THE COMPACTIFICATIONS OF MODULI SPACES OF BURNIAT SURFACES WITH $2 \leq K^2 \leq 5$

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Abstract. We describe compactifications of moduli spaces of Burniat surfaces with $2 \leq K_X^2 \leq 5$ obtained by adding KSBA surfaces, i.e. slc surfaces $X$ with ample canonical class $K_X$.

Contents

Introduction 1
1. Burniat Surfaces 4
1.1. Preliminaries 4
1.2. Abelian covers 6
1.3. The construction of the compactified moduli space $\overline{M}^{d}_{Bur}$. 7
2. Burniat surfaces with $K^2 = 5$ 8
2.1. Burniat surfaces with $K^2 = 5$. 8
2.2. Degenerations of Burniat surfaces with $K^2 = 5$. 9
2.3. Log canonical degenerations. 13
3. Burniat surfaces with $K^2 = 4$ 16
4. Burniat surfaces with $K^2 = 3$ 23
5. Matroid tilings of polytopes $\triangle_{Bur,d}^d, d \leq 5$ 26
References 32

Introduction

Burniat surfaces are special cases of surfaces of general type with $p_g = q = 0$, $2 \leq K_X^2 \leq 6$. They were first introduced by Burniat in [Bu66]. Peters [Pet77] reinterpreted Burniat’s construction using the modern language of branched abelian covers. In [LP01], Lopes and Pardini proved that a minimal surface $S$ of general type with $p_g(S) = 0$, $K_S^2 = 6$, and bicanonical map of degree 4 is a Burniat surface. Moreover, they showed that minimal surfaces $S$ with $p_g = 0$, $K_S^2 = 6$ and bicanonical map of degree 4 form a four-dimensional irreducible component of the moduli space of surfaces of general type.
In [KSB88], Kollár and Shepherd-Barron introduced stable surfaces and proposed a way to compactify the moduli space of surfaces of general type by adding stable surfaces (also called KSBA surfaces). They showed that the appropriate singularities to permit for the surfaces at the boundaries of moduli spaces are semi log canonical (slc) and classified all the semi log canonical surface singularities. The boundedness of slc surfaces with a fixed canonical class $K^2$ was settled in [Ale94].

In [Ale96a, Ale96b], Alexeev extended Kollár and Shepherd-Barron’s construction to stable pairs and stable maps.

In [AP09], Alexeev and Pardini constructed an explicit compactification of the moduli space of Burniat surfaces with $K^2 = 6$ by adding KSBA surfaces, i.e. slc surfaces $X$ with ample canonical class $K_X$, on the boundary. They also gave a constructive algorithm for computing all stable Burniat surfaces (not necessarily from degenerations of smooth surfaces), which reduced them to computing certain tilings by matroid polytopes.

The aim of this paper is to extend the results and methods in [AP09] from the case $K^2 = 6$ to all the remaining cases $2 \leq K^2 \leq 5$. The moduli space $M_{d\text{Bur}}$ of Burniat surfaces with $K^2 = d$ is a subset of dimension $d - 2$ in the moduli space $\mathcal{M}^{\text{can}}$ of canonical surfaces, where a point in $M_{d\text{Bur}}^d$ corresponds to the canonical model of a smooth Burniat surface. When $d = 6, 5$, the moduli space $M_{d\text{Bur}}^d$ is an irreducible component in $\mathcal{M}^{\text{can}}$. Bauer and Catanese [BC10b] showed that $M_{4\text{Bur}}^4$ is a union of two irreducible subvarieties $M_{4\text{Bur},1}^4$ and $M_{4\text{Bur},2}^4$, where a general point of $M_{4\text{Bur},1}^4$ corresponds to a smooth Burniat surface, while a general surface in $M_{4\text{Bur},2}^4$ has an $A_1$-singularity (nodal case). Moreover, $M_{4\text{Bur},1}^4$ is an irreducible component in $\mathcal{M}^{\text{can}}$, whereas $M_{4\text{Bur},2}^4$ is contained in an irreducible component of dimension 3 in $\mathcal{M}^{\text{can}}$. The moduli space $M_{3\text{Bur}}^3$ is irreducible and is contained in an irreducible component of dimension 4 in $\mathcal{M}^{\text{can}}$. $M_{2\text{Bur}}^2$ is just one point so already compact. Thus we will restrict ourselves to compactifying the moduli space $M_{d\text{Bur}}^d$, $3 \leq d \leq 5$.

We reduce the problem of compactifying $M_{d\text{Bur}}^d$ to the one of compactifying the moduli space of certain stable pairs $(Y, \mathcal{D})$. A point in $M_{d\text{Bur}}^d$ corresponds to a Burniat surface $X$ with $K^2 = d$, that is the canonical model of a $\mathbb{Z}_2^2$-cover of $Y = Bl_{9-d}\mathbb{P}^2$ branched along $12 + d$ irreducible curves consisting of 9 strict preimages of lines and $3 + d$ exceptional divisors. The branch data is encoded in the Hurwitz divisor $\mathcal{D}$ (see Section 1.2). An abelian cover of a variety $Y$ with group $G$ or a $G$-cover is a finite map $\pi : X \to Y$ together with a faithful action of a finite abelian group $G$ on $X$ such that $\pi$ exhibits $Y$ as the quotient of $X$ by $G$. In the case $Y$ is smooth and $X$ is normal, Pardini...
in [Par91] described the general structure of abelian covers \( \pi : X \to Y \) using the building data which we will discuss in Section 1.2. The work was extended to the case of non-normal abelian covers in [AP12]. In Sections 2, 3, 4, we list all the interesting degenerate configurations of stable pairs \((Y, \frac{1}{2}D)\) with \(K^2 = 3, 4, 5\), up to symmetry, and find their canonical models using the minimal model program for 3-folds. Here, interesting degenerate configurations are the ones with reducible canonical models.

The stable surfaces appearing on the boundary are quite nontrivial and provide examples of many interesting features of the general case. The construction of the compactified coarse moduli spaces \( \overline{M}_{\text{Bur}} \) of Burniat surfaces is an application of [Ale08], which provides a stable pair compactification \( \overline{M}_\beta(r, n) \) for the moduli space of weighted hyperplane arrangements \((\mathbb{P}^{r-1}, \sum b_iB_i)\) with arbitrary weight \( \beta = (b_1, \ldots, b_n) \), \( 0 \leq b_i \leq 1 \) and \( b_i \in \mathbb{Q} \). In this paper, we apply [Ale08] in the case of \( \mathbb{P}^2 \) and \( n = 9 \) with \( \beta = \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \).

Several new phenomena happen in the case \( K^2 \leq 5 \) as compared to the case \( K^2 = 6 \) in [AP09]. Most importantly, when running the minimal model program, in addition to divisorial contractions occurring in the case \( K^2 = 6 \), flips and flops also occur. It is also surprising that some non log canonical degenerations in the case \( K^2 = 6 \) correspond to log canonical degenerations in the cases \( K^2 \leq 5 \).

We first study degenerations of stable pairs \((Y, \frac{1}{2}D)\) and apply the minimal model program to find the stable limit. We summarize our main results below.

**Theorem 1.** The main component of the compactified coarse moduli space \( \overline{M}_{\text{Bur}} \) of stable Burniat surfaces, or equivalently, of stable pairs \((Y, \frac{1}{2}D)\), is of dimension \( d - 2 \), irreducible for \( d \neq 4 \), and with two components for \( d = 4 \). The types of degenerations, up to symmetry, are listed as below.

(i) There are 6 types of degenerate configurations of stable pairs with reduced log canonical models in the moduli space of stable pairs \((Y, \frac{1}{2}D)\) for \( K^2 = 5 \) case up to the symmetry group \( \mathbb{Z}_6 \) described in Section 2.

(ii) There are 5 types of degenerations with reducible lc models in the moduli space of stable pairs \((Y, \frac{1}{2}D)\) for \( K^2 = 4 \) nodal case and 3 types of degenerations for \( K^2 = 4 \) non-nodal case up to the symmetry group \( \mathbb{Z}_2 \) described in Section 3.

(iii) There are only 2 types of degenerations with reducible lc models in the moduli space of stable pairs \((Y, \frac{1}{2}D)\) for \( K^2 = 3 \) described in Section 4.
According to the general theory of [Ale08], the unweighted stable hyperplane arrangements are described by matroid tilings of the hypersimplex $\Delta(r, n)$. Their weighted counterparts are described by partial tilings of the hypersimplex $\Delta(r, n)$ that cover a $\beta$-cut hypersimplex $\Delta_\beta(r, n)$.

