General Neveu-Schwarz Correlators in Super Liouville Theory

E. Abdalla\textsuperscript{1}, M.C.B. Abdalla\textsuperscript{2},
D. Dalmazi\textsuperscript{2}, Koji Harada\textsuperscript{3}

\textsuperscript{1}Instituto de Física, Univ. São Paulo, CP 20516, São Paulo, Brazil

\textsuperscript{2}Instituto de Física Teórica, UNESP, Rua Pamplona 145,
CEP 01405, São Paulo, Brazil

\textsuperscript{3}Department of Physics, Kyushu University, Fukuoka 812, Japan

Abstract

In this paper we compute the N-point correlation functions of the tachyon operator from the Neveu Schwarz sector of super Liouville theory coupled to matter fields (with \( \hat{c} \leq 1 \)) in the super Coulomb gas formulation, on world sheets with spherical topology. We first integrate over the zero mode assuming that the \( s \) parameter takes an integer value, subsequently we continue the parameter to an arbitrary real number. We included an arbitrary number of screening charges (s.c.) and as a result, after renormalizing the s.c., the external legs and the cosmological constant, the form of the final amplitudes do not modify. Remarkably, the result is completely parallel to the bosonic case. We also completed a discussion on the calculation of bosonic correlators including arbitrary screening charges.
1- Introduction

Matrix models proved to be an efficient means to obtain information about non-critical string theory, specially in computations of correlation functions\(^1\). However, supersymmetric extensions\(^2\) are, up to now, not constructed, due to some technical difficulties.

On the other hand, in the continuum approach, we have to deal with Liouville theory\(^3\). Unfortunately, we still do not understand it very well in spite of much effort. In particular, it has been difficult to calculate correlation functions in a reliable way because perturbation theory does not apply. Recently, however, several authors\(^4\)–\(^10\) succeeded in taming the difficulties of the Liouville theory and computed exactly correlation functions in the continuum approach to conformal fields coupled to two dimensional gravity. The technique is based on the integration over the zero mode of the Liouville field. The resulting amplitude is a function of a parameter \(s\) which depends on the central charge and on the external momenta. The amplitudes can be computed when the above parameter is a non-negative integer. Later on, one analytically continues that parameter to real (or complex) values. The results thus obtained agree with the matrix model approach.

An advantage of the Liouville approach is that it is very easy to extend it to supersymmetric theories\(^12\). Since matrix models seem to be less powerful (up to now) in this point, it is natural to investigate supersymmetric Liouville theory in order to know something about 2-D supergravity (or off-critical NSR string).

Our aim here is to investigate the supersymmetric Liouville theory. We shall compute supersymmetric correlation functions on world sheets with spherical topology in the Neveu-Schwarz sector, where the super-Liouville is coupled to superconformal matter with central charge \(\hat{c} \leq 1\), represented as a super Coulomb gas\(^12\),\(^13\). The results are remarkable, and very parallel to the bosonic case; it is equivalent to a redefinition of the cosmological constant, and of the primary superfields, the resulting amplitudes have the same form as those of the bosonic theory obtained by Di Francesco and Kutasov\(^5\). The super Liouville theory\(^14\) has also been studied along similar lines. Our present results generalize those presented in a recent paper\(^15\), as well as others recently obtained in the literature\(^16\).

This paper is divided as follows: in section 2 we review the computations of bosonic correlators, and include the case with an arbitrary number of screening charges. In section 3 we calculate the \(N\)-point tachyon (Neveu-Schwarz) correlation functions also with screening charges. Results are remarkably similar to the bosonic case\(^5\). In section 4 we draw some conclusions. In Appendix A, we calculate the supersymmetric generalization of equation (B.10) of Dotsenko and Fateev.

2- Bosonic Correlators

2.1- The 3-point tachyon amplitude

In a recent paper Di Francesco and Kutasov\(^6\) calculated the \(N\)-point tachyon correlation functions in Liouville theory on world sheets with spherical topology, coupled to
$c \leq 1$ conformal matter in a Coulomb gas representation. They worked in the DDK’s framework\textsuperscript{17} where the total action is given by:

$$S = \frac{1}{2\pi} \int d^2 w \sqrt{g} \left[ g^{ab} \partial_a \phi \partial_b \phi - \frac{Q}{4} \hat{R} \phi + 2\mu e^{\alpha \phi} + \hat{g}^{ab} \partial_a X \partial_b X + \frac{i\alpha_0}{2} \hat{R} X \right] , \quad (2.1)$$

where $\phi$ is the Liouville mode and $X$ represents the matter with $c = 1 - 12\alpha_0^2$. Following DDK, the constant $Q$ is determined by imposing a vanishing total central charge,

$$Q = \sqrt{\frac{25 - c}{3}} = 2\sqrt{2 + \alpha_0^2} , \quad (2.2)$$

and $\alpha$ is determined by requiring $e^{\alpha \phi}$ to be a (1,1) conformal operator, i.e., $-\frac{1}{2} \alpha (\alpha + Q) = 1$. We define the solutions to this equation by:

$$\alpha_\pm = -\frac{Q}{2} \pm |\alpha_0| , \quad \alpha_+ \alpha_- = 2 ; \quad (2.3)$$

the semiclassical limit ($c \to \infty$) fixes $\alpha = \alpha_+$. In the following we calculate gravitationally-dressed tachyon amplitudes:

$$\langle T_{k_1} \cdots T_{k_N} \rangle = \left\langle \prod_{j=1}^N \int d^2 z_j e^{ik_j X(z_j) + \beta(k_j) \phi(z_j)} \right\rangle \quad (2.4)$$

the dressing parameter $\beta$ is fixed imposing $e^{ik_j X + \beta_j \phi}$ to be a (1,1) conformal operator:

$$\beta_j = \beta(k_j) = -\frac{Q}{2} + |k_j - \alpha_0| . \quad (2.5)$$

An important ingredient in the calculation of $\langle T_{k_1} \cdots T_{k_N} \rangle$ is the integration over the matter ($X_0$) and the Liouville ($\phi_0$) zero modes. We make the following split\textsuperscript{4,5}

$$\phi = \phi_0 + \tilde{\phi}$$

$$X = X_0 + \tilde{X} \quad (2.6)$$

where the fields $\tilde{\phi}$ and $\tilde{X}$ obey the condition

$$\int d^2 w \tilde{\phi} = \int d^2 w \tilde{X} = 0 . \quad (2.7)$$

The integration over the zero modes $X_0$ and $\phi_0$ gives the following results:

$$\int_{-\infty}^{\infty} DX_0 e^{iX_0 \left( \sum_{i=1}^N k_i - 2\alpha_0 \right)} = 2\pi \delta \left( \sum_{i=1}^N k_i - 2\alpha_0 \right) , \quad (2.8a)$$
\[
\int_{-\infty}^{\infty} D\phi_0 e^{i\phi_0 (\sum_{j=1}^{N} \beta_j + Q)} - e^{\alpha_+ \phi_0 \left( \frac{\mu}{\pi} \int d^2 w e^{\alpha_+ \phi} \right)^s} = \frac{\Gamma(-s)}{-\alpha_+} \left( \frac{\mu}{\pi} \int d^2 w e^{\alpha_+ \phi} \right)^s,
\]

where
\[
s = -\frac{1}{\alpha_+} \left( \sum_{j=1}^{N} \beta_j + Q \right).
\]

And we have used that on the sphere \( \frac{1}{8\pi} \int d^2 w \sqrt{g} \hat{R} = 1 \). We thus obtain
\[
\langle T_{k_1} \cdots T_{k_N} \rangle = 2\pi \delta \left( \sum_{j=1}^{N} k_j - 2\alpha_0 \right) A_N(k_1 \cdots k_N)
\]

\[
A_N(k_1 \cdots k_N) = \frac{\Gamma(-s)}{-\alpha_+} \left( \frac{\mu}{\pi} \right)^s \left( \prod_{j=1}^{N} \int d^2 z_j e^{i k_j + \beta_j \phi(z_j)} \left( \int d^2 w e^{\alpha_+ \phi} \right)^s \right)_0
\]

where \( \langle \cdots \rangle_0 \) means that now the correlation functions are calculated as in the free theory \( (\mu = 0) \). The strategy to obtain \( A_N \) is to assume first that \( s \) is a non-negative integer and to continue the result to any real \( s \) at the end. Thus, using free propagators:

\[
\langle X(w)X(z) \rangle_0 = \langle \phi(w)\phi(z) \rangle_0 = \ln |w - z|^{-2}
\]

and fixing the residual \( SL(2 \mathbb{C}) \) invariance of the conformal gauge on the sphere by choosing \( (z_1 = 0, z_2 = 1, z_3 = \infty) \), we have in the case of the 3-point function:

\[
A_3(k_1, k_2, k_3) = \frac{\Gamma(-s)}{-\alpha_+} \left( \frac{\mu}{\pi} \right)^s \int \prod_{j=1}^{3} d^2 w_j |w_j|^{2\alpha} |1 - w_j|^{2\beta} \prod_{i<j}^{3} |w_i - w_j|^{4\rho}
\]

Above, we defined \( \alpha = -\alpha_+ \beta_1, \beta = -\alpha_+ \beta_2, \rho = -\alpha_+^2 / 2 \). The above integral has been calculated by Dotsenko and Fateev\(^1\) (see formula (B.9), of the second paper). Using their result Di Francesco and Kutasov obtained:

\[
A_3(k_1, k_2, k_3) = \frac{\Gamma(-s)}{-\alpha_+} \left( \frac{\mu}{\pi} \right)^s \Gamma(s + 1)[\Delta(1 - \rho)]^s \prod_{i=1}^{s} \Delta(i\rho)
\]

\[
\times \prod_{i=0}^{s-1} \Delta(1 + \alpha + i\rho)\Delta(1 + \beta + i\rho)\Delta(-1 - \alpha - \beta - (s - 1 + i)\rho)
\]

where \( \Delta(x) = \Gamma(x)/\Gamma(1 - x) \). Choosing the kinematics\(^2\) \( k_1, k_3 \geq \alpha_0 \), \( k_2 < \alpha_0 \) we can eliminate \( \beta \) using (2.6), (2.9) and the momentum conservation: \( \sum_{i=1}^{3} k_i = 2\alpha_0 \):.

\[
\beta = \begin{cases} 
\rho(1 - s), & \alpha_0 > 0 \\
-1 - \rho, & \alpha_0 < 0
\end{cases}
\]

\(^1\) Hereafter we drop the tilde in the fields defined by (2.9), since no confusion can occur.

