The theory of max-min \((K, L)\)-eigenvectors

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Abstract

We develop a new approach to the max-min eigenproblem, in which the max-min eigenspace is split into several regions according to the order relations between the eigenvalue and the components of \(x\). The resulting theory of \((K, L)\)-eigenvectors, being based on the fundamental results of Gondran and Minoux, allows to describe the whole max-min eigenspace explicitly and in more detail.

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1. Introduction

By max-min algebra we mean the unit interval \(\mathbb{B} = <0, 1>\) equipped with the arithmetic operations of "addition" \(a \oplus b = \max(a, b)\) and "multiplication" \(a \otimes b = \min(a, b)\). Algebraically speaking, max-min algebra is a semiring where both arithmetic operations are idempotent. Let us also note that, algebraically, max-min algebra is an example of incline algebra of \([4]\).

The arithmetic operations \(\oplus\) and \(\otimes\) can be extended to matrices and vectors in the usual way so that for any matrices \(A = (a_{ij})\) and \(B = (b_{ij})\) of appropriate dimensions we can define their "sum" \(A \oplus B\) and "product" \(A \otimes B\) by the usual rules: \((A \oplus B)_{ij} = a_{ij} \oplus b_{ij}\) and \((A \otimes B)_{ij} = \bigoplus_k a_{ik} \otimes b_{kj}\). For a square matrix \(A \in \mathbb{B}^{n \times n}\) we can also define its max-min matrix powers: \(A^k = A \otimes \ldots \otimes A\), where \(k\) is a natural number. Note that \(A^0 = I\), the usual identity matrix. Further we will systematically omit the product sign \(\otimes\), for brevity.

As usual in tropical/idempotent algebras, to each matrix \(A \in \mathbb{B}^{n \times n}\) we associate a digraph \(\mathcal{D}(A) = (N, E)\) with the node set \(N = \{1, \ldots, n\}\) and edge set \(E = \{(i, j)\colon a_{ij} > 0\}\). Each edge has a weight \(a_{ij}\). A sequence of

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edges \( \{(i_0, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k)\} \) where each edge belongs to \( E \) is called a
walk whose length is \( k \) and whose weight is given by the max-min product
\( a_{i_0i_1} \cdots a_{i_{k-1}i_k} \).

It is easy to see that each entry \((A^k)_{ij}\) is the greatest max-min weight of
a walk connecting node \( i \) to node \( j \) and having length \( k \). This gives rise to
the following example of algebraic optimal path problem \([5, 20]\) to which the
theory of max-min matrix powers can be applied. Suppose that we are given
a network of roads with one bridge on each road. The capacity of each bridge
(i.e., the biggest weight of a vehicle that can go over it) is given. The capacity
of a sequence of such roads with bridges (i.e., a walk in this network) is then
determined as the minimal capacity of the bridges in it. The problem then is,
given a starting point and an end point and (possibly) the number of bridges
to be passed, to find a sequence of roads with the greatest capacity.

Another motivation to study max-min algebra comes from the theory of
fuzzy sets where the operation \( \min(a, b) \) is one of the most useful examples of
triangular norms \([17]\). See also \([15]\) for more on semirings and other algebraic
models relevant to the theory of fuzzy sets.

The main goal of the present paper is to develop a new approach to the
max-min eigenproblem. For a given matrix \( A \in \mathbb{B}^{n \times n} \) and a number \( \lambda \in \mathbb{B} \), it
consists in finding the set of vectors \( x \in \mathbb{B}^{n \times 1} \) such that
\[
Ax = \lambda x. \tag{1}
\]
The set of these vectors forms a max-min \( \lambda \)-eigenspace of \( A \). Note that it is
indeed a space in the sense of max-min algebra: for any \( \alpha, \beta \in \mathbb{B} \) and any \( u \) and
\( v \) satisfying \([1]\) \( \alpha u \oplus \beta v \) also satisfies this equation.

Let us now give a motivation for studying the eigenproblem in max-min
algebra related to the area of data transfer security is described in the following
text. Consider a simple network consisting of several network devices \( D_i \).
The devices are connected by lines \( E_{ij} \), so that we are able to send the data from
device \( D_i \) to \( D_j \). Weight of every line \( E_{ij} \) is denoted as \( a_{ij} \) and represents
the security level of this connection. The values \( a_{ij} \in \{0, 1\} \), where \( a_{ij} = 0 \)
means completely unsecured connection, whereas \( a_{ij} = 1 \) stands for connection
absolutely secured. When exploring the data transfer security, the security of
connections should not be the only one aspect. The data passing through the
network may possibly enter the network with some security level \( x_i \in \{0, 1\} \).
The level of expected or given technological security level is also considered.
The technological security level is influenced by the accessible technological
possibilities and is also dependent on the available budget. This level is denoted
by \( \lambda \in \{0, 1\} \).

The formula \( A \otimes x \) indicates the maximal security level for the data going
out from particular devices. The solution of \( A \otimes x = x \) describes the data with
security level unchanged by passing through the network. On the other hand,
solving \([1]\) means to take into account the possibility that the security level of
some data can be reduced by the technological security level \( \lambda \).

Problem \([1]\) has been studied in max-min algebra at least since \([13, 14]\).
The approach taken in these works resembles that of max-plus algebra, where eigenspaces are characterized as particular subspaces of the column span of Kleene star. There is also a number of works where a different approach is taken. The paper [6] focuses on characterizing two particular eigenvectors (called lower and upper basic eigenvectors) in terms of the associated graph. The structure of the eigenspace of increasing max-min eigenvectors $x_1 \leq x_2 \leq \ldots \leq x_n$ in max-min algebra has been described in [10]. Various types of max-min interval eigenvectors have been studied in [12].

