EQUIVARIANT BUNDLES AND ADAPTED CONNECTIONS

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Abstract. Given a complex manifold \( M \) equipped with a holomorphic action of a connected complex Lie group \( G \), and a holomorphic principal \( H \)-bundle \( E_H \) over \( X \) equipped with a \( G \)-connection \( h \), we investigate the connections on the principal \( H \)-bundle \( E_H \) that are (strongly) adapted to \( h \). Examples are provided by holomorphic principal \( H \)-bundles equipped with a flat partial connection over a foliated manifold.

1. Introduction

Let \( X \) be a complex manifold, \( G \) a connected complex Lie group and \( \rho : G \times X \to X \) a holomorphic action of \( G \) on \( X \). The Lie algebra of \( G \) is denoted by \( \mathfrak{g} \). Let \( p : E_H \to X \) be a holomorphic principal \( H \)-bundle, where \( H \) is a complex Lie group. A \( G \)-connection on \( E_H \) is a \( \mathbb{C} \)-linear map \( h : \mathfrak{g} \to H^0(E_H, T E_H)^H \) such that for every \( v \in \mathfrak{g} \), the vector field \( dp \circ h(v) \) on \( X \) coincides with the one defined by \( v \) using the above action \( \rho \) (see Section 2.2). In [BP], \( G \)-connections were investigated, in particular, a criterion was given for the existence of a \( G \)-connection.

Here we continue the investigations of \( G \)-connections. More precisely, we study the interactions of \( G \)-connections on \( E_H \) with the holomorphic connections on the principal \( H \)-bundle \( E_H \). There are two possible compatibility conditions between them which are called “adapted” and “strongly adapted” (see Section 3.1). To explain these conditions, if \( h \) is given by a holomorphic action \( \rho_E \) of \( G \) on \( E_H \), then a holomorphic connection \( \eta \) on the principal \( H \)-bundle \( E_H \) is adapted to \( h \) if and only if \( \eta \) is preserved by \( \rho_E \); such an adapted connection \( \eta \) is called strongly adapted if the image of the homomorphism \( h \) is contained in the horizontal subbundle of \( T E_H \) for the connection \( \eta \).

The property of a holomorphic connection \( \eta \) on a holomorphic principal \( H \)-bundle \( E_H \) that it is strongly adapted to a \( G \)-connection \( h \) on \( E_H \) can

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also be formulated in the context of foliated manifolds and principal \( H \)-bundles on them equipped with a flat partial connection; the details are in Section 5.

2. Preliminaries

2.1. Atiyah bundle. Let \( H \) be a complex Lie group. Its Lie algebra will be denoted by \( \mathfrak{h} \). Let \( X \) be a connected complex manifold and

\[
p : E_H \to X
\]

a holomorphic principal \( H \)-bundle over \( X \). This means that \( E_H \) is a complex manifold equipped with a holomorphic right action of \( H \)

\[
a : E_H \times H \to E_H
\]

such that

- \( p \circ a = p \circ p_{E_H} \), where \( p_{E_H} \) is the projection of \( E_H \times H \) to \( E_H \), and
- the map \( (p_{E_H}, a) : E_H \times H \to E_H \times_X E_H \) is an isomorphism.

Note that the first condition means that the action of \( H \) takes a fiber of \( p \) to itself, so the image of the map \( (p_{E_H}, a) \) is contained in the fiber product \( E_H \times_X E_H \). The second condition above means that the action of \( H \) on a fiber of \( p \) is free and transitive.

The adjoint bundle for \( E_H \)

\[
ad(E_H) := E_H \times^H \mathfrak{h} \to X
\]

is the holomorphic vector bundle over \( X \) associated to \( E_H \) for the adjoint action of \( H \) on the Lie algebra \( \mathfrak{h} \).

The holomorphic tangent (respectively, cotangent) bundle of a complex manifold \( Y \) will be denoted by \( T_Y \) (respectively, \( T^*_Y \)). The tangent bundle of a real manifold \( Y \) will be denoted by \( T^R_Y \).

The Atiyah bundle for \( E_H \)

\[
At(E_H) := (T E_H)/H \to E_H/H = X
\]

is a holomorphic vector bundle over \( X \) whose rank is \( \text{dim } X + \text{dim } \mathfrak{h} \); see [At]. Let

\[
T_{E_H/X} \subset T E_H
\]

be the relative tangent bundle for the projection \( p \) in (2.1). The subbundle

\[
(T_{E_H/X})/H \subset (T E_H)/H = At(E_H)
\]

is identified with the adjoint vector bundle \( \text{ad}(E_H) \). This identification is a consequence of the isomorphism of \( T_{E_H/X} \) with the trivial vector bundle
$E_H \times \mathfrak{h} \rightarrow E_H$ given by the action of $H$ on $E_H$. Therefore, the short exact sequence

$$0 \rightarrow T_{E_H/X} \rightarrow TE_H \xrightarrow{dp} p^*TX \rightarrow 0,$$

where $dp$ is the differential of $p$, produces a short exact sequence on $X$

$$0 \rightarrow \text{ad}(E_H) \rightarrow \text{At}(E_H) \xrightarrow{dp} TX \rightarrow 0,$$

