Manifolds of algebraic elements in the algebra $\mathcal{L}(H)$ of bounded linear operators.

José M. Isidro ∗
Facultad de Matemáticas,
Universidad de Santiago,
Santiago de Compostela, Spain.
jmisidro@zmat.usc.es

October 10, 2001

Abstract

Given a complex Hilbert space $H$, we study the differential geometry of the manifold $\mathcal{A}$ of normal algebraic elements in $Z = \mathcal{L}(H)$. We represent $\mathcal{A}$ as a disjoint union of connected subsets $M$ of $Z$. Using the algebraic structure of $Z$, a torsionfree affine connection $\nabla$ (that is invariant under the group $\text{Aut}(Z)$ of automorphisms of $Z$) is defined on each of these connected components and the geodesics are computed. In case $M$ consists of elements that have a fixed finite rank $r$, $(0 < r < \infty)$, $\text{Aut}(Z)$-invariant Riemann and Kähler structures are defined on $M$ which in this way becomes a totally geodesic symmetric holomorphic manifold.

Keywords. JB$^*$-triples, Grassmann manifolds, Riemann manifolds.

AMS 2000 Subject Classification. 48G20, 72H51.

1 Introduction

In this paper we are concerned with the differential geometry of some infinite-dimensional Grassmann manifolds in $Z = \mathcal{L}(H)$, the space of bounded linear operators $z: H \to H$ in a complex Hilbert space $H$. Grassmann manifolds are a classical object in Differential Geometry and in recent years several authors have considered them in the Banach space setting. Besides the Grassmann structure, a Riemann and a Kähler structure has sometimes been defined even in the infinite-dimensional setting. Let us recall some aspects of the topic that are relevant for our purpose.

The study of the manifold of minimal projections in a finite-dimensional simple formally real Jordan algebra was made by U. Hirzebruch in [6], who proved that such a manifold is a compact symmetric Riemann space of rank 1, and that every such a space arises in this way. Later on, Nomura in [18, 19] established similar results for the manifold of fixed finite rank projections in a topologically simple real Jordan-Hilbert algebra. In [8], the authors studied the Riemann and Kähler structure of the manifold of finite rank projections in $Z$ without the use of any global scalar product. As pointed out there, the Jordan-Banach structure of $Z$ encodes information about the differential geometry of some manifolds naturally associated to it, one of which is the manifold of algebraic elements in $Z$. On the other hand, the Grassmann manifold of all projections in $Z$ has been discussed by Kaup in [10] and [13]. See also [1, 7] for related results.

It is therefore reasonable to ask whether a Riemann structure can be defined in the set of algebraic elements in $Z$, and how does it behave when it exists. We restrict our considerations to the set $\mathcal{A}$ of all normal algebraic elements in $Z$ that have finite rank. Remark that the assumption concerning the finiteness of the rank can not be dropped, as proved in [8]. Normality allows us to use spectral theory which is an essential tool. In the case $H = \mathbb{C}^n$, all elements in $Z$ are algebraic (as any square matrix is a root of its characteristic polynomial) and have finite rank. Under the above restrictions $\mathcal{A}$ is represented as a disjoint union of connected subsets $M$ of $Z$, each of which is

*Supported by Ministerio de Educación y Cultura of Spain, Research Project PB 98-1371.
invariant under Aut (Z) (the group of all C*–automorphisms of Z). Using algebraic tools, a holomorphic manifold structure and an Aut (Z)-invariant affine connection ∇ are introduced on M and its geodesics are calculated. One of the novelties is that we take JB*-triple system approach instead of the Jordan-algebra approach of [12, 19]. As noted in [1] and [7], within this context the algebraic structure of JB*-triple acts as a substitute for the Jordan algebra structure. In case M consists of elements that have a fixed finite rank r, (0 < r < ∞), the JB*-triple structure provides a local scalar product known as the algebraic metric of Harris ([3], prop. 9.12). Although Z is not a Hilbert space, the use of the algebraic scalar product allows us to define an Aut (Z)-invariant Riemann and a Kähler structure on M. We prove that ∇ is the Levi-Civita and the Kähler connection of M, and that M is a symmetric holomorphic manifold on which Aut (Z) acts transitively as a group of isometries.

The role that projections play in the study of the algebra Z = L(H) is taken by tripotents in the study of a JB*-triple system. A spectral calculus and a notion of algebraic element is available in the setting of JB*-triples, and the manifold of all finite rank algebraic elements in a JB*-triple Z is studied in the final section.

2 Algebraic preliminaries.

For a complex Banach space X denote by XR the underlying real Banach space, and let L (X) and LXR (X) respectively be the Banach algebra of all bounded complex-linear operators on X and the Banach algebra of all bounded real-linear operators on XR. A complex Banach space Z with a continuous mapping (a, b, c) → {abc} from Z × Z × Z to Z is called a JB*-triple if the following conditions are satisfied for all a, b, c, d ∈ Z, where the operator a□b ∈ L(Z) is defined by z → {abs} and [, ] is the commutator product:

1. {abc} is symmetric complex linear in a, c and conjugate linear in b.
2. [a□b, c□d] = {abc □ d − c□{dab}.
3. a□a is hermitian and has spectrum ≥ 0.
4. ∥{aaa}∥ = ∥a∥3.

If a complex vector space Z admits a JB*-triple structure, then the norm and the triple product determine each other. For x, y, z ∈ Z we write L(x, y)(z) = (x□y)(z) and Q(x, y)(z) := {xyz}. Note that L(x, y) ∈ L(Z) whereas Q(x, y) ∈ LXR(Z), and that the operators Lα = L(a, a) and Qα = Q(a, a) commute. A derivation of a JB*-triple Z is an element δ ∈ L(Z) such that δ{zzz} = {δz}zz + {z(δz)}z + {zz(δz)} and an automorphism is a bijection φ ∈ L(Z) such that φ{zzz} = {φz}{φz}(φz) for z ∈ Z. The latter occurs if and only if φ is a surjective linear isometry of Z. The group Aut (Z) of automorphisms of Z is a real Banach-Lie group whose Banach-Lie algebra is the connected component of the identity in Aut (Z) is denoted by Aut (Z). Two elements x, y ∈ Z are orthogonal if x□y = 0 and e ∈ Z is called a tripotent if {eee} = e, the set of which is denoted by Tri (Z). For e ∈ Tri (Z), the set of eigenvalues of e□e ∈ L(Z) is contained in {0, 1, 1} and the topological direct sum decomposition, called the Peirce decomposition of Z,

\[ Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e). \]  

holds. Here Zk(e) is the k- eigenspace of e□e and the Peirce projections are

\[ P_1(e) = Q^2(e), \quad P_{1/2}(e) = 2(e□e − Q^2(e)), \quad P_0(e) = \text{Id} − 2e□e + Q^2(e). \]

We will use the Peirce rules \{Z_l(e) | Z_{l+1}(e) \} ⊂ Z_{l−1}(e) where Z_l(e) = \{0\} for l ≠ 0, 1/2, 1. In particular, every Peirce space is a JB*-subtriple of Z and Z1(e)□Z0(e) = \{0\}. We note that Z1(e) is a complex unital JB*-algebra in the product a □ b: = \{aeb\} and involution a# := \{eae\}. Let

\[ A(e): = \{z ∈ Z_1(e) : z# = z\}. \]

