Israel–Wilson–Perjés Solutions in Heterotic String Theory

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Abstract
We present a simple algorithm to obtain solutions that generalize the Israel–Wilson–Perjés class for the low-energy limit of heterotic string theory toroidally compactified from $D = d + 3$ to three dimensions. A remarkable map existing between the Einstein–Maxwell (EM) theory and the theory under consideration allows us to solve directly the equations of motion making use of the matrix Ernst potentials connected with the coset matrix of heterotic string theory [1]. For the particular case $d = 1$ (if we put $n = 6$, the resulting theory can be considered as the bosonic part of the action of $D = 4, N = 4$ supergravity) we obtain explicitly a dyonic solution in terms of one real $2 \times 2$–matrix harmonic function and $2n$ real constants ($n$ being the number of Abelian vector fields). By studying the asymptotic behaviour of the field configurations we define the charges of the system. They satisfy the Bogomol’nyi–Prasad–Sommerfeld (BPS) bound.
1 Introduction

Recently much work has been devoted to the construction of stationary solutions for the low energy limit of heterotic string theory (or to equivalent extended supergravity models) \[2\]–\[12\]. These solutions were obtained by using duality symmetries, by taking as ansatz harmonic functions or by directly solving the associated Killing spinor equations in the supersymmetric case. In this paper we obtain a class of extremal stationary solutions by directly solving the equations of motion of the theory. In order to characterize these classes of solutions can be used two approaches: the criteria of unbroken supersymmetries \[12\] or the criteria of saturation of the BPS bound \[13\]–\[14\]. Classical solutions that saturate this bound are stable and usually associated with solitons. In the framework of heterotic string theory, the BPS bound was studied in \[15\]–\[17\]. The importance of BPS bounds in extended supergravities was pointed out by Gibbons in \[14\]; it is related to the existence of unbroken supersymmetry \[18\]. Since quantum corrections to supersymmetric backgrounds are controllable, the BPS solutions constitute an important tool to investigate the underlying quantum theory from a non–perturbative point of view \[19\].

The study of duality symmetries of compactified string theories can yield non–perturbative information about the full string theory (for a review see for example \[20\] and references therein). These symmetries relate different backgrounds which define essentially the same quantum conformal field theories.

In this work we will make use of another kind of “duality” between the effective action of the low–energy limit of heterotic string theory toroidally compactified to three dimensions and the stationary EM theory (below we will present the map existing between both theories). Actually, this is not a duality in the usual sense of the word, but the similarity of the group structure of both theories enables us to express the effective action of the heterotic string, as well as the equations of motion, in an EM form. We see that this fact also allows us to obtain exact solutions that generalize the solutions for the EM theory. On the other hand, the effective action of heterotic string theory can be considered as a generalization of the EM one (if we set to zero the dilaton and axion fields ‘in the action’, we get the action for the EM theory). Thus, the study of the hidden symmetries that arise from dimensional reduction of EM theory provides many information about the rich internal symmetries of heterotic string theory. A further reduction to two dimensions leads to completely integrable theories (see for instance \[21\]–\[22\]), which entails infinite dimensional symmetries (the Geroch group comes into play).

This letter is organized as follows: in Sec. 2 we summarize the three–dimensional effective action of heterotic string compactified on a seven torus. In Sec. 3 we express the action and the equations of motion of the theory in terms of a pair of matrix Ernst potentials. In Sec.
we indicate the procedure to obtain stationary extremal solutions that generalize the IWP class for the effective action of the low–energy limit of heterotic string theory. In Sec. 5 we obtain explicitly a BPS saturated dyonic solution for the simplest case of the formalism: $d = 1$. There we define as well the set of charges of the system and show that they satisfy the BPS bound. In Sec. 6 we summarize our results and discuss on their implications.

