Uniqueness of Solution for Nonlinear Implicit Fractional Differential Equation

H. L. Tidke

Department of Mathematics, North Maharashtra University, Jalgaon-425 001, India
tharibhau@gmail.com

ABSTRACT
We study the uniqueness of solution for nonlinear implicit fractional differential equation with initial condition involving Caputo fractional derivative. The technique used in our analysis is based on an application of Bihari and Medved inequalities.

Keywords:
Caputo fractional derivative, fractional integral, Implicit fractional differential equation, Bihari and Medved inequalities.

Subject Classification: 26D10, 26A33, 34A08, 34A09, 34A12, 34A34.

1. INTRODUCTION
Fractional calculus is generalization of ordinary differential equations and integrations to arbitrary non-integer orders. One can describe many physical phenomena arising in engineering, physics, economics and science more accurately through the fractional derivative formulation, see [4, 5, 7, 9, 11, 18, 19].

Within the last years the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional (i.e., noninteger) differential operators were re-introduced: the Grunwald–Letnikow, Riemann–Liouville, Hadamard, Caputo, Riesz (Hilfer 2000; Kilbas et al. 2006; Podlubny 1999; Samko et al. 1993) and the more recent notions of Cresson (2007), and Katugampola (2011) or variable order fractional operators introduced by Samko and Ross (1993). Several authors have studied the existence, uniqueness and qualitative properties of the initial value problems of fractional order by different techniques, refer [5, 7, 10, 14, 18, 21].

Recently, Pachpatte D. B. and Chinchane V. L. [1] have discussed the uniqueness of solution of fractional differential equation with the Riemann–Liouville derivative.

By motivation of the recent works [1, 15], we extend the results presented by Pachpatte D. B. and Chinchane V. L. for nonlinear implicit fractional differential equation involving Caputo fractional derivative. In this paper we consider the initial value problem of the type

\[ cD^\alpha x(t) = f(t, x(t), \ldots, x^{(n-1)}(t)), \quad t \in J = [0, b] \]
\[ x(0) = x_0 \in \mathbb{R}, \quad (2.1) \]

where \( cD^\alpha (0 < \alpha < 1) \) denotes the Caputo fractional derivative operator and \( f \) is real continuous valued function on \( J \times \mathbb{R} \times \mathbb{R} \) into \( \mathbb{R} \); \( \mathbb{R} \) denotes the real space.

The paper is organized as follows. In Section 2, some definitions, lemmas and preliminary results are introduced to be used in the sequel. Section 3 will deals the results and proofs of uniqueness problem of (1.1).

2. PRELIMINARIES

Let us recall the concepts of the fractional calculus [7, 8, 13, 15, 18, 19, 20] for more details.

Definition 2.1 A real valued function \( f(t) (t > 0) \) is said to be in space \( C_\mu \) (\( \mu \in \mathbb{R} \)) if there exists a real number \( p(> \mu) \) such that \( f(t) = t^p g(t), \) where \( g \in C[0, \infty). \)

Definition 2.2 A real function \( f(t) (t > 0) \) is said to be in the space \( C_\mu^n, \quad n \in \mathbb{N} \cup \{0\}, \) iff \( (n) \in C_\mu. \)

Definition 2.3 Let \( f \in C_\mu \) (\( \mu \geq -1 \)), then the (left-sided) Riemann–Liouville fractional integral of order \( \alpha > 0 \) of the function \( f \) is given by

\[ I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0 \quad \text{and} \quad I^0 f(t) = f(t) \]  
\[ (2.1) \]

where \( \Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx. \)

Definition 2.4 The (left-sided) Caputo fractional derivative of order \( \alpha > 0 \) of the function \( f \in C_\mu^n \) (\( n \in \mathbb{N} \cup \{0\} \)), if \( n \neq 0 \), is given by:

\[ cD^\alpha f(t) = f(x) = \begin{cases} -f^{(n)}(x), & \text{if } \alpha = n \\ \int_{t-n}^{t} f(x) dx, & \text{if } n - 1 < \alpha < n, \quad n \in \mathbb{N}. \end{cases} \]

Note that \( I^\alpha cD^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} t^k, \quad n - 1 < \alpha \leq n, \quad n \in \mathbb{N}. \)
Definition 2.5. The (left-sided) Riemann–Liouville fractional derivative of order $\alpha > 0$ of the function $f \in C^0_{\alpha,1}$ $(n \in \mathbb{N} \cup \{0\})$, is given by:

$$D^\alpha f(t) = \frac{d^n}{dt^n} [t^{n-\alpha} f(t)], \quad n-1 < \alpha \leq n, \ n \in \mathbb{N}, (2.3)$$

In [15], the author have been studied the existence and uniqueness of the initial value problem (1.1)-(1.2), first, let us reduce the problem (1.1)-(1.2) into equivalent fractional integral equation, we obtain

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad (2.4)$$

The results of our work in the present paper differ substantially from the work of [8, 15] in terms of assumptions and methods of proof.

