Loop Groups and Discrete KdV Equations

Jeremy Schiff
Department of Mathematics, Bar-Ilan University
Ramat Gan 52900, Israel
email: schiff@math.biu.ac.il

September 2002

Abstract

A study is presented of fully discretized lattice equations associated with the KdV hierarchy. Loop group methods give a systematic way of constructing discretizations of the equations in the hierarchy. The lattice KdV system of Nijhoff et al. arises from the lowest order discretization of the trivial, lowest order equation in the hierarchy, \( b_t = b_x \). Two new discretizations are also given, the lowest order discretization of the first nontrivial equation in the hierarchy, and a “second order” discretization of \( b_t = b_x \). The former, which is given the name full lattice KdV has the (potential) KdV equation as a standard continuum limit. For each discretization a Bäcklund transformation is given and soliton content analyzed. The full lattice KdV system has, like KdV itself, solitons of all speeds, whereas both other discretizations studied have a limited range of speeds (being discretizations of an equation with solutions only of a fixed speed).

1 Introduction

Despite the fact that numerical simulations of PDEs of KdV type can be done quickly and accurately these days using standard spectral methods, it is still of interest to look at discretizations of such PDEs, and see how “integrability properties” (elastic soliton scattering, existence of conserved quantities etc.) are affected by discretization, and in particular to see if there are “integrable discretizations”, that exhibit all the special properties of the underlying PDE. One can consider both “partial” and “full” discretizations; in the former only the spatial coordinate is discretized, in the latter time is discretized too. This paper focuses on full discretizations.
The difference equation usually known as discrete KdV was first studied by Hirota Hirota. Using a slightly different notation from that of Hirota, discrete KdV is the equation

\[
\frac{1}{1 + u_{n+1,m+1}} - \frac{1}{1 + u_{n,m}} = c(u_{n+1,m} - u_{n,m+1}) \quad (c \text{ constant}).
\]

(1)

This is a discretization of KdV, but in a rather unusual sense. The main justification for the name “discrete KdV” is that it has a bilinear formulation and a family of soliton solutions very similar to those of KdV (see also for rational solutions). The study of discrete KdV was taken further by Nijhoff and collaborators (see for a review and references). The work of Nijhoff et al. focuses on the equation

\[
\left(1 + \frac{b_{n,m+1} - b_{n,m}}{p - q}\right)\left(1 + \frac{b_{n,m} - b_{n+1,m+1}}{p + q}\right) = 1 \quad (p, q \text{ constant}),
\]

(2)

which they call lattice KdV. In fact this equation is a “potential form” of discrete KdV, in the sense that if \(b_{n,m}\) satisfies \(\hat{d}p\), then it is easy to check that

\[
u_{n,m} = \frac{b_{n-1,m-1} - b_{n,m}}{p + q}
\]

satisfies \(\hat{d}h\) with \(c = (p + q)/(p - q)\). (This is supposed to be an analog of the fact that if \(b(x,t)\) satisfies the “potential KdV” equation \(b_t = \frac{1}{4}b_{xxx} + \frac{3}{2}b_x^2 + \delta(t)\) for some function \(\delta(t)\), then \(u = b_x\) satisfies the KdV equation \(u_t = \frac{1}{4}u_{xxx} + 3uu_x\).)

Nijhoff et al.’s lattice KdV equation has an advantage over Hirota’s discrete KdV in that it is easier to see its continuum limit (in the usual sense, to be explained shortly) as well as at least one nonstandard continuum limit in which it reduces to the potential KdV equation. On substituting \(p = 1/h\) and \(q = 1/k\), \(\hat{d}p\) becomes

\[
- \frac{b_{n+1,m+1} - b_{n,m+1} + b_{n+1,m} - b_{n,m}}{h} + \frac{b_{n+1,m+1} - b_{n,m+1} + b_{n,m+1} - b_{n,m}}{k} + (b_{n,m} - b_{n+1,m+1})(b_{n,m+1} - b_{n+1,m}) = 0.
\]

(4)

Taking the standard continuum limit will be taken to mean replacing \(b_{n,m}\) by \(b(x,t)\), \(b_{n+1,m}\) by \(b(x+h,t)\), \(b_{n,m+1}\) by \(b(x,t+k)\), \(b_{n+1,m+1}\) by \(b(x+h,t+k)\), expanding in powers of \(h\) and \(k\) and ignoring all but leading order terms. It is clear that in this limit the first term in \(\hat{d}p^2\) gives \(-2b_x\), the second \(2b_t\) and the third 0. Thus in the standard continuum limit, lattice KdV is simply a discretization of \(b_t = b_x\). A nonstandard continuum limit of \(\hat{d}p^2\) that gives the potential KdV equation is as follows: Make the same replacements as before, expand in powers of \(h\) and \(k\), but keep not only the leading order terms but also all terms of order \(h\) and \(h^2\). This gives

\[
- 2 \left(b_x + \frac{h}{2}b_{xx} + \frac{h^2}{6}b_{xxx}\right) + b_t + \left(b_t + hb_{tx} + \frac{h^2}{2}b_{txx}\right) + h^2b_x^2 = 0
\]

(5)
Now write \( b = \tilde{b} - \frac{h}{2} \tilde{b}_x \). Ignoring terms of order \( h^3 \) and above, the last equation can be written

\[
\tilde{b}_t = \tilde{b}_x - \frac{h^2}{3} \left( \frac{1}{4} \tilde{b}_{xxx} + \frac{3}{2} \tilde{b}_x^2 \right). \tag{6}
\]

This is a “linear combination” of the flow obtained in the standard continuum limit with the potential KdV flow.

The foregoing discussions raise a variety of questions. The relationship of KdV/potential KdV and discrete KdV/lattice KdV as it stands is rather cryptic and requires some clarification. It would also be good to have another integrable lattice equation from which KdV/potential KdV can be obtained by taking a standard continuum limit. If this is possible, then it would be good to know just what freedoms there are in constructing integrable discretizations. Finally, though this is a question that will not be addressed in the current paper, given an integrable lattice equation, just how much freedom is there in taking the continuum limit?

This paper discusses the subject of discretizations of KdV using loop group methods. The basic fact behind the loop group approach to KdV is that the KdV equation (or, more precisely, the Lax pair for the KdV equation) is simply a “disguised” version of the Frobenius-integrable pair of linear first-order constant-coefficient ODEs

\[
\partial_x U = \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} U, \quad \partial_t U = l \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} U \tag{7}
\]

(here \( U \) is a \( 2 \times 2 \) matrix function of \( x, t, l \)). The relation of the above system with KdV will be explained fully in section 2 below. In greater generality, the \( N \)th flow \((N = 1, 3, 5, \ldots)\) in the KdV hierarchy is associated with the system

\[
\partial_x U = \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} U, \quad \partial_t U = \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix}^N U, \tag{8}
\]

which reduces to the standard system \( \text{lin1} \) when \( N = 3 \). The approach proposed in this paper for constructing integrable discretizations of KdV is simply to discretize the system \( \text{lin1} \) or \( \text{lin2} \) (any explicit scheme for numerical integration of ODEs can be used) and then to apply the necessary “disguise” to translate this system into a discrete KdV. Section 3 is devoted to the simplest discretization of \( \text{lin2} \) with \( N = 1 \), namely

\[
U_{n+1,m} = \left[ I + h \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right] U_{n,m}, \quad U_{n,m+1} = \left[ I + k \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right] U_{n,m}. \tag{9}
\]

This is just a first-order Euler scheme with different step sizes in the \( x \) and \( t \) directions. This scheme gives rise to the lattice KdV equation, which, as shown above, is a first-order discretization of the \( N = 1 \) flow in the potential KdV hierarchy, \( b_t = b_x \). As an application of the loop group formulation, a Bäcklund transformation for \( \tilde{b}_p \) is given, and soliton solutions are derived.
A brief analysis of the soliton solutions is given, which helps clarify the rather schizophrenic nature of the lattice KdV equation, which on one hand is a (nonlinear) discretization of $b_t = b_x$, and on the other displays features of potential KdV.

Section 4 is devoted to the simplest discretization of lin1, namely

$$U_{n+1,m} = \left[ I + h \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right] U_{n,m}, \quad U_{n,m+1} = \left[ I + kl \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right] U_{n,m}. \quad (10)$$

As expected, this gives rise to a system which is, in a natural way, a first-order discretization of the potential KdV equation. The system is a little complicated, involving two auxiliary fields (reminiscent of the discretization of the sinh-Gordon equation given in bobenko), but it seems this is the price that has to be paid to have an integrable lattice equation that has potential KdV as a natural continuum limit. The Bäcklund transformation and soliton solutions are derived for this system too.

Section 5 considers another discretization of lin2 for $N = 1$, namely

$$U_{n+1,m} = \left[ I + h \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} + \frac{h^2}{2} \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix}^2 \right] U_{n,m}, \quad U_{n,m+1} = \left[ I + k \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right] U_{n,m}. \quad (11)$$

This example is worked out mainly to illustrate that the method can be extended to arbitrary order discretizations of lin1 and lin2, establishing that there is quite a lot of freedom in constructing integrable discretizations. Section 6 contains some concluding remarks.