\(^2\) Notice that our notation differs from Ref.[6] by the exchange of \( k_2 \) and \( k_3 \).
Back in (2.13) it is easy to see that for $\alpha_0 > 0$ there appears a factor $\Gamma(0)$ in the denominator of $A_3$ and the amplitude vanishes identically. For $\alpha_0 < 0$, using

$$\frac{1}{2}(\beta^2_1 - k_1^2) = \rho - \alpha \quad ,$$
$$\frac{1}{2}(\beta^2_2 - k_2^2) = 1 + \alpha(s - 1)\rho \quad ,$$
$$\frac{1}{2}(\beta^2_3 - k_3^2) = -s \quad ,$$

we can write the 3-point amplitude in a rather compact form

$$A_3 = [\mu \Delta(-\rho)]^s \prod_{j=1}^{3} \Delta\left(\frac{1}{2}(\beta^2_j - k_j^2)\right) .$$

Thus, after the redefinitions of the cosmological constant and of the external fields as

$$\mu \rightarrow \frac{\mu}{\Delta(-\rho)} \quad , \quad T_{k_j} \rightarrow \frac{T_{k_j}}{\Delta\left(\frac{1}{2}(\beta^2_j - k_j^2)\right)} \quad ,$$

Di Francesco and Kutasov\(^6\) obtained for the three-point function

$$A_3 = \mu^s \quad ,$$

which is also obtained in the matrix model approach. In the next sub-sections we shall see that this expression holds for general N-point tachyon amplitudes with an arbitrary number of s.c..

**2.2- The 3-point tachyon amplitude with an arbitrary number of screening charges**

Now we show explicitly how one generalizes the previous calculation to the case which includes an arbitrary number of screening charges in the matter sector. We introduce $n$ operators $e^{id_1X}$ and $m$ operators $e^{id_2X}$, with $d_\pm$ solutions of: $\frac{1}{2}d(d - 2\alpha_0) = 1$, $(d_+ d_- = -\alpha_+ \alpha_- = -2)$. Integrating over the zero-modes again we get:

$$\left\{ T_{k_1} T_{k_2} T_{k_3} \left( \frac{1}{n!} \prod_{i=1}^{n} \int d^2 t_i e^{id_1 X(t_i)} \right) \left( \frac{1}{m!} \prod_{i=1}^{m} \int d^2 r_i e^{id_2 X(r_i)} \right) \right\} = 2\pi \delta\left( \sum_{i=1}^{3} k_i + nd_+ + md_- - 2\alpha_0 \right) A_3^{nm}(k_1, k_2, k_3)$$

(2.19)
where the amplitude $A_{3}^{nm}(k_1, k_2, k_3)$ is given by the expression

$$
A_{3}^{nm}(k_1, k_2, k_3) = \frac{\Gamma(-s)}{-\alpha_+} \left( \frac{\mu}{\pi} \right)^s \prod_{i=1}^{n} \int d^2 t_i |t_i|^{2\hat{\alpha}} |1 - t_i|^{2\hat{\beta}} \prod_{i<j}^{n} |t_i - t_j|^{4\hat{\rho}} \\
\times \prod_{i=1}^{m} \int d^2 r_i |r_i|^{2\tilde{\alpha}'} |1 - r_i|^{2\tilde{\beta}'} \prod_{i<j}^{m} |r_i - r_j|^{4\tilde{\rho}'}
$$

(2.20)

The parameters $\alpha, \beta$ and $\rho$ are defined as before, and the remaining parameters are

$$\tilde{\alpha} = d_+ k_1 \quad , \quad \tilde{\beta} = d_+ k_2 \quad , \quad \tilde{\rho} = \frac{1}{2} d_+^2
$$

$$\tilde{\alpha}' = d_- k_1 \quad , \quad \tilde{\beta}' = d_- k_2 \quad , \quad \tilde{\rho}' = \frac{1}{2} d_-^2
$$

(2.21)

Notice that the gravitational part of the amplitude (integrals over $z_i$) is the same as in the case without screening charges. The integrals over $t_i$ and $r_j$ (matter contributions) have been also calculated by Dotsenko and Fateev12 (See their formula (B.10)); the result turns out to be

$$
A_{3}^{nm} = \left( \frac{\mu}{\pi} \right)^s \Gamma(-s) \Gamma(s+1) \pi^{s+n+m} \tilde{\rho}^{-4nm}[\Delta(1 - \tilde{\rho})]^n[\Delta(1 - \tilde{\rho}')]^m \prod_{i=1}^{m} \Delta(n - \tilde{\rho}') \prod_{i=1}^{n} \Delta(n - \tilde{\rho})
$$

$$\times \prod_{i=0}^{m-1} \Delta(1 + \tilde{\alpha} + i\tilde{\rho}) \Delta(n - \tilde{\beta} + i\tilde{\rho}) \Delta(1 + 2\tilde{\rho} - \tilde{\alpha} - \tilde{\beta} - (n - 1 + i)\tilde{\rho})
$$

$$\times \prod_{i=0}^{n-1} \Delta(1 + \tilde{\alpha}' + i\tilde{\rho}') \Delta(n - \tilde{\beta}' + i\tilde{\rho}') \Delta(1 + 2\tilde{\rho}' - \tilde{\alpha}' - \tilde{\beta}' - (n - 1 + i)\tilde{\rho}')$$

$$\times \left[ \Delta(1 - \rho) \right]^s \prod_{i=1}^{s} \Delta(i\rho) \prod_{i=0}^{s-1} \Delta(1 + \alpha + i\rho) \Delta(1 + \beta + i\rho) \Delta(-1 - \alpha - \beta - (s - 1 + i)\rho)
$$

(2.22)

Using the same kinematics ($k_1, k_3 \geq \alpha_0, k_2 < \alpha_0$) we eliminate $\beta$ again using (2.5), (2.9) and momentum conservation $\sum_{i=1}^{3} k_i + nd_+ + md_- = 2\alpha_0$,

$$\beta = \begin{cases} 
\rho(m + 1 - s) + n , & \alpha_0 > 0 \\
1 - m - (s + n)\rho , & \alpha_0 < 0
\end{cases}
$$

It is easy to see, assuming $s \geq m + 2$, that for $\alpha_0 > 0$ the amplitude vanishes again due to a factor $\Gamma(-n)$ in the denominator in the gravitational part of the amplitude. Therefore we concentrate now on the $\alpha_0 < 0$ case where we have

$$\tilde{\alpha} = \alpha - 2\rho \quad , \quad \tilde{\alpha}' = -2 + \rho\alpha$$

$$\tilde{\beta} = m - 1 + (s + n)\rho \quad , \quad \tilde{\beta}' = s + n + \rho^{-1}(m - 1)
$$

$$\tilde{\rho} = -\rho \quad , \quad \tilde{\rho}' = -\rho^{-1}
$$

(2.23)
Substituting in (2.22) we obtain:

\[ A_{3}^{nm} = \left( \frac{\mu}{\pi} \right)^{s} \Gamma(-s) \Gamma(s + 1) \pi^{s+n+m}(\bar{\rho})^{-4n} (C_{M+G}D_{M}E_{M+G}) \]  

(2.24)

where

\[ C_{M+G} = [\Delta(1 + \rho^{-1})]^{m} \Delta(1 + \rho)^{n} \prod_{i=1}^{m} \Delta(i\rho^{-1} - n) \prod_{i=1}^{n} \Delta(-i\rho) \]

\[ \times [\Delta(1 - \rho)]^{s} \prod_{i=1}^{s} \Delta(i\rho) \]

\[ \times \prod_{i=0}^{n-1} \Delta(m + (s + n - i)\rho) \prod_{i=0}^{s-1} \Delta(-m + (s + n - i)\rho) \]

\[ \times \prod_{i=0}^{m-1} \Delta(1 + s + (m - 1 - i)\rho^{-1}) \]  

(2.25a)

\[ D_{M} = \prod_{i=0}^{n-1} \Delta(-1 - n + \rho^{-1}\alpha - i\rho^{-1}) \Delta(1 - s - \rho^{-1}\alpha + i\rho^{-1}) \]  

(2.25b)

\[ E_{M+G} = \prod_{i=0}^{s-1} \Delta(1 + \alpha - (i + 2)\rho) \Delta(m - \alpha - (s - 1 - i)\rho) \]

\[ \prod_{i=0}^{s-1} \Delta(1 + \alpha + i\rho) \Delta(m - \alpha + (n + 1 - i)\rho) \]  

(2.25c)

To get a simple expression for \( A_{3}^{nm} \) we look for \( \Delta(\rho - \alpha)\Delta(\rho(s-n+1)+\alpha-m+1)\Delta(-m\rho^{-1} - (s+n)) \) which corresponds to \( \prod_{i=1}^{3} \Delta(\frac{1}{2}(\beta_{i}^{2} - k_{i}^{2})) \). We expect that these terms show up in the result. For example:

\[ E_{M+G} = \prod_{i=2}^{n+1} \Delta(1 + \alpha - i\rho) \prod_{i=0}^{s-1} \Delta(1 + \alpha + i\rho) \prod_{i=0}^{s-1} \Delta(m - \alpha - i\rho) \prod_{i=0}^{s-2} \Delta(m - \alpha + i\rho) \]  

(2.25d)

Using \( \Delta(x)\Delta(1 - x) = 1 \), and \( \Delta(x+1) = -x^{2}\Delta(x) \), we easily get:

\[ E_{M+G} = (-)^{m(s+n+1)} \Delta(1 - m + \alpha + (s - n + 1)\rho) \Delta(\rho - \alpha) \]

\[ \times \prod_{1-s}^{n+1} (m - 1 + \alpha + i\rho)^{2}(m - 2 + \alpha + i\rho) \cdots (\alpha + i\rho)^{2} \]  

(2.26a)

Analogously, we also arrive at

\[ D_{M} = (-)^{m(n+s+1)} \rho^{2m(n+s+1)} \prod_{1-s}^{n+1} \left[ \frac{\Gamma(-\alpha + i\rho)}{\Gamma(m - \alpha + i\rho)} \right]^{2} \]  

(2.26b)

\[ C_{M+G} = (-)^{s}\rho^{-2(s+n)+2m(n-s)} \frac{\Delta(1 + \rho^{-1})^{m} \Delta(1 + \rho)^{n} \Delta(1 - \rho)^{s}}{\Gamma(-s)\Gamma(s + 1)} \]  

(2.26c)
Now substituting (2.26) into (2.24) we have

\[
\mathcal{A}_{nm}^3 = [\mu \Delta(-\rho)]^s \left[ -\pi \Delta(\rho^{-1}) \right]^m \left[ -\pi \Delta(\rho) \right]^m \prod_{i=1}^3 (-\pi) \Delta \left( \frac{1}{2} (\beta_i^2 - k_i^2) \right) .
\]  

(2.27)

Therefore redefining the screening operators as

\[
e^{id+X} \rightarrow \frac{1}{\Delta(\rho)} e^{id+X} \\
e^{id-X} \rightarrow \frac{1}{\Delta(\rho^{-1})} e^{id-X},
\]

(2.29a, 2.29b)

the operators \( T_{k_i} \) and the cosmological constant \( \mu \) as before (see (2.17)), we get the very simple result:

\[
\mathcal{A}_{nm}^3 = \mu^s ,
\]

(2.29)

which should be compared to (2.18). This result has been also obtained by Di Francesco and Kutasov\(^6,16\). Note that the factors \( \Delta(\rho) \) and \( \Delta(\rho^{-1}) \) can be easily understood; the screening operators are renormalized like the tachyon vertex operators \( T_k \) with vanishing dressing \( \beta(k) \).

