Localized Exotic Smoothness

Carl H. Brans

Institute for Advanced Study
Princeton, NJ 08540
and
Physics Department
Loyola University
New Orleans, LA 70118
e-mail:brans@music.loyno.edu

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Abstract

Gompf’s end-sum techniques are used to establish the existence of an infinity of non-diffeomorphic manifolds, all having the same trivial $\mathbb{R}^4$ topology, but for which the exotic differentiable structure is confined to a region which is spatially limited. Thus, the smoothness is standard outside of a region which is topologically (but not smoothly) $B^3 \times \mathbb{R}^1$, where $B^3$ is the compact three ball. The exterior of this region is diffeomorphic to standard $\mathbb{R}^1 \times S^2 \times \mathbb{R}^1$. In a space-time diagram, the confined exoticness sweeps out a world tube which, it is conjectured, might act as a source for certain non-standard solutions to the Einstein equations. It is shown that smooth Lorentz signature metrics can be globally continued from ones given on appropriately defined regions, including the exterior (standard) region. Similar constructs are provided for the topology, $S^2 \times \mathbb{R}^2$ of the Kruskal form of the Schwarzschild solution. This leads to conjectures on the existence of Einstein metrics which are externally identical to standard black hole ones, but none of which can be globally diffeomorphic to such standard objects. Certain aspects of the Cauchy problem are also discussed in terms of $\mathbb{R}_0^4$ models which are “half-standard”, say for all $t < 0$, but for which $t$ cannot be globally smooth.

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This paper is concerned with smooth manifold models for space-time which have relatively trivial topology, e.g., $\mathbb{R}^4$, or $\mathbb{R}^2 \times S^2$, but non-standard, or “exotic” smoothness structures. By definition, such manifolds are not diffeomorphic to their standard smooth form, and hence, from the basic principles of general relativity, cannot be physically equivalent to any previously studied manifold with the corresponding simple topology. In the non-compact cases an important feature of many of these examples is that they require that the exotic part extend “to infinity”, as illustrated, for example, in figure 1. This fact has served as a deterrent to the consideration of such spaces as space-time models since classical observations of space-time are “large scale” in some sense, with resulting expectation of asymptotic regularity, including smoothness. In fact, all that the mathematics requires is that the exotic region not be contained in a compact set. However, to my knowledge, no example of an exotic manifold which is standard at spatial infinity has ever been published before now. The main result of this paper can be summarized informally:

**Result 1** There exists exotic smooth manifolds with $\mathbb{R}^4$ topology which are standard at spatial infinity, so that the exoticness can be regarded as spatially confined.

A more precise statement of this result is provided in Theorem 1 below. The resulting manifold structures are illustrated in examples such as those shown in figures 3 and 4 where everything looks normal at space-like infinity but the standard structure cannot be continued all the way in to spatial origin. This work is based on the remarkable mathematical breakthroughs of Milnor, Freedman, Donaldson, Gompf [1],[2],[3],[4], and others, establishing the surprising existence of such exotic structures on topologically trivial spaces, including $\mathbb{R}^4$, together with the end-sum techniques of Gompf[7].

This result could have great significance in all fields of physics, not just relativity. Some model of space-time underlies every field of physics. It has now been proven that we cannot infer that space is necessarily smoothly standard from investigating what happens at space-like infinity, even for topologically trivial $\mathbb{R}^4$. It seems very clear that this is potentially very important to all of physics since it implies that there is another possible obstruction, in addition to material sources and topological ones, to continuing external vacuum solutions for any field equations from infinity to the origin. Of course, in the absence of any explicit coordinate patch presentation, no example can be displayed. However, this leads naturally to a conjecture, informally stated:

**Conjecture 1** This localized exoticness can act as a source for some externally regular field, just as matter or a wormhole can.

