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ENERGY DIFFUSION IN HARMONIC SYSTEM WITH
CONSERVATIVE NOISE

GIADA BASILE AND STEFANO OLLA

Abstract. We prove diffusive behaviour of the energy fluctuations in a system of harmonic oscillators with a stochastic perturbation of the dynamics that conserves energy and momentum. The results concern pinned systems or lattice dimension $d \geq 3$, where the thermal diffusivity is finite.

1. Introduction

Lattice networks of oscillators have been considered for a long time as good models for studying macroscopic energy transport and its diffusion, i.e. for obtaining, on a macroscopic space-time scale, heat equation and Fourier law of conduction ([1]). It is well understood that the diffusive behavior of the energy is due to the non-linearity of the interactions, and that purely deterministic harmonic systems have a ballistic transport of energy (cf. [14]). On the other hand, non-linear dynamics are very difficult to study and even the convergence of the Green-Kubo formula defining the macroscopic thermal conductivity is an open problem. In fact in some cases, like in one dimensional unpinned systems, it is expected (and observed numerically) an infinite conductivity and a superdiffusion of the energy (cf. [12, 11]).

In order to model the phonon scattering due to the dynamics, various stochastic perturbation of the harmonic dynamics have been proposed, where the added random dynamics conserves the energy. In [7] Fourier Law is proven for an harmonic chain with stochastic dynamics that conserves only energy. In [2, 3] is studied the Green-Kubo formula for a stochastic perturbation that conserves energy and momentum. It is proven there that conductivity is finite for pinned systems, or unpinned if dimension is greater than 2. Qualitatively this agrees to what is expected for deterministic anharmonic dynamics.

In this article we consider the same stochastic dynamics as in [2, 3] and we prove that in the cases when conductivity is finite, the energy
behave diffusively in the sense that energy fluctuations of the system in equilibrium evolve according to a linear heat equation. The key point in proving such diffusive behavior is to obtain a fluctuation-dissipation decomposition of the microscopic energy currents $j_{x,y}$ between two adjacent atoms $x, y$. This means to be able to approximate $j_{x,y}$ by a function of the form $\kappa(e_x - e_y) + LF$, where $e_x$ is the energy of atom $x$, $L$ is the generator of the dynamics and $F$ a function in its domain, possibly local. With such decomposition it is possible to close macroscopically the evolution of the energy and identify the thermal diffusivity.

In the harmonic chain with noise that conserves only energy, there is an exact fluctuation-dissipation decomposition with a local function $F$ (cf. [7]). For anharmonic system this decomposition is non-local, and much harder to obtain. In fact there exist only results with an elliptic noise that acts also on the positions of the atoms (cf. [13]).

In the model considered here, the noise is of hypoelliptic type, i.e. acts only on the velocity. But because of the additional conservation of momentum, the fluctuation-dissipation decomposition of the currents is non-local. Thanks to the linearity of the interaction, we can use resolvent approximations instead of local approximations for this decomposition.

The corresponding results for the cases when thermal conductivity is infinite, i.e. the unpinned model in dimension 1 and 2, remains open problems. In dimension 2 the divergence of the conductivity is of logarithmic type, and we expect a diffusive behavior of the energy fluctuations under the proper space-time scale.

In dimension 1, also to establish a conjecture is hard, as the behavior is really superdiffusive. Under a weak noise limit the local spectral energy density (Wigner distribution) behaves following a linear Boltzmann type equation (cf. [5]). Under proper rescaling of this equation, is proven the convergence to a superdiffusive Levy process governed by a fractional laplacian (cf. [4, 8]. See also [6] for diffusive behaviour in dimension 2). A possible guess is that the fractional laplacian behavior will govern energy fluctuations without taking first a weak noise limit. For what concerns this problem in one dimensional unpinned anharmonic chains, see the recent article by Herbert Spohn [15].

2. The dynamics

The Hamiltonian is given by

$$ H = \frac{1}{2} \sum_x \left[ p_x^2 + q_x \cdot (-\nu I - \alpha \Delta) q_x \right]. $$

The atoms are labeled by $x \in \mathbb{Z}^d$ and $\{q_x\}$ are the displacements of the atoms from their equilibrium positions. We denote with $\nabla, \nabla^*$
ENERGY DIFFUSION IN HARMONIC SYSTEM WITH CONSERVATIVE NOISE

and $\Delta = \nabla^* \cdot \nabla$ respectively the discrete gradient, its adjoint and the discrete Laplacian on $\mathbb{Z}^d$. These are defined as

\begin{equation}
\nabla_{e_j} f(x) = f(x + e_j) - f(x)
\end{equation}

and

\begin{equation}
\nabla^*_{e_j} f(x) = f(x - e_j) - f(x).
\end{equation}

The parameter $\alpha > 0$ is the strength of the interparticles springs, and $\nu \geq 0$ is the strength of the pinning (on-site potential).

We consider the stochastic dynamics corresponding to the Fokker-Planck equation

\begin{equation}
\frac{\partial P}{\partial t} = (-A + \gamma S) P = LP.
\end{equation}

where $A$ is the usual Hamiltonian vector field

\begin{equation*}
A = \sum_x \{ p_x \cdot \partial q_x + [(\alpha \Delta - \nu I) q_x] \cdot \partial p_x \}
\end{equation*}

while $S$ is the generator of the stochastic perturbation and $\gamma > 0$ is a positive parameter that regulates its strength. The operator $S$ acts only on the momentums $\{p_x\}$ and generates a diffusion on the surface of constant kinetic energy and constant momentum. This is defined as follows. For every nearest neighbor atoms $x$ and $z$, consider the $d-1$ dimensional surface of constant kinetic energy and momentum

\begin{equation*}
S_{e,p} = \left\{ (p_x, p_z) \in \mathbb{R}^{2d} : \frac{1}{2} (p_x^2 + p_z^2) = e ; \ p_x + p_z = p \right\}.
\end{equation*}

The following vector fields are tangent to $S_{e,p}$

\begin{equation*}
X_{x,z}^{i,j} = (p_z^j - p_x^i)(\partial_{p_x} - \partial_{p_z}) - (p_z^i - p_x^j)(\partial_{p_z} - \partial_{p_x}).
\end{equation*}

so $\sum_{i,j=1}^d (X_{x,z}^{i,j})^2$ generates a diffusion on $S_{e,p}$. In $d \geq 2$ we define

\begin{equation*}
S = \frac{1}{2(d-1)} \sum_{x} \sum_{i,j,k} (X_{x,x+e_k}^{i,j})^2
= \frac{1}{4(d-1)} \sum_{x \in \mathbb{Z}^d} \sum_{i,j} \sum_{\|x-z\|=1} (X_{x,z}^{i,j})^2
\end{equation*}

where $e_1, \ldots, e_d$ is canonical basis of $\mathbb{Z}^d$.

