The origin of degree correlations in the Internet and other networks

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It has been argued that the observed anticorrelation between the degrees of adjacent vertices in the network representation of the Internet has its origin in the restriction that no two vertices have more than one edge connecting them. Here we introduce a formalism for modeling ensembles of graphs with single edges only and derive values for the exponents and correlation coefficients characterizing them. Our results confirm that the conjectured mechanism does indeed give rise to correlations of the kind seen in the Internet, although only a part of the measured correlation can be accounted for in this way.

I. INTRODUCTION

The statistical properties of networks have been the topic of considerable attention in the physics literature in recent years \cite{1,2,3,4}. Motivated by the availability of large-scale structural data for networks including the Internet, the World Wide Web, and social and biological networks of various kinds, researchers have created a wide selection of models of networks and processes taking place on networks. One topic of particular current interest is the issue of degree correlations in networks. A network or graph is in general composed of some set of nodes or “vertices” joined together by lines or “edges,” and the degree of a vertex is defined to be the number of edges connected to the vertex. It has been found that for many real-world networks the degrees of the vertices at either end of an edge are not independent, but are correlated with one another, either positively or negatively \cite{5,6,7}. A network in which the degrees of adjacent vertices are positively correlated is said to show assortative mixing by degree, whereas a network in which they are negatively correlated is said to show disassortative mixing. A striking pattern that emerges when networks of different types are compared is that most social networks appear to be assortatively mixed, whereas most technological and biological networks appear to be disassortative \cite{8,7}.

Of particular interest to us in this paper is the Internet. At the time of writing, the Internet forms a network of about 11,000 vertices and 32,000 edges, and, as first pointed out by Pastor-Satorras et al. \cite{6}, the degrees of adjacent vertices have significant anticorrelation. This is demonstrated by calculating the mean degree $\bar{d}$ of the neighbors of a vertex $v$ in the network as a function of the degree $k$ of that vertex. The resulting function is found to fall off with increasing $k$ roughly as a power-law $k^{-\nu}$ with exponent $\nu \approx 0.5$, so that the higher the degree $k$ of the one vertex, the lower the mean degree of its neighbors.

In a recent paper, Maslov et al. \cite{6} have proposed a possible explanation for this result. Rather than supposing the anticorrelation of vertex degrees to be the result of some specific social or engineering constraints on the construction of data networks, they suggest instead a topological explanation. Using computer simulations, they show for a network of the size and degree sequence of the Internet that the requirement that there is at most one edge between any pair of vertices induces degree anticorrelations very similar to those observed. And indeed there are no double edges in the Internet, a statistically unlikely occurrence were we given complete freedom about how vertices were connected.

The physical intuition behind the suggestion of Maslov et al. is that the restriction to single edges causes high-degree vertices to have fewer connections between them than they would if edges were assigned purely at random, and hence there must be more connections between high-degree/low-degree vertex pairs instead. A similar explanation could apply in the case of other types of networks as well, such as directed networks. The World Wide Web and foodwebs are two examples of directed networks that appear to be disassortative and usually have no double edges \cite{21}.

In this paper we study the mechanism proposed by Maslov et al. analytically, and demonstrate that it does indeed produce disassortative mixing by degree of precisely the type observed by Pastor-Satorras et al. \cite{2}. The particular model chosen by Maslov et al. to test their idea turns out to be difficult to treat analytically. They studied the ensemble of all graphs with a particular degree sequence and at most one edge between any vertex pair, in which each allowed graph appears with equal probability. Calculating the correlations in this ensemble requires us to enumerate binary matrices with given row and column sums. No closed-form solution for such an enumeration is known at present despite decades of study by mathematicians \cite{9,10}. In this paper, therefore, we take a different approach, borrowing a trick from statistical mechanics. We study an expanded “grand canonical” ensemble of graphs in which the number of edges is allowed to vary under the action of a chemical potential. As network size becomes large, the number of edges becomes narrowly peaked and the predictions of the model become similar to those of the model of Maslov et al., while the calculations are far easier. (A grand canonical ensemble of graphs has also been studied recently by Dorogovtsev et al. \cite{11}, although using a different formal-
isim and to a different purpose.)

