On violation of the Pauli principle and dimension of the Universe at very large distances

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Abstract

We discuss a Modified Field theory (MOFT) in which the number of fields can vary. It is shown that in MOFT fermions obey the so-called parastatistics in which the order of the parastatistics depends on scales. In particular, at very large distances $r > r_{\text{min}}$ fermions violate the Pauli principle. It is also shown that in MOFT in some range of scales $r_{\text{max}} > r > r_{\text{min}}$ the Universe acquires features of a two-dimensional space whose distribution in the observed 3-dimensional volume has an irregular character. This provides a natural explanation to the observed fractal distribution of galaxies and the logarithmic behavior of the Newton’s potential for a point source.
The Modified Field Theory (MOFT) was suggested in Ref. [1] to account for spacetime foam effects, and it was recently demonstrated in Ref. [2] that MOFT provides a reasonable alternative to the dark matter paradigm. It was shown that MOFT possesses a nontrivial vacuum state which leads to a scale-dependent renormalization of all interaction constants $\alpha^2 \rightarrow \alpha^2 N(k)$ (where $k$ is the wave number and $\alpha$ is either the electron charge $e$, the gauge charge $g$, or the gravitational constant $\sqrt{G}$) with some structural function $N(k)$ which reflects the topology of the momentum space (for details we send readers to Ref. [2]). This means that particles lose their point-like character in MOFT and acquire a specific distribution in space, i.e., each point source is surrounded with a dark halo which carries charges of all sorts. In the simplest case properties of the vacuum and that of the dark halos are described by two characteristic scales $r_{\text{min}}$ and $r_{\text{max}}$ between which interaction constants increase $\alpha^2(\ell) \sim \alpha^2_0 \ell/r_{\text{min}}$ ($\ell = 2\pi/k$) and the Newton’s and Coulomb’s interaction energies show the logarithmic $V \sim \ln r$ (instead of $1/r$) behavior. Thus, MOFT reproduces the basic phenomenological feature of the Modified Newtonian Dynamics (MOND) proposed by Milgrom [3] to model the dark matter effects (see also the list of references devoted to MOND at the site [4]) but conceptually MOFT differs from MOND very much.

In addition to logarithmic potentials, MOFT seems to require that the Universe has a fractal structure in the same range of scales with dimension $D \approx 2$ [5], which is supported by the observed fractal distribution of galaxies (e.g., see Refs. [6] - [9]). In particular, the fractal distribution of luminous matter is the only possible picture of the Universe which appears in the case when the maximal scale is absent (i.e., $r_{\text{max}} \rightarrow \infty$). At the same time, it is shown in Ref. [5] that the theoretical scheme of MOFT provides a full agreement of the observed fractal distribution of luminous matter with homogeneity of the Universe and observational limits on $\Delta T/T$ in the microwave background.

However, in the present form MOFT is not a self-consistent theory, regardless of the good features pointed out. The problem is that in MOFT fermions and bosons are described in essentially different ways, for MOFT was originally developed only for bosonic fields [1]. In fact, there exists an argument which shows that the situation has to be somewhat different. Indeed, the basic idea of MOFT is that the nontrivial topology of space displays itself in the multivalued nature of all observable fields, i.e. the number of fields represents an additional dynamical variable which was shown to depend on the position in the momentum space [1, 2]. The topological origin of this additional variable means that the number of fields $N(k)$ should represent a certain function of the wave number $k$, which must be the same for all types of fields and which serves as a geometric characteristic of the momentum space. In fact, this function defines the spectral number of modes in the interval between $k$ and $k + dk$

$$g \frac{k^3 N(k) \, dk}{4\pi^2 k},$$

where $g$ is the number of polarization states. Since $N(k)$ has a pure geometrical nature, it is clear that the measure (1) characterizes the number of degrees of freedom of a particular point particle and must hold for all types of particles, regardless of their statistics, contrary to the claim of Ref. [1] that the modification should involve only bosonic fields. In fact, a self-consistent consideration of fermions also requires a description in terms of variable number of fields.

