ASYMPTOTIC AND EXACT RESULTS ON THE COMPLEXITY OF THE NOVELLI–PAK–STOYANOVSKII ALGORITHM

CARSTEN SCHNEIDER AND ROBIN SULZGRUBER

Abstract. The Novelli–Pak–Stoyanovskii algorithm is a sorting algorithm for Young tableaux of a fixed shape that was originally devised to give a bijective proof of the hook-length formula. We obtain new asymptotic results on the average case and worst case complexity of this algorithm as the underlying shape tends to a fixed limit curve. Furthermore, using the summation package Sigma we prove an exact formula for the average case complexity when the underlying shape consists of only two rows. We thereby answer questions posed by Krattenthaler and Müller.

The Novelli–Pak–Stoyanovskii algorithm (NPS algorithm) transforms (sorts) an arbitrary filling of a Young diagram \( \lambda \) of size \( n \) with the numbers \( \{1, 2, \ldots, n\} \) into a standard Young tableau of the shape \( \lambda \). From a sorters point of view, the algorithm is best described as a “two-dimensional insertion sort”. Following certain rules, at each step two entries of adjacent cells are compared and possibly exchanged. The remarkable property of the NPS algorithm is that, when applied to all possible fillings of a fixed diagram, it produces every standard Young tableau of this shape equally often as an output. Thus it provides a uniformly distributed random sampler for standard Young tableaux of a given shape. The algorithm was originally defined by Novelli, Pak and Stoyanovskii in \[ \text{NPS97, PS92} \] to give a bijective proof of the hook-length formula due to Frame, Robinson and Thrall \[ \text{FRT54} \].

The study of the complexity of the NPS algorithm on a partition \( \lambda \) was initiated by Krattenthaler and Müller. They define the average case complexity \( C(\lambda) \) and the worst case complexity \( W(\lambda) \) as the average, respectively the maximal number of exchanges performed by the NPS algorithm applied to an arbitrary filling of shape \( \lambda \). Their analysis lead to multiple interesting conjectures that are the main motivation for the present paper. Three of these conjectures are listed below. The first two conjectures concern the asymptotic behaviour of \( C(\lambda(n)) \) and \( W(\lambda(n)) \), where \( (\lambda(n))_{n \in \mathbb{N}} \) is a sequence of partitions that approach a fixed limit curve \( \gamma \) after a suitable rescaling. (This is made precise in Section 2.)

**Conjecture 1.** The order of magnitude of \( C(\lambda(n)) \) lies between \( |\lambda(n)|^{3/2} \) and \( |\lambda(n)|^2 \) where \( |\lambda(n)| \) denotes the size of \( \lambda(n) \).

**Conjecture 2.** The average case complexity \( C(\lambda(n)) \) is asymptotically one half of the worst case complexity \( W(\lambda(n)) \).

The third conjecture that we are interested in is an exact formula for the average case complexity when the partition \( \lambda = (\lambda_1, \lambda_2) \) consists of two parts.

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Conjecture 3. The average case complexity of the NPS algorithm on \( \lambda = (\lambda_1, \lambda_2) \) is given by

\[
C(\lambda) = \frac{\lambda_1 (\lambda_1 - 1)}{4} + \frac{\lambda_2 (\lambda_2 - 3)}{4} - 2 \sum_{k=1}^{\lambda_2} \binom{\lambda_2}{k} \frac{(-1)^k (2k - 2)!}{(\lambda_1 - \lambda_2 + 2)_{2k-1}}.
\]

In this paper we prove Conjectures 1 and 3. While we were unable to prove Conjecture 2 in full generality, we do provide a proof for a large class of sequences of partitions, adding further evidence to its validity. The article is structured as follows:

In Section 1 we review the combinatorics of the NPS algorithm. In Sections 2–4 we engage sequences of partitions that converge under a balanced scaling, that is, by a factor of \( \sqrt{n} \) in both dimensions. Section 2 contains mostly preparatory results and includes a precise definition of convergence. Section 3 treats the worst case complexity. We derive an exact combinatorial formula for \( W(\lambda) \) for any partition \( \lambda \) in Theorem 3.1 by proving that a trivial upper bound is tight. Moreover Theorem 3.2 provides an asymptotic result on \( W(\lambda(n)) \) in the balanced case. Section 4 treats the average case complexity in the balanced case. Here we give an asymptotic lower bound for \( C(\lambda(n)) \) in Theorem 4.3. It is a consequence of our results that both \( C(\lambda(n)) \) and \( W(\lambda(n)) \) are of order \( |\lambda(n)|^{3/2} \), which is in accordance with Conjecture 1.

In Section 5 we turn to sequences that converge when subjected to an imbalanced scaling, that is, by a factor of \( n^{1/p} \) in one direction and a factor of \( n^{1/q} \) in another. Theorem 5.1 verifies both Conjectures 1 and 2 in the imbalanced setting. More precisely, we show that \( C(\lambda(n)) \) and \( W(\lambda(n)) \) are both of order \( n^{(p+1)/p} \) if \( p < q \) and of order \( n^{(q+1)/q} \) if \( p > q \), and that the leading coefficient of the average case complexity is one half of the leading coefficient of the worst case complexity.

In Section 6 we prove Conjecture 3 employing the summation package Sigma \( \text{Sch07} \) in a non-trivial fashion. Here we first provide an alternative representation \( C(\lambda) \) in terms of five non-trivial double sums and show that this expression corresponds to the single sum expression given in Conjecture 3. The underlying machinery is based on the summation paradigms of creative telescoping, recurrence solving and the zero-recognition problem for the class of (indefinite) nested sums over hypergeometric products. As a by-product we provide alternative representations of \( C(\lambda) = C(\lambda_1, \lambda_2) \) that enable one to calculate \( C(\lambda) \) efficiently if one keeps \( \lambda_1 \) symbolic and specializes \( \lambda_2 \) to a concrete value, or if one keeps \( \lambda_1 \) symbolic and specializes the distance \( \lambda_1 - \lambda_2 \) to a specific non-negative integers. In particular, we discover a particularly nice formula for the special case \( \lambda_1 = \lambda_2 \).

1. The NPS algorithm

\begin{figure}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
12 & 7 & 5 & 1 \\
\hline
2 & 10 & 9 & 11 \\
\hline
13 & 4 & 8 & 6 \\
\hline
3 & 8 & 6 & 3 \\
\hline
\end{tabular}
\caption{The partition \( \lambda = (4, 4, 2, 1, 1, 1) \) in English convention, and a tableau of shape \( \lambda \).}
\end{figure}

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1Here \((x)_k\) stands for the Pochhammer symbol which is 1 for \( k = 0 \) and which equals \( x(x+1) \ldots (x+k-1) \) for positive integers \( k \).
In this section we recall some definitions concerning partitions and Young tableaux as well as the needed facts about the NPS algorithm.

Let \( n \in \mathbb{N} \) be a non-negative integer. A partition \( \lambda \) of \( n \) is a weakly decreasing sequence \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \) of positive integers such that \( |\lambda| = \sum \lambda_i = n \). We call \( |\lambda| \) the size of \( \lambda \). The length \( l(\lambda) \) is the number of summands \( \lambda_i \). We identify a partition with its Young diagram \( \lambda = \{(i, j) : 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\} \). The elements \((i, j)\) are called the cells of the partition \( \lambda \).

The conjugate \( \lambda' \) of \( \lambda \) is given by the Young diagram \( \{(j, i) : (i, j) \in \lambda\} \). Visually we imagine a partition as a left justified array of \( n \) boxes, with \( \lambda_i \) boxes in the \( i \)-th row counting from top to bottom as in Figure 1. Thus \( \lambda' \) is obtained from \( \lambda \) by a flip along the main diagonal.

Define the arm of a cell \( \text{arm}_\lambda(i, j) = \lambda_i - j \) as the number of cells in the same column as \((i, j)\) and strictly to the right of \((i, j)\). Define the leg of a cell \( \text{leg}_\lambda(i, j) = \lambda'_j - i \) as the number of cells in the same column as \((i, j)\) and strictly below \((i, j)\). Furthermore, define the hook length of a cell as \( h_\lambda(i, j) = \lambda_i + \lambda'_j - i - j + 1 \). We call a cell \((i, j)\) a corner of \( \lambda \) if \( h_\lambda(i, j) = 1 \).

An integer filling of a partition is a map \( T : \lambda \to \mathbb{Z} \) assigning an entry \( T(i, j) \) to each cell \((i, j)\). The partition \( \lambda \) is called the shape of \( T \). A tableau is a bijection \( T : \lambda \to \{1, 2, \ldots, n\} \). A standard Young tableau (SYT) is a tableau that increases along rows from left to right as well as down columns, that is, \( T(i, j) \leq T(i', j') \) whenever \( i \leq i' \) and \( j \leq j' \). A hook tableau is a map \( H : \lambda \to \mathbb{Z} \) such that for each cell \((i, j)\) we have \( -\text{leg}_\lambda(i, j) \leq H(i, j) \leq \text{arm}_\lambda(i, j) \). Note that somewhat counter-intuitively a hook tableau is not a tableau. We denote the set of all tableaux, standard Young tableaux and hook tableaux of shape \( \lambda \) by \( \text{SYT}(\lambda) \), \( \text{SYT}(\lambda) \) and \( H(\lambda) \), respectively.

The celebrated hook-length formula \([\text{FRT}54]\) determines the number of standard Young tableaux of a fixed shape:

\[
\# \text{SYT}(\lambda) = \prod_{(i, j) \in \lambda} h_\lambda(i, j).
\]

Since \( \# T(\lambda) = n! \) and \( \# H(\lambda) = \prod_{(i, j) \in \lambda} h_\lambda(i, j) \) it is possible to prove \([11]\) bijectively by constructing a bijection

\[
\Phi : T(\lambda) \to \text{SYT}(\lambda) \times H(\lambda).
\]

Such a construction was found by Novelli, Pak and Stoyanovskii \([\text{PS92}, \text{NPS97}]\). We are now going to describe the map \( \Phi \). See Figure 2 for an example.

First note that given a permutation \( \sigma \in \mathfrak{S}_n \) and a tableau \( T \in T(\lambda) \), we obtain a new tableau \( T' = \sigma \circ T \) by setting \( T'(i, j) = \sigma(T(i, j)) \). In particular if \( \sigma = (k, m) \) is a transposition, then \((k, m) \circ T \) arises from \( T \) by exchanging the two entries \( k \) and \( m \).

Impose the reverse lexicographic order \( \preceq \) on the cells of \( \lambda \), that is, \((i, j) \preceq (i', j')\) if \( j < j' \) or if \( j = j' \) and \( i \leq i' \). Let \((i_1, j_1) \succ (i_2, j_2) \succ \cdots \succ (i_n, j_n)\) be the cells of \( \lambda \) in decreasing order and set \( k_r = T(i_r, j_r) \). Moreover, set \( T_0 = T \) and let \( H_0 \) be the hook tableau with all entries equal to zero.

Given a pair tableau \((T_{r-1}, H_{r-1})\) we first construct the tableau \( T_r \) from \( T_{r-1} \) as follows:

**E0:** Set \( T = T_{r-1} \).

**E1:** Set \((i, j) = T^{-1}(k_r)\).

**E2:** If \((i, j)\) is a corner of \( \lambda \), then return \( T_r = T \).

**E3:** Otherwise set \( m = \min T^{-1}(\lambda \cap \{(i + 1, j), (i, j + 1)\}) \).

**E4:** If \( k_r < m \) then return \( T_r = T \).

**E5:** Otherwise \( m < k_r \). In this case exchange the entries \( m \) and \( k_r \), that is, set \( T = (k_r, m) \circ T \), and return to step **E1**.

