Gravitational waves and massless particle fields

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Abstract

These notes address the planar gravitational wave solutions of general relativity in empty space-time, and analyze the motion of test particles in the gravitational wave field. Next we consider related solutions of the Einstein equations for the gravitational field accompanied by long-range wave fields of scalar, spinor and vector type, corresponding to massless particles of spin \( s = (0, \frac{1}{2}, 1) \). The motion of test masses in the combined gravitational and scalar, spinor or vector wave fields is discussed.

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1 Planar gravitational waves

a. Planar wave solutions of the Einstein equations

Planar gravitational wave solutions of the Einstein equations have been known since a long time [1]-[3]. In the following I discuss unidirectional solutions of this type, propagating along a fixed light-cone direction; thus the fields depend only on one of the light-cone co-ordinates \((u, v)\), here taken transverse to the \(x\)-\(y\)-plane:

\[
u = ct - z, \quad v = ct + z.
\]

(1)

Such gravitational waves can be described by space-time metrics

\[
\gamma_{\mu\nu} dx^\mu dx^\nu = -du dv - K(u, x, y) du^2 + dx^2 + dy^2 = -c^2 d\tau^2,
\]

(2)
or similar solutions with the roles of \(v\) and \(u\) interchanged. If the space-time is asymptotically minkowskian. With the metric (2), the connection co-efficients become

\[
\Gamma_{u u}^v = K_{,u}, \quad \Gamma_{u u}^x = \frac{1}{2} \Gamma_{u v}^x = \frac{1}{2} K_{,x}, \quad \Gamma_{u u}^y = \frac{1}{2} \Gamma_{y v}^x = \frac{1}{2} K_{,y}.
\]

(3)

All other components vanish. The corresponding Riemann tensor has non-zero components

\[
R_{uuxx} = -\frac{1}{2} K_{,xx}, \quad R_{uyuy} = -\frac{1}{2} K_{,yy},
\]

(4)

\[
R_{uxuy} = R_{uyux} = -\frac{1}{2} K_{,xy}.
\]

The only non-vanishing component of the Ricci tensor then is

\[
R_{uu} = -\frac{1}{2} (K_{,xx} + K_{,yy}) \equiv -\frac{1}{2} \Delta_{trans} K.
\]

(5)

Here the label \(trans\) refers to the transverse \((x, y)\)-plane, with the \(z\)-axis representing the longitudinal direction. In complex notation

\[
\zeta = x + iy, \quad \bar{\zeta} = x - iy,
\]

(6)

the Einstein equations in vacuo become

\[
R_{\mu\nu} = 0 \quad \iff \quad K_{,\zeta\bar{\zeta}} = 0.
\]

(7)

The general solution of this equation reads

\[
K(u, \zeta, \bar{\zeta}) = f(u; \zeta) + \bar{f}(u; \bar{\zeta}) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left( \epsilon_n(k) e^{-iku\zeta^n} + \bar{\epsilon}_n(k) e^{iku \bar{\zeta}^n} \right).
\]

(8)

Note that the terms with \(n = 0, 1\) correspond to vanishing Riemann tensor: \(R_{\mu\nu\rho\lambda} = 0\); therefore they represent flat Minkowski space-time in a non-standard choice of co-ordinates. For this reason we adopt the convention that \(\epsilon_0 = \epsilon_1 = 0\), which is just a choice of gauge.

b. Geodesics of planar-wave space-times

We proceed to solve the geodesic equation in the gravity-wave space-time (2) along the lines of ref.[3]:

\[
\ddot{x}^\mu + \Gamma_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda = 0
\]

(9)
Here the overdot denotes a proper-time derivative. The proper-time Hamiltonian satisfies a constraint imposed by eq.(9):

\[ H = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \]

\[ = - \dot{u} \dot{v} - K(u, x, y) \dot{u}^2 + \dot{x}^2 + \dot{y}^2 = -c^2. \]

Because the metric is covariantly constant, the Hamiltonian is a constant of motion:

\[ \dot{H} = 0. \] (11)

This can be checked directly from the geodesic equation (9). Also, as \( v \) is a cyclic coordinate, its conjugate momentum is conserved:

\[ \ddot{u} = 0, \] (12)

with the simple solution \( \dot{u} \equiv \gamma = \text{constant} \). Again, this agrees with the geodesic equation, as there is no non-vanishing connection component in the \( u \)-direction: \( \Gamma^u_{\nu\lambda} = 0 \).