Then we have Z1(e) = A(e) ⊕ iA(e). A tripotent e in a JB*-triple Z is said to be minimal if e ≠ 0 and P1(e)Z = C e, and we let Min (Z) be the set of them. If e ∈ Min (Z) then ∥e∥ = 1. A JB*-triple Z may have no non-zero tripotents.
Let \( e = (e_1, \cdots, e_n) \) be a finite sequence of non-zero mutually orthogonal tripotents \( e_j \in Z \), and define for all integers \( 0 \leq j, k \leq n \) the linear subspaces
\[
\begin{align*}
Z_{j,j}(e) &= Z_1(e_j) \quad 1 \leq j \leq n, \\
Z_{j,k}(e) &= Z_{k,j}(e) = Z_{1/2}(e_j) \cap Z_{1/2}(e_k) \quad 1 \leq j, k \leq n, \ j \neq k, \\
Z_{0,j}(e) &= Z_{1}(e_j) \cap \bigcap_{k \neq j} Z_0(e_k) \quad 1 \leq j \leq n, \\
Z_{0,0}(e) &= \bigcap_j Z_0(e_j).
\end{align*}
\]

Then the following topologically direct sum decomposition, called the Peirce decomposition relative to \( e \), holds
\[
Z = \bigoplus_{0 \leq j \leq k \leq n} Z_{j,k}(e).
\]

The Peirce spaces multiply according to the rules \( \{Z_{j,m}Z_{m,n}Z_{n,k}\} \subseteq Z_{j,k} \), and all products that cannot be brought to this form (after reflecting pairs of indices if necessary) vanish. In terms of this decomposition, the Peirce spaces of the tripotent \( e : = e_1 + \cdots + e_n \) are
\[
\begin{align*}
Z_1(e) &= \bigoplus_{j,k} Z_{j,k}(e) = \left( \bigoplus_{1 \leq j \leq n} Z_{j,j}(e) \right) \oplus \left( \bigoplus_{1 \leq j,k \leq n, j \neq k} Z_{j,k}(e) \right), \\
Z_{1/2}(e) &= \bigoplus_{1 \leq j \leq n} Z_{0,j}(e), \\
Z_0(e) &= Z_{0,0}(e).
\end{align*}
\]

Recall that every C*-algebra \( Z \) is a JB*-triple with respect to the triple product \( 2\{abc\} : = (ab^*c + cb^*a) \). In that case, every projection in \( Z \) is a tripotent and more generally the tripotents are precisely the partial isometries in \( Z \). C*-algebra derivations and C*–automorphisms are derivations and automorphisms of \( Z \) as a JB*-triple though the converse is not true.

We refer to [11], [13], [16], [20] and the references therein for the background of JB*-triples theory.

3 Manifolds of algebraic elements in \( \mathcal{L}(H) \).

From now on, \( Z \) will denote the C*-algebra \( \mathcal{L}(H) \). An element \( a \in Z \) is said to be algebraic if it satisfies the equation \( p(a) = 0 \) for some non identically null polynomial \( p \in \mathbb{C}[X] \). By elementary spectral theory \( \sigma(a) \), the spectrum of \( a \) in \( Z \), is a finite set whose elements are roots of the algebraic equation \( p(\lambda) = 0 \). In case \( a \) is normal we have
\[
a = \sum_{\lambda \in \sigma(a)} \lambda e_{\lambda}
\]
where \( \lambda \) and \( e_{\lambda} \) are, respectively, the spectral values and the corresponding spectral projections of \( a \). If \( 0 \in \sigma(a) \) then \( e_0 \), the projection onto \( \ker(a) \), satisfies \( e_0 \neq 0 \) but in \( \{3\} \) the summand \( 0e_0 \) is null and will be omitted. In particular, in [3] the numbers \( \lambda \) are non-zero pairwise distinct complex numbers and the \( e_{\lambda} \) are pairwise orthogonal non-zero projections. We say that \( a \) has finite rank if \( \dim a(H) < \infty \), which always occurs if \( \dim(H) < \infty \). Set \( r_{\lambda} : = \dim( e_{\lambda} \{0\} ) \). Then \( a \) has finite rank if and only if \( r_{\lambda} < \infty \) for all \( \lambda \in \sigma(a) \setminus \{0\} \) (the case \( 0 \in \sigma(a) \) and \( \dim \ker a = \infty \) may occur).

Thus, every finite rank normal algebraic element \( a \in Z \) gives rise to: (i) a positive integer \( n \) which is the cardinal of \( \sigma(a) \setminus \{0\} \), (ii) an ordered n-uple \((\lambda_1, \cdots, \lambda_n)\) of numbers in \( \mathbb{C} \setminus \{0\} \) which is the set of the pairwise distinct non-zero spectral values of \( a \), (iii) an ordered n-uple \((e_1, \cdots, e_n)\) of non-zero pairwise orthogonal projections, and (iii) an ordered n-uple \((r_1, \cdots, r_n)\) where \( r_k \in \mathbb{N} \setminus \{0\} \).

The spectral resolution of \( a \) is unique except for the order of the summands in [3], therefore these three n-uples are uniquely determined up to a permutation of the indices \((1, \cdots, n)\). The operator \( a \) can be recovered from the set of the first two ordered n-uples, \( a \) being given by [3].
Given the n-uples $\Lambda: = (\lambda_1, \cdots, \lambda_n)$ and $R: = (r_1, \cdots, r_n)$ in the above conditions, we let

$$M(n, \Lambda, R): = \{ \sum_{k} \lambda_k e_k : e_j e_k = 0 \text{ for } j \neq k, \text{ rank } (e_k) = r_k, 1 \leq j, k \leq n \}$$

be the set of the elements $[5]$ where the coefficients $\lambda_k$ and ranks $r_k$ are given and the $e_k$ range over non-zero, pairwise orthogonal projections of rank $r_k$. For instance, for $n = 1$, $\Lambda = \{1\}$ and $R = \{r\}$ we obtain the manifold of projections with a given finite rank $r$, that was studied in [8]. For the n-uple $\Lambda = (\lambda_1, \cdots, \lambda_n)$ we set $\Lambda^* = (\bar{\lambda}_1, \cdots, \bar{\lambda}_n)$. The involution $z \mapsto z^*$ on $Z$ induces a map $M(n, \Lambda, R) \rightarrow M(n, \Lambda^*, R)$ where $M(n, \Lambda, R)^* = \{z^*: z \in M\} = M(n, \Lambda^*, R)$, and $\Lambda \subset \mathbb{R}$ if and only if $M(n, \Lambda, R)$ consists of hermitian elements.

For a normal algebraic element $a = \sum_{\lambda \in [\sigma(a) \setminus \{0\}]} \lambda e_\lambda$ we define its support to be the projection $a = \text{supp } a: = \sum_{\lambda \in [\sigma(a) \setminus \{0\}]} e_\lambda = e_1 + \cdots + e_n$.

It is clear that $h(\text{supp } a) = \text{supp } h(a)$ holds for all $h \in \text{Aut}^0(Z)$, which combined with the $\text{Aut}^0(Z)$-invariance of Peirce projectors $P_k$ gives the following useful formula

$$P_k(\text{supp } h(a)) = P_k(\text{supp } h(a)) = h P_k(\text{supp } a) h^{-1}, \quad (k = 1, 1/2, 0).$$

**Proposition 3.1** Let $\mathcal{A}$ be the set of all normal algebraic elements of finite rank in $Z$, and let $M(n, \Lambda, R)$ be defined as in [2]. Then

$$\mathcal{A} = \bigcup_{n, \Lambda, R} M(n, \Lambda, R)$$

is a disjoint union of $\text{Aut}^0(Z)$-invariant connected subset of $Z$ on which the group $\text{Aut}^0(Z)$ acts transitively.