2 Compactification to Three Dimensions

Our starting point is the effective field theory of heterotic string in $D$ dimensions. The action of this theory describes gravity coupled to matter fields [23]–[24]:

$$S^{(D)} = \int d^{(D)}x \left| G^{(D)} \right|^\frac{1}{2} e^{-\phi^{(D)}} (R^{(D)} + \phi^{(D)}_{;M} \phi^{(D)};M - \frac{1}{12} H^{(D)}_{MNP} H^{(D)MNP} - \frac{1}{4} F^{(D)I}_{MN} F^{(D)IMN} \right),\tag{1}$$

where

$$F^{(D)I}_{MN} = \partial_M A^{(D)I}_N - \partial_N A^{(D)I}_M,$$

$$H^{(D)}_{MNP} = \partial_M B^{(D)NP}_N - \frac{1}{2} A^{(D)I}_M F^{(D)I}_{NP} + \text{cycl. perms. of } M,N,P.$$

Here $G^{(D)}_{MN}$ is the $D$-dimensional metric, $B^{(D)}_{MN}$ is the anti–symmetric Kalb-Ramond field, $\phi^{(D)}$ is the dilaton and $A^{(D)I}_M$ denotes a set ($I = 1, 2, ..., n$) of $U(1)$ gauge fields. At a generic point of the moduli space, vector fields should form an Abelian multiplet of dimension $d + n$, where $n$ is the number of initial vector fields and $d$ is the number of compactified dimensions (in order to get a self–consistent theory we must put $D = 10$ and $n = 16$ [24], although we shall leave these parameters arbitrary for the sake of generality).

In [23]-[24] it was shown that after the Kaluza-Klein compactification of $d = D - 3$ dimensions on a torus, the resulting theory is

$$S^{(3)} = \int d^3x \left| g \right|^\frac{1}{2} \left[ R + \phi_{;\mu} \phi^{;\mu} - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - e^{-2\phi} F^{T}_{\mu\nu} M^{-1} F^{\mu\nu} - \frac{1}{8} \text{Tr} \left( J^M \right)^2 \right].\tag{2}$$

Here the symmetric matrix $M$ has the following structure

$$M = \begin{pmatrix}
G^{-1} & G^{-1}(B + C) & G^{-1}A \\
(-B + C)G^{-1} & (G - B + C)G^{-1}(G + B + C) & (G - B + C)G^{-1}A \\
A^T G^{-1} & A^T G^{-1}(G + B + C) & I_n + A^T G^{-1}A
\end{pmatrix}\tag{3}$$
with block elements defined by
\[ G = (G_{pq} \equiv G^{(D)}_{p+2,q+2}), \quad B = (B_{pq} \equiv B^{(D)}_{p+2,q+2}), \quad A = (A^I_p \equiv A^{(D)}_{p+2}), \quad C = \frac{1}{2} AA^T \] and \( p, q = 1, 2, ..., d \). Matrix \( M \) satisfies the \( O(d, d+n) \) group relation
\[ M^T L M = L, \quad \text{where} \quad L = \begin{pmatrix} O & I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & -I_n \end{pmatrix}; \quad (5) \]
thus \( M \in O(d, d+n)/O(d) \times O(d+n) \).

The remaining 3-fields are defined in the following way: for dilaton and metric fields one has
\[ \phi = \phi^{(D)} - \frac{1}{2} \ln |\det G|, \quad g_{\mu\nu} = e^{-2\phi} \left( G^{(D)}_{\mu\nu} - G^{(D)}_{p+2,\mu} G^{(D)}_{q+2,\nu} G_{pq} \right). \]

Then, the set of Maxwell strengths \( F^{(a)}_{\mu\nu} \) \((a = 1, 2, ..., 2d+n)\) is constructed on the basis of \( A^{(a)}_{\mu} \), where
\[ A^p_{\mu} = \frac{1}{2} G^{pq} G^{(D)}_{q+2,\mu} A_{\mu+2d} = -\frac{1}{2} A^{(D)}_{p+2,\mu} + A^q_{\mu} A^p_{\mu}, \]
\[ A^{p+d}_{\mu} = \frac{1}{2} B^{(D)}_{p+2,\mu} - B_{pq} A^q_{\mu} + \frac{1}{2} A^I_{p} A^{I+2d}_{\mu}. \]

