Constructing uniform 2-factorizations via row-sum matrices: solutions to the Hamilton-Waterloo problem

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Abstract

In this paper, we formally introduce the concept of a row-sum matrix over an arbitrary group $G$. When $G$ is cyclic, these types of matrices have been widely used to build uniform 2-factorizations of small Cayley graphs (or, Cayley subgraphs of blown-up cycles), which themselves factorize complete (equipartite) graphs.

Here, we construct row-sum matrices over a class of non-abelian groups, the generalized dihedral groups, and we use them to construct uniform 2-factorizations that solve infinitely many open cases of the Hamilton-Waterloo problem, thus filling up large parts of the gaps in the spectrum of orders for which such factorizations are known to exist.

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1 Introduction

In this paper we denote by \( L = [\alpha^1 a_1, \ldots, \alpha^\ell a_\ell] \) the multiset containing \( \alpha_i \geq 0 \) copies of the element \( a_i \in A \), for each \( i \in \{1, \ldots, \ell\} \). Note that the \( a_i \)'s need not be distinct. We will call such a multiset a list, even though order is not important, as we will be dealing extensively with so called \( v \)-lists, which are multisets as above where \( \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = v \) (see Section 2).

Given a simple graph \( G \), we denote by \( V(G) \) and \( E(G) \) its sets of vertices and edges, respectively. As usual, we denote by \( C_\ell \) a cycle of length \( \ell \) (briefly, an \( \ell \)-cycle), and by \( (x_0, x_1, \ldots, x_{\ell-1}) \) the \( \ell \)-cycle with edges \( x_0 x_1, x_1 x_2, \ldots, x_{\ell-1} x_0 \). A factor of \( G \) is a spanning subgraph \( F \) of \( G \); when \( F \) is \( i \)-regular, we speak of an \( i \)-factor. In particular, a 1-factor (resp. a 2-factor) of \( G \) is a vertex-disjoint union of edges (cycles) whose vertices cover \( V(G) \). A 2-factor \( F \) of \( G \) containing only cycles of length \( \ell \) will be called a \( C_\ell \)-factor or \( \text{uniform factor} \).

By \( K_v^* \) we mean the complete graph \( K_v \) on \( v \) vertices when \( v \) is odd and \( K_v - I \), that is, \( K_v \) minus the edges of the 1-factor \( I \), when \( v \) is even. Also, by \( K_t[z] \) we denote the complete equipartite graph with \( t \) parts of size \( z \geq 1 \). Note that \( K_t[1] = K_t \).

A 2-factorization of a simple graph \( G \) is a set \( G \) of 2-factors of \( G \) whose edge sets partition \( E(G) \). It is well known that \( G \) has a 2-factorization if and only if it is regular of even degree. However, if we require the factors of \( G \) to have a specific structure then the problem becomes much harder. For example, the existence of a 2-factorization of \( G \) into copies of a given 2-factor \( F \) is an open problem even when \( G = K_v^* \). This is the well-known Oberwolfach Problem, originally posed by Ringel in 1967 for odd \( v \). A survey of the most relevant results on this problem, updated to 2006, can be found in [13, Section VI.12]. For more recent results we refer the reader to [12].

A factorization of the simple graph \( G \) into copies of a \( C_\ell \)-factor is briefly called a \( C_\ell \)-factorization or \( \text{uniform factorization} \) of \( G \). The problem of factoring \( K_v^* \) into copies of a uniform 2-factor, that is, the uniform Oberwolfach Problem, has been solved [11, 2, 18, 25].

**Theorem 1.1** ([11, 2, 18, 25]). Let \( v, \ell \geq 3 \) be integers. There is a \( C_\ell \)-factorization of \( K_v^* \) if and only if \( \ell \mid v \), except that there is no \( C_3 \)-factorization of \( K_6^* \) or \( K_{12}^* \).

A similar result when \( G = K_t[z] \) has more recently been obtained in [22, 23].
Theorem 1.2 \([22, 23]\). Let \(\ell, t\) and \(z\) be positive integers with \(\ell \geq 3\). There exists a \(C_\ell\)-factorization of \(K_t[z]\) if and only if \(\ell \mid tz\), \((t-1)z\) is even, further \(\ell\) is even when \(t = 2\), and \((\ell, t, z) \not\in \{(3, 3, 2), (3, 6, 2), (3, 3, 6), (6, 2, 6)\}\).

We may generalise this problem to the Generalized Oberwolfach Problem, denoted GOP\((G; \mathcal{R})\), where \(\mathcal{R} = \{\alpha_1 R_1, \ldots, \alpha_t R_t\}\) is a list of 2-factors of \(G\), where each \(R_i\) is repeated \(\alpha_i\) times (with \(\alpha_i\) a positive integer) and the \(R_i\) are pairwise non-isomorphic. The Generalized Oberwolfach Problem then requires that the edges of \(G\) be factored into a union of \(\alpha_i\) copies of \(R_i\), \(1 \leq i \leq t\). If each \(R_i\) is uniform, with cycles of length \(a_i\), we speak of GOP\((G; [a_1, \ldots, a_t])\). Since the \(R_i\) are factors and every edge of \(G\) is in one of the factors \(R_i\), this requires that \(G\) is regular with each vertex having degree \(2 \sum_{i=1}^{t} \alpha_i\) and that if \(R_i\) is a \(C_{a_i}\)-factor, then \(a_i\) divides the order of \(G\). Despite recent probabilistic results which show eventual existence, these results are non-constructive and give no lower bounds for their implementation and so this problem remains wide open; see [10] for more details on the Generalized Oberwolfach Problem.

When \(\mathcal{R} = \{\alpha R_1, \beta R_2\}\), then GOP\((G; \mathcal{R})\) represents the most studied variant of the Oberwolfach Problem, known as the Hamilton-Waterloo Problem, and denoted by HWP\((G; R_1, R_2; \alpha, \beta)\), or HWP\((v; R_1, R_2; \alpha, \beta)\) when \(G\) is \(K^*_v\). This problem asks for a factorization of \(G\) into \(\alpha\) copies of \(R_1\) and \(\beta\) copies of \(R_2\). In the case where \(R_1\) and \(R_2\) are a \(C_M\)-factor and \(C_N\)-factor, respectively, we refer to HWP\((G; M, N; \alpha, \beta)\), or HWP\((v; M, N; \alpha, \beta)\) when \(G\) is \(K^*_v\), and speak of the uniform Hamilton-Waterloo Problem. Clearly, when \(\alpha = 0\) or \(\beta = 0\) we obtain the uniform Oberwolfach problem which is completely solved (Theorem 1.1). Therefore, from now on we will assume that both \(\alpha\) and \(\beta\) are positive integers. Well-known obvious necessary conditions for the solvability of HWP\((G; M, N; \alpha, \beta)\) are given the following theorem.

Theorem 1.3. Let \(G\) be a graph of order \(v\), and let \(M, N, \alpha\) and \(\beta\) be non-negative integers. In order for a solution of HWP\((G; M, N; \alpha, \beta)\) to exist, \(M\) and \(N\) must be divisors of \(v\) greater than 2, and \(G\) must be regular of degree \(2(\alpha + \beta)\).

We are interested in constructing solutions to the uniform Hamilton-Waterloo Problem. We point out that this case (as well as the general problem) is still open, and this is quite surprising considering that the equivalent problem of factoring \(K^*_v\) into uniform factors (the uniform OP) was solved in the nineties (see Theorem 1.1).
For more details and some history on the problem, we refer the reader to [8]. That paper deals with the case where both $M$ and $N$ are odd positive integers and provides an almost complete solution to $HWP(v; M, N; \alpha, \beta)$ for odd $v$. If $M$ and $N$ are both even, then $HWP(v; M, N; \alpha, \beta)$ has a solution except possibly when $\alpha = 1$ or $\beta = 1$; $\beta = 3$, $v \equiv 2 \pmod{4}$ and $\gcd(M, N) = 2$; or $v = MN/\gcd(M, N) \equiv 2 \pmod{4}$ [5, 11]. However, the problem is completely solved when $M$ and $N$ are even and $M$ is a divisor of $N$ [6]. The case where $M$ and $N$ have different parities is the most challenging. Indeed, the only case where $M \not\equiv N \pmod{2}$ that has been completely solved is when $(M, N) = (3, 4)$ [14, 21, 27]. The only other cases which have been considered are when $M$ is a divisor of $N$ [3, 9, 21]; $M = 4$ [19, 24]; $M = 8$ [20]; and when $M$ and $N$ are not coprime, $M$ is odd, $N = 2^k n$ and $4^k$ divides $v$ [20]. However, possible exceptions remain in all of these cases. The following theorem summarizes the results in [5, 6, 8, 9, 11, 20].

**Theorem 1.4** (5, 6, 8, 9, 11, 20). There is a solution to $HWP(v; M, N; \alpha, \beta)$ when

1. $M, N \geq 3$ are odd, $M \geq N$, $MN/\gcd(M, N)$ divides $v$ and $v \neq MN/\gcd(M, N)$, except possibly if $\beta \in \{1, 3\}$.

2. $M$ and $N$ are even, with $M > N$, and $M$ and $N$ divide $v$, except possibly if $N$ does not divide $M$, and $1 \in \{\alpha, \beta\}$; if $\beta = 3$, $v \equiv 2 \pmod{4}$ and $\gcd(M, N) = 2$; or if $v = MN/\gcd(M, N) \equiv 2 \pmod{4}$, and $\alpha$ and $\beta$ are odd.

