CERTAIN PROPERTIES OF THE POWER GRAPH ASSOCIATED WITH A FINITE GROUP*

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Abstract
There are a variety of ways to associate directed or undirected graphs to a group. It may be interesting to investigate the relations between the structure of these graphs and characterizing certain properties of the group in terms of some properties of the associated graph. The power graph \( P(G) \) of a group \( G \) is a simple graph whose vertex-set is \( G \) and two vertices \( x \) and \( y \) in \( G \) are adjacent if and only if \( y = x^m \) or \( x = y^m \) for some positive integer \( m \). We also pay attention to the subgraph \( P^*(G) \) of \( P(G) \) which is obtained by deleting the vertex 1 (the identity element of \( G \)). In the present paper, we first investigate some properties of the power graph \( P(G) \) and the subgraph \( P^*(G) \). We next prove that many of finite groups such as finite simple groups, symmetric groups and the automorphism groups of sporadic simple groups can be uniquely determined by their power graphs among all finite groups. We have also determined up to isomorphism the structure of any finite group \( G \) such that the graph \( P^*(G) \) is a strongly regular graph, a bipartite graph, a planar graph or an Eulerian graph. Finally, we obtained some infinite families of finite groups such that the graph \( P^*(G) \) containing some cut-edges.

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1 Introduction

Given an algebraic structure \( S \), there are different ways to associate a directed or undirected graph to \( S \) in such a way the vertices are associated with families of elements or subsets of \( S \) and in which two vertices are joined by an arc or by an edge if and only if they satisfy a certain relation. Since a graph (directed or undirected) can be investigated in terms of the results from Graph Theory, one can obtain some information about the structure of \( S \). In other words, we are interested in characterizing certain properties of \( S \) in terms of some properties of the associated graph. This has been a fruitful topic in the last years.

Notation and Definitions. We begin by introducing some well-known graphs associated with semigroups or groups.

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• The Power graph. Let $S$ be a semigroup and $X$ a non-empty subset of $S$. The directed power graph on $X$, denoted by $\overparen{P}(S, X)$, has $X$ as its vertex-set and for two distinct vertices $x, y \in X$ there is an arc from $x$ to $y$ if and only if $y = x^m$ for some positive integer $m$. Similarly, the (undirected) power graph $P(S, X)$, where $S$ is a semigroup and $X$ a non-empty subset of $S$, is the graph with vertex-set $X$ and such that two vertices $x, y \in X$ are joined by an edge (and we write $x \sim y$) if $x \neq y$ and $y = x^m$ or $x = y^m$ for some positive integer $m$. In the case when $X = S$, we denote the directed (resp. undirected) power graphs by $\overparen{P}(S)$ (resp. $P(S)$). The graphs $\overparen{P}(S)$ and $P(S)$ have been introduced and studied for the first time in [20] and [8], respectively. Given a group $G$, we denote by $P^*(G) = P(G, G \setminus \{1\})$ (resp. $\overparen{P}^*(G) = \overparen{P}(G, G \setminus \{1\})$), the graph (resp. direct graph) obtained by deleting the vertex 1 (identity element) from $G$.

Generally, one can study the power graph $P(S, X)$ for different choices of $S$ and $X$, and from a number of different perspectives. In particular, some other algebraic structures such as: groups, rings, fields etc., are interesting choices for $S$. In recent years, the directed and undirected power graphs have been investigated by many authors. For earlier results concerning these graphs we refer to [5, 6, 8, 20, 21, 22]. In particular, in [5], the power graph $P(G)$ is studied in the case when $G$ is an abelian group.

• The commuting graph and its complement. Another graph that has attracted the attention of many authors is the commuting graph associated with a finite group. For a finite group $G$ and $X$ a non-empty subset of $G$, the commuting graph on $X$ denoted $C(G, X)$ has $X$ as its vertex-set with $x, y \in X$ joined by an edge whenever $xy = yx$. Many authors have studied $C(G, X)$ for different choices of $G$ and $X$. For instance, in the case when $X$ is a set of involutions, then $C(G, X)$ is called a commuting involution graph. In particular, commuting involution graphs for arbitrary involution conjugacy classes of symmetric groups were considered in [2]. Moreover, in the case when $G$ is a finite nonabelian group and $X = G \setminus Z(G)$, the non-central elements of $G$, we call $C(G, X)$ the commuting graph of $G$. In [34] and [35], Segev and Seitz apply the commuting graph $\Delta(G) := C(G, G \setminus \{1\})$ of $G$, with $G$ a nonabelian simple group, in order to prove the Margulis-Platonov conjecture on arithmetic groups.

The noncommuting graph of a nonabelian group $G$, denoted $N(G)$, is defined as follows: its vertices are the non-central elements of $G$, and two vertices are adjacent when they do not commute. Noncommuting graphs have been investigated by many authors (for instance, see [1, 28, 29, 31]).

• The prime graph or Gruenberg-Kegel graph. Another graph which has deserved a lot of attention is the prime graph or Gruenberg-Kegel graph $\Gamma(G)$ of a finite group $G$. In this graph, the vertices are the prime numbers dividing the order of the group $G$ and two different vertices $p$ and $q$ are joined when $G$ possesses an element of order $pq$. As a matter of example, for a finite nilpotent group, this graph is complete, but for the alternating groups of degree 5 or 6, this graph has three isolated vertices. The first references of the prime graph known to Gruenberg and Kegel in an unpublished manuscript, and Williams, Kondratev, Iiyori and Yamaki, who studied the number of connected components of the prime graph of finite groups (see [14, 18, 19, 24, 50, 51]).

Graph notation. Two vertices which are incident with a common edge are adjacent, and two distinct adjacent vertices are neighbours. The set of neighbours of a vertex $v$ in a graph $\Gamma$ is denoted by $N_\Gamma(v)$. The degree (or valency) of a vertex $v$ in a graph $\Gamma$, denoted by $d_\Gamma(v)$, is the number of neighbours of $v$. When the graph $\Gamma$ is clear from the context we simply denote $N(v)$ and $d(v)$. A graph $\Gamma$ is $k$-regular if $d(v) = k$ for all $v \in V$. A spanning subgraph of a graph $\Gamma$ is a subgraph obtained by edge deletions only, in other words, a subgraph whose vertex-set is the entire vertex set of $\Gamma$. Recall that a null graph (empty graph) is a graph without edges. A star graph consists of one central vertex having edges to other vertices in it. An independent set is a set of vertices in a graph, no two of which are adjacent. A clique in a graph is a set of vertices all pairwise adjacent. Two graphs are disjoint if they have no vertex in common. The union of simple graphs $\Gamma_1$ and $\Gamma_2$ is the graph $\Gamma_1 \cup \Gamma_2$ with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$. If $\Gamma_1$ and $\Gamma_2$ are disjoint, we refer to their union as a disjoint union, and generally denote it by.
\[\Gamma_1 \oplus \Gamma_2.\]

By starting with a disjoint union of two graphs \(\Gamma_1\) and \(\Gamma_2\) and adding edges joining every vertex of \(\Gamma_1\) to every vertex of \(\Gamma_2\), one obtains the join of \(\Gamma_1\) and \(\Gamma_2\), denoted \(\Gamma_1 \vee \Gamma_2\). A digraph is an ordered pair of sets \(D = (V, A)\), where \(V\) is the set of vertices and \(A\) is the set of ordered pairs (called arcs) of vertices of \(V\). Given a digraph \(D = (V, A)\) and \(x \in V\) we will denote by \(N_D^+(x)\) (resp. \(N_D^-(x)\)) the set of the out-neighbours (resp. in-neighbours) of \(x\), i.e., \(\{y \in V \mid (x, y) \in A\}\) (resp. \(\{y \in V \mid (y, x) \in A\}\)). The out-degree \(d_D^+(v)\) (resp. in-degree \(d_D^-(v)\)) of \(v\) is the number of in-neighbours (resp. out-neighbours) of \(v\). In other words, \(d_D^+(v) = |N_D^+(v)|\), where \(e \in \{+, -\}\). A vertex of out-degree zero is called a sink, one of in-degree zero a source. Note that, as with graphs, we will drop the letter \(D\) from our notation whenever possible, thus we will write \(N^+(v), N^-(v)\), \(d^+(v)\) and \(d^-(v)\) for \(N_D^+(v), N_D^-(v), d_D^+(v)\) and \(d_D^-(v)\), respectively.

**Group Notation.** We denote by \(\pi(n)\) the set of the prime divisors of a positive integer \(n\). Given a group \(G\), we shall write \(\pi(G)\) instead of \(\pi(|G|)\). We denote the order of an element \(x\) in \(G\) by \(o(x)\). A group \(G\) is periodic if every element in \(G\) has finite order. The spectrum of a periodic group \(G\) is a subset \(\pi_e(G)\) of the set of natural numbers consisting of all orders of elements of \(G\). This set is closed and partially ordered by divisibility; hence, it is uniquely determined by \(\mu(G)\), a subset of its elements that are maximal under the divisibility relation. A group is called an EPPO-group if every element of the group has prime power order, i.e., \(\pi_e(G) = \{p^s, q^t, \ldots\}\), where \(p^s, q^t, \ldots\) are primes powers (see [45]). Especially, a group in which every non-trivial element has prime order, i.e., \(\pi_e(G) = \{1, p, q, \ldots\}\), where \(p, q, \ldots\) are primes, is also called an EPO-group (see [44]). Moreover, a group \(G\) is called an EPMO-group if \(\pi_e(G) = \{1, p, q, \ldots, r, m\}\), where \(p, q, \ldots, r\) are primes (see [46]), and a group \(G\) is called an EPPMO-group if \(\pi_e(G) = \{p^s, q^t, \ldots, r^k, m\}\), where \(p^s, q^t, \ldots, r^k\) are prime powers (see [17]). Denote by \(A_m\) and \(S_m\) the alternating and symmetric groups of degree \(m\), respectively. The notation \(\mathbb{Z}_m^n\) means that the direct product if \(n\) copies of \(\mathbb{Z}_m\) (the cyclic group of order \(m\)) and \(G = [N]K\) denotes the split extension of a normal subgroup \(N\) of \(G\) by a complement \(K\).

