Energy relaxation of a superconducting charge qubit via Andreev processes

R. M. Lutchyn and L. I. Glazman
W.I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA
(Dated: February 1, 2008)

We study fundamental limitations on the energy relaxation rate of a superconducting charge qubit with a large-gap Cooper-pair box, \( \Delta_b > \Delta_r \). At a sufficiently large mismatch between the gap energies in the box \( \Delta_b \) and in the reservoir \( \Delta_r \), “quasiparticle poisoning” becomes ineffective even in the presence of nonequilibrium quasiparticles in the reservoir. The qubit relaxation still may occur due to higher-order (Andreev) processes. In this paper we evaluate the qubit energy relaxation rate \( T_1^{-1} \) due to Andreev processes.

I. INTRODUCTION

A large number of recent experimental studies\(^{1,2,3,4,5,6,7,8,9}\) indicates the presence of quasiparticles in superconducting single-charge devices at low temperatures. The operation of devices, of which the best known is Cooper-pair box qubit, requires 2\( e \)-periodic dependence of the charge of the box on its gate voltage, and thus, an introduction of an unpaired electron/quasiparticle in the Cooper-pair box (CPB) is a significant problem. The superconducting charge qubit operates at the degeneracy point for Cooper-pairs, \( N_g = 1 \), with \( N_g \) being the dimensionless gate voltage. For equal gap energies in the Cooper-pair box and reservoir, \( \Delta_b = \Delta_r \), the states of the qubit at \( N_g = 1 \) are unstable with respect to quasiparticle tunneling to the box. The quasiparticle changes the charge state of CPB from even to odd, and lowers the charging energy. This phenomenon, commonly referred to as “quasiparticle poisoning”, is well-known from the studies of the charge parity effect in superconductors, see, for example, Matveev et al.\(^{10}\) and references therein. “Quasiparticle poisoning” can degrade the performance of the charge qubit in two ways. First, it causes the operating point of the qubit to shift stochastically on the time scale comparable with the measurement time.\(^{11}\) Second, it contributes to the decoherence.\(^{11}\) One of the approaches to improve the performance of charge qubits is to use superconducting gap engineering. In most single-charge superconducting devices “quasiparticle poisoning” can be suppressed even in the presence of nonequilibrium quasiparticles in the reservoir by engineering a large mismatch between \( \Delta_b \) and \( \Delta_r \). Gap energies in superconductors can be modified by oxygen doping,\(^{12}\) applying a magnetic field,\(^{13,14}\) and adjusting layer thickness.\(^{15}\) In this paper we study the fundamental limitations on the energy relaxation rate in a charge qubit with a large gap in the box, \( \Delta_b > \Delta_r \).

For equal gap energies in the box and reservoir, \( \Delta_b = \Delta_r \), the energy relaxation rate due to “quasiparticle poisoning”\(^{11}\) is

\[
\frac{1}{T_1} \propto \frac{g_r n_{qp}}{\hbar \nu_F} \sqrt{\frac{T}{E_j}} \tag{1}
\]

