BRASSELET NUMBER AND NEWTON POLYGONS

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ABSTRACT. We present a formula to compute the Brasselet number of \( f : (Y, 0) \to (\mathbb{C}, 0) \) where \( Y \subset X \) is a non-degenerate complete intersection in a toric variety \( X \). As applications we establish several results concerning about invariance of the Brasselet number for families of non-degenerate complete intersections. Moreover, when \( (X, 0) = (\mathbb{C}^n, 0) \) we derive sufficient conditions to obtain the invariance of the Euler obstruction for families of complete intersections with isolated singularity which are contained on \( X \).

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1. INTRODUCTION

Given a germ of an analytic function \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) with isolated critical point at the origin, an important invariant of this germ is its Milnor number [Mil68], denoted by \( \mu(f) \). The Milnor number is considered a central invariant, since it provides algebraic, topological and geometric information from the germ \( f \). For instance, the Milnor number coincides with the number of Morse points of a morseification of \( f \).

Initially the Milnor number was associated to germs of analytic functions \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) with isolated critical point, and consequently to study isolated hypersurfaces singularities. However this invariant is well defined in many other contexts, for example curves [BuGr80], isolated complete intersection singularities, or ICIS [Ham71], and determinantal varieties with codimension two [SRDSP14], to name just a few.

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Let us now denote by \((X, 0)\) a germ of analytic singular space embedded in \(\mathbb{C}^n\) and \(f : (X, 0) \to (\mathbb{C}, 0)\) a germ of analytic function with isolated critical point at the origin. In this situation, Brasselet, Massey, Parameswaran and Seade introduced an invariant associated to \(f\) called the Euler obstruction of \(f\) \([BMPS04]\) and denoted by \(E_{u,f,X}(0)\). Roughly speaking, \(E_{u,f,X}(0)\) is the obstruction to extending a lifting of the conjugate of the gradient vector field of \(f\) as a section of the Nash bundle of \((X, 0)\). This invariant is closely related with the Euler obstruction of \(X\), what explains its name. The Euler obstruction was defined by MacPherson in \([MAC74]\) for the construction of characteristic classes of singular complex algebraic varieties. Thereafter, the Euler obstruction has been deeply investigated by many authors such as Brasselet and Schwartz \([BrSc81]\), Dutertre \([Dut16]\), Gaffney, Grulha and Ruas \([GGJR]\), Gonzalez-Sprinberg \([GS79]\), Lê and Teissier \([LêTe81]\), Massey \([Mas]\), Seade, Tibăr and Verjovsky \([STV05]\), among others.

An important result, proved first in \([BrSc81]\) and then by several other authors, is that the local Euler obstruction is a constructible function, which means that, it is constant along the strata of a Whitney stratification of \(X\). This is essentially a consequence of the topological triviality of \(X\) on Whitney strata. An important consequence of this fact is that the Euler obstruction does not depend on the Whitney stratification of \(X\). Indeed, this can be concluded, for example, from the following Lefschetz-type formula proved by Brasselet, Lê and Seade \([BLS00]\).

**Theorem 1.1.** Let \((X, 0) \subset (\mathbb{C}^n, 0)\) be an equidimensional complex analytic singularity germ with a Whitney stratification \(\{V_i\}\), then given a generic linear form \(l\), there is \(\varepsilon_0\) such that for any \(\varepsilon \) with \(0 < \varepsilon < \varepsilon_0\), we have

\[
E_{u,f,X}(0) = \sum_i \chi(V_i \cap B_\varepsilon \cap f^{-1}(\delta)) \cdot E_{u,X}(V_i),
\]

where \(\chi\) is the Euler-Poincaré characteristic, \(E_{u,X}(V_i)\) is the value of the Euler obstruction of \(X\) at any point of the stratum \(V_i\), and \(0 < |\delta| \ll \varepsilon \ll 1\).

The previous Theorem says that the Euler obstruction, as a constructible function on \(X\) satisfies the Euler condition relatively to a generic linear function.

For the Euler obstruction of an analytic function \(f : (X, 0) \to (\mathbb{C}, 0)\) with isolated critical point at the origin, there is also a Lefschetz-type formula. This formula was proved in \([BMPS04]\). The purpose of the authors was to understand what prevents the local Euler obstruction from satisfying the local Euler condition with respect to functions which are singular at the origin.

**Theorem 1.2.** Let \((X, 0) \subset (\mathbb{C}^n, 0)\) be an equidimensional complex analytic singularity germ with a Whitney stratification \(\{V_i\}\), and let \(f : (X, 0) \to (\mathbb{C}, 0)\) be a function with an isolated singularity at 0. Then,

\[
E_{u,f,X}(0) = E_{u,X}(0) - \left( \sum_i \chi(V_i \cap B_\varepsilon \cap f^{-1}(\delta)) \cdot E_{u,X}(V_i) \right),
\]

where \(0 < |\delta| \ll \varepsilon \ll 1\).
We point out that this formula gives the relation between the Euler obstruction of $X$ and the Euler obstruction of $f$.

Seade, Tibăr and Verjovsky continued the study of the properties of $\text{Eu}_{f,X}(0)$ in [STV05]. The authors compared $\text{Eu}_{f,X}(0)$ to the highest Betti number of the Milnor fiber of $f$ obtaining the following equality.

**Proposition 1.3.** Let $(X, 0)$ be an equidimensional complex analytic singularity germ of dimension $d$ and $f : (X, 0) \to (\mathbb{C}, 0)$ a germ of an analytic function with isolated critical point at the origin. Then,

$$\text{Eu}_{f,X}(0) = (-1)^d n_{\text{reg}}$$

where $n_{\text{reg}}$ is the number of Morse points in the regular part of $X$ appearing in a stratified morsefication of $f$.

Therefore, the Euler obstruction of $f$ is the number of Morse points of a morsefication of $f$ on the regular part of $X$, up to sign. Hence this invariant can be seen as a generalization of the Milnor number of $f$.

Another invariant associated with a germ of an analytic function $f : (X, 0) \to (\mathbb{C}, 0)$ is the Brasselet number introduced by Dutertre and Grulha in [DuGr14] and denoted by $B_{f,X}(0)$. If $f$ has an isolated critical point, the Brasselet number satisfies the equality

$$B_{f,X}(0) = \text{Eu}_X(0) - \text{Eu}_{f,X}(0).$$

If $f$ is linear and generic, it gives $\text{Eu}_X(0)$, hence it can be viewed as a generalization of the Euler obstruction. Moreover, it has interesting meanings, even if $f$ has a non-isolated singularity. For instance, it has a Lê-Greuel type formula, that affirms that, under some conditions, the difference of the Brasselet number between two functions $f$ and $g$ is measured by the number of Morse critical points on the top stratum of the Milnor fiber of $f$ [DuGr14, Theorem 4.4].

Although their importance, the invariants mentioned above are not easily computed using their definition. Many authors proposed formulas which make the computation easier [BLS00, BMPS04, LêTe81, Dut16].

In order to understand better these ideas, some authors worked on more specific situations. For instance, in the special case of toric surfaces, an interesting formula for the local Euler obstruction was proved by Gonzalez-Sprinberg [GS76], this formula was generalized by Matsui and Takeuchi [MATA11A] for normal toric varieties of any dimension.

Toric varieties are particularly interesting objects, we can, for example, highlight its relation with elemental convex geometry. We have on these varieties an action of the algebraic torus $(\mathbb{C}^*)^n$, with this action we have a finite decomposition of the variety in orbits, all homeomorphic to torus.

In [VAR76] Varchenko described the topology of the Milnor fiber of a function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ using the geometry of the Newton polygon of $f$, and consequently, the Milnor number can be expressed by volumes of polytopes related to the Newton polygon of $f$. In his prove, he constructed a toric modification of $\mathbb{C}^n$ on which the pull-back of $f$ defines a hypersurface with only normal crossing
singularities. Whereas \( \mathbb{C}^n \) is a very special smooth toric variety, it would be natural to generalize his formula to Milnor fibers over general singular toric varieties. This was done by Matsui and Takeuchi in [MaTa11b].

We use [MaTa11b] to establish several combinatorial formulas for the computation of the Brasselet number of \( f : (Y, 0) \rightarrow (\mathbb{C}, 0) \) where \( Y \subset X \) is a non-degenerate complete intersection in a toric variety \( X \). Those formulas, will be given in terms of volumes of Newton polygons associated to \( f \).

This paper is organized as follows. In Section 2 we present some background material concerning the Brasselet number and toric varieties, which will be used in the entire work. In Section 3, we compute the Brasselet number of a polynomial function \( f : (X, 0) \rightarrow (\mathbb{C}, 0) \), where \( X \subset \mathbb{C}^n \) is a toric variety. Moreover, we compute this invariant for functions defined on \( X^g \), where \( g : X \rightarrow \mathbb{C}^k \) is a non-degenerate complete intersection. As a consequence, assuming that \( g \) have isolated critical point on \( X \) and on \( X^f \), we also obtain a formula for the number of stratified Morse critical points on the top stratum of the Milnor fiber of \( f \) appearing in a morseification of \( g : X \cap f^{-1}(\delta) \cap B_\varepsilon(0) \rightarrow \mathbb{C} \). As applications we establish several results concerning about constance of these invariants. In Section 4 we consider the case where \( (X, 0) = (\mathbb{C}^n, 0) \) and we derive sufficient conditions to obtain the constance of the Euler obstruction for families of complete intersections with isolated singularity which are contained on \( X \). We use this result to study the constance of the Bruce-Roberts’s Milnor number for families of functions defined on hypersurfaces. In Section 5 we work in the case of surfaces, i.e., in the case where \( X \) is a toric variety 2-dimensional. In this situation, we present a characterization of the polynomials functions \( g : X \rightarrow \mathbb{C} \) which have stratified isolated singularity at the origin. We use this characterization to present some examples for a class of toric surface that is also determinantal.

2. Preliminaries notions and results

For convenience of the reader and to fix some notations we present some general facts in order to establish our results.

2.1. Stratifications and Brasselet number. Considering a complex analytic germ \( (X, 0) \subset (\mathbb{C}^n, 0) \), an analytic function \( f : (X, 0) \rightarrow (\mathbb{C}, 0) \), which is the restriction of an analytic function \( F : U \rightarrow \mathbb{C} \), where \( U \subset \mathbb{C}^n \) is an open set which contains \( X \), and assuming that \( X \) is equipped with a good stratification relative to \( f \), Dutertre and Gruilha [DuGr14] defined the Brasselet number, \( B_{f,X}(0) \), and, under some conditions, obtained a Lê-Greuel formula for this invariant. If \( f \) has an isolated critical point, the Brasselet number satisfies the equality

\[
B_{f,X}(0) = Eu_X(0) - Eu_{f,X}(0).
\]

If \( f \) is linear and generic, it gives \( Eu_X(0) \), hence it can be viewed as a generalization of the Euler obstruction.

