SHARPENING SOME CLASSICAL NUMERICAL RADIUS INEQUALITIES

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ABSTRACT. New upper and lower bounds for the numerical radii of Hilbert space operators are given. Among our results, we prove that if \( A \in \mathcal{B}(\mathcal{H}) \) is a hyponormal operator, then for all non-negative non-decreasing operator convex \( f \) on \([0, \infty)\), we have

\[
f(\omega(A)) \leq \frac{1}{2} \left\| f \left( \frac{1}{1 + \xi_{|A|}} |A| \right) \right\| + f \left( \frac{1}{1 + \xi_{|A|}} |A^*| \right),
\]

where \( \xi_{|A|} = \inf_{\|x\|=1} \left\{ \frac{(\|A\| - |A^*|)\|x\|}{(\|A\| + |A^*|)\|x\|} \right\} \). Our results refine and generalize earlier inequalities for hyponormal operator.

1. Introduction

Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a complex Hilbert space and \(\mathcal{B}(\mathcal{H})\) denote the \(C^*\)-algebra of all bounded linear operators on \(\mathcal{H}\). For \(A \in \mathcal{B}(\mathcal{H})\), we denote by \(|A|\) the absolute value operator of \(A\), that is, \(|A| = (A^*A)^{\frac{1}{2}}\), where \(A^*\) is the adjoint operator of \(A\). A continuous real-valued function \(f\) defined on an interval \(I\) is said to be operator convex if \(f(\lambda A + (1 - \lambda) B) \leq \lambda f(A) + (1 - \lambda) f(B)\) for all self-adjoint operators \(A, B\) with spectra contained in \(I\) and all \(\lambda \in [0, 1]\).

The numerical range of an operator \(A\) in \(\mathcal{B}(\mathcal{H})\) is defined as \(W(A) = \{ \langle Ax, x \rangle : \|x\| = 1 \}\). For any \(A \in \mathcal{B}(\mathcal{H})\), \(\overline{W(A)}\) is a convex subset of the complex plane containing the spectrum of \(A\) (see [5, Chapter 2]).

Recall that \(\omega(A) = \sup_{\|x\|=1} |\langle Ax, x \rangle|\) and \(\|A\| = \sup_{\|x\|=1} \|Ax\|\). It is well-known that \(\omega(\cdot)\) defines a norm on \(\mathcal{B}(\mathcal{H})\), which is equivalent to the usual operator norm \(\|\cdot\|\). Namely, for \(A \in \mathcal{B}(\mathcal{H})\), we have

\[
\frac{1}{2} \|A\| \leq \omega(A) \leq \|A\|.
\]

Other facts about the numerical radius that we use can be found in [6].

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The inequalities in (1.1) have been improved considerably by many authors, (see, e.g., [1, 8, 9, 15, 16, 17]), Kittaneh [12, 14] has shown the following precise estimates of \( \omega(A) \) by using several norm inequalities and ingenious techniques:

\[ \omega(A) \leq \frac{1}{2} \left( \|A\| + \|A^2\|^\frac{1}{2} \right), \]

and

\[ \frac{1}{4} \left( \|A\|^2 + \|A^*\|^2 \right) \leq \omega^2(A) \leq \frac{1}{2} \left( \|A\|^2 + \|A^*\|^2 \right). \]

In [3], Dragomir gave the following estimate of the numerical radius which refines the second inequality in (1.1): For every \( A \),

\[ \omega^2(A) \leq \frac{1}{2} \left( \omega(A^2) + \|A\|^2 \right). \]

In this paper, we establish a considerable improvement of the second inequality in (1.3). We also propose a new upper bound for \( \omega(\cdot) \) for the hyponormal operators. Next, we will give a refinement of the first inequality in (1.1).

2. Upper bounds for the numerical radii

The following lemma is known as the mixed Schwarz inequality (see [7, pp. 75-76]).

**Lemma 2.1.** If \( A \in B(\mathcal{H}) \), then

\[ |\langle Ax, y \rangle| \leq \langle |A| x, x \rangle^{\frac{1}{2}} \langle |A^*| y, y \rangle^{\frac{1}{2}}, \]

for all \( x, y \in \mathcal{H} \).

The second lemma is a norm inequality for the sum of two positive operators, which can be found in [13].

**Lemma 2.2.** If \( A \) and \( B \) are positive operators in \( B(\mathcal{H}) \), then

\[ \|A + B\| \leq \max(\|A\|, \|B\|) + \left\| A^{\frac{1}{2}} B^{\frac{1}{2}} \right\|. \]

The following lemma contains a simple inequality, which will be needed in the sequel.

