QUATERNIONIC INNER AND OUTER FUNCTIONS

ALESSANDRO MONGUZZI, GIULIA SARFATTI, AND DANIEL SECO

ABSTRACT. We study properties of inner and outer functions in the Hardy space of the quaternionic unit ball. In particular, we give sufficient conditions as well as necessary ones for functions to be inner or outer.

1. INTRODUCTION

In the function theory of the unit disk of the complex plane, an essential role is played by the property that any function in the Hardy space factorizes (uniquely) as the product of an inner and an outer function. The connections of inner and outer functions are ubiquitous in mathematical analysis, ranging from operator theory to dynamical systems and PDEs (see [4] and [7], for instance). One of the main reasons for this is the fact that invariant subspaces (for the shift operator in the Hardy space) can be described via an identification with inner functions whereas outer functions contain information about approximation properties, and in fact, coincide with cyclic functions. In the recent paper [11], our first two authors proved an inner-outer factorization theorem for the Hardy space of slice regular functions on the quaternionic unit ball $H^2(\mathbb{B})$.

Thus, it seems natural to investigate the properties of inner and outer functions in the quaternionic setting more deeply, and the present paper is a first step in this direction. We will see that some properties of holomorphic inner and outer functions are straightforwardly generalized to the quaternionic setting, whereas some other properties are more peculiar of slice regular functions.

The paper is organized as follows: In Section 2 we fix the notation and we recall some basic definitions and properties of slice regular functions and the quaternionic Hardy space $H^2(\mathbb{B})$. We devote Section 3 to properties of inner functions, whereas in Section 4 we focus on outer ones. Then, in Section 5 we will dedicate some time to cyclicity and properties of optimal approximant polynomials in the quaternionic setting. We conclude formulating some open problems in Section 6.

Date: October 25, 2018.

2010 Mathematics Subject Classification. Primary 30G35; Secondary 30H10, 30J05.
2. Notation and Basic Definitions

In this section we recall a few definitions and properties of slice regular functions and the quaternionic Hardy space $H^2(\mathbb{B})$. We do not include any proofs here; we refer the reader to the monograph [8] for the basics on slice regular functions, and to [5] for results concerning $H^2(\mathbb{B})$.

Let $\mathbb{H}$ denote the skew field of quaternions, let $\mathbb{B} = \{ q \in \mathbb{H} : |q| < 1 \}$ be the quaternionic unit ball and let $\partial \mathbb{B}$ be its boundary, containing elements of the form $q = e^{it} = \cos t + \sin t I$, $I \in \mathbb{S}$, $t \in \mathbb{R}$, where $\mathbb{S} = \{ q \in \mathbb{H} : q^2 = -1 \}$ is the two dimensional sphere of imaginary units in $\mathbb{H}$. Then,

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} (\mathbb{R} + \mathbb{R} I), \quad \mathbb{R} = \bigcap_{I \in \mathbb{S}} (\mathbb{R} + \mathbb{R} I),$$

where the slice $L_I := \mathbb{R} + \mathbb{R} I$ can be identified with the complex plane $\mathbb{C}$ for any $I \in \mathbb{S}$.

A function $f : \mathbb{B} \to \mathbb{H}$ is a slice regular function if the restriction $f_I$ of $f$ to $\mathbb{B}_I := \mathbb{B} \cap L_I$ is holomorphic, i.e., it has continuous partial derivatives and it is such that

$$\partial_I f_I(x + yI) = \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all $x + yI \in \mathbb{B}_I$. The relationship between slice regular functions and holomorphic functions of one complex variable is the following:

**Lemma 2.1** (Splitting Lemma). If $f$ is a slice regular function on $\mathbb{B}$, then for every $I \in \mathbb{S}$ and for every $J \in \mathbb{S}$, $J$ orthogonal to $I$, there exist two holomorphic functions $F, G : \mathbb{B}_I \to L_I$ such that for every $z = x + yI \in \mathbb{B}_I$,

$$f_I(z) = F(z) + G(z)J.$$

It is a well-known fact that every slice regular function on the unit ball $\mathbb{B}$ admits a power series expansion of the form

$$f(q) = \sum_{n \geq 0} q^n a_n,$$

with all $a_n \in \mathbb{H}$. The conjugate of $f$, which we denote by $f^c$, is the function defined by

$$f^c(q) := \sum_{n \in \mathbb{Z}} q^n \overline{a_n}. \quad (1)$$

Moreover, we denote by $\widetilde{f}$ the function

$$\widetilde{f}(q) := f(\overline{q}).$$

The function $\widetilde{f}$ is not slice regular but it is a slice function. The class of slice functions was introduced in [9] in a more general setting than the present
one. In this paper we adopt as definition of slice functions the following: a function $f : \mathbb{B} \to \mathbb{H}$ is a slice function if for any $I, J \in \mathbb{S}$ it holds
\begin{equation}
    f(x + yI) = \frac{1 - IJ}{2} f(x + yJ) + \frac{1 + IJ}{2} f(x - yJ).
\end{equation}
Slice regular functions are examples of slice functions. Moreover formula (2) furnishes a tool to uniquely extend a holomorphic function defined on the complex disk $\mathbb{B}_J$ to a slice regular function defined on the whole unit ball $\mathbb{B}$ (see [8]). Given $f_J : \mathbb{B}_J \to \mathbb{H}$ holomorphic function of the complex variable $x + yJ$, the function $\text{ext}(f_J) : \mathbb{B} \to \mathbb{H}$ defined, for any $x + yI \in \mathbb{B}$, as
\begin{equation}
    \text{ext}(f_J)(x + yI) = \frac{1 - IJ}{2} f_J(x + yJ) + \frac{1 + IJ}{2} f_J(x - yJ)
\end{equation}
is slice regular on $\mathbb{B}$.
Formula (2) can be also used to prove the following result concerning the zeros of a slice regular function.

**Proposition 2.2.** Let $f$ be a slice regular function on $\mathbb{B}$ such that $f(\mathbb{B}_I) \subseteq L_I$ for some $I \in \mathbb{S}$. If $f(x + yJ) = 0$ for some $J \in \mathbb{S} \setminus \{\pm I\}$, then $f(x + yK) = 0$ for any $K \in \mathbb{S}$.

The structure of the zero set of a slice regular function is completely understood.

**Theorem 2.3.** Let $f$ be a slice regular function on $\mathbb{B}$. If $f$ does not vanish identically, then its zero set consists of the union of isolated points and isolated 2-spheres of the form $x + yS$ with $x, y \in \mathbb{R}, y \neq 0$.

A 2-dimensional sphere $x + yS \subseteq \mathbb{B}$ of zeros of $f$ is called a spherical zero of $f$. Any point $x + yI$ of such a sphere is called a generator of the spherical zero $x + yS$. Any zero of $f$ that is not a generator of a spherical zero is called an isolated zero of $f$. Moreover, on each sphere $x + yS$ contained in $\mathbb{B}$, the zeros of $f$ are in one-to-one correspondence with the zeros of $f^c$, see [8, Proposition 3.9].

In general, the pointwise product of two slice regular functions is not a slice regular function, thus a suitable product must be considered, namely, the so-called slice or $\ast$-product. If $f(q) = \sum_{n \in \mathbb{N}} q^n a_n$ and $g(q) = \sum_{n \in \mathbb{N}} q^n b_n$ are two slice regular functions on $\mathbb{B}$, then
\begin{equation}
    f \ast g(q) := \sum_{n \in \mathbb{N}} q^n \sum_{k \in \mathbb{N}} a_k b_{n-k}.
\end{equation}
This product is related to the pointwise product by the formula
\begin{equation}
    f \ast g(q) = \begin{cases} 
        0 & \text{if } f(q) = 0 \\
        f(q)g(T_f^c(q)) & \text{if } f(q) \neq 0,
    \end{cases}
\end{equation}
where \( T_{fc}(q) := f(q)^{-1}q f(q) \).