Only the equations of motion in the \( x-y \)-plane depend on the specific wave potential \( K(u, x, y) \):

\[ \ddot{x} = - \frac{1}{2} K_x \dot{u}^2 = - \frac{\gamma^2}{2} K_x, \]

\[ \ddot{y} = - \frac{1}{2} K_y \dot{u}^2 = - \frac{\gamma^2}{2} K_y, \] (13)

Eqs. (10)-(13) specify completely the motion of a test particle, with the conservation of \( H \) taking the place of the equation for the acceleration in the \( v \)-direction:

\[ \gamma \dot{v} + \gamma^2 K(u, x, y) = \dot{x}^2 + \dot{y}^2 + c^2. \] (14)

If we now add \( \dot{z}^2 \) to the left- and right-hand side, and remember that

\[ \gamma \dot{v} = \dot{u} \dot{v} = c^2 \dot{t}^2 - \dot{z}^2, \] (15)

we can rewrite the Hamiltonian conservation law as

\[ c^2 \dot{t}^2 + \gamma^2 K = c^2 + \dot{t}^2. \] (16)

Finally, with \( v = d\mathbf{r}/dt = \dot{\mathbf{r}}/\dot{t} \), the equation can be cast into the form

\[ \dot{t} = \frac{dt}{d\tau} = \sqrt{\frac{1 - \gamma^2 K/c^2}{1 - v^2/c^2}} \] (17)

This equation describes relativistic time-dilation as resulting from two effects:
(i) the usual special-relativistic time-dilation from the relative motion of observers in the rest- and laboratory frame, whose time co-ordinates are \( \tau \) and \( t \), respectively;
(ii) the gravitational redshift resulting from the non-trivial potential \( K \).

Now from the conservation of \( \gamma = \dot{u} = c\dot{t} - \dot{z} \) it follows, that

\[ \gamma = c\dot{t} \left( 1 - \frac{v_z}{c} \right), \] (18)

with \( v_z = dz/dt \). Eqs. (17), (18) can then be solved for \( \gamma \):

\[ \frac{\gamma^2}{c^2} = \frac{1}{K + \frac{1 - v^2/c^2}{(1 - v_z/c)^2}}. \] (19)
Thus, for a particle starting at rest at infinity in an asymptotically minkoskian space-time, we find \( \gamma = c \). At the same time we observe that

\[
\gamma = c
\]

is conserved. Now we recall that in our conventions \( K \) is at least quadratic in the transverse co-ordinates; hence the components \( \dddot{x} \) and \( \dddot{y} \) of the transverse acceleration vanish for \( x = y = 0 \).

Furthermore \( K(u, 0, 0) = 0 \), with the result that the origin of the transverse plane moves at constant velocity along the \( z \)-axis:

\[
\gamma^2 c^2 = 1 - \frac{v_z}{c} \quad \Rightarrow \quad \frac{v_z}{c} = \frac{1 - \gamma^2/c^2}{1 + \gamma^2/c^2}.
\]

In particular, the point at rest in the origin moves along the simple geodesic

\[
x^\mu(\tau) = (c\tau, 0, 0, 0).
\]

Taking this geodesic as our reference, the solution for the geodesic motion \( \dddot{x}^\mu(\tau) \) of any other test particle at the same time presents a measure for the geodesic deviation between the worldlines of the two particles.

### 2 Einstein-scalar waves

Having discussed the planar gravitational waves \( \mathcal{G} \) in empty space we now turn to discuss similar unidirectional wave solutions of the combined system of Einstein gravity and a set of massless self-interacting scalar fields. The solutions of the inhomogeneous and non-linear Einstein equations, with the energy-momentum tensor that of the right- (or left-) moving scalar waves, nevertheless turn out to be a linear superposition of the gravitational field of the scalar waves and the free gravitational wave solutions discussed in the first paragraph.

