Examples of surfaces with canonical map of degree 4

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Abstract. We give two examples of surfaces with canonical map of degree 4 onto a canonical surface.

Keywords. algebraic surface; surface of general type; abelian cover; canonical map

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1. Introduction

Let $S$ be a smooth minimal surface of general type with geometric genus $p_g \geq 3$. Denote by $\phi : S \rightarrow \mathbb{P}^{p_g-1}$ the canonical map and let $d := \deg(\phi)$. The following result of Beauville is well-known.

**Theorem 1.1** ([Bea79]). If the canonical image $\Sigma := \phi(S)$ is a surface, then either:

(A) $p_g(\Sigma) = 0$, or

(B) $\Sigma$ is a canonical surface (i.e. it is the canonical image of a surface with birational canonical map), in particular $p_g(\Sigma) = p_g(S)$.

Moreover, in case (A) $d \leq 36$ and in case (B) $d \leq 9$.

The question of which pairs $(d, p_g)$ can actually occur has been object of study for some authors. Several examples were given for case (A), but case (B) is still mysterious. It is known that if $d > 3$, then $p_g \leq 12$, but so far only the case $(d, p_g) = (5, 4)$ has been shown to exist (independently by Tan [Tan92] and by Pardini [Par91b]). We refer the recent preprint by Mendes Lopes and Pardini [MLP21] for a more detailed account on the subject. They leave some open problems, this note is motivated by their last question.

**Question.** For what pairs $(d, p_g)$, with $d > 3$, are there examples of surfaces in case (B) of Theorem 1.1?

Here we give examples for the cases $(d, p_g) = (4, 5)$ and $(4, 7)$, with canonical images a 40-nodal complete intersection surface in $\mathbb{P}^4$ and a 48-nodal complete intersection surface in $\mathbb{P}^6$, respectively (Beauville also paid some attention to such nodal surfaces, see [Bea17]).

The strategy for the construction is the following. If $X$ is a surface with nodes admitting a Galois covering $Y \rightarrow X$ ramified over the nodes and with Galois group $G$, a group with a “big” number of subgroups, then we have a “big” number of intermediate coverings of $X$. By computing the geometric genus $p_g$ of all involved surfaces, we may hope to find some $\rho : W \rightarrow Z$ with $p_g(W) = p_g(Z)$, hence such that the canonical map of $W$ factors through $\rho$.

We work explicitly with the equations of a 40-nodal surface from [RRS19], all computations are implemented with Magma [BCP97].

**Notation**

As usual the holomorphic Euler characteristic of a surface $S$ is denoted by $\chi(S)$, the geometric genus by $p_g(S)$, the irregularity by $q(S)$, and a canonical divisor by $K_S$. A $(-m)$-curve is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-m$. A node of $S$ is an ordinary double point of $S$. We say that a set of nodes of $S$ is 2-divisible if the sum $\sum A_i$ of the corresponding $(-2)$-curves in the smooth minimal model of $S$ is 2-divisible in the Picard group.
2. \((\mathbb{Z}/2)\)\(^r\)-coverings

The following result is taken from [Cat08, Proposition 7.6]. See also [Par91a].

**Proposition 2.1.** A normal finite \(G \cong (\mathbb{Z}/2)^r\)-covering \(\pi : Y \to X\) of a smooth variety \(X\) is completely determined by the datum of

1. reduced effective divisors \(D_\sigma\), for all \(\sigma \in G\), with no common components;
2. divisor linear equivalence classes \(L_{\chi_1}, \ldots, L_{\chi_r}\) for \(\chi_1, \ldots, \chi_r\) a basis of the group of characters \(G^\vee\), such that

\[
2L_{\chi_i} \equiv \sum_{\chi_i(\sigma) = 1} D_\sigma
\]

(with additive notation for the characters).

Conversely, given (1) and (2), one obtains a normal scheme \(Y\) with a finite \(G \cong (\mathbb{Z}/2)^r\)-covering \(Y \to X\), with branch curves the divisors \(D_\sigma\).

