UNIQUENESS OF SOLUTIONS TO THE SCHRÖDINGER EQUATION
ON THE HEISENBERG GROUP

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ABSTRACT. This paper deals with the Schrödinger equation $i\partial_s u(z, t; s) - \mathcal{L} u(z, t; s) = 0$, where $\mathcal{L}$ is the sub-Laplacian on the Heisenberg group. Assume that the initial data $f$ satisfies $|f(z, t)| \leq Cq_s(z, t)$, where $q_s$ is the heat kernel associated to $\mathcal{L}$. If in addition $|u(z, t; s_0)| \leq Cq_b(z, t)$, for some $s_0 \in \mathbb{R}^*$, then we prove that $u(z, t; s) = 0$ for all $s \in \mathbb{R}$ whenever $ab < s_0^2$. This result also holds true on $H$-type groups.

1. Introduction

Let $\mathbb{H}^n$ be the $(2n + 1)$-dimensional Heisenberg group, and denote by $\mathcal{L}$ the sub-Laplacian for $\mathbb{H}^n$. In this paper we consider the following initial value problem for the Schrödinger equation for $\mathcal{L}$:

$$
i\partial_s u(z, t; s) - \mathcal{L} u(z, t; s) = 0, \quad (z, t) \in \mathbb{H}^n, \quad s \in \mathbb{R},$$

$$u(z, t; 0) = f(z, t)$$

where $f$ is assumed to be in $L^2(\mathbb{H}^n)$. Our goal is to find sufficient conditions on the behavior of the solution $u$ at two different times 0 and $s_0$ which guarantee that $u \equiv 0$ is the unique solution to the above initial data problem. More precisely, under some conditions, we prove that if the function $f$ has sufficient decay and if in addition the solution $u(z, t; s_0)$ has sufficient decay at a fixed $s_0 \in \mathbb{R} \setminus \{0\}$, then the solution must be trivial.

Uniqueness theorems of this kind were first proved by Chanillo [2] where he considered the Schrödinger equation associated to the standard Laplacian on $\mathbb{R}^n$. Using Hardy’s theorem for the Euclidean Fourier transform he proved a uniqueness theorem for solutions of the Schrödinger equation. Until then Hardy’s theorem was considered only in the context of heat equation and Chanillo’s work triggered a lot of attention on the Schrödinger equation. Chanillo himself treated the Schrödinger equation on complex Lie groups where the initial condition was assumed to be $K$-bi-invariant. However, if we use Radon transform the problem can be reduced to the Euclidean case and his result holds without any restriction either on the group or on the initial condition.

Similar uniqueness results for other Schrödinger equations and for the Korteweg-de Vries equation have received a good deal of attention in recent years (see for instance [4, 5, 8, 10, 13, 17]). These authors have developed powerful PDE techniques to deal with uniqueness results. Completing a full circle, in a recent work Cowling et al. [3] have used a uniqueness theorem for the Schrödinger equation to give a ‘real variable proof’ of Hardy’s theorem. See also the works [6, 7].

In this article we prove a uniqueness theorem for the Schrödinger equation on the Heisenberg group which is similar to what Chanillo has proved in the Euclidean case. Our approach
uses Hardy’s theorem for the Hankel transform obtained in [16], which says that a function and its Hankel transform both cannot have arbitrary Gaussian decay at infinity unless, of course, the function is identically zero. It is interesting to note that we do not need to use Hardy’s theorem for the Heisenberg group proved in [14].

In the last section we extend our main result to a class of groups that generalizes the Heisenberg group, namely $H$-type groups. This class was introduced in [9]. The list of $H$-type groups includes the Heisenberg groups and their analogues built up with quaternions or octonions in place of complex numbers, as well as many other groups.

2. Background

The $(2n + 1)$-dimensional Heisenberg group, denoted by $\mathbb{H}^n$, is $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z, t)(w, s) = (z + w, t + s + \frac{1}{2} \text{Im}(z \cdot \bar{w})).$$

Under this multiplication $\mathbb{H}^n$ becomes a nilpotent unimodular Lie group, the Haar measure being the Lebesgue measure $dzdt$ on $\mathbb{C}^n \times \mathbb{R}$. The corresponding Lie algebra is generated by the vector fields

$$X_j := \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n,$$

$$Y_j := \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n,$$

and $T := \frac{\partial}{\partial t}$. The sub-Laplacian

$$\mathcal{L} := -\sum_{j=1}^{n} X_j^2 + Y_j^2$$

can be written as

$$\mathcal{L} = -\Delta_{\mathbb{R}^{2n}} - \frac{1}{4} |z|^2 \frac{\partial^2}{\partial t^2} + N \frac{\partial}{\partial t},$$

where

$$N = \sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}.$$ 

This second order differential operator $\mathcal{L}$ is hypoelliptic, self-adjoint and nonnegative. It generates a semigroup with kernel $q_s(z, t)$, called the heat kernel. In particular, $q_s(z, t)$ is nonnegative and has the property

$$q_s(z, t) = r^{-2(n+1)} q_s(r^{-1} z, r^{-2} t), \quad r \neq 0.$$ 

Moreover,

$$\int_{\mathbb{R}} e^{iH} q_s(z, t) dt = (4\pi)^{-n} \left( \frac{\lambda}{\sinh \lambda s} \right)^n e^{-\frac{1}{4}(\cosh s) |z|^2}$$

(see [14]). Henceforth, for $f \in L^1(\mathbb{H}^n)$ and $\lambda \in \mathbb{R}$, we will write

$$f^\lambda(z) := \int_{\mathbb{R}} e^{iH} f(z, t) dt.$$
We now collect some properties of the heat kernel \( q_s(z, t) \).

