Expander Graphs – A Study

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Abstract. Expander graphs are highly connected graphs that have numerous applications in statistical physics, pure mathematics and in computer science. The increased connectivity in expanders is useful to model connections between interconnecting systems which can be considered as a graph composed of particles as vertices and edges represent interactions. This paper focuses on the fascinating and highly active area of research on expander graphs. In this article, different classes of expander graphs such as Schreier graphs, Ramanujan graphs and Lp-expanders are categorized and various constructions of an explicit family of expanders are explored. Based on their construction the chromatic number of these graphs is obtained.

1. Introduction

Graph theoretical concepts are omnipresent in statistical physics. They are the straight-forward and flexible considerations of discrete, local interactions in a global domain, and an effective way to model real interacting systems. The vibrational physics of real-world systems can be comprehended through such interactions. Statistical physics especially uses graphs that can be modelled as interconnecting parts of a system and is also used in social networks such as Facebook and Twitter and communication network topology for VLSI design etc. Graph theory is greatly associated with other subjects, such as group theory and topology and additionally, as mathematical objects, they are paramount ideas in combinatorics. The last several decades have taught us that, graphs with which we are more and more often faced in applications can look very excitingly different from the small pictures we can draw and comprehend. Amid the explicit asymptotic behaviour of graphs, expansions, the property of being sparse but highly connected is one of the most ubiquitous.

Expander graphs were first defined by Bassalygo and Pinsker, and their existence was first proved by Pinsker [14]. The property of being an expander seems significant in many of these mathematical, computational and physical contexts. Expander graphs have found multiple applications in physics and they are used today as a standard null model in simulating many physical processes on graphs and networks. Interacting systems are represented by a graph $G$, where each particle being represented by a single isolated vertex and the interactions are considered as edges. If no interaction is present, the edge is omitted. A graph with vertex set $V$ and edge set $E$ is denoted by $G(V, E)$. The cardinality of the
vertex set is denoted by $|V| = n$. For other definitions concerning graph theory not mentioned here, one can refer [7, 8, 9].

If $A$ is the adjacency matrix of the graph $G$ and $D$ is the diagonal matrix of $G$ then, the Laplacian matrix of order $n$ is denoted by $L = D - A$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$ be the eigen values of $L$. Normalized Laplacian matrix is defined as $L_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } \deg(v_i) \neq 0 \\ \frac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & \text{if } i \neq j \text{ and } (v_i, v_j) \in E. \\ 0 & \text{otherwise} \end{cases}$

This article is divided into three sections namely, section 2 deals with the definitions of different classes of expander graphs such as Schreier graphs, Ramanujan graphs and $L_p$ –expanders. In section 3, these different classes of expanders are categorized and various constructions of an explicit family of expanders are explored. Section 4 concerns with the chromatic number of these expander graphs.

2. Preliminaries

2.1 Expander graphs:
There are multiple ways of defining expander graphs. Expanders are graphs which are deeply connected and for this reason one has to detach a lot of edges to disconnect a huge segment of the graph. Algebraically, expander graphs may be defined as the $r$ – regular graphs for which all non-zero Laplacian eigen values are bounded away from zero [4]. These graphs possess a lot of wonderful properties. Since the least possible set of vertices contain remarkably large set of neighbours, that is, since the neighbourhoods expand, these graphs are called expanders. Expander graphs are families of graphs, becoming larger and larger, possessing the following emulating characteristics: (i) they have moderately less of number of edges, compared to the number of vertices (ii) but are highly connected and are indeed extraordinarily tough systems [6].

2.2 Ramanujan graphs:
Ramanujan graphs are examples of very nice expanders. The Ramanujan graphs were introduced by Lubotzk et al. [10]. These graphs possess number theory properties and so named Ramanujan graph. If $r > 2$ is an integer then a $r$ – regular connected finite graph $G$ is called a Ramanujan graph if all the eigen values $\lambda$ of $G$ satisfy either $\lambda \in \{-r, r\}$ or $|\lambda| \leq \sqrt{r - 1}$ [10]. The complete graph $K_n$, the complete bipartite graph $K_{n,n}$ and the Petersen graph are Ramanujan. If $r > 3$ then there exists a family of bipartite $r$ – regular Ramanujan graphs [11].

2.3 Schreier graphs:
Algebraically Schreier graphs are defined as follows: If $G$ is a group and $S \subseteq G$ a symmetric subset and $H$ be a subgroup of the group $G$ then, the Schreier graph of $H$ acting on a set $S$ is a graph with the vertex set $S$ where $s \in S$ is connected [12].

