Effects of near-zero Dirac eigenmodes on axial U(1) symmetry at finite temperature

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1. Introduction
Chiral symmetry breaking in QCD (N_f=2, m_{ud}=0)

\[ T = 0 \]

\[ \frac{SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A}{SSB} \rightarrow SU(2)_V \times U(1)_V \] (Remains)

\[ T > T_c \]

\[ SU(2)_V \rightarrow SU(2)_L \times SU(2)_R \] (Restored)

\[ U(1)_A \rightarrow ?? \]

Susceptibilities, Dirac Spectrum

Cossu’s talk

This Talk
Dirac Spectrum and Symmetry

\[ SU(2)_L \times SU(2)_R \]

Banks-Casher Relation

\[ |\rho(0)| = \frac{\Sigma}{\pi} \]

\[ U(1)_A \]

Atiyah-Singer Index Theorem

\[ n_+ - n_- = \nu \]

\[ n_\pm : \# \text{ of chiral zero-modes} \]

Dirac low modes are important for both symmetries
Dirac Spectrum and Symmetry

Aoki-Fukaya-Taniguchi (2012) argued that, if we assume

- SU(2) x SU(2) is restored ( $T > T_c$ )
- Ginsparg-Wilson relation is satisfied
- Analyticity in mass

→ Spectrum starts from cubic power* at least

\[ \rho = c_3 \lambda^3 + \cdots \]

→ U(1)$_A$ anomaly is invisible in the (pseudo) scalar correlators

\( \begin{align*}
&\text{Vol} \to \infty \\
&m_{ud} \to 0
\end{align*} \)

*G.Cossu et al (JLQCD 2013) reported a gap in the Dirac spectrum
Cohen(1996) argued that:

If the chiral zero-mode's effect is ignored, and if there is a gap in the Dirac spectrum

$\rightarrow U(1)_A$ breaking susceptibility

$$= \chi_{\pi} - \chi_{\delta}$$

$$= \int_{0}^{\infty} d\lambda \frac{4m^2 \rho(\lambda)}{(m^2 + \lambda^2)^2} = 0$$
(Controversial) Previous lattice studies

| Group                  | Action                        | Vol.    | Gap      | U(1)$_A$     |
|------------------------|-------------------------------|---------|----------|--------------|
| JLQCD(2013)            | Overlap                       | L=16    | Yes      | Restored     |
|                        | Fixed Topology                |         |          |              |
| Chiu et al (2013)      | Optimized                     | L=16    | Yes?     | Restored     |
|                        | Domain-wall                   |         | $\rho \sim \lambda^3 + \ldots$ |              |
| Ohno et al (2011)      | HISQ                          | L=32    | No       | Violated     |
| LLNL/RBC (2013)        | Domain-wall                   | L=16, 32| No       | Violated     |

What makes the difference: Finite V effects ? Fixed topology ? Chiral symmetry ?
This Work

| Finite volume | Larger volume |
|---------------|---------------|
| Fixed Topology| Tunneling Allowed |
| Chiral symmetry| OV/DW reweighting |
## Whats’ New in This work?

|                | G.Cossu et al (2013) | This Work                           |
|----------------|----------------------|-------------------------------------|
| Fermion        | Overlap              | Mobius Domain-wall                  |
| GW-rel.        | Exact                | $m_{\text{res}} \sim 1 \text{MeV} \text{ or lower}$ |
| Cost           | 16                   | 16, 32                              |
| Lat. Size      | Frozen               | Allowed                             |
| Topology       |                      |                                     |
| tunneling      |                      | We also try reweighting to OV      |
| Comment        |                      |                                     |
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2. Mobius DW
Mobius Domain Wall

Edwards-Heller (2000)

Overlap: 
\[ D_N(m) = \frac{1 + m}{2} + \frac{1 - m}{2} \gamma_5 \text{sgn}(H_K). \]