In the present paper we suggest an extension of MOFT to the case of fermionic fields as well. From the mathematical standpoint the generalization of the quantum field theory to the
case of a variable number of fields does not depend on the statistics of particles and remains 
the same as in the case of bosonic fields (e.g., see Ref. [1]). Fermions, however, involve two 
modifications. First, the number of particles in every mode takes now only two values \( n = 0, 1 \) 
(according to the Fermi statistics) and, secondly, the fundamental operators of creation and 
annihilation of fermionic modes have to obey the Bose statistics (otherwise the total number of 
modes will be always restricted by \( N(k) \leq 2 \) for all \( k \)). At very large scales corresponding to 
\( k < 2\pi/r_{\text{min}} \), the number of modes \( N(k) \) was shown to take values \( N(k) > 1 \) [2] and, therefore, 
the Pauli principle is violated (up to \( N(k) \) fermions can occupy the same quantum state which 
corresponds to a wave number \( k \)). Thus, we see that particles obey in MOFT the so-called 
parastatistics (or the Green statistics) where the function \( N(k) \) plays the role of the order of 
the parastatistics. We note that the parastatistics was first suggested by Green in Ref. [10] 
and it has been studied in many papers since then (e.g., see Refs. [11] - [15] and, for more 
recent discussions, Refs. [16]). In this manner, we see that MOFT gives a specific realization 
of the parafield theory in which the order of the statistics represents a new variable related to 
the topology of space.

The fact that the number of degrees of freedom for any point particle is described by the 
measure (1) means that the number of fields \( N(k) \) characterizes the density of physical space. 
This allows us to give a common, pure geometrical interpretation to both observational phenom-
enia: logarithmic potentials and the fractal distribution of galaxies, as we briefly demonstrate 
in the present paper. Namely, we show that in the range of scales \( r_{\text{min}} < r < r_{\text{max}} \) some kind 
of reduction of the dimension of space happens, i.e. an effective dimension of the Universe is 
\( D \approx 2 \).

2 Description of particles in MOFT

Let \( \psi \) be an arbitrary field which, upon the expansion in modes, is described by a set of 
creation and annihilation operators \( \{a_{\alpha,k}, a_{\alpha,k}^+\} \), where the index \( \alpha \) numerates polarizations and 
distinguishes between particles and antiparticles. In what follows, for the sake of simplicity, we 
ignore the presence of the additional discrete index \( \alpha \). These operators are supposed to satisfy the 
relations

\[
a_k a_p^+ \pm a_p^+ a_k = \delta_{kp},
\]

where the sign \( \pm \) depends on the statistics of particles. The expression (1) implies that in 
MOFT the number of fields is a variable (which must be the same for all types of particles) and, 
therefore, the set of field operators \( \{a_k, a_k^+\} \) is replaced with the expanded set \( \{a_k(j), a_k^+(j)\} \), 
where \( j \in [1, ..., N(k)] \). For a free field, the energy is an additive quantity, so it can be written as

\[
H_0 = \sum_k \sum_j^{N(k)} \omega_k a_k^+(j) a_k(j),
\]

where \( \omega_k = \sqrt{k^2 + m^2} \). In the general case, the total Hamiltonian \( H = H_0 + V \) (where the 
potential term \( V \) is responsible for interactions) can be expanded in the set of operators (e.g., 
see for details Ref. [1])

\[
A_{m,n}(k) = \sum_{j=1}^{N(k)} (a_k^+(j))^m (a_k(j))^n.
\]

In a complete theory with a variable number of fields, quantum states can be classified by 
occupation numbers. To this end, we consider the set of creation and annihilation operators for
field modes \( \{ C (n, k), C^+ (n, k) \} \), where \( k \) is the wave number and \( n \) is the number of particles in the given mode. These operators obey the standard relations

\[
C (n, k) C^+ (m, k') \mp C^+ (m, k') C (n, k) = \delta_{nm} \delta_{kk'}.
\]

and should be used to construct the Fock space in MOFT. The sign \( \mp \) in (5) depends on the symmetry of the wave function under field permutations, i.e., on the statistics of fields. In the case of the so-called parafield theory there exists a spin-statistics theorem [15] (analogous to the Pauli theorem) which states that bosonic modes should be quantized according to the Fermi statistics, while fermionic modes should obey the Bose statistics. Since MOFT represents a specific generalization of the parafield theory, we will assume the same rule for MOFT.

In terms of \( C \) and \( C^+ \), operators (4) can be expressed as follows

\[
\hat{A}_{m_1, m_2} (k) = \sum_n \frac{\sqrt{(n + m_1)!(n + m_2)!}}{n!} C^+ (n + m_1, k) C (n + m_2, k)
\]

where the sum is taken over the values \( n = 0, 1, \ldots \) in the case of bosons, and \( n = 0, 1 \) in the case of fermions. Thus, the eigenvalues of the Hamiltonian of a free field take the form

\[
\hat{H}_0 = \sum_k \omega_k \hat{A}_{1, 1} (k) = \sum_{k, n} n \omega_k N (n, k),
\]

where \( N (n, k) \) is the number of modes for fixed values of the wave number \( k \) and the number of particles \( n \) (i.e., \( N (n, k) = C^+ (n, k) C (n, k) \)).