In order to introduce the definition and properties of the Brasselet number, we need of some notions concerning about stratifications, that we present below. For more details, we refer to Massey [Mas96, Mas07].
We denote by $X^f$ the set $X \cap f^{-1}(0)$.

**Definition 2.1.** A good stratification of $X$ relative to $f$ is a stratification $\mathcal{V}$ of $X$ which is adapted to $X^f$, such that $\{V_\alpha \in \mathcal{V} \mid V_\alpha \not\subset X^f\}$ is a Whitney stratification of $X \setminus X^f$, and besides that, for any pair of strata $(V_\alpha, V_\beta)$ such that $V_\alpha \not\subset X^f$ and $V_\beta \subset X^f$, the $(\alpha ; \beta )$-Thom condition is satisfied. We call the strata included in $X^f$ the good strata.

By [Lê 73], given a stratification, $\mathcal{S}$ of $X$, one can refine $\mathcal{S}$ to obtain a Whitney stratification $\mathcal{V}$ of $X$ which is adapted to $X^f$. Moreover, by [Mas 07], the refinement $\mathcal{V}$ is a $(\alpha _f)$ stratification. This means that, good stratifications always exist.

Given a stratification $\mathcal{V} = \{V_i\}$ of $X$ we define the following:

**Definition 2.2.** Let $f : X \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $F : U \to \mathbb{C}$. The critical locus of $f$ relative to $\mathcal{V}$, denoted by $\Sigma_\mathcal{V}f$, is the union of the critical locus of $f$ restricted to each of the strata, i.e., $\Sigma_\mathcal{V}f = \bigcup_i \Sigma(f|_{V_i})$.

We remark that the definition of critical locus of $f$ relative to a complex analytic stratification $\mathcal{V}$ is the same even if $\mathcal{V}$ is not necessarily Whitney.

A critical point of $f$ relative to $\mathcal{V}$ is a point $p \in \Sigma_\mathcal{V}f$. If the stratification $\mathcal{V}$ is clear, we refer to the elements of $\Sigma_\mathcal{V}f$ simply as stratified critical points of $f$.

If $p$ is an isolated point of $\Sigma_\mathcal{V}f$, we call $p$ a stratified isolated critical point of $f$ (with respect to $\mathcal{V}$).

For instance, if $\mathcal{V}$ is a Whitney stratification of $X$ and $f : X \to \mathbb{C}$ has a stratified isolated critical point at the origin, then

$$\left\{V_\alpha \setminus X^f, V_\alpha \cap X^f \setminus \{0\}, \{0\}; \ V_\alpha \in \mathcal{V}\right\},$$

is a good stratification for $f$. We call it the good stratification induced by $f$.

Durtertre and Grulha [DuGr14] defined the Brasselet number as follows.

**Definition 2.3.** Suppose that $X$ is equidimensional. Let $\mathcal{V} = \{V_i\}_{i=0}^q$ be a good stratification of $X$ relative to $f$. The Brasselet number, $B_{f,X}(0)$, is defined by

$$B_{f,X}(0) = \sum_{i=1}^q \chi(V_i \cap B_\varepsilon(0) \cap f^{-1}(\delta)) \cdot \text{Eu}_X(V_i),$$

where $0 < |\delta| < \varepsilon \ll 1$.

If $f$ has a stratified isolated critical point at the origin and $X$ is equidimensional the Theorem 1.2 implies that

$$B_{f,X}(0) = \text{Eu}_X(0) - \text{Eu}_{f,X}(0). \quad (2.1)$$

The Brasselet number has many interesting properties. For instance, it has several multiplicity formulas, which enable the authors to establish in [DuGr14] a relative version of the local index formula and a Gauss-Bonnet formula for $B_{f,X}(0)$. However, one of the most important properties of this invariant, is that it has a Lê-Greuel type formula, which affirms that the difference of the Brasselet number between two functions $f$ and $g$ is measure by the number of Morse critical points on the top stratum of the Milnor fiber of $f$ [DuGr14, Theorem 4.4]. For this last
property, we need to impose some conditions in the functions to ensure that $X_0$ meets $X'$ in a nice way. So it is necessary to define.

**Definition 2.4.** Let $\mathcal{V}$ be a good stratification of $X$ relative to $f$. We say that $g : (X, 0) \to (\mathbb{C}, 0)$ is prepolar with respect to $\mathcal{V}$ at the origin if the origin is an isolated critical point of $g$.

The condition of $g$ be prepolar means that it has an isolated critical point (in the stratified sense), both on $X$ and on $X'$, and that $X_0$ transversely intersects each stratum of $\mathcal{V}$ in a neighbourhood of the origin, except perhaps at the origin itself. However, its important to note that, while $X_0$ meets $X'$ in a nice way, $X'$ may have arbitrarily bad singularities when restricted to $X_0$. The $(v_1)$-Thom condition in the Definition 2.3 together with the hypothesis of $g$ be prepolar guarantee that $g : X \cap f^{-1}(\delta) \cap B_\varepsilon(0) \to \mathbb{C}$ has no critical points on $(g = 0)$ [Mas96, Proposition 1.12] and so the number of stratified Morse critical points on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\varepsilon(0)$ appearing in a morseification of $g : X \cap f^{-1}(\delta) \cap B_\varepsilon(0) \to \mathbb{C}$ does not depend on the morseification.

The following result shows that the Brasselet number satisfy a Lê-Greuel type formula [DuGr14, Theorem 4.4].

**Theorem 2.5.** Suppose that $X$ is equidimensional and that $g$ is prepolar with respect to $\mathcal{V}$ at the origin. Then,

$$B_{t,X}(0) - B_{t,X_0}(0) = (-1)^{d-1}n_q,$$

where $n_q$ is the number of stratified Morse critical points on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\varepsilon(0)$ appearing in a morseification of $g : X \cap f^{-1}(\delta) \cap B_\varepsilon(0) \to \mathbb{C}$, and $0 < |\delta| \ll \varepsilon \ll 1$. In particular, this number is independent on the morseification.

### 2.2. Toric Varieties.

The theory of Toric Varieties can be seen as a cornerstone for the interaction between combinatorics and algebraic geometry, which relates the combinatorial study of convex polytopes with algebraic torus actions. Moreover, for polynomial functions defined in such varieties, it is possible to obtain a combinatorial description of the topology of their Milnor fibers in terms of their Newton polygon. The reader may consult [FUL93, Oda88] for an overview about Toric Varieties and for the description of Milnor fibers in Newton polygon see [MA111B, Oka97].

Let $N \cong \mathbb{Z}^d$ be a $\mathbb{Z}$-lattice of rank $d$ and $\sigma$ a strongly convex rational polyhedral cone in $N_\mathbb{R} = \mathbb{R} \otimes \mathbb{Z} N$. We denote by $M$ the dual lattice of $N$ and the polar cone $\sigma^\ast$ of $\sigma$ in $M_\mathbb{R} = \mathbb{R} \otimes \mathbb{Z} M$ by

$$\sigma^\ast = \{v \in M_\mathbb{R}; \langle u, v \rangle \geq 0 \text{ for any } u \in \sigma\},$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^d$. Then the dimension of $\sigma^\ast$ is $d$ and we obtain a semigroup $S_\sigma := \sigma \cap M$

**Definition 2.6.** A $d$-dimensional affine Toric Variety $X_\sigma$ is defined by the spectrum of $\mathbb{C}[S_\sigma]$, i.e., $X = \text{Spec}(\mathbb{C}[S_\sigma])$.

The algebraic torus $T = \text{Spec}(\mathbb{C}[M]) \cong (\mathbb{C}^*)^d$ acts naturally on $X_\sigma$ and the $T$-orbits in $X_\sigma$ are indexed by the faces $\Delta$ of $\sigma^\ast (\sigma \prec \sigma^\ast)$. We denote by $\mathbb{L}(\Delta)$ the
smallest linear subspace of $M_{\mathbb{R}}$ containing $\Delta$. For a face $\Delta$ of $\sigma$, denote by $T_{\Delta}$ the $T$-orbit in $\text{Spec}(\mathbb{C}[M \cap L(\Delta)])$ which corresponds to $\Delta$. We observe that the $d$-dimensional affine toric varieties are exactly those $d$-dimensional affine, normal varieties admitting a $(\mathbb{C}^*)^d$-action with an open, dense orbit homeomorphic to $(\mathbb{C}^*)^d$. Besides, each $T$-orbit $T_{\Delta}$ is homeomorphic to $(\mathbb{C}^*)^r$, where $r$ is the dimension of $L(\Delta)$.

Therefore we obtain a decomposition $X_{\sigma} = \bigsqcup_{\Delta \prec \sigma} T_{\Delta}$ into $T$-orbits, which are homeomorphic to algebraic torus $(\mathbb{C}^*)^r$. Due to this fact, and also the informations coming from the combinatorial residing in these varieties, many questions that was originally studies for functions defined on $\mathbb{C}^d$ can be extended to functions defined on Toric Varieties. In this work we will study some invariants of functions defined on $X_{\sigma}$. Then, let us introduce some concepts that will be necessary.

Consider $f : X_{\sigma} \to \mathbb{C}$ a polynomial function on $X_{\sigma}$, i.e., a function that corresponds to an element $f = \sum_{v \in S_\sigma} a_v \cdot v$ of $\mathbb{C}[S_\sigma]$, where $a_v \in \mathbb{C}$.

**Definition 2.7.** Let $f = \sum_{v \in S_\sigma} a_v \cdot v$ be a polynomial function on $X_{\sigma}$.

(a) The set $\{v \in S_\sigma; \ a_v \neq 0\} \subset S_\sigma$ is called the support of $f$ and we denote it by $\text{supp} f$;

(b) The Newton polygon $\Gamma_+(f)$ of $f$ is the convex hull of

$$\bigcup_{v \in \text{supp} f} (v + \delta) \in \delta.$$

Now let us fix a function $f \in \mathbb{C}[S_\sigma]$ such that $0 \not\in \text{supp} f$, i.e., $f : X_{\sigma} \to \mathbb{C}$ vanishes at the $T$-fixed point $0$. Considering $M(S_\sigma)$ the $\mathbb{Z}$-sublattice of rank $d$ in $M$ generated by $S_\sigma$ we have that each element $v$ of $S_\sigma \subset M(S_\sigma)$ is identified with a $\mathbb{Z}$-vector $v = (v_1, \ldots, v_d)$ and to any $g = \sum_{v \in S_\sigma} b_v \cdot v \in \mathbb{C}[S_\sigma]$ we can associate a Laurent polynomial $L(g) = \sum_{v \in S_\sigma} b_v \cdot x^v$ on $T = (\mathbb{C}^*)^d$, where $x^v := x_1^{v_1}x_2^{v_2}\cdots x_d^{v_d}$.

