DC to AC converter on Abrikosov vortices in a washboard pinning potential

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Abstract. The nonlinear dynamics of Abrikosov vortices in a cosine dc-biased washboard pinning potential at nonzero temperature is theoretically investigated. The problem is treated relying upon the exact solution of the Langevin equation for non-interacting vortices by using the Fokker-Planck method combined with the scalar continued fractions technique. The time variation of the local mean vortex velocity $v(t)$ is calculated. The time voltage $E(t) \propto v(t)$ is predicted to oscillate with a dc current-dependent frequency and a tunable pulse shape. Formulas for $v(t)$ are discussed as functions of dc transport current and temperature, in a wide range of the corresponding dimensionless parameters. The derived expressions can be adapted for a number of physical applications utilizing the overdamped motion of a Brownian particle in a tilted cosine potential, e.g., the resistively shunted Josephson junction model.

1. Introduction

Described by the Langevin equation or by the equivalent Fokker-Planck equation, the problem of Brownian motion in a periodic potential arises in different fields of science and engineering, such as solid-state physics, chemical physics, and communication theory. Exemplary systems include Josephson junctions (JJs) [1], superionic conductors [2], ring laser gyroscopes [3], charge density waves [4], phase-locking loops [7] in radioengineering, the magnetization dynamics [6], the diffusion of colloidal particles in periodic structures [5], and the dynamics of Abrikosov vortices in superconductors [8, 9, 10]. The ubiquitous appearance of pulsed signal in response to constant drive in the presence of a periodic potential is generic to all these domains. For instance, the appearance of the ac voltage in the resistively shunted JJ problem at zero temperature was theoretically analyzed as early as forty years ago [1]. However, at non-zero temperature this problem has not been solved in either domain so far.

The objective of this work is to report the results of a theoretical treatment of the time-dependent nonlinear stochastic dynamics of the Abrikosov vortex in a tilted cosine washboard pinning potential (WPP). We will show that for the cosine WPP, in the approximation of non-interacting vortices, it is possible to derive exact expressions for the ac voltage arising in response to a dc transport current. The predicted results can be experimentally examined, e.g., on superconductor thin films with a WPP landscape [11], e.g., similar to that shown in figure 1(a). Besides, the derived expressions can be adapted for many of the aforementioned research domains.
For the distribution function $\eta$ represented by a Gaussian white noise and follows. First, we decompose the probability function $P$ the driving force, whereas the second term is the contribution due to the thermal fluctuations.

In the general case, the transport current density $j$ is tilted at an angle $\alpha$ with respect to $y$. $F_p$ is the average pinning force provided by the WPP, $F_L$ is the Lorentz force for a vortex, and $B$ is the magnetic field vector directed perpendicular to the film plane. A typical layout of contacts in the thin-film geometry is shown in the inset.

2. Main results

The geometry of the considered system is sketched in figure 1(b). Our objective is to calculate the time voltage arising due to vortices on move under the action of the dc transport current $j$. We treat that system on the basis of the Langevin equation for a vortex moving with an actual stochastic velocity $v$ in a magnetic field $B = nB$. Here $B \equiv |B|, n = nz, n = \pm 1,$ and $z$ is the unit vector in the $z$ direction. Neglecting the Hall effect the Langevin equation reads [8, 9, 10]

$$\eta v = F_L + F_p + F_{th}, \quad (1)$$

where $F_L = n(\Phi_0/c)j \times z$ is the Lorentz force, $\Phi_0$ is the magnetic flux quantum, $c$ is the speed of light, and $j$ is the dc transport current density. $F_p = -\nabla U_p(x)$ is the anisotropic pinning force, where $U_p(x)$ is a WPP. As usually [9, 10], the WPP is modeled by $U_p(x) = (U_p/2)(1 - \cos kx)/2$, where $k = 2\pi/a$, $a$ is the period, and $U_p$ is its depth. $F_{th}$ is the thermal fluctuation force represented by a Gaussian white noise and $\eta$ is the vortex viscosity constant.

In the general case, equation (1) can be reduced to the system of Fokker-Planck equations [7] for the distribution function $P = P(r, t)$ associated with the probability density of finding the vortex at the point $r = (x, y)$ at the time $t$. This stationary distribution function allows one to calculate the probability flux density of the vortex $S(r, t) \equiv P(r, t)v(r, t) = \text{const}$ and to derive expressions for the experimentally observable ac voltage in terms of a continued fraction. For the sake of simplicity, in the following we consider the one-dimensional vortex motion along the $x$ axis, i.e., when $j$ flows along the WPP channels as in the inset of figure 1(b). Then, the stationary dimensionless Fokker-Planck equation reads [12]

$$S = P(\xi - \sin x) - (1/g)(dP/dx), \quad (2)$$

where $S = P(x)v(x), \xi = F_{Lx}/F_p$ is the dimensionless dc driving force, $kx \to x$ is the new dimensionless coordinate, and $g = U_p/2T$ is the dimensionless inverse temperature. In the right part of equation (2) the first term stands for the deterministic vortex drift in the direction of the driving force, whereas the second term is the contribution due to the thermal fluctuations.

