Article

Continuity and Analyticity for the Generalized Benjamin–Ono Equation

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Abstract: This work mainly focuses on the continuity and analyticity for the generalized Benjamin–Ono (g-BO) equation. From the local well-posedness results for g-BO equation, we know that its solutions depend continuously on their initial data. In the present paper, we further show that such dependence is not uniformly continuous in Sobolev spaces $H^s(\mathbb{R})$ with $s > 3/2$. We also provide more information about the stability of the data-solution map, i.e., the solution map for g-BO equation is Hölder continuous in $H^r$-topology for all $0 \leq r < s$ with exponent $\alpha$ depending on $s$ and $r$. Finally, applying the generalized Ovsyannikov type theorem and the basic properties of Sobolev–Gevrey spaces, we prove the Gevrey regularity and analyticity for the g-BO equation. In addition, by the symmetry of the spatial variable, we obtain a lower bound of the lifespan and the continuity of the data-to-solution map.

Keywords: generalized Benjamin–Ono equation; non-uniform dependence; Hölder continuous; symmetry; analyticity; Gevrey regularity

MSC: 35G2; 35L05; 35Q50

1. Introduction

In this paper, we study the Cauchy problem for the generalized Benjamin–Ono equation

$$\begin{align*}
\partial_t u + H\partial_x^2 u + u^k \partial_x u &= 0, \quad t > 0, x \in \mathbb{R}, \\
u(x, 0) &= u_0(x), \quad t = 0, x \in \mathbb{R},
\end{align*}$$

(1)

where $H$ is the spatial symmetrical Hilbert transform

$$H(f)(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$
spaces $H^s(\mathbb{R})$ were investigated, cf. [5–11]. More precisely, the local well-posedness for initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ was shown in [7], and the globally well-posed in $H^s(\mathbb{R})$ for $s \geq \frac{3}{4}$ was also obtained in [10]. By the half Strichartz estimates for linear problems with variable coefficients, Koch and Tzvetkov [9] obtained the local well-posedness when $s > \frac{5}{4}$.

Subsequently, Kenig and Koenig [8] extended this result to $s > \frac{9}{8}$: Tao [11] obtained global well-posedness in $H^s(\mathbb{R})$ for $s \geq 1$ by a gauge transformation as for the derivative Schrödinger equation. Recently, the Benjamin–Ono equation was proved to be locally well-posed in $H^s(\mathbb{R})$ with $s > \frac{1}{4}$ in [5] and global well-posedness in $H^s(\mathbb{R})$ with $s \geq 0$ in [6].

More interestingly, based on the well-posedness results, Koch and Tzvetkov [12] showed that the solution mapping was not even locally uniformly continuous in $H^s(\mathbb{R})$ for $s > 0$. Fonseca and Ponce [13] established persistence properties and proved some unique continuation properties of the solution flow in the weighted Sobolev spaces $Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^r dx)$. For $k \geq 2$, the g-BO equation presents the interesting fact that the dispersive effect is stronger than the linear Schrödinger equation. Recently, the Benjamin–Ono equation was proved to be local well-posed for the initial-value problem (1) satisfying

$$\lim_{t \to \infty} \|u(t)\|_{H^s} < \infty.$$
In [12], Koch and Tzvetkov prove that the flow map of the Benjamin–Ono equation cannot be uniformly continuous on bounded sets of $H^s(\mathbb{R})$ for $s > 0$. We compare with the Benjamin–Ono equation, and the g-BO equation has a higher order nonlinear term $u^k \partial_x u$. If taking the similarly approximate solutions as Koch and Tzvetkov [12], we cannot successively estimate the error in suitable Sobolev norm, instead we must select a more complicated form of the low and high frequency parts for the approximate solutions (see (13) and (15)).

Motivated by the results obtained in [18–21], we use the interpolation properties of the Sobolev spaces and commutator estimates to present that the data-to-solution map as continuous but not uniformly continuous in Sobolev spaces $H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Our results extend the work of Koch and Tzvetkov [12] to more general equations with higher-order nonlinearities. Our main result is stated as follows:

**Theorem 1.** If the initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, then the data-to-solution map $u_0 \rightarrow u(t)$ for the g-BO Equation (1) is not uniformly continuous from any bounded subset of $H^s(\mathbb{R})$ into $C([0,T]; H^s(\mathbb{R})) \times C([0,T]; H^{s-1}(\mathbb{R}))$.

Theorem 1 shows that the data-solution map depends on the initial data being continuous but not uniformly continuous. Our next result will provide information about the stability of the data-solution map. Our next result establishes the stability of the data-solution map, i.e., the solution map for g-BO equation is Hölder continuous in $H^s$-topology.

**Theorem 2.** Let $s > \frac{3}{2}$ and $0 \leq r < s$. Then, the data-to-solution map for the g-BO Equation (1) is Hölder continuous in $H^s(\mathbb{R})$ equipped with $H^r(\mathbb{R})$-norm. In particular, the solutions $u(t), v(t)$ to the g-BO Equation (1) corresponding to the initial data $u_0, v_0$ in the ball $B(0,\rho) = \{\psi \in H^s(\mathbb{R}) : \|\psi\|_{H^s(\mathbb{R})} \leq \rho\}$ of $H^s(\mathbb{R})$ satisfy the following inequality

$$\|u(t) - v(t)\|_{H^r(\mathbb{R})} \leq C \|u_0 - v_0\|^\alpha_{H^s(\mathbb{R})},$$

where the parameter $\alpha$ is given by

$$\alpha = \begin{cases} 1, & (s,r) \in A_1 = \{(s,r) : s > \frac{3}{2}, 0 \leq r \leq s - 1, r + s \geq 2\}; \\ \frac{2(s-1)}{s-r}, & (s,r) \in A_2 = \{(s,r) : 2 > s > \frac{3}{2}, 0 \leq r < 2 - s\}; \\ s - r, & (s,r) \in A_3 = \{(s,r) : s > \frac{3}{2}, s - 1 < r < s\}. \end{cases}$$

The lifespan $T$ and the constant $C$ only depend on $r, s$ and $\rho$ (see Figure 1).

![Figure 1](image-url)
Many researchers have studied the analyticity of solutions to g-BO, cf. [12]. However, to our best knowledge, the Gevrey regularity of solutions to the BO equation is still an open problem. The definition of Sobolev–Gevrey spaces is stated as follows.

**Definition 1.** Let $s$ be a real number and $\sigma, \delta > 0$. A function $f \in C^2_{\sigma, \delta}(\mathbb{R})$ if and only if $f \in C^\infty(\mathbb{R})$ and satisfies

$$
\|f\|_{C^2_{\sigma, \delta}(\mathbb{R})} = \left( \int_{\mathbb{R}} (1 + |\xi|^2)^{\sigma} e^{2\delta|\xi|^2} \left| \hat{f}(\xi) \right|^2 d\xi \right)^{1/2} < \infty.
$$

Denoting the Fourier multiplier $e^{\delta(-\Delta)^{\frac{1}{2}}} \hat{f}$ by $e^{\delta(-\Delta)^{\frac{1}{2}}} f = \mathcal{F}^{-1}(e^{\delta|\xi|^2} \hat{f})$, we deduce that

$$
\|f\|_{C^2_{\sigma, \delta}(\mathbb{R})} = \|e^{\delta(-\Delta)^{\frac{1}{2}} f}\|_{L^2(\mathbb{R})}.
$$

For $0 < r < 1$, it is called ultra-analytic function. If $r = 1$, it is a usual analytic (or holomorphic) function, and $\delta$ is called the radius of analyticity. If $r > 1$, it is the Gevrey class function.

By the generalized Ovsyannikov theorem [22] (see Theorem 6 in the Section 5), we can obtain the Gevrey regularity and analyticity of the g-BO equation.

