Abstract

In extensions of the Standard Model with extra scalars, the electroweak phase transition can be very strong, and the bubble walls can be highly relativistic. We revisit our previous argument that electroweak bubble walls can “run away,” that is, achieve extreme ultrarelativistic velocities $\gamma \sim 10^{14}$. We show that, when particles cross the bubble wall, they can emit transition radiation. Wall-frame soft processes, though suppressed by a power of the coupling $\alpha$, have a significance enhanced by the $\gamma$-factor of the wall, limiting wall velocities to $\gamma \sim 1/\alpha$. Though the bubble walls can move at almost the speed of light, they carry an infinitesimal share of the plasma’s energy.
I. INTRODUCTION

A first order electroweak phase transition in the expanding Universe proceeds through the expansion of bubbles into the high temperature phase [1, 2]. Potentially observable consequences of this phase transition depend critically on the bubble wall velocity $v$ [3-6]. It is in general quite difficult to compute $v$. Most existing microscopic calculations are for the case that the transition occurs due to radiative corrections [2, 7-13]. Generally these studies find that the bubble wall velocity is less than the speed of sound. However, in the presence of singlet scalars, the electroweak phase transition can be very strong, with a mean-field analysis already predicting a first-order transition [6]. In [14] we argued that, in this case, the bubble wall velocity typically becomes large, with the gamma factor $\gamma \equiv (1-v^2)^{-1/2} \gg 1$. We presented a leading-order calculation of the “friction” (backwards force) on an advancing bubble wall, and showed that the calculation simplifies tremendously in the $\gamma \to \infty$ limit, and that the “friction” on the bubble wall approaches a constant. This occurs because of two compensating effects. First, the density of particles (as viewed in the wall’s rest frame) rises due to Lorentz contraction of the plasma, $n \propto \gamma$. But the retarding force from each particle diminishes as $\gamma^{-1}$, leading to a finite $\gamma \to \infty$ limit. This limit could be expressed in terms of a modified effective potential, where the thermal part of $V_{\text{eff}}$ is replaced by a certain mean-field approximation. This gives a simple criterion for whether the wall would speed up even further (‘run away’) or not: if the phase transition is first order with the mean field potential, then the wall will generally run away, with the $\gamma$-factor growing linearly with propagation distance.

The difference between nonrelativistic and relativistic bubble walls is important to baryogenesis. The difference between relativistic bubble walls and true runaway walls is relevant for possible gravitational wave signatures of the transition [15-17], and deserves more careful investigation. The analysis of [14] considered only the leading order in the couplings, which means that it neglected any interactions of medium excitations other than those with the electroweak bubble wall. In particular, we did not take into account that particles may radiate (split) when they hit the wall. Such radiation can occur because crossing the bubble wall causes particles to accelerate, as their masses change from one side to the other. Furthermore, even if a particle remains massless, it can radiate if it interacts with a particle (such as the $W$ boson) with a phase-dependent mass, or if an interaction vertex changes in strength from one phase to the other (for instance, a 3-point scalar vertex proportional to the vacuum value of a field). In these cases, the “radiation cloud” surrounding (“dressing”) the particle is not the correct cloud for the other phase, and the difference is emitted as radiation.

This is the same phenomenon as transition radiation: when an electron moves from one medium into another medium with a different index of refraction, the mismatch in the electromagnetic “cloud” it should carry in each medium causes a radiation associated with the transition between media – transition radiation [18].

In the current context, the emission of transition radiation induces a force on the bubble wall. The force depends on the momentum of the emitted particle. We will show that this

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1 In footnote 8 of [14] it was claimed that particle splitting does not significantly modify the conclusions of the main text. We will find that this is not true.
force is dominated by radiated particles which are soft in the wall frame. We will show that the probability to emit a gauge boson with a phase-dependent mass, and the emitted spectrum of soft gauge bosons, approach a finite $\gamma \gg 1$ limit. Therefore the force on the wall per particle, arising from transition radiation, approaches a constant in the large $\gamma$ limit. Since the density of particles rises as $n \propto \gamma$, the friction rises linearly in $\gamma$. Therefore some $\gamma$ is sufficient to resist the forward pressure driving the wall’s acceleration. Typically this occurs for $\gamma \sim 1/\alpha$.

The remaining sections of the paper fill in the details in the argument presented above. The next section sets the stage with a review of bubble wall kinematics. Then Section III explains how to compute the rate of transition radiation for a bubble wall. We follow this with an outline of the rate of hard emissions in IV and of the complications associated with soft emissions in V. We end with a short conclusion.

