COERCIVE INEQUALITIES AND U-BOUNDS

E. BOU DAGHER AND B. ZEGARLIŃSKI

Abstract. We prove Poincaré and Log$\beta$-Sobolev inequalities for probability measures on step-two Carnot groups.

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1. Introduction

Although the question of obtaining coercive inequalities such as the Poincaré or the Logarithmic Sobolev inequalities for a probability measure on a metric measure space has been a subject of numerous works, the literature on this topic in the setup Carnot groups is scarce.

In [13], L. Gross obtained the following Logarithmic Sobolev inequality:

\begin{equation}
\int_{\mathbb{R}^n} f^2 \log \left( \frac{f^2}{\int_{\mathbb{R}^n} f^2 d\mu} \right) d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,
\end{equation}

where $\nabla$ is the standard gradient on $\mathbb{R}^n$ and $d\mu = e^{-|x|^2/2} d\lambda$ is the Gaussian measure.

In a setup of a more general metric space, a natural question would be to try to find similar inequalities with different measures of the form $d\mu = e^{-U(d)} Z d\lambda$, where $U$ is a function of a metric $d$, and where the Euclidean gradient is replaced by a more general sub-gradient in $\mathbb{R}^n$.

Aside from their theoretical importance, such inequalities are needed because of their applications, some of which will be discussed briefly. L.Gross also pointed out ([13]) the importance of the inequality (1.1) in the sense that it can be extended to infinite dimensions with additional useful results. (See also works: [14, 5, 30, 27, 6, 34, 29].) He proved that if $\mathcal{L}$ is the non-positive self-adjoint operator on $L^2(\mu)$ such that

\begin{equation}
(\mathcal{L}f, f)_{L^2(\mu)} = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,
\end{equation}

then (1.1) is equivalent to the fact that the semigroup $P_t = e^{t\mathcal{L}}$ generated by $\mathcal{L}$ is hypercontractive: i.e. for $q(t) \leq 1 + (q - 1) e^{2t}$ with $q > 1$, we have $\| P_t f \|_q \leq \| f \|_q$ for all $f \in L^q(\mu)$. ([13])

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In [8], D. Bakry and M. Emery extended the Logarithmic Sobolev inequality for a larger class of probability measures defined on Riemannian manifolds under an important Curvature-Dimension condition. More generally, if $(\Omega, F, \mu)$ a probability space, and $L$ is a non-positive self-adjoint operator acting on $L^2(\mu)$, we say that the measure $\mu$ satisfies a Logarithmic Sobolev inequality if there is a constant $c$ such that, for $f \in D(L)$,

$$\int f^2 \log \frac{f^2}{\mu} \, d\mu \leq c \int (-Lf) \, d\mu.$$ 

In this general setting, the connection between this inequality and the property of hypercontractivity was shown in [13].

Another generalisation, the so-called q-Logarithmic Sobolev inequality, in the setting of a metric measure space, was obtained by S. Bobkov and M. Ledoux in [19], in the form:

$$\int f^q \log \frac{f^q}{\mu} \, d\mu \leq c \int \|\nabla f\|^q \, d\mu,$$

where $q \in (1, 2]$. Here, on a metric space, the magnitude of the gradient is defined by

$$|\nabla f|(x) = \limsup_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

In [5], S. Bobkov and B. Zegarliński showed that the q-Logarithmic Sobolev inequality is better than the classical $q = 2$ inequality in the sense that one gets a stronger decay of tail estimates. In addition, when the space is finite, and under weak conditions, they proved that the corresponding semigroup $P_t$ is ultracontractive i.e.

$$\|P_t f\|_\infty \leq \|f\|_p$$

for all $t \geq 0$ and $p \in [1, \infty)$.

We point out, that in [20], M. Ledoux made a connection between the Logarithmic Sobolev inequality and the isoperimetric problem. (See also: [3, 17, 8])

The important q-Poincaré inequality

$$\int |f - \int f \, d\mu|^q \, d\mu \leq c \int \|\nabla f\|^q \, d\mu$$

can be obtained from the q-Logarithmic Sobolev inequality by simply replacing $f$ by $1 + \varepsilon f$ in that inequality, and letting $\varepsilon \to 0$.

In this paper, our primary interest is to prove the existence of coercive inequalities for different measures in the setting of step-two nilpotent Lie groups, whose tangent space at every point is spanned by a family of degenerate and non-commuting vector fields $\{X_i, i \in \mathcal{R}, 1 < |\mathcal{R}| < \infty\}$, where $|\mathcal{R}|$ is the cardinality of the index set $\mathcal{R}$. These inequalities, when satisfied, give us information about the spectra of the associated generators of the form

$$L = \sum_{i \in \mathcal{R}} X_i^2$$

where $|\mathcal{R}|$ is strictly less than the dimension of the space. (See also [12] and references therein)

Thus, by Hörmander’s result in [7], the sub-Laplacian (1.2) is hypoelliptic; in other words, every distributional solution to $Lu = f$ is of class $C^\infty$ whenever $f$ is of class $C^\infty$.

We point out that, according to [32], if we have a uniqueness of solution in the space of square integrable functions for the Cauchy problem

$$\begin{cases}
\frac{d}{dt} u = Lu \\
u|_{t=0} = f,
\end{cases}$$

then the solution of the heat equation will be given by $u = P_t f$. 
In the setting of step-two nilpotent Lie groups, since the Laplacian is of Hörmander type and has some degeneracy, D. Bakry and M. Emery’s Curvature-Dimension condition in [8] will no longer hold true. In [15], a method of studying coercive inequalities on general metric spaces that does not require a bound on the curvature of space was developed. Working on a general metric space equipped with non-commuting vector fields \( \{X_1, \ldots, X_n\} \), their method is based on U-bounds, which are inequalities of the form:

\[
\int f^q U(d) \, d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu
\]

where \( d\mu = \frac{e^{-U(d)}}{Z} \, d\lambda \) is a probability measure, \( U(d) \) and \( \mathcal{U}(d) \) are functions having a suitable growth at infinity, \( \lambda \) is a natural measure like the Lebesgue measure for instance (which is the Haar measure for nilpotent Lie groups), \( d \) is a metric related to the gradient \( \nabla = (X_1, \ldots, X_n) \), and \( q \in (1, \infty) \).

It is worth mentioning that in the setting of nilpotent Lie groups, heat kernel estimates were studied to get a variety of coercive inequalities ([33, 21, 2, 28, 22, 23, 24, 25, 26, 10]). In our setting, we study coercive inequalities involving sub-gradients and probability measures on the group which is a difficult and much less explored subject. An approach, pioneered in [15], was used by J. Inglis to get Poincaré inequality in the setting of the Heisenberg-type group with measure as a function of Kaplan distance [16] and by M. Chatzakou et al. to get Poincaré inequality in the setting of the Engel-type group with a measure as a function of some homogeneous norm [9].

In section 2 we define the step-two Carnot group, and introduce \( N \), the homogeneous norm we are working with, that is of the form of the Kaplan norm in the Heisenberg-type group. Section 3 contains the main theorem, which is a proof of a U-Bound of the form

\[
\int g^p(N) f^q d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu,
\]

where \( g(N) \) satisfies some growth conditions. In section 4, we apply this U-bound together with some results of [15] to get the \( q \)-Poincaré inequality with \( q \geq 2 \) for the measures \( d\mu = \frac{e^{-g(N)}}{Z} \, d\lambda \).