Observe that this noise conserves the total momentum $\sum_x p_x$ and energy $\mathcal{H}_N$, i.e.

\begin{equation*}
S \sum_x p_x = 0 , \quad S \mathcal{H}_N = 0
\end{equation*}

In dimension 1, in order to conserve total momentum and total kinetic energy, we have to consider a random exchange of momentum
between three consecutive atoms, and we define
\[ S = \frac{1}{6} \sum_{x \in T_x} (Y_x)^2 \]
where
\[ Y_x = (p_x - p_{x+1})\partial_{p_{x-1}} + (p_{x+1} - p_{x-1})\partial_{p_x} + (p_{x-1} - p_x)\partial_{p_{x+1}} \]
which is vector field tangent to the surface of constant energy and momentum of the three particles involved. The Fokker-Planck equation (2.3) gives the time evolution of the probability distribution \( P(q, p, t) \), given an initial distribution \( P(q, p, 0) \). It correspond to the law at time \( t \) of the solution of the following stochastic differential equations:
\[ dq_x = dp_x dt \]
\[ dp_x = -(\nu I - \alpha \Delta)q_x dt + 2\gamma \Delta p_x dt + \frac{\sqrt{\gamma}}{2\sqrt{d-1}} \sum_{x:||x-x||=1} \sum_{i,j} (X_{x,x}^{ij}p_x) dw_{x,x}^{ij}(t) \]
where \( \{w_{x,y}^{ij} = w_{y,x}^{ij}, x, y \in \mathbb{Z}^d; i, j = 1, \ldots, d; ||y-x|| = 1\} \) are independent standard Wiener processes. In \( d = 1 \) the sde are:
\[ dp_x = -(\nu I - \alpha \Delta)q_x dt + \frac{\gamma}{6} \Delta (4p_x + p_{x-1} + p_{x+1}) dt + \sqrt{\frac{\gamma}{3}} \sum_{k=-1,0,1} (Y_{x+k}p_x) dw_{x+k}(t) \]
where here \( \{w_x(t), x = 1, \ldots, N\} \) are independent standard Wiener processes.

Defining the energy of the atom \( x \) as
\[ e_x = \frac{1}{2} p_x^2 + \frac{\alpha}{4} \sum_{y:||y-x||=1} (q_y - q_x)^2 + \frac{\nu}{2} q_x^2 , \]
the energy conservation law can be read locally as
\[ e_x(t) - e_x(0) = \sum_{k=1}^{d} (J_{x-e_k,x}(t) - J_{x,x+e_k}(t)) \]
where \( J_{x,x+e_k}(t) \) is the total energy current between \( x \) and \( x + e_k \) up to time \( t \). This can be written as
\[ J_{x,x+e_k}(t) = \int_0^t j_{x,x+e_k}(s) ds + M_{x,x+e_k}(t) . \]
In the above \( M_{x,x+e_k}(t) \) are martingales that can be written explicitly as Ito stochastic integrals
\[ M_{x,x+e_k}(t) = \sqrt{\frac{\gamma}{(d-1)}} \sum_{i,j} \int_0^t (X_{x,x+e_k}^{ij}e_x) (s) dw_{x,x+e_k}^{ij}(s) \]
The stationary equilibrium probability measures for this dynamics are given by the corresponding Gibbs measure, defined through the DLR equations. Because of the linearity of the interaction and the conservation laws of the stochastic perturbation, these are gaussian measures on \((\mathbb{R}^{2d})^2\) with covariance given by

\[
\begin{align*}
< (p^i_x - v^i)(p^j_y - v^j) > &= \beta^{-1} \delta_{i,j} \delta_{x,y}, \\
< (p^i_x - v^i)q^j_y > &= 0, \\
< q^i_x q^j_y > &= \frac{1}{d} \Gamma(x - y) \delta_{i,j}
\end{align*}
\]

where \(\Gamma(x) = (\nu I - \alpha \Delta)^{-1}(x)\). In the pinned case, \(\nu = 0\), momentum is not conserved and we have to set \(v = 0\). Since in the unpinned case the parameter \(v\) represent a trivial translation invariance, for simplicity we choose \(v = 0\).

We consider this dynamics starting with an equilibrium distribution at a given temperature \(\beta^{-1}\). The existence of the infinite dynamics under this initial distribution can be proven by standard techniques (for example see [10]).

Given two continuous functions \(F, H\) on \(\mathbb{R}^d\) with compact support, and \(\varepsilon > 0\), we look at the evolution of the following quantity

\[
\sigma_{\varepsilon,t}(F, H) = \varepsilon^d \sum_{x,y} F(\varepsilon x) H(\varepsilon y) \langle (e_x(\varepsilon^{-2}t) - \beta^{-1})(e_0(0) - \beta^{-1}) \rangle
\]

where \(\langle \cdot \rangle\) is the expectation value with respect to the equilibrium measure.

**Theorem 2.1.**

\[
\lim_{\varepsilon \to 0} \sigma_{\varepsilon,t}(F, H) = \int\int d\mathbf{u} d\mathbf{v} F(\mathbf{u}) \int G(\mathbf{v}) \frac{e^{-|u-v|^2/2\kappa}}{(2\pi \kappa)^{d/2}}
\]

where the diffusion coefficient \(\kappa\) is given by:

\[
\kappa = \frac{1}{8\pi^2 \gamma} \int_{\mathbb{T}^d} d\xi \left( \frac{\partial_1 \omega(\xi)}{\Phi(\xi)} \right)^2 + \gamma
\]

Here \(\omega\) is the dispersion relation

\[
\omega(\xi) = (\nu + 4\alpha \sum_{j=1}^{d} \sin^2(\pi \xi_j))^{1/2}
\]

and \(\Phi\) is the scattering rate

\[
\Phi(k) = \begin{cases} 
8 \sum_{j=1}^{d} \sin^2(\pi k_j) & d \geq 2 \\
\frac{4}{3} \sin^2(\pi k)(1 + 2 \cos^2(\pi k)) & d = 1
\end{cases}
\]
With a little more work one can prove that the fluctuation field
\[
Y^\varepsilon_t(F) = \varepsilon^{d/2} \sum_y F(\varepsilon y) \left[ e_x(\varepsilon^{-2}t) - \beta^{-1} \right]
\]
converges in law to the infinite dimensional Ornstein Uhlenbeck \(Y_t\) solution of the linear stochastic PDE:
\[
\partial_t Y = \frac{\kappa}{2} \Delta Y + \beta^{-1/2} \nabla W
\]
where \(W(x, t)\) is the standard space time white noise on \(\mathbb{R}^{d+1}\). This extension is standard and we will expose here only the proof of Theorem 2.1.