**Theorem 3.** Let $\sigma \geq 1$ and $s > \frac{3}{2}$. Assume that $u_0 \in C^2_{\sigma, \delta}(\mathbb{R})$. Then, for every $0 < \delta < 1$, there exists a $T_0 > 0$ such that the g-BO equation has a unique solution $u$, which is holomorphic in $|t| < \frac{T_0 (1 - \delta)^\frac{3}{2}}{2 - \frac{3}{2} \sigma}$ with values in $C^\infty_{\sigma, \delta}(\mathbb{R})$. Moreover, there is a positive constant $C$ such that

$$
T_0 = \frac{C}{\|u_0\|_{C^2_{\sigma, \delta}(\mathbb{R})}}.
$$

Theorem 3 tells us that solutions of g-BO equation are analytic in both space and time variables. Moreover, we give a lower bound of the analytic lifespan. Then, we continue to study the continuity of the data-to-solution.

**Definition 2.** Let $\sigma \geq 1$ and $s > \frac{3}{2}$. We say that the data-to-solution map $u_0 \rightarrow u(t)$ of the g-BO is continuous, if for a given $u_0^0 \in C^2_{\sigma, \delta}(\mathbb{R})$ there exists a $T = T(\|u_0^0\|_{C^2_{\sigma, \delta}})$ such that, for any sequence $u_0^n \in C^2_{\sigma, \delta}$ and $\|u_0^n - u_0^0\|_{C^2_{\sigma, \delta}} \rightarrow 0$ for $n \rightarrow \infty$, the corresponding solutions $u^n$ of g-BO satisfy $\|u^n - u^0\|_{E_T} \rightarrow 0$ for $n \rightarrow \infty$, where

$$
\|f\|_{E_T} = \sup_{|t| < \frac{T_0 (1 - \delta)^\frac{3}{2}}{2 - \frac{3}{2} \sigma}} \left( \|f\|_{C^2_{\sigma, \delta}} (1 - \delta)^x \sqrt{1 - \frac{|t|}{T_0 (1 - \delta)^\frac{3}{2}}} \right).
$$

**Theorem 4.** Let $\sigma \geq 1$ and $s > \frac{3}{2}$. Assume that $u_0 \in C^2_{\sigma, \delta}(\mathbb{R})$. Then, the data-to-solution map $u_0 \rightarrow u$ of the g-BO equation is continuous from $C^2_{\sigma, \delta}(\mathbb{R})$ into the solutions space.

This paper is organized as follows. In Section 2, we recall some notation, give a prior well-posedness estimate for g-BO Equation (1), and determine a lower bound on the existence time of the solution in $H^s(\mathbb{R})$. In Section 3, adopting the method of approximate solutions and the well-posedness estimates, we show that the data-to-solution map fails to be locally uniformly continuous. In Section 4, we prove that the solution map for g-BO Equation (1) is Hölder continuous in $H^s$-topology for all $0 \leq r < s$. Finally, applying generalized Ovsyannikov type theorem and properties of Sobolev–Gevrey spaces, we establish the Gevrey regularity and analyticity of the g-BO equation and obtain the continuity of the data-to-solution map.

2. Priori Estimates and Lifespan of Solution

For any $s \in \mathbb{R}$, we take the operator $D^s = (1 - \partial_x^2)^{s/2}$ to be defined by

$$
\hat{D}^s f(\xi) = (1 + \xi^2)^{s/2} \hat{f}(\xi),
$$
where \( \hat{f}(\xi) \) is the Fourier transform,
\[
\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) \, dx, \quad \xi \in \mathbb{R}.
\]

Let \( H^s \) be the Sobolev space consisting of all tempered distributions \( f \) such that
\[
\|f\|_{H^s} \doteq \|f\|_{H^s(\mathbb{R})} = \left( \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} < \infty.
\]

**Theorem 5.** Assume \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{3}{2} \). Let \( T \) be the maximal existence time of the solution \( u \) to g-BO Equation (1) with the initial data \( u_0 \). Then, \( T \) satisfies
\[
T \geq T_0 := \frac{2^k - 1}{2^k C_s \|u_0\|_{H^s}^2},
\]
where \( C_s \) is a constant depending only on \( s \). We have
\[
\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad 0 \leq t \leq T_0.
\]

**Proof.** Applying the operator \( D^s \) to g-BO Equation (1), it can be rewritten as follows
\[
\partial_t D^s u + D^s \mathcal{H}(\partial_x^2 u) + \left( D^s (u^k \partial_x u) - u^k D^s \partial_x u \right) + u^k \partial_x D^s u = 0.
\]

Multiplying the g-BO Equation (5) by \( D^s u \) and then integrating it with respect to \( x \in \mathbb{R} \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^s}^2 = - \int_{\mathbb{R}} D^s \mathcal{H}(\partial_x^2 u) \cdot D^s u \, dx - \int_{\mathbb{R}} \left( |D^s u|^2 + u^k \partial_x \right) \cdot D^s u \, dx - \int_{\mathbb{R}} u^k \partial_x D^s u \cdot D^s u \, dx.
\]

Noting that \( \int_{\mathbb{R}} D^s \mathcal{H}(\partial_x^2 u) \cdot D^s u \, dx = 0 \). To estimate the second integral on the right-hand side of (6), we need the following lemma, which is derived from [23,24].

**Lemma 1.** If \( r > 0 \), then
\[
\|[D^r, f]g\|_{L^2} \leq C_r (\|f\|_{L^\infty} \|D^{r-1} g\|_{L^2} + \|D^r f\|_{L^2} \|g\|_{L^\infty}),
\]
where \( C_r \) is a positive constant depending only on \( r \).

Using the Cauchy–Schwarz inequality and Lemma 1, we can estimate the second integral of (6)
\[
\left| \int_{\mathbb{R}} \left( |D^s u|^2 + u^k \partial_x u \right) \cdot D^s u \, dx \right| \leq \left\| |D^s u|^2 + u^k \partial_x u \right\|_{L^2} \|D^s u\|_{L^2}
\leq C_s (\|\partial_x u\|_{L^\infty} \|D^{s-1} \partial_x u\|_{L^2} + \|D^s u\|_{L^2} \|\partial_x u\|_{L^\infty}) \|D^s u\|_{L^2}
\leq C \|u\|_{H^s}^{k+2},
\]
where we have used the equality \( \|D^s u\|_{L^2} = \|u\|_{H^s} \) and the Sobolev embedding theorem \( H^s \hookrightarrow L^\infty \) for \( s > \frac{3}{2} \).

