Non conformal gauge theories from D branes

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We use fractional and wrapped branes to describe perturbative and nonperturbative properties of the gauge theories living on their worldvolume.

1 Introduction

One of the most important ideas developed in recent years has been the one that goes under the name of gauge-gravity correspondence. According to it one can either use the low-energy dynamics of branes to study the properties of the gauge theory living on them or, if one knows the properties of the gauge theory living on a brane, one can deduce its low-energy dynamics. This idea is also at the basis of the Maldacena conjecture that, by using it, has established a complete equivalence between a gauge theory (\(N = 4\) super Yang-Mills) and a superstring (supergravity) theory (type IIB string theory compactified on \(AdS_5 \times S^5\)). In this paper we want to use the gauge-gravity correspondence for studying the properties of less supersymmetric and non-conformal gauge theories. We will not try to establish an exact duality between these gauge theories and some superstring theory as in the case of the Maldacena conjecture, but we will use classical supergravity solutions corresponding to fractional and wrapped branes having supersymmetric non-conformal gauge theories living on them in order to study their perturbative and nonperturbative properties. In particular we will use the expression of the gauge coupling constant and of the \(\theta\) angle in terms of the supergravity fields in order to compute them when a consistent classical solution is found.

For both wrapped D5 and fractional D3 branes of the orbifold \(C^2/Z_2\) the gauge coupling constant is given by:

\[
\frac{1}{g_{YM}^2} = \frac{\tau_5 (2\pi \alpha')^2}{2} \int C_2 e^{-\phi} \sqrt{\det (G_{AB} + B_{AB})}, \quad \tau_5 = \frac{1}{g_s \sqrt{\alpha' (2\pi \sqrt{\alpha'})^5}}. \tag{1}
\]

In the case of wrapped branes that we will consider in this paper we have to put \(B = 0\), while for fractional D3 branes, that for the sake of simplicity we take those of the \(Z_2\) orbifold having only one vanishing two cycle, we get:

\[
\frac{1}{g_{YM}^2} = \frac{\tau_5 (2\pi \alpha')^2}{2} \int C_2 e^{-\phi} B_2 = \frac{1}{4\pi g_s (2\pi \sqrt{\alpha'})^2} \int C_2 e^{-\phi} B_2. \tag{2}
\]

Finally the \(\theta\) angle both in the case of fractional D3 branes and wrapped D5 branes is given by:

\[
\theta_{YM} = \tau_5 (2\pi \alpha')^2 (2\pi)^2 \int C_2 + C_0 B_2. \tag{3}
\]
The paper is organized as follows. In the next section we will consider the case of fractional branes, while in Sect. 3 we will use wrapped branes for studying the properties of the gauge theory living on them.

2 Fractional branes

In this section we will consider fractional D3 and D7 branes of the orbifolds \( C^2/Z_2 \) and \( C^3/(Z_2 \times Z_2) \) in order to study the properties of respectively \( N = 2 \) and \( N = 1 \) supersymmetric gauge theories. The orbifold group acts on the directions \( x^4, \ldots, x^9 \) transverse to the worldvolume of the D3 brane where the gauge theory lives. In particular in the case of the first orbifold the nontrivial generator \( h \) of \( Z_2 \) acts as \( z_2, z_3 \to -z_2, -z_3 \) while in the case of the second orbifold the three nontrivial generators act as follows on the transverse coordinates:

\[
\begin{align*}
  h \times 1 & \Rightarrow z_1 \to z_1, \quad z_2 \to -z_2, \quad z_3 \to -z_3, \\
  1 \times h & \Rightarrow z_1 \to -z_1, \quad z_2 \to z_2, \quad z_3 \to -z_3, \\
  h \times h & \Rightarrow z_1 \to -z_1, \quad z_2 \to -z_2, \quad z_3 \to z_3.
\end{align*}
\]

