A QUESTION OF DOCTOR MALTE WANDEL ON AUTOMORPHISMS
OF THE PUNCTURED HILBERT SCHEMES OF K3 SURFACES

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ABSTRACT. We present a sufficient condition for the punctured Hilbert scheme of length two of a K3 surface with finite automorphism group to have automorphism group of infinite order in geometric terms, and give some concrete examples. This affirmatively answers a question of Doctor Malte Wandel at the conference, "Arithmetic and Algebraic Geometry" held at University of Tokyo, January 2014.

1. INTRODUCTION

We work in the category of projective varieties defined over the complex number field \( \mathbb{C} \). We denote by \( S^{[n]} \) the Hilbert scheme of 0-dimensional closed subschemes of length \( n \geq 2 \) on a surface \( S \) and by \( S^{(n)} \) the Chow variety of 0-dimensional cycles of length \( n \geq 2 \). If \( S \) is a K3 surface, then \( S^{[n]} \) is a hyperkähler manifold, that is, an irreducible holomorphic symplectic manifold, of dimension \( 2n \) ([Fu83], [Be84]).

The aim of this short note is to give an affirmative answer for the following question asked by Doctor Malte Wandel at the conference, "Arithmetic and Algebraic Geometry" held at University of Tokyo, January 27-30, 2014, in my talk relevant to this topic ([Og14]):

**Question 1.1.** Is it possible that \( |Aut (S^{[2]})| = \infty \) for a K3 surface \( S \) with \( |Aut (S)| < \infty \)?

Note that \( Aut (S) \) naturally and faithfully acts on \( S^{[n]} \) so that \( |Aut (S^{[n]})| = \infty \) if \( |Aut (S)| = \infty \).

Throughout this note, we denote by \( \Lambda \) the even hyperbolic lattice of rank 2 and of discriminant 17, defined by:

\[
\Lambda := \left( \mathbb{Z}^2, \begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix} \right).
\]

Our main result is the following:

**Theorem 1.2.** Let \( S \) be a K3 surface with \( NS(S) \simeq \Lambda \). Then \( |Aut (S^{[2]})| = \infty \) and \( |Aut (S)| < \infty \).

See also Theorem (2.1) in Section 2 for a geometric criterion behind this theorem. The essential point of the proof is that \( S \) has at least two different quartic surface structures with no line, and therefore, \( (S^{[2]}) \) has at least two different biregular Beauville involutions ([Be83], Section 6), See also Section 2).

The first author is supported by JSPS Grant-in-Aid (S) No 25220701, JSPS Grant-in-Aid (S) No 22224001, JSPS Grant-in-Aid (B) No 22340009, and by KIAS Scholar Program.
Remark 1.3. K3 surfaces $S$ in Theorem (1.2) form a dense subset of 18-dimensional family of $\Lambda$-polarized K3 surfaces in the sense of Dolgachev [Do96] (See also [Mo84, Corollary 2.9] for the existence).

Remark 1.4. (1) If $S$ is a K3 surface with $\rho(S) = 1$, then both $\text{Aut}(S)$ and $\text{Bir}(S)$ are finite groups ([Og12, Corollary 5.2]). Thus, $\rho(S) = 2$ in Theorem (1.2) is the smallest Picard number for Question (1.1) to be affirmative.

(2) Theorem (2.1) in Section 2 and our proof of Theorem (1.2) in Section 3 suggest that there will be many other ways to construct examples similar to the ones in Theorem (1.2).

Relevant to Theorem (1.2), Professor Shinnosuke Okawa kindly suggested the following:

Remark 1.5. The Hilbert-Chow morphism
$$\mu : S^{[2]} := \text{Hilb}^2 S \to S^{(2)} ,$$
for $S$ in Theorem (1.2), provides an example of an extremal crepant resolution such that the source is not a Mori dream space, MDS for short, but the target is MDS. See Theorem (4.1) in Section 4 for a slightly more general result. For the definition of MDS, we refer to the original paper [HK00, Definition 11], which we follow.

It is natural to ask a similar question for $S^{[n]}$ with $n \geq 3$, also in connection with the following open question of Professor Alessandra Sarti, called the naturality question:

Question 1.6. Is $\text{Aut}(S^{[n]}) = \text{Aut}(S)$ for any K3 surface $S$ and any $n \geq 3$?