**Proof.**

We have seen before that $\mathcal{A} \subset \bigcup_{n, \Lambda, R} M(n, \Lambda, R)$. Conversely, let $a$ belong to some $M(n, \Lambda, R)$ hence we have $a = \sum_k \lambda_k e_k$ for some orthogonal projections $e_k$. Then $\text{Id} = (e_1 + \cdots + e_n) + f$ where $f$ is the projection onto $\ker(a)$ in case $0 \in \sigma(a)$ and $f = 0$ otherwise. The above properties of the $e_k, f$ yield easily $ap(a) = 0$ or $p(a) = 0$ according to the cases, where $p \in \mathbb{R}[X]$ is the polynomial $p(z) = (z - \lambda_1) \cdot \cdots \cdot (z - \lambda_n)$. Hence $a \in \mathcal{A}$. Clearly [2] is union of disjoint subsets.

Fix one of the sets $M: = M(n, \Lambda, R)$ and take any pair $a, b \in M$. Then

$$a = \lambda_1 p_1 + \cdots + \lambda_n p_n, \quad b = \lambda_1 q_1 + \cdots + \lambda_n q_n.$$ 

In case $0 \in \sigma(a)$, set $p_0: = \text{Id} - \sum_k p_k$ and $q_0: = \text{Id} - \sum_k q_k$. Since rank $p_k = \text{rank } q_k$, the projections $p_k$ and $q_k$ are unitarily equivalent and so are $p_0$ and $q_0$. Let us choose orthonormal basis $B_k^0$ and $B_k^0$ in the ranges $p_k(H)$ and $q_k(H)$ for $k = 0, 1, \cdots, n$. Then $\bigcup_k B_k^0$ and $\bigcup_k B_k^0$ are two orthonormal basis in $H$. The unitary operator $U \in Z$ that exchanges these basis satisfies $Ua = b$. In particular $M$ is the orbit of any of its points under the action of the unitary group of $H$. Since this group is connected and its action on $Z$ is continuous, $M$ is connected. \hfill \Box

Let $a \in Z$ be a normal algebraic element with finite rank and $a = \text{supp } a$ its support. In the Peirce decomposition

$$Z = Z_{1/2}(a) \oplus Z_{1/2}(a) \oplus Z_0(a)$$

every Peirce space $Z_k(a)_s$ is invariant under the natural involution $*$ of $Z$, and we let $Z_k(a)_s$ denote its selfadjoint part, $(k = 1, 1/2, 0)$. In what follows, the map $Z \times Z \rightarrow Z$ given by $(x, y) \mapsto g(a, x)y$, and the partial maps obtained by fixing one of the variables, will play an important role. For every fixed value $x \in Z_{1/2}(a)$, we get an operator $g(a, x)\cdot(\cdot)$ which is an inner JB$^*$-triple derivation of $Z$, hence we have an operator-valued continuous real-linear map $Z_{1/2}(a) \rightarrow \text{Der}(Z)$. Moreover $g(a, x)\cdot(\cdot)$ is a $C^*$-algebra derivation if and only if $x \in Z_{1/2}(a)_s$ (see [3,4]). For $y = a$ fixed, we get the map $x \mapsto g(a, x)a$ for which we introduce the notation

$$\Phi_a(x): = g(a, x)a = \{a x a\} - \{x a a\} = (Q(a, a) - L(a, a))x, \quad x \in Z.$$ 

First we discuss $Z_{1/2}(a)$.
Proposition 3.2. Let \( a \in Z \) be a normal algebraic element of finite rank, and let \( a = e_1 + \cdots + e_n \) be its support. Then \( Z_{1/2}(a) \) consists of the operators

\[
u = \sum_k u_k, \quad u_k \in Z_{1/2}(e_k), \quad e_k u_j = u_j e_k = 0, \quad j \neq k, \quad (1 \leq j, k \leq n).
\] (9)

If \( u \in Z_{1/2}(a) \), then we have the additional condition \( u_k \in Z_{1/2}(e_k) \).

**Proof.**

Let \( u \in Z \) be selfadjoint. The relation \( u \in Z_{1/2}(a) \) is equivalent to \( u = 2\{aa_u\} \) which now reads

\[
u = 2\{aa_u\} = aa^*u + uaa = \sum_k (e_k u + u e_k) = \sum_k u_k
\]

where

\[
u_k := e_k u + u e_k \quad \text{for} \quad 1 \leq k \leq n.
\] (10)

Note that \( e_j, e_k \in Z_1(a) \), hence by the Peirce multiplication rules \( \{e_j u e_k\} \in \{Z_1(a)Z_{1/2}(a)Z_1(a)\} = \{0\} \), that is \( e_j u e_k + e_k u e_j = 0 \) for all \( 1 \leq j, k \leq n \). Multiplying the latter by \( e_j \) with \( j \neq k \) yields \( e_j u e_k = 0 \) for \( j \neq k \), \( (1 \leq j, k \leq n) \). Therefore by (11)

\[
u = 2\{e_k e_k u_k\} = e_k (e_k u + u e_k) + (e_k u + u e_k) e_k + (e_k u + u e_k) = u_k
\]

which shows \( u_k \in Z_{1/2}(e_k) \) and clearly \( u_k = u_k^* \) for \( 1 \leq k \leq n \). Multiplying in (10) by \( e_j \) with \( j \neq k \) we get \( u_k e_j = e_j u_k = 0 \) and in particular \( e_j u e_k + u_k e_j = 0 \) for \( j \neq k \).

Conversely, let \( u_k \) satisfy the properties in (11). Then \( u = \sum_k u_k \) is selfadjoint and \( e_k u = e_k (\sum_j u_j) = e_k u_k \). Similarly \( u e_k = u_k e_k \), hence \( 2\{u u\} = aa^*u + u a a = (\sum_j e_j) u + u (\sum_j e_j) = (\sum_j (e_j u + u e_j)) = u \).

Using the \( \ast \)-invariance of \( Z_{1/2}(a) \) every element in this space can be written in the form \( u = u_1 + i u_2 \) with \( u_1, u_2 \in Z_{1/2}(a)_s \) and the result follows easily. \(\square\)

The following result should be compared with (11), th. 3.1

**Proposition 3.3** Let \( a \in Z \) be a normal algebraic element of finite rank and \( a := \text{supp}(a) \). Then for any \( u \in Z_{1/2}(a) \), the operator \( g(a, u) := a u - u a \) is an inner \( C^* \)-derivation of \( Z \) if and only if \( u \) is selfadjoint.

**Proof.**

Let \( a = \sum_k \lambda_k e_k \) and \( a = \sum_k e_k \) be the spectral resolution and the support of \( a \). Suppose \( u = u^* \). By (12) \( u \) has the form \( u = \sum k u_k \) with \( u_k \in Z_{1/2}(e_k)_s \) and \( e_k u_j = u_j e_k = 0 \) for all \( j \neq k \). Therefore

\[
u(g(a, u)) = \sum_k (e_k u_k - u_k e_k) = \sum_k g(e_k, u_k).
\] (11)

Here the \( e_k \) are projections in \( Z \) and \( u_k \in Z_{1/2}(e_k)_s \), hence by (12), th. 3.1) each \( g(e_k, u_k) \) is an inner \( C^* \)-derivation of \( Z \) and so is the sum. Conversely, since \( a \) is a projection, whenever \( g(a, u) \) is a \( C^* \)-algebra derivation we have \( u \in Z_{1/2}(a)_s \) again by (12), th. 3.1).