Gondran and Minoux obtained fundamental results for (1) also over more general semirings with idempotent multiplication, see [16] Section 6.3 for one of the latest accounts. We are going to use these results. However, we observe that the theory as presented in that monograph is incomplete. In particular, although [16] Ch. 6, Corollary 3.5 (if understood literally) claims to describe the whole set of solutions to (1), it only describes the solutions whose all components are less than or equal to $\lambda$. Several examples when (1) also admits other solutions can be found in the present paper. To describe those other solutions we adopt an approach which is similar to that of [11]. Namely, given $A$ and $\lambda$ we consider a partition of $N = \{1, \ldots, n\}$ into two disjoint subsets $K$ and $L$ such that $K \cup L = N$ and pose a problem of describing all vectors $x$ that satisfy $x_i \leq \lambda$ and hence $Ax_i = x_i$ for all $i \in K$ and $x_i \geq \lambda$ and hence $\lambda x_i = \lambda$ for all $i \in L$. When $K = N$ we call such vectors "pure eigenvectors" since (1) becomes $Ax = x$, and when $L = N$ we call such vectors "background eigenvectors", in analogy with [11]. In the latter case (1) becomes $Ax = \lambda 1$, where 1 denotes the vector of all 1's. Pure eigenvectors were described in [16] Ch. 6, Corollary 3.5, which we revisit here in Corollary 3.1. Background eigenvectors are easy to obtain: see Proposition 3.1 and Proposition 3.2 below. Pure and background eigenvectors are fundamental for describing the $(K, L)$-eigenvectors in the case of general $K$ and $L$. Their description is stated in Theorem 3.1, which can be considered as our main result.

All new results of this paper are obtained in Section 3. Preliminary notions and results from [3] and [16], which provide the necessary algebraic tools, are given in Section 2.

2. Some problems of max-min algebra

In this section we will give some necessary notions and facts from max-min algebra on which our study of max-min $(K, L)$-eigenvectors will be based. We will start with defining the notions of metric matrix and Kleene star and (following [16]) giving a description of the set of principal eigenvectors ($x$ such that $Ax = x$). This will be followed by describing the solution set to max-min Bellman ($Z$-matrix) equation (following [3] or [19]) and solving a special type of max-min equation (11).
2.1. Metric matrix, Kleene star and the principal eigenproblem

For a square matrix \( A \in \mathbb{B}^{n \times n} \) let us define its metric matrix \( A^+ = (a^+_{ij})_{i,j=1}^n \) and Kleene star \( (a^*_{ij})_{i,j=1}^n \) by the following series:

\[
A^+ = A \oplus A^2 \oplus A^3 \oplus \ldots \\
A^* = I \oplus A \oplus A^2 \oplus \ldots
\]  

(2)

It is well-known that in max-min algebra these series always converge and, moreover, can be truncated:

\[
A^+ = A \oplus \ldots \oplus A^n, \\
A^* = I \oplus A \oplus \ldots \oplus A^{n-1}.
\]  

(3)

The following properties of metric matrix and Kleene star are well-known, see [16][Chapter 6]:

\[
A^* = AA^* \oplus I = A^*A \oplus I, \\
A^+ = AA^* = A^*A.
\]  

(4)

In terms of the associated graph, \( a^+_{ij} \), being equal to \( a^*_{ij} \) when \( i \neq j \), is the maximal (max-min) weight of all walks connecting \( i \) to \( j \) with unrestricted length. In terms of the network with bridges, this is the maximal capacity of all sequences of roads connecting \( i \) to \( j \). So is the optimal walk interpretation of metric matrices and Kleene stars.

Metric matrices, Kleene stars and associated digraphs provide some of the basic tools for the max-min eigenproblem. Let us start with the principal eigenproblem: the problem of identifying all vectors \( x \) that satisfy \( Ax = x \) for a given matrix \( A \). Such vectors will be called principal eigenvectors of \( A \).

For each principal eigenvector \( x \), following the terminology of [1], define its saturation graph \( \text{Sat}(A, x) \) as the graph consisting of all edges \((i,j)\) that satisfy \( a_{ij} \otimes x_j = x_i \) and all nodes on these edges. This graph in general has several maximal strongly connected components, and let \( C(A, x) \) denote a subset of \( N = \{1, \ldots, n\} \) that contains one node from each strongly connected component of \( \text{Sat}(A, x) \). We now state a description of the set of principal eigenvectors, which is essentially due to [16]. The proof is also given here, for convenience of the reader.

**Theorem 2.1 (Gondran-Minoux [16] Section 6.3).** The set of principal eigenvectors is a max-min space generated by vectors

\[
a^+_{ii}(A^*), \quad i = 1, \ldots, n.
\]  

(5)

More precisely, each vector of (5) is a principal eigenvector, and each principal eigenvector \( x \) can be represented as

\[
x = \bigoplus_{i \in C(A, x)} x_i a^+_{ii}(A^*).  
\]  

(6)
Proof: We first show that each vector in (7) is a principal eigenvector. We have:

\[ A(a_{ii}^{+}(A^*)_i) = a_{ii}^{+}(A^*)_i, \tag{7} \]

and we need to show that \( a_{ii}^{+}(A^*)_i = a_{ii}^{+}(A^*)_i \). By their definition, matrices \( A^* \) and \( A^+ \) differ only on the diagonal and therefore \( a_{ii}^{+}a_{ii}^{*} = a_{ii}^{+}a_{ii}^{*} \) for \( k \neq i \), and for \( k = i \) we have \( a_{ii}^{+} = (a_{ii}^{+})^2 \) by the idempotency of multiplication. This implies \( a_{ii}^{+}(A^*)_i = a_{ii}^{+}(A^*)_i \).

