \leq S_0,
\]

\[
f^{\circ,\pm}(x,p) = 0 \quad \text{if} \quad |x| \geq R_0 \quad \text{or} \quad |p| \geq R_0.
\]

Here \(\|\cdot\|_{W^{k,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)}\) denotes the usual Sobolev-norm. For further notations also in the following theorem, see the beginning of Section 4. Now we can state the main result of this paper.

**Theorem 3.2.** Fix a positive integer \(k\), constants \(R_0\), \(S_0\) and initial data \(f^{\circ,\pm}\) according to (11). Then for every \(0 \leq \varepsilon \leq 1\) the problem (rVPRD\(_\varepsilon\)) has a unique classical solution

\[
\left( f^{\pm}_T, E, D^{[3]} \right) \in C^k([0,T_\varepsilon] \times \mathbb{R}^3 \times \mathbb{R}^3) \times C^k([0,T_\varepsilon] \times \mathbb{R}^3) \times C^k([0,T_\varepsilon])
\]

with \(0 < T_\varepsilon \leq \infty\). Furthermore, there is a constant \(0 < \hat{T} \leq \infty\) and for every constant \(0 < T < \hat{T}\) there is a constant \(M(T)\) such that for all \(x \in \mathbb{R}^3\), \(p \in \mathbb{R}^3\), \(0 \leq t \leq T\), \(0 \leq \varepsilon \leq 1\) and all non-negative integers \(l \leq k\)

(i) \(\hat{T} \leq T\),

(ii) \(f^{\pm}(t,x,p) = 0 \quad \text{if} \quad |x| \geq M(T) \quad \text{or} \quad |p| \geq M(T)\),

(iii) \(\|\partial_{t,x,p} f^{\pm}(t)\|_\infty + \|\partial_{t,x,p} E(t)\|_\infty + |\frac{d}{dt} D^{[3]}(t)| \leq M(T)\).

In addition

(iv) \(\hat{D}^{[2]}(t) = D^{[3]}(t) \quad \text{for all} \quad 0 \leq t < T_\varepsilon\).

The constants \(\hat{T}\) and \(M(T)\) are only depending on \(k\), \(R_0\) and \(S_0\). In particular they are independent of \(\varepsilon\).

The proof is elaborated in Section 4.

### 3.3. Global solutions

Next we turn to the question of existence of global in time classical solutions. Motivated by a preprint version of this paper global solvability for small initial data has already been proved in [6].

One crucial ingredient of every proof of existence of global solutions with unrestricted initial data is an a-priori bound on the second velocity momentum.\(^3\) This bound is usually obtained by some energy conservation equation. We introduce the notation

\[
D^{[1]} := \int j^+ - j^- \, dx = \hat{D}
\]

and compute \(\hat{D}^{[1]} = D^{[2]} + \varepsilon MD^{[3]}\) for a solution of (rVPRD\(_\varepsilon\)), where \(M\) is the time-independent bare mass, see (4). We recall \(\hat{D}^{[2]} = D^{[3]}\) and define the energy of the system (rVPRD\(_\varepsilon\)) by

\[
\dot{\mathcal{E}}_S = \frac{1}{2} \int p^2 (f^+ + f^-) \, dp \, dx + \frac{1}{8\pi} \int |E|^2 \, dx - \varepsilon D^{[1]} \cdot D^{[2]} + \frac{\varepsilon^2 M}{2} |D^{[2]}|^2.
\]

\(^2\)For the approximation we also need some correction terms of ‘Darwin-order’

\(^3\)Very recently in [7] a proof of existence of global classical solutions of the Vlasov-Poisson system has been published, which does not rely on an a-priori bound of the second velocity momentum. It remains to investigate wether this proof can be applied to the system at hand.
This energy is decreasing, more precisely we have
\[ \frac{d}{dt} \mathcal{E}_S = -\varepsilon |D[2](t)|^2. \]

Obviously, this energy is indefinite. This situation is well known from the Vlasov-Poisson equation in the gravitational case, where the potential energy is negative. Here the question is, whether or not it is possible to estimate \(|D[1] \cdot D[2]|\) against one of the immanently positive terms in a suitable way. Of course we can estimate \([17, \text{Lemma 1.8}]\), we only get the estimate
\[ | D^2 | \leq \frac{|D[2]|^2}{\eta} + \eta |D[2]|^2 \text{ for all } \eta > 0; \]
therefore, it would be sufficient to bound \(|D[1]|^2\). Unfortunately, using the usual bounds for velocity momenta, see e.g. \([17, \text{Lemma 1.8}]\), we only get the estimate
\[ \|j^\pm\|_1 \leq C \|f\|_1^{1/2} \left( \int \int p^2 f^\pm \, dp \, dx \right)^{1/2}, \tag{12} \]
which is just not sufficient for our purposes.

We also can’t make use of the smallness of \(\varepsilon\); in order to swallow the \(|D[2]|^2\) term we have to choose \(\eta = \varepsilon M^2\) and end with the factor \(M^2\) in front of the \(|D[1]|^2\) term which is just cancelled by the square of \(|f^\pm|^{1/2}\) in (12). It is because of this that the existence of global classical solutions seems to be out of reach at the moment. If we take the boundedness of these second velocity momenta for granted, there are no obvious obstacles to prove the existence of global solutions following the lines of the proof of Lions and Perthame in \([16]\).4

4. Proof of Theorem 3.2. Theorem 3.2 will be proved in two steps. After collecting some well known estimates in Subsection 4.1 we define a modification of the usual iteration scheme in Subsection 4.2 and establish certain bounds on the iterates. In Theorem 4.8 in Subsection 4.3, we prove that the iterates converge to a solution of (rVPRD) and that this solution is unique. Finally, in Subsection 4.4 we shall prove higher regularity of solutions, see Theorem 4.9. Together Theorem 4.8 and Theorem 4.9 give Theorem 3.2.

