New Zero-Density Results for Automorphic L-Functions of $GL(n)$

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Abstract: Let $L(s, \pi)$ be an automorphic L-function of $GL(n)$, where $\pi$ is an automorphic representation of group $GL(n)$ over rational number field $\mathbb{Q}$. In this paper, we study the zero-density estimates for $L(s, \pi)$. Define $N_\pi(\sigma, T_1, T_2) = \# \{p = \beta + i\gamma : L(\rho, \pi) = 0, 0 < \sigma < \beta < 1, T_1 \leq \gamma \leq T_2\}$, where $0 \leq \sigma < 1$ and $T_1 < T_2$. We first establish an upper bound for $N_\pi(\sigma, T, 2T)$ when $\sigma$ is close to 1. Then we restrict the imaginary part $\gamma$ into a narrow strip $[T, T + T^a]$ with $0 < a \leq 1$ and prove some new zero-density results on $N_\pi(\sigma, T, T + T^a)$ under specific conditions, which improves previous results when $\sigma$ near $\frac{1}{2}$ and 1, respectively. The proofs rely on the zero detecting method and the Halász-Montgomery method.

Keywords: zero density; automorphic $L$-function; automorphic representation

MSC: 11F66; 11M26; 11M41

1. Introduction

Let $\pi$ be an automorphic representation of group $GL(n)$ over rational number field $\mathbb{Q}$. Then the automorphic (finite-part) $L$-function related to $\pi$ can be defined as

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{A_\pi(n)}{n^s} \quad (\Re s > 1).$$

Furthermore, $L(s, \pi)$ satisfies the Euler product

$$L(s, \pi) = \prod_{p<\infty} \prod_{j=1}^{\infty} (1 - a_\pi(p, j)p^{-s})^{-1} \quad (\Re s > 1),$$

where $a_\pi(p, j)$ are the Langlands parameters of $\pi$. The automorphic $L$-function $L(s, \pi)$ can be analytically continued to the whole complex plane and has a standard functional equation.

The zero estimates of $L(s, \pi)$ is an important topic in number theory and has many applications in various problems, for instance, the applications in the composition of integers and integral ideals (see [1–4], etc.) in divisor problems (see [5–8], etc.) and in mean value estimates involving Hecke eigenvalues (see [9–12], etc.). Many scholars have established zero-density estimates of $L(s, \pi)$ (see [13–15], etc.). We know that all non-trivial zeros of $L(s, \pi)$ are included in the critical strip $0 < \Re s < 1$. As we all know, the Generalized Riemann Hypothesis (GRH) conjectures that all these non-trivial zeros are on the critical line $\Re s = \frac{1}{2}$. Now the GRH is still open. Then it is natural to study the zero-density estimates of $L(s, \pi)$ in a rectangle including the critical line.
where $0 \leq \sigma < 1$ and $T_1 < T_2$. Then the famous “density hypothesis” is

$$N_\pi(\sigma, T, 2T) \ll T^{2(1-\sigma)+\varepsilon}, \quad \frac{1}{2} \leq \sigma < 1.$$ 

Here we take the Riemann zeta function $\zeta(s)$ as an example of the automorphic $L$-function $L(s, \pi)$. To specify $\pi$ by cusp forms and Maass forms, we refer to the references [16–18] and references therein. From the works of Ingham [19] and Huxley [20] we know that

$$N_\pi(\sigma, T, 2T) \ll \begin{cases} T \frac{3(1-\sigma)+\varepsilon}{2-\sigma}, & \text{if } \frac{1}{2} \leq \sigma < \frac{3}{4}, \\ T \frac{3(1-\sigma)+\varepsilon}{3-2\sigma}, & \text{if } \frac{3}{4} \leq \sigma < 1. \end{cases}$$

About the results on the density hypothesis, in 1977, Jutila [21] proved that

$$N_\pi(\sigma, T, 2T) \ll T^{2(1-\sigma)+\varepsilon}, \quad \frac{11}{14} \leq \sigma < 1.$$ 

In 2000, Bourgain [22] improved Jutila’s result to

$$N_\pi(\sigma, T, 2T) \ll T^{2(1-\sigma)+\varepsilon}, \quad \frac{25}{32} \leq \sigma < 1.$$ 

For $\frac{1}{2} \leq u \leq 1$, let

$$M(u, T) = \max_{1 \leq \gamma \leq T} |\zeta(u + i\gamma)|.$$ 

When $\sigma$ is close to 1, Ivić (see [23], Theorem 11.3) proved that for $\frac{9}{10} \leq \sigma \leq 1$,

$$N_\pi(\sigma, -T, T) \ll (M(5\sigma - 4, 3T))^{\frac{3}{2}} \log^{169} T.$$ 

In this paper, motivated by Ivić’s work, we first establish an upper bound of $N_\pi(\sigma, T, 2T)$, when $\sigma$ is close to 1 in the following theorem.

**Theorem 1.** Let $M_1(u, T) = \max_{1 \leq \gamma \leq T} |\zeta(u + i\gamma)|$ and $M_2(u, T) = \max_{1 \leq \gamma \leq T} |L(u + i\gamma, \pi)|$ for $\frac{1}{2} \leq u < 1$, $T \geq 3$. Then for $\frac{9}{10} \leq \sigma \leq 1$ we have

$$N_\pi(\sigma, T, 2T) \ll \left(M_1(5\sigma - 4, 3T)\right)^{\frac{3}{2}} \left(M_2(5\sigma - 4, 4T)\right)^{\frac{1}{2}} T^\varepsilon. \quad (2)$$

**Remark 1.** Since $L(s, \pi)$ is a general automorphic $L$-function in Theorem 1, the mean value estimate now is worse than the case of the Riemann zeta function, which results in the $T^\varepsilon$ (see the argument around (22) for details).