which is known as the Atiyah exact sequence for $E_H$. For simplicity, we have used the same notation $dp$ for the differential $TE_H \rightarrow p^*TX$ over $E_H$ as well as its descent $\text{At}(E_H) \rightarrow TX$ to $X$. A holomorphic connection on $E_H$ is a holomorphic homomorphism

$$\eta : TX \rightarrow \text{At}(E_H)$$

such that $(dp) \circ \eta = \text{Id}_{TX}$, where $dp$ is the homomorphism in (2.2). For a holomorphic connection $\eta$ on $E_H$, the homomorphism

$$\bigwedge^2 TX \rightarrow \text{ad}(E_H), \ v \otimes w - w \otimes v \mapsto 2([\eta(v), \eta(w)] - \eta([v, w])), \ $$

where $v$ and $w$ are locally defined holomorphic sections of $TX$, produces a holomorphic section of $(\bigwedge^2 T^*X) \otimes \text{ad}(E_H)$. This holomorphic section of $(\bigwedge^2 T^*X) \otimes \text{ad}(E_H)$ is called the curvature of the connection $\eta$.

The vector bundle $TE_H \otimes p^*(TX)^*$ on $E_H$ has a natural action of $H$ given by the action of $H$ on $TE_H$ and the tautological action of $H$ on $p^*(TX)^*$. We note that a holomorphic connection on $E_H$ is an $H$–invariant holomorphic section of $TE_H \otimes p^*(TX)^*$.

2.2. $G$–connections on $E_H$. Let $G$ be a connected complex Lie group; its Lie algebra will be denoted by $\mathfrak{g}$. The identity element of $G$ will be denoted by $e$. Let

$$\rho : G \times X \rightarrow X$$

be a holomorphic action of $G$ on $X$. Consider the holomorphic homomorphism

$$\rho' : \text{At}(E_H) \oplus (X \times \mathfrak{g}) \rightarrow TX, \ (v, w) \mapsto dp(v) - d'\rho(w),$$

where $dp$ is the homomorphism in (2.2), and

$$d'\rho : X \times \mathfrak{g} \rightarrow TX, \ (x, v) \mapsto (dp)(e, x)(v, 0),$$

with $(dp)(e, x) : \mathfrak{g} \oplus T_xX \rightarrow T_xX$ being the differential of $\rho$ at $(e, x) \in G \times X$. Define the subsheaf

$$\text{At}_\rho(E_H) := (\rho')^{-1}(0) \subset \text{At}(E_H) \oplus (X \times \mathfrak{g}).$$
Since the differential $dp$ is surjective, it follows that $\rho'$ is surjective. This implies that $A_{\rho'(E_H)}$ is a holomorphic subbundle of $A(E_H) \oplus (X \times g)$. The vector bundle $A_{\rho'(E_H)}$ fits in a commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \rightarrow & \text{ad}(E_H) & \rightarrow & A_{\rho'(E_H)} & \frac{q}{\rightarrow} & X \times g & \rightarrow & 0 \\
& & \| & & \downarrow J & & \downarrow d'\rho \\
0 & \rightarrow & \text{ad}(E_H) & \rightarrow & A(E_H) & \frac{dp}{\rightarrow} & TX & \rightarrow & 0
\end{array}
$$

(2.7)

where $J$ (respectively, $q$) is given by the projection of $A(E_H) \oplus (X \times g)$ to $A(E_H)$ (respectively, $X \times g$). (See [BP].)

A holomorphic $G$–connection on $E_H$ is a holomorphic homomorphism of vector bundles

$$
h : X \times g \rightarrow A_{\rho'(E_H)}
$$

(2.8)
such that $q \circ h = \text{Id}_{X \times g}$, where $q$ is the homomorphism in (2.7). The curvature of a $G$–connection $h$

$$(s, t) \mapsto [h(s), h(t)] - h([s, t])$$

is a holomorphic section

$$\mathcal{K}(h) \in H^0(X, \text{ad}(E_H) \otimes \bigwedge^2 (X \times g)^*) = H^0(X, \text{ad}(E_H)) \otimes \bigwedge^2 g^*.$$ (2.9)

We will give examples of $G$–connection.

Let $a : E_H \times H \rightarrow E_H$ be the action of $H$ on the principal $H$–bundle $E_H$.

A $G$–action on the principal bundle $E_H$ is a holomorphic action of $G$ on the total space of $E_H$

$$\rho_E : G \times E_H \rightarrow E_H
$$

(2.10)
such that

1. $p \circ \rho_E = \rho \circ (\text{Id}_G \times p)$, where $p$ and $\rho$ are the maps in (2.1) and (2.4) respectively, and
2. $\rho_E \circ (\text{Id}_G \times a) = a \circ (\rho_E \times \text{Id}_H)$ as maps from $G \times E_H \times H$ to $E_H$

(this condition means that the actions of $G$ and $H$ on $E_H$ commute).

An equivariant principal $H$–bundle is a holomorphic principal $H$–bundle with a $G$–action.