Thus, the field state vector \( \Phi \) is a function of the occupation numbers \( \Phi (N (n, k), t) \), and its evolution is described by the Shrödinger equation

\[
i \partial_t \Phi = H \Phi.
\]

Consider the operator

\[
N (k) = \hat{A}_{0, 0} (k) = \sum_n C^+ (n, k) C (n, k)
\]

which characterizes the total number of modes for a fixed wave number \( k \). In standard processes when the number of fields is conserved (e.g., when topology transformations are suppressed) \( N (k) \) is a constant of the motion \([N (k), H] = 0\) and, therefore, this operator can be considered as an ordinary fixed function of wave numbers.

Consider now the particle creation and annihilation operators. Among the operators \( \hat{A}_{n, m} (k) \) are some which change the number of particles by one

\[
b^-_m (k) = \hat{A}_{m, m+1} (k), \quad b^+_m (k) = \hat{A}_{m+1, m} (k),
\]

and which replace the standard operators of creation and annihilation of particles, i.e., they satisfy the relations

\[
[\hat{n}, b^{(\pm)}_m (k)] = \pm b^{(\pm)}_m (k), \quad [H_0, b^{(\pm)}_m (k)] = \pm \omega_k b^{(\pm)}_m (k),
\]

where

\[
\hat{n} = \sum_k \hat{n}_k = \sum_{k, n} n N (n, k).
\]
In the case of fermions there exist only two such operators \( b_0^+ (k) \) and \( b_0^- (k) \), while in the case of bosons the total number of creation/annihilation operators is determined by the structure of the interaction term \( V \). In the simplest case (e.g., in the electrodynamics) the interaction term is expressed solely via \( b_0^+ (k) \) and \( b_0^- (k) \) and, therefore, we can introduce creation/annihilation operators for the effective field

\[
\tilde{a}_k = \frac{1}{\sqrt{N(k)}} b_0^-(k), \quad \tilde{a}_k^+ = \frac{1}{\sqrt{N(k)}} b_0^+(k), \quad (13)
\]

which satisfy the standard commutation relations, i.e., \( [\tilde{a}_k, \tilde{a}_p^+] = \delta_{kp} \). This restores the standard theory but new features, however, appear. First, the renormalization (13) results in the renormalization of interaction constants (e.g., see for details [2]) and, secondly, in the region of wave numbers in which \( N(k) > 1 \) fermions violate the Pauli principle (up to \( N(k) \) fermions can be created at the given wave number \( k \)).

3 Vacuum state in MOFT

In this section we describe the structure of the vacuum state for bosons and fermions. The true vacuum state in MOFT is defined by the relation

\[
C(n, k) |0\rangle = 0. \quad (14)
\]

In the true vacuum state all modes are absent \( N(k) = 0 \), hence no particles can be created and all observables related to the field are absent. Thus, the true vacuum state corresponds to the absence of physical space and, in reality, cannot be achieved. Assuming that upon the quantum period of the evolution of the Universe topology transformations are suppressed, we should require that the number of fields conserves \( N(k) = \text{const} \) in every mode. Then we can define the ground state of the field \( \psi \) (which is the vacuum from the standpoint of particles) which is the vector \( \Phi_0 \) satisfying the relations

\[
b_m (k) \Phi_0 = 0 \quad (15)
\]

for all values \( k \) and \( m = 0, 1, \ldots \). However, these relations still do not define a unique ground state and should be completed by relations which specify the distribution of modes \( N(k) \). In the case of bosons, additional relations can be taken in the form

\[
b_{N(k)+m}^+ (k) \Phi_0 = 0 \quad (16)
\]

where \( m = 0, 1, \ldots \). Then the state \( \Phi_0 \) corresponds to the minimum energy for a fixed mode distribution \( N(k) \) and is characterized by the occupation numbers

\[
N(n, k) = \theta (\mu_k - n \omega_k) \quad (17)
\]

where \( \theta (x) \) is the Heaviside step function and \( \mu_k \) is the chemical potential which can be expressed via the number of modes \( N(k) \) as

\[
N(k) = \sum_n \theta (\mu_k - n \omega_k) = 1 + \left[ \frac{\mu_k}{\omega_k} \right]. \quad (18)
\]
In particular, from (17) we find that the ground state for bosonic fields contains real particles

$$\tilde{n}_k = \sum_{n=0}^{\infty} nN(n,k) = \frac{1}{2}N(k)(N(k) + 1)$$

(19)

and, therefore, corresponds to a finite energy $$E_0 = \sum \omega_k \tilde{n}_k$$. These particles, however, are “dark” for they correspond to the ground state.