Throughout this section we fix a positive integer \(k\) and some constants \(R_0\) and \(S_0\). Generic constants, which may be computed from \(k, S_0\) and \(R_0\), are denoted by \(C, C_1, C_2\) and so on. Constants \(C\) may change from line to line. For \(1 \leq p \leq \infty\) we denote the spatial-space and the phase-space \(L^p\) norms by \(\| \cdot \|_p\). If we consider a pair of functions then \(\| \cdot \|_p\) gives the sum of the norms of the pair, e.g. \(\|f^\pm\|_p = \|f^+\|_p + \|f^-\|_p\). For \(t \in \mathbb{R}\) we denote by \(f(t)\) the function
\[ f(t) : \mathbb{R}^3 \times \mathbb{R} \ni (x, p) \mapsto f(t, x, p) \]
and in the same way we use \(g(t), j(t)\) and \(E(t)\). By \(C(\mathbb{R}^n)\) and \(C^k_c(\mathbb{R}^n)\) we denote the space of \(k\) times continuously differentiable functions on \(\mathbb{R}^n\), the subscript \(c\) indicates compactly supported functions. For differentiable functions \(f\) we denote (the vector of) partial derivatives with respect to \(t, x, p\) or combinations of it by \(\partial_t f, \partial_x f, \partial_p f, \partial_{x,p} f\) and so on. Vectors of partial derivatives of order \(l\) will be denoted by \(\partial^l_{x,p} f, \partial^l_{t,x,p} f\) and so on. The subscripts indicate which partial derivatives are included.

4 Note that for a solution of (6) the usual total energy \(\mathcal{E} = \frac{1}{2} \int \int p^2 (f^+ + f^-) \, dp \, dx + \frac{1}{2} \int |E|^2 \, dx\) is decreasing, \(\partial_t \mathcal{E}(t) = -\varepsilon |D[2]|^2\). Using the resulting a-priori bound on \(\int \int p^2 (f^+ + f^-) \, dp \, dx\) global existence of smooth solutions has been shown in [12].
4.1. Preparations. For \( \varrho \in C_c(\mathbb{R}^3) \) we define

\[ E_\varrho(x) = -\nabla_x \int \varrho(y) \frac{1}{|x-y|} \, dy = \int \frac{x-y}{|x-y|^3} \varrho(y) \, dy. \]

Furthermore, we set \( \ln^+(a) := 1 + \ln(a) \) if \( 1 \leq a \) and \( \ln^+(a) = a \) if \( 0 \leq a < 1 \). For the convenience of the reader we cite the following propositions from [1, Prop. 1, Prop. 2].

**Proposition 1.** Let \( \varrho \in C^1(\mathbb{R}^3) \). Then \( E_\varrho \in C^1(\mathbb{R}^3) \) and for all \( 0 < d \leq R \)

(i) \[ \|E_\varrho\|_\infty \leq 3(2\pi)^{2/3} \|\varrho\|_1^{1/3} \|\varrho\|_2^{2/3}, \]

(ii) \[ \|\partial_x E_\varrho\|_\infty \leq C \left[ R^{-3} \|\varrho\|_1 \right. + \left. d \|\partial_x \varrho\|_\infty + (1 + \ln(R/d)) \|\varrho\|_\infty \right] , \]

(iii) \[ \|\partial_\varrho E_\varrho\|_\infty \leq C \left[ (1 + \|\varrho\|_\infty) (1 + \ln^+ \|\partial_x \varrho\|_\infty) + \|\varrho\|_1 \right] . \]

For \( \varrho \in C_c(\mathbb{R}^3) \) and \( j \in C_c(\mathbb{R}^3; \mathbb{R}^3) \) we define

\[ D^{[3]}_{\varrho j} := 2 \int \varrho(x) [H(j)](x) \, dx . \]

**Lemma 4.1.** Let \( \varrho \in C_c(\mathbb{R}^3) \) and \( j \in C_c(\mathbb{R}^3; \mathbb{R}^3) \). Then

\[ |D^{[3]}_{\varrho j}| \leq 2 c_{CZ} \|\varrho\|_2 \|j\|_2 \leq 2 c_{CZ} \left( \|\varrho\|_\infty \|\varrho\|_1 \|j\|_\infty \|j\|_1 \right)^{1/2} , \]

where \( c_{CZ} \) is the Calderón-Zygmund constant. Let \( J \subset \mathbb{R} \) an intervall and furthermore \( \varrho \in C(J; C_c(\mathbb{R}^3)) \) as well as \( j \in C(J; C_c(\mathbb{R}^3; \mathbb{R}^3)) \). Then the function \( J \ni t \mapsto D^{[3]}_{\varrho j}(t) := D^{[3]}_{\varrho j(t)j(t)} \) is continuous. If additionally \( \varrho \) and \( j \) are continuously differentiable with respect to \( t \), then \( D^{[3]}_{\varrho j} \) is differentiable and the usual product rule holds true.

