’t Hooft surface operators in five dimensions and elliptic Ruijsenaars operators

Yutaka Yoshida

Department of Physics, Tokyo Institute of Technology, Tokyo, 152-8551, Japan

Abstract

We introduce codimension three magnetically charged surface operators in five-dimensional (5d) $\mathcal{N} = 1$ supersymmetric gauge on $T^2 \times \mathbb{R}^3$. We evaluate the vacuum expectation values (vevs) of surface operators by supersymmetric localization techniques. Contributions of Monopole bubbling effects to the path integral are given by elliptic genera of world volume theories on D-branes. Our result gives an elliptic deformation of the SUSY localization formula [1] (resp. [2, 3]) of BPS ’t Hooft loops (resp. bare monopole operators) in 4d $\mathcal{N} = 2$ (resp. 3d $\mathcal{N} = 4$) gauge theories. We define deformation quantizations of vevs of surface operators in terms of the Weyl-Wigner transform, where the $\Omega$-background parameter play the role of the Planck constant. For 5d $\mathcal{N} = 1^{*}$ gauge theory, we find that the deformation quantization of the surface operators in the anti-symmetric representations agrees with the type A elliptic Ruijsenaars operators. The mutual commutativity of these difference operators is related to the commutativity of products of ’t Hooft surface operators.
1 Introduction

An 't Hooft loop operator [4] in a four-dimensional (4d) gauge theory is an example of disorder operator defined by a boundary condition of the gauge field with a prescribed singularity along the loop. In supersymmetric (SUSY) gauge theories, BPS analogues of disorder operators preserving a part of the supersymmetry have interesting properties in a variety of situations.

In three dimensions, BPS monopole operators defined by singular boundary conditions at a point in the spacetime [5, 6] play crucial roles in the study of quantum corrections and dualities in supersymmetric gauge theories. In 3d $\mathcal{N} = 4$ non-abelian gauge theories, the moduli
space of Coulomb branch vacua receives non-perturbative corrections from ’t Hooft-Polyakov monopoles. In general, it is difficult to exactly evaluate the non-perturbative corrections to the hyperKähler metric of the moduli space of the Coulomb branch vacua. An algebra consisting of the vacuum expectation values (vevs) of the monopole operators and Coulomb branch scalars called Coulomb branch chiral ring conjecturally gives the coordinate ring of the moduli space of Coulomb branch vacua as an algebraic variety. Here an important point is that the quantum corrections to a Coulomb branch chiral ring is easier to handle than the corrections to the hyperKähler metric. In fact, the Coulomb branch chiral rings in quiver gauge theories of ADE-type were determined in [7] with certain assumptions; abelianization and the mirror symmetry of abelian gauge theories. In [2], and also [3], the exact computation of vevs of monopole operators was developed in terms of supersymmetric localization methods. Then it was found that the algebras of monopole operators and Coulomb branch obtained by the supersymmetric localization formula agrees with the Coulomb branch chiral rings and their deformation quantizations in [7, 8, 9].

In four dimensions, BPS ’t Hooft loops have attracted a lot of attention from the mathematical physics viewpoints for more than a decade. For example, there exists the S-duality between Wilson loops and ’t Hooft loops in 4d \( \mathcal{N} = 4 \) super Yang-Mills theory [10, 11]. In AGT correspondence [12], ’t Hooft loops in 4d \( \mathcal{N} = 2 \) gauge theories belonging to the class \( S \) [13] conjecturally agree with Verlinde loops in Toda theories [14, 15, 16, 17, 1]. On the \( \Omega \)-background, the algebra of the ’t Hooft loops, Wilson loops, Dyonic loops and equivalently the algebra of Verlinde loops gives a deformation quantization of Coulomb branch of 4d \( \mathcal{N} = 2 \) gauge theory on \( S^1 \times \mathbb{R}^3 \) and also gives a deformation quantization of the moduli space of flat connections on a punctured Riemann surface [18, 19, 20, 1]. In this story, supersymmetric localization method provides a powerful method to checks the correspondence between BPS ’t Hooft loops and Verlinde operators.

Although quantum field theories in five dimensions are non-renormalizable by the power counting argument, some class in 5d \( \mathcal{N} = 1 \) supersymmetric gauge theories have non-trivial fixed points in the renormalization group flow and make sense as quantum field theories [21, 22]. Supersymmetric localization formulas of partition functions and supersymmetric indices on five-dimensional manifolds give non-trivial quantitative tests of predictions for quantum aspects of 5d \( \mathcal{N} = 1 \) supersymmetric theories. In this paper we introduce a BPS analogue of disorder operators in 5d \( \mathcal{N} = 1 \) supersymmetric gauge theories on \( T^2 \times \mathbb{R}^3 \) by imposing boundary conditions that has a Dirac monopole singularity extending along a two dimensional torus \( T^2 \). The BPS disorder operators which we call as BPS ’t Hooft surface operators are five-dimensional analogues of BPS ’t Hooft loops in 4d \( \mathcal{N} = 2 \) gauge theories on \( S^1 \times \mathbb{R}^3 \), and BPS monopole operators in 3d \( \mathcal{N} = 4 \) gauge theories on \( \mathbb{R}^3 \). We evaluate the vevs of ’t Hooft surface operators by supersymmetric localization techniques, and study their properties.

This article is organized as follows. In Section 2, we introduce the BPS ’t Hooft surface
operators in the path integral formalism by imposing certain boundary conditions for the fields in the five-dimensional supermultiplets. In Section 3, we study the supersymmetric localization computation of vevs of 't Hooft surface operators and evaluate the classical and the one-loop contribution to the vevs of 't Hooft surface operators. In the path integral, there exists non-perturbative corrections coming from the monopole bubbling effect; the path integral over the moduli space of certain monopole solutions. In Section 4, we evaluate monopole bubbling effect in the vacuum expectation values (vevs) of 't Hooft surface operators in terms of D-brane realizations of monopole bubbling. The monopole bubbling effects contributing to the vevs of 't Hooft surface operators are given by elliptic genera of the low energy world volume theories on D-branes. In Section 5, we define the products of vevs of 't Hooft surface operators in terms of the Moyal product and also define the deformation quantization of vevs of the surface operators in terms of the Weyl-Wigner transform. We find that the deformation quantization of surface operators in 5d $\mathcal{N} = 1^*$ gauge theory coincides with simultaneously commuting difference operators appearing in an integrable system, called elliptic Ruijsenaars operators. In Section 6, we study the algebra of surface operators with respect to the Moyal product in 5d $\mathcal{N} = 1^*$ gauge theory. In Section 7, we discuss our results and future problems.

2 Monopole surface operator in 5d $\mathcal{N} = 1$ SUSY gauge theories

In this section we explain the decomposition of the ten-dimensional (10d) vector multiplet on the spacetime $\mathbb{R}^{10}$ to a 5d vector multiplet and a 5d hypermultiplet in the adjoint representation on the spacetime $\mathbb{R}^5$ by the dimensional reduction. Next we define the vevs of an 't Hooft surface operator as a supersymmetric indices on $T^2 \times \mathbb{R}^3$.

2.1 5d SUSY gauge theory from 10d super Yang-Mills theory

The convention of the gauge covariant derivative is $D_M = \partial_M + iA_M$. The indices $\mu, \nu, \cdots \in \{0, 1, 2, 3, 4\}$ express the subscripts for the five-dimensional spacetime. The indices $M, N, \cdots \in \{0, 1, \cdots, 9\}$ label the subscripts for the ten-dimensional spacetime. The spacetimes $\mathbb{R}^5$ and $\mathbb{R}^{10}$ have the Euclidean signature metrics $\delta_{\mu\nu}$ for $\mu, \nu = 0, 1, 2, 3, 4$ and $\delta_{MN}$ for $M, N = 0, 1, \cdots, 9$, respectively. $x^M$ for $M = 0, 1, \cdots, 9$ denotes the coordinate of $\mathbb{R}^{10}$. The definition and properties of the $16 \times 16$ gamma matrices $\Gamma^M$ and $\tilde{\Gamma}^M$ are summarized in Appendix A.

By the dimensional reduction, the gauge field $A_M$ for $M = 0, 1, \cdots, 9$ and the gaugino $\Psi$ in the 10d maximal super Yang-Mills theory are decomposed to the 5d $\mathcal{N} = 1$ supermultiplets as follows. In the dimensional reduction in the directions $x^i$ for $i = 5, 6, 7, 8, 9$, the five
dimensional gauge fields $A_\mu$ for $\mu = 0, 1, 2, 3, 4$, an adjoint scalar $\sigma = A_9$ and a fermion $\lambda = \frac{1}{2}(1 - \Gamma^{5678})\Psi$ form a 5d $\mathcal{N} = 1$ vector multiplet. Scalars $\Phi_i := A_i$ for $i = 5, 6, 7, 8$ and a fermion $\psi := \frac{1}{2}(1 + \Gamma^{5678})\Psi$ form a 5d $\mathcal{N} = 1$ hypermultiplet in the adjoint representation.

The action of the 5d $\mathcal{N} = 1$ super Yang-Mills theory is given by

$$S_{\text{vec}} = \frac{1}{g^2} \int_{\mathbb{R}^5} d^5x \text{Tr} \left[ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \sigma)^2 - \lambda \Gamma^\mu D_\mu \lambda + i \lambda \Gamma^9 [\lambda, \sigma] \right]. \quad (2.1)$$

Here $g$ is the Yang-Mills coupling constant. A symbol $\text{Tr}$ is a trace taken over the Lie algebra $\mathfrak{g}$ of the gauge group $G$. The action of the 5d hypermultiplet in the adjoint representation is given by

$$S_{\text{hyp}} = \int_{\mathbb{R}^5} d^5x \text{Tr} \left[ (D_\mu \Phi_i)^2 - \frac{1}{2} [\Phi_i, \Phi_j]^2 - [\sigma, \Phi_i]^2 \right.\left. - \psi \Gamma^\mu D_\mu \psi - i\psi \Gamma^9 [\sigma, \psi] - i\psi \Gamma^9 [\Phi_i, \psi] \right]. \quad (2.2)$$

We can introduce a fugacity (mass) $m_{\text{ad}}$ for $U(1)$ flavor symmetry of the adjoint hypermultiplet, which breaks the $\mathcal{N} = 2$ supersymmetry to an $\mathcal{N} = 1$ supersymmetry in five dimensions. In particular, the 5d $\mathcal{N} = 1$ supersymmetry obtained by the mass deformation of 5d $\mathcal{N} = 2$ supersymmetry is called 5d $\mathcal{N} = 1^*$ supersymmetry.

To apply supersymmetric localization method, we need at least one off-shell supercharge. For the supersymmetric gauge theories, by adding the action of auxiliary fields $\sum_{i=1}^7 K_i^2$ to the above actions (2.1) and (2.2), one can keep some of the supercharges without using the equation of motion, i.e. off-shell level $[23]$. To write a supersymmetry transformation, we choose a supersymmetric variation parameter $\varepsilon$ and introduce parameters $\nu_j$ for $j = 1, \cdots, 7$ given by

$$\varepsilon = \frac{1}{\sqrt{2}} (1, 0^7, 1, 0^7) = \frac{1}{\sqrt{2}} (1, 0, \cdots, 0, 1, 0, \cdots, 0), \quad (2.3)$$

$$\nu_j = \begin{cases} 
\Gamma^{8,j+4} \varepsilon & \text{for } j = 1, 2, 3, \\
\Gamma^{89} \varepsilon & \text{for } j = 4, \\
\Gamma^{8,j-4} \varepsilon & \text{for } j = 5, 6, 7. 
\end{cases} \quad (2.4)$$

Then the actions (2.1) and (2.2) with the action of auxiliary fields are invariant under the following off-shell supersymmetry transformation:

$$Q \cdot A_M = \varepsilon \Gamma_M \Psi, \quad (2.5)$$

$$Q \cdot \Psi = \frac{1}{2} \Gamma^{MN} F_{MN} \varepsilon + i K^i \nu_i, \quad (2.6)$$

$$Q \cdot K_i = i\nu_i \Gamma^M D_M \Psi. \quad (2.7)$$
Here the action of the off shell supercharge $Q$ is defined by $[Q, X]$ (resp. $\{Q, X\}$) for the Grassmann even fields $X = A_M, K_i$ (resp. the Grassmann odd field $X = \Psi$). The square of the SUSY transformation generates

\begin{align}
Q^2 \cdot A_M &= -2F_{zM}, \\
Q^2 \cdot \Psi &= -2D_{z}\bar{\Psi}, \\
Q^2 \cdot K_i &= -2D_zK_i.
\end{align}

Here $z = x^4 + ix^0$ and $\bar{z} = x^4 - ix^0$. If we replace the representation for the fields in the adjoint hypermultiplet by a symplectic representation $\mathcal{R} \oplus \overline{\mathcal{R}}$ of the gauge group $G$, we obtain the supersymmetry transformation and the action for the hypermultiplet in a symplectic representation $\mathcal{R} \oplus \overline{\mathcal{R}}$.