**Fact 2.1.** The heat kernel satisfies the semigroup property \( q_s \ast q_b(z, t) = q_{s+b}(z, t) \).

The following is a slight modification of [14, Proposition 2.8.2].

**Fact 2.2.** The heat kernel \( q_s(z, t) \) satisfies the following estimate

\[
q_s(z, t) \leq Cs^{-n-1}e^{-\frac{\|z\|^2}{2s}}e^{-\frac{1}{2s}|z|^2}, \quad s > 0.
\]

Indeed, for \( s = 1 \) by [14, (2.8.9-2.8.10)], we have

\[ q_1(z, t) \leq Ce^{-\frac{1}{2}|z|^2}. \]

Now Fact 2.2 follows from the fact that \( q_s(z, t) = s^{-n-1}q_1(s^{-1/2}z, s^{-1}t) \) for all \( s > 0 \).

Let \( f \) and \( g \) be two functions on \( \mathbb{H}^n \). The convolution of \( f \) with \( g \) is defined by

\[ (f \ast g)(z, t) = \int_{\mathbb{H}^n} f((z, t)(-w, s))g(w, s)dwds. \]

An easy calculation shows that

\[ (f \ast g)^1(z) = \int_{\mathbb{H}^n} f^1(z - w)g^1(w)e^{i\frac{1}{2}Im(z\bar{w})}dw. \]

The right hand side is called the \( \lambda \)-twisted convolution of \( f^1 \) with \( g^1 \) denoted by \( f^1 \ast_\lambda g^1 \).

Let \( \mathcal{P} \) be the set of all polynomials of the form \( P(z) = \sum_{|\alpha|+|\beta|\leq m} a_{\alpha,\beta}z^\alpha \bar{z}^\beta \). For each pair of nonnegative integers \( (p, q) \), we define \( \mathcal{P}_{p,q} \) to be the subspace of \( \mathcal{P} \) consisting of all polynomials of the form \( P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} a_{\alpha,\beta}z^\alpha \bar{z}^\beta \).

Let \( \mathcal{H}_{p,q} := \{ P \in \mathcal{P}_{p,q} \mid \Delta P = 0 \} \), where \( \Delta \) denotes the Laplacian on \( \mathbb{C}^n \). The elements of \( \mathcal{H}_{p,q} \) are called bigraded solid harmonics of degree \( (p, q) \). We will denote by \( \mathcal{I}_{p,q} \) the space of all restrictions of bigraded solid harmonics of degree \( (p, q) \) to the sphere \( S^{2n-1} \). By [14], the space \( L^2(S^{2n-1}) \) is the orthogonal direct sum of the spaces \( \mathcal{I}_{p,q} \), with \( p, q \geq 0 \). We choose an orthonormal basis \( \{ Y^j_{p,q} \mid 1 \leq j \leq d(p, q) \} \) for \( \mathcal{I}_{p,q} \). Then by standard arguments it follows that every continuous function \( f \) on \( \mathbb{C}^n \) can be expanded as

\[
f(r\omega) = \sum_{p,q \geq 0} \sum_{j=1}^{d(p,q)} f_{p,q,j}(r)Y^j_{p,q}(\omega), \quad r > 0, \ \omega \in S^{2n-1},
\]

where

\[
f_{p,q,j}(r) := \int_{S^{2n-1}} f(r\omega)\overline{Y^j_{p,q}(\omega)}d\sigma(\omega). \quad (2.2)
\]

For \( k \in \mathbb{N} \), we write \( L^{n-1}_k \) for the Laguerre polynomial defined by

\[
L^{n-1}_k(t) = \sum_{j=0}^{k} \frac{(-1)^j\Gamma(n+k)}{(k-j)!\Gamma(n+j)}t^j.
\]

For \( \lambda \in \mathbb{R}^* \), define the Laguerre functions \( \varphi^{n-1}_{k,\lambda} \) by

\[
\varphi^{n-1}_{k,\lambda}(z) = L^{n-1}_k\left(\frac{\lambda}{2}|z|^2\right)e^{-\frac{\lambda}{2}|z|^2}, \quad (2.3)
\]
for \( z \in \mathbb{C}^n \). Suppose that \( f \) if a radial function in \( L^1(\mathbb{H}^n) \). Then \( f(r) \) is in \( L^1(\mathbb{R}^+, r^{2n-1} dr) \), where \( f(r) \) stands for \( f(w) \) with \( |w| = r \). For the following Hecke-Bochner formula we refer to [14, Theorem 2.6.1].

**Theorem 2.3.** Let \( f(z) = P(z)g(|z|) \), where \( P \in \mathcal{H}_{p,q} \) and \( g \in L^1(\mathbb{R}^+, r^{2n-1} dr) \). Then for \( \lambda \in \mathbb{R}^+ \), we have

\[
\mathcal{H}_\lambda F(s) = \int_0^\infty F(r) \frac{J_\alpha(rs)}{(rs)^\alpha} r^{2n+1} dr,
\]

where \( J_\alpha(w) \) is the Bessel function of order \( \alpha \) defined by

\[
J_\alpha(w) = \left( \frac{w}{2} \right)^\alpha \sum_{k=0}^\infty \frac{(-1)^k (w/2)^{2k}}{k! \Gamma(\alpha + k + 1)}.
\]

**Theorem 2.4.** (Hardy’s theorem, [16]) Let \( F \) be a measurable function on \( \mathbb{R}^+ \) such that

\[
F(r) = O(e^{-ar^2}), \quad \mathcal{H}_\alpha F(s) = O(e^{-bs^2})
\]

for some positive \( a \) and \( b \). Then \( F = 0 \) whenever \( ab > 1/4 \) and \( F(r) = Ce^{-ar^2} \) whenever \( ab = 1/4 \).