Let $\rho_E : G \times E_H \rightarrow E_H$ be a $G$–action on $E_H$. Consider the homomorphism

$$\tilde{h} : E_H \times g \rightarrow TE_H$$

given by the differential $d\rho_E$ of the action $\rho_E$; more precisely, $\tilde{h}(z, v) = d\rho_E(e, z)(v, 0)$, so $\tilde{h}$ is the homomorphism in (2.5) when $X$ is substituted by
Since the actions of $G$ and $H$ on $E_H$ commute, this homomorphism $\tilde{h}$ produces a $G$–connection

$$h_0 : X \times \mathfrak{g} \longrightarrow \mathfrak{At}_\rho(E_H)$$

on $E_H$; the curvature of this $G$–connection $h_0$ vanishes identically [BP, Lemma 4.1].

Let $Y$ be a connected compact complex manifold such that $TY$ is holomorphically trivial. Then $Y$ is holomorphically isomorphic to $G/\Gamma$, where $G$ is a connected complex Lie group and $\Gamma \subset G$ is a cocompact lattice [Wa]; in fact, $G$ is the connected component, containing the identity element, of the group of all holomorphic automorphisms of $Y$. Consider the left–translation action of $G$ on $G/\Gamma = Y$. A $G$–connection on a holomorphic principal $H$–bundle $E_H$ on $Y$ is an usual holomorphic connection on the principal $H$–bundle.

2.3. Distributions under a flow. Let $Y$ be a connected $C^\infty$ manifold and $D \subset T_R Y$ a $C^\infty$ subbundle. In other words, $D$ is a distribution on $Y$. The fiber of $D$ over any point $z \in Y$ will be denoted by $D_z$.

Let $\xi$ be a $C^\infty$ vector field on $Y$. Given any point $x \in Y$, there is an open neighborhood $x \in U_x \subset Y$ and an open interval $0 \in I_x \subset \mathbb{R}$, such that $\xi$ integrates to a flow

$$\Phi_x : U_x \times I_x \longrightarrow Y.$$

For any $t \in I_x$, define

$$\Phi_{x,t} : U_x \longrightarrow Y, \quad z \mapsto \Phi_x(z,t).$$

Lemma 2.1. The following two are equivalent:

1. For every $x \in Y$ and $z \in U_x$ as above,

$$(d\Phi_{x,t})(z)(D_z) = D_{\Phi_x(z)},$$

where $d\Phi_{x,t}(z) : T^R_z Y \longrightarrow T^R_{\Phi_x(z)}Y$ is the differential of the map $\Phi_{x,t}$ at $z$.

2. $[\xi, D] \subset D$.

Proof. Let $\mathcal{W}$ denote the space of all $C^\infty$ 1–forms on $Y$ that vanish on $D$. The first statement is equivalent to the statement that

$$L_\xi(w) \in \mathcal{W} \quad \forall \ w \in \mathcal{W},$$

where $L_\xi$ denotes the Lie derivative with respect to the vector field $\xi$. 
First assume that 
\[ [\xi, D] \subset D. \quad (2.13) \]
To prove that (2.12) holds, take any \( w \in W \) and any \( C^\infty \) section \( \theta \) of \( D \). We have 
\[
(L_\xi (w))(\theta) = \xi (w(\theta)) - w(L_\xi \theta) = \xi (w(\theta)) - w([\xi, \theta]).
\]
Now, \( w(\theta) = 0 \), and \([\xi, \theta]\) is section of \( D \) by (2.13). Hence \((L_\xi (w))(\theta)=0\), which implies that (2.12) holds.

Now assume that (2.12) holds. To prove (2.13), let \( \theta \) be any \( C^\infty \) section of \( D \). Take any \( v \in W \). We have 
\[
w([\xi, \theta]) = w(L_\xi \theta) = \xi (w(\theta)) - (L_\xi w)(\theta).
\]
Now, \( w(\theta) = 0 \), and also \((L_\xi w)(\theta)=0\) because \( L_\xi w \in W \) by (2.12). Hence (2.13) holds. \( \square \)

3. Connections and (strongly) adapted connections

3.1. Definitions. Let \( E_H \) be a holomorphic principal bundle on \( X \) equipped with a holomorphic connection 
\[ \eta : TX \longrightarrow \text{At}(E_H) \]
(see (2.3)). Since \( \text{At}(E_H) = (TE_H)/H \), the image of \( \eta \) is a holomorphic distribution on \( E_H \); it is known as the horizontal distribution for the connection \( \eta \).

As before, a connected complex Lie group \( G \) acts holomorphically on \( X \).

Given a holomorphic \( G \)-connection \( h : X \times g \longrightarrow \text{At}_\rho(E_H) \) on \( E_H \) (see (2.8)), the connection \( \eta \) is said to be adapted to \( h \) if 
\[
[J \circ h(X \times \{v\}), \eta(TX)] \subset \eta(TX) \quad \forall \; v \in g, \quad (3.1)
\]
where \( J \) is the homomorphism in (2.7). Note that a \( C^\infty \) section of \( \text{At}(E_H) \) defines a \( H \)-invariant vector field on \( E_H \) of type \((1, 0)\).

The connection \( \eta \) is said to be strongly adapted to \( h \) if it is adapted to \( h \), and furthermore 
\[
\text{image}(J \circ h) \subset \text{image}(\eta). \quad (3.2)
\]

We will now give examples to show that the conditions in (3.1) and (3.2) are independent.