Now consider the joint Peirce decomposition of \( Z \) relative to the family \( (e_1, \cdots, e_n) \) where \( a = \lambda_1 e_1 + \cdots + \lambda_n e_n \) is the spectral resolution of \( a \). Remark that

\[
\sum_{1 \leq k \leq n} i A(e_k) \subset Z_1(a) \text{ is a direct summand of } Z, \text{ hence so is the space } X := \bigoplus_{1 \leq k \leq n} i A(e_k) \oplus Z_{1/2}(a).
\]

**Proposition 3.4** Let \( a \in Z \) be a normal algebraic element of finite rank and \( a := \text{supp}(a) \). Then \( \Phi_a \) is a surjective complex linear homeomorphism of \( Z_{1/2}(a) \). If \( a \) is hermitian then \( \Phi_a \) is a surjective real homeomorphism of \( X \) that preserves the subspace \( \bigoplus_{1 \leq k \leq n} i A(e_k) \).
PROOF.
Let $x = iv + u \in X$ where $v \in \bigoplus_{1 \leq k \leq n} A(e_k)$ and $u \in Z_{1/2}(a)$. The Peirce multiplication rules give for
$v = \sum_j v_j$ with $v_j \in A(e_j)$ and $u = \sum_k u_k$ according to \([3.2]\)
\[
\{a Z_{1/2}(a) a\} = \{0\},
\]
\[
\{a i v a\} = -i \{\sum_j e_j \sum_k v_k \sum_l \lambda_l e_l\} = -i \sum_k \lambda_k v_k,
\]
\[
\{u a a\} = i \{\sum_j u_j \sum_k e_k \sum_l \lambda_l e_l\} = \frac{i}{2} \sum_k \lambda_k u_k.
\]

Therefore
\[
\Phi_a(x) = -2i \sum_k \lambda_k v_k - \frac{1}{2} \sum_k \lambda_k u_k \in \left( \bigoplus_{1 \leq k \leq n} Z_{1}(e_k) \right) \oplus Z_{1/2}(a).
\] (12)

It is now clear that $\Phi_a$ preserves $Z_{1/2}(a)$. If $a$ is hermitian then $\Lambda \subset \mathbb{R}^n$ and $\Phi_a$ also preserves $\bigoplus_{1 \leq k \leq n} i A(e_k)$. Moreover $\Phi_a(x) = 0$ with $x \in X$ is equivalent to $\sum \lambda_k v_k = 0 = \sum \lambda_k u_k$ which is equivalent to $v = 0 = u$ since the coefficients satisfy $\lambda_k \in \sigma(a) \setminus \{0\}$. We can recover $x$ from $\Phi_a(x)$, hence the result follows. \( \square \)

Recall that a subset $M \subset Z$ is called a real analytic (respectively, holomorphic) submanifold if to every $a \in M$ there are open subsets $P, Q \subset Z$ and a closed real-linear (resp. complex) subspace $X \subset Z$ with $a \in P$ and
\[\phi(P \cap M) = Q \cap X\] for some bianalytic (resp. biholomorphic) map $\phi : P \to Q$. If to every $a \in M$ the linear subspace $X = T_a M$, called the tangent space to $M$ at $a$, can be chosen to be topologically complemented in $Z$ then $M$ is called a direct submanifold of $Z$.

Fix one of the sets $M = M(n, \Lambda, R)$ and a point $a \in M$ with spectral resolution $a = \sum_k \lambda_k e_k$. By the orthogonality properties of the $e_k$, the successive powers of $a$ have the expression
\[
a^l = \lambda_1^l e_1 + \cdots + \lambda_n^l e_n, \quad 1 \leq l \leq n,
\]
where the determinant $\det(\lambda_i^j) \neq 0$ does not vanish since it is a Vandermonde determinant and the $\lambda_k$ are pairwise distinct. Thus the $e_k$ are polynomials in $a$ whose coefficients are rational functions of the $\lambda_k$. Suppose $M$ is a differentiable manifold, and let us obtain its tangent space $T_a M$. Consider a smooth curve $t \mapsto a(t)$ through $a \in M$, $t \in I$, for a neighbourhood $I$ of $0 \in \mathbb{R}$ and $a(0) = a$. Each $a(t)$ has a spectral resolution
\[
a(t) = \lambda_1 e_1(t) + \cdots + \lambda_n e_n(t),
\]
therefore the maps $t \mapsto e_k(t)$, $(1 \leq k \leq n)$, are smooth curves in the manifolds $\mathbb{M}(r_k)$ of the projections in $Z$ that have fixed finite rank $r_k = \text{rank} (e_k)$, whose tangent spaces at $e_k = e_k(0)$ are $Z_{1/2}(e_k)$ (see \([4]\) or \([5]\)). Therefore
\[
\frac{d}{dt}|_{t=0} e_k(t) \in Z_{1/2}(e_k), \quad 1 \leq k \leq n.
\]
Since the spectral projections of $a(t)$ corresponding to different spectral values $\lambda_k \neq \lambda_j$ are orthogonal, we have $e_j(t) e_k(t) = 0$ for all $t \in I$, and taking the derivative at $t = 0$,
\[
e_j u_k = u_k e_j = 0, \quad j \neq k, \quad 1 \leq j, k \leq n.
\] (13)

By \([5]\), the tangent vector to $t \mapsto a(t)$ at $t = 0$, that is, $u : = \frac{d}{dt}|_{t=0} a(t) = \sum_k \lambda_k u_k$ satisfies
\[
\{a a u\} = \left\{ \sum_j e_j \sum_k e_k \sum_l \lambda_l u_l \right\} = \sum_{j,k,l} \lambda_l \left\{ e_j e_k u_l \right\} = \left( \sum_k \lambda_k \sum_l \lambda_l \sum_{j,k} \lambda_l e_j e_k u_l \right) = \frac{1}{2} \sum_{k,l} \lambda_l \{ e_k u_l \} + \frac{1}{2} \sum_{k,l} \lambda_l \{ e_l u_k \} = \frac{1}{2} \sum_{j,k,l} \lambda_l \{ e_j e_k u_l \} = \frac{1}{2} \sum_l \lambda_l u_l = \frac{1}{2} u
\]
hence $u \in Z_{1/2}(a)$, and $T_a M$ can be identified with a vector subspace of $Z_{1/2}(a)$. In fact $T_a M = Z_{1/2}(a)$ as it easily follows from the following result that should be compared with \([4]\) th. 3.3)
Theorem 3.5  The sets \( M = M(n, \Lambda, R) \) defined in (3) are holomorphic direct submanifolds of \( Z \). The tangent space at the point \( a \in M \) is the Peirce subspace \( Z_{1/2}(a) \) where \( a = \text{supp} (a) \), and a local chart at \( a \) given by

\[
f : u \mapsto f(u) := (\exp g(a, u))a
\]

with \( g(a, u) = a \Box u - u \Box a \).

Proof.

\( M \subset Z \) is invariant under \( \text{Aut}^\circ(Z) \). Fix any \( a \in M \) and let \( X := \left( \bigoplus_{1 \leq k \leq n} i A(e_k) \right) \oplus Z_{1/2}(a) \). Thus \( Z = X \oplus Y \) for a certain subspace \( Y \). The mapping \( X \oplus Y \to Z \) defined by \((x, y) \mapsto F(x, y) := (\exp g(a, x))y \in Z \) is a real-analytic and its Fréchet derivative at \((0, a)\) is invertible. In fact this derivative is

\[
\begin{align*}
\frac{\partial F}{\partial x}|_{(0, a)}(u, v) &= g(a, u) a = \Phi_a(u), \\
\frac{\partial F}{\partial y}|_{(0, a)}(u, v) &= (\exp g(a, 0))v = v,
\end{align*}
\]

which is invertible according to (3.3). By the implicit function theorem there are open sets \( U, V \) with \( 0 \in U \subset X \) and \( a \in V \subset Y \) such that \( W := F(U \times V) \) is open in \( Z \) and \( F : U \times V \to W \) is bianalytic.

To simplify notation set \( X_1 := Z_{1/2}(a) \subset X \). Then \( f = F|X_1 \) establishes a real analytic homeomorphism between the sets \( N_1 := U \setminus X_1 \) and \( M_1 := f(N_1) \). Since \( X_1 \) is a direct summand in \( X \) (hence also in \( Z \)), the image \( M_1 = f(N_1) \) is a direct submanifold.

