Adjointness relations as a criterion for choosing an inner product

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1. Sufficient conditions for uniqueness

In the quantisation of constrained systems it can happen that one obtains a representation of an algebra of quantum operators on a vector space without a preferred inner product. Since an inner product is necessary for the probabilistic interpretation of quantum theory, some way needs to be found of introducing an appropriate inner product on this vector space. If the classical system being quantised possesses some background structure then it may be possible to use this to fix an inner product. In the case of gravity, where no background structure is present, this is not an option. The algebra of quantum observables usually admits a preferred *-operation, often related to complex conjugation of functions on the classical phase space. It has been suggested by Ashtekar that a preferred inner product could be fixed by the requirement that this *-operation is mapped by the representation into the operation of taking the adjoint of an operator with respect to the inner product in question. Discussions of this proposal can be found in [1-3]. The purpose of the following is to discuss the circumstances under which this idea suffices to determine the inner product uniquely.

Let $A$ be an associative algebra with identity over the complex numbers. This is to be interpreted as the algebra of quantum observables. Suppose that a representation $\rho$ of $A$ on a complex vector space $V$ is given. Suppose further that a *-operation $a \mapsto a^*$ is given on $A$. The defining properties of a *-operation are that it is conjugate linear $((\lambda a + \mu b)^* = \overline{\lambda}a^* + \overline{\mu}b^*)$, that $(ab)^* = b^*a^*$ and that $(a^*)^* = a$. In this paper the origin of these various objects will not be discussed; information on that can be found in [1] and [2]. Instead we take this collection of objects as starting point. The condition which is supposed to characterise the inner product is that

$$\langle \rho(a)x, y \rangle = \langle x, \rho(a^*)y \rangle, \tag{1}$$

for all $a \in A$ and $x, y \in V$. Note that there is a trivial non-uniqueness due to the possibility of multiplying the inner product by a non-zero real constant but this is not physically significant since it leaves expectation values unchanged. In order to ensure the uniqueness of the inner product it is necessary to require that the representation $\rho$ be irreducible in some appropriate sense. In [3] it was argued that the correct concept of irreducibility to use is that of topological irreducibility and this leads to the following definition:

**Definition 1** Let $A$ be a complex *-algebra with identity and $\rho$ a representation of $A$ on a complex vector space $V$. An inner product $\langle \ , \ \rangle$ on $V$ is called strongly admissible if

(i) $\rho$ is a *-representation with respect to this inner product i.e. equation (1) is satisfied

(ii) for each $a \in A$ the operator $\rho(a)$ is bounded with respect to the norm associated to the given inner product so that $\rho$ extends uniquely by continuity to a representation $\hat{\rho}$ on the Hilbert space completion $\hat{V}$ of $V$ with respect to this norm.
(iii) $\hat{\rho}$ is topologically irreducible i.e. it leaves no non-trivial closed subspaces of $\hat{V}$ invariant

This definition is tailored to the case of representations by bounded operators; the unbounded case will be discussed later. In [3] it was claimed that if $\langle \ , \rangle_1$ and $\langle \ , \rangle_2$ are two inner products which are strongly admissible for a given representation then there exists a positive real number $c$ such that $\langle \ , \rangle_2 = c\langle \ , \rangle_1$. In fact this is incorrect. An explicit example where the claim fails is given in section 3 below. An additional condition which ensures uniqueness will now be presented but first some terminology is required. Let $\langle \ , \rangle_1$ and $\langle \ , \rangle_2$ be two admissible inner products. The inner product $\langle \ , \rangle_2$ will be said to be compatible with $\langle \ , \rangle_1$ if any sequence $\{x_n\}$ in $V$ which satisfies $\langle x_n, x_n \rangle_1 \to 0$ as $n \to \infty$ and which is a Cauchy sequence with respect to $\langle \ , \rangle_2$ also satisfies $\langle x_n, x_n \rangle_2 \to 0$. (This is not a standard definition. It is only introduced for convenience in this paper.) The modified claim is now:

**Theorem 1** Let $\langle \ , \rangle_1$ and $\langle \ , \rangle_2$ be inner products on a complex vector space $V$ which are strongly admissible with respect to a representation $\rho$ of a complex *-algebra $A$. Suppose that $\langle \ , \rangle_2$ is compatible with $\langle \ , \rangle_1$. Then $\langle \ , \rangle_2 = c\langle \ , \rangle_1$ for some positive real number $c$.