Also, in [17] Medved defined a special class of nonlinear function and developed a method to estimate solution for nonlinear integral inequalities with singular kernel. The class of function defined as follows:

Definition 2.6. [17] Let $q > 0$ be a real number and $0 < b \leq \infty$. The function $w : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies the following condition

$$e^{-\delta t}[w(u)]^q \leq R(t)w(e^{-\delta t}u^q), (2.5)$$

for all $u \in \mathbb{R}^+$, $t \in [0, b]$, where $R(t)$ is a continuous, nonnegative function.

Remark 2.1. If $w(u) = u^m$, $m > 0$, then

$$e^{-\delta t}[w(u)]^q = e^{(m-1)q}\delta t w(e^{-\delta t}u^q), (2.6)$$

for any $q > 1$, i.e. the condition (2.5) is satisfied with $R(t) = e^{(m-1)q}$. For $w(u) = u + au^m$, where $0 \leq a \leq 1, m \geq 1$, the function $w$ satisfies the condition (2.5) with $q > 1$ and $R(t) = 2^{q-1}e^{qmt}$, see [12].

In [17], Medved introduced the following inequalities which are the best widely and known in the study of many qualitative properties of solution of differential equations.

Lemma 2.1. Let $0 \leq T \leq \infty$, $u(t), b(t), a(t), a'(t) \in C([0, T), \mathbb{R}^+)$; $w \in C(\mathbb{R}^+, \mathbb{R})$ be nondecreasing function, $w(0) = 0, w(u) > 0$ on $(0, T)$, and

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s)w(u(s))ds, (2.7)$$

for $t \in [0, T]$ where $\beta > 0$ is constant. Then following hold:

(i). Suppose $\beta > \frac{1}{2}$, and if $w$ satisfies the condition (2.5) with $q = 2$, then

$$u(t) \leq e^{\frac{1}{2}}(\Omega^{-1}\Omega(2a(t)^2) + g_1(t))^\frac{1}{2}, (2.8)$$

for $t \in [0, T]$, where

$$g_1(t) = \frac{\Gamma(2\beta-1)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} b(s)^2ds, (2.9)$$

where $\Gamma$ is gamma function, $\Omega(v) = \int_{v_0}^v \frac{ds}{w(s)}, v_0 > 0$, $\Omega^{-1}$ is the inverse of $\Omega$, and $t \in \mathbb{R}^+$ is such that $\Omega(2a(t)^2) + g_1(t) \in \text{Dom}(\Omega^{-1})$ for all $t \in [0, T]$. (ii). Let $\beta = (0, \frac{1}{2}]$ and $w$ satisfies the condition (2.5) with $q = 2$, where $z = \frac{1-\beta}{\beta}$, i.e. $\beta = \frac{1}{z+1}$. Let $\Omega$, $\Omega^{-1}$ be as in part (i). Then

$$u(t) \leq e^{\frac{1}{2}}(\Omega^{-1}[\Omega(2z^{-1}a(t)^2) + g_2(t)])^\frac{1}{2}, (2.10)$$

for $t \in [0, T]$, where

$$g_2(t) = 2^{-z-1}K_2^\frac{1}{2} \int_0^t (t-s)^{\beta-1} b(s)^2ds, (2.11)$$

$$K_2 = \frac{\Gamma(1-\alpha)}{\beta^{-1}+\alpha}, \quad \alpha = \frac{z}{z+1}, \quad p = \frac{z+2}{z+1} (2.12)$$

and $T \in \mathbb{R}^+$ is such that $\Omega(2z^{-1}a(t)^2) + g_2(t) \in \text{Dom}(\Omega^{-1})$ for all $t \in [0, T]$. (iii). Let $0 \leq T \leq \infty$, $u(t), b(t), a(t), a'(t) \in C([0, T), \mathbb{R}^+)$; $w \in C(\mathbb{R}^+, \mathbb{R})$ and

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s)w(u(s))ds, (2.13)$$

for $t \in [0, T]$ where $\beta > 0$ is constant. Then following hold:

(i). Suppose $\beta > \frac{1}{2}$, then
\[
\begin{align*}
u(t) & \leq (\sqrt{2})a(t)\exp\left[\frac{2^{(2q-1)} - 1}{q} \int_{0}^{t} b(s)^2 ds + t\right], \\
\text{for } t & \in [0, T].
\end{align*}
\]

(ii) If \( \beta = \frac{1}{z+1} \) for some \( z \geq 1 \), then
\[
u(t) \leq (2q^{-1})^{\frac{1}{z+1}} a(t)\exp\left[\frac{2^{(2q-1)} - 1}{q} z \int_{0}^{t} b(s)^2 ds + t\right],
\]

for \( t \in [0, T] \), where \( k_2 \) is defined as in (2.12), \( q = z + 2 \).

For detail proof of above two theorems see [16].

**Lemma 2.3** ([2, 16], p. 152) (Bihari inequality) Let \( u \) and \( f \) be nonnegative defined on \( \mathbb{R}^+ \), let \( w(u) \) be continuous nondecreasing function defined on \( \mathbb{R}^+ \), and \( w(u) > 0 \), on \( (0, \infty) \). If
\[
u(t) \leq k + \int_{0}^{t} f(s)w(u(s))ds, \tag{2.16}
\]
for \( t \in \mathbb{R}^+ \) where \( k \) is nonnegative constant, for \( 0 \leq t \leq T \),
\[
u(t) \leq G^{-1}[G(k) + \int_{0}^{t} f(s)ds], \tag{2.17}
\]
where
\[
G(r) = \int_{0}^{r} \frac{ds}{w(s)}, r > 0, r_0 > 0
\]
and \( G^{-1} \) is the inverse function of \( G \) and \( t_1 \) is chosen so that \( G(k) + \int_{0}^{t_1} f(s)ds \in \text{Dom}(G^{-1}) \) for all \( t \in \mathbb{R}^+ \) laying in the interval \( 0 \leq t \leq t_1 \).

**3. RESULTS**

In this section, we establish the uniqueness of solution of the following initial value problem.

**Theorem 3.1** If \( f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and satisfies condition
\[
|f(t, x, y) - f(t, x', y')| \leq M\Phi(|x - x'|) + L|y - y'|, \tag{3.1}
\]
where \( M \) is positive constant, \( L \in (0, 1) \) and \( \Phi \) is a continuous nondecreasing function on \( 0 < u \leq A \), with \( \Phi(0) = 0 \) and
\[
\int_{0}^{A} \frac{du}{\Phi(u)} \tag{3.2}
\]
then the problem (1.1)-(1.2) has unique solution on \( J \).

**Proof.** Assume that there exists two solutions \( x(t) \) and \( y(t) \) of (1.1)-(1.2). Then we have
\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s, x(s), D^\alpha x(s))ds \tag{3.3}
\]
and
\[
y(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s, y(s), D^\alpha y(s))ds. \tag{3.4}
\]
Hence we have
\[
|x(t) - y(t)| \leq \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} |f(s, x(s), D^\alpha x(s)) - f(s, y(s), D^\alpha y(s))|ds
\]
\[
\leq \epsilon + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} [M\Phi(|x(s) - y(s)|) + L|D^\alpha (x(s) - y(s))|]ds. \tag{3.5}
\]
But by hypothesis (3.1) for any \( t \in [0, b] \) and any \( x, y \in \mathbb{R} \),
\[
|D^\alpha (x(t) - y(t))| = |f(t, x(t), D^\alpha x(t)) - f(t, y(t), D^\alpha y(t))|
\leq M\Phi(|x(t) - y(t)|) + L|D^\alpha (x(t) - y(t))|.
\]
This implies
\[
|D^\alpha (x(t) - y(t))| \leq \frac{M}{1-L} \Phi(|x(t) - y(t)|). \tag{3.6}
\]
Using above estimation in (3.5), we get
\[
|x(t) - y(t)| \leq \epsilon + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \left[ M\Phi(|x(s) - y(s)|) + \frac{ML}{1-L} \Phi(|x(s) - y(s)|) \right]ds
\]
\[
\leq \epsilon + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \left[ M + \frac{ML}{1-L} \right] \Phi(|x(s) - y(s)|)ds
\]
Now an application of Lemma 2.3 to (3.7) which yields
\[
|x(t) - y(t)| < \Phi^{-1}\left(\Phi(\epsilon) + \frac{M}{\Gamma(\alpha)} (1 - L) \int_0^b (t-s)^{a-1}ds\right)
\]
\[
< \Phi^{-1}\left(\Phi(\epsilon) + \frac{M}{\Gamma(\alpha)} (1 - L) \frac{(t-s)^a}{a} \right)
\]
\[
< \Phi^{-1}\left(\Phi(\epsilon) + \frac{M}{\Gamma(\alpha)} (1 - L) \frac{(b^a - (t-b)^a)}{a} \right)
\]
\[
< \Phi^{-1}\left(\Phi(\epsilon) + \frac{M}{\Gamma(\alpha)} (1 - L) \frac{(b^a - (1-L)^a)}{a} \right)
\]
\[
(3.8)
\]
where \(\Phi(x)\) is primitive for \(\frac{1}{\Phi(x)}\). We shall prove that the right-hand side of (3.8) tends toward zero as \(\epsilon \to 0\). As \(|x(t) - y(t)|\) is independent of \(\epsilon\), it follows that \(x(t) = y(t)\), which we need. Let us remark that condition (3.2) implies \(\Phi(\epsilon) \to -\infty\) for \(\epsilon \to 0\), no matter how we choose the primitive of \(\frac{1}{\Phi(x)}\). Thus \(\Phi^{-1}(x) \to 0\) as \(x \to -\infty\). Consequently, \(\epsilon \to 0\) in the inequality (3.8), the right-hand side tends toward zero. This completes the proof of the theorem.

