A Schrödinger singular perturbation problem

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

Consider the equation $-\varepsilon^2 \Delta u_\varepsilon + q(x)u_\varepsilon = f(u_\varepsilon)$ in $\mathbb{R}^3$, $|u(\infty)| < \infty$, $\varepsilon = \text{const} > 0$. Under what assumptions on $q(x)$ and $f(u)$ can one prove that the solution $u_\varepsilon$ exists and $\lim_{\varepsilon \to 0} u_\varepsilon = u(x)$, where $u(x)$ solves the limiting problem $q(x)u = f(u)$? These are the questions discussed in the paper.

1 Introduction

Let

$$-\varepsilon^2 \Delta u_\varepsilon + q(x)u_\varepsilon = f(u_\varepsilon) \text{ in } \mathbb{R}^3, \quad |u_\varepsilon(\infty)| < \infty, \quad \varepsilon = \text{const} > 0,$$

$$f \text{ is a nonlinear smooth function, } q(x) \in C(\mathbb{R}^3) \text{ is a real-valued function}$$

$$a^2 \leq q(x), \quad a = \text{const} > 0. \quad (1.2)$$

We are interested in the following questions:

1) Under what assumptions does problem (1.1) have a solution?
2) When does $u_\varepsilon$ converge to $u$ as $\varepsilon \to 0$?

Here $u$ is a solution to

$$q(x)u = f(u). \quad (1.3)$$

The following is an answer to the first question.

**Theorem 1.1.** Assume $q \in C(\mathbb{R}^3)$, (1.2) holds, $f(0) \neq 0$, and $a$ is sufficiently large. More precisely, let $M(R) := \max_{|u| \leq R} |f(u)|$, $M_1(R) = \max_{|\xi| \leq R} |f'(\xi)|$, $p := q(x) - a^2$, and assume that $\frac{\|p\|}{a^2} \leq R$, and $\frac{\|p\| + M_1(R)}{a^2} \leq \gamma < 1$, where $\gamma > 0$ is a constant and $\|p\| := \sup_{x \in \mathbb{R}^3} |p(x)|$. Then equation (1.1) has a solution $u_\varepsilon \neq 0$, $u_\varepsilon \in C(\mathbb{R}^3)$, for any $\varepsilon > 0$.

Math subject classification: 35J60, 35B25
key words: nonlinear elliptic equations, singular perturbations
In Section 4 the potential $q$ is allowed to grow at infinity.

An answer to the second question is:

**Theorem 1.2.** If $\frac{f(u)}{u}$ is a monotone, growing function on the interval $[u_0, \infty)$, such that $\lim_{u \to \infty} \frac{f(u)}{u} = \infty$, and $\frac{f(u)}{u_0} < a^2$, where $u_0 > 0$ is a fixed number, then there is a solution $u_\varepsilon$ to (1.1) such that

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u(x),$$

(1.4)

where $u(x)$ solves (1.3).

Singular perturbation problems have been discussed in the literature [1], [3], [5], but our results are new.

In Section 2 proofs are given.

In Section 3 an alternative approach is proposed.

In Section 4 an extension of the results to a larger class of potentials is given.

## 2 Proofs

**Proof of Theorem 1.1.** The existence of a solution to (1.1) is proved by means of the contraction mapping principle.

Let $g$ be the Green function

$$(-\varepsilon^2 \Delta + a^2)g = \delta(x-y) \text{ in } \mathbb{R}^3, \quad g := g_a(x, y, \varepsilon) \quad \text{as } |x| \to \infty, \quad g = \frac{e^{-\frac{\varepsilon^2 |x-y|}{4\pi|x-y|\varepsilon^2}}}{4\pi|x-y|\varepsilon^2}. \quad (2.1)$$

Let $p := q - a^2 \geq 0$. Then (1.1) can be written as:

$$u_\varepsilon(x) = -\int_{\mathbb{R}^3} g p u_\varepsilon dy + \int_{\mathbb{R}^3} g f(u_\varepsilon) dy := T(u_\varepsilon). \quad (2.2)$$

Let $X = C(\mathbb{R}^3)$ be the Banach space of continuous and globally bounded functions with the sup-norm: $\|v\| := \sup_{x \in \mathbb{R}^3} |v(x)|$. Let $B_R := \{v : \|v\| \leq R\}$.