3. $N = 2^k n$ with $k \geq 1$, $M$ and $n$ are odd, and either $M$ divides $n$, $v > 6N > 36M$ and $s \geq 3$; or $\gcd(M, n) \geq 3$, $4^k$ divides $v$, $v/(4^k \text{lcm}(M, n))$ is at least 3 and $1 \not\in \{\alpha, \beta\}$.

The results in [8, 9, 20] were obtained using solutions to $HWP(C_g[u], M, N, \alpha, \beta)$, where $C_g[u]$ is the graph obtained from $C_g$ by replacing every vertex in the cycle with $u$ copies of it. In other words, it is the graph with vertices of the form $x_{i,j}$, $0 \leq i \leq g - 1$, $0 \leq j \leq u - 1$, and edges of the form $x_{i,a}x_{i+1,b}$ with addition done modulo $u$, and $0 \leq a, b \leq u - 1$.

In this paper, we make further progress when $M$ and $N$ are not coprime in two regards. On the one hand, we improve the result for the case when $M$ and $N$ have different parities, changing the condition that $4^k$ divides $v$ into the condition that $2^{k+2}$ divides $v$. On the other hand, our results put no restrictions on $\alpha$ and $\beta$, covering the difficult case when $M$ and $N$ have the
The same parity and 1 ∈ \{α, β\} that was previously left open. More precisely, our main result is the following.

**Theorem 1.5.** Let \( v, M \) and \( N \) be integers greater than 3, and let \( \ell = \text{lcm}(M, N) \). A solution to HWP\((v; M, N; \alpha, \beta)\) exists if and only if \( M \mid v \) and \( N \mid v \), except possibly when

- \( \gcd(M, N) \in \{1, 2\} \);
- \( 4 \) does not divide \( v/\ell \);
- \( v/4\ell \in \{1, 2\} \);
- \( v = 16\ell \) and \( \gcd(M, N) \) is odd;
- \( v = 24\ell \) and \( \gcd(M, N) = 3 \).

In Section 2 we introduce the concept of row-sum matrices and prove some preliminary results. In particular, we show how to use said matrices to obtain solutions to HWP\((C_g[u], M, N, \alpha, \beta)\). Next, in Section 3 we prove the existence of the matrices we need for our main result. Finally, in Section 4 we complete the proof of Theorem 1.5.

## 2 Preliminary results

Recall that given a group \( \Gamma \), and an integer \( v \), a \( v \)-list of \( \Gamma \) is a list \( \Delta = [\delta_1, \ldots, \delta_v] \) of \( v \) (not necessarily distinct) elements of \( \Gamma \). Given an integer \( g \), set \( g\Delta = [g\delta_1, \ldots, g\delta_v] \). It will be helpful to refer to the list \( \omega(\Delta) \) of element orders associated to \( \Delta \), defined as follows: \( \omega(\Delta) = [\omega(\delta_1), \ldots, \omega(\delta_v)] \), where each \( \omega(\delta_i) \) is the order of the element \( \delta_i \) of the group \( \Gamma \), and hence a divisor of the order of \( \Gamma \).

### 2.1 \( \Delta \)-Permutations

Let \( \Gamma \) be an arbitrary group of order \( v \), and let \( \Delta \) be a \( v \)-list of \( \Gamma \). We say that a permutation \( \varphi \) of \( \Gamma \) is a \( \Delta \)-permutation if the following condition holds:

\[
[\varphi(a_1) - a_1, \varphi(a_2) - a_2, \ldots, \varphi(a_v) - a_v] = \Delta,
\]

where \( \{a_1, a_2, \ldots, a_v\} = \Gamma \).
Remark 2.1. Given any fixed $x \in \Gamma$ and $\delta \in \Delta$, we can assume that $\varphi(x) - x = \delta$. Otherwise, take $a_j \in \Gamma$ such that $\varphi(a_j) - a_j = \delta$, and set $b_i = a_i - a_j + x$ and $\phi(b_i) = \varphi(a_i) - a_j + x$, for every $i = 1, \ldots, v$. Clearly, $\Gamma = \{b_1, b_2, \ldots, b_v\}$ and $\phi$ is a $\Delta$-permutation of $\Gamma$, since $\phi(b_i) - b_i = \varphi(a_i) - a_i$ for each $i = 1, \ldots, v$. Also, $b_j = x$ and $\phi(x) - x = \varphi(a_j) - a_j = \delta$.

Given an arbitrary $v$-list $\Delta$ of an abelian group $G$ of order $v$ a necessary condition for a $\Delta$-permutation of $G$ to exist is that $\sum \Delta = 0$, where $\sum \Delta$ denotes the sum of the elements in $\Delta$. M. Hall [16] proved that this condition is also sufficient.

Theorem 2.2 ([16]). Let $G$ be an abelian group of order $v$, and let $\Delta$ be a $v$-list of $G$. There exists a $\Delta$-permutation of $G$ if and only if $\sum \Delta = 0$.

The following special $\Delta$-permutations will be useful in the constructions of Section 4.

Theorem 2.3. Let $m$ and $n$ be positive integers with $m$ odd. Then there exists a $\Delta$-permutation $\psi$ of $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ such that

1. $\Delta = \left\{ [1](0, 0) \right\} \cup \left\{ [\gamma] \mid \gamma \in \mathbb{Z}_m \times \mathbb{Z}_{2n}, \gamma \neq (0, n) \right\}$,

2. $\psi$ fixes $(0, 0)$ and $(-m^{-1}, \lfloor n + 1 \rfloor + m^{-1}n)$.

Proof. Assume that $V(K_{2mn}) = \mathbb{Z}_m \times \mathbb{Z}_{2n}$. We are going to construct a suitable matching $H$ of $K_{2mn}$ with $mn - 1$ edges. We leave to the reader the check that the permutation that swaps every pair of adjacent vertices in $H$ and fixes the remaining two is the desired $\Delta$-permutation of $\mathbb{Z}_m \times \mathbb{Z}_{2n}$.

We consider the following matching of $K_{2n}$,

$$F = \left\{ (j, -j) \mid j = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \cup \left\{ (j, -j + 1) \mid j = \left\lfloor \frac{n+3}{2} \right\rfloor, \ldots, n \right\},$$

and note that $V(F) = \mathbb{Z}_{2n} \setminus \{0, u\}$ with $u = \left\lfloor \frac{n+1}{2} \right\rfloor$.

For every non-negative integer $i$, we lift $F$ to a matching $F(i)$ with vertex-set $\{\pm i\} \times (\mathbb{Z}_{2n} \setminus \{0, u\})$ defined as follows:

$$F(i) = \left\{ ((x, y_1), (-x, y_2)) \mid x = \pm i, \{y_1, y_2\} \in E(F) \right\}.$$ 

Consider also the following two matchings of $K_{2mn}$:

$$F' = \left\{ ((i, in), -(i, in)) \mid i = 1, \ldots, \frac{m-1}{2} \right\},$$
\[ F'' = \left\{ \{(-i + 1, u + in + n), (i, u + in)\} \mid i = 1, \ldots, \frac{m-1}{2} \right\}, \]

and set

\[ H = \bigcup_{i=0}^{(m-1)/2} (F(i) + (0, in)) \cup F' \cup F''. \]

It is not difficult to check that \( H \) is a matching of \( K_{2mn} \) with \( 2mn - 1 \) edges, missing the vertices \((0, 0)\) and \((-\frac{m-1}{2}, u + \frac{m-1}{2}n)\).

**Theorem 2.4.** Let \( m \geq 1 \) and \( n \geq 3 \) be odd integers. Then there exists a \( \Delta \)-permutation \( \psi \) of \( \mathbb{Z}_m \times \mathbb{Z}_{2n} \) such that

1. \( \Delta = [2^{mn-6}(1, 0), 3(2, 0), 1(0, 2), 1(0, n-2), 1(0, n)] \),
2. \( \psi(0, 0) = (0, n), \psi(0, n) = (0, n + 2) \).

**Proof.** Let \( \psi \) be the permutation of \( \mathbb{Z}_m \times \mathbb{Z}_{2n} \) defined as follows

\[ \psi(0, 0) = (0, n), \quad \psi(0, n) = (0, n + 2), \quad \psi(0, n + 2) = (0, 0), \]

and for every \( z \in \mathbb{Z}_m \times \mathbb{Z}_{2n} \setminus \{(0, 0), (0, n), (0, n + 2)\} \), let

\[ \psi(z) = \begin{cases} 
   z + (2, 0) & \text{if } z \in \{(-1, 0), (-1, n), (-1, n + 2)\}, \\
   z + (1, 0) & \text{otherwise.}
\end{cases} \]

One can check that \( \psi \) is the desired permutation. \( \square \)

### 2.2 Row-sum matrices and 2-factorizations of \( C_g[n] \)

Let \( \Gamma \) be a group, and let \( S \subset \Gamma \). Also, let \( \Sigma \) be an \( |S| \)-list of elements of \( \Gamma \). A \textit{row-sum matrix} \( RSM_{\Gamma}(S, g; \Sigma) \) is an \( |S| \times g \) matrix, whose \( g \geq 2 \) columns are permutations of \( S \) and such that the list of (left-to-right) row-sums is \( \Sigma \). We write \( RSM_{\Gamma}(S, g; \omega(\Sigma)) \) whenever we are just interested in the list \( \omega(\Sigma) \) of orders of the row-sums. Notice that an \( RSM_{\Gamma}(\Gamma, 2; \Sigma) \) is equivalent to a \( \Sigma \)-permutation of \( \Gamma \).