The scheme \(Y\) is irreducible if \(\{\sigma | D_\sigma > 0\}\) generates \(G\). We have a splitting

\[
\pi_* O_Y = \bigoplus_{\chi \in G^\vee} L_{\chi}^{-1}.
\]

From now on, we assume that \(X\) and \(Y\) are surfaces. If each \(D_\sigma\) is smooth and \(\sum D_\sigma\) has simple normal crossings, then \(Y\) is smooth and its invariants are

\[
\chi(O_Y) = 2^r \chi(O_X) + \frac{1}{2} \sum_{\chi \in G^\vee} \left( t^2_\chi + K_X \cdot L_\chi \right),
\]

\[
p_g(Y) = p_g(X) + \sum_{\chi \in G^\vee} h^0(X, O_X(K_X + L_\chi)).
\]

Let \(R_\sigma\) be the support of \(\pi^*(D_\sigma)\). The Hurwitz formula gives

\[
K_Y \equiv \pi^*(K_X) + \sum_{\sigma \in G} R_\sigma.
\]

Now assume that the \(D_\sigma\) are disjoint \((-2)\)-curves. Then the \(R_\sigma\) are disjoint \((-1)\)-curves, the canonical map of \(Y\) factors through the covering \(Y \to X\) if and only if \(p_g(Y) = p_g(X)\), and one has a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & Y' \\
\downarrow & \downarrow \\
X & \longrightarrow & X'
\end{array}
\]

where \(Y \to Y'\) is the contraction of the \((-1)\)-curves \(R_\sigma\), the surface \(X'\) has nodes corresponding to the \((-2)\)-curves of \(X\), and \(Y' \to X'\) is a \((\mathbb{Z}/2)^r\)-covering ramified on those nodes. In this case Equation (2.1) becomes

\[
\chi(O_Y) = 2^r (\chi(O_X) - m/8)
\]

where \(m\) is the number of nodes of \(X'\).

3. Construction

Let \(X_{40}\) be the surface in \(\mathbb{P}^4\) given by the equations

\[
\begin{align*}
5 \left( x^2 + y^2 + z^2 + w^2 + t^2 \right) - 7 \left( x + y + z + w + t \right)^2 & = 0 \\
4 \left( x^4 + y^4 + z^4 + w^4 + t^4 + h^4 \right) - \left( x^2 + y^2 + z^2 + w^2 + t^2 + h^2 \right)^2 & = 0
\end{align*}
\]

\[(3.1)\]
where
\[ h := -(x + y + z + w + t). \]

It is the canonical model of a surface with invariants \( p_g = 5, \ q = 0 \) and \( K^2 = 8 \). The above quartic \( I \) is classically known as the Igusa quartic; its singular set is the union of 15 lines. The quadric meets these lines transversally, and is tangent to \( I \) at 10 smooth points, thus the singular set of \( X_{40} \) is the union of 40 nodes \( N_1, \ldots, N_{40} \) (for more details see [RRS19]).

Let \( \tilde{X}_{40} \) be the smooth minimal model of \( X_{40} \) and denote by \( A_i \) the \((-2)\)-curves in \( \tilde{X}_{40} \) corresponding to the nodes \( N_i, \ i = 1, \ldots, 40 \). Let \( a, b, c \) be the canonical generators of the group \((\mathbb{Z}/2)^3\) and, for \( i, j, k \in \mathbb{Z}/2 \), let \( \chi_{ijk} \) denote the character which takes the value \( i, j, k \) on \( a, b, c \), respectively. We show in Section 4.1 that one can write
\[ A_1 + \cdots + A_{40} = D_a + D_b + D_c + D_{abc} + D_{bc} + D_{ac} + D_{ab} \]
where each of \( D_a, D_b, D_c, D_{abc} \) is a sum of 4 \((-2)\)-curves, each of \( D_{bc}, D_{ac}, D_{ab} \) is a sum of 8 \((-2)\)-curves, and such that there exist divisors \( L_{100}, L_{010}, L_{001} \) satisfying:
\[
\begin{align*}
D_a + D_{abc} + D_{ac} + D_{ab} &\equiv 2L_{100} \\
D_b + D_{abc} + D_{bc} + D_{ab} &\equiv 2L_{010} \\
D_c + D_{abc} + D_{bc} + D_{ac} &\equiv 2L_{001}.
\end{align*}
\]

