Anisotropies of Gravitational Wave Backgrounds: A Line Of Sight Approach.

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In the weak field regime, gravitational waves can be considered as being made up of collisionless, relativistic tensor modes that travel along null geodesics of the perturbed background metric. We work in this geometric optics picture to calculate the anisotropies in gravitational wave backgrounds resulting from astrophysical and cosmological sources. Our formalism yields expressions for the angular power spectrum of the anisotropies. We show how the anisotropies are sourced by intrinsic, Doppler, Sachs-Wolfe, and Integrated Sachs-Wolfe terms in analogy with Cosmic Microwave Background photons.

Introduction. The first direct detection of gravitational waves [1] has heralded a new era of gravitational wave astronomy. Future improvements in ground [2] and space-based detectors [3, 4] and pulsar timing arrays promise to increase sensitivity to the level required to carry out detailed studies of stochastic and relic backgrounds of gravitational waves. Pulsar timings arrays, in particular, probe the longest baselines providing a window at frequencies not accessible to ground and space-based detector (see eg. [5–7]). Advanced, space-based interferometer detectors such as the proposed Big Bang Observer [8, 9] will target the same frequency window as ground based detectors but may reach the sensitivity required to detect the relic background left over from inflation.

Stochastic backgrounds of gravitational waves are made up of the superposition of astrophysical signals from unresolved sources. A number of different source mechanisms may result in stochastic backgrounds including the merger of compact objects, emission from cosmic string networks or phase transitions in the early universe (see eg. [10, 11]. Although these sources can be at cosmological distances we differentiate them from the relic, or cosmological, background due to inflation or another mechanism operating at much higher redshifts. The different backgrounds can be distinguished, in principle, by their different frequency scalings and statistical properties [12].

The energy flux of any gravitational wave background will not be constant across the sky. These anisotropies will contain information about the mechanism that generated the gravitational waves and about the nature of the spacetime along the line of propagation of the waves. If backgrounds will be detected in future it is interesting to consider what anisotropic signal we should expect. Various proposals have been made for how to map anisotropies in the backgrounds [13–16]. However, little progress has been made to calculate the expected anisotropy. This is distinct from the calculation of the anisotropy due to clustering of resolved sources [17] which may be of interest in luminosity distance constraints. An isotropic background has angular distance dependence but no luminosity distance dependence.

Although extremely challenging measurements of even just the largest angular scales of the anisotropies will provide interesting astrophysical and/or cosmological constraints. If anisotropies in the relic background were measured directly they would provide a unique window onto the Planckian epoch.

In this letter we introduce a new formalism for calculating gravitational wave background anisotropies. Our assumption will be that some future detectors will have the sensitivity to determine the energy flux in the gravitational wave background as a function of direction in the sky. We start by developing a Boltzmann equation for the perturbation in the distribution function of the tensor, or “graviton”, modes propagating the energy flux of the gravitational waves. This is in analogy with Cosmic Microwave Background (CMB) calculations. We extend this analogy further by employing the line-of-sight method to obtain a form for the angular power spectrum of the anisotropies that consist of a time integral of generic Legendre expanded source functions. We show how the formalism can be employed to calculate the signal from different generic emission mechanisms responsible for the backgrounds.

Our approach takes into account both source and line-of-sight effects and shows how gravitational waves could, in principle, be used as a non-electromagnetic probe of the universe over cosmological distances. This is very counter-intuitive since it suggests using tensor perturbations to probe scalar-dominated perturbations to the background metric. However the astonishing sensitivity being forecast in the field of direct detection of gravitational waves make this a concrete, albeit long-term, possibility.

Graviton distribution function. Gravitons, like photons, are assumed to be massless modes, or fluctuations, that make up the energy and momentum flux carried by the coherent oscillations known as gravitational or electromagnetic waves respectively. In analogy with photons we can treat of gravitons as propagating along null geodesics of the background spacetimes. We must be careful in making this assumption since gravitons, unlike photons, are a direct manifestation of the perturbation
of the spacetime in a non-linear theory. However, the shortwave formalism, developed over fifty years ago \[18\], shows that this is a good approximation in the weak field limit even when the curvature of the background is large. In this picture we can take a geometric optics approach to the propagation of gravitational waves by considering them as made up of a stream of massless, collisionless, gravitons following null geodesics. The null geodesics are determined by the perturbed background.

The energy flux carried by gravitational waves is conserved. In the geometric optics approach this is included by defining an adiabatic invariant such as the graviton number density, or phase-space distribution function. This is the starting point for our calculation, in direct analogy with that of CMB anisotropies \[19\].

