A connection between Lyapunov exponents and sensitive dependence on parameters of chaotic systems

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Abstract

The sensitive dependence of chaos on parameters is a topic of great interest in the study of integrability and stability of dynamical systems. Previous work has proposed ways to identify the sensitive dependence on parameters by topological criteria or large numerical simulations. In this paper, we show that when the Lyapunov exponents of the system vary with a change in the parameters, the system diverges exponentially in the orbits associated with the considered parameters. We use this result to explore the sensitive dependence on parameters in an uncertainty interval and conclude that the characterization of this phenomenon is directly related to our ability to determine the Lyapunov exponents of the system for different parameters.

Keywords: Parameters; Lyapunov Exponents; Power Laws.

1. Introduction

Bifurcation theory describes how the qualitative behavior of dynamical systems is affected by changes in the parameters of the equations of motion. When the parameter space has a bifurcation set, it is possible to call the system sensitive to a variation in the parameters, so that small variations produce topological changes that significantly change the evolution of the orbits [1].

Since 1985, many works have developed topological, dynamical or experimental techniques to identify the sensitive dependence on parameters [2]. Grebogi et al. [3], presents a probabilistic formulation for the characterization of this phenomenon by defining an $\alpha$ value such that computing $\alpha = [1 - P(\varepsilon)]$ provides the probability that, given a $\varepsilon$ measurement precision on a parameter $r$, the simulated asymptotic behavior is consistent with the true dynamics of the system, where $P(\varepsilon)$ indicates the probability that two parameters $r$ and $r' = r + \varepsilon$ lead to different asymptotic behavior considering different random choices of $r$ in the interval of interest [4].

Other works have argued that the use of topological criteria are efficient tools for determining parameter sensitivity [2] [5] [6], especially in the context of quadratic maps, and evidenced that this phenomenon is sufficient to produce divergence patterns close or even equivalent to those caused by variations in the initial conditions of chaotic systems [7]. On the other hand, investigations involving the dependence of Lyapunov

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exponents on parameters show that variations in system parameters can be reflected in simple power laws \[8, 9, 10\], which suggests the existence of some kind of connection between the sensitivity dependence on parameters and the way that Lyapunov exponents vary.

In this sense, we show that the divergence process defined by Lyapunov exponents can be used to study the sensitive dependence on parameters as a technique to identify the consistency of simulations of chaotic processes, given uncertainties associated with the system parameters. Thus, we can estimate the time required for the effects of a variation in the parameters to cause divergence between the trajectories in chaotic systems. The paper is organized as follows: In Section 2, we define the linearity condition that implies a homogeneous dependence of the Lyapunov exponents on the parameters and, consequently, allows one to obtain power laws associated with the dependence on the parameters in Lagrangian systems. In Section 3, we show how the sensitive dependence on parameters is related to the Lyapunov exponents and implies an upper bound for the time in which two trajectories certainly remain within an uncertainty interval around the state of the system in phase space. Lastly, in Section 4, we use these concepts to analyze how two Lagrangian chaotic systems behave as the system parameters vary.

2. Parameters linearity

Consider the Lagrangian function given by:

\[
\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t, \omega) = T(\dot{\mathbf{q}}, t, \mu) - U(\mathbf{q}, t, \nu),
\]

where \(\omega = (\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m), \mu_i \neq \nu_j\), defines the parameters that appear in the description of kinetic and potential energies, such as mass, charge, elasticity constant, etc.

Here, we are interested in the case where the parameters appear linearly in the Lagrangian description, which can be characterized by Lagrangians functions that satisfy:

\[
\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t, \alpha \omega) = \alpha \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t, \omega), \forall \alpha \in \mathbb{R}.
\]

Thus, the Lagrange equations associated to \(\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t, \omega)\) and \(\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t, \alpha \omega)\) are identical, so an analysis of the properties that arise from a variation in the parameters can be used to determine how the trajectories vary within the interval of measurement uncertainty of these same parameters.

