The stability problem and special solutions for the 5-components Maxwell-Bloch equations

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AMS Classification: 34D20, 34C37, 34C45
Keywords: stability, Hamilton-Poisson system, bi-Hamiltonian structure, homoclinic orbit, periodic solution, invariant set.

Abstract
For the 5-components Maxwell-Bloch system the stability problem for the isolated equilibria is completely solved. Using the geometry of the symplectic leaves, a detailed construction of the homoclinic orbits is given. Studying the problem of invariant sets for the system, we discover a rich family of periodic solutions in explicit form.

1 Introduction
After averaging and neglecting non-resonant terms, the unperturbed Maxwell-Bloch dynamics in the rotating wave approximation (RWA) is given by

\[
\begin{align*}
\dot{X} &= Y \\
\dot{Y} &= XZ \\
\dot{Z} &= -\frac{1}{2}(XY^* + X^*Y),
\end{align*}
\]

where \(X, Y\) are complex scalar functions, that are denoting the self-consistent electric field and respectively the polarizability of the laser-matter, \(Z\) is a real scalar function, which denotes the difference of its occupation numbers. The superscript * stands for the complex conjugate. For more details about the history and physical interpretations of this system see [9], [10], [11].

Writing \(X = x_1 + ix_2, Y = y_1 + iy_2\) and \(Z = z\) the above system transforms into the 5-components Maxwell-Bloch system

\[
\begin{align*}
\dot{x}_1 &= y_1 \\
\dot{y}_1 &= x_1z \\
\dot{x}_2 &= y_2 \\
\dot{y}_2 &= x_2z \\
\dot{z} &= -(x_1y_1 + x_2y_2).
\end{align*}
\]

The Maxwell-Bloch system in the form [11] has the advantage of a rich underlying geometrical structure that can be used in the study of its dynamical behavior.

The system [11] admits a Hamilton-Poisson formulation, where the Poisson tensor is given by

\[
J(x_1, y_1, x_2, y_2, z) =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & x_1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & x_2 \\
0 & -x_1 & 0 & -x_2 & 0
\end{bmatrix}
\]
and the Hamiltonian function is given by
\[ H(x_1, y_1, x_2, y_2, z) = \frac{1}{2} (y_1^2 + y_2^2 + z^2). \]

The system has two additional constants of motion, namely the Casimir of the Poisson structure \( J \), given by
\[ C(x_1, y_1, x_2, y_2, z) = \frac{1}{2} (x_1^2 + x_2^2) + z \]
and a constant of motion derived from a bi-Hamiltonian structure of the system (1.1) (see [10]) given by
\[ I(x_1, y_1, x_2, y_2, z) = x_2 y_1 - x_1 y_2. \]

A commuting property of the constants of motion \( H \) and \( I \) holds, i.e. \( \{H, I\} = 0 \), where \( \{\cdot, \cdot\} \) is the Poisson bracket associated to the Poisson tensor \( J \).

## 2 Stability of equilibria

By a direct computation we obtain three families of equilibria for the system (1.1):
\[ E_1 = \{(0, 0, 0, 0, M) | M \in \mathbb{R}^* \}; \quad E_2 = \{(M, 0, N, 0, 0) | M, N \in \mathbb{R}, M^2 + N^2 \neq 0 \}; \quad E_3 = \{(0, 0, 0, 0, 0) \}. \]

It is a well known fact that the dynamics of a Hamilton-Poisson system is foliated by the symplectic leaves associated to the Poisson structure. In our case the regular symplectic leaves are given by the connected components corresponding to pre-images of regular values of the Casimir function \( C \). We denote by \( O_c = C^{-1}(c), c \in \mathbb{R} \) the regular symplectic leaves of the Poisson structure \( J \).

The restriction of the dynamics (1.1) to a regular leaf \( O_c \) becomes a completely integrable Hamiltonian system
\[ (O_c, \omega_{O_c}, H|_{O_c}), \tag{2.1} \]
where the second commuting constant of motion is \( I|_{O_c} \). We will study the stability problem of equilibria on regular leaves \( O_c \) analogously to the approach used in [2].