This generalises the result by J. Inglis [16] who, in the setting of the Heisenberg-type group, proved the \( q \)-Poincaré inequality for the measure \( d\mu = \frac{e^{-\alpha N^p}}{Z} d\lambda \), where \( p \geq 2, \alpha > 0 \), \( q \) is the finite index conjugate to \( p \), and with \( \tilde{N} \) the Kaplan norm. In section 5, we extend J. Inglis et al.’s Theorem 2.1 [17] who proved a \( \log^\beta – \text{Sobolev} \), \( \beta \in (0, 1) \) inequality in the context of the Heisenberg group. Recall that for density defined with a smooth homogeneous norm, \( \beta = 1 \) is not allowed ([15]). We extend the corresponding results to a \( \phi \)-Logarithmic Sobolev inequality, where \( \phi \) is concave, on step-two Carnot groups. Finally, we utilise the U-Bound to get to a \( \log^\beta – \text{Sobolev} \) inequality for \( d\mu = \frac{e^{-\alpha N^p}}{Z} d\lambda \), where \( p \geq 4, q \geq 2 \), and \( 0 < \beta \leq \frac{p-3}{p} \), indicating also no-go zone of parameters where the corresponding inequality fails.

### 2. Setup

Carnot groups are geodesic metric spaces that appear in many mathematical contexts like harmonic analysis in the study of hypoelliptic differential operators ([18, 32]) and in geometric measure theory (see extensive reference list in the survey paper [19]).
We will be working in the setting of $\mathbb{G}$, a step-two Carnot group, i.e. a group isomorphic to $\mathbb{R}^{n+m}$ with the group law

$$(x, z) \circ (x', z') = \left( x_i + x'_i, z_j + z'_j + \frac{1}{2} < \Lambda^{(j)} x, x' > \right)_{i=1, \ldots, n; j=1, \ldots, m}$$

for $x, x' \in \mathbb{R}^n, z, z' \in \mathbb{R}^m$, where the matrices $\Lambda^{(j)}$ are $n \times n$ skew-symmetric and linearly independent and $<, >$ stands for the inner product on $\mathbb{R}^n$. One can verify that $\mathbb{G}$ is a Lie group whose identity is the origin and where the inverse is given by $(x, z)^{-1} = -(x, z)$.

The dilation

$$\delta_\lambda : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}, \quad \delta_\lambda (x, z) = (\lambda x, \lambda^2 z)$$

is an automorphism of $(\mathbb{R}^{n+m}, \circ)$ for any $\lambda > 0$. Then, $\mathbb{G} = (\mathbb{R}^{n+m}, \circ, \delta_\lambda)$ is a homogeneous Lie group.

The Jacobian matrix at $(0, 0)$ of the left translation $\tau_{(x, z)}$ i.e the map

$$\mathbb{G} \ni (x', z') \rightarrow \tau_{(x, z)}((x', z')) := (x, z) \circ (x', z') \in \mathbb{G}$$

for fixed $(x, z) \in \mathbb{G}$ takes the following form

$$J_{\tau_{(x, z)}}(0, 0) = \begin{pmatrix} I_n & 0_{n \times m} \\ \frac{1}{2} \sum_{l=1}^n \Lambda^{(1)}_{1l} x_l & \cdots & \frac{1}{2} \sum_{l=1}^n \Lambda^{(1)}_{nl} x_l \\ \vdots & \ddots & \vdots \\ \frac{1}{2} \sum_{l=1}^n \Lambda^{(m)}_{1l} x_l & \cdots & \frac{1}{2} \sum_{l=1}^n \Lambda^{(m)}_{nl} x_l \\ I_m & 0_{m \times n} \end{pmatrix}.$$ 

Then, the Jacobian basis of $\mathfrak{g}$, the Lie algebra of $\mathbb{G}$, is given by

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{k=1}^m \sum_{l=1}^n \Lambda^{(k)}_{il} x_l \frac{\partial}{\partial z_k} \quad \text{and} \quad Z_j = \frac{\partial}{\partial z_j}$$

for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$.

Let $\nabla \equiv (X_i)_{i=1, \ldots, n}$ and $\Delta \equiv \sum_{i=1, \ldots, n} X_i^2$ denote the associated sub-gradient and sub-Laplacian, respectively. We consider the following smooth homogeneous norm on $\mathbb{G}$

$$N \equiv (|x|^4 + a|z|^2)^{\frac{1}{4}}$$

with $a \in (0, \infty)$.

The motivation behind choosing such a norm is that in the setting of the Heisenberg-type groups (where we assume in addition that $\Lambda^{(j)}$ are orthogonal matrices and that $\Lambda^{(i)} \Lambda^{(j)} = -\Lambda^{(j)} \Lambda^{(i)}$ for every $i, j \in \{1, \ldots, m\}$ with $i \neq j$), $N \equiv (|x|^4 + 16|z|^2)^{\frac{1}{4}}$ is the Kaplan norm which arises from the fundamental solution of the sub-Laplacian. In other words, $\Delta N^{2-n-2m} = 0$ in $\mathbb{G} \setminus \{0\}$. Recall that J. Inglis, in [16], proved the $q$-Poincaré inequality in the setting of the Heisenberg-type group for the measure $d\mu = e^{-\alpha N^p} Z^{-1} d\lambda$, where $p \geq 2$, $\alpha > 0$, $q$ is the finite index conjugate to $p$, and...
\( N \equiv (|x|^4 + 16|z|^2)^{\frac{1}{4}} \). We extend, giving a simpler proof, the result of J. Inglis by obtaining a q-Poincaré inequality in the setting of step-two Carnot groups for the measures \( d\mu = \frac{e^{-g(N)}}{Z} d\lambda \), with \( \frac{g''(N)}{N^2} \) an increasing function, \( q \geq 2 \), and where \( N \equiv (|x|^4 + a|z|^2)^{\frac{1}{4}} \).

Our first key result this paper is obtaining the following U-Bound (section 3):

\[
\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu,
\]

under certain growth conditions for \( g(N) \). This U-bound is a useful tool to get a q-Poincaré inequality (section 4) and a \( \log^\beta \)-Sobolev inequality (section 5) for \( q \geq 2 \). We expect that this U-bound can be used to extend those coercive inequalities to (nonproduct) measures in an infinite dimensional setting. [35]

3. U-Bound

**Theorem 1.** Let \( N = (|x|^4 + a|z|^2)^{\frac{1}{4}} \) with \( a \in (0, \infty) \), and let \( g: [0, \infty) \rightarrow [0, \infty) \) be a differentiable increasing function such that \( g''(N) \leq g'(N)^3 N^3 \) on \( \{N \geq 1\} \). Let \( d\mu = \frac{e^{-g(N)}}{Z} d\lambda \) be a probability measure, where \( Z \) is the normalization constant. Then, given \( q \geq 2 \),

\[
\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu
\]

holds for all locally Lipschitz functions \( f \), supported outside the unit ball \( \{N < 1\} \), with \( C \) and \( D \) positive constants independent of \( f \).

The proof of Theorem 1 uses the following properties of a smooth norm \( N \) proven in the Appendix.

**Lemma 2.** There exist constants \( A, C \in (0, \infty) \)

\[
(3.1) \quad A \frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C \frac{|x|^2}{N^2}
\]

and there exists a constant \( B \in (0, \infty) \) such that

\[
(3.2) \quad |\Delta N| \leq B \frac{|x|^2}{N^3}
\]

and

\[
(3.3) \quad \frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}.
\]

The other main tools we use are Hardy’s inequality (see [31] and references therein) and the Coarea formula (page 468 of [7]).