### 2.3- N-Point tachyon amplitude (\( N \geq 4 \)) with an arbitrary number of screening charges

Repeating the zero-mode trick in the most general case of an \( N \)-point function with arbitrary screening charges we have\(^3\):

\[
\mathcal{A}_{nm}^N = (-\pi)^3 \left( \frac{\mu}{\pi} \right)^s \Gamma(-s) \prod_{i=1}^N \int d^2 z_i \prod_{j=1}^n \int \frac{d^2 t_j}{n!} \prod_{k=1}^m \int \frac{d^2 r_k}{m!} \times \prod_{l=1}^s \int d^2 w_l \left( e^{ik_i X(z_i)} e^{id+X(t_j)} e^{id-X(r_n)} \right) \left( e^{\beta_i \phi(z_i)} e^{\alpha_+ \phi(w_l)} \right)_0 ,
\]

(2.30)

\(^3\) We have absorbed a factor \( \pi^3/\alpha_+ \) in the measure of the path integral.
where \( s = -\frac{1}{\alpha_+} (\sum_{i=1}^{N} \beta_i + Q) \). Fixing the \( SL(2, \mathbb{C}) \) symmetry we get:

\[
A_N^{nm} = (-\pi)^3 \left( \frac{\mu}{\pi} \right)^s \Gamma(-s) I_N^{nm} ,
\]

\[
I_N^{nm} = \int \prod_{j=4}^{N} d^2 z_j |z_j|^{2\alpha_j'} |1 - z_j|^{2\beta_j} \prod_{i<j}^{N} |z_i - z_j|^{4\rho_{ij}} \times \int \prod_{i=1}^{8} d^2 w_i |w_i|^{2\alpha_i} |1 - w_i|^{2\beta} \prod_{i<j}^{s} |w_i - w_j|^{4\tilde{\rho}} \prod_{i=1}^{s} \prod_{j=4}^{N} |w_i - z_j|^{2\beta_j} \times \prod_{i=1}^{n} d^2 t_i |t_i|^{2\tilde{\alpha}_i} |1 - t_i|^{2\tilde{\beta}_i} \prod_{i<j}^{n} |t_i - t_j|^{4\tilde{\rho}} \prod_{i=1}^{n} \prod_{j=4}^{N} |z_j - z_i|^{2\tilde{\alpha}_j} \times \int \prod_{i=1}^{m} d^2 r_i |r_i|^{2\tilde{\alpha}'_i} |1 - r_i|^{2\tilde{\beta}'_i} \prod_{i<j}^{m} |r_i - r_j|^{4\tilde{\rho}'} \prod_{i=1}^{m} \prod_{j=4}^{N} |z_j - r_i|^{2\tilde{\alpha}'_j} \times \prod_{i=1}^{n} \prod_{j=1}^{m} |t_i - r_j|^{-4} ,
\]

(2.31)

where \( \alpha, \beta, \rho, \tilde{\alpha}, \tilde{\beta}, \tilde{\rho}, \tilde{\alpha}', \tilde{\beta}', \tilde{\rho}' \) are defined as

\[
\alpha_j = k_1 k_j - \beta_1 \beta_j , \quad \tilde{\alpha}_j = d_+ k_j \]

\[
\beta_j = k_2 k_j - \beta_2 \tilde{\beta}_j , \quad \tilde{\alpha}_j = d_- k_j
\]

\[
\rho_{ij} = \frac{1}{2} (k_l k_j - \beta_l \beta_j) , \quad 4 \leq j, l \leq N .
\]

(2.32)

The integral \( I_N^{nm} \) for the case \( n = m = 0 \) has been calculated by Di Francesco and Kutasov\(^6\). We shall use the same technique for arbitrary \( n, m \). First we notice that translation invariance (\( w_j \rightarrow 1 - w_i \), \( z_i \rightarrow 1 - z_i \), \( t_i \rightarrow 1 - t_i \), \( r_i \rightarrow 1 - r_i \)) implies (\( \alpha \leftrightarrow \beta \), \( \alpha' \leftrightarrow \beta' \), \( \tilde{\alpha} \leftrightarrow \tilde{\beta} \), \( \tilde{\alpha}' \leftrightarrow \tilde{\beta}' \)) so after the elimination of the remaining parameters as a function of \( \alpha, \beta, p_j \) and \( \rho \) (\( j = 4, 5, \ldots, N - 1 \)), \( I_N^{nm} \) exhibits an \( \alpha-\beta \) symmetry

\[
I_N^{nm}(\alpha, \beta, p_j, \rho) = I_N^{nm}(\beta, \alpha, p_j, \rho) .
\]

(2.33)

Similarly by the inversion of all variables \( w_i, z_i, t_i, r_i \) we have:

\[
I_N^{nm}(\alpha, \beta, p_j, \rho) = I_N^{nm}(-2 - \alpha - \beta - 2\rho(s - 1) - p_N - P, \beta, p_j, \rho)
\]

(2.34)

where \( P = \sum_{j=4}^{N-1} p_j \). Further information about \( I_N^{nm} \) can be obtained in the limit \( \alpha \rightarrow \infty \) (or \( \beta \rightarrow \infty \)), by using a technique applied by Dotsenko and Fateev\(^12\) in the case of contour integrals. Take for instance the simple case:

\[
I(\alpha, \beta) = \int d^2 w |w|^{2\alpha} |1 - w|^{2\beta} = \pi \Delta(1 + \alpha) \Delta(1 + \beta) \Delta(-1 - \alpha - \beta)
\]

(2.35)
by making a change of variables $w \rightarrow e^{-\frac{a}{\alpha}} \left( w^* \rightarrow e^{-\frac{a^*}{\alpha}} \right)$ we have:

$$I(\alpha \rightarrow \infty, \beta) \approx \alpha^{-2-2\beta} \tilde{I}(\beta). \quad (2.36)$$

This large-$\alpha$ behaviour can be checked by using Stirling’s formula ($\Gamma(\alpha+B) \sim \alpha^B \Gamma(\alpha)$) on the r.h.s. of equation (2.35). Applying this technique to $I_{N}^{nm}$ we get:

$$I_{N}^{nm} \approx \alpha^{2\beta+2\rho(s-N-n+3)+2P-2m} \quad (2.37)$$

where we have used the kinematics:

$$k_1, k_2, \cdots, k_{N-1} \geq \alpha_0, \ k_N < \alpha_0 \quad (2.38)$$

and assumed $\alpha_0 < 0$, which permits us to eliminate the remaining parameters in terms of $\alpha, \beta, p_j$ and $\rho$ as follows:

$$p_N = -1 - m - \rho(N + s + n - 3)$$

$$\beta' = \beta + p_j - 2\rho$$

$$\beta' = m - 1 + (\rho - \beta)(N + s + n - 3) - m\rho^{-1}\beta$$

$$\rho' = \frac{1}{2}(p_j + p_l) - \rho$$

$$\rho' = \frac{m - 1}{2} + \frac{(\rho - p_j)}{2}(N + s + n - 3) - \frac{m\rho^{-1}}{2}p_j$$

$$\tilde{\alpha} = \alpha - 2\rho$$

$$\tilde{\alpha}' = \alpha\rho^{-1} - 2$$

$$\tilde{\rho} = -\rho$$

$$\tilde{\beta} = \beta - 2\rho$$

$$\tilde{\beta}' = \beta\rho^{-1} - 2$$

$$\tilde{\rho}' = -\rho^{-1}$$

$$\tilde{\alpha}_j = p_j - 2\rho$$

$$\tilde{\alpha}_j' = \rho^{-1}p_j - 2$$

$$\tilde{\alpha}_N = m - 1 + \rho(N + s + n - 3)$$

$$\tilde{\alpha}_N' = N + s + n - 3 + \rho^{-1}(m - 1)$$

where $4 \leq j, l \leq N - 1$. Notice that eliminating $p_N$ the symmetry under inversion implies:

$$I_{N}^{nm}(\alpha, \beta, p_j, \rho) = I_{N}^{nm}(m - 1 - P - \alpha - \beta + \rho(N + n - 1 - s), \beta, p_j, \rho) \quad (2.40)$$

It is not difficult to check (using Stirling’s formula) that the following Ansatz is consistent with (2.33), (2.37) and (2.40):

$$A_{N}^{nm} = f_{N}^{nm}(\rho, p_j) \Delta(\rho - \alpha)\Delta(\rho - \beta)\Delta(1 - m + P + \alpha + \beta + \rho(s + 2 - N - n)) \quad (2.41a)$$

$$A_{N}^{nm} = f_{N}^{nm}(\rho, p_j) \prod_{j=1}^{3} \Delta \left( \frac{1}{2}(\beta_{j}^{2} - k_{j}^{2}) \right) \quad (2.41b)$$

Following Di Francesco and Kutasov\textsuperscript{6}, we can fix $f_{N}^{nm}(\rho, p_j)$ by using the 3-point function $A_{3}^{nm}$ through\textsuperscript{4}:

$$A_{N}^{nm}(k_1, k_2, k_j \rightarrow 0, k_N) = (-\pi)^{N-3} \frac{\partial}{\partial \mu} A_{3}^{nm}(k_1, k_2, k_N) \quad 3 \leq j \leq N - 1 \quad (2.42)$$

\textsuperscript{4} Notice that $\lim_{k_j \to 0} \beta_j = \alpha_+$, thus $s = -\frac{1}{\alpha_+} \left( \sum_{j=1}^{N} \beta_j + Q \right) \rightarrow \tilde{s} + 3 - N$ where $\tilde{s} = -\frac{1}{\alpha_+} \left( \sum_{j=1,2,N} \beta_j + Q \right)$.
Now using the result for $A_{3}^{nm}$ (formula (2.27)) we get:

$$f_{N}^{nm}(\rho, p) = [-\pi \Delta(\rho^{-1})]^{m}[-\pi \Delta(\rho)]^{n} \left( \frac{\partial^{N-3}}{\partial \mu} \right)_{s+N-3} \[\Delta(-\rho)]^{s} \prod_{j=4}^{N} (-\pi) \Delta\left(\frac{1}{2}(\beta_{j}^{2} - k_{j}^{2})\right). (2.43)$$

Back in (2.41) we have

$$A_{N}^{nm} = (s + N - 3)(s + N - 4) \cdots (s + 1) [\mu \Delta(-\rho)]^{s} \left( \frac{-\pi \Delta}{\Delta(\rho - 1)} \right)^{m} \left[ -\pi \Delta(\rho)^{n} \prod_{j=1}^{N} (-\pi) \Delta\left(\frac{1}{2}(\beta_{j}^{2} - k_{j}^{2})\right) \right], (2.44)$$

therefore, redefining the screening operators, $T_{kj}$ and $\mu$ as before, we have:

$$A_{N}^{nm} = \frac{\partial^{N-3}}{\partial \mu} \mu^{s+N-3} (2.45)$$

which is a remarkable result. We generalize this technique to the NS sector of the supersymmetric theory in the next section.

