A linear program for the finite block length converse of Polyanskiy-Poor-Verdú via non-signalling codes

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Abstract—Motivated by recent work on entanglement-assisted codes for sending messages over classical channels, the larger, easily characterised class of non-signalling codes is defined. Analysing the optimal performance of these codes yields an alternative proof of the finite block length converse of Polyanskiy, Poor and Verdú, and shows that they achieve this converse. This provides an explicit formulation of the converse as a linear program which has some useful features. For discrete memoryless channels, it is shown that non-signalling codes attain the channel capacity with zero error probability if and only if the dispersion of the channel is zero.

I. INTRODUCTION

A key goal of information theory is to quantify the extent to which reliable communication is possible over a noisy channel. A code of size $M$ and block length $n$ allows communication of one of $M$ messages via $n$ uses of the channel. The fundamental tradeoff between these quantities and the reliability of communication, is captured by $M_\epsilon(E^n)$ - the largest size of code with error probability $\epsilon$ (for equiprobable messages). While emphasis is often placed on quantifying asymptotics of the large $n$ limit (by computing channel capacities or reliability functions, for example) but this information isn’t necessarily useful if one wishes to compute a lower bound on the block length needed for a certain rate and error probability, for instance.

While actually computing $M_\epsilon(E^n)$ is intractable, it is possible to obtain lower (achievability) and upper (converse) bounds on it from which the asymptotic quantities derived, but which also give useful answers for questions concerning finite block lengths. In their recent paper [1], Polyanskiy, Poor and Verdú prove a very general converse bound (the ‘PPV converse’ for the purposes of this article)

$$M_\epsilon(E^n) \leq M_{\epsilon, PPV}^2(E^n),$$

where $M_{\epsilon, PPV}^2(E^n)$ is given by a maximin optimisation of the reciprocal of the minimum type II error over a set of hypothesis tests. They go on to show how many existing converse results can be easily derived from theirs.

Recent work has shown that it can be advantageous in classical coding over classical channels for the sender and receiver to share entangled quantum systems [2, 3, 4, 5]. While the capacity cannot be increased, the number of messages possible for a given error bound can be. Entanglement assistance can even increase the zero-error capacity [6]. This raises questions about the extent to which entanglement can assist in general.

In an entanglement-assisted code, the output of the decoder is conditionally independent of the input to the encoder given the input to the decoder, and vice-versa. A non-signalling (NS) code is any code with this property, and $M_{\epsilon, NS}^2(E^n)$ is largest size of NS code with error probability $\epsilon$. Any upper bound $M_{\epsilon, NS}^2(E^n)$ clearly applies to entanglement-assisted codes.

From the elegant proof of the PPV converse [1], it can be seen that it also applies to non-signalling codes, that is

$$M_{\epsilon, NS}^2(E^n) \leq M_{\epsilon, PPV}^2(E^n).$$

This fact, combined with lower bounds on $M_\epsilon(E^n)$, provides quite stringent limits on the advantage from entanglement assistance. Section III precisely defines the concepts and quantities of interest, and recaps the proof of the PPV converse.

Section III analyses performance of non-signalling codes directly, deriving a linear program for a quantity $M_\epsilon^*(E^n)$ whose integer part $\lfloor M_\epsilon^*(E^n) \rfloor$ is precisely $M_{\epsilon, NS}^2(E^n)$. Clearly this quantity is an upper bound on $M_\epsilon(E^n)$ and, as mentioned, no larger than $M_{\epsilon, PPV}^2(E^n)$. Remarkably, it turns out that $M_\epsilon^*(E^n)$ is precisely equal to $M_{\epsilon, PPV}^2(E^n)$. This provides an alternative proof of the PPV converse (for discrete channels), which shows that it is achieved by NS codes, and provides primal and dual linear programs (LPs) for it, which are useful for computing the bound: The duality theorem for LPs means that any feasible point for the dual LP gives a valid converse bound, and can allow for certification of optimality. There is also an operationally intuitive way to use symmetry of the channel to reduce the size of the linear programs, from exponential to polynomial in $n$ in the case of discrete memoryless channels (DMCs).

Section IV shows that DMCs where non-signalling codes can attain the channel capacity with zero-error, are precisely those with zero channel dispersion, and thus also admit particularly efficient classical codes. The final section concludes with some suggested directions for future research.

II. DEFINITIONS AND BACKGROUND

We consider a single use of a discrete channel with input alphabet $A$ and output alphabet $B$. The channel input and output are random variables (RVs) $X$ and $Y$, respectively. Our

1My thanks to an anonymous referee for pointing this out.
description of the channel use $\mathcal{E}$ determines the probabilities $\mathcal{E}(y|x) := \Pr(Y = y|X = x, \mathcal{E})$. A message $W$ is selected from a set of $M$ possible messages $\{1, 2, \ldots, M\}$ by a source $S$, which determines the probabilities $S(w) := \Pr(W = w|S)$. A code $Z$ consists of an encoder, which takes input $W$ and whose output is the channel input $X$ from $\mathcal{A}$, and a decoder whose input is the channel output $Y$ (in $\mathcal{B}$) and which produces a decoding $\hat{W}$ of the message. The code $Z$ determines the probabilities

$$Z(x, \hat{w}|w, y) := \Pr(X = x, \hat{W} = \hat{w}|W = w, Y = y, Z).$$

(3)

An error has occurred if $W \neq \hat{W}$.

**Remark 1.** When considering $n$ uses of a channel, the alphabets are $\mathcal{A}^n$ and $\mathcal{B}^n$ and the channel use is $\mathcal{E}^n$, which gives the conditional probabilities of output strings $x = x_1 \ldots x_n \in \mathcal{A}^n$ given each input string $y = y_1 \ldots y_n \in \mathcal{B}^n$. A discrete channel is fully described by specifying $\mathcal{E} = \{ \mathcal{E}^n : n \in \mathbb{N}\}$. A discrete memoryless channel (DMC), is one where $\mathcal{E}^n(y|x) = \mathcal{E}^n(y|x) := \prod_{i=1}^n \mathcal{E}(y_i|x_i)$, for all $n$.

In a classical code, the encoder and decoder are uncorrelated, in the sense that

$$Z(x, \hat{w}|w, y) = F(x|w)G(\hat{w}|y)$$

(4)

for some conditional probability distributions $F$ and $G$. This property defines the class NC of codes with No Correlation (in the absence of a channel) between the encoder and the decoder.