3. **Schrödinger equation on \( \mathbb{H}^n \times \mathbb{R} \)**

Let us consider the Schrödinger equation on \( \mathbb{H}^n \times \mathbb{R} \)

\[
i \partial_s u(z, t; s) = \mathcal{L} u(z, t; s),
\]

with the initial condition \( u(z, t; 0) = f(z, t) \). As the closure of \( \mathcal{L} \) on \( C_c^\infty(\mathbb{H}^n) \) is a self-adjoint operator, \(-i\mathcal{L}\) generates a unitary semi-group \( e^{-is\mathcal{L}} \) on \( L^2(\mathbb{H}^n) \), and the solution of the above Schrödinger equation is given by

\[
u(z, t; s) = e^{-is\mathcal{L}} f(z, t).
\]

The main result of the paper is:

**Theorem 3.1.** Let \( u(z, t; s) \) be the solution to the Schrödinger equation for the sub-Laplacian \( \mathcal{L} \) with initial condition \( f \). Suppose that

\[
|f(z, t)| \leq Cq_0(z, t), \quad (3.1 \text{a})
\]

\[
|u(z, t; s_0)| \leq Cq_0(z, t), \quad (3.1 \text{b})
\]
for some $a, b > 0$ and for a fixed $s_0 \in \mathbb{R}^*$. Then $u(z, t; s) = 0$ on $\mathbb{H}^n \times \mathbb{R}$ whenever $ab < s_0^2$.

The remaining part of this section is devoted to the proof of the above statement.

The heat kernel $q_s(z, t)$ has an analytic continuation in $s$ as long as real part of $s$ is positive. However, due to the zeros of the sine function, the kernel $q_{i\lambda}(z, t)$ does not exist as can be seen from the formula for $q_{i\lambda}(z)$. Hence the solution $u(z, t; s)$ does not have an integral representation. We will therefore consider the following regularised problem on $\mathbb{H}^n \times \mathbb{R}$:

\[
i\partial_s u(z, t; s) = \mathcal{L} u(z, t; s), \quad \epsilon > 0,
\]

where $f_\epsilon(z, t) := e^{-\epsilon \mathcal{L}} f(z, t)$. The solution $u_\epsilon$ on $\mathbb{H}^n \times \mathbb{R}$ is given by

\[
u_\epsilon(z, t; s) = e^{-i\epsilon \mathcal{L}} f_\epsilon(z, t) = f * q_\epsilon(z, t),
\]

where $\epsilon = \epsilon + is$ and

\[
q_\epsilon(z, t) := \frac{1}{(8\pi^2)^n} \int_\mathbb{R} e^{-i\lambda t} \left( \frac{\lambda}{\sinh \lambda \epsilon} \right)^n e^{-\frac{i}{4}(\cosh \lambda \epsilon) |\xi|^2} d\lambda.
\]

Observe that the kernel $q_\epsilon(z, t)$ is well defined.

**Lemma 3.2.** Under the assumptions (3.1 a) and (3.1 b), we have

\[
|f_\epsilon(z, t)| \leq C q_{a+\epsilon}(z, t), \quad (3.2 \text{ a})
\]

\[
|u_\epsilon(z, t; s_0)| \leq C q_{b+\epsilon}(z, t). \quad (3.2 \text{ b})
\]

**Proof.** For the first estimate, we have

\[
|f_\epsilon(z, t)| = |e^{-\epsilon \mathcal{L}} f(z, t)| = |f * q_\epsilon(z, t)|
\]

\[
\leq C q_{a+\epsilon}(z, t).
\]

Above we have used the fact that $q_\epsilon$ is nonnegative and Fact 2.1. Similarly we have

\[
|u_\epsilon(z, t; s_0)| = |u(\cdot, \cdot; s_0) * q_\epsilon(z, t)|
\]

\[
\leq C q_{b+\epsilon}(z, t).
\]

Recall that for $\lambda \in \mathbb{R}$, the notation $f^\lambda(z)$ stands for the inverse Fourier transform of $f(z, t)$ in the $t$-variable. In view of the hypothesis (3.1 a) on $f$ and the estimate (2.1) on the heat kernel, one can see that the function $\lambda \mapsto f^\lambda(z)$ extends to a holomorphic function of $\lambda$ on the strip $|\text{Im}(\lambda)| < \frac{\pi}{2\alpha}$. Thus the following statement is true.

**Lemma 3.3.** Under the hypothesis (3.1 a) on $f$, the inverse Fourier transform $f^\lambda(z)$ of $f(z, t)$ in the $t$-variable extends to a holomorphic function of $\lambda$ in a tubular neighborhood in $\mathbb{C}$ of the real line.

We point out that the above lemma also holds for the function $\lambda \mapsto f_\epsilon^\lambda$.