All further unexplained notation and terminology for graphs, semigroups, and groups are standard and we refer the reader to [9, 10, 12, 13, 16, 32, 49], for instance.

In [5], Cameron and Ghosh asked the following question:

**Question 1** Is it true two groups with isomorphic power graphs must themselves be isomorphic?

They showed this question has an affirmative answer for finite abelian groups [5]. In other words, a finite abelian group is determined up to isomorphism by its power graph. However, the answer to the Question 1 is negative in general, as the following examples show (see [5]):

- **Infinite abelian groups.** In fact, the power graph of the Prüfer group \(C_{p^{\infty}}\) is a countable complete graph, for each prime \(p\).
- **Finite groups of order 27 and exponent 3.** Consider the elementary abelian group \(P\) of order 27, and the group of order 27 with the following presentation:

\[
G = \langle x, y \mid x^3 = y^3 = [x, y]^3 = 1 \rangle,
\]

where \([x, y]\) is the commutator \(x^{-1}y^{-1}xy\). It is easy to see that, these groups have isomorphic power graphs, while they are non-isomorphic groups.

In this paper we will continue the study of the power graph \(P(G)\) and \(P^*(G)\) as well. Among the other results, we classify the finite groups whose power graphs \(P(G)\) or \(P^*(G)\) has a certain graph property.

**A few words about the contents.** The paper consists of seven sections. In Section 2, we first provide some elementary results used throughout the paper, and also we present a brief discussion on the connectivity of the graph \(P^*(G)\) and the relation between power graph and the other graphs. In Section 3, we show that a finite simple group, a symmetric group and the automorphism group of a sporadic simple group can be uniquely determined by their power graphs among all finite groups.
In the rest of paper, we determine up to isomorphism the structure of any finite group $G$ for which $\mathcal{P}^*(G)$ is strongly regular, bipartite, planar or Eulerian, and we obtained some infinite families of finite groups such that the graph $\mathcal{P}^*(G)$ containing some cut-edges.

2 Preliminaries

In this section we collect some of the results that will be needed later.

2.1 Some Elementary Results

We begin with the following lemma which is taken from [8].

Lemma 1 [8, Theorem 2.12] Let $G$ be a finite group. Then $\mathcal{P}(G)$ is complete if and only if $G$ is a cyclic group of order 1 or $p^m$ for some prime number $p$ and for some natural number $m$.

We next give some elementary facts concerning power graphs which follows from the definition.

Lemma 2 Let $G$ be a finite group. Then the following conditions hold.

(a) If $H$ is a subgroup of $G$, then $\mathcal{P}(H) \leq \mathcal{P}(G)$. In particular, if $x$ is a $p$-element of $G$, where $p$ is a prime, then $\langle x \rangle$ is a clique in $\mathcal{P}(G)$.

(b) If $x \in G$ and $o(x) \in \mu(G)$, then as a vertex in the power graph $\mathcal{P}(G)$ we have $d(x) = o(x) - 1$. (Note that, in general, if $x \in G$, then $o(x) - 1 \leq d(x) \leq |C_G(x)| - 1$.) Therefore, the power graph $\mathcal{P}(G)$ of an elementary $p$-group $G$ of order $p^m$ consist of

$$
(p^m - 1)/(p - 1) = p^{m-1} + p^{m-2} + \cdots + p + 1,
$$

complete subgraphs on $p$ vertices sharing a common vertex (the identity element). In particular, when $p = 2$ the power graph $\mathcal{P}(G)$ is a star graph. In general, the power graph $\mathcal{P}(G)$ of an EPPO-group $G$ is the union of complete subgraphs with exactly one common vertex, i.e., the identity element.

(c) If $x, y \in G$ are elements of coprime orders, or if $x, y \in G$ are involutions, then $x \sim y$ in the power graph $\mathcal{P}(G)$. In particular, the set of involutions of $G$ is an independent set in $\mathcal{P}(G)$.

Proof.

(a) The assertion follows immediately from Proposition 4.5 in [8] and Lemma 1.

(b) Evidently $x \sim x^i$ for each $2 \leq i \leq o(x)$, hence $d(x) \geq o(x) - 1$. Now, let $y \in G \setminus \langle x \rangle$ and $y \sim x$. By the definition of power graph we have $x \in \langle y \rangle$ or equivalently $x = y^k$ for some $k$. But then $o(x)|o(y)$, which implies that $o(y) = o(x)$ because $o(x) \in \mu(G)$, and so $\langle y \rangle = \langle x \rangle$, an impossible. Therefore $d(x) = o(x) - 1$. The rest is obvious.

(c) Indeed, if $x \sim y$, then by the definition $x \in \langle y \rangle$ or $y \in \langle x \rangle$, hence $o(y)|o(x)$ or $o(x)|o(y)$, which is a contradiction. If $x, y \in G$ are distinct involutions, then $x \notin \langle y \rangle$ and $y \notin \langle x \rangle$, hence $x \sim y$.

The proof is complete. $\square$

Similarly, the following simple lemma is a direct consequence of the definition.

Lemma 3 Let $G$ be a finite group and $D = \vec{\mathcal{P}}(G)$. Then the following conditions hold.

(a) If $H$ is a subgroup of $G$, then $\vec{\mathcal{P}}(H) \leq D$. 

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(b) If \( x \in G \) and \( o(x) \in \mu(G) \), then as a vertex in the direct power graph \( D \) we have
\[
N_D^+(x) = \{1, x^2, x^3, \ldots, x^{o(x)-1}\} \quad \text{and} \quad N_D^-(x) = \{x^n \mid (n, o(x)) = 1, n \neq 1\}.
\]
In particular, we have \( d_D^+(x) = o(x) - 1 \) and \( d_D^-(x) = \phi(o(x)) - 1 \).

(c) If \( x \in G \) is an involution, then \( d_D^+(x) = 1 \). In other words, every involution of \( G \) is a sink in \( \mathcal{P}^*(G) \).

(d) \( D \) does not contain any source.

(e) If \( x^m \) and \( x^n \) are two generators of \( \langle x \rangle \), then
\[
N_D^+(x^m) \setminus \{x^m\} = N_D^+(x^n) \setminus \{x^n\} \quad \text{and} \quad N_D^-(x^m) \setminus \{x^m\} = N_D^-(x^n) \setminus \{x^n\}.
\]

Proof. These are straightforward verifications. \( \square \)

Next two results are elementary but important for further investigations.

**Lemma 4** Let \( a \) and \( b \) be two elements of \( G \) such that the order of one of them does not divide the order of the other one and they commute, then \( a \) and \( b \) lie in the same component of \( \mathcal{P}^*(G) \).

Proof. Without loss of generality we may assume that \( o(a) < o(b) \) and \( o(b) \equiv n \pmod{o(a)} \). Then, since
\[
an = (ab)^{o(b)} \in \langle ab \rangle \quad \text{and} \quad b^{o(a)} = (ab)^{o(a)} \in \langle ab \rangle,
\]
we easily conclude that
\[
a \sim a^n \sim ab \sim b^{o(a)} \sim b,
\]
which shows that the vertices \( a \) and \( b \) belong to the same component of \( \mathcal{P}^*(G) \), as claimed. \( \square \)

**Lemma 5** Let \( G = \langle x \rangle \) be a cyclic group of order \( n \), \( D = \mathcal{P}(G) \) and \( \Gamma = \mathcal{P}(G) \). Then the out-degree and in-degree of \( x^m \in G \) in the direct graph \( D \) is given by
\[
d_D^+(x^m) = o(x^m) - 1 = \frac{n}{(m, n)} - 1 \quad \text{and} \quad d_D^-(x^m) = \sum_{d|(m,n)} \phi\left(\frac{n}{d}\right),
\]
respectively. Furthermore, the degree of \( x^m \in G \) in the power graph \( \Gamma \) is given by
\[
d_\Gamma(x^m) = \frac{n}{(m, n)} - 1 + \sum_{d|(m,n)} \phi\left(\frac{n}{d}\right).
\]

Proof. Suppose \( x^k \) is an arbitrary element in \( G \). Then, by the definition we have
\[
x^m \sim x^k \iff x^m \in \langle x^k \rangle \quad \text{or} \quad x^k \in \langle x^m \rangle
\]
\[
\iff x^m = (x^k)^r \quad \text{or} \quad x^k = (x^m)^s \quad \text{for some} \ r, s
\]
\[
\iff kr \equiv m \pmod{n} \quad \text{or} \quad ms \equiv k \pmod{n}
\]
\[
\iff (k, n)|m \quad \text{or} \quad (m, n)|k.
\]

First, we determine the number of elements \( x^k \) such that \( 1 \leq k \leq n \) and \( (k, n)|m \), which is equal to \( d_D^-(x^m) \). Suppose \( d_1 = 1, d_2, \ldots, d_t \) are common divisors of \( m \) and \( n \). Now, using the fact that \( (k, n) = d_i \) if and only if \( \left(\frac{k}{d_i}, \frac{n}{d_i}\right) = 1 \), we conclude that
\[
d_D^-(x^m) = \sum_{d|(m,n)} \phi\left(\frac{n}{d}\right).
\]
Next, we determine the number of elements \( x^k \) such that \( 1 \leq k \leq n \) and \((m, n)|k\), which is equal to \( d_D^+(x^m) \). In fact, from \((m, n)|k\) and \( ms \equiv k \pmod{n} \) (for some \( s \)), it follows that \( s = 1, 2, 3, \ldots, o(x^m) = \frac{n}{(m, n)} \). Therefore, since \( D \) is loopless, we obtain

\[
d_D^+(x^m) = o(x^m) - 1 = \frac{n}{(m, n)} - 1.
\] (2)

Evidently \( d_{\Gamma}(G) \leq d_D^+(x^m) + d_D^-(x^m) \). In the sequel to calculate the degree of \( x^m \) in \( \Gamma \), we have to count the bidirected arcs in \( \overrightarrow{P}(\langle x \rangle) \) (see Figure 1). In fact, the number of such arcs is precisely

\[
\phi(o(x^m)) = \frac{n}{(m, n)},
\]

and so from Eqs. (1) and (2) we obtain

\[
d_{\Gamma}(x^m) = \left(d_D^+(x^m) - \phi\left(\frac{n}{(m, n)}\right)\right) + d_D^-(x^m) = o(x^m) - 1 - \phi\left(\frac{n}{(m, n)}\right) + \sum_{d|(m, n)} \phi\left(\frac{d}{d}\right) = \frac{n}{(m, n)} - 1 + \sum_{d|(m, n)} \phi\left(\frac{d}{d}\right).
\]

This completes the proof. □

![Figure 1. The out-degree and in-degree of vertex \( x^m \) in \( D = \overrightarrow{P}(\langle x \rangle) \). Here \( d = d_D^+(x) - \phi(o(x^m)) \).](image)

The following corollary is an immediate consequence of Lemma 5.