3. Energy currents

Let \(e_x\) be the energy of the atom \(x\), which is equal to
\[
e_x = \frac{1}{2} p_x^2 - \frac{1}{2} q_x \cdot (\alpha \Delta - \nu) q_x.
\]
When \(\nu = 0\) there is no pinning. We consider cases \(\nu > 0, d \geq 1\) and \(\nu = 0, d = 3\). The instantaneous energy currents \(j_{x,x+e_i}, i = 1, \ldots, d\), satisfy the equation
\[
Le_x = \sum_{i=1}^{d} \left( j_{x-e_i,x} - j_{x,x+e_i} \right),
\]
and can be written as
\[
j_{x,x+e_i} = j^a_{x,x+e_i} + \gamma j^{s}_{x,x+e_i}.
\]
The first term is the Hamiltonian contribution to the energy current, namely
\[
j^a_{x,x+e_i} = -\frac{\alpha}{2} (q_{x+e_i} - q_x) \cdot (p_{x+e_i} + p_x),
\]
while the noise contribution in \(d \geq 2\) is
\[
\gamma j^{s}_{x,x+e_i} = -\gamma \nabla_{e_i} p_x^2.
\]
In one dimension
\[
\gamma j^{s}_{x,x+1} = -\gamma \nabla \phi(p_{x-1}, p_x p_{x+1}),
\]
\[
\phi(p_{x-1}, p_x p_{x+1}) = \frac{1}{6} \left[ p_{x+1}^2 + 4p_x^2 + p_{x-1}^2 + p_{x+1}p_{x-1} - 2p_{x+1}p_x - 2p_{x-1}p_x \right]
\]
We denote with \(\phi_x := \phi(p_{x-1}, p_x, p_{x+1})\).
ENERGY DIFFUSION IN HARMONIC SYSTEM WITH CONSERVATIVE NOISE

Given $F, H \in C^2_c(\mathbb{R}^d)$ (twice differentiable functions with compact support), and $\varepsilon > 0$, we look at the evolution of the following quantity

$$\sigma_{\varepsilon,t} (F, H) = \varepsilon^d \sum_{x,y} F(\varepsilon x) H(\varepsilon y) \langle (e_x (\varepsilon^{-2} t) - \beta^{-1}) (e_y (0) - \beta^{-1}) \rangle$$

$$\quad = \varepsilon^d \sum_{y,z} F(\varepsilon (y + z)) H(\varepsilon y) \langle e_z (\varepsilon^{-2} t) (e_0 (0) - \beta^{-1}) \rangle,$$

where $\langle \cdot \rangle$ is the expectation value with respect to the equilibrium measure. For $d \geq 1$ we have

$$\sigma_{\varepsilon,t} (F, H) = \sigma_{\varepsilon,0} (F, H)$$

$$\quad + \varepsilon^d \sum_{y,z} \sum_{i=1}^d \nabla_{e_i} \langle \varepsilon (y + z) \rangle H(\varepsilon y) \frac{1}{\varepsilon} \int_0^t \frac{dF_j}{ds} \left( j_{x,z+e_i} (s/\varepsilon^2) (e_0 (0) - \beta^{-1}) \right).$$

Accordingly to (3.1), we decompose the energy current $j_{x,z+e_i}$ in two parts and we treat separately the two integrals.

**Noise current.** For $d \geq 2$, using (3.3) we have

$$\gamma \varepsilon^d \sum_{y,z} \sum_{i=1}^d \nabla_{e_i} \langle \varepsilon (y + z) \rangle H(\varepsilon y) \frac{1}{\varepsilon} \int_0^t \frac{dF_j}{ds} \left( j_{x,z+e_i} (s/\varepsilon^2) (e_0 (0) - \beta^{-1}) \right)$$

$$\quad = \gamma \varepsilon^d \sum_{x,y} \Delta F(\varepsilon (y + z)) H(\varepsilon y) \int_0^t \frac{dG}{ds} \left( p_{x}^2 (s/\varepsilon^2) (e_0 (0) - \beta^{-1}) \right) + O(\varepsilon).$$

In order to replace in the last expression $p_x^2$ with $e_z$, we use the following Lemma:

**Lemma 3.1.** For every $G \in L^2(\mathbb{R}^d)$, $\forall d \geq 1$

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \left\langle \left( \int_0^t \frac{dG}{ds} \varepsilon^d/2 \sum_y G(\varepsilon y) \left[ p_y^2 - e_y \right] \left( s/\varepsilon^2 \right) \right)^2 \right\rangle = 0.$$

**Proof.** Observe that

$$p_x^2 - e_x = \frac{1}{2} p_x^2 + \frac{1}{2} q_x \cdot [\alpha \Delta - \nu I] q_x = \frac{1}{2} L[q_x \cdot p_x] - \frac{1}{2} \gamma S[q_x \cdot p_x]\]$$

Therefore

$$\left\langle \left( \int_0^t \frac{dG}{ds} \varepsilon^d/2 \sum_y G(\varepsilon y) \left[ p_y^2 - e_y \right] \left( s/\varepsilon^2 \right) \right)^2 \right\rangle \leq \varepsilon^4 \left\langle \left( \int_0^t \frac{dG}{ds} \varepsilon^d/2 \sum_y G(\varepsilon y) \varepsilon^{-2} L[q_y \cdot p_y] \left( s/\varepsilon^2 \right) \right)^2 \right\rangle$$

$$\quad + \gamma^2 \left\langle \left( \int_0^t \frac{dG}{ds} \varepsilon^d/2 \sum_y G(\varepsilon y) S[q_y \cdot p_y] \left( s/\varepsilon^2 \right) \right)^2 \right\rangle.$$ (3.6)
In the first term of the rhs of the above inequality, we can perform explicitly the time integration and we have

$$
\frac{\varepsilon^{d/2}}{2} \sum_y G(\varepsilon y) \int_0^t ds \varepsilon^{-2} L[q_y \cdot p_y](s/\varepsilon^2) 
$$

$$
= \frac{\varepsilon^{d/2}}{2} \sum_y G(\varepsilon y) \left( [q_y \cdot p_y](t/\varepsilon^2) - [q_y \cdot p_y](0) \right) + \mathcal{M}_\varepsilon(G, t)
$$

where $\mathcal{M}_\varepsilon(G, t)$ is a martingale (given by a stochastic integral) whose quadratic variation is bounded by

$$
[\mathcal{M}_\varepsilon(G, t)^2] \leq \frac{\varepsilon^{d}}{4} \sum_y |\nabla G(\varepsilon y)|^2 \int_0^t \langle p_y^2(s/\varepsilon^2) q_y^2(s/\varepsilon^2) \rangle ds 
$$

$$
\leq \varepsilon \|G\|^2 \beta^{-1} \langle q_0^2 \rangle.
$$

Since $\langle q_0^2 \rangle$ is bounded in the pinned case and for $d \geq 3$, we conclude that the first term on the RHS of (3.6) converges to 0 as $\varepsilon \to 0$.

