CONSTRUCTING SYNCHRONOUSLY ROTATING DOUBLE WHITE DWARF BINARIES

WESLEY EVEN AND JOEL E. TOHLINE
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA

Received 2009 March 31; accepted 2009 August 21; published 2009 September 16

ABSTRACT

We have developed a self-consistent-field (SCF) technique similar to the one described by Hachisu et al. in 1986 that can be used to construct detailed force-balanced models of synchronously rotating, double white dwarf (DWD) binaries that have a wide range of total masses, mass ratios, and separations. In addition to providing a computational tool that can be used to provide quiet initial starts for dynamical studies of the onset of mass transfer in DWD systems, we show that this SCF technique can be used to construct model sequences that mimic the last portion of the detached inspiral phase of DWD binary evolutions, and semi-detached model sequences that mimic a phase of conservative mass transfer.

Key words: binaries: close – hydrodynamics – methods: numerical – supernovae: general – white dwarfs

Online-only material: color figures

1. INTRODUCTION

As Webbink (1984) and Iben & Tutukov (1984, 1986) have pointed out, double white dwarf (DWD) binaries are expected to be the end product of the thermonuclear evolution of a sizeable fraction of all binary systems. The subset of DWD binaries that are born with orbital periods $P_{orb} \lesssim 5$ hr is of particular astrophysical interest because they will be driven into contact within a Hubble time via the loss of angular momentum through gravitational radiation (Paczynski 1967). Even this short-period component of the DWD population is expected to be quite large in our Galaxy. Sufficiently large that, according to Hils et al. (1990) and Cornish & Larson (2003), DWDs are likely to be a dominant source of background noise for the proposed Laser Interferometer Space Antenna (LISA; Faller & Bender 1984; Evans et al. 1987; Bender 1998) in its lower gravitational-wave frequency band, $f_{GW} = 2/P_{orb} \lesssim 3 \times 10^{-3}$ Hz. Ironically, it is difficult to detect DWDs via traditional observational techniques because the intrinsic photon luminosity of white dwarfs (WDs) is very low.

Broadly speaking, our Galaxy’s DWD binary population should be dominated by systems that are in two distinctly different evolutionary phases: the inspiral phase alluded to above, during which both stars are detached from their respective Roche lobes; and a semi-detached, stable mass transfer phase, during which the less-massive star fills its Roche lobe and is slowly transferring mass to its more-massive companion. To date ~100 detached systems have been identified (Napiwotzki et al. 2001, 2004). Orbital periods and component masses have been determined for approximately one quarter of this sample (Marsh 2000; Maxted et al. 2000; Napiwotzki et al. 2002; Karl et al. 2003; Nelemans et al. 2005). AM CVn is the prototype of semi-detached DWD systems that are undergoing a phase of stable mass transfer (Warner 1995). In the immediate solar neighborhood, ~20 such systems are known (Woudt & Warner 2003; Nelemans 2005; Anderson et al. 2005; Roelofs et al. 2005; Ramsay et al. 2007).

Theoretical arguments suggest that the ultimate fate of a DWD binary will depend on the system’s total mass, $M_{tot}$, and mass ratio $q_0$ at the onset of mass transfer (Marsh et al. 2004; Gokhale et al. 2007; Frank 2008). For example, a DWD will likely only be able to enter an extended phase of stable mass transfer as characterized by AM CVn systems if $q_0$ is less than some critical value $q_{stable}$—where $q_{stable}$ is almost certainly $\lesssim 2/3$ and may be closer to $1/4$. If $q_0 > q_{stable}$, the mass transfer rate is expected to diverge in a finite time, ultimately implying tidal disruption of the donor and/or a catastrophic merger of the two components. Consistent with the theoretical ideas presented by Webbink (1984) and Iben et al. (1996), recent observations strongly suggest that the end product of some DWD mergers is R Coronae Borealis (RCB) stars and hydrogen-deficient carbon (HdC) stars (Clayton et al. 2007). Also, DWD systems with $M_{tot}$ in excess of the Chandrasekhar mass have long been considered likely progenitors of Type Ia supernovae (Webbink 1984; Iben et al. 1996; Livio 2000; Yoon et al. 2007).

Over the past couple of decades, various groups have employed smoothed particle hydrodynamics (SPH) techniques to illustrate the dynamical behavior of DWD systems that violently merge after encountering an unstable mass-transfer event (Benz et al. 1999; Rasio & Shapiro 1995; Segretain et al. 1997; Fryer et al. 1999; Guerrero et al. 2004; Yoon et al. 2007). Typically, initial states for these simulations have been constructed in such a way that the merger process is completed in $\lesssim 5$ orbits after first contact. D’Souza et al. (2006) and Motl et al. (2007) recently employed a grid-based finite-volume (FV) computational fluid dynamic technique to also study the onset and nonlinear development of mass transfer in strongly interacting binary systems. Their primary objective was to ascertain the value of $q_{stable}$ in systems, such as DWDs, that undergo a phase of direct impact accretion following the onset of mass transfer. They were able to follow the evolution of a couple of different systems through $\gtrsim 30$ orbits. Instead of merging, these systems appeared to be entering a long-term phase of stable mass transfer. Presumably these are the types of binary configurations that serve as the progenitors of AM CVn systems.

Motl et al. (2007) found that the outcome of their simulations—for example, whether a system survives the onset of mass transfer or merges—can be sensitive to initial conditions. In particular if, rather than making only marginal contact with its Roche lobe, the donor star is in relatively deep contact at the onset of a simulation, the mass-transfer rate will initially be artificially high and it may be difficult for the system to avoid merger. As a result, a DWD system that should be categorized by numerical simulation as an AM CVn progenitor may be
incorrectly categorized as a likely progenitor of an RCB star or a Type Ia supernova. It is therefore important to start such simulations from initial states that are very quiet—that is, from initial configurations that are in detailed force balance throughout—and to perform each simulation with a grid (or particle) resolution that is sufficient to resolve marginal, or at least very shallow, contact between the donor and its Roche lobe. In two very recent reports, Fryer & Diehl (2008) and Dan et al. (2008) have confirmed this finding. Both groups have shown that SPH techniques also can be used to follow stable mass-transfer events through $\gtrsim 30$ orbits if sufficiently quiet initial states are used and if the simulations are carried out using a sufficiently large number of SPH particles.

Quiet initial states were constructed for the hydrodynamic simulations reported by D’Souza et al. (2006) and Motl et al. (2007) using a self-consistent-field (SCF) technique very similar to the one described by Hachisu (1986b). However, these initial models were constructed using a polytropic equation of state. Here we use a modified version of the SCF technique developed by Hachisu et al. (1986b) for the zero-temperature white dwarf (ZTWD) equation of state derived by Chandrasekhar (1935). We show how this SCF technique can be used to construct synchronously rotating, unequal-mass DWD binaries in which the less-massive (donor) star is in marginal contact with its Roche lobe and thereby provides excellent initial models for dynamical studies of interacting DWD systems that have a realistic equation of state.

To demonstrate the broad utility of this SCF technique, we construct a sequence of detached, synchronously rotating binaries of varying separation, but fixed mass ratio and constant total mass, to mimic the last portion of the inspiral phase of evolution of DWD binary systems. This enables us to determine, for example, the degree to which the functional dependence of the orbital frequency on orbital separation $\Omega(a)$ deviates from a pure Keplerian behavior. This type of sequence also can be used to follow stable mass-transfer events through $\gtrsim 30$ orbits if sufficiently quiet initial states are used and if the simulations are carried out using a sufficiently large number of SPH particles.

In the ZTWD equation of state (Chandrasekhar 1935, 1967; Hachisu 1986a), the electron degeneracy pressure $P$ varies with the mass density $\rho$ according to the relation

$$P = A[x(2x^2 - 3)(x^2 + 1)^{1/2} + 3\sinh^{-1} x],$$

(1)

where the dimensionless parameter,

$$x \equiv \left(\frac{\rho}{B}\right)^{1/3},$$

(2)

and the constants $A$ and $B$ are (see the Appendix and Table A1 for elaboration),

$$A = \frac{\pi m_e^4 e^5}{3 h^3} = 6.00228 \times 10^{22} \text{ dynes cm}^{-2},$$

(3)

$$B = \frac{8\pi m_p}{3} \left(\frac{m_e c}{\hbar}\right)^3 = 9.81011 \times 10^5 \text{ g cm}^{-3}.\,$$

(4)

According to Chandrasekhar (1967; see again our Appendix), a natural length scale associated with models of ZTWDs is

$$\mu_e \ell_1 = \left(\frac{2A}{\pi G}\right)^{1/2} \frac{\mu_e}{B} = 7.71395 \times 10^8 \text{ cm} = 0.0111 R_\odot,$$

(5)

and the associated limiting WD mass is

$$\mu_e^2 M_{\text{ch}} = 4\pi (2.01824) \left(\frac{2A}{\pi G}\right)^{3/2} \left(\frac{\mu_e}{B}\right)^2 = 1.14205 \times 10^{34} \text{ g} = 5.742 M_\odot.$$  

(6)

Throughout this work, we will assume that the average ratio of nucleons to electrons throughout each WD is $\mu_e = 2$. Hence, $B = 1.96202 \times 10^6 \text{ g cm}^{-3}$, $\ell_1 = 5.55 \times 10^{-3} R_\odot$, and $M_{\text{ch}} = 1.435 M_\odot$.