Of course, the exploration of this conjecture will require more detailed knowledge of the global metric structure than is available at present. The notions of domains of dependence, Cauchy surfaces, etc., necessary for such studies cannot be fully explored with present differential geometric information on exotic manifolds. However, a beginning can be made with certain general existence results as established and discussed below.
In order to provide some background for these matters, let us begin with a brief review of the relevant mathematical facts. The apparently innocuous question of whether or not the set of differentiable structures (modulo diffeomorphisms) on \( \mathbb{R}^n \) is trivial has long been of mathematical interest. As of about ten years ago, this question had been settled in the expected affirmative for all \( n \neq 4 \), and probably most people expected the exceptional case \( n = 4 \) to ultimately resolve to the same conclusion. After all, there is certainly no interesting topology in \( \mathbb{R}^4 \) to provide a basis for any other expectation. It was thus of considerable interest when the existence of counter-examples began to appear around 1982, \[2\], \[3\], \[4\]. Our paper \[6\] provides a brief survey of this problem and some conjectures on the possible physical implications of these results. In this paper, certain questions raised in \[6\] are at least partially answered.

Since the existence of non-trivial differentiable structures on topologically trivial spaces is so strikingly counter-intuitive, it is important to clarify several issues relating to differential topology. Specifically we must distinguish the case of merely differentiable structures from non-diffeomorphic ones. The former are physically indistinguishable, but the latter are definitely not physically equivalent as space-time models. These issues are discussed in \[6\]. For all \( n \neq 4 \) it is possible to show that all smoothness structures on \( \mathbb{R}^n \) are diffeomorphic.

However, precisely the opposite is true for the surprising case, \( n = 4 \), \[2\], \[3\]. A smooth manifold homeomorphic to \( \mathbb{R}^4 \) but not diffeomorphic to it is called “exotic” (or “fake”) and denoted here by \( \mathbb{R}_\Theta^4 \). Such a manifold consists of a set of points which can be globally topologically identified with the ordered set of four numbers, say \((t, x, y, z)\). While these may be smooth coordinates locally over some neighborhood, they cannot be globally continued as smooth functions. Furthermore, in no diffeomorphic image of this \( \mathbb{R}_\Theta^4 \) can the global topological coordinates be extended as smooth beyond some compact set.

Also, note that certain \( \mathbb{R}_\Theta^4 \) have the property that they contain compact sets which cannot themselves be contained in the interior of any smooth \( S^3 \). Thus, for some \( R_0 \), the topological three-sphere, \( t^2 + x^2 + y^2 + z^2 = R^2 \), cannot be smooth if \( R > R_0 \). This is illustrated in Figure 1. Notice that in Figures 1 through 5 one space dimension has been suppressed, so each point is actually a z-axis, while in Figure 6 two dimensions are suppressed and each point is an \( S^2 \).

As interesting as these \( \mathbb{R}_\Theta^4 \) are in their own right, a technique developed by Gompf\[7\] allows the construction of a large topological variety of exotic four-manifolds, some of which would appear to have considerable potential for physics. Gompf’s “end-sum” process provides a straightforward technique for constructing an exotic version, \( M \), of any non-compact four-manifold whose standard version, \( M_0 \), can be smoothly embedded in standard \( \mathbb{R}^4 \). Recall that we want to construct \( M \) which is homeomorphic to \( M_0 \), but not diffeomorphic to it. First construct a tubular neighborhood, \( T_0 \), of a half ray in \( M_0 \). \( T_0 \) is thus standard \( \mathbb{R}^4 = [0, \infty) \times \mathbb{R}^3 \). Now consider a diffeomorphism, \( \phi_0 \) of \( T_0 \) onto \( N_0 = [0, 1/2) \times \mathbb{R}^3 \) which is the identity on the \( \mathbb{R}^3 \) fibers. Do the same thing

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1In general, the subscript \( \Theta \) will indicate a non-standard object or process. So \( M \times_\Theta N \) means a smooth manifold which is the topological, but not smooth, cartesian product of the two manifolds.
for some exotic $\mathbb{R}^4_\Theta$ with the important proviso that it cannot be smoothly embedded in standard $\mathbb{R}^4$. Such manifolds are known in infinite abundance \[5\]. Then construct a similar tubular neighborhood for this $\mathbb{R}^4_\Theta$, $T_1$, with diffeomorphism, $\phi_1$, taking it onto $N_1 = [1, 1/2) \times \mathbb{R}^3$. The desired exotic $M$ is then obtained by forming the identification

$$M = M_0 \cup_{\phi_0} ([0, 1] \times \mathbb{R}^3) \cup_{\phi_1} \mathbb{R}^4_\Theta$$

(1)

The techniques of forming tubular manifolds and defining identification manifolds can be found in standard differential topology texts, such as \[8\] or \[9\].

