On the Asymptotic Behavior of Solutions to the Einstein Vacuum Equations in Wave Coordinates

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Abstract: We give asymptotics for the Einstein vacuum equations in wave coordinates with small asymptotically flat data. We show that the behavior is wave like at null infinity and homogeneous towards time like infinity. We use the asymptotics to show that the outgoing null hypersurfaces approach the Schwarzschild ones for the same mass and that the radiated energy is equal to the initial mass.

1. Introduction

The Einstein vacuum equations \( R_{\mu \nu} = 0 \) in wave coordinates become a system of nonlinear wave equations for the metric, called the reduced Einstein equations

\[
\tilde{\Box}_g g_{\mu \nu} = F_{\mu \nu}(g)[\partial g, \partial g], \quad \text{where} \quad \tilde{\Box}_g = g^{\alpha \beta} \partial_{\alpha} \partial_{\beta}, \quad g^{\alpha \beta} = (g^{-1})^{\alpha \beta},
\]

is the reduced wave operator and \( F_{\mu \nu} \) are quadratic in \( \partial g \). The metric is assumed to have signature \((-1, 1, 1, 1)\) and satisfy the wave coordinate condition

\[
\partial_{\alpha} (\sqrt{|g|} g^{\alpha \beta}) = 0, \quad \text{where} \quad |g| = |\det g|.
\]

This is preserved by (1.1) if data satisfies the constraint equations. The initial data are assumed to be asymptotically flat, i.e., for some small \( M > 0 \) and \( 0 < \gamma < 1 \)

\[
g_{ij} |_{t=0} = (1 + Mr^{-1}) \delta_{ij} + o(r^{-1-\gamma}), \quad \partial_t g_{ij} |_{t=0} = o(r^{-2-\gamma}), \quad r = |x|.
\]

Choquet-Bruhat [CB1] proved local existence for Einstein’s equations in wave coordinates. Christodoulou–Klainerman [CK] proved global existence of small solutions for Einstein’s equations in a coordinate invariant way. It was assumed that the wave coordinates behaved badly for large times. Nevertheless, in Lindblad–Rodnianski [LR3] we proved global existence in wave coordinates. In this paper we are studying the precise asymptotic behaviour of solutions to (1.1)–(1.2) that are small perturbations \( g = m + h \) of the Minkowski metric \( m \).
The decay we prove is $\varepsilon(t+r)^{-1}$ for tangential components of $h$ and for all components with a logarithmic loss close to the light cone, see Sect. 1.3. The asymptotics we give can roughly be written in the form

$$h(t, r \omega) \sim H(r^* - t, \omega)/(t + r) + K\left(\frac{r^* - t}{r + r^*}, \omega\right)/(t + r), \quad r^* \sim r + M \ln r,$$

where $\omega = x/|x|$. $H$ is concentrated close to the outgoing light cones $r^* - t$ constant, $|H(q^*, \omega)| \leq \varepsilon (1 + |q^*|)^{-\gamma'}$, and $K$ is homogeneous of degree 0 with a log singularity at the light cone $|K(s, \omega)| \leq \varepsilon \ln |s|$ for the nontangential components, see Sect. 1.5. $H$ is the radiation field of a free curved wave operator and $K$ is the backscattering of the wave operator with quadratic source terms.

The estimates can be used for proving sharp decay of the curvature, weak Penrose peeling properties, as in [CK]. We use the asymptotics to prove a Bondi type mass loss law, that the radiated energy equals the initial mass. The radiated energy is what is detected in the gravitational wave detectors [HN,C2]. For coupling to matter fields or for scattering from infinity one needs to know the precise decay or asymptotics also in the interior. It is plausible our methods can be used for studying gravitational radiation from post-Newtonian sources [B] and polyhomogeneous expansions at null infinity [CW]. The method also works for other wave equations with semilinear terms that satisfy a weak null condition.

Below we give heuristics, present the results and explain the structure of the proof. We start by reviewing the null structure and the global existence result of [LR3] in Sect. 1.1. In Sect. 1.2 we give a heuristic explanation of the nonlinear effects on the asymptotic behavior. In Sect. 1.3 we give Einstein’s equations in asymptotic characteristic coordinates and we state the sharp decay estimates that we prove in Sect. 2 through 6 (assuming the decay estimates of [LR3]). In Sect. 1.5 we give a heuristic explanation of the weak null condition and the asymptotic expansion along outgoing characteristics towards null infinity. In Sect. 1.6 we state the asymptotics that we prove in Sect. 7 through 9, first for tangential components at null infinity and later in the interior (which depends on the former). In Sect. 1.6 we state the asymptotics of the characteristic surfaces and a Bondi type mass law we prove in Sects. 10 and 11.

### 1.1. Einstein’s equation in wave coordinates, the weak null structure and global existence.

Einstein’s equations in wave coordinates form a system for $h = g - m$;

$$\Box_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h), \quad \text{where} \quad \Box_g = \Box + \tilde{h}^{\alpha\beta} \partial_\alpha \partial_\beta. \quad (1.5)$$

Here $\Box = \partial_t^2 - \Delta_x$, $\tilde{h}^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta} = -h^{\alpha\beta} + O(h^2)$, where $h^{\alpha\beta} = m^{\alpha\mu} m^{\beta\nu} h_{\mu\nu}$, and

$$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial h, \partial h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$

where $Q_{\mu\nu}$ satisfies the standard null condition and $G_{\mu\nu}$ is cubic, and by [LR1]

$$P(D, E) = D^\alpha E^\beta / 4 - D^{\alpha\beta} E_{\alpha\beta}/2.$$

General wave equations with quadratic nonlinearities may blow up as shown in [J1,J2] for $\Box \phi = \phi_\gamma$ or $\Box \phi = \phi_t \Box \phi$. The null condition, e.g. for $\Box \phi = \phi_\gamma - |\nabla_x \phi|^2$, guarantees small data global existence [C1,K1]. Einstein’s equations in wave coordinates do not, however, satisfy the null condition. For the quasilinear equation $\Box \phi = c^{\alpha\beta} \phi^{\alpha\beta} \partial_\alpha \partial_\beta \phi$, which resembles the quasilinear terms in Einstein’s equations, global existence was
proven in [L3,L4,A]. A simple semilinear system that violates the null condition yet trivially has global solutions is
\[ \Box \phi_1 = 0, \quad \Box \phi_2 = (\partial_t \phi_1)^2. \] (1.7)

In [LR1] we observed that the semilinear terms of (1.5) in a null frame \( \mathcal{N} \),
\[ L = \partial_t - \partial_r, \quad L = \partial_t + \partial_r, \quad S_1, S_2 \in S^2, \quad \langle S_i, S_j \rangle = \delta_{ij} \] (1.8)
can be modeled by such a system. In fact, it is well known that for solutions of wave equations derivatives tangential to the outgoing light cones \( \tilde{\mathcal{O}} \in \mathcal{T} = \{ L, S_1, S_2 \} \) decay faster. Modulo tangential derivative \( \tilde{\mathcal{O}} \) we have
\[ \partial_\mu h \sim L_\mu \partial_q h, \quad \text{where} \quad \partial_q = (\partial_r - \partial_t)/2, \quad L_\mu = m_\mu L^\nu, \] (1.9)
and modulo quadratic terms with at least one tangential derivative or cubic
\[ \tilde{\mathcal{O}}_g h_{\mu \nu} \sim L_\mu L_\nu P(\partial_q h, \partial_q h), \quad \text{where} \quad \tilde{\mathcal{O}}_g h_{\mu \nu} \sim \Box h_{\mu \nu} - h_{LL} \partial_\mu^2 h_{\mu \nu}, \] (1.10)
and \( h_{LL} = h_{\alpha \beta} L^\alpha L^\beta \). In a null frame the semilinear terms become
\[ \tilde{\mathcal{O}}_g h_{TU} \sim 0, \quad T \in \mathcal{T}, \quad U \in \mathcal{N} \quad (\tilde{\mathcal{O}}_g h)_{UU} \sim 4P_\mathcal{N}(\partial_q h, \partial_q h), \] (1.11)
since \( T^\mu L_\mu = 0 \), \( T \in \mathcal{T} \). Here by [LR2], (1.6) in a null frame becomes
\[ P_\mathcal{N}(D, E) = -(D_{LL} E_{LL} + D_{LL} E_{LL})/8 - (2D_{AB} E^{AB} - D^A E_B^B)/4 \] (1.12)
\[ + (2D_{AL} E^A_L + 2D_{AL} E^A_L - D^A E_{LL} - D_{LL} E^A_A)/4. \] (1.13)
Hence the right of (1.11) only contain \( \partial_q h_{LL} \) through the term \( \partial_q h_{LL} \partial_q h_{LL} \). However, using (1.9) the wave coordinate condition (1.2) in a null frame becomes
\[ \partial_q h_{LT} \sim 0, \quad T \in \mathcal{T}, \quad \delta^{AB} \partial_q h_{AB}^T \sim 0, \quad A, B \in \mathcal{S} = \{ S_1, S_2 \}, \] (1.14)
modulo tangential derivatives, see [LR2], so \( P_\mathcal{N}(\partial_q h, \partial_q h) \sim P_N(\partial_q h, \partial_q h) \), where
\[ P_N(D, E) = -D_{AB} E^{AB}/2, \quad A, B \in \mathcal{S}. \] (1.15)
Hence the right in (1.11) only depends on components we have better control on. The main quasilinear term (1.10) is controlled integrating (1.14) from data (1.3)
\[ h_{LL} \sim 2M/r. \] (1.16)

In [LR3] we found solutions to Einstein’s equations in wave coordinates:
\[ g = m + h, \quad h = h^0 + h^1, \quad h^0_{\alpha \beta} = \tilde{\chi} \left( \frac{r}{r_{\text{inf}}} \right)^M \delta_{\alpha \beta}, \] (1.17)
where \( m \) is the Minkowski metric, \( h^0 \) is the leading term at space-like infinity. Here \( \tilde{\chi}(s) = 1 \), when \( s > 1/2 \) and \( \tilde{\chi}(s) = 0 \), when \( s < 1/4 \), and \( \tilde{\chi}'(s) \geq 0 \). The mass \( M \) is assumed to be small and \( h_1 \) and its derivatives are assumed to be small and satisfying the asymptotic flatness condition (1.3) initially, i.e., decay like \( r^{1-\gamma} \). We showed that the solution exists globally and satisfies the decay estimates
\[ |Z^I h^1(I, x)| \leq C_N \varepsilon (1 + t)^{-1 + C_N \varepsilon (1 + q_\pm)^{-\gamma}}, \quad |I| \leq N - 2, \] (1.18)
where \( q = r - t \) and \( q_\pm = \max(\pm q, 0) \). Here \( N \geq 6 \) and \( 0 < \gamma < 1 \) and \( Z^I \) stands for a product of \( |I| \) of the vector fields that commute with \( \Box \) and the scaling vector field, i.e. \( \partial_t, \partial_q, x^i \partial_j - x^j \partial_i, x^i \partial_t + t \partial_i, \) and \( t \partial_t + x^i \partial_i, i, j = 1, 2, 3. \)

Our first result says that one can almost remove the \( C_N \varepsilon \) in the exponent in (1.18) by changing to asymptotically characteristic coordinates, see Sect. 1.3.
1.2. Nonlinear effects on the asymptotic behavior. There are two types of nonlinear distortions of the linear asymptotic behaviour related to the quasilinear terms $g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu}$ and the semilinear terms $F_{\mu\nu}(g)[\delta g, \partial g]$, respectively.

1.2.1. Asymptotic Schwarzschild coordinates. In order to unravel the effect of the quasilinear terms one can change to characteristic coordinates as in [CK], but this loses regularity and is not explicit. Instead we use the asymptotic behavior of the metric to determine the characteristic surfaces asymptotically and use this to construct coordinates. Due to (1.16) the outgoing light cones of a solution with asymptotically flat data (1.3) approach those of the Schwarzschild metric

$$-\frac{r-M/2}{r+M/2} \ dt^2 + \frac{r-M/2}{r+M/2} \ dr^2 + (r + \frac{M}{2})^2 (d\theta^2 + \sin^2 \theta \ d\phi^2) \sim (m_{\alpha\beta} + \frac{M}{r} \delta_{\alpha\beta}) \ dx^\alpha \ dx^\beta.$$  

The outgoing light cones for the Schwarzschild metric satisfy $t \sim r^* - q^*$, where $r^* = r + M \ln r$. We show that there is a solution to the eikonal equation that approaches the one for Schwarzschild

$$g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0, \quad u \to u^* = t - r^*, \quad \text{when} \quad r > t/2 \to \infty.$$  

In fact the terms that could cause the largest deviation are controlled by the wave coordinate condition. We therefore make the change of variables $x = r \omega \to x^* = r^* \omega$, for large $r$, and the wave operator $\square_g$ asymptotically becomes the constant coefficient wave operator $\square^*$ in the $(t, x^*)$ coordinates. We use the vector fields $Z^*$, which commute with this wave operator $\square^*$ and the scaling.

1.2.2. Sources on light cones. The inhomogeneous terms in (1.1) cause a more serious distortion of the asymptotic behaviour. A solution of a linear homogeneous wave equation $\square \phi = 0$ decays like $t^{-1}$ and has a radiation field

$$\phi(t, x) \sim F(r - t, \omega)/r.$$  

The same is true if only $|\square \phi| \lesssim (t + r)^{-2-\epsilon} (t - r)^{-1}$ decay sufficiently fast. However, quadratic inhomogeneous terms do not decay sufficiently

$$\phi_q(t, x)^2 \sim F_q(r - t, \omega)^2/r^2, \quad \text{where} \quad |F_q(q, \omega)| \lesssim (1 + |q|)^{-1}.$$  

The asymptotics for the wave equation with such sources was studied in [L1]:

$$-\square \psi = n(r - t, \omega)/r^2, \quad \text{(1.19)}$$  

where $n(q, \omega)$ has compact support in $q$. The solution is given by the formula

$$\psi(t, r \omega) = \int_{r-t}^\infty \frac{1}{4\pi} \int_{S^2} \frac{n(q, \sigma)}{t + q - r \langle \omega, \sigma \rangle} \ dS(\sigma) \ dq. \quad \text{(1.20)}$$  

Close to the light cone $t \sim r$ the integrand is concentrated when $n(q, \sigma) \sim n(q, \omega)$:

$$\psi(t, r \omega) \sim \int_{r-t}^\infty \frac{1}{2r} \ln |\frac{t + q + r}{t + q - r}| \ n(q, \omega) \ dq, \quad \text{when} \quad r \sim t, \quad \text{(1.21)}$$  

and this leads to a log correction to the asymptotic behavior. In fact an explicit calculation in spherical coordinates $\square \psi = r^{-1}(\partial_r - \partial_\tau)(\partial_r + \partial_\tau)(r \psi) + r^{-2} \Delta_\omega \psi$ shows that (1.21) satisfies (1.19) up to angular derivatives that decay faster. (1.20) holds also in the interior and gives $t^{-1}$ decay when $r < t/2$.  

1.3. Sharp decay in asymptotic Schwarzschild coordinates. We now summarize the decay results from Sect. 6 in the coordinates in Sect. 1.2. With $\tilde{\chi}$ as in (1.17) let

$$r^* = r + M \tilde{\chi} \left( \frac{r}{1 + r} \right) \ln |r|, \quad t^* = t, \quad \omega^* = \omega, \quad x^* = r^* \omega. \quad (1.22)$$

1.3.1. Einstein equations in asymptotic Schwarzschild coordinates. We express Einstein’s equation in the good coordinates up to an error controlled by [LR3]:

**Proposition 1.** Let $P_\mathcal{F}$ be $P_N$ or $P_\mathcal{S}$ as in (1.12) and (1.15) and let

$$P_{\mu\nu}^* = P_\mathcal{F}(\partial_\mu h, \partial_\nu h) \quad \text{or} \quad P_{\mu\nu}^* = L_\mu L_\nu P_\mathcal{F}(\partial_\mu h, \partial_\nu h).$$

Let $Z^*$ stand for the vector fields $\partial_t^*, \partial_\mu^* \partial_\nu^* - x^i \partial_j^* - x^j \partial_i^*, \partial^* + t \partial_i^*$, and $t \partial_t^* + x^i \partial_i^*$, $i, j = 1, 2, 3$. Let $\gamma$ be as in (1.18). Then for $|I| \leq N - 5$ and $\gamma' < \gamma - C\varepsilon$

$$|Z^* [\Box^* h_{\mu\nu} - P_{\mu\nu}^*]| \lesssim \varepsilon^2 (1 + t + |q^*|)^{-2} - \gamma' (1 + |q^*|)^{-2 + \gamma}, \quad \Box^* = m^\alpha^\beta \partial_\alpha \partial_\beta.$$

1.3.2. The sharp decay estimates for tangential components of the metric. The estimate for tangential components follows from commuting the wave equation above with the tangential frame using that $P_{TU}^* \sim \partial_t h \partial_t h$ decay faster. (A difficulty is that the commutator with the angular part of $\Box$ is large in the interior.)

**Proposition 2.** For $T \in \{L, S_1, S_2\}$, $U \in \{L, L, S_1, S_2\}$, $h^1$ as in (1.17) we have (Here (1.24) holds also for $Z_{TA}^* (h_{UV}^1)$ replaced by the Lie derivatives $(L_T^I, h_{UV}^1)$)

$$|Z^* h^1_{TU}| \lesssim \varepsilon^2 (1 + t + r^*)^{-1} (1 + q^*_t)^{-\gamma'}, \quad \gamma' < \gamma - C\varepsilon, \quad (1.23)$$

$$|Z^* h^1_{LT}| + |\delta^{AB} Z^* h^1_{AB}| \lesssim \frac{\varepsilon}{1 + t + r^*} \left( \frac{1 + q^*_t}{1 + t + r^*} \right)^{\gamma'}, \quad |I| \leq N - 6. \quad (1.24)$$

1.3.3. The sharp decay estimates for all components. The estimate for all components follows from using the estimates for tangential components in $P_{\mu\nu}^*$:

**Proposition 3.** With $\gamma' = \gamma - C\varepsilon$ we have for $|I| \leq N - 6$

$$|Z^* h^1| \lesssim \frac{\varepsilon + \varepsilon^2 S^0(t, r^*)}{(1 + t + r^*)(1 + q^*_t)^{\gamma'}} \quad \text{where} \quad S^0(t, r^*) = \frac{t}{r^*} \ln \left( \frac{\langle t + r^* \rangle}{\langle t - r^* \rangle} \right).$$

1.4. The asymptotics at null infinity. Here we give the heuristics for the asymptotics along the surfaces $q^* = r^* - t$ constant towards null infinity.

1.4.1. The weak null condition and the asymptotic system for wave equations. Consider a general system of quasilinear wave equations in 3 space dimensions:

$$\Box \phi_I = \sum_{|\alpha| \leq |\beta| \leq 2, |\beta| \geq 1} A^{IK}_{\alpha\beta} \partial^\alpha \phi_J \partial^\beta \phi_K + \text{cubic terms}, \quad (1.25)$$

with small initial data. If we neglect derivatives tangential to the outgoing light cones and cubic terms, which decay faster, we get

$$\Box \phi = r^{-1} (\partial_t + \partial_r)(\partial_r - \partial_t) (r \phi) + r^{-2} \times \text{angular derivatives},$$

$$\partial_\mu = \frac{1}{r} \partial_\mu (\partial_r - \partial_t) + \text{tangential derivatives}, \quad \dot{\omega} = (-1, \omega),$$
where \( x = r \omega, \omega \in \mathbb{S}^2 \) are polar coordinates. We see that asymptotically
\[
(\partial_t + \partial_r)(\partial_r - \partial_t)(r \phi_I) \sim r^{-1} \sum A^K_{I, nm} 2^{-n-m} (\partial_r - \partial_t)^n (r \phi_I) (\partial_r - \partial_t)^m (r \phi_K),
\]
where \( A^K_{I, nm} (\omega) := \sum_{|\alpha| = n, |\beta| = m} A^K_{I, \alpha \beta} \hat{\omega}^\alpha \hat{\omega}^\beta \) and \( \hat{\omega}^\alpha = \hat{\omega}_\alpha \cdot \cdots \hat{\omega}_\alpha \).

[H] proposed an asymptotic expansion as \( r \to \infty \) and \( r \sim t \) of the form
\[
\phi_I(t, x) \sim \Phi_I(q, s, \omega)/r, \quad \text{where} \quad q = r - t, \quad s = \ln r,
\]
and \( \Phi_I \) satisfies the asymptotic system:
\[
2 \partial_s \partial_q \phi_I = \sum_{n \leq m \leq 2, m \geq 1} A^K_{I, mn} (\omega) \partial_q^m \phi_I \partial_q^n \phi_K. \tag{1.26}
\]
Solutions to linear wave equations have such an expansion independent of \( s \). The null condition, which guarantees small data global existence [C1,K1] is \( A^K_{I, nm} \equiv 0 \), e.g. for \( \square \phi = \phi_t^2 - |\nabla \phi|^2 \). On the other hand [J2] showed that solutions to \( \square \phi = \phi_t \Delta_x \phi \) blowup and [H] used the blow up of the corresponding asymptotic systems to predict the precise exponential blow up time. For the quasilinear equation \( \square \phi = c^{\alpha \beta} \phi \partial_\alpha \partial_\beta \phi \), [L3] observed that the asymptotic system has global exponentially growing solutions \( \Phi \sim e^{c s} \). For the simpler semilinear system (1.7) that violates the null condition, the solution to the asymptotic system
\[
2 \partial_s \partial_q \phi_I = 0, \quad 2 \partial_s \partial_q \phi_2 = (\partial_q \phi_1)^2, \tag{1.27}
\]
is \( \Phi_1 = F_1(q, \omega), \Phi_2 = s F_2(q, \omega) + F_3(q, \omega), F_2(q, \omega) = \int (\partial_q F_1(q, \omega))^2 dq/2 \) so
\[
\phi_1 \sim F_1(t - r, \omega)/r, \quad \phi_2 \sim \ln r F_2(t - r, \omega)/r + F_3(t - r, \omega)/r.
\]

In view of these examples we say that (1.25) satisfies the weak null condition [LR1] if (1.26) has global solutions growing at most exponentially in \( s \). The methods here work for the subclass where it grows at most polynomially in \( s \).

1.4.2. The asymptotic system for Einstein’s equations in wave coordinates. With
\[
h_{\mu \nu}(t, x) \sim H_{\mu \nu}(q, s, \omega)/r, \quad \text{where} \quad q = r - t, \quad s = \ln r, \quad r \sim t,
\]
the asymptotic system for Einstein’s equations in a null frame takes the form:
\[
(2 \partial_s - H_{LL} \partial_q) H_{TU} = 0, \tag{1.28}
\]
\[
(2 \partial_s - H_{LL} \partial_q) H_{LL} = 4 P_{S^2}(\partial_q H, \partial_q H), \tag{1.29}
\]
by (1.11). By (1.14) the wave coordinate condition takes the asymptotic form
\[
\partial_q H_{LT} = 0, \quad T \in \{ L, S_1, S_2 \}, \quad \delta^{AB} \partial_q H_{AB} = 0, \quad A, B \in \{ S_1, S_2 \},
\]
and because the solution for large \( r \) asymptotically is Schwarzschild
\[
H_{LL} = 2M, \quad H_{LA} = 0, \quad \delta^{AB} H_{AB} = 2M, \quad A, B \in \{ S_1, S_2 \}.
\]
Here the right hand side of (1.29) only depends on tangential components \( P_S(D, E) = -D_{AB} E^{A\bar{B}}/2 \) and the quasilinear term simplifies if we introduce the integral curves to
the vector field \((2\partial_t - 2M\partial_q)\) given by \(r - t = q(s) = q^* - Ms = q^* - M\ln(1 + r)\). Hence if we change variables to \(q^*\) and integrate we get

\[
\begin{align*}
H^*_T(q^*, \omega, s) &= H^*_T(q^*, \omega), \\
H^*_L(q^*, \omega, s) &= H^*_L(q^*, \omega) - s \int P_S(\partial_{q^*}H^\infty, \partial_{q^*}H^\infty)(q^*, \omega) dq^*. 
\end{align*}
\]

We get an asymptotic radiation field as for a linear homogeneous wave equation for all but one component which is multiplied by a logarithm \(s = \ln(1 + r)\).

1.5. The full asymptotics of the metric. Using the decay estimates we will prove the asymptotics stated below in Sects. 7–9. First we prove the asymptotics for the tangential components in Sect. 1.5.1. First after that can one define the asymptotic source term in Sect. 1.5.2 since it depends on the tangential components. Next we make a different decomposition into a part that comes from the asymptotic source and a remainder. In Sect. 1.5.3 we show that the remainder has the asymptotics of a free wave. For the backscattering of the source we first give the asymptotics close to the light cone in Sect. 1.5.4 and in Sect. 1.5.5 we give a formula that also gives the leading interior behaviour.