**Definition 2.8.** We say that $f = \sum_{v \in S_\sigma} a_v \cdot v \in \mathbb{C}[S_\sigma]$ is non-degenerate if for any compact face $\gamma$ of $\Gamma_+(f)$ the complex hypersurface

$$\{ x = (x_1, \ldots, x_d) \in (\mathbb{C}^*)^d; \ L(f_\gamma)(x) = 0 \}$$

in $(\mathbb{C}^*)^d$ is smooth and reduced, where $f_\gamma := \sum_{v \in \gamma \cap S_\sigma} a_v \cdot v$.

We can also study non-degeneracy in case of complete intersections defined on $X_{\sigma}$. Let $f_1, f_2, \ldots, f_k \in \mathbb{C}[S_\sigma] \ (1 \leq k \leq d = \dim X_\sigma)$ and consider the following subvarieties of $X_{\sigma}$:

$$V := \{ f_1 = \cdots = f_{k-1} = f_k = 0 \} \subset W := \{ f_1 = \cdots = f_{k-1} = f_k = 0 \}.$$

Assume that $0 \in V$. For each face $\Delta \prec \sigma$ such that $\Gamma_+(f_k) \cap \Delta \neq \emptyset$, we set

$$I(\Delta) = \{ j = 1, 2, \ldots, k-1; \Gamma_+(f_j) \cap \Delta \neq \emptyset \} \subset \{1, 2, \ldots, k-1 \}$$

and $m(\Delta) = \sharp I(\Delta) + 1$.

Let $L(\Delta)$ and $M(S_\sigma \cap \Delta)$ be as before and $L(\Delta)^*$ the dual vector space of $L(\Delta)$. Then $M(S_\sigma \cap \Delta)^*$ is naturally identified with a subset of $L(\Delta)^*$ and the polar cone $\Delta = \{ u \in L(\Delta)^*; \ \langle u, v \rangle \geq 0 \ \text{for any} \ v \in \Delta \}$ of $\Delta$ in $L(\Delta)^*$ is a rational polyhedral convex cone with respect to the lattice $M(S_\sigma \cap \Delta)^*$ in $L(\Delta)^*$.
Definition 2.9. (i) For a function \( f = \sum_{v} a_v \cdot v \in \mathbb{C}[S_\sigma] \) on \( X_\sigma \) and \( u \in \Delta \), we set
\[
f_{\Delta} = \sum_{v \in \Gamma_{+}(f) \cap \Delta} a_v \cdot v \in \mathbb{C}[S_\sigma \cap \Delta]
\]
d and for any face \( \Delta \prec \bar{\Delta} \) such that \( \Gamma_{+}(f) \cap \Delta \neq \emptyset \), we define the \( \Gamma_{+}(f) \cap \Delta \)-content of \( f \) by
\[
\Gamma(f_{\Delta}; u) = \{ v \in \Gamma_{+}(f) \cap \Delta; \langle u, v \rangle = \min \langle u, w \rangle, \text{ for } w \in \Gamma_{+}(f) \cap \Delta \}.
\]
We call \( \Gamma(f_{\Delta}; u) \) the supporting face of \( u \) in \( \Gamma_{+}(f) \cap \Delta \).

(ii) For \( j \in I(\Delta) \cup \{ k \} \) and \( u \in \bar{\Delta} \), we define the \( u \)-part \( f_{\Delta}^{u} \in \mathbb{C}[S_\sigma \cap \Delta] \) of \( f_{\Delta} \) by
\[
f_{\Delta}^{u} = \sum_{v \in \Gamma(f_{\Delta}^{u}; u)} a_v \cdot v \in \mathbb{C}[S_\sigma \cap \Delta],
\]
where \( f_{\Delta} = \sum_{v \in \Gamma_{+}(f)} a_v \cdot v \in \mathbb{C}[S_\sigma] \).

By taking a \( \mathbb{Z} \)-basis of \( M(S_\sigma) \) and identifying the \( u \)-parts \( f_{\Delta}^{u} \) with Laurent polynomials \( L(f_{\Delta}^{u}) \) on \( T = (\mathbb{C}^*)^{d} \) as before, we have the following definition which does not depend on the choice of the \( \mathbb{Z} \)-basis of \( M(S_\sigma) \).

Definition 2.10. We say that \( (f_{1}, \ldots, f_{k}) \) is non-degenerate if for any face \( \Delta \prec \bar{\Delta} \) such that \( \Gamma_{+}(f_{k}) \cap \Delta \neq \emptyset \) (including the case where \( \Delta = \bar{\Delta} \)) and any \( u \in \text{Int}(\Delta) \cap M(S_\sigma \cap \Delta)^{\ast} \) there are two subvarieties of \( (\mathbb{C}^*)^{d} \) are non-degenerate complete intersections
\[
\left\{ x \in (\mathbb{C}^*)^{d}; L(f_{i}^{u})(x) = 0, \forall j \in I(\Delta) \right\};\ \left\{ x \in (\mathbb{C}^*)^{d}; L(f_{i}^{u})(x) = 0, \forall j \in I(\Delta) \cup \{ k \} \right\}.
\]

For these non-degenerate singularities, it is possible describe their geometrical and topological properties by the combinatorics. This is done in [MaTa11B] using mixed volume as follows.

For each face \( \Delta \prec \bar{\Delta} \) of \( \bar{\Delta} \) such that \( \Gamma_{+}(f_{k}) \cap \Delta \neq \emptyset \), let us set
\[
f_{\Delta} = \left( \prod_{j \in I(\Delta)} f_{j} \right) \cdot f_{k} \in \mathbb{C}[S_\sigma]
\]
and consider its Newton polygon \( \Gamma_{+}(f_{\Delta}) = \left\{ \sum_{j \in I(\Delta)} \Gamma_{+}(f_{j}) \right\} \). Let \( \gamma_{1}, \ldots, \gamma_{d}(\Delta) \) be the compact faces of \( \Gamma_{+}(f_{\Delta}) \) such that \( \dim \gamma_{i}^{\Delta} = \dim \Delta - 1 \). For each \( 1 \leq i \leq \nu(\Delta) \) there exists a unique primitive vector \( u_{i}^{\Delta} \in \text{Int}(\Delta) \cap M(S_\sigma \cap \Delta)^{\ast} \) which takes its minimal in \( \Gamma_{+}(f_{\Delta}) \cap \Delta \) exactly on \( \gamma_{i}^{\Delta} \).

For \( j \in I(\Delta) \cup \{ k \} \), set \( \gamma_{(j)}^{\Delta} := \Gamma(f_{j}^{u} \cap \Delta; u_{i}^{\Delta}) \) and \( \mathfrak{d}_{\Delta} := \min_{w \in \Gamma_{+}(f_{k}) \cap \Delta} \langle u_{i}^{\Delta}, w \rangle \). Note that we have
\[
\gamma_{i}^{\Delta} = \left\{ \sum_{j \in I(\Delta) \cup \{ k \}} \gamma_{(j)}^{\Delta} \right\}
\]
for any face \( \Delta \prec \bar{\Delta} \) such that \( \Gamma_{+}(f_{k}) \cap \Delta \neq \emptyset \) and \( 1 \leq i \leq \nu(\Delta) \). For each face \( \Delta \prec \bar{\Delta} \) such that \( \Gamma_{+}(f_{k}) \cap \Delta \neq \emptyset \), \( \dim \Delta \geq \mathfrak{m}(\Delta) \) and \( 1 \leq i \leq \nu(\Delta) \), we set
\[
I(\Delta) \cup \{ k \} = \{ j_{1}, j_{2}, \ldots, j_{\mathfrak{m}(\Delta)}-1, k = j_{\mathfrak{m}(\Delta)} \}
\]
and
\[
K_{\Delta} := \sum_{\alpha_{1}+\cdots+\alpha_{\mathfrak{m}(\Delta)} = \dim \Delta - 1} \text{Vol}_{Z}(\gamma_{(f_{j_{1}}^{\Delta})}^{\Delta}, \ldots, \gamma_{(f_{j_{\mathfrak{m}(\Delta)}}^{\Delta})}^{\Delta}, \gamma_{(f_{j_{1}}^{\Delta})}^{\Delta}, \ldots, \gamma_{(f_{j_{\mathfrak{m}(\Delta)}}^{\Delta})}^{\Delta}).
\]
Here
\[
\text{Vol}_Z(\gamma(f_{j_1})^\Delta_{i_1}, \ldots, \gamma(f_{j_1})^\Delta_{i_1}, \ldots, \gamma(f_m(\Delta))_{i_1}, \ldots, \gamma(f_m(\Delta))_{i_1})
\]
\[\vdots
\]}
\[\text{is the normalized } (\dim \Delta - 1)\text{-dimensional mixed volume with respect to the lattice } M(S_\sigma \cap \Delta) \cap L(\gamma_\Delta^f) \text{ (see Definition 2.6, pg 205 from [GKZ08]). For } \Delta \text{ such that } \dim \Delta - 1 = 0, \text{ we set}
\]
\[
K_\Delta^f = \text{Vol}_Z(\gamma(f_k)^\Delta_{i_1}, \ldots, \gamma(f_k)^\Delta_{i_1}) := 1
\]
(in this case \(\gamma(f_k)^\Delta_{i_1}\) is a point).

3. The Brasselet number and Torus actions

Considering a complex analytic germ \((X, 0)\) and an analytic function \(f : (X, 0) \to (\mathbb{C}, 0)\), the Brasselet number \(B_{f,X}(0)\) can be viewed as a generalization of the Euler obstruction \(\text{Eu}_X(0)\). It has a Lê-Greuel type formula, which affirms that, the difference of the Brasselet number between two functions \(f\) and \(g\) is measure by the number of Morse critical points on the top stratum of the Milnor fiber of \(f\) (see Theorem 2.5), where \(g\) is a prepolar function. Using Newton polygons, we present formulas for the computation of the Brasselet number of a function defined on non-degenerate complete intersections contained in toric varieties. As applications we establish several results concerning about its invariance for families of non-degenerate complete intersections.

Let \(X_\sigma \subset \mathbb{C}^n\) be a \(d\)-dimensional toric variety and \((f_1, \ldots, f_k) : (X_\sigma, 0) \to (\mathbb{C}^k, 0)\) a non-degenerate complete intersection, with \(1 \leq k \leq d\). From now on we will denote by \(g\) the complete intersection \((f_1, \ldots, f_{k-1})\) and by \(f\) the function \(f_k\).

