Rank of ordinary webs in codimension one

An effective method

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Abstract:

We are interested by holomorphic $d$-webs $W$ of codimension one in a complex $n$-dimensional manifold $M$. If they are ordinary, i.e. if they satisfy to some condition of genericity (whose precise definition is recalled below), we proved in [CL] that their rank $\rho(W)$ is upper-bounded by a certain number $\pi'(n, d)$ (which, for $n \geq 3$, is strictly smaller than the Castelnuovo-Chern’s bound $\pi(n, d)$).

In fact, denoting by $c(n, h)$ the dimension of the space of homogeneous polynomials of degree $h$ with $n$ unknowns, and by $h_0$ the integer such that

$$c(n, h_0 - 1) < d \leq c(n, h_0),$$

$\pi'(n, d)$ is just the first number of a decreasing sequence of positive integers

$$\pi'(n, d) = \rho_{h_0 - 2} \geq \rho_{h_0 - 1} \geq \cdots \geq \rho_h \geq \rho_{h+1} \geq \cdots \geq \rho_\infty = \rho(W) \geq 0$$

becoming stationary equal to $\rho(W)$ after a finite number of steps. This sequence is an interesting invariant of the web, refining the data of the only rank.

The method is effective: theoretically, we can compute $\rho_h$ for any given $h$; and, as soon as two consecutive such numbers are equal ($\rho_h = \rho_{h+1}$, $h \geq h_0 - 2$), we can construct a holomorphic vector bundle $R_h \to M$ of rank $\rho_h$, equipped with a tautological holomorphic connection $\nabla^h$ whose curvature $K^h$ vanishes iff the above sequence is stationary from there. Thus, we may stop the process at the first step where the curvature vanishes.

Examples will be given.

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1 Introduction

Recall that a totally decomposable\(^1\) holomorphic \(d\)-web of codimension one without singularity on a complex \(n\)-dimensional manifold \(M\) is defined by the data of \(d\) holomorphic regular foliations \(F_i\) of codimension one on \(M\), \((1 \leq i \leq d)\), any one of them being transverse to each other at any point.

We assume \(d > n\) and the web to be at least in weak general position.\(^2\)

An abelian relation on an open set \(U\) (assumed to be connected and simply connected) of \(M\) is then the data of a family \((F_i)_i\) of holomorphic functions on \(U\), \(1 \leq i \leq d\), such that

- for any \(i\), \(F_i\) is a first integral of \(F_i\) (maybe with singularities),
- the sum \(\sum_{i=1}^{d} F_i\) is a constant on \(U\).

These first integrals being defined up to an additive constant, we are only interested by their differential

\[ m = \text{rank at a point.} \]

The germs of abelian relations at a point \(m\) constitute a vector space, whose dimension is called the rank of the web at this point.\(^3\)

It will be useful to give an equivalent definition in words of differential operator. Denote by \(T F_i \subset TM\) the vector bundle of vectors tangent to \(F_i\), and \(A_i \subset T^*M\) the dual vector bundle of \(TM/T F_i\) (i.e. the vector bundle of holomorphic 1-forms vanishing on \(T F_i\)). Let

\[ Tr : \oplus_{i=1}^{d} A_i \to T^*M \]

be the morphism of vector bundles (the Trace), defined by \(Tr((\omega_i)_i) = \sum_{i=1}^{d} \omega_i\). The assumption of “at least weak general position” means that \(Tr\) has maximal rank \(n\) : its kernel

\[ A := \text{Ker} Tr \]

is therefore a holomorphic vector bundle of rank \(d - n\). We define a linear differential operator of order one

\[ D : J^1 A \to B, \]

where \(B = (\wedge^2 T^*M)^{\oplus d}\), by mapping any section \(s = (\omega_i)_i\) of \(A\) onto the family \((d\omega_i)_i\) of the differentials. Then, an abelian relation may be identified with a holomorphic section \(s\) of \(A\) such that \(D(j^1 s) = 0\).

The kernel \(R_1 = \text{Ker}(D : J^1 A \to B)\) is the vector bundle of formal abelian relations at order one. More generally, the space \(R_h\) of formal abelian relations at order \(h\) is the kernel of the \((h - 1)^{th}\)-prolongation \(D_h\) of the differential operator \(D\) (= \(D_1\)):

\[ R_h = \text{Ker}(D_h : J^h A \to J^{h-1}B). \]

For any \(h\) \((h \geq 1)\), abelian relations may still be identified with holomorphic sections \(s\) of \(A\) such that \(j^h s\) belong to \(R_h\).

Denoting by \(\pi_h : R_h \to R_{h-1}\) the natural projection, we shall see that the elements of \(R_h\) which are mapped by \(\pi_h\) onto a given element \(a_{h-1}\) of \(R_{h-1}\) are the solutions of a linear system \(\Sigma_h(a_{h-1})\)

\(^1\) More generally, the web is defined by one foliation \(\mathcal{F}\) on a covering space \(\tilde{M} \to M\) with \(d\) sheets. On an open set \(U\) of \(M\) on which this covering space is trivial, the data of \(\mathcal{F}\) is equivalent to that of the \(d\)'s \(F_i\) which are the projections of \(\mathcal{F}\). All fiber bundles and connections studied below may be defined globally. But all computations being done locally, we shall recall the definitions only in the case of a totally decomposable web.

\(^2\) The web is said to be in weak (resp. strong) general position if, at any point \(m\), there exists at least \(n\) of the foliations among the \(d\)'s, whose tangent spaces at \(m\) are in general position (resp. if any family of \(n\) foliations among the \(d\)'s have this property).