Moreover, by the estimate in appendix A

$$
\left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_y G(\varepsilon y) S[q_y \cdot p_y](s/\varepsilon^2) \right)^2 \right\rangle \leq C t \varepsilon^2
$$

which vanishes as $\varepsilon \to 0$.

\[\square\]

Using Cauchy-Schwartz, the quantity

$$
\gamma \varepsilon^d \sum_{z,y} \Delta F(\varepsilon(y + z)) H(\varepsilon y) \int_0^t ds \langle [p_z^2 - e_z](s/\varepsilon^2) \rangle \left( (\varepsilon_0(0) - \beta^{-1}) \right)
$$

$$
= \gamma \varepsilon^d \sum_{y'} \Delta F(\varepsilon y') H(\varepsilon y) \int_0^t ds \langle [p_{y'}^2 - e_{y'}](s/\varepsilon^2) \rangle \left( (\varepsilon_y(0) - \beta^{-1}) \right)
$$

is bounded in absolute value by

$$
\gamma \|H\| \langle (\varepsilon_0 - \beta^{-1})^2 \rangle^{1/2} \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_y \Delta F(\varepsilon y) [p_y^2 - e_y](s/\varepsilon^2) \right)^2 \right\rangle^{1/2},
$$

which vanishes as $\varepsilon \to 0$ in view of the previous lemma. Then the contribution of the noise in the evolution of the energy fluctuations is
given by
\begin{equation}
\gamma \epsilon^d \sum_{y,z} \sum_{i=1}^{d} \nabla_{e_i} F(\epsilon(y + z)) H(\epsilon y) \frac{1}{\epsilon} \int_0^t ds \left\langle J^{z+e_i}(s/\epsilon^2)(e_0(0) - \beta^{-1}) \right\rangle
\end{equation}
\begin{align}
= \gamma \epsilon^d \sum_{x,y} \Delta F(\epsilon(y + z)) H(\epsilon y) \int_0^t ds \left\langle e_z(s/\epsilon^2) (e_0(0) - \beta^{-1}) \right\rangle + O(\epsilon) \\
= \sigma_{\epsilon,t}(\gamma \Delta F, G) + O(\epsilon).
\end{align}

In $d = 1$ we have
\begin{align}
\gamma \epsilon \sum_{y,z} \nabla^2 F(\epsilon(y + z)) H(\epsilon y) \frac{1}{\epsilon} \int_0^t ds \left\langle J^{z+1}(s/\epsilon^2)(e_0(0) - \beta^{-1}) \right\rangle \\
= \gamma \epsilon \sum_{y,z} \sum_{i=1}^{d} F''(\epsilon(y + z)) H(\epsilon y) \int_0^t ds \left\langle \phi_z(s/\epsilon^2) (e_0(0) - \beta^{-1}) \right\rangle + O(\epsilon),
\end{align}
with $\phi$ defined in (3.4). The last quantity can be written as
\begin{align}
&= \gamma \epsilon \sum_{y,z} \sum_{i=1}^{d} F''(\epsilon(y + z)) H(\epsilon y) \int_0^t ds \left\langle \left( p^2_z(s/\epsilon^2) (e_0(0) - \beta^{-1}) \right) \right\rangle \\
&+ \frac{1}{6} \left\langle (p_{z+1}p_{z-1} - 2p_zp_{z+1} - 2p_zp_{z-1})(s/\epsilon^2)(e_0(0) - \beta^{-1}) \right\rangle + O(\epsilon),
\end{align}

where we can replace $p^2_z$ with $e_z$ using same arguments as in $d \geq 2$. We will prove in Section 4 that the second integral vanishes as $\epsilon \to 0$, then formula (3.7) holds in all dimensions.

**Hamiltonian currents.** Notice that
\begin{equation}
J_{0,e_i}^z = \frac{\alpha}{2} (p_{e_i} - p_{-e_i}) \cdot q_{e_i} - \frac{\alpha}{2} \nabla \cdot (p_0 \cdot q_0 + p_{-e_i} \cdot q_{-e_i})
\end{equation}
it is easy to see that the contribution of the gradient terms in (3.5) vanish as $\epsilon \to 0$, so we have only to consider the first term.

Let $g^{2,j}_\lambda$, $j = 1, \ldots, d$ be the solution of the equation
\begin{equation}
(\lambda - 2\Delta) g^{2,j}_\lambda(z) = \delta(z + e_j) - \delta(z - e_j), \quad d \geq 2,
\end{equation}
with $\lambda > 0$. In $d = 1$ the equation reads
\begin{equation}
-\frac{1}{3} \Delta [4g_{\lambda}(z) + g_{\lambda}(z+1) + g_{\lambda}(z-1)] + \lambda g_{\lambda}(z) = \delta(z+1) - \delta(z-1).
\end{equation}
We observe that $g^{2,j}_\lambda$ decays exponentially fast, and in particular the $\ell^2$-norm is finite. We by $\hat{g}^{2,j}_\lambda$ its Fourier transform, namely
\begin{equation}
\hat{g}^{2,j}_\lambda(k) = \frac{2i \sin(2\pi k_j)}{\Phi(k) + \lambda},
\end{equation}
where $\Phi$ is defined by (2.11). We define

$$(3.12) \quad u^i_\lambda = \sum_x g^i_\lambda(x) \mathbf{q}_0 \cdot \mathbf{p}_x, \quad i = 1, \ldots, d.$$ 

so that

$$\alpha \left( p_{e_i} - p_{-e_i} \right) \cdot \mathbf{q}_{e_i} = \alpha \left( \lambda - S \right) u^i_\lambda = \alpha \lambda u^i_\lambda - \frac{\alpha}{\gamma} L u^i_\lambda + \frac{\alpha}{\gamma} A u^i_\lambda.$$ 

Now we show that the contribution of the first two terms of the above to (3.5) will vanish as $\epsilon \to 0, \lambda \to 0$. We will use the following Lemma.