**Lemma 2.3.** For each \( \alpha \geq 1 \), we have

\[ \frac{\alpha - 1}{\alpha + 1} \leq \ln \alpha. \]

**Proof.** Taking \( f(\alpha) \equiv \ln \alpha - \frac{\alpha - 1}{\alpha + 1} \), where \( \alpha \geq 1 \). By an elementary computation we have \( f'(\alpha) \geq 0 \), so \( f(\alpha) \) is an increasing function for \( \alpha \geq 1 \). On the other hand \( f(\alpha) \geq f(1) = 0 \). \( \square \)
Now, we are ready to present our new improvement of the second inequality in (1.3). Recall that, an operator $A$ defined on a Hilbert space $\mathcal{H}$ is said to be hyponormal if $A^*A - AA^* \geq 0$, or equivalently if $\|A^*x\| \leq \|Ax\|$ for every $x \in \mathcal{H}$.

**Theorem A.** Let $A \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator. Then, for all non-negative non-decreasing operator convex $f$ on $[0, \infty)$, we have

\begin{equation}
(2.2) \quad f(\omega(A)) \leq \frac{1}{2} \left\| f \left( \frac{1}{1 + \frac{\xi_{|A|}}{8}} \right) + f \left( \frac{1}{1 + \frac{\xi_{|A^*|}}{8}} \right) \right\|
\end{equation}

where $\xi_{|A|} = \inf_{\|x\|=1} \left\{ \frac{\langle |A| - |A^*|, x, x \rangle}{\langle |A| + |A^*|, x, x \rangle} \right\}$.

**Proof.** Since $A$ is a hyponormal operator we have $1 \leq \frac{\langle |A|, x, x \rangle}{\langle |A^*|, x, x \rangle}$, for each $x \in \mathcal{H}$. On choosing $\alpha = \frac{\langle |A|, x, x \rangle}{\langle |A^*|, x, x \rangle}$ in (2.1) we get

\begin{equation}
(0 \leq \frac{\langle |A| - |A^*|, x, x \rangle}{\langle |A| + |A^*|, x, x \rangle} \leq \ln \frac{\langle |A|, x, x \rangle}{\langle |A^*|, x, x \rangle}.
\end{equation}

Whence

\begin{equation}
(2.3) \quad \inf_{\|x\|=1} \frac{\langle |A| - |A^*|, x, x \rangle}{\langle |A| + |A^*|, x, x \rangle} \leq \ln \frac{\langle |A|, x, x \rangle}{\langle |A^*|, x, x \rangle}.
\end{equation}

We denote the expression on the left-hand side of (2.3) by $\xi_{|A|}$. On the other hand Zou et al. in [18] proved that for each $a, b > 0$,

\begin{equation}
\left( 1 + \frac{(\ln a - \ln b)^2}{8} \right) \sqrt{ab} \leq \frac{a + b}{2}.
\end{equation}

By taking $a = \langle |A|, x, x \rangle$ and $b = \langle |A^*|, x, x \rangle$ and taking into account that $\xi_{|A|} \leq \ln \frac{\langle |A|, x, x \rangle}{\langle |A^*|, x, x \rangle}$, we infer that

\begin{equation}
\sqrt{\langle |A|, x, x \rangle \langle |A^*|, x, x \rangle} \leq \frac{1}{2 \left( 1 + \frac{\xi_{|A|}}{8} \right)} \langle |A| + |A^*|, x, x \rangle.
\end{equation}

By using Lemma 2.1, we get

\begin{equation}
\langle Ax, x \rangle \leq \frac{1}{2 \left( 1 + \frac{\xi_{|A^*|}}{8} \right)} \langle |A| + |A^*|, x, x \rangle.
\end{equation}

Now, by taking supremum over $x \in \mathcal{H}, \|x\| = 1$, we get

\begin{equation}
\omega(A) \leq \frac{1}{2 \left( 1 + \frac{\xi_{|A|}}{8} \right)} \| |A| + |A^*| \|. 
\end{equation}
Therefore,
\[
f(\omega(A)) \leq f \left( \frac{1}{2 \left( 1 + \frac{\xi^2 |A|}{8} \right)} \| |A| + |A^*| \| \right)
\]
\[
= \left\| f \left( \frac{1}{2 \left( 1 + \frac{\xi^2 |A|}{8} \right)} |A| + \frac{1}{2 \left( 1 + \frac{\xi^2 |A|}{8} \right)} |A^*| \right) \right\|
\]
\[
\leq \frac{1}{2} \left\| f \left( \frac{1}{1 + \frac{\xi^2 |A|}{8}} |A| \right) + f \left( \frac{1}{1 + \frac{\xi^2 |A|}{8}} |A^*| \right) \right\|.
\]
This completes the proof. □

**Remark 2.1.** Notice that, if \( A \) is a normal operator, then \( \xi_{|A|} = 0 \).

An important special case of Theorem A, which leads to an improvement and a generalization of inequality (1.3) for hyponormal operators, can be stated as follows.