**Proof of Theorem 1.** First, we prove the result for \( q = 2 \):

We note that using integration by parts, one gets

\[
\int (\nabla N) \cdot (\nabla f) e^{-g(N)} d\lambda = - \int \nabla \left( \nabla Ne^{-g(N)} \right) f d\lambda = - \int \Delta N f e^{-g(N)} d\lambda + \int |\nabla N|^2 f g'(N) e^{-g(N)} d\lambda.
\]
Net, using (3.1) and (3.2),

\[
\int (\nabla N) \cdot (\nabla f) e^{-g(N)} d\lambda \geq -B \int \frac{|x|^2}{N^2} f e^{-g(N)} d\lambda + A \int \frac{|x|^2}{N^2} f g' (N) e^{-g(N)} d\lambda.
\]

Replacing \(f\) by \(\frac{f^2}{|x|^2}\):

\[
(3.4) \quad \int (\nabla N) \cdot \left( \nabla \left( \frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda \geq \int f^2 \left( \frac{Ag' (N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda.
\]

As for the left-hand side of (3.4),

\[
\int (\nabla N) \cdot \left( \nabla \left( \frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda = \int (\nabla N) \cdot \left[ 2 f \frac{\nabla f}{|x|^2} - \frac{2 f^2 \nabla |x|}{|x|^3} \right] e^{-g(N)} d\lambda
\]

\[
= \int \frac{2 f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda - 2 \int \frac{f^2}{N^3} e^{-g(N)} d\lambda.
\]

Using the calculation of \(\nabla N \cdot x\), from (3.3), we get:

\[
= \int \frac{2 f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda - 2 \int \frac{f^2}{N^3} e^{-g(N)} d\lambda
\]

\[
\leq \int \frac{2 f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda.
\]

Combining with (3.4),

\[
\int f^2 \left( \frac{Ag' (N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda \leq \int (\nabla N) \cdot \left( \nabla \left( \frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda
\]

\[
\leq 2 \left| \int \frac{f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda \right|
\]

\[
\leq 2 \int \frac{f}{|x|^2} \nabla |N| \nabla f e^{-g(N)} d\lambda
\]

using (3.1),

\[
\leq 2 \sqrt{C} \int \frac{|f|}{|N||x|} \nabla f e^{-g(N)} d\lambda.
\]

Let \(E = \{ (x, z) : |x| \geq \frac{1}{\sqrt{g'(N)}} \}\) and \(F = \{ (x, z) : |x| < \frac{1}{\sqrt{g'(N)}} \}\).

Applying Cauchy’s inequality with \(\epsilon : ab \leq \frac{a^2}{2} + \frac{b^2}{2\epsilon}\) with \(a = \frac{|f|}{N|x|} e^{-\frac{g(N)}{2}}\) and \(b = \sqrt{C} |\nabla f| e^{-\frac{g(N)}{2}}\), to obtain

\[
(3.5) \quad \int f^2 \left( \frac{Ag' (N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda \leq \epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda
\]

\[
= \epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda
\]

\[
\leq \epsilon \int \frac{|f|^2 e^{-g(N)} |x|^2}{N^2|x|^2} d\lambda + \epsilon \int \frac{g'(N)|f|^2}{N^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda.
\]
where (3.5) is the consequence of $E = \{(x, z) : |x| \geq \frac{1}{\sqrt{g(N)}}\}$.

The aim now is to estimate the first term on the right-hand side of (3.5). Consider $F_r = \{ |x| > \sqrt{g(N)} \}$, where $1 < r < 2$. Integrating by parts:

$$
eq \frac{\epsilon}{n-2} \int_{F_r} \nabla \left( \frac{f - g(N)}{N} \right)^2 \frac{x}{|x|^2} d\lambda + \frac{\epsilon}{n-2} \int_{\partial F_r} f^2 e^{-g(N)} \sum_{j=1}^{n} x_j < X_j I, \nabla_{\text{euc}} \left( |x| \sqrt{g(N)} \right) > dH^{n+m-1}$$

$$\leq \frac{2\epsilon}{n-2} \int_{F_r} \left( \frac{f - g(N)}{N} \right)^2 \frac{x}{|x|^2} d\lambda + \frac{2\epsilon}{(n-2)^2} \int_{F_r} \left| \frac{f - g(N)}{N} \right|^2 d\lambda$$

Integrating both sides of the inequality from $r = 1$ to $r = 2$, we get:

$$\epsilon \int_1^2 \int_{F_1} \left( \frac{f - g(N)}{N} \right)^2 \frac{x}{|x|^2} d\lambda d\lambda \leq \frac{4 \epsilon}{(n-2)^2} \int_{F_1} \left( \frac{f - g(N)}{N} \right)^2 \frac{x}{|x|^2} d\lambda$$

To recover the full measure in the boundary term, we use the Coarea formula:

$$\epsilon \int_{F_1} \left( \frac{f - g(N)}{N} \right)^2 \frac{x}{|x|^2} d\lambda \leq \frac{4 \epsilon}{(n-2)^2} \int_{F_2} \left( \frac{f - g(N)}{N} \right)^2 \frac{x}{|x|^2} d\lambda$$
\[ A = \frac{4\varepsilon}{(n-2)^2} \int_{F_2} \left| \nabla \left( \frac{f e^{-\frac{g(N)}{2}}}{N} \right) \right|^2 \, d\lambda = \frac{4\varepsilon}{(n-2)^2} \int_{F_2} \left| \nabla f \left( \frac{e^{-\frac{g(N)}{2}}}{N} \right) - \frac{f g'(N) \nabla e^{-\frac{g(N)}{2}}}{2} \right|^2 \, d\lambda \]
\[ \leq \frac{16\varepsilon}{(n-2)^2} \int_{F_2} |\nabla f|^2 e^{-g(N)} \, d\lambda + \frac{4\varepsilon}{(n-2)^2} \int_{F_2} f^2 g'(N)^2 \frac{|\nabla N|^2}{N^2} e^{-g(N)} \, d\lambda + \frac{16\varepsilon}{(n-2)^2} \int_{F_2} f^2 \frac{|\nabla N|^2}{N^4} e^{-g(N)} \, d\lambda \]

Using (3.1) and taking into consideration that \( N > 1 \) and on \( F_2, |\nabla N|^2 \leq \frac{C|x|^2}{N^2} \leq \frac{4C}{N^2 g'(N)} \),
\[ A \leq \tilde{C} \int_{F_2} |\nabla f|^2 e^{-g(N)} \, d\lambda + \tilde{B} \int_{F_2} f^2 e^{-g(N)} \, d\lambda + \frac{16\varepsilon C}{(n-2)^2} \int_{F_2} f^2 \frac{g'(N)}{N^2} e^{-g(N)} \, d\lambda \]

We do not worry about the third term in this inequality since it is dominated by \( \int f^2 \frac{g'(N)}{N^2} e^{-g(N)} \, d\lambda \) for \( N > 1 \). For the second term of (3.6),
\[ B = \frac{2\varepsilon}{n-2} \int_{\{1 < |x|/\sqrt{g(N)} \leq 2\}} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n x_j < X_j I, \nabla_{\text{euc}} \left( |x|/\sqrt{g(N)} \right) \, d\lambda. \]

For \( e_i \) the standard Euclidean basis on \( \mathbb{R}^{n+m} \),
\[ X_j I \cdot e_i = \begin{cases} 0 & \text{for } i \neq j \text{ and } i \leq n \\ 1 & \text{for } i = j \text{ and } i \leq n \\ \frac{1}{2} \sum_{l=1}^n A_{jl} x_l & \text{for } n+1 \leq i \leq n+m. \end{cases} \]

\[ \nabla_{\text{euc}} \left( |x|/\sqrt{g(N)} \right) \cdot e_i = \begin{cases} \frac{x_i \sqrt{g(N)}}{|x|} + \frac{|x|^3 g''(N)x_i}{2 \sqrt{g(N)} N^3} & \text{for } i = j \text{ and } i \leq n \\ \frac{a|x|g''(N)x_i}{2 \sqrt{g(N)} N^3} & \text{for } n+1 \leq i \leq n+m \end{cases} \]