By means of the \(*\)-product, we can associate to a function \( f \) its symmetrization \( f^s \), that is,

\[
  f^s(q) := f^c \ast f(q) = f \ast f^c(q). \tag{5}
\]

We remark here that the symmetrization \( f^s \) is a slice preserving function, namely \( f^s(B_I) \subseteq L_I \) for all \( I \in \mathbb{S} \). In particular, this is equivalent to the fact that the coefficients in the power series expansion of \( f^s \) are all real numbers.

Finally, we denote by \( f^{-*} \) the inverse of \( f \) with respect to the \(*\)-product, which is given by

\[
  f^{-*}(q) = (f^s(q))^{-1}f^c(q).
\]

The function \( f^{-*} \) is defined on \( \{q \in \mathbb{B} \mid f^s(q) \neq 0\} \) and \( f \ast f^{-*} = f^{-*} \ast f = 1 \). All of \( f^c, f^s \) and \( f^{-*} \) are slice regular functions if \( f \) is.

In [5] the basic theory of Hardy spaces \( H^p(\mathbb{B}) \) was established; here we only recall some facts for the Hilbert case \( p = 2 \) and for the extremal case \( p = \infty \) since that is enough for our purposes. Denoting \( \ell^2 := \ell^2(\mathbb{N}, \mathbb{H}) \), the Hardy space \( H^2(\mathbb{B}) \) can be described as follows:

\[
  H^2(\mathbb{B}) := \left\{ f \text{ slice regular on } \mathbb{B} : f(q) = \sum_{n \in \mathbb{N}} q^n a_n, \{a_n\}_{n \in \mathbb{N}} \in \ell^2 \right\}.
\]

Each function \( f \in H^2(\mathbb{B}) \) admits a boundary value function, defined almost everywhere (with respect to a suitable measure \( \Sigma \)) in a canonical sense. We still denote this function by \( f \). With this in mind, if \( f \in H^2(\mathbb{B}) \), whenever we write \( f(q) \) with \( |q| = 1 \), we are implicitly evaluating the boundary value function associated to \( f \).

The space \( H^2(\mathbb{B}) \) is a right quaternionic Hilbert space with respect to the inner product

\[
  \langle \sum_{n \in \mathbb{N}} q^n a_n, \sum_{n \in \mathbb{N}} q^n b_n \rangle := \sum_{n \in \mathbb{N}} \overline{b_n} a_n. \tag{6}
\]

This inner product on \( H^2(\mathbb{B}) \) also admits an integral representation: we endow \( \partial \mathbb{B} \) with the measure

\[
  d\Sigma(e^{I t}) = d\sigma(I) dt, \tag{7}
\]

where \( dt \) is the Lebesgue measure on \([0, 2\pi)\) and \( d\sigma \) is the standard area element of \( \mathbb{S} \), normalized so that \( \sigma(\mathbb{S}) = \Sigma(\partial \mathbb{B}) = 1 \). Then,

\[
  \langle f, g \rangle = \int_{\partial \mathbb{B}} \overline{g(q)} f(q) d\Sigma(q). \tag{8}
\]

The measure \( d\Sigma \), and not the induced Lebesgue measure on \( \partial \mathbb{B} \), is naturally associated to the Hardy space \( H^2(\mathbb{B}) \), as reasoned in [5 2]. An important
feature of this inner product is that it can be actually computed by restricting it to any slice \( L_I \). In more detail, given any \( I \in \mathbb{S} \) we set
\[
\langle f, g \rangle_I = \frac{1}{2\pi} \int_0^{2\pi} g(e^{\theta I}) f(e^{\theta I}) d\theta,
\]
where \( d\theta \) is the Lebesgue measure on \([0, 2\pi)\). For any \( I \in \mathbb{S} \), we have
\[
\langle f, g \rangle = \int_{\partial \mathbb{B}} g(q) f(q) \ d\Sigma(q) = \langle f, g \rangle_I.
\]

By \( H^\infty(\mathbb{B}) \), we denote the space of bounded slice regular functions on the unit ball. Notice that \( H^\infty(\mathbb{B}) \subseteq H^2(\mathbb{B}) \).

Definitions of inner and outer functions in the quaternionic setting are similar to the classical ones for holomorphic functions and appeared in [5].

Definition 2.4. We say that \( \varphi \in H^\infty(\mathbb{B}) \) is inner if \( |\varphi(q)| \leq 1 \) on \( \mathbb{B} \) and \( |\varphi(q)| = 1 \ \Sigma\)-almost everywhere on \( \partial \mathbb{B} \).

Definition 2.5. A function \( g \in H^2(\mathbb{B}) \) is outer if given \( f \in H^2(\mathbb{B}) \) such that \( |g(q)| = |f(q)| \) for \( \Sigma \)-almost every \( q \in \partial \mathbb{B} \), then \( |g(q)| \geq |f(q)| \) for \( q \in \mathbb{B} \).

We point out that the definition of inner and outer functions were given in terms of the induced Lebesgue measure \( m \) on \( \partial \mathbb{B} \). It is not difficult to show that \( \Sigma \) and \( m \) are mutually absolutely continuous.

The following theorem was proved in [11] by the first two authors.

Theorem 2.6 (Inner-outer factorization). Let \( f \in H^2(\mathbb{B}) \), \( f \neq 0 \). Then \( f \) has a factorization \( f = \varphi \ast g \) where \( \varphi \) is inner and \( g \) is outer.

Moreover, this factorization is unique up to a unitary constant in the following sense: if \( f = \varphi \ast g = \varphi_1 \ast g_1 \), then \( \varphi_1 = \varphi \ast \lambda \) and \( g_1 = \lambda \ast g \) for some \( \lambda \in \mathbb{H} \) such that \( |\lambda| = 1 \).

The proof of this theorem makes use of the concept of cyclicity.

Definition 2.7. A function \( g \) is cyclic (in \( H^2(\mathbb{B}) \)) if
\[
[g] := \overline{\text{span} \{ q^n \ast g, n \geq 0 \} } = H^2(\mathbb{B}).
\]

We stress out that \([g]\) is the smallest closed invariant subspace containing \( g \). Thus, \( g \) is cyclic if the smallest closed subspace containing \( g \) is the whole space \( H^2(\mathbb{B}) \). In [11] it is firstly proved that each function \( f \in H^2(\mathbb{B}) \) admits a factorization \( f = \varphi \ast g \) with \( \varphi \) inner function and \( g \) a cyclic function. Afterwards, cyclic is proved equivalent to outer in the sense of Definition 2.5.

We remark that we work with right quaternionic Hilbert spaces, therefore the left-hand side of (11) denotes the closure in \( H^2(\mathbb{B}) \) of elements of the
form
\[ \sum_{n=0}^{m} (q^n \ast g) \alpha_n = \sum_{n=0}^{m} (g \ast q^n) \alpha_n = g \ast p_m \]
where \( p_m \) is a quaternionic polynomial with scalar coefficients \( \alpha_n \in \mathbb{H} \).

3. INNER FUNCTIONS

Let us next focus on inner functions. To start, we would like to better understand any connection between \( f \) being an inner function in \( H^2(\mathbb{B}) \) and the properties of \( f_I \) (the restriction of \( f \) to the slice \( L_I = \mathbb{R} + \mathbb{R}I \)), or of the splitting components of \( f \) (see Lemma 2.1). Some of the results we include in this section are somehow hidden in \cite{11}; here we state them explicitly and make some remarks.