II. REVIEW OF BUBBLE-WALL KINEMATICS

We start with a lightning review of the main ideas in our first paper [14]. Consider an electroweak phase interface (bubble wall) moving through the plasma. This means that there is a scalar field with a spacetime varying expectation value $\phi(t, r)$. Also there will be one or more particles whose masses arise from this expectation value, $m \propto \phi(t, r)$. We assume that the wall reaches a steady-state velocity $v$ and gamma-factor $\gamma = 1/\sqrt{1 - v^2} \gg 1$ relative to the plasma in front of it, which we will try to determine self-consistently. Since bubbles are macroscopic, we can take the wall to be planar and choose coordinates where $\phi = \phi(t, z)$; in the wall’s rest frame $\phi = \phi(z)$, and $v$ is the velocity of the plasma approaching the wall from in front. The geometry is summarized in Fig. I

![Fig. 1: Bubble wall geometry, and particle trajectories depending on energy and direction.](image)

The wall is pushed forward (to the left in the figure) by the vacuum energy difference between phases. There is a restraining force from particles, which must balance the forward force for the wall to reach a steady state. Therefore it is essential to compute this restraining force...
force, which we do by computing the impulse from a single particle and integrating over the flux of particles. By assumption the wall is moving very fast relative to the plasma, $\gamma \gg 1$, so almost all particles have an energy $E \sim \gamma T$ much larger than the mass change in crossing the wall. Furthermore the particles have much shorter wavelength than the wall’s thickness, so the WKB approximation is good and we can neglect reflection. Another simplification due to $\gamma \gg 1$ is that particles only approach the wall from the symmetric-phase side.

We will choose coordinates such that the Higgs field varies along the $z$ axis from a large “h” value at positive $z$ to a small “s” value at negative $z$. We work in the wall rest frame, meaning the frame obtained from the s-phase plasma rest frame by a purely $z$-directed boost. Consider a particle $a$ of energy $E_a \sim \gamma T$ in the s phase impinging on the wall from the left. Our frame is time and $r_\perp$ independent, so $E$ and $p_\perp$ are conserved. But the wall breaks $z$-translation invariance, so $p_z$ is not conserved. Any change to a particle’s $p_z$ must be supplied by an impulse from the bubble wall; so we need the particle’s $p_z$-change in crossing the wall. Before hitting the wall, the energy is related to transverse momentum $p_\perp$ and $z$-momentum $p_z$ as $E^2 = p_z^2 + p_\perp^2 + m_a^2$, where the subscript $s$ refers to the symmetric phase and $p_\perp \equiv |p_\perp|$. Since $E \sim \gamma T$ and $p_\perp^2 \sim T^2$, we can expand in $E, p_z \gg p_\perp, m$, in which case the momentum transferred to the wall is

$$\Delta p_{1\rightarrow 1} = p_{z,\text{in}} - p_{z,\text{out}} \simeq \frac{m_{a,h}^2 - m_{a,s}^2}{2E}. \quad (1)$$

This should be integrated over the phase space and occupancy of particles impinging on the wall. Because the reflection coefficient is approximately zero and no particles hit the wall from the h-phase, no “news” of the wall’s approach has reached the s-phase, and the s-phase occupancies take their equilibrium values. So the pressure on the wall is

$$P_{1\rightarrow 1} = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_a(p) \times \frac{m_{a,h}^2 - m_{a,s}^2}{2E}$$

$$= \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3 2E} f_a(p) \left( m_{a,h}^2 - m_{a,s}^2 \right). \quad (2)$$

Here the sum $\sum_a$ is over all particle types, and $\nu_a$ is the degeneracy (number of spin and color or other group labels). The notation $1 \rightarrow 1$ means that we are considering a single particle approaching the wall, remaining a single particle after crossing the wall. The combination $d^3p/E$ is frame-independent, leading to a $\gamma$ independent pressure (in the large-$\gamma$ limit, used to reach Eq. (2)). This result – that the pressure restraining the bubble wall approaches an asymptotic value rather than growing with increasing $\gamma$ – was the main result of our previous paper [14].

### III. TRANSITION SPLITTING

Transition radiation, or transition splitting, can occur if an initial particle species $a$ couples to final species $b, c$ and either the strength of their interaction, or one or more particle mass, differs between the two phases. The best known case is the emission of light by a charged particle, say an electron, via the process $e^- \rightarrow e^-\gamma$, when crossing an interface which
changes the photon dispersion.\textsuperscript{2} Several species change mass in going from the symmetric to the Higgs phase, and they all have gauge and/or Yukawa interactions allowing \( a \to bc \) type processes; so transition radiation can certainly occur when a particle hits the electroweak interface. Such a splitting changes the kinematics and increases the \( p_z \) transferred to the wall. Some of the incident particle’s energy goes into the mass of the additional particle, and some into the relative transverse momentum of \( b \) and \( c \), and there is less energy left for the longitudinal momenta of the produced particles. So we need to consider these processes to see whether they have an important effect on the previous arguments.

