Solving Topological 2D Quantum Gravity Using Ward Identities

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Abstract

A topological procedure for computing correlation functions for any \((1, q)\) model is presented. Our procedure can be used to compute any correlation function on the sphere as well as some correlation functions at higher genus. We derive new and simpler recursion relations that extend previously known results based on \(W\) constraints. In addition, we compute an effective contact algebra with multiple contacts that extends Verlindes’ algebra. Computational techniques based on the KdV approach are developed and used to compute the same correlation functions. A simple and elegant proof of the puncture equation derived directly from the KdV equations is included. We hope that this approach can lead to a deeper understanding of \(D = 1\) quantum gravity and non-critical string theory.

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1. Introduction

In recent years a class of models of two-dimensional gravity has been solved exactly. These models could be variously interpreted as \((p, q)\) conformal field theories coupled to two-dimensional gravity (i.e., Liouville field) or the continuum limit of a spin system on a random lattice. These interpretations correspond to the KdV and matrix model approaches, respectively. It was hoped that these solutions would provide insight into higher dimensional quantum gravity or non-critical string theory. Though these models are in principle exactly solvable, there were technical complications in providing explicit general solutions for correlation functions; i.e., solving the non-linear differential equations of the KdV hierarchy. Thus, it was difficult to develop intuition about the solutions. In this paper we provide a simple geometrical method for computing correlation functions for the \((1, q)\) models. Any correlation function on the sphere can be computed; partial results for higher genus will also be presented. The advantage of our method over that of the \(W\)-constraints is that our constraints (or Ward identities) can be written instantly for theories involving any number of primary fields. \(W\)-constraints are not known explicitly for anything greater than \(W_3\).

It has been suggested that the partition function for \(c < 1\) quantum gravity is given by the square of a \(\tau\) function of the KdV hierarchy satisfying the string equation. This followed the work on the matrix model formulation of two dimensional gravity\cite{1-6} and its double scaling limit\cite{7-9}. Douglas suggested that the KdV hierarchy along with the string equation may be used to describe minimal matter coupled to two dimensional gravity\cite{10}. The work of Dijkgraaf et al.\cite{11} and Fukuma et al.\cite{12} proved that the \(\tau\) function for 2D gravity satisfies a set of constraints given by generators of the Virasoro algebra. These generators act on the infinite dimensional space of all flow parameters of the KdV hierarchy. The constraints were used to derive recursion relations for the \((1, q)\) models of 2D gravity. These models are the critical topological points of minimal matter coupled to 2D gravity. For the \((1, 2)\) models, which have only one primary field, the Virasoro constraints are sufficient to completely solve the model. They are entirely consistent with the KdV approach.

The Virasoro constraints are not sufficient in themselves to completely solve the models with more than one primary field. It was then suggested that since the only set of operators forming a closed algebra with the Virasoro constraints are the \(W_n\) generators, perhaps the recursion relations necessary to completely solve the \((1, q)\) models could be derived by imposing the \(W_n\) constraints. Goeree\cite{13} has derived the \(W_3\) constraints from the string equation and obtained the corresponding recursion relations. These recursion relations are difficult to derive and provide very little intuition into the structure of quantum gravity. To
obtain explicit recursion relations for models with more than two primary fields using $W_n$ constraints is an extremely tedious process which has not yet been done.

Shortly after the matrix model and double scaling limit breakthrough, an alternative approach was suggested by Witten based on topological field theory ideas\[14\]. Subsequently, the so-called topological gravity, was developed further and it was shown to be related to the ‘regular’ version in a direct way\[15–20\].

We will present a very simple and intuitive procedure for computing any correlation function (including that of primary fields) in topological gravity with an arbitrary number of primary fields. Some higher genus results are also derived, including the one point function on the torus. This is a very interesting result because it is not computable from the $W_n$ constraints.

Our procedure involves imposing a Ward identity of a clearly geometrical origin (though we cannot derive it from a field theory). This Ward identity is superficially similar to the $W$-constraints. It involves the evaluation of a correlation function by summing over all possible surface degenerations. At each degeneration, a “complete” set of states is inserted. Contact terms between the operators are then evaluated at the degeneration point. Multiple contacts and degenerations occur for the $(1, q)$ models (for $q > 2$). We will show that the actual numerical values for the contact terms are irrelevant. All that matters is the number of contacts (that depends on the primary field “doing” the contact), ghost number conservation and one- and two-point correlation functions. (From the two-point function (“metric”) an “identity” operator will be constructed.) In order to achieve this we had to introduce “anti-states” or operators of negative dimension. These are, in our interpretation, unphysical operators whose higher point correlation functions (greater than two) vanish. This will be formulated more precisely in the body of the paper.

Our method relies on intuition from both topological and KdV approaches to quantum gravity. Some techniques for arriving at certain exact results directly from the KdV approach are presented in the appendices. We review our derivation of ghost number conservation for the $(1, q)$ models from the KdV and derive the metric used in the body of the paper. Perhaps, one of the more elegant results is a simple proof of the puncture equation for topological gravity directly from the KdV equations.

This paper is then organized as follows. In Section 2 we introduce the anti-states and discuss the “metric” and the “identity operator”. In section 3 we present our method on the sphere and derive the correspondence with previous known results. In Section 4 we compute effective contact terms and derive the Virasoro and $W_3$ constraints. In Section 5 we investigate the degeneration equation for the torus and higher genus. In the conclusion open
questions and directions for future research are discussed. A few appendices are included. In Appendix A we review the KdV approach to quantum gravity, in Appendix B we review the (1, q) models and the ghost number conservation rule. Appendix C is devoted to the extension of the KdV approach to anti-states and the derivation of the metric. In Appendix D we prove the puncture equation from the KdV and in appendix E we review the $W_3$ constraints.

2. Anti-States

In order to describe the Ward identities (or recursion relations) systematically, we first need to introduce the concept of an “anti-state” and identity operator. By anti-states we mean the states of negative dimension which are conjugate to the physical states on the sphere. These are necessary in order to have a non-vanishing two-point function or metric. Typically, of course, in a field theory the anti-states are also physical states. The appropriate analogy is that of momentum and position states. In quantum gravity there is a difficulty: the anti-states are not physical. Two-point functions of physical states vanish. Nevertheless, we will introduce these conjugate states in order to make sense of the theory. They will be unphysical states whose higher point functions (greater than two) will be set to zero. This will then allow for an elegant prescription for deriving a complete set of Ward identities that will determine all of the correlation functions of the (1, q) models of two-dimensional gravity.

Our procedure has many features reminiscent of a field theory interpretation. It is also analogous with the $W$-constraints. The results are in complete agreement with the KdV equations. Following the Verlinde's[16] we attempted to determine recursion relations from the possible contact terms and surface degenerations. We found to our surprise that if we included only all possible surface degenerations, we could reproduce the Virasoro constraints and recursion relations for the other primary fields consistent with the KdV. At each surface degeneration we would insert a “complete” set of states which would come in contact with the operator at the degeneration. Contact terms would arise effectively whenever one of the degenerating surfaces corresponded to a trivial correlation function. This procedure proved general enough to compute any correlation function on the sphere, including those of primary fields. Partial results for higher genus are also computable.

2.1. Defining the Hilbert Space

We will now discuss the basic structure of the correlation functions of the (1, q) models.*

*The (1, q) models are defined in Appendix B.
Our construction will be based in analogy to standard quantum mechanics (without gravity) where one can define a Hilbert space with an adjoint, metric, and identity operator such that correlation functions can be evaluated as follows,

$$\langle \alpha | \beta \rangle = \sum_{n,m} \langle \alpha | m \rangle \eta^{mn} \langle n | \beta \rangle,$$

where $|n\rangle$ are a complete set of states such that $I = \sum_{m,n} |m\rangle \eta^{mn} \langle n|$ is the identity and $\langle m|n\rangle = \eta_{mn}$ is the invertible metric. In quantum gravity there are difficulties in defining a meaningful Hilbert space. In the $(1, q)$ models of two-dimensional gravity it turns out that if one chooses the two-point function to be the metric, then the adjoint is not in the spectrum of original states. Indeed, all two point functions vanish. Nevertheless, one can define a non-vanishing two-point function by extending the set of allowed states. These new unphysical states will appear only on the boundaries of moduli space and simulate the interaction of the physical states with the boundaries. They never appear in the final correlation functions. In the following we will show precisely how this procedure works. But first we will develop the necessary tools for evaluating the correlation functions.

