Parton Transverse Momentum and Orbital Angular Momentum Distributions

Abha Rajan,1 Aurore Courtoy,2 Michael Engelhardt,3 and Simonetta Liuti4

1University of Virginia - Physics Department, 382 McCormick Rd., Charlottesville, Virginia 22904 - USA
2Universidad Autónoma de Aguascalientes, A. P. 101, Aguascalientes, Mexico
3New Mexico State University - Department of Physics, Box 30001 MSC 3D, Las Cruces, NM 88003 - USA
4University of Virginia - Physics Department, 382 McCormick Rd., Charlottesville, Virginia 22904 - USA

The quark orbital angular momentum component of proton spin, $L_q$, can be defined in QCD as the integral of a Wigner phase space distribution weighting the cross product of the quark’s transverse position and momentum. It can also be independently defined from the operator product expansion for the off-forward Compton amplitude in terms of a twist-three generalized parton distribution. We provide an explicit link between the two definitions, connecting them through their dependence on partonic intrinsic transverse momentum. Connecting the definitions provides the key for correlating direct experimental determinations of $L_q$, and evaluations through Lattice QCD calculations. The direct observation of quark orbital angular momentum does not require transverse spin polarization, but can occur using longitudinally polarized targets.

PACS numbers:

Orbital Angular Momentum (OAM), $L_{q,g}$, is generated inside the proton as a consequence of the quark and gluon transverse motion about the system’s center of momentum. It has been identified as a critical component in the resolution of the proton spin puzzle [1], since the seminal EMC experiments demonstrated that quark spin alone cannot account for the proton spin [2, 3]. Understanding OAM in the proton was the original motivation for introducing Generalized Parton Distributions (GPDs) in Refs. [4, 5], in that they provided a novel way of accessing angular momentum through a class of exclusive reactions including Deeply Virtual Compton Scattering (DVCS), Deeply Virtual Meson Production (DVMP), and related experiments. Through Ji’s sum rule [5], one can, in fact, relate the components of the Energy Momentum Tensor (EMT) known as the gravitomagnetic form factors, $A_{q,g}$ and $B_{q,g}$, to the quark and gluon total angular momenta, $J_{q,g}$. The pivotal observation made in [5] is that $A_{q,g}$ and $B_{q,g}$ correspond to $n = 2$ Mellin moments of GPDs which, in turn, define the matrix elements for DVCS. These important developments rendered total angular momentum a measurable quantity. Although the decomposition of $J_q$ into its spin and orbital components has proven difficult to define gauge invariantly, the orbital angular momentum of quarks is well defined through $J_q = L_q + S_q$. Even so, the direct observability of $L_q$ remains a challenging question: the framework defined so far does not tell us how to access the dynamics of quark orbital motion since $L_q$ is only obtained through the difference of the total angular momentum and spin components.

$L_q$ has more recently been associated with precise operators and structure functions, given within two alternative approaches. On one side, a dynamical picture of quark orbital motion was given in terms of a Generalized Transverse Momentum Distribution (GTMD), i.e., an unintegrated over transverse momentum GPD, in Refs. [6-8]. The GTMD-based definition of quark OAM is

$$I^\Gamma_q(x) = \int d^2 k_T \int d^2 b_T (b_T \times k_T) \beta W^\Gamma(x, k_T, b_T) \quad (1)$$

where $W^\Gamma$ is a Wigner distribution derived from the quark-quark off-forward correlator in a longitudinally polarized nucleon moving in the 3-direction.

$$\Phi^\Gamma_{\Lambda', \Lambda}(p'; p; z', z) = \langle p', \Lambda' | \bar{\psi}(z') \Gamma U \psi(z) | p, \Lambda \rangle \quad (2)$$

where $\Gamma$ denotes an arbitrary $\gamma$-matrix structure. $W^\Gamma$ is obtained by Fourier-transforming [2] for $\Gamma = \gamma^+$ from $z - z'$ to struck quark intrinsic momentum $k$, projecting onto $(z - z')^+ = 0$, as well as from (transverse) momentum

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1 Throughout this paper we consider zero skewness, i.e., the plus component of the momentum transfer vanishes, $\Delta^+ = 0$. Moreover, we omit writing explicitly the $Q^2$ dependence which is, however, present in all expressions.
transfer $\Delta_T$ to transverse position $b_T$. If one foregoes
the transformation to $b_T$, one can relate $L_q^T$ to the $k_T^2$
moment of the GTMD $F_{14}$ [8,10] for $\Delta_T \to 0$,

$$L_q^T(x) = -F_{14}^{(1)} = -\int d^2k_T \frac{k_T^2}{M^2} F_{14}(x, k_T^2, k_T \cdot \Delta_T, \Delta_T^2).$$  (3)

$F_{14}$ is a GTMD describing an unpolarized quark inside a
longitudinally polarized proton [10]. Finally, $\mathcal{U}$ in Eq. (2)
denotes the gauge link, i.e., the Wilson path-ordered
exponential connecting the coordinates $z$ and $z'$. We will
restrict the discussion in the present Letter to the case
of a straight gauge link, corresponding to what is known
as Ji’s decomposition of angular momentum [11], and defer
the analogous treatment of other relevant gauge link
structures to an expanded exposition.

