On the critical points of random matrix characteristic polynomials and of the Riemann $\xi$-function

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A one-parameter family of point processes describing the distribution of the critical points of the characteristic polynomial of large random Hermitian matrices on the scale of mean spacing is investigated. Conditionally on the Riemann hypothesis and the multiple correlation conjecture, we show that one of these limiting processes also describes the distribution of the critical points of the Riemann $\xi$-function on the critical line.

We prove that each of these processes boasts stronger level repulsion than the sine process describing the limiting statistics of the eigenvalues: the probability to find $k$ critical points in a short interval is comparable to the probability to find $k + 1$ eigenvalues there. We also prove a similar property for the critical points and zeros of the Riemann $\xi$-function, conditionally on the Riemann hypothesis but not on the multiple correlation conjecture.

1 Introduction

1.1 Let $\mathcal{E}$ be the sine point process, i.e. a random locally finite subset of $\mathbb{R}$ the distribution of which is determined by

\[ E \sum_{x_1, \ldots, x_k \in \mathcal{E}} f(x_1, \ldots, x_k) = \int d^k x f(x_1, \ldots, x_k) \det \left( \frac{\sin \pi (x_j - x_l)}{\pi (x_j - x_l)} \right)_{j,l=1}^k \]  

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The sine process describes the eigenvalue distribution of random Hermitian matrices on the scale of mean eigenvalue spacing. For complex Wigner matrices (a class of high-dimensional Hermitian random matrices with independent entries above the main diagonal, cf. Section 2.1), this is expressed by the following relation, which is part of a series of results obtained by Erdős–Yau, Tao–Vu and coworkers, see [18, 62]: if $H_N$ is a sequence of random matrices of growing dimension satisfying the assumptions listed in Section 2.1, and $\lambda_{j,N}$ are the eigenvalues of $H_N/\sqrt{N}$, then for $E \in (-2, 2)$

$$
\left\{ \left( \lambda_{j,N} - E \right) \frac{N\sqrt{4 - E^2}}{2\pi} \right\}_{j=1}^{N} \xrightarrow{\text{dist}} \mathcal{S}_i
$$

with respect to the topology defined by continuous test functions of compact support. The factor $\frac{2\pi}{N\sqrt{4-E^2}}$ by which the eigenvalues are scaled is the (approximate) mean spacing between eigenvalues near $E$. Similar results are available for other random matrix ensembles, see the monographs [2, 53] and references therein.

The correlation conjecture of Montgomery [50] in the extended version of Rudnick–Sarnak [57] and Bogomolny–Keating [7, 8] states that a similar relation holds for the zeros of the Riemann $\zeta$-function on the critical line: if $t$ is chosen uniformly at random in $[0, T]$, then

$$
\left\{ \left( \gamma - t \right) \frac{\log T}{2\pi} \left| \zeta\left( \frac{1}{2} + i\gamma \right) = 0 \right. \right\} \xrightarrow{??} \mathcal{S}_i \quad \text{in distribution (3)}
$$

(the question marks are put to emphasise that this relation is still conjectural). The scaling factor $\frac{2\pi}{\log T}$ is the approximate mean spacing between the zeros with imaginary part near $T$. The results of [50, 37, 57] imply that, conditionally on the Riemann hypothesis, convergence holds for a restricted family of test functions.

Following Aizenman and Warzel [1] and Chhaibi, Najnudel and Nikeghbali [12], consider the random entire function

$$
\Phi(z) = \lim_{R \to \infty} \prod_{x \in \mathcal{S}_i \cap (-R, R)} (1 - z/x).
$$

(4)

We study the one-parameter family of point processes

$$
\mathcal{S}_i^a = \left\{ z \in \mathbb{C} \left| \Phi'(z) = a\Phi(z) \right. \right\} \subset \mathbb{R}, \quad a \in \mathbb{R}.
$$

For any $a$, the points of $\mathcal{S}_i^a$ interlace with those of $\mathcal{S}_i$. Therefore the statistical properties of $\mathcal{S}_i^a$ on long scales are very close to those of $\mathcal{S}_i$. Also, $\mathcal{S}_i^a \to \mathcal{S}_i$ as $a \to \infty$. 
On the other hand, on short scales the processes \( \mathcal{G}_i' \) are much more rigid. To quantify this, introduce the events

\[
\Omega_k(\mathcal{G}, \epsilon) = \{ \# [\mathcal{G} \cap (-\epsilon, \epsilon)] \geq k \}, \quad \Omega_k(\mathcal{G}_i', \epsilon) = \{ \# [\mathcal{G}_i' \cap (-\epsilon, \epsilon)] \geq k \}.
\]

From the special case

\[
\mathbb{E} \left( \frac{\#[\mathcal{G} \cap (-\epsilon, \epsilon)]}{\#[\mathcal{G} \cap (-\epsilon, \epsilon)] - k} \right) = \int_{(-\epsilon, \epsilon)^k} d^k x \ det \left( \frac{\sin \pi(x_j - x_l)}{\pi(x_j - x_l)} \right)_{j,l=1}^k
\]

of (1), the sine process boasts the following repulsion property: for any \( k \geq 1 \),

\[
\mathbb{P}(\Omega_k(\mathcal{G}, \epsilon)) = c_k \epsilon^{k^2} + o(\epsilon^{k^2}), \quad \epsilon \to +0, \quad \text{where } 0 < c_k < \infty.
\] (5)

For comparison, the probability of the corresponding event in the standard Poisson process decays as \( \epsilon^k \). Our first result is

**Theorem 1.** For any \( a \in \mathbb{R}, k \geq 2 \) and \( 0 < \epsilon < 1 \)

\[
\mathbb{P}(\Omega_k(\mathcal{G}_i', \epsilon) \setminus \Omega_k+1(\mathcal{G}_i, 5\epsilon)) \leq C_k \left( \epsilon \log \frac{1}{\epsilon} \right)^{(k+2)^2}.
\] (6)

That is, \( k \)-tuples of critical points in a short interval (for a fixed value of \( k \)) are mostly due to \( (k + 1) \)-tuples of zeros in a slightly larger interval. From (6) and (5) we have

\[
(c_{k+1} + o(1)) \epsilon^{(k+1)^2} \leq \mathbb{P}(\Omega_k(\mathcal{G}_i', \epsilon)) \leq (5(k+1)^2 c_{k+1} + o(1)) \epsilon^{(k+1)^2}, \quad \epsilon \to +0.
\]

A slightly more careful argument shows that

**Corollary 1.1.** For any \( k \geq 2 \) there exists a limit

\[
c'_k = \lim_{\epsilon \to +0} \frac{\mathbb{P}(\Omega_k(\mathcal{G}_i', \epsilon))}{\epsilon^{(k+1)^2}} \in [c_{k+1}, (1 + \frac{4}{k-1})^{(k+1)^2} c_{k+1}],
\]

independent of \( a \in \mathbb{R} \).

Combining Corollary 1.1 with a result of Aizenman–Warzel [1] which is stated as Proposition 2.2 below, we obtain

**Corollary 1.2.** Let \( (H_N) \) be a sequence of complex Wigner matrices satisfying the assumptions listed in Section 2.1 and let \( (\lambda'_j)_{j=1}^{N-1} \) be the critical points of the characteristic polynomial \( P_N(\lambda) = \det(H_N/\sqrt{N} - \lambda) \). For \( E \in (-2, 2) \) and \( k \geq 2 \),

\[
\lim_{N \to \infty} \mathbb{P} \left\{ \left| \left| \lambda'_j - E \right| < \frac{2\pi \epsilon}{N \sqrt{4 - E^2}} \right| \geq k \right\} = (c'_k + o(1)) \epsilon^{(k+1)^2}.
\]
Figure 1.1: The critical points of the characteristic polynomial of GUE$_{40}$, the eigenvalues, and the eigenvalues of a principal submatrix of dimension 39.

The stronger repulsion between the critical points can be seen on Figures 1.1 and 1.2.

In the number-theoretic setting, we consider the Riemann $\xi$-function

$$\xi(s) = \frac{s(s-1)}{2\pi s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s).$$

This is an entire function which is real on the critical line; its zeros coincide with the non-trivial zeros of the $\zeta$-function. Conditionally on the Riemann hypothesis, the zeros of $\xi'$ lie on the critical line $\Re s = \frac{1}{2}$ and interlace with the zeros of $\xi$ (cf. Section 2.3). Assuming the Riemann hypothesis together with the multiple correlation conjecture, we prove (see Corollary 2.3) that

$$\left\{ (\gamma' - t) \frac{\log T}{2\pi} \mid \xi'(\frac{1}{2} + i\gamma') = 0 \right\} \xrightarrow{??} \Xi' \text{ in distribution.} \quad (7)$$

See Figure 1.3.