Row-sum matrices are useful to build factorizations of suitable Cayley subgraphs of \( C_g[n] \). More precisely, we denote by \( C_g[\Gamma, S] \) (\( g \geq 3 \)) the graph with point set \( \mathbb{Z}_g \times \Gamma \) and edges \((i, x)(i + 1, d + x), i \in \mathbb{Z}_g, x \in \Gamma\) and \( d \in S \). In other words, \( C_g[\Gamma, S] = \text{cay}(\mathbb{Z}_g \times \Gamma, \{1\} \times S) \); hence, it is \( 2|S| \)-regular. It is straightforward to see that if \( \Gamma \) has order \( n \), then \( C_g[n] \cong C_g[\Gamma, \Gamma] \); hence, \( C_g[\Gamma, S] \) is a subgraph of \( C_g[n] \).

The following result, proven in [10] Theorem 2.1 when \( \Gamma \) has odd order, shows that row-sum matrices can be used to build factorizations of \( C_g[\Gamma, S] \).
Theorem 2.5. If there exists an $RSM_\Gamma(S, g; \Sigma)$, then $GOP(C_2[\Gamma, S]; g_\omega(\Sigma))$ has a solution.

We skip the proof of Theorem 2.5 when $|\Gamma|$ is even, as it is identical to the odd case.

We now show that row-sum matrices can be easily extended over the columns whenever $S$ is closed under taking negatives (that is, $S = -S$) or when $S = \Gamma$ has a complete mapping. We recall that a complete mapping of $\Gamma$ is a permutation $\pi$ of $\Gamma$ such that $\rho(x) = x + \pi(x)$ is also a permutation.

Theorem 2.6. If there is an $RSM_\Gamma(S, g; \Sigma)$, then there exists $RSM_\Gamma(S, g + i; \Sigma)$ in each of the following cases:

1. $S = -S$ and $i \geq 2$ is even, or
2. $S = \Gamma$ has a complete mapping and $i \geq 1$.

Proof. Let $A$ be an $RSM_\Gamma(S, g; \Sigma)$. We first assume that $S = -S$ and $i \geq 2$ is even. The existence of an $RSM_\Gamma(S, g + i; \Sigma)$ was essentially proven in [10, Theorem 2.1] as follows: it is enough to extend $A$ by adding $i/2$ copies of an $RSM_\Gamma(S, 2; \Sigma')$, say $B$, where $\Sigma'$ is the $|S|$-list of zeros (i.e., each row of $B$ sums to 0). Clearly, $B$ can be easily built by choosing its second column to be the negative of the first one, which must be a permutation of $S$ by assumption.

Now let $i \geq 1$, assume that $S = \Gamma$ has a complete mapping $\pi$, and set $\rho(x) = x + \pi(x)$. In view of the first part of the proof, it suffices to show the existence of an $RSM_\Gamma(S, g + 1; \Sigma)$. To do so, it is enough to replace each element of the last column of $A$, say $y$, with the pair $(x, \pi(x))$ where $x = \rho^{-1}(y)$.

Throughout the paper we will only deal with solvable groups, and we will make use of a result by Hall and Paige who proved that a finite solvable group $\Gamma$ has a complete mapping if and only if its 2-Sylow subgroup is either trivial or non-cyclic. This result was then extended to an arbitrary group (see [15]).

Theorem 2.7 ([17]). A finite solvable group $\Gamma$ has a complete mapping if and only if its 2-Sylow subgroup is either trivial or non-cyclic.

This, combined with Theorems 2.5 and 2.6 means that when $S = \Gamma$ is as in Theorem 2.7 it is enough to construct row-sum matrices with 3 columns.
Corollary 2.8. Let $\Gamma$ be a solvable group of order $n$ whose 2-Sylow subgroup is either trivial or non-cyclic. If there is an $RSM_{\Gamma}(\Gamma, 3; \Sigma)$, then there exist

1. an $RSM_{\Gamma}(\Gamma, g; \Sigma)$, and
2. a solution to $GOP(C_g[n]; g\omega(\Sigma))$,

for every $g \geq 3$.

We end this section by constructing row-sum matrices over an abelian group having a given list of associated row-sum orders.

Theorem 2.9. Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2^\ell \times \mathbb{Z}_{mn}$ with $n$ odd and $\ell \geq 1$. Then, there is an $RSM_{\Gamma}(\Gamma, g; [2^\gamma m, 2^k 2^\ell n])$ whenever $\gamma + \delta = 2^\ell mn$, $g \geq 3$ and $0 \leq k \leq \ell$.

Proof. Since $\ell \geq 1$, the Sylow 2-subgroup of $\Gamma$ is non-cyclic. Hence, by Corollary 2.8 it is enough to prove the assertion for $g = 3$. Set

$$\Delta = [\gamma(0, 0, n), \gamma(0, 0, -n), \delta(0, 2^\ell - k, m), \delta(0, -2^\ell - k, -m)] .$$

Since $\sum \Delta = 0 = \sum \Gamma$, by Theorem 2.2 there are a $\Delta$-permutation $\varphi$ of $\Gamma$ and a $\Gamma$-permutation $\psi$ of $\Gamma$. Now consider the $2^\ell + 1 mn \times r$ matrix $A$ whose rows, indexed over $\Gamma$, have the following form

$$A_x = \begin{pmatrix} -\psi(x) & x & \varphi(\psi(x) - x) \end{pmatrix} .$$

One can easily check that each column of $A$ is a permutation of $\Gamma$, and

$$[\sum A_x \mid x \in \Gamma] = \Delta .$$

Since $\Delta$ contains $2\gamma$ elements of order $m$ and $2\delta$ elements of order $2^k n$, the assertion follows.

2.3 Matrices over a generalized dihedral group

From now on, $g, m$ and $n$ will denote positive integers with $g \geq 3$, $m$ and $n$ odd, and $G = \mathbb{Z}_m \times \mathbb{Z}_{2^{k+1}n}$ with $k \geq 2$. Notice that we allow both $m$ and $n$ to be equal to 1. In the case $m = 1$ this means that we work with $\mathbb{Z}_1 \times \mathbb{Z}_{2^{k+1}n} \simeq \mathbb{Z}_{2^{k+1}n}$. Recall that the coordinate-wise multiplication by any element $x \in G$ is a homomorphism of the group $G$; in particular, the multiplication by $\epsilon = (-1, -1)$ is a group automorphism, and its order is 2 since $\epsilon^2 = (1, 1)$. 

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Therefore, we can define the semidirect product $G \rtimes \mathbb{Z}_2$ whose underlying set is $G \times \mathbb{Z}_2$, and the group operation, still denoted by $+$, is defined as follows:

$$(x, \tau) + (x', \tau') = (x + e^\tau x', \tau + \tau').$$

It is not difficult to check that $G \rtimes \mathbb{Z}_2 \simeq \text{Dih}(\mathbb{Z}_m \times \mathbb{Z}_{2^{k+1}n})$ is the generalized dihedral group over $\mathbb{Z}_m \times \mathbb{Z}_{2^{k+1}n}$.

From now on, $\Gamma = G \rtimes \mathbb{Z}_2$ and for every subset $S \subseteq \Gamma$, we simply write $C_g[S]$ in place of $C_g[\Gamma, S]$. Note that $2\Gamma \cong 2G = \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$ and $|2\Gamma| = 2^k mn$. Consider the interval $I = \{0, \ldots, 2^k n - 1\}$ and the function $\rho : 2\mathbb{Z}_{2^{k+1}n} \to I$ with $\rho(y)$ being defined by $2\rho(y) = y$. Similarly, for every $x \in \mathbb{Z}_m$ ($m$ odd) we denote by $x/2$ the unique element of $\mathbb{Z}_m$ such that $2(x/2) = x$. Now let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, where each $\varphi_h$ is a permutation of $\mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$, and define the following five bijections:

$$a_h : \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n} \to \mathbb{Z}_m \times (2\mathbb{Z}_{2^{k+1}n} + 1), \quad (x, y) \mapsto \varphi_h(x, y) + (0, 1),$$

for $h \in \{1, 2, 3\}$, and

$$b : \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n} \to \mathbb{Z}_m \times (-I), \quad (x, y) \mapsto (-x/2, -\rho(y)),
$$

$$c : \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n} \to \mathbb{Z}_m \times (I + 1), \quad (x, y) \mapsto (x/2, \rho(y) + 1).$$

Finally, for every $(x, y) \in \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$, let $A(x, y)$ and $A'(x, y)$ be the $3 \times 3$ matrices with entries from $\Gamma$ defined as follows:

$$A(\varphi, (x, y)) = \begin{bmatrix}
(a_1(x, y), 0) & (b(x, y), 1) & (c(x, y), 1) \\
(c(x, y), 1) & (a_2(x, y), 0) & (b(x, y), 1) \\
(b(x, y), 1) & (c(x, y), 1) & (a_3(x, y), 0)
\end{bmatrix},$$

$$A'(\varphi, (x, y)) = \begin{bmatrix}
(a_1(x, y), 0) & (c(x, y), 1) & (b(x, y), 1) \\
(c(x, y), 1) & (b(x, y), 1) & (a_2(x, y), 0) \\
(b(x, y), 1) & (a_3(x, y), 0) & (c(x, y), 1)
\end{bmatrix}.$$

Note that $A'(\varphi, (x, y))$ is obtained from $A(\varphi, (x, y))$ by swapping columns 2 and 3. From now on, given a matrix $M$ over an arbitrary group, $\sum M_h$ represents the (left-to-right) sum of the $h$-th row of $M$ denoted by $M_h$.