It follows from Proposition 2.1 that these data define a \((\mathbb{Z}/2)^3\)-covering \( \pi : \tilde{Y} \rightarrow \tilde{X}_{40} \) branched on the \((-2)\)-curves \( A_i \), equivalently a \((\mathbb{Z}/2)^3\)-covering \( \psi : Y \rightarrow X_{40} \) branched on the nodes of \( X_{40} \) (the surface \( Y \) is minimal because \( X_{40} \) is minimal and \( \psi \) is étale in codimension 1). In particular there exist divisors \( L_{111}, L_{110}, L_{101}, L_{011} \) such that:
\[
\begin{align*}
D_a + D_b + D_c + D_{abc} &\equiv 2L_{111} \\
D_a + D_b + D_{bc} + D_{ac} &\equiv 2L_{110} \\
D_a + D_c + D_{bc} + D_{ab} &\equiv 2L_{101} \\
D_b + D_c + D_{ac} + D_{ab} &\equiv 2L_{011}.
\end{align*}
\]

One has
\[ 2L_{ijk} = \sum_{X_{ijk}(\alpha)=1} D_{\alpha}. \]

Since \( \psi \) is ramified only on nodes, we have \( K_Y \equiv \psi^*(K_{X_{40}}) \) and then \( K_Y^2 = 8K_{X_{40}}^2 = 64 \). We show in Section 4.1 that
\[ h^0(\tilde{X}_{40}, \mathcal{O}_{\tilde{X}_{40}}(K_{\tilde{X}_{40}} + L_{111})) = 2 \]
and
\[ h^0(\tilde{X}_{40}, \mathcal{O}_{\tilde{X}_{40}}(K_{\tilde{X}_{40}} + L_{ijk})) = 0 \quad \text{for} \quad ijk \neq 111, \]
thus
\[ p_g(Y) = p_g(X_{40}) + 2 + 0 + \cdots + 0 = 7. \]

We get from (2.2) that \( \chi(Y) = 8(6 - 5) = 8 \), thus \( q(Y) = 0 \).

The covering \( \psi \) factors as
\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & Y_{48} \\
\downarrow & & \downarrow \\
X_{16} & \xrightarrow{\chi} & X_{40}
\end{array}
\]
with \( Y_{48} \) and \( X_{16} \) given by the quotients by the groups \langle ab, ac \rangle \ and \( \langle c \rangle \), respectively (the subscript \( n \) means a surface with singular set the union of \( n \) nodes). All these surfaces are regular because \( q(Y) = 0 \).

It follows from (2.2) that \( \chi(X_{16}) = 4(6 - 36/8) = 6 \), thus \( p_g(X_{16}) = p_g(X_{40}) = 5 \), and we conclude that the \((\mathbb{Z}/2)^2\)-covering \( X_{16} \rightarrow X_{40} \) is the canonical map of \( X_{16} \).
Analogously, $p_g(Y) = p_g(Y_{48}) = 7$ and we claim that

the $(\mathbb{Z}/2)^2$-covering $Y \to Y_{48}$ is the canonical map of $Y$.

For this it suffices to show that $Y_{48}$ is a canonical surface.

Since the canonical system of $Y_{48}$ contains the pullback of the canonical system of $X_{40}$ and since $p_g(Y_{48}) > p_g(X_{40})$, the canonical map of $Y_{48}$ must be birational. But we can be more precise. We follow Beauville [Bea17] and show that $Y_{48}$ can be embedded in $\mathbb{P}^6$ as a complete intersection of 4 quadrics in the following way. The linear system $L$ of quadrics through the branch locus of the covering $Y_{48} \to X_{40}$ (16 nodes) is of dimension 2. Using computer algebra it is not difficult to show that $L$ contains quadrics $B, C, D$ such that the surface $X_{40}$ is given by $Q = 0$, $B^2 - CD = 0$, where $Q$ is the quadric from (3.1) (we write the quadrics as general elements of $L$, thus depending on some parameters; then we obtain a variety on these parameters by imposing that the hypersurfaces $Q = 0$ and $B^2 - CD = 0$ are tangent at the 24 nodes of $X_{40}$ which are disjoint from the 16 nodes of $B^2 - CD = 0$; finally we compute points in this variety).

Then $Y_{48}$ is given in $\mathbb{P}^6(x, y, z, w, t, u, v)$ by equations

$$u^2 - C = v^2 - D = uv - B = Q = 0.$$

We give these equations in Section 4.2 and verify that $Y_{48}$ is as stated.