Denote

$$\Omega_k(\xi, T, e) = \left\{ 0 \leq t \leq T \mid \# \left\{ \gamma \in (t - \frac{2\pi e}{\log T}, t + \frac{2\pi e}{\log T}), \xi(\frac{1}{2} + i\gamma) = 0 \right\} \geq k \right\}$$

$$\Omega_k(\xi', T, e) = \left\{ 0 \leq t \leq T \mid \# \left\{ \gamma' \in (t - \frac{2\pi e}{\log T}, t + \frac{2\pi e}{\log T}), \xi'(\frac{1}{2} + i\gamma') = 0 \right\} \geq k \right\}$$
From Corollary 1.1 and (7), we obtain

**Corollary 1.3.** Assume the Riemann hypothesis and the multiple correlation conjecture (3). Then

\[
\lim_{T \to \infty} \frac{1}{T} \text{mes}(\Omega_k(\xi', T, \epsilon)) = (c'_k + o(1))e^{(k+1)^2}.
\]

We also prove the following less conditional result with a similar message: \(k\)-tuples of critical points of the \(\xi\)-function crowding short intervals are mostly a consequence of \((k + 1)\)-tuples of zeros crowding slightly larger intervals.

**Theorem 2.** Assume the Riemann hypothesis. For any \(k \geq 2\), \(0 < \epsilon < 1\), \(R \geq 5\)

\[
\frac{1}{T} \text{mes}(\Omega_k(\xi', T, \epsilon) \setminus (\Omega_{k+1}(\xi, T, 5\epsilon) \cup \Omega_{k+2}(\xi, T, Re))) \leq \frac{C}{e^{ckR}}.
\]

1.2 Let us discuss the motivation for these results. The traditional object of study in random matrix theory is the joint distribution of the eigenvalues. The
local eigenvalue statistics, i.e. the study of eigenvalues on the scale of mean eigenvalue spacing, is of particular interest due to the robust (universal) nature of the limiting objects.

Recently, the value distribution of the characteristic polynomial of a random matrix also received significant attention. While the characteristic polynomial is determined by the eigenvalues, its restriction to an interval depends both on the eigenvalues inside the interval and those outside it. Therefore the statistical properties of the characteristic polynomial on the scale of mean eigenvalue spacing are not necessarily determined by the local eigenvalue statistics.

As one varies the argument over an interval containing many eigenvalues for a given realisation of the random matrix, the value of the polynomial shows huge variations by the orders of magnitude. We refer to the works [26,28] for a discussion and references, in particular, for a statistical mechanics perspective.
on the absolute value of the characteristic polynomial as a disordered landscape (a Boltzmann weight with a log-correlated potential). It was found that characteristic polynomials of random matrices can be used to model the value distribution of the Riemann zeta function on the critical line \([39, 38, 35]\). In particular, the statistics related to the global maximum of the modulus of characteristic polynomial has been studied for matrices drawn from the circular ensemble, and these results were used to study (on the physical and mathematical levels of rigour) the properties of the global maximum of \(|ζ(1/2 + it)|\) in various intervals \([27, 28, 3, 4, 52, 11, 12, 51, 5]\). Parallel questions for the characteristic polynomial of Hermitian random matrices were investigated in \([30, 31, 29]\).

The sequence of positions of the local maxima (and minima) of the characteristic polynomial which we study in this paper is one of the natural characteristics of the random landscape.

Second, the zeros and the critical points of the characteristic polynomial form an interlacing pair of sequences. As put forth by Kerov \([41, 40]\), such pairs naturally appear in numerous problems of analysis, probability theory, and representation theory, and their limiting properties in various asymptotics regimes are of particular importance.

In the recent work \([59]\) we studied the statistical properties of the zeros and the critical points from the point of view of the global regime, namely, the fluctuations of linear statistics. In particular, the fluctuations differ from those of another natural interlacing pair, formed by the eigenvalues of a random matrix and those of a principal submatrix; see Erdős and Schröder \([17]\).

Here we study the joint distribution of the zeros and the critical points in the local regime, i.e. on the scale of the mean spacing. See further Corollary 2.2 and Paragraph 3.4.2, and also Figure 1.1.

Finally, the strong repulsion between the critical points is an instance of a general phenomenon: the zeros of the derivative of a polynomial with real zeros (or of an analytic function in the Laguerre–Pólya class) are more evenly spaced than the zeros of the original polynomial. In the deterministic setting, this phenomenon goes back to the work of Stoyanoff and M. Riesz \([61]\), see \([21]\) for a historical discussion. Theorem 1 and its corollaries provide additional examples in the random setting.

Here we remark that, under repeated differentiation, the zeros become more and more rigid and approach an arithmetic progression (after proper rescaling and with the right order of limits). This was established in various settings in \([54, 22, 42]\) (of which \([42]\) is applicable to the \(ξ\)-function), and is probably true for \(Φ(z)\) of \([4]\) as well.
1.3 The second source of motivation comes from number theory (disclaimer: there are no new number-theoretic ingredients in our arguments).

The study of the zeros of \( \xi' \) goes back to the work of Levinson [46] and Conrey [13, 14], who obtained unconditional lower bounds on the fraction of zeros lying on the critical line (see [58, 45], the more recent [23] and references therein for the corresponding results pertaining to the zeros of \( \xi \)).

More recently, Farmer, Gonek and Lee [19] and further Bian [6] and Bui [10] studied the correlations between the zeros of \( \xi' \), arguing that a detailed understanding of the joint statistics of the zeros of \( \xi \) and \( \xi' \) may allow to rule out the so-called Alternative Hypothesis, according to which the spacings between nearest high-lying zeros of \( \xi \) are close to half-integer multiples of the mean spacing. The main result in [19] asserts that, conditionally on the Riemann hypothesis,

\[
\frac{1}{N(T)} \sum_{0 < \gamma', \tilde{\gamma}' < T, \xi'(\frac{1}{2} + i\gamma') = \xi'(\frac{1}{2} + i\tilde{\gamma}') = 0} 4 e^{ia(\gamma' - \tilde{\gamma}')} \log T \frac{1}{4 + (\gamma' - \tilde{\gamma}')^2} \rightarrow |\alpha| - 4|\alpha|^2 + \sum_{k=1}^{\infty} \frac{(k-1)!}{(2k)!} (2|\alpha|)^{2k+1} \quad (8)
\]

as \( T \to \infty \), for \( 0 < \alpha < 1 \), where

\[
N(T) = \# \{ 0 \leq t \leq T \mid \xi(1/2 + it) = 0 \} = \frac{T}{2\pi} \log T (1 + o(1)) . \quad (9)
\]

For comparison, Montgomery showed [50] that (conditionally on the Riemann hypothesis) for \( 0 < \alpha < 1 \)

\[
\frac{1}{N(T)} \sum_{0 < \gamma, \tilde{\gamma} < T, \xi(\frac{1}{2} + i\gamma) = \xi(\frac{1}{2} + i\tilde{\gamma}) = 0} 4 e^{ia(\gamma - \tilde{\gamma})} \log T \frac{1}{4 + (\gamma - \tilde{\gamma})^2} \rightarrow |\alpha| , \quad T \to \infty . \quad (10)
\]

The \( \alpha \to 0 \) asymptotics of form factors on the left-hand side of (8) and (10) capture the behaviour of the spacings between zeros on long scales. The short scale behaviour roughly corresponds to the \( \alpha \to \infty \) asymptotics of the form factor.

The pair correlation conjecture of Montgomery [50] states that for \( \alpha \geq 1 \) the limit of the left-hand side of (10) is equal to 1, similarly to the form factor of the sine process. Further, Hejhal [37] and Rudnick and Sarnak [57] extended the result (10) to higher correlations, which, together with the arguments of Bogomolny–Keating [7, 8], led to the conjecture [7, 8, 57] that the full asymptotic distribution of the zeros of \( \xi \) on the scale of mean spacing is described by the sine process of random matrix theory, (3).

