Jacobi fields, conjugate points and cut points on timelike geodesics in special spacetimes

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Abstract

Several physical problems such as the ‘twin paradox’ in curved spacetimes have purely geometrical nature and may be reduced to studying properties of bundles of timelike geodesics. The paper is a general introduction to systematic investigations of the geodesic structure of physically relevant spacetimes. The investigations are focussed on the search of locally and globally maximal timelike geodesics. The method of dealing with the local problem is in a sense algorithmic and is based on the geodesic deviation equation. Yet the search for globally maximal geodesics is non-algorithmic and cannot be treated analytically by solving a differential equation. Here one must apply a mixture of methods: spacetime symmetries (we have effectively employed the spherical symmetry), the use of the comoving coordinates adapted to the given congruence of timelike geodesics and the conjugate points on these geodesics. All these methods have been effectively applied in both the local and global problems in a number of simple and important spacetimes and their outcomes have already been published in three papers. Our approach shows that even in Schwarzschild spacetime (as well as in other static spherically symmetric ones) one can find a new unexpected geometrical feature: instead of one there are three different infinite sets of conjugate points on each stable circular timelike geodesic curve. Due to problems with solving differential equations we are dealing solely with radial and circular geodesics.
1 Introduction

This paper serves as a generic introduction to and a formulation of a systematic research programme for studying the geodesic structure of a class (as wide as possible) of curved spacetimes which are physically interesting and relevant. Actually the present paper had first appeared as an arXiv preprint (arXiv:1402.3976v1[gr-qc]) and preceded two published papers [1], [2] containing detailed results on the structure in a few simplest spacetimes. After publishing these two papers and after some discussions we have realized that some items in the formulation of the programme and first of all, in the applied methods, should be made more precise. These concern the use of Gaussian normal geodesic (GNG) coordinate system; we have also found that our investigation of circular timelike geodesics in Schwarzschild spacetime, published in our first work initiating the programme, [3], was incomplete. For these reasons we revise the preprint version. We have endeavoured to make the paper self-contained and readable, therefore it is written in an expository style and contains a lot of auxiliary material which may be found elsewhere. A substantially abridged version of the paper, presenting only the original results, has been published as [4].

The programme of investigating the geodesic structure of various spacetimes has originally been motivated by the famous ‘twin paradox’ (being obviously a historical misnomer since there is no contradiction at all). The paradox may be considered on three levels of comprehending: on the lowest level one merely asks of why there is at all the asymmetry between the twins and most textbooks on special relativity do not go beyond this question, on the higher level one tries to explain why the accelerated twin is younger than the twin staying all the time in one inertial frame and here one invokes the reverse triangle inequality. Ultimately one may study the paradox in curved spacetimes and this problem has recently been discussed [5], [6], [7], [8], [9]. It turns out that when the gravitational field is present, then contrary to the conjecture stated in [9], no general rule is valid concerning of which twin is younger and one must study each case separately. The problem is of purely geometrical nature and consists in computing the lengths of various timelike
curves having common points. Assuming that these curves are worldlines of a number of twins (or more adequately ‘siblings’) one easily sees that if the number of worldlines connecting the two given endpoints is unlimited, then there is no youngest twin because the shortest timelike curve does not exist —the lower limit of their lengths is zero and it is inaccessible. Yet the problem: which timelike curve with the given endpoints is the longest one, is meaningful and directly leads to searching the geodesic structure of the spacetime and this is why it is worth studying.

According to global Lorentzian geometry we only deal with timelike curves (though some global theorems also include null geodesics) and we will not mark it each time. The problem of maximally long curves actually consists of two separate problems: local and global. In the local version of the problem one considers a bunch of infinitesimally close timelike curves emanating from the given initial point $p$ and intersecting at the endpoint $q$. If the bunch contains a timelike geodesic denoted by $γ$, this geodesic is the longest curve in the bunch provided the segment $pq$ of $γ$ does not contain a point conjugate to $p$. Here the key notion is that of conjugate points and in paper [3] we present four propositions relevant to the problem, taken from two advanced textbooks [10], [11]. Applying them one finds the locally longest timelike curve between the given endpoints, i.e. in the bunch of neighbouring timelike curves.

The global problem consists in searching for the longest curve in the whole space of all timelike curves with common endpoints. If a geodesic $γ$ is locally the longest, it needs not be globally the longest one since there may exist a timelike curve $σ$ which beyond the common endpoints is far from $γ$ and longer than it. A timelike geodesic $γ$ is globally maximal on a segment $pq$ if the segment does not contain a future cut point of $p$.

The conclusion one draws from the brief summary given in Section 3 of the global Lorentzian geometry concerning the maximal length curves is that the current knowledge of the subject provides no analytic tools to establish if the given (geodesic) curve is globally maximal or to find out the maximal geodesic emanating from the given point. This is clearly the direct consequence of the nonlocal nature of the maximal curve: one cannot use a local tool, such as a differential equation, to identify it. Only in spacetimes with some high symmetries one can directly apply a global theorem to recognize the cut points (or their absence) and identify the maximal curves. In most cases one must take into account all geodesics with the given endpoints. For this reason we mainly deal with a more tractable problem of finding out locally maximal curves since there is a well developed analytic method of searching them applying Jacobi vector fields and conjugate points on time-
like geodesics. In some cases, which are presented in detail below, we indicate which segments of special timelike geodesics are globally maximal.

The general question concerning Jacobi fields and cut points was put forward by Steven Harris: ‘What are the conditions on the various ingredients that go to make up a static spacetime, that guarantee the existence (or absence) of conjugate points along timelike geodesics?’

Our results up to now, based on a number of special spacetimes, apparently do not indicate that a general and unique answer to this question does exist. Even in the class of static spherically symmetric (SSS) spacetimes we find diverse properties and a general rule remains elusive. This may be decided only after the research programme is completed by investigations of a sufficiently large number of diverse spacetimes. (This does not suggest that the programme will last infinitely long.)

We emphasize that in the search for locally maximal worldlines one must solve the geodesic deviation equation (GDE) and to this end one must know an explicit parametric representation of the given geodesic, \( x^\alpha = x^\alpha(\tau) \), \( \alpha = 0, 1, 2, 3 \), where \( \tau \) is a scalar parameter, possibly in terms of elementary functions and this occurs rather exceptionally. Complete sets of analytic solutions to the geodesic equation are known in very few spacetimes, e.g., for Schwarzschild metric [12], [13] and only recently these solutions were found in Schwarzschild–(anti)–de Sitter spacetimes [14], [15] (and references therein).

It is not explicitly stated in these papers, nevertheless one concludes from them that at least in the case of Schwarzschild metric the timelike geodesics (both the bound and unbound orbits) may be given in the parametric form with \( x^\alpha(\tau) \) being known transcendental functions. The radial and circular geodesics (also in a general static spherically symmetric spacetime) are exceptional in that the parametric description is in terms of simple elementary functions. Besides these two special cases the geodesic deviation equation is either intractable or so difficult that it is reasonable to first learn about the geodesic structure of a wide class of spacetimes by investigating the radial and circular geodesics and only after that to attempt to deal with generic geodesic curves. We also notice that the problem for general geodesics is more tractable in the very special class of ultrastatic spherically symmetric spacetimes and in [2] we discussed the general formalism in this case and presented one example. We shall come back to these spacetimes in a forthcoming paper. In this work we focus our attention on radial and circular geodesics in general SSS spacetimes.

The paper is organized as follows. In section 2 we present the analytic

\[ ^1 \text{Presented in a private communication to L.M.S.} \]
method for determining of which segments of the given geodesic are locally maximal: the GDE for Jacobi vector fields is recasted in the form of three ordinary equations for the three Jacobi scalars together with their first integrals generated by Killing vectors. The following two sections are devoted to the global problem. Section 3 contains six theorems in global Lorentzian geometry which are relevant in the search for globally maximal timelike curves, quoted from the fundamental monograph [16]. Our experience shows that this subject is rather little known in the community of relativists. We do not directly employ all the theorems in our present calculations, rather we quote some of them just to give the ‘flavour’ of what is expected and what may be effectively done in the global problem. And we expect that these theorems will be useful in our future work. In section 4 we apply the fact that it is easy to show that if the given geodesic may be presented in the Gaussian normal geodesic (GNG) coordinates as a line of the coordinate time (spatial coordinates along it are constant), then the segment of the geodesic which lies in the coordinate domain is globally maximal (provided the chart domain is sufficiently large); this makes the GNG coordinates a useful tool in the search for these segments. These coordinates exist in any spacetime, however in most cases it is rather hard to find out the transformation from the coordinate system in which the metric is given to the GNG coordinates that are adapted to the given geodesic. The transformation may be effectively found in static spherically symmetric (SSS) spacetimes for radial geodesics (which do exist in these manifolds) and in section 4 we derive it and find the domain of the GNG chart for a number of metrics. In sections 5 and 6 we return to the local problem. The equations for the Jacobi scalars on the radial geodesics in SSS spacetimes are studied in section 5. Section 6 is devoted to Jacobi fields and conjugate points on circular geodesics in these spacetimes. Our approach allows one to find two new infinite sets of conjugate points on stable timelike circular geodesics. In a generic SSS spacetime timelike circular and radial geodesics are geometrically different, whereas in de Sitter and anti–de Sitter spaces their difference vanishes. Brief conclusions are contained in section 7.

For concreteness and as a trace of the original twin paradox, we assume that a circular geodesic is followed by the twin B and the radial one is a worldline of the twin C. When considering circular geodesics in static spherically symmetric spacetimes we shall also introduce the static nongeodesic twin A which appeared in the previous papers.
2 Locally maximal timelike curves: Jacobi fields and conjugate points

A timelike curve connecting points \( p \) and \( q \) is locally maximal in a set of nearby curves if it is a geodesic and if there are no conjugate points to \( p \) on its segment \( pq \). The conjugate points are determined by zeros of any Jacobi vector field on the geodesic. All the necessary propositions concerning the existence and properties of the conjugate points are contained in the books \[10\] and \[11\] and are briefly summarized in \[3\].

We recall that a Jacobi field on a given timelike geodesic \( \gamma \) with a unit tangent vector field \( u^\alpha (s) \) is any vector field \( Z^\mu (s) \) being a solution of the geodesic deviation equation on \( \gamma \),

\[
\frac{D^2}{ds^2} Z^\mu = R^\mu_{\alpha \beta \gamma} u^\alpha u^\beta Z^\gamma,
\]

which is orthogonal to the geodesic, \( Z^\mu u_\mu = 0 \). Geometrically \( Z^\mu \) is a connecting vector joining \( \gamma \) to an infinitesimally close geodesic \( \gamma_\varepsilon \) given by \( \bar{x}^\mu (s, \varepsilon) = x^\mu (s) + \varepsilon Z^\mu (s) \), where \( x^\mu (s) \) are coordinates of points of \( \gamma \) and \( |\varepsilon| \ll 1 \). The GDE is derived in the linear approximation in \( \varepsilon \). If \( Z^\mu (0) = 0 = Z^\mu (s_0) \) for \( s_0 \neq 0 \) and \( Z^\mu \) does not vanish identically, then it is said that \( \gamma_\varepsilon \) intersects \( \gamma \) at points \( \gamma (0) \) and \( \gamma (s_0) \). Actually \( \gamma_\varepsilon \) needs not to intersect \( \gamma \) at \( \gamma (s_0) \) and \( Z^\mu (s_0) = 0 \) means that for \( s = s_0 \) the two geodesics are close of order higher than \( \varepsilon \). A geodesic nearby to the given \( \gamma \) for not very small \( \varepsilon \) may be determined by expanding the difference between its coordinates and the coordinates of \( \gamma \) in a series of deviations,

\[
\bar{x}^\mu (s, \varepsilon) = x^\mu (s) + \varepsilon Z^\mu (s) + \frac{1}{2!} \delta^\mu x^\mu (s) + \ldots + \frac{1}{n!} \delta^n x^\mu (s) + \ldots,
\]

where the \( n \)–th deviation \( \delta^n x^\mu \) is of order \( \varepsilon^n \) and for \( n > 1 \) it is not a vector. Using this expansion analytic expressions for perturbed circular geodesics (geodesics close to a circular one) in Schwarzschild and Kerr spacetimes were found \[17\], \[18\]. Yet in the search for locally maximal curves the equation (1) derived in the lowest approximation and the conjugate points determined by its solutions are fully sufficient. (It is also worth noticing that this equation also describes in a similar way the motion of nearby free test particles in spacetimes of any dimension \( D > 4 \) \[19\].) Due to the presence of the second absolute derivative \( D^2 / ds^2 \) the GDE is very complicated and one can simplify it by removing this derivative and replacing it by the ordinary ones. To this end one expands \( Z^\mu \) in a basis consisting of three spacelike orthonormal vector
fields \(e_a^\mu(s), a = 1, 2, 3\) on \(\gamma\), which are orthogonal to \(\gamma\) and are parallelly transported along the geodesic, i.e.