The equilibria of the Hamiltonian system (2.1) can be divided in two types:
\[ \mathcal{K}_0 := \{(x_1, y_1, x_2, y_2, z) \in O_c | d(H|_{O_c}) (x_1, y_1, x_2, y_2, z) = 0, d(I|_{O_c}) (x_1, y_1, x_2, y_2, z) = 0 \}; \]
\[ \mathcal{K}_1 := \{(x_1, y_1, x_2, y_2, z) \in O_c | d(H|_{O_c}) (x_1, y_1, x_2, y_2, z) = 0, d(I|_{O_c}) (x_1, y_1, x_2, y_2, z) \neq 0 \}. \]

**Proposition 2.1.** On a regular symplectic leaf \( O_c \) we have the following characterization for the equilibria:
\[ \mathcal{K}_0 = O_c \cap (E_1 \cup E_3); \quad \mathcal{K}_1 = O_c \cap E_2. \]

**Proof.** Because (2.1) is a Hamiltonian system on a symplectic manifold the condition \( d(H|_{O_c}) (e) = 0 \) is verified for any equilibrium point \( e \in O_c \).

Let \( e_2 \in O_c \cap \mathcal{E}_2 \). Then
\[ T_{e_2} O_c = \{ \tilde{v} = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5 | <\tilde{v}, \nabla C(e_2)> = 0 \} = \{ \tilde{v} \in \mathbb{R}^5 | v_1 M + v_3 N + v_5 = 0 \}. \]

We also have \( d(I|_{O_c})(e_2) = N d y_1 - M d y_2 \). Taking, for example, \( \tilde{v} = (-N, N, M, -M, 0) \) \( \in T_{e_2} O_c \) we have
\[ d(I|_{O_c})(\tilde{v}) = M^2 + N^2 \neq 0, \]
which proves that \( d(I|_{O_c})(e_2) \neq 0 \).

For the equilibria \( e \in E_1 \cup E_3 \) the condition \( d(I|_{O_c})(e) = 0 \) is trivially verified. \( \square \)

The commutativity of the constants of motion \( H|_{O_c} \) and \( I|_{O_c} \) with respect to the symplectic form \( \omega_{O_c} \) implies that at an equilibrium point \( e \in O_c \) we have
\[ [DX_{H|_{O_c}}(e), DX_{I|_{O_c}}(e)] = 0, \]
where \( DX_{H|_{O_c}}(e) \) and \( DX_{I|_{O_c}}(e) \) are the derivatives of the vector fields \( X_{H|_{O_c}} \) and \( X_{I|_{O_c}} \) at the equilibrium \( e \) and consequently \( DX_{H|_{O_c}}(e), DX_{I|_{O_c}}(e) \) are infinitesimally symplectic relative to the symplectic form \( \omega_{O_c}(e) \) on the vector space \( T_e O_c \).
Definition 2.1. An equilibrium point \( e \in \mathcal{K}_0 \) is called non-degenerate if \( DX_{H|\omega_c}(e) \) and \( DX_{I|\omega_c}(e) \) generate a Cartan subalgebra of the Lie algebra of infinitesimal linear transformations of the symplectic vector space \((T_\omega \mathcal{O}_c, \omega_\mathcal{O}_c(e))\). A Cartan subalgebra of the Lie algebra \( sp(4, \mathbb{R}) \) is a two dimensional commutative sub-algebra which contains an element whose eigenvalues are all distinct.