For networks with power-law degree distributions, we will show that indeed \( k_v^n \) falls off as a power of \( k_v \) and derive the value of the exponent \( \nu \). We also calculate the value of the degree correlation coefficient for adjacent vertices, which measures the amount of disassortative mixing in the network. We show that the mechanism of Maslov et al. can account for some, but not all, of the disassortativity seen in the Internet, suggesting that there are also other mechanisms contributing to the observed degree correlations.

II. DEFINITIONS

The classic model in the study of graphs with arbitrary degree sequences is the so-called configuration model \[ \{12\ 13\ 14\ 15\ 16\}, \] in which one specifies the degree \( k_v \) of each vertex \( v = 1\ldots n \) in a network, which also fixes the total number of edges to be \( m = \frac{1}{2} \sum_v k_v \). Subject to the given degree sequence, the vertices are randomly wired to one another. The combinatorics of this model are however awkward and so Chung and Lu \[ 17\] have recently proposed an alternative model that is in many ways more convenient. (Models similar to that of Chung and Lu have also been introduced independently by Dorogovtsev et al. \[ 11\] and Caldarelli et al. \[ 18\].) As we will show, by making use of an extension of their model we can make tractable the problem of counting graphs with single edges only. The model of Chung and Lu deals with undirected networks, and we consider that case first. A fairly straightforward generalization to directed networks will be dealt with briefly.

A. The network model of Chung and Lu

In the model of Chung and Lu \[ 17\] one specifies the desired degrees \( \tilde{k}_v \) of vertices \( v \) and then places edges between vertex pairs \( (v, w) \) with probability

\[
f_{vw} = \frac{\tilde{k}_v \tilde{k}_w}{2\tilde{m}},
\]

where \( \tilde{m} = \frac{1}{2} \sum_v \tilde{k}_v \) is the desired number of edges in the graph. The expected degree of vertex \( v \) is then

\[
\tilde{k}_v = \sum_w f_{vw} = \frac{\tilde{k}_v}{2\tilde{m}} \sum_w \tilde{k}_w = \tilde{k}_v.
\]

Thus the expected degree of each vertex is equal to its desired degree and the expected degree distribution is asymptotically equal to the distribution of the desired degree sequence, although any individual vertex may have a degree that differs from its desired value. (Throughout this paper, we denote desired values of quantities by a tilde (e.g., \( \tilde{k} \)), expected values or ensemble means by a bar (e.g., \( \bar{k} \)), and actual values in a particular graph by undecorated characters (e.g., \( k \)).

However, this approach is not entirely satisfactory. For some degree distributions the probability \( f_{vw} \) can exceed one. Physically, this means that there can be more than one edge between a pair of vertices, precisely the situation that we will want to exclude in our calculations. Chung and Lu circumvent this problem by specifying an additional constraint on the distribution of desired degrees, \( \tilde{k}_v \leq \sqrt{2\tilde{m}} \) for all \( v \). While this condition ensures that \( f_{vw} \leq 1 \), it is strongly violated by networks like the Internet that have power-law degree distributions. Here, therefore, we adopt an alternative strategy, and adapt the model of Chung and Lu to incorporate an explicit condition that there is only one edge between every vertex pair. As we will see, this leads to some interesting new physics, and in particular to an explanation of the origin of disassortativity.

B. Ensemble of networks with single edges

We consider explicitly an ensemble of networks in which there is only a single edge between any pair of vertices. There will be an edge between the pair \( (v, w) \) with probability \( f_{vw} \) or not with probability \( 1 - f_{vw} \). Then the probability of occurrence of a particular graph \( G \) can be written

\[
\Gamma(G) = \prod_{(v, w)} (1 - f_{vw}) \prod_{\text{edges}} \frac{f_{vw}}{1 - f_{vw}}
\]

where the first product is over all unique vertex pairs \( (v, w) \) and the second is over only those pairs between which there is an edge. For convenience, we will write \( P_{vw} = f_{vw}/(1 - f_{vw}) \), \( \Gamma_0 = \prod_{(v, w)} (1 - f_{vw}) \), and define \( \delta_{vw} \) to be 1 if there is an edge between \( v \) and \( w \) and zero otherwise. Then

\[
\Gamma(G) = \Gamma_0 \prod_{(v, w)} P_{vw}^{\delta_{vw}}.
\]

To progress, we need to choose a form for \( P_{vw} \), or equivalently for \( f_{vw} \). We here make the particular choice of the factorizable form

\[
P_{vw} = \beta \lambda_v \lambda_w,
\]

where \( \beta \) is a free parameter that will control the total number of edges in the graph, and the fugacity \( \lambda_v \) is a real number assigned to each vertex \( v \) that will control the expected degree of that vertex. (Note that this choice is not the same as that of Chung and Lu, Eq. \[ 11\], although in the “classical limit” of graphs with few double edges it becomes the same. See below.)