In another approach [12–14], it was observed that
OAM is associated with a twist-three GPD, $G_2$. Similar
to the treatment of the forward case [15–17], one can
write the Mellin moments of $G_2$, which appears in the
parametrization of the off-forward amplitude, in terms of
both twist-two operators and (genuine) twist-three
operators. For the second moment, the genuine twist-three
contribution vanishes and one obtains, for $\Delta_T \to 0$,

$$\int_0^1 dx x G_2 = -\frac{1}{2} \int_0^1 dx x (H + E) + \frac{1}{2} \int_0^1 dx \tilde{H} = -J_q + S_q = -L_q^{(2)}$$  (4)

where only a straight gauge link structure applies in
such a relation involving only GPDs. This result can be
viewed as an extension of the Efremov-Leader-Teryaev
(ELT) sum rule [18], written for the polarized structure functions, to off-forward kinematics.

Notwithstanding these developments, two main problems
remain to be solved: 1) relating the two distinct
structures, one ($F_{14}$) appearing at twist two, and one
($G_2$) at twist three, both describing OAM within the
same gauge invariant framework; 2) singling out an
experimental measurement to access directly OAM, possibly
through the newly defined structures. In this Letter,
we provide a direct link between the $k_T^2$ moment of the
GTMD and the twist-three GPD describing OAM, eluci-
dating the underlying dependence on partonic intrinsic
transverse momentum and off-shellness. The GTMD-
based definition is calculable in Lattice QCD using the
techniques of Ref. [19]. On the other side, the twist-three
GPD-based definition can be measured directly in DVCS-
type experiments, through the azimuthal angle modula-
tions which are sensitive to twist-three GPDs in DVCS off
a longitudinally polarized target [20]; this is at variance
with the notion that transverse polarization, or proton
spin-flip processes are necessary to obtain information
on quark OAM.

Our central result is the following integral relations
connecting $F_{14}, G_2, \tilde{E}_{2T}, H, E$ and $\tilde{H}$ in the limit
$\Delta_T \to 0$, where $\tilde{E}_{2T}$ is a twist-three GPD in the classi-
fication of [10] related to the GPD $G_2$ in the classification of [13] by

$$\int dx x \tilde{E}_{2T} = - \int dx x (H + E + G_2),$$  (5)

(LIR)

$$F_{14}^{(1)} = -\int_0^1 dy \left( \tilde{E}_{2T} + H + E \right) \Rightarrow -L_q^{(2)} = \int_0^1 dx F_{14}^{(1)} = \int_0^1 dx x G_2$$  (6)

(EoM)

$$x(\tilde{E}_{2T} + H + E) = x \left[ (H + E) - \int_0^1 dy \left( (H + E) - \frac{1}{x} H + \int_0^1 \frac{dy}{y^2} \tilde{H} \right) + G^{(3)} = x(\tilde{E}_{2T} + H + E)^{WW} + G^{(3)} \right]$$  (7)

Eq. (6) is a Lorentz Invariance Relation (LIR), obtained
from the analysis of the most general Lorentz decompo-
sition of the quark-quark correlation function. It states
a remarkable equivalence between the GTMD and twist-
three GPD-based definitions.

Eq. (7), which integrates over $x$ to give (4), is an Equa-
tions of Motion (EoM) relation derived by applying the
QCD EoM for the quark fields to the correlation function.
Together with Eq. (9), it allows one to connect OAM
defined through a Wigner distribution, Eq. (3), to the
sum rule definition, Eq. (4). In Eq. (7), the superscript
“WW” denotes the Wandzura-Wilczek part, in analogy
to the derivation for the polarized structure functions $g_1$.
and $g_2$ [21]. On the other hand,

$$G^{(3)} = -\tilde{M} + x \int_x^1 \frac{dy}{y} \tilde{M}$$  \hspace{1cm} (8)

with $\tilde{M}$ given in Eq. (17) below, is a genuine twist-three term – a quark-gluon-quark correlation – whose contribution to angular momentum can be explicitly seen to vanish in the case treated here as a consequence of the underlying structure of Eq. (15).

