Critical Behaviour of Superfluid $^4$He in Aerogel

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We report on Monte Carlo studies of the critical behaviour of superfluid $^4$He in the presence of quenched disorder with long-range fractal correlations. According to the heuristic argument by Harris, uncorrelated disorder is irrelevant when the specific heat critical exponent $\alpha$ is negative, which is the case for the pure $^4$He. However, experiments on helium in aerogel have shown that the superfluid density critical exponent $\zeta$ changes. We hypothesize that this is a cross-over effect due to the fractal nature of aerogel. Modelling the aerogel as an incipient percolating cluster in 3D and weakening the bonds at the fractal sites, we perform XY-model simulations, which demonstrate an increase in $\zeta$ from $0.67 \pm 0.005$ for the pure case to an apparent value of $0.722 \pm 0.005$ in the presence of the fractal disorder, provided that the helium correlation length does not exceed the fractal correlation length.

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It is generally believed that the superfluid transition ($\lambda$-point) of pure $^4$He belongs to the classical 3D-XY model universality class. Near the critical temperature $T_c$, the superfluid density scales as $|T - T_c|^\zeta$, where $\zeta$ is measured to be $0.674 \pm 0.003$. There has been considerable interest in studying the superfluid transition for helium in a variety of porous media. One such system is Vycor which is a random glass with a porosity of order 30%. Remarkably, the critical exponent $\zeta$ is found to be unchanged by the seemingly huge perturbation represented by the glass. This however is the result expected according to heuristic arguments of Harris and more rigorous recent work which shows that weak, uncorrelated randomness is irrelevant at the unperturbed critical point provided that the specific heat exponent is negative. The exponent $\alpha \approx -0.026 \pm 0.004$ is indeed negative for $^4$He, however Narayan and Fisher have argued that since $\alpha$ is only slightly negative, the crossover to the pure 3D XY critical regime is almost logarithmically slow. Until recently Vycor was the only porous medium in which $\zeta$ was observed to be unchanged. New experiments however appear to have added a second material, porous gold (which also has short-range correlations), to the list.

Remarkable results have been obtained for the aerogel system. Aerogel is a fractal silica ‘dust’ with porosities of 95-98% or higher. Despite the fact that aerogel is almost entirely empty space, and that it (unlike Vycor) has only a tiny effect on the critical temperature, the exponent $\zeta$ is apparently quite distinctly shifted to approximately 0.75 (or larger), indicating that the nominally weak aerogel perturbation is relevant at the 3D XY critical point and aerogel (perhaps) produces a new universality class. The specific heat data suggests that either hyperscaling is violated or the amplitude ratio is exceptionally small. In addition to shifting $\zeta$ and $\alpha$, aerogel dramatically changes the topology of the $^3$He-$^4$He mixture phase diagram.

It is known that long-range correlations can make disorder irrelevant, even when $\alpha$ is negative. Weinrib and Halperin have demonstrated this for the special case of weak, gaussian distributed disorder with long-range correlations. Li and Teitel have looked at a model of non-weak (broadly distributed) but uncorrelated disorder and find an apparent increase in the exponent $\zeta$ to a value which depends on the exponent in the algebraic divergence of the disorder strength distribution at weak disorder.

Machta has considered a model of aerogel as a relatively uniform medium filled with pores on many different length scales. This model was motivated by early experiments which saw two transitions, one at the usual bulk temperature and one at a slightly lower temperature. Recent improvements in aerogel synthesis techniques appear however to have eliminated the larger pores and inhomogeneities so that now only a single transition is observed at a temperature slightly below the bulk value. Recently, Huang and Meng have examined a mean-field theory in a percolating cluster system.

In order to investigate this fascinating subject, we have performed extensive Monte Carlo simulations on the 3D XY model for three cases: i) no disorder, ii) uncorrelated disorder, and iii) fractal disorder. We consider lattice sizes up to $24^3$, and use the Wolff algorithm to minimize the otherwise severe effects of critical slowing down. The model is defined by compact phase variables $\{\theta\}$ on sites of a simple cubic lattice

$$ S[\theta] = -\sum_{r, \delta} \frac{K_r}{T} \cos \left[ \theta(r) - \theta(r + \delta) \right]. \quad (1) $$

We measured the disorder-averaged superfluid density $\rho_s$ as a function of temperature and the system size using the usual Kubo formula expression. Defining one ‘sweep’
as growing and reorienting a cluster of spins (on the order the system size when near the critical point) with the Wolf algorithm, typical runs involved \(5 \times 10^4\) warm-up sweeps, and 2 \(\times 10^6\) production sweeps, with measurements taken every 200 sweeps. Results were averaged over typically 100 disorder realizations.