We introduce a set of massless scalar fields \( \sigma^i(x), \; i = 1, \ldots, N \), taking values in a manifold with the dimensionless metric \( G_{ij}[\sigma] \). In four-dimensional space-time the fields themselves have dimension \( [\sigma] = \sqrt{E/l} \); thus, introducing an appropriate length scale \( 1/f \), in the context of quantum field theory we could write \( \sigma^i = \sqrt{E/l} f \eta^i \), with \( \eta^i(x) \) a dimensionless field.

The starting point of our analysis is given by the gravitational and \( \sigma \)-model field equations

\[
\Box^\text{cov} \sigma^i + \Gamma_{jk}^i [\sigma] g^{\mu\nu} \partial_\mu \sigma^j \partial_\nu \sigma^k = 0,
\]

\[
R_{\mu\nu} = -\frac{8\pi G}{c^4} G_{ij}[\sigma] \partial_\mu \sigma^i \partial_\nu \sigma^j.
\]

Here the covariant d’Alembertian is defined on scalar fields in the standard fashion

\[
\Box^\text{cov} = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu,
\]

whilst \( \Gamma_{ij}^k[\sigma] \) denotes the Riemann-Christoffel connection in the target manifold of the scalar fields. These equations can be derived straightforwardly from the combined Einstein-\( \sigma \)-model action, but we will skip the details of that procedure here. Our aim is to construct simultaneous traveling wave solutions of the full set of equations (23). Such solutions are actually quite easy to find. First, the scalar field equation is solved by taking right-moving fields

\[
\sigma^i = \sigma^i(u),
\]
with no dependence on any other co-ordinate. Next we substitute this solution of the scalar field into the second equation for the corresponding gravitational field. As before, only the uu-component of this equation survives, reading

\[ R_{uu} = - \frac{1}{2} \Delta_{trans} K = - \frac{8\pi G}{c^4} G_{ij}[\sigma] \partial_u \sigma^i \partial_u \sigma^j. \]  

(25)

As this is a linear equation, the general solution consists of a linear superposition of a particular solution and the general free gravitational wave of the previous section:

\[ K(u, \zeta, \bar{\zeta}) = \frac{8\pi G}{c^4} G_{ij}[\sigma] \partial_u \sigma^i \partial_u \sigma^j \zeta \zeta + f(u, \zeta) + \bar{f}(u, \bar{\zeta}). \]  

(26)

Now any specific solution \( \sigma^i(u) \) is a map from the real line into the target manifold of the scalar fields. Consider the special case that this curve in the target manifold is a geodesic:

\[ d^2 \sigma^i du^2 + \Gamma_{jk}^i d\sigma^j du d\sigma^k du = 0. \]  

(27)

Then the quantity

\[ I = G_{ij}[\sigma] \frac{d\sigma^i}{du} \frac{d\sigma^j}{du}, \]  

(28)

generating translations in \( u \), is constant along this curve: \( dI/du = 0 \). Moreover, for Euclidean manifolds with non-degenerate metric it is positive definite: \( I > 0 \). Observe, that for manifolds with compact directions (like spheres) the geodesics may be closed; then the corresponding scalar field configurations are periodic.

The special solution for the accompanying gravitational field now becomes

\[ K_{scalar}(u, x, y) = \frac{4\pi G I}{c^4} (x^2 + y^2), \]  

(29)

to which an arbitrary free gravitational wave solution can be added. In this special case, upon inserting \( K_{scalar} \) into eqs.(13) the transversal equations of motion of a test mass take the particularly simple form:

\[ \ddot{x} = - \frac{4\pi G I \gamma^2}{c^4} x, \quad \ddot{y} = - \frac{4\pi G I \gamma^2}{c^4} y. \]  

(30)

Thus the test mass executes a simple harmonic motion in the transverse plane, with frequency

\[ \omega = \frac{\gamma}{c^2} \sqrt{4\pi GI}. \]  

(31)

The solutions for the coupled Einstein-scalar field equations discussed here are not the only ones of interest. For example, the gravitational waves accompanying expanding domain walls in a theory with a spontaneously broken global symmetry can be calculated and have been discussed e.g. in [4, 5].

3 Einstein-Dirac waves

In this section we construct wave-solutions for massless chiral fermions coupled to Einstein gravity. As before the waves are unidirectional, and both left- and righthanded fermion solutions, associated with helicity \( \pm 1 \) quantum states, exist.