They are both non compact orbifolds with respectively one and three fixed points at the origin corresponding to the point \( z_2, z_3 = 0 \) and to the three points \( z_1, z_2, z_3 = 0 \) and \( z_2, z_3 = 0 \). Each fixed point corresponds to a vanishing 2-cycle. Fractional Dp branes are D(p+2) branes wrapped on the vanishing two-cycle and therefore are, unlike bulk branes, stuck at the orbifold fixed point. By considering \( N \) fractional D3 and \( M (2M) \) fractional D7 branes of the two previous orbifolds we are able to study \( N = 2 \) \((N = 1)\) super QCD with \( M \) hypermultiplets. In order to do that we need to determine the classical solution corresponding to the previous brane configuration. For the case of the orbifold \( C^2/Z_2 \) the complete classical solution has been found in [1] \(^2\). In the following we write it explicitly for a system of \( N \) D3 fractional branes with worldvolume along the directions \( x^5, x^1, x^2 \), and \( x^3 \) and \( M \) D7 fractional branes containing the D3 branes and having the remaining four worldvolume directions along the orbifolded ones. The metric, the 5-form field strenght, the axion and the dilaton are given by \(^3\):

\[
ds^2 = H^{-1/2} g_{\alpha \beta} dx^\alpha dx^\beta + H^{1/2} \left( \delta_{in} dx^i dx^m + e^{-\phi} \delta_{ij} dx^i dx^j \right),
\]

\[
\tilde{F}^{(5)} = d \left( H^{-1} dx^0 \wedge \ldots \wedge dx^3 \right) \wedge d \left( H^{-1} dx^0 \wedge \ldots \wedge dx^3 \right),
\]

\[
\tau \equiv C_0 + i e^{-\phi} = i \left( 1 - \frac{N g_s}{2 \pi} \log \frac{z}{\rho_s} \right), \quad z \equiv x^4 + i x^5 = \rho e^{i \theta}
\]

where the warp factor \( H \) is a function of all coordinates that are transverse to the D3 brane \((x^4, \ldots, x^9)\). The twisted fields are instead given by \( B_2 = \omega_2 b \), \( C_2 = \omega_2 c \) where \( \omega_2 \) is the volume form corresponding to the vanishing 2-cycle and

\[
b e^{-\phi} = \frac{(2\pi \sqrt{\alpha'})^2}{2} \left[ 1 + \frac{2N - M}{\pi} g_s \log \frac{\rho}{\rho_s} \right], \quad c + C_0 b = -2\pi \alpha' g_s (2N - M).
\]

It can be seen that the previous solution has a naked singularity of the repulson type at short distances. But, on the other hand, if we probe it with a brane probe approaching the stack of branes corresponding to the classical solution from infinity, it can also be seen that the tension of the probe vanishes at a certain distance

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\(^1\) We denote \( z_1 = x^4 + i x^5 \), \( z_2 = x^6 + i x^7 \) and \( z_3 = x^8 + i x^9 \)

\(^2\) See also [2–5] and [6] for a review on fractional branes.

\(^3\) We denote with \( \alpha \) and \( \beta \) the four directions corresponding to the worldvolume of the fractional D3 brane, with \( \ell \) and \( m \) those along the four orbifolded directions \( x^5, x^1, x^2 \) and \( x^3 \) and with \( i \) and \( j \) the directions \( x^5 \) and \( x^7 \) that are transverse to both the D3 and the D7 branes.
from the stack of branes that is larger than that of the naked singularity. The point where the probe brane becomes tensionless is called in the literature enhancer [7] and at this point the classical solution cannot be used anymore to describe the stack of fractional branes.

Inserting in eqs. (2) and (3) the classical solution we get the gauge coupling constant and the \( \theta \) angle [1]:

\[
\frac{1}{g_{YM}^2} = \frac{1}{8\pi g_s} + \frac{2N - M}{8\pi^2} \log \frac{\rho}{\epsilon}, \quad \theta_{YM} = - \theta(2N - M).
\] (9)

Actually in the case of an \( \mathcal{N} = 2 \) supersymmetric theory one gets in the gauge multiplet also a complex scalar field \( \Psi \). This means that, when we derive the Yang-Mills action from the Born-Infeld action we also get a contribution from the kinetic terms of the brane coordinates \( x^4 \) and \( x^5 \) that are transverse to the brane and transverse to the orbifolded ones. This implies that the complex scalar field of the gauge supermultiplet is related to the coordinate \( z \) of supergravity through the following gauge-gravity relation

\[
\Psi \sim z^{2\pi\alpha'/g_s}. 
\]