2.2. The Metric on the Boundary

From the KdV formulation of two-dimensional gravity one can show that the two-point function on the sphere for the $(1, q)$ models is given by (C.17), (see Appendix C)

$$\langle P_i P_j \rangle_0 = |i| \delta_{i+j}. \hspace{1cm} (2.2)$$

Since the operators, $P_i$, exist only for $i > 0$, we see that the two-point function always vanishes for physical operators. Thus, we seem to be unable to write a metric. Let us circumvent this problem by going ahead and defining a formal adjoint operation as follows,

$$P_i^\dagger = P_{-i}, \hspace{1cm} (2.3)$$

and, thus, extend the space of operators to include $P_i$ with $i < 0$. These operators have negative dimension. We now have an invertible metric defined by the two point function,

$$\langle P_i P_j \rangle_0 = \eta_{ij}, \hspace{1cm} (2.4)$$

where $\eta_{ij} = |i| \delta_{i+j}$. We still have to find an identity operator to complete our Hilbert space. We also need to show that the operators, $P_i$ with $i < 0$, decouple (except for the metric).
2.3. Definition of the Identity Operator

We must now define an identity operator. This is a more subtle problem. The naive choice,

\[ I = \sum_{j,i \neq 0 \mod q} |P_i \rangle \eta^{ij} \langle P_j |, \tag{2.5} \]

where \( \eta^{ij} \) is the inverted metric ("propagator"), does not reproduce the correct structure of the \((1,q)\) gravity. In some sense this problem cannot be resolved. An identity operator cannot be defined globally in a theory of gravity. But an identity as in (2.5) can be defined "locally" which is, in fact, all that we need. Because the theory is topological what we mean by local is a degeneration of the surface marked by a point. Thus, in computing correlation functions, (2.5) must be inserted at every possible degeneration of the surface consistent with the compactification of the moduli space. These degenerations correspond to two points coming in contact to pinch off a sphere or a handle (of a higher genus surface). In general, \((1,q)\) models permit multiple degenerations (up to \( q \) contacts). Then, at each degeneration one inserts (2.5) and evaluates the contact terms. The precise procedure will be explained in the next section.

2.4. The One-Point and other Correlations with Anti-States

Although introducing negative dimension states enabled us to write a non-vanishing two-point function, one has to check the consequences for other correlation functions. Indeed, the one-point function on the sphere can be non-vanishing only if the operator is \( P_{-q-1} \). Thus, in a regular \( i.e., \) without anti-states) theory, all one-point functions vanish on the sphere. In the extended model, however, this is not necessarily true. Since the operator \( P_{-q-1} \) exists, we cannot set the this one-point function to zero arbitrarily. One has to make sure that the theory remains consistent.

From the KdV approach, one can compute this correlation and get

\[ \langle P_{-q-1} \rangle_0 = -q. \tag{2.6} \]

We see that, indeed, this function is not zero and, as we will see later, is very important in the construction of the degeneration equation (Ward identities).

Higher point correlation functions with anti-states will be set to zero. This can be done consistently. They thus never contribute to the recursion relations. From the KdV approach

*This is a consequence of ghost number conservation rule. This was derived in [21,22] and is reviewed in the appendices.
we know that any correlation function with (at least one) puncture operator and more then three fields will always vanish if one of the fields in of negative dimension (anti-state). It is natural to extend this to any higher point function. A direct computation involving analytic continuation of KdV results supports this conclusion.

3. Computation of Correlation Functions on the Sphere

In this section the method for evaluating correlation functions on the sphere will be presented. We will introduce a diagrammatic notation that will help clarify the method.

3.1. Notation and Diagrams

We begin the computation of a correlation function by specifying an operator at the location in which the degenerations take place. This operator will conveniently be taken to be the first one and will be referred to as the marked operator. Fig. 1 represents the correlation function \( \langle \mathcal{P}_n \mathcal{P}_{i_1} \cdots \mathcal{P}_{i_m} \rangle_0 \). The marked operator will be represented by an ‘\( \times \)’ in the figure.

![Figure 1. The Correlation Function \( \langle \mathcal{P}_n \mathcal{P}_{i_1} \cdots \mathcal{P}_{i_m} \rangle_0 \).](image)

The marked operator \( \mathcal{P}_n \) is a descendant of the primary field \( \mathcal{P}_\alpha \), where \( \alpha = (n \mod q) \). This determines the number of contacts it can have. Descendants of \( \mathcal{P}_\alpha \) will have contacts with \( \alpha \) operators. We will represent a contact as an overbrac e in the correlation function and in figures it will be represented by a line connecting the marked operator with the other operator in contact. For multiple contact, the overbrace will be over all the operators and in the figures we will have a line connecting the marked operotor with each operator it comes in contact with. For example, Fig. 2 represents the correlation function \( \langle \mathcal{P}_s \mathcal{P}_t \mathcal{P}_j \mathcal{P}_k \mathcal{P}_l \rangle_0 \), where \( \mathcal{P}_s \) is in contact with both \( \mathcal{P}_j \) and \( \mathcal{P}_j \).
Figure 2. A Correlation Function with Contacts: $\langle \overline{P}_2 P_i P_j P_k P_l \rangle_0$.

A degeneration of a sphere results in a split to two spheres. Multiple degenerations result in multiple spheres. In our method, each degeneration involves the insertion of the "identity" operator. In the figures we will represent the degeneration in an obvious way, and the operators from the identity will be subscripted with integers to distinguish them from the original operators in the correlation function. Also, a summation will be assumed on integer subscripts. Fig. 3, for example, represent the double degeneration of $\langle P_a P_b \rangle_0$ into

$$\sum_{i_1,j_1,i_2,j_2} \langle P_a P_{i_1} P_{i_2} \rangle_0 \langle P_b P_{j_1} \rangle_0 \langle P_b P_{j_2} \rangle_0 \eta^{i_1 j_1} \eta^{i_2 j_2}.$$

Figure 3. A Double Degeneration of $\langle P_a P_b \rangle$.

The combination of degenerations and contacts is the only contribution to correlation functions in our method. The marked operator comes in contact with degenerations, the number of which is determined by the primary field from which the marked operator descends. In the figures, we combine the notations for contacts and degenerations and get, for example, Fig. 4. In Figure 4 the operator $P_2$ in the correlation function $\langle P_2 P_a P_b \rangle_0$ comes in contact with a double degeneration.
3.2. The Degeneration Equation

We are now in a position to present the method. The main idea is the following statement:

Given a correlation function \( \langle \mathcal{P}_n \mathcal{P}_i \ldots \mathcal{P}_{i_m} \rangle_0 \), with the marked operator \( \mathcal{P}_n \), then the summation over all the contacts with degenerations of this operator vanishes.

Formally, we will write

\[
\sum_\Delta \langle \mathcal{P}_n \mathcal{P}_i \ldots \mathcal{P}_{i_m} \rangle_0 = 0, \tag{3.1}
\]

where the symbol \( \sum_\Delta \) means ‘sum over all degenerations when the first operator performs the contacts’. We will call this equation the degeneration equation.

Let us demonstrate the technique by computing a simple correlation function. Let \( q = 4 \) (i.e., three primary fields), and we will compute \( \langle \mathcal{P}_2 \mathcal{P}_2 \mathcal{P}_1 \rangle_0 \). \( \mathcal{P}_2 \) is the marked operator (it is first) and since it’s a descendant of \( \mathcal{P}_2 \) (2 mod 4 = 2)*, it will come in contact with two operators. Thus we have a double degeneration. The degeneration equation is then

\[
0 = \sum_\Delta \langle \mathcal{P}_2 \mathcal{P}_2 \mathcal{P}_1 \rangle_0 = \\
\langle \mathcal{P}_2 \mathcal{P}_i \mathcal{P}_i \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{j_1} \rangle_0 \langle \mathcal{P}_{j_2} \rangle_0 \eta^{i_1 j_1} \eta^{i_2 j_2} + \langle \mathcal{P}_2 \mathcal{P}_i \mathcal{P}_i \mathcal{P}_2 \rangle_0 \langle \mathcal{P}_{j_1} \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{j_2} \rangle_0 \eta^{i_1 j_1} \eta^{i_2 j_2} + \\
\langle \mathcal{P}_2 \mathcal{P}_i \mathcal{P}_i \mathcal{P}_2 \rangle_0 \langle \mathcal{P}_{j_1} \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{j_2} \rangle_0 \eta^{i_1 j_1} \eta^{i_2 j_2} + \langle \mathcal{P}_2 \mathcal{P}_i \mathcal{P}_i \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{j_1} \mathcal{P}_2 \rangle_0 \langle \mathcal{P}_{j_2} \rangle_0 \eta^{i_1 j_1} \eta^{i_2 j_2} + \\
\langle \mathcal{P}_2 \mathcal{P}_i \mathcal{P}_i \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{j_1} \mathcal{P}_2 \rangle_0 \langle \mathcal{P}_{j_2} \rangle_0 \eta^{i_1 j_1} \eta^{i_2 j_2} + \langle \mathcal{P}_2 \mathcal{P}_i \mathcal{P}_i \mathcal{P}_2 \rangle_0 \langle \mathcal{P}_{j_1} \rangle_0 \langle \mathcal{P}_{j_2} \mathcal{P}_1 \rangle_0 \eta^{i_1 j_1} \eta^{i_2 j_2} + \\
\langle \mathcal{P}_2 \mathcal{P}_i \mathcal{P}_i \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{j_1} \rangle_0 \langle \mathcal{P}_{j_2} \mathcal{P}_2 \rangle_0 \eta^{i_1 j_1} \eta^{i_2 j_2}, \tag{3.2}
\]

*Obviously, \( \mathcal{P}_2 \) is the primary field and as such it’s the 0th descendant of itself.
where an implicit sum over $i_1, j_1, i_2$ and $j_2$ is understood. Basically, we have a situation similar to Figure 4. All we do is distribute the non-marked operators in the correlation function in all possible ways. Note the two inverse metrics is each term: those are from the double insertion of the identity operator.