Remark 1.7. It might be interesting to check the naturality question for K3 surfaces $S$ in Theorem (1.2): If it would be affirmative, then $\text{Aut}(S^{[n]}) = \infty$ only when $n = 2$. For this, a version of Torelli theorem in [Ma11] and a recent work of Markman and Yoshioka ([MY14]) might be of some help.

Acknowledgement. I would like to express my thanks to Doctor Malte Wandel and Professors Shinnosuke Okawa and Alessandra Sarti for valuable questions, suggestions and discussions, to Professors Yujiro Kawamata, Toshiyuki Katsura, Iku Nakamura and Tomohide Terasoma for invitation to the conference to talk and warm hospitality there.

2. A geometric criterion

In this section, we shall prove the following:

Theorem 2.1. Let $S$ be a K3 surface of Picard number $\rho(S) = 2$ such that:

(1) $S$ has either a smooth elliptic curve or a smooth rational curve; and

(2) $S$ has very ample divisors $H_1, H_2$ such that $(H_1^2) = (H_2^2) = 4$, $H_1 \neq H_2$ in $\text{NS}(S)$, and $S$ has no smooth rational curve $C_k$ such that $(C_k.H_k) = 1$ for each $k = 1, 2$.

Then $|\text{Aut}(S)| < \infty$ and $|\text{Aut}(S^{[2]})| = \infty$.

Proof. By (1), the nef cone $\text{Amp}(S) \subset \text{NS}(S) \otimes \mathbb{R}$ has at least one boundary defined over $\mathbb{Q}$. Since $\rho(S) = 2$, this implies $|\text{Aut}(S)| < \infty$ (see eg. [Og12, Proposition 2.4]).

By (2), we have the embedding
$$\Phi_k := \Phi_{|H_k|} : S \to \mathbb{P}^3 \quad (k = 1, 2) .$$
The characteristic polynomial $F$.

Observe then that $m$

Here, we put $f$

where $\iota$

representations of $4.1$]. Then, with respect to the $Q$

This formula is first proved by Debarre ([De84, Théorème 4.1], see also [OGr05, Proposition

This identification is the natural one induced by the Hilbert-Chow morphism

Consider the action of the Beauville involution $i_k$ on NS $(S^{[2]}):

This is the anti-involution with respect to the invariant vector $H_k - e$ with $((H_k - e)^2) = 2$:

This formula is first proved by Debarre ([De84, Théorème 4.1], see also [OGr05, Proposition

Then, with respect to the $Q$-basis $\langle H_1 - e, H_2 - e, e \rangle$, we have the following matrix representations of $i_k^\ast$:

Here, we put $m := (H_1,H_2)$. Note that $m \geq 5$ by the Hodge index theorem:

Observe then that

The characteristic polynomial $F(t)$ of $(\iota_2\iota_1)^\ast$ is

where $f(t) = t^2 - ((m - 2)^2 - 2)t + 1$. By $m \geq 5$, we have

$(m - 2)^2 - 2 \geq (5 - 2)^2 - 2 \geq 7$, $D := ((m - 2)^2 - 2)^2 - 4 > 0$. 

\textbf{A Question of Doctor Malte Wandel}
Hence \( f(t) \) has two real zeros, say, \( \alpha < \beta \). Since 
\[
2\beta > \alpha + \beta = (m - 2)^2 - 2 \geq 7,
\]
it follows that \( \beta > 1 \). Thus \( \nu_2 \nu_1 \in \text{Aut}(S^{[2]}) \) is of infinite order. \( \square \)

3. Proof of Theorem (1.2)

We shall prove Theorem (1.2) by reducing to Theorem (2.1). We proceed by dividing into several steps. Set 
\[
\text{NS}(S) = \mathbb{Z}\langle L, H \rangle, \quad (L^2) = 2, \quad (H^2) = 4, \quad (L.H) = 5.
\]

**Claim 3.1.** There is no \( C \in \text{NS}(S) \setminus \{0\} \) such that \( (C^2) = 0 \). In particular, \( S \) has no curve \( C \) with arithmetic genus \( p_a(C) = 1 \).