The justification for our choice of \( P_{vw} \) is that we would like all graphs with a given degree sequence to appear in our ensemble with equal probability. This is the same criterion applied by Maslov et al. \[ 3\] in their simulations, and allows us to compare our results with theirs. To see
that the criterion is satisfied in this case, we need only observe that
\[ \Gamma(G) = \Gamma_0 \beta^n \prod_v k_v^{k_v}, \]  
(6)
where \( k_v \) is the actual degree of \( v \) in the particular graph \( G \) and \( m = \frac{1}{2} \sum_v k_v \) is the actual number of edges. Clearly for given \( \beta \) and \( \{ \lambda_v \} \) this expression is a function only of the degree sequence \( \{ k_v \} \).

III. PREDICTIONS AND RESULTS

We now define a grand partition function
\[ Z = \sum_G \Gamma(G) = \Gamma_0 \sum_{\{ k_{vw} \}} \prod_{v,w} P_{vw}^{k_{vw}}, \]  
(7)
Interchanging the order of sum and product this gives
\[ Z = \prod_{v,w} \sum_{k_{vw}} P_{vw}^{k_{vw}} = \prod_{v,w} (1 + P_{vw}) \]
\[ = \prod_{v,w} (1 + \beta \lambda_v \lambda_w), \]  
(8)
where we have dropped the factor of \( \Gamma_0 \). (As is typically the case with partition functions, leading factors of this type cancel out of all observable quantities in the theory.)

From Eq. (8) we can now see that the expected degree \( \bar{k}_v \) of vertex \( v \) will be given by
\[ \bar{k}_v = \lambda_v \frac{\partial Z}{\partial \lambda_v} = -\lambda_v \frac{\partial F}{\partial \lambda_v}, \]  
(9)
where \( F \) is the free energy
\[ F = -\log Z = -\sum_{(v,w)} \log(1 + \beta \lambda_v \lambda_w). \]  
(10)
Combining Eqs. (9) and (10), we then get
\[ \bar{k}_v = \sum_w \frac{\beta \lambda_v \lambda_w}{1 + \beta \lambda_v \lambda_w}. \]  
(11)
The expected number of edges \( \bar{m} \) is the ensemble mean of the exponent of \( \beta \) in the partition function, which is given by
\[ \bar{m} = -\beta \frac{\partial F}{\partial \beta} = \sum_{(v,w)} \frac{\beta \lambda_v \lambda_w}{1 + \beta \lambda_v \lambda_w}. \]  
(12)
The mean degree of the entire system \( \bar{z} \) is simply \( 2\bar{m}/n \), where \( n \) is the total number of vertices.

There are clear parallels between these results and the familiar Fermi ensemble of elementary statistical mechanics. The quantity \( f_{vw} \) introduced earlier, which we can now write in the form
\[ f_{vw} = \frac{\beta \lambda_v \lambda_w}{1 + \beta \lambda_v \lambda_w}, \]  
(13)
lies strictly in the range from 0 to 1, and represents the probability that an edge lies between a particular pair of vertices. This is the equivalent of the Fermi function of statistical mechanics.