Let us explain how we find 2-divisible sets of nodes in $X_{40}$. The surface $X_{40}$ contains 40 tropes, which are hyperplane sections $H_i = 2T_i$ with $T_i \subset X_{40}$ a reduced curve through 12 nodes of $X_{40}$, and smooth at these points. Thus in $\overline{X}_{40}$ the pullback of such a trope can be written as

$$\overline{H}_i = 2\overline{T}_i + \sum_{j \in J} A_j,$$

with $\# J = 12$.

Thus for each pair of tropes the sum of nodes contained in their union and not contained in their intersection is 2-divisible.

Using these 2-divisibilities, the strategy for finding configurations as in (3.2) is simple: we have used a computer algorithm to list and check possibilities.

4. Computations

The computations below are implemented with Magma V2.26-5.

4.1. The covering $Y \to X_{40}$

We start by defining the surface $X_{40}$ and its singular set.

```magma
K:=Rationals();
R<r>:=PolynomialRing(K);
K<r>:=ext<K|r^2 + 15>;
P<x,y,z,w,t>:=ProjectiveSpace(K,4);
h:=-x-y-z-w-t;
Q:=5*(x^2+y^2+z^2+w^2+t^2)-7*(x+y+z+w+t)^2;
I:=4*(x^4+y^4+z^4+w^4+t^4+h^4)-(x^2+y^2+z^2+w^2+t^2+h^2)^2;
X40:=Surface(P,[Q,I]);
SX40:=SingularSubscheme(X40);
```

The partition of the 40 nodes:

```magma
Da:={P![3,3,-2,-2,3],P![4,-r+1,r-5,-r+1,4],
P![-r+1,4,r-5,-r+1,4],P![r+1,r+1,-r-5,4,4]};
Db:={P![2,-3,-3,-3,2],P![4,r+1,r+1,-r-5,4],
P![-r-5,r-5,-r-5,10],P![-r-5,-r+1,-r+1,4,4]};
```
Dc:=\{\text{points}\};
Dabc:=\{\text{points}\};
Dbc:=\{\text{points}\};
Dac:=\{\text{points}\};
Dab:=\{\text{points}\};

Verification that these are in fact the nodes:
&\text{join}[Da,Db,Dc,Dabc,Dbc,Dac,Dab] eq SingularPoints(X40);
HasSingularPointsOverExtension(X40) eq false;

Some of the tropes of $X_{40}$:
tropes:=
\begin{align*}
6x + (-r - 9)y + (r - 9)z + (r - 9)w + (-r - 9)t, \\
16x + (-r - 9)y + 16z + (3r + 11)w + (3r + 11)t, \\
16x + (r - 9)y + 16z + (-3r + 11)w + (-3r + 11)t, \\
6x + (r - 9)y + (-r - 9)z + (r - 9)w + (-r - 9)t, \\
16x + (3r + 11)y + 16z + (3r + 11)w + (-r - 9)t, \\
16x + (-3r + 11)y + (-3r + 11)z + (r - 9)w + 16t, \\
x + y + w, \\
16x + (r - 9)y + (-3r + 11)z + (-3r + 11)w + 16t, \\
x + z + w
\end{align*}

The reduced subscheme of these tropes:
red:=\{\text{reduced scheme}\};
&\text{Dimension}(\text{SingularSubscheme}(q) meet SX40) eq -1:q in red];

They are smooth at the nodes of $X_{40}$:

Two 2-divisible disjoint sets of 20 nodes, which confirm that the 40 nodes are 2-divisible:
s1:=Points(Scheme(SX40,\text{tropes}[1]*\text{tropes}[2])) diff
Points(Scheme(SX40,\text{tropes}[1],\text{tropes}[2]));
s2:=Points(Scheme(SX40,\text{tropes}[6]*\text{tropes}[7])) diff
Points(Scheme(SX40,\text{tropes}[6],\text{tropes}[7]));
&\#s1 eq 20,\#s2 eq 20,(s1 join s2) eq 40];