Finally, the 3-dimensional axion
\[ H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + 2 A^a_{\mu} L_{ab} F^b_{\nu\rho} + \text{cycl. perms. of } \mu, \nu, \rho \]
depends on the 3-dimensional Kalb-Ramond field
\[ B_{\mu\nu} = B^{(D)}_{\mu\nu} - 4 B_{pq} A^p_{\mu} A^q_{\nu} - 2 \left( A^p_{\mu} A^{p+d}_{\mu} - A^p_{\mu} A^{p+d}_{\nu} \right). \]

In three dimensions this system can be simplified because the Kalb-Ramond field \( B_{\mu\nu} \) has no physical degrees of freedom. Moreover, the fields \( A^a_{\mu} \) are dualized on-shell as follows
\[ e^{-2\phi} M L F_{\mu\nu} = \frac{1}{2} E_{\mu\nu\rho} \nabla^\rho \psi; \quad (6) \]
so, the final system is defined by the quantities \( M, \phi \) and \( \psi \). As it had been established by Sen in [24], it is possible to introduce the matrix \( \mathcal{M}_S \) in terms of which the action of the system adopts the standard chiral form
\[ S^{(3)} = \int d^3 x \ | g |^{-\frac{1}{2}} \left[ R - \frac{1}{8} Tr \left( J^{\mathcal{M}_S} \right)^2 \right], \quad (7) \]
where \( J^{M_S} = \nabla M_S M_S^{-1} \). This matrix is symmetric \( M_S = M_S^T \) and satisfies the \( O(d + 1, d + n + 1) \)-group relation

\[
M_S L_S M_S = L_S \quad \text{with} \quad L_S = \begin{pmatrix} L & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

so that \( M_S \) belongs to the coset \( O(d + 1, d + n + 1)/O(d + 1) \times O(d + n + 1) \).

In [1] it was constructed another chiral matrix \( M \in O(d + 1, d + n + 1)/O(d + 1) \times O(d + n + 1) \) that possesses the same structure that \( M \) with block components \( G, B \) and \( A \) of dimensions \( (d + 1) \times (d + 1), (d + 1) \times (d + 1) \) and \( (d + 1) \times n \), respectively. In order to obtain this matrix it was necessary to express the column \( \psi \) in the form

\[
L \psi = \begin{pmatrix} u \\ v \\ s \end{pmatrix},
\]

where \( u \) and \( v \) are columns of dimension \( d \), whereas the dimension of the column \( s \) is \( n \). Then, the simple formulae

\[
G = \begin{pmatrix} -e^{-2\phi} + v^T G & v^T G \\ G v & G \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -w^T \\ w & B \end{pmatrix}, \quad A = \begin{pmatrix} s^T + v^T A \\ A \end{pmatrix},
\]

where \( w = u + B v + \frac{1}{2} A s \), actually solved the task. In terms of these three matrices the matter part of the action reads

\[
S^{(3)}_{\text{matter}} = -\frac{1}{8} \int d^3 x | g |^\frac{1}{2} Tr \left\{ \frac{1}{4} \left[ (J^G)^2 - (J^B)^2 \right] + \frac{1}{2} \nabla A^T G^{-1} \nabla A \right\},
\]

where \( J^G = \nabla G G^{-1} \) and \( J^B = \left( \nabla B + \frac{1}{2} (A \nabla A^T - \nabla A A^T) \right) G^{-1} \), which exactly corresponds to the chiral one

\[
S^{(3)}[M] = -\frac{1}{8} \int d^3 x | g |^\frac{1}{2} Tr \left( J^M \right)^2
\]

where the matrix \( M \) is defined by the block components \( G, B \) and \( A \) in the same way that the matrix \( M \) is defined by \( G, B \) and \( A \):