with \( n_{qp} \), \( g_r \) and \( \nu_F \) being the density of quasiparticles in the reservoir, dimensionless conductance of the junction and density of states at the Fermi level, respectively. The relaxation rate \( 1/T_1 \) in Eq. (1) was derived under the assumption that an unpaired electron tunnels from the reservoir to the box to minimize the energy of the system. Indeed, for \( \Delta_b = \Delta_r \), the odd-charge state of the CPB has lower energy at \( N_g = 1 \) due to the Coulomb blockade effect. By properly engineering superconducting gap energies (i.e. inducing large gap mismatch, \( \Delta_b > \Delta_r \)), one can substantially reduce quasiparticle tunneling rate to the Cooper-pair box. Suppose initially the qubit is in the excited state with energy \( E_{i+} \), and the quasiparticle is in the reservoir with energy \( E_p \). Upon quasiparticle tunneling to the box, the minimum energy of the final state is \( E_{f_{\min}} = \Delta_b + E_{N+1} \) with \( E_{N+1} \) being the energy of the CPB in the odd-charge state. Therefore, the threshold energy for a quasiparticle to tunnel to the box is \( E_{p_{\min}} = \Delta_b + E_{N+1} - E_{i+} \), see also Fig. 1. If \( E_{p_{\min}} - \Delta_r > E_j > T \), only exponentially small fraction of quasiparticles are able to tunnel into the island. (Note that the energy difference between excited and ground state of a charge qubit is \( E_j \), while the energy of the qubit in the excited state is \( E_{i+} = E_c + E_j/2 \). Here \( E_c \), \( E_j \) and \( T \) are the charging energy of the CPB, the Josephson energy associated with the tunnel junction, and the temperature, respectively.) Thus, the contribution to the qubit relaxation rate \( T_1^{-1} \) from the processes involving real quasiparticle tunneling to the island becomes

\[
\frac{1}{T_1} \propto \frac{g_r n_{qp}}{\hbar \nu_F} \exp \left( -\frac{\Delta_b - \Delta_r - E_c - E_j/2}{T} \right), \tag{2}
\]

and is much smaller than the one of Eq. (1). (To obtain Eq. (2), we used the fact that \( E_{N+1} = 0 \) at \( N_g = 1 \).) However, there is also a mechanism of energy relaxation originating from the higher order tunneling processes (Andreev reflection). The contribution of these processes to the qubit relaxation is activationless, and can be much larger than the one of Eq. (2). In the rest of the paper we study qubit energy relaxation due to Andreev processes in detail.
where \( x \) and \( T \) are the coordinates in the plane of the tunnel junction and perpendicular to it, respectively. The Hamiltonian \( E \) along with the above definition of \( T(x, x') \) properly takes into account the fact that in the tunnel-Hamiltonian approximation the wavefunctions turn to zero at the surface of the junction\(^\text{13,14}\). In terms of the transmission coefficient \( T \), the dimensionless conductance of the tunnel junction \( g_T \) can be defined as 
\[
g_T = \frac{T S J k_B^2/4\pi}{2 TN_{\text{ch}}} \]
where \( S_J \) is the area of the junction, and \( N_{\text{ch}} \) is the number of transverse channels in the junction.

The energy relaxation rate of the qubit due to higher-order processes is given by
\[
\Gamma_A = \frac{2\pi}{\hbar} \sum_{p,p'} 2 |A_{p'p}|^2 \delta(E_{p'} - E_p - E_j) f_p(E_p)(1 - f_p(E_{p'})).
\]
Here \( f_p(E_p) \) is the Fermi distribution function with \( E_p = \sqrt{E_0^2 + \Delta_p^2} \) being the energy of a quasiparticle in the reservoir. The amplitude \( A_{p'p} \) is given by the second order perturbation theory in \( V \),
\[
A_{p'p} = \langle -, E_{p'} | V \frac{1}{E_i - H_0} V | +, E_{p} \rangle.
\]

At \( E_c \gg E_j \) and \( N_g = 1 \), the eigenstates of the qubit are given by the symmetric and antisymmetric superposition of two charge states, \( \langle - \rangle = \frac{|N\rangle - |N+2\rangle}{\sqrt{2}} \) and \( \langle + \rangle = \frac{|N\rangle + |N+2\rangle}{\sqrt{2}} \) with the corresponding eigenvalues \( E_{\langle \pm\rangle} = E_c \pm E_j/2 \). In the initial moment of time the qubit is prepared in the excited state and the quasiparticle is in the reservoir, \( \langle +, E_{p} \rangle \equiv |+\rangle \otimes |E_{p}\rangle \). The energy of the initial state is \( E_i = E_0 + E_{p} \). The denominator in the amplitude \( \gamma_{n,\sigma} \) corresponds to the formation of the virtual intermediate state when the quasiparticle has tunnelled to the island from the reservoir. Since a quasiparticle is a superposition of a quasi-electron and quasi-hole, the contributions to \( A_{p'p} \) come from two interfering paths:
\[
A_{p'p} = \frac{1}{2} \langle N + 2, E_{p'} | V \frac{1}{E_i - H_0} V | N, E_{p} \rangle - \frac{1}{2} \langle N, E_{p'} | V \frac{1}{E_i - H_0} V | N + 2, E_{p} \rangle.
\]