I conclude the introduction with a brief mention of some relevant literature. The approach to discretization taken in this paper is closely related to the approach of discretizing the scattering problem, which was first proposed by Ablowitz and Ladik al, and recently has been revisited by Boiti et al. BPPS. Several potentially interesting applications of discretizations of equations of KdV type have emerged recently. Nijhoff et al. Nijhoff were the first to notice the link between lattice KdV and the discrete conformal map equation

$$\frac{(z_{n,m} - z_{n+1,m})(z_{n,m+1} - z_{n+1,m+1})}{(z_{n,m} - z_{n,m+1})(z_{n+1,m} - z_{n+1,m+1})} = s \quad (s \text{ constant}), \quad (13)$$

which in the case $s = -1$ is a natural discretization of the Cauchy-Riemann conditions. Techniques related to those of this paper have been applied to dcm in dcpaper. dcm may well play a significant role in the field of numerical conformal mapping. Discretizations of KdV and related equations have also been shown to have a role in a variety of other numerical algorithms numapps.
2 KdV as a linear constant-coefficient flow

This section contains a summary of results from me1, relating the (potential) KdV equation with the linear constant coefficient flow lin1. A rather more mathematical description can be found in sw.

The general solution of lin1 is

\[ U(x, t, l) = \exp \left( x \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right) U(0, 0, l) . \] (14)

Assume that the function \( U(0, 0, l) \) is defined for \( |l| = 1 \), and has nonzero determinant; in other words it is an element of the loop group \( LGL_2(\mathbb{C}) \) PS. Then evidently so is \( U(x, t, l) \). Now a typical element \( g(l) \) of the loop group \( LGL_2(\mathbb{C}) \) can be written as a product \( S^{-1}(l)Y(l) \) where \( Y(l) \) is holomorphic for \( |l| < 1 \) and \( S(l) \) is holomorphic for \( |l| > 1 \) with \( S(\infty) = I \). This is the so-called Birkhoff factorization theorem, see PS, chapter 8. So let us write

\[ U(x, t, l) = S^{-1}(x, t, l)Y(x, t, l) , \] (15)

(with \( Y \) holomorphic in \( |l| < 1 \), \( S \) holomorphic in \( |l| > 1 \) and \( S(x, t, \infty) = I \)) and let us try to find differential equations satisfied by the two “components” \( Y \) and \( S \) of \( U \). Substituting fact into lin1, multiplying on the left by \( S \) and on the right by \( Y^{-1} \) gives

\[ - S_x S^{-1} + Y_x Y^{-1} = S \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} S^{-1}, \quad - S_t S^{-1} + Y_t Y^{-1} = l S \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} S^{-1} . \] (16)

Now, if

\[ S = I + \frac{1}{l} \begin{pmatrix} a_1(x, t) & b_1(x, t) \\ c_1(x, t) & d_1(x, t) \end{pmatrix} + \frac{1}{l^2} \begin{pmatrix} a_2(x, t) & b_2(x, t) \\ c_2(x, t) & d_2(x, t) \end{pmatrix} + \ldots \] (17)

then a brief calculation shows

\[ S \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} -b & 1 \\ l-v & b \end{pmatrix} + O\left(l^{-1}\right) , \] (18)

\[ l S \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} -bl-B & l+v-b^2 \\ l^2-vl-V & bl+B \end{pmatrix} + O\left(l^{-1}\right) , \] (19)

where \( b = -b_1, v = a_1 - d_1, B = c_1 - b_2 + a_1 b_1, V = a_2 - d_2 + a_1 d_1 - b_1 c_1 - a_1^2 \). Substitute these results in 16. On the left-hand side of the equations in 16, since \( Y \) is holomorphic in \( |l| < 1 \), \( Y_x Y^{-1} \) and \( Y_t Y^{-1} \) can be written as power series in \( l \). And since \( S \) is holomorphic in \( |l| > 1 \) with \( S(x, t, \infty) = I \), \( S_x S^{-1} \) and \( S_t S^{-1} \) can be written as power series in \( 1/l \) with no constant term. Thus from the non-negative powers of \( l \) in 16, after substituting 18-19, it follows that

\[ Y_x Y^{-1} = \begin{pmatrix} -b & 1 \\ l-v & b \end{pmatrix} , \] (20)

\[ Y_t Y^{-1} = \begin{pmatrix} -bl-B & l+v-b^2 \\ l^2-vl-V & bl+B \end{pmatrix} . \] (21)
If $X = Y_x Y^{-1}, T = Y_t Y^{-1}$ then $X, T$ must satisfy the zero-curvature equation

$$X_t - T_x + [X, T] = 0 .$$

(22)

Substituting the forms $\hat{\text{lp}}^1-\hat{\text{lp}}^2$ into the zero-curvature equation, required to be true for all $l$, gives the following system of equations:

\begin{align*}
v &= b_x + b^2 , \\
B &= \frac{1}{2} b_{xx} + bb_x , \\
V_x &= \left( \frac{1}{4} b_{xxx} + \frac{1}{2} b_x^2 + b b_{xx} + b^2 b_x \right)_x , \\
b_t &= \frac{1}{2} b_{xxx} + 2 b_x^2 + b b_{xx} + b^2 b_x - V .
\end{align*}

(23) (24) (25) (26)

The third equation can be integrated to give $V = \frac{1}{4} b_{xxx} + bb_{xx} + \frac{1}{2} b_x^2 + b^2 b_x - \delta(t)$, where $\delta$ is an arbitrary function of $t$ alone. Using this in the last equation gives

$$b_t = \frac{1}{4} b_{xxx} + \frac{3}{2} b_x^2 + \delta(t) .$$

(27)

All this can be summarized in the following result:

**Proposition 2.1:** Let $U(0, 0, l)$ be an element of $LGL_2(\mathbb{C})$, let $S^{-1}Y$ be the Birkhoff decomposition of $U(x, t, l) = \exp \left( x \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} + tl \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right) U(0, 0, l)$, and let $b(x, t)$ be $(-1)$ times the component of $1/l$ in the 1,2-entry of $S$. Then $b(x, t)$ is a solution, possibly with singularities, of the potential KdV equation $\hat{\text{pKdV}}$ for some function $\delta$.

The reason for the phrase “possibly with singularities” here is because for some values of $x$ and $t$, $U(x, t, l)$ might leave the dense open set of $LGL_2(\mathbb{C})$ where Birkhoff decomposition is possible (it can be proved that these values are isolated). It is important for the purposes of this paper to note that although the above proposition makes no mention of the linear constant-coefficient flow $\hat{\text{lin1}}$, the heart of its proof is that the this flow induces, via Birkhoff decomposition, the matrix Lax pair $\hat{\text{lp}}^1-\hat{\text{lp}}^2$ for the potential KdV equation. Note also that the first equation of the Lax pair $\hat{\text{lp}}^1$ gives the usual relation of KdV with the Schrödinger equation. Writing either column of $Y$ as $\begin{pmatrix} \psi \\ \phi \end{pmatrix}$, $\hat{\text{lp}}^1$ gives $\psi_{xx} = (l - 2b_x) \psi$.

There are many applications of the above result, of which only one will be discussed here, the construction of the standard Bäcklund transformation for potential KdV. The idea behind this Bäcklund transformation is as follows: Suppose the element $U(0, 0, l)$ of the loop group gives solution $U(x, t, l)$ of the linear system $\hat{\text{lin1}}$ with Birkhoff decomposition $S^{-1}Y$ and corresponding potential KdV solution $b(x, t)$. Let us now try to find the potential KdV solution corresponding to the element

$$\sqrt{\frac{l - \theta}{l}} \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} U(0, 0, l) \begin{pmatrix} 0 & \frac{1}{l - \theta} \\ 1 & 0 \end{pmatrix}$$

(28)
with $0 < \theta < 1$. The new solution of the linear system $\hat{\text{lin1}}$ is

$$
\sqrt{\frac{l-\theta}{l}} \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} U(x,t,l) \begin{pmatrix} 0 & \frac{1}{l-\theta} \\ 1 & 0 \end{pmatrix}
$$

$$= \sqrt{\frac{l-\theta}{l}} \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} S^{-1}(x,t,l) Y(x,t,l) \begin{pmatrix} 0 & \frac{1}{l-\theta} \\ 1 & 0 \end{pmatrix}.
$$

(29)

To perform the new Birkhoff decomposition, a certain matrix and its inverse must be inserted as follows:

$$
\sqrt{\frac{l-\theta}{l}} \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} S^{-1}(x,t,l) \left( \begin{array}{cc} \alpha & 1 \\ l-\theta+\alpha\beta & \beta \end{array} \right)^{-1} \left( \begin{array}{cc} \alpha & 1 \\ l-\theta+\alpha\beta & \beta \end{array} \right) Y(x,t,l) \begin{pmatrix} 0 & \frac{1}{l-\theta} \\ 1 & 0 \end{pmatrix}.
$$

(30)