### 2.2 Monopole surface operators as SUSY indices on $T^2 \times \mathbb{R}^3$

We consider a twisted compactification of 5d supersymmetric gauge theories on the two-dimensional torus $T^2$ defined by

\begin{align}
T^2 := \{(x^4, x^0) | x^4 + ix^0 \equiv x^4 + ix^0 + 2\pi \equiv x^4 + ix^0 + 2\pi\tau\} \\
= \{(z, \bar{z}) | z \equiv z + 2\pi \equiv z + 2\pi\tau\}.
\end{align}

Here $\tau = \tau_1 + i\tau_2$ is the moduli of the torus $T^2$. A symbol “≡” denotes the identification. We also introduce another coordinate $0 \leq s, t \leq 2\pi$ of $T^2$ defined by

\begin{align}
x^4 = s + \tau_1 t, \quad x^0 = \tau_2 t.
\end{align}

We impose the following twisted boundary condition along the $x^0$ and $x^4$ directions for the fields:

\begin{align}
X(z) \equiv X(z + 2\pi), \quad X(z + 2\pi\tau) \equiv e^{-2\pi i(J_3 + \mathbf{1}_3)} \prod_f e^{-2\pi i m_f F_f} X(z)
\end{align}

where $X(z)$ denotes a field in the 5d supermultiplets. Here we have suppressed the coordinates dependence on $x^i$ for $i = 1, 2, 3$ and $\bar{z}$ to shorten the notation.

A symbol $J_3$ denotes a generator of the rotation in the $x^1, x^2$-plane around the origin. When we emphasize the presence of a fugacity $\epsilon$ called an $\Omega$-background parameter, we write a spacetime $\mathbb{R}^3$ in the $x^{1,2,3}$-directions as $\mathbb{R}^2 \times \mathbb{R}$. A symbol $\mathbf{1}_3$ is a generator of $\mathfrak{u}(1) \subset \mathfrak{su}(2)_H$, where $\mathfrak{su}(2)_H$ is the Lie algebra of the R-symmetry group for the 5d $\mathcal{N} = 1$ supersymmetry algebra. $F_f$’s are generators (charges) of the Cartan subalgebra of the flavor symmetry group acting on the hypermultiplets in the representation $\mathcal{R} \oplus \overline{\mathcal{R}}$. $m_f$’s are fugacities for these generators. If $\mathcal{R}$ is the adjoint representation, we have a single flavor fugacity $m_{\text{ad}} := m_1$. 

5
We introduce the vev of a BPS 't Hooft surface operator \( S_B \) for 5d \( \mathcal{N} = 1 \) supersymmetric gauge theory on \( T^2 \times \mathbb{R}^3 \) as a supersymmetric index:

\[
\langle S_B \rangle := \text{Tr}_{\mathcal{H}_B} (-1)^F e^{-2\pi \mathcal{H}} e^{2\pi i(J_3 + I_3)} \prod_f e^{2\pi i m_f F_f} .
\] (2.14)

To be more precise, the vev is defined by the path integral in the presence of singular monopoles, which will mentioned later. A magnetic charge \( B \) is an element of the coweights lattice \( \Lambda_{cw} \) of the Lie algebra \( \mathfrak{g} \). Since a Weyl group action of \( B \) defines a same operator, we may assume \( B \) in \( S_B \) as a dominant coweight. \( \mathcal{H}_B \) is the Hilbert space of the supersymmetric theory on \( S^1_s \times \mathbb{R}^3 \), where \( S^1_s \) is the circle in the \( s \)-direction. A symbol \( F \) denotes the Fermion number operator. We take a coordinate \( t \) as the time direction and define the Hamiltonian \( H \) by the generator of translation in the direction \( t \). In the path integral formalism, the vev of an 't Hooft surface operator is given by

\[
\langle S_B \rangle = \int_{\text{B.C.}} DA D\Psi D\mathcal{K} \exp (-S_{\text{vec}} - S_{\text{hyp}} - S_{\text{b.d}}) .
\] (2.15)

Here we add a boundary term \( S_{\text{b.d}} \) to regularize the singularity coming from the Dirac monopole, see (3.30). In the path integral, the twisted boundary condition (2.13) is given by the shift of the time derivative:

\[
\partial_t \mapsto \partial_t - i\epsilon(J_3 + I_3) - i \sum_f m_f F_f .
\] (2.16)

Then the fugacities are given by background gauge field in the path integral formalism. In the rest of this paper, we include these background gauge fields in the definition of covariant derivative in the \( t \)-direction, i.e., \( D_t = \partial_t + iA_t - i\epsilon(J_3 + I_3) - im_f F_f \).

"B.C." in (2.15) denotes the following boundary conditions of the fields in the path integral. At the infinitesimal neighborhood of \(( x^0, 0, 0, x^4 )\) with \( \forall (x^0, x^4) \in T^2 \), we impose a boundary condition admitting a singular Dirac monopole with the magnetic charge \( B \):

\[
\sum_{i=1,2,3} A_i dx^i \sim \frac{B}{2} (1 - \cos \theta) d\phi, \quad \sigma \sim \frac{B}{2r}, \quad \text{for } r \to 0 .
\] (2.17)

Here \(( r, \theta, \phi )\) is the polar coordinates of the space \(( x^1, x^2, x^3 )\). In the path integral we also sum over the boundary conditions, where the elements of \( B \) in (2.17) are permuted by an arbitrary Weyl group action of \( W_G \).

We also have to specify the boundary conditions of the fields at a sufficiently large \( r \). At the spatial infinity, the gauge fields \( A_i \) for \( i = 0, 4 \) and a scalar \( \sigma \) have definite values. We assume these values are in the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \). Since gauge fields \( A_i \) for \( i = 1, 2, 3 \) have a magnetic charge \( B \) near the origin \(( x^1, x^2, x^3 ) = (0, 0, 0) \), a natural choice of boundary value \( A_i \) for \( i = 1, 2, 3 \) at the spatial infinity is

\[
\sum_{i=1,2,3} A_i dx^i \sim \frac{B}{2} (1 - \cos \theta) d\phi, \quad \text{for } r \to \infty .
\] (2.18)
But an important point here is that we have to take into account not only the boundary condition \((2.18)\), but all the boundary conditions associated with monopole bubbling (a.k.a. monopole screening) \([11]\). A monopole bubbling is a phenomenon that an ’t Hooft-Polyakov monopole screen out the charge of the Dirac monopole \(B\) and reduced it to \(p\) with \(||p|| < ||B||\), when a smooth ’t Hooft-Polyakov monopole with a magnetic charge \(p - B\) in the coroot lattice \(\Lambda_{cr}\) exists at the infinitesimal neighborhood of the center of the singular Dirac monopole with the charge \(B \in \Lambda_{cw}\). Here \(||B|| = \sqrt{\text{Tr}(B^2)}\).

Monopole bubbling effects in the SUSY localization formula of BPS ’t Hooft loops for \(U(N)\) and \(SU(N)\) gauge theories were originally studied in \([17, 1]\) and were further studied in \([24]\). It turned out that ’t Hooft loops agree with Verlinde loop operators by including monopole bubbling effects. Moreover monopole bubbling effects are also necessary to reproduce the correct properties of operator product expansions (OPEs) of BPS ’t Hooft loops \([11, 1, 25, 26]\). In three dimensions, monopole bubbling effects for BPS monopole operators on \(\mathbb{R}^3\) were studied in \([2, 3]\). It found that monopole bubbling effects are necessary in order for monopole operators with higher charges to be generated by the products of operators with smaller charges. From these observations, we consider all the boundary conditions with monopole bubblings for the surface operator specified by

\[
\sum_{i=1,2,3} A_i dx^i \sim \frac{p}{2}(1 - \cos \theta) d\phi, \quad \text{for } r = \text{finite}, \quad (2.19)
\]

if \(p \in B + \Lambda_{cr}\) with \(||p|| < ||B||\) exists. We will study the contribution from the moduli of monopole bubbling to the path integral in terms of brane constructions in Section 4.

### 3 SUSY localization of surface operator

In this section, we perform the path integral for the vev of surface operators in terms of supersymmetric localization. Before we explain the technical details of the localization computation, we first summarize the result of our supersymmetric localization formula. We consider that the gauge group is \(G\) and \(N_F\) hypermultiplets are in a symplectic representation \(\mathcal{R} \oplus \overline{\mathcal{R}}\) of the gauge group. The localization formula for the vev of monopole surface operator is given by

\[
\langle S_B \rangle = \sum_{p \in B + \Lambda_{cr} \atop ||p|| \leq ||B||} e^{p \cdot b} Z_{1\text{-loop}}^{5d}(a, m, p, \epsilon, \tau) Z_{\text{mono}}(a, m, p, B, \epsilon, \tau). \quad (3.1)
\]

\(Z_{\text{mono}}\)'s in \((3.1)\) are the contributions from the monopole bubbling effects. Note that \(Z_{\text{mono}}(p, B) = 1\) for \(||p|| = ||B||\). If \(p \in B + \Lambda_{cr}\) with \(||p|| < ||B||\) does not exist, the localization formula is completely determined by the one-loop computations for \(||p|| = ||B||\). The right hand side of \((3.1)\) is given as follows. \(b\) is defined in \((3.32)\). A pairing \(p \cdot b\) of \(p\) and \(b\) is induced by
the trace over $\mathfrak{g}$:

$$p \cdot b = \text{Tr}(pb).$$  \tag{3.2}$$

$Z_{1\text{-loop}}^{5d}(a, m, p, \epsilon, \tau)$ is the one-loop determinant of the Q-exact action around the saddle point specified by a magnetic charge $p$, and factorizes to the one-loop determinant of the 5d $\mathcal{N} = 1$ vector multiplet $Z_{1\text{-loop}}^{5d, \text{vec}}$ and the one of the 5d $\mathcal{N} = 1$ hypermultiplet $Z_{1\text{-loop}}^{5d, \text{hyp}}$:

$$Z_{1\text{-loop}}^{5d}(a, m, p, \epsilon, \tau) = Z_{1\text{-loop}}^{5d, \text{vec}}(a, p, \epsilon, \tau)Z_{1\text{-loop}}^{5d, \text{hyp}}(a, m, p, \epsilon, \tau),$$  \tag{3.3}$$

where

$$Z_{1\text{-loop}}^{5d, \text{vec}}(a, p, \epsilon, \tau) = \left[ \prod_{\alpha \in \text{rt}(\mathfrak{g})} \prod_{k=0}^{[\alpha \cdot p]-1} \vartheta_1 \left( \alpha \cdot a + \left( \frac{[\alpha \cdot p]}{2} - k \right) \epsilon; \tau \right) \right]^{-\frac{1}{2}},$$  \tag{3.4}$$

$$Z_{1\text{-loop}}^{5d, \text{hyp}}(a, m, p, \epsilon, \tau) = \left[ \prod_{w \in \Delta(\mathcal{R})} \prod_{f=1}^{N_F} \prod_{k=0}^{[w \cdot p]-1} \vartheta_1 \left( w \cdot a - m_f + \left( \frac{[w \cdot p]-1}{2} - k \right) \epsilon; \tau \right) \right]^{\frac{1}{2}},$$  \tag{3.5}$$

rt($\mathfrak{g}$) is the set of roots of $\mathfrak{g}$ and $\Delta(\mathcal{R})$ is the set of weights of a representation $\mathcal{R}$.

