MESOSCOPIC FLUCTUATIONS IN MODELS OF CLASSICAL
AND QUANTUM DIFFUSION

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Abstract

It is shown that the characteristics of the mesoscopic fluctuations in the conventional quantum-diffusion model and the model of the non-coherent (‘classical’) diffusion in media with long-range correlated disorder are quite similar in the weak-disorder limit. The relative values of the variance and of the high-order moments of the fluctuations in one model are obtained from those in another one by substituting a proper weak-disorder parameter. As behaviour of the ensemble-averaged diffusion coefficient is quite different in these models, it suggests that the mesoscopic properties, including a possible universality in the distribution functions, could be independent of the behaviour of the averaged quantities.

I. INTRODUCTION

The absence of self-averaging in mesoscopic conductors (see recent collections of review papers [1,2]) is known to result from the interference effects in quantum disordered systems. The universal conductance fluctuations (UCF) in such systems are the most famous example of phenomena caused by the absence of the self-averaging. Conductance (or resistance) fluctuations are also characteristic of mesoscopic semiconductors with hopping conductivity. In this case, the absence of self-averaging has nothing to do with the quantum coherence, and is due to the fact that conductance is determined by the most favourable channel which should be totally different for different realizations of disorder, and could be easily
changed by applying external fields. These fluctuations are by no means universal, and their amplitude could be much larger than an appropriate average value, in contrast to the UCF in the mesoscopic conductors whose amplitude is much smaller than the average conductance. In the absence of the quantum interference, transport properties are governed by the probabilities rather than by amplitudes of wave functions, so that the mesoscopic effects may be described in the framework of classical models. In general, mesoscopics of systems with and without quantum coherence are totally different.

In this paper, however, we will consider some models of classical transport in disordered media [3]–[8] whose “mesoscopic” properties show remarkable similarity [6] with those of the conductance fluctuations in metals [7,9]. Firstly, the fractional value of the variance of the fluctuations has the same dependence on the effective disorder parameter as that in the quantum diffusion problem, although the fluctuations in the former case are not universal, in contrast to the latter. Moreover, all the higher moments of the fluctuations have the same (nontrivial) dependence on the appropriate weak-disorder parameter. It makes the distribution functions of the fluctuating transport coefficients to have the same shape in the both cases: almost Gaussian in the bulk of the distribution but with slowly decreasing lognormal tails. Such a similarity is even more striking, taking into account that average values of the transport coefficients (conductance, diffusion constant, etc.) have absolutely different dependence on the disorder. The reason is that the “mesoscopic properties” could be unrelated to those particular features of a model which determine behaviour of the ensemble-averaged quantities (e.g. the conductance). Instead, they could be determined by some general symmetry that might appear independent of details of this or that model, and even of the presence or absence of the quantum coherence. We shall discuss this in the conclusion but now let us begin with describing the classical model under consideration.

II. MODELS OF CLASSICAL DIFFUSION IN WEAKLY DISORDERED MEDIA

First of all, to be able to compare transport properties in models of classical and quantum diffusion, we need to define a classical model for diffusion in a weakly disordered medium. In a generic quantum diffusion model (the Anderson model, or the model of free electrons in a random potential), the disorder parameter is directly related to the transport properties [12]: it is just the inverse dimensionless conductance, $1/g \sim 1/(p_e l)^{d-1}$, and the dimensionless conductance $g$ is related to the diffusion coefficient $D = v_F \tau / d$ by the Einstein relation, $g = c_d \nu D$ ($l$ is the mean free path, $c_d$ is the constant depending only on the dimensionality $d$). In contrast to this, the parameter of quenched disorder in classical transport can be totally unrelated to the diffusion coefficient. We consider a generic model for non-coherent hopping transport, and discuss different ways of introducing a weak disorder into it.
A. Lattice hopping models and continuum limit

The lattice hopping model may be defined by the master equation for the probability $\rho_r(t)$ of finding the particle at the $r$-th site at the moment $t$:

$$\frac{\partial \rho_r}{\partial t} = \sum_{r'} (W_{rr'} \rho_{r'} - W_{r'r} \rho_r). \tag{1}$$

Here $W_{rr'}$ is the probability of hopping from site $r'$ to site $r$. We call this model classical just because transport is described in terms of the probabilities rather than amplitudes, although the origin of hopping could be quantum as well.

In the continuous limit, this model goes over to the classical Fokker–Plank (FP) equation:

$$\left[ \frac{\partial}{\partial t} + \partial_\alpha (v_\alpha - D \partial_\alpha) \right] \rho(r, t) = 0, \tag{2}$$

where $\alpha$ is a vector index in the $d$-dimensional space. The parameters of this equation are related to the hopping probabilities $W$ as follows:

$$D(r) = \frac{1}{2} \sum_{r'} (r - r')^2 W_{rr'} \tag{3}$$

$$v(r) = \sum_{r'} (r - r') W_{rr'} \tag{4}$$

On a regular lattice in the absence of external fields, the hopping probability $W^0$ is symmetric, $W^0_{r-r'} = W^0_{r'-r}$, so that the drift velocity $v$, Eq. (4), vanishes and only the diffusion term survives in the FP equation (2), with the diffusion coefficient $D$ given by Eq. (3).