**Corollary 1** All non-trivial elements of a cyclic group with the same orders have the same degrees in its power graph.

**Lemma 6** Suppose \( \Gamma \) is a strongly regular graph with parameters \((n, k, \lambda, \mu)\). Then, the following are equivalent:

(a) \( \Gamma \) is disconnected.

(b) \( \Gamma \) is isomorphic with \( tK_{k+1} \), for some \( t > 1 \).

**Proof.** See Lemma 10.1.1 in [11]. □

### 2.2 Power Graphs and Connectivity

For a finite group \( G \), the power graph \( P(G) \) is always connected, because the identity element of \( G \) is adjacent to all other vertices of \( P(G) \). In this section, we will focus our attention on the connectivity of \( P^*(G) \), the subgraph of \( P(G) \) obtained by deleting the vertex 1 (the identity element of \( G \)). In the next result, we provide necessary and sufficient conditions for the subgraph \( P^*(G) \) to be connected, when \( G \) is a \( p \)-group for some prime \( p \).
Lemma 7 Let $G$ be a finite $p$-group. Then $P^*(G)$ is connected if and only if $G$ has unique minimal subgroup.

Proof. Let $G$ be a finite $p$-group with a unique minimal subgroup of order $p$ for a prime $p$, say $\langle x \rangle$. Now let $g \in G \setminus \{1\}$ be an arbitrary $p$-element. Then, it is clear that $\langle x \rangle \subseteq \langle g \rangle$, and hence $x \sim g$. This shows that $P^*(G)$ is connected, as required. 

Conversely, assume that $G$ is a finite $p$-group such that $P^*(G)$ is connected. We claim that $G$ possesses a unique minimal subgroup of order $p$. Suppose on the contrary that there are two distinct minimal subgroups of $G$ of order $p$, say $\langle x \rangle \neq \langle y \rangle$. Since $P^*(G)$ is connected, there is a path between the vertices $x$ and $y$. Assume that 

$$x = x_0 \sim x_1 \sim x_2 \sim \ldots \sim x_n = y,$$

is a path with the least length from $x$ to $y$. Certainly $n \geq 2$ and $x \sim x_i$, for each $i = 2, 3, \ldots, n$. Since $x \sim x_1$, by the definition we have $x \in \langle x_1 \rangle$ or $x_1 \in \langle x \rangle$. We only consider the first case, and the second one goes similarly. Since $x_1 \sim x_2$, it follows that $x_1 \in \langle x_2 \rangle$ or $x_2 \in \langle x_1 \rangle$. If $x_1 \in \langle x_2 \rangle$, then $x \in \langle x_2 \rangle$, which is a contradiction. Therefore $x_2 \in \langle x_1 \rangle$. But then $x, x_2 \in \langle x_1 \rangle$ and since $\langle x_1 \rangle$ is a $p$-group, Lemma 2 (i) shows that $\langle x_1 \rangle \setminus \{1\}$ is a clique in $P^*(G)$. Hence $x \sim x_2$, an impossible. 

The proof of this lemma is complete. \qed

Remark 1. The finite $p$-groups with a unique minimal subgroup of order $p$ are well characterized: If $G$ is a $p$-group which has a unique minimal subgroup of order $p$, then $G$ is either a cyclic group or a generalized quaternion group [12, Theorem 5.4.10. (ii)]. Notice that a generalized quaternion group is not necessarily a $2$-group.

An immediate consequence of Lemma 7 is the following.

Corollary 2 Let $G$ be a finite $p$-group. Then $P^*(G)$ is connected if and only if $G$ is either cyclic or generalized quaternion.

Some Examples. The generalized quaternion group $Q_{4n}$, is defined by 

$$Q_{4n} = \langle x, y \mid x^{2n} = 1, y^2 = x^n, x^y = x^{-1} \rangle.$$ 

For $n = 2$, there is another well-known representation:

$$Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \mid i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j \}.$$ 

We recall some elementary facts about generalized quaternion groups below without proof, because they can be calculated in straightforward ways.

- Every element $g \in Q_{4n}$ can be written uniquely as $g = x^i y^j$ where $0 \leq i \leq 2n - 1$ and $0 \leq j \leq 1$. In particular, the order of $Q_{4n}$ is exactly $4n$.

- We have 

$$\mu(Q_{4n}) = \begin{cases} \{4, 2n\} & n \text{ odd}, \\ \{2n\} & n \text{ even}. \end{cases}$$

In fact, it is easy to see that 

$$o(x^i y^j) = \begin{cases} 2n & j = 0, \ 1 \leq i \leq 2n, \\ 4 & j = 1, \ 1 \leq i \leq 2n. \end{cases}$$

- The center of $Q_{4n}$ is $\langle x^n \rangle = \langle y^2 \rangle \cong \mathbb{Z}_2$. 

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• The generalized quaternion group $Q_{4n}$ has a unique minimal subgroup if and only if $n$ is a power of 2.

• If $n$ is a power of 2, then the power graph $P^*(Q_{4n})$ has the following form:

$$P^*(Q_{4n}) = K_1 \vee (K_{2n-2} \oplus K_2 \oplus K_2 \oplus \cdots \oplus K_2).$$

Actually, $P^*(Q_{4n})$ consists of a complete graph on $2n - 1$ vertices and $n$ triangles sharing a common vertex (the unique involution). Note that, for every element $g \in Q_{4n} \setminus \{1\}$, the subgroup $\langle g \rangle$ contains the unique involution $x^n$, and so $x^n \sim g$ in $P^*(Q_{4n})$. On the other hand, all vertices of $P^*(G)$ with degree $|G| - 2$ lie in the center of $G$, hence in $P^*(Q_{4n})$ the unique involution $x^n$ is the only vertex of degree $4n - 2$. For instance, the power graph $P^*(Q_8)$ is depicted in Figure 2.

![Figure 2. The power graph $P^*(Q_8)$](image)

![Figure 3. The power graph $P^*(\mathbb{Z}_6)$](image)

We next note that, if there are some vertices in the graph $P^*(G)$ which are adjacent to all other vertices, then $P^*(G)$ is connected. In particular, if $G$ is a finite cyclic group with $|G| > 1$, then $P^*(G)$ is connected, because each generator of $G$ is adjacent to all other nontrivial elements of $G$ in $P^*(G)$. In what follows, a necessary and sufficient condition is established for the existence of such vertices in $P^*(G)$. The proof of the next lemma is similar to the proof of Proposition 4 in [6] and is included for the sake of completeness.

**Lemma 8** If $G$ is a finite group, then $P^*(G)$ contains a vertex which is joined to all other vertices if and only if $G$ is either cyclic or generalized quaternion.

**Proof.** We need only prove the necessity part. (Note that, the unique involution in a generalized quaternion 2-group is joined to all other vertices in its power graph, and in the case when $G$ is a cyclic group every generator of $G$ has the same property.) In the sequel, we will consider separately two cases: $G$ is a $p$-group or not.

**Case 1.** $G$ is a $p$-group.

In this case, since $P^*(G)$ is connected, the conclusion follows immediately from Corollary 2.

**Case 2.** $G$ is not a $p$-group.

In this case, we claim that $G$ is always cyclic. Assume to the contrary that $G$ is not a cyclic group. Let $x \in G \setminus \{1\}$ be a vertex in $P^*(G)$ which is joined to all others. Clearly $\langle x \rangle$ is a proper subgroup of $G$. Suppose $y \in G \setminus \langle x \rangle$. Since $x \sim y$, it follows by the definition that $\langle x \rangle \nsubseteq \langle y \rangle$, and so

$$o(x) \nmid o(y) \text{ while } o(x) < o(y). \quad (3)$$

Moreover, from part (c) of Lemma 2, we deduce that $\pi(o(x)) = \pi(G)$ and so $x$ is not a $p$-element for some prime $p \in \pi(G)$. 

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Suppose now that $\pi(G) = \{p_1, p_2, \ldots, p_k\}$. Let $\langle g_i \rangle$ be a maximal cyclic $p_i$-subgroup of $G$ for each $1 \leq i \leq k$. Since $x \sim g_i$, $g_i \in \langle x \rangle$ otherwise $x$ would be a $p_i$-element of $G$, which is a contradiction. Therefore, $o(x)$ is divisible by $o(g_i)$, for each $1 \leq i \leq k$, which shows by the maximality property of $o(g_i)$ that $\mu(G) = \{o(x)\}$. But then $o(y)|o(x)$, which is a contradiction from Eq. (3). □

**Lemma 9** If $G$ is a finite group such that $|\pi(Z(G))| \geq 2$, then $\mathcal{P}^*(G)$ is connected.

**Proof.** Since $Z(G)$ is abelian, it follows that $|\mu(Z(G))| = 1$. Let $\mu(Z(G)) = \{m\}$ and $o(g) = m$ for some $g \in Z(G)$. In the sequel, we will show that for each element $x \in G \setminus \{1\}$ there is a path between $x$ and $g$, which means that $\mathcal{P}^*(G)$ is connected. we distinguish two cases:

Case 1. $\pi(o(x)) = \pi(m)$.