In terms of the enthalpy of the gas,$^1$

$$H = \int \frac{dP}{\rho},$$

(7)

the ZTWD equation of state shown in Equation (1) can also be written in the form

$$H = \frac{8A}{B} [x^2 + 1]^{1/2}.$$

(8)

Inverting this gives the dependence of $\rho$ on $H$, namely,

$$\rho B = x^3 = \left[\left(\frac{BH}{8A}\right)^{2} - 1\right]^{3/2}.$$

(9)

$^1$ As defined here, $H$ is actually enthalpy per unit mass.
Figure 1. Mass–radius relationship is shown for spherical stars with our adopted ZTWD equation of state. Diamonds represent results derived using our three-dimensional SCF scheme applied to nonrotating, isolated configurations (see Table 2); asterisks show previously published results for the same equation of state taken from Hachisu (1986a); the solid curve shows the analytic mass–radius relation, Equation (A14), derived by Nauenberg (1972).

As a foundation for both constructing and understanding the structures of the synchronously rotating and tidally distorted stars in ZTWD binary systems, we have regenerated Chandrasekhar’s spherical WD sequence using a variation of the SCF technique outlined by Hachisu (1986a). As discussed in Section 3.1, Table 2 details key properties of the ZTWD structures that lie along this spherical model sequence. The WD mass–radius relationship that is derived from models along this sequence is illustrated by the diamonds in Figure 1. For comparison, results from the published spherical sequence of Hachisu (1986a) are represented in this figure by asterisks and the solid curve shows the approximate, analytic mass–radius relationship, Equation (A14), derived for ZTWD stars by Nauenberg (1972). (As explained in the Appendix, it is more appropriate for us to compare our results to this “Nauenberg” mass–radius relation than to the more widely used “Eggleton” mass–radius relation, shown in Equation (A16).)

### Table 1
Convergence of the SCF Method: Binary Model B3

| $N_{a}$ | $N_{b}$ | $N_{c}$ | $\Delta$ | VE |
|---------|---------|---------|---------|----|
| 64      | 128     | 33      | $1.0 \times 10^{-2}$ | 4.5 $\times 10^{-1}$ |
|         |         |         | $1.0 \times 10^{-3}$ | 2.6 $\times 10^{-3}$ |
|         |         |         | $1.4 \times 10^{-4}$ | 2.2 $\times 10^{-4}$ |
| 128     | 256     | 65      | $1.0 \times 10^{-2}$ | 4.0 $\times 10^{-3}$ |
|         |         |         | $1.0 \times 10^{-3}$ | 9.1 $\times 10^{-4}$ |
|         |         |         | $1.0 \times 10^{-4}$ | 5.7 $\times 10^{-4}$ |
|         |         |         | $3.5 \times 10^{-5}$ | 5.4 $\times 10^{-4}$ |

### Table 2
Sequence of Single, Nonrotating ZTWDs

| $(M, R)$ | $\rho_{\text{max}}$ | $x_{\text{max}}$ | $J$ | $k$ | VE |
|---------|---------------------|------------------|-----|-----|----|
| $(10^8 \text{ cm})$ | $(10^6 \text{ g cm}^{-3})$ | | $(10^{50} \text{ g cm}^2)$ | | | |
| 0.0844  | 19.7673             | $3.1623 \times 10^6$ | 0.2526 | 1.3317 | 0.2036 | 1.4 $\times 10^{-5}$ |
| 0.1113  | 17.9368             | $5.6234 \times 10^6$ | 0.3960 | 1.4422 | 0.2031 | 6.1 $\times 10^{-5}$ |
| 0.1460  | 16.2672             | 1.0000 $\times 10^5$ | 0.3708 | 1.5508 | 0.2024 | 1.4 $\times 10^{-5}$ |
| 0.1903  | 14.7421             | 1.7783 $\times 10^5$ | 0.4492 | 1.6517 | 0.2015 | 5.8 $\times 10^{-5}$ |
| 0.2457  | 13.3464             | $3.1623 \times 10^5$ | 0.5442 | 1.7360 | 0.2001 | 1.4 $\times 10^{-5}$ |
| 0.3134  | 12.0666             | $5.6234 \times 10^5$ | 0.6593 | 1.7936 | 0.1983 | 5.8 $\times 10^{-5}$ |
| 0.3938  | 10.8906             | 1.0000 $\times 10^5$ | 0.7988 | 1.8133 | 0.1958 | 5.9 $\times 10^{-5}$ |
| 0.4859  | 9.0878              | 1.7783 $\times 10^5$ | 0.9678 | 1.7852 | 0.1926 | 6.0 $\times 10^{-5}$ |
| 0.5873  | 8.0907              | $3.1623 \times 10^5$ | 1.1725 | 1.7051 | 0.1887 | 6.1 $\times 10^{-5}$ |
| 0.6942  | 7.8896              | $5.6234 \times 10^5$ | 1.4205 | 1.5754 | 0.1839 | 5.8 $\times 10^{-5}$ |
| 0.8018  | 7.0419              | 1.0000 $\times 10^5$ | 1.7209 | 1.4058 | 0.1783 | 7.6 $\times 10^{-5}$ |
| 0.9058  | 6.2628              | 1.7783 $\times 10^5$ | 2.0850 | 1.2122 | 0.1721 | 6.6 $\times 10^{-5}$ |
| 1.0022  | 5.5487              | $3.1623 \times 10^5$ | 2.5260 | 1.0111 | 0.1653 | 5.1 $\times 10^{-5}$ |
| 1.0882  | 4.8961              | $5.6234 \times 10^5$ | 3.0603 | 0.8176 | 0.1581 | 6.3 $\times 10^{-5}$ |
| 1.1624  | 4.3023              | 1.0000 $\times 10^5$ | 3.7076 | 0.6429 | 0.1507 | 6.3 $\times 10^{-5}$ |
| 1.2246  | 3.7643              | 1.7783 $\times 10^5$ | 4.4919 | 0.4929 | 0.1433 | 7.0 $\times 10^{-5}$ |
| 1.2753  | 3.2793              | $3.1623 \times 10^5$ | 5.4421 | 0.3698 | 0.1368 | 7.4 $\times 10^{-5}$ |
| 1.3155  | 2.8443              | $5.6234 \times 10^5$ | 6.5932 | 0.2723 | 0.1290 | 9.0 $\times 10^{-5}$ |
| 1.3469  | 2.4560              | 1.0000 $\times 10^5$ | 7.9879 | 0.1972 | 0.1225 | 1.1 $\times 10^{-4}$ |
| 1.3708  | 2.1116              | 1.7783 $\times 10^5$ | 9.6775 | 0.1410 | 0.1164 | 1.2 $\times 10^{-4}$ |
| 1.3887  | 1.8078              | $3.1623 \times 10^5$ | 11.7246 | 0.0918 | 0.1108 | 1.3 $\times 10^{-4}$ |
| 1.4020  | 1.5414              | $5.6234 \times 10^5$ | 14.2047 | 0.0699 | 0.1058 | 1.4 $\times 10^{-4}$ |
| 1.4116  | 1.3092              | 1.0000 $\times 10^5$ | 17.2094 | 0.0486 | 0.1013 | 1.4 $\times 10^{-4}$ |

Table 3
Selected Single, Nonrotating ZTWDs

| $M$ | $R$ | $\rho_{\text{max}}$ | $x_{\text{max}}$ | $k$ | $W$ | $U$ | $\Pi$ |
|-----|-----|--------------------|------------------|-----|-----|-----|------|
| $(M_{\odot})$ | $(10^8 \text{ cm})$ | $(10^6 \text{ g cm}^{-3})$ | | | $(10^{50} \text{ erg})$ | $(10^{50} \text{ erg})$ | |
In Figure 2, the more-massive, primary star is shown on the left and the less-massive, secondary star is on the right; the centers of the stars are located at distances \( \sigma_1 \) and \( \sigma_2 \), respectively, from the center of mass of the binary system and the binary separation \( a = \sigma_1 + \sigma_2 \). Because we are using a ZTWD equation of state, the central density \( \rho_{\text{max}} \) and radius \( R_c \) of each star \( (i = 1, 2) \) cannot be specified independently of each star’s chosen mass. As an initial guess for our SCF technique, the values of \( \rho_{\text{max}}^{i=1}, \rho_{\text{max}}^{i=2}, R_1, \) and \( R_2 \) are drawn from Table 2, that is, they are given by values that correspond to spherical ZTWDs having masses \( M_1 \) and \( M_2 \).

For various values of the three principal system parameters \( M_1, M_2, \) and \( a \), our specific aim is to determine in a self-consistent fashion on a cylindrical coordinate mesh \( (\sigma, \vartheta, Z) \), the values and spatial distribution of the scalar fields \( \rho(\sigma, \vartheta, Z), H(\sigma, \vartheta, Z), \) and \( P(\sigma, \vartheta, Z) \), for both stars \( (i = 1, 2) \) in synchronously rotating, ZTWD binaries. Following Hachisu (1986a, 1986b) and Hachisu et al. (1986a, 1986b), in order to construct these desired binary configurations we need to solve the following five equations simultaneously:

\[
\nabla^2 \Phi(\sigma, \vartheta, z) = 4\pi G \sum_i \rho^i(\sigma, \vartheta, z), \tag{10}
\]

\[
C^i = H^i(\sigma, \vartheta, z) + \Phi(\sigma, \vartheta, z) - \frac{1}{2} \Omega^2 \sigma^2, \tag{11}
\]

\[
\rho^i(\sigma, \vartheta, z) = B[x^i(\sigma, \vartheta, z)]^3 = B \left[ \frac{B}{8A} H^i(\sigma, \vartheta, z) \right]^2 - 1 \right]^{3/2}, \tag{12}
\]

where \( \Phi(\sigma, \vartheta, Z) \) is the Newtonian gravitational potential of the combined stellar system, and \( C^i=1 \) and \( C^i=2 \) are constants that specify the conditions of the Bernoulli flow inside each star.