1.5.1. Asymptotics for tangential components close to the light cone. Let

\[
h^*_\mu\nu(t, r^*\omega) = h_{\mu\nu}(t, r\omega), \quad H^*_{TU}(q^*, \omega, r^*) = r^*H^*_{TU}(r^* - q^*, r^*\omega).
\]

With similar estimates used to prove decay for tangential components we prove:

**Proposition 4. The limit**

\[
H^\infty_T(q^*, \omega) = \lim_{r^*\to\infty} H^*_T(q^*, \omega, r^*), \quad T \in \{L, A, B\}, \quad U \in \{L, L, A, B\},
\]

exists and satisfies \(H^\infty_T = H^\infty_U\),

\[
H^\infty_L(q^*, \omega) = 0, \quad H^\infty_L(q^*, \omega) = \delta^{AB}H^\infty_{AB}(q^*, \omega) = 2M.
\]

For \(|\alpha| + k \leq N - 6\), we have

\[
|\partial^\alpha(1 + |q^*|)\partial_{q^*})^k H^\infty_T(q^*, \omega)| \lesssim \varepsilon,
\]

\[
|\partial^\alpha(1 + |q^*|)\partial_{q^*})^k [H^*_T(q^*, \omega, r^*) - H^\infty_T(q^*, \omega)]| \lesssim \varepsilon\left(\frac{1 + q^*}{1 + t + r^*}\right)^{\gamma'}.
\]

1.5.2. The asymptotics source. Now, once we have shown that the limit of the tangential components exist we can define

\[
n(q^*, \omega) = -P_S(\partial_{q^*}H^\infty, \partial_{q^*}H^\infty)(q^*, \omega).
\]

It follows that

\[
|\partial^\alpha(q^*\partial_{q^*})^k n(q^*, \omega)| \lesssim \varepsilon^2(1 + |q^*|)^{-2},
\]

for \(|\alpha| + k \leq N - 7\). (Here \(P_S\) given by (1.15).) Let \(k_{\mu\nu}\) be the solution to

\[
-\Box^*k_{\mu\nu} = \mu_\mu(\omega)L_\nu(\omega)n(r^* - t, \omega)r^{s-2}\chi\left(\frac{r^* - t}{4r^*}\right)
\]

with vanishing data, where \(\chi(s) = 1\), when \(|s| \leq 1/2\) and \(\chi(s) = 0\), when \(|s| \geq 3/4\).
Proposition 5. We have
\[ |Z^I (\square^* h_{\mu\nu} - \square^* k_{\mu\nu})| \lesssim \varepsilon^2 (1 + t + r^*)^{-2-\gamma+C\varepsilon (1 + |q^*|)^{-2+\gamma}}, \quad |I| \leq N - 7. \]

Let
\[ h^{1e} = h - h^{0e}, \quad \text{where} \quad h^{0e}_{\mu\nu} = \delta_{\mu\nu} \chi^e(r^* - t)/r^*. \]

Here \( \chi^e(s) = 1 \), when \( s \geq 2 \) and \( \chi^e(s) = 0 \), when \( s \leq 1 \). Then
\[ |Z^I (h^{1e}_{\mu\nu} - k_{\mu\nu})| \lesssim \varepsilon^2 (1 + t + r^*)^{-1} (1 + |q^*|)^{-\gamma'}, \quad |I| \leq N - 7. \]

These estimates tell us that the leading behavior in the exterior is determined by the Schwarzschild metric and the leading behavior in the interior by the solution to the wave equation with a source given by the far field \( n(q^*, \omega) \).

1.5.3. The asymptotics for all components of the metric in the interior. Let
\[ h^e_{\mu\nu}(t, r^* \omega) = h^{1e}_{\mu\nu}(t, r^* \omega) - k_{\mu\nu}(t, r^* \omega), \quad H^e_{\mu\nu}(q^*, \omega, r^*) = r^* h^e_{\mu\nu}(r^* - q^*, r^* \omega). \]

With similar estimates used to prove decay for all components we prove:

Proposition 6. The limit
\[ H^e_{\mu\nu}(q^*, \omega, r^*) = \lim_{r^* \to \infty} H^e_{\mu\nu}(q^*, \omega, r^*), \]
exists and satisfies \( H^e_{\mu\nu} = H^e_{\nu\mu} \). Moreover for \( |\alpha| + k \leq N - 7 \) we have
\[ |\partial^\alpha \left( (1 + |q^*|)^{\beta} \right) H^e_{\mu\nu}(q^*, \omega) | \lesssim \varepsilon (1 + |q^*|)^{-\gamma'}, \]
\[ |\partial^\alpha \left( (1 + |q^*|)^{\beta} \right) [H^e_{\mu\nu}(q^*, \omega, r^*) - H^e_{\mu\nu}(q^*, \omega)] | \lesssim \varepsilon (1 + t + r^*)^{-\gamma'}. \]

1.5.4. Null infinity asymptotics of the solution with the asymptotic source. Let
\[ k^1_{\mu\nu}(t, r^* \omega) = L_\mu(\omega) L_\nu(\omega) \int_{r^* - t}^\infty \frac{1}{2r^*} \ln \left( \frac{t + r^* + q^*}{r^* - t + q^*} \right) n(q^*, \omega) dq^* \chi \left( \frac{r^* - t}{1 + r^*} \right). \]

An explicit calculation in spherical coordinates show that

Proposition 7. For any \( a < 1 \) we have
\[ |Z^I (k_{\mu\nu}^1 - k_{\mu\nu}^0)(t, r^* \omega) | \lesssim \varepsilon (t + r)^{-3} \ln \left( \frac{t + r^*}{t - r^* + q^*} \right) (r^* - t)^{-a}, \]
\[ |Z^I (k_{\mu\nu} - k_{\mu\nu}^1)(t, r^* \omega) | \lesssim \varepsilon (1 + t + r^*)^{-1} (r^* - t)^{-a}. \]

Let
\[ k^0_{\mu\nu}(t, r^* \omega) = k^1_{\mu\nu}(t, r^* \omega) - k_{\mu\nu}(t, r^* \omega), \quad K^0_{\mu\nu}(q^*, \omega, r^*) = r^* k^0_{\mu\nu}(r^* - q^*, r^* \omega). \]

With similar estimates used to prove decay for all components we prove:

Proposition 8. The limit
\[ K^0_{\mu\nu}(q^*, \omega, r^*) = \lim_{r^* \to \infty} K^0_{\mu\nu}(q^*, \omega, r^*), \]
exists and satisfies \( K^0_{\mu\nu} = K^0_{\nu\mu} \) and for \( a < 1 \) and \( |\alpha| + k \leq N - 7 \)
\[ |\partial^\alpha \left( (1 + |q^*|)^{\beta} \right) K^0_{\mu\nu}(q^*, \omega) | \lesssim \varepsilon (1 + q^*)^{-a}, \]
\[ |\partial^\alpha \left( (1 + |q^*|)^{\beta} \right) [K^0_{\mu\nu}(q^*, \omega, r^*) - K^0_{\mu\nu}(q^*, \omega)] | \lesssim \varepsilon \left( \frac{1 + q^*}{1 + t + r^*} \right)^a. \]
1.5.5. Interior asymptotics of the solution with the asymptotic source. The results so far suffice to prove existence of the radiation field, but to get more precise behavior we can subtract off a better approximation using formulas from [L1]. Note that (1.31) is asymptotically homogenous in any region \( r/t < c < 1 \).

**Proposition 9.** Let \( n \) and \( k \) be as in Sect. 1.5.2 and set

\[
k_{\mu\nu}(x^*, t) = \int_0^{\infty} \frac{1}{4\pi} \int_{S^2} \frac{L_\mu(\sigma)L_\nu(\sigma)n(q^*, \sigma)}{t + q^* - (x^*, \sigma)} dS(\sigma) \chi(\langle q^* \rangle_{t+r^*}) dq^*.
\]

Then for any \( a < 1 \) and \( |I| + |J| + |K| \leq N - 7 \)

\[
|\Omega^I S^J \partial^K (k - k^2)| \lesssim \varepsilon^2 (1 + t + r^*)^{-1} (1 + |r^* - t|)^{-a}.
\]

1.6. Applications. Here we use the decay estimates and asymptotics to study the asymptotic surfaces and null foliation as in [KN] and the radiated energy and mass as in [C2]. The estimates and asymptotics will also be used for proving global existence with matter fields and scattering from infinity in future work.

1.6.1. Asymptotics of the characteristic surfaces. We show that the eikonal eq.

\[
g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0, \quad \text{in } r > |t|/2,
\]

has a unique solution with asymptotic data at infinity \( u \sim u^* = t - r^* \), as \( t \to \infty \):

**Proposition 10.** The eikonal equation (1.32) has a solution \( u = \tilde{u} + u^* \) satisfying

\[
\sum_{|I|\leq 2} |Z^I \tilde{u}| + |Z^I \tilde{v}| \leq C_1 \varepsilon \left( \frac{1 + (r^* - |t|)}{1 + t + |q^*|} \right)^{\gamma'}, \quad r > |t|/2.
\]

By time reflection there is also a solution \( v = \tilde{v} + v^* \), so \( v \sim v^* = t + r^* \), as \( t \to -\infty \). [CK] used \( u \) to define modified vector fields. This shows why it works with \( u^* \).

1.6.2. The mass loss law. Using that the asymptotic of \( H_{LL} \) close to the light cone is \( 2M \) and on the other hand is given by the source there has to be a relation between \( M \) and the source formula above determined by \( n \). We have

**Proposition 11.** Let \( n(q^*, \omega) \) given by (1.30) and let \( M \) be as in (1.3). Then

\[
\frac{1}{2} \int_{-\infty}^{+\infty} \int_{S^2} n(q^*, \omega) \frac{dS(\omega)}{4\pi} dq^* = M.
\]

The proposition in particular implies that, if \( n = 0 \) then \( M = 0 \), and then by the positive mass theorem the space time is Minkowski space. The proposition can be interpreted as that the outgoing radiation is equal to the initial mass.
2. The Weak Decay of the Metric in the Minkowski Coordinates and Decay from the Wave Coordinate Condition

We start from decay estimates for the metric from [LR3]. There we constructed metrics $g$ satisfying Einstein’s equations in wave coordinates of the form

$$ g = m + h^0 + h^1, \quad \text{where} \quad h^0_{\alpha\beta} = \tilde{\chi}(\frac{r}{1+r}) \frac{M}{r} \delta_{\alpha\beta}, $$

where $m$ is the Minkowski metric and $h^0$ is picking up the decay at space-like infinity. Here $\tilde{\chi}(s) = 1$, when $s > 1/2$, $\tilde{\chi}(s) = 0$, when $s < 1/4$, and $\tilde{\chi}'(s) \geq 0$. We proved that for any $N \geq 6$ and $0 < \gamma < 1$ there are solutions satisfying

$$ E_N(t) = \sum_{|I| \leq N} \| w^{1/2} \partial I h^1(t, \cdot) \|_{L^2} \leq C_N \alpha(1 + t)^{CN}, \quad w^{1/2} = (1 + q_+)^{1/2q_+}, $$

where $q = r - t$ and $q_{\pm} = \max(\pm q, 0)$, provided that this norm is small initially and $M$ is small. (Here $Z^I$ stands for a product of $|I|$ of the vector fields that commute with the constant coefficient wave operator and the scaling vector field, i.e. $\partial_t$, $\partial_i$, $x^i \partial_j - x^j \partial_i$, $x^i \partial_t + t \partial_i$, and $t \partial_0 + x^i \partial_i$, $i = 1, 2, 3$.) In [LR3] we proved:

**Proposition 12 (Weak decay).** For $|I| \leq N - 4$ we have

$$ | \partial Z^I h^1 | + | Z^I h^1 | + | \tilde{\partial} Z^I h^1 | + | \tilde{\partial} h^1 | 1 + r + r 1 + |q| \leq \epsilon(1 + q_+)^{-\gamma} (1 + r + r)^{1-\epsilon} (1 + |q|). \quad (2.1) $$

This estimates, with $(1 + q_+)^{-\gamma}$ replaced by $(1 + q_+)^{-C\epsilon}$, hold also for $h^0 + h^1$.

Here $\tilde{\partial}$ stands for derivatives tangential to the outgoing light cones, i.e. linear combinations of $L$, $S_1$, $S_2$ where $L = \partial_t + \partial_r$, $L = \partial_i - \partial_r$, and $S_1$, $S_2$ are orthonormal vectors (in the Minkowski metric) that span the tangent space of the sphere $S^2$ whose components are independent of $t, r$ (i.e. $S_i = S^k_i(\omega) \partial_k$). $A, B, C, D$ will denote any of the vector fields $S_1, S_2$. Repeated use of these are summed over.

Similarly we express the inverse of the metric as

$$ g^{\mu\nu} = m^{\mu\nu} + h_0^{\mu\nu} + h_1^{\mu\nu}, \quad h_0^{\mu\nu} = -\tilde{\chi}(\frac{r}{1+r}) \frac{M}{r} \delta^{\mu\nu}. $$

Then $m^{\mu\nu} + h_0^{\mu\nu} - h_1^{\mu\nu}$ is an approximate inverse to $g_{\mu\nu} = m_{\mu\nu} + h_0^{\mu\nu} + h_1^{\mu\nu}$ up to $O(h^2)$ so $h_1^{\mu\nu} = -h_1^{\nu\mu} + O(h^2)$. Therefore $h_1$ will satisfy the same estimates (2.1). Certain components of $h^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu}$ expressed in a null frame $h_{UV} = U\mu V_\nu h^{\mu\nu}$, where $V_\mu = m_{\mu\nu} V^\nu$ and $U, V \in N = \{L, L, S_1, S_2\}$, have improved decay. This comes from the wave coordinate condition; $\partial_\mu (g^{\mu\nu} \sqrt{|\det g|}) = 0$ that can be expressed

$$ \partial_\nu h^{\mu\nu} = \Lambda^\nu(h, \partial h), \quad \text{where} \quad \Lambda^\nu = h^{\mu\nu} - m^{\mu\nu} \text{tr} h/2, \quad \text{tr} h = m_{\alpha\beta} h^{\alpha\beta}, \quad (2.2) $$

and $\Lambda^\nu(h, \partial h) = (m^{\mu\nu} m_{\alpha\beta} - g^{\mu\nu} g_{\alpha\beta}) \partial_\mu g^{\alpha\beta}/2 = O(h \partial h)$. Using this we get

**Proposition 13 (Weak wave coordinate decay).** For $|I| \leq N - 4$ and $r^8 > t/8$:

$$ | \partial Z^I h_{1LT} | + | \partial Z^I \delta^{AB} h_{1AB} | \lesssim \epsilon(1 + r + r)^{-2\epsilon} (1 + q_+)^{-\gamma}, \quad (2.3) $$

$$ | Z^I h_{1LT} | + | Z^I \delta^{AB} h_{1AB} | \lesssim \epsilon(1 + r + r)^{-1-\gamma+\epsilon} (1 + q_+) + \epsilon(1 + r + r)^{-\epsilon} (1 + q_+). \quad (2.4) $$

(2.3)–(2.4) also hold for $Z^I(h_{LT})$ replaced by $(LZ^I)_L T$ and $Z^I(h_{AB})$ by $(LZ^I)_L A B$, where $LZ h^{\mu\nu} = Z h^{\mu\nu} - \partial_\alpha Z h^{\mu\nu} - \partial_\alpha Z^\nu h^{\mu\nu}$ is the Lie derivative, $Z \in N$. 

Remark 1. In general \((L_Z h)_{LT} \neq Z(h_{LT})\) but \((L_{W} h)_{LL} = \Omega(h_{LL})\).

Lemma 1. With \(\partial_q = (\partial_r - \partial_t)/2\), \(\partial_s = (\partial_r + \partial_t)/2\) and sum over \(A = S_1, S_2\),
\[
\partial_q \left( L_\mu U_\nu k^{\mu\nu} \right) = \partial_s \left( L_\mu U_\nu k^{\mu\nu} \right) - A_\mu U_\nu \delta_A k^{\mu\nu} + U_\nu \partial_\mu k^{\mu\nu}, \quad U \in \mathcal{N}. \tag{2.5}
\]

Moreover
\[
\partial_q (r^2 L_\mu L_\nu k^{\mu\nu}) = \partial_s (r^2 L_\mu L_\nu k^{\mu\nu}) + r \delta^{AB} A_\mu B_\nu k^{\mu\nu} + L_\nu r^2 \partial_\mu k^{\mu\nu}
- r^2 \partial_A (A_\mu L_\nu k^{\mu\nu}) + r^2 (\partial_t A^i) A_\mu L_\nu k^{\mu\nu}. \tag{2.6}
\]

Proof. (2.5) follows from expressing the divergence in the null frame, \(\partial_\mu F^\mu = L_\mu \partial_q F^\mu - L_\mu \partial_s F^\mu + A_\mu \partial_A F^\mu\), and using that \(\partial_s\) and \(\partial_q\) commute with the frame.

(2.6) follows from (2.5) since \(L_\mu \partial_s r - L_\mu \partial_q r = R_\mu\), \(R = (0, \omega), r A^i \partial_t L_\mu = A_\mu\) and
\[
A^i \partial_t A_j = \partial_t (A_j A^i) - A_j \partial_t A^i = \partial_t (\delta^i_j - \omega_j \omega^i) - A_j \partial_t A^i = -2\omega_j r - A_j \partial_t A^i. \tag{2.7}
\]

Lemma 2. If \(Z = Z^\alpha \partial_\alpha\), where \(\partial_\beta Z^\alpha\) are constants then \(L_Z \partial_\mu k^{\mu\nu} = \partial_\mu L_Z k^{\mu\nu}\).

Proof (of Proposition 13). (2.4) follows from integrating (2.3) in the \(t - r\) direction from initial data. (2.3) follows from (2.1) and
\[
|\partial_q Z^J h_{1LT}| + |\partial_q Z^J \delta^{AB} h_{1AB}| \lesssim \sum_{|J| \leq |I| + 1} \frac{|Z^J h_1|}{1 + t + |q|} + \sum_{|J| + |K| \leq |I|} |Z^J h| \cdot |\partial Z^K h|. \tag{2.8}
\]

It suffices to prove (2.8) for \(r > 3t/4\) using that \(|\partial \phi| \leq C (1 + |t - r|)^{-1} \sum_{|J| = 1} |Z^J \phi|\).

With \(\hat{h}_{1}^\mu = h_{1}^{\mu} - m^{\mu\nu} \text{tr } h_1/2\), where \(\text{tr } h = m_{\alpha\beta} h^{\alpha\beta} = -h_{LL} + \delta^{CD} h_{CD}\), we have
\[
\partial_\mu \hat{h}_{0}^\mu = -\partial_\mu (\check{\chi} (\frac{r}{1+ t}) M r^{-1} (\delta^{\mu\nu} - m^{\mu\nu})) = 2 \check{\chi}' (\frac{r}{1+ t}) M (1 + t)^{-2} \delta^{0} \nu,
\]
and hence
\[
\partial_\mu \hat{h}_{1}^\mu = -\frac{1}{2} (\delta^{\mu\nu} g_{\alpha\beta} - m^{\mu\nu} m_{\alpha\beta}) \partial_\mu g^{\alpha\beta} - 2 \check{\chi}' (\frac{r}{1+ t}) M (1 + t)^{-2} \delta^{0} \nu.
\]

We are now going to commute vector fields through the equations in Lemma 1:
\[
[Z, \partial_q] = C^Z_L \partial_q + C^Z_A \partial_A + C^Z_C \partial_C, \quad t/2 < r < 2t,
\]
for some smooth homogeneous of degree 0 functions \(C^Z_U\). On the other hand
\[
\partial_s = \sum_{|J| = 1} \frac{c_J^L Z_J}{t + r}, \quad \partial_C = \sum_{|J| = 1} \frac{c_J^C Z_J}{t + r}, \quad t/2 < r < 2t,
\]
with some smooth \(c_J^U\) homogeneous of degree 0. Using these identities for the terms in right of (2.5) and terms generated in the commutator \([Z, \partial_q]\) we obtain
\[
|\partial_q Z^J \hat{h}_{1LT}| \lesssim \sum_{|J| \leq |I| + 1} \frac{|Z^J \hat{h}_1|}{1 + t + r} + \sum_{|J| + |K| \leq |I|} |Z^J h| \cdot |\partial Z^K h|.
\]
(2.8) follows since \(\hat{h}_{LT} = h_{1LT}\) and \(\hat{h}_{1LL} = \delta^{AB} h_{1AB}\).

There is an improvement in the \(L\) component of the quadratic terms in (2.2)
\[
\Lambda_L = O(h \partial h) + O(h_{LL} \partial tr h) + O(h^2 \partial h). \tag{2.9}
\]
3. The Asymptotic Approximation of Einstein’s Equations

Using the decay estimates from Sect. 2 we can neglect terms that decay faster (even if they depend on higher derivatives). From [LR3] we know that

\[ \square_g h_{\mu\nu} = F_{\mu\nu}(h) (\partial h, \partial h) \sim P (\partial h, \partial h), \]

where \( \square_g = g^{\alpha\beta} \partial_\alpha \partial_\beta \) and

\[ P (h, k) = (m^{\alpha\beta} m^{\alpha'\beta'}/4 - m^{\alpha\alpha'} m^{\beta\beta'}/2) h_{\alpha\beta} k_{\alpha'\beta'} \]

and \( F_{\mu\nu}(h)(\partial h, \partial h) \) are quadratic forms in \( \partial h \) depending on \( h \) such that

\[ |Z^I (F_{\mu\nu}(h)(\partial h, \partial h) - P (\partial h, \partial h))| \]
\[ \lesssim \sum_{|J| + |K| \leq |I|} |\overline{\partial} Z^J h| |\partial Z^K h| + \sum_{|J_1| + \cdots + |J_k| + |K| + |L| \leq |I|, 1 \leq k \leq |I|} |Z^J h| \cdots |Z^h| |\partial Z^K h| |\partial Z^L h|. \]  

(3.1)

We can express the tensors \( h_{\mu\nu} \) in the null frame \( h_{UV} = U^\mu V^\nu h_{\mu\nu} \) and we can express the metric \( m^{\alpha\beta} \) and hence \( P \) in terms of the null frame. By [LR2];

\[ P_N (h, k) = -\frac{1}{8} (h_{LLLL} + h_{LLLL}) - \frac{1}{4} \delta^{CD} \delta^{C'D'} (2h_{CC'} k_{DD'} - h_{CD} k_{CD'}) \]
\[ + \frac{1}{4} \delta^{CD} (2h_{CCLL} + 2h_{CLCL} - h_{CD} k_{CD}) \]

(3.2)

The special structure is important; the worst component \( h_{LL} \) is multiplied with a good component \( h_{LL} \) that can be controlled by the wave coordinate condition:

\[ P_N (h, k) = -\frac{1}{8} (h_{LLLL} + h_{LLLL}) + \sum_{S,T \in T, U,V \in N} c^{US VT} h_{US} k_{VT}. \]  

(3.3)

where the sum is over \( S, T \in T = \{ L, S_1, S_2 \} \) and \( U, V \in N = \{ L, L, S_1, S_2 \} \).

**Remark 2.** \( P_N (h, k) \) is a bilinear form on tensors expressed in frame \( N \) whereas \( P (h, k) \) is a bilinear form on tensors in the original coordinates. Now \( P_N (h, k) = P (h, k) \), and \( P_N (\partial h, \partial q) = P (\partial h, \partial q) \) since \( \partial q \) commutes with contractions with the frame. However, by \( P_N (\partial h, \partial h) \) we mean the form acting on the tensors \( \partial h_U V = \partial h (h_{\alpha\beta} U^\alpha V^\beta) \neq U^\alpha V^\beta \partial h_{\alpha\beta} \) which is different from \( P (\partial h, \partial h) \):

\[ P_N (\partial h, \partial h) = -\frac{1}{8} (\partial h_{LL} \partial h_{LL} + \partial h_{LL} \partial h_{LL}) + \sum_{S,T \in T, U,V \in N} c^{US VT} \partial h_{US} k_{VT}. \]

**Proposition 14** (Asymptotic Approximate Einstein’s equations). Let

\[ P_{\mu\nu} = \chi (\frac{r-t}{r+\epsilon}) P_N (\partial h, \partial h), \text{ or } P_{\mu\nu} = \chi (\frac{r-t}{r+\epsilon}) L^\mu L^\nu P_N (\partial h, \partial h). \]

where \( \chi \in C_0^\infty \) satisfies \( \chi (q) = 0, \) when \( |q| \geq 3/4 \) and \( \chi (q) = 1, \) when \( |q| \leq 1/2. \) Then

\[ |Z^{I J} (\square_0 h_{\mu\nu} - P_{\mu\nu})| \lesssim \frac{\epsilon^2}{(1+r+|q|) (1+r+|q|)} + \frac{\epsilon^2}{(1+r+|q|) (1+r+|q|)} \]

for \( |I| \leq N - 4. \) Here the asymptotic Schwarzschild wave operator is given by

\[ \square_0 = (m^{\alpha\beta} + h_0^{\alpha\beta}) \partial_\alpha \partial_\beta, \text{ where } h_0^{\alpha\beta} = -\frac{M}{r} \tilde{\chi} (\frac{r}{1+\epsilon}) \delta^{\alpha\beta}. \]  

(3.4)
The proof will be a consequence of the following lemmas and previous estimates.

**Lemma 3.** We have $P_N(\partial_q h, \partial_q k) = P(\partial_q h, \partial_q k)$ and

$$\left| Z^I \left[ P(\partial_\mu h, \partial_\nu h) - P_N(\partial_\mu h, \partial_\nu h) \right] \right| \lesssim \sum_{|J| + |K| \leq |I| + 1} \frac{|Z^J h|}{1 + t + |q|} |\partial Z^K h|,$$

$$\left| Z^I \left[ P(\partial_\mu h, \partial_\nu h) - L_\mu L_\nu P_N(\partial_q h, \partial_q h) \right] \right| \lesssim \sum_{|J| + |K| \leq |I| + 1} \frac{|Z^J h|}{1 + t + |q|} |\partial Z^K h|.$$ 

**Proof.** The first inequality follows since if $U \in \mathcal{N} = \{L, L, S_1, S_2\}$ then

$$\partial_\mu U^\nu = c_\mu^\nu(\omega, r/t)/(t + |q|), \quad t/8 < r < 8t, \quad (3.5)$$

for some smooth functions $c_\mu^\nu(\omega, r/t)$ homogeneous of degree 0. The second inequality follows from using (3.5) after expanding in the null frame

$$\partial_\mu = L_\mu \partial_q - L_\mu \partial_s + A_\mu \partial_A, \quad \partial_q = (\partial_r - \partial_t)/2, \quad \partial_s = (\partial_r + \partial_t)/2. \quad (3.6)$$

We can further decompose $g^{\alpha\beta} = m^{\alpha\beta} + h_0^{\alpha\beta} + h_1^{\alpha\beta}$, where $h_0^{\alpha\beta}$ is given by (3.4). With the asymptotic Schwarzschild wave operator given by (3.4) we write

$$\Box_0 h_{\mu\nu} = F_{\mu\nu} - F_{1\mu\nu}, \quad F_{1\mu\nu} = h_1^{\alpha\beta} \partial_\alpha \partial_\beta h_{\mu\nu}. \quad (3.7)$$

To estimate $F_1$ we use the following;

**Lemma 4.** We have

$$\left| Z^I (k^{\alpha\beta} \partial_\alpha \partial_\beta \phi) \right| \lesssim \sum_{|J| + |K| \leq |I| + 1} \frac{|Z^J k_{LL}| |\partial Z^K \phi|}{1 + |q|} + \frac{|Z^J k| |\partial Z^K \phi|}{1 + t + r}.$$

**Proof.** If we expand in null frame, $k^{\alpha\beta} \partial_\alpha \partial_\beta = k^{UV} U^\alpha V^\beta \partial_\alpha \partial_\beta$, and use that

$$k^\alpha \partial_\alpha = \sum_{|J| = 1} \frac{c_J^L(\omega, r/t)}{t - r} Z^J, \quad T^\alpha \partial_\alpha = \sum_{|J| = 1} \frac{c_J^T(\omega, r/t)}{t + r} Z^J, \quad T \in T,$$

for some smooth functions $c_J^U(\omega, r/t)$, when $t/8 < r < 8t$, we can write

$$k^{\alpha\beta} \partial_\alpha \partial_\beta = \sum_{|J| = 1} k_{LL} C_J^{\alpha\beta}(\omega, r/t) \frac{Z^J \partial_\gamma}{t - r} + \sum_{|J| = 1} k^{\alpha\beta} C_J^{\gamma\alpha\beta}(\omega, r/t) \frac{Z^J \partial_\gamma}{t + r},$$

for some smooth functions $C_J^{\alpha\beta}(\omega, r/t)$ and $C_J^{\gamma\alpha\beta}(\omega, r/t)$, where $k_{LL} = k_{LL}/4$. When $|t - r| < 1$ we replace the first sum with $k_{LL} L^\alpha \partial_\alpha \partial_\beta$ and similarly for the second when $t + r < 1$. This proves the lemma when $|I| = 0$. To prove the lemma in general we just have to note that $Z(t - r) = c_Z(\omega)(t - r)$. 

