Large–order behavior of non–decoupling effects and triviality

Kenichiro Aoki†

Department of Physics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo, JAPAN 152

Abstract

We compute some non–decoupling effects in the standard model, such as the $\rho$ parameter, to all orders in the coupling constant expansion. We analyze their large order behavior and explicitly show how it is related to the non–perturbative cutoff dependence of these non–decoupling effects due to the triviality of the theory.

† email: ken@phys.titech.ac.jp
1. Introduction

Non–decoupling effects, such as the $\rho$ parameter, have played a crucial role in restricting the parameters of the standard model using the precision electroweak measurements \[1\]. In these effects, the quantum effects of heavy particles in low energy physical observables are not suppressed by the heavy particle mass. In contrast to the cases where the decoupling theorem applies, these cases arise when the increase in the particle mass is accompanied by an increase in the strength of interactions, as it is in the standard model \[2\].

In another direction, non–perturbatively, it has been established that for weak gauge couplings, the standard model is a theory defined with a physical cutoff at high energies, namely the triviality scale \[3\][\[4\][\[5\][\[6\]]. This scale is always above the symmetry breaking scale for consistency and becomes lower as the top (or the Higgs) becomes heavier. The quantities measured at low energies should be insensitive to these high energy effects if the decoupling theorem applies. In non–decoupling effects, this does not necessarily hold since it actually sees the physics of the symmetry breaking scale even though they are measured at low energies. Indeed, it has been shown non–perturbatively using the $1/N_F$ expansion that when the cutoff is of the order of the symmetry breaking scale, the non–decoupling effects inevitably exhibit non–universal behavior \[7\][\[8\]. (See also \[9\] where a similar conclusion is drawn from another approach.) This ambiguity is an inherent limitation of the standard model which needs to be defined with a high energy cutoff. The cutoff dependence of non–decoupling effects is the sensitivity at low energies to the physics beyond the standard model.

In this work, we will explicitly compute the perturbative behavior of some non–decoupling effects in the standard model to all orders, to leading order in the $1/N_F$ expansion. Then we will show, in some cases, how the large order behavior is related to the aforementioned cutoff dependence of these non–decoupling effects necessitated by the triviality of the standard model. There has been considerable amount of work on the large order behavior of perturbation theory. (For a general review and references on the subject, see, for instance \[10\][\[11\].) To our knowledge, our work is novel in that we explicitly show how the large order behavior of perturbation theory is related to the physical effects of triviality using a controlled non–perturbative approximation. In non–decoupling effects, these non–perturbative effects become substantial when the top or the Higgs mass is large.

First, let us consider the top contribution to the $\rho$ parameter in the standard model. The one loop expression, to leading order in the gauge couplings, is

$$\delta \rho \big|_{\text{top},1\text{–loop}} = \frac{3}{(4\pi)^2} \sqrt{2} G_F m_t^2 = \frac{3 y_t^2}{32 \pi^2}$$

Here, $y_t$ is the Yukawa coupling for the top and $G_F$ is the Fermi’s constant $(\sqrt{2} G_F)^{-1} \equiv v^2 = (246 \text{GeV})^2$. We see that this is a typical non–decoupling effect wherein the contribution to a low energy measurable parameter from a virtual top increases as a power of the top mass rather than fall off. The top contribution to the $\rho$ parameter to leading order in the $1/N_F$ expansion is \[7\]

$$\delta \rho \big|_{\text{top}} = \frac{N_C}{v^2} \int_{k^2 < \Lambda^2} \frac{d^4 k}{(2\pi)^4} \frac{m_t^4}{k^2 \left[ (1 - \alpha_y \log(k^2/s_0)) k^2 + m_t^2 \right]^2}$$
$N_F$ is the number of flavors and $N_C$ is the number of colors; in the standard model, $N_F = 2$ and $N_C = 3$. The Yukawa coupling constant, $y_t$, renormalized at the scale $s_0$ has been used in the expression and the notation $\alpha_y \equiv y_t^2 N_F/(32\pi^2)$ has been used for brevity. Also, $m_t^2 \equiv y_t^2 v^2/2$. The renormalization point $s_0$ is arbitrary. The integral is not finite unless we cutoff the integral at the scale $\Lambda^2$. The cutoff is $\Lambda^2$ is smaller than, but is of the same order as the triviality scale, $s_{\text{triv}}^t = s_0 \exp(1/\alpha_y)$. This triviality scale has a typical non–perturbative dependence on the coupling and is a physical parameter independent of the renormalization scale $s_0$. We note that the cutoff for the non–perturbative expression for the $\rho$ parameter in (2) is not only natural from the point of view of triviality, but necessary. The existence of the cutoff leads to a cutoff dependence of the $\rho$ parameter. These non–universal effects can be substantial when the cutoff is of the order of the symmetry breaking scale, $v^2$.