**Definition 2.** SE (Shared Entanglement) is the class of entanglement-assisted codes which can be implemented by local operations of the encoder and decoder on quantum systems (with finite Hilbert spaces) in a shared entangled state.

A positive operator valued measure (POVM) $L$ for a Hilbert space $\mathcal{H}$, and finite set of outcomes $R$, assigns positive (semidefinite) operators $L(r)$ on $\mathcal{H}$ to the outcomes $r \in R$ such that $\sum_{r \in R} L(r) = I$, where $I$ is the identity operator on $\mathcal{H}$. A code $Z$ is in SE iff there exist finite dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, POVMs $D_w$ for $\mathcal{H}_A$, with outcomes in $\mathcal{A}$ for $w \in \{1, \ldots, M\}$, POVMs $F_y$ for $\mathcal{H}_B$, with outcomes in $\mathcal{B}$ for $y \in \mathcal{B}$, and a density operator $\rho_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, such that

$$Z(x, \hat{w}|w, y) = Tr_E L_w(x) \otimes D_w(\hat{w}) \rho_{AB}.$$

The class SE contains the class NC, and is itself contained in the class of non-signalling codes:

**Definition 3.** A non-signalling (NS) code is any one for which the marginal distribution of the output of the decoder is conditionally independent of the input to the decoder, and vice-versa. That is, for all $x \in \mathcal{A}, y \in \mathcal{B}, w, \hat{w} \in \{1, \ldots, M\}$,

$$\Pr(\hat{W} = \hat{w}|W = w, Y = y, Z) = \Pr(\hat{W} = \hat{w}|Y = y, Z),$$

\hspace{1cm} (5)

$$\Pr(X = x|W = w, Y = y, Z) = \Pr(X = x|W = w, Z).$$

\hspace{1cm} (6)

These conditions define the class NS of Non-Signalling-assisted codes.

From Bayes’ rule and (6),

$$Z(x, \hat{w}|w, y) = \Pr(\hat{W} = \hat{w}|W = w, Y = y, X = x, Z) \Pr(Y = y, X = x, Z)p(x|w, Z).$$

(7)

where $p(x|w, Z) := \Pr(X = x|W = w, Z)$. This can be interpreted operationally as indicating that if (6) holds, then $Z$ could be implemented by having the encoder stochastically generate $X$ according to the value of $W$, and then send the values of $X$ and $W$ to the decoder (using additional communication) which would use these, in addition to $Y$, to determine how to generate $\hat{W}$. Using (7) and the fact that

$$\Pr(Y = y, X = x|W = w, Z, \mathcal{E}) = \mathcal{E}(y|x)p(x|w, Z)$$

(8)

it is easy to show that

$$\Pr(\hat{W} = \hat{w}, Y = y, X = x|W = w, \mathcal{E}, Z, S) = Z(x, \hat{w}|w, y)\mathcal{E}(y|x).$$

(9)

**Proposition 4.** The conditional probabilities (9) are clearly non-negative. To form a valid conditional distribution, they must also satisfy

$$\forall w : \sum_{\hat{w}, x, y} Z(x, \hat{w}|w, y)\mathcal{E}(y|x) = 1.$$  

(10)

This is true for all channels $\mathcal{E}$ if and only if $Z$ is non-signalling from the receiver to the sender (this is the condition expressed by (6)).

**Proof:** (10) is a straightforward consequence of (6) via (7). For the other direction, if $Z$ is signalling from Bob to Alice then there exist $w' \in \{1, \ldots, M\}, x' \in \mathcal{A}$ and $y_0, y_1 \in \mathcal{B}$ such that $\sum_{\hat{w}} Z(x, \hat{w}|w', y_0) > \sum_{\hat{w}} Z(x, \hat{w}|w', y_1)$. Choosing the channel $\mathcal{E}$ with $\mathcal{E}(y_0|x') = 1$ and, for all $x \neq x'$, $\mathcal{E}(y_1|x) = 1$,

$$\sum_{x, \hat{w}} \mathcal{E}(y_0|x)Z(x, \hat{w}|w', y_0) > \sum_{x, \hat{w}} \mathcal{E}(y_0|x)Z(x, \hat{w}|w', y_1)$$

(11)

Since $\forall x : \mathcal{E}(y_0|x) = 1 - \mathcal{E}(y_1|x)$, this implies that

$$\sum_{\hat{w}, x, y} Z(x, \hat{w}|w', y)\mathcal{E}(y|x) > 1.$$  

(12)

For the rest of this paper, the source is taken to be $S_M$, which assigns equal probability to each message: $\forall w : S_M(w) = 1/M$.

**Definition 5.** For channel $\mathcal{E}$, the minimum average probability of error which can be achieved by a code in class $\Omega$ is

$$\epsilon^\Omega(M, \mathcal{E}) := \min\{\Pr(W \neq \hat{W}|Z, \mathcal{E}, S_M) : Z \in \Omega\}$$

and the largest local code with error no larger than $\epsilon$ has size

$$M^\epsilon(\mathcal{E}) := \max\{M : \Pr(W \neq \hat{W}|Z, \mathcal{E}, S_M) \leq \epsilon, Z \in \Omega\}.$$  

When the superscript $\Omega$ is omitted, it is intended that $\Omega = NC$.

**Remark 6.** By the inclusions of the classes of codes,

$$\epsilon(M, \mathcal{E}) \geq \epsilon^\text{SE}(M, \mathcal{E}) \geq \epsilon^\text{NS}(M, \mathcal{E}),$$

and

$$M_\epsilon(\mathcal{E}) \leq M_\epsilon^\text{SE}(\mathcal{E}) \leq M_\epsilon^\text{NS}(\mathcal{E}).$$

(14)
Remark 7. Note that if \( E(y|x) = q(y) \) then, using (5),
\[
\epsilon_{\text{NS}}(M, \mathcal{E}) = 1 - \frac{1}{M} \sum_{w,x,y} Z(x, w|w, y)q(y)
\]
\[
= 1 - \frac{1}{M} \sum_{w,y} \Pr(W = w|Y = y)q(y) = 1 - 1/M.
\]

The direct part of Proposition 4 guarantees that this is a valid error
\( \sum \) with \( P \mathcal{E} \). The fact that the PPV converse applies to NS codes has
some immediate consequences:

Remark 11. Since the information spectrum converse that
Verdú and Han use to derive their general formula for channel
capacity \( [2] \) can be derived from the PPV converse, this
formula also gives the capacity for NS codes.