Taking the dot product and summing,
\[ \sum_{j=1}^n x_j < X_j I, \nabla_{\text{euc}} \left( |x|/\sqrt{g(N)} \right) > = \left| x \sqrt{g'(N)} + \frac{|x|^5 g''(N)}{2 \sqrt{g'(N)} N^3} \right| + \sum_{j=1}^n x_j \sum_{i=n+1}^{n+m} \left( \frac{a|x|g''(N)}{8 \sqrt{g'(N)} N^3} \right) z_i \sum_{l=1}^n A_{jl} x_l \]
\[ = \left| x \sqrt{g'(N)} + \frac{|x|^5 g''(N)}{2 \sqrt{g'(N)} N^3} \right| + \sum_{j=1}^n \sum_{i=n+1}^{n+m} z_i A_{jl} x_l x_j \]
\[ = \left| x \sqrt{g'(N)} + \frac{|x|^5 g''(N)}{2 \sqrt{g'(N)} N^3} \right|, \]

where \( \sum_{j=1}^n \sum_{l=1}^n A_{jl} x_l x_j = 0 \) since \( A_{jl} \) is skew symmetric.
Therefore, replacing,

\[
B = \frac{2\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 e^{-g(N)}}{N^2|x|^2} \sum_{j=1}^{n} x_j < X_j I, \nabla euc (|x|\sqrt{g'(N)}) > d\lambda
\]

\[
= \frac{2\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 \sqrt{g'(N)}}{N^2|x|} e^{-g(N)} d\lambda + \frac{2\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2|x|^3 g''(N)}{2N^3 \sqrt{g'(N)}} e^{-g(N)} d\lambda.
\]

Using the fact that we are integrating over \(\{1<|x|\sqrt{g'(N)}<2\}\),

\[
B \leq \frac{2\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda + \frac{8\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 g''(N)}{N^3 g'(N)^2} e^{-g(N)} d\lambda.
\]

Using the condition of the theorem that \(g''(N) \leq g'(N)^3 N^3\), we get

\[
B \leq \frac{10\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda.
\]

Inserting bounds on \(A\) and \(B\) in (3.6), we get:

\[
e \int F_1 |f e^{-\frac{e}{2} g(N)}|^2 d\lambda \leq \tilde{C} \int F_2 |\nabla f|^2 e^{-g(N)} d\lambda + \tilde{D} \int F_2 f^2 e^{-g(N)} d\lambda
\]

\[
+ \frac{16eC}{(n-2)^2} \int F_2 \frac{f^2 g'(N)}{N^4} e^{-g(N)} d\lambda + \frac{10\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda.
\]

Using this last bound to estimate (3.5), we get:

\[
\int f^2 \left( \frac{Ag'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda \leq \tilde{C} \int |\nabla f|^2 e^{-g(N)} d\lambda + \tilde{D} \int f^2 e^{-g(N)} d\lambda + \frac{16eC}{(n-2)^2} \int \frac{f^2 g'(N)}{N^4} e^{-g(N)} d\lambda + \left( \frac{10\epsilon}{n-2} + \epsilon \right) \int \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda.
\]

On \(\{N > 1\}\), \(\int f^2 \left( \frac{B}{N^3} \right) e^{-g(N)} d\lambda\) and \(\frac{16eC}{(n-2)^2} \int \frac{f^2 g'(N)}{N^4} e^{-g(N)} d\lambda\) are of lower order. So, choosing \(\left( \frac{10\epsilon}{n-2} + \epsilon \right) \leq A\) we get

\[
\int f^2 \left( \frac{g'(N)}{N^2} \right) e^{-g(N)} d\lambda \leq C \int |\nabla f|^2 e^{-g(N)} d\lambda + D \int f^2 e^{-g(N)} d\lambda.
\]

Secondly, for \(q > 2\), replacing \(|f|\) by \(|f|^q\), we get:

\[
(3.7) \quad \int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla |f|^\frac{q}{2}|^2 d\mu + D \int |f|^q d\mu.
\]

Calculating,

\[
\int \left| \nabla |f|^\frac{q}{2} \right|^2 d\mu = \int \left| \frac{q}{2} |f|^\frac{q-2}{2} \text{sgn}(f) \nabla f \right|^2 d\mu
\]

\[
\leq \frac{q^2}{4} \int |f|^{q-2} |\nabla f|^2 d\mu.
\]

**Remark:** We note that at this point we get the inequality which implies the necessary and sufficient condition for exponential decay in \(\mathbb{L}_p\) as described in [30].
Using Hölder’s inequality,

\[
\int |\nabla f|^\frac{2}{q} \, d\mu \leq \frac{q^2}{4} \left( \int |f|^q \, d\mu \right)^\frac{2}{q^2} \left( \int |\nabla f|^q \, d\mu \right)^\frac{2}{q} \tag{3.8}
\]

\[
\leq \frac{q(q - 2)}{4} \int |f|^q \, d\mu + \frac{q}{2} \int |\nabla f|^q \, d\mu.
\]

Where the last inequality uses \( ab \leq a^{p'} + b^{q'} \), with \( a = \left( \int |f|^q \, d\mu \right)^\frac{1}{q} \), \( b = \left( \int |\nabla f|^q \, d\mu \right)^\frac{1}{q} \), and \( p' \) and \( q' \) are conjugates.

Choosing \( p' = \frac{q}{q - 2} \), we obtain \( \frac{1}{q'} = 1 - \frac{q - 2}{q} \), so \( q' = \frac{q}{2} \). Using the inequalities (3.7) and (3.8), we get,

\[
\int \frac{g'(N)}{N^2} |f|^q \, d\mu \leq C \int |\nabla f|^\frac{2}{q} \, d\mu + D \int |f|^q \, d\mu
\]

\[
\leq C' \int |\nabla f|^q \, d\mu + D' \int |f|^q \, d\mu.
\]

\[\square\]

4. Poincaré Inequality

We now have the U-Bound (2.1) at our disposal and are ready to prove the q-Poincaré inequality using the method [15]:

Let \( \lambda \) be a measure satisfying the q-Poincaré inequality for every ball \( B_R = \{ x : N(x) < R \} \), i.e. there exists a constant \( C_R \in (0, \infty) \) such that

\[
\frac{1}{|B_R|} \int_{B_R} \left| f - \frac{1}{|B_R|} \int_{B_R} f \right|^q \, d\lambda \leq C_R \left| \frac{1}{|B_R|} \int_{B_R} \nabla f \right|^q \, d\lambda,
\]

where \( 1 \leq q < \infty \).

Note that we have this Poincaré inequality on balls in the setting of Nilpotent lie groups thanks to J. Jerison’s celebrated paper [18]. With this we can use the following result:

**Theorem 3** (Hebisch, Zegarliński [15]). Let \( \mu \) be a probability measure on \( \mathbb{R}^n \) which is absolutely continuous with respect to the measure \( \lambda \) and such that

\[
\int f^q \, d\mu \leq C \int |\nabla f|^q \, d\mu + D \int f^q \, d\mu
\]

with some non-negative function \( \eta \) and some constants \( C, D \in (0, \infty) \) independent of a function \( f \). If for any \( L \in (0, \infty) \) there is a constant \( A_L \) such that \( \frac{1}{A_L} \leq \frac{d\mu}{d\lambda} \leq A_L \) on the set \( \{ \eta < L \} \) and, for some \( R \in (0, \infty) \) (depending on \( L \)), we have \( \{ \eta < L \} \subset B_R \), then \( \mu \) satisfies the q-Poincaré inequality

\[
\mu |f - \mu f|^q \leq c \mu |\nabla f|^q
\]

with some constant \( c \in (0, \infty) \) independent of \( f \).
The role of \( \eta \) in Theorem 3 is played by \( \frac{g'(N)}{N^2} \) from the U-Bound of Theorem 1. Hence, we get the following corollaries:

**Corollary 4.** The Poincaré inequality for \( q \geq 2 \) holds for the measure \( \int f^q \, d\mu = \frac{\exp\left(-\cosh\left(N \lambda^k\right)\right)}{Z} d\lambda \), where \( \lambda \) is the Lebesgue measure, and \( k \geq 1 \) in the setting of the step-two Carnot group.

