An observation on the relation between the fine structure constant and the Gibbs phenomenon in Fourier analysis

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Abstract

A value given by a simple mathematical expression is proposed which is close to the fine structure constant given by 1998 CODATA recommended values of the fundamental physical constants up to relative accuracy $10^{-7}$. This expression relates closely with the value of the overshoot of the Gibbs phenomenon in Fourier analysis.

1 Introduction

The fine structure constant $\alpha$ is the most important dimensionless universal physical constant. Since it is dimensionless and universal, it is very interesting to know whether it can be expressed by a simple expression of universal mathematical constants. This may help understanding the nature.

As is known, the definition of fine structure constant $\alpha$ is

$$\alpha^{-1} = \frac{2ce_0h}{e^2} = \frac{2h}{c\mu_0\varepsilon_0}$$ (1)

where $c$ is the speed of light in vacuum, $\mu_0$ and $\varepsilon_0 = 1/c^2\mu_0$ are the permeability and permittivity of vacuum respectively, $e$ is the elementary charge and $h$ is the Planck constant.

Here we propose

$$\alpha_z^{-1} = \frac{1}{\sqrt{2}} \left(\frac{3\pi}{2}\right)^3 \int_0^{\pi} \frac{\sin x}{x} dx \approx 137.03598260.$$ (2)

This value coincides with the value

$$\alpha^{-1} = 137.03599976 \pm 0.00000050$$ (3)
given by the 1998 CODATA recommended values of the fundamental physical constants up to relative accuracy $1.26 \times 10^{-7}$.

In the expression (2), the integral $\text{Si}(\pi) = \int_0^{\pi} \frac{\sin x}{x} dx$ is related to a universal mathematical constant — the overshoot of the Gibbs phenomenon in Fourier analysis, which is the same to all the jump discontinuities of all piecewisely smooth functions. The expression $\alpha_z^{-1}/\text{Si}(\pi) = (3\pi/2)^3/\sqrt{2}$ is so simple that it seems that there may be physical essence behind the relation $\alpha_z \approx \alpha$, although it is not known presently.

In the next section, the universality of the constant $\text{Si}(\pi)$ in Fourier analysis is reviewed briefly. Then, the value $\alpha_z$ is compared with the values of $\alpha$ given by experiments and physical theories, which were used to determine $\alpha$ for the 1998 CODATA recommended values of the fundamental physical constants. At last, some discussions are presented.

2 A brief review of the Gibbs phenomenon

The Gibbs phenomenon is a universal phenomenon at all the jump discontinuities for Fourier series or Fourier transformations. First, let us look at a simple example. Let $f(x)$ be a square wave of period $2\pi$
with \( f(x) = 1 \) for \( 0 < x < \pi \) and \( f(x) = -1 \) for \( -\pi < x < 0 \). Let \( s_n \) be the partial sum of its Fourier series, i.e.

\[
s_n(x) = \frac{4}{\pi} \sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}.
\]

(4)

Then when \( n \to \infty \), there are overshoots and undershoots in the graph of \( s_n \) (Fig. 1).

The limit of the amplitude of \( s_n \) near 0 as \( n \to \infty \) is \( 2C_G \), which is \( C_G \) times the jump of \( f(x) \) at 0. Here

\[
C_G = \frac{2}{\pi} \text{Si}(\pi) = \frac{2}{\pi} \int_0^\pi \frac{\sin x}{x} \, dx \approx 1.17897974447217.
\]

(5)

This is the famous Gibbs phenomenon \([2, 3, 4, 5]\), which was discovered more than a century ago.

The Gibbs phenomenon appears not only in the square wave, but also at all jump discontinuities generally. For the Fourier series of any piecewisely smooth period function \( f(x) \), the limit of the amplitude of the partial sum at a jump discontinuity \( x_0 \) of \( f(x) \) equals to \( C_G \) times the jump of \( f(x) \) at that point. That is, for the partial sum \( s_n(x) \),

\[
\lim_{\delta \to 0^+} \lim_{n \to \infty} \text{osc}_{|x-x_0| \leq \delta} s_n(x) = C_G \lim_{\delta \to 0^+} \lim_{n \to \infty} \text{osc}_{|x-x_0| \leq \delta} s_n(x)
\]

(6)

where \( \text{osc} \) refers to the difference of the maximum and the minimum of a function. This fact is also true for the Fourier transformation of a non-periodic function, if the function is absolutely integrable and piecewisely smooth \([5]\). In all cases, the constant \( C_G \) is the same.

3 Comparison with experimental values

In Fig. 2, \( \alpha^{-1}_z \) given by \([3]\) was compared with the values of \( \alpha^{-1} \) given by the experimental data together with various physical theories. Each line segment in the figure represents a datum in Table XV of \([1]\). That table was used to determine the value \( \alpha \) for the 1998 CODATA recommended values of the fundamental physical constants. For each value \( x \) with uncertainty \( \sigma \), the line segment extends from \( x - \sigma \) to \( x + \sigma \) in Fig. 2. The identifications to the right of the line segments are the same as those in Table XV of \([1]\). The vertical line marked \( \alpha^{-1}_z \) represents the value given by \([3]\) and that marked \( \alpha^{-1} \) represents the value given by the 1998 CODATA recommended values of the fundamental physical constants.

4 Discussions

There are other theoretical values of \( \alpha \), such as those given by the string theory \([3, 7]\). For example, the simplest expression is \( \alpha^{-1} = \phi^{10-\phi^3} = 137.7880938 \) where \( \phi = (\sqrt{5} - 1)/2 \) is the golden mean. More accurate values with accuracy \( 3 \times 10^{-7} \) were also given by a complicated expansion \([3]\). Although the expression \( \alpha^{-1}_z \) in this paper is simply an observation, it is much simpler than the other theoretical results with similar accuracy.
There is a small possibility that $\alpha^{-1} \approx \alpha^{-1}$ is simply an accidental coincidence. Actually it is not difficult to construct complicated relations of $2, 3, \pi$ etc. to approximate $\alpha^{-1}$ because any real number can be approximated by a rational number to any accuracy. For example, $2^{-19/6} \frac{3157}{24} \pi^{-1/16} = 137.0360046$. (There is also a number $2^{19/3} - 7/4 \pi^{11/4} = 137.036082$ given by Wyler [6].) However, the probability is very small to get accidentally a simple relation like $\alpha^{-1}/\sin(\pi) = (3\pi/2)^{3/\sqrt{2}}$ in this paper. The internal relation between $\alpha$ and the fine structure constant is to be revealed.

References

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