To compute the force due to transition radiation, we have to understand the flux of incident particles, the probability for each to undergo transition radiation, and the impulse to the wall if it does. We start with the flux of particles, which is simple. In the wall’s rest frame, the density of particles in front of the wall, and the flux impinging on the wall, are

\[
\text{density} = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_a(p), \quad \text{flux} = \sum_a \nu_a \int \frac{d^3p^z}{(2\pi)^3} p^z f_a(p). \tag{3}
\]

The backwards pressure due to splittings is found by inserting the differential probability to split, times the momentum transferred by the splitting process, inside this integrand:

\[
P_{1\to2} = \sum_a \nu_a \int \frac{p_z d^3p}{p^0 (2\pi)^3} f_a(p) \times \sum_{bc} \int dP_{a\to bc} \times (p_{z,s} - k_{z,h} - q_{z,h}). \tag{4}
\]

Here \( k \) and \( q \) are the final momenta of the \( b \) and \( c \) species respectively. The notation \( 1 \to 2 \) means we are computing the pressure arising when 1 incoming particle becomes 2 final particles after crossing the wall.

To be more precise about the differential splitting probability \( dP_{a\to bc} \), consider a single particle impinging on the wall. To work with a properly normalized state, we integrate the (improperly normalized) momentum-space states,

\[
\langle p' | p \rangle = 2p^0 (2\pi)^3 \delta^3(p - p'),
\]

over a wave packet which builds a properly normalized single particle initial state,

\[
|\phi \rangle \equiv \int \frac{d^3p'}{(2\pi)^3 2p_0^3} \phi(p') | p' \rangle \quad \text{with} \quad \int \frac{d^3p}{(2\pi)^3 2p_0} |\phi(p)\rangle^2 = 1. \tag{6}
\]

We are labeling momentum space states in terms of their incoming symmetric-phase momentum; because there is a bubble wall, they actually vary nontrivially in \( z \)-direction,

\[
\langle r | p \rangle = \sqrt{2p^0} e^{i p_z z} \chi_p(z) \quad \text{(for scalars)}
\]

with \( \chi_p(z) = \exp(i p_z z) \) in the s phase, neglecting any reflected wave; but the behavior of \( \chi_p \) near the wall and in the broken phase must be found by explicitly solving the associated free particle evolution equation in the presence of the bubble wall. For spinor or vector fields,

\textsuperscript{2} This is distinct from Cherenkov radiation. In particular, transition radiation occurs even if the electron velocity is below the Cherenkov velocity in each medium.
one should replace $\chi_p$ with an appropriate spinor or vector solution to the linearized Dirac or Yang-Mills equation in the wall background.

An integral over the final state momenta, with momentum-basis states, does constitute a properly normalized treatment of the final states.\(^3\) So the splitting probability alluded to above is the integral over final state phase space of the squared $T$-matrix element of the initial state with the multiparticle final state:

$$\int dP_{a\rightarrow bc} \equiv \int \frac{d^3 k \, d^3 q}{(2\pi)^3 2 \hbar^0 (2\pi)^3 2 q^0} \langle \phi | T | k q \rangle \langle k q | T | \phi \rangle,$$

implicitly summing over final state spin and color indices.

The bubble wall is invariant in time and the transverse directions, ensuring that energy and transverse momentum are conserved, so the transition matrix element between momentum states is

$$\langle k q | T | p \rangle = \int d^4 x \, \langle k q | \mathcal{H}_{\text{int}} | p \rangle$$

$$= (2\pi)^3 \delta^2 (p_\perp - k_\perp - q_\perp) \delta (p^0 - k^0 - q^0) \mathcal{M},$$

with

$$\mathcal{M} \equiv \int dz \, \chi_\check{q}^*(z) \chi_q(z) V(z) \chi_p(z).$$

Here $V(z)$ is the contraction of the interaction Hamiltonian density with all other state information (spinors and polarizations), which would be the same as the interaction matrix element if we were considering simple plane wave states. Note that $V(z)$ has dimensions of energy, while $\mathcal{M}$ is dimensionless.

Applying Eq. (6) and Eq. (9) to Eq. (8), we find

$$\frac{p_z}{p} \int dP_{a\rightarrow bc} = \int \frac{p_1 d^3 p_1 d^3 p_2}{(2\pi)^6 2 (2 \hbar^0)^2 2 p_2^0} \phi^*(p_1) \phi(p_2) (2\pi)^3 \delta^2 (p_{1\perp} - p_{2\perp}) \delta (p_1^0 - p_2^0)$$

$$\times \int \frac{d^3 k d^3 q}{(2\pi)^3 2 \hbar^0 (2\pi)^3 2 q^0} (2\pi)^3 \delta^2 (p_{1\perp} - k_\perp - q_\perp) \delta (p_1^0 - k^0 - q^0) |\mathcal{M}|^2.$$

Note that $p_z \delta (p_1^0 - p_2^0) / p^0 = \delta (p_{1z} - p_{2z})$, so the $p_2$ integration can be performed using the delta functions, leaving a factor of $1/2 p^0$. The remaining $p_1$ integral over the wave packets gives 1, and replaces $p_1 \rightarrow p$, the central value of the wave packet, in the remaining expressions. Also note that $|\mathcal{M}|^2$ in Eq. (11) is final state spin and color summed.