**Proof.** Applying Hölder’s inequality, the Calderón-Zygmund inequality, see [20, pp.39, Thm.3], and Hölder’s inequality again proves the estimate. The latter statements follow by standard theorems on Lebesgue-integrable functions. \( \square \)

For an intervall \( J \subset [0, \infty) \), \( 0 \in J \), a mapping \( F \in C(J \times \mathbb{R}^3; \mathbb{R}^3) \) being continuously differentiable in the second variable \( \mathcal{X} \in \mathbb{R}^3 \), for \( t \in J \) and \( x, p \in \mathbb{R}^3 \times \mathbb{R}^3 \) we define \( Z_F = (\mathcal{X}_F, \mathcal{P}_F) = (\mathcal{X}_F, \mathcal{P}_F)(s; t, x, p) \) as the unique solution of

\[ \frac{d}{ds} \mathcal{X}_F(s) = \mathcal{P}_F(s), \quad \mathcal{X}_F(t; t, x, p) = x, \]

\[ \frac{d}{ds} \mathcal{P}_F(s) = F(s, \mathcal{X}_F(s)), \quad \mathcal{P}_F(t; t, x, p) = p. \]

Furthermore, we set

\[ P_Z(t) := \sup \{ |\mathcal{P}_F(s; 0, \tilde{x}, \tilde{p})| : 0 \leq s \leq t, |\tilde{x}| \leq R_0, |\tilde{p}| \leq R_0 \} , \]

\[ X_Z(t) := R_0 + \int_0^t P_Z(\sigma) \, d\sigma \quad \text{and} \quad C_Z(t) := \sup \{ |\partial_{x,p} Z_F(s; t, x, p)| : 0 \leq s \leq t, x, p \in \mathbb{R}^3 \} . \]

For an initial datum \( f^* \) according to (11) we define

\[ f_F(t, x, p) := f^*(Z_F(0; t, x, p)). \]

We collect some well known facts about the flow generated by a smooth bounded field. For a proof see e.g. [17, Lemma 1.2, Lemma 1.3].
Lemma 4.2. Assume that $F$ is additionally bounded on $I \times \mathbb{R}^3$ for every compact interval $I \subset J$. Then

(i) $Z_F \in C^1(J \times J \times \mathbb{R}^3 \times \mathbb{R}^3)$ and for all $(s,t) \in J \times J$ the map $Z_F(s,t,\cdot,\cdot)$ is a measure preserving $C^1$-diffeomorphism with inverse

$$Z_F(s,t,\cdot,\cdot)^{-1} = Z_F(t,s,\cdot,\cdot).$$

(ii) $P_Z(t) \leq R_0 + \int_0^t \|F(\sigma)\|_\infty \, d\sigma$ for all $t \in J$.

(iii) $C_Z(t) \leq 6 \exp\left(\int_0^t 1 + \|\partial_2 F(\sigma)\|_\infty \, d\sigma\right)$ for all $t \in J$.

(iv) $f_F \in C^1(J \times \mathbb{R}^3 \times \mathbb{R}^3)$, $f_F(t,x,p) = 0$ if $|x| > X_F(t)$ or $|p| > P_Z(t)$, $\|f_F(t)\|_\infty = \|f^0\|_\infty$ and $\|f_F(t)\|_1 = \|f^0\|_1$ for all $t \in J$.

(v) $f_F$ is the unique $C^1$-solution of the Vlasov equation

$$\partial_t f + p \cdot \partial_x f + F \cdot \partial_p f = 0 \quad \text{with} \quad f(0) = f^0.$$
Proof. Using Proposition 1, Lemma 4.1 and Lemma 4.2 this Lemma follows immediately by induction on $n \in \mathbb{N}$.

Next we shall give an estimate of the size of the support uniformly in $n$. Let $P : [0, a) \to \mathbb{R}$ denote the maximal solution of the equation

$$P(t) = R_0 + C_3 \int_0^t P^2(\sigma) + P^4(\sigma) \, d\sigma \quad \text{with} \quad C_3 := \max\{C_1, C_2\} \quad (13)$$

and define

$$X(t) := R_0 + \int_0^t P(\sigma) \, d\sigma .$$

**Lemma 4.4.** For all $0 \leq t < a$, $n \in \mathbb{N}$ and $\varepsilon \in [0, 1]$ we have $P_n(t) \leq P(t)$ and $X_n(t) \leq X(t)$.

**Proof.** Using Lemma 4.2(ii) and Lemma 4.3(iv) and (v) this follows by induction on $n \in \mathbb{N}$.

In the next steps we prove that a bound on the $p$-support on an interval $J \subset [0, \infty)$, $0 \in J$, implies bounds on all other quantities as well as the existence of a smooth solution of $\left( rVPRD_x \right)_n$ on the interval $J$. We assume that we are furnished with such an interval $J$ and a continuous monotonously increasing positive function, called $P$ again, such that for all $n \in \mathbb{N}$, $\varepsilon \in [0, 1]$, $t \in J$ and $x, p \in \mathbb{R}^3$

$$P_n(t) \leq P(t) .$$

In fact, this is at least true for $J = (0, a)$. In addition we introduce the following helpful conventions. Henceforward, continuous monotonously increasing positive functions, which can be calculated by the knowledge of $P$ and the basic constants $k, S_0$ and $R_0$ alone, are denoted by $C(\cdot)$, $C_1(\cdot), C_2(\cdot)$. Especially they are independent of $n \in \mathbb{N}$ and $\varepsilon \in [0, 1]$. While $C_1(\cdot), C_2(\cdot)$ will be fixed functions, the definition of $C(\cdot)$ may change from line to line, e.g. we have

$$C(t) = P(t) = P(t) \exp(C(t)) + C(t) .$$

**Lemma 4.5.** For all $t \in J$ we have $\|\partial_x y_n^\pm(t, x)\|_{\infty} + \|\partial_x E_n(t)\|_{\infty} \leq C(t)$.