The polytope $\Delta_{\text{Bur}}^d, d \leq 6$ is the polytope in $\mathbb{R}^{12}$ that corresponds to the stable pairs $(Y, \frac{1}{2}D)$ with $K^2 = d$, where $Y = B\text{bl}_{9-d}\mathbb{P}^2$. Inductively, we restrict the matroid tilings of the polytope $\Delta_{\text{Bur}}^d$ for each $d = 6, 5, 4$ to the polytope $\Delta_{\text{Bur}}^{d-1}$ and find all possible stable pairs in the main component of the compactified moduli space of stable pairs with $K^2 = d - 1$. The possible surfaces produced by this computation exactly coincide with the degenerations listed in Sections 2, 3, 4. This also shows that the stable pairs listed in Sections 2, 3, 4 are all the degenerations for the main components of the compactified moduli space of stable pairs with $K^2 = d \leq 5$. All the tilings of $\Delta_{\text{Bur}}^{d-1}$ corresponding to degenerations are restrictions of some tilings of $\Delta_{\text{Bur}}^d$. However, not all restrictions of the tilings of $\Delta_{\text{Bur}}^d$ to $\Delta_{\text{Bur}}^{d-1}$ correspond to degenerations of Burniat surfaces with $K^2 = d - 1$. For example, the tiling #1 of the polytope $\Delta_{\text{Bur}}^6$ listed in table 2 [AP09] is a tiling of the polytope $\Delta_{\text{Bur}}^5$ as well. Tiling #1 corresponds to the non log canonical degeneration Case 1 with $K^2 = 6$ in [AP09], but it does not correspond to any non log canonical degenerations in the case $K^2 = 5$.

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1. Burniat Surfaces

1.1. Preliminaries. We say that a variety is d.c. (double crossings) if every point is either smooth or analytically isomorphic to $xy = 0$. We say that a variety is g.d.c. (has generically double crossings) if it is d.c. outside a closed subset of codimension $\geq 2$.

Let $X$ be a projective variety. Let $B = \sum b_i B_i$ be a linear combination of effective divisors, where $b_i$ is the weight of $B_i$ which is allowed to be an arbitrary rational number with $0 < b_i \leq 1$. The divisors $B_i$’s are possibly reducible and possibly have irreducible components in common. We recall some basic definitions.

Definition 2. Assume that $X$ is a normal variety. A pair $(X, B)$ is called log canonical (lc) if
(1) \(m(K_X + B)\) is a Cartier divisor for some integer \(m > 0\),
(2) for every proper birational morphism \(\pi : X' \to X\) with normal \(X'\),
\[
K_{X'} + \pi^{-1}_* B = \pi^*(K_X + B) + \sum a_i E_i
\]
one has \(a_i \geq -1\). Here the \(E_i\)'s are the irreducible exceptional divisors of \(\pi\), and the pullback \(\pi^*\) is defined by extending \(\mathbb{Q}\)-linearly the pullback on Cartier divisors; \(\pi^{-1}_* B\) is the strict preimage of \(B\).

**Definition 3.** A pair \((X, B)\) is called **semi log canonical (slc)** if
(1) \(X\) satisfies Serre’s condition \(S_2\),
(2) \(X\) is g.d.c., and no divisor \(B_i\) contains any component of the double locus of \(X\),
(3) \(m(K_X + B)\) is a Cartier divisor for some integer \(m > 0\),
(4) for the normalization \(\nu : X'' \to X\), the pair \((X'', (\text{double locus}) + \nu^{-1}_* B)\) is log canonical.

**Definition 4.** Let \((X, B)\) be a semi log canonical pair and \(f : X \to S\) a proper morphism. A pair \((X^c, B^c)\) sitting in a diagram
\[
\begin{array}{ccc}
X & \phi \hookrightarrow & X^c \\
\downarrow f & & \downarrow f^c \\
S & &
\end{array}
\]
is called a **log canonical model** of \((X, B)\) if
(1) \(f^c\) is proper,
(2) \(\phi\) is a birational contraction, that is, \(\phi^{-1}\) has no exceptional divisors,
(3) \(B^c = \phi_* B\),
(4) \(K_{X^c} + B^c\) is \(\mathbb{Q}\)-Cartier and \(f^c\)-ample, and
(5) \(a(E, X, B) \leq a(E, X^c, B^c)\) for every \(\phi\)-exceptional divisor \(E \subset X\).

**Definition 5.** The pair \((X, B)\) is called **stable** if it satisfies the following conditions
(1) on singularities: the pair \((X, B)\) is semi log canonical, and
(2) numerical: the divisor \(K_X + B\) is ample.

Let \(\beta = (b_1, \ldots, b_n)\), \(0 < b_i \leq 1\), \(b_i \in \mathbb{Q}\) be a weight. A hyperplane arrangement is a pair \((\mathbb{P}^{r-1}, \sum b_i B_i)\) with weight \(\beta\), where \(B_1, \ldots, B_n\) are
hyperplanes in $\mathbb{P}^{r-1}$. The pair $(\mathbb{P}^{r-1}, \sum b_i B_i)$ is lc if for each intersection $\cap_{i \in I} B_i$ of codimension $k$, one has $\sum_{i \in I} b_i \leq k$, where $I \subset \{1, \ldots, n\}$. The pair $(\mathbb{P}^{r-1}, \sum b_i B_i)$ is stable if and only if it is lc (slc being an analog of lc for nonnormal pairs) and $|\beta| = \sum_{i=1}^n b_i > r$.

1.2. Abelian covers. We will recall some definitions and theorems from [Par91, AP12] first.

**Definition 6.** Let $G$ be a finite abelian group. An abelian cover with Galois group $G$, or a $G$-cover, is a finite morphism $\pi: X \to Y$ of varieties which is the quotient map for a generically faithful action of a finite abelian group $G$.

An isomorphism of $G$-covers $\pi_1: X_1 \to Y$, $\pi_2: X_2 \to Y$ is an isomorphism $\phi: X_1 \to X_2$ such that $\pi_1 = \pi_1 \circ \phi$.

Let $Y$ be a smooth variety and $X$ be a normal variety. Let $G$ be a finite abelian group and $G^* = \text{Hom}(G, \mathbb{C}^*)$ is the group of characters of $G$. The $G$-action on $X$ with $X/G = Y$ is equivalent to the decomposition:

$$\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{L}_\chi^{-1}, \quad \mathcal{L}_1 = \mathcal{O}_Y$$

where the $\mathcal{L}_\chi$ are line bundles on $Y$ and $G$ acts on $\mathcal{L}_\chi^{-1}$ via the character $\chi$.

In this paper we will only discuss the case when $G = \mathbb{Z}_r^2$. A set of building data $(L_\chi, D_g)$ for the case $G = \mathbb{Z}_r^2$ described in [Par91] can be simplified as

- effective Cartier divisors $D_g$, $g \in G \setminus \{0\}$ (possibly not distinct),
- line bundles $L_\chi$, $\chi \in G^*$.

Moreover the building data for the case $G = \mathbb{Z}_r^2$ need only satisfy the fundamental relations:

$$L_\chi + L_{\chi'} \equiv L_{\chi \chi'} + \sum_{g \in G} \epsilon_g^{\chi, \chi'} D_g$$

where $\epsilon_g^{\chi, \chi'} = 1$ if both $\chi(g) = \chi'(g) = -1$ and $\epsilon_g^{\chi, \chi'} = 0$ otherwise.

In particular, let $G = \mathbb{Z}_r^2 = \{e, a, b, c\}$ and $G^* = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ be the character group with $\chi_0 \equiv 1$, $\chi_1(b) = \chi_1(c) = -1$, $\chi_2(a) = \chi_2(c) = -1$, $\chi_3(a) = \chi_3(b) = -1$, and assume that Pic $Y$ has no $2$-torsion. Then the building data only needs to satisfy
The general theory of abelian covers was extended to the case of non-normal \(X\) in [AP12]; it is used in [AP09]. For details of the abelian covers for the case of non-normal \(X\) we will refer to [AP12]. Now we will recall a theorem in [AP12] which is needed for our paper.

For every building data \((L, D_g)\), [Par91, Def. 2.2] defines a standard abelian cover explicitly, by equations.

**Definition 7.** For a standard \(G\)-cover \(\pi : X \to Y\), the Hurwitz divisor of \(\pi\) is the \(\mathbb{Q}\)-divisor 
\[ D_{Hur} := \sum_i m_i^{-1} D_i, \]
where \(m_i\) is the ramification index of \(D_i\).

The Hurwitz formula
\[ K_X \sim_{\mathbb{Q}} \pi^*(K_Y + D_{Hur}) \]
shows that \(X\) is of general type if and only if \(K_Y + D_{Hur}\) is big.

**Theorem.** [AP12, Proposition 2.5]. Let \(\pi : X \to Y\) be a \(G\)-cover and let \(D\) be the Hurwitz divisor of \(\pi\). Then
(i) The divisor \(K_X\) is \(\mathbb{Q}\)-Cartier if and only if \(K_Y + D\) is \(\mathbb{Q}\)-Cartier.
(ii) \(K_X = \pi^*(K_Y + D)\).
(iii) The variety \(X\) is slc if and only if so is the pair \((Y, D)\).

**Corollary 8.** For a \(G\)-cover \(\pi : X \to Y\) with Hurwitz divisor \(D\), \(X\) is stable if and only if the pair \((Y, D)\) is stable.