### 3- Supersymmetric Correlators

#### 3.1- The 3-point fermionic NS correlator

In a recent paper\(^{15}\) we have calculated the 3- and 4-point NS correlations functions using DHK formulation\(^{13}\) of super Liouville theory coupled to superconformal matter on the sphere without screening charge. The total action $S$ is given by $S = S_{SL} + S_{M}$ where

$$S_{SL} = \frac{1}{4\pi} \int d^{2}z \hat{E} \left( \frac{1}{2} \hat{D}_{\alpha} \Phi_{SL} \hat{D}^{\alpha} \Phi_{SL} - Q \hat{Y} \Phi_{SL} - 4i \mu e^{\alpha_{+} \Phi_{SL}} \right),$$

$$S_{M} = \frac{1}{4\pi} \int d^{2}z \hat{E} \left( \frac{1}{2} \hat{D}_{\alpha} \Phi_{M} \hat{D}^{\alpha} \Phi_{M} + 2i \alpha_{0} \hat{Y} \Phi_{M} \right), (3.1)$$

where $\Phi_{SL}, \Phi_{M}$ are super Liouville and matter superfields respectively. The matter sector has the central charge $\hat{c}_{m} = 1 - 8\alpha_{0}$. Analogous to the bosonic case the parameters $Q$ and $\alpha_{\pm}$ are given by (compare with (2.2))

$$Q = 2\sqrt{1 + \alpha_{0}^{2}}, \quad \alpha_{\pm} = -\frac{Q}{2} \pm \frac{1}{2} \sqrt{Q^{2} - 4} = -\frac{Q}{2} \pm |\alpha_{0}|, \quad \alpha_{+} \alpha_{-} = 1. (3.2)$$

The (gravitationally dressed) primary superfields $\tilde{\Psi}_{NS}$ are given by

$$\tilde{\Psi}_{NS}(z_{i}, k_{i}) = d^{2}z \hat{E} e^{ik \Phi_{M}(z)} e^{\frac{i}{2} \hat{D}_{\alpha} \Phi_{SL}(z)}$$
where \( \beta(k) = -\frac{Q}{2} + |k - \alpha_0| \).

In what follows we review the calculation of the three-point function of the primary superfield \( \tilde{\Psi}_{NS} \), that is:

\[
\langle \prod_{i=1}^{3} \int \tilde{\Psi}_{NS}(z_i, k_i) \rangle \equiv \int [D\hat{\Phi}_{SL}] [D\hat{\Phi}_{M}] \prod_{i=1}^{3} \tilde{\Psi}_{NS}(z_i, k_i) e^{-\mathcal{S}}.
\]

The method will closely parallel the bosonic case. After integrating over the bosonic zero modes we get

\[
\langle \prod_{i=1}^{3} \int \tilde{\Psi}_{NS}(z_i, k_i) \rangle \equiv 2\pi \delta \left( \sum_{i=1}^{3} k_i - 2\alpha_0 \right) \mathcal{A}(k_1, k_2, k_3),
\]

where \( \mathcal{A}(k_1, k_2, k_3) = \Gamma(-s) \left( \frac{-\pi}{2} \right)^3 (\frac{i\mu}{\pi})^s \left( \int d^2 \tilde{z}_i e^{ik_i \Phi_{M}(\tilde{z}_i)} e^{\beta_i \Phi_{SL}(\tilde{z}_i)} \left( \int d^2 \mathcal{Z} e^{\alpha_i + \Phi_{SL}(\mathcal{Z})} \right)^s \right) \)

(3.5)

where \( \langle \cdots \rangle_0 \) denotes again the expectation value evaluated in the free theory \( (\mu = 0) \) and we have absorbed the factor \( [\alpha_{+}(-\pi/3)]^{-1} \) into the normalization of the path integral.

The \( \hat{SL}_2 \) volume is treated analogously to the bosonic case. Indeed, the generators of the superconformal transformations on the \( (z, \theta) \) variables

\[
L_0 = z \partial_z + \frac{1}{2} \theta \partial_\theta - j
\]

\[
L_1 = \partial_z
\]

\[
L_{-1} = z^2 \partial_z + z \theta \partial_\theta - 2jz
\]

\[
Q^{1/2} = i \sqrt{\frac{1}{2}} (\partial_\theta + \theta \partial_z)
\]

\[
Q^{-1/2} = i \sqrt{\frac{1}{2}} (z \partial_\theta + z \theta \partial_z - 2j \theta)
\]

(3.7)
imply that we can fix \( z_1 = 0, z_2 = 1, z_3 = \infty, \theta_2 = \theta_3 = 0, \theta_1 = \theta \). The integral is the supersymmetric generalization of (B.9) of Ref.[12]. Alternatively, it is expressed in components (\( \Phi_{SL} = \phi + \theta \psi + \theta \bar{\psi} \)):

\[
\mathcal{A}(k_1, k_2, k_3) = \Gamma(-s) \left( -\frac{\pi}{2} \right)^3 \left( \frac{i\alpha_+^2 \mu}{\pi} \right)^s \beta_1^2 \int \prod_{i=1}^{s} d^2 z_i \prod_{i=1}^{s} |z_i|^{-2\alpha_+ \beta_1} |1 - z_i|^{-2\alpha_+ \beta_2} \prod_{i<j}^{s} |z_i - z_j|^{-2\alpha_+ \beta_2} \left( \bar{\psi} \psi(0) \bar{\psi}(z_1) \cdots \bar{\psi}(z_s) \right)_0 .
\]

(3.8)

We first observe that this is non-vanishing only for \( s \) odd \( (s = 2l + 1) \). One may evaluate \( \langle \bar{\psi} \cdots \bar{\psi} \rangle_0 \) and \( \langle \psi \cdots \psi \rangle_0 \) independently. Since the rest of the integrand is symmetric, one may write the result in a simple form by relabelling coordinates: (compare with \( A_3 \) bosonic formula (2.12)).

\[
\mathcal{A}(k_1, k_2, k_3) = \Gamma(-s) \left( -\frac{\pi}{2} \right)^3 \frac{1}{\alpha_+^2} \left( \frac{i\alpha_+^2 \mu}{\pi} \right)^s \alpha^2 (-1)^{s+1} s!! \int \prod_{i=1}^{s} d^2 z_i \prod_{i=1}^{s} |z_i|^{2\alpha} |1 - z_i|^{2\beta} \prod_{i<j}^{s} |z_i - z_j|^{4\rho} \prod_{i=1}^{s} |z_{2i-1} - z_{2i}|^{-2} |z_s|^{-2} .
\]

With \( z_s \equiv w, z_{2i-1} \equiv \zeta_i, z_{2i} \equiv \eta_i \), we have:

\[
\mathcal{A}(k_1, k_2, k_3) = -i \left( -\frac{\pi}{2} \right)^3 \Gamma(-s) \Gamma(s + 1) \frac{1}{\alpha_+^2} \left( \frac{\alpha_+^2 \mu}{\pi} \right) I^l(\alpha, \beta; \rho)
\]

(3.9)

where

\[
I^l(\alpha, \beta; \rho)
= \frac{1}{2\alpha^2} \int d^2 w \prod_{i=1}^{m} d^2 \zeta_i d^2 \eta_i |w|^{2\alpha - 2} |1 - w|^{2\beta} \prod_{i=1}^{m} |w - \zeta_i|^{4\rho} |w - \eta_i|^{4\rho} \times \prod_{i=1}^{m} |\zeta_i|^{2\alpha} |1 - \zeta_i|^{2\beta} |1 - \eta_i|^{2\beta} \prod_{i,j}^{m} |\zeta_i - \eta_j|^{4\rho} \prod_{i<j}^{m} |\zeta_i - \zeta_j|^{4\rho} |\eta_i - \eta_j|^{4\rho} \prod_{i=1}^{m} |\zeta_i - \eta_i|^{-2} ,
\]

and \( \alpha = -\alpha_+ \beta_1, \beta = -\alpha_+ \beta_2, \rho = -\frac{1}{2} \alpha_+^2 \). In the Ref.[15] we calculated \( I^l \) in detail by using the symmetries \( I^l(\alpha, \beta; \rho) = I^l(\beta, \alpha; \rho) \), \( I^l(\alpha, \beta; \rho) = I^l(-1 - \alpha - \beta - 4\rho, \beta; \rho) \) and
looking at its large $\alpha$ behavior (see Ref.[15]) we obtained:

$$I^l(\alpha, \beta; \rho) = -\frac{\pi^{2l+1}}{2^{2l}} \left[\Delta \left(\frac{1}{2} - \rho\right)\right]^{2l+1} \prod_{i=1}^{l} \Delta(2 + \rho) \prod_{i=1}^{l} \Delta \left(\frac{1}{2} + (2i + 1)\rho\right) \times \prod_{i=0}^{l} \Delta(1 + \alpha + 2i\rho) \Delta(1 + \beta + 2i\rho) \Delta(-\alpha - \beta + (2i - 4l)\rho)$$

$$\times \prod_{i=1}^{l} \Delta(\frac{1}{2} + \alpha + (2i - 1)\rho) \Delta(\frac{1}{2} + \beta + (2i - 1)\rho) \Delta(-\frac{1}{2} - \alpha - \beta + (2i - 4l - 1)\rho)$$

(3.11)

We can choose, without loss of generality $k_1, k_3 \geq \alpha_0, k_2 \leq \alpha_0$. We proceed now as in the bosonic case and we have (compare with (2.14))

$$\beta = \begin{cases} 
\rho(1 - s) & (\alpha_0 > 0) \\
-\frac{1}{2} - \rho s & (\alpha_0 < 0)
\end{cases}$$

(3.12)

Now we are ready to write down the amplitude. For $\alpha_0 < 0$ we have the non-trivial amplitude:

$$A(k_1, k_2, k_3) = \left[\frac{-i\pi}{2}\right]^3 \left[\mu \Delta \left(\frac{1}{2} - \rho\right)\right]^s \Delta \left(\frac{1}{2} - \frac{s}{2}\right) \Delta(1 + \alpha - (s - 1)\rho) \Delta \left(\frac{1}{2} - \alpha + \rho\right)$$

$$= \left[\frac{\mu \Delta \left(\frac{1}{2} - \rho\right)}{\left(-\frac{i\pi}{2}\right)^3} \Delta(1 + \beta^2 - k_j^2)\right] \prod_{j=1}^{3}$$

(3.13)

By redefining the cosmological constant and the primary superfield $\tilde{\Psi}_{NS}$

$$\mu \rightarrow \frac{2}{\Delta \left(\frac{1}{2} - \rho\right)\mu}, \quad \tilde{\Psi}_{NS}(k_j) \rightarrow \frac{1}{\left(-\frac{i\pi}{2}\right) \Delta \left(\frac{1}{2} \left[1 + \beta_j^2 - k_j^2\right]\right)} \tilde{\Psi}_{NS}(k_j)$$

(3.14)

we get

$$A(k_1, k_2, k_3) = \mu^s$$

(3.15)

As in the bosonic case we have a remarkably simple result. The only differences with respect to the bosonic case are in the details of the renormalization factors. Compare (3.14) with (2.17). Note that the singular point at the renormalization of the cosmological constant is $\rho = -1$ in the bosonic case, which corresponds to $c = 1$, and $\rho = -\frac{1}{2}$ in the supersymmetric case, corresponding to $\hat{c} = 1$ or $c = 3/2$, as it should.