**Proof.** Using Lemma 4.4 and Lemma 4.2(iii) we compute

$$|\partial_x y_n^\pm(t, x)| \leq \int_{|p| \leq P_n(t)} |\partial_x \left( f^{\pm} \left( Z_n^\pm(0; t, x, p) \right) \right) dp \leq P^3(t) C \left\| \partial_x Z_n^\pm(0; t, \cdot, \cdot) \right\|_{\infty} \leq C(t) \exp \left( \int_0^t 1 + \|\partial_x E_{n-1}(\sigma)\|_{\infty} \, d\sigma \right) .$$

Plugging this estimate together with Lemma 4.3(ii) in Proposition 1(iii) we have

$$|\partial_x E_n(t)|_{\infty} \leq C(t) \left( 1 + \ln \left( \exp \left( \int_0^t \|\partial_x E_{n-1}(\sigma)\|_{\infty} \, d\sigma \right) \right) \right) \leq C_1(t) \left( 1 + \int_0^t |\partial_x E_{n-1}(\sigma)|_{\infty} \, d\sigma \right) ,$$

where we choose $C_1(\cdot)$ in such a way that additionally the following inequality holds:

$$C_1(0) \geq \|\partial_x E_0(0)\|_{\infty} = |\partial_x E_0(t)|_{\infty} .$$

We define $C_2(\cdot)$ as the solution of

$$C_2(t) = C_1(t) \left( 1 + \int_0^t C_2(\sigma) \, d\sigma \right) .$$
By induction on \( n \) we conclude that
\[
\|\partial_x E_n(t)\|_\infty \leq C_2(t) \quad \text{and} \quad \|e_n(t)\|_\infty \leq C(t)
\]
for all \( t \in J \) and all \( n \in \mathbb{N} \).

For \( n \in \mathbb{N} \) we define
\[
\alpha_n(t) := \sup \{ |Z_n^\pm(s; t, x, p) - Z_n^\pm(s; t, x, p)| : 0 \leq s \leq t, x, p \in \mathbb{R}^3 \}.
\]

**Lemma 4.6.** For all \( n \in \mathbb{N} \) and all \( t \in J \) we have
\[
\begin{align*}
\text{(i)} & \quad \|f_{n+1}^\pm(t) - f_n^\pm(t)\|_\infty \leq C\frac{C_3(t)^n}{n!} t^n \\
\text{(ii)} & \quad \alpha_n(t) \leq C(t) \frac{C_3(t)^{n-1}}{n!} t^n.
\end{align*}
\]

**Proof.** In the first step we compute
\[
|f_{n+1}^\pm(t, x, p) - f_n^\pm(t, x, p)| \leq C|Z_{n+1}^\pm(0; t, x, p) - Z_n^\pm(0; t, x, p)|. \tag{14}
\]
Using the characteristic equations we also have for all \( 0 \leq s \leq t \) and all \( x, p \in \mathbb{R}^3 \)
\[
|\chi_{n+1}^\pm(s; t, x, p) - \chi_{n}^\pm(s; t, x, p)| \leq \int_s^t |\mathcal{P}_{n+1}^\pm(\sigma; t, x, p) - \mathcal{P}_n^\pm(\sigma; t, x, p)| d\sigma,
\]
\[
|\mathcal{P}_{n+1}^\pm(s; t, x, p) - \mathcal{P}_n^\pm(s; t, x, p)| \leq \int_s^t |E_n(\sigma, \chi_{n+1}(\sigma)) - E_{n-1}(\sigma, \chi_n(\sigma))|
\]
\[
\quad + \left| \frac{D_3[\sigma]}{n} - \frac{D_3[\sigma]}{n-1} \right| d\sigma.
\]
Utilizing Lemma 4.5 the electric fields can be estimated by
\[
|E_n(\sigma, \chi_{n+1}(\sigma)) - E_{n-1}(\sigma, \chi_n(\sigma))| 
\leq |\partial_x E_n(\sigma)|_\infty |\chi_{n+1}(\sigma) - \chi_{n}(\sigma)| + |E_n(\sigma) - E_{n-1}(\sigma)|_\infty 
\leq C(\sigma)|\chi_{n+1}(\sigma) - \chi_{n}(\sigma)| + C(\sigma) \|f_{n}^\pm(\sigma) - f_{n-1}^\pm(\sigma)\|_\infty.
\]
Employing Lemma 4.1 we have for the dipoles
\[
|D_3[\sigma] - D_3[\sigma]| 
\leq C \left( \left\| q_{n}(\sigma) - q_{n-1}(\sigma) \right\|_2 \left\| j_+^\pm(\sigma) \right\|_2 + \left\| q_{n+1}(\sigma) - q_{n-1}(\sigma) \right\|_2 \left\| j_\mp(\sigma) \right\|_2 
\right.
\]
\[
+ \left. \left\| q_{n}(\sigma) - q_{n-1}(\sigma) \right\|_2 \left\| j_+^\pm(\sigma) \right\|_2 + \left\| q_{n+1}(\sigma) - q_{n-1}(\sigma) \right\|_2 \left\| j_\mp(\sigma) \right\|_2 \right)
\leq C(\sigma) \|f_{n}^\pm(\sigma) - f_{n-1}^\pm(\sigma)\|_\infty.
\]
Collecting these two estimates we conclude
\[
|Z_{n+1}^\pm(s) - Z_n^\pm(s)| \leq \int_s^t C(\sigma)|Z_{n+1}^\pm(\sigma) - Z_n^\pm(\sigma)| + C(\sigma) \|f_{n}^\pm(\sigma) - f_{n-1}^\pm(\sigma)\|_\infty d\sigma
\]
and using Gronwall’s Lemma
\[
|Z_{n+1}^\pm(s) - Z_n^\pm(s)| \leq \int_s^t C(\sigma) \|f_{n}^\pm(\sigma) - f_{n-1}^\pm(\sigma)\|_\infty d\sigma \exp \left( \int_s^t C(\sigma) d\sigma \right). \tag{15}
\]
Combining (14) and (15) we have
\[
\|f_{n+1}^\pm(t) - f_n^\pm(t)\|_\infty \leq C_3(t) \int_0^t \|f_{n}^\pm(\sigma) - f_{n-1}^\pm(\sigma)\|_\infty d\sigma.
\]
By induction it follows that
\[
\| f_{n+1}^{\pm} (t) - f_n^{\pm} (t) \|_{\infty} \leq \sup \left\{ \| f_s^{\pm} (s) - f_0^{\pm} (s) \|_{\infty} : 0 \leq s \leq t \right\} \frac{C_4(t)}{n!} \frac{t^n}{n^n} = C \frac{C_4(t)}{n!} t^n.
\]
Plugging this inequality into (15) we get
\[
\alpha_n (t) \leq C (t) \frac{C_3^{n-1} (t)}{n!} t^n.
\]