In the case of fermions the only nontrivial operator $$b_{m}^{\pm}(k)$$ corresponds to $$m = 0$$, and the additional relations read

$$\left(b_{0}^{+}(k)\right)^{N(k)+1} \Phi_0 = 0.$$ 

(20)

These relations mean that the total number of particles $$n_k$$ which can be created at the given wave number $$k$$ takes values $$n_k = 0, 1, ..., N(k)$$ (i.e., it cannot exceed the number of modes). In this case the ground state $$\Phi_0$$ corresponding to the fixed mode distribution $$N(k)$$ is characterized by the occupation numbers

$$N(n,k) = N(k) \delta_{n,0}.$$ 

(21)

In contrast to the case of bosonic fields, the ground state (21) contains no particles and corresponds to the zero energy $$E_0 = 0$$. Formally the ground state (21) can be constructed from the true vacuum state as follows

$$\Phi_0 = |N(k),0\rangle = \prod_k \frac{(C^+(0,k))^{N(k)}}{\sqrt{N(k)!}} |0\rangle,$$

(22)

while the basis of the Fock space consists of vectors of the type

$$|N(k) - m_i, m_i\rangle = \prod_i \sqrt{\frac{(N(k_i) - m_i)!}{N(k_i)! m_i!}} (b_{m_i}^+(k_i))^{m_i} |N(k),0\rangle,$$

(23)

where $$m_i = 0, 1, ..., N(k)$$.

We interpret the function $$N(k)$$ as a geometric characteristic of the momentum space which has formed during the quantum period in the evolution of the Universe, when topology transformations took place. Then assigning a specific value for the function $$N(k)$$, expressions (17) and (21) define the ground state for respective particles.

### 4 Origin of the spectral number of fields

A rigorous derivation of properties of the function $$N(k)$$ requires studying processes involving topology changes which took place during the quantum stage of the evolution of the Universe. At the moment, we do not have an exact model describing the formation of $$N(k)$$ and, therefore, our consideration will have a phenomenological character. We assume that upon the quantum period of the Universe the matter was thermalized with a very high temperature. Then, as the temperature dropped during the early stage of the evolution of the Universe, the topological structure of the space (and the spectral number of fields) has tempered and the subsequent evolution resulted only in the cosmological shift of the physical scales.

There exist at least two possibilities. The first and the simplest possibility is the case when a transformation in the topology of space results in an equal transformation of the number of modes for all fields (regardless of the type of the fields). In this case processes involving
topology changes generate a unique function $N(k)$ which is the same for all fields. This case was considered in Ref. [2]. However, the mathematical structure of MOFT reserves a more general possibility when the formation of the spectral distribution of modes goes in independent ways for different fields. In this case every particular field $\psi_a$ will be characterized by its own function $N_a(k)$ which can possess specific features. Which case is realized in the nature can be determined only by means of confrontation with observations and below we consider both cases.

Upon the quantum period the Universe is supposed to be described by the homogeneous metric of the form

$$ds^2 = dt^2 - a^2(t) \, dl^2,$$

where $a(t)$ is the scale factor, and $dl^2$ is the spatial interval. It is expected that the state of fields was thermalized with a very high temperature $T > T_{Pl}$ where $T_{Pl}$ is the Planck temperature. Then the state of fields was characterized by the thermal density matrix with mean values for occupation numbers

$$\langle N(k, n) \rangle = \left( \exp \left( \frac{n\omega_k - \mu_k}{T} \right) \pm 1 \right)^{-1},$$

where the signs $+$ and $-$ corresponds to bosonic and fermionic fields respectively and the chemical potential $\mu_k$ can be expressed via the spectral number of fields as

$$N(k) = \sum_n \left( \exp \left( \frac{n\omega_k - \mu_k}{T} \right) \pm 1 \right)^{-1}.$$  \hspace{1cm} (26)

In particular, from (26) we find that for fermions in the region $\omega_k - \mu_k \ll T$ the relation between the spectral number of fields and $\mu_k$ takes the form

$$N(k) \sim \frac{T}{(-\mu_k) + T/(\omega_k - \mu_k)}.$$  \hspace{1cm} (27)