We choose $R$ such that

$$T(B_R) \subset B_R \quad (2.3)$$

and

$$\|T(v) - T(w)\| \leq \gamma \|v - w\|, \quad v, w \in B_R, \quad 0 < \gamma < 1. \quad (2.4)$$

If (2.3) and (2.4) hold, then the contraction mapping principle yields a unique solution $u_\varepsilon \in B_R$ to (2.2), and $u_\varepsilon$ solves problem (1.1).

The assumption $f(0) \neq 0$ guarantees that $u_\varepsilon \neq 0$.

Let us check (2.3). If $\|v\| \leq R$, then

$$\|T(v)\| \leq \|v\||p| \int_{\mathbb{R}^3} g(x, y) dy \leq \frac{M(R)}{a^2} \leq \frac{\|p\|R + M(R)}{a^2}, \quad (2.5)$$

3
where \( M(R) := \max_{|u| \leq R} |f(u)| \). Here we have used the following estimate:

\[
\int_{\mathbb{R}^3} g(x, y)dy = \int_{\mathbb{R}^3} \frac{e^{-\frac{1}{2}|x-y|}}{4\pi|x-y|} dy = \frac{1}{a^2}. \tag{2.6}
\]

If \( \|p\| < \infty \) and \( a \) is such that

\[
\frac{\|p\| R + M(R)}{a^2} \leq R, \tag{2.7}
\]

then (2.3) holds.

Let us check (2.4). Assume that \( v, w \in B_R, v - w : = z \). Then

\[
\|T(v) - T(w)\| \leq \frac{\|p\|}{a^2} \|z\| + \frac{M_1(R)}{a^2} \|z\|, \tag{2.8}
\]

where, by the Lagrange formula, \( M_1(R) = \max_{|\xi| \leq R} |f'(\xi)| \). If

\[
\frac{\|p\|}{a^2} + \frac{M_1(R)}{a^2} \leq \gamma < 1, \tag{2.9}
\]

then (2.4) holds. By the contraction mapping principle, (2.7) and (2.9) imply the existence and uniqueness of the solution \( u_\varepsilon(x) \) to (1.1) in \( B_R \) for any \( \varepsilon > 0 \).

Theorem 1.1 is proved.

Proof of Theorem 1.2. In the proof of Theorem 1.1 the parameters \( R \) and \( \gamma \) are independent of \( \varepsilon > 0 \). Let us denote by \( T_\varepsilon \) the operator defined in (2.2). Then

\[
\lim_{\varepsilon \to 0} \|T_\varepsilon(v) - T_0(v)\| = 0, \tag{2.10}
\]

for every \( v \in C(\mathbb{R}^3) \), where the limiting operator \( T_0 \), corresponding to the value \( \varepsilon = 0 \), is of the form:

\[
T_0(v) = \frac{-pv + f(v)}{a^2}. \tag{2.11}
\]

To calculate \( T_0(v) \) we have used the following formula

\[
\lim_{\varepsilon \to 0} g_a(x, y, \varepsilon) = \frac{1}{a^2} \delta(x - y), \tag{2.12}
\]

where convergence is understood in the following sense: for every \( h \in C(\mathbb{R}^3) \) one has:

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} g_a(x, y, \varepsilon) h(y)dy = \frac{h(x)}{a^2}, \quad a > 0. \tag{2.13}
\]

Indeed, one can easily check that

\[
\lim_{\varepsilon \to 0} \int_{|x-y| \geq c>0} g_a(x, y, \varepsilon)dy = 0, \quad \lim_{\varepsilon \to 0} \int_{|x-y| \leq c} g_a(x, y, \varepsilon)dy = \frac{1}{a^2}, \quad a > 0, \tag{2.14}
\]
where $c > 0$ is an arbitrary small constant. These two relations imply (2.13).

We claim that if (2.10) holds for every $v \in X$, and $\gamma$ in (2.4) does not depend on $\varepsilon$, then (1.4) holds, where $u$ solves the limiting equation (2.2):

$$u = T_0(u) = \frac{-pu + f(u)}{a^2}. \quad (2.15)$$

Equation (2.15) is equivalent to (1.3). The assumptions of Theorem 1.2 imply that equation (1.3) has a unique solution.

Let us now prove the above claim.

