Logarithmic Extensions to Inflation Universality Classes

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The values of, and connection between, the cosmological observables of the primordial power spectrum tilt $n_s$ and the inflationary tensor to scalar ratio $r$ are key guideposts to the physics of inflation. Universality classes can be defined for the tilt from the scale free value proportional to $1/N$, where $N$ is the number of e-folds. We examine the consequences of a $\ln N$ next to leading order correction rather than an expansion in $1/N$, or introducing a new parameter. While nominally this can lower $r$ for some too-high $r$ simple inflation models (e.g. large field models), there is an interesting cancellation preventing such models from coming back into favor. On the other branch of the universality class, near Starobinsky inflation, $r$ can be raised, making it easier to detect.

I. INTRODUCTION

Inflation – a period of cosmic acceleration in the very early and energetic universe – is on the brink of being tested in new detail through the detection of primordial gravitational waves. These tensor modes reveal the energy scale of inflation, and together with the scalar (density) perturbations give key indications of the inflationary physics. The tensor to scalar ratio $r$ and the scalar power spectrum slope $n_s$ form a parameter space with different classes of models lying in different regions of it.

A useful and intriguing approach is to explore universality classes rather than individual models of the inflationary potential or slow roll parameters. This has the philosophy that the scalar tilt $n_s - 1$ should be a function purely of the number of e-folds of inflation $N$, without additional scales entering.

Traditionally the ansatz has meant taking an expansion such that $n_s - 1 \sim 1/N$ plus higher order terms in powers of $1/N$. Here we keep the philosophy of $N$ being the determining factor, but allow for $\ln N$ corrections to the leading order – still without introducing any other scale. We explore the effects of the next to leading order term, in particular on the $n_s$-$r$ plane, and implications for experimental limits.

II. THE $n_s$-$r$ RELATION

For slow roll inflation one has a hierarchy of derivatives of the scalar field potential, or Hubble parameter, and from these one can compute the observables of the scalar tilt $n_s - 1$ and tensor to scalar ratio $r$. While one can work within a specific model of the potential $V(\phi)$ or expansion $H(\phi)$, many classes of inflation theory exhibit a universality relation $n_s(N)$. One can start instead with that more model independent approach as the ansatz, as advocated early by [1–5] and many others since then. (But see [6] for the limitations of such an ansatz.)

The tensor to scalar ratio $r$ is given by a differential equation under the slow roll assumption,

$$\frac{d \ln r}{dN} - \frac{r}{8} = n_s - 1. \tag{1}$$

The standard universality relation has

$$n_s - 1 = \frac{\alpha}{N}, \tag{2}$$

where $\alpha$ is of order one, and with this we can solve Eq. (1) for $r$.

A. $\alpha = \text{constant}$

When $\alpha$ is constant, one has the well known solution

$$r = \frac{8(\alpha - 1)}{N + cN^\alpha}, \tag{3}$$
where $c$ is a constant of integration. This gives two asymptotic branches, where $r \sim N^{-1}$ and where $r \sim N^{-\alpha}$. Since under slow roll $r = 16\epsilon$ and the slow roll parameter

$$
\epsilon = -d\ln H/dN = \frac{1}{2} \left( \frac{dV/d\phi}{V} \right)^2,
$$

then by converting $N$ to $\phi$,

$$
\phi = \int dN \sqrt{2}\epsilon,
$$

one can see that the first case corresponds to

$$
V(\phi) \sim \phi^{2(\alpha - 1)},
$$

and the second case is

$$
\begin{align*}
V(\phi) & \sim \phi^{(2-2\alpha)/(2-\alpha)} \quad [\alpha \neq 2] \\
V(\phi) & \sim \left(1 - e^{-\phi/2c} \right) \quad [\alpha = 2],
\end{align*}
$$

(rolling off the nearly flat plateau at large $\phi$ in the $\alpha = 2$ case).

### B. Logarithmic running of $\alpha$

However, $\alpha$ generally gets terms beyond the leading order constant term. For $\alpha(N)$, Eq. (1) can be solved to give

$$
\frac{r}{s} = \left[ -e^{f(dN'/N')} \alpha(N') \int dN' e^{-f(dN''/N'')} \alpha(N'') + c e^{f(dN'/N') \alpha(N')} \right]^{-1},
$$

where $c$ is again an integration constant and the integrals are evaluated up to a $N$ e-folds.

The point of universality is to not introduce any time dependence or scale other than the e-fold scale $N$. Thus we expect that higher order terms inducing a variation of $\alpha(N)$ should be in some series expansion in $N$ with coefficients of order one. The natural ansatz is $\alpha(N) = \alpha_0 + \alpha_1/N + \alpha_2/N^2 + \ldots$. This has been studied for many cases and arises from, for example, hilltop inflation [2].

One could also consider ln $N$ effects, and indeed Starobinsky inflation [7] has $\alpha(N) = 2 - 3 \ln N/N$. Since $\ln N/N > 1/N$ then we expect this correction to play a larger role in altering the relation in the $n_s - r$ plane than a simple $1/N$ next to leading order term. However, there is potentially an even larger next to leading order term: $1/\ln N > \ln N/N$. This is the term we consider in this paper:

$$
\alpha(N) = \alpha_0 + \alpha_1/(\ln N)^s,
$$

where the next to leading order term is suppressed by $(\ln N)^s$.