Consider now the first case when the spectral number of fields $N(k)$ is a unique function for all fields. It is well known that near the singularity the evolution of the Universe is governed by a scalar field (responsible for a subsequent inflationary phase), while all other fields can be neglected. We assume that the same field is responsible for topology transformation processes which took place in the early Universe. Thus, we can expect that the state of the scalar field was characterized by the thermal density matrix (25) with $\mu = 0$ (for the number of fields varies). On the early stage $m \ll T$, and the temperature and the energy of scalar particles depend on time as $T = \tilde{T}/a(t)$, $k = \tilde{k}/a(t)$. When the temperature drops below a critical value $T_*$, which corresponds to the moment $t_* \sim t_{pl}$, topological structure (and the number of fields) tempers. This generates the value of the chemical potential for scalar particles $\mu \sim T_*$.  

Let us neglect the temperature corrections, which are essential only at $t \sim t_*$ and whose role is in smoothing the real distribution $N_k$. Then at the moment $t \sim t_*$ the ground state of the scalar field will be described by (17) with $\mu_k = \mu = const \sim T_*$. During the subsequent evolution the physical scales are subjected to the cosmological shift, however the form of this distribution in the commoving frame must remain the same. Thus, on the later stages $t \geq t_*$, we find

$$N(k) = 1 + \left[ \frac{\tilde{k}_1}{\Omega_k(t)} \right],$$

where $\Omega_k(t) = \sqrt{a^2(t)k^2 + \tilde{k}_2^2}$, $\tilde{k}_1 \sim a_0\mu$, and $\tilde{k}_2 \sim a_0m$ ($a_0 = a(t_*)$). From (28) we see that there is a finite interval of wave numbers $k \in [k_{\text{min}}(t), k_{\text{max}}(t)]$ on which the number of fields
$N_k$ changes its value from $N_k = 1$ (at the point $k_{\text{max}}$) to the maximal value $N_{\text{max}} = 1 + \left[ \tilde{k}_1/\tilde{k}_2^2 \right]$ (at the point $k_{\text{min}}$) where the boundary points of the interval of $k$ depend on time and are expressed via the free phenomenological parameters $\tilde{k}_1$ and $\tilde{k}_2$ as follows

$$k_{\text{max}} = \frac{1}{a(t)} \sqrt{\tilde{k}_1^2 - \tilde{k}_2^2}, \quad k_{\text{min}} = \frac{1}{a(t)} \sqrt{\tilde{k}_1^2 / (N_{\text{max}} - 1)^2 - \tilde{k}_2^2}.$$

Out of this interval the number of fields remains constant i.e., $N(k) = N_{\text{max}}$ for the range $k \leq k_{\text{min}}(t)$ and $N(k) = 1$ for the range $k \geq k_{\text{max}}(t)$. From restrictions on parameters of inflationary scenarios we get $m \lesssim 10^{-5}m_{\text{pl}}$ which gives $N_{\text{max}} \gtrsim 10^5 T_*/m_{\text{pl}}$, where $T_*$ is the critical temperature at which topology has been tempered. Thus, substituting (28) in (26) we define the values of the chemical potentials $\mu_k$ for all other particles.

Consider now the second case when the spectral number of modes $N(k)$ forms independently for different fields. In this case bosonic fields are described by the same distribution (28) in which, however, the parameters $\tilde{k}_1$ and $\tilde{k}_2$ are free phenomenological parameters which are specific for every particular field. In particular, for massless fields we find $\tilde{k}_2 = k_{\text{min}} = 0$ and $N(k) = 1 + (k_{\text{max}}/k)$. In the case of fermions the chemical potential $\mu_k$ cannot vanish and for $T \geq T_*$ it should take some rest value $\mu_k = \epsilon_0$ (otherwise $N(k,0) \to \infty$). Thus, in the same way as in the case of bosons, we find $N(k) = 1 + \theta (k_{\text{max}} - k)$, where $\theta(x)$ is the Heaviside step function and $k_{\text{max}} \sim T_1 a(t) / a(t)$. In this case the spectral number of fermions is characterized by the only phenomenological parameter and $N(k) = N_{\text{max}} = 2$ as $k < k_{\text{max}}$.

In this manner we have shown that properties of the spectral number of fields can be different, depending on which case is realized in the nature. However, we note that if in the first case the spectral number of fields $N(k)$ can be considered as a new geometric characteristics which straightforwardly defines properties of space and hence of all matter fields, in the second case we, rigorously speaking, cannot use such an interpretation. Moreover, if the last case is really realized in the nature, it should relate to yet unknown processes. Therefore, in the next section we will discuss the first possibility only.