To continue we need to evaluate the contact terms. Using ghost number conservation, we translate the contact term into one operator of the same ghost number, with some yet to be determined coefficient. Thus, for example,

$$\overline{\mathcal{P}_2 \mathcal{P}_a \mathcal{P}_b} = \beta_{2ab} \mathcal{P}_{2+a+b-2(q+1)}.$$  

Assume that $\beta_{2ab}$ is proportional to $|a| |b|$. This assumption, that the contact term is proportional to the dimensions of the other (non-marked) operators, is not surprising if we remember that the two-point function is proportional to the dimension. In this way the contact coefficient will cancel the contributions from the inverse metric $\eta^{i_1 j_1}$.

Now, looking at equation 3.2, we see that the proportionality constant in the contact terms, if not zero, will cancel out of the equation. Of course, if it is zero, then the theory is trivial. Thus, the exact numerical value of the contact coefficient is irrelevant. Only the fact that it’s proportional to the dimensions of the non-contact operators is important.

One can easily convince oneself the the cancellation of the numerical coefficient in the contact term happens for any $(1, q)$ model. We thus will define the constant to be 1 and we get the following contact algebra,

$$\overline{\mathcal{P}_n \mathcal{P}_{i_1} \cdots \mathcal{P}_{i_m}} = |i_1| \cdots |i_m| \mathcal{P}_{n+\sum_{j=1}^{m} i_j-m(q+1)}.$$  

Returning to our example, using the ghost number conservation rule, the contact term value and the metric, we can sum over the indices in equation 3.2. We get,

$$0 = \langle \mathcal{P}_2 \mathcal{P}_2 \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 + \langle \mathcal{P}_{-2} \mathcal{P}_2 \rangle_0 \langle \mathcal{P}_{-1} \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 + \langle \mathcal{P}_{-2} \mathcal{P}_2 \rangle_0 \langle \mathcal{P}_{-1} \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 + \langle \mathcal{P}_{-2} \mathcal{P}_2 \rangle_0 \langle \mathcal{P}_{-1} \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 + \langle \mathcal{P}_{-2} \mathcal{P}_2 \rangle_0 \langle \mathcal{P}_{-1} \mathcal{P}_1 \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 + \langle \mathcal{P}_{-5} \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 + \langle \mathcal{P}_{-5} \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 + \langle \mathcal{P}_{-5} \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 + \langle \mathcal{P}_{-5} \rangle_0 \langle \mathcal{P}_{-5} \rangle_0 \langle \mathcal{P}_{-5} \rangle_0,$$

and substituting the values $\langle \mathcal{P}_1 \mathcal{P}_{-1} \rangle_0 = \langle \mathcal{P}_{-1} \mathcal{P}_1 \rangle_0 = 1$, $\langle \mathcal{P}_2 \mathcal{P}_{-2} \rangle_0 = \langle \mathcal{P}_{-2} \mathcal{P}_2 \rangle_0 = 2$ and $\langle \mathcal{P}_{-5} \rangle_0 = -4$, one gets the answer,

$$\langle \mathcal{P}_2 \mathcal{P}_2 \mathcal{P}_1 \rangle_0 = 1.$$
It is instructive to compute the same correlation function when \( P_1 \) is the marked operator. In this case there is only one degeneration (\( P_1 \) is the puncture operator!), and the degeneration equation is

\[
0 = \sum_\Delta \langle P_1 P_2 P_2 \rangle_0 =
\sum_{i_1, j_1} \langle \widehat{P_1 P_{i_1}} P_2 P_2 \rangle_0 \langle P_{j_1} \rangle_0 \eta^{i_1 j_1} + \\
2 \sum_{i_1, j_1} \langle \widehat{P_1 P_{i_1}} P_2 P_2 \rangle_0 \langle P_{j_1} P_2 \rangle_0 \eta^{i_1 j_1} + \\
\sum_{i_1, j_1} \langle \widehat{P_1 P_{i_1}} P_2 P_2 \rangle_0 \langle P_{j_1} P_2 \rangle_0 \eta^{i_1 j_1}
\]

(3.7)

Notice the combinatorial factor in the second term. Continuing in a similar fashion to equation (3.5), one get the same result.

It is clear that the anti-states are very important here. The fact that the one-point correlation \( \langle P_{-q-1} \rangle_0 \) is non-vanishing guarantees that by performing the sum over degenerations on a correlation function, the same correlation function will reappear in the sum. There will always be terms in the sum in which all the other (\( i.e. \), non-marked) operators will be on one of the surfaces (spheres). By ghost number conservation, the last operator on this surface that either came from the contact term or from the identity operator is guaranteed to be identical to the original marked operator. In equation (3.5), for example, the first, eighth and ninth terms are of this kind.

3.3. The General Skeletal Degeneration Equation

As discussed above, one notes that the marked operator is the only element that specifies the number of degenerations. Also, one notes, that that operator comes in contact only with the degenerations, which are represented by the identity operator. Thus we can write a skeletal degeneration equation that depends only on the first operator. We will write, for the general \((1, q)\) model with \( q - 1 \) primary fields,

\[
\sum_\Delta \langle \sigma_n(\mathcal{P}_\alpha) X \rangle_0 = \sum_{\bigcup_{i=0}^q X_i = X} \langle [\sigma_n(\mathcal{P}_\alpha) \mathcal{P}_{-i_1} \cdots \mathcal{P}_{-i_m}] X \rangle_0 \langle \mathcal{P}_{i_1} X_1 \rangle_0 \cdots \langle \mathcal{P}_{i_m} X_\alpha \rangle_0 = 0,
\]

(3.8)

where we defined the normalized contact

\[
[\mathcal{P}_n \mathcal{P}_{i_1} \cdots \mathcal{P}_{i_m}] = \mathcal{P}_{n+\sum_{j=1}^m i_j - m(q+1)},
\]

(3.9)
Note that we didn’t write the inverse metric factors in equation (3.8). Those cancel with the regular contact term and yield the normalized contact term. That is
\[ \tilde{p}_n \tilde{p}_{i_1} \cdots \tilde{p}_{i_m} = |i_1| \cdots |i_m| [p_n p_{i_1} \cdots p_{i_m}]. \] (3.10)

There is an implicit summation in equation 3.8 over the $i_j$’s. However, because of ghost number conservation, there will be only one possible $i_j$. The ‘X’ in equation (3.8) is some arbitrary collection of operators and the various $X_j$ have no element in common.

4. Recursion Relations and Effective Contacts

In equation (3.8), there are $(\alpha + 1)$ terms in the sum that are proportional to the original correlation function, as explained in the end of subsection 3.2. Computing these terms separately we can transform the degeneration equation into a recursion relation. We will get the following skeletal recursion relation
\[ \langle \sigma_n (p_\alpha) X \rangle_0 = -\frac{1}{(\alpha + 1)(-q)^\alpha} \sum_{\bigcup_{X_i = X} X_i \neq X} \langle [\sigma_n (p_\alpha) \tilde{p}_{i_1} \cdots \tilde{p}_{i_\alpha}] X_0 \rangle_0 \langle p_{i_1} X_1 \rangle_0 \cdots \langle p_{i_\alpha} X_\alpha \rangle_0. \] (4.1)

For $\alpha = 1$, for example, the above equation is
\[ \langle \sigma_n (p_1) X \rangle_0 = \frac{1}{2q} \sum_{\bigcup_{X_i = X} X_i \neq X} \langle [\sigma_n (p_1) \tilde{p}_{i_1}] X_0 \rangle_0 \langle p_{i_1} X_1 \rangle_0, \] (4.2)
which is identical to the Virasoro constraints restricted to the sphere.