On the other hand, a conjectural description of the full limiting distribution of the critical points seems to have been missing. Our results (Corollary 2.3
below) provide such a conjectural description: it turns out that \( (3) \) formally implies that

\[
\left\{ (\gamma' - t) \frac{\log T}{2\pi} \mid \xi'(\frac{1}{2} + i\gamma') = 0 \right\} \xrightarrow{??} \xi_0'
\]

in distribution. (11)

Then, Corollary 1.3 provides a conditional description of the left tail of the spacing distribution, whereas Theorem 2 provides some less conditional information.

Here we also mention the works devoted to the zeros of \( \zeta' \), particularly, \([47, 48, 15, 43]\) (and references therein). The zeros of \( \zeta' \) do not lie on the critical line, and are therefore thematically more distant from the current study; their counterparts in random matrix setting (in a sense made precise in the aforementioned works) are the critical points of the characteristic polynomial of a circular ensemble.

1.4 Let us briefly discuss the derivation of Corollaries 1.2 and 1.3 from Theorem 1; see Sections 2.2 and 2.3 for the precise definition and proofs. The collection of all the critical points of the characteristic polynomial is determined by the collection of all the eigenvalues. However, it is not a priori clear whether this relation persists in the local limit regime: that is, whether the conditional distribution of the critical points in an interval of length, say, \((5 \times \text{mean spacing})\), conditioned on the eigenvalues outside a concentric interval of length \((R \times \text{mean spacing})\) degenerates in the limit \(R \to \infty\), uniformly in the matrix size.

Technically, (2) amounts to the convergence in distribution of linear statistics of the form

\[
\sum_{j=1}^{N} f \left( (\lambda_{j,N} - E) \frac{\sqrt{4 - E^2}}{2\pi} \right), \quad f \in C_0(\mathbb{R})
\]

(12)

to the corresponding statistics of the sine process. It is possible to extend this to other integrable test functions satisfying mild regularity conditions. On the other hand, the critical points are controlled by linear statistics corresponding to functions of the form \( f(\lambda) = \frac{1}{\lambda - z}, z \in \mathbb{C} \setminus \mathbb{R} \); the asymptotics of such linear statistics is not a formal consequence of (2).

Recently, Aizenman and Warzel [1] put forth a general condition which ensures that (2) can be upgraded to the convergence of such linear statistics. In the setting of Wigner matrices, they verified the condition using the local semicircle law of Erdős–Schlein–Yau [16] and the universality results of Erdős–Yau, Tao–Vu and coworkers [18, 62], and obtained:
Proposition 2.2 (Aizenman–Warzel). Let \( P_N(z) = \det(H_N/\sqrt{N} - z) \), where \( H_N \) is a sequence of complex Wigner matrices satisfying the assumptions listed in Section 2.1. Then for \( E \in (-2, 2) \)

\[
\frac{2\pi}{N \sqrt{4 - E^2}} \frac{P_N'(E + \frac{2\pi z}{N \sqrt{4 - E^2}})}{P_N(E)} \xrightarrow{\text{distr}} \Phi'(z) \exp \left( \frac{\pi E z}{\sqrt{4 - E^2}} \right), \quad N \to \infty \tag{13}
\]

\[
\frac{P_N(E + \frac{2\pi z}{N \sqrt{4 - E^2}})}{P_N(E)} \xrightarrow{\text{distr}} \Phi(z) \quad N \to \infty \tag{14}
\]

with respect to the topology of locally uniform convergence on \( \mathbb{C} \setminus \mathbb{R} \) (in the first relation) and of locally uniform convergence on \( \mathbb{C} \) (in the second relation).

The second relation follows from the first one by (careful) integration. We mention that Chhaibi, Najnudel and Nikeghbali established a counterpart of (13)–(14) for the Circular Unitary Ensemble; see further Paragraph 3.4.1.

We show that a similar result holds for the Riemann \( \xi \)-function, conditionally on the Riemann hypothesis; see Proposition 2.3. Somewhat similar statements have been proved for the \( \zeta \)-function, cf. [20, 34, 55]. Here we quote the following corollary (the non-trivial statement is \( \implies \)):

Corollary 2.3. Conditionally on the Riemann hypothesis, the multiple correlation conjecture (3) is equivalent to each of the following relations:

\[
\frac{2\pi i}{\log T} \frac{\xi'(\frac{1}{2} + i(t + \frac{2\pi z}{\log T}))}{\xi(\frac{1}{2} + i(t + \frac{2\pi z}{\log T}))} \xrightarrow{\text{distr}} \Phi'(z), \quad T \to \infty \tag{15}
\]

\[
\frac{\xi(\frac{1}{2} + i(t + \frac{2\pi z}{\log T}))}{\xi(\frac{1}{2} + it)} \xrightarrow{\text{distr}} \Phi(z), \quad T \to \infty \tag{16}
\]

when \( t \) is chosen uniformly at random in \( [0, T] \).

The combination of Theorem 1 with Proposition 2.2 and Corollary 2.3 implies Corollaries 1.2 and 1.3. Roughly speaking, the asymptotics of the statistics which depend on ratios of the characteristic polynomial [or the \( \xi \)-function], for example, the joint distribution of the zeros of the first \( k \) derivatives, are determined by the sine process.

Here we remark that the fluctuations of linear statistics corresponding to functions such as \( f(\lambda) = \log(\lambda - z) \) contain a component that depends on the

\[\text{... some care is required to upgrade uniform convergence on compact subsets of } \mathbb{C} \setminus \mathbb{R} \text{ to uniform convergence on compact subsets of } \mathbb{C} \text{ not containing the poles of the limiting meromorphic function.} \]
eigenvalues [or zeros] outside the microscopic window. This component is not universal, and, for the $\xi$-function, contains an arithmetic piece; see Gonek, Hughes and Keating [35] and references therein. Therefore the limiting value distribution of the characteristic polynomial differs from that of the $\xi$-function (for any deterministic regularisation).

2 Convergence of random analytic functions

2.1 Let $H_N = (H(i,j))_{i,j=1}^N$ be a complex Wigner matrix, which for us is a Hermitian random matrix such that

a. $\{(\Re H(i,j))_{i<j}, (\Im H(i,j))_{i<j}, (H(i,i))_i\}$ are independent random variables;

b. $(\Re H(i,j))_{i<j}, (\Im H(i,j))_{i<j}$ are identically distributed, and

$$\mathbb{E}H(i,j) = 0, \quad \mathbb{E}|H(i,j)|^2 = 1, \quad \mathbb{E}\exp(\delta |H(i,j)|^2) < \infty$$

for some $\delta > 0$;

c. $(H(i,i))_i$ are identically distributed, and

$$\mathbb{E}H(i,i) = 0, \quad \mathbb{E}|H(i,i)|^2 < \infty, \quad \mathbb{E}\exp(\delta |H(i,i)|^2) < \infty.$$  

The main example is the Gaussian Unitary Ensemble (GUE), in which the joint probability density of the matrix elements of $H$ is proportional to $\exp\left(-\frac{1}{2} \operatorname{tr} H^2\right)$. Denote by $(\lambda_{j,N})_{j=1}^N$ the eigenvalues of $H/\sqrt{N}$. The global statistics of the eigenvalues is described by Wigner’s law:

$$\frac{1}{N} \sum_{j=1}^N \delta(E - \lambda_{j,N}) \longrightarrow \rho(E) dE$$  

(17)

weakly in distribution, where $\rho(E) = \frac{1}{2\pi} \sqrt{(4 - E^2)_+}$ is the semicircular density. Due to the interlacing between the critical points $\lambda'_{j,N}$ and the zeros $\lambda_{j,N}$ of $P_N(\lambda) = \det(H_N/\sqrt{N} - \lambda)$, one also has:

$$\frac{1}{N-1} \sum_{j=1}^{N-1} \delta(E - \lambda'_{j,N}) \longrightarrow \rho(E) dE .$$

The local statistics of $\lambda_{j,N}$ are described by the sine point process $\mathbb{E}i$ defined in (1) (see e.g. [60] for general properties of determinantal point processes): for
any $E \in (-2, 2)$ one has the convergence in distribution:

$$\sum_j \delta \left(u - (\lambda_{j,N} - E)N\rho(E)\right) \to \sum_{x \in \Xi} \delta(u - x), \quad N \to \infty.$$  \hfill (18)

This result was proved in the 1960-s for the Gaussian Unitary Ensemble (the eigenvalues of which form a determinantal point process); see [49]. In a series of works by Erdős–Yau, Tao–Vu, and coworkers, (18) was generalised to Wigner matrices satisfying assumptions such as a.–c. above; see [18, 62] and references therein.

