Fundamental groups, homology equivalences and one-sided $h$-cobordisms

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Abstract

We give a sufficient and necessary condition for the fundamental group homomorphism of a map between CW complexes (manifolds) to induce partial homology equivalences. As applications, we obtain characterizations of fundamental groups of homology spheres and Moore manifolds. Moreover, a classification of one-sided $h$-cobordism of manifolds up to diffeomorphisms is obtained, based on Quillen’s plus construction with Whitehead torsions.

1 Introduction

When studying manifold version of Quillen’s plus construction, Guilbault and Tinsley [6] introduce the notion of one-sided $h$-cobordism. This is important to their study of ends of non-compact manifolds (see Guilbault and Tinsley [7]). Recall that a one-sided $h$-cobordism $(W; X, Y)$ is a compact cobordism between closed manifolds such that the inclusion $Y \hookrightarrow W$ is a homotopy equivalence. In [15], the second author introduces the notion of one-sided homology cobordism. Let $(W; X, Y)$ be a compact cobordism between closed manifolds and $R$ be a $\mathbb{Z}[\pi_1(W)]$-module. We call $(W; X, Y)$ a one-sided $R$-homology cobordism if the inclusion $Y \hookrightarrow W$ induces isomorphisms $\pi_1(Y) \cong \pi_1(W)$ and $H_q(Y; R) \cong H_q(W; R)$ for all $q \geq 0$. When $R = \mathbb{Z}[\pi_1(W)]$, the one-sided $R$-homology cobordism is a one-sided $h$-cobordism. There are two aims in this article. The first is to give a sufficient and necessary condition for the fundamental group homomorphism of a map between CW complexes (or manifolds) to induce a one-sided $h$-cobordism of manifolds up to diffeomorphisms.

We study the case of CW complexes first. Let $f : X \to Y$ be a map between CW complexes inducing $\mathbb{Z}$-homology equivalence. When $X$ is fixed and $f$ induces an epimorphism of fundamental groups, Rodríguez and Scevenels [13] show that the kernel $\ker := \ker(\pi_1(f) : \pi_1(X) \to \pi_1(Y))$ is a relative perfect subgroup of $\pi_1(X)$ i.e. $[\ker, \pi_1(X)] = \ker$. Moreover, the maximal such kernel is the intersection of the transfinite lower central series of $\pi_1(X)$. When $\pi_1(f)$ is not necessarily an epimorphism, Bousfield [2] Lemma 6.1] shows that there exists a
CW complex $Y$ such that $f$ is $\mathbb{Z}$-homology equivalent and $\pi_1(Y) = G$ if and only if $H_1(f) : H_1(X; \mathbb{Z}) \rightarrow H_1(G; \mathbb{Z})$ is an isomorphism and $H_2(f) : H_2(X; \mathbb{Z}) \rightarrow H_2(G; \mathbb{Z})$ is epimorphic. We consider high-dimensional homology equivalences, as follows. Assume that for each integer $q \geq 2$, $f_q : H_q(X; \mathbb{Z}) \rightarrow H_q(Y; \mathbb{Z})$ is an isomorphism (high-dimensional homology equivalences). An immediate consequence is that $f$ induces an epimorphism $H_2(f) : H_2(\pi_1(X); \mathbb{Z}) \rightarrow H_2(\pi_1(Y); \mathbb{Z})$ of second homology groups of fundamental groups, which could be obtained from the Hopf exact sequence. If we fix the CW complex $X$ and a group homomorphism $\alpha : \pi_1(X) \rightarrow G$, we show that the necessary condition just mentioned is also sufficient for the existence of a CW complex $Y$ with $\pi_1(Y) = G$ and a cellular map $f : X \rightarrow Y$ inducing $\alpha$ and high-dimensional homology equivalences. More precisely, we have the following result.

**Theorem 1.1** Let $X$ be a (finite, resp.) CW complex and $R$ a subring of rationals or the finite ring $\mathbb{Z}/p$ for some prime number $p$. Suppose that $\alpha : \pi_1(X) \rightarrow G$ is a group homomorphism from the fundamental group of $X$ to a (finely presented, resp.) group $G$. Then the following are equivalent:

(i) $\alpha$ induces an epimorphism $H_2(\alpha) : H_2(\pi_1(X); R) \rightarrow H_2(G; R)$.

(ii) There exists a (finite, resp.) CW complex $Y$ and a cellular map $f : X \rightarrow Y$ such that $\pi_1(Y) = G$, $\pi_1(f) = \alpha : \pi_1(X) \rightarrow \pi_1(Y)$ and for any integer $q \geq 2$, $f$ induces an isomorphism

$$f_q : H_q(X; R) \rightarrow H_q(Y; R).$$

When $f$ is a homology equivalence, this clearly recovers the Bousfield’s result mentioned above (cf. [2] Lemma 6.1).

In Ye [17], the second author shows that when $H_1(\alpha) : H_1(\pi_1(X); \mathbb{Z}) \rightarrow H_1(G; \mathbb{Z})$ is injective and $H_2(\alpha) : H_2(\pi_1(X); \mathbb{Z}) \rightarrow H_2(G; \mathbb{Z})$ is surjective, there is a CW complex $Y$ obtained by adding low-dimensional cells to $X$ such that the fundamental group $\pi_1(Y) = G$ and the inclusion map $\tilde{f} : X \rightarrow Y$ induces the same fundamental group homomorphism as $\alpha$ and for any integer $q \geq 2$, the map $f_q : H_q(X; \mathbb{Z}) \rightarrow H_q(Y; \mathbb{Z})$ is an isomorphism. Actually in [17], more general coefficients are considered. Such a construction gives a unified approach to Quillen plus construction, Bousfield’s integral homology localization, the existence of Moore spaces $M(G, 1)$, Bousfield and Kan’s partial $k$-completion of spaces and some examples in the zero-in-the-spectrum conjecture. Compared with [17, Theorem 1.1], in Theorem 1.1 we drop the condition that $H_1(\alpha)$ is injective, but only for the cases when the coefficients $R$ are subrings of the rationals or $\mathbb{Z}/p$ for some prime number $p$. In these cases, $R$ are principal ideal domains (PID). Therefore, all the applications in [17] when $R$ is a PID are corollaries of Theorem 1.1 as well. These include Bousfield’s integral homology localization and the existence of Moore spaces $M(G, 1)$. In [12, Proposition 4.4], Levin proves, as emphasized by Dranishnikov [3, that for every connected CW-complex $K$ there is a simply connected CW-complex $K^+$ obtained from $K$ by attaching cells of dimension 2 and 3 such that the inclusion $K \rightarrow K^+$
induces isomorphisms of homology groups in dimension $> 1$. This is a special case of Theorem 1.1 when $R = \mathbb{Z}$ and $G = 0$.