Now we restrict the imaginary part $\gamma$ into a narrow strip $[T, T + T^a]$ and suppose

$$\int_T^{T + T^a} \left|L\left(\frac{1}{2} + it, \pi\right)\right|^{2l} dt \ll \varepsilon, T^{\theta + \varepsilon}, \quad (3)$$

for certain $0 < a \leq 1$ and $\theta \geq a$, where $l \geq 1$ is an integer and $T \geq 3$. Ye and Zhang [24] established some bounds for $N_\pi(\sigma, T, T + T^a)$ with $\frac{1}{2} \leq \sigma < 1$. Later, Dong, Liu and Zhang [25] obtained a sharper bound for $N_\pi(\sigma, T, T + T^a)$ when $\sigma$ is close to 1 and show a range of $\sigma$ for which the density hypothesis holds. We shall keep on studying zero-density estimates for $L(s, \pi)$ in this strip and improve previous results when $\sigma$ near $\frac{3}{4}$ and 1, respectively.
Theorem 2. Let $L(s, \pi)$ be an automorphic $L$-function satisfying (3). Then for $\frac{3}{4} \leq \sigma \leq \frac{10}{13}$ and $2\sigma(2\alpha - 16\theta + 3\alpha^2 - 6\alpha^2) \geq 4\alpha \theta - 20\theta + 3\alpha^2 l - 12\alpha^2$ we have

$$N_{\pi}(\sigma, T, T^\alpha) \ll T^{\frac{3\alpha^2(1-\epsilon)}{2(1-\sigma)}} + \epsilon,$$

and for $\frac{10}{13} < \sigma < 1$ and $2\sigma(6\theta - 22\theta + 9\alpha^2 - 18\alpha^2) \geq 12\theta - 20\theta + 9\alpha^2 - 36\alpha^2$, we have

$$N_{\pi}(\sigma, T, T^\alpha) \ll T^{\frac{9\alpha^2(1-\epsilon)}{2(1-\sigma)}} + \epsilon.$$

Theorem 3. Let $L(s, \pi)$ be an automorphic $L$-function satisfying (3). Then for $\frac{3}{4} \leq \sigma \leq \frac{10}{13}$ and $2\sigma(\alpha \theta - 4\theta + 2\alpha^2 l - 2\alpha^2) \geq 2\alpha \theta - 5\theta + 2\alpha^2 l - 4\alpha^2$, we have

$$N_{\pi}(\sigma, T, T^\alpha) \ll T^{\frac{2\alpha^2(1-\epsilon)}{2(1-\sigma)}} + \epsilon,$$

and for $\frac{10}{13} < \sigma < 1$ and $\sigma(6\alpha - 12\alpha^2 - 11\theta + 12\alpha^2) \geq 6\alpha - 12\alpha^2 - 5\theta + 6\alpha^2$, we have

$$N_{\pi}(\sigma, T, T^\alpha) \ll T^{\frac{12\alpha^2(1-\epsilon)}{2(1-\sigma)}} + \epsilon.$$

Notation 1. Throughout this paper, the letter $\epsilon$ represents a sufficiently small positive number, whose value may change from statement to statement. Constants, both explicit and implicit, in Vinogradov symbols $\ll$ may depend on $\epsilon$ and $\pi$.

2. Some Lemmas

Lemma 1. For $L(s, \pi)$ we have

$$N_{\pi}(0, T, T + 1) = \frac{m}{2\pi} \log T(1 + o(1)).$$

Proof. We can get this lemma by a standard winding number argument on (3) as in Davenport [26] and Rudnick and Sarnak [15].

Lemma 2 ([27], Lemma 1.7). Let $\xi, \varphi_1, \ldots, \varphi_R$ be arbitrary vectors in an inner-product vector space over $\mathbb{C}$. If $a = \{a_n\}_{n=1}^\infty$ and $b = \{b_n\}_{n=1}^\infty$ are two vectors of $\mathbb{C}$, then the inner-product of $a$ and $b$ is defined as

$$(a, b) = \sum_{n=1}^\infty a_n b_n.$$

Then we have the inequality

$$\sum_{r \leq R} |(\xi, \varphi_r)| \leq \|\xi\| \left( \sum_{r, s \leq R} |(\varphi_r, \varphi_s)| \right)^{\frac{1}{2}},$$

where $\|a\|^2 = (a, a)$.

Lemma 3 ([23], (4.60)). Suppose that $Y, h > 0$ and $k \geq 1$ is an integer, we have

$$\sum_{n=1}^\infty e^{-(\xi + y)h} d_k(n) n^{-s} = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} e^s \Gamma(1 + \frac{\nu}{h}) w^{-1} dw.$$

Lemma 4. For a fixed $\theta$ satisfying $\frac{1}{2} < \theta < 1$, define

$$S(\theta) = \sum_{r, s \leq R} |\xi(\theta + it_r - it_s + iv')|^2,$$
where \( T \leq t_r \leq T + T^a \) (1 \( \leq r \leq R \)) and \( v' \) is defined by

\[
|\zeta(\theta + it_r - it_s + iv')| = \max_{-\log^2 T \leq v \leq \log^2 T} |\zeta(\theta + it_r - it_s + iv)|.
\]

Suppose that \(|t_r - t_s| \geq 3 \log^4 T\) for \( r \neq s \leq R\), then we have

\[
S\left(\frac{3}{4}\right) \ll T^\varepsilon\left(R^2 + R^{\frac{11}{8}}T^\frac{4}{3}\right).
\]

**Proof.** We can get this lemma by following the argument of ([23], Lemma 11.6). The main difference is that the interval of \( t_r \) now is \( T \leq t_r \leq T + \varepsilon T^a \).