The correlation conjecture of Montgomery [50] in the extended version of Rudnick–Sarnak [57] and Bogomolny–Keating [7, 8] asserts that a similar statement holds for the non-trivial zeros of the $\zeta$-function:

$$\sum_{\xi(1/2+i\gamma)=0} \delta \left(u - (\gamma - t)^{\log T}/2\pi\right) \to \sum_{x \in \Xi} \delta(u - x), \quad T \to \infty$$  \hfill (19)

in distribution, when $t$ is uniformly chosen from $[0, T]$.

The relations (18) and (19) mean that

$$\sum_j f \left((\lambda_{j,N} - E)N\rho(E)\right) \to \sum_{x \in \Xi} f(x)$$  \hfill (20)

$$\sum_{\xi(1/2+i\gamma)=0} f \left((\gamma - t)^{\log T}/2\pi\right) \to \sum_{x \in \Xi} f(x)$$  \hfill (21)

in distribution for continuous test functions $f$ of compact support. It is possible to extend this to integrable test functions satisfying mild regularity conditions. On the other hand, going beyond integrable functions requires additional information about the zeros lying far away from the microscopic window. It turns out that the second relation, (21) is (conditionally) valid for test functions $f(x) = 1/(x - z)$, if the sums are properly regularised, whereas the first relation, (20) requires a deterministic correction depending on $E$; see Sections 2.3 and 2.2 (relying on the works [34] and [1], respectively). These properties imply that the critical points depend quasi-locally, so to speak, on the zeros / eigenvalues.

2.2 We recall the construction of Aizenman–Warzel [1]. Recall that a function $w : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ belongs to the Nevanlinna [= Herglotz = Pick] class ($w \in \mathcal{R}$) if it is analytic and

$$\overline{w(z)} = w(\overline{z}) , \quad \Im w(z) / \Im z > 0.$$
The class $\mathcal{R}$ is equipped with the topology of pointwise convergence on compact subsets of $\mathbb{C} \setminus \mathbb{R}$.

Denote

$$W(z) = \lim_{R \to \infty} \sum_{x \in \mathbb{S}^i} \left[ \frac{1}{x - z} - \frac{1}{x - iR} \right] + i\pi = \lim_{R \to \infty} \sum_{x \in \mathbb{S}^i \cap (-R,R)} \frac{1}{x - z}. \quad (22)$$

The two limits exist and coincide according to a general criterion of [1], and $W(z)$ is a random element of the Nevanlinna class. Also note that $-W(z)$ is the logarithmic derivative of the function $\Phi(z)$ from (4).

As before, let $P_N(z) = \det(H_N/\sqrt{N} - z)$ be the characteristic polynomial of $H_N/\sqrt{N}$, and denote

$$W_N(z;E) = -\frac{1}{N\rho(E)} \frac{p'_N(E + \frac{z}{N\rho(E)})}{p_N(E + \frac{z}{N\rho(E)})} = \sum_j \frac{1}{(\lambda_j,N - E)N\rho(E) - z}. \quad (23)$$

**Proposition 2.2 (Aizenman–Warzel).** For $|E| < 2$,

$$W_N(z;E) \xrightarrow{\text{distr}} W(z) - \frac{\pi E}{\sqrt{4 - E^2}} \quad (23)$$

$$P_N\left( E + \frac{2\pi z}{N\sqrt{4 - E^2}} \right) \xrightarrow{\text{distr}} \Phi(z) \exp \left[ \frac{\pi E z}{\sqrt{4 - E^2}} \right]. \quad (24)$$

**Proof.** The first statement is proved in [1, Corollary 6.5], their argument relies on the results obtained in the works [16,18,62] on the local eigenvalue statistics of Wigner matrices, and on the general theory of random Nevanlinna functions which was developed in [1]. The second statement follows from the first one by (carefully) integrating from 0 to $z$. \qed

Denote by $w^{-1}(a)$ the collection of solutions of $w(z) = a$. Observe that the map $w \mapsto w^{-1}(a)$ from $\mathcal{R} \cap \{\text{meromorphic functions}\}$ to locally finite (multi-)subsets of $\mathbb{R}$ is continuous. From this observation and (23) we deduce:

**Corollary 2.2.** Let $(H_N)$ be a sequence of random matrices satisfying the assumptions listed in Section 2.1. Then for any $E \in (-2,2)$

$$\left( \sum_j \delta \left( u - (\lambda_{j,N} - E)N\rho(E) \right) , \sum_j \delta \left( u - (\lambda'_{j,N} - E)N\rho(E) \right) \right) \xrightarrow{\text{distr}} (W^{-1}(\infty), W^{-1}(-a)) = (\mathbb{S}i, \mathbb{S}i'_a) \quad (25)$$

in distribution, where $a = -\frac{\pi E}{\sqrt{4 - E^2}}$. 

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To recapitulate, the non-obvious part of the statement is that the zeros $\lambda_{j,N}$ which are not in a $\Theta(1/N)$-neighbourhood of $E$ influence the critical points $\lambda'_{j,N}$ near $E$ only via the deterministic quantity $\pi E (4 - E^2)^{-1/2}$. Colloquially, the conditional distribution of the critical points in $(E - r/N, E + r/N)$ given the eigenvalues in $(E - R/N, E + R/N)$ degenerates in the limit $R \to \infty$.

Proof of Corollary 1.2. Follows from Theorem 1 and Corollary 2.2. □

2.3 Denote

$$W_{t,T}(z) = -\frac{2\pi i}{\log T} \frac{\xi'(\frac{1}{2} + i[t + \frac{2\pi z}{\log T}])}{\xi(\frac{1}{2} + i[t + \frac{2\pi z}{\log T}])}, \quad 0 \leq t \leq T.$$ 

Assuming the Riemann hypothesis, $W_{t,T}$ belongs to the Nevanlinna class. \(^2\)

We shall treat $t$ as a random variable uniformly chosen in $[0, T]$, and denote the corresponding random function by $W_T$. The next proposition is close to the results for the $\zeta$-function which were proved in [34] and [20].

Proposition 2.3. Assume the Riemann hypothesis. Let $T_n \to \infty$ be a sequence and $\Psi$ — a point process such that, for $t$ uniformly chosen in $[0, T_n]$, 

$$\sum_{\xi(\frac{1}{2} + iy) = 0} \delta(u - (\gamma - t) \frac{\log T_n}{2\pi}) \to \Psi \quad \text{in distribution,} \quad n \to \infty.$$ 

Then $W_{T_n} \to W_{\Psi}$ in distribution, where $W_{\Psi}$ is the unique random Nevanlinna function such that

$$W_{\Psi}^{-1}(\infty) = \Psi, \quad W_{\Psi}(i\infty) = i\pi,$$

and in particular,

$$\left( \sum_{\xi(\frac{1}{2} + iy) = 0} \delta(u - (\gamma - t) \frac{\log T_n}{2\pi}), \sum_{\xi'(\frac{1}{2} + iy) = 0} \delta(u - (\gamma' - t) \frac{\log T_n}{2\pi}) \right) \to (\Psi, \Psi'_0),$$

where $\Psi'_0 = W_{\Psi}^{-1}(0)$.

Remark. Assuming the Riemann hypothesis, it follows from the results of Fujii [24][25] that

$$\frac{1}{T} \int_{0}^{T} \left( N(t + \frac{2\pi R}{\log T}) - N(t) - R \right)^2 dt \leq C \log(e + R), \quad (26)$$

\(^2\)the property $W_{t,T} \in \mathcal{R}$ is independent of $t$ and $T$, and is in fact equivalent to the Riemann hypothesis.
therefore the family of point processes $W_T^{-1}(\infty)$ is precompact (i.e. for any finite interval $I$ the family of random variables $(W_T^{-1}(\infty) \cap I)$ is tight), and, moreover, any limit point $\Psi$ satisfies:

$$\mathbb{E}\# [\Psi \cap [-R, R]] = 2R, \quad \mathbb{E}\# [\Psi \cap [0, R]] - R^2 \leq C \log(e + R).$$

In particular, the conditions of [1] Theorem 4.1 are satisfied, and therefore the function $W_\Psi$ (uniquely) exists.