**Lemma 2.10.** For every $(x, y) \in \mathbb{Z}_m \times 2\mathbb{Z}_{2^{k+1}n}$, we have that

$$\sum A(\varphi, (x, y))_h = \begin{cases}
((\varphi_h(x, y) - (x, y)), 0) & \text{if } h = 1, 3, \\
((x, y) - \varphi_h(x, y), 0) & \text{if } h = 2,
\end{cases}$$

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\[ \sum A'(\varphi, (x, y))_h = \begin{cases} 
(\varphi_h(x, y) + (x, y), 0) + (0, 2, 0) & \text{if } h = 1, 2, \\
-((\varphi_h(x, y) + (x, y)), 0) - (0, 2, 0) & \text{if } h = 3. 
\end{cases} \]

Proof. Note that \[ \sum A(\varphi, (x, y))_h = (-1)^h(\sigma(x, y), 0), \] where \[ \sigma = c - b - a_h, \] for every \( h = 1, 2, 3. \) Also,
\[ \sigma(x, y) = (x/2, \rho(y) + 1) - (-x/2, -\rho(y)) - (\varphi_h(x, y) + (0, 1)) \]
\[ = (x, 2\rho(y)) - \varphi_h(x, y) = (x, y) - \varphi_h(x, y). \]

The result for \( A'(\varphi, (x, y)) \) is similar. \( \square \)

3 Row-sum matrices over a generalized dihedral group

In this section, we build row-sum matrices over the group \( \Gamma \simeq \text{Dih}(\mathbb{Z}_m \times \mathbb{Z}_{2^{k+1}n}) \) (defined in Section 2.3) and prove the following.

Theorem 3.1. Let \( g \geq 3, k \geq 0, \) and let \( m, n \geq 1 \) be odd integers. Then \( RSM_\Gamma(\Gamma, g; [^a m, \beta 2^k n]) \) exists if and only if \( \alpha + \beta = 2^{k+2} mn \) except possibly in the following cases

1. \( k \geq 2 \) and \( \beta = 0, \) or
2. \( k = 1 \) and \( \alpha, \beta \in \{0, 2, 4\}. \)

Its proof is given in Sections 3.1, 3.2 and 3.3 which respectively deal with the cases \( k \geq 2, k = 1 \) and \( k = 0. \) It mostly relies on Theorem 2.9 and the following Theorems 3.2 and 3.3.

Recall that \( g, m, n \) are positive integers with \( g \geq 3, \) and \( m \) and \( n \) odd.

Theorem 3.2. \( RSM_\Gamma(\Gamma \setminus 2\Gamma, g; [^a m, \beta 2^k n]) \) exists when \( \alpha \neq 1 \) and \( \beta \neq 1, \) with \( \alpha + \beta = 3 \cdot 2^k mn. \)

Proof. We first deal with the case where \( \alpha \) and \( \beta \) are even, with \( 0 \leq \alpha, \beta \leq 3 \cdot 2^k mn. \) Letting \( \alpha = 2a \) and \( \beta = 2b, \) and recalling that \( |\Gamma \setminus 2\Gamma| = 3 \cdot 2^k mn, \) we can write
\[ a = \sum_{h=1}^3 a_h \text{ and } b = \sum_{h=1}^3 b_h, \]
where $2(a_h + b_h) = 2^k mn = |2\Gamma|$, for every $h \in \{1, 2, 3\}$. By Theorem 2.2, there exists a $\Delta_h$-permutation $\varphi_h$ of $Z_m \times 2Z_{2k+1,n}$, where

$$\Delta_h = [a_h(1, 0), a_h(-1, 0), b_h(0, 2), b_h(0, -2)].$$

Notice that $|\Delta_h| = [2a_h m, 2b_h 2^k n]$, for every $h \in \{1, 2, 3\}$. Let $A = [A(\varphi, (x, y))]$ be the column block-matrix whose blocks are the matrices $A(\varphi, (x, y))$ for $(x, y) \in Z_m \times 2Z_{2k+1,n}$. Note that $A$ is a $(3 \cdot 2^kn) \times 3$ matrix whose columns are permutations of $\Gamma \setminus 2\Gamma$. Also, letting $L_A$ be the list of row-sums of $A$, by Lemma 2.10 we have that

$$\omega(L_A) = \omega(\Delta_1) \cup \omega(\Delta_2) \cup \omega(\Delta_3) = [2a m, 2b 2^k n],$$

and the result follows.

It is left to deal with the case where $\alpha$ and $\beta$ are odd, with $3 \leq \alpha, \beta \leq 3 \cdot 2^k mn$. Let $A$ be the matrix built above with $2a = \alpha + 3 \geq 6$ and $2b = \beta - 3$. Since $2a \geq 6$, we can take $a_1, a_2, a_3 \geq 1$. Also, by Remark 2.1, we can assume that for $z = (\frac{m-1}{2}, 0)$ we have $\varphi_h(z) = z + (1, 0)$, for every $h = 1, 2, 3$. We denote by $A'$ the $(3 \cdot 2^k mn) \times 3$ matrix that we obtain by replacing the block $A(\varphi, z)$ of $A$ with $A'(\varphi, z)$. Clearly, the columns of $A'$ are still permutations of $\Gamma \setminus 2\Gamma$. Also, note that by Lemma 2.10, each $\sum A(\varphi, z)_h$ has order $m$, whereas $\sum A'(\varphi, z)_h = (0, 2, 0)$ has order $2^k n$, for every $h = 1, 2, 3$. Therefore, denoting by $L_A'$ the list of row sums of $A'$, we have that

$$\omega(L_A') = [2a - 3 m, 2b + 3 2^k n] = [\alpha m, \beta 2^k n].$$

The result then follows by applying Corollary 2.8 to $A'$.

**Theorem 3.3.** \(\text{RSM}_I(2\Gamma, g; [\alpha m, \beta 2^k n])\) exists whenever $\alpha + \beta = 2^k mn$ and one of the following conditions holds:

1. $k = 0$, $\alpha \neq 1$ and $\beta \neq 1$, or
2. $k = 1$ and $\alpha, \beta \geq 3$ are odd, or
3. $k \geq 2$, $\alpha, \beta$ are even and $\beta \geq 2$.

**Proof.** Recall that $2\Gamma = 2G \times \{0\}$ and $2G = Z_m \times 2Z_{2k+1,n}$. Let

$$(\alpha, \beta) = \begin{cases} (2a + 3, 2b + 3) & \text{if } k = 0, 1, \text{ and } \alpha, \beta \text{ are odd}, \\ (2a, 2b) & \text{if } k \neq 1, \text{ and } \alpha, \beta \text{ are even}. \end{cases}$$

By Theorem 2.2 and Remark 2.1 there are $\Lambda_i$-permutations $\psi_i$ of $2G$, with $i \in \{1, 2\}$, such that

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1. \( \Lambda_1 = \begin{cases} 2G \\ \{1 (0, 0) \} \cup \{z \mid z \in 2G, z \neq (0, 2^k n)\} \end{cases} \) if \( k = 0 \),
\[ \begin{cases} \{1(2, 0), a(1, 0), a+2(-1, 0), 1(0, 4), b(0, 2), b+2(0, -2)\} \end{cases} \] if \( k \geq 1 \);
\[ \begin{cases} \{a(1, 0), a(-1, 0), b(0, 2), b(0, -2)\} \end{cases} \] if \( k \geq 2 \);

2. \( \Lambda_2 = \begin{cases} \{1(2, 0), a(1, 0), a+2(-1, 0), 1(0, 4), b(0, 2), b+2(0, -2)\} \end{cases} \) if \( k = 1 \),
\[ \begin{cases} \{a(1, 0), a(-1, 0), b(0, 2), b(0, -2)\} \end{cases} \] if \( k \geq 2 \);

3. if \( k \geq 1 \), then \( \psi_2(0, 2^k n) = (0, 2^k n) + \begin{cases} (0, 4) \text{ if } k = 1, \\ (0, -2) \text{ if } k \geq 2. \end{cases} \)

Since \((0, 0) \in \Lambda_1\), there exists a pair \( \bar{z} \in 2G \) such that \( \psi_1(\bar{z}) = \bar{z} \). Let \( B \) denote the \( 2^k \cdot mn \times 3 \) matrix (with entries from \( 2\Gamma \)) whose rows \( B_z \), indexed over \( 2G \), are defined as follows:
\[ B_z = \begin{pmatrix} (z, 0) \\ (-\psi_1(z), 0) \\ (\psi_2(w), 0) \end{pmatrix} \]
where \( w = \begin{cases} (0, 2^k n) \text{ if } z = \bar{z} \text{ and } k \neq 0, \\ \psi_1(z) - z \text{ otherwise.} \end{cases} \)

Note that the columns of \( B \) are permutations of \( 2\Gamma \). Also, one can check that for the list \( L_B \) of row sums of \( B \) we have
\[ \omega(L_B) = [\alpha m, \beta 2^k n]. \]

The result then follows by applying Lemma 2.8 to \( B \).