3. **Proof of Theorem 1**

We first consider the number of zeros \( \rho = \beta + i\gamma \) of \( L(s, \pi) \) in the rectangle

\[
\sigma < \beta < 1, \quad T \leq \gamma \leq T + T^a,
\]

where \( \sigma \geq \frac{1}{2}, T \geq 3, \) and \( 0 < \alpha \leq 1 \). We define \( \mu_\pi(n) \) by

\[
\frac{1}{L(s, \pi)} = \sum_{n=1}^{\infty} \frac{\mu_\pi(n)}{n^s},
\]

for \( \Re s > 1 \). By the Euler product of \( L(s, \pi) \) in (1), we have

\[
\frac{1}{L(s, \pi)} = \prod_{p<\infty} \prod_{j=1}^{m} (1 - a_\pi(p, j) p^{-s}).
\]

Consequently \( \mu_\pi(n) \) is a multiplicative function and

\[
\mu_\pi(1) = 1,
\]

\[
\mu_\pi(p^k) = (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq m} \prod_{v=1}^{k} a_\pi(p, j_v)
\]

for \( k = 1, \ldots, m \) and \( \mu_\pi(p^k) = 0 \) for \( k > m \). Then from \( L(s, \pi) \cdot \frac{1}{L(s, \pi)} = 1 \), we get

\[
\sum_{d|n} \mu_\pi(d) A_\pi\left(\frac{n}{d}\right) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}
\]

Assume further that \( s = \sigma + it, T \leq t \leq T + T^a, 1 \ll X \ll Y \ll T^A \) for some \( A > 0 \), and \( X = X(T) \) and \( Y = Y(T) \) are two parameters to be decided later. Let

\[
M_X(s, \pi) = \sum_{n \leq X} \frac{\mu_\pi(n)}{n^s}.
\]

Then we have

\[
L(s, \pi) M_X(s, \pi) = \sum_{n=1}^{\infty} \frac{b_\pi(n)}{n^s},
\]

for \( \Re s > 1 \), where

\[
b_\pi(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } 2 \leq n \leq X, \\ \sum_{d|n, d \leq X} \mu_\pi(d) A_\pi\left(\frac{n}{d}\right), & \text{if } n > X. \end{cases}
\]
In terms of (7), for \( \sigma > 1 \) and \( \Re s > 0 \), we obtain
\[
\frac{1}{2\pi i} \int_{\sigma} L(s + \omega, \pi) M_X(s + \omega, \pi) \Gamma(\omega) \gamma^s d\omega = \frac{1}{2\pi i} \int_{\sigma} \sum_{n \geq X} \frac{b_\pi(n)}{n^s} \Gamma(\omega) \gamma^s d\omega.
\]

From the inverse Mellin transform of \( \Gamma(\omega) \), the right-hand side of the above formula can be written as
\[
\frac{1}{2\pi i} \int_{\sigma} \Gamma(\omega) \gamma^s d\omega + \sum_{n \geq X} \frac{b_\pi(n)}{n^s} \frac{1}{2\pi i} \int_{\sigma} \Gamma(\omega) \left( \frac{Y}{n} \right)^\omega d\omega = e^{-\frac{\gamma}{\beta}} + \sum_{n \geq X} \frac{b_\pi(n)}{n^s} e^{-\frac{\gamma}{\beta}}.
\]

Hence we have, for \( \sigma > 1 \) and \( \Re s > 0 \),
\[
e^{-\frac{\gamma}{\beta}} + \sum_{n \geq X} \frac{b_\pi(n)}{n^s} e^{-\frac{\gamma}{\beta}} = \frac{1}{2\pi i} \int_{\sigma} L(s + \omega, \pi) M_X(s + \omega, \pi) \Gamma(\omega) \gamma^s d\omega.
\]

Now we move the line of integration of (8) to \( \Re \omega = u - \beta < 0 \) for some suitable \( \frac{1}{2} \leq u < 1 \). Then integral on the right hand of (8) becomes
\[
L(s, \pi) M_X(s, \pi) + \frac{1}{2\pi i} \int_{(u-\beta)} L(s + \omega, \pi) M_X(s + \omega, \pi) \Gamma(\omega) \gamma^s d\omega,
\]
where \( L(s, \pi) M_X(s, \pi) \) is the residue of the integrand at \( \omega = 0 \). Setting \( \omega = u - \beta + iv \) and taking \( s \) equal to a non-trivial zero \( \rho = \beta + iv \), we have
\[
e^{-\frac{\gamma}{\beta}} + \sum_{n \geq X} \frac{b_\pi(n)}{n^\rho} e^{-\frac{\gamma}{\beta}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} L(u + iv, \pi) M_X(u + iv, \pi) \Gamma(u - \beta + iv) Y^{u-\beta+iv} dv.
\]

Since \( e^{-\frac{\gamma}{\beta}} \to 1 \) and \( \sum \frac{b_\pi(n)}{n^\rho} e^{-\frac{\gamma}{\beta}} \) contribute \( o(1) \) as \( Y \to \infty \), a non-trivial zero \( \rho = \beta + iv \) counted in \( N_\pi(\sigma, T, T^*) \) satisfies either
\[
1 \ll \left| \sum_{X < n \leq Y \log^2 Y} \frac{b_\pi(n)}{n^\rho} e^{-\frac{\gamma}{\beta}} \right| (10)
\]
or
\[
1 \ll \int_{-\infty}^{+\infty} \left| L(u + iv, \pi) M_X(u + iv, \pi) \Gamma(u - \beta + iv) Y^{u-\beta+iv} \right| dv. \tag{11}
\]