Now one introduces a weak disorder into the model, making the hopping probability $W$ slightly fluctuating around its mean value $W^0$ on the regular lattice,

$$W_{rr'} = W^0_{r-r'} + \delta W_{rr'}, \tag{5}$$

with $\delta W_{rr'}$ describing the quenched weak disorder on the lattice. It contributes to both $D$ and $v$ in the continuous limit of the model, Eq. (2). Were the hopping probabilities $W = W^0 + \delta W$ kept symmetric, $v$ would vanish and the weak disorder would enter the model only via slightly fluctuating diffusion coefficient $D$. We will show that such a weak disorder is irrelevant, in a sense that it does not effect large-scale (or long-time) properties of the hopping model. The only source of a relevant weak disorder could be in the drift velocity $v$, i.e. in the asymmetry of the hopping probabilities $\delta W$.

The random asymmetry that leads to the fluctuating $v$ could be due to the presence of magnetic or charged impurities. Following ref. [7], we will show that the presence of a random electric field $\mathbf{E}(\mathbf{r})$ of charged impurities leads to the potential random drifts (with $\text{curl} \mathbf{v} = 0$) in the continuum-limit model, Eq. (4). If $\mathbf{E}$ is the only source of randomness, the probability of thermally activated hops at a distance $b$ is...
\[ W_{r,r+b} = W_0^b \exp \left[ -\frac{eE(r)b}{2kT} \right]. \] (6)

Assuming then a random distribution of the charged impurities on the lattice and a global electro-neutrality, and representing the Fourier transform of \( E(r) \) as

\[ E(q) = \frac{2\pi i q}{\varepsilon q} \sum_j e_j \exp(i q \cdot r_j), \]

with \( e_j = \pm e \) being the charges of impurities with random coordinates \( r_j \) and \( \varepsilon \) being the dielectric constant, one finds

\[ \langle E_\alpha(q)E_\beta(q') \rangle = \frac{(2\pi e)^2 C}{\varepsilon} \frac{q_\alpha q_\beta}{q^2} \delta_{qq'} \] (7)

where \( C \) is the concentration of the charged impurities. Let us also assume for simplicity the nearest-neighbor hopping only, and take the weak-disorder limit which allows to expand the exponent in Eq. (6). Then, substituting \( \delta W \propto E \) into Eq. (4) that define the drift velocity in the continuum limit, one arrives at the model described by the FP equation (2) with random drifts \( v \) with zero average and the correlation function

\[ \langle v_\alpha(r)v_\beta(r') \rangle = \gamma_0 F_{\alpha\beta}(r-r') \] (8)

where the Fourier transform of \( F_{\alpha\beta}(r-r') \) is given by \( F_{\alpha\beta}(q) = \frac{q_\alpha q_\beta}{q^2} \). Obviously, \( \text{curl} \mathbf{v} = 0 \) so that we will refer to this model as one with the potential random drifts, or just the potential disorder model. The strength of disorder \( \gamma_0 \) in Eq.(8) is related to the parameters of the hopping model as

\[ \gamma_0 = \frac{\pi^2 e^4 C}{\varepsilon^2(kT)^2} \left( \frac{W_0 z a^2}{2} \right)^2 \equiv \frac{\pi^2 e^4 C}{\varepsilon^2(kT)^2} D_0^2, \] (9)

where \( z \) is the coordination number of the lattice, \( a \) is the lattice constant, \( W_0 \equiv W_0^a \) is a regular part of the nearest-neighbor hopping probability, Eq. (3), and \( D_0 \) is the average diffusion coefficient. The randomness in \( W \), both symmetric and asymmetric, results also in spatial variations of the diffusion coefficient, \( D(r) = D_0 + \delta D(r) \). The variations can be described in terms of some distribution \( P(D) \) governed by the cumulants

\[ \langle \delta D(r_1) \delta D(r_2) \rangle = 2! g^{(2)}(r_1 - r_2), \]

\[ \ldots \ldots \]

\[ \langle \delta D(r_1) \ldots \delta D(r_s) \rangle = s! g^{(s)}(r_1 - r_2) \ldots \delta(r_1 - r_s), \] (10)

which may be directly expressed via the cumulants of \( \delta W \). The absence of spatial correlations of \( D(r) \) in the continuum limit results from taking into account next-neighbor hopping only. In the next section we show that it is the presence of the random drifts, Eq. (8), that
changes drastically the long-time (or long-range) asymptotic properties of the transport coefficients while the disorder in the diffusion coefficients is totally irrelevant for the behavior of the averaged quantities. On the other hand, we will show that whatever were the initial distribution of the diffusion coefficient characterized by the couplings $g^{(s)}$, Eq. (11), its asymptotic characteristics would also be drastically changed due to the influence of the random drifts.