**Theorem 3.2** If the function \(f\) is continuous and satisfies the condition
\[
|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq M |x - \bar{x}| + L |y - \bar{y}|,(3.9)
\]
for some positive constant \(M\) and \(L \in (0,1)\) then the initial value problem (3.1)-(3.2) has unique solution in the interval \(J\).

**Proof.** Assume that there exists two solutions \(x(t)\) and \(y(t)\) of (1.1)-(1.2). Then we have
\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1} f(s, x(s), \ 1D^a x(s))ds
\]
and
\[
y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1} f(s, y(s), \ 1D^a y(s))ds.
\]
Therefore, using these (3.10), (3.11) and hypothesis (3.9), we have
\[
|x(t) - y(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1} |f(s, x(s), \ 1D^a x(s)) - f(s, y(s), \ 1D^a y(s))|ds
\]
\[
\leq \epsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1} M |(x(s) - y(s))| + L |1D^a(x(s) - y(s))|ds
\]
(3.12)
But by hypothesis (3.9) for any \(\epsilon \in [0, b]\) and any \(x, y \in \mathbb{R}\),
\[
|1D^a(x(t) - y(t))| = \left| f(t, x(t), \ 1D^a x(t)) - f(t, y(t), \ 1D^a y(t)) \right|
\]
\[
\leq M |(x(t) - y(t))| + L |1D^a(x(t) - y(t))|.
\]
This implies
\[
|1D^a(x(t) - y(t))| \leq \frac{M}{\Gamma(\alpha)} |x(t) - y(t)|.
\]
(3.13)
Using (3.13) in (3.12), we obtain
\[
|x(t) - y(t)| \leq \epsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1} \left(M |(x(s) - y(s))| + \frac{ML}{1-L} |(x(s) - y(s))| \right)ds
\]
\[
\leq \epsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1} \left(M + \frac{ML}{1-L} \right) |(x(s) - y(s))|ds
\]
\[
\leq \epsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1} \left(1 + \frac{L}{1-L} \right) |(x(s) - y(s))|ds
\]
\[
\leq \epsilon + \int_0^t (t-s)^{a-1} \frac{M}{\Gamma(\alpha) (1-L)} |(x(s) - y(s))|ds.
\]
(3.14)
Now, (a) suppose that \(\alpha > \frac{1}{2}\), then applying Lemma 2.2 (i) to (3.14), we have
\[
|x(t) - y(t)| < \sqrt{2} \epsilon \exp \left[ \frac{2 \sqrt{2} \alpha - 1}{4a} \int_0^t \left(\frac{1}{\Gamma(\alpha) (1-L)} \right)^2 ds + t \right]
\]
\[
< \sqrt{2} \epsilon \exp \left[ \frac{2 \sqrt{2} \alpha - 1}{4a} \int_0^t \left(\frac{1}{\Gamma(\alpha) (1-L)} \right)^2 ds + t \right]
\]
\[
< \sqrt{2} \epsilon \exp \left[ \frac{2 \sqrt{2} \alpha - 1}{4a} \int_0^t \left(\frac{1}{\Gamma(\alpha) (1-L)} \right)^2 ds + t \right],
\]
(3.15)
for \(\epsilon \in J\). Since \(\epsilon\) was arbitrary, as \(\epsilon \to 0\) the inequality (3.13) implies that \(x(t) = y(t)\) on \(J\).
(b) Let $\alpha > \frac{1}{s+1}$ for some $s \geq 1$. Then by Lemma 2.2(ii) to (3.14), again we have,