\(^3\) A. Hénaut proved that this rank doesn’t depend on \(m\), as far as the web satisfies to the assumption of strong general position ([H2]). In case we have only weak general position, we shall define the rank of the web as being the highest of the rank at a point.
of \( c(n, h + 1) \) equations with \( d \) unknowns, whose homogeneous part doesn’t depend on \( a_{h-1} \), with notation:

\[
c(n, h) := \frac{(n - 1 + h)!}{(n - 1)! \ h!}.
\]

Then ordinary webs are those for which all of these systems have maximal rank \( \inf (d, c(n, h + 1)) \). Denoting by \( h_0 \) the integer such that

\[
c(n, h_0 - 1) < d \leq c(n, h_0),
\]

it is in fact sufficient that this rank be maximal for \( h \leq h_0 \), for being maximal for any \( h \).

We proved in [CL] that the rank of an ordinary web is at most equal to the integer

\[
\pi'(n, d) := \sum_{h=1}^{h_0-1} (d - c(n, h)), \quad (=(h_0-1)d - c(n+1, h_0-1) + 1).
\]

which, for \( n \geq 3 \), is strictly smaller than the Castelnuovo’s number \( \pi(n, d) \).

Since the linear system \( \Sigma_h(a_{h-1}) \) of \( c(n, h + 1) \) equations with \( d \) unknowns has rank \( c(n, h + 1) \) for \( h \leq h_0 - 2 \), the projection \( \pi_h : R_h \rightarrow R_{h-1} \) is then surjective, and \( R_h \rightarrow M \) is a holomorphic vector bundle of rank \( \sum_{h=1}^{k+1} (d - c(n, h)) \) for \( k \leq h_0 - 2 \). In particular

\[
R_{h_0-2} \rightarrow M \text{ is a holomorphic vector bundle of rank } \pi'(n, d).
\]

For \( h \geq h_0 - 1 \), \( \Sigma_h(a_{h-1}) \) has rank \( d \), and has at most one solution (since it contains a cramerian sub-system), but may be no one (since it is overdetermined). In general, \( R_h \) will still be a vector bundle (denoting by \( \rho_h \) its rank), but it may happen that the projection \( \pi_{h+1} : R_{h+1} \rightarrow R_h \) be no more surjective, hence:

\[
\rho_h \geq \rho_{h+1}.
\]

When \( \rho_h = \rho_{h+1} \), \( (h \geq h_0 - 1) \), the projection \( \pi_{h+1} : R_{h+1} \rightarrow R_h \) is now an isomorphism of vector bundles. The inverse isomorphism \( R_h \xrightarrow{\pi_{h+1}} R_{h+1} \) composed with the natural inclusion \( R_{h+1} \subset J^1R_h \) defines a connection \( \nabla^h \) on \( R_h \), and abelian relations may be identified with sections \( s \in \mathcal{A} = R_0 \) such that \( j^h(s) \) be a section of \( R_h \) with vanishing covariant derivative: \( \nabla^h(j^h(s)) \equiv 0 \). Hence the rank \( \rho(W) \) of the web will be at most equal to the rank \( \rho_h \) of \( R_h \), and equal if the curvature \( K_h \) of \( \nabla^h \) vanishes. This proves in particular the inequalities

\[
\sup \left( 0, (h+1)d - c(n+1, h+1) + 1 \right) \leq \rho_h \leq (h_0-1)d - c(n+1, h_0-1) + 1 = \pi'(n, d)
\]

for \( h \geq h_0 - 2 \). We shall see how to compute effectively \( \rho_h \) and \( K_h \).

When \( d = c(n, h_0) \) (then, \( d \) is said to be calibrated), \( \rho_{h_0-2} = \rho_{h_0-1} \), and \( \nabla^{h_0-2} \) is the connection defined in [CL], computed with Maple in [DL], generalizing to any \( n \) the connection previously defined for planar webs \( (n=2) \) by Hénaut ([H1]), and independently by Pirio ([Pi]) who related the corresponding curvature to invariants defined previously by Pantazi ([Pa]) (the curvature of which being the Blaschke-Dubourdieu curvature ([BB]) for \( n = 2, d = 3 \)). In this case of planar webs, Ripoll ([R]) computed the rank of the web by another method (corank of some matrix deduced from \( \nabla^{h_0-2} \), from the curvature \( K^{h_0-2} \) and its derivatives).

**Remark about non-ordinary webs:** If the web is not ordinary, its rank may be bigger than \( \pi'(n, d) \), and we may not affirm anymore that the sequence of the \( \rho_h \)'s increases for \( h \leq h_0 - 2 \) and decreases for \( h \) bigger. However, if -by chance- we can find some \( h \) such that \( \pi_{h+1} : R_{h+1} \rightarrow R_h \) is an isomorphism of vector bundles of same rank \( \rho_h = \rho_{h+1} \), then we still can define the connection \( \nabla^h \), and it is still true that the vanishing of the curvature \( K^h \) implies the equality \( \rho(W) = \rho_h \).

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4We prefer this notation to the usual one \( \binom{n-1+h}{h} \) for the binomial coefficient, mainly because it suggests explicitly the dimension of the vector space of homogeneous polynomials of degree \( h \) with \( n \) unknowns, and also because it needs less space.

5The Castelnuovo’s number

\[
\pi(n, d) := \sum_{h \geq 1} (d - h(n-1) - 1) + , \text{ where } a^+ \text{ denotes the number } \sup (a, 0)
\]

is the maximal arithmetical genus of irreducible algebraic curves of degree \( d \) in the complex \( n \)-dimensional projective space \( \mathbb{P}_n \). It is also, after Chern ([C]), the maximal rank of the \( d \)-webs in codimension one, verifying only the assumption of strong general position (but not necessarily ordinary for \( n \geq 3 \)).
Notation: Denote by

\[ i \text{ an index from 1 to } d, \]
\[ \lambda, \mu, \ldots \text{ an index from 1 to } n, \]
\[ L = (\ell_1, \ell_2, \cdots, \ell_n) \text{ a multi-index } \ell_\lambda \geq 0 \text{ of } n \text{ integers, and } |L| := \sum \ell_\lambda \text{ its degree.} \]

If \( L = (\ell_1, \ell_2, \cdots, \ell_n) \), and \( L' = (\ell'_1, \ell'_2, \cdots, \ell'_n) \), \( L + L' \) (resp. \( L - L' \)) denotes \( (\ell_1 + \ell'_1, \cdots, \ell_n + \ell'_n) \) (resp \( (\ell_1 - \ell'_1, \cdots, \ell_n - \ell'_n) \)).