Informally, what is being done is that the tubular neighborhoods are being smoothly glued across their “ends”, each $\mathbb{R}^3$. The proof that the resulting $M$ is indeed exotic is then easy: $M$ contains $\mathbb{R}^4_\Theta$ as a smooth sub-manifold. If $M$ were diffeomorphic to $M_0$ then $M$, and thus $\mathbb{R}^4_\Theta$, could be smoothly embedded in standard $\mathbb{R}^4$, contradicting the assumption on $\mathbb{R}^4_\Theta$. Finally, it is clear that the constructed $M$ is indeed homeomorphic to the original $M_0$ since all that has been done topologically is the extension of $T_0$. See figure 2 for a visualization of this process when $M_0$ is $\mathbb{R}^4$. Smoothly “stuffing” the upper $\mathbb{R}^4_\Theta$ into the tube results in another visualization of the new manifold as shown in figure 3. A natural doubling of this process leads to figure 4. Finally, smoothly spreading out the exotic tube in figure 3 leads to figure 5.

The smoothness properties of the $\mathbb{R}^4_\Theta$ in figure 4 can be summarized by saying the global $C^0$ coordinates, $(t, x, y, z)$, are smooth in the exterior region $[a, \infty) \times S^2 \times \mathbb{R}^1$ given by $x^2 + y^2 + z^2 > a^2$ for some positive constant $a$, while the closure of the complement of this is clearly an exotic $B^3 \times \Theta R^1$. Since the exterior component is standard, a wide variety, including flat, of Lorentz metrics can be imposed. Picking only those for which $\partial/\partial t$ is timelike in this region provides a natural sense in which the world-tube confining the exotic part is “spatially localized.” The smooth continuation of such a metric to the full metric is then guaranteed by Lemma 1 and the discussion following it below. Thus, we can state

**Theorem 1** There exists smooth manifolds which are homeomorphic but not diffeomorphic to $\mathbb{R}^4$ and for which the global topological coordinates $(t, x, y, z)$ are smooth for $x^2 + y^2 + z^2 \geq a^2 > 0$, but not globally. Smooth metrics exists for which the boundary of this region is timelike, so that the exoticness is spatially confined.

We can also use this technique to generate an infinity of non- diffeomorphic manifolds, $\mathbb{R}^2 \times_\Theta S^2$, each having the topology of the Kruskal presentation of the Schwarzschild metric. Using the standard Kruskal notation $\{(u, v, \omega); u^2 - v^2 < 1, \omega \in S^2\}$ constitute global topological coordinates, but $(u, v)$ cannot be continued as smooth functions over the entire range: $u^2 - v^2 < 1$. However, by techniques discussed in \[3\], these coordinates can be smooth over some closed submanifold, say $A$, as illustrated in Figure 6. Over $A$ then we can solve the vacuum Einstein equations as usual to get the Kruskal form. However, this metric cannot be extended over the full manifold, not for any reasons associated with

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\[2\]Here the “exotic” can be understood as referring to the product which is continuous but cannot be smooth. See the discussion around Lemma 2 below.
the development of singularities in the coordinate expression of the metric, or for any topological reasons, but simply because the coordinates, \((u,v,\omega)\), cannot be continued smoothly beyond some proper subset, \(A\), of the full manifold, thus establishing

**Theorem 2** On some smooth manifolds which are topologically \(\mathbb{R}^2 \times S^2\), the standard Kruskal metric cannot be smoothly continued over the full range, \(u^2 - v^2 < 1\).