It follows that for a non-degenerate equilibrium belonging to \( \mathcal{K}_0 \) the matrices \( DX_{H|\omega_c}(e) \) and \( DX_{I|\omega_c}(e) \) can be simultaneously conjugated to one of the following four Cartan sub-algebras

\[
\begin{align*}
\text{Type 1:} & \quad \begin{bmatrix} 0 & 0 & -A & 0 \\ 0 & 0 & 0 & -B \\ A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{bmatrix} & \quad \text{Type 2:} & \quad \begin{bmatrix} -A & 0 & 0 & 0 \\ 0 & 0 & 0 & -B \\ 0 & 0 & A & 0 \\ 0 & B & 0 & 0 \end{bmatrix} \\
\text{Type 3:} & \quad \begin{bmatrix} -A & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & B & 0 & B \end{bmatrix} & \quad \text{Type 4:} & \quad \begin{bmatrix} -A & -B & 0 & 0 \\ B & -A & 0 & 0 \\ 0 & 0 & A & -B \\ 0 & 0 & B & A \end{bmatrix}
\end{align*}
\]

(2.2)

where \( A, B \in \mathbb{R} \) (see, e.g., [4], Theorems 1.3 and 1.4).

Equilibria of type 1 are called center-center with the corresponding eigenvalues for the linearized system: \( iA, -iA, iB, -iB \).

Equilibria of type 2 are called center-saddle with the corresponding eigenvalues for the linearized system: \( A, -A, iB, -iB \).

Equilibria of type 3 are called saddle-saddle with the corresponding eigenvalues for the linearized system: \( A, -A, B, -B \).

Equilibria of type 4 are called focus-focus with the corresponding eigenvalues for the linearized system: \( A + iB, A - iB, -A + iB, -A - iB \).

Theorem 2.2. We have the following stability behavior for the equilibria in \( \mathcal{O}_c \cap \mathcal{E}_1 \):

(i) The equilibrium point \( \mathcal{O}_c \cap \mathcal{E}_1 = \{(0, 0, 0, 0, c)\} \) for \( c > 0 \) is a non-degenerate equilibrium of type focus-focus and consequently unstable.

(ii) The equilibrium point \( \mathcal{O}_c \cap \mathcal{E}_1 = \{(0, 0, 0, 0, c)\} \) for \( c < 0 \) is a non-degenerate equilibrium of type center-center and consequently stable.

Proof. (i) For the linearized systems at the equilibrium \((0, 0, 0, 0, c)\) we have:

\[
DX_{H|\omega_c}(0, 0, 0, 0, c) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & 0 \end{bmatrix}
\]

and its characteristic polynomial has the non-distinct eigenvalues \( \sqrt{c}, \sqrt{c}, -\sqrt{c}, -\sqrt{c} \) and respectively

\[
DX_{I|\omega_c}(0, 0, 0, 0, c) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\]

and its characteristic polynomial has the non-distinct eigenvalues \( i, -i, -i, i \).

To decide the type of stability we need to determine the non-degeneracy of the equilibrium \((0, 0, 0, 0, c)\), i.e. we have to find a linear combination \( DX_{H|\omega_c}(0, 0, 0, 0, c) + \alpha DX_{I|\omega_c}(0, 0, 0, 0, c) \), where \( \alpha \) is a non-zero real number, that has distinct eigenvalues. The characteristic polynomial of such a linear combination is given by

\[
t^4 + (2\alpha^2 - 2c)t^2 + (\alpha^2 + c)^2.
\]
After the substitution $t^2 = s$ we obtain the quadratic polynomial
\[s^2 + (2\alpha^2 - 2c)s + (\alpha^2 + c)^2,\]
which has the discriminant $\Delta = -16\alpha^2 < 0$ and therefore has two distinct complex roots. It follows that the characteristic polynomial $t^4 + (2\alpha^2 - 2c)t^2 + (\alpha^2 + c)^2$ has four distinct complex eigenvalues of the form $A + iB, A - iB, -A + iB, -A - iB$ with $A, B \in \mathbb{R}^*$. Consequently, the equilibrium $(0, 0, 0, 0, c)$ for $c > 0$ is a non-degenerate equilibrium of focus-focus type for the dynamics (2.1) and therefore unstable for this dynamics.