The operator \( g(a, x) = a \Box x - x \Box a \) is an inner JB\(^*\) triple derivation of \( Z \), hence \( h := \exp g(a, u) \) is a JB\(^*\) triple automorphism of \( Z \). Actually \( h \) lies in \( \text{Aut}^\circ(Z) \), the identity connected component. But it is known (\( [\text{i}] \)) that \( \text{Aut}(Z) \) has two connected components and that the elements in the identity component are C\(^*\)-algebra automorphisms of \( Z \) since they have the form \( z \mapsto u z U^* \) for some \( U \) in the unitary group of \( H \). In particular \( h \) preserves normality, spectral values and ranks hence it preserves \( M \) and so

\[
M_1 = f(N_1) = \{ (\exp g(a, u))a : u \in N_1 \} \subset M.
\]

To complete the proof, it suffices to show that \( f = F|X_1 \) is a biholomorphic mapping. The Fréchet derivative of \( f \) at \( a \) is

\[
f'|_a(u) = g(a, u) a = \{ a, u, a \} - \{ u, a, a \}, \quad u \in Z_{1/2}(a).
\]

Therefore \( \overline{f'} u = \{ a, u, a \} \) and \( \partial f'u = -\{ u, a, a \} \) are the (uniquely determined) complex-linear and complex-antilinear components of \( f' \). The Peirce rules give \( \{ a, u, a \} = 0 \) for all \( u \in Z_{1/2}(a) \), hence \( f \) is holomorphic and the same argument holds for the inverse \( f^{-1} \) map. \( \square \)

Remark that if the algebraic element \( a \) is a projection then \( a = a^* \) and \( M \) as a differentiable manifold is the one constructed in (\( [\text{i}] \) th. 3.3) and (1).

4  The Jordan connection on \( M(n, \Lambda, R) \)

Let \( a \in M := M(n, \Lambda, R) \) and set \( a = \text{supp} (a) \). Recall that a vector field \( X \) on \( M \) is a map from \( M \) to the tangent bundle \( TM \). Thus \( X_a \), the value of \( X \) at \( a \in M \), satisfies \( X_a \in T_a M \approx Z_{1/2}(a) \). We let \( \mathfrak{D}(M) \) be the Lie algebra of smooth vector fields on \( M \). Since the tangent space \( T_a M \) at \( a \in M \) has been identified with \( Z_{1/2}(a) \), we shall consider every vector field on \( M \) as a \( Z \)-valued function such that the value at \( a \) is contained in \( Z_{1/2}(a) \).

Let \( Y' \) be the Fréchet derivative of \( Y \in \mathfrak{D}(M) \) at \( a \). Thus \( Y'_a \) is a bounded linear operator \( Z_{1/2}(a) \to Z \), hence \( Y'_a X_a \in Z \) and it makes sense to take the projection \( P_{1/2}(a) Y'_a X_a \in Z_{1/2}(a) \approx T_a M \).

Definition 4.1  We define a connection \( \nabla \) on \( M \) by

\[
(\nabla_X Y)_a := P_{1/2}(\text{supp} (a)) Y'_a X_a, \quad X, Y \in \mathfrak{D}(M), \quad a \in M.
\]

Note that if \( a \) is a projection, then \( a = \text{supp} (a) \) and \( \nabla \) coincides with the affine connection defined in (\( [\text{i}] \) def 3.6) and (1). It is a matter of routine to check that \( \nabla \) is an affine connection on \( M \), that it is \( \text{Aut}^\circ(Z) \)- invariant and torsion-free, i. e.,

\[
g(\nabla_X Y) = \nabla_{g(X)} g(Y), \quad g \in \text{Aut}^\circ(Z),
\]
where \((gX)_a := g_a' (X_{g_a})\) for all \(X \in \mathfrak{D}(M)\), and \[T(X,Y) := \nabla_X Y - \nabla_Y X - [XY] = 0, \quad X,Y \in \mathfrak{D}(M).\]

**Theorem 4.2** Let the manifolds \(M\) be defined as in [1]. Then the \(\triangledown\)-geodesics of \(M\) are the curves
\[\gamma(t) := (\exp t g(a,u))a, \quad t \in \mathbb{R},\]
where \(a \in M\) and \(u \in Z_{1/2}(a)\).

**Proof.**
Recall that the geodesics of \(\triangledown\) are the curves \(t \mapsto \gamma(t) \in M\) that satisfy the second order ordinary differential equation
\[(\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t))_{\gamma(t)} = 0.\]
Let \(u \in Z_{1/2}(a)\). Then \(g(a,u) = a \triangleleft u - u \triangleleft a\) is an inner JB*-derivation of \(Z\), and, as established in the proof of [5,3], \(h(t) := \exp t g(a,u)\) is an inner C*-automorphism of \(Z\). Thus \(h(t)a \in M\) and \(t \mapsto \gamma(t)\) is a curve in the manifold \(M\). Clearly \(\gamma(0) = a\) and taking the derivative with respect to \(t\) at \(t = 0\) we get by the Peirce rules
\[
\dot{\gamma}(t) = g(a,u)\gamma(t) = h(t)g(a,u)a, \quad \dot{\gamma}(0) = g(a,u)a \in Z_{1/2}(a),
\]
\[
\dot{\gamma}(t) = g^2(a,u)\gamma(t) = h(t)g(a,u)^2a, \quad \dot{\gamma}(0) = g(a,u)^2(0) \in Z_{1}(a) \oplus Z_{0}(a).
\]
In particular \(P_{1/2}(a)g(a,u)^2a = 0\). The definition of \(\triangledown\) and the relation [7] give
\[
(\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t))_{\gamma(t)} = P_{1/2}(\text{supp } \gamma(t)) (\dot{\gamma}(t), \dot{\gamma}(t)) P_{1/2}(\text{supp } \gamma(t)) \dot{\gamma}(t) = P_{1/2}(\text{supp } h(t)a) h(t)g(a,u)a = h(t)P_{1/2}(\text{supp } a) g(a,u)^2a = 0
\]
for all \(t \in \mathbb{R}\). Using the representation \(u = \sum_k u_k\) given by [3] one gets \(g(a,u)a = -\frac{1}{2} \sum_k \lambda_k u_k\), and as \(\lambda \in \sigma(a) \setminus \{0\}\) the mapping \(u \mapsto g(a,u)a\) is a linear homeomorphism of \(Z_{1/2}(a)\). Since geodesics are uniquely determined by the initial point \(\gamma(0) = a\) and the initial velocity \(\dot{\gamma}(0) = g(a,u)a\), the above shows that family of curves in [3] with \(a \in M\) and \(u \in T_a M \approx Z_{1/2}(a)\) are all geodesics of the connection \(\triangledown\).

Recall that \(a = \text{supp } (a)\) is a finite rank projection, hence by ([8], th. 1.1) the JB*-subtriple \(Z_{1/2}(a)\) has finite rank and the tangent space \(T_a M \approx Z_{1/2}(a)\) is linearly homeomorphic to a Hilbert space under an \(\mathfrak{Aut}^{\gamma}(Z)\)-invariant scalar product (say \(\langle \cdot, \cdot \rangle\)). Thus we can define a Riemann metric on \(M\) by
\[
g_a(X,Y) := \langle X_a, Y_a \rangle, \quad X,Y \in \mathfrak{D}(M), \quad a \in M.
\]
Remark that \(g\) is hermitian, i.e. we have \(g_a(iX, iY) = g_a(X,Y)\), and that it has been defined in algebraic terms, hence it is \(\mathfrak{Aut}^{\gamma}(Z)\)-invariant. Moreover, \(\triangledown\) is compatible with the Riemann structure, i.e.
\[
X g(Y,W) = g(\nabla_X Y, W) + g(Y, \nabla_X W), \quad X,Y,W \in \mathfrak{D}(M).
\]
Therefore, \(\triangledown\) is the only Levi-Civita connection on \(M\). On the other hand, let the map \(J : Z_{1/2}(a) \to Z_{1/2}(a)\) be given by \(Jz := iz\). Clearly \(J^2 = -Id\), hence \(J\) defines (the usual) complex structure on the tangent space to \(M\) and \(\triangledown\) is \(J\)-hermitian
\[
\nabla_X (iY) = i \nabla_X Y, \quad X,Y \in \mathfrak{D}(M).
\]
hence \(\triangledown\) is the only hermitian connection on \(M\). Thus the Levi-Civita and the hermitian connection are the same in this case, and so \(\triangledown\) is the Kähler connection on \(M\).