A theta function $\vartheta_1(u; \tau)$ and eta function $\eta(\tau)$ are defined by

$$\vartheta_1(u) = \vartheta_1(u; \tau) := 2e^{\frac{i\pi\tau}{4}}\sin(\pi u) \prod_{i=1}^{\infty} (1 - e^{2\pi i n \tau})(1 - e^{2\pi in \tau} e^{2\pi i u})(1 - e^{2\pi i n \tau} e^{-2\pi i u}),$$  \tag{3.6}$$

$$\eta(\tau) := e^{\frac{2\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).$$  \tag{3.7}$$

To shorten expressions, we introduce the following notation for the theta function:

$$\vartheta_1(\pm x + y) = \prod_{\alpha=\pm 1} \vartheta_1(\alpha x + y), \quad \vartheta_1(\pm x \pm y + w) = \prod_{\alpha, \beta=\pm 1} \vartheta_1(\alpha x + \beta y + w).$$  \tag{3.8}$$

$Z_{\text{mono}}(p, B)$ in (3.1) is interpreted as a contribution from the path integral over the moduli space of Bogomolny equation (3.21) with a monopole bubbling explained in Section 4. We give explicit computations of $\langle S_B \rangle$ for small magnetic charges in Section 6.

### 3.1 Zero locus of Q-exact term in 5d $\mathcal{N} = 1$ SUSY gauge theory

To apply supersymmetric localization procedure, we introduce the Faddeev–Popov ghosts $c, \bar{c}$, the Nakanishi-Lautrup B-field $\bar{b}$, and a BRST charge $Q_B$. We will explain the definition of the BRST charge $Q_B$ and the gauge fixing term in the next subsection. We add one-parameter family of Q-exact action $t \hat{Q} \cdot V$ to the original action, and take a limit $t \to \infty$:

$$\langle S_B \rangle = \lim_{t \to \infty} \int \mathcal{D}A \mathcal{D}\Psi \mathcal{D}K \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\bar{b} \exp \left( -S_{\text{vec}} - S_{\text{hyp}} - S_{b.d} - t \hat{Q} \cdot V \right),$$  \tag{3.9}$$

8
with
\[ \hat{Q} := Q + Q_B. \] (3.10)

Then the principle of supersymmetric localization [27] tells us that the path integral in (2.15) is exactly evaluated in terms of the one-loop integral around \( \hat{Q} \cdot V = 0 \) and integrals over certain moduli spaces of equations associated with \( \hat{Q} \cdot V = 0 \) in (3.9).

In this section we focus on the following matter part in the \( Q \)-exact action \( \hat{Q} \cdot V \):
\[ Q \cdot (\Psi, \tilde{Q} \cdot \tilde{\Psi}), \] (3.11)

and study the zero locus (saddle point locus) of (3.11). Here \( \tilde{Q} \cdot \tilde{\Psi} \) is the complex conjugate of \( Q \cdot \Psi \). \( (A, B) = \int_{T^2 \times \mathbb{R}^3} AB \), where the spinor indices and Lie algebra indices in five dimensional fields \( A \) and \( B \) are contracted. To write down explicitly the saddle equation \( Q \cdot \Psi = 0 \) in (3.11), it is convenient to decompose the fermion \( \Psi \) with the sixteen components into Grassmann odd functions \( \Psi_M \) for \( M = 1, \cdots, 9 \) and \( \Upsilon^i \) for \( i = 1, \cdots, 7 \) as
\[ \Psi = \sum_{M=1}^{9} \Psi_M \tilde{\Gamma}^M \varepsilon + i \sum_{i=1}^{7} \Upsilon^i \nu_i. \] (3.12)

Here \( \Psi_M \) and \( \Upsilon^i \) are defined by
\[ \Psi_M = \varepsilon \Gamma^M \Psi, \quad i \Upsilon^i = \bar{\nu}_i \Psi, \] (3.13)
with \( \bar{\nu}_i = -\nu_i \). Then, we have the supersymmetry transformation of \( \Psi_M \) and \( \Upsilon_i \) as follows.
\[ Q \cdot \Psi_M = \varepsilon \Gamma^M Q \cdot \Psi = iF_{M0} + F_{M4}. \] (3.14)

Here we used the following properties; \( \Gamma^M \Gamma^{NL} = \Gamma^{[M} \Gamma^{NL]}, \varepsilon \Gamma^{[M} \Gamma^{NL]} \varepsilon = 0 \) and \( (\varepsilon \Gamma^0 \varepsilon, \varepsilon \Gamma^1 \varepsilon, \cdots, \varepsilon \Gamma^9 \varepsilon) = (i, 0^3, 1, 0^5) \).

\[ iQ \cdot \Upsilon_i = \bar{\nu}_i Q \cdot \Psi \]
\[ = \frac{1}{2} \left( \sum_{j,k=1}^{3} F_{jk} \bar{\nu}_i \Gamma^{jk} \varepsilon + 2 \sum_{j=1}^{3} \sum_{k=5}^{8} F_{jk} \bar{\nu}_i \Gamma^{jk} \varepsilon + 2 \sum_{j=1}^{3} F_{j9} \bar{\nu}_i \Gamma^{j9} \varepsilon + 2 \sum_{j=5}^{8} F_{j9} \bar{\nu}_i \Gamma^{j9} \varepsilon \right) + iK^i. \] (3.15)

Here we used \( \bar{\nu}_i \nu_j = -\delta_{ij} \) and \( \bar{\nu}_i \Gamma^{M4} \varepsilon = 0 \) for \( i = 1 \cdots, 7 \) and \( M = 0, 1, \cdots, 9 \). Then
\[ iQ \cdot \Upsilon_{i=1,2,3} = -\frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} F_{jk} + D_i \sigma - iK^i \] (3.16)
\[ iQ \cdot \Upsilon_{i=4,5,6,7} = 3 \sum_{j=1}^{3} \sum_{k=5}^{8} D_j \Phi_k \bar{\nu}_i \Gamma^{jk} \varepsilon + \sum_{j=5}^{8} i[\Phi_j, \sigma] \bar{\nu}_i \Gamma^{j9} \varepsilon - iK^i \] (3.17)
Here we used
\[ \bar{\nu}_i \Gamma^{jk} \varepsilon = 0 \quad \text{for} \quad i \in \{5, 6, 7\}, \ j, k \in \{1, 2, 3\} \quad (3.18) \]
\[ \bar{\nu}_i \Gamma^{ij} \varepsilon = 0 \quad \text{for} \quad i \in \{4, 5, 6, 7\}, \ j \in \{1, 2, 3\}. \quad (3.19) \]

From (3.14) and (3.16), we find that the saddle point equation \( Q \cdot \Psi = 0 \) for the bosonic fields in the 5d vector multiplet are decomposed to
\[ F_{04} = 0, \quad (3.20) \]
\[ D_i \sigma = \frac{1}{2} \epsilon_{ijk} F_{jk} \quad (i, j, k = 1, 2, 3), \quad (3.21) \]
\[ K_i = 0. \quad (3.22) \]

From (3.17), the saddle point equation \( Q \cdot \Psi = 0 \) for the 5d hypermultiplet (3.17) is written as
\[ \sum_{i=1}^{3} \sigma^i D_i q + [\sigma, q] = 0, \quad (3.23) \]
where \( \sigma^i, i = 1, 2, 3 \) are the Pauli matrices, and \( q = (q_1, q_2)^T \) is defined by
\[ q_1 := \Phi_5 - i\Phi_6 + i\Phi_7 + \Phi_8, \quad q_2 := i\Phi_5 - \Phi_6 - \Phi_7 - i\Phi_8. \quad (3.24) \]

The invariance under the flavor symmetry rotation require the saddle point value of \( \Phi_i \) for \( i = 5, \cdots, 9 \) is zero.

We evaluate the saddle point value of the super Yang-Mills action and the boundary term. \( F_{04} = 0 \) means that the saddle point configuration for the gauge fields \( A_i \) for \( i = 0, 4 \) are flat connections which are fixed by the value at \( r \to \infty \). Let \( \bar{A}_i, i = 0, 4 \) be the boundary values \( A_i |_{r=\infty} = \text{constant for} i = 0, 4 \) and define \( a \) as a holomorphic combination of the constant gauge fields:
\[ a := \bar{A}_0 + i\bar{A}_4 \in \mathfrak{h} \otimes \mathbb{C}. \quad (3.25) \]

Next let \( (\bar{A}_i, \bar{\sigma}) \) for \( i = 1, 2, 3 \) be a solution of the Bogomolny equation (3.21). From the boundary condition of the path integral (2.17), \( (\bar{A}_i, \bar{\sigma}) \) behaves as
\[ \sum_{i=1,2,3} \bar{A}_i dx^i \sim \frac{B}{2} (1 - \cos \theta) d\phi, \quad \bar{\sigma} \sim \frac{B}{2r}; \quad \text{for} \ r \to 0, \quad (3.26) \]
\[ \sum_{i=1,2,3} A_i dx^i \sim \frac{p}{2} (1 - \cos \theta) d\phi, \quad \bar{\sigma} \sim \frac{p}{2r} + \sigma_0 \quad \text{for} \ r \text{ finite}. \quad (3.27) \]
Here we assume that $\sigma_0$ is a Cartan valued constant. We substitute a saddle point value $(\bar{A}_{i=0,\ldots,4},\bar{\sigma})$ into the 5d super Yang-Mills action. Then saddle point value of the 5d super Yang-Mills action is given

$$S_{\text{vec}}|_{\text{saddle}} = \frac{1}{g^2} \int_{T^2 \times \mathbb{R}^3 \setminus B^3_R} d^5x \text{Tr} \left[ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} + D^\mu \sigma D_\mu \sigma - \lambda \Gamma^\mu D_\mu \lambda + i \lambda \Gamma^9 [\lambda, \Phi_9] \right] |_{\text{saddle}}$$

$$= \frac{2\pi \text{vol}(T^2)}{g^2 R} \text{Tr}(p^2).$$  \hspace{1cm} (3.28)

where

$$B^3_R := \{(x^1, x^2, x^3) | |x^1|^2 + |x^2|^2 + |x^3|^2 \leq R^2 \}.$$  \hspace{1cm} (3.29)

In (3.28), we removed the three-dimensional ball $B^3_R$ from the integration region to regularize the divergence coming from the center of the Dirac monopole. The divergence coming from $R \to 0$ is compensated by the saddle point value of the boundary term $S_{\text{b.d}}$ \cite{1} as follows.

$$S_{\text{b.d}}|_{\text{saddle}} = \frac{2}{g^2} \int_{T^2 \times \partial B^3_R} d\text{vol}(T^2) \text{Tr}(\sigma F_A) |_{\text{saddle}}$$

$$= -\frac{1}{g^2} \text{vol}(T^2) \text{Tr} \left( \frac{2\pi p^2}{R} + 4\pi^2 R \sigma_0 p \right).$$  \hspace{1cm} (3.30)

Here $d\text{vol}(T^2)$ is the volume form of $T^2$, and $F_A$ is the restriction of the field strength $\frac{1}{2} \sum_{i,j=1}^3 F_{ij} dx^i \wedge dx^j$ on the boundary $\partial B^3_R$. The the saddle point value of the sum of the super Yang-Mills action and the boundary term has a finite value:

$$(S_{\text{vec}} + S_{\text{b.d}})|_{\text{saddle}} = -\frac{4\pi^2 R}{g^2} \text{vol}(T^2) \text{Tr} (p \sigma_0) =: p \cdot b,$$  \hspace{1cm} (3.31)

with

$$b = -\frac{4\pi^2 R}{g^2} \text{vol}(T^2) \sigma_0.$$  \hspace{1cm} (3.32)

On the other hand, the saddle point values of the fields in the hypermultiplet are zero, which means that the saddle point values of the action of hypermultiplet is zero.