1.3. The construction of the compactified moduli space \(\overline{M}_{Bur}^d\).

The compactified moduli space \(\overline{M}_{Bur}^d\) is constructed in [AP09, Section 5.3] as an adaption of the construction of the moduli space \(\overline{M}_b(3, 9)\) of weighted hyperplane arrangements of 9 lines in \(\mathbb{P}^2\) with weight \(b = (\frac{1}{2}, \ldots, \frac{1}{2})\). This construction carries over verbatim to the \(K^2 \leq 5\) case. We refer to [AP09] for details.

Fix weight \(b = (\frac{1}{2}, \ldots, \frac{1}{2})\) and a polytope \(\triangle_{Bur}^d\) (see Section 5). We define \(\overline{M}_{Bur}^d\) to be the moduli space of stable toric varieties over \(G^d_{Bur,b}\) of topological type \(\triangle_{Bur}^d\), where \(G_{Bur,b}^d\) is the \(b\)-cut of certain subvariety \(G^d_{Bur} \subset G(3, 9)\) (see [AP09, Section 5.3]). Thus \(\overline{M}_{Bur}^d\) parametrizes stable toric varieties \(Z \to G^d_{Bur,b}\), and the moment polytopes of the irreducible components of \(Z = \bigcup Z_s\) give a tiling of \(\triangle_{Bur}^d\). For a stable
toric variety $Z \to C^d_{Bur,b}$, one recovers the stable pair $(Y, \frac{1}{2}D)$ as a GIT quotient $P^d_{Bur,Z}//bT$, where $P^d_{Bur,Z} = P \times G^d_{Bur,b}$ $Z$ is the pullback of the universal family $P$.

2. Burniat surfaces with $K^2 = 5$

2.1. Burniat surfaces with $K^2 = 5$. We will use the construction of Burniat surfaces in [Pet77]. To construct a Burniat surface $X$ with $K_X^2 = 5$, we start with an arrangement of 9 distinct lines $A_0, A_1, A_2, B_0, B_1, B_2, C_0, C_1, C_2$ in $\mathbb{P}^2$. The lines $A_0, B_0, C_0$ form a non-degenerate triangle with the vertices $P_A, P_B, P_C$. Lines $A_1, A_2$ pass through $P_B, B_1, B_2$ pass through $P_C$, and $C_1, C_2$ pass through $P_A$. Moreover, $A_1, B_1, C_1$ meet at one point $P$. The other lines are in general position otherwise.

Blow up $\mathbb{P}^2$ at $P_A, P_B, P_C, P$. We denote the exceptional divisors on $\text{Bl}_4\mathbb{P}^2$ by $A_3, B_3, C_3, E$ and by $A_i, B_i, C_i$, $i = 0, 1, 2$ the strict preimages of $A_i, B_i, C_i$ on $\mathbb{P}^2$. The blowup morphism is as follows

![Blowup Diagram](image)

**Definition 9.** A Burniat surface $X$ with $K_X^2 = 5$ is a $\mathbb{Z}_2^3$-cover of $\Sigma = \text{Bl}_4\mathbb{P}^2$ for the building data $D_a = \sum_{i=0}^3 A_i$, $D_b = \sum_{i=0}^3 B_i$, $D_c = \sum_{i=0}^3 C_i$, where $a, b, c$ are the 3 nonzero elements of $\mathbb{Z}_2^3$.

For a $\mathbb{Z}_2^3$-cover, the Hurwitz divisor is $D = \frac{1}{2}(D_a + D_b + D_c)$. Using the Riemann-Hurwitz formula $K_X = \pi^*(K_\Sigma + D)$, we have

$$K_X^2 = (\pi^*(K_\Sigma + D))^2 = 4 \cdot (K_\Sigma + D)^2 = 4 \left(\frac{1}{2}K_\Sigma\right)^2 = 5.$$ 

By the theorem in Section 1.2, we can reduce the problem of compactifying the moduli space of stable Burniat surfaces with $K^2 = 5$ to compactifying the moduli space of stable pairs $(\Sigma, D)$ described above.
2.2. Degenerations of Burniat surfaces with $K^2 = 5$. We consider degenerations of Burniat arrangements of curves on $\Sigma = Bl_4\mathbb{P}^2$. When the arrangement on $\Sigma$ is not log canonical, choose a generic one-parameter family of degenerating arrangements on $\Sigma$ degenerating to it. Then the limit stable surface splits into several irreducible components. Below, we consider such generic degenerations. Let $\mathcal{Y}$ be the total space of the one parameter family of surfaces isomorphic to $\Sigma$ with the central fiber being the degenerating arrangement. Write $\Sigma_0$ for the central fiber of $\mathcal{Y}$.

**Case 1.** When the curve $A_2$ degenerates to $A_0 + C_3$, $B_2$ degenerates to $A_0 + B_3$, and $C_2$ degenerates to $B_3 + C_0$ (the first figure below). Let $L_P$ be the curve in $\mathcal{Y}$ consisting of the points $P$ in each fiber, which is the intersection of the curves $A_1, B_1, C_1$. We first blow up the total space along $L_P$, then blow up the resulting total space along $A_0$ in the central fiber. The central fiber $\Sigma_0$ becomes $Bl_4\mathbb{P}^2 \cup F_1$ (the second figure below), where $A_0$ is the (-1)-curve in $F_1$. Finally we blow up the total space along the proper transform of $B_3$ in the component $Bl_4\mathbb{P}^2$ of $\Sigma_0$. The resulting central fiber is a union of three components $Bl_4\mathbb{P}^2 \cup Bl_1F_1 \cup F_0$.

We can use the triple point formula to compute $(K_{\Sigma_0} + D)|_{Y_i}C$, where $Y_i$ is a component of $\Sigma_0$ and $C$ is a curve in the component $Y_i$.

Let us recall the triple point formula: let $\Sigma_0 = \bigcup Y_i$ be the central fiber in a smooth one-parameter family, and assume that $\Sigma_0$ is reduced and has simple normal crossing. Let $C$ be the intersection $Y_i \cap Y_j$ and assume that it is a smooth curve. Denote by $p_3$ the number of the triple points of $\Sigma_0$ contained in $C$, then

$$(C|_{Y_i})^2 + (C|_{Y_j})^2 + p_3 = 0.$$  

By the adjunction formula, we also have

$$(K_{\Sigma_0} + D)|_{Y_i} = K_{Y_i} + D|_{Y_i} + (\text{the double locus}).$$  

The intersection number $(K_{\Sigma_0} + D)|_{Bl_4\mathbb{P}^2}C$ is 0 when the curve $C$ is $A_1$, $C_0$, $C_1$ or $C_3$, and positive for the other curves in $Bl_4\mathbb{P}^2$. In the component $Bl_1F_1$, $(K_{\Sigma_0} + D)|_{Bl_1F_1}B_3 = 0$ and $(K_{\Sigma_0} + D)|_{Bl_1F_1}C > 0$. 

for the other curves. We also have \((K_{\Sigma_0} + D)|_{F_0} C > 0\) for all the curves in the component \(F_0\). Thus \(K_{\Sigma_0} + D\) is big, nef and vanishes on \(B_3, C_1\) and \(C_3\). The 3-fold is the minimal model of the degenerate family. Using the inversion of adjunction in [Ka07], we see that the pair \((\mathcal{Y}, \mathcal{D})\) is log terminal and \(\mathcal{D}\) is an effective divisor on \(\mathcal{X}\) such that \(K_{\mathcal{Y}} + \mathcal{D}\) is nef and big. By Base Point Free theorem, the linear system \(|n(K_{\mathcal{Y}} + \mathcal{D})|\) is base point free for all sufficiently large \(n \in \mathbb{N}\). Then we can define a birational morphism by the linear system \(|n(K_{\mathcal{Y}} + \mathcal{D})|\), which contracts \(A_1B_3, C_0, C_1, C_3\) labeled in the third figure above. The image of the birational morphism is the lc model of the degenerate family.

The surface \(Bl_4\mathbb{P}^2\) becomes \(\mathbb{P}^2\) after contracting \(A_1, C_0, C_1, C_3\). The component \(Bl_1F_1\) becomes \(F_0\) after contracting \(B_3\). The central fiber of the resulting log canonical model is \(F_0 \cup F_0 \cup \mathbb{P}^2\), which is the first figure below. For \(\mathbb{P}^1 \times \mathbb{P}^1\), there is a further degeneration that splits to \(\mathbb{P}^2 \cup \mathbb{P}^2\). We list the three possible further degenerations below which are the rest three figures. The second and third figures differ only by a permutation of colors. Thus there are only two different degenerations, we call them Case 2 and Case 3.

Case 1 could be obtained from another degeneration when \(B_1\) goes to \(A_0 + B_3\) and \(C_1\) degenerates to \(B_3 + C_0\) (the first figure below). We first blow up the total space \(\mathcal{Y}\) along the line \(B_3\) and then blow up along the strict image of \(A_0\) in the component \(Bl_3\mathbb{P}^2\) of the central fiber. Finally we blow up the resulting total space along the proper transform \(\tilde{L}_P\) of the line \(L_P\). The central fiber becomes \(Bl_3\mathbb{P}^2 \cup F_0 \cup Bl_2F_0\) (the second figure blow). Running the minimal model program, we obtain the lc model with the central fiber \(\mathbb{P}^2 \cup F_0 \cup F_0\) (the third figure below), which is the same as Case 1 above by changing the color of the building data due to the symmetry. Both degenerations could come from Case 2 for \(K^2 = 6\) in [AP09], with \(A_1, B_1, C_1\) meeting at a point \(P\). Case 2 could
be obtained from the degeneration of Case 7 for $K^2 = 6$ with the point $P$ on the boundary of the hexagon.