Inserting the resulting $dP_{a\rightarrow bc}$ into Eq. (11), and restoring the final state Pauli blocking or Bose stimulation factors which we neglected to write so far, we find that the backwards pressure on the bubble wall from splitting processes is

$$\mathcal{P}_{1\rightarrow 2} = \sum_{a,b,c} \nu_a \int \frac{d^3 p}{(2\pi)^3 2 p^0} \int \frac{d^3 k d^3 q}{(2\pi)^6 2 \hbar^0 2 q^0} f_p [1 \pm f_k] [1 \pm f_q] (p_{1,s} - k_{z,h} - q_{z,h})$$

$$\times (2\pi)^3 \delta^2 (p_{1\perp} - k_\perp - q_\perp) \delta (p_1^0 - k^0 - q^0) |\mathcal{M}|^2.$$

\(^3\) With the normalization of Eq. (5), $(2\pi)^{-3} \int d^3 k (2 \hbar^0)^{-1} |k\rangle \langle k|$ is a properly normalized projection operator.
The matrix element $M$ can be nontrivial if either $V(z)$ or one of the $\chi(z)$ is a nontrivial function of $z$. If both $V$ and all $\chi$ are $z$-independent, it reduces to $M \propto \delta(p_z - k_z - q_z)$, indicating that no momentum is transferred to the wall. In this case the process is either kinematically forbidden, or it represents a kinematically allowed decay $a \rightarrow bc$ which occurs anywhere in space, regardless of the wall. Therefore the interesting cases are when either the vertex, or one or more particle masses, differ between phases.

IV. SEMI-SOFT EMISSION

To make further progress, we will specialize to a kinematic case which will be sufficient to determine where the dominant friction arises. We will consider

$$p_z \sim \gamma T, \quad p_z \gg k_z \gg L^{-1}, m,$$

that is, $k_z$ intermediate between the “hard” scale $p_z$ and the soft scale set by masses. Here $L^{-1}$ is the inverse wall thickness, which is related to the scalar (Higgs) field mass. We will also assume

$$k_\perp^2 \sim m^2 \quad (\ll k_z m \text{ by Eq.}(13)),$$

which will prove to be the most important $k_\perp$ range. These kinematical approximations will simplify things enough to get explicit expressions, which we can use to determine the most relevant processes and momentum ranges. We will thereby find that the most important processes are those which emit vector bosons with phase-dependent masses, and that the most important kinematic range for this process is in fact “soft” $k$, with $k_\perp \sim k_z \sim m$. Our analysis will identify this momentum range as the most important, but its quantitative evaluation then lies outside of the range of validity of the analysis.

The $z$-dependent longitudinal momenta can now be approximated as

$$k_z(z) = \sqrt{k_0^2 - m^2(z) - k_\perp^2 \approx k^0 - \frac{m^2(z) + k_\perp^2}{2k^0}}.$$

Using $q^0 \approx p^0$ the integral (12) for the pressure can be simplified to

$$p_{1 \rightarrow 2} = \sum_{a,b,c} \nu_a \int \frac{d^3p}{(2\pi)^3 4p^0} f_p \int \frac{d^2k_\perp}{(2\pi)^2} \int_0^\infty \frac{dk^0}{2\pi 2k^0} [1 \pm f_k] [1 \pm f_{p-k}] \frac{k^2_\perp + m_{k,b}^2}{2k^0} |M|^2.$$

Because we assume $p^0, k^0, q^0 \gg L^{-1}$, we may treat the mode functions $\chi(z)$ in the WKB approximation,

$$\chi_k(z) \approx \sqrt{\frac{k_{z,s}}{k_z(z)}} \exp \left(i \int_0^z k_z(z') dz'\right) \approx e^{ik^0 z} \exp \left(-\frac{i}{2k^0} \int_0^z (m^2(z') + k_\perp^2) dz'\right).$$

Making a small transverse boost so $p_\perp = 0$, and introducing the energy fraction

$$x \equiv k^0/p^0 \ll 1$$
In the two cases of most interest we find

$$|\mathcal{M}|^2 \simeq 4p_0^2 |V_h - V_s|^2 A^2$$ \quad (V_h \neq V_s), \quad (22)$$

$$|\mathcal{M}|^2 \simeq 4p_0^2 |V|^2 (A_h - A_s)^2 A_h^2 A_s^2$$ \quad (A_h \neq A_s). \quad (23)$$