(Satisfy Ginsparg-Wilson relation)

| Domain Wall | Mobius DW |
|-------------|-----------|
| 4-dim eff. operator |  |
| \[ D^4 = \frac{1 + m}{2} + \frac{1 - m}{2} \gamma_5 \frac{T^{-L_s} - 1}{T^{-L_s} + 1}. \] | \[ D^4 = \frac{1 + m}{2} + \frac{1 - m}{2} \gamma_5 \frac{\prod_{s} L_{s} T_{s}^{-1} - 1}{\prod_{s} L_{s} T_{s}^{-1} + 1}. \] |
| \[ T^{-1} = \frac{1 + H_T}{1 - H_T} \] \[ H_T = \gamma_5 \frac{D_W}{2 + D_W}. \] | \[ T^{-1} = \frac{1 + \omega_s H_M}{1 - \omega_s H_M}. \] \[ H_M = \gamma_5 \frac{bD_W}{2 + cD_W}. \] |

Parameter | \[ L_s \] | \[ L_s, b, c \] |
|-----------|--------|---------------|
| (L_s → ∞ : OV) | \[ (L_s → ∞ : OV) \] | 

b and c make \( m_{\text{res}} \) small  
(b=2, c=1, 10^{-1}-10^{-3} smaller \( m_{\text{res}} \) for \( L_s=12 \))
Lattice set up

Gauge action: tree level **Sym anzik**
Fermion : Mobius DW\( (b=2, \ c=1, \ \text{Scaled Shamir} \ + \ \text{Tanh}) \)
w/ **Stout** smearing(3)
code : lrolro++ (G. Cossu et al.)
Resource : BG/Q (KEK)

| \( L^3 \times L_t \) | \( \beta \) | \( m_{ud}(\text{MeV}) \) | \( L_s \) | \( m_{res}(\text{MeV}) \) | Temp. (MeV) | Note |
|----------------------|-----------|-----------------|--------|-----------------|------------|------|
| \( 16^3 \times 8 \)  | 4.07      | 30              | 12     | 2.5             | 180        | 488 Conf. every 50 Trj. |
| \( 16^3 \times 8 \)  | 4.07      | 3.0             | 24     | 1.4             | 180        | 319 Conf. every 20 Trj. |
| \( 16^3 \times 8 \)  | 4.10      | 32              | 12     | 1.2             | 200        | 480 Conf. every 50 Trj. |
| \( 16^3 \times 8 \)  | 4.10      | 3.2             | 24     | 0.8             | 200        | 538 Conf. every 50 Trj. |
| \( 32^3 \times 8 \)  | 4.10      | 32              | 12     | 1.7             | 200        | 175 Conf. every 20 Trj. |
| \( 32^3 \times 8 \)  | 4.10      | 16              | 24     | 1.7             | 200        | 294 Conf. every 20 Trj. |
| \( 32^3 \times 8 \)  | 4.10      | 3.2             | 24     | -               | 200        | 88 Conf. every 10 Trj.  |
Topological charge changes along HMC

\[ L = 16, \ \beta = 4.10, \ m = 0.01, \ L_s = 12 \]

Index of DW
Wilson flow cooling

\[ Q_{\text{top}} \]

\[ \text{Conf} \]

20000 21000 22000 23000 24000 25000
Tc Estimation

Polyakov & Chiral condensate

Chiral Condensate

Vol. dependence of Polyakov loop
Decreasing of Chiral condensate

Above Tc (T=200MeV)
Around Tc (T=180MeV)

Polyakov loop
3. Domain-wall Dirac spectrum
Observable

Histogram of Dirac operator

\[ H_m \psi_i = \lambda_i^m \psi_i \]

\[ H_m = \gamma_5 \left[ (1 - m_{ud}) D^4 + m_{ud} \right] \]

\[ D^4 = \left[ \mathcal{P}^{-1} \left( D_{DW}^5 (m = 1) \right)^{-1} D_{DW}^5 (m_{ud}) \mathcal{P} \right]_{11} \]
3. Histogram for DW(T ~ Tc)

