Sequential Analysis of Chunks of Pure Qubits

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We investigate an advantage for information processing of ordering a set of states over making a global quantum measurement with a fixed number of copies. The setting that we want to investigate is clearer if we introduce Alice and Bob, as usual. Suppose Alice has $N < \infty$ copies of one of two quantum states $\sigma_0$ or $\sigma_1$ and she gives this states to Bob. We find that there are situations where Bob has to make chunks of $l$ states and process them sequentially to optimally distinguish the two hypotheses. We find an expression for the optimal $l$ given some desired error constraints in the case of pure qubits. The results presented here also have relevance in the “it from qubit” program in the sense that they suggest that time arises from information processing.

I. INTRODUCTION

One of the most obvious questions in contemporary physics is that of quantum gravity. A universal phenomenon as gravitation should have a description in terms of the most fundamental theory at hand: quantum theory \cite{1}. Despite big effort, it has remained a mystery how to build a successful quantum theory of gravitation. One of the challenges resides in the fact that quantum phenomena is taken with an absolute causal structure whereas General Relativity does not have one \cite{2, 3, 4, 5}.

This points towards the abandonment of the use of absolute time frames to describe phenomena in quantum systems \cite{6, 7}. If absolute time is not necessary it is however a powerful concept to describe quantum phenomena. Therefore, there is a need to understand the conditions when absolute time would be necessary or convenient in the description of quantum phenomena. This paper shows the necessity of time ordering in a specific setting: optimal communication of two parties using quantum systems. Related to these inquiries is the “it from qubit” program which investigates the possibility that quantum information processing is relevant in the interactions of physical systems such that spacetime properties arise naturally \cite{6, 7, 8, 9}.

The specific task we investigate is one very fundamental to quantum information: quantum state discrimination \cite{10, 12}. In its simplest form, this task consists in being given a state $\rho$ with the promise of being one of two possible states: $\rho = \sigma_0$ or $\rho = \sigma_1$ and construct a measurement which distinguishes them with the lowest possible error, therefore, the erroneous events deserve special attention. We build a positive operator-valued measure (POVM) taking into consideration the error probability $\alpha$ of guessing $\rho = \sigma_1$ when in reality we have $\rho = \sigma_0$ that is called type-I error and, respectively interchanging $\sigma_0$ and $\sigma_1$ we get the type-II error with a probability $\beta$. The POVM that minimizes the average (total) error was found by Helstrom \cite{13}. Now, we can consider $N$ copies of states and form tensor states $\rho^\otimes N$. The probability of success will be higher with more copies, as more resources are available \cite{14}. Then, the problem changes to distinguish between two hypotheses with the lowest possible average error using the given number of copies. Not until recently it has been addressed a dual approach: fixing the desirable $\alpha$ and $\beta$ and letting the number of copies be a random variable \cite{15}. The problem consists in minimizing the average number of copies needed to make a decision with the desired error probabilities $\alpha$ and $\beta$. The classical version of this problem was addressed by A. Wald \cite{16}, he introduced the Sequential Probability Ratio Test (SPRT) that achieves precisely this minimum \cite{17}.

Here we treat the problem of optimal state discrimination with a fixed number of copies $N$ and ask for optimal strategies that fulfill imposed error requirements. We find that there are instances where an ordered set in time is beneficial. This means, we find an advantage of using the SPRT for distinguishing two quantum hypotheses. We have a setting as in Fig. (1). Alice gives a state $\rho^\otimes N$ to Bob and we investigate if slicing this set into $N/l$ chunks of the state $\rho^\otimes l$ and measuring them in an ordering given by a function $f$ is beneficial for Bob in terms of distinguishing which state he was given: $\sigma_0^\otimes N$ or $\sigma_1^\otimes N$.

There should be a trade-off, as measurements with more copies yield less error, however, if we make the chunks too large we will run out of copies for the SPRT, as Alice handles a finite number of copies $N$. Therefore,
The reason of the simple substitution on Eq. (3) is that the global performance of the unambiguous protocol is achieved by an online strategy \[15\]. The online strategy is to apply an unambiguous POVM for each available copy. Being an unambiguous measurement then the probability of success with \(N\) copies coincides with probability of stopping at step \(N\) because this measurement yields a zero error answer. Only if we get an inconclusive outcome we would have to keep on measuring. However, we can wait to have all the \(N\) copies and make a global unambiguous measurement and have a result with the same success probability, therefore there is no gain by the ordering strategy by Bob in the unambiguous protocol.

A drawback for using an unambiguous protocol is that despite that it yields an exact answer, the whole protocol has, in general, a lower probability of success than a two outcome POVM. The reason for this is that is a very restrictive protocol: it allows no errors. Also, for fixed states unambiguous discrimination is possible only in very restrictive cases. It is therefore of interest to study a two-outcome POVM scenario.