Similarly, in the presence of a random magnetic field one arrives [6] at the model of solenoidal random drifts with zero average and the correlation function (8) with the Fourier transform of $F_{\alpha\beta}(\mathbf{r} - \mathbf{r}')$ given by $F_{\alpha\beta}(\mathbf{q}) = \delta_{\alpha\beta} - q_\alpha q_\beta/q^2$. Finally, one may consider a model with the short-range correlation function [4], $F_{\alpha\beta}(\mathbf{r} - \mathbf{r}') = \delta_{\alpha\beta}\delta(\mathbf{r} - \mathbf{r}')$, although the latter one is not directly related to the lattice hopping models (besides the case where a specific relation is imposed between an external magnetic field and a disorder parameter [13]). So we can consider the models described by the FP equation (2) with the irrelevant weak disorder in the diffusion coefficient, Eq. (10), and the relevant one in the drift term, Eq. (8), which is characterized by three possible choices for the correlation function:

$$F_{\alpha\beta} = \begin{cases} 
q_\alpha q_\beta/q^2, & \text{potential disorder} \\
\delta_{\alpha\beta} - q_\alpha q_\beta/q^2, & \text{solenoidal disorder} \\
\delta_{\alpha\beta}, & \text{mixed disorder}
\end{cases} \tag{11}$$

The FP description above is the basis for the further considerations. An appropriate Langevin description is also quite useful as it provides a clear qualitative picture. A diffusing particle can be considered as a random walker exposed to a strong thermal noise (‘random winds’) and a relatively weak quenched disorder (‘random drifts’):

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{r}) + \mathbf{\eta}(\mathbf{r}, t) \tag{12}$$

In order to reproduce the FP equation (2), the thermal noise $\mathbf{\eta}$ is chosen to be the Gaussian white noise with a zero average and

$$\overline{\eta_\alpha(\mathbf{r}, t) \eta_\beta(\mathbf{r}, t')} = 2D(\mathbf{r}) \delta_{\alpha\beta} \delta(t - t'), \tag{13}$$

where $D(\mathbf{r})$ is the weakly fluctuating diffusion coefficient with the distribution (10). Were the quenched disorder strong, the random walker would mainly follow the drift lines of the field $\mathbf{v}$ with rare hops from one line to another due to the presence of the random winds.

We consider, however, only the case of the weak disorder when the motion is mainly governed by the random winds. In the absence of $\mathbf{v}$, such a motion would be pure diffusive. The weak random drifts correct this diffusive motion as the random walker experiences also some almost ballistic motion along the drift lines, albeit short and frequently interrupted by hops between the lines induced by the random winds. The potential model is characterized
by the presence of sinks and sources in the field lines. Without the thermal noise, the particle would end at some sink (i.e. would be confined in a restricted region of space). Then it is natural to expect that the influence of the weak potential drifts lead to sub-diffusion behaviour in the long-time limit. In the solenoidal field, there are only closed drift lines whose presence could help the particle diffusion, so that one could expect a super-diffusion behaviour. And in the case of the mixed disorder, these two trends could partially cancel each other. We show further how to confirm this qualitative picture.

B. The effective functional

A standard way to describe the long-range and long-time asymptotic processes is to derive and solve the renormalization group (RG) equations for the quantities of interest. To this end, we introduce the effective field-theoretical functional describing the properties of the model.

All the transport properties may be expressed in terms of the Green’s function of the FP equation (2). In the absence of the drift fields, the Green’s function $G_0$ is the usual diffusion propagator:

$$G_0(\omega, q) = \left(-i\omega + Dq^2\right)^{-1}. \quad (14)$$

To calculate the disorder-averaged Green’s function $G$ in the presence of the random drifts one could develop a perturbation theory in terms of $v$, as we consider only the small disorder. However, the long-range character of the disorder correlations in Eq. (11) (which is effectively present even in the model (c) as $F(q)$ does not vanish when $q \to 0$) makes all the diagrams diverging as $\omega \to 0$ in the case $d \leq 2$. The situation is similar to that in the quantum-diffusion problem where the appropriate divergence is due to the interference effects. More convenient way to perform the ensemble averaging is to represent $G$ as a functional integral over the conjugate complex fields $\varphi(r)$ and $\varphi^*(r)$:

$$G(r, r'; \omega) = \frac{i \int \overline{\varphi}(r) \varphi(r') e^{iS[\varphi, \varphi^*]} D\varphi D\varphi^*}{\int e^{iS[\varphi, \varphi^*]} D\varphi D\varphi^*}, \quad (15)$$

where the effective action functional is given by

$$S[\varphi, \varphi^*] = \int d^d r \left[i\omega \varphi \varphi + v_\alpha (\partial_\alpha \varphi) \varphi - D\partial_\alpha \varphi \partial_\alpha \varphi\right] \quad (16)$$

The averaging is performed by the standard replica trick: the fields $\varphi(r)$ and $\varphi^*(r)$ and the functional integration in Eq. (13) are $N$-replicated and the independent averaging over the numerator and denominator in Eq. (13) is justified in the replica limit $N = 0$ (that should be taken in the final results). Taking into account the disorder both in the random drifts $v$
and in the diffusion coefficients $D$ which is defined by Eqs. (8) and (10), one deduces the effective action

$$S[\phi, \phi] = \{S_0 + S_{int} + S_{cum}\} [\phi, \phi] \quad (17)$$

which should be substituted for that given by Eq. (16) into Eq. (15) where the functional integration should be performed over all components of the fields $\phi = \phi_1 \ldots \phi_N$ and $\phi = \phi_1 \ldots \phi_N$. Here

$$S_0[\phi, \phi] = \int d^d r (i\omega + D_0 \partial^2) \phi \quad (18)$$

$$S_{int}[\phi, \phi] = \frac{i\gamma}{2} \int d^d r d^d r' \left( \partial_\alpha \phi \phi_r \right) F_{\alpha\beta}(r - r') \left( \partial_\beta \phi \phi_{r'} \right) \quad (19)$$

$$S_{cum}[\phi, \phi] = i \sum_{s=2}^\infty g^{(s)} S^{(s)}; \quad S^{(s)} = \int d^d r \prod_{i=1}^s \left( \partial_\alpha \phi \partial_\alpha \phi \right)_r \quad (20)$$