Estimating the third integral of the right-hand side of (6), integrating by parts, we deduce
\[
\left| \int_{\mathbb{R}} u^k \partial_x D^s u \cdot D^s u \, dx \right| \leq \|\partial_x u\|_{L^\infty} \|D^s u\|_{L^2}^2 \leq C \|u\|_{H^s}^{k+2}.
\]
Combining (6)–(8) we can obtain the following inequality
\[ \frac{1}{2} \frac{d}{dt} \| u(t) \|_{H^s}^2 \leq C \| u \|_{H^s}^{k+2}. \]  
(9)

Solving differential inequality (9) yields
\[ \| u \|_{H^s} \leq \frac{\| u_0 \|_{H^s}}{(1 - Ckt \| u_0 \|_{H^s})^{1/k}}. \]  
(10)

Letting \( T_0 := \min \left\{ \frac{1}{Ct \| u_0 \|_{H^s}} \cdot \frac{2^k - 1}{2^Ct \| u_0 \|_{H^s}} \right\} \), then \((1 - CkT_0 \| u_0 \|_{H^s})^{1/k} \geq \frac{1}{2}\). From (10), the solution \( u \) exists for \( 0 \leq t \leq T_0 \) with the following bound
\[ \| u \|_{H^s} \leq \frac{\| u_0 \|_{H^s}}{(1 - CkT_0 \| u_0 \|_{H^s})^{1/k}} \leq 2 \| u_0 \|_{H^s}, \quad 0 \leq t \leq T_0. \]  
(11)

This completes the proof of Theorem 5. \( \square \)

3. Nonuniform Dependence for the Solution to g-BO

3.1. Approximate Solutions

In this section, we consider approximate solutions of the Equation (1) of the form
\[ u^\omega,\lambda = u_l + u_h, \]  
(12)

where \( \omega = \pm 1 \) and \( \lambda > 0 \). The high frequency part is given by
\[ u_h = -\lambda^{-(1+\delta)/2k-s} \phi_\lambda \cos \Phi, \]  
(13)

where \( \phi_\lambda = \phi(\frac{x}{\alpha^{1+\delta} \lambda k}), \Phi = -\omega^2 \lambda^{1/k} t + \omega \lambda^{1/2k} x - \omega^{k+1} \lambda^{-1/2} t \) and \( \phi \) is \( C^\infty(\mathbb{R}) \) cutoff functions such that
\[ \phi(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 2. \end{cases} \]  
(14)

The low frequency part \( u_l \) is the solution to system (1) with initial data
\[ \begin{cases} \partial_t u_l + \mathcal{H}\partial_x^2 u_l + u_l \partial_x u_l = 0, & t > 0, x \in \mathbb{R}, \\ u_l(x,0) = \omega \lambda^{-1/2} \tilde{\phi}_\lambda, & x \in \mathbb{R}, \end{cases} \]  
(15)

where \( \tilde{\phi} \) is \( C^\infty(\mathbb{R}) \) functions such that
\[ \tilde{\phi}(x) = 1 \text{ if } x \in \text{supp} \phi. \]

Lemma 2 (See [12]). Let \( \psi \in \mathcal{S}(\mathbb{R}), 0 < \delta < 2 \) and \( \alpha \in \mathbb{R} \). Then, for any \( s \geq 0 \), we have that
\[ \lim_{\lambda \to \infty} \lambda^{-\frac{1}{2} - s} \left\| \psi \left( \frac{X}{\lambda^s} \right) \cos(\lambda x - a) \right\|_{H^s(\mathbb{R})} = \frac{1}{\sqrt{2}} \| \psi \|_{L^2(\mathbb{R})}. \]  
(16)

Relation (16) is also true if \( \cos \) is replaced by \( \sin \).

Lemma 3. Let \( 0 < \delta < 1 \) and \( \phi \in C^\infty_c(\mathbb{R}) \). Then, for any \( N > 0 \), there exists a positive constant \( C_N \) such that for every \( \alpha \in \mathbb{R} \)
\[ \left\| [\mathcal{H}, \phi_\lambda] \cos(\lambda x + \alpha) \right\|_{H^s(\mathbb{R})} \leq C_N \lambda^{-N}. \]  
(17)

Since the proofs of Lemma 3 are quite similar to lemma 2.2 in [12], they are omitted to make the paper concise.
Lemma 4. Let \( \omega = \pm 1, 0 < \delta < 1 \) and \( \lambda \gg 1 \). Then, the initial-value problem (1) has a unique solution \( u(t) \in C((0,T); H^s(\mathbb{R})) \), \( s > \frac{1}{2} \). For all \( \sigma \geq 0 \), this solutions satisfies the estimate

\[
\| u(t) \|_{H^s(\mathbb{R})} \leq C_s \lambda^{(\delta-1)/2k}.
\] (18)

Proof. Clearly, for any function \( \phi \in S(\mathbb{R}) \), we can easily check that

\[
\| \phi \left( \frac{x}{\lambda} \right) \|_{H^s} \leq \lambda^{k/2} \| \phi \|_{H^s}.
\] (19)

As per the relation \( \phi \left( \frac{x}{\lambda} \right) (\xi) = \rho \phi(\rho \xi) \), making the change of variables \( \eta = \lambda^{k/2} \xi \) yields

\[
\| \phi \left( \frac{x}{\lambda} \right) \|_{H^s} = \frac{1}{2\pi} \int_{\mathbb{R}} \left( 1 + \xi^2 \right)^{\sigma/2} |\lambda^{k/2} \phi(\lambda^{k/2})| d\xi
\]

\[
= \frac{\lambda^{k/2}}{2\pi} \int_{\mathbb{R}} \left( 1 + \eta^2 \right)^{\sigma/2} |\phi(\eta)| d\eta
\]

\[
\leq \lambda^{k/2} \| \phi \|_{H^s}.
\] (20)

According to (19), we know that the initial data \( u_0(0) \) satisfy the following estimate

\[
\| u_0(0) \|_{H^s} \leq \lambda^{(\delta-1)/2k},
\]

which decays if \( 0 < \delta < 1 \). Furthermore, the estimate (3) from Theorem 5 yields the lifespan \( T = \frac{\phi_{\lambda}^1 - 1}{2C_k |a_0|_{H^s}} \geq 1 \) for \( \lambda \gg 1 \) and \( 0 < \delta < 1 \). If \( \sigma \geq 0 \), then the estimate (4) of Theorem 5 implies

\[
\| u(t) \|_{H^s} \leq \| u(t) \|_{H^{s+2}} \leq C_s \| u_0(0) \|_{H^{s+2}} \leq C_s \lambda^{(\delta-1)/2k},
\]

which achieves Lemma 4. \( \square \)

Now, we estimate the error in \( H^s \)-norm of these approximate solutions. Substituting the approximate solution \( u^{\omega_\lambda}(x,t) \) into Equation (1), we find the following error:

\[
F = \partial_t u_h + H \partial_x^2 u_h + u_t \partial_x u_h + \left( \sum_{i=1}^k c_i u_{t_i}^{k-i} u_h \right) \partial_x (u_t + u_h)
\]

\[
= \omega \lambda^{-\delta/2k-s} \phi_\lambda \sin \Phi \left( u_{t^k}^k(x,t) - u_{x}^k(x,0) \right) - \lambda^{-3(1+\delta)/2k-s} u_{t^k}^{k} \phi_\lambda' \cos \Phi
\]

\[
+ \lambda^{-(1+\delta)/2k-s} \left[ 2\omega \lambda^{-(1+\delta)/2k} \mathcal{H}(\phi_\lambda' \sin \Phi) - \lambda^{-2(1+\delta)/2k} \mathcal{H}(\phi_\lambda'' \cos \Phi) \right] (21)
\]

\[
+ \omega^2 \lambda^{(1-\delta)/2k-s} \partial_x (u^k, u^k) \cos \Phi + \left( \sum_{i=1}^k c_i u_{t_i}^{k-i} u_h \right) \partial_x (u_t + u_h)
\]

\[
:= F_1 + F_2 + \cdots + F_5.
\]

Estimating the \( H^s \)-norm of \( F_1 \). Apparently, applying the Cauchy–Schwarz inequality yields

\[
\| F_1 \|_{H^s} \lesssim \lambda^{-(\delta/2k-s)} \left\| u_{t^k}^k(x,t) - u_{x}^k(x,0) \right\|_{H^s} \| \phi_\lambda \sin \Phi \|_{H^s}
\]

\[
\lesssim \lambda^{1/2k+s-\sigma} \left\| u_{t^k}^k(x,t) - u_{x}^k(x,0) \right\|_{H^s}.
\] (22)
To estimate the $H^s$-norm of the difference $u^k(x,t) - u^k(x,0)$, we adopt the fundamental theorem of calculus in time variable to obtain
\begin{equation}
\left\| u^k_t(x,t) - u^k_t(x,0) \right\|_{H^s} \leq \int_0^t \left\| u^{k-1}_t(x,\tau) \right\|_{H^s} d\tau.
\end{equation}