In particular \( 1_\lambda \) denotes the multi-index obtained with 1 at the place \( \lambda \) and 0 elsewhere.

Relatively to local coordinates \( x = (x_1, \cdots, x_n) \) in \( M \), we shall denote by \( \partial_{x_i} a \) or \( a'_i \) the partial derivative \( \frac{\partial a}{\partial x_i} \) of a holomorphic function \( a \) or of a matrix with holomorphic coefficients.

More generally, \( \partial_{L} a \) or \( a'_L \) denotes the partial derivative \( \frac{\partial^{\ell_1} \cdots \partial^{\ell_n} a}{\partial x_1^{\ell_1} \cdots \partial x_n^{\ell_n}} \) of order \( |L| \).

2 Computation of \( R_h \)

We assume each foliation \( F_i \) defined by a first integral \( u_i \) without singularity. The data of another first integral \( F_i = G_i(u_i) \) up to an additive constant is equivalent to the data of the derivative \( g_i = (G_i)' \). Each vector bundle \( A_i \) being now trivialized by \( du_i \), we set \( \omega_i = g_i(u_i) \ du_i \) (such a 1-form is automatically closed). The data of an abelian relation is now equivalent to the data of a family \((g_i)\) of holomorphic functions of one variable \(1 \leq i \leq d\) such that \( \sum_i g_i(u_i) \ du_i \equiv 0 \), or equivalently:

\[ (E_\lambda) \quad \sum_i (u_i)'_\lambda g_i(u_i) \equiv 0 \quad \text{for any } \lambda, \]

which can still be written

\[ < P_1, f > = 0, \]

where \( P_1 := \frac{\partial (u_1, \cdots, u_d)}{\partial (x_1, \cdots, x_n)} \) denotes the jacobian matrix and \( f \) the \( d \)-vector \( (g_1 \circ u_1, \cdots, g_d \circ u_d). \) [The functions \( u_i \) are given, and the functions \( g_i \) unknown].

Coefficients \( C^h_{i,L}(u) \) and matrices \( M^{(h)}_j \)

For any \( h \geq 0 \), \( g_i^{(h)} \) will denote the \( h \)-th derivative of \( g_i \) (with the convention \( g_i^{(0)} := g_i \));

We set:

\[ f_i := g_i \circ u_i \text{ and } f_i^{(h)} := g_i^{(h)} \circ u_i, \]
\[ f := d \text{-vector } (f_1, f_2, \cdots, f_d), \text{ and } f^{(h)} := d \text{-vector } (f_1^{(h)}, f_2^{(h)}, \cdots, f_d^{(h)}). \]

For any integer \( k \) \( k \geq 0 \), a \( k \)-jet of abelian relation at a point \( m \) of \( M \) is defined by the family

\[ \left( f_i^{(h)}(m) = (g_i^{(h)} \circ u_i)(m) \right)_{i,h}, \quad (0 \leq h \leq k, \ 1 \leq i \leq d). \]

The partial derivatives of the relations \((E_\lambda)\) will make us able to compute locally \( R_h \). In fact, the functions \( C^h_{i,L} \) will be defined by iteration on \(|L|\) so that

\[ \left( f_i^{(h)}(u_i)^{L}_{\lambda} \right)^{'} = \sum_{h=0}^{|L|} C^h_{i,L+1 \lambda} \cdot f_i^{(h)} \]

as far as \( (f_i \ du_i)_i \) be an abelian relation.

Lemma 2-1: For any holomorphic function \( u \) of \( n \) variables, and any holomorphic function \( g \) of one variable,

(i) The derivatives \( (g \circ u)^{' \lambda}_{L} \) are linear combinations

\[ (g \circ u)^{' \lambda}_{L} = \sum_{h=0}^{|L|} C^h_{L+1 \lambda}(u) \cdot (g^{(h)} \circ u) \]
of the successive derivatives $g^{(h)}$ of $g$ (we set $g^{(0)} = g$), whose coefficients $C^h_{L}(u) = C^h_{L+1\lambda}(u)$ depend only on $u$ and on the multi-index $L' = L + 1\lambda$, and not on its decomposition under the shape $L + 1\lambda$.

(ii) They can be computed by iteration on $|L|$ using the formula

$$
C^0_{1\lambda}(u) = u_\lambda',
$$
$$
C^0_{L+1\mu}(u) = \partial_\mu C^0_{L}(u),
$$
$$
C^h_{L+1\mu}(u) = \partial_\mu C^h_{L}(u) + C^{h-1}_{L}(u) \cdot u'_\mu \text{ for } 1 \leq h \leq |L| - 1,
$$
$$
C_{L+1\mu}(u) = C^{[L]-1}_{L}(u) \cdot u'_\mu.
$$

This is due to the fact that the 1-form $d(G(u))$ is closed, $G$ denoting a primitive of $g$.