A further application is the following: let $n$ be a positive integer and $R$ be a subring of the rationals or the finite ring $\mathbb{Z}/p$ for some prime number $p$. We define an $R$-homology $n$-sphere to be a CW complex $Y$ with the same homology groups as those of the standard sphere $S^n$, i.e. $H_i(Y; R) = H_i(S^n; R)$. When $R = \mathbb{Z}$, $n \geq 5$ and $Y = Y^n$ is a manifold, Kervaire [11] proves that a finitely presented group $G$ is the fundamental group of a $\mathbb{Z}$-homology $n$-sphere $Y$ if and only if $H_1(G; \mathbb{Z}) = H_2(G; \mathbb{Z}) = 0$. The $\mathbb{Z}$-homology spheres are also studied by Dror [4].

**Corollary 1.2** Suppose that $R$ is a subring of the rationals or the finite ring $\mathbb{Z}/p$ for some prime number $p$. Let $G$ be a (finitely presented, resp.) group satisfying $H_2(G; R) = 0$ and $X$ a (finite, resp.) CW complex. There exists a (finite, resp.) CW complex $Y$ with $\pi_1(Y) = G$ and the homology group $H_i(Y; R) \cong H_i(X; R)$, $i \geq 2$, obtained from $X$ by attaching 1-cells, 2-cells and 3-cells. In particular, we have the following:

(i) A (finitely presented, resp.) group $G$ is the fundamental group of an (finite, resp.) $R$-homology circle (i.e. 1-sphere) if and only if $H_1(G; R) = R$ and $H_2(G; R) = 0$.

(ii) A group $G$ is the fundamental group of an $R$-homology 2-sphere if and only if $H_1(G; R) = 0$ and $H_2(G; R)$ is a quotient of $R$.

(iii) When $n > 2$, a (finitely presented, resp.) group $G$ is the fundamental group of an (finite, resp.) $R$-homology $n$-sphere if and only if $H_1(G; R) = H_2(G; R) = 0$.

We now study the case of manifolds. The following result is a manifold version of Theorem 1.1. (In the remainder of this paper, we assume all manifolds are smooth manifolds, but our results hold in the PL and topological categories as well.)

**Theorem 1.3** Let $X$ be a closed manifold of dimension $n$ ($n \geq 5$), $G$ be a finitely presented group and $\alpha: \pi_1(X) \to G$ a group homomorphism. Assume that $X$ is spin or that $\ker\{H_1(\alpha): H_1(\pi_1(X); \mathbb{Z}) \to H_1(G; \mathbb{Z})\}$ is torsion free. The following are equivalent:

(i) $\alpha$ induces an epimorphism $H_2(\alpha): H_2(\pi_1(X); \mathbb{Z}) \to H_2(G; \mathbb{Z})$.

(ii) There exists a cobordism $(W; X, Y)$ such that $\pi_1(W) = G$ and the inclusion map $g: X \hookrightarrow W$ induces the same fundamental group homomorphism as $\alpha$, and for any integer $q \geq 2$, the map $g$ induces an isomorphism $H_q(X; \mathbb{Z}) \cong H_q(W; \mathbb{Z})$. 

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Compared with [18, Theorem 1.1], in Theorem 1.3 we drop the spin condition on $X$ when $H_1(\alpha)$ is injective for the coefficients $R = \mathbb{Z}$. Therefore, all the applications in [18] when $R = \mathbb{Z}$ and $\ker(H_1(\alpha))$ is a free abelian group are corollaries of Theorem 1.3 as well. These include existence of homology spheres, characterizations of high-dimensional knot groups and integral homology localization of manifolds. As another application of Theorem 1.3, we give a characterization of fundamental groups of Moore manifolds (see Section 2.2 for more details).

The following corollary of Theorem 1.3 gives a characterization of the fundamental groups of a one-sided $\mathbb{Z}$-homology cobordism.

**Corollary 1.4** Let $X$ be a closed manifold of dimension $n$ ($n \geq 5$), $G$ a finitely presented group and $\alpha: \pi = \pi_1(X) \to G$ a group homomorphism. The following are equivalent.

1. $H_1(\alpha): H_1(\pi; \mathbb{Z}) \to H_1(G; \mathbb{Z})$ is an isomorphism and $H_2(\alpha): H_2(\pi; \mathbb{Z}) \to H_2(G; \mathbb{Z})$ is an epimorphism;
2. There exists a one-sided $\mathbb{Z}$-homology cobordism $(W; X, Y)$ with $\pi_1(W) = G$ and the inclusion $X \hookrightarrow W$ induces the same fundamental group homomorphism as $\alpha$.

When $\alpha$ is an epimorphism, Corollary 1.4 is the integral localization of manifolds (cf. Corollary 2.8), which was first proved by the second author in [18].

While it seems complicated to give a classification of one-sided $R$-homology cobordisms for a general module $R$, we give a classification of one-sided $h$-cobordisms up to diffeomorphisms. Two one-sided $h$-cobordisms $(W; M, N)$ and $(W'; M, N')$ are equivalent if there exists a diffeomorphism $f: W \to W'$ such that $f|_M = \text{id}_M$ and $f(N) = N'$. Clearly this is an equivalence relation. Denote by $S_h(M)$ the set of all equivalence classes of one-sided $h$-cobordism $(W; M, N)$ on $M$. We have the following result.

**Theorem 1.5** Let $M^n$ be a manifold of dimension $n \geq 5$. Denote by $\text{Pf}(\pi_1(M))$ the set of all normally finitely generated perfect normal subgroups in $\pi_1(M)$. Then there is a bijection of sets

$$S_h(M) \cong \bigcup_{P \in \text{Pf}(\pi_1(M))} \text{Wh}(\pi_1(M)/P),$$

where $\text{Wh}(\cdot)$ is the Whitehead group.

The proof of Theorem 1.5 is based on a manifold version of Quillen’s plus construction with a given Whitehead torsion (see Section 3 for more details).

The paper is organized as follows. In Section 2, we prove Theorems 1.1, 1.3 and list some applications. In Section 3, we introduce Quillen’s plus construction with Whitehead torsions for CW complexes and manifolds. Theorem 1.5 is proved at the end of Section 3.
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2 Homology equivalences and fundamental groups

In this section, we prove Theorem 1.1 and 1.3. Some applications are also given. These include the integral homology localizations, existence of Moore spaces, homology spheres and high-dimensional knot groups.