3.1 The proof of Theorem 3.1 when \( k \geq 2 \)

Let \( \alpha \) and \( \beta \) be non-negative integers such that \( \alpha + \beta = 2^k + 2mn \). First, we assume that both \( \alpha \neq 1 \) and \( \beta \not\in \{0, 1, 3\} \) and let \( \beta_1 = 2^k mn - \alpha_1 \)
\( \beta_2 = 3 \cdot 2^k mn - \alpha_2 \) where \( \alpha_1 \) and \( \alpha_2 \) are defined as follows:
\[ \alpha_1 = \begin{cases} \min \left\{ 2^k \cdot mn - 2, 2 \left\lfloor \frac{\alpha}{2} \right\rfloor \right\} \text{ if } \alpha \geq 2^k mn + 1, \\ \alpha \text{ if } \alpha \leq 2^k mn \text{ is even,} \\ \alpha - 3 \text{ if } 3 \leq \alpha < 2^k mn \text{ is odd,} \end{cases} \]
\[ \alpha_2 = \alpha - \alpha_1. \]

Clearly, \( \alpha_1 \) and \( \beta_1 \) are even, hence Theorem 3.3 guarantees the existence of an \( RSM_{\overline{\Gamma}}(2\Gamma, g, [\alpha_1 m, \beta_1 2^k n]) \), say \( B \). Furthermore, one can check that \( \alpha_2 \neq 1 \) and \( \beta_2 \neq 1 \). Hence, by Theorem 3.2 there is an \( RSM_{\overline{\Gamma}}(\Gamma \setminus 2\Gamma, g, [\alpha_2 m, \beta_2 2^k n]) \), say \( A \). Therefore, \( C = \begin{pmatrix} A \\ B \end{pmatrix} \) and Lemma 2.8 provide the desired RSM.
It is left to deal with the cases $\alpha = 1$ and $\beta = 1, 3$.

Case 1: $\alpha = 1$. Let $C = \begin{bmatrix} A & B \end{bmatrix}$ where $A$ is the matrix defined in Theorem 3.2 with $a = 0$ and $b = 3 \cdot 2^k mn$, and $B$ is the matrix defined in Theorem 3.3 with $a = 0$ and $b = 2^k mn$. It follows that for the list $L_C$ of row sums of $C$ we have

$$\omega(L_C) = [2^{k+2mn}2^k n].$$

By Remark 2.1, we can assume that the permutations $\varphi_1$ and $\varphi_3$ used to define $A$ in Theorem 3.2 satisfy the condition $\varphi_h(z') = z' - (0, 2) = (1, 0)$, with $z' = (1, 2)$, hence

$$a_h(z') = \varphi_h(z') + (0, 1) = (1, 1),$$

for $h = 1$ or 3. Furthermore, since $\Lambda_2 = [2^{k-1} nm (0, 2), 2^{k-1} nm (0, -2)]$, we can assume that $\psi_2(0, 0) = (0, 2)$. Consider the following matrices:

$$U = \begin{bmatrix} A(\varphi, z')_1 & B_{(0,2)} & A(\varphi, z')_3 \end{bmatrix} = \begin{bmatrix} (a_1(z'), 0) & (b(z'), 1) & (c(z'), 1) \\ (0, 2, 0) & (0, -2, 0) & (0, 2, 0) \\ (b(z'), 1) & (c(z'), 1) & (a_3(z'), 0) \end{bmatrix}$$

$$U' = \begin{bmatrix} (0, 2, 0) & (c(z'), 1) & (c(z'), 1) \\ (a_1(z'), 0) & (0, -2, 0) & (a_3(z'), 0) \\ (b(z'), 1) & (b(z'), 1) & (0, 2, 0) \end{bmatrix}.$$ 

Note that $U$ is a submatrix of $C$, while each column of $U'$ is a permutation of the corresponding column of $U$. Therefore, by replacing the block $U$ of $C$ with $U'$, we obtain a new $2^{k+2mn} \times 3$ matrix $C'$ whose columns are still permutations of $\Gamma$. Denote by $L'$ the list of row sums of $C'$. Taking into account (2) and considering that

$$\sum U_h = \sum U'_j = \pm (0, 2, 0) \text{ has order } 2^k n \text{ for } h = 1, 2, 3 \text{ and } j = 1, 3, \text{ and } \sum U'_2 = (2, 0, 0) \text{ has order } m,$$

we have $\omega(L_{C'}) = [^1m, 2^{k+2mn-1}2^k n]$. The result follows by applying Lemma 2.8 to $C'$. 

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Case 2: \( \beta = 1, 3 \). Let \( C = \begin{bmatrix} A \\ B \end{bmatrix} \) where \( A \) is the matrix defined in Theorem 3.2 with \( a = 3 \cdot 2^k m n \) and \( b = 0 \), and \( B \) is the matrix defined in Theorem 3.3 with \( a = 2^k m n - 2 \) and \( b = (\beta + 1)/2 \). It follows that for the list \( L_C \) of row sums of \( C \) we have

\[
\omega(L_C) = [2^{k+2} m n - \beta - 1 m, \beta + 1 2^k n].
\]

(3)

By Remark 2.1, we can assume that the permutations \( \varphi_1 \) and \( \varphi_2 \) (used in Theorem 3.2 to define \( A \)) satisfy the condition \( \varphi_h(z') = z' + (1, 0) = (1, 2^{k-1} n) \), with \( z' = (0, 2^{k-1} n) \), hence

\[
a_h(z') = \varphi_h(z') + (0, 1) = (1, 2^{k-1} n + 1),
\]

for \( h = 1 \) or 2.

Again, by Remark 2.1, we can assume that the permutation \( \psi_1 \) (used in Theorem 3.3 to define \( B \)) fixes \( \overline{z} = (1, 0) \).

Consider the following matrices:

\[
U = \begin{bmatrix} A(\varphi, z')_1 \\ A(\varphi, z')_2 \\ B_{\overline{z}} \end{bmatrix} = \begin{bmatrix} (a_1(z'), 0) & (b(z'), 1) & (c(z'), 1) \\ (c(z'), 1) & (a_2(z'), 0) & (b(z'), 1) \\ (1, 0, 0) & -(1, 0, 0) & (0, 2^k n - 2, 0) \end{bmatrix}
\]

\[
U' = \begin{bmatrix} (1, 0, 0) & (b(z'), 1) & (b(z'), 1) \\ (c(z'), 1) & -(1, 0, 0) & (c(z), 1) \\ (a_1(z'), 0) & (a_2(z'), 0) & (0, 2^k n - 2, 0) \end{bmatrix}.
\]

Note that \( U \) is a submatrix of \( C \), while each column of \( U' \) is a permutation of the corresponding column of \( U \). Therefore, by replacing the block \( U \) of \( C \) with \( U' \), we obtain a new \( 2^{k+2} m n \times 3 \) matrix \( C' \) whose columns are still permutations of \( \Gamma \). Denote by \( L' \) the list of row sums of \( C' \). Taking into account (3) and considering that

\[
\sum U_h = \sum U_j = \pm (1, 0, 0) \text{ has order } m, \text{ for } h, j = 1, 2,
\]

\[
\sum U_3 = (0, 2^k n - 2, 0) \text{ has order } 2^k n, \text{ and}
\]

\[
\sum U'_3 = (2, 0, 0) \text{ has order } m,
\]

we have \( \omega(L_{C'}) = [2^{k+2} m n - \beta m, \beta 2^k n] \). The result follows by applying Lemma 2.8 to \( C' \).
3.2 The proof of Theorem 3.1 when \( k = 1 \)

Let \( \alpha \) and \( \beta \) be non-negative integers such that \( \alpha + \beta = 2k^2 + \alpha \beta = 8mn \).

We first deal with the case where \( \alpha, \beta \not\in \{0, 1, 2, 4\} \), and let \( \beta_1 = 2mn - \alpha_1 \beta_2 = 6mn - \alpha_2 \), where \( \alpha_1 \) and \( \alpha_2 \) are defined as follows:

\[
(\alpha_1, \alpha_2) = \begin{cases} 
(3, \alpha - 3) & \text{if } 3 \leq \alpha \leq 6mn + 3, \\
(2mn - 3, \alpha - 2mn + 3) & \text{if } 6mn + 4 \leq \alpha \leq 8mn - 3.
\end{cases}
\]

Clearly, \( \alpha_1 \geq 3 \) and \( \beta_1 \geq 3 \) are odd, hence Theorem 3.3 guarantees the existence of an \( RSM_{\Gamma}(2\Gamma, g, [\alpha_1m, \beta_12k]) \), say \( B \). Furthermore, \( \alpha_2 \neq 1 \) and \( \beta_2 \neq 1 \), hence Theorem 3.2 provides an \( RSM_{\Gamma}((\Gamma \setminus 2\Gamma), g, [\alpha_2m, \beta_22k]) \), say \( A \). Therefore, \( C = [A \ B] \) and Lemma 2.8 give the desired \( RSM_{\Gamma} \).

The case \( \alpha = 1 \) is dealt with in Theorems 3.4 and 3.5, while the case \( \beta = 1 \) is proven in Theorems 3.6 and 3.7.

**Theorem 3.4.** Let \( m \geq 1 \) and \( n \geq 3 \) be odd integers and let \( k = 1 \). Then a \( RSM_{\Gamma}(\Gamma, g; [8mn - 1, m, 12n]) \) exists.