For the first figure above, if moreover the curve $C_2$ degenerates to $C_3 + B_0$, then the lc model is the same as Case 3 with $K^2 = 5$.

**Case 4.** When the curve $A_2$ degenerates to $A_0 + C_3$ and $B_2$ degenerates to $B_3 + A_0$. We first blow up the total space $\mathcal{Y}$ along the line $A_0$ in the central fiber, then blow up along the curve $\tilde{L}_P$, which is the proper transform of $L_P$. The central fiber $\Sigma_0$ becomes $Bl_4\mathbb{P}^2 \cup \mathbb{F}_1$ and the 3-fold is the minimal model of the degenerate family. This case could be obtained from Case 6 for $K^2 = 6$ in [AP09] with $A_1, B_1, C_1$ meeting at a point $P$.

We get the lc model and call it Case 4, by contracting $C_3$ and $B_3$ in the component $Bl_4\mathbb{P}^2$. In the component $\mathbb{F}_1$, the curve $A_0$ is the $(-1)$-curve $s$, curves $B_3, C_1, C_2, C_3$ are fibers $f$, and curves $A_2, B_2$ are sections of the numerical type $s + f$. In the component $\mathbb{F}_0$, the double locus is the diagonal $s + f$ and all of the other curves are fibers.
There exists some further degenerations of Case 4. Take a one-parameter family with general fibers $\mathbb{F}_1 \cup \mathbb{F}_0$ we obtained above. In the central fiber, the curve $C_2$ coincides with $C_3$, $A_2$ degenerates to $A_0 + C_3$ in the component $\mathbb{F}_1$ and the curve $A_3$ coincide with $B_1$ in the component $\mathbb{P}^1 \times \mathbb{P}^1$. The total space of the one-parameter family is a union of two nonsingular three dimensional spaces $A^1 \times \mathbb{F}_1$ and $A^1 \times \mathbb{F}_0$. Blowing up the total space along a line in the central fiber is the same as blowing up the line in each three dimensional space first and then gluing the two resulting surfaces together in the central fiber. Now let’s see the degenerations in each component.

In the central fiber, the surface $\mathbb{F}_1$ splits into $\mathbb{P}^2 \cup \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ becomes $\mathbb{P}^2 \cup \mathbb{P}^2$. Gluing the two resulting surfaces together, we obtain a further degeneration as in the first figure below. Another possible degeneration is that $C_2$ moves to $B_3$, and $B_2$ degenerates to $A_0 + B_3$ (the central fiber of lc model is the second figure below). Since both the first and second figures contain a component $\mathbb{P}^1 \times \mathbb{P}^1$, the diagonals can again be degenerated to the section $s + f$, and the limits are the two remaining figures below, each of which are just the previous cases 2 and 3.
Case 6. When the five lines $A_1, A_2, B_1, B_2, C_1, C_2$ meet at the point $P$. We first blow up the total space along the curve $L_P$, then blow up the point $P$ in the central fiber, which is the intersection of $A_1, B_1, C_1$ in the exceptional divisor $\mathbb{P}^2$ of the blowup. The resulting central fiber contains two components $Bl_4\mathbb{P}^2 \cup F_1$, which is the central fiber of the minimal model.

Running the minimal model program, we contract $A_1, B_1, C_1$ and get the log canonical model with the central fiber $\mathbb{P}^1 \times \mathbb{P}^1 \cup F_1$, where $E$ is the (-1)-curve in $F_1$. There is no further degeneration for this case.

Case 6 can be obtained from Case 9 with $K^2 = 6$, by taking $A_1, B_1, C_1$ to have a common intersection. For the above surface, if $B_2, C_2$ in the second component $F_1$ degenerate to $B_1 + E, C_1 + E$, then it is the central fiber of lc model of the degeneration comes from Case 10 with $K^2 = 6$.

2.3. Log canonical degenerations. Case 1, 8 and 5 with $K^2 = 6$ are special. Case 5 with $K^2 = 6$ does not produce any degenerations with $K^2 = 5$. 

13
For Case 5 with $K^2 = 6$ (the left figure below), there is no corresponding degeneration with $K^2 = 5$. Since $A_1, B_1, C_1$ must intersect, the resulting degeneration has an infinite automorphism group, and therefore does not correspond to an irreducible component of a stable pair.

Case 1 and 8 with $K^2 = 6$ produce degenerations with $K^2 = 5$. But it is surprising that the lc models of the degenerations are irreducible and are the same as some lc degenerations. We elaborate on the special cases 1 and 8 as following.

We first look at Case 8 with $K^2 = 6$ which is also a degeneration with $K^2 = 5$. When all of the five lines $A_1, A_2, B_1, B_2, C_1$ meet at a point $P$, we blow up the total space along the curve $L_P$, and the resulting central fiber contains two components $Bl_4 \mathbb{P}^2 \cup F_1$. Running the minimal model program, the whole component $Bl_1 \mathbb{P}^2$ is contracted and the central fiber of the lc model is irreducible, which is $Bl_4 \mathbb{P}^2$.

For Case 1 with $K^2 = 6$, we can degenerate $B_1$ to $A_0 + B_3$ to produce the degeneration with $K^2 = 5$, which is the first figure below. We first blow up the total space along the curve $A_0$ in the central fiber, then blow up the total space along the strict preimage of $C_3$ in the central fiber. The resulting central fiber is $Bl_3 \mathbb{P}^2 \cup Bl_1 \mathbb{F}_1 \cup \mathbb{F}_0$, which is the second figure below.
Consider the curve $B_3$ in the component $Bl_3\mathbb{P}^2$ of the central fiber, we have $\left( K_\mathcal{Y} + \frac{1}{2}D \right) \cdot B_3 = -\frac{1}{2} < 0$ and $K_{\Sigma_0}|_{Bl_3\mathbb{P}^2} \cdot B_3 = 0$. When run the minimal model program, there will be a flip for $\left( \mathcal{Y}, \frac{1}{2}D \right)$. The normal bundle of $B_3$ in the total space is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The flip for $\left( \mathcal{Y}, \frac{1}{2}D \right)$ is the Atiyah flop for $\mathcal{Y}$. The process is as follows.

After applying the flip to $\left( \mathcal{Y}, \frac{1}{2}D \right)$, the central fiber of the resulting 3-fold space is $Bl_4\mathbb{P}^2 \cup \mathbb{P}_0 \cup Bl_3\mathbb{P}^2$. Finally we blow up the total space along the strict preimage of $L_P$. The general fibers and the central fiber are all blown up at one point. Now we have $K_{\Sigma_0} \cdot C \geq 0$ for all the curve $C$ in $\Sigma_0$. Running the minimal model program, both the components $\mathbb{P}_0$ and $Bl_3\mathbb{P}^2$ in the central fiber are contracted. The central fiber becomes $Bl_4\mathbb{P}^2$, which is a lc degeneration of the general fibers $Bl_4\mathbb{P}^2$. 


From the cases discussed above, we conclude that there are 6 types of degenerate configurations with reduced log canonical models for the moduli space of Burniat surfaces with $K^2 = 5$, up to the symmetry group $\mathbb{Z}_6$. All of the 6 cases could be obtained from the degenerating cases for $K^2 = 6$ listed in [AP09], with the additional condition that $A_1, B_1, C_1$ meet at a point $P$. All the lc models of the degenerate configurations come from Case 1 and 8 for $K^2 = 6$ are irreducible. Case 5 for $K^2 = 6$ does not produce any degenerations for $K^2 = 5$. We give a table with the relations between cases for $K^2 = 5$ and cases for $K^2 = 6$. This table describes how to get cases with $K^2 = 5$ possibly from cases with $K^2 = 6$ with $A_1, B_1, C_1$ meeting at one point $P$.

| $K^2 = 6$ | $K^2 = 5$ | $K^2 = 5$, further degenerations |
|-----------|-----------|----------------------------------|
| Case 1    | lc        | Case 2, 3                        |
| Case 2    | Case 1    | Case 3                           |
| Case 3    | Case 2    | Case 3                           |
| Case 4    | Case 3    |                                  |
| Case 6    | Case 4    | Case 5, 3                        |
| Case 5    | none      |                                  |
| Case 7    | Case 5    | Case 3                           |
| Case 8    | lc        |                                  |
| Case 9,10 | Case 6    |                                  |

3. Burniat surfaces with $K^2 = 4$

We consider $\mathbb{P}^2$ with 9 lines. There are two cases with two distinct points $P_1, P_2$ which are the intersections of three lines inside the triangle. We denote these two cases as a “nodal case” and a “non-nodal case”.  