T = 180MeV ~ Tc (L = 16)

\[ \rho(\lambda) \]

\[ \rho(\lambda)_{a^3} \]

\[ \rho(\lambda)_{a^3} \]

\[ m_{ud} = 30 \text{MeV} \]

\[ m_{res} = 2.5 \text{MeV} \]

\[ m_{ud} = 3.0 \text{MeV} \]

\[ m_{res} = 1.4 \text{MeV} \]

Gap? Finite V effect?
3. Histogram for DW (above Tc)

\( T=200 \text{MeV} > T_c \) (\( L=16 \))

\[ \rho(\lambda) \]

\[ \rho(\lambda) \]

\[ \lambda \]

\[ \lambda^{|m_{\text{ud}} - m_{\text{uda}}} | \]

\[ m_{\text{ud}}=32 \text{MeV} \]

\[ m_{\text{res}}=1.2 \text{MeV} \]

\[ m_{\text{ud}}=3.2 \text{MeV} \]

\[ m_{\text{res}}=0.8 \text{MeV} \]

Gap? Finite V effect?
3. Histogram for DW (above Tc)

$T = 200\text{MeV} > T_c \quad (L = 32)$

$m_{ud} = 32\text{MeV}$
$m_{res} = 1.7\text{MeV}$

$m_{ud} = 3.2\text{MeV}$

Very small but non-zero => Gap is not apparent
U(1) looks broken
Short summary

$L=32$, $T=200$ MeV $m_{ud}=3.2$ MeV No clear Gap

$U(1)_{A}$ looks broken

Consistent with LLNL/RBC(2013). Then, What is the difference from OV(JLQCD)?

Finite $V$?

topology tunneling?

Violation of Ginsparg-Wilson relation?
4. Violation of Ginsparg-Wilson relation
Violation of Ginsparg-Wilson relation for each mode

\[ g_i \equiv \frac{\psi_i^\dagger \gamma_5 [D\gamma_5 + \gamma_5 D - 2D\gamma_5 D]\psi_i}{\lambda_i^m} \left[ \frac{(1 - m_{ud})^2}{2(1 + m_{ud})} \right] \]

\( g_i \) should be zero if GW is satisfied

Cf.

\[ m_{\text{res}} = \sum_i \frac{\lambda_i^m (1 + m_{ud})}{(1 - m_{ud})^2 (\lambda_i^m)^2} \frac{1}{\sum_i \frac{1}{(\lambda_i^m)^2}} g_i \]
Low-modes have significant violation of Ginsparg Wilson relation

$|g_i| \text{ vs } |\lambda_m|$

$|GW\text{-violation}|$

$g_i \text{ vs E-val DW-sHtTanh-16x8x24-b4.10-M1.00-m0.001}$
5. (Reweighted) Overlap Dirac spectrum
Reweighting to OV

\[
\langle \mathcal{O} \rangle_{ov} = \left\langle \mathcal{O} \frac{\det D_{ov}^2(m_{ud})}{\det D_{DW}^2(m_{ud})} \frac{\det D_{DW}^2(1/2a)}{\det D_{ov}^2(1/2a)} \right\rangle_{DW}
\]

We can measure OV quantity by using DW configuration

\[
\begin{align*}
\langle \rho(\lambda_{DW}) \rangle_{DW} \\
\langle \rho(\lambda_{ov}) \rangle_{DW} \\
\langle \rho(\lambda_{ov}) \rangle_{ov}
\end{align*}
\]

partially quenched OV
rewighted overlap

Let’s compare them!
T = 200 MeV, $m_{ud} = 32$ MeV

Domain-wall and overlap: visible difference.
T=200MeV, \( m_{ud}=3.2\)MeV

Consistent with LLNL/RBC 2013

Partially Quenched OV

Reweighted OV

Isolated chiral zero-modes
**T=200MeV, m_{ud}=3.2MeV**

| Observation                                                                 |                                                                 |                                                                 |
|----------------------------------------------------------------------------|-----------------------------------------------------------------|-----------------------------------------------------------------|
| • Strong violation of Ginsparg-Wilson relation in the low lying mode        | • the histograms(DW vs OV) look different                       |                                                                 |
| • Overlap Dirac operator has isolated chiral zero-modes + gap. (DW vs pqOV) | • Exactly chiral zero-modes should disappear in the large volume limit |                                                                 |
| • The gap looks stable as Volume increases. (Partially quenched OV L=16 vs L32) | • This gap may suggest U(1)_A symmetry restoration              |                                                                 |