We now take an arbitrary principal eigenvector \( x \). Since \( Ax = x \) we also have \( A^kx = x \) for any \( k \geq 1 \) and therefore also \( A^*x = x \), adding up all these equalities and using the idempotency of \( \oplus \). Writing this in terms of columns of \( A^* \) we have

\[ x = \bigoplus_{i=1}^{n} x_i(A^*)_i. \tag{8} \]

To show that this is the same as (6) we first take \( i \in C(A,x) \). Since \( i \) is on a cycle of \( \text{Sat}(A,x) \), for some \( i_1, \ldots, i_k \) where \( i = i_1 \) we have that

\[ a_{i_1}x_{i_2} = x_{i_1}, a_{i_2}x_{i_3} = x_{i_2}, \ldots, a_{i_k}x_{i_1} = x_{i_k}. \]

These equations imply that

\[ a_{i_1}a_{i_2}x_{i_3} \cdots a_{i_k}x_{i_1} = x_{i_1} \]

and hence \( a_{i_1}^{+}x_{i} \geq x_{i} \), which is equivalent to \( a_{i_1}^{+}x_{i} = x_{i} \). Therefore \( x_{i} \) can be replaced in (8) with \( x_{i}a_{i_1}^{+} \), for such \( i \).

Since each node has an outgoing edge in \( \text{Sat}(A,x) \), for each \( j \notin C(A,x) \) there exists a walk in \( \text{Sat}(A,x) \) connecting it to a (maximal) strongly connected component of \( \text{Sat}(A,x) \) and hence a walk connecting to a node \( i \in C(A,x) \), which is a sequence \( i_1, \ldots, i_k \) with \( j = i_1 \) and \( i = i_k \) such that

\[ a_{i_1}x_{i_2} = x_{i_1}, a_{i_2}x_{i_3} = x_{i_2}, \ldots, a_{i_k-1}x_{i_k} = x_{i_{k-1}}. \]

These equations imply that

\[ a_{i_1}a_{i_2} \cdots a_{i_k-1}x_{i_k} = x_{i_1}. \]

Using this we obtain that for arbitrary index \( \ell \)

\[ a_{\ell_1}^{+}x_{j} = a_{\ell_1}^{+}a_{\ell_2}^{+}a_{\ell_3}^{+} \cdots a_{i_{k-1}}^{+}x_{i_k} \leq a_{\ell_1}^{+}x_{k} = a_{\ell_2}^{+}x_{i} \]

where the last inequality is due to the optimal walk interpretation of Kleene star. Hence we have shown that for each \( j \notin C(A,x) \) there exists \( i \in C(A,x) \) with \( a_{\ell_1}^{+}x_{j} \leq a_{\ell_1}^{+}x_{i} \). Hence we can omit the terms in (8) where \( i \notin C(A,x) \), while the terms for which \( i \in C(A,x) \) can be multiplied by \( a_{\ell_1}^{+} \). This shows (6) and hence the whole claim. \( \Box \)
2.2. Bellman equation

We will also use the theory of algebraic Bellman equation

\[ x = Ax \oplus b, \]  

(9)

studied over general semirings, e.g., in [5], [20]. In nonnegative linear algebra this equation is also known as Z-matrix equation [3].

Although (9) has been known for decades, the following fundamental result was formulated only recently in [3, 18, 19]. The solution set of (9) will be denoted by \( S(A, b) \). A short proof that is close to the one given in [11] is presented for reader’s convenience.

**Theorem 2.2.** Let \( A \in \mathbb{B}^{n \times n} \) and \( b \in \mathbb{B}^n \). Equation (9) always has nontrivial solutions, and the set of these solutions is given by

\[ S(A, b) = \{ A^*b \oplus v : Av = v \}. \]

(10)

**Proof:** First it can be verified that any vector like on the r.h.s. of (10) satisfies \( x = Ax \oplus b \), using that \( A^* = AA^* \oplus I \) (4). In particular, a solution of (9) always exists since \( A^* \) always converges in the max-min case.

Iterating the equation \( x = Ax \oplus b \) we obtain

\[ x = Ax \oplus b = A(Ax \oplus b) \oplus b = \ldots = A^kx \oplus (A^{k-1} \oplus \ldots \oplus I)b = A^kx \oplus A^*b \]

for all \( k \geq n \).

This implies \( x \geq A^*b \). Further, \( x \) satisfies \( Ax \leq x \), hence \( x \geq Ax \geq \ldots \geq A^kx \geq \ldots \).

In max-min algebra, the orbit \( \{ A^kx \}_{k \geq 1} \) always starts to cycle from some \( k \). Indeed, in this algebra every entry of \( A^kx \) for any \( k \) is an entry of \( A \) or an entry of \( x \), and therefore there exist \( k_1 \) and \( k_2 \) for which \( A^{k_1}x = A^{k_2}x \). But as \( Ax \leq x \), we have \( x \geq Ax \geq \ldots \geq A^kx \geq \ldots \), and it is only possible that the sequence \( \{ A^kx \}_{k \geq 1} \) stabilizes starting from some \( k \). That is, starting from some \( k \), vector \( v = A^kx \) satisfies \( Av = v \). The proof is complete. \( \Box \)

2.3. Special type of equation

We also need to describe the solution set for the system

\[ Az \oplus b = \lambda 1. \]

(11)

where \( A \in \mathbb{B}^{m \times n} \), \( z \in \mathbb{B}^n \) and \( b \in \mathbb{B}^m \) (for arbitrary natural numbers \( m \) and \( n \)).

We will study this system under the condition that all coefficients of \( A \) and \( b \) are less or equal to \( \lambda \):

\[ a_{ij} \leq \lambda, \quad b_i \leq \lambda \quad \forall i, j \in N. \]

(12)
The description will be obtained in terms of coverings and minimal coverings, following the known solution method for systems $A \otimes x = b$ and $A \otimes x \geq b$ in max-plus and max-min algebra (see, e.g., [2], [8] and [9]).

Note that if the system (11) is solvable, then in every row of the equation, we have to obtain $\lambda$ either from $Ax$ or from $b$.