\[
e_a^\mu e_b^\mu = -\delta_{ab}, \quad e_a^\mu u^\mu = 0, \quad \frac{D}{ds} e_a^\mu = 0. \tag{2}
\]

(Since we are dealing with timelike curves it is convenient to apply the metric signature \(+ - - -\).) Then \(Z^\mu = \sum_a Z_a e_a^\mu\) and the covariant vector equation (1) is reduced to three scalar second order ODEs for the scalar functions \(Z_a(s)\) (Jacobi scalars),

\[
\frac{d^2}{ds^2} Z_a = -e_a^\mu R^\mu_{\alpha\beta\gamma} u^\alpha u^\beta \sum_{b=1}^3 Z_b e_b^\gamma. \tag{3}
\]

A general Jacobi field depends on 6 integration constants appearing as a result of solving (3).

Any Killing vector field \(K^\mu\) of the spacetime generates a first integral of eq. (1) of the form \(^2\)

\[
K_\mu \frac{D}{ds} Z^\mu - Z^\mu \frac{D}{ds} K_\mu = \text{const}. \tag{4}
\]

One verifies by a direct calculation that the function on the LHS of (4) is constant along the given geodesic. The integral of motion may be recast in terms of the scalars \(Z_a\). To this end one introduces a spacetime tetrad \(e_A^\mu\), \(A = 0, 1, 2, 3\), along \(\gamma\) consisting of the spacelike vectors \(e_a^\mu(s)\) supplemented by \(e_0^\mu \equiv u^\mu\). The tetrad is orthonormal,

\[
e_A^\mu e_B^\mu = \eta_{AB} = \text{diag}(1, -1, -1, -1) \tag{5}
\]

and parallelly transported along \(\gamma\). Expanding \(Z^\mu\) and \(K^\mu = \sum_{A=0}^3 K_A e_A^\mu\) in the tetrad and inserting them into (4) one gets

\[
\sum_{a=1}^3 \left( Z_a \frac{dK_a}{ds} - \frac{dZ_a}{ds} K_a \right) = \text{const}, \tag{6}
\]

where \(K_a = -K^\mu e_{a\mu}\). If the spacetime admits \(n\) linearly independent Killing vector fields one gets \(n\) integrals of motion (6). In some cases we find that some of these integrals generated by independent Killing vectors turn out to be dependent. In general, besides few simple spacetimes, such as the maximally symmetric ones, the first integrals (6) are essential in solving equations

\[^2\]The vector index of a Jacobi vector field will always be written as a superscript and the number of the Jacobi scalar — as a subscript.
There are two approaches to finding the Jacobi vector fields. Bažański [21] gave a generic algorithm for solving the geodesic deviation equation in cases where one knows a complete integral of the Hamilton-Jacobi equation for timelike geodesics. In a subsequent work [22] the formalism was applied in Schwarzschild spacetime. This case shows that this beautiful formalism is of restricted practical use: it does not apply to circular geodesics. If one wishes to apply the algorithm to a particular type of geodesic lines, e.g. radial ones, it is necessary to first find the general solution of the geodesic deviation equation and then carefully take appropriate limits in it to this type, what makes the procedure rather cumbersome. Furthermore, at least in the Schwarzschild metric, the algorithm works in the case of radial geodesics only for worldlines escaping to the spatial infinity, what excludes finite geodesics, such as worldlines considered in the twin paradox [3]. This is why our approach is closer to that of Fuchs, who directly solved the geodesic deviation equation in static spherically symmetric spacetimes [23]. A general formula for the Jacobi field is given in his work in terms of four integrals of expressions made up of Killing vectors and constants of motion they generate. It is our experience that employing this formula is not considerably simpler than solving the equation for radial geodesics from the very beginning. Also the Fuchs’ formula does not apply to the circular geodesics and this case must be dealt with separately [24]. We therefore have not employed the Fuchs’ integral solutions and solve the GDE independently in each case under study.

To summarize, the procedure is as follows.

— Choose an interesting spacetime with some isometries (Killing vectors).
— Choose a geometrically interesting (and possibly simple, e.g. radial or circular) timelike geodesic $\gamma$ explicitly given, $x^a = x^a(\tau)$, where $\tau$ is a scalar parameter. In SSS spacetimes $\tau$ is the arc length $s$ for circular geodesics, whereas $\tau$ is different from $s$ on radial curves.
— Choose the spacelike triad $e^a_\mu(s)$ on $\gamma$ with the properties (2). It is clear that the triad is not uniquely determined by eqs. (2) and should be properly chosen as to render the equations (3) as simple as possible.
— Solve the GDE (3) applying the first integrals and find a generic solution $Z_a(\tau)$. If $\tau \neq s$ one must appropriately transform the LHS of (3).
— Consider all possible special solutions with $Z_a(0) = 0$ and seek for their zeros, $Z_a(\tau_0) = 0$ for $\tau_0 > 0$.

Then the geodesic $\gamma$ with $x^a = x^a(\tau)$ is uniquely locally maximal on the segment $0 \leq \tau < \tau_0$ and is non–uniquely locally maximal on the segment $0 \leq \tau \leq \tau_0$. If $\tau_1 > \tau_0$, then there is a timelike curve (not necessarily
geodesic) joining the points $\gamma(0)$ and $\gamma(\tau_1)$ which is longer than $\gamma$. This is an algorithmic and effective procedure for checking whether the given geodesic is the unique locally longest curve between its fixed endpoints. We emphasize that the procedure is algorithmic in the sense that one has to do a finite number of definite steps culminating with solving GDE and it is effective providing that one is capable to solve the concrete GDE. Clearly solving this equation is not an algorithmic process and limitations in finding out the solution are the main obstacle in determining locally maximal curves.

3 Global versus local

Some confusion might have arisen due to the fact that the Proposition 4.5.8 in [10] actually deals with timelike geodesics which attain a local maximum of length while we have quoted it as Proposition 2 in [3] in a version suggesting that it establishes a necessary and sufficient condition for the global maximum of length. In consequence what has been shown there is that the radial timelike geodesic in Schwarzschild spacetime is locally maximal, while that it is globally maximal is proved in the present work in section 4.

The difference between the global and local maximum of length of a timelike curve is essential both conceptually and in practice, i.e. in our ability to computationally establish a maximal curve. We recall that propositions 4.4.2 and 4.5.8 in [10] establish under what conditions conjugate points exist on a geodesic (provided it can be sufficiently extended) and that a geodesic segment free of conjugate points is locally the longest one. Yet the case of globally maximal length is quite different. Here one takes into account all timelike curves connecting $p$ and $q$ in the spacetime (actually the rigorous definitions and theorems require to take all the future directed nonspacelike curves from $p$ to $q$; for our purposes it is usually sufficient to include only future directed timelike curves). Let $\Omega_{p,q}$ denote the path space of all future directed timelike piecewise smooth curves between $p$ and $q$; each curve $\lambda$ has then the well defined length $s(\lambda) > 0$. Here the key notion is that of Lorentzian distance function $d(p,q)$ of any two points. It is defined as follows ([16], Chap. 4). If $q$ does not lie in the causal future $J^+(p)$ of $p$, then $d(p,q) = 0$ and if $q$ is in $J^+(p)$, then $d(p,q) \equiv \sup\{s(\lambda) : \lambda \in \Omega_{p,q}\}$. The distance is nonzero, $d(p,q) > 0$, if and only if $q$ is in the chronological future $I^+(p)$ of $p$. The distance function is nonsymmetric, $d(p,q) \neq d(q,p)$ and if $0 < d(p,q) < \infty$, then $d(q,p) = 0$; in some spacetimes, e.g. Reissner-Nordström one, there are points such that $d(p,q) = \infty$ and in totally vicious spacetimes there is $d(p,p) = \infty$ for all $p$. The curve $\lambda \in \Omega_{p,q}$ is said to be globally maximal (or shortly maximal) if it is the longest one in the set
\( \Omega_{p,q} \), i.e. if \( s(\lambda) = d(p,q) \). The maximal curve (usually non unique) is always a timelike geodesic (Theorem 4.13 of [16]). The definition does not imply that in an arbitrary spacetime the maximal geodesic does exist between any chronologically related points, as the counterexample of anti–de Sitter spacetime shows. Yet in globally hyperbolic spacetimes for any pair of chronologically related points \( p \) and \( q \) \((p \ll q)\) there is a maximal future directed geodesic segment \( \gamma \in \Omega_{p,q} \) with \( s(\gamma) = d(p,q) \) (Theorem 6.1 in [16]); usually it is not unique.

If a timelike geodesic is complete (it is defined for all real values of the canonical length parameter, \(-\infty < s < +\infty\)), it usually is not maximal beyond some segment from \( p \) to \( q \). A Riemannian example: a great circle arc on a sphere emanating from the north pole is maximal (in this case ‘maximal’ means ‘globally the shortest’) on the half-circle up to the south pole since points on the arc lying beyond this segment may be connected to the north pole by a shorter geodesic. This gives rise to the notion of the cut point on a geodesic. Let \( \gamma : [0,a) \to M \) be a future directed, future inextendible, timelike geodesic parameterized by its proper length \( s \) in a spacetime \((M, g)\).

Set
\[
s_0 \equiv \sup \{s \in [0,a) : d(\gamma(0), \gamma(s)) = s\}.
\]

If \( 0 < s_0 < a \), then \( \gamma(s_0) \) is said to be the future timelike cut point of \( \gamma(0) \) along \( \gamma \). For all \( 0 < s < s_0 \) the geodesic \( \gamma \) is the unique globally maximal timelike curve from \( \gamma(0) \) to \( \gamma(s) \) and is globally maximal (not necessarily unique) on the segment from \( \gamma(0) \) to \( \gamma(s_0) \), while for \( s_1 > s_0 \) there exists a future directed timelike curve \( \sigma \) from \( \gamma(0) \) to \( \gamma(s_1) \) with \( s(\sigma) > s(\gamma) \). In other terms \( s_0 \) is the length of the longest maximal segment of the given geodesic (for a fixed initial point).

**Theorem 1 (Theorem 9.10 in [16])** A timelike geodesic is not maximal beyond the first conjugate point, or equivalently: the future cut point of \( p = \gamma(0) \) along \( \gamma \) comes no later than the first future conjugate point to \( p \).

A closer connection between conjugate and cut points is revealed in

**Theorem 2 (Theorem 9.12 in [16])** Let \((M, g)\) be globally hyperbolic. If \( q = \gamma(s_0) \) is the future cut point of \( p = \gamma(0) \) along the timelike geodesic \( \gamma \) from \( p \) to \( q \), then either one or possibly both of the following hold:

i) the point \( q \) is the first future conjugate point to \( p \);

ii) there exist at least two future directed maximal timelike geodesic segments from \( p \) to \( q \).