We compute three 2-divisible sets of 24 nodes:
Sets:=\{\};
for q in [[2,5],[1,4],[3,8]] do
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\[ \text{pts:=Points(Scheme(SX40,tropes[q[1]]*tropes[q[2]])) diff Points(Scheme(SX40,[tropes[q[1]],tropes[q[2]]]));} \]
\[ \text{Append("Sets,SingularPoints(X40) diff pts); end for;} \]

and use these sets to check the divisibilities in (3.2):
\[ \text{Da join Dabc join Dac join Dab eq Sets[1];} \]
\[ \text{Db join Dabc join Dbc join Dab eq Sets[2];} \]
\[ \text{Dc join Dabc join Dbc join Dac eq Sets[3];} \]

Now we show that
\[ h^0(\tilde{X}_{40}, O_{\tilde{X}_{40}}(K_{\tilde{X}_{40}} + L_{111})) = 2. \]

Let \( N_1, \ldots, N_{16} \) be the nodes in \( D_a + D_b + D_c + D_{abc} \) and \( A_1, \ldots, A_{16} \) be the corresponding \((-2)\)-curves. Let \( H_1, H_2 \) be the tropes whose pullback to \( \tilde{X}_{40} \) is
\[ \tilde{H}_1 + \tilde{H}_2 = 2\tilde{T}_1 + 2\tilde{T}_2 + \sum_{i=1}^{16} A_i + 2\sum_{i=17}^{20} A_i, \]
with \( A_{17}, \ldots, A_{20} \in \tilde{H}_1 \cap \tilde{H}_2 \).

We compute below that the system of quadrics through the curves \( T_1, T_2 \subset \mathbb{P}^4 \) is generated by 2 elements, modulo the quadric \( Q \). For \( i = 17, \ldots, 20 \), the fact \( (2\tilde{H} - \tilde{T}_1 - \tilde{T}_2) \cdot A_i < 0 \) implies that \( A_i \) is contained in the base component of the linear system \( |2\tilde{H} - \tilde{T}_1 - \tilde{T}_2| \). This gives
\[ h^0(\tilde{X}_{40}, O_{\tilde{X}_{40}}(K_{\tilde{X}_{40}} + L_{111})) = 2. \]

\[ \text{T1:=ReducedSubscheme(Scheme(X40,tropes[2]));} \]
\[ \text{T2:=ReducedSubscheme(Scheme(X40,tropes[9]));} \]
\[ \text{pts:=Points(SX40 meet (T1 join T2)) diff Points(SX40 meet T1 meet T2);} \]
\[ \text{L:=LinearSystem(LinearSystem(P,2),T1 join T2);} \]
\[ \text{#Sections(LinearSystemTrace(L,X40)) eq 2;} \]

Let us show that
\[ h^0(\tilde{X}_{40}, O_{\tilde{X}_{40}}(K_{\tilde{X}_{40}} + L_{ijk})) = 0 \]
for \( ijk \neq 111 \). Suppose the opposite. Let \( A_1, \ldots, A_{24} \) be the corresponding \((-2)\)-curves. Then there is a curve \( E \in |K_{\tilde{X}_{40}} + L_{ijk}| \), and \( E \cdot A_i = -1 \) implies that the linear system \( |K_{\tilde{X}_{40}} + L_{ijk} - \sum_{j=1}^{24} A_j| = |K_{\tilde{X}_{40}} - L_{ijk}| \) is nonempty. Therefore \( 2K_{\tilde{X}_{40}} - \sum_{j=1}^{24} A_j \) is nonempty, which implies that there is at least one quadric in \( \mathbb{P}^4 \) through the corresponding nodes \( N_1, \ldots, N_{24} \) (modulo the quadric \( Q \)). We show below that this does not happen.

\[ \text{Sets:=[} \]
\[ \text{Da join Dabc join Dac join Dab,} \]
\[ \text{Db join Dabc join Dbc join Dab,} \]
\[ \text{Dc join Dabc join Dbc join Dac,} \]
\[ \text{Da join Db join Dbc join Dac,} \]
\[ \text{Da join Dc join Dbc join Dab,} \]
\[ \text{Db join Dc join Dac join Dab;} \]
\[ \text{for q in Sets do} \]
4.2. The surface $Y_{48}$

Here we give the equations of $Y_{48}$ as a complete intersection of 4 quadrics in $\mathbb{P}^6$. We start by defining $\mathbb{P}^6$ over a certain number field.