- We need to confirm this in L=32 overlap (or DW with better chirality) simulations.
6. Summary
Summary

We have studied eigenvalue distribution of DW and (reweighted)overlap Dirac operators above Tc

1. Mobius Domain-wall spectrum => U(1)$_A$ is broken. consistent with LLNL/RBC(2013)

2. We found significant violation of chiral symmetry of low-lying modes even when m$_{\text{res}}$ is small.

3. OV/DW reweighting shows gap for lighter mass => U(1)$_A$ restoration? consistent with JLQCD(2013)

4. More study of finite volume effect is necessary. (OV/DW reweighting works only for smaller lattice)
Backup
T=200MeV, $m_{ud}=16$MeV
(beta=4.10 m=0.005)

**L32**

- DW
  - DW-sHtTanh-32x8x24-b4.10-M1.00-m$_{ud}$0.005

- Partially Quenched OV
  - L32HovTanhthre0.24m0.005

- Reweighted OV
  - Reweighting not available

**L16**

- DW-sHtTanh-16x8x12-b4.10-M1.00-m$_{ud}$0.01
Reweighting to OV
with UV suppressing determinant

\[ R_{UVS}^{UVS} = \left( \frac{\det \gamma_5 D_{ov}(m_{ud})}{\det \gamma_5 D_{DW}(m_{ud})} \right)^2 \left( \frac{\det \gamma_5 D_{DW}(M)}{\det \gamma_5 D_{ov}(M)} \right)^2 \]

DW/OV reweighting is UV surpassing determinant. unphysical mode suppressed by heavy unphysical modes M~O(1/a).
ma=0.01
m~30MeV

ma=0.001
m~3MeV

L16
b4.07

L16
b4.10

L32
b4.10

m_{\text{res}}=2.5\text{MeV}

m_{\text{res}}=1.4\text{MeV}

m_{\text{res}}=1.2\text{MeV}

m_{\text{res}}=0.8\text{MeV}

m_{\text{res}}=1.7\text{MeV}

m_{\text{res}}=1.7\text{MeV}
f(x) = a + c \times x^3

variance of residuals (reduced chisquare) = \frac{WSSR}{ndf} : 1.33016

Final set of parameters

| Parameter | Value       | Error        | Relative Error |
|-----------|-------------|--------------|----------------|
| a         | 0.000132414 | +/- 6.752e-05 | (50.99%)       |
| b         | 6.76224     | +/- 1.104    | (16.32%)       |
Large violation of GW-rel

$m_{\text{res}}$(Next to lowest) history

History of $m_{\text{res}}$ from $g_{ii}$: plot CP-smeared-SymDW-sHtTanh-16x8x24-b4.10-M1.00-mud0.001
The construction of the 4D effective operator (22) and the propagator (23) can be done exactly as before. Namely, we seek for parameters that match the Shamir (or DW) kernel with new parameters.

If we set the parameters as $\omega = 1$, the Zolotarev approximation, which is shown in Figure 7, is now a good idea especially when the distribution of the eigenvalues is limited in a narrow range, since the tanh approximation is exponentially good in the region $|\tan(x)| < 0.163$. $(1, 2, 3)$. (For instance, for $L_s = 6, 8, 12, 16$ are plotted as a function of the scale parameter $b$. $c = 1$.

$$H_M = \gamma_5 \frac{bD_W}{2 + cD_W}$$

Figure 9: Residual mass with the scaled-Shamir kernel and tanh approximation. The results with $L_s = 6, 8, 12, 16$ are plotted as a function of the scale parameter $b$. $c = 1$. 