### 2.3. Solution Strategy

Our solution strategy closely follows the methods described in Hachisu (1986b) and Hachisu et al. (1986b) so only the differences between our methods will be given in detail here. The variables with a carat (\( ^* \)) above them are the dimensionless version of the variables as defined in Equations (22)–(27) of Hachisu (1986b).

To begin an SCF iteration two spherical stars are initially placed on the computational grid in such a way that the center of mass of the system falls at the origin of the coordinate system and the outer edge of the secondary star (point \( O_a \) in Figure 2) is at \( \sigma = 1 \). The centers of the stars are therefore located, respectively, at

\[
\sigma_1 = \frac{\sigma_1}{\sigma_a} = \frac{q}{1 + \ell(1 + q)}, \tag{13}
\]

\[
\sigma_2 = \frac{\sigma_2}{\sigma_a} = \frac{1}{1 + \ell(1 + q)}, \tag{14}
\]

where the dimensionless ratio \( \ell \equiv R_2/a \) is known once \( M_2 \) and \( a \) have been chosen. These two expressions make sense because \( \sigma_i = (\sigma_i + a\ell) = [(\sigma_i + a\ell)]^\ell \) and, for a point-mass binary whose center of mass is at the origin of the grid, \( \sigma_1 = q \sigma_2 \). With \( \rho^i \) defined everywhere on the grid, \( \Phi(\sigma, \vartheta, z) \) is calculated via Equation (10). In this work the boundary values for \( \Phi \) are calculated using the compact cylindrical Green’s function expansion described in Cohl & Tohline (1999), and the values of the potential throughout the interior volume of the

### 2.2. Binary System Geometry and Governing Equations

Our objective is to determine the 3D structure of a pair of ZTWD stars that are in a tight, circular orbit under the condition that both stars are synchronously rotating with the binary orbital frequency \( \Omega \). We begin by specifying the masses \( M_1 \) and \( M_2 \) of the primary and secondary stars, respectively, such that \( M_2 \leq M_1 \). Alternatively, we can specify the total system mass \( M_{\text{tot}} \equiv M_1 + M_2 \) and the system mass ratio \( q \equiv M_2/M_1 \leq 1 \), in which case,

\[
M_1 = \left( \frac{1}{1 + q} \right) M_{\text{tot}},
\]

\[
M_2 = \left( \frac{q}{1 + q} \right) M_{\text{tot}}.
\]

Figure 2 shows a slice through the equatorial plane of such a system under the assumption that both stars are spherically symmetric. For our final equilibrium models in which the effects of tidal and rotational distortions are taken into account in a fully self-consistent fashion, this figure provides only a schematic illustration of the binary system’s equatorial-plane structure. However, it provides an accurate depiction of the equatorial-plane structure of the initial stellar models that are fed into our iterative SCF scheme (see Section 2.3).
computational grid are calculated using the Krylov subspace methods provided by the PETSc software library (Balay et al. 2004).

During each iteration, the interior structure of the secondary star is calculated using the same strategy as outlined in Section 2b of Hachisu et al. (1986b). To update the structure of the primary star, however, the location of $H_{max}^{i=1}$ is used instead of specifying the inner edge of the star, as was done in Hachisu et al. (1986b). This choice eliminates the extra iterative steps that were needed when the third condition was specified at the inner edge of the primary star. Using Equation (8), we determine the value of the normalized enthalpy at the center of the primary star from the values of $x_{max}$ selected for both stars and the value of $H_{max}^{i=2}$ just derived for the secondary star. Specifically, we set

$$H_{max}^{i=1} = H_{max}^{i=2} \left\{ \left( \frac{x_{max}^{i=1}}{x_{max}^{i=2}} \right)^{2} + 1 \right\}^{1/2}.$$  \hspace{1cm} (15)

Then we determine the value of the Bernoulli constant inside the primary star by examining the distribution of the variable $\hat{F}$ as defined in Equation (48) of Hachisu (1986a), that is,

$$\hat{F}^{i} = -\Phi + \frac{1}{2} \Omega^{i} \sigma^{2} = (\hat{H}^{i} - \hat{C}^{i}).$$  \hspace{1cm} (16)

In the vicinity of the original center of the primary star, that is, in the vicinity of point $O_{1}$ as illustrated in Figure 2, the function $\hat{F}$ should exhibit a local maximum. We associate the location of this local maximum with the updated position of point $O_{1}$ and we set $\hat{F}_{max}^{i=1}$ equal to the value of the function at this local maximum. We therefore deduce from Equation (16) that

$$\hat{C}_{i=1}^{i=1} = \hat{H}_{max}^{i=1} - \hat{F}_{max}^{i=1},$$  \hspace{1cm} (17)

With this constant in hand, the normalized enthalpy throughout the primary star can be determined via the expression

$$\hat{H}_{i=1}^{i=1} = \hat{C}_{i=1}^{i=1} + \hat{F}_{i=1}^{i=1},$$  \hspace{1cm} (18)

and we obtain an updated “guess” for the normalized density distribution inside the primary star via the expression

$$\hat{\rho}_{i=1}^{i=1} = \frac{1}{(x_{max}^{i=1})^{2}} \left\{ \frac{\hat{H}_{i=1}^{i=1}}{\hat{H}_{max}^{i=1}} \right\}^{1/2} \left[ \left( \frac{x_{max}^{i=1}}{x_{max}^{i=2}} \right)^{2} + 1 \right]^{3/2}.$$  \hspace{1cm} (19)

\subsection{2.4. Global Properties and Convergence}

Our iterative scheme is judged to be operating well if various calculated model parameters—such as the dimensionless stellar masses $\hat{M}$ and Bernoulli constants $\hat{C}$—converge toward well-defined values. We also have found it useful to track the
convergence of various global energy parameters. Specifically, at the end of each iteration cycle we calculate the dimensionless rotational kinetic energy \( \hat{K} \), gravitational potential energy \( \hat{W} \), total internal energy \( \hat{U} \) (see, for example, Equation (75) in Chapter XI of Chandrasekhar 1967), and globally averaged pressure \( \hat{P} \) of the model, defined as follows:

\[
\hat{K} = \int \frac{1}{2} \hat{\Omega}^2 \hat{\sigma}^2 \hat{\rho} d\hat{V},
\]

\[
\hat{W} = \int \frac{1}{2} \hat{\phi} \hat{\rho} d\hat{V},
\]

\[
\hat{U} \equiv \int \left[ \left( \hat{H} - \frac{8A}{B} \right) \hat{\rho} - \hat{P} \right] d\hat{V},
\]

where \( d\hat{V} = \hat{\sigma} d\hat{\sigma} d\hat{\phi} d\hat{d} \) is the dimensionless differential volume element on our cylindrical grid. Then the system’s dimensionless total energy is given by the sum,

\[
\hat{E}_{tot} \equiv \frac{E_{tot}}{G \rho_{\text{max}}^2 \hat{\sigma}^2} = \hat{K} + \hat{W} + \hat{U},
\]

and, if the model has converged to a proper equilibrium state, according to the virial theorem we should expect

\[
2\hat{K} + \hat{W} + 3\hat{\Pi} = 0.
\]

In general, at each iteration step the condition of virial equilibrium, Equation (25), will not be satisfied, but if our iteration scheme is well behaved, convergence toward the virial condition should be achieved. With this in mind, we have found that the virial error (VE),

\[
\text{VE} \equiv \left| \frac{2\hat{K} + \hat{W} + 3\hat{\Pi}}{\hat{W}} \right|,
\]

provides a meaningful measure of the quality of each model.

We declare that satisfactory convergence to a given model has been achieved when the absolute value of the fractional change between iterations has dropped below a specified convergence criterion, \( \delta \sim 10^{-4} \), for all of the following quantities: \( \hat{C}_i \), \( \hat{M}_i \), \( \Omega \), \( \hat{K} \), \( \hat{W} \), \( \hat{\Pi} \), and the physical value of \( \sigma_x \). In addition, the converged model is judged to be a good equilibrium state if VE is sufficiently small. Table 1 illustrates how we were able to achieve a lower VE and, hence, a more accurate representation of an equilibrium configuration, by improving the grid resolution and/or by specifying a tighter convergence criterion. Specifically, the table shows that as we were constructing binary model B3 (see discussion associated with Table 6, below) we were able to push the VE down from a value \( \sim 5 \times 10^{-3} \) to a value \( \sim 5 \times 10^{-4} \) by increasing the grid resolution from (64, 128, 33) to (128, 256, 65) zones in \( (\hat{\sigma}, \hat{\phi}, \hat{z}) \) and by pushing \( \delta \) from \( 10^{-2} \) to \( 3.5 \times 10^{-5} \).