### 3.2 Gauge fixing term and BRST transformation

To evaluate the one-loop determinant, we introduce the action for the ghosts and the gauge fixing term:

$$S_{\text{g.f}} = \int_{T^2 \times \mathbb{R}^3} d^5x Q_B \cdot \text{Tr} \left( \bar{c} \left( \sum_{i=1,2,3,9} \bar{D}_M \bar{A}_M + \frac{\xi}{2} \bar{b} \right) \right)$$

$$= \int_{T^2 \times \mathbb{R}^3} d^5x \text{Tr} \left( -i \bar{c} \bar{D}_M D_M c + \bar{b} \left( i \bar{D}^M \bar{A}_M + \frac{\xi}{2} \bar{b} \right) \right)$$  \hspace{1cm} (3.33)
Here the g-valued ghost fields $c, \bar{c}$ are Grassmann odd, and the g-valued B-field $\bar{b}$ is a Grassmann even. $D_M = \partial_M + i \tilde{A}_M$, where $\tilde{A}_M$ denotes the saddle point values of the gauge fields and the vector multiplet scalar given by (3.25), (3.26), (3.27), and $\tilde{A}_{i=5,6,7,8} = 0$. $\tilde{A}_M$ is the fluctuation of around $\tilde{A}_M$ mentioned in the beginning of Section 3.3. We define the BRST transformation by

$$Q_B \cdot A_M = D_M c, \quad Q_B \cdot \Psi = -i[c, \Psi], \quad Q_B \cdot K_i = -i[c, K_i],$$
$$Q_B \cdot c = -\frac{i}{2}[c, c], \quad Q_B \cdot \bar{c} = \bar{b}, \quad Q_B \cdot \bar{b} = 0. \quad (3.34)$$

Note that the BRST charge is nilpotent $\{Q_B, Q_B\} = 0$. The supersymmetry transformation for $(c, \bar{c}, \bar{b})$ is defined by

$$Q \cdot c = -i\tilde{A}_0 - \tilde{A}_4, \quad Q \cdot \bar{c} = 0, \quad Q \cdot \bar{b} = -(iD_0 + D_4)\bar{c}. \quad (3.35)$$

We define the Q-exact term by

$$\hat{Q} \cdot V := \hat{Q} \left( (\Psi, \bar{Q} \cdot \Psi) + V_{g.f} \right) \quad (3.36)$$

with $\hat{Q} = Q + Q_B$.

### 3.3 Evaluation of one-loop determinants

In the localization computation of the path integral in (3.9), we decompose the fields to $X = \bar{X} + \bar{X}/\sqrt{t}$, where $\bar{X}$ denotes the fields satisfy $\hat{Q} \cdot \bar{V} = 0$, and $\bar{X}$ is called as the fluctuation fields around the saddle point value. In the limit $t \to \infty$, the quadratic part of the fluctuation fields in the Q-exact action only contributes to the path integral, and the higher order interaction terms in the Q-exact action are negligible, i.e., the one-loop computation for the fluctuation fields gives the exact answer. Then we may take the first order approximation of $\hat{Q} \cdot \bar{X}$ in the Q-exact action with respect to the fluctuation fields.

In order to evaluate the one-loop determinant in terms of the index theorem of the transversally elliptic operators, let us define collections of the fluctuation fields $X_i, i = 0, 1$ by

$$X_0 = (X_{0,1}, X_{0,2}, \cdots, X_{0,9}) = (\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_9),$$
$$X_1 = (X_{1,1}, X_{1,2}, \cdots, X_{1,9}) = (\Upsilon_1, \cdots, \Upsilon_7, c, \bar{c}). \quad (3.37)$$

To make the expression concise, we omitted $\bar{\cdot}$ for the fluctuation fields $\Upsilon_i$ and the ghosts. Note that the saddle point configuration for the ghost and B-field is trivial. The square of
\[ \hat{Q}^2 = -\frac{i}{\tau_2} \left( \partial_t - \tau \partial_s + i a - i c (J_3 + I_3) - i \sum_f m_f F_f \right). \] (3.38)

At the quadratic order of \( X_i \), \( V \) can be written as
\[
V = (\Psi, \hat{Q} \cdot \Psi) + V_{g.f} = \left\langle \left( \hat{Q} \cdot X_0, X_1 \right), \left( D_{00} D_{01} \right) \left( X_0 \hat{Q} \cdot X_1 \right) \right\rangle,
\]
(3.39)
where \( D_{ij} \) for \( i, j = 0, 1 \) are linear differential operators which depend on the saddle point field configuration, but are independent of the fluctuations field \( X_i \). Then the Q-exact action \( \hat{Q} \cdot V \) at the quadratic order is given by
\[
\hat{Q} \cdot V = \left\langle \left( \hat{Q} \cdot X_0, X_1 \right), \left( -\hat{Q}^2, 0 \right) \right\rangle \left\langle \left( D_{00} D_{01} \right) \left( X_0 \hat{Q} \cdot X_1 \right) \right\rangle
+ \left\langle \left( \hat{Q} \cdot X_0, X_1 \right), \left( D_{00} D_{01} \right) \left( -1, 0 \right) \left( \hat{Q} \cdot X_0 \hat{Q} \cdot X_1 \right) \right\rangle.
\]
(3.40)

Following the argument in [27], one can show that the one-loop determinant is reduced to the ratio of the functional determinants over the spaces \( \text{Det}_{\text{coker}(D_{10})} \) and \( \text{Det}_{\text{ker}(D_{10})} \):
\[
Z_{1\text{-loop}}^{5d} = \frac{\text{Det} \left( \left( D_{00} D_{01} \right) \left( -1, 0 \right) \left( -\hat{Q}^2, 0 \right) \right) \text{Det}_{\text{coker}(D_{10})} \hat{Q}^2}{\text{Det}_{\text{ker}(D_{10})} \hat{Q}^2} \left[ \frac{\text{Det}_{\text{ker}(D_{10})} \hat{Q}^2}{\text{Det}_{\text{coker}(D_{10})} \hat{Q}^2} \right]^{\frac{1}{2}}.
\]
(3.41)

Here \( \text{ker}(D_{10}) \) is the space of the kernel of the differential operator \( D_{10} \), and \( \text{coker}(D_{10}) \) is the space of the cokernel of the differential operator \( D_{10} \). Since the differential operator \( D_{10} \) is decomposed to \( D_{10} = D_{\text{vec}} \oplus D_{\text{hyp}} \), where \( D_{\text{vec}} \) acting on the fields in the vector multiplet and \( D_{\text{hyp}} \) acting on the ones in the hypermultiplet, the one-loop determinant factorizes to the product of the one-loop determinants of the vector multiplet and the hypermultiplet:
\[
Z_{1\text{-loop}}^{5d} = Z_{1\text{-loop}}^{5d, \text{vec}} Z_{1\text{-loop}}^{5d, \text{hyp}},
\]
(3.42)
with
\[
Z_{1\text{-loop}}^{5d, \text{vec}} \left[ \frac{\text{Det}_{\text{coker}(D_{\text{vec}})} \hat{Q}^2}{\text{Det}_{\text{ker}(D_{\text{vec}})} \hat{Q}^2} \right]^{\frac{1}{2}}, \quad Z_{1\text{-loop}}^{5d, \text{hyp}} = \left[ \frac{\text{Det}_{\text{coker}(D_{\text{hyp}})} \hat{Q}^2}{\text{Det}_{\text{ker}(D_{\text{hyp}})} \hat{Q}^2} \right]^{\frac{1}{2}}.
\]
(3.43)
The important point here is that the ratio of the functional determinants is evaluated in terms of the index theorem of transversally elliptic operators. Let \( \text{ind}(D) \) be the equivariant index of a transversally elliptic operator \( D = D_{\text{vec}} \) or \( D_{\text{hyp}} \) defined by

\[
\text{ind}(D) = \text{tr}_{\ker(D)}(e^{\hat{Q}^2}) - \text{tr}_{\coker(D)}(e^{\hat{Q}^2}).
\]  

(3.44)

Since \( \hat{Q}^2 \) generates a linear combination of torus actions, \( e^{\hat{Q}^2} \) is thought as a group element. When the equivariant index is written as a form \( \text{ind}(D) = \sum_i n_i e^{w_i} \), where \( w_i \) is a weight of a representation of \( \hat{Q}^2 \) and \( n_i \) is the multiplicity of the representation, the ratio of the functional determinants is obtained by the following rule:

\[
\text{ind}(D) = \sum_i n_i e^{w_i} \rightarrow \frac{\text{Det}_{\coker(D)}(\hat{Q}^2)}{\text{Det}_{\ker(D)}(\hat{Q}^2)} = \prod_i w_i^{n_i}.
\]  

(3.45)

Thus remained task to obtain the one-loop determinant is to evaluate the weights \( w_i \) and the multiplicities \( n_i \) for the representations of \( D_{\text{vec}} \) and \( D_{\text{hyp}} \). The expression of \( D_{10} \) evaluated in Appendix C shows that the differential operators \( D_{\text{vec}} \) and \( D_{\text{hyp}} \) are same type as the ones appeared in the evaluation of the one-loop determinant for \( \text{\textquoteleft\textquoteprime} \text{t} \text{Hooft} \) loops on \( S^1 \times \mathbb{R}^3 \) [1]. The difference between the differential operators for \( \text{\textquoteprime} \text{t} \text{Hooft} \) surface operators here and the ones for BPS \( \text{\textquoteprime} \text{t} \text{Hooft} \) loop operators in [1] only comes from the spacetime derivative in \( \hat{Q}^2 \); \( \hat{Q}^2 \) in (3.38) contains the derivative of the \( T^2 \)-coordinate, whereas \( \hat{Q}^2 \) for \( \text{\textquoteprime} \text{t} \text{Hooft} \) loops contains the derivative of the \( S^1 \)-coordinate [1]. Therefore, for the each Kaluza-Klein (KK) mode along the \( T^2 \)-direction, the index is same as the equivariant indices \( \text{ind}(D_{\text{Bogo}}) \) and \( \text{ind}(D_{\text{DH},\mathcal{R}}) \) evaluated in [17, 1]:

\[
\text{ind}(D_{\text{Bogo}},\mathcal{C}) = -\frac{e^{\pi i e} + e^{-\pi i e}}{2} \sum_{\alpha \in \Lambda(g)} e^{2\pi i \alpha \cdot a} \sum_{k=0}^{\lfloor (\alpha \cdot p) - 2k - 1 \rfloor} e^{\pi i (\alpha \cdot p - 2k - 1) \epsilon},
\]  

(3.46)

\[
\text{ind}(D_{\text{DH},\mathcal{R}}) = -\frac{1}{2} \sum_{w \in \Delta(\mathcal{R})} e^{2\pi i w \cdot a} \sum_{k=0}^{\lfloor (w \cdot p) - 2k - 1 \rfloor} e^{\pi i (w \cdot p - 2k - 1) \epsilon},
\]  

(3.47)

Here we choose a saddle point specified by a magnetic charge \( p \). By summing over the KK modes along \( T^2 \) we obtain the indices of \( D_{\text{vec}} \) and \( D_{\text{hyp}} \) on \( T^2 \times \mathbb{R}^3 \):

\[
\text{ind}(D_{\text{vec}}) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m + n \tau)} \text{ind}(D_{\text{Bogo}},\mathcal{C}),
\]  

(3.48)

\[
\text{ind}(D_{\text{hyp}}) = -\sum_{m,n \in \mathbb{Z}} e^{2\pi i (m + n \tau)} \sum_{f=1}^{N_F} (e^{-2\pi i m_f} \text{ind}(D_{\text{DH},\mathcal{R}}) + e^{2\pi i m_f} \text{ind}(D_{\text{DH},\mathcal{R}})|_{a \rightarrow -a})).
\]  

(3.49)

Note that if the KK momentum contributions \( \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m + n \tau)} \) for \( T^2 \) in (3.48) and (3.49) are replaced by \( \sum_{m \in \mathbb{Z}} e^{2\pi i m} \) for \( S^1 \), we obtain the equivariant indices for the one-loop determinants of the BPS \( \text{\textquoteprime} \text{t} \text{Hooft} \) loop operators on \( S^1 \times \mathbb{R}^3 \). By using the relation (3.45) with
|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| D5 | × | × | × | × | × | × |   |   |   |
| NS5 | × | × | × | × | × | × |   |   |   |
| D3 | × | × | × | × | × | × |   |   |   |
| NS5' | × | × | × | × | × | × |   |   |   |

Table 1: The symbol × denotes the directions in which D-branes and NS5-branes extend.