In this case, since $|\pi(m)| \geq 2$, there are two distinct primes $p$ and $q$ in $\pi(m)$ such that $o(x^r) = p$ and $o(g^s) = q$ for some natural numbers $r$ and $s$. Evidently $x^r$ and $g^s$ are elements of coprime order and commute, hence from Lemma 4 we deduce that $x^r$ and $g^s$ lie in the same component of $\mathcal{P}^*(G)$. On the other hand $x \sim x^r$ and $g \sim g^s$, hence $x$ and $g$ lie in the same component of $\mathcal{P}^*(G)$ too.

Case 2. $\pi(o(x)) \neq \pi(m)$.

In this case, there exists a prime $q \in \pi(o(x)) \setminus \pi(m)$ (or $q \in \pi(m) \setminus \pi(o(x)))$. Assume that $q \neq p \in \pi(m)$. Again there are two elements $x^r$ and $g^s$ for some natural numbers $r$ and $s$ such that $o(x^r) = q$ and $o(g^s) = p$. Using a similar argument as previous paragraph we see that these elements belong to the same component of $\mathcal{P}^*(G)$ and since $x \sim x^r$ and $g \sim g^s$, hence $x$ and $g$ lie in the same connected component of $\mathcal{P}^*(G)$ too. □

**Lemma 10** Let $G$ be a finite group with $|\pi(G)| \geq 2$ and let its center is a $p$-group for some prime $p \in \pi(G)$. Then, the power graph $\mathcal{P}^*(G)$ is connected if and only if for every non-central element $x$ of order $p$ there exists a non $p$-element $g$ such that $x \sim g$ in $\mathcal{P}^*(G)$.

**Proof.** ($\Longrightarrow$) Suppose to the contrary that there exists a non-central $p$-element $x$ of order $p$ such that for every non $p$-element $g$, $x \not\sim g$. Let $z \in Z(G)$ be an element of order $p$. In the sequel we will show that there is no path between $x$ and $z$. Suppose the contrary that there is a path between $x$ and $z$, say $P$, with minimum possible length:

$$P: \quad x \sim c_1 \sim c_2 \sim \cdots \sim c_k \sim z.$$ 

We claim that $x \sim z$. If so then we obtain $\langle x \rangle = \langle z \rangle$, which is a contradiction because $x \not\in Z(G)$.

Therefore, we assume that $k \geq 1$. Note that by our assumption $c_1$ is a $p$-element of $G$. Since $x \sim c_1$, by the definition $x \in \langle c_1 \rangle$ or $c_1 \in \langle x \rangle$. We distinguish two cases separately.

Case 1. $x \in \langle c_1 \rangle$.

In this case, since $c_1 \sim c_2$ and $P$ has minimum length, it follows that $c_2 \in \langle c_1 \rangle$. Since $\langle c_1 \rangle$ is a $p$-group, $\mathcal{P}^*(\langle c_1 \rangle)$ is a complete graph by Lemma 1. Hence $x \sim c_2$, which is a contradiction by minimality of length of $P$.

Case 2. $c_1 \in \langle x \rangle$.

This case can be proven in a similar way as previous case.

($\Longleftarrow$) Since $Z(G)$ is a proper subgroup of $G$, it is enough to show that for each $g \in G \setminus Z(G)$, there is a path from $g$ to each non-trivial element of $Z(G)$. Therefore, we assume that $g \in G \setminus Z(G)$ is an arbitrary element. Again, we will consider two cases separately.

Case 1. $g$ is not a $p$-element.

Suppose $q \in \pi(G) \setminus \{p\}$ divides $o(g)$ and $o(g^l) = q$ for some positive integer $l$. Then by Lemma 4, there is a path between $g^l$ and every non-trivial element in $Z(G)$, and so there is a path between $g$ and all non-trivial elements in $Z(G)$.
Case 2. $g$ is a $p$-element.
Suppose $o(g^l) = p$ for some positive integer $l$. Assume first that $g^l \in Z(G)$. In this case, if $x$ is a $p'$-element of $G$, then there is a path between $x$ and $g^l$ by Lemma 4. On the other hand, $g \sim g^l$ and from Case 1 there is a path between $x$ and every non-trivial element in $Z(G)$, as required. Assume next that $g^l \notin Z(G)$. Now, from our assumption there is a non-$p$-element in $G$, say $h$, such that $g^l \sim h$. Again $g \sim g^l \sim h$ and by Case 1 there is a path between $h$ and so $g$ and all non-trivial elements in $Z(G)$. The proof of lemma is now complete. □

2.3 The Relation Between Power Graph and the Other Graphs

Lemma 11 Let $G$ be a finite group. If the power graph $P^*(G)$ is connected, then the prime graph $\Gamma(G)$ is connected too.

Proof. Let $x$ and $y$ be two arbitrary adjacent vertices in the power graph $P^*(G)$. Then by the definition we have $x \in \langle y \rangle$ or $y \in \langle x \rangle$, and so $o(x)|o(y)$ or $o(y)|o(x)$. However, this means that all primes in $\pi(o(x)) \cup \pi(o(y))$ lie in the same connected component of the prime graph $\Gamma(G)$. Since $P^*(G)$ is connected we may proceed successively in this manner, eventually showing that all primes in

$$\pi(G) = \bigcup_{x \in G} \pi(o(x)),$$

lie in the same connected component of $\Gamma(G)$, and hence $\Gamma(G)$ is connected, as required. □

Lemma 12 Let $G$ be a group. Then $P^*(G)$ is a spanning subgraph of $\Delta(G)$.

Proof. First of all, one can easily see that both graphs $P^*(G)$ and $\Delta(G)$ have the same vertex set, i.e., $G \setminus \{1\}$. Also from the definition of the power graph, it follows that for any $x, y \in G \setminus \{1\}$, they are adjacent in $P^*(G)$ if $x \in \langle y \rangle$ or $y \in \langle x \rangle$, and in both cases they commute which shows that they are adjacent in $\Delta(G)$. Thus $P^*(G)$ is an induced subgraph of $\Delta(G)$. □

3 Recognizing Some Almost Simple Groups by Power Graph

In [6], Cameron showed that the power graph of a finite group determines its spectrum.

Lemma 13 (Corollary 3, [6]) Two finite groups whose power graphs are isomorphic have the same numbers of elements of each order. In particular, they have the same spectra.

In 1987, the third author of the paper, Shi posed that every finite simple group is uniquely determined by its spectrum and order in the class of all groups (see [39]). In fact, this conjecture is Question 12.39 in the Kourovka Notebook [47], and is stated as follows:

Shi’s Conjecture. A finite group and a finite simple group are isomorphic if they have the same orders and spectra.

The answer to Shi’s Conjecture is obviously ‘yes’ for abelian simple groups. In a series of papers [7, 39, 40, 41, 42, 43, 48, 52], an affirmative answer was given for all nonabelian simple groups. Thus, Shi’s Conjecture is confirmed and the following proposition holds.

Proposition 1 If $S$ is a finite simple group, and $G$ is a finite group with $\pi_e(G) = \pi_e(S)$ and $|G| = |S|$, then $G \cong S$.

Besides the finite simple groups one can characterize some non-solvable groups using their spectra and orders. For instance, Bi proved the following result concerning symmetric groups (see [3]).
Proposition 2 Let $G$ be a finite group and $n \geq 3$ an integer. Then $G \cong S_n$ if and only if $|G| = |S_n|$ and $\pi_e(G) = \pi_e(S_n)$.

In [23, 36], it is proved that Shi’s Conjecture is valid for the automorphism groups of sporadic simple groups. In fact, we have the following.

Proposition 3 Let $G$ be a finite group and $A$ the automorphism group of a sporadic simple group. Then $G \cong A$ if and only if $|G| = |A|$ and $\pi_e(G) = \pi_e(A)$.

Given a group $M$, denote by $h_P(M)$ the number of isomorphism classes of groups $G$ such that $P(G) \cong P(M)$. A group $M$ is called $k$-fold $P$-characterizable if $h_P(M) = k$. Usually, a 1-fold $P$-characterizable group is simply called $P$-characterizable. We are now ready to return to Question 1. By combining Lemma 13 and Propositions 1-3, we obtain the following theorem.

Theorem 1 Let $M$ be one of the following groups:
(a) A finite simple group,
(b) A symmetric group,
(c) The automorphism group of a sporadic simple group.
Then, $h_P(M) = 1$, in other words, the group $M$ is $P$-characterizable.

Proof. The proof follows in a straightforward way from Lemma 13 and Propositions 1, 2 and 3. Note that, by Lemma 13, two groups whose power graphs are isomorphic have, in fact, the same order and spectra. □

It is well-known that for each positive integer $n$ there are only finitely many non-isomorphic groups of order $n$ normally denoted by $\nu(n)$. In fact, the number of $n \times n$ arrays with entries chosen from a set of size $n$ is $n^{n^2}$. So certainly this is an upper bound for $\nu(n)$. For another upper bound we have $\nu(n) \leq (n!)^{log_2 n}$ (see [33], page 109). Therefore the following result follows immediately.

Theorem 2 All finite groups are $k$-fold $P$-characterizable for some natural number $k$.

