A remark on monotonicity in Bernoulli bond
Percolation

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Abstract

Consider an anisotropic independent bond percolation model on the
d-dimensional hypercubic lattice, \( d \geq 2 \), with parameter \( p \). We show
that the two point connectivity function \( P_p(\{(0,\ldots,0) \leftrightarrow (n,0,\ldots,0)\}) \)
is a monotone function in \( n \) when the parameter \( p \) is close enough to 0.
Analogously, we show that truncated connectivity function
\( P_p(\{(0,\ldots,0) \leftrightarrow (n,0,\ldots,0),(0,\ldots,0) \leftrightarrow \infty\}) \) is also a monotone function in \( n \) when
\( p \) is close to 1.

Keywords: percolation; monotonicity of connectivity; Orstein-Zernike behavior.

1 Introduction and main result

Consider an ordinary Bernoulli bond percolation model, with parameter
\( p \in [0, 1] \) on the graph \( \mathbb{L}^d = (\mathbb{Z}^d, \mathcal{E}) \) for \( d \geq 2 \), with \( \mathcal{E} = \{ e = (x,y) : x,y \in \mathbb{Z}^d, |x-y|_1 = 1 \} \). That is each bond is open independently with probability \( p \), otherwise it is closed with probability \( 1-p \). Thus, this model is described by the probability space \( (\Omega, \mathcal{F}, P_p) \) where \( \Omega = \{0,1\}^\mathcal{E} \), \( \mathcal{F} \) is the \( \sigma \)-algebra generated by the cylinder sets in \( \Omega \) and \( P_p = \prod_{e \in \mathcal{E}} \mu(e) \) is the product of Bernoulli measures with parameter \( p \).

Given two vertices \( x,y \in \mathbb{Z}^d \), we use the standard notation \( (x \leftrightarrow y) \) to
denote the set of configurations \( \omega \in \Omega \) such that \( x \) is connected to \( y \) by a path of open bonds. Given the parameter \( p \in [0,1] \) and \( n \in \mathbb{N} \), we define the two-point connectivity function \( \tau_p(n) =: P_p((0,\ldots,0) \leftrightarrow (n,0,\ldots,0)) \). It is still an open question to prove if \( \tau_p(n) \) is monotone in \( n \in \mathbb{N} \) for all values of \( p \); this problem was told to one of us (B.N.B.L.) by J. van den Berg [4]. We also define the truncated two-point function \( \tau_p^f(n) =: P_p((0,\ldots,0) \leftrightarrow (n,0,\ldots,0),(0,\ldots,0) \leftrightarrow \infty) \) as the probability of the set of configurations where the origin is connected by open paths to the vertex \( (n,0,\ldots,0) \) but
the origin is to connected by open paths to at most finitely many vertices.

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From now on, as usual \( p_c = p_c(d) \) is the percolation threshold for ordinary Bernoulli percolation on \( \mathbb{L}^d \). We remind that \( \tau^f_p(n) \) is the interesting quantity in the supercritical phase since \( \tau_p(n) \) does not decay at all when \( p > p_c \).

The main result of this note is the following theorem:

**Theorem 1.** Consider the Bernoulli bond percolation on \( \mathbb{L}^d \) with \( d \geq 2 \), then:

i) There exists \( p' > 0 \) (depending upon the dimension \( d \)), such that
\[
\tau_p(n) > \tau_p(n + 1), \quad \forall n \in \mathbb{N}, \forall p < p'.
\]

ii) There exists \( p'' < 1 \) (depending upon the dimension \( d \)), such that
\[
\tau^f_p(n) > \tau^f_p(n + 1), \quad \forall n \in \mathbb{N}, \forall p > p''.
\]

An analogous question was conjectured by Hammersley and Welsh [12] for the monotonicity of expected passage times in the context of first-passage percolation, and this question is also still open. There are some partial results like [10], [13], [2] and [1]. One very interesting negative result is due van den Berg [5], he consider first-passage percolation on the graph \( \mathbb{Z}_+ \times \mathbb{Z} \) and proves that the expect passage time from the origin to \((2, 0)\) is less than the expect passage time from the origin to \((1, 0)\).