We continue from a characterization of inner functions in \( H^2(\mathbb{B}) \). In the following statement the \( \ast \)-product is the extension of (3) to the more general setting of slice \( L^2 \) functions on \( \partial \mathbb{B} \), that is the space \( L^2(\partial \mathbb{B}) \) of functions of the form \( q \mapsto \sum_{k \in \mathbb{Z}} q^k a_k \) with \( q \in \partial \mathbb{B} \) and \( \{a_k\}_{k \in \mathbb{Z}} \in \ell^2 \), see \cite{11}. If \( f(q) = \sum_{n \in \mathbb{Z}} q^n a_n \) and \( g(q) = \sum_{n \in \mathbb{Z}} q^n b_n \), then
\[ f \ast g(q) := \sum_{n \in \mathbb{Z}} q^n \sum_{k \in \mathbb{Z}} a_k b_{n-k}. \]

**Proposition 3.1.** Let \( f \in H^2(\mathbb{B}) \). Then the following are equivalent

(i) \( f \) is inner;

(ii) \( \tilde{f} \ast f^c = f^c \ast \tilde{f} = 1 \) \( \Sigma \)-almost everywhere on \( \partial \mathbb{B} \);

(iii) there exists \( I \in \mathbb{S} \) such that \( (\tilde{f} \ast f^c)_I = (f^c \ast \tilde{f})_I = 1 \) almost everywhere with respect to the induced Lebesgue measure on \( \partial \mathbb{B} \).

**Proof.** Recalling Lemma 2.4 in \cite{11}, we know that \( \tilde{f} \ast f^c = f^c \ast \tilde{f} = 1 \) \( \Sigma \)-almost everywhere on \( \partial \mathbb{B} \) is equivalent to \( |f| = 1 \) \( \Sigma \)-almost everywhere on \( \partial \mathbb{B} \). Then, if \( f \) is inner, we immediately get the first implication of the statement.

Suppose now that \( \tilde{f} \ast f^c = f^c \ast \tilde{f} = 1 \) \( \Sigma \)-almost everywhere on \( \partial \mathbb{B} \), i.e. that \( |f| = 1 \) \( \Sigma \)-almost everywhere on \( \partial \mathbb{B} \). Let \( g \in H^2(\mathbb{B}) \), \( g \neq 0 \), and denote by \( Z_{g^*} = \{q \in \mathbb{B} : g^*(q) = 0\} \). Proposition 2.3 in \cite{11} guarantees that \( \Sigma(Z_{g^*}) = 0 \), and Proposition 5.32 in \cite{8} that the map \( T_g : \partial \mathbb{B} \setminus Z_{g^*} \to \partial \mathbb{B} \setminus Z_{g^*} \) is a bijection, so we have
\[
\|f \ast g\|^2_{H^2} = \|g^* \ast f^c\|^2_{H^2} = \int_{\partial \mathbb{B} \setminus Z_{g^*}} |g^c(q)|^2 |f^c(T_g(q))|^2 d\Sigma(q)
\]
\[
= \int_{\partial \mathbb{B} \setminus Z_{g^*}} |g^c(q)|^2 d\Sigma(q) = \|g^c\|^2_{H^2} = \|g\|^2_{H^2},
\]
where we used that $|f| = 1 \Sigma$-almost everywhere on $\partial \mathbb{B}$ if and only if the same holds true for $|f^c|$ (see [6, Proposition 5]). This implies that $f$ is a multiplier for $H^2(\mathbb{B})$. Thanks to [1, Corollary 3.5] we can conclude that $f \in H^\infty(\mathbb{B})$ and thus that $f$ is an inner function. Therefore conditions (i) and (ii) are equivalent.

Clearly (ii) implies (iii). Suppose now that condition (iii) holds. Then, for almost every $t \in [0, \pi)$ we have both
\[ 1 = \tilde{f} \ast f^c(e^{it}) \quad \text{and} \quad 1 = \tilde{f} \ast f^c(e^{(t+\pi)i}) = \tilde{f} \ast f^c(e^{-it}). \]

Using Formula (2) we obtain then that for any $J \in \mathbb{S}$
\[ \tilde{f} \ast f^c(e^{iJ}) = \frac{1 - JI}{2} + \frac{1 + JI}{2} = 1. \]

Recalling that $d\Sigma(e^{iJ}) = dt d\sigma(I)$, see (7), we immediately get (ii). \qed

**Remark 3.2.** We remark that the previous proof actually showed that condition (iii) implies that $(\tilde{f} \ast f^c)_I = (f^c \ast \tilde{f})_I = 1$ almost everywhere on $\partial \mathbb{B}_I$ for any $I \in \mathbb{S}$.

By means of the previous result we obtain another characterization of inner functions in $H^2(\mathbb{B})$ which is often used as the definition of inner in more abstract Hilbert spaces (see [12]). Recall the notations from (6) and (9) and denote by $\delta_k(j)$ the Kronecker delta.

**Theorem 3.3.** Let $f \in H^2(\mathbb{B})$. The following are equivalent:

(i) $f$ is inner;

(ii) For all $k \in \mathbb{N}$, we have
\[ \langle q^k \ast f, f \rangle = \delta_k(0); \]

(iii) There exists $I \in \mathbb{S}$ such that, for all $k \in \mathbb{N}$, we have
\[ \langle q^k \ast f, f \rangle_I = \delta_k(0). \]

**Proof.** Let $f(q) = \sum_{n \in \mathbb{N}} q^n a_n \in H^2(\mathbb{B})$. Extending the notion of $\ast$-product to the setting of slice $L^2$ functions, and by the definition of the inner product, we can compute, for $q \in \partial \mathbb{B}$,
\[ \langle q^k \ast f, f \rangle = \sum_{n \geq 0} q^{n+k} a_n \sum_{n \geq 0} q^n a_n = \sum_{n \geq \max\{0,k\}} a_n a_{n-k}. \]

Moreover, for $\Sigma$-almost any $q \in \partial \mathbb{B}$, it holds
\[ f^c(q) = \sum_{n \geq 0} q^n \overline{a_n} \quad \text{and} \quad \tilde{f}(q) = \sum_{n \leq 0} q^n a_{-n}. \]
Hence
\[ f^c \ast \tilde{f}(q) = \sum_{k \in \mathbb{Z}} q^k \sum_{n \in \mathbb{Z}} a_n a_{n-k} = \sum_{k \in \mathbb{Z}} q^k \sum_{n \geq \max\{0, k\}} a_n a_{n-k}. \]

Therefore, we obtain that for any \( k \in \mathbb{N} \), \( \langle q^k \ast f, f \rangle \) is the \( k \)-th coefficient in the power series expansion of \( f^c \ast \tilde{f} \). From this fact and Proposition 3.1, it is now easy to deduce that (ii) follows from (i). The reverse implication can be proved using the natural extension of the \( H^2 \) inner product to the bigger space of slice \( L^2 \) functions
\[ \langle \sum_{n \in \mathbb{Z}} q^n a_n, \sum_{n \in \mathbb{Z}} q^n b_n \rangle_{L^2} = \sum_{n \in \mathbb{Z}} b_n a_n. \]

In fact suppose (ii) holds, then the \( k \)-th coefficient in the power series expansion of \( f^c \ast \tilde{f} \) vanishes for \( k > 0 \) and equals 1 for \( k = 0 \). To show that all the coefficients with \( k = -n < 0 \) equal zero consider
\[ \langle q^{-n} \ast f, f \rangle_{L^2} = \langle f, q^n \ast f \rangle_{L^2} = \langle q^n \ast f, f \rangle = 0, \]
thus the first part of the theorem is proved. From (10), (ii) is equivalent to (iii): the inner product of \( H^2(\mathbb{B}) \) can be computed on a single slice and does not depend on the choice of slice.