Note that \( T_r \) is obtained from \( T_{r-1} \) by applying a jeu-de-taquin-like move to the entry \( k_r \). Next we construct \( H_r \) from \( H_{r-1} \) as follows:
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{nps_algorithm.png}
\caption{The NPS algorithm in action.}
\end{figure}

\begin{enumerate}
\item[H0:] Set $H = H_{r-1}$, $(i, j) = T_{r-1}^{-1}(k_r)$ and $(i', j') = T_{r}^{-1}(k_r)$.
\item[H1:] For each $s = i, \ldots, i' - 1$ set $H(s, j) = H(s + 1, j) - 1$.
\item[H2:] Set $H(i', j) = j' - j$.
\item[H3:] Return $H_r = H$.
\end{enumerate}

These rules give rise to a sequence $(T_0, H_0), (T_1, H_1), \ldots, (T_n, H_n)$ of pairs of a tableau and a hook tableau. We define $\Phi(T) = (T_n, H_n)$. While it is not too difficult to show that $T_n$ is a standard Young tableau and that $H_n$ is a hook tableau, it takes considerably more effort to prove that $\Phi$ is a bijection. For details we refer to [Kra99, NPS97, Sag01].

Given a tableau $T$ we denote by $n(T)$ the number of exchanges performed during the application of the NPS algorithm, that is, the number of times step E5 is visited during the construction of all tableaux $T_1, \ldots, T_n$. The average case complexity of the NPS algorithm is now defined as

$$C(\lambda) = \frac{1}{n!} \sum_{T \in T(\lambda)} n(T).$$

The worst case complexity is defined as

$$W(\lambda) = \max_{T \in T(\lambda)} n(T).$$

Let $\lambda$ and $\mu$ be two partitions. We write $\mu \subseteq \lambda$ if the Young diagram of $\mu$ is contained in the Young diagram of $\lambda$. In this case the skew diagram $\lambda/\mu$ consists of the cells of $\lambda$ that are not cells of $\mu$. A tableau of skew shape $\lambda/\mu$ is simply a bijection $T : \lambda/\mu \to \{1, \ldots, \abs{\lambda} - \abs{\mu}\}$. A standard Young tableau of skew shape $\lambda/\mu$ is a tableau of this shape that increases along rows and down columns. In the proof of Theorem 3.1 we apply the NPS algorithm to a tableau of skew shape. We remark that this poses no difficulties. The output will even be a pair of a standard Young

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**Figure 2.** The NPS algorithm in action.

- **H0:** Set $H = H_{r-1}$, $(i, j) = T_{r-1}^{-1}(k_r)$ and $(i', j') = T_{r}^{-1}(k_r)$.
- **H1:** For each $s = i, \ldots, i' - 1$ set $H(s, j) = H(s + 1, j) - 1$.
- **H2:** Set $H(i', j) = j' - j$.
- **H3:** Return $H_r = H$. 

These rules give rise to a sequence $(T_0, H_0), (T_1, H_1), \ldots, (T_n, H_n)$ of pairs of a tableau and a hook tableau. We define $\Phi(T) = (T_n, H_n)$. While it is not too difficult to show that $T_n$ is a standard Young tableau and that $H_n$ is a hook tableau, it takes considerably more effort to prove that $\Phi$ is a bijection. For details we refer to [Kra99, NPS97, Sag01].

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tableau and a hook tableau of the same skew shape. The bijectivity of the NPS algorithm is, however, lost.

2. CONVERGENCE AND HOOK COORDINATES

We shall work with the following coordinate system corresponding to a rotation by $\pi/2$ and a rescaling by $\sqrt{n}$

$$u = u(x, y) = \frac{\sqrt{n}}{\sqrt{2}}(x + y), \quad v = v(x, y) = \frac{\sqrt{n}}{\sqrt{2}}(y - x), \quad \left| \frac{\partial(u, v)}{\partial(x, y)} \right| = n. \quad (2.1)$$

Given a partition $\lambda$ of $n$ define

$$D_\lambda = \{(x, y) \in \mathbb{R}^2 : 0 < v, 0 < u \leq \lambda[u] \text{ or } 0 \leq u, 0 \leq v, uv = 0 \}. \quad (2.2)$$

We define the boundary function $\gamma : \mathbb{R} \to \mathbb{R}$ of $\lambda$ via

$$\gamma(x) = \sup\{y \in \mathbb{R} : (x, y) \in D_\lambda \}. \quad (2.3)$$

The set $D_\lambda$ and function $\gamma$ describe the (rescaled) partition $\lambda$ in the so-called Russian convention.

Let $(\lambda^{(n)})_{n \in \mathbb{N}}$ be a sequence of partitions, such that $\lambda^{(n)}$ is a partition of $n$ and has the boundary function $\gamma_n$. Denote by $\Gamma$ the set of all 1-Lipschitz functions $\gamma : \mathbb{R} \to \mathbb{R}$ such that there exists an interval $(a, b)$ with $\gamma(x) = |x|$ for all $x /\in (a, b)$. Moreover, denote by $\Gamma_1$ the set of all functions $\gamma \in \Gamma$ such that

$$\int_{\mathbb{R}} \gamma(x) - |x| \, dx = 1.$$ 

Clearly $\gamma_n \in \Gamma_1$ for all $n \in \mathbb{N}$. We say the sequence $\lambda^{(n)}$ converges uniformly to a limit curve $\gamma \in \Gamma_1$ if

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |\gamma(x) - \gamma_n(x)| = 0$$

and there exists a uniform interval $(a, b)$ such that $\gamma_n(x) = |x|$ for all $x /\in (a, b)$ and all $n \in \mathbb{N}$.

For any $\gamma \in \Gamma$ set

$$D_\gamma = \{(x, y) \in \mathbb{R}^2 : |x| \leq y \leq \gamma(x) \},$$

and let $(x, y)$ be an interior point of $D_\gamma$. We define three functions via

$$a_\gamma(x, y) = \sqrt{2} \sup\{t \in \mathbb{R} : (x + t, y + t) \in D_\gamma \},$$

$$t_\gamma(x, y) = \sqrt{2} \sup\{t \in \mathbb{R} : (x - t, y + t) \in D_\gamma \},$$

$$d_\gamma(x, y) = \sqrt{2} \sup\{s + t : s, t > 0, (x + s - t, y + s + t) \in D_\gamma \}. $$
Geometrically \(a_\ast(x, y)\) is the distance from \((x, y)\) to the curve \(\gamma\) in \(u\)-direction, \(\ell_\ast(x, y)\) is the distance from \((x, y)\) to the curve \(\gamma\) in \(v\)-direction, and \(d_\ast(x, y)\) is half of the maximal perimeter among all rectangles confined in \(D_\gamma\) with sides parallel to the \(u\)- and \(v\)-axes, whose lower corner is \((x, y)\). Extend \(a_\ast, \ell_\ast\) and \(d_\ast\) from the interior of \(D_\gamma\) to \(\mathbb{R}^2\) by assigning zero to all other points.

With regard to the subsequent sections we need to give the following question some thought. Suppose \(\gamma, \eta \in \Gamma\) are close with respect to the supremum norm, then what can be said about the relationship between \(a_\gamma\) and \(a_\eta\) on their common domain \(D_\gamma \cap D_\eta\)? The following example demonstrates that \(\|a_\gamma - a_\eta\|_\infty\) does not need to be small. Let

\[
\gamma(x) = \begin{cases} \sqrt{2n} + x & \text{if } -\frac{n}{\sqrt{2}} \leq x \leq -\frac{1}{\sqrt{2}}(n - \frac{1}{n}) \\
\sqrt{2} \frac{n}{n} - x & \text{if } -\frac{1}{\sqrt{2}}(n - \frac{1}{n}) \leq x \leq \frac{1}{n\sqrt{2}}, \quad \text{and } \eta(x) = \gamma(-x).
\end{cases}
\]

Then \(|\gamma - \eta|\|_\infty = \sqrt{2}\) but \(\|a_\gamma - a_\eta\|_\infty = \|\ell_\gamma - \ell_\eta\|_\infty = n - \frac{1}{n}\).

We show, however, in Lemma 2.2 that when \(\gamma\) and \(\eta\) agree outside of a fixed interval \((a, b)\), then the exceptional set of points on which \(a_\ast\) and \(a_\eta\) diverge is small when \(|\gamma - \eta|\|_\infty\) is small. The proof of Lemma 2.2 is based on a geometric argument. In Lemma 2.2 and Lemma 2.2 we deduce analogous results for \(\ell_\ast\) and \(d_\ast\), which causes little effort once Lemma 2.2 is established.

For a (measurable) subset \(A\) of \(\mathbb{R}^n\), let \(|A|\) denote the Lebesgue measure of \(A\).

**Lemma 2.1.** Let \(\gamma, \eta \in \Gamma\) such that \(\gamma(x) = |x| = \eta(x)\) for all \(x \notin (a, b)\). Then the Lebesgue measure of the symmetric difference of the sets \(D_\gamma\) and \(D_\eta\) is bounded by

\[
|(D_\gamma - D_\eta) \cup (D_\eta - D_\gamma)| < (b - a)|\gamma - \eta|\|_\infty.
\]

**Proof.** It follows immediately from the definition of \(D_\gamma\) that

\[
|(D_\gamma - D_\eta) \cup (D_\eta - D_\gamma)| = \int_a^b |\gamma(x) - \eta(x)| \, dx \leq (b - a)|\gamma - \eta|\|_\infty.
\]

The inequality is strict since \(\gamma\) and \(\eta\) agree at \(a\) and \(b\) and are continuous. \(\square\)

**Lemma 2.2.** Let \(\varepsilon > 0\) and \((a, b)\) be an interval. Then there exists a constant \(K\) such that for all functions \(\gamma, \eta \in \Gamma\) with \(\gamma(x) = |x| = \eta(x)\) for all \(x \notin (a, b)\) we have

\[
\left|\left\{(x, y) \in D_\gamma \cap D_\eta : |a_\gamma(x, y) - a_\eta(x, y)| > \varepsilon\right\}\right| \leq \frac{K}{\varepsilon}\|\gamma - \eta\|\|_\infty.
\]

**Proof.** Let \(|\gamma - \eta|\|_\infty = \delta\).

We first prove the claim under the assumption that \(\gamma(x) \leq \eta(x)\) for all \(x \in (a, b)\). To this end let \(k = \left\lfloor \frac{b - a}{\delta} \right\rfloor\) and \(x_0, x_1, \ldots, x_k\) be a subdivision of the interval \([a, b]\) such that \(x_0 = a\), \(x_k = b\) and \(x_i - x_{i-1} = \frac{\delta}{k} \leq \delta\). Furthermore, subdivide the curve of \(\gamma\) into segments \(\sigma_1, \ldots, \sigma_k\) such that

\[
\sigma_i = \{(x, \gamma(x)) : x \in [x_{i-1}, x_i]\} \quad \text{for } i \in \{1, \ldots, k\}.
\]

Define

\[
\tau_i = \{(x, \gamma(x) + \delta) : x \in [x_{i-1}, x_i]\} \quad \text{for } i \in \{1, \ldots, k\},
\]

and add \(\tau_0 = \{(x, a + \delta) : x \in [a - \delta, a]\}\) and \(\tau_{k+1} = \{(x, b + \delta) : x \in [b, b + \delta]\}\).

A crucial observation is that \(\gamma(x) \leq \eta(x) \leq \tau(x)\) for all \(x \in \mathbb{R}\) and consequently \(a_\gamma(x, y) \leq a_\eta(x, y) \leq a_\tau(x, y)\) for all \((x, y) \in D_\gamma\). Here \(\tau \in \Gamma\) denotes the concatenation of the segments \(\tau_i\) extended by \(\tau(x) = |x|\) for all \(x \notin (a - \delta, b + \delta)\).