4. Asymptotic Schwarzschild Coordinates

Recall from the previous section, the asymptotic Schwarzschild wave operator:

\[ \square_0 = \square - \frac{\chi_0}{r} (\partial_r^2 + \Delta_\omega) = -(1 + \frac{\chi_0}{r}) \partial_r^2 + (1 - \frac{\chi_0}{r}) (\partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \Delta_\omega). \quad (4.1) \]

where \( \chi_0 = M \tilde{X}(\frac{r}{M}) \) and \( \Delta_\omega \) is the Laplacian on the sphere. The wave operator \( \square_0 \) is better expressed in asymptotic Schwarzschild coordinates. When \( r > t/2 \) these are the Regge-Wheeler coordinates that transforms the wave operator in the Schwarzschild metric in the radial case to the constant coefficient operator. In the interior \( r < t/4 \) these are the regular coordinates. Specifically,

\[ r^* = r + \chi_0 \ln r, \quad t^* = t, \quad \omega^* = \omega, \quad x^* = r^* \omega. \]

We show that \( \square_0 \) is close to the flat wave operator in Schwarzschild coordinates:

\[ \square^* = -\partial_t^2 + \Delta_\chi t = -\partial_t^2 + \partial_r^2 + 2r^{-1} \partial_r + r^{-2} \Delta_\omega. \quad (4.2) \]

Since \( \partial_r r/\partial r \sim 1 + \chi_0 r/r \) and \( r^*/r = 1 + \chi_0 \ln r/r \) it follows that

\[ \square^* \sim -\partial_t^2 + (1 + \chi_0/r)^{-2} \partial_r^2 + 2r^{-1} \partial_r + r^{-2} \Delta_\omega \sim \square_0. \]

**Proposition 15** (Asymptotic Schwarzschild coordinates). We have

\[ |Z^I(\square^* - \square_0)| \lesssim \frac{M \ln |1 + r|}{1 + t + r} \sum_{|J| \leq |I| + 1} \frac{|\partial Z^J \phi|}{1 + t + r}, \quad (4.3) \]

\[ |\partial^\alpha Z^I (\partial^*_\mu - \partial_\mu) \phi| \lesssim \frac{M \ln |1 + r|}{1 + t + r} \sum_{|J| \leq |I|, |\beta| \leq |\alpha|} |\partial_\beta \partial Z^J \phi|. \quad (4.4) \]

Moreover, with \( N = 1 + \frac{M \ln |1 + r|}{1 + |q|} \) and \( N_\ast = 1 + \frac{M \ln |1 + r|}{1 + |q^\ast|} \), and \( Z \) homogeneous,

\[ N_\ast^{-k} \sum_{|I| \leq k, |\alpha| \leq \ell} |\partial^\alpha Z^I \phi| \lesssim \sum_{|I| \leq k, |\alpha| \leq \ell} \sum_{|J| \leq |I|, |\beta| \leq |\alpha|} |\partial^\beta \partial Z^J \phi|, \quad (4.5) \]

\[ N_\ast^{-1} (1 + |q|) \lesssim 1 + |q^\ast| \lesssim N (1 + |q|). \quad (4.6) \]

Here \( M \ln |1 + r| \lesssim \varepsilon (1 + r)^\varepsilon \) if \( M \lesssim \varepsilon^2 \). Since the estimates are clearly true when \( t \) is bounded we can translate the operators in time to reduce to the case when \( t \geq 1 \) and \( \chi_0 = \chi_0(\frac{t}{M}) = M \tilde{X}(\frac{t}{M}) \) is homogeneous. In the proof \( \chi(\frac{y}{t}) \) respectively \( \chi'(\frac{y}{t}) \) with some indices, and possibly depending on other variables, will denote homogeneous smooth functions of \( y = x/t \) such that

\[ \chi(y) = 0, \quad |y| \leq 1/4, \quad \text{respectively} \quad \chi'(y) = 0, \quad |y| - 1/2 \geq 1/4. \quad (4.7) \]

Throughout the proof we will also use the estimate

\[ |\partial \phi| \lesssim \sum_{|I| \leq 1} |Z^I \phi|/(1 + |q|) \quad (4.8) \]

and its consequence using that \( \chi' \) is supported in a set where \( q \sim t \sim r \)

\[ |Z^I (r^{-1} \chi' \partial r^2 \phi)| \lesssim \sum_{|J| \leq |I| + 1} |\partial Z^J \phi|/(1 + t + r)^2. \quad (4.9) \]
We start by making reductions of the operators to estimate (4.3). Using
\[ \partial_t^2 - \partial_r^2 = (t + r)^{-1} ((\partial_t - \partial_r) S + \omega^j (\partial_t - \partial_r) \Omega_{ij}), \quad \Delta_\omega = \sum \Omega_{ij}^2, \] (4.10)
we see that modulo terms controlled by the right of (4.3) we can replace \[ \square_0 \] by
\[ \square_0 = -\partial_t^2 + c^{-2} \partial_r^2 + 2 r^{-1} \partial_r + r^{-2} \Delta_\omega, \quad \text{where} \quad c = 1 + \chi_0/r. \] (4.11)

Let us first collect some identities for the change of variables:

**Lemma 5.** Let \( \kappa = r^* - r = \chi_0 (\frac{r}{t}) \ln |r| \), where \( \chi_0 = M \bar{\chi} \). We have
\[ \frac{\partial}{\partial r} - \frac{\partial}{\partial r^*} = \frac{\partial \kappa}{\partial r} \frac{\partial}{\partial r^*}, \quad \frac{\partial}{\partial t} - \frac{\partial}{\partial t^*} = \frac{\partial \kappa}{\partial t^*}, \] (4.12)
where
\[ \frac{\partial \kappa}{\partial r} = \frac{\chi_0 (\frac{r}{t})}{r} + \chi_0' (\frac{r}{t}) \ln |r|, \quad \frac{\partial \kappa}{\partial t} = -\frac{\chi_0' (\frac{r}{t})}{t} \ln |r|. \]

The following quantities will be \( \sim 1 \)
\[ a = 1 + \kappa/r = 1 + \chi_0/r + \chi_0' \ln r/t, \quad b = 1 + \kappa/r = 1 + \chi_0 \ln r/r. \] (4.13)

For some smooth functions \( f_p(b) \)
\[ \frac{1}{r^p} - \frac{1}{r^{p+1}} = \frac{\chi_0 \ln r}{r^{p+1}} f_p(b), \quad \partial_{r^*} - \partial_r = \frac{1}{a} \frac{\chi_0 + \chi_0' \ln r}{r} \partial_r. \] (4.14)

It follows that to show that (4.3) holds for the difference of (4.2) and (4.11) it only remains to show that it holds for the difference of the principal radial parts:

**Lemma 6.** With \( a = 1 + \kappa_r = 1 + \chi_0/r + \chi_0' \ln r/t \) we have
\[ -\partial_{r^*}^2 + \partial_r^2 = p (\partial_t, \partial_r) + a^{-1} (p (\partial_t, \partial_r) \kappa) \partial_r, \] (4.15)
where with \( c = 1 + \chi_0/r \)
\[ p (\partial_t, \partial_r) = -\partial_t^2 + \frac{1}{a^2} \partial_r^2 + \frac{2 \kappa_t}{a} \partial_t \partial_r = -\partial_t^2 + \frac{1}{c^2} \partial_r^2 + \frac{\chi_0' \ln r}{t} q (\partial_t, \partial_r). \]

Here
\[ q (\partial_t, \partial_r) = \frac{-(c + a) + (c - a) r^2/t^2}{a^2 c^2} \partial_r^2 - \frac{2r/t}{a} \partial_r \partial_t. \] (4.16)

**Proof.** We have
\[ -\partial_{r^*}^2 + \partial_r^2 = - (\partial_t - \kappa_r a^{-1} \partial_r)^2 + (a^{-1} \partial_r)^2 \]
\[ = -\partial_t^2 + \frac{1}{a^2} \partial_r^2 + \frac{2 \kappa_t}{a} \partial_t \partial_r + \left( (\partial_t - \kappa_r a^{-1} \partial_r) \left( \frac{\kappa_r}{a} + \frac{1}{a} \partial_t \left( \frac{1}{a} \right) \right) \partial_r \right) \]
\[ = -\partial_t^2 + \frac{1}{a^2} \partial_r^2 + \frac{2 \kappa_t}{a} \partial_t \partial_r - \frac{1}{a} \left( -\partial_t^2 + \frac{1}{a^2} \partial_r^2 + \frac{2 \kappa_t}{a} \partial_r \partial_r \right) \partial_r. \]
It follows from the lemma that
\[
(-\partial_r^2 + \partial_r^2)\phi - (-\partial_r^2 + c^{-2}\partial_r^2)\phi = \frac{\chi_0'\ln r}{t} q(\partial_t, \partial_r)\phi + \frac{1}{a} (p(\partial_t, \partial_r)\kappa)\partial_r\phi.
\] (4.17)

The first term can be estimated using (4.9) since it can be written
\[
t^{-1}\chi_0' \ln r q(\partial_t, \partial_r)\phi = r^{-1} \ln r \chi^{\tau\alpha\beta}(\frac{r}{t}, a, b, c)\partial_\alpha \partial_\beta \phi,
\] (4.18)

for some smooth functions $\chi^{\tau\alpha\beta}$ satisfying the second support condition in (4.7). Since any of the vector fields $Z$ applied to functions of this form produces functions of the same form it follows from (4.9) that (4.19) can be estimated by the right hand side of (4.3). Similarly the second term can be written
\[
(p(\partial_t, \partial_r)\kappa)\partial_r\phi / a = (\chi^{\alpha\beta\mu}(\frac{r}{t}, a, b, c)\partial_\alpha \partial_\beta \kappa)\partial_\mu \phi = \chi^\mu (\frac{r}{t}, a, b, c)\partial_\mu \phi / t^2,
\] (4.19)

for some smooth $\chi^{\alpha\beta\mu}$ and $\chi^{\alpha\beta\mu}$ satisfying the first support condition in (4.7). This can be estimated by the right hand side of (4.3). We will use the following

**Lemma 7.** We have
\[
\frac{\partial}{\partial r} - \frac{\partial}{\partial r^*} = \frac{\partial \kappa}{\partial r} \frac{\partial}{\partial r^*} + \frac{\partial}{\partial x^i} \left( \frac{\partial \kappa}{\partial x^i} \right) r \frac{\partial}{\partial r^*} = \frac{\partial \kappa}{\partial r} \frac{\partial}{\partial r^*} - \left( \frac{\partial \kappa}{\partial r} - \frac{\kappa}{r} \right) \Pi_{ij}^r \frac{\partial}{\partial x^j}. \tag{4.20}
\]
where $\Pi_{ij} = \delta_{ij} - \omega_i \omega_j$ is the projection to the tangent space of $S^2$. Moreover
\[
\frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^{*i}} = \frac{\kappa}{r} \frac{\partial}{\partial x^i} \left( \frac{\partial \kappa}{\partial x^i} \right) r \frac{\partial}{\partial r^*} = \frac{\partial \kappa}{\partial r} \frac{\partial}{\partial r^*} - \left( \frac{\partial \kappa}{\partial r} - \frac{\kappa}{r} \right) \Pi_{ij}^r \frac{\partial}{\partial x^j}. \tag{4.22}
\]

where $\Pi_{ij}^r \frac{\partial}{\partial x^i} = r^{* - 1} \omega_j \Omega_{ji}^r$, and det $(\partial x^*/\partial x) = (1 + \kappa/r)^2 (1 + \kappa r)$. Furthermore
\[
\Omega_{ij} = \Omega_{ij}^*, \quad S = S^* + S(\frac{r}{t}) \partial_r, \quad \Omega_{0i} = \Omega_{0i}^* + \Omega_{0i}(\frac{r}{t}) \partial_r + \frac{\kappa}{r} (t \partial_{r^*} - x_i \partial_t). \tag{4.23}
\]

Moreover, with $a = \partial r^*/\partial r = 1 + \partial \kappa / \partial r$ and $b = r^*/r = 1 + \kappa / r$
\[
\frac{\partial}{\partial x^i} = \frac{1}{a} \frac{\partial}{\partial x^i} + \frac{1}{a b} \left( \frac{\partial \kappa}{\partial r} - \frac{\kappa}{r} \right) \omega_j \Omega_{ji}^r \frac{\partial}{\partial r^*}, \quad \frac{\partial}{\partial t^*} = \frac{\partial}{\partial t} - \frac{1}{a} \frac{\partial \kappa}{\partial t} \frac{\partial}{\partial t^*}, \quad \frac{\partial}{\partial r} = \frac{1}{a} \frac{\partial}{\partial r}. \tag{4.24}
\]

We start by proving (4.5). First we will show that
\[
|\partial^{\gamma} Z^J \phi| \lesssim \sum |J| \leq |I|, |\beta| \leq |\alpha| (1 + M \ln |1 + r|)^{|\gamma|} |\partial^\beta \partial^\gamma Z^J \phi|. \tag{4.25}
\]

The first inequality in (4.5) follows from this using $|\partial^\beta \phi| \lesssim \sum |I| \leq 1 |Z^J| \phi| (1 + |q^*|)$. We will use induction to prove that $Z^J$ is a sum of terms of the form $\chi_{J, \gamma} \partial^\gamma Z^J$, with $|\gamma| + |J| = |I|$, where $\chi_{J, \gamma} = \chi_{J, \gamma, 0} + \cdots + \chi_{J, \gamma, |J|} (\ln r)^{2 J}$ and $\chi_{J, \gamma}$ are homogeneous of degree 0. Here $\chi_{i, 0} = 1$ and $\chi_{J, j}$ are supported in $r \geq t/2$ for $|J| < |I|$. By the Lemma 7 we have for some $\chi^Z_{J, \gamma}$ homogeneous of degree 0
\[
Z (\chi_{J, \gamma} \partial^\gamma Z^J) = (Z \chi_{J, \gamma}) \partial^\gamma Z^J + \kappa_{J, \gamma} (Z^* + (\chi_{Z, 0} + \chi_{Z, 1}) \ln r) \partial^\gamma Z^J.
\]
Here $\mathcal{Z}_{\kappa_{1,\gamma}}$ is of the form $\chi_0 + \cdots + \chi_{\gamma}(\ln r)^{\gamma}$ and $\kappa_{1,\gamma}(\chi_{Z,0} + \chi_{Z,1} \ln r)$ is of the form $\chi_0 + \cdots + \chi_{\gamma}(\ln r)^{\gamma + 1}$. This proves the assertion and (4.25) for $|\alpha| = 0$. To prove it for $|\alpha| > 0$ we claim that $\partial_{\alpha}^\gamma$ applied to $\kappa_k \psi$, where $\kappa_k$ is of the form $\chi_0 + \cdots + \chi_k(\ln r)^k$ is a sum of terms of the form $\kappa_{k+\ell} t^{-\ell} \partial_{\gamma}^\beta \psi$ with $|\beta| + \ell = |\alpha|$. In fact by Lemma 7 we have for some $\chi_{\mu,0}^v$ homogeneous of degree 0

$$
\partial_{\mu}(\kappa_{k+\ell} t^{-\ell} \partial_{\gamma}^\beta) = (\partial_{\mu}(\kappa_{k+\ell} t^{-\ell})) \partial_{\gamma}^\beta + \kappa_{k+\ell} t^{-\ell}(\partial_{\mu}^\gamma + (\chi_{\mu,0}^v + \chi_{\mu,1}^v \ln r) t^{-1} \partial_{\gamma}^\beta).
$$

Here $\partial_{\mu}(\kappa_{k+\ell} t^{-\ell})$ and $\kappa_{k+\ell} t^{-\ell}(\chi_{\mu,0}^v + \chi_{\mu,1} \ln r) t^{-1}$ are both of the form $(\chi_0 + \cdots + \chi_{k+\ell+1}(\ln r)^{k+\ell+1}) t^{-\ell-1}$ which proves the assertion and (4.25) follows.

We now prove the second inequality in (4.5) which would follow from

$$
|\partial_{\alpha}^{\gamma} Z_{\mu}^I \phi| \lesssim \sum_{|I| + |\gamma| \leq |I|, |\beta| \leq |\alpha|} (1 + M \ln |1 + r|)^{|\gamma|} |\partial_{\beta}^\gamma Z^I \phi|.
$$

The proof uses the argument above with the inverse identities (4.24) that give

$$
Z^* = Z + (f_{Z,0}(a, b) \chi_{Z,0} + f_{Z,1}(a, b) \chi_{Z,1} \ln r) \partial_{\gamma},
$$

(4.27)

$$
\partial_{\mu}^* = \partial_{\mu} + (f_{\mu,0}(a, b) \chi_{\mu,0}^v + f_{\mu,1}(a, b) \chi_{\mu,1} \ln r) t^{-1} \partial_{\gamma},
$$

(4.28)

where $\chi_{Z, j}^v$ and $\chi_{\mu, j}^v$ are homogenous of degree 0, $f_{Z, j}(a, b)$ and $f_{\mu, j}(a, b)$ are smooth functions when $a = 1 + \kappa / \partial r > 0$ and $b = 1 + \kappa > 0$. The only difference is when derivatives fall on $a$ or $b$ that just produces lower order terms. (4.4) follows directly from applying vector fields and derivatives to (4.27).

5. Decay Estimates for the Inhomogeneous Wave Equation

Lemma 8. Let $\Box_r \phi = r^{-1}(\partial_r^2 - \partial_t^2)(r \phi)$. Then with $q = r - t$, we have for $r > t/2$:

$$
(1 + t + r) |\partial_{\phi}(t, r \omega)| \lesssim \max_{\xi = t+r, 3|q|} \sum_{|I| \leq 1} |Z^I \phi(\frac{\xi-q}{2}, \frac{\xi+q}{2})| = \int_{3|q|} |\Box_r \phi(\frac{\xi-q}{2}, \frac{\xi+q}{2})| d\xi.
$$

Proof. If we integrate $(\partial_r + \partial_t)(\partial_r - \partial_t)(r \phi) = r \Box_r \phi$ in the $t+r$ direction from the intersection with the line $r = t/2$ ($q < 0$) or the line $r = 2t$ ($q > 0$) we get

$$
|\partial_r - \partial_t|(r \phi)(t, r \omega) \leq |(\partial_r - \partial_t)(r \phi)(\frac{3|q|-q}{2}, \frac{3|q|+q}{2})| + \int_{3|q|} r \Box_r \phi(\frac{\xi-q}{2}, \frac{\xi+q}{2})| d\xi.
$$

Now

$$
(1 + t + r) |\partial_{\phi}(t, r \omega)| \lesssim |(\partial_r - \partial_t)(r \phi)(t, r \omega)| + \sum_{|I| \leq 1} |Z^I \phi(\frac{3|q|-q}{2}, \frac{3|q|+q}{2})|,
$$

and

$$
|\partial_r - \partial_t|(r \phi)(\frac{3|q|-q}{2}, \frac{3|q|+q}{2})| \lesssim \sum_{|I| \leq 1} |Z^I \phi(\frac{3|q|-q}{2}, \frac{3|q|+q}{2})|.
$$
Lemma 9. If \(-\Box \phi = F\), with vanishing data, where

\[ |F| \leq \frac{C}{(1 + r)(1 + t + r)(1 + |t - r|)^{1+\delta}}, \quad \delta > 0 \]  

(5.1)

then with \(\langle q \rangle = \sqrt{1 + q^2}\)

\[ |\phi| \leq \frac{C S^0(t, r)}{(1 + t + r)(1 + q_+)^\delta}, \quad \text{where } S^0(t, r) = \frac{t}{r} \ln \left( \frac{t + r}{t - r} \right). \]  

(5.2)

Here \(q_+ = r - t\), when \(r \geq 0\) and \(q_+ = 0\), when \(r \leq t\). On the other hand if

\[ |F| \leq \frac{C}{(1 + r)(1 + t + r)^{1+\mu}(1 + |t - r|)^{1-\mu}(1 + q_+)^\delta}(1 + q_-)^{\delta_-}, \]

with \(0 < \delta_+ < \mu, \quad 0 \leq \delta_- \leq \delta_+\), then

\[ |\phi| \leq \frac{C}{(1 + t + r)(1 + q_+)^{\delta_+}(1 + q_-)^{\delta_-}}. \]  

(5.3)

Proof. Let \(\overline{F}(t, r) = \sup_{\omega \in S^2} |F(t, t, \omega)|\) and let \(F_0 = FH\) where \(H = 1\), when \(t > 0\) and \(H = 0\), when \(t < 0\). Since \(|F_0| \leq \overline{F}_0\) it follows from the positivity of the fundamental solution that \(|\phi| \leq |\overline{\phi}|\) where \(\overline{\phi}\) is the solution of \(-\Box \overline{\phi} = \overline{F}_0\) with vanishing initial data. Since the wave operator is invariant under rotations it follows that \(\overline{\phi}\) is independent of the angular variables so \((\partial_t - \partial_r)(\partial_t + \partial_r)\overline{\phi}(t, r) = r \overline{F}_0\). If we now introduce new variables \(\xi = t + r\) and \(\eta = t - r\) and integrate over the region \(R = \{ (\xi, \eta); -\infty \leq \eta \leq t - r, t - r \leq \xi \leq t + r \}\) using that \(r \overline{\phi}(t, r)\) vanishes when \(\eta = -\infty\) and when \(r = 0\), i.e. \(\xi = \eta = t - r\) we obtain

\[ r \overline{\phi}(t, r) = 4 \int_{t-r}^{t+r} \int_{-\infty}^{-t} \rho \overline{F}_0(s, \rho) H(s) \, d\eta d\xi, \quad s = \frac{\xi + \eta}{2}, \quad \rho = \frac{\xi - \eta}{2}. \]

In the first case we have

\[ r \overline{\phi}(t, r) \leq 4 \int_{t-r}^{t+r} \int_{-\xi}^{-t} \frac{H(\xi + \eta)}{(1 + |\xi|)(1 + |\eta|)^{1+\delta}} \, d\eta d\xi. \]

If \(t > r\) (5.2) follows from integrating this, since \(\frac{1}{t} \log \left( \frac{1+t+r}{1+t-r} \right) \leq \frac{C}{1+t+r} S^0(t, r)\). If \(r > t\) then we integrate first in the \(\xi\) direction

\[ r \overline{\phi}(t, r) \leq \int_{-t}^{-(r-t)} \int_{-\xi}^{-t} d\xi d\eta \leq \int_{-t}^{-(r-t)} \log \left( \frac{1+t+r}{1+t-r} \right) \, d\eta \]

\[ \leq C \int_{1+t+r}^{1+\xi \delta} \int_{1+t+r}^{1+\xi \delta} \frac{1}{\xi^\delta} \, ds d\xi, \]

and (5.2) for \(r > t\) follows from this. To prove (5.3) we must estimate

\[ r \overline{\phi}(t, r) \leq 4 \int_{t-r}^{t+r} \int_{-\xi}^{-t} \frac{H(\xi + \eta)}{(1 + |\xi|)^{1+\mu}(1 + |\eta|)^{1-\mu}(1 + \eta_-)^{\delta_-}} \, d\eta d\xi. \]

If \(r > t\) then we integrate first in the \(\xi\) direction
\[ r \bar{\phi}(t, r) \lesssim \int_{-t+r}^{t+r} \frac{d\xi d\eta}{(1 + |\xi|)(1 + |\eta|)^{1+\delta_{+}}} \lesssim \int_{-t+r}^{t-r} \frac{d\eta}{(1 + |\eta|)^{1+\delta_{+}}} \]

which is \( \lesssim (1 + |t - r|)^{-\delta_{+}} \) so \((5.3)\) for \( r > t \) follows. If \( t > r \) it follows from integrating in the \( \eta \) direction that if \( 1 + \delta_{+} - \mu < 1 \) we have with \( \delta = \min(\delta_{-}, \delta_{+}) \)

\[ r \bar{\phi}(t, r) \leq 4 \int_{t-r}^{t+r} \frac{1}{(1 + |\xi|)^{1+\delta_{+}}} + \frac{(1 + |t - r|)^{\mu - \delta_{-}}}{(1 + |\xi|)^{1+\mu}} d\xi \leq C r \frac{1}{(1 + t + r)(1 + |t - r|)^{\delta}}. \]

**Lemma 10.** If \( w \) is the solution of

\[ -\Box w = 0, \quad w|_{t=0} = w_0, \quad \partial_t w|_{t=0} = w_1 \]

then for any \( 0 < \gamma < 1 \);

\[ (1 + t + r)(1 + |r - t|)^{\gamma} |w(t, x)| \lesssim \sup_x \left( (1 + |x|)^{2+\gamma} (|w_1(x)| + |\partial_0 w_0(x)|) + (1 + |x|)^{1+\gamma} |w_0(x)| \right). \] \( (5.4) \)

**Proof.** The proof is an immediate consequence of Kirchoff’s formula

\[ w(t, x) = t \int \left( w_1(x + t \omega) + \langle w_0(x + t \omega), \omega \rangle \right) \frac{dS(\omega)}{4\pi} + \int w_0(x + t \omega) \frac{dS(\omega)}{4\pi}, \]

where \( dS(\omega) \) is the measure on \( S^2 \). If \( x = re_1 \), where \( e_1 = (1, 0, 0) \) then for \( k = 1, 2 \)

\[ \int \frac{dS(\omega)/4\pi}{1 + |r e_1 + t \omega|^{k+\gamma}} = \int_{-1}^{1} \frac{d\omega_1/2}{1 + ((r - t \omega_1)^2 + t^2(1 - \omega_1^2))^{(k+\gamma)/2}} \]

\[ = \int_{0}^{2} \frac{ds/2}{1 + ((r - t)^2 + 2r ts)^{(k+\gamma)/2}}. \]

(5.4) follows directly if \( |r - t| \geq t/2 \). If \( t/2 < r < 2t \) we change variables \( \tau = rts \). If \( k = 2 \) it can be bounded by \( (rt)^{-1} (1 + |r - t|)^{-\gamma} \) and if \( k = 1 \) by \( (rt)^{-1} (1 + rt)^{(1-\gamma)/2} \).