Let us analyze $\delta \rho|_{\text{top}}$ in the perturbative context: To leading order in the $1/N_F$ expansion, we may obtain the perturbative series to all orders in the coupling constant expansion as

$$
\delta \rho|_{\text{top}} = \frac{1}{N_F} \sum_{n=0}^{\infty} A_n^\gamma \alpha_y^{n+1}, \quad A_n^\gamma \equiv (n+1) \int_0^\infty dx \frac{(x \log x)^n}{(x+1)^{n+2}}
$$

(3)

In the perturbative context, there is no triviality scale; it is both unnatural and unnecessary to cutoff the momentum integrals and we shall not do so. We have chosen to renormalize at the scale $s_0 = m_t^2$ for convenience.

The large order behavior of the coefficients of the perturbative series may be obtained using a saddle point approximation as

$$
A_n^\gamma = (n+1)! \left( 1 + O\left( \frac{1}{n^2} \right) \right)
$$

(4)

Numerically, the agreement is better than 0.1% for $n \geq 14$. A couple of comments are in order: First, we see that the perturbative series has zero radius of convergence. This is not surprising since we do not expect the theory to make sense when the Yukawa coupling constant is imaginary [12]. Second, the perturbative series is not even Borel summable[13]. That is, using the integral expression for $n!$, we may derive an integral expression incorporating the higher order behavior of the perturbative series for $\delta \rho|_{\text{top}}$ as

$$
\delta \rho|_{\text{top}} \simeq \frac{1}{\alpha_y N_F} \int_0^\infty dz \frac{z e^{-z/\alpha_y}}{1 - z}
$$

(5)

This expression is ill–defined due to the existence of a pole on the positive real axis in the integrand. From this, we see that there is no obvious way to obtain an unambiguous prediction from the perturbative result. Within the perturbative context, it is not clear whether this is a limitation of the perturbation theory or something deeper.

In view of the non–perturbative understanding explained previously, we may understand the physics underlying the breakdown of the perturbation theory in $\delta \rho|_{\text{top}}$. Using the resummed expression (3), we could try to make sense out of the whole perturbative series by somehow regulating the integral, for instance, by cutting off the integral at $z < 1$. 

2
This, however, leads to inherent ambiguities in the result, which we can estimate to be of order $1/\alpha_y \exp(-1/\alpha_y)$. These ambiguities are none other than the cutoff dependence of the non–perturbative result for $\delta \rho |_{\text{top}}$ in (2) introduced by triviality. Indeed, these cutoff effects may be obtained from (2) to be $O(v^2/s_{\text{triv}}^0) = O(1/\alpha_y \exp(-1/\alpha_y))$. As previously mentioned, the renormalization scale $s_0$ was chosen to be at $m_t^2$. Had we chosen another scale, the coupling constant expansion would have been slightly more complicated, but it would not have changed the properties of the perturbative series, in particular, its non–Borel summability and the consequent ambiguity. This is consistent with the fact that this ambiguity, or equivalently, the cutoff effects in $\delta \rho |_{\text{top}}$ are physical so that it should not depend on the choice of the renormalization scheme at all.

Next, we consider the Higgs contribution to the $\rho$ parameter in the standard model. The one loop expression is

\[
\delta \rho |_{\text{Higgs},1\text{--loop}} = -\frac{3}{4 g'^2} \frac{M_W^2}{(4\pi)^2} \log \frac{M_H^2}{M_W^2} \tag{6}
\]

In this case, the low energy observable grows logarithmically with respect to the heavy particle mass, in accordance with the screening theorem [14]. The non–perturbative expression for $\delta \rho |_{\text{Higgs}}$, to leading order in the $1/N_F$ expansion, is [7]

\[
\delta \rho |_{\text{Higgs}} = \left(-\frac{3}{4 g'^2}\right) \int_{k^2 < \Lambda^2} \frac{d^4k}{(2\pi)^4} \frac{M_W^2}{(k^2 + M_W^2)(k^2 + M_Z^2) \left[(1 - \alpha\lambda \log k^2/s_0)k^2 + M_H^2\right]} \tag{7}
\]