**Theorem 3.1.** Let \(X_\sigma \subset \mathbb{C}^n\) be a \(d\)-dimensional toric variety and \((g, f) : (X_\sigma, 0) \to (\mathbb{C}^k, 0)\) a non-degenerate complete intersection. Then,
\[
B_{f,X_\sigma}(0) = \sum_{\Gamma = \{f(\Delta) \neq 0 \mid \dim \Delta \geq m(\Delta)}} (-1)^{\dim \Delta - m(\Delta)} \left( \sum_{i=1}^{\nu(\Delta)} d_i \cdot K_\Delta^f \right) \cdot \text{Eu}_{X_\sigma}^g(T_\Delta \cap X_\sigma^f).
\]

**Proof.** Let \(\mathcal{T}\) be the decomposition of \(X_\sigma = \bigsqcup_{\Delta \subset \partial} T_\Delta\) into \(T\)-orbits, and \(\mathcal{T}_g\) the decomposition of \(X_\sigma^g = \bigsqcup_{\Delta \subset \partial} T_\Delta \cap X_\sigma^f\), which is a Whitney stratification of \(X_\sigma^g\), since \(\mathcal{T}\) is a Whitney stratification of \(X_\sigma\) and \(g\) is a non-degenerate complete intersection on \(X_\sigma\). Besides that, as \((g, f)\) is a non-degenerate complete intersection, \(f\) is a polynomial function non-degenerate, then
\[
\mathcal{T}_{(g,f)} = \left\{ (T_\Delta \cap X_\sigma^g) \cap X_\sigma^f, (T_\Delta \cap X_\sigma^g) \setminus X_\sigma^f, \{0\}; \ T_\Delta \prec \partial \right\}
\]
is a Whitney stratification of \(X_\sigma^g\) which is adapted to \(X_\sigma^g \cap f^{-1}(0)\). Then, \(\mathcal{T}_{(g,f)}\) is a good stratification of \(X_\sigma^g\) relative to \(f|_{X_\sigma^g}\). Lastly, let us observe that \(f \equiv 0\) on \(T_\Delta\) for any face \(\Delta\) such that \(g_+(\Delta) \cap \Delta = \emptyset\), then we can neglect those faces. Therefore, the result follows from [MATA11B, Theorem 3.12].
If \( f : X^d_\sigma \to \mathbb{C} \) has stratified isolated critical point, then Equation (2.1) holds. Altogether, we have.

**Corollary 3.2.** Let \( X_\sigma \subset \mathbb{C}^n \) be a d-dimensional toric variety and \((g, f) : (X_\sigma, 0) \to (\mathbb{C}^k, 0)\) a non-degenerate complete intersection. If \( f : X^d_\sigma \to \mathbb{C} \) has an isolated singularity at the origin, then

\[
\text{Eu}_{f, X^d_\sigma}(0) = \text{Eu}_{X^d_\sigma}(0) - \sum_{\Gamma_\sigma \cap \Delta \neq \emptyset} (-1)^{\dim \Delta} \left( \sum_{i=1}^{\nu(\Delta)} d^\Delta_i \cdot K^\Delta_i \right),
\]

When \( k = 1 \), using exactly the same argument we used in Theorem 3.1, we obtain \( B_{f, X^d_\sigma}(0) \). In fact, for each face \( \Delta \subset \sigma \) such that \( \Gamma_\sigma \cap \Delta \neq \emptyset \), let \( \beta_1^\Delta, \beta_2^\Delta, \ldots, \beta_{\mu(\Delta)}^\Delta \) be the compact faces of \( \Gamma_\sigma \cap \Delta \) such that \( \dim \beta_i^\Delta = \dim \Delta - 1 \). Let \( \Gamma_i^\Delta \) be the convex hull of \( \beta_i^\Delta \cup (0) \) in \( \mathbb{R} \langle \Delta \rangle \) and consider the normalized \((\dim \Delta)\)-dimensional volume \( \text{Vol}_\mathbb{Z}(\Gamma_i^\Delta) \in \mathbb{Z} \) of \( \Gamma_i^\Delta \) with respect to the lattice \( M(S_\sigma \cap \Delta) \), where \( M(S_\sigma \cap \Delta) \) denote the sublattice of \( M(S_\sigma) \) generated by \( S_\sigma \cap \Delta \). Then we have the following result.

**Proposition 3.3.** Assume that \( f = \sum_{\nu \in S_\sigma} a_\nu \cdot \nu \in \mathbb{C}[S_\sigma] \) is non-degenerate. Then

\[
B_{f, X^d_\sigma}(0) = \sum_{\Gamma_\sigma \cap \Delta \neq \emptyset} (-1)^{\dim \Delta - 1} \left( \sum_{i=1}^{\mu(\Delta)} \text{Vol}_\mathbb{Z}(\Gamma_i^\Delta) \right) \cdot \text{Eu}_{X^d_\sigma}(T_\Delta).
\]

**Proof.** For each \((\dim \Delta - 1)\)-compact face \( \beta_i^\Delta \) of \( \Gamma_\sigma \cap \Delta \neq \emptyset \), with \( 1 \leq i \leq \mu(\Delta) \) we have

\[
K_i^\Delta := \text{Vol}_\mathbb{Z}(\beta_i^\Delta),
\]

and from [GKZ08, Proposition 2.7] we now that

\[
\text{Vol}_\mathbb{Z}(\beta_i^\Delta) = \text{Vol}_\mathbb{Z}(\Gamma_i^\Delta).
\]

Therefore, the result follows from the fact that

\[
\text{Vol}_\mathbb{Z}(\Gamma_i^\Delta) = d_i^\Delta \cdot K_i^\Delta,
\]

for \( 1 \leq i \leq \mu(\Delta) \).

We will apply Theorem 3.1 in order to show that the Brasselet number is invariant for some families of complete intersections. For this we will introduce some new concepts and notations.

**Definition 3.4.** A deformation of a map germ \( f : (X, 0) \to (\mathbb{C}^k, 0) \) is another map germ \( F : (\mathbb{C} \times X) \to (\mathbb{C}^k, 0) \) such that \( F(0, x) = f(x), \) for all \( x \in X \).

We assume that \( F \) is origin preserving, that is, \( F(t, 0) = 0 \) for all \( t \in \mathbb{C} \), so we have a 1-parameter family of map germs \( f_t : (X, 0) \to (\mathbb{C}^k, 0) \) given by \( f_t(x) = F(t, x) \). Moreover, associated to the family \( f_t : (X, 0) \to (\mathbb{C}^k, 0) \) we have the family \( X^{f_t} = X \cap f_t^{-1}(0) \) of subvarieties of \( X \).
In the particular case of a polynomial function $f : (X, 0) \to (C, 0)$, any polynomial deformation $f_t$ can be written as:

$$f_t(x) = f(x) + \sum_{i=1}^{r} \theta_i(t) \cdot h_i(x)$$  \hspace{1cm} (3.2)

for some polynomials $h_i : (X, 0) \to (C, 0)$ and $\theta_i : (C, 0) \to (C, 0)$, where $\theta_i(0) = 0$, for all $i = 1, \ldots, r$.

Given a polynomial function $f : X_\sigma \to C$ defined on a toric variety $X_\sigma$, and a family as in Eq. (3.2), if

$$\Gamma_+(h_i) \subset \Gamma_+(f), \text{ for all } i = 1, \ldots, r$$

and besides that, if for each face $\Delta \prec \delta$, which satisfy $\Gamma_+(h_i) \cap \Delta \neq \emptyset$, we have $\gamma^\Delta_{i_1} \cap \beta^\Delta_{j_1} = \emptyset$, for all $l = 1, \ldots, i_\nu(\Delta)$ and $j = 1, \ldots, \mu(\Delta)$, where $\gamma^\Delta_{i_1}, \gamma^\Delta_{i_2}, \ldots, \gamma^\Delta_{i_\nu(\Delta)}$ and $\beta^\Delta_{j_1}, \beta^\Delta_{j_2}, \ldots, \beta^\Delta_{j_{\mu(\Delta)}}$ are the compact faces of $\Gamma_+(h_i) \cap \Delta$ and of $\Gamma_+(f) \cap \Delta$, respectively, such that $\dim \gamma^\Delta_{i_1} = \dim \beta^\Delta_{j_1} = \dim \Delta - 1$, then

$$\Gamma_{f_t} = \Gamma_f, \text{ for all } t \in C.$$

In this case, we fix the notation

$$\Gamma_+(h_i) \subseteq \Gamma_+(f),$$  \hspace{1cm} (3.3)

for all $i = 1, \ldots, r$.

In the sequence we present some applications of Theorem 3.1.

**Corollary 3.5.** Let $X_\sigma \subset C^n$ be a $d$-dimensional toric variety and $(g, f) : (X_\sigma, 0) \to (C^k, 0)$ a non-degenerate complete intersection. If $(g, f_t)$ is a family of non-degenerate complete intersections, where, for all $t \in C$,

$$f_t(x) = f(x) + \sum_{i=1}^{r} \theta_i(t) \cdot h_i(x)$$

is a polynomial function on $X_\sigma$, and $h_i$ satisfies the condition Eq. (3.3) for all $i = 1, \ldots, r$.

Then $B_{f_t, X_\sigma}(0)$ is constant for the family.

**Proof.** As we have already noted, for each face $\Delta \prec \delta$ such that $\Gamma_+(f) \cap \Delta \neq \emptyset$, the Newton polygon $\Gamma_+(f_\Delta)$ of the function

$$f_\Delta = \left( \prod_{j \in I(\Delta)} f_j \right) \cdot f \in C[S_\sigma]$$

is

$$\left\{ \sum_{j \in I(\Delta)} \Gamma_+(f_j) \right\} + \Gamma_+(f) \subset \delta.$$ Therefore, $\Gamma_+(f_\Delta) = \Gamma_+(f_{\Delta t})$, for all $t \in C$, where

$$f_{\Delta t} = \left( \prod_{j \in I(\Delta)} f_j \right) \cdot f_{t_1},$$

since $\Gamma_+(f) = \Gamma_+(f_t)$ and the result follows by Theorem 3.1. \hfill \square
Roughly speaking the Brasselet number depends only of the monomials of smallest degree in each variable.

Given $g$ and $f$ non-degenerate polynomial functions on $X_\sigma$, in general, we have no way to relate the Euler obstructions $E_{u_{X_\sigma}}(T_\Delta \cap X_\sigma^0)$ to the Euler obstructions $E_{u_{X_\sigma}}(T_\Delta)$. However, if we assume the additional hypothesis that $g$ has isolated critical point at $0$ both in $X_\sigma$ and in $X_\sigma^f$ (in stratified sense), the following result holds.