This theorem will be proved in section 2. A question which comes up immediately is: what is the interpretation of the property of compatibility and is it a reasonable condition from the point of view of the original motivation, namely quantisation of certain systems? To make contact with known mathematics, let $\hat{V}_1$ be the completion of $V$ with respect to $\langle \ , \rangle_1$ and consider $\langle \ , \rangle_2$ as an unbounded sesquilinear form on the Hilbert space $\hat{V}_1$ with domain $V$. In general, if $S$ is an unbounded positive sesquilinear form on a Hilbert space $H$ with domain $D$ then a new inner product can be defined on $D$ by $\langle x, y \rangle_S = \langle x, y \rangle_H + S(x, y)$. The sesquilinear form $S$ is called closed if $D$ is complete with respect to $\langle \ , \rangle_S$. More generally $S$ is called closable if it has an extension to a domain $\hat{D}$ such that the extension is closed. It turns out that $\langle \ , \rangle_2$ is compatible with $\langle \ , \rangle_1$ if and only if $\langle \ , \rangle_2$ is closable when considered as an unbounded sesquilinear form on $\hat{V}_1$ (see chapter 6 of [4]). It is useful to have this information since it puts at our disposal known results on closable sesquilinear forms.

Concerning the relation of the compatibility condition with the original motivation, note first that the kind of situation which is likely to occur in practice is that one inner product is already known and we would like to know if it is characterised by the condition that the *-relations go over to adjointness relations. When the requirement of compatibility is added to the other hypotheses of the theorem the result is not to restrict the class of representations covered but rather to narrow the class of alternative inner products within which uniqueness is shown to hold. In examples it is usually the case that the vector space $V$ is a space of functions on some set $X$ and that the inner product which is known is the $L^2$ inner product corresponding to some measure $d\mu$ on that set. It is desired to have uniqueness within the class of inner products corresponding to the $L^2$ norms defined by measures of the form $f d\mu$, where $f$ is a non-negative real-valued function on $X$. There are general theorems which make it reasonable to expect that these inner products will all be compatible with the original one for conventional choices of the space of functions $V$. Thus the compatibility condition does not restrict the applicability of Theorem 1 too drastically. To illustrate this some examples will now be discussed.
First let \((X, d\mu)\) be a general measure space and \(f\) a non-negative measurable real-valued function on \(X\). Let

\[
D_f = \left\{ g \in L^2(X, d\mu) : \int |g|^2 d\mu < \infty \right\}.
\]

Then \(D_f\) is dense in \(L^2(X, d\mu)\) and the formula

\[
S_f(g, h) = \int f \overline{g} h d\mu
\]

defines a closed sesquilinear form with domain \(D_f\). It is compatible with the restriction of the \(L^2\) norm to \(D_f\) and so this provides a wide class of examples where compatibility is satisfied. It is not necessary to choose \(V = D_f\) in this example. Any linear subspace of \(D_f\) which is dense in \(L^2\) would also suffice. For instance in the case that \(X = \mathbb{R}^n\) and \(d\mu\) is Lebesgue measure \(V\) could be chosen to be the space of smooth functions with compact support. More generally, it is possible to consider two non-negative real-valued locally integrable functions \(f_1\) and \(f_2\) on \(\mathbb{R}^n\). Let \(\langle \cdot, \cdot \rangle_1\) and \(\langle \cdot, \cdot \rangle_2\) be the \(L^2\) inner products defined by the measures \(f_1 d\mu\) and \(f_2 d\mu\), restricted to \(V = C^\infty_c(\mathbb{R}^n)\). A sufficient condition that these sesquilinear forms be inner products is that the zero sets of \(f_1\) and \(f_2\) have zero measure. Assume that this is the case. \(V\) is dense in \(L^2(\mathbb{R}^n, f_1 d\mu)\) and it follows from the above that \(\langle \cdot, \cdot \rangle_2\) is compatible with \(\langle \cdot, \cdot \rangle_1\).