Now consider the set of all future directed timelike geodesics emanating
from any point $p$. In general each of them has the cut point. The *future timelike cut locus* $C_i^+(p)$ of $p$ in $(M, g)$ is defined to be the set of cut points along all future directed timelike geodesic segments issuing from $p$. One may ask whether the cut locus contains a point $q$ which is the closest one to $p$, i.e. $d(p, q) \leq d(p, r)$ for all $r \in C_i^+(p)$. It turns out that

**Theorem 3 (Theorem 9.24 in [16])** If a point $p$ in a globally hyperbolic spacetime has a closest cut point $q$, then $q$ must be a point conjugate to $p$ on a geodesic.

In a noncompact complete Riemannian manifold at each point there is a direction (a tangent vector) such that the geodesic emanating from this point in this direction has no cut points. Something analogous occurs in specific spacetimes.

**Theorem 4 (Theorem 9.23 in [16])** i) In a strongly causal $(M, g)$ at each point there is a future directed nonspacelike direction such that the geodesic issuing in this direction has no cut point.

ii) In a globally hyperbolic spacetime given any point $p$, there is no farthest nonspacelike cut point of $p$.

Some Riemannian manifolds are distinguished by satisfying the topological condition of being simply connected. For Lorentzian manifolds one introduces an analogous notion of a spacetime being *future one-connected* if for all pairs of chronologically related points, $p \ll q$, any two future directed timelike curves from $p$ to $q$ are homotopic through smooth future directed timelike curves with fixed endpoints $p$ and $q$. An example (R. Geroch, quoted in [16]) shows that the topological simple connectedness does not imply that the spacetime is one-connected.

Finally one deals with properties of Jacobi vector fields on a geodesic. Let $J_t(\gamma)$ denote the vector space of smooth Jacobi vector fields $Z^\mu(s)$ along the timelike geodesic $\gamma : [a, b] \to M$ with $Z^\mu(a) = Z^\mu(t) = 0$ for some $a < t \leq b$. Then the *order* of the conjugate point $\gamma(t)$ to $p$ on the timelike geodesic $\gamma$ with $\gamma(a) = p$ is defined as $\dim J_t(\gamma)$. Applying these two notions two theorems were proved.

**Theorem 5 (Theorem 10.30 in [16])** Let $(M, g)$ be future one-connected and globally hyperbolic. Suppose that for some $p$ in $M$ the first future conjugate point on every timelike geodesic emanating from $p$ is of order two or greater. Then the future timelike cut locus of $p$ and the locus of first future timelike conjugate points to $p$ coincide. Equivalently: all future timelike...
geodesics from \( p \) are maximal up to the first future conjugate point.

**Theorem 6 (Theorem 11.16 in [16])** Let \((M, g)\) be a future one-connected globally hyperbolic spacetime with no future nonspacelike conjugate points. Then given any \( p, q \in M \) with \( p \prec\prec q \), there is exactly one future directed timelike geodesic from \( p \) to \( q \) (and is clearly maximal).

These six theorems express our basic current knowledge about maximal timelike geodesics in various spacetimes. These are mathematical ‘existence theorems’ stating the presence of some global properties if some global conditions are satisfied. They are not ‘constructive’ in the sense that they do not indicate a computationally effective procedure for obtaining the interesting object, as is seen in the two most important cases. Firstly, given two chronologically related points \( p \) and \( q \), one may indicate a geodesic connecting them and its segment is maximal if and only if the cut point of \( p \) is at \( q \) or farther. However, the location of the cut point cannot be found by investigating solely this geodesic. One then may find by geometrical and/or physical arguments a number or a continuous narrow class of geodesics joining \( p \) and \( q \) and by the direct computation get the longest curve (one or more), that free of conjugate points on the segment. In this way one determines the locally longest curve and nothing more, even if the set under consideration contains curves distant from this one. In fact, the absence of conjugate points on the locally longest segment does not imply that it does not contain cut points and the distant curves belonging to the set may not include the maximal geodesic. There is no general way out of the problem and in the search for the maximal curve from \( p \) to \( q \) one must deal with the whole space \( \Omega_{p,q} \) and the space cannot be examined in a finite number of steps.

Secondly, given point \( p \), one may ask of which timelike geodesic emanating from \( p \) contains the longest maximal segment, i.e. which cut point \( q \in C^t_+ (p) \) is farthest from \( p \). In an arbitrary spacetime in this problem again there is no shortcut and one must study all geodesics emanating from the point.

In conclusion, the difficulty lies in that there is no analytic tool, such as a differential equation, allowing one to find the cut point on the given geodesic in a finite number of steps and this is due to the very nonlocal nature of the notion. Quite the opposite, one should first study all geodesics in the space \( \Omega_{p,q} \), compute their lengths, find points where they intersect and in this way determine their cut points. Then Theorems 2 and 6 in the first problem and Theorems 3, 4 and 5 in the second problem (given the initial point) will turn out to be a compact and geometrically elegant description of the results of all the computations. Without this huge work being done, the theorems are practically useless for any quantitative problem, e.g. the twin paradox.
If one restricts the research to spacetimes which not only are both globally hyperbolic and future one-connected, but also have some high isometries, the problem of globally maximal worldlines is no more hopeless. Our experience up to now shows that spherical symmetry is useful. Another approach is based on the use of the Gaussian normal geodesic coordinates wherein timelike geodesics may be expressed in a very simple form. By joining the spherical symmetry to the use of the GNG coordinates one gets an effective tool in the research.

4 Maximal segments of radial timelike geodesics in comoving coordinates in static spherically symmetric spacetimes

The comoving, i.e. Gaussian normal geodesic (GNG) coordinates allow one to easily establish that some segments of some timelike geodesics are globally maximal. Consider a congruence of timelike geodesics which are orthogonal to a spacelike hypersurface in a spacetime. These curves may be interpreted as worldlines of freely falling point particles; the particles may be either test ones in an external gravitational field or may be forming a self–gravitating dust. The swarm of the particles determines their own rest frame — the comoving frame. In this frame the particles’ worldlines coincide with the coordinate time lines. In fact, the metric is

\[ ds^2 = d\tau^2 + g_{ij}(\tau, x^k)\, dx^i\, dx^j, \]

where \( g_{ij} \) is the negative definite 3–metric of the spacelike hypersurfaces \( x^0 \equiv \tau = \text{const} \) and \( \tau \) is the time coordinate. The congruence of timelike geodesics which are orthogonal to these hypersurfaces is given by \( x^i = \text{const} \) and their velocities, i.e. the tangent vector field, are \( u^\alpha = dx^\alpha/d\tau = \delta^\alpha_0 \), hence \( u_\alpha = \delta^0_\alpha = \partial_\alpha \tau \). Denote by \( D \) the GNG chart domain. The extent of the domain is crucial for our purpose since, as it is shown below, it is rather obvious that some segments of the congruence geodesics lying in \( D \) are globally maximal. As the GNG coordinates may be introduced in any spacetime and for any hypersurface orthogonal timelike geodesic congruence, one gets a universal tool for determining maximal segments in this class of geodesic curves. (Clearly the method works only for hypersurface orthogonal congruences and cannot be applied to a single geodesic.) The method is interesting if it provides sufficiently large segments which are globally maximal. Let a segment of the future directed geodesic \( \gamma \) belonging to the congruence,
which lies in the chart domain $D$, be parameterized by the time $\tau$ in the
interval $(\tau_1, \tau_2)$. Clearly the length of the segment is $s(\gamma) = \int ds = \tau_2 - \tau_1$. Assume that the domain $D$ is so large that any timelike future directed curve $\sigma$ joining points $\gamma(\tau_1)$ and $\gamma(\tau_2)$ lies in $D$. Let $\sigma$ also be parameterized by $\tau$, $y^\alpha = y^\alpha(\tau)$, then its length is

$$s(\sigma) = s(\sigma, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} ds = \int_{\tau_1}^{\tau_2} \left[ 1 + g_{ij}(\tau, y^k) \frac{dy^i}{d\tau} \frac{dy^j}{d\tau} \right]^{1/2} d\tau < \tau_2 - \tau_1,$$

(8)

since $g_{ij} dy^i/d\tau \ dy^j/d\tau < 0$ along $\sigma$ if the curve is different from $\gamma$.

The GNG chart domain depends both on the metric and the congruence. Depending on the initial spacelike hypersurface, the geodesic lines of the time eventually cross and develop coordinate singularities, therefore apart from exceptional cases the comoving coordinates cannot cover the whole spacetime. Yet from the proof above it is clear that one is interested in using the domain $D$ which is the largest possible part of the manifold. This means that the congruence must be carefully chosen since for most congruences the globally maximal segments are uninterestingly small (it is obvious that for any two sufficiently close points on any geodesic, the segment between them is globally maximal). This imposes severe limitations on the applicability of the method. Another restriction arises from the fact that the spacetime metric is usually given in the coordinates that exhibit geometrical properties (isometries) of the manifold. For the chosen hypersurface orthogonal geodesic congruence one must find the transformation from these coordinates (the „standard” ones) to the GNG ones which are adapted to the congruence. Usually finding out the transformation is not easy. Then one determines the domain $D$ either directly from the transformation or by determining possible coordinate singularities of the analytically extended metric in the comoving system.

In this work we consider static spherically symmetric spacetimes; in these manifolds timelike radial and circular (if exist) geodesics (they are defined in the coordinate system adapted to this symmetry) are singled out by their simplicity and physical relevance. That the radial geodesics form a congruence orthogonal to the constant time hypersurfaces is physically obvious and may be verified by a direct calculation. In this section we show that one can effectively determine the transformation law from the standard coordinates in these spacetimes to the GNG ones adapted to the radial congruence, then one can determine the GNG domain and ultimately the extent of the geodesic segments. The case of circular geodesics is more complicated and it turns out that it is more practical to determine first their locally longest segments:
there are three infinite sequences of conjugate points showing (from Theorem 1) that globally maximal segments are rather short (see section 6).

The metric of any SSS spacetime in the standard coordinates \((t, r, \theta, \phi)\) is \((c = 1)\)

\[
\begin{align*}
\text{ds}^2 &= e^{\nu(r)} \text{dt}^2 - e^{\lambda(r)} \text{dr}^2 - F^2(r) \left( \sin^2 \theta \, d\phi^2 + d\theta^2 \right),
\end{align*}
\]

(9)

where \(t \in (-\infty, \infty)\), functions \(\nu\) and \(\lambda\) are real for \(r \in (r_m, r_M)\) and \(t\) and \(r\) have dimension of length; we assume \(r_m \geq 0\). \(\nu(r)\) and \(\lambda(r)\) are given functions and \(F(r) = r\) for a generic SSS metric and \(F(r) = \text{const} = a\) for Bertotti–Robinson spacetime. The timelike Killing vector is \(K^\alpha = \kappa \delta^\alpha_0\) and \(\kappa = \text{const}\) is a normalization factor (chosen either at \(r = 0\) or at spatial infinity) depending on the spacetime. Let a timelike geodesic be the worldline of a particle of mass \(m\), then the integral of energy per unit rest mass of the particle is \(k \equiv E/(mc^2) > 0\) and is dimensionless and one finds

\[
\dot{t} \equiv \frac{dt}{ds} = \frac{k}{\kappa} e^{-\nu}.
\]

(10)

We construct the comoving system by generalizing to any SSS spacetime the method applied to Schwarzschild metric [25] (Lemaître coordinates). First one chooses a constant \(A > 0\) which depends on the specific metric, then the method applies in the interval \((r_1, r_2) \subset (r_m, r_M)\) such that \(e^{\nu(r)} \leq A^2 < \infty\). The transformation to the comoving coordinates \((x'^\mu) = (\tau, R, \theta, \phi)\) is

\[
\begin{align*}
\tau &= t + \int e^{-\nu} f(r) \, dr, \\
R &= t + \int e^\lambda \frac{dr}{f(r)},
\end{align*}
\]

(11)

where \(f(r)\) is a disposable function. The metric takes the form

\[
\text{ds}^2 = e^{\nu} (1 - e^{-\nu-\lambda} f^2)^{-1} \text{dt}^2 - e^{-\lambda} f^2 (1 - e^{-\nu-\lambda} f^2)^{-1} \text{dR}^2 - F^2(r) \text{d}\Omega^2,
\]