\[
M = \begin{pmatrix} G^{-1} (B + C) \\ (-B + C) G^{-1} \\ A^T G^{-1} \end{pmatrix}, \quad \begin{pmatrix} G^{-1} \quad (G - B + C) G^{-1} (G + B + C) \\ G^{-1} A \quad (G - B + C) G^{-1} A \\ I_n + A^T G^{-1} A \end{pmatrix}.
\]
This matrix is symmetric and satisfies the $O(d + 1, d + n + 1)$-group relation
\[
\mathcal{MLM} = \mathcal{L}, \quad \text{where} \quad \mathcal{L} = \begin{pmatrix}
0 & I_{d+1} & 0 \\
I_{d+1} & 0 & 0 \\
0 & 0 & -I_n
\end{pmatrix},
\tag{14}
\]
so it belongs to the coset $O(d + 1, d + n + 1)/O(d + 1) \times O(d + n + 1)$.

3 Matrix Ernst Potentials

At this stage one can introduce the matrix potential \[25\]
\[
\mathcal{X} = \mathcal{G} + \mathcal{B} + \frac{1}{2} \mathcal{A} \mathcal{A}^T = \begin{pmatrix}
-e^{-2\phi} + v^T X v + v^T A s + \frac{1}{2} s^T s & v^T X - u^T \\
X v + u + A s
\end{pmatrix},
\tag{15}
\]
where the $d \times d$ matrix potential $X = G + B + \frac{1}{2} A A^T$ was defined for the first time by Maharana and Schwarz in the case when $A = 0$ \[23\]. Thus, the pair of potentials $\mathcal{X}$ and $\mathcal{A}$ allows us to express the 3-dimensional action as follows
\[
S^{(3)} = \int d^3 x \left| g \right|^{\frac{1}{2}} \left\{ -R + \text{Tr} \left[ \frac{1}{4} \left( \nabla \mathcal{X} - \nabla \mathcal{A} \mathcal{A}^T \right) \mathcal{G}^{-1} \left( \nabla \mathcal{X}^T - \mathcal{A} \nabla \mathcal{A}^T \right) \mathcal{G}^{-1} + \frac{1}{2} \nabla \mathcal{A}^T \mathcal{G}^{-1} \nabla \mathcal{A} \right] \right\},
\tag{16}
\]
where $\mathcal{G} = \frac{1}{2} \left( \mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T \right)$. Its form is very similar to the stationary Einstein–Maxwell system (see for instance \[21\] and \[26\]). Actually the map
\[
\Phi \rightarrow i \mathcal{A}/\sqrt{2}, \quad \mathcal{E} \rightarrow \mathcal{X}
\tag{17}
\]
establishes a direct relation between EM theory and string gravity, i.e., the matrix $\mathcal{X}$ formally plays the role of the gravitational Ernst potential $\mathcal{E}$, whereas the matrix $\mathcal{A}$ corresponds to the electromagnetic potential $\Phi$ of EM theory \[27\]. At the same time, one can notice a direct correspondence between the transposition of $\mathcal{X}$ and $\mathcal{A}$ on the one hand, and the complex conjugation of $\mathcal{E}$ and $\Phi$, on the other. This analogy was useful to study the symmetry group of string gravity in \[1\].

The action (16) leads to the following equations of motion
\[
\nabla^2 \mathcal{X} - 2(\nabla \mathcal{X} - \nabla \mathcal{A} \mathcal{A}^T)(\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T)^{-1} \nabla \mathcal{X} = 0,
\nabla^2 \mathcal{A} - 2(\nabla \mathcal{X} - \nabla \mathcal{A} \mathcal{A}^T)(\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T)^{-1} \nabla \mathcal{A} = 0.
\tag{18}
\]
It is worth noticing that these equations are quite symmetric each other as in the case of EM theory.
4 Generalized Israel–Wilson–Perjés Solutions