Choosing the dimensionality of the fields $\phi$ and $\phi$ so that the action $S_0$ is dimensionless, one sees that the random drift term, Eq. (19), is relevant in dimensionalities $d \leq 2$. The RG analysis of the functional (18), (19) in the upper critical dimensionality $d = 2$ describes the anomalous long-time behaviour of the average transport coefficients. On the other hand, the naïve counting of the scaling dimensions shows that the higher-order gradient operators, Eq. (20), are irrelevant in any dimensionality $d > 0$. Indeed, under rescaling $L \rightarrow \lambda L$, $g^{(s)} \rightarrow \lambda^{(s-1)d} g^{(s)}$ so that the naïve dimension of $g^{(s)}$ is always negative. Further on, we will show that, similar to the case of the quantum diffusion [14], in the case of the potential model, Eq. (11a), the one-loop RG corrections overturn this conclusion and make the scaling dimensions of the high-order gradient operators positive which will lead to nontrivial asymptotic properties [9] of the distribution $P(D)$.

III. LONG-TIME BEHAVIOUR OF THE TRANSPORT COEFFICIENTS

We begin with renormalizing the functional (18), (19) to find the long-time behaviour of the transport coefficients. The standard RG procedure is performed by expanding $\exp(iS_{int})$ in a power series, integrating with the weight $\exp(iS_0[\phi_0, \phi_0])$ over the “fast” components of the fields and exponentiating the results of the integration. Here $\phi(r)$ (and $\phi(r)$) is decomposed into the sum of “slow”, $\phi(r)$, and “fast”, $\phi_0(r)$ components where

$$\phi(r) = \sum_{q<\lambda q_0} \phi(q)e^{iq\cdot r}, \quad \phi_0(r) = \sum_{\lambda q_0<q<q_0} \phi(q)e^{iq\cdot r},$$

$0 < \lambda < 1$ is the scaling parameter, $q_0$ is the ultraviolet cutoff, e.g. the inverse lattice constant in the lattice realization of the model. The actual (loop) expansion is carried out in powers of the dimensionless (at $d = 2$) disorder parameter.
\[
g = \frac{\gamma}{4\pi D^2},
\]

which takes a part of the effective coupling constant of the model. (Here we omit the subscript 0 in \( \gamma \) and \( D \), and will keep it only to denote the bare, unrenormalized values of appropriate quantities). In the weak-disorder approximation considered here \( g_0 \ll 1 \).

In the two-loop approximation, one comes in the replica limit \( (N = 0) \) to the following RG equations [7] for the two parameters of the functional (18), (19) (as usual, the frequency \( \omega \) is not renormalized due to the conservation of the total probability):

\[
d\ln D/d\ln \lambda^{-1} = \alpha g - 2(1 - \alpha^2)g^2,
\]

\[
d\ln \gamma/d\ln \lambda^{-1} = -(1 - \alpha)g - 2(1 - \alpha^2)g^2,
\]

which lead to the following RG equation for the coupling constant \( g \):

\[
d\ln g/d\ln \lambda^{-1} = -(1 + \alpha)g + 2(1 - \alpha^2)g^2.
\]

Here, for the models defined in Eq. (11 a–c),

\[
\alpha = \begin{cases} 
-1, & (a) \\
1, & (b) \\
0. & (c)
\end{cases}
\]

For the models of the solenoidal and mixed disorder, Eq. (11 b,c), one faces the “zero-charge” situation: the coupling constant that is initially small \( (g_0 \ll 1 \text{ in the weak-disorder case}) \) is further decreased under the RG transformations, Eq. (23). Therefore, the RG solutions of these model are asymptotically exact. In the potential-disorder model, Eq. (11a), the coupling constant is not renormalized at all. This is obvious within the two-loop accuracy of Eq. (23). Moreover, this statement has been proved to the all orders of the loop expansion [8] which, naturally, does not rule out a possibility of non-perturbative contributions to the coupling constant). The absence of renormalization of the coupling constant does not make the model to be trivial, as both the diffusion coefficient, \( D \), and the strength of disorder, \( \gamma \), do change nontrivially with the change of the scale \( \lambda \), according to Eq. (22). Solving this equation and setting the logarithmic RG variable to \( \ln(Dq_0^2/\omega)^{1/2} \), or equivalently, in the time representation, to \( \ln(t/\tau)^{1/2} \) \( (q_0 \text{ is the ultraviolet cutoff which is of order the inverse lattice constant in the lattice realization of the model, and } \tau \sim 1/Dq_0^2) \), one finds the long-time behaviour of the diffusion coefficient as

\[
D(t) \equiv \frac{\partial}{\partial t} \langle r^2(t) \rangle \propto \begin{cases} 
D_0(t/\tau)^{-g/2}, & \text{potential disorder} \hspace{1cm} (a) \\
D_0 \ln^{1/2}(t/\tau), & \text{solenoidal disorder} \hspace{1cm} (b) \\
D_0[1 + A/\ln(t/\tau)], & \text{mixed disorder} \hspace{1cm} (c)
\end{cases}
\]

8
So, indeed, the potential and solenoidal models show sub- and super-diffusion behaviour, respectively, while the long-time asymptotics of $D$ in the mixed-disorder model is practically unaffected.