(3.48) and (3.49), up to overall normalization constants, we obtain the non-regularized expression of the one-loop determinants around a saddle point specified by a magnetic charge $p$:

$$Z_{5d, \text{vec}}^{1\text{-loop}}(\alpha, p, \epsilon, \tau) = \left( \prod_{\alpha \in rt(g)} \prod_{k=0}^{n_{\alpha}} \prod_{m,n \in \mathbb{Z}} \left( m + n\tau + \alpha \cdot a + \left( \frac{|\alpha \cdot p|}{2} - k \right) \epsilon \right) \right)^{-\frac{1}{2}}, \quad (3.50)$$

$$Z_{5d, \text{hyp}}^{1\text{-loop}}(a, m, p, \epsilon, \tau) = \left( \prod_{w \in \Delta(R)} \prod_{k=0}^{N_F} \prod_{m,n \in \mathbb{Z}} \left( m + n\tau + w \cdot a - m_f + \left( \frac{|w \cdot p| - 1}{2} - k \right) \epsilon \right) \right)^{\frac{1}{2}}, \quad (3.51)$$

By using the zeta function regularization, we obtain the one-loop determinants (3.4), (3.5).

4 Monopole bubbling effects via branes and elliptic genera

For BPS 't Hooft loops in 4d $\mathcal{N} = 2 U(N)$ and $SU(N)$ gauge theories on $S^1 \times \mathbb{R}^3$, monopole bubbling effects are given by Witten indices for supersymmetric quantum mechanics (SQM) arising from the low energy world volume theories on D1-branes [24, 28, 29] in the type IIB string theory. Monopole bubbling effects were also studied in [26] in terms of the type IIA string theory. By taking a T-dual of the brane configuration of [26] in the $x^4$-direction, we obtain monopole bubbling effects $Z_{\text{mono}}$ for the surface operators on $T^2 \times \mathbb{R}^3$ which are given by elliptic genera for 2d gauged linear sigma models (GLSMs) living on $T^2$.

4.1 Brane construction for 't Hooft surface operators and monopole bubbling

First we explain a 5d $\mathcal{N} = 1 U(N)$ gauge theory arising from the low energy world volume theory on D5-NS5-brane system. The brane configuration is specified by Figure 1. When $N$ D5-branes are suspended between two NS5-branes in the $x^6$-direction, a 5d $\mathcal{N} = 1 U(N)$ vector multiplet arises from the low energy world volume theory on the D5-branes. When
Figure 1: (a): A brane configuration for a magnetic charge $B = e_1$. (b): A brane configuration for a magnetic charge $B = -e_N$. (c): A brane configuration for $B = e_1 + e_2$. (d): A brane configuration for a magnetic charge $B = 2e_1$.

$N$ D5-branes and $N'$ D5-branes are attached to an NS5-brane from the left side and the right side in the $x^6$-direction, respectively, a 5d $\mathcal{N} = 1$ hypermultiplet in the $U(N) \times U(N')$ bi-fundamental representation arises from the open strings ended on the $N$ D5-branes and the $N'$ D5-branes separated by the NS5-brane. If these $N$ (resp. $N'$) D5-branes are suspended between two NS5-branes, $U(N)$ (resp. $U(N')$) is a gauge group. On the other hand, if the $N$ (resp. $N'$) D5-branes are semi-infinite in the $x^6$-direction, $U(N)$ (resp. $U(N')$) is a flavor symmetry group. If the $x^6$-direction is compactified by a circle, we obtain circular quiver gauge theories with $U(N)$ gauge groups. In this paper, we compute explicitly monopole bubbling effects for 5d $U(N)$, $N' = 1^*$ gauge theories. The $N' = 1^*$ $U(N)$ gauge theory is obtained by the $N$ D5-branes are ended on a single NS5-brane and $x^6$-direction is compactified by the circle. Monopole bubbling effects for other gauge theories will be studied in elsewhere.

An 't Hooft surface operator is described by D3-branes stretched between D5-branes and NS5'-branes in the $x^5$-direction. Here we write an NS5-brane extending in the $x^{0,4,6,7,8,9}$-directions as an NS5'-brane to distinguish it from an NS5-brane extending in the $x^{0,1,2,3,4,5}$-directions. For later convenience, we call $N$ D5-branes as the first, the second, ..., the $N$-th D5-brane from the right to the left in the $x^5$-direction. The magnetic charge $B$ is read off from brane configurations as
Figure 2: (a): The brane configuration for the surface operator with the magnetic charge \( B = e_1 - e_N \). (b) A D3-brane suspended between the leftmost D5-brane and the rightmost D5-brane is introduced to describe the ’t Hooft-Polyakov monopole with the charge \( B = -e_1 + e_N \). (c): When the positions of the three D3-branes in the \( x^3 \)-direction coincide, the D3-branes reconnect and form a single D3-brane. Then the charge is reduced to \( p = 0 \) which is the brane interpretation of the monopole bubbling effect. (d): The quiver diagram for the world volume theory on the D3-brane in Figure (c). The circle with 1 denotes 2d \( \mathcal{N} = (4,4) \) \( U(1) \) vector multiplet. The solid line connected between the circle and the box with \( N \) denotes the \( N \)-tuple 2d \( \mathcal{N} = (4,4) \) hypermultiplets with the gauge charge +1.

- A D3-brane attached to the \( i \)-th D5-brane from the right side corresponds to a magnetic charge \( B = e_i \).
- A D3-brane attached to the \( i \)-th D5-brane from the left side corresponds to a magnetic charge \( B = -e_i \).

Here \( e_i = \text{diag}(0, \ldots, 0, 1, 0 \cdots, 0) \). Examples of brane configurations for surface operators and their magnetic charges are listed in Figure 1.

To describe monopole bubbling effects in the brane set up, we introduce a D3-brane suspended between the \( i \)-th D5-brane and the \( j \)-th D5-brane which is interpreted as a smooth ’t Hooft-Polyakov monopole with \( B = e_i - e_j \) [30], see Figure 2(b). When the \( x^3 \)-coordinate of the D3-branes for the surface operators and the one of a D3-brane for the ’t Hooft-Polyakov monopole coincide, the magnetic charge is screened and monopole bubbling occurs, see 2(c). The low energy world volume theory on D3-branes suspended between NS5'-branes is 2d supersymmetric GLSM. This T-dual picture of the proposal in [28, 26] suggests that the supersymmetric indices, i.e., elliptic genera for the world volume theories of D3-branes ended
on NS5'-branes give the monopole bubbling effects $Z_{\text{mono}}$.

### 4.2 Elliptic genera for monopole bubbling effects

By using the localization formula for elliptic genera in [31, 32], we can express the elliptic genus for the monopole bubbling effects in terms of residue integrals called Jeffrey-Kirwan (JK) residues:

$$Z^{(\eta)}_{\text{mono}}(a, m, p, B, \epsilon, \tau) = \frac{1}{|W_{G_{2d}}|} \sum_{u_* \in \mathcal{M}_{\text{sing}}} \text{JK-Res}(Q_*, \eta)$$

$$\times \prod Z_{2d, \text{vec}} \prod Z_{2d, \text{hyp}} \, du_1^1 \wedge \cdots \wedge du_{\text{rk}(g_{2d})}.$$  (4.1)

Here $G_{2d}$ and $g_{2d}$ are the gauge group and its Lie algebra of 2d GLSM, respectively. $\text{rk}(g)$ is the rank of the Lie algebra $g$.

For 5d $\mathcal{N} = 1^*$ $U(N)$ gauge theory, the GLSMs for the monopole bubbling preserve the 2d $\mathcal{N} = (4, 4)$ supersymmetry (more precisely the mass deformation of $\mathcal{N} = (4, 4)$ supersymmetry). For $G_{2d} = U(n)$ 2d $\mathcal{N} = (4, 4)$ GLSM, the one-loop determinant for the vector multiplet is

$$Z_{2d, \text{vec}} = (2\pi \eta(\tau))^3 \prod_{1 \leq a \neq b \leq n} \vartheta_1(u_a - u_b) \cdot \prod_{a, b = 1}^{n} \frac{\vartheta_1(u_a - u_b + \epsilon)}{\vartheta_1(u_a - u_b \pm m_{ad} + \frac{1}{2} \epsilon)}.$$  (4.2)

A fugacity $m_{ad}$ corresponds to the flavor fugacity for 5d $\mathcal{N} = 1^*$ theory. The one-loop determinant for a 2d $\mathcal{N} = (4, 4)$ hypermultiplet in the bifundamental representation of $U(n) \times U(n')$ is

$$Z_{2d, \text{hyp}} = \prod_{a=1}^{n} \prod_{b=1}^{n'} \vartheta_1(\pm (u_a - u'_b) - m_{ad}) \frac{\vartheta_1(\pm (u_a - u'_b) + \frac{1}{2} \epsilon)}{\vartheta_1(\pm (u_a - u'_b) + \frac{1}{2} \epsilon)}.$$  (4.3)

When $U(n')$ is a flavor symmetry of 2d GLSM, fugacity $u'$ corresponds to components of the flat connection $a$ of the five-dimensional gauge field.

JK-Res$_{u=u_*}(Q_*, \eta)$ in (4.1) denotes the Jeffrey–Kirwan residue at a point $u_*$ defined as follows. We consider a situation that $\text{rk}(g_{2d})$ hyperplanes of codimension one, called singular hyperplanes $Q_i \cdot (u - u_*) = \sum_{a=1}^{\text{rk}(g_{2d})} Q_i^a (u_a - u_a^*) = 0$ ($i = 1, \cdots, \text{rk}(g_{2d})$) intersect at a point $u_* = (u_1^*, \cdots, u_{\text{rk}(g_{2d})}^*)$ in the $u$-space. In our case, $Q_i = (Q_1^1, \cdots, Q_i^{\text{rk}(g_{2d})}) \in \mathbb{R}^{\text{rk}(g_{2d})}$ is a weight vector appearing in the denominators of the one-loop determinants (4.2) and (4.3).

Then the JK residue at the point $u_*$ is defined by

$$\text{JK-Res}_{u=u_*}(Q_*, \eta) = \begin{cases} \frac{du_1^1 \wedge \cdots \wedge du_{\text{rk}(g_{2d})}}{Q_1 \cdot (u - u_*) \cdots Q_{\text{rk}(g_{2d})} \cdot (u - u_*)} & \text{if } \eta \in \text{Cone}(Q_1, \cdots, Q_{\text{rk}(g_{2d})}), \\ 0 & \text{otherwise}. \end{cases}$$  (4.4)
Here \( \text{Cone}(Q_1, \ldots, Q_{\text{rk}(\mathfrak{g}_{2d})}) = \sum_{i=1}^{\text{rk}(\mathfrak{g}_{2d})} \mathbb{R}_{>0} Q_i \) and each \( Q_i \) is a \( \text{rk}(\mathfrak{g}_{2d}) \)-dimensional vector. The sum \( \sum_{u_*} \) runs over all the points \( u_* \), where \( N' \) with \( N' = \text{rk}(\mathfrak{g}_{2d}) \) singular hyperplanes meet at a point and the condition \( \eta \in \text{Cone}(Q_1, \ldots, Q_{\text{rk}(\mathfrak{g}_{2d})}) \) is satisfied. If \( N' > \text{rk}(\mathfrak{g}_{2d}) \) singular hyperplanes with \( N' > \text{rk}(\mathfrak{g}_{2d}) \) intersect at a point, we apply the constructive definition of the JK residue in [32]. All the examples treated in Section 6 satisfy the condition \( N' = \text{rk}(\mathfrak{g}_{2d}) \).

5 Deformation quantization and elliptic Ruijsenaars operators

In [2] we have shown that the deformation quantization of SUSY localization formula of the ‘t Hooft loops in the anti-symmetric representation in 4d \( \mathcal{N} = 2^* \) \( U(N) \) gauge theory can be identified with trigonometric Macdonald operators by redefinitions of variables. We also gave the identification between SUSY localization formulas for monopole operators and representation of monopole operators as difference operators of rational type in [9]. These observations motivate us to relate a deformation quantization of vevs of ‘t Hooft surface operators to difference operator of elliptic type.