$$|x(t) - y(t)| < (2^q-1)^\frac{1}{q} \epsilon \exp \left[ \int_0^t \left( \frac{M}{L} \right)^q \Gamma (-q) \right] ds + \left( \frac{1}{L} \right)^q |x(0) - y(0)|$$

for $t \in [0, T]$ where $K_i$ is defined by (2.12). Since $\epsilon$ was arbitrary in (3.16), it implies that $x(t) = y(t)$ as $\epsilon \to 0$. This completes the proof of the theorem.

REFERENCES

[1] Chinchane Vajianath L., Pachpatte Deepak B., Uniqueness of Fractional Differential Equation using the Inequalities, International Journal of Pure and Applied Mathematics, Volume 88 No. 4 (2013), 529-536.

[2] Corduneanu C., Principal of Differential and Integral Equation, Allyn and Bacon, Boston, 1971.

[3] Cresson J. Fractional embedding of differential operators and Lagrangian systems. J Math Phys 48(3) (2007), 34 pp.

[4] El-Borai M., Some probability densities and fundamental solutions of fractional evolution equations, Chaos Solitons Fractals 14(2002), No. 3, 433–440.

[5] Hilfer R., Application of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[6] Katugampola UN, New approach to a generalized fractional integral. Appl. Math. Comput. 218(3) (2011), 860–865.

[7] Kilbas A. A., Srivastava H. M. and Trujillo J. J., Theory and Application of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[8] Kucche K. D., Nieto Juan J., Venktesh Venktesh, Theory of nonlinear implicit fractional differential equations, Differ Equ Dyn Syst, DOI 10.1007/s12591-016-0297-7.

[9] Lakshmikantam V. and VatsalaA.S., Basic theory of fractional Differential Equations, Nonlinear Analysis, 69 No. 8 (2008), 2677-2682.,doi:10.1016/j.na.2007.08.042.

[10] Lakshmikantam V., Theory of Fractional functional Differential Equations, Nonlinear Analysis, 69 No. 10 (2008), 3337-3343.,doi:10.1016/j.na.2007.09.025.

[11] Mainardi F., Fractional Calculus: some basic problem in continium and statistal mechanics, in factals and fractional calculus in contiuam mechanics, (A.Carpinteri and F.Mainardi, Eds)Springer-Verlag, Wien, (1997) 291-348.

[12] Medved’ M., A new approach to an analysis of Henry type integral inequalities and their Bihari type versions, J. Math.Anal.Appl. 214(1997), 349-366.

[13] Miller K. S., Ross B., An Introduction to the Fractional Calculus and Differential Equations; John Wiley: New York, NY, USA, 1993.

[14] Momani S. M., Local and global uniqueness theorem on differential equations of non-integer order via Bihari’s and Gronwall inequalities , Rev.Tec. Ing. Univ. Zulia. Vol. 23, No.1, (2000) 66-69.

[15] Nieto Juan J., Abelghani Ouahab and VenkteshVenktesh, Implicit Fractional Differential Equations via the Liouville–Caputo Derivative, Mathematics 2015, 3, 398-411; doi:10.3390/math3020398.

[16] Pachpatte B. G., Inequalities for Differential and Integral Equations, Academic Press, New York, 2005.

[17] Pachpatte B. G., Integral and finite difference inequalities and applications,North Holland Mathematics Studies, 205, Elsevier Science, 2006

[18] Podlubny I., Fractional Differential Equations, Academic Press, New York, 1999.

[19] Somko S. G., Kilbas A. A. and Marichev O. I., Fractional Integral and Derivative Theory and Application, Gordon and Breach, Yverdon, Switzerland, 1993.

[20] Haping Ye, Jiaming Gao and Yongs Ding, A generalized Gronwall inequality and it’s application to fractional differential equation, Journal of Mathematical Analysis and Application. 328 No 2 (2007), 1075-1081.,doi:10.1016/j.jmaa.2006.05.061.

[21] Yong. Zhou, Existence and uniqueness of solution for system of fractional differential equations, Fractional Calculus and Applied Analysis. 12, No.2,2009.