**Proof.** \( g(N) = \cosh\left(N \lambda^k\right) \), so \( g'(N) = kN^{k-1} \sinh\left(N \lambda^k\right) \), and \( g''(N) = k(k-1)N^{k-2} \sinh(N \lambda^k) + k^2 N^{2k-2} \cosh(N \lambda^k) \).

First, on \( \{N \geq 1\} \), \( g''(N) \leq k^3N^{3k} \sinh^3(N \lambda^k) = g'(N)^3N^3 \), so the condition of Theorem 1 is satisfied. Second,

\[
\int \frac{g'(N)}{N^2} f^q \, d\mu = \int f^q \left[ kN^{k-3} \sinh(N \lambda^k) \right] \, d\mu \\
= \int \{N < 1\} f^q \left[ kN^{k-3} \frac{e^{N \lambda^k} - e^{-N \lambda^k}}{2} \right] \, d\mu + \int \{N \geq 1\} f^q \left[ kN^{k-3} \frac{e^{N \lambda^k} - e^{-N \lambda^k}}{2} \right] \, d\mu \\
\leq \int \{N < 1\} f^q \left[ \frac{k e^{-1}}{2} \right] \, d\mu + C' \int \{N \geq 1\} \left| f \right|^q \, d\mu + D' \int \{N \geq 1\} \left| f \right|^q \, d\mu \\
\leq C \left| \nabla f \right|^q \, d\mu + D \int \left| f \right|^q \, d\mu.
\]

Thus, the conditions of Theorem 3 are satisfied for \( \eta = kN^{k-3} \sinh(N \lambda^k) \), and \( k \geq 1 \). So, the Poincaré inequality holds for \( q \geq 2 \).

The following corollary was proven in the setting of the Heisenberg-type group for the Kaplan norm \( N = (|x|^4 + 16|z|^2)^{\frac{1}{4}} \) by J. Inglis, Theorem 4.5.5 of [16].

In the setting of the step-two Carnot group, we obtain a generalised version for a similar homogeneous norm \( N = (|x|^4 + a|z|^2)^{\frac{1}{4}} \).

**Corollary 5.** The Poincaré inequality for \( q \geq 2 \) holds for the measure \( \int f^q \, d\mu = \frac{\exp\left(-N \lambda^k\right)}{Z} d\lambda \), where \( \lambda \) is the Lebesgue measure, and \( k \geq 4 \) in the setting of the step-two Carnot group.

**Proof.** Let \( g(N) = N^k \), so \( g'(N) = kN^{k-1} \), and \( g''(N) = k(k-1)N^{k-2} \). First, on \( \{N \geq 1\} \), \( g''(N) \leq k^3N^{3k} = g'(N)^3N^3 \), so the condition of Theorem 1 is satisfied. Second,

\[
\int \frac{g'(N)}{N^2} f^q \, d\mu = \int f^q \left[ kN^{k-3} \right] \, d\mu \\
= \int \{N < 1\} f^q \left[ kN^{k-3} \right] \, d\mu + \int \{N \geq 1\} f^q \left[ kN^{k-3} \right] \, d\mu \\
\leq \int \{N < 1\} kf^q \, d\mu + C' \int \{N \geq 1\} \left| f \right|^q \, d\mu + D' \int \{N \geq 1\} \left| f \right|^q \, d\mu \\
\leq C \left| \nabla f \right|^q \, d\mu + D \int \left| f \right|^q \, d\mu.
\]
Thus, the conditions of Theorem 3 are satisfied for \( \eta = kN^{k-3} \), and \( k \geq 4 \). So, the Poincaré inequality holds for \( q \geq 2 \). \( \square \)

The following corollary improves Corollary 5 in an interesting way. Namely, at a cost of a logarithmic factor, we now get the Poincaré inequality for polynomial growth of order \( k \geq 3 \).

**Corollary 6.** The Poincaré inequality for \( q \geq 2 \) holds for the measure \( \mu = \exp\left(-\frac{N^k \log(N + 1)}{Z}\right) d\lambda \), where \( \lambda \) is the Lebesgue measure, and \( k \geq 3 \) in the setting of the step-two Carnot group.

**Proof.** Let \( g(N) = N^k \log(N + 1) \), so \( g'(N) = kN^{k-1} \log(N + 1) + \frac{N^k}{N + 1} \), and

\[
g''(N) = (k - 1)N^{k-2}\log(N + 1) + \frac{2kN^{k-1}}{N + 1} - \frac{N^k}{(N + 1)^2}.
\]

First, on \( \{ N \geq 1 \} \),

\[
g''(N) \leq k^3 N^{3k+3}(N + 1) + \frac{N^{2k+3}}{(N + 1)^3}\]

so the condition of Theorem 1 is satisfied. Second,

\[
\int \frac{g''(N)}{N^2} |f|^q d\mu = \int f^q \left[ kN^{k-3}\log(N + 1) + \frac{N^{k-2}}{N + 1} \right] d\mu
\]

\[
= \int_{\{ N < 1 \}} f^q \left[ kN^{k-3}\log(N + 1) + \frac{N^{k-2}}{N + 1} \right] d\mu + \int_{\{ N \geq 1 \}} f^q \left[ kN^{k-3}\log(N + 1) + \frac{N^{k-2}}{N + 1} \right] d\mu
\]

\[
\leq \int_{\{ N < 1 \}} (k\log(2) + 1) f^q d\mu + C' \int_{\{ N \geq 1 \}} |\nabla f|^q d\mu + D' \int_{\{ N \geq 1 \}} |f|^q d\mu
\]

\[
\leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu.
\]

Hence, the Poincaré inequality holds for \( q \geq 2 \). \( \square \)

### 5. \( \phi \)-Logarithmic Sobolev Inequality

After proving the \( q \)-Poincaré inequality for measures as a function of the homogeneous norm \( N = \langle |x|^4 + a|z|^2 \rangle ^{\frac{1}{4}} \), a natural question would be if one could obtain other coercive inequalities.

J.Inglis et al.’s Theorem 2.1 [17] proved that, for the measure \( \mu = \exp\left(-\frac{e^{-U} d\lambda}{Z}\right) \), provided we have the U-Bound

\[
\mu(|f|)|U|^\beta + |\nabla U|) \leq \mu|\nabla f| + 
\]

one obtains

\[
\mu \left( |f| \frac{|\log \frac{|f|}{\mu|f|}|^\beta}{\mu|f|} \right) \leq C \mu|\nabla f| + B\mu|f|.
\]

We will first extend their theorem, and then we will use Theorem 1 to get more general coercive inequalities.
Theorem 7. Let $U$ be a locally lipschitz function on $\mathbb{R}^N$ which is bounded below such that $Z = \int e^{-U} d\lambda < \infty$ and $d\mu = \frac{e^{-U}}{Z} d\lambda$. Let $\phi : [0, \infty) \to \mathbb{R}^+$ be a non-negative, non-decreasing, concave function such that $\phi(0) > 0$, and $\phi'(0) > 0$. Assume the following classical Sobolev inequality is satisfied:

$$\left( \int |f|^q d\lambda \right)^\frac{1}{q} \leq a \int |\nabla f|^q d\lambda + b \int |f|^q d\lambda$$

for some $a, b \in [0, \infty)$, and $\epsilon > 0$. Moreover, if for some $A, B \in [0, \infty)$, we have:

$$\mu(|f|^q(\phi(U) + |\nabla U|^q)) \leq A\mu|\nabla f|^q + B\mu|f|^q,$$

Then, there exists constants $C, D \in [0, \infty)$ such that:

$$\mu\left(|f|^q \left( \log \frac{|f|^q}{\mu|f|^q} \right) \right) \leq C\mu|\nabla f|^q + D\mu|f|^q,$$

for all locally Lipschitz functions $f$.