Here, we used the notations $M_W \equiv gv/2$, $M_Z = gv/(2 \cos \theta_W)$, $M_H^2 \equiv 2\lambda Hu^2$, $\alpha\lambda \equiv \lambda H N_F/(8\pi^2)$ and $\lambda_H$ is the renormalized coupling constant at the scale $s_0$. As in the top case, the non–perturbative expression is not finite unless a cutoff $\Lambda^2$ is imposed, which leads to the cutoff dependence of the result. The triviality scale is $s_{\text{triv}}^H \equiv s_0 \exp(1/\alpha\lambda)$ and the cutoff is $\Lambda^2 \lesssim s_{\text{triv}}^H$.

We may obtain the perturbative series to all orders to leading order in the $1/N_F$ expansion as

\[
\delta \rho |_{\text{Higgs}} = -\frac{3}{4 g'^2} \left(\sum_{n=0}^{\infty} A_n^\lambda \alpha\lambda^n\right), \quad A_n^\lambda = \int_0^\infty dx \frac{x^{n+1} \log^n x}{(x + \beta_1)(x + \beta_2)(x + 1)^{n+1}} \tag{8}
\]

where $\beta_1 \equiv M_W^2/M_H^2$, $\beta_2 \equiv M_Z^2/M_H^2$. The large order behavior of the coefficients in the series is

\[
A_n^\lambda = n! \left(1 + O(\frac{1}{n^2})\right) \tag{9}
\]

This is similar to the top case and the resummed series is

\[
\delta \rho |_{\text{Higgs}} \simeq \left(-\frac{3}{4 g'^2}\right) \frac{1}{\alpha\lambda} \int_0^\infty dz \frac{e^{-z/\alpha\lambda}}{1 - z} \tag{10}
\]

which again is ill–defined due to the pole at $z = 1$. This leads to the ambiguity in the resummed series of order $g'^2 v^2/s_{\text{triv}}^H$ which again may be identified with the cutoff effects in
\[\delta \rho_{\text{top}} \text{ due to triviality. As we can see from the one loop results (1), (6), the top and the Higgs cases differ qualitatively. However, the non–perturbative ambiguity due to triviality and its relation to the large order behavior of their perturbative expansion is essentially the same, apart from the overall factor of } g^2.\]

The singularity in the Borel transform we have found is ultraviolet in origin and is sometimes called a “renormalon” effect [10] [13]. This should be distinguished from the divergence of perturbation series which is infrared in origin, such as the behavior that is associated with the classical solutions in the massless scalar theory [14]. In the theories which have no asymptotic freedom, this second kind of divergence result in alternating signs for the coefficient of the perturbation series, which can produce singularities on the negative real axis in the Borel transform. This will not prevent us from resumming the perturbation theory to obtain an unambiguous number. The role of the infrared and the ultraviolet are thought to be reversed in asymptotically free theories.

To summarize, the perturbative series for \[\delta \rho_{\text{top}}, \delta \rho_{\text{Higgs}}\] are not only divergent but also not Borel summable. This leads to ambiguities in the resummed series for \[\delta \rho_{\text{top}}, \delta \rho_{\text{Higgs}}\] which can be identified with the non–universal effects in these non–decoupling effects introduced due to triviality of the standard model. It is important to note that this argument cannot be reversed. It is possible for additional non–perturbative contributions to exist that is not apparent within the perturbation theory, even if we compute the perturbative series to all orders. In particular, even if the series is convergent, there could be non–perturbative contributions. This is clearly illustrated in the next example.

Let us consider the heavy fermion contribution to the so–called $S$ parameter [17] from the longitudinal modes of the gauge bosons. This parameter which we call $\tilde{S}$, is to one loop,

\[\tilde{S} = \frac{N_C}{12\pi}\]

per one heavy fermion multiplet. In this non–decoupling effect, the contribution from the heavy particles is a constant with respect to the heavy particle mass. To leading order in the $1/N_F$ expansion, the non–perturbative expression for this parameter is [8]

\[\tilde{S} = \frac{N_C}{12\pi} \left[ 1 + \gamma^2 \left( \frac{2\log(\Lambda^2/s_{\text{triv}}^H) + 3}{(-\log(\Lambda^2/s_{\text{triv}}^H) + 1)^4} \right) \right] \quad \text{where } \gamma \equiv \frac{(4\pi)^2 \alpha^2}{N_F \Lambda^2} \]

The perturbative expansion for this parameter is exactly (11) to all orders. The perturbative series is not only Borel summable, but it has an infinite radius of convergence. However this observable also has the same kind of cutoff effects as \[\delta \rho_{\text{top}}, \delta \rho_{\text{Higgs}}\]. The cutoff dependence of $\tilde{S}$ is of $O(v^4/(s_{\text{triv}}^H)^2) = O(\alpha_y^{-2} \exp(-1/(2\alpha_y)))$, which is completely hidden from perturbation theory. It should perhaps be pointed out, however, that in the same theory, there also exist non–decoupling effects whose perturbation series are non–Borel summable, such as \[\delta \rho_{\text{top}}\] in (3).