At zeroth order in perturbations, the energy and the magnitude of its three momentum of a massless graviton are set by a single parameter \(p\) that is also proportional to the frequency \(\nu\). For gravitational waves emitted isotropically and homogeneously, with an energy spectrum \(dE/d\nu\), the distribution function will be a function of time \(t\) and frequency only

\[
f(\nu, t) = \frac{1}{\nu^3} \frac{dE}{d\nu}.
\]

The total energy density carried by a gravitational wave can be obtained by integrating the distribution function over the three momentum using the definition of the infinitesimal momentum volume element \(p^2dpd\Omega = \nu^2d\nu d\Omega\), where \(d\Omega\) is the momentum space infinitesimal angular element. For the isotropic case we then have

\[
\rho_{gw}(t) \equiv \int d\nu d\Omega \nu^3 f(\nu, t) = 4\pi \int d\nu \frac{dE}{d\nu}.
\]

The spectrum \(dE/d\nu\) is specific to the mechanism generating the gravitational waves (see eg. \[12\] for a review) and \(\rho_{gw}\) is related to the strain power measured by detectors, related to the square of the wave amplitude. Alternatively \(f(\nu, t)\) can be considered as proportional to the the specific intensity of the gravitational waves \[14\]. The polarization of the gravitational waves is of great interest but for simplicity we will assume the measurement is not polarization sensitive in this work.

**Anisotropies.** We now introduce anisotropies by allowing the distribution function to depend on the arrival direction \(\hat{n}\) with \(f = f(\nu, \hat{n}, t) = f(p, \hat{n}, t)\). The anisotropies are due to inhomogeneities in either the source mechanism or the propagation of the gravitational waves. We consider first order perturbations around a Friedmann-Robertson-Walker (FRW) metric with scale factor \(a(t)\) and coordinates \[20\] \(x^\mu = (ct, \mathbf{x})\). The metric in the Newtonian gauge, with scalar, first order perturbations, is given by:

\[
g_{00} = -(1 + 2\Psi), \quad g_{ij} = \delta_{ij}a^2(t)(1 + 2\Phi), \quad \text{and} \quad g_{0i} = g_{i0} = 0,
\]

with \(\Psi(x)\) and \(\Phi(x)\) the first order scalar potential and curvature perturbations. The four momentum of the gravitons is defined with respect to the affine parameter \(\lambda\) along the particle’s trajectory \(P^\mu = dx^\mu/d\lambda\). The energy of the massless graviton is now perturbed and we have:

\[
P^2 = -(1 + 2\Psi)(P^0)^2 + p^2 = 0,
\]

with three momentum magnitude \(p^2 = g_{ij}P^iP^j\). Then at first order in the perturbation we have:

\[
P^0 = P(1 - \Phi) \quad \text{and} \quad P^i = p\hat{p}^i(1 - \Phi)/a,
\]

where components \(p^i\) define the instantaneous unit vector for the propagation.

Liouville’s theorem states that \(df/d\lambda = 0\) in the absence of collisions and injection of modes. Adding collision and source operators, \(C[...]\) and \(J[...]\), on the right hand side of Liouville’s equation, we obtain a Boltzmann-type equation

\[
\frac{df}{d\lambda} = C[f(\lambda)] + J[f(\lambda)].
\]

In the case of gravitational waves the collisional term is not present. The emission term is present however and for astrophysical source it will be extended in time. This is distinct from the CMB case where collisions are present until last scattering and the injection is included simply as thermal initial conditions.

The left hand side of (3), given the perturbed metric, can be expanded using the perturbed geodesic equation and re-written in terms of physical time \(t\) \[21\].

For the right hand side of (3) we introduce a source term of the form \(df/dt = j(f)\) defined by an emissivity rate per comoving volume \(j(t)\).

The distribution function itself must be expanded in the perturbations. On the left hand side of the Boltzmann equation we consider perturbations in the energy of the modes via an expansion to first order in a dimensionless perturbation \[22\] \(\Gamma(\mathbf{x}, \hat{p}, t)\)

\[
f(p[1 + \Gamma]) \approx f(p) + p \frac{\partial f}{\partial p} \Gamma.
\]

To expand on the the emission side we introduce a perturbation due to the peculiar velocity \(v(x, t)\) of the emitter, with respect to the rest frame of the observer, and an inhomogeneity in the emission \(\Pi(x, t)\)

\[
f(p[1 + \hat{p}^i v_i + \Pi]) \approx f(p) + p \frac{\partial f}{\partial p} (\hat{p}^i v_i + \Pi).
\]

Notice that we chosen to introduce the inhomogeneity of emission as an inhomogeneous perturbation of the energy per mode rather than an inhomogeneity in the emissivity. Our choice simplifies the formalism and the two give equivalent perturbations to the energy density of the gravitational waves. We have also assumed that the emission mechanism is still isotropic apart from the Doppler-like term due to the peculiar velocity of the emitter.