In this Sec. we point out a simple algorithm to obtain real stationary extremal (with a flat background metric) solutions for the equations of motion (18) in terms of a real constant matrix of dimension \(d + 1\) and a set of \(2n\) real constants \([28]\). As in EM theory, we shall consider that there exists a linear dependence between \(\mathcal{A}\) and \(\mathcal{X}\). Indeed, if one requires the matrix Ernst potentials to satisfy the asymptotic flatness conditions \(\mathcal{X}_\infty \to \Sigma\), where \(\Sigma = \text{diag}(-1, -1, 1, 1, ..., 1)\), and \(\mathcal{A}_\infty \to 0\), the “electromagnetic” potential can be expressed as follows

\[
\mathcal{A} = (\Sigma - \mathcal{X})b\mathcal{H},
\]

where \(b\) is an arbitrary constant \((d + 1) \times n\)-matrix and the matrix \(\mathcal{H} \in O(n)\) plays the role of a normalization factor which does not affect the geometry of the theory since \(\mathcal{H}\mathcal{H}^T = I_n\), we shall omit this factor in what follows. By performing the replacement (19) in the action (16) and setting the Lagrangian of the system to zero (it implies that \(R_{ij} = 0\)), we get the following condition to be satisfied

\[
bb^T = -\Sigma/2;
\]

it means that \(b_{\tilde{p}n}b_{n\tilde{q}} = \delta_{\tilde{p}\tilde{q}}/2\) if \(\tilde{p}, \tilde{q} = 1, 2\); and \(b_{\tilde{p}n}b_{n\tilde{q}} = -\delta_{\tilde{p}\tilde{q}}/2\) if \(\tilde{p}, \tilde{q} = 3, 4, ..., d + 1\). First of all we can conclude that equation (20) is solvable if and only if \(n \geq d + 1\), because for \(n < d + 1\), the equation system that arises from it (for the components \(b_n\)) turns out to be incompatible.

By substituting relation (19) in the equations of motion (18), both of them reduce to the Laplace equation in Euclidean 3-space

\[
\nabla^2[(\Sigma + \mathcal{X})^{-1}] = 0
\]

which can be directly solved. For instance, we can immediately write down a simple solution in terms of the harmonic function \(^{[1]}\)

\[
\frac{2}{\Sigma + \mathcal{X}} = \Sigma + \frac{\bar{M}}{r},
\]

\(^{[1]}\)This solution correspond to a single object and can be simply generalized to a multicenter solution by the following relation \(\frac{2}{\Sigma + \mathcal{X}} = \Sigma + \sum_j \frac{\bar{M}_j}{r_j}\), where \(r_j = \sqrt{x_j^2 + y_j^2 + (z_j - i\alpha_j)^2}\) and the subscript \(j\) labels an arbitrary number of sources.
where $\tilde{M}$ is a real $(d + 1)$-dimensional arbitrary constant matrix, $r = \sqrt{x^2 + y^2 + (z - i\alpha)^2}$ and $\alpha$ is a real constant. We choose $r$ in this way in order to deal with rotating black hole solutions [3]–[4] (in this case we have a ring singularity). This is because general IWP solutions include both NUT charge and angular momentum. We see that solution (22) is complex in general. On the other hand, all variables entering the action (16) are real. In this paper we shall restrict ourselves to a real class of solutions, leaving the study of the complex solutions for the future. First of all we notice that we could take into account the real or imaginary part of the harmonic function (22); we shall put $1/R = \text{Re}1/r$; secondly, we must require the constants $\tilde{M}_{\tilde{p}\tilde{q}}$ and $b_{\tilde{p}n}$ to be real. However, because of the indefinite character of $\Sigma$, matrix $b$ is complex in general as well. We can proceed as follows: Let us require just the first two rows of $b$ to be real (leaving the remaining components imaginary), then we perform the matrix product (19) and set the factors that multiply the imaginary components of $b$ to zero. It turns out that this condition imposes some restrictions on the matrix $\tilde{M}$ (some of its components vanish) leading to real solutions for the potentials $A$ and $X$.

The procedure for obtaining the explicit form of the fields is the following: once a solution of the Laplace equation is written down (a solution of the form (22), for example), one solves the system of matrix equations that arise from it and (19) in order to express the 3-fields $G, B, A, \phi, u, v$ and $s$ in terms of $R$ and the real arbitrary constants $\tilde{M}_{\tilde{p}\tilde{q}}$ and $b_{\tilde{p}n}$. Then one constructs the scalar column $\psi$ and with the aid of the dualization formula (6) one calculates the components of the “true” gauge fields $A_{\mu}{}^{(a)}$. This enables us to explicitly write down the $D$-fields (see Sec. 2) as well as the $D$-dimensional interval.