For the integral in (11) we have
\[
\int_{-\infty}^{-\log^2 T} + \int_{\log^2 T}^{+\infty} = o(1)
\]
due to the fact that they are absolutely convergent. By Stirling’s formula again, we can further remove the \( \Gamma \) function in (11) and get that a non-trivial zero \( \rho = \beta + iv \) counted in \( N_\pi(\sigma, T, T^*) \) satisfies either (10) or
\[
1 \ll Y^{u-\beta} \int_{-\log^2 T}^{\log^2 T} \left| L(u + iv, \pi) M_X(u + iv, \pi) \right| dv. \tag{12}
\]

We divide the elongated rectangle (6) into successive rectangles of length \( 3 \log^4 T \) noting Lemma 1. These rectangles start at
\[ \sigma < \beta < 1, \ T \leq \gamma \leq T + 3 \log^4 T \] (13)

and the last one may have a shorter length. Call these smaller rectangles as \( I_1, I_2, \ldots \). The number of zeros are denoted by \( A_{ij} \) in all odd-numbered rectangles if \( j = 1 \) or in all even-numbered rectangles if \( j = 2 \), which satisfy (10) if \( i = 1 \), or (12) if \( i = 2 \). Then, we have

\[ N_\pi(\sigma, T, T + T^a) \leq A_{11} + A_{12} + A_{21} + A_{22}. \]

We can get a sequence of zeros \( \rho_r^{(ij)} = \beta_r^{(ij)} + \gamma_r^{(ij)} \) counted in \( A_{ij} \) for \( r = 1, \ldots, R_{ij} \), if we choose a zero from each rectangle which contains at least one zero. Here, \( R_{ij} \) is the number of rectangles that contains at least one zero counted in \( A_{ij} \). By Lemma 1, we note that each rectangle contains at most \( \frac{3\pi}{2^r} \log^5 T (1 + o(1)) \) zeros. Consequently, we have

\[ N_\pi(\sigma, T, T + T^a) \leq (R_{11} + R_{12} + R_{21} + R_{22}) \log^5 T. \] (14)

Now we begin to estimate \( R_{1j} \). By a dyadic subdivision on the sum in (10), we know that each \( \rho_r^{(ij)} = \beta_r^{(ij)} + \gamma_r^{(ij)} \) counted in \( R_{1j} \) satisfies

\[ 1 \ll \cdots + \left| \sum_{m_{ij} < k \leq 2m_{ij}} \frac{b_\pi(n)}{n^{\rho^{(ij)}}} e^{-\frac{n}{Y}} \right| + \sum_{\sqrt{Y} < n < 2\sqrt{Y}} \frac{b_\pi(n)}{n^{\rho^{(ij)}}} e^{-\frac{n}{Y}} + \cdots, \] (15)

where there are \( O(\log(Y \log^2 Y)) \) terms. Then there exists \( M_r = 2^{\sigma'} \sqrt{Y} \) for some \( \sigma' \in \mathbb{Z} \) and \( X \leq M_r \leq Y \log^2 Y \) such that

\[ \sum_{M_r < n < 2M_r} \frac{b_\pi(n)}{n^{\rho^{(ij)}}} e^{-\frac{n}{Y}} \gg \frac{1}{\log Y}. \] (16)

Raising (16) to \( k \)th power we get

\[ \sum_{M_r^k < n \leq (2M_r)^k} \frac{c_\pi(n)}{n^{\rho^{(ij)}}} \gg \frac{1}{\log^k Y}, \] (17)

with

\[ c_\pi(n) = \sum_{n = n_1^{(ij)} \cdots n_k^{(ij)}} \prod_{M_r < n_j \leq 2M_r} b_\pi(n_j) e^{-\frac{n_1 + n_2 + \cdots + n_k}{Y}} \]

and \( k \) is a natural number depending on \( M_r \) such that \( M_r^k = N, \ (2M_r)^k = P \leq T^{\sigma'} \) from where \( k \ll 1 \) and \( P \ll N \). We split the sum in (17) into subsums of length \( N \) and choose \( k \) so that \( M_r^k \leq Y^r \log^{2r} Y \leq M_r^{k+1} \), where \( r \) is a fixed integer and \( k \geq r \geq 1 \) is satisfied. Then (17) can be written as

\[ \sum_{N < n \leq 2N} \frac{c_\pi(n)}{n^{\rho^{(ij)}}} \gg \frac{1}{\log^D Y} \] (18)

for some \( D \approx 1 \) and

\[ Y^{\frac{1}{\sigma'} \log \log^2 Y} \ll N \ll Y^r \log^{2r} Y. \] (19)

By partial summation and (18), we have

\[ R_{1j} \ll \log^D T \left\{ \sum_{\rho^{(ij)}} \left( \sum_{N < n \leq 2N} c_\pi(n)n^{-\sigma - i\gamma^{(ij)}} \right) N^{\sigma - \beta^{(ij)}} \right\} \]

\[ + \int_N^{2N} \left( \sum_{N < n \leq u} c_\pi(n)n^{-\sigma - i\gamma^{(ij)}} \right) N^{\sigma - \beta^{(ij)} - 1} du \right\}. \] (20)
We relabel \( c_\pi(n) \) to satisfy \( c_\pi(n) = 0 \) for \( n > u \). Then simplifying (20), we can get

\[
R_{ij} \ll \log^D T \left| \sum_{r=1}^{R_{ij}} \sum_{R_{ij} < n < 2N} c_\pi(n)e^{-i\eta_1(n)} \right|.
\]  

(21)