For a compact interval $I \subset J$ we define
\[
\Delta_I := \{(s, t) : t \in I, 0 \leq s \leq t\}.
\]

**Corollary 1.** For every compact interval $I \subset J$ the sequences $(f_n^{\pm})_n$, $(Z_n^{\pm})_n$, $(g_n^{\pm})_n$, $(j_n^{\pm})_n$, $(E_n)_n$ and $(D_n^{[3]})_n$ are Cauchy sequences uniformly in $I \times \mathbb{R}^3 \times \mathbb{R}^3$, resp. $\Delta_I \times \mathbb{R}^3 \times \mathbb{R}^3$, resp. $I \times \mathbb{R}^3$, resp. $I$ and $\epsilon \in [0, 1]$.

**Proof.** This is an immediate consequence of Lemma 4.6 in combination with Lemma 4.3. □

**Lemma 4.7.** The sequence $(\partial_x E_n)_n$ is a Cauchy sequence uniformly in $I \times \mathbb{R}^3$ and $\epsilon \in [0, 1]$ for every compact interval $I \subset J$.

**Proof.** Using Proposition 1(ii) for every $0 < d \leq R$ we have
\[
\| \partial_x E_n (t) - \partial_x E_m (t) \|_{\infty} \leq C \left[ R^{-3} \| g_n (t) - g_m (t) \|_{\infty} + d \| \partial_x g_n (t) - \partial_x g_m (t) \|_{\infty} + (1 + \ln (R/d)) \| \partial_x g_n (t) - \partial_x g_m (t) \|_{\infty} \right].
\]
Choosing $d$ sufficiently small proves the claim. □

**4.3. $C^1$-solution and estimates.** We define
\[
\lim_{n \to \infty} f_n^{\pm} := f^{\pm} \in C (J \times \mathbb{R}^3 \times \mathbb{R}^3) \quad \lim_{n \to \infty} Z_n^{\pm} := Z^{\pm} \in C (\Delta_J \times \mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3)
\]
\[
\lim_{n \to \infty} g_n^{\pm} := g^{\pm} \in C (J \times \mathbb{R}^3) \quad \lim_{n \to \infty} j_n^{\pm} := j^{\pm} \in C (J \times \mathbb{R}^3; \mathbb{R}^3)
\]
\[
\lim_{n \to \infty} E_n := E \in C (J \times \mathbb{R}^3; \mathbb{R}^3) \quad \lim_{n \to \infty} D_n^{[3]} := D^{[3]} \in C (J; \mathbb{R}^3).
\]
Additionally, we have $\lim_{n \to \infty} \partial_x E_n = \partial_x E \in C (J \times \mathbb{R}^3; \mathbb{R}^3)$.