(12)

where as usual \(d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2\). The coordinates \(\tau\) and \(R\) are comoving if \(g'_{00} = \text{const} \equiv C^2 > 0\) and \(g'_{11} < 0\). By solving the equation \(g'_{00} = C^2\) one gets that \(f\) must be

\[
f^2(r) = e^{\nu+\lambda} \left(1 - \frac{e^{\nu}}{C^2}\right) \geq 0.
\]

(13)

In the interval \((r_1, r_2)\) the simplest choice is \(C = A\) and for it one gets \(g'_{11} = -(A^2 - e^\nu) \leq 0\) ensuring that \(R\) is a spatial coordinate. Finally one makes a linear transformation of time, \(\tau = \tau'/A\) and denoting the new
coordinate again by $\tau$ one gets the explicit form of the transformation from
the standard coordinates to the comoving ones,

$$\tau = At + \int \left[ e^{\lambda - \nu} (A^2 - e^\nu) \right]^{1/2} \, dr, \quad (14)$$

$$R = t + A \int \left[ e^{\lambda - \nu} (A^2 - e^\nu)^{-1} \right]^{1/2} \, dr. \quad (15)$$

In the comoving coordinates the SSS metric is

$$ds^2 = d\tau^2 - (A^2 - e^\nu) dR^2 - F^2(r) d\Omega^2, \quad (16)$$

where $e^{\nu(r)} \leq A^2$. The metric does not explicitly depend on $\lambda(r)$ since
this function has been absorbed in the transformation; however the metric
depends on $\lambda$ implicitly and explicitly on the time $\tau$—via the inverse trans-
formation $r = r(\tau, R)$. The latter arises from the following difference:

$$AR - \tau = \int e^{\frac{1}{2}(\nu + \lambda)} (A^2 - e^\nu)^{-1/2} \, dr \equiv W(r, A) \quad (17)$$

and this means that $r$ is a function of $AR - \tau$. The function $W$ may be
either positive or negative. The transformation (14) and (15) is mathemati-
cally correct if it is reversible and this requires that $W$ be monotonic. (The
the time $t$ may be easily recovered from $\tau$ and $R$ when the function $r = r(\tau, R)$
is known.) $W(r, A)$ is monotonic in some interval $(r_{1W}, r_{2W}) \subset (r_1, r_2)$ with
$A = \sup \{ e^{\nu(r)}, r \in (r_{1W}, r_{2W}) \}$ and varies from $W_1$ to $W_2$ in it. This implies
that $\tau$ and $R$ vary in the strip $W_1 \leq AR - \tau \leq W_2$. One concludes that in
the given SSS spacetime the transformation to the comoving coordinates is
valid in the interval $r_{1W} < r < r_{2W}$, which in general is smaller than $(r_1, r_2)$. Yet
the metric (16) may be analytically extended to a larger domain with
boundaries on which coordinate singularities develop.

Notice that the transformations (14) and (15) make sense only if $\nu(r)$ is
not identically zero (or a constant). In ultrastatic spherically symmetric
spacetimes one has $\nu(r) \equiv 0$ and the coordinates $(t, r, \theta, \phi)$ are already the
comoving ones, then $A = 1$ and (14) is reduced to $\tau = t$ whereas (15) be-
comes meaningless.

Below we give four examples of the function $W$, its reverse and the maxi-
mal analytic extension of (16) in the GNG coordinates. Clearly the first two
examples merely show of how the method works in the simplest cases.

1. De Sitter space. In the standard static coordinates covering one half of
the manifold up to the event horizon, one has

$$e^\nu = 1 - H^2 r^2 = e^{-\lambda}, \quad (18)$$
with \(0 < r < 1/H\). Since \(e^\nu \leq 1\), hence \(A = 1\) and \(W(r, 1) = \frac{1}{H} \ln(Hr)\) and \(W\) is monotonically growing from \(-\infty\) to 0 in the entire interval. Its reverse is \(r = (1/H) \exp[H(R - \tau)]\) and the domain of the transformation coincides with that of the standard coordinates, implying that the comoving ones are valid for all \(\tau > R\). The metric is then

\[
d s^2 = d\tau^2 - e^{2H(R-\tau)}(dR^2 + \frac{1}{H^2} d\Omega^2) \tag{19}
\]

and since \(g_{11} = -(Hr)^2\) in it, the domain of the GNG chart may be extended to the entire \((\tau, R)\) plane, i.e. also for \(Hr > 1\); this extension is useless. However applying the method in this case is impractical: usually the metric of dS space is given in other comoving coordinates which cover the whole manifold and, as is shown in [1], neither timelike nor null geodesics contain conjugate points and one concludes from Theorem 11.16 in [16] (cited here as Theorem 6) that each timelike geodesic (radial or not) is globally maximal between its endpoints.

2. Anti–de Sitter space. Actually we consider the covering anti–de Sitter (CAdS) space with the standard coordinates for the complete manifold with

\[
e^\nu = \frac{1}{a^2}(r^2 + a^2) = e^{-\lambda}, \tag{20}
\]

\(-\infty < t < \infty\) and \(0 \leq r < \infty\). \(e^\nu\) grows monotonically from 1 to \(\infty\) and we choose \(A\) arbitrarily large. \(W(r) = a \arcsin(r/a)\), where \(a^2 = a^2(A^2 - 1)\), and is monotonic for \(0 \leq r/a \leq 1\). From \(e^\nu \leq A^2\) one gets the maximal value of \(r\) equal \(r_2 = a\sqrt{A^2-1} = \alpha\) and the condition \(r_{2W}/a \leq 1\) yields \(r_{2W} = r_2 = \alpha\). Then \(r = \alpha \sin([AR - \tau]/a)\) and the metric reads

\[
d s^2 = d\tau^2 - (A^2 - 1) \left[ \cos^2 \left( \frac{AR - \tau}{a} \right) dR^2 + a^2 \sin^2 \left( \frac{AR - \tau}{a} \right) d\Omega^2 \right]. \tag{21}
\]

The transformation is valid for \(0 \leq r \leq \alpha\) and maps this interval onto the strip \(0 \leq AR - \tau \leq \pi a/2\). Its boundary lines are coordinate singularities of the metric (21) and the strip cannot be extended. Any timelike radial geodesic line \(R = R_0\) intersects this strip at points \(\tau_1 = AR_0 - \pi a/2\) and \(\tau_2 = AR_0\), hence the segment of the curve belonging to the coordinate domain has the length \(\Delta \tau = \tau_2 - \tau_1 = \pi a/2\). Again applying the general method to this spacetime is unnecessary: other, more convenient comoving coordinates are well known and they show that all timelike radial geodesics emanating from one point reconverge at a point \(\Delta s = \pi a\) away ([1], par. 5.2). In [1] we show that the circular and all radial (which cross \(r = 0\)) geodesics emanating from a point do meet again at a distance \(\Delta s = \pi a\) and this intersection point
is their common future cut point.

3. Reissner–Nordström black hole, $M^2 > Q^2$,

$$e^{\nu} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = e^{-\lambda}, \quad (22)$$

here $r \in (r_+, \infty)$ with $r_+ = M + \sqrt{M^2 - Q^2}$. We do not consider the maximally extended spacetime and assume the existence of only one exterior asymptotically flat region. One assumes $A = 1$ since outside the outer event horizon $0 < e^{\nu} < 1$, then

$$W(r, 1) = \frac{2}{3}(2M)^{-1/2} \left( r - \frac{Q^2}{2M} \left( r + \frac{Q^2}{M} \right) \right). \quad (23)$$

$W > 0$ and is monotonically increasing to infinity for $r \to \infty$, yet it cannot be effectively reversed since one should solve an algebraic cubic equation. For a given timelike radial geodesic $R = R_0 = \text{const}$ the proper time varies from $\tau = -\infty$ to $\tau = R_0 - W(r_+)$, where

$$W(r_+) = \frac{r_+}{3M} \left( r_+ + \frac{Q^2}{M} \right).$$

The domain of the comoving coordinates is the same as that of the standard ones, i.e. $r_+ < r < \infty$. The timelike radial geodesics $R = R_0$ are maximal outside the outer event horizon $r = r_+$. Clearly the same holds for these curves in Schwarzschild spacetime.

4. Kottler (Schwarzschild–de Sitter) black hole for $\Lambda > 0$,

$$e^{\nu} = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 = e^{-\lambda}. \quad (24)$$

The spacetime is static in some interval $0 < r_m < r < r_M$ if $e^{\nu} > 0$ there and this is possible if and only if $9M^2\Lambda < 1$. Then $e^{\nu} = 0$ has two different positive roots given implicitly by

$$(r_m, r_M) = \frac{1}{\sqrt{\Lambda}} \left( \cos \alpha/3 \mp \sqrt{3} \sin \alpha/3 \right), \quad (25)$$

where $\cos \alpha \equiv 3M\sqrt{\Lambda} < 1$, what implies $0 < \alpha/3 < \pi/6$. $e^{\nu}$ has maximum for $r = r_e = (3M/\Lambda)^{1/3}$, hence one should separately study the resulting two intervals.
i) $r \in (r_m, r_e)$, where $e^\nu$ monotonically grows. One assumes $A^2 \equiv e^{\nu(r_e)} = 1 - (9M^2\Lambda)^{1/3} > 0$. Setting $x \equiv r/r_e$ one finds

$$
\left( \frac{\Lambda}{3} \right)^{1/2} W = \frac{1}{\sqrt{3}} \ln \left[ \frac{1}{|1-x|} \left( 2x + 1 - \sqrt{3}\sqrt{x(x+2)} \right) \right] + \ln [x+1+\sqrt{x(x+2)}],
$$

(26)

where $r_m/r_e < x < 1$. In this interval $W$ monotonically decreases from $W(r_m/r_e)$ to $-\infty$ for $r = r_e$. Along the radial geodesic $R = R_0$ the proper time grows from $AR_0 - W(r_m/r_e)$ to $\tau = +\infty$.

ii) $r \in (r_e, r_M)$ and $e^\nu$ decreases from $A^2$ to 0. As previously $x \equiv r/r_e$, now $1 \leq x < r_M/r_e$ and $W(x)$ is again given in (26) with $|1-x| = x-1$. $W$ monotonically grows from $-\infty$ to $W(r_M/r_e)$ and radial geodesics $R = R_0$ extend from $\tau = AR_0 - W(r_M/r_e)$ to $\tau = +\infty$.

The function $W = AR - \tau$ should be separately inverted to $r = W^{-1}(AR - \tau)$ in $(r_m, r_e)$ and in $(r_e, r_M)$, therefore actually there exist two different and non–overlapping Gaussian normal geodesic charts for the spacetime in the interval $r_m < r < r_M$. The equation (26) cannot be effectively solved with respect to $x$.

5 Jacobi fields on timelike radial geodesics in static spherically symmetric spacetimes

The method of the comoving coordinates applies only to radial geodesics, furthermore the domain of these coordinates is usually smaller than that of the standard spherical ones. The case of the covering anti–de Sitter space (the standard time coordinate varies from $-\infty$ to $+\infty$) shows that the radial geodesics contain conjugate and cut points $[\text{II}]$. We are therefore interested here in locally maximal curves and in SSS spacetimes we consider two classes of distinguished geodesics: radial and circular ones. In this section we derive the geodesic deviation equation for the timelike radial geodesics; the detailed form of the equation (and in consequence, the Jacobi vector field) depends on the spacetime under consideration. We begin with deriving the equations describing any timelike geodesic. We assume the metric (9) with $F(r) = r$ and postpone discussing the case $F(r) = \text{const} = a$ to the next section and recall that the special case of Bertotti–Robinson spacetime has already been studied separately $[\text{II}]$.