Let $u = T_\varepsilon(u)$, $u = u_\varepsilon$, $v = T_0(v)$, and $\lim_{\varepsilon \to 0} ||T_\varepsilon(w) - T_0(w)|| = 0$ for all $w \in X$. Assume that $||T_\varepsilon(v) - T_\varepsilon(w)|| \leq \gamma||v - w||$, $0 < \gamma < 1$, where the constant $\gamma$ does not depend on $\varepsilon$, so that $T_\varepsilon$ is a contraction map. Consider the iterative process $u_{n+1} = T_\varepsilon(u_n)$, $u_0 = v$. The usual estimate for the elements $u_n$ is: $||u_n - v|| \leq \frac{1}{1-\gamma}||T_\varepsilon v - v||$. Let $u = \lim_{n \to \infty} u_n$. This limit does exist because $T_\varepsilon$ is a contraction map. Taking $n \to \infty$, one gets $||u - v|| \leq \frac{1}{1-\gamma}||T_\varepsilon(v) - T_0(v)|| \to 0$ as $\varepsilon \to 0$. The claim is proved.

Theorem 1.2 is proved. \hfill \Box

**Remark 2.1.** Conditions of Theorem 1.1 and of Theorem 1.2 are satisfied if, for example, $q(x) = a^2 + 1 + \sin(\omega x)$, where $\omega = \text{const} > 0$, $f(u) = (u + 1)^m$, $m > 1$, or $f(u) = e^u$. If $R = 1$, and $f(u) = e^u$, then $M(R) = e$, $M_1(R) = e$, $||p|| \leq 2$, so $\frac{2+\varepsilon}{a^2} \leq 1$ and $\frac{2+\varepsilon}{a^2} \leq \gamma < 1$ provided that $a > \sqrt{5}$. For these $a$, the conditions of Theorem 1.1 are satisfied and there is a solution to problem (1.1) in the ball $B_\varepsilon$ for any $\varepsilon > 0$.

### 3 A different approach

Let us outline a different approach to problem (1.1). Set $x = \xi + \varepsilon y$. Then

$$-\Delta_y w_\varepsilon + a^2 w_\varepsilon + p(\varepsilon y + \xi) w_\varepsilon = f(w_\varepsilon), \quad |w_\varepsilon(\infty)| < \infty, \quad (3.1)$$

$w_\varepsilon := u_\varepsilon(\varepsilon y + \xi)$, $p := q(\varepsilon y + \xi) - a^2 \geq 0$. Thus

$$w_\varepsilon = -\int_{\mathbb{R}^3} G(x, y)p(\varepsilon y + \xi) w_\varepsilon dy + \int_{\mathbb{R}^3} G(x, y)f(w_\varepsilon)dy, \quad (3.2)$$

where

$$(-\Delta + a^2)G = \delta(x - y) \text{ in } \mathbb{R}^3, \quad G = \frac{e^{-a|x-y|}}{4\pi|x-y|}, \quad a > 0. \quad (3.3)$$

One has

$$\int_{\mathbb{R}^3} G(x, y)dy = \frac{1}{a^2}. \quad (3.4)$$

Using an argument similar to the one in the proofs of Theorem 1.1 and Theorem 1.2 one concludes that for any $\varepsilon > 0$ and any sufficiently large $a$, problem (3.1) has a unique
solution \( w_\varepsilon = w_\varepsilon(y, \xi) \), which tends to a limit \( w = w(y, \xi) \) as \( \varepsilon \to 0 \), where \( w \) solves the limiting problem
\[
-\Delta_y w + q(\xi)w = f(w), \quad |w(\infty, \xi)| < \infty. 
\]
(3.5)

Problem (3.5) has a solution \( w = w(\xi) \), which is independent of \( y \) and solves the equation
\[
q(\xi)w = f(w). 
\]
(3.6)

The solution to (3.5), bounded at infinity, is unique if \( a \) is sufficiently large. This is proved similarly to the proof of (2.9). Namely, let \( b^2 := q(\xi) \). Note that \( b \geq a \). If there are two solutions to (3.5), say \( w \) and \( v \), and if \( z := w - v \), then \( ||z|| \leq b^{-2}M_1(R)||z|| < ||z|| \), provided that \( b^{-2}M_1(R) < 1 \). Thus \( z = 0 \), and the uniqueness of the solution to (3.5) is proved under the assumption \( q(\xi) > M_1(R) \), where \( M_1(R) = \max_{|\xi| \leq R}|f'(\xi)| \).

Replacing \( \xi \) by \( x \) in (3.6), we obtain the solution found in Theorem 1.2.

4 Extension of the results to a larger class of potentials

Here a method for a study of problem (1.1) for a larger class of potentials \( q(x) \) is given.