2.1 Proof of Theorems 1.1 and 1.3

Proof of Theorem 1.1. Suppose that \(f : X \to Y\) is a map such that for any integer \(q \geq 2\), the map \(f^q : H^q(X; R) \to H^q(Y; R)\) is an isomorphism. According to Hopf’s exact sequence (cf. [8, Lemma 2.2]), we have the following diagram:

\[
\begin{array}{ccc}
H_2(X; R) & \to & H_2(\pi_1(X); R) \\
\downarrow & & \downarrow \\
H_2(Y; R) & \to & H_2(\pi_1(Y); R).
\end{array}
\]

Therefore, the group homomorphism \(\alpha\) induces a surjection \(H_2(\pi_1(X); R) \to H_2(G; R) = H_2(\pi_1(Y); R)\).

Conversely, suppose that \(\alpha\) induces an epimorphism \(H_2(\pi_1(X); R) \to H_2(G; R)\). The strategy of constructing \(Y\) is similar to that of [17, Theorem 1.1]. For the group homomorphism \(\alpha : \pi_1(X) \to G\), we can construct a CW complex \(W\) by adding 1-cells and 2-cells to \(X\) such that \(\pi_1(W) = G\), just as in the proof of [17, Theorem 1.1]. We consider the homology groups of the pair \((W, X)\). By Hopf’s exact sequence, there is a commutative diagram:

\[
\begin{array}{ccc}
H_2(\tilde{X}) \otimes_{\mathbb{Z}[G]} R & \to & H_2(\tilde{W}) \otimes_{\mathbb{Z}[G]} R \\
\downarrow j_2 & & \downarrow j_4 \\
H_2(X; R) & \to & H_2(W; R) \to H_2(W, X; R) \to H_1(X; R) \to H_1(W; R).
\end{array}
\]

Since \(R\) is a principal ideal domain, the relative homology group \(H_2(W, X; R)\) is a free \(R\)-module and the image \(\text{im} j_1\) is also a free \(R\)-module. By diagram chasing (cf. [17, Theorem 1.1], proof of Theorem 1.1), there is a surjection

\[j_1 \circ j_4 : H_2(\tilde{W}) \otimes_{\mathbb{Z}[G]} R \to \text{im} j_1.\]
Note that $R$ is a $G$-dense ring in the sense of [17]. Therefore, we can find a basis $S$ for $\text{im} j_1$ in the image of $H_2(W) \otimes 1$. Then there are maps $b_\lambda : S_\lambda^2 \to W$ with $\lambda \in S$ such that the composition of maps

$$H_2(\bigvee_{\lambda \in S} S_\lambda^2; R) \to H_2(W; R) \to \text{im} j_1$$

is an isomorphism.

For each such $\lambda$, attach a 3-cell $(D^3, S^2)$ to $W$ along $b_\lambda$. Let $Y$ denote the resulting space. We see that the diagram

$$\begin{array}{ccc}
\bigvee_{\lambda} S^2 & \longrightarrow & W \\
\downarrow & \searrow & \downarrow \\
\bigvee_{\lambda} D^3 & \longrightarrow & Y
\end{array}$$

is a pushout diagram. By the van Kampen theorem, the fundamental group of $Y$ is still $G$. We have a commutative diagram

$$\begin{array}{ccc}
H_2(X; R) & \to & H_2(W; R) \\
\downarrow & \searrow & \downarrow \\
H_2(X; R) & \to & H_2(Y; R)
\end{array}$$

Since the relative homology group $H_2(Y, W; R) = H_2(\bigvee_{\lambda} D^3, \bigvee_{\lambda} S^2; R) = 0$, the map $H_2(W; R) \to H_2(Y; R)$ is surjective. Therefore, the right vertical map induces a surjection $\text{im} j_1 \to \text{im} b$. Denoting by $H_*(\cdot; R)$ the homology groups $H_*(-; R)$, we have the following commutative diagram:

$$\begin{array}{ccc}
\cdots & \to & H_2(\bigvee D^3, \bigvee S^2) \\
\downarrow & & \downarrow \\
\cdots & \to & H_2(\bigvee S^2, \text{pt}) \\
\downarrow & & \downarrow \\
\cdots & \to & H_2(\bigvee D^3, \text{pt}) \\
\downarrow & & \downarrow \\
\cdots & \to & H_2(\bigvee D^3, \bigvee S^2)
\end{array}$$

Since $H_2(\bigvee S^2, \text{pt}) \to \text{im} j_1$ is an isomorphism and $H_2(\bigvee D^3, \text{pt}) = 0$, the image $\text{im} b = 0$. By a five lemma argument, for any $i \geq 3$ the relative homology group $H_i(Y, X; R) = 0$. The vanishing of these relative homology groups and $\text{im} b$ shows that for any $q \geq 2$, there is an isomorphism $H_q(X; R) \cong H_q(Y; R)$. ■

The proof of Theorem [1.3] is parallel to that of Theorem [1.1] in the sense that one adds handles instead of cells. However, in this situation more efforts are needed to control the normal bundle of the attaching spheres of the 3-handles.

**Proof of Theorem [1.3]** First we may attach 1- and 2-handles to the right hand boundary of $X \times [0, 1]$ to obtain an $(n+1)$-dimensional manifold $W_1$ such that $\pi_1(W_1) = G$ and the homomorphism $\pi_1(X) \to \pi_1(W_1)$ induced by the inclusion $X \hookrightarrow W_1$ is $\alpha$. Note that $W_1$ is homotopy equivalent to the complex $W$ constructed in the proof of Theorem [1.1]. From the argument of the proof of Theorem [1.1] we see that $\text{im} \{ j : H_2(W_1) \to H_2(W_1, X) \}$ is a free abelian group, and there is a basis of $\text{im} j$ whose elements are spherical, i.e. in the image of $\pi_2(W_1) \to H_2(W_1) \to \text{im} j$. 

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Denote by \(X_1\) the other boundary component of \(W_1\). Clearly \(\pi_2(W_1) \cong \pi_2(X_1)\).

If \(X\) is spin, then it’s well-known that we may choose appropriate framings of the attaching spheres of the handles such that \(W_1\) is spin, thus \(X_1\) is also spin. Therefore any embedded 2-sphere in \(X_1\) has trivial normal bundle and we may attach 3-handles to obtain \(W\) and \(Y\) as desired.

In the following, we deal with the case where \(X\) is not necessarily spin but \(\ker\{H_1(\alpha): H_1(\pi_1(X); \mathbb{Z}) \to H_1(\pi_1(G); \mathbb{Z})\}\) is torsion free. The key point is to choose appropriate framings of the attaching spheres of the 2-handles to ensure that we may attach the 3-handles.