To treat fermions in interaction with gravity, it is necessary to introduce the vierbein and spin connection into the formalism. With the local minkowski metric \( \eta = \text{diag}(+1, +1, +1, -1) \),
the vierbein is a local lorentz vector of 1-forms \( E^a(x) = dx^\mu e^a_\mu(x) \) satisfying the symmetric product rule
\[
\eta_{ab} E^a E^b = \eta_{ab} e^a_\mu e^b_\nu dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu.
\] (32)

In a convenient local lorentz gauge, the vierbein corresponding to the metric (2) takes the form
\[
E^a = \left( dx, dy, \frac{1}{2} ((K - 1) du + dv), \frac{1}{2} ((K + 1) du + dv) \right).
\] (33)

The inverse vierbein is defined by the differential operator \( \nabla_a = e^\mu_a \partial_\mu \) such that
\[
E^a \nabla_a = dx^\mu \partial_\mu
\] (34)

In components it reads
\[
\nabla_a = (\partial_x, \partial_y, -\partial_u + (K + 1) \partial_v, \partial_u - (K - 1) \partial_v).
\] (35)

Next we compute the components of the spin connection \( \omega^a_b = dx^\mu \omega_{\mu}^a_b \) from the identity
\[
dE^a = \omega^a_b \wedge E^b.
\] (36)

With the vierbein (33) the spin connection has only one component
\[
\omega^a_b = -\omega^b_a = \frac{1}{2} \begin{pmatrix}
0 & 0 & K_x & K_x \\
0 & 0 & K_y & K_y \\
-K_x & -K_y & 0 & 0 \\
-K_x & -K_y & 0 & 0
\end{pmatrix}.
\] (37)

In order to construct the dirac operator we introduce a basis for the dirac matrices satisfying
\[
\{ \gamma^a, \gamma^b \} = 2 \eta^{ab},
\]
and define a set spinor generators for the lorentz algebra by \( \sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] \). Then the dirac operator is
\[
\gamma \cdot D = \gamma^a \left( \nabla_a - \frac{1}{2} \omega_{\mu}^{bc} \sigma_{bc} \right).
\] (38)

The results we need all depend on the property of the light-cone components of the dirac algebra:
\[
\gamma^u = \gamma^a e^a_u = -\gamma_3 + \gamma_0.
\] (39)

This element of the dirac algebra is nilpotent:
\[
(\gamma^u)^2 = 0.
\] (40)

The same is true for \( \gamma_v = e^a_v \gamma_a = \frac{1}{2} \gamma^u \). Because of the form of the spin connection (37), the dirac-algebra valued form \( \omega_{\mu}^{ab} \sigma_{ab} \) itself proportional to \( \gamma^u \); its nilpotency then guarantees that the spin-connection term in the covariant derivative (38) vanishes by itself:
\[
\gamma^a \omega_{\mu}^{bc} \sigma_{bc} = \gamma^u \omega_{\mu}^{bc} \sigma_{bc} = 0.
\] (41)

Hence the only vestige of curved space-time left in the dirac operator is the inverse vierbein in the contraction of dirac matrices and differential operators:
\[
\gamma \cdot D = \gamma^a \nabla_a = \gamma^\mu \partial_\mu
\]
\[
= i \begin{pmatrix}
\partial_u - (K - 1) \partial_v & -\sigma_1 \partial_x - \sigma_2 \partial_y \\
-\sigma_3 (-\partial_u + (K + 1) \partial_v) & -\sigma_2 (\partial_x + \sigma_2 \partial_y) \\
\sigma_1 \partial_x + \sigma_2 \partial_y & -\partial_u + (K - 1) \partial_v \\
+\sigma_3 (-\partial_u + (K + 1) \partial_v) & -\partial_x - \sigma_1 \partial_y
\end{pmatrix}.
\] (42)
Here we have introduced the following basis for the Dirac algebra:

\[
\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3; \quad \gamma^0 = \begin{pmatrix} i1 & 0 \\ 0 & -i1 \end{pmatrix},
\]