**Proof.** Write \( C = xL + yH \) with \( x, y \in \mathbb{Z} \). Then 
\[
(C^2) = 2x^2 + 10xy + 4y^2.
\]
So, if \( (C^2) = 0 \), then \( x = (-5 \pm \sqrt{17})y/2 \). Since \( x, y \in \mathbb{Z} \), this implies that \( x = y = 0 \), i.e., \( C = 0 \). \( \square \)

**Claim 3.2.** There is no \( C \in \text{NS}(S) \) such that \( (L.C) = 0 \) and \( (C^2) = -2 \).

**Proof.** Write \( C = xL + yH \) with \( x, y \in \mathbb{Z} \). Then 
\[
(L.C) = 2x + 5y.
\]
So, if \( (L.C) = 0 \), then \( x = 5z, y = -2z \) for some \( z \in \mathbb{Z} \). Then 
\[
(C^2) = ((5L - 2H)^2)z^2 = -34z^2 \neq -2,
\]
This implies the claim. \( \square \)

**Claim 3.3.** Set \( P(S) := \{ x \in \text{NS}(S) \otimes \mathbb{R} \mid (x^2) > 0 \} \) and denote by \( P^+(S) \) the \( \mathbb{Q} \)-rational hull of \( P(S) \) in \( \text{NS}(S) \otimes \mathbb{R} \). Let \( \text{Amp}(S) \subset \text{NS}(S) \otimes \mathbb{R} \) be the ample cone of \( S \) and \( \text{Amp}^+(S) \) be the \( \mathbb{Q} \)-rational hull of \( \text{Amp}(S) \) in \( \text{NS}(S) \otimes \mathbb{R} \). Then \( \text{Amp}^+(S) \) is the fundamental domain of the natural action of \( W(\text{NS}(S)) \times \{ \pm \text{id}_{\text{NS}(S)} \} \) on \( P^+(S) \). Here \( W(\text{NS}(S)) \) is the reflection group generated by the reflections with respect to all \( C \in \text{NS}(S) \) such that \( (C^2) = -2 \):
\[
r_C : x \mapsto x + (x.C)C.
\]

**Proof.** This is a version of the Nakai-Moishezon criterion for the ampleness of line bundles on a projective K3 surface (See eg., [BHPV, Chapter VIII]). \( \square \)

**Claim 3.4.** We may and will assume that \( L \) is ample.

**Proof.** By (3.3), we may assume that \( L \in \text{Amp}^+(S) \cap \text{NS}(S) \). Then \( L \) is nef and big. Thus \( L = K_S + L \) is semi-ample ([Ka84]). So, if \( L \) would not be ample, then there would be an irreducible curve \( C \subset S \), being necessarily isomorphic to \( \mathbb{P}^1 \), such that \( (L.C) = 0 \), a contradiction to (3.2). \( \square \)

**Claim 3.5.** The complete linear system \( |L| \) is free and the associated morphism 
\[
\Phi := \Phi_{|L|} : S \to \mathbb{P}^2
\]
is a finite double cover. (Remember that \( L \) is now assumed to be ample.)
Proof. By the Riemann-Roch formula, we have \( h^0(S, L) = 3 \). For the claim, it is sufficient to prove that \(|L|\) is free. Assume to the contrary that \(|L|\) is not free. Then, \(|L| = |M| + F\), where \(|M|\) is the movable part and \(F \neq 0\) is the fixed component, which is effective and non-zero ([SD74] Corollary 3.2). Then
\[
2 = (L^2) > (L.M) \geq (M^2) \geq 0.
\]
Since \((M^2) \in 2\mathbb{Z}\), it follows that \((M^2) = 0\), a contradiction to (3.1). \(\square\)

Claim 3.6. Let \(\tau\) be the covering involution of \(\Phi : S \to \mathbb{P}^2\). Then \(\tau \in \text{Aut}(S)\) and
\[
\tau^* = \begin{pmatrix} 1 & 5 \\ 0 & -1 \end{pmatrix}
\]
on \(\text{NS}(S)\) with respect to the \(\mathbb{Z}\)-basis \(\langle L, H \rangle\).