In general, the restriction \( f_I \) of \( f \) to the slice \( L_I \) is a function of one complex variable, but still quaternion-valued. If \( f_I \) were a complex-valued function, then \( f_I \) would be a truly inner function of \( H^2(\mathbb{D}) \). Therefore, Theorem 3.3 guarantees that the restriction to any slice \( L_I \) of a slice regular inner function of \( H^2(\mathbb{B}) \) is “almost” an inner function of \( H^2(\mathbb{D}) \). At this point it is natural to question about a simple converse: we wonder whether any inner function \( F \in H^2(\mathbb{D}) \) admits a slice regular extension \( f := \text{ext}(F) \) to the whole unit ball \( \mathbb{B} \) such that \( f \) is inner for \( H^2(\mathbb{B}) \). Here we are identifying \( \mathbb{D} \) with \( \mathbb{B}_i \). Then, the following corollary of Theorem 3.3 holds.

**Corollary 3.4.** Let \( F \in H^2(\mathbb{D}) \) be an inner function. Then, the slice regular extension \( f = \text{ext}(F) \) is an inner function of \( H^2(\mathbb{B}) \).

**Proof.** Since (10) guarantees that the inner product of \( H^2(\mathbb{B}) \) can be computed by restricting to any slice, it is clear that \( f = \text{ext}(F) \in H^2(\mathbb{B}) \) whenever \( F \in H^2(\mathbb{D}) \). The conclusion now follows from Theorem 3.3 since condition (iii) is satisfied for \( I = i \). \( \square \)

We point out that there exist inner functions of \( H^2(\mathbb{B}) \) that are truly of the quaternionic setting: they are not just the slice regular extension of some inner function of \( H^2(\mathbb{D}) \). In fact, if a slice regular function \( f \) is the extension of a complex inner function \( F \), then it necessarily preserves the slice \( L_i \), i.e. \( f(\mathbb{B}_i) \subset L_i \). Recalling Proposition 2.2, we have that if a function preserves
a slice, then all its isolated, non-spherical, zeros are contained in that slice. It is enough to take a slice regular Blaschke product which has at least two zeros that are not on the same slice (and neither on the same sphere). See [5] for an explicit construction of such a function.

So far we looked at how \( f \) being inner in \( H^2(\mathbb{B}) \) reflects on a restriction of \( f \) to any slice. We would also want to understand how being inner affects the splitting components of the function (see Lemma 2.1). The following is our best result in that sense.

**Theorem 3.5.** Let \( f \in H^2(\mathbb{B}) \), \( I, J \in \mathbb{S} \), \( J \) orthogonal to \( I \), and \( F, G : \mathbb{B}_I \to \mathbb{C} \) holomorphic functions so that for any \( z \in \mathbb{B}_I \),

\[
f_I(z) = F(z) + G(z)J.
\]

(12)

Then, \( f \) is inner if and only if for almost every \( z \in \partial \mathbb{B}_I \) (with respect to the Lebesgue measure of \( \partial \mathbb{B}_I \)) the following conditions hold:

\[
\begin{align*}
|F(z)|^2 + |G(z)|^2 &= 1 \\
F(z)G(\overline{z}) &= F(\overline{z})G(z)
\end{align*}
\]

(13)

**Proof.** If \( F, G \) are the splitting components of \( f_I \) as in (12), then, following [8, Chapter 1], we get

\[
f^c_I(z) = \overline{F(\overline{z}) - G(z)J} \quad \text{and} \quad \tilde{f}_I(z) = F(\overline{z}) + G(\overline{z})J.
\]

(14)

Consider now the power series expansion

\[
f(q) = \sum_{n \in \mathbb{N}} q^n a_n = \sum_{n \in \mathbb{N}} q^n (\alpha_n + \beta_n J)
\]

where \( \alpha_n, \beta_n \in L_I \). Then

\[
F(z) = \sum_{n \in \mathbb{N}} z^n \alpha_n, \quad \overline{F(z)} = \sum_{n \in \mathbb{N}} z^n \overline{\alpha_n}, \quad G(z) = \sum_{n \in \mathbb{N}} z^n \beta_n, \quad \overline{G(z)} = \sum_{n \in \mathbb{N}} z^n \overline{\beta_n},
\]

and hence, for almost every \( z \in \partial \mathbb{B}_I \),

\[
(\tilde{f} * f^c)_I(z) = \left( \sum_{n \leq 0} z^n (\alpha_{-n} + \beta_{-n} J) \right) \ast \left( \sum_{n \geq 0} z^n (\overline{\alpha_n} - \beta_n J) \right)
\]

\[
= \sum_{n \in \mathbb{Z}} z^n \sum_{k \leq \min\{0, n\}} (\alpha_{-k} + \beta_{-k} J)(\overline{\alpha_n - k} - \beta_{n-k} J)
\]

\[
= \sum_{n \in \mathbb{Z}} z^n \sum_{k \leq \min\{0, n\}} (\alpha_{-k} \overline{\alpha_n - k} + \beta_{-k} \beta_{n-k}) + (\beta_{-k} \alpha_{n-k} - \alpha_{-k} \beta_{n-k}) J
\]

\[
= F(\overline{z})F(\overline{z}) + G(\overline{z})\overline{G(z)} + (G(\overline{z})F(z) - F(\overline{z})G(z))J.
\]

(15)
Combining (15) with Remark 3.2 we get
\[ F(\overline{z})F(z) + G(\overline{z})G(z) + (G(\overline{z})F(z) - F(\overline{z})G(z))J = 1 \]
for almost every \( z \in \partial B_I \). This holds if and only if (13) is satisfied. \( \square \)

We conclude this section showing that the characterization of inner functions in [12] involving their \( H^2 \) and \( H^\infty \) norms works as well in the quaternionic setting.

**Theorem 3.6.** Let \( f \in H^2(\mathbb{B}) \). The following are equivalent:

(i) \( f \) is inner;

(ii) \( \|f\|_{H^2} = \|f\|_{H^\infty} = 1 \);

(iii) \( \|f\|_{H^2} = 1 \) and for all \( k \in \mathbb{N} \) and \( \lambda \in \mathbb{H} \) we have
\[ \|(q^k + \lambda) * f\|_{H^2} \leq \|q^k + \lambda\|_{H^2} \]

**Proof.** Clearly, if \( f \) is inner, then its \( H^\infty \) norm equals 1 and
\[ \|f\|^2_{H^2} = \int_{\partial \mathbb{B}} |f|^2 d\Sigma = 1. \]
and so (i) implies (ii). It is also clear that (ii) implies (iii), if we bear in mind that the multiplier space of \( H^2 \) coincides isometrically with \( H^\infty \), since condition \( \|f\|_{H^\infty} = 1 \) means then that \( \|g * f\| \leq \|g\| \) for any \( g \in H^2 \). To see that (iii) implies (i), apply Theorem 3.3 and if \( \|f\|_{H^2} = 1 \) and \( f \) is not inner, then there must be \( k \in \mathbb{N} \setminus \{0\} \) such that
\[ \langle q^k \ast f, f \rangle \neq 0. \]
Choose such \( k \) and notice that for all \( \lambda \in \mathbb{H} \), we have
\[ \|q^k + \lambda\|^2_{H^2} = 1 + |\lambda|^2. \]