It is interesting to note that in equation 4.2 we recover the contact algebra as written by the Verlindes. (Actually, this is slightly more general, since the collection of operators, $X$, can contain operators that are not descendants of the puncture operator, $p_1$.) Explicitly writing $X$ and evaluating the possible two-point functions, we get
\[ \langle \sigma_n (p_1) p_{i_1} \cdots p_{i_m} \rangle_0 = \frac{1}{2q} \sum_{X_0 \bigcup_{X_1 = X} X_i \neq X} \langle [\sigma_n (p_1) \tilde{p}_{i_1}] X_0 \rangle_0 \langle p_{i_1} X_1 \rangle_0. \] (4.3)

The first sum on the right hand side is a result of splitting $X$ into a single operator and the rest, and since this single operator can be in any of the two surfaces, we get the factor of two, and hence the value $1/q$ in front. Also note the factor of $i_j$ in the first term. That is a
result of evaluating the two-point function. The second term on the right hand side is the rest of the possible divisions of the set $X$. In the language of the Verlindes we identify the first sum as the contribution from the contact terms and the second sum as degenerations of the surface. We will refer to these (that is-Verlindes) contact terms as effective.

Similarly, for $\alpha = 2$ we can get an effective contact algebra for the second primary field. It is interesting to note that although in the degeneration equation (3.8) the operator $P_2$ has contacts with 2 other operators (no more and no less), the effective contact algebra has single and double contacts.

Let us demonstrate this by calculating a less trivial example. Consider the $(1,3)$ model which has two primary fields, $P_1$ and $P_2$, with descendants, $P_{3i+1}$ and $P_{3i+2}$, respectively. Now using the above prescription, we can write a degeneration equation to compute a correlation function involving $P_{3i+2}$ and $P_{1}$.

$$\sum_{a=0}^{i+2} \sum_{b=0}^{i+2-a} \left( \begin{array}{c} i+2 \\ a \\ b \end{array} \right) \langle [P_{3i+2}, P_{-t_1}, P_{-t_2}]P_{1}^{a} \rangle_{0} \langle P_{t_1}^{b} P_{1}^{i+2-a-b} \rangle_{0} = 0. \quad (4.4)$$

Separating the sum to the cases for which $a$, $b$ and/or $i+2-a-b$ are zero or one, we get a recursion relation without anti-states,

$$\langle P_{3i+2} P_{1}^{i+2} \rangle_{0} = \frac{2}{3} (i+2) \langle P_{3(i-1)+2} P_{1}^{i+1} \rangle_{0} - \frac{1}{9} (i+2)(i+1) \langle P_{3(i-2)+2} P_{1}^{i} \rangle_{0}$$

$$+ \frac{2}{9} \sum_{a=2}^{i} \left( \begin{array}{c} i+2 \\ a \end{array} \right) \langle P_{3(a-2)+2} P_{1}^{a} \rangle_{0} \langle P_{3(i-a)+2} P_{1}^{i+2-a} \rangle_{0}$$

$$- \frac{2}{27} (i+2) \sum_{a=2}^{i+1} \left( \begin{array}{c} i+1 \\ a \end{array} \right) \langle P_{3(a-2)+2} P_{1}^{a} \rangle_{0} \langle P_{3(i-a-1)+2} P_{1}^{i+1-a} \rangle_{0}$$

$$- \frac{1}{27} \sum_{a=2}^{i} \sum_{b=2}^{i-a} \left( \begin{array}{c} i+2 \\ a \\ b \end{array} \right) \langle P_{3(a-2)+2} P_{1}^{a} \rangle_{0} \langle P_{3(b-2)+2} P_{1}^{b} \rangle_{0} \langle P_{3(i-a-b)+2} P_{1}^{i+2-a-b} \rangle_{0}. \quad (4.5)$$

The terms in this equation are interpreted in the following way. The first is an effective contact between the marked operator and (each) one of the other operators. The second term is a double contact with two of them. The third term is a single contact of the marked operator with a single surface degeneration and the forth is a double contact with one operator and one degeneration. The last term is a contact with double degeneration.

One can extract the effective contact between $P_{3i+2}$ and $P_{1}$. It is

$$\langle P_{3i+2} P_{1} \rangle = \frac{2}{3} P_{3(i-1)+2}. \quad (4.6)$$
where we introduced a notation for the effective contact. Similarly, the effective double contact between $P_{3i+2}$ and two $P_1$'s is

$$\{ \{ P_{3i+2} P_1 P_1 \} \} = -\frac{2}{9} P_{3(i-2)+2}.$$  \hspace{1cm} (4.7)

It is instructive to compare equations 4.4 and 4.5. The second appears to be far more complicated, although it can be derived from the first one. The $W_3$ constraints will give rise to such an equation. (See Appendix E.)

Following the same procedure, we can compute more general recursion relations. We cannot compare them with the results from $W_q$ constraints ($q > 3$), since no explicit results are available.

A generalization of equation (4.5) to the $(1, q)$ model is given by the following,

$$\langle P_{qi} P_{i+2} \rangle_0 = \alpha(2) \sum_{a=0}^{i+2} \binom{i+2}{a} \langle P_{qa-q-1} P_{a}^1 \rangle_0 \langle P_{q(i+2-a)-q-1} P_{1}^{i+2-a} \rangle_0$$

$$+ \alpha(3) \sum_{a=0}^{i+2} \sum_{b=0}^{i+2-a} \binom{i+2}{a} \binom{i+2-a}{b} \langle P_{qa-q-1} P_{a}^1 \rangle_0 \langle P_{qb-q-1} P_{b}^1 \rangle_0 \langle P_{q(i+2-a-b)-q-1} P_{1}^{i+2-a-b} \rangle_0$$

$$+ \ldots$$

$$+ \alpha(k) \sum_{a_1=0}^{i+2} \sum_{a_2=0}^{i+2-a_1} \cdots \sum_{a_{k-1}=0}^{i+2-a_{k-2}} \binom{i+2-a_{a_1}}{a_1} \binom{i+2-a_{a_2}}{a_2} \cdots \binom{i+2-a_{a_{k-1}}}{a_{k-1}} \times$$

$$\langle P_{qa_{1}-q-1} P_{a_1}^1 \rangle_0 \langle P_{qa_{2}-q-1} P_{a_2}^1 \rangle_0 \cdots \langle P_{qa_{k-1}-q-1} P_{a_{k-1}}^1 \rangle_0 \langle P_{q(i+2-a_{\sum_{j=1}^{k-2} a_j})-q-1} P_{1}^{i+2-a_{\sum_{j=1}^{k-1} a_j}} \rangle_0$$

$$+ \ldots$$  \hspace{1cm} (4.8)

where the coefficients $\alpha(k)$ are

$$\alpha(k) = \frac{(-1)^k}{k!} \prod_{j=1}^{k-1} \left( 1 - \frac{j}{q} \right).$$  \hspace{1cm} (4.9)

Another formula we will need in order to derive the effective contacts is one involving
three distinct primary fields. This can be similarly derived,

\[
\langle \mathcal{P}_{q_i+\alpha} \mathcal{P}_{q_j+q-\alpha} \mathcal{P}_{1}^{i+j+1} \rangle_0 = \\
+ 2\beta(2) \sum_{a=0}^{i+j+1} \binom{i+j+1}{a} \langle \mathcal{P}_{q_a-q_i-q-\alpha} \mathcal{P}_{q_j+q-\alpha} \mathcal{P}_{1}^a \rangle_0 \langle \mathcal{P}_{q(i+j+1-a)-q-1} \mathcal{P}_{1}^{i+j+1-a} \rangle_0 \\
+ 3\beta(3) \sum_{a_1=0}^{i+j+1} \sum_{a_2=0}^{i+j+1-a_1} \binom{i+j+1}{a_1} \binom{i+j+1-a_1}{a_2} \times \\
\langle \mathcal{P}_{a_1,q-q_j-q-\alpha} \mathcal{P}_{q_j+q-a} \mathcal{P}_{1}^{a_1} \rangle_0 \langle \mathcal{P}_{q_{a_2}-q-1} \mathcal{P}_{1}^{a_2} \rangle_0 \langle \mathcal{P}_{q(i+j+1-a_1-a_2)-q-1} \mathcal{P}_{1}^{i+j+1-a_1-a_2} \rangle_0 \\
+ \ldots \\
+ k\beta(k) \sum_{a_1=0}^{i+j+1} \sum_{a_{k-1}=0}^{i+j+1-\sum_{j=1}^{k-2} a_j} \binom{i+j+1-\sum_{j=1}^{k-2} a_j}{a_1} \ldots \binom{i+j+1-\sum_{j=1}^{k-2} a_j}{a_{k-1}} \times \\
\langle \mathcal{P}_{a_1,q-q_j-q-\alpha} \mathcal{P}_{q_j+q-a} \mathcal{P}_{1}^{a_1} \rangle_0 \langle \mathcal{P}_{q_{a_2}-q-1} \mathcal{P}_{1}^{a_2} \rangle_0 \ldots \langle \mathcal{P}_{q_{a_{k-1}}-q-1} \mathcal{P}_{1}^{a_{k-1}} \rangle_0 \times \\
\langle \mathcal{P}_{q(i+j+1-\sum_{j=1}^{k-2} a_j)-q-1} \mathcal{P}_{1}^{i+j+1-\sum_{j=1}^{k-1} a_j} \rangle_0 + \ldots \ .
\]

where

\[
\beta(k) = \frac{(-1)^k}{k!q^{k-1}} \prod_{l=0}^{k-2} (\alpha - l).
\]