Next we investigate the form of $V$, the $1 \to 2$ scattering matrix element. For the cases with $|V|^2 \propto k_1^2$, it has a simple relationship to the standard DGLAP splitting kernel, with $P_{b\to q}(x) = |V|^2 x(1-x)/16\pi^2 k_1^2$. The additional factors arise from the phase space integration involved in treating the $1 \to 2$ process in the DGLAP setting. We list the value of $|V|^2$ for the most interesting processes in Table I. In computing the expressions in the table we have again expanded systematically in large $p_z \gg k_z \gg m, k_\perp$. We have also considered a simple group, so for instance for $W, Z$ boson emission from a doublet matter field one should use $C_2[R] = 3/4$ if we neglect the hypercharge interaction. To properly take into account the mixing between SU(2) and U(1) interactions to find $Z$-boson radiation at finite weak mixing angle requires a little more work.

The vertices divide into two groups. Some, such as those involving longitudinal gauge bosons or the $S \to SS$ process, have a phase dependent value, which however does not grow with $p^0$ at fixed $p^0/k^0 = 1/x$ ratio. Others scale with $k_\perp$ and therefore increase as all momenta are raised simultaneously; but these terms are the same in the two phases. This makes sense. Any dependence of $|V|^2$ on the phase must arise from an interaction with the scalar expectation value, and should therefore be proportional to $m^2$. The only way for $|V|^2$ to carry no powers of the expectation value is for it to be built out of an invariant of the momenta, such as $p_\mu k^\mu \sim (p^0/k^0)k_\perp^2$. The quantity $p_0^2$ itself involves the dot of $p^\mu$ with the wall normal vector and cannot appear without an $m^2$ factor.

The most small-$x$ singular terms are those involving the emission of a soft vector boson, with $|V|^2 \sim x^{-2}$. All other terms scale as $x^{-1}$ or less. We will now see that this leads to soft vector boson emission dominating the friction on the bubble wall, and that the $x^{-2}$ behavior predicts that the friction is controlled by the small-$x$ region. To see this, insert the matrix
TABLE I: Squared vertex functions for the most interesting transition radiation processes, averaged/summed over initial/final state spins and group indices. Here $S$ is a scalar, $F$ is a spin 1/2 fermion, and $V_T, V_L$ are transverse and longitudinal vector bosons. $C_2[R]$ is the second Casimir of the representation of the incoming particle, and $T[R]$ is the trace normalization (Dynkin index) of the final particle, and $\varphi$ is the phase-dependent background value of the scalar field.

The overall dependence is as 4

$$|\mathcal{M}|^2 \simeq 16g^2C_2[R] p_0^2 \frac{m_{V,h}^4}{k_{\perp}^4 (k_{\perp}^2 + m_{V,h}^2)^2}. \hspace{1cm} (25)$$

The $k_{\perp}$ integral in Eq. (16) is now dominated by $k_{\perp}^2 \sim m^2$, justifying Eq. (14) and giving approximately

$$\int \frac{d^2k_{\perp}}{(2\pi)^2} \frac{1}{k_{\perp}^2 (k_{\perp}^2 + m_{V,h}^2)} \sim \frac{1}{24\pi m^2}, \hspace{1cm} (26)$$

4 In deriving Eq. (24) we have neglected thermal masses. This is a good approximation if the phase transition is strong so $m^2 \gg g^2T^2$. Including thermal masses raises some subtle issues, but it would be necessary in a careful quantitative treatment if the phase transition is not very strong.
and the $p, k$ integrals are of the form
\[ \int \frac{d^3p}{(2\pi)^3} f_p \times m_{V,h}^4 \int \frac{dk^0}{k^0} [1 \pm f_k][1 \pm f_{p-k}] . \] (27)

The $p$ integral counts the wall-frame density of incoming particles, and scales as $\gamma T^3$. The factor $[1 \pm f_{p-k}]$ is order 1, and the factor $[1 \pm f_k]$ becomes 1 when we consider the difference between emission and absorption processes. The $k^0$ integral is small-$k^0$ divergent. It should be cut off by the mass scale, giving a pressure of form
\[ P_{1\rightarrow2} \sim g^2 C_2[R] n_p m \propto \gamma g^2 m T^3 . \] (28)

The important feature is the scaling with the incoming particle density, and therefore the linear proportionality to $\gamma$.

If we consider instead the emission of a longitudinal vector boson, with $|V|^2 \sim g^2 m_{V,h}^2 / x^2$, we find
\[ |M|^2 \simeq 4p^2 g^2 C_2[R] m_{V,h}^2 \frac{x^2}{(k^2 + m^2)^2} . \] (29)

The $k_\perp$ integral is now only logarithmically small-$k_\perp$ dominated, but the result is parametrically the same if one neglects logarithms.