16
Let $\Sigma = Bl_5 \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at 5 points $P_A, P_B, P_C$ and $P_1, P_2$.

**Definition 10.** A Burniat surface $X$ in $M_{3, Bur}^3$ is the canonical model of a $\mathbb{Z}_2^3$-cover of $\Sigma = Bl_5 \mathbb{P}^2$ for the building data $D_a = \sum_{i=0}^{3} A_i, D_b = \sum_{i=0}^{3} B_i, D_c = \sum_{i=0}^{3} C_i$, where $a, b, c$ are the 3 nonzero elements of $\mathbb{Z}_2^3$.

Let $D = \frac{1}{2} (D_a + D_b + D_c)$, then we have

$$K_X^2 = (\pi^*(K_\Sigma + D))^2 = \left(\pi^*(-\frac{1}{2}K_\Sigma)\right)^2 = 4 \left(\frac{1}{4}K_\Sigma^2\right) = 4.$$  

For the nodal case, the curve $A_1$ in $\Sigma$ is a $(-2)$-curve and $K_\Sigma.A_1 = 0$. The anti-canonical divisor $-K_\Sigma$ is nef but not ample, so $K_\Sigma + D = -\frac{1}{2}K_\Sigma$ is not ample which implies $X$ is not ample. Stable Burniat surfaces $X$ with $K_X^2 = 4$ are $\mathbb{Z}_2^3$-covers of the canonical models $\Sigma^c$ of $\Sigma$ with the building data $\frac{1}{2}D$.

For the non-nodal case, we have that $X$ is stable as that $-K_\Sigma$ is ample, and stable Burniat surfaces $X$ with $K_X^2 = 4$ are $\mathbb{Z}_2^3$-covers of $\Sigma$ with the building data $\frac{1}{2}D$.

To compactify the moduli space of stable pairs $(Y, \frac{1}{2}D)$, we will study one-parameter families of configurations in the moduli space. For the nodal case, the general fiber $\Sigma^c$ is the blown down of a $(-2)$-curve $A_1$ of $\Sigma = Bl_5 \mathbb{P}^2$; for the non-nodal case, the general fiber $\Sigma$ is $Bl_5 \mathbb{P}^2$. 

\begin{align*}
K_X^2 &= 4 \text{ nodal} & K_X^2 &= 4 \text{ non-nodal} \\
\Sigma &\twoheadrightarrow \Sigma^c = \text{contract } A_1 \text{ in } \Sigma \\
\Sigma &\twoheadrightarrow \Sigma
\end{align*}
The general fiber $\Sigma^c$ is a singular surface with an $A_1$-singularity, which is obtained from $Bl_5\mathbb{P}^2$ by contracting the $(-2)$-curve. To see the degenerating arrangements with $K^2 = 4$, we will start with surfaces $Bl_3\mathbb{P}^2$ which are shown in the following figures.

We first consider the nodal case with $K^2 = 4$.

**Case 1.** When the curve $A_2$ degenerates to $A_0 + C_3$, $B_2$ degenerates to $A_0 + B_3$, and $C_2$ degenerates to $B_3 + C_0$. Blowing up the total space $\mathcal{Y}$ along the curve $L_{P_3}$ and the curve $A_0$ in $\Sigma_0$, we see the general fibers are $Bl_4\mathbb{P}^2$ and the central fiber is $\Sigma_0 = Bl_4\mathbb{P}^2 \cup \mathbb{F}_1$. Next we blow up the total space along the strict preimage of $B_3$ in the component $Bl_4\mathbb{P}^2$ of $\Sigma_0$ and along the strict transform $L_{P_2}$, which results in the central fiber becoming a union of three components $Bl_4\mathbb{P}^2 \cup Bl_1\mathbb{F}_1 \cup \mathbb{F}_0$ (see the first figure of the second row below). Running the minimal model program, we get the lc model with central fiber $\mathbb{F}_0 \cup \mathbb{P}^2 \cup \mathbb{P}^2$ and we call it Case 1. The further degeneration is 4 copies of $\mathbb{P}^2$ and we call it Case 2.
Case 3. When one of the two points $P_1, P_2$ is on $B_0$ or $B_3$. Each degenerating arrangement is the same up to rotation. WLOG, we can assume that $P_2$ is on $B_0$. To get the minimal model, we first blow up the total space $\mathcal{Y}$ along the curve $B_0$ in the central fiber. Let curves $\widetilde{L_{P_1}}$ and $\widetilde{L_{P_2}}$ be the proper transform of $L_{P_1}$ and $L_{P_2}$. Then blow up $\mathcal{Y}_1 = Bl_1 \mathcal{Y}$ along $\widetilde{L_{P_1}}$ and $\widetilde{L_{P_2}}$. The central fiber $\Sigma_0$ becomes $Bl_4 \mathbb{P}^2 \cup Bl_1 \mathbb{F}_1$.

The canonical model of $\Sigma_0$ is $\Sigma^c_0 = \mathbb{F}_0 \cup \mathbb{F}_0$ which we denote as Case 3. The first figure below is obtained from the component $Bl_4 \mathbb{P}^2$ by contracting 3 curves $A_1, A_3, C_3$. The second figure below is obtained from the component $Bl_1 \mathbb{F}_1$ by contracting the curve $A_1$. This case could be obtained from Case 4 for $K^2 = 5$.

There are further degenerations; however, the further degenerations do not produce any new cases. For example, when the point 3 on the double locus goes to the point 4 in the above figures, the lc model of the further degeneration is the same as Case 2.

Case 4. When $P_1, P_2$ coincide. Blow up the total space $\mathcal{Y}$ at the point $P$ in the central fiber, the central fiber is as follows.
Then we blow up the total space along the proper transform \( \tilde{L}_{P_1} \) and \( \tilde{L}_{P_2} \). The central fiber of the minimal model is \( Bl_4\mathbb{P}^2 \cup Bl_2\mathbb{P}^2 \). Running the minimal model program, we contract \( A_1, A_2, B_1, B_2, C_1, C_2 \) in the component \( Bl_4\mathbb{P}^2 \) and \( A_1 \) in the component \( Bl_2\mathbb{P}^2 \) in the central fiber. The central fiber of the canonical model is \( F_0 \cup F_0 \) as follows.

**Case 5.** The further degeneration of Case 4 above. After blowing up the total space at \( P \), the points \( P_1, P_2 \) could still coincide in the exceptional divisor \( \mathbb{P}^2 \) of the blowup. We need to blow up the total space at the point \( P \) first, then \( P_1, P_2 \) will be distinct. Now we can blow up the total space along the lines \( \tilde{L}_{P_1} \) and \( \tilde{L}_{P_2} \). The following figures are only the second component of the lc model, with the first component \( Bl_4\mathbb{P}^2 \), which is the same as the first figure above.

Consider the line \( A_1 \) in the component \( Bl_2\mathbb{P}^2 \) of the central fiber, we have

\[
(K_{Y} + D).A_1 = -\frac{1}{2} < 0
\]

and

\[
K_{Y_0|Bl_2\mathbb{P}^2}.A_1 = (K_{Bl_2\mathbb{P}^2} + \Delta).A_1 = 0
\]
According to the minimal model program, there will be a flip for \( (\mathcal{Y}, \frac{1}{2}D) \). The normal bundle of \( A_1 \) in the total space \( \mathcal{Y}_2 \) is \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). The flip for \( (\mathcal{Y}, \frac{1}{2}D) \) is the Atiyah flop for \( \mathcal{Y} \).

When we apply the flip, the component \( \mathbb{F}_1 \) is blown up at one point on the double locus. The central fiber becomes a union of three components \( Bl_4 \mathbb{P}^2 \cup Bl_1 \mathbb{F}_1 \cup \mathbb{F}_0 \).

We have \( (K_{\mathcal{Y}} + D) |_{\Sigma_0} C \geq 0 \) for all the curves \( C \) in \( \Sigma_0 \), and in particular \( (K_{\mathcal{Y}} + D) |_{\mathbb{P}^1 \times \mathbb{P}^1} C = 0 \) for all the curves \( C \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). Running the minimal model program, we obtain the canonical model by contracting \( A_1, B_1, C_1 \) in \( Bl_4 \mathbb{P}^2 \), \( A_1, \triangle \) in \( Bl_1 \mathbb{F}_1 \) and the whole component \( \mathbb{P}^1 \times \mathbb{P}^1 \), where \( \triangle \) is the double locus. The central fiber of the resulting canonical model is \( \mathbb{P}^1 \times \mathbb{P}^1 \cup S \), where \( S \) is obtained from \( \mathbb{F}_2 \) by contracting the (-2)-curve. So \( S \) is a surface with an \( A_1 \)-singularity. We call it Case 5.
The following correspond to the non-nodal case with $K^2 = 4$.

**Case 6.** Similar to case 3, but the point $P_1$ is on $B_0$ instead.

![Diagram of Case 6]

The central fiber of the resulting lc model is $\mathbb{P}^1 \times \mathbb{P}^1 \cup Bl_4 \mathbb{P}^2 \cup B_1 \mathbb{P}^1$. It is not isomorphic to Case 3, which is also $\mathbb{P}^1 \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \mathbb{P}^1$, since the line arrangements are not isomorphic.