Let
\[
V_1 = X \times [0, 1] \cup \bigcup_i h^1_i
\]
be the manifold obtained by attaching 1-handles and \(X'\) the right hand boundary of \(V_1\). Let
\[
V_2 = X' \times [0, 1] \cup \bigcup_k h^2_k
\]
be the result of attaching 2-handles and \(X_1\) the the right hand boundary. Then we get \(W_1 = V_1 \cup_X V_2\). In the long exact sequence
\[
H_2(V_1, X) \to H_2(W_1, X) \to H_2(W_1, V_1) \to H_1(V_1, X)
\]
of the triple \((W_1, V_1, X)\), we know that \(H_2(V_1, X) = 0\) and \(H_1(V_1, X)\) is torsion free. Therefore, the relative homology group \(H_2(W_1, X)\) can be viewed as a direct summand of \(H_2(W_1, V_1) \cong H_2(V_2, X')\). In the long exact sequence
\[
H_2(W_1) \overset{j}{\to} H_2(W_1, X) \to H_1(X) \to H_1(W_1),
\]
by assumption, \(\text{coker} j \cong \ker\{H_1(X) \to H_1(W_1)\} = \ker\{H_1(\alpha): H_1(\pi_1(X)) \to H_1(\pi_1(G))\}\) is torsion free. Therefore, the image \(\text{im} j\) is isomorphic to a direct summand of \(H_2(W_1, X)\) and hence a direct summand of \(H_2(V_2, X')\).

Let the attaching maps of the 2-handles be
\[
D^2_k \times D^{n-1} \supset S^1 \times D^{n-1} \xrightarrow{\varphi_k} X'.
\]
Then
\[
X_1 = (X' - \bigcup_k \varphi_k(S^1 \times D^{n-1})) \cup \bigcup_k D^2_k \times S^{n-2}
\]
and we have a canonical basis \(\{b_i\} i = 1, \ldots, m\) of \(H_2(V_2, X')\) represented by \(D^2_k \times \{p\}\), where \(p \in \partial D^{n-1}\) is a fixed point.

Recall that we have elements \(x_1, \ldots, x_m \in \pi_2(W_1)\) such that \(j(x_1), \ldots, j(x_m)\) form a basis of \(\text{im} j\). Let \(j(x_i) = \sum_k a_{ik} b_k\). We may assume that each \(x_i\) is represented by an embedded 2-sphere \(S^2_i\) in \(X_1\), and the intersection of \(S^2_i\) with the 2-handle \(h^2_k\) consists of \(a'_{ik}\) copies of disks \(D^2_{ik}(1), \ldots, D^2_{ik}(a'_{ik})\) parallel to the core disk \(D^2_{ik} \times \{0\}\), where \(a'_{ik} \equiv a_{ik} \mod 2\).

As seen from the proof of Theorem 1.1, we need to attach 3-handles along the 2-spheres \(S^2_i\), which can be done if the normal bundle of these embedded
2-spheres are trivial. Note that a stable vector bundle over $S^2$ is determined by its second Stiefel-Whitney class $w_2$. Hence for an embedded 2-sphere in $W_1$, the triviality of its normal bundle is determined by the evaluation of $w_2(W_1)$ on the homology class represented by this sphere. That is the following ($\nu$ denotes the normal bundle of this sphere)

$$\langle w_2(\nu), [S^2_1] \rangle = \langle w_2(\nu \oplus TS^2), [S^2_1] \rangle = \langle w_2(W_1), x_i \rangle.$$

Define a homomorphism $f: \text{im} j \to \mathbb{Z}_2$ by $f(j(x_i)) = \langle w_2(W_1), x_i \rangle$. Since $\text{im} j$ is a direct summand of $H_2(V_2, X')$, we can extend $f$ to a homomorphism $f: H_2(V_2, X') \to \mathbb{Z}_2$. Now we rechoose the framing of attaching spheres of the 2-handles according to $f(b_k)$, as follows. If $f(b_k) = 0$, we keep $\varphi_k$ unchanged. If $f(b_k) = 1$, we use the other framing. Denote by $W'_i$ the manifold obtained by using these framing data. Now for the normal bundle $\nu$, the clutching function along the boundary of $D'_{ik}(j)$ ($j = 1, \cdots, a_{ik}$) changes if $f(b_k) = 1$ and remains unchanged if $f(b_k) = 0$. If a clutching function changes, the evaluation $\langle w_2(\nu), [S^2_1] \rangle$ will change by 1. Therefore

$$\langle w_2(W'_i), x_i \rangle = \langle w_2(W_1), x_i \rangle + \sum_k a'_{ik} f(b_k) = f(j(x_i)) + f(j(x_i)) = 0.$$

Therefore, the normal bundles of the embedded 2-spheres representing $x_i$ ($i = 1, \cdots, m$) are trivial and we can attach 3-handles in the same manner employed in the proof of Theorem 1.1.

\textbf{Remark 2.1} The proof only works for the coefficient $R = \mathbb{Z}$. For other coefficients $R$, even though we know that a basis of $\text{im} j \subset H_2(W_1, X_1) \otimes R$ is represented by spheres, we don’t know whether these spheres form a basis of $\text{im} j \subset H_2(W_1, X_1)$ or not. If not, the argument in the above doesn’t work any more.

Corollary 1.4 directly follows from Theorem 1.3 by noting that

$$H_1(f) = H_1(\alpha): H_1(X) = H_1(\pi_1(X)) \to H_1(W) = H_1(G).$$

\subsection{Applications to homology spheres and Moore manifolds}

In this subsection, we give some applications of Theorem 1.1 and 1.3.

Recall the definition of $R$-homology spheres from the introduction. Corollary 1.2 gives a characterization of the fundamental groups of $R$-homology spheres. In order to prove Corollary 1.2 we need a lemma. The following result also shows that the CW complex $Y$ in Theorem 1.1 is not unique in general.

\textbf{Lemma 2.2} Let $X$ be a simply connected CW complex and $R = \mathbb{Z}/p$ a finite field for some prime $p$. There exists a simply connected CW complex $Y$ and an inclusion map $f: X \to Y$ such that $H_2(Y; \mathbb{Z})$ is $p$-torsion-free, i.e. $px = 0$ implies $x = 0$ for $x \in H_2(Y; \mathbb{Z})$, and $f$ induces isomorphism $H_i(X; R) \to H_i(Y; R)$ for any $i \geq 0$.

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Proof. Let $S$ be a set of generators for the $p$-torsion elements in $H_2(X; \mathbb{Z})$. For each $x \in S$, attach a 3-cell to $X$. We get a new space $W = X \cup_{\lambda \in S} e^3_\lambda$. Since $H_2(X; R) \cong H_2(X; \mathbb{Z}) \otimes Z R$, the boundary map $\partial = 0$ in the relative homology exact sequence

$$0 \to H_3(X; R) \to H_3(W; R) \xrightarrow{j} H_3(W, X; R) \xrightarrow{\partial} H_2(X; R) \to H_2(W; R) \to \cdots.$$  

Therefore, $j$ is split surjective as an $R$-module homomorphism. Since $H_2(W; \mathbb{Z})$ is $p$-torsion-free, the universal coefficient theorem implies that $H_3(W; R) \cong H_3(W; \mathbb{Z}) \otimes Z R$.