Apply Lemma 4 and (15) to imply that
\begin{equation}
\| \partial_t u^k_t(x,\tau) \|_{H^s} \lesssim \| H\partial_x^2 u^k_t \|_{H^s} + \| u^k_t \partial_x u^k_t \|_{H^s} \lesssim \lambda^{(\delta-1)/2k}.
\end{equation}

Substitute (17) and (24) into (23) to yield
\begin{equation}
\left\| u^k_t(x,t) - u^k_t(x,0) \right\|_{H^s} \lesssim \lambda^{(\delta-1)/2}.
\end{equation}

Finally, combining (22) and (25) gives
\begin{equation}
\| F_1 \|_{H^s} \lesssim \lambda^{(k\delta-k+1)/2k+\sigma-s}.
\end{equation}

**Estimating the $H^s$-norm of $F_2$.** Applying Lemma 2, we can easily estimate $F_2$
\begin{equation}
\| F_2 \|_{H^s} = \| \lambda^{-3(1+\delta)/2k-s} u^k t \phi_0 \cos \Phi \|_{H^s} \lesssim \lambda^{-1+\delta/k+k+\sigma-s}.
\end{equation}

**Estimating the $H^s$-norm of $F_3$.** Similar to the type, we readily check
\begin{equation}
\| F_3 \|_{H^s} = \left\| \lambda^{-3(1+\delta)/2k-s} \left[ 2\omega \lambda^{-2(1+\delta)/k} \mathcal{H} (\phi_0' \sin \Phi) - \lambda^{-2(1+\delta)/k} \mathcal{H} (\phi_0'' \cos \Phi) \right] \right\|_{H^s} \lesssim \lambda^{-\delta/k+k+\sigma-s}.
\end{equation}

**Estimating the $H^s$-norm of $F_4$.** Using Lemma 3, we achieve
\begin{equation}
\| F_4 \|_{H^s} = \| \omega^2 \lambda^{-\delta/2k-1} [\mathcal{H}, \phi] \cos \Phi \|_{H^s} \lesssim \lambda^{-\delta/k+k+\sigma-s}.
\end{equation}

**Estimating the $H^s$-norm of $F_5$.** To estimate $F_5$, we need the following lemma.

**Lemma 5 (see [23,24]).** If $\sigma > 0$, then $H^s \cap L^\infty$ is an algebra. Moreover,
\begin{enumerate}[(i)]
\item $\| fg \|_{H^s} \leq c_{\sigma} \left( \| f \|_{L^\infty} \| g \|_{H^s} + \| g \|_{L^\infty} \| f \|_{H^s} \right)$, for $\sigma > 0$;
\item $\| fg \|_{H^s} \leq c_{\sigma} \| f \|_{H^s} \| g \|_{H^s}$, for $\sigma > \frac{1}{2}$.
\end{enumerate}

Apply the Lemma 5 to obtain
\begin{equation}
\| F_5 \|_{H^s} \lesssim \lambda^{(\delta-1)/2k+\sigma-s} + \lambda^{-1+\delta/2k+\sigma-2s}.
\end{equation}

Collecting the estimates above, we can obtain the following proposition.

**Proposition 1.** For $s > \frac{3}{2}, \frac{1}{2} < \sigma \leq 1$ and $0 < \delta < \frac{1}{2}$, we can find the following estimate
\begin{equation}
\| F \|_{H^s} \lesssim \lambda^{-\delta/k+k+\sigma-s}.
\end{equation}

### 3.2. Error Estimation between Approximate and Actual Solutions

Let $u_{w,\lambda}(t,x)$ be the solution to the Cauchy problem (1)—that is, $u_{w,\lambda}(t,x)$ satisfies
\begin{equation}
\begin{cases}
\partial_t u_{w,\lambda} + \mathcal{H}\partial_x^2 u_{w,\lambda} + u^k \partial_x u_{w,\lambda} = 0, \\
u_{w,\lambda}(x,0) = u^w(\lambda, x, 0) = \omega \lambda^{-2/\delta} \phi_0 - \lambda^{-1+\delta/2k} \phi_1 \cos \omega \lambda^{1/k} x.
\end{cases}
\end{equation}

To estimate the error between approximate and actual solutions, let
\begin{equation}
\nu = u_{w,\lambda} - u^w.\lambda.$
where $F$ is defined by (21) and satisfying the $H^s$-estimate (31).

**Proposition 2.** If $\lambda > 1$, $s > \frac{3}{2}$ and $\frac{1}{2} < \sigma \leq s$, then

$$\|v(t)\|_{H^r} = \|u_{\omega,\lambda}(t)\|_{H^r} \lesssim \lambda^{-\delta/k + \sigma - s}, \text{ for } 0 \leq t \leq T.$$  

**Proof.** Applying the operator $D^\sigma$ to both sides of Equation (33), and multiplying the resulting equation by $D^\sigma v$, then integrating it with respect to $x \in \mathbb{R}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^r}^2 = \int_\mathbb{R} D^\sigma v \cdot D^\sigma F dx - \int_\mathbb{R} D^\sigma v \cdot \mathcal{H} \partial_x^2 D^\sigma v dx - \int_\mathbb{R} D^\sigma v \cdot D^\sigma (u_{\omega,\lambda} \partial_x v) dx$$

$$- \int_\mathbb{R} D^\sigma v \cdot D^\sigma \left( \nu \partial_x \frac{k}{\omega,\lambda} \sum_{i+j=k-1} (u_{\omega,\lambda})_j (u_{\omega,\lambda})^i \right) dx$$

$$\leq E_1 + E_2 + E_3 + E_4.$$  

Noting that $E_2 = \int_\mathbb{R} D^\sigma v \cdot \mathcal{H} \partial_x^2 D^\sigma v dx = 0$.

**Estimating the $H^r$-norm of $E_1$.** Referring to $E_1$ from (35) and using the Cauchy–Schwarz inequality gives us

$$|E_1| = \left| \int_\mathbb{R} D^\sigma v D^\sigma F dx \right| \leq \|v\|_{H^r} \|F\|_{H^r}. \quad (36)$$

**Estimating the $H^r$-norm of $E_3$.** Clearly, $E_3$ can be rewritten as

$$E_3 = - \int_\mathbb{R} D^\sigma v \cdot [D^\sigma, u_{\omega,\lambda}^k] \partial_x v dx - \int_\mathbb{R} D^\sigma v \cdot u_{\omega,\lambda}^k \partial_x \partial_x v dx. \quad (37)$$

Using Lemma 1, we can estimate the first integral of the right-hand side of (37)

$$\int_\mathbb{R} D^\sigma v \cdot [D^\sigma, u_{\omega,\lambda}^k] \partial_x v dx \leq \|v\|_{H^r} \left( \|D^\sigma, u_{\omega,\lambda}^k\|_{L^2} \|D^\sigma \partial_x v\|_{L^2} \right)$$

$$\lesssim \|v\|_{H^r} \left( \|\partial_x u_{\omega,\lambda}^k\|_{L^\infty} + \|D^\sigma u_{\omega,\lambda}^k\|_{L^2} + \|D^\sigma \partial_x v\|_{L^\infty} \right) \quad (38)$$

Integrating by parts, we can estimate the second integral of the right-hand side of (37)

$$\int_\mathbb{R} D^\sigma v \cdot u_{\omega,\lambda}^k \partial_x \partial_x v dx = \frac{1}{2} \int_\mathbb{R} u_{\omega,\lambda}^k \partial_x (D^\sigma v)^2 dx$$

$$= \frac{1}{2} \int_\mathbb{R} \partial_x u_{\omega,\lambda}^k (D^\sigma v)^2 dx \lesssim \|u_{\omega,\lambda}\|_{H^r}^2 \|v\|_{H^r}^2. \quad (39)$$