However, Table II shows that only these processes give rise to a $|V|^2 \propto 1/x^2$ behavior. This is familiar from the DGLAP kernels, where only soft vector emission has a $1/x$ enhancement. Substituting a less $x$-singular expression, such as $|V|^2$ for the $F \rightarrow F V$ process, in Eq. (24), we find that Eq. (27) becomes
\[ \int \frac{d^3p}{(2\pi)^3} f_p \times m^2 \int \frac{dk^0}{k^0} \ldots \] (30)

which is only logarithmically small-$k$ singular. More importantly, the $p$ integration gives the invariant particle density $\int d^3p f_p / p \propto T^2$, rather than the wall-frame density $\int d^3p f_p \propto \gamma T^3$. The result is not enhanced at large $\gamma$, or at most it is enhanced by logarithms of $\gamma$, rather than a power. Therefore such processes are subdominant.

Note also that the emission of a vector boson which is massless in each phase (say, the gluon) also does not give rise to a linear-in-$\gamma$ contribution. Consider again the $F \rightarrow VF$ process, but with $m_F^2 = 0$ in each phase, and $m_F^2$ changing between phases. Then $A_h - A_s \sim m_F^2$ without a $1/x$ factor, making Eq. (24) less singular by $x^2$.

Therefore the friction from transition radiation is dominated by the emission of vector bosons with phase-dependent masses, with backwards pressure of parametric form $P_{1\rightarrow2} \sim \gamma g^2 m_V T^3$.

V. INFRARED BEHAVIOR

We have shown above that the friction from transition radiation is dominated by gauge bosons receiving a mass in the transition, with the most important momentum range being $k_\perp \sim k_\perp \sim m_V \equiv m$ in the wall frame. Here we will outline the challenges associated with a complete calculation. However, we will not carry out such a calculation, as it is technically
complicated and sensitive to the details of the (beyond the Standard Model) physics giving rise to the transition.

Since the dominant emitted momentum is \( k_z \sim m \), we must revisit two approximations we made above. First, in Eq. (20), we approximated the phase change in traversing the wall to be small. This amounts to the approximation

\[
\text{small phase change: } \quad L \frac{k_{\perp}^2 + m^2}{k_z} \ll 1, \quad \text{or} \quad mL \ll 1 \quad \text{if} \quad k_{\perp}, k_z \sim m. \tag{31}
\]

Second, in treating the wave functions of all particles in the WKB approximation, we assumed

\[
\text{WKB approximation: } \quad k_z L \gg 1, \quad \text{or} \quad mL \gg 1 \quad \text{if} \quad k_{\perp}, k_z \sim m. \tag{32}
\]

Clearly, in the soft regime, one or the other of these approximations will break down. To see what kind of numerical factors can be involved we do an explicit computation assuming that the wall is thick,

\[
L m \gg 1. \tag{33}
\]

and we assume \( z \)-independent \( V = C g k_{\perp} / x \). The masses in Eq. (33) have contributions from the Higgs expectation value \( h \), and also \( h \)-independent ones, which are proportional to \( T \). As before, we take \( x \ll 1 \), so that

\[
\mathcal{M} \simeq V \int dz \exp \left( i z \frac{k_{\perp}^2}{2k_0} + \frac{i}{2k_0} \int_0^z dz' m^2(z') \right). \tag{34}
\]

This gives

\[
\mathcal{M} \simeq \frac{VL}{2} \exp \left( -i \log 2 \frac{L}{4k} \Delta m^2 \right) \frac{\Gamma \left( -i \frac{L}{4k} (k_{\perp}^2 + m_h^2) \right) \Gamma \left( i \frac{L}{4k} (k_{\perp}^2 + m_s^2) \right)}{\Gamma \left( -i \frac{L}{4k} \Delta m^2 \right)} \tag{35}
\]

with \( \Delta m^2 = m_h^2 - m_s^2 \). Note that for \((Lm)/(m/k) \ll 1, k_{\perp} \gg m\) this agrees with (25), as it should. Using that \(|\Gamma(iy)|^2 = \pi/|y \sinh(\pi y)|\) for real \( y \) [19] we obtain

\[
|\mathcal{M}|^2 = \frac{\pi L |V|^2 k \Delta m^2}{(k_{\perp}^2 + m_h^2)(k_{\perp}^2 + m_s^2)} \frac{\sinh \left( \frac{\pi L}{4k} \Delta m^2 \right)}{\sin \left( \frac{\pi L}{4k} (k_{\perp}^2 + m_h^2) \right) \sin \left( \frac{\pi L}{4k} (k_{\perp}^2 + m_s^2) \right)}. \tag{36}
\]