For a tripotent \(e \in \text{Tri } (Z)\), the Peirce reflection around \(e\) is the linear map \(S_e := \text{id} - P_{1/2}(e)\) or in detail \(z = z_1 + z_{1/2} + z_0 \mapsto S_e(z) = z_1 - z_{1/2} + z_0\) where \(z_k\) are the Peirce \(e\)-projections of \(z\), \((k = 1, 1/2, 0)\). Recall that \(S_e\) is an involutory triple automorphism of \(Z\) with \(S_e(e) = e\), and that if \(e\) is a projection (taken as a tripotent) then \(S_e\) is a C*-algebra automorphism of \(Z\). This applies to \(a = \text{supp } (a)\), hence to each \(a \in M\) we get \(S_a\), an involutory automorphism of the manifold \(M\) which in this way becomes a symmetric holomorphic Riemann (Kähler) manifold. Note that in general \(a \notin M\) even if \(a \in M\), hence \(S_a\) may have no fixed points in \(M\).

It would be interesting to know if any two points \(a, b\) in \(M\) can be joined by a geodesic and whether geodesics are minimizing curves for the Riemann distance. The answers to these questions are affirmative when \(M\) consists of projections of the same finite rank (see [8]).
5 Algebraic elements in JB*-triples

The role that projections play in the study of algebras is taken by tripotents in the study of triple systems. A spectral calculus and a notion of algebraic elements is available in the setting of JB*-triples. In what follows we shall consider the manifold of all finite rank algebraic elements in a JB*-triple \( Z \).

**Definition 5.1** An element \( a \in Z \) is called algebraic if there exists a decomposition

\[
a = \lambda_1 e_1 + \cdots + \lambda_n e_n
\]

where \((e_k)\) is a family of pairwise orthogonal tripotents in \( Z \) and \((\lambda_k)\) are complex coefficients.

For an algebraic element \( a \in Z \) the above decomposition can always be chosen in such a way that every \( e_k \) is non-zero and the \( \lambda_k \) are real numbers with \( 0 < \lambda_1 < \cdots < \lambda_n \), and under these additional conditions the spectral representation of \( a \) is unique. Clearly \( a \) has finite rank if and only if every \( e_k \) does.

Remark that for \( Z = \mathcal{L}(H) \), normal algebraic elements in the C*-algebra \( Z \) are algebraic elements in \( Z \) as a JB*-triple. Given a positive integer \( n \in \mathbb{N} \), an increasing \( n \)-uple of non-zero real numbers \( \Lambda = (\lambda_1, \cdots, \lambda_n) \) and an \( n \)-uple \( R = (r_1, \cdots, r_n) \) where \( 0 < r_k \in \mathbb{N} \), we define

\[
N(n, \Lambda, R): = \{ \sum_k \lambda_k e_k : e_j \square e_k = 0 \text{ for } j \neq k, \text{ rank } (e_k) = r_k, \ 1 \leq j, k \leq n \}
\]

to be the set of the elements \((17)\) where the coefficients \( \lambda_k \) and ranks \( r_k \) are given and the \( e_k \) range over non-zero, pairwise orthogonal tripotents in \( Z \) such that rank \( (e_k) = r_k \). The set \( \mathcal{A} \) of finite rank algebraic elements in \( Z \) is the disjoint union \( \mathcal{A} = \bigcup_{n, \Lambda, R} N(n, \Lambda, R) \).

**Lemma 5.2** Let \( Z \) be an irreducible JBW*-triple. Then each of sets \( N = N(n, \Lambda, R) \) is an \( \mathcal{Aut}^\circ (Z) \)-invariant connected subset of \( Z \) on which the group \( \mathcal{Aut}^\circ (Z) \) acts transitively.

**Proof.**

Irreducible JBW*-triples are Cartan factors and we may assume that \( Z \) is a not special as otherwise \( \dim Z < \infty \) and the result is known \([16]\). Thus \( Z \) is a J*-algebra in the sense of Harris \([4]\) that is, a weak*-operator closed complex linear subspace of \( \mathcal{L}(H, K) \) that is closed under the operation of taking triple products, for suitable complex Hilbert spaces \( H, K \) with \( \dim H \leq \dim K \). Tripotents are the partial isometries \( e: H \to K \) that lie in \( Z \).

We make a type by type proof. Let \( Z = \mathcal{L}(H, K) \) be a type I Cartan factor and let \( a, b \in N \). In particular

\[
a = \lambda_1 e_1 + \cdots + \lambda_n e_n, \quad b = \lambda_1 e_1' + \cdots + \lambda_n e_n'
\]

Let \( H_k, H'_k \subset H \) be the domains of the partial isometries \( e_k \) and \( e_k' \), and similarly let \( K_k, K'_k \subset K \) denote their respective ranges. Since \( e_k \) and \( e_k' \) have the same finite rank \( r_k \), they are unitarily equivalent, that is there are unitary operators \( U_k: H_k \to H'_k \) and \( V_k: K_k \to K'_k \) such that \( e_k' = V_k e_k U_k \). Since the \( e_k \) are pairwise orthogonal we have \( H_k \perp H_j \) and \( K_k \perp K_j \) for \( k \neq j \) and \( \bigoplus U_k, \bigoplus V_k \) are unitary operators on \( \bigoplus H_k \) and \( \bigoplus K_k \) that can be extended to unitary operators \( U: H \to H \) and \( V: K \to K \) if needed. The mapping \( Z \to Z \) given by \( z \mapsto VzU \) is a JB*-triple automorphism that lies in \( \mathcal{Aut}^\circ (Z) \) \([10]\) and clearly satisfies \( b = V a U \). Hence \( \mathcal{Aut}^\circ (Z) \) acts transitively on \( N, N \) is connected and invariant under that group.

Cartan factors of types II and III can treated in the same way. The case of spin factors may be discussed with a different approach, but we shall not go into details.

Now consider the joint Peirce decomposition of \( Z \) relative to the family \( (e_1, \cdots, e_n) \) where \( a = \lambda_1 e_1 + \cdots + \lambda_n e_n \) is the spectral resolution of \( a \). Let the support of \( a \) be tripotent \( a = \text{supp } a = e_1 + \cdots + e_n \), and note that

\[
X: = (\bigoplus_{1 \leq k \leq n} i \mathcal{A}(e_k)) \oplus Z_{1/2}(a).
\]

is a topologically complemented subspace in \( Z \).