6. Sharp Decay in Asymptotic Schwarzschild Coordinates

Using Proposition 15 to change to asymptotic coordinates Proposition 14 become

**Proposition 16 (Asymptotic Einstein in Schwarzschild coordinates).** Let

\[ P^*_{\mu\nu} = \chi \left( \frac{(r - t)}{4t^2} \right) P_{\mu\nu}(\partial_{\mu}^{*} h, \partial_{\nu}^{*} h), \]

or \( P^*_{\mu\nu} = \chi \left( \frac{(r - t)}{4t^2} \right) L_{\mu\nu} P(\partial_{\mu}^{*} h, \partial_{\nu}^{*} h), \)

where \( \chi \in C_0^\infty \) satisfies \( \chi(q) = 0, \) when \( |q| \geq 3/4, \) \( \chi(q) = 1, \) when \( |q| \leq 1/2. \) We have

\[ |Z^* I[h_{\mu\nu} - P^*_{\mu\nu}]| \lesssim \frac{\varepsilon^2 (1 + |q^*|)^{-1}}{(1 + t + r^*)^{3 - C\varepsilon}} + \frac{\varepsilon^2 (1 + |q^*|)^{-2}}{(1 + t + r^*)^{2+\gamma - C\varepsilon}}, \]

\[ |I| \leq N - 5. \] \( (6.1) \)

and Propositions 12 and 13 become
Lemma 11 (Weak decay). For $|I| \leq N - 4$ we have

$$ (1 + |q^*|) |\partial^* Z^I h^{1*}| + |Z^I h^{1*}| \lesssim \varepsilon (1 + q^*)^{-\gamma} (1 + t + |q^*|)^{-1+C\varepsilon}, \quad (6.2) $$

where we have written $h = h^{0*} + h^{1*}$, where $h^{0*}_{\alpha\beta} = \tilde{X}(\frac{r}{1+t})^M \delta_{\alpha\beta}$. When $r^* \geq t/8$

$$ |\partial^* Z^I h^{1*}_{LT}| \lesssim \frac{\varepsilon (1 + q^*)^{-\gamma}}{(1 + t + |q^*|)^{2-C\varepsilon}}, \quad |Z^I h^{1*}_{LT}| \lesssim \frac{\varepsilon (1 + |q^*|)^\gamma}{(1 + t + |q^*|)^{1+\gamma-C\varepsilon}}. \quad (6.3) $$

Moreover (6.3) also hold for $h^{1*}_{LT}$ replaced by $\delta^{AB} h^{1*}_{AB}$.

In this section we will prove the following decay estimates:

Proposition 17 (Sharp decay). For $|I| \leq N - 6$ with $\gamma' = \gamma - C\varepsilon$ we have

$$ |Z^I h^{1*}| \lesssim \frac{\varepsilon^2 S^0(t, r^*)}{(1 + t + r^*) (1 + q^*)^{1-C\varepsilon}} + \frac{\varepsilon}{1 + t + r^*} \frac{1}{(1 + |q^*|)^{\gamma'}}. $$

where

$$ S^0(t, r^*) = \frac{t}{r^*} \ln \left( \frac{(t + r^*)}{(t - r^*)} \right) \approx \frac{1}{\varepsilon} \left( \frac{(t + 0)}{(t - 0)} \right)^{\varepsilon}, \quad \langle q \rangle = \sqrt{1 + q^2}. $$

For $r^* \geq t/2$ we have

$$ |Z^I h^{1*}_{TU}| \lesssim \frac{\varepsilon}{(1 + t + r^*) (1 + q^*)^{\gamma'}}. \quad (6.4) $$

$$ |\partial Z^I h^{1*}_{LT}| + |\partial Z^I \delta^{AB} h^{1*}_{AB}| \lesssim \frac{\varepsilon}{(1 + t + r^*)^{2-\varepsilon} (1 + |q^*|)^\varepsilon (1 + q^*)^{\gamma'}}. \quad (6.5) $$

$$ |Z^I h^{1*}_{LT}| + |Z^I \delta^{AB} h^{1*}_{AB}| \lesssim \frac{\varepsilon}{(1 + t + r^*)^{1+\gamma} + \varepsilon (1 + q^*)^{1-\varepsilon}}. \quad (6.6) $$

Moreover (6.5)–(6.6) hold for $Z^I (h^1_{UV})$ replaced by the Lie derivatives $(\mathcal{L}^I_Z, h^1_{UV})$.

Remark 3. Using (2.6), (2.9) and (6.16) one can show (6.5) for $h^{1*}_{LT}$ with $\varepsilon = 0$ in the exponents.

We will successively improve the estimates starting with those in Lemma 11. We now want to contract with a null frame, which does not quite commute with the wave equation. The null frame only depends on the angular variables so it commutes with the radial part of the wave operator but not the angular:

$$ \square^* = \square^*_r + r^*-2 \Delta_\omega, \quad \text{where} \quad \square^*_r \phi = r^*-1 \{ \partial_{r^*} + \partial_r \} (\partial_{r^*} - \partial_r) (r^* \phi). $$

Since $\Delta_\omega = \sum \Omega_{ij}^2$ and $|Z^* U| \leq C$, for $U \in \{ A, B, L, L \}$, it follows that

$$ |Z^I (\square^* h^1_{UV} - U^\mu V^\nu \square^* h_{\mu\nu})| \leq r^*-2 \sum_{|J| \leq |I| + 1} |Z^J h|. $$

Since $[Z^*, \square^*]$ is either 0 or 2 $\square^*$ we have $|\square^* Z^I h^1_{TU}| \lesssim \sum_{|J| \leq |I|} |Z^J \square^* h^1_{TU}|$ so

$$ |\square^* Z^I h^1_{TU}| \lesssim r^*-2 \sum_{|J| \leq |I| + 1} |Z^J h| + \sum_{|J| \leq |I|} |Z^J (T^\mu U^\nu \square^* h_{\mu\nu})|, \quad (6.7) $$

$$ \square^* = \square^*_r + r^*-2 \Delta_\omega, \quad \text{where} \quad \square^*_r \phi = r^*-1 \{ \partial_{r^*} + \partial_r \} (\partial_{r^*} - \partial_r) (r^* \phi). $$

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$$ |Z^I (\square^* h^1_{UV} - U^\mu V^\nu \square^* h_{\mu\nu})| \leq r^*-2 \sum_{|J| \leq |I| + 1} |Z^J h|. $$

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$$ |\square^* Z^I h^1_{TU}| \lesssim r^*-2 \sum_{|J| \leq |I| + 1} |Z^J h| + \sum_{|J| \leq |I|} |Z^J (T^\mu U^\nu \square^* h_{\mu\nu})|, \quad (6.7) $$
where since $T^\mu L_\mu = 0$ for $T \in \{L, A, B\}$

$$|Z^sJ(T^\mu U^v \square^s h_{\mu v})| \leq |Z^sJ(T^\mu U^v[\square^s h_{\mu v} - L_\mu L_v P_N (\partial_q^s h, \partial_q^s h)/4])|.$$  \hspace{1cm} (6.8)

Moreover

$$|\square^s Z^s I h_{TU}| \lesssim r^{*-2} \sum_{|J| \leq |I|+2} |Z^sJ h| + |\square^s Z^s I h_{TU}|.$$  \hspace{1cm} (6.9)

It follows from using the estimates (6.1) in (6.8) and (6.2) in (6.7) and (6.9):

**Lemma 12.** For $|I| \leq N - 6$ we have for $r^* \geq t/2$

$$|\square^s Z^s I h_{TU}| + |\square^s Z^s I h_{TU}| \lesssim \frac{\varepsilon (1 + q_+)^{-C \varepsilon}}{r^{*-2}(1 + t + |q^*|)^{1-C\varepsilon}} + \frac{\varepsilon^2 (1 + |q^*|)^{-2+\gamma}}{(1 + t + |q^*|)^{2+\gamma-C\varepsilon}}.$$  

The second estimate follows since $\square^s h_{\mu \nu} = \square^* h^1_{\mu \nu}$, when $r^* \geq t/2$. Using these estimates we obtain the following two lemmas:

**Lemma 13.** For $|I| \leq N - 6$ and $r^* \geq t/8$ we have

$$(1 + t + r^*)|\partial^s Z^s I h^1_{TU}(t, r^*, \omega)| \lesssim \varepsilon (1 + q_+)^{-\gamma} (1 + |q^*|)^{-1+C\varepsilon},$$  \hspace{1cm} (6.10)

$$(1 + t + r^*)|Z^s I h^1_{TU}(t, r^*, \omega)| \lesssim \varepsilon (1 + q_+)^{C\varepsilon-\gamma} + ((1 + q^*)^{C\varepsilon} - 1).$$  \hspace{1cm} (6.11)

The last estimate for $q^* < 0$ can be replaced by $C_\mu \varepsilon (1 + |q^*|)^{\mu}$, for any $\mu > 0$. The same estimates hold for $h$ in place of $h^1$, if $\gamma$ is replaced by $C\varepsilon$.

**Proof.** If we apply Lemma 8 to $Z^s I h^1_{TU}$ using Lemmas 11, 12 we get for $r^* \geq t/2$

$$(1 + t + r^*)|\partial^s Z^s I h^1_{TU}(t, r^*, \omega)|$$

$$\lesssim \frac{\varepsilon (1 + q_+)^{-\gamma}}{1 + |q^*|)}^{1-C\varepsilon} + \int_{3}^{t+r^*} \left( \frac{\varepsilon (1 + q_+)^{-\gamma}}{1 + \xi} \frac{[1 + q^*]^{1-C\varepsilon} + \varepsilon^2 (1 + |q^*|)^{-2+\gamma}}{(1 + \xi)^{1-C\varepsilon+\gamma}} \right) d\xi,$$

which proves (6.10) for $r^* \geq t/2$ but for $t/8 \leq r^* \leq t/2$ it follows from Lemma 11. (6.11) follows by integrating (6.10) in the $t - r^*$ direction from the initial surface.

Let

$$h^1 = h - h^{0e}, \quad h^{0e}_{\mu \nu} = \delta_{\mu \nu} M \chi^e(r^* - t)/r^*,$$  \hspace{1cm} (6.12)

where $\chi^e(s) = 1$, when $s \geq 2$ and $\chi^e(s) = 0$, when $s \leq 1$. Then $\square^* h^{0e} = 0$.

**Lemma 14.** For $|I| \leq N - 5$ have

$$|Z^s I h^1 = | \lesssim \frac{\varepsilon^2 S^0(t, r^*)}{(1 + t + r^*)(1 + q_+)^{1-C\varepsilon}} + \frac{\varepsilon}{1 + t + r^* (1 + |q^*|)^{\gamma-C\varepsilon}}.$$  \hspace{1cm} (6.13)
Proof. For $|I| \leq N - 5$ have

$$\left| \square^* Z^I h_{\mu \nu} \right| \leq \sum_{|J| \leq |I|} \left| Z^J \chi \left( \frac{(r-t)}{t+r} \right) P_N (\partial_{\mu}^* h, \partial_{\nu}^* h) \right| + \sum_{|J| \leq |I|} \left| Z^J \chi \left( \frac{(r-t)}{t+r} \right) \left[ \square^* h_{\mu \nu} - P_N (\partial_{\mu}^* h, \partial_{\nu}^* h) \right] \right|,$$

where the second term is estimated by Proposition 16. Using (3.3), Lemmas 11 and 13 the first is estimated by

$$\left| \sum_{|J|+|K| \leq |I|} \sum_{S,T \in T, U,V \in N} \left| \partial^* Z^J h_{TU} \right| \left| \partial^* Z^K h_{SV} \right| \right| \leq \sum_{|J|+|K| \leq |I|} \sum_{S,T \in T, U,V \in N} \frac{\varepsilon^2}{(1+t+r^*)^2 (1+|q^*|)^{2-C \varepsilon}} ,$$

when $r^* \geq t/8$. Hence by Proposition 16

$$\left| \square^* Z^I h_{\mu \nu} \right| \lesssim \varepsilon^2 (1+t+r^*)^{-2} (1+|q^*|)^{-2+C \varepsilon}.$$

The same estimate holds for $h_{\mu \nu}^I = h - h_{E}^0$, since $\square^* h_{\mu \nu}^0 = 0$. We now write $h_{\mu \nu}^I = \phi + w$, where $\square w = 0$ with data $(w, \partial w)|_{t=0} = (h_{\mu \nu}^I, \partial_t h_{\mu \nu}^I)|_{t=0}$, and apply Lemma 9 to $\phi$ and Lemma 10 to $w$, using (6.2) to estimate the initial conditions.

Since $\square^* h_{\mu \nu} = \square^* h_{\mu \nu}^I$, it follows from (6.7)–(6.9), (6.13) and Proposition 16

**Lemma 15.** For $|I| \leq N - 6$ and $r^* \geq t/8$ we have

$$\left| \square^* Z^I h_{TU}^{1e} \right| + \left| \square^* Z^I h_{TU}^e \right| \leq \frac{\varepsilon^2 S_0(t,r^*) (1+q^*_I)^{-1+C \varepsilon}}{r^* (1+t+r^*)} + \frac{\varepsilon (1+|q^*|)^{-\gamma+C \varepsilon}}{r^* (1+t+r^*)} + \frac{\varepsilon^2 (1+|q^*|)^{-2+C \varepsilon}}{(1+t+r^*)^2 (1+|q^*|)^{2-C \varepsilon}}. \quad (6.14)$$

Using this improved estimate we get an improvement of Lemma 13:

**Lemma 16.** For $|I| \leq N - 6$ we have for $r^* \geq t/2$

$$\left| Z^I h_{TU}^{1e} \right| \lesssim \varepsilon (1+t+r^*)^{-1} (1+q^*_I)^{-\gamma+C \varepsilon}.$$

**Proof.** By Lemma 14 the commutator satisfies

$$\left| [\square^*, \chi \left( \frac{(r-t)}{t+r} \right)] Z^I h_{TU}^{1e} \right| \leq \frac{\varepsilon \left( 1+q^*_I \right)^{-\gamma+C \varepsilon}}{(1+t+r^*)^3}.$$
Using the improved estimates in the wave coordinate condition (2.8) we get:

**Lemma 17.** For \( |I| \leq N - 6 \) we have

\[
|\partial Z^* h_{LT}^1| + |\partial Z^* (h_{AA}^1 + h_{BB}^1)| \leq \varepsilon (1 + t + r^*)^{-2+\varepsilon} (1 + |q^*|)^{-\varepsilon} (1 + q^*_*)^{-\gamma + C\varepsilon}.
\]

The estimate holds also for \( Z^* (h_{UV}^1) \) replaced by the Lie derivatives \( (L^*_L h)_{UV} \).

**Proof.** For \( q^* \geq 0 \) this is a direct consequence of (6.3). To get the sharp estimate for \( q^* \leq 0 \) we need to reexpress the divergence in the \( x^* \) coordinates and repeat the proof of Proposition 13 in these coordinates. By the invariance of the divergence under change of coordinates we have \( \partial_\mu F^\mu = |D| \partial_\mu (F^\mu D^\mu_{\mu)/|D|} \), if \( D^\mu_{\mu} = \partial x^\mu / \partial x^\mu \) and \( |D| = \text{det} D \).

By Lemma 7 \( |D| = (1 + M \ln r/r)^2 (1 + M/r) \). By (2.2)

\[
\partial_\mu (\hat{h}^\mu^\nu D^\nu_{\mu}) = -\frac{1}{2} \left( g^{\mu\nu} g_{\alpha\beta} - m^{\mu\nu} m_{\alpha\beta} \right) \partial_\mu g^{\alpha\beta} + \hat{h}^\mu^\nu |D|^{-1} \partial_\mu |D|
\]

and expressing the divergence in a null frame as in the proof of Lemma 1

\[
\partial^\mu (L^*_\gamma U^\nu k^\nu_{\nu}) = \partial^\mu (L^*_\gamma U^\nu k^\nu_{\nu} - A^\mu_{\nu} U^\nu k^\nu_{\nu} + U^\nu \partial^\mu k^\nu_{\nu}, \ U^\nu = m_{\nu\mu} U^\mu, \ U^\nu \in N^\nu.
\]

where \( a^\mu = A^k(\omega) a^k \) and \( k^\nu = \hat{h}^{\mu^\nu} D^\nu_{\mu} \). Here \( L^*_\gamma D^\nu_{\nu} = L^*_\gamma, L^*_\gamma D^\nu_{\nu} = L^*_\gamma \) and \( A^\mu_{\nu} \hat{D}^\nu_{\mu} = A^\mu_{\nu} r^*/r \), by Lemma 7. The rest of the proof is as in Proposition 13 but with the \( x^* \) coordinates replaced by the \( x^* \) coordinates everywhere. Note that the difference \( L^*_\gamma - L = O(r^{-1}) \) is lower order so \( L^*_\gamma U k^\nu_{\nu} = L^*_\gamma U^\nu h^{\mu\nu} \sim L^*_\gamma U^\nu h^{\mu\nu} \).

This concludes the proof of Proposition 17. For the \( L \) derivative we also have

**Proposition 18.** With \( \delta^L U^*_U V_L = 1 \) if \( U = V = L^*_\gamma \) and 0 otherwise we have

\[
\frac{1}{r} |\partial L^*_r (r^* Z^* h_{UV}^1)| \lesssim \frac{\varepsilon (1 + q^*_*)^{-\gamma - C\varepsilon}}{(1 + t + r^*)^{2+\gamma - C\varepsilon}} + \delta^L L^*_r L^*_L \frac{\varepsilon (1 + q^*_*)^{-\gamma}}{(1 + t + r^*)^2}.
\]

(6.16)

\[
|\partial L^*_r Z^* h_{LT}^1| \lesssim \frac{\varepsilon (1 + q^*_*)^{-\gamma}}{(1 + t + r^*)^2} \left( \frac{1 + q^*_*}{1 + t + r^*} \right)^{\gamma}.
\]

(6.17)

**Proof.** Integrating \( L^*_\gamma L^*(r^* \phi) = r^* \Box^* \phi \) along the flow lines of \( L^*_\gamma \) from initial data

\[
\partial L^*_r (r^* Z^* h_{UV}^1) = \frac{1}{2} \int_{q^*}^{t+q^*} r^* \Box^* Z^* h_{UV}^1 dq^* + \partial L^*_r (r^* Z^* h_{UV}^1) \bigg|_{r=t+q^*},
\]

where \( (|Z^* h_{UV}^1| + (1 + r)|\partial Z^* h_{UV}^1|) \big|_{r=0} \lesssim \varepsilon (1 + r)^{-1-\gamma} \) and

\[
|\Box^* Z^* h_{UV}^1| \lesssim \frac{\varepsilon (1 + q^*_*)^{-\gamma}}{(1 + t + |q^*|)^3 - C\varepsilon} + \frac{\varepsilon^2 (1 + |q^*|)^{-2\gamma}}{(1 + t + |q^*|)^{2\gamma} - C\varepsilon} + \delta^L L^*_r L^*_L \frac{\varepsilon^2 (1 + |q^*|)^{-2}}{(1 + t + |q^*|)^2}.
\]

Integrating gives (6.16), and (6.17) follows from also using Proposition 17.
7. Asymptotics for the Wave Equation with Inhomogeneous Sources

It is well known [H], that a solution of a linear wave equation $\Box u = 0$, with sufficiently fast decaying smooth initial data have an asymptotic expansion

$$u(t, x) \sim U_0(r - t, \omega)/r + U_1(r - t, \omega)/r^2 + \ldots,$$

where $U_0$ is the Friedlander radiation field. In fact $U(r - t, \omega, 1/r) = ru(t, x)$ is an analytic function of $1/r$ and $U_0$ can be calculated from data. Data for Einstein’s equations are however not fast decaying and the equations are non-linear. Still if the right hand side and tangential derivatives decay fast the limit exists:

**Lemma 18.** Suppose that for some $0 \leq \delta < 1 - \gamma$ and $0 \leq \gamma' \leq \gamma$

$$\left| \Box^I S^J \partial^K u \right| \lesssim \frac{\epsilon}{(1 + t + r)^{3-\delta}(1 + |r - t|)^{\delta}(1 + (r - t)_)^{\gamma}(1 + (r - t)_)^{\gamma'}},$$

$$\left| \triangle_{\omega} \Box^I S^J \partial^K u \right| \lesssim \frac{\epsilon}{(1 + t + r)^{1-\delta}(1 + |r - t|)^{\delta}(1 + (r - t)_)^{\gamma}(1 + (r - t)_)^{\gamma'}},$$

for $|I| + |J| + |K| \leq N$ and $r > t/4$, and

$$\left| (\partial_t + \partial_r)(\Box^I S^J \partial^K ru) \right| \lesssim \epsilon(1 + r)^{-1-\gamma}, \quad \text{when } t = 0.$$

Then

$$\left| (\partial_t + \partial_r)(\Box^I S^J \partial^K ru)(t, r\omega) \right| \lesssim \epsilon(1 + (r - t) - \gamma' - (1 + r + t)^{-1-\gamma}), \quad r \geq t/4.$$

Moreover, limit

$$U^\infty(q, \omega) = \lim_{r \to \infty} U(q, \omega, r), \quad U(q, \omega, r) = ru(r - q, r\omega),$$

exists and satisfies, for $r > t/4$ and $|I| + |J| + |K| \leq N$,

$$\left| \Box^I S^J \partial^K ru(t, r\omega) - \Box^I (q \partial_q)J(-\partial_q)^K U^\infty(q, \omega) \right| \lesssim \epsilon \frac{(1 + q_-)^{\gamma - \gamma'}}{(1 + r)^{\gamma}}, \quad (7.1)$$

$$\left| \Box^I (q \partial_q)^J \partial^K U^\infty(q, \omega) \right| \lesssim \epsilon \frac{(1 + q_-)^{\gamma - \gamma'}}{(1 + |q|)^{\gamma'}}, \quad (7.2)$$

**Proof.** We prove the result for $N = 0$ as the case $N > 0$ follows from the same argument. This follows from expressing the wave operator in spherical coordinates

$$(\partial_t - \partial_r)(\partial_t + \partial_r)(ru) = r^{-1} \triangle_{\omega} u - r \Box u$$

and integrating first in the $r - t$ direction from $(t, r)$ to initial data when $t = 0$

$$\left| (\partial_t + \partial_r)(ru)(t, r\omega) \right| \lesssim \int_{r - t}^{t + r} \left| r^{-1} \triangle_{\omega} u - r \Box u \right| dq + \left| (\partial_t + \partial_r)(ru)(0, (t + r)\omega) \right|$$

$$\lesssim \int_{r - t}^{t + r} \frac{\epsilon dq}{(1 + t + r)^{2-\delta}(1 + |q|)^{\delta}(1 + q_-)^\gamma(1 + q_-)^\gamma'} + \frac{\epsilon}{(t + r)^{1+\gamma}}$$

$$\lesssim \frac{\epsilon}{(t + r)^{1+\gamma}} + \frac{\epsilon(r - t)^{-1-\gamma}}{(t + r)^{2-\delta}} \lesssim \epsilon \frac{(1 + (r - t) - \gamma' - (1 + r + t)^{-1-\gamma})}{(1 + t + r)^{1+\gamma}} \quad r > t/2.$$

For fixed $q = r - t$ integrating this in $t + r$ between $2r_1 - q \leq r + t \leq 2r_2 - q$ gives

$$\left| U(q, \omega, r_2) - U(q, \omega, r_1) \right| \lesssim \epsilon(1 + q_-)^{\gamma' - \gamma'}(1 + r_1)^{-\gamma}, \quad r > t/2,$$

from which it follows that the limit exists and satisfies (7.1)–(7.2).
For Einstein’s equations we have the extra difficulty that it is a system and the components do not separate due to angular derivatives on the frame:

\[
\square(T^\mu U^\nu h_{\mu\nu}) - T^\mu U^\nu \square h_{\mu\nu} = r^{-2} \Delta_\omega (T^\mu U^\nu h_{\mu\nu}) - T^\mu U^\nu r^{-2} \Delta_\omega h_{\mu\nu}.
\]

It can be estimated in terms of tangential derivatives of all components. This procedure will give us the existence of the radiation field for all components in a null frame except for \( h_{LL} \). This component will in fact not have as simple radiation field but there will also be a logarithm in its radiation field. However, the asymptotics of the source \( P(\partial_\mu h, \partial_\nu h) \) can be calculated in terms of the radiation field of the components we already calculated. It will be of the form

\[
P_N(\partial_\mu h, \partial_\nu h) \sim C^{TSV} L_\mu(\omega)L_\nu(\omega)r^{-2} \partial_q H_{TL}^\infty(r-t, \omega)\partial_q H_{SV}^\infty(r-t, \omega), \quad r > \frac{t}{2}.
\]

We now want to find an approximate solution to \( \square \phi = P(\partial_\mu h, \partial_\nu h) \). Formulas for the solution of the wave equation with such sources were obtained in [L1]. First we use a simplified version which is sufficient for asymptotics in null directions.