Now, (37)–(39) imply

$$|E_3| \lesssim \|u_{\omega,\lambda}\|_{H^r}^2 \|v\|_{H^r}^2. \quad (40)$$
Estimating the $H^r$-norm of $E_4$. Apply the Cauchy–Schwarz inequality and Lemma 5 (ii) to obtain

$$|E_4| \lesssim \|v\|_{H^r} \left\| \partial_x u_{\alpha,\lambda} \right\|_{H^r} \sum_{i+j=k-1} (u_{\alpha,\lambda})^i(u_{\alpha,\lambda})^j \left\| \partial_x u_{\alpha,\lambda} \right\|_{H^r} \lesssim \|v\|_{H^r}^2 \left\| \partial_x u_{\alpha,\lambda} \right\|_{H^r} \sum_{i+j=k-1} (u_{\alpha,\lambda})^i(u_{\alpha,\lambda})^j \left\| \partial_x u_{\alpha,\lambda} \right\|_{H^r} \lesssim \|v\|_{H^r}^2.$$  \hspace{1cm} (41)

Combining the estimates of (35)–(41) yields the ODE

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^r}^2 \lesssim \|v\|_{H^r}^2 + \lambda^{-\delta/k+\sigma-s} \|v\|_{H^r},$$

and thus

$$\frac{d}{dt} \|v\|_{H^r} \lesssim \|v\|_{H^r} + \lambda^{-\delta/k+\sigma-s},$$

which gives rise to the following estimate

$$\|v(t)\|_{H^r} \lesssim \lambda^{-\delta/k+\sigma-s}, \text{ for } \lambda \gg 1, 0 \leq t \leq T.$$  \hspace{1cm} (42)

This proves the Proposition 2. \hfill \Box

3.3. Proof of Theorem 1  

In this subsection, with the error estimation between approximate and actual solutions in hand, and using the interpolation properties of the Sobolev spaces, we can prove Theorem 1.

Proof. Let $s > \frac{3}{4}$ and define $u_{1,\lambda}(x,t)$ and $u_{-1,\lambda}(x,t)$ as the unique solutions to Equation (15) with the initial data $u_{1,\lambda}(x,0)$ and $u_{-1,\lambda}(x,0)$, respectively. From Lemma 2 and (4), we can obtain the following inequality

$$\|u_{1,\lambda}(t)\|_{H^r} + \|u_{-1,\lambda}(t)\|_{H^r} \leq 2(\|u^{1,\lambda}(0)\|_{H^r} + \|u^{-1,\lambda}(0)\|_{H^r}) \lesssim 1.$$  

At time $t = 0$, we deduce

$$\lim_{\lambda \to \infty} \|u_{1,\lambda}(0) - u_{-1,\lambda}(0)\|_{H^r} = \lim_{\lambda \to \infty} 2\omega\lambda^{-1/k} \|\varphi\|_{H^r} = 0.$$  

Next, we examine the $H^s$-norm of the difference when $t > 0$. Using the triangle inequality, we find

$$\|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^r} \geq \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^r} - \|u^{1,\lambda}(t) - u_{1,\lambda}(t)\|_{H^r} - \|u^{-1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^r}. \hspace{1cm} (43)$$

With $k$-even, using the identity $\cos \alpha - \cos \beta = -2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}$, we find that

$$\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^r} = \|u_{1,\lambda} - u_{-1,\lambda} - 2\lambda^{-(1+\delta)/2k-s} \varphi \lambda \sin \lambda^{1/2k} x \sin (\lambda^{1/k} t + \lambda^{-1+1/2k} t)\|_{H^r}. \hspace{1cm} (44)$$

With $k$-odd, we deduce

$$\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^r} = \|u_{1,\lambda} - u_{-1,\lambda} - 2\lambda^{-(1+\delta)/2k-s} \varphi \lambda \sin (\lambda^{1/2k} x - \lambda^{-1+1/2k} t) \sin \lambda^{1/k} t\|_{H^r}. \hspace{1cm} (45)$$
Letting \( m = [s] + 2 > 2 \), apply Lemma 2 and (11) to find
\[
\|u^{t,\lambda}(t) - u_{0,\lambda}(t)\|_{H^m} \lesssim \|u^{t,\lambda}(t)\|_{H^m} + \|u^{t,\lambda}(0)\|_{H^m} \lesssim \lambda^{m-s}, \quad 0 < t \leq T. \tag{46}
\]

**Lemma 6.** Suppose \( s_1 < s < s_2 \) and \( f \in H^s(\mathbb{R}) \). Then,
\[
\|f\|_{H^s} \leq \|f\|_{H^{s_1}} \|f\|_{H^{s_2}}. \tag{47}
\]

Employing the interpolation inequality in Lemma 6 with \( s_1 = \sigma \) and \( s_2 = [s] + 2 = m \) and Equations (42) and (46), we obtain
\[
\|u^{t,\lambda}(t) - u_{0,\lambda}(t)\|_{H^m} \leq \|u^{t,\lambda}(t) - u_{0,\lambda}(t)\|_{H^m} - \|u^{t,\lambda}(t) - u_{0,\lambda}(t)\|_{H^m} \\
\lesssim \lambda^{m-s} \quad (48)
\]

Taking the limit infimum to both sides of (43) gives us
\[
\liminf_{\lambda \to \infty} \|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^m} \\
\geq \liminf_{\lambda \to \infty} \|u^{t,\lambda}(t) - u_{0,\lambda}(t)\|_{H^m} - \|u^{t,\lambda}(t) - u_{0,\lambda}(t)\|_{H^m} - \|u^{t,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^m} \\
\geq \liminf_{\lambda \to \infty} |\sin \lambda^{1/k} t|,
\]

apparently \( \liminf_{\lambda \to \infty} \|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^m} \neq 0 \); thus, we complete the proof of Theorem 1.


4. Hölder Continuous in \( H^s \)-Topology

In this section, we continue to study the continuity properties for the solution map in Hölder spaces \( H^s \). More precisely, we consider two solutions of Equation (1), \( u \) and \( v \), which emanate from the initial data \( u_0 \) and \( v_0 \), respectively. We expect that if the initial data \( u_0 \) and \( v_0 \) are assigned in a ball with radius \( \rho \) in \( H^s \), i.e.,
\[
\|u_0\|_{H^s} \leq \rho, \quad \|v_0\|_{H^s} \leq \rho, \quad s > \frac{3}{2r}, \tag{50}
\]
and then we obtain
\[
\|u(t) - v(t)\|_{H^s} \lesssim \|u_0 - v_0\|_{H^s}, \quad 0 \leq r < s,
\]
where the Hölder exponent \( \alpha \) is to be determined.