with the \( \sigma_k \) the standard Pauli matrices. The zero modes of this operator with the property that the energy-momentum tensor only has a non-zero \( T_{uu} \) component are flat spinor fields \( \psi(u) \) of the form

\[
\psi(u) = i\gamma^u \left( \begin{array}{c} \chi(u) \\ 0 \end{array} \right) = \left( \begin{array}{c} 1 \\ -\sigma_3 \\ -\sigma_3 \end{array} \right) \left( \begin{array}{c} \chi(u) \\ 0 \end{array} \right) = \left( \begin{array}{c} \chi(u) \\ -\sigma_3\chi(u) \end{array} \right),
\]

where \( \chi(u) \) is a 2-component (Pauli) spinor. Indeed, first of all spinors of this type are zero-modes of the Dirac operator:

\[
\gamma \cdot D\psi = 0.
\]

This follows by direct application of the expression (42) to the spinor (44), using the nilpotency of \( \gamma^u \). Moreover, with this property it also follows that the energy-momentum tensor takes the form

\[
T_{\mu \nu} = \frac{1}{8} \overline{\psi} (\gamma_\mu D_{\nu} + \gamma_\nu D_{\mu}) \psi = \frac{1}{4} \delta^u_\mu \delta^v_\nu \overline{\psi} \gamma_u \partial_u \psi.
\]

To see this, first note that the \( u \)-component of the covariant derivative \( D_\mu \) is the only one that does not vanish on \( \psi(u) \) in general. We then only have to check that in all remaining cases with \( \gamma_\mu \neq \gamma_u \) the spinor \( \psi(u) \) (44) gets multiplied by a Dirac matrix which can be factorized such as to have a right multiplicator of the form \( \gamma^u \). Again, as \( (\gamma^u)^2 \neq 0 \), \( T_{\mu \nu} \) necessarily is of the required form (46).

Finally we remark, that the upper- and lower component of the Pauli spinor \( \chi(u) \) in our conventions correspond to a negative and positive helicity state, respectively. Thus we find as solutions of the Dirac operator in the metric (4) two massless spinor states, corresponding to right-moving zero-modes of the Dirac operator with helicity \( \pm 1 \), respectively.

This solution is self-consistent as the only non-zero component of the energy-momentum tensor is

\[
T_{uu}(u) = -\frac{1}{2} \left[ \chi^1 \chi' \right](u),
\]

where the prime denotes a derivative w.r.t. \( u \), and the dagger on \( \chi \) indicates Hermitian conjugation of the 2-component spinor. It is then straightforward to solve the Einstein equation for \( K \) in the presence of the energy momentum distribution of the spinor field:

\[
K_{\text{spinor}}(u, x, y) = -\frac{2\pi G}{c^4} \left[ \chi^1 \chi' \right](u) (x^2 + y^2).
\]

Again, to this particular solution an arbitrary free gravitational wave can be added. It should be mentioned here, that consistency requires the spinors in the energy momentum tensor (46), (47) to be anti-commuting objects, i.e. if the spinor fields \( \chi(u) \) are expanded in a Fourier series of massless matter waves, the coefficients take values in an infinite-dimensional Grassmann algebra. Thus the expression can be given an operational meaning only in the context of quantum theory, by performing some averaging procedure. For example, if the spinors form a condensate such that the kinetic energy \( \Sigma = -\langle [\chi^1 \chi'] \rangle = \text{constant} > 0 \), then such a condensate would generate gravitational waves in which test-masses perform harmonic motion of the type (30), (31) with frequency

\[
\omega = \frac{\gamma}{c^2} \sqrt{2\pi G \Sigma}.
\]
4 Einstein-Maxwell waves

As the last example we consider coupled Einstein-Maxwell fields. We look for solutions of wave-type, using the metric (2). In the absence of masses and charges, the field equations are:

\[ R_{\mu\nu} = -\frac{8\pi\varepsilon_0 G}{c^2} \left( F_{\mu\lambda} F^{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F^2 \right), \quad D_\lambda F^{\lambda\mu} = 0. \] (50)

With the same metric (2), we also find the same expressions for the components of the connection (3), and the Riemann and Ricci curvature tensors (4), (5). Therefore the left-hand side of the Einstein eqn. (50) is fixed in terms of the potential \( K(u, y, z) \).