4 The Power Graph of Finite Group as an Strongly Regular Graph

A strongly regular graph (henceforth SRG) with parameters $(n, k, \lambda, \mu)$, which will be denoted by $\text{sr}(n, k, \lambda, \mu)$, is a regular graph on $n$ vertices of valency $k$ such that each pair of adjacent vertices has exactly $\lambda$ common neighbours, and each pair of non-adjacent vertices has exactly $\mu$ common neighbours.

Theorem 3 Let $G$ be a nontrivial finite group. Then the following conditions are equivalent.
(a) $P^*(G)$ is a strongly regular graph.
(b) $G$ is a $p$-group of order $p^m$ for which $\exp(G) = p$ or $p^m$, for some prime $p$.

Proof. (a) $\implies$ (b). Let $G$ be a group such that $P^*(G)$ is a strongly regular graph with parameters $(n, k, \lambda, \mu)$. We distinguish two cases separately according to it is connected or disconnected.

Case 1. $P^*(G)$ is disconnected.
In this case, by Lemma 12, we have
$$P^*(G) \cong K_{k+1} \oplus K_{k+1} \oplus \cdots \oplus K_{k+1} = tK_{k+1},$$
for some $t > 1$.

Suppose first that $x_1 \in G \setminus \{1\}$ such that $o(x_1) = \max \pi_e(G)$. Then by Lemma 2 (b), $d(x_1) = o(x_1) - 2$. Therefore, $o(x_1) = k + 2$ and the connected component containing $x_1$ is $P^*(\langle x_1 \rangle)$.
Suppose next that \( x_2 \in G \setminus \langle x_1 \rangle \) with \( o(x_2) = \max \{ o(g) \mid g \in G \setminus \langle x_1 \rangle \} \). Again, by Lemma 2 (b), we conclude that \( d(x_2) = o(x_2) - 2 \). Hence \( o(x_2) = k + 2 \), and the connected component containing \( x_2 \) is \( \mathcal{P}^*(\langle x_2 \rangle) \).

We continue this process until we obtain the disjoint cyclic subgroups \( \langle x_1 \rangle, \langle x_2 \rangle, \ldots, \langle x_t \rangle \), such that

\[
G = \langle x_1 \rangle \cup \langle x_2 \rangle \cup \cdots \cup \langle x_t \rangle.
\] (4)

Evidently \( o(x_1) = o(x_2) = \ldots = o(x_t) = k + 2 \).

In the sequel, we show that \( G \) is a \( p \)-group for some prime \( p \). For this purpose, it is enough to show that \( |\pi(k + 2)| = 1 \). Assume to the contrary that \( |\pi(k + 2)| > 1 \) and let \( p, q \in \pi(k + 2) \) be two distinct primes. Further we assume that \( x_i^{k_1} \) and \( x_i^{k_2} \) are two vertices in the \( i \)th connected component of order \( p \) and \( q \), respectively. Since each connected component is clique, \( x_i^{k_1} \sim x_i^{k_2} \), which is a contradiction by Lemma 2 (c).

Now we show that \( \exp(G) = p \). To do this, it is enough to show that \( o(x_i) = p \). Assume to the contrary that \( o(x_i) = p^l \), where \( l > 1 \). Since \( G \) is a nontrivial \( p \)-group, \( Z(G) > 1 \) and so we can choose an element of order \( p \), say \( z \), in \( Z(G) \setminus \{1\} \). From (4), it follows that \( z \in \langle x_i \rangle \) for some \( i \). Since \( t > 1 \), we can consider the element \( zx_j \), where \( j \neq i \). Evidently, \( zx_j \notin \langle x_i \rangle \cup \langle x_j \rangle \). Therefore \( t \geq 3 \), and so \( zx_j \in \langle x_k \rangle \), where \( k \neq i, j \). But then, we obtain

\[(zx_j)^p = x_j^p \in \langle x_k \rangle \setminus \{1\},\]

since \( o(x_j) = o(x_1) = p^l > p \). This means that there exists a path between \( x_j \) and \( x_k \), which is a contradiction.

**Case 2.** \( \mathcal{P}^*(G) \) is connected.

In this case, we first claim that:

"\( G \) is a cyclic group if and only if \( G \) is a \( p \)-group for some prime \( p \)."

(†)

First, if \( G \) is a cyclic group, say \( G = \langle x \rangle \), then \( o(x) = \max \pi_e(G) \) and from Lemma 2 (b), we have

\[d(x) = o(x) - 2 = |G| - 2,\]

hence \( \mathcal{P}^*(G) \) is a \( (|G| - 2) \)-regular graph. Thus \( \mathcal{P}^*(G) \) and so \( \mathcal{P}(G) \) is a complete graph and by Lemma 1 it follows that \( G \) is a \( p \)-group for some prime \( p \).

Conversely, we assume that \( G \) is a \( p \)-group of order \( p^m \) and \( x \in G \setminus \{1\} \) such that \( o(x) = p^l \in \mu(G) \). It is enough to show that \( l = m \). Again by Lemma 1, we see that \( \mathcal{P}(\langle x \rangle) \) and so \( \mathcal{P}^*(\langle x \rangle) \) is a complete graph. Hence \( \mathcal{P}^*(G) \) is a connected \( (p^l - 2) \)-regular graph which includes a clique of size \( p^l - 1 \), this forces \( G = \langle x \rangle \).

In what follows, we will show that \( G \) is a \( p \)-group for some prime \( p \), which implies by (†) that \( G \) is cyclic, as required. Assume that \( x \in G \setminus \{1\} \) such that \( o(x) = \max \pi_e(G) \). Then \( d(x) = o(x) - 2 \) and so \( \mathcal{P}^*(G) \) is a \( (o(x) - 2) \)-regular graph. Assume first that \( |\pi(o(x))| \geq 2 \) and let \( p, q \in \pi(o(x)) \) be two distinct primes. Since \( o(x) > 2 \), the cyclic group \( \langle x \rangle \) has at least two generators, say \( x \) and \( x^t \) for some \( t \). Actually \( x \) and \( x^t \) are two adjacent vertices with \( d(x) = d(x^t) \) and each of them joint to all vertices in \( \langle x \rangle \setminus \{1, x, x^t\} \), which forces \( \lambda = o(x) - 3 \). Suppose now that \( x^r \) and \( x^s \) are two elements in \( \langle x \rangle \) of order \( p \) and \( q \), respectively. Evidently \( x^r \sim x^s \). On the other hand, the vertices \( x \) and \( x^t \) are adjacent and hence they have \( o(x) - 3 \) common neighbours. However, since \( N(x) = \langle x \rangle \setminus \{1, x\} \), we conclude that

\[N(x) \cap N(x^r) \subseteq N(x) \setminus \{x^s\} = \langle x \rangle \setminus \{1, x, x^s\},\]

and since \( |N(x) \cap N(x^r)| = o(x) - 3 \), we get \( N(x) \cap N(x^r) = \langle x \rangle \setminus \{1, x, x^s\} \), which is a contradiction because \( x^r \notin N(x^r) \).

Next, suppose that \( |\pi(o(x))| = 1 \). In this case, \( \langle x \rangle \) is a \( p \)-group for some prime \( p \), and \( \mathcal{P}^*(G) \) is \( (o(x) - 2) \)-regular. Moreover, by Lemma 1, \( \langle x \rangle \) is a clique in \( \mathcal{P}^*(G) \), which forces \( G = \langle x \rangle \). Therefore \( G \) is a \( p \)-group, and the proof is complete.
(b) $\implies$ (a). The proof is straightforward. $\square$

5 The Power Graph Which is Bipartite or Planar

An independent set in a graph is a set of pairwise nonadjacent vertices. A graph $\Gamma$ is called bipartite if whose vertex set can be partitioned into two independent sets called partite sets. In the following result we recognize the groups $G$ for which the graph $\mathcal{P}^*(G)$ is bipartite.

**Theorem 4** Let $G$ be a nontrivial finite group. Then the following conditions are equivalent.

(a) $\mathcal{P}^*(G)$ is a bipartite graph.

(b) $\pi_e(G) \subseteq \{1, 2, 3\}$.

**Proof.** (a) $\implies$ (b). Let $\mathcal{P}^*(G)$ be a bipartite graph. Clearly, if $n \in \pi_e(G)$, then $G$ always has at least $\phi(n)$ elements of order $n$, where $\phi(n)$ signifies the Euler’s totient function. Suppose now that $G$ contains an element of order $\geq 4$, say $x$. Certainly $\phi(o(x)) > 1$, so the cyclic group $\langle x \rangle$ has at least two generators, say $x$ and $x^t$. Now, we have the 3-cycle

$$x \sim x^j \sim x^t \sim x,$$

where $j \neq 1, t$. This shows that $\mathcal{P}^*(G)$ cannot be a bipartite graph, which is a contradiction.

(b) $\implies$ (a). Let $G$ be a nontrivial finite group such that $\pi_e(G) \subseteq \{1, 2, 3\}$. In the case when $\pi_e(G) = \{1, 2\}$, $G$ is an elementary abelian 2-group and by the definition of power graph, it is easy to see that the graph $\mathcal{P}^*(G)$ consists of only isolated vertices and hence it can be considered as a bipartite graph. On the other hand, if $3 \in \pi_e(G) \subseteq \{1, 2, 3\}$, then $3 = \max \pi_e(G)$ and for all elements of order $3$, say $x$, we have the singleton edge $x \sim x^2$ and so $d(x) = 1$. In fact, the power graph $\mathcal{P}^*(G)$ is isomorphic to a graph as in the following:

$$\mathcal{P}^*(G) \cong K_1 \oplus K_1 \oplus \cdots \oplus K_1 \oplus K_2 \oplus K_2 \oplus \cdots \oplus K_2,$$

consequently it contains no cycles and so $\mathcal{P}^*(G)$ is bipartite, as desired. $\square$

An immediate consequence of Theorem 4 is the following.