**Theorem 3.6.** Let $X_\sigma \subset \mathbb{C}^n$ be a $d$-dimensional toric variety and $(g, f) : (X_\sigma, 0) \to (\mathbb{C}^2, 0)$ a non-degenerate complete intersection. If $g$ is prepolar with respect to $T_f$ at the origin, then

$$B_{f,X_\sigma^f}(0) = \sum_{\Gamma_+(f) \cap \Delta \neq \emptyset} (-1)^{\dim \Delta - 2} \sum_{i=1}^{\nu(\Delta)} d_i^\Delta \cdot K_i^\Delta \cdot E_{u_{X_\sigma}}(T_\Delta).$$

**Proof.** Given the good stratification $\mathcal{T}_{(g,f)}$ of $X_\sigma^0$ relative to $f$, we know that,

$$B_{f,X_\sigma^f}(0) = \sum_{\Gamma_+(f) \cap \Delta \neq \emptyset} \chi(W_\Delta \cap B_{\varepsilon}(0) \cap f^{-1}(\delta)) \cdot E_{u_{X_\sigma}}(W_\Delta),$$

where $W_\Delta$ are the strata $(T_\Delta \cap X_\sigma^f) \setminus X_\sigma^f$ (which are not contained in $\{f = 0\}$), and $0 < |\delta| < \varepsilon \ll 1$. Moreover, for $\Delta < \delta$, we have $E_{u_{X_\sigma}}(T_\Delta) = E_{u_{X_\sigma}}(W_\Delta)$, since $X_\sigma^0$ intersects the strata of $T_f$ transversally (see [DuGr14, pg 137]). Hence,

$$B_{f,X_\sigma^f}(0) = \sum_{\Gamma_+(f) \cap \Delta \neq \emptyset} \chi(X_\sigma^0 \cap T_\Delta \cap B_{\varepsilon}(0) \cap f^{-1}(\delta)) \cdot E_{u_{X_\sigma}}(T_\Delta).$$

Besides, $\Gamma_+(g) \cap \Delta \neq \emptyset$ for any face $0 \preceq \Delta < \delta$, since $g$ is prepolar with respect to $T_f$ at the origin, then $m(\Delta) = 2$, for all face $\Delta < \delta$ such that $\Gamma_+(f) \cap \Delta \neq \emptyset$. Finally, applying [MATa11b, Theorem 3.12], we obtain the result.

Therefore, if $g$ is prepolar with respect to

$$\mathcal{T}_f = \left\{ T_\Delta \cap X_\sigma^f, T_\Delta \setminus X_\sigma^f, \{0\}; \Delta \prec \delta \right\},$$

we can obtain a more general version of Corollary 3.5, since we can relate the Euler obstructions $E_{u_{X_\sigma}}(T_\Delta \cap X_\sigma^0)$ to the Euler obstructions $E_{u_{X_\sigma}}(T_\Delta)$.

**Corollary 3.7.** Let $X_\sigma \subset \mathbb{C}^n$ be a $d$-dimensional toric variety and $(g, f) : (X_\sigma, 0) \to (\mathbb{C}^2, 0)$ a non-degenerate complete intersection. If

$$\left( g_s(x), f_t(x) \right) = \left( g(x) + \sum_{i=1}^{m} \ell_i(s) \cdot l_i(x), f(x) + \sum_{j=1}^{r} \theta_j(t) \cdot h_j(x) \right)$$

is a family of non-degenerate complete intersections with $l_i$ and $h_j$ satisfying the condition Eq. (3.3) for all $i = 1, \ldots, m$ and $j = 1, \ldots, r$, and if $g_s$ is prepolar with respect to $T_{f_t}$ at the origin, for all $s, t \in \mathbb{C}$. Then, $B_{f_t,X_\sigma^f}(0)$ is constant for all $t, s \in \mathbb{C}$.
**Proof.** Since \( \Gamma_+(t_i) \subset \Gamma_+(g) \) and \( \Gamma_+(h_j) \subset \Gamma_+(f) \), for each face \( \Delta \prec \check{\sigma} \) such that \( \Gamma_+(f) \cap \Delta \neq \emptyset \), the Newton polygon \( \Gamma_+(\{f\}) \) of the function

\[
f_{\Delta} = \left( \prod_{\Gamma_+(f) \cap \Delta \neq \emptyset} g \right) \cdot f \in \mathbb{C}[S_\sigma]
\]

is equals to \( \Gamma_+(\{f\}) \), where

\[
f_{t\Delta} = \left( \prod_{\Gamma_+(f) \cap \Delta \neq \emptyset} g_s \right) \cdot f_t \in \mathbb{C}[S_\sigma].
\]

Then, by [MAT11b] we can conclude that the Euler characteristic

\[
\chi\left( X^{\delta_1} \cap \Delta \cap B_{\varepsilon}(0) \cap f_t^{-1}(\delta) \right)
\]

is constant for all \( s, t \in \mathbb{C} \). Moreover, as \( g_s \) is prepolar with respect to \( \mathcal{T}_t \), we can proceed exactly in the same way as in Theorem 3.6. Hence,

\[
\text{Eu}_{X_\sigma}(T_\Delta) = \text{Eu}_{X^{\delta_1}}(T_\Delta \cap X^{\delta_1}).
\]

This conclude the proof of the result. \( \square \)

As a consequence from Theorem 3.6 and Proposition 3.3, if \( g : X_\sigma \to \mathbb{C} \) is prepolar with respect to \( \mathcal{T}_t \), we can give the number of stratified Morse critical points on the stratum of maximum dimension appearing in a morsefication of \( g : X_\sigma \cap f^{-1}(\delta) \cap B_{\varepsilon}(0) \to \mathbb{C} \) in terms of volumes of convex polytopes. More precisely, on Theorem 2.5, we have

\[
(-1)^{d-1}n_d = \sum_{\Gamma_+(f) \cap \Delta \neq \emptyset} (-1)^{\dim \Delta - 1} \left( \sum_{i=1}^{\mu(\Delta)} \text{Vol}_2(\Gamma_i^\Delta) \right) \cdot \text{Eu}_{X_\sigma}(T_\Delta)
\]

\[
- \sum_{\Gamma_+(f) \cap \Delta \neq \emptyset \cap \dim \Delta \geq 2} (-1)^{\dim \Delta - 2} \left( \sum_{i=1}^{\nu(\Delta)} d_i^\Delta \cdot K_i^\Delta \right) \cdot \text{Eu}_{X_\sigma}(T_\Delta)
\]

where \( n_d \) is the number of stratified Morse critical points on the top stratum \( T_{\Delta_\sigma} \cap f^{-1}(\delta) \cap B_{\varepsilon}(0) \) appearing in a morsefication of \( g : X_\sigma \cap f^{-1}(\delta) \cap B_{\varepsilon}(0) \to \mathbb{C} \).

Therefore, if \( f_t(x) = f(x) + \sum_{j=1}^{r} \theta_j(t) \cdot h_j(x) \) is a family of non-degenerate polynomial functions on \( X_\sigma \) and if \( (g_s, f_t) : (X_\sigma, 0) \to (\mathbb{C}^2, 0) \) is a family of non-degenerate complete intersections which satisfy the same hypotheses from Corollary 3.7, then

\[
(-1)^{d-1}n_d = B_{f_t, X_\sigma}(0) - B_{f_t, X^{\delta_1}}(0)
\]

is constant for all \( s, t \in \mathbb{C} \). Hence, we can state the following.

**Corollary 3.8.** Let \( X_\sigma \subset \mathbb{C}^n \) be a \( d \)-dimensional toric variety and \( (g, f) : (X_\sigma, 0) \to (\mathbb{C}^2, 0) \) a non-degenerate complete intersection. If

\[
\left( g_s(x), f_t(x) \right) = \left( g(x) + \sum_{i=1}^{m} \xi_i(s) \cdot l_i(x), f(x) + \sum_{j=1}^{r} \theta_j(t) \cdot h_j(x) \right)
\]
is a family of non-degenerate complete intersections with $l_i$ and $h_j$ satisfying the condition Eq. (3.3) for all $i = 1, \ldots, m$ and $j = 1, \ldots, r$, and if $g_s$ is prepolar with respect to $\mathcal{T}_i$, at the origin. Then $(-1)^{d-1}n_d$ is constant for all $s, t \in \mathbb{C}$.

We will give nice examples of the results presented before in Section 5.

4. The Euler obstruction and Bruce-Roberts’s Milnor number

In this section, considering $(X, 0) = (\mathbb{C}^n, 0)$, we derive sufficient conditions to obtain the constance of the Euler obstruction for families of complete intersections with isolated singularity at the origin which are contained on $X$. As an application of this fact, we study the invariance of the Bruce-Roberts’s Milnor number for families of functions defined on hypersurfaces. The Bruce-Roberts’s Milnor number was defined in [BrRo88] and we will denote it by $\mu_{BR}(f, X)$.

4.1. Euler obstruction of non-degenerate ICIS. As observed in [BrGr10, Remark 2.5] the Euler obstruction is not a topological invariant. However, for non-degenerate complete intersections with isolated singularity (ICIS) we have the following result.

**Theorem 4.1.** Let $S_\theta = \mathbb{Z}_\theta^n$ and $X_\theta = \mathbb{C}^n$ be the smooth $n$-dimensional toric variety and $g = (f_1, \ldots, f_{k-1}) : \mathbb{C}^n \to \mathbb{C}^{k-1}$ a non-degenerate complete intersection with isolated singularity at $0$, where $1 \leq k \leq n$. If

$$g_s(x) = \left( f_1(x) + \sum_{i=1}^{m_1} \theta_{i_1}(s) \cdot h_{i_1}(x), \ldots, f_{k-1}(x) + \sum_{i_{k-1}=1}^{m_{k-1}} \theta_{i_{k-1}}(s) \cdot h_{i_{k-1}}(x) \right)$$

is a family of non-degenerate complete intersections with isolated singularity at $0$ where $h_{i_p}$ satisfies the condition Eq. (3.3) for all $p \in \{1, \ldots, k-1\}$ and $i_p \in \{1, \ldots, m_p\}$, and assuming that there is a linear form $L : \mathbb{C}^n \to \mathbb{C}$ which is generic with respect to $X_\theta^{g_s}$, for all $s \in \mathbb{C}$. Then, $\Eu_{X_\theta^{g_s}}(0)$ is invariant for the family $\{g_s\}_{s \in \mathbb{C}}$.