By the Hurewicz theorem (cf. Hu [10, Theorem 8.1, p.305]) and the fact that tensor product is right exact, the Hurewicz map $\pi_3(W) \otimes Z R \to H_3(W; \mathbb{Z}) \otimes Z R \cong H_3(W; R)$ is surjective. Using the fact that $\mathbb{Z}/p$ is a $G$-dense ring for the trivial group (cf. Lemma 2.2 in Ye [17]), there exists a set $S'$ of maps $[g_\lambda : S^3 \to W] \in \pi_3(W)$ such that the composition

$$H_3(\vee_{\lambda \in S'} S^2_\lambda; R) \to H_3(W; R) \to H_3(W, X; R)$$  

is isomorphic. For each such a map $g_\lambda$, attach a 4-cell to $W$, getting a space $Y$. By the exact relative homology sequence

$$\cdots \to H_{i+1}(Y, X; R) \to H_i(W, X; R) \to H_i(W, Y; R) \to H_i(Y, X; R) \to \cdots$$  

and a similar diagram chase as that in the proof Theorem 1.1 we see that $H_i(Y, X; R) = 0$ for any $i \geq 0$. This shows that the inclusion $X \hookrightarrow Y$ induces a homology equivalence with coefficients $R$. \hfill \blacksquare

Proof of Corollary 1.2 The first part follows directly from Theorem 1.1 with $f : \pi_1(X) \to G$ the trivial group homomorphism. When $n > 2$, the fundamental group $G$ of an $R$-homology $n$-sphere satisfies the condition that $H_1(G; R) = H_2(G; R) = 0$, by the Hopf exact sequence (cf. [8, Lemma 2.2]).

The $R$-homology 1-sphere is a special kind of a generalized Moore space $M(G, 1; R)$ defined in Ye [17]. It is proved that a group $G$ is the fundamental group of a Moore space $M(G, 1; R)$ if and only if $H_2(G; R) = 0$ (cf. [17, Proposition 4.6]). It follows that a group $G$ is the fundamental group of an $R$-homology 1-sphere if and only if $H_1(G; R) = R$ and $H_2(G; R) = 0$.

We consider the case of $n = 2$. By the Hopf exact sequence again, we see that the condition that $H_1(G; R) = 0$ and $H_2(G; R)$ is a quotient of $R$ is necessary. Conversely, let $X = K(G, 1)$ be a classifying space of $G$ and $\alpha : G \to 1$ a trivial group homomorphism. By Theorem 1.1 there is a simply connected CW complex $Y$ and a map $f : X \to Y$ inducing an $R$-homology equivalence. Note that the coefficients $R$ is a principal ideal domain and there is an $R$-epimorphism $R \to H_2(G; R) \cong H_2(Y; R) \cong \pi_2(Y) \otimes Z R$. 

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By applying Lemma 2.2 if necessary, we may assume that $\pi_2(Y) \cong H_2(Y;\mathbb{Z})$ is a cyclic group. Choose $\eta : S^2 \to Y$ as a generator of $\pi_2(Y)$. Viewing $f$ as a fibration by replacing $X$ by the path space $E_f$ (still denoted by $X$ without confusions), we let $K$ denote the pullback the following diagram

$$
\begin{array}{ccc}
K & \to & X \\
\downarrow & & \downarrow f \\
S^2 & \to & Y.
\end{array}
$$

Denote by $F$ the homotopy fiber of $f$. By the commutative diagram

$$
\begin{array}{cccc}
\cdots \to \pi_2(S^2) & \to & \pi_1(F) & \to & \pi_1(K) & \to & \pi_1(S^2) = 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to \pi_2(Y) & \to & \pi_1(F) & \to & \pi_1(X) = G & \to & \pi_1(Y) = 1,
\end{array}
$$

we see that $\pi_1(K) \to G$ is an isomorphism. By the Serre spectral sequence, we see that $H_*(F;R) = H_*(pt;R)$. Using the Serre spectral sequence again, the map $K \to S^2$ induces an $R$-homology equivalence. This finishes the proof.

**Remark 2.3**

(i) The existence of $\mathbb{Z}$-homology 2-sphere is actually already contained in Dror [4, proof of Theorem 3.2, p.122].

(ii) Although Kervaire [11] proves that every finitely presented group $G$ with $H_1(G;\mathbb{Z}) = H_2(G;\mathbb{Z}) = 0$ could be realized as the fundamental group of a $\mathbb{Z}$-homology sphere $M^n$ (a closed manifold) when $n \geq 5$, Hausmann and Weinberger [9] show that it is not true for $n = 4$.

(iii) The authors don’t know whether every finitely presented group with the condition in Corollary 1.2 (ii) could be realized as a finite $R$-homology 2-sphere.

Recall from [16] that for a given integer $n \geq 1$ and a group $G$ (abelian if $n \geq 2$), a Moore space $M(G;n)$ is a CW complex $X$ such that the homotopy group $\pi_j(X) = 0$ for $j < n$, $\pi_n(X) = G$ and the homology group $H_i(X;\mathbb{Z}) = 0$ for each $i > n$. As analogues of Moore spaces, we define Moore manifolds as follows. Let $k$ be a positive integer and $G$ a finitely presented group. When $k \geq 2$, we assume further that $G$ is abelian.

**Definition 2.4** Let $n, k$ be two positive integers. An $n$-dimensional Moore manifold $M^n(G,k)$ is an orientable closed manifold $X$ such that for any integer $i < k$, the homotopy group $\pi_i(X) = 0$, $\pi_k(X) = G$ and for any integer $k < i \leq (n+1)/2$, the homology group $H_i(X;\mathbb{Z}) = 0$.

When $k > [(n+1)/2]$, by Poincaré duality, Moore manifold $M^n(G,k)$ only exists when $G = 1$, the trivial group. In this case, $M^n(G,k)$ is the standard sphere. Therefore, in the remainder of this article, we always assume $k \leq (n+1)/2$.

Similar to the existence of Moore spaces $M(G,1)$ in Varadarajan [16], we give a characterization of Moore manifolds $M^n(G,1)$, as follows.
**Proposition 2.5** Let \( n \geq 5 \) and \( G \) a finitely presented group. There exists a Moore manifold \( M^n(G,1) \) if and only if \( H_2(G;\mathbb{Z}) = 0 \).