In conclusion of this section we point out to a formal analogy between MOFT and the Hagedorn theory [17]. This analogy, however, is not complete, for in MOFT the number of sorts of particles is always finite and the total energy density $\varepsilon$ behaves like in the standard theory, e.g., for $T \gg m$ we can show that $\varepsilon = \kappa(T) T^4$, where $\kappa \to \text{const}$ as $T \to \infty$. We also note that the real distribution can be different from (28), which depends on the specific picture of topology transformations in the early Universe and requires the construction of the exact theory (in particular, thermal corrections smoothen the step-like distribution (28)). However we believe that the general features of $N_k$ will remain the same.

## 5 Effective dimension of the Universe

The growth of the spectral number of fields which takes place in the range of wave numbers $k_{\text{min}}(t) < k < k_{\text{max}}(t)$ leads to the fact that in this range our Universe has to demonstrate nontrivial geometric properties. Indeed, it was recently shown that in the same range of scales a stable equilibrium distribution of baryons requires a fractal behavior with dimension $D \approx 2$ [5], while the Newton’s and Coulomb’s energies of interaction between point particles show the logarithmic behavior [2]. We recall that the logarithmic potential $\ln(r)$ gives the solution of the Poisson equation with a point source for two dimensions. Both these phenomena are in agreement with the observed picture of the Universe and it turns out that they have a common
pure geometrical interpretation. Namely, we can say that in the range of scales \( r_{\text{min}} < r < r_{\text{max}} \) (where \( r_{\text{min}} = 2\pi/k_{\text{max}} \)) some kind of reduction of the dimension of space happens.

Indeed, the simplest way to demonstrate this is to compare the spectral number of modes in the interval between \( k \) and \( k + dk \) in MOFT, which is given by the measure (1) (i.e., \( N(k) d^3k/(2\pi)^3 \)), with the spectral number of modes for \( n \) dimensions in the standard field theory (i.e., \( d^n k/(2\pi)^n \)). Hence we can define the effective dimension \( D \) of space as follows

\[
k^3 N(k) \sim k^D. \tag{29}\]

In the standard field theory \( N(k) = 1 \) and we get \( D = 3 \), while in MOFT the properties of the function \( N(k) \) were formed during the quantum period in the evolution of the Universe and depend on specific features of topology transformation processes. Thus, in general case, the effective number of dimensions \( D \) may take different values for different intervals of scales. If we take the value (28) we find that in the range of wave numbers \( k_{\text{max}} \geq k \geq k_{\text{min}} \) (where the function \( N(k) \) can be approximated by \( N(k) \sim k_{\text{max}}/k \) the effective dimension of space is indeed \( D \approx 2 \). The scale \( r_{\text{min}} = 2\pi/k_{\text{max}} \) can be called an effective scale of compactification of \( 3 - D \) dimensions, while the scale \( r_{\text{max}} \), if it really exists, represents the boundary after which the dimension \( D = 3 \) restores. We note also that after this scale the fractal picture of the Universe crosses over to homogeneity and the standard Newton’s law restores.

In MOFT, \( N(k) \) represents the operator of the number of fields (9) which is common for all types of fields and, therefore, plays the role of the density operator for the momentum space. We note that \( N(k) \) is an ordinary function only in the case when topology changes are suppressed. In the coordinate representation the number of fields is described by an operator \( N(x) \) (e.g., see Ref. [2]) which defines the density of physical space, i.e., the volume element is given by \( dV = N(x) d^3x \).

Consider the relation between these two operators. In what follows we, for the sake of convenience, consider a box of the length \( L \) and will use the periodic boundary conditions (i.e., \( k = 2\pi n/L \) and, as \( L \to \infty \), \( \sum_k \to \int (L/2\pi)^3 d^3k \)). From the dynamical point of view the operator \( N(k) \) has a canonically conjugated variable \( \vartheta(k) \) such that

\[
[\vartheta(k), N(k')] = \vartheta(k) N(k') - N(k') \vartheta(k) = i\delta_{k,k'}. \tag{30}\]