For a web locally defined by the functions $u_i$, we set :

$$
C^h_{i,L} = C^h_{L}(u_i).
$$

We check in particular

$$
C^0_{i,L} = (u_i)'_L,
$$
$$
\text{and } C^{[L]-1}_{i,L} = \prod_{\lambda=1}^n ((u_i)'_\lambda)_{\lambda} \text{ for } L = (\ell_1, \ell_2, \cdots, \ell_n).
$$

We set :

$\Theta^r$ denotes the trivial holomorphic bundle of rank $r$,

$\beta_k := c(n+1,k) - 1 \ (= \sum_{h=1}^k c(n,h))$,

$M^{(h)}_j$ denotes the matrix $([C^h_{i,L}]_{(i,|L|—j)}$ of size $c(n,j) \times d$, $(1 \leq j, 0 \leq h)$,

(with $M^{(h)}_j = 0$ for $h \geq j$),

$P_j := M^{(j-1)}_j$.

$\mathcal{M}_k$ denotes the matrix of size $\beta_k \times kd$ built with the blocks $M^{(h)}_j$ for $1 \leq j \leq k$ and $0 \leq h \leq k - 1$, where the block $M^{(h+1)}_j$ is on the right of $M^{(h)}_j$, and $M^{(h)}_{j+1}$ below,

$Q_{k+1}$ denotes the sub matrix of size $c(n,k+1) \times kd$ in $\mathcal{M}_{k+1}$ built with the blocks $M^{(h)}_{k+j}$ for $0 \leq h \leq k - 1$ :

$$
\mathcal{M}_k = 
\begin{pmatrix}
M^{(0)}_1 = P_1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
M^{(1)}_2 = P_2 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
M^{(k-1)}_{k-1} & M^{(k-2)}_{k-1} & \cdots & 0 & 0 & 0 & 0 \\
M^{(0)}_k & M^{(1)}_k & M^{(2)}_k & M^{(3)}_k & \cdots & M^{(k-2)}_{k-1} & 0 & M^{(k-1)}_{k-1} = P_k
\end{pmatrix}
$$

$$
Q_{k+1} = 
\begin{pmatrix}
M^{(0)}_{k+1} & M^{(1)}_{k+1} & M^{(2)}_{k+1} & M^{(3)}_{k+1} & \cdots & M^{(k-2)}_{k+1} & M^{(k-1)}_{k+1}
\end{pmatrix}
$$

Theorem 2-2 :

(i) Locally, $R_k$ is the kernel of $\mathcal{M}_{k+1}$ (included into the trivial bundle $\Theta^{(k+1)d}$). Hence, when the matrix $\mathcal{M}_{k+1}$ has constant rank (always true for $k \leq h_0 - 2$), $R_k \rightarrow M$ is a holomorphic vector bundle of rank

$$
\rho_k = (k+1)d - \text{rank}(\mathcal{M}_{k+1}).
$$

(ii) If an element $a_{k-1} \in R_{k-1}$ (imbedded into $\Theta^{kd}$) is defined by the family $f$ of $d$-vectors $(f^{(h)})_h$, $(0 \leq h \leq k - 1)$, the elements of $R_k$ which project onto $a_{k-1}$ are those whose the last component $f^{(k)}$
is solution of the linear system $\Sigma_h(a_{h-1})$:

$$< P_{k+1} , f^{(k)} > = - < Q_{k+1} , f >$$

$$= - \sum_{h=0}^{k-1} < M^{(h)}_{k+1} , f^{(h)} > .$$

**Proof**: A $k$-jet of abelian relation at a point $m \in M$ is then represented by its components $j^h_m(u_i \circ g_i)$ in $J^hA_i$, and each of them is completely defined by the family of the numbers

$$\left( j^h_i = (g^h_i \circ u_i)(m) \right)_{0 \leq h \leq k} .$$

Thus a family of numbers $(w^{(h)}_i)_{i,h}$ belongs to $R_k$ if it satisfies to any of the equations

$$(E_L) \quad \sum_{i=1}^{d} \sum_{h=0}^{\lfloor L \rfloor - 1} C^h_{iL} \cdot w^{(h)}_i = 0 .$$

QED

**Estimation of the ranks**

The assumption for the web to be ordinary means that the matrices $P_j := M^{(j-1)}_j$ have all maximal rank, that is $c(n,j)$ for $j \leq h_0 - 1$, and $d$ for $j \geq h_0$.

**Lemma 2-3**

If $P_j$ has maximal rank $c(n,j)$ for $1 \leq j \leq h_0$, it has maximal rank $d$ for any $j \geq h_0$.

**Proof**: The meaning of the lemma not depending on the local coordinates, we may assume that all foliations are transversal to the $x_n$-axis near a point; therefore all derivatives $(u_i)'_n$ are not zero. The formula

$$(P_{j+1})_{i,L+1} = (u_i)'_n \cdot (P_j)_{i,L}$$

proves that the rank of $P_{j+1}$ is at least equal to that of $P_j$, thus is equal if $j$ is big enough for this rank to be stationary equal to $d$.

QED

Let $\rho_k$ be the rank of $R_k$.

**Theorem 2-4** ([CL])

For $k \leq h_0 - 2$, $R_k \to M$ is a holomorphic vector bundle of rank

$$\rho_k = (k + 1)d - \beta_{k+1} .$$

In particular, $\rho_{h_0 - 2} = \pi'(n,d)$.

**Proof**: In fact, the matrices $M_h$, of size $\beta_h \times hd$, are triangular by blocks, and the diagonal blocks are the $P_j$’s. Since the rank of $P_j$ is $c(n,j)$ for $j \leq h_0 - 1$, $M_{k+1}$ has maximal rank $\beta_{k+1} = \sum_{h=1}^{k+1} c(n,h)$ in this range. Thus $R_k$ (= Ker $M_{k+1}$) has there rank $(k + 1)d - \beta_{k+1}$.