Consider the trivial action of the multiplicative group \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) on \( X \). Let \( E \) be a holomorphic principal \( GL(r, \mathbb{C}) \)-bundle on \( X \) admitting a holomorphic connection, for example \( E \) can be the trivial holomorphic principal \( GL(r, \mathbb{C}) \)-bundle \( X \times GL(r, \mathbb{C}) \) on \( X \). The center of \( GL(r, \mathbb{C}) \) is
identified with \( \mathbb{C}^* \) by sending any \( c \in \mathbb{C}^* \) to \( c \cdot \text{Id}_{E'} \in \text{GL}(r, \mathbb{C}) \). Using this identification, the action of the center of \( \text{GL}(r, \mathbb{C}) \) on \( E \) produces an action of \( \mathbb{C}^* \) on \( E \). Since \( \mathbb{C}^* \) is in the center of \( \text{GL}(r, \mathbb{C}) \), the actions of \( \mathbb{C}^* \) and \( \text{GL}(r, \mathbb{C}) \) on \( E \) commute. If \( E' \) is the vector bundle of rank \( r \) associated to \( E \) by the standard representation of \( \text{GL}(r, \mathbb{C}) \), then this action of \( \mathbb{C}^* \) on \( E \) corresponds to the action of \( \mathbb{C}^* \) on \( E' \) as scalar multiplications. Let \( h \) be the holomorphic \( \mathbb{C}^* \)--connection on \( E \) given by this action of \( \mathbb{C}^* \) on \( E \) (see (2.11)). Any holomorphic connection on the principal \( \text{GL}(r, \mathbb{C}) \)--bundle \( E \) is adapted to \( h \). But (3.2) fails for every holomorphic connection on \( E \).

Now take \( X = \mathbb{C}^2 \) and \( G = \mathbb{C} = H \). Let \( E_H \) be the trivial principal \( \mathbb{C} \)--bundle \( \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2 \). Take \( \rho \) to be the action of \( \mathbb{C} \) on \( \mathbb{C}^2 \) defined by \((z, (x, y)) \mapsto (x + z, y), \ z \in \mathbb{C}, \ (x, y) \in \mathbb{C}^2\).

This action of \( \mathbb{C} \) on \( X \) and the trivial action of \( \mathbb{C} \) on \( \mathbb{C} \) together define an action of \( \mathbb{C} \) on \( E_H = X \times \mathbb{C} \). Let \( h \) be the holomorphic \( \mathbb{C} \)--connection on \( E_H \) associated to this action of \( \mathbb{C} \) on \( E_H \) (see (2.11)). Let \( D \) be the holomorphic connection on the principal \( H \)--bundle \( E_H \) defined by \( f \mapsto df + xf \cdot dy \), where \( f \) is any holomorphic function on \( \mathbb{C}^2 \) (holomorphic sections of \( E_H \) are holomorphic functions) and \( d \) denotes the standard de Rham differential. Then (3.2) holds while (3.1) fails.

### 3.2. Equivariant bundles and adaptable connections.

As in (2.10), take a \( G \)--action \( \rho_E \) on \( E_H \). As mentioned earlier, there is a natural \( G \)--connection on \( E_H \)

\[
h_0 : X \times \mathfrak{g} \to \text{At}_\rho(E_H)
\]  
(3.3)

corresponding to \( \rho_E \).

Let \( p_X : G \times X \to X \) be the natural projection. The action \( \rho_E \) produces a holomorphic isomorphism of principal \( H \)--bundles

\[
\beta : p_X^*E_H \to \rho^*E_H, \ \beta(g, x)(z) = \rho_E(g, z) \ \forall \ g \in G, \ x \in X, \ z \in (E_H)_x,
\]  
(3.4)

where \( \rho \) is the map in (2.4).

For any \( g \in G \), let

\[
j_g : X \to G \times X, \ x \mapsto (g, x)
\]

be the embedding. For all \( g \in G \), the isomorphism \( \beta \) in (3.4) produces a holomorphic isomorphism of principal \( H \)--bundles

\[
\beta^g : E_H \to (\rho \circ j_g)^*E_H, \ z \mapsto \beta(g, x)(z) = \rho_E(g, z) \ \forall \ x \in X, \ z \in (E_H)_x.
\]  
(3.5)
The map from the holomorphic connections on $E_H$ to the holomorphic connections on $(\rho \circ j_g)^*E_H$ induced by the above isomorphism $\beta^g$ will be denoted by $\beta^g_*$; note that $\beta^g_*$ is a bijection.

**Proposition 3.1.** A holomorphic connection $\eta$ on $E_H$ is adapted to the $G$–connection $h_0$ in (3.3) associated to $\rho_E$ if and only if for all $g \in G$,

$$(\rho \circ j_g)^*\eta = \beta^g_*(\eta) \quad (3.6)$$

(both are connections on the principal $H$–bundle $(\rho \circ j_g)^*E_H$).