5 BPS Saturated Dyonic Solution

In this Sec. we shall write down explicit solutions for the simplest case of the formalism: $d = 1$ (hence $n \geq 2$). If we put $n = 6$, the resulting theory can be considered as the bosonic part of the action of $D = 4, N = 4$ supergravity [12], where the axion field can be introduced on-shell as follows:

$$\partial_\mu a = \frac{1}{3} e^{-4\phi} E_{\mu\nu\lambda\sigma} H^{\nu\lambda\sigma}. \quad (23)$$

where $E_{\mu\nu\lambda\sigma} = \sqrt{-g} \epsilon_{\mu\nu\lambda\sigma}$. Thus, in this case $\tilde{M}$ becomes a $2 \times 2$-matrix with 4 real parameters $m_{kl}$, $k, l = 1, 2$; and $b$ contains $2n$ real constants constrained by the equality $b_{kn} b_{nk'} = \delta_{kk'}/2$. So, we have at our disposal $2 + 2n$ (taking into account the parameter $\alpha$) integration constants with $n \geq 2$ in the generic case. On the other hand, the physical
parameters of the theory are: The ADM mass, the NUT, dilaton, and axion charges, the angular momentum and two sets of \( n \) electric and magnetic charges. There exist a relation between the charges of the theory to be satisfied: the Bogomol’nyi bound. We see that the number of integration constants does not match the number of physical parameters of the theory. This is because the condition \( b_{kn}b_{nk'} = \delta_{kk'}/2 \) imposes 3 constrains on the set of electric and magnetic charges of the system. Thus, electric and magnetic charges are not independent each other. Bellow we shall study the dependence that exists between them.

In [2] general SWIP solutions were obtained using as ansatz two arbitrary harmonic functions. As we have said above, we suppose a linear dependence between the matrix Ernst potentials. This fact allowed us to reduce the equations of motion to a matrix Laplace equation and then to write down the solution (22).

Since both theories are given in two very different settings, it is not easy to make accurate comparisons of the obtained solutions. Another paper, where we choose the solution of the Laplace equation in terms of two complex harmonic functions, is in progress. In this case the settings of both theories are more similar and we shall try to compare their solutions in a simple way.

But let us keep in obtaining the solution, from (19) and (22) we have

\[
\mathcal{X} = 2Y - \Sigma = \begin{pmatrix} 2y_{11} + 1 & 2y_{12} \\ 2y_{21} & 2y_{22} + 1 \end{pmatrix} = \begin{pmatrix} -e^{-2\phi} + v^2X + vAs + \frac{1}{2}s^Ts & vX - u \\ Xv + u + As \end{pmatrix},
\]

\[
A = -2(Y + I_2)b = \begin{pmatrix} -2(y_{1k} + \delta_{1k}b_{kn}) \\ -2(y_{2k} + \delta_{2k}b_{kn}) \end{pmatrix} = \begin{pmatrix} s^T + vA \\ A \end{pmatrix};
\]

(24)

here we have introduced the matrix \( Y^{-1} = \left( \Sigma - \frac{\bar{M}}{R} \right) \) with \( det(Y^{-1}) = 1 - \frac{m_{11}m_{22} + det\bar{M}}{R^2} \neq 0 \).

