A new identity linking coefficients of a certain class of homogeneous polynomials and Gauss hypergeometric functions

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Abstract

In investigating the properties of a certain class of homogeneous polynomials, we discovered an identity satisfied by their coefficients which involves simple \(^2F_1\) Gauss hypergeometric functions. This result appears to be new and we supply a direct proof. The simplicity of the identity is suggestive of a deeper result.

Key words: hypergeometric function, homogeneous polynomial, Taylor’s constraint

The results that we report here arose when investigating an applied mathematics problem ostensibly unrelated to combinatorics: the self-generation of magnetic fields in a spherical geometry as applicable, for instance, to the Earth’s fluid core. It happens that a certain set of homogeneous conditions apply to such magnetic fields, the so-called Taylor constraints, which arise in the physically interesting low-viscosity fast-rotation limit of the governing equations \(^5\). These constraints take the form of the vanishing of certain integrals over surfaces of constant cylindrical radius of a quantity involving the magnetic field. We have recently proven that a necessary condition for any solution is the vanishing of a certain function of cylindrical radius on both inner and outer spherical boundaries of the core \(^2\). On adopting a truncated spatial discretisation based on spherical harmonics in solid angle and certain regular polynomials in radius, this amounts to the vanishing of

\[ Q(\rho, s) = \sum_{j,k} a_{jk} I_{jk}(\rho, s) \]

regarded as a function of \(s\) at fixed values of \(\rho = 7/20\) and \(\rho = 1\). The coefficients \(a_{jk}\) are real and

\[ I_{jk}(\rho, s) = \int_0^\infty \sqrt{\rho^2 - s^2} \frac{z^j}{(z^2 + s^2)^k} \, dz. \]

In \(^1\), \(j\) and \(k\) belong to a prescribed finite set (whose size depends on the spectral truncation adopted in the numerical scheme) and \(s\) is the nondimensional cylindrical radius; the given values of \(\rho\) represent respectively the spherical radii of the inner and

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outer boundaries of the Earth’s fluid core. We note two points associated with any given 
\((j, k)\). Firstly, the integrand in (2) is a homogeneous polynomial in \((s, z)\) of degree \(2(j + k)\); 
secondly, its limits are homogeneous of degree one. Thus \(I_{jk}(\rho, s)\) is homogeneous of 
degree \(2(j + k) + 1\) and can be written

\[
I_{jk}(\rho, s) = \sqrt{\rho^2 - s^2} \sum_{l=0}^{j+k} B_l(\rho) s^{2l}. \tag{3}
\]

It follows immediately that, for each choice of \(\rho\), (1) becomes

\[
Q(\rho, s) = \sqrt{\rho^2 - s^2} \sum_{l=0}^{L} A_l(\rho) s^{2l} \tag{4}
\]

where \(L\) is the maximum value of \(j + k\).

We may therefore impose the required conditions, the vanishing of both \(Q(7/20, s)\) 
and \(Q(1, s)\), simply by setting each coefficient \(A_l\) appearing in (4) individually to zero. 
This procedure yields a set of constraints of size \(2L\). However, it happens that this set 
contains (in general) many degeneracies. Empirically, we found that the \(A_l\), produced 
by choosing one of the two values of \(\rho\) as unity, are related by

\[
A_N(1) = A_N(\rho) + \frac{\rho^2}{2} \sum_{m=1}^{L-N} \mu_m(\rho) A_{N+m}(\rho), \quad N \geq \max\{j\} \tag{5}
\]

where \(\mu_m\) is proportional to a \(\text{$_2F_1$}\) Gauss hypergeometric, given below. In our Taylor-
state problem, we will fix \(\rho = 7/20\), but the above statement is true more generally. 
Thus, for any \(N \geq \max\{j\}\), if the values of \(A_l\) appearing on the right hand side above 
are set to zero, then \(A_N(1)\) automatically vanishes and explicitly enforcing \(A_N(1) = 0\) is 
unnecessary and introduces a degeneracy in the set of constraints.

The source of the linear homogeneous condition (5) is immediately traced to the same 
condition on each \(I_{jk}\) individually, that is,

\[
B_N(1) = B_N(\rho) + \frac{\rho^2}{2} \sum_{m=1}^{j+k-N} \mu_m(\rho) B_{N+m}(\rho), \quad N \geq j \tag{6}
\]

where \(B_l\), defined in (3), is proportional to \(\rho^{2(j+k-l)}\) as follows from homogeneity. Note 
that the upper limit on the summation is the tightest possible since \(B_l = 0\) if \(l > k + j\). 
It is immediate that (5) follows from (6) because the summand is independent of \((j, k)\).