**Proof.** First assume that $\eta$ is adapted to $h_0$. Take any $v \in g$. The flow on $E_H$ generated by $v$ sends any $t \in \mathbb{R}$ to the biholomorphism

$$F_t : E_H \to E_H, \quad z \mapsto \rho_E(\exp(tv), z).$$

Note that $F_t$ coincides with $\beta^{\exp(tv)}$ constructed in (3.5). Consider the $H$-invariant distribution

$$D^\eta := \text{image}(\eta) \subset T E_H.$$ 

Its fiber over any point $z \in E_H$ will be denoted by $D^\eta_z$. Since $\eta$ is adapted to $h_0$, from Lemma 2.1 it follows that

$$(dF_t)(z)(D^\eta_z) = D^\eta_{F_t(z)} \quad (3.7)$$

for all $z \in E_H$ and $t \in \mathbb{R}$, where $(dF_t)(z) : T_z E_H \to T_{F_t(z)} E_H$ is the differential of the map $F_t$. Since the subset $\{\exp(tv)\}_{v \in g, t \in \mathbb{R}} \subset G$ is dense in the analytic topology (recall that $G$ is connected), and also $F_t = \beta^{\exp(tv)}$, from (3.7) we conclude that (3.6) holds for all $g \in G$.

Now assume that (3.6) holds for all $g \in G$. This implies that (3.7) holds for all $z \in E_H$ and $t \in \mathbb{R}$. Consequently, from Lemma 2.1 we conclude that $\eta$ is adapted to $h_0$. 

□

Take any point $x \in X$. Define

$$\rho_x : G \to X, \quad g \mapsto \rho \circ j_g(x) = \rho(g, x).$$

Consider the map

$$\rho_{E,x} : G \times (E_H)_x \to \rho_x^*E_H, \quad (g, z) \mapsto \rho_E(g, z).$$

Since this $\rho_{E,x}$ is $H$–equivariant (recall that the actions of $G$ and $H$ on $E_H$ commute), it identifies the pulled back principal $H$–bundle $\rho_x^*E_H$ with the trivial principal $H$–bundle $G \times (E_H)_x \to G$. Let $D^0_x$ be the holomorphic connection on the principal $H$–bundle $\rho_x^*E_H$ induced by the trivial connection on $G \times (E_H)_x$ using the above isomorphism $\rho_{E,x}$. Note that $\rho_x^*E_H$ is
identified with the restriction of $\rho^*E_H$ to $G \times \{x\}$, because $\rho_x$ is the restriction of $\rho$ to $G \times \{x\}$. Therefore, $\rho^*\eta|_{G \times \{x\}}$ is also a connection on $\rho^*E_H$.

**Proposition 3.2.** A holomorphic connection $\eta$ on $E_H$ is strongly adapted to the $G$–connection $h_0$ in (3.3) if and only if the following two hold:

1. For all $g \in G$,
   $$ (\rho \circ j_g)^*\eta = \beta^g_\eta. $$
2. For every $x \in X$, the connection $D^0_x$ on $\rho^*E_H$ coincides with the connection $\rho^*\eta|_{G \times \{x\}}$.

**Proof.** First assume that $\eta$ is strongly adapted to $h_0$. Since $\eta$ is adapted to $h_0$, Proposition 3.1 says that $(\rho \circ j_g)^*\eta = \beta^g_\eta$ for all $g \in G$. The given condition (3.2) implies that the connection $D^0_x$ coincides with $\rho^*\eta|_{G \times \{x\}}$.

The converse is similarly proved. Assume that the two statements in the proposition hold. From Proposition 3.1 we know that $\eta$ is adapted to $h_0$. The second condition in the proposition implies that (3.2) holds. □

**4. CRITERION FOR ADAPTED CONNECTION**

Let $\eta : TX \rightarrow \text{At}(E_H)$ be a holomorphic connection on $E_H$. Let

$$ \tilde{\eta} : X \times \mathfrak{g} \rightarrow \text{At}(E_H) \oplus (X \times \mathfrak{g}) $$

be the $O_X$–linear homomorphism defined by

$$ (x, v) \mapsto (\eta(d'\rho(x, v)), (x, v)), $$

where $d'\rho$ is the homomorphism in (2.5). Since we have $(dp) \circ \eta = \text{Id}_{TX}$, where $dp$ is the homomorphism in (2.2), it follows immediately that the image of $\tilde{\eta}$ is contained in $\text{At}_\rho(E_H) := (\rho')^{-1}(0)$ (see (2.6)). The homomorphism $\tilde{\eta}$ evidently is a $G$–connection on $E_H$.

Let $\mathcal{K}(\eta) \in H^0(X, \Omega^2_X \otimes \text{ad}(E_H))$ be the curvature of the connection $\eta$, where $\Omega^2_X = \wedge^2 T^*X$. For any $w \in T_xX$, let

$$ i_w(\mathcal{K}(\eta)(x)) \in (T^*X)_x \otimes \text{ad}(E_H)_x = (T^*X \otimes \text{ad}(E_H))_x $$

be the contraction of $\mathcal{K}(\eta)(x) \in (\Omega^2_X \otimes \text{ad}(E_H))_x$ by the tangent vector $w \in T_xX$.