**Proof.** Recall that \( 2\Gamma = 2G \times \{0\} \) where \( 2G = \mathbb{Z}_m \times 2\mathbb{Z}_{4n} \).

By Theorem 2.2 there exists a \( \Delta_h \)-permutation \( \varphi_h \) of \( 2G \), where \( \Delta_h = \{2mn(1, 0)\} \), for every \( h \in \{1, 2, 3\} \). Let \( A = [A(\varphi, z)] \) be the row block-matrix whose blocks are the \( 3 \times 3 \) matrices \( A(\varphi, z) \) for \( z \in 2G \). Note that \( A \) is a \( 6mn \times 3 \) matrix whose columns are permutations of \( \Gamma \setminus 2\Gamma \). Also, letting \( L_A \) be the list of row-sums of \( A \), by Lemma 2.10 we have that

\[ \omega(L_A) = \omega((\Delta_1 \cup \Delta_2 \cup \Delta_3) = \omega([6mn(1, 0)]) = [6mnm]. \]

By Theorems 2.3 and 2.4 there are \( \Lambda_i \)-permutation \( \psi_i \) of \( 2G \), with \( i \in \{1, 2\} \), such that

1. \( \Lambda_1 = [1(0, 0)] \cup [1\gamma | \gamma \in 2G, \gamma \neq (0, 2n)], \)
2. \( \Lambda_2 = [2mn-6(1, 0), 3(2, 0), 1(0, 4), 1(0, 2n - 4), 1(0, 2n)], \)
3. \( \psi_1 \) fixes \( (0, 0) \) and \( \gamma = (-m-1 \over 2, mn + 1), \)
4. \( \psi_2(0, 0) = (0, 2n), \psi_2(0, 2n) = (0, 2n + 4). \)
Let $B$ denote the $2^kmn \times 3$ matrix whose rows $B_\gamma$, indexed over $2\Gamma$, are defined as follows: $B_\gamma = [(\gamma, 0) \ (\psi_1(\gamma), 0) \ (\psi_2(\delta), 0)]$ where

$$\delta = \begin{cases} 
(0, 2n) & \text{if } \gamma = (0, 0), \\
\psi_1(\gamma) - \gamma & \text{otherwise}.
\end{cases}$$

Note that the columns of $B$ are permutations of $2\Gamma$. Also, one can check that for the list $L_B$ of row sums of $B$ we have

$$L_B = \begin{bmatrix} 2mn - 6(1, 0, 0), 3(2, 0, 0), 1(0, 2n + 4, 0), 1(0, 2n - 4, 0), 1(0, 2n, 0) \end{bmatrix}$$

hence, $\omega(L_B) = [2mn - 3, 2n, 12]$. It follows that each column of $C = \begin{bmatrix} A \\ B \end{bmatrix}$ is a permutation of $\Gamma$. Clearly, the list of row sums of $C$ is $L_C = L_A \cup L_B$ and $\omega(L_C) = [8mn - 3, 2n, 12]$.

Now, let $\mu \in \{\pm 1\}$ such that $\mu \equiv m \pmod{4}$, and take the following four elements $\gamma_i$ of $2\Gamma$, whose second entry depends on $\mu$:

$$\gamma_1 = \begin{cases} 
\left(\frac{m-1}{2}, 3n - 1\right), & \text{if } m = 1 \text{ and } n = 3, \\
\left(\frac{m-1}{2}, n + \mu - 2\right), & \text{otherwise},
\end{cases}$$

$$\gamma_2 = \begin{cases} 
\left(\frac{-m-1}{2}, 3n - 1\right), & \text{if } m = 1 \text{ and } n = 3, \\
\left(\frac{-m-1}{2}, n + \mu - 2\right), & \text{otherwise},
\end{cases}$$

$$\gamma_3 = \left(\frac{-m-1}{4}, (3 + \mu)n - \mu - 1\right),$$

$$\gamma_4 = \left(\frac{m-1}{4}, (3 - \mu)n - \mu - 3\right).$$

Consider the submatrix $U = \begin{bmatrix} S \\ T \end{bmatrix}$ of $C = \begin{bmatrix} A \\ B \end{bmatrix}$, where $S$ and $T$ are submatrices of $A$ and $B$, respectively:

$$S = \begin{bmatrix} 
(a_1(\gamma_1), 0) & (b(\gamma_1), 1) & (c(\gamma_1), 1) \\
(c(\gamma_2), 1) & (a_2(\gamma_2), 0) & (b(\gamma_2), 1) \\
(a_1(\gamma_3), 0) & (b(\gamma_3), 1) & (c(\gamma_3), 1) \\
(c(\gamma_4), 1) & (a_2(\gamma_4), 0) & (b(\gamma_4), 1)
\end{bmatrix},$$

$$T = \begin{bmatrix} 
(0, 0, 0) & (0, 0, 0) & (0, 2n + 4, 0) \\
(\bar{\gamma}, 0) & -(\bar{\gamma}, 0) & (0, 2n, 0)
\end{bmatrix}. $$
Considering that $\gamma_1 \neq \gamma_3$ and $\gamma_2 \neq \gamma_4$, then $S$ is well-defined, that is, $S$ contains four distinct rows of $A$. We denote by $C'$ the matrix obtained from $C$ by replacing $U$ with the matrix $U'$ defined below

$$U' = \begin{bmatrix}
(0, 0, 0) & (b(\gamma_1), 1) & (b(\gamma_2), 1) \\
(c(\gamma_2), 1) & (0, 0, 0) & (c(\gamma_1), 1) \\
(\bar{\gamma}, 0) & (b(\gamma_3), 1) & (b(\gamma_4), 1) \\
(c(\gamma_4), 1) & -(\bar{\gamma}, 0) & (c(\gamma_3), 1) \\
(a_1(\gamma_1), 0) & (a_2(\gamma_2), 0) & (0, 2n - 2\mu + 2, 0) \\
(a_1(\gamma_3), 0) & (a_2(\gamma_4), 0) & (0, 2n + 2\mu + 2, 0)
\end{bmatrix}.$$ 

Note that each column of $U'$ is a permutation of the corresponding column of $U$. Therefore, each column of $C'$ is a permutation of $\Gamma$.

Considering that

$$\sum S_h \text{ has order } m, \text{ for every } h = 1, \ldots, 4,$$
$$\sum T_1 = (0, 2n + 4, 0) \text{ has order } 2n,$$
$$\sum T_2 = (0, 2n, 0) \text{ has order } 2, \text{ and}$$
$$\sum U'_h \text{ has order } m, \text{ for every } h = 1, \ldots, 6$$

and denoting by $L_{C'}$ the list of row sums of $C'$, we have that $\omega(L_{C'}) = [8mn^{-1}m, 12n]$. The result follows by applying Lemma 2.8 to $C'$. \hfill \Box

**Theorem 3.5.** Let $m \geq 1$ be an odd integer and let $k = n = 1$. Then a $RSM_{\Gamma}(\Gamma, g; [8mn^{-1}m, 12])$ exists.

**Proof.** Recall that $2\Gamma = 2G \times \{0\}$ where $2G = \mathbb{Z}_m \times 2\mathbb{Z}_4$. Also, $\Gamma = (\Gamma \setminus 2\Gamma) \cup 2\Gamma$. By Theorem 3.2 there exists a $RSM_{\Gamma}(\Gamma \setminus 2\Gamma, g; [6mn])$. Therefore, it is left to show that a $RSM_{\Gamma}(2\Gamma, g; [2m^{-1}m, 12])$ exists.

By Theorem 2.2 and Remark 2.1 there are $\Lambda_i$-permutations $\psi_i$ of $2\Gamma$, with $i \in \{1, 2\}$, such that

1. $\Lambda_1 = [1(0, 0)] \cup [1z | z \in 2G, z \neq (0, 2)]$, with $\psi_1(0, 0) = (0, 0)$, and
2. $\Lambda_2 = [2m^{-6}(1, 0), 3(2, 0), 2(0, 2), 1(0, 0)]$. 

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It is easy to see that $\psi_2$ can be chosen so that it fixes $(2, 0)$ and swaps $(0, 0)$ and $(0, 2)$.

Denote by $z_0, z_1$ the elements of $2G$, with $z_0 \neq (0, 0)$, such that

$$\psi_1(z_0) = z_0, \quad \text{and} \quad \psi_1(z_1) - z_1 = (2, 0),$$

and let $B$ denote the $2^k m n \times 3$ matrix (with entries from $2\Gamma$) whose rows $B_z$, indexed over $2G$, are defined as follows:

$$B_z = [(z, 0) \ (-\psi_1(z), 0) \ (\psi_2(w), 0)]$$

where

$$w = \begin{cases} (2, 0) & \text{if } z = z_0, \\ (0, 2) & \text{if } z = z_1, \\ \psi_1(z) - z & \text{otherwise.} \end{cases}$$

One can check that the columns of $B$ are permutations of $2\Gamma$, and for the list $L_B$ of row sums of $B$ we have

$$\omega(L_B) = [2^{m-1} m, 12].$$

The result then follows by applying Lemma 2.8 to $C$.

**Theorem 3.6.** Let $m \geq 1$ and $n \geq 3$ be odd integers, and let $k = 1$. Then a RSM$_F(\Gamma, g; [1 m, 8 m-1 2])$ exists.

**Proof.** Recall that $2\Gamma = 2G \times \{0\}$ where $2G = \mathbb{Z}_m \times 2\mathbb{Z}_{4n}$.

By Theorem 2.2 there exists a $\Delta_h$-permutation $\varphi_h$ of $2G$, where $\Delta_h = [2mn(0, 2)]$, for every $h \in \{1, 2, 3\}$. Let $A = [A(\varphi, z)]$ be the row block-matrix whose blocks are the $3 \times 3$ matrices $A(\varphi, z)$ for $z \in 2G$. Note that $A$ is a $6mn \times 3$ matrix whose columns are permutations of $\Gamma \setminus 2\Gamma$. Also, letting $L_A$ be the list of row-sums of $A$, by Lemma 2.10 we have that

$$\omega(L_A) = \omega(\Delta_1 \cup \Delta_2 \cup \Delta_3) = \omega([6mn(0, 2)]) = [6mn-2n].$$

By Theorems 2.3 and 2.2 there are $\Lambda_i$-permutations $\psi_i$ of $2G$, with $i \in \{1, 2\}$, such that

1. $\Lambda_1 = [1(0, 0)] \cup [1\gamma \mid \gamma \in 2G, \gamma \neq (0, 2n)]$,
2. $\Lambda_2 = [2mn(0, 2)]$,
3. $\psi_1$ fixes $(0, 0)$ and $\varphi = (-\frac{m-1}{2}, mn + 1)$. 