**Corollary 2.1.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be a hyponormal operator. Then, for all \( 1 \leq r \leq 2 \) we have
\[
\omega^r (A) \leq \frac{1}{2 \left( 1 + \frac{\xi^2 |A|}{8} \right)} \| |A|^r + |A^*|^r \|,
\]
where \( \xi_{|A|} = \inf \left\{ \frac{\langle |A| - |A^*| \rangle x, x \rangle}{\langle |A| + |A^*| \rangle x, x \rangle} \right\} \). In particular,
\[
(2.4) \quad \omega(A) \leq \frac{1}{2 \left( 1 + \frac{\xi^2 |A|}{8} \right)} \| |A| + |A^*| \|,
\]
and
\[
\omega^2 (A) \leq \frac{1}{2 \left( 1 + \frac{\xi^2 |A|}{8} \right)^2} \| A^* A + A A^* \|.
\]

An operator norm inequality which will be used in next corollary says that for any positive operators \( A, B \in \mathcal{B}(\mathcal{H}) \), we have (see [2])
\[
(2.5) \quad \| A^r B^r \| \leq \| A B \|^r, \quad \text{for all} \quad 0 \leq r \leq 1.
\]
The following result refines and generalizes inequality (1.2) for hyponormal operators.

**Corollary 2.2.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be a hyponormal operator. Then
\[
\omega^r (A) \leq \frac{1}{2 \left( 1 + \frac{\xi^2 |A|}{8} \right)^r} \left( \| |A|^r + \| |A^*|^r \| \right),
\]
where \( \xi_{|A|} = \inf \left\{ \frac{\langle |A| - |A^*| \rangle x, x \rangle}{\langle |A| + |A^*| \rangle x, x \rangle} \right\} \).
for all $1 \leq r \leq 2$. In particular

$$
\omega^r (A) \leq \frac{1}{2 \left( 1 + \frac{\epsilon^2}{8} \right)} \left( \|A\|^r + \|A^2\|^r \right),
$$

for $1 \leq r \leq 2$.

**Proof.** Applying Corollary 2.1 and Lemma 2.2, we have

$$
\omega^r (A) \leq \frac{1}{2 \left( 1 + \frac{\epsilon^2}{8} \right)} \left( \|A\|^r + \|A^r\|^r \right)
$$

$$
\leq \frac{1}{2 \left( 1 + \frac{\epsilon^2}{8} \right)} \left( \max \left( \|A\|^r, \|A^r\|^r \right) + \left\| \|A^r\| A^r \right\| \right)
$$

$$
= \frac{1}{2 \left( 1 + \frac{\epsilon^2}{8} \right)} \left( \|A\|^r + \left\| \|A^r\| A^r \right\| \right).
$$

For the particular applying inequality (2.5), we have

$$
\left\| \|A^r\| A^r \right\| \leq \|A\| \|A^r\| = \|A^2\|^r,
$$

for $1 \leq r \leq 2$. □

Recently, Kian [11] improved Jensen’s operator inequality via superquadratic functions. As an application, he showed that the following inequality is valid:

**Lemma 2.4.** [11, Example 3.6] Let $A_1, \ldots, A_n$ be positive operators, then

$$
\left\| \sum_{i=1}^n w_i A_i \right\|^r \leq \left\| \sum_{i=1}^n w_i A_i^r \right\| - \inf_{\|x\|=1} \left\{ \sum_{i=1}^n w_i \left\langle A_i - \sum_{j=1}^n w_j \langle A_j x, x \rangle \right\| x, x \right\}, \quad r \geq 2,
$$

for each $w_1, \ldots, w_n$ with $\sum_{i=1}^n w_i = 1$.