Proposition 2.3 implies

**Corollary 2.3.** Assume the Riemann hypothesis and the multiple correlation conjecture (3). Then

$$W_T \xrightarrow{\text{distr}} W, \quad \frac{\xi\left(\frac{1}{2} + it + \frac{2\pi z}{\log T}\right)}{\xi\left(\frac{1}{2} + it\right)} \xrightarrow{\text{distr}} \Phi(z), \quad T \to \infty$$

and consequently

$$\left( \sum_{\xi\left(\frac{1}{2} + iy\right) = 0} \delta(u - (\gamma - t) \frac{\log T}{2\pi}), \sum_{\xi'(\frac{1}{2} + iy') = 0} \delta(u - (\gamma' - t) \frac{\log T}{2\pi}) \right) \xrightarrow{\text{distr}} (\Xi, \Xi') \ .$$

**Proof of Corollary 1.3.** Follows from Theorem 1 and Corollary 2.3. □

The proof of Proposition 2.3 relies on the following lemma. Related results go back to the work of Selberg [58]. We essentially follow the argument in [34], relying on the work of Montgomery [50].

**Lemma.** Assuming the Riemann hypothesis, one has for any $R > 0$:

$$\lim_{T \to \infty} \int_0^T \frac{dt}{T} W_{t,T}(iR) = i\pi$$  \hspace{1cm} (27)

$$\limsup_{T \to \infty} \int_0^T \frac{dt}{T} |W_{t,T}(iR) - i\pi|^2 \leq \frac{C}{R^2} \ .$$  \hspace{1cm} (28)

**Proof of Proposition 2.3.** The convergence of $W_T$ to $\Phi$ follows from the lemma, in combination with the general criterion of Aizenman and Warzel [1, Theorem 6.1]; it implies the other two statements. □

**Proof of Lemma.** To prove (27), consider the integral of $\xi'/\xi$ along the closed contour $\Gamma$ composed of the segments connecting the points

$$\frac{1}{2} - \frac{R}{\log T' \frac{1}{2}} + \frac{R}{\log T'} \frac{1}{2} + \frac{R}{\log T} + Ti, \frac{1}{2} + \frac{R}{\log T} + T'i, \frac{1}{2} - \frac{R}{\log T} + Ti$$
Figure 2.1: The contour $\Gamma$ from the proof of Lemma 2.3

counterclockwise (see Figure 2.3), where $T'$ is the real number closest to $T$ such that there are no zeros of the $\xi$-function in the $1/(100 \log T)$-neighbourhood of $\frac{1}{2} + iT$. By the residue theorem and the asymptotics (9) of $N(T)$, the integral is equal to

$$2\pi i \# \{ \text{zeros of } \xi \text{ with imaginary part in } [0, T'] \} = 2\pi i \frac{T \log T}{2\pi} (1 + o(1)).$$

On the other hand, from the functional equation $\xi(1 - z) = \xi(z)$, the integral along the left vertical line is equal to the integral along the right vertical line, and the integral along the bottom horizontal line is zero; the integral along the two segments on the top is negligible (as one can see, for example, from (30) below). Therefore

$$\int_0^T \left( \frac{1}{2} + i(t + iR/\log T) \right) dt = -\pi \frac{T \log T}{2\pi} (1 + o(1)).$$

which is equivalent to (27). We note for the sequel that (27) implies the following smoothened version:

$$\lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{T dt}{\pi(t^2 + T^2)} W_{t,T}(iR) = i\pi. \quad (29)$$

To prove (28), we use the Hadamard product representation (cf. [63, 2.12])

$$\xi(\frac{1}{2} + iz) = \xi(\frac{1}{2}) e^{\hat{b}z} \prod_{\xi(\frac{1}{2} + iy) = 0} (1 - z/\gamma) e^{z/\gamma},$$

which implies that

$$W_{t,T}(iR) = \sum_{\gamma} \left[ \frac{1}{(\gamma - t)^{\log T} - iR} - \frac{1}{\gamma^{\log T} - iR} \right] - \frac{2\pi \hat{b}}{\log T}. \quad (30)$$
Integrating with the weight \( T/(\pi(t^2 + T^2)) \) and using (29) and the Cauchy theorem, we obtain:

\[
\lim_{T \to \infty} \sum_{\gamma} \left[ \frac{1}{(\gamma - iT) \log T - iR} - \frac{1}{\gamma \log T} \right] = \frac{i}{2}.
\]  

(31)

Note that (31) holds for any real \( R \) (positive or negative).

Now we compute

\[
I(R, T) = \int_{-\infty}^{\infty} \frac{T dt}{\pi(t^2 + T^2)} \left| W_{t,T}(iR) \right| \frac{\hat{b}}{2 \log T} \right|^2 = \sum_{\gamma, \tilde{\gamma}} I_{\gamma, \tilde{\gamma}}(R, T),
\]

where

\[
I_{\gamma, \tilde{\gamma}}(R, T) = \int_{-\infty}^{\infty} \frac{T dt}{\pi(t^2 + T^2)} \left[ \frac{1}{(\gamma - t) \log T - iR} - \frac{1}{\gamma \log T} \right]
\]

\[
\times \left[ \frac{1}{(\tilde{\gamma} - t) \log T + iR} - \frac{1}{\tilde{\gamma} \log T} \right].
\]

By the Cauchy theorem,

\[
I_{\gamma, \tilde{\gamma}}(R, T) = I'_{\gamma, \tilde{\gamma}}(R, T) + I''_{\gamma, \tilde{\gamma}}(R, T),
\]

where

\[
I'_{\gamma, \tilde{\gamma}}(R, T) = \left[ \frac{1}{(\gamma - iT) \log T - iR} - \frac{1}{\gamma \log T} \right] \left[ \frac{1}{(\tilde{\gamma} - iT) \log T + iR} - \frac{1}{\tilde{\gamma} \log T} \right]
\]

\[
I''_{\gamma, \tilde{\gamma}}(R, T) = \frac{-2iT}{T^2 + (\gamma + \frac{iR}{\log T})^2} \left[ \frac{1}{(\gamma - \tilde{\gamma}) \log T - 2iR} - \frac{1}{\gamma \log T} \right] \frac{1}{\log T}.
\]

In view of (31),

\[
\lim_{T \to \infty} \sum_{\gamma, \tilde{\gamma}} I'_{\gamma, \tilde{\gamma}}(R, T) = (i/2)^2 = -1/4.
\]  

(32)

To estimate the sum of \( I''_{\gamma, \tilde{\gamma}}(R, T) \), let

\[
J_{\gamma, \tilde{\gamma}}(R, T) = \frac{2T}{T^2 + (\gamma + \frac{iR}{\log T})^2} \left( \frac{2R}{(\gamma - \tilde{\gamma})^2} \right) \frac{1}{\log T}.
\]

\[
J'_{\gamma, \tilde{\gamma}}(R, T) = \Re I''_{\gamma, \tilde{\gamma}}(R, T) - J_{\gamma, \tilde{\gamma}}(R, T).
\]

Using the estimates

\[
\left| \frac{1}{T^2 + (\gamma + \frac{iR}{\log T})^2} - \frac{1}{T^2 + (\gamma)^2} \right| \leq \frac{3R}{(T^2 + (\gamma)^2)^2}
\]

\[
\left| \frac{1}{(\gamma - \tilde{\gamma}) \log T - 2iR} - \frac{1}{\gamma \log T} \right| \leq \frac{\sqrt{2}(|\gamma| \log T + 2R)}{|\gamma| \log T(|\gamma - \tilde{\gamma}| \log T + 2R)}.
\]
we deduce that
\[
\lim_{T \to \infty} \sum_{\gamma, \tilde{\gamma}} J'_{\gamma, \tilde{\gamma}}(R, T) = 0 .
\] (33)

Now we turn to \( \sum_{\gamma, \tilde{\gamma}} J'_{\gamma, \tilde{\gamma}}(R, T) \) and show that
\[
\limsup_{T \to \infty} \left| \sum_{\gamma, \tilde{\gamma}} J'_{\gamma, \tilde{\gamma}}(R, T) - \frac{1}{2} \right| \leq \frac{\text{Const}_{R^2}}{R^2} .
\] (34)

It will suffice to prove that
\[
\limsup_{T \to \infty} \left| \frac{2\pi}{T \log T} \sum_{0 \leq \gamma, \tilde{\gamma} \leq T} \frac{2R}{(\gamma - \tilde{\gamma})^2 \log^2 T + 4R^2} - 1 \right| \leq \frac{\text{Const}_{R^2}}{R^2} .
\] (35)