Fix one of the sets \( N = N(n, \Lambda, R) \) and a point \( a \in N \) with spectral resolution \( a = \sum_k \lambda_k e_k \). From the properties \( e_k \square e_j = 0 \) for \( j \neq k \), the successive odd powers of \( a \) have the expression

\[
a^l = \lambda_1^{2l} e_1 + \cdots + \lambda_n^{2l} e_n, \quad 0 \leq l \leq n - 1,
\]

9
where the determinant \( \det(\lambda_k^{2i+1}) \neq 0 \) does not vanish since it is a Vandermonde determinant and the \( \lambda_k \) are pairwise distinct. Thus the \( e_k \) are polynomials in \( a \) whose coefficients are rational functions of the \( \lambda_k \). Suppose \( N \) is a differentiable manifold, and let us obtain its tangent space \( T_aN \). Consider a smooth curve \( t \mapsto a(t) \) in \( N \), through \( a \), for a neighbourhood \( I \) of \( 0 \in \mathbb{R} \) and \( a(0) = a \). Each \( a(t) \) has a spectral resolution

\[
a(t) = \lambda_1 e_1(t) + \cdots + \lambda_n e_n(t),
\]

therefore the maps \( t \mapsto e_k(t), (1 \leq k \leq n) \), are smooth curves in the manifolds \( \mathcal{N}(r_k) \) of the tripotents in \( Z \) that have fixed finite rank \( r_k = \text{rank } (e_k) \), whose tangent spaces at \( e_k = e_k(0) \) are respectively \( iA(e_k) \oplus Z_{1/2}(e_k) \) (see \([4]\) or \([8]\)). Therefore

\[
z_k : = \frac{d}{dt}|_{t=0} e_k(t) = i\lambda_k + u_k : \in iA(e_k) \oplus Z_{1/2}(e_k), \quad 1 \leq k \leq n.
\]

Set \( v : = \sum_k \lambda_k v_k \) and \( u : = \sum_k \lambda_k u_k \). From \( Z_1(e_k) \cap Z_0(e_j) = \{0\} \), we get

\[
\{a a iv \} = i \sum_{j,k,l} \lambda_j e_j e_k u_l = i \sum_k \lambda_k v_k = iv \in i \bigoplus_k A(e_k)
\]

The spectral tripotents of \( a(t) \) corresponding to different spectral values \( \lambda_k \neq \lambda_j \) are orthogonal, hence \( e_j(t) \square e_k(t) = 0 \) for all \( t \in I \), and taking the derivative at \( t = 0 \) we get

\[
e_j \square u_k = u_k \square e_j = 0, \quad j \neq k, \quad 1 \leq j,k \leq n.
\]

Hence

\[
\{a a u \} = \{ i \sum_k \lambda_k v_k \} = \{ i \sum_k \lambda_k u_k \} = \frac{1}{2} \sum_k \lambda_k u_k = \frac{1}{2} u
\]

which shows that \( u \in Z_{1/2}(a) \). By \([9]\) the tangent vector to \( t \mapsto a(t) \) at \( t = 0 \) is \( z : = \frac{d}{dt}|_{t=0} a(t) = \sum_k \lambda_k (iv_k + u_k) = iv + u \) hence it satisfies

\[
\{a a z\} = iv + \frac{1}{2} u \in i \bigoplus_k A(e_k) \oplus Z_{1/2}(a),
\]

hence \( T_aN \) can be identified with a vector subspace of \( i \bigoplus_k A(e_k) \oplus Z_{1/2}(a) \). In fact \( T_aN \) coincides with that space as it easily follows from the following result that should be compared with \([4]\) th. 3.3

**Theorem 5.3** The sets \( N = N(n, \Lambda, R) \) defined in \([8]\) are real analytic direct submanifolds of \( Z \). The tangent space at the point \( a \in N \) is the Peirce subspace \( X \), where \( a = \text{supp } (a) \), and a local chart at a given by

\[
f : z \mapsto f(z) : = (\exp g(a, z)) a
\]

with \( g(a, z) = a \square z - z \square a \).

**Proof.**

\( N \subset Z \) is invariant under \( \text{Aut}^0(Z) \). Fix any \( a \in N \) and let \( X : = (\bigoplus_{1 \leq k \leq n} iA(e_k)) \oplus Z_{1/2}(a) \). Thus \( Z = X \oplus Y \) for a certain subspace \( Y \). The mapping \( X \oplus Y \to Z \) defined by \( (x, y) \mapsto F(x, y) : = (\exp g(a, x)) y \in Z \) is a real-analytic and its Fréchet derivative at \((0, a)\) is invertible as proved in \([5, 4]\). By the implicit function theorem there are open sets \( U, V \) with \( 0 \in U \subset X \) and \( a \in V \subset Y \) such that \( W : = F(U \times V) \) is open in \( Z \) and \( F : U \times V \to W \) is bialyptic and the image \( F(U) \) is a direct real analytic submanifold of \( Z \).

The operator \( g(a, z) = a \square z - z \square a \) is an inner JB*-triple derivation of \( Z \), hence \( h : = \exp g(a, z) \) is a JB*-triple automorphism of \( Z \). Actually \( h \) lies in \( \text{Aut}^0(Z) \), the identity connected component. In particular \( h \) preserves the algebraic character and the spectral decomposition, hence it preserves \( N \) and so

\[
F(N) = \{(\exp g(a, z)) a : z \in U\} \subset N.
\]

This completes the proof. \( \square \)

**Definition 5.4** For the tripotents \( e, e' \) we set \( e \sim e' \) if and only if \( e \) and \( e' \) have the same \( k \)-Peirce projectors for \( k = 0, 1/2, 1 \).
This notion was introduced by Neher who proved ([17], th.2.3) that

\[ e \sim e' \iff e \in Z_1(e') \text{ and } e' \in Z_1(e), \]  

or equivalently if and only if \( e \square e = e' \square e' \). Next we extend this relation to an equivalence in the manifold \( N \).

**Definition 5.5** Let \( a, b \) be elements in \( N \) with spectral resolutions \( a = \sum_k \lambda_k e_k \) and \( b = \sum_k \lambda_k f_k \) respectively. We say that \( a \) and \( b \) are equivalent (and write \( a \sim b \)) if and only if the joint Peirce decompositions of \( Z \) relative to the orthogonal families \( E = (e_k) \) and \( F = (f_k) \) are the same.

Note that \( \sim \) coincides with the equivalence of Neher when the algebraic elements \( a \) and \( b \) are tripotents. By ([16], th. 3.14), the Peirce spaces of the tripotent \( e_k \) can be expressed in terms of the joint Peirce decomposition of \( Z \) relative to \( E \), hence \( a \sim b \) if and only if \( e_k \sim f_k \) for \( 1 \leq k \leq n \).

**Proposition 5.6** Let \( a, b \) be points in \( N \) such that \( a = \sum \lambda_k e_k \) and \( b = (\exp g(a, z) a \) for some tangent vector \( z = iv + u \in (\bigoplus_{1 \leq k \leq n} i A(e_k)) \oplus Z_{1/2}(a) \). Then \( a \sim b \) if and only if \( u = 0 \).

**Proof.** Let \( b = (\exp g(a, z) a = \sum_k \lambda_k f_k \) be the spectral resolution of \( b \). Then each \( f_k \) is an odd polynomial in \( b \), say \( f_k = p_k(b), 1 \leq k \leq n \). To simplify the notation, consider the index \( k = 1 \) and omit the reference to it in the rest of the proof. If \( a \sim b \) then \( e \sim f \) hence by (21) we must have \( f = \{ee\} \) that is

\[ p(b) = \{ee(b)\} = p(\{ee\}) \]  

Clearly we have \( \rho b \sim a \) for all \( \rho \in \mathbb{T} \), which replaced above yields an identity between two polynomials in \( \rho \). Let \( X^m \), for some positive odd integer \( m \), be the term of \( p \) of lowest degree whose coefficient is not zero. Then (22) entails \( b^m = \{eeb\} \), that is \( (\exp g(a, z))^m a = \{ee(\exp g(a, z))^m a \} \). Taking the Fréchet derivative at the origin \( g(a, \cdot) a = \{ee g(a, \cdot) a \} \), which evaluated at the tangent vector \( z = iv + u = i \sum_k v_k + \sum_k u_k \) and using the Peirce rules as in the proof of ([3,4]) yields \( u = 0 \). The converse is easy.