To calculate the amplitude \( A_{p'p} \), we use particle-conserving Bogoliubov transformation\(^\text{15,16,17}\).
\[
\gamma_{n,\sigma} = \int dx \left[ U_n(x) \Psi_\sigma(x) - \sigma V_n(x) \Psi_{-\sigma}(x) R^\dagger \right]
\]
\[
\gamma_{n,\sigma} = \int dx \left[ U_n(x) \Psi_\sigma(x) - \sigma V_n(x) \Psi_{-\sigma}^\dagger(x) R \right]
\]

The operators \( R^\dagger \) and \( R \) transform a given state in an \( N \)-particle system into the corresponding state in the \( N + 2 \)
and $N-2$ particle system, respectively, leaving the quasiparticle distribution unchanged, \textit{i.e.} $R^1|N \rangle = |N+2\rangle$. Thus, quasiparticle operators $\gamma^\dagger_{i\sigma}$ and $\gamma_{i\sigma}$ defined in Eq. (18) do conserve particle number. The transformation coefficients $U_n(x)$ and $V_n(x)$ are given by the solution of Bogoliubov-de Gennes equation. For spatially homogenous superconducting gap $\Delta$, the functions $U_n(x)$ and $V_n(x)$ can be written as $U_n(x) = u_n \phi_n(x)$ and $V_n(x) = v_n \phi_n(x)$. The coherence factors $u_n$ and $v_n$ are given by

$$ u_n^2 = \frac{1}{2} \left( 1 + \frac{\varepsilon_n}{E_n} \right) \quad \text{and} \quad v_n^2 = \frac{1}{2} \left( 1 - \frac{\varepsilon_n}{E_n} \right). $$

Here $E_n = \sqrt{\varepsilon_n^2 + \Delta^2}$; $\varepsilon_n$ and $\phi_n(x)$ are exact eigenvalues and eigenfunctions of the single-particle Hamiltonian, which may include random potential $V(x)$, \textit{e.g.}, due to impurities. The single-particle energies $\varepsilon_n$ and wavefunctions $\phi_n(x)$ are defined by the following Shrödinger equation:

$$ \frac{-\hbar^2}{2m} \nabla^2 + V(x) \phi_n(x) = \varepsilon_n \phi_n(x). $$

In the presence of time-reversal symmetry $u_n$, $v_n$ and $\phi_n(x)$ can be taken to be real. Then with the help of Eq. (10), we obtain the amplitude of the process $A_{p'p}$:

$$ A_{p'p} = \frac{1}{2} \int d^2x_1d^2x_1'd^2x_2d^2x_2' \langle u_{p'}(x_1') | V_p(x_2') | u_p(x_1) \rangle \sum_{n} \frac{U_n(x_1)V_n(x_2)}{E_n + \delta E_+ - E_k}, $$

(11)

where $\delta E_+ \equiv E_{|+} - E_{N+1} = E_+ + E_{p}/2$. The minus sign in the parenthesis here reflects the destructive interference between quasi-electron and quasi-hole contributions, see also Eq. (9).