A finite-size scaling analysis is crucial to the accurate determination of the critical temperature and exponents. The scaling ansatz assumes that \(\rho_s\) has the form \[\rho_s = TL^{-\Omega}G(L/\xi) = TL^{-\Omega}G(T - T_c)L^{1/\nu},\] (2)

where \(\Omega = (2 - \alpha)/\nu - 2\). Hence, the dimensionless combination \(\rho_sL^{\Omega}/T\) is scale invariant at the critical point \(T_c\). If we assume that hyperscaling holds, then we have \(\Omega = d - 2 = 1\), otherwise \(\Omega\) is a priori unknown. In the inset of Fig.(1) we determine \(T_c = 2.156 \pm 0.001\) by plotting \(\rho_sL/T\) vs. temperature for uncorrelated disorder

\(K_{\epsilon,\delta} = 1 + \delta K_{\epsilon,\delta}; \delta K_{\epsilon,\delta} \in [\Delta - \Delta, \Delta]\) and \(\Delta < 1\). (3)

of relatively large (but bounded) strength \(\Delta = 0.7\). Similar calculations for the pure (disorder-free) case yield distinctly larger value \(T_c \approx 2.203 \pm 0.001\). In the main part of Fig.1, \(\tilde{G}\) is plotted as a function of the scaling variable \((T - T_c)L^{1/\nu}\) with the value \(\nu = 0.667 \pm 0.005\) which gives the best data collapse onto a single universal scaling curve. Clearly, this unchanged exponent is consistent with the Harris criterion and the experimental observation in Vycor that uncorrelated randomness does not change the universality class. In order to cross check the above result, we have also measured the magnetization \(m\) and computed the Binder ratio, \(U_4\), which is automatically scale invariant at the critical point,

\(U_4 = 1 - \left[\frac{<m^4>}{3<m^2>^2}\right]_{\text{ave}}.\) (4)

This allows one to estimate the critical exponent \(\nu\) without assuming a value for \(\Omega\) as was necessary in the case of the superfluid density. Plotting \(U_4\) against the variable \((T - T_c)L^{1/\nu}\), we found good scaling for the same values of \(T_c\) and \(\nu\) obtained from the scaling of \(\rho_s\).

We turn now to a discussion of fractal disorder. A material with fractal dimension \(D_t\) has a mass that scales with length like \(M \sim L^{D_t}\). If \(D_t < 3\) then (real) objects can never be fractal beyond some finite correlation length \(\xi\) because otherwise the density would vanish. Static structure factor measurements indicate that acid-catalyzed aerogel has fractal dimension \(D_t \sim 2.4 - 2.5\) over a wide range of length scales from \(\sim 6\) Å out to roughly \(\xi \sim 600\) Å. Base-catalyzed aerogel (typically used in the helium experiments) is believed to have a somewhat lower fractal dimension and a lower range of length scales. Various measurements of the ‘fracton’ vibrational properties of aerogels however place a lower bound on the correlation length for the connectivity that is at least an order of magnitude larger. The connectivity structure may be important because the closed vortex loops in the helium are presumably attracted to the aerogel strands and are thus sensitive to the connectivity. One of the central mysteries shown up by the experiments is the following. In the critical regime with reduced temperature \(t \sim 10^{-5}\) the correlation length of the helium is expected to be larger (\(\sim 1\) µm) than even the (estimated) connectivity length scale. Nevertheless no evidence of a cross-over to the uncorrelated disorder regime is evident in the full-pore experiments. Very recently however, Crowell et. al. have performed experiments in the regime of lower helium densities where even larger correlation lengths can be obtained. They find evidence of a possible cross-over to the uncorrelated disorder regime with a lower value of \(\zeta\).

We have considered the possibility that the deformability of the tenuous aerogel structure is relevant. On length scales beyond the fracton correlation length, aerogel acts to sound waves like a relatively homogeneous system with a low speed of sound (\(\sim 100\) m/s) despite its very low mass density, indicating that its compressibility is nearly \(10^6\) times that of ordinary glass. The aerogel is known to be sufficiently flexible that it modifies the collective sound mode dispersion. On scales beyond the fracton correlation length, it is reasonable to argue that the aerogel density fluctuations \(\delta \Phi\) act as, simple, uncorrelated local annealed disorder coupling to the magnitude of the helium order parameter in a Ginsburg-Landau theory with action

\[S = \frac{1}{2k}(\delta \Phi)^2 + \left[\frac{1}{2} - h \delta \Phi\right](\nabla \psi)^2 + [\alpha + g \delta \Phi]|\psi|^2 + [\lambda + k \delta \Phi]|\psi|^4.\] (5)

Integrating out \(\delta \Phi\) for small \(g, h, k\) produces only irrelevant couplings. For stronger couplings however, the system can, in the right circumstances, be driven to a tricritical point, beyond which the transition is first order. This is precisely what happens in \(^3\)He-\(^4\)He mixtures where it is a good approximation to treat the \(^3\)He impurities as annealed disorder. This confirms the idea that deformability of the aerogel should be irrelevant. It should be noted however that it is probably more physically correct to include the constraint \(\int d^3r \delta \Phi = 0\) which would lead to the slow logarithmic case of Fisher renormalization of the critical exponents. The possibility that this may account for the peculiar features of the specific heat data should be looked into in more detail.