Proof. Note that \(\text{Bir}(S) = \text{Aut}(S)\) by \(K_S = 0\). Hence \(\tau \in \text{Aut}(S)\). By the definition, \(L = \Phi^*\mathcal{O}_{\mathbb{P}^2}(1)\). Thus \(\tau^*L = L\).

We have
\[
\Phi^*\Phi_*H = H + \tau^*H,
\]
where \(\Phi_*H\) is the pushforward as 1-cycles. Then \(\Phi_*H \in |\mathcal{O}_{\mathbb{P}^2}(m)|\) for some \(m \in \mathbb{Z}_{>0}\) and
\[
(\Phi^*\Phi_*H.L) = (\Phi^*\Phi_*H.\Phi^*\mathcal{O}_{\mathbb{P}^2}(1)) = 2(\Phi_*H.\mathcal{O}_{\mathbb{P}^2}(1)) = 2m.
\]
On the other hand,
\[
(\Phi^*\Phi_*H.L) = (H + \tau^*H.L) = (H.L) + (\tau^*H.L) = (H.L) + (\tau^*H.\tau^*L) = 2(H.L) = 10.
\]
Hence \(m = 5\), and therefore \(\tau^*H = 5L - H\). With \(\tau^*L = L\), we obtain the result. \(\square\)

Claim 3.7. If \(C\) is a smooth rational curve on \(S\), then \(|C| = \{C\}\).

Proof. This is well-known and easily follows from the irreducibility plus \((C^2) = -2 < 0\) (See eg. [BHPV, Chapter VIII]). \(\square\)

Claim 3.8. (1) \(S\) has exactly two smooth rational curves and their classes are
\[
-L + 2H, \quad \tau^*(-L + 2H) = 9L - 2H.
\]
(2) Moreover, the cone of effective curves on \(S\) is
\[
\text{NE}(S) = \mathbb{R}_{\geq 0}(-L + 2H) + \mathbb{R}_{\geq 0}(9L - 2H).
\]
(3) In particular, \(S\) satisfies the condition (1) in Theorem (2.1).

Proof. The assertion (3) follows from (2). Recall that \(\rho(S) = 2\). Then (2) follows from (1) and (3.7). That \(\tau^*(-L + 2H) = 9L - H\) follows from (3.3). By (3.7), it now suffices to show that \(|-L + 2H|\) contains the class of a smooth rational curve. Set
\[
C := -L + 2H
\]
in \(\text{NS}(S)\). Then
\[
(C^2) = -2, \quad (C.L) = 8 > 0.
\]
Hence, by the Riemann-Roch formula and the Serre duality, it follows that \(|C| \neq \emptyset\). Remember here that \(L\) is ample. Noticing that \((C^2) = -2\), we can then choose a smooth rational curve \(C_1\) and an effective curve \(C_2\), possibly 0, such that
\[
C_1 + C_2 \in |C|.
\]
Since \((L.C) = (L - L + 2H) = 8\) and \(L\) is ample, it follows that
\[
1 \leq k := (L.C_1) \leq 8.
\]
Consider the sublattice
\[
M := \mathbb{Z} \langle L, C_1 \rangle \subset \text{NS} \ (S).
\]
Then the Gram matrix of \(M\) is
\[
\begin{pmatrix}
2 & k \\
-k & -2
\end{pmatrix}
\]
and the discriminant of \(M\) is \(k^2 + 4\). On the other hand, the discriminant of \(\text{NS} \ (S)\) is 17.
Thus, by the elementary divisor theorem, we have
\[
k^2 + 4 = 17l^2
\]
for some \(l \in \mathbb{Z}\). Since \(1 \leq k \leq 8\), this implies that \(k = 8\). Hence \(C_2 = 0\), that is, \(C = -L + 2H\) is the class of a smooth rational curve \(C_1\).
\[\square\]
From now, we shall show that \(H\) and \(\tau^*H\) satisfy the condition (2) in Theorem (2.1). We proceed by dividing into several steps.

Claim 3.9. \(H\) is ample and \(h^0(S, H) = 4\).