**Lemma 4.1.** The connection $\eta$ on $E_H$ is strongly adapted to the above constructed $G$–connection $\tilde{\eta}$ if and only if for all $v \in \mathfrak{g}$ and $x \in X$,

$$ i_{d'\rho(x, v)}(\mathcal{K}(\eta)(x)) = 0, $$

where $d'\rho$ is defined in (2.5) (see (4.2) for the contraction).
Proof. From the construction of $\tilde{\eta}$ in (4.1) it follows immediately that the condition in (3.2) holds. We need to show that (3.1) holds if and only if (4.3) holds.

To prove this, we recall a construction of the curvature $K(\eta)$. Given a point $x \in X$ and holomorphic tangent vectors $v, w \in T_x X$, extend $v, w$ to vector fields $\tilde{v}, \tilde{w}$ of type $(1, 0)$ on some open neighborhood of the point $x$. Let $\hat{v} = \eta(\tilde{v})$ and $\hat{w} = \eta(\tilde{w})$ be the horizontal lifts of $\tilde{v}$ and $\tilde{w}$ respectively, for the connection $\eta$. Then

$$K(\eta)(x)(v, w) = ([\hat{v}, \hat{w}]_{\text{Vert}})|_{p^{-1}(x)},$$

where $[\hat{v}, \hat{w}]_{\text{Vert}}$ is the component of the Lie bracket $[\hat{v}, \hat{w}]$ in the vertical direction (for the projection $p$). We note that the section $([\hat{v}, \hat{w}]_{\text{Vert}})|_{p^{-1}(x)}$ of $T_{E_H/X}$ over $p^{-1}(x)$ is $H$–invariant and hence it defines an element of the fiber $\text{ad}(E_H)_x$ over $x$; recall that $\text{ad}(E_H)$ is identified with $(T_{E_H/X})/H$. The element $([\hat{v}, \hat{w}]_{\text{Vert}})|_{p^{-1}(x)} \in \text{ad}(E_H)_x$ does not depend on the choice of the extensions $\tilde{v}$ and $\tilde{w}$ of $v$ and $w$ respectively. From this description of $K(\eta)$ it follows immediately that (3.1) holds if and only if (4.3) holds. □

From the proof of Lemma 4.1 we have the following:

**Corollary 4.2.** The connection $\eta$ on $E_H$ is adapted to the above constructed $G$–connection $\tilde{\eta}$ if and only if the condition in (4.3) holds. In other words, the connection $\eta$ on $E_H$ is strongly adapted to $\tilde{\eta}$ if $\eta$ is adapted to $\tilde{\eta}$.

Take a $\mathbb{C}$–linear map

$$\varphi_0 : \mathfrak{g} \longrightarrow H^0(X, \text{ad}(E_H)).$$

(4.4)

For any $v \in \mathfrak{g}$, the section $\varphi_0(v) \in H^0(X, \text{ad}(E_H))$ defines a holomorphic vertical tangent vector field on $E_H$ for the projection $p$. This vertical tangent vector field on $E_H$ will be denoted by $\varphi(v)$. Let $U \subset X$ be an open subset and $V$ a $C^\infty$ vector field on $U$ of type $(1, 0)$. Let $V' = \eta(V)$ be the horizontal lift of $V$ on $p^{-1}(U)$ for the holomorphic connection $\eta$ on $E_H$. Let $f_0$ be any $C^\infty$ function on $U$. Then $V'(f_0 \circ p)$ is a $H$–invariant function on $p^{-1}(U)$, and hence

$$\varphi(v)(V'(f_0 \circ p)) = 0.$$  

(4.5)

On the other hand,

$$\varphi(v)(f_0 \circ p) = 0$$

(4.6)

because $\varphi(v)$ is a vertical vector field. From (4.5) and (4.6) we conclude that

$$[\varphi(v), V'](f_0 \circ p) = 0.$$
In other words,

$$[\varphi(v), V'] = [\varphi(v), V']_{\text{Vert}}, \quad (4.7)$$

where $$[\varphi(v), V']_{\text{Vert}}$$ is the vertical component of $$[\varphi(v), V']$$. The vector field $$[\varphi(v), V']$$ is $$H$$–invariant because both $$\varphi(v)$$ and $$V'$$ are $$H$$–invariant. If $$f_1$$ is a $$C^\infty$$ function on $$U$$, then note that

$$[\varphi(v), (f_1 \circ p) \cdot V'] = (f_1 \circ p) \cdot [\varphi(v), V']$$

because $$\varphi(v)(f_1 \circ p) = 0$$. Clearly, the vector field $$(f_1 \circ p) \cdot V'$$ is the horizontal lift of the vector field $$f_1 \cdot V$$ on $$U$$ for the connection $$\eta$$. From these observations we conclude that there is a homomorphism

$$\tilde{\varphi} : g \otimes_C TX \longrightarrow \text{ad}(E_H) \quad (4.8)$$

that sends $$v \otimes w \in g \otimes T_x X$$ to $$[\varphi(v), V'](x)$$, where $$V' = \eta(V)$$ is the horizontal lift, with respect to the connection $$\eta$$, of a vector field $$V$$ defined on a neighborhood of the point $$x \in X$$ with $$V(x) = w$$. Note that $$[\varphi(v), V'](x)$$ does not depend on the choice of the extension $$V$$ of $$w$$.