In a similar way, one finds the long-time dependence of the mobility, $\mu$, by substituting the drift field in the presence of an external electric field $E$, $\mathbf{V} = \mu \mathbf{E} + \mathbf{v}(\mathbf{r})$ for $\mathbf{v}$ into the FP equation (2) and the effective functional (18) and solving the appropriate RG equations. It yields

$$
\mu(t) \propto \begin{cases} 
\mu_0(t/\tau)^{-g/2}, & \text{potential disorder (a)} \\
\mu_0, & \text{solenoidal disorder (b)} \\
\mu_0/\ln(t/\tau), & \text{mixed disorder (c)}
\end{cases}
$$

(26)

Note that the Einstein relation between the renormalized diffusion coefficient and the mobility holds only for the potential model. Its violation for the other two models can be ascribed to the renormalization of the effective temperature [7]. In the potential case temperature remains unrenormalized. Applying the results (25), (26) to the hopping model above and substituting $g$ as in Eqs. (21) (9), one finds that the mobility (and thus conductivity) acquires a characteristic temperature dependence $\exp(-T^{-2})$.

In general, all the three models show the scale-dependence of the averaged transport coefficients on the disorder which is totally different from that in the quantum-diffusion problem. In the latter, for $d = 2$ the effective disorder parameter, the inverse dimensionless conductance $g^{-1}$, increases with the scale [13] so that the system is driven towards the strong disorder (localization), while in the former the effective disorder parameter, $g$, either decreases with the scale (solenoidal or mixed disorder) or remains unchanged (potential disorder). This makes the classical-diffusion models considered to have asymptotically exact solutions but, unfortunately, prevents even qualitative considerations of the strong-disorder limit within the perturbative RG approach which proved to be so fruitful in the case of the quantum diffusion (see for reviews Ref. [12]).

Although the disorder-dependence of the average quantities in the models considered has nothing to do with that in the quantum diffusion problem, their mesoscopic properties show quite a striking resemblance to those of the quantum model. We demonstrate this, limiting further considerations to the potential-disorder case which is most interesting of the three.

**IV. MESOSCOPICS FLUCTUATIONS IN THE POTENTIAL-DISORDER MODEL**

In considering the “mesoscopic properties, we should answer for the two important questions. First, whether the model considered shows the absence of self-averaging which is
characteristic for the quantum-diffusion problem at $T = 0$, so that the fractional value of the fluctuation does not vanish in the thermodynamic limit, $L \to \infty$. We will show that this happens due to the long-range character of the correlations (11a). Then, what is a natural “mesoscopic” scale beyond which the self-averaging is restored? In the quantum-diffusion problem, such a scale is a coherence length $\xi$. There is no direct analog to that in the problem of non-coherent transport but the “mesoscopic” scale arises naturally within the lattice hopping model considered above.

A. Mesoscopic scale

In real lattices the correlation function (7) becomes non-singular ($\propto q^\alpha q^\beta$) for $q \gg r_0$ where $r_0$ is a screening radius. The continuum model with the correlation function (11a) could then describe only the random walks at a distance not exceeding $r_0$. In general, $r_0$ takes the part of the infrared cutoff similar to that of the phase-breaking length in the weak-localization theory and defines the mesoscopic scale for the classical-diffusion problem considered. The screening radius can be “mesoscopically” large for systems where the carrier density is much lower than the density of the charged impurities which are responsible for the asymmetric disorder in the hopping probability that leads to the presence of the random potential drifts in Eq. (2). Among numerous examples are diffusion of a charged particle injected into a disordered dielectric or inversion layer [16], or that in weakly doped semiconductors with a very high or a very low degree of compensation, $K$. In the latter case, the screening radius measured in the length of elementary hops is proportional to a large parameter [17], $(1 - K)^{-2/3}$ for strong compensation $(1 - K \ll 1)$, and $K^{-1/2}$ for weak compensation $(K \ll 1)$. Note that in all the cases above disorder is not expected to be weak in contrast to our model. Nevertheless, by focusing on the influence of the asymmetric disorder which is dominant in the model, we hope to shed some light at the processes in real system. Our main aim, however, is to use the model as a toy one to demonstrate a possibility of deep similarity between coherent and non-coherent mesoscopics.

B. The absence of self-averaging in conductivity

We start with the continuum model with the unscreened correlations (11a). To make an analogy with the quantum-diffusion problem most transparent, we consider by way of example the fluctuations of the conductivity $\sigma_{\alpha\beta}$. It can be defined in terms of the Green’s function of the FP equation as

$$\sigma_{\alpha\beta} = \sigma_0 \left( \delta_{\alpha\beta} - \frac{i\omega}{L^2} \int v_{\alpha}(\mathbf{r}) \partial^\prime_{\beta} G(\mathbf{r}, \mathbf{r}'; \omega) G(\mathbf{r'}, \mathbf{r''}; \omega) \, d\mathbf{r} \, d\mathbf{r'} \, d\mathbf{r''} \right).$$