Using Eq. (9) this gives, for $s \neq 1$,

$$
\frac{r}{s} = \left[ -N^{\alpha_0} e^{[\alpha_1/(1-s)][\ln N]^{1-s}} \int dN N^{-\alpha_0} e^{-[\alpha_1/(1-s)][\ln N]^{1-s}} + c N^{\alpha_0} e^{[\alpha_1/(1-s)][\ln N]^{1-s}} \right]^{-1}.
$$

The $s = 1$ case is simpler, with

$$
\frac{r}{s} = \left[ -N^{\alpha_0} (\ln N)^{\alpha_1} \int dN N^{-\alpha_0} (\ln N)^{-\alpha_1} + c N^{\alpha_0} (\ln N)^{\alpha_1} \right]^{-1}.
$$

For simplicity, we consider two analytic cases: Case A with $s = 1$ and $\alpha_1 = -1$, and Case B with arbitrary $s$ and $\alpha_0 = 1$.

Case A yields

$$
\frac{r}{s} = \left[ \frac{N}{(\alpha_0 - 1)} + \frac{N}{(\alpha_0 - 1)^2 \ln N} + c N^{\alpha_0} \ln N \right]^{-1}.
$$
This is interesting: the first branch gains an extra (positive) term in the denominator, lowering \( r \) for a given \( \alpha_0 \), while the second branch has a \( \ln N \) suppression, raising \( r \).

Figure 1 shows the behavior in the \( n_s-r \) plane for the standard relation (leading order term only), and for Case A with the next to leading order \( 1/\ln N \) term, i.e. Eq. (13). In the region of greatest interest, \( 0.96 \lesssim n_s \lesssim 0.97 \), the second branch dominates for \( c \sim 1 \) (here we took \( c = 1 \)). Indeed the next to leading order term enhances the value of \( r \) for a given \( n_s \), making inflationary gravitational waves easier to detect.

The next to leading order term gives a multiplicative enhancement of \( r \) on the second branch by a factor \( \ln N \approx 4 \) (we show results for \( N = 60 \)) for fixed \( \alpha_0 \). However, it shifts \( n_s \) as well, so the vertical and horizon displacements combine to give a smaller gain in \( r \), by \( (\ln N)/e \sim 1.5 \). Since the first branch has shallower slope, the horizontal shift actually erases the expected lowering of \( r \), giving a slight enhancement. In any case, this region of relatively high \( r \) is disfavored by data, especially after the new BICEP/KECK results, \( r < 0.036 \) at 95% confidence level [8].

To clarify the shift induced in \( r \) we plot the \( r-\alpha_0 \) plane for Case A in Fig. 2; this gives a purer picture of \( r \) but note that a given value of \( \alpha_0 \) will have differing values of \( n_s \) between the curves using leading order alone and those including the next to leading order. The factor of four enhancement for the second branch and the reduction by a factor \( 1 + 1/[(\alpha_0 - 1) \ln N] \) for the first branch are clear.

Turning to Case B, with arbitrary \( s \) but \( \alpha_0 = 1 \), gives

\[
\frac{r}{8} = \left[ \frac{N(\ln N)^s}{\alpha_1} + cN e^{[\alpha_1/(1-s)][(\ln N)^{1-s}]} \right]^{-1},
\]

or when \( s = 1 \),

\[
\frac{r}{8} = \left[ \frac{N \ln N}{\alpha_1 - 1} + cN (\ln N)^{\alpha_1} \right]^{-1}.
\]

For integer \( s > 1 \), we find \( n_s \) too high to be viable, and a low \( r \), so the \( s = 1 \) case is most relevant, with a large enough \( \alpha_1 \) to give a viable \( n_s - 1 = -1/N - \alpha_1/(N \ln N) \).
FIG. 2. As Fig. 1 for Case A but now in the $r$–$\alpha_0$ plane, so $n_s$ runs along the curves. Values $\alpha_0 \approx 2$ give $n_s \approx 0.96$–0.97.

Figure 3 shows the results for Case B ranging over $\alpha_1 = 1.05$–5. Again the second branch is the dominant one for viably small $r$. In fact, the constraint on $n_s$ forces smaller $r$ than in the previous models, with $n_s = 0.970$ (0.973) having $\alpha_1 = 3.2$ (2.5) and $r = 1.4 \times 10^{-3}$ (3.6 $\times 10^{-3}$).

III. CONCLUSIONS

The ansatz $n_s - 1 \sim 1/N$ is an attractive quasi model independent, or universality class, approach to inflation. It has the virtue that no scale enters different from order one other than the number of e-folds of inflation. Beyond leading order terms that also depend only on $N$ have the same property. Here we examined the largest possible next to leading order term, $1/\ln N$.

Logarithmic terms in the expansion also appear, though in a different form, in well known inflation theories such as Starobinsky inflation. Here we explore what impact this larger correction can have on the power spectrum tilt $n_s$ and tensor to scalar ratio $r$. Equation (10) gives our basic ansatz, and Eqs. (13) and (15) the solutions of most interest. For viable values of $n_s$ and $r$, the second branch is most relevant, and we find that for Case A the next to leading order term can make inflationary gravitational waves more detectable by increasing $r$ by a factor $\sim 1.5$ for a given measured value of $n_s$. 
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