**Theorem 4.8.** The triple $(f^{\pm}, E, D^{[3]})$ is the unique $C^1$-solution of (vPRD$_e$) and for all $t \in J$ and $l \in \{0, 1\}$ we have
\[
\begin{align*}
\text{(i)} & \quad \| \partial_{t,x,p}^{\pm} (t) \|_{\infty} + \| \partial_{t,x} g^{\pm} (t) \|_{\infty} + \| \partial_{t,x} j^{\pm} (t) \|_{\infty} \leq C (t) \\
\text{(ii)} & \quad \| \partial_{t,x} E (t) \|_{\infty} + \| \frac{dt}{dt} D^{[3]} (t) \| \leq C (t) \\
\text{(iii)} & \quad f^{\pm} (t, x, p) = 0 \quad \text{if} \ |x| \geq X (t) \quad \text{or} \ |p| \geq P (t).
\end{align*}
\]

**Proof.** Clearly the densities $f^{\pm} (t)$, $g^{\pm} (t)$ and $j^{\pm} (t)$ inherit the bounds on the supports $X(t)$ and $P(t)$. By standard theorems of integration we conclude $f^{\pm} = f_{0}^{\pm} (Z^{\pm} (0; \cdot, \cdot, \cdot))$, $g^{\pm} = g_{f^{\pm}}$, $j^{\pm} = j_{f^{\pm}}$, as well as $E = E_{g^{\pm}} - E_{g^{\pm}}$ and $D^{[3]} = D^{[3]}$. □
Remark 1. The usual continuation property also holds for \((\text{rVPRD}_\varepsilon)\): an a-priori bound on the velocity support yields global existence of solutions. This can be shown in the same way as in [17, Step 7, p. 401]. The crucial point is that the constants \(C_1 \) and \(C_2 \) in (13) do only depend on the \(L^1 \) and the \(L^\infty \) norm and not on the size of the support of the initial data.
4.4. $C^k$-solutions and $C^k$-estimates.

**Theorem 4.9.** $f^\pm, \varphi^\pm, j^\pm, E$ and $D^{[3]}$ are $C^k$ and for all integers $0 \leq j \leq k$ we have the estimate

$$\left\| \partial_{x,p}^j f^\pm(t) \right\|_\infty + \left\| \partial_{x,p}^j \varphi^\pm(t) \right\|_\infty + \left\| \partial_{x,p}^j j^\pm(t) \right\|_\infty + \left\| \partial_{x,p}^j E(t) \right\|_\infty + \left| \frac{d^j}{dt^j} D^{[3]}(t) \right| \leq C(t). \quad (16)$$

We shall prove this theorem by induction on $1 \leq l \leq k$. Before starting we give a precise statement of the induction hypothesis.

**Definition 4.10.** For $1 \leq l \leq k$ we say $\mathcal{H}(l)$ holds true, iff the following holds:

(i) For all $1 \leq j \leq l$ we have

$$\left\| \partial_{x,p}^j Z^\pm_n(0; t, \cdot, \cdot) \right\|_\infty + \left\| \partial_{x,p}^j \varphi^\pm_n(0; t, \cdot, \cdot) \right\|_\infty + \left\| \partial_{x,p}^j E_n(0; t, \cdot, \cdot) \right\|_\infty \leq C(t)$$

(ii) For all $0 \leq j \leq l-1$ the sequences $(\partial_{x,p}^j f^\pm)_n, (\partial_{x,p}^j \varphi^\pm_n)_n$ and $(\partial_{x,p}^j j^\pm)_n$ converge to $\partial_{x,p}^j f^\pm, \partial_{x,p}^j \varphi^\pm$ and $\partial_{x,p}^j j^\pm$ uniformly in $\varepsilon \in [0, 1]$ and uniformly on $I \times \mathbb{R}^3 \times \mathbb{R}^3, \Delta I \times \mathbb{R}^3 \times \mathbb{R}^3$ and $I \times \mathbb{R}^3$, respectively, for all compact subintervals $I \subset J$.

(iii) For all $0 \leq j \leq l$, $(\partial_{x,p}^j E_n)_n$ converges to $\partial_{x,p}^j E$ uniformly in $\varepsilon \in [0, 1]$ and on $I \times \mathbb{R}^3$, for all compact subintervals $I \subset J$.

(iv) The functions $f^\pm, \varphi^\pm, j^\pm, E$ and $D^{[3]}$ are $C^3$ and for all integers $0 \leq j \leq l$ holds true.

**Proof.** For $l = 1$, $\mathcal{H}(1)$ is already proved by means of Theorem 4.8. Now we assume that $\mathcal{H}(l)$ holds true for some $1 \leq l \leq k - 1$. We need a suitable representation of higher derivatives of composed functions. Consider a multi-index $\alpha = (\alpha_x, \alpha_p) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2$, and a higher order partial derivative $\partial^n = \partial_{x,p}^n \partial_{x,p}^n$. Let $G : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, resp. $\mathbb{R}^3$ and $H : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ be sufficiently smooth. By induction on $1 \leq |\alpha| \leq k$ the following representation can be verified.

$$\partial^n(G(H(x, p))) = \sum_{1 \leq |\beta| \leq |\alpha|} \partial^{3\gamma}(G)(H(x, p)) \cdot Q_{\alpha, \beta}(\partial^\gamma H)(x, p) \quad (17)$$

where $Q_{\alpha, \beta}$ are certain homogeneous polynomials of degree $|\beta|$ in the variables $\partial^\gamma H$, where $\gamma = (\gamma_x, \gamma_p)$ varies in $\mathbb{N}_0^2 \times \mathbb{N}_0^2$ and $1 \leq |\gamma| \leq |\alpha| + 1 - |\beta|$. In addition, if $|\beta| = 1$, then $Q_{\alpha, \beta}$ is a polynomial in the variables $\partial^\gamma H$, $|\gamma| = |\alpha|$. An explicit expression of these polynomials can be found in [8].