**Proposition 19.** Let \( \chi \in C_0^\infty \) satisfy \( \chi(q) = 0 \), when \( |q| \geq 3/4 \) and \( \chi(q) = 1 \), when \( |q| \leq 1/2 \). Set

\[
F[n](t, x) = n(r-t, \omega)r^{-2}\chi(\frac{r-t}{t+r}), \quad \langle q \rangle = \sqrt{1 + q^2}
\]

where \( n \) is a smooth function satisfying

\[
\sum_{|\alpha| + k \leq N} |(\langle q \rangle \partial_q)^k \partial^\alpha \Omega(q, \omega)| \lesssim \langle q \rangle^{-1-a}, \quad 0 < a < 1.
\]

Let \( \Phi[n] \) be the solution of \( -\square \Phi[n] = F[n] \) with vanishing initial data and let

\[
\Phi_1[n](t, r, \omega) = \int_{r-1}^{\infty} \frac{1}{2r} \ln \left( \frac{t+r+q}{t-r+q} \right)n(q, \omega) dq \chi(\frac{r-t}{t+r}).
\]

Set \( \Phi_0[n] = \Phi[n] - \Phi_1[n] \), \( u = \Phi[n] \), \( u_1 = \Phi_1[n] \) and \( u_0 = \Phi_0[n] = u - u_1 \). Then

\[
|\partial_t + \partial_r| (\Omega^I S^J \partial^K r u_0) \lesssim \frac{(1 + (t-r)_+)^a}{(1 + t + r)^{1+a}}, \quad |I| + |J| + |K| \leq N.
\]

Moreover, limit

\[
U_0^\infty(q, \omega) = \lim_{r \to \infty} U_0(q, \omega, r), \quad U_0(q, \omega, r) = ru_0(r-q, r\omega),
\]

exists and satisfies

\[
|\Omega^I S^J \partial^K r u_0(t, r, \omega) - \Omega^I (q \partial_q)^J (-\partial_q)^K U_0^\infty(q, \omega)| \lesssim \varepsilon(1 + q_-)^a,
\]

\[
|\Omega^I (q \partial_q)^J \partial^K q U_0^\infty(q, \omega)| \lesssim \varepsilon(1 + q_-)^a,
\]

for \( r > t/4 \) and \( |I| + |J| + |K| \leq N \). Furthermore

\[
|\Omega^I S^J \partial^K u - \Phi_1[\Omega^I (q \partial_q)^J (-\partial_q)^K n]| \lesssim \frac{1}{(1 + t + r)(1 + (r-t)_+)^a},
\]
and
\[ |(\partial_t + \partial_r)(r t^I S^J \partial^K_r u) - \Phi_{1+}[\Omega^I (q \partial_q) J (-\partial_q) K n]| \lesssim \frac{(1 + (t - r) +)^a}{(1 + t + r)^{1+\sigma}}, \]
where
\[ \Phi_{1+}[n](t, r \omega) = \frac{1}{2r} \int_{r-t}^{\infty} n(q, \omega) dq \chi(r-t). \]

For the proof we need the following technical lemma:

**Lemma 19.** Suppose that \( m \) is a smooth function satisfying. We have
\[ \sum_{|\alpha| + k \leq N} |\langle q \rangle^{\alpha} \partial^\alpha q m(q, \omega)| \lesssim \langle q \rangle^{-1-b}. \]
Let \( \delta_{b1} = 1 \), when \( b = 1 \) and 0 otherwise. Then if \( 0 < b \leq 2 \)
\[ |\int_{r-t}^{\infty} \ln \left| \frac{t+r+q}{t-r+q} \right| - \ln \left| \frac{t+r}{t-r} \right| m(q, \omega) dq | \lesssim \frac{1}{\langle t-r \rangle^b} + \frac{H(t-r)}{\langle t-r \rangle} (1 + \delta_{b1} \ln \langle t-r \rangle). \]

**Proof (of Lemma 19).** The integral over \( q \geq t + r \) is easily bounded by
\[ (1 + \ln \left( \frac{t+r}{t-r} \right)) \frac{1}{\langle t+r \rangle^b} \lesssim \frac{1}{\langle t-r \rangle^b}, \]
so we may integrate over \( q \leq t + r \) only. We have
\[ \ln \left( \frac{t+r+q}{t-r+q} \right) - \ln \left( \frac{t+r}{t-r} \right) = \ln \left( \frac{t+r}{t-r} + \frac{q}{t-r+q} \right). \]
The integral of the first term is also easy to bound. If \( r > t \) it can be bounded by
\[ \int_{r-t}^{t+r} \left( \frac{q}{t+r} \right) \frac{dq}{\langle q \rangle^{1+b}} \lesssim \frac{1}{t+r} \left( (t+r)^{1-b} + (t-r)^{1-b} + \delta_{b1} \ln \left( \frac{t+r}{t-r} \right) \right) \lesssim \frac{1}{\langle t-r \rangle^b}, \]
where \( \delta_{b1} = 1 \) if \( b = 1 \) and 0 otherwise, and if \( r > t \) by
\[ \int_{r-t}^{t+r} \left( \frac{q}{t+r} \right) \frac{dq}{\langle q \rangle^{1+b}} \lesssim \frac{1}{t+r} \left( 1 + (t-r)^{1-b} + \delta_{b1} \ln \langle t-r \rangle \right). \]

We may therefore concentrate on the second term. If \( |r - t| \leq 1 \) the integral is easily bounded so we may as well assume that \( |r - t| > 1 \). Moreover, in that case
\[ \left| \ln \left( \frac{t-r}{t-r} \right) \right| \lesssim \frac{1}{(t-r)^2} \]
so we are left with estimating
\[ \int_{r-t}^{t+r} \left| \ln \left( \frac{|t-r|}{t-r+q} \right) \right| \frac{dq}{(1 + |q|)^{1+b}} \lesssim \frac{1}{|t-r|^b} \int_0^{2r} \left| \ln s \right| ds \left( 1/|t-r| + |s + 1| \right)^{1+b}, \]
where the sign \( \pm \) is the same as the sign of \( r - t \). If \( r > t \) or \( b > 1 \) this is bounded by \( |t-r|^{-b} \). If \( r < t \) and \( b > 1 \) its bounded by \( |t-r|^{-1} \) and if \( b = 1 \) its bounded by \( |t-r|^{-1} \ln |t-r| \).
Proof (Proof of Proposition 19). We have

\[(\partial_t - \partial_r)(\partial_t + \partial_r)\int_{t-r}^{\infty} \ln \left| \frac{t+r+q}{t-r+q} \right| \frac{n(q, \omega)dq}{2} = (\partial_t - \partial_r)\int_{t-r}^{\infty} \frac{n(q, \omega)dq}{t+r+q} = n(r-t, \omega) \frac{n(r-t, \omega)}{r}.\]

We can write

\[\Phi_1[n] = \frac{1}{2t} \int_0^{\infty} \ln \left( \frac{2t+s}{s} \right) n(s+ r - t, \omega) ds \, \chi \left( \frac{r-t}{t+r} \right).\]

Hence

\[-\Box \Phi_1[n] - F[n] = -\frac{1}{2r^3} \int_0^{\infty} \ln \left( \frac{2r+s}{s} \right) \Delta_\omega n(s+ r - t, \omega) ds \, \chi \left( \frac{r-t}{t+r} \right) \]
\[+ \frac{1}{2r} (\partial_t + \partial_r) \int_0^{\infty} \ln \left( \frac{2r+s}{s} \right) n(s+ r - t, \omega) ds \, (\partial_t + \partial_r) \chi \left( \frac{r-t}{t+r} \right) \]
\[+ \frac{1}{2r} (\partial_t - \partial_r) \int_0^{\infty} \ln \left( \frac{2r+s}{s} \right) n(s+ r - t, \omega) ds \, (\partial_t - \partial_r) \chi \left( \frac{r-t}{t+r} \right) \]
\[+ \frac{1}{2r} \int_0^{\infty} \ln \left( \frac{2r+s}{s} \right) n(s+ r - t, \omega) ds \, (\partial_t - \partial_r)(\partial_t + \partial_r) \chi \left( \frac{r-t}{t+r} \right) \]

so

\[-\Box \Phi_0[n] = -\frac{1}{2r^3} \int_0^{\infty} \ln \left( \frac{2r+s}{s} \right) \Delta_\omega n(s+ r - t, \omega) ds \, \chi \left( \frac{r-t}{t+r} \right) \]
\[+ \frac{2}{r^2} \int_0^{\infty} \frac{1}{2r+s} n(s+ r - t, \omega) ds \, \chi' \left( \frac{r-t}{t+r} \right) \left( \frac{r-t}{r+t} - \frac{r-t}{t+r} \right) \]
\[+ \frac{2}{r^2} \int_0^{\infty} \ln \left( \frac{2r+s}{s} \right) n'_q(s+ r - t, \omega) ds \, \chi' \left( \frac{r-t}{t+r} \right) \left( \frac{r-t}{r+t} - \frac{r-t}{t+r} \right) \]
\[+ \int_0^{\infty} \ln \left( \frac{2r+s}{s} \right) n(s+ r - t, \omega) ds \, \chi'' \left( \frac{r-t}{t+r} \right) \left( \frac{r-t}{r+t} - \frac{r-t}{t+r} \right) \]
\[\times \left( \frac{t-r}{t+r} - \frac{r-t}{t+r} \right) \frac{r^2}{r+\frac{t-r}{t+r}}. \tag{7.3}\]

By Lemma 19

\[\left| \int_0^{\infty} \ln \left| \frac{2r+s}{s} \right| n'_q(s+ r - t, \omega) ds + \ln \left| \frac{t+r}{t-r} \right| n(r-t, \omega) \right| \lesssim \frac{1}{(r-t)_+^3} \frac{1}{(r-t)_+^a},\]

and

\[\left| \int_0^{\infty} \ln \left| \frac{2r+s}{s} \right| n(s+ r - t, \omega) ds + \ln \left| \frac{t+r}{t-r} \right| \int_0^{\infty} n(s+ r - t, \omega) ds \right| \lesssim \frac{1}{(r-t)_+^3} \frac{1}{(r-t)_+^a},\]
\[\left| \int_0^{\infty} \ln \left| \frac{2r+s}{s} \right| \Delta_\omega n(s+ r - t, \omega) ds + \ln \left| \frac{t+r}{t-r} \right| \int_0^{\infty} \Delta_\omega n(s+ r - t, \omega) ds \right| \lesssim \frac{1}{(r-t)_+^3} \frac{1}{(r-t)_+^a}.\]

In view of this and that \( t - r \sim t + r \) in the support of \( \chi' \) it follows that

\[\Box \Phi_0[n] \lesssim \frac{1}{(t+r)^3} \ln \left| \frac{t+r}{t-r} \right| \frac{1}{(r-t)_+^a}.\]
We claim that
\[ \left| \Delta_\omega \Phi_1[n] \right| + \left| \Delta_\omega \Phi[n] \right| \lesssim \frac{1}{(t + r)} \ln \left( \frac{t + r}{t - r} \right) \frac{1}{(r - t + r)^a}. \]

For \( \Phi_1[n] \) this follows from using Lemma 19 applied to
\[ \Phi_1[n] = \frac{1}{2r} \int_0^\infty \ln \left( \frac{2r + s}{s} \right) n(s + r - t, \omega) \, ds \chi \left( \frac{r - t}{t + r} \right) \quad (7.4) \]
with \( n \) replaced by \( \Delta_\omega n \). For \( \Phi[n] \) this follows from Lemma 9 applied to
\[ -\square \Phi[n] = F[n] = n(r - t, \omega) r^{-2} \chi \left( \frac{t - r}{t + r} \right) \quad (7.5) \]
with \( n \) replaced by \( \Delta_\omega n \). Hence the same estimate holds for \( \Delta_\omega \Phi_0 \). It therefore follows from Lemma 18 that the \( u_0 \) satisfies the required estimates for \( N = 0 \) and that the limit \( U^\infty \) exists and satisfies the estimates for \( N = 0 \). It remains to prove these estimates for \( N > 0 \). For angular derivatives this is clear since then \( \Omega^I \Phi[n] = \Phi[\Omega^I n], \Omega^I \Phi_1[n] = \Phi_1[\Omega^I n] \) and \( \Omega^I \Phi_0[n] = \Phi_0[\Omega^I n] \). For time derivatives we have that modulo terms with the time derivative falling on the cutoffs and whenever a time derivative falls on cutoff we get terms of the same form but with additional decay of \( (t + r)^{-1} \) and moreover \( t - r \sim t + r \) in the support of the cutoff so we have
\[ \left| \square \partial_t \Phi_0[n] - \square \Phi_0[-\partial_q n] \right| \lesssim (t + r)^{-4} (r - t + r)^{-a}. \]

Now \( \square S \Phi_0[n] = (S + 2) \square \Phi_0[n] \). Recall that \( S f(t, r) = \partial_q f(at, ar) \big|_{q = 1} \). Let \( I[n](t, r) \) denote any of the integrals in \( (7.3) \) with the factors of \( r \) in front. Then changing variables we see that \( I[n(q, \omega)](at, ar) = a^{-2} I[n(aq, \omega)] \) so \( (S + 2) I[n] = I[n \partial_q n] \). Any of the factors that multiply the integrals would be homogeneous of degree 0 if \( (t - r) \) was replaced by \( t - r \) and hence they would have vanished in that case when \( S \) is applied to them. However the error is of lower order because
\[ S \frac{t - r}{t - r} = \frac{t - r}{2(t - r)^2} S \frac{(t - r)^2}{t - r} = \frac{1}{2} \frac{t - r}{(t - r)^2} S \frac{1}{t - r} = -\frac{1}{t - r} \]
Hence
\[ \left| \square S \Phi_0[n] - \square \Phi_0[q \partial_q n] \right| \lesssim (t + r)^{-5} (r - t + r)^{-a}. \]

If \( S \) falls on the cutoff functions we get errors that decay like \( (t + r)^{-2} \) and if another \( S \) falls on these errors the errors still decays like \( (t + r)^{-2} \). If a \( t \) derivative first falls of the cutoffs then we get an error that decay like \( (t + r)^{-1} \) and again \( S \) applied to it still decays like \( (t + r)^{-1} \). We conclude that
\[ \left| \square \Omega^I S^J \partial^K \Phi_0[n] - \square \Phi_0[\Omega^I (q \partial_q)^J (-\partial_q)^K n] \right| \lesssim (t + r)^{-4} (r - t + r)^{-a} \]
and hence
\[ \left| \square \Omega^I S^J \partial^K \Phi_0[n] \right| \lesssim \frac{1}{(t + r)^3} \ln \left( \frac{t + r}{t - r} \right) \frac{1}{(r - t + r)^a}. \]
By the same argument if we apply these operators to \( (7.4) \) we get
\[ \left| \partial_t \Phi_1[n] - \Phi_1[-\partial_q \Delta_\omega n] \right| \lesssim (t + r)^{-2} (r - t + r)^{-a}, \]
\[ \left| \triangle_\omega \Phi_1[n] - \Phi_1[q \partial_q \Delta_\omega n] \right| \lesssim (t + r)^{-3} (r - t + r)^{-a}. \]
Repeating these arguments give
\[ |\Omega^l S^j \partial_i^K \triangle \omega \Phi_1[n] - \Phi_1[\Omega^l (q \partial_q^j)(-\partial_q)^K \triangle \omega n]| \lesssim (t+r)^{-2}(r-t)^{-a} \]
and hence
\[ |\Omega^l S^j \partial_i^K \triangle \omega \Phi_1[n]| \lesssim \frac{1}{(t+r)} \ln \left( \frac{t+r}{t-r} \right) \frac{1}{(r-t)^a}. \]
Applying the vector fields to (7.5) gives the error term when the vector fields fall on the cutoff function
\[ |\square \partial_t \Phi[n] - \square \Phi[-\partial_q n]| \lesssim (t+r)^{-4-a} \]
since \( t-r \sim t+r \) in the support of the derivative \( \chi' \) and since \( n \) decays. Similarly
\[ |\square S \Phi[n] - \square \Phi[q \partial_q n]| \lesssim (t+r)^{-5-a}. \]
In general we get as above
\[ |\square \Omega^l S^j \partial_i^K \Phi[n] - \square \Phi[\Omega^l (q \partial_q^j)(-\partial_q)^K n]| \lesssim (t+r)^{-4-a}, \]
and
\[ |\triangle \omega \Omega^l S^j \partial_i^K \triangle \omega \Phi[n]| \lesssim \frac{1}{(t+r)} \ln \left( \frac{t+r}{t-r} \right) \frac{1}{(r-t)^a}. \]
This finishes the proof of the first part of the theorem also for \( N > 0 \).
It remains to prove the last estimate which since we already have an estimate for vector fields of \((\partial_t + \partial_r)(r \Phi_0[n])\) would follow from
\[ \sum_{|I|+|J|+|K| \leq N} |(\partial_t + \partial_r)(r \Omega^l S^j \partial_i^K \Phi_1[n]) - \Phi_1[\Omega^l (q \partial_q^j)(-\partial_q)^K n]| \lesssim \frac{(1+(t+r)_+)^a}{(1+t+r)^{1+a}}. \]
We have
\[
(\partial_t + \partial_r)(r \Phi_1[n]) = (\partial_t + \partial_r) \frac{1}{2} \int_0^\infty \ln \left( \frac{2r+s}{s} \right) n(s+r-t, \omega) ds \chi' \left( \frac{r-t}{t+r} \right)
= \int_0^\infty \frac{1}{2r+s} n(s+r-t, \omega) ds \chi' \left( \frac{r-t}{t+r} \right)
- \frac{1}{2r} \int_0^\infty \ln \left( \frac{2r+s}{s} \right) n(s+r-t, \omega) ds \chi' \left( \frac{r-t}{t+r} \right) 2r \left( \frac{r-t}{t+r} \right) \frac{2r}{t+r}.
\]
Here the first integral on the right is
\[
\int_0^\infty \frac{n(s+r-t, \omega)}{2r+s} ds = \frac{1}{2r} \int_{r-t}^\infty n(q, \omega) dq + \frac{1}{2r} \int_{r-t}^\infty \frac{(r-t-q)}{t+r+q} \frac{n(q, \omega)}{t+r+q} dq,
\]
so we obtain
\[
(\partial_t + \partial_r)(r \Phi_1[n]) - \Phi_{1+}[n] = \frac{1}{2r} \int_{r-t}^\infty \frac{(r-t-q)}{t+r+q} n(q, \omega) dq \chi' \left( \frac{r-t}{t+r} \right)
- \frac{1}{2r} \int_0^\infty \ln \left( \frac{2r+s}{s} \right) n(s+r-t, \omega) ds \chi' \left( \frac{r-t}{t+r} \right) 2r \left( \frac{r-t}{t+r} \right) \frac{2r}{t+r}.
\]
Here
\[
\int_{r-t}^{\infty} \frac{|q| n(q, \omega)}{t + r + q} \, dq \lesssim \frac{1}{t + r} \int_{r-t}^{t+r} |q| n(q, \omega) \, dq + \int_{t+r}^{\infty} |n(q, \omega)| \, dq \lesssim \frac{1}{(t + r)^{a}}
\]
\[
\int_{r-t}^{\infty} \frac{|t-r| |n(q, \omega)|}{t + r + q} \, dq \lesssim \frac{|t-r|}{t + r} \int_{r-t}^{t+r} |n(q, \omega)| \, dq \lesssim \frac{|t-r|}{t + r} \left(1 + (r - t)\right)^{a},
\]
and by Lemma 19
\[
\int_{0}^{\infty} \ln \left(\frac{2r + s}{s}\right) n(s + r - t, \omega) \, ds \lesssim \ln \left(\frac{\langle t + r \rangle}{\langle t - r \rangle} \right) \frac{1}{(t - r)^{a}}.
\]
However, this term is multiplied with the derivative of the cutoff function that is supported when \( t - r \sim t + r \) so we can multiply \( \frac{(t - r)^{a}}{(t + r)^{a}} \) and we get that all remainder terms are bounded by
\[
| (\partial_{t} + \partial_{r})(r \Phi_{1}[n]) - \Phi_{1+}[n] | \lesssim (1 + (t - r))^{(1 + t + r)^{-1}}.
\]
We now need to apply vector fields. Note that \( S(\partial_{t} + \partial_{r})(r \Phi) = (\partial_{t} + \partial_{r})(r \Phi) \) so
\[
\Omega^{I} S^{J} \partial_{K}^{I} (\partial_{t} + \partial_{r})(r \Phi_{1}[n]) = (\partial_{t} + \partial_{r})(r \Omega^{I} S^{J} \partial_{K}^{I} \Phi_{1}[n]).
\]
As before we have
\[
\left| \Omega^{I} S^{J} \partial_{K}^{I} ((\partial_{t} + \partial_{r})(r \Phi_{1}) - \Phi_{1+})[n] - ((\partial_{t} + \partial_{r})(r \Phi_{1}) - \Phi_{1+})[\Omega^{I} (q \partial_{q})^{J} (-\partial_{q})^{K}]n] \right| \lesssim \frac{(1 + (t - r)\right)^{a}}{1 + t + r^{2+a}}
\]
so
\[
\left| \Omega^{I} S^{J} \partial_{K}^{I} ((\partial_{t} + \partial_{r})(r \Phi_{1}[n]) - \Phi_{1+}[n]) \right| \lesssim \langle t - r \rangle^{-a} (1 + t + r)^{-1-a},
\]
which concludes the proof of the proposition.

8. The Asymptotic of the Metric in Schwarzschild Coordinates

Let
\[
h_{\mu\nu}^{*}(t, r^{*} \omega) = h_{\mu\nu}(t, r \omega), \quad H_{TU}^{*}(q^{*}, \omega, r^{*}) = r^{*} h_{TU}^{*}(r^{*} - q^{*}, r^{*} \omega),
\]
where \( r^{*} = r + M \ln |1 + r| \), and similarly define \( h^{1*}, H^{1*}, h^{0*} \) and \( H^{0*} \).

**Proposition 20.** The limit
\[
H_{TU}^{1*}(q^{*}, \omega) = \lim_{r^{*} \to \infty} H_{TU}^{1*}(q^{*}, \omega, r^{*}),
\]
exists and satisfies \( H_{TU}^{1*} = H_{TU}^{1*} \), and \( H_{L}^{1*}(q^{*}, \omega) = \delta^{AB} H_{AB}^{1*}(q^{*}, \omega) = 0 \). Moreover, for \( |\alpha| + k \leq N - 6 \) and \( r > t/2 \)
\[
| \partial_{\omega}^{\alpha} ((1 + |q^{*}|)\partial_{q^{*}})^{k} H_{TU}^{1*}(q^{*}, \omega) | \lesssim \varepsilon (1 + q^{*})^{-r},
\]
\[
| \partial_{\omega}^{\alpha} ((1 + |q^{*}|)\partial_{q^{*}})^{k} [H_{TU}^{1*}(q^{*}, \omega, r^{*}) - H_{TU}^{1*}(q^{*}, \omega)] | \lesssim \varepsilon \left(\frac{1 + q^{*}}{1 + r^{*}}\right)^{-r}.
\]
Proof. Note first that
\[ |\partial^{\alpha}_{\omega}(1 + |q|^*)\partial_{q^*} H^1_{TU}(q^*, \omega, r^*)| \lesssim \sum_{|\alpha| \leq k} |r^* Z\ast l h^1_{TU}|, \]
since |Z\ast l r^*| \lesssim 1 + t + r^*. By Lemma 15 for |I| \leq N - 6 we have the estimate
\[ \left| (\partial_{r^*} + \partial_{r^*})(\partial_{r^*} - \partial_{r^*})(r^* Z\ast l h^1_{TU}) \right| \lesssim \frac{\epsilon(1 + |q|^*)^{-\gamma}}{(1 + t + r^*)^{2-\epsilon}(1 + |q|^*)^{\epsilon}} + \frac{\epsilon^2(1 + |q|^*)^{-2}}{(1 + t + r^*)^{1+\gamma-\epsilon}}. \]
Integrating this over the set \(2r^*_k - q^* \leq r^* + t \leq 2r^*_2 - q^*, r^* - t \geq q^*, t \geq 0 \) gives
\[ |\partial^{\alpha}_{\omega}(1 + |q|^*)\partial_{q^*} \left( H^1_{TU}(q^*, \omega, r^*_2) - H^1_{TU}(q^*, \omega, r^*_1) \right)| \lesssim \frac{\epsilon}{(1 + r^*_1)^{1+\gamma-\epsilon}} + \epsilon \left( \frac{1 + q^*}{1 + r^*_1} \right)^{1-\epsilon}, \]
from which it follows that the limit exists. Moreover, by Proposition 17
\[ |\partial^{\alpha}_{\omega}(1 + |q|^*)\partial_{q^*} H^1_{TU}| \lesssim \epsilon(1 + q^*)^{-\gamma'}, \]
so that is true for the limit as well. That \( H^1_{LT}(q^*, \omega) = \delta^{AB} H^1_{AB}(q^*, \omega) = 0 \) follows from passing to the limit in the wave coordinate condition.