The solution of equation (2) can be found in terms of an ordinary continued fraction as follows. First, we decompose the probability function $P(x)$ into a series of Fourier amplitudes $c_m$ as $P(x) = \sum_{m=-\infty}^{\infty} c_m e^{imx}$, where $c_m = c_{-m}^*$ and the asterisk denotes the complex conjugate. Then, equation (2) can be reduced to the following tridiagonal recurrence equation

$$(\xi - im/g)c_m + (i/2)c_{m-1} - (i/2)c_{m+1} = S\delta_{m0}, \quad (3)$$

Figure 1. (a) Atomic force microscope image of an exemplary WPP landscape fabricated by focused ion beam milling [11] on the surface of a Nb thin film. (b) The system of coordinates $xy$ with the unit vectors $x$ and $y$ is associated with the WPP channels which are parallel to the vector $y$. The geometry of the considered system is sketched in figure 1(b). The contacts in the thin-film geometry is shown in the inset.
Figure 2. The coordinate dependence of the probability distribution function \( P(x) \) calculated by equation (5) at (a) subcritical, (b) critical and (c) overcritical transport currents for a set of inverse temperatures, as indicated. At subcritical tilts, the maximum in \( P(x) \) corresponds to the minimum in the effective pinning potential \( U_{\text{eff}}(x) \equiv U_p(x) - F_L x \).

where \( \delta_{m0} \) is Kronecker’s delta and we have used the relation \[ \langle \sin x \rangle = i \left( c_m + 1 - c_{m-1} \right) / 2 \] with \( \langle \ldots \rangle \) meaning the statistical averaging. For \( m \geq 1 \) equation (3) gives an infinite continued fraction for \( c_1/c_0 = c_1^*/c_0 \). If one then uses this equation for \( m = 0 \) and the normalization condition for \( P(x) \) in the form \[ \int_0^{2\pi} P(x) dx = 1 \] corresponding to \( c_0 = 1/2\pi \), it is possible to express \( S_m \equiv 2\pi c_m \) as the following continued fraction

\[
S_m = \frac{1/4}{i\xi + m/g + \frac{1/4}{i\xi + (m+1)/g + \frac{1/4}{i\xi + (m+2)/g + \ldots}}}, \tag{4}
\]

From equations (3) and (4) follows the next expression for the stationary distribution function

\[
P(x, g, \xi) = 1/2\pi + (1/\pi) \sum_{m=1}^{\infty} \text{Re}[e^{im\xi}(S_m\ldots S_1)], \tag{5}
\]

whose graphs for various values of the parameters \( g \) and \( \xi \) are shown in figure 2. Using the stationary \( P(x) \) one can treat the time-dependent periodic dynamics for a vortex moving in the cosine WPP under the influence of a dc drive. Due to the stationarity of the probability current \( v(x) = S/P(x) \) we conclude that \( t = \int_0^x dx'/v(x') \) or

\[
t = \frac{1}{S} \int_0^x P(x') dx', \tag{6}
\]

where the time \( t \) is scaled in units of the relaxation time \( \hat{\tau} \equiv \eta/k_Fp \).

From equation (6) one finds \( t = t(x) \) and its inversion results in \( x = x(t) \). Finally, we arrive at the main result of our study, \( v(t) = v[x(t)] \), which determines the time variation of the voltage \( E(t) \propto v(t) \). In equation (6), \( 0 < x < 2\pi \) and \( 0 < t < \theta \), where \( 2\pi \) and \( \theta = 1/S \) are the dimensionless spatial and temporal periods of the vortex motion. The dependences \( v(t) \) are plotted in figure 3 in the different dc current regimes for a wide range of \( g \). Consider these curves in more detail. Depending on the bias value \( \xi \) the behavior of the curves differs substantially. In contrast to the well-known zero-temperature results in the resistively shunted JJ problem [1], accounting for the thermal fluctuations allows the vortex to overcome the WPP barrier at subcritical dc biases. Overcoming of each WPP barrier is accompanied by a voltage pulse, as depicted in figure 3(a). With increasing bias the time between two neighboring pulses becomes
Figure 3. The time dependence of the mean local vortex velocity $v(t)$ at (a) small, (b) critical, and (c) overcritical dc currents for a set of inverse temperatures, as indicated. In (a) the curves are cut off to simplify reading the data.

shorter and a notable “voltage floor” appears in figure 3(b) with decreasing $g$. Averaging the time voltage over the oscillation period and plotting it versus $\xi$ results in the conventional current-voltage characteristics [1]. With further increasing $\xi$ the “voltage floor” is rising [figure 3(c)] and the pulses become harmonic in the limit of very strong dc currents.

3. Conclusion
In this work we have theoretically treated the nonlinear dynamics of Abrikosov vortices in a cosine dc-biased washboard pinning potential at nonzero temperature. The corresponding Langevin equation for non-interacting vortices has been solved by using the Fokker-Planck method combined with the scalar continued fractions technique. The main result of this work is the derived time variation of the local mean vortex velocity $v(t)$ responsible for the observable ac voltage $E(t)$. The latter is predicted to oscillate with a dc current-dependent frequency and a tunable pulse shape ranging from very short, delta function-like peaks in the limit of small dc current $\xi$ to the well-known harmonic form in the limit of strong $\xi$. The reported model structure can be adapted for a number of physical applications (see Introduction) utilizing the overdamped motion of a Brownian particle in a tilted cosine potential.

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