Let \(P : \mathbb{R}^2 \to \mathbb{R}^2\) be the orthogonal projection onto the line \(\{(x, -x) : x \in \mathbb{R}\}\). Set \(P(\sigma_i) = \{P(x, y) : (x, y) \in \sigma_i\}\) and define \(P(\tau_i)\) analogously. If \((x, y), (x', y')\) are points in \(D_\gamma\) such that
By definition of the sequences \((2.6)\) hence choose sequences \((2.4)\). In this case observe that \(P(\sigma_i) \cap P(\sigma_{i+1})\) consists of a single point, and \(P(\sigma_i) \cap P(\sigma_j)\) is empty unless all sets \(P(\sigma_{i+1}), \ldots, P(\sigma_{j-1})\) collapse to a single point.

Suppose that \(\{x, |x|\} \in P(\sigma_i)\) with \(|a_\gamma(x, |x|) - a_\eta(x, |x|)| > \varepsilon\) then \(\{x, |x|\} \in P(\sigma_i) \cap P(\tau_{i+m})\) for some \(m\) with

\[
(2.4) \\
|P(\sigma_i) \cap P(\tau_{i+m})| < 5 \delta > \varepsilon.
\]

In this case observe that

\[
(2.5) \\
\left| \bigcup_{j=0}^{m} P(\sigma_{i+j}) \right| < \frac{5}{\sqrt{2}} \delta.
\]

Choose sequences \((i_1, i_2, \ldots)\) and \((m_1, m_2, \ldots)\) as follows. Let \(i_1\) be the minimal \(i \in \{1, \ldots, k\}\) such that for some \(\{x, |x|\} \in P(\sigma_i)\) we have \(|a_\gamma(x, |x|) - a_\eta(x, |x|)| > \varepsilon\). Given \(i_j\) let \(m_j\) be maximal such that \(P(\sigma_{i_j}) \cap P(\tau_{i_j+m}) \neq \emptyset\). Given \(i_j\) and \(m_j\), if there is an \(i \in \{i_j + m_j + 1, \ldots, k\}\) such that \(|a_\gamma(x, |x|) - a_\eta(x, |x|)| > \varepsilon\) for some \(\{x, |x|\} \in P(\sigma_i)\), then let \(i_{j+1}\) be the minimal \(i\) with this property. Otherwise terminate both sequences.

Clearly the sequence \((i_1, i_2, \ldots, i_r)\) is finite. Because of \((2.4)\) we have \(m_j > \frac{a}{\sqrt{2}}\) for all \(j\), and hence

\[
(j - 1) \frac{\varepsilon}{\sqrt{2}} < i_j \leq k < \frac{b - \alpha + 1}{\delta}.
\]

In particular,

\[
(2.6) \\
r < \sqrt{2}(b - \alpha + 1) \frac{\varepsilon}{\delta} + 1.
\]

By definition of the sequences \((i_j)\) and \((m_j)\) for every point \((x, y) \in D_\gamma\) with \(|a_\gamma(x, y) - a_\eta(x, y)| > \varepsilon\) the projection \(P(x, y)\) is contained in \(P(\sigma_{i_1}) \cup \cdots \cup P(\sigma_{i_r})\) for some \(j \in \{1, \ldots, r\}\). But now, using \((2.5)\) and \((2.6)\) we obtain

\[
(2.7) \\
\left| \left\{ (x, y) : |a_\gamma(x, y) - a_\eta(x, y)| > \varepsilon \right\} \right| < \frac{5}{\sqrt{2}} \delta.
\]

Now drop the condition \(\gamma \leq \eta\) and assume \(||\gamma - \eta|| < \frac{\delta}{2}\). Consider the function \(\rho\) defined by \(\rho(x) = \max\{\gamma(x) - \frac{\delta}{2}, |x|\}\). Clearly \(\rho \leq \gamma, ||\rho - \gamma|| < \delta\) and \(\rho \leq \eta, ||\rho - \eta|| < \delta\). Thus, appealing to \((2.7)\) twice, the inclusion

\[
\left\{ (x, y) : |a_\gamma(x, y) - a_\eta(x, y)| > \varepsilon \right\} \subseteq \left\{ (x, y) : |a_\rho(x, y) - a_\gamma(x, y)| > \varepsilon \right\} \cup \left\{ (x, y) : |a_\rho(x, y) - a_\eta(x, y)| > \varepsilon \right\}
\]

holds and Lemma 2.1 completes the proof. \(\square\)

The analogous result on \(\ell_{\gamma}\) follows easily.
Lemma 2.3. Let $\varepsilon > 0$ and $(a, b)$ be an interval. Then there exists a constant $K$ such that for all $\gamma, \eta \in \Gamma$ with $\gamma(x) = |x| = \eta(x)$ for all $x \notin (a, b)$ we have

$$\left| \{(x, y) \in D_\gamma \cap D_\eta : |\ell_\gamma(x, y) - \ell_\eta(x, y)| > \varepsilon \} \right| < \frac{K}{\varepsilon} \|\gamma - \eta\|_\infty.$$ 

Proof. The claim follows directly from Lemma 2.2 and the symmetry

$$a_\gamma(x, y) = \ell_\rho(-x, y),$$

where $\rho \in \Gamma$ is the function defined by $\rho(x) = \gamma(-x)$.

Finally there is a similar result for the function $d_\gamma$.

Lemma 2.4. Let $\varepsilon > 0$ and $(a, b)$ be an interval. Then there exists a constant $K$ such that for all $\gamma, \eta \in \Gamma$ with $\gamma(x) = |x| = \eta(x)$ for all $x \notin (a, b)$ we have

$$\left| \{(x, y) \in D_\gamma \cap D_\eta : |d_\gamma(x, y) - d_\eta(x, y)| > \varepsilon \} \right| < \frac{K}{\varepsilon} \|\gamma - \eta\|_\infty.$$ 

Proof. The claim follows from Lemma 2.2 and Lemma 2.3 and the estimation

$$|d_\gamma(x, y) - d_\eta(x, y)| \leq |a_\gamma(x, y) - a_\eta(x, y)| + |\ell_\gamma(x, y) - \ell_\eta(x, y)| + \sqrt{2} \|\gamma - \eta\|_\infty.$$ 

We conclude this section by introducing the so called hook coordinates, which were named (to the best of the authors’ knowledge) by Dan Romik, and appear naturally in the study of limit shapes of partitions, see for example [LS77]. Namely, we set

$$s = x - \frac{\ell_\gamma(x, y)}{\sqrt{2}}, \quad t = x + \frac{a_\gamma(x, y)}{\sqrt{2}}.$$ 

(2.8)

Note that

$$s + \gamma(s) = x + y, \quad t - \gamma(t) = x - y,$$

which yields

$$x = \frac{1}{2}(s + t + \gamma(s) - \gamma(t)), \quad y = \frac{1}{2}(s - t + \gamma(s) + \gamma(t)),$$

and

$$\left| \frac{\partial(x, y)}{\partial(s, t)} \right| = \frac{1}{2}(1 + \gamma'(s))(1 - \gamma'(t)),$$

where the derivative $\gamma'$ is defined almost everywhere since $\gamma$ is 1-Lipschitz.
3. Worst case complexity

In this section we analyse the asymptotic behaviour of the worst case complexity of the NPS algorithm. We first demonstrate in Theorem 3.1 that a trivial combinatorial upper bound for the worst case complexity of the NPS algorithm on a fixed shape $W(\lambda)$ is in fact tight. Furthermore, Theorem 3.2 provides the first order asymptotics of $W(\lambda^{(n)})$, where $(\lambda^{(n)})_{n \in \mathbb{N}}$ converges uniformly, in terms of the limit curve $\gamma$.

For a cell $(i, j) \in \lambda$ denote by
\begin{equation}
    w(i, j) = \max \left\{ i' - i + j' - j : (i', j') \in \lambda, i \leq i' \text{ and } j \leq j' \right\}
\end{equation}
the maximal distance of the cell $(i, j)$ to a cell $(i', j') \in \lambda$ South-East of $(i, j)$. Let $W(\lambda)$ denote the worst case complexity of the NPS algorithm on $\lambda$. Clearly
\[ W(\lambda) \leq \sum_{(i, j) \in \lambda} w(i, j). \]
We first show that this upper bound is tight.

**Theorem 3.1.** Let $T$ be a standard Young tableau of skew shape $\lambda/\mu$. Then there exists a tableau $F$ of skew shape $\lambda/\mu$ such that the NPS algorithm transforms $F$ into $T$ using exactly
\[ \sum_{(i, j) \in \lambda/\mu} w(i, j) \]
exchanges.

**Proof.** Let $(i, j)$ be the minimal cell of $\lambda/\mu$ with respect to the order $\preceq$, let $(i', j')$ be a corner of $\lambda/\mu$ such that $i' \geq i$, $j' \geq j$ and $i' - i + j' - j = w(i, j)$, and set $k' = T(i', j')$. It suffices to show that there is a tableau $F$ of shape $\lambda/\mu$ with $F(i, j) = k'$ that induces a SYT on $\lambda/\mu - \{(i, j)\}$, which is transformed into $T$ by the NPS algorithm using $w(i, j)$ exchanges. We prove this claim by induction on $n = |\lambda/\mu|$.

The case $n = 1$ is trivial. To demonstrate the induction step we distinguish two cases.

If there is a second corner $(i'', j'')$ of $\lambda/\mu$ then let $k'' = T(i'', j'')$. By induction there exists a tableau $F'$ of shape $\lambda/\mu - \{(i'', j'')\}$ with $F'(i, j) = k'$ which induces a SYT on $\lambda/\mu - \{(i, j), (i'', j'')\}$ such that NPS transforms $F'$ into the SYT $T'$ in $w(i, j)$ steps, where $T'$ is obtained from $T$ by deleting the cell $(i'', j'')$ and reducing each entry in $\{k'' + 1, \ldots, n\}$ by one. Let $F$ be the tableau of shape $\lambda/\mu$ obtained from $F'$ by increasing each entry in $\{k', \ldots, n - 1\}$ by one, and then setting $F(i'', j'') = k''$. By construction $F$ induces a SYT on $\lambda/\mu - \{(i, j)\}$ and NPS transforms $F$ into $T$ in $w(i, j)$ steps.

If $(i', j')$ is the only corner of $\lambda/\mu$ then $k' = n$. Let $F'$ be a tableau of shape $\lambda/\mu - \{(i', j')\}$ with $F'(i, j) = n - 1$, which induces a SYT on $\lambda/\mu - \{(i, j), (i', j')\}$, and is transformed into $T'$ by the NPS algorithm using $w(i, j) - 1$ exchanges, where $T'$ is obtained from $T$ by deleting the corner $(i', j')$. Then the desired skew tableau $F$ is obtained from $F'$ by setting $F(i, j) = n$ and $F(i', j') = n - 1$.

Approximating the right hand side of (3.1) by an integral and making use of the preparatory results in Section 2 we are able to draw conclusions on the asymptotics of the worst case complexity.

**Theorem 3.2.** Let $\lambda^{(n)}$ be a sequence of partitions converging uniformly to the limit shape $\gamma \in \Gamma_1$. Then
\begin{equation}
    W(\lambda^{(n)}) = n^{3/2} \int_{D_\gamma} d_r(x, y) \, dx \, dy + o(n^{3/2}) \quad \text{as } n \to \infty.
\end{equation}
Before we turn to the proof let us state some remarks.