### III. Disorder Averaging

It is well-known that Andreev conductance is sensitive to disorder, see, for example, Refs. [19,20]. Similarly, the rate $\Gamma_A$ is affected by electron backscattering to the tunnel junction, see Fig. 2. If a quasiparticle bounces off the walls of the box or impurities many times, it is reasonable to expect the chaotization of its motion. Thus, one is prompted to consider ensemble-averaged quantities rather than their particular realization. Using Eqs. (7) and (11), we obtain

$$ \langle \Gamma_A \rangle = \frac{2}{\hbar} \sum_{p,p'} \prod_{i=1.4} dx_i dx_i' T(x_1,x_1')T(x_2,x_2')T(x_3,x_3')T(x_4,x_4') (u_{p'}v_{p'}\phi_{p'}(x_1')\phi_p(x_2') - u_p v_{p'} \phi_p(x_1') \phi_{p'}(x_2')) $$

$$ \times (u_{p'}v_{p'}\phi_{p'}(x_3')\phi_p(x_4') - u_p v_{p'} \phi_p(x_3') \phi_{p'}(x_4')) \sum_{k} \frac{u_k v_k \phi_k(x_1) \phi_k(x_2)}{E_p + \delta E_+ - E_k} \sum_{k'} \frac{u_{k'} v_{k'} \phi_{k'}(x_3) \phi_{k'}(x_4)}{E_{p'} + \delta E_+ - E_{k'}} \delta(E_{p'} - E_p - E_j) f_F(E_p) (1 - f_F(E_{p'})). $$

(12)

Here the brackets $\langle \ldots \rangle$ denote averaging independently over different realizations of the random potential in the box and reservoir. In order to average over the disorder in the CPB, one has to calculate the following correlation
function:

\[ I = \sum_{k,k'} \frac{u_k u_k' \phi_k(x_1) \phi_k(x_2) u_k' u_k' \phi_{k'}(x_3) \phi_{k'}(x_4)}{E_p - \Delta E_+ - E_k} \]

\[ = \int \frac{d\xi d\xi'}{4E(\xi) E(\xi')} \frac{\langle K_{\xi_1}(x_1,x_2) \rangle \langle K_{\xi_2}(x_3,x_4) \rangle}{E_p + \delta E_+ - E(\xi)} \]

where \( \langle K_{\xi_1}(x_1,x_2) \rangle \) is given by the universal limit,

\[ \langle K_{\xi_1}(x_1,x_2) \rangle = \frac{\nu_F}{\Delta_b} \delta(\xi - \xi_1) \delta(\xi_2) \]

In this case the irreducible part in Eq. (14) is given by the universal limit,

\[ \langle K_{\xi_1}(x_1,x_2) \rangle \langle K_{\xi_2}(x_3,x_4) \rangle \]

\[ = \frac{\nu_F}{\Delta_b} \delta(\xi - \xi_1) \delta(\xi_2) \delta(\xi_1 - \xi_2) \]

The reducible part can be easily calculated by relating \( \langle K_{\xi_1}(x_1,x_2) \rangle \) to the ensemble-averaged Green function:

\[ \langle K_{\xi_1}(x_1,x_2) \rangle \equiv -\frac{d}{d\xi} \text{Im}(G^{D}_{\xi}(x_1,x_2)) = \nu_F f_{12}. \] (12)

The function \( f_{12} \) is given by \( f_{12} = \langle e^{iK(x_1-x_2)} \rangle_{FS} \) with \( \langle ... \rangle_{FS} \) being the average over electron momentum on the Fermi surface. For 3D system the function \( f_{12} \) is equal to \( f_{12} = \frac{\sin(KF|x_1-x_2|)}{K|x_1-x_2|} \).