3. Characterization of the sensitive dependence on parameters

3.1. Homogeneity of Lyapunov exponents

Given a differential equation:

\[
\dot{x} = f(x, t, \phi),
\]

where \(x \in \mathbb{R}^N\) is the state vector and \(\phi \in \mathbb{R}^k\) characterizes the parameters of the equation of motion, it is possible to define over space \(E : x_1 \times \ldots \times x_N\) a number \(\lambda\) which evaluates the magnitude of the divergence of two initially close trajectories. The value \(\lambda\) is defined so that, given two trajectories \(x(t, \phi)\) and
\( y(t, \phi) = x(t, \phi) + \delta(t, \phi) \), the difference \( \delta(t, \phi) \) evolves exponentially from the initial state, and \( \| \delta(t, \phi) \| = \| \delta(t_0, \phi) \| e^{\lambda t} \). Thus:

\[
\lambda(\phi) = \lim_{t \to \infty} \lim_{\| \delta(t, \phi) \| \to 0} \frac{1}{t} \ln \left( \frac{\| \delta(t, \phi) \|}{\| \delta(t_0, \phi) \|} \right),
\]

defines the Lyapunov exponent associated with the system, for the set of parameters \( \phi \).

In general, studying the chaotic behavior of dynamical systems involves analyzing how the equilibrium points of (3) depend on the parameter set \( \phi \), which characterizes the Bifurcation theory, and then evaluating how the divergence of the system changes at bifurcation points via Lyapunov exponents. However, since \( \phi \) characterizes a set of parameters that in physical systems depend on an experimental measurement to be defined, we can imagine that if any time series is affected by a variation in the parameters so that this dependence is significant over the interval of experimental measurement of the parameters, then the system can be sensitive to experimental uncertainty in the parameters, regardless of the initial conditions.

Indeed, note that if the analyzed system satisfies the equation (2), then:

\[
\frac{\partial L(q, \dot{q}, t, \alpha \omega)}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L(q, \dot{q}, t, \alpha \omega)}{\partial \dot{q}_i} \right) = \alpha \left[ \frac{\partial L(q, \dot{q}, t, \omega)}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L(q, \dot{q}, t, \omega)}{\partial \dot{q}_i} \right) \right],
\]

that is, the equations of motion are invariant by a proportionality relation between the parameters, and as \( \mathcal{E}(q, \dot{q}, t, \omega) = 0 \) implies \( \mathcal{E}(q, \dot{q}, t, \alpha \omega) = 0 \), then the state vectors \( (q(t, \alpha \omega), \dot{q}(t, \alpha \omega)) \) and \( (q(t, \omega), \dot{q}(t, \omega)) \) are identical. Therefore, given two initial conditions \( Q_1(t_0, \omega) = (q(t_0, \omega), \dot{q}(t_0, \omega)) \) and \( Q_2(t_0, \omega) = Q_1(t_0, \omega) + \delta(t_0, \omega) \), equation (4) implies:

\[
\lambda(\omega) = \lim_{t \to \infty} \lim_{\| \delta(t, \omega) \| \to 0} \frac{1}{t} \ln \left( \frac{\| \delta(t, \omega) \|}{\| \delta(t_0, \omega) \|} \right) = \lambda(\alpha \omega).
\]

This analysis shows that \( \lambda(\omega) \) is a homogeneous function of zero degree, for which Euler’s Theorem for homogeneous functions has the form:

\[
\sum_{i=1}^{n} \frac{\partial \lambda}{\partial q_i} \mu_i + \sum_{j=1}^{m} \frac{\partial \lambda}{\partial \nu_j} \nu_j = 0.
\]

Note that finding homogeneous functions that satisfy equation (5) allows us to evaluate how the Lyapunov exponents should depend on each of the parameters.

### 3.2. Power laws

A natural consequence of the homogeneity of the Lyapunov exponents is that power laws associated with the parameters should appear naturally in many physical systems, and in particular in systems satisfying equation (2). A linear combination of separable solutions to the equation (5) can be written as:
\[ \lambda = \sum_k C_k \left( \mu_{a_1}^{a_{1k}} \cdots \mu_{a_n}^{a_{nk}} \cdot \nu_{b_1}^{b_{1k}} \cdots \nu_{b_m}^{b_{mk}} \right). \] (6)

and it is possible to observe this type of solution in numerical works such as Safitri et al. (2020) [11], who evidenced a dependence of Lyapunov exponents with the mass ratio of a double pendulum \( \lambda \sim \frac{m_2}{m_1} \), or Delis et al. (2015) [9], who explored the dependence of Lyapunov exponents with a mass parameter \( \lambda \sim m^p \) in a galactic potential. The latter also searched for physical origins for the power law associated with the mass parameter, however, according to the equation (6), it is possible to verify that this type of dependence appears simply due to the linearity of the parameters in the analyzed lagrangian function.