This is a relation between a quantity of the gauge theory living on the fractional D3 branes and the coordinate \( z \) of supergravity. This identification allows one to obtain the gauge theory anomalies from the supergravity background. In fact, since we know how the anomalous scale and \( U(1) \) transformations act on \( \Psi \), from the previous gauge-gravity relation we can deduce how they act on \( z \), namely

\[
\Psi \to s e^{2i\alpha} \Psi \iff z \to s e^{2i\alpha} z \Rightarrow \rho \to s \rho, \quad \theta \to \theta + 2\alpha.
\] (10)

Those transformations do not leave unchanged the supergravity background in eqs. (8) and, as a consequence, they generate the anomalies of the gauge theory living on the fractional D3 branes. Acting with those transformations on eqs. (9) we get:

\[
\frac{1}{g_{YM}^2} \to \frac{1}{g_{YM}^2} + \frac{2N - M}{8\pi^2} \log s, \quad \theta_{YM} \to \theta_{YM} - 2\alpha(2N - M).
\] (11)

The first equation implies that the \( \beta \)-function of \( \mathcal{N} = 2 \) super QCD with \( M \) hypermultiplets is given by:

\[
\beta(g_{YM}) = - \frac{2N - M}{16\pi^2} g_{YM}^3
\] (12)

while the second one reproduces the chiral \( U(1) \) anomaly [8,9]. In particular, if we choose \( \alpha = \frac{2\pi}{2(2N - M)} \), then \( \theta_{YM} \) is shifted by a multiple of \( 2\pi \). Since \( \theta_{YM} \) is periodic of \( 2\pi \), this means that the subgroup \( \mathbb{Z}_{2(2N - M)} \) is not anomalous in perfect agreement with gauge theory results.

Using eqs. (9) it is easy to compute the combination:

\[
\tau_{YM} = \frac{\theta_{YM}}{2\pi} + i \frac{4\pi}{g_{YM}^2} = i \frac{2N - M}{2\pi} \log \frac{z}{\rho_e}, \quad \rho_e = \epsilon e^{\pi/(2N - M) g_s}.
\] (13)

where \( \rho_e \) is called in the literature the enhancer radius and corresponds in the gauge theory to the dimensional scale \( \Lambda \) generated by dimensional transmutation. Eq. (13) reproduces the perturbative moduli space of \( \mathcal{N} = 2 \) super QCD, but not the instanton corrections. This corresponds to the fact that the classical solution is reliable for large distances in supergravity corresponding to short distances in the gauge theory, while it cannot be used below the enhancer radius where nonperturbative physics is expected to show up. This means that in order to study nonperturbative effects in the gauge theory we need to find a classical solution free from enhancers and naked singularities. This will be done in the next section. Before doing that let us first extend the previous results to \( \mathcal{N} = 1 \) super QCD that can be obtained as a particular case of the general one studied in [10]. In this case only the asymptotic behaviour for large distances of the classical solution has been explicitly obtained and this is sufficient for computing the gauge coupling constant and the \( \theta \) angle of \( \mathcal{N} = 1 \) super QCD. As explained in [10], together with \( N \) fractional D3 branes of the same
type, one must also consider two kinds of $M$ fractional D7 branes in order to avoid gauge anomalies and one gets the following expressions for the gauge coupling constant and the $\theta$ angle ($z_i = \rho_i e^{i \theta_i}$) [9–11]:

\[
\frac{1}{g^2_{YM}} = \frac{1}{16 \pi g_s} + \frac{1}{8 \pi^2} \left( N \sum_{i=1}^{M} \log \frac{\rho_i}{\epsilon} - M \log \frac{\rho_1}{\epsilon} \right), \quad \theta_{YM} = -N \sum_{i=1}^{3} \theta_i + M \theta_1. \tag{14}
\]