In the first step we shall establish bounds on $\partial_{x,p}^{1+} Z_n^\pm, \partial_{x,p}^{1+} \varphi_n^\pm$ and $\partial_{x,p}^{1+} E_n$. By formula (17) and $\mathcal{H}(l)$ we estimate for $x \in \mathbb{R}^3$

$$|\partial_{x,p}^{1+} \varphi_n^\pm(t, x, p)| \leq \int_{|p| \leq P(t)} |\partial_{x,p}^{1+} \varphi_n^\pm(0; t, x, p)| \, dp \leq C(t) + C(t) \left\| \partial_{x,p}^{1+} Z_n^\pm(0; t, \cdot, \cdot) \right\|_\infty. \quad (18)$$

Differentiating the characteristic equation we compute for $0 \leq s \leq t$ and $x, p \in \mathbb{R}^3$, using the short forms $(s) = (s; t, x, p)$ and $(\sigma) = (\sigma; t, x, p)$

$$|\partial_{x,p}^{1+} Z_n^\pm(s)| \leq \int_s t \left\| \partial_{x,p}^{1+} \varphi_n^\pm(\sigma)| + |\partial_{x,p}^{1+} (E_n(\sigma, X_n^{\pm}(\sigma))) \right\| \, d\sigma \quad (19)$$

With formula (17) and the induction hypothesis $\mathcal{H}(l)$ we estimate

$$|\partial_{x,p}^{1+} (E_n(\sigma, X_n^{\pm}(\sigma)))| \leq C(\sigma) + C(\sigma) \left\| \partial_{x,p}^{1+} E_n(\sigma) \right\|_\infty + C(\sigma) |\partial_{x,p}^{1+} Z_n^\pm(\sigma)|.$$
Furthermore, employing Proposition 1(i) and Lemma 4.4 we also have
\[ \|\partial_x^{i+1} E_{n-1}(\sigma)\|_{\infty} \leq C(\sigma) \|\partial_x^{i+1} v_{n-1}(\sigma)\|_{\infty}. \] (20)
Inserting these two estimates into (19) we have
\[ |\partial_x^{i+1} Z_n^\pm(s)| \leq \int_s^t C(\sigma) \left( 1 + \|\partial_x^{i+1} v_{n-1}(\sigma)\|_{\infty} + |\partial_x^{i+1} Z_n^\pm(\sigma)| \right) d\sigma. \]
Thus, using Gronwall’s Lemma,
\[ \|\partial_x^{i+1} Z_n^\pm(0; t, \cdot, \cdot)\|_{\infty} \leq C(t) + C(t) \int_0^t \|\partial_x^{i+1} v_{n-1}(\sigma)\|_{\infty} d\sigma. \] (21)
Plugging this inequality into (18) we arrive at
\[ \|\partial_x^{i+1} v_n^\pm(t)\|_{\infty} \leq C_4(t) + C_5(t) \int_0^t \|\partial_x^{i+1} v_{n-1}(\sigma)\|_{\infty} d\sigma, \]
where we chose \( C_4(\cdot) \) in such a way that \( \|\partial_x^{i+1} f_{\pm, \cdot}\|_{\infty} \leq C_4(0) \). Furthermore we define \( C_6(t) = C_4(t) \exp(tC_5(t)) \). By induction on \( n \)
\[ \|\partial_x^{i+1} v_n^\pm(t)\|_{\infty} \leq C_6(t) \quad \text{for all } n \in \mathbb{N}. \]
Combining this with (20) and (21) proves \( \mathcal{H}(l + 1)(i) \).
In the second step we proof \( \mathcal{H}(l + 1)(ii) \). Using formula (17) we have
\begin{align*}
|\partial_{x,p} f_n^\pm(t, x, p) - \partial_{x,p} f_n^\pm(t, x, p)| &= |\partial_{x,p} f_{n, \pm}(Z_n^\pm(0)) - \partial_{x,p} f_{n, \pm}(Z_n^\pm(0))| \\
&\leq \sum_{|\alpha|=1} \sum_{1 \leq |\beta| \leq l} |\partial^n f_{n, \pm}(Z_n^\pm(0))Q_{\alpha\beta} (\partial^n Z_n^\pm(0)) - \partial^n f_{n, \pm}(Z_n^\pm(0))Q_{\alpha\beta} (\partial^n Z_n^\pm(0))| \\
&\leq \sum_{|\alpha|=1} \sum_{1 \leq |\beta| \leq l} |\partial^n f_{n, \pm}(Z_n^\pm(0)) - \partial^n f_{n, \pm}(Z_n^\pm(0))||Q_{\alpha\beta} (\partial^n Z_n^\pm(0))| \\
&\quad + |\partial^n f_{n, \pm}(Z_n^\pm(0))||Q_{\alpha\beta} (\partial^n Z_n^\pm(0)) - Q_{\alpha\beta} (\partial^n Z_n^\pm(0))| \\
&\leq \varepsilon_n(t)C(t) + C|\partial_{x,p} Z_n^\pm(0) - \partial_{x,p} Z_n^\pm(0)| \quad (22)
\end{align*}
with a null sequence \( \varepsilon_n(t) \) which is independent of \( \varepsilon \in [0, 1] \). Next we have to estimate the derivatives of the characteristics: Using the convenient abbreviation \( \zeta_{n, l}(s) := |\partial_{x,p} Z_n^\pm(s; t, x, p) - \partial_{x,p} Z_n^\pm(s; t, x, p)| \) we compute
\[ \zeta_{n, l}(s) \leq \int_s^t \zeta_{n, l}(\sigma) + \big| \partial_{x,p} (E_{n-1}(\sigma, \lambda_n^\pm(\sigma))) - \partial_{x,p} (E(\sigma, \lambda^\pm(\sigma))) \big| d\sigma. \]
With formula (17) we have for the second summand in the integral
\begin{align*}
&|\partial_{x,p} (E_{n-1}(\sigma, \lambda_n^\pm(\sigma))) - \partial_{x,p} (E(\sigma, \lambda^\pm(\sigma)))| \\
&\leq \sum_{|\alpha|=1} \sum_{1 \leq |\beta| \leq l} |\partial^n E_{n-1}(\lambda_n^\pm)Q_{\alpha\beta}(\partial^n \lambda_n^\pm) - \partial^n E(\lambda^\pm)Q_{\alpha\beta}(\partial^n \lambda^\pm)| \\
&\leq \sum_{|\alpha|=1} \sum_{1 \leq |\beta| \leq l} |\partial^n E_{n-1}(\lambda_n^\pm) - \partial^n E_{n-1}(\lambda^\pm)| |Q_{\alpha\beta}(\partial^n \lambda_n^\pm)| \\
&\quad + |\partial^n E_{n-1}(\lambda^\pm) - \partial^n E(\lambda^\pm)||Q_{\alpha\beta}(\partial^n \lambda_n^\pm)| \\
&\quad + |\partial^n E(\lambda^\pm)||Q_{\alpha\beta}(\partial^n \lambda_n^\pm) - Q_{\alpha\beta}(\partial^n \lambda^\pm)|
\end{align*}
With regard to the summands in the first line we have
\[
|\partial^\beta E_{n-1}(X_n^\pm) - \partial^\beta E_n(X_n^\pm)| |Q_{\alpha,\beta}(\partial^\gamma Z_n^\pm)|
\leq \left\| \partial_x^{[\beta]+1} E_{n-1}(\sigma) \right\|_\infty \left| Z_n^\pm(\sigma) - Z^\pm(\sigma) \right| |Q_{\alpha,\beta}(\partial^\gamma Z_n^\pm)|
\leq \varepsilon_n(t) C(t).
\]