QED

The sequence $(hd - \beta_k)_h$ becomes decreasing for $h \geq h_0$. Then, for $h \geq h_0$, it may be no more true that $M_{h+1}$ has maximal rank $\beta_{h+1}$, so that the rank $\rho_h$ of $R_h$ may be now bigger than $(h+1)d - \beta_{h+1}$ (but remains at most equal to $\pi'(n,d) = (h_0 - 1) d - \beta_{h_0 - 1}$). Thus we get :

**Theorem 2-5**

Assuming $\pi_{h+1}$ to have a constant rank for $h \geq h_0 - 2$, the sequence $(\rho_h)_{h \geq h_0 - 2}$ is decreasing from $\pi'(n,d)$ to the rank $\rho(W)$ of the web, and satisfies to the inequalities

$$(h+2)d - \beta_{h+2} \leq \rho_{h+1} \leq \rho_h \leq \pi'(n,d).$$

6Be careful not to confuse $R_h$ with the set $J^h(\mathcal{A}b)$ of the $h$-jets of the true abelian relations (which may be smaller).
3 The connections $\nabla^h$

In this section, we assume:

$$ h \geq h_0 - 2, $$

and $\rho_h = \rho_{h+1}, \quad (\pi_{h+1} : R_{h+1} \xrightarrow{\cong} R_h \text{ being then an isomorphism of vector bundles}).$

If $\rho_h = 0$, then $\rho(W) = 0$. If $\rho_h > 0$, we shall define a connection $\nabla^h$ on $R_h$, whose curvature vanishes iff $\rho_h = \rho(W)$.

We recall that $R_{h+1}$ is the intersection of $J^1(R_h)$ and $J^{h+1}R_0$ into $J^1(J^hR_0)$:

$$ R_{h+1} = J^1R_h \cap J^{h+1}R_0. $$

Denote by

$$ \epsilon_h : R_{h+1} \hookrightarrow J^1R_h \text{ the natural inclusion}, $$

and by $v_h : R_h \rightarrow R_{h+1}$ the isomorphism inverse of the projection $\pi_{h+1}$.

The composed map $\xi_h := \epsilon_h \circ v_h$ is a splitting of the exact sequence

$$ 0 \rightarrow T^*(V) \otimes R_h \rightarrow J^1R_h \xrightarrow{\xi_h} R_h \rightarrow 0 $$

and defines consequently a holomorphic connection on $R_h$, whose covariant derivative is:

$$ \nabla^h\sigma = j^1\sigma - \langle \xi_h, \sigma \rangle. $$

Since the abelian relations may be identified by the map $s \rightarrow j^{h+1}s$ to the sections $s$ of $R_0 (= A)$ such that $j^{h+1}s$ belong to $R_{h+1}$, and since $\xi_h$ factorizes through $R_{h+1}$, the following assertions are equivalent:

(i) $s$ is an abelian relation,

(ii) $\nabla^h(j^h s) \equiv 0$.

Since the framework is holomorphic, $\rho_\infty = \rho(W)$, and we get therefore the

Theorem 3-1

(i) A section $s$ of $A (= R_0)$ is an abelian relation iff $j^h s$ is a section of $R_h$ and $\nabla^h(j^h s) \equiv 0$.

(ii) The rank $\rho(W)$ of the web is at most equal to the rank $\rho_h$ of the bundle $R_h$.

(iii) There exists an integer $h_1$ such that

- or $\rho_{h_1} = 0$ and then $\rho(W) = 0$,

- or $\rho_{h_1} = \rho_{h_1+1} (\neq 0)$, the curvature $K^{h_1}$ vanishes, and then $\rho(W) = \rho_{h_1}$.

Remark: If the web is not-ordinary, we still may define the connection $\nabla^h$, as soon as we can find some $h$ for which the projection $R_{h+1} \rightarrow R_h$ is an isomorphism of vector bundles, whatever be $h$ (no more necessarily $\geq h_0 - 2$). And it remains true that the vanishing of its curvature $K^h$ implies $\rho(W) = \rho_h$.

4 Algorithm

Theoretically, the following algorithm always works for any ordinary web. But it may need a long time of computer. Practically, in some cases, considerations specific to each example may be used for shortening the process, some of them being sketched in the remark at the end of the section.

- explicit $P_1, \cdots, P_{h_0}$;

\footnote{It can be explicitly computed by mean of the generalized inverse $IP := (P^*P)^{-1}P^*$ of $P$, where $P := P_{h+2}$ and $P^*$ means the transposed matrix of $P$.}
- check :
  \[ \text{Rank } (P_j) = c(n, j) \text{ for any } j \ (1 \leq j \leq h_0 - 1), \text{ and Rank } (P_{h_0}) = d; \]

if this condition is not realized, \( W \) is not ordinary ;
- STOP ;
- else, compute \( M_{h_0} \);

**Loop \( L(h) \) from \( h = h_0 - 2 \) :**
- compute \( M_{h+2} \) (and \( M_{h+1} \) sub-matrix of \( M_{h+2} \)) ;
- compute \( \rho_h = (h + 1) \) \( d - \text{Rank } (M_{h+1}) \) and \( \rho_{h+1} = (h + 2) \) \( d - \text{Rank } (M_{h+2}) \) ;
  - if \( \rho_h > \rho_{h+1} \), go to \( L(h+1) \) ;
  - else (i.e. when \( \rho_h = \rho_{h+1} \)), compute \( \nabla^h \) and \( K^h \) ;
  - if \( K^h \neq 0 \), go to \( L(h+1) \) ;
  - else (i.e. when \( K^h = 0 \)),
    \[ \rho(W) = \rho_h. \]
  - STOP .

Thus, we have an effective procedure to compute the rank, even when it is not maximal, without having to exhibit explicit abelian relations.

**Remarks :**

1- When \( \rho_h = \rho_{h+1} \), it is often useful to check immediately if there would not be some \( k, k > h \), such that \( \rho_k > \rho_{k+1} \). In this case, we know a priori that \( K^h \) doesn’t vanish, without to have to compute it.