3.3. Variation of the trajectories in the uncertainty interval

The dependency of the function \( \lambda(\omega) \) with parameters that may be associated with an experimental uncertainty provides evidence that, given a measurement \( \omega \pm \Delta \omega \), the chaotic behavior of the system is affected in some way. The variation of the physical parameters of the equation of motion is independent of the initial positions and velocities, so if the system has its chaotic pattern significantly affected, given the interval of experimental uncertainty, then we can say that the system is sensitive to the uncertainty with the parameters are measured, establishing a sensitivity that is not directly related to the initial conditions of the system.

In fact, note that if \( \omega^\dagger \in [\omega^* - \Delta \omega^*, \omega^* + \Delta \omega^*] \), the Lyapunov exponent definition implies that:

\[
\| \delta(t, \omega^\dagger) \| = \| \delta(t_o, \omega^\dagger) \| e^{\lambda(\omega^\dagger)t} \tag{7}
\]
\[
\| \delta(t, \omega^*) \| = \| \delta(t_o, \omega^*) \| e^{\lambda(\omega^*)t} \tag{8}
\]

Thus, if the system starts from the same initial conditions for the two parameters considered, we have \( \| \delta(t_o, \omega^\dagger) \| = \| \delta(t_o, \omega^*) \| \) and, dividing the equation (7) by (8):

\[
\| \delta(t, \omega^\dagger) \| = \| \delta(t, \omega^*) \| e^{[\lambda(\omega^\dagger) - \lambda(\omega^*)]t}. \tag{9}
\]

Equation (9) implies that if the experimental uncertainty of the measurement of a parameter is sufficient to obtain different Lyapunov exponents, then the evolution of the trajectories associated with these parameters depart exponentially in phase space with the magnitude of the difference between the Lyapunov exponents considered:
Therefore, we can assign the difference between the Lyapunov exponents a quantity that determines the order of sensitivity, i.e., the maximum divergence that the evolution of a trajectory can assume, given an interval of uncertainty. In this sense, the order of sensitivity of a measurement \( \omega^* \pm \Delta \omega^* \) is determined inside the Ball:

\[
B(\omega^*, \Delta \omega^*) = \{ \omega : \| \omega - \omega^* \| \leq \Delta \omega^* \},
\]

and given by:

\[
\mathcal{O}(\omega^*, \Delta \omega^*) = \max \{ \Omega \} - \lambda(\omega^*),
\]

where \( \Omega = \{ \lambda(\omega) : \omega \in B(\omega^*, \Delta \omega^*) \} \). This definition implies that for any measure \( \omega \) around the center that is in the uncertainty interval satisfies:

\[
\| \delta(t, \omega) \| \leq \| \delta(t, \omega^*) \| e^{\mathcal{O}(\omega^*, \Delta \omega^*) t},
\]

where equality is obtained by evaluating the trajectory that produces the largest possible divergence in the uncertainty interval.

Hence, parameter sensitivity depends on a beating effect between Lyapunov exponents calculated in the proximities of a parameter and different relationships between dynamics and parameter sensitivity can arise (Table 1):

| \( \lambda(\omega) \) | \( \mathcal{O}(\omega^*, \Delta \omega^*) \) | Dynamics | Parameter variation |
|------------------------|-------------------------------|----------|---------------------|
| <0                     | -                             | Dissipative | Non-sensitive       |
| 0                      | -                             | Conservative | Non-sensitive       |
| >0                     | 0                             | Chaotic    | Non-sensitive       |
| >0                     | >0                            | Chaotic    | Sensitive           |

Thus, conservative or dissipative systems will present a dynamic that prevents parameter sensitivity, since the distances in phase space approach or remain constant without the possibility of a divergence process,
as presented in Figure [1], arising in these cases. For the chaotic behavior, sensitivity to parameters may not appear in cases where the Lyapunov exponents do not vary with a change in parameters, such as in a simple nonlinear pendulum when the parameter evaluated is mass\textsuperscript{2}. Finally, if the system exhibits chaotic behavior and variation in Lyapunov exponents on a set of parameters, then the system is subject not only to sensitivity to initial conditions, but also to the divergence process associated with sensitivity to parameters.

3.4. Predictability in Time Series

As the experimental uncertainty of the parameters can produce divergences, even considering two systems starting from identical initial conditions, it is interesting to evaluate how fast is the influence of this phenomenon on the evolution of a time series, in order to analyze how long a series can be evolved without information about the true dynamics being lost.