**Proof.** The necessary condition follows easily Hopf’s exact sequence

\[
\pi_2(M^n(G,1)) \to H_2(M^n(G,1);\mathbb{Z}) \to H_2(G;\mathbb{Z}) \to 0.
\]

Conversely when \( H_2(G;\mathbb{Z}) = 0 \), we choose \( X = S^n \) and \( \alpha : 1 \to G \) the trivial group homomorphism. By Theorem 1.3, there exists an orientable closed manifold \( Y \), which is obtained from \( X \) by adding 1-handles, 2-handles and 3-handles, such that \( \pi_1(Y) = G \) and the inclusion \( X \hookrightarrow W \); the cobordism between \( X \) and \( Y \), induces that for any integer \( q \geq 2 \), the relative homology group \( H_q(W;X;\mathbb{Z}) = 0 \). By the universal coefficients theorem and Poincaré duality, for each \( q \geq 2 \), there is an isomorphism \( H^q(W,X;\mathbb{Z}) = H_{n+1-q}(W,Y;\mathbb{Z}) = 0 \).

This implies that for any integer \( 2 \leq i \leq n-2 \), the homology group \( H_i(Y;\mathbb{Z}) = H_i(X;\mathbb{Z}) = 0 \). By the assumption that \( n \geq 5 \), we have \( n-2 \geq [(n+1)/2] \).

Therefore, such \( Y \) is a Moore manifold \( M^n(G,1) \).

**Proposition 2.6** Let \( n \geq 5 \), \( k < (n-1)/2 \) and \( G \) a finitely generated abelian group. There exists a Moore manifold \( M^n(G,k) \).

**Proof.** Without loss of generality, suppose that \( G = \mathbb{Z}/t \) for some integer \( t \). Then \( t = 0 \), assume \( G = \mathbb{Z} \). The general Moore manifold can be obtained as connected sum of such manifolds \( M^n(\mathbb{Z}/t,k) \). Take \( X = S^k \times S^{n-k} \). Let \( f : S^k \to X \) be an embedding representing the element \( [t] \in \mathbb{Z} = \pi_k(X) \). Since \( X \) is parallelizable, \( f(S^k) \) has a trivial normal bundle in \( X \). Extend \( f \) to an embedding \( \tilde{f} : S^k \times D^{n-k} \to X \). Doing surgery on \( X \) along \( \tilde{f} \), the resulting manifold is denoted by \( Y \). Suppose that the surgery trace is \( W \). It is not hard to see that \( H_k(W;\mathbb{Z}) = G \), and the homology group \( H_i(W) = 0 \) for any integer \( k < i < (n+1)/2 \). Since \( W \simeq Y \cup e^{n-k} \) and \( n-k > (n+1)/2 > k+1 \), the manifold \( Y \) has the same homology groups as \( W \) up to the middle dimension. This shows that \( Y \) is a Moore space \( M^n(G,k) \).

**Remark 2.7**

1. For an integer \( k \) close to \( (n+1)/2 \) the manifold \( M^n(G,k) \) may not exit, see the Corollary before Lemma F in Barden [1].

2. Hausmann and Weinberger [2] constructed a superperfect group \( G \) for which any 4-manifold \( Y \) with \( \pi_1(Y) = G \) satisfies \( \chi(Y) > 2 \). This implies that Proposition 2.5 does not hold for \( n = 4 \).

As another application of Theorem 1.3, the following result is an improvement of Corollary 1.3 in [18, Theorem 1.1].

**Corollary 2.8** Let \( n \geq 5 \) and \( X \) be a closed \( n \)-dimensional manifold with fundamental group \( \pi \) and \( N \) a normal subgroup of \( \pi \). The following are equivalent:

(i) There is a closed manifold \( Y \) obtained from \( X \) by adding 2-handles and 3-handles with \( \pi_1(Y) = \pi/N \) such that for any \( q \geq 0 \) there is an isomorphism

\[
H_q(Y;\mathbb{Z}) \cong H_q(X;\mathbb{Z}).
\]
(ii) The group $N$ is normally generated by some finite number of elements and is a relatively perfect subgroup of $\pi$, i.e. $[\pi, N] = N$.

**Proof.** Compared with Corollary 1.3 of [13], we drop the condition that $X$ is spin here, since $H_1(\pi; \mathbb{Z}) \to H_1(\pi/N; \mathbb{Z})$ is an isomorphism and Theorem 1.3 applies. ■

3 Quillen’s plus construction with a given Whitehead torsion

In this section, we introduce Quillen’s plus constructions with given Whitehead torsions for both CW complexes and manifolds. Theorem 1.5 is proved at the end of this section.

3.1 Plus construction with torsions for CW complexes

Let $X$ be a CW-complex and $P \triangleleft \pi_1(X)$ a perfect normal subgroup, normally generated by finitely many elements. Then Quillen’s plus construction is a CW-complex $X^+$ containing $X$ as a subcomplex such that $i_\ast: \pi_1(X) \to \pi_1(X^+)$ is the projection $\pi_1(X) \to \pi_1(X)/P$ and the pair $(X^+, X)$ is homologically acyclic, i.e. $H_\ast(X^+, X; A) = 0$ for any $\pi_1(X)/P$-module $A$. Especially, we have $H_\ast(X^+, X) = 0$, where $X^+$ is the universal cover of $X^+$ and $\tilde{X}$ is the corresponding covering space of $X$. Therefore, there is a well-defined torsion $\tau(X^+, X) \in \text{Wh}(G)$ ($G = \pi_1(X)/P$) of the pair $(X^+, X)$ (cf. Remark 2 of [13, p. 378]).

**Theorem 3.1** Given an element $\tau \in \text{Wh}(G)$, there exists a plus construction $X^+$ of $X$ such that $\tau(X^+, X) = \tau$. If there is another $X^+_1$ with the same property, then there is a simply homotopy equivalence $f: X^+ \to X^+_1$ which is homotopic to the identity on $X$.

This is a stronger version of the existence and uniqueness of the plus construction (cf. [15, Theorem 5.2.2]).