3.2- The 3-point NS correlator with arbitrary s.c. ($A_{3}^{nm}$)

Now we shall generalize the above result to the case which includes screening charges in the supermatter sector. We consider $n$ charges $e^{i\theta_+ \Phi_M}$ and $m$ charges $e^{i\theta_- \Phi_M}$, where
$d_\pm$ are solutions of the equation $\frac{1}{2}d(d-2\alpha_0) = \frac{1}{2}$. After integrating over the matter and Liouville zero modes we get

$$
\left\langle \prod_{i=1}^3 \int \tilde{\psi}_{NS}(\tilde{z}_i, k_i) \prod_{i=1}^n \int \frac{d^2t_i}{n!} e^{i\tilde{d}_+ \Phi_M(t_i)} \prod_{i=1}^m \int \frac{d^2r_i}{m!} e^{i\tilde{d}_- \Phi_M(r_i)} \right\rangle
$$

$$
\equiv 2\pi\delta \left( \sum_{i=1}^3 k_i + nd_+ + md_- - 2\alpha_0 \right) \mathcal{A}_3^{nm}(k_1, k_2, k_3)
$$

$$
\mathcal{A}_3^{nm}(k_1, k_2, k_3) = \Gamma(-s) \left( \frac{-\pi}{2} \right)^3 \left( \frac{i\mu\alpha_+^2}{\pi} \right)^s \left( \frac{-d_+^2}{n!} \right)^n \left( \frac{-d_-^2}{m!} \right)^m
$$

$$
\times \prod_{i=1}^n \int d^2t_i |t_i|^{-2d_+k_1} |1-t_i|^{-d_+k_2} \prod_{i<j} |t_i-t_j|^{2d_+}
$$

$$
\times \prod_{i=1}^m \int d^2r_i |r_i|^{-2d_-k_1} |1-r_i|^{-d_-k_2} \prod_{i<j} |r_i-r_j|^{2d_-} \prod_{i=1}^n \prod_{i=1}^m |t_i-t_j|^{-2}
$$

$$
\times \prod_{i=1}^s \int d^2z_i |z_i|^{-2\alpha_+^2} |1-z_i|^{-2\alpha_+} \prod_{i<j} |z_i-z_j|^{-2\alpha_-^2}
$$

$$
\times \left\langle (\beta_1^2 \tilde{\psi}\psi(0) - k_2^2 \xi\xi(0)) \prod_{i=1}^n \xi(t_i) \prod_{i=1}^m \xi(r_i) \prod_{i=1}^s \tilde{\psi}\psi(z_i) \right\rangle_0
$$

(3.16)

Integrating over the Grassmann variables and fixing the $SL(2)$ symmetry as before ($\tilde{z}_1 = 0$, $\tilde{z}_2 = \infty$, $\tilde{\theta}_1 = \theta$, $\tilde{\theta}_2 = \tilde{\theta}_3 = 0$) we obtain (using $d_+d_- = -\alpha_+\alpha_- = -1$)

$$
\mathcal{A}_3^{nm}(k_1, k_2, k_3) = \Gamma(-s) \left( \frac{-\pi}{2} \right)^3 \left( \frac{i\mu\alpha_+^2}{\pi} \right)^s \left( \frac{-d_+^2}{n!} \right)^n \left( \frac{-d_-^2}{m!} \right)^m
$$

$$
\times \prod_{i=1}^n \int d^2t_i |t_i|^{-2d_+k_1} |1-t_i|^{-d_+k_2} \prod_{i<j} |t_i-t_j|^{2d_+}
$$

$$
\times \prod_{i=1}^m \int d^2r_i |r_i|^{-2d_-k_1} |1-r_i|^{-d_-k_2} \prod_{i<j} |r_i-r_j|^{2d_-} \prod_{i=1}^n \prod_{i=1}^m |t_i-t_j|^{-2}
$$

$$
\times \prod_{i=1}^s \int d^2z_i |z_i|^{-2\alpha_+^2} |1-z_i|^{-2\alpha_+} \prod_{i<j} |z_i-z_j|^{-2\alpha_-^2}
$$

$$
\times \left\langle (\beta_1^2 \tilde{\psi}\psi(0) - k_2^2 \xi\xi(0)) \prod_{i=1}^n \xi(t_i) \prod_{i=1}^m \xi(r_i) \prod_{i=1}^s \tilde{\psi}\psi(z_i) \right\rangle_0
$$

(3.17)

Since the vacuum expectation value of an odd number of $\tilde{\psi}\psi$ (or $\tilde{\xi}\xi$) operators is zero we have only two non-trivial cases: in the first case $n + m = \text{odd}$, $s = \text{even}$ and in the second one $n + m = \text{even}$, $s = \text{odd}$. Thus we have

$$
\mathcal{A}_3^{nm}(k_1, k_2, k_3) = \Gamma(-s) \left( \frac{-\pi}{2} \right)^3 \left( \frac{i\mu\alpha_+^2}{\pi} \right)^s \left( \frac{-d_+^2}{n!} \right)^n \left( \frac{-d_-^2}{m!} \right)^m
$$

$$
\times \left\{ \begin{array}{l}
I_M^{nm}(\tilde{\alpha}, \tilde{\beta}; \tilde{\rho}) \times J_G^s(\alpha, \beta; \rho), n + m = \text{even}, s = \text{odd} \\
J_M^{nm}(\tilde{\alpha}, \tilde{\beta}; \tilde{\rho}) \times J_G^s(\alpha, \beta; \rho), n + m = \text{odd}, s = \text{even}
\end{array} \right.
$$

(3.18)

where

$$
I_M^{nm}(\tilde{\alpha}, \tilde{\beta}; \tilde{\rho}) = \prod_{i=1}^n \int d^2t_i |t_i|^2\tilde{\alpha} |1-t_i|^2\tilde{\beta} \prod_{i<j} |t_i-t_j|^{2\tilde{\rho}}
$$
\[
\times \prod_{i=1}^{m} \int d^{2}r_{i} |r_{i}|^{2\tilde{\alpha}'}|1 - r_{i}|^{2\tilde{\beta}'} \prod_{i<j}^{m} |r_{i} - r_{j}|^{2\tilde{\rho}'} \prod_{i=1}^{n} \prod_{j=1}^{m} |t_{i} - r_{j}|^{-2} \\
\times \left\langle \prod_{i=1}^{n} \bar{\xi}(t_{i}) \prod_{i=1}^{m} \xi(r_{i}) \right\rangle_{0},
\]

(3.19)

\[
I_{G}^{n}(\alpha, \beta; \rho) = \alpha^{2} \int d^{2}z_{i} \prod_{i=1}^{s} |z_{i}|^{2\alpha}|1 - z_{i}|^{2\beta} \prod_{i<j}^{s} |z_{i} - z_{j}|^{4\rho} \left\langle \bar{\psi}\psi(0) \prod_{i=1}^{s} \bar{\psi}\psi(z_{i}) \right\rangle_{0} (3.20a)
\]

with

\[
\tilde{\alpha} = -d_{+}k_{1} \quad , \quad \tilde{\beta} = -d_{+}k_{2} \quad , \quad \tilde{\rho} = d_{+}^{2} \\\n\tilde{\alpha}' = -d_{-}k_{1} \quad , \quad \tilde{\beta}' = -d_{-}k_{2} \quad , \quad \tilde{\rho}' = d_{-}^{2}
\]

(3.20b)

and \(\alpha, \beta, \rho\) are defined as before. Note that \(I_{M}^{nm}\) is the supersymmetric generalization of (B.10) of Ref.[12]. The integral \(J_{M}^{nm}\) differs from \(I_{M}^{nm}\) by the introduction of a factor \(\bar{\xi}\xi(0)\) and \(J_{G}^{n}\) can be obtained from \(I_{G}^{n}\) by dropping \(\bar{\psi}\psi(0)\). Henceforth we assume, for simplicity, \(n + m = \text{even}, s = \text{odd}\). We will work out explicitly only the case \(n, m = \text{even}\). However, the final result for the amplitude does not depend on which case we choose. In the Appendix A we calculate \(I_{M}^{nm}\) for \(n, m\) even and we get:

\[
I_{M}^{nm}(\tilde{\alpha}, \tilde{\beta}; \tilde{\rho}) = (-\frac{n+m}{2})^{n+m} \pi^{n+m} n! m! \left[ -\frac{\tilde{\rho}}{2} \right]^{-2nm} \left[ \Delta \left( \frac{1}{2} - \frac{\tilde{\rho}'}{2} \right) \right]^{n} \left[ \Delta \left( \frac{1}{2} - \frac{\tilde{\rho}'}{2} \right) \right]^{m} \\\n\times \prod_{i=0}^{\frac{m}{2}} \Delta(i\tilde{\rho}) \Delta \left( \frac{1}{2} + \tilde{\rho} \left( i - \frac{1}{2} \right) \right) \prod_{i=1}^{\frac{m}{2}} \Delta(i\tilde{\rho}' - \frac{n}{2}) \Delta \left( \frac{1}{2} - \frac{n}{2} - \tilde{\rho}' \left( i - \frac{1}{2} \right) \right) \\\n\times \prod_{i=0}^{\frac{n}{2}} \Delta(1 + \tilde{\alpha} + i\tilde{\rho}) \Delta(1 + \tilde{\beta} + i\tilde{\rho}) \Delta(m - \tilde{\alpha} - \tilde{\beta} + \tilde{\rho}(i - n + 1)) \\\n\times \prod_{i=1}^{\frac{m}{2}} \Delta(\frac{1}{2} + \tilde{\alpha} + (i - \frac{1}{2})\tilde{\rho}) \Delta(\frac{1}{2} + \tilde{\beta} + (i - \frac{1}{2})\tilde{\rho}) \Delta(-\frac{1}{2} - \tilde{\alpha} + m - \tilde{\beta} + \tilde{\rho}(i - n + 1)) \\\n\times \prod_{i=0}^{\frac{n}{2}} \Delta(1 + \tilde{\alpha}' - \frac{n}{2} + i\tilde{\rho}') \Delta(1 - \frac{n}{2} + \tilde{\beta}' + i\tilde{\rho}') \Delta(\frac{n}{2} - \tilde{\alpha}' - \tilde{\beta}' + \tilde{\rho}'(i - m + 1)) \\\n\times \prod_{i=1}^{\frac{m}{2}} \Delta(\frac{1}{2} - \frac{n}{2} + \tilde{\alpha}' + (i - \frac{1}{2})\tilde{\rho}') \Delta(\frac{1}{2} - \frac{n}{2} + \tilde{\beta}' + (i - \frac{1}{2})\tilde{\rho}') \Delta(-\frac{1}{2} + \frac{n}{2} - \tilde{\alpha}' - \tilde{\beta}' + \tilde{\rho}'(i - m + 1))
\]

(3.21)