**Proof.** As $g_s : \mathbb{C}^n \to \mathbb{C}^{k-1}$ is a non-degenerate complete intersection and $L$ is generic with respect to $X_\theta^{g_s}$, for all $s$, then $(g_s, L) : \mathbb{C}^n \to \mathbb{C}^k$ is a non-degenerate complete intersection. Then, for all face $\Delta \neq \{0\}$ of $\sigma$, the Newton polygon $\Gamma_+(L_\Delta)$ of the function

$$L_\Delta = \left( \prod_{j \in I(\Delta)} f_j \right) \cdot L \in \mathbb{C}[S_\theta]$$

is equals to $\Gamma_+(L_\Delta)$, where

$$L_\Delta = \left( \prod_{p \in I(\Delta)} f_p + \sum_{i_p=1}^{m_p} \theta_{i_p} \cdot h_{i_p} \right) \cdot L \in \mathbb{C}[S_\theta].$$

By [MaTa11b, Theorem 3.12] we can conclude that the Euler characteristic of the Milnor fiber of $L|_{X_\theta^{g_s}} : X_\theta^{g_s} \to \mathbb{C}$ is invariant to the family. Then, since each $X_\theta^{g_s}$ is an ICIS, $\Eu_{X_\theta^{g_s}}(0)$ is invariant to the family. \qed
With the same assumptions of the previous Theorem, and consider \( f_t(x) = f_k(x) + \sum_{i_k=1}^{m_k} \theta_{i_k}(t) \cdot h_{i_k}(x) \) a family of polynomial functions such that \((g, f_t)\) is a family of non-degenerate complete intersections where \( h_{i_k} \) satisfies the condition Eq. (3.3), i.e.,

\[
\Gamma_+(h_{i_k}) \subseteq \Gamma_+(f_k), \quad \text{for all } i_k = 1, \ldots, m_k,
\]

and in a such way that \( f_t : X^0_{\delta} \to \mathbb{C} \) has a stratified isolated critical point at 0. For each face \( \Delta < \delta \) satisfying \( \Gamma_+(f_k) \cap \Delta \neq \emptyset \), the Newton polygon \( \Gamma_+(f_\Delta) \) of the function

\[
f_\Delta = \left( \prod_{j \in I(\Delta)} f_j \right) \cdot f_k \in \mathbb{C}[S_\delta]
\]
is equals to \( \Gamma_+(f_\Delta) \), where

\[
f_\Delta = \left( \prod_{p \in I(\Delta)} f_p + \sum_{i_p=1}^{m_p} \theta_{i_p} \cdot h_{i_p} \right) \cdot \left( f_t = f_k + \sum_{i_k=1}^{m_k} \theta_{i_k} \cdot h_{i_k} \right) \in \mathbb{C}[S_\delta].
\]

Using [MATA11B, Theorem 3.12] we conclude that the Euler characteristic of the Milnor fiber of \( f_t : X^0_{\delta} \to \mathbb{C} \) is invariant to the family. Therefore, \( \text{Eu}_{f_t, X^0_{\delta}}(0) \) is invariant to the family.

### 4.2. Bruce-Roberts’s Milnor number.

In [BrRo88], Bruce and Roberts introduced a Milnor number for functions germs on singular varieties.

Let \( X \) be a sufficiently small representative of the germ \((X, 0)\) and let \( I(X) \) denote the ideal in \( O_{n,0} \) consisting of the germs of functions vanishing on \( X \). We say that two germs \( f \) and \( g \) in \( O_{n,0} \) are \( R_X \)-equivalent if there exists a germ of diffeomorphism \( \phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) such that \( \phi(X) = X \) and \( f \circ \phi = g \). Let \( \theta_n \) denote the \( O_{n,0} \)-module of germs of vector fields on \((\mathbb{C}^n, 0)\). Each vector field \( \xi \in \theta_n \) can be seen as a derivation \( \xi : O_{n,0} \to O_{n,0} \). We denote by \( \theta_X \) those vector fields that are tangent to \( X \), i.e.,

\[
\theta_X := \{ \xi \in \theta_n : dg(\xi) = \xi g \in I(X), \forall g \in I(X) \}.
\]

**Definition 4.2.** Let \( f \) be a function in \( O_{n,0} \) and let \( df(\theta_X) \) be the ideal \( \{ \xi f : \xi \in \theta_X \} \) in \( O_{n,0} \). The number

\[
\mu_{BR}(X, f) = \dim_C \frac{O_{n,0}}{df(\theta_X)}
\]
is called the Bruce-Roberts number of \( f \) with respect to \( X \).

We refer to [BrRo88] for more details and properties about \( \mu_{BR}(X, f) \). In particular, \( \mu_{BR}(X, f) \) is finite if and only if \( f \) is \( R_X \)-finitely determined.

An interesting open problem is to know whether the Bruce-Roberts number is or is not a topological invariant. In [GRU09, GRU12, Corollary 5.19] Grulha gave a partial answer to this problem. The author proved that, if \((X, 0)\) is a hypersurface whose logarithmic characteristic variety \( LC(X) \) [BrRo88, Definition 1.13], is Cohen-Macaulay and if \( f_t \) is a \( C^0 \)-\( R_X \)-trivial deformation of \( f \), then \( \mu_{BR}(f_t, X) \) is constant.

For any hypersurface \( X \) the problem of \( LC(X) \) be Cohen-Macaulay remains open. When \( X \) is a quasihomogeneous hypersurface with isolated singularity,
LC(\(X\)) is Cohen-Macaulay by [NnBOT13, Theorem 4.2]. Using this result, in [NnBOOT17] the authors proved the following theorem.

**Theorem 4.3.** [NnBOOT17, Theorem 3.6] Let \(S_\sigma = \mathbb{Z}_+^n\) and \(X_\sigma = \mathbb{C}^n\) be the smooth \(n\)-dimensional toric variety, \((X_\sigma, 0) \subset (\mathbb{C}^n, 0)\) a weighted homogeneous hypersurface with isolated singularity at the origin, \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) a weighted homogeneous \(R_X\)-finitely determined function and let \(f_1(x) = f(x) + \sum_{j=1}^r \theta_j(t) \cdot h_j(x)\) be a family of polynomial functions with \(h_j\) satisfying the condition Eq. (3.3). The following statements are equivalent:

1. the family \(f_1\) is \(C^0\)-\(R_X\)-trivial;
2. \(f_1\) is a non-negative deformation;
3. \(\mu_{BR}(f_1, X)\) is constant.

Let us recall that \(\mu(f)\) denotes the Milnor number of a germ of an analytic function \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) with an isolated critical point at the origin and it is defined as

\[
\mu(f) = \dim_{\mathbb{C}} \mathcal{O}_{n,0} \frac{f}{J(f)},
\]

where \(\mathcal{O}_{n,0}\) is the ring of germs of analytic functions at the origin, and \(J(f)\) is the Jacobian ideal of \(f\).

**Proposition 4.4.** Let \(S_\sigma = \mathbb{Z}_+^n\) and \(X_\sigma = \mathbb{C}^n\) be the smooth \(n\)-dimensional toric variety. Let \((g, f) : (X_\sigma) \to (\mathbb{C}^2, 0)\) be a non-degenerate complete intersection, and

\[
(g_s(x), f_t(x)) = (g(x) + \sum_{i=1}^m \xi_i(s) \cdot l_i(x), f(x) + \sum_{j=1}^r \theta_j(t) \cdot h_j(x))
\]

a family of non-degenerate complete intersections with \(h_j\) and \(l_i\) satisfying the condition Eq. (3.3). If we assume that, for all \(s, t \in \mathbb{C}\), \(X_\sigma^{g_s} \subset \mathbb{C}^n\) is a weighted homogeneous hypersurface with isolated singularity at the origin, \(f_t : X_\sigma^{g_s} \to \mathbb{C}\) has an isolated critical point at the origin, and that there is a linear form \(L : \mathbb{C}^n \to \mathbb{C}\) which is generic with respect to \(X_\sigma^{g_s}\). Then, \(\mu_{BR}(f_1, X_\sigma^{g_s})\) is constant to all \(s, t \in \mathbb{C}\).

**Proof.** From [Gru09, Gru12] and [NnBOT13] we have

\[
\mu_{BR}(f_1, X_\sigma^{g_s}) = \mu(f_1) + \mathrm{Eu}_{X_\sigma^{g_s}}(0) + (-1)^{n-1} \left(\mathrm{Eu}_{f_t, X_\sigma^{g_s}}(0) + 1\right).
\]

Therefore the result follow from Theorem 4.1.

5. **The case of Toric Surfaces**

Let \(f\) be a polynomial function defined on a 2-dimensional toric variety \(X_\sigma \subset \mathbb{C}^n\). In this section, we present a characterization of the polynomials functions \(g : X_\sigma \to \mathbb{C}\) which are prepolar with respect to \(T_f\) at the origin. Using this characterization and the results of the last sections we present some examples of computation of the Brasselet number \(B_{f_1, X_\sigma}\), for a class of toric surfaces \(X_\sigma\) that are also determinantal.

Let us remember that a strongly convex cone in \(\mathbb{R}^2\) has the following normal form.
Proposition 5.1 ([FUL93]). Let \( \sigma \subset \mathbb{R}^2 \) be a strongly convex cone, then \( \sigma \) is isomorphic to the cone generated by the vectors \( v_1 = pe_1 - qe_2 \) and \( v_2 = e_2 \), for some integers \( p, q \in \mathbb{Z}_{>0} \) such that \( 0 < q < p \) and \( p, q \) are coprime.

Given a cone \( \sigma \subset \mathbb{R}^2 \), Riemenschneider proved in [RIE74, RIE81] that the binomials which generate the ideal \( I_\sigma \) are given by quasiminors of a quasimatrix, where \( X_\sigma = V(I_\sigma) \). In the following we recall the definition of quasimatrix.

Definition 5.2. Given \( A_i, B_i, C_{l,l+1} \in \mathbb{C} \) with \( i = 1, \ldots, n \) and \( l = 1, \ldots, n - 1 \), a quasimatrix with entries \( A_i, B_i, C_{l,l+1} \) is written as

\[
A = \begin{pmatrix}
A_1 & A_2 & \cdots & A_{n-1} & A_n \\
B_1 & B_2 & \cdots & B_{n-1} & B_n \\
C_{1,2} & \cdots & C_{n-1,n}
\end{pmatrix}.
\]

The quasiminors of the quasimatrix \( A \) are defined by

\[
A_i \cdot B_j - B_i \cdot (C_{i,i+1} \cdot C_{i+1,i+2} \cdots C_{j-1,j}) \cdot A_j
\]

for \( 1 \leq i < j \leq n \).