From now on, we let \( a = 1 \) in (6). Now we apply Lemma 2 to (21) and take \( \zeta = \{ \zeta_n \}_{n=1}^\infty \) with \( c_\pi(n) \ll n^s \), where

\[
\zeta_n = \begin{cases} 
    c_\pi(n)(e^{-\frac{n}{\pi}} - e^{-\pi n})^{-\frac{1}{2}} n^{-s}, & N < n \leq 2N, \\
    0, & \text{otherwise},
\end{cases}
\]

and \( \phi_r = \{ \phi_{r,n} \}_{n=1}^\infty \) with \( \phi_{r,n} = (e^{-\frac{n}{\pi}} - e^{-\pi n})^{\frac{1}{2}} n^{-i\gamma_r(n)} \) \( (n = 1, 2, 3 \ldots) \). Denoting the imaginary part of representative zeros of \( R_{ij} \) by \( \gamma_{r_1}^{(j)}, \ldots, \gamma_{r_k}^{(j)} \), we then have

\[
R_{ij}^2 \ll \log^{2D} T \left| \sum_{r \leq R_{ij}} \sum_{N < n \leq 2N} \zeta_n \phi_{r,n} \right|^2 \ll \log^{2D} T \| \zeta \|^2 \sum_{r \leq R_{ij}} \| (\phi_r, \phi_s) \|
\]

\[
\ll \log^{2D} T \left( \sum_{N < n \leq 2N} c_\pi^2(n)e^{-\frac{2\pi n}{\pi}} n^{-2s} \right) \left( \sum_{r \leq s} \| (\phi_r, \phi_s) \| + \sum_{r \neq s} \| (\phi_r, \phi_s) \| \right)
\]

\[
\ll \log^{2D} T \left( \sum_{N < n \leq 2N} c_\pi^2(n)n^{-2s} e^{-\frac{2\pi n}{\pi}} \right) \left( R_{ij}N + \sum_{r \neq s \leq R_{ij}} (H(i\gamma_r^{(j)} - i\gamma_s^{(j)})) \right)
\]

\[
\ll T^s \left( N^{2-2s} R_{ij} + \sum_{r \neq s \leq R_{ij}} (H(i\gamma_r^{(j)} - i\gamma_s^{(j)})) \right),
\]

where

\[
H(it) = \sum_{n=1}^\infty \left( e^{-\frac{n}{\pi}} - e^{-\pi n} \right) n^{-it}
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^{2+\infty} \zeta(w + it)((2N)^w - 1) \Gamma(w)dw,
\]

since \( 1 \ll e^{-\frac{n}{\pi}} - e^{-\pi n} \ll 1 \) for \( N < n \leq 2N \). Thus, we have

\[
H(i\gamma_r^{(j)} - i\gamma_s^{(j)}) = \frac{1}{2\pi i} \int_{-\infty}^{2+\infty} \zeta(w + it)(1 - (2N)^w) \Gamma(w)dw.
\]

(24)

Moving the line of integration in (24) to \( \Re w = u < 1 \), we encounter a simple pole at \( w = 1 - i\gamma_r^{(j)} + i\gamma_s^{(j)} \) with residue \( \ll N \epsilon^{-\gamma_r^{(j)} - \gamma_s^{(j)}} \), so that

\[
H(i\gamma_r^{(j)} - i\gamma_s^{(j)})
\]

\[
= \frac{1}{2\pi i} \int_{u-\infty}^{u+\infty} \zeta(w + it)(1 - (2N)^w - N^w) \Gamma(w)dw + O(N \epsilon^{-\gamma_r^{(j)} - \gamma_s^{(j)}}).
\]

(25)

In view of \( |\Gamma(s)| = (2\pi)^{\frac{1}{2}|s|} |\epsilon^{\frac{1}{2}} |^{|s|} (1 + O(|\epsilon|^{-1})) \), the integral in (25) is \( o(1) \) for \( N \ll T^s \) if \( |\Im w| \geq \log^2 T \), which gives

\[
\sum_{r \neq s \leq R_{ij}} |H(i\gamma_r^{(j)} - i\gamma_s^{(j)})| \ll N \sum_{r \neq s \leq R_{ij}} e^{-|\gamma_r^{(j)} - \gamma_s^{(j)}|} + o(R_{ij}^2)
\]

\[
+ Nu \int_{-\log^2 T}^{log^2 T} \sum_{r \neq s \leq R_{ij}} \left| \zeta(u + i\gamma_r^{(j)} - i\gamma_s^{(j)} + iv) \right| dv.
\]

(26)
Since the \( \gamma_r^{(1)} \)'s are at least \( 3 \log^4 T \) apart, the first sum on the right side of (26) is \( O(R_{ij}) \).

For the second sum on the right side of (26), we fix each \( s \) and let \( \tau_r = \gamma_r^{(1)} - \gamma_s^{(1)} + v \).

Then we have \( |\tau_r| \leq 3T \) for \( r = 1, 2, \ldots, R_{ij} \) and \( |\tau_r - \tau_{r'}| \geq 3 \log^4 T \) for \( r_1 \neq r_2 \). Hence we get

\[
\sum_{r \neq s \leq R_{ij}} \left( H(i\gamma_r^{(1)} - i\gamma_s^{(1)}) \right) \ll R_{ij}^2 N^u M_1(u, 3T) \log^2 T. \tag{27}
\]

Inserting (27) into (22) we obtain

\[
R_{ij}^2 \ll T^2 \left( R_{ij} N^{2-2\sigma} + R_{ij} N^{1+\sigma-2\sigma} M_1(u, 3T) \right).
\]

Then we have for \( \sigma \geq \frac{u+1}{2} \),

\[
R_{ij} \ll \max_{X \leq N \leq Y \log^2 Y} N^{2-2\sigma} T^2 \ll \gamma^{2-2\sigma} T^2, \tag{28}
\]

if \( X^{2\sigma - 1 - u} \gg M_1(u, 3T) T^2 \).