Let us compute the left hand side in the condition from (iii) with a \( \lambda \) to be chosen later:
\[ \|(q^k + \lambda) \ast f\|^2_{H^2} = \|q^k f\|^2_{H^2} + |\lambda|^2 \|f\|^2_{H^2} + 2 \text{Re}(\overline{\lambda} \langle q^k \ast f, f \rangle). \]

Since the shift is an isometry over its image on \( H^2(\mathbb{B}) \), we see that the first two components of the right-hand side sum to \( \|q^k + \lambda\|^2 \). Therefore, choosing \( \lambda = \langle q^k \ast f, f \rangle \) will contradict our hypothesis in (iii). \( \square \)

4. Outer functions

Already in [11], it was shown that the Definition 2.5 of outer functions is equivalent to the concept of cyclicity, extending a celebrated result of Beurling. Recall that a function \( g \in H^2(\mathbb{B}) \) is cyclic if \([g] \), the smallest (closed) subspace of \( H^2(\mathbb{B}) \) invariant under the action of the shift, is the whole \( H^2(\mathbb{B}) \). What is currently missing in the quaternionic setting is an
analogous of the classical characterization of outer functions on the unit disk in terms of the logarithm; namely, \( f \in H^2(\mathbb{D}) \) is outer if and only if
\[
  f(z) = \alpha \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \left| \frac{e^{i\theta} + z}{e^{i\theta} - z} \right| d\theta \right\}
\]
for any \( z \in \mathbb{D} \). In this section we prove some preliminary results that go in the direction of finding an analogous characterization for quaternionic outer functions. We provide some necessary conditions as well as sufficient ones for functions to be outer. Some of these conditions are in terms of the symmetrization \( f^s \) of the function \( f \). We will see that in some cases \( f \) being outer is equivalent to \( f^s \) being outer. However \( f^s \) is slice preserving, hence a logarithm characterization for \( f^s \) to be outer is available.

Most of our proofs rely on cyclicity rather than outerness, thus similar proofs could work for spaces in which outer and cyclic functions do not coincide necessarily.

Our first finding is rather unsurprising but needed:

**Lemma 4.1.** Let \( f \in H^2(\mathbb{B}) \). Then, \( f \) is outer if and only if \( f^c \) is outer.

**Proof.** Let \( f = f_i \ast f_o \) be the inner-outer factorization of \( f \) where \( f_i \) denotes the inner part of \( f \) and \( f_o \) denotes the outer part. Assume that \( f \) is outer, that is, \( f = f_o \), and suppose that \( f^c \) is not cyclic, that is, \( f^c = (f^c)_i \ast (f^c)_o \). Then,
\[
  f = f_o = ((f^c)_o)^c \ast ((f^c)_i)^c.
\]
In particular, thanks to [11, Proposition 2.3], for \( \Sigma \)-almost every \( q \in \partial \mathbb{B} \),
\[
  |f(q)| = |f_o(q)| = |((f^c)_o)^c \ast ((f^c)_i)^c(q)| = |((f^c)_o)^c(q)|.
\]
Since \( f \) is outer, we get
\[
  |f(q)| = |f_o(q)| \geq |((f^c)_o)^c(q)|
\]
for any \( q \in \mathbb{B} \). On the other hand, recall that a function is inner if and only if its conjugate function is inner ([11, Proposition 2.1]), hence \( |((f^c)_i)^c(q)| \leq 1 \) inside the ball and we get also
\[
  |f(q)| = |f_o(q)| = |((f^c)_o)^c(q)| |((f^c)_i)^c(T_{f^c}(q))| \leq |((f^c)_o)^c(q)|
\]
where \( T_{f^c}(q) = ((f^c)_o)^c(q)^{-1} q ((f^c)_o)^c(q) \). We remark here that \( (f^c)_o \) never vanishes as well. As a consequence, \( T_{f^c} \) is a bijection of \( \mathbb{B} \) to itself; see [8, Proposition 5.32].

Therefore, we obtain that \( |((f^c)_i)^c| = 1 \) in \( \mathbb{B} \), hence, for the maximum modulus principle in the quaternionic setting, we conclude that \( (f^c)_i)^c \equiv \alpha \) where \( \alpha \) is a quaternion of modulus 1. Thus,
\[
  f^c = \overline{\alpha} \ast (f^c)_o,
\]
hence $f^c$ is outer and the proof is concluded.

Let us now introduce the concept of optimal approximants which will be needed in what follows.

**Definition 4.2.** Let $f \in H^2(\mathbb{B})$ and $\mathcal{P}_n := \{p(q) = \sum_{k=0}^{n} q^k a_k : a_k \in \mathbb{H}\}$. A polynomial $p_n \in \mathcal{P}_n$ is an optimal approximant of degree $n$ of $f^*$ if $p_n$ is such that $\|f \ast p_n - 1\|_{H^2(\mathbb{B})} = \min \{\|f \ast p - 1\|_{H^2(\mathbb{B})} : p \in \mathcal{P}_n\}$.

The existence and uniqueness of such a minimizer is guaranteed by the projection theorem for quaternionic Hilbert space, see [10]. In particular, the minimizer $f \ast p_n$ is given by the orthogonal projection of the constant function 1 on the closed subspace $f \ast \mathcal{P}_n \subseteq H^2(\mathbb{B})$.

The constant function 1 plays a special role because it is cyclic. Then, to check cyclicity of a function $f$ is equivalent to show that its optimal approximants satisfy

$$\|f \ast p_n - 1\|_{H^2} \to 0 \quad \text{as} \quad n \to \infty. \quad (17)$$

In fact, if the constant function 1 satisfies equation (17), then it belongs to $[f]$. The fact that this is a closed and invariant subspace, guarantees then that $H^2(\mathbb{B}) = [1] \subseteq [f]$, that is that $f$ is cyclic.

The following result states the relationship between the invariant subspace generated by an $H^2$ function $f$ and the inner-outer factorization of $f$.

**Lemma 4.3.** Let $f \in H^2(\mathbb{B})$ factorize as $f = f_i \ast f_o$, where $f_i$ is inner and $f_o$ is outer. Then $[f] = [f_i]$.

*Proof.* Let $\{p_n\}_{n \in \mathbb{N}}$ be a sequence of polynomials such that $\|f_o \ast p_n - 1\|_{H^2} \to 0$ as $n$ tends to $\infty$. Since $f_i$ is bounded, it is a multiplier and, thus, we have that

$$\|f \ast p_n - f_i\|_{H^2} \leq \|f_i\|_{H^\infty} \|f_o p_n - 1\|_{H^2}.$$ 

This shows that $f_i \in [f]$ and hence $[f_i] \subseteq [f]$. The fact that $[f] \subseteq [f_i]$ follows taking into account that $f_i$ is inner and hence the invariant subspace $[f_i] = f_i \ast H^2(\mathbb{B})$, which clearly contains $f$, since $f_o \in H^2(\mathbb{B})$. \qed

**Lemma 4.4.** Let $f, g \in H^\infty(\mathbb{B})$. Then $f \ast g$ is cyclic if and only if both $f$ and $g$ are cyclic.

Before the proof, notice that even in the complex case, the assumption that $f, g \in H^\infty(\mathbb{D})$ cannot be discarded since there are functions in $H^2(\mathbb{D})$ whose square is not an element of $H^2(\mathbb{D})$.