Let
\[
F(\alpha, T) = \frac{2\pi}{T \log T} \sum_{0 \leq \gamma, \tilde{\gamma} \leq T} T^{i\alpha(\gamma - \tilde{\gamma})} w(\gamma - \tilde{\gamma}) , \quad w(u) = \frac{4}{4 + u^2} .
\] (36)

Montgomery showed \[50\] that
\[
F(\alpha, T) = |\alpha| + T^{-2|\alpha|} \log T (1 + o(1)) + o(1) , \quad |\alpha| < 1
\] (37)
(which implies \([10]\)). Therefore
\[
\sup \int_x^{x+1} F(\alpha, T) d\alpha \leq C
\] (38)
(see \[32,33\], where this is proved with \( C = \frac{8}{3} + \epsilon \) and \( \frac{29}{12} + \epsilon \), respectively). From the definition \( (36) \) of \( F \) we have (cf. \[34\], (2.11)):
\[
\frac{2\pi}{T \log T} \sum_{0 \leq \gamma, \tilde{\gamma} \leq T} \frac{2R}{(\gamma - \tilde{\gamma})^2 \log^2 T + 4R^2} w(\gamma - \tilde{\gamma}) = \int_0^\infty F(\alpha, T) e^{-2R|\alpha|} d\alpha ,
\] (39)
and thus, from \( (37) \) and \( (38) \),
\[
\limsup_{T \to \infty} \left| \frac{2\pi}{T \log T} \sum_{0 \leq \gamma, \tilde{\gamma} \leq T} \frac{4R}{(\gamma - \tilde{\gamma})^2 \log^2 T + 4R^2} w(\gamma - \tilde{\gamma}) - 1 \right| \leq \frac{C}{R^2} .
\] (40)

Observing that
\[
\lim_{T \to \infty} \frac{2\pi}{T \log T} \sum_{0 \leq \gamma, \tilde{\gamma} \leq T} \frac{4R}{(\gamma - \tilde{\gamma})^2 \log^2 T + 4R^2} (1 - w(\gamma - \tilde{\gamma})) = 0 ,
\]
we obtain (35) and thus (34). The relations (32), (33) and (34) imply that
\[
\lim_{R \to \infty} \limsup_{T \to \infty} |I(R, T) - 1/4| = 0
\]
and hence (28) holds. □

3 Repulsion

In this section we prove Theorems 1 and 2 and Corollary 1.1, the (re-)formulation of which we recall for the convenience of the reader. Let
\[
\Omega_k(\Xi_i, e) = \{ \# [\Xi_i \cap (-e, e)] \geq k \}, \quad \Omega_k(\Xi_i', e) = \{ \# [\Xi_i' \cap (-e, e)] \geq k \}.
\]

**Theorem 1'.** For any \( a \in \mathbb{R} \), \( k \geq 2 \), \( 0 < \epsilon < \frac{1}{8 \max(|a|, 2e)} \) and \( R \geq 5 \)
\[
\mathbb{P} \left( \Omega_k(\Xi_i', e) \setminus \left( \Omega_{k+1}(\Xi_i, (1 + \frac{4}{k-1})e) \cup \Omega_{k+2}(\Xi_i, Re) \right) \right) \leq 2 \exp\left( -\frac{kR}{64} \right).
\]

Taking \( R = 1000k \log \frac{1}{\epsilon} \) and using (5), we obtain the version of Theorem 1 stated in the introduction.

**Corollary 1.1.** For any \( k \geq 2 \) there exists a limit
\[
e_k' = \lim_{\epsilon \to 0} \frac{\mathbb{P}(\Omega_k(\Xi_i', e))}{e^{(k+1)^2}} \in (e_{k+1}, (1 + \frac{4}{k-1})e_{k+1}),
\]
independent of \( a \in \mathbb{R} \).

Next, denote
\[
\Omega_k(\xi, T, e) = \left\{ 0 \leq t \leq T \left| \# \left[ \gamma \in (t - \frac{2\pi e}{\log T}, t + \frac{2\pi e}{\log T}), \xi(\frac{1}{2} + iy) = 0 \right] \geq k \right. \right\}
\]
\[
\Omega_k(\xi', T, e) = \left\{ 0 \leq t \leq T \left| \# \left[ \gamma' \in (t - \frac{2\pi e}{\log T}, t + \frac{2\pi e}{\log T}), \xi'(\frac{1}{2} + iy') = 0 \right] \geq k \right. \right\}
\]

**Theorem 2'.** Assume the Riemann hypothesis. For any \( k \geq 2 \), \( 0 < \epsilon < 1 \), \( R \geq 1 + \frac{4}{k-1} \)
\[
\frac{1}{T} \text{mes} \left( \Omega_k(\xi', T, e) \setminus \left( \Omega_{k+1}(\xi, T, (1 + \frac{4}{k-1})e) \cup \Omega_{k+2}(\xi, T, Re) \right) \right) \leq \frac{C}{e^{ckR}}.
\]

**Remark.** The coefficient \( 1 + \frac{4}{k-1} \) in these results can be further improved to \( 1 + \frac{2}{k} + o(1) \).
3.1

Proof of Theorem 3.1. By Proposition 2.2, the theorem is equivalent to Corollary 1.2, and, moreover, to its special case pertaining to one (arbitrary) ensemble of random matrices; we choose the Gaussian Unitary Ensemble. (We could equally work directly with \( \Xi i \); in that case we would need to regularise all the sums.) Denote

\[ x_{j,N} = (\lambda_{j,N} - E)N\rho(E), \quad x'_{j,N} = (\lambda'_{j,N} - E)N\rho(E), \]

where \( a = -\frac{\pi E}{\sqrt{4 - E^2}} \). Let us first show that for any \( R \geq 1 + 4/(k - 1) \)

\[ \Omega_k(\Xi i'_a, \epsilon) \setminus \left( \Omega_{k+1}(\Xi i_a, (1 + \frac{4}{k-1})\epsilon) \cup \Omega_{k+2}(\Xi i_a, R\epsilon) \right) \]

\[ \subset \left\{ \sum_{|x_{j,N} - \epsilon| \geq (R-1)\epsilon} \frac{1}{x_{j,N} - \epsilon} \geq \frac{k - 1}{4\epsilon} \right\} \cup \left\{ \sum_{|x_{j,N} + \epsilon| \geq (R-1)\epsilon} \frac{1}{x_{j,N} + \epsilon} \leq -\frac{k - 1}{4\epsilon} \right\} \] (41)

Indeed, assume that

\[ \# [ |x'_{j,N}| < \epsilon ] \geq k, \quad \# [ |x_{j,N}| < (1 + \frac{4}{k-1})\epsilon ] \leq k. \] (42)

Then we have by interlacing:

\[ \# \left[ x_{j,N} \in (-\epsilon, \epsilon) \right] = k - 1, \] (43)

and on the other hand there are no \( x_{j,N} \) at least in one of the intervals

\[ (-(1 + \frac{4}{k-1})\epsilon, -\epsilon), \quad (+\epsilon, +(1 + \frac{4}{k-1})\epsilon); \]

for example, in the second one. Then

\[ \sum_{j=1}^{N} \frac{1}{x_{j,N} - \epsilon} \geq 0, \]

whence by (43) and (42)

\[ \sum_{|x_{j,N} - \epsilon| \geq \frac{1}{k-1}\epsilon} \frac{1}{x_{j,N} - \epsilon} \geq \sum_{|x_{j,N}| < \epsilon} \frac{-1}{x_{j,N} - \epsilon} \geq \frac{k - 1}{2\epsilon}. \]
Every $x_{j,N}$ contributes at most $(k - 1)/(4\epsilon)$ to the sum on the left-hand side, therefore either $\#[(1 + \frac{4}{k-1})\epsilon \leq x_{j,N} < R\epsilon] \geq 2$ or

$$\sum_{|x_{j,N}-\epsilon|\geq(R-1)\epsilon} \frac{1}{x_{j,N}-\epsilon} \geq \frac{k-1}{4\epsilon}.$$  

This proves (41), and it remains to bound the probability of the two terms on the right-hand side. By symmetry, we can focus on the first term, for which we use the following lemma, proved below (see e.g. Breuer–Duits [9, Theorem 3.1] for more sophisticated bounds):

**Lemma.** For any determinantal process $\mathcal{D}$ with self-adjoint kernel of finite rank and any (bounded Borel measurable) test function $f$,

$$\mathbb{P}\left\{ \sum_{x \in \mathcal{D}} f(x) \geq \mathbb{E} \sum_{x \in \mathcal{D}} f(x) + r \right\} \leq \exp(-A^*(r)), \quad r > 0,$$

where

$$A^*(r) = \sup_{t \leq \|f\|_\infty^{-1}} (rt - A(t)), \quad A(t) = \frac{et^2}{2} \mathbb{E} \sum_{x \in \mathcal{D}} f(x)^2. \quad (44)$$