We now sketch the derivation of Eqs. (9) [10], highlighting the role of quark $k_T$ and, thus, the off-shellness of partons in generating proton spin. The completely unintegrated off-forward quark-quark correlation function $W_{TT}^{A\Lambda}$, i.e., (half) the four-dimensional Fourier transform of (2) from $z-z'$ to $k$ [10] [15] [17] [22], can be parametrized [10] in terms of invariant functions $A_i$. On the other hand, its $k^-$ integral $\tilde{W}_{TT}^{A\Lambda} = \int dk^- W_{TT}^{A\Lambda}$ is parametrized by the GTMDs. This implies the following twist-two relations already given in [10], adapted to the straight gauge link, zero skewness case considered here,

$$\frac{k_T \cdot \Delta_T}{\Delta_T^2} F_{12} + F_{13} = 2 P^+ \int dk^- \left( \frac{k_T \cdot \Delta_T}{\Delta_T^2} A_5 + A_6 \right) - \frac{x P^2 - k \cdot P}{M^2} (A_8 + x A_9)$$  \hspace{1cm} (9)

$$F_{14} = 2 P^+ \int dk^- (A_8 + x A_9)$$  \hspace{1cm} (10)

which we supplement by the twist-three relation

$$\frac{k_T \cdot \Delta_T}{\Delta_T^2} F_{27} + F_{28} = 2 P^+ \int dk^- \left( \frac{k_T \cdot \Delta_T}{\Delta_T^2} A_5 + A_6 \right) + \frac{1}{M^2} \left( \frac{(k_T \cdot \Delta_T)^2}{\Delta_T^2} - k_T^2 \right) A_9 \right)$$  \hspace{1cm} (11)

Combining integrals over transverse $k_T$ of these relations, one arrives at the LIR

$$\frac{d}{dx} \int d^2 k_T \frac{k_T^2}{M^2} F_{14} = \tilde{E}_{2T} + H + E$$  \hspace{1cm} (12)

in the limit $\Delta_T \to 0$, having identified the GPD combinations $H + E$ and $\tilde{E}_{2T}$ resulting after $k_T$ integration of the GTMD combinations appearing in [9] and [11] [10]. Finally, integrating over $x$, one arrives at Eq. (6).

The EoM relation in Eq. (7) was obtained by considering [2] for $\Gamma = i\sigma^{++\gamma_5}$, $(i = 1, 2)$, and inserting the equation of motion for the quark operator (the symmetrized form serving to cancel the mass terms),

$$0 = \int \frac{dz^2 d^2 z}{(2\pi)^3} e^{ix\cdot z - ik_T \cdot z_T} \times \hspace{1cm} (13)$$

\( (p', \Lambda' | \bar{\psi}(z/2)(\Gamma U i\tilde{D} + i\tilde{D} \Gamma U)\psi(z/2) | p, \Lambda)'_{z^+ = 0}. \)

This yields the relation between $k_T$ integrated correlators

$$- x P^+ i \epsilon^{ij}_T \tilde{W}_{TT}^{ij} = \frac{\Delta_T}{2} \tilde{W}_{TT}^{\gamma^+ \gamma^-} - k_T^2 \epsilon^{ij}_T \tilde{W}_{TT}^{\gamma^+ \gamma^-} + \mathcal{M}_{TT}^{ij} \text{,}$$  \hspace{1cm} (14)

with the genuine twist-three quark-gluon-quark correlator (still denoting $\Gamma = i\sigma^{++\gamma_5}$),

$$\mathcal{M}^{ij}_{TT} = \frac{1}{4} \int \frac{dz^2 d^2 z}{(2\pi)^3} e^{ix\cdot z - ik_T \cdot z_T} \langle p', \Lambda' | \bar{\psi}(z/2) \left[ \left( \tilde{\phi} - ig A \right) U \right]_{z'^-, z'^+} + \Gamma U \left( \tilde{\phi} + ig A \right) | z/2 \rangle \psi(z/2) | p, \Lambda \rangle_{z^+ = 0} \hspace{1cm} (15)$$

The genuine twist-three term $\tilde{M}$ is given by

$$\tilde{M} = 2 M \frac{\Delta_T}{\Delta_T^2} \int d^2 k_T \left[ \mathcal{M}_{TT}^{ij} - \mathcal{M}_{TT}^{ij} \right] \hspace{1cm} (17)$$

The final expression defining the EoM relation in Eq. (7) is obtained by taking the derivative in $x$, and replacing the $k_T^2$ moment of $F_{14}$ with the expression from the LIR in Eq. (6).

It should be noted that the relations discussed here are perturbatively divergent and require consistent reg-
ularization/renormalization at each step. An interesting aspect, e.g., of the LIR \([6]\) is that it connects a GTMD which does not have a GPD limit, \(F_{14}\), to GPDs. To treat both sides on an equal footing implies utilizing a transverse momentum-dependent regularization and renormalization scheme, and thus interpreting the GPDs in terms of the underlying GTMDs of which they are the GPD limit. On the other hand, it seems tempting to speculate that relations of the type \([6]\) may ultimately be useful to connect the renormalization of quantities which are intrinsically defined as transverse momentum-dependent, such as \(F_{14}\), to the more standard schemes employed for GPDs.