In this note we provide a proof of a generalization of (6), and it is then a simple 
matter to use (5) to count the number of independent conditions related to enforcing 
Taylor’s constraint.

The particular structure of the integrand arises in the following manner. The mono-
nomial dependence of \(z\) stems from the alignment of the cylindrical integrals, defining 
Taylor’s constraint, with the \(z\)-axis. The appearance of \(s^2\) only through \(s^2 + z^2\) comes 
about due to assumed \(C^\infty\) behaviour of the magnetic field: since both \(z\) and the square 
of the spherical radial distance, \(r^2 = s^2 + z^2\), are both \(C^\infty\), any functional relation of 
the given form also inherits this property. Lastly, the form of the upper limit on in-
tegration comes from the height of the intersection of a cylinder of cylindrical radius
with a sphere of radius $\rho$. The elementary structure of the identity is suggestive of a deeper result, although the statement cannot be readily generalized, for instance, to odd exponents of $z$. It is possible that this result arises in other circumstances involving cylindrically symmetric constraints in fluid dynamics, for instance, the Taylor-Proudman condition in a rotating sphere [1].

Immediately below we state the main result, but defer proof until two necessary Lemmas are stated and proved.

**Theorem 1.** Let $j$ and $k$ be positive integers and let the coefficients $B_l(\rho)$ be defined by

$$I_{jk}(\rho, s) = \int_0^{\sqrt{\rho^2 - s^2}} z^{2j}(z^2 + s^2)^k \, dz = \sqrt{\rho^2 - s^2} \sum_{l=0}^{j+k} B_l(\rho) s^{2l}. \quad (7)$$

Then, for any positive integer $N$ with $k + j \geq N \geq j$, the quantity

$$B_N(\rho) + \frac{1 - \rho^2}{2} \sum_{m=0}^{j+k-N} \mu_m(\rho) B_{N+m}(\rho) \quad (8)$$

is independent of $\rho$ where

$$\mu_m(\rho) = \frac{2\Gamma(m + 1/2)}{\sqrt{\pi} \Gamma(m + 1)} \binom{\alpha}{\beta} \binom{\alpha + 1}{\beta + n} \binom{n}{k}.$$  

**Remarks 1.** The statement in the theorem is considerably stronger than that of equation (6) and relates expressions of the form (2) between any two values of $\rho$, not necessarily including unity. Additionally, since $\mu_0 = 2(1 - \rho^2)^{-1}$, the theorem is equivalent to the $\rho$-independence of

$$\frac{1 - \rho^2}{2} \sum_{m=0}^{j+k-N} \mu_m B_{N+m}.$$  

Although this is the most succinct form of the result, the statement in the theorem is easier to prove as $\mu_m(\rho)$, for $m \geq 1$, are simply polynomials. The $\mu_m$ have been arbitrarily normalised such that $\mu_0 = 1$.

**Lemma 2.** For any real values $\alpha, \beta$, the following are identities

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{\alpha + k}{\beta + n} \binom{n}{k} = \binom{\alpha}{\beta}, \quad (9)$$

$$\sum_{k=0}^{n} \frac{1}{\beta + k} \binom{n - k - 1/2}{-1/2} \binom{k - 1/2}{-1/2} = \frac{\Gamma(\beta) \Gamma(n + \beta + 1/2)}{\Gamma(\beta + 1/2) \Gamma(\beta + n + 1)}. \quad (10)$$

where $\binom{n}{\beta}$ is a binomial coefficient (extended to non-integer values of its arguments).

**Proof.** We rewrite the left hand side of the above two equations in a hypergeometric form,

$$\frac{(-1)^n \Gamma(\alpha + 1)}{\Gamma(\alpha - \beta - n + 1) \Gamma(\beta + n + 1)} \binom{-n, \alpha + 1}{\alpha - \beta - n + 1} \binom{n}{1}, \quad (11)$$

$$\frac{\Gamma(n + 1/2)}{\sqrt{\pi} \Gamma(n + 1) \beta} \binom{-n, 1/2, \beta}{\beta + n + 1} \binom{n}{1}. \quad (12)$$
Using the theorems of Gauss and Saalschütz \[4\], we may evaluate the \(2F_1\) and \(3F_2\) functions and the results follow easily.

**Proof of Main Result.** We prove the theorem directly in two steps. First, we find a general expression for the coefficients \(B_l\) and second, we show the required relation holds between them.