We apply the lemma to

$$f(x) = \mathbb{1}_{|x-\epsilon|\geq(R-1)\epsilon} \frac{1}{x-\epsilon}, \quad F_N = \sum f(x_{j,N}),$$

then, denoting by $\rho_N(E) = \frac{1}{N} \frac{d}{dE} \mathbb{E} \# \{ \lambda_{j,N} \leq E \}$ the mean density of eigenvalues, we have:

$$\mathbb{E} F_N = \int_{|x|\geq(R-1)\epsilon} \frac{\rho_N(E + \frac{x+\epsilon}{N\rho(E)})}{x} dx = \int_{|x|\geq(R-1)\epsilon} \left[ \mathbb{1}_{|x|\geq(R-1)\epsilon} \frac{x}{x^2+1} - \frac{x}{x^2+1} \right] \rho_N(E + \frac{x+\epsilon}{N\rho(E)}) dx + \int \frac{x}{x^2+1} \rho_N(E + \frac{x+\epsilon}{N\rho(E)}) dx.$$

The first addend tends to zero since $\rho_N \to \rho$ uniformly, whereas the second addend tends to $-\frac{\pi E}{\sqrt{4-E^2}}$ by (23). Therefore

$$\lim_{N \to \infty} \mathbb{E} F_N = -\frac{\pi E}{\sqrt{4-E^2}}.$$

Next, for $t \leq R\epsilon$

$$\lim_{N \to \infty} A(t) = \frac{et^2}{2} \lim_{N \to \infty} \mathbb{E} \sum_{x \in \mathcal{D}} f(x_{j,N})^2 = \frac{et^2}{2} \lim_{N \to \infty} \int_{|x|\geq R\epsilon} \frac{dx}{|x-\epsilon|^2} \leq \frac{et^2}{(R-1)\epsilon}.$$
whence, taking $t = (R - 1)e$ in the definition (44) of $A^*$ and assuming that $e < 1/(16e)$,

$$\lim_{N \to \infty} A^* \left( \frac{k - 1}{8e} \right) \geq \frac{k - 1}{8e} (R - 1)e - e(R - 1)e \geq \frac{(k - 1)(R - 1)}{16}.$$ 

According to the lemma, we have for $e \leq \min\left(\frac{\sqrt{4 - E^2}}{8\pi|E|}, \frac{1}{16e}\right)$:

$$P\left\{ F_N \geq \frac{k - 1}{4e} \right\} \leq \exp\left\{ -\frac{(k - 1)(R - 1)}{16} \right\} \leq \exp\left\{ -\frac{kR}{64} \right\},$$

as claimed.

\begin{proof}[Proof of Lemma] Let $K$ denote the operator defining the determinantal process; then for $t\|f\|_\infty \leq 1$

$$\log \mathbb{E} \exp \left\{ t \sum_{x \in \mathcal{D}} f(x) \right\} = \log \det(1 + (e^t f - 1)K) = \text{tr} \log(1 + (e^t f - 1)K)$$

$$\leq \text{tr}(e^t f - 1)K \leq \frac{e t^2}{2} \text{tr} f^2 K + t \text{ tr} f K$$

$$\leq A(t) + t \mathbb{E} \sum_{x \in \mathcal{D}} f(x).$$

Therefore by the Chebyshev inequality

$$P\left\{ \sum_{x \in \mathcal{D}} f(x) \geq \mathbb{E} \sum_{x \in \mathcal{D}} f(x) + r \right\} \leq \exp(A(t) - rt).$$

\end{proof}

### 3.2

\begin{proof}[Proof of Corollary 1.1'] Let us show that the limit

$$c'_k = \lim_{e \to 0} \frac{P(\Omega_k(\Xi_{\mathcal{D}}', \epsilon))}{e^{(k+1)^2}}, \quad k \geq 2$$

exists and does not depend on $a$. Choose $\alpha_k > \alpha'_k > 0$ sufficiently small to ensure that

$$(1 - \alpha_k)(k + 2)^2 > (k + 1)^2.$$
Denote by $\mathcal{Y}(\varepsilon)$ the event

\[ \exists \text{ pairwise distinct distinct } X = (x_1, \ldots, x_{k+1}) \in (\Xi i \cap (-\varepsilon^{1-a_k}, \varepsilon^{1-a_k}))^{k+1} , \]

such that all the zeros of $P_X(x) = \frac{d}{dx} \prod_{j=1}^{k+1} (x - x_j)$ lie in $(-\varepsilon, \varepsilon)$ . (47)

Let us show that

\[ \mathbb{P}(\mathcal{Y}(\varepsilon - e^{1+a'_k}) \setminus \Omega_k(\Xi i', e)) \), \quad \mathbb{P}(\Omega_k(\Xi i', e) \setminus \mathcal{Y}(\varepsilon + e^{1+a'_k})) = o(e^{(k+1)^2}) . \] (48)

Indeed, on the event $\mathcal{Y}(\varepsilon - e^{1+a'_k}) \setminus (\Omega_k(\Xi i', e) \cup \Omega_{k+2}(\Xi i, e^{1-a_k}))$ at least one of the $x_j$ lies outside $(-\varepsilon, \varepsilon)$; assume for example that $x_{k+1} > \varepsilon$ and decompose

\[ 0 > W(\varepsilon) = \sum_{j=1}^{k+1} \frac{1}{x_j - \varepsilon} + \lim_{r \to \infty} \sum_{x \in \Xi i : e^{1-a_k} \leq |x| \leq r} \frac{1}{x - \varepsilon} . \]

Then the first term is bounded from below by

\[ \sum_{j=1}^{k+1} \frac{1}{x_j - \varepsilon} \geq \sum_{j=1}^{k+1} \frac{1}{x_j - \varepsilon - e^{1+a'_k}} + \frac{(k+1)e^{1+a'_k}}{4\varepsilon^2} \geq \frac{k+1}{4e^{1-a'_k}} . \]

The probability of the event

\[ \lim_{r \to +\infty} \sum_{x \in \Xi i : e^{1-a_k} \leq |x| \leq r} \frac{1}{x - \varepsilon} \leq -\frac{k+1}{4e^{1-a'_k}} \]

is $o(e^{(k+1)^2})$ by a tail estimate which follows from the Lemma in Section 3.1 (similarly to the proof of Theorem 1), whereas $\mathbb{P}(\Omega_{k+2}(\Xi i, e^{1-a_k})) = o(e^{(k+1)^2})$ by the condition (46) on $a_k$ and (5). This proves the first part of (48); the second part is proved in a similar way.

From (1) and the asymptotics

\[ \det \left( \frac{\sin \pi (x_j - x_m)}{\pi (x_j - x_m)} \right)_{j,m=1}^{k} = (1 + o(1))c_{k,1} \prod_{j < m} (x_j - x_m)^2 \quad x \to 0 \]

(where $c_{k,1}$ as well as $c_{k,2}$ and $c_{k,3}$ below are numerical constants), we obtain

\[ \mathbb{P}(\mathcal{Y}(\varepsilon)) = (c_{k,2} + o(1))e \int_{\sum x_j = 0} d^k X \prod_{j < m} (x_j - x_m)^2 \mathbbm{1} \{ \text{the zeros of } P_X \text{ lie in } (-\varepsilon, \varepsilon) \} \]

\[ = (c_{k,2} + o(1))c_{k,3} e^{(k+1)^2} , \]
which implies, with (48), that (45) holds with \(c_k' = c_{k,2}c_{k,3}\). Note that the indicator under the integral is compactly supported.

The bound

\[
e_k' \in \left[ e_{k+1}, (1 + \frac{4}{k-1})e_{k+1}(k+1)^2 \right]
\]

follows from Theorem 1' and (5).

3.3 The proof of Theorem 2 relies on a bound, proved by Rodgers [56], on the moments of the logarithmic derivative of the \(\zeta\)-function; see (52) below. The results of Farmer, Gonek, Lee and Lester [20] imply a more precise bound under additional hypotheses.

