AN UPPER BOUND FOR THE REGULARITY OF POWERS OF EDGE IDEALS

Abstract. For a finite simple graph $G$ we give an upper bound for the regularity of the powers of the edge ideal $I(G)$.

In this note we provide an upper bound for the regularity of the powers of the edge ideal $I(G)$ of a finite simple graph $G$. A general lower bound is known by Beyarslan, Hà and Trung, see [1, Theorem 4.5], while upper bounds are only known under additional assumptions, for example when $G$ is bipartite ([7, Theorem 1.1]. By Cutkosky, Herzog and Trung [3] and Kodiyalam [9] it is known that for any graded ideal $I$ in $S = k[x_1, \ldots, x_n]$ there exist integers $a > 0$ and $b \geq 0$ such that $\text{reg } I^s = as + b$ for all $s \gg 0$. In the case that $I$ is the edge ideal of a graph, the constant $a$ is equal to 2, so that $\text{reg } I^s(G) \leq 2s + b$ for all $s \gg 0$. This result implies that there exists an integer $c$ with $\text{reg } I^s(G) \leq 2s + c$ for all $s$.

In the following theorem we will see that one can choose $c$ to be the dimension of the complex $\Delta(G)$ of stable sets of $G$. Recall that a subset $S$ of the vertex set $V(G)$ of $G$ is called a stable (or independent) set, if no 2-element subset of $G$ is an edge of $G$.

Theorem 1. Let $G$ be a finite simple graph, and let $c$ be the dimension of the stable complex of $G$. Then

$$\text{reg } I^s(G) \leq 2s + c \text{ for all } s.$$  

Proof. The proof depends very much on a result by Jayanthan and Selvaraja [8] for very well-covered graphs which says that for any very well-covered graph $G$ one has $\text{reg } I^s(G) = s + \nu(G) - 1$ for all $s \geq 1$, where $\nu(G)$ is the induced matching number of $G$. The same result was proved before by Norouzi, Seyed Fakhari, and Yassemi [10] with an additional assumption.

Here we apply their theorem to the whisker graph $G^*$ of $G$. Recall that $G^*$ is obtained from $G$ by adding to each vertex of $G$ a leaf. Thus if $V(G) = [n]$, then $I(G^*) = (I(G), x_1 y_1, \ldots, x_n y_n) \subset K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. It is obvious that $G^*$ is a very well-covered graph. Indeed, since it is the polarization of the ideal $J = (I(G), x_1^2, \ldots, x_n^2)$, it is a Cohen-Macaulay ideal, so that all maximal stable sets of $G^*$ have the same cardinality. On the other hand, the vertices of $G$ form a maximal stable sets of $G^*$. This shows that all maximal stable sets of $G$ have cardinality $n = |V(G^*)|/2$. Being very well-covered means exactly this. Thus by the above mentioned theorem Jayanthan et al we have $\text{reg } I^s(G^*) = 2s + \nu(G^*) - 1$.

Next we use the restriction lemma as given in [6, Lemma 4.4]: let $I$ be a monomial ideal with multigraded (minimal) free resolution $\mathbb{F}$, and let $c \in \mathbb{N}_\infty$, where $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Then the restricted complex $\mathbb{F}^c$, which is the subcomplex of $\mathbb{F}$ for which

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Corollary 2. Let $K$ have a element of $S/J$ modulo $\ell$ is a regular sequence and since $S/J$ shown that $\nu$ lemma implies that reg components equal to 0. Then $(x$ which is generated by all monomials $x^b \in J$ with $b \leq c$, componentwise.

We choose $c = (\infty, \ldots, \infty, 0, \ldots, 0) \in \mathbb{N}_\infty^n$ with $n$ components equal to $\infty$ and $n$ components equal to 0. Then $(I(G^s))^c = I(G^s)$ for all $s$. Hence the restriction lemma implies that reg $I(G^s) \leq \text{reg } I(G^s)^s = 2s + \nu(G^s) - 1$ for all $s$. It remains to be shown that $\nu(G^s) = c+1$, where $c = \dim \Delta(G^s)$. In fact, since $\ell = x_1 - y_1, \ldots, x_n - y_n$ is a regular sequence and since $S/J$ is isomorphic to $K[x_1, \ldots, x_n, y_1, \ldots, y_n]/I(G^s)$ modulo $\ell$, it follows that reg $J = \text{reg } I(G^s)$. Let $\sigma$ be the maximal degree of a socle element of $S/J$, then reg$(J) = \deg \sigma + 1$. The desired conclusion follows since $S/J$ has a $K$-basis consisting elements $u + J$, where $u = \prod_{i \in F} x_i$ and $F \in \Delta(G)$. □