First, let us argue the existence of the integral in (3.2). Since $d_\gamma(x, y)$ is bounded and $D_\gamma$ is the union of a compact set and a null set, the integral is proper. Furthermore, the function $d_\gamma(x, y)$ is decreasing in $y$ and hence integrable. The function $\int_{|x|} d_\gamma(x, y) \, dy$ is even continuous in $x$.

Secondly, since the right hand side of (3.2) is \textit{a priori} not straight forward to compute, we offer the estimation

$$\iint_{D_\gamma} d_\gamma(x, y) \, dx \, dy \leq \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (t - s)(1 + \gamma'(s))(1 - \gamma'(t)) \, dt \, ds,$$

which is obtained from $d_\gamma(x, y) \leq a_\gamma(x, y) + \ell_\gamma(x, y)$ by a substitution of hook coordinates.

\textit{Proof of Theorem 3.2.} First rewrite $W(\lambda^{(n)})$ as an integral as follows: A cell $(i, j) \in \lambda^{(n)}$ corresponds to the square

$$Z(i, j) = \{ (x, y) : i - 1 \leq v \leq i, j - 1 \leq u \leq j \} \subseteq D_n.$$

Let $(x, y)$ be an interior point of $Z(i, j)$. Then

$$\sqrt{n} d_\gamma(x, y) = w(i, j) + i + j - u - v.$$

Hence

$$n^{3/2} \iint_{Z(i, j)} d_\gamma(x, y) \, dx \, dy = n \iint_{Z(i, j)} w(i, j) + i + j - u - v \, dx \, dy$$

$$= \int_{i-1}^{i} \int_{j-1}^{j} w(i, j) + i + j - u - v \, du \, dv$$

and

$$\sum_{(i,j)\in \lambda^{(n)}} w(i, j) = -n + n^{3/2} \iint_{D_n} d_\gamma(x, y) \, dx \, dy.$$

Now fix $\varepsilon > 0$. It suffices to show that

$$\left| \iint_{D_n} d_\gamma(x, y) \, dx \, dy - \iint_{D_n} d_{\gamma_n} \, dx \, dy \right| < \varepsilon$$

for all sufficiently large $n$. In order to do so choose an interval $(a, b)$ such that $\gamma_n(x) = |x|$ for all $x \not\in (a, b)$ and all $n \in \mathbb{N}$. It follows that also $\gamma(x) = |x|$ outside of $(a, b)$. By Lemma 2.1 the Lebesgue measure of the symmetric difference of the sets $D_\gamma$ and $D_n$ tends to zero as $n$ tends to infinity. Since both $d_\gamma$ and $d_{\gamma_n}$ are bounded by the constant $(b - a)\sqrt{2}$,

$$\int_{D_{\gamma_n}} d_{\gamma_n}(x, y) \, dx \, dy + \int_{D_{\gamma} - D_{\gamma_n}} d_\gamma(x, y) \, dx \, dy < \varepsilon$$

when $||\gamma - \gamma_n||_\infty < \varepsilon/((b - a)^2\sqrt{2})$. On the other hand by Lemma 2.3 there exist sets $A$ and $B$ and a constant $K$ such that $D_\gamma \cap D_n = A \cup B$, $|d_\gamma(x, y) - d_{\gamma_n}(x, y)| < \varepsilon/2$ for all $(x, y) \in A$ and $|B| < K||\gamma - \gamma_n||_\infty/\varepsilon$. Hence,

$$\int_{D_\gamma \cap D_n} |d_\gamma(x, y) - d_{\gamma_n}(x, y)| \, dy \, dx < \frac{\varepsilon}{2} + (b - a)\sqrt{2} K \frac{||\gamma - \gamma_n||_\infty}{\varepsilon} < \varepsilon$$

if $||\gamma - \gamma_n||_\infty$ is sufficiently small. \hfill \qed
4. Average case complexity

The main result of this section is an asymptotic lower bound for the average case complexity of the NPS algorithm, which we obtain in three steps. Proposition 4.1 obtains a combinatorial bound for the average case complexity of the NPS algorithm on a fixed shape $C(\lambda)$. Proposition 4.2 approximates this combinatorial bound by an integral. Finally, in Theorem 4.3 we derive an asymptotic bound for $C(\lambda(n))$, where $(\lambda(n))_{n \in \mathbb{N}}$ converges uniformly, in terms of the limit curve $\gamma$.

Given a hook tableau $H$ of shape $\lambda$ denote

$$|H| = \sum_{(i,j) \in \lambda} |H(i,j)|.$$ 

**Proposition 4.1.** Let $\lambda$ be a partition of $n$, and $H$ be a hook tableau of shape $\lambda$ chosen uniformly at random. Then

$$C(\lambda) > E(|H|).$$

**Proof.** During the application of the NPS algorithm to a tableau $T$ of shape $\lambda$ a hook tableau of the same shape is built from the zero tableau, that is, the hook tableau with $|H| = 0$. This is done by applying the following transformations. Suppose the entry of the cell $(i, j)$ drops to the cell $(i', j')$. Then $H(s, j)$ is set to $H(s + 1, j) - 1$ for $s = i, \ldots, i' - 1$, and $H(i', j)$ is set to $j' - j$. Thereby $|H|$ is increased by no more than $i' - i + j' - j$ which is exactly the number of performed exchanges. Since the NPS algorithm produces each hook tableau equally often as $T$ ranges over all possible tableaux of shape $\lambda$, we conclude the following lower bound on the average number of exchanges

$$C(\lambda) > \frac{\lambda^2}{n!} \sum_{H} |H|,$$

where the sum is taken over all hook tableaux of shape $\lambda$. The number of hook tableaux is given by the hook product

$$\prod_{(i,j) \in \lambda} h_\lambda(i,j).$$

By use of the hook-length formula, the right hand side of (4.1) is just the expected value of the random variable $|H|$. □

In a next step we replace the combinatorial lower bound $C(\lambda) > E(|H|)$ by an integral.

**Proposition 4.2.** Let $\lambda$ be a partition of $n$ with boundary $\gamma \in \Gamma_1$. Then

$$C(\lambda) > \frac{n^{3/2}}{2} \iint_{D_\gamma} \frac{a_\gamma(x,y)^2 + \ell_\gamma(x,y)^2}{a_\gamma(x,y) + \ell_\gamma(x,y) + \frac{1}{\sqrt{n}}} \, dx \, dy + \frac{n}{2}.$$  

**Proof.** Recall that $C(\lambda) > E(|H|)$, where $H$ ranges over the hook tableaux of shape $\lambda$ by Proposition 4.1. By linearity

$$E(|H|) = \sum_{(i,j) \in \lambda} E(|H(i,j)|)$$

$$= \sum_{(i,j) \in \lambda} \frac{\text{arm}(i,j)^2 + \text{arm}(i,j) + \text{leg}(i,j)^2 + \text{leg}(i,j)}{2h(i,j)}$$

$$= \left( \sum_{(i,j) \in \lambda} \frac{\text{arm}(i,j)^2 + \text{leg}(i,j)^2}{2h(i,j)} \right) + \frac{n}{2} - \sum_{(i,j) \in \lambda} \frac{1}{2h(i,j)}.$$
For any cell \((i, j) \in \lambda\) let \(Z(i, j) = \{(x, y) : i - 1 \leq v \leq i, j - 1 \leq u \leq j\}\) as in the proof of Theorem 3.2. If \((x, y)\) is an interior point of \(Z(i, j)\) then
\[ a_3(x, y)\sqrt{n} = \text{arm}(i, j) + j - u \quad \text{and} \quad \ell_3(x, y)\sqrt{n} = \text{leg}(i, j) + i - v. \]

Throughout the remainder of this proof denote \(a_{ij} = \text{arm}(i, j)\), \(l_{ij} = \text{leg}(i, j)\), \(h_{ij} = h(i, j)\), \(w = i - v\) and \(z = j - u\). A straightforward computation yields
\[
\begin{align*}
\frac{n}{\sqrt{n}} a_3(x, y)^2 &= \frac{a^2_{ij}}{h_{ij}} = \frac{(a_{ij} + z)^2}{h_{ij} + w + z} - \frac{a^2_{ij}}{h_{ij}} \\
&= \frac{(a_{ij} + l_{ij} + 1)(a^2_{ij} + 2a_{ij}z + z^2) - (a_{ij} + l_{ij} + w + z + 1)a^2_{ij}}{h_{ij}(h_{ij} + w + z)} \\
&= \frac{a^2_{ij}z + a_{ij}z^2 + 2a_{ij}l_{ij}z + l_{ij}z^2 + 2a_{ij}z + z^2 - a^2_{ij}w}{h_{ij}(h_{ij} + w + z)} \\
&= \frac{a^2_{ij}z - w}{h_{ij}(h_{ij} + w + z)} + \frac{a_{ij}l_{ij} + a_{ij}}{h_{ij}} \frac{2z}{h_{ij} + w + z} + \frac{a_{ij} + l_{ij} + 1}{h_{ij}} \frac{z^2}{h_{ij} + w + z}.
\end{align*}
\]

Thus
\[
\begin{align*}
\frac{a^2_{ij}}{h_{ij}} &= n \int_{Z(i, j)} \frac{a_3(x, y)^2}{h_{ij}} \, dx \, dy \\
&= n^{3/2} \int_{Z(i, j)} a_3(x, y)^2 \, dx \, dy \\
&- n \int_{Z(i, j)} \frac{a^2_{ij}}{h_{ij}} \int_{h_{ij} + w + z} w - z \, dx \, dy \\
(4.2) &- n \frac{a_{ij}l_{ij} + a_{ij}}{h_{ij}} \int_{Z(i, j)} \frac{2h_{ij}z}{h_{ij} + w + z} \, dx \, dy \\
(4.3) &- n \int_{Z(i, j)} \frac{z^2}{h_{ij} + w + z} \, dx \, dy. \\
(4.4)
\end{align*}
\]

The error term \(4.2\) vanishes by symmetry in \(z\) and \(w\). We have
\[
n \int_{Z(i, j)} \frac{w - z}{h_{ij} + w + z} \, dy \, dx = \int_0^1 \int_0^1 \frac{w - z}{h_{ij} + w + z} \, dz \, dw = 0.
\]
The other two error terms, \((4.3)\) and \((4.4)\), could be neglected as the integrands are non-negative. However, there is no harm in showing that they are also small. This can be seen easily since the integrands converge uniformly to \(2z\) and \(0\) respectively. Thus
\[
\lim_{m \to \infty} \int_0^1 \int_0^1 \frac{2mz}{m + w + z} \, dz \, dw = 1, \quad \lim_{m \to \infty} \int_0^1 \int_0^1 \frac{z^2}{m + w + z} \, dz \, dw = 0.
\]
In particular, the integrals in \((4.3)\) and \((4.4)\) are uniformly bounded no matter how large \(h_{ij}\) is.

Approximating \(\ell_3^2/h_{ij}\) by an analogous integral and summing over all cells of \(\lambda\) yields the claim.
\qed

Using the preparatory results of Section 2 we obtain the main theorem of this section.
Theorem 4.3. Let $\gamma(n)_{n \in \mathbb{N}}$ be a sequence of partitions converging uniformly to the limit curve $\gamma \in \Gamma_1$. Then
\begin{equation}
C(\lambda(n)) > \frac{\sqrt{2}n^{3/2}a}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( (t - s) + \frac{\gamma(t) - \gamma(s)}{t - s} \right) \left( 1 + \gamma'(s)(1 - \gamma'(t)) \right) dt ds + o(n^{3/2})
\end{equation}
as $n \to \infty$.