2- There are usually two ways for computing \( \rho_h \) : the first one, used in the algorithm above, consists in computing the kernel of the matrix \( M_{h+1} \) of size \( (c(n + 1, h + 1) - 1) \times (h + 1) d \) :

\[ \rho_h = (h + 1) \) \( d - \text{Rank}(M_{h+1}). \]

This size increases more rapidly with \( h \) than the size \( c(n, h + 1) \times d \) of the matrix \( P_{h+1} \) of the linear system \( \Sigma_h(a) \) giving the elements of \( R_h \) above a given element \( a \in R_{h-1} \) (essentially because the process uses the knowledge of \( R_{h-1} \) that we got previously, which is not true for the first process). Thus, for \( h \) big enough, knowing already \( \rho_{h-1} \) and a trivialization \( (\epsilon_s)_s \) of \( R_{h-1} \), the following process may need a shorter time of computer than the previous one, despite of the fact that there are more operations to be done :

- choose a \( d \times d \) invertible sub-matrix \( P^0_{h+1} \) of \( P_{h+1} \),
- solve the corresponding cramerian sub-system of \( \Sigma_h(a) \),
- for each line \( \ell \) among the \( c(n, h + 1) - d \) deleted for getting \( P^0_{h+1} \) from \( P_{h+1} \), and for each \( \epsilon_s \) belonging to the trivialization of \( R_{h-1} \), build the characteristic determinant \( \Delta(s, \ell) \) whose vanishing asserts the compatibility of the new equation \( \ell \) with the cramerian sub-system,
- then the kernel of the matrix \( \Delta^h := (\Delta(s, \ell)) \) of size \( (c(n, h + 1) - d) \times \rho_{h-1} \) defines the projection of \( R_h \) onto \( R_{h-1} \), which is an isomorphism, and

\[ \rho_h = \rho_{h-1} - \text{Rank}(\Delta^h). \]

5 **Examples**

The process described in the algorithm above works for any \((n, d, h)\). However, most of our examples are relative to low values of these integers : in fact, the size of the involved matrices becomes very rapidly huge, and would need in practice more powerful computers than our small portable.
5.1 Case $n = 2$, $d = 3$ :

There is no hope to refine the classification of the non-hexagonal planar 3-webs by the order of the step from which the sequence of the $\rho_i$’s vanishes. In fact, we can prove easily that the sequence of the $\rho_i$’s becomes immediately stationary after the first step, and there are only two possibilities:

- sequence $(1, 1, \ldots, 1 = \rho_\infty)$ if the Blaschke-Dubourdieu curvature $K^0$ vanishes (hexagonal case),
- sequence $(1, 0, \ldots, 0 = \rho_\infty)$ if $K^0 \neq 0$.

5.2 Example $n = 2$, $d = 4$ ($\pi'(2, 4) = 3$):

We recall that all planar webs are ordinary, and calibrated with $h_0 = d - 1$. Moreover $\pi'(2, d)$ is then equal to $\pi(2, d) = \frac{(d-1)(d-2)}{2}$.

For the planar 4-web $(x, y, x + y, x - y, xy, x^2 + y^2, x^2 - y^2, x^4 + y^4)$, we have an obvious abelian relation $f \circ u_1 - u_2 - u_4 \equiv 0$, with $f(x) := x + x^5$. Thus, we know already $1 \leq \rho(W) \leq 3$.

Computing $\rho_k$, we get

$$\rho_1 = \rho_2 = 3 > \rho_3 = \rho_4 = 2.$$  

Since $\rho_3 < \rho_2$, we are sure that the curvature $K^1$ doesn’t vanish, without to have to compute it. We get $K^3 = 0$. Thus the sequence of the $\rho_i$’s is necessarily stationary equal to 2 from $\rho_3$:

$$\rho(W) = 2.$$  

We are sure that there is another abelian relation independant on the obvious one, without to have to exhibit it.

5.3 Example $n = 2$, $d = 8$ ($\pi'(2, 8) = 21$):

Let $W$ be the planar 8-web

$$x, y, x + y, x - y, xy, x^2 + y^2, x^2 - y^2, x^4 + y^4.$$  

We observed in [DL] that its curvature $K^5$ did not vanish, but that its connection form $\omega^5$ relative to some “adapted” trivialization (matrix of size $21 \times 21$, whose coefficients are scalar 1-forms) had only zero’s in the 19th first columns. Thus, we deduced that the rank of $W$ was at least 19, and at most 20: in fact, $\rho_5 = 21$, $\rho_6 = 20$ and $\rho_7 = 19$. Thus

$$\rho(W) = 19,$$  

while the 7-sub-web defined by deleting $x^4 + y^4$ has maximal rank 15 (see [Pi]).

5.4 Case $d = n + 1$, $n > 2$ ($\pi'(n, n + 1) = 1$):

Denoting by $(x_1, \ldots, x_n)$ local coordinates, we consider the $(n + 1)$-web $W$ defined by the functions $(x_1, \ldots, x_n, F(x_1, \ldots, x_n))$.

For a convenient order of the multi-indices $L$, the matrix $M_2$ has the shape

$$
\begin{pmatrix}
I_n & F^{(1)} \\
0 & F^{(2)} \\
0 & G^{(2)}
\end{pmatrix},
$$

where

$I_n$ is the identity $n \times n$-matrix,
$F^{(1)}$ is the column of the $(F^{(1)}_i)_{1 \leq i \leq n}$, (with $n$ rows),
$F^{(2)}$ is the column of the $(F^{(2)}_i)_{1 \leq i \leq n}$, (with $n$ rows),
\( F^2 \) is the column of the \(((F_i')^2)_{1 \leq i \leq n}\), (with \( n \) rows),
\( G^{(2)} \) is the column which has coefficients \( F''_{ij} \) \((i \neq j)\), (with \( c(n, 2) - n \) rows),
and \( G^2 \) is the column which has coefficients \( F'_i F'_j \) \((i \neq j)\), (with \( c(n, 2) - n \) rows, same order of the pairs \((i, j)\) as in \( G^{(2)} \)).