Corollary 2. Let $G$ be a finite simple graph with $n$ vertices and $e$ edges. Then

$$\text{reg } I(G)^s \leq 2s + [1/2 + \sqrt{1/4 + n^2 - n - 2e}] - 1 \text{ for all } s.$$  

Proof. For the proof we use Theorem 1 and a famous formula of Hansen [3] who showed that the size of a maximal stable set is bounded by $[1/2 + \sqrt{1/4 + n^2 - n - 2e}]$.

There are many other upper bounds for the size of a maximal stable set of a graph. Well known is the bound given by Kwok which is given as an exercise in [11]. Kwok’s upper bound is $n - e/\Delta$, where $\Delta$ is the maximal degree of a vertex of $G$. A survey on the known upper bounds can be found in the thesis of Willis [12].

Even though $f(s) = \text{reg } I^s$ is linear function of $s$ for $s \gg 0$ when $I$ is a graded ideal of the polynomial ring, the initial behaviour of $f(s)$ is not so well understood. In [2], Conca gives some examples for the unexpected behaviour of the function $f(s)$. On the positive side, Eisenbud and Harris [4, Proposition 1.1] showed that for a graded ideal $I \subset S = K[x_1, \ldots, x_n]$ with height $I = n$ which is generated in a single degree, say $d$, one has $f(s) = ds + b_i$ with $b_1 \geq b_2 \geq \cdots$. We will use this result in the proof of the next theorem.

For a monomial ideal $L \subset S$ we denote by $L^{\text{pol}}$ the polarization of $L$, and by $S^{\text{pol}}$ the polynomial ring in the variables which are needed to define $L^{\text{pol}}$.

Theorem 3. Let $G$ be a finite simple graph with $n$ vertices, and let

$$J = (I(G), x_1^2, \ldots, x_n^2).$$

Then $\text{reg } I(G)^s \leq \text{reg } J^s = \text{reg } (J^{\text{pol}})^s$ for all $s$.

Proof. The inequality $\text{reg } I(G)^s \leq \text{reg } J^s$ follows from the equality $\text{reg } J^s = \text{reg } (J^{\text{pol}})^s$ and the proof of Theorem 1. Thus it remains to prove these equalities. For $s = 1$, the equality holds, since polarization of an ideal does not change its graded Betti numbers. Now since $J^{\text{pol}} = I(G^s)$, [5] implies that $\text{reg } (J^{\text{pol}})^s - 2s$ is a constant function on $s$. Thus the desired result follows once we have shown that $\text{reg } J^s - 2s$ is also a constant function on $s$. Indeed we will show that $\text{reg } J^s - \text{reg } J \geq 2(s - 1)$ for all $s \geq 1$. Then, together with the result of Eisenbud and Harris, the desired conclusion follows.
In order to prove $\text{reg} J^s - \text{reg} J \geq 2(s - 1)$ for all $s \geq 1$, we show the following: let $F \in \Delta(G)$ a facet with $|F| = c + 1$, and set $u = \prod_{i \in F} x_i$. We may assume that $x_1$ divides $u$, and consider $w = ux_i^{2(s-1)}$. Let $m = (x_1, \ldots, x_n)$. Since $mu \in J$ it follows that $mw \in J^s$. It remains to be shown that $w \notin J^s$. Indeed, suppose that $w = v_1v_2 \cdots v_s$ with $v_i \in J$ for $i = 1, \ldots, s$. Since $x_1^{2s}$ does not divide $w$, one of the factors, say $v_1$, must be squarefree. Since $v_1$ divides $w$, it then follows that $v_1 \notin J$, a contradiction.

It should be noted that the equalities $\text{reg} J^s = \text{reg}(J^{\text{pol}})^s$ are quite surprising because in general polarization and taking powers are not very well compatible operations.

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