The mean sum of the degrees of the neighbors of a vertex \( v \), which we denote \( K_{nn} \), is given by
\[ K_{nn} = \sum_w f_{vw} \tilde{k}_w = \sum_w \frac{\beta \lambda_v \lambda_w \tilde{k}_w}{1 + \beta \lambda_v \lambda_w}, \]  
(14)
with \( \tilde{k}_w \) given by Eq. (11), and the mean degree of a neighbor of \( v \) is equal to \( k_{nn} = K_{nn}/k_v \). We will also want to calculate the correlation coefficient of the degree of vertices at either end of an edge \( \tilde{r} \), whose value is given by
\[ \tilde{r} = \frac{\sum_v \tilde{k}_v K_{nn} -(2\bar{m})^{-1} \left[ \sum_v \tilde{k}_v^2 \right]^2}{\sum_v \tilde{k}_v^2 -(2\bar{m})^{-1} \left[ \sum_v \tilde{k}_v^2 \right]^2}. \]  
(15)
Although in this paper we will be dealing primarily with undirected networks, generalization of the theory to directed networks is straightforward. If \( f_{vw} \) denotes the probability of existence of a directed edge from \( v \) to \( w \) and \( P_{vw} \), defined as before, then the expected out-degree (number of outgoing edges) of a vertex \( v \) will be
\[ \bar{k}_v^{\text{out}} = \sum_w f_{vw} = \sum_w \frac{P_{vw}}{1 + P_{vw}}, \]  
(16)
the expected in-degree (number of incoming edges) will be
\[ \bar{k}_v^{\text{in}} = \sum_w f_{wv} = \sum_w \frac{P_{wv}}{1 + P_{wv}}, \]  
(17)
and the obvious generalizations of Eqs. (12) and (14) apply.

A. Example: power-law degree distribution

We are here particularly interested in the case of the Internet, which, like a number of other networks, has a degree distribution that approximately follows a power law
\[ p_k \propto k^{-\tau}, \]  
(18)
with \( \tau = 2.2 \pm 0.3 \) \[14,20\]. The long tail of the power law means that the highest degree vertex pairs in the network would be quite likely to have more than one edge running between them were edges assigned at random, and the behavior of the network changes substantially when these multiple edges are disallowed. This is the origin of the effects observed by Maslov et al. \[8\] in their simulations.

As we now show, the power-law degree distribution can be reproduced in our model by choosing the fugacity \( \lambda \)
also to have a power-law distribution with the same exponent \( \tau \), so that the number of vertices with fugacity between \( \lambda \) and \( \lambda + d\lambda \) is \( p(\lambda) d\lambda \), where

\[
p(\lambda) = \begin{cases} C\lambda^{-\tau} & \text{for } \lambda \geq \lambda_0 \\ 0 & \text{for } \lambda < \lambda_0. \end{cases}
\]

(19)

The lower cutoff makes the distribution normalizable, and \( C \) is a normalizing constant given by

\[
C^{-1} = \int_{\lambda_0}^{\infty} \lambda^{-\tau} d\lambda = \frac{\lambda_0^{-\tau+1}}{\tau - 1}. \quad (20)
\]

(Bear in mind that \( \lambda \) is not restricted, as the degree is, to integer values.) Let us consider the case \( \tau = \frac{5}{2} \), for which the expressions for the quantities of interest take particularly simple forms. For this choice, the expected degree \( \bar{k}(\lambda) \) of a vertex with fugacity \( \lambda \) is

\[
\bar{k}(\lambda) = n \int_{\lambda_0}^{\infty} \frac{\beta \lambda \lambda'}{1 + \beta \lambda \lambda'} p(\lambda') d\lambda'
\]

\[
= 3n \left[ \beta \lambda_0 \lambda - (\beta \lambda_0 \lambda)^{3/2} \arctan((\beta \lambda_0 \lambda)^{-1/2}) \right], \quad (21)
\]

and the mean degree \( \bar{z} \) of the system is

\[
\bar{z} = \frac{2\pi}{n} = 2 \int_{\lambda_0}^{\infty} \bar{k}(\lambda) p(\lambda) d\lambda
\]

\[
= 9n \beta \lambda_0^2 \left[ 1 - \frac{1}{\beta \lambda_0^2} \Phi \left( \frac{1}{\beta \lambda_0^2}, 2, \frac{3}{2} \right) \right], \quad (22)
\]

where \( \Phi(x, a, b) \) denotes the analytic continuation of the Lerch transcendent.

The parameter \( \beta \) is to some extent redundant in these expressions, since we are free to choose \( \lambda \) as we wish, but it proves convenient nonetheless. If we choose \( \beta = (2\bar{m})^{-1} \), where \( \bar{m} \) is the desired number of edges as before, then for graphs in which there are few double edges we have

\[
f_{vw} \approx (\lambda_v \lambda_w)/(2\bar{m}),
\]

and

\[
\bar{k}_v = \sum_w f_{vw} \approx \sum_w (\lambda_v \lambda_w)/(2\bar{m}) \approx \lambda_v,
\]

so that the fugacity is simply equal to the desired degree of a vertex, as in the model of Chung and Lu \cite{17}.