There is a further degeneration as follows, where the central fiber of the resulting lc model is a union of four copies of $\mathbb{P}^2$.

![Diagram of Case 7]

**Case 7**

**Case 8.** For non-nodal case, when $A_1, A_2, B_1, B_2, C_1, C_2$ intersect at one point. The central fiber of the lc model is $\mathbb{P}^1 \times \mathbb{P}^1 \cup Bl_2 \mathbb{P}^2$. 

22
In total, there are 5 types of degenerations with reducible lc models for $K^2 = 4$ nodal case and 3 types of degenerations for $K^2 = 4$ non-nodal case up to the symmetry group $\mathbb{Z}_2$. All of them could be obtained from the cases with $K^2 = 5$.

| $K^2 = 5$ | $K^2 = 4$ | further degenerations |
|-----------|-----------|-----------------------|
| Case 1    | Case 1    | Case 2               |
| Case 2    | Case 2    |                       |
| Case 3    | Case 6    |                       |
| Case 4    | Cases 3,6 | Cases 2,7            |
| Case 5    | Case 2,7  |                       |
| Case 6    | Case 4,8  | Case 5               |

4. Burniat surfaces with $K^2 = 3$

Consider the surface $\mathbb{P}^2$ with 9 lines and 3 points $P_1, P_2$ and $P_3$ which are the intersections of 3 lines. A Burniat surface $X$ in $M^3_{\text{Bur}}$ is the canonical model of a $\mathbb{Z}_2^3$-cover of $\Sigma = Bl_6 \mathbb{P}^2$ for the building data $D_a, D_b$ and $D_c$. Here $\Sigma$ is the blown up of $\mathbb{P}^2$ at the six points $P_A, P_B, P_c$ and $P_1, P_2, P_3$.

There are three $(-2)$-curves $A_1, B_1, C_1$ in $\Sigma$ and $K_\Sigma + D = -\frac{1}{2}K_\Sigma$ is nef but not ample. The canonical model $\Sigma^c$ of $\Sigma$ is obtained from $\Sigma$ by contracting the three $(-2)$-curves. Stable Burniat surfaces $X$ with $K^2_X = 3$ are $\mathbb{Z}_2^3$-covers of the canonical models $\Sigma^c$ of $\Sigma$ with the building data $\frac{1}{2}D$. The general fiber of a one-parameter family is $\Sigma^c$ and it contains three $A_1$-singularities. We denote the singularities, by contracting from $A_1, B_1, C_1$, by $Q_1, Q_2, Q_3$. 

Case 1. The three points $P_1, P_2, P_3$ coincide. Take a one-parameter family of $\Sigma$ with the general fiber $\text{Bl}_5\mathbb{P}^2$ with the central fiber the degenerating arrangement $\Sigma_0$. Denote the one parameter family space by $Y$.

We first blow up $Y$ at the point $P$ on the central fiber. The central fiber becomes $\text{Bl}_4\mathbb{P}^2 \cup \mathbb{P}^2$.

Now we blow up the total space $Y_1 = \text{Bl}_pY$ along the curves $\tilde{L}_{P_1}$, $\tilde{L}_{P_2}$, and $\tilde{L}_{P_3}$, which are the proper transformation of $L_{P_1}$, $L_{P_2}$, and $L_{P_3}$. The central fiber turns to be $\text{Bl}_4\mathbb{P}^2 \cup \text{Bl}_3\mathbb{P}^2$. The component $\text{Bl}_3\mathbb{P}^2$ is the blowup of $\mathbb{P}^2$ at three points $P_1, P_2, P_3$. When we run the minimal model program, in the central fiber, the curves $A_1, B_1, C_2$ in the component $\text{Bl}_3\mathbb{P}^2$ are contracted. In the general fiber, the curves $A_1, B_1, C_1$ are contracted as well. Clearly we also have that $\text{Bl}_3\mathbb{P}^2$ goes back to $\mathbb{P}^2$ in the central fiber. The general fiber of the lc model is $\Sigma^c$, which we described at the beginning of this section, and the central fiber is $\mathbb{P}^1 \times \mathbb{P}^1 \cup \mathbb{P}^2$. 

24
Case 1

Case 2. When $P_1$ is on $B_0$ and $P_2$ is on $A_3$. We first blow up the total space along the curve $A_3$, then blow up along the strict preimage of $B_0$ in $Bl_3 \mathbb{P}^2$. Finally we blow up the total space along the three curves $\tilde{L}_{P_1}$, $\tilde{L}_{P_2}$ and $\tilde{L}_{P_3}$. The central fiber of the minimal model is $Bl_3 \mathbb{P}^2 \cup Bl_3 \mathbb{F}_1 \cup Bl_1 \mathbb{F}_0$. We obtain the central fiber $\mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2$ of the canonical model by contracting the lines labeled in the fourth figure below. The general fiber of the lc model is again $\Sigma^\vee$.

There are only 2 types of degenerations with reducible lc models for $K^2 = 3$. Both of them could be obtained from cases with $K^2 = 4$.

We summarize the above computations in the following statement:

Theorem [1]. The main component of the compactified coarse moduli space $\overline{M}_{\text{Bur}}^{d}$ of stable Burniat surfaces, or equivalently, of stable pairs $(Y, \frac{1}{2}D)$, is of dimension $d - 2$, irreducible for $d \neq 4$, and with two components for $d = 4$. The types of degenerations, up to symmetry, are listed as below.
(i) There are 6 types of degenerate configurations of stable pairs with reduced log canonical models in the moduli space of stable pairs \((Y, \frac{1}{2}D)\) for \(K^2 = 5\) case up to the symmetry group \(\mathbb{Z}_6\) described in Section 2.

(ii) There are 5 types of degenerations with reducible lc models in the moduli space of stable pairs \((Y, \frac{1}{2}D)\) for \(K^2 = 4\) nodal case and 3 types of degeneration for \(K^2 = 4\) non-nodal case up to the symmetry group \(\mathbb{Z}_2\) described in Section 3.

(iii) There are only 2 types of degenerations with reducible lc models in the moduli space of stable pairs \((Y, \frac{1}{2}D)\) for \(K^2 = 3\) described in Section 4.

There is only one surface with \(K^2 = 2\), thus the moduli space of Burniat surfaces with \(K^2 = 2\) is just a single point.

5. Matroid tilings of polytopes \(\triangle_{Bur}^d, d \leq 5\)

According to the general theory of [Ale08], the unweighted stable hyperplane arrangements are described by matroid tilings of the hypersimplex \(\triangle(r, n)\). Their weighted counterparts are described by partial tilings of the hypersimplex \(\triangle(r, n)\) in \(\mathbb{R}^n\). In this section, we will discuss the matroid tiling of the certain polytopes \(\triangle_{Bur}^d, d \leq 5\) corresponding to Burniat surfaces with \(K^2 = d\).

In [AP09], Alexeev-Pardini defined the polytope \(\triangle_{Bur}^6\) corresponding to Burniat surfaces with \(K^2 = 6\), which is a subpolytope of a hypersimplex \(\triangle(3, 9)\). They computed all stable Burniat surfaces with \(K^2 = 6\) by computing matroid tilings of a certain polytope \(\triangle_{Bur}^6\). We define the corresponding polytopes \(\triangle_{Bur}^d, d \leq 5\) similarly to \(\triangle_{Bur}^6\). We restrict the matroid tilings of the polytope \(\triangle_{Bur}^d\) to \(\triangle_{Bur}^{d-1}\) for \(d \leq 6\) to find all possible stable surfaces in the main component of the compactified moduli space of Burniat surfaces with \(K^2 = 5\).

Let’s recall some definitions and results in [AP09][Ale08]. A hyper-simplex \(\triangle(r, n)\) is defined to be a convex hull

\[
\triangle(r, n) = \text{Conv}(e_I | I \in \pi, |I| = r) = \{(x_1, ..., x_n) \in \mathbb{R}^n | 0 \leq x_i \leq 1, \sum x_i = r\}
\]

A matroid polytope \(BP_V \subset \triangle(r, n)\) is the polytope corresponding to the toric variety \(\overline{T V}\) for some geometric point \([V \subset \mathbb{A}^n] \in G(r, n)(k)\).

One can also describe the matroid polytopes in terms of hyperplane arrangements. Let \(\mathbb{P}V \simeq \mathbb{P}^r\) and assume that it is not contained in the \(n\) coordinate hyperplane \(H_i\) (i.e. all \(z_i \neq 0\) on \(\mathbb{P}V\)); let \(B_1, ..., B_n \subset \mathbb{P}V\) be \(H_i \cap \mathbb{P}V\). Then for the hyperplane arrangement \((\mathbb{P}V, \sum B_i)\), the
matroid polytope $BP_V$ is the convex hull of the points $v_I \in \mathbb{Z}^n$ for all $I \subseteq \pi$ such that $\cap_{i \in I} B_i = \emptyset$, or in terms of inequalities as

$$BP_V = \left\{ (x_1, ..., x_n) \in \Delta(r, n) \mid \sum_{i \in I} x_i \leq \text{codim} \cap_{i \in I} B_i, \forall I \subset \pi \right\}.$$

For a hyperplane arrangement in general position, one has $BP_V = \Delta(r, n)$.