We assume that \( q(x) \geq a^2 \) and can grow to infinity as \( |x| \to \infty \). Note that in Sections 1 and 2 the potential was assumed to be a bounded function. Let \( g_\varepsilon \) be the Green function
\[
-\varepsilon^2\Delta_x g_\varepsilon + q(x)g_\varepsilon = \delta(x - y) \quad \text{in } \mathbb{R}^3, \quad |g_\varepsilon(\infty, y)| < \infty. 
\]
(4.1)

As in Section 2, problem (1.1) is equivalent to
\[
u_\varepsilon = \int_{\mathbb{R}^3} g_\varepsilon(x, y)f(u_\varepsilon(y))dy, 
\]
(4.2)
and this equation has a unique solution in \( B_R \) if \( a^2 \) is sufficiently large. The proof, similar to the one given in Section 2, requires the estimate
\[
\int_{\mathbb{R}^3} g_\varepsilon(x, y)dy \leq \frac{1}{a^2}. 
\]
(4.3)

Let us prove inequality (4.3). Let \( G_j \) be the Green function satisfying equation (4.1) with \( q = q_j, j = 1, 2 \). Estimate (4.3) follows from the inequality
\[
G_1 \leq G_2 \quad \text{if } q_1 \geq q_2. 
\]
(4.4)

This inequality can be derived from the maximum principle.

If \( q_2 = a^2 \), then \( G_2 = \frac{e^{-\frac{|x-y|}{a\varepsilon}}}{4\pi|x-y|\varepsilon^2} \), and the inequality \( g_\varepsilon(x, y) \leq \frac{e^{-\frac{|x-y|}{a\varepsilon}}}{4\pi|x-y|\varepsilon^2} \) implies (4.3).

Let us prove the following relation:
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} g_\varepsilon(x, y)h(y)dy = \frac{h(x)}{q(x)} \quad \forall h \in \mathcal{C}^\infty(\mathbb{R}^3), 
\]
(4.5)
where $C^\infty$ is the set of $C^\infty(\mathbb{R}^3)$ functions vanishing at infinity together with their derivatives. This formula is an analog to (2.12).

To prove (4.5), multiply (4.1) by $h(y)$, integrate over $\mathbb{R}^3$ with respect to $y$, and then let $\varepsilon \to 0$. The result is (4.5). More detailed argument is given at the end of the paper.

Thus, Theorem 1.1 and Theorem 1.2 remain valid for $q(x) \geq a^2$, $a > 0$ sufficiently large, provided that $\frac{f(u)}{u}$ monotonically growing to infinity and $\frac{f(u_0)}{u_0} < a^2$ for some $u_0 > 0$. Under these assumptions the solution $u(x)$ to the limiting equation (1.3) is the limit of the solution to (4.2) as $\varepsilon \to 0$.

Let us give details of the proof of (4.5). Denote the integral on the left-hand side of (4.5) by $w = w_\varepsilon(x)$. From (4.1) it follows that

$$-\varepsilon^2 \Delta w_\varepsilon + q(x)w_\varepsilon = h(x).$$

Multiplying (4.6) by $w_\varepsilon$ and integrating by parts yields the estimate $\|w_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq c$, where $c > 0$ is a constant independent of $\varepsilon$. Consequently, one may assume that $w_\varepsilon$ converges weakly in $L^2(\mathbb{R}^3)$ to an element $w$. Multiplying (4.6) by an arbitrary function $\phi \in C^\infty_0(\mathbb{R}^3)$, integrating over $\mathbb{R}^3$, then integrating by parts the first term twice, and then taking $\varepsilon \to 0$, one obtains the relation:

$$\int_{\mathbb{R}^3} q(x)w(x)\phi(x)dx = \int_{\mathbb{R}^3} h(x)\phi(x)dx,$$

which holds for all $\phi \in C^\infty_0(\mathbb{R}^3)$. It follows from (4.7) that $qw = h$. This proves formula (4.5).

References

[1] Berger, M., Nonlinearity and functional analysis, Acad. Press, New York, 1977.

[2] Kantorovich, L., Akilov, G., Functional analysis, Pergamon Press, New York, 1982.

[3] Lomov, S., Introduction into the theory of singular perturbations, AMS, Providence RI, 1992.

[4] Ramm, A.G., Wave scattering by small bodies of arbitrary shapes, World Sci. Publishers, Singapore, 2005.

[5] Vishik, M., Lusternik, L., Regular degeneration and boundary layer for linear differential equations with small parameter, Uspekhi Mat. Nauk, 12, N5 (1957), 3-122.