*Proof.* To show that $f \ast g$ cyclic implies $g$ cyclic, thanks to Lemma 4.1 it will be enough to show that $f \ast g$ cyclic implies $f$ cyclic and then apply the result to $g^c \ast f^c$. Suppose now that $f \ast g$ is cyclic and consider the inner-outer
factorization of \( f = f_i * f_o \) with \( f_i \) inner and \( f_o \) outer. Then \( [f] = f_i * H^2(\mathbb{B}) \).
Hence \( f * g = f_i * (f_o * g) \) is an element of \([f]\), recalling that the fact that \( f_o \in H^2(\mathbb{B}) \) and that \( g \in H^\infty(\mathbb{B}) \) guarantees that \( f_o * g \in H^2(\mathbb{B}) \) (see [5]). Then \([f * g] \subseteq [f]\) but \([f * g] = H^2(\mathbb{B})\). Thus, \( f \) is cyclic.

Now suppose that \( f \) and \( g \) are cyclic, and let \( \{p_n\}_{n \in \mathbb{N}} \) and \( \{r_m\}_{m \in \mathbb{N}} \) be respectively the sequences of optimal approximants of \( f \) and \( g \). Then for each \( n, m \in \mathbb{N} \), from the triangle inequality we have
\[
\|(f * g) * (r_m * p_n) - 1\|_{H^2} \leq \|(f * g) * (r_m * p_n) - f * p_n\|_{H^2} + \|f * p_n - 1\|_{H^2}.
\]

The last term on the right hand side will be arbitrarily small whenever \( n \) is large enough. The other one may be bounded using the fact that \( \|\epsilon \| \leq \|c\| \). The previous result will prove particularly useful when applied to the symmetrization \( f^s = f * f^c = f^c * f \) of a function \( f \in H^2(\mathbb{B}) \).

For each \( p \in [1, \infty] \), the function \( f^s \) is in \( H^p \) provided that \( f \) is in \( H^{2p}(\mathbb{B}) \); in particular if \( f \in H^\infty(\mathbb{B}) \), then \( f^s \in H^\infty(\mathbb{B}), \) see [5].

**Corollary 4.5.** Let \( f \in H^\infty(\mathbb{B}) \). Then \( f \) is cyclic if and only if \( f^s \) is cyclic.

**Proof.** If \( f \) is bounded, so is \( f^c \) (see [6]) and so we can apply both Lemma 4.4 and 4.1.

As we mentioned, the importance of \( f^s \) comes from the fact that it preserves slices, that is, it can be seen as a holomorphic complex-valued function on each slice. This is important because we can transfer the theory from the disk to the quaternionic ball. In the following theorem we denote by \( H^2(\mathbb{B}_I) \), \( I \in \mathbb{S} \), the function space defined as
\[
H^2(\mathbb{B}_I) = \left\{ f \in \mathbb{B}_I \to L_I : f(z) = \sum_{n=0}^{\infty} z^n \alpha_n, \{\alpha_n\} \subseteq \ell^2(\mathbb{N}, L_I) \right\}.
\]
It is clear that \( H^2(\mathbb{B}_I) \) can be identified with \( H^2(\mathbb{D}) \).

**Theorem 4.6.** Let \( f \in H^\infty(\mathbb{B}) \). The following are equivalent:

(i) \( f \) is cyclic in \( H^2(\mathbb{B}) \);
(ii) \( f \) is outer in \( H^2(\mathbb{B}) \);
(iii) \( f^s \) is cyclic in \( H^2(\mathbb{B}) \);
(iv) \( f^s \) is outer in \( H^2(\mathbb{B}) \);
(v) \( f_i^* \) is cyclic in \( H^2(\mathbb{B}_I) \) for all \( I \in \mathbb{S} \);
(vi) there exists \( I \in \mathbb{S} \) such that \( f_i^* \) is cyclic \( H^2(\mathbb{B}_I) \);
(vii) \( f_i^* \) is outer in \( H^2(\mathbb{B}_I) \) for all \( I \in \mathbb{S} \);
(viii) there exists \( I \in \mathbb{S} \) such that \( f_i^* \) is outer in \( H^2(\mathbb{B}_I) \).

**Proof.** The equivalence between (i), (ii), (iii) and (iv) is guaranteed by Corollary 4.5 and by [11, Theorem 4.2]. Also, since \( f_i^* \) is a holomorphic function, the equivalence between (v) and (vii) and that of (vi) and (viii) is a well known result, consequence of the Beurling Theorem.

Let us now prove that (iii) implies (v) and (vi). Let \( f^s \) be cyclic in \( H^2(\mathbb{B}) \) and let \( I \in \mathbb{S} \). Then, for any \( g \in H^2(\mathbb{B}_I) \) there exists a sequence of quaterniionic polynomials \( \{p_n\}_{n \in \mathbb{N}} \) such that \( \|f^s \ast p_n - \text{ext } g_I\|_{H^2} \) tends to zero, as \( n \) goes to infinity. Let now \( p_n(z) = P_n(z) + Q_n(z)J \) be the splitting of \( p_n \) with respect to \( J \in \mathbb{S} \), orthogonal to \( I \). Then, evaluating the \( H^2 \) norm on the slice \( L_I \) we get that

\[
\|f^s \ast p_n - \text{ext } g_I\|^2_{H^2} = \|f_i^*(P_n + Q_nJ) - g_I\|^2_{H^2(\mathbb{B}_I)}
= \|f_i^*P_n - g_I\|^2_{H^2(\mathbb{B}_I)} + \|Q_n\|^2_{H^2(\mathbb{B}_I)}
\]

where the last equality is due to the orthogonality of \( I \) and \( J \). Then the sequence of complex polynomials \( \{P_n\}_{n \in \mathbb{N}} \) of the variable \( z \in \mathbb{B}_I \) is such that \( \|f_i^*P_n - g_I\|_{H^2(\mathbb{B}_I)} \) tends to zero, as \( n \) goes to infinity, that is \( f_i^* \) is cyclic in \( H^2(\mathbb{B}_I) \). To conclude, it suffices to show that (vi) implies (iii), since clearly (v) implies (vi). Suppose then that \( f_i^* \) is cyclic in \( H^2(\mathbb{B}_I) \), for some \( I \in \mathbb{S} \). Consider now \( g \in H^2(\mathbb{B}) \) and let \( g(z) = F(z) + G(z)J \) be its splitting on \( \mathbb{B}_I \) with respect to \( J \in \mathbb{S} \), \( J \) orthogonal to \( I \). By hypothesis, there exist two sequences \( \{P_n\}_{n \in \mathbb{N}} \) and \( \{Q_n\}_{n \in \mathbb{N}} \) of complex polynomials in \( \mathbb{B}_I \) such that \( \|f_i^*P_n - F\|_{H^2(\mathbb{B}_I)} \) and \( \|f_i^*Q_n - G\|_{H^2(\mathbb{B}_I)} \) tend to zero as \( n \) goes to infinity. Then, using again the fact that the \( H^2 \) norm can be computed on any slice, and the orthogonality of \( I \) and \( J \), we have that

\[
\|f^s \ast \text{ext } (P_n + Q_nJ) - g\|^2_{H^2} = \|f_i^*(P_n + Q_nJ) - g_I\|^2_{H^2(\mathbb{B}_I)}
= \|f_i^*(P_n + Q_nJ) - (F + GJ)\|^2_{H^2(\mathbb{B}_I)}
= \|f_i^*P_n - F\|^2_{H^2(\mathbb{B}_I)} + \|f_i^*Q_n - G\|^2_{H^2(\mathbb{B}_I)}
\]

tends to zero as \( n \) goes to infinity, thus we can conclude that \( f^s \) is cyclic in \( H^2(\mathbb{B}) \). \qed

Notice that the concept of outer in (vi) and (viii) is the classical one and therefore may be expressed as in (16), that is, in terms of the mean value properties of the logarithm. Again, the assumption on \( f \) being in \( H^\infty(\mathbb{B}) \) cannot simply be dropped, not only because we need \( f^s \) to be defined as a \( H^2(\mathbb{B}) \) function so that we can apply Beurling’s theorem to it (and this
would be guaranteed if \( f \in H^4(\mathbb{B}) \), see [5]), but also because of the applicability of Lemma 4.4, for which we need \( f \in H^\infty(\mathbb{B}) \).