**Corollary 3** Let $G$ be a nontrivial finite group. Then $\mathcal{P}^*(G)$ is a tree if and only if $G = \mathbb{Z}_2$ or $\mathbb{Z}_3$.

A graph is planar if it has a drawing in the plane without crossing edges. In the following result we characterize the groups $G$ for which the graph $\mathcal{P}^*(G)$ is planar.

**Theorem 5** Let $G$ be a nontrivial finite group. Then the following conditions are equivalent.

(a) $\mathcal{P}^*(G)$ is planar.

(b) $\pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6\}$.

**Proof.** (a) $\implies$ (b). Let $x$ be an arbitrary element of $G$ and set $H := \langle x \rangle$. Then, by Lemma 2 (a), $\mathcal{P}^*(H)$ is a subgraph of $\mathcal{P}^*(G)$. On the one hand, if $x$ has prime-power order and $o(x) \geq 7$, then $\mathcal{P}^*(H)$ is complete, thus $\mathcal{P}^*(H)$ and so $\mathcal{P}^*(G)$ contains $K_5$, which shows that $\mathcal{P}^*(G)$ is not planar (Theorem 6.2.2, [49]). With the similar argument, we can verify that if $\phi(o(x)) \geq 5$, then $H$ has at least 5 generators, which forces $\mathcal{P}^*(H)$ and so $\mathcal{P}^*(G)$ again contains $K_5$, and hence $\mathcal{P}^*(G)$ is not planar. In particular, we conclude that $\pi(G) \subseteq \{2, 3, 5\}$ and $\pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6, 10, 12\}$.

Furthermore, if $G$ contains an element of order 10 or 12, say $x$, then the cyclic subgroup $\langle x \rangle$ has four generators, each of them is adjacent to all other elements of $\langle x \rangle$ in $\mathcal{P}^*(\langle x \rangle)$. Now the induced subgraph on these generators and the vertex $x^2$ is a clique which is isomorphic to $K_5$. Thus $\mathcal{P}^*(\langle x \rangle)$

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and so \( \mathcal{P}^*(G) \) contains \( K_5 \), which is not planar as before. Therefore \( \pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6\} \), as required.

(b) \( \implies \) (a). In the case when \( \pi_e(G) \subseteq \{1, 2, 3, 4, 5\} \), \( G \) is an EPPO-group and from Lemma 2 (b) it follows that the power graph \( \mathcal{P}(G) \) is the union of complete subgraphs \( K_2, K_3, K_4 \) and \( K_5 \) with exactly one common vertex, i.e., the identity element. Hence the graph \( \mathcal{P}^*(G) \) is a disjoint union of complete subgraphs \( K_1, K_2, K_3 \) and \( K_4 \), and so it does not contain a subdivision of \( K_5 \) or \( K_{3,3} \), which shows that \( \mathcal{P}^*(G) \) is planar (Theorem 6.2.2, [49]).

Now, we may assume that \( 6 \in \pi_e(G) \). In this case the only possibilities for \( \pi_e(G) \) are:

\[
\{1, 2, 3, 6\}, \ \{1, 2, 3, 4, 6\}, \ \{1, 2, 3, 5, 6\} \ \text{and} \ \{1, 2, 3, 4, 5, 6\},
\]
or equivalently \( 6 \in \mu(G) \subseteq \{4, 5, 6\} \). Note that, if \( 5 \in \pi(G) \), then \( G \) is \( C_{5,5} \)-group (i.e., a group whose order is divisible by 5 and in which the centralizer of a 5-element is a 5-group). Thus for every 5-element of \( G \), say \( x \), the subgraph \( \mathcal{P}^*(\langle x \rangle) \cong K_4 \) is a connected component of \( \mathcal{P}^*(G) \) (see Lemmas 1 and 2 (b)). In the case when \( G \) contains an element of order 4, say \( y \), noting \( 4 \in \mu(G) \) a similar reasoning shows that the subgraph \( \mathcal{P}^*(\langle y \rangle) \cong K_3 \) is a connected component of \( \mathcal{P}^*(G) \). In general case we have

\[
\mathcal{P}^*(\mathbb{Z}_n) \cong K_{n-1}, \quad n = 2, 3, 4, 5,
\]
each of them is planar. The power graph \( \mathcal{P}^*(\mathbb{Z}_6) \) is also planar, indeed we have the following planar drawing of the power graph \( \mathcal{P}^*(\mathbb{Z}_6) \):

![Figure 4. The power graph \( \mathcal{P}^*(\mathbb{Z}_6) \)](image)

As discussed above, we have

\[
\mathcal{P}^*(G) = \bigcup_{x \in G} \mathcal{P}^*(\langle x \rangle) = K_4 \oplus K_4 \oplus \cdots \oplus K_4 \oplus \bigcup_{o(x) \neq 5} \mathcal{P}^*(\langle x \rangle),
\]

On the other hand, since

\[
|\mathbb{Z}_2 \cap \mathbb{Z}_4| \leq 2, \quad |\mathbb{Z}_2 \cap \mathbb{Z}_6| \leq 2, \quad |\mathbb{Z}_4 \cap \mathbb{Z}_6| \leq 2 \quad \text{and} \quad |\mathbb{Z}_3 \cap \mathbb{Z}_6| \leq 3,
\]

the subgraph

\[
\bigcup_{o(x) \neq 5} \mathcal{P}^*(\langle x \rangle)
\]

of \( \mathcal{P}^*(G) \) consist of some planar graphs sharing a common vertex or a common edge, which implies that \( \bigcup_{o(x) \neq 5} \mathcal{P}^*(\langle x \rangle) \) and so \( \mathcal{P}^*(G) \) is planar. This completes the proof. \( \square \)

In [17, 44, 45, 46], under the assumption of finiteness of a group the authors study the structure of \( EPPO, EPO, EPPOM \) and \( EPOM \)-groups. In [15, 25, 26, 27], the authors determined also groups with small orders of elements.

**Remark 2.** Let \( G \) be a non-trivial finite group with \( \pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6\} \). The classification of all such groups are listed below:
(1) \( \pi_e(G) = \{1, 2\} \) and \( G \) is an elementary abelian 2-group.

(2) \( \pi_e(G) = \{1, 3\} \) and \( G \) is nilpotent of class at most 3 (\([25]\))

(3) \( \pi_e(G) = \{1, 5\} \) and \( G \) is a 5-group of exponent 5.

(4) \( \pi_e(G) = \{1, 2, 3\} \) and \( G = [N]K \) is a Frobenius group where either \( N \cong \mathbb{Z}_3^t \), \( K \cong \mathbb{Z}_2 \)

\( \text{or } N \cong \mathbb{Z}_3^{2t}, K \cong \mathbb{Z}_3 \) (\([30]\)).

(5) \( \pi_e(G) = \{1, 2, 4\} \) and \( G \) is a 2-group of exponent 4.

(6) \( \pi_e(G) = \{1, 2, 5\} \) and \( G \) is a \( EPO \)-group. (\([44]\))

(7) \( \pi_e(G) = \{1, 3, 5\} \) and \( G = [N]K \) is a Frobenius group where either

\( \textit{N is a 5-group which is nilpotent of class at most 2 and } K = 3 \), or

\( \textit{N is a 3-group which is nilpotent of class at most 3 and } K \textit{ is a 5-group. } ([15]) \)

(8) \( \pi_e(G) = \{1, 2, 3, 4\} \) and \( G = [N]K \) and one of the following occurs: \([4, 26]\)

\( \textit{N is } \mathbb{Z}_3^{2t} \textit{ and } K \cong \mathbb{Z}_4 \textit{ or } K \cong \mathbb{Q}_8 \textit{, and } G \textit{ is a Frobenius group.}

\( \textit{N is } \mathbb{Z}_3^{2t} \textit{ and } K \cong S_3. \)

\( \textit{N is a 2-group with exponent 4 and of class } 2 \textit{ and } K \cong \mathbb{Z}_3. \)

(9) \( \pi_e(G) = \{1, 2, 4, 5\} \) and \( G = [N]K \) and one of the following occurs: \([15]\)

\( \textit{N is an elementary abelian 2-group and } K \textit{ is a non-abelian group of order 10.}

\( \textit{N is an elementary abelian 5-group and } K \textit{ is isomorphic to a subgroup of } Q_8. \)

\( \textit{N is a 2-group which is nilpotent of class at most 6 and } K \textit{ is a 5-group.} \)

(10) \( \pi_e(G) = \{1, 2, 3, 5\} \) and \( G \cong A_5. ([44, 54]) \)

(11) \( \pi_e(G) = \{1, 2, 3, 6\} \) and \( G = PQ \) is a \{2, 3\}-group of exponent 6, where \( P \) is an elementary abelian 2-group and \( Q \) is a 3-group with exponent 3.

(\([46\), Theorem 1 (III)\])

(12) \( \pi_e(G) = \{1, 2, 3, 4, 5\} \) and one of the following holds: \([4, 27]\)

\( \textit{G } \cong A_6. \)

\( \textit{G } = [N]K, \textit{ where } N \textit{ is an elementary abelian 2-group and a direct sum of natural SL}(2, 4)-\textit{modules, and } K \cong A_5. \)

(13) \( \pi_e(G) = \{1, 2, 3, 4, 6\} \) and \( G \) is a solvable \( EPPMO-\textit{group}. \) In particular, \( G \) is a \{2, 3\}-group of exponent 12. ([17], Theorem 2.1 (I))

(14) \( \pi_e(G) = \{1, 2, 3, 5, 6\} \) and \( G \) is a solvable group. ([53], Theorem 2).