Now we turn to estimate \( R_{2j} \). We may suppose first that \( \sigma < 1 - \frac{C}{\log T} \) in view of the zero-free region. Note that in this case the zeros \( \rho_{r}^{(2j)} = \beta_{r}^{(2j)} + i \gamma_r^{(2j)}, j = 1, 2 \) satisfy

\[
1 \ll Y^{u-\beta} \int_{\log^2 T} \left| L(u + i\gamma_r^{(2j)} + iv, \pi) M_X(u + i\gamma_r^{(2j)} + iv, \pi) \right| dv. \tag{29}
\]

From the mean value theorem for integration, we can see that there exists \( l_r^{(2j)} = \gamma_r^{(2j)} + v \), which makes (29) become

\[
1 \ll Y^{u-\beta} \left| L(u + il_r^{(2j)}, \pi) M_X(u + il_r^{(2j)}, \pi) \right| \log^2 T. \tag{30}
\]

Thus, from (30) we get

\[
M_X(u + il_r^{(2j)}, \pi) \gg Y^{u-\sigma} (M_2(u, 4T) \log^2 T)^{-1},
\]

for \( R_{2j} \) points \( l_r^{(2j)} \) such that \( |l_r^{(2j)}| \leq 4T \). Then there is a number \( N (1 \ll N \leq X) \) such that

\[
\sum_{N < n \leq 2N} \frac{\mu_n(n)}{n^{u + il_r^{(2j)}}} \gg Y^{u-\sigma} (M_2(u, 4T) \log^2 T)^{-1},
\]

hence it is easy to get

\[
1 \ll Y^{u-\sigma} M_2(u, 4T) \log^2 T \sum_{N < n \leq 2N} \frac{\mu_n(n)}{n^{u + il_r^{(2j)}}}.
\]

We can apply Lemma 2 as in the previous case, but now the choice of \( \xi \) and \( \varphi_r \) are different. We shall take \( \xi = \{ \xi_n \}_{n=1}^{\infty} \) with \( \xi_n = \mu_n(n) (e^{-\frac{\pi}{n}} - e^{-\frac{\pi}{n+1}}) - \frac{1}{2} n^{-u} \) for \( N < n \leq 2N \) and zero otherwise, \( \varphi_r = \{ \varphi_{r,n} \}_{n=1}^{\infty} \) with \( \varphi_{r,n} = (e^{-\frac{\pi}{n}} - e^{-\frac{\pi}{n+1}}) \frac{1}{2} n^{-u} \) for \( n = 1, 2, 3, \ldots \). Then
We get estimates of $R_4$. Proof of Theorem 2 follows from (31) with $u = 5\sigma - 4$. 

Recall the functional equation $X = C_1 M_1(u, 3T) \frac{1}{\sigma + 1 - \frac{1}{2}} T^\epsilon$, 

$Y = C_2 M_2(u, 4T) \frac{1}{\sigma + 1 - \frac{1}{2}} T^\epsilon$.

We see that the bound for $R_{2j}$ is smaller than the one for $R_{1j}$. Recalling (14) we have

$$N_{ij}(\sigma, T, 2T) \ll \sigma^{2 - 2\epsilon} T^\epsilon \tag{31}$$

where $\frac{1}{2} \leq u \leq 1, \frac{u+1}{2} \leq \sigma < 1 - \frac{c}{\log T}$. Let $u = k\sigma - (k - 1) (k > 2)$. Then in view of (31), we see that it is appropriate to take $k = 5$. Hence $u = 5\sigma - 4$, $\sigma \geq \frac{u+1}{2}$ is satisfied and $u \geq \frac{1}{2}$ for $\sigma \geq \frac{9}{10}$. Thus, Theorem 1 follows from (31) with $u = 5\sigma - 4$.

4. Proof of Theorem 2

Throughout the proof of Theorem 2, we restrict the range of zeros to (6). From the estimates of $R_{1j}^2$ in (22), we obtain now

$$R_{1j}^2 \ll T^4 \left( N^{2 - 2\epsilon} R_{1j} + N^{1 - 2\epsilon} \sum_{r \neq s \leq R_{1j}} |H(i\gamma_r^{(1j)} - i\gamma_s^{(1j)})| \right). \tag{32}$$

Recall the functional equation $\zeta(s) = \chi(s)\zeta(1 - s)$ with

$$\chi(s) = \left( \frac{2\pi}{T} \right)^{\sigma + \frac{u-1}{2}} e^{\left((\sigma + \frac{u}{2})T\right)} (1 + O|t|^{-1}).$$

Moving the line of integration in $H(it)$ to $\Re w = \frac{1}{2}$ and applying the Cauchy–Schwarz inequality, we get
\[
\sum_{r \neq s \leq R_{ij}} |H(i_{r}^{(1)} - i_{s}^{(1)})| \ll N \sum_{r \neq s \leq R_{ij}} e^{-|i_{r}^{(1)} - i_{s}^{(1)}|} + R_{ij}^{2} + N^{\frac{1}{4}} T^\frac{\varepsilon}{2}
\times \int_{-\log^{2}T}^{\log^{2}T} \sum_{r \neq s \leq R_{ij}} |\xi(\frac{3}{4} + i_{r}^{(1)} - i_{s}^{(1)} + iv)|dv
\]

(33)

where the \( S(\frac{3}{4}) \) is from Lemma 4.