However, given any Lorentzian metric on a closed submanifold, \(A\), some smooth continuation of the metric to all of \(M\) can be guaranteed to exist under certain conditions. For example, we have

**Lemma 1** If \(M\) is any smooth connected 4-manifold and \(A\) is a closed submanifold for which \(H^4(M, A; \mathbb{Z}) = 0\), then any smooth, time-orientable Lorentz signature metric defined over \(A\) can be smoothly continued to all of \(M\).

Proof: This is basically a question of the continuation of cross sections on fiber bundles. Standard obstruction theory is usually done in the continuous category, but it has a natural extension to the smooth class, \([10]\). First, we note that any time-orientable Lorentz metric is decomposable into a Riemannian one, \(g\), plus a non-zero vector field, \(v\). The continuation of \(g\) follows from the fact that the fiber, \(Y_s\), of non-degenerate symmetric four by four matrices is \(q\)-connected for all \(q\). From standard obstruction theory, this implies that \(g\) can be continued from \(A\) to all of \(M\) without any topological restrictions. On the other hand, the fiber of non-zero vector fields is the three-sphere which is \(q\)-connected for all \(q < 3\), but certainly not 3-connected (\(\pi_3(S^3) = \mathbb{Z}\)). Again from standard results, \([10]\), any obstruction to a continuation of \(v\) from \(A\) to all of \(M\) is an element of \(H^4(M, A; \mathbb{Z})\). Thus, the vanishing of this group is a sufficient condition for the continuation of \(v\), establishing the Lemma.

In the applications in this paper, \(M\) is non-compact, so \(H^4(M; \mathbb{Z}) = 0\). Using the exact cohomology sequence generated by the inclusion \(A \to M\),

\[
\cdots \to H^3(M; \mathbb{Z}) \to H^3(A; \mathbb{Z}) \to H^4(M, A; \mathbb{Z}) \to H^4(M; \mathbb{Z}) \to \cdots
\]

we see that one way to guarantee the condition of the Lemma is to have \(H^3(A; \mathbb{Z}) = 0\). Another would be to establish that the map, \(H^3(M; \mathbb{Z}) \to H^3(A; \mathbb{Z})\), is an epimorphism. For example, if \(A\) is simply a closed miniature version of \(\mathbb{R}^2 \times S^2\) itself, i.e., \(A = D^2 \times S^2\), then \(H^3(A; \mathbb{Z}) = 0\) so the continuation of a smooth Lorentzian metric is ensured. Whatever this metric is, it cannot be the Kruskal one, since otherwise the manifold would be diffeomorphic to standard \(\mathbb{R}^2 \times S^2\). In the case discussed in Theorem 1, \(A = [a, \infty) \times S^2 \times R^1\), for which it is also easy to see that \(H^3(A; \mathbb{Z}) = 0\), guaranteeing the global smooth metric continuation.

An interesting variation of the situation described in Figure 6 occurs when \(A\) intersects the horizon. Thus it contains a trapped surface, so a singularity will inevitably develop from well-known theorems. However, if \(A\) does not contain a trapped surface what will happen is not known.

What is missing from this result, of course, is that the continued metric satisfy the vacuum Einstein equations and that it be complete in the Lorentzian sense. Of course, any
smooth Lorentzian metric satisfies the Einstein equation for some stress-energy tensor, but this tensor must be shown to be physically acceptable. Unfortunately, these issues cannot be resolved without more explicit information on the global exotic structure than is presently available.