We solve these equations and found the following expressions for the 3-fields

\[
G = -\frac{1 - \frac{2m_{11}}{R} + \frac{m_{11}^2}{R^2}}{\left(1 - \frac{m_{11}m_{22}}{R} + \frac{det\bar{M}}{R^2}\right)^2} = -\frac{R^2(R^2 - 2m_{11}R + m_{k1}^2)}{[R^2 - (m_{11} + m_{22})R + det\bar{M}]^2},
\]

\[e^{2\phi} = 1 - \frac{2m_{11}}{R} + \frac{m_{k1}^2}{R^2},\]

\[
A = 2\left(\frac{m_{21}R}{R^2 - (m_{11} + m_{22})R + det\bar{M}}b_{1n} + \frac{m_{22}R - det\bar{M}}{R^2 - (m_{11} + m_{22})R + det\bar{M}}b_{2n}\right), \quad B = 0,
\]

\[
L_\psi = \begin{pmatrix} u \\ v \\ s \end{pmatrix} = \begin{pmatrix} (m_{12} - m_{21})R - m_{k1}m_{k2}/(R^2 - 2m_{11}R + m_{k1}^2) \\ (m_{12} + m_{21})R - m_{k1}m_{k2}/(R^2 - 2m_{11}R + m_{k1}^2) \\ 2[(m_{11}R - m_{k1}^2)b_{n1} + (m_{12} + m_{21})R - m_{k1}m_{k2}]b_{n2}/(R^2 - 2m_{11}R + m_{k1}^2) \end{pmatrix},
\]

(25)
After some algebraic calculations, from (6) we have
\[
\left(\nabla \times \hat{A}^{(a)}\right)^\lambda = m^{(a)}\nabla^\lambda \left(\frac{1}{R}\right),
\]
(26)
where \(m^{(1)} = -(m_{12} - m_{21})/2\), \(m^{(2)} = -(m_{12} + m_{21})/2\) and \(m^{(2+n)} = (m_{11}b_{n1} + m_{12}b_{n2})\).

The relation between physical parameters and integration constants becomes evident when we switch from Cartesian to oblate spheroidal coordinates defined by
\[
x = \sqrt{\rho^2 + \alpha^2 \sin^2 \theta \cos \varphi}, \quad y = \sqrt{\rho^2 + \alpha^2 \sin^2 \theta \sin \varphi}, \quad z = \rho \cos \theta,
\]
(27)
In terms of these coordinates the 3–interval adopts the form
\[
ds_3^2 = (\rho^2 + \alpha^2 \cos^2 \theta)(\rho^2 + \alpha^2)^{-1}d\rho^2 + (\rho^2 + \alpha^2 \cos^2 \theta)d\theta^2 + (\rho^2 + \alpha^2)\sin^2 \theta d\varphi^2
\]
(28)
and only the \(A_{\varphi}^{(a)}\) does not vanish\(^2\):\n\[
A_{\varphi}^{(a)} = m^{(a)} \cos \theta \frac{\rho^2 + \alpha^2}{\rho^2 + \alpha^2 \cos^2 \theta} = m^{(a)} \epsilon.
\]
(29)
Studying their asymptotic behaviour we see that the integration constants and the physical parameters of the theory are related by
\[
-g_{tt} = -G \sim 1 + \frac{2m_{22}}{\rho} = 1 - \frac{2m}{\rho},
\]
\[
\phi \sim -\frac{m_{11}}{\rho} = \frac{D}{\rho},
\]
\[
u \sim \frac{m_{12} + m_{21}}{\rho} = \frac{N_u}{\rho},
\]
\[
v \sim \frac{m_{12} + m_{21}}{\rho} = \frac{Q_B}{\rho},
\]
\[
s \sim 2\frac{m_{11}b_{n1} + m_{12}b_{n2}}{\rho} = \frac{Q_m^{(n)}}{\rho},
\]
\(^2\)In fact we have imposed the axial symmetry with respect to \(z\).
\[ A_t^{(n)} \sim 2 \frac{m_{21} b_{1n} + m_{22} b_{2n}}{\rho} = \frac{Q_e^{(n)}}{\rho}, \]

where the \( m \) is the ADM mass of the black hole configuration, \( D \), \( N_u \), \( N_B \) are the dilaton, NUT and axion charges, respectively; \( Q_m^{(n)} \) and \( Q_e^{(n)} \) are two sets of \( n \) electric and magnetic charges. These charges satisfy the BPS bound

\[ 4(D^2 + m^2) + 2(Q_B^2 + N_u^2) = \sum_n (Q_m^{(n)})^2 + \sum_n (Q_e^{(n)})^2. \]