**Strategy.** To prove the main theorem, our strategy is to show that $f = 0$ on $\mathbb{H}^n$ whenever $ab < s_0^2$. However, by the above lemma, showing that $f^\lambda = 0$ on $\mathbb{C}^n$ for $0 < \lambda < \delta$, for some $\delta > 0$, will force $f^\lambda = 0$ on $\mathbb{H}^n$ for all $\lambda \in \mathbb{R}$ and hence $f = 0$ on $\mathbb{H}^n$. Furthermore, since $f_\epsilon^\lambda = f^\lambda * q_\epsilon^\lambda$, then proving that $f^\lambda = 0$ on $\mathbb{C}^n$ for $0 < \lambda < \delta$ is equivalent to show the same
statement for \( f^1_\epsilon \). On the other hand, in order to prove that \( f^1_\epsilon(z) = 0 \) for \( 0 < \lambda < \delta \), for some \( \delta > 0 \), it is enough to prove that the spherical harmonic coefficients

\[
(f^1_\epsilon)_{p,q,j}(r) = \int_{S^{2n-1}} f^1_\epsilon(r\omega)Y^j_{p,q}(\omega)d\sigma(\omega)
\]

vanish for \( 0 < \lambda < \delta \), for all \( p, q \geq 0 \) and \( 1 \leq j \leq d(p,q) \). In conclusion, the proof of the main theorem reduces to prove that if \( ab < s_0^2 \), then \((f^1_\epsilon)_{p,q,j} = 0 \) on \( \mathbb{R}^+ \) for \( 0 < \lambda < \delta \), for all \( p, q \geq 0 \) and \( 1 \leq j \leq d(p,q) \).

The following theorem will be of crucial importance to us.

**Theorem 3.4.** Let us fix \( p_0, q_0 \geq 0 \) and \( 1 \leq j_0 \leq d(p_0,q_0) \). For all \( r > 0 \), there exists a constant \( c_1 \) which depends only on \( \lambda \) such that

\[
\int_{S^{2n-1}} u^1_\epsilon(r\omega; s_0)Y^j_{p_0,q_0}(\omega)d\sigma(\omega) = c_1r^{p_0+q_0}e^{i\frac{r^2}{2}\cotg(\lambda s_0)}\mathcal{H}_{\alpha}(r\omega; s_0)\left( e^{i\frac{r^2}{2}\cotg(\lambda s_0)}(f^1_\epsilon)_{p_0,q_0,j_0}(\omega) \right) \left( \frac{\lambda r}{2\sin(\lambda s_0)} \right),
\]

where \( u^1_\epsilon(z; s_0) \) denotes the inverse Fourier transform of \( u_\epsilon(z,t; s_0) \) in the \( t \)-variable, \( \mathcal{H}_\alpha \) denotes the Hankel transform of order \( \alpha \) (see (2.5)), and \((f^1_\epsilon)_{p_0,q_0,j_0} = \mathcal{H}_{\alpha}(r\omega; s_0)\left( e^{i\frac{r^2}{2}\cotg(\lambda s_0)}(f^1_\epsilon)_{p_0,q_0,j_0}(\omega) \right) \left( \frac{\lambda r}{2\sin(\lambda s_0)} \right) , \)

**Proof.** In what follows \( c_1 \) will stand for constants depending only on \( \lambda \) which will vary from one line to another. Using Fact (2.2) we can rewrite \( u^1_\epsilon(z; s_0) \) as

\[
u^1_\epsilon(z; s_0) = f^1_\epsilon * \lambda q^1_{\lambda s_0}(z),
\]

where

\[
q^1_{\lambda s_0}(z) = (4\pi)^{-n} \left( \frac{\lambda}{i\sin(\lambda s_0)} \right)^n e^{i\lambda(\cotg(\lambda s_0)|z|^2)}
\]

which exists for all but a countably many values of \( \lambda \). Thus

\[
\int_{S^{2n-1}} u^1_\epsilon(r\omega; s_0)Y^j_{p_0,q_0}(\omega)d\sigma(\omega)
= \int_{S^{2n-1}} \int_{\mathbb{R}^n \times \mathbb{R}^n} f^1_\epsilon(r\omega - w)q^1_{\lambda s_0}(w)e^{i\frac{r^2}{2}\cotg(\lambda s_0)}d\text{w}Y^j_{p_0,q_0}(\omega)d\sigma(\omega)
= \int_{S^{2n-1}} \int_{\mathbb{R}^n \times \mathbb{R}^n} f^1_\epsilon(w)q^1_{\lambda s_0}(r\omega - w)e^{i\frac{r^2}{2}\cotg(\lambda s_0)}d\text{w}Y^j_{p_0,q_0}(\omega)d\sigma(\omega).
\]