Substituting (33) into (32), we can get

\[ R_{ij} \ll T^{\varepsilon} N^{2-2\sigma} + R_{ij}^{2} T^{\varepsilon+\varepsilon} N^{\frac{5\alpha - 5}{2}} + T^{2\varepsilon + \varepsilon} N^{\frac{5\alpha - 5}{2}}, \]  

(34)

where we used Lemma 4 to \( S(\frac{3}{4}) \). For \( R_{0} \) points lying in an interval of length \( T = T_{0} = N^{\frac{5\alpha - 5}{2}} - \varepsilon \) we have

\[ R_{0} \ll T^{\varepsilon} (N^{2-2\sigma} + T_{0}^{6 \varepsilon} N^{\frac{20 - 32\sigma}{3}}) \ll T^{\varepsilon} N^{2-2\sigma} \text{ for } \frac{3}{4} \leq \sigma \leq \frac{10}{13}. \]

In (19), we take \( r = 2 \) to get \( Y^{\frac{5}{2}} \log^{\frac{5}{2}} Y \ll N \ll Y^{2} \log^{4} Y \) and then

\[ R_{ij} \ll R_{0} \left(1 + \frac{T^{\alpha}}{T_{0}}\right) \ll T^{\varepsilon} N^{2-2\sigma} \left(1 + \frac{T^{\alpha}}{T_{0}}\right) \ll T^{\varepsilon} \left(Y^{4-4\sigma} + T^{\alpha} Y^{\frac{5\alpha(1-\sigma) + 20 - 32\sigma}{3\alpha}}\right). \]

It follows from ([24], (5.21)) that

\[ R_{2j} \ll T^{\theta+\varepsilon} Y^{l(1-2\sigma)}. \]

Consequently, we have

\[ N_{\pi}(\sigma, T, T + T^{\alpha}) \ll T^{\varepsilon} \sum_{j=1}^{2} (R_{1j} + R_{2j}) \ll T^{\varepsilon} \left(T^{\theta} Y^{l(1-2\sigma)} + Y^{4-4\sigma} + T^{\alpha} Y^{\frac{5\alpha(1-\sigma) + 20 - 32\sigma}{3\alpha}}\right). \]

(36)

We set \( Y^{4-4\sigma} = T^{\alpha} Y^{\frac{5\alpha(1-\sigma) + 20 - 32\sigma}{3\alpha}} \), which is equivalent to \( Y = T^{\frac{3\alpha^{2}(1-\sigma)}{4(1+\theta)+3\alpha^{2}}} \). Thus, from (36) we get

\[ N_{\pi}(\sigma, T, T + T^{\alpha}) \ll T^{\varepsilon} (Y^{4-4\sigma} + T^{\theta} Y^{l(1-2\sigma)}) \]

\[ = T^{\varepsilon} \left(T^{\frac{3\alpha^{2}(1-\sigma)}{4(1+\theta)+3\alpha^{2}}} + T^{\theta + \frac{4\alpha^{2}(1-\sigma)}{4(1+\theta)+3\alpha^{2}}}, \right) \]

(37)

Comparing the two terms in (37), we can get

\[ N_{\pi}(\sigma, T, T + T^{\alpha}) \ll T^{\frac{3\alpha^{2}(1-\sigma)}{4(1+\theta)+3\alpha^{2}} + \varepsilon} \]

for \( 2\sigma(2\alpha \theta - 16\theta + 3\alpha^{2}l - 6\alpha^{2}) \geq 4\alpha \theta - 20\theta + 3\alpha^{2}l - 12\alpha^{2} \), from which we complete the proof of the first result of Theorem 2.

For \( R_{0} \) points lying in an interval of length \( T = T_{0} = N^{\frac{10\pi - 5}{3\alpha}} \), we have

\[ R_{0} \ll T^{\varepsilon} (N^{2-2\sigma} + R_{0} T_{0}^{\frac{5}{2}} N^{\frac{5-8\sigma}{3\alpha}}) = T^{\varepsilon} (N^{2-2\sigma} + R_{0} N^{\frac{10 - 13\pi}{2\alpha}}), \]
which implies

$$R_0 \ll T^\varepsilon \frac{N^{2-2\sigma}}{1 - N^{\frac{5\alpha - 8\alpha}{4}}} \ll T^\varepsilon N^{2-2\sigma} \text{ for } \frac{10}{13} < \sigma < 1.$$  

Then we have

$$R_{1j} \ll R_0 \left(1 + \frac{T^\alpha}{T_0^\alpha} \right) \ll T^\varepsilon \left( N^{2-2\sigma} + T^\alpha N^{6\alpha(1-\sigma) - 11\sigma} + 5 \right).$$

(38)

Consequently, recalling (35) we have

$$N_\pi(\sigma, T + T^\alpha) \ll \sum_{j=1}^2 \left( R_{1j} + R_{2j} \right) \log^5 T$$

$$\ll T^\varepsilon \left( Y^{4-4\sigma} + T^\alpha Y^{24\alpha(1-\sigma) - 44\sigma + 20} \right).$$

(39)

We choose $Y^{4-4\sigma} = T^\alpha Y^{24\alpha(1-\sigma) - 44\sigma + 20}$, which is equivalent to $Y = T^{12\alpha(1-\sigma) + 4(1\sigma - 5)}$. Thus, from (39) we have

$$N_\pi(\sigma, T + T^\alpha) \ll T^\varepsilon \left( Y^{4-4\sigma} + T^\alpha Y^{24\alpha(1-\sigma) - 44\sigma + 20} \right)$$

$$= T^\varepsilon \left( T^{5\alpha(1-\sigma) + 11\sigma - 5} + T^\alpha + T^{12\alpha(1-\sigma) + 4(1\sigma - 5)} \right).$$

(40)

Comparing the two terms in (40), we can get

$$N_\pi(\sigma, T + T^\alpha) \ll T^{\frac{5\alpha(1-\sigma)}{5\alpha(1-\sigma) + 11\sigma - 5} + \varepsilon}$$

for $2\sigma(18\alpha^2 - 6\alpha + 22\theta - 9\alpha^2) \leq 36\alpha^2 - 12\theta + 20\theta - 9\alpha^2$, from which we complete the proof of the second result of Theorem 2.