The coordinates are so chosen that a timelike geodesic lies in the two–surface $\theta = \pi/2$, moreover there are two integrals of motion. These are the integrals of energy $k$ and of angular momentum. The rotational Killing field $\partial/\partial \phi$ with components $\xi^a = \delta_0^a$ is normalized as in Minkowski space and gives rise
to conserved \( J = -\xi^\alpha p_\alpha \) with \( p^\alpha = mcx^\alpha \). Introducing a constant \( L > 0 \) of dimension of length by \( J = mcL \), one gets \( \dot{\phi} \equiv d\phi/ds = L/r^2 \). The latter expression together with (10) are inserted into the radial component of the geodesic equation which then reads (\( \dot{r} = dr/ds \))

\[
\ddot{r} + \frac{1}{2} \lambda r^2 + \frac{k^2}{2\kappa^2} \nu e^{-(\nu+\lambda)} - \frac{L^2}{r^3} e^{-\lambda} = 0, \tag{27}
\]

\( f' \equiv df/dr \) for any \( f(r) \). The universal integral of motion \( g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 1 \) yields

\[
\dot{r}^2 = \frac{k^2}{\kappa^2} e^{-(\nu+\lambda)} - e^{-\lambda} \left( \frac{L^2}{r^2} + 1 \right). \tag{28}
\]

In this section we investigate radial timelike geodesics, \( \theta \) and \( \phi \) constant, and as mentioned in Introduction, we assume that these are possible worldlines of the twin C. Its angular momentum vanishes, \( L = 0 \), and (28) is reduced to

\[
\dot{r}^2 = e^{-\lambda} \left( \frac{k^2}{\kappa^2} e^{-\nu} - 1 \right). \tag{29}
\]

The starting point of C is \( r = r_0, r_m < r_0 < r_M \), and the initial radial velocity is \( \dot{r}(r_0) \equiv u \) with \( u \geq 0 \) or \( u < 0 \). The following motion depends on the behaviour of \( e^\nu \).

i) \( e^\nu \) is decreasing for \( r > r_0 \) (e. g. dS metric). If \( u > 0 \), then the twin C flies upwards and since \( (k^2/\kappa^2)e^{-\nu} \) is always greater than 1, it will escape to the domain boundary \( r = r_M \) and will never return. The same occurs for the start from the rest, \( u = 0 \). The gravitational field is repulsive. If \( u < 0 \) the twin falls down, then in general there is the minimal height \( r = \rho \) for which \( \dot{\phi}(\rho) = 0 \). At \( r = \rho \) the trajectory C turns back and flies away to the boundary \( r = r_M \).

ii) \( e^\nu \) increases for \( r > r_0 \) (CAdS and R–N). For \( r < r_0 \) one sees from (29) that \( \dot{r}^2(r) \) is positive and for \( u \leq 0 \) the twin C falls down towards the lower boundary \( r = r_M \). If \( u > 0 \) the twin flies upwards and reaches the maximal height \( r = R \) (not to be confused with the radial coordinate in the comoving system) for which \( \dot{\phi}(R) = 0 \), then it turns back and radially falls down to \( r_m \) and further. The gravity is attractive.

The case of Kottler spacetime, where \( e^\nu \) is not monotonic, is more complicated and requires a separate study; the motion of C there depends on the starting point and the initial velocity (or the integral of energy \( k \)).

In this section we consider the cases i) and ii), i. e. \( e^\nu \) is monotonic between \( r_m \) and \( r_M \). In both the cases we study the more general situation: the geodesic C consists of two segments, the incoming segment from \( r_0 \) to \( \rho \).
(and possibly lower) and the outgoing one from $r_0$ to $R$ (and possibly to $r_M$). It is convenient to parameterize the geodesic and its length with a suitably chosen variable $\eta$, $x^\alpha = x^\alpha(\eta)$ via $r = f(\eta)$. $f(\eta)$ is proportional to $\cos^2 \eta$ for R–N and CAdS metrics and to $\cosh \eta$ for de Sitter space. The vector tangent to the geodesic $C$ is

$$u^\alpha = \dot{x}^\alpha = (\dot{t}, \dot{r}, 0, 0) = \left[ \frac{k}{\kappa} e^{-\nu}, \varepsilon e^{-\lambda/2} \left( \frac{k^2}{\kappa^2} e^{-\nu} - 1 \right)^{1/2}, 0, 0 \right], \quad (30)$$

where $\varepsilon = +1$ for the outgoing segment and $\varepsilon = -1$ for the incoming one. The spacetime interval along the geodesic yields

$$\left( \frac{d s}{d \eta} \right)^2 = e^\nu \left( \frac{d t}{d s} \frac{d s}{d \eta} \right)^2 - e^\lambda \left( \frac{d r}{d \eta} \right)^2,$$

this may be solved with respect to $d s/d\eta$ giving rise to

$$\frac{d s}{d \eta} = \left| \frac{d f}{d \eta} \right| e^{\lambda/2} \left( \frac{k^2}{\kappa^2} e^{-\nu} - 1 \right)^{-1/2}. \quad (31)$$

Since $d t/d\eta = (d t/d s)(d s/d\eta)$, from (10) and (31) one gets

$$\frac{d t}{d \eta} = \frac{k}{\kappa} \left| \frac{d f}{d \eta} \right| e^{(\lambda-\nu)/2} \left( \frac{k^2}{\kappa^2} - e^{\nu} \right)^{-1/2}. \quad (32)$$

The spacelike orthonormal triad which is orthogonal to the geodesic $C$ and is parallelly transported along it, i.e. satisfies (2), is clearly non–unique and we choose it in the possibly simplest form,

$$e_1^\alpha = \left[ \varepsilon e^{-\nu/2} \left( \frac{k^2}{\kappa^2} e^{-\nu} - 1 \right)^{1/2}, \frac{k}{\kappa} e^{-\nu/2}, 0, 0 \right],$$

$$e_2^\alpha = \left[ 0, 0, \frac{1}{r}, 0 \right], \quad e_3^\alpha = \left[ 0, 0, 0, \frac{1}{r} \right]. \quad (33)$$

with $\varepsilon = \pm 1$ as above.

The Riemann tensor of any SSS spacetime is block–diagonal, i.e. has six nonvanishing components $R_{\mu\nu\mu\nu}$. The geodesic deviation equation for the Jacobi scalars consists of three separated equations,

$$\frac{d^2}{d s^2} Z_1 = \frac{1}{4} \left( \nu' \lambda' - 2 \nu'' - \nu'^2 \right) e^{-\lambda} Z_1, \quad (34)$$

$$\frac{d^2}{d s^2} Z_2 = - \left[ \frac{k^2}{2 \kappa^2 r^2} e^{-(\nu+\lambda)} (\nu' \lambda' - \frac{\lambda'}{r} e^{-\lambda}) \right] Z_2. \quad (35)$$

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and the equation for \( Z_3 \) is identical with that for \( Z_2 \). For a generic SSS spacetime these equations depend on the energy \( k \). In special spacetimes wherein \( \nu + \lambda = 0 \) this dependence disappears. On the RHS of these equations one has derivatives w.r.t. \( r \), whereas on the LHS —w.r.t. the proper time and it is here that the use of the suitably chosen variable \( \eta \) is necessary. Applying (31) one finds more complicated equations,

\[
\frac{d^2 Z_1}{d\eta^2} - \frac{d f}{d\eta} \left[ \left( \frac{d f}{d\eta} \right)^{-2} \frac{d^2 f}{d\eta^2} + \frac{\chi}{2} + \frac{k^2}{2\kappa^2} \nu' \left( \frac{k^2}{\kappa^2} - e^\nu \right)^{-1} \right] \frac{dZ_1}{d\eta} = 0,
\]

\[
= \frac{1}{4} (\nu' \chi' - 2\nu'' - \nu^2) \left( \frac{k^2}{\kappa^2} e^{-\nu} - 1 \right)^{-1} \left( \frac{d f}{d\eta} \right)^2 Z_1,
\]

\[
\frac{d^2 Z_2}{d\eta^2} - \frac{d f}{d\eta} \left[ \left( \frac{d f}{d\eta} \right)^{-2} \frac{d^2 f}{d\eta^2} + \frac{\chi}{2} + \frac{k^2}{2\kappa^2} \nu' \left( \frac{k^2}{\kappa^2} - e^\nu \right)^{-1} \right] \frac{dZ_2}{d\eta} = -e^{\lambda} \left( \frac{k^2}{\kappa^2} e^{-\nu} - 1 \right)^{-1} \left( \frac{d f}{d\eta} \right)^2 \frac{1}{2r} \left[ \frac{k^2}{\kappa^2} e^{(\nu+\lambda)}(\nu' + \chi') - \lambda' e^{\nu} \right] Z_2
\]

and the equation for \( Z_3 \) is identical with (37); in the equations one sets \( r = f(\eta) \). The first integrals (6) for these equations are generated by the timelike Killing vector \( K^\alpha_t = \kappa \delta^\alpha_0 \) and the three spacelike rotational Killing fields, which at the points of the geodesic \( C \) take the form

\[
K^\alpha_x = (0, 0, -\sin \phi_0, 0), \quad K^\alpha_y = (0, 0, \cos \phi_0, 0), \quad K^\alpha_z = \delta^\alpha_3;
\]

obviously \( K^\alpha_x \) and \( K^\alpha_y \) generate the same integral. In eqs. (6) one replaces \( dZ_a/ds \) by \( dZ_a/d\eta \) and similarly for other derivatives. The three first integrals are also separated. \( K^\alpha_x \) gives rise to the following integral for \( Z_1 \),

\[
\frac{1}{2} \chi' \frac{d f}{d\eta} Z_1 + \left( \frac{k^2}{\kappa^2} - e^\nu \right) \frac{dZ_1}{d\eta} = C_1 \frac{d f}{d\eta} \left( e^{(\nu+\lambda)/2}, \right),
\]

whereas \( K^\alpha_y \) generates

\[
f(\eta) \frac{dZ_2}{d\eta} - \frac{d f}{d\eta} Z_2 = C_2 \frac{d f}{d\eta} \left( e^{\lambda/2} \left( \frac{k^2}{\kappa^2} e^{-\nu} - 1 \right)^{-1/2} \right)
\]

and \( K^\alpha_z \) gives rise to

\[
f(\eta) \frac{dZ_3}{d\eta} - \frac{d f}{d\eta} Z_3 = C_3 \frac{d f}{d\eta} \left( e^{\lambda/2} \left( \frac{k^2}{\kappa^2} e^{-\nu} - 1 \right)^{-1/2} \right)
\]

which is the same as (40); \( C_1, C_2 \) and \( C_3 \) are arbitrary constants. These equations together with their first integrals may be solved only if the functions \( \nu(r), \lambda(r) \) and \( r = f(\eta) \) are explicitly given. The solutions for the R–N metric are given in [2] and for Schwarzschild field in [3].

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6 Jacobi fields on timelike circular geodesics in static spherically symmetric spacetimes

First we should check the very existence of the circular geodesic for some \( r = r_0, \) \( r_m < r_0 < r_M \). If it exists, we assume that it is the worldline of the twin \( B \). We denote \( \nu_0 = \nu(r_0), \lambda_0 = \lambda(r_0), \) then \( \nu'_0 = d\nu(r_0)/dr \) and \( \lambda'_0 = d\lambda(r_0)/dr \). First we consider the metric (9) with \( F(r) = a \) (an example is provided by the Bertotti–Robinson spacetime). It is easy to show that the equation replacing in this case eq. (27) implies for a circular geodesic at \( r = r_0 \) that \( \nu'_0 = 0 \) for any \( r_0 \). If \( \nu'(r) \neq 0 \) as is in the B–R case, then circular geodesics do not exist. On the other hand, if \( \nu' \equiv 0 \), then \( g_{00} = 1 \) and one deals with ultrastatic spherically symmetric spacetime, whose metric depends on one arbitrary function \( \lambda(r) \). These spacetimes do admit circular geodesics. Moreover, in these spacetimes the geodesic equation may be explicitly integrated for any timelike geodesic providing functions \( t(s), \phi(s) \) and \( s(r) \) [2]. In what follows we assume the generic case: \( \nu'(r) \neq 0 \) and \( F(r) = r \) in (9).