Remark 12. For DMCs, a converse derived from the PPV
converse and an achievability bound for classical codes, can
be used to prove [1] the result of Strassen [8],
\[
\log M(e^{\text{SN}}) = nC - \sqrt{nVQ^{-1}(e)} + O(\log n),
\]
where \( C \) is the channel capacity, \( V \) is the channel dispersion (see Section [1])
and \( Q(x) := (2\pi)^{-1/2} \int_x^\infty e^{-t^2/2} dt \).

Since the PPV converse also applies to NS codes, (27)
applies to these too, and the difference in the rates achieved
by classical and NS codes (for fixed \( e \)) is only of order \( O(\log n)/n \).

III. THE PERFORMANCE OF NON-SIGNALLING CODES

The optimisation over codes that yields \( \epsilon_{\text{NS}}(M, \mathcal{E}) \) in Definition (5)
is already a linear program (LP): The variable is the code \( \mathcal{E} \) (considered as a \(|A| \times |B| \times M^2 \) dimensional real vector),
the objective function \( \Pr(W \neq \hat{W}|Z, \mathcal{E}, S_M) \) is
\[
1 - \frac{1}{M} \sum_{w,x,y} \mathcal{E}(y|x)Z(x, w|w, y),
\]
and the constraints are simply the linear equalities comprising
the non-signalling conditions (5) and (6), in addition to the
non-negativity and normalisation of \( Z \).

Fig. 1. Operational interpretation of the code \( \mathcal{E} \) which results from the
symmetrisation [29] of a non-signalling code. The boxes marked ‘e’ and ‘d’
are the encoder and decoder for the original non-signalling code \( \mathcal{E} \). The
permutations are coordinated by a shared random variable \( \pi \) drawn uniformly
at random from the symmetric group on \( \{1, \ldots, M\} \).
If $Z$ is an NS code, then let

$$Z(x, \hat{w}|w, y) = \frac{1}{|G|} \sum_{\pi \in G} Z(x, \pi(\hat{w})|\pi(w), y)$$

where $G$ is the symmetric group on $\{1, \ldots, M\}$, $\pi(w)$ denotes the action of a permutation in $G$ on $w \in \{1, \ldots, M\}$. This symmetrized code $Z(x, \hat{w}|w, y)$ has an operational interpretation given in Fig 1 from which it is clear that it is also non-signalling and since

$$\Pr(W = \hat{W}|Z, E, S_M) = \frac{1}{|G|} \sum_{x,y} \sum_{\pi \in G} E(y|x)Z(x, \pi(w)|\pi(w), y)$$

$$= \frac{1}{|G|} \sum_{\hat{w}} \sum_{x,y} \sum_{\pi \in G} E(y|x)Z(x, \hat{w}|w, y)$$

$$= \Pr(W = \hat{W}|Z, E, S_M),$$

the optimisation over NS codes for $\epsilon^{NS}(M, E)$, can be restricted to symmetrized codes. These are precisely those codes with the form

$$Z(x, \hat{w}|w, y) = \begin{cases} R_{xy} & \text{if } \hat{w} = w, \\ Q_{xy} & \text{if } \hat{w} \neq w. \end{cases}$$

In these terms, the non-signalling condition (29) is equivalent to saying that there exists $p : A \rightarrow \mathbb{R}$ such that $R_{xy} + (M-1)Q_{xy} = p(x)$, and so

$$Z(x, \hat{w}|w, y) = \begin{cases} R_{xy} & \text{if } \hat{w} = w, \\ p(x) - R_{xy} & \text{if } \hat{w} \neq w. \end{cases}$$

With this simplification, the conditional probabilities in $Z$ are non-negative if $R_{xy} \geq 0$ and $p(x) \geq R_{xy}$ for all $x, y$, and the normalisation condition $\forall w, y, \sum_{x, \hat{w}} Z(x, \hat{w}|w, y) = 1$ is equivalent to $\sum_x p(x) = 1$. The condition (5) of no signalling from encoder to decoder is $\forall y : \sum_x R_{xy} = \sum_x (p(x) - R_{xy})/(M-1)$ which, in light of the normalisation condition, is equivalent to

$$\forall y : \sum_x R_{xy} = 1/M.$$  

Proposition 13.

$$1 - \epsilon^{NS}(M, E) = \max \sum_{x \in A} \sum_{y \in B} E(y|x)R_{xy}$$

subject to

$$\forall y \in B : \sum_{x \in A} R_{xy} \leq 1/M,$$

$$\forall x \in A, y \in B : p(x) \geq R_{xy},$$

$$\sum_{x \in A} p(x) = 1,$$

$$\forall x \in A, y \in B : R_{xy} \geq 0, p_x \geq 0.$$  

Introducing Lagrange multipliers $D_{xy}$, $z_y$, $\alpha$ for the constraints (41), (40), (42) respectively, the Lagrangian function is

$$\sum_{x \in A} \sum_{y \in B} E(y|x)R_{xy} + \sum_{x \in A} \sum_{y \in B} D_{xy}(p(x) - R_{xy})$$

$$+ \sum_{y \in B} z_y \left( \frac{1}{M} - \sum_{x \in A} R_{xy} \right) + \alpha \left( 1 - \sum_{x \in A} p(x) \right)$$

$$= \sum_{x \in A} \sum_{y \in B} R_{xy}(E(y|x) - D_{xy} - z_y)$$

$$+ \sum_{x \in A} \sum_{y \in B} p(x) \left( \sum_{y \in B} D_{xy} - \alpha \right) + \alpha + \frac{1}{M} \sum_{y \in B} z_y.$$  