III. TWO OUTCOME SEQUENTIAL SHARP POVM

A. Classic SPRT

We now consider a two outcome sharp POVM, i.e., a minimum error scenario \[11\]. Here we restrict to fixed strategies, therefore the POVM that we consider can be written in terms of projectors on two qubits \(\{\omega(0)|\omega(0)\rangle, |\omega(1)\rangle, |\omega(1)\rangle\}\), conveniently written \[19\] as

\[
|\omega(0)\rangle = \cos(\phi - \frac{a\pi}{2}) |x\rangle + \sin(\phi - \frac{a\pi}{2}) |y\rangle.
\]

We have a probability of the outcome \(x\) with the state \(b\) given by

\[
p(x|b) = \cos^2\left[\phi - \frac{a\pi}{2} - (-1)^b\theta\right].
\]

Following the supplemental material of \[15\] the angle \(\phi\) is fixed as the one that minimizes the average sample number, explicitly, \(\phi = \theta/2\). As we are considering fixed measurements then we have independent and identically distributed (i.i.d.) samples and we can apply the theory of sequential analysis by Wald \[16\]. Given a fixed \(N_0 \in \mathbb{N}\), we can calculate lower bounds for the probability of an SPRT to stop with a number of samples equal or less to \(N_0\) \[21\]. Here, the probability of stopping with less than or equal \(N_0\) copies is not equivalent to the probability of success (as in the unambiguous case) because we are still within error bounds meanwhile the unambiguous case admits no error for stopping. We can define a useful variable

\[
z(x) = \log \frac{p(x|0)}{p(x|1)}.
\]
Let us denote for simplicity $p(0|0) = p$ and $p(0|1) = q$. Thus, with a set of outputs $\{x_1, \ldots, x_n\}$, $x_i \in \{0, 1\}$ we have a set of values $\{z(x_1), \ldots, z(x_n)\}$ that we will denote as $\{z_1, \ldots, z_n\}$ for simplicity. Observe that the SPRT consists on studying how the sum

$$Z_n := \sum_{i=1}^n z_i$$

(7)
behaves. The SPRT will consist in observing the behavior of $Z_n$, when this quantity goes out of an interval, the process will stop [16]. The boundaries of the interval are given, in general, by the error constraints that are asked for [22], we define

$$A = \frac{1 - \beta}{\alpha}.$$ (8)

We have chosen to label $n$ in equation [7] as the number when the SPRT stops, which is a random variable, in general different from $N_0$. Suppose we make an experiment $N_0$ times and we get $k$ times the outcome 0, then we would have

$$Z_{N_0} = k \log \frac{p}{q} + (N_0 - k) \log \frac{1 - p}{1 - q}. \quad (9)$$

From where we can get an expression for $k$ in terms of the rest,

$$k = \frac{Z_{N_0} - N_0 \log \frac{1-p}{1-q}}{\log \left(\frac{p(1-q)}{q(1-p)}\right)}. \quad (10)$$

Let us take $N_0$ as fixed positive integer. Remembering the stopping rule, the SPRT will stop when

$$Z_n \geq \log A \quad \text{or} \quad Z_n \leq \log B. \quad (11)$$

Therefore, the interesting probabilities to calculate will be

$$\Pr(Z_{N_0} \geq \log A) \leq \Pr(n \leq N_0) \quad \text{and} \quad \Pr(Z_{N_0} \leq \log B) \leq \Pr(n \leq N_0). \quad (12)$$

We can rewrite the stopping condition for $N_0$, $Z_{N_0} \geq \log A$ as,

$$\frac{Z_{N_0} - N_0 \log \frac{1-p}{1-q}}{\log \left(\frac{p(1-q)}{q(1-p)}\right)} \geq \log A - N_0 \log \frac{1-p}{1-q} := k(N_0) \quad (13)$$

where we have defined $k(N_0)$ as the right-hand side of the inequality [13]. If we denote $B_{n,p}(k)$ the probability that a binomial random variable with $n$ trials and bias $p$ to have a number of successes less than or equal to $\lambda$ then, the probability that such random variable takes a value $\geq \lambda$ is given by $1 - B_{n,p}(\lambda)$. Therefore, the probability that equation [13] holds is given by

$$\Pr(Z_{N_0} \geq \log A) = 1 - B_{N_0,p}(k(N_0)) \leq \Pr(n \leq N_0). \quad (14)$$

recalling equation [12] for the last inequality. Therefore, $1 - B_{N_0,p}(k(N_0))$ is a lower bound for the probability that $n \leq N_0$ when $H_0$ is true.

There is a correspondent bound for when the $Z_{N_0}$ hits the other boundary and the analysis is similar, this would correspond to $\Pr(Z_{N_0} \leq \log B) \leq \Pr(n \leq N_0)$. The condition $Z_{N_0} \leq \log B$ is rewritten as

$$\frac{Z_{N_0} - N_0 \log \frac{1-p}{1-q}}{\log \left(\frac{p(1-q)}{q(1-p)}\right)} \leq \frac{\log B - N_0 \log \frac{1-p}{1-q}}{\log \left(\frac{p(1-q)}{q(1-p)}\right)} := r(N_0). \quad (15)$$

Then, following the definitions above the probability that a binomially distributed random variable to take a value less than or equal $r$ is $B_{n,p}(r)$, therefore

$$\Pr(Z_{N_0} \leq \log B) = B_{N_0,p}(r(N_0)) \leq \Pr(n \leq N_0). \quad (16)$$

B. Collective measurements: chunks

Now we can ask for collective measurements. Returning to our original question: Bob has $N$ copies of pure states and makes collective measurements such that he has $N_0 = N/l$ chunks of $l$ copies. Bob uses an ordering given by a function $f : \mathbb{N} \to \mathbb{N}$ that for our purposes can be any function that assigns order to a finite set. The probabilities $p$ and $q$ depend also on the chunk size $l$. The reason for this dependence comes from the fact that the effective “angle” between chunks is a function of $l$, i.e. $\Theta(l)$. Recall that the angle $\Theta$ goes exponentially fast to $\pi/2$ with respect to $l$ [19].

$$\cos 2\Theta = \cos^2 2\theta =: c