We evaluate the effective contact terms derived from equations (4.8) and (4.10). First, using equation (4.8) we can show that,

\[
\{ \{ \mathcal{P}_{q_i+q-1} \mathcal{P}_{1} \} \} = \frac{q-1}{q} \mathcal{P}_{q_i-1},
\]

\[
\{ \{ \mathcal{P}_{q_i+q-1} \mathcal{P}_{1} \} \} = -\frac{(q-1)(q-2)}{q^2} \mathcal{P}_{q_i-1},
\]

\[
\vdots
\]

\[
\{ \{ \mathcal{P}_{q_i+q-1} \mathcal{P}_{1} \} \} \ldots \{ \{ \mathcal{P}_{1} \} \} = (-1)^k \prod_{j=1}^{k-1} (1 - \frac{j}{q}) \mathcal{P}_{q(i-k+2)-1}.
\]

From equation (4.10) we get the following effective contact terms,

\[
\{ \{ \mathcal{P}_{q_i+\alpha} \mathcal{P}_{q_j+q-\alpha} \} \} = \frac{\alpha}{q} (qj + q - \alpha) \mathcal{P}_{q_i+q_j-1},
\]

\[
\{ \{ \mathcal{P}_{q_i+\alpha} \mathcal{P}_{1} \} \} = \frac{\alpha}{q} \mathcal{P}_{q_i+\alpha-q},
\]

\[
\{ \{ \mathcal{P}_{q_i+\alpha} \mathcal{P}_{q_j+q-\alpha} \mathcal{P}_{1} \} \} = -\frac{\alpha(\alpha-1)}{q^2} (qj + q - \alpha) \mathcal{P}_{q(i+j)-q-1},
\]

and so forth.
By looking at more complicated recursion relations with more distinct primary fields, we conjecture that in general the effective contact algebra is given as follows,

\[
\{ \{ P_i P_j \} \} = 1! \left( \frac{\alpha}{q} \right)^j P_{i+j-(q+1)},
\]

\[
\{ P_i P_j P_k \} = -2! \left( \frac{\alpha}{q^2} \right)^{jk} P_{i+j+k-2(q+1)},
\]

\[
\vdots
\]

\[
\{ P_i \prod_{k=1}^n P_{j_k} \} = (-1)^{n-1} n! \left( \frac{\alpha}{q^n} \right) \prod_{k=1}^n j_k P_{i+\sum_{k=1}^n j_k-n(q+1)},
\]

where \( \alpha = i \mod q (> 0) \).

One may explore what kind of algebra one gets by considering the commutators of the two-term contacts in (4.14). We get the following result,

\[
[ P_i, P_j ] = \frac{1}{q} ((i \mod q) j - (j \mod q) i) P_{i+j-q-1}.
\]  

(4.15)

If \( q = 2 \), we simply get

\[
[ P_i, P_j ] = \frac{1}{2} (j - i) P_{i+j-3}.
\]

For \( q = 3 \) let us define \( \hat{Q}_i \equiv P_{3i+2} \) and \( \hat{P}_j \equiv P_{3j+1} \). Then,

\[
[ \hat{P}_i, \hat{Q}_j ] = \frac{1}{3} (2j - i) \hat{Q}_{i+j-4};
\]

\[
[ \hat{P}_i, \hat{P}_j ] = \frac{2}{3} (j - i) \hat{P}_{i+j-4};
\]

\[
[ \hat{Q}_i, \hat{Q}_j ] = 0,
\]

(4.16)

The last commutator follows from the fact that there is no operator with the correct ghost number.

5. The Torus and Higher Genus

It is natural to investigate the generalization to higher genus. It turns out that there is a natural extension of the degeneration equation in higher genus. One problem, however, does arise because of a counting ambiguity in ordering the operators in the identity insertion. Nevertheless, we will be able to derive partial results, whenever this ambiguity does not occur. Our discussion will be mostly restricted to genus 1, but we will also make some comments on higher genus at the end.
5.1. Notation and Diagrams

We begin by naturally extending the various diagrams of Section 3.1 to tori. A correlation function on a genus one surface, \( \langle P_n P_{i_1} \ldots P_{i_m} \rangle_1 \), is depicted in Fig. 5. Again we use an \( \times \) symbol for the marked operator, \( P_n \).

![Figure 5. The Correlation Function \( \langle P_n P_{i_1} \ldots P_{i_m} \rangle_1 \).](image)

Contacts between operators will be represented by lines, as before (see Fig. 2). The difference in higher genus arises in the degenerations. In genus 1 there are two possible types of degenerations. The first is the splitting of a sphere off the torus, and the second is a handle being pinched off. One should notice, however, that the marked operator can end up in the sphere being split off (or, of course, remain in the torus). Fig. 6 is the degeneration of the first kind, a sphere being split off. The correlation function \( \langle P_1 X \rangle_1 \) has a sphere split off and the marked operator can end up in two surfaces.

![Figure 6. Two Possibilities for the Marked Operator after a Split.](image)

A handle pinch in the same correlation function is shown in Fig. 7. Notice the insertion of the identity operator.
One notes, that when a handle is pinched, there two operator from the identity insertion end up in the same surface (in our case, a sphere). This actually increases the number of operators in the correlation function, but decreases the genus.

5.2. The Degeneration Equation at Genus One

Formally, the degeneration equation is exactly the same as before; i.e., the sum over all degenerations in contact with the marked operator is zero. However, one has to remember that there is also the possibility of a handle pinch. The easiest way to visualize the degeneration equation is to draw a skeletal diagram, in which the marked operator will be in contact with the appropriate identity operator, and the sum over different types of degenerations will be explicitly present. All there is to do then is to distribute the other (non-marked) operators and calculate.

Let us demonstrate it for the simplest case for which the marked operator is a descendant of \( P_1 \). Such an operator, as explained above, can have only one contact. Thus, the skeletal degeneration equation will be,

\[
\sum_{\Delta} \langle P_{q+1} \ast \rangle_1 = \langle [P_{q+1} P_{-\ast}] \ast \rangle_1 \langle P_{\ast \ast} \rangle_0 + \langle [P_{q+1} P_{-\ast}] \ast \rangle_0 \langle P_{\ast \ast} \rangle_1 + \langle [P_{q+1} P_{-\ast}] P_{\ast \ast} \rangle_0 = 0. \tag{5.1}
\]

In this equation, the “\( \ast \)” in the left hand side is any collection of physical operators and the “\( \ast \)” on the right hand side is an implicit summation over all the possible distributions of these operators into the various surfaces. Diagrammatically, the first two parts are depicted in Fig. 6 and the third part in Fig. 7.

In general, for descendants of higher primary fields, there will be more then one contact (and degeneration). In that case the specified number of degenerations could be a
combination of a pinch and multiple splittings or only splittings.

5.3. An Example Computation

As an example, let us compute the correlation function $\langle P_{q(i+1)+1} P_1^i \rangle_1$ when $i > 0$. Using the degeneration equation (5.1), we get,

$$\sum_{\Delta} \langle P_{q(i+1)+1} P_1^i \rangle_1 = \sum_{a=0}^{i} \binom{i}{a} \langle [P_{q(i+1)+1} P_{j-1}^a] P_1^i \rangle_1 \langle P_j P_1^{i-a} \rangle_0 +$$

$$\sum_{a=0}^{i} \binom{i}{a} \langle [P_{q(i+1)+1} P_{j-1}^a] P_1^i \rangle_0 \langle P_j P_1^{i-a} \rangle_1 + \langle [P_{q(i+1)+1} P_{j-1}^a] P_1^i \rangle_0 = 0. \quad (5.2)$$

As before, there is an implicit summation over the index $j$ (from the identity operator). In the first two terms, as before, there is only one value of $j$ for which the correlation function will not vanish (from ghost number conservation). In the third term, however, there is a range of possible $j$’s for which the correlation will not vanish. This is exactly the point where we have the ambiguity problem with normal ordering. It is true that as long as the range of possible $j$’s is finite, the computation can proceed as usual and the result is totally consistent with the KdV approach. It is when the range of $j$ is infinite that a problem arises. At first glance the counting ambiguity seems never to occur, but, alas, this is not true. In the above example it occurs for $i = 0$ (only!). In fact, one can easily convince oneself that only when the marked operator is a descendant of $P_1$ do we have an ambiguity, since this is the only one point correlation function. For higher primary fields, however, there can be a handle pinch together with sphere splitting. In that case the infinity problem will arise again. Our method works whenever there is no such ambiguity. It breaks down otherwise.