After the SCF code has converged to the desired equilibrium model, the various dimensionless variables are converted back to proper physical units following, for example, the scalings presented in Equations (22)–(27) of Hachisu (1986a). We note in particular that the value of the scale length \( \sigma_x = \sigma_{\text{max}} \) is obtained by evaluating a dimensionless version of Equation (8) for the secondary star in combination with Equation (27) from Hachisu (1986a), which gives

\[
\sigma_x = \left[ \frac{8A}{B \rho_{\text{max}}^2} \right]^{1/2} \left( \hat{H}_{\text{max}} - \hat{W}_{\text{max}} \right)^{-1/2} \left( \hat{x}_{\text{max}^2} + 1 \right)^{1/4}.
\]

In addition to the physical variables already identified, for each converged model we have found it useful to evaluate the system’s total angular momentum,

\[
J_{\text{tot}} \equiv \int \hat{\sigma} \hat{\Omega} \hat{\rho} d\hat{V},
\]
as well as the spin angular momentum of each component star, \( J_{\text{spin}}^i \), and each star’s Roche-lobe filling factor, \( f_{\text{KL}}^i \). As with the determination of quantities such as \( M^i \) and \( R^i \), these latter two quantities are obtained by performing volume integrals over appropriate subdomains of the computational grid, determined as follows. Let the origin of a Cartesian grid coincide with the center of mass of the binary system and align the x-axis of that grid with the line that connects the centers of the two stars as illustrated in Figure 2. Between points \( O_1 \) and \( O_2 \) along this axis, the effective potential,

\[
\Phi_{\text{eff}}(x) \equiv \Phi(x) - \frac{1}{2} \Omega^2 x^2,
\]

will exhibit a maximum at position \( x_{L1} \) associated with the inner “L1” Lagrange point. We define the subdomain \( D_{x_{L1}}^i \) as the volume of the grid for which \( x \equiv r \cos \theta \geq x_{L1} \) and \( r > 0 \), that is, the region occupied by the secondary star; and we define the subdomain \( D_{x_{L1}}^{<i} \) as the volume of the grid for which \( x < x_{L1} \) and \( r > 0 \), that is, the region occupied by the primary star. Then the mass of each star is determined by the integral,

\[
M^i = \int_{D_{x_{L1}}^i} \rho^i dV, \tag{30}
\]
As mentioned earlier, we initially used a simplified version of our SCF code to construct a large number of single, nonrotating WDs in order to compare our solutions with previous results (see Figure 1) and to provide initial guesses for the density distributions inside both stars in each binary system. Table 2 details the properties of single, nonrotating WDs that have central densities ranging from $10^{2.5}$ g cm$^{-3}$ to $10^{10}$ g cm$^{-3}$ as determined from our model calculations; the 23 selected models are equally spaced in units of log $\rho_{\text{max}}$. These spherical models were constructed on a uniform cylindrical mesh with resolution (128, 128, 128) in $(\tilde{\sigma}, \tilde{\vartheta}, \tilde{z})$ using a convergence criterion $\delta = 10^{-4}$. For each converged model, the first six columns of Table 2 list, respectively, the star’s mass $M$ in solar masses, radius $R$ in units of $10^8$ cm, central density $\rho_{\text{max}}$ in g cm$^{-3}$, corresponding value of $x_{\text{max}} = (\rho_{\text{max}}/B)^{1/3}$, moment of inertia,

$$I = \int \sigma^2 \rho dV,$$  

(36)

in units of $10^{40}$ g cm$^2$, and the radius of gyration, $k \equiv I/(MR^2)$. As shown in the last column of Table 2, a typical VE for these converged models was $10^{-7}$ to $10^{-6}$. The values tabulated for the radius of gyration vary smoothly from $k = 0.2036$ for $M = 0.0844 M_\odot$ to $k = 0.1013$ for $M = 1.4081 M_\odot$. This is consistent with our understanding that low-mass WDs have structures similar to $n = 3/2$ polytropes for which $k = 0.205$ (Ruciński 1988), while high-mass WDs display structures similar to $n = 3$ polytropes for which $k = 0.0758$ (Ruciński 1988). Our values of $k$ over this range of stellar masses are also consistent with the analytic function for $k(M)$ that Marsh et al. (2004) fit through similar spherical model data. Knowledge of the radius of gyration of these spherical ZTWD models has assisted us in analyzing the tidally distorted structures that arise in our models of synchronously rotating WDs in close binary systems (see further discussion, below).

Using this same three-dimensional, cylindrical coordinate grid we constructed nonrotating models with central densities above $10^{10}$ g cm$^{-3}$, that is, with masses above $1.4 M_\odot$. We have not included these higher mass models in Table 2 or Figure 1, however, because they did not converge to satisfactorily accurate structures. In particular, as the mass was steadily increased above $1.4 M_\odot$, the models converged to structures with steadily increasing (rather than decreasing) values of $k$. By contrast, models constructed using a one-dimensional spherical code with much higher spatial resolution displayed values of $k$ that decreased steadily to a value of 0.0755 at masses approaching $M_\odot$. If desired, the three-dimensional computational grid resolution could be increased to produce more accurate models of the WD structure near the Chandrasekhar mass limit.

### 3.2. White Dwarf Binary Sequences

The slow inspiral evolution of a DWD binary can be mimicked by constructing a sequence of detached binaries having

$$f_{RL} = \frac{V^4_i}{V^4_{RL}},$$  

(35)

and each star’s Roche-lobe filling factor is obtained from the ratio

$$I = \int \sigma^2 \rho dV,$$  

(36)

in units of $10^{40}$ g cm$^2$, and the radius of gyration, $k \equiv I/(MR^2)$. As shown in the last column of Table 2, a typical VE for these converged models was $10^{-7}$ to $10^{-6}$. The values tabulated for the radius of gyration vary smoothly from $k = 0.2036$ for $M = 0.0844 M_\odot$ to $k = 0.1013$ for $M = 1.4081 M_\odot$. This is consistent with our understanding that low-mass WDs have structures similar to $n = 3/2$ polytropes for which $k = 0.205$ (Ruciński 1988), while high-mass WDs display structures similar to $n = 3$ polytropes for which $k = 0.0758$ (Ruciński 1988). Our values of $k$ over this range of stellar masses are also consistent with the analytic function for $k(M)$ that Marsh et al. (2004) fit through similar spherical model data. Knowledge of the radius of gyration of these spherical ZTWD models has assisted us in analyzing the tidally distorted structures that arise in our models of synchronously rotating WDs in close binary systems (see further discussion, below).

Using this same three-dimensional, cylindrical coordinate grid we constructed nonrotating models with central densities above $10^{10}$ g cm$^{-3}$, that is, with masses above $1.4 M_\odot$. We have not included these higher mass models in Table 2 or Figure 1, however, because they did not converge to satisfactorily accurate structures. In particular, as the mass was steadily increased above $1.4 M_\odot$, the models converged to structures with steadily increasing (rather than decreasing) values of $k$. By contrast, models constructed using a one-dimensional spherical code with much higher spatial resolution displayed values of $k$ that decreased steadily to a value of 0.0755 at masses approaching $M_\odot$. If desired, the three-dimensional computational grid resolution could be increased to produce more accurate models of the WD structure near the Chandrasekhar mass limit.

### 3.2. White Dwarf Binary Sequences

The slow inspiral evolution of a DWD binary can be mimicked by constructing a sequence of detached binaries having
fixed $M_{\text{tot}}$ and fixed $q$ but varying separation, down to the separation at which the less-massive star first makes contact with its Roche lobe. In an effort to illustrate the capabilities of our code, we have constructed three binary sequences having the same total mass—namely, $M_{\text{tot}} = 1.5 M_\odot$—but three separate mass ratios. Specifically, sequence “A” has $q = 1$, sequence “B” has $q = 2/3$, and sequence “C” has $q = 1/2$. As detailed in Table 3, spherical models were constructed with the desired primary and secondary masses for these three sequences—specifically, $M = 0.5 M_\odot, 0.6 M_\odot, 0.75 M_\odot, 0.9 M_\odot,$ and $1.0 M_\odot$—to provide good “guesses” for the initial binary star density distributions to start each SCF iteration. In addition to listing the values of $M, R, \rho_{\text{max}}, s_{\text{max}}$, and $k$ for each of these converged spherical models, as was done for a wider range of spherical models in Table 2, Table 3 also lists values for the global energies $W, U$, and $\Pi$ in units of $10^{50} \text{ erg}$.

Along each sequence, all the binary models were constructed using a uniform cylindrical grid with (128, 256, 65) zones in $(\hat{\phi}, \hat{\theta}, \hat{z})$; by implementing reflection symmetry through the equatorial plane, only half as many zones were needed in the vertical direction as in the radial direction to achieve the same resolution in both. No additional symmetries were assumed in constructing the sequence, although, for the models shown here, symmetry through the $x-z$ plane also could have been implemented for additional savings. The convergence criterion was set to $\delta = 2.5 \times 10^{-4}$; in most models, $\Omega$ was the last variable to converge to this desired level. We note that, because the same number of grid zones was used for each model and each binary was scaled to fit entirely within the grid, the effective resolution of each star decreased as the binary separation increased along each sequence.

Two tables have been produced for each DWD inspiral sequence in order to detail the properties of the models that lie along each sequence. For sequence “A” ($q = 1$), for example, Table 4 lists the values of six global binary system parameters ($a, \Omega, M_\text{ini}, q, J_\text{tot}, L_\text{tot}$) and the VE obtained for 35 models (numbered A1 through A35) whose binary separations vary from 2.0956 $\times 10^{10}$ cm at contact (model A1) to 3.0911 $\times 10^{10}$ cm (model A35). For this same group of models, Table 5 lists calculated values of five parameters ($M_1, R_1, \rho_{\text{max}}', f_{\text{RL}}', J_{\text{spin}}'$) for the individual stellar components ($i = 1, 2$). Tables 6 and 7 provide the same detailed information for models along sequence “B” ($q = 2/3$), and Tables 8 and 9 provide this information for models along sequence “C” ($q = 1/2$).