Consider the maximum possible divergence given by the equation [11]:

\[
\|\delta(t, \omega)\| = \|\delta(t, \omega^*)\| e^{O(\omega^*, \Delta \omega^*) t}.
\] (12)

From a configuration of the distances in phase space, we can obtain the time interval associated with the evolution of the series by doing:

\[
\tau = \frac{1}{O(\omega^*, \Delta \omega^*)} \ln \left( \frac{\|\delta(\tau, \omega)\|}{\|\delta(\tau, \omega^*)\|} \right).
\] (13)

In this case, it is interesting to estimate the time required for the distances between the orbits associated with the parameter set \(\omega\) to move away from the trajectory associated with \(\omega^*\), such that:

\[
\|\delta(\tau, \omega) - \delta(\tau, \omega^*)\| = \|\sigma_\delta\|,
\] (14)

where \(\sigma_\delta\) is the uncertainty measure associated with the state variables around \(\delta(\tau, \omega^*)\), i.e:

\[
\sigma_\delta = (\sigma_{q_1}, \ldots, \sigma_{q_n}, \sigma_{\dot{q}_1}, \ldots, \sigma_{\dot{q}_n}).
\]

Equation [14] fixes \(\tau\) as the instant from which the distance associated the trajectories for the different parameters is greater than the uncertainty of a direct measurement of the state variables, that is, the instant from which an experimental measurement of the trajectory evolved from a set of parameters \(\omega^*\) does not bound the trajectory evolved from \(\omega\).

Note, however, that taking the equation [14] from the order of sensitivity actually creates a limiting for \(\tau\), but does not explicitly determine its value. Since \(O(\omega^*, \Delta \omega^*)\) is the largest possible exponent within the parameter uncertainty interval, then the time interval required to satisfy the equation [14] will always be greater than or equal to \(\tau\). In this way, it can be taken as a predictability criterion in a time series associated with an uncertainty \(\sigma_\delta\), where we can obtain, with certainty, the series up to a time interval \(\tau\) when evolved

\textsuperscript{2}Since the equations of motion do not depend on mass, its variation does not affect the chaotic pattern of the system.
from a parameter set \( \omega \in \omega^* \pm \Delta \omega^* \). After this interval, the predictability of the series will be affected by the divergence of the trajectories, being maximum for the parameters that define the order of sensitivity.

To make \( \tau \) explicit in equation (13), we can use the triangle inequality in equation (14) to write:

\[
\left\| \delta(\tau, \omega) \right\| \leq \left\| \sigma \delta \right\| + \left\| \delta(\tau, \omega^*) \right\|.
\]

Replace (15) in (13):

\[
\tau \leq \frac{1}{\mathcal{O}(\omega^*, \Delta \omega^*)} \ln \left( \frac{\left\| \sigma \delta \right\| + \left\| \delta(\tau, \omega^*) \right\|}{\left\| \delta(\tau, \omega^*) \right\|} \right)
\]

and it follows, from \( \mathcal{O}(\omega^*, \Delta \omega^*) \) and the fact that \( \ln(x) \leq x, x \geq 0 \):

\[
\tau \leq \frac{\left\| \sigma \delta \right\|}{\mathcal{O}(\omega^*, \Delta \omega^*)} \left\| \delta(\tau, \omega^*) \right\| e^{\lambda(\omega^*) \tau}.
\]

Thus, we have:

\[
\tau e^{\lambda(\omega^*) \tau} \leq \frac{\left\| \sigma \delta \right\|}{\mathcal{O}(\omega^*, \Delta \omega^*)} \left\| \delta(t_0, \omega^*) \right\|
\]

which implies:

\[
\tau \leq \frac{W_0(\chi)}{\lambda(\omega^*)} \to \tau_{max} = \frac{W_0(\chi)}{\lambda(\omega^*)},
\]

where \( W_0 \) is the Lambert \( W_0 \) function and \( \chi = \frac{\left\| \sigma \delta \right\|}{\mathcal{O}(\omega^*, \Delta \omega^*)} \left\| \delta(t_0, \omega^*) \right\|. \) This analysis shows that the order of sensitivity is so important to characterizing chaotic behavior and determining the predictability of the time series as the Lyapunov exponents, because even though the divergence caused by an uncertainty in the parameters is slower than that caused by a variation in initial conditions, both lead to a divergence process when \( t \to \infty \).