The sub-matrix \( M_1 \) has always rank \( n \), hence \( \rho_0 = 1 \), while \( M_2 \) has generally rank \( 2n + 2 \); hence, in general \( \rho_1 \) \((= \rho(W))\) = 0, and there is no abelian relation.

The exceptional case (Rank\((M_2) = 2n + 1\), and \( \rho_1 = 1 \)) happens iff \( G^{(2)} \) and \( G^2 \) are collinear. This means the set of relations
\[
\frac{F''_{ij}}{F'_i F'_j} = \frac{F''_{rs}}{F'_r F'_s}
\]
for any \( i, j, r \) and \( s \) with \( j \neq i \) and \( s \neq r \).

We shall now study this case by mean of the connection \( \nabla^0 \). Then, a trivialization of \( R_0 = \text{Ker} \ M_1 \) is given by the \((n + 1)\)-vector
\[
f^{(0)} = (-F'_1, -F'_2, \ldots, -F'_n, 1)
\]
and a trivialization of \( R_1 = \text{Ker} \ M_2 \) is given by some \((n + 1)\)-vector
\[
f^{(1)} = \left( X_1, X_2, \ldots, X_{n+1} \right)
\]
satisfying in particular to the identities
\[
X_{n+1} = -\frac{F''_{ij}}{F'_i F'_j} \quad \text{whenever be } i, j, \ (i \neq j).
\]

Denoting by \( \Delta_i \) the \((n + 1) \times (n + 1)\) diagonal matrix built on the \((n + 1)\)-vector
\[
(0, \ldots, 0, 1, 0, \ldots, 0, F'_i),
\]
with 1 as \( i^{th} \) component, \( F'_i \) as \((n + 1)^{th}\) component and 0 elsewhere, the connexion \( \nabla^0 \) on \( R_0 \) is then defined by
\[
\nabla^0_i f^{(0)} = \partial_i f^{(0)} - \langle \Delta_i, f^{(1)} \rangle,
\]
where \( \partial_i \) (resp. \( \nabla^0_i \)) means the partial derivative (resp. the covariant derivative) with respect to \( \partial x_i \).

Thus, we get :
\[
\nabla^0_i f^{(0)} = -F'_i X_{n+1} f^{(0)}.
\]
The curvature has then components
\[
K^0_{ij} = \partial_i (F'_j X_{n+1}) - \partial_j (F'_i X_{n+1}).
\]
Fix a pair \((i, j)\) and choose an index \( k \) different from \( i \) and \( j \). We get :
\[
F'_i X_{n+1} = -\frac{F''_{ik}}{F'_k},
\]
hence
\[
\partial_j (F'_i X_{n+1}) = -\partial_j (F''_{ik}/F'_k) = -\partial_j (\partial_i (F'_i))' = \partial_{ij} (F'_i).
\]
This gives \( K^0_{ij} = 0 \). So, if we set
\[
L_{ij} = \ln(F'_i/F'_j),
\]
we have proved the following proposition.

**Proposition 5-2 :**

The web \( W \) has a non-trivial abelian relation if and only iff
\[
(L_{ij})'_{k} = 0,
\]
for any triple \( i, j, k \) of indices, each one different to each other.

Notice that, when \( W \) is in strong general position, the existence of an abelian relation is equivalent to the fact that we can choose new coordinates \((\tau_i)_i\) such that
\[
F(x_1, \ldots, x_n) \equiv \tau_1 + \cdots + \tau_n.
\]

Thus, the existence of an abelian relation is equivalent for the web to be “parallelisable”.
5.5 An example $n = 3$, $d = 5$ $(\pi'(3, 5) = 2)$:

Denoting by $(x, y, z)$ local coordinates, and defining the web by the functions $(x, y, z, x+y+z, F(x, y, z))$, assume that the function $F$ depends only on $x + y$ and $z$:

$$F(x, y, z) \equiv g(x + y, z)$$

for some function $g$.

We set $u := x + y$,

$$p := g_u', \quad q := g_z',$$

$$r := g_{uu}'', \quad s := g_{uz}'', \quad t := g_{zz}'',$$

$$a := g_{uuu}''', \quad b := g_{uu}'', \quad c := g_{uuu}''', \quad e := g_{uuuu}''' .$$

We consider $M_1, M_2, Q_3$ and $P_3$ as sub-matrices of the matrix $M_3$ described below for a convenient order of the multi-indices $L$:

$$M_3 = \begin{pmatrix}
1 & 0 & 0 & 1 & p \\
0 & 1 & 0 & 1 & p \\
0 & 0 & 1 & 1 & q
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 1 & p^2 \\
0 & 1 & 0 & 1 & p^2 \\
0 & 0 & 1 & 1 & q^2 \\
0 & 0 & 0 & 1 & pq \\
0 & 0 & 0 & 1 & pq
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & a \\
0 & 0 & 0 & 0 & 3pr \\
0 & 0 & 0 & 0 & 3qt \\
0 & 0 & 0 & 0 & 2ps + rq \\
0 & 0 & 0 & 0 & 2qs + pt \\
0 & 0 & 0 & 0 & 3pr \\
0 & 0 & 0 & 0 & 3pr
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 1 & p^3 \\
0 & 1 & 0 & 1 & p^3 \\
0 & 0 & 1 & 1 & q^3 \\
0 & 0 & 0 & 1 & p^2q \\
0 & 0 & 0 & 1 & pq^2 \\
0 & 0 & 0 & 1 & pq^2 \\
0 & 0 & 0 & 1 & p^2q
\end{pmatrix}.$$

We can check that $M_1, M_2$ above have respectively rank $3, 8$; thus

$$\rho_0 = 2 (= 5 - 3), \quad \rho_1 = 2 (= 10 - 8).$$

In general $M_3$ has rank $14$ and $\rho_2 = 1 (= 15 - 14)$. But it may happen that $M_3$ has rank $13$ and $\rho_2 = 2$ for exceptional $g$'s. This can be seen by computing the curvature $K_0$.