Let \( V^*_{TU} = \partial_{q^*} H^1_{TU} \) and \( V^*_{LU} = \partial_{q^*} H^1_{TU} \). Note that this is the same as with \( H^1_{*} \) and \( H^1_{\infty} \). Then by [LR3] we have
\[ P(V, V) = -\frac{1}{4} V_{LL} V_{LL} - \frac{1}{4} \delta_{CD} \delta^{CD'} (2 V_{CC'} V_{DD'} - V_{CD} V_{CD'}) + \frac{1}{2} \delta_{CD} (2 V_{CL} V_{DL} - V_{CD} V_{LL}). \]
Since we shown that \( V^*_{LU} = 0 \) it follows that this expression can be calculated from just knowing \( V^*_{LU} \) and by the previous proposition \( V^*_{LT} = \delta^{AB} V^*_{AB} = 0 \) so
\[ n(q^*, \omega) = -P(V^\infty, V^\infty)(q^*, \omega) = \frac{1}{2} \delta_{CD} \delta^{CD'} V^\infty_{CC'} V^\infty_{DD'}(q^*, \omega) \leq 0. \]
By the previous proposition; for \(|\alpha| + k \leq N - 7\)
\[ |\partial^{\alpha}_{\omega}(1 + |q^*|)\partial_{q^*} n(q^*, \omega) | \lesssim \frac{\epsilon^2}{(1 + |q^*|)^2 - \gamma}(1 + q^*_{*})^{2\gamma'}, \]

Proposition 21. Let \( k_{\mu\nu} \) be the solution with vanishing initial data to
\[ -\Box^* k_{\mu\nu} = L_{\mu}(\omega) L_{\nu}(\omega)n(r^* - t, \omega)r^{* - 2} \chi \left( \frac{r^* - t}{r^* + r^*_{0}} \right), \]
where \( \chi(s) = 1 \), when \(|s| \leq 1/2 \) and \( \chi(s) = 0 \), when \(|s| \geq 3/4 \). We have
\[ |Z^l(\Box^* h_{\mu\nu} - \Box^* k_{\mu\nu})| \lesssim \frac{\epsilon^2}{(1 + t + r^*)^{2+\gamma-\epsilon}(1 + |q|^*)^{2-\gamma}}, \quad |I| \leq N - 7, \]
and with \( h^1_{\mu\nu} \) as in (6.12) we have for any \( \gamma' < \gamma - \epsilon \)
\[ |Z^l(h^1_{\mu\nu} - k_{\mu\nu})| \lesssim \frac{\epsilon^2}{(1 + t + r^*)^{(1 + |q|^*)^{\gamma'}}}, \quad |I| \leq N - 7. \]
The proof is just an application of Proposition 16 and Lemma 9. By Lemma 18
\[
| (\partial_t + \partial_r^*)(\Omega^I S^J \partial^K r^*(h_{\mu \nu}^1 - k_{\mu \nu})) | \lesssim \frac{1}{(1 + t + r^*)^{1+\gamma'}},
\]
for $|I| + |J| + |K| \leq N - 7$, and by Proposition 19
\[
| (\partial_t + \partial_r^*) \Omega^I S^J \partial^K r^*(k_{\mu \nu} - k_{\mu \nu}^1) | \lesssim \frac{1}{(1 + t + r^*)^{1+a}},
\]
where
\[
k_{\mu \nu}^1(t, r^* \omega) = L_{\mu}(\omega) L_{\nu}(\omega) \int_{r^*-t}^{\infty} \frac{1}{2r^*} \ln \left( \frac{t + r^* + q^*}{t - r^* + q^*} \right) n(q^*, \omega) \, dq^* \, \chi(\frac{t - r^*}{t + r^*}),
\]
and $0 < a < 1$ is number such that
\[
\sum_{|\alpha| + k \leq N} | (q^*)^k \partial_{\alpha} q^* n(q^*, \omega) | \lesssim \varepsilon^2 (q^*)^{-1-a}.
\]
This is true for any $a \leq 1$, in particular for $a = \gamma'$. It therefore follows that
\[
| (\partial_t + \partial_r^*) (\Omega^I S^J \partial^K r^*(h_{\mu \nu}^1 - k_{\mu \nu}^1)) | \lesssim \varepsilon \left( \frac{1 + (t - r^*)}{1 + t + r^*} \right)^{1+\gamma'},
\]
for $|I| + |J| + |K| \leq N - 7$. Set
\[
H_{\mu \nu}^{1e*}(q^*, \omega, r^*) = r^*(h_{\mu \nu}^1 - k_{\mu \nu}^1)(r^* - q^*, r^* \omega).
\]
We now have an improvement of Proposition 20:

**Proposition 22.** The limit
\[
H_{\mu \nu}^{1e\infty}(q^*, \omega) = \lim_{r^* \to \infty} H_{\mu \nu}^{1e*}(q^*, \omega, r^*),
\]
exists and satisfies $H_{UU}^{1e\infty} = H_{UU}^{1e\infty}$, and $H_{LT}^{1e\infty}(q^*, \omega) = \delta^{AB} H_{AB}^{1e\infty}(q^*, \omega) = 0$, and
\[
| \partial_{\alpha}^k (1 + |q^*|) \partial_{q^*}^k H_{\mu \nu}^{1e\infty}(q^*, \omega) | \lesssim \varepsilon (1 + q^*)^{-\gamma'},
\]
\[
| \partial_{\alpha}^k (1 + |q^*|) \partial_{q^*}^k \left[ H_{\mu \nu}^{1e*}(q^*, \omega, r^*) - H_{\mu \nu}^{1e\infty}(q^*, \omega) \right] | \lesssim \varepsilon \left( \frac{1 + q^*}{1 + t + r^*} \right)^{1+\gamma'},
\]
for $|\alpha| + k \leq N - 7$ and when $r^* > t/2$. Moreover
\[
| (t + r^*)(\partial_t + \partial_r^*) \partial_{\alpha}^k (1 + |q^*|) \partial_{q^*}^k \left[ H_{\mu \nu}^{1e*}(q^*, \omega, r^*) \right] | \lesssim \varepsilon \left( \frac{1 + q^*}{1 + t + r^*} \right)^{1+\gamma'},
\]
\[
| (\partial_t + \partial_r^*) (r \Omega^I S^J \partial^K r^*(h_{\mu \nu}^1 - k_{\mu \nu}^1)) - \frac{1}{2r} \int_{r-t}^{\infty} \Omega^I(q \partial_q) (-\partial_q) K n(q, \omega) \, dq | \lesssim \varepsilon \left( \frac{1 + (t - r^*)}{1 + t + r^*} \right)^{1+\gamma'},
\]
9. Interior Asymptotics for the Wave Equation with Sources

The results so far sufficed to prove existence of the radiation field. To get more precise behavior towards time-like infinity we use formulas from [L1]:

**Proposition 23.** Let $F$ and $\phi = \Phi[n]$ be as in the Proposition 19 and set

$$
\phi_2(x, t) = \Phi_2[n](x, t) = \int_{r-t}^{\infty} \frac{1}{4\pi} \int \frac{n(q, \omega)}{t + q - \langle x, \omega \rangle} dS(\omega) \chi_{\frac{\langle q \rangle}{t+r}} dq. 
$$

(9.1)

Then for $|I| + |J| + |K| \leq N$

$$
|\Omega^I S^J \partial_t^K (\phi - \phi_2)| \lesssim \frac{1}{(1 + t + r)(1 + |r - t|)^a}, \quad 0 < a < 1.
$$

(9.2)

$$
| (\partial_t + \partial_r)(r \Omega^I S^J \partial_t^K (\phi - \phi_2))| \lesssim \frac{1}{(1 + t + r)^{1+a}}.
$$

(9.3)

**Proof.** Following [L1] we write the convolution of $F(t, x) = \eta(|x| - t, x)/|x|^2$,

with the fundamental solution $E$ of $\Box$

$$
\phi(t, x) = E * F(t, x) = \int_{|x| - t}^{\infty} \phi_q(t, x) dq,
$$

where

$$
\phi_q(t, x) = \frac{1}{4\pi} \int \frac{\eta(q, \rho \omega)}{t + q - \langle x, \omega \rangle} dS(\omega) H(t + q - |x|).
$$

Here $H(s) = 1, s > 0$ and 0 otherwise, and

$$
\rho = \rho(q, \omega) = \frac{1}{2} \frac{(t + q)^2 - r^2}{t + q - \langle x, \omega \rangle}; \quad s = \rho - q,
$$

satisfy

$$
0 \leq t + q - r \leq 2\rho \leq t + q + r, \quad \text{and} \quad t - r \leq \rho + s \leq t + r.
$$

In our case

$$
\eta(q, y) = n(q, \omega)\psi(q, \rho), \quad \text{and} \quad \psi(q, \rho) = \chi_{\frac{\langle q \rangle}{s+\rho}}, \quad y = \rho \omega.
$$

We have

$$
2\langle q \rangle \leq \rho + s \quad \Rightarrow \quad 2\langle q \rangle \leq t + r.
$$

Hence

$$
\chi_{\frac{\langle q \rangle}{s+\rho}} = \chi_{\frac{\langle q \rangle}{t+r}} = 1, \quad \text{when} \quad \frac{\langle q \rangle}{s+\rho} \leq \frac{1}{2}.
$$

(9.4)

We write

$$
\phi(t, x) = \int_{r-t}^{t+r} \frac{1}{4\pi} \int \frac{n(q, \omega)}{t + q - \langle x, \omega \rangle} \psi(q, \rho(q, \omega)) dS(\omega) dq = \phi_2(t, x) - \mathcal{E}(t, x),
$$
where
\[ \mathcal{E}(t, x) = \int_{r-t}^{t+r} \frac{1}{4\pi} \int \frac{n(q, \omega)}{t + q - \langle x, \omega \rangle} \left( \chi\left(\frac{q}{1 + r}\right) - \chi\left(\frac{q}{2 + r}\right) \right) \, dS(\omega) \, dq. \]

It follows from (9.4) that the integrand is nonvanishing only if \( 2\langle q \rangle \geq 2\rho - q \), i.e.
\[ \rho \geq \langle q \rangle + q/2, \quad \text{i.e.} \quad t + q - \langle x, \omega \rangle \geq (t + q)^2 - r^2)/(2\langle q \rangle + q). \]

Hence we want to estimate the integral
\[ |\mathcal{E}(t, x)| \leq \int_{r-t}^{t+r} \frac{1}{4\pi} \int H(t + q - \langle x, \omega \rangle) \geq \frac{(t + q)^2 - r^2}{2\langle q \rangle + q} \frac{dS(\omega)}{t + q - \langle x, \omega \rangle} \frac{dq}{\langle q \rangle^{1+a}}, \]

where \( H(f) \) is the characteristic function of the set where \( f \geq 0 \). We can choose coordinate so that \( \langle x, \omega \rangle = r\omega_1 \) and integrate over the other angular variables:
\[ |\mathcal{E}(t, x)| \leq \int_{r-t}^{t+r} \frac{1}{2} \int_{t+q-r}^{t+q+r} H(u) \geq \frac{(t + q)^2 - r^2}{2\langle q \rangle + q} \frac{du}{ur} (1 + |q|)^{1+a}. \]

We have
\[ [(t + q)^2 - r^2)/(2\langle q \rangle + q) \leq t + q + r \quad \iff \quad t - r \leq 2\langle q \rangle. \]

If \( t - r > 2 \) then we must also have \( 2|q| \geq \sqrt{(t-r)^2 - 4} \), in which case
\[ 2|r \, \mathcal{E}(t, x)| \leq \int_{r-t}^{t+r} \int_{(t+q-r)^2}^{(t+q+r)^2} \frac{du}{u} \frac{dq}{\langle q \rangle^{1+a}} = \int_{r-t}^{t+r} \ln \left( \frac{2\langle q \rangle + q}{t + q - r} \right) \frac{H(2\langle q \rangle \geq t - r) \, dq}{\langle q \rangle^{1+a}}. \]

If \( |t - r| \leq 4 \), this integral is bounded. If \( q \geq t - r \geq 4 \) then \( 2\langle q \rangle + q \leq 4q \) and if we change variables \( q = (r - t)z \) we see that the integral is bounded by
\[ \frac{1}{|t - r|^a} \int_{1}^{\infty} \ln \left( \frac{4z}{z - 1} \right) \frac{dz}{z^{1+a}} \lesssim \frac{1}{|t - r|^a}. \]

If \( t - r \geq 4 \) then \( |q| \geq \sqrt{(t-r)^2 - 4}/2 \geq |t - r|/4 \) and the integral
\[ \frac{1}{|t - r|^a} \int_{-1}^{\infty} \ln \left( \frac{4|z|}{|1 + z|} \right) \frac{H(|z| \geq 1/4) \, dz}{z^{1+a}} \lesssim \frac{1}{|t - r|^a}. \]

This proves (9.2) for \( |I| = |J| = 0 \) and the fact that we can get the same bounds for higher \( |I| \) follows from rewriting \( x = rQe, e = (1, 0, 0), \) \( QT Q = I, \) and making an angular change of variables \( \omega' = Q\omega \) so that any angular derivative on \( x \) becomes and angular derivative of \( n \) which we control by assumption:
\[ \mathcal{E}(t, rQe) = \frac{1}{r} \int_{r-t}^{t+r} \frac{1}{4\pi} \int \frac{n(q, Q\omega)}{(q + t - r)/r + 1 - \omega_1} \left( \chi\left(\frac{q}{1 + r}\right) - \chi\left(\frac{q}{2 + r}\right) \right) \, dS(\omega) \, dq. \]
The result for the scaling vector fields follows similarly by noting that a scaling \((t, r) \to (at, ar)\) by a change of variables corresponds to scaling \(n(q, \omega) \to n(aq, \omega)\) so \(S(\phi - \phi_2)\) corresponds to replacing \(n(q, \omega)\) by \(q \partial_q n(q, \omega)\) plus an error

\[
\partial_a \left( \left( \frac{(aq)}{t+q} \right) - \chi \left( \frac{(aq)}{s+\rho} \right) \right) \bigg|_{a=1} = -\frac{1}{(q)^2} \left( \chi' \left( \frac{(aq)}{t+q} \right) - \chi' \left( \frac{(aq)}{s+\rho} \right) \right),
\]

decaying faster. Let us now estimate \((\partial_r + \partial_r) (r E)\). The terms where derivatives fall on \(\chi\) are easily bounded separately whereas for the main term where \(\chi\) is not differentiated we need to use the cancellation between the two cutoff functions, as we did in the estimates above. Let us assume that \(x = r(1, 0, 0)\) and let us replace \(n(q, \omega)\) by \(n(q, \omega_1)\) its average over \(\omega_0^2 + \omega_3^2 = 1 - \omega_1^2\). Writing

\[
\rho = \frac{1}{2} \left( \frac{(t + q)^2 - r^2}{t + q - r \omega_1} \right) = \frac{t + q - r}{t + q + r} \frac{t + q + r}{t + q - r \omega_1}
\]

we see that

\[
(\partial_r + \partial_r) \rho = \frac{t + q - r}{2} \frac{2(t + q - r \omega_1) - (t + q + r)(1 - \omega_1)}{(t + q - r \omega_1)^2} = \frac{(t + q - r)^2 (1 + \omega_1)}{2(t + q - r \omega_1)^2} = \frac{2 \rho^2 (1 + \omega_1)}{(t + q + r)^2}.
\]

Since \(s = \rho - q\) we also have \((\partial_t + \partial_r)(s + \rho) = 2(\partial_t + \partial_r) \rho\) and hence

\[
(\partial_t + \partial_r) \chi \left( \frac{q}{s+\rho} \right) = \chi' \left( \frac{q}{s+\rho} \right) \frac{2}{(s + \rho)^2} \frac{(1 + \omega_1)}{(t + q + r)^2},
\]

which is bounded by a constat times \((q)/(t + r)^2\) in the support of \(\chi' \left( \frac{q}{s+\rho} \right)\):

\[
f \left( \frac{t+r}{s+\rho}, \frac{q}{s+\rho}, \omega_1 \right) = \frac{(t + r)^2 2 \rho^2 (1 + \omega_1)}{(s + \rho)^2 (t + q + r)^2} = \frac{(1 + \frac{q}{s+\rho})^2}{2} \frac{1 + \omega_1}{2} \leq C, \quad \text{if} \quad \frac{q}{s+\rho} \leq \frac{3}{4}.
\]

We have

\[
(\partial_t + \partial_r)(r E) = (\partial_t + \partial_r) \int_{r-t}^{t+r} \int_{-1}^{1} n(q, \omega_1) \left( \chi \left( \frac{q}{t+r} \right) - \chi \left( \frac{q}{s+\rho} \right) \right) \frac{d\omega_1}{2} dq = E_+ + E'_+,
\]

where

\[
E_+ = \int_{r-t}^{t+r} \int_{-1}^{1} \frac{n(q, \omega_1)}{(t + q - r)/(r + 1 - \omega_1)^2} \left( \frac{t + q - r}{r^2} \chi \left( \frac{q}{t+r} \right) - \chi \left( \frac{q}{s+\rho} \right) \right) \frac{d\omega_1}{2} dq,
\]

\[
E'_+ = -2 \int_{r-t}^{t+r} \int_{-1}^{1} \frac{n(q, \omega_1)}{(t + q - r)/(r + 1 - \omega_1)(t + r)^2} \left( \chi' \left( \frac{q}{t+r} \right) - \chi' \left( \frac{q}{s+\rho} \right) f \left( \frac{q}{t+r}, \frac{q}{s+\rho}, \omega_1 \right) \right) \frac{d\omega_1}{2} dq,
\]
since the boundary term vanishes at $q = t + r$, since $\chi(s) = 0$ if $s \geq 1$. We claim

$$|\mathcal{E}_+| + |\mathcal{E}_+^t| \lesssim (t + r)^{-1 - a}, \quad 0 < a < 1.$$  

Since

$$\chi'(\frac{q}{s + \rho}) = \chi'(\frac{q}{t + r}) = 0, \quad \text{when } \frac{q}{s + \rho} \leq \frac{1}{2},$$

the previous argument applied to $n(q, \omega)(1 + q^2)^{1/2}$ gives the bound for $\mathcal{E}_+^t$. The same bound also hold for $\mathcal{E}_+$ if we estimate differences of cutoff functions using

$$\frac{1}{s + \rho} - \frac{1}{t + r} = \frac{r}{t + r s + \rho} \left(1 - \omega_1\right).$$

This estimate in turn follows since

$$t + r - (s + \rho) = t + r + q - \frac{t + q - r \omega_1}{t + q - r \omega_1} = (t + q + r) \frac{r(1 - \omega_1)}{t + q - r \omega_1} = r \rho (1 - \omega_1).$$

We have

$$(\partial_t - \partial_r) \frac{n(\tau + r - t, \omega_1) \tau}{(\tau + r (1 - \omega_1))^2} = - \frac{2 n'(\tau + r - t, \omega_1) \tau}{(\tau + r (1 - \omega_1))^2} + \frac{2 (1 - \omega_1) n(\tau + r - t, \omega_1) \tau}{(\tau + r (1 - \omega_1))^3}.$$  

10. The Asymptotics of the Characteristic Surfaces

We solve the eikonal equation

$$g^{\alpha \beta} \partial_{\alpha} u \partial_{\beta} u = 0, \quad \text{in } r > t/2 > 0,$$  

(10.1)

with asymptotic data as $t \to \infty$:

$$u \sim u^* = t - r^*, \quad r^* = \mathcal{Q}(r) = r + M \ln r + O(M/r).$$  

(10.2)

Here we have modified the definition of $r^*$ slightly from before:

$$\frac{dr^*}{dr} = \mathcal{Q}'(r) = \left(\frac{1 + M/r}{1 - M/r}\right)^{1/2} = 1 + \frac{M}{r} + O(M^2/r^2),$$  

(10.3)

so that $u^*$ is a solution of the eikonal equation for the metric $g^{\alpha \beta}_0 = m^{\alpha \beta} + h^{\alpha \beta}_0$, where

$h^{\alpha \beta}_0 = - \frac{M}{r} g^{\alpha \beta} \tilde{\chi}(\frac{r}{1 - r^2})$ and $\tilde{\chi}(s) = 1$, when $s > 1/2$ and $0$ when $s < 1/4$;

$$g^{\alpha \beta}_0 \partial_{\alpha} u^* \partial_{\beta} u^* = 0, \quad r > t/2.$$  

By the local existence theorem (10.1) with data $u = u^*$ on $N_T = \{t = T, \; r \geq T/2\}$, has a local solution that extends to $[0 < t \leq T, \; r > t/2 + 1]$, as long as we have bounds for the first order derivatives and $g^{\alpha \beta} \partial_{\alpha} u \partial_{\beta}$ is close to $L$. We simply define $u$ to be constant along the null geodesics from $N_T$ in the direction $g^{\alpha \beta} \partial_{\alpha} u \partial_{\beta}$ (where on $N_T$, $u$ is determined by data $u^*$ and $\partial_{\alpha} u$ by that this vector should be null and ingoing backwards). This gives a well defined solution as long as the derivatives are bounded, which we control by integrating the system for the first order derivatives obtained by differentiating the eikonal equation (10.1).
Let \( \gamma(t) = \gamma(t, q^*, \omega) = (t, x(t)) \) denote the integral curve of the vector field 
\[
F^a(g, \partial u) = g^{a\beta} \partial_\beta u / g^{0\beta} \partial_\beta u,
\]
going through \( \gamma(T) = \gamma_0(T) \), where \( \gamma_0(t) = (t, r_0) \), \( t = \varphi(r) + q^* \), is an integral curve of \( \tilde{F}(g_0, u^*) \), when \( g, u \) is replaced by \( g_0, u^* \). Set \( \tilde{\gamma}(t) = \gamma(t) - \gamma_0(t) \).

We will first show that we have uniform bounds independent of \( T \) and then that the solution \( u \) will converge to a limit as \( T \to \infty \) satisfying (10.1)–(10.2).

**Proposition 24.** Let \( \tilde{u} = u - u^* \). For \( r > t/2 \) and \( 0 \leq t \leq T \) we have 
\[
(1 + |q^*|)|\partial \tilde{u}| \leq C_1 \varepsilon \left( \frac{1 + |q^*|}{1 + t + |q^*|} \right)^{1 - \gamma' - \varepsilon} \left( \frac{1 + q^*}{1 + t + |q^*|} \right)^{\gamma'}, \tag{10.4}
\]
\[
(1 + t + |q^*|)|\partial \tilde{v}| + |\tilde{u}| \leq C_1 \varepsilon \left( \frac{1 + q^*}{1 + t + |q^*|} \right)^{\gamma'}, \tag{10.5}
\]
for some constant \( C_1 \) independent of \( T \). Here \( q^* = r^* - t \). Moreover,
\[
|\tilde{\gamma}(t, q^*, \omega)| \leq C_1 \varepsilon \left( \frac{1 + q^*}{1 + t + |q^*|} \right)^{\gamma'}. \tag{10.6}
\]
The convergence is proved by similar estimates for differences of solutions with data at \( T_1, T_2 \). The differences satisfy homogeneous equations with data at \( T = \min(T_1, T_2) \) satisfying (10.4)–(10.5) that tend to 0 as \( T \to \infty \), see Proposition 27.

By time reflection we also have a solution \( v \) to the eikonal equation in \( r \geq |t|/2, t \leq 0 \) satisfying \( v \sim v^* = r^* + t \), as \( t \to -\infty \) and \( (1 + r)|\partial \tilde{v}| + |\tilde{v}| \leq (1 + r)^{-\gamma'} \) when \( t = 0 \), \( \tilde{v} = v - v^* \). This solution can be extended into the region \( r \geq t/2 + 1, t > 0 \): 

**Proposition 25.** Suppose that \( (1 + r)|\partial \tilde{v}| + |\tilde{v}| \leq C_1 \varepsilon (1 + r)^{-\gamma'} \) when \( t = 0 \). Then
\[
(1 + t + |q^*|)|\partial \tilde{v}| + (1 + |q^*|)|\partial \tilde{v}| + |\tilde{v}| \lesssim 2C_1 \varepsilon \left( \frac{1 + q^*}{1 + t + |q^*|} \right)^{\gamma'}, \tag{10.7}
\]
for \( r \geq t/2 > 0 \), where \( q^* = r^* - t \). If \( \tilde{\sigma} \) is the corresponding characteristic deviation:
\[
|\tilde{\sigma}(t, q^*, \omega)| \leq C_1 \varepsilon \left( \frac{1 + q^*}{1 + t + |q^*|} \right)^{\gamma'}. \tag{10.8}
\]

**Proposition 26.** We have \( |Z^* \tilde{u}| + |Z^* \tilde{v}| \lesssim \varepsilon \left( \frac{1 + q^*}{1 + t + |q^*|} \right)^{\gamma'} \) for \( |I| \leq 2 \) and \( r \geq |t|/2 \).

We will now derive the system for the derivatives. Differentiating (10.1) gives
\[
2g^{a\beta} \partial_\alpha u \partial_\beta Z u = -g^{a\beta} \partial_\alpha u \partial_\beta u, \tag{10.9}
\]
where the Lie derivative \( g_Z^{a\beta} = \mathcal{L}_Z g^{a\beta} \) is given by
\[
g_Z^{a\beta} \partial_\alpha u \partial_\beta w = (Z g^{a\beta}) \partial_\alpha u \partial_\beta w + g^{a\beta} \partial_\alpha u [Z, \partial_\beta] w + g^{a\beta} [Z, \partial_\alpha] u \partial_\beta w. \tag{10.10}
\]
Hence with the notation \( g_Z(U, V) = g_Z^{a\beta} U_\alpha V_\beta \) \( (10.9) \) respectively \( (10.1) \) become
\[
\partial_\beta Z u = -\frac{1}{2} g_Z(\partial u, \partial u), \quad \partial_\beta u = 0 \text{ where } \tilde{L}^\beta = g^{a\beta} \partial_\alpha u. \tag{10.11}
\]

We will now give a sequence of lemmas used to estimate this system.
**Lemma 20.** If $Z \in Z = \{\Omega_{ij}, \partial_i\}$ then $g_{0Z}(U, V) = 0$ and $\partial Z u^a = 0$.

Using the eikonal system (10.11) and the lemma above we obtain:

**Lemma 21.** If $Z \in \{\Omega_{ij}, \partial_i\}$ then with $h_1^{\alpha\beta} = g^{\alpha\beta} - g^{00}$ and $L_0^\alpha = g^{0\beta} \partial_\beta u^a$ we have

$$\partial_\lambda Z \tilde{u} = -\frac{1}{2} h_{1Z}(\partial u, \partial u),$$  \hspace{1cm} (10.12)

$$\partial_{L_0} \tilde{u} + \frac{1}{2} g_0(\partial \tilde{u}, \partial \tilde{u}) = -\frac{1}{2} h_1(\partial u, \partial u).$$  \hspace{1cm} (10.13)

This system gives control of all derivatives of $\tilde{u}$, first the rotations $\Omega$, by integrating (10.12) and the derivative along the outgoing light cones $L_0^\alpha$ directly from (10.13).

This gives good control of all the tangential derivatives $\tilde{\partial}$ and then we control the time derivatives by integrating (10.12), and hence all derivatives.