Proof. First of all, we remark that for a concave function $\phi$ as in our assumptions, we have

$$\phi(y) - \phi(x) \leq \phi'(0)(y - x). \tag{5.1}$$

Suppose first that $\int |f|^q = 1$, and let $E = \{ x \in \mathbb{R}^N : |\log|f|| > U \}$.

$$\int |f|^q \phi(|\log|f||) d\mu = \int_E |f|^q \phi(|\log|f||) d\mu + \int_{E^c} |f|^q \phi(|\log|f||) d\mu$$

$$= \int_E |f|^q \phi(|\log|f||) - \phi(U) d\mu + \int_E |f|^q \phi(U) d\mu + \int_{E^c} |f|^q \phi(|\log|f||) d\mu$$

$$\leq \phi'(0) \int_E |f|^q (|\log|f|| - U) d\mu + \int_E |f|^q \phi(U) d\mu + \int_{E^c} |f|^q \phi(U) d\mu,$$

where the last inequality uses (5.1) on $E$, and uses the fact that $\phi$ is non-decreasing on $E^c$, hence, $|\log|f|| < U$. Let $E_1 = \{ |f|^q > U \}$, $E_2 = \{ |f|^q < -U \}$, and $c = \int_{E_1} |f|^q e^{-U} d\lambda$.

$$\int |f|^q \phi(|\log|f||) d\mu \leq \frac{c\phi'(0)}{e} \int_{E_1} \left( \frac{|f|^q}{e^U} \right)^q \log \left( \frac{|f|^q}{e^U} \right) d\lambda + \phi'(0) \int_{E_2} e^{-U} d\mu + \int |f|^q \phi(U) d\mu$$

Using Jensen’s inequality,

$$\leq \frac{c\phi'(0)(q + \epsilon)}{Ze} \log \left( \int_{E_1} \left( \frac{|f|^q}{e^U} \right)^{q+\epsilon} d\lambda \right)^\frac{1}{q+\epsilon} + \phi'(0) \int_{E_2} 1 d\mu + \int |f|^q \phi(U) d\mu$$

$$\leq \frac{c\phi'(0)(q + \epsilon)}{Ze} \log \left( \int \left( |f|^q \right)^{q+\epsilon} d\lambda \right)^\frac{1}{q+\epsilon} + \phi'(0) Z + \int |f|^q \phi(U) d\mu$$
Using classical Sobolev inequality,

\[ \leq a \int \|f\|^q d\mu + b \int \left| \nabla (f e^{-\frac{u}{q}}) \right|^q d\lambda + \phi'(0) Z + \int |f|^q \phi(U) d\mu \]

\[ = a + \phi'(0) Z + b \int \frac{f}{q} \nabla (f) e^{-\frac{u}{q}} d\lambda + \int |f|^q \phi(U) d\mu \]

\[ \leq a + \phi'(0) Z + b Z \int |\nabla f|^q d\mu + \frac{Z b^q - 1}{q^q} \int |f|^q \nabla U d\mu + \int |f|^q \phi(U) d\mu \]

Using the U-bound in the Theorem’s condition

\[ \leq A + B \int |\nabla f|^q d\mu \]

Finally, replace \(|f|^q\) by \(\frac{|f|^q}{\mu|f|^q}\) to get the desired inequality. \(\square\)

**Corollary 8.** \(\phi(x) = (1 + x)^2\), for \(\beta \in (0, 1)\) is non-negative, non-decreasing, and concave function satisfying \(\phi(0) = 1 > 0\), and \(\phi'(0) = \beta > 0\). Therefore, Theorem 7 applies, and

\[ \mu \left( \left| f \right|^q \log \left( \frac{|f|^q}{\mu|f|^q} \right)^\beta \right) \leq \mu \left( \left| f \right|^q \left( 1 + \log \left( \frac{|f|^q}{\mu|f|^q} \right) \right)^\beta \right) \leq C \mu|\nabla f|^q + D \mu|f|^q. \]

**Corollary 9.** Let \(h^{(1)}(x) = \log(\alpha + x)\), where \(\alpha > 1\). Define recursively

\(h^{(n)}(x) = \log(\alpha + h^{(n-1)}(x))\). Then, for all \(n \geq 1\), \(h^{(n)}(x) = \phi(x)\) of Theorem 7. Therefore, we obtain

\[ \mu \left( \left| f \right|^q \log^{*(n)} \left( \frac{|f|^q}{\mu|f|^q} \right) \right) \leq \mu \left( \left| f \right|^q h^{(n)} \left( \frac{|f|^q}{\mu|f|^q} \right) \right) \leq C \mu|\nabla f|^q + D \mu|f|^q, \]

where \(\log^{*(n)}\) is the positive part of \(\log^{(n)}\).

**Proof.** The proof proceeds by induction. For \(n = 1\), \(h^{(1)}(x) = \log(\alpha + x)\), so \(h^{(1)}(x)' = \frac{1}{\alpha + x}\), and \(h^{(1)}(x)'' = \frac{-1}{(\alpha + x)^2}\). \(h^{(1)}(0) = \log(\alpha) > 0\), and \(h^{(1)}(0)' = \frac{1}{\alpha} > 0\). In addition, \(h^{(1)}(x)\) is non-negative, non-decreasing, and concave; hence the conditions of Theorem 7 are satisfied.

Assume it is true for \(n = k\), prove it is true for \(n = k + 1\) : \(h^{(k+1)}(x) = \log(\alpha + h^{(k)}(x))\), so

\(h^{(k+1)}(x)' = \frac{h^{(k)}(x)'}{\alpha + h^{(k)}(x)}\), and \(h^{(k+1)}(x)'' = \frac{h^{(k)}(x)''}{\alpha + h^{(k)}(x)} - \frac{h^{(k)}(x)'}{(\alpha + h^{(k)}(x))^2}\). The result follows directly. \(\square\)

Returning to the measure as a function of the homogeneous norm \(N = (|x|^4 + a|z|)^\frac{1}{2}\), \(d\mu = e^{-\frac{Np}{Z}} d\lambda\), we will prove using Theorems 1 and 7, that the \(L^p\)-Sobolev inequality \((0 < \beta \leq 1)\) (Corollary 8) holds for \(q \geq 2\), yet fails for \(1 < q < \frac{2p\beta}{p - 1}\). To start with, we will show why the \(L^p\)-Sobolev inequality fails for \(1 < q < \frac{2p\beta}{p - 1}\). The proof uses the idea of Theorem 6.3 of [15].
Theorem 10. Let $G$ be a stratified group, and $N$ be a smooth homogenous norm on $G$. For $\alpha > 0$, $p \geq 1$, let $d\mu = \frac{e^{-\alpha N^p}}{Z}d\lambda$, where $Z$ is the normalization constant. The measure $\mu$ satisfies no Log$^\beta$-Sobolev inequality ($0 < \beta \leq 1$) for $1 < q < \frac{2p\beta}{p-1}$.