(27)
In the average, \( \langle \sigma_{\alpha \beta} \rangle = \rho \mu \delta_{\alpha \beta} \) where \( \rho \) is the particle density. The fractional value of the conductivity fluctuations is determined by the correlation function

\[
K_{\alpha \beta; \gamma \delta} = \frac{\langle \delta \sigma_{\alpha \beta} \delta \sigma_{\gamma \delta} \rangle}{\sigma^2_0}, \tag{28}
\]

where \( \delta \sigma_{\alpha \beta} = \sigma_{\alpha \beta} - \langle \sigma_{\alpha \beta} \rangle \). In the lowest nonvanishing order of the perturbation theory, one obtains for \( \omega \to 0 \):

\[
K_{\alpha \beta; \gamma \delta} = \left( \frac{\gamma}{L} \right)^2 \int \frac{d^2 q}{(2\pi)^2} \frac{q_\beta q_\delta q_\mu q_\nu}{(Dq^2)^4} \left[ F_{\alpha \gamma}(q) F_{\mu \nu}(q) + F_{\alpha \nu}(q) F_{\gamma \mu}(q) \right]. \tag{29}
\]

The long-range character of the correlations (11a) leads to a divergence in \( K \) in the limit \( L \to \infty \) that exactly compensates the \( L^{-2} \) factor attached to the integral:

\[
K_{\alpha \beta; \gamma \delta} = g^2 s_{\alpha \beta \gamma \delta} L^{-2} \int \frac{d^2 q}{q^4}, \tag{30}
\]

where \( s_{\alpha \beta \gamma \delta} = \delta_{\alpha \beta} \delta_{\gamma \delta} + \delta_{\alpha \gamma} \delta_{\beta \delta} + \delta_{\alpha \delta} \delta_{\beta \gamma} \).

After the substitution

\[
g \longrightarrow \overline{g}^{-1} \tag{31}
\]

where \( \overline{g} \) is the average dimensionless conductance, this expression exactly corresponds to that for the appropriate correlation function in the quantum-diffusion problem \[18\]. Not only the proportionality to the proper disorder parameter is the same but also both the tensor structure of this expression, and the dependence on a geometric shape of the sample which is hidden in the same diverging integral. Naturally, the scale beyond which Eq. (30) is no longer valid is totally different for the two cases (the screening radius of charged impurities in the former, and the coherence length in the latter), as well as behaviour at larger scales. Nevertheless, the similarity is quite striking, and even more so in much more subtle properties of the fluctuations like the long-tail asymptotics of the distribution functions.

**C. Anomalous dimensionality of the high-gradient operators**

If the distribution of the fluctuations were Gaussian, only the average and the second moment would be relevant. So one should find the higher moments of the distribution to determine its shape. We show here following Ref. [9] that the high moments of the distribution function becomes large as compared to the second one so that only the tails of distribution are distinctly non-Gaussian similar to those in the quantum-diffusion problem. The distribution of the diffusion coefficient is easiest for considerations as the couplings \( g^{(s)} \) that define its higher moments are directly included into the effective action (20). To find the scale-dependent moments, we perform the one-loop renormalization of these couplings.
We will not discuss this renormalization in detail (see Ref. [9]) but outline the main steps. First, the index structure of the action (20) is not conserved under the renormalization. A set of additional operators is generated having the structure
\[ \partial_\alpha \varphi^a \partial_\beta \varphi^b \partial_\gamma \varphi^c \partial_\delta \varphi^d \ldots \] (32)
with all possible permutations of the vector indices \( \alpha, \beta, \ldots \) and the replica indices \( a, b, \ldots \), each index being repeated twice, where the summation over repeated replica indices from 1 to \( N \) and over repeated vector indices from 1 to \( d (= 2) \) is implied. These operators are “unphysical” in a sense that the distribution \( P(D) \) is defined by the renormalization of the initial operators (20) only. Then, only those operators are of interest which have the RG feedback to the initial ones.

To classify all the additional operators one introduces matrix notations
\[ Q^{ab} = \partial_\alpha \varphi^a \partial_\alpha \varphi^b, \quad P^{ab} = \partial_\alpha \varphi^a \partial_\alpha \varphi^b, \quad \overline{P}^{ab} = \partial_\alpha \varphi^a \partial_\alpha \varphi^b. \] (33)
In these notations, the initial action (20) is \( \int d^d r (\text{Tr} Q)^s \). It is possible to show that the RG equations have a triangular structure: the operators containing only the matrices \( Q \) are not influenced on by those containing \( P \). Therefore, all the relevant operators may be written down as
\[ S_s[Q] = g^{(s)} \int d^d r \left\{ \left[ \text{Tr} Q \right]^{s_1} \ldots \left[ \text{Tr} (Q) \right]^{s_m} \ldots \right\}. \] (34)
Here the set of integers \( s = s_1 \ldots s_m \ldots \) obeys the constraint
\[ \sum_{m \geq 1} m s_m = s, \] (35)
the initial operator (20) corresponding to \( s = (0, \ldots, 0, s) \). The renormalization of the \( s \)-th cumulants results from solving the RG equations for the whole set of \( g^{(s)} \), the bare values of all the additional charges being equal to zero.