We now expand \( f^1_\epsilon \) in terms of bigraded spherical harmonics as

\[
f^1_\epsilon(\eta) = \sum_{p,q \geq 0} \sum_{j=1}^{d(p,q)} (f^1_\epsilon)_{p,q,j}(t)Y^j_{p,q}(\eta),
\]

where \((f^1_\epsilon)_{p,q,j}\) is as in (2.2). Further, by [14] (2.8.7) we have

\[
q^1_{\lambda s_0}(r\omega - \eta) = (2\pi)^{-n} |\lambda|^n \sum_{k=0}^{\infty} e^{-i(2k+n)|\lambda|s_0^{k+1}} \varphi^{n-1}_{k,\lambda}(r\omega - \eta),
\]
where $\varphi^{n-1}_{k,\lambda}$ is given by (2.3). Now the Hecke-Bochner formula for the $\lambda$-twisted convolution (see Theorem 2.3) gives us

$$\int_0^\infty \int_{\mathbb{S}^{2n-1}} (f_\varepsilon^j)_{p,q,j}(t) Y_{p,q}(\eta) \varphi^{n-1}_{k,\lambda}(r \omega - t \eta) e^{-\frac{i}{2} \lambda t \text{Im} (\omega \cdot \eta) t^{2n-1}} dt d\sigma(\eta)$$

where $\varphi^{n-1}_{k,\lambda}$ is defined by (2.4). Above we have used the fact that $\varphi^{n-1}_{k,\lambda} = \varphi^{n-1}_{k,-\lambda}$. Using the orthogonality of the basis $\{ Y_{p,q}^j : 1 \leq j \leq d(p,q) \}$ we obtain:

$$\int_0^\infty \int_{\mathbb{S}^{2n-1}} (f_\varepsilon^j)_{p,q,j}^\lambda(r \omega - w) e^{-\frac{i}{2} \lambda t \text{Im} (\omega \cdot w) t^{2n-1}} d\omega d(w)$$

On the other hand, by (2.4) we have

$$(f_\varepsilon^j)_{p,q,j}^\lambda(r \omega) = c_\lambda \frac{\Gamma(k+1)}{\Gamma(k+n+p_0+q_0)}$$

Hence we obtain

$$\int_0^\infty u_\varepsilon^j(r \omega; s_0) Y_{p,q}^{j_0}(\omega) d\sigma(\omega)$$

$$= c_\lambda r^{p_0+q_0} \sum_{k=0}^\infty \frac{\Gamma(k+1)}{\Gamma(k+n+p_0+q_0)} e^{-i(2k+n+2p_0) |\lambda| \omega |t_0|} L_k^{n+p_0+q_0-1} \left( \frac{|\lambda|}{2} r^2 \right) e^{-\frac{|\lambda|}{2} r^2}$$

where

$$K_{\lambda}(r, t; s_0) := \sum_{k=0}^\infty \frac{\Gamma(k+1)}{\Gamma(k+n+p_0+q_0)} e^{-i(2k+n+2p_0) |\lambda| \omega |t_0|} L_k^{n+p_0+q_0-1} \left( \frac{|\lambda|}{2} r^2 \right) L_k^{n+p_0+q_0-1} \left( \frac{|\lambda|}{2} r^2 \right).$$
Now we can use the following Hille-Hardy identity (see for instance [15])

\[
\sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(x) L_k^\alpha(y) w^k = (1-w)^{-(\alpha+1)} e^{-\frac{w}{x+y}} \tilde{J}_\alpha \left( \frac{2(1-wy)^{1/2}}{1-w} \right),
\]

where \( \tilde{J}_\alpha(w) := \left( \frac{w}{2} \right)^\alpha J_\alpha(w) \) and \( J_\alpha \) is the Bessel function of order \( \alpha \). Thus we may rewrite the kernel \( K_\lambda \) as

\[
K_\lambda(r, t; s_0) = e^{\|\nabla\|_{(q_0-p_0)}}(2i \sin(\|\nabla\|_{s_0}))^{-n+p_0+q_0} e^{i2r^2} \cotg(\lambda s_0) \tilde{J}_{n+p_0+q_0-1} \left( \frac{\lambda}{2 \sin(\lambda s_0)} \right) \cdot \frac{rt}{2 \sin(\lambda s_0)}
\]

Thus we arrive at

\[
\int_{S^{2n-1}} u_\lambda^1(r, s_0) \bar{Y}_{\lambda s_0}(\omega) d\sigma(\omega) = c_1 r^{p_0+q_0} \int_0^{\infty} e^{i2r^2} \cotg(\lambda s_0) \langle f_\lambda^{(1)} \rangle_{p_0+q_0} (t) \tilde{J}_{n+p_0+q_0-1} \left( \frac{\lambda}{2 \sin(\lambda s_0)} \right) \cdot \frac{rt}{2 \sin(\lambda s_0)} - 1 dt
\]

\[
= c_1 r^{p_0+q_0} e^{i2r^2} \cotg(\lambda s_0) \mathcal{H}_{n+p_0+q_0-1} \left( e^{i2r^2} \cotg(\lambda s_0) \langle f_\lambda^{(1)} \rangle_{p_0+q_0} \right) \left( \frac{\lambda r}{2 \sin(\lambda s_0)} \right).
\]

Hence Theorem 3.4 has been proved.

We are ready to complete the proof of the main result.

The estimate (3.2 a) on \( f_\lambda(z, t) \) together with Fact [1.1] lead us to

\[
|f_\lambda^{(1)}(z)| \leq c e^{-\frac{1}{\pi} r^2},
\]

for some constant \( c \). Thus, the spherical harmonic coefficient \( \langle f_\lambda^{(1)} \rangle_{p_0+q_0,j_0} \) satisfies

\[
|\langle f_\lambda^{(1)} \rangle_{p_0+q_0,j_0}(t)| \leq c t^{-(p_0+q_0)} e^{-\frac{1}{\pi} r^2}.
\]