$$K:=\text{Rationals}(); \quad R<x>:=\text{PolynomialRing}(K);$$
$$K<r,m>:=\text{ext}<K|x^2 + 15,x^2 - 95/42*x + 2855/2646>; \quad R<n>:=\text{PolynomialRing}(K);$$
$$K<n>:=\text{ext}<K|n^2 + 443889777/20639124080000*r - 46942774543/619173642264000>; \quad P6<x,y,z,w,t,u,v>:=\text{ProjectiveSpace}(K,6);$$

The three quadrics $B, C, D$:

$$B:=(675/4802*r+334125/33614)*n*x*z+(-389475/67228*r+3266325/67228)*n*x*w+(-62100/16807*r+348300/16807)*n*w^2+(239625/33614*r+1541025/33614)*n*x*t+(-8100/2401*r+137700/16807)*n*y*t+(239625/33614*r+1541025/33614)*n*z*t+(6075/9604*r+3007125/67228)*n*w*t+(71550/16807*r+319950/16807)*n*t^2; \quad C:=x*y+1/154*(126*m-181)*y^2+1/42*(-42*m+95)*x*z+y*z+(1/1540*(-14*m+25)*r+1/924*(-798*m+1997))*x*w+(1/420*(-42*m+65)*r+1/308*(-294*m+767))*y*w+(1/1540*(-14*m+25)*r+1/924*(-798*m+1997))*z*w+(1/385*(-119*m+185)*r+1/462*(-168*m+311))*w^2+(1/1540*(-14*m+25)*r+1/924*(-798*m+1997))*x*t+(1/420*(-42*m+65)*r+1/308*(-294*m+767))*y*t+(1/1540*(-14*m+25)*r+1/924*(-798*m+1997))*z*t+1/154*(126*m-71)*w*t+(1/2310*(-21*m+10)*r+1/154*(133*m+32))*x*u+(1/70*(-7*m+5)*r+1/154*(147*m+51))*y*u+(1/2310*(-21*m+10)*r+1/154*(133*m+32))*z*u+(1/2310*(714*m-505)*r+1/154*(56*m-23))*w^2+(1/2310*(21*m-10)*r+1/154*(133*m+32))*x*t+(1/70*(-7*m+5)*r+1/154*(147*m+51))*y*t+(1/2310*(21*m-10)*r+1/154*(133*m+32))*z*t+1/77*(-63*m+107)*w*t+(1/2310*(-714*m+505)*r+1/154*(56*m-23))*t^2; \quad D:=x*y+1/77*(-63*m+52)*y^2+m*x*z+y*z+(1/2310*(-21*m+10)*r+1/154*(133*m+32))*x*u+(1/70*(-7*m+5)*r+1/154*(147*m+51))*y*u+(1/2310*(-21*m+10)*r+1/154*(133*m+32))*z*u+(1/2310*(714*m-505)*r+1/154*(56*m-23))*w^2+(1/2310*(21*m-10)*r+1/154*(133*m+32))*x*t+(1/70*(-7*m+5)*r+1/154*(147*m+51))*y*t+(1/2310*(21*m-10)*r+1/154*(133*m+32))*z*t+1/77*(-63*m+107)*w*t+(1/2310*(-714*m+505)*r+1/154*(56*m-23))*t^2;$$

We obtain alternative equations for $X_{40}$:

$$F:=B^2-C*D; \quad Q:=5*(x^2+y^2+z^2+w^2+t^2)-7*(x+y+z+w+t)^2; \quad X:=%\text{Scheme}(P6,[F,Q,u,v]);$$
$$h:=-x-y-z-w-t; \quad I:=4*(x^2+y^2+z^2+w^2+t^2)-7*(x+y+z+w+t)^2; \quad X40:=%\text{Scheme}(P6,[Q,I,u,v]);$$
$$X \text{ eq } X40; \quad Y_{48}:=%\text{Surface}(P6,[u^2-C,v^2-D,u*v-B,Q]); \quad SY_{48}:=%\text{SingularSubscheme}(Y_{48});$$
$$\text{Dimension}(SY_{48}) \text{ eq } 0; \quad \text{Degree}(SY_{48}) \text{ eq } 48; \quad \text{Degree}(%\text{ReducedSubscheme}(SY_{48})) \text{ eq } 48;$$
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