Taking the supremum over non-negative $R$ and $u$ and restricting the multipliers to the region where it is finite yields the dual LP, whose solution is equal to that of the primal LP by the strong duality theorem for linear programming:

$$\epsilon^{NS}(M, E) = \max \left( 1 - \alpha - \frac{1}{M} \sum_{y \in B} z_y \right)$$

subject to

$$\forall x \in A, y \in B : E(y|x) \leq D_{xy} + z_y,$$

$$\forall x \in A : \sum_{y \in B} D_{xy} \leq \alpha,$$

$$\forall x \in A, y \in B : D_{xy} \geq 0.$$  

Fixing $z$, one should pick $D_{xy} = \max\{E(y|x) - z_y, 0\}$ and $\alpha = \max_{x \in A} \sum_{y \in B} D_{xy}$ so that the objective function is

$$1 - \max_{x \in A} \sum_{y \in B} \max\{E(y|x) - z_y, 0\} - \frac{1}{M} \sum_{y \in B} z_y$$

$$= \min_{x \in A} \sum_{y \in B} (E(y|x) - \max\{E(y|x) - z_y, 0\}) - \frac{1}{M} \sum_{y \in B} z_y$$

$$= \min_{x \in A} \sum_{y \in B} \min\{E(y|x), z_y\} - \frac{1}{M} \sum_{y \in B} z_y.$$  

It remains to maximise over $z$:

**Proposition 14.** The minimum error probability which can be attained by an NS code is

$$\epsilon^{NS}(M, E) = \min_z \sum_{x \in A} \min_{y \in B} (\min\{z_y, E(y|x)\} - z_y/M).$$
Allowing $M$ to take on real values in Proposition 13 and defining $\mu := 1/M$, it is evident that $c^{\text{NS}}(M, \mathcal{E})$ is a piecewise linear, non-increasing, concave function of $\mu$ for $\mu \in [0, 1]$. What’s more, this can be inverted to obtain a linear program which gives the smallest value of $1/M$ such that there exists an NS code of size $\lfloor M \rfloor$ with error probability $\epsilon$ for $\mathcal{E}$. That is, $M^{\text{NS}}_{\epsilon}(\mathcal{E}) = \lfloor M^{\ast}(\mathcal{E}) \rfloor$ where

$$M^{\ast}(\mathcal{E})^{-1} = \min \mu,$$

subject to

$$\forall y \in B : \sum_{x \in A} R_{xy} \leq \mu,$$

$$\sum_{x \in A} \sum_{y \in B} \mathcal{E}(y|x) R_{xy} \geq 1 - \epsilon,$$

and the constraints (44, 43).

At this point, it is quite straightforward to show the claimed equivalence to the PPV converse:

**Proposition 15.**

$$M^{\ast}(\mathcal{E}) = M^{\text{PPV}}_{\epsilon}(\mathcal{E})$$

**Proof:** Writing out the optimisation that determines the PPV converse (Theorem 9) explicitly (with the shorthands $p(x) := P_{X}(x)$, $q(x) := Q_{Y}(y)$), it is clear that the function being optimised is bilinear in $T$ and $q$, both of which are constrained to finite dimensional polytopes. Using von Neumann’s minimax theorem 9,

$$M^{\text{PPV}}_{\epsilon}(\mathcal{E})^{-1} = \min_{p} \max_{q} \min_{T} \sum_{x \in A} \sum_{y \in B} T_{xy} p(x) q(y)$$

$$= \min_{p} \max_{q} \min_{T} \sum_{x \in A} \sum_{y \in B} T_{xy} p(x) q(y)$$

$$= \min_{p} \max_{q} \min_{T} \sum_{x \in A} \sum_{y \in B} T_{xy} p(x)$$

subject to

$$\sum_{x \in A} \sum_{y \in B} \mathcal{E}(y|x) p_{y} T_{xy} \geq 1 - \epsilon,$$

$$\sum_{y \in B} p(x) = 1, \sum_{y \in B} q(y) = 1,$$

$$\forall x \in A, y \in B : 0 \leq T_{xy} \leq 1,$$

$$\forall x, y : p(x) \geq 0, q(y) \geq 0.$$

Writing $R_{xy} = p(x) T_{xy}$, this linear program is equivalent to

$$\min \mu,$$

subject to

$$\forall y \in B : \sum_{x \in A} R_{xy} \leq \mu,$$

$$\sum_{x \in A} \sum_{y \in B} \mathcal{E}(y|x) R_{xy} \geq 1 - \epsilon,$$

$$\sum_{x \in A} p(x) = 1,$$

$$\forall x \in A, y \in B : 0 \leq R_{xy} \leq p(x),$$

which is exactly the primal LP for $M^{\ast}(\mathcal{E})^{-1}$. ■

Since the maximisation of $1/\mu$ under the constraints (72, 75), which yields $M^{\ast}_{p}(\mathcal{E})$ directly, is a linear-fractional program [10], the Charnes-Cooper transformation [11]

$$F_{xy} := R_{xy}/\mu, \quad v_{x} := p(x)/\mu, \quad t := 1/\mu,$$  

(76)

can be used to transform it into a linear program for $M^{\ast}_{p}(\mathcal{E})$, from which $t$ can be eliminated by using the transformed version of (74). $\sum_{x \in A} v_{x} = t$, to obtain

**Theorem 16.** $M^{\text{NS}}_{\epsilon}(\mathcal{E}) = \lfloor M^{\ast}(\mathcal{E}) \rfloor$, where

$$M^{\ast}(\mathcal{E}) = \max_{x} v_{x},$$

subject to

$$\forall x \in A, y \in B : F_{xy} \leq v_{x},$$

$$\forall y \in B : \sum_{x \in A} F_{xy} \leq 1,$$

$$\forall x \in A, y \in B : F_{xy} \geq 0, v_{x} \geq 0.$$

Since the main goal is to obtain upper bounds on $M$, the dual of this linear program is more useful. Introducing Lagrange multipliers $V_{xy}$, $c_{y}$ and $\xi$ for the constraints (79), (80) and (81) respectively, taking the infimum of the resulting Lagrangian over non-negative $F$ and $v$, and restricting the multipliers to the finite region gives us the dual program:

**Theorem 17.** $M^{\text{NS}}_{\epsilon}(\mathcal{E}) = \lfloor M^{\ast}(\mathcal{E}) \rfloor$, where

$$M^{\ast}(\mathcal{E}) = \min_{y \in B} c_{y},$$

subject to

$$\forall x \in A, y \in B : V_{xy} + c_{y} \geq \xi \mathcal{E}(y|x),$$

$$\forall x \in A : \sum_{y \in B} V_{xy} \leq (1 - \epsilon) \xi - 1,$$

$$\forall x \in A, y \in B : V_{xy} \geq 0, c_{y} \geq 0.$$

At any feasible point of this dual LP, the value of the objective function is an upper bound on $M^{\text{NS}}_{\epsilon}(\mathcal{E})$.