The couplings \( g^{(s)} \) may be represented as some ket-vector defined by the “occupation numbers” \( s_m \)
\[ g^{(s)} \equiv |s\rangle \equiv |1^{s_1} 2^{s_2} \ldots m^{s_m} \ldots \rangle. \] (36)
The matrix set of the RG equations involving all the couplings (36) can be diagonalized exactly [9]. The eigenvectors are given by
\[ |\rho\rangle = \sum_{\{s\}} g(s) \chi_{\rho}(s)|s\rangle \] (37)
where the summation is performed over all the partitions \( \{s\} \equiv s_1 \ldots s_m \ldots \) of the integer \( s \) obeying the constraint (35), \( g(s) = s! / \prod_m m^{s_m} s_m! \) is the number of elements in the class
defined by the partition \( s \), and \( \chi_\rho(s) \) are the characters of irreducible representation of the group of permutations characterized by the Young frame \( \rho \) having boxes of length \( \rho_1 \ldots \rho_m \ldots \) where \( |s\rangle u m_m \rho_m = s \). The appropriate eigenvalues are given by

\[
\alpha_s(\rho) = \frac{s(s-1)}{2} + \sum_m \rho_m(\rho_m - 2m + 1) \tag{38}
\]

The maximum eigenvalue corresponds to the eigenvector characterized by the one-line Young frame with \( \rho_1 = s, \rho_m = 0 \) for \( m > 1 \) for which \( \chi_\rho(s) = 1 \) for all \( s \), so that it is equal to \( s(s-1) \), as in the case of the quantum diffusion. Note that it could be verified, without any reference to the representations of the permutation group, by mapping the renormalization group operator onto a certain one-dimensional model of bosons with nontrivial cubic interaction \[14,9\]. Thus, with the one-loop accuracy, the dimension of the operators coupled to the moments of the diffusion coefficient is given by

\[
\alpha_s = -(s-1)d + gs(s-1), \tag{39}
\]

so that for large enough \( s \left( s \gtrsim g^{-1} \right) \) the one-loop correction overtakes the negative naïve dimensionality of the operators \[20\].

Note that there is a deep technical analogy with the quantum-diffusion problem. In the latter, the high-order moments of the conductance fluctuations are described with the help of the high-gradient operators in the nonlinear \( \sigma \) model \[14\]. Their renormalization involves either the mapping onto the one-dimensional model of interacting bosons \[14\] or the analysis in terms of irreducible representations of the group of permutations \[14\], similar to the procedure outlined above, and leads to the anomalous dimension of the operators given by Eq. \[38\] after the same substitution \[31\] as for the variance of the conductance fluctuations \[30\]. This results in nontrivial similarity between the properties of the fluctuations in the two systems.

V. COMPARISON OF THE RESULTS FOR THE FLUCTUATIONS IN THE COHERENT AND NON-COHERENT DIFFUSION PROBLEMS

There are two types of contributions into the fluctuations of the diffusion coefficient in the random-walks model considered, similar to the quantum diffusion problem \[14\]. The “normal” one is given only by the functional \[14\]. It diverges in the infrared limit thus making the fractional fluctuation \( \left( \langle \delta D \rangle^2 \right)^{1/2} / D^2 \propto g^2 \) to be independent of the size of the system, analogous to the UCF in metals. The appropriate contribution to the conductance fluctuation in the potential-disorder model is given by Eq. \[30\]. The additional contribution to the diffusion cumulants \[10\] is governed by the dimensions of the couplings \( g^{(s)} \) in the
high-gradient functional \([24]\). Keeping only the maximum eigenvalue, as in Eq.\((39)\), one finds in the critical dimensionality \(d = 2\)

\[
\langle \langle (\delta D)^s \rangle \rangle \propto \left( \frac{l}{L} \right)^{2(s-1)-gs(s-1)},
\]

(40)

where \(l\) is some microscopic length that could be of the order of the lattice spacing, etc. For \(s \gtrsim g^{-1}\), these cumulants increase very fast with the system size \(L\). It is quite similar to the “additional” contribution to the conductance cumulants (or density of states, or diffusion ones) in the quantum-diffusion problem which are proportional \([14]\) to \(\left( \frac{l}{L} \right)^2 e^{us(s-1)}\) with \(u = \ln(g_0/\overline{g})\) where \(g_0\) is the value of the average dimensionless conductance \(\overline{g}\) in the square of the size \(l^2\), and \(l\) in this case is the mean free path. To make the analogy more striking, one substitutes here the value of the parameter \(u\) in the weak-localization limit at \(d = 2\), using \(g = g_0 - \ln(L/l)\) \([12]\). It gives for the conductance cumulants in the quantum-diffusion problem

\[
\langle \langle (\delta g)^s \rangle \rangle \propto \left( \frac{l}{L} \right)^{2(s-1)-\overline{g}^{-1}s(s-1)}. \tag{41}
\]

Therefore, the same substitution \((31)\) relates not only the variance of the fluctuations in the coherent and non-coherent problems, but also the high moments of the fluctuations, Eqs. \((40)\) and \((41)\).

The increase with \(L\) of the high-order moments leads to the lognormal asymptotic tails of the distribution functions \([14, 9]\) which are naturally identical after a proper definition of the parameters:

\[
f(\delta X) \propto \exp \left[ -\frac{1}{4u^2} \ln^2 \left( \delta X \alpha^2 \right) \right] \tag{42}\]

Here \(\delta X\) stands for either \(\delta D\) or for \(g\) in the classical or quantum problems, respectively, \(\alpha \equiv L/l\), and \(u\) is given above for the quantum problem, and equals \(g \ln(L/l)\) for the classical one.