The equatorial-plane density distributions displayed in Figures 3–5 illustrate the degree to which both WD components are distorted by tides for various binary separations along each sequence. Labels in the upper right-hand corner of each figure
Figure 3. Density contours in the equatorial plane are shown for eight separate ZTWD binary models with increasing separation along inspiral sequence “A” ($M_{\text{tot}} = 1.5 M_\odot; q = 1$). Labels in the upper right-hand corner of each panel identify each model by number according to its corresponding position along the sequence as itemized in Tables 4 and 5.
(A color version of this figure is available in the online journal.)

Figure 4. Density contours in the equatorial plane are shown for eight separate ZTWD binary models with increasing separation along inspiral sequence “B” ($M_{\text{tot}} = 1.5 M_\odot; q = 2/3$). Labels in the upper right-hand corner of each panel identify each model by number according to its corresponding position along the sequence as itemized in Tables 6 and 7.
(A color version of this figure is available in the online journal.)

Figure 5. Density contours in the equatorial plane are shown for eight separate ZTWD binary models with increasing separation along inspiral sequence “C” ($M_{\text{tot}} = 1.5 M_\odot; q = 1/2$). Labels in the upper right-hand corner of each panel identify each model by number according to its corresponding position along the sequence as itemized in Tables 8 and 9.
(A color version of this figure is available in the online journal.)
Figure 6. (Top panel) Normalized total angular momentum $J_{\text{tot}}/J_{\text{norm}}$, (second panel) normalized total energy $E_{\text{tot}}/E_{\text{norm}}$, (third panel) the Roche-lobe filling factor, $f_{RL}$, for the secondary (asterisks) and primary (diamonds) stars, and (bottom panel) the normalized orbital angular velocity, $\Omega/\Omega_K$, are plotted as a function of binary separation for models A1 through A35 along inspiral sequence “A” ($M_{\text{tot}} = 1.5 M_\odot; q = 1$). Data for the individual models are drawn from Tables 4 and 5; the separation $a$ is labeled in units of 10^9 cm along the bottom axis and as a ratio to $R_{0.75}$ along the top axis. The solid curves in the top two panels display the analytic functions $J_{pm}(a)/J_{\text{norm}}$ and $E_{pm}(a)/E_{\text{norm}}$ given, respectively, by Equations (40) and (44) for a “point-mass” sequence of the specified total mass and mass ratio; and the dot-dashed curves display the analytic functions $J_{ss}(a)/J_{\text{norm}}$ and $E_{ss}(a)/E_{\text{norm}}$ appropriate for a “spinning spheres” sequence given, respectively, by Equations (41) and (45).

Along sequence “A” (Figure 3), both components of the binary system are of equal size and display identical degrees of tidal distortion because the mass ratio $q = 1$. Along sequences “B” and “C” (Figures 4 and 5, respectively), however, the primary star (on the left in each figure panel) is noticeably smaller and less distorted than the secondary star.

Figure 6 has been constructed from the data detailed in Tables 4 and 5 for binary sequence “A”. Specifically, the diamond symbols in the top two panels and in the bottom panel of this figure show, respectively, how the binary system’s total angular momentum, $J_{\text{tot}}$, total energy, $E_{\text{tot}}$, and orbital angular velocity, $\Omega$, vary with orbital separation along this sequence; and the third panel from the top shows how the Roche-lobe filling factor, $f_{RL}$, varies with orbital separation for both the primary star (diamonds) and the secondary star (asterisks). Figures 7 and 8 have been similarly constructed from the data detailed, respectively, in Tables 6 and 7, and in Tables 8 and 9.

Following the lead of New & Tohline (1997), in constructing Figures 6–8 we have normalized our tabulated values of $J_{\text{tot}}$ and $E_{\text{tot}}$ to the quantities,

$$J_{\text{norm}} \equiv (GM_{0.75}^2 R_{0.75})^{1/2} = 4.0735 \times 10^{50} \text{ g cm}^2 \text{ s}^{-1}, \quad (37)$$

$$E_{\text{norm}} \equiv \frac{GM_{0.75}^2}{R_{0.75}} = 2.0119 \times 10^{50} \text{ erg}, \quad (38)$$

where $R_{0.75} = 7.4244 \times 10^8 \text{ cm}$ is the radius of a spherical ZTWD whose mass is $M_{0.75} = 0.7522 M_\odot$ as tabulated in

Figure 7. Same as Figure 6 but for models B1 through B49 along the inspiral sequence “B” ($M_{\text{tot}} = 1.5 M_\odot; q = 2/3$), as tabulated in Tables 6 and 7; along the top axis, the separation $a$ is labeled as a ratio to $R_{0.60}$.

Figure 8. Same as Figure 6 but for models C1 through C36 along the inspiral sequence “C” ($M_{\text{tot}} = 1.5 M_\odot; q = 1/2$), as tabulated in Tables 8 and 9; along the top axis, the separation $a$ is labeled as a ratio to $R_{0.50}$. 

$\text{E}_{\text{norm}} \equiv \frac{GM_{0.75}^2}{R_{0.75}} = 2.0119 \times 10^{50} \text{ erg}, \quad (38)$

where $R_{0.75} = 7.4244 \times 10^8 \text{ cm}$ is the radius of a spherical ZTWD whose mass is $M_{0.75} = 0.7522 M_\odot$ as tabulated in
Table 3. Also, at each separation our tabulated values of \( \Omega \) have been normalized to the Keplerian orbital frequency,

\[
\Omega_K = \left( \frac{2GM_{0.75}}{a^3} \right)^{1/2}.
\]  

(39)

In all three figures, values of the orbital separation have been specified (bottom horizontal axis) in units of \( 10^9 \) cm and (top horizontal axis) as normalized to the radius of a spherical ZTWD having the mass of the system’s secondary star as tabulated in Table 3, that is, \( R_{0.75} \equiv 7.424 \times 10^9 \) cm, \( R_{0.60} \equiv 8.671 \times 10^9 \) cm, and \( R_{0.50} \equiv 9.638 \times 10^9 \) cm.

New & Tohline (1997) have previously constructed inspiral sequences for equal-mass DWD binary systems in which the structure of the individual component stars is governed by the Chandrasekhar ZTWD equation of state (1). The sequences published by New & Tohline (1997) cover a wide range of total masses. The one that most closely resembles our sequence “A” (our only equal-mass sequence) has \( M_{\text{tot}} = 1.63 M_\odot \); the functional behavior of \( E_{\text{tot}}(a) \) and \( J_{\text{tot}}(a) \) for this sequence is presented in Figure 16 of New & Tohline (1997). Along this \( M_{\text{tot}} = 1.63 M_\odot \) sequence, the two stars first make contact with their respective Roche lobes at a normalized separation of approximately 2.825 (see also Figure 5 of New & Tohline 1997). This is completely consistent with the behavior of our sequence “A”, where contact occurs (model A1) when \( a/R_{0.75} = 2.823 \).

The DWD sequences constructed by New & Tohline (1997) all extend to separations smaller than the point of first contact, as their SCF technique allowed them to build overcontact (common envelope) equal-mass binaries. Their functions \( E_{\text{tot}}(a) \) and \( J_{\text{tot}}(a) \) display a quadratic behavior along the overcontact segment of each sequence, passing through a minimum at a binary separation smaller than the point of first contact. None of our three sequences show this behavior because we have not attempted to construct models past the initial point of contact.

Indeed, it seems unlikely that equilibrium configurations exist at smaller separations except when the system mass ratio is precisely \( q = 1 \).

For each of our DWD binary sequences, it is useful to compare the displayed functional behavior of \( J_{\text{tot}}(a) \) from our numerical models against the behavior predicted by two simplified models. In the case of two point masses in circular orbit, the total angular momentum \( J_{\text{pm}} \) is given simply by the system’s orbital angular momentum, that is,

\[
J_{\text{pm}} = J_{\text{orb}} = M_1 \sigma^2 \Omega_K + M_2 \sigma^2 \Omega_K = \frac{q}{(1+q)^2} \left[ GM_{\text{tot}} \right]^{1/2}.
\]  

(40)

This function, normalized to \( J_{\text{norm}} \), is displayed by the solid curve in the top panels of Figures 6–8. An even more realistic representation of the function \( J_{\text{pm}}(a) \) can be obtained by adding an approximate representation for the spin angular momentum, \( I_{i}\Omega_i \), of both stars to the point-mass expression for \( J_{\text{pm}} \). If we assume that both stars retain a spherical structure while spinning at the Keplerian orbital frequency, \( \Omega_K \), the appropriate expression for the total “spinning sphere” system angular momentum is

\[
J_{\text{ss}} = J_{\text{orb}} + (I_1 + I_2)\Omega_K = J_{\text{orb}} + (k_1 M_1 R_1^2 + k_2 M_2 R_2^2)\Omega_K
= J_{\text{pm}} \left[ \frac{1}{q} \left( \frac{R_1}{a} \right)^2 + \frac{R_2}{a} \right].
\]  

(41)

where, in addition to \( q \), values of (the constants) \( R_i \) and \( k_i \) appropriate for each binary sequence can be obtained from Table 3. Function (41), normalized to \( J_{\text{norm}} \), is displayed by the dot-dashed curve in the top panels of Figures 6–8. Analytic expression (40) predicts that \( J_{\text{pm}} \propto a^{1/2} \). Through a correction factor, Equation (41) displays a somewhat more complex behavior. Overall, our SCF model sequences match Equation (41) particularly well. The largest deviation arises...
in all cases at the smallest separations; the slope of the SCF-generated $J_{\text{tot}}(a)$ function flattens somewhat as the secondary star approaches contact with its Roche lobe, that is, as $\frac{f_{\text{rel}}}{a} \rightarrow 1$.