Let $b = (b_1, ..., b_n)$ be a weight, a $b$-cut hypersimplex is

$$\Delta_b(r, n) = \left\{ (x_1, ..., x_n) \mid 0 \leq x_i \leq b_i, \sum x_i = r \right\} = \left\{ \alpha \in \Delta(r, n) \mid \alpha \leq b \right\}.$$

We have the theorem in [Ale08]

**Theorem.** [Ale08 2.12] The matroid polytope $BP_V$ is the set of points $(x_i) \in \mathbb{R}^n$ such that the pair $(PV, \sum x_i B_i)$ is lc and $K_{PV} + \sum x_i B_i = 0$; the interior $\text{Int}BP_V$ is the set of points such that $(PV, \sum x_i B_i)$ is klt and $K_{PV} + \sum x_i B_i = 0$.

A tiling of the $b$-cut hypersimplex $\Delta_b$ is a partial matroid tiling of $\Delta(r, n)$ such that $\cup BP_{M_j} \supset \Delta_b$ and such that all base polytopes $BP_{M_j}$ intersect the interior of $\Delta_b$.

Let $(PV, \sum b_i B_i)$ be a hyperplane arrangement. For a point $p \in PV$, we denote by $I(p)$ the set of $i \in \pi$ such that $p \in B_i$. We define $\Delta^p_0$ to be the face (possibly empty) of $\Delta_b$, where $x_i = b_i$ for all $i \in I(p)$.

**Theorem.** [Ale08 6.6] Let $(PV, \sum b_i B_i)$ be a hyperplane arrangement of general type. Suppose $BP_M \cap \Delta_b \neq \emptyset$. Then $(PV, \Sigma b_i B_i)$ is lc at $p$ if and only if $BP_M \cap \Delta^p_0 \neq \emptyset$.

Now let us look at Burniat surfaces with $K^2 = 5$.

A Burniat surface with $K^2 = 5$ is a $\mathbb{Z}_2^2$-cover of $Bl_4\mathbb{P}^2$ for the data

$$D = \sum (a_iA_i + b_iB_i + c_iC_i + eE)$$

where $D_a, D_b, D_c$ are branched divisors of the Galois cover and $E$ is not.

Denote by $\Delta^d_{Bar}$ the polytope corresponding to Burniat surfaces with $K^2 = d$. This is a subpolytope of the hypersimplex $\Delta(3, 9)$ with weight $b = (\frac{1}{2}, ..., \frac{1}{2})$. In [AP09], the polytope $\Delta^6_{Bar}$ is defined to be
\[ \triangle(3, 9) \supset \triangle_{Bur}^{9} = \{(a_0, a_1, a_2, b_0, b_2, c_0, c_2, c_3) \in \mathbb{R}^9 \text{ satisfying} \]
\[
\begin{align*}
0 &\leq a_i, b_i, c_i \leq \frac{1}{2}, e \leq 0; \\
\sum_{i=0}^{2} (a_i + b_i + c_i) &= 3; \\
0 &\leq a_3 = c_0 + c_1 + c_2 + b_0 - 1 \leq 1/2; \\
0 &\leq b_3 = a_0 + a_1 + a_2 + c_0 - 1 \leq 1/2; \\
0 &\leq c_3 = b_0 + b_1 + b_2 + a_0 - 1 \leq 1/2; \\
e &\leq a_1 + b_1 + c_1 - 1 \leq 0\}
\]

For the case \( K^2 = 5 \), the divisor \( D \) on \( \Sigma \) satisfies \( K_{\Sigma} + D = 0 \) and we got an extra equation \( e = a_1 + b_1 + c_1 - 1 \) comparing to \( \triangle_{Bur}^{6} \). Since the cover \( \pi : X \to Y \) is unramified over \( E \), the coefficient \( e \leq \frac{r-1}{r} = 0 \), where \( r = 1 \) is the ramification index. Then we define

\[ \triangle(3, 9) \supset \triangle_{Bur}^{5} = \{(a_0, a_1, a_2, b_0, b_2, c_0, c_2, c_3) \in \mathbb{R}^9 \text{ satisfying} \]
\[
\begin{align*}
0 &\leq a_i, b_i, c_i \leq \frac{1}{2}, e \leq 0; \\
\sum_{i=0}^{2} (a_i + b_i + c_i) &= 3; \\
0 &\leq a_3 = c_0 + c_1 + c_2 + b_0 - 1 \leq 1/2; \\
0 &\leq b_3 = a_0 + a_1 + a_2 + c_0 - 1 \leq 1/2; \\
0 &\leq c_3 = b_0 + b_1 + b_2 + a_0 - 1 \leq 1/2; \\
e &\leq a_1 + b_1 + c_1 - 1 \leq 0\}
\]

We need to classify all matroid tilings of the polytope \( \triangle_{Bur}^{5} \). In [AP09], the authors listed all the nonempty intersection of maximal-dimensional matroid polytopes \( BP_M \) with the interior of \( \triangle_{Bur}^{6} \) in Table 1. It is easy to see that all the base polytopes \( BP_M \) listed in Table 1 [AP09] still intersect \( \triangle_{Bur}^{5} \).

Now we restrict the matroid tilings of the polytope \( \triangle_{Bur}^{6} \) to \( \triangle_{Bur}^{5} \) and list the tilings corresponding to degenerations of stable Burniat surfaces of degree 5, with reducible lc models \((Y, \frac{1}{2} D)\). We will give the explanation below the table where we use \( a_i b_j c_k \leq 1 \) as the abbreviation of \( a_i + b_j + c_k \leq 1 \).
In Table 1, tiling #1 for $K^2 = 5$ is the union of 3 matroid polytopes $BP_{M_1} \cup BP_{M_2} \cup BP_{M_3}$, where

$$BP_{M_1} = \{a_0 + a_2 + b_2 \leq 1, b_2 + b_3 + c_2 \leq 1\} \cap \triangle(3, 9)$$

$$BP_{M_2} = \{a_1 + c_0 + c_2 \leq 1, a_1 + a_3 + b_1 \leq 1\} \cap \triangle(3, 9)$$

$$BP_{M_3} = \{a_2 + c_1 + c_3 \leq 1, b_0 + b_3 + c_1 \leq 1\} \cap \triangle(3, 9)$$

Tiling #1 for $K^2 = 5$ is the same as the tiling #2 in [AP09] for $K^2 = 6$.

We compare all the tilings of $\triangle_{Bur}^6$ in [AP09] with tilings of $\triangle_{Bur}^5$ listed above. The tiling #2 of $\triangle_{Bur}^6$ is

$$BP_{M_1} = \{a_0a_3a_2 \leq 1, c_3a_1a_2 \leq 1\}$$

$$BP_{M_2} = \{b_0b_1b_2 \leq 1, a_3b_1b_2 \leq 1\}$$

$$BP_{M_3} = \{c_0c_1c_2 \leq 1, b_3c_1c_2 \leq 1\}$$

The restriction of the tiling #2 of $\triangle_{Bur}^6$ is the tiling #1 of $\triangle_{Bur}^5$. All the tilings of $\triangle_{Bur}^5$ above come from the restriction of the tilings of $\triangle_{Bur}^6$. However, not all the restrictions of the tilings of $\triangle_{Bur}^6$ correspond to stable Burniat surfaces of degree 5. There are several special cases to consider.
The tiling \#8 of \( \Delta_{Bur}^6 \) is the union of two base polytopes
\[
BP_{M_1} = \{a_1 + a_2 + b_1 + b_2 + c_1 + c_2 \leq 2\} \cap \Delta(3,9) \\
BP_{M_2} = \{a_0 + b_0 + c_0 + c_2 \leq 1\} \cap \Delta(3,9)
\]
For the polytope \( \Delta_{Bur}^5 \), we have the inequalities \( a_1 + b_1 + c_1 \leq 1 \) and \( 0 \leq a_2, b_2 \leq \frac{1}{2} \). These two inequalities imply \( a_1 + a_2 + b_1 + b_2 + c_1 \leq 2 \). Hence \( \Delta_{Bur}^5 \subset BP_{M_1} \) and the corresponding line arrangement for \( BP_{M_1} \) is lc. This coincides with what we got in Section 2.

The tiling \#10 of \( \Delta_{Bur}^6 \) composes of 3 matroid polytopes,
\[
BP_{M_1} = \{a_1 + a_2 + b_1 + b_2 + c_1 + c_2 \leq 2\} \cap \Delta(3,9) \\
BP_{M_2} = \{a_0 + b_0 + c_0 \leq 1, a_1 + a_2 + b_1 + b_2 + c_1 \leq 2\} \cap \Delta(3,9) \\
BP_{M_3} = \{a_0 + b_0 + c_0 + c_2 \leq 1\} \cap \Delta(3,9)
\]
The hypersimplex \( \Delta(3,9) \) lies in the \( \left\{ \sum_{i=0}^{2} a_i + b_i + c_i = 3 \right\} \), and the complement of \( BP_{M_3} \) in \( \Delta(3,9) \) is \( \{a_1 + a_2 + b_1 + b_2 + c_1 \geq 2\} \). For the polytope \( \Delta_{Bur}^5 \), the conditions \( a_1 + b_1 + c_1 \leq 1 \) and \( a_2 \leq \frac{1}{2}, b_2 \leq \frac{1}{2} \) imply \( a_1 + a_2 + b_1 + b_2 + c_1 \geq 2 \). Therefore, \( BP_{M_3} \cap \text{int}(\Delta_{Bur}^5) = \emptyset \). The restriction of the tiling \#10 of \( \Delta_{Bur}^6 \) and the tiling \#9 of \( \Delta_{Bur}^6 \) are the same and is our tiling \#6 of \( \Delta_{Bur}^5 \).

