How to find the holonomy algebra of a Lorentzian manifold

Anton S. Galaev

Abstract. If the holonomy algebra \( g \subset \mathfrak{so}(1, n - 1) \) of a locally indecomposable Lorentzian manifold \((M, g)\) of dimension \(n\) is different from \(\mathfrak{so}(1, n - 1)\), then it is contained in the similitude algebra \(\mathfrak{sim}(n - 2)\). There are 4 types of such holonomy algebras. We give criterion how to find the type of \(g\). To each \(g\) there is a canonically associated subalgebra \(h \subset \mathfrak{so}(n - 2)\). We provide an algorithm how to find \(h\). We also give algorithms for obtaining the de Rham-Wu decomposition for Riemannian and Lorentzian manifolds. These results show how one can find the holonomy algebra of an arbitrary Lorentzian manifold.

Keywords: Lorentzian manifold, holonomy group, holonomy algebra, de Rham-Wu decomposition

MSC codes: 53C29, 53B30, 53C50

1. Introduction

The classification of connected holonomy groups (equivalently, of holonomy algebras) of Riemannian manifolds is a classical result that has many applications both in geometry and theoretical physics, see \([6, 10, 19, 21]\) and references therein. In particular, it gives rise to different special geometries, e.g. Kählerian manifolds or Spin(7)-manifolds. To find the holonomy algebra of an indecomposable Riemannian manifolds, one may use the fact that Riemannian manifolds with different holonomy algebras have different geometric properties, in particular, they admit different parallel forms, see \([6]\) and Section 4 below. Another approach using the curvature tensor can be found in \([28]\).

By the de Rham decomposition Theorem, any Riemannian manifold \((M, g)\) can be at least locally decomposed into a product of a flat Riemannian manifolds and of Riemannian manifolds with irreducible holonomy groups. This decomposition is trivial, i.e. \((M, g)\) is the only manifold in the product, if and only if the holonomy group of \((M, g)\) is irreducible. In Section 3 we give an algorithm how to find the de Rham decomposition of a Riemannian manifold, then at the end of Section 4 we explain how one can find the holonomy algebra of an arbitrary Riemannian manifold.

The main object of this paper are holonomy algebras of Lorentzian manifolds. The Wu decomposition Theorem is the analog of the de Rham Theorem for pseudo-Riemannian manifolds. We provide an algorithm how to obtain it for a Lorentzian manifold in Section 5. After that we assume that the Lorentzian manifold \((M, g)\) under the consideration is locally indecomposable, then its holonomy algebra is weakly irreducible, and it does not must be irreducible. The only irreducible holonomy algebra is \(\mathfrak{so}(1, n + 1), \dim M = n + 2 \geq 2\). Other holonomy algebras are contained in the similitude algebra \(\mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \rtimes \mathbb{R}^n\), and this is the most interesting case. These holonomy algebras are classified recently in \([5, 26, 13, 14]\).

Lorentzian manifolds with special holonomy algebras are discussed e.g. in \([2, 3, 7, 17, 24, 25, 27]\). Holonomy algebras of 4-dimensional Lorentzian manifolds and their relation to General Relativity are studied e.g. in \([22]\). Recently an attention to the holonomy algebras of Lorentzian manifolds of arbitrary dimension is paid in \([8, 10, 18, 20]\).

It is natural to ask the question: How to find the holonomy algebra of a Lorentzian manifold? In the paper we give a complete answer to this question.
To a holonomy algebra $\mathfrak{g} \subset \mathfrak{sim}(n)$ one associates its $\mathfrak{so}(n)$-projection $\mathfrak{h}$, which must be a holonomy algebra of a Riemannian manifold \cite{26} and it is called the orthogonal part of $\mathfrak{g}$. We provide an algorithm how to find $\mathfrak{h}$ in Section \[9\].

Next, there are 4 types of holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}(n)$. The algebras of type 1 and 2 have simple structure and they are of the form $(\mathbb{R} \oplus \mathfrak{h}) \times \mathbb{R}^n$ and $\mathfrak{h} \times \mathbb{R}^n$, respectively. The algebras of types 3 and 4 are more exotic, and they can be obtained from these two by some twisting. In Section \[10\] we provide criteria that allow to find the type of the holonomy algebra. Similar criteria are given in \[7\]. Our criteria are more concrete: we show how the type of the holonomy algebra can be found using the local coordinates and it becomes computable.

Thus we provide the complete algorithm that allows to find the holonomy algebra of an arbitrary Lorentzian manifold. This algorithm can be computerized, since it requires computations of certain parallel tensors (e.g. parallel symmetric bilinear forms and certain differential forms that can be found as the solutions to some systems of partial differential equations), some computations in linear algebra, and computations in local coordinates.

### 2. Holonomy group; holonomy algebra

The theory of holonomy algebras of pseudo-Riemannian manifolds can be found e.g. in \[6, 21, 15\].

Let $(M, g)$ be a connected pseudo-Riemannian manifold of signature $(r, s)$ ($r$ is the number of minuses in the signature of the metric $g$). We will be interested in the case of Riemannian manifolds ($r = 0$, i.e. $g$ is positive definite) and in the case of Lorentzian manifolds ($r = 1$).

Denote by $\nabla$ the Levi-Civita connection on $M$ defined by the metric $g$: $\nabla g = 0$. Let $\gamma: [a, b] \subset \mathbb{R} \rightarrow M$ be a piecewise smooth curve on $M$. The connection $\nabla$ defines the parallel transport $\tau_\gamma: T_{\gamma(a)}M \rightarrow T_{\gamma(b)}M$, which is an isomorphism of the pseudo-Euclidean spaces $(T_{\gamma(a)}M, g_{\gamma(a)})$ and $(T_{\gamma(b)}M, g_{\gamma(b)})$.

The holonomy group $G_x$ of $(M, g)$ at a point $x \in M$ is the Lie group that consists of the pseudo-orthogonal transformations given by the parallel transports along all piecewise smooth loops at the point $x$. It can be identified with a Lie subgroup of the pseudo-orthogonal Lie group $O(r, s) = O(T_xM, g_x)$. The corresponding subalgebra $\mathfrak{g}_x$ of $\mathfrak{so}(r, s) = \mathfrak{so}(T_xM, g_x)$ is called the holonomy algebra of $(M, g)$ at the point $x \in M$.

The Ambrose-Singer Theorems states that the holonomy algebra $\mathfrak{g}_x$ is spanned by the following endomorphisms of $T_xM$:

$$\tau_\gamma^{-1} \circ R_y(\tau_\gamma X, \tau_\gamma Y) \circ \tau_\gamma,$$

where $\gamma$ is a piecewise smooth curve starting at the point $x$ with an end-point $y \in M$, and $X, Y \in T_xM$.

Since the manifold $M$ is connected, the holonomy groups (holonomy algebras) of $(M, g)$ at different points are isomorphic, and one may speak about the holonomy group $G \subset O(r, s)$ (the holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(r, s)$) of $(M, g)$.

Recall that a tensor field $T$ on $(M, g)$ is parallel if $\nabla T = 0$, or equivalently $T$ is preserved by parallel transports: for any piecewise smooth curve starting at a point $y \in M$ with an end-point $z \in M$ it holds $\tau_\gamma T_y = T_z$, where $\tau_\gamma$ is the extension of the parallel transport along $\gamma$ to tensors.

The fundamental principle states that there exists a one-to-one correspondences between parallel tensor fields $T$ on $M$ and tensors $T_0$ of the same type at $x$ preserved by the tensor extension of the representation of the holonomy group.

If the manifold $M$ is simply connected, then the holonomy group is connected and it is uniquely defined by the holonomy algebra. In this case there exists a one-to-one correspondences between parallel tensor fields $T$ on $M$ and tensors $T_0$ of the same type at $x$ annihilated by the tensor extension of the representation of the holonomy algebra.

Since we are interested in holonomy algebras, in what follows we will assume that the manifold $M$ is simply connected. In general case one can pass to the universal covering $(\tilde{M}, \tilde{g})$.

In general it is impossible to find the holonomy group using the definition, and it is impossible to find the holonomy algebra using the Ambrose-Singer Theorem, since then one should consider
parallel transports along all piecewise smooth loops at a point or parallel transports along all piecewise smooth curves starting at a point. Below we will show how to compute the holonomy algebra of any Riemannian or Lorentzian manifold. For that we will use the classification of the holonomy algebras for these manifolds and the geometric properties of manifolds with each possible holonomy algebra.

3. The de Rham decomposition for Riemannian manifolds

In this section we will see that any Riemannian manifold can be decomposed at least locally in the product of a flat Riemannian manifold and of Riemannian manifolds with irreducible holonomy groups. We will give an algorithm how to obtain this decomposition. This decomposition allows to restrict attention to Riemannian manifolds with irreducible holonomy groups.

We consider a Riemannian manifold \((M, g)\) of dimension \(n\) with the holonomy group \(G \subset O(n)\).

If \((N, h)\) is another Riemannian manifold of dimension \(m\) with the holonomy group \(H \subset O(m)\), then the product \((M \times N, g + h)\) is a Riemannian manifold with the holonomy group \(G \times H \subset O(n + m)\). This statement can be inverted.

The de Rham decomposition Theorem states that if \((M, g)\) is simply connected and complete, then \((M, g)\) can be decomposed into the product of a flat Riemannian manifold \((M_0, g_0)\) and of Riemannian manifolds \((M_1, g_1), \ldots, (M_r, g_r)\) with irreducible holonomy groups. For general \((M, g)\) such decomposition exists only locally.

If the de Rham decomposition for \((M, g)\) is trivial, i.e. \((M, g) = (M_1, g_1)\), then \((M, g)\) is called locally indecomposable. This is equivalent to the irreducibility of the holonomy group of \((M, g)\).

Let us explain where the de Rham decomposition comes from. Let \(x \in M\). Since the holonomy group \(G \subset O(n)\) is totally reducible, the tangent space \(T_x M\) can be decomposed into an orthogonal direct sum

\[
T_x M = E_{0x} \oplus E_{1x} \oplus \cdots \oplus E_{rx},
\]

where \(E_{0x}\) is the subspace consisting of \(G\)-invariant vectors, each subspace \(E_{\alpha x} \subset T_x M, 1 \leq \alpha \leq r,\) is \(G\)-invariant and the induced representation is irreducible.

The subsets \(E_{0x}, \ldots, E_{rx} \subset T_x M\) define parallel distributions \(E_0, \ldots, E_r\) on \(M\), i.e. these distributions are preserved by the parallel transports. These distributions are involutive, and the manifolds \(M_0, \ldots, M_r\) from the above decomposition are maximal integral manifolds of these distributions passing through the point \(x\). The metrics \(g_\alpha\) are the restrictions of \(g\) to these distributions.

The holonomy group of \((M, g)\) is the product

\[
G = G_1 \times \cdots \times G_r,
\]

where \(G_\alpha\) is the restriction of \(G\) to \(E_{\alpha x}\). If the decomposition of \((M, g)\) is global, then \(G_\alpha\) is the holonomy group of the manifold \((M_\alpha, g_\alpha)\). However, if the decomposition is not global, then the holonomy group of \((M_\alpha, g_\alpha)\) is a subgroup of \(G_\alpha\), and these groups must not coincide; in that case \(G_\alpha\) is the holonomy group of the induced connection on the distribution \(E_\alpha\) considered as a vector bundle over \(M_\alpha\).

The task is to find the distributions \(E_\alpha\). We may find \(E_0\) as the distribution consisting of all parallel vector fields. Then we may work with \(E_0^g\) and \(g\) restricted to it. This allows us to assume that \(E_0 = 0\).

Note that if \((M, g)\) is indecomposable, then the dimension of parallel symmetric bilinear forms on \((M, g)\) equals to one. This follows from the Fundamental principle and from the fact that any element in the second symmetric power \(\bigcirc^2 \mathbb{R}^n\) of \(T_x M \cong \mathbb{R}^n\) preserved by the irreducible subgroup \(G \subset O(n)\) is proportional to the metric \(g_x\) at the point \(x\). In general, the dimension of parallel symmetric bilinear forms on \((M, g)\) equals to \(r\) (we assume that \(E_0 = 0\)) and this real vector space is generated by \(g_1, \ldots, g_r\) (here we assume that \(g_\alpha|_{E_\alpha \times E_\alpha} = g|_{E_\alpha \times E_\alpha}\) and \(g_\alpha|_{E_i \times E_j} = 0\) if \(i \neq j\)).