This, in turn, leads to the following:

**Theorem B.** Let $A \in B (\mathcal{H})$, then

$$
(2.6) \quad \omega^2 (A) \leq \frac{1}{2} \left( \|A\|^2 + \|A^*\|^2 \right) - \inf_{\|x\|=1} \xi (x),
$$

where $\xi (x) = \left\langle \left( \|A\| - \frac{1}{2} \left( \|A\| + \|A^*\| \right) x, x \right)^2 + \|A^*\| - \frac{1}{2} \left( \|A\| + \|A^*\| \right) x, x \right\rangle x, x$. 
Proof. One can easily see that for each \( A \in \mathcal{B}(\mathcal{H}) \) we have
\[
\omega(A) \leq \frac{1}{2} \| |A| + |A^*|| ,
\]
we can also write
\[
(2.7) \quad \omega^2(A) \leq \frac{1}{4} \| |A| + |A^*||^2.
\]
Choosing \( n, r = 2, \ w_1 = w_2 = \frac{1}{2}, \ A_1 = |A| \) and \( A_2 = |A^*| \) in Lemma 2.4, we infer
\[
\| |A| + |A^*||^2 \leq 2 (\| |A| + |A^*||^2) - \inf_{\|x\| = 1} \left\{ \left( |A| - \frac{1}{2} (|A| x, x) + |A^*| x, x \right) |^2 x, x \right\}.
\]

It now follows from (2.7) that
\[
\omega^2(A) \leq \frac{1}{4} \| |A| + |A^*||^2
\leq \frac{1}{2} (\| |A| + |A^*||^2 - \inf_{\|x\| = 1} \left\{ \left( |A| - \frac{1}{2} (|A| x, x) + |A^*| x, x \right) |^2 x, x \right\}
\leq \left( |A^*| - \frac{1}{2} (|A| x, x) + |A^*| x, x \right) |^2 x, x \right\}.
\]

The validity of this inequality is just Theorem B.

Remark 2.2. Notice that
\[
\inf_{\|x\| = 1} \xi(x) > 0 \iff 0 \not\in W \left( |A| - \frac{1}{2} (|A| + |A^*|) x, x \right) + |A^*| - \frac{1}{2} (|A| + |A^*|) x, x \right) ^2\right).
\]

To make things a bit clearer, we consider the following example:

Example 2.1. Taking \( A = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} \). By an easy computation we find that
\[
\left( |A| - \frac{1}{2} (|A| + |A^*|) x, x \right) ^2 + |A^*| - \frac{1}{2} (|A| + |A^*|) x, x \right) ^2 = \begin{pmatrix} 4.5 & 0 \\ 0 & 4.5 \end{pmatrix}.
\]

It is well-known that, \( A = \lambda I \) if and only if \( W(A) = \{ \lambda \} \) (see, e.g., [10, Section 18]). So we get
\[
\inf_{\|x\| = 1} \xi(x) = 4.5 > 0.
\]

This shows that the inequality (2.6) provides an improvement for the second inequality in (1.3).
3. Lower bounds for the numerical radii

The next theorem is slightly more intricate.

**Theorem C.** Let \( A \in B(\mathcal{H}) \), then

\[
\|A\| \left( 1 - \frac{1}{2} \left\| I - \frac{A}{\|A\|} \right\|^2 \right) \leq \omega(A).
\]

**Proof.** It is easy to check that

\[
1 - \frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \leq \frac{1}{\|x\| \|y\|} |\langle x, y \rangle|,
\]

for every \( x, y \in \mathcal{H} \).

If we choose \( \|x\| = \|y\| = 1 \) in (3.2) we get

\[
1 - \frac{1}{2} \|x - y\|^2 \leq |\langle x, y \rangle|.
\]

This is an interesting inequality in itself as well. Now taking \( y = \frac{Ax}{\|Ax\|} \) in (3.3), we infer

\[
\|Ax\| \left( 1 - \frac{1}{2} \left\| I - \frac{A}{\|A\|} \right\|^2 \right) \leq |\langle Ax, x \rangle|.
\]

Since \( \|x\| = 1 \), \( \|Ax\| \) does not exceed \( \|A\| \). Hence we get from (3.4) that

\[
\|Ax\| \left( 1 - \frac{1}{2} \left\| I - \frac{A}{\|A\|} \right\|^2 \right) \leq |\langle Ax, x \rangle|.
\]

Now by taking supremum over \( x \in \mathcal{H} \) with \( \|x\| = 1 \), we deduce the desired inequality (3.1). \( \square \)

**Remark 3.1.** It is striking that if \( \|A - \|A\|| \leq \|A\| \), then inequality (3.1) provides an improvement for the first inequality in (1.1).