Let us denote $I_0 = \{ i : b_i < \lambda \}$ and $C_j = \{ i \in I_0 : a_{ij} = \lambda \}$ for $j \in N$. Furthermore, for $W \subseteq N$ we denote $C^W$ and $z^W$ by putting

$$C^W = \{ C_j : j \in W \}$$

$$z^W_j = \begin{cases} \lambda & \text{if } j \in W \\ 0 & \text{otherwise} \end{cases}$$

We say that $C^W$ is a covering of $I_0$, if $I_0 = \bigcup C^W$ (for $j \in W$). Moreover, $C^W$ is a minimal covering of $I_0$, if $I_0 \neq \bigcup C^V$ for every $V \subseteq N$, $V \neq W$.

**Proposition 2.1.** Vector $z \in B^n$ is a solution of (11) with condition (12) if and only if $z \geq z^W$ for some $W \subseteq N$ such that $C^W$ is a minimal covering of $I_0$.

**Proof:** Suppose $z \in B^n$ is a solution of (11). Then $b_i < \lambda$ implies $(A_i)z = \lambda$. Then, also using (12), we obtain that for every $i \in I_0$ there is $j = j(i) \in N$ with $a_{ij(i)} = \lambda$ and $z^W_{j(i)} \geq \lambda$. Denoting $W = \{ j(i) : i \in I_0 \}$, we get $I_0 = \bigcup C^W$ and $z \geq z^W$. Without loss of generality, we can assume that the covering $C^W$ of $I_0$ is a minimal one.

For the converse implication, suppose $z \geq z^W$ for some $W \subseteq N$, with $C^W$ being a minimal covering of $I_0$. Then, for any $i \in I_0$, there is $j = j(i) \in W$ such that $i \in C^W$. That is, $z^W_{j(i)} \geq \lambda$ and $a_{ij(i)} = \lambda$, which gives $(A_i)z = \lambda$.

□

**Corollary 2.1.** Equation (11) with condition (12) is solvable if and only if there exists a covering $C^W$ of $I_0$. In this case, solution set to (11) can be represented as $\bigcup W S^W$ where the union is taken over $W \subseteq N$ such that $C^W$ is a minimal covering of $I_0$ and

$$S^W := \{ z : z^W \leq z \leq 1 \}$$

Note that if $I_0 = \emptyset$ then the minimal covering of $I_0$ is $C^W$ with $W = \emptyset$ and the unique minimal (and hence the least) solution of (11) is $0$ and the solution set is $\{ z : 0 \leq z \leq 1 \}$.

In general, solution set $S^W = \{ z : z^W \leq z \leq 1 \}$ can be algebraically expressed as follows.
Proposition 2.2. Suppose that
\[ S^W = \{ z : \lambda \leq z_k \leq 1 \text{ for } k \in W, \ z_k \geq 0 \text{ for } k \in N \setminus W \}, \]  
where \( W \subseteq N \). Then
\[ S^W = \{ z^W \oplus \Lambda^W v_N : v_N \in \mathbb{B}^{|N|} \} \]  
where the entries of \( \Lambda^W \) are defined as
\[ (\Lambda^W)_{ij} = \begin{cases} 1, & \text{if } i = j, \\ z^W_i, & \text{if } i \neq j \end{cases} \]  
i, j \in N.

Example 2.1. We shall illustrate the solution of (11) by the following system:
\[ A = \begin{pmatrix} .3 & .5 & .3 \\ .6 & .6 & .2 \\ .6 & .3 & .6 \end{pmatrix}, \quad b = \begin{pmatrix} .6 \\ .3 \\ .2 \end{pmatrix} \]  
and consider \( \lambda = .6 \).

Then the set \( I_0 = \{ 2, 3 \} \). From the entries of \( A \) we can derive sets \( C_1 = \{ 2, 3 \}, C_2 = \{ 2 \} \) and \( C_3 = \{ 3 \} \).

For \( W = \{ 1 \} \), we have \( C^W = \{ C_1 \} \). This set is minimal covering of \( I_0 \), minimal solution here is \( z^W = z^{(1)} = (0, .6, 0, 0) \) and every \( z \geq z^{(1)} \) is also a solution, i.e. \( S^{(1)} = \{ z^{(1)} \oplus \Lambda^W v_N : v_N \in \mathbb{B}^{|N|} \} \).

Similarly, for \( W = \{ 2, 3 \} \) we have the set \( C^W = \{ C_2, C_3 \} \) which is minimal covering of \( I_0 \) with minimal solution \( z^W = z^{(2,3)} = (0, .6, .6) \). Again, also vectors \( z \geq z^{(2,3)} \) are solutions to the system, i.e. \( S^{(2,3)} = \{ z^{(2,3)} \oplus \Lambda^W v_N : v_N \in \mathbb{B}^{|N|} \} \).

Final solution set for the system (11) is then represented by the union of particular solution sets, \( \bigcup_W S^W \).

3. Max-min eigenproblem

Let us first consider some two-dimensional examples that show the solution to the max-min eigenproblem (1).

That means two-dimensional max-min \((K,L)\)-eigenspaces (where \( K \) ranges over all subsets of \( \{ 1, 2 \} \) and \( L \) is the complement of \( K \)).

Example 3.1. Take
\[ A = \begin{pmatrix} .7 & .3 \\ .2 & .5 \end{pmatrix} \]  
and consider \( \lambda = .5 \). Then the solution of (11) is equivalent to the system
\[ \max(\min(.7, x_1), \min(.3, x_2)) = \min(.5, x_1), \]  
(21)
Figure 1: Sets $X', X''$ and their intersection: the 0.5-eigenspace of $A$

\[ \max(\min(0.2, x_1), \min(0.5, x_2)) = \min(0.5, x_2) \]  

The solution set for (21) is

\[ X' = \{(x_1, x_2): (x_1 \geq 0.3) \lor (x_1 \geq x_2) \land (x_1 \leq 0.5)\}, \]

and the solution set for (22) is

\[ X'' = \{(x_1, x_2): (x_2 \geq 0.2) \lor (x_1 \leq x_2)\}. \]

Then, the solution set to the eigenproblem is $X' \cap X''$. Sets $X', X''$ and their intersection are displayed in Figure 1.