Proof. The proof is by contradiction. Let $x_0$ be such that $(\nabla N)(x_0) = 0$. For $t > 0$ put $r = t^{\frac{p+1}{p-1}}$, and

$$f = \max \left[ \min \left( \frac{2 - d(x,tx_0)}{r}, 1 \right), 0 \right].$$

On $B(tx_0, 2r) = \{ x : d(x,tx_0) \leq 2r \}$, by homogeneity, by Lemma 6.3 of [15], and by the fact that $(\nabla N)(x_0) = 0$, we have $|N(x) - N(tx_0)| \leq c_1 r^2$, so $|N(x)^p - N(tx_0)^p| \leq c_2$. Consequently, the exponential factor in $\mu$ is comparable to a constant on the support of $f$. Also, $|\nabla f| \leq \frac{1}{t}$, and

\begin{align*}
\mu[|f|^q] &\approx r^q \left( e^{-\alpha N^p(tx_0)} \right) \\
log(\mu[|f|^q]) &\approx -t^p \\
\mu[|\nabla f|^q] &\approx r^{-q} r^q e^{-\alpha N^p(tx_0)}
\end{align*}

Choose $t$ large enough so that $r < \frac{2}{t}$. On $B(tx_0, 2r)$, $2 - d(x,tx_0) \geq 2 - 2r \geq r$. Thus, we have $f = \max \left[ \min \left( \frac{2 - d(x,tx_0)}{r}, 1 \right), 0 \right] = 1$, and consequently $log|f|^q = 0$.

$$\mu \left( \left| f \right|^q \left| \log \left( \frac{|f|^q}{\mu[|f|^q]} \right) \right|^{\beta} \right) = \mu \left( \left| f \right|^q \left( |log|f|^q - \log \mu[|f|^q] \right) \right) = \mu \left( \left| f \right|^q \left( |log\mu[|f|^q] \right) \right)$$

by (5.3),

$$\approx \mu[|f|^q t^{p\beta}]$$

by (5.2)

$$\approx t^{p\beta} r^q e^{-\alpha N^p(tx_0)}.$$

Assuming we have $\beta$–logarithmic Sobolev inequality, and using (5.4), we get:

$$t^{p\beta} r^q e^{-\alpha N^p(tx_0)} \leq M r^{-q} r^q e^{-\alpha N^p(tx_0)}$$

since $r = t^{\frac{p+1}{p-1}}$,

$$t^{p\beta} \leq M t^{-q} \left( \frac{p+1}{p-1} \right).$$

For $t$ large enough, we get a contradiction when $p\beta > \frac{q(p-1)}{2}$ i.e. for $q < \frac{2p\beta}{p-1}$. So, the measure $\mu$ satisfies no $\beta$–logarithmic Sobolev inequality for $1 < q < \frac{2p\beta}{p-1}$. \hfill \Box

Now we prove that for $q \geq 2$, Log$^\beta$-Sobolev inequality holds true for $d\mu = \frac{e^{-\alpha N^p}}{Z}d\lambda$, where $N = (|x|^4 + a|x|^2)^\frac{1}{2}$ and $0 < \beta \leq \frac{p-3}{p}$. 
Theorem 11. Let $\mathcal{G}$ be an step-two Carnot group. Consider the probability measure given by

$$d\mu = \frac{e^{-g(N)}}{Z} d\lambda,$$

where $Z$ is the normalization constant and $N = (|x|^4 + a|x|^2)^{\frac{1}{4}}$ with $a \in (0, \infty)$. Let $g : [0, \infty) \to [0, \infty)$ be a differentiable increasing function such that $g'(N)$ is increasing, $g(N) \leq \left(\frac{g'(N)}{N^2}\right)^{\frac{1}{2}}$, and $g''(N) < dg'(N)^2$ on $\{N \geq 1\}$, for some constants $c, d \in (0, \infty)$. Then

$$\mu \left(|f|^q \left\|\log \left(\frac{|f|^q}{\|f\|^q}\right)\right\|^2\right) \leq C \mu|\nabla f|^q + D \mu|f|^q,$$

for $C$ and $D$ positive constants and for $q \geq 2$.

Proof. On $\{N \geq 1\}$, $g''(N) < dg'(N)^2 \leq g'(N)^3 N^3$, so the condition of Theorem 1 is satisfied. Thus, on $\{N \geq 1\}$, we have the U-bound (2.1):

$$\mu \left(\frac{g'(N)}{N^2} |f|^q \right) \leq C \mu|\nabla f|^q + D \mu|f|^q.$$

By the condition $g(N) \leq \left(\frac{g'(N)}{N^2}\right)^{\frac{1}{2}}$, we obtain $\phi(g(N)) = (1 + g(N))^2 \leq \frac{g'(N)}{N^2}$ on $\{N \geq 1\}$. Hence, since $g(N)$ is increasing and using the U-bound, we have

$$\mu (\phi(g(N)) |f|^q) \leq \int_{\{N \geq 1\}} \left(\frac{g'(N)}{N^2} |f|^q\right) d\mu + \int_{\{N < 1\}} \phi(g(N)) |f|^q d\mu$$

$$\leq \int_{\{N \geq 1\}} \left(\frac{g'(N)}{N^2} |f|^q\right) d\mu + \int_{\{N < 1\}} (1 + g(1))^2 |f|^q d\mu$$

$$\leq C \mu|\nabla f|^q + D \mu|f|^q.$$

In order to use Theorem 7, it remains to prove:

$$\mu(|f|^q |\nabla g(N)|^q) \leq C \mu|f|^q + D \mu|\nabla f|^q.$$

On $\{N < 1\}$, since $g'(N)$ is increasing and using (3.1),

$$\int_{\{N < 1\}} (|f|^q |\nabla g(N)|^q) d\mu = \int_{\{N < 1\}} (|f|^q g'(N) |\nabla N|^q) d\mu \leq C^2 \int_{\{N < 1\}} |f|^q |g'(1)|^q d\mu.$$

We now need to consider $\{N \geq 1\}$:

$$\int |f|^q |\nabla g(N) \cdot V - \nabla \cdot V| d\mu = \int \nabla |f|^q \cdot V d\mu \leq \frac{\epsilon}{p} \int |f|^q |V|^p d\mu + \frac{1}{\epsilon} q^{q-1} \int |\nabla f|^q d\mu,$$

where the last inequality uses $ab \leq \frac{\epsilon a^p}{p} + \frac{b q^{q-1}}{q}$, where $a = |f|^q |V|$, and $b = q |\nabla f|$. Let $V = \nabla N |x|^{-2} g'(N)^q - 1$. Since $\nabla g(N) = g'(N) \nabla N$, then $\nabla g(N) \cdot V = |\nabla g(N)|^q$, which is the term on the left hand side of
(5.6). Using the inequality (3.1) on the first term on the right hand side of (5.7) we get
\[ \frac{\epsilon}{p} \int |f|^q |V|^p d\mu = \frac{\epsilon}{p} \int |\nabla N|^p |f|^q |x|^{(q-2)p} g'(N)^q \frac{N_{q-2}}{N_{q-2}p} d\mu \]
\[ \leq \frac{\epsilon C^2}{p} \int |f|^q |x|^{q} g'(N)^q \frac{N_{q}}{N_{q}} d\mu \]
which can subtracted from the left hand side of (5.7) since by choosing \( \epsilon \) small enough and noting that using (3.1), one has
\[ \nabla g(N) \cdot V = |\nabla N|^2 |x|^{q-2} g'(N)^q \geq A \frac{|x|^q}{N_q} g'(N)^q. \]

It remains to compute \( \nabla \cdot V \). Using \( |\Delta N| \leq B \frac{|x|^2}{N_q} \), (3.2), and \( \frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N_q} \), (3.3), we have
\[ \nabla \cdot V = \Delta N \frac{|x|^{q-2}}{N_q} g'(N)^{q-1} + (q-2) \frac{|x|^q g'(N)^{q-1}}{N_{q+1}} - (q-2) \frac{|\nabla N|^2 |x|^{q-2} g'(N)^{q-1}}{N_{q-2}} \]
\[ + (q-1) g'(N)^{q-2} g''(N) \frac{|x|^{q-2} |\nabla N|^2}{N_{q-2}} \]
and hence
\[ |\nabla V| \leq B \frac{|x|^q}{N_{q+1}} g'(N)^{q-1} + (q-2) \frac{|x|^q g'(N)^{q-1}}{N_{q+1}} + (q-2) \frac{C|\nabla N|^2 |x|^q g'(N)^{q-1}}{N_{q+1}} + C(q-1) g'(N)^{q-2} g''(N) \frac{|x|^q}{N_q}. \]