These two operators can be used to define a new set of creation and annihilation operators

\[
\Psi^+(k) = \sqrt{N(k)} e^{i\vartheta(k)}, \Psi(k) = e^{-i\vartheta(k)} \sqrt{N(k)} \tag{31}\]

which obey the standard commutation relations

\[
[\Psi(k), \Psi^+(k')] = \delta_{k,k'} \tag{32}\]

and have the meaning of the creation/annihilation operators for field modes, e.g., the density operator for the momentum space is defined simply as \( N(k) = \Psi^+(k) \Psi(k) \). In the case when topology transformations are suppressed the operator \( \Psi \) can be considered as a classical scalar field \( \varphi \) which characterizes the density of physical space\(^1\) and, in general, depends on time and space coordinates (in what follows we consider the spatial dependence only). Indeed, in applying to the ground state \( \Phi_0 = |N(k)\rangle \) (which is expressed by the occupation numbers (17)\(^1\)}

\(^1\) We note that if every field is characterized by its own spectral number density \( N(k) \), we have to introduce a set of independent classical scalar fields (one for every quantum field). This gives a good chance to explain origin of Higgs fields in particle theory.
and (21)) the operators $\Psi$ and $\Psi^+$ change the number of modes by one, while the total number of modes $N \to \infty$. In this sense the state $\Phi'_0 = \Psi \Phi_0 \approx \Phi_0$. Thus, the classical field $\varphi$ may be defined by relations

$$\varphi (k) = \langle N (k) | \Psi (k) | N (k) + 1 \rangle, \quad \varphi^* (k) = \langle N (k) + 1 | \Psi^+ (k) | N (k) \rangle,$$

which gives $\varphi (k) = \varphi^* (k) = \sqrt{N (k)}$. The coordinate dependence of this field can be found in the standard way (e.g., see Ref. [18])

$$\Psi (\mathbf{r}) = e^{-i \mathbf{r} \cdot \mathbf{P}} \Psi (0) e^{i \mathbf{r} \cdot \mathbf{P}}$$

where $\mathbf{P}$ is the total momentum operator. If in states $N_i$ and $N_f$ the system possesses fixed momenta $\mathbf{P}_i$ and $\mathbf{P}_f$, then

$$\langle N_i | \Psi (\mathbf{r}) | N_f \rangle = \exp (-i \mathbf{k}_{if} \cdot \mathbf{r}) \langle N_i | \Psi (0) | N_f \rangle$$

where $\mathbf{k}_{if} = \mathbf{P}_i - \mathbf{P}_f$. We note that the creation/annihilation of a single mode is accompanied with the increase/decrease in the number of ”dark” particles (19) by $N (k)$ and hence in the total momentum by $\mathbf{P}_k = z N (k) \mathbf{k}$, where $z$ is the number of different types of bosonic fields. Thus, from (35) we find that coordinate dependence will be described by the sum of the type

$$\varphi (\mathbf{x}) \sim \sum_k c_k \exp (-i \mathbf{P}_k \mathbf{x}).$$

where $c_k \sim \sqrt{N (k)}$. In the range of wave numbers $k_{\text{max}} \geq k \gg k_{\text{min}}$ we can use the approximation $N (k) \simeq 1 + k_{\text{max}}/k$ and let the length of the box be $L < r_{\text{max}} = 2\pi/k_{\text{min}}$. Then we find that in the sum (36) the maximal wavelength is indeed restricted by

$$\ell \leq \ell_0 < L,$$

where $\ell_0 = 2\pi/zk_{\text{max}} = r_{\text{min}}/z$, which means that at least in one direction the space box is effectively compactified to the size $\ell_0$. From the physical standpoint such a compactification will be displayed in irregularities of the function $N (x) \sim | \varphi (x) |^2$ (we point out that the time dependence will randomize phases in (36)). E.g., the function $N (x)$ may take essential values

$N (x) \simeq < N >$ only on thin two dimensional surfaces of the width $\sim \ell_0$ and rapidly decay outside, which may explain the formation of the observed fractal distribution of galaxies and the logarithmic behavior of the Newton’s potential. In considering the box of the larger and larger size $L \gg r_{\text{max}}$ we find that $N (k) \to N_{\text{max}} \sim r_{\text{max}}/r_{\text{min}}$ and the effect of the compactification disappears (no restrictions on possible values of wavelengths emerge). Thus at scales $\ell > r_{\text{max}}$ dimension $D = 3$ restores which restores the standard Newton’s law and the distribution of galaxies crosses over to homogeneity.