**Example 3.1.** Taking \( A = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix} \). Then \( \|A\| \simeq 4.1594 \) and \( \|A - \|A\|| \| \simeq 2.3807 \). We obtain by easy computation

\[
\frac{1}{2} \|A\| \simeq 2.079, \quad \|A\| \left( 1 - \frac{1}{2} \left\| I - \frac{A}{\|A\|} \right\|^2 \right) \simeq 2.968, \quad \omega(A) \simeq 4.118,
\]

whence

\[
\frac{1}{2} \|A\| \leq \|A\| \left( 1 - \frac{1}{2} \left\| I - \frac{A}{\|A\|} \right\|^2 \right) \leq \omega(A),
\]

which shows that if \( \|A - \|A\|| \leq \|A\| \), then inequality (3.1) is really an improvement of the first inequality in (1.1).
The following basic lemma is essentially known as in [4, Lemma 1], but our expression is a little bit different from those in [4]. For the sake of convenience, we give it a slim proof.

**Lemma 3.1.** Let \( x, y, z_i, i = 1, \ldots, n \) be nonzero vectors and \( \langle z_j, z_i \rangle \neq 0 \), then

\[
\left| \left\langle x - \sum_i \frac{\langle x, z_i \rangle}{\sum_j |\langle z_j, z_i \rangle|} z_i, y \right\rangle \right|^2 \leq \|y\|^2 \left( \|x\|^2 - \sum_i \frac{|\langle x, z_i \rangle|^2}{\sum_j |\langle z_i, z_j \rangle|} \right).
\]  

**Proof.** Define

\[
u = x - \sum_i \frac{\langle x, z_i \rangle}{\sum_j |\langle z_j, z_i \rangle|} z_i.
\]

Whence

\[
\|\nu\|^2 = \left\| x - \sum_i a_i z_i \right\|^2 \leq \|x\|^2 - \sum_i \frac{|\langle x, z_i \rangle|^2}{\sum_j |\langle z_i, z_j \rangle|}.
\]

By multiplying both sides (3.6) by \( \|y\|^2 \) and then utilizing the Cauchy Schwarz inequality we get

\[
|\langle \nu, y \rangle|^2 \leq \|y\|^2 \left( \|x\|^2 - \sum_i \frac{|\langle x, z_i \rangle|^2}{\sum_j |\langle z_i, z_j \rangle|} \right),
\]

which is exactly desired inequality (3.5). \( \square \)

Finally, we state the last result.

**Theorem D.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be an invertible operator, then

\[
\inf_{\|x\|=1} \xi^2(x) + \omega^2(A) \leq \|A\|^2,
\]

where \( \xi(x) = \frac{|\langle A^2 x, x \rangle - \langle Ax, x \rangle^2|}{\|A^* x\|} \).

**Proof.** Simplifying (3.5) for the case \( n = 1 \), we find that

\[
\left| \langle x, y \rangle - \frac{\langle x, z \rangle}{\|z\|^2} \langle z, y \rangle \right|^2 + \frac{|\langle x, z \rangle|^2}{\|z\|^2} \|y\|^2 \leq \|x\|^2 \|y\|^2.
\]

Apply these considerations to \( x = Ax, y = A^* x \) and \( z = x \) with \( \|x\| = 1 \) we deduce

\[
\left( \frac{|\langle A^2 x, x \rangle - \langle Ax, x \rangle^2|}{\|A^* x\|} \right)^2 + |\langle Ax, x \rangle|^2 \leq \|Ax\|^2.
\]

We denote the first expression on the left-hand side of (3.7) by \( \xi(x) \). Whence (3.7) implies that

\[
\inf_{\|x\|=1} \xi^2(x) + |\langle Ax, x \rangle|^2 \leq \|Ax\|^2.
\]

Now, the result follows by taking the supremum over all unit vectors in \( \mathcal{H} \). \( \square \)
Remark 3.2. Of course, if $A$ is a normal operator we must have $\xi(x) = 0$. In this regard, we have:

(i) If $A$ is a normal matrix and $x$ is an eigenvector of $A$ with the eigenvalue $e$, then $\langle A^2x, x \rangle - \langle Ax, x \rangle^2 = e^2 - e^2 = 0$.

(ii) Let $\sigma(A)$ and $\sigma_{\text{ap}}(A)$ be the spectrum and approximate spectrum of $A$, respectively. It is well-known that the spectrum of a normal operator has a simple structure. More precisely, if $A$ is normal, then we have $\sigma(A) = \sigma_{\text{ap}}(A)$. If we assume that $e$ is in the approximate point spectrum of normal operator $A$, then there is a sequence $x_n \in \mathcal{H}$ with $\|x_n\| = 1$ and $\langle Ax_n, x_n \rangle \to e$ as $n \to \infty$. Therefore $\lim_{n \to \infty} |\langle A^2x_n, x_n \rangle - \langle Ax_n, x_n \rangle^2| = 0$.

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