Inserting these into (3) we obtain a zeroth order equation

\[
\frac{\partial f}{\partial t} - H p \frac{\partial f}{\partial p} = j f,
\]

where
where $H$ is the background Hubble rate. This describes the redshifting of the spectrum of the gravitational waves due to the expanding background and the growth of the monopole of the background due to any time dependent emission mechanism. In essence, integrating (6) determines $\rho_{gw}(\nu)$.

At first order, after rearranging and expanding in plane waves with wavevectors $k$ with $k \cdot p = k \mu$ and changing to conformal time $\eta$, we obtain a differential equation for the dimensionless perturbation

$$\dot{\Gamma} + (i k \mu + \dot{\sigma}) \Gamma = \dot{\sigma} (\dot{\rho}^i v_i + \Pi) + \dot{\Phi} + i k \mu \Psi , \quad (7)$$

where an over dot represents a derivative with respect to $\eta$ and we have introduced the conformal emissivity rate $\dot{\sigma} \equiv a \dot{\sigma}$. The perturbation does not depend on $p$ so the anisotropies will have the same frequency dependence as the monopole $\rho_{gw}(\nu)$.

**Streaming of gravitational waves.** Equation (7) describes the evolution of the anisotropy in the specific intensity of gravitational waves given their streaming along perturbed geodesics and the injection of waves with a given spectrum and rate. Its form is intentionally similar to the equivalent equation for CMB anisotropies.

We now use the line-of-sight integration method [23] to determine the anisotropy at our location today, $\eta = \eta_0$ by integrating (7). Just as with CMB calculations, we can make use of the fact that the directional dependence is determined purely by the inner product of the gravitational wave momentum vector $p$ (the line-of-sight) with the plane wavevector $k = k \hat{k}$. The $\mu$ dependence can be isolated through integration by parts and the perturbation can be Legendre expanded to obtain a multipole expansion of the anisotropy

$$\Gamma_\ell(k, \eta_0) = \int_{\eta}^{\eta_0} d\eta_j \, j_\ell |k(\eta_0 - \eta)| \, e^{-\Delta \sigma} \, S(k, \eta) . \quad (8)$$

Here $\eta_j$ is an initial time, $\Delta \sigma(\eta_j) \equiv \sigma(\eta_0) - \sigma(\eta_j)$, $j_\ell$ are spherical Bessel functions, and we have assumed that $\Gamma(\eta_j) \to 0$. The direction independent source function is given by

$$S(k, \eta) = \dot{\Phi} - \dot{\Psi} + \dot{\sigma} \big( \dot{\rho}^i v_i + \Pi - \Psi \big) . \quad (9)$$

This expression is the main result of this work. Each term in (9) is due to well understood physical effects with counterparts in the CMB source function. The first two terms and last term in the brackets are the Integrated Sachs-Wolfe (ISW) and Sachs-Wolfe (SW) effects respectively [24–26]. The remaining terms are a Doppler contribution due to the peculiar velocity of the emitter and an intrinsic contribution due to the inhomogeneous distribution of emitters. The emissivity rate $\dot{\sigma}$ defines an emission “depth” in analogy to the optical depth parameter $\tau$ for CMB photons.

The SW effect arises from the gravitational redshift caused by the local curvature at emission. This effect is somewhat ambiguous for the case of gravitational waves since there may be strong, non-linear effects from the dynamics involved in the emission mechanism but we may interpret it as the effect of the local curvature perturbation in the asymptotic spacetime at a certain distance from the source. In the following we will assume vanishing anisotropic stresses in the scalar perturbations by setting $\Psi = -\Phi$.

A gravitational wave transfer function can be defined by dividing the perturbation by the primordial, scalar curvature perturbation $\Delta^s_\ell(k, \eta_0) = \Gamma_\ell(k, \eta_0)^s/\Phi_0(k)$. It may seems unnatural to normalize the modes by the scalar amplitude but we have done this in anticipation that only in the case of a relic background would the primordial tensor amplitude appear in the emission contribution to the source function. The appearance of a primordial tensor amplitude can always be accounted for by using the primordial tensor-to-scalar ratio $r$.

By considering the spherical harmonic coefficients of the perturbation $a_{\ell m}^h(\eta_0)$ we can obtain an expression for the angular power spectrum of the gravitational wave background anisotropies

$$C_{\ell}^h = 2 \pi \int k^2 dk P_{\Phi}(k) |\Delta^h_\ell(k, \eta_0)|^2 , \quad (10)$$

where we have introduced the power spectrum of primordial curvature perturbations $k^3 P_{\Phi}(k) = A_s k^{n_s - 1}$.