Proof. Observe that
\[
(H - L + 2H) = 3 > 0, \quad (H.9L - 2H) = 37 > 0.
\]
Thus \(H\) is ample by the Kleiman's criterion and (3.8)(2). Since \((H^2) = 4\), the second assertion now follows from the Riemann-Roch formula. \[\square\]

Claim 3.10. \(|H|\) has no fixed component.

Proof. Assume to the contrary that \(|H|\) has a fixed component. Then \(|H| = |M| + F\), where \(|M|\) is the movable part and \(F\) is a non-zero effective divisor. Using (3.1), we have
\[
4 = (H^2) > (H.M) \geq (M^2) > 0, \quad 5 = (H.L) > (M.L) > 0
\]
and therefore \((M^2) = 2\) (also by the evenness) and \(1 \leq k := (M.L) \leq 4\). Then the discriminant of \(\mathbb{Z} \langle L, M \rangle\) is \(|k^2 - 4|\), possibly 0. For the same reason as in (3.8), we have
\[
|k^2 - 4| = 17l^2
\]
for some \(l \in \mathbb{Z}\). Since \(1 \leq k \leq 4\), it follows that \(k = 2\) and \(|k^2 - 4| = 0\). Thus \(M = mL\) for some \(m \in \mathbb{Q}\). Then, by \((M^2) = 2 = (L^2)\), it follows that \(M = L\) in \(\text{Pic} \ (S) \simeq \text{NS} \ (S)\), a contradiction to the fact that \(h^0(S, M) = h^0(S, H) = 4\) and \(h^0(S, L) = 3\). \[\square\]

Claim 3.11. \(H\) is very ample.

Proof. By [SD74, Theorem 5.2] and by (3.10), it suffices to check the following two:
(i) there is no irreducible curve \(E \subset S\) such that \(p_a(E) = 1\) and \((E.H) = 2\);
(ii) there is no irreducible curve \(B \subset S\) such that \(p_a(B) = 2\) and \(H = 2B\) in \(\text{NS} \ (S)\).
The assertion (i) follows from (3.11). The assertion (ii) follows from the fact that \((B^2) = 2\) when \(p_a(B) = 2\). \[\square\]

Claim 3.12. There is no \(C \simeq \mathbb{P}^1\) on \(S\) such that \((H.C) = 1\).
Proof. Recall that the class of $C$ is either $-L + 2H$ or $\tau^*(-L + 2H) = 9L - 2H$ by (3.8). The result now follows from
\[(H. - L + 2H) = 3 \neq 1, \quad (H.9L - 2H) = 37 \neq 1\].

Claim 3.13. $H_1 := H$ and $H_2 := \tau^*H = 5L - H$ satisfy the condition (2) in Theorem (2.1).

Proof. Since $H_1 \neq H_2$ in $\text{NS}(S)$, $(H^2) = 4$ and $\tau \in \text{Aut}(S)$, the result follows from (3.11) and (3.12).

Theorem (1.2) now follows from Theorem (2.1), Claims (3.8), (3.13).

4. An Application for MDS

In this section, we shall prove the following:

Theorem 4.1. Let $S$ be a K3 surface such that $|\text{Aut}(S)| < \infty$ and $|\text{Aut}(S^{[n]})| = \infty$. Then $S^{[n]}$ is not MDS but $S^{(n)}$ is MDS. In particular, the Hilbert-Chow morphism
\[\nu : S^{[n]} \to S^{(n)}\]
is a crepant projective resolution, which is also extremal in the sense of the minimal model program, but does not preserve MDS property.

Remark 4.2. (1) K3 surfaces $S$ in Theorem (1.2) provide concrete examples of $S$ in Theorem (4.1) for $n = 2$.

(2) There is no known example of $S$ for $n \geq 3$ and such $S$ will clearly be a counter-example of Question (1.6) if it exists.

Proof. That $\nu$ is a crepant extremal resolution is proved by Beauville ([Be84, Proposition 5]), being based on the fundamental result of Forgaty.

Claim 4.3. $S^{[n]}$ is not MDS. More strongly, the nef cone of $S^{[n]}$ is not a finite rational polyhedral cone.