Before we give a proof let us argue that the right hand side of (4.5) is well-defined. First note that the integral is really taken over a compact set. Suppose that $\gamma(x) = |x|$ for all $x \notin (a, b)$ then $\gamma'(s) = -1$ for all $s < a$ and $\gamma'(t) = 1$ for all $t > b$, and the integrand vanishes. Because $\gamma$ is Lipschitz continuous, its derivative exists almost everywhere, is Lebesgue integrable and fulfills $\int_a^b \gamma'(s) ds = \gamma(b) - \gamma(a)$. Thus the limit of the quotient $(\gamma(t) - \gamma(s))/(t - s)$ as $t$ tends to $s$ exists almost everywhere. The integrand is therefore essentially bounded and integrable.

Proof of Theorem 4.3. Let $\gamma_n \in \Gamma_1$ be the boundary function of $\lambda(n)$ and choose an interval $(a, b)$ such that $\gamma_n(x) = |x|$ for all $x \notin (a, b)$. We begin by noting that
\begin{equation}
\int D_{\gamma_n} \frac{a_{\gamma_n}(x, y)^2 + \ell_{\gamma_n}(x, y)^2}{a_{\gamma_n}(x, y) + \ell_{\gamma_n}(x, y)} dx dy - \int \frac{a_{\gamma}(x, y)^2 + \ell_{\gamma}(x, y)^2}{a_{\gamma}(x, y) + \ell_{\gamma}(x, y)} dx dy \to 0
\end{equation}
as $n$ tends to infinity. To see this, first note that the functions $a_{\gamma_n}$, $a_{\gamma}$, $\ell_{\gamma_n}$ and $\ell_{\gamma}$ are all non-negative and bounded by $(b - a)\sqrt{2}$. Thus also the integrands in (4.6) are non-negative and bounded. By Lemma 2.1, it suffices to consider the common domain $D_{\gamma_n} \cap D_{\gamma}$. It then follows from the Lemmata 2.2 and 2.3 that
\begin{equation}
C(\lambda(n)) > \frac{n^{3/2}}{2} \int \frac{a_{\gamma}(x, y)^2 + \ell_{\gamma}(x, y)^2}{a_{\gamma}(x, y) + \ell_{\gamma}(x, y)} dx dy + o(n^{3/2}) \quad \text{as } n \to \infty.
\end{equation}

To finish the proof we use our hook coordinates. Substitution gives
\begin{align*}
\frac{1}{2} & \int D_{\gamma} \frac{a_{\gamma}(x, y) + \ell_{\gamma}(x, y)}{a_{\gamma}(x, y) + \ell_{\gamma}(x, y)} dx dy \\
&= \frac{1}{2} \int D_{\gamma} \frac{2t - 2(x - s)^2}{\sqrt{2}(t - s)} dx dy \\
&= \frac{\sqrt{2}}{2} \int D_{\gamma} \frac{(t - s + \gamma(s) - \gamma(t))^2 + (s + t + \gamma(s) - \gamma(t) - s)^2}{t - s} dx dy \\
&= \frac{\sqrt{2}}{8} \int D_{\gamma} \frac{((t - s) + (\gamma(t) - \gamma(s)))^2 + ((t - s) - (\gamma(t) - \gamma(s)))^2}{t - s} dx dy \\
&= \frac{\sqrt{2}}{8} \int D_{\gamma} \frac{2(t - s)^2 + 2(\gamma(t) - \gamma(s))^2}{t - s} dx dy \\
&= \frac{\sqrt{2}}{8} \int_{-\infty}^{\infty} \frac{(t - s) + (\gamma(t) - \gamma(s))^2}{t - s} (1 + \gamma'(s)(1 - \gamma'(t))) dt ds.
\end{align*}
Perhaps the only step that needs comment is the last one. Each pair $(s, t)$ with $s \leq t$ gives rise to a unique point $(x, y) \in D_{\gamma}$ unless $\gamma'(s) = -1$ or $\gamma'(t) = 1$. This follows from the Lipschitz property of $\gamma$. However, the integrand vanishes when either of the two cases $\gamma'(s) = -1$ or $\gamma'(t) = 1$ occurs. Hence, we can relax the limits of the integral without altering its evaluation. □
5. Imbalanced scaling

Some types of partitions, for example partitions with a fixed number of parts, do not converge to a limit curve \( \gamma \in \Gamma_1 \) in the sense of Section 2. However, they might converge if an alternative scaling, that is, not by a factor of \( \sqrt{n} \) in both \( u \)- and \( v \)-direction, is chosen. In this section we study the asymptotic behaviour of the average case and worst case complexity of the NPS algorithm when the partitions under consideration converge after an imbalanced scaling.

For \( p, q \in \mathbb{N} \cup \{ \infty \} \) such that

\[
\frac{1}{p} + \frac{1}{q} = 1
\]

consider the coordinates given by

\[
(5.1) \quad u = u(x, y) = \frac{n^{1/p}}{\sqrt{2}}(x+y), \quad v = v(x, y) = \frac{n^{1/q}}{\sqrt{2}}(y-x), \quad \left| \frac{\partial(u, v)}{\partial(x, y)} \right| = n.
\]

To each partition \( \lambda \) of \( n \) we associate a set \( D_\lambda \) and a \( p,q \)-boundary function \( \gamma \), defined exactly as in (2.2) and (2.3) but with \( u \) and \( v \) now given by (5.1).

Let \( (\lambda^{(n)})_{n \in \mathbb{N}} \) be a sequence of partitions such that \( \lambda^{(n)} \) is a partition of \( n \) and has \( p,q \)-boundary \( \gamma_n \). We say \( (\lambda^{(n)})_{n \in \mathbb{N}} \) converges \( p,q \)-uniformly to the limit curve \( \gamma \in \Gamma_1 \) if

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |\gamma(x) - \gamma_n(x)| = 0
\]

and there exists an interval \((a,b)\) such that \( \gamma_n(x) = |x| \) for all \( x \notin (a,b) \) and all \( n \in \mathbb{N} \).

The main result of this section provides the leading terms of \( C(\lambda^{(n)}) \) and \( W(\lambda^{(n)}) \), where \( (\lambda^{(n)})_{n \in \mathbb{N}} \) converges \( p,q \)-uniformly, in terms of the limit curve \( \gamma \). They turn out to be of order \( n^{1+\max(1/p,1/q)} \) as \( n \) tends to infinity.

**Theorem 5.1.** Let \( (\lambda^{(n)})_{n \in \mathbb{N}} \) be a sequence of partitions converging \( p,q \)-uniformly to the limit curve \( \gamma \in \Gamma_1 \). If \( p < q \) then

\[
(5.2) \quad C(\lambda^{(n)}) = n^{(p+1)/p} \frac{I_1}{2} + o(n^{(p+1)/p}) \quad \text{as } n \to \infty,
\]

and

\[
(5.3) \quad W(\lambda^{(n)}) = n^{(p+1)/p} I_1 + o(n^{(p+1)/p}) \quad \text{as } n \to \infty,
\]

where

\[
I_1 = \iint_{D_\lambda} a_\gamma(x,y) \, dx \, dy = \frac{\sqrt{2}}{4} \int_{-\infty}^{\infty} \int_{t}^{\infty} (s - \gamma(t) - \gamma(s))(1 + \gamma'(s))(1 - \gamma'(t)) \, dt \, ds.
\]

On the other hand, if \( p > q \) then

\[
(5.4) \quad C(\lambda^{(n)}) = n^{(p+1)/p} \frac{I_2}{2} + o(n^{(p+1)/p}) \quad \text{as } n \to \infty,
\]

and

\[
(5.5) \quad W(\lambda^{(n)}) = n^{(p+1)/p} I_2 + o(n^{(p+1)/p}) \quad \text{as } n \to \infty,
\]

where

\[
I_2 = \iint_{D_\lambda} \ell_\gamma(x,y) \, dx \, dy = \frac{\sqrt{2}}{4} \int_{-\infty}^{\infty} \int_{t}^{\infty} (s - \gamma(t) + \gamma(s))(1 + \gamma'(s))(1 - \gamma'(t)) \, dt \, ds.
\]
The reason why constructing the hook tableau during the application of the NPS algorithm described in Section 1.

In order to obtain an upper bound for the average case complexity, recall the algorithm for which is a trivial consequence of Theorem 3.1. The right hand side of (5.7) equals

\[ n^{1/p} a_{\gamma_n}(x, y) = \text{arm}(i, j) + j - u \quad \text{and} \quad n^{1/q} \ell_{\gamma_n}(x, y) = \text{leg}(i, j) + i - v, \]

where \( \gamma_n \) denotes the \( p, q \)-boundary of \( \lambda^{(n)} \). The asymptotically dominant term in the left hand side of (5.6) can be approximated by an integral

\[ \sum_{(i, j) \in \lambda} \frac{\text{arm}(i, j)^2}{2h(i, j)} = \frac{n^{(p+1)/p}}{2} \int_{D_{\gamma_n}} \frac{a_{\gamma_n}(x, y)^2}{a_{\gamma_n}(x, y) + n^{1/q-1/p}\ell_{\gamma_n}(x, y) + n^{-1/p}} \, dx \, dy. \]

By use of the Lemmata 2.1 and 2.2 it follows that

\[ \left| \int_{D_{\gamma_n}} \frac{a_{\gamma_n}(x, y)^2}{a_{\gamma_n}(x, y) + n^{1/q-1/p}\ell_{\gamma_n}(x, y) + n^{-1/p}} \, dx \, dy - \int_{D_{\gamma_n}} a_{\gamma}(x, y) \, dx \, dy \right| \to 0 \]

as \( n \to \infty \). This establishes the asymptotic lower bound

\[ C(\lambda^{(n)}) > \frac{n^{(p+1)/p}}{2} \int_{D_{\gamma_n}} a_{\gamma}(x, y) \, dx \, dy + o(n^{(p+1)/p}) \quad \text{as} \quad n \to \infty. \]

In order to obtain an upper bound for the average case complexity, recall the algorithm for constructing the hook tableau during the application of the NPS algorithm described in Section 1. The reason why \( |H| \) can be less than \( C(\lambda^{(n)}) \) is that there might be a cancellation in step H1. However, the total cancellation cannot exceed \( \sum_{(i, j) \in \lambda^{(n)}} \text{leg}(i, j) \) such that

\[ C(\lambda^{(n)}) < \mathbb{E}(|H|) + \sum_{(i, j) \in \lambda^{(n)}} \text{leg}(i, j). \]

Since the term

\[ \sum_{(i, j) \in \lambda^{(n)}} \text{leg}(i, j) = n^{(q+1)/q} \int_{D_{\gamma_n}} \ell_{\gamma_n}(x, y) \, dx \, dy + \frac{n}{2} \]

is of order less than \( n^{(p+1)/p} \) as \( n \to \infty \), we conclude (5.2).

Our starting point for the analysis of the worst case complexity is the inequality

\[ (5.7) \quad \sum_{(i, j) \in \lambda} \text{arm}(i, j) \leq W(\lambda^{(n)}) \leq \sum_{(i, j) \in \lambda^{(n)}} \text{arm}(i, j) + \text{leg}(i, j), \]

which is a trivial consequence of Theorem 4.1. The right hand side of (5.7) equals

\[ n^{(p+1)/p} \int_{D_{\gamma_n}} a_{\gamma_n}(x, y) \, dx \, dy + n^{(q+1)/q} \int_{D_{\gamma_n}} \ell_{\gamma_n}(x, y) \, dx \, dy + n. \]

Again the term corresponding to the leg function is of lower order and can be dropped. Thus \( W(\lambda^{(n)}) \) is asymptotically equivalent to the left hand side of (5.7). Lemmata 2.1 and 2.2 imply that

\[ \left| \int_{D_{\gamma_n}} a_{\gamma_n}(x, y) \, dx \, dy - \int_{D_{\gamma}} a_{\gamma}(x, y) \, dx \, dy \right| \to 0 \quad \text{as} \quad n \to \infty, \]
which yields (5.3). The alternative formula in terms of hook coordinates is simply obtained by substitution, which completes the first part of the proof. The second part, that is, the case $p > q$, follows similarly.