Similar computations lead to the proof of (ii). \(\square\)

Although the equilibrium $(0, 0, 0, 0, 0) \in \mathcal{O}_0$ belongs to $\mathcal{X}_0$, it is a degenerate equilibrium in the sense of Definition 2.1. Indeed, any linear combination $aDX_{\mathcal{H}(0)}(0, 0, 0, 0, 0) + \beta DX_{\mathcal{H}(0)}(0, 0, 0, 0, 0)$ has the characteristic polynomial $(t^2 + \beta)^2$, which has non-distinct eigenvalues. Its stability property can be established using an algebraic method (see [1], [2], [6], [7]). More precisely, the system of algebraic equations
\[H(x_1, y_1, x_2, y_2, z) = H(0, 0, 0, 0, 0), I(x_1, y_1, x_2, y_2, z) = I(0, 0, 0, 0, 0), C(x_1, y_1, x_2, y_2, z) = C(0, 0, 0, 0, 0)\]
has as unique solution the equilibrium $(0, 0, 0, 0, 0)$, leading to the following stability result.

**Theorem 2.3.** The equilibrium $(0, 0, 0, 0, 0)$ is degenerate and stable with respect to the dynamics (1.1).

### 3 Homoclinic orbits

In this section we will give an explicit form of the homoclinic orbits for the unstable equilibria of focus-focus type. This type of equilibria belong to symplectic orbits $\mathcal{O}_c$ with $c > 0$.

In order to compute the homoclinic orbits, we introduce a local system of coordinates around the equilibrium point $e_c = (0, 0, 0, 0, c) \in \mathcal{O}_c$. The local system of coordinates is given by
\[\Phi : \mathbb{R}^5 \rightarrow \mathbb{R}^5, \quad (r_1, \theta, y_1, y_2, c) \mapsto \begin{cases} x_1 = r_1 \cos \theta \\ x_2 = r_1 \sin \theta \\ y_1 = y_1 \\ y_2 = y_2 \\ z = c - \frac{1}{2}r_1^2. \end{cases}\]

Freezing the parameter $c$ we obtain the local system of coordinates on the symplectic orbit $\mathcal{O}_c$ around the equilibrium point $e_c$:
\[\Phi_c : \mathbb{R}^4 \rightarrow \mathcal{O}_c \setminus \{e_c\}, \quad (r_1, \theta, y_1, y_2) \mapsto \begin{cases} x_1 = r_1 \cos \theta \\ x_2 = r_1 \sin \theta \\ y_1 = y_1 \\ y_2 = y_2. \end{cases}\]

As we have excluded the equilibrium point $e_c$ we can work under the assumption that $r_1 \neq 0$. The advantage of using polar coordinates in the study of bi-focal homoclinic orbits in four dimensions can be ascertained in [8]. By a straightforward computation we obtain that the reduced system on the symplectic leaf $\mathcal{O}_c$ is given by
\[
\begin{align*}
\dot{r}_1 &= y_1 \cos \theta + y_2 \sin \theta \\
\dot{\theta} &= y_2 \cos \theta - y_1 \sin \theta \\
\dot{y}_1 &= r_1 \cos \theta \left( c - \frac{1}{2}r_1^2 \right) \\
\dot{y}_2 &= r_1 \sin \theta \left( c - \frac{1}{2}r_1^2 \right).
\end{align*}
\] (3.1)

Using a continuity argument and the fact that $I(x_1, y_1, x_2, y_2, z) = x_2y_1 - x_1y_2$ is a constant of motion we obtain that if there exists a homoclinic it should belong to the connected component of level set...
$I^{-1}(I(c_c)) = I^{-1}(0)$ that contains $c_c$. If a curve $c(t) = (r(t), \theta(t), y_1(t), y_2(t))$ is a homoclinic, then it has to be a solution for the system (3.1) and to satisfy the following equation for all $t$:

$$r_1(t)(y_1(t) \sin \theta(t) - y_2(t) \cos \theta(t)) = 0.$$  

This implies that $\dot{\theta}(t) = 0$ and thus $\theta(t) = \theta_0$ constant for all $t$. By differentiation and substitution we obtain the following second order equation

$$\dot{r}_1 = r_1 \left( c - \frac{1}{2} r_1^2 \right).$$

Making the change of variable $r_1 = 2\sqrt{c} \, \tilde{r}_1$ and the time re-parametrization $\sqrt{c} \, t = \tilde{t}$ we obtain the equation

$$\tilde{r}_1(\tilde{t}) = \tilde{r}_1(\tilde{t}) - 2\tilde{r}_1^2(\tilde{t}).$$

It is well known that this second order differential equation has as solutions $\pm \cosh(\tilde{t})$, $1 = \pm \text{sech}(\tilde{t})$. Consequently, we obtain $r_1(t) = \pm \sqrt{c} \, \text{sech}(\sqrt{c} \, t)$.