As concerns the Maxwell equations, the covariant derivative

\[ D_\lambda F^{\lambda\mu} = \partial_\lambda F^{\lambda\mu} + \Gamma_\lambda^{\lambda\nu} F^{\nu\mu} + \Gamma_\mu^{\lambda\nu} F^{\lambda\nu} \] (51)

reduces to the first term on the r.h.s., an ordinary four-divergence; this happens because in the last term the even connection is contracted with the odd field-strength tensor, whilst the middle term contains a trace over an upper and a lower index of the connection, which vanishes in our case.

Thus the Maxwell equation reduces to the same expression as in minkowski space-time, and it has the same wave solutions. We consider an elementary wave solution, which in terms of the co-ordinate system (2) is described by the vector potential

\[ A_\mu = (a \sin k(ct - z), 0, 0), \] (52)

with the light-cone components vanishing, and with \( a \) a constant transverse vector: \( a_z = 0 \). Of course, arbitrary solution can be constructed from the elementary waves (52) by linear superposition. With \( u = ct - z \) and \( \omega = kc \) the angular frequency of the wave, the electric and magnetic fields are

\[ E_k(u) = \omega a \cos ku, \quad B_k(u) = k \times a \cos ku. \] (53)

As usual for e.m. waves, \( |E_k(0)| = c|B_k(0)| \), and \( E_k \cdot B_k = 0 \). Indeed, the only non-zero components of the full field strength are

\[ F_{ui} = -F_{iu} = ka_i \cos ku, \quad i = (x, y), \] (54)

all others vanishing. It is now straightforward to compute the stress-energy tensor components of the electro-magnetic field, with the result

\[ T_{uu} = \varepsilon_0 c^2 k^2 a^2 \cos^2 ku, \] (55)

and all other components zero. The Einstein-Maxwell equations then reduce to

\[ \Delta_{\text{trans}} K = \frac{16\pi\varepsilon_0 G}{c^2} k^2 a^2 \cos^2 ku. \] (56)

This has the special solution

\[ K_{em} = \frac{4\pi\varepsilon_0 G}{c^2} k^2 a^2 \cos^2 ku \left( x^2 + y^2 \right) = \frac{4\pi\varepsilon_0 G}{c^2} E^2_k(u) \zeta \zeta. \] (57)

In view of the linearity of eq. (56), the general solution is a superposition of such special solutions and arbitrary free gravitational waves of the type (3):

\[ K(u, \zeta, \bar{\zeta}) = K_{em}(u, \zeta, \bar{\zeta}) + f(u, \zeta) + \bar{f}(u, \bar{\zeta}). \] (58)
Next we turn to the motion of a test particle with mass \( m \) and charge \( q \) in the background of these gravitational and electro-magnetic fields. These equations are modified to take into account the Lorentz force on the test charge:

\[ \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda = \frac{q}{m} F^\mu_{\nu} \dot{x}^\nu. \]  

(59)

With the only non-zero covariant components of \( F_{\mu\nu} \) given by eq.(54), there are no contravariant components in the lightcone direction \( u \). As a result the equation for \( u \) is not modified, and we again find

\[ \dot{u} = \gamma = \text{const.} \]  

(60)

This also follows, because the electro-magnetic forces do not change the proper-time hamiltonian:

\[ H = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \]

(61)

except that \( K(u, x, y) \) now is given by the modified expression (58). Therefore \( v \) is still a cyclic co-ordinate and equation (14) for \( \dot{v} \) again follows from the conservation of \( H \):

\[ \gamma \dot{v} + \gamma^2 K(u, x, y) = \dot{x}^2 + \dot{y}^2 + c^2. \]  

(62)

As a result we find in this case the same formal expressions for the solution of the equations of motion in the time-like and longitudinal directions:

\[ \dot{t} = \frac{dt}{d\tau} = \sqrt{\frac{1 - \gamma^2 K/c^2}{1 - v^2/c^2}}, \]

(63)

whilst

\[ h = K + \frac{1 - v^2/c^2}{(1 - v_z/c)^2} \]

(64)

is again a constant of motion. In both cases of course \( K \) now is the full solution (58).