**Anisotropies from compact object mergers.** For backgrounds arising from mergers of compact objects such as black hole collisions (BHBH) or black hole, neutron star (BHNS) collisions it is reasonable to assume that the perturbation to the density of sources is a biased tracer of the perturbation to the background matter with the

![FIG. 1. Angular power spectrum of anisotropies in $\rho_{gw}$ for merger and relic backgrounds and the CMB for standard inflationary primordial spectra. The merger model uses a simple merger rate peaking at redshift $z = 1$ to model the emission.](image-url)
form $\Pi(\vec{k}, \eta) = b(k, \eta)\delta_m(\vec{k}, \eta)$ where $b \sim O(1)$ is the linear bias parameter and $\delta_m$ is the dark matter density contrast. The statistics of the dark matter distribution is determined by the late-time matter power spectrum $\langle \delta_m(\vec{k}, \eta_0)\delta_m(\vec{k}', \eta_0) \rangle = (2\pi)^3\delta(\vec{k} - \vec{k}') P(k, \eta_0)$. 

The matter power spectrum can itself be related to the spectrum of primordial curvature perturbations via the Poisson equation $k^2 \Phi(\vec{k}, \eta) = 4\pi a^2 \delta_m \rho_m$. Thus when considering correlations in the anisotropy $\Gamma$ we should be able to relate these to the statistics of the underlying curvature perturbation, albeit via a heavily biased tracer in the case of merger sources. The Doppler term in (9) can also be related to the primordial curvature via linear perturbation theory but we shall omit it here for simplicity.

We can now use linear growth of structure to relate the primordial and late-time curvature perturbations using the Poisson equation $k^2 \Phi(\vec{k}, \eta) = 4\pi a^2 \delta_m \rho_m$. Thus when $k \gg a\delta_m \rho_m$, we can use linear growth of structure to relate the primordial and late-time curvature perturbations using the Poisson equation $k^2 \Phi(\vec{k}, \eta) = 4\pi a^2 \delta_m \rho_m$.

The matter power spectrum can itself be related to the spectrum of primordial curvature perturbations via the Poisson equation $k^2 \Phi(\vec{k}, \eta) = 4\pi a^2 \delta_m \rho_m$. Thus when considering correlations in the anisotropy $\Gamma$ we should be able to relate these to the statistics of the underlying curvature perturbation, albeit via a heavily biased tracer in the case of merger sources. The Doppler term in (9) can also be related to the primordial curvature via linear perturbation theory but we shall omit it here for simplicity.

The dominant term in (11) is the intrinsic one. This is determined by the late-time matter power spectrum $\langle \delta_m(\vec{k}, \eta_0)\delta_m(\vec{k}', \eta_0) \rangle = (2\pi)^3\delta(\vec{k} - \vec{k}') P(k, \eta_0)$. In this case contributions from the combined SW and ISW terms will be of comparable magnitude on large angular scales, if the ISW is present, as with the CMB case.

It is instructive to compare with the CMB photon transfer function on large angular scales, assuming instantaneous recombination at $\eta = \eta_s$.

\[
\Delta^\gamma(k, \eta_s) = \frac{3}{10}(\frac{k}{\eta_s})^2 - \frac{3}{5} \int_0^{\eta_s} \frac{\eta}{\eta_s} \frac{d\eta}{\eta_s} \exp(-\frac{\eta}{\eta_s}) d\eta.
\]

For an inflationary background the CMB and gravitational wave background anisotropies will be highly correlated. The relic anisotropies will have a larger amplitude than the CMB since gravitational waves “last scattered” long before recombination. The CMB also suffers from attenuation from rescattering. The anisotropies from low redshift mergers are orders of magnitude larger than either relic or CMB anisotropies but we should still expect a correlation on large scales from ISW and SW terms although this will be difficult to model precisely in practice.

In Fig. 1 we show the $C_\ell$ obtained from the expressions above for a standard $\Lambda$CDM model with power-law primordial spectra. For the merger background we use a very simple toy model for the merger rate that peaks at redshift $z = 1$ and use a bias factor $b = 3$. The relic background uses $r = 0.1$ and $n_t = 0.16$.

**Discussion.** We have introduced a line-of-sight approach for the calculation of anisotropies in gravitational wave backgrounds. The feasibility of future measurements of this kind remains unclear but it is important to quantify the information in these signals. Our result shows how anisotropies are induced by inhomogeneities at the source and along the line-of-sight. We have used our expressions to make some preliminary estimates of the anisotropies about the monopole in the backgrounds but further work is needed to model the signal accurately, particularly for the compact source case.

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