If one inner product is continuous with respect to the other, \(\| \|_2 \leq C \| \|_1\) for some constant \(C\), then it is obviously compatible with it. However it is in any case rather easy to prove a uniqueness result for inner products in that case. If \(V\) is finite dimensional then the continuity is automatic. It is natural to ask what happens when \(V\) has a countable basis. (The word basis is used here in the algebraic and not in the Hilbert space sense.) Suppose that an inner product \(\langle \cdot, \cdot \rangle_1\) is given on \(V\). Using the Gram-Schmidt process it is possible to go over to an orthonormal basis. Hence there is an isomorphism of \(V\) with the space of complex sequences with finitely many non-zero entries such that the inner product takes the form \(\langle \{a_n\}, \{b_m\}\rangle_1 = \sum a_n \overline{b}_n\). The completion of \(V\) can then be identified with the space \(l^2\) of square summable sequences and this will be done from now on. Any other inner product on \(V\) takes the form \(\langle \{a_n\}, \{b_m\}\rangle_2 = \sum a_n k_{mn} \overline{b}_m\) for some \(k_{mn}\). Deciding whether a given set of coefficients \(k_{mn}\) defines an inner product compatible with the original one is a concrete problem on the convergence of sequences. Nevertheless it does not seem easy to give a general solution. It will be shown by example in section 3 that the compatibility condition does not always hold.

An analogue of Theorem 1 for unbounded operators will now be presented. In [3] a procedure was given for reducing the problem of uniqueness of the inner product in the case of a representation by unbounded operators to the corresponding problem for bounded operators under certain circumstances. This reduction process combined with Theorem 1 gives a theorem in the case of unbounded operators which will now be stated.

**Definition 2** Let \(A\), \(\rho\) and \(V\) be as in Definition 1. Let \(S\) be a set of elements of \(A\) which satisfy \(a^* = a\) and which generate \(A\). An inner product \(\langle \cdot, \cdot \rangle\) on \(V\) is said to be **admissible** if:
(i) $\rho$ is a $*$-representation with respect to this inner product
(ii) for each $a \in S$ the operator $\hat{\rho}(a)$ is essentially self-adjoint
(iii) $\hat{\rho}$ is irreducible
(iv) $\hat{\rho}$ is closed

Here $\hat{\rho}$ is obtained from $\rho$ by considering the linear maps $\rho(a)$ on $V$ as unbounded operators on $\hat{V}$ with domain $V$. The meanings of the words ‘closed’ and ‘irreducible’ in this context are explained in [3]; suffice it to say that this definition reduces to the definition of ‘strongly admissible’ in the case that all $\rho(a)$ are bounded.

**Theorem 2** Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be inner products on a complex vector space $V$ which are admissible with respect to a representation $\rho$ of a complex $*$-algebra $A$. Suppose that $\langle \cdot, \cdot \rangle_2$ is compatible with $\langle \cdot, \cdot \rangle_1$. Then $\langle \cdot, \cdot \rangle_2 = c\langle \cdot, \cdot \rangle_1$ for some positive real number $c$.

2. **Proof of the uniqueness theorem**

Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be inner products satisfying the assumptions of Theorem 1. Let $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$. Define $\hat{V}$ and $\hat{V}_1$ to be the completions of $V$ with respect to $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_1$ respectively. The representation $\rho$ extends uniquely by continuity to representations $\hat{\rho}$ and $\hat{\rho}_1$ on $\hat{V}$ and $\hat{V}_1$ respectively. Now some facts proved in [3] will be recalled. It was shown there under the hypotheses that the inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are admissible that there exists a bounded self-adjoint operator $L_1$ on $\hat{V}$ such that $\langle x, y \rangle_1 = \langle x, L_1 y \rangle$ for all $x, y$ in $V$. It was also shown that unless the two inner products are proportional the operator $L_1$ has a non-trivial kernel. Now the sesquilinear form $\langle \cdot, \cdot \rangle_1$ on $V$ is bounded with respect to $\langle \cdot, \cdot \rangle$. Hence it extends uniquely by continuity to a sesquilinear form $S$ on $\hat{V}$. Using continuity again shows that $S(x, y) = \langle x, L_1 y \rangle$ for all $x$ and $y$ in $\hat{V}$. It follows that any vector $x$ in the kernel of $L_1$ satisfies $S(x, x) = 0$ and hence that $S$ is degenerate. In [3] it was claimed that this is incompatible with the fact that $\langle \cdot, \cdot \rangle_1$ is an inner product. However this is not true, as can be seen explicitly in the example given in section 3 below. The facts that $\langle \cdot, \cdot \rangle_1$ is non-degenerate and that $S$ is an extension by continuity of $\langle \cdot, \cdot \rangle_1$ do not together imply that $S$ is non-degenerate.