In closing, we would like to comment on non–decoupling effects in supersymmetric theories. In general, the quantum properties of supersymmetric theories qualitatively differ from those of the non–supersymmetric theories, as exemplified by the so–called “non–renormalization theorems”. It has been shown, however, that the supersymmetric standard
model is also a theory defined with a cutoff, namely the triviality scale, in a manner similar to the standard model \[18\]. The $\rho$ parameter has been computed non-perturbatively in some cases and it has cutoff dependence due to triviality as in the non-supersymmetric case \[19\]. The contribution of the top supermultiplet to the $\rho$ parameter is essentially the same as $\delta \rho|_{\text{top}}$ analyzed above when there are no soft breaking terms and no mixing of the Higgs supermultiplets. Therefore, at least in this case, the relation between the large order behavior of perturbation theory and the non-universal effects due to triviality may be understood in a manner identical to the non-supersymmetric case. We expect the above understanding of the relation between the perturbation theory and the non-perturbative effects due to triviality to apply in general to non-decoupling effects in supersymmetric theories with triviality.

**Acknowledgments:** We would like to thank Norisuke Sakai and Hidenori Sonoda for encouragement and discussions.
References

[1] For a review on radiative corrections in the standard model, see for instance, K. Aoki, Z. Hioki, R. Kawabe, M. Konuma, T. Muta, Suppl. Progr. Theor. Phys., 73 (1982) 1. For a recent comparison with experiments, see for instance, P. Langacker, M. Luo, A. Mann, Rev. Mod. Phys. 64 (1992) 87.

[2] T. Appelquist, J. Carazzone, Phys. Rev. 11 (1975) 2856.

[3] L.D. Landau, I. Pomeranchuk, Dokl. Akad. Nauk. USSR, 102 (1955) 489.

[4] A. Hasenfratz, K. Jansen, C.B. Lang, T. Neuhaus, H. Yoneyama, Phys. Lett. 199B (1988) 531.

[5] M.B. Einhorn, G. Goldberg, Phys. Rev. Lett. 57 (1986) 2115.

[6] J. Shigemitsu, Phys. Lett. 226B (1989) 364.

[7] K. Aoki, S. Peris, UCLA preprint, UCLA/92/TEP/23 (1992).

[8] K. Aoki, S. Peris, Phys. Rev. Lett. 70 (1993) 1743.

[9] S. Cortese, E. Pallante, R. Petronzio, Phys. Lett. 301B (1993) 203.

[10] G. ’t Hooft, lectures given at Erice (1977).

[11] J.C. Le Guillou, J. Zinn-Justin, (eds.), “Large order perturbation theory”, North Holland (1990).

[12] F.J. Dyson, Phys. Rev. 85 (1952) 631.

[13] E.T. Whittacker, G.N. Watson, “A course in modern analysis”, Cambridge University Press (1902).

[14] M. Veltman, Acta Phys. Pol. B8 (1977) Phys. Lett. 70B, (1977) 253.

M.B. Einhorn, J. Wudka, Phys. Rev. 39D (1989) 39.
[15] D.J. Gross, A. Neveu, *Phys. Rev.* **D10** (1974) 3235
    B. Lautrup, *Phys. Lett.* **69B** (1977) 438
    S. Chadha, P. Olesen, *Phys. Lett.* **72B** (1977) 87
    G. Parisi, *Phys. Lett.* **73B** (1978) 65
[16] L.N. Lipatov, *JETP Lett.* **25** (1977) 104, *Sov. Phys. JETP* **45** (1977) 216
[17] M. Peskin, T. Takeuchi, *Phys. Rev. Lett.* **65** (1990) 964
[18] K. Aoki, *Phys. Rev.* **D46** (1992) 1123
[19] K. Aoki, Tokyo Institute of Technology preprint, TIT/HEP–233