The presence in Theorem 4.6 of the hypothesis \( f \in H^\infty(\mathbb{B}) \) is likely unsatisfactory. We have however not been able to show any characterization in terms of mean value properties for more general \( f \), but sufficient conditions may be shown. For \( \omega \in \mathbb{B} \) let \( \tau_\omega \) denote the automorphism of the unit ball taking 0 to \( \omega \), see [13],

\[
\tau_\omega(q) = (1 - q\overline{\omega})^{-*}(\omega - q),
\]

and let \( I_\omega \) be the imaginary unit identified by \( \omega \), \( I_\omega = \begin{cases} \frac{\omega - \text{Re}\omega}{|\omega - \text{Re}\omega|} & \text{if } \omega \text{ is not real}, \\ I_\omega & \text{otherwise} \end{cases} \) so that

\[
(\tau_\omega)_I_\omega(z) = (1 - z\overline{\omega})^{-1}(\omega - z)
\]

for any \( z \in \mathbb{B} \cap L_{I_\omega} \).

**Proposition 4.7.** Let \( f \in H^2(\mathbb{B}) \). Suppose that for all \( \omega \in \mathbb{B} \) we have

\[
\frac{1}{2\pi} \int_{\partial B_{I_\omega}} \log |f_{I_\omega} \circ \tau_\omega(e^{\theta I_\omega})| d\theta = \log |f(\omega)|,
\]

(18)

Then \( f \) is outer.

**Proof.** Suppose that \( g \) is a function such that on \( \partial \mathbb{B} \) we have \( |f| = |g| \) \( \Sigma \)-almost everywhere, and let \( \omega \in \mathbb{B} \). Then we have

\[
\log |f(\omega)| = \frac{1}{2\pi} \int_{\partial B_{I_\omega}} \log |f_{I_\omega} \circ \tau_\omega(e^{\theta I_\omega})| d\theta.
\]

Since \( |f| \) is equal to \( |g| \) on the boundary, the right-hand side is equal to

\[
\frac{1}{2\pi} \int_{\partial B_{I_\omega}} \log |g_{I_\omega} \circ \tau_\omega(e^{\theta I_\omega})| d\theta.
\]

Notice that the composition \( g_{I_\omega} \circ \tau_\omega \) is well defined on the slice \( L_{I_\omega} \) and it is indeed the restriction of the slice regular function \( \text{ext}(g_{I_\omega} \circ \tau_\omega) \). Recalling that the logarithm of the modulus of a slice regular function is subharmonic (see [5]), we get that

\[
\frac{1}{2\pi} \int_{\partial B_{I_\omega}} \log |g_{I_\omega} \circ \tau_\omega(e^{\theta I_\omega})| d\theta \geq \log |g(\omega)|.
\]

All this together, yields that \( |f(\omega)| \geq |g(\omega)| \). Since \( \omega \) was arbitrary, \( f \) is outer. \( \square \)
5. Optimal approximants

We pose ourselves the task now of extending as much as possible the theory of optimal approximants to the quaternionic setting. A good account of the theory of such polynomials in the classical holomorphic setting is given in [3].

Recall that \( \mathcal{P}_n = \{ p(q) = \sum_{k=0}^{n} a_k q^k : a_k \in \mathbb{H} \} \). The reproducing kernel of the subspace \( f \star \mathcal{P}_n \) exists since it is a closed subspace of \( H^2(\mathbb{B}) \) which is itself a reproducing kernel Hilbert space, the kernel function being \( k(q, w) = (1 - q\bar{w})^{-1} \).

Let \( \{ f \star \varphi_k \}_{k=0}^{n} \) be an orthonormal basis of \( f \star \mathcal{P}_n \), where \( \varphi_k \) is a polynomial of degree \( k \) for any \( k = 0, \ldots, n \). Then, by a standard argument in quaternionic reproducing kernel Hilbert spaces theory, the reproducing kernel of \( f \star \mathcal{P}_n \) is given by

\[
K_n(q, w) = \sum_{k=0}^{n} f \star \varphi_k(q) \langle K_n(q, w), f \star \varphi_k \rangle_{H^2(\mathbb{B})} = \sum_{k=0}^{n} f \star \varphi_k(q) \overline{f \star \varphi_k(w)}.
\]

Notice that, since \( f \star p_n \) is the orthogonal projection of \( 1 \),

\[
f \star p_n(q) = \sum_{k=0}^{n} f \star \varphi_k(q) \langle 1, f \star \varphi_k \rangle_{H^2(\mathbb{B})} = \sum_{k=0}^{n} f \star \varphi_k(q) \overline{f \star \varphi_k(0)},
\]

i.e.

\[
f \star p_n(q) = K_n(q, 0). \tag{19}
\]

In particular, if \( f(0) = 0 \) then \( p_n \equiv 0 \) for all \( n \in \mathbb{N} \).

**Theorem 5.1.** Let \( f \in H^2(\mathbb{B}) \) be such that \( f(0) \neq 0 \) and let \( p_n \) be the optimal approximant of \( f^{-*} \) of degree \( n \). Then all the zeros of \( p_n \) lie outside the closed unit ball \( \mathbb{B} \).

**Proof.** First let us show that we can reduce the problem to optimal approximants of degree 1. Let \( \lambda \) be a zero of an optimal approximant \( p_n \) for the function \( f \in H^2(\mathbb{B}) \). Then there exists \( \hat{\lambda} \) on the same two dimensional sphere of \( \lambda \), such that \( p_n^c(q) = (q - \lambda) \cdot \hat{p}_n^c \), so that \( p_n(q) = \hat{p}_n \ast \lambda \). Then the optimality of \( p_n \) guarantees that \( (q - \hat{\lambda}) \) is the optimal approximant of degree 1 for the function \( f \ast p_n \in H^2(\mathbb{B}) \) which implies that \( \hat{\lambda} \) is also a zero of degree 1 optimal approximant. Therefore to understand the possible positions of any such zero it is enough to understand the same question for \( n = 1 \). Now, suppose that \( p_1(q) = (q - \lambda) c \) is the optimal approximant for \( f \in H^2(\mathbb{B}) \). Then, by the definition of the orthogonal projection, \( f \ast p_1 - 1 \) must be orthogonal to \( f \ast q \), which translates easily in the equation

\[
0 = \langle f \ast p_1, f \ast q \rangle = \langle f \ast qc - f \lambda c, f \ast q \rangle,
\]
which implies that
\[ \langle f * q, f * q \rangle c = \langle f, f * q \rangle \lambda c \]
and therefore that
\[ |\lambda| = \frac{\|f * q\|^2}{|\langle f, f * q \rangle|}. \tag{20} \]
Notice that \( f * q \) is never a multiple of \( f \) unless \( f \equiv 0 \) (which is against our hypothesis), and hence we can apply Cauchy-Schwarz inequality as a strict inequality to \( \langle f, f * q \rangle \) in (20) to get
\[ |\lambda| > \frac{\|f * q\|}{\|f\|}. \]
Since \( f * q = q * f \) and the shift is an isometry, the right-hand side is equal to 1.