**Lemma 3.2.** For every $F \in L^2(\mathbb{R}^d)$, $\forall i = 1, \ldots, d$

$$(3.13) \quad \left\langle \left( \epsilon^{d/2} \sum_x F(\epsilon x) \tau_x u^i_\lambda \right)^2 \right\rangle \leq \beta^{-1} \|F\|^2 \int d\mathbf{k} \hat{\Gamma}(\mathbf{k}) |\hat{g}^i_\lambda(\mathbf{k})|^2.$$ 

**Proof.** By Schwarz inequality

$$\left\langle \left( \epsilon^{d/2} \sum_x F(\epsilon x) \tau_x u^i_\lambda \right)^2 \right\rangle \leq \|F\|^2 \sum_x \left\langle u^i_\lambda \tau_x u^i_\lambda \right\rangle,$$

where

$$\sum_x \left\langle u^i_\lambda \tau_x u^i_\lambda \right\rangle = \beta^{-1} \sum_x \Gamma(x) \sum_i \sum_z g^i_\lambda(z) g^i_\lambda(z + x) = \beta^{-1} \int d\mathbf{k} \hat{\Gamma}(\mathbf{k}) |\hat{g}^i_\lambda(\mathbf{k})|^2.$$

Notice that if $d \geq 3$ or $\nu > 0$ the quantity $\int d\mathbf{k} \hat{\Gamma}(\mathbf{k}) |\hat{g}^i_\lambda(\mathbf{k})|^2$ is bounded.

In order to to prove that $Lu^i_\lambda$ does not contribute in the limit we have just to show that

$$\left\langle \left( \epsilon^{-1} \int_0^t \epsilon^{d/2} \sum_x F(\epsilon x) \tau_x Lu^i_\lambda(s/\epsilon^2) \right)^2 \right\rangle \to 0, \quad \epsilon \to 0.$$ 

We can perform the time integration and we deduce

$$\epsilon^{-1} \int_0^t \epsilon^{d/2} \sum_x F(\epsilon x) \tau_x Lu^i_\lambda(s/\epsilon^2)$$

$$= \epsilon \epsilon^{d/2} \sum_x F(\epsilon x) \tau_x \left[ u^i_\lambda(t/\epsilon^2) - u^i_\lambda(0) \right] + \epsilon \mathcal{M}_\epsilon(F, t),$$

where $\mathcal{M}_\epsilon(F, t)$ is a martingale given by a stochastic integral. By Lemma 3.2 the first term on the rhs of the previous equality is of order $\epsilon$. Moreover, $\mathcal{M}_\epsilon(F, t)$ has bounded quadratic variation, therefore the contribution of $\epsilon \mathcal{M}_\epsilon(F, t)$ vanishes as $\epsilon \to 0$. 


Lemma 3.3. For every \( F \in L^2(\mathbb{R}^d) \), \( \forall i = 1, \ldots, d \)
\[
\lim_{\lambda \to 0} \lim_{\varepsilon \to 0} \left( \frac{\lambda}{\varepsilon} \int_0^t ds \varepsilon^{d/2} \sum_z F(\varepsilon y) \tau_y u_\lambda^i(s/\varepsilon^2) \right)^2 = 0
\]

Proof. The following inequality holds:
\[
\sup_{t \in [0,T]} \left\langle \left( \frac{\lambda}{\varepsilon} \int_0^t ds \varepsilon^{d/2} \sum_z F(\varepsilon y) \tau_y u_\lambda^i s(\varepsilon^2) \right)^2 \right\rangle \leq C T \sup_f \left\{ 2 \frac{\lambda}{\varepsilon} \varepsilon^{d/2} \sum_y F(\varepsilon y) \langle f \tau_y u_\lambda^i \rangle - \frac{1}{\varepsilon^2} \langle f(-S f) \rangle \right\}.
\]

Observe that
\[
u^i_\lambda = \frac{1}{d-1} \sum_x \sum_{k,j} G_\lambda(x) q_0^j \left[ X_{x-e_0}^k p_{x-e_0}^k + X_{x+e_0}^k p_{x+e_0}^k \right]
\]
for \( d \geq 2 \), where \( G_\lambda(x) \) solves \((\lambda - 2\Delta)G_\lambda(x) = \delta(x)\). A similar formula holds for \( d = 1 \). Then
\[
\frac{\lambda}{\varepsilon} \varepsilon^{d/2} \sum_y F(\varepsilon y) \langle f \tau_y u_\lambda^i \rangle
= \frac{1}{d-1} \frac{\lambda}{\varepsilon} \varepsilon^{d/2} \sum_y F(\varepsilon y) \sum_x \sum_{k,j} G_\lambda(x - y) \left\{ \langle X_{x-e_0}^k f \rangle q_0^j p_{x-e_0}^k \right\} + \left\{ \langle X_{x-e_0}^k f \rangle q_0^j p_{x-e_0}^k \right\}
\]
which is bounded in absolute value by
\[
C \left( \lambda^2 \beta^{-1} \sum_x \sum_j \left\langle \varepsilon^{d/2} \sum_y F(\varepsilon y) G_\lambda(x - y) q_0^j \right\rangle^2 \right)^{1/2} \times \left( \varepsilon^{-2} \langle f(-S f) \rangle \right)^{1/2}.
\]

Observe that
\[
\frac{\lambda^2}{\varepsilon^d} \sum_{y,z} \left\langle \varepsilon^{d/2} \sum_y F(\varepsilon y) G_\lambda(x - y) q_0^j \right\rangle^2
= \lambda^2 \varepsilon^d \sum_{y,z} F(\varepsilon y) F(\varepsilon (y + z)) \sum_x G_\lambda(x) G_\lambda(x - z) \langle q_z \cdot q_0 \rangle
\leq \lambda^2 \|F\|^2 \sum_{x,z} G_\lambda(x) G_\lambda(x - z) \Gamma(z),
\]
where in he last step we used Schwarz inequality. In the Fourier space
\[
\lambda^2 \sum_{x,z} G_\lambda(x) G_\lambda(x - z) \Gamma(z) = \lambda^2 \int_{T^d} dk \frac{\hat{\Gamma}(k)}{(\lambda + \Phi(k))^2},
\]
which vanishes as $\lambda \to 0$ for $\nu > 0$ or $d \geq 3$. Therefore

$$\sup_{f} \left\{ 2\lambda \varepsilon^{d/2} \sum_y F(\varepsilon y) (f g^i_{\lambda y}^i) - \frac{1}{\varepsilon^2} (f (-Sf)) \right\}$$

$$\leq C_0 \|F\|^{2} \lambda^{2} \sum_{x,z} G_{\lambda}(x) G_{\lambda}(x - z) \Gamma(z)$$

which vanishes for $\lambda \to 0$.

\[\Box\]