In particular, there is a neighbourhood of \( a \) in \( N \) in which the algebraic elements \( b \) equivalent to \( a \) are those of the form \( b = (\exp g(a, iv)) a \) with \( v = \sum_k v_k \in \bigoplus_k A(e_k) \), which gives the expression of the fibre of \( N \) through \( a \).

**Proposition 5.7** Let \( a \in N \) be an algebraic element in \( Z \) with spectral resolution \( a = \sum_k \lambda_k e_k \). Then the fibre of \( N \) through \( a \) is the set of the elements \( \sum \lambda_k z_k \) where \( z_k \) lies in the unit circle of the \( JB^* \)-algebra \( Z_1(e_k) \) for \( 1 \leq k \leq n \).

**Proof.** Let \( v = \sum_k \lambda_k v_k \in \bigoplus_k A(e_k) \), and consider the curves in \( Z \)

\[ \phi(t): = (\exp tg(a, iv)) a, \quad \psi(t): = \sum_k \lambda_k (\exp tg(e_k, iv_k)) e_k: = \sum_k \lambda_k \psi_k(t), \quad t \in \mathbb{R}. \]  

They are the solutions of the differential equations

\[ \frac{d\phi(t)}{dt} = g(a, \phi(t)), \quad \frac{d\psi(t)}{dt} = \sum_k \lambda_k g(e_k, \psi_k(t)) \]

with the initial conditions \( \phi(0) = a \) and \( \psi(0) = \sum \lambda_k e_k = a \) respectively. From \( Z_1(e_k) \cap Z_1(e_j) = \{0\} \) for \( k \neq j \) we get

\[ g(a, iv) = g(\sum_k e_k, i \sum_j \lambda_j v_j) = \sum_k \lambda_k g(e_k, iv_k) \]

and the uniqueness of solutions of differential equations gives \( \phi(t) = \sum_k \lambda_k \psi_k(t) \) for all \( t \in \mathbb{R} \). But it is known ([16] th. 5.6) that for fixed \( k, 1 \leq k \leq n \), the set \( z_k = (\exp tg(e_k, iv_k)) e_k, \quad t \in \mathbb{R}, v_k \in A(e_k), \) is the unit circle of the \( JB^* \)-algebra \( Z_1(e_k) \), that is the set of those \( w \in Z_1(e_k) \) that satisfy \( w^* = w^{-1} \). This completes the proof. \( \square \)
By restricting the local charts in 10 to the direct summand \( Z_{1/2}(\mathfrak{a}) \subset T_{\mathfrak{a}} N \) we get a direct submanifold \( B = B(n, \Lambda, R) \) of \( Z \), and we refer to \( B \) as the base manifold of \( N \). Clearly \( B \) is a holomorphic submanifold of the real analytic manifold \( N \), and as in section 3

\[
\left( \nabla_X Y \right)_a = P_{1/2}(\mathfrak{a}) Y'_a X_a, \quad X, Y \in \mathfrak{D}(B), \quad a \in B,
\]

is an \( \text{Aut}^\circ (Z) \)-invariant torsionfree affine connection on \( B \) whose geodesics are the curves \( \gamma(t) = (\exp t g(\mathfrak{a}, u)) a, t \in \mathbb{R}, \) for \( a \in B \) and \( u \in Z_{1/2}(\mathfrak{a}) \). Moreover, for \( a \in B \) the Peirce reflection with respect to \( \mathfrak{a} \) is an involutory triple automorphisms of \( Z \) that fixes \( \mathfrak{a} \), hence it fixes \( i \sum_k A(\mathfrak{e}_k) \) and \( Z_{1/2}(\mathfrak{a}) \). It is easy to see that this reflection commutes with the exponential mapping, hence it fixes \( B(n, \Lambda, R) \) and os it defines a holomorphic symmetry of \( B \). In general \( \mathfrak{a} \) does not belong to \( B \) hence this symmetry in general has no fixed points in \( B \). When the algebraic element \( a \in Z \) has finite rank, that is when rank \( (a) = \sum_i \text{rank} \ (\mathfrak{e}_k) < \infty \), the subtriple \( Z_{1/2}(\mathfrak{a}) \) is linearly equivalent to a complex Hilbert space by 12 and by using the algebraic metric of Harris one can introduce an \( \text{Aut}^\circ (Z) \)-invariant Riemann structure and a Kähler structure on the base manifold in exactly the same way we did in section 3, and the connection \( \nabla \) turns out to be the Levi-Civita and the Kähler connection on \( B \).

References

[1] CHU, C.H., & ISIDRO, J. M. Manifolds of tripotents in JB*-triples Math. Z. 233 2000, 741-754.
[2] DINEEN, S. The Schwarz lemma, Oxford Mathematical Monographs, Clarendon Press, Oxford 1989.
[3] HANSCHE-OLSEN, H & STORMER, E. Jordan Operator Algebras, Monographs and Studies in Mathematics vol 21, Pitman, Boston 1984.
[4] HARRIS, L. A. Bounded symmetric homogeneous domains in infinite dimensional spaces. In: Proceedings on Infinite dimensional Holomorphy, Lecture Notes in Mathematics 364 1973, 13-40, Springer-Verlag Berlin 1973.
[5] HARRIS, L. A. & KAUP, W. Linear algebraic groups in infinite dimensions. Illinois J. Math. 21 1977, 666-674.
[6] HIRZEBRUCH, U. Über Jordan-Algebren und kompakte Riemannsche symmetrische Räume von Rang 1. Math. Z. 90 1965, 339-354.
[7] ISIDRO, J. M. The manifold of minimal partial isometries in the space \( L(H, K) \) of bounded linear operators. Acta Sci. (Szeged) 66 2000, 793-808.
[8] ISIDRO, J. M. & MACKEY, M. The manifold of finite rank projections in the algebra \( L(H) \) of bounded linear operators. To appear in Exp. Math.
[9] ISIDRO, J. M. & STACHÓ, L. L. The manifold of finite rank tripotents in JB*-triples To appear.
[10] KAUP, W. Über die Automorphismen Grassmannscher Mannigfaltigkeiten unendlicher Dimension. Math. Z. 144 1975, 75-96.
[11] KAUP, W. A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. Math. Z. 183 1983, 503-529.
[12] KAUP, W. Über die Klassifikation der symmetrischen Hermiteschen Mannigfaltigkeiten unendlicher Dimension, I, II. Math. Ann. 257 1981, 463-483 and 262 1983, 503-529.
[13] KAUP, W. On Grassmannians associated with JB*-triples. Math. Z. 236 2001, 567-584.
[14] KAUP, W. Cauchy-Riemann structures associated with bounded symmetric domains. preprint 2001.
[15] KLINGENBERG W. Riemannian Geometry, Walter der Gruyter 1982.
[16] LOOS, O. Bounded symmetric domains and Jordan pairs Mathematical Lectures, University of California at Irvine 1977.
[17] NEHER, E. Grids in Jordan triple systems. Lecture Notes in Math. 1280. Springer-Verlag, Berlin 1987.
[18] NOMURA, T. Manifold of primitive idempotents in a Jordan-Hilbert algebra. J. Math. Soc. Japan 45 1993, 37-58.
[19] NOMURA, T. Grassmann manifold of a JH-algebra. Annals of Global Analysis and Geometry 12 1994, 237-260
[20] UPMEIER, H. Symmetric Banach manifolds and Jordan C*-algebras, North Holland Math. Studies vol 104, Amsterdam 1985.