It can be seen, that the value $\lambda$ has some effect on the final solution set. The eigenvectors can be thus studied in individual areas (subsets) defined by $\lambda$.

We can observe from Figure 2 that the boundaries defined by $\lambda$ value (represented by the dashed line) divided the solution set of our two-dimensional example into four areas (in the figure quadrants $Q_1 - Q_4$).

For the eigenvectors in $Q_1$ it holds that all $x_i \geq \lambda$ and thus we say that all $i \in L$ and $K = \emptyset$. We call these eigenvectors the background eigenvectors of $A$. For $Q_2$ and $Q_3$ it holds that $x_i \leq \lambda$ for some $i \in K$ and some $x_i \geq \lambda$ for some $i \in L$. In $Q_4$ all $x_i \leq \lambda$, it means that all $i \in K$ and $L = \emptyset$. We call these vectors the pure eigenvectors of $A$.

Note that in this example we have some “genuine” $(K,L)$-eigenvectors in the interior of $Q_2$, which are neither pure nor background eigenvectors.

**Example 3.2.** In this example, we take

\[ B = \begin{pmatrix} 0.4 & 0.5 \\ 0.2 & 0.5 \end{pmatrix} \]

and consider the same $\lambda = 0.5$. To solve for $A = B$ we need to solve the system

\[ \max(\min(0.4, x_1), \min(0.5, x_2)) = \min(0.5, x_1), \]
Figure 2: \((K, L)\)-eigenvectors

\[
\text{max}(\min(2, x_1), \min(5, x_2)) = \min(5, x_2) \tag{27}
\]

Note that (27) is the same as (22). The solution set for (26) is

\[
Y' = \left\{ (x_1, x_2): ((x_1 = x_2 \leq .5) \lor (x_2 \leq x_1 \leq .4) \lor ((x_1 \geq .5) \land (x_2 \geq .5)) \right\}, \tag{28}
\]

and the solution set for (27) is \(Y'' = X''\) expressed in (24). Solution sets \(Y'\) and \(Y''\) and their intersection with their intersection are depicted in the Figure 3.

We are now going to give a theoretical description of background eigenvectors, pure eigenvectors and \((K, L)\)-eigenvectors in max-min algebra.

3.1. Background \(\lambda\)-eigenvectors

These are the vectors that satisfy \(Ax = \lambda 1\) and \(x_i \geq \lambda\) for all \(i\). Let us introduce the following notation:

\[
N^{>\lambda} = \{k: \max_{1 \leq i \leq n} a_{ik} > \lambda\}, \quad N^{\leq\lambda} = \{k: \max_{1 \leq i \leq n} a_{ik} \leq \lambda\}. \tag{29}
\]

The set of background eigenvectors can be described as follows.
**Proposition 3.1.** Let $A \in B^{n \times n}$ and $\lambda \in B$. Then the set of background $\lambda$-eigenvectors of $A$ is nonempty if and only if

$$\max_{1 \leq j \leq n} a_{ij} \geq \lambda \quad \forall i. \quad (30)$$

If (30) holds then the set of background eigenvectors is given by

$$\{x: x_k = \lambda \text{ for } k \in N^{>\lambda}, \lambda \leq x_k \leq 1 \text{ for } k \in N^{\leq\lambda}\}. \quad (31)$$

**Proof:** Observe first that if there exist $i$ with $a_{ij} < \lambda$ for all $j$, then also $a_{ij}x_j < \lambda$ for all $j$ implying that $\sum_{j=1}^{n} a_{ij}x_j = \lambda$ cannot hold and the set of background eigenvectors is empty. If (30) holds then the constant vector $x = \lambda 1$ satisfies $Ax = \lambda 1$ hence the set of background eigenvectors is nonempty.

If $k \in N^{>\lambda}$ then there exists $i$ that $a_{ik} > \lambda$, and we need $x_k = \lambda$ to make sure that $\sum_{j=1}^{n} a_{ij}x_j \leq \lambda$. This shows that the set of background $\lambda$-eigenvectors is a subset of (31).

Now take a vector from (31). Obviously, it satisfies $x_i \geq \lambda$ for each $i$.

Since also $\max_j a_{ij} \geq \lambda$ for all $i$, we have $\sum_{j=1}^{n} a_{ij}x_j \geq \lambda$. But we also have $\sum_{j=1}^{n} a_{ij}x_j \leq \lambda$. Indeed, since $x_j = \lambda$ in (31) whenever $j \in N^{>\lambda}$, that is, whenever there exists $i$ with $a_{ij} > \lambda$, we have $a_{ij}x_j \leq \lambda$ for all such $j$ and all $i$. For $j \in N^{\leq\lambda}$, inequality $a_{ij}x_j \leq \lambda$ follows from $\max_i a_{ij} \leq \lambda$ (by the definition of $N^{\leq\lambda}$).

The proof is complete. $\square$

The following proposition will be helpful when describing sets of the form (31) by column spaces of some matrices.

**Proposition 3.2.** Suppose that

$$S = \{x: x_k = \lambda \text{ for } k \in N_1, \lambda \leq x_k \leq 1 \text{ for } k \in N_2\}, \quad (32)$$

where $N_1$ and $N_2$ are such that $N_1 \cup N_2 = N(= \{1, \ldots, n\})$ and $N_1 \cap N_2 = \emptyset$.