The complete solution is given by the following relations

\[ ds^2 = G_{MN} dx^M dx^N = G (dt + \omega d\varphi)^2 + e^{2\phi} g_{\mu\nu} dx^\mu dx^\nu, \]

where

\[ G = - \frac{(P^2 - Q^2) \left[ \rho^2 - (m_{11} + m_{22}) \rho + m_{k1}^2 - \text{det} \tilde{M} \right] (\rho^2 - \alpha^2 \cos^2 \theta)}{(P^2 + Q^2)^2} + \]

\[ \frac{2PQ \left[ (3\rho^2 - \alpha^2 \cos^2 \theta)(m_{22} - m_{11}) + 2\rho (m_{k1}^2 - \text{det} \tilde{M}) \right]}{(P^2 + Q^2)^2} - \frac{P(\rho^2 - \alpha^2 \cos^2 \theta) + 2\alpha \cos \theta Q \rho}{P^2 + Q^2}, \]

\[ \omega = 2A^{(1)} = 2m^{(1)} \epsilon, \]

\[ e^{2\phi} = \frac{(\rho^2 + \alpha^2 \cos^2 \theta)^2 - 2m_{11} \rho (\rho^2 + \alpha^2 \cos^2 \theta) + m_{k1}^2 (\rho^2 - \alpha^2 \cos^2 \theta)}{(\rho^2 + \alpha^2 \cos^2 \theta)^2}, \]

\[ \phi^{(4)} = \ln \left( \frac{P(\rho^2 - 2m_{11} \rho + m_{k1}^2 - \alpha^2 \cos^2 \theta) + 2\alpha \cos \theta Q (\rho - m_{11})}{P^2 + Q^2} \right), \]

\[ A_t^I = 2 \left( \frac{(P \rho + Q)m_{21} b_{1n} + (P(m_{22} \rho - \text{det} \tilde{M}) + m_{22} Q)b_{2n}}{P^2 + Q^2} \right) m^{(1)} - m^{(2+n)} \epsilon, \]

\[ A_t^I = 2 \left( \frac{(P \rho + Q)m_{21} b_{1n} + (P(m_{22} \rho - \text{det} \tilde{M}) + m_{22} Q)b_{2n}}{P^2 + Q^2} \right) \]

\[ B_{t\varphi} = 2 \left( \frac{m^{(2)} - (P \rho + Q)m_{21} b_{1n} + (P(m_{22} \rho - \text{det} \tilde{M}) + m_{22} Q)b_{2n}}{P^2 + Q^2} \right) m^{(2+n)} \epsilon, \]

\[ B_{\mu\nu} = 0, \]

where \( P = \rho^2 - (m_{11} + m_{22}) \rho + \text{det} \tilde{M} - \alpha^2 \cos^2 \theta \) and \( Q = (2\rho - (m_{11} + m_{22})) \alpha \cos \theta. \)
6 Conclusions

In this work we have presented a simple algorithm to obtain general exact solutions that
generalize the IWP class of EM theory for the effective action of the low–energy limit of
heterotic string theory compactified to three dimensions in a $d = D - 3$ torus. The method
allows one to reduce the motion equations to a matrix Laplace equation just requiring the
matrix Ernst potentials $A$ and $X$ to satisfy the asymptotic flatness conditions (assuming a
linear dependence between them). Their solution is then expressed in terms of one matrix
harmonic function. For the simplest case of the formalism ($d = 1$) a BPS saturated dyonic
solution was explicitly constructed. This solution is parametrized by a real $(2 \times 2)$–matrix $\tilde{M}$,
a set of $2n$ real constants $b_{kn}$ constrained by $b_{kn}b_{nk} = \delta_{kk'}/2$ and the constant $\alpha$ which defines
the rotation of the configuration. The charges of the field system saturate the Bogomol’nyi
bound (31).

Among these solutions we identify rotating black hole–like solutions. However, if we
require the asymptotic flatness condition for the black hole configuration to be satisfied,
both the NUT parameter and the rotation one vanish, leading to a static class of solutions.

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