Finding all parallel symmetric bilinear forms on \((M, g)\) (e.g. with Maple), we get an answer in the form

\[
c_1 \tilde{g}_1 + \cdots + c_r \tilde{g}_r,
\]
where $c_1,\ldots,c_r \in \mathbb{R}$ are arbitrary and $\tilde{g}_1,\ldots,\tilde{g}_r$ is a basis of the space of all parallel symmetric bilinear forms on $(M, g)$. Since $g$ is parallel, we may assume that $\tilde{g}_1 = g$. Indeed, we may find a linear independent subsystem in \{g, \tilde{g}_1,\ldots,\tilde{g}_r\} that contains $g$.

We may write
\begin{equation}
\tilde{g}_\alpha = \sum_{\beta=1}^r A_{\beta\alpha} g_\beta, \quad A_{\beta\alpha} \in \mathbb{R}.
\end{equation}

Since we need to find the numbers $A_{\beta\alpha} \in \mathbb{R}$, we may work with a fixed point $x \in M$.

We will decompose $T_x M$. Consider $\tilde{g}_{2x}$, then
\[ T_x M = F \oplus F^\perp, \]
where
\[ F = \{ X \in T_x M | \tilde{g}_{2x}(X,Y) = 0 \text{ for all } Y \in T_x M \} \]
is the kernel of $\tilde{g}_{2x}$ and $F^\perp$ is its orthogonal complement with respect to $g$. Both $F$ and $F^\perp$ consist of some of $E_{\alpha x}$, i.e. this decomposition is orthogonal with respect to all tensors $g_{\alpha x}$. Consider the decomposition $T_x M = F \oplus F^\perp$, take the restrictions of $\tilde{g}_{2x}$ to each of these spaces and decompose $F$ and $F^\perp$ in the same manner. Continue this process for all $\tilde{g}_{\alpha x}$, then we get a decomposition
\begin{equation}
T_x M = F_1 \oplus \cdots \oplus F_s
\end{equation}
such that the restriction of each $\tilde{g}_{\alpha x}$ to any of $F_k$ is either zero or non-degenerate. Now we continue to subdivide this decomposition. Let $\alpha$ run from 2 to $r$. Consider the restrictions $\tilde{g}_{\alpha x}|_{F_k \times F_k}$ and $g_x|_{F_k \times F_k}$ of $\tilde{g}_{\alpha x}$ and $g_x$ to each $F_k$. If the restriction $\tilde{g}_{\alpha x}|_{F_k \times F_k}$ is non-zero (i.e. $\tilde{g}_{\alpha x}|_{F_k \times F_k}$ is non-degenerate), and $g_{\alpha x}|_{F_k \times F_k}$ is not proportional to $g_x|_{F_k \times F_k}$, then instead of $\tilde{g}_{\alpha}$ consider $\tilde{g}_{\alpha} - bg$, where $b \in \mathbb{R}$ is a number such that the restriction of $(\tilde{g}_{\alpha} - bg_x)|_{F_k \times F_k}$ to $F_k$ is degenerate. To find such $b$, take any vector $X \in F_k$ such that $g_x(X,X), \tilde{g}_{\alpha x}(X,X) \neq 0$ (in the case of Riemannian manifolds, any non-zero $X$ satisfies this condition) and set $b = \frac{\tilde{g}_{\alpha x}(X,X)}{g_x(X,X)}$. Using the new tensor $\tilde{g}_{\alpha x}$ we may subdivide $F_k$ (since now $\tilde{g}_{\alpha x}|_{F_k \times F_k}$ is degenerate and non-zero), i.e. we subdivide the decomposition (3). Continue this process. At the end we will get that if the restriction of any $g_{\alpha x}$ to any $F_k$ is non-zero, then it is proportional to $g_x|_{F_k \times F_k}$. This means that the number $s$ in decomposition (3) equals $r$, that is decomposition (3) is the decomposition (1) at the point $x$, i.e. $F_s = E_{\alpha x}$ (up to a renumbering). We may find each tensor $g_{\alpha x}$ as the restriction of $g_x$ to $E_{\alpha x}$. Using (2) considered at the point $x$, we find the matrix $(A_{\beta\alpha})$. Then using (2) and the inverse matrix, we find the metrics $g_{\alpha}$. Now for any $y \in M$ find
\[ E_{\alpha y} = \{ X \in T_y M | g_{\beta y}(X,\cdot) = 0 \text{ for all } \beta \neq \alpha \}. \]

Thus we know the the distributions $E_{\alpha}$.

4. Riemannian manifolds with irreducible holonomy algebras

The results that we review in this section are the major achievements of the holonomy theory, they can be found e.g. in [6] [21].

Possible connected irreducible holonomy groups (i.e. possible irreducible holonomy algebras) of not locally symmetric Riemannian manifolds classified Berger in [4]. Later it was proved that all these algebras can be realized as the holonomy algebras of Riemannian manifolds [9].

The holonomy algebra of a locally symmetric Riemannian space $(M, g)$ at a point $x$ coincides with $\{ R_x(X,Y) | X,Y \in T_x M \}$. Locally $(M, g)$ is isometric to a simply connected symmetric space $H/G$, where $H$ is the group of transvections of that space; the holonomy group of that space coincides with the isotropy representation of the stabilizer $H$ of a point. The list of indecomposable simply connected Riemannian spaces can be found e.g. in [6].

Here we list irreducible holonomy algebras of not locally symmetric Riemannian manifolds and we give the description of the corresponding geometries including the Einstein condition and parallel forms (we do not include trivial parallel forms i.e. constant function and the volume form on an orientable manifold):

- $so(n)$: generic Riemannian manifolds, no parallel forms;
• $u(m) \subset so(2m)$: Kählerian manifolds, parallel Kählerian 2-form and its powers, not Ricci-flat;
• $su(m) \subset so(2m)$: special Kählerian manifolds or Calabi-Yau manifolds, parallel Kählerian 2-form, its powers, parallel complex volume form and its conjugate, Ricci-flat;
• $sp(k) \subset so(4k)$: hyper-Kählerian manifolds, 3 independent parallel Kählerian 2-forms and forms obtained from their combinations, Ricci-flat;
• $sp(k) \oplus sp(1) \subset so(4k)$: quaternionic-Kählerian manifolds, parallel 4-form and its powers, Einstein and not Ricci-flat;
• $spin(7) \subset so(8)$: Ricci-flat, parallel 4-form;
• $G_2 \subset so(7)$: Ricci-flat, a parallel 3-form and its dual.

Compact Riemannian manifolds with the holonomy groups $SU(2)$, $SU(3)$, $G_2$ and $Spin(7)$ are extremely useful in theoretical physics, see [10] [19] [21] and references therein.

Irreducible Riemannian holonomy algebras $\mathfrak{g} \subset so(n)$ that appear as the holonomy algebras of symmetric Riemannian spaces and are different from $so(n)$, $u(m)$, $sp(k) \oplus sp(1)$ are called symmetric Berger algebras.

Now we may easily find the holonomy algebra $\mathfrak{g}$ of any Riemannian manifold $(M, g)$. First suppose that $(M, g)$ is locally indecomposable, i.e. its holonomy algebra is irreducible. If $\nabla R = 0$, then the manifold is locally symmetric and its holonomy algebra at a point $x \in M$ coincides with $\{R_x(X,Y)|X,Y \in T_xM\}$. If $\nabla R \neq 0$, then there are only 7 possibilities for $\mathfrak{g}$. According to the list of the holonomy algebras and to the geometric properties of the corresponding Riemannian manifolds, to find $\mathfrak{g}$ it is enough to compute the Ricci tensor of $(M, g)$ and to find parallel 2,3,4-forms on $(M, g)$ (all that can be done using e.g. Maple); of course, one should also analyze the dimension $n$ of $(M, g)$, e.g. if $n = 9$, then $\mathfrak{g} = so(9)$; if $n = 7$, then $\mathfrak{g} = G_2$ if there exists a parallel 3-form and $\mathfrak{g} = so(7)$ otherwise. Another approach that uses the computation of the curvature tensor is proposed in [28].

Now suppose that $(M, g)$ is locally decomposable. If $(M, g)$ is a global product of Riemannian manifolds, then its holonomy algebra is the direct sum of the corresponding holonomy algebras. If the de Rham decomposition is not global, then the direct sum of the holonomy algebras of the induced connections on the distribution $E_{\alpha}$, $1 \leq \alpha \leq r$ considered as a vector bundle over $M$, see Section 3. The holonomy algebra of each distribution is irreducible and it can be found in the same way as the holonomy algebra of a locally indecomposable Riemannian manifold above. Let $\nabla$ and $R$ be the connection and the curvature of the manifold $(M, g)$. If $\nabla R|_{E_{\alpha} \times E_{\alpha} \times E_{\alpha}} = 0$, then the holonomy algebra of $E_{\alpha}$ coincides with $\{R_x(X,Y)|X,Y \in E_{\alpha}\}$; otherwise it is one of the holonomy algebras from the above list and it can be found analyzing $\text{Ric}|_{E_{\alpha} \times E_{\alpha}}$ and parallel sections of the bundle $\Lambda^k E_{\alpha}$, $k = 2, 3, 4$.

5. The Wu decomposition for Lorentzian manifolds

The Wu decomposition Theorem [30] generalizes the de Rham Theorem for the case of pseudo-Riemannian manifolds. It states that any pseudo-Riemannian manifold $(M, g)$ can be decomposed at list locally into the product of pseudo-Riemannian manifolds $(M_0, g_0), ..., (M_r, g_r)$, but now the holonomy groups of $(M_1, g_1), ..., (M_r, g_r)$ must not be irreducible any more; these groups are weakly irreducible, i.e. each of them does not preserve any proper non-degenerate subspace of the corresponding tangent space. Consequently, a pseudo-Riemannian manifold $(M, g)$ is locally indecomposable if and only if its holonomy group is weakly irreducible. In that case $E_0$ is not the distribution that consists of all parallel vector fields on $(M, g)$, but it is a non-degenerate subdistribution of the last one and in general it is not defined uniquely.

The algorithm of Section 3 works also for pseudo-Riemannian manifolds if we know that the holonomy algebra of each factor in the decomposition is irreducible. In that case the dimension of parallel symmetric bilinear forms equals to the number of the manifolds in the decomposition (without loss of generality we assume that $E_0 = 0$), then (2) holds. The problem is that a locally indecomposable pseudo-Riemannian manifold may admit a parallel light-like vector field $p$, in this case we have an extra parallel symmetric bilinear form $\theta \otimes \theta$, where $\theta = g(p, \cdot)$ is the 1-form
corresponding to $p$. Hence if in the Wu decomposition one of the manifolds satisfies this property, then (2) does not take the place.

Now we consider a Lorentzian manifold $(M, g)$. And we will obtain the Wu decomposition of that manifold.

A.V. Aminova [1] proved that if a locally indecomposable Lorentzian manifold $(M, g)$ admits a parallel bilinear form not proportional to the metric, then $(M, g)$ admits a parallel light-like vector field $p$, the space of parallel bilinear forms is 2-dimensional, and it is spanned by $g$ and $\theta \otimes \theta$, where $\theta$ is the 1-form corresponding to $p$.

It is clear that in the Wu decomposition of a Lorentzian manifold only one manifold is Lorentzian, and all the other are Riemannian. The Lorentzian part is locally indecomposable and admits a parallel light-like vector field if and only if the restriction of $g$ to the space of parallel vector fields is degenerate (this property may be checked at a single point if we restrict the parallel vector fields to that point). If the Lorentzian part is contained in the distribution $E_0$, or it does not admit a parallel light-like vector field, then the algorithm of Section 3 works.