**Proof of Theorem 2.** Lipschitz continuity in region \( A_1 \). Let \( v \) be another solution to the Cauchy problem for (1) corresponding to the initial data \( v_0(x) \in H^s(\mathbb{R}) \), i.e.
\[
\begin{aligned}
\partial_t v + \mathcal{H} \partial_x^2 v + v^k \partial_x v = 0, & \quad t > 0, \ x \in \mathbb{R}, \\
v(x,0) = v_0(x), & \quad t = 0, \ x \in \mathbb{R}.
\end{aligned} \tag{51}
\]

Subtracting (51) from (1) yields the Cauchy problem for \( w \) to
\[
\partial_t w + \mathcal{H} \partial_x^2 w + \partial_x (v^k w) - \omega \partial_x v^k + \omega \partial_x u \sum_{i+j=k-1} (u^i v^j) = 0, \tag{52}
\]
where \( w = u - v \) and \( i, j \in \). For a fixed \( 0 \leq r \leq s - 1 \) with \( r + s \geq 2 \), estimating the \( H^r \) energy of \( w \) leads us to the following equation

\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2, H^r}^2 = - \int_R D^r w \cdot \mathcal{H} \partial_x^2 D^r w \, dx - \int_R D^r w \cdot D^r \partial_x (v^k w) \, dx
\]

\[
+ \int_R D^r w \cdot D^r (w \partial_x v^k) \, dx - \int_R D^r w \cdot D^r \left( w \partial_x u \sum_{i+j=k-1} (u^i u^j) \right) \, dx. \tag{53}
\]

Noting that \( \int_R D^r w \cdot \mathcal{H} \partial_x^2 D^r w \, dx = 0 \). Clearly, the second integral of the right-hand side of (53) can be rewritten as

\[
- \int_R D^r w \cdot D^r \partial_x (v^k w) \, dx = - \int_R D^r w \cdot [D^r \partial_x, v^k] w \, dx - \int_R D^r w \cdot v^k (D^r \partial_x w) \, dx. \tag{54}
\]

To estimate the first term on the right-hand sides of (57), we need the following lemma (see [23,24]).

**Lemma 7.** If \( \sigma + 1 \geq 0, s > \frac{3}{2}, \sigma + 1 \leq s \), then

\[
\| [D^r \partial_x, f] g \|_{L^2} \leq C_{\sigma,s} \| f \|_{H^r} \| g \|_{H^r}.
\]

Using Lemma 7 and (50), we find

\[
\left| \int_R D^r w \cdot [D^r \partial_x, v^k] w \, dx \right| \leq C_{\sigma,r} \| w \|_{H^r} \| [D^r \partial_x, v^k] w \|_{L^2} \leq C_{\sigma,r} \| w \|_{H^r} \| w \|_{H^r}^2
\]

\[
\leq C_{\sigma,r} \| v(0) \|_{H^r} \| w \|_{H^r}^2 \leq C_{\sigma,r} \rho^k \| w \|_{H^r}^2, \tag{55}
\]

and we can easily yield the following estimates

\[
\left| \int_R D^r w \cdot v^k (D^r \partial_x w) \, dx \right| \leq C_{\sigma,r} \| v \|_{H^r} \| w \|_{H^r}^2 \leq C_{\sigma,r} \rho^k \| w \|_{H^r}^2. \tag{56}
\]

Combining (55) and (56) yields the estimates

\[
\left| \int_R D^r w \cdot D^r \partial_x (v^k w) \, dx \right| \leq C_{\sigma,r} \rho^k \| w \|_{H^r}^2. \tag{57}
\]

For the third term on the right-hand sides of (53), we readily check

\[
\left| \int_R D^r w \cdot D^r (w \partial_x u^k) \, dx \right| \leq C_{\sigma,r} \| w \|_{H^r} \| w \partial_x v^k \|_{H^r} \leq C_{\sigma,r} \rho^k \| w \|_{H^r}^2, \tag{58}
\]

where, in the second inequality, we used following lemma.

**Lemma 8** (see [25,26]). If \( \sigma > -\frac{1}{2} \), then

\[
\| f g \|_{H^r} \leq C_{\sigma} \| f \|_{H^{r+1}} \| g \|_{H^r}.
\]

Similarly, we can estimate the last term of (53)

\[
\left| \int_R D^r w \cdot D^r \left( w \partial_x u \sum_{i+j=k-1} (u^i u^j) \right) \, dx \right| \leq C_{\sigma,r} \rho^k \| w \|_{H^r}^2. \tag{59}
\]

**End of Lipschitz Continuity in** \( A_1 \). Combining the above estimates generates the following energy inequality

\[
\frac{d}{dt} \| w(t) \|_{H^r} \leq C_{r,s} \| w(t) \|_{H^r},
\]
which implies
\[ \|w(t)\|_{H^r} \leq e^{C_{r,p}T}\|w(0)\|_{H^r}. \]
Clearly, it is equivalent to
\[ \|u(t) - v(t)\|_{H^r} \leq e^{C_{r,p}T}\|u(0) - v(0)\|_{H^r}, \]
which is the desired Hölder continuity in \( A_1 \).

**Hölder Continuity in \( A_2 \).** As per the Lipschitz continuity in \( A_1 \) and the assumption \( r < 2 - s \), we deduce
\[ \|u(t) - v(t)\|_{H^r} \leq \|u(0) - v(0)\|_{H^{2-s}} \leq e^{C_{r,p}T}\|u(0) - v(0)\|_{H^{2-s}}. \]
Since \( r < 2 - s < s \), by the interpolation between the \( H^r \) and \( H^s \) norms described in Lemma 6, we find
\[ \|u(0) - v(0)\|_{H^{2-s}} \leq \|u(0) - v(0)\|_{H^r}^{2(1-s)} \|u(0) - v(0)\|_{H^s}^{2-s} \leq C_{r,s,p}\|u(0) - v(0)\|_{H^r}^{2(1-s)}, \]
which guarantees the Hölder continuity in \( A_2 \).

**Hölder Continuity in \( A_3 \).** For \( s - 1 < r < s \), by the interpolation between \( H^{s-1} \) and \( H^s \) norms, we have
\[ \|u(t) - v(t)\|_{H^r} \leq \|u(t) - v(t)\|_{H^{s-1}}^{s-r} \|u(t) - v(t)\|_{H^r}^{r-s+1}. \]
By the well-posedness size estimate (50), we find
\[ \|u(t) - v(t)\|_{H^r} \lesssim \|u_0\|_{H^r} + \|v_0\|_{H^r} \lesssim \rho, \]
which, therefore, gives
\[ \|u(t) - v(t)\|_{H^r} \leq C_{r,s,p}\|u(t) - v(t)\|_{H^{s-1}}^{s-r}. \]
The Lipschitz continuity in \( A_1 \) and the condition \( s - 1 < r \) admit
\[ \|u(t) - v(t)\|_{H^r} \leq C_{r,s,p}\|u(0) - v(0)\|_{H^{s-1}}^{s-r} \leq C_{r,s,p}\|u(0) - v(0)\|_{H^r}^{s-r}, \]
which is the desired Hölder continuity in \( A_3 \). \( \square \)

5. Gevrey Regularity and Analyticity for g-BO System

5.1. Analytic Solutions for g-BO in \( G^\delta \)

In this section, by applying nonlinear Cauchy-Kowalevski theory, we will establish the Gevrey regularity and analyticity of solutions to g-BO system.

**Theorem 6** (see [22]). Let \( (X_{\delta}, \|\cdot\|_{G^\delta})_{0 < \delta < 1} \) be a scale of decreasing Banach spaces, namely, for any \( \delta' < \delta \) we have \( X_{\delta} \subset X_{\delta'} \) and \( \|\cdot\|_{G^\delta} \leq \|\cdot\|_{G^\delta'}. \) Consider the Cauchy problem

\[
\begin{cases} 
\frac{du}{dt} = F(t, u(t)), \\
u|_{t=0} = 0.
\end{cases}
\]
\( (60) \)

Let \( T, R > 0 \) and \( \sigma \geq 1 \). For given \( u_0 \in X_{\delta} \), assume that \( F \) satisfies the following conditions:
(1) If for any \( 0 < \delta' < \delta < 1 \), the function \( t \mapsto u(t) \) is holomorphic on \( |t| < T \) and continuous on \( |t| < T \) with values in \( X_{\delta} \) and
\[ \sup_{|t| < T}\|u(t)\|_{G^\delta} < R, \]
and then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{\delta}$.