A basis for $R_0 = \text{Ker} \ M_1$ is

$$f_1 = (-1, -1, -1, 1, 0), \quad f_2 = (-p, -p, -q, 0, 1).$$

The lines 7 and 8 of $M_2$ being the same, we may ignore the line 8 in the computation of $R_1 = \text{Ker} \ M_2$. We assume $p \neq q$, in such a way that the sub-matrix $P_2^{(0)}$ of $P_2$ that we get in forgetting its line 5 is invertible. Thus, $R_1$ has rank $\rho_1 = 2$, and we can lift $f_1$, and $f_2$ in $R_1$, defining $f_1^{(1)} = - < (P_2^{(0)})^{-1} \ M_2^{(0)}, f_1 >$, and $f_2^{(1)} = - < (P_2^{(0)})^{-1} \ M_2^{(0)}, f_2 >$. We get:

$$f_1^{(1)} = (0, 0, 0, 0, 0), \quad f_2^{(1)} = (0, 0, Z, T, U),$$

where $Z, T$ and $U$ are solution of the cramerian linear system

$$T + p^2 U + r = 0$$
$$T + pq U + s = 0$$
$$Z + T + q^2 U + t = 0.$$
Denoting respectively by $\Delta_x$, $\Delta_y$, and $\Delta_z$, the $5 \times 5$ diagonal matrices built with $(1,0,0,1,p)$, $(0,1,0,1,p)$, and $(0,0,1,1,q)$, the connection $\nabla^0$ on $R_0$ is then given by the formulae:
\[
\nabla^0 f_1 = 0,
\]
\[
\nabla^0 f_2 = \frac{\partial}{\partial x} f_2 - \langle \Delta_x, f_2 \rangle, \quad \nabla^0 f_2 = \frac{\partial}{\partial y} f_2 - \langle \Delta_y, f_2 \rangle, \quad \nabla^0 f_2 = \frac{\partial}{\partial z} f_2 - \langle \Delta_z, f_2 \rangle,
\]
where $\nabla^0_x$, $\nabla^0_y$ and $\nabla^0_z$ denote the covariant derivative with respect to $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$. The connection form relative to $(f_1, f_2)$ is then
\[
\omega_0 = \begin{pmatrix}
0 & -T(du + dz) \\
0 & -U(p \, du + q \, dz)
\end{pmatrix},
\]
and the curvature writes
\[
K^0 = \begin{pmatrix}
0 & T_z' - T_u' + (q - p)TU \\
0 & pU_z' - qU_u'
\end{pmatrix}(dx + dy) \wedge dz.
\]
If this curvature vanishes (according to $g$), $\rho(W) = 2$. Otherwise, $\rho(W) = 1$. (The rank may not be zero, due to the obvious non-trivial abelian relation $(x) + (y) + (z) - (x + y + z) \equiv 0$). For example, if $g(u, z) = u^2 + 2\mu uz + \mu z^2$, $(\lambda, \mu \in \mathbb{C})$, we can affirm that there is no other independent abelian relation if $\lambda \neq 1$. If $\lambda = 1$, we have a vanishing curvature, corresponding to the second abelian relation $u_5 \equiv (u_4)^2 + (\mu - 1)(u_3)^2$.

5.6 An example $n = 3$, $d = 11$ ($\pi'(3, 11) = 14$):
Let $W$ be the 11-web (quasi-parallel: all $u_i$'s except one are affine functions):

$x, y, z, x + y + z, x + 2y + z, x + 3y + z, x + y + 5z, x + y + 7z, x + 11y + z, 19x + y + z, x + yz$.

We get $\rho_2 = 14 > \rho_3 = \rho_4 = 13$, and $K^3 = 0$. Hence

$\rho(W) = 13$.

5.7 Parallelisable webs:
These are webs such that all $u_i$'s are affine functions relatively to some system of local coordinates. Then, with these coordinates, the only blocks $M^{(h)}_k$ which are not zero in the matrices $M_k$ are the diagonal blocks $F_k = M^{(k-1)}_k$, and $\text{Rank}(M_k) = \sum_{h=1}^{k} \text{Rank}(P_h)$. Thus

$\rho_{h+1} = \rho_h + (d - \text{Rank}(P_{h+2}))$.

In particular, if the web is ordinary, $\rho_{h+1} = \rho_h$ for $h \geq h_0 - 2$. Hence, all ordinary parallelisable webs have maximal rank $\pi'(n, d)$ ($= \rho_{h_0 - 2}$).

If they are not ordinary, and if there exists some $h_1$ $(\geq h_0 - 2)$ such that $\rho_{h_1 + 1} = \rho_{h_1}$, then the sequence of the $\rho_i$'s is stationary from there because of the lemma 2-3 above, and then

$\rho(W) = \rho_{h_1} > \pi'(n, d)$.

Such an example is given below.

5.8 Non ordinary example $n = 3$, $d = 10$:
Let $W_{10}$ be the parallel 10-sub-web of the ordinary 11-web above, obtained by deleting $u_{11}$. It is not ordinary (since $P_3$ has rank 9, not 10). We get:

$\pi'(3, 10) = 11 = \rho_3 < \rho_4 = \rho_5 = 12 = \rho(W_{10}) < \pi(3, 10) = 16$.

This has already been quoted in [CL] (theorem 6-5) by other considerations.
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