A NOTE ON THE DIFFUSIVITY OF FINITE-RANGE ASYMMETRIC EXCLUSION PROCESSES ON $\mathbb{Z}$

JEREMY QUASTEL 1 AND BENEDEK VALKÓ 1,2

Abstract. The diffusivity $D(t)$ of finite-range asymmetric exclusion processes on $\mathbb{Z}$ with non-zero drift is expected to be of order $t^{1/3}$. Seppäläinen and Balázs recently proved this conjecture for the nearest neighbor case. We extend their results to general finite range exclusion by proving that the Laplace transform of the diffusivity is of the conjectured order. We also obtain a pointwise upper bound for $D(t)$ of the correct order.

1. Introduction

A finite-range exclusion process on the integer lattice $\mathbb{Z}$ is a system of continuous time, rate one random walks with finite-range jump law $p(\cdot)$, i.e. $p(z) \geq 0$, and $p(z) = 0$ for $|z| > R$ for some $R < \infty$, $\sum_z p(z) = 1$, interacting via exclusion: Attempted jumps to occupied sites are suppressed. We consider asymmetric exclusion process (AEP) with non-zero drift,

$$\sum_z z p(z) = b \neq 0.$$ (1.1)

The state space of the process is $\{0,1\}^\mathbb{Z}$. Particle configurations are denoted by $\eta$, with $\eta_x \in \{0,1\}$ indicating the absence, or presence, of a particle at $x \in \mathbb{Z}$. The infinitesimal generator of the process is given by

$$L f(\eta) = \sum_{x,z \in \mathbb{Z}} p(z) \eta_x (1 - \eta_{x+z})(f(\eta^{x,x+z}) - f(\eta))$$ (1.2)

where $\eta^{x,y}$ denotes the configuration obtained from $\eta$ by interchanging the occupation variables at $x$ and $y$.

Bernoulli product measures $\pi_\rho$, $\rho \in [0,1]$, with $\pi_\rho(\eta_x = 1) = \rho$ form a one-parameter family of invariant measures for the process. The process starting from $\pi_0$ and $\pi_1$ are trivial and so we consider the stationary process obtained by starting with $\pi_\rho$ for some $\rho \in (0,1)$. Although the fixed time marginals of this stationary process is easy to understand (since the $\eta$'s are just independent Bernoulli$(\rho)$ random variables), there are still lots of open questions about the full (space-time) distribution. Information about this process (and about the appropriate scaling limit) would be very valuable to understand such elusive objects as the Stochastic Burgers and the Kardar-Parisi-Zhang equations (see [QV] for a more detailed discussion).

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We consider the two-point function,
\[ S(x,t) = E[(\eta_x(t) - \rho)(\eta_0(0) - \rho)], \]  
where the expectation is with respect to the stationary process obtained by starting from one of the invariant measures \( \pi_\rho \). \( S(x,t) \) satisfies the sum rules (see [PS])
\[ \sum_x S(x,t) = \rho(1 - \rho) = \chi, \quad \frac{1}{\chi} \sum_x xS(x,t) = (1 - 2\rho)bt. \]  
The diffusivity \( D(t) \) is defined as
\[ D(t) = (\chi t)^{-1} \sum_{x \in \mathbb{Z}} (x - (1 - 2\rho)bt)^2 S(x,t). \]  
Using scaling arguments one conjectures [S],
\[ S(x,t) \simeq t^{-2/3} \Phi(t^{-2/3}(x - (1 - 2\rho)bt)) \]  
for some scaling function \( \Phi \), as \( t \to \infty \). A reduced conjecture is that
\[ D(t) \simeq Ct^{1/3}, \]  
as \( t \to \infty \). Note that this means that the process has a superdiffusive behavior, as the usual diffusive scaling would lead to \( D(t) \to D \). It is known that the mean-zero jump law would lead to this case, see [V].

If \( f(t) \simeq t^\rho \) as \( t \to \infty \) then as \( \lambda \to 0 \),
\[ \int_0^\infty e^{-\lambda t} f(t) dt \simeq \lambda^{-(2+\rho)}. \]  
If \( f \) satisfies [LS] then we will say that \( f(t) \simeq t^\rho \) in the weak (Tauberian) sense. Without some extra regularity for \( f \) (for example, lack of oscillations as \( t \to \infty \) ), such a statement will not imply a strong version of \( f(t) \simeq t^\rho \) as \( t \to \infty \). However, it does capture the key scaling exponent. The weak (Tauberian) version of the conjecture ([7]) is \( \int_0^\infty e^{-\lambda t} D(t) dt \simeq \lambda^{-7/3} \).

The first non-trivial bound on \( D(t) \) was given in [LQSY] using the so-called resolvent approach: the authors proved that \( D(t) \geq C t^{1/4} \) in a weak (Tauberian) sense. (They also proved the bound \( D(t) \geq C(\log t)^{1/2} \) in \( d = 2 \), which was later improved to \( D(t) \simeq C(\log t)^{2/3} \) in [Y].) This result shows that the stationary process is indeed superdiffusive, but does not provide the conjectured scaling exponent 1/3.