**Proof.** We just need to modify the ordinary plus construction to take into account the torsion issue. Let $\tau \in \text{Wh}(G)$ be represented by a matrix $A = (a_{ij})$ of size $N$ for a larger integer $N$. Let $\tilde{Y}$ be obtained by attaching $k$ 2-cells $e^2_i$ on $X$ to have the fundamental group $G$ and $(N - k)$ 2-cells with trivial attaching maps. Let $\tilde{X}$ be the universal cover of $\tilde{Y}$ and $\tilde{X}$ be the corresponding cover of $X$. Then the relative homology groups of $(\tilde{Y}, \tilde{X})$ concentrate in dimension 2 and the homomorphism

$$j: \pi_2(Y) \cong H_2(\tilde{Y}) \to H_2(\tilde{Y}, \tilde{X}) = \bigoplus_{i=1}^N \mathbb{Z}[G]/[e_i^2]$$

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is surjective since $H_1(X) = H_1(\pi) = 0$. Therefore we may choose $x_1, \ldots, x_N \in \pi_2(Y)$ such that when expressed in the canonical basis $[e_i^2]$, the coefficients of $j(x_i)$ are the $i$-th row of $A$. Using these $x_i$ as attaching data we form

$$X^+ = Y \cup \cup e_i^3.$$ 

Now the chain complex $C_\bullet(\tilde{X}^+, \tilde{X})$ concentrates in dimension 2 and 3, and

$$C_3(\tilde{X}^+, \tilde{X}) = H_3(\tilde{X}^+, \tilde{Y}) = \bigoplus \mathbb{Z}[G][e_i^3]$$

$$C_2(\tilde{X}^+, \tilde{X}) = H_2(\tilde{Y}, \tilde{X}) = \bigoplus \mathbb{Z}[G][e_i^2]$$

and the boundary map $\partial: C_3(\tilde{X}^+, \tilde{X}) \to C_2(\tilde{X}^+, \tilde{X})$ is just the boundary map $\partial: H_3(\tilde{X}^+, \tilde{Y}) \to H_2(\tilde{Y}, \tilde{X})$ in the long exact sequence of the triple $(\tilde{X}^+, \tilde{Y}, \tilde{X})$. Hence by construction, the pair $(X^+, X)$ is homologically acyclic and the torsion of $C_\bullet(\tilde{X}^+, \tilde{X})$ is represented by $A$, which equals to $\tau$.

For the uniqueness of $X^+$, it is shown that there exists a homotopy equivalence $f: X^+ \to X_1^+$ which is homotopic to the identity on $X$ (cf. [15, Theorem 5.2.2]). There is a short exact sequence of chain complexes

$$0 \to C_*(\tilde{X}^+, \tilde{X}) \to C_*(\tilde{X}_1^+, \tilde{X}) \to C_*(\tilde{X}_1^+, \tilde{X}^+) \to 0$$

obtained from the triple $(X_1^+, X^+, X)$. From the additivity of the Whitehead torsion [13, Theorem 3.1], we have $\tau(X_1^+, X) = \tau(X^+, X) + \tau(X_1^+, X^+)$. Therefore $\tau(X_1^+, X^+) = 0$, which implies that $f$ is a simple homotopy equivalence.

### 3.2 Embedding manifold plus construction with torsions

In the case that $X$ is a manifold $M^n$ ($n \geq 5$), it is shown in [6, Theorem 4.1] that the plus construction can be done in the world of manifolds and one obtains a one-sided $h$-cobordism $(W; M, M_2)$ (i.e. the inclusion $M_2 \hookrightarrow W$ is a homotopy equivalence) such that the Whitehead torsion of $(W; M_2)$ is trivial. In this section we generalize the manifold plus construction as following.

**Theorem 3.2** Let $M^n$ be a manifold of dimension $n \geq 5$, $P \triangleleft \pi_1(M)$ a perfect normal subgroup normally generated by finitely many elements. Let $\tau \in \text{Wh}(G)$ ($G = \pi_1(M)/P$) be an arbitrary element. Then there is a one-sided $h$-cobordism $(W; M, M_2)$ such that $W$ is the plus construction of $M$ corresponding to $P$ and the Whitehead torsion $\tau(W, M_2) = \tau$. Furthermore, $W$ is unique up to diffeomorphism rel $M$.

Actually the existence part of this theorem can be deduced from a combination of [6, Theorem 4.1] and [13, Existence Theorem 11.1]. However, the proof given below shows that in the process of attaching handles, we can control the homotopy type and the Whitehead torsion simultaneously. Therefore, it can
Lemma 3.3 There exists a framing of the normal bundle of $S$ such that for the resulting manifold $W$, the evaluation of the second Stiefel-Whitney class $w_2(W)$ on $B$ is trivial. (Since $W$ is canonically homotopy equivalent to $M \cup e^2$, we may identify the homology groups of $W$ obtained using different framings.)

Proof. We start by choosing one framing and get an embedding $\varphi: S^1 \times D^{n-1} \rightarrow M$ with $S = S^1 \times \{0\}$ and attach a 2-handle $h^2$ via $\varphi$

$$W = M \times [0,1] \cup_\varphi D^2 \times D^{n-1}.$$ 

Let $M_0 = M - \varphi(S^1 \times D^{n-1})$ and $M_1 = M_0 \cup \varphi D^2 \times S^{n-2}$, the other end of $W$. Clearly, we have that $H_2(W) = H_2(M_1)$.

The Hurewicz map $\Omega^S_2(W) \rightarrow H_2(W)$ from the cobordism group of $W$ to the homology group of $W$ is surjective (easily seen by a standard Atiyah-Hirzebruch spectral sequence argument), therefore a generator of $B$ is represented by an embedded closed oriented surface $F^2 \rightarrow M_1 \subset W$. After an isotopy of $F^2$, we may assume that the intersection of $F^2$ with $\varphi(D^2 \times S^{n-2})$ consists of $m$ disks $D^2 \times \{x_1\}, \ldots, D^2 \times \{x_m\}$ ($x_i \in S^{n-2}$) parallel to the core disk $D^2 \times \{0\}$. By surgering away extraneous pairs of algebraically opposite 2-disks, we get a new surface, still denoted by $F$, whose intersection with $\varphi(D^2 \times S^{n-2})$ is $D^2 \times \{x\}$.

Let $\nu$ be the normal bundle of the embedded surface $F$. It’s known that an orientable stable vector bundle $\nu$ over a closed surface $F$ is trivial if and only if $\langle w_2(\nu), [F] \rangle = 0$. Also since the stable tangent bundle of a closed orientable surface is trivial, we have that

$$\langle w_2(W), [F] \rangle = \langle w_2(TF \oplus \nu), [F] \rangle = \langle w_2(\nu), [F] \rangle.$$ 

Therefore if $\nu$ is trivial, then we are done. If $\nu$ is nontrivial, then we use the other framing of $S$. In this case, along the boundary of $D^2 \times \{x\}$, the clutching function of the normal bundle $\nu$ changes, the new normal bundle is trivial.