For the circular geodesic the radial equation (27) reduces to an algebraic equation expressing the integral of energy \( k \) as a function of the angular momentum \( L \). On the other hand the universal integral of motion (28) expresses for the curve \( B \) the value of \( k^2 \) in terms of \( \nu_0 \) and \( \nu'_0 \). The result is

\[
\begin{align*}
  k^2 &= \frac{2\kappa^2 e^{\nu_0}}{2 - r_0 \nu'_0}, \\
  L^2 &= \frac{r_0^3 \nu'_0}{2 - r_0 \nu'_0}.
\end{align*}
\]

Since \( k^2 > 0 \) and \( L^2 > 0 \) one gets that the necessary and sufficient conditions for circular geodesics to exist are respectively \( r_0 \nu'_0 < 2 \) and \( \nu'_0 > 0 \), what implies that \( g_{00} = e^{\nu(r)} \) is an increasing function around \( r = r_0 \); these conditions were found in a different way in [24]. From (10) and \( \dot{\phi} = L/r^2 \) one immediately gets for \( B \)

\[
  t - t_0 = \frac{k}{\kappa} e^{-\nu_0} s \quad \text{and} \quad \phi - \phi_0 = \frac{L}{r_0^2} s.
\]

The length of the worldline \( B \) corresponding to one full circle is determined by \( \phi - \phi_0 = 2\pi \) and equals

\[
  s_B = \frac{2\pi}{L} r_0^2
\]

and the corresponding interval of the coordinate time is

\[
  \Delta t = t(s_B) - t_0 = \frac{2\pi k}{\kappa L} r_0^2 e^{-\nu_0}.
\]
At this point a subtle problem arises in maximally symmetric (i.e., both homogeneous and spherically symmetric) spacetimes: is it possible to discriminate between radial and circular geodesics in these spacetimes? Applying an embedding flat five-dimensional space it was shown by Calabi and Markus \[26\] that both in de Sitter and anti–de Sitter spaces the two curves are identical and their apparent distinction is coordinate dependent: it is entirely due to the choice of the origin of the standard spherical coordinates. In all other spherically symmetric spacetimes the distinction between radial and circular curves is geometrically meaningful.

### 6.1 Stable circular orbits

Here we collect for completeness some facts on stability of particles’ trajectories. As is well known from the Schwarzschild case the circular orbits may be stable or unstable. To establish a condition for the existence of stable circular orbits we apply to a generic SSS spacetime the standard method used in the case of Reissner–Nordström metric \[27\]. One interprets \(r^2\) as a ‘kinetic energy’ and expresses the integral of motion (28) as a difference between the total energy and a ‘potential energy’, to this end one introduces an effective potential \(V\),

\[
r^2 = \frac{k^2}{\kappa^2} - V(r, k, L), \quad \text{where}
\]

\[
V(r, k, L) \equiv e^{-\lambda} \left( \frac{L^2}{r^2} + 1 \right) - \frac{k^2}{\kappa^2} (e^{-(\nu+\lambda)} - 1).
\]  

(46)

For a circular orbit \(r = r_0\) the constants of motion \(k^2\) and \(L^2\) are determined by \(r = r_0\) and expressed in (42) and for these values the point is a stationary one, \(dV/dr(r_0) = 0\). The orbit is stable if the effective potential reaches minimum at this point, or

\[
d^2V/dr^2(r_0) = -\frac{e^{-\lambda_0}}{2 - r_0\nu_0} \left( 2\nu_0' - 2\nu_0'' - \frac{6\nu_0'}{r_0} \right) > 0,
\]  

(47)

what amounts to

\[
\nu_0'' - \nu_0'^2 + \frac{3\nu_0'}{r_0} > 0.
\]  

(48)

For the R–N metric stable circular orbits exist for sufficiently large \(r_0\) and there is a lower limit to \(r_0\) and one expects that the same holds for other SSS spacetimes. The minimum radius \(r_I\) represents the innermost stable
circular orbit (ISCO) and is determined by a point of inflection of the effective potential,
\[ \frac{d^2V}{dr^2}(r_I) = 0, \quad \text{or} \quad \nu''_I - \nu'_I^2 + \frac{3\nu'_I}{r_I} = 0, \quad (49) \]
here \( \nu'_I = \frac{dv}{dr}(r_I) \). For R–N metric this is a cubic equation; for the charge \( Q^2 = 0 \) (the Schwarzschild case) one gets the well known result \( r_I = 6M \) and for the extreme R–N black hole, \( Q^2 = M^2 \), there are two solutions: \( r_I = M \), which coincides with the outer (and inner) event horizon and should be rejected and \( r_I = 4M \), which gives the unique ISCO. For \( 0 < Q^2 < M^2 \) the equation has one real solution \( r_I = 2M + w + v \), where \( w = (P + \sqrt{D})^{1/3}, \quad v = (P - \sqrt{D})^{1/3} \) and
\[ P = 8M^3 + 2\frac{Q^4}{M} - 9MQ^2, \quad D = 4\frac{Q^4}{M^2}(M^2 - Q^2) \left( \frac{5}{4}M^2 - Q^2 \right) > 0. \]
The function \( r_I(Q) \) monotonically decreases from \( 6M \) to \( 4M \) and the unique ISCO exists outside the outer event horizon [2].

For a generic SSS spacetime the condition (49) for the existence of ISCO may be satisfied if \( \nu''_I \) is sufficiently negative since \( \nu'_I > 0 \) and \( r_I\nu'_I < 2 \). The solution, if exists, is unique on physical grounds. For CAdS metric the LHS of eq. (49) is always positive implying that each circular orbit is stable.

### 6.2 Equations for the Jacobi scalars

First we introduce, as in [3], a third twin, a static twin A moving on a nongeodesic worldline \( r = r_0, \theta = \pi/2, \phi = \phi_0 \). The twins A and B start from the point \( P_0 \) \( (t = t_0, r = r_0, \theta = \pi/2, \phi = \phi_0) \) and meet again at \( P_1(t = t_0 + \Delta t, r = r_0) \), being the same point in the space. The length of the worldline A between \( P_0 \) and \( P_1 \) is
\[ s_A(\Delta t) = \int_{t_0}^{t_0+\Delta t} e^{\nu_0/2} \, dt = \frac{2\pi k}{\kappa L} r_0^2 e^{-\nu_0/2} \quad (50) \]
and the ratio of their worldline lengths is
\[ \frac{s_A(\Delta t)}{s_B} = \left( \frac{2}{2 - r_0\nu'_0} \right)^{1/2} > 1, \quad (51) \]
this implies that the geodesic B has a point conjugate to \( P_0 \) in the segment \( P_0P_1 \).

As for the radial geodesic C, the spacelike basis triad on B satisfying (2) is non–unique and we choose it in the form
\[ e_1^\alpha = [-T \sin qs, X \cos qs, 0, -Y \sin qs], \quad e_2^\alpha = [0, 0, \frac{1}{r_0}, 0], \]
\[ e_3^\alpha = \frac{\sin qs}{r_0} \cos qs, 0, -\frac{\cos qs}{r_0}, 0]. \]
\[ e_3^\alpha = -\frac{1}{q} \frac{d}{ds} e_1^\alpha, \]  
\[ q = \left( \frac{\nu_0^2}{2 r_0 e^{-\lambda_0}} \right)^{1/2} \]  
(52)

where the constants are
\[ T = \left( \frac{r_0 \nu_0 e^{-\nu_0}}{2 - r_0 \nu_0'} \right)^{1/2}, \quad X = e^{-\lambda_0/2}, \quad Y = \frac{1}{r_0} \left( \frac{2}{2 - r_0 \nu_0'} \right)^{1/2} \]  
and
\[ \frac{d}{ds} Z_1 = q^2 [(b \cos^2 qs - 1) Z_1 + b Z_3 \sin qs \cos qs], \]  
(55)
\[ \frac{d}{ds} Z_2 = - \frac{2q^2}{2 - r_0 \nu_0'} e^{\lambda_0} Z_2, \]  
(56)
\[ \frac{d}{ds} Z_3 = q^2 [b Z_1 \sin qs \cos qs + (b \sin^2 qs - 1) Z_3], \]  
(57)

where
\[ b = \frac{2}{2 - r_0 \nu_0'} (1 - r_0 \nu_0' - r_0 \frac{\nu_0'}{\nu_0}). \]  
(58)

The basis triad and the velocity vector differ from the corresponding four vectors on the radial geodesics in that they do not depend on the metric functions and depend only on constants determined by the metric components. Since \( Z^\mu \) is a vector field connecting nearby curves, one sees from (52) and (53) that in the spherical coordinates the Jacobi scalars \( Z_\alpha \) have dimension of length and the solutions to the equations below should be multiplied by a length scale. Applying these four vectors one finds, after a longer computation, the geodesic deviation equations for the three scalars (3),

One sees that on the circular geodesics the equations for the Jacobi scalars are universal, i. e. are the same in all SSS spacetimes, only the numerical coefficients depend on \( r_0 \) and values of \( \lambda_0, \nu_0' \) and \( \nu_0'' \). The range of \( b \) depends on the spacetime. We exclude the case \( b = 0 \) (CAdS space) and assume \( b > 0 \), e. g. for R–N metric \( 3 < b < \infty \). The equations for \( Z_1 \) and \( Z_3 \) are similar, but not exactly symmetric. All the functions explicitly depend on the proper time \( s \) on the curve B.
Again the first integrals (6) of the equations are generated by the four Killing fields of the SSS spacetime and the vectors on the geodesic B are

\[
K^\alpha_t = \delta^\alpha_0, \quad K^\alpha_x = (0, 0, -\sin \phi(s), 0), \quad K^\alpha_y = (0, 0, \cos \phi(s), 0), \quad K^\alpha_z = \delta^\alpha_3,
\]

for simplicity we put \( \kappa = 1 \) and apply (43). The following integrals of motion are also universal. The vectors \( K^\alpha_t \) and \( K^\alpha_z \) generate the same first integral of the coupled equations (55) and (57),

\[
-dZ_1 \sin qs + Z_1 q \cos qs + \frac{dZ_3}{ds} \cos qs + Z_3 q \sin qs = C_1,
\]

whereas vectors \( K^\alpha_x \) and \( K^\alpha_y \) give rise to two independent first integrals for eq. (56),

\[
\begin{align*}
 r_0 \frac{dZ_2}{ds} \sin \phi - \left( \frac{r_0 \nu'_0}{2 - r_0 \nu'_0} \right)^{1/2} Z_2 \cos \phi &= C_2, \\
 r_0 \frac{dZ_2}{ds} \cos \phi + \left( \frac{r_0 \nu'_0}{2 - r_0 \nu'_0} \right)^{1/2} Z_2 \sin \phi &= C_3,
\end{align*}
\]

\( C_1, C_2, C_3 \) are arbitrary constants. Eq. (56) may be immediately integrated, yet its two first integrals allow one to solve it without any integration,

\[
Z_2 = C' \sin \frac{Ls}{r_0} + C'' \cos \frac{Ls}{r_0},
\]

arbitrary \( C' \) and \( C'' \) have dimension of length. The universality of the equations implies universality (modulo the values of the constants) of conjugate points on B. Solutions giving rise to two of the three sequences of conjugate points on B were previously found in [3] and in [2] we presented some properties of nearby timelike geodesics intersecting B at these points.