The contraction in (4.2) produces a homomorphism

$$\Pi : g \otimes_C TX \longrightarrow \text{ad}(E_H) \quad (4.9)$$

that sends $$v \otimes w \in g \otimes T_x X$$ to

$$i_w i_{d'\rho(x,v)}(K(\eta)(x)) \in \text{ad}(E_H)_x,$$

which is the contraction of $$i_{d'\rho(x,v)}(K(\eta)(x)) \in (T^*X)_x \otimes \text{ad}(E_H)_x$$ (see (2.5), (4.2)) by the tangent vector $$w \in T_x X$$.

**Theorem 4.3.** Let $$X$$ be a complex manifold equipped with a holomorphic action of $$G$$ and $$E_H$$ a holomorphic principal $$H$$–bundle on $$X$$ equipped with a holomorphic connection $$\eta$$. Then there is a $$G$$–connection $$h$$ on $$E_H$$ such that $$\eta$$ is adapted to $$h$$ if and only if there is a homomorphism $$\varphi_0$$ as in (4.4) such that the homomorphism $$\tilde{\varphi}$$ in (4.8) coincides with the homomorphism $$-\Pi$$, where $$\Pi$$ is constructed in (4.9).

**Proof.** Let $$h : g \longrightarrow H^0(X, \text{At}_\rho(E_H))$$ be a $$G$$–connection on $$E_H$$ such that $$\eta$$ is adapted to $$h$$. For any $$v \in g$$, consider

$$J \circ h(v) - \eta(v') \in H^0(X, \text{At}(E_H)),$$

where $$J$$ is the homomorphism in (2.7) and $$v'$$ is the holomorphic vector field on $$X$$ defined by $$x \longmapsto d'\rho(x,v)$$ (see (2.5)). Note that $$dp \circ J \circ h(v) = v'$$, where $$dp$$ is the homomorphism in (2.2). Therefore, we have

$$J \circ h(v) - \eta(v') \in H^0(X, \text{ad}(E_H)) \subset H^0(X, \text{At}(E_H))$$
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(see (2.7)). Now define
\[ \varphi_0 : \mathfrak{g} \to H^0(X, \text{ad}(E_H)), \quad v \mapsto J \circ h(v) - \eta(v'). \]

We will show that the homomorphism \( \tilde{\varphi} \) in (4.8) for this \( \varphi_0 \) coincides with the homomorphism \( -\Pi \).

Take any \( v \in \mathfrak{g} \). Given any \( x \in X \) and any \( w \in T_xX \), let \( V \) be any \( C^\infty \) vector field of type \((1,0)\), defined on an open neighborhood of \( x \in X \), such that
\[ [v', V] = 0. \]

Since \( \eta \) is adapted to \( h \), the Lie bracket \([J \circ h(v), \eta(V)]\) lies in the horizontal subbundle \( \eta(TX) \subset TE_H \). In other words, the vertical component of \([J \circ h(v), \eta(V)]\) vanishes identically.

The Lie bracket \([\eta(v'), \eta(V)]\) is vertical because \( dp([\eta(v'), \eta(V)]) = [v', V] = 0 \). From (4.7) we know that the Lie bracket \([\varphi(v), \eta(V)]\) is vertical, where \( \varphi(v) \) is the vertical vector field corresponding to \( \varphi_0(v) \in H^0(X, \text{ad}(E_H)) \).

This and the fact that \([\eta(v'), \eta(V)]\) is vertical together imply that
\[ [\varphi(v) + \eta(v'), \eta(V)] = [J \circ h(v), \eta(V)] \]
(4.10)
is vertical. But it was shown above that the vertical component of \([J \circ h(v), \eta(V)]\) vanishes identically. Hence we conclude that
\[ [J \circ h(v), \eta(V)] = 0. \]

Consequently, we have
\[ [\varphi(v), \eta(V)] = -[\eta(v'), \eta(V)] \]
for all \( v \in \mathfrak{g} \). Since \([\varphi(v), \eta(V)] = \tilde{\varphi}(v \otimes V) \) and \([\eta(v'), \eta(V)] = \Pi(v \otimes V) \), from (4.11) it follows that
\[ \tilde{\varphi} = -\Pi. \]

To prove the converse, take any homomorphism \( \varphi_0 \) as in (4.4) such that
\[ \tilde{\varphi} = -\Pi. \]

Now define a \( G \)-connection
\[ h : \mathfrak{g} \to H^0(X, \text{At}_p(E_H)), \quad v \mapsto (\varphi_0(v) + \eta(v'), X \times \{v\}). \]

We will show that \( \eta \) is adapted to \( h \).