6 Conclusions

In this manner we have shown that MOFT predicts a rather interesting physics for the range of scales $r_{\text{min}} < r < r_{\text{max}}$. First of all we point out to the fact that in this range the Universe acquires features of a two-dimensional space whose distribution in the observed 3-dimensional volume has an irregular character. This can provide a natural explanation to the observed fractal distribution of galaxies and the dark matter problem. We note that such properties
originates from a primordial thermodynamically equilibrium state and are in agreement with the homogeneity and isotropy of the Universe. Thus, such a picture of the Universe represents a homogeneous background, while gravitational potential fluctuations should be considered in the same way as in the standard model. Nevertheless, we can state that in the range of wavelengths \( r_{\min} < \lambda < r_{\max} \) the propagation of perturbations will correspond to the 2-dimensional law.

The two-dimensional character of the Universe in the range of scales means that we should expect that the standard Newtonian kinematics breaks also down. Some realization of such a possibility is contained in MOND (the Modified Newtonian Dynamics) proposed by Milgrom in Ref. [3]. We believe that MOFT provides a rigorous way to derive such a modification which, however, requires a separate investigation.

In the present paper we have supposed that upon the quantum period in the evolution of the Universe the spectral number of fields \( N(k) \) is conserved, which means that \( N(k) \) depends on time via only the cosmological shift of scales (i.e., \( k(t) \sim 1/a(t) \), where \( a(t) \) is the scale factor). However, there are reasonable arguments which show that the number of fields should eventually decay. Indeed, if we take into account thermal corrections to \( N(k) \), then instead of (28) we get the expression of the type \( N(k) \sim 1 + T_s/k + \ldots \). In this case the temperature \( T_s \) plays the role of the maximal wave number \( k_{max} \). Why at present this temperature is so small \( T_s \ll T_\gamma \) (where \( T_\gamma \) is the CMB temperature) requires a separate explanation. The situation is different when we assume that the number of modes may decay. At the moment, we do not have an exact model describing the decay of fields, however, phenomenologically the decay can be described by the expression of the type \( N(k) \sim N_0(k) e^{-\Gamma_k t} \) (where \( \Gamma_k \) is the period of the half-decay which, in general, can depend on wave numbers \( k \)). In the simplest case \( \Gamma_k = const \) and this would lead to an additional monotonic increase of the minimal scale \( r_{\min} \sim a(t) e^{\Gamma t} \), while the maximal scale changes according to the cosmological shift only \( r_{\max} \sim a(t) \). We note that the ground state of fields \( \Phi_0 \) contains hidden bosons (19). The decay of modes transforms them into real particles and, therefore, the decay is accompanied by an additional reheating which should change the temperature of the primordial plasma (which can explain the difference between \( T_s \) and \( T_\gamma \)). Besides, the additional increase in \( r_{\min} \) means that the number of baryons contained within a radius \( r_{\min} \) depends on time \( N_b(r_{\min}) \sim e^{3\Gamma t} \). Thus, on early stages of the evolution of the Universe we find that \( N_b(r_{\min}) \ll 1 \) and the stable equilibrium distribution of baryons had the fractal character \( N_b(r) \sim r^\delta \) for all scales \( r < r_{\max} \). Eventually, \( r_{\min} \) increases, the fractal distribution of baryons below the scale \( r_{\min} \) becomes unstable, and the larger and larger number of baryons \( N_b(r_{\min}) \sim e^{2\Gamma t} \) are involved into the structure formation. In such a picture the smallest objects appear first and then they form groups, galaxies, clusters, etc.. The advantage of such a scenario is the fact that it does not require the existence of primeval perturbations of the metric.

We also point out to the violation of the Pauli principle for wavelengths \( \lambda > r_{\min} \) (more than one fermion can have a wavelength \( \lambda \)). Such particles are located in the volume \( \gtrsim r_{\min}^3 \) and at laboratory scales the portion of states violating the statistics is extremely suppressed \( P \lesssim (L/r_{\min})^3 \), where \( L \) is a characteristic spatial scale of a system under measurement (e.g., if such a scale \( L \) we take the Earth radius, this factor will be still extremely small \( P \sim 10^{-32} \)). We note that for a system of particles which is in the thermal equilibrium state the characteristic scale \( L \sim 1/T \) (where \( T \) is the temperature). Thus, if the number of fields is conserved, MOFT will not change the predictions of the standard model of nucleosynthesis, e.g., corrections will have the factor \( P \sim (1/Tr_{\min})^3 \) which does not depend on time. However, the nucleosynthesis can introduce some restrictions on the possible rate of the decay of modes \( \Gamma \). This problem, however, requires the further investigation.

In conclusion we point out to the fact that, according to MOFT, the violation of the Pauli
principle serves as an indicator of a nontrivial topology of space and, therefore, the possibility to detect such a violation in laboratory experiments represents an extremely intriguing problem.

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