The aim is to choose $\alpha$ and $\beta$ so that this is written in Birkhoff factorized form, i.e. so that

$$
\tilde{S}(x,t,l) = \sqrt{\frac{l}{l-\theta}} \begin{pmatrix} \alpha & 1 \\ l-\theta+\alpha\beta & \beta \end{pmatrix} S(x,t,l) \begin{pmatrix} 0 & \frac{1}{l} \\ 1 & 0 \end{pmatrix}
$$

(31)

is holomorphic in $|l| > 1$ and satisfies $\tilde{S}(x,t,\infty) = I$, and

$$
\tilde{Y}(x,t,l) = \begin{pmatrix} \alpha & 1 \\ l-\theta+\alpha\beta & \beta \end{pmatrix} Y(x,t,l) \begin{pmatrix} 0 & \frac{1}{l-\theta} \\ 1 & 0 \end{pmatrix}
$$

(32)

is holomorphic in $|l| < 1$. Inserting the expansion $\tilde{\text{sexp}}$ in $\tilde{s\text{con}}$, the former condition requires $\beta = b$. For the latter condition it is just necessary to check $\tilde{Y}$ does not have a pole at $l = \theta$, and this requires $\alpha = -Y_{21}(x,t,\theta)/Y_{11}(x,t,\theta)$. Finally, it is necessary to compute the new solution of the potential KdV equation, i.e. the component of $1/l$ in the 1,2-entry of $\tilde{S}$. A brief calculation shows this is simply $-\alpha$. Utilizing the Lax pair $\hat{\text{lp1}}-\hat{\text{lp2}}$ it is straightforward to determine properties of $\alpha$ leading to the following result:

**Proposition 2.2:** If $b$ is a solution of the potential KdV equation $\hat{\text{pkdv}}$ and $\psi$ satisfies

$$
\psi_{xx} = (\theta - 2b_x)\psi, \quad \psi_t = -\frac{1}{2}b_{xx}\psi + (\theta + b_x)\psi_x
$$

(33)

then $\tilde{b} = b + \psi_x/\psi$ is also a solution of potential KdV, for the same function $\delta$.

Equations $\hat{\text{lp}}$ comprise the standard scalar Lax pair for the KdV equation. Applying the Bäcklund transformation to the $x$-independent solution $b(t) = \int \delta(t) dt$ gives the 1-soliton solutions

$$
b(x,t) = \int \delta(t) dt + \sqrt{\theta}\tanh \left( \sqrt{\theta}(x + \theta t) + C \right)
$$

(34)

and the singular solutions

$$
b(x,t) = \int \delta(t) dt + \sqrt{\theta}\coth \left( \sqrt{\theta}(x + \theta t) + C \right)
$$

(35)
where in both formulae $C$ is a constant. The easiest way to apply the Bäcklund transformation again to this solution is to use the Bianchi permutability theorem that states that the two-parameter family of solutions obtained by applying first the Bäcklund transformation with parameter $\theta_1$ and then the Bäcklund transformation with parameter $\theta_2$ is the same as the two-parameter family of solutions obtained by applying the two Bäcklund transformations in the reverse order.

See m1 for a detailed discussion of this. The Bianchi permutability theorem can be used to derive an algebraic expression for the solutions obtained by applying two Bäcklund transformations (see dj section 5.4.3):

**Proposition 2.3:** If $b$ is a solution of the potential KdV equation $\hat{p}_{kdv}$, and $b_1$ and $b_2$ are solutions obtained by applying Bäcklund transformations with parameters $\theta_1$ and $\theta_2$ respectively to $b$, then

$$B = b + \frac{\theta_1 - \theta_2}{b_1 - b_2}$$

is a solution obtained by applying the two Bäcklund transformations successively to $b$, in either order.

Applying this result using a 1-soliton solution for $b_1$, a singular solution for $b_2$ and $\theta_2 > \theta_1$ gives the 2-soliton solution

$$b(x,t) = \int \delta(t) dt + \frac{\theta_1 - \theta_2}{\sqrt{\theta_1 \tanh \alpha_1 - \sqrt{\theta_2 \coth \alpha_2}}}\left\{ \begin{array}{l}
\alpha_1 = \sqrt{\theta_1} (x + \theta_1 t) + C_1 \\
\alpha_2 = \sqrt{\theta_2} (x + \theta_2 t) + C_2
\end{array} \right.$$  \hspace{1cm} (36)

This concludes our presentation of the basic theory of the KdV equation and its relation with the linear system $\hat{\text{lin1}}$ which will be imitated for discrete systems in later sections.

### 3 Discretizations I: Lattice KdV

The aim in this section is to follow the procedures of the last section as closely as possible, but replacing the solution $U(x,t,l)$ of $\hat{\text{lin1}}$ by the solution $U_{nm}(l)$ of the lattice equation $\hat{\text{disc1}}$, which has general solution

$$U_{n,m}(l) = \left[ I + h \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right]^n \left[ I + k \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right]^m U_{0,0}(l)$$

$$= \frac{1}{4} \begin{pmatrix} (1 + h \sqrt{l})^n + (1 - h \sqrt{l})^n \\ (1 + k \sqrt{l})^m + (1 - k \sqrt{l})^m \end{pmatrix} \begin{pmatrix} 1 + h \sqrt{l} & 1 - h \sqrt{l} \\ 1 + k \sqrt{l} & 1 - k \sqrt{l} \end{pmatrix} U_{0,0}(l)$$

\hspace{1cm} (38)
Suppose $U_{n,m}(l)$ has a Birkhoff factorization $S_{n,m}^{-1}(l)Y_{n,m}(l)$. Substituting in $\hat{\text{disc}}1$ and rearranging gives

\begin{align*}
Y_{n+1,m}Y_{n,m}^{-1} &= S_{n+1,m} \left[ I + h \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right] S_{n,m}^{-1}, \tag{39} \\
Y_{n,m+1}Y_{n,m}^{-1} &= S_{n,m+1} \left[ I + k \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right] S_{n,m}^{-1}. \tag{40}
\end{align*}

Writing

\begin{equation}
S_{n,m} = I + \frac{1}{l} \begin{pmatrix} a_{n,m} & -b_{n,m} \\ c_{n,m} & d_{n,m} \end{pmatrix} + \ldots \tag{41}
\end{equation}

and comparing non-negative powers of $l$ on both sides of $\hat{\text{int}}1-\hat{\text{int}}2$ gives

\begin{align*}
Y_{n+1,m} &= \begin{pmatrix} 1 - hb_{n+1,m} \\ hl + h(d_{n+1,m} - a_{n,m}) \end{pmatrix} \begin{pmatrix} h \\ 1 + hb_{n,m} \end{pmatrix} \left( 1 - hb_{n+1,m} + h(l + h(d_{n+1,m} - a_{n,m})) \right) Y_{n,m}, \tag{42} \\
Y_{n,m+1} &= \begin{pmatrix} 1 - kb_{n,m+1} \\ kl + k(d_{n+1,m} - a_{n,m}) \end{pmatrix} \begin{pmatrix} k \\ 1 + kb_{n,m} \end{pmatrix} \left( 1 - kb_{n,m+1} + k(l + k(d_{n+1,m} - a_{n,m})) \right) Y_{n,m}. \tag{43}
\end{align*}

There is one further simplification that can be made in these equations. $\hat{\text{int}}1$ (and similarly $\hat{\text{int}}2$) can be written in the form

\begin{equation}
S_{n,m}S_{n+1,m}^{-1} \cdot Y_{n+1,m}Y_{n,m}^{-1} = S_{n,m} \left[ I + h \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right] S_{n,m}^{-1}. \tag{44}
\end{equation}

Taking the determinant gives

\begin{equation}
\det \left( S_{n,m}S_{n+1,m}^{-1} \right) \det \left( Y_{n+1,m}Y_{n,m}^{-1} \right) = 1 - h^2 l. \tag{45}
\end{equation}

The Birkhoff factorization theorem applies for scalars (elements of $LGL_1\mathbf{C}$) too, so from this it can be deduced that $\det \left( Y_{n+1,m}Y_{n,m}^{-1} \right) = 1 - h^2 l$ (and $\det \left( S_{n,m}S_{n+1,m}^{-1} \right) = 1$). Applying this to $\hat{\text{int}}3$ (and the corresponding result $\det \left( Y_{n,m+1}Y_{n,m}^{-1} \right) = 1 - k^2 l$ to $\hat{\text{int}}4$) gives

\begin{align*}
Y_{n+1,m} &= \begin{pmatrix} 1 - hb_{n+1,m} \\ hl + b_{n,m} - b_{n+1,m} - hb_{n,m}b_{n+1,m} \end{pmatrix} \begin{pmatrix} h \\ 1 + hb_{n,m} \end{pmatrix} \left( 1 - hb_{n+1,m} + h(l + h(b_{n,m} - b_{n+1,m} - hb_{n,m}b_{n+1,m})) \right) Y_{n,m}, \tag{46} \\
Y_{n,m+1} &= \begin{pmatrix} 1 - kb_{n,m+1} \\ kl + b_{n,m} - b_{n+1,m} - kb_{n,m}b_{n+1,m} \end{pmatrix} \begin{pmatrix} k \\ 1 + kb_{n,m} \end{pmatrix} \left( 1 - kb_{n,m+1} + k(l + k(b_{n,m} - b_{n+1,m} - kb_{n,m}b_{n+1,m})) \right) Y_{n,m}. \tag{47}
\end{align*}