**A. The zero-error case.** In [8], it was shown that $M^{\text{NS}}_{\epsilon}(\mathcal{E})$ is given by a linear program which is determined by a combinatorial object associated with $\mathcal{E}$, namely its hypergraph. This subsection recovers that result as a special case of the results developed here. First, some definitions: The hypergraph $H(\mathcal{E})$ of $\mathcal{E}$ has vertex set $V(H) = A$ and hyperedges

$$E(H(\mathcal{E})) := \{ e_{y} := \{ x : \mathcal{E}(y|x) > 0 \} : \forall y \in Y \}$$

(88)

capturing the equivocation of each output symbol $y \in B$. (Note that since the set of hyperedges is defined by its members, these being subsets of $A$, the number of hyperedges may be less than the number of output symbols.) A fractional packing of a hypergraph $H$ is an assignment of non-negative weights $v(x) \leq 1$ to all vertices $x \in V(H)$ such that

$$\forall e \in E(H) : \sum_{x \in e} v_{x} \leq 1.$$

(89)
A fractional covering of a hypergraph $H$ is an assignment of non-negative weights $c(e) \leq 1$ to all hyperedges $e \in E(H)$ such that
\[
\forall x \in A : \sum_{e \ni x} c_e \geq 1. \tag{90}
\]
(Restricting the weights to $\{0,1\}$ recovers the combinatorial notions of packing and covering.)

The fractional packing number $\alpha^*(H)$ is the maximum total weight allowed in a fractional packing of $H$ and the fractional covering number $\omega^*(H)$ is the minimum total weight required for a fractional covering of $H$. These are clearly dual linear programs, which for a channel hypergraph $H(\mathcal{E})$ have the formulation
\[
\alpha^*(H(\mathcal{E})) = \max \left\{ \sum_{x \in A} v_x : \forall x \in A, v(x) \geq 0, \sum_{y \in B} [\mathcal{E}(y|x)] v_x \leq 1 \right\},
\]
\[
\omega^*(H(\mathcal{E})) = \min \left\{ \sum_{y \in B} c_y : \forall y \in B, c_y \geq 0, \sum_{x \in A} [\mathcal{E}(y|x)] c_y \geq 1 \right\},
\]
(note that $[\mathcal{E}(y|x)]$ is 0 if $\mathcal{E}(y|x) = 0$ and is otherwise 1.)

In [3] it was shown that $M^0_{\text{NS}}(\mathcal{E}) = [\alpha^*(H(\mathcal{E}))]$. Given Theorem 16 this is equivalent to

**Proposition 18.**
\[
M^0_{\text{NS}}(\mathcal{E}) = \omega^*(H(\mathcal{E})) = \alpha^*(H(\mathcal{E})). \tag{91}
\]

**Proof:** In the primal LP for $M^0_{\text{NS}}(\mathcal{E})$ (Theorem 16), let $v_x$ be any fractional packing of $H(\mathcal{E})$, and let
\[
F_{xy} = \begin{cases} v_x & \text{if } \mathcal{E}(y|x) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{92}
\]
Now, the constraints (79) are trivially satisfied and the constraint (81) is satisfied because $\sum_{x \in A} \sum_{y \in B} \mathcal{E}(y|x) F_{xy} = \sum_{x \in A} \sum_{y \in B} \mathcal{E}(y|x) v_x = \sum_{x \in A} v_x$. For all $y \in B$, $\sum_{x \in A} F_{xy} = \sum_{x: \mathcal{E}(y|x) > 0} v_x$ which is less than or equal to one because $v_x$ is a fractional packing, so the constraints (80) are satisfied. Therefore,
\[
\alpha^*(H(\mathcal{E})) \leq M^0_{\text{NS}}(\mathcal{E}). \tag{93}
\]

In the dual LP for $M^0_{\text{NS}}(\mathcal{E})$ (Theorem 17), let $c_y$ be any fractional covering of $H(\mathcal{E})$, choose the smallest $\zeta$ such that $\forall x, y : \zeta \mathcal{E}(y|x) \geq c_y$, and let $V_{xy} = \max\{0, \zeta \mathcal{E}(y|x) - c_y\}$. Clearly the constraints (85) are satisfied, and for all $x \in A$,
\[
\sum_{y \in B} V_{xy} = \sum_{y: \mathcal{E}(y|x) > 0} (\zeta \mathcal{E}(y|x) - c_y) \leq \zeta - 1, \tag{94}
\]
as required for (86). Therefore,
\[
M^0_{\text{NS}}(\mathcal{E}) \leq \omega^*(H(\mathcal{E})). \tag{95}
\]

Since $\omega^*(H(\mathcal{E})) = \alpha^*(H(\mathcal{E}))$, the result follows.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Operational interpretation of the code $\mathcal{Z}$ which results from the symmetrisation (77) of a non-signalling code. The boxes marked ‘$e’ and ‘$d’ are the encoder and decoder for the original non-signalling code $\mathcal{Z}$. The transformations of the channel input and output are coordinated by a shared random variable $g$ drawn uniformly at random from the group $G$.}
\end{figure}

**B. Taking advantage of symmetry**

Let $G$ be a group with an action on the input alphabet $A$ and on the output alphabet $B$ (inducing a joint action on $A \times B$), such that
\[
\forall g \in G : \mathcal{E}(g \circ y | g \circ x) = \mathcal{E}(y|x). \tag{96}
\]
For any non-signalling code $\mathcal{Z}$ define the code
\[
\mathcal{Z}(x, \hat{w}|w, y) := \frac{1}{|G|} \sum_{g \in G} \mathcal{Z}(g \cdot x, \hat{w}|w, g \cdot y), \tag{97}
\]
whose operational interpretation is given in Fig 2 and which is also non-signalling. The value of $\mathcal{Z}(x, \hat{w}|w, y)$ depends only on $G(x, y)$, that is, the orbit of $(x, y)$ under the joint action of $G$, and since
\[
\Pr(\hat{W} = \hat{w}|W = w, \mathcal{Z}, \mathcal{E}, S_M) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in A} \sum_{y \in B} \mathcal{Z}(\mathcal{E}(g \circ y | g \circ x) \circ x, \hat{w}|w, g^{-1} \cdot y) \mathcal{E}(y|x) \tag{98}
\]
\[
= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in A} \sum_{y \in B} \mathcal{Z}(g^{-1} \circ x, \hat{w}|w, g^{-1} \cdot y) \mathcal{E}(y|x) \tag{100}
\]
\[
= \Pr(\hat{W} = \hat{w}|W = w, \mathcal{Z}, \mathcal{E}, S_M), \tag{101}
\]
the optimisations for $\epsilon_{\text{NS}}(M, \mathcal{E})$ and $M^0_{\text{NS}}(\mathcal{E})$ in Definition 5 can be restricted to codes with this symmetry.