In the case where \(s = 2l + 1\) the gravitational contribution to \(A^{nm}(k_{1}, k_{2}, k_{3})\), i.e, \(I_{G}^{n}\) is just the same as in the case without screening charges, thus from the last section we have
the supersymmetric generalization of (B.9) of Ref.[12]:

\[ I^s_G = (-)^{s+1} \pi^s \frac{1}{2s-1} s! \left[ \Delta \left( \frac{1}{2} - \rho \right) \right]^{\frac{s-1}{2}} \prod_{i=1}^{\frac{s-1}{2}} \Delta(2i\rho) \prod_{i=0}^{\frac{s-1}{2}} \Delta(\frac{1}{2} + (2i + 1)\rho) \]
\[ \times \prod_{i=0}^{\frac{s-1}{2}} \Delta(1 + \tilde{\alpha} + 2i\tilde{\rho}) \Delta(1 + \tilde{\beta} + 2i\tilde{\rho}) \Delta(-\tilde{\alpha} - \tilde{\beta} + 2i\tilde{\rho}(i - s + 1)) \]
\[ \times \prod_{i=0}^{\frac{s-1}{2}} \Delta\left( \frac{1}{2} + \tilde{\alpha} + (2i - 1)\tilde{\rho} \right) \Delta\left( \frac{1}{2} + \tilde{\beta} + (2i - 1)\tilde{\rho} \right) \Delta\left( -\frac{1}{2} - \tilde{\alpha} - \tilde{\beta} + \tilde{\rho}(2i - 2s + 1) \right) \]

(3.22)

To obtain \( A^{nm}(k_1, k_2, k_3) \) (see (3.18)) we have to calculate \( I^{nm} \times I^s_G \). Using the same kinematics as in the case without screening charges, it is easy to deduce:

\[ \beta = \begin{cases} \frac{n}{2} + \rho(m - s + 1) & , \alpha_0 > 0 \\ -\frac{1}{2} - \frac{m}{2} - (n + s)\rho & , \alpha_0 < 0 \end{cases} \]  
(3.23)

As before, in the case \( \alpha_0 > 0 \) the amplitude vanishes if we assume\(^5\) \( s \geq m + 2 \) because there appears \( \Gamma(-n) \) in the denominator of \( I^s_G \) after the substitution of \( \beta \) above and consequently \( I^s_G \) vanishes. Therefore we have to look at the \( \alpha_0 < 0 \) case to have a non-trivial amplitude. In this case we have (remember that \( d_+ = -\alpha_+ , d_- = \alpha_- \), if \( \alpha_0 < 0 \)):

\[ \tilde{\alpha} = \alpha - 2\rho \quad , \quad \tilde{\alpha}' = -1 + \frac{\rho^{-1}\alpha}{2} \]
\[ \tilde{\rho} = -2\rho \quad , \quad \tilde{\rho}' = -\frac{\rho^{-1}}{2} \]
\[ \tilde{\beta} = -\beta - 1 = (n + 1)\rho + \frac{m}{2} - \frac{1}{2} \]
\[ \tilde{\beta}' = \frac{(n + s)}{2} + \frac{\rho^{-1}}{4}(m - 1) \quad . \]

---

\(^5\) Actually the kinematics chosen here is self only if \( s >> n, m \).
Substituting in (3.21) and (3.22) and using (3.18) we have a very involved expression:

\[
\mathcal{A}_3^{nm}(k_1, k_2, k_3) = \Gamma(-s) \left(\frac{-\pi}{2}\right)^3 \left(\frac{i\mu}{\pi}\right)^s \alpha_{+}^{2(s-1)}(2\rho)^{n-m-\frac{a+m+s-1}{2}} \frac{\pi^{s+n+m}}{2m+n+s-1} \rho^{-2mn}s!
\]

\[
\times \left[\Delta(\frac{1}{2} + \rho)\right]^n \left[\Delta(\frac{1}{2} + \frac{\rho^{-1}}{4})\right]^m
\]

\[
\times \prod_{i=1}^{\frac{s}{2}-1} \Delta(-2i\rho)\Delta\left(\frac{1}{2} + \rho(1-2i)\right) \prod_{i=1}^{\frac{m}{2}} \Delta\left(-\frac{n}{2} - i\rho^{-1}\right)\Delta\left(\frac{1}{2} - \frac{n}{2} + \left(\frac{1}{4} - \frac{i}{2}\right)\rho^{-1}\right)
\]

\[
\times \prod_{i=0}^{\frac{s}{2}} \Delta\left(\frac{1}{2} + \frac{m}{2} + \rho(n+s-2i)\right) \prod_{i=1}^{\frac{s}{2}} \Delta\left(\frac{m}{2} + \rho(n+s+1-2i)\right)
\]

\[
\times \prod_{i=0}^{\frac{m}{2}} \Delta\left(1 + \frac{s}{2} - \frac{\rho^{-1}}{2}(i + \frac{1}{2} - \frac{n}{2})\right) \prod_{i=1}^{\frac{m}{2}} \Delta\left(\frac{1}{2} + \frac{s}{2} - \frac{\rho^{-1}}{2}(i - \frac{m}{2})\right)
\]

\[
\times \left[\Delta(\frac{1}{2} - \rho)\right]^n \prod_{i=1}^{\frac{s}{2}-1} \Delta(2i\rho) \prod_{i=0}^{\frac{s}{2}-1} \Delta\left(\frac{1}{2} + (2i+1)\rho\right)
\]

\[
\times \prod_{i=0}^{\frac{s}{2}} \Delta\left(\frac{1}{2} - \frac{m}{2} + (2i-n-s)\rho\right) \prod_{i=1}^{\frac{s}{2}} \delta(-\frac{m}{2} + (2i-1-n-s)\rho)
\]

\[
\times \prod_{i=1}^{\frac{s}{2}} \Delta\left(\frac{1}{2} + \alpha - \rho(2i+1)\right)\Delta\left(\frac{1}{2} + \frac{m}{2} - \alpha - \rho(s-n-2+2i)\right)
\]

\[
\times \prod_{i=0}^{\frac{s}{2}} \Delta\left(\frac{s}{2} - \alpha + (2i-s+n+2)\rho\right) \prod_{i=1}^{\frac{s}{2}} \Delta\left(\frac{1}{2} + \alpha + (2i-1)\rho\right)
\]

\[
\times \prod_{i=1}^{\frac{s}{2}} \Delta\left(\frac{m}{2} - \alpha - \rho(2i-n+s-1)\right)\Delta(1 + \alpha - 2i\rho)
\]

\[
\times \prod_{i=0}^{\frac{s}{2}} \Delta(1 + \alpha + 2i\rho) \prod_{i=1}^{\frac{s}{2}} \Delta\left(\frac{m}{2} - \alpha + (2i-s+n+1)\rho\right)
\]

\[
\times \prod_{i=1}^{\frac{s}{2}} \Delta\left(\frac{1}{2} - \frac{s}{2} - \frac{\rho^{-1}}{2}(i + \alpha - \frac{m}{2})\right)\Delta\left(-\frac{n}{2} - \frac{\rho^{-1}}{2}(i - 1 - \alpha)\right)
\]

\[
\times \prod_{i=1}^{\frac{s}{2}} \Delta\left(-\frac{n}{2} - \frac{1}{2} - \frac{\rho^{-1}}{2}(i - \alpha - \frac{1}{2})\right)\Delta\left(1 - \frac{s}{2} - \frac{\rho^{-1}}{2}(i - \frac{(m+1)}{2} + \alpha)\right)
\]

In order to obtain a simple expression for the amplitude we have to combine in each term the matter and the gravitational parts as in the bosonic case. The calculation is more
complicate now and we finally get

\[ A_{3}^{nm}(k_1, k_2, k_3) = (-\frac{\pi}{2})^{3} \left[ \frac{\mu}{2} \Delta \left( \frac{1}{2} - \rho \right) \right]^{s} \left[ -\frac{i\pi}{2} \Delta \left( \frac{1}{2} + \rho^{-1} \right) \right]^{m} \left[ -\frac{i\pi}{2} \Delta \left( \frac{1}{2} + \rho \right) \right]^{n} \]

\times \Delta \left( \rho - \alpha + \frac{1}{2} \right) \Delta \left( \frac{1}{2} - \frac{n + s - m\rho^{-1}}{2} \right) \Delta(1 - \frac{m}{2} + \alpha + (s - n - 1)\rho) \]

\[ = \left[ \frac{\mu}{2} \Delta \left( \frac{1}{2} - \rho \right) \right]^{s} \left[ -\frac{i\pi}{2} \Delta \left( \frac{1}{2} + \rho^{-1} \right) \right]^{m} \left[ -\frac{i\pi}{2} \Delta \left( \frac{1}{2} + \rho \right) \right]^{n} \]

\[ \times \prod_{i=1}^{3} \left( -\frac{i\pi}{2} \right) \Delta \left( \frac{1}{2} + \frac{1}{2}(\beta_i^2 - k_i^2) \right) \] (3.26)

Therefore after redefining the cosmological constant, the NS operators and the screening charges

\[ e^{i\mu_+\Phi_M(t_i)} \rightarrow \left[ -\frac{i\pi}{2} \Delta \left( \frac{1}{2} + \rho \right) \right]^{-1} e^{i\mu_+\Phi_M(t_i)} \] (3.27a)

\[ e^{i\mu_-\Phi_M(t_i)} \rightarrow \left[ -\frac{i\pi}{2} \Delta \left( \frac{1}{2} + \rho^{-1} \right) \right]^{-1} e^{i\mu_-\Phi_M(t_i)} \] (3.27b)

\[ \Psi_{NS} \rightarrow \left[ -\frac{i\pi}{2} \Delta \left( \frac{1}{2} + \frac{1}{2}(\beta_i^2 - k_i^2) \right) \right]^{-1} \Psi_{NS} \] (3.27c)

\[ \mu \rightarrow \left[ \frac{1}{2} \Delta \left( \frac{1}{2} - \rho \right) \right]^{-1} \mu \] (3.27d)

we obtain the very simple result:

\[ A_{3}^{nm}(k_1, k_2, k_3) = \mu^{s} \] (3.28)

In view of the complexity of (3.25), the simplicity of the result is remarkable.