Up to a rescaling this is precisely Nijhoff et al.’s scalar Lax pair for the lattice KdV equation Nijhoff. Writing

\begin{align*}
L_{n,m} &= \begin{pmatrix} 1 - hb_{n+1,m} \\ hl + b_{n,m} - b_{n+1,m} - hb_{n,m}b_{n+1,m} \end{pmatrix} \begin{pmatrix} h \\ 1 + hb_{n,m} \end{pmatrix}, \tag{48} \\
M_{n,m} &= \begin{pmatrix} 1 - kb_{n,m+1} \\ kl + b_{n,m} - b_{n+1,m} - kb_{n,m}b_{n+1,m} \end{pmatrix} \begin{pmatrix} k \\ 1 + kb_{n,m} \end{pmatrix}, \tag{49}
\end{align*}
equations \( \tilde{d}lp1-\tilde{d}lp2 \) are just
\[
Y_{n+1,m} = L_{n,m}Y_{n,m}, \quad Y_{n,m+1} = M_{n,m}Y_{n,m},
\] (50)
and for consistency \( L_{n,m+1}M_{n,m} = M_{n+1,m}L_{n,m} \). (This last equation plays the role of the zero-curvature equation in the continuous case.) Substituting the forms found for \( L_{n,m}, M_{n,m} \) in the consistency condition gives lattice KdV \( \tilde{d}p2 \). Thus the analog of Proposition 2.1 is obtained:

**Proposition 3.1:** Let \( U_{0,0}(l) \) be an element of \( LGL_2(\mathbb{C}) \), let \( S^{-1}_{n,m}(l)Y_{n,m}(l) \) be the Birkhoff decomposition of
\[
U_{n,m}(l) = \left[ I + h \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right]^n \left[ I + k \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right]^m U_{0,0}(l),
\] (51)
and let \( b_{n,m} \) be \((-1)\) times the component of \( 1/l \) in the \( 1,2 \)-entry of \( S_{n,m} \). Then \( b_{n,m} \) is a solution, possibly with singularities, of the lattice KdV equation \( \tilde{d}p2 \).

In fact there is no reason why \( U_{0,0}(l) \) should not, in this case, be dependent on \( h \) and \( k \). So in principle the class of solutions of lattice KdV occuring this way is much larger than the corresponding class of solutions of (potential) KdV.

Let us attempt to find a Bäcklund transformation and soliton solutions for lattice KdV proceeding as in section 2. Making the replacement
\[
U_{0,0}(l) \to \sqrt{\frac{l - \theta}{l}} \left( \frac{0}{l} \frac{1}{0} \right) U_{0,0}(l) \left( \frac{0}{1} \frac{1}{0} \right)
\] (52)
gives
\[
U_{n,m}(l) \to \sqrt{\frac{l - \theta}{l}} \left( \frac{0}{l} \frac{1}{0} \right) U_{n,m}(l) \left( \frac{0}{1} \frac{1}{0} \right)
\] (53)
and
\[
S_{n,m}(l) \to \sqrt{\frac{l}{l - \theta}} \left( \frac{\alpha_{n,m}}{l - \theta + \alpha_{n,m} b_{n,m}} \frac{1}{b_{n,m}} \right) S_{n,m}(l) \left( \frac{0}{1} \frac{1}{0} \right)
\] (54)
\[
Y_{n,m}(l) \to \left( \frac{\alpha_{n,m}}{l - \theta + \alpha_{n,m} b_{n,m}} \frac{1}{b_{n,m}} \right) Y_{n,m}(l) \left( \frac{0}{1} \frac{1}{0} \right)
\] (55)
where \( \alpha_{n,m} = -(Y_{n,m})_{21}(\theta)/(Y_{n,m})_{11}(\theta) \). The new solution of lattice KdV is simply \(-\alpha_{n,m}\). Using \( \tilde{d}lp1-\tilde{d}lp2 \) to find properties of \( \alpha_{n,m} \) gives the Bäcklund transformation:

**Proposition 3.2:** If \( b_{n,m} \) is a solution of the lattice KdV equation \( \tilde{d}p2 \) and \( \psi_{n,m} \) satisfies
\[
\frac{\psi_{n+2,m} - 2\psi_{n+1,m} + \psi_{n,m}}{h^2} = \frac{\theta}{h^2} \psi_{n,m} - \left( \frac{b_{n+2,m} - b_{n,m}}{h} \right) \psi_{n+1,m}
\] (56)
\[
\frac{\psi_{n,m+1} - \psi_{n,m}}{k} = \frac{\psi_{n+1,m} - \psi_{n,m}}{h} + (b_{n+1,m} - b_{n,m+1}) \psi_{n,m}
\] (57)
then \( b_{n,m} = b_{n+1,m} + (\psi_{n+1,m} - \psi_{n,m})/(h\psi_{n,m}) \) is also a solution of lattice KdV.

The first equation here is a natural discretization of the first equation in \( \hat{\text{ip}} \), and is the discretization of the Schrödinger equation studied in BPPS. The second equation is, however, completely unrelated to that in \( \text{ip} \). To get 1-soliton and singular solutions the Bäcklund transformation can be applied to the trivial solution \( b_{n,m} = 0 \). This gives solutions of the form

\[
b_{n,m} = \sqrt{\theta} A(1 + h\sqrt{\theta})^n(1 + k\sqrt{\theta})^m - B(1 - h\sqrt{\theta})^n(1 - k\sqrt{\theta})^m \quad (A, B \text{ constants}).
\]  

(58)

If \( A : B \) is positive and \( h, k < 1/\sqrt{\theta} \) this gives 1-soliton solutions

\[
b_{n,m} = \sqrt{\theta} \tanh \left( n \tanh^{-1}(h\sqrt{\theta}) + m \tanh^{-1}(k\sqrt{\theta}) + C \right) \quad (C \text{ constant}).
\]  

(59)

If \( A : B \) is negative and \( h, k < 1/\sqrt{\theta} \) gensol gives singular solutions

\[
b_{n,m} = \sqrt{\theta} \coth \left( n \tanh^{-1}(h\sqrt{\theta}) + m \tanh^{-1}(k\sqrt{\theta}) + C \right).
\]  

(60)

The Bianchi permutability theorem applies equally here in the discrete case, and this can be used to give the discrete version of Proposition 2.3:

**Proposition 3.3:** If \( b_{n,m} \) is a solution of the lattice KdV equation \( \hat{\text{dp}}2 \), and \( b^{(1)}_{n,m} \) and \( b^{(2)}_{n,m} \) are solutions obtained by applying Bäcklund transformations with parameters \( \theta_1 \) and \( \theta_2 \) respectively to \( b_{n,m} \), then

\[
B_{n,m} = b_{n,m} + \frac{\theta_1 - \theta_2}{b^{(1)}_{n,m} - b^{(2)}_{n,m}}
\]  

(61)

is a solution obtained by applying the two Bäcklund transformations successively to \( b_{n,m} \), in either order.

**Proof.** Writing \( q_{n,m} = (\psi_{n+1,m} - \psi_{n,m})/(h\psi_{n,m}) \), the Bäcklund transformation can be written as \( b_{n,m} \to b_{n+1,m} + q_{n,m} \) where \( q_{n,m} \) satisfies the discrete Riccati equation \( \hat{\text{ss}} \)

\[
q_{n+1,m} = \frac{q_{n,m}(1 + hb_{n+2,m} + hb_{n,m}) + (h\theta - b_{n+2,m} + b_{n,m})}{1 + b&q_{n,m}}
\]  

(62)

(for the sake of brevity I only look at the first equation in \( \hat{\text{dsl1-sdl2}} \)). Alternatively, after a little algebra, the transformation can be written \( b_{n,m} \to \tilde{b}_{n,m} \) where \( b_{n,m}, \tilde{b}_{n,m} \) are related by

\[
\frac{\tilde{b}_{n+1,m} - \tilde{b}_{n,m} + b_{n+1,m} - b_{n,m}}{h} = \theta + (b_{n,m} - \tilde{b}_{n+1,m})(\tilde{b}_{n,m} - b_{n+1,m}).
\]  

(63)

Using the Bianchi permutability theorem and the premises of the theorem, it is known that applying the BT with parameter \( \theta_1 \) to \( b_{n,m} \) gives \( b^{(1)}_{n,m} \), applying the BT with parameter \( \theta_2 \) to \( b_{n,m} \),
gives \( b_{n,m}^{(2)} \), and applying either the BT with parameter \( \theta_2 \) to \( b_{n,m}^{(1)} \) or the BT with parameter \( \theta_1 \) to \( b_{n,m}^{(2)} \) gives the same solution \( B_{n,m} \). This implies 4 equations:

\[
\frac{b_{n+1,m}^{(1)} - b_{n,m}^{(1)} + b_{n+1,m} - b_{n,m}}{h} = \theta_1 + (b_{n,m}^{(1)} - b_{n+1,m}^{(1)})(b_{n,m}^{(1)} - b_{n+1,m}), \quad (64)
\]

\[
\frac{b_{n,m}^{(2)} - b_{n+1,m}^{(2)} + b_{n,m} - b_{n+1,m}}{h} = \theta_2 + (b_{n,m}^{(2)} - b_{n+1,m}^{(2)})(b_{n,m}^{(2)} - b_{n+1,m}), \quad (65)
\]

\[
\frac{B_{n+1,m} - B_{n,m} + b_{n+1,m}^{(1)} - b_{n,m}^{(1)}}{h} = \theta_2 + (b_{n,m}^{(1)} - B_{n+1,m})(B_{n,m} - b_{n+1,m}^{(1)}), \quad (66)
\]

\[
\frac{B_{n+1,m} - B_{n,m} + b_{n,m}^{(2)} - b_{n,m}^{(2)}}{h} = \theta_1 + (b_{n,m}^{(2)} - B_{n+1,m})(B_{n,m} - b_{n+1,m}^{(2)}). \quad (67)
\]