In order to estimate the system we need to express the above quadratic forms in a null frame: $S_1, S_2, L^* = \partial_t + \partial_r^*, L^- = \partial_t - \partial_r^*$. With respect to the dual frame

$$L^*_{\alpha} = -\partial_\alpha u^a, \quad L^-_{\alpha} = -\partial_\alpha u^a, \quad \text{where } u^a = t + r^*, \quad \text{and } A_i = \delta_{ij} A^j.$$

(10.14)

we have with $w_U = U^\alpha w_\alpha$

$$\partial_\mu = -\frac{1}{2} L^\mu_{\alpha} \partial_L^\alpha - \frac{1}{2} L^\mu_{\alpha} \partial_L^\alpha + A_\mu \partial_A, \quad w_\mu = -\frac{1}{2} L^\mu_{\alpha} w_L^\alpha - \frac{1}{2} L^\mu_{\alpha} w_L^\alpha + A_\mu w_A.$$

(10.15)

Previously we defined $k_{UV} = k^{\alpha\beta} U^\alpha V^\beta$, for $U, V \in N$. This is however equal to $k_{UV} = k^{\alpha\beta} U_\alpha V_\beta$, where $k_{\alpha\beta} = m_{\alpha\mu} m_{\beta\nu} k^{\mu\nu}$ and $U_\alpha = m_{\alpha\beta} V^\beta$ are the corresponding covectors in the dual frame. We now use the later as definition. In particular for $U^\alpha, V^\alpha \in N^\alpha$, we define $k_{U^\alpha V^\alpha} = k^{\alpha\beta} U^\alpha V^\beta$, where the corresponding vectors in the dual frame are given by (10.14). Note that the corresponding covectors in the new null frame differs at most by a factor of $dr^*/dr$ from the original null frame: $U - U^\alpha \sim M/r$. Since the good components decay at most a factor of $1/r$ better we have the same estimates for $k_{U^\alpha V^\alpha}$ as we do for $k_{UV}$. For some coefficients $k^{U^\alpha V^\alpha}$ to be calculated we have $k_{\alpha\beta} w_\alpha v_\beta = k_{U^\alpha V^\alpha} W^\alpha_{U^\alpha} V^\alpha_{V^\alpha}$:

**Lemma 22.** Let $k_{UV} = k_{\alpha\beta} U_\alpha V_\beta$ and $W_U = U^\alpha w_\alpha$. We have $k^{BD} = \delta^{AB} \delta^{CD} k_{AC}$.

$$k^{\alpha\beta} w_\alpha v_\beta = (k_{L^\alpha L^\beta} W_L^\alpha W_L^\beta + k_{L^\alpha L^\beta} (W_L^\alpha W_L^\beta + W_L^\alpha W_L^\beta)) / 4 - \delta^{AB} (k_{L^\alpha A} (W_L^\alpha W_B + W_B W_L^\alpha) + k_{L^\alpha A} (W_L^\alpha W_B + W_B W_L^\alpha)) / 2 + k^{BD} W_B V_D,$$

(10.16)

$$g^{\alpha\beta} w_\alpha w_\beta = -\frac{1}{2} (1 + \frac{M}{r}) (W_L^\alpha W_L^\beta + W_L^\alpha W_L^\beta) + (1 - \frac{M}{r}) \delta^{AB} W_A W_B,$$

(10.17)

$$g^{\alpha\beta} \partial_\alpha u \partial_\beta = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha L^\beta \partial_L^\alpha - \frac{1}{2} g^{\alpha\beta} \partial_\alpha L^\beta \partial_L^\alpha + \delta^{AB} g^{\alpha\beta} \partial_\alpha u A_\beta \partial_B.$$

(10.18)

**Lemma 23.** If $|\partial \tilde{u}| \leq 1/16$ and $|h_1| \leq 1/16$ we have with $|h_{L^\alpha A}| = |h_{L^\alpha S_1}| + |h_{L^\alpha S_1}|$

$$|\partial_\alpha \tilde{u}| \leq |h_{L^\alpha A}| + |h_{L^\alpha A}| |\partial \tilde{u}| + |\partial \tilde{u}|^2; \quad \text{where } |\partial \tilde{u}|^2 = \sum |\partial_i \tilde{u}|^2; \quad (10.19)$$

$$|k(\partial u, \partial u)| \leq |k_{L^\alpha L^\beta}| + |k_{L^\alpha A}| |\partial \tilde{u}| + (|k_{L^\alpha L^\beta}| + |k||h_{L^\alpha L^\beta}||h_{L^\alpha L^\beta}|}(|k_{L^\alpha L^\beta}| + |k||h_{L^\alpha L^\beta}| |\partial \tilde{u}|) + (|k_{L^\alpha L^\beta}| + |k_{L^\alpha A}| + |k||h_{L^\alpha L^\beta}||\partial \tilde{u}|^2 + |k||\partial \tilde{u}|^3.$$

(10.20)
Proof. Using (10.17), (10.13) becomes

\[(1 + \frac{M}{r})\partial_{L}u \partial_{L} \tilde{u} + \frac{1}{2}(1 - \frac{M}{r})|\delta \tilde{u}|^2 = -\frac{1}{2}h_1(\partial u, \partial u), \quad \partial_{L}u = 2 + \partial_{L} \tilde{u}. \quad (10.21)\]

from which it follows that \(|\partial_{L} \tilde{u}| \leq |\delta \tilde{u}|^2 + |h_1(\partial u, \partial u)|). By (10.16)

\[|k(\partial u, \partial u)| \leq |k_{L} | \partial_{L}u|^2|/4 + (2|k_{L} | \partial_{L}u| + |k_{L} | \partial_{L} \tilde{u}|)| \partial_{L}u| + 2| \partial_{L} \tilde{u}||\partial_{L} \tilde{u} + |\delta \tilde{u}| + |k_{AB} ||\delta \tilde{u}|^2. \quad (10.22)\]

(10.19) follows from this applied to \(h_1\). (10.20) follows from (10.22) and (10.19).

We now turn to estimating the quadratic terms in the right of (10.12).

Lemma 24. If \(\Omega = x^i \partial_j - x^j \partial_i\) then with \(k^a \Omega /r = k^a \omega_j - k^a \omega_i\) we have

\[
(L_{\Omega}k)(\partial u, \partial v) = (\Omega k)(\partial u, \partial v) + k(\Omega, [\partial u, \partial v]) + k(\partial u, [\Omega, \partial v]),
\]

\[k^\beta [\partial_\beta, \Omega]u = k^\beta \Omega /r \partial_r u + (k^a \partial_\beta - k^a \partial_\beta)u. \quad (10.24)\]

If \(|\partial u| \leq 1\) and \(|\partial v| \leq 1\) we have

\[|k|(\Omega, [\partial u, \partial v]| \leq |k_{L} | + |k_{L} U^\gamma ||\partial u| + (|k_{A} | + |k_{B} U^\gamma ||\partial u)|\partial v| + (|k_{L} | + |k_{L} U^\gamma ||\partial u)|\partial v|. \quad (10.25)\]

Proof. (10.24) follows from \([\partial_k, \Omega] = \delta_k \partial_j - \delta_j \partial_i\) and \(\partial_i = \omega_i \partial_r + \tilde{\partial}_i\).

Lemma 25. Suppose that \(|\delta \tilde{u}| \leq 1/16\) and \(|h_1| \leq 1/16\) and \(|Z h_1| \leq 1/16\). Then

\[|h_1 \partial (\partial u, \partial u)| \leq |h_1 \partial L_{L} + \partial_1 h_1 L_{L} + \partial_1 h_1 L_{L} + \partial_1 h_1 (h_1 L_{L} + h_1 L_{A}) |\delta \tilde{u}| + |h_1 L_{A} ||\delta \tilde{u}| + (|h_1 L_{A} + h_1 A B + |h_1 h_1 A B|| h_1 A B)| \delta \tilde{u}|^2 + |h_1 ||\delta \tilde{u}|^3. \quad (10.26)\]

\[|h_1 \partial (\partial u, \partial u)| \leq |\Omega h_1 L_{L} + h_1 L_{T} + \partial_1 |(\Omega h_1 L_{A} + h_1 A B + h_1 L_{A})| \delta \tilde{u}| + (|\Omega h_1 L_{T} + \partial_1 h_1 A B + h_1 A B|)| \delta \tilde{u}|^2 + (|\Omega h_1 + h_1|) \delta \tilde{u}|^3. \quad (10.27)\]

Proof. (10.26) follows from (10.20). (10.27) follows from (10.20) and (10.25).

We are now going to substitute our estimates for \(h_1\) into the previous lemma. Note that the estimates in Proposition 17 hold if we replace \(L\) by \(L^*\) and \(L^*\) by \(L^*_*\), since the difference is \(\sim 1/r\). Here \((\Omega h_1) U^*_V V^* \neq (h_1 U^*_V V^*)\) but the lower order terms generated are exactly the ones that show up in (10.27). (In fact we are estimating the Lie derivative which satisfy the same estimates.) We have

Lemma 26. Suppose that \(|\delta \tilde{u}| + |h_1| + |\Omega h_1| + |\partial h_1| + M \leq c_0\). Then with \(q^* = r^* - t\)

\[|h_1 \Omega (\partial u, \partial u)| \leq \frac{\epsilon}{1 + t + r^*} \left(1 + q^*_1 \right)^{\gamma'} \frac{\epsilon(1 + |q^*|)^{-\gamma} |\delta \tilde{u}|^2}{(1 + t + r^*)^{1 - \gamma}(1 + q^*_1)^{\gamma'}} + \frac{\epsilon(1 + |q^*|)^{-\gamma} |\delta \tilde{u}|^2}{(1 + t + r^*)^{1 - \gamma}(1 + q^*_1)^{\gamma'}} \quad (10.28)\]

\[|h_1 \partial_1 (\partial u, \partial u)| \leq \frac{\epsilon(1 + |q^*|)^{-\gamma} |\delta \tilde{u}|^2}{(1 + t + r^*)^{2 - \gamma}(1 + q^*_1)^{\gamma'}} + \frac{\epsilon(1 + |q^*|)^{-\gamma} |\delta \tilde{u}|^2}{(1 + t + r^*)^{1 - \gamma}(1 + q^*_1)^{\gamma'}} \quad (10.29)\]
10.1. Proof of the Uniform Bounds for $\tilde{u}$ in Proposition 24. Integrating
\[ 2\partial_T^* \Omega \tilde{u} = -h_{1\Omega}(\partial u, \partial u), \]
backwards from $t = T$ where $\Omega \tilde{u} = 0$ gives an estimate for $r |\partial \tilde{u}| = c(\sum_\Omega |\Omega \tilde{u}|^2)^{1/2}$ independent of $T$. At first we assume that $|\tilde{u}| \leq 1$ so $q^*$ changes at most by 1 along the integral curves. The integral of the first term in (10.28) is bounded by
\[
\int_t^T \frac{\varepsilon}{1 + \xi + |q^*|^2} \left( \frac{1 + q^*}{1 + \xi + |q^*|^2} \right)^{\gamma'} d\xi \leq \frac{\varepsilon}{\gamma'} \left( \frac{1 + q^*}{1 + t + |q^*|^2} \right)^{\gamma'}. \tag{10.30}
\]
Assuming that $r |\partial \tilde{u}|$ is bounded by a constant times this, the integral of the other term in (10.28) is smaller than half this if $\varepsilon$ is small, and we get back a better bound which proves the bound by continuity. Dividing by $r$ this proves
\[
|\partial \tilde{u}| \lesssim \frac{\varepsilon}{1 + t + r^*} \left( \frac{1 + q^*}{1 + t + r^*} \right)^{\gamma'}.
\]
Since the same estimate holds for $|h_{1L^* L'}|$ it follows from (10.19) that also $|\partial L^* \tilde{u}|$ is bounded by this. The estimate for $|\partial \tilde{u}|$ follows in a similar way integrating
\[ 2\partial_T^* \partial \tilde{u} = -h_{1\Omega} (\partial u, \partial u), \]
from $t = T$ where now $|\partial_t \tilde{u}| \lesssim |\partial L^* \tilde{u}|$. Using the estimate for $\partial \tilde{u}$ we see that both terms in (10.29) can be estimate by the first and the integral is estimated by
\[
\int_t^T \frac{\varepsilon}{(1 + \xi + |q^*|^2)^2-\varepsilon} (1 + |q^*|^2)^{-\varepsilon} (1 + q^*)^{\gamma'} \lesssim \frac{\varepsilon}{(1 + t + |q^*|^2)^{1-\varepsilon} (1 + |q^*|^{\epsilon} (1 + q^*)^{\gamma'}}.
\]

10.2. Proof of the Estimates for $\gamma$ in Proposition 24. Recall that $\gamma(t) = \gamma(t, q^*, \omega) = (t, x(t))$ denotes the integral curve of the vector field
\[ F^\alpha(g, \partial u) = g^{\alpha\beta} \partial \beta u / g^{0\beta} \partial \beta u, \tag{10.31} \]
going through $\gamma(T) = \gamma_0(T)$, where $\gamma_0(t) = (t, \omega_0, t) = \phi(r) + q^*$, is an integral curve of $F(g_0, u^*)$, when $g, u$ is replaced by $g_0, u^*$. Then $\tilde{\gamma}(t) = \gamma(t) - \gamma_0(t)$ satisfies
\[ d\tilde{\gamma}/dt = F(g, \partial u)^\alpha - F(g_0, \partial u^*)^\alpha = (g^{\alpha\beta} - F_0^\alpha g^{0\beta}) \partial \beta u / g^{0\beta} \partial \beta u, \tag{10.32} \]
where $F_0 = F(g_0, \partial u^*) = (1, \omega / \omega')$, where $\omega' = dr^*/dr$. Here
\[ (g^{\alpha\beta} - F_0^{\alpha\beta} g^{0\beta}) \partial \beta u = (h_1^{\alpha\beta} - F_0^{\alpha\beta} h_1^{0\beta}) \partial \beta u + (h_1^{\alpha\beta} - F_0^{\alpha\beta} h_1^{0\beta}) \partial \beta \tilde{u} + (g_0^{\alpha\beta} - F_0^{\alpha\beta} g_0^{0\beta}) \partial \beta \tilde{u}. \tag{10.33} \]
If $\alpha = 0$ this is 0 and if $\alpha = i > 0$ then $F_0^i = \omega_i / \omega'$ so
\[ (g_0^{ij} - F_0^{ij} g_0^{0\beta}) \partial \beta = \omega_j Q_i^{1-1} (1 + M/r) \partial \gamma + \delta^{ij} (1 - M/r) \tilde{\partial}^j, \quad i > 0, \tag{10.34} \]
\[ h_1^{ij} - F_0 h_1^{ij} = (\omega^i \omega_j + A^i A_k) h_1^{kj} - \omega^i Q_i^{1-1} h_1^{0\beta} = \omega^i Q_i^{1-1} h_1^\alpha L^* _\alpha + A^i A_k h_1^{kj}. \tag{10.35} \]
Hence we get the following, that integrated with respect to $t$ gives (10.6).
\[ |F(g, \partial u) - F(g_0, \partial u^*)| \lesssim |h_1 L^T| + |h_1 T^* U^*||\partial u| + |\partial u| \lesssim \frac{\varepsilon}{1 + t + |q^*|} \left(\frac{1 + q^*}{1 + t + |q^*|}\right)^{\gamma'} . \] (10.36)

10.3. Proof of Proposition 25. We can also commute with scaling \[ S^\alpha \partial_\alpha \] for components. Integrating the first equation from \[ 0 \] to \[ t \] gives (10.37).

Lemma 27. The Lie derivative \[ \mathcal{L}_{S^\alpha} k^{\alpha \beta} = S^\alpha \partial_\alpha k^{\beta} - \partial_\gamma S^\alpha k^{\gamma \beta} \] satisfies

\[ \left( \mathcal{L}_{S^\alpha} k^{\alpha \beta} + 2k^{\alpha \beta} - S^\alpha k^{\alpha \beta} \right) \partial_\alpha u \partial_\beta v = -\left( \kappa_1 - \kappa_2 \right) k^{i} \left( \partial_i u \partial_\beta v + \partial_\beta u \partial_i v \right) \]

where \[ \kappa_1 = r^* g^{i} - 1, \kappa_2 = (1 + \kappa_1) M(r - M^2/r), r^* = g(r). \]

Proof. Writing \[ S^* = S^\alpha \partial_\alpha \] we have \[ \partial_0 S^* = \partial_i S^* = 0, \partial_0 S^* = 1 \] and using (10.3)

\[ \partial_0 S^* = \partial_0 \left( \frac{r^*}{r \partial_r} x^j \right) = \frac{r^*}{r \partial_r} \partial_0 \frac{r^*}{r \partial_r} = \frac{r^*}{r \partial_r} \left( \delta^i_k - \delta^i_k \right) + \omega^i_k \partial_r \]

This proves (10.37). (10.38) follows from this noting that \[ S^* \partial_0 = M(1 + \kappa_1) \delta^i_k / r. \]

We need to estimate the scaling \[ S^* \partial_0 \] since we do not get an estimate for \[ L^* \partial_0 \partial_r \].

It follows from (10.24) and (10.37), (10.38) that with \[ L = g^{\alpha \beta} \partial_\alpha v \partial_\beta \]

\[ |L \partial_0 \partial_r | \lesssim |O h_1| + |h_1|, \quad |L S^* \partial_0 | \lesssim |S^* h_1| + |h_1| + |\partial_0 v|^2 M \ln r. \] (10.39)

There is no gain in components. Integrating the first equation from \[ t = 0 \] gives

\[ \left| \Omega \partial_0 \partial_r | \lesssim |O \partial_0 v| \right|_{t=0} + \int_{\partial_0}^{t+t^*} \frac{\varepsilon^{(1 + \eta)^{-\gamma' - \epsilon} d \eta}}{(1 + t + t^*)^{1 - \epsilon}} + \int_{t^*}^{\Omega} \frac{(1 + |\eta|)^{-\epsilon} d \eta}{(1 + t + r^*)^{1 - \epsilon}} \lesssim \frac{\varepsilon^{(1 + q^*)^1 - \epsilon}}{(1 + t + r^*)^{1 - \epsilon}} \lesssim \frac{\varepsilon}{(1 + t + r^*)^{\gamma' - \epsilon}}. \] (10.40)

The estimate for \[ S^* \partial_0 \] is the same. Integrating \( \tilde{\partial}_0 \partial_0 \) does not work. The bounds for \[ L^* \partial_0 \partial_r \] and \[ \partial_0 \] follows from \[ \tilde{\partial}_0 \partial_0 = 0 \] since \[ \tilde{\partial}_0 \partial_0 \sim g_0^{\alpha \beta} \partial_\beta v \sim L^* \partial_0 \partial_r \]. As for (10.19) we have

\[ |L^* \partial_0 \partial_r | \lesssim |h_1| + |\partial_0 v|^2, \quad \text{if} \quad |\partial_0 v| \leq 1/16. \] (10.41)

As in Sect. 10.2 \[ |d\tilde{\partial}/dt| \lesssim |h_1| + |\partial_0 \partial_0 v| \] and (10.8) follows as in (10.40).
10.4. Proof of Proposition 26 for \(|Z^{\ast} \hat{u}|\). For \(X \in \mathcal{X} = \{S^\ast \Omega_{ij}, (q^\ast) \partial_j\}\) let \(\hat{X} = X - \delta_{XS^\ast}\) and \(\hat{L}_X = L_X + 2 \delta_{XS^\ast}\), where \(\delta_{XS^\ast} = 1\) if \(X = S^\ast\), and \(= 0\) otherwise. Then \(X(k(\partial u, \partial v) = (\hat{L}_X k)(\partial u, \partial v) + k(\partial \hat{X} u, \partial v) + k(\partial u, \partial \hat{X} v)\) and \(\partial X u = 0\). Since \(g(\partial u, \partial u) = 0\) we get
\[
\partial Z \hat{u} = g(\partial u, \partial \hat{X} u) = -H(g, u)/2, \quad \text{where} \quad \hat{L}^\nu = g^{\mu \nu} \partial \nu u \text{and}
\]
\[
H(g, u) = \hat{L}_X \hat{Z}_Z g(\partial u, \partial u) + 2 \hat{L}_X g(\partial u, \partial \hat{Z} \hat{u}) + 2 \hat{Z}_Z g(\partial u, \partial \hat{X} u) + 2 g(\partial \hat{Z} \hat{u}, \partial \hat{X} u).
\]
Here \(\hat{L}_X g = \hat{L}_X g_0 + \hat{L}_X h_1\), where \(\hat{L}_X g_0 = 0\) and \(\hat{L}_X h_1 = \kappa_3 g_0 - 2(\kappa_1 - \kappa_2) g_0\).

Here \(\kappa_1 \sim M \ln r/r, \kappa_2 \sim \kappa_3 \sim M/r\) and \(g_0(\partial u, \partial v) = g_{0ij} \partial_i u \partial_j v\). If we apply this to \(g_0(\partial u^*, \partial u^*) = 0\) we get \(H(g, u^*) = (\hat{L}_X \hat{Z}_Z g_0)(\partial u^*, \partial u^*) = 0\). Moreover, \(g_0(\partial u^*, \partial w) = 0\) so \(\hat{L}_X g_0(\partial u^*, \partial w) = -g_0(\partial \hat{X} u^*, \partial w) - g_0(\partial u^*, \partial \hat{X} w) = 0\). It follows that \(\hat{L}_X g_0(\partial u^*, \partial w) \lesssim \kappa_3 |\partial L^w u|\), for \(|I| \leq 2\). Hence
\[
|H(g, u)| \lesssim (\kappa_3 + |\partial \hat{u}| + |\partial \hat{X} u| + |\partial \hat{Z} \hat{u}|)(|\partial L^u u| + |\partial \hat{L} \hat{X} u| + |\partial \hat{L} \hat{Z} \hat{u}|) + (|\partial \hat{u}| + |\partial \hat{Z} \hat{u}|)(|\partial \hat{u}| + |\partial \hat{X} u|), \quad (10.42)
\]
\[
|H(h, u)| \lesssim |(\hat{L}_X \hat{Z}_Z h_1)|_{L^w u},
\]
\[
\begin{align*}
L_X g(\partial, \partial u) &= 2 \hat{L}_X g_0(\partial u^*, \partial u) + \hat{L}_X g_0(\partial u, \partial u) \\
&= \hat{L}_X h_1(\partial u^*, \partial u^*) + \hat{L}_X h_1(\partial u, \partial u) + \hat{L}_X h_1(\partial u, \partial u).
\end{align*}
\]
Using that \(|\partial \hat{u}| + |h_1| + |X h_1| + |\hat{L}_X h_1| \leq c_0\) and \(|\partial L^u u| \lesssim |h_1 L^u u| + |h_1 L^A u| + |\partial \hat{u}| + |\partial \hat{Z} \hat{u}|^2\)
\[
|\hat{L}_X g(\partial, \partial u)| \lesssim \kappa_3 (|\partial L^u u| + |\partial \hat{u}|^2) + (|\hat{L}_X h_1|_{L^u u} + |\hat{L}_X h_1|_{L^w u})(|\partial \hat{u}| + |\partial \hat{u}|) + |\hat{L}_X h_1|_{L^w u}|\partial \hat{u}|^2
\]
\[
\lesssim (|\hat{L}_X h_1|_{L^w u} + |h_1 L^w u|) + |\partial \hat{u}|^2.
\]
By \((10.18)\) \(\partial \hat{L} \hat{X} \hat{u} = F_{L^w u} + F\partial L^u u + F^A \partial A\), where \(|F_{L^w} - 1| \leq 1/2, |F^A| \lesssim |\partial \hat{u}| + |h_1 L^A u| + |\partial L^u u| \lesssim |\partial \hat{u}| + |h_1 L^w u|\) and \(|F_{L^w} \lesssim |h_1 L^w u| + |\partial L^u u| \lesssim |h_1 L^w u| + |\partial \hat{u}|^2\) so
\[
|\partial \hat{L} \hat{X} \hat{u}| \lesssim ((\hat{L}_X h_1)_{L^w u} + (h_1 L^w u) + |\partial \hat{u}|^2)(1 + |\partial \hat{X} u|) + |\partial \hat{u}|(|\partial \hat{X} u|). \quad (10.44)
\]
If we make the inductive assumption \(|\partial \hat{X} \hat{u}| \leq c_0\) above we get from \((10.42)\)–\((10.44)\)
\[
|\partial \hat{L} \hat{X} \hat{u}| \lesssim \sum_{|I| \leq 2} ((\hat{L}_X h_1)_{L^w u} + (|\partial \hat{u}| + |\partial \hat{X} u|)(|\partial \hat{u}| + |\partial \hat{Z} \hat{u}|)), \quad (10.45)
\]
and using the estimates from Sect. 6 and the fact that \(r |\partial v| = c(|\Omega_\nu|^2)^{1/2}\)
\[
|\partial \hat{L} \hat{X} \hat{u}| \lesssim \frac{e}{1 + t + r^*} \left(1 + \frac{q^\ast}{1 + t + r^*}\right)^{r'} + \sum_{\Omega} \frac{(|\Omega \hat{u}| + |\Omega \hat{X} u| + |\Omega \hat{Z} \hat{u}|)^2}{(1 + t + r^*)^2}. \quad (10.46)
\]
The first term is the same as in (10.28)–(10.30) and there is room in the second.

10.5. Proof of Proposition 26 for \(|Z^u v^*\rangle\). To generalize the bounds in Sect. 10.3 to two vector fields we replace \(u, u^*, v, L^*\) and \(L\) in Sect. 10.4 by \(v, v^*, \tilde{v}, L^*\) and \(\tilde{L}\) respectively. We have \(|H(h_1, v)| \leq |\tilde{L} \tilde{X} h_1| + |\tilde{L} \xi h_1| + |\tilde{L} h_1| + |h_1|\) and

\[
|H(g_0, v)| \lesssim \left( \kappa_2 |\partial \tilde{v}| + |\partial \tilde{X} \tilde{v}| + |\partial \tilde{Z} \tilde{v}| \right) (|\partial \tilde{v}| + |\partial \tilde{X} \tilde{v}| + |\partial \tilde{Z} \tilde{v}|)
\]

where \(|\partial \tilde{v}| \lesssim |\tilde{L} \tilde{X} h_1| + |h_1| + |\partial \tilde{v}|^2 + |\partial \tilde{v}| |\partial \tilde{X} \tilde{v}|\). Hence

\[
|\partial \tilde{v}| \lesssim \left( |\tilde{L} \tilde{X} h_1| + |\tilde{L} \xi h_1| + |\tilde{L} h_1| + |h_1| \right) + \left( |\partial \tilde{v}| + |\partial \tilde{X} \tilde{v}| \right) (|\partial \tilde{v}| + |\partial \tilde{X} \tilde{v}|).
\]

This is of the form (10.39) and can be integrated as in (10.40). This gives the estimate in Proposition 26 for \(XZ \tilde{v}\) if \(X, Z \in \{S, \Omega\}\) and for \(\langle q^* \rangle |\hat{L}^2 \tilde{v}|\) we apply \(L^*\) to \(\partial \tilde{v} = 0\). By (10.18) \(\partial \tilde{v} = F \tilde{L} \partial \tilde{v} + F \tilde{L} \partial \hat{L} + F \partial A\), where \(|F \tilde{L}^2 - 1| \leq 1/2\) so

\[
|\partial \tilde{v}| \lesssim |\partial \tilde{L}^2 |F \tilde{T}| |\partial \tilde{T} v| + |F \tilde{T}| |\partial \tilde{v}| + |\partial \tilde{L}^2 F \tilde{L}^2| |\partial \tilde{v}|
\]

and we repeat it for the terms containing \(\partial \tilde{L}^2\). The terms generated decay at least like \((1 + t)^{-s'}(q^*)^{-2+\gamma'}\) so multiplying by \(\langle q^* \rangle^2\) gives the desired bound.