Observe that

$$\frac{\alpha}{\gamma} A u^i_{\lambda} = \frac{\alpha}{\gamma} \sum_{x} \sum_{y,z} \nabla_{e_i} F(\varepsilon (y + z)) H(\varepsilon y) \tau_{z} A u^i_{\lambda}$$

$$= \frac{\alpha}{\gamma} \varepsilon^{d-1} \sum_{i} \sum_{y,z} \nabla_{e_i} F(\varepsilon (y + z)) H(\varepsilon y) g^i_{\lambda}(x) \left[ p_{z} \cdot p_{x} + q_{z} \cdot (\alpha \Delta - \nu I) q_{x} \right]$$

$$= \frac{\alpha}{\gamma} \varepsilon^{d-1} \sum_{i} \sum_{y,z} \nabla_{e_i} F(\varepsilon (y + z)) H(\varepsilon y) g^i_{\lambda}(x) \left[ p_{z} \cdot p_{x} + q_{z} \cdot (\alpha \Delta - \nu I) q_{x} \right]$$

So we have to look at

$$\frac{\alpha}{\gamma} \varepsilon^{d-1} \sum_{i} \sum_{y,z} \nabla_{e_i} F(\varepsilon (y + z)) H(\varepsilon y) \tau_{z} A u^i_{\lambda}$$

$$= \frac{\alpha}{\gamma} \varepsilon^{d-1} \sum_{i} \sum_{y,z} \nabla_{e_i} F(\varepsilon (y + z)) H(\varepsilon y) g^i_{\lambda}(x) \left( p_{z} \cdot p_{x} + q_{z} \cdot (\alpha \Delta - \nu I) q_{x} \right)$$

$$= \frac{\alpha}{\gamma} \varepsilon^{d-1} \sum_{i} \sum_{y,z} \nabla_{e_i} F(\varepsilon (y + z)) H(\varepsilon y) g^i_{\lambda}(x) \left( p_{z} \cdot p_{x} + q_{z} \cdot (\alpha \Delta - \nu I) q_{x} \right)$$

We are left to study:

$$K_{2} = \sum_{i} \frac{\alpha}{\gamma} \varepsilon^{d-1} \sum_{y,z} H(\varepsilon y) \nabla_{e_i} F(\varepsilon (y + z)) \sum_{x} g^i_{\lambda}(x)$$

$$\times \frac{1}{\varepsilon} \int_{0}^{t} ds \langle \tau_{x}[p_{0} \cdot p_{x} + q_{0} \cdot (\alpha \Delta - \nu I) q_{x}](s/\varepsilon^{2})(e_{0}(0) - \beta^{-1}) \rangle.$$

Remark that

$$A \frac{1}{2} \left[ p_{0} \cdot q_{x} + q_{0} \cdot p_{x} \right] = \frac{1}{2} \left[ q_{0} \cdot (\alpha \Delta - \nu I) q_{x} + (\alpha \Delta - \nu I) q_{0} \cdot q_{x} \right] + p_{0} \cdot p_{x},$$

that implies

$$p_{0} \cdot p_{x} + q_{0} \cdot (\alpha \Delta - \nu I) q_{x}$$

$$= \frac{1}{2} \left[ q_{0} \cdot (\alpha \Delta - \nu I) q_{x} - (\alpha \Delta - \nu I) q_{0} \cdot q_{x} \right] + \frac{1}{2} A \left[ p_{0} \cdot q_{x} + q_{0} \cdot p_{x} \right]$$

$$= \frac{1}{2} \left[ \nabla_{e_j} \left[ q_{x} \cdot q_{x-e_j} - q_{0} \cdot q_{x-e_j} \right] + \frac{1}{2} (L - \gamma S) \left[ p_{0} \cdot q_{x} + q_{0} \cdot p_{x} \right] \right]$$
The last term on the right hand side gives a negligible contribution as \( \varepsilon \to 0 \). The first term is a gradient and by summation by part gives

\[
(3.14) \quad \frac{\alpha}{\gamma} \varepsilon^d \sum_{i,j} \sum_{y,z} H(\varepsilon y) \nabla_j^\varepsilon \nabla_i^\varepsilon F(\varepsilon (y+z)) \sum_x g_i^j(x) \\
\int_0^t ds \langle \tau_z[q_x \cdot q_{-e_i} - q_0 \cdot q_{x-e_i}] (s/\varepsilon^2) (\varepsilon_0(0) - \beta^{-1}) \rangle \\
= \frac{\alpha}{\gamma} \varepsilon^d \sum_{i,j} \sum_{y,z} H(\varepsilon y) \nabla_j^\varepsilon \nabla_i^\varepsilon F(\varepsilon z) \sum_x g_i^j(x) \\
\int_0^t ds \langle \tau_z[q_x \cdot q_{-e_i} - q_0 \cdot q_{x-e_i}] (s/\varepsilon^2) (\varepsilon_0(y) - \beta^{-1}) \rangle
\]

Observe that

\[
(3.15) \quad \sum_z G(\varepsilon z) \sum_x g_i^j(x) \tau_z[q_x \cdot q_{-e_i} - q_0 \cdot q_{x-e_i}] \\
= \sum_z G(\varepsilon z) \sum_x [g_i^j(x - e_i) - g_i^j(x + e_i)] \tau_z[q_x \cdot q_0](1 + O(\varepsilon)).
\]

We denote by \( c_i^j(x) := g_i^j(x - e_i) - g_i^j(x + e_i) \). Let \( f(z) \) be the function satisfying

\[
(\alpha \Delta - \nu I) f(z) = \delta(z).
\]

By direct computation

\[
\sum_x c_i^j(x) q_x \cdot q_0 = (L - \gamma S)[q_0 \cdot \sum_z \sum_x c_i^j(x) f(z-x) p_z] \\
- p_0 \cdot \sum_z \sum_x c_i^j(x) f(z-x) p_z.
\]

The function \( \tilde{\kappa}_\lambda^j(x) := -\sum_x c_i^j(x) f(z-x) \) decay exponentially fast, since the Fourier transform is given by

\[
\frac{4 \sin(2\pi k_i) \sin(2\pi k_j)}{\omega^2(k)} \frac{1}{\Phi(k) + \lambda^2}.
\]