The reducible part of the correlation function (14) reads

\[ \langle K_{\xi_1}(x_1,x_2) \rangle \langle K_{\xi_2}(x_3,x_4) \rangle \]

\[ = \frac{\nu_F}{\pi} \text{Re} \left[ f_{14} f_{24} \mathcal{P}_{\xi_2-\xi_1}(x_1,x_3) + f_{13} f_{23} \mathcal{P}_{\xi_2-\xi_1}(x_1,x_3) \right]. \]

The spectral expansion of \( \mathcal{P}_{\omega}(x_1,x_2) \) for the diffusive system is

\[ \mathcal{P}_{\omega}(x_1,x_2) = \sum_n f_n^*(x_1) f_n(x_2) \frac{e^{-i\omega + \gamma_n}}{-i\omega + \gamma_n}. \]

The expressions above are valid for \( y < 1 \). The function \( L_2(y) \) has the following asymptotes

\[ L_2(y) \approx \begin{cases} \frac{\pi}{4} + \frac{4}{3}y, & y \ll 1, \\ \frac{\pi}{2\sqrt{y(1-y)^{3/2}}}, & 1 - y \ll 1. \end{cases} \]

After substituting Eq. (18) into Eq. (12) and averaging over disorder in the reservoir, we obtain the following expression for \( \langle \Gamma_A \rangle \):
the integrals over energies $E_q(14)$ and $E_q(15)$. Using Eq. $(6)$ and evaluating the answer for

\begin{align}
\Gamma \approx \frac{\pi e^2}{2h} \int d\xi \int \delta(E(q^2) - E(q')) f_f(E(q')) (1 - f_f(E(q'))) \int_{i=1}^{N} dx_i dx'_i T(x_1, x'_1) T(x_2, x'_2) T(x_3, x'_3) T(x_4, x'_4) \\
\times \left[ f_{12} f_{34} L_1 \left( \frac{E(q') + \Delta E_+}{\Delta_b} \right) + \frac{\delta_b}{2\Delta_b} \left(f_{14} f_{23} + f_{13} f_{24} \right) L_2 \left( \frac{E(q') + \Delta E_+}{\Delta_b} \right) \right] \left(1 - \frac{\Delta^2}{E(q') E(q')} \right) \langle K_q \langle x'_1, x'_3 \rangle K_q \langle x'_2, x'_4 \rangle \rangle.
\end{align}

Here $E(q') = \sqrt{q^2 + \Delta^2}$.

The correlation function in the reservoir $(K_q \langle x'_1, x'_3 \rangle K_q \langle x'_2, x'_4 \rangle)$ follows from Eqs. (14) and (15). Using Eq. (6) and evaluating the spatial integrals over the area of the junction as well as the integrals over energies $q'_1$ and $q'_2$, we finally obtain the answer for $(\Gamma_1)$:

\begin{align}
(\Gamma) &= \Gamma_1 + \Gamma_2
\end{align}

with $\Gamma_1$ and $\Gamma_2$ being defined as

\begin{align}
\Gamma_1 &= \frac{2\pi}{h} \frac{3 C_1}{(4\pi^2)^2 N_{ch}} \sqrt{\frac{E_j}{2\Delta_0 + E_j}} n_{qp} L_1 \left[ \frac{\Delta_x + \Delta E_+}{\Delta_b} \right], \tag{23}
\end{align}

and

\begin{align}
\Gamma_2 &= \frac{2\pi}{h} \frac{g^2}{8(4\pi^2)^2 \Delta_b} \sqrt{\frac{E_j}{2\Delta_0 + E_j}} n_{qp} L_2 \left[ \frac{\Delta_x + \Delta E_+}{\Delta_b} \right]. \tag{24}
\end{align}

Here $C_1$ is a numerical constant of the order of one:

\begin{align}
C_1 &= \frac{1}{\pi^{3/2} k_B^2 S_j} \int_{k_B^2 S_j} dy_1 dy_2 dy_3 dy_4 P_{12} P_{13} P_{24} P_{34}
\end{align}

with $y$ being a dimensionless coordinate in the plane of a tunnel junction, and $P_{12} = \frac{\sin(y_1 - y_2) - [y_1 - y_2 \cos(y_1 - y_2)]}{y_1^2 - y_2^2}$.