We regard \( \mathbf{a} = (a_1, \ldots, a_{\text{rk}(\mathfrak{g})}), \mathbf{b} = (b_1, \ldots, b_{\text{rk}(\mathfrak{g})}) \) as the coordinate and momentum variables, respectively, and define the quantization by

\[
[\hat{a}_i, \hat{a}_j] = 0, \quad [\hat{b}_i, \hat{b}_j] = 0, \quad [\hat{b}_i, \hat{a}_j] = \epsilon C_{ij}.
\] (5.1)

where, and \( C_{ij} \) is the \((i,j)\)-component of the inverse matrix of the Killing form \( C_{ij} = \text{Tr}(H_i H_j) \), where \( \{H_i\}_{i=1}^{\text{rk}(\mathfrak{g})} \) is bases of the Cartan subalgebra of \( \mathfrak{g} \).

We define the quantization of a function \( f(\mathbf{a}, \mathbf{b}) \) by the Weyl-Wigner transform:

\[
\hat{f}(\hat{a}, \hat{b}) := \left. \exp \left( \frac{\epsilon}{2} \sum_{i,j=1}^{N} C_{ij} \partial_{b_i} \partial_{a_j} \right) f(\mathbf{a}, \mathbf{b}) \right|_{a \rightarrow \hat{a}, b \rightarrow \hat{b}}.
\] (5.2)

After the derivative is taken in (5.2), the ordering of \( \hat{a}_i \) and \( \hat{b}_i \) is defined as follows. All the \( a_i \)’s should be on left side and all \( b_i \)’s should be on the right side. Next we define the Moyal product \( f \ast g \) of \( f \) and \( g \) by

\[
f(\mathbf{a}, \mathbf{b}) \ast g(\mathbf{a}, \mathbf{b}) := \exp \left( \frac{\epsilon}{2} \sum_{i,j=1}^{N} C^{ij}(\partial_{a_i} \partial_{b_j} - \partial_{a_j} \partial_{b_i}) \right) f(\mathbf{a}, \mathbf{b}) g(\mathbf{a}, \mathbf{b}) \big|_{\mathbf{a} = \mathbf{a}, \mathbf{b} = \mathbf{b}}.
\] (5.3)

Since the Weyl-Wigner transform satisfies the following relation

\[
\hat{f} \ast \hat{g} = \hat{f} \hat{g},
\] (5.4)

1For the simple Lie algebras, we normalize \( H_i \) so that \( C^{ij} \) agrees with the Cartan matrix. For \( U(N) \) gauge theory, \( C^{ij} \) is normalized as \( C^{ij} = \delta^{ij} \).
the algebra of the deformation quantization of surface operators can be studied in term of the Moyal products. For example, the commutativity of the operators $f$ and $g$ is equivalent to that of $\hat{f}$ and $\hat{g}$ with respect to the Moyal product:

$$[\hat{f}, \hat{g}] = 0 \leftrightarrow f \ast g - g \ast f = 0.$$  \hspace{1cm} (5.5)

In particular, we will see that the commutativity of elliptic Ruijsenaars operators is rephrased as the absence of wall-crossing phenomena ($\eta$-independence) of elliptic genera for the monopole bubbling effects. Recently the author of \[33\] observed that the localization formula \[1\] of face operators with specific magnetic charge in 5d $\mathcal{N} = 1$ theory agrees with a trigonometric limit of type-A elliptic Ruijsenaars operators \[34\]. Since the KK modes along the $T^2$-direction give an elliptic deformation of the localization formula of the 't Hooft loops, we expect the deformation quantization of 't Hooft surface operator itself agrees with elliptic Ruisenaars operators. In fact, we show that the deformation quantization of 't Hooft surface operators with specific magnetic charge in 5d $\mathcal{N} = 1^*$ theory is identified with elliptic Ruijsenaars operators. We also study the algebra of 't Hooft surface operators defined by the Moyal product.

From the the localization formula \[3.1\], the vevs of 't Hooft surface operators $S_B$ with $B = \sum_{k=1}^\ell e_k$ and $B = -\sum_{k=\ell+1}^N e_k$ in the $\mathcal{N} = 1^*$ $U(N)$ gauge theory are given by\[2\]

$$\langle S_{\sum_{k=1}^\ell e_k} \rangle = \sum_{I \subset \{1, \cdots, N\} \atop |I| = \ell} \left( \prod_{i \in I, j \notin I} \frac{\vartheta(a_i - a_j - m_{ad})}{\vartheta(a_i - a_j + \frac{\epsilon}{2})} \frac{\vartheta(a_j - a_i - m_{ad})}{\vartheta(a_j - a_i + \frac{\epsilon}{2})} \right) \frac{1}{2} \prod_{i \in I} e^{b_i}, \hspace{1cm} (5.6)$$

$$\langle S_{-\sum_{k=\ell+1}^N e_k} \rangle = \sum_{I \subset \{1, \cdots, N\} \atop |I| = \ell} \left( \prod_{i \in I, j \notin I} \frac{\vartheta(a_i - a_j - m_{ad})}{\vartheta(a_i - a_j + \frac{\epsilon}{2})} \frac{\vartheta(a_j - a_i - m_{ad})}{\vartheta(a_j - a_i + \frac{\epsilon}{2})} \right) \frac{1}{2} \prod_{i \in I} e^{-b_i}. \hspace{1cm} (5.7)$$

Here $m_{ad}$ is the flavor fugacity for the adjoint hypermultiplet in five dimensions. $I$ is a subset of $\{1, \cdots, N\}$ with the cardinality $|I| = \ell$. Then the Weyl-Wigner transform of the vevs of surface operators are written as

$$\hat{S}_{\sum_{k=1}^\ell e_k} = \sum_{I \subset \{1, \cdots, N\} \atop |I| = \ell} \left( \prod_{i \in I, j \notin I} \frac{\vartheta(a_i - a_j - m_{ad} + \frac{\epsilon}{2})}{\vartheta(a_i - a_j + \epsilon)} \frac{\vartheta(a_j - a_i - m_{ad} - \frac{\epsilon}{2})}{\vartheta(a_j - a_i - \frac{\epsilon}{2})} \right) \frac{1}{2} \prod_{i \in I} e^{\varphi \frac{b_i}{m_{ad}}},$$

$$\hat{S}_{-\sum_{k=\ell+1}^N e_k} = \sum_{I \subset \{1, \cdots, N\} \atop |I| = \ell} \left( \prod_{i \in I, j \notin I} \frac{\vartheta(a_i - a_j - m_{ad} - \frac{\epsilon}{2})}{\vartheta(a_i - a_j - \epsilon)} \frac{\vartheta(a_j - a_i - m_{ad} + \frac{\epsilon}{2})}{\vartheta(a_j - a_i + \frac{\epsilon}{2})} \right) \frac{1}{2} \prod_{i \in I} e^{-\varphi \frac{b_i}{m_{ad}}}. \hspace{1cm} (5.8)$$

\[2\] A dominant coweight $B$ of $U(N)$ is $\text{diag}(B_1, B_2, \cdots, B_N)$ with $B_1 \geq B_2 \geq \cdots \geq B_N$ and $B_i \in \mathbb{Z}$. 

20
Here we use a representation \( \hat{a}_i = a_i \) and \( \hat{b}_i = e^{\epsilon \frac{\partial}{\partial a_i}} \), and find that the deformation quantization of the 't Hooft surface operators (5.8) and (5.9) agree with type-A elliptic Ruijsenaars operators \[34\].

6 Products of 't Hooft surface operators and monopole bubbling

In this section, we evaluate explicitly elliptic genera for monopole bubbling effects with small values of magnetic charges and compare them with results from the Moyal product. We consider monopole bubbling effects in the monopole surface operator \( S_B \) for \( B = (1, 0, \cdots, 0, -1) = \text{diag}(1, 0, \cdots, 0, -1) \) and \( B = (1, 1, 0, \cdots, 0, -1, -1) = \text{diag}(1, 1, 0, \cdots, 0, -1, -1) \) in \( \mathcal{N} = 1^* U(N) \) gauge theory.

6.1 Surface operator \( S_{(1,0,\cdots,0,-1)} \)

From the localization formula \[3.1\] and the definition of the Moyal product \[5.3\], two different orderings of the Moyal product of \( \langle S_{e_1} \rangle \) and \( \langle S_{-e_N} \rangle \) are evaluated as

\[
\langle S_{e_1} \rangle \ast \langle S_{-e_N} \rangle = \sum_{1 \leq i \neq j \leq N} e^{b_i - b_j} Z_{1\text{-loop}}^{5d}(p = e_i - e_j) \left[ \sum_{i=1}^{N} \prod_{j=1 \atop j \neq i}^{N} \frac{\vartheta_1(a_i - a_j - m_{ad} + \frac{1}{2} \epsilon) \vartheta_1(a_j - a_i - m_{ad} - \frac{1}{2} \epsilon)}{\vartheta_1(a_i - a_j + \epsilon) \vartheta_1(a_j - a_i)} \right], \tag{6.1}
\]

\[
\langle S_{-e_N} \rangle \ast \langle S_{e_1} \rangle = \sum_{1 \leq i \neq j \leq N} e^{b_i - b_j} Z_{1\text{-loop}}^{5d}(p = e_i - e_j) \left[ \sum_{i=1}^{N} \prod_{j=1 \atop j \neq i}^{N} \frac{\vartheta_1(a_i - a_j - m_{ad} - \frac{1}{2} \epsilon) \vartheta_1(a_j - a_i - m_{ad} + \frac{1}{2} \epsilon)}{\vartheta_1(a_i - a_j + \epsilon) \vartheta_1(a_j - a_i)} \right]. \tag{6.2}
\]

Here the explicit form of the one-loop determinant \( Z_{1\text{-loop}}^{5d}(p = e_i - e_j) \) is summarized in Appendix \[B\]. Although the second line in (6.1) looks different from the one in (6.2), we will show these terms are actually same by using the contour integral expression of an elliptic genus. This leads to the Moyal products of \( \langle S_{-e_N} \rangle \) and \( \langle S_{e_1} \rangle \) commute each other.

\[ \langle S_{e_1} \rangle \ast \langle S_{-e_N} \rangle = \langle S_{e_1} \rangle \ast \langle S_{-e_N} \rangle. \]

We evaluate the monopole bubbling effect in the expectation value of \( S_B \) with \( B = e_1 - e_N = (1, 0, \cdots, 0, -1) \) and study the algebra of 't Hooft surface operators. From the localization formula \[3.1\], \( \langle S_{e_1-e_N} \rangle \) is given by

\[
\langle S_{e_1-e_N} \rangle = \sum_{1 \leq i \neq j \leq N} e^{b_i - b_j} Z_{1\text{-loop}}^{5d}(p = e_i - e_j) + Z_{\text{mono}}(p = 0, B = e_1 - e_N). \tag{6.3}
\]
The monopole bubbling effect in (6.3) is evaluated by the brane construction and localization formula for the elliptic genus. As explained in Section 4.1, the brane configuration for the monopole bubbling effect $B = e_1 - e_N$ and $p = 0$ is depicted in Figure 2. The quiver diagram of the bubbling GLSM is specified by Figure 2(d). From the localization formula, the JK residue for the elliptic genus of the GLSM in Figure 2(d) is same as the following contour integral:

$$Z_{\text{mono}}^{(\eta)}(p = 0, B = e_1 - e_N) = \text{sign}(\eta) \frac{2\eta(\tau)^3}{\vartheta_1(\vartheta)} \oint du \prod_{i=1}^{N} \vartheta_1(\vartheta u - a_i + \frac{1}{2}\epsilon) \vartheta_1(\vartheta u - a_i - \frac{1}{2}\epsilon).$$

(6.4)

Here $\text{sign}(\eta)$ is $+1$ for $\eta > 0$ and $-1$ for $\eta < 0$, respectively. When $\eta > 0$, the JK residue operation is the residues at poles $u = a_i - \frac{1}{2}\epsilon$ for $i = 1, \ldots, N$. On the other hand, when $\eta < 0$, the JK residue operation is the residues at poles $u = a_i + \frac{1}{2}\epsilon$ for $i = 1, \ldots, N$. The elliptic genus for the the monopole bubbling effect is given by

$$Z_{\text{mono}}^{(\eta>0)}(p = 0, B = e_1 - e_N) = \frac{\prod_{i=1}^{N} \vartheta_1(a_i - a_j - m_{\text{ad}} - \frac{1}{2}\epsilon) \vartheta_1(a_j - a_i - m_{\text{ad}} + \frac{1}{2}\epsilon)}{\vartheta_1(a_i - a_j) \vartheta_1(a_j - a_i + \epsilon)}.$$

(6.5)

$$Z_{\text{mono}}^{(\eta<0)}(p = 0, B = e_1 - e_N) = \frac{\prod_{i=1}^{N} \vartheta_1(a_j - a_i - m_{\text{ad}} + \frac{1}{2}\epsilon) \vartheta_1(a_i - a_j - m_{\text{ad}} - \frac{1}{2}\epsilon)}{\vartheta_1(a_i - a_j + \epsilon) \vartheta_1(a_j - a_i)}.$$

(6.6)

Note that $Z_{\text{mono}}^{(\eta>0)} = Z_{\text{mono}}^{(\eta<0)}$. This follows from the fact that the sum of the residues of a rational form on the torus is zero.