The regime in which there are few double edges can be thought of as the “classical limit” of our Fermi ensemble, and corresponds to the case where the first terms in Eqs. (21) and (22) dominate. As \( \lambda \) becomes large, however, encouraging vertices to have high degree, we enter the quantum regime, where it becomes harder and harder for vertices to find others to connect to. This is reflected in Eq. (21) also. Expanding the inverse tangent as \( \arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - O(x^7) \), we find that the leading term cancels and

\[
\bar{k}(\lambda) = n - \frac{3n}{5\beta \lambda_0} + nO((\beta \lambda_0 \lambda)^{-2}). \quad (23)
\]

Thus, as \( \lambda \to \infty \) the degree tends to \( n \), as we would expect, since this is the largest degree a vertex can have on a network with no double edges.

![FIG. 1: The ensemble mean \( \bar{k} \) of the degree of a vertex in our model as a function of the fugacity \( \lambda \) of the vertex. The numerical results are averaged over 1000 repetitions of the simulation. The dotted line indicates the form \( \bar{k} = \lambda \), which the curve is expected to approximate for small \( \lambda \).](image)

The mean sum \( \bar{K}^{nn}(\lambda) \) of the degrees of the neighbors of a vertex with fugacity \( \lambda \) is

\[
\bar{K}^{nn}(\lambda) = n \int_{\lambda_0}^{\infty} \frac{\beta \lambda \lambda'}{1 + \beta \lambda \lambda'} \bar{k}(\lambda') p(\lambda') d\lambda'
\]

\[
= 9n^2 (\beta \lambda_0 \lambda)^{3/2} \left[ \frac{\lambda_0}{\lambda} \arctan((\beta \lambda_0 \lambda)^{-1/2}) \right.
\]

\[
- \frac{\pi}{4} \left( \frac{\lambda_0}{\lambda} \right)^{3/2} \left[ 2\log(1 + (\lambda/\lambda_0)^{1/2}) \right.
\]

\[
\left. - \log(1 + \beta \lambda_0 \lambda) + O(\lambda_0 \beta \lambda^{1/2}) \right] \right], \quad (24)
\]

and from this we can calculate \( \bar{K}^{nn} \) \cite{22}.

These results can be extended to other values of \( \tau \) also, although the formulas are not as elegant as for the case \( \tau = \frac{5}{2} \). For example, for general \( \tau > 1 \) the equivalent of Eq. (21) is

\[
\bar{k}(\lambda) = n_2 F_1 \left( 1, -1 + \tau; \tau; -\frac{1}{\beta \lambda_0 \lambda} \right), \quad (25)
\]

where \( _2 F_1 \) is a hypergeometric function. This form is used in some of the calculations in the next section.

### B. Comparison with the Internet

We now compare our model quantitatively with the Internet graph. To do this, it is important that we make the size \( n \) and number of edges \( \bar{m} \) the same as the real Internet, since our predictions, Eqs. (21) and (22), are dependent on these quantities. For the purposes of comparison, we use the data of Chen et al. \cite{20} from 2001.
Consider first Fig. 1, which is a plot of the mean degree of a vertex as a function of its fugacity. As the figure shows, the degree is closely linear in the fugacity for small $\lambda$ and flattens off as degree approaches $n$, as expected.

The same behavior is evident in Fig. 2 also, which shows the cumulative distribution function of degrees in simulations of the model for power-law distributed fugacity, Eq. (19). The distribution of degrees also follows a power law (a straight line on the logarithmic axes used), until degree approaches $n$, where the distribution is cut off. This is eminently sensible behavior: given the constraint of single edges only, presumably the real Internet must deviate from power-law behavior for large degree, and our model should and does reflect this behavior.

The fundamental result of this paper is shown in Fig. 3, where we have plotted the mean degree $\bar{k}_{nn}$ of the neighbors of a vertex, calculated from Eq. (24), against the degree of that vertex. This is the comparison used by Pastor-Satorras et al. to demonstrate degree anticorrelation in the Internet. As the figure shows, there is a clear decline in the value of $\bar{k}_{nn}$ as degree increases, just as in the real Internet, confirming that the single-edge constraint does indeed give rise to anticorrelations, as conjectured by Maslov et al. Moreover, the decline appears to be approximately power-law in form $\bar{k}_{nn} \sim \bar{k}^{-\nu}$, as found by Pastor-Satorras et al. We can deduce approximate values for the exponent $\nu$ from our results. We find for $\tau = 2.1$, $\nu \approx 0.65$, for $\tau = 2.3$, $\nu \approx 0.55$, and for $\tau = 2.5$, $\nu \approx 0.42$. The slopes are shown as the dotted lines in Fig. 2. The values for $\nu$ are all close to the value $\nu \approx 0.5$ observed for the real Internet.