Another similar inequality in the context of oriented percolation was obtained by E. Andjel and M. Sued in [3].

## 2 Proof of Theorem 1

Essentially the proof combines well established estimates of the Ornstein-Zernike behavior for the correlation functions for large values of \( n \) and classical estimates via polymer expansion for small values of \( n \). Here, as usual \( \xi(p) \) and \( \xi^f(p) \) are the correlation length and the truncated correlation length defined as (see equations (6.54) and (8.56) in [11]):

\[
\xi(p) = \left[ \lim_{n \to \infty} - \frac{\log \tau_p(n)}{n} \right]^{-1}
\]

and

\[
\xi^f(p) = \left[ \lim_{n \to \infty} - \frac{\log \tau^f_p(n)}{n} \right]^{-1}.
\]

### 2.1 Proof of i)

The proof is based on two results. The first one concerns the Ornstein-Zernike decay of the two-point connectivity function \( \tau_p(n) \) in the whole
subcritical phase and was originally proved by and Campanino, Chayes and Chayes [7]. The second result concerning upper and lower bounds for the two-point connectivity function $\tau_p(n)$ in the highly subcritical phase, was obtained in [14] via polymer expansion. Hereafter all constants depend on $d$.

**Lemma 1.** [Theorem 6.2 (II) of [7]] Consider independent bond percolation on $\mathbb{Z}^d$, $d \geq 2$. For all $p < p_c$ there exists $\alpha(p) > 0$, $K_2(p) > 0$ such that

$$
\tau_p(n) = \frac{K_2(p)}{(\alpha(p)\pi n)^{d/2}} \exp \left\{ -\frac{n}{\xi(p)} \right\} \left[ 1 + O(n^{-1}) \right], \forall n \in \mathbb{N}.
$$

**Lemma 2.** [Eq. (6.3) of [14]] Consider independent bond percolation on $\mathbb{Z}^d$, $d \geq 2$. There exist $p_0 > 0$ close enough to zero and constants $C_1, C_2 > 0$, such that $\forall p < p_0$, it holds:

$$
p^n(1-p)^{C_1 n} \leq \tau_p(n) \leq p^n(1+C_2 p)^{n/2}, \forall n \in \mathbb{N}.
$$

By Lemma 1, it holds that

$$
\frac{\tau_p(n)}{\tau_p(n+1)} = (1 + 1/n)^{d-1} e^\xi(p) \left( \frac{1 + O(n^{-1})}{1 + O((n+1)^{-1})} \right), \forall n \in \mathbb{N}.
$$

Observe that $-\frac{C(p)}{n} \leq |O(n^{-1})| \leq \frac{C(p)}{n}$, where $C(p)$ is a bounded function of $p$, at least in some interval $[0, p_1]$, for $p_1$ small. Therefore,

$$
\frac{1 + O(n^{-1})}{1 + O((n+1)^{-1})} \geq \frac{1 - C/n}{1 + C/(n+1)} \geq 1 - 2C/n,
$$

where $C = \sup \{C(p); 0 \leq p \leq p_1\}$.

Hence, there exists $n_0 = n_0(p_1)$ such that, for $n \geq n_0$, we have

$$
(1 + 1/n)^{d-1} (1 - 2C/n) > e^{-\frac{1}{\xi(p)}}, \forall p \in [0, p_1].
$$

Therefore the monotonicity of $\tau_p(n)$ is proved for all $n \geq n_0$ and all $p \in [0, p_1]$. Now we prove the monotonicity of $\tau_p(n)$ for all $n \leq n_0$. By Lemma 2, for all $n \leq n_0$ and $p \in [0, p_0]$, it holds that

$$
\frac{\tau_p(n)}{\tau_p(n+1)} \geq \frac{1}{p} \frac{(1-p)^{C_1 n}}{(1+C_2 p)^{(n+1)/2}} \geq \frac{1}{p} \left( \frac{(1-p)^{C_1}}{(1+C_2 p)^{1/2}} \right)^{n_0+1},
$$

where $n_0$ is the positive integer defined above.