Proposition 13 was obtained by showing that one can already to NS codes of the form
\[
\mathcal{Z}(x, \hat{w}|w, y) = \begin{cases} R_{x,y} & \text{if } \hat{w} = w, \\ R_{x,y} \frac{M - 1}{M - 1} & \text{if } \hat{w} \neq w, \end{cases} \tag{102}
\]
without increasing the optimal error probability. Applying the symmetrisation (77) to this expression, $R_{x,y}$ and $p(x)$ will only depend on $G(x, y)$ and $Gx$, respectively.

An example where symmetry can be used to great effect is where $\mathcal{E}^{(n)}$ (with input alphabet is $A^n$ and output alphabet $B^n$) is invariant under the actions of the symmetric group $S^n$ that permutes the symbols in the input and output strings. This is true for any DMC, for example.

Following [12] and [13], the joint type of a pair of sequences $x = x_1 \ldots x_n \in A^n$ and $y = y_1 \ldots y_n \in B^n$ is the distribution $F_{x,y}$ on $A \times B$ defined by $F_{x,y}(a, b) = N(a,b|x,y)$ where $N(a,b|x,y)$ is the number of values of $i$ for which $(x_i, y_i) = (a, b)$. $P_n(A \times B)$ denotes the set of all such joint types.
Likewise, the type of a sequence $x \in A^n$ is the distribution $P_x$ on $A$ with $N(a|x) = n P_x(a)$ and $P_n(A)$ is the set of these. Given a joint type $\tau_{AB}$, the joint type class $T^n_{\tau_{AB}}$ is the set of all pairs of strings $(x, y)$ with joint type $\tau_{AB}$. Similarly, for a type $\tau_A$, $T^n_{\tau_A} := \{ x \in A^n : P_x = \tau_A \}$.

As is well known, the orbits of $A^n \times B^n$ under the joint action of the symmetric group described above, are precisely the joint type classes $T^n_{\tau_{AB}}$, for each joint type in $P_n(A \times B)$, and $\mathcal{E}^n(y|x)$ is a function only of the joint type of $x$ and $y$:

$$\mathcal{E}^n(y|x) = \mathcal{E}^n(P_{x,y}).$$  \hspace{1cm} (103)

For a DMC, and joint type $\tau_{AB} \in P_n(A \times B)$,

$$\mathcal{E}^n(P_{x,y}) = \prod_{a \in A, b \in B} \mathcal{E}(b|a)^{n \tau_{AB}(a,b)}.$$  \hspace{1cm} (104)

Therefore, in the primal formulation of $i^{NS}(M, \mathcal{E}^n)$ (Proposition 13) one can take $R_{xy} = R(P_{x,y})$ and $p(x) = p(P_x)$ for all $x$, $y$, and replace the sums over the input and output strings with sums over joint types (or types) which incorporate the correct multiplicity factors.

The objective function in (38) becomes

$$\sum_{\tau_{AB} \in P_n(A \times B)} |T^n_{\tau_{AB}}| R(\tau_{AB}) \mathcal{E}(\tau_{AB}),$$  \hspace{1cm} (105)

where $|T^n_{\tau_{AB}}| = n! / \prod_{a \in A, b \in B} (n \tau_{AB}(a,b) !)$. Similarly, the normalisation of $u$ in (42) becomes

$$\sum_{\sigma \in P_n(A)} |T^n_{\sigma}| u(\sigma) = 1.$$  \hspace{1cm} (106)

where $|T^n_{\sigma}| = n! / \prod_{a \in A} (n \sigma(a) !)$. In (40) there is a constraint on a sum over $A^n$ for each output string in $B^n$. The number of pairs $(x, y)$ with joint type equal to $\tau_{AB}$ for fixed $y$, depends only on $P_y$, and is equal to

$$m(\tau; P_y) := \left\{ \begin{array}{ll} \prod_{b \in B} \frac{(n \tau_B(b) !)}{\prod_{a \in A} (n \tau_{AB}(a, b) !)} & \text{if } \tau_B = P_y, \\ 0 & \text{otherwise}. \end{array} \right.$$  \hspace{1cm} (107)

(Note that if the joint type of $(x, y)$ is $\tau_{AB}$, then the marginal distribution $\tau_A$ is the type of $x$, and $\tau_B$ is the type of $y$.) Therefore, (40) can be replaced by

$$\forall \sigma_B \in P_n(B) : \sum_{\tau_{AB} \in P_n(A \times B)} m(\tau_{AB}; \sigma_B) R(\tau_{AB}) \leq 1/M.$$  \hspace{1cm} (108)

The remaining constraints are equivalent to

$$\forall \tau_{AB} \in P_n(A \times B) : 0 \leq R(\tau_{AB}) \leq p(\tau_A).$$  \hspace{1cm} (109)

Since the number of (joint) types is polynomial in $n$ [12, 13], the number of variables and constraints in the simplified LP given above is polynomial in $n$, and this is also true of dual of this program. The linear programs derived for $M^{NS}_n(\mathcal{E}^n)$ can be simplified similarly.