The functional dependence of each system’s total energy, $E_{\text{tot}}(a)$, can be understood in a similar fashion. Considering only the kinetic and gravitational potential energy of two point masses in circular orbit, we obtain

$$E_{\text{orb}} = K_{\text{orb}} + W_{\text{orb}} = -K_{\text{orb}} = -\frac{1}{2} \left[ \frac{q}{(1+q)^2} \right] \frac{GM_{\text{tot}}^2}{a}, \quad (42)$$

where we have used the virial relation ($2K_{\text{orb}} + W_{\text{orb}} = 0$). While this $a^{-1}$ functional dependence explains the general $E_{\text{tot}}(a)$ behavior outlined by diamonds in the same panel of these three figures shown in Figures 6–8 by our numerically constructed model sequences, expression (42) is missing a nontrivial shift in the overall energy scale that is set by the binding energies of the two stars, namely,

$$E_b = \sum_{i=1}^{2} (W_{i} + U_{i}). \quad (43)$$

Based on the properties of the spherical stellar models provided in Table 3, the appropriate energy shift for sequences “A”, “B”, and “C” is, respectively, $E_b = -1.551 \times 10^{50}$ erg, $-1.698 \times 10^{50}$ erg, and $-1.963 \times 10^{50}$ erg. Adding $E_b$ to $E_{\text{orb}}$ provides what we will refer to as the “point mass” total system energy

$$E_{\text{pm}} = \frac{1}{2} \left[ \frac{q}{(1+q)^2} \right] \frac{GM_{\text{tot}}^2}{a} + E_b. \quad (44)$$

This analytic function, normalized to $E_{\text{norm}}$, is displayed as a dot-dashed curve in the plots of $E_{\text{tot}}$ versus $a$ shown in Figures 6–8. Expression (45) describes particularly well the variation of $E_{\text{tot}}$ with separation displayed by our numerically constructed binary sequences “B” and “C”. We note, however, that all three of our sequences show that the total system energy drops slightly below the behavior predicted by Equation (45) at the smallest separations.

The curve outlined by asterisks in the third panel from the top of Figures 6–8 shows that $f_{\text{rel}}^{-1}$ steadily increases from a value $\sim 0.2$ to a value of 1.0 at the smallest separation along all three inspiral sequences, implying that the secondary star has made contact with its Roche lobe. For comparison, the curve outlined by diamonds in the same panel of these three figures shows how the Roche-lobe filling factor of the primary star varies along each sequence. The value of $f_{\text{rel}}^{-1}$ does not climb above 0.063 for sequence “C” or above 0.191 for sequence “B”, reflecting the fact that in both cases the primary star is significantly more massive—and, hence, it has a significantly smaller radius—than the secondary star. For inspiral sequence “A”, $f_{\text{rel}}^{-1}(a)$ displays an identical behavior to $f_{\text{rel}}^{-1}(a)$ because the primary and secondary stars have equal masses.

The bottom panel of Figures 6–8 displays the behavior of the normalized orbital frequency $\Omega/\Omega_{K}$ as a function of binary separation derived from our three numerically constructed inspiral sequences. At the smallest separations, our models show that the orbital frequency is always $\sim 0.5\%$ higher than predicted by the “point-mass” Keplerian frequency. Our equal-mass sequence exhibits the largest deviation at contact; specifically, for model “A1”, we find $\Omega = 1.0085\Omega_{K}$. As the separation is increased along each sequence, the figures show that $\Omega/\Omega_{K}$ approaches unity, as expected. However, at a sufficiently wide separation, each of our sequences displays a tiny discontinuous drop in the orbital frequency, followed by further decline that ultimately falls below the local Keplerian value. We suspect that this odd behavior at wide separations arises from the discrete nature of our grid calculations coupled with progressively fewer grid zones falling inside both stars—resulting in progressively poorer numerical resolution—at wider separations.

### 3.3. Conservative Mass-transfer Sequences

During a phase of stable mass transfer, a DWD binary system will evolve in such a way that the secondary star remains in marginal contact with its Roche lobe while it slowly transfers mass to the primary star. If the total mass of the system is conserved, then the evolution should proceed along
a sequence of synchronously rotating configurations in which $M_{\text{tot}}$ is constant, $f_{RL}^{i+2} = 1$, and $q$ is steadily decreasing. Models A1, B1, and C1 can be viewed as representing three such configurations along a sequence whose total system mass is $M_{\text{tot}} = 1.5 M_{\odot}$. In evolving from an initially equal mass, quasi-equilibrium configuration (model A1) to a semi-detached configuration with $q = 2/3$ (model B1), then on to a semi-detached configuration with $q = 1/2$ (model C1), the separation of such a system (measured in units of $10^7$ cm) will increase from $a_0 = 2.10$ to $a_0 = 2.67$, and then to $a_0 = 3.18$; and the system’s orbital period ($P_{\text{orb}} = 2\pi/\Omega$) will increase from 42.3 s to 61.0 s, and then to 79.5 s.

It is clear, therefore, that our new SCF code can be used to construct model sequences that mimic the evolution of DWD systems undergoing slow, conservative mass transfer. The models detailed in Tables 10 (sequence “D”) and 11 (sequence “E”) trace two such semi-detached sequences as the system mass ratio evolves from $q = 1$ to $q \lesssim 0.5$. For sequence “D”, $M_{\text{tot}} = 1.5 M_{\odot}$ and for sequence “E”, $M_{\text{tot}} = 1.0 M_{\odot}$. In the top two panels of Figures 9 and 10, data from Tables 10 and 11 have been plotted as diamond symbols to illustrate how $a$ and $\Omega$ vary with $q$ while $f_{RL}^{i+2}$ is held to a value of unity (definition of a semi-detached binary) along these two fixed-mass sequences.

Up to now, the community has relied upon some relatively simple analytic expressions to approximate the behavior of, for example, $a(q)$ along conservative mass-transfer evolutionary trajectories. For example, by setting the radius of the secondary star as given by the Nauenberg mass–radius relation (A14) equal to the Roche-lobe radius $R_{RL}$ as defined in terms of $a$ and $q$ by the approximate relation provided by Eggleton (1983), namely,

$$R_{RL} = a \left[ \frac{0.49 q^{2/3}}{0.6 q^{2/3} + \ln(1 + q^{1/3})} \right]. \quad (46)$$

one obtains

$$a \approx \frac{0.0229 (n_2 q^3)^{-1/3} (1 - n_2^{4/3})^{1/2}}{[0.6 q^{2/3} + \ln(1 + q^{1/3})]}, \quad (47)$$

where

$$n_2 \equiv \frac{q}{(1 + q)} \left( \frac{M_{\text{tot}}}{M_{\odot}} \right). \quad (48)$$

The function $a(q)$, defined by Equation (47) for a given $M_{\text{tot}}$, has been plotted as a solid curve in the top panels of Figures 9 and 10, and the Keplerian orbital frequency associated with this separation (and relevant $M_{\text{tot}}$, has been plotted as a solid curve in the second panel of Figures 9 and 10. For both sequences “D” and “E”, the analytically derived curves are consistently offset by 3%–5% from our numerical model results. But overall, the analytically predicted functional behavior of $a(q)$ and $\Omega(q)$ is in very good agreement with our results. This is reassuring as it provides a degree of validation for both our numerical code and the approximations that were adopted by earlier investigators when deriving the more easily manipulated analytic expressions.

Finally, in the bottom two panels of Figures 9 and 10, the diamond symbols display the variation of $J_{\text{tot}}$ and $E_{\text{tot}}$ with $q$ along sequence “D” and sequence “E”, respectively. The solid curve drawn in the $J_{\text{tot}}(q)$ panel of both figures shows the behavior predicted by our “spinning sphere” expression for the total system angular momentum (41) when used in conjunction with the $a(q, M_{\text{tot}})$ behavior prescribed by Equation (47). Again, for a given $M_{\text{tot}}$, there appears to be very good agreement between the functional behavior of $J_{\text{tot}}(q)$ displayed by our numerical model results and the analytic expressions. There is also a systematic offset between the two. In either case it is clear that, unlike the behavior displayed by $a(q)$ and $\Omega(q)$, the system’s total angular momentum does not vary monotonically with $q$ along a conservative mass-transfer evolutionary trajectory. Note, in particular, that if the system mass ratio $q$ is initially close to unity, $J_{\text{tot}}$ increases as $q$ decreases along the displayed trajectory. This result is unphysical. It signifies that slow evolution along a synchronously rotating, conservative mass-transfer trajectory can occur only if, at the onset of mass transfer, $q < q_{\text{crit}}$, where the value of $q_{\text{crit}}$ for a given $M_{\text{tot}}$ is prescribed by the location of the maximum of the $J_{\text{tot}}(q)$ curve. For our model sequences “D” and “E”, we see that $q_{\text{crit}} \lesssim 2/3$, consistent with the mass-transfer stability limit that has already received much attention in the literature (Frank 2008).