4. Applications

4.1. Henon-Heiles system

The first chaotic system analyzed to see how sensitivity in parameters can be studied from the variation of Lyapunov exponents is the Henon-Heiles system, described by Lagrangian:

\[
\mathcal{L} = \frac{1}{2} M \left( \dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} \left( \omega_1 x^2 + \omega_2 y^2 \right) + axy^2 - \frac{1}{3} b y^3,
\]

where \( M \) is the mass of an object subject to a Galactic potential characterized by the parameters \( \omega_1, \omega_2, a \) and \( b \). The Lagrangian satisfies the equation (2), so it can be assumed that the dependence with the parameter \( b \), which characterizes the cubic term of the potential, is of the form:

\[
\lambda(b) = Cb^k
\]

(17)
where $C$ and $k$ are constants, according to equation (6). This behavior can be seen numerically by considering a interval $b \pm 0.001$ with initial conditions $x(0) = 0.2$, $v_x(0) = 0$, $y(0) = -0.3$ and $v_y(0) = -0.1$, in a time interval $\Delta t = 500$ with step 0.005, where the other parameters are fixed at $M = \omega_1 = \omega_2 = a = 1$. In this case, Figure 2(a) shows the evolution of the central trajectory ($b = 1$) associated with a perturbed orbit at 1% under initial conditions and Figure 2(b) the variation of the Lyapunov exponents with parameter $b$.

From integrating the equations of motion, it is possible to fit the data obtained into a power law so that the equation (17) takes the form:

$$\lambda (b) = 0.0132b^{0.7137}$$  \hspace{1cm} (18)

which reproduces the obtained Lyapunov exponents with a precision of the order of $10^{-6}$. Furthermore, simulations in which accuracy is a determining factor for the analysis of the evolution of orbits can only be obtained for a short time interval, so that optimal accuracies $\|\sigma\| << 0.1$ cannot be obtained by considering the uncertainty interval of $b$ at the same time that the simulations are extended enough to observe the qualitative behavior of the system, as shown in Figure 3(a). Note that since the function (18) is increasing, then the order of sensitivity is determined by considering the difference $\mathcal{O}(1, 0.001) = \lambda(1.001) - \lambda(1) \sim 10^{-6}$.

In Figure 3(b) it is possible to visualize how, since the system is chaotic and presents a positive order of sensitivity, the variation in the parameters implies a divergence phenomenon between the trajectories quite similar, although slower, to the one caused by variations in the initial conditions:
We can also consider varying other parameters to see how these properties change. As the Henon-Heiles system is related to the description of the motion of galaxies interacting by gravitational force, it is convenient to evaluate how the Lyapunov exponents depend on the mass parameter. So, consider $M$ in the interval $M = 1 \pm 0.001$, with $b$ fixed at $b = 1$, for the same initial conditions as in the previous IVP. In this case, Figure 4(a) shows how the predictability criterion for varying the parameter $M$ allows one to obtain accurate trajectories for a longer period of time, when compared to Figure 3(a). Moreover, for mass, the Lyapunov exponents decrease as $M$ increases, having the opposite behavior than the one verified for the parameter $b$, as seen in Figure 4(b).

Again, we can associate the variation of the exponents to a power law, which for the parameter $M$ has the form:

$$\lambda(M) = 0.132 M^{-0.331}$$
with similar precision as the previous case $\sim 10^{-6}$. It is easy to see, therefore, that uncertainties associated with the mass of the system produce smoother divergences than uncertainties associated with the parameter $b$, which implies closer trajectories for longer periods of time, as it is possible to verify by comparing Figure 3(b) with Figure 5:

![Figure 5: Trajectory for different values of M.](image)

Note that using the order of sensitivity associated with the predictability criterion we can obtain very practical results, considering a parameter of interest, in terms of consistency of the simulations and the phenomenology attributed to the sensitivity on parameters. In other words, while Lyapunov exponents measure the magnitude of the distance between two trajectories in phase space given a variation in initial conditions, the order of sensitivity determines how distances in phase space vary, relative to each other, given a variation in a parameter. This implies that equation (16) provides a way to simultaneously visualize how sensitivity to initial conditions and sensitivity dependence on parameters influence the dynamics of the system, requiring only to find out how the Lyapunov exponents vary with the parameters.