On the other hand, by means of Theorem 3.4 and the estimate (3.2 b) on \( u_\lambda(z, t; s_0) \), we deduce that

\[
\mathcal{H}_{n+p_0+q_0-1} \left( e^{i2r^2} \cotg(\lambda s_0) \langle f_\lambda^{(1)} \rangle_{p_0+q_0,j_0} \right) \left( \frac{\lambda r}{2 \sin(\lambda s_0)} \right) \leq c d r^{-p_0+q_0} e^{-\frac{1}{\pi} r^2}.\]

That is

\[
\mathcal{H}_{n+p_0+q_0-1} \left( e^{i2r^2} \cotg(\lambda s_0) \langle f_\lambda^{(1)} \rangle_{p_0+q_0,j_0} \right) (r) \leq c d r^{-p_0+q_0} e^{-\frac{1}{\pi} r^2}.\]

Given \( a, b > 0 \) such that \( ab < s_0^2 \), we can choose \( \varepsilon > 0 \) such that \( (a + \varepsilon)(b + \varepsilon) < s_0^2 \). We can also choose \( \delta > 0 \) small enough in such a way that for \( 0 < \lambda < \delta \) we have \( (a + \varepsilon)(b + \varepsilon) < s_0^2 \left( \frac{\sin(\lambda s_0)}{\lambda s_0} \right)^2 \). This inequality can be written as

\[
\frac{1}{4(a + \varepsilon)(b + \varepsilon)} \left( \frac{2 \sin(\lambda s_0)}{\lambda s_0} \right)^2 > \frac{1}{4}.\]

Therefore, by Hardy's theorem for the Hankel transform (see Theorem 2.4), we deduce that for \( 0 < \lambda < \delta \) we have \( \langle f_\lambda^{(1)} \rangle_{p_0+q_0,j_0} = 0 \), for all \( p_0, q_0 \geq 0 \) and \( 1 \leq j_0 \leq d(p_0, q_0) \). That is \( f_\lambda^{(1)} = 0 \) on \( \mathbb{C}^n \) for \( 0 < \lambda < \delta \), which forces \( f_\lambda^{(1)} = 0 \) for all \( \lambda \) and hence \( f_\lambda = 0 \) on \( \mathbb{H}^n \). That is \( f = 0 \) on \( \mathbb{H}^n \). This finishes the proof Theorem 3.4.
4. **The main result for $H$-type groups**

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra over $\mathbb{R}$ with an inner product $\langle \cdot, \cdot \rangle$. The corresponding simply connected Lie group is denoted by $G$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is called an $H$-type algebra if for every $v \in \mathfrak{v}$, the map $\text{ad}_v : \mathfrak{v} \to \mathfrak{z}$ is a surjective isometry when restricted to the orthogonal complement of its kernel.

For the $H$-type algebra $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, let $\dim(\mathfrak{v}) = 2n$ and $\dim(\mathfrak{z}) = k$. The class of groups of $H$-type includes the Heisenberg group $\mathbb{H}^n$ when $k = 1$. Let $\eta$ be a unit element in $\mathfrak{z}$ and denote its orthogonal complement in $\mathfrak{z}$ by $\eta^\perp$. The quotient algebra $\mathfrak{g}/\eta^\perp$ is a Lie algebra with Lie bracket $[X, Y]_\eta = \langle [X, Y], \eta \rangle$.

The quotient $\mathfrak{g}/\eta^\perp$ is an $H$-type algebra with inner product $\langle \cdot, \cdot \rangle_\eta$ given by $\langle (v_1, t_1), (v_2, t_2) \rangle_\eta = \langle v_1, v_2 \rangle + t_1 t_2$, where $v_1, v_2 \in \mathfrak{v}$, $t_1, t_2 \in \mathbb{R}$, and $\langle v_1, v_2 \rangle$ is the inner product in $\mathfrak{g}$. Here $(v, t)$ stands for the coset of $v + \eta t$ in $\mathfrak{g}/\eta^\perp$. Moreover, if we denote by $G_\eta$ the simply connected Lie group with Lie algebra $\mathfrak{g}/\eta^\perp$, then by [12], the Lie group $G_\eta$ is isomorphic to the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$. We refer to [11] for more details on the theory of $H$-type groups.

We fix an orthonormal basis $X_1, \ldots, X_{2n}$ for $\mathfrak{v}$, and define the sub-Laplacian by

$$\mathcal{L} = -\sum_{j=1}^{2n} X_j^2.$$ 

It is known that $\mathcal{L}$ generates a semigroup which is given by convolution with the heat kernel for $G$. As in the case of Heisenberg group, the kernel is explicitly known and is given by

$$h_\zeta(v, t) = \frac{1}{2^n(2\pi)^n s^{k/2}} \int_{0}^{\infty} \frac{\lambda^{k/2}}{|t|^{k/2-1} J_{k/2-1}(\lambda t)} \left( \frac{\lambda}{\sinh(s\lambda)} \right)^n e^{-\frac{1}{2}(\coth s \lambda)|v|^2} d\lambda,$$

for $(v, t) \in G$ and $s > 0$. Here $J_\alpha$ denotes the Bessel function of order $\alpha$. This formula has been proved in [11], where the author also obtains the integral expression for the analytic continuation, $h_\zeta$, of the heat kernel $h_\zeta$ as long as $\text{Rel}(\zeta) > 0$.