Adding the first and last of these equations and subtracting the other two gives

\[
2(\theta_1 - \theta_2) = (B_{n,m} - b_{n,m})(b_{n,m}^{(1)} - b_{n,m}^{(2)}) + (B_{n+1,m} - b_{n+1,m})(b_{n+1,m}^{(1)} - b_{n+1,m}) \quad (68)
\]

The general solution of this is clearly

\[
(B_{n,m} - b_{n,m})(b_{n,m}^{(1)} - b_{n,m}^{(2)}) = (\theta_1 - \theta_2) + (-1)^n F(m), \quad (69)
\]

where \( F \) is an arbitrary function of \( m \). Using the second equation in \( \ddsl1-\ddsl2 \) is is possible to show that \( F(m) = 0 \).

All that remains to do in this section is to briefly discuss the nature of soliton solutions of lattice KdV, and in particular how they compare to those of continuum KdV. From dsol the speed of the soliton with parameter \( \theta \) is

\[
c = \frac{h \tanh^{-1}(k \sqrt{\theta})}{k \tanh^{-1}(h \sqrt{\theta})}. \quad (70)
\]

(The formal definition of the “speed” is the number \( c \) such that the solution depends on \( m, n \) only through the combination \( nh + cmk \).) Recall that the parameter \( \theta \) is limited by the requirements \( h \sqrt{\theta}, k \sqrt{\theta} < 1 \). Thus:

**Proposition 3.4:** For \( h = k \) the soliton solutions of lattice KdV \( \ddsl2 \) all have speed 1. For \( h < k \) there are solitons with all speeds greater than 1. For \( h > k \) there are solitons with all speeds between 0 and 1.

**Proof.** The result for \( h = k \) is obvious. Switching \( h \) and \( k \) switches \( c \) and \( 1/c \) so it is just necessary to check the result for, say, \( h < k \). As \( \theta \) tends to 0 \( c \) tends to 1, and as \( \theta \) tends to \( 1/k^2 \) (which is less than \( 1/h^2 \) \( c \) tends to \( \infty \). So the result will be proved if we can establish that \( c \) is a monotonic increasing function of \( \theta \) for \( 0 < \theta < 1/k^2 \). Writing \( z = k \sqrt{\theta} \) and \( \alpha = h/k < 1 \),

\[
c = \frac{\alpha \tanh^{-1} z}{\tanh^{-1} \alpha z}, \quad (71)
\]
and it is necessary to check this is a monotonic function of \( z \) on \( 0 < z < 1 \) for \( \alpha \) fixed between 0 and 1. Differentiating gives

\[
\frac{dc}{dz} = \frac{\alpha}{[\tanh^{-1}(\alpha z)]^2 (1-z^2)(1-\alpha^2 z^2)} \left[ (1-\alpha^2 z^2) \tanh^{-1}(\alpha z) - \alpha (1-z^2) \tanh^{-1}(z) \right].
\] (72)

All the terms except the last are evidently positive. The last term can be written \( g(\alpha z) - \alpha g(z) \) where \( g(z) = (1-z^2) \tanh^{-1}(z) \). Thus it is necessary to show \( g(\alpha z) > \alpha g(z) \). But this follows immediately from the convexity of \( g \), which is trivial as

\[
\frac{d^2 g}{dz^2} = -2 \tanh^{-1} z - \frac{2z}{1-z^2} < 0 \quad \text{for } 0 < z < 1.
\] (73)

Proposition 3.4 does a lot to clarify the relationship of lattice KdV \( \hat{\text{dp2}} \) with its standard continuum limit \( b_t = b_x \) on the one hand, and potential KdV \( \hat{\text{pkdv}} \) on the other. The linear equation \( b_t = b_x \) admits solitons of speed 1, but, since it is linear, the solitons can be of arbitrary amplitude. The indirect method of discretization used has given rise to a nonlinear discretization, except when \( k = h \) (when \( \text{dp2} \) can be written as a product of linear factors). The family of speed 1 solitons with arbitrary amplitude is perturbed, after discretization, into a family of solitons with a nontrivial speed-amplitude relation. For small \( h, k \) the low amplitude solitons (those with \( \sqrt{\theta} \ll 1/h, 1/k \)) must have speed close to 1, and indeed this is the case. For larger amplitudes the speeds can change substantially, giving a range of speeds ranging from 1 to either 0 or \( \infty \). Since there now are solitons of different speeds, and the necessary algebraic structure has been preserved, the phenomena associated with KdV will emerge, in particular elastic soliton scattering. Thus from a phenomenological viewpoint, lattice KdV is closer to potential KdV than the linear equation \( b_t = b_x \). There are, however, several fundamental differences: First, the range of soliton speeds in lattice KdV is limited to speeds either less than or greater than 1. Second, there are many solutions of lattice KdV that do not have natural continuum limits; for example, solutions \( \hat{\text{gensol}} \) in the case where \( \theta \) exceeds \( 1/h, 1/k \) (or both).

### 4 Discretizations II: The simplest natural discretization

This section is devoted to the simple discretization \( \hat{\text{disc2}} \) of \( \hat{\text{lin1}} \), which, as explained in the introduction, should give an integrable lattice equation which has potential KdV as its standard continuum limit. The general solution of \( \hat{\text{disc2}} \) is given by \( \hat{\text{huge}} \) on replacing \( k \) with \( kl \).

Once again suppose \( U_{n,m}(l) \) has a Birkhoff factorization \( S_{n,m}^{-1}(l)Y_{n,m}(l) \), and substitute in \( \hat{\text{disc2}} \) to get

\[
Y_{n+1,m}Y_{n,m}^{-1} = S_{n+1,m} \left[ I + h \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \right] S_{n,m}^{-1},
\] (74)
Writing
\[ S_{n,m} = I + \frac{1}{l} \left( \begin{array}{cc} a_{n,m} & -b_{n,m} \\ c_{n,m} & d_{n,m} \end{array} \right) + \frac{1}{l^2} \left( \begin{array}{cc} \bar{a}_{n,m} & \bar{b}_{n,m} \\ \bar{c}_{n,m} & \bar{d}_{n,m} \end{array} \right) + \ldots, \]
and employing the relations \( \det \left( Y_{n+1,m}Y_{n,m}^{-1} \right) = 1 - h^2l, \det \left( Y_{n,m+1}Y_{n,m}^{-1} \right) = 1 - k^2l^3 \) (obtained by left-multiplying \( \tilde{t}m_1 \) and \( \tilde{t}m_2 \) respectively by \( S_{n,m}S_{n+1,m}^{-1} \) and \( S_{n,m}S_{n+1,m+1}^{-1} \), taking the determinant and factorizing) gives

\[ Y_{n+1,m} = L_{n,m}Y_{n,m}, \quad Y_{n,m+1} = M_{n,m}Y_{n,m}, \]

where
\[ L_{n,m} = \begin{pmatrix} 1 - hb_{n+1,m} + \lambda \beta_{n,m} & h \\ hl + b_{n,m} - b_{n+1,m} - hb_{n,m}b_{n+1,m} & 1 + hb_{n,m} \end{pmatrix}, \]
\[ M_{n,m} = k \begin{pmatrix} \lambda^2 - \lambda(\beta_{n,m} + b_{n,m}b_{n+1,m}) - b_{n,m}b_{n+1,m}\beta_{n,m} + \lambda b_{n,m} + \Sigma_{n,m} & \lambda b_{n,m} + \Sigma_{n,m} - \beta_{n,m}b_{n,m} \\ \beta_{n,m}^2 + \Delta_{n,m}(b_{n,m} + b_{n+1,m}) + \Sigma_{n,m}(b_{n,m} - b_{n+1,m}) & \Delta_{n,m} + \beta_{n,m}b_{n,m} \end{pmatrix}. \]

The matrix \( M \) depends on three lattice fields \( \beta, \Delta, \Sigma \) in addition to the basic lattice field \( b \), but \( \Sigma \) is determined via the relation
\[ \Sigma_{n,m} = \sqrt{\frac{1}{k^2} + \Delta^2_{n,m} + \beta^3_{n,m}}. \]