Suppose that the Lorentzian part (that we assume to be $(M, g_r)$) in the Wu decomposition is indecomposable and admits a parallel vector field $p$. Let $E_0 \subset TM$ be the subbundle spanned by all parallel vector fields on $(M, g)$. Then, $p \in \Gamma(E_0)$. Let $E_{0x} \subset E_{0x}$ be any subspace complementary to $\mathbb{R} p_x$. Let $E_0 \subset E_0$ be the subbundle spanned by parallel vector fields with values in $E_{0x}$ at the point $x$. Then $E_0 \subset TM$ is a parallel subbundle and the restriction of $g$ to $E_0$ is non-degenerate. We consider $E_0^\perp$ and the restriction of $g$ to it. Hence we again may assume that $E_0 = 0$. Then the space of parallel bilinear forms on $(M, g)$ is spanned by $g_1, \ldots, g_r, \theta \otimes \theta$ and it is of dimension $r + 1$.

Finding all parallel symmetric bilinear forms on $(M, g)$, we get an answer in the form $c_1 \bar{g}_1 + \cdots + c_{r+1} \bar{g}_{r+1}$, where $c_1, \ldots, c_{r+1} \in \mathbb{R}$ are arbitrary and $\bar{g}_1, \ldots, \bar{g}_{r+1}$ is a basis of the space of all parallel symmetric bilinear forms on $(M, g)$. Since $g$ is parallel, we may assume that $\bar{g}_1 = g$.

There exist real numbers $(C_{\beta \alpha})_{\beta, \alpha = 1}^{r + 1}$ such that

$$\bar{g}_\alpha = \sum_{\beta = 1}^{r} C_{\beta \alpha} g_\beta + C_{r+1} \alpha \theta \otimes \theta.$$  \hspace{1cm} (4)

In particular, $(C_{\beta \alpha})_{\beta = 1}^{r + 1} = (1, 0, \ldots, 0)$.

Let $q_x \in T_x M$ be a light-like vector not proportional to $p_x$, i.e. $\theta_x(q_x) = g(p_x, q_x) \neq 0$. To find such vector it is enough to take any vector $X \in T_x M$ such that $g_x(p_x, X) \neq 0$, and if $g_x(X, X) \neq 0$, then take $q_x = p_x - \frac{2g_x(p_x, X)}{g_x(X, X)} X$. It is clear that

$$\bar{g}_\alpha(q_x, p_x) = C_{r \alpha} g_x(p_x, q_x) = C_{r \alpha} g_x(p_x, q_x), \hspace{1cm} 2 \leq \alpha \leq r + 1.$$

This allows to find the coefficients $C_{r \alpha}$. Changing each $\bar{g}_\alpha$ to $\bar{g}_\alpha - C_{r \alpha} g$, we obtain that $C_{r \alpha} = 0$ for $2 \leq \alpha \leq r + 1$.

The following three lemmas will allow to get the algorithm.

**Lemma 1.** Let $2 \leq \alpha \leq r + 1$. If $C_{r+1} \alpha = 0$, then

$$\ker \bar{g}_\alpha = \oplus_{1 \leq \beta \leq r, C_{\beta \alpha} = 0} E_{\beta x},$$

and

$$\ker g_x |_{\ker \bar{g}_\alpha \times \ker \bar{g}_\alpha} = 0.$$  \hspace{1cm} If $C_{r+1} \alpha \neq 0$, then

$$\ker \bar{g}_\alpha = \oplus_{1 \leq \beta \leq r-1, C_{\beta \alpha} = 0} E_{\beta x} \oplus \{ X \in E_{rx} | \theta(X) = 0 \},$$

and

$$\ker g_x |_{\ker \bar{g}_\alpha \times \ker \bar{g}_\alpha} = \mathbb{R} p_x.$$

**Proof.** Suppose that $C_{r+1} \alpha = 0$. If $C_{\beta \alpha} = 0$ then it is clear that $E_{\beta x} \subset \ker \bar{g}_\alpha$. Let $X \in \ker \bar{g}_\alpha$. We may write $X = X_1 + \cdots + X_r$, where $X_7 \in E_{7x}$. Suppose that $C_{\beta \alpha} \neq 0$. Let $Y \in E_{\beta x}$ be any vector. Then, $0 = \bar{g}_\alpha(X, Y) = C_{\beta \alpha} g_\beta(X_\beta, Y)$ for any $Y \in E_{\beta x}$, i.e. $X_\beta = 0$. The proof is complete.
This implies the first equality of the first statement. The second equality is obvious, since \( g_x \) is non-degenerate on each \( E_{\beta x} \).

Suppose that \( C_{r+1} \neq 0 \). The inclusion \( \supset \) in the first equality is obvious. Let \( X \in \ker \bar{g}_{ax} \). We write \( X = X_1 + \cdots + X_r \), where \( X_\gamma \in E_{\gamma z} \). Suppose that \( C_\beta \neq 0 \), \( 1 \leq \beta \leq r - 1 \). As above, this implies \( X_\beta = 0 \). Let \( Y \in E_{rx} \) be a vector such that \( \theta(Y) \neq 0 \). Then \( 0 = \bar{g}_a(X, Y) = C_{r+1} \theta(X_r) \theta(Y) \), i.e. \( \theta(X_r) = 0 \). This proves the second statement.

This lemma allows us easily indicate whether \( C_{r+1} = 0 \) or not: we should compute \( \ker \bar{g}_{ax} \) and \( \ker g_x |_{\ker \bar{g}_{ax} \times \ker \bar{g}_{ax}} \). If \( C_{r+1} = 0 \) for some \( \alpha \geq 2 \), then we add \( \theta \otimes \theta \) to \( g_a \). Then by the lemma we get

\[
\ker \bar{g}_{ax} = \bigoplus_{1 \leq \beta \leq r - 1} C_{\beta \alpha} = 0 E_{\beta x} \oplus \{ X \in E_{rx} | \theta(X) = 0 \}, \quad 2 \leq \alpha \leq r + 1.
\]

Let us consider the following vector space:

\[
W = \bigcap_{\alpha=2}^{r+1} \ker \bar{g}_{ax}.
\]

**Lemma 2.** It holds

\[
W = \{ X \in E_{rx} | \theta(X) = 0 \},
\]

\[
W^{\perp_s} = E_{1x} \oplus \cdots \oplus E_{r-1x} \oplus \mathbb{R} p_x.
\]

**Proof.** We claim that for any \( \beta, 1 \leq \beta \leq r-1 \) there exists an \( \alpha, 2 \leq \alpha \leq r + 1 \), such that \( C_{\beta \alpha} \neq 0 \). Indeed, if the claim is wrong then there exists a \( \beta, 1 \leq \beta \leq r - 1 \) such that it holds

\[
(C_{\beta \alpha})_{a=1}^{r+1} = (C_{\gamma \alpha})_{a=1}^{r+1} = (1, 0, \ldots, 0),
\]

i.e. the matrix \((C_{\beta \alpha})_{\beta, \alpha=1}^{r+1}\) is degenerate, that gives a contradiction. The first equality follows from the claim. The second equality is obvious. \( \square \)

**Lemma 3.** The intersection \( \bigcap_{\alpha=2}^{r+1} (W^{\perp_s})^{\perp_{g_\alpha}} \) that can be written as

\[
\{ X \in T_x M | g_{ax}(X, Y) = 0 \text{ for all } 2 \leq \alpha \leq r + 1, \ Y \in W^{\perp_s} \}
\]

coincides with \( E_{rx} \).

**Proof.** Suppose that \( X \in \bigcap_{\alpha=2}^{r+1} (W^{\perp_s})^{\perp_{g_\alpha}} \). Then for any \( \alpha, 2 \leq \alpha \leq r + 1 \) and all \( Y \in W^{\perp_s} \) it holds \( g_{ax}(X, Y) = 0 \). Consider the decomposition \( X = X_1 + \cdots + X_r \). Let \( 1 \leq \beta \leq r - 1 \). Above we have seen that there exist \( \alpha, 2 \leq \alpha \leq r + 1 \) such that \( C_{\beta \alpha} \neq 0 \). Let \( Y \in E_{\beta x} \subset W^{\perp_s} \). Then,

\[
0 = g_{ax}(X, Y) = C_{\beta \alpha} g_{\beta x}(X, Y) = C_{\beta \alpha} g_{\beta x}(X_\beta, Y).
\]

Consequently, \( X_\beta = 0 \), and \( X = X_r \in E_{rx} \).

Conversely, if \( X \in E_{rx} \), \( 2 \leq \alpha \leq r + 1 \), \( Y \in W^{\perp_s} \), then \( g_{ax}(X, Y) = C_{r+1} \theta(X) \theta(Y) = 0 \), since \( \theta(Y) = 0 \). This proves the lemma. \( \square \)

Thus in order to find the space \( E_{rx} \), it is enough to compute the spaces

\[
W = \bigcap_{\alpha=2}^{r+1} \ker \bar{g}_{ax}, \quad W^{\perp_s}, \quad \bigcap_{\alpha=2}^{r+1} (W^{\perp_s})^{\perp_{g_\alpha}}.
\]

Now we consider \((E_{rx})^{\perp_s}\) and the restrictions of the forms \( \bar{g}_{1x} = g_x, \bar{g}_{2x}, \ldots, \bar{g}_{r+1x} \) to \((E_{rx})^{\perp_s}\). Clearly, the rank of this system equals to \( r - 1 \). Let \( \bar{g}_{1x}, \ldots, \bar{g}_{r-1x} \) be a linearly independent subsystem such that \( \bar{g}_{1x} = g_x \). There exist real numbers \((A_{\beta \alpha})_{\beta, \alpha=1}^{r-1}\) such that

\[
(5) \quad \bar{g}_{ax} = \sum_{\beta=1}^{r-1} A_{\beta \alpha} g_{\beta x}.
\]

These numbers can be found in the same way as in Section 2. Using (4) and the inverse matrix to \( A_{\beta \alpha} \), we find the forms \( \bar{g}_{1x}, \ldots, \bar{g}_{r-1x} \). Next, \( g_{rx} = g_x - g_{1x} - \cdots - g_{r-1x} \) and \( \theta_x = g_x |_{p_x} = g_{rx} |_{p_x} \). Evaluating (4) at the point \( x \), we find the matrix \((C_{\beta \alpha})\). Then using (4) and the inverse matrix to \((C_{\beta \alpha})\), we find the metrics \( g_a \). Now for any \( y \in M \) find

\[
E_{\beta y} = \{ X \in T_y M | g_{\beta y}(X, Y) = 0 \text{ for all } \alpha \neq \beta, 1 \leq \beta \leq r \}, \quad 1 \leq \alpha \leq r.
\]

Thus we know the the distributions \( E_{\alpha} \).
6. Classification of holonomy algebras of Lorentzian manifolds

Here we review results from [5, 20, 13, 14]. Let \((M, g)\) be a simply connected Lorentzian manifold of dimension \(n+2\), \(n \geq 0\). Fix a point \(x \in M\). The tangent space \((T_xM, g_x)\) can be identified with the Minkowski space \(\mathbb{R}^{1,n+1}\). Then the holonomy algebra \((M, g)\) at the point \(x\) is identified with a subalgebra \(\mathfrak{g} \subset \mathfrak{so}(1, n+1)\). From the above it follows that we may assume that \(\mathfrak{g} \subset \mathfrak{so}(1, n+1)\) is weakly irreducible. If \(\mathfrak{g} \subset \mathfrak{so}(1, n+1)\) is irreducible, then \(\mathfrak{g} = \mathfrak{so}(1, n+1)\). Suppose that \(\mathfrak{g} \subset \mathfrak{so}(1, n+1)\) is not irreducible, the \(\mathfrak{g}\) preserves an isotropic line in \(\mathbb{R}^{1,n+1}\).