Here one can explicitly see how \( |\mathcal{M}|^2 \) cuts off the integrals over \( x \) and \( k_{\perp} \), both at small and at large values. The result of these integrations will be a complicated function of the thermal
masses, and the result for the friction force will not be as simple as without radiation [14]. We find
\[
\int dP_{\text{split}} \Delta p_{1\rightarrow2} = \left| C \right|^2 g^2 L^{-1} \frac{\Delta m^2}{m_s^2} f \left( \frac{\Delta m^2}{m_s^2} \right)
\]
where the function \( f \) is given by the integral
\[
f(\xi) \equiv \int_0^{\infty} dz \frac{z}{(1 + z)^3} \int_0^{\infty} du \frac{\sinh \left( \frac{u \xi}{1 + z} \right)}{\sinh(u) \sinh \left( u + \frac{u \xi}{1 + z} \right)}.
\]
For small and large \( \xi \)
\[
f(\xi) \simeq \frac{\pi^2}{36} \xi \quad (\xi \ll 1), \quad f(\xi) \rightarrow \frac{\pi^2}{24} \simeq 0.41 \quad (\xi \rightarrow \infty).
\]
Thanks to this and to the factor \((2\pi)^{-3}\) in (37) the numerical prefactor in the splitting contribution to the friction is small. The splitting therefore only starts to dominate at quite large \( \gamma \).

In the opposite limit of \( mL \ll 1 \), it is the WKB approximation which breaks down. In this case it is necessary to compute the full \( \chi(k) \) mode function in the presence of the wall, including the finite reflection amplitude. These mode functions were found explicitly by Farrar and McIntosh [20], again assuming a tanh wall profile. We will not pursue this approach further here.

Finally we should point out another effect which can limit the emission of the gauge bosons responsible for the friction. Let us estimate the phase-space density of the most important emitted gauge bosons. We found around Eq. (28) that the pressure is
\[
P_{1\rightarrow2} \sim \gamma g^2 m T^3,
\]
arising from particles with \( k_\perp \sim k_z \sim m \). Looking at Eq. (15) we find the force per particle is \( p_z - k_z - q_z \sim m \). So the density of emitted particles is \( n \sim \gamma g^2 T^3 \). This is to be compared to the phase space \( d^3k \sim m^3 \) they fill. The mean occupancy is
\[
f(k) \sim \frac{n}{\Delta k^3} \sim \frac{g^2 \gamma T^3}{m^3}.
\]
For large \( \gamma > m^3/g^4 T^3 \), this can exceed the “saturation” occupancy \( 1/g^2 \). In this case, we expect that it is no longer safe to consider different radiations as independent processes; there are nonperturbatively large interactions between emitted quanta which should suppress the emission process such that no phase space region has occupancy higher than \( \sim 1/g^2 \).

With this in mind, let us rewrite Eq. (25), Eq. (16) as follows. For the emission of transverse \( W \) bosons, dropping order-1 constants but keeping the parametric dependence on \( T, m, \gamma, g \), the pressure is
\[
\mathcal{P}_{1\rightarrow2} \sim \int d^2k_\perp dk_z \left( k^0 - k_z \right) \times \left[ \frac{g^2 k_\perp^4 m^4}{k_z (k_\perp^2 + m^2)^4} \right] \int f(p) d^3p
\]
\[
\sim \int d^2k_\perp dk_z \left( k^0 - k_z \right) \times \left[ \frac{g^2 m^4 \gamma T^3}{k_z k_\perp^6} \right],
\]
(41)
where \( \int d^2 k_\perp dk_z \) is the phase space of emitted particles, \((k^0 - k_z)\) is the momentum transfer per particle, and the quantity in square brackets is the occupancy (phase space density) of emitted particles. Roughly speaking, saturation tells us to cut off the quantity in square brackets when it exceeds \(1/g^2\). Doing so, the integral is dominated when \(k_z \sim k_\perp\). The occupancy reaches \(g^{-2}\) when

\[
k_z \sim k_\perp \sim (\gamma g^4 T^3 m^4)^{1/7}, \tag{42}
\]

\[
P_{1 \to 2} \sim \frac{k_\perp^4}{g^2} \sim \gamma^{\frac{2}{7}} g^2 T^{\frac{12}{7}} m^{\frac{4}{7}}. \tag{43}
\]

This quantity rises with increasing \(\gamma\) as \(\gamma^{4/7}\), rather than \(\gamma^1\) as we found before. Nevertheless, this is sufficient to ensure that the pressure is enough to prevent a “runaway” wall, \(\gamma \ll 10^{10}\) so that the walls make up a negligible fraction of the total energy in the Universe.