(2) For any $0 < \delta' < \delta < 1$ and $u, v \in B(u_0, R) \subset X_{\delta}$—that is, $\|u\|_{G^0_{\delta', \delta}} < R$, $\|v\|_{G^0_{\delta', \delta}} < R$, there exists a positive constant $L$ depending on $u_0$ and $R$ such that

$$\sup_{|t| < T} \|F(t, u) - F(t, v)\|_{G^0_{\delta', \delta}} \leq \frac{L}{(\delta - \delta')^{2\sigma}} \|u - v\|_{G^0_{\delta', \delta}}.$$ 

(3) There exists a $M > 0$ depending on $u_0$ and $R$ such that, for any $0 < \delta < 1$,

$$\sup_{|t| < T} \|F(t, u_0)\|_{G^0_{\delta', \delta}} \leq \frac{M}{(1 - \delta)^{2\sigma}}.$$ 

Then, there exists a $T_0 \in (0, T)$ and a unique function $u(t)$ to the Cauchy problem (60) that is holomorphic in $|t| < \frac{(1 - \delta')^{2\sigma} T_0}{2\sigma - 1}$ with values in $X_{\delta}$ for every $\delta \in (0, 1)$.

**Proposition 3** (see [22]). Let $0 < \delta' < \delta, 0 < \sigma' < \sigma$ and $s' < s$. From Definition 1, one can check that $G^0_{\delta', \delta} \hookrightarrow G^0_{\delta', \delta'} \hookrightarrow G^0_{s', \delta}$ and $G^0_{\delta', \delta} \hookrightarrow G^0_{s', \delta'}$.

**Proposition 4.** Let $s$ be a real number and $\sigma > 0$. Assume that $0 < \delta' < \delta$. Then, we have

$$\|\partial_s f\|_{G^0_{\delta', \delta}(R)} \leq \frac{e^{-\sigma} e^{\sigma}}{(\delta - \delta')^{2\sigma}} \|f\|_{G^0_{\delta', \delta}(R)}, \|\partial^2_s f\|_{G^0_{\delta', \delta}(R)} \leq \frac{e^{-2\sigma} (2\sigma)^{2\sigma}}{(\delta - \delta')^{2\sigma}} \|f\|_{G^0_{\delta', \delta}(R)}.$$ 

**Proof.** The first inequality can be found in [22], so we only prove the second inequality. Since $\partial_s f = i\zeta \hat{f}$, it follows that

$$\|\partial^2_s f\|_{G^0_{\delta', \delta}(R)}^2 = \int_R (1 + |\zeta|^2)^{2\sigma} |\zeta|^{1/\sigma} |\hat{f}(\zeta)|^2 d\zeta$$

$$= \frac{1}{(\delta - \delta')^{4\sigma}} \int_R (1 + |\zeta|^2)^{2\sigma} |\zeta|^{1/\sigma} \int_R e^{-2|\zeta|^2/\sigma} |\hat{f}(\zeta)|^2 d\zeta$$

$$\leq \|f\|_{G^0_{\delta', \delta}(R)}^2 \sup_{\zeta \in \mathbb{R}} \{ e^{-2((\delta - \delta')^{4\sigma} - 1)|\zeta|^{1/\sigma}} (\delta - \delta')^{4\sigma} |\zeta|^{1/\sigma} \}. \quad (61)$$

Let $z = \{(\delta - \delta')^{4\sigma} - 1\}^{1/\sigma} \geq 0$ and consider the function $g(z) = e^{-2z^{1/\sigma}}$. By directly calculating, we have $\lim_{z \to 0} g(z) = 0, \lim_{z \to \infty} g(z) = 0$ and $g'(z) = 2z^{-1/\sigma}(2z - 1)$. By solving $g'(z) = 0$, we obtain that $z = 2\sigma$, which implies that $g(z) \leq g(2\sigma) = e^{-4\sigma(2\sigma)^{4\sigma}}$. Then, we deduce from (61) that

$$\|\partial^2_s f\|_{G^0_{\delta', \delta}(R)}^2 \leq \frac{e^{-2\sigma} (2\sigma)^{2\sigma}}{(\delta - \delta')^{2\sigma}} \|f\|_{G^0_{\delta', \delta}(R)}.$$ 

□

**Proposition 5** (see [22]). Let $s > \frac{1}{2}, \sigma \geq 1$ and $\delta > 0$. Then, $G^0_{\delta, \delta}(\mathbb{R})$ is an algebra. Moreover, there exists a constant $C_s$ such that

$$\|f g\|_{G^0_{\delta', \delta}(R)} \leq C_s \|f\|_{G^0_{\delta', \delta}(R)} \|g\|_{G^0_{\delta', \delta}(R)}.$$ 

**Proof of Theorem 3.** We rewrite (g-BO) as follows:

$$\begin{cases}
    u_t = F(u) = -\mathcal{H} \partial_s u - u^k \partial_k u, \\
    u|_{t=0} = 0.
\end{cases} \quad (62)$$
For a fixed $\sigma \geq 1$ and $s > \frac{3}{2}$. By virtue of Propositions 3, 4 and 5, we deduce that, for any $0 < \delta' < \delta < 1$,}
\[
\| F(u) \|_{G_{\delta,s}} \leq \| \nabla^2 u \|_{G_{\delta,s}} + \| u^k \|_{G_{\delta,s}}
\leq \frac{e^{-2\nu(2\sigma)^{2\nu}} \| u \|_{G_{\delta,s}}^k + C e^{-\sigma} \| u \|_{G_{\delta,s}}^{k+1}}{\delta (2\sigma)^{2\nu}} \times (\| u \|_{G_{\delta,s}}^k + \| u \|_{G_{\delta,s}}^{k+1})
\]  
(63)

which implies that $F$ satisfies the condition (1) of Theorem 6. By the same token, we obtain that $\| F(u_0) \|_{G_{\delta,s}} \leq \frac{C e^{-\sigma}(2\sigma)^{2\nu}}{1-\delta} \left( \| u_0 \|_{G_{\delta,s}} + \| u_0 \|_{G_{\delta,s}}^{k+1} \right)$. Thus, $F$ satisfies the condition (3) of Theorem 6 with $M = C e^{-\sigma}(2\sigma)^{2\nu} (\| u_0 \|_{G_{\delta,s}} + \| u_0 \|_{G_{\delta,s}}^{k+1})$. Finally, we will show that $F$ satisfies the condition (2) of Theorem 6. Assume that $\| u - u_0 \|_{G_{\delta,s}} \leq R$, $\| v - u_0 \|_{G_{\delta,s}} \leq R$ and $w = u - v$. Applying Proposition 4, we find
\[
\| F(u) - F(v) \|_{G_{\delta,s}} = \left\| H \nabla^2 w + \varepsilon (\nabla^k w) - \omega \varepsilon^k + \omega \varepsilon \sum_{i+j=k-1} (\varepsilon^l \varepsilon^j) \right\|_{G_{\delta,s}}
\leq \frac{e^{-2\nu(2\sigma)^{2\nu}} \| w \|_{G_{\delta,s}}^k + C e^{-\sigma} \| w \|_{G_{\delta,s}}^k \| \varepsilon \|_{G_{\delta,s}}^k + C e^{-\sigma} \| w \|_{G_{\delta,s}}^k \| \varepsilon \|_{G_{\delta,s}}^k}{\delta (2\sigma)^{2\nu}} \sum_{i+j=k-1} (\varepsilon^l \varepsilon^j) \right\|_{G_{\delta,s}}
\leq \frac{C e^{-2\nu(2\sigma)^{2\nu}} + C e^{-\sigma} \nu \left( \| u_0 \|_{G_{\delta,s}} + R \right) \| w \|_{G_{\delta,s}}^k}{(\delta - \Omega)^{2\nu}} \times \right\| w \|_{G_{\delta,s}}^k \leq \frac{C e^{-\sigma}(2\sigma)^{2\nu} \left( 1 + \left( \| u_0 \|_{G_{\delta,s}} + R \right) \right)}{(\delta - \Omega)^{2\nu}} \frac{\| w \|_{G_{\delta,s}}^k}{(\delta - \Omega)^{2\nu}}.
\]