Proof of Theorem 3.2. Let $\bar{\tau}$ be the conjugate of $\tau$ (for the definition, see Section 6 of [13]). First we attach 2-handles to $M \times [0,1]$ to kill $P$. We also attach some trivial 2-handles such that the total number of 2-handles is $N$ if $(-1)^n \bar{\tau} \in \text{Wh}(G)$ is represented by a matrix $A$ of size $N$. Denote by $W_1$ the
surgery trace and by $M_1$ the right hand boundary of $W_1$. Suppose that $\tilde{W}_1$ is the universal covering space of $W_1$ and $\overline{M}$ is the corresponding covering space of $M$. As in the proof of Theorem 3.1 we have a surjection

$$\tilde{j}: \pi_2(W_1) \cong H_2(\tilde{W}_1) \to H_2(\tilde{W}_1, \overline{M}) = \bigoplus_{i=1}^{N} \mathbb{Z}[G_i[h_i^2]].$$

We choose $x_1, \cdots, x_N \in \pi_2(W_1)$ such that the coefficients of $\tilde{j}(x_i)$ in the basis $[h_i^2]$ consist of the $i$-th row of $A$. $\pi_2(M_1) \cong \pi_2(W_1)$.

Note that a stable vector bundle $\xi$ over $S^2$ is determined by its second Stiefel-Whitney class $w_2(\xi)$. Hence for an embedded 2-sphere in $W_1$, the triviality of its normal bundle is determined by the evaluation of $w_2(W)$ on the homology class represented by this sphere. From the commutative diagram

$$\begin{array}{ccc} H_2(\tilde{W}_1) & \overset{j}{\rightarrow} & H_2(\tilde{W}_1, \overline{M}) \\ \downarrow & & \downarrow \\ H_2(W_1) & \overset{j}{\rightarrow} & H_2(\tilde{W}_1, M), \end{array}$$

it’s seen that under the Hurewicz map, the image of $x_i$ ($i = 1, \cdots, N$) generate a direct summand $B$ of $H_2(W_1)$, which is mapped isomorphically to $H_2(W_1, M)$ under $j$. Now by Lemma 3.3 we may choose appropriate framings of the attaching spheres of the 2-handles such that the evaluation of $w_2(W_1)$ on $B$ is zero.

Therefore, we may attach 3-handles to $M_1$ (since $\pi_2(M_1) \cong \pi_2(W_1)$) using embedded 2-spheres representing $x_i$ ($i = 1, \cdots, N$). Denote by $W$ the resulting manifold with right hand boundary $M_2$. From the construction, we see that $W \simeq M^+$. Hence, $(W, M)$ is homologically acyclic. By Poincaré duality, we get $H_*(\tilde{W}, \tilde{M}_2) = 0$, which implies $W \simeq M_2$. Also from the construction it’s seen that $\tau(W, M) = [A] = (-1)^n\bar{\tau}$. By the duality of Whitehead torsion (cf. [13, p. 394]), we have $\tau(W, M_2) = \tau$.

In order to prove the uniqueness of $W$, we modify the construction in [5, p. 197]. Let $(W', M, M')$ be another such one-sided $h$-cobordism with base $M$. Suppose that $X = W \cup_M W'$. Then $X$ is an $h$-cobordism with two ends $M_2$ and $M'$. For the Whitehead torsions, we have that

$$\tau(M_2 \to X) = \tau(M_2 \to W) + \tau(W \to X)$$

$$= \tau + \tau(M_2 \to W')$$

$$= \tau + (-1)^n\bar{\tau}.$$

Suppose that $(V; X, X_1)$ is an $h$-cobordism rel boundary with base $X$ and Whitehead torsion $\tau(X \to V) = (-1)^{n-1}\bar{\tau}$ (see the figure below).

Then we have that

$$\tau(M_2 \to V) = \tau(M_2 \to X) + \tau(X \to V) = \tau$$

and

$$\tau(M_2 \to V) = \tau(M_2 \to X_1) + \tau(X_1 \to V) = \tau(M_2 \to X_1) + \tau.$$
These imply that $\tau(M_2 \to X_1) = 0$ and $X_1$ is an $s$-cobordism. On the other hand, since the Whitehead torsion $\tau(W \to V) = \tau(W \to X) + \tau(X \to V) = 0$, $V$ is an $s$-cobordism relative to the boundary from $W$ to $W'$. Therefore, by the $s$-cobordism theorem, the diffeomorphism $X_1 \cong M_2 \times [0,1]$ extends to a diffeomorphism $V \cong W \times [0,1]$ and $W$ is diffeomorphic to $W'$ relative to $M$.

The embedding plus construction for manifolds is considered by Guilbault and Tinsley [6, 7]. This is important to their study of ends of non-compact manifolds. We give an embedding plus construction with a given Whitehead torsion as follows.

**Theorem 3.4** Let $W^n$ be a connected manifold of dimension $n \geq 6$ and $M$ a closed component of the boundary of $W$. Suppose that $P$ is a normal subgroup of the kernel $\ker(\pi_1(M) \to \pi_1(W))$, which is normally generated by a finite set of elements. Then for any element $\tau \in \text{Wh}(\pi_1(M)/P)$, there exists a one-sided $h$-cobordism $(W'; M, M')$ embedded in $W$ fixing $M$ such that $\pi_1(W') = \pi_1(M)/P$ and $\tau(W', M') = \tau$.

**Proof.** By Theorem [3.2] there exists a one-sided $h$-cobordism $(W'; M, M')$ such that $\tau(W', M') = \tau$. According to Theorem 11.1 of Milnor [13], there exists a cobordism $(W_1; M', N)$ such that $\tau(W_1, N) = -\tau$. Glue $W'$ and $W_1$ together along $M'$ to get a new manifold $W' \cup_{M'} W_1$. Then $\tau(W' \cup_{M'} W_1, N) = 0$. Note that $(W' \cup_{M'}, W_1; M, N)$ is a one-sided $h$-cobordism with the inclusion map $N \hookrightarrow W' \cup_{M'} W_1$ a simple homotopy equivalence. By Theorem 1.1 in [6], there is an embedding $W' \cup_{M'} W_1 \to W$ fixing $M$. As $W'$ is a subset of $W' \cup_{M'} W_1$, we finish the proof.

**Proof of Theorem 1.5** For each one-sided $h$-cobordism $(W; M, N)$, the inclusion map $M \hookrightarrow W$ induces a homology equivalence with coefficients $\mathbb{Z}[\pi_1(W)]$ by Poincaré duality. This shows that the inclusion $M \hookrightarrow W$ is a Quillen plus
construction. Therefore, the kernel \( \ker(\pi_1(M) \to \pi_1(W)) \) is a perfect normal subgroup. Since both \( \pi_1(M) \) and \( \pi_1(N) \) are finitely presented, this kernel is normally finitely generated. Assign \((W; M, N)\) the Whitehead torsion \( \tau(W, N) \in \text{Wh}(\pi_1(W)) \). Since a diffeomorphism has trivial Whitehead torsion, this gives a well-defined map from \( S_h(M) \) to the right hand. Theorem 3.2 shows that this map is both surjective and injective.

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