Therefore one can show that the the term \( (L - \gamma S)(\cdot) \) gives a contribution of order \( \varepsilon \) and (3.14) is equal to

\[
(3.17) \quad \frac{1}{8\pi^2 \gamma} \varepsilon^d \sum_{y,z} \sum_{j,\ell} H(\varepsilon y) \nabla_j^\varepsilon \nabla_\ell^\varepsilon F(\varepsilon (y+z)) \int_0^t ds \sum_x \kappa_i^j(\lambda)(x) \\
\times \langle [p_z \cdot p_{x+z}] (s/\varepsilon^2) (\varepsilon_0(0) - \beta^{-1}) \rangle + O(\varepsilon),
\]

where

\[
\kappa_i^j(\lambda)(x) = \int_{T^d} d\xi \frac{\partial_j \omega^2(\xi) \partial_\ell \omega^2(\xi)}{\omega^2(\xi)} \frac{1}{\Phi(\xi) + \lambda^2} e^{2i\pi \cdot \xi}, \quad \ell, i = 1, \ldots, d.
\]
Consequently the quantity (3.17) is asymptotic to
\[
\frac{1}{8\pi^2\gamma^d} \sum_{y,z} H(\varepsilon y) \nabla_j^o \nabla_k^o F(\varepsilon (y + z))
\times \int_0^t ds \kappa_\lambda^{j,\ell}(0) \langle \varepsilon (s/\varepsilon^2)(e_0(0) - \beta^{-1}) \rangle ^{\ell, j, k}
\]
\[
+ \frac{1}{8\pi^2\gamma^d} \sum_{y,z} H(\varepsilon y) \nabla_j^o \nabla_k^o F(\varepsilon (y + z))
\times \int_0^t ds \sum_{x \neq 0} \kappa_\lambda^{j,\ell}(x) \langle [p_x \cdot p_{x+z}] (s/\varepsilon^2)(e_0(0) - \beta^{-1}) \rangle ^{\ell, j, k}.
\]

We observe that \(\kappa_\lambda^{j,\ell}(0) = 0\) if \(j \neq \ell\) and \(\kappa_\lambda^{j,j}(0) = \kappa_\lambda^{1,1}(0) \forall j = 1, \ldots, d\).
We set
\[
(3.19) \quad \kappa_\lambda := \kappa_\lambda^{1,1}(0) = \int_{\mathbb{T}^d} d\xi \left( \frac{\partial \omega^2(\xi)}{\omega(\xi)} \right)^2 \frac{1}{\Phi(\xi) + \lambda}.
\]
Then (3.18) is equal to
\[
(3.20) \quad \frac{1}{8\pi^2\gamma^d} \sum_{y,y'} H(\varepsilon y) \kappa_\lambda \Delta F(\varepsilon y') \int_0^t ds \langle \varepsilon (s/\varepsilon^2)(e_0(0) - \beta^{-1}) \rangle ^{\ell, j, k}
\times \int_0^t ds \sum_{x \neq 0} \kappa_\lambda^{j,\ell}(x) \langle [p_x \cdot p_{x+z}] (s/\varepsilon^2)(e_0(0) - \beta^{-1}) \rangle ^{\ell, j, k}.
\]

We will prove in the next section that the second term vanishes as \(\varepsilon \to 0, \lambda \to 0\).

4. BOLTZMANN-GIBBS PRINCIPLE

By using as above the Schwarz inequality, all we need to prove is that
\[
(4.1) \quad \sum_{x \neq 0} |\kappa_\lambda^{j,\ell}(x)| \left( \int_0^t ds \varepsilon^{d/2} \sum_{y} F(\varepsilon y)p_y \cdot p_{y+x}(s/\varepsilon^2) \right)^{1/2}
\]
is negligible as \(\varepsilon \to 0\).

We denote the cube in \(\mathbb{Z}^d\) of size \(2\ell + 1\) by \(\Lambda_{\ell} := \{z \in \mathbb{Z}^d : |z| \leq \ell, j = 1, \ldots, d\}\), and for every \(x \in \mathbb{Z}\) we define \(\Psi_{\ell,x} := \frac{1}{|\Lambda_{\ell}|} \sum_{z \in \Lambda_{\ell}} p_z \cdot p_{z+x}\). We observe that in (4.1) we can replace \(p_y \cdot p_{y+x}\) with \(\tau_y \Psi_{\ell,x}\), with a difference \(\sim \varepsilon |\Lambda_{\ell}| \|\kappa_\lambda\|_{\ell}\). We denote by \(\langle \cdot \rangle_{\Lambda_K} = \langle \cdot \rangle_{\Lambda_K, T_K, P_K}\) the micro-canonical expectation in the box \(\Lambda_K\), with \(T_K = \sum_{x \in \Lambda_K} P_x^2\) and
\( P_K = \sum_{x \in \Lambda_K} p_x \). We define \( \tilde{\Psi}_{t,x} := \Psi_{t,x} - \langle \Psi_{t,x} \rangle_{\Lambda_{t+|x|}} \). Then

\[
\left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_y F(\varepsilon y) \tau_y \tilde{\Psi}_{t,x}(s/\varepsilon^2) \right)^2 \right\rangle \leq Ct \sup_f \left\{ \varepsilon^{d/2} \sum_y F(\varepsilon y) \left\langle f \tau_y \tilde{\Psi}_{t,x} \right\rangle - \varepsilon^{-2} \langle f (-S f) \rangle \right\}.
\]

(4.2)

Introduce \( S_{\Lambda_K} = \frac{1}{4(d-1)} \sum_{x,a \in \Lambda_K} \varepsilon_{i,j} \sum_{x = \varepsilon^{d/2} \sum_y F(\varepsilon y) \left\langle f \tau_y \tilde{\Psi}_{t,x} \right\rangle - \varepsilon^{-2} \langle f (-S f) \rangle \}

By the spectral gap of \( S_K \), there exists \( \tilde{\Psi}_{t,x} = S_{\Lambda_{t+|x|}} U_{t,x}, \forall x, \forall \ell \). Moreover, since the spectral gap of \( S_{\Lambda_K} \) is bounded below by \( CK^{-2} \), we have that \( \langle U_{t,x} \tilde{\Psi}_{t,x} \rangle^2 \leq C(\ell + |x|)^2 \langle \tilde{\Psi}_{t,x}^2 \rangle \leq C(\ell + |x|)^2 \beta^{-1} \ell^{-d} \).