VI. CONCLUSIONS

In conclusion, if the plasma induces insufficient friction on the bubble wall to prevent runaway, the wall accelerates to large \(\gamma\). At leading order, the backwards friction on the bubble wall from the medium scales as \(P_{1 \to 1} \sim m^2 T^2\), reaching a finite limit at large \(\gamma\). The pressure driving the wall forward is generically of the same scale, but may be numerically larger, in which case this analysis alone would suggest a “runaway” situation with \(\gamma\) growing without bound.

But at the next order, transition radiation of wall-frame soft, massive vector bosons introduces an additional friction term with \(P_{1 \to 2} \sim \gamma g^2 m T^3\). Since this term rises linearly with \(\gamma\), it will limit the \(\gamma\)-factor of the wall and prevent a true “runaway.” However, under criteria where the leading-order calculation predicts runaway, we still expect a parametrically large \(\gamma\)-factor, \(\gamma \sim m/g^2 T\). This velocity is fast enough that, from the point of view of electroweak baryogenesis, the wall can be considered to move with \(v = 1\). For the production of gravitational waves, one can treat \(v \simeq 1\) but one can neglect the energy accumulated by the bubble walls.

In some cases, the transition radiation may lead to a very high occupancy of wall-frame soft \(W, Z\) bosons, interacting nonperturbatively with each other. It is not clear to us what physics might arise after the wall’s passage, as these particles thermalize with the broken-phase medium. The occupancies are sufficient that they could in principle generate sphaleron transitions immediately after the passage of the wall. We leave further considerations of this problem to future investigations.

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[1] D. A. Kirzhnits and Andrei D. Linde. Symmetry Behavior in Gauge Theories. *Annals Phys.*, 101:195–238, 1976.
[2] Michael Dine, Robert G. Leigh, Patrick Y. Huet, Andrei D. Linde, and Dmitri A. Linde. Towards the theory of the electroweak phase transition. *Phys. Rev.*, D46:550–571, 1992.
[3] Andrew G. Cohen, D. B. Kaplan, and A. E. Nelson. Progress in electroweak baryogenesis. *Ann. Rev. Nucl. Part. Sci.*, 43:27–70, 1993.
[4] Andrew G. Cohen, D. B. Kaplan, and A. E. Nelson. Diffusion enhances spontaneous electroweak baryogenesis. *Phys. Lett.*, B336:41–47, 1994.
[5] Michael Joyce, Tomislav Prokopec, and Neil Turok. Nonlocal electroweak baryogenesis. Part 2: The Classical regime. *Phys. Rev.*, D53:2958–2980, 1996.
[6] S. J. Huber and M. G. Schmidt. Electroweak baryogenesis: Concrete in a SUSY model with a gauge singlet. *Nucl. Phys.*, B606:183–230, 2001.
[7] N. Turok. Electroweak bubbles: Nucleation and growth. *Phys. Rev. Lett.*, 68:1803–1806, 1992.
[8] Bao-Hua Liu, Larry D. McLerran, and Neil Turok. Bubble nucleation and growth at a baryon number producing electroweak phase transition. *Phys. Rev.*, D46:2668–2688, 1992.
[9] S. Yu. Khlebnikov. Fluctuation - dissipation formula for bubble wall velocity. *Phys. Rev.*, D46:R3222–R3226, 1992.
[10] Peter Brockway Arnold. One loop fluctuation - dissipation formula for bubble wall velocity. *Phys. Rev.*, D48:1539–1545, 1993.
[11] Guy D. Moore and Tomislav Prokopec. How fast can the wall move? A Study of the electroweak phase transition dynamics. *Phys. Rev.*, D52:7182–7204, 1995.
[12] P. John and M. G. Schmidt. Do stops slow down electroweak bubble walls? *Nucl. Phys.*, B598:291–305, 2001. [Erratum: Nucl. Phys.B648,449(2003)].
[13] Guy D. Moore. Electroweak bubble wall friction: Analytic results. *JHEP*, 03:006, 2000.
[14] Dietrich Bodeker and Guy D. Moore. Can electroweak bubble walls run away? *JCAP*, 0905:009, 2009.
[15] Jose R. Espinosa, Thomas Konstandin, Jose M. No, and Geraldine Servant. Energy Budget of Cosmological First-order Phase Transitions. *JCAP*, 1006:028, 2010.
[16] Leonardo Leitao and Ariel Megevand. Gravitational waves from a very strong electroweak phase transition. *JCAP*, 1605(05):037, 2016.
[17] Andrey Katz and Antonio Riotto. Baryogenesis and Gravitational Waves from Runaway Bubble Collisions. *JCAP*, 1611(11):011, 2016.
[18] John David Jackson. *Classical Electrodynamics*. Wiley, 1998.
[19] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, ninth dover printing, tenth gpo printing edition, 1964.
[20] Glennys R. Farrar and John W. McIntosh, Jr. Scattering from a domain wall in a spontaneously broken gauge theory. *Phys. Rev.*, D51:5889–5904, 1995.