Suppose then that $x \in \hat{V}$ satisfies the condition that $S(x, x) = 0$. Since $V$ is dense in $\hat{V}$ there exists a sequence $x_n$ of vectors in $V$ with $\|x - x_n\| \to 0$ as $n \to \infty$. On the other hand, the continuity of $S$ implies that $S(x_n, x_n) \to 0$ and, due to the fact that all $x_n$ belong to $V$, this is equivalent to the condition that $\|x_n\|_1 \to 0$. The sequence $x_n$ is a Cauchy sequence with respect to $\langle \cdot, \cdot \rangle$ and hence with respect to $\langle \cdot, \cdot \rangle_2$. Now the hypothesis that $\langle \cdot, \cdot \rangle_2$ is compatible with $\langle \cdot, \cdot \rangle_1$ implies that $x_n \to 0$ with respect to $\langle \cdot, \cdot \rangle_2$. We already know that it tends to zero with respect to $\langle \cdot, \cdot \rangle_1$. It follows that $x = 0$. This means that the kernel of $L_1$ is trivial and, in conjunction with what was said above, completes the proof of the theorem.

3. **A cautionary example**

Let $H$ be the Hilbert space $L^2([-1, 1])$ and denote its inner product by $\langle \cdot, \cdot \rangle$. Let $V$ be the space of functions on $[-1, 1]$ which extend analytically to a neighbourhood of that interval. If $f \in V$ let $M_f$ be the multiplication operator on $H$ defined by $g \mapsto fg$. This is a bounded operator. If $\phi$ is an orientation preserving diffeomorphism of $[-1, 1]$ with $\phi(0) = 0$
which extends to an analytic mapping on a neighbourhood of $[-1, 1]$ define an operator $T_\phi$ on $H$ by $T_\phi f = f \circ \phi$. This operator is also bounded. Define $A$ to be the algebra of bounded operators on $H$ generated by all the $M_f$ and $T_\phi$. If $\phi$ is a diffeomorphism as above define a function $\tilde{\phi}$ by

$$\tilde{\phi}(x) = d/dx(\phi^{-1}(x)) \quad (4)$$

Then the adjoints of the operators of interest are given by $M_f^* = M_{\tilde{f}}$ and $T_\phi^* = M_{\tilde{\phi}}T_{\phi^{-1}}$. It follows that the operation of taking the adjoint defines a $*$-operation on $A$. Each operator belonging to $A$ maps $V$ into itself and so restricting the elements of $A$ to $V$ defines a representation of $A$ on $V$. It will be shown that there exist two inner products on $V$ which are strongly admissible with respect to $\rho$ and which are not proportional. These are defined as follows.

$$\langle f, g \rangle_+ = \int_0^1 f(x)\bar{g}(x)dx,$$

$$\langle f, g \rangle_- = \int_{-1}^0 f(x)\bar{g}(x)dx.$$ 

These expressions obviously define sesquilinear forms but it needs to be checked that they are non-degenerate on $V$. If $\langle f, f \rangle_+ = 0$ then $f$ vanishes almost everywhere on $[0, 1]$. But by analyticity this implies that $f$ vanishes identically. The proof for $\langle , \rangle_-$ is similar.

It remains to show that both inner products are admissible. Because of the symmetry of the situation it suffices to do this for $\langle , \rangle_+$. First note that the completion of $V$ with respect to $\langle , \rangle_+$ can be identified with $L^2([0, 1])$. The computations which show that $\rho$ is a $*$-representation with respect to $\langle , \rangle_+$ and that the operators $M_f$ and $T_\phi$ are bounded with respect to the corresponding norm $\| \cdot \|_+$ are essentially the same as those which are needed to show that they are bounded on $H$ and to compute their adjoints there. Next the irreducibility of $\hat{\rho}_+$ will be examined. Let $\Pi$ be a projection in the Hilbert space $\hat{V}_+$ which commutes with all operators in the image of $\hat{\rho}_+$. The aim is to show that $\Pi$ must be zero or the identity. Let $p = \Pi(1)$. This is an $L^2$ function on $[0, 1]$. If $f$ belongs to $V$ then