Then since \( \langle f \tau_y \tilde{\Psi}_{t,x} \rangle \leq \langle U_{t,x} \tilde{\Psi}_{t,x} \rangle^{1/2} \langle \tilde{\tau}_y f (-S_{\Lambda_{t+|x|}} \tilde{\tau}_y f) \rangle^{1/2} \), we can bound the right hand side of (4.2) by

\[
Ct \sum_{y} \sup_f \left\{ \varepsilon^{d/2} F(\varepsilon y) \left\langle f \tau_y \tilde{\Psi}_{t,x} \right\rangle - \varepsilon^{-2} |\Lambda_{t+|x|}|^{-1} \left\langle \tilde{\tau}_y f (-S_{\Lambda_{t+|x|}} \tilde{\tau}_y f) \right\rangle \right\}
\]

\[
\leq Ct \sum_{y} \sup_f \left\{ \varepsilon^{d/2} F(\varepsilon y) C(\ell + |x|) \langle \tilde{\Psi}_{t,x}^2 \rangle^{1/2} \langle \tilde{\tau}_y f (-S_{\Lambda_{t+|x|}} \tilde{\tau}_y f) \rangle^{1/2} - \varepsilon^{-2} |\Lambda_{t+|x|}|^{-1} \left\langle \tilde{\tau}_y f (-S_{\Lambda_{t+|x|}} \tilde{\tau}_y f) \right\rangle \right\}
\]

\[
\leq Ct \varepsilon^{d/2} \sum_{y} F(\varepsilon y)^2 (\ell + |x|)^{d+2} \langle \tilde{\Psi}_{t,x}^2 \rangle.
\]

Since \( \sum_x \kappa_{\ell}^d(x) (\ell + |x|)^{(d+2)/2} < \infty \) we conclude that

\[
\lim_{\varepsilon \to 0} \sum_{x \neq 0} |\kappa_{\ell}^d(x)| \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_y F(\varepsilon y) \tau_y \tilde{\Psi}_{t,x}(s/\varepsilon^2) \right)^2 \right\rangle^{1/2} = 0.
\]

Setting \( \tilde{\Psi}_{t,x} := \langle \Psi_{t,x} \rangle_{\Lambda_{t+|x|}} \), now we have just to show that

\[
\lim_{\varepsilon \to 0} \sum_{x \neq 0} |\kappa_{\ell}^d(x)| \left\langle \left( \int_0^t ds \varepsilon^{d/2} \sum_y F(\varepsilon y) \tau_y \tilde{\Psi}_{t,x}(s/\varepsilon^2) \right)^2 \right\rangle^{1/2} = 0.
\]

The previous expression is bounded by

\[
t \sum_{x \neq 0} |\kappa_{\ell}(x)| \left\langle \left( \varepsilon^{d/2} \sum_y F(\varepsilon y) \tau_y \tilde{\Psi}_{t,x} \right)^2 \right\rangle^{1/2}.
\]
We observe that

\[
\left\langle \left( \varepsilon^d/2 \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \bar{\Psi}_{\ell,x} \right)^2 \right\rangle \leq \varepsilon^d \sum_{\mathbf{y}, \mathbf{y}'} F(\varepsilon \mathbf{y}) F(\varepsilon \mathbf{y}') \left\langle \tau_{\mathbf{y} - \mathbf{y}'} \bar{\Psi}_{\ell,x} \bar{\Psi}_{\ell,x} \right\rangle \leq \frac{\varepsilon^d}{2} \sum_{\mathbf{y}, \mathbf{y}'} \left( F(\varepsilon \mathbf{y})^2 + F(\varepsilon \mathbf{y}')^2 \right) \left\langle \tau_{\mathbf{y} - \mathbf{y}'} \bar{\Psi}_{\ell,x} \bar{\Psi}_{\ell,x} \right\rangle \leq \|F\|_{L^2(\mathbb{R}^d)}^2 \sum_{|\mathbf{x}| \leq 2(\ell + |\mathbf{x}|)} \left\langle \tau_{\mathbf{z}} \bar{\Psi}_{\ell,x} \bar{\Psi}_{\ell,x} \right\rangle \leq C \|F\|_{L^2(\mathbb{R}^d)}^2 (\ell + |\mathbf{x}|)^d \bar{\Psi}_{\ell,x}^2.
\]

By the properties of the microcanonical measure it holds \( \left\langle \bar{\Psi}_{\ell,x}^2 \right\rangle \leq C_0 \beta^{-2} (\ell + |\mathbf{x}|)^{-2d} \), and therefore

\[
t \sum_{\mathbf{x}, \mathbf{x} \neq \mathbf{0}} |\kappa_{\lambda}(\mathbf{x})| \left\langle \left( \varepsilon^d/2 \sum_{\mathbf{y}} F'(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \bar{\Psi}_{\ell,x} \right)^2 \right\rangle^{1/2} \leq C_1 T \beta^{-1} \|F'\|_{L^2(\mathbb{R}^d)} \|\kappa_{\lambda}\|_{L^1} \frac{1}{\ell^{d/2}},
\]

which vanishes as \( \ell \to \infty \).

5. Appendix A

Let \( \phi(\mathbf{x}, \mathbf{y}) \) a local function on \( \mathbb{Z}^d \times \mathbb{Z}^d \) and define

\[ \Phi = \sum_{\mathbf{x}, \mathbf{y}} \phi(\mathbf{x}, \mathbf{y}) p_{\mathbf{x}} \cdot q_{\mathbf{y}} \]

Consider \( F \in L^2(\mathbb{R}^d) \). We want to find a good upper bound for the variance:

\[
(5.1) \quad \left\langle \left( \int_0^t ds \varepsilon^d/2 \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} S \Phi(s/\varepsilon^2) \right)^2 \right\rangle
\]

By lemma (2.4) in [9], pag. 48, we have

\[
\left\langle \left( \sup_{0 \leq s \leq T} \int_0^t ds \varepsilon^d/2 \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} S \Phi(s/\varepsilon^2) \right)^2 \right\rangle \leq 24 T \varepsilon^2 \left\langle \left( \varepsilon^d/2 \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \Phi \right) (-S) \left( \varepsilon^d/2 \sum_{\mathbf{y}} F(\varepsilon \mathbf{y}) \tau_{\mathbf{y}} \Phi \right) \right\rangle \leq \frac{48 T}{\beta} \varepsilon^{d+2} \sum_{\mathbf{y}, \mathbf{y}'} F(\varepsilon \mathbf{y}) F(\varepsilon \mathbf{y}') \Xi(\mathbf{y} - \mathbf{y}')
\]
where

\[ \Xi(y) = \sum_{x,x',z} \phi(x,x')(\Delta_1 \phi)(z + y, y,x') \Gamma(x + y, x' - z) \]

where \( \Delta_1 \phi \) indicates the discrete laplacian on the first variable of \( \phi \).

Then by Schwarz inequality we can bound (5.1) by

\[ \|F\|^2 \varepsilon^2 \sum_y |\Xi(y)|. \]

6. FORMULAIRE

Some formulas we use:

\begin{align*}
(6.1) \quad S p_x &= 2 \Delta p_x \quad d \geq 2 \\
(6.2) \quad S p_x &= \frac{1}{6} \Delta (4p_x + p_{x+1} + p_{x-1}) \quad d = 1 \\
(6.3) \quad S e_x &= S p_x^2 / 2 = \Delta p_x \quad d \geq 2 \\
(6.4) \quad S e_x &= S p_x^2 / 2 = \frac{1}{6} \Delta (4p_x^2 + p_{x+1}^2 + p_{x-1}^2) \quad d = 1
\end{align*}

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