3.3- \( N \)-point \((N \geq 4)\) supersymmetric correlation functions with an arbitrary number of screening charges

In this subsection we show that it is possible to obtain a simple result for the most general case of a \( N \)-point amplitude with an arbitrary number of screening charges \((A_{N}^{nm})\). In that general case we have to calculate the following integral\(^6\)

\[ A_{N}^{nm}(k_1, \cdots, k_N) = \Gamma(-s) \left( \frac{i\mu}{\pi} \right)^{s} \left\langle \prod_{i=1}^{N} \int d^{2}\tilde{z}_i e^{ik_i\Phi_M(\tilde{z}_i)} + \beta_i \Phi_{SL}(\tilde{z}_i) \right\rangle \]

\[ \times \prod_{i=1}^{n} \int d^{2}t_i e^{i\Phi_M(t_i)} \prod_{j=1}^{m} \int d^{2}\tilde{r}_j e^{i\mu_-\Phi_M(\tilde{r}_j)} \prod_{j=1}^{s} \int d^{2}z_j e^{i\alpha_+\Phi_{SL}(z_j)} \] (4.1)

\(^6\) We are computing 1PI amplitudes; see in this respect ref.[6]
where \( s = -\frac{1}{\alpha} (\sum_{i=1}^{N} \beta_i + Q) \) and \( \sum_{i=1}^{N} k_i + nd_+ + md_- = 2\alpha_0 \). After fixing the \( S_\infty \) symmetry as before and integrating over the Grassmann variables the amplitude becomes

\[
\mathcal{A}_{N}^{nm} = \Gamma(-s) \left( -\frac{\pi}{2} \right)^3 \left( \frac{i\mu\alpha^2_+}{\pi} \right)^s (-d_+^2)^n (-d_-^2)^m \\
\times \prod_{j=4}^{N} \int d^2 z_j \prod_{i=1}^{n} \int d^2 t_i \prod_{i=1}^{m} \int d^2 r_i \prod_{i=1}^{s} d^2 w_i |w_i|^{-2\alpha+\beta_1} |1 - w_i|^{-2\alpha+\beta_2} \\
\times \prod_{i<j} \left| w_i - w_j \right|^{-2\alpha_+} \prod_{i=1}^{N} \prod_{j=4}^{N} \left| w_i - z_j \right|^{-2\alpha+\beta_j} \\
\times \prod_{j=4}^{N} |\bar{z}_j|^{2(k_1 k_j - k_1 \beta_j)} |1 - \bar{z}_j|^{2(k_2 k_j - k_2 \beta_j)} \prod_{j<l=4}^{N} |\bar{z}_j - \bar{z}_l|^{2(k_j k_l - k_1 \beta_0)} \\
\times \prod_{i=1}^{n} \left| t_i \right|^{2k_1 d_+ + 1 - t_i} \prod_{i<j}^{m} \left| t_i - t_j \right|^{2d_+} \prod_{i=1}^{s} \prod_{j=4}^{N} |t_i - r_j|^{-2} \\
\times \prod_{i=1}^{m} \left| r_i \right|^{2k_2 d_- - 1 - r_i} \prod_{i<j}^{n} \left| r_i - r_j \right|^{2d_-} \prod_{i=1}^{n} \prod_{j=4}^{N} |t_i - z_j|^{2d_+ k_j} \prod_{i=1}^{m} \prod_{j=4}^{N} |r_i - z_j|^{2d_- k_j} \\
\times \left( \langle \beta_+^{2} \bar{\psi}(0) - k_1^{2} \xi(0) \rangle \prod_{j=4}^{N} \langle \beta_+^{2} \psi_j(\bar{z}_j) - k_2^{2} \xi(\bar{z}_j) \rangle \prod_{i=1}^{n} \xi(t_i) \prod_{i=1}^{s} \bar{\psi}(w_i) \right)_{0} \tag{4.2}
\]

Now we have several terms which give non-trivial amplitudes, in the following we assume \( m + n \) and \( N + s \) even, so we have, for instance:

\[
\mathcal{A}_{N}^{nm} = \Gamma(-s) \left( -\frac{\pi}{2} \right)^3 \left( \frac{i\mu\alpha^2_+}{\pi} \right)^s (-d_+^2)^n (-d_-^2)^m \left( \prod_{j=4}^{N} \beta_+^2 \right) (\alpha_+)^{-2} \\
\times \prod_{j=4}^{N} \int d^2 z_j \prod_{i=1}^{n} \int d^2 t_i \prod_{i=1}^{m} \int d^2 r_i \prod_{i=1}^{s} d^2 w_i \left| w_i \right|^{2\alpha} \left| 1 - w_i \right|^{2\beta} \\
\times \prod_{i<j} \left| w_i - w_j \right|^{4\rho} \prod_{i=1}^{N} \prod_{j=4}^{N} \left| w_i - z_j \right|^{2\alpha} \prod_{j=4}^{N} \left| z_j \right|^{2\alpha} \left| 1 - z_j \right|^{2\beta} \prod_{j<l=4}^{N} \left| z_j - z_l \right|^{2\rho} \\
\times \prod_{i=1}^{n} \left| t_i \right|^{2\alpha} \left| 1 - t_i \right|^{2\beta} \prod_{i<j}^{m} \left| t_i - t_j \right|^{2\beta} \prod_{i=1}^{s} \prod_{j=4}^{N} \left| t_i - r_j \right|^{-2} \\
\times \prod_{i=1}^{m} \left| r_i \right|^{2\alpha} \left| 1 - r_i \right|^{2\beta} \prod_{i<j}^{n} \left| r_i - r_j \right|^{2\beta} \prod_{i=1}^{n} \prod_{j=4}^{N} \left| t_i - z_j \right|^{2\alpha} \prod_{i=1}^{m} \prod_{j=4}^{N} \left| r_i - z_j \right|^{2\alpha} \\
\times \left( \prod_{i=1}^{n} \xi(t_i) \prod_{i=1}^{m} \xi(r_i) \right)_{0} \left( \bar{\psi}(0) \prod_{j=4}^{N} \bar{\psi}(w_i) \prod_{j=4}^{N} \bar{\psi}(z_j) \right)_{0} \tag{4.3}
\]
with $\alpha, \beta, \rho, \tilde{\alpha}, \tilde{\beta}, \tilde{\rho}, \tilde{\beta}', \tilde{\rho}'$ defined as before and

$$
\begin{align*}
\alpha_j' &= k_1 k_j - \beta_1 \beta_j, & \tilde{\alpha}_j &= d_+ k_j, \\
\beta_j' &= k_2 k_j - \beta_2 \beta_j, & \tilde{\alpha}_j' &= d_- k_j, \\
\rho_j &= \frac{1}{2} (k_j k_l - \beta_j \beta_l), & p_j &= -\alpha_j \beta_j, & 4 \leq j, l \leq N
\end{align*}
(4.4)

Using the kinematics: $k_1, k_2, \cdots, k_{N-1} \geq \alpha_0$, $k_N < \alpha_0$ it is possible to eliminate all parameters in terms of $\alpha, \beta, \rho$ and $p_j (4 \leq j \leq N-1)$:

$$
\begin{align*}
p_N &= -\frac{m+1}{2} - \rho(N + s + n - 3) \\
\alpha_j' &= \alpha + p_j - 2\rho, & \beta_j' &= \beta + p_j - 2\rho, & 4 \leq j \leq N - 1 \\
\alpha_N' &= \frac{(m-1)}{2} + (\rho - \alpha)(N + s + n - 3) - \frac{m\rho^{-1} \beta}{2} \\
\rho_j &= \frac{(m-1)}{4} - \frac{m\rho^{-1}}{4} p_j + \frac{(\rho - p_j)}{2} (N + s + n - 3) \\
\rho_j &= \frac{1}{2} (p_l + p_j) - \rho \\
\tilde{\alpha} &= \alpha - 2\rho, & \tilde{\beta} &= \beta - 2\rho, \\
\tilde{\alpha}' &= -1 + \frac{\rho^{-1}}{2} \alpha, & \tilde{\beta}' &= -1 + \frac{\beta}{2} \rho^{-1}
\end{align*}
(4.5)

where $4 \leq j, l \leq N - 1$. Using the symmetries:

$$
\mathcal{A}_{N}^{nm}(\alpha, \beta, \rho, p_1, p_2, \cdots, p_{N-1}) = \mathcal{A}_{N}^{nm}(\beta, \alpha, \rho, p_1, p_2, \cdots, p_{N-1})
(4.6)
$$

$$
\mathcal{A}_{N}^{nm}(\alpha, \beta, \rho, p_1, \cdots, p_{N-1}) = \mathcal{A}_{N}^{nm}(-\alpha - \beta + \frac{(m-1)}{2} - P + \rho(N + n - s - 1), \beta, \rho, p_1, \cdots, p_{N-1})
(4.7)
$$

(with $P = \sum_{j=4}^{N-1} p_j$), and the large-$\alpha$ behaviour:

$$
\mathcal{A}_{N}^{nm}(\alpha \to \infty) \sim \alpha^{1-m+2\beta+2\rho(s-N-n+3)+2P}
(4.8)
$$

we have the ansatz:

$$
\mathcal{A}_{N}^{nm} = f_{N}^{nm}(\rho, p_1, \cdots, p_{N-1}) \Delta \left( \frac{1}{2} + \rho - \alpha \right) \Delta \left( \frac{1}{2} + \rho - \beta \right) \Delta (1 - \frac{m}{2} - P + \alpha + \beta + \rho(2 + s - n - N))
(4.9)
$$

By sending $k_i (3 \leq i \leq N - 1)$ to zero, which implies $p_j \to 2\rho (4 \leq j \leq N - 1)$, we can determine $f_{N}^{nm}(\rho, p_1, \cdots, p_{N-1})$ using:

$$
\mathcal{A}_{N}^{nm}(\alpha, \beta, \rho, k_i \to 0) = \left(-\frac{i\pi}{2}\right)^{N-3} \frac{\partial^{N-3}}{\partial \mu} \mathcal{A}_{N}^{nm}(k_1, k_2, k_N)
(4.10)
$$
and the result for $A_{nm}$ (see (3.26)). We get

$$f_N(\rho, p_1, \cdots, p_{N-1}) = \left[ -\frac{i\pi}{2} \Delta(\frac{1}{2} + \rho) \right]^n \left[ -\frac{i\pi}{2} \Delta(\frac{1}{2} + \frac{\rho}{4}) \right]^m \left( -\frac{i\pi}{2} \right)^N \times \left[ \Delta(\frac{1}{2} - \rho) \right] \left( \prod_{i=4}^{N-1} \Delta(\frac{1}{2} + \rho - p_j) \right) \Delta\left( -\frac{(s+n+N-4)}{2} - \frac{m}{4} \rho^{-1} \right) \times \frac{\partial}{\partial \mu} \left[ \mu \right]^{s+N-3} \right]_{µ=\mu/\Delta(-κ)} \right).$$

(4.11)

So, the final result for the general $N$-point function with arbitrary screening charges can be written in a simple form:

$$A_{nm}^N = (s+N-3)(s+N-4)\cdots(s+1)\left[ \mu \Delta(\frac{1}{2} - \rho) \right]^s \times \left[ -\frac{i\pi}{2} \Delta(\frac{1}{2} + \rho) \right]^n \left[ -\frac{i\pi}{2} \Delta(\frac{1}{2} + \rho) \right]^m \prod_{i=1}^{N} \left( -\frac{i\pi}{2} \right)^N \Delta(\frac{1}{2} \left( 1 + \beta^2_i - k_i^2 \right)) \right) \right) (4.12)$$

Redefining $Ψ_{NS}$, $\mu$ and the screening charges we have our final result:

$$A_{nm}^N = \frac{\partial}{\partial \mu} \left[ \mu \right]^{s+N-3}$$

(4.13)