As explained in [11] the anomalous scale and $U(1)$ transformations act on $z_i$ as $z_i \to s e^{2 \alpha/3} z_i$. This implies that the gauge parameters are transformed as follows:

\[
\frac{1}{g^2_{YM}} \to \frac{1}{g^2_{YM}} + \frac{3N - M}{8 \pi^2} \log s, \quad \theta_{YM} \to \theta_{YM} - 2\alpha(N - M/3) \tag{15}
\]

that reproduce the anomalies of $\mathcal{N} = 1$ super QCD. The differences between the anomalies in the $\mathcal{N} = 2$ (eq. (11)) and $\mathcal{N} = 1$ (eq. (15)) super QCD can be easily understood in terms of the different structure of the two orbifold considered. If we consider the two gauge coupling constants there is a factor $\frac{3}{2}$ between the contributions coming from the pure gauge part, while the contribution of the matter is the same. The factor 3 is a consequence of the fact that the orbifold $C^4/(Z_2 \times Z_2)$ has three sectors, while the factor $\frac{1}{2}$ follows from an additional factor $\frac{1}{2}$ in the orbifold projection for the orbifold $C^3/(z_2 \times Z_2)$ with respect to the orbifold $C^2/Z_2$. This explains the factor $\frac{3}{2}$ in the gauge field contribution to the $\beta$-function. The matter part is the same because in the orbifold $C^2/Z_2$ we have only one kind of fractional branes, while in the other orbifold, in order to cancel the gauge anomaly [10], we need two kinds of fractional branes. This factor 2 cancels the factor $\frac{1}{2}$ coming from the orbifold projection. Similar considerations can also be used to relate the two chiral anomalies.

In conclusion, by using the fractional branes we have reproduced the one-loop perturbative behaviour of both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super QCD, but, because of the enhançon and naked singularities we are not able to enter the nonperturbative region in the gauge theory corresponding to short distances in supergravity. In order to do this we must find a classical solution free of singularities. That is why in the next section we turn to wrapped branes.

### 3 Running coupling constant from wrapped branes

In this section we turn to the case of wrapped branes and in particular we will focus on a D5 brane wrapped on $S^2$ whose corresponding solution, found in [12] in four dimensions, was rinterpreted as a ten dimensional one corresponding to a wrapped D5 brane and used in [13] for describing $\mathcal{N} = 1$ super Yang-Mills. A more detailed and pedagogical derivation of the classical solution is presented in [14] where the classical solution was used for determining the running coupling constant of $\mathcal{N} = 1$ super Yang-Mills as a function of the renormalization group scale $\mu$. In particular, inserting the classical solution in eq. (1), one can determine how the gauge coupling constant depends on the distance from the branes. One gets:

\[
\frac{4 \pi^2}{N g^2_{YM}} = F(\rho). \tag{16}
\]

But in order to determine the behaviour of the gauge coupling constant as a function of the renormalization scale $\mu$ one must also give a relation between $\rho$ and $\mu$. This was obtained in [14] by connecting a certain function of $\rho$, called in [14] $a(\rho)$, to the gaugino condensate following the suggestion of [15]. The result was:

\[
a(\rho) = \frac{2 \rho}{\sinh 2 \rho} = \frac{\Lambda^3}{\mu^3}. \tag{17}
\]

The running coupling constant is determined once we fix the function $F(\rho)$ that depends on the two-cycle on which we wrap the 5 brane. On the other hand it is important to stress that the gauge coupling constant
depends on the renormalization scheme chosen and therefore two different choices of the two-cycle can be interpreted to correspond to two different renormalization schemes. In [14] the brane was wrapped on the \( S^2 \) spanned by the coordinates \( \theta \) and \( \tilde{\phi} \) having chosen the other coordinates \( \psi, \theta' \) and \( \phi \) at constant values \(^4\). This choice gave the following result:

\[
F(\rho) = \frac{1}{4} E \left( \frac{Y(\rho) - 1}{Y(\rho)} \right), \quad Y(\rho) = 4\rho \coth 2\rho - 1
\]  

(18)

where \( E \) is the elliptic integral and \( F \) behaves as \( \rho \) for large values of \( \rho \). In [14], by considering only the leading asymptotic behaviour of eq. (18) and by combining it with eq. (17), it was derived that the \( \beta \)-function of \( \mathcal{N} = 1 \) super Yang-Mills was exactly the NSVZ \( \beta \)-function [16] plus non perturbative corrections due to fractional instantons \(^5\). This result was questioned in [18] where it was shown that, if one also includes the first non leading logarithmic correction, one gets an extra contribution to the \( \beta \)-function that modifies the one derived in [16] already at two-loop level. Then, in order to recover the correct two-loop behaviour, it was suggested in [18] to add in eq. (17) an extra function \( f(g_{\text{YM}}) \) of the coupling constant that can be fixed by requiring agreement with the correct two-loop result. Of course it turns out that \( f(g_{\text{YM}}) \) must be singular at \( g_{\text{YM}} \sim 0 \) as the transformation that is needed in going from the holomorphic to the wilsonian \( \beta \)-function [19]. But, if we are prepared to recover the correct two-loop behaviour by simply changing the renormalization scheme as was implicitly done in [14], this way of thinking eliminates a problem that seems to appear if we perform a gauge transformation on the non-abelian gauge field of gauged supergravity. In fact, if one performs a gauge transformation in such a way that the gauge field is vanishing in the deep infrared (\( \rho = 0 \)), one gets a function \( F(\rho) \) that is different from the one in eq. (16). One gets [20]