### 6.3 Conjugate points generated by the Jacobi scalar \( Z_2 \)

The deviation vector field generated by \( Z_2 \) is \( Z^\mu = Z_2(s)e^\mu_2 \) with \( e^\mu_2 = (1/r_0)\delta^\mu_2 \) and is directed off the 2-surface \( \theta = \pi/2 \). To determine points on B conjugate to \( P_0(s = 0) \) one takes the vector field vanishing at \( P_0 \),

\[
Z^\mu = \frac{C''}{r_0} \delta^\mu_2 \sin \frac{Ls}{r_0^2}.
\]

The field has infinite number of zeros at points \( Q_n(s_n) \) with

\[
s_n = n\pi \frac{r_0^2}{L} = n\pi \left[ \frac{r_0}{\nu'_0}(2 - r_0 \nu'_0) \right]^{1/2}, \quad n = 1, 2, \ldots
\]
The location of these points is found by comparing their distances to \(P_0\) with the distance from \(P_0\) to \(P_1\), \(s_n/s_B = n/2\). Thus for \(n\) even the points \(Q_n\) coincide in the space with \(P_0\) and \(P_1\), whereas for \(n\) odd they are points antipodic in the space to \(P_0\) on the circle (they are distant by \(\Delta \phi = \pi\) from \(P_0\)). This result is geometrically and physically quite obvious: if one rotates in the space the \(2\)-surface \(\theta = \pi/2\) by a small angle about the axis joining the spatial projections of \(P_0\) and \(Q_1\), then the nearby circular timelike geodesics emanating from \(P_0\) will successively intersect at points \(Q_n, n = 1, 2, \ldots,\) in the spacetime. This effect was earlier found for Schwarzschild \[3\]. According to Theorem 1 of section 3 the conjugate points \(Q_n\) are also future cut points to \(Q_{n-1}\).

### 6.4 Jacobi fields spanned on the basis vectors \(e_1\) and \(e_3\) — an infinite sequence of conjugate points

Surprisingly, there exist other points conjugate to the arbitrary point \(P_0\) besides the sequence \(\{Q_n\}\). The coupled equations (55) and (57) have a complete system of basis solutions consisting of four independent pairs of solutions \((Z_{1N}, Z_{3N}), N = 1, 2, 3, 4\) and the general solution to these equations is

\[
Z_1 = \sum_{N=1}^{4} A_N Z_{1N} \quad \text{and} \quad Z_3 = \sum_{N=1}^{4} A_N Z_{3N}
\]

(64)

with arbitrary constants \(A_N\). Since the equations for \(Z_1\) and \(Z_3\) are identical for all SSS spacetimes, their solutions were found while investigating the simplest (nonhomogeneous) of these, the Schwarzschild metric \[3\]. For the reader’s convenience we present them here in a different, more readable order. The third and fourth pair of the basis solutions show that the value \(b = 4\) of the parameter is distinguished. For \(b < 4\) it appears in the argument of trigonometric functions in the form \(\sqrt{4 - bqs}\) and for \(b > 4\) in the argument of corresponding hyperbolic functions as \(\sqrt{b - 4qs}\). This implies that these two pairs of solutions are non-analytic in \(b\) at \(b = 4\) and appropriate solutions for \(b = 4\), i. e. \(Z_{13}, Z_{14}, Z_{33}\) and \(Z_{34}\) cannot be found from these by taking the limit \(b \rightarrow 4\). The first pair of solutions is independent of \(b\) and denoting \(x \equiv qs\) it reads

\[
Z_{11}(s) = \sin x, \quad Z_{31}(s) = -\cos x
\]

(65)

and the second pair is valid for all values of the parameter,

\[
Z_{12}(b, s) = 2 \cos x + bx \sin x, \quad Z_{32}(b, s) = 2 \sin x - bx \cos x.
\]

(66)
The third and fourth pair actually consist of three distinct solutions valid for different intervals of \( b \),

\[
Z_{13}(b, s) = 2 \sin x \cos(\sqrt{4-b}x) + \sqrt{4-b} \cos x \sin(\sqrt{4-b}x), \\
Z_{33}(b, s) = -2 \cos x \cos(\sqrt{4-b}x) - \sqrt{4-b} \sin x \sin(\sqrt{4-b}x)
\]  
for \( b < 4 \),

\[
Z_{13}(4, s) = x \cos x + x^2 \sin x, \\
Z_{33}(4, s) = x \sin x - x^2 \cos x \text{ for } b = 4,
\]  
(68)

\[
Z_{14}(b, s) = 2 \sin x \sinh(\sqrt{b-4}x) + \sqrt{b-4} \cos x \cosh(\sqrt{b-4}x), \\
Z_{34}(b, s) = \sqrt{b-4} \sin x \cosh(\sqrt{b-4}x) - 2 \cos x \sinh(\sqrt{b-4}x)
\]  
for \( b > 4 \). Finally the fourth pair,

\[
Z_{14}(b, s) = 2 \sin x \sin(\sqrt{4-b}x) + \sqrt{4-b} \cos x \cos(\sqrt{4-b}x), \\
Z_{34}(b, s) = -2 \cos x \sin(\sqrt{4-b}x) + \sqrt{4-b} \sin x \cos(\sqrt{4-b}x)
\]  
(70)

for \( b < 4 \),

\[
Z_{14}(4, s) = 4x^3 \sin x + 3(1 + 2x^2) \cos x, \\
Z_{34}(4, s) = 3(1 + 2x^2) \sin x - 4x^3 \cos x \text{ for } b = 4,
\]  
(71)

\[
Z_{14}(b, s) = 2 \sin x \cosh(\sqrt{b-4}x) + \sqrt{b-4} \cos x \sinh(\sqrt{b-4}x), \\
Z_{34}(b, s) = \sqrt{b-4} \sin x \sinh(\sqrt{b-4}x) - 2 \cos x \cosh(\sqrt{b-4}x)
\]  
(72)

for \( b > 4 \). From the definition (58) it follows that the critical value \( b = 4 \) corresponds to (49), i.e. the point of inflection of the effective potential, or ISCO. The condition for a stable circular orbit, (48), implies \( b < 4 \). For physical reasons we are interested in seeking for conjugate points on stable orbits and expect that there are no conjugate points on unstable orbits. The solutions show that this is the case.

The relevant Jacobi fields must vanish for \( s = 0 \) and in the case under consideration this implies \( Z_1(0) = 0 = Z_3(0) \); these conditions impose restrictions on the coefficients \( A_N \). One separately studies the cases \( b > 4 \), \( b = 4 \) and \( b < 4 \). For the ISCO, \( b = 4 \), the two conditions applied to (65), (66), (68) and (71) imply \( A_1 = 0 \), \( A_4 = -2A_2/3 \) with arbitrary \( A_2 \) and \( A_3 \). The resulting Jacobi scalars \( Z_1 \) and \( Z_3 \) do not have common roots for \( s \neq 0 \),
hence they do not determine conjugate points to \( s = 0 \). The analogous procedure applied to the unstable orbits, \( b > 4 \) provides the same outcome: no common roots for \( s \neq 0 \). In the most interesting case of stable orbits, \( b < 4 \), it turns out that the analysis performed in [3] was incomplete and here we present its complete version. The deviation field vanishing at \( s = 0 \) depends on arbitrary \( A_1 \) and \( A_4 \) whereas
\[
A_2 = -\frac{1}{2} \sqrt{4 - b} A_4, \quad A_3 = -\frac{1}{2} A_1,
\]
(73)
then \( Z_1 \) and \( Z_3 \) are linear combinations of all the basis solutions \( Z_{1N} \) and \( Z_{3N} \) respectively. By substituting their explicit forms and denoting \( y = \sqrt{4 - b x} = \sqrt{4 - b \nu s} \), one gets the deviation vector \( Z^\mu(s) \),
\[
Z^0 = T \left[ -A_1(1 - \cos y) + A_4 \left( \frac{1}{2} by - 2 \sin y \right) \right],
\]
\[
Z^1 = X \sqrt{4 - b} \left[ \frac{1}{2} A_1 \sin y - A_4 (1 - \cos y) \right], \quad Z^2 = 0,
\]
\[
Z^3 = Y \left[ -A_1(1 - \cos y) + A_4 \left( \frac{1}{2} by - 2 \sin y \right) \right] = \frac{Y}{T} Z^0.
\]
(74)
The vector \( \varepsilon Z^\mu(s) \) connects the circular geodesic \( B \equiv \gamma(0) \) to a geodesic \( \gamma(\varepsilon) \) which is at \( \varepsilon \)-distance from it and which emanates from \( P_0 \); the spatial orbit of this geodesic entirely lies in the surface \( \theta = \pi/2 \). \( \gamma(\varepsilon) \) is parametrically given by \( x^\mu(s, \varepsilon) = x^\mu(s, 0) + \varepsilon Z^\mu(s) \), where \( x^\mu(s, 0) \) describes \( B \) and is given in (43). In the search for conjugate points to \( P_0 \) one considers three cases depending on values of \( A_1 \) and \( A_4 \). In this subsection we study two of these. In the first case, \( A_1 = 0 \) and \( A_4 \neq 0 \), the vector components \( Z^0 \) and \( Z^1 \) do not have common roots for \( s \neq 0 \) and do not indicate conjugate points. In the second case, \( A_1 \neq 0 \) and \( A_4 = 0 \), one immediately sees from (74) that \( Z^\mu(s) \) is zero at the infinite sequence of points \( Q'_n(s'_n) \) on \( B \), where
\[
s'_n = \frac{2n\pi}{q\sqrt{4 - b}}, \quad n = 1, 2, \ldots.
\]
(75)
The expression is divergent for \( b \to 4 \) indicating that ISCOs do not contain conjugate points. To see whether the first conjugate point \( Q'_1 \) lies within the arc \( P_0P_1 \) we compute the ratio
\[
\frac{s'}{s_B} = \frac{L}{q\sqrt{4 - b r_0^2}} = \left( \frac{\nu_0^3 e^{\lambda_0}}{3 \nu_0^3 r_0 \nu_0^2 + r_0 \nu_0^2} \right)^{1/2}.
\]
(76)
For Schwarzschild metric the ratio is \( s'/s_B = [r_0(r_0 - 6M)^{-1}]^{1/2} > 1 \) and qualitatively the same holds for the R–N spacetime [2]; due to arbitrariness
of $\lambda(r)$ the ratio may be arbitrary and for each SSS spacetime it should be separately computed. The geometrical interpretation of the second infinite sequence of conjugate points $\{Q'_n(s'_n)\}$ on the circular geodesics is unclear. For CAdS space $b = 0$ and the sequence coincides with that of conjugate points $\{Q_n(s_n)\}$ generated by $Z_2 e^{\mu_2}$, hence $s_n = s'_n = n\pi a$ and $s_1/s_B = 1/2$.