By comparing the expressions of $\langle S_{e_1} \rangle \langle S_{-e_N} \rangle$, $\langle S_{-e_N} \rangle \langle S_{e_1} \rangle$ and $\langle S_{e_1 - e_N} \rangle$, we find that the algebraic relation of the ’t Hooft surface operators:

$$\langle S_{e_1 - e_N} \rangle = \langle S_{e_1} \rangle \langle S_{-e_N} \rangle = \langle S_{-e_N} \rangle \langle S_{e_1} \rangle = \langle S_{e_1} \rangle \langle S_{-e_N} \rangle$$

(6.7)

Therefore we find that a surface operator with a higher charge is generated by surface operators with the minimal charges. Applying the Weyl-Wigner transform to $\langle S_{e_1} \rangle \langle S_{-e_N} \rangle = \langle S_{-e_N} \rangle \langle S_{e_1} \rangle$, we obtain the commutativity of two elliptic Ruijsenaars operators $[\hat{S}_{e_1}, \hat{S}_{-e_N}] = 0$.

### 6.2 Surface operator $S_{(1,1,0,\ldots,0,-1,-1)}$

Next we study the monopole bubbling effects in $\langle S_{(1,1,0,\ldots,0,-1,-1)} \rangle = \langle S_{e_1 + e_2 - e_{N-1} - e_N} \rangle$, and study the algebra of ’t Hooft surface operators. The Moyal products of $\langle S_{e_1 + e_2} \rangle$ and
Figure 3: (a) The brane configuration for the surface operator with the magnetic charge $B = e_1 + e_2 - e_{N-1} - e_N$. (b) A D3-brane suspended between the second D5-brane and the $(N - 1)$-th D5-brane is introduced to describe the 't Hooft-Polyakov monopole with the charge $B = -e_2 + e_{N-1}$. (c) After Hanany-Witten effect, we obtain the brane configuration for describing the monopole bubbling effect $p = e_1 - e_N$.

\[
\langle S_{-e_{N-1}-e_N} \rangle \text{ are evaluated as}
\]

\[
\langle S_{e_1+e_2} \rangle \cdot \langle S_{-e_{N-1}-e_N} \rangle = \sum_{1 \leq i \neq j \leq N, 1 \leq k \leq N} \epsilon^{b_i+b_j-b_k-b_l} Z_{1\text{-loop}}^{5d} (p = e_i + e_j - e_k - e_l)
\]

\[
\sum_{1 \leq i \neq j \leq N} \epsilon^{b_i-b_j} Z_{1\text{-loop}}^{5d} (p = e_i - e_j) \sum_{l=1}^{N} \prod_{k=1}^{N} \frac{\vartheta_1(a_l - a_k - m_{ad} + \frac{1}{2} \epsilon) \vartheta_1(a_k - a_l - m_{ad} - \frac{1}{2} \epsilon)}{\vartheta_1(a_l - a_k + \epsilon) \vartheta_1(a_k - a_l)},
\]

\[
\langle S_{-e_{N-1}-e_N} \rangle \cdot \langle S_{e_1+e_2} \rangle = \sum_{1 \leq i \neq j \leq N, 1 \leq k \leq N} \epsilon^{b_i+b_j-b_k-b_l} Z_{1\text{-loop}}^{5d} (p = e_i + e_j - e_k - e_l)
\]

\[
\sum_{1 \leq i \neq j \leq N} \epsilon^{b_i-b_j} Z_{1\text{-loop}}^{5d} (p = e_i - e_j) \sum_{l=1}^{N} \prod_{k=1}^{N} \frac{\vartheta_1(a_l - a_k - m_{ad} - \frac{1}{2} \epsilon) \vartheta_1(a_k - a_l - m_{ad} + \frac{1}{2} \epsilon)}{\vartheta_1(a_l - a_k + \epsilon) \vartheta_1(a_k - a_l + \epsilon)}.
\]

We evaluate $\langle S_{e_1+e_2-e_{N-1}-e_N} \rangle$ and compare it with the Moyal products (6.8) and (6.9). The localization formula for $\langle S_{e_1+e_2-e_{N-1}-e_N} \rangle$ has the following expression.

\[
\langle S_{e_1+e_2-e_{N-1}-e_N} \rangle = \sum_{1 \leq i \neq j \leq N, 1 \leq k \leq N} \epsilon^{b_i+b_j-b_k-b_l} Z_{1\text{-loop}}^{5d} (p = e_i + e_j - e_k - e_l)
\]

\[
+ \sum_{1 \leq i \neq j \leq N} \epsilon^{b_i-b_j} Z_{1\text{-loop}}^{5d} (p = e_i - e_j) Z_{\text{mono}} (p = e_i - e_j).
\]
Here we suppressed \( B = e_1 + e_2 - e_{N-1} - e_N \) in \( Z_{\text{mono}} \) s to shorten the expression. The one-loop determinants \( Z_{1\text{-loop}}^{5\text{d}} \) are given by \((B.1)\) and \((B.2)\). In this case, there are two monopole bubbling sectors specified by \( p = e_i - e_j \) and \( p = 0 \).

First let us evaluate the monopole bubbling effect specified by \( p = e_i - e_j \). The brane configuration for the ‘t Hooft surface operator with the magnetic charge \( B = e_1 + e_2 - e_{N-1} - e_N \) is depicted by Figure 3(a). We introduce a D3-brane suspended between two D5-branes as Figure 3(b), which correspond to an ‘t Hooft-Polyakov monopole. After Hanany-Witten transition, the matter contents of the low energy world volume theory on the D3-brane is that the gauge group is \( U(1) \) and the number of \( \mathcal{N} = (4, 4) \) hypermultiplets is \( N = 2 \). For example the brane configuration for \( i = 1, j = N \) depicted by Figure 3(c). Then the monopole bubbling effect for \( B = e_1 + e_2 - e_{N-1} - e_N \) and \( p = e_i - e_j \) is given by

\[
Z_{\text{mono}}^{(\eta)}(p = e_i - e_j) = \text{sign}(\eta) \frac{2\eta(\tau)^3}{\vartheta_1(\pm m_{\text{ad}} + \frac{1}{2} \epsilon)} \int du \prod_{k=1, k \neq i, j}^N \frac{\vartheta_1(\pm(u-a_k) - m_{\text{ad}})}{\vartheta_1(\pm(u-a_k) + \frac{1}{2} \epsilon)}.
\]

(6.10)

(6.11) is evaluate in similar way as (6.4).

\[
Z_{\text{mono}}^{(\eta>0)}(p = e_i - e_j) = \sum_{\ell = 1, \ell \neq i, j}^N \prod_{k=1, k \neq i, j}^N \frac{\vartheta_1(a_l - a_k - m_{\text{ad}} - \frac{1}{2} \epsilon) \vartheta_1(a_k - a_l - m_{\text{ad}} + \frac{1}{2} \epsilon)}{\vartheta_1(\pm(u-a_k) - m_{\text{ad}})}
\]

(6.12)

\[
Z_{\text{mono}}^{(\eta<0)}(p = e_i - e_j) = \sum_{\ell = 1, \ell \neq i, j}^N \prod_{k=1, k \neq i, j}^N \frac{\vartheta_1(a_l - a_k - m_{\text{ad}} + \frac{1}{2} \epsilon) \vartheta_1(a_k - a_l - m_{\text{ad}} - \frac{1}{2} \epsilon)}{\vartheta_1(\pm(u-a_k) + m_{\text{ad}})}
\]

(6.13)

Again we have \( Z_{\text{mono}}^{(\eta>0)}(p = e_i - e_j) = Z_{\text{mono}}^{(\eta<0)}(p = e_i - e_j) \).

Next we evaluate the monopole bubbling effect specified by \( p = 0 \) from the brane construction. To achieve the monopole bubbling effect, we introduce two D3-branes suspended between D5-branes depicted as Figure 4(a). When the positions of the D3-branes coincides, we obtain the brane configuration in Figure 4(b). The low energy world volume theory on two D3-branes in 4(b) is the 2d \( \mathcal{N} = (4, 4) \) \( U(2) \) GLSM with \( N \) hypermultiplets. The localization formula for the elliptic genus is given by

\[
Z_{\text{mono}}^{(\eta)}(p = 0) = \frac{(2\pi \eta (\tau)^3)^2}{2} \sum_{u_+ \in \mathfrak{M}_{\text{sing}}} \text{JK-Res}(Q_+, \eta) \prod_{1 \leq a \neq b \leq 2} \vartheta_1(u_a - u_b)
\]

\[
\times \prod_{a,b=1}^2 \vartheta_1(u_a - u_b + \epsilon) \cdot \prod_{b=1}^2 \prod_{i=1}^N \frac{\vartheta_1(\pm(u_b - a_i) - m_{\text{ad}})}{\vartheta_1(\pm(u_b - a_i) + \frac{1}{2} \epsilon)} du_1 \wedge du_2.
\]

(6.14)
Figure 4: (a): We added two D3-branes suspended between D5-branes to Figure 3(a). When the positions of the segments of D3-branes coincide, the screening of monopole charges charge occurs. The world volume theory on two D3-branes give the monopole bubbling effect for \( \mathbf{p} = \mathbf{0} \). (b): The quiver diagram denotes the matter content of the low energy world volume theory on the D3-branes of Figure 4(b). The circle with 2 denotes \( N = (4, 4) \) \( U(2) \) vector multiplet, and the solid line denotes the \( N \) hypermultiplets in the fundamental representation of \( U(2) \).