Notice that all points that are outside the closed unit ball are indeed zeros of some optimal approximants: indeed, if \( p_1(q) = q - \lambda \) with \( |\lambda| > 1 \) then \( p_1^* \in H^2(\mathbb{B}) \) and hence \( \|p_1^* \cdot p_1 - 1\| = 0 \) so \( p_1 \) must be the only optimal approximant.

We can further understand the relationship between optimal approximants and orthogonal polynomials in the spirit of [3]:

**Theorem 5.2.** Let \( f \in H^2(\mathbb{B}) \) and let \( p_n \) be the optimal approximant of degree \( n \) of \( f^{-*} \). Let \( \{f * \varphi_k\}_{k=0}^n \) be an orthonormal basis of \( f * \mathcal{P}_n \), where \( \varphi_k \in \mathcal{P}_k \). Then, the following are equivalent:

(i) \( f \) is cyclic;
(ii) \( p_n(0) \) converges to \( f^{-*}(0) \) as \( n \to \infty \);
(iii) \( \sum_{k=0}^\infty |\varphi_k(0)|^2 = |f^{-*}(0)|^2 \).

**Proof.** Bearing in mind that \( f * p_n \) is the orthogonal projection of 1, we can see that
\[ \|f * p_n - 1\|^2 = \langle 1 - f * p_n, 1 - f * p_n \rangle = \langle 1 - f * p_n, 1 \rangle = 1 - f(0) * p_n(0). \]
This shows that (i) (the left-hand side tends to 0) is equivalent to (ii) (the right-hand side does). Also, from (19) we see that either (i) or (ii) is equivalent to \( 1 - K_n(0, 0) \to 0 \) as \( n \to \infty \). However, \( K_n(0, 0) \) tending to 1 is equivalent to (iii). \( \square \)

6. **Some open problems**

We consider that the topic needs more development. We propose a few questions that seem natural from where we stand, beyond the obvious elimination of the boundedness hypothesis in Theorem 4.6.
(A) If \( f \in H^2(\mathbb{B}) \), let \( f = f_i * f_o \) and \( f^c = (f^c)_i * (f^c)_o \) be the inner-outer factorization of \( f \) and on its conjugate function. Then we also have \( f = f_i * f_o = [(f^c)_o]^c * [(f^c)_i]^c \). Is there any relationship between these two factorization? Is there something that can be said about the inner-outer factorization of \( f^s \)?

(B) Suppose that the symmetrization \( f^s \) of a function \( f \in H^2(\mathbb{B}) \) is inner. Is it true that \( f \) (or \( f^c \)) is inner?

(C) Is the sufficient condition on Proposition \[4,7\] necessary for a function to be outer? This can be shown for \( f^s \) under the assumption that \( f \) is a multiplier.

(D) The boundary values of slice components of a quaternionic inner function form what is usually called a \textit{pythagorean pair}, a special situation in which two functions have modulus 1 everywhere when seen as one function in \( \mathbb{T}^2 \). Such pairs arise in connections with so-called de Branges-Rovnyak spaces and other areas of mathematics. Can anything else be said about this relation at all?

**Acknowledgements.** The first author is partially supported by the 2015 PRIN grant \textit{Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis} of the Italian Ministry of Education (MIUR).

The second author is partially supported by INdAM-GNSAGA, by the 2014 SIR grant \textit{Analytic Aspects in Complex and Hypercomplex Geometry} and by \textit{Finanziamento Premiale FOE 2014 Splines for accurate NumeRics: adaptIve models for Simulation Environments} of the Italian Ministry of Education (MIUR).

The third author is grateful for the financial support by the Severo Ochoa Programme for Centers of Excellence in R&D (SEV-2015-0554) at ICMAT, and by the Spanish Ministry of Economy and Competitiveness, through grant MTM2016-77710-P.

Part of this project was carried out during a visit of the first and third author at the University of Firenze, and we wish to thank the Department of Mathematics and Computer Sciences for the financial support and the warm hospitality.

**References**

[1] \textsc{Alpay, D.}, \textsc{Bolotnikov, V.}, \textsc{Colombo, F.}, \textsc{Sabadini, I.}, Self-mappings of the quaternionic unit ball: multiplier properties, the Schwarz-Pick inequality, and the Nevanlinna-Pick interpolation problem, \textit{Indiana Univ. Math. J.} \textbf{64} (2015) no. 1, 151–180.

[2] \textsc{Arcozzi, N.} and \textsc{Sarfatti, G.}, Invariant metrics for the quaternionic Hardy space, \textit{J. Geom. Anal.} \textbf{25} (2015) no. 3, 2028–2059.

[3] \textsc{Bénétaleau, C.}, \textsc{Khabinson, D.}, \textsc{Liaw, C.}, \textsc{Seco, D.}, and \textsc{Sola, A. A.}, Orthogonal polynomials, reproducing kernels, and zeros of optimal approximants, \textit{J. Lond. Math. Soc.} \textbf{94} (2016) no. 3, 726–746.

[4] \textsc{Chalendar, I.}, \textsc{Gorkin, P.}, and \textsc{Partington, J. R.}, Inner functions and operator theory, \textit{North-West. Eur. J. Math.} \textbf{1} (2015) 7–22.
[5] DE FABRITIS, C., GENTILI, G., and SARFATTI, G., Quaternionic Hardy spaces, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 18 (2018) no. 2, 697–733.

[6] DELLA ROCCHETTA, C., GENTILI, G., SARFATTI, G., A Bloch-Landau theorem for slice regular functions, in *Advances in Hypercomplex Analysis*, ed. by G. Gentili, I. Sabadini, M. V. Shapiro, F. Sommen, D. C. Struppa, Springer INdAM Series, Springer, Milan, 2013, pp. 55-74.

[7] GARNETT, J. B., *Bounded analytic functions*, Academic Press Inc., 1981.

[8] GENTILI, G., STOPPATO, C., and STRUPPA, D. C., *Regular functions of a quaternionic variable*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013.

[9] GHIOLONI, R., PEROTTI, A., Slice regular functions on real alternative algebras, *Adv. Math.*, 226 (2011) 1662-1691.

[10] JAMISON, J. E., *Extension of some theorems of complex functional analysis to linear spaces over the quaternions and Cayley numbers*, Thesis (Ph.D.)–University of Missouri - Rolla, ProQuest LLC, Ann Arbor, MI, 1970, 178 pp.

[11] MONGUZZI, A. and SARFATTI, G., Shift invariant subspaces of slice $L^2$ functions, *Ann. Acad. Sci. Fenn. Math.* 43 (2018) 1045–1061.

[12] SECO, D., A characterization of Dirichlet inner functions, *Complex Anal Oper. Th.*, online first.

[13] STOPPATO, C., Regular Moebius transformations of the space of quaternions, *Ann. Global Anal. Geom.*, 39 (2010) 387-401.

**DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA C. SALDINI 50, 20133 MILANO, ITALY**

*E-mail address*: alessandro.monguzzi@unimi.it

**DIPARTIMENTO DI MATEMATICA E INFORMATICA “U. DINI”, UNIVERSITÀ DI FIRENZE, VIALE MORGAGNI 67/A, 50134 FIRENZE, ITALY**

*E-mail address*: giulia.sarfatti@unifi.it

**INSTITUTO DE CIENCIAS MATEMÁTICAS, CALLE NICOLÁS CABRERA 13-15, 28049 MADRID, SPAIN.**

*E-mail address*: dsf_cm@yahoo.es