As an application of the relations between the different ways to access angular momentum, we compile and correlate in Fig. 1 determinations of \(J_q\), \(L_q\) and \(S_q\) from several sources, including experiment, lattice QCD, and models. The value of \(J_{u-d} = J_u - J_d\) is plotted versus \(L_{u-d} = L_u - L_d\). The horizontal bands represent measurements/calculations of \(J_{u-d}\) using DVCS data \([23]\)/GPD evaluations; the slanted band is given by the relation \(J_q = L_q + \Delta S_q/2\), where the experimental value for \(\Delta S_{u-d}\) was taken from Ref. \([23]\). The vertical bands correspond to preliminary data for \(L_q\) obtained in a lattice QCD calculation at an artificially high pion mass of \(m_\pi = 518\) MeV using an approach related to the GTMD \(F_{14}\) from Eq. \([3]\) \([28]\), and to a calculation of \(\bar{E}_{2T}\) in the reggeized diquark model \([25, 26]\). The lattice result is expected to be enhanced by roughly 30% as one goes to the physical pion mass. The reggeized diquark model produces a parametrization of the GPDs \(H\) and \(E\), which is fitted to both the nucleon unpolarized PDFs for the \(u\) and \(d\) quarks, and to the flavor-separated nucleon electromagnetic form factors \([29]\). An independent experimental constraint on the normalization of the genuine twist-three part of \(\bar{E}_{2T}\) is obtained by using its third Mellin moment, which can be related to

\[
d_2 = 3 \int_0^1 dx x^2 g_2^{tw^3}(x) ,
\]

where \(g_2\), the transverse spin-dependent structure function, is obtained in double-spin asymmetry measurements of longitudinally polarized electrons scattering from longitudinally and transversely polarized nucleons. We used, in particular, the SLAC data for the \(u\) and \(d\) quark values of \(d_2\) at a common \(Q^2\) value of 5 GeV\(^2\) \([30]\). With the normalization of the twist-three part of \(\bar{E}_{2T}\) obtained from Eq. \([18]\) we then evaluated \(L_q\). The result is the vertical green band. This is consistent, although with a large error, with the values extracted from the lattice. No experimental determinations of \(L_q\) to corroborate our analysis can be placed on the graph at this point, although future extractions will be possible from analyses of the \(\sin 2\phi\) modulation of DVCS data \([20]\).

In Fig. 2 we exhibit in more detail the contributions in Eq. \([7]\) as a function of \(x\), i.e., the behavior of \(x(\bar{E}_{2T} + H + E)^{WW}\), the genuine twist-three term, and their sum at the initial scale of the model. As for \(g_2\), the genuine twist-three part is predicted to be large. Due to the Regge
behavior of the functions, we expect measurements at low $x$, i.e., in a regime which would be best accessible at an Electron Ion Collider to be important.

Finally, future developments will include the extension of our study to the Jaffe-Manohar \cite{1} decomposition of angular momentum, which, as shown in Ref. \cite{11}, involves a final state interaction (encoded in a staple-shaped gauge link), and is related to the Ji decomposition by

$$L_q^{IM} = L_q^{3J} + \langle \tau_3 \rangle$$ \hspace{1cm} (19)

where $\langle \tau_3 \rangle$ is an off-forward extension of a Qiu-Sterman term \cite{31}. $\langle \tau_4 \rangle$ has been interpreted physically as a change in OAM due to a torque - a final state interaction exerted on the outgoing quark by the color-magnetic field produced by the spectators \cite{11}.

In conclusion, understanding quark OAM entails cross-correlating phenomenology, theory and lattice QCD efforts to bring them to bear simultaneously on the subject. We provided relations that are key for realizing such a coordinated approach, utilizing directly non-local, $k_T$-unintegrated quark-quark correlation functions. This approach opens up an avenue to explore the role of partonic transverse momentum and off-shellness for OAM, while providing a formalism which connects to lattice QCD calculations on one side and to experiment on the other. A first, exploratory direct calculation of quark OAM in lattice QCD using an approach related to the GTMD $F_{1g}$ was incorporated into the analysis, and confronted with independent determinations, e.g., via Ji’s sum rule, and through $d_2$ measurements. Our relations bring to the fore the intricacies of connecting a twist-two GTMD moment and a twist-three GPD, before the backdrop of a field theoretic rendition of OAM.

Acknowledgments: Discussions with M. Burkardt, M. Diehl, G. Goldstein, A. Klein , S. Pate and the Theory Group at Jefferson Lab are gratefully acknowledged. This work was supported by the U.S. DOE and the Office of Nuclear Physics through grants DE-FG02-96ER40965 (M.E.), DE-AC05-06OR23177, and DE-FG02-01ER41200 (A.R. and S.L.).