5. Proof of Theorem 3

The proofs of Theorems 2 and 3 are similar, and the main difference is the range of $N$. Here for completeness, we state some critical details. We let $r = 1$ in (19) and get $Y^{\frac{1}{2}} \log Y \ll N \ll Y \log^2 Y$.

Unlike the previous (34), we now have

$$R_{1j} \ll T^\varepsilon N^{2-2\sigma} + R_{1j} T^\frac{\varepsilon + \varepsilon}{\varepsilon} N^{\frac{3\varepsilon}{4}} + T^{\frac{\varepsilon}{4} + \varepsilon} N^{\frac{5\alpha - 3\varepsilon}{4}}.$$  

For $R_0$ points lying in an interval of length $T = T_0 = N^{\frac{5\alpha - 3\varepsilon}{4} - \varepsilon}$, we have

$$R_0 \ll T^\varepsilon \left( N^{2-2\sigma} + T_0^{\frac{6\varepsilon}{4}} N^{\frac{3\varepsilon}{4}} \right) \ll T^\varepsilon N^{2-2\sigma} \text{ for } \frac{3}{4} \leq \sigma \leq \frac{10}{13}.$$  

Then

$$R_{1j} \ll R_0 \left(1 + \frac{T^\alpha}{T_0^\alpha} \right) \ll T^\varepsilon N^{2-2\sigma} \left(1 + \frac{T^\alpha}{T_0^\alpha} \right) \ll T^\varepsilon \left( Y^{2-2\sigma} + T^\alpha Y^{24\alpha(1-\sigma) - 5 - 8\varepsilon} \right).$$

Consequently, recalling (35), we have

$$N_\pi(\sigma, T + T^\alpha) \ll \sum_{j=1}^2 \left( R_{1j} + R_{2j} \right)$$

$$\ll T^\varepsilon \left( T^\theta Y^{1-2\sigma} + Y^{2-2\sigma} + T^\alpha Y^{24\alpha(1-\sigma) - 5 - 8\varepsilon} \right).$$

(41)
We set $Y^{2-2\sigma} = T^a Y^{\frac{2a(1-\sigma) + 5 - 8\sigma}{6a}}$ which is equivalent to $Y = T^{\frac{2a^2}{6a(1-\sigma) + 11\sigma - 5}}$. Thus, (41) becomes

$$N_\Pi(\sigma, T, T + T^a) \ll T^e (Y^{2-2\sigma} + \sigma Y^{(1-2\sigma)})$$

Comparing the two terms in (42), we can get

$$N_\Pi(\sigma, T, T + T^a) \ll T^e (T^{\frac{2a^2}{6a(1-\sigma) + 11\sigma - 5}} + T^{\theta Y^{(1-2\sigma)}}).$$

For $2\sigma(\theta - 4\theta + 2\alpha^2 - 2\alpha^2) \geq 2\alpha\theta - 5\theta + 2\alpha^2 - 4\alpha^2$, from which we complete the proof of the first result of Theorem 3.

Comparing the two terms in (42), we can get

$$N_\Pi(\sigma, T, T + T^a) \ll T^e (T^{\frac{2a^2}{6a(1-\sigma) + 11\sigma - 5}} + T^{\theta Y^{(1-2\sigma)}}).$$

which implies

$$R_0 \ll T^e (N^{2-2\sigma} + R_0 T^a) = T^e (N^{2-2\sigma} + R_0 N^{\frac{10-11\alpha}{2}}),$$

Then we have

$$R_{1j} \ll R_0 (1 + \frac{T^a}{T_0}) \ll T^e (N^{2-2\sigma} + T^a \frac{N^{6a(1-\sigma) - 11\sigma + 5}}{6a}).$$

Consequently, recalling (35) we have

$$N_\Pi(\sigma, T, T + T^a) \ll \sum_{j=1}^{2} (R_{1j} + R_{2j})$$

$$\ll T^e (Y^{2-2\sigma} + T^a Y^{\frac{6a(1-\sigma) - 11\sigma + 5}{6a}} + \sigma Y^{(1-2\sigma)}).$$

We choose $Y^{2-2\sigma} = T^a Y^{\frac{6a(1-\sigma) - 11\sigma + 5}{6a}}$ which is equivalent to $Y = T^{\frac{6a^2}{6a(1-\sigma) + 11\sigma - 5}}$. Thus, (44) becomes

$$N_\Pi(\sigma, T, T + T^a) \ll T^e (Y^{2-2\sigma} + \theta Y^{(1-2\sigma)})$$

Comparing the two terms in (45), we can get

$$N_\Pi(\sigma, T, T + T^a) \ll T^e (T^{\frac{12a^2}{6a(1-\sigma) + 11\sigma - 5}} + T^{\theta Y^{(1-2\sigma)}}).$$

for $\sigma(6\theta\alpha - 12\alpha^2 - 11\theta + 12\alpha^2) \geq 6\theta\alpha - 12\alpha^2 - 5\theta + 6\alpha^2 l$, from which we complete the proof of the second result of Theorem 3.

Author Contributions: Conceptualization, W.D. and H.L.; Methodology, D.Z.; Software, W.D.; Validation, W.D., H.L., and D.Z.; Formal Analysis, W.D.; Investigation, W.D.; Resources, H.L.; Data Curation, D.Z.; Writing—Original Draft Preparation, W.D.; Writing—Review and Editing, H.L.; Visualization, D.Z.; Supervision, D.Z.; Project Administration, D.Z.; Funding Acquisition, H.L. and D.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China (Grant Nos. 11771256 and 11801327).
Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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