VI. CONCLUSION

We have shown that there exist a very deep and surprising similarity between the characteristics of the mesoscopic fluctuations in the conventional quantum-diffusion model and the model of the non-coherent (‘classical’) diffusion which is described in the continuum limit with the Fokker–Plank equation \((2)\) with the quenched potential random drifts, Eq. \((11a)\). All such characteristics of one model may be obtained from those of another one with the help of the substitution \((31)\), i.e. by substituting a proper weak-disorder parameter. Such a parameter is defined in a very different way for the two models. In the quantum-diffusion problem it equals to the inverse conductance while in the classical one it is proportional to
the inverse square of the diffusion coefficient (and, thus, to the inverse square of the conductance), Eq. (21). The similarity between the high moments of the fluctuations leads to the distribution of the diffusion coefficient in the classical model to be very similar to the conductance distribution in a weakly disordered metal. In both cases, the distributions turn out to be almost Gaussian in the weak-disorder limit but have slowly decreasing lognormal tails, and the part of the tails increase with increasing the disorder. The mathematical reason for the similarity is that the RG equations governing the cumulants of the distributions in both cases are classified according to the same irreducible representations of the group of permutations. However, the derivation of the RG equations for the high-gradient operators proved to be much easier in the classical-diffusion model than in the quantum one.

This similarity occurs in spite of the fact that the average transport coefficients behave absolutely differently in the two models. Mathematically, this is due to a different behaviour of the coupling constants in the field-theoretical models describing the quantum and classical diffusion. The asymptotic freedom of the nonlinear $\sigma$ model that describes the quantum-diffusion problem \cite{20}, i.e. the increase of the coupling constant (inversely proportional to the conductance) with increasing a scale, is believed to govern the Anderson transition. No transition occurs in the classical diffusion problem where in the case of the potential disorder a perturbative renormalization of the coupling constant proves to be absent in all orders \cite{8} thus leading to the sub-diffusion, Eq. (25a).

Then one can hope that it is possible to separate in some way description of the average values from that of the fluctuations. A very simple, if not too simple, assumption is a possibility to use the one-loop (i.e. the lowest-order) RG results for the distribution, Eq. (12), by substituting more rigorous (or exact) results for the average quantities. Surprisingly, it allows to reproduce \cite{14} exact one-dimensional results for lognormal distributions by substituting the exact one-dimensional value of $u$ into the formulae similar to (12). It provides a basis for the conjecture that one can obtained a reasonable description of the distributions near the transition just by substituting $e^u \sim |g - g_c|^\nu$ with a proper choice of the critical exponent $\nu$.

Furthermore, there is a hope that studying the classical-diffusion problem described here gives a possibility to learn more about the mesoscopic properties of the quantum diffusion in disordered media.
REFERENCES

[1] *Mesoscopic Phenomena in Solids*, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb, Elsevier Sci. Publishers, Amsterdam (1991).

[2] *Quantum Coherence in Mesoscopic Systems*, edited by B. Kramer, NATO ASI Series, B:254, Plenum, NY & London (1991).

[3] Y. G. Sinai, Russ. Math. Survey 25 (1970) 137.

[4] J. M. Luck, Nucl. Phys. B 225 (1983) 169; L. Peliti, Phys.Repts. 103 (1984) 225; D. S. Fisher, Phys. Rev. A 30 (1984) 960.

[5] J. A. Aronovitz and D. R. Nelson, Phys. Rev. A 30 (1984) 1948.

[6] D. S. Fisher, D. Friedan, Z. Qiu, S. J. Shenker, and S. H. Shenker, Phys. Rev. A 31 (1985) 3841.

[7] V. E. Kravtsov, I. V. Lerner, and V. I. Yudson, J. Phys. A 18 (1985) L703; Zh. Eksp. Teor. Fiz. 91 (1986) 569 [Sov. Phys. JETP 64 (1986) 336].

[8] V. E. Kravtsov, I. V. Lerner, and V. I. Yudson, Phys. Lett. 119A (1986) 203; J. Honkonen, Y. M. Pismak, and A. N. Vasilev, J.Phys. A 21 (1988) No17.

[9] I. V. Lerner, Nucl. Phys. A 560 (1993) 274.

[10] B. L. Altshuler, Pis’ma v ZhETF 41 (1985) 530 [JETP Letters 41 (1985) 648].

[11] P. A. Lee and A. D. Stone, Phys. Rev. Lett. 55 (1985) 1622.

[12] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. 57 (1985) 287.

[13] V. E. Kravtsov, I. V. Lerner, and V. I. Yudson, Phys. Lett. 114A (1986) 58.

[14] B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, in Ref. [1], p.449; I. V. Lerner, Ref. [2], p.279.

[15] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42 (1979) 673.

[16] N. F. Mott and E. A. Davis, *Electronic Properties of Doped Semiconductors*, Springer, Heidelberg (1984).

[17] B. I. Shklovskii and A. L. Efros, *Electronic Processes in Non-Crystalline Materials*, Clarendon Press, Oxford (1971).

[18] B. L. Altshuler and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. 91 (1986) 220 [Sov. Phys. JETP 64 (1986) 127].
[19] F. Wegner, Nucl. Phys. B 354 (1991) 441; I. V. Lerner and F. Wegner, Z. Phys. B 81 (1990) 95.

[20] F. Wegner, Z. Phys. B 35 (1979) 207; F. Wegner, Phys. Repts. 67 (1980) 15; K. B. Efetov, A. I. Larkin, and D. E. Khmel’nitskii, Zh. Eksp. Teor. Fiz. 79 (1980) 1120 [Sov. Phys. JETP 52 (1980) 568].