4.2. Double Pendulum

The next system with interesting properties exhibited from this approach is the Double Pendulum, whose Lagrangian function is given by:

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2(l_2\dot{\varphi}_2)^2 + m_2l_1l_2\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + (m_1 + m_2)gl_1 \cos(\varphi_1) + m_2gl_2 \cos(\varphi_2)$$

In this case, the Lagrangian function does not satisfy the equation (2) for the pendulum lengths, so that the behavior produced by variations in $l_1$ and $l_2$ must be more complex than the power laws exemplified earlier for the Henon-Heiles system. However, for the masses, linearity is satisfied so that the equation (5) takes the form:
\[
\frac{\partial \lambda}{\partial m_1} m_1 + \frac{\partial \lambda}{\partial m_2} m_2 = 0
\]
whose solution is given by any function \( \lambda : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \lambda = \lambda \left( \frac{m_1}{m_2} \right) \). Therefore, we can evaluate how the chaotic behavior of the system varies, in both variables, defining \( \epsilon = \left( \frac{m_1}{m_2} \right) \) and considering the initial conditions \( \varphi_1(0) = \frac{3}{4} \varphi_2(0) = \frac{\pi}{2} \) and \( \dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0 \) for \( l_1 = l_2 = 1 \) and \( m_1 = m_2 = 1 \pm 0.05 \). Figure 6(a) shows the evolution of the predictability criterion and Figure 6(b) the different Lyapunov exponents for the range of variation of the masses:

\[\text{(a) Function } \tau_{\text{max}}(\|\sigma\|) \text{ for the IVP.} \]
\[\text{(b) Lyapunov exponents associated with the IVP, for the parameters } m_1 \text{ and } m_2.\]

Due to the nuances associated with the variation of the Lyapunov exponents, it is reasonable to assume that the power law associated with the \( \epsilon \) dependence is more complicated with respect to the previous case for the Henon-Heiles system. Since the Lyapunov exponents were obtained by subdividing the intervals \( m_1 \pm 0.05 \) and \( m_2 \pm 0.05 \) into 50 parts, it is possible to assume that the function \( \lambda(\epsilon) \) has the form:

\[\lambda(\epsilon) = \sum_{i=1}^{6} C_i \epsilon^{a_i},\]
so that Least Squares Method (LSM) fitting of the data results in Figure 7.
This implies that the order of sensitivity is given by $\mathcal{O}(1, 0.05) = \lambda (\frac{0.95}{100}) - \lambda (1) = 0.0155$, and indicates a significantly higher sensitivity in the parameters when compared to the results obtained for the Henon-Heiles system. Obviously, this difference depends essentially on the initial conditions and the dynamics of the systems, but in both cases it is possible to see how the parameters directly affect the evolution of the system and produce divergence processes very similar to those caused by variations in the initial conditions. For the case of the Double Pendulum, this divergence process quickly affects the system, as shown in Figure 8, and makes the uncertainty interval associated with the masses a determining factor for the unpredictability of the system:

These results are particularly interesting because the problem of estimating the parameters of dynam-
ical systems is recurrent in the study of nonlinear phenomena. Several works have developed and tested independent methods for parameter estimation, which may involve nonlinear regressions, objective function minimization, or probabilistic approaches [12, 13, 14, 15, 16]. However, since these methods are committed only to approximating the parameters from an observational data set or time series simulations, parameter estimation can be only a necessary but not sufficient condition for adequate prediction, regardless of the method chosen. Indeed, it may be the case that any uncertainty $\Delta \omega^*$ is sufficient to contain a variation in the Lyapunov exponents, so that the order of sensitivity is positive and, given the chaotic behavior of the system, a noise effect appears as a consequence of the divergence process presented in section 3.3. In this sense, a suitable formulation for the problem of estimating the parameters of a chaotic system can be made by taking as a starting point not only the minimization of the error associated with time series evolution, but also the minimization of the difference between the Lyapunov exponents in the neighborhood of the convergence region of the estimation method, so that it is guaranteed that $\tau_{\text{max}}$ assumes a value at least as large as the analyzed time series, for uncertainties $\sigma$ as small as those associated with the observational data.

5. Concluding Remarks

In this paper, we explore the sensitive dependence on parameters of chaotic systems. We show that a linearity condition on Lagrangian functions of chaotic systems can lead to power laws associated with dependence of the Lyapunov exponents with the parameters.

The variation of Lyapunov exponents with parameters allows us to characterize sensitivity in parameters as an effect arising from the distinct evolution of distances in phase space associated with distinct parameters. We use this result to estimate the time required for the divergence effect caused by variation in the parameters to separate two initially identical orbits, allowing us to directly visualize the predictability of the system.

Finally, we present two chaotic systems for which both the linearity properties of the Lagrangian function and the sensitivity effects on the parameters significantly affect the evolution of the trajectories. In addition, we discuss how the approach presented here can be used to assist parameter identification methods from time series by defining objective functions that include minimizing the variance of the Lyapunov exponents in the convergence neighborhood of the estimated parameters.

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