Continuing with the above example (and limiting ourselves to $i > 0$), we can evaluate the contact terms and compute the possible $j$’s. We get,

$$\sum_{a=0}^{i} \binom{i}{a} \langle P_{q(a+1)+1} P_1^a \rangle_1 \langle P_{q(i-2)+q-1} P_1^{i-a} \rangle_0 +$$

$$\sum_{a=0}^{i} \binom{i}{a} \langle P_{q(a+2)+q-1} P_1^a \rangle_0 \langle P_{q(i-1)+1} P_1^{i-a} \rangle_1 + \sum_{j=0}^{qi} \langle P_{qi-j} P_j P_1^i \rangle_0 = 0. \quad (5.3)$$

Transforming this equation into a recursion relation, as was done on the sphere, we can compute the result explicitly. It agrees with the KdV result which is

$$\langle P_{q(i+1)+1} P_1^i \rangle_1 = \frac{q - 1}{24} \prod_{j=1}^{i+1} (j + \frac{1}{q}). \quad (5.4)$$

*It might be obvious here how to generalize this to higher genus.*
Note, though, that in equation (5.4) one can set \( i = 0 \). The result is

\[
\langle \mathcal{P}_{q+1} \rangle_1 = \frac{q^2 - 1}{24q^2}.
\]  

(5.5)

It leads one to think that there might be a way to regularize the infinite summation in the degeneration equation and get a meaningful result. We have been able to get some partial result, but we haven’t been able to solve the general problem.

5.4. Higher Primary Fields on the Torus

When the marked operator is a descendant of a higher primary field, there are two points that one has to be careful with. The first is combinatorics. One has to count carefully the number of possible degenerations. The second point to notice is the distribution of the operators from the identity operator over the different surfaces when there is a pinch and a split. There are two possibilities to perform a pinch and a degeneration on a torus. The first is to first pinch the handle and then to pinch a sphere from the side. The second is to pinch the handle in two places at once. In Fig. 8 those two possibilities are depicted for the correlation function \( \langle \mathcal{P}_2 \mathcal{P}_k \ldots \mathcal{P}_l \rangle_1 \).

![Figure 8. Two Ways for a Pinch and a Split.](image)

It is interesting to note that for the (1, 3) model, we can compute many correlation functions at genus one, as we do not get infinite summations.

5.5. Higher Genus

In principle, the degeneration equation can be generalized to higher genera (larger than 1). However, the regularization ambiguity limits the results, as the probability of encountering it grows at higher genus.
It is true, nevertheless, that if the marked operator is a descendant of the puncture operator $\mathcal{P}_1$, then the degeneration equation exists. Of course, after the first recursion, one might find that one needs to compute correlation functions with other marked operators that may have the ambiguity problem. For the (1, 2) model which has only one primary field any correlation can be computed. For the (1, 3) model, we also suspect that any correlation function can be computed, but this is due to a coincidence that the infinite summation does not occur. An example of a correlation function with normal ordering problem is $\langle \mathcal{P}_3 \mathcal{P}_7 \rangle_1$ in the (1, 4) model.

6. Conclusions

In this paper we used some of the general results about the correlation functions of the (1, $q$) models derived from the KdV hierarchy[21] in order to achieve a better understanding of the structure of quantum gravity. A simple underlying structure for the correlation functions of the (1, $q$) models on the sphere was discovered. It is applicable to models with an arbitrary finite number of primary fields. We introduced an “identity” operator (which in quantum theories of gravity is a difficult concept to define) and computed correlation functions by inserting it at the surface degenerations. This is intuitively how one would have expected to compute correlation functions in a topological quantum theory of gravity. We demonstrated the immense simplicity of this procedure over that based on the $W_n$ constraints. Any correlation function on the sphere could be computed directly including those of primary fields.

The introduction of a formal adjoint operation and a local identity operator may hint at possible extensions to models of gravity with a continuous number of “primary fields.” The operators of negative dimension which were introduced in Section 2 to simplify computations may turn out to have some physical interpretation. They certainly cannot be interpreted as forms on the relevant moduli space (that of Riemann surfaces for $q = 2$).

Since we provide explicit expressions for correlation functions as functions of $q$, an interesting open question which may be considered using the results of this paper and References [21] and [22] is to consider the limit $q \to \infty$. This should correspond to some semi-classical limit with $c = -\infty$, depending on the way the limit is taken. It is clear that if the limit is taken appropriately some of the correlation functions will converge to a finite value. It is not transparent, however, how the fields $\mathcal{P}_i$ relate to the Liouville field if $i$ becomes a continuous parameter as $q \to \infty$.

Another problem is the counting ambiguity at genus one (and higher). A few possibil-
ities to cure this problem are currently under investigation. One is to find an appropriate regularization scheme. Another is to investigate the possibility that the anti-states do not commute with the regular physical states. The analogy might be with an infinite set of harmonic oscillators. For each oscillator we have a creation and annihilation operators denoted by $a_i^\dagger$ and $a_i$ that do not commute. The physical operators $P_i$ can be thought of as creation operators and the anti-states as annihilation ones. All the creation operators mutually commute as do the annihilation operators. There is, however, a non-zero commutator between a creation operator and its conjugate. A third possibility is to extend the definition of the identity operator and hence the metric. We used only the two-point function on the sphere to construct the identity. It is possible that two-point functions on higher genus will contribute to the identity. This should not change the genus 0 results, but may change higher genus answers. All these options are currently under investigation.

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The computations for this paper were done using Mathematica and C programs on a NeXTstation. The co-author of the programs (Alice Quillen) has agreed to make them available to interested readers upon request. They include a Mathematica program that performs operations on general pseudo-differential operators, another Mathematica program that computes pseudo-differential operators in the $(1,q)$ models and a C program with a Mathematica interface (using MathLink) that computes correlation functions according to the method presented in this paper.

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Appendix A. KdV Gravity

The KdV approach to two-dimensional quantum gravity is a recipe for computing correlation functions using the generalized KdV hierarchy of differential equations. It is based
on a $q$th order differential operator,

$$Q = D^q + u_{q-2}(x,t)D^{q-2} + u_{q-3}(x,t)D^{q-3} + \cdots + u_0(x,t), \quad (A.1)$$

where $u_i$ are formal power series in $x$ and an infinite number of ‘time’ variables, $t = (t_1, t_2, \ldots)$. $D$ is the differentiation operator $\partial / \partial x$. The dependence of these functions on the time variables are determined by the KdV hierarchy (the KdV ‘flows’)

$$\frac{\partial Q}{\partial t_i} = [R_i^+, Q], \quad (A.2)$$

where $R$ is a the pseudo-differential operator satisfying $R^q = Q$, and $R_+$ is the differential part of $R$. The dependence on $x$ is specified by the so-called ‘string equation’,

$$[P, Q] = 1, \quad (A.3)$$

where $P$ is a $p$th order differential operator. One can show that $P$ must be a linear combination of various $R_i^+$. If $P = R_i^+$, then we have the $(p, q)$ model.

To compute the correlation functions one maps the fields in the theory to powers of $R$,

$$P_i \leftrightarrow R_i^+, \quad (A.4)$$

and identifies the two puncture correlation function to be proportional to $u_{q-2}$. That is,

$$\langle P_1 P_1 \rangle = \text{Res}_{-1} R, \quad (A.5)$$

where $\text{Res}_i O$ is the coefficient of $D^i$ in the operator $O$. One can imagine that the original Lagrangian is perturbed by $\sum t_i P_i$ where the sum is taken over all possible fields. Taking a partial derivative with respect to $t_i$ will insert the operator $P_i$ into the correlation function. The basic idea is then to identify these parameters with the flows of the KdV hierarchy! Thus one can compute,

$$\langle P_1 P_1 P_m \rangle = \frac{\partial}{\partial t_m} \langle P_1 P_1 \rangle = \text{Res}_{-1} [R_+^m, R]. \quad (A.6)$$

After some manipulation one gets

$$\langle P_1 P_1 P_m \rangle = \left( \text{Res}_{-1} R^m \right)’, \quad (A.7)$$

and integrating

$$\langle P_1 P_m \rangle = \text{Res}_{-1} R^m + \alpha_m. \quad (A.8)$$
This integration constant is usually set to zero. However, when discussing the \((1,q)\) models, we will return to this point.

Proceeding, one can compute

\[
\langle \mathcal{P}_1 \mathcal{P}_m \mathcal{P}_n \rangle = \text{Res}_{-1} \left[ R^m_+ , R^n \right],
\]  

(A.9)

and similarly

\[
\langle \mathcal{P}_1 \mathcal{P}_l \mathcal{P}_m \mathcal{P}_n \rangle = \text{Res}_{-1} \left( \left[ \left[ \left[ R^l_+ , R^m \right]_+ , R^n \right] + \left[ R^m_+ , \left[ R^l_+ , R^n \right] \right] \right) \right).
\]  

(A.10)

In a generic model there are some features which are independent of the exact choice of the string equation (A.3). First, as one can easily see, inserting the operator \(\mathcal{P}_{mq}\) into the correlation function is trivial, \textit{i.e.}, the correlation function vanishes because \(R^m_{+q} = Q^m_+ \equiv Q^n\) and thus commutes with any power of \(R\). Thus, naturally, the operators come in bands of \(q-1\). They are identified with primary and descendant fields in the following way: The primaries are \(P_\alpha\) for \(1 \leq \alpha \leq q-1\) and the \(m\)th descendant of \(P_\alpha\) is \(\sigma_m(P_\alpha) = P_{mq+\alpha}\).