Manifest changes in the equations of motion are obtained in the transverse directions:

\[ \ddot{\xi} = -\frac{\gamma^2}{2} \nabla_\xi K - \frac{q\gamma}{m} ka \cos ku, \]

(65)

where \( \xi = (x, y) \) is a transverse vector and \( \nabla_\xi \) is the gradient in the transverse plane. If we take for \( K \) the special solution (57), we find the conservation law

\[ \frac{4\pi \varepsilon_0 G}{c^2} k^2 a^2 \xi^2 \cos^2 ku + \frac{1 - v^2/c^2}{(1 - v_z/c)^2} = h = \text{const.} \]

(66)

Inserting the explicit form of \( u(\tau) = \gamma \tau \), eqs.(65) then take the form

\[ \ddot{\xi} = -\frac{4\pi \varepsilon_0 G}{c^2} \gamma^2 k^2 a^2 \cos^2(\gamma k \tau) \xi - \frac{q\gamma}{m} ka \cos(\gamma k \tau). \]

(67)

Equivalently, we can use \( u \) instead of \( \tau \) as the independent variable:

\[ \frac{d^2 \xi}{du^2} = -\frac{4\pi \varepsilon_0 G}{c^2} k^2 a^2 \cos^2(\gamma k u) \xi - \frac{q}{m\gamma} ka \cos ku. \]

(68)

Clearly, it is useful to decompose \( \xi \) into components parallel and orthogonal to the electric field \( E_k \), which in our choice of electro-magnetic gauge is the same as that of the vector potential \( a \):

\[ \xi = \xi_\parallel + \xi_\perp. \]

(69)
with
\[
\xi_\parallel = \frac{\xi \cdot a}{|a|^2}, \quad \xi_\perp = \frac{\xi \times a}{|a|}.
\] (70)

It follows that
\[
\frac{d^2 \xi_\parallel}{du^2} = -\frac{4\pi \varepsilon_0 G}{c^2} k^2 a^2 \cos^2(ka - qm\gamma k a \cos ku),
\] (71)
\[
\frac{d^2 \xi_\perp}{du^2} = -\frac{4\pi \varepsilon_0 G}{c^2} k^2 a^2 \cos^2(ka - qm\gamma k a \cos ku).
\] (72)

Transforming to the cosine of the double argument, the last equation can be seen to reduce to the standard Mathieu equation:
\[
\frac{d^2 \xi_\perp}{du^2} + 2\pi \varepsilon_0 G k^2 a^2 (1 + \cos 2ku) \xi_\perp = 0,
\] (72)

whilst the other equation becomes an inhomogeneous Mathieu equation, with the Lorentz force representing the inhomogeneous term:
\[
\frac{d^2 \xi_\parallel}{du^2} + 2\pi \varepsilon_0 G k^2 a^2 (1 + \cos 2ku) \xi_\parallel = -\frac{q m \gamma}{1 mc \gamma} E_0.
\] (73)

Obviously, one may try to find a particular solution to this equation by making an expansion in powers of \( \cos ku \). The general solution is a superposition of this special one plus the general solution of the Mathieu equation (72).

A special case is that of static crossed electric and magnetic fields, obtained in the limit \( k \to 0 \). Then the eqs. (72) and (73) reduce to ordinary homogeneous and inhomogeneous harmonic equations:
\[
\frac{d^2 \xi_\perp}{du^2} + 4\pi \varepsilon_0 G k^2 a^2 E_0^2 \xi_\perp = 0,
\] (74)
\[
\frac{d^2 \xi_\parallel}{du^2} + 4\pi \varepsilon_0 G k^2 a^2 E_0^2 \xi_\parallel = -\frac{q m \gamma}{1 mc \gamma} E_0.
\] (75)

The angular frequency of this harmonic motion is
\[
\omega = \sqrt{\frac{4\pi \varepsilon_0 G}{c^2}} E_0 = 0.29 \times 10^{-18} E_0 \text{ (V/m)}.
\] (75)

Clearly, the Lorentz force due to the constant electric field produces a constant proper-time acceleration of the test charge, but the harmonic gravitational component of the motion is very slow for practically realistic electric fields: periods of a year or less require a field strength of the order of \( 10^{10} \text{ V/m} \) or more.

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