Substituting these ansätze into the consistency equation \( \tilde{c} \) cons gives the following 3 equations for the 3 fundamental fields \( b, \beta, \Delta \):

\[ \beta_{n+1,m} + \beta_{n,m} = \frac{b_{n+1,m} + b_{n+1,m+1} - b_{n,m} - b_{n+1,m}}{h} + (b_{n+1,m+1} - b_{n,m})(b_{n,m+1} - b_{n+1,m}), \]
\[ \Delta_{n+1,m} + \Delta_{n,m} = \left( \frac{\beta_{n+1,m} - \beta_{n,m}}{h} \right) \left( -1 + \frac{h}{2}(b_{n+1,m+1} + b_{n+1,m} - b_{n,m+1} - b_{n,m}) \right), \]
\[ = \sqrt{1 + k^2(\Delta^2_{n+1,m} + \beta^3_{n+1,m}) - \frac{1}{h^2} - \frac{1}{k^2}(\Delta^2_{n,m} + \beta^3_{n,m})}. \]

Note the equations involve \( b \) at 4 points \((b_{n,m}, b_{n+1,m}, b_{n,m+1}, b_{n+1,m+1})\) but \( \beta \) and \( \Delta \) at only 2 \((\beta_{n,m}, \beta_{n+1,m}, \Delta_{n,m}, \Delta_{n+1,m})\). The system \( \tilde{f}kdv1-\tilde{f}kdv3 \) will be given the title \( \text{full lattice KdV} \); as will shortly be shown, unlike standard lattice KdV, full lattice KdV displays, for certain choices of \( h \) and \( k \), solitons with the full range of speeds. Full lattice KdV also has, as expected, potential
KdV as a standard continuum limit: Replacing $b_{n,m}$ by $b(x,t)$, $b_{n+1,m}$ by $b(x+h,t)$, $b_{n,m+1}$ by $b(x,t+k)$, $b_{n+1,m+1}$ by $b(x+h,t+k)$, and similarly for $\beta$ and $\Delta$, and then taking the limit $h,k \to 0$, the equations $\text{KdV}_1-\text{KdV}_3$ become

$$2\beta = 2b_x, \quad 2\Delta = -\beta_x, \quad 2\beta_x b_t = (\Delta^2 + \beta^3)_x. \quad (84)$$

Eliminating $\beta$ and $\Delta$ from these yields potential KdV $b_t = \frac{1}{4}b_{xxx} + \frac{3}{2}b_x^2$.

There are analogs for full lattice KdV of all the results of the previous sections:

**Proposition 4.1:** Let $U_{0,0}(l)$ be an element of $LGL_2(\mathbb{C})$, let $S_{n,m}(l)Y_{n,m}(l)$ be the Birkhoff decomposition of

$$U_{n,m}(l) = \left[ I + h \left( \begin{array}{cc} 0 & 1 \\ l & 0 \end{array} \right) \right] \left[ I + kl \left( \begin{array}{cc} 0 & 1 \\ l & 0 \end{array} \right) \right]^m U_{0,0}(l), \quad (85)$$

and let $b_{n,m}$ be $(-1)$ times the component of $1/l$ in the 1,2-entry of $S_{n,m}$. Then $b_{n,m}$ is a solution, possibly with singularities, of the full lattice KdV system $\text{KdV}_1-\text{KdV}_3$.

By “$b_{n,m}$ is a solution of full lattice KdV,” I mean that there exist fields $\beta, \Delta$ for which equations $\text{KdV}_1-\text{KdV}_3$ hold. In practice, once $b$ is known, the easiest way to determine $\beta, \Delta$ will be directly from equations $\text{KdV}_1$ and $\text{KdV}_2$. In the previous proposition the other fields can actually be determined from $S$ if this is known in full. If the expansion of $S$ in powers of $1/l$ is as in $\hat{S}$, then

$$\beta_{n,m} = a_{n,m+1} - d_{n,m} - b_{n,m}b_{n,m+1} \quad (86)$$

$$\Delta_{n,m} = \frac{1}{2} \left( b_{n,m+1} + \tilde{b}_{n,m} - c_{n,m+1} - c_{n,m} - b_{n,m}d_{n,m+1} - b_{n,m+1}d_{n,m} \right. \left. + (b_{n,m+1} + b_{n,m})(a_{n,m} + a_{n,m+1} - b_{n,m}b_{n,m+1}) \right) \quad (87)$$

$$\Sigma_{n,m} = \frac{1}{k} + \frac{1}{2} \left( \tilde{b}_{n,m+1} - \tilde{b}_{n,m} + c_{n,m+1} - c_{n,m} + b_{n,m}d_{n,m+1} - b_{n,m+1}d_{n,m} \right. \left. + (b_{n,m+1} - b_{n,m})(a_{n,m} + a_{n,m+1} - b_{n,m}b_{n,m+1}) \right) \quad (88)$$

The Bäcklund transformation takes the following form:

**Proposition 4.2:** If $b_{n,m}, \beta_{n,m}, \Delta_{n,m}$ is a solution of the full lattice KdV equation $\text{KdV}_1-\text{KdV}_3$ and $\psi_{n,m}$ satisfies

$$\frac{\psi_{n+2,m} - 2\psi_{n+1,m} + \psi_{n,m}}{h^2} = \theta \psi_{n,m} - \left( \frac{b_{n+2,m} - b_{n,m}}{h} \right) \psi_{n+1,m} \quad (89)$$

$$\frac{\psi_{n,m+1} - \psi_{n,m}}{k} = (\theta + \beta_{n,m}) \frac{\psi_{n+1,m} - \psi_{n,m}}{h} \quad (90)$$

then $b^\text{new}_{n,m} = b_{n+1,m} + (\psi_{n+1,m} - \psi_{n,m})/(hb\psi_{n,m})$ is also a solution of full lattice KdV. The fields $\beta, \Delta$ are replaced by $\beta^\text{new}, \Delta^\text{new}$ respectively, which are given by the following algebraic equations:

$$\beta^\text{new}_{n,m} + \beta_{n,m} = \theta + (b_{n,m} - b^\text{new}_{n,m+1})(b^\text{new}_{n,m} - b_{n,m+1}) \quad (91)$$

$$\Delta^\text{new}_{n,m} + \Delta_{n,m} = \frac{1}{2} \left( \beta^\text{new}_{n,m} - \beta_{n,m} \right) \left( b^\text{new}_{n,m+1} + b^\text{new}_{n,m} - b_{n,m+1} - b_{n,m} \right) \quad (92)$$
The Bäcklund transformation for continuum KdV (Proposition 2.2) is recovered in the limit \( h,k \to 0 \), since \( \beta_{n,m} \to b_x, \Delta_{n,m} \to -\frac{1}{2}b_{xx}, \Sigma_{n,m} - \frac{1}{k} \to 0 \). Note the difference between the second equation in Īsl1-Īsl2 and the discrete evolution proposed in BPPS. The solutions obtained using the Bäcklund transformation on the vacuum solution \( b_{n,m} = \beta_{n,m} = \Delta_{n,m} = 0 \) are given by gensol with \( k \) replaced by \( k \theta \). In particular, writing \( t(n,m) \) in place of \( \tanh^2(n \tanh^{-1}(h \sqrt{\theta}) + m \tanh^{-1}(k \theta \sqrt{\theta}) + C) \), it is straightforward to verify that the soliton solution is given by

\[
\begin{align*}
 b_{n,m} &= \sqrt{\theta} t(n,m) , \\
 \beta_{n,m} &= \frac{t(n,m + 1) - t(n,m)}{k \sqrt{\theta}} , \\
 \Delta_{n,m} &= \frac{t(n,m + 1)^2 - t(n,m)^2}{2k} , \\
 \Sigma_{n,m} &= \frac{1}{k} + \frac{(t(n,m + 1) - t(n,m))^2}{2k} .
\end{align*}
\]

(93) (94) (95) (96)

Before exploring the phenomenology of these solitons, note that since the proof of Proposition 3.3 is based almost entirely on the first equation of the scalar Lax pair Īsl1-Īsl2, it is no surprise that it goes through verbatim to full lattice KdV, i.e.

**Proposition 4.3:** If \( b_{n,m} \) is a solution of the full lattice KdV system Īkdv1-Īkdv3, and \( b^{(1)}_{n,m} \) and \( b^{(2)}_{n,m} \) are solutions obtained by applying Bäcklund transformations with parameters \( \theta_1 \) and \( \theta_2 \) respectively to \( b_{n,m} \), then

\[
B_{n,m} = b_{n,m} + \frac{\theta_1 - \theta_2}{b^{(1)}_{n,m} - b^{(2)}_{n,m}}
\]

(97)

is a solution obtained by applying the two Bäcklund transformations successively to \( b_{n,m} \), in either order.

Since the formulae ālg1-ālg2 for applying the Bäcklund transformation to the fields \( \beta, \Delta \) are already pure algebraic there is no need to consider them in proposition 4.3.