IV. ASSISTED ZERO-ERROR CAPACITIES OF DISCRETE MEMORYLESS CHANNELS WITH ZERO DISPERSION

As shown in [3], for any DMC, it follows from Proposition 18 and the multiplicativity of the fractional packing number, that

$$C_{0}^{NS}(\mathcal{E}) = \log \alpha(\mathcal{E})^*.$$  \hspace{1cm} (110)

A simulation $\mathcal{L}$ of size $\kappa$ for the channel use $\mathcal{E}$ of size $\kappa$ consists of an encoder which takes an input $X$ from $A$ and produces a message $J$ in $\{ 1, \ldots, \kappa \}$, and a decoder which takes a message $\hat{J}$ in $\{ 1, \ldots, \kappa \}$ and produces an output $Y$ from $B$. The $\mathcal{L}$ determines the probabilities

$$\mathcal{L}(j, y|x, J) := \Pr(J = j, Y = y|X = x, \hat{J} = \hat{j}, \mathcal{L}).$$  \hspace{1cm} (111)

We assume that the message is perfectly transmitted from the encoder to the decoder, so $J = \hat{J}$. The simulation is exact if

$$\Pr(Y = y|X = x, \mathcal{L}) = \sum_{j=1}^{\kappa} \mathcal{L}(j, y|x, j) = \mathcal{E}(y|x).$$  \hspace{1cm} (112)

A non-signalling (NS) simulation is one where

$$\Pr(Y = y|X = x, \hat{J} = \hat{j}, \mathcal{L}) = \Pr(Y = y|\hat{J} = \hat{j}, \mathcal{L}),$$  \hspace{1cm} (113)

$$\Pr(J = j|X = x, \hat{J} = \hat{j}, \mathcal{L}) = \Pr(J = j|X = x, \mathcal{L}).$$  \hspace{1cm} (114)

$k_0^{NS}(\mathcal{E})$ denotes the minimum size of an exact NS simulation of $\mathcal{E}$, and

$$K_0^{NS}(\mathcal{E}) := \lim_{n \to \infty} \frac{1}{n} \log k_0^{NS}(\mathcal{E}^n)$$  \hspace{1cm} (115)

is the (asymptotic) exact simulation cost of $\mathcal{E}$. In [3] it was shown that, for any DMC,

$$K_0^{NS}(\mathcal{E}) = \log \sum_{y \in B} \max_{x \in A} \mathcal{E}(y|x).$$  \hspace{1cm} (116)

In what follows, $\mathcal{E}$ is omitted as an argument, since it refers to some fixed channel. For any discrete channel,

$$C_0 \leq C_0^{SE} \leq C_0^{NS} \leq C \leq K_0^{NS}.$$  \hspace{1cm} (117)

From now on, let $\mathcal{E}$ be a DMC with $\mathcal{E}^n = \mathcal{E}^{\otimes n}$. Proposition 26 of [3] shows that, given any requirement on which transition probabilities in $\mathcal{E}$ must be zero, it is possible to find an $\mathcal{E}$ that satisfies that requirement and has all three quantities in (117) equal. In [3] it was shown that there are DMCs where even the entanglement-assisted zero-error capacity $C_0^{SE}$ reaches $C$ (and with a block length one entanglement-assisted code) despite the unassisted zero-error capacity $C_0$ being strictly smaller.

For a DMC, $V$ is the minimum variance of the information density of the channel for the joint distribution induced by any capacity achieving input distribution for a single channel use. The information density is

$$i(x; y) = \log \frac{\mathcal{E}(y|x)}{q(y)},$$  \hspace{1cm} (118)

where $q \in P(B)$ is the output distribution. The capacity is the expectation of the information density. Therefore, $V = 0$ iff there exists a capacity achieving input distribution $p \in P(A)$ (with induced output distribution $q$) such that (118) is equal
to $C$ when $\mathcal{E}(y|x)p(x)$ is non-zero, i.e. if and only if, for $x$ s.t. $p(x) > 0$
\[ \mathcal{E}(y|x) = [\mathcal{E}(y|x)]q(y)2^C. \] (119)

If $V$ is zero, then the $\sqrt{n}$ term vanishes in the asymptotic expansion. In this sense, a channel with zero dispersion admits qualitatively more efficient codes (in terms of approaching capacity with increasing block length) than a channel with positive variance does. It turns out that a channel has zero dispersion if and only if its capacity can be achieved with zero-error by NS codes.

**Theorem 19. For a DMC $\mathcal{E}$ the three conditions**

1) $C_{0}^{NS}(\mathcal{E}) = C(\mathcal{E})$,
2) $K_{0}^{NS}(\mathcal{E}) = C(\mathcal{E})$,
3) $V(\mathcal{E}) = 0$.

**are equivalent.**

**Proof:** The following propositions show that (3) implies (1) and (2); that (1) implies (3); and that (2) implies (3). ■

**Proposition 20. If $V = 0$ then $C_{0}^{NS}$ and $K_{0}^{NS}$ are both equal to $C$.**

**Proof:** We show that if $V(\mathcal{E}) = 0$ then the opposite inequalities to those in (117) also hold. Using (119),
\[ K_{0}^{NS} = \log \sum_{y \in \mathcal{B}} \max_{x} \mathcal{E}(y|x) \] (120)
\[ = \log \sum_{y \in \mathcal{B}} \max_{x} [\mathcal{E}(y|x)]q(y)2^C \leq C. \] (121)

For the other part, when $q(y)$ is non-zero
\[ \sum_{x \in A} [\mathcal{E}(y|x)]p(x)2^C = \sum_{x \in A} \mathcal{E}(y|x)p(x) = 1, \] (122)
and when $q(y)$ is zero we must have $[\mathcal{E}(y|x)]p(x) = 0$ for all $x \in A$ and
\[ \sum_{x \in A} [\mathcal{E}(y|x)]p(x)2^C = 0. \] (123)

Therefore $p(x)2^C$ is a fractional packing, and $C^{NS} \geq C$. ■

**Definition 21.**
\[ \alpha^*(\mathcal{E}, p) := \max \{ \alpha : \forall y \sum_{x} [\mathcal{E}(y|x)]\alpha p(x) \leq 1 \}, \] (124)
which is equivalent to
\[ \alpha^*(\mathcal{E}, p) = \frac{1}{\max_{p} \sum_{x \in A} [\mathcal{E}(y|x)]p(x)}. \] (125)

Clearly the fractional packing number is given by $\alpha^*(\mathcal{E}) = \max_{p} \alpha^*(\mathcal{E}, p)$, where the maximum is over probability distributions $p$ on the input alphabet.