We fix a basis \(p, X_1, ..., X_n, q\) of \(\mathbb{R}^{1,n+1}\) such that \(p\) and \(q\) are light-like vectors, \(g(p,q) = g(q,p) = 1\) and \(g(e_i,e_i) = 1\) and the subspace \(E \subset \mathbb{R}^{1,n+1}\) spanned by \(X_1, ..., X_n\) is an Euclidean subspace orthogonal to \(p\) and \(q\). We obtain the decomposition
\[
T_xM = \mathbb{R}^{1,n+1} = \mathbb{R}p \oplus E \oplus \mathbb{R}q.
\]

Denote by \(\text{sim}(n)\) the subalgebra of \(\mathfrak{so}(1, n+1)\) that preserves the isotropic line \(\mathbb{R}p\). The Lie algebra \(\text{sim}(n)\) can be identified with the following matrix algebra:
\[
(7) \quad \text{sim}(n) = \left\{ \begin{pmatrix} a & (GX)^t & 0 \\ 0 & A & -X \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R}, X \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\}.
\]

The above matrix can be identified with the triple \((a, A, X)\). We get the decomposition
\[
\text{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n,
\]
which means that \(\mathbb{R} \oplus \mathfrak{so}(n) \subset \text{sim}(n)\) is a subalgebra and \(\mathbb{R}^n \subset \text{sim}(n)\) is an ideal, and the Lie brackets of \(\mathbb{R} \oplus \mathfrak{so}(n)\) with \(\mathbb{R}^n\) are given by the standard representation of \(\mathbb{R} \oplus \mathfrak{so}(n)\) in \(\mathbb{R}^n\). The Lie algebra \(\text{sim}(n)\) is isomorphic to the Lie algebra of the Lie group of similarity transformations of \(\mathbb{R}^n\). The explicit isomorphism on the group level is constructed in [12]. We may assume that \(\mathfrak{g} \subset \text{sim}(n)\). We identify \(\mathbb{R}^n\) and \(E\).

Let \(\mathfrak{h} \subset \mathfrak{so}(n)\) be a subalgebra. Recall that \(\mathfrak{h}\) is a compact Lie algebra and we have the decomposition \(\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{h})\), where \(\mathfrak{h}'\) is the commutant of \(\mathfrak{h}\), and \(\mathfrak{z}(\mathfrak{h})\) is the center of \(\mathfrak{h}\). If \(\mathfrak{h} \subset \text{sim}(n)\) is irreducible, then \(\mathfrak{z}(\mathfrak{h}) \neq 0\) if and only if \(\mathfrak{h} \subset \mathfrak{u}(\mathbb{R}^2)\); in this case \(\mathfrak{h}' \subset \mathfrak{su}(\mathbb{R}^2)\) and \(\mathfrak{z}(\mathfrak{h}) = \mathbb{R}J\), where \(J\) is the complex structure.

The next theorem gives the classification of weakly-irreducible not irreducible holonomy algebras of Lorentzian manifolds.

**Theorem 1.** A subalgebra \(\mathfrak{g} \subset \text{sim}(n)\) is the weakly-irreducible holonomy algebra of a Lorentzian manifold if and only if it is conjugated to one of the following subalgebras:
\[
\text{type 1: } \mathfrak{g}^{1,\mathbb{R}} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n, \text{ where } \mathfrak{h} \subset \mathfrak{so}(n),
\]
\[
\text{type 2: } \mathfrak{g}^{2,\mathbb{R}} = \mathfrak{h} \ltimes \mathbb{R}^n,
\]
\[
\text{type 3: } \mathfrak{g}^{3,\mathbb{R},\varphi} = \{ (\varphi(A), A, 0) | A \in \mathfrak{h} \} \ltimes \mathbb{R}^n,
\]
\[
\text{type 4: } \mathfrak{g}^{4,\mathbb{R},m,\psi} = \{ (0, A, X + \psi(A)) | A \in \mathfrak{h}, X \in \mathbb{R}^m \},
\]
where \(\mathfrak{h} \subset \mathfrak{so}(n)\) is the holonomy algebra of a Riemannian manifold; for \(\mathfrak{g}^{3,\mathbb{R},\varphi}\) it holds \(\mathfrak{z}(\mathfrak{h}) \neq \{0\}\), and \(\varphi : \mathfrak{h} \to \mathbb{R}\) is a non-zero linear map with \(\varphi|_{\mathfrak{h}'} = 0\); for \(\mathfrak{g}^{4,\mathbb{R},m,\psi}\) it holds \(0 < m < n\) is an integer, \(\mathfrak{h} \subset \mathfrak{so}(m)\), \(\dim \mathfrak{z}(\mathfrak{h}) \geq n - m\), a decomposition \(\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}\) is fixed, and \(\psi : \mathfrak{h} \to \mathbb{R}^{n-m}\) is a surjective linear map with \(\psi|_{\mathfrak{h}'} = 0\).

The subalgebra \(\mathfrak{h} \subset \mathfrak{so}(n)\) associated to a weakly-irreducible Lorentzian holonomy algebra \(\mathfrak{g} \subset \text{sim}(n)\) is called the orthogonal part of \(\mathfrak{g}\). For \(\mathfrak{h} \subset \mathfrak{so}(n)\) there exist the decompositions
\[
(8) \quad E = E_0 \oplus E_1 \oplus \cdots \oplus E_r, \quad \mathfrak{h} = \{0\} \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r
\]
such that \(\mathfrak{h}\) annihilates \(E_0, \mathfrak{h}_\alpha(E_\beta) = 0\) for \(\alpha \neq \beta\), and \(\mathfrak{h}_\alpha \subset \mathfrak{so}(E_\alpha)\) is an irreducible subalgebra for \(1 \leq \alpha \leq r\). Let \(n_\alpha = \dim E_\alpha\).

Let us give a more precise descriptions for algebras of type 3. Let \(K \subset \{1, ..., r\}\) be the set of indices such that \(\varphi|_{\mathfrak{h}_\alpha} \neq 0\). If \(\alpha \in K\), then \(\mathfrak{h}_\alpha \subset \mathfrak{u}(\mathbb{R}^2)\); in this case \(\mathfrak{h}_\alpha' \subset \mathfrak{su}(\mathbb{R}^2)\) and \(\mathfrak{z}(\mathfrak{h}_\alpha) = \mathbb{R}J_\alpha\), where \(J_\alpha\) is the complex the structure on \(E_\alpha\). Let \(c_\alpha = \varphi(J_\alpha)\). Then
\[
(9) \quad \mathfrak{g}^{3,\mathbb{R},\varphi} = \bigoplus_{\alpha \in K} \mathbb{R}(c_\alpha + J_\alpha) \oplus \mathfrak{h}'_\alpha \oplus \bigoplus_{\alpha \notin K} \mathfrak{h}_\alpha \ltimes \mathbb{R}^n,
\]
where \( c_α + J_α \) denotes \( (c_α, J_α, 0) \).

Similarly we may write

\[
\mathfrak{g}^{4,h,m,ψ} = (\oplus_α \in K \mathbb{R} (J_α + ψ(J_α)) \oplus \mathfrak{h}_α' \oplus \oplus_α \not{\in} K \mathfrak{h}_α) \otimes \mathbb{R}^m,
\]

where \( K \) is defined in the same way as for \( \mathfrak{g} \) of type 3.

### 7. Walker and adapted coordinates

Let \((M, g)\) be a Lorentzian manifold with the holonomy algebra \( \mathfrak{g} \subset \mathfrak{sl}(n) \). Then \((M, g)\) admits a parallel distribution of isotropic lines \( ℓ \). According to [29], locally there exist so called Walker coordinates \( v, x^1, \ldots, x^n, u \) such that the metric \( g \) has the form

\[
g = 2dvdu + h + 2Adu + H(du)^2,
\]

where \( h = h_{ij}(x^1, \ldots, x^n, u)dx^idx^j \) is an \( u \)-dependent family of Riemannian metrics, \( A = A_i(x^1, \ldots, x^n, u)dx^i \) is an \( u \)-dependent family of one-forms, and \( H = H(v, x^1, \ldots, x^n, u) \) is a local function on \( M \). Consider the local frame

\[
p = \partial_v, \quad X_i = \partial_i - A_i\partial_v, \quad q = \partial_u - \frac{1}{2} H\partial_0.
\]

Let \( E \) be the distribution generated by the vector fields \( X_1, \ldots, X_n \). Clearly, the vector fields \( p, q \) are light-like, \( g(p, q) = 1 \), the restriction of \( g \) to \( E \) is positive definite, and \( E \) is orthogonal to \( p \) and \( q \). The vector field \( p \) defines the parallel distribution of null lines \( ℓ \) and it is recurrent, i.e. \( \nabla p = \mu \otimes p \), where \( \mu = \frac{1}{2} \partial_v Hdu \). Since the manifold is locally indecomposable, any other recurrent vector field is proportional to \( p \). Next, \( p \) is proportional to a parallel vector field if and only if \( dθ = 0 \), which is equivalent to \( \partial^2_v H = \partial_i\partial_0 H = 0 \). In the last case the coordinates can be chosen in such a way that \( \partial_i H = 0 \) and \( \nabla p = \nabla \partial_0 = 0 \), see e.g. [14].

Boubel [7] proved that there exist Walker coordinates

\[
v, x_0 = (x_0^1, \ldots, x_0^n), \ldots, x_r = (x_r^1, \ldots, x_r^n), \quad u
\]

adapted to the decomposition (5). This means that

\[
h = h_0 + h_1 + \cdots + h_r, \quad h_0 = \sum_{i=1}^{n_0} (dx^i_0)^2, \quad h_α = \sum_{i,j=1}^{n_α} h_{αij}dx^iαdx^jα,
\]

\[
A = \sum_{α=0}^{r} A_α, \quad A_0 = 0, \quad A_α = \sum_{k=1}^{n_α} A^kαdx^k,
\]

and one has

\[
\frac{∂}{∂x^kβ} h_{αij} = \frac{∂}{∂x^kβ} A^i_α = 0, \quad \text{if } β ≠ α.
\]

We call these coordinates adapted. The coordinates can be chosen so that in addition \( A = 0 \), see [17].

### 8. The curvature tensor

Since the Ambrose-Singer Theorem provides the relation of the holonomy algebra and the curvature tensor, we describe here the curvature tensor of a Walker manifold \((M, g)\) with the holonomy algebra \( \mathfrak{g} \subset \mathfrak{sl}(n) \) at the point \( x \in M \). For that it is convenient to consider the space \( \mathcal{R}(\mathfrak{g}) \) of algebraic curvature tensors of type \( \mathfrak{g} \), i.e. the space of linear maps from \( \Lambda^2 \mathbb{R}^{1,n+1} \) to \( \mathfrak{g} \) satisfying the first Bianchi identity. The curvature tensor of \((M, g), R = R_x \) at the point \( x \in M \) belongs to the space \( \mathcal{R}(\mathfrak{g}) \). This space is found in [11], [16]. Consider the decomposition (8). For a subalgebra \( \mathfrak{h} \subset \mathfrak{so}(n) \) consider the space

\[
\mathcal{P}(\mathfrak{h}) = \{ P \in (\mathbb{R}^n)^* \otimes \mathfrak{h} | g(P(X)Y, Z) + g(P(Y)Z, X) + g(P(Z)X, Y) = 0 \quad \text{for all } X, Y, Z \in \mathbb{R}^n \}.
\]
Define the map \( \widetilde{\text{Ric}} : \mathcal{P}(h) \to \mathbb{R}^n, \) \( \widetilde{\text{Ric}}(P) = P_i^j g_{ik} X_j. \) It does not depend on the choice of the basis \( X_1, \ldots, X_n. \) The tensor \( R \in \mathcal{R}(g, h) \) is uniquely given by elements \( \lambda \in \mathbb{R}, e \in E, R_0 \in \mathcal{R}(h), P \in \mathcal{P}(h), T \in \otimes^2 E \) in the following way:

\[
R(p, q) = (\lambda, 0, e), \quad R(X, Y) = (0, R_0(X, Y), P(Y)X - P(X)Y), \quad R(X, q) = (g(e, X), P(X), T(X)), \quad R(p, X) = 0
\]

for all \( X, Y \in \mathbb{R}^n. \) We write \( R = R(\lambda, e, R_0, P, T). \) The Ricci tensor \( \text{Ric}(R) \) of \( R \) is given by \( \text{Ric}(R)(U, V) = \text{tr}(Z \mapsto R(Z, U)V) \) and it satisfies