When a vector \( \eta \) is proportional to \((1, 1)\), From the definition of the JK residues, \( \mathfrak{M}_{\text{sing}} \) in the \((u_1, u_2)\) space, where JK residues are evaluated are specified by the intersection point of the following singular hyperplanes:

\[
\begin{align*}
\left\{ u_1 - a_i + \frac{1}{2} \epsilon = 0 \right\} \cap \left\{ u_2 - a_j + \frac{1}{2} \epsilon = 0 \right\} & \quad \text{for } i, j = 1, \ldots, N, \quad (6.15) \\
\left\{ u_1 - u_2 \pm m_{\text{ad}} + \frac{1}{2} \epsilon \right\} \cap \left\{ u_2 - a_i + \frac{1}{2} \epsilon = 0 \right\} & \quad \text{for } i = 1, \ldots, N, \quad (6.16) \\
\left\{ u_1 - u_2 \pm m_{\text{ad}} + \frac{1}{2} \epsilon \right\} \cap \left\{ u_2 - a_i + \frac{1}{2} \epsilon = 0 \right\} & \quad \text{for } i = 1, \ldots, N. \quad (6.17)
\end{align*}
\]

The JK residues at the intersection points of \((6.16)\) and \((6.17)\) are zero and the monopole bubbling effect is given by the JK residue at the intersection points of \((6.15)\) as

\[
Z_{\text{mono}}^{(\eta=(1,1))}(\mathbf{p} = \mathbf{0}, \mathbf{B} = \mathbf{e}_1 - \mathbf{e}_N) = \sum_{i,j=1}^{N} \prod_{l=i,j}^{N} \prod_{k=1}^{N} \frac{\vartheta_1(a_l - a_k - m_{\text{ad}} - \frac{1}{2} \epsilon) \vartheta_1(a_k - a_l - m_{\text{ad}} - \frac{1}{2} \epsilon)}{\vartheta_1(a_l - a_k) \vartheta_1(a_k - a_l + \epsilon)}. \quad (6.18)
\]

In the similar way, when \( \eta \) is proportional to \((-1,-1)\), JK residues are evaluated at

\[
\begin{align*}
\left\{ -(u_1 - a_i) + \frac{1}{2} \epsilon = 0 \right\} \cap \left\{ -(u_2 - a_j) + \frac{1}{2} \epsilon = 0 \right\} & \quad \text{for } i, j = 1, \ldots, N, \quad (6.19) \\
\left\{ u_1 - u_2 \pm m_{\text{ad}} + \frac{1}{2} \epsilon \right\} \cap \left\{ -(u_1 - a_i) + \frac{1}{2} \epsilon = 0 \right\} & \quad \text{for } i = 1, \ldots, N, \quad (6.20) \\
\left\{ u_2 - u_1 \pm m_{\text{ad}} + \frac{1}{2} \epsilon \right\} \cap \left\{ -(u_2 - a_i) + \frac{1}{2} \epsilon = 0 \right\} & \quad \text{for } i = 1, \ldots, N. \quad (6.21)
\end{align*}
\]
the monopole bubbling effect are given by

\[ Z_{\text{mono}}^{(\eta=(-1,-1))}(p = 0, B = e_1 - e_N) = \sum_{i,j=1}^{N} \prod_{k=1}^{N} \prod_{l \neq i,j}^{N} \left( a_l - a_k - m_{ad} + \frac{1}{2} \epsilon \right) \partial_1(a_k - a_l - m_{ad} - \frac{1}{2} \epsilon) \partial_1(a_l - a_k + \epsilon) \partial_1(a_k - a_l) \]. \tag{6.22}

For the same reason as before, \( Z_{\text{mono}}^{(\eta=(1,1))} = Z_{\text{mono}}^{(\eta=(-1,-1))} \). By comparing the Moyal products with the SUSY localization formula for (6.10), we find the algebraic relation between vevs of 't Hooft surface operators:

\[ \langle S_{e_1+e_2-e_{N-1}-e_N} \rangle = \langle S_{e_1+e_2} \rangle \ast \langle S_{-e_{N-1}-e_N} \rangle = \langle S_{-e_{N-1}-e_N} \rangle \ast \langle S_{e_1+e_2} \rangle. \tag{6.23} \]

By applying the Weyl-Wigner transform to (6.23), we obtain the commutation relation of elliptic Ruijsenaars operators: \( \left[ \hat{S}_{e_1+e_2}, \hat{S}_{-e_{N-1}-e_N} \right] = 0. \)

7 Discussion

We discuss the results in our paper and the future directions. We introduced magnetically charged surface operators on \( T^2 \times \mathbb{R}^3 \) and evaluated the expectation values in terms of supersymmetric localization. The SUSY localization formula obtained in this paper gives an elliptic deformation of localization formula for BPS 't Hooft loops on \( S^1 \times \mathbb{R}^3 \)[1] and the one for BPS bare monopole operators in [2, 3]. We concentrated on 5d \( \mathcal{N} = 1 \ast U(N) \) gauge theory and concretely studied the algebra of surface operators and monopole bubbling effects with small magnetic charges. For general 5d \( \mathcal{N} = 1 \) gauge theories, monopole bubbling effects are described by elliptic genera of 2d \( \mathcal{N} = (0,4) \) GLSMs, which are obtained by the T-dual picture of the brane configuration studied in [26]. The gauge anomaly cancellation condition for 2d \( \mathcal{N} = (0,4) \) GLSMs give constraints on the matter contents of the five dimensional theory and also magnetic charge \( B \). So far we do not have a clear understanding of the role of gauge anomaly cancellation condition from the five-dimensional view point.

In Section 5, we found that the deformation quantization of 't Hooft surface operators in 5d \( \mathcal{N} = 1 \ast \) gauge theory agrees with the type-A elliptic Ruijsenaars operators. Although the integrable structure appears in the 't Hooft surface operators is not manifest in the supersymmetric gauge theory, by using dualities in string theory, the brane configuration for 't Hooft operators without the monopole bubbling effect is interpreted as defects in four-dimensional Chern-Simons theory, where the quantum integrable structure naturally appear [33].

In Section 6, we studied monopole bubbling effects and algebraic relation of 't Hooft surface operators. The physical interpretation of the Moyal product and Weyl-Wigner transform are as follows. For 't Hooft loop operator on \( S^1 \times \mathbb{R}^2 \times \mathbb{R} \), the Moyal product of vevs of \( n \)
't Hooft loops is identified with \( n \)-point correlation functions of 't Hooft loop operators \([1][2][5]\). Then we expect a similar result holds, i.e.,

\[
\langle S_{B_1} \cdot S_{B_2} \cdots S_{B_{n-1}} \cdot S_{B_n} \rangle = \langle S_{B_1} \rangle * \langle S_{B_2} \rangle * \cdots * \langle S_{B_{n-1}} \rangle * \langle S_{B_n} \rangle.
\]

(7.1)

Here the center of Dirac monopole for the surface operator \( S_{B_i} \) for \( i = 1, \cdots, n \) locates at \((x^1, x^2, x^3) = (0, 0, x^3_i)\) with \( x^3_1 > x^3_2 > \cdots > x^3_n \). If an algebraic relation we find is written as \( \langle S_{B_1} + B_2 \rangle = \langle S_{B_1} \rangle \ast \langle S_{B_2} \rangle \), (7.1) implies that the operator product expansion of 't Hooft surface operators \( S_{B_2} S_{B_2} = S_{B_1+B_2} \) holds in the correlation function. It is desirable to evaluate correlation functions for surface operators in terms of supersymmetric localization and to establish the relation between the correlation functions and the Moyal products of vevs of a single 't Hooft surface operator.

Another future direction is to study an elliptic deformation of the Coulomb branch chiral rings for three dimensional gauge theories. In 3d \( \mathcal{N} = 4 \) gauge theories, the algebra of the BPS bare monopole operators, the coulomb branch scalars and the BPS dressed monopole operators is defined in terms of the Moyal product and the Weyl-Wigner transform \([2]\). Then it found that the algebra agrees with the Coulomb branch chiral ring and its deformation quantization (a.k.a quantized Coulomb branch) in \([7]\). Here the three dimensional version of algebraic relations studied in Section 6 are identified with ring relations of bare monopole operators in the quantized Coulomb branch. Thus the algebraic relation we find is expected to be related to ring relations in an elliptic deformation of the quantized Coulomb branches. It is interesting to study the five-dimensional uplifts of the Coulomb branch scalars and the dressed monopole operators and define an elliptic deformation of the quantized Coulomb branches.

**Acknowledgements**

The author would like to thank Hirotaka Hayashi, Kazunobu Maruyoshi, Hiraku Nakajima, and Takuya Okuda for helpful discussions.

**A Gamma matrices**

We summarize the definition of gamma matrices and their useful properties. The \( 16 \times 16 \) gamma matrices \( \Gamma^M \) are defined as follows.

\[
\Gamma^M = \begin{pmatrix} 0 & E_{M+1}^T \\ E_{M+1} & 0 \end{pmatrix}, \quad M = 1, 2, 3, 5, 6, 7,
\]

(A.1)

\[
\Gamma^4 = \begin{pmatrix} 0 & E_1^T \\ E_1 & 0 \end{pmatrix}, \quad \Gamma^8 = \begin{pmatrix} 0 & E_5^T \\ E_5 & 0 \end{pmatrix},
\]

(A.2)
\[
\Gamma^0 = \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & 1_{8 \times 8} \end{pmatrix}, \quad \Gamma^0 = \begin{pmatrix} i1_{8 \times 8} & 0 \\ 0 & i1_{8 \times 8} \end{pmatrix}.
\] (A.3)

Here \( E_i \) for \( i = 1, \cdots, 8 \) are defined by
\[
E_i = \begin{pmatrix} J^\mu \\ 0 \\ J^\mu \end{pmatrix}, \quad \mu = 1, 2, 3, 4, \quad E_A = \begin{pmatrix} 0 & -J_A^T \\ J_A & 0 \end{pmatrix}, \quad A = 5, 6, 7, 8.
\] (A.4)

with
\[
(J_1, J_2, J_3, J_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\tilde{J}_1, \tilde{J}_2, \tilde{J}_3, \tilde{J}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (J_5, J_6, J_7, J_8) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\] (A.5)

Another gamma matrix \( \tilde{\Gamma}^M \) is defined by
\[
\tilde{\Gamma}^M = \begin{cases} -\Gamma^0 & \text{for } M = 0, \\
\Gamma^M & \text{for } M = 1, 2, \cdots, 9.
\end{cases}
\] (A.8)

Note that \( \Gamma^M \) and \( \tilde{\Gamma}^M \) satisfy the relation
\[
\tilde{\Gamma}^M \Gamma^N + \tilde{\Gamma}^N \Gamma^M = 2\delta^{MN}, \quad \Gamma^M \tilde{\Gamma}^N + \Gamma^N \tilde{\Gamma}^M = 2\delta^{MN},
\] (A.9)
\[
(\Gamma^M)^T = \Gamma^M, \quad (\tilde{\Gamma}^M)^T = \tilde{\Gamma}^M.
\] (A.10)

\( \Gamma^{MN}, \tilde{\Gamma}^{MN} \) and \( \Gamma^{KLMN} \) are defined by
\[
\Gamma^{MN} = \tilde{\Gamma}^{[M} \Gamma^{N]}, \quad \tilde{\Gamma}^{MN} = \Gamma^{[M} \tilde{\Gamma}^{N]},
\] (A.11)
\[
\Gamma^{KLMN} = \tilde{\Gamma}^{[K} \Gamma^{L} \Gamma^{M} \Gamma^{N]} = \frac{1}{4!} \sum_{\sigma \in \mathfrak{S}_4} \text{sgn}(\sigma) \tilde{\Gamma}^{\sigma(K)} \Gamma^{\sigma(L)} \Gamma^{\sigma(M)} \Gamma^{\sigma(N)}
\] (A.12)

where \( \mathfrak{S}_4 \) is the permutation group of four elements and \( \text{sgn}(\sigma) \) is the signature of \( \sigma \in \mathfrak{S}_4 \). Here \( [KL] \) and \( [KLMN] \) denote the anti-symmetrization of products of gamma matrices under the exchanges of any two indices.

### B One-loop determinants in 5d \( \mathcal{N} = 1^* \) gauge theory

We summarize the explicit forms of the one-loop determinants of the surface operators studied in Section 6. From the localization computation of the one-loop determinant (3.3),
In order to compute the differential operator $D_{10}$, we explicitly write (3.39).
From (C.1), the action of $D_{10}$ on the fluctuation fields $X_0$ are read off as

$$(D_{10} \cdot X_0)_i = -2i \sum_{j,k=1}^{3} (\nu_i \tilde{\Gamma}^{jk} \varepsilon) \tilde{D}_j X_{0,k} + 2i \tilde{D}_i X_{0,9} - 2i \tilde{D}_9 X_{0,i} \quad i \in \{1, 2, 3\}. \quad (C.2)$$

$$(D_{10} \cdot X_0)_i = 2i \sum_{j=1}^{3} \sum_{k=5}^{8} (\nu_i \tilde{\Gamma}^{jk} \varepsilon) D_k X_{0,k} + 2i \sum_{k=5}^{9} (\nu_i \tilde{\Gamma}^{9k} \varepsilon) D_9 X_{0,k} \quad i \in \{4, 5, 6, 7\}, \quad (C.3)$$

$$(D_{10} \cdot X_0)_8 = \tilde{D}_M (2 \tilde{D}_M X_{0,4} - \tilde{D}_4 X_{0,M} - i \tilde{D}_0 X_{0,M}), \quad (C.4)$$

$$(D_{10} \cdot X_0)_9 = i \sum_{i=1,2,3,9} \tilde{D}_i X_{0,i}. \quad (C.5)$$

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