2.4 $L_p$ –expanders:
Graph expansion related to the behavior of $L_p$ –functions was recently introduced in [2]. Here $p = n$, since in this paper we denote $|V| = n$. It was proved that the eigen values of the adjacency matrix of a $(|E| + 1) –$ regular graph is bounded by $L_p$ –norm of the matrix. The result was generalized to operators on edges and to bipartite graphs [2]. For $n \geq 2$, a graph is an $L_p$ – expander if it satisfies one of the equivalent conditions given in [2]. In particular, $L_2$ – expander graph is isomorphic to the Ramanujan graph.
3. Construction of Expanders

A family of expander graphs were first explicitly constructed by Margulis [4]. This method of construction is still among the ultimate and ingenious method of generating expander graphs, of all known constructions. Initially explicit constructions of expanders were mostly performed by algebraic techniques and pure mathematics. Later on, the zig-zag product of graph was introduced that allowed to give an elementary explicit construction of expander families and has been applied in various contexts to give interesting families of expanders in which the construction is based on lifts. Most of the known explicit constructions of expander graphs depend on deep ideas from number theory [14]. Recently, degree 3 Ramanujan graphs which are biparite, are constructed in [13].

4. Chromatic number of Expanders

In the graph colouring problem [3], to each vertex of a graph we attribute a label called colour, so that with a minimum number of colours, every edge is connected with vertices of different colours. Replacing colours with numbers, \( k \) — colouring of a graph \( G = (V, E) \) is defined to be a function \( c(G): V(G) \to \{1, \ldots, k\} \) such that no two adjacent vertices have the same colour. That is, \( c(i) = c(j) \) for all \((i, j) \in E\). The least value of \( k \) for which \( G \) has a \( k \)—colouring is called the chromatic number of a graph \( G \) denoted by \( \chi(G) \).

It was proved that the chromatic number of a graph may possibly be bounded above, by its largest adjacency eigen value [4]. In the following theorem we calculate the exact value of the chromatic number of a graph in terms of the eigen value of the normalised Laplacian matrix. These results are pertinent for all expanders (Ramanujan graphs, Schreier graphs, Lp-expanders) satisfying the bipartiteness property.

4.1 Theorem: If \( G \) is a \( r \)—regular bipartite expander graph with \( r \geq 3 \) then the second largest eigen value of \( G \) is equal to the chromatic number of \( G \).

Proof. Let \( G \) be a \( r \)—bipartite expander graph. Such graphs exist for \( r \geq 3 \) [1]. Let \( L \) be the normalised Laplacian matrix of \( G \). Since \( G \) is bipartite, the vertex set \( V \) can be partitioned as two disjoint subsets of \( V \) (say \( A \) and \( B \) ) such that every edge starts in \( A \) and ends in \( B \). For proving the result, we use the construction of expanders based on combinatorial algebraic techniques. Let the second largest eigen value of the normalized Laplacian matrix be denoted by \( \lambda_{n-1} \). Since the graph \( G \) is bipartite there will be no odd cycles in \( G \). Therefore, \( \chi(G) = 2 \). It was proved that all the eigenvalues of the normalized Laplacian matrix are less than or equal to two and if \( G \) is bipartite then the second largest eigen value of \( G \) is exactly equal to two \([11]\). Therefore, we conclude that \( \lambda_{n-1} = \chi(G) \).

The next theorem yields the relation between coverage of vertices by edges, and the chromatic number of the complement of a graph \( G \).

4.2 Theorem: If \( G \) is a \( r \)—regular bipartite expander graph, then the minimum number of edges for covering the vertices of \( G \) is equal to the chromatic number of the complement graph \( G \).

Proof. Given that \( G \) is a \( r \)—regular bipartite expander graph. Accordingly, if we take the complement graph of any bipartite expander graph, we get a graph which is said to be perfect. Consequently, \( G \) is a perfect graph. For this reason, the clique number of the complement of \( G \) equals the chromatic number of the complement of \( G \)\([8]\). Since the clique number of the complement of \( G \) equals the maximum independent set, we have the result that, the minimum number of edges for covering the vertices of \( G \) is equal to the chromatic number of the complement graph \( G \).
5. Conclusion

In this paper, different classes of expander graphs such as Schreier graphs, Ramanujan graphs and Lp-expanders are discussed. Based on their construction, the chromatic number of these graphs are obtained. Because of their high connectivity and sparseness these graphs can be used to model many physical systems. We hope that this work motivates other approaches to utilize results from graph theory to develop efficient network architectures in statistical physics.

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