Substituting $r_1(t)$ in the expression of the local parametrization $\Phi_c$, and for $z$ in the expression of local parametrization $\Phi$ and integrating for $y_1$ and $y_2$ in (3.1) we obtain the homoclinic solutions

$$\begin{cases}
  x_1(t) = \pm 2\sqrt{c} \text{sech}(\sqrt{c} t) \cos \theta_0 \\
  x_2(t) = \pm 2\sqrt{c} \text{sech}(\sqrt{c} t) \sin \theta_0 \\
  y_1(t) = \mp 2c \text{sech}(\sqrt{c} t) \tanh(\sqrt{c} t) \cos \theta_0 \\
  y_2(t) = \mp 2c \text{sech}(\sqrt{c} t) \tanh(\sqrt{c} t) \sin \theta_0 \\
  z(t) = c(1 - 2 \text{sech}^2(\sqrt{c} t)).
\end{cases}$$

The above homoclinic orbits, using different parametrization and arguments, have been discussed in [9], [10].

4 Invariant sets and periodic orbits

We will look for invariant sets of the system (1.1) using the technique presented in [3]. We have the following vectorial conserved quantity $F : \mathbb{R}^3 \to \mathbb{R}^3$, $F(p) = (H(p), I(p), C(p))$. In [3], Theorem 2.3, it has been proved that the set $M^F_p = \{ p \in \mathbb{R}^3 : \text{rank} \, \nabla F(p) = 2 \}$ is invariant under the dynamics of the system. By direct computation we obtain that $M^F_{(2)} = M_1 \cup M_2$, where

$$M_1 := \left\{ (x_1, y_1, x_2, -\frac{x_1 y_1}{x_2}, -\frac{y_1^2}{x_2^2}) \mid x_2 \neq 0 \right\};$$

$$M_2 := \left\{ (x_1, 0, 0, y_2, -\frac{y_2^2}{x_1}) \mid x_1 \neq 0 \right\}.$$  

The union $M_1 \cup M_2$, which is a connected set in $\mathbb{R}^5$, is invariant under the dynamics (3.1), but neither the set $M_1$ nor the set $M_2$ are invariant under this dynamics. The vector field corresponding to (1.1) is tangent to the sub-manifold $M_1$ and the restricted dynamics on $M_1$ is given by

$$\begin{cases}
  \dot{x}_1 = y_1 \\
  \dot{y}_1 = -\frac{x_1 y_1^2}{x_2} \\
  \dot{x}_2 = -\frac{x_1 y_2}{x_2^2}.
\end{cases}$$

We notice that the above dynamical system has two conserved quantities, $f_1, f_2 : M_1 \to \mathbb{R}$, $f_1(x_1, y_1, x_2) = x_1^2 + x_2^2$ and $f_2(x_1, y_1, x_2) = y_1^2$. Using these conserved quantities and choosing an initial condition $x_1^0, y_1^0, x_2^0$ with $x_2^0 \neq 0$ and $y_1^0 \neq 0$ we can explicitly solve the system (1.1):