Let \( V \) be a \( C^\infty \) vector field of type \((1,0)\) defined on an open subset \( U \subset X \). Take any \( v \in \mathfrak{g} \). The Lie bracket \([\varphi(v), \eta(V)]\) is vertical (see
(4.7), where $\varphi(v)$, as before, is the vertical vector field for the projection $p$ corresponding to the section $\varphi_0(v)$ of $\text{ad}(E_H)$. We have
\[
\bar{\varphi}(v \otimes V) = [\varphi(v), \eta(V)],
\]
and $\Pi(v \otimes V)$ is the vertical component of $[\eta(v'), \eta(V)]$. Consequently, from (4.12) and the definition of $h$ it follows that the vertical component of $[J \circ h(v), \eta(V)]$ vanishes. This implies that $\eta$ is adapted to $h$. □

Let $h : g \rightarrow H^0(X, \text{At}_{\rho}(E_H))$ be a $G$–connection on $E_H$. Take any section $\theta \in C^\infty(X, \text{At}(E_H)^a \otimes (\text{At}(E_H)^*)^b)$, where $a$ and $b$ are nonnegative integers. Note that $\theta$ defines a $H$–invariant section of the vector bundle $(T^*E_H)^a \otimes (T^*E_H)^b$ on $E_H$; this section of $(T^*E_H)^a \otimes (T^*E_H)^b$ will be denoted by $\Theta$. We say that $\theta$ is preserved by $h$ if
\[
L_{J \circ h(v)} \Theta = 0 \quad \forall \quad v \in g,
\]
where $L_{J \circ h(v)}$ is the Lie derivative with respect to the vector field $J \circ h(v)$ on $E_H$ (the homomorphism $J$ is constructed in (2.7)).

If $h$ is the $G$–connection associated to a $G$–action $\rho_E$ on $E_H$, then it is straightforward to check that $\theta$ is preserved by $h$ if and only if the section $\Theta$ is preserved by the action $\rho_E$ on $E_H$.

5. Holomorphic foliations and strongly adapted connections

As before, $X$ is a complex manifold. Let
\[
F \subset TX
\]
be a holomorphic foliation on $X$, which means that $F$ is a holomorphic subbundle of $TX$ such that for any two sections $s$ and $t$ of $F$ defined over some open subset of $X$, the Lie bracket $[s, t]$ is also a section of $F$. Let $E_H$ be a holomorphic principal $H$–bundle on $X$.

Consider the Atiyah exact sequence for $E_H$ in (2.2). Define
\[
\text{At}_F(E_H) := (dp)^{-1}(F) \subset \text{At}(E_H).
\]
So, from (2.2) we have the short exact sequence of holomorphic vector bundles
\[
0 \rightarrow \text{ad}(E_H) \rightarrow \text{At}_F(E_H) \xrightarrow{\widetilde{dp}} F \rightarrow 0,
\]
where $\widetilde{dp}$ is the restriction of $dp$ to $\text{At}_F(E_H)$. A holomorphic partial connection on $E_H$ is a homomorphism
\[
D : F \rightarrow \text{At}_F(E_H)
\]
such that $\widetilde{dp} \circ D = \operatorname{Id}_\mathcal{F}$ \cite{La}.

Given such a holomorphic partial connection $D$, the homomorphism

$$\wedge^2 \mathcal{F} \to \operatorname{ad}(E_H), \quad v \otimes w - w \otimes v \mapsto 2([D(v), D(w)] - D([v, w])), \quad v, w \in \mathcal{F}_x,$$

where $v$ and $w$ are locally defined holomorphic sections of $\mathcal{F}$, produces a holomorphic section of $(\wedge^2 \mathcal{F}^*) \otimes \operatorname{ad}(E_H)$. This holomorphic section of $(\wedge^2 \mathcal{F}^*) \otimes \operatorname{ad}(E_H)$ is called the curvature of the partial connection $D$. A holomorphic partial connection is called flat if its curvature vanishes identically.

Let $\eta : TX \to \operatorname{At}(E_H)$ be a holomorphic connection on the principal $H$–bundle $E_H$. As before, the curvature of $\eta$ will be denoted by $\mathcal{K}(\eta)$. Let $D : \mathcal{F} \to \operatorname{At}_\mathcal{F}(E_H)$ be a flat holomorphic partial connection on $E_H$.

The connection $\eta$ is said to be strongly adapted to $D$ if

- the restriction $\eta|_\mathcal{F} : \mathcal{F} \to \operatorname{At}(E_H)$ coincides with $D$, and
- for any $x \in X$ and $w \in \mathcal{F}_x$, the contraction

$$i_w \mathcal{K}(\eta)(x) \in T^*_x X \otimes \operatorname{ad}(E_H)_x$$

vanishes.

**Corollary 5.1.** Suppose that $\mathcal{F}$ is given by a holomorphic action $\rho$ of a connected complex Lie group $G$ on $X$ (so the leaves of $\mathcal{F}$ are the orbits of $G$), and also assume that $D$ is given by a $G$–action $\rho_E$ on $E_H$ (so the tangent spaces to the leaves in $E_H$ are the horizontal subspaces). Then $\eta$ is strongly adapted to $D$ if and only if $\eta$ is strongly adapted to the $G$–connection on $E_H$ given by $\rho_E$.

**Proof.** The above condition that $\eta|_\mathcal{F} = D$ is equivalent to the condition that the $G$–connection $\widetilde{\eta}$ constructed in \cite{BP} from $\eta$ coincides with the $G$–connection on $E_H$ given by the above $G$–action $\rho_E$. Therefore, the result follows from Lemma \cite{BP}.

**References**

\begin{itemize}
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