The tiling \#1 of \( \Delta_{Bur}^6 \) composes of 3 matroid polytopes,
\[
BP_{M_1} = \{a_0 + a_1 + a_2 \leq 1, a_1 + a_2 + c_3 \leq 1\} \cap \Delta(3,9) \\
BP_{M_2} = \{b_0 + b_1 + b_2 \leq 1, a_3 + b_1 + b_2 \leq 1\} \cap \Delta(3,9) \\
BP_{M_3} = \{c_0 + c_1 + c_2 \leq 1, b_3 + c_1 + c_2 \leq 1\} \cap \Delta(3,9)
\]
This three matroid polytopes correspond to three degenerations with \( K^2 = 6 \). The restriction of the tiling is a tiling of \( \Delta_{Bur}^5 \) as well, but it does not correspond to any stable Burniat surface of degree 5. To get the further degeneration with \( A_1, B_1, C_1 \) intersecting at a point, we can degenerate for instance \( B_1 \) to \( A_0 + C_3 \). The corresponding base polytope is
\[
BP_M = \{a_0 + a_1 + a_2 + b_1 \leq 1, a_1 + a_2 + c_3 \leq 1, b_1 + b_0 \leq 1\} \cap \Delta(3,9)
\]
Since we have \( \Sigma_{i=0}^{3} a_i + b_i + c_i = 3 \), the inequality \( a_0 + a_1 + a_2 + b_1 \leq 1 \) is equivalent to \( b_0 + b_2 + c_0 + c_1 + c_2 \geq 2 \). For \( \Delta_{Bur}^5 \), we have \( a_3 = c_0 + c_1 + c_2 + b_0 - 1 \), so \( b_0 + b_2 + c_0 + c_1 + c_2 \geq 2 \) is the same as \( b_2 + a_3 \geq 1 \). But \( 0 \leq b_2, a_3 \leq \frac{1}{2} \) for \( \Delta_{Bur}^5 \), hence \( BP_M \cap \text{int}(\Delta_{Bur}^5) = \emptyset \) and the base polytope \( BP_M \) does not correspond to a degeneration with \( K^2 = 5 \). This means the further degeneration is a lc degeneration for \( K^2 = 5 \).
with reducible lc model. We check all the possibility of the degenerations coming from Case 1 in [AP09] and find out that the restriction of the tiling #1 does not correspond to any degenerations with $K^2 = 5$.

The tiling #5 of $\Delta^6_{Bur}$ consists of 6 matroid polytopes. WOLG, we pick one matroid polytope in the tiling $BP_M = \{a_0 + a_1 + b_2 \leq 1, a_1 + a_2 + c_3 \leq 1, b_2 + b_3 + c_1 \leq 1\} \cap \triangle(3,9)$

If we force the condition that $A_1, B_1, C_1$ intersect at one point, to the line arrangement corresponding to the polytope with $K^2 = 6$, then the resulting degeneration has a finite automorphism group. Therefore the tiling #5 of $\Delta^5_{Bur}$ is still a tiling of $\Delta^5_{Bur}$, but it does not correspond to any degenerations with $K^2 = 5$.

Table 1 tells us that the 6 types of degenerations listed in Section 2 are all the degenerations up to symmetry in the main component of the compactified moduli space of Burniat surfaces with $K^2 = 5$.

We perform the same process for the cases $K^2 = 4$ and $K^2 = 3$. The following tables are for $K^2 = 4$ and $K^2 = 3$ cases. There is no need to look at tilings for $K^2 = 2$, as the moduli space for $K^2 = 2$ is just a point.

Table 2

| # | Tilings for $K^2 = 4$ | From $K^2 = 5$ |
|---|---------------------|----------------|
| 1 | $a_1c_0c_2 \leq 1, a_1a_3b_1 \leq 1; a_1a_2b_2 \leq 1, b_2b_3c_2 \leq 1; b_0b_1c_1 \leq 1, a_2c_1c_3 \leq 1$ | Case 1 |
| 2 | $b_0b_1c_1 \leq 1, a_1a_3b_1 \leq 1, a_1a_2b_2 \leq 1, c_0c_2 \leq 1; a_1a_3b_1 \leq 1, a_1a_3c_2 \leq 1, b_2b_3c_2 \leq 1$ | Case 2 |
| 3 | $b_0b_1c_1 \leq 1; b_2b_3c_2 \leq 1$ | Case 4 |
| 4 | $a_1a_2b_1b_2c_1c_2 \leq 2; a_0b_0c_0 \leq 1$ | Case 6 |
| 5 | $a_1a_2b_1b_2c_1c_2 \leq 2$ | Case 6 |
| 6 | $b_0b_1c_1 \leq 1; b_2b_3c_2 \leq 1$ | Case 3,4 |
| 7 | $b_0b_1c_1 \leq 1, a_1a_3b_1 \leq 1; a_0a_2b_2 \leq 1, b_2b_3c_2 \leq 1; a_0a_2b_2 \leq 1, a_2c_1c_3 \leq 1, b_0b_1c_1 \leq 1; a_1a_3b_1 \leq 1, a_1a_3c_2 \leq 1, b_2b_3c_2 \leq 1$ | Case 5 |
| 8 | $a_1a_2b_1b_2c_1c_2 \leq 2; a_0b_0c_0 \leq 1$ | Case 6 |
Table 3

| #  | Tilings for $K^2 = 3$                                                                 | From $K^2 = 4$ |
|----|--------------------------------------------------------------------------------------|----------------|
| 1  | $a_1a_2b_1b_2c_1c_2 \leq 2; a_0b_0c_0 \leq 1$                                       | Case 4         |
| 2  | $b_0b_2c_2 \leq 1, a_2a_3b_2 \leq 1; a_1c_1c_2 \leq 1, b_1b_3c_1 \leq 1; \quad a_0a_1b_1 \leq 1, a_1c_1c_3 \leq 1$ | Case 1,3       |

REFERENCES

[Ale94] V. Alexeev, *Boundedness and $K^2$ for log surfaces*, Internat. J. Math. 5 (1994), no. 6., 91-116.
[Ale96a] V. Alexeev, *Log canonical singularities and complete moduli of stable pairs*, arXiv:alg-geom/9608013 (1996).
[Ale96b] V. Alexeev, *Moduli spaces $M_{g,n}(W)$ for surfaces*, Higher-dimensional complex varieties on (Trento, 1994), de Gruyter, Berlin, 1996, pp. 1-22.
[Ale08] V. Alexeev, *Weighted grassmannians and stable hyperplane arrangements*, arXiv:0806.0881.
[AP09] V. Alexeev and R. Pardini, *Explicit compactifications of moduli spaces of Campedelli and Burniat surfaces*, arXiv:0901.4431 [math.AG] (2009).
[AP12] V. Alexeev and R. Pardini, *Non-normal abelian covers*, Compositio Mathematica 148 (2012), no. 4, 1051-1084.
[BC09b] I. Bauer, F. Catanese, *Burniat surfaces I: fundamental groups ans moduli of primary Burniat surfaces*, European Mathematical Society (EMS). EMS Series of Congress Reports, (2011), 49-76.
[BC10a] I. Bauer, F. Catanese, *Burniat surfaces II:secondary Burniat surfaces form three connected components of the moduli space*, Invent. Math. 180 (2010), 559-588.
[BC10b] I. Bauer, F. Catanese, *Burniat surfaces III: deformations of automorphisms and extended Burniat surfaces*, arXiv:1012.3770.
[BHPV] Barth, Hulek, Peters, Van De Ven, *Compact Complex Surfaces*, 2ed.
[Bu66] P. Burniat, *Sur les surfaces de genre $P_1^2 > 1$*, Ann. Math. Pura Appl. 71 (1966), 1-24.
[Ka07] M.Kawakita, *Inversion of adjunction on log canonicity*, Invent. Math. 167 (2007), 129-133.
[KSB88] J.Kollár and N.I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math 91 (1998), no. 2, 299-338.
[LP01] M.M. Lopes, R. Pardini, *A connected component of the moduli space of surfaces with $p_g = 0$*, Topology 40 (2001), no. 5, 977-991.
[Par91] Rita Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. 417 (1991), 191-213.
[Pet77] Peters, C.A.M. *on certain examples of surfaces with $p_g = 0$ due to Burniat*, Nagoya Math. J. 66 (1977), 109-119.

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