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Let \( B \) denote the \( 2^{kn} \times 3 \) matrix whose rows \( B_\gamma \), indexed over \( 2G \), are defined as follows:

\[
B_\gamma = \begin{bmatrix}
(\gamma, 0) & (-\psi_1(\gamma), 0) & (\psi_2(\delta), 0)
\end{bmatrix}
\]

where

\[
\delta = \begin{cases}
(0, 2n) & \text{if } \gamma = (0, 0), \\
\psi_1(\gamma) - \gamma & \text{otherwise}.
\end{cases}
\]

Note that the columns of \( B \) are permutations of \( 2\Gamma \). Also, one can check that for the list \( L_B \) of row sums of \( B \) we have

\[
L_B = \begin{bmatrix}
2^{mn} - 1 & 1 & 1
\end{bmatrix},
\]

hence, \( \omega(L_B) = [2^{mn-1}2n, 1u] \), where \( u \) is the order of \( 2n + 2 \) in \( 2\mathbb{Z}_{4n} \). It follows that each column of \( C = \begin{bmatrix} A \\ B \end{bmatrix} \) is a permutation of \( \Gamma \). Clearly, the list of row sums of \( C \) is \( L_C = L_A \cup L_B \) and \( \omega(L_C) = [8^{mn-1}2n, 1u] \).

Now, take the following four elements \( \gamma_i \) of \( 2G \):

\[
\gamma_1 = (1, 2n - 2),
\]

\[
\gamma_2 = (1, 2n - 6),
\]

\[
\gamma_3 = \begin{cases}
(-\frac{m-1}{2}, 3n - 3), & \text{if } mn \equiv 1 \pmod{4}, \\
(-\frac{m-1}{2}, 3n - 5), & \text{if } mn \equiv 3 \pmod{4},
\end{cases}
\]

\[
\gamma_4 = \begin{cases}
\left(\frac{m-1}{2}, 3n - 3\right), & \text{if } mn \equiv 1 \pmod{4}, \\
\left(\frac{m-1}{2}, 3n - 1\right), & \text{if } mn \equiv 3 \pmod{4}.
\end{cases}
\]

Consider the submatrix \( U = \begin{bmatrix} S \\ T \end{bmatrix} \) of \( C = \begin{bmatrix} A \\ B \end{bmatrix} \), where \( S \) and \( T \) are submatrices of \( A \) and \( B \), respectively:

\[
S = \begin{bmatrix}
(a_1(\gamma_1), 0) & (b(\gamma_1), 1) & (c(\gamma_1), 1) \\
(c(\gamma_2), 1) & (a_2(\gamma_2), 0) & (b(\gamma_2), 1) \\
(a_1(\gamma_3), 0) & (b(\gamma_3), 1) & (c(\gamma_3), 1) \\
(c(\gamma_4), 1) & (a_2(\gamma_4), 0) & (b(\gamma_4), 1)
\end{bmatrix},
\]

\[
T = \begin{bmatrix}
(0, 0, 0) & (0, 0, 0) & (0, 2n + 2, 0) \\
(\bar{\gamma}, 0) & -(\bar{\gamma}, 0) & (0, 2, 0)
\end{bmatrix}.
\]
Considering that $\gamma_1 \neq \gamma_3$ and $\gamma_2 \neq \gamma_4$, then $S$ contains four distinct rows of $A$. We denote by $C'$ the matrix obtained from $C$ by replacing $U$ with the matrix $U'$ defined below

\[
U' = \begin{pmatrix}
(0, 0, 0) & (b(\gamma_1), 1) & (b(\gamma_2), 1) \\
(c(\gamma_2), 1) & (0, 0, 0) & (c(\gamma_1), 1) \\
(\bar{\gamma}, 0) & (b(\gamma_3), 1) & (b(\gamma_4), 1) \\
(c(\gamma_4), 1) & -(\bar{\gamma}, 0) & (c(\gamma_3), 1) \\
(a_1(\gamma_3), 0) & (a_2(\gamma_4), 0) & (0, 2n + 2, 0) \\
(a_1(\gamma_1), 0) & (a_2(\gamma_2), 0) & (0, 2, 0)
\end{pmatrix}.
\]

Note that each column of $U'$ is a permutation of the corresponding column of $U$. Therefore, each column of $C'$ is a permutation of $\Gamma$.

Considering that
\[
\sum S_h \text{ has order } 2n, \text{ for every } h = 1, \ldots, 4, \\
\sum T_1 = (0, 2n + 2, 0) \text{ has order } u, \\
\sum T_2 = (0, 2, 0) \text{ has order } 2n, \text{ and} \\
\sum U'_h \text{ has order } 2n, \text{ for every } h = 1, \ldots, 5 \\
\sum U'_6 \text{ has order } m,
\]

and denoting by $L_{C'}$ the list of row sums of $C'$, we have that $\omega(L_{C'}) = [8mn - 12n, 1m]$. The result follows by applying Lemma 2.8 to $C'$.

**Theorem 3.7.** Let $m \geq 1$ be an odd integer and let $k = n = 1$. Then a $RSM_{\Gamma}(\Gamma, g; [1m, 8m-12])$ exists.

**Proof.** Recall that $2\Gamma = 2G \times \{0\}$ where $2G = \mathbb{Z}_m \times 2\mathbb{Z}_4$. Also, $C_g[\Gamma \setminus 2\Gamma]$ and $C_g[2\Gamma]$ decompose $C_g[8m]$.

By Theorem 2.2 there exists a $\Delta_h$-permutation $\varphi_h$ of $2G$, where $\Delta_h = [2mn(0, 2)]$, for every $h \in \{1, 2, 3\}$. Let $A = [A(\varphi, z)]$ be the row block-matrix whose blocks are the $3 \times 3$ matrices $A(\varphi, z)$ for $z \in 2G$. Note that $A$ is a $6mn \times 3$ matrix whose columns are permutations of $\Gamma \setminus 2\Gamma$. Also, letting $L_A$ be the list of row-sums of $A$, by Lemma 2.10 we have that
\[
\omega(L_A) = \omega(\Delta_1 \cup \Delta_2 \cup \Delta_3) = \omega([6mn(0, 2, 0)]) = [6m, 2].
\]
By Theorem 2.2 and Remark 2.1, there are Λ_i-permutations ψ_i of 2G, with i ∈ {1, 2}, such that

1. \( \Lambda_1 = [1(0, 0)] \cup [z | z \in 2G, z \neq (0, 2)] \), with \( \psi(0, 0) = (0, 0) \), and
2. \( \Lambda_2 = [2m(0, 2)] \).

Let \( B \) denote the \( 2^km \times 3 \) matrix whose rows \( B_γ \), indexed over \( 2G \), are defined as follows:

\[
B_γ = \begin{bmatrix}
(γ, 0) & (−ψ_1(γ), 0) & (ψ_2(δ), 0)
\end{bmatrix}
\]

where

\[
δ = \begin{cases}
(0, 2) & \text{if } γ = (0, 0), \\
ψ_1(γ) - γ & \text{otherwise}.
\end{cases}
\]

Note that the columns of \( B \) are permutations of \( 2Γ \). Also, one can check that for the list \( L_B \) of row sums of \( B \) we have

\[
L_B = [2^{m-1}(0, 2, 0), 1(0, 0, 0)],
\]

hence, \( \omega(L_B) = [2^{m-1}, 11] \). It follows that each column of \( C = \begin{bmatrix} A & B \end{bmatrix} \) is a permutation of \( Γ \). Clearly, the list of row sums of \( C \) is \( L_C = L_A \cup L_B \) and \( \omega(L_C) = [8^{m-1}, 11] \).

Set \( γ_1 = (1, 0) \) and \( γ_2 = (1, 2) \), and note that each \( φ_i \) swaps \( γ_1 \) and \( γ_2 \). Consider the submatrix \( U = \begin{bmatrix} S & T \end{bmatrix} \) of \( C = \begin{bmatrix} A & B \end{bmatrix} \), where \( S \) and \( T \) are submatrices of \( A \) and \( B \), respectively defined as follows:

\[
S = \begin{bmatrix}
(a_1(γ_1), 0) & (b(γ_1), 1) & (c(γ_1), 1) \\
(c(γ_2), 1) & (a_2(γ_2), 0) & (b(γ_2), 1)
\end{bmatrix},
\]

\[
T = \begin{bmatrix}
(0, 0, 0) & (0, 0, 0) & (0, 0, 0)
\end{bmatrix}.
\]

We denote by \( C' \) the matrix obtained from \( C \) by replacing \( U \) with the matrix \( U' \) defined below

\[
U' = \begin{bmatrix}
(a_1(γ_1), 0) & (a_2(γ_2), 0) & (0, 0, 0) \\
(c(γ_2), 1) & (b(γ_1), 1) & (b(γ_2), 1)
\end{bmatrix}.
\]

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Note that each column of $U'$ is a permutation of the corresponding column of $U$. Therefore, each column of $C'$ is a permutation of $\Gamma$.