The power law is only approximate however—the functional form of Eq. (24) is not just a simple power law, and we can see from the figure that the slope of $\bar{k}_{nn}$ is smaller for smaller $\bar{k}$. The same behavior is visible in both the real Internet data and the simulation results of Maslov et al.

Finally, in Table I we show values for the mean degree $\bar{z}$ and degree correlation coefficient $r$ for our model. As we see, the theoretical calculations and numerical results again agree well. Since the Internet is disassortative, we expect the degree correlation coefficient to be negative in the real network, and its value has been measured to be $r = -0.189$ [7]. In the model we also see negative values of $r$, whose magnitude depends quite strongly on the degree of the exponent $\tau$. A detailed comparison of model and real-world data may therefore have to wait on more precise measurements of the degree distribution (about which there is at present some dispute [20]). However it is interesting to note that none of the cases in Table I
is as strongly anticorrelated as the real Internet. Thus our calculations appear to indicate that some of the disassortativity in the Internet can be accounted for by the mechanism proposed by Maslov et al., but probably not all of it. The remainder of the disassortativity is presumably due to engineering or social constraints on the structure of the network. One possibility, which has been discussed elsewhere, is that the Internet is divided into connectivity providers such as phone companies and ISPs, who tend to have very high degree, since they have lots of customers, and end users of connectivity, who typically have a degree of only one or two. Most connections in the network run from the providers to the end users and are therefore from high to low degree, giving a social reason for disassortativity in addition to the purely topological one considered here.

IV. DISCUSSION AND CONCLUSIONS

In this paper we have studied analytically ensembles of networks where there is at most one edge between any pair of vertices. By making use of an enlarged ensemble in which the number of edges is allowed to vary, in a manner reminiscent of the Fermi ensemble of traditional statistical mechanics, we have been able to find closed form expressions for ensemble averages of a number of quantities of interest. In particular, we have confirmed the previous numerical finding that graph ensembles with single edges have negative correlations between the degrees of adjacent vertices. This has been proposed as a possible explanation for the anticorrelation or disassortativity observed in the topology of the Internet. We find that the restriction to single edges only can account for some but not all of the correlations observed in real Internet data.

The same mechanism could be responsible for disassortativity in other networks also. Many networks, including citation networks, the World Wide Web, social networks, collaboration networks, metabolic and genetic regulatory networks, and food webs have, at least in their most common representations, only single edges between vertex pairs. Thus it is reasonable to suppose that these networks would be disassortative also, and indeed this appears to be the case for most networks that have been studied. There is one important exception to this rule however: social networks almost all appear to be significantly assortative in their mixing patterns. We conjecture, therefore, that disassortativity by degree is the normal state of affairs for a network, as a result of the mechanisms described in this paper, with social networks being assortative probably because of additional social effects that are absent from other network types; for one reason or another, it appears that gregarious people prefer to associate with other gregarious people. Furthermore, when assessing the level of assortativity in a social network, one should take into account the natural tendency for networks to be disassortative, since this tendency implies that to reach a level even of neutral assortativity would take a moderately strong bias in favor of positive degree correlation, and reaching a substantially assortative state would take a very strong such bias.

Finally, we point out that the general analytical technique employed in this paper, of enlarging an ensemble of graphs to create a “grand canonical” ensemble, may have applications to other problems in the study of networks also. It is well known among statistical physicists that using such an ensemble often makes the analytic treatment of a problem easier, and the results presented here offer hope that this approach may prove useful in other settings.

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[21] The Web can have more than one link from one page to another, but in most studies of the topology of the Web graph duplicate links have been neglected.

[22] To be precise, we note that our formulas give $\bar{k}_{nn}$ as a function of the expected degree of a vertex, rather than its actual degree. However in the large-degree tail, where the power-law behavior of interest is observed, we expect that actual degree will be narrowly peaked around the expected value.