Another way to study this metric is in terms of the original Schwarzschild \((r, t)\) coordinates, as seen in figure 4. For this model the coordinates \((t, r, \omega)\) are smooth for all of the closed sub-manifold \(A\) defined by \(r \geq a > 2M\) but cannot be continued as smooth over the entire \(M\) or over any diffeomorphic (physically equivalent) copy. In this case \(A\) is topologically \([a, \infty) \times S^2 \times \mathbb{R}^1\), so again \(H^3(A; \mathbb{Z}) = 0\) and the conditions of lemma 1 are met. Hence there is some smooth continuation of any exterior Lorentzian metric in \(A\), in particular, the Schwarzschild metric, over the full \(\mathbb{R}_4^4\). Whatever this metric is, it cannot be Schwarzschild since the manifolds are not diffeomorphic. An interesting feature of this model is that the manifold is “asymptotically” standard in spite of the well known fact that exotic manifolds are badly behaved “at infinity”. However, we note that this model is asymptotically standard only as \(r \to \infty\), but certainly not as \(t \to \infty\).

These models, especially as visualized in figures 3 and 4 are clearly highly suggestive for investigation of alternative continuation of exterior solutions into the tube near \(r = 0\). We often discover an exterior, vacuum solution, and look to continue it back to some source. This is a standard problem. In the stationary case, we typically have a local, exterior solution to an elliptic problem, and try to continue it into origin but find we can’t as a vacuum solution unless we have a topology change (e.g., a wormhole), or unless we add a matter source, changing the equation. Now, looking at figures 3 and 4, we are led to consider a third alternative.

Of course, the discussion of stationary solutions involves the idea of time foliations, which cannot exist globally for these exotic manifolds, at least not into standard factors. In fact,

**Lemma 2** \(\mathbb{R}_4^4\) cannot be written as a smooth product, \(\mathbb{R}^1 \times_{\text{smooth}} \mathbb{R}^3\). Similarly \(\mathbb{R}^2 \times_{\emptyset} S^2\) cannot be written as \(\mathbb{R}^1 \times_{\text{smooth}} (\mathbb{R}^1 \times S^2)\).

Clearly, if either factor decomposition were smooth, the original manifold would be standard, since the factors are necessarily standard from known lower dimensional results, establishing the lemma. I am indebted to Robert Gompf and Duane Randall for pointing out to me that because of still open questions it is not now possible to establish the more general result for which the second factor is simply some smooth three manifold without restriction. Also, note that the question of factor decomposition of \(\mathbb{R}^4\) into Whitehead spaces was considered by McMillan[1].

Of course, the lack of a global time foliation of these manifolds means that such models are inconsistent with canonical approach to gravity, quantum theory, etc. However, it is worth noting that all experiments yield only local data, so we have no a priori basis for excluding such manifolds.

These discussions lead naturally to a consideration of what can be said about Cauchy problems. Consider then the manifold in figure 5. The global \((t, x, y, z)\) coordinates are smooth for all \(t < 0\) but not globally. Now consider, the Cauchy problem \(R_{\alpha\beta} = 0\),
with flat initial data on $t = -1$. This is guaranteed to have the complete flat metric as solution in the standard, $\mathbb{R}^4$ case. However, the similar problem cannot have a complete flat solution for $\mathbb{R}^4_\theta$ since then the exponential geodesic map would be a diffeomorphism of $\mathbb{R}^4_\theta$ onto its tangent space, which is standard $\mathbb{R}^4$. This is discussed in [6]. What must go wrong in the exotic case, of course, is that $t = -1$ is no longer a Cauchy surface. However, Lemma 1 can again be applied here to guarantee the continuation of some Lorentzian metric over the full manifold since here $A = (-\infty, -1] \times \mathbb{R}^3$ so clearly $H^3(A; \mathbb{Z}) = 0$.

Finally, consider the cosmological model, $\mathbb{R}^1 \times_\theta \mathbb{S}^3$ discussed in [6]. In this case, assume a standard cosmological metric for some time, so here $A = (-\infty, 1] \times \mathbb{S}^3$. Clearly, $H^3(A; \mathbb{Z})$ does not vanish in this case, but it can be shown that the inclusion induced map $H^3(M; \mathbb{Z}) \to H^3(A; \mathbb{Z})$ is onto, so the conditions of Lemma 1 are met. Thus some smooth Lorentzian continuation will indeed exist, leading to some exotic cosmology on $\mathbb{R}^1 \times \mathbb{S}^3$.

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