Proof of Theorem 2'. Let \(\frac{1}{2} + iy_j\) be the zeros of \(\xi(z)\); rescale them as follows:

\[
x_{j,T} = (y_j - t)\frac{\log T}{2\pi}.
\]

We start with the following counterpart of (41):

\[
\frac{1}{T} \text{mes} \left( \Omega_k(\xi', T, e) \setminus \left( \Omega_{k+1}(\xi, T, (1 + \frac{4}{k-1})e) \cup \Omega_{k+2}(\xi, T, R e) \right) \right)
\]

\[
\leq \frac{1}{T} \text{mes} \left( \left\{ W_{t,T}(e) - \sum_{|x_{j,T}-\epsilon|<(R-1)e} \frac{1}{x_{j,t,T}-\epsilon} \geq \frac{k-1}{4\epsilon} \right\} \setminus \Omega_{k+2}(\xi, T, R e) \right)
\]

\[
+ \frac{1}{T} \text{mes} \left( \left\{ W_{t,T}(-\epsilon) - \sum_{|x_{j,T}+\epsilon|<(R-1)e} \frac{1}{x_{j,t,T}+\epsilon} \leq -\frac{k-1}{4\epsilon} \right\} \setminus \Omega_{k+2}(\xi, T, R e) \right).
\]

To show that each of the two terms is bounded by \(\frac{C}{e^{ckR}}\), it will suffice to prove that

\[
\frac{1}{T} \text{mes} \left( \left\{ \left| W_{t,T}(0) - \sum_{|x_{j,t}|<Re} \frac{1}{x_{j,t,T}} \right| \geq \frac{k-1}{4\epsilon} \right\} \setminus \Omega_{k+2}(\xi, T, R e) \right) \leq \frac{C}{e^{ckR}}. \tag{49}
\]

Decompose

\[
W_{t,T}(0) - \sum_{|x_{j,T}|<Re} \frac{1}{x_{j,t,T}}
\]

\[
= \Re W_{t,T}(20iRe) - \sum_{|x_{j,T}|<Re} \frac{x_{j,t,T}}{x_{j,t,T}^2 + 400R^2\epsilon^2} + \sum_{|x_{j,T}|\geq Re} \frac{400R^2\epsilon^2}{x_{j,t,T}(x_{j,t,T}^2 + 400R^2\epsilon^2)}
\]

and observe that on the complement of \(\Omega_{k+2}(\xi, T, R e)\)

\[
\left| \sum_{|x_{j,T}|<Re} \frac{x_{j,t,T}}{x_{j,t,T}^2 + 400R^2\epsilon^2} \right| \leq \frac{k+1}{20Re} \leq \frac{k-1}{6\epsilon}. \tag{50}
\]
whereas

\[ |\Re W_{t,T}(20iRe)|, \quad \frac{1}{20} \left| \sum_{|x_i| \geq k-1} \frac{400R^2e^2}{x_i^2(x_i^2 + 400R^2e^2)} \right| \leq |W_{t,T}(20iRe)|, \]

whence we need to show that

\[ \frac{1}{T} \text{mes} \left\{ 0 \leq t \leq T \mid |W_{t,T}(20iRe)| \geq \frac{k-1}{480e} \right\} \leq C \exp(-ckR). \quad (51) \]

By a result of Rodgers \[56\], Theorem 2.1,

\[ \int_0^T \left| \frac{\xi'(1/2 + i(t + \frac{2\pi i\delta}{\log T}))}{\xi(1/2 + i(t + \frac{2\pi i\delta}{\log T}))} \right|^m dt \leq C^m \log^m T \log^m T, \quad (52) \]

hence

\[ \int_0^T \left| \frac{\xi'(1/2 + i(t + \frac{2\pi i\delta}{\log T}))}{\xi(1/2 + i(t + \frac{2\pi i\delta}{\log T}))} \right|^m dt \leq C^m \log^m \delta^{-m} T \log^m T, \]

and, finally,

\[ \frac{1}{T} \int_0^T |W_{t,T}(20iRe)|^m dt \leq C^m \log^m (R\epsilon)^{-m}, \quad (53) \]

from which (51) follows by the Chebyshev inequality. \(\square\)

3.4 We conclude with several questions and comments.

3.4.1. Other random matrix ensembles. The results of this paper have counterparts for other random matrix ensembles (and other \(\xi\)-functions). In particular, Chhaibi, Najnudel and Nikeghbali \[12\] showed that the characteristic polynomial \(Z_N(z) = \det(1_N - zU_N^*)\) of the Circular Unitary Ensemble satisfies:

\[ \frac{Z_N(\exp(2\pi iz/N))}{Z_N(1)} \xrightarrow{\text{distr}} \exp(i\pi z)\Phi(z). \]

Hence the collection of critical points of \(Z_N(z)z^{-N/2}\) converges, after rescaling by \(N\), to the same process \(\mathcal{S}_0\) as in Corollaries 2.2 and 2.3. This is consistent with the discussion in \[19\] Sections 2.3 and 6.2.
3.4.2. Critical points versus submatrix eigenvalues. Let

\[ W^{\text{dec}}(z) = \lim_{R \to \infty} \sum_{x \in \mathbb{Z} \cap (-R, R)} \frac{|g_x|^2}{x - z}, \]

where, conditionally on \( \mathbb{Z} \), the random variables \( g_x \) are independent complex standard Gaussian. Then Proposition 2.2 has the following counterpart:

\[
\left( \{ \lambda_{j,N} - E \} N \rho(E) \right), \{ \{ \lambda_{j,N-1} - E \} N \rho(E) \}
\rightarrow \left( (W^{\text{dec}})^{-1}( -\frac{\pi E}{\sqrt{4 - E^2}} ), W^{-1}(\infty) \right).
\]

(Amusingly, the distribution of the first term does not depend on \( E \).) We are not sure whether (54) has a number-theoretic analogue.

3.4.3. Form factors. Suppose a point process \( \Psi \) is a limit point of the rescaled zeros of \( \xi \), as in Proposition 2.3. Then the rescaled critical points converge, along the same subsequence, to the point process \( \Psi' = W^{-1}_\Psi(0) \), the distribution of which is uniquely determined by that of \( \Psi \). By a result of Montgomery [50] and its extension by Hejhal [37] and Rudnick and Sarnak [57], the (multiple) form factors of \( \Psi \) coincide, in a restricted domain of momenta, with those of the sine process. In view of the research programme suggested by Farmer, Gonek and Lee in [19], it is natural to ask which constraints does this impose on the form factors of \( \Psi' \), and to compare these with the results of [19, 6].

It would also be interesting to compute the form factor of \( \mathbb{Z} \)'s and to check whether it coincides with the right-hand side of (8) for \( a = 0 \) and \( |\alpha| \leq 1 \). A possible starting point is the identity \( e_k(\lambda'_1, \cdots, \lambda'_{N-1}) = (1 - \frac{k}{N}) e_k(\lambda_1, \cdots, \lambda_N) \) relating the elementary symmetric functions in the critical points of a polynomial to the elementary symmetric functions in the zeros.

3.4.4. Zeros of higher derivatives. The result (8) of [19] was extended to higher derivatives of \( \xi \) in the Ph.D. thesis of Bian [6]. In this context, we mention that, if the Riemann hypothesis and the multiple correlation conjecture hold, Corollary 2.3 implies that for any \( k \geq 1 \)

\[
\sum_{\xi^{(k)}(1/2 + i\gamma''') = 0} \delta(u - (\gamma''' - t) \frac{\log T}{2\pi}) \xrightarrow{\text{distr}} \mathbb{Z}^{(k)}_0 = \{ \Phi^{(k)} = 0 \}.
\]

3.4.5. Logarithmic derivative. From Proposition 3.4 and [1] Theorems 2.3 and 6.2 of Aizenman–Warzel it follows, conditionally on the Riemann hypothesis,
that (for $t$ chosen uniformly in $[0, T]$) the rescaled logarithmic derivatives

$$
\frac{2i}{\log T} \frac{\xi'(1/2 + it)}{\xi(1/2 + it)}, \quad \frac{2i}{\log T} \frac{\zeta'(1/2 + it)}{\zeta(1/2 + it)} + i \xrightarrow{T \to \infty} \text{Cauchy},
$$

(55)

where the right-hand side is a standard (real) Cauchy random variable. For comparison, it was proved by Lester [44], following earlier results by Guo [36], that the distribution of $(\zeta'/\zeta)(\sigma(T) + it)$, where $|\sigma(T) - 1/2| \log T \to \infty$ and $|\sigma(T) - 1/2| \to 0$, is approximately (complex) Gaussian for large $T$.

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