Then

$$S = \{\lambda 1 \oplus \Lambda_{N_1,N_2} z_{N_2}: z_{N_2} \in B^{|N_2|}\} \quad (33)$$

where $\Lambda$ is defined by

$$(\Lambda)_{ij} = \begin{cases} 1, & \text{if } i = j, \\ \lambda, & \text{if } i \neq j, \end{cases} \quad (34)$$

$$i,j \in N.$$

**3.2. Pure $\lambda$-eigenvectors**

These are the vectors that satisfy $Ax = x$ and $x_i \leq \lambda$ for all $i$. Description of a generating set of the space of pure max-min $\lambda$-eigenvectors is given below. Observe that any pure $\lambda$-eigenvector is a principal eigenvector and therefore we can apply Theorem 2.1.
Corollary 3.1 (Gondran-Minoux [16], Ch. 6, Corollary 3.5). The set of pure $\lambda$-eigenvectors of $A \in \mathbb{B}^{n \times n}$ is the max-min column space $\{ A^*_\lambda y : y \in \mathbb{B}^n \}$, where the columns of $A^*_\lambda$ are defined by

$$(A^*_\lambda)_{i} = \lambda (A^*)_{i} : i = 1, \ldots, n.$$  

More precisely, each vector of (35) is a pure $\lambda$-eigenvector, and each pure $\lambda$-eigenvector $x$ can be represented as

$$x = \bigoplus_{i \in C(A, x)} \lambda x_i (A^*)_{i},$$

where $C(A, x) \subseteq N$ is a set containing a node from each strongly connected component of $\text{Sat}(A, x)$.

**Proof:** The claim follows as an easy corollary of Theorem 2.1. Indeed, since $a_i^+(A^*)_{i}$ satisfies $Ax = x$, so does $\lambda a_i^+(A^*)_{i}$. As components of this vector do not exceed $\lambda$, it is a pure $\lambda$-eigenvector. Letting $x$ be a pure $\lambda$-eigenvector, we see that it satisfies (6) since it satisfies $Ax = x$. Equation (36) follows from (6) after multiplying both parts of (6) by $\lambda$ and observing that $\lambda x_i = x_i$ since $x_i \leq \lambda$ for all $i$. \qed

3.3. $(K, L)$ $\lambda$-eigenvectors

Now we consider $(K, L)$ max-min $\lambda$-eigenvectors, i.e., $x \in \mathbb{B}^n$ such that $Ax = \lambda x$, $x_i \leq \lambda$ for $i \in K$ and $x_i \geq \lambda$ for $i \in L$, where $K, L \subseteq \{1, \ldots, n\}$ are such that $K \cup L = \{1, \ldots, n\}$ and $K \cap L = \emptyset$.

By definition, $(K, L)$ $\lambda$-eigenvectors satisfy

$$A_{KK} x_K + A_{KL} x_L = x_K, \quad x_K \leq \lambda 1_K$$

$$A_{LK} x_K + A_{LL} x_L = \lambda 1_L, \quad x_L \geq \lambda 1_L$$

We start by writing out the solution of (37). Applying Theorem 2.2, the set of $x_K$ satisfying this equation (together with $x_K \leq \lambda 1_K$) is nonempty if and only if

$$(A_{KK})^* A_{KL} x_L \leq \lambda 1_K,$$

and then it is given by

$$S_K(x_L, \lambda) = \{(A_{KK})^* A_{KL} x_L + v : A_{KK} v = v, \ v \leq \lambda 1_K \}$$

$$= \{(A_{KK})^* A_{KL} x_L + (A_{KK})^* z_K : z_K \in \mathbb{B}^{\|K\|} \}$$

where $(A_{KK})^*$ is defined as in (35).

As for (38), denote

$$L_1 = \{i \in L : \bigoplus_{j \in L} a_{ij} \geq \lambda\}, \quad L_2 = \{i \in L : \bigoplus_{j \in L} a_{ij} < \lambda\}$$

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In terms of this notation (38) is written as follows:

\[ A_{L_1}x_K + A_{L_1}x_L = \lambda_1 L_1, \]  
\[ A_{L_2}x_K + A_{L_2}x_L = \lambda_2 L_2, \]  
\[ x_L \geq \lambda_1 L_1, \quad x_K \leq \lambda_1 K. \]  

and this is equivalent to the following system

\[ A_{L_1}x_L = \lambda_1 L_1, \quad x_L \geq \lambda_1 L, \]  
\[ A_{L_2}x_K = \lambda_1 L_2, \quad x_K \leq \lambda_1 K. \]  

Indeed, \( x_L \geq \lambda_1 L \) together with \( \bigoplus_{j \in L} a_{ij} \geq \lambda \) for \( i \in L_1 \) imply that \( A_{L_1}x_L \geq \lambda_1 L_1 \) holds for any feasible \( x \), and this makes the first term in (42) redundant, also since \( x_K \leq \lambda_1 K \) implies \( A_{L_1}x_K \leq \lambda_1 L_1 \). As for the second term in (43), we have \( \bigoplus_{j \in L} a_{ij} \leq \lambda \) for \( i \in L_2 \) implying that \( A_{L_2}x_L \leq \lambda_1 L_2 \) and making this term redundant as well.

To deduce the last expression we observed that

\[ (A_{K\ell}^*)A_{K\ell}z_L \leq \lambda_1 L \]  
\[ (A_{K\ell}^*)A_{K\ell}z_L \leq \lambda_1 L \]  

Following Proposition 3.1, the set of solutions to (45) is

\[ \{x_L: x_\ell = \lambda \text{ for } \ell \in L^{>\lambda,L_1}, \, \lambda \leq x_\ell \leq 1 \text{ for } \ell \notin L^{>\lambda,L_1}\}. \]  

However, we also have (39), which is satisfied whenever \( x_\ell = \lambda \) for all \( \ell \in L^{>\lambda,K} \) where

\[ L^{>\lambda,K} = \{\ell \in L: \max_{k \in K}(A_{K\ell})^*A_{K\ell} > \lambda\} \]  

Set of solutions to (45) that satisfy this condition can be written as

\[ S_L(\lambda) = \{x_L: x_\ell = \lambda \text{ for } \ell \in L', \, \lambda \leq x_\ell \leq 1 \text{ for } \ell \notin \tilde{L}\}, \]  

where \( L' = L^{>\lambda,L_1} \cup L^{>\lambda,K} \) and \( \tilde{L} = L \setminus L' \).