We now consider the solution of the Schrödinger equation on $G \times \mathbb{R}$

$$i\partial_t u(v, t; s) = \mathcal{L} u(v, t; s),$$

$$u(v, t; 0) = f(v, t),$$

which is given by $u(v, t; s) = e^{-is\mathcal{L}} f(v, t)$. When we replace the initial condition $f$ by $e^{-\epsilon \mathcal{L}} f$, for some $\epsilon > 0$, then the solution is given by

$$u_\epsilon(v, t; s) = f * h_\zeta(v, t), \quad \zeta = \epsilon + is.$$

We claim that the uniqueness Theorem 3.1 for the Schrödinger equation on $\mathbb{H}^n \times \mathbb{R}$ is true in the more general setting $G \times \mathbb{R}$. The rest of this section is devoted to the proof of the following theorem.

**Theorem 4.1.** Let $u(v, t; s)$ be the solution of the Schrödinger equation on $G \times \mathbb{R}$, with initial data $f$. Assume that $|f(v, t)| \leq C_{a_0}(v, t)$ for some $a > 0$. Further, suppose that there exists $s_0 \in \mathbb{R} \setminus \{0\}$ such that $|u(v, t; s_0)| \leq C_{b_0}(v, t)$ for some $b > 0$. If $ab < s_0^2$, then $u(v, t; s) = 0$ for all $(v, t) \in G$ and for all $s \in \mathbb{R}$.
For a suitable function \( f \) on \( G \) we define its partial Radon transform \( \mathcal{R}_\eta f(v, t) \) on \( G_\eta \) by

\[
\mathcal{R}_\eta f(v, t) = \int_{\eta^+} f(v, t\eta + v) dv
\]

where \( dv \) is the Lebesgue measure on \( \eta^+ \). Since \( G_\eta \) can be identified with the Heisenberg group \( \mathbb{H}^n \), we can think of \( \mathcal{R}_\eta f \) as a function on \( \mathbb{H}^n \). With this identification it has been proved in [11] that \( \mathcal{R}_\eta h_s(v, t) = q_s(v, t) \), for \( s > 0 \), where \( q_s(v, t) \) is the heat kernel from section 2. The above identity between the heat kernels holds true even when \( s \) is complex with \( \text{Re}(s) > 0 \).

In view of the assumptions on \( f(v, t) \) and \( u(v, t; s_0) \) it follows that \( \mathcal{R}_\eta f(v, t) \) and \( \mathcal{R}_\eta u(v, t; s_0) \) satisfy

\[
|\mathcal{R}_\eta f(v, t)| \leq Cq_a(v, t),
\]

\[
|\mathcal{R}_\eta u(v, t; s_0)| \leq Cq_b(v, t).
\]

Moreover, using the fact that under the Radon transform \( \mathcal{R}_\eta \), the sub-Laplacian \( \mathcal{L} \) on \( G \) goes into the sub-Laplacian \( \mathcal{L} \) on \( \mathbb{H}^n \) (see [12]), it follows that \( \mathcal{R}_\eta u \) solves the Schrödinger equation on \( \mathbb{H}^n \times \mathbb{R} \) with initial data \( \mathcal{R}_\eta f(v, t) \). Hence we can appeal to Theorem 3.1 to conclude that

\[
\mathcal{R}_\eta u(v, t; s) = 0 \quad \text{for all } s \in \mathbb{R}
\]

and for all \( \eta \in \mathbb{R} \) whenever \( ab < s_0^2 \). Now the injectivity of the Radon transform implies that if \( ab < s_0^2 \), then \( u(v, t; s) = 0 \) for all \( (v, t) \in G \) and \( s \in \mathbb{R} \).

This establishes Theorem 4.1.

5. Some concluding remarks

It would be interesting to see if Theorem 3.1 is sharp. Though we believe it is sharp we are not able to prove it. The main reason for the difficulty lies in the fact that the heat kernel \( q_a(z, t) \) does not have Gaussian decay in the central variable. For the same reason the equality case of Hardy’s theorem for the group Fourier transform on the Heisenberg group is still an open problem. However, if we assume conditions on \( f^1 \) and \( u^1 \) instead of on \( f \) and \( u \) we can prove the following result.

**Theorem 5.1.** Let \( u(z, t; s) \) be the solution to the Schrödinger equation for the sub-Laplacian \( \mathcal{L} \) with initial condition \( f \). Fix \( \lambda \neq 0 \) and suppose that

\[
|f^1(z)| \leq Cq_a^1(z), \quad |u^1(z; s_0)| \leq Cq_b^1(z)
\]

for some \( a, b > 0 \) and for a fixed \( s_0 \in \mathbb{R}^* \). Then we have \( f^1(z) = c_\lambda q_a^1(z)e^{-\frac{i}{2}|z|^2\cotg(\lambda s_0)} \) whenever \( \tan(\alpha) \tan(b, \lambda) = \sin^2(\lambda s_0) \).

To prove this theorem, we can proceed as in the proof of Theorem 3.1. We end up with the estimates

\[
\left| H_n^{\alpha}(p_0 + q_0, j_0) \left( f^1_{p_0, q_0, j_0}(r) \right) \right| \leq c_\lambda r^{-(p_0 + q_0)} e^{-\frac{1}{2} \coth(\lambda s_0)^2 \frac{2\sin(\lambda s_0)}{s_0}}
\]

and

\[
\left| (f^1_{p_0, q_0, j_0})^{-1}(r) \right| \leq c_\lambda e^{-\frac{1}{2} \coth(\lambda s_0)^2}.
\]

We can now appeal to the equality case of Hardy’s theorem for the Hankel transform (Theorem 2.4) to conclude that

\[
f^1_{p_0, q_0, j_0}(r) = c_\lambda (p_0, q_0, j_0)^{p_0 + q_0} e^{-\frac{1}{2} \coth(\lambda s_0)^2} e^{-\frac{1}{2} r^2 \cotg(\lambda s_0)}.
\]
But this is not compatible with the hypothesis on $f^4$ unless $c_1(p_0, q_0, j_0) = 0$ for all $(p_0, q_0) \neq (0, 0)$. Hence $f^4$ is radial and equals $c_1 q_n^4(z)e^{-\frac{i}{2} \|x\|^2 \cotg(\lambda z_0)}$. This proves Theorem [5,1].