\[
F(\rho) = e^{2h} + \frac{1}{4} (a - 1)^2 = \rho \tanh \rho
\]  

(19)

that, when put in eq. (16), gives a Landau-pole singularity at \( \mu = \Lambda \) unlike the function in eq. (18) that gave a smooth behaviour at \( \rho = 0 \). This is, however, not a problem if one also interprets the gauge transformation in supergravity as a change of renormalization scheme in the gauge theory.

A natural and elegant way to get directly the SNVZ \( \beta \)-function without having to change the renormalization scheme as was implicitly done in [14], is presented in [21] and is based on the proposal of choosing the same cycle used in [14] if one uses the solution after having performed the previous discussed gauge transformation or equivalently use the original solution and integrate on any of the two following cycles:

\[
\tilde{\theta} = -\theta, \tilde{\phi} = -\phi, \psi = 0 \quad \text{or} \quad \tilde{\theta} = \theta, \tilde{\phi} = -\phi, \psi = \pi.
\]

In both cases one gets precisely the expression in eq. (19) [20, 21]. This means that the definition of the two-cycle depends on which gauge we use for the gauge field of gauged supergravity and if one takes into account these changes one gets always the same result for the gauge coupling constant of the gauge theory living on the wrapped brane.

In conclusion if one follows the proposal of [21] the two equations that determine the running gauge coupling constant of \( \mathcal{N} = 1 \) super Yang-Mills as a function of the renormalization scale \( \mu \) are the following:

\[
\frac{4\pi^2}{Ng_{\text{YM}}^2} = \rho \tanh \rho; \quad \frac{2\rho}{\sinh 2\rho} = \frac{\Lambda^3}{\mu^3}.
\]  

(20)

It is easy to check that they imply the NSVZ \( \beta \)-function plus corrections due to fractional instantons. In fact from the previous two equations after some simple calculation one gets:

\[
\frac{\partial g_{\text{YM}}}{\partial \log (\mu/\Lambda)} \equiv \beta(g_{\text{YM}}) = -\frac{3Ng_{\text{YM}}^3}{16\pi^2} \left[ \frac{1 + 2\rho/\sinh 2\rho}{\coth^2 \rho - (Ng_{\text{YM}}^2)/(8\pi^2)} - 1/(2\sinh^2 \rho) \right].
\]  

(21)

\(^4\) We use the notation of [14].

\(^5\) An extension to the noncommutative case was done in [17].
This equation is exact and should be used together with the first equation in (20) in order to get the $\beta$-function as a function of $g_{YM}$. It does not seem possible, however, to trade $\rho$ with $g_{YM}$ in an analytic way. It can be done in the ultraviolet where, from the first equation in (20), it can be seen that $\rho$ can be approximated with

$$\rho = \frac{4\pi^2}{Ng_{YM}^2} \coth \frac{4\pi^2}{Ng_{YM}^2}$$

obtaining the following $\beta$-function:

$$\beta(g_{YM}) = -\frac{3Ng_{YM}^3}{16\pi^2} \frac{1 + (4\pi^2)/(Ng_{YM}^2) \sinh^{-2}((4\pi^2)/(Ng_{YM}^2))}{\coth^2((4\pi^2)/(Ng_{YM}^2)) - (Ng_{YM}^2)/(8\pi^2) - \frac{1}{2} \sinh^{-2}((4\pi^2)/(Ng_{YM}^2))}$$

(22)

that is equal to the NSVZ $\beta$-function plus nonperturbative corrections due to fractional instantons.

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