The functions $L_1$ and $L_2$ are defined in Eq. (19), and their dependence on the ratio $(\Delta_x + \Delta E_+)/\Delta_b$ is shown in Fig. 3. The rate $\Gamma_1$ describes the contribution from the reducible terms, see Eq. (14), and is similar to the ballistic case when electron scattering from the impurities or boundaries is negligible. The other term, $\Gamma_2$, reflects the enhancement of $(\Gamma_A)$ in the diffusive limit due to the quantum interference of quasiparticle return trajectories, and originates from the irreducible contributions, see Fig. 4.

In the case of $N_{ch} \Delta_b/\Delta_b \gg 1$, the contribution of this interference term becomes dominant, $\Gamma_2 \gg \Gamma_1$. The contribution of the interference in the reservoir to the rate $\Gamma_2$, see Fig. 2(b), is geometry dependent. For a typical charge qubit with the small junction connected to a large electrode, backscattering of electrons to the junction from the reservoir side gives much smaller contribution to $\Gamma_2$ than the similar one for the box side of the junction. In particular, for the layout of the qubit shown in Fig. 3, the contribution of the interference in the reservoir to $\Gamma_2$ is smaller than the one in the box by a factor $d_{h(r)} \Delta_b / E_q \ln \left[ \frac{KD}{\Delta_x S_j} \right] L_1(a_0) / L_2(a_0) \ll 1$. Here $a_0 = (\Delta_x + \Delta E_+)/\Delta_b$, and $d_{h(r)}$ is the thickness of the superconducting film in the box(reservoir). Therefore, we neglected the terms corresponding to the interference in the reservoir in Eq. (21).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{The dependence of the functions $L_1(a_0)$ and $L_2(a_0)$ (normalized by $L_1(0)$ and $L_2(0)$, respectively) on the dimensionless parameter $a_0 = (\Delta_x + \Delta E_+)/\Delta_b$. The solid and dashed lines correspond to $L_1$ and $L_2$, respectively, and reflect the increase of the rates $\Gamma_1$ and $\Gamma_2$ with $a_0$. The expressions for $L_1(a_0)$ and $L_2(a_0)$ given by Eq. (21) are valid for $a_0 \ll 1 - T/\Delta_b$.}
\end{figure}

\section{Conclusion}

We have studied the fundamental limitations on the energy relaxation time in a charge qubit with a large-gap Cooper-pair box, $\Delta_b > \Delta_q$. For sufficiently large $\Delta_b$, real quasiparticle transitions can be exponentially suppressed, and the dominant contribution to the charge qubit energy relaxation time $T_1$ comes from the higher-order (Andreev) processes, see Eq. (22). For realistic geometry of the charge qubits and the density of non-equilibrium quasiparticles in the reservoir $n_{qp} \sim 10^{19} - 10^{18} m^{-3}$, we estimate the Andreev relaxation rate to be $(\Gamma_A) \sim 10^{-1} - 10^{-2} Hz$. Thus, in the absence of other relaxation channels, the mismatch of gap energies leads to extremely long $T_1$-times. (For comparison, the
quasiparticle-induced $T_1$ found in Ref. [11] for the charge qubit with equal gap energies was $T_1^{-1} \sim 10^5 - 10^3 \text{Hz}$.)

The charge qubit with a large gap in the box also permits to reduce quasiparticle-induced decoherence. Since real quasiparticle transitions into the island are suppressed, see Eq. (2), the dephasing time of the qubit is limited by the energy relaxation processes, i.e. $T_2 \approx 2/\langle \Gamma_A \rangle$.

Acknowledgments

This work was supported by NSF grants DMR 02-37296, and DMR 04-39026.