The identification of this exponent was given in the breakthrough paper of Ferrari and Spohn [PS]. They treated the case of the totally asymmetric simple exclusion process (TASEP) where the jump law is \( p(1) = 1, p(z) = 0, z \neq 1 \). The focus of [PS] is not the diffusivity, their main result is a scaling limit for the fluctuation at time \( t \) of a randomly growing discrete one dimensional interface \( h_t(x) \) connected to the equilibrium process of TASEP. This random interface (the so-called height function) is basically the discrete integral of the function \( \eta_x(t) \) in \( x \). The scaling factor in their result is \( t^{1/3} \) and the limiting distribution is connected to the Tracy-Widom distribution. The proof of Ferrari and Spohn is through a direct mapping between TASEP and a particular last passage percolation problem, using a combination of results from [BDJ], [J], [BR], [PS].
The diffusivity $D^{TASEP}(t)$ can be expressed using the variance of the height function (see [QV]):

$$D^{TASEP}(t) = (4\chi t)^{-1} \sum_{x \in \mathbb{Z}} \text{Var}(h_t(x)) - 4\chi|x - (1 - 2\rho)t|.$$  \hspace{1cm} (1.9)

This identity, the results of [FS] and some additional tightness bounds would imply the existence of the limit $D^{TASEP}(t)^{-1/3}$ and even the limiting constant can be computed (see [FS] and [QV] for details). Unfortunately, the needed estimates are still missing, but from [FS] one can at least obtain a lower bound of the right order:

$$D^{TASEP}(t) > Ct^{1/3}$$  \hspace{1cm} (1.10)

with a positive constant $C$ (see [QV] for the proof).

In [QV] the resolvent approach is used to prove the following comparison theorem.

**Theorem 1** (QV, 2006). Let $D_1(t), D_2(t)$ be the diffusivities of two finite range exclusion processes in $d = 1$ with non-zero drift. There exists $0 < \beta, C < \infty$ such that

$$C^{-1} \int_0^\infty e^{-\beta t} D_1(t) dt \leq \int_0^\infty e^{-\beta t} D_2(t) dt \leq C \int_0^\infty e^{-\beta t} D_1(t) dt$$

\hspace{1cm} (1.11)

Combining Theorem 1 with (1.10) it was shown in [QV] that

**Theorem 2** (QV, 2006). For any finite range exclusion process in $d = 1$ with non-zero drift, $D(t) \geq Ct^{1/3}$ in the weak (Tauberian) sense.

[QV] also converts the Tauberian bound into pointwise bound in the nearest neighbor case to get $D(t) \geq Ct^{1/3}(\log t)^{-7/3}$.

Just a few months later Balázs and Seppäläinen [BS], building on ideas of Ferrari and Fontes [FF], Ferrari, Kipnis and Saada [FKS], and Cator and Groeneboom [CG], proved the following theorem:

**Theorem 3** (Balázs–Seppäläinen, 2006). For any nearest neighbor asymmetric exclusion process in $d = 1$ there exists a finite constant $C$ such that for all $t \geq 1$,

$$C^{-1} t^{1/3} \leq D(t) \leq Ct^{1/3}.$$  \hspace{1cm} (1.12)

Their proof uses refined and ingenious couplings to give bounds on the tail-probabilities of the distribution of the second class particle.

The aim of the present short note is to show how one can extend the results of [QV] using Theorem 3. Once we have the correct upper and lower bounds for $D(t)$ from (1.12), in the nearest neighbor case, we can strengthen the statement of Theorem 2 using the comparison theorem:

**Theorem 4.** For any finite range exclusion process in $d = 1$ with non-zero drift, $D(t) = O(t^{1/3})$ in the weak (Tauberian) sense: there exists a constant $0 < C < \infty$ such that

$$C^{-1} \lambda^{-7/3} \leq \int_0^\infty e^{-\lambda t} D(t) dt \leq C \lambda^{-7/3}.$$  \hspace{1cm} (1.13)
Getting strict estimates for a function using the asymptotic behavior of its Laplace transform usually requires some regularity and unfortunately very little is known qualitatively about $D(t)$. However, in our case (as noted in [QV]), one can get an upper bound for $D(t)$ using an inequality involving $H^{-1}$ norms:

**Theorem 5.** For any finite range exclusion process in $d = 1$ with non-zero drift, there exists $C > 0$ such that for all $t \geq 1$,

$$D(t) \leq Ct^{1/3}. \quad (1.14)$$

Note that in all these statements the constant $0 < C < \infty$ depends on the jump law $p(\cdot)$ as well as the density $0 < \rho < 1$.