(14) \[
\text{Ric}(p, q) = \lambda, \quad \text{Ric}(X, Y) = \text{Ric}(R_0)(X, Y),
\]

(15) \[
\text{Ric}(X, q) = g(X, e - \widetilde{\text{Ric}}(P)), \quad \text{Ric}(q, q) = -\text{tr} T.
\]

Decomposition \( \mathcal{R} \) defines the decomposition \( P = P_1 + \cdots + P_r, P_\alpha \in \mathcal{P}(h_\alpha) \) and \( R_0 = R_{01} + \cdots + R_{0r}, R_{00} \in \mathcal{R}(h_0). \)

For the above tensor \( R, \) the condition \( R \in \mathcal{R}(g, h, \varphi) \) is equivalent to the following conditions:

\[
\lambda = 0, \quad g(e, X) = \varphi(P(X)), \quad X \in E, \quad R_0 \in \mathcal{R}(\ker \varphi).
\]

The condition \( R \in \mathcal{R}(g, h, m, \varphi) \) is equivalent to the following conditions:

\[
\lambda = 0, \quad e = 0, \quad pr_{\mathbb{R}^{n-m}} \circ T = \psi \circ P, \quad R_0 \in \mathcal{R}(\ker \varphi).
\]

Note that a weakly irreducible holonomy algebra \( g \subset \text{im}(n) \) defines canonically only the isotropic line \( \mathbb{R}p. \) Let us take a real number \( \mu \neq 0, \) the vector \( p' = \mu p, \) and any light-like vector \( q' \) with \( g(p', q') = 1. \) There exists a unique vector \( w \in E \) such that \( q' = \frac{1}{\mu}(-\frac{1}{2}g(w, w)p + w + q). \) The corresponding \( E' \) has the form \( E' = \{ -g(x, w)p + x | x \in E \}. \) We will consider the map \( x \in E \mapsto x' = -g(x, w)p + x \in E' \). Using this, we obtain that \( R = R(\lambda, \tilde{e}, R_0, P, T). \) For example, it holds

(16) \[
\tilde{\lambda} = \lambda, \quad \tilde{e} = \frac{1}{\mu}(e - \lambda w)', \quad \tilde{P}(x') = \frac{1}{\mu}(P(x) + R_0(x, w))', \quad \tilde{R}_0(x', y')z' = (R_0(x, y)z}'.
\]

This shows e.g. that if \( \lambda = 0, \) then the projection of the vector \( e \) to \( p^\perp / \mathbb{R}p \) is defined up to a non-zero real multiple.

Let \( n = 2m \geq 2 \) and consider the space \( \mathcal{P}(u(m)) \), note that \( u(1) = \mathfrak{so}(2). \) In \( \mathcal{P}(u(m)) \) it is shown that the \( u(m) \)-module \( \mathcal{P}(u(m)) \) admits the decomposition

\[
\mathcal{P}(u(m)) = \mathcal{P}_0(u(m)) \oplus \mathcal{P}_1(u(m))
\]

into the direct sum of irreducible submodules. It holds \( \mathcal{P}_0(u(m)) = \{ P \in \mathcal{P}(u(m)) | \widetilde{\text{Ric}}(P) = 0 \} \) and \( \mathcal{P}_1(u(m)) \simeq \mathbb{R}^n. \) The last isomorphism has the form

\[
Z \in \mathbb{R}^n \, \mapsto \, P, \quad P(X) = RC^{Pn}(X, Z),
\]

where \( RC^{Pn} \) is the curvature tensor at a point of the complex projective space,

\[
RC^{Pn}(X, Z) = \frac{1}{2}g(JX, Z)J + \frac{1}{4}(X \wedge Z + JX \wedge JZ),
\]

where \( (X \wedge Z)Y = g(X, Y)Z - g(Z, Y)X. \)

**Lemma 4.** For \( P \in \mathcal{P}_1(u(m)) \) corresponding to \( Z \in \mathbb{R}^n \) it holds

\[
\tilde{\text{Ric}}(P) = \frac{m+1}{2}Z, \quad pr_{\mathbb{R}^r}P(X) = \frac{m+1}{2m}g(JX, Z)J.
\]

**Proof.** Using the complex structure \( J, \) we identify the space \( \mathbb{R}^{2m} \) with \( \mathbb{C}^m. \) Let \( \tilde{g} \) be the Hermitian form on \( \mathbb{C}^m \) corresponding to \( g, \) i.e.

\[
\tilde{g}(X, Y) = g(X, Y) + ig(X, JY).
\]

Let \( e_1, \ldots, e_m \) be an orthogonal basis of \( \mathbb{C}^m. \) For the trace of any element \( L \in u(m) \) acting on \( \mathbb{C}^m \) it holds

\[
\text{tr}_\mathbb{C} L = \sum_{k=1}^m \tilde{g}(Le_k, e_k) = \sum_{k=1}^m (g(Le_k, e_k) + ig(Le_k, Je_k)) = i \sum_{k=1}^m g(Le_k, Je_k).
\]
Recall that for $L \in \mathfrak{su}(m)$ it holds $\text{tr}_C L = 0$, and $\text{tr}_C J = mi$. Note that
\[(X \wedge Z + JX \wedge JZ)Y = \tilde{g}(Y, X)Z - \tilde{g}(Y, Z)X.\]
This implies that
\[
\text{tr}_C(X \wedge Z + JX \wedge JZ) = \sum_{k=1}^m \tilde{g}((X \wedge Z + JX \wedge JZ)e_k, e_k)
\]
\[
= \sum_{k=1}^m \tilde{g}(\tilde{g}(e_k, X)Z - \tilde{g}(e_k, Z)X, e_k) = \tilde{g}\left(Z, \sum_{k=1}^m \tilde{g}(e_k, e_k)\right) - \tilde{g}\left(X, \sum_{k=1}^m \tilde{g}(Z, e_k), e_k\right)
\]
\[
= \tilde{g}(Z, X) - \tilde{g}(X, Z) = 2i\tilde{g}(Z, JX).
\]
We conclude that
\[
\text{pr}_E J = \text{pr}_E J = \frac{m + 1}{2m} g(JX, Z)J.
\]
In [14] it is shown that
\[
\tilde{g}(\text{Ric} P, X) = -\sum_{k=1}^m \tilde{g}(P(JX)e_k, JY_k).
\]
for all $X \in E$. Consequently,
\[
\tilde{g}(\text{Ric} P, X) = i \text{tr}_C P(JX) = i \text{tr}_C R^{Cp_m}(JX, Z) = \frac{m + 1}{2} g(X, Z),
\]
i.e. $\text{Ric} = \frac{m+1}{2} Z$. \qed

The above considerations easily imply the following

Lemma 5. Suppose that $\mathfrak{h} \subset \mathfrak{su}(\frac{n}{2})$. If $\lambda = 0$ and $\text{Ric}(R_0) = 0$, then the projections of the vectors $e$ and $Z$ to $p^\perp/\mathfrak{k}_0$ are defined up to a non-zero real multiple.

If we fix on $(M, g)$ Walker coordinates as in Section [4] then we get vector fields $p$, $q$, a distribution $E$ over an open subset of $M$. Consequently, the curvature tensor of $(M, g)$ over this subset is defined by some tensor fields $\lambda, v, R_0, P, T$. It can be checked that $R_0 = R(h)$ is the curvature tensor of the Riemannian metric $h$, and it holds
\[
\lambda = -\frac{1}{2} \partial_0^2 H, \quad e = -\frac{1}{2} (\partial_i \partial_0 H - A_0 \partial_i^2 H) h^{ij} X_j.
\]

9. Finding the orthogonal part of a Lorentzian holonomy algebra

Let $(M, g)$ be a Lorentzian manifold with the holonomy algebra $\mathfrak{g} \subset \mathfrak{sl}(n)$. In this section we give an algorithm how to find the orthogonal part $\mathfrak{h} \subset \mathfrak{so}(n)$ of $\mathfrak{g}$.

The subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ coincides with the holonomy algebra of the induced connection on the so-called screen bundle $\mathcal{E} = \mathfrak{t}^+ / \mathfrak{l}$ [27]. If we choose a decomposition [6] over an open subset of $M$, then $\mathcal{E}$ restricted to this subset may be identified with the distribution $E$. For the curvature tensor of the connection on $\mathcal{E}$ we get
\[
R(p, \cdot) = 0, \quad R(X, Y) = R_0(X, Y), \quad R(X, q) = P(X), \quad X, Y \in \Gamma(E).
\]
Recall that $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold. The decomposition
\[
\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r
\]
into the direct sum of a flat subbundle $\mathcal{E}_0 \subset \mathcal{E}$ and parallel subbundles $\mathcal{E}_1, \ldots, \mathcal{E}_r \subset \mathcal{E}$, corresponding to the decompositions [9], can be obtained exactly in the same way as in Section [3]. Hence we may assume that the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is irreducible.

The subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ can not be found exactly in the same way, as the holonomy algebra of a Riemannian manifold in Section [4], since we can not distinguish symmetric Berger subalgebras from the very beginning. By this reason we use a dipper analysis.

If we already know that $\mathfrak{h} \subset \mathfrak{so}(n)$ is a symmetric Berger subalgebra, then it can be found in the following way. Let $y \in M$ be a point such that either $R_q(h) \neq 0$, or $P_q \neq 0$, such point exists since $\mathfrak{h} \neq 0$. Since $\mathfrak{h}$ does not contain any proper Berger subalgebra, and each of the subsets
\{R_{g}(h)(X,Y)|X,Y \in E_{g}\} and \{P_{g}(X)|X \in E_{g}\} generates a Berger subalgebra in \(\mathfrak{h}\), \(\mathfrak{h}\) is generated either by \(\{R_{g}(h)(X,Y)|X,Y \in E_{g}\}\), or by \(\{P_{g}(X)|X \in E_{g}\}\).

First we compute Ric\((h)\) and Ric\((P)\) (for the last object, the formula (21) given below can be used). From the results of [16] it follows that if Ric\((h)\) = 0 and Ric\((P)\) = 0, then the subalgebra \(\mathfrak{h} \subset \mathfrak{so}(n)\) is one of \(\mathfrak{so}(n)\), \(\mathfrak{su}(\frac{n}{2})\), \(\mathfrak{sp}(\frac{n}{2})\), \(\mathfrak{spin}(7)\), \(G_{2}\). In this case, \(\mathfrak{h} \subset \mathfrak{so}(n)\) can be found simply analyzing the parallel sections of \(\Lambda^{2}\mathcal{E}\) \((k = 2, 3, 4)\).

Now we may assume that Ric\((h)\) \(\neq 0\) or Ric\((P)\) \(\neq 0\), then \(\mathfrak{h} \subset \mathfrak{so}(n)\) is either one of \(\mathfrak{so}(n)\), \(\mathfrak{u}(\frac{n}{2})\), \(\mathfrak{sp}(\frac{n}{2}) \oplus \mathfrak{sp}(1)\), or \(\mathfrak{h} \subset \mathfrak{so}(n)\) is a symmetric Berger subalgebra.

If there exists a parallel section of \(\Lambda^{2}\mathcal{E}\), then \(\mathfrak{h}\) is contained in \(\mathfrak{u}(\frac{n}{2})\). Next, the subspace of the \(\mathfrak{u}(\frac{n}{2})\)-module

\[\circ^{2}\mathfrak{u}(\frac{n}{2}) \cong \circ^{2}\mathfrak{su}(\frac{n}{2}) \oplus \mathfrak{su}(\frac{n}{2}) \oplus \mathbb{R}\]

annihilated by \(\mathfrak{u}(\frac{n}{2})\) clearly is of dimension two. This subspace is spanned by the curvature tensor of the complex projective space and by the subset \(\mathbb{R}\). On the other hand, any symmetric Berger subalgebra \(\mathfrak{h} \subset \mathfrak{u}(\frac{n}{2})\) annihilates in addition the curvature tensor valued at a point of the corresponding symmetric space, which is an element of the space \(\circ^{2}\mathfrak{u}(\frac{n}{2})\). Consequently, if the space of parallel sections of \(\circ^{2}\mathfrak{u}(\mathcal{E})\) equals 2, then \(\mathfrak{h} = \mathfrak{u}(\frac{n}{2})\). Otherwise, \(\mathfrak{h} \subset \mathfrak{u}(\frac{n}{2})\) is a symmetric Berger subalgebra.