Considering that 

$$\sum S_h \text{ has order } 2, \text{ for } h = 1, 2,$$

$$\sum T = (0, 0, 0) \text{ has order } 1,$$

$$\sum U'_1 = (2, 0, 0) \text{ has order } m,$$

$$\sum U'_h \text{ has order } 2, \text{ for } h = 2, 3$$

and denoting by $L_{C'}$ the list of row sums of $C'$, we have that $\omega(L_{C'}) = [8mn-2, 1m]$. The result follows by applying Lemma 2.8 to $C'$. \(\square\)

### 3.3 The proof of Theorem 3.1 when $k = 0$

Let $\alpha$ and $\beta$ be non-negative integers such that $\alpha + \beta = 4mn$. First, we assume that both $\alpha \neq 1$ and $\beta \neq 1$ and let $\beta_1 = mn - \alpha_1, \beta_2 = 3mn - \alpha_2$ where $\alpha_1$ and $\alpha_2$ are defined as follows:

$$\alpha_1 = \begin{cases} 
 mn & \text{if } mn + 2 \leq \alpha \leq 4mn - 2 \text{ or } \alpha = 4mn, \\
 \alpha - 4 & \text{if } \alpha = mn + 1, \\
 \alpha - 2 & \text{if } \alpha = mn - 1, \\
 \alpha & \text{if } 2 \leq \alpha \leq mn - 2 \text{ or } \alpha \in \{0, mn\}, 
\end{cases}$$

$$\alpha_2 = \alpha - \alpha_1.$$ 

Since $\alpha_1 \neq 1$ and $\beta_1 \neq 1$, Theorem 3.3 guarantees the existence of an $RSM_{\Gamma}(2\Gamma, g, [\alpha_1m, \beta_1n])$. Furthermore, one can check that $\alpha_2 \neq 1$ and $\beta_2 \neq 1$. Hence, by Theorem 3.2 there is an $RSM_{\Gamma}(\Gamma \setminus 2\Gamma, g, [\alpha_2m, \beta_2n])$.

Therefore, $C = \begin{bmatrix} A \\ B \end{bmatrix}$ and Lemma 2.8 provide the desired RSM.

It is then left to deal with the case $\alpha = 1$; indeed, since both $m$ and $n$ are odd, the case $\beta = 1$ can be obtained by exchanging the roles of $m$ and $n$.

Let $\alpha = 1$. By Theorem 2.2 there exists a $\Delta_h$-permutation $\varphi_h$ of $\mathbb{Z}_m \times 2\mathbb{Z}_{2n}$, with

$$\Delta_h = \left[\frac{(mn+1)}{2}(0, 2), \frac{(mn-3)}{2}(0, -2), (0, -4)\right],$$

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where \( \omega(\Delta_h) = \binom{mn}{n} \), for every \( h \in \{1, 2, 3, 4\} \). Now let \( A = [A(\varphi, (x, y))] \) be the column block-matrix whose blocks are the matrices \( A(\varphi, (x, y)) \) for \((x, y) \in \mathbb{Z}_m \times 2\mathbb{Z}_{2n}\), where \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \). Note that \( A \) is a \( (3 \cdot 2^h mn) \times 3 \) matrix whose columns are permutations of \( \Gamma \setminus 2\Gamma \). Also, letting \( L_A \) be the list of row-sums of \( A \), by Lemma 2.10 we have that

\[
\omega(L_A) = \omega(\Delta_1) \cup \omega(\Delta_2) \cup \omega(\Delta_3) = \binom{3mn}{n}.
\]

Now let \( B \) be the matrix whose rows are indexed over \( 2G \) such that

\[
B_{(x,y)} = \begin{bmatrix} (x,y,0) & (-2x,-2y,0) & (\varphi_4(x,y),0) \end{bmatrix}.
\]

Notice that each column of \( B \) is a permutation of \( 2G \) and \( \sum B(x,y) = (\varphi_4(x,y) - (x,y),0) \). Hence, letting \( L_B \) be the list of row-sums of \( B \) we have that \( \omega(L_B) = \omega(\Delta_4) = \binom{mn}{n} \). Therefore, the matrix \( C = \begin{bmatrix} A & B \end{bmatrix} \) is an \( RSM_{\Gamma}(\Gamma, 3; \binom{4mn}{n}) \).

By Remark 2.1, we can assume that the permutations \( \varphi_2 \) and \( \varphi_3 \) used to define \( A \) satisfy the condition \( \varphi_h(1, 2) = (1, 0) \), hence

\[
a_h(1,2) = \varphi_h(1,2) + (0,1) = (1,1),
\]

for \( h = 2 \) or 3. Furthermore, we can assume that

\[
B_{(0,-2)} = \begin{bmatrix} (0,-2,0) & (0,4,0) & (0,-4,0) \end{bmatrix}.
\]

Now, consider the following matrices:

\[
U = \begin{bmatrix} B_{(0,-2)} & A(\varphi, (1, 2))_2 & A(\varphi, (1, 2))_3 \end{bmatrix} = \begin{bmatrix} (0,-2,0) & (0,4,0) & (0,-4,0) \\
(c(1,2), 1) & (a_2(1,2), 0) & (b(1,2), 1) \\
(b(1,2), 1) & (c(1,2), 1) & (a_3(1,2), 0) \end{bmatrix}
\]

\[
U' = \begin{bmatrix} (0,-2,0) & (a_2(1,2), 0) & (a_3(1,2), 0) \\
(b(1,2), 1) & (0,4,0) & (b(1,2), 1) \\
(c(1,2), 1) & (c(1,2), 1) & (0,-4,0) \end{bmatrix} = \begin{bmatrix} (0,-2,0) & (1,1,0) & (1,1,0) \\
(b(1,2), 1) & (0,4,0) & (b(1,2), 1) \\
(c(1,2), 1) & (c(1,2), 1) & (0,-4,0) \end{bmatrix}.
\]

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Notice that \( \sum U'_1 = (2, 0, 0) \) and \( \sum U'_h = (0, -4, 0) \) for \( h = 2, 3 \). Note that \( U \) is a submatrix of \( C \), while each column of \( U' \) is a permutation of the corresponding column of \( U \). Therefore, by replacing the block \( U \) of \( C \) with \( U' \), we obtain a new \( 4mn \times 3 \) matrix \( C' \) whose columns are still permutations of \( \Gamma \). Denoting by \( L' \) the list of row sums of \( C' \), and taking into account the values \( \sum U'_h \), we have that \( \omega(L'C') = [1m, 4mn−1n] \). Therefore, \( C' \) is an \( RSM_R(\Gamma, 3; [1m, 4mn−1n]) \), and the result follows by applying Lemma 2.8 to \( C' \).

4 The proof of Theorem 1.5

Theorem 4.1. Let \( g \geq 3 \), \( k \geq 0 \), and let \( m, n \geq 1 \) be odd integers. Then \( HWP(C_g[2k+2mn]; gm, 2^k gn; \alpha, \beta) \) has a solution if and only if \( \alpha + \beta = 2^k + 2^{k+1} mn \).

Proof. It is a straightforward consequence of Theorems 2.5, 2.9 and 3.1.

We are now ready to prove the main result of this paper.

Theorem 1.5. Let \( v \), \( M \) and \( N \) be integers greater than 3, and let \( \ell = \text{lcm}(M,N) \). then a solution to \( HWP(v; M, N; \alpha, \beta) \) exists if and only if \( \ell \mid v \), except possibly when

- \( \gcd(M, N) \in \{1, 2\} \);
- 4 does not divide \( v/\ell \);
- \( v = 4\ell, 8\ell \);
- \( v = 16\ell \) and \( \gcd(M, N) \) is odd;
- \( v = 24\ell \) and \( \gcd(M, N) = 3 \).

Proof. Let \( g = \gcd(M, N) > 2 \). We may assume that \( M = gm \) and \( N = 2^k gn \), where both \( m \) and \( n \) are odd positive integers and \( k \geq 0 \); hence, \( \ell = \text{lcm}(M,N) = 2^k gmn \). By assumption, we also have that \( v = 4\ell s \) with \( s \geq 3 \), \( g \) is even when \( s = 4 \), and \( (g, s) \neq (3, 6) \).

Let \( (t, \epsilon) = (s/2, 2) \) if \( s \geq 6 \) is even, otherwise set \( (t, \epsilon) = (s, 1) \). We start by factorizing \( K_v \) into two graphs \( G_0 \) and \( G_1 \) where \( G_o \) is the vertex disjoint union of \( t \) copies of \( K_{4\ell \epsilon} \), while \( G_1 \simeq K_t[4\ell \epsilon] \). Also, set \( (\alpha_0, \beta_0) = \)
(2\ell\varepsilon - 1, 0) if \( \alpha \geq 2\ell\varepsilon - 1 \), otherwise, set \((\alpha_0, \beta_0) = (0, 2\ell\varepsilon - 1)\), and set \((\alpha_1, \beta_1) = (\alpha, \beta) - (\alpha_0, \beta_0)\).

By Theorem 1.2 there is a \( C_g \)-factorization of \( K_t[ge] \), hence there exists a \( C_g[4\ell/g] \)-factorization of \( K_t[ge][4\ell/g] \cong K_t[4\ell/e] \cong G_1 \). Note that \( 4\ell/g = 2^{k+2}mn \), therefore by applying Theorem 4.1 to each component of every \( C_g[4\ell/g] \)-factor, we obtain a solution to HWP\((G_1; gm, 2^k n; \alpha_1, \beta_1)\). By adding a solution to HWP\((G_0; gm, 2^k n; \alpha_0, \beta_0)\) we obtain the assertion.

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