Using Proposition 3.2 we can express \( S_L(\lambda) \) and further \( S_K(x_L, \lambda) \) for \( x_L \in S_L(\lambda) \) using (40) as follows:

\[ S_L(\lambda) = \{\lambda_1 L \oplus \Lambda_{L\tilde{L}}z_L: z_L \in \mathbb{B}^{[\tilde{L}]}\}, \]  
\[ S_K(x_L, \lambda) = \{(A_{KK}^*)A_{KL}x_L \oplus (A_{KK})^*z_K: z_K \in \mathbb{B}^{[K]}\} \]  
\[ = \{(A_{KK})^*A_{KL}1_L \oplus (A_{KL})^*A_{KL}z_L \oplus (A_{KK})^*z_K: z_K \in \mathbb{B}^{[K]}\} \]  
\[ = \{(A_{KK})^*A_{KL}1_L \oplus (A_{KL})^*A_{KL}z_L \oplus (A_{KK})^*z_K: z_K \in \mathbb{B}^{[K]}\} \]

To deduce the last expression we observed that

\[ (A_{KK})^*A_{KL}z_L \leq (A_{KK})^*A_{KL}z_L \leq (A_{KK})^*A_{KL}1_L, \]

\[ \leq (A_{KK})^*A_{KL}1_L \]
using the definition \[34\].

If \(L_2 = \emptyset\) then the solution set is just

\[
\{ x : x_L \in S_L(\lambda), x_K \in S_K(x_L, \lambda) \}
\]

and every \(x = (x_L, x_K)\) is determined by parameters \(z_L \in \mathbb{B}^{\tilde{L}}, z_K \in \mathbb{B}^{\tilde{K}}\) with arbitrary values, unlike in the following subcase.

If \(L_2 \neq \emptyset\) then, recalling \[46\] and seeing that \(x_K \leq \lambda \mathbf{1}_K\) is satisfied for all \(x_K \in S_K(x_L, \lambda)\), we need to take the intersection of \[62\] with

\[
\{ x : A_{L_2} x_K = \lambda \mathbf{1}_{L_2} \}.
\]

Expressing \[52\] by means of \[51\], we see that to find this intersection we need to solve

\[
\lambda A_{L_2} (A_{KK})^* A_{KL} \mathbf{1}_L \oplus A_{L_2} (A_{KK})^* A_{KL} A_{LL} z_L \oplus A_{L_2} (A_{KK})^* z_K = \lambda \mathbf{1}_{L_2}
\]

In this case the unknown vectors \(z_L\) and \(z_K\) must be computed, and the system is of the form \(A'z' \oplus b' = \lambda \mathbf{1}\) where

\[
A' = (A_{L_2} (A_{KK})^* A_{KL} A_{LL}, A_{L_2} (A_{KK})^* A_{KL}), b' = \lambda A_{L_2} (A_{KK})^* A_{KL} \mathbf{1}_L, z' = (z_L, z_K)^T.
\]

Observe that all entries of \(A_{KK}^* A_{KL}\) do not exceed \(\lambda\) by \[49\] and the definition of \(\tilde{L}\), and that all coefficients of \((A_{KK})^*_L\) do not exceed \(\lambda\) by \[55\]. Hence the entries of \(A'\) and \(b'\) do not exceed \(\lambda\), and all solutions of \(A'z' \oplus b' = \lambda \mathbf{1}\) can be found as in Subsection \[2.3\] with \(A'\) and \(b'\) instead of \(A\) and \(b\), \(\tilde{N} = \tilde{L} \cup K\) instead of \(N\), and

\[
I_0 = \{ i \in L_2 : A_{ik} (A_{KK})^* A_{KL} \mathbf{1}_L < \lambda \}.
\]

Finding all minimal solutions \((z')^W = (z_K^W, z_L^W)\), which correspond to minimal coverings of \(I_0\) by

\[
C_j = \{ i \in I_0 : (A')_{ij} = \lambda \} \quad \text{for} \quad j \in \tilde{N},
\]

and using Proposition \[2.2\] to express the solution sets \(S^W := \{ z' : (z')^W \leq z' \leq W \}\) algebraically, we can substitute the result back in \[51\] thus obtaining the following description of \((K, L)\)-eigenvectors.

**Theorem 3.1.** Let \(A \in \mathbb{B}^{n \times n}\), \(\lambda \in \mathbb{B}\) and \(K, L\) such that \(K \cup L = N\) and \(K \cap L = \emptyset\) be given. Then \((K, L)\)-eigenvector exists if and only if \[54\] is solvable, which happens if and only if \(I_0 = \cup_{j \in \tilde{N}} C_j\), with \(I_0\) and \(C_j\) defined as in \[56\] and \[57\] and \(\tilde{N} = \tilde{L} \cup K\).

In this case, \(x = (x_K, x_L)^T\) is a \((K, L)\)-eigenvector of \(A\) with eigenvalue \(\lambda\) if
and only if $x_K$ and $x_L$ can be expressed as follows:

$$x_L = \lambda_1 L \oplus \Lambda_{LL}(z^W_L \oplus \Lambda_{LL}v_K)$$
$$x_K = \lambda(A_{KK})^*A_{KL}1_L \oplus (A_{KK})^*A_{LL}(z^W_L \oplus \Lambda_{LL}v_N) \oplus (A_{KK})^*(z^W_K \oplus \Lambda_{KL}v_K) \}
$$

(58)

where $v_K \in \mathbb{B}[\mathcal{N}]$ and $(z^W_K, z^W_L)$ is a minimal solution of (54).

For the purposes of computation note that the minimal solutions of (54), which correspond to minimal coverings of $I_0$ by unions of $C_j$, can be found using the methods described in [9], and that the set of all $(K, L)$-eigenvectors arising from such minimal solution can be efficiently described using (58).

**Example 3.3.** The following three-dimensional example shows how to find the eigenspace for given matrix $A$ and eigenvalue $\lambda$.