Proof. Assume to the contrary that the nef cone $\text{Amp}(S^{[n]})$ is a finite rational polyhedral cone. Then there are only finitely many 1-dimensional rays of $\text{Amp}(S^{[n]})$, say $L_k$ ($1 \leq k \leq m$). Choose the primitive integral generator $e_k$ of $L_k$. Then $H := \sum_{k=1}^{m} e_k$ is an ample class invariant under $\text{Aut}(S^{[n]})$. Since $H^0(T_{S^{[n]}}) = 0$, this already implies that $|\text{Aut}(S^{[n]})| < \infty$ (see eg. [Og12, Proposition 2.4]), a contradiction to our assumption.

Proposition 4.4. Let $S_k$ ($1 \leq k \leq n$) be K3 surfaces such that $|\text{Aut}(S_k)| < \infty$. Set
\[X := S_1 \times S_2 \times \cdots \times S_n\].

Then $X$ is MDS. More strongly, $X$ satisfies that $\text{Pic}(X) \simeq \text{NS}(X)$, $\text{Bir}(X) = \text{Aut}(X)$, $\text{Mov}(X) = \text{Amp}(X)$ is a finite rational polyhedral cone and any rational nef divisor is semi-ample.
Proof. Choose points $Q_k \in S_k$ ($1 \leq k \leq m$). Using the exponential sequence, the Kunneth formula and the fact that $h^{1,0}(S_k) = h^{0,1}(S_k) = 0$, we have

$$\text{Pic}(X) \cong \text{NS}(X) = \text{NS}(S_1) \times \text{NS}(S_2) \times \cdots \times \text{NS}(S_n)$$

under the natural identifications $\text{NS}(S_k) = p^*_k \text{NS}(S_k)$ through the projections $p_k : X \to S_k$.

Let $\iota_k : S_k \to X$ be an inclusion given by $x \mapsto (Q_1, Q_2, \ldots, Q_{k-1}, x, Q_{k+1}, \ldots, Q_n)$.

Under the equality of the Néron-Severi groups above, if $h = (h_1, h_2, \ldots, h_n)$ then $h = \sum_{k=1}^n p^*_k h_k$. Therefore, under the equality above, we obtain

$$\text{Amp}(X) = \text{Amp}(S_1) \times \text{Amp}(S_2) \times \cdots \times \text{Amp}(S_n).$$

By $|\text{Aut}(S_k)| < \infty$, the nef cone $\overline{\text{Amp}(S_k)}$ is a finite rational polyhedral cone, by the solution of the cone conjecture for K3 surfaces ([St85, Lemma 2.4], [Ka97, Theorem 2.1]; see also [To10, Theorem 3.3, Corollary 5.1], [AHL10, Theorem 2.7] for closely related results). Recall also that all rational nef classes on a K3 surface is effective by the Riemann-Roch formula, and therefore, they are semi-ample by the log abundance theorem in dimension 2 ([Fj84]). Hence $\text{Amp}(X)$ is also a finite rational polyhedral cone and all rational nef classes on $X$ are semi-ample. Moreover, every projective birational contraction of $X$ is of the form

$$\mu_1 \times \mu_2 \times \cdots \times \mu_n : X \to V_1 \times V_2 \times \cdots \times V_n,$$

where $\mu_k : S_k \to V_k$ are birational projective contractions, possibly some of them are isomorphisms. In particular, $X$ has no small projective contraction. Hence $X$ has no flop. Therefore

$$\text{Bir}(X) = \text{Aut}(X), \quad \overline{\text{Mov}(X)} = \overline{\text{Amp}(X)}$$

by the fundamental result due to Kawamata ([Ka08]). Now we are done.

Claim 4.5. $S^{(n)}$ is MDS.

Proof. By Proposition (4.4), $S^n$ is MDS. We have a surjective morphism

$$S^n \to S^{(n)}, \quad (P_1, P_2, \ldots, P_n) \mapsto P_1 + P_2 + \cdots + P_n,$$

in which $S^{(n)}$ is $\mathbb{Q}$-factorial. Actually $S^{(n)}$ is the global quotient of $S^n$ by the symmetric group of $n$-factors. Since $S^n$ is MDS by Proposition (4.4), so is $S^{(n)}$ by the general result of Okawa ([Ok11, Theorem 1.1], see also [Ba11, Theorem 1.1] in our situation).

Claims (4.3), (4.5) imply the result.

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