$$pf = M_f p = M_f \Pi(1) = \Pi M_f(1) = \Pi f \quad (5)$$

Thus on $V$ the operator $\Pi$ is given by multiplication by $p$. It can easily be seen by approximating an arbitrary continuous function on $[0, 1]$ uniformly by elements of $V$ that in fact $\Pi f = pf$ for any continuous function $f$. It will now be shown that the function $p$ must be essentially bounded. Let $E_n$ be the set where $|p| \geq n$ and let $\epsilon_n$ be the measure of $E_n$. By Lusin’s theorem [5] there exists a continuous function $f_n$ with $|f_n| \leq 1$ such that the measure of the set where $f_n$ is not equal to the characteristic function of $E_n$ is less than $\epsilon_n/2$. Hence $\|pf_n\|_2^2 \geq n^2\epsilon_n/2$. On the other hand $\|pf_n\|_2^2 = \|\Pi f_n\|_2^2 \leq \|f_n\|_2^2 \leq 3\epsilon_n/2$. Hence $\epsilon_n = 0$ for $n \geq 2$ and $p$ is essentially bounded. It follows that multiplication by $p$ defines a bounded operator $M_p$ on $L^2([0, 1])$. Since this operator agrees with $\Pi$ on a dense subspace it follows that $\Pi = M_p$. Now $\Pi^2 = \Pi$ implies that $M_p = M_p^2 = M_{p^2}$. Hence $(p^2 - p)f = 0$ for all $f \in L^2([0, 1])$. It follows that $p = 0$ or 1 almost everywhere and that $p$ is equal to the characteristic function $\chi_E$ of some measurable subset $E$ of $[0, 1]$. The condition $\Pi T_\phi = T_\phi \Pi$ will now be used. When worked out explicitly it gives

$$[p(x) - p(\phi(x))]f(\phi(x)) = 0 \quad (6)$$
for any $L^2$ function $f$. It follows that $p(\phi(x)) = p(x)$ i.e. that $\phi(E) = E$ up to set of measure zero. It will now be shown that if $E$ has non-zero measure then it must differ from $[0,1]$ by a set of zero measure. If $E$ has non-zero measure there must exist a point $x \in E \cap (0,1)$ which is a point of density. A point of density is roughly speaking a point of $E$ which is almost entirely surrounded by other points of $E$; the exact definition can be found in [6] where it is proved that any set of non-zero measure contains such a point. This notion is invariant under diffeomorphisms and is insensitive to altering the set $E$ by a set of measure zero. We can therefore conclude that the set of points of density of $E \cap (0,1)$ is invariant under all diffeomorphisms of the type under consideration here. Consider now the vector field on $\mathbb{R}$ given by $X = (1 - \cos(2\pi x))\partial/\partial x$. Exponentiating it gives a one-parameter group of analytic diffeomorphisms $\phi_t$. The restriction of each $\phi_t$ to the interval $[-1,1]$ belongs to the class of diffeomorphisms used in defining the operators $T_\phi$. Moreover, if $x_1$ and $x_2$ are any two points of $(0,1)$ there is some $t$ for which $\phi_t(x_1) = x_2$. It follows that all points of $(0,1)$ are points of density of $E$. The definition of points then implies that no point of $(0,1)$ is a point of density of the complement of $E$. Hence the complement of $E$ has measure zero. This completes the proof of the irreducibility of $\hat{\rho}_+^{\ast}$ and hence of the strong admissibility of $\langle , \rangle_+^{\ast}$.

A small modification of this example can be used to establish another point, namely the existence of inner products on a vector space with a countable basis which are not compatible. Let $V'$ be the vector space of polynomials on $[-1,1]$. Define two inner products $\langle , \rangle_+^{\ast}$ and $\langle , \rangle_-^{\ast}$ on $V'$ as the restrictions of the corresponding inner products on $V$ defined above. Then $\langle , \rangle_-^{\ast}$ is not compatible with $\langle , \rangle_+^{\ast}$. To see this, let $f$ be a continuous function on $[-1,1]$ which vanishes identically on $[0,1]$ but not on $[-1,0]$. By Weierstrass’ theorem there exists a sequence of polynomials converging uniformly to $f$ on the interval $[-1,1]$. This converges to zero with respect to the norm $\| \|_+$ and is Cauchy with respect to $\| \|_-$. However it does not converge to zero with respect to $\| \|_-$. The operator algebra defined above does not act on $V'$ and this example is only meant to illustrate the notion of compatibility of inner products. It is not known to the author whether Theorem 1 remains true for a vector space $V$ with countable basis if the compatibility hypothesis is dropped.

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