The above result can be viewed as a uniqueness theorem for solutions of the Schrödinger equation associated to the twisted Laplacian $L_4$ defined by $\mathcal{L}(e^{itf(z)}) = e^{it} L_4 f(z)$. Indeed, $q_n^4(z)$ is the heat kernel associated to this operator. We refer to [14, (2.3.7)] for the explicit expression of $L_4$. We can also consider the result as an analogue of Hardy’s theorem for fractional powers of the symplectic Fourier transform. In fact, the unitary operator $e^{ins} e^{-isL_4}$ with $s = \frac{\pi}{2}$ is just the symplectic Fourier transform. Thus the above theorem for $s_0 = \frac{\pi}{2}$ follows immediately from Hardy’s theorem for the Fourier transform whereas for other values of $s_0$ we require a longwinding proof.

For the sake of completeness we state another result which can be considered as a theorem for fractional Fourier transform as well as a theorem for solutions of the Schrödinger equation associated to the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. This elliptic operator generates the Hermite semigroup whose kernel is known explicitly. We also know that $e^{i\pi \epsilon} e^{-\frac{4}{\pi} \lambda H}$ is the Fourier transform on $\mathbb{R}^n$.

**Theorem 5.2.** Let $u(x, s) = e^{-iH} f(x)$ be the solution to the Schrödinger equation

$$i\partial_s u(x, s) - Hu(x, s) = 0,$$

with initial condition $f$. Suppose

$$|f(x)| = O(e^{-a|x|^2}), \quad |u(x, s_0)| = O(e^{-b|x|^2})$$

for some $a, b > 0$. Then $u = 0$ on $\mathbb{R}^n \times \mathbb{R}$ whenever $ab \sin^2(2s_0) > \frac{1}{4}$.

The theorem follows from Hardy’s theorem for $\mathbb{R}^n$ once we realise $u$ as the Fourier transform of a function. But this is easy to check in view of the Mehler’s formula (see [14]) for the Hermite functions. In view of this formula, the kernel of $e^{-itH}$ is given by

$$K_r(x, y) = \pi^{-n/2} (1 - r^2)^{-n/2} e^{-\frac{1}{4} \frac{1}{\sin^2(2s_0)} (|x|^2 + |y|^2) + \frac{2\pi}{\sin^2(2s_0)} xy}$$

where $r = e^{-2it}$. Using this formula the theorem can be easily proved.

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**References**

[1] J. Berndt, F. Tricerri, and L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture Notes in Mathematics, **1598** Springer-Verlag, Berlin (1995).

[2] S. Chanillo, *Uniqueness of solutions to Schrödinger equations on complex semi-simple Lie groups*, Proc. Indian Acad. Sci. Math. Sci. **117** (2007), 325–331.

[3] M. Cowling, L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, *The Hardy uncertainty principle revisited*, (preprint), 2010.

[4] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, *Hardy’s uncertainty principle, convexity and Schrödinger evolutions*, J. Euro. Math. Soc. **10** (2008), 883-907.

[5] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, *Convexity properties of solutions to the free Schrödinger equation with Gaussian decay*, Math. Res. Lett. **15** (2008), 957-971.
[6] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, The sharp Hardy uncertainty principle for Schrodinger evolutions (preprint).

[7] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, Uncertainty principles of Morgan type and Schrödinger evolutions (preprint).

[8] I. D. Ionescu and C. E. Kenig, Uniqueness properties of solutions of Schrodinger equations, J. Funct. Anal. 232 (2006), 90–136

[9] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms, Trans. Amer. Math. Soc. 258 (1980), 147–153.

[10] C. E. Kenig, G. Ponce, and L. Vega, On the support of solutions of nonlinear Schrodinger equations, Comm. Pure Appl. Math. 56 (2002), 1247–1262

[11] J. Randall, The heat kernel for generalized Heisenberg groups, J. Geom. Anal. 6 (1996), 287–316.

[12] F. Ricci, Commutative algebras of invariant functions on groups of Heisenberg type, J. London Math. Soc. 32 (1985), 265–271.

[13] L. Robbiano, Unicité forte à l’infini pour KdV, Control Opt. and Cal. Var. 8 (2002), 933–939

[14] S. Thangavelu, An introduction to the uncertainty principle. Hardy’s theorem on Lie groups, Progress in Mathematics, 217. Birkhauser Boston, Inc., Boston, MA, 2004.

[15] S. Thangavelu, Lectures on Hermite and Laguerre expansions, Mathematical Notes 42, Princeton University Press, Princeton, 1993.

[16] V.-K. Tuan, Uncertainty principles for the Hankel transform, Integral Transforms Spec. Funct. 18 (2007), 369–381.

[17] B.-Y. Zhang, Unique continuation for the Korteweg-de Vries equation, SIAM J. Math. Anal. 23 (1992), 55–71.

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