Step 1 By exploiting equation B6 in \[3\], we can write

\[ I_{jk}(\rho, s) = \frac{\rho^{2k}}{2j + 2k + 1} (\rho^2 - s^2)^{j+1/2} \binom{[1,-k]}{[1/2-j-k]} \left(\frac{s}{\rho}\right)^{2n} \]

The hypergeometric function appearing is simply a polynomial in \(s^2/\rho^2\) of degree \(k\),

\[ 2F_1 \left( \binom{[1,-k]}{[1/2-j-k]} ; \frac{s^2}{\rho^2} \right) = \sum_{n=0}^{k} \frac{\Gamma(k+1) \Gamma(k+j-n+1/2)}{\Gamma(k-n+1) \Gamma(k+j+1/2)} \left(\frac{s}{\rho}\right)^{2n} \]

and \(I_{jk}(\rho, s)\) can be written

\[ I_{jk}(\rho, s) = (\rho^2 - s^2)^{1/2} \frac{\Gamma(k+1)}{2 \Gamma(k+j+3/2)} (\rho^2 - s^2)^j \sum_{n=0}^{k} \frac{\Gamma(k+j-n+1/2) \Gamma(j+1) (-1)^i}{\Gamma(k-n+1) \Gamma(j-i+1) \Gamma(i+1)} 2^{(n+i)} \rho^{2(k-n+j-i)} \]  \hspace{1cm} (13)

Up to the prefactor of \((\rho^2 - s^2)^{1/2}\), \(I_{jk}(\rho, s)\) is simply a polynomial in \(s^2\) of degree \(k+j\). To express this in the form of \(3\), we may write \(l = n + i\) and re-order the sum over dummy indices \(l\) and \(n\). However, care must be taken with the limits. Noting that \(i \geq 0\) and therefore \(n \leq l\), coupled with the upper bound \(n \leq k\), leads to \(n \leq \min(k, l)\). Additionally, since \(i \leq j\) and therefore \(n \geq l - j\), coupled with the lower bound \(n \geq 0\), leads to \(n \geq \max(0, l-j)\). Hence

\[ B_l(\rho) = \frac{\Gamma(k+1) \Gamma(j+1)}{2 \Gamma(k+j+3/2)} \sum_{n=\max(0, l-j)}^{\min(k, l)} \frac{(-1)^i \Gamma(k+j-n+1/2)}{\Gamma(k-n+1) \Gamma(j-l+1) \Gamma(l-n+1)} \rho^{2(k+j-l)}. \]  \hspace{1cm} (14)

Shifting the dummy indices by writing \(m = n - (l-j)\) and defining \(T = k + j - l\), this expression can be simplified to

\[ B_l(\rho) = \frac{\Gamma(k+1) \Gamma(j+1/2) \rho^{2(k+j-l)}}{2 \Gamma(k+j+3/2)} \sum_{m=\max(0, j-l)}^{\min(T, j)} (-1)^m \binom{T - 1/2 + j - m}{j - 1/2} \binom{j}{j - m}. \]  \hspace{1cm} (15)
Note that if \( m > T \) or \( m > j \) then respectively, the first or second binomial term vanishes. Hence, without loss of generality, we can replace the upper limit of the sum by \( j \). If \( l \geq j \) then the lower limit is zero and a trivial reordering of the sum, setting \( m' = j - m \), and using (9), leads to

\[
B_l(\rho) = \frac{\Gamma(k + j - l + 1/2) \Gamma(k + 1) \Gamma(j + 1/2) (-1)^j}{2\sqrt{\pi} \Gamma(k + j + 1 - l) \Gamma(k + j + 3/2)} \rho^{2(k+j-l)}. \tag{16}
\]

If \( l < j \) then the summation in (15) has a nonzero lower limit and cannot be evaluated in such a simple closed form.

**Step 2** We now are in a position to prove that, if \( l \geq j \) then the \( B_l(\rho) \) satisfy (8). We shall show that

\[
\sum_{m=1}^{k+j-N} B_{N+m}(\rho) \mu_m(\rho) = \frac{\Gamma(k+j-N+1/2) \Gamma(k+1) \Gamma(j+1/2)}{\sqrt{\pi} \Gamma(k + j + 1 - N) \Gamma(k + j + 3/2)} \sum_{b=0}^{k+j-N-1} \rho^{2b} \tag{17}
\]

and it follows immediately that

\[
B_N(\rho) + \left(1 - \rho^2\right) \frac{k+j-N}{2} \sum_{m=1}^{k+j-N} B_{N+m}(\rho) \mu_m(\rho)
= \frac{\Gamma(k+j-N+1/2) \Gamma(k+1) \Gamma(j+1/2)}{2\sqrt{\pi} \Gamma(k + j + 1 - N) \Gamma(k + j + 3/2)} \left(\rho^{2(k+j-N)} + (1 - \rho^2) \sum_{b=0}^{k+j-N-1} \rho^{2b}\right)
= \frac{\Gamma(k+j-N+1/2) \Gamma(k+1) \Gamma(j+1/2)}{2\sqrt{\pi} \Gamma(k + j + 1 - N) \Gamma(k + j + 3/2)} \tag{18}
\]

independent of \( \rho \). Note that (8) involves \( B_l \) with \( l \geq N \). Coupled with the initial hypothesis \( N \geq j \), this is consistent with \( l \geq j \), required in the derivation of the closed form for \( B_l \).