4. SUMMARY AND CONCLUSIONS

Based on the earlier work of the Hachisu (1986a, 1986b) and Hachisu et al. (1986a) we have developed an SCF technique that can be used to construct equilibrium models of synchronously rotating DWD binaries having a range of total masses, mass ratios, and binary separations. In addition to effects introduced by synchronous rotation, the distorted structure of both stars in each converged model is governed by the ZTWD equation of state (1) and a self-consistently determined, Newtonian gravitational field. In an effort to illustrate the technique’s capabilities, we have constructed a set of models along five sequences: three sequences (“A”, “B”, and “C”) mimic the last segment of the detached “inspiral” phase of DWD binary evolutions during which both $M_{\text{tot}}$ and $q$ are held constant as $a$ decreases; and two sequences (“D” and “E”) mimic a semi-detached “conservative mass transfer” phase of evolution during which $M_{\text{tot}}$ is held fixed and the less-massive star stays in marginal contact with its Roche lobe, but $q$ steadily decreases while $a$ steadily increases.

Along each inspiral sequence, the functional dependence of $J_{\text{tot}}$ and $E_{\text{tot}}$ on the orbital separation can be well understood in terms of simple analytical expressions that describe two spinning spherical WDs in circular orbit about one another. For a given total mass and separation, the calculated orbital frequencies along each inspiral sequence deviate measurably from associated Keplerian frequencies only in models for which the Roche-lobe filling factor of the less-massive star is $\lesssim 60\%$. But, at least for the sequences examined here, the deviation from Keplerian frequencies is never more than 1% even at contact. Along both conservative mass-transfer sequences, we have documented how $a$, $\Omega$, $J_{\text{tot}}$, and $E_{\text{tot}}$ vary with the system mass ratio as $q$ decreases by roughly a factor of 2, from $q = 1.0$ down to $q \lesssim 0.5$. Along each sequence we have compared our numerically determined values of $a$ at various values of $q$ with the analytic $a(q)$ function (47) that is derived by setting the radius of the less-massive star, as specified by the Nauenberg (1972) mass–radius relation, equal to the Roche-lobe radius, as approximated by Eggleton (1983). Qualitatively, our results show the same $a(q)$ behavior that is predicted by this analytic expression. However, at a given $q$ the value of $a$ derived from our models is consistently $\sim 8\%$ larger than the value obtained from Equation (47). The analytic expression could be brought into closer quantitative agreement with our numerical results if the leading coefficient in Equation (47) is increased by 8%, that is, if the expression’s leading coefficient

$J_{\text{tot}} \approx 0.0229 (n_2 q^3)^{-1/3} (1 - n_2^{4/3})^{1/2} / [0.6 q^{2/3} + \ln(1 + q^{1/3})]$.

$J_{\text{tot}} \approx 0.0229 (n_2 q^3)^{-1/3} (1 - n_2^{4/3})^{1/2} / [0.6 q^{2/3} + \ln(1 + q^{1/3})]$. 

$J_{\text{tot}} \approx 0.0229 (n_2 q^3)^{-1/3} (1 - n_2^{4/3})^{1/2} / [0.6 q^{2/3} + \ln(1 + q^{1/3})]$. 

$J_{\text{tot}} \approx 0.0229 (n_2 q^3)^{-1/3} (1 - n_2^{4/3})^{1/2} / [0.6 q^{2/3} + \ln(1 + q^{1/3})]$.
is changed from 0.0229 to 0.0247. This modification will, in turn, decrease the Keplerian frequency obtained from the analytic \(a(q)\) expression by \(\sim 9\%\), simultaneously bringing the analytically predicted orbital frequency into much closer agreement with our numerically determined values of \(\Omega\). Along both of our conservative mass-transfer sequences, the plot of \(J_{\text{ot}}(q)\) displays an extremum at a value of \(q \lesssim 2/3\). The location of this extremum is almost certainly identifying the value of \(q_{\text{crit}}\) that is relevant along both sequences.

The development of this SCF technique was originally motivated by our desire to build models that would serve as good, “quiet” initial conditions for hydrodynamical simulations that are designed to probe the onset and nonlinear development of mass-transfer instabilities in close, unequal-mass DWD binaries. The new computational tool that we have described in this paper achieves this objective.

We acknowledge valuable interactions that we have had with B. Bourdin, J. Frank, D. Marcello, P. M. Mott, and S. Ou over the course of this project. We also thank an anonymous referee for pointing us to a key reference from I. Hachisu’s collection of work during the mid-1980s. This work has been supported, in part, by grants AST-0708551 and DGE-0504507 from the U.S. National Science Foundation and, in part, by grants AST-0708551 and DGE-0504507 from the U.S. National Science Foundation. (4) Grant NNX07AG84G from NASA’s ATP program. This research has been supported by grants of high-performance computing time on the TeraGrid (MCA98N043), at LSU, and across LONI (Louisiana Optical Network Initiative).

APPENDIX

WHITE DWARF MASS–RADIUS RELATIONSHIP

A.1. The Chandrasekhar Mass

Chandrasekhar (1935) was the first to construct models of spherically symmetric stars using the equation of state defined by Equation (1) and, in so doing, demonstrated that the maximum mass of an isolated, nonrotating WD is \(M_{\text{ch}} = 1.44(\mu_e/2)M_\odot\), where \(\mu_e\) is the number of nucleons per electron and, hence, depends on the chemical composition of the WD. A concise derivation of \(M_{\text{ch}}\) (although, at the time, it was referred to as \(M_1\)) is presented in Chapter XI of Chandrasekhar (1967), where we also find that the expressions for the two key coefficients in Equations (1) and (2)

\[
A \equiv \frac{\pi m_3^3 c^5}{3h^4}, \quad (A1)
\]

\[
B \mu_e^{-1} \equiv \frac{8\pi m_p}{3} \left(\frac{m_e c}{h}\right)^3. \quad (A2)
\]

Numerical values for \(A\) and \(B \mu_e^{-1}\) are given here in Table A1 along with values of the physical constants \(c\), \(h\), \(m_e\), and \(m_p\) that we have used (Column 2) and that Chandrasekhar (1967) used (Column 3) to determine the values of \(A\) and \(B \mu_e^{-1}\). The derived analytic expression for the limiting mass is

\[
\mu_e^2 M_{\text{ch}} = 4\pi m_3 \left(\frac{2A}{\pi G}\right)^{3/2} \left(\frac{\mu_e}{B}\right)^2 = 1.14205 \times 10^{34}\text{ g}, \quad (A3)
\]

where the coefficient

\[
m_3 \equiv \left(-\frac{\xi^2 d\theta_3}{d\xi}\right)_{\xi=\ell_1(\theta_3)} = 2.01824, \quad (A4)
\]

\[\xi =\]

| Table A1 |
| --- |
| **Physical Constants** |
| | **This Paper** | Chandrasekhar (1967)** |
| | (1) | (2) | (3) |
| \(c\) (cm s\(^{-1}\)) | 2.99792 \times 10^{10} | 2.9978 \times 10^{10} | 2.9978 \times 10^{10} |
| \(h\) (erg s\(^{-1}\)) | 6.62608 \times 10^{-27} | 6.62 \times 10^{-27} | 6.62 \times 10^{-27} |
| \(m_e\) (g) | 9.10939 \times 10^{-28} | 9.105 \times 10^{-28} | 9.105 \times 10^{-28} |
| \(m_p\) (g) | 1.67262 \times 10^{-24} | 1.672 \times 10^{-24} | 1.672 \times 10^{-24} |
| \(\mu_e\) (g) | 1.66054 \times 10^{-24} | \ldots | \ldots |
| \(G\) (cm\(^3\) g\(^{-1}\) s\(^{-2}\)) | 6.6726 \times 10^{-8} | 6.62 \times 10^{-8} | 6.62 \times 10^{-8} |
| \(M_1\) (g) | 1.9891 \times 10^{33} | 1.985 \times 10^{33} | 1.985 \times 10^{33} |
| \(R_1\) (cm) | 6.955 \times 10^{10} | 6.951 \times 10^{10} | 6.951 \times 10^{10} |
| \(A\) (dynes cm\(^{-2}\)) | 6.00228 \times 10^{22} | 6.01 \times 10^{22} | 6.01 \times 10^{22} |
| \(B \mu_e^{-1}\) (g cm\(^{-3}\)) | 9.81011 \times 10^{6} | 9.82 \times 10^{7} | 9.82 \times 10^{7} |
| \(\ell_1\) (cm) | 7.71395 \times 10^{8} | 7.67 \times 10^{8} cm. |

Notes.

\(a\) Speed of light, \(c\); Planck’s constant, \(h\); mass of the electron, \(m_e\); mass of the proton \(m_p\); atomic mass unit, \(\mu_e\); universal gravitational constant, \(G\); solar mass, \(M_\odot\); solar radius, \(R_\odot\); as used in the ZTWD equation of state (1), \(A = \pi m_e^2 c^3/3h^4\) and \(B \mu_e^{-1} = 8\pi m_3^2 \mu_e m_p/3h^4\); the characteristic WD length scale, \(\ell_1 = (2A/\pi G)^{1/2} / (\mu_e/3)\).

\(b\) Drawn from Appendix I, Table 32 of Chandrasekhar (1967).