Take

$$A = \begin{pmatrix}
.1 & .5 & .7 \\
0 & .4 & .8 \\
.1 & .1 & .5
\end{pmatrix}, \quad \lambda = .5$$

First, let us introduce the notation of the vector with interval entries. The vector where entries are of form $a \leq x_1 \leq b$ and $c \leq x_2 \leq d$ is denoted in further text as

$$x = \begin{pmatrix}
(a, b) \\
(c, d)
\end{pmatrix}.$$

As the eigenspace is a union of background, pure and $(K, L)$ eigenvectors, we are going to compute the solution for each individual case.

For the case of background eigenvectors, first, we have to verify the existence of this eigenvector-type. From (30) we see that this set is nonempty. According to (31), background eigenvectors are all vectors of form

$$x = \begin{pmatrix}
(5, 1) \\
(5, 1) \\
.5
\end{pmatrix}.$$  

(59)

Pure eigenvectors are computed using Corollary 3.1. We find that generators of max-min eigenspace for $x_i \leq \lambda$ are $u = (.1, 1, 1)$, $v = (.4, .4, 1)$ and $w = (.5, .5, 5)$.

When computing $(K, L)$ eigenvectors, first we have to determine particular $(K, L)$ partition. For partition $K = \{1, 3\}$, $L = \{2\}$ we have

$$A_{KK} = \begin{pmatrix}
1 & .7 \\
1 & .5
\end{pmatrix}, A_{KL} = \begin{pmatrix}
.5 \\
.1
\end{pmatrix}, A_{LL} = \begin{pmatrix}
.8 \\
.4
\end{pmatrix}, A_{LL} = (.4),$$

$$(A_{KK})^{*} = \begin{pmatrix}
1 & .7 \\
1 & .5
\end{pmatrix}, (A_{KK})_{\lambda}^{*} = \begin{pmatrix}
1 & .5 \\
1 & .5
\end{pmatrix}.$$
We are solving the system

\[
\begin{pmatrix}
.1 & .7 \\
.1 & .5 \\
\end{pmatrix} \odot \begin{pmatrix} x_1 \\
x_3 \end{pmatrix} \oplus \begin{pmatrix} .5 \\
.1 \end{pmatrix} \odot x_2 = \begin{pmatrix} x_1 \\
x_3 \end{pmatrix}
\] (60)

\[
(0 .8) \odot \begin{pmatrix} x_1 \\
x_3 \end{pmatrix} \oplus (4) \odot x_2 = \lambda
\] (61)

By verifying (39), we find out that (39) holds for all values \(x_2\), and thus a solution to (60) exists.

We can also express the sets \(L_1 = \emptyset, L_2 = \{2\}, L' = \emptyset\) and \(\tilde{L} = \{2\}\). As \(L_2 \neq \emptyset\), we need to find the solution set to (54):

\[
.5 \odot (0 .8) \odot \begin{pmatrix} 1 .7 \\
.1 & 1 \end{pmatrix} \odot (0 .8) \odot \begin{pmatrix} 1 .5 \\
.1 & 1 \end{pmatrix} \odot \begin{pmatrix} z_1 \\
z_3 \end{pmatrix} = .5,
\] (62)

which is the same as

\[
.1 \oplus (1 .1 .5) \odot \begin{pmatrix} z_1 \\
z_2 \\
z_3 \end{pmatrix} = .5
\] (63)

The only minimal (and hence the least) solution to this system is \(z^{(3)} = (0, 0, .5)\), so the solution set is \(\{z: z^{(3)} \leq z \leq 1\}\). It corresponds to

\[
\Lambda^{(3)} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
.5 & .5 & 1 \\
\end{pmatrix}
\]

We can also express \(z_L^{(3)} = z_{L'}^{(3)} = (0), z_K^{(3)} = (0, .5)\).

Following Theorem 3.1 and using equations (58) we obtain

\[
x_2 = .5 \oplus 1 \begin{pmatrix} 0 \oplus (0 1 0) \begin{pmatrix} v_1 \\
v_2 \\
v_3 \end{pmatrix} \end{pmatrix} = .5 \oplus v_2,
\]

\[
\begin{pmatrix} x_1 \\
x_3 \end{pmatrix} = .5 \begin{pmatrix} 1 .7 \\
.1 & 1 \end{pmatrix} \oplus .5 \begin{pmatrix} 1 .7 \\
.1 & 1 \end{pmatrix} \odot (0 .8) \oplus (0 .8) \odot \begin{pmatrix} 1 .5 \\
.1 & 1 \end{pmatrix} \odot \begin{pmatrix} z_1 \\
z_3 \end{pmatrix} = .5 (1 \oplus v_2) \oplus (.5) \odot (.5) \odot (v_1 v_2 v_3) = (.5),
\] (64)
which can be written as

\[
x = \begin{pmatrix} .5 \\ .5 \\ .5 \end{pmatrix}
\]

(65)

For the partition \(K = \{2, 3\}, \, L = \{1\}\) we are solving the system

\[
\begin{pmatrix} .4 & .8 \\ .1 & .5 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ .1 \end{pmatrix} \otimes x_1 = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}
\]

(66)

\[
\begin{pmatrix} .5 & .7 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} .1 \end{pmatrix} \otimes x_1 = \lambda
\]

(67)

Similarly to previous procedures we compute eigenvectors for this partition

\[
x = \begin{pmatrix} (5, 1) \\ .5 \\ .5 \end{pmatrix}
\]

(68)

Considering any other \((K, L)\) partition we will find out that the solution set is empty. The final solution set is then a union of computed parts: background, pure and \((K, L)\) eigenvectors for \(L = \{1\}\) and \(L = \{2\}\), see Figure 4. Note however that all \((K, L)\)-eigenvectors are also background eigenvectors in this case. This is different from Example 3.1 where we have \((K, L)\)-eigenvectors are neither background nor pure.
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