We seem to be left only with the possibility of aerogel as quenched disorder whose fractal character extends beyond the lower bound set by the fracton cutoff. A variety of schemes have been used to model the aerogel structure. We have chosen a simple percolation model for the fractal structure (probably more appropriate for acid-catalyzed than base-catalyzed aerogel). We generate a critical percolation backbone on the 3D lattice by
randomly occupying lattice sites with probability \( p = p_c \) and keep only the largest connected cluster. In addition, we selected only fractal realizations with porosity in a narrow window centered on the median value in order to reduce the sample-to-sample fluctuations in the disorder strength. We confirmed that these objects had the known fractal dimension \( D_t \sim 2.5 \).

In order to be able to do finite size scaling, while avoiding the scale dependence of the porosity, a single large cluster was generated on an \( L_0 = 48 \) lattice (giving a porosity of about 95\%) and divided into smaller subsystems of size \( L = 8, 12, 16, 24 \). Simulations were performed for different subsystems with periodic boundary conditions and averaged over the subsystems and different fractal realizations. The bond strength \( K \) on the fractal was arbitrarily reduced from unity to 0.26. The inset of Fig. (2) shows \( \rho_s L/T \) vs. \( T_c \), and appears to give a clear fixed point with \( T_c \) estimated to be \( 2.1834 \pm 0.001 \) which is closer to the pure critical point than for the uncorrelated disorder model, since the porosity of the fractal is so high. In the main part of Fig. (2), we plot \( G \) vs. the scaling variable and find that the (apparent) critical exponent \( \zeta \) increases to \( 0.722 \pm 0.005 \). We have confirmed this result with measurements of the Binder ratio. Unlike the case of uncorrelated disorder, we observed a slow drift downward of \( U_4 \) with system size at the previously determined \( T_c \). Taking this out by scalings \( U_4(T_c, 0)/U_4(T_c, 1/L) \) yields essentially perfect data collapse with \( \nu = 0.72 \pm 0.007 \), as shown in Fig. (3). Assuming a violation of hyperscaling gives an anomalous dimension to the superfluid density \( \rho_s \propto L^{-1+\delta}G(T-T_c)L^{1/\nu} \), we can place an approximate upper bound \(|\delta| \leq 0.06\).

It is enlightening to compare the present results to a model with disorder of lower dimension, namely infinitely long columnar defects. Recent work on this model indicates that \( \nu_\perp = \zeta_\perp \sim 1 \) is even larger than for the present model. This must be the case in order to satisfy the rigorous lower bound \( \nu \), since this model does truly represent a new universality class, and not simply a crossover. The superfluid density measured parallel to the columns has an even larger exponent \( (z = \nu_\parallel/\nu_\perp \sim 1.07) \).

In conclusion, we have argued that the apparent increase in the superfluid density exponent in aerogel cannot be due to a true change of universality class but must be a cross over effect in the regime where the helium correlation length is less than the (apparently large, but necessarily finite) correlation length for the disorder. We have performed Monte Carlo simulations in this regime for a percolation cluster model of fractal disorder and find an increase in the effective exponent to \( \zeta = 0.722 \pm 0.005 \) which appears to be roughly consistent with experiment. However, we see no apparent violation of hyperscaling, and attempts to confirm the unusual behavior of the experimental specific heat have proved too difficult computationally at this time.

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It is possible that there is a logarithmic correction to scaling visible in our data, but it is very weak.

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FIG. 1. Universal scaling function $\tilde{G}$ vs. the scaling variable $(T - T_c)L^{1/\nu}$ for the case of uncorrelated disorder. The critical exponent $\nu$ is estimated to be $0.667 \pm 0.005$. The inset shows the dimensionless superfluid density $\rho_s L/T$ plotted against temperature.

FIG. 2. Universal scaling function $\tilde{G}$ for the case of 95% porosity fractal disorder. The critical exponent $\nu$ is estimated to be $0.722 \pm 0.005$. The inset shows the dimensionless superfluid density $\rho_s L/T$ plotted against temperature.

FIG. 3. Universal scaling curve for the (size-corrected) Binder ratio for the case of fractal disorder. The choice of the critical exponent $\nu = 0.72 \pm 0.007$ makes the data for different system sizes collapse onto a single curve and agrees with the conclusion drawn from analysis of the superfluid density.
$T_c \sim 2.156$

$\nu \sim 0.667$

Moon Girvin PRL Fig. 1
\( \rho_s^* \frac{L}{T} \) vs. \( (T - T_c)^{1/\nu} \nu \)

\( T_c \approx 2.183 \)

\( \nu \approx 0.722 \)

Moon Girvin PRL Fig.2
\[ X = (T - T_c) \times L^{1/\nu} \]

\[ U_4^0(X) \]

\[ T_c \sim 2.183 \]
\[ \nu \sim 0.722 \]

Moon Girvin PRL Fig. 3