It just remains to investigate the speed-amplitude relation of the soliton solutions. The soliton speed is

\[
c = \frac{h \tanh^{-1}(k \theta \sqrt{\theta})}{k \tanh^{-1}(h \sqrt{\theta})} ,
\]

(98)

where the range of the parameter \( \theta \) is limited by the requirements \( k \theta \sqrt{\theta}, h \sqrt{\theta} < 1 \). Writing \( \alpha = hk^{-1/3} \) and \( z = \sqrt{\theta} k^{1/3} \) gives

\[
c = k^{-2/3} \frac{\alpha \tanh^{-1}(z^3)}{\tanh^{-1}(\alpha z)} .
\]

(99)

See figure 1. For \( \alpha < 1 \Leftrightarrow h^3 < k \) the speed is a monotonic increasing function of \( z \) (or \( \theta \)), going from 0 as \( z \to 0 \) to \( \infty \) as \( z \to 1 \). Thus for this range of parameters the soliton content exactly
mirrors that of continuum KdV. For $\alpha = 1 \Leftrightarrow h^3 = k$ the speed is a monotonic increasing function of $z$ (or $\theta$), going from 0 as $z \to 0$ to 1 as $z \to 1$. For $\alpha > 1 \Leftrightarrow h^3 > k$ there is an interesting effect that $c$ increases from 0, reaches a maximum value, and then decreases again to 0 as $z$ approaches $1/\alpha$. Thus for these choices of $h, k$ there is a limited set of speeds, but for all but the fastest there are solitons of 2 different amplitudes; furthermore these can be superposed to give other types of soliton solution. Note that if our interest in discretizations of KdV were for the purposes of numerical simulation, we would presumably want both $h$ and $k$ small and of the same order of magnitude, and thus be in the $h^3 < k$ regime, where the soliton phenomenology is correct.

5 Discretizations III: A second-order discretization

In this section our method is applied to the discretization $\hat{\text{sol}}$-$\hat{\text{so}}2$ of $\hat{\text{lin}}2$ with $N = 1$. The resulting system is of limited intrinsic interest, the main point here is to illustrate that our methods can in principle be extended to give a whole range of integrable discretizations of equations in the KdV hierarchy. One interesting point that emerges is the form of the related discretization of the Schrödinger equation.

Following the usual procedure, assuming $U_{n,m}(l)$ has a Birkhoff decomposition $S_{n,m}^{-1}(l)Y_{n,m}(l)$, writing

$$S_{n,m} = I + \frac{1}{l} \left( \begin{array}{cc} a_{n,m} & -b_{n,m} \\ c_{n,m} & d_{n,m} \end{array} \right) + \ldots$$

(100)
etc., gives the system

\[ Y_{n+1,m} = L_{n,m} Y_{n,m} , \quad Y_{n,m+1} = M_{n,m} Y_{n,m} , \] (101)

where

\[ L_{n,m} = \left( 1 + \frac{1}{2} h^2 l - h b_{n+1,m} + \frac{1}{2} h^2 (a_{n+1,m} - a_{n,m}) \right) h l + b_{n,m} - b_{n+1,m} - h \Delta + \frac{1}{2} h^2 (b_{n,m} - b_{n+1,m}) \), (102)

\[ M_{n,m} = \left( \frac{1 - k b_{n,m+1}}{k} \right) (k l + b_{n,m} - b_{n+1,m} - k b_{n,m} b_{n+1,m} + 1 + k b_{n,m}) , \] (103)

and

\[ \Delta = \frac{1}{4} b_{n,m}^2 + b_{n+1,m}^2 - \frac{1}{2} h (a_{n+1,m} - a_{n,m}) (b_{n+1,m} + b_{n,m}) + \frac{1}{4} h^2 (a_{n,m} - a_{n+1,m})^2 \] (104)

The consistency condition \( L_{n,m+1} M_{n,m} = M_{n+1,m} L_{n,m} \) unravels to two equations for the fields \( a, b \).

Introducing the combinations

\[ \Sigma_{n,m} = \frac{1}{2} (a_{n+1,m+1} - a_{n,m+1} + a_{n+1,m} - a_{n,m}) , \] (105)

\[ \Delta_{n,m} = \frac{1}{2} (a_{n+1,m+1} - a_{n,m+1} - a_{n+1,m} + a_{n,m}) , \] (106)

the equations can be written

\[ \Sigma_{n,m} = \frac{1}{2} \left( \frac{b_{n+1,m+1} - b_{n,m+1} - b_{n+1,m} + b_{n,m}}{k} + b_{n+1,m+1} b_{n+1,m} - b_{n,m+1} b_{n,m} \right) , \] (107)

\[ 0 = \left( \frac{h \Delta_{n,m}}{2} + \frac{1}{k} \right) (b_{n+1,m+1} + b_{n,m+1} - b_{n+1,m} - b_{n,m}) \] (108)

\[ + \left( \frac{h^2 \Delta_{n,m}}{4k} - \frac{1}{h} \right) (b_{n+1,m+1} - b_{n,m+1} + b_{n+1,m} - b_{n,m}) \]

\[ + \frac{h}{k} \left( b_{n,m}^2 + b_{n+1,m}^2 - b_{n+1,m+1}^2 - b_{n,m+1}^2 + 4 b_{n+1,m+1} b_{n,m} - 4 b_{n+1,m} b_{n,m+1} \right) \]

\[ + \frac{h}{8k} \left( -b_{n+1,m+1} b_{n+1,m} + b_{n+1,m} b_{n,m+1} - b_{n+1,m+1} b_{n,m} + b_{n+1,m} b_{n,m+1} \right) \]

\[ + \frac{1}{2} \left( 3 b_{n+1,m} b_{n+1,m+1} + 3 b_{n,m} b_{n+1,m+1} - b_{n+1,m+1} b_{n,m} \right) \]

\[ + \frac{h}{4} \left( 3 b_{n+1,m+1} b_{n+1,m} + 3 b_{n,m+1} b_{n+1,m+1} - 3 b_{n+1,m} b_{n,m+1} \right) \]

\[ + \frac{h}{4} \left( -b_{n+1,m+1} b_{n+1,m} + b_{n+1,m} b_{n,m+1} - b_{n+1,m} b_{n,m+1} + b_{n+1,m+1} b_{n,m} \right) \]

\[ + \frac{h^2}{8} \left( b_{n,m+1} b_{n+1,m} + b_{n+1,m} b_{n,m+1} + 2 b_{n+1,m} b_{n+1,m+1} b_{n,m} b_{n,m+1} \right) \]

\[ - b_{n+1,m+1} b_{n+1,m+1} b_{n,m} + b_{n+1,m+1} b_{n,m+1} b_{n,m} - b_{n+1,m+1} b_{n+1,m} b_{n,m} \right) \].
Since the field $a$ only appears in the equations through the combinations $\Delta$ and $\Sigma$, which only depend on $a$ through differences, solutions of this system are only defined up to addition of a constant to $a$. The analog of propositions 2.1, 3.1 and 4.1 is

**Proposition 5.1:** Let $U_{0,0}(l)$ be an element of $LGL_2(C)$, let $S_{n,m}^{-1}(l)Y_{n,m}(l)$ be the Birkhoff decomposition of

$$U_{n,m}(l) = \left(1 + \frac{h^2}{2l} \right)^n \left( \begin{array}{cc} 1 & k \frac{m}{kl} \\ k & 1 \end{array} \right) U_{0,0}(l),$$  \hspace{1cm} (109)

and let $a_{n,m}$ and $b_{n,m}$ be, respectively, the 1,1-entry and $(-1)$ times the 1,2-entry in the 1/l component of $S_{n,m}$. Then $a_{n,m}, b_{n,m}$ is a solution, possibly with singularities, of the system $\ddot{m}dkdv1-\ddot{m}dkdv2$.

The system $\ddot{m}dkdv1-\ddot{m}dkdv2$, despite its algebraic complexity, is an integrable discretization of the equation $b_t = b_x$ in every sense that lattice KdV is. The soliton solutions are given as follows:

**Proposition 5.2:** The system $\ddot{m}dkdv1-\ddot{m}dkdv2$ has soliton solutions

$$b_{n,m} = \sqrt{\theta} \tanh \left(n \tanh^{-1} \left( \frac{h\sqrt{\theta}}{1 + \frac{h^2}{2}\theta} \right) + m \tanh^{-1}(k\sqrt{\theta}) + C \right),$$  \hspace{1cm} (110)

$$a_{n,m} = \text{constant},$$  \hspace{1cm} (111)

with speed

$$c = \frac{h \tanh^{-1}(k\sqrt{\theta})}{k \tanh^{-1} \left( \frac{h\sqrt{\theta}}{1 + \frac{h^2}{2}\theta} \right)}.$$  \hspace{1cm} (112)

In greater generality, it can be shown that if instead of equation 80 1 a pth order approximation

$$U_{n+1,m} = \sum_{i=0}^{p} \frac{h^i}{i!} \left( \begin{array}{cc} 0 & 1 \\ l & 0 \end{array} \right) U_{n,m}$$  \hspace{1cm} (113)

is used, then the speed of the soliton solution is

$$c = \frac{h \tanh^{-1}(k\sqrt{\theta})}{k \tanh^{-1} \left( \frac{s_p(h\sqrt{\theta})}{c_p(h\sqrt{\theta})} \right)},$$  \hspace{1cm} (114)

where $c_p(x)$ and $s_p(x)$ are, respectively, the order $p$ truncations of the Taylor series for $\cosh(x)$ and $\sinh(x)$ (ignoring terms of order $x^{p+1}$ and higher). It is straightforward to verify that for small $x$

$$\frac{1}{x} \tanh^{-1} \left( \frac{s_p(x)}{c_p(x)} \right) = \left\{ \begin{array}{ll} 1 + O(x^p) & x \text{ even} \\ 1 + O(x^{p+1}) & x \text{ odd} \end{array} \right.$$

Thus for small $h$ the dependence of the soliton speed on $h$ becomes weaker as $p$ increases. Likewise the order of accuracy in $k$ can be increased. (The distinction between the even and odd cases in
tanap, that for odd \( p \) there is a “free” extra order of magnitude accuracy, means that \( \text{mdkdv1-mdkdv2} \), for which \( p = 2 \), is actually no more accurate in this regard than standard lattice KdV, with \( p = 1 \). The equations obtained from \( p = 3 \) can be written down, but due to their length I have restricted the discussion to the \( p = 2 \) case.