$$\begin{cases}
  x_1(t) = x_1^0 \sin \left( \frac{y_1^0}{x_2^0} t \right) + x_1^0 \cos \left( \frac{y_1^0}{x_2^0} t \right) \\
  y_1(t) = -\frac{y_1^0}{x_2^0} \left( x_1^0 \sin \left( \frac{y_1^0}{x_2^0} t \right) - x_2^0 \cos \left( \frac{y_1^0}{x_2^0} t \right) \right) \\
  x_2(t) = -x_2^0 \sin \left( \frac{y_1^0}{x_2^0} t \right) + x_2^0 \cos \left( \frac{y_1^0}{x_2^0} t \right).
\end{cases}$$
Notice that if \( y_0 = 0 \) we obtain as constant solutions the equilibrium points from \( E_2 \subset M_1 \). The above solution is defined on time intervals \((t_k, t_{k+1})\), where \( t_k = x^0_1 \vartheta + k \pi y^0_1 \) with \( k \in \mathbb{Z} \) and \( \vartheta \in [0, 2\pi) \) is the unique real number such that \( \sin \vartheta = \frac{x^0_1}{\sqrt{(x^0_1)^2 + (x^0_2)^2}} \) and \( \cos \vartheta = \frac{x^0_2}{\sqrt{(x^0_1)^2 + (x^0_2)^2}} \). For time values \( t_k \) the above solution exits the set \( M_1 \) and punctures the set \( M_2 \), thus making the union \( M_1 \cup M_2 \) an invariant set. Although the solution starting from \( M_1 \) is not complete, we can construct a complete periodic solution for the initial system \((1.1)\) given by

\[
\begin{align*}
x_1(t) &= x^0_1 \sin \left( x^0_1 t \right) + x^0_2 \cos \left( x^0_2 t \right), \\
y_1(t) &= -y^0_1 \left( x^0_1 \sin \left( x^0_2 t \right) - x^0_2 \cos \left( x^0_2 t \right) \right), \\
x_2(t) &= -x^0_1 \sin \left( x^0_2 t \right) + x^0_2 \cos \left( x^0_2 t \right), \\
y_2(t) &= -y^0_1 \left( x^0_2 \sin \left( x^0_1 t \right) + x^0_1 \cos \left( x^0_1 t \right) \right), \\
z(t) &= -\frac{y^0_1}{x^0_1^2 + x^0_2^2}.
\end{align*}
\]

Acknowledgements. Petre Birtea was supported by a grant of the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-RU-TE-2011-3-0006. Ioan Cașu was supported by a grant of the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-ID-PCE-2011-3-0571.

References

[1] D. Aeyels, On stabilization by means of the Energy-Casimir method, Systems & Control Letters 18 (1992) 325–328.

[2] P. Birtea, I. Cașu, T. Ratiu, M. Turhan, Stability of equilibria for the so(4) free rigid body, Journal of Nonlinear Science 22 (2012) 187–212.

[3] P. Birtea, D. Comănescu, Invariant critical sets of conserved quantities, Chaos, Solitons and Fractals 44 (2011) 693–701.

[4] A.V. Bolsinov, A.T. Fomenko, Integrable Hamiltonian Systems: Geometry, Topology, Classification, Chapman & Hall/CRC Press, 2004.

[5] D. Comănescu, The stability problem for the torque-free gyrostat investigated by using algebraic methods, Applied Mathematics Letters 25 (2012) 1185–1190.

[6] D. Comănescu, Stability of equilibrium states in the Zhukovski case of heavy gyrostat using algebraic methods, Mathematical Methods in the Applied Sciences (2012) DOI: 10.1002/mma.2595.

[7] D. Comănescu, A note on stability of the vertical uniform rotations of the heavy top, ZAMM - Journal of Applied Mathematics and Mechanics - Zeitschrift für Angewandte Mathematik und Mechanik (2012) DOI: 10.1002/zamm.201200162.

[8] A.C. Fowler, C.T. Sparrow, Bifocal homoclinic orbits in four dimensions, Nonlinearity 4 (1991) 1159–1182.

[9] D.D. Holm, G. Kovacic, B. Sundaram, Chaotic laser-matter interaction, Physics Letters A 154 (1991) 346–352.

[10] D. Huang, Bi-Hamiltonian structure and homoclinic orbits of the Maxwell-Bloch equations with RWA, Chaos, Solitons and Fractals 22 (2004) 207–212.

[11] A. Nath, D.S. Ray, Horseshoe-shaped maps in chaotic dynamics of atom-field interaction, Physical Review A 36 (1987) 431–434.