Second, the fact the the KdV hierarchy is completely integrable guarantees that the correlation functions are independent of the order of the operators, since the flows (A.2) commute.

Third, the correlation functions are non-perturbative. That is, they include the sum over all genera.

Fourth, usually one cannot write the solution in a closed analytic form. This is because, generically, the string equation is horribly non-linear.

However, there is a set of models, the \((1,q)\) series, which are solvable analytically. Moreover, each correlation function gets a contribution from only one genus. This is an indication that these models may be identified with those of topological gravity.

Appendix B. \((1,q)\) Models

The \((1,q)\) models are specified by the choice,

\[
P = R_+ \equiv D.
\]  

(B.1)

This enables one to solve the string equation explicitly for the \(x\) dependence,

\[
Q = D^q + x.
\]  

(B.2)
As discussed in detail in [21,22], this allows one to write exact formulae for the $R^i$

\begin{align}
R^i = & D^i + \frac{i}{q}D^{i-q} + \frac{i(i-q)}{2q}D^{i-q-1} \\
& + \frac{i(i-q)}{2q^2}x^2D^{i-2q} + \frac{i(i-q)(i-2q)}{2q^2}xD^{i-2q-1} \\
& + \frac{i(i-q)(i-2q)(3i-5q-4)}{24q^2}D^{i-2q-2} + \frac{i(i-q)(i-2q)}{6q^3}xD^{i-3q-2} + ...
\end{align}

(B.3)

and for various correlation functions. We refer the reader to those papers for details. We showed that one may associate a ‘ghost’ number to each operator,

$$\text{gh} (\mathcal{P}_i) = q + 1 - i,$$

(B.4)

and that the correlation functions obey certain selection rules. The first one states that the correlation function $\langle \mathcal{P}_{i_1} \ldots \mathcal{P}_{i_n} \rangle$ will vanish unless,

$$\sum_{j=1}^{n} \text{gh} (\mathcal{P}_{i_j}) \leq 2(1 + q).$$

(B.5)

This is true even for $x \neq 0$. The other selection rule which is true only for $x = 0$ (i.e., in the topological limit) is,

$$\sum_{j=1}^{n} \dim (\mathcal{P}_{i_j}) = \sum_{j=1}^{n} i_j = 0 \pmod{q+1}.$$  

(B.6)

Combining the two together one derives the ghost number conservation law,

$$\sum_{j=1}^{n} \text{gh} (\mathcal{P}_{i_j}) = 2(1 + q)(1 - g).$$

(B.7)

where $g$ is a positive integer that is identified with the genus.

Now, as promised earlier, we will discuss the integration constants. As explained in our previous paper, the reason $Q$ is equal to $D^q + x$ is because one can associate a scaling dimension to the operators. The result is that the function $u_0 = x$ has dimension $q$. Derivatives contribute dimension 1 and thus constants have dimension 0 (mod $q+1$). Thus, when integrating an equation, one can add constants only when the dimension of the correlation function is 0 (mod $q+1$). Thus, for example, in the transition from (A.6) to (A.8) the constant $\alpha_m$ may not be zero only for $m = q$ (mod $q+1$)! In general a ‘constant’ means only that it’s a constant with respect to $x$ (or $t_1$). So, when taking flows with respect to the other $t_i$’s one has to be careful and try to compute those constants. Only correlation functions that have two punctures (i.e., two insertions of $\mathcal{P}_1$) do not involve integration.
Appendix C. Extension of (1,q) Models and the Metric

Since the operators $R^i$ exist for negative $i^*$, one may extend the range of fields $P_i$ to include negative indices. Doing this, one easily sees that since $\text{ord } R^i = i$ then $R^i_+ = 0$ for $i < 0$. Thus, the KdV flows with respect to these operators are trivial. That is

$$\frac{\partial R^i}{\partial t_j} = [R^j_+, R^i] = 0 \quad \text{for} \quad j \leq 0. \quad (C.1)$$

This yields the not too surprising result that correlations functions that contain these fields vanish. For example,

$$\langle P_1 P_1 P_i P_j \rangle = 0 \quad \text{for} \quad i < 0. \quad (C.2)$$

However, there are some correlation functions that do not vanish. For example,

$$\langle P_1 P_m \rangle = \text{Res}_{-1} R^m \quad (C.3)$$

does not vanish for $m = -1$. Actually one can evaluate it exactly and get

$$\langle P_1 P_{-1} \rangle = 1. \quad (C.4)$$

It is clear that correlation functions that involve ‘flowing’ of $\langle P_1 P_m \rangle$ with a negative flow will vanish (such as (C.2)), but some that involve direct integration might be non-zero.

The general two-point correlation function on the sphere, $\langle P_i P_j \rangle_0$, will vanish unless $i + j = 0$. This is a consequence of the ghost number conservation rule (B.7). In the regular theory this correlation function vanishes always because there are no negative indexed fields. However, after adding the new fields some two-point functions may be non-zero, and we now wish to compute them. That will enable us to define a metric in a way similar to regular (i.e., without gravity) topological field theory. We conjecture that the result is

$$\langle P_i P_j \rangle = |i| \delta_{i+j,0}. \quad (C.5)$$

We will prove it for low values of $i$ and $j$.

Starting with $\langle P_1 P_i \rangle$ we flow with $t_j$ and then integrate out the $P_1$. For $j < q$ one has

$$\langle P_1 P_i P_j \rangle = \text{Res}_{-1} [D^j, R^i]$$

$$= \sum_{k=0}^{j-1} \binom{j}{k} \text{Res}_{-1} (R^i)^{(j-k)} D^k \quad (C.6)$$

$$= \sum_{k=0}^{j-1} \binom{j}{k} \left( \text{Res}_{-1-k} R^i \right)^{(j-k)}. \quad (C.6)$$

The explicit formula in reference [21] is valid for negative powers.
Integrating, we get
\[ \langle P_i P_j \rangle = \sum_{k=0}^{j-1} \binom{j}{k} \left( \text{Res} R^i \right)^{(j-k-1)} \left( \text{Res} R^i \right)^{(j-k-1)}, \]  
(C.7)
and substituting \( i = -j \),
\[ \langle P_{-j} P_j \rangle = j \quad \text{for} \quad q > j > 0. \]  
(C.8)

To compute the metric for non-primary fields we need to introduce a trick. This involves the computation of derivatives of \( R^i \). It is based on the observation that (note that this is only true for the \((1, q)\) models),
\[ Q' = (R^q)' = 1. \]  
(C.9)

Thus, we get that
\[ R' = \frac{1}{q} R^{1-q}, \]  
(C.10)
and, generalizing, we get*
\[ \left( R^i \right)^{(j)} = \frac{i(i-q)(i-2q) \cdots (i-(j-1)q)}{q^j} R^{i-jq}. \]  
(C.11)

Another useful identity is
\[ Q \left( R^i \right)^{(j)} = \frac{i-(j-1)q}{i+q} \left( R^{i+q} \right)^{(j)}. \]  
(C.12)

So, for \( q > j > 0 \) we can compute
\[ \langle P_1 P_i P_{j+q} \rangle = \text{Res}_{-1} \left[ R^{q+j}, R^i \right] \]
\[ = \text{Res}_{-1} \left[ D^{q+j} + \frac{q+j}{q} x D^j + \frac{j(q+j)}{2q} D^{j-1}, R^i \right] \]
\[ = \text{Res}_{-1} \left[ \frac{q+j}{q} Q D^j - \frac{j}{q} D^{q+j} + \frac{j(q+j)}{2q} D^{j-1}, R^i \right] \]
\[ = \text{Res}_{-1} \left[ \frac{q+j}{q} \sum_{k=0}^{j-1} \binom{j}{k} Q \left( R^i \right)^{(j-k)} D^k - \frac{j}{q} \sum_{k=0}^{q+j-1} \binom{q+j}{k} \left( R^i \right)^{(q+j-k)} D^k \right. \]
\[ \left. + \frac{j(q+j)}{2q} \sum_{k=0}^{j-2} \binom{j-1}{k} \left( R^i \right)^{(j-1-k)} D^k \right]. \]  
(C.13)

*This identity can be used to compute the partition function and various one-point functions (by integrating \( P_1 \)) and also to prove the ‘puncture equation’ directly from KdV gravity. We will do it later.
Thus,
\[
\langle P_i P_{j+q} \rangle = \frac{q + j}{q} \sum_{k=0}^{j-1} \binom{j}{k} \frac{i - (j - k - 1)q}{i + q} \left( \operatorname{Res}_{-1-k} R^{i+q} \right)^{(j-k-1)} 
\]
\[
- \frac{j}{q} \sum_{k=0}^{q+j-1} \binom{q+j}{k} \left( \operatorname{Res}_{-1-k} R^i \right)^{(q+j-k-1)}
\]
\[
+ \frac{j(q + j)}{2q} \sum_{k=0}^{j-2} \binom{j-1}{k} \left( \operatorname{Res}_{-1-k} R^i \right)^{(j-2-k)}
\]  
\[(C.14)\]
and setting \( j = -i - q \) we get
\[
\langle P_{-i-q} P_{i+q} \rangle = q + i \quad \text{for} \quad q > i > 0.
\]  
\[(C.15)\]
We computed the metric (C.5) for \( 2q < i < 3q \) as well, and the result still holds. The computation is not illuminating and will be omitted. Through induction it may now be possible to prove (C.5) for all \( i \).