Returning to the formula for new speed, note that if \( v = k\sqrt{\theta} \), \( w = h\sqrt{\theta} \), then

\[
c = \frac{\frac{1}{v} \tanh^{-1} v}{\frac{1}{w} \tanh^{-1} \left( \frac{w}{1 + \frac{1}{w^2}} \right)}.
\]

The function in the numerator increases monotonically from 1 to \( \infty \) as \( v \) goes from 0 to 1. The function in the denominator decreases monotonically from 1 to 0 as \( v \) goes from 0 to \( \infty \). Thus for the current discretization \( c \) can only take values greater than 1.

The soliton solutions just presented can be found using the Bäcklund transformation, which is obtained as in previous sections:

**Proposition 5.3:** If \( b_{n,m}, a_{n,m} \) is a solution of \( \text{mdkdv1-mdkdv2} \) and \( \psi_{n,m} \) satisfies

\[
\begin{align*}
\frac{\psi_{n+2,m} - 2\psi_{n+1,m} + \psi_{n,m}}{h^2} &= \theta \psi_{n+1,m} \left( 1 + \frac{h}{4}(b_{n,m} - b_{n+2,m}) \right) \\
&+ \frac{(b_{n+1,m} - b_{n,m})\psi_{n+2,m} - 3(b_{n+2,m} - b_{n,m})\psi_{n+1,m} + (b_{n+2,m} - b_{n+1,m})\psi_{n,m}}{2h} \\
&- \frac{h^2\theta^2}{4} \left( 1 + \frac{h(b_{n+1,m} - b_{n+2,m})}{2} \right) \psi_{n,m} + \frac{a_{n+2,m} - 2a_{n+1,m} + a_{n,m}}{2} \psi_{n+1,m} \\
&+ \frac{(b_{n+2,m} + b_{n,m})b_{n+1,m} - 2b_{n,m}b_{n+2,m}}{2} \psi_{n+1,m} \\
&+ \frac{h}{4} \left( a_{n,m}b_{n+1,m} + a_{n+1,m}b_{n+2,m} + a_{n+2,m}b_{n,m} \\
&- a_{n,m}b_{n+2,m} + a_{n+1,m}b_{n,m} - a_{n+2,m}b_{n+1,m} \right) \psi_{n+1,m}
\end{align*}
\]

\[
\psi_{n,m+1} - \psi_{n,m} = \frac{\left( \frac{\psi_{n+1,m} - \psi_{n,m}}{h} + \frac{b_{n+1,m} - b_{n,m+1}}{\psi_{n,m}} \cdot \frac{\psi_{n+1,m} - \psi_{n,m}}{h} + \frac{b_{n,m} - b_{n+1,m}}{\psi_{n,m}} \cdot \frac{\psi_{n+1,m} - \psi_{n,m}}{h} \right)}{1 + \frac{1}{2}h(b_{n,m} - b_{n+1,m})}
\]

then

\[
b_{n,m}^{\text{new}} = \frac{b_{n+1,m} + \frac{\psi_{n+1,m} - \psi_{n,m}}{h\psi_{n,m}} + \frac{1}{2}h^2(a_{n,m} - a_{n+1,m} - \theta)}{1 + \frac{1}{2}h(b_{n,m} - b_{n+1,m})}
\]

\[
a_{n,m}^{\text{new}} = b_{n,m}b_{n,m}^{\text{new}} - a_{n,m} + \text{constant}
\]

is also a solution of \( \text{mdkdv1-mdkdv2} \).

All formulae in the previous proposition have been written in a manner that hopefully makes it clear in what sense they are modifications of the corresponding formulae in proposition 3.2. The
surprising feature of the discretization of the Schrödinger equation in proposition 3.2, equation dsl1, is that in it the parameter $\theta$ multiplies $\psi_{n,m}$, not $\psi_{n+1,m}$, which would seem more natural. The new discretization just presented, equation mdkdv1, has $\theta$ multiplying $\psi_{n+1,m}$. But the cost of this is the introduction of many new terms, including a term proportional to $\theta^2$, multiplying $\psi_{n,m}$. It can be checked that the new discretization mdkdv1 is a second order approximation to the Schrödinger equation, while dsl1 is only first order. This is the justification for the title of this section.

6 Concluding Remarks

In this paper I have presented a systematic approach towards integrable discretizations, based on the loop group approach to integrable systems. Three integrable discretizations have been examined in detail, one known, the lattice KdV system of Nijhoff et al., and two new, one of which I have called full lattice KdV, as it would seem to be the first discrete integrable system with (potential) KdV as its standard continuum limit. For each integrable discretization a Bäcklund transformation has been given and soliton solutions have been analyzed. Unlike the lattice KdV system of Nijhoff et al., which only displays solitons with speeds below, above or equal to 1, full lattice KdV has the full range of soliton speeds (for suitable choices of $h$ and $k$).

Full lattice KdV would seem to merit further attention. Our plans include conducting numerical studies, and to try to work out a suitable inverse scattering formalism. Another issue that has not been touched upon in this paper is the subject of tau functions for discretizations. The linear flows on a loop group that underlie KdV can be extended to the central extension of the group, and one would expect the same to be true for the discretizations looked at in this paper.

The formalism developed here can also be extended to look at integrable discretizations of KdV on non-rectangular lattices, see ħex.

Acknowledgments

I would like to thank David Kessler for discussions. This work was supported by the Israel National Science Foundation.
References

[1] R.Hirota, *Nonlinear partial difference equations. I. A difference analogue of the Korteweg-de Vries equation*, J.Phys.Soc.Jap. 43 (1977) 1424-1433.

[2] X.-B.Hu and P.A.Clarkson, *Rational solutions of a differential difference KdV equation, the Toda equation and the discrete KdV equation*, J. Phys. A 28 (1995) 5009–5016.

[3] F.Nijhoff and H.Capel, *The Discrete Korteweg-de Vries Equation*, Acta Appl.Math. 39 (1995) 133-158.

[4] A.I.Bobenko and U.Pinkall, *Discretization of surfaces and integrable systems*, in Discrete integrable geometry and physics Oxford Univ. Press, New York (1999).

[5] M.J.Ablowitz and J.F.Ladik, *Nonlinear differential-difference equations*, J.Math.Phys. 16 (1975) 598-603. *Nonlinear differential-difference equations and Fourier analysis*, J.Math. Phys. 17 (1976) 1011-1018.

[6] M.Boiti, F.Pempinelli, B.Prinari and A.Spire, *An integrable discretization of KdV at large times*, Inverse Prob. 17 (2001) 515-526.

[7] U.Hetrich-Jeromin, I.McIntosh, P.Norman and F.Pedit, *Periodic Discrete Conformal Maps*, J.Rein.Ange.Math. 534 (2001) 129-153.

[8] A.Mukaihira and Y.Nakamura, *Integrable Discretizations of the Modified KdV Equation and Applications*, Inv.Prob. 26 (2000) 413-424; A.Nagai and J.Satsuma, *Discrete soliton equations and convergence acceleration algorithms*, Phys.Lett.A 209 (1995) 305-312; V.Papageorgiou, B.Grammaticos and A.Ramani, *Integrable lattices and convergence acceleration algorithms*, Phys.Lett.A 179 (1993) 111-115.

[9] J.Schiff, *Symmetries of KdV and Loop Groups*, arXiv:math-ph/9606004.

[10] G.Segal and G.Wilson, *Loop Groups and Equations of KdV Type*, Pub.Math.I.H.E.S 61 (1983) 5-65; G.Wilson, *Habillage et fonctions τ*, C.R. Acad.Sc.Paris 299 (1984) 587-590; G.Wilson, *Infinite-dimensional Lie groups and algebraic geometry in soliton theory*, Phil. Trans.Roy.Soc.Lond. 315 (1985) 393-404.

[11] A.Pressley and G.Segal, *Loop Groups*, Oxford Univ. Press, New York (1990).

[12] P.G.Drazin and R.S.Johnson, *Solitons: An Introduction*, Cambridge Univ. Press, Cambridge (1989).
[13] J.Schiff and S.Shnider, *A Natural Approach to the Numerical Integration of Riccati Differential Equations*, SIAM J.Numer.Anal. **36** (1999) 1392-1413.

[14] J.Schiff, *HexaKdV*, in preparation.