6. Partitions with two parts

The main result of this section is a nice formula for the average case complexity of the NPS algorithm when the partition consists of only two rows.

**Theorem 6.1.** Let $\lambda = (\lambda_1, \lambda_2)$ be a partition with two parts. Then the average case complexity of the NPS algorithm on $\lambda$ is given by

$$C(\lambda) = \frac{\lambda_1(\lambda_1 - 1)}{4} + \frac{\lambda_2(\lambda_2 - 3)}{4} - 2 \sum_{k=1}^{\lambda_2} \binom{\lambda_2}{k} \frac{(-1)^k(2k-2)!}{(\lambda_1 - \lambda_2 + 2)2k-1}.$$  

Our starting point for proving Theorem 6.1 is the following formula for the average case complexity of the NPS algorithm for general partitions.

**Theorem 6.2.** ([Sul14, Thm 2.4]) Let $n \in \mathbb{N}$ and $\lambda$ be a partition of $n$. Then

$$C(\lambda) = \sum_{x \in \lambda} \sum_{k=1}^{n} |x| \frac{f^\lambda(x,k)}{f^\lambda} (H_n - H_{n-k} - 1),$$

where the outer sum is taken over all cells $x$ of the Young diagram of $\lambda$, $|x| = i + j - 2$ denotes the distance of a cell $x = (i, j)$ to the top left cell, $f^\lambda$ denotes the number of SYT of shape $\lambda$, $f^\lambda(x,k)$ denotes the number of SYT of shape $\lambda$ such that the cell $x$ contains the entry $k$, and $H_n = \sum_{i=1}^{n} \frac{1}{i}$ denotes the $n$-th harmonic number.

While the appearance of harmonic numbers in (6.2) is quite surprising, the most challenging expressions are the numbers $f^\lambda(x,k)$. For partitions with only two parts we derive a first explicit expression for the average case complexity in terms of five double sums, each of which contains a harmonic number in the summand.

**Lemma 6.3.** Let $\lambda = (\lambda_1, \lambda_2)$ be a partition with two parts. Then

$$f^\lambda = \frac{\lambda_1 - \lambda_2 + 1}{\lambda_1 + 1} \binom{\lambda_1 + \lambda_2}{\lambda_2}$$

and

$$C(\lambda) = \left( \frac{\lambda_1}{2} + \frac{\lambda_2 + 1}{2} \right) (H_{\lambda_1 + \lambda_2} - 1)$$

$$- \frac{1}{f^\lambda} \sum_{j=1}^{\lambda_1} \sum_{k=j}^{2j-1} \frac{(j-1)(2j-k)}{k} \binom{k}{j} \binom{\lambda_1 + \lambda_2 - k}{\lambda_1 - j} H_{\lambda_1 + \lambda_2 - k}$$

$$+ \frac{1}{f^\lambda} \sum_{j=1}^{\lambda_2} \sum_{k=j}^{2j-1} \frac{(j-1)(2j-k)}{k} \binom{k}{j} \binom{\lambda_1 + \lambda_2 - k}{\lambda_2 - j - 1} H_{\lambda_1 + \lambda_2 - k}$$

$$- \frac{1}{f^\lambda} \sum_{j=\lambda_1 + j}^{\lambda_2 + j} \sum_{k=j}^{\lambda_2 + j} \frac{(j-1)(2j-k)}{k} \binom{k}{j} \binom{\lambda_1 + \lambda_2 - k}{\lambda_1 - j} H_{\lambda_1 + \lambda_2 - k}$$

$$- \frac{1}{f^\lambda} \sum_{j=1}^{\lambda_2} \sum_{k=2j}^{\lambda_2 + j} \frac{(k-2j+2)}{k} \binom{k}{j-1} \binom{\lambda_1 + \lambda_2 - k}{\lambda_2 - j} H_{\lambda_1 + \lambda_2 - k}.$$
SYT of shape $\tilde{\lambda}$ computed using Aitken's determinant formula \[Ait43\].

While $f$.

In the present case
\[
\sum_{x, k} f^\lambda(x, k) = f^\tilde{\mu} f^{\lambda/\mu}.
\]

While $f^\tilde{\mu}$ is given by the hook-length formula, the number of SYT of skew shape $f^{\lambda/\mu}$ can be computed using Aitken’s determinant formula \[Ait43\].

\[
f^{\lambda/\mu} = ([\lambda]! [\mu]! \cdot \det_{i,j} \left( 1/((\lambda_i + j - i + j)! \right).
\]

In the present case
\[
\begin{align*}
f^{\lambda/\mu} &= (\lambda_1 + \lambda_2 - k)! \cdot \det \begin{pmatrix} 1/(\lambda_1 - j)! & 1/(\lambda_1 + j - k + 1)! \\ 1/(\lambda_2 - j - 1)! & 1/(\lambda_2 + j - k)! \end{pmatrix} \\ &= \left( \begin{array}{c} \lambda_1 + \lambda_2 - k \\ \lambda_2 - j - 1 \end{array} \right) - \left( \begin{array}{c} \lambda_1 + \lambda_2 - k \\ \lambda_2 - j - 1 \end{array} \right)
\end{align*}
\]

Summing
\[
\sum_{x \in S(\lambda)} |x| f^\tilde{\mu} f^{\lambda/\mu}(x, k) = (-\det_{i,j} H_{\lambda_i + \lambda_j - k})
\]

over all cells $x = (1, j)$ of the first row of $\lambda$, accounts for the first three double sums in \[6.3\]. The case where $x = (2, j)$ is a cell of the second row of $\lambda$ is treated in the same way, except that now $\tilde{\mu} = (k - j, j - 1)$ and $\mu = (k - j, j)$, and accounts for the fourth and fifth double sums in \[6.3\].

While the proof of Lemma \[6.3\] is not too complicated, we begin to appreciate how remarkably simple the expression in \[6.1\] really is, consisting of a single sum devoid of harmonic numbers.
In the following we will prove Theorem 6.1. More precisely, denoting the right hand sides of (6.1) and (6.3) by $A(\lambda_1, \lambda_2)$ and $B(\lambda_1, \lambda_2)$, respectively, we will show that

(6.4) \[ A(\lambda_1, \lambda_2) = B(\lambda_1, \lambda_2) \]

holds for all $\lambda_1, \lambda_2 \in \mathbb{N}$ with $0 \leq \lambda_2 \leq \lambda_1$.

Looking at the given problem, one could be tempted to try the following summation tactic: compute for each of the sums a homogeneous recurrence relation in one of the discrete parameters, say $\lambda_2$ (using, e.g., the package MultiSum [Weg97]), and combine the found recurrences to one linear homogeneous recurrence for the expression

(6.5) \[ T(\lambda_1, \lambda_2) := A(\lambda_1, \lambda_2) - B(\lambda_1, \lambda_2) \]

(using, e.g., the Mathematica package GeneratingFunctions [Mal96]). However, in this particular situation this tactic seems rather clumsy: already the calculation of the linear recurrences for each single sum is a hard nut, and assembling the recurrences to a big recurrence for (6.5) is rather hopeless.

Therefore we will follow an alternative and more suitable tactic using various features of the summation package Sigma [Sch07]. Namely, exploiting a constructive difference ring and field theory [Kar81, Sch16a, Sch16b], that is incorporated within Sigma, we will proceed as follows.

(DEF): Using Sigma’s definite summation toolbox, we will find alternative sum representations where the occurring sums are indefinite nested w.r.t. to the discrete parameter $\lambda_2$. In a nutshell, we will rewrite the expression $T(\lambda_1, \lambda_2)$ given in terms of 6 definite sums to an expression in terms of indefinite nested sums w.r.t. $\lambda_2$.

(IND): Using Sigma’s indefinite summation toolbox, we will rewrite the expression (6.5) further such that no algebraic relations exist among the arising indefinite nested sums and products.

As we will see below, the derived expression of $T(\lambda_1, \lambda_2)$ will collapse to zero, which will prove (6.4) and thus will establish Theorem 6.1.

We emphasize that this tactic is justified by the following central result stated in [Sch16b, Prop. 7.5]: Suppose that we are given an expression $T(n)$ in terms of indefinite nested sums w.r.t. an integer parameter $n$ such that there exist no algebraic relations among the sums and products. Then we can solve the zero-recognition problem trivially: there exists a $\delta \in \mathbb{N}$ such that $T(\nu) = 0$ holds for all $\nu \in \mathbb{N}$ with $\nu \geq \delta$ if and only if the expression $T(n)$ collapses to zero.

We start our proposed summation tactic by loading in the package

\begin{verbatim}
<< Sigma
\end{verbatim}

into the Mathematica system. First we tackle the definite sum

\[ S_0(\lambda_1, \lambda_2) = \sum_{k=1}^{\lambda_2} \binom{\lambda_2}{k} \frac{(-1)^k (2k-2)!}{(\lambda_1 - \lambda_2 + 2)_{2k-1}} \]

given in $A(\lambda_1, \lambda_2)$, i.e., given in the right hand side of (6.1). After entering this sum into Mathematica

\begin{verbatim}
In[2]:= S0 = SigmaSum\[ (-1)^k \text{Binomial}[\lambda_2, k] \frac{(-2 + 2k)!}{(\lambda_1 - \lambda_2 + 2)_{2k-1}}, \{k, 1, \lambda_2\}]\n\end{verbatim}

\begin{verbatim}
Out[2]= \[ \sum_{k=1}^{\lambda_2} \frac{(-1)^k \binom{\lambda_2}{k} (-2 + 2k)!}{(\lambda_1 - \lambda_2 + 2)_{2k-1}} \]
\end{verbatim}

we use Sigma’s recurrence finder to calculate the following recurrence relation:

\begin{verbatim}
In[3]= rec = GenerateRecurrence[S0, \lambda_2][[1]]
\end{verbatim}
This means that \( S_0(\lambda_1, \lambda_2) = \text{SUM}[\lambda_2] \) is a solution of Out[3]. Internally, Zeilberger’s creative telescoping paradigm is used: given the summand

\[
f(\lambda_1, \lambda_2, k) = \binom{\lambda_2}{k} \frac{(-1)^k (2k - 2)!}{(\lambda_1 - \lambda_2 + 2)_{2k-1}}
\]

of \( S_0(\lambda_1, \lambda_2) \). Sigma computes the constants \( c_0(\lambda_1, \lambda_2) = 2(\lambda_2 + 1) \), \( c_1(\lambda_1, \lambda_2) = -3 - \lambda_1 - 3\lambda_2 \) and \( c_2(\lambda_1, \lambda_2) = 2 + \lambda_1 + \lambda_2 \) together with the expression

\[
g(\lambda_1, \lambda_2, k) = \frac{k(\lambda_2 + 1)(2k + \lambda_1 - \lambda_2)(-1 + 2k + \lambda_1 - \lambda_2)(2 + \lambda_1 + \lambda_2)}{(2 - k - \lambda_2)(1 - k + \lambda_2)(-1 - \lambda_1 + \lambda_2 + 2(\lambda_1 + \lambda_2)_{2k-1})}
\]

such that the summand recurrence

\[
c_0(\lambda_1, \lambda_2) f(\lambda_1, \lambda_2, k) + c_1(\lambda_1, \lambda_2) f(\lambda_1, \lambda_2 + 1, k) + c_2(\lambda_1, \lambda_2) f(\lambda_1, \lambda_2 + 2, k) = g(\lambda_1, \lambda_2, k) - g(\lambda_1, \lambda_2, k)
\]

holds. Note that the special case \( \lambda_1 = \lambda_2 \) is problematic and will be treated separately. For all other cases, i.e., for all \( \lambda_1, \lambda_2, k \in \mathbb{N} \) with \( \lambda_1 > \lambda_2 \geq k \geq 0 \) the correctness of this relation can be checked easily by simple polynomial arithmetic. Hence summing (6.6) over \( k \) yields the recurrence Out[3]. In particular, since (6.6) has been verified, we proved that \( S_0(\lambda_1, \lambda_2) \) is a solution of Out[3] for all \( \lambda_1, \lambda_2 \in \mathbb{N} \) with \( \lambda_1 > \lambda_2 \). We remark further that in this particular instance also simpler algorithms, like any implementation of Zeilberger’s creative telescoping algorithm [PWZ96] for hypergeometric terms, could have been used.