By an alternative method, we can compute the metric for any \( i \). This method is not as convincing as the previous one because it involves analytic continuation of integers to real numbers. The starting point is the following correlation functions derived from the KdV approach.
\[
\langle P_{qi+\alpha} P_{qj+q-\alpha} P_{1+i+j+1} \rangle_0 = q \binom{i+j}{i} \frac{\Gamma(i + 1 + \frac{\alpha}{q})}{\Gamma(1 + \frac{\alpha}{q})} \frac{\Gamma(j + 1 + \frac{q-\alpha}{q})}{\Gamma(1 + \frac{q-\alpha}{q})}.
\]  
\[(C.16)\]
This equation is true for all non-negative (integer) \( i \) and \( j \). By analytically continuing the result to \( i + j + 1 = 0 \) one can show, after some algebra, that
\[
\langle P_{qi+\alpha} P_{-qi-\alpha} \rangle = qi + \alpha.
\]  
\[(C.17)\]

Appendix D. Integration and the Puncture Equation

Using equation (C.11) we can integrate the two-point correlation function (C.3) and get the one-point function. By using the identity \( R^m = q/(q + m)(R^{m+q})' \), we get
\[
\langle P_m \rangle = \frac{q}{q + m} \operatorname{Res}_{-1} R^{i+m}.
\]  
\[(D.1)\]
For example, the dilaton expectation value is
\[
\langle P_{q+1} \rangle = \frac{q^2 - 1}{24q}
\]  
\[(D.2)\]
Similarly, we can integrate (A.5) twice and get the partition function

$$
\langle 1 \rangle = \frac{q^2}{(q + 1)(2q + 1)} \text{Res } R^{2q+1} = \frac{q-1}{24}.
$$

One should note that one can not flow these equations because we have set all the flow variables to zero by using the explicit form of $Q$ (except for $x$, of course).

To get the puncture equation we use a variant of (C.11):

$$
(R^i)^{(j)} = \frac{i}{q} (R^{i-q})^{(j-1)}.
$$

Thus, using also the KdV flows, one can show that

$$
\langle P_{kn+1} P_{i_1} P_{i_2} \cdots P_{i_n} \rangle = \sum_{j=1}^{n} \frac{i_j}{q} \langle P_{1} P_{i_1} \cdots P_{i_{j-1}} P_{i_{j-1}-q} P_{i_{j+1}} \cdots P_{i_n} \rangle
$$

which is the puncture equation. It was the first example of a recursion relation in topological gravity and, as we have seen, can be proven directly from the KdV.

Appendix E. The $W_3$ Recursion Relations

In this appendix we will review the $W_3$ constraints and present the corresponding recursion relations on the sphere. These results should be compared with those of Section 4.

Let us consider $(1, q)$ models. For correlation functions involving the $n$th descendant of the puncture operator as the first operator$^*$,

$$
\langle P_{nq+1} \prod_{i \in S} P_i \rangle,
$$

the Virasoro constraints can be used to derive recursion relations$^{[11,12]}$. Since we actually have $q - 1$ primary fields, more information is needed to completely solve the models.

The $W_3$ constraints can be used to derive recursion relations for correlation functions involving descendants of the second primary field as the first operator,

$$
\langle P_{nq+2} \prod_{i \in S} P_i \rangle.
$$

$^*$The notation is defined in the appendices.
For the (1, 3) model K. Li\textsuperscript{[19]} has shown that the $W_3$ constraints are given by $W_m \tau = 0$ with $m \geq -2$ and

\[ W_m = \frac{16}{9} \frac{\partial}{\partial t_{m+2}} - \frac{8}{3} \sum_{r-p=m+1} (p + \frac{1}{3}) t_p t_{r,1} \frac{\partial}{\partial t_{r,2}} - \frac{4\lambda^2}{3} \sum_{r+q=m} \frac{\partial^2}{\partial t_{r,2} \partial t_{q,2}} \]

+ \sum_{p+q-r=-m} (p + \frac{2}{3}) (q + \frac{2}{3}) t_p t_{q,2} \frac{\partial}{\partial t_{r,1}}

+ \lambda^2 \sum_{p-q-r=-m+1} (p + \frac{1}{3}) t_p \frac{\partial^2}{\partial t_{q,2} \partial t_{r,2}}

+ \sum_{p-q-r=-m} (p + \frac{2}{3}) t_{p,2} \frac{\partial^2}{\partial t_{q,1} \partial t_{r,1}}

+ \frac{2\lambda^4}{3} \left\{ \sum_{p+q+r=-m-1} \frac{\partial^3}{\partial t_{p,1} \partial t_{q,1} \partial t_{r,1}} + \sum_{p+q+r=-m-2} \frac{\partial^3}{\partial t_{p,2} \partial t_{q,2} \partial t_{r,2}} \right\}

+ \frac{1}{27} \lambda^{-2} ((8t_{2,0}^3 - 4t_{0,1}^2) \delta_{m,2}^3 + t_{0,1}^3 \delta_{m,1}),

where $\lambda$ is the string coupling constant. This was derived by requiring the consistency of the commutation relations of the $W_3$ algebra. From the above one can now read off the recursion relations. In particular, we are interested in the recursion relations on the sphere.

Let us consider the following set of correlation functions:

\[ \langle \mathcal{P}_{3i+2} \mathcal{P}_1^{i+2} \rangle_0 = \prod_{j=0}^{i} (j + \frac{2}{3}) \quad (E.2) \]

This result can be obtained from either the KdV\textsuperscript{[21]} or the Virasoro constraints. From the $W_3$ constraint, (E.1), the recursion relation for the above correlation functions is

\[ \langle \mathcal{P}_{3i+2} \mathcal{P}_1^{i+2} \rangle_0 = \]

+ $a_2 \sum_{a=2}^{i} \left( \frac{i + 2}{a} \right) \langle \mathcal{P}_{3(a-2) + 2} \mathcal{P}_1^a \rangle_0 \langle \mathcal{P}_{3(i-a) + 2} \mathcal{P}_1^{i+2-a} \rangle_0$

+ $a_3 \sum_{a=2}^{i-a} \sum_{b=2}^{i} \left( \frac{i + 2 - a}{b} \right) \left( \frac{i + 2 - a}{b} \right) \langle \mathcal{P}_{3(a-2) + 2} \mathcal{P}_1^a \rangle_0 \langle \mathcal{P}_{3(b-2) + 2} \mathcal{P}_1^b \rangle_0 \langle \mathcal{P}_{3(i-a-b) + 2} \mathcal{P}_1^{i+2-a-b} \rangle_0$

+ $a_4 \sum_{a=3}^{i} \left( \frac{i + 2}{a} \right) a \frac{1}{3} \langle \mathcal{P}_{3(a-3) + 2} \mathcal{P}_1^{a-1} \rangle_0 \langle \mathcal{P}_{3(i-a) + 2} \mathcal{P}_1^{i+2-a} \rangle_0 + a_5 \delta_{i,1} + a_6 \delta_{i,2}.

(E.3)
The different terms in this equation correspond to double contact terms, triple contact terms, a single surface degeneration, a double surface degeneration, a surface degeneration plus a contact term, and the \( \delta \) functions in equation (E.1), respectively. With a normalization consistent with the KdV results (as presented in the appendices):

\[
\begin{align*}
  a_0 &= 2, & a_1 &= -2, & a_2 &= \frac{1}{3}, & a_3 &= -\frac{1}{27}, \\
  a_4 &= -\frac{1}{3}, & a_5 &= -\frac{2}{9} & \text{and} & a_6 &= \frac{2}{3}.
\end{align*}
\]

(E.4)

Equation (E.3) is not enlightening and tedious to extend to general \((1, q)\) models; in particular, the \( \delta \) functions are somewhat difficult to explain. This should be contrasted with equation (4.5).
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