Now \(\mathfrak{h} \subset \mathfrak{so}(n)\) is either one of \(\mathfrak{so}(n)\), \(\mathfrak{sp}(\frac{n}{2}) \oplus \mathfrak{sp}(1)\), or it is a symmetric Berger subalgebra not contained in \(\mathfrak{u}(\frac{n}{2})\).

In [23] it is shown that if an indecomposable simply connected Riemannian symmetric space admits a non-trivial parallel 4-form, then its holonomy algebra is not simple. This and the list of indecomposable simply connected Riemannian symmetric space [6] show that such a space admits a parallel 4-form, then it is either Kählerian, or quaternionic-Kählerian, or its holonomy algebra is one of \(\mathfrak{so}(r) \oplus \mathfrak{so}(s) \subset \mathfrak{so}(rs)\) \((r,s \neq 2)\) and \(\mathfrak{sp}(r) \oplus \mathfrak{sp}(s) \subset \mathfrak{sp}(rs)\) \((r,s \neq 1)\).

If there are non non-trivial parallel sections in \(\Lambda^{4}\mathcal{E}\), then either \(\mathfrak{h} = \mathfrak{so}(n)\), or \(\mathfrak{h} \subset \mathfrak{so}(n)\) is a simple symmetric Berger algebra, which is not contained neither in \(\mathfrak{u}(\frac{n}{2})\), nor in \(\mathfrak{sp}(\frac{n}{2}) \oplus \mathfrak{sp}(1)\). The Lie algebra \(\mathfrak{so}(n)\) annihilates exactly one 1-dimensional subspace \(\circ^{2}\mathfrak{so}(n)\), while for symmetric Berger subalgebras \(\mathfrak{h} \subset \mathfrak{so}(n)\) this subspace is at least of dimension 2. Hence, if the dimension of parallel sections of the bundle \(\Lambda^{2}\mathfrak{so}(\mathcal{E})\) equals to 1, then \(\mathfrak{h} = \mathfrak{so}(n)\). Otherwise \(\mathfrak{h} \subset \mathfrak{so}(n)\) is a symmetric Berger algebra.

Suppose that there is a non-trivial parallel sections in \(\omega \in \Lambda^{4}\mathcal{E}\). Note that the stabilizer of the KRAINES 4-form evaluated at a point of a quaternionic-Kählerian manifold of dimension \(n\) coincides with \(\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)\) [6]. Let \(\mathfrak{f} \subset \mathfrak{so}(n)\) be the stabilizer of \(\omega\) at some point. Clearly, \(\mathfrak{h} \subset \mathfrak{f}\). If \(\mathfrak{f} \neq \mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)\), then \(\mathfrak{h} \subset \mathfrak{so}(n)\) is a symmetric Berger algebra and it is one of \(\mathfrak{so}(r) \oplus \mathfrak{so}(s) \subset \mathfrak{so}(rs)\) \((r,s \neq 2)\) and \(\mathfrak{sp}(r) \oplus \mathfrak{sp}(s) \subset \mathfrak{sp}(rs)\) \((r,s \neq 1)\). Otherwise, \(\mathfrak{h} \subset \mathfrak{f} = \mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)\). Again, the \(\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)\)-module

\[\circ^{2}(\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)) \cong \circ^{2}\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1) \oplus \circ^{2}\mathfrak{sp}(1)\]

annihilated by \(\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)\) of dimension two, while for a symmetric Berger subalgebra \(\mathfrak{h} \subset \mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)\) this dimension is bigger. It is enough to find parallel sections of the bundle \(\circ^{2}(\mathfrak{sp}(\mathcal{E}) \oplus \mathfrak{sp}(1))\).

10. Finding the type of a Lorentzian holonomy algebra

Let \((M,g)\) be a locally indecomposable Lorentzian manifold with the holonomy algebra \(\mathfrak{g} \subset \mathfrak{so}(1,n+1)\). If \((M,g)\) admits a parallel distribution of isotropic lines, or equivalently, locally it admits recurrent light-like vector fields that are proportional on the intersections of the domains of their definition, then \(\mathfrak{g} \subset \mathfrak{sim}(n)\); otherwise, \(\mathfrak{g} = \mathfrak{so}(1,n+1)\).

We consider the case \(\mathfrak{g} \subset \mathfrak{sim}(n)\). The following statement has been already discussed in Section [7]
Proposition 1. Let \((M, g)\) be a locally indecomposable Lorentzian manifold with the holonomy algebra \(\mathfrak{g} \subset \text{sim}(n)\), then \(\mathfrak{g}\) is of type 2 or 4 if and only if for any Walker coordinate system it holds \(\partial^2_\alpha H = \partial_\alpha \partial_\beta H = 0\), equivalently, there exists a Walker coordinate system in a neighbourhood of each point such that \(\partial_\alpha H = 0\). If \(M\) is simply connected, then these conditions are equivalent to the existence of a parallel light-like vector field. Otherwise, \(\mathfrak{g}\) is of type 1 or 3.

Now we should be able to decide between types 1 and 3 and between types 2 and 4. We will do that in the following two theorems.

Suppose that \(\mathfrak{g}\) is of type 1 or 3. The following theorem allows to find the type of \(\mathfrak{g}\).

Theorem 2. Let \((M, g)\) be a simply connected locally indecomposable Lorentzian manifold with the holonomy algebra \(\mathfrak{g} \subset \text{sim}(n)\). Suppose that \((M, g)\) does not admit any parallel light-like vector field. Then \(\mathfrak{g}\) is of type 3 if and only if the following conditions hold:

1. for any Walker coordinate system it holds \(\partial^2_\alpha H = 0\);
2. there is a non-empty subset \(K \subset \{1, \ldots, r\}\) of indexes \(\alpha\) such that
   2.a. if \(\alpha \in K\), then \(\mathfrak{h}_\alpha\) is contained in \(\mathfrak{u}\left(\frac{2n}3\right)\), i.e. the bundle \(\mathcal{E}_\alpha\) admits a parallel complex structure \(J_\alpha\);
   2.b. the Riemannian metric \(h_\alpha\) is Ricci-flat for all \(\alpha \in K\);
   2.c. if \(\alpha \notin K\), then \(\partial_\alpha \partial_\alpha H = 0\) for any adapted coordinate system;
   2.d. for each \(\alpha \in K\) there exists a non-zero constant \(c_\alpha \in \mathbb{R}\) such that for any adapted coordinate system it holds

\[
\text{pr}_{\mathcal{E}_\alpha} \dot{e} = -\frac{2c_\alpha}{n_\alpha} J_\alpha \widetilde{\text{Ric}} P_\alpha,
\]

where
\[
e = -R(p, q)g, \quad \widetilde{\text{Ric}} P_\alpha = \sum_{i,j=1}^{n_\alpha} h^{ij}_\alpha R(\partial_{x^i}, q) \partial_{x^j}
\]

and \(\text{pr}_{\mathcal{E}_\alpha} \dot{e} \neq 0\) for some adapted coordinate system.

Equation (18) has the following coordinate form

\[
\partial_{x^i} \partial_{x^j} h_{\alpha k} = -\frac{2c_\alpha}{n_\alpha} \left(\nabla^j F^\alpha_{ij} + \nabla^i \dot{h}_{\alpha j} - \partial_{x^j} h_{\alpha k} \dot{h}_{\alpha j k} \right) J^k_{\alpha j},
\]

where the indexes \(i, j, k, \ell\) run from 1 to \(n_\alpha\), \(\dot{h} = \partial_\alpha h\), \(F^\alpha_{ij} = \partial_{x^j} A_\alpha^i - \partial_{x^i} A_\alpha^j\) (no sum over \(\alpha\)). It holds \(\partial_{x^i} \partial_{x^j} h_{\alpha k} \neq 0\) for some adapted coordinate system and some \(i, j, k\).

Otherwise, \(\mathfrak{g}\) is of type 1.

Proof of Theorem 2. Suppose that the holonomy algebra \(\mathfrak{g}\) of \((M, g)\) at a point \(x \in M\) is of type 3, i.e. \(\mathfrak{g} = \mathfrak{g}^{3, b, \cdot}\). Let \(K \subset \{1, \ldots, r\}\) be as in Section 6. Condition 2.a follows from the definition of the algebra \(\mathfrak{g}^{3, b, \cdot}\).

Let \(y \in M\) and let \(\gamma\) be a piecewise smooth curve beginning at \(x\) and ending at \(y\). Note that the holonomy algebra at the point \(y\) equals \(\tau_{-1, \lambda} \circ \mathfrak{g} \circ \tau_{\gamma}\), and it is isomorphic to \(\mathfrak{g} = \mathfrak{g}^{3, b, \cdot}\). Clearly, for the curvature tensor at the point \(y\) it holds \(R_y \in \mathcal{R}(\tau_{-1, \lambda} \circ \mathfrak{g} \circ \tau_{\gamma}) \simeq \mathcal{R}(\mathfrak{g})\). We will identify \(\tau_{-1, \lambda} \circ \mathfrak{g} \circ \tau_{\gamma}\) and \(\mathfrak{g}\). Fix a coordinate system in a neighborhood of the point \(y\). Then \(R_y\) can be decomposed as in Section 8. Condition 1 of the theorem follows from (17) and the fact that \(\lambda = 0\) for any element from \(\mathcal{R}(\mathfrak{g})\). Let \(\alpha \in K\) and \(X, Y \in E_{a_0}\). We have \(R_0(X, Y) = R(h)(X, Y) = R(h)(X, Y) \in \mathfrak{h}_\alpha\). The fact that \(R_0 = R(h) \in \mathcal{R}(\mathfrak{h}_\alpha)\) implies condition 2.b.

Let \(\alpha \notin K\) and \(X \in E_{a_0}\). Then \(R(X, q) = (g(e, X), P(X), T(X))\) and \(P(X) \in \mathfrak{h}_\alpha\). Since \(\mathfrak{g}\) contains \(\mathfrak{h}_\alpha\) and \(\mathbb{R}^n\), we obtain \((g(e, X), 0, 0) \in \mathfrak{g}\). Consequently, \(g(e, X) = 0\) for any \(X \in E_{a_0}\). This and (17) imply 2.c.