Next, we exploit Sigma’s recurrence solver to compute all d’Alembertian solutions [AP94, PWZ96] by executing the function call

\[
\text{In}[4]:= \text{recSol} = \text{SolveRecurrenceRec, SUM}[\lambda_2]//\text{ToSpecialFunction}
\]

This means that

\[
h_1(\lambda_1, \lambda_2) = -1 - \lambda_1 + \lambda_2;
\]

\[
h_2(\lambda_1, \lambda_2) = \frac{2\lambda_2 \lambda_2!}{(\lambda_2 + 2)_{\lambda_2}} + \frac{1}{2} (1 + \lambda_1 - \lambda_2) \sum_{i=1}^{\lambda_2} \frac{2^i i!}{i(2 + \lambda_1)^i}
\]

are two linearly independent solutions of the homogeneous version of the recurrence Out[3] and that

\[
p(\lambda_1, \lambda_2) = \left( - \sum_{i=1}^{\lambda_2} \frac{1}{i - 1 + \lambda_1} - \sum_{i=1}^{\lambda_2} \frac{1}{i} + \frac{1}{2} \sum_{i=1}^{\lambda_2} \frac{2^i i! \sum_{j=1}^{i} 2^{-j}(2 + \lambda_1)}{j i(2 + \lambda_1)^i} \right) - \frac{2\lambda_2 \lambda_2!}{(\lambda_2 + 2)_{\lambda_2}} \sum_{i=1}^{\lambda_2} \frac{2^{-i}(2 + \lambda_1)}{i!}
\]

is a particular solution of the recurrence Out[3] itself. In other words, letting \( d_1(\lambda_1) \) and \( d_2(\lambda_1) \) be arbitrary constants we obtain the general solution

\[
(6.7) \quad d_1(\lambda_1) h_1(\lambda_1, \lambda_2) + d_2(\lambda_1) h_2(\lambda_1, \lambda_2) + p(\lambda_1, \lambda_2)
\]
of the recurrence Out[3]. By the underlying algorithms the d’Alembertian solutions are produced in a rather complicated form and Sigma’s built-in simplifier worked hard to drop the nice solutions; in addition, the arising sums within the output are algebraically independent among each other. We emphasize further that one can verify straightforwardly with simple polynomial arithmetic that (6.7) is indeed a solution of the recurrence Out[3] for all \( \lambda_1, \lambda_2 \in \mathbb{N} \) with \( \lambda_1 > \lambda_2 \).

Finally, we determine the values \( d_1(\lambda_1) = -\frac{1}{\lambda_1+1} \) and \( d_2(\lambda_1) = -1 \) such that \( S_0(\lambda_1, \lambda_2) \) and (6.7) agree for \( \lambda_2 = 1, 2 \). Since both expressions are a solution of Out[3] by construction, it follows that they agree for all \( \lambda_2 \in \mathbb{N} \). This last step can be performed by computing the first two initial values at \( \lambda_2 = 1, 2 \), namely

\[
\text{In}[5]:= \text{initial} = \text{Table}[[S0[\lambda_1+1], \text{\{1, initial\}, \lambda_2, 2}]
\]

and using Sigma’s function call to combine the solutions accordingly:

\[
\text{In}[6]:= \text{FindLinearCombination}[[\text{recSol}, \text{\{1, initial\}, \lambda_2, 2}]
\]

Summarizing, we showed that \( S_0(\lambda_1, \lambda_2) \) equals the expression calculated in Out[6] for all \( \lambda_1, \lambda_2 \in \mathbb{N} \) with \( 1 \leq \lambda_2 < \lambda_1 \).

This “hand calculation” with the computer is reasonable if one treats the simplest of the sums in (6.5). However, if one is faced with one of the double sums in (6.3), this mechanical task gets more and more tedious. Luckily, we can use the package EvaluateMultiSums that has been originally designed for similar and even worse expressions arising in the field of particle physics, see [ABRS05] and references therein. Namely, after loading in the package

\[
\text{In}[7]:= \text{<< EvaluateMultiSums.m}
\]

we can carry out the above calculations completely automatically using internally the functionality of Sigma. For instance, typing in the command

\[
\text{In}[8]:= \text{solSum0} = \text{EvaluateMultiSum}[[(-1)^k(\lambda_2)(-2 + 2k)!/(2 + \lambda_1 - \lambda_2) - 1 + 2k], \text{\{1, 1, \lambda_2\}, \text{\{lambda_2, lambda_1\}, \text{\{1, 1\}, \text{\{lambda_1, infinity\}}}}]
\]

we arrive at an equivalent result SolSum0 in terms of factorials and the Pochhammer symbol. Within the function call the input \{\lambda_2, \lambda_1\}, \{1, 1\}, \{\lambda_1, \infty\} specifies that the discrete parameters \( \lambda_1, \lambda_2 \) fulfill the constraints \( 1 \leq \lambda_2 \leq \lambda_1 \) and \( 1 \leq \lambda_1 \leq \infty \).

Finally, we rewrite the objects in solSum0 in terms of the harmonic numbers and the binomial coefficient. This rewriting can be accomplished by the function call

\[
\text{In}[9]:= \text{solSum0} = \text{SigmaReduce}[[\text{solSum0}, \lambda_2, \text{\{\lambda_1 + \lambda_2\}, H_{\lambda_1 + \lambda_2}, H_{\lambda_1 - \lambda_2}, H_{\lambda_2}},]]
\]

Summarizing, we end up at the following alternative representation...
\[ S_0(\lambda_1, \lambda_2) = -\frac{2\lambda_2}{\lambda_2 + \lambda_1} \left( \sum_{i=1}^{\lambda_2} \frac{(i+\lambda_1)}{2^i} + 1 \right) + \frac{-1 - \lambda_1 + \lambda_2}{2} \sum_{i=1}^{\lambda_2} 2^i \frac{\sum_{j=1}^{i} \frac{(i+\lambda_1)}{2^j}}{i} \]

where all the sums are indefinite nested w.r.t. the outer most summation index \( \lambda_2 \).

Finally, we have to address the special case \( \lambda_1 = \lambda_2 \). In this particular case the sum \( S_0(\lambda_2, \lambda_2) \) simplifies to
\[ \text{Out}[10] = \text{Out}[11] = -\frac{1 + \lambda_1 - \lambda_2}{2} \sum_{i=1}^{\lambda_2} 2^i \frac{\sum_{j=1}^{i} \frac{(i+\lambda_1)}{2^j}}{i} + \left( 1 + \lambda_1 - \lambda_2 \right) \left( H_{\lambda_1} + \frac{1}{2} H_{\lambda_2} - H_{\lambda_1 - \lambda_2} \right) + 1 \]

Namely, for all \( \lambda \)
\[ \sum_{i=1}^{\lambda_2} 2^i \frac{\sum_{j=1}^{i} \frac{(i+\lambda_1)}{2^j}}{i} \]

Observe further that the arising sums in \( \text{solSum0Subst} \) can be simplified with our machinery.
Namely, for all \( \lambda_2 \in \mathbb{N} \) we calculate the right hand sides of
\[ \sum_{i=1}^{\lambda_2} 2^i \frac{\sum_{j=1}^{i} \frac{(i+\lambda_1)}{2^j}}{i} = 2^{\lambda_2} - 1, \]
\[ \sum_{i=1}^{\lambda_2} 2^i \frac{\sum_{j=1}^{i} \frac{(i+\lambda_2)}{2^j}}{i} = 2^{\lambda_2} \sum_{i=1}^{\lambda_2} \frac{2^i}{i} \]
\[ \sum_{i=1}^{\lambda_2} 2^i \frac{\sum_{j=1}^{i} \frac{(i+\lambda_2)}{2^j}}{i} \sum_{j=1}^{\lambda_2} 2^{-j} \frac{(j+\lambda_2)}{j} = -2^{2\lambda_2} \sum_{i=1}^{\lambda_2} \frac{2^i}{i} + 2H_{\lambda_2}. \]

Inserting this simplification in \( \text{solSum0Subst} \) we arrive at the same result as given in \( \text{Out}[10] \).
Note that the described calculations can be carried out straightforwardly by simply typing in
\[ \text{Out}[12] = \text{solSum0Subst} = \text{EvaluateMultiSum}\left[ \{ \lambda_2 \} \right] \rightarrow \{ \{ 2\lambda_2 \} \} \]
\[ \text{Out}[12] = -\frac{2^{2\lambda_2}}{(\lambda_2)^2} + \frac{H_{\lambda_2}}{2} + 1 \]

Since this result is precisely the same expression as given on the right hand side of (6.9), it follows that our identity (6.8) holds also for the special case \( \lambda_2 = \lambda_1 \), i.e., it holds for all \( \lambda_1, \lambda_2 \in \mathbb{N} \) with

1 \leq \lambda_2 \leq \lambda_1.
Next, we turn to the first double sum

\[ S_1(\lambda_1, \lambda_2) = \sum_{j=1}^{\lambda_2} \sum_{k=0}^{2j-1} \frac{(j-1)(2j-k)}{k} \binom{\lambda_1 + \lambda_2 - k}{\lambda_1 - j} H_{\lambda_1 + \lambda_2 - k} \]

of the expression \( B(\lambda_1, \lambda_2) \), i.e., of the right hand side of (6.3). For convenience, we adapt the sum so that the lower bounds of the summation quantifiers are non-negative integers. This yields

\[(6.10) \quad S_1(\lambda_1, \lambda_2) = \sum_{j=1}^{\lambda_2} h(\lambda_1, \lambda_2, j) \]

with

\[ h(\lambda_1, \lambda_2, j) = \sum_{k=0}^{j-1+j} \frac{(-1+j)(j-k)}{j} \binom{j+k}{j} \binom{-j-k+\lambda_1 + \lambda_2}{-j+\lambda_1} H_{-j-k+\lambda_1 + \lambda_2}. \]