(15) \( \pi_e(G) = \{1, 2, 3, 4, 5, 6\} \) and one of the following holds: \([4]\)

\( \textit{G/}[Z_3^8]K \textit{ is a Frobenius group, where } K \cong [\mathbb{Z}_3^8] \textit{ or } K \cong SL(2, 3). \)

\( \textit{G/}O_2(G) \cong A_5 \textit{ and } O_2(G) \textit{ is elementary abelian and a direct sum of natural and orthogonal SL}(2, 4)-\textit{modules.} \)

\( \textit{G } \cong S_5 \textit{ or } G = S_6. \)

6 The Power Graph of Finite Group Which is Eulerian

Recall that a trail in a graph \( \Gamma \) is a walk with no repeated edge. A trail that traverses every edge and every vertex of \( \Gamma \) is called an Eulerian trail. A closed Eulerian trail is called an Eulerian circuit. A connected graph is said to be Eulerian if it contains an Eulerian circuit, and non-Eulerian otherwise. A well-known theorem due to Euler states a connected graph \( \Gamma \) is Eulerian if and only if all the vertices of \( \Gamma \) are of even degree. We would like to examine now the cyclic groups \( G \) for which the graph \( \mathcal{P}^*(G) \) is Eulerian. Our principal result in this section is the following.

**Proposition 4** Let \( G \) be a cyclic group of order \( n \). Then \( \mathcal{P}^*(G) \) is Eulerian if and only if \( n \) is a power of 2.

**Proof.** Suppose first that \( n \) is a power of 2. Then it follows from Lemma 13 that \( \mathcal{P}^*(G) \) is a complete graph, and so the degree of all vertices of \( \mathcal{P}^*(G) \) is \( n - 2 \), which is even. This shows that \( \mathcal{P}^*(G) \) is Eulerian.
Conversely, we assume that $\mathcal{P}^*(G)$ is Eulerian. We want to prove that $n$ is a power of 2. First of all, if $x$ is a generator of $G$, then by Lemma 2 (b), $d(x) = o(x) - 2 = n - 2$, which shows that $n$ must be even. We claim now that $n$ is a power of 2. Assume the contrary. Let $n$ have prime-power factorization

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $p_1, p_2, \ldots, p_k$ are distinct primes and $k \geq 2, \alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. Then

$$G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}.$$ 

Let $\mathbb{Z}_{p_i^{\alpha_i}} = \langle x_i \rangle$, $i = 1, 2, \ldots, k$ and $\bar{x} = (1, \bar{x}_2, \ldots, \bar{x}_k)$. Clearly $o(\bar{x}) = p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. In the sequel, we will show that

$$d(\bar{x}) = \prod_{i=2}^{k} p_i^{\alpha_i} - 2 + (2^{\alpha_1} - 1) \prod_{i=2}^{k} \phi(p_i^{\alpha_i}),$$

(5)

which is an odd number, because $\phi(p_i^{\alpha_i})$ is even for all $i \geq 2$. Therefore in this case we arrive to a contradiction, since the degree of all vertices in $\mathcal{P}^*(G)$ must be even.

The proof of the claim (5) requires some calculations. Let $\bar{z} = (\bar{x}_1^{m_1}, \bar{x}_2^{m_2}, \ldots, \bar{x}_k^{m_k})$ be an arbitrary element in $G \setminus \{1\}$, where $0 \leq m_i < p_i^{\alpha_i}$, $i = 1, \ldots, k$, and $m_1 + m_2 + \cdots + m_k \neq 0$. It will be convenient to consider among three cases:

(a) $m_1 \neq 0$ and $m_2 = m_3 = \cdots = m_k = 0$,
(b) $m_1 = 0$ and $m_2 + m_3 + \cdots + m_k \neq 0$,
(c) $m_1 \neq 0$ and $m_2 + m_3 + \cdots + m_k \neq 0$.

Suppose first that (a) holds, that is $\bar{z} = (\bar{x}_1^{m_1}, 1, \ldots, 1)$, where $1 \leq m_1 < p_1^{\alpha_1}$. Then $o(\bar{z})$ is a power of 2, and since $(o(\bar{x}), o(\bar{z})) = 1$, $\bar{x} \sim \bar{z}$ in $\mathcal{P}^*(G)$ by Lemma 2 (c).

Suppose next that (b) holds, that is $\bar{z} = (1, \bar{x}_2^{m_2}, \ldots, \bar{x}_k^{m_k})$, where $0 \leq m_i < p_i^{\alpha_i}$, $i = 2, \ldots, k$, and $m_2 + \cdots + m_k \neq 0$. In what follows, we will prove that $\bar{x} \sim \bar{z}$. To do this, we show that $\bar{z} \in \langle \bar{x} \rangle$, or equivalently $\bar{z} = \bar{x}^m$ for some $m$. Let us consider the following system of simultaneous congruences

\[
\begin{align*}
  x &\equiv m_2 \pmod{p_2^{\alpha_2}} \\
  x &\equiv m_3 \pmod{p_3^{\alpha_3}} \\
  &\vdots \\
  x &\equiv m_k \pmod{p_k^{\alpha_k}}.
\end{align*}
\]

(6)

Since $p_2^{\alpha_2}, p_3^{\alpha_3}, \ldots, p_k^{\alpha_k}$ are pairwise relatively prime, the Chinese Remainder Theorem tells us that there is a solution for the system of congruences (6), say $x = m$. Now, it is easy to see that $\bar{z} = \bar{x}^m$, as required.

Finally, assume that (c) holds, that is $\bar{z} = (\bar{x}_1^{m_1}, \bar{x}_2^{m_2}, \ldots, \bar{x}_k^{m_k})$, where $0 \leq m_i < p_i^{\alpha_i}$, $m_1 \neq 0$ and $m_2 + \cdots + m_k \neq 0$. In this case, we claim that

$$\bar{x} \sim \bar{z} \text{ if and only if } (m_i, p_i^{\alpha_i}) = 1 \text{ for each } i = 2, 3, \ldots, k.$$

For the proof of the claim, we assume first that $(m_i, p_i^{\alpha_i}) = 1$ for each $i = 2, 3, \ldots, k$. Let $o(\bar{x}_1^{m_1}) = l$ which is a power of 2, and let $n_i := lm_i$ for each $i = 2, 3, \ldots, k$. Then we obtain

$$\bar{z}^l = (1, \bar{x}_2^{n_2}, \ldots, \bar{x}_k^{n_k}).$$

Since for each $i = 2, 3, \ldots, k$, we have $(n_i, p_i^{\alpha_i}) = 1$, $n_i$ has a unique multiplicative inverse modulo $p_i^{\alpha_i}$, which is denoted by $n_1^*$, that is $n_i n_i^* \equiv 1 \pmod{p_i^{\alpha_i}}$. Now, we consider the system of simultaneous congruences

\[
\begin{align*}
  x &\equiv n_2^* \pmod{p_2^{\alpha_2}} \\
  x &\equiv n_3^* \pmod{p_3^{\alpha_3}} \\
  &\vdots \\
  x &\equiv n_k^* \pmod{p_k^{\alpha_k}}.
\end{align*}
\]

(7)
Again, by the Chinese Remainder Theorem there exists a solution for the system of congruences (7), say \(x = n\). Therefore, we obtain

\[ z^{ln} = (1, \bar{x}_2^{n_2}, \ldots, \bar{x}_k^{n_k}) = (1, \bar{x}_2, \ldots, \bar{x}_k) = \bar{x}, \]

which shows that \(\bar{x} \in \langle \bar{z} \rangle\), and so \(\bar{x} \sim \bar{z}\), as claimed.

Now suppose that there exists an integer \(i \in \{2, 3, \ldots, k\}\) such that \((m_i, p_i^{\alpha_i}) \neq 1\). Hence \(p_i\) divides \(m_i\). We want to show that \(\bar{x} \sim \bar{z}\). Suppose to the contrary that \(\bar{x} \sim \bar{z}\). Then by the definition we have \(\bar{x} \in \langle \bar{z} \rangle\) or \(\bar{z} \in \langle \bar{x} \rangle\). Clearly \(\bar{z} \notin \langle \bar{x} \rangle\), because \(\bar{x}_1^{m_1} \neq 1\). On the other hand, if \(\bar{x} \in \langle \bar{z} \rangle\), then \(\bar{x} = \bar{z}^l\) for some \(l\). In particular, we conclude that \(\bar{x}_i = \bar{x}_i^{m_i}\), and this means that \(lm_i \equiv 1 \pmod{p_i^{\alpha_i}}\), or equivalently \(p_i^{\alpha_i}\) divides \(lm_i - 1\). But this is contrary to the fact that \(p_i|m_i\).

Finally, we are now ready to calculate the degree of vertex \(x\) in \(\mathcal{P}^*(G)\). By what observed above (cases \((a) - (c)\)) we obtain

\[
d(\bar{x}) = |\langle \bar{x} \rangle \setminus \{1, \bar{x}\}| + |\langle \bar{x}_1^{m_1}, \bar{x}_2^{m_2}, \ldots, \bar{x}_k^{m_k} \rangle| = 0 < m_1 < p_i^{\alpha_i}, (m_i, p_i^{\alpha_i}) = 1, 2 \leq i \leq k |
\]

\[= \prod_{i=2}^k p_i^{\alpha_i} - 2 + (2^{\alpha_1} - 1) \prod_{i=2}^k \phi(p_i^{\alpha_i}),\]

which completes the proof. \(\square\)

### 7 Cut-edges in Power Graphs

For any edge \(e\) of a graph \(\Gamma\), if \(c(\Gamma - e) = c(\Gamma) + 1\), the edge \(e\) is called a cut edge of \(\Gamma\), where \(c(\Gamma)\) denotes the number of connected components of \(\Gamma\). Note that the following characterization of cut edges is well-known: An edge \(e\) of a graph \(\Gamma\) is a cut edge if and only if \(e\) belongs to no cycle of \(\Gamma\).

**Theorem 6** Let \(G\) be a finite group and \(\Gamma = \mathcal{P}^*(G)\). An edge \(e = xy \in \Gamma\) is a cut edge if and only if \(d_{\Gamma}^-(x) = d_{\Gamma}^+(x) = 1\).