4- Conclusion

We have computed exactly $N$-point correlators in the NS sector of super Liouville theory conformally coupled to $c \leq 3/2$ supermatter in a supercoulomb gas representation including an arbitrary number of screening charges in the matter sector. We also generalized previous results for bosonic amplitudes to the case including arbitrary s.c. We have learned that in all those cases the final $N$-point amplitude with $n, m$ s.c. has the same (rather simple) form given by (4.13) above in terms of the renormalized cosmological constant $\mu$ and the parameter $s$ which is a function of the matter central charge and the external momenta. This confirms suspicions18,19 in that direction using a proposal of the matrix model approach. The close connection to the bosonic amplitude was suggested from super KP systems as well19. This demonstration is however more direct (see also [15] and [16]). The similarity is striking, and the hope is that a full treatment of the theory by the super matrix model approach could be checked against our results. Furthermore, the 4-point NS amplitude at $s = 0$ recently obtained by Di Francesco and Kutasov16 can be also compared with our result (in that limit) and the check is positive. It should be stressed, however, that our results for arbitrary $s$ permits to visualize the role of the barrier at $c = 1$ (in the bosonic case), since the renormalization $\mu \rightarrow \mu/\Delta(-\rho)$ is infinity at
$c = 1$ (where $\rho = -1$). In the supersymmetric case, the barrier occurs at $c = 3/2$, the renormalization $\mu \to \mu/\Delta(1/2 - \rho)$ is infinity at $c = 3/2$ (where $\rho = -1/2$). It would be interesting to see whether this barrier indeed disappears for N=2 supersymmetry, and we hope to obtain also the N=2 super correlators. The next interesting question concerns the Ramond sector. Work in this direction is in progress. There are, however, new difficulties in that case.
Appendix A

In this appendix we calculate (for \( n \) and \( m \) even) the following integral:

\[
I_{nm}^{\alpha \beta \rho} = \prod_{i=1}^{n} \int d^2 t_i |t_i|^{2\alpha} |1-t_i|^{2\beta} \prod_{i<j}^{n} |t_i-t_j|^{2\rho} \\
\times \prod_{i=1}^{m} \int d^2 r_i |r_i|^{2\alpha'} |1-r_i|^{2\beta'} \prod_{i<j}^{m} |r_i-r_j|^{2\rho'} \prod_{i=1}^{n} \prod_{j=1}^{m} |t_i-r_j|^{-2} \\
\times \left\langle \prod_{i=1}^{n} \xi_i(t_i) \prod_{i=1}^{m} \xi_i(r_i) \right\rangle_0 , \tag{A.1}
\]

where \( \langle \xi(t) \xi(r) \rangle_0 = |t-r|^{-2} \) and \( \rho' = 1/\rho, \alpha' = -\rho' \alpha, \beta' = -\rho' \beta \). The above integral is the supersymmetric generalization of (B.10) of the second reference of [12]. In order to obtain \( I_{nm} \) we first notice that by making translations \( (t_i \to 1-t_i, r_i \to 1-r_i) \) and inversions \( (r_i \to 1/r_i, t_i \to 1/t_i) \) we have the symmetries, respectively:

\[
I_{mn}(\alpha, \beta, \rho) = I_{mn}(\beta, \alpha, \rho) \tag{A.2}
\]

\[
I_{mn}(\alpha, \beta; \rho) = I_{mn}(m-1-\alpha-\beta-\rho(n-1), \beta; \rho) \tag{A.3}
\]

Changing variables we can analyse the asymptotic behaviour of \( I_{mn} \) for large \( \alpha \):

\[
I_{mn}(\alpha \to \infty, \beta; \rho) = \alpha^{2nm-n-m-2n\beta-2m\beta'-\rho(n-1)n-\rho'(m-1)m} . \tag{A.4}
\]

Another information can be used, namely that for \( \rho = \rho' = -1 (\alpha' = \alpha, \beta' = \beta) \) the integral must be a function of \( n \) and \( m \) through the combination \( n+m \). It is not difficult to check (using Stirling’s formula \( \Gamma(\alpha + c) \sim \alpha^c \Gamma(\alpha) \), for large-\( \alpha \)), that the following Ansatz is consistent with all requirements given above:

\[
I_{mn}(\alpha, \beta; \rho) = C_{mn}(\rho) \\
\times \prod_{i=0}^{n-1} \Delta(1+\alpha+ip)\Delta(1+\beta+ip)\Delta(m-\alpha-\beta+\rho(i-n+1)) \\
\times \prod_{i=1}^{\frac{m-1}{2}} \Delta(\frac{1}{2}+\alpha+(i-\frac{1}{2})p)\Delta(\frac{1}{2}+\beta+(i-\frac{1}{2})p)\Delta(-\frac{1}{2}-\alpha+m-\beta+\rho(i-n+\frac{1}{2})) \\
\times \prod_{i=0}^{\frac{m-1}{2}} \Delta(1+\alpha'-\frac{n}{2}+ip')\Delta(1-\frac{n}{2}+\beta'+ip')\Delta(-\frac{n}{2}-\alpha'-\beta'+\rho'(i-m+1)) \\
\times \prod_{i=1}^{\frac{m-1}{2}} \Delta(\frac{1}{2}-\frac{n}{2}+\alpha'+(i-\frac{1}{2})p')\Delta(\frac{1}{2}-\frac{n}{2}+\beta'+(i-\frac{1}{2})p')\Delta(-\frac{1}{2}+\frac{n}{2}-\alpha'-\beta'+\rho'(i-m+\frac{1}{2})) \tag{A.5}
\]
The coefficient $C_{mn}(\rho)$ can be basically obtained by noticing that the integral $I_{mn}$ reduces to a known integral when $m$ or $n$ vanishes:

\[ I_{0n} = (-)^{n/2}(n!!)J^{n/2}(\alpha, \beta, \gamma = -1/2, \rho/2) \] (A.6)

\[ I_{m0} = (-)^{m/2}(m!!)J^{m/2}(\alpha', \beta', \gamma = -1/2, \rho'/2) \] (A.7)

We have calculated the integral $J^{m}(\alpha, \beta; \rho)$ in Ref. [15]. For the reader’s convenience we give the result:

\[ J^{m}(\alpha, \beta; \gamma; \rho) = \pi^2 m^2 \prod_{i=1}^{m} \Delta (1 + \gamma + (2i - 1)\rho) \]

\[ \times \prod_{i=0}^{m-1} \Delta (1 + \alpha + 2i\rho) \Delta (1 + \beta + 2i\rho) \Delta (-1 - \alpha - \beta - 2\gamma + (2i - 4m + 2)\rho) \]

\[ \times \prod_{i=1}^{m} \Delta (1 + \alpha + \gamma + (2i - 1)\rho) \Delta (1 + \beta + \gamma + (2i - 1)\rho) \Delta (-1 - \alpha - \beta - \gamma + (2i - 4m + 2)\rho) \] (A.8)

Using the above result in eq.(A.6) and (A.7) with the Ansatz (A.5) we obtain $C_{mn}(\rho)$:

\[ C_{mn}(\rho) = (-)^{n+m} \pi^{n+m} 2^{n+m} \frac{m!}{2n+m} \left( -\frac{\rho}{2} \right)^{2nm} \left[ \Delta \left( \frac{1}{2} - \frac{\rho}{2} \right) \right]^n \left[ \Delta \left( \frac{1}{2} - \frac{\rho'}{2} \right) \right]^m \]

\[ \times \prod_{i=1}^{n} \Delta (ip) \Delta \left( \frac{1}{2} + \rho \left( i - \frac{1}{2} \right) \right) \prod_{i=1}^{m} \Delta (ip' - \frac{n}{2}) \Delta \left( \frac{1}{2} - \frac{n}{2} - \rho' \left( i - \frac{1}{2} \right) \right) \] (A.9)

which determines $I_{mn}(\alpha, \beta; \rho)$ completely.

**Acknowledgments**

The work of K.H. (contract # 90/1799-9) and D.D. (contract # 90/2246-3) was supported by FAPESP while the work of E.A. and M.C.B.A. is partially supported by CNPq.

**References**

[1] E. Brézin and V. A. Kazakov, Phys. Lett. B236 (1990)144; M. R. Douglas and S. H. Shenker, Nucl. Phys. B335(1990)635; D. J. Gross and A. A. Migdal, Phys. Rev. Lett. 64(1990)127.

[2] A. M. Polyakov, Phys. Lett. B103(1981)207.

A. M. Polyakov, Phys. Lett. B103 (1981)211.
[3] T. L. Curtright and C. B. Thorn, Phys. Rev. Lett. 48, (1982)1309; E. Braaten, T. L. Curtright and C. B. Thorn, Phys. Lett. B118(1982)115; Ann. Phys. 147(1983)365; E. Braaten, T. L. Curtright, G. Gandour and C. B. Thorn, Phys. Rev. Lett. 51(1983)19; Ann. Phys. 153(1984)147; J. L. Gervais and A. Neveu, Nucl. Phys. B199(1982)50; B209(1982)125; B224 (1983)329; B238(1984)123,396; E. D'Hoker and R. Jackiw, Phys. Rev. D26(1982)3517; T. Yoneya, Phys. Lett. B148(1984)111. N. Seiberg, Lecture at 1990 Yukawa Int. Sem. Common Trends in Math. and Quantum Field Theory, and Cargese meeting Random Surfaces, Quantum Gravity and Strings, May 27, June 2, 1990; J. Polchinski, Strings ’90 Conference, College Station, TX, Mar 12-17, 1990, Nucl. Phys. B357(1991)241.

[4] M. Goulian and M. Li, Phys. Rev. Lett. 66(1991)2051.

[5] N. Sakai and Y. Tanii, Tokyo Institute of Technology preprint TIT/HEP-169 (STUPP-91-116) (1991).

[6] P. Di Francesco and D. Kutasov, Phys. Lett. B261(1991)385.

[7] Vl. S. Dotsenko, Paris VI preprint PAR-LPTHE 91-18 (1991).

[8] Y. Kitazawa, Harvard Univ. preprint HUTP-91/A013 (1991).

[9] A. Gupta, S. P. Trivedi and M. B. Wise, Nucl. Phys. B340 (1990)475.

[10] M. Bershadsky and I. R. Klebanov, Phys. Rev. Lett. 65, 3088 (1990); Harvard Univ. preprint HUTP-91/A002 (PUPT-1236) (1991); A. M. Polyakov, Mod. Phys. Lett. A6(1991)635.

[11] J. F. Arvis, Nucl. Phys. B212(1983)151; B218(1983)309; O. Babelon, Nucl. Phys. B258(1985)680; Phys. Lett. 141B(1984)353; T. Curtright and G. Ghandour, Phys. Lett. B136(1984)50.

[12] Vl. S. Dotsenko and V. Fateev, Nucl. Phys. B240 (1984)312; B251(1985)691.

[13] J. Distler, Z. Hlousek and H. Kawai, Int. J. Mod. Phys. A5 (1990)391.

[14] E. Martinec, Phys. Rev. D28(1983)2604.

[15] E. Abdalla, M.C.B. Abdalla, D. Dalmazi and K. Harada, “ Correlation functions in super Liouville theory”, preprint IFT-P.029/91 (1991).

[16] P. Di Francesco and D. Kutasov, Princeton Univ. preprint PUPT-1276 (1991). K. Aoki and E. D’Hoker, preprint UCLA/91/TEP/33(1991).

[17] F. David, Mod. Phys. Lett. A3(1988)1651; J. Distler and H. Kawai, Nucl. Phys. B321(1989)509.

[18] L. Alvarez-Gaumé and J. L. Mañez, Mod. Phys. Lett. A6, 2039 (1991).

[19] P. Di Francesco, J. Distler and D. Kutasov, Mod. Phys. Lett. A5, 2135 (1990).