1. J. Mannik and J. E. Lukens, Phys. Rev. Lett. 92, 057004 (2004).
2. J. Aumentado, M. W. Keller, J. M. Martinis, M. H. Devoret, Phys. Rev. Lett. 92, 66802 (2004).
3. A. Guillame, J. F. Schneiderman, P. Delsing, H. M. Bozler, and P. M. Echtner, Phys. Rev. B 69, 132504 (2004).
4. D. Gunarsson, T. Duty, K. Bladh, and P. Delsing, Phys. Rev. B 70, 224523 (2004).
5. B. A. Turek, K. W. Lehnmert, A. Clerk, D. Gunarsson, K. Bladh, P. Delsing, and R. J. Schoelkopf, Phys. Rev. B 71, 193304 (2005).
6. O. Naaman and J. Aumentado, Phys. Rev. B 73, 172504 (2006).
7. A. J. Ferguson, N. A. Court, F. E. Hudson, R. G. Clark, Phys. Rev. Lett. 97, 106603 (2006).
8. T. Yamamoto, Y. Nakamura, Yu. A. Pashkin, O. Astafiev, and J.S. Tsai, Appl. Phys. Lett. 88, 212509 (2006).
9. J. Könnemann, H. Zangerle, B. Mackrodt, R. Dolata, and A.B. Zorin, arXiv:cond-mat/0701144.
10. K.A. Matveev, L.I. Glazman, and R.I. Shekhter, Mod. Phys. Lett. B 8, 1007 (1994).
11. R. M. Lutchyn, L. I. Glazman, and A. I. Larkin, Phys. Rev. B 74, 064515 (2006).
12. R. Lutchyn, L. Glazman, A. Larkin, Phys. Rev. B 72, 014517 (2005).
13. E. Prada and F. Sols, Eur. Phys. J. B 40, 379 (2004).
14. M. Houzet, D. A. Pesin, A.V. Andreev, and L.I. Glazman, Phys. Rev. B 72, 104507 (2005).
15. J.R. Schrieffer, Theory of Superconductivity, (Oxford: Advanced Book Program, Perseus, 1999).
16. J. Bardeen, Phys. Rev. Lett. 9, 147 (1962).
17. B. D. Josephson, Phys. Lett., 1, 251 (1962).
18. We apply this transformation to a Cooper-pair box assuming that it is not very small, i.e. $\Delta_b \gg T \gg \delta_b$. Here $\delta_b$ is the mean level spacing in the box.
19. F. W. J. Hekking and Yu. V. Nazarov, Phys. Rev. B 49, 6847 (1994).
20. H. Pothier, S. Gueron, D. Esteve, and M.H. Devoret, Physica B 203, 226 (1994).
21. I. L. Aleiner and P. W. Brouwer and L. I. Glazman, Physics Reports 358, 309 (2002).
22. For an island shown in Fig. 3 $E_T = 14.7\pi hD/S_b$. The superconducting gap $\Delta_b$ can be expressed in terms of the diffusion constant $D$ and the coherence length in a system with disorder $\xi_{dirty}$: $\Delta_b = \frac{1}{2\pi} \frac{\hbar D}{\xi_{dirty}^2}$. Thus, the ratio $\Delta_b/E_T \approx 6 \cdot 10^{-3} S_b/\xi_{dirty}^2 \ll 1$ sets constraints on the size of the box.
23. P. Santhanam and D. E. Prober, Phys. Rev. B 29, 3733 (1984).
24. D. V. Averin and Yu. V. Nazarov, Phys. Rev. Lett. 65, 2446 (1990).
25. In the presence of a strong magnetic field, $B \gg B_c$, the quantum interference pattern is altered leading to a suppression of the rate $\Gamma_2$. Here $B_c$ is the correlation field $B_c \sim \Phi_0/S_b \sqrt{g}$ with $\Phi_0$, $S_b$ and $g$ being the flux quantum, the area and dimensionless conductance of the island, respectively. (See for details Aleiner et. al. [21] and references therein.) At the same time, rate $\Gamma_1$ remains unchanged. However, for a weak magnetic field $B \ll B_c$, Eq. (24) still holds.