It remains to show (17). By expanding the hypergeometric component of \( \mu_m(\rho) \), a polynomial of degree \( m - 1 \) in \( \rho \), it follows easily that

\[
\mu_m(\rho) = \sum_{n=0}^{m-1} \frac{2 \Gamma(m) \Gamma(n + 1/2) \Gamma(m - n + 1/2)}{\Gamma(m+1) \Gamma(m-n) \Gamma(n+1)} \rho^{2n}.
\]

Hence

\[
\sum_{m=1}^{k+j-N} B_{N+m}(\rho) \mu_m(\rho) = \frac{(-1)^j \Gamma(k+1) \Gamma(j+1/2)}{\pi^{3/2} \Gamma(k + j + 3/2)} \times
\sum_{m=1}^{k+j-N} \sum_{n=0}^{m-1} \frac{\Gamma(k+j-N-m+1/2) \Gamma(m) \Gamma(n + 1/2) \Gamma(m - n + 1/2)}{\Gamma(k + j + 1 - N - m) \Gamma(m+1) \Gamma(m-n) \Gamma(n+1)} \rho^{2(k+j+n-N-m)}.
\tag{19}
\]

This is a polynomial in \( s^2 \) of degree \( k + j - N - 1 \). By introducing a new dummy variable \( b = k + j - N - m \) the double sum above can be rewritten as

\[
\sum_{m=1}^{k+j-N} B_{N+m}(\rho) \mu_m(\rho) = \sum_{b=0}^{k+j-N-1} \nu_b \rho^{2b}
\]
where
\[
\nu_b = \frac{(-1)^j \Gamma(k+1) \Gamma(j+1/2) \Gamma(k+j-N-b+1/2)}{\pi^{3/2} \Gamma(k+j+3/2) \Gamma(k+j-N-b)} \times 
\sum_{m=k+j-N}^{b} \frac{\Gamma(k+j-N-m+1/2) \Gamma(m) \Gamma(b-k-j+N+m+1/2)}{\Gamma(k+j+1-N-m) \Gamma(m+1) \Gamma(b-k-j+N+m+1)}.
\]

(20)

The change in the lower limits on \(m\) arises since \(b - k - j + m + N = n \geq 0\) in the original sum and hence \(m \geq k + j - N - b\). Note that, since \(b \leq k + j - N - 1\), \(m \geq 1\), consistent with the original lower limit on \(m\) from (19). Defining new variables \(c = m - \beta, \beta = k + j - N - b\), the summation can be written
\[
\sum_{c=0}^{b} \frac{\Gamma(b+c+1/2) \Gamma(c+1/2)}{\Gamma(b+c+1) \Gamma(c+1)} \frac{1}{c + \beta} = \sum_{c=0}^{b} \binom{b+c-1/2}{-1/2} \frac{\Gamma(1/2)^2}{c + \beta}
\]
and by using (10) it is immediate that
\[
\nu_b = (-1)^j \frac{\Gamma(k+j-N+1/2) \Gamma(k+1) \Gamma(j+1/2)}{\sqrt{\pi} \Gamma(k+j+1-N) \Gamma(k+j+3/2)}
\]
which is independent of \(b\), and (17) follows.

References

[1] G. K. Batchelor, An introduction to fluid dynamics, CUP, 1967.
[2] P. Livermore, G. Ierley, and A. Jackson, The construction of Taylor states II: the dynamical influence of an inner core., Phys. Earth Planet. Int. (2008), submitted.
[3] The structure of Taylor’s constraint in three dimensions, Proc. R. Soc. A (2008a), DOI 10.1098/rspa.2008.0091.
[4] M. Petkovšek, H. Wilf, and D. Zeilberger, A=B, A K Peters Ltd, Wellesley, MA, USA, 1997.
[5] J. B. Taylor, The magneto-hydrodynamics of a rotating fluid and the Earth’s dynamo problem, Proc. R. Soc. A 9 (1963), 274–283.