Now we start with the inner sum \( h(\lambda_1, \lambda_2, j) \) and repeat the symbolic summation approach from above: we compute a linear recurrence of \( h(\lambda_1, \lambda_2, j) \) in order of 2, and we compute two linearly independent solutions plus one particular solution of the found recurrence. Finally, taking the first two initial values yields a rather huge expression (which is too big to print it here) in terms of indefinite nested sums w.r.t. the integer parameter \( j \). Precisely this form enables one to apply \texttt{Sigma}'s summation toolbox in order to tackle (6.10) where \( j \) is the main summation variable: we can calculate a linear recurrence of \( h(\lambda_1, \lambda_2, j) \) in \( \lambda_2 \), solve the recurrence in terms of \( \text{d'Alembertian solutions} \), and obtain finally a representation where the occurring sums are indefinite nested w.r.t. the parameter \( \lambda_2 \). In order to carry out all the calculations steps automatically, we execute simply the function call

\[ \text{In[14]} := \text{solSum1} = \text{EvaluateMultiSum}[(-1+j)(\lambda_1 + \lambda_2)(j-k)(\lambda_1 + \lambda_2, -j-k+\lambda_1 + \lambda_2) \{\{0, j\}, \{j, 1, \lambda_2\}, \{\lambda_2, \lambda_1\}, \{1, 1\}, \{\lambda_1, \infty\}]; \]

and get the result in terms of \( \text{Pochhammer symbols} \) and \( \text{factorials} \). This result (which we did not print here) can be easily rewritten in terms of the binomial coefficient \( \binom{\lambda_1 + \lambda_2}{\lambda_2} \) by executing the following command:

\[ \text{In[15]} := \text{solSum1} = \text{SigmaReduce}[\text{solSum1}, \lambda_2, \text{Tower} \to \{\binom{\lambda_1 + \lambda_2}{\lambda_2}\}] \]

\[ \text{Out[15]} := \binom{\lambda_1 + \lambda_2}{\lambda_2} \left( \frac{1}{16} \left( 7 \lambda_1 - 3 \lambda_1^2 - 7 \lambda_2 + 4 \lambda_1 \lambda_2 - \lambda_2^2 \right) - \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} - \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \right. \]

\[ + 2^{\lambda_2 - 1} \left( 8 - 3 \lambda_1 + \lambda_1^2 - 2 \lambda_2 \right) \left( \frac{1}{16} \left( 7 \lambda_1 - 3 \lambda_1^2 - 7 \lambda_2 + 4 \lambda_1 \lambda_2 - \lambda_2^2 \right) - \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \right. \]

\[ + \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \left( \frac{1}{16} \left( 7 \lambda_1 - 3 \lambda_1^2 - 7 \lambda_2 + 4 \lambda_1 \lambda_2 - \lambda_2^2 \right) - \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \right. \]

\[ + \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \left( \frac{1}{16} \left( 7 \lambda_1 - 3 \lambda_1^2 - 7 \lambda_2 + 4 \lambda_1 \lambda_2 - \lambda_2^2 \right) - \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \right. \]

\[ + \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \left( \frac{1}{16} \left( 7 \lambda_1 - 3 \lambda_1^2 - 7 \lambda_2 + 4 \lambda_1 \lambda_2 - \lambda_2^2 \right) - \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \right. \]

\[ + \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \left( \frac{1}{16} \left( 7 \lambda_1 - 3 \lambda_1^2 - 7 \lambda_2 + 4 \lambda_1 \lambda_2 - \lambda_2^2 \right) - \sum_{i=1}^{\lambda_2} \frac{2^i \sum_{j=1}^{\lambda_2} 2^{-j} \binom{\lambda_1 + \lambda_2}{j} \binom{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - j} H_{\lambda_1 + \lambda_2 - j} \right) \right. \]

We emphasize that each calculation step of this transformation can be verified in the same fashion as it has been worked out for the sum \( S_0(\lambda_1, \lambda_2) \) from above. Completely, analogously
we obtain the representations solSum2, solSum3, solSum4, solSum5 (together with correctness proofs) of the remaining four sums in (6.3) in terms of indefinite nested sums and products which are all valid for all \( \lambda_1, \lambda_2 \) with \( 1 \leq \lambda_2 \leq \lambda_1 \). This completes step (DEF) of our summation tactic.

Now we are ready to carry out step (IND). With the function call

\[
\text{SigmaReduce}[(\text{solSum0}, \text{solSum1}, \text{solSum2}, \text{solSum3}, \text{solSum4}, \text{solSum5}) \rightarrow \text{solSum2}, \text{solSum3}, \text{solSum4}, \text{solSum5})]
\]

we synchronize the arising sums and products within the representations of the six sums: all expressions depend on the indefinite nested sums and products

\[
(6.11) \quad 2^{\lambda_2} \left( \frac{\lambda_1 + \lambda_2}{\lambda_2} \right) H_{\lambda_1}, H_{\lambda_1 - \lambda_2}, H_{\lambda_2}, H_{\lambda_1 + \lambda_2}, \sum_{i=1}^{\lambda_2} 2^{-i} \left( \frac{i + \lambda_1}{i} \right) H_{i + \lambda_1}, \sum_{i=1}^{\lambda_2} 2^{-i} \left( \frac{i + \lambda_1}{i} \right) H_{i + \lambda_1}.
\]

Furthermore, it follows by the underlying difference ring algorithms that the sequences produced by the objects in (6.11) are algebraically independent among each other.

Putting all the building blocks together

\[
\text{In}[17] = \text{solSum0} - 2 \text{solSum0} - \left( \left( \frac{\lambda_1}{2} \right) + \left( \frac{\lambda_2 + 1}{2} \right) \right) (H_{\lambda_1 + \lambda_2} - 1) + \frac{\lambda_1 - \lambda_2 + 1}{\lambda_1 + 1} \left( \frac{\lambda_1 + \lambda_2}{\lambda_2} \right) (- \text{solSum1} + \text{solSum2} - \text{solSum3} - \text{solSum4} + \text{solSum5})
\]

yields an alternative representation of our expression \( P(\lambda_1, \lambda_2) \) given in terms of (6.5) which is valid for all \( \lambda_1, \lambda_2 \in \mathbb{N} \) with \( 1 \leq \lambda_2 \leq \lambda_1 \). Finally, we observe with

\[
\text{In}[18] = \text{Together}[P]
\]

\[
\text{Out}[18] = 0
\]

that \( P(\lambda_1, \lambda_2) = 0 \) holds or equivalently that (6.4) holds for all \( \lambda_1, \lambda_2 \in \mathbb{N} \) with \( 1 \leq \lambda_2 \leq \lambda_1 \). This completes the proof of Theorem 6.1.

As a reward for all our calculations we obtain besides a proof of Conjecture 3 in addition new representations of \( C(\lambda) \). Using (6.4) together with (6.3) yields

\[
C(\lambda) = C(\lambda_1, \lambda_2) = \frac{\lambda_1(\lambda_1 - 1)}{4} + \frac{\lambda_2(\lambda_2 - 3)}{4} - 2 \left( - \frac{2^{\lambda_2}}{(\lambda_1 + \lambda_2)^2} \left( \sum_{i=1}^{\lambda_2} \frac{i + \lambda_1}{2^i} + 1 \right) + \frac{-1 - \lambda_1 + \lambda_2}{2} \sum_{i=1}^{\lambda_2} 2^i \left( \sum_{j=1}^{i} \frac{j + \lambda_1}{i} \right) \right)
\]

\[
- \left( 1 + \frac{\lambda_1 - \lambda_2}{2} \sum_{i=1}^{\lambda_2} \frac{2^i}{i + \lambda_1} \right) \left( 1 + \frac{\lambda_1 - \lambda_2}{2} \left( H_{\lambda_1} + \frac{1}{2} H_{\lambda_2} - H_{\lambda_1 - \lambda_2} \right) + 1 \right).
\]

This expression is of particular interest if one wants to calculate \( C(\lambda_1, \lambda_2) \) efficiently for \( \lambda_2 = 0, 1, 2, 3, \ldots \) and keeping \( \lambda_1 \) symbolic. In addition, we obtain the specially nice formula

\[
C(\lambda_2, \lambda_2) = \frac{\lambda_2(\lambda_2 - 2)}{2} - 2 \left( - \frac{2^{\lambda_2}}{(\lambda_2)^2} + \frac{H_{\lambda_2}}{2} + 1 \right)
\]
using the identity \([5.9]\). Similarly, if one is interested in an efficient evaluation of \(C(\lambda)\) for a symbolic \(\lambda_2\) and a fixed distance \(\delta = \lambda_1 - \lambda_2 \geq 0\), we can derive with our summation toolbox the following representation for the sum \(S_0(\lambda_1, \lambda_2)\):

\[
\begin{align*}
\text{In}[16] &= \text{resDelta} = \text{EvaluateMultiSum}(\frac{(-1)^k(\lambda_2^k)}{(\lambda_2)}(-2 + 2k)^{\frac{\lambda}{\delta}}/\lambda_1 \rightarrow \delta + \lambda_2, \\
&\quad \{\{k, 1, \lambda_2\}, \{\delta, \lambda_2\}, \{0, 1\}, \{\infty, \infty\}\};
\end{align*}
\]

\[
\begin{align*}
\text{Out}[20] &= \left( -1 - \delta + \frac{(\delta + 1)2^{2\lambda_2}}{(\lambda_2)} \sum_{i=1}^{\delta} \frac{2^i(\delta + \lambda_2)}{i(\delta + 2\lambda_2)} \right) \left( \frac{2^{\delta + 1}2^{2\lambda_2}}{\delta} \sum_{i=1}^{\delta} \frac{2^i(\delta + \lambda_2)}{i(\delta + 2\lambda_2)} \right) \\
&\quad \times \left( -2^{\lambda_2} + \frac{(2\lambda_2)}{\lambda_2} \right) \delta + 1 + \frac{(\delta + 1)H_{\delta + 2\lambda_2} + \frac{1}{2}(\delta + 1)H_{\lambda_2} - (\delta + 1)H_{2\lambda_2} - (\delta + 1)H_{\delta}}{2\lambda_2 + 1}.
\end{align*}
\]

This yields

\[
C(\lambda) = C(\lambda_1, \lambda_2) = \frac{\lambda_1(\lambda_1 - 1)}{4} + \frac{\lambda_2(\lambda_2 - 3)}{4} - 2\left( -1 - \delta + \frac{(\delta + 1)2^{2\lambda_2}}{(\lambda_2)} \sum_{i=1}^{\delta} \frac{2^i(\delta + \lambda_2)}{i(\delta + 2\lambda_2)} \right) \left( \frac{2^{\delta + 1}2^{2\lambda_2}}{\delta} \sum_{i=1}^{\delta} \frac{2^i(\delta + \lambda_2)}{i(\delta + 2\lambda_2)} \right) \\
&\quad + \left( \frac{2^{\lambda_2}(\delta + 1 + \lambda_2)}{\delta} \sum_{i=1}^{\delta} \frac{2^i(\delta + \lambda_2)}{i(\delta + 2\lambda_2)} \right) \left( \frac{2^{\delta + 1}(\delta + 1 + \lambda_2)}{\delta} \sum_{i=1}^{\delta} \frac{2^i(\delta + \lambda_2)}{i(\delta + 2\lambda_2)} \right) - 2(\delta + 1)\lambda_2 \sum_{i=1}^{\delta} \frac{2^i(\delta + \lambda_2)}{i(\delta + \lambda_2)} \left( \frac{2^{\lambda_2}(\delta + 1)}{\lambda_2} \right) \\
&\quad + \left( \frac{2^{\lambda_2}(\delta + 1)}{\lambda_2} \sum_{i=1}^{\delta} \frac{2^i(\delta + \lambda_2)}{i(\delta + \lambda_2)} \right) \left( \frac{2^{\delta + 1}(\delta + 1 + \lambda_2)}{\delta} \sum_{i=1}^{\delta} \frac{2^i(\delta + \lambda_2)}{i(\delta + 2\lambda_2)} \right) - \frac{1}{2}(\delta + 1)H_{\delta + 2\lambda_2} + \frac{1}{2}(\delta + 1)H_{\lambda_2} - \frac{1}{2}(\delta + 1)H_{2\lambda_2} - \frac{1}{2}(\delta + 1)H_{\delta}
\]

with \(\lambda_1 = \lambda_2 + \delta\) for a positive integer parameter \(\lambda_2\) and a non-negative integer \(\delta\).

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(Carsten Schneider) Research Institute for Symbolic Computation, J. Kepler University Linz, A-4040 Linz, Austria  
E-mail address: Carsten.Schneider@risc.jku.at

(Robin Sulzgruber) Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria  
E-mail address: robin.sulzgruber@univie.ac.at