From the above inequality, we verify that $F$ satisfies the condition (2) of Theorem 6 with $L = C e^{-\sigma}(2\sigma)^{2\nu} \left( 1 + \left( \| u_0 \|_{G_{\delta,s}} + R \right) \right)$. Moreover, $T_0 = \min \left\{ \frac{1}{24^{k+1}}, \frac{(2\sigma)^{2\nu} R}{2 \left( (2\sigma)^{2\nu} - 1 \right) L R} \right\}$, by setting $R = \| u_0 \|_{G_{\delta,s}}$, we see that $L = C e^{-\sigma}(2\sigma)^{2\nu} \left( 1 + \left( 2k \| u_0 \|_{G_{\delta,s}} \right) \right)$ and $M \leq 2^{2\sigma+3} L R$. Then, we have $T_0 = \frac{1}{C e^{-\sigma}(2\sigma)^{2\nu} \left( 1 + 2k \| u_0 \|_{G_{\delta,s}} \right)}$.

5.2. Continuity of the Data-to-Solution Map in $G_{\delta,s}$

**Proof of Theorem 4.** Without loss of generality, we may assume that $t \geq 0$. Define that
\[
T_n = \frac{1}{C e^{-\sigma}(2\sigma)^{2\nu} \left( 1 + 2k \| u_0 \|_{G_{\delta,s}} \right)}, \quad T^n = \frac{1}{C e^{-\sigma}(2\sigma)^{2\nu} \left( 1 + 2k \| u_0 \|_{G_{\delta,s}} \right)}
\]
(64)

where $C$ is given in Proposition 5. Since $\| u_0^n - u_0^n \|_{G_{\delta,s}} \rightarrow 0$; therefore, there exists a constant $N$ such that
\[
\| u_0^n \|_{G_{\delta,s}} \leq \| u_0 \|_{G_{\delta,s}} + 1, \quad \text{for } n \geq N.
\]
(65)

Then,
\[
T \leq \frac{1}{C e^{-\sigma}(2\sigma)^{2\nu} \left( 1 + 2k \| u_0 \|_{G_{\delta,s}} \right)} < \min \{ T_n, T^n \}, \quad \text{for } n \geq N.
\]
(66)
Furthermore, as in the proof of Theorem 3, we see that $T^n$ and $T^\infty$ are the existence time corresponding to $\|u_0^n\|_{\mathcal{G}_{\delta,s}}$ and $\|u_0^\infty\|_{\mathcal{G}_{\delta,s}}$, respectively, which implies that, for any $n \geq N$

$$u^n(t,x) = u^n_0(x) + \int_0^t F(u^n(t,\tau))d\tau, \quad 0 \leq t < \frac{T(1-\delta)^{2\sigma}}{2^{2\sigma} - 1},$$

(67)

$$u^\infty(t,x) = u^\infty_0(x) + \int_0^t F(u^\infty(t,\tau))d\tau, \quad 0 \leq t < \frac{T(1-\delta)^{2\sigma}}{2^{2\sigma} - 1},$$

where $F$ is given in (62). Therefore, we verify that, for any $0 \leq t < \frac{T(1-\delta)^{2\sigma}}{2^{2\sigma} - 1}$ and $0 < \delta < 1$

$$\|u^n - u^\infty\|_{\mathcal{G}_{\delta,s}} \leq \|u^n_0 - u^\infty_0\|_{\mathcal{G}_{\delta,s}} + \int_0^t \|F(u^n(t,\tau))-F(u^\infty(t,\tau))\|_{\mathcal{G}_{\delta,s}}d\tau.$$  

(68)

Choosing $\delta(\tau) = \frac{1+\sqrt{2}}{2} + \frac{12+1/2\sigma}{2}\left\{[(1-\delta)^{2\sigma} - \frac{t}{T}]^{1/2\sigma} - [(1-\delta)^{2\sigma} + (2^{2\sigma+1} - 1)\frac{t}{T}]^{1/2\sigma}\right\}$, we find

$$\|F(u^n(t,\tau))-F(u^\infty(t,\tau))\|_{\mathcal{G}_{\delta,s}} \leq \frac{L||u^n - u^\infty||_{\mathcal{G}_{\delta,s}}}{(\delta(\tau)-\delta(\tau'))}$$

with

$L = C_\delta e^{-\tau}(2\sigma)^{2\sigma}\left(1 + 2^k\|u_0\|^k\right)$ and $0 < \delta < \delta(\tau) < 1$. Using this in (68) yields

$$\|u^n - u^\infty\|_{\mathcal{G}_{\delta,s}} \leq \|u^n_0 - u^\infty_0\|_{\mathcal{G}_{\delta,s}} + \int_0^t \frac{2^{4\sigma+3}LT||u^n - u^\infty||_{\mathcal{G}_{\delta,s}}}{(1-\delta)^{2\sigma}} d\tau.$$  

(69)

where, in the last inequality, we used lemma 3.7 in [22]. Since $T = \frac{1}{C_\delta 2^{2\sigma} e^{-\tau}(2\sigma)^{2\sigma}\left(1 + 2^k\|u_0\|^k\right)}$ and $L = C_\delta e^{-\tau}(2\sigma)^{2\sigma}\left(1 + 2^k\|u_0\|^k\right)$, this yields that $2^{4\sigma+3}LT < \frac{1}{2}$. Then, we have

$$\|u^n - u^\infty\|_{\mathcal{G}_{\delta,s}} \leq \|u^n_0 - u^\infty_0\|_{\mathcal{G}_{\delta,s}} + \frac{\|u^n - u^\infty\|_{\mathcal{G}_{\delta,s}}}{2(1-\delta)^{2\sigma}} \frac{T(1-\delta)^{2\sigma}}{T(1-\delta)^{2\sigma} - t}.$$  

(70)

This leads to

$$\|u^n - u^\infty\|_{\mathcal{G}_{\delta,s}} (1-\delta)^{2\sigma} \sqrt{1 - \frac{t}{T(1-\delta)^{2\sigma}}} \leq \|u^n_0 - u^\infty_0\|_{\mathcal{G}_{\delta,s}} (1-\delta)^{2\sigma} \sqrt{1 - \frac{t}{T(1-\delta)^{2\sigma}}} + \frac{1}{2} \|u^n - u^\infty\|_{\mathcal{G}_{\delta,s}}.$$  

(71)

Note that the right hand side of the above inequality is independent of $t$ and $\delta$. By the definition of $E_T$, we have

$$\|u^n - u^\infty\|_{E_T} \leq 2\|u^n_0 - u^\infty_0\|_{\mathcal{G}_{\delta,s}}.$$  

(72)

The above inequality holds true for any $n \geq N$, which leads to our desired result.

6. Conclusions

The local well-posedness results in [14–16] imply that the existence, uniqueness and continuously dependence on their initial data of the solutions to the g-BO Equation (1) in $H^p(\mathbb{R})$ with $s > \frac{3}{2}$. We showed that such a data-to-solution map is not uniformly continuous in Theorem 1 but Hölder continuous in $H^p$-topology. On the other hand, in
Sobolev–Gevrey spaces, we proved that the solutions of g-BO equation are analytic in both space and time variables in Theorem 3. In addition, the continuity of the data-to-solution in Sobolev–Gevrey spaces was also obtained.

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