It is interesting to see whether some of the geodesics $\gamma(\varepsilon)$ which infinitely many times intersect B (i.e. $A_4 = 0$) have closed orbits. To this end we notice that all the orbits are contained between the minimal and maximal value of the radius, $r_{\text{min}} = r_0 - \frac{1}{2}\varepsilon A_1X\sqrt{4-b}$ and $r_{\text{max}} = r_0 + \frac{1}{2}\varepsilon A_1X\sqrt{4-b}$. The successive maxima of $r$ are for $y_n = \sqrt{4-b}q\tilde{s}_n = (2n + \frac{1}{2})\pi$ and the arc length of $\gamma(\varepsilon)$ between two successive maxima of $r$ is

$$Ds \equiv \tilde{s}_{n+1} - \tilde{s}_n = \frac{2\pi}{q\sqrt{4-b}}.$$  \hspace{1cm} (77)

On the other hand the angular distance between the two successive maxima is, from (43) and (74),

$$D\phi \equiv \phi(\tilde{s}_{n+1}) - \phi(\tilde{s}_n) = \frac{2\pi L}{r_0^2 q\sqrt{4-b}}.$$  \hspace{1cm} (78)

Yet the successive conjugate points on B, $Q'_n$ and $Q'_{n+1}$, are at the distance $\Delta s \equiv s'_{n+1} - s'_n = 2\pi(q\sqrt{4-b})^{-1} = Ds$, hence the angular distance between these two points is, from (43), $\Delta \phi \equiv \phi(s'_{n+1}) - \phi(s'_n) = L\Delta s/r_0^2 = D\phi$, that is, the angular and spacetime distances between the conjugate points on B and the points of maximal radius of the orbit of $\gamma(\varepsilon)$ are respectively equal. The orbit of $\gamma(\varepsilon)$ is closed if $D\phi = 2\pi l/m$ for some integers $l$ and $m$. Then after $m$ periods of change from $r_{\text{max}}$ to $r_{\text{min}}$ and back to $r_{\text{max}}$ the angle $\phi$ increases by $2\pi l$ and the orbit returns to the same point in the surface $\theta = \pi/2$. Hence the orbit is closed if

$$\frac{L}{r_0^2 q\sqrt{4-b}} = \frac{l}{m},$$

or inserting the values of the parameters,

$$\frac{\nu'_0 e^{\lambda_0}}{3\nu'_0 - r_0\nu'_0^2 + r_0\nu''_0} = \frac{l^2}{m^2};$$  \hspace{1cm} (79)

for every SSS spacetime it is an algebraic equation for the radius of B. One gets an infinite discrete set of values $r_0(l/m)$; for Schwarzschild metric it is

$$r_0 \left( \frac{l}{m} \right) = \frac{6l^2 M}{l^2 - m^2},$$

clearly $l > m$ and $r_0 > 6M$. 31
6.5 Jacobi fields spanned on $e_1^\mu$ and $e_3^\mu$ — infinite set of single conjugate points

Finally we study the third, general, case of search for zeros of the deviation vector, $A_1 \neq 0$ and $A_4 \neq 0$ (this case was not studied in [3]). Since $Z^\mu$ is determined up to a constant factor, we put $A_1 = 2$, then

$$Z_1 = 2Z_{11} - \frac{1}{2}\sqrt{4 - b}A_4Z_{12} - Z_{13} + A_4Z_{14} \quad (80)$$

and $Z_3$ is given by the same combination of $Z_{3\nu}$. In search of solutions of the equations $Z_1 = 0$ and $Z_3 = 0$ for $x \neq 0$ we apply (65), (66), (67) and (70) and replace the two equations by an equivalent simpler system (as above $y = \sqrt{4 - bx}$),

$$\sin y + A_4(\cos y - 1) = 0,$$

$$2A_4\sin y - 2\cos y - \frac{1}{2}A_4by + 2 = 0, \quad (81)$$

these are equations for $A_4$ and $y$; clearly they are satisfied by $y = 0$ and any value of $A_4$, what corresponds to the initial point. We seek for roots $y \neq 0$. For $y = 2n\pi$ one gets $A_4 = 0$ and returns to the second case and the sequence $\{Q_n(s'_n)\}$. One computes $A_4$ from the first equation, $A_4 = (1 - \cos y)^{-1}\sin y$ for $y \neq 2n\pi$, $n = 1, 2, \ldots$, and inserts it into the other of (81). After simple manipulations one gets a crucial equation,

$$\cos y + \frac{b}{8}y\sin y - 1 = 0. \quad (82)$$

All positive roots (excluding $y_n = 2n\pi$) form an infinite sequence $y_n(r_0) = (2n + 1)\pi - \delta_n(b(r_0))$, $n = 1, 2, \ldots$, where $\delta_n > 0$ are found numerically. The term $\delta_1(b)$ is of order unity for $0 < b < 4$ and decreases for increasing $b$. The sequence $\{\delta_n(b)\}$ is decreasing and for large $n$ its terms behave as

$$\delta_n \to \frac{16(2n + 1)\pi}{(2n + 1)^2\pi^2b - 16}. \quad (83)$$

Each root $y_n(r_0)$ determines a separate deviation vector field

$$Z^n(n, r_0, s) = Z_1(n, r_0, s) e_1^n(s) + Z_3(n, r_0, s) e_3^n(s) \quad (84)$$

connecting the circular curve $B(r_0)$ to the nearby geodesic $\gamma(\varepsilon, n, r_0)$ which emanates from $P_0(s = 0)$ on $B$, entirely lies on the 2–surface $\theta = \pi/2$ and intersects $B$ once at $s = \bar{s}_n$, where

$$\bar{s}_n = \frac{y_n(r_0)}{q(r_0)\sqrt{4 - b(r_0)}} = \left( \frac{r_0(2 - r_0\nu_0')e^{\lambda_0}}{3\nu_0' - r_0\nu_0'^2 + r_0\nu_0''} \right)^{1/2} y_n(r_0). \quad (85)$$
The denominator in (85) (and in (76)) is positive since \( b < 4 \). The ratio of the distance to the first conjugate point of the sequence, \( \bar{s}_1 \), to the length \( s_B \) of one revolution of \( B \) is, from (44) and (42),

\[
\frac{\bar{s}_1}{s_B} = \frac{s'_1}{s_B} = \frac{3\nu_0^\prime \nu_0 \lambda_0}{3\nu_0^\prime r_0 \nu_0^2 + r_0 \nu_0''} \left( \frac{3\pi - \delta_1}{2}\right)^{1/2},
\]

hence it is always larger than \( s'_1/s_B \). As an example we take Schwarzschild metric:

i) for \( r_0 = 6, 26087M \) one has \( b = 3.92 \), then \( \bar{s}_1 = 497,249M \) and \( \bar{s}_1/s_B = 7 - 4 \cdot 10^{-5} \), what corresponds to the angular distance (from (43)) \( \phi - \phi_0 = 14\pi - 6 \cdot 10^{-4} \);

ii) for \( r_0 = 78M \) one has \( b = 3.04 \) then \( \bar{s}_1 = 6219,826M \) and \( \bar{s}_1/s_B = 1,4655 \) and \( \phi - \phi_0 = 2\pi + 2.9246 \);

the larger \( r_0 \) is, the closer (in terms of the angular distance) to \( P_0 \) the conjugate point \( \bar{s}_1 \) is, but always \( \phi - \phi_0 > 2\pi \).

7 Conclusions

The main result of the method developed here is that in a general static spherically symmetric spacetime admitting circular timelike geodesics, each stable circular geodesic contains, besides the trivial infinite sequence of conjugate points arising directly from the spherical symmetry, two other infinite sets of conjugate points, whose geometrical and physical interpretation is unclear. This outcome has already been mentioned (without derivation) in [1], [2]. In some spacetimes, such as anti–de Sitter one, the three sets merge into the first sequence. At least in the Schwarzschild case, the first conjugate point of each of the two additional sequences appears after making more than one full revolution. This unexpected result shows that the general method for searching for locally maximal timelike curves is effective at least for SSS spacetimes.

Due to difficulties with solving complicated differential equations, we deal here solely with radial and circular timelike geodesics.

This paper contains no other concrete geometrical/physical conclusions since it is a theoretical introduction to the research programme of investigations of the geodesic structure of physically interesting spacetimes. In the search for locally maximal geodesics one applies an ‘algorithmic’ method consisting of a finite number of steps; the method is effective if and only if the geodesic deviation equation is solvable on the given geodesic curve. Yet in the global problem an analogous procedure cannot exist and we apply a suitably chosen
Gaussian normal geodesic coordinate system. The use of this system, supplemented by spacetime isometries such as in static spherically symmetric manifolds and conjugate points found in solving the local problem, allows one to determine globally maximal segments of some classes of geodesics.

At present the only general conclusion that can be drawn from our work is that the geodesic structure of curved spacetimes, even those quite simple (high symmetry), is richer and more complicated than it might be expected.

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References

[1] L.M. Sokolowski and Z.A. Golda, The local and global geometrical aspects of the twin paradox in static spacetimes: I. Three spherically symmetric spacetimes. Acta Phys. Polon. B 45(2014) 1051-1075 [arXiv:1402.6511v2 [gr-qc]].

[2] L.M. Sokolowski and Z.A. Golda, The local and global geometrical aspects of the twin paradox in static spacetimes: II. Reissner–Nordström and ultra-static metrics. Acta Phys. Polon. B 45(2014) 1713-1741 [arXiv:1404.5808 [gr-qc]].

[3] L.M. Sokolowski, On the twin paradox in static spacetimes: I. Schwarzschild metric. Gen. Rel. Grav. (2012) 44:1267-1283 [arXiv:1203.0748 [gr-qc]].

[4] L.M. Sokolowski and Z.A. Golda, Jacobi fields and conjugate points on timelike geodesics in special spacetimes. Acta Phys. Polon. B 46(2015) 773-787 (published version of this paper).

[5] L. Iorio, An analytical treatment of the clock paradox in the framework of the special and general theories of relativity. Found. Phys. Lett. 18, 1 (2005) [arXiv:physics/0405038].

[6] L. Iorio, On the clock paradox in the case of circular motion of the moving clock. Eur. J. Phys. 26, 535 (2005) [arXiv:physics/0406139].
[7] P. Jones and L.F. Wanex, The clock paradox in a static homogeneous gravitational field, Found. Phys. Lett. 19, 75 (2006) [arXiv:physics/0604025].

[8] C.E. Dolby and S.F. Gull, On radar time and the twin paradox, Amer. J. Phys. 69, 1257 (2001) [arXiv:gr-qc/0104077v2].

[9] M.A. Abramowicz, S. Bajtlik and W. Kluzniak, The twin paradox on the photon sphere. Phys. Rev. A75, 044101 (2007)

[10] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-Time. Cambridge University Press, Cambridge (1973), chapter 4.

[11] R.M. Wald, General Relativity. University of Chicago Press, Chicago (1984), chapter 9.

[12] Y. Hagihara, Theory of the relativistic trajectories in a gravitational field of Schwarzschild. Japan. J. Astron. Geophys. 8(1931), 67-176.

[13] F.T. Hioe and D. Kuebel, Characterization of all possible orbits in the Schwarzschild metric revisited. [arXiv:1207.7041 [gr-qc]].

[14] N. Cruz, M. Olivares and J.R. Villanueva, The geodesic structure of the Schwarzschild anti–de Sitter black hole. Class. Quantum Grav. 22 (2005) 1167-1190.

[15] E. Hackmann and C. Lämmerzahl, Geodesic equation in Schwarzschild-(anti)-de Sitter spacetimes: Analytical solutions and applications. Phys. Rev. D78, 024035(2008), 1-21.

[16] J.K. Beem, P.E. Ehrlich and K.L. Easley, Global Lorentzian Geometry, second edition. Marcel Dekker, New York (1996).

[17] R. Kerner, J.W. van Holten and R. Colistete Jr., Relativistic epicycles: another approach to geodesic deviation. Class. Quantum Grav. 18 (2001) 4725-4742.

[18] R. Colistete Jr., C. Leygnac, and R. Kerner, Higher–order geodesic deviations applied to the Kerr metric. Class. Quantum Grav. 19 (2002) 4573-4590.

[19] J. Podolsky and R. Švarc, Interpreting spacetimes of any dimension using geodesic deviation. Phys. Rev. D85, 044057 (2012) [arXiv:1201.4790v2[gr-qc]].
[20] H. Fuchs, Solutions of the equations of geodesic deviation for static spherical symmetric space-times. Ann. d. Physik 40, 231-233 (1983).

[21] S.L. Bażański, Hamilton-Jacobi formalism for geodesics and geodesic deviations. J. Math. Phys. 30, 1018-1029 (1989).

[22] S.L. Bażański and P. Jaranowski, Geodesic deviation in the Schwarzschild spacetime. J. Math. Phys. 30, 1794-1803 (1989).

[23] H. Fuchs, Paralleltransport and geodesic deviation in static spherically symmetric space-times. Astron. Nachr. 311(1990) 219-222.

[24] H. Fuchs, Deviation of circular geodesics in static spherically symmetric space-times. Astron. Nachr. 311(1990) 271-276.

[25] L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields. 4th revised English edition, Butterworth–Heinemann, Oxford 1999, par. 102.

[26] E. Calabi and L. Markus, Relativistic space forms. Annals of Mathematics 75 (1962) 63–76.

[27] S. Chandrasekhar, Mathematical Theory of Black Holes. Oxford Univ. Press, Oxford 1983, chap. 5, par.40.