**Lemma 22. Let $I(\mathcal{E}, p)$ denote the mutual information between channel input and output when the input has probability mass function $p$.**
\[ I(\mathcal{E}, p) \geq \log \alpha^*(\mathcal{E}, p) \] (126)

**Proof:** Let $q(y) = \sum_{x \in A} p(x)\mathcal{E}(y|x)$.
\[ \log \alpha^*(\mathcal{E}, p) = -\max_{y \in \mathcal{B}} \sum_{x \in A} [\mathcal{E}(y|x)]p(x) \] (127)
\[ \leq -\sum_{y \in \mathcal{B}} q(y) \log \sum_{x \in A} [\mathcal{E}(y|x)]p(x) \] (128)
\[ = -\sum_{y \in \mathcal{B}} \sum_{x \in A} \mathcal{E}(y|x)p(x) \log \sum_{x} [\mathcal{E}(y|x')]p(x') \] (129)

Subtracting $I(\mathcal{E}, p)$ from this last expression one obtains
\[ \sum_{y \in \mathcal{B}} \sum_{x \in A} \mathcal{E}(y|x)p(x) \log \sum_{x} [\mathcal{E}(y|x')]p(x') \] (130)
which is never larger than zero because, using $\log x \leq (x - 1)/(\ln 2)$,
\[ \sum_{x : \mathcal{E}(y|x) > 0} \mathcal{E}(y|x)p(x) \log \frac{\sum_{x} [\mathcal{E}(y|x')]p(x')}{\mathcal{E}(y|x) \sum_{x} [\mathcal{E}(y|x')]p(x')} \] (131)
\[ \leq \sum_{x : \mathcal{E}(y|x) > 0} \mathcal{E}(y|x)p(x) \log \left( \frac{\sum_{x} [\mathcal{E}(y|x')]p(x')}{\mathcal{E}(y|x) \sum_{x} [\mathcal{E}(y|x')]p(x')} - 1 \right) \] (132)
\[ = \left( \sum_{x} \mathcal{E}(y|x)p(x) \right) \left( \sum_{x} [\mathcal{E}(y|x')]p(x') \right) \] (133)
\[ - \sum_{x \in A} [\mathcal{E}(y|x)]p(x) \right) \right] / (\ln 2) \] (134)
\[ = 0. \] (135)

**Proposition 23. If $C_{0}^{NS} = C$ then $V = 0$.**

**Proof:** Suppose that $\mathcal{E}(y|x)$ is a channel with $C_{0}^{NS} = C$. Let $w : A \to [0, 1]$ be any optimal fractional packing for the channel and let $\alpha^*$ be the fractional packing number. By the preceding lemma, $p(x) = w(x)/\alpha^*$ defines a capacity achieving input probability mass function for the channel. Let $q$ be the corresponding output probability mass function. It was shown in [14] that if $p$ is capacity achieving then
\[ D(\mathcal{E}(\cdot|x)||q) \begin{cases} = C & \text{when } p(x) > 0, \\ \leq C & \text{when } p(x) = 0. \end{cases} \] (135)

Since, $C = \log \alpha^*$ by assumption, these conditions imply that, for all $x \in A$,
\[ 0 \leq \log \alpha^* - D(\mathcal{E}(\cdot|x)||q) = \sum_{y \in \mathcal{B}} \mathcal{E}(y|x) \log \frac{q(y)\alpha^*}{\mathcal{E}(y|x)} \] (136)
and, using $\log x \leq (x - 1)/(\ln 2)$ again,
\[ 0 \leq \sum_{y \in \mathcal{B}} \mathcal{E}(y|x) \mathcal{E}(y|x) \left( \frac{q(y)\alpha^*}{\mathcal{E}(y|x)} - 1 \right) \] (137)
\[ = \sum_{y \in \mathcal{B}} \alpha^* q(y) \mathcal{E}(y|x) - \sum_{y \in \mathcal{B}} \mathcal{E}(y|x) \] (138)
\[ = \sum_{y \in \mathcal{B}} \alpha^* q(y) \mathcal{E}(y|x) - 1. \] (139)
Therefore, \( v_g := \alpha^* q(y) \) is a fractional covering for the channel hypergraph, and it is optimal. Furthermore, the complementary slackness condition demands that when \( p(x) > 0 \) the corresponding inequality must be saturated. Therefore, when \( \mathcal{E}(y|x)p(x) > 0 \) we must have \( \frac{\mathcal{E}(y|x)}{\sum x' \mathcal{E}(y|x')} p(x') - 1 = 0 \) or \( \log \mathcal{E}(y|x)/\alpha = \log \alpha^* \) so the variance of the information density is zero for this capacity achieving distribution. □

**Proposition 24.** If \( K^0_{\text{NS}} = C \) then \( V = 0 \).

**Proof:** Let \( p \) be a capacity achieving probability mass function.

\[
C = \sum_{x \in A} \sum_{y \in B} \mathcal{E}(y|x)p(x) \log \frac{\mathcal{E}(y|x)}{\sum x' \mathcal{E}(y|x') p(x')}
\]

\[
\leq \log \sum_{x \in A} \sum_{y \in B} \mathcal{E}(y|x)p(x) \mathcal{E}(y|x) \sum x' \mathcal{E}(y|x') p(x')
\]

\[
\leq \log \sum_{y \in B} \max_{x' \in A} \mathcal{E}(y|x') \mathcal{E}(y|x) \sum x' \mathcal{E}(y|x') p(x')
\]

\[
= \log \max_{y \in B} \mathcal{E}(y|x)
\]

\[
= K^0_{\text{NS}}.
\]

For equality to hold, Jensen’s inequality \([141]\) must be saturated. This happens if and only if

\[
\frac{\mathcal{E}(y|x)}{\sum x' \mathcal{E}(y|x') p(x')} = C
\]

for all \( x, y \) such that \( \mathcal{E}(y|x)p(x) > 0 \), which is equivalent to \( V = 0 \). □

**V. Conclusion**

It was shown that maximum size of non-signalling code with a given error probability is given by the integer part of the solution to a linear program, and that this is equal to the converse bound of Polyanskiy, Poor and Verdú [1], thus giving an alternative proof of that result. When \( n \) uses of the channel are symmetric under simultaneous permutations of the input and output strings, the LP can be simplified to one with \( \text{poly}(n) \) variables and constraints.

It was also proven that the capacity of a DMC is achieved with zero-error by NS codes, if and only if the channel has zero dispersion, and therefore already admits especially efficient classical codes.

It would be interesting to see if the dual linear programming formulation of the converse given in this paper can help in extending the finite block length results given in [1]. The technique of using non-signalling assistance to obtain linear program converses for classical coding protocols extends naturally to multi-terminal situations like broadcast or multiple access channels, and may prove useful in this context.

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