Given \( \sigma \subset \mathbb{R}^2 \) generated by \( v_1 = pe_1 - qe_2 \) and \( v_2 = e_2 \), with \( p \) and \( q \) as above, let us consider the Hirzebruch-Jung continued fraction

\[
\frac{p}{p - q} = a_2 - \frac{1}{a_3 - \frac{1}{\ddots - \frac{1}{a_{n-1}}}} = [[a_2, a_3, \ldots, a_{n-1}]]
\]

where the integers \( a_2, \ldots, a_{n-1} \) satisfies \( a_i \geq 2 \), for \( i = 2, \ldots, n - 1 \). By [RIE81] we have:

Proposition 5.3. The ideal \( I_\sigma \) is generated by the quasiminors of the quasimatrix

\[
\begin{pmatrix}
z_1 & z_2 & z_3 & \cdots & z_{n-2} & z_{n-1} \\
z_2 & z_3 & z_4 & \cdots & z_{n-1} & z_n \\
z_2^{a_2-2} & z_3^{a_3-2} & \cdots & z_{n-1}^{a_{n-1}-2}
\end{pmatrix}.
\]

Where the \( a_i \) are given by the Hirzebruch-Jung continued fraction of \( \frac{p}{p - q} \). Moreover, this set of generators is minimal.

Then, if \( a_i = 2 \) for \( i = 3, \ldots, n - 2 \), we have that \( X_\sigma \) is a determinantal surface [GGJR, NXBOOT13, SRDSP14], in particular if the minimal dimension of embedding of \( X_\sigma \) is 4, i.e., if

\[
\frac{p}{p - q} = a_2 - \frac{1}{a_3}
\]

then \( X_\sigma \) is always determinantal and the ideal \( I_\sigma \) is generated by the \( 2 \times 2 \) minors of the matrix

\[
\begin{pmatrix}
z_1 & z_2 & z_3^{a_3-1} \\
z_2^{a_2-1} & z_3 & z_4
\end{pmatrix}.
\]
Moreover, as in Section 5.1, the integers coming from the Hirzebruch-Jung continued fraction of \( \frac{p}{q} \), we will denote by

\[
\mu_1 = (\mu_1^1, \mu_1^2) = (1, 0), \quad \mu_2 = (\mu_2^1, \mu_2^2) = (1, 1), \quad \mu_{i+1} = a_i \cdot \mu_i^1 - \mu_{i-1}^1,
\]

the minimal set of generators of \( S_\sigma \), with \( i = 2, \ldots, n - 1; \ j = 1, 2 \), where it is possible to show that, \( \mu_n = (\mu_n^1, \mu_n^2) = (q, p) \) (see \[Rie74, Rie81\]). Then, \( \phi : (\mathbb{C}^*)^2 \times X_\sigma \to X_\sigma \) given by

\[
\phi(t = (t_1, t_2), (z_1, \ldots, z_n)) = (t_1 \cdot z_1, t_1 \cdot t_2 \cdot z_2, t_1^{\mu_1^3} \cdot t_2^{\mu_2^3} \cdot z_3, \ldots, t_1^q \cdot t_2^p \cdot z_n)
\]
is an action of \((\mathbb{C}_*)^2\) in \( X_\sigma \). Each orbit of \( \phi \) is a embedding of a \( d \)-dimensional torus, \( 0 \leq d \leq 2 \), in \( X_\sigma \). The action \( \phi \) has 4 orbits, that are

\[
T_{\Delta_0} = (0, \ldots, 0), \quad T_{\Delta_1} = (t_1, 0, \ldots, 0), \quad T_{\Delta_2} = (0, t_2, \ldots, 0), \quad T_{\Delta_3} = (t_1, t_2, \ldots, 0).
\]

Moreover, as in Section 5.1,

\[
X_\sigma = \bigsqcup_{\Delta_i < \sigma} T_{\Delta_i},
\]

with \( i = 0, 1, 2, 3 \), is a decomposition of \( X_\sigma \) in strata satisfying the Whitney conditions. Moreover, if \( f = \sum_{v \in S_\sigma} a_v \cdot v \in \mathbb{C}[S_\sigma] \) is a non-degenerate polynomial function on \( X_\sigma \), then

\[
T_f = \left\{ T_{\Delta_i} \setminus X_\sigma^f, \quad T_{\Delta_i} \cap X_\sigma^f, \quad \{0\}; \quad i = 0, 1, 2, 3 \right\},
\]
is a good stratification of \( X_\sigma \) relative to \( f \).

Next, we characterize the polynomial functions which have stratified isolated singularity at the origin.

**Lemma 5.4.** Let \( \sigma \subset \mathbb{R}^2 \) be a strongly convex cone and \( T \) the Whitney stratification of \( X_\sigma \subset \mathbb{C}^n \) whose strata are \( T_{\Delta_0}, T_{\Delta_1}, T_{\Delta_2} \) and \( T_{\Delta_3} \). Then, a non-degenerate polynomial function \( g \) on \( X_\sigma \) has isolated singularity at the origin if, and only if,

\[
g(z_1, \ldots, z_n) = c_1 z_1^{p_1^1} + h(z_1, \ldots, z_n) + c_n z_n^{p_n^1},
\]

where \( h \) is a polynomial function on \( X_\sigma \), \( c_1, c_n \in \mathbb{C}^* \) and \( p_1, p_n \in \mathbb{Z}_{>0} \).

**Proof.** Let us write \( g \) as follows

\[
g(z_1, \ldots, z_n) = \sum_{l=1}^{m} c_l z_1^{p_1^1} \cdot z_2^{p_2^1} \cdots z_n^{p_n^1},
\]

where \( l = 1, \ldots, m, \ p_i^1 \in \mathbb{Z}_{>0} \) and \( c_1 \in \mathbb{C} \).

Suppose that \( g \) has a stratified isolated singularity at the origin \( 0 \in \mathbb{C}^n \), with respect to the stratification \( T \), then there must be \( l_1, l_n \in \{1, \ldots, m\} \) such that

\[
\begin{align*}
c_{l_1} & \in \mathbb{C}^*; \quad p_1^{l_1} \neq 0 \quad \text{and} \quad p_1^{l_1} = 0, \quad \text{for} \ i \in \{2, \ldots, n\} \\
c_{l_n} & \in \mathbb{C}^*; \quad p_n^{l_n} \neq 0 \quad \text{and} \quad p_n^{l_n} = 0, \quad \text{for} \ i \in \{1, \ldots, n - 1\}
\end{align*}
\]
we obtain information about the singular set $\Gamma_+(g)$ of polynomial function $g$ in the ideal generated by the function $g$.

Example 5.5. Let $(g, f) : (X_\sigma, 0) \to (\mathbb{C}^2, 0)$ be a non-degenerate complete intersection. The polynomial function $g$ is prepolynomial with respect to $T_f$ if, and only if,

$$g(z_1, \ldots, z_n) = c_1z_1^{p_1} + h(z_1, \ldots, z_n) + c_nz_n^{p_n},$$

where $h$ is a polynomial function on $X_\sigma$, $c_1, c_n \in \mathbb{C}^*$ and $p_1, p_n \in \mathbb{Z}_{>0}$.

Proposition 5.5. Let $(g, f) : (X_\sigma, 0) \to (\mathbb{C}^2, 0)$ be a non-degenerate complete intersection. The polynomial function $g$ on $X_\sigma$ has isolated singularity at the origin if, and only if, $\Gamma_+(g) \cap T_{\Delta_1} \neq \emptyset$ and $\Gamma_+(g) \cap T_{\Delta_2} \neq \emptyset$. Therefore, the result follows from the fact that $g$ is non-degenerate.

As a consequence of Lemma 5.4 we obtain information about the singular set of $g$ just looking to its Newton polygon $\Gamma_+(g)$. More precisely, a non-degenerate polynomial function $g$ on $X_\sigma$ has isolated singularity at the origin if, and only if, $\Gamma_+(g)$ intersects $T_{\Delta_1}$ and $T_{\Delta_2}$, exactly on the same way as the classic case, i.e., in the case where $X_\sigma = \mathbb{C}^2$.

Example 5.6. Let $\sigma \subset \mathbb{R}^2$ be the cone generated by the vectors $v_1 = e_2$ and $v_2 = ne_1 - e_2$. The toric surface associated to $\sigma$ is $X_\sigma = V(I_\sigma) \subset \mathbb{C}^{n+1}$, where $I_\sigma$ is the ideal generated by the $2 \times 2$ minors of the matrix

$$\begin{pmatrix}
1 & z_2 & \cdots & z_{n-1} & z_n \\
z_2 & z_3 & \cdots & z_n & z_{n+1}
\end{pmatrix}$$

i.e., $X_\sigma$ is a codimension $n - 1$ determinantal surface. Consider $f : X_\sigma \to \mathbb{C}$ the function given by $f(z_1, \ldots, z_{n+1}) = z_1^{d_1} + z_2^{d_2} + \cdots + z_{n+1}^{d_{n+1}}$, where

$$g(z_1, \ldots, z_{n+1}) = \sum_{l=1}^{m} z_1^{p_{1_1}}z_2^{p_{1_2}}\cdots z_{n+1}^{p_{1_{n+1}}}$$

is a polynomial function on $X_\sigma$ satisfying $p_{1_1} + p_{1_2} + \cdots + p_{1_{n+1}} > d$ for every $l = 1, \ldots, m$, where $p_{1_i} \in \mathbb{Z}_{>0}$. If $f$ is a non-degenerate polynomial function, then

$$B_{f(X_\sigma)}(0) = 2d - nd^2.$$ 

Indeed, consider $h : X_\sigma \to \mathbb{C}$ the function given by $h(z_1, \ldots, z_{n+1}) = z_1^{d_1} + z_2^{d_2} + \cdots + z_{n+1}^{d_{n+1}}$. The Newton polygon $\Gamma_+(h)$ has an unique 1-dimensional compact face $\beta_1$, that is the straight line segment connecting the points $(d, 0)$ and $(d, nd)$ in $\delta$. Then, $\Gamma^{\Delta_1}_1$ is the straight line segment connecting the points $(0, 0)$ and $(d, 0)$, $\Gamma^{\Delta_2}_1$ is the straight line segment connecting the points $(0, 0)$ and $(d, nd)$ and $\Gamma^{\Delta_3}_1$ is the triangle of vertices $(0, 0)$, $(d, 0)$ and $(d, nd)$. Therefore,

$$\text{Vol}_{\mathbb{Z}}(\Gamma^{\Delta_1}_1) = \text{Vol}_{\mathbb{Z}}(\Gamma^{\Delta_3}_1) = d \quad \text{and} \quad \text{Vol}_{\mathbb{Z}}(\Gamma^{\Delta_3}_1) = nd^2.$$ 

Then,

$$B_{h(X_\sigma)}(0) = 2d - nd^2.$$
Now, note that $\Gamma_+(g) \subset \Gamma_+(h)$, since $S_\sigma$ is the semigroup generated by
\[\{(1,0), (1,1), (1,2) \ldots, (1,n)\}\]
Moreover, we can conclude by Lemma 5.4, that $f$ has an isolated singularity at the origin, thus
\[B_{f,X_\sigma}(0) = Eu_{X_\sigma}(0) - Eu_{f,X_\sigma}(0).\]
However, using [GS79] we obtain the equality, $Eu_{f,X_\sigma}(0) = 3 - (n+1) - 2d + nd^2$. Therefore, a morseification of $f$ has $3 - (n+1) - 2d + nd^2$ Morse points on the regular part of $X_\sigma$.