Let \(\alpha \in K\) and \(X \in E_{a_0}\). The projection of \(R(X, q)\) on \(\mathcal{R} \oplus \mathcal{R}J_\alpha \subset \text{sim}(n)\) must belong to \(\mathbb{R} (c_\alpha + J_\alpha)\). On the other hand, \(R(X, q) = (g(e, X), P(X), T(X))\) and \(\text{pr}_{\mathcal{R} \oplus \mathcal{R} J_\alpha} = g(e, X) + \frac{2c_\alpha}{n_\alpha} J_\alpha P_\alpha\). From Lemma 4 it follows that \(\text{pr}_{\mathcal{R} J_\alpha} P_\alpha(X) = -\frac{2c_\alpha}{n_\alpha} g(X, J_\alpha \widetilde{\text{Ric}} P_\alpha) J_\alpha\). We conclude that \(\text{pr}_{\mathcal{E}_\alpha} \dot{e} = -\frac{2c_\alpha}{n_\alpha} J_\alpha \widetilde{\text{Ric}} P_\alpha\).
Let us find the coordinate form of the last equality. For simplicity we assume that $n_\alpha = n_1 = n$, i.e. $\mathfrak{h} \subset \mathfrak{u}(\mathfrak{g})$ is irreducible. Since $\partial_\mathfrak{h}^2 \mathbf{H} = 0$, from $[\mathfrak{g}]$ it follows that $e = -\frac{1}{2}(\partial_\mathfrak{h} \partial_\mathfrak{e} \mathbf{H}) \mathbf{h}^{ij} X_j$. In Section 8 we have seen that $\text{Ric}(X, q) = g(e - \text{Ric} P, X)$ for all $X \in E$. In $[\mathfrak{g}]$ it is shown that

$$\text{Ric}(\partial_\mathfrak{e}, q) = -\frac{1}{2}(\partial_\mathfrak{e} \partial_\mathfrak{e} \mathbf{H} + \nabla^j F_{ij} + \nabla^j \mathbf{h}_{ij} - \partial_\mathfrak{e} \mathbf{h}^{jk} \mathbf{h}_{jk})$$

Recall that $X_i = \partial_\mathfrak{e} - A_{ij}$. We obtain that

$$\text{Ric}(\partial_\mathfrak{h}, q) = \frac{1}{2} \left(\nabla^j F_{ij} + \nabla^j \mathbf{h}_{ij} - \partial_\mathfrak{h} \mathbf{h}^{jk} \mathbf{h}_{jk}\right).$$

The equation under consideration takes the form

$$(\partial_\mathfrak{h} \partial_\mathfrak{h} \mathbf{H}) \mathbf{h}^{ij} X_j = \frac{2\alpha \mathbf{c}}{n_\alpha} J(\text{Ric} P) \mathbf{h}^{ij} X_j.$$ 

Let $JX_j = J^1_j X_j$. Then

$$(\partial_\mathfrak{e} \partial_\mathfrak{h} \mathbf{H}) \mathbf{h}^{ij} X_j = \frac{2\alpha \mathbf{c}}{n_\alpha} (\text{Ric} P) \mathbf{h}^{ij} J^1_j X_i.$$ 

This implies

$$\partial_\mathfrak{e} \partial_\mathfrak{h} \mathbf{H} = \frac{2\alpha \mathbf{c}}{n_\alpha} (\text{Ric} P) \mathbf{h}^{ij} J^1_j h_{ii}.$$ 

Since $J$ is a Kählerian structure, it holds $J^1_j h_{ii} = -h_{ij} J^1_i$. Now it is easy to obtain $[\mathfrak{g}]$. Thus condition 2 is proved.

Conversely, suppose that for a Lorentzian manifold $(M, g)$ the conditions 1 and 2 hold. We should prove that the holonomy algebra of $(M, g)$ at a point $x \in M$ coincides with $\mathfrak{g}^{3,h,\psi}$. Let $\gamma$ be a piecewise smooth curve beginning at $x$; let $y \in M$ be its end-point. Fix a decomposition $[\mathfrak{g}]$ of $T_y M$. It defines the decomposition

$$(\partial_\mathfrak{e} \partial_\mathfrak{e} \mathbf{H}) \mathbf{h}^{ij} X_j = \frac{2\alpha \mathbf{c}}{n_\alpha} J(\text{Ric} P) \mathbf{h}^{ij} X_j.$$ 

Since we know that $\mathfrak{g}$ is either of type 1 or of type 3, it contains the ideal $\mathbb{R}^n$. Consider the tensor $R_\gamma = \tau_\gamma^{-1} - \mathbf{R}_\gamma(\tau_\gamma, \tau_\gamma) \in \mathcal{R}(\mathfrak{g})$. As in Section 8, it is defined by elements $\lambda, e, P, R_0$ and $T$. Since we have the isomorphism $\tau_\gamma : T_x M \to T_y M$ and consider the decomposition $[\mathfrak{g}]$, the tensor $R_\gamma$ is defined by the above elements mapped by the isomorphism $\tau_\gamma$ to the point $y$. Fix an adapted coordinate system in a neighborhood of the point $y$. Let $\lambda_y, ..., T_y$ be the elements defining $R_\gamma$ and corresponding to these coordinates. The condition 1 and the results of Section 8 imply that $\lambda = \lambda_y = 0$. Suppose that $\alpha \notin K$. Condition 2 implies that $\text{pr}_{E_0} e_y = 0$. From Lemma 5 it follows that $\text{pr}_{E_0} e = 0$, i.e. $\text{pr}_{E_0} R_\gamma(X, q_x) = 0$ for any $X \in E_\alpha$. Condition 2 implies that $R_\gamma(X, Y) = 0$ for all $X, Y \in E_\alpha$. We have only to consider the projection $\text{pr}_{\mathbb{R} \oplus \mathbb{R} E_0} R_\gamma(X, q_x) | \alpha \in K, X \in E_\alpha$. Lemma 5 condition 2.d and the above proof show that this projection coincides with $\mathbb{R}(\mathcal{c}_0 + J_\alpha)$. Thus $\mathfrak{g} = \mathfrak{g}^{3,h,\psi}$. The theorem is true.

Suppose that we have a local Walker metric that satisfies conditions 2.a and 2.b. According to Section 9 these conditions depend only on $h$ and $A$. We may ask if the function $H$ can be found in such a way that the holonomy of this metric is of type 3. The condition 1 of the theorem can be easily satisfied and we are left with Equations $[\mathfrak{g}]$. The integrability condition for this system of equations is of the form $\partial_{\mathfrak{e} \mathfrak{h}} B_i = \partial_{\mathfrak{h} \mathfrak{e}} B_i$, where $B_i$ is the right hand side of $[\mathfrak{g}]$. Thus a priori the function $H$ can not be changed to make the holonomy of the metric to be of type 3.

Next suppose that the holonomy algebra $\mathfrak{g}$ is of type 2 or 4. Suppose that $E_0 \neq 0$ (this is true if $\mathfrak{g}$ is of type 4). Since the connection on the vector bundle $E_0$ is flat and $M$ is simply connected, there exist orthonormal parallel sections $e_1, ..., e_{n_0}$ spanning $E_0$. We will assume that all adapted coordinate systems are chosen in such a way that $\partial_{\mathfrak{e} \mathfrak{h}} e_0 = e_x$. Suppose that $\mathfrak{g} = \mathfrak{g}^{4,h,m,\psi}$. Suppose that $e_1, ..., e_{n-m}$ is a basis of $\psi(h)$. Then there exists a matrix $(\mathcal{c}_0 \mathbf{e}_0)$ such that $\psi(J_\alpha) = \sum_{\alpha=1}^{n-m} \mathcal{c}_0 \alpha \mathbf{e}_0$, $\alpha \in K$. Note that $m = n - \text{rk}(\mathcal{c}_0 \mathbf{e}_0)$.

**Theorem 3.** Let $(M, g)$ be a simply connected locally indecomposable Lorentzian manifold with the holonomy algebra $\mathfrak{g} \subset \mathfrak{sim}(n)$. Suppose that $(M, g)$ admits a parallel light-like vector field. Then $\mathfrak{g}$ is of type 4 if and only if only if there exists a number $m$, $1 \leq m < n$ such that the following conditions hold:
1. rank of the subbundle $E_0 \subset E$ is not smaller then $n - m$; for any adapted coordinate system it holds $\partial_{x^s} \partial_{x^t} H = 0$, $1 \leq s, t \leq n - m$;

2. there is a non-empty subset $K \subset \{1, \ldots, r\}$ of indexes $\alpha$ such that

2.a. if $\alpha \in K$, then $h_\alpha$ is contained in $u \left( \frac{\lambda}{2} \right)$, i.e. the bundle $E_\alpha$ admits a parallel complex structure $J_\alpha$;

2.b. the Riemannian metric $h_\alpha$ is Ricci-flat for all $\alpha \in K$;

2.c. if $\alpha \notin K$, then $\partial_{x^s} \partial_{x^t} H = 0$ for any adapted coordinate system and $1 \leq s \leq n - m$;

2.d. for each $\alpha \in K$ there exists numbers $(c_{s\alpha})_{s=1}^{n-m} \in \mathbb{R}$ such that $m = n - \text{rk}(c_{s\alpha})$ and for any adapted coordinate system it holds

\begin{equation}
pr_{E_\alpha} T(e_s) = \frac{2c_{s\alpha}}{n_\alpha} J_\alpha \widetilde{\text{Ric}}_{P_\alpha},
\end{equation}

where

\[ T(e_s) = -R(e_s, q)q, \quad \widetilde{\text{Ric}}_{P_\alpha} = \sum_{i,j=1}^{n_\alpha} h^{ij}_\alpha R(\partial_{x^i}, q)\partial_{x^j} \]

and for each $s$, $1 \leq s \leq n - m$, $pr_{E_\alpha} T(e_s) \neq 0$ for some adapted coordinate system. Equation (23) has the following coordinate form

\begin{equation}
\partial_{x^s} \partial_{x^t} H = -\frac{2c_{s\alpha}}{n_\alpha} \left( \nabla^j F^i_{tj} + \nabla^j h_{\alpha lj} - \partial_{x^i} h^{jk}_\alpha h_{\alpha jk} \right) J^s_{\alpha t},
\end{equation}

where the indexes $j, k, l$ run from $1$ to $n_\alpha$, $h = \partial_{x^s} h$, $F^i_{tj} = \partial_{x^s} A^i_t - \partial_{x^t} A^i_s$ (no sum over $\alpha$). For each $s$, $1 \leq s \leq n - m$, it holds $\partial_{x^s} \partial_{x^t} H \neq 0$ for some adapted coordinate system and some $i$, $1 \leq i \leq n_\alpha$.

Otherwise, $\alpha$ is of type $2$.

**Proof of Theorem 3.** The proof is similar to the proof of Theorem 2. The description of an element $R \in R(g^{1, h, m, \psi})$ implies $pr_{E_{R-m}} T|_{E_{R-m}} = 0$, this gives the condition $\partial_{x^s} \partial_{x^t} H = 0$, $1 \leq s, t \leq n - m$. Next, if $\alpha \notin K$, then $pr_{E_{R-m}} T|_{E_\alpha} = 0$, this gives condition 2.c. Let us consider condition 2.d. As in the same way as in the proof of Theorem 2 we get

\[ pr_{E_\alpha} T(X) = -\frac{2}{n_\alpha} g(X, J_\alpha \widetilde{\text{Ric}} P)\psi(J_\alpha), \quad \alpha \in K, X \in E_{\alpha x}. \]

Substituting $\psi(J_\alpha) = \sum_{t=1}^{n-m} \epsilon_t \epsilon_t$, multiplying the obtained equality by $\epsilon_s$, and using the facts that $T$ is symmetric and $\epsilon_1, \ldots, \epsilon_{n_\alpha}$ is an orthonormal basis, we obtain (23).

Thus, in order to find the type of the holonomy algebra $\mathfrak{g}$ knowing the orthogonal part of $\mathfrak{g}$, it is enough to apply Proposition 1 and to check the conditions 1 and 2 of one of Theorems 2 or 3. For that it is necessary to find the parallel complex structures $J_\alpha$ on the bundles $E_\alpha$, or the corresponding parallel 2-forms. To compute the right hand side of the condition 2.d, one can use the fact that by (20) it mostly coincides with the one part of the Ricci tensor of $(M, g)$.

In [7], Boubel proved theorems similar to Theorems 2 and 3 where conditions 2.d are changed to equivalent conditions on the curvature tensor. Our conditions 2.d are more precise, they can be checked using the local coordinates and the give the following geometric description of the manifolds with the holonomy algebra of type 3 and 4.

Let us explain the geometric properties of the manifold with the holonomy algebra $\mathfrak{g}$ of type 3. Equality (17) shows that the first condition of the theorem is equivalent to the equation $\lambda = 0$, where $\lambda$ is the canonically defined function from Section 6. In Section 8 we shown also that if $\lambda = 0$ and a section $H$ of the parallel distribution $\ell$ is fixed, then we obtain a sections $e$ of $E = \ell^\perp/\ell$ and a section $Z_\alpha$ of $E_\alpha \subset E$ for each $\alpha \in K$. Condition 2.d is equivalent to the equality

\[ pr_{E_\alpha} e = -\frac{(n_\alpha + 1)c_\alpha}{n_\alpha} J_\alpha Z_\alpha, \quad \alpha \in K. \]

Results of Section 8 show that if we choose $p' = \mu p$ for some non-zero function $\mu$, then $e$ and $Z_\alpha$ change to $\mu e$ and $\mu Z_\alpha$, respectively, i.e. the last equality does not depend on the choice of $p$. 