### 2. Proofs

Theorem 4 immediately follows from Theorem 1 using TASEP and a general finite range exclusion, together with Theorem 3. To prove Theorem 5 one needs the Green-Kubo formula which relates the diffusivity to the time integral of current-current correlation functions:

$$D(t) = \sum_z z^2 p(z) + 2\chi t^{-1} \int_0^t \int_0^s \langle \langle w_0, e^{\lambda L} w_0 \rangle \rangle duds. \quad (2.1)$$

Here $w_x$ is the normalized microscopic flux

$$w_x = \frac{1}{\rho(1-\rho)} \sum_z p(z)(\eta_{x+z} - \rho)(\eta_x - \rho), \quad (2.2)$$

$L$ is the generator of the exclusion process and the inner product $\langle \langle \cdot, \cdot \rangle \rangle$ is defined for mean zero local functions $\phi, \psi$ as

$$\langle \langle \phi, \psi \rangle \rangle = E\left[ \phi \sum_x \tau_x \psi \right], \quad (2.3)$$

with $\tau_x$ being the appropriate shift operator.

(2.1) is proved in [LOY] (in the special case $p(1) = 1$, but the proof for general AEP is the same.) A useful variant is obtained by taking the Laplace transform,

$$\int_0^\infty e^{-\lambda t} D(t) dt = \lambda^{-2} \left( \sum_z z^2 p(z) + 2\chi \| w \|_{-1,\lambda}^2 \right) \quad (2.4)$$

where the $H_{-1}$ norm corresponding to $L$ is defined on a core of local functions by

$$\| \phi \|_{-1,\lambda} = \langle \langle \phi, (\lambda - L)^{-1} \phi \rangle \rangle^{1/2}. \quad (2.5)$$

We also need the following inequality which has appeared in several similar versions in the literature (e.g. [LY], [KL]).

**Lemma 1.** Let $w$ be the current for a finite range exclusion process in $d = 1$ with non-zero drift. Then,

$$t^{-1} \sum_x E\left[ \int_0^t w_0(s) ds \right] \int_0^t w_x(s) ds \leq 12 \| w_0 \|_{-1,t-1}^2. \quad (2.6)$$
Proof. We will use the notation
\[ \| \phi \|^2 = \langle \phi, \phi \rangle, \]  
(2.7)
note that the right hand side of (2.6) is equal to \( t^{-1} \| \int_0^t w_0(s) ds \|^2 \). Let \( \lambda > 0 \) and \( u_\lambda = (\lambda - L)^{-1} w_0 \). Using Ito’s formula together with the identity \( Lu_\lambda = \lambda u_\lambda - w_0 \) we get
\[ u_\lambda(t) = u_\lambda(0) - \int_0^t (\lambda u_\lambda - w_0)(s) ds + M_t \]  
(2.8)
where \( M_t \) is a mean zero martingale with
\[ \| M_t \|^2 = \int_0^t \langle u_\lambda(s), -Lu_\lambda(s) \rangle \, ds = t \langle u_\lambda, -Lu_\lambda \rangle = t \langle u_\lambda, (\lambda - L)u_\lambda \rangle - t \lambda \| u_\lambda \|^2. \]  
(2.9)
Rearranging (2.8), applying Schwarz Lemma and using stationarity
\[ \| \int_0^t w_0(s) ds \|^2 \leq 8 \| u_\lambda \|^2 + 4 \| M_t \|^2 + 4 \lambda^2 \| \int_0^t u_\lambda(s) ds \|^2 \]  
(2.10)
To bound the last term of (2.10) we again use Schwartz Lemma and stationarity to get
\[ \| \int_0^t u_\lambda(s) ds \|^2 \leq t \int_0^t \| u_\lambda(s) \|^2 ds = t^2 \| u_\lambda \|^2. \]  
(2.11)
Putting our estimates together,
\[ \| \int_0^t w_0(s) ds \|^2 \leq (8 - 4t \lambda + 4t^2 \lambda^2) \| u_\lambda \|^2 + 4t \langle u_\lambda, (\lambda - L)u_\lambda \rangle \]  
(2.12)
Setting \( \lambda = t^{-1} \) and dividing the previous inequality by \( t \):
\[ t^{-1} \| \int_0^t w_0(s) ds \|^2 \leq 8 \lambda \| u_\lambda \|^2 + 4 \langle u_\lambda, (\lambda - L)u_\lambda \rangle \]
\[ \leq 12 \langle u_\lambda, (\lambda - L)u_\lambda \rangle = 12 \| w_0 \|_{-1, \lambda}^2, \]
which proves the lemma. \( \square \)

Proof of Theorem 3. Theorem 4 and identity (2.4) gives
\[ \| w_0 \|_{-1, \lambda}^2 \leq C \lambda^{-1/3} \]  
(2.13)
if \( \lambda > 0 \) is sufficiently small. To bound \( D(t) \) we use the Green-Kubo formula (2.1) noting that the second term on the right is equal to \( \chi t^{-1} \| \int_0^t w_0(s) ds \|^2 \). Using Lemma 1 and then (2.13) to estimate this term we get that
\[ D(t) \leq \sum_z z^2 p(z) + 12 \chi \| w_0 \|_{-1, t^{-1}}^2 \leq Ct^{1/3} \]  
(2.14)
for large enough \( t \). \( \square \)
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Departments of Mathematics and Statistics, University of Toronto
E-mail address: quastel@math.toronto.edu

Departments of Mathematics and Statistics, University of Toronto
E-mail address: valko@math.toronto.edu