\(c\) Drawn from Chapter XI, Table 4 of Chandrasekhar (1967). We note as well that Chandrasekhar (1967) identified a characteristic radius, \(\ell_1\), for WDs given by the expression

\[
\ell_1 \mu_e = \left(2A / (\pi G)\right)^{1/2} / (\mu_e/3) = 7.71395 \times 10^{8} \text{ cm.} \quad (A5)
\]

A.2. The “Nauenberg” Mass–Radius Relationship

Nauenberg (1972) derived an analytic approximation for the mass–radius relationship exhibited by isolated, spherical WDs that obey the ZTWD equation of state given in Equation (1). Specifically, he offered an expression of the form

\[
R = R_0 \left[\frac{1 - n^{4/3}}{n}\right]^{1/2}, \quad (A6)
\]

where

\[
n = \frac{M}{(\mu m_p)N_0}, \quad (A7)
\]

\[
N_0 = \frac{(3\pi^2 \xi)^{1/2}}{v^{3/2}} \left[\frac{hc}{2\pi G (\mu m_p)^{1/2}}\right]^{3/2}
\]

\[
= \frac{\mu_e^2 m_p}{(\mu m_p)^{3/2}} \left[\frac{4\pi \xi}{m_3^2 v}\right]^{1/2} M_{\text{ch}}, \quad (A8)
\]

\[
R_0 = \frac{(3\pi^2 \xi)^{1/2}}{v^{1/2}} \left[\frac{h}{2\pi m_e c}\right] N_0^{1/3}
\]

\[
= \frac{m}{(\mu m_p)} \left[\frac{4\pi \xi}{v}\right]^{1/2} \ell_1, \quad (A9)
\]

\(m_3\) is the atomic mass unit (see Table A1), \(\mu\) is the mean molecular weight of the gas, and \(\xi\) and \(v\) are two adjustable parameters in Nauenberg’s analytic approximation, both of which are expected to be of order unity. By assuming that
the average particle mass denoted by Chandrasekhar (1967) as \((\mu_m, m_p)\) is identical to the average particle mass specified by Nauenberg (1972) as \((\mu_m)\) and, following Nauenberg’s lead, by setting \(v = 1\) and,\(^2\)
\[
\zeta = \frac{m^2}{4\pi} = 0.324142, \quad \text{(A10)}
\]
in Equation (A8) we see that
\[
(\mu_m)N_0 = M_{ch}. \quad \text{(A11)}
\]
Hence, the denominator in (A7) becomes the Chandrasekhar mass. Furthermore, expressions (A9) and (A6) become, respectively,
\[
\mu_eR_0 = m_3(\ell_1\mu_e) = 1.55686 \times 10^{19}\text{cm}, \quad \text{(A12)}
\]
and
\[
R = R_0 \left\{\frac{1 - (M/M_{ch})^{4/3}}{(M/M_{ch})^{1/3}}\right\}. \quad \text{(A13)}
\]
Finally, by adopting the values of \(M_{ch}\) and \(R_0\) listed in Table A1, we obtain essentially\(^3\) the identical approximate, analytic mass–radius relationship for ZTWDs presented in Equations (27) and (28).

\section*{A.3. The “Eggleton” Mass–Radius Relationship}

Verbunt & Rappaport (1988) introduced the following approximate, analytic expression for the mass–radius relationship of a “completely degenerate . . . star composed of pure helium” (i.e., \(\mu_e = 2\)), attributing its origin to P. P. Eggleton (1986, private communication):
\[
\frac{R}{R_\odot} = 0.0224 \left\{\frac{1 - (M/M_{ch})^{4/3}}{(M/M_{ch})^{1/3}}\right\}. \quad \text{(A14)}
\]
where
\[
\frac{M_{ch}}{M_\odot} = \frac{5.742}{\mu_e^3}. \quad \text{(A15)}
\]

\section*{REFERENCES}

Anderson, S. F., et al. 2005, \textit{AJ}, 130, 2230
Balay, S., et al. 2004, PETSc Users Manual, ANL-95/11, Revision 2.1.5
Bender, P. L. 1998, \textit{BAAS}, 30, 1326
Benz, W., Cameron, A. G. W., Press, W. H., & Bowers, R. L. 1990, \textit{ApJ}, 348, 647
Chandrasekhar, S. 1935, \textit{MNRAS}, 95, 207
Chandrasekhar, S. 1967, An Introduction to the Study of Stellar Structure (New York: Dover)
Clayton, G. C., Geballe, T. R., Herwig, F., Fryer, C., & Asplund, M. 2007, \textit{ApJ}, 662, 1220
Cohl, H., & Tohline, J. 1999, \textit{ApJ}, 527, 86
Comish, N. J., & Larson, S. L. 2003, \textit{Phys. Rev. D}, 67, 103001
Cox, A. N. 2000, Allen’s Astrophysical Quantities (4th ed.; New York: Springer)
Dan, M., Rosswog, S., & Brüggen, M. 2009, \textit{J. Phys. Conf. Ser.}, 172, 012034
D’Souza, M. C. R., Motl, P. M., Tohline, J. E., & Frank, J. 2006, \textit{ApJ}, 643, 381
Eggleton, P. P. 1983, \textit{ApJ}, 268, 368
Evron, C. R., Iben, I., Jr, & Smarr, L. 1987, \textit{ApJ}, 323, 129
Faller, J. E., & Bender, P. L. 1984, in Precision Measurement and Fundamental Constants II, NBS Spec. Pub. 617, ed. B. N. Taylor & W. D. Phillips (Washington, DC: NBS), 689
Franq, J. 2008, \textit{New Astron. Rev.}, 51, 878
Fryer, C. L., & Diehl, S. 2008, in ASP Conf. Ser. 391, Hydrogen Deficient Stars, ed. K. Werner & T. Rauch (San Francisco, CA: ASP), 335
Fryer, C. L., Woosley, S. E., Herant, M., & Davies, M. B. 1999, \textit{ApJ}, 520, 650
Gokhale, V., Peng, X.-M., & Frank, J. 2007, \textit{ApJ}, 665, 1010
Gueguen, J., Garcia-Berro, E., & Isern, J. 2004, \textit{A&A}, 413, 257
Hachisu, I. 1986a, \textit{ApJ}, 61, 479
Hachisu, I. 1986b, \textit{ApJS}, 62, 641
Hachisu, I., Eriguchi, Y., & Nomoto, K. 1986a, \textit{ApJ}, 308, 161
Hachisu, I., Eriguchi, Y., & Nomoto, K. 1986b, \textit{ApJ}, 311, 214
Hils, D., Bender, P. L., & Webbink, R. F. 1990, \textit{ApJ}, 360, 75
Iben, I. J., & Tutukov, A. V. 1984, \textit{ApJS}, 55, 335
Iben, I. J., & Tutukov, A. V. 1986, \textit{ApJ}, 311, 753
Iben, I. J., Tutukov, A. V., & Yungelson, L. R. 1996, \textit{ApJ}, 465, 750
Karl, C. A., Napiwotzki, R., Nelemans, G., Christlieb, N., Koester, D., Heber, U., & Reimers, D. 2003, \textit{A&A}, 410, 663
Kopparapu, R. K., & Tohline, J. E. 2007, \textit{ApJ}, 655, 1025
Livio, M. 2000, in \textit{Type Ia Supernovae: Theory and Cosmology}, ed. J. C. Iben, I. J., Tutukov, A. V. 1986, \textit{ApJ}, 311, 753
Maxted, P. F. L., Marsh, T. R., & Moran, C. K. 2000, \textit{MNRAS}, 319, 305
Motl, M. P., Frank, J., Tohline, J. E., & D’Souza, M. C. R. 2007, \textit{ApJ}, 670, 1314
Napiwotzki, R., et al. 2001, \textit{Astron. Nachr.}, 322, 411
Napiwotzki, R., et al. 2002, \textit{A&A}, 386, 957
Napiwotzki, R., et al. 2004, in ASP Conf. Ser. 318, Spectroscopically and Spatially Resolving the Components of Close Binary Stars, ed. R. W. Hilditch, H. Hensberge, & K. Pavlovski (San Francisco, CA: ASP), 402
Nelemans, G. 1997, \textit{ApJ}, 175, 417
Nelemans, G. 2005, in ASP Conf. Ser. 330, The Astrophysics of Cataclysmic Variables and Related Objects, ed. J.-M. Hameury & J.-P. Lasota (San Francisco, CA: ASP), 27
Nelemans, G., et al. 2005, \textit{A&A}, 440, 1087
New, K. C. B., & Tohline, J. E. 1997, \textit{ApJ}, 490, 311
Ostriker, J. P., & Mark, J. W. 1964, \textit{ApJ}, 141, 1057
Paczyński, B. 1967, \textit{Acta Astron.}, 17, 287
Ramsay, G., et al. 2007, in ASP Conf. Ser. 372, 15th European Workshop on White Dwarfs, ed. R. Napiwotzki & M. R. Burleigh (San Francisco, CA: ASP), 425
Rasio, F. A., & Shapiro, S. L. 1995, \textit{ApJ}, 438, 887
Roelofs, G. H. A., Groot, P. J., Marsh, T. R., Steeghs, D., Barros, S. C. C., & Nelemans, G. 2005, \textit{MNRAS}, 361, 487
Ruciński, S. M. 1988, \textit{AJ}, 95, 1895
Segretain, L., Chabrier, G., & Mochnick, R. 1997, \textit{ApJ}, 481, 355
Verbunt, F., & Rappaport, S. 1988, \textit{ApJ}, 332, 193
Warner, B. 1995, \textit{ApBS}, 225, 249
Webbink, R. F. 1984, \textit{ApJ}, 277, 555
Woudt, P. A., & Warner, B. 2003, \textit{MNRAS}, 345, 1266
Yoon, S.-C., Podsiadlowski, Ph., & Rosswog, S. 2007, \textit{MNRAS}, 380, 933
Zapolsky, H. S., & Salpeter, E. E. 1969, \textit{ApJ}, 158, 809