Manifold with the holonomy algebra \( \mathfrak{g} \) of type 4 have the following geometric properties. First of all, there exists a parallel subbundle \( U \subseteq \mathcal{E}_0 \) of rank \( n - m \). It can be checked that if \( X \in \Gamma(U) \), and a parallel light-like vector \( p \) is fixed, then the projection of \( T(X) \) to \( \mathcal{E} = \mathcal{E}/\ell \) does not depend on the choice of distribution \( E \). We obtain \( n - m \) sections \( T(e_1), \ldots, T(e_{n-m}) \) of \( \mathcal{E} \). Condition 1 shows that these sections belong to \( U^{\perp} \); condition 2.c. shows that the projections of these sections to \( \mathcal{E}_\alpha \) are trivial for \( \alpha \notin K \). Thus if a vector field \( p \) is fixed, then we obtain sections \( T(e_1), \ldots, T(e_{n-m}) \) of \( \mathcal{E} = \mathcal{E}/\ell \) and a section \( Z_\alpha \) of \( \mathcal{E}_\alpha \subset \mathcal{E} \) for each \( \alpha \in K \). Condition 2.d is equivalent to the equality

\[
\text{pr}_{E_\alpha} T(e_s) = -\frac{(n_\alpha + 1)e_{\alpha s}}{n_\alpha} J_\alpha Z_\alpha, \quad \alpha \in K, \quad 1 \leq s \leq n - m.
\]

If we choose \( p' = \mu p \) for some non-zero function \( \mu \), then \( T(e_1), \ldots, T(e_{n-m}) \) and \( Z_\alpha \) change to \( \mu T(e_1), \ldots, \mu T(e_{n-m}) \) and \( \mu Z_\alpha \), respectively, i.e. the last equality does not depend on the choice of \( p \).

Now suppose that \( (M, g) \) is a locally decomposable Lorentzian manifold. If \( (M, g) \) is a global product of Riemannian manifolds and of a Lorentzian manifold, then its holonomy algebra is the direct sum of the corresponding holonomy algebras. If the Wu decomposition is not global, then the holonomy algebra of \( (M, g) \) is the direct sum of the holonomy algebras of the induced connections on the distributions \( E_\alpha \), \( 1 \leq \alpha \leq r \) considered as a vector bundle over \( M \). The holonomy algebra of each distribution \( E_\alpha \), \( 1 \leq \alpha \leq r - 1 \), is irreducible Riemannian holonomy algebra and it can be found as it is explained at the end of Section 4. If the Lorentzian part is flat, then the holonomy is found. Otherwise, by our convention, the holonomy algebra of the distribution \( E_\alpha \) coincides with the holonomy algebra of a Lorentzian manifold, and it can be found using the statements of Proposition 1 and Theorems 2 and 3 applied to local coordinates on the integral submanifolds of the distribution \( E_\alpha \).

11. Example

Let us compare the statement of Theorem 2 with the construction from 13.

Let us fix an irreducible subalgebra \( \mathfrak{h} \subset \mathfrak{u}(m) \), \( n = 2m \), such that \( \mathfrak{h} \) contains the complex structure \( J_0 \) on \( \mathbb{R}^{2m} \). Let \( c \neq 0 \) be a real number. Now we construct a metric \( g \) with the holonomy algebra

\[
\mathfrak{g} = \mathfrak{g}^{3, h, \varphi} = (\mathbb{R}(c + J_0) \oplus \mathfrak{h}') \ltimes \mathbb{R}^n, \quad \varphi(J_0) = c
\]

following 13. Consider the metric

\[
g = 2dvdu + \sum_{i=1}^{n} (dx^i)^2 + 2Adu + h(du)^2, \quad A = A_i(x^1, \ldots, x^n, u)dx^i.
\]

We should consider elements \( P_1, \ldots, P_N \in P(\mathfrak{h}) \) such that their images generate \( \mathfrak{h} \). In fact, it is enough to consider a single \( P \in P(\mathfrak{h}) \); if \( \mathfrak{h} = \mathfrak{u}(m) \), we take \( P = R^{c, \varphi = c}(\cdot, Z) \) for some non-zero vector \( Z \); if \( \mathfrak{h} \subset \mathfrak{u}(m) \) is a symmetric Berger algebra, take \( P = R^h(\cdot, Z) \), where \( R^h \) is the curvature tensor of a symmetric Riemannian space with the holonomy algebra \( \mathfrak{h} \).

Let \( e_1, \ldots, e_n \) be the standard basis of \( \mathbb{R}^n \). Define the numbers

\[
P(e_ke_i) = P^j_{ik}e_j, \quad a^j_{ik} = \frac{1}{3}(P^j_{ik} + P^j_{ki}),
\]

then the metric is given by

\[
A_j = a^j_{ik}x^ix^k, \quad H = 2vx^j\varphi(P(e_i)).
\]

Consider the conditions of Theorem 2. Let \( J_0 e_i = J^i_0 e_j \). Recall that we consider the distribution \( E \) spanned by the vector fields \( X_i = \partial_i - A_i \partial_u \) and the induced connection on \( E \); the holonomy algebra of this connection coincides with \( \mathfrak{h} \). We claim that the complex structure \( J \) on \( E \) defined by \( JX_i = J^0_iX_j \) is parallel. From 13 it follows that the only nonzero Christoffel symbols of the induced connection on \( E \) are of the form

\[
\Gamma^i_{u,j} = P^i_{jk}x^k.
\]
The condition that $h$ commutes with $J_0$ implies
\[ P_{jk}^i J_0^l = J_0^j P_{lk}^i. \]
Consequently,
\[ \nabla_v J = \nabla_i J = 0, \quad (\nabla_u J)_j^i = \partial_u A_j^i + J_0^j \Gamma^i_{al} - J_0^i \Gamma^j_{al} = 0, \]
i.e. $J$ is parallel. Next,
\[
\nabla^i F_{lj} = \sum_{j=1}^{n} \partial_j (\partial_i A_j - \partial_j A_i) = \sum_{j=1}^{n} \partial_j (2a_{jk}^l x^k - 2a_{jk}^l x^k) = 2 \sum_{j=1}^{n} (a_{jl}^j - a_{jj}^l) = 2 \sum_{j=1}^{n} (P_{jl}^l + P_{lj}^l - 2P_{jj}^l) = -2 \sum_{j=1}^{n} P_{jj}^l = -2(\tilde{\text{Ric}} P)_i.
\]
Condition 2.d takes the form
\[
\partial_l \partial_v H = \frac{2c}{m} \sum_{l,j=1}^{n} P_{jj}^l J_0^l.
\]
Clearly, the function
\[
(26) \quad H = \frac{4c}{m} v \sum_{l,j=1}^{n} P_{jj}^l J_0^l x^l
\]
satisfies this and the first conditions. From the proof of Theorem 2 it follows that
\[
\varphi(P(e_i)) = \frac{2c}{m} \sum_{l,j=1}^{n} P_{jj}^l J_0^l.
\]
Using this and comparing (26) with (26), we see that Theorem 2 is in accord with the construction from [13].

References

[1] A. V. Aminova, *Pseudo-Riemannian manifolds with general geodesics*, Russian Math. Surveys 48 (1993), no. 2, 105–160.
[2] H. Baum, O. Müller, *Codazzi spinors and globally hyperbolic manifolds with special holonomy*, Math. Z. 258 (2008), no. 1, 185–211.
[3] Ya. V. Bazakın *Globally hyperbolic Lorentzian spaces with special holonomy groups*, Siberian Mathematical Journal, 50 (2009), no. 4, 567–579.
[4] M. Berger, *Sur les groupers d’holonomie des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France 83 (1955), 279–330.
[5] L. Bérard Bergery, A. Ikemakhen, *On the Holonomy of Lorentzian Manifolds*, Proceeding of symposia in pure math., volume 54 (1993), 27–40.
[6] A. L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin-Heidelberg-New York, 1987.
[7] Ch. Boubel, *On the holonomy of Lorentzian metrics*, Ann. Fac. Sci. Toulouse Math. (6) 16 (2007), no. 3, 427–475.
[8] J. Brannlund, A. Coley, S. Hervik, *Supersymmetry, holonomy and Kundt spacetimes*, Class. Quantum Grav. 25 (2008) 195007 (10pp).
[9] R. Bryant, *Metrics with exceptional holonomy*. Ann. of Math. (2) 126 (1987), 525–576.
[10] S. Cecotti, *A Geometric Introduction to F-Theory*, Lectures Notes, SISSA, 2010, available at [http://people.sissa.it/~cecotti](http://people.sissa.it/~cecotti)
[11] A. S. Galaev, *The spaces of curvature tensors for holonomy algebras of Lorentzian manifolds*, Diff. Geom. and its Applications 22 (2005), 1–18.
[12] A. S. Galaev, *Isometry groups of Lobachevskian spaces, similarity transformation groups of Euclidean spaces and Lorentzian holonomy groups*, Rend. Circ. Mat. Palermo (2) Suppl. No. 79 (2006), 87–97.
[13] A. S. Galaev, *Metrics that realize all Lorentzian holonomy algebras*, Int. J. Geom. Methods Mod. Phys. 3 (2006), nos. 5–6, 1025–1045.
[14] A. S. Galaev, T. Leistner, *Holonomy groups of Lorentzian manifolds: classification, examples, and applications*, Recent developments in pseudo-Riemannian geometry, 53–96, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008.
[15] A. Galaev, T. Leistner, *Recent developments in pseudo-Riemannian holonomy theory*. Handbook of pseudo-Riemannian geometry and supersymmetry, 581627, IRMA Lect. Math. Theor. Phys., 16, Eur. Math. Soc., Zürich, 2010.
[16] A. S. Galaev, *One component of the curvature tensor of a Lorentzian manifold*, J. Geom. Phys 60 (2010), 962–971.
[17] A. S. Galaev, T. Leistner, *On the local structure of Lorentzian Einstein manifolds with parallel distribution of null lines*, Class. Quantum Grav. 27 (2010), 225003 (16pp.).
[18] G. W. Gibbons, C. N. Pope, *Time-Dependent Multi-Centre Solutions from New Metrics with Holonomy Sim(n − 2)*, Class. Quantum Grav. 25 (2008) 125015 (21pp).
[19] S. S. Gubser, *Special holonomy in string theory and M-theory*, Strings, branes and extra dimensions. TASI 2001, 197233, World Sci. Publ., River Edge, NJ, 2004.
[20] J. M. Figueroa-O’Farrill, *Breaking the M-waves*, Class. Quantum Grav. 17 (2000), no. 15, 2925–2947.
[21] D. Joyce, *Riemannian holonomy groups and calibrated geometry*. Oxford University Press (2007).
[22] G. S. Hall, D. P. Lonie, *Holonomy groups and spacetimes*, Class. Quantum Grav. 17 (2000), 1369–1382.
[23] B. Kostant, *On invariant skew-tensors*, Proc. Nat. Acad. Sci. U. S. A. 42 (1956), 148–151.
[24] T. Krantz, *Kaluza-Klein-type metrics with special Lorentzian holonomy*, J. Geom. Phys. 60 (2010), no. 1, 74–80.
[25] K. Lärz, *Global Aspects of Holonomy in Pseudo-Riemannian Geometry*, PhD thesis, Humboldt-Universität zu Berlin, 2011.
[26] T. Leistner, *On the classification of Lorentzian holonomy groups*, J. Differential Geom. 76 (2007), no. 3, 423–484.
[27] T. Leistner, *Screen bundles of Lorentzian manifolds and some generalisations of pp-waves*. J. Geom. Phys. 56 (2006), no. 10, 2117–2134.
[28] B. McInnes, *Obtaining holonomy from curvature*, J. Phys. A: Math. Gen. 30 (1997), 661–671.
[29] A. G. Walker, *On parallel fields of partially null vector spaces*, Quart. J. Math., Oxford Ser., 20 (1949), 135–145.
[30] H. Wu, *On the de Rham decomposition theorem*, Illinois J. Math., 8 (1964), 291–311.