Thus, there exists $p_2 \in [0, p_0]$, small enough, such that for all $p < p_2$, we have

$$
\frac{1}{p} \left( \frac{(1-p)^{C_1}}{(1+C_2 p)^{1/2}} \right)^{n_0+1} > 1.
$$

We have thus proved the monotonicity of $\tau_p(n)$ for all $n \leq n_0$ in the interval $[0, p_2]$. Then, part i) of Theorem 1 is proved taking $p' = \min\{p_1, p_2\}$. 

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2.2 Proof of ii)

The proof for \(d \geq 3\) is based on analogous results for the highly supercritical phase. The Ornstein-Zernike decay of the truncated two-point function for the Bernoulli bond percolation in the highly supercritical phase (Lemma 3 below) was originally obtained in [6], while the upper and lower bounds for the truncated two-point function \(\tau_p(n)\) in the highly supercritical phase (Lemma 4 below) was once again given in [14].

The proof of ii) \(d \geq 3\) then follows from the two lemmas below performing the same steps of part i) with minor modifications. From now on, let \(\lambda = \lambda(p) = \frac{1-p}{p}\).

**Lemma 3.** [Theorem 1.1 of [6]] For \(d \geq 3\), there exists \(p_3 < 1\), close enough to 1, such that for all \(p \in [p_3, 1)\), it holds that

\[
\tau_p^f(n) = \frac{(1-p)^{4d-2d^2-1}}{(2\pi)^{d-1}} e^{-\frac{n^2}{2}(1 + g(p))} \left(1 + O(n^{-1})\right)
\]

where \(g(p)\) is an analytic function for \(p \in [p_3, 1]\).

**Lemma 4.** [Theorem 5.1 of [14]] For \(d \geq 3\), there exists a constant \(C > 0\) and \(p_4 < 1\), close enough to 1, such that, for all \(p \in [p_4, 1)\), it holds that

\[
\left(\frac{\lambda}{(1+\lambda)^2}\right)^{2(d-1)(n+1)+2} \leq \tau_p^f(n) \leq 2(\lambda \sqrt{1+C\lambda})^{2(d-1)(n+1)+2}.
\]

Finally, for the case \(d = 2\), we use two analogous results recently obtained. The first one, about Ornstein-Zernike behaviour, was given in [8] and it is stated below as Lemma 5. The second one, on bounds upper and lower bonds for truncated two-point function was obtained in [9], is stated below as Lemma 6.

Using these two lemmas here below, the proof of ii) for \(d = 2\) follows the same lines as in the subcritical case.

**Lemma 5.** [Theorem 1.1 of [8]] For \(d = 2\), there are \(\psi(p) > 0\) and \(p_5 < 1\), close to 1, such that

\[
\tau_p^f(n) = \psi(p) e^{-\frac{\xi^2(p)}{n^2}} (1 + f(p,n))
\]

where \(-c(p)f(n) \leq f(p,n) \leq c(p)f(n)\), with \(f(n) \to 0\) as \(n \to \infty\) and \(c(p)\) is bounded in \(p \in [p_5, 1]\).

**Lemma 6.** [Proposition 2 of [9]] For \(\mathbb{Z}^2\), there exists \(p_6 < 1\), close enough to 1, such that, for all \(p \in [p_6, 1)\), it holds that

\[
\lambda^{2n+2}p^{2n} \leq \tau_p^f(n) \leq \lambda^{2n+2} \left[\frac{(4^3\lambda)^{n/2+1}}{1-4^3\lambda} + (1 + 12\lambda)^n\right].
\]
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