All terms can be absorbed by the first term in (5.7). Using (5.5) and (5.6), the condition of Theorem 7 is satisfied, and we obtain \( \log \alpha \)-Sobolev inequality:
\[ \mu \left( |f|^q \log \left( \frac{|f|^q}{\mu|f|^q} \right) \right) \leq C \mu|f|^q + D \mu|\nabla f|^q \]
for \( C \) and \( D \) positive constants. \( \square \)

**Corollary 12.** Let \( G \) be a step-two Carnot group and \( N = \left( |x|^4 + a|x|^2 \right)^{\frac{1}{2}} \) with \( a \in (0, \infty) \). Let the probability measure be \( d\mu = e^{-\beta N} \frac{Z}{Z} d\lambda \), where \( Z \) is the normalization constant. Then, for \( p \geq 4 \) and \( 0 < \beta \leq \frac{p-3}{p} \),
\[ \mu \left( |f|^q \log \left( \frac{|f|^q}{\mu|f|^q} \right) \right) \leq C \mu|f|^q + D \mu|\nabla f|^q, \]
for \( C \) and \( D \) positive constants and for \( q \geq 2 \).

6. **Appendix: Proof of Lemma 2**

**Proof.** We first compute \( \nabla N = \left( X_i N \right)_{i=1,...,n} \)
\[ X_i N = N^{-3} \left( |x|^2 x_i + \frac{a}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda^{(k)}_{il} x_{i} x_{l} \right). \]
Therefore,

\[(6.1)\]
\[
|\nabla N|^2 = N^{-6} \left( |x|^6 + \frac{a_2}{2} \sum_{i=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} |x|^2 x_i x_l z_k + \sum_{i=1}^{n} \frac{a_1^2}{16} \sum_{k,k'=1}^{m} \sum_{l,l'=1}^{n} \Lambda_{kl}^{(k)} \Lambda_{l'l'}^{(k')} x_i x_l z_k z_{k'} \right)
\]

= \[
N^{-6} \left( |x|^6 + \frac{a_2^2}{16} \sum_{i=1}^{m} \sum_{k,k'=1}^{m} \sum_{l,l'=1}^{n} \Lambda_{kl}^{(k)} \Lambda_{l'l'}^{(k')} x_i x_l z_k z_{k'} \right)
\]

= \[
\frac{|x|^2}{N^2} N^{-4} \left( |x|^4 + \frac{a^2}{16} \sum_{i=1}^{n} \sum_{k,k'=1}^{m} \sum_{l,l'=1}^{n} \Lambda_{kl}^{(k)} \Lambda_{l'l'}^{(k')} x_i x_l z_k |x| ) \right),
\]

where we used that for each skew-symmetric matrix \(\Lambda^{(k)}\), all \(k \in \{1, ..., m\}\), we have that

\[
\sum_{i=1}^{n} \sum_{k=1}^{m} \Lambda_{kl}^{(k)} x_l x_i = 0.
\]

From (6.1), that with some constants \(A, C \in (0, \infty)\), we have

\[
A \frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C \frac{|x|^2}{N^2}.
\]

By choosing \(a \in (0, \infty)\) sufficiently small, we can ensure that \(C \leq 1\). We note that using antisymmetry of matrices \(\Lambda_{kl}^{(k)}\) we get

\[
\frac{x}{|x|} \cdot \nabla N = \sum_{i=1}^{n} \frac{x_i}{|x|} N^{-3} \left( |x|^2 x_i + \frac{a}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} x_l z_k \right)
\]

Next we compute

\[
X_i^2 N = \left( \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} x_l \frac{\partial}{\partial z_k} \right) \left( N^{-3} \left( |x|^2 x_i + \frac{a}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} x_l z_k \right) \right)
\]

= \[
-3 \left( N^{-7} \left( |x|^2 x_i + \frac{a}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} x_l z_k \right) \right)
\]

+ \[
\left( N^{-3} \left( |x|^2 + 2x_i^2 + \frac{a}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} \delta_{kl} z_k \right) \right)
\]

+ \[
\left( N^{-3} \left( \frac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} x_l z_k \right) \right)
\]

Hence we obtain

\[
\Delta N = \sum_{i=1}^{n} X_i^2 N
\]

= \[
-3 \left( N^{-7} \left( |x|^6 + \frac{2a}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} |x|^2 x_l x_i z_k \right) \right)
\]

+ \[
\frac{a^2}{16} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{k'=1}^{m} \sum_{l'=1}^{n} \Lambda_{kl}^{(k')} \Lambda_{l'l'}^{(k')} x_l x_l z_k z_{k'} \right)
\]

+ \[
\left( N^{-3} \left( (n+2)|x|^2 + \frac{a}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} \delta_{kl} z_k \right) \right)
\]

+ \[
\left( N^{-3} \left( \frac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} \Lambda_{kl}^{(k)} x_l z_k \right) \right)
\].
which after simplifications yields
\[
\Delta N = \sum_{i=1}^{n} X_i^2 N
\]
\[
= -3 \left( N^{-7} \left( |x|^6 + \frac{a^2}{16} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{k'=1}^{m} \sum_{l'=1}^{n} \Lambda_{ll'}^{(k)} \Lambda_{l'l}^{(k')} x_l x_{l'} x_k x_{k'} \right) \right) + (n-2) \left( N^{-3} \left( (n+2)|x|^2 \right) \right) + \left( N^{-3} \left( \frac{a}{8} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{k'=1}^{m} \sum_{l'=1}^{n} \Lambda_{ll'}^{(k)} \Lambda_{l'l}^{(k')} x_l x_{l'} \right) \right).
\]
Thus we get
\[
\Delta N = \frac{|x|^2}{N^3} \left[ -3 \left( N^{-4} \left( |x|^4 + \frac{a^2}{16} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{k'=1}^{m} \sum_{l'=1}^{n} \Lambda_{ll'}^{(k)} \Lambda_{l'l}^{(k')} x_l x_{l'} |x|^2 z_k z_{k'} \right) \right) + (n+2) \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{k'=1}^{m} \sum_{l'=1}^{n} \Lambda_{ll'}^{(k)} \Lambda_{l'l}^{(k')} \right].
\]
which can be represented as follows
\[
\Delta N = (n-1) \frac{|x|^2}{N^3} + \frac{|x|^2}{N^3} \left[ -3 \left( N^{-4} \left( -a|x|^2 + \frac{a^2}{16} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{k'=1}^{m} \sum_{l'=1}^{n} \Lambda_{ll'}^{(k)} \Lambda_{l'l}^{(k')} x_l x_{l'} |x|^2 z_k z_{k'} \right) \right) \right] + \frac{a}{8} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{k'=1}^{m} \sum_{l'=1}^{n} \Lambda_{ll'}^{(k)} \Lambda_{l'l}^{(k')} \frac{x_l x_{l'}}{|x||x|}.
\]
Hence, there exists a constant \( B \in (0, \infty) \) such that
\[
|\Delta N| \leq B \frac{|x|^2}{N^3}
\]
**Remark:** If \( a > 0 \) is small, \( \Delta N \geq 0 \). For large \( a \), in some directions \( \Delta N \) can be negative. \( \square \)

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Esther Bou Dagher:
Department of Mathematics
Imperial College London
180 Queen’s Gate, London SW7 2AZ
United Kingdom
Email address: esther.bou-dagher17@imperial.ac.uk

Bogusław Zegarliński:
Department of Mathematics
Imperial College London
180 Queen’s Gate, London SW7 2AZ
United Kingdom
Email address: b.zegarlinski@imperial.ac.uk