10.6. Convergence. Suppose that \(u_1\) and \(u_2\) are solutions such that \(u_1 = u^*\) when \(t = T_i\), for \(i = 1, 2\). Let \(X_1 = X_1(t, q^*, \omega) = (t, x_i(t))\) be the integral curve of \(F(g, u^*)\) in (10.31), going through \(X_i = X_0\), when \(t = T_i\), where \(X_0(t) = (t, r\omega), t = \varrho(r) + q^*\) is an integral curve of \(F(g_0, u^*)\). Then \(u_1 = u_2 = q^*\) along the curves, but \(\partial u_1 \neq \partial u_2\). Let \(W_i\) denote the covector \(w_{ia}(t) = (\partial_a u) \circ X_i(t)\) expressed in the frame \(\mathcal{N}_i = (L^*_i, \Omega_i, S_i, S_2)\), i.e \(W_i = w_{ia} U^* a\), for \(U^* \in \mathcal{N}_i\). Similarly, let \(W_i = \partial_a u\) be the covector \(w_{ia} = (\partial_a u) \circ X_i(t)\) expressed in this frame. We also write \(W = (W_{L^*}, \overline{W})\), where \(\overline{W} = (W_{L^*}, \overline{W})\) is the tangential part, and \(\overline{W} = (W_{S_1}, W_{S_2})\). The frame is chosen so that \(W_{L^*} = 2\) and \(\overline{W} = 0\). Let \(W_Z = Z^a w_{a}\). We rewrite

\[
dX^a/dt = F^a(g(X), W) = g^{aU^*}(X)W_{U^*}/g^{U^*}(X)W_{U^*},
\]

\[
dW_Z/dt = H_Z(g(X), W) = -h_{1Z}^{V^*}(X)W_{U^*}W_{V^*}/g^{U^*}(X)W_{U^*},
\]

for \(Z = \Omega_{ij}, \partial_i\). Convergence as \(T \to \infty\) follows from:

**Proposition 27.** Let \(0 < \gamma'' < \gamma'\). Then if \(\varepsilon > 0\) is sufficiently small we have for \(r > t/2\) with constants independent of \(T = \min(T_1, T_2),

\[
(1 + |q^*|)|W_2 - W_1| \leq C_2 \varepsilon \left( \frac{1 + q^*_2}{1 + t + |q^*|} \right)^{\gamma''} \left( \frac{1 + q^*_2}{1 + t + |q^*|} \right)^{\gamma''},
\]

\[
(1 + t + |q^*|)|\overline{W}_2 - \overline{W}_1| \leq C_2 \varepsilon \left( \frac{1 + q^*_2}{1 + t + |q^*|} \right)^{\gamma''} \left( \frac{1 + q^*_2}{1 + t + |q^*|} \right)^{\gamma''},
\]

\[
|X_2 - X_1| \leq C_3 \varepsilon \left( \frac{1 + q^*_2}{1 + t + |q^*|} \right)^{\gamma''} \left( \frac{1 + q^*_2}{1 + t + |q^*|} \right)^{\gamma''}.
\]
Note that $h_i^{qU^*}$ satisfy the same estimates as the corresponding components in the Minkowski null frame $h_i^{U^*}$ since the difference $U - U^* \sim 1/r$. By (10.6) $X_i(t) \in B(X_0(t), C_1 \varepsilon)$, the ball of radius $C_1 \varepsilon$ at $X_0(t)$ so estimating with $L^\infty$ norm

$$|h \circ X_2(t) - h \circ X_1(t)| \leq \|\partial h(t, \cdot)\|_{B(X_0(t), C_1 \varepsilon)} |X_2(t) - X_1(t)|.$$ 

(10.52)

**Lemma 28.** We have

$$|F(X_2, W_2) - F(X_1, W_1)| \leq \frac{C_{0 \varepsilon}|X_2 - X_1|}{(1 + t + |q^*|)^{2-\varepsilon}(1 + |q^*|)^\varepsilon} + 2|W_2 - W_1| + \frac{C_{0 \varepsilon}|W_2 - W_1|}{(1 + t + |q^*|)^{2-\varepsilon}(1 + |q^*|)^\varepsilon}. $$

(10.53)

**Proof.** Let $g_i = g \circ X_i$, $g_i^* = g_0 \circ X_i$ and $h_{1i} = h_1 \circ X_i$. We have

$$F^\alpha(g, W) - F^\alpha(g, W - V) = (g^{\alpha \beta} - F^\alpha(g, W)g^{0 \beta})V_\beta/g^{0 \beta}(W - V)_\beta. $$

(10.54)

$$F^\alpha(g + h, W) - F^\alpha(g, W) = (h^{\alpha \beta} - F^\alpha(g, W)h^{0 \beta})W_\beta/(g + h)^{0 \beta}W_\beta. $$

(10.55)

We want to use the above to estimate

$$F(g_2, W_2) - F(g_1, W_1) = F(g_2, W_2) - F(g_2, W_1) + F(g_2, W_1) - F(g_1, W_1).$$

(10.56)

We can replace $g_2 - g_1$ by $\delta h_1 = h_1 \circ X_2 - h_1 \circ X_1$ since $g_2^* - g_1^*$ is bounded by $C M r^{-2}|X_2 - X_1|$. Moreover, when applying (10.54)–(10.55) we can replace $F(g_i, W_i)$ in the right by $F(g_i^*, W_i^*)$ since the error is bounded by:

$$|F(g_2, W_2) - F(g_1, W_1)| \leq |F(g_2, W_2) - F(g_2, W_1)| + |F(g_1, W_1) - F(g_1^*, W_1^*)| |g_2 - g_1|,$$

where $F(g_i, W_i) - F(g_i^*, W_i^*)$ is controlled by (10.36), and modulo lower order $g_2 - g_1$ by $\delta h_1$, which by (10.52) is controlled by $\|\partial h_1\|_{B(X_0, C_1 \varepsilon)}$ times $\delta X = X_2 - X_1$. With $\delta W = W_2 - W_1$ it therefore remains to estimate

$$(g^{\alpha \beta}_2 - F^\alpha(g_2^*, W_2^*)g^{0 \beta}_2)\delta W_\beta \quad \text{and} \quad (\delta h_1^{\alpha \beta} - F^\alpha(g_1^*, W_1^*)\delta h_1^{0 \beta})W_\beta.$$

Moreover, using (10.35) we see using (10.16) that modulo errors controlled by $|h_{1LT}| |\delta W| + |h_{1LTE}| |\delta \mathbf{W}|$ respectively $|\delta h_{1LT}| |W| + |\delta h_{1LTE}| |\mathbf{W}|$ we can replace $g_2$ by $g^*_2$ respectively $W_1$ by $W_1^*$ so we are left with the main parts

$$(g^{\alpha \beta}_2 - F^\alpha(g_2^*, W_2^*)g^{0 \beta}_2)\delta W_\beta \quad \text{and} \quad (\delta h_1^{\alpha \beta} - F^\alpha(g_1^*, W_1^*)\delta h_1^{0 \beta})W_1^\beta.$$

Using (10.34) the first can be estimated by $\delta \mathbf{W}$ and using (10.35) the second can be estimated by $\delta h_{1LT}$, which in turn is bounded by $\|\partial h_{1LT}\|_{B(X_0, C_1 \varepsilon)} |\delta X|$. 

**Lemma 29.** If $|W_i - W_i^*| \leq 1/4$ and $|h_1| \leq 1/16$ then

$$|W_2 L^* - W_1 L^*| \lesssim \frac{C_{0 \varepsilon}(1 + q^*_\varepsilon)^{-\varepsilon}}{(1 + t)^{2-\varepsilon}(1 + |q^*|)^\varepsilon}|X_2 - X_1|. $$

(10.57)
Proof. Let $\widetilde{W}_i = W_i - W_i^*$. By (10.21)

$$(1 + \frac{M}{r})(2 + \widetilde{W}_iL^*)W_iL^* = -2h_iU^*W_iU^* - \frac{1}{2}h_{ii}(\widetilde{W}_i, \widetilde{W}_i) - \frac{1}{2}(1 - \frac{M}{r})|W|^2,$$

where $h_{ii} = h_i \circ X_i$. Subtracting the case $i = 2$ from $i = 1$ we see using (10.16) that the difference of the linear terms in the right is bounded by

$$\begin{align*}
(|h_{11L^*L^*}| + |h_{12L^*L^*}|)|\delta W| &+ (|h_{11L^*T^*}| + |h_{12L^*T^*}|)|\delta W| \\
&+ |\delta h_{1L^*T^*}| + (|W_1L^*| + |W_2L^*|)|\delta h_{1L^*L^*}|,
\end{align*}$$

where $\delta W = W_2 - W_1$ and $\delta h_1 = h_{12} - h_{11}$. The first term is small compared to the left of (10.57) and the second is bounded by the first term in the right. By (10.52) the term $|\delta h_{1L^*T^*}| \leq \|\delta h_{1L^*T^*}\|_{B(X_0, C_1)} |\delta X|$ is bounded by the second term in (10.57) and so is the last. The difference in the quadratic terms is bounded

$$((|\tilde{W}_1| + |\tilde{W}_2|)|W_2L^* - W_1L^*| + (|\tilde{W}_1| + |\tilde{W}_2|) + |h_{11L^*T^*}|)|\delta \tilde{W}_1| + |h_{12L^*T^*}|)|\delta \tilde{W}_2| |W_2 - W_1|,$$

where the first term is small compared to the left of (10.57) and the second can be estimated by the first term on the right of (10.57). In the above we neglected the change in the coefficients $M/r^2$ which is bounded by $M/r^2$ times $\delta X$ multiplied by $|W|$ which is bounded by the second term in the right of (10.57).

We will estimate $\delta H_Z = H_Z(X_2, W_2) - H_Z(X_1, W_1)$ by the $L^\infty$ norms over $D_e(t)$:

$$|\delta H_Z(t)| \leq \|\frac{\partial H_Z}{\partial X}(t, \cdot)\|_{D_e(t)}|\delta X(t)| + \sum_{U^*\in \mathcal{N}_s} \|\frac{\partial H_Z}{\partial U^*}(t, \cdot)\|_{D_e(t)}|\delta U^*(t)|,$$

where $D_e(t) = \{X \in B(X_0(t), C_1), (1 + |q^*|)|\tilde{W}| + (1 + t + |q^*|)|\tilde{W}| \leq C_1 \varepsilon \left(\frac{1 + q^*}{1 + t + |q^*|}\right)^\gamma\}$,

$$H_Z(X, W) = -\frac{4h_{1L^*L^*}}{l^2} (X) + 4h_{1L^*U^*} (X) \tilde{W}_{U^*} + h_{1U^*V^*} (X) \tilde{W}_{U^*V^*} - 2 + \frac{h_iL^*}{l^2} (X) + 4g_{0U^*}(X) \tilde{W}_{U^*}. $$

Lemma 30. We have

$$|H_{\theta_i}(X_2, W_2) - H_{\theta_i}(X_1, W_1)| \leq C_0 \varepsilon \left(\frac{|X_2 - X_1| + (1 + |q^*|)|W_2 - W_1|}{(1 + t + |q^*|)^{2-\varepsilon}(1 + |q^*|)^{1+\varepsilon}(1 + q^*_\beta)\gamma^\prime}\right),$$

Proof. Substituting $h_{1\theta_i}(V, W) = \partial_i h_{1L^*} v_\theta W_{\theta \beta}$ into (10.60) using (10.16) we get

$$|H_{\theta_i}| \leq \left(\frac{\varepsilon (1 + q^*_\beta)^{-\gamma^\prime}}{(1 + t)^{2-\varepsilon}(1 + |q^*|)^\varepsilon}\right),$$

$$|\frac{\partial H_{\theta_i}}{\partial X}| \leq \left(\frac{\varepsilon (1 + q^*_\beta)^{-\gamma^\prime}}{(1 + t)^{2-\varepsilon}(1 + |q^*|)^\varepsilon}\right),$$

$$|\frac{\partial H_{\theta_i}}{\partial W}| \leq \left(\frac{\varepsilon (1 + q^*_\beta)^{-\gamma^\prime}}{(1 + t)^{2-\varepsilon}(1 + |q^*|)^\varepsilon}\right),$$

$$|\frac{\partial H_{\theta_i}}{\partial W^*}| \leq \left(\frac{\varepsilon (1 + q^*_\beta)^{-\gamma^\prime}}{(1 + t)^{2-\varepsilon}(1 + |q^*|)^\varepsilon}\right).$$
\[ \frac{\partial H_0}{\partial W_t} \lesssim |\partial h_{1TU}| + |\partial h_1| |W_{L^*}| + |H_0| \lesssim \frac{\varepsilon(1 + |q_\ast^*|)^{-\gamma'}}{(1 + t)(1 + |q_\ast|)}. \]

**Lemma 31.** We have

\[ \sum_\Omega |H_\Omega(X_2, W_2) - H_\Omega(X_1, W_1)| \leq \frac{C_0 \varepsilon(1 + q_\ast^*)^{-\gamma'}}{(1 + t + |q_\ast|)^{2-\varepsilon}(1 + |q_\ast|)^{\varepsilon}} |X_2 - X_1| 
+ \frac{C_0 \varepsilon |W_2 - W_1|}{(1 + t + |q_\ast|)} \left( \frac{1 + q_\ast^*}{1 + t + |q_\ast|} \right)^{\gamma'} + \frac{C_0 \varepsilon(1 + q_\ast^*)^{-\gamma'}}{1 + t + |q_\ast|} |\overline{W_2} - \overline{W_1}|. \]  

(10.62)

**Proof.** If \( \Omega = \Omega_{ij} = x^i \partial_j - x^j \partial_i \), then with \( R = \omega \), we have by (10.24)

\[ \begin{aligned} h_{1\Omega}(V, W) &= (\Omega h_{1\beta}^\alpha) V_\alpha W_{\beta} + h_1^{\alpha/\beta} (V_\alpha W_\beta + V_{\beta} W_\alpha) + h_1^{\alpha R} (V_\alpha \overline{W}_{\beta/\gamma} + \overline{W}_{\gamma/\beta} W_\alpha) 
+ &h_1^{\alpha C} (V_\alpha (C^i \overline{W}_j - C^j \overline{W}_i) + (C^i \overline{W}_j - C^j \overline{W}_i) W_\alpha). \end{aligned} \]  

(10.63)

Hence

\[ |h_{1\Omega}(V, W)| \lesssim |\Omega h_1| |\overline{V}| |\overline{W}| + |\Omega h_{1TU}| |\overline{V}| |\overline{W}| + |\Omega h_1| |\overline{W}|^2 \lesssim \frac{\varepsilon (1 + q_\ast^*)^{1+}}{1 + t} \gamma'. \]

By (10.63) and (10.15) we have with \( \rho' = dr^*/dr \)

\[ \begin{aligned} h_{1\Omega}(W^*, W^*) &= \Omega h_{1\beta}^{L^*} + 2h_1^{\alpha/\beta} \rho' \overline{W}_{\beta} 
+ h_1^{C/\beta} (\overline{W}_{\beta} - \overline{W}_{\beta^*} - h_1^{\alpha R} \overline{W}_{\beta/\beta^*} + h_1^{\alpha C} \overline{W}_{\beta} (C^i \overline{W}_j - C^j \overline{W}_i), 
+ &2h_1^{\alpha C} \overline{W}_{\beta^*} (C^i \overline{W}_j - C^j \overline{W}_i), 
- \frac{\varepsilon (1 + q_\ast^*)^{1+}}{1 + t} \gamma'. \end{aligned} \]  

(10.65)

Since \( h_{1\Omega}(W, W) = h_{1\Omega}(W^*, W^*) + h_{1\Omega}(W, \overline{W}) + h_{1\Omega}(\overline{W}, \overline{W}) \) it follows that

\[ \left| \frac{\partial H_\Omega}{\partial X} \right| \lesssim \sum_{k \leq 1} |\partial \Omega^k h_{1L^* T}| + |\partial \Omega^k h_{1TU}| |\overline{W}| + |\partial \Omega^k h_1| |\overline{W}|^2 
+ |H_\Omega| \left( |\partial h_{1TU}| + |\partial h_1| |W_{L^*}| + \frac{M}{(1 + t)^2} \right) \lesssim \frac{\varepsilon (1 + q_\ast^*)^{1+}}{(1 + r)^{2-\varepsilon}(1 + |q_\ast|)^{\varepsilon}}, \]

when \( r \geq t/2 \). Moreover, since \( |W| \leq 1 \)

\[ \begin{aligned} \left| \frac{\partial H_\Omega}{\partial W} \right| &\lesssim |\Omega h_{1TU}| + |h_{1TU}| + |h_1| |W_{L^*}| + |H_\Omega| |h_{1TU}| \lesssim \frac{\varepsilon (1 + q_\ast^*)^{1+}}{1 + t}, 
\left| \frac{\partial H_\Omega}{\partial W_{L^*}} \right| &\lesssim |\Omega h_{1TU}| + |h_{1TU}| + |\Omega h_1| |W_{L^*}| + |h_1| |\overline{W}| + |H_\Omega| \lesssim \frac{\varepsilon (1 + q_\ast^*)^{1+}}{1 + t}, 
\left| \frac{\partial H_\Omega}{\partial W_{L^*}} \right| &\lesssim \sum_{k \leq 1} |\partial \Omega^k h_{1L^* T}| + |\partial \Omega^k h_{1TU}| |\overline{W}| + |H_\Omega| \lesssim \frac{\varepsilon (1 + q_\ast^*)^{1+}}{1 + t} \gamma' \end{aligned} \]
10.7. Proof of Proposition 27. By Proposition 24 these estimates are true when $t = T$ for any constant $C_2 \geq 2C_1$ and $C_3 \geq 2C_1$. We claim that if $\varepsilon > 0$ is sufficiently small they are true for $q^*_\gamma \leq t \leq T$, with $C_2 = 8C_1$ and $C_3 = 8C_2C_\gamma\nu$, for some universal constant $C_{\gamma\nu}$. Since as we shall see below we have differential equations for these quantities they are continuous functions so we can prove this by assuming that these estimates are true for $t \geq t_1$ and show that they imply better estimates as long as $t_1 \geq q^*_\gamma$. If we integrate \((10.48)\) with $Z = \Omega$ we get

$$|r^n_2 W_2 - r^n_1 W_1| \leq |r^n_2 W_2 - r^n_1 W_1|_{t = T} + \int_t^T \sum_{\Omega} |H_\Omega(X_2, W_2) - H_\Omega(X_1, W_1)| \, dt.$$  

If we use \((10.5)\) at $t = T$, \((10.62)\), \((10.57)\) and the assumed bounds \((10.49)-(10.51)\):

$$(1 + t + |q^*|)|W_2 - W_1| \leq 2C_1 \varepsilon \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma'} + C(C_2 + C_3)\varepsilon^2 \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma''} \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma''},$$

if $\varepsilon$ is sufficiently small which proves \((10.50)\). If we integrate \((10.48)\) with $Z = \partial_t$:

$$|W_2 - W_1| \leq |W_2 - W_1|_{t = T} + \int_t^T |H_\partial(X_2, W_2) - H_\partial(X_1, W_1)| \, dt.$$  

If we use \((10.4)\) at $t = T$, \((10.61)\) with the bounds \((10.49)-(10.51)\) we get as above

$$(1 + |q^*|)|W_2 - W_1| < 8C_1 \varepsilon \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma'} \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma''},$$

if $\varepsilon > 0$ is small which proves \((10.49)\). Integrating \((10.47)\) we get

$$|X_2 - X_1| \leq |X_2 - X_1|_{t = T} + \int_t^T |F(X_2, W_2) - F(X_1, W_1)| \, dt.$$  

\((10.51)\) follows if we use \((10.6)\) at $t = T$ and \((10.53)\) with \((10.49)-(10.51)\)

$$|X_2 - X_1| \leq 2C_1 \varepsilon \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma'} + (C(C_3 + C_2)\varepsilon^2 + C_2\varepsilon C_{\gamma\nu}) \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma''} \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma''},$$

$$< 2C_2 \varepsilon C_{\gamma\nu} \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma'} \left(\frac{1 + q^*_\gamma}{1 + t + |q^*|}\right)^{\gamma''}.$$

11. The Mass Loss Law

We have seen that the asymptotics of $H_{LL}$ close to the light cone is $2M$. On the other hand it is also given by the asymptotics for the wave equation with the source determined by $n$. Therefore there has to be a relation between $M$ and $n$:

**Proposition 28.** We have

$$\frac{1}{2} \int_{-\infty}^{+\infty} \int_{S^2} n(q^*, \omega) \frac{dS(\omega)}{4\pi} \, dq^* = M. \quad (11.1)$$
Proof. Since \(-L^\mu(\omega)L_\mu(\sigma) = 1 - \langle\omega,\sigma\rangle\) we have

\[
k_{LL}^2(t, r^\ast \omega) = \frac{1}{r^*} \int_{S^2} \left(1 - \langle\omega, \sigma\rangle\right)^2 n(\rho^*, \sigma) \frac{dS(\sigma)}{4\pi} \chi\left(\frac{\rho^*}{i+r^*}\right) d\rho^*,
\]

where \(q^* = r^* - t << 0\). Hence

\[
\int_{S^2} r^* k_{LL}^2(t, r^* \omega) \frac{dS(\omega)}{4\pi} = \int_{q^*}^{\infty} \int_{S^2} \frac{\omega^2 n(\rho^*, \sigma)}{2} \frac{d\omega_1}{4\pi} \chi\left(\frac{\rho^*}{i+r^*}\right) d\rho^* \\
= \int_{q^*}^{\infty} \int_{S^2} (2 - 2a + a^2 \log \left|\frac{2 + a}{a}\right|) n(\rho^*, \sigma) \frac{dS(\sigma)}{8\pi} \chi\left(\frac{\rho^*}{i+r^*}\right) d\rho^*,
\]

where \(a = (\rho^* - q^*)/r^*\). Since \(|n(\rho^*, \sigma)| \lesssim \varepsilon^2 (1 + |\rho^*|)^{-2} (1 + \rho^*)^{-\gamma'}\) it follows that

\[
\left| \int_{S^2} r^* k_{LL}^2(r^* - q^*, r^* \omega) \frac{dS(\omega)}{4\pi} - \int_{q^*}^{\infty} \int_{S^2} n(\rho^*, \sigma) \frac{dS(\sigma)}{4\pi} d\rho^* \right| \lesssim \varepsilon^2 |q^*|/r^*.
\]

Let \(\Phi_{LL}^2(q^*, \omega, r^*) = r^* k_{LL}^2(r^* - q^*, r^* \omega)\). By the previous arguments \(\Phi_{LL}^{2\infty}(q^*, \omega, r^*)\) exists and satisfies \(\Phi_{LL}^{2\infty}(q^*, \omega) - 2M \lesssim \varepsilon (1 + q^\gamma)^{-\gamma'}\). Hence if we pass to the limit \(r^* \to \infty\) in the above

\[
2M - \int_{q^*}^{\infty} \int_{S^2} n(\rho^*, \sigma) \frac{dS(\sigma)}{4\pi} d\rho^* \lesssim \frac{\varepsilon}{(1 + q^\gamma)^{\gamma'}}.
\]

Taking \(q^* \to -\infty\) proves the theorem.

The proposition in particular implies that, if \(n = 0\) then \(M = 0\), and then by the positive mass theorem the space time is Minkowski space. The proposition can be interpreted as that the total radiated energy equals the initial mass, if there is no black hole. By [BBM, C2] the radiated energy density is equal to the limit along outgoing null hypersurfaces of the square of the trace less part of the conjugate null second fundamental form of surfaces in the null hypersurface.

We define the radius of a surface \(S\) to be \(r(S) = \sqrt{\text{Area}(S)/4\pi}\). Let \(\hat{L}\) and \(\hat{\hat{L}}\) be the outgoing respectively incoming null normals to \(S\) satisfying \(g(\hat{L}, \hat{\hat{L}}) = -2\). \(\hat{L}\) and \(\hat{\hat{L}}\) are unique up to the transformation \(\hat{L} \to a\hat{L}\) and \(\hat{\hat{L}} \to a^{-1}\hat{\hat{L}}\). The null second fundamental form and the conjugate null second fundamental form are defined to be the tensors

\[
\chi(X, Y) = g(\nabla_X \hat{L}, Y) \text{ respectively } \hat{\chi}(X, Y) = g(\nabla_X \hat{\hat{L}}, Y) \text{ for any vectors } X, Y \text{ tangent to } S\text{ at a point, where } \nabla_X \text{ is covariant differentiation. Under the transformation above } \chi \to a\chi \text{ and } \hat{\chi} \to a^{-1}\hat{\chi} \text{ so the Hawking mass of } S, M_H(S) = r(S) \left(1 + \int_S \text{tr} \chi \text{ tr} \chi \text{ dS}/16\pi\right)/2 \text{ is invariant. If tr} \chi \text{ tr} \chi < 0 \text{ we can fix } \hat{L}\text{ and } \hat{\hat{L}} \text{ by tr} \chi + \text{tr} \chi = 0. \text{ Let } \hat{\chi}\text{ and } \hat{\hat{\chi}}\text{ be the traceless parts. The incoming respectively outgoing energy flux through } S\text{ are } E(S) = \int_S \hat{\chi}^2 dS/32\pi \text{ and } \hat{E}(S) = \int_S \hat{\chi}^2 dS/32\pi.
\]

Let \(C_u\) and \(C_v\) be the characteristic surfaces of constant \(u \sim -r^*\) respectively \(v \sim t + r^*\) as in Sect. 10. Let \(S_{u,v} = C_u \cap C_v\). For fixed \(v\), \(E(C_v) = \int \hat{E}(S_{u,v})\text{d}u\) is the norm of characteristic initial data on \(C_v\). The energy at null infinity is \(\int \hat{E}(u)\text{d}u\) where \(E(u)\) is the limit of \(E(S_{u,v})\) as \(v \to \infty\). The Bondi mass \(M(u)\) is the limit of \(M_H(S_{u,v})\) as
\( v \to \infty \). By the Bondi mass loss law \( dM(u)/du = -E(u) \). For asymptotically flat data \( M(u) \to M \), the ADM mass, as \( u \to -\infty \), and in the absence of a black hole, \( M(u) \to 0 \) as \( u \to +\infty \). If \( E(q^\star) = \int_{S_2^n} n(q^\star, \omega) dS(\omega)/8\pi \) then (11.1) says that \( \int_{-\infty}^{+\infty} E(u) du = M \). This will be explored in forthcoming papers.

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