Observe that we already have established the bound on \(\|\partial_x^{[\beta]+1} E_n\|_\infty\). For the second line we have
\[
|\partial^\beta E_{n-1}(X_n^\pm) - \partial^\beta E(X_n^\pm)| |Q_{\alpha,\beta}(\partial^\gamma Z_n^\pm)|
\leq \left\| \partial_x^{[\beta]} E_{n-1}(\sigma) - \partial_x^{[\beta]} E(\sigma) \right\|_\infty |Q_{\alpha,\beta}(\partial^\gamma Z_n^\pm)| \leq \varepsilon_n(t) C(t).
\]

In the third line summands with \(2 \leq |\beta| \leq l\) can be estimated against \(\varepsilon_n(t) C(t)\), whereas summands with \(|\beta| = 1\) can be estimated against \(C(\sigma) \zeta_{n,t}(\sigma)\). Collecting everything we conclude
\[
\zeta_{n,t}(s) \leq \varepsilon_n(t) C(t) + C(t) \int_s^t \zeta_{n,t}(\sigma) d\sigma
\]
and using Gronwall’s Lemma
\[
\zeta_{n,t}(0) \leq \varepsilon_n(t) C(t).
\]

Combining this inequality with (22) proves \(\mathcal{H}(l+1)(ii)\).

For the third step we apply Proposition 1(iii) on \(\partial_{t,x}^k E_n\) and \(\partial_{t,x}^k g_n\) and chose \(d\) sufficiently small again. Thus, \(\mathcal{H}(l+1)(iii)\) is shown.

For the last step we note that by formally differentiating the characteristic equations
\[
\partial_{t,x}^{[\beta]+1} X^\pm(s) = \partial_{t,x}^{[\beta]+1} \left( x + \int_t^s P^\pm(\sigma) d\sigma \right)
\]
\[
\partial_{t,x}^{[\beta]+1} P^\pm(s) = \partial_{t,x}^{[\beta]+1} \left( p + \int_t^s E(\sigma, X^\pm(\sigma)) + D^{[3]}(\sigma) d\sigma \right)
\]
we end with an initial value problem for an inhomogeneous linear ODE in \(\partial_{t,x}^{[\beta]+1} Z^\pm\)
\[
\frac{d}{ds} \partial_{t,x}^{[\beta]+1} Z^\pm = A(s; t, x, p) \partial_{t,x}^{[\beta]+1} Z^\pm + B(s; t, x, p) \quad \partial_{t,x}^{[\beta]+1} Z^\pm(t; t, x, p) = D(t, x, p)
\]
where the coefficient functions are continuous and
\[
|A(s; t, x, p)| + |B(s; t, x, p)| + |D(t, x, p)| \leq C(t)
\]
for all \(0 \leq s \leq t\) and \(x, p \in \mathbb{R}^3\). For this step it is crucial that the only derivates of order \(l+1\) applied on \(E\) and \(D^{[3]}\) are purely spatial and thus continuous. Therefore, we have \(Z^\pm \in C^{l+1}(J \times J \times \mathbb{R}^3 \times \mathbb{R}^3)\) and using a standard estimate of ODEs, see e.g. [21, 16.VI],
\[
|\partial_{t,x}^{[\beta]+1} Z^\pm(s)| \leq C(t).
\]

Using this together with the support properties of \(f^\pm\) gives \(\mathcal{H}(l+1)(iv)\) and Theorem 4.9 is completely proved.
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