**Proof.** \((\Longrightarrow)\) Assume that \(e = xy\) is a cut edge of \(\Gamma\). Then by definition we have \(y \in \langle x \rangle\) or \(x \in \langle y \rangle\), and hence \(o(y)o(x)\) or \(o(x)o(y)\). Without loss of generality we may assume that \(y \in \langle x \rangle\) and so \(o(y)o(x)\). If \(o(x) \geq 4\), then \(o(y)o(x) \geq 2\), this means that the cyclic group \(\langle x \rangle\) has at least two generators which forces \(e\) lies in a cycle of \(\Gamma\). Therefore \(o(x) \leq 3\). If \(o(x) = 2\), then \(x\) and \(y\) would be two involutions which is joined be an edge, a contradiction by Lemma 2 (c). Finally, we conclude that \(o(x) = 3\), \(y = x^2\) and from Lemma 3 (e) it follows that \(N_{\Gamma}^-(x) \setminus \{x^2\} = N_{\Gamma}^-(x^2) \setminus \{x\} = \{y\} = \emptyset\), otherwise \(e = xx^2\) lies on a cycle of \(\Gamma\) and \(N_{\Gamma}^+(x) \setminus \{x^2\} = N_{\Gamma}^+(x^2) \setminus \{x\} \neq \emptyset\), because \(x\) an element of order 3). Therefore \(d_{\Gamma}^-(x) = d_{\Gamma}^+(x) = 1\).

\((\Longleftarrow)\) Conversely, assume that \(d_{\Gamma}^-(x) = d_{\Gamma}^+(x) = 1\). Then \(N_{\Gamma}^+(x) = \{1, x^2\}\), \(N_{\Gamma}^-(x) = \{x^2\}\) and so \(o(x) = 3\). This means that \(y = x^2\) and \(e = xx^2\) is a singleton edge in \(\Gamma\), and hence the vertices \(x\) and \(x^2\) are two isolated vertices in \(\Gamma - e\), which implies that \(c(\Gamma - e) = c(\Gamma) + 1\). Therefore \(e\) is a cut edge in \(\Gamma\), as claimed. \(\square\)

The condition \(d_{\Gamma}^-(x) = d_{\Gamma}^+(x) = 1\) in Theorem 6 is equivalent to the following condition:

“The group \(G\) has an element \(x\) of order 3 such that for all elements \(y \in G \setminus \langle x \rangle\), \(x \notin \langle y \rangle\).”

There are many examples of such groups, for instance:

\[S_3 = \langle Z_3 \rangle Z_2, \quad F_{21} = [Z_7]Z_3, \quad S_4, \quad S_3 \times Z_3, \quad S_4 \times Z_3 \quad \text{and} \quad S_3 \times S_3.\]

Moreover, an infinite family of such groups is

\[\langle Z_7 \rangle Z_3, \quad [Z_7 \times Z_7]Z_3, \quad [Z_7 \times Z_7 \times Z_7]Z_3, \quad [Z_7 \times Z_7 \times Z_7 \times Z_7]Z_3, \ldots\]
It is worth noting that this family of groups can be regarded as a special case of EPPO-groups. As a matter of fact, since the power graph associated with an EPPO-group is the union of complete subgraphs with exactly one common vertex (Lemma 2 (b)), all EPPO-groups \( G \) with spectrum \( \pi_e(G) = \{1, 3, p^q, q^t, \ldots\} \) are such examples. Therefore, we focus our attention on EPPO-groups.

The complete classification of finite EPPO-groups is given in [45, Theorems 2.4 and 3.1].

**Theorem 7** Let \( G \) be a finite EPPO-group. Then we have

1. If \( G \) is solvable, then \( |\pi(G)| \leq 2 \). Moreover, if \( |G| = p^a q^b \), \( P_1 \) is the maximal normal \( p \)-subgroup of \( G \) and \( |P_1| = p^3 \), then the Sylow subgroups of \( G/P_1 \) are cyclic or generalized quaternion, \( p^a - \gamma | q - 1 \) and \( \gamma = kb \) where \( b \) is the exponent of \( p \) (mod \( q^b \)) for the case of cyclic or \( p \) (mod \( 2^b - 1 \)) for the case of generalized quaternion (in this case \( \gamma = \alpha \)).

2. If \( G \) is non-solvable, then one of the following hold:
   - \( G \) is simple and \( G \cong L_2(q) \), \( q = 5, 7, 9, 17 \); \( L_3(4) \), \( Sz(8) \) or \( Sz(32) \),
   - \( G \cong M_{10} \), or
   - \( G \) has an elementary abelian 2-subgroup \( P \), \( P \triangleleft G \) and \( \frac{G}{P} \cong L_2(5) \), \( L_2(8) \), \( Sz(8) \) or \( Sz(32) \).

An immediate consequence of Theorem 7 is the following which gives the structure of finite EPPO-groups \( G \) with spectrum \( \pi_e(G) = \{1, 3, p^q, q^t, \ldots\} \).

**Corollary 4** Let \( G \) be an EPPO-group with \( \pi_e(G) = \{1, 3, p^q, q^t, \ldots\} \), where \( p, q, \ldots \) are primes not equal to 3. Then \( G \) is isomorphic to one of the following groups:

1. If \( G \) is solvable and non-nilpotent, then \( |G| = 3^a q^b \). If \( Q \) is the maximal normal \( q \)-subgroup \((q \neq 3)\) of \( G \), then
   1.1. \( q \neq 2 \), \( G \) has the chief factors \( 3, 3, \ldots, 3; q^b, q^t, \ldots, q^b, \beta = kb \) and \( b \) is the exponent of \( q \) (mod \( 3^b \)), \( Q \) is the Sylow \( q \)-subgroup of \( G \)
   1.2. \( q = 2 \) and the Sylow 2-subgroups of \( G \) are not generalized quaternion, \( G \) has the chief factors \( 2; 3; 3, \ldots, 3; 2^b_1, 2^b_2, \ldots, 2^b_k, b|b_i \) and \( b \) is the exponent of 3 (mod \( 2^b \)).
   If the maximal normal subgroup of \( G \) is a 3-subgroup, then
   1.3. \( q = 2 \) and the Sylow 2-subgroups of \( G \) are generalized quaternion, \( G \) has a chief factors \( 2, 2, \ldots, 2; 3^b_1, 3^b_2, \ldots, 3^b_k, b|b_i \) and \( b \) is the exponent of 3 (mod \( 2^b - 1 \)).

2. If \( G \) is not solvable, then \( G \) is one of the following groups:
   2.1. \( G \) is simple and \( G \cong L_2(q) \), \( q = 5, 7, 9 \); or \( L_3(4) \).
   2.2. \( G \cong M_{10} \), or
   2.3. \( G \) has an elementary abelian 2-subgroup \( Q \), \( Q \triangleleft G \) and \( G/Q \cong L_2(5) \).

### 8 On the Number of Edges in \( \mathcal{P}^*(G) \) and Related Results

**Lemma 14** Let \( G \) be a finite group. Then the number of edges \( e^* \) of \( \mathcal{P}^*(G) \) is given by

\[
2e^* = \sum_{g \in G^\#} (2\sigma(g) - \phi(\sigma(g)) - 3),
\]

where \( G^\# = G \setminus \{1\} \). Especially, if \( G \) is an elementary \( p \)-group of order \( p^m \), then

\[
e^* = (p^m - 1)(p - 2)/2.
\]

In particular, if \( G \) is an elementary abelian 2-group, then the graph \( \mathcal{P}^*(G) \) is a null graph.
Proof. The first assertion follows immediately from [8, Theorem 4.2]. The rest of lemma can be verified by direct computations. □

An immediate consequence of Lemma 14 is the following.

Corollary 5 Let $G$ be a finite group. Then, there holds

$$2e^* = \sum_{n \in \pi_e(G) \setminus \{1\}} s_n (2n - \phi(n) - 3),$$

where $s_n$ is the number of elements with order $n$.

The next result is also a simple consequence of Theorem 5 and Lemma 14.

Corollary 6 Let $G$ be a nontrivial finite group such that the graph $P^*(G)$ is 2-partite. Then the number of edges of $P^*(G)$ is equal to half the number of elements of order 3 in $G$.

Below is a number-theoretic proposition, whose validity is verified by direct computations. However, we will prove it using a graph approach.

Proposition 5 Let $p$ be a prime number and $n$ a positive integer. Then there holds

$$\sum_{i=1}^{n} \phi(p^i)(2p^i - \phi(p^i) - 3) = 2 \left( p^n - 1 \right).$$

Proof. Let $G$ be a cyclic group of order $p^n$. Then, on the one hand, the graph $P^*(G)$ is a complete graph on $p^n - 1$ vertices and then the number of its edges is equal to

$$e^* = \left( p^n - 1 \right). \quad (8)$$

On the other hand, we know that $\pi_e(G) = \{1, p, p^2, \ldots, p^n\}$ and by Corollary 5, we obtain that

$$2e^* = \sum_{i=1}^{n} s_{p^i} (2p^i - \phi(p^i) - 3), \quad (9)$$

where $s_m$ signifies the number of elements with order $m$. Note that $s_m = k\phi(m)$, where $k$ is number of cyclic subgroups of order $m$ and $\phi(m)$ Euler totient function. Moreover, since $G$ is a cyclic group, $k = 1$ for all $m \in \pi_e(G)$. Thus, $s_{p^i} = \phi(p^i)$ for each $i = 1, 2, \ldots, n$. If this is substituted in Eq. (9), then we obtain

$$e^* = \frac{1}{2} \sum_{i=1}^{n} \phi(p^i)(2p^i - \phi(p^i) - 3). \quad (10)$$

The result now follows by comparing Eqs. (8) and (10). □

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