**Example 5.7.** Let $\sigma \subset \mathbb{R}^2$ be the cone generated by the vectors $v_1 = e_2$ and $v_2 = 2e_1 - e_2$. The toric surface associated to $\sigma$ is $X_\sigma = V(I_\sigma) \subset \mathbb{C}^3$, with $I_\sigma$ the ideal generated by $z_1z_3 - z_2^2$. Consider $f : X_\sigma \to \mathbb{C}$ the function given by $f(z_1, z_2, z_3) = z_2^2 - z_3^3$, which is a non-degenerate polynomial function, whose the singular set is
\[\Sigma f = \{(0,0,z_3); \; z_3 \in \mathbb{C}\} \subset X_\sigma.
\]
Besides, $\Gamma_+(f)$ has an unique 1-dimensional compact face $\beta_1$, which is the straight line segment connecting the points $(3,0)$ and $(2,2)$ in $\sigma$. Thus, $\Gamma_1^{\Delta_1}$ is the straight line segment connecting the points $(0,0)$ and $(3,0)$. $\Gamma_1^{\Delta_2}$ is the triangle of vertices $(0,0)$, $(3,0)$ and $(2,2)$, and $\Gamma_1^{\Delta_3} = \emptyset$. Therefore, $B_{f,X_\sigma}(0) = 3 - 6 = -3$. Now let $g : X_\sigma \to \mathbb{C}$ be the non-degenerate polynomial function given by $g(z_1, z_2, z_3) = z_1 - z_2^2$, which is prepolar with respect to $T_f$. Moreover, $(g,f)$ is a non-degenerated complete intersection. The Newton polygon $\Gamma_+(g,f)$ has two 1-dimensional compact faces $\gamma_1$ and $\gamma_2$, which are the straight line segment connecting the points $(4,0)$ and $(3,2)$ and the straight line segment connecting the points $(3,2)$ and $(4,6)$, respectively. Thus, the primitives vectors
\[u_1^{\Delta_3}, u_2^{\Delta_3} \in Int(\Delta_3) \cap M(S_\sigma \cap \Delta_3)^*\]
which takes its minimal in $\Gamma_+(g,f) \cap \Delta_3$ exactly on $\gamma_1$ and $\gamma_2$, respectively, are $u_1^{\Delta_3} = (2,1)$ and $u_2^{\Delta_3} = (4,-1)$. Now let us observe that
\[
\begin{align*}
g(\gamma_1)^{\Delta_3} := \Gamma(g|_{\Delta_3}; u_1^{\Delta_3}) &= \{(1,0)\} \\
g(\gamma_2)^{\Delta_3} := \Gamma(g|_{\Delta_3}; u_2^{\Delta_3}) &= \alpha_1 \\
d_1^{\Delta_3} := d_2^{\Delta_3} &= 6 \\
\kappa_1^{\Delta_3} := \kappa_2^{\Delta_3} &= 1
\end{align*}
\]
where $\alpha_1$ is the 1-dimensional compact face of $\Gamma_+(g)$. Applying Theorem 3.6, we have
\[B_{f,X_\sigma}(0) = 12.\]
Therefore, we obtain the equality
\[B_{f,X_\sigma}(0) - B_{f,X_\sigma}(0) = -3 - 12 = -15,\]
which means that, the number of stratified Morse critical points on the top stratum $T_{d_1} \cap f^{-1}(1) \cap B_t(0)$ appearing in a morseification of $g : X_\sigma \cap f^{-1}(1) \cap B_t(0) \to \mathbb{C}$ is 15. Moreover, considering $h, l : X_\sigma \to \mathbb{C}$ the polynomial functions given by
\[h(z_1, z_2, z_3) = -z_1^2 z_2^2, \quad l(z_1, z_2, z_3) = z_3^3,\]
and observing that
\[ \Gamma_+(h) \subseteq \Gamma_+(f), \quad \Gamma_+(l) \subseteq \Gamma_+(g), \]
by Corollary 3.5 we have
\[ B_{f_t,X_\sigma}(0) = B_{l,X_\sigma}(0) = -3, \quad B_{f_t,X_\sigma^g}(0) = B_{l,X_\sigma^g}(0) = 12, \]
where \( f_t(x) = f(x) + t \cdot h(x) \) is a deformation of the cusp \( f_0(z_1, z_2, z_3) = z_2^2 - z_1^3 \)
(See Figure (1)) and \( g_s(x) = g(x) + s \cdot l(x) \). Consequently
\[ B_{f_t,X_\sigma}(0) - B_{f_t,X_\sigma^g}(0) = -3 - 12 = -15, \]
for all \( t, s \in \mathbb{C} \).

5.1. **Indices of vector fields.** Let us observe that a toric surface \( X_\sigma \), which is a
cyclic quotient singularity, always possesses a smoothing [Rie74, Satz 10]. Therefore,
when we consider a radial continuous vector field \( v \) on \( X_\sigma \) with isolated
singularity at 0, we can relate the Euler characteristic of a fiber of this smoothing
with the GSV index of \( v \) in \( X_\sigma \). The definition of this index for smoothable isolated
singularity can be find in Section 3 of [BSS09].

In the particular case of toric surfaces which are also isolated determinantal
singularities, we have the following result concerning GSV index.

Let \( X_\sigma \subset \mathbb{C}^n \) be a toric surface that is also an isolated determinantal singularity,
\( i.e., \sigma \) is generated by the vectors \( v_1 = p e_1 - q e_2 \) and \( v_2 = e_2 \), where \( 0 < q < p \),
p, q are coprime, and whose the Hirzebruch-Jung continued fraction is
\[ \frac{p}{p - q} = [a_2, 2, 2, \ldots, 2, a_{n-1}]. \]
Given \( f_t(x) = f(x) + \sum_{j=1}^r \theta_j(t) : h_j(x) \) a family of non-degenerate polynomial
functions on \( X_\sigma \), which satisfies the conditions
\[ \Gamma_+(h_j) \not\subseteq \Gamma_+(f), \quad \text{for all} \quad j = 0, \ldots, r. \]
If we assume that this family has isolated singularity at the origin, the following
result holds.

**Proposition 5.8.** Let \( v_t \) be the vector field given by the gradient of the function \( f_t \). Then,
the following are equivalent:
(a) \( \text{Eu}_{f_t,X_\sigma}(0) \) is constant for the family;
(b) \( \text{Ind}_{GSV}(v_t,X_\sigma,F) \) is constant for the family, where \( F \) is the flat map associated to the
smoothing of \( X_\sigma \).
\textbf{Proof.} By [NNBOOT13] the determinantal Milnor number of the function \(f\) on the Isolated Determinantal Singularity \(X_\sigma\), is 
\[
\mu(f|_{X_\sigma}) = \#\Sigma(\tilde{f}|_{X_\sigma}),
\]
where \(X_\sigma\) is a fiber of a smoothing of \(X_\sigma\), \(\tilde{f}|_{X_\sigma}\) is a morseicification of \(f\) and \(\#\Sigma(\tilde{f}|_{X_\sigma})\) denote the number of Morse points of \(\tilde{f}\) on \(X_\sigma\). From the definition of the GSV index in the case of smoothable varieties (see [BSS09]) we have 
\[
\mu(f|_{X_\sigma}) = \text{Ind}_{GSV}(v, X_\sigma, F).
\]
Moreover, in [ANNBOOT16] the authors proved that \(\mu_{f|_{X_\sigma}}(\emptyset)\) is constant for the family if and only if \(\mu(f|_{X_\sigma})\) is constant for the family. \(\square\)

In [CBMSS16], the authors extended the concept of GSV index and proved a Lê-Greuel formula (see [CBMSS16, Theorem 3.1]) which holds in the same general setting of Theorem 2.5. More precisely, if we consider \(X, f\) and \(g\) as in Theorem 2.5 they express the difference 
\[
\sum_{i=1}^{q} (\chi(V_i \cap B_\varepsilon(0) \cap f^{-1}(\delta)) - \chi(X^g \cap V_i \cap B_\varepsilon(0) \cap f^{-1}(\delta)))
\]
as a index of vector fields: 
\[
\sum_{i=1}^{q} (\chi(V_i \cap B_\varepsilon(0) \cap f^{-1}(\delta)) - \chi(X^g \cap V_i \cap B_\varepsilon(0) \cap f^{-1}(\delta))) = \text{Ind}_{GSV}(g, 0; f),
\]
where \(\text{Ind}_{GSV}(g, 0; f)\) is the GSV-index of \(g\) on \(X^f\) relative to the function \(f\) (see [CBMSS16, Definition 2.5]).

In the toric case, we have the following result.

\textbf{Proposition 5.9.} Let \(S_\sigma = \mathbb{Z}_+^n\) and \(X_\sigma = \mathbb{C}^n\) be the smooth \(n\)-dimensional toric variety. Let \((g, f) : (X_\sigma, 0) \rightarrow (\mathbb{C}^2, 0)\) be a non-degenerate complete intersection. If 
\[
\left( g_s(x), f_t(x) \right) = \left( g(x) + \sum_{i=1}^{m} \xi_i(s) \cdot l_i(x), f(x) + \sum_{j=1}^{r} \theta_j(t) \cdot h_j(x) \right)
\]
is a family of non-degenerate complete intersections with \(h_j\) and \(l_i\) satisfying the condition Eq. (3.3) for all \(i = 1, \ldots, m\) and \(j = 1, \ldots, r\), and if \(g_s\) is prepolar with respect to \(T_\sigma\) at the origin. Then, \(\text{Ind}_{GSV}(g_s, 0; f_t)\) is invariant to the family.

\textbf{Example 5.10.} Consider the toric surface \(X_\sigma = V(I_\sigma) \subset \mathbb{C}^3\), with \(I_\sigma\) the ideal generated by \(z_1z_3 - z_2^2\). Let \(f_t\) and \(g_s\) be the same families of functions from Example 5.7, then 
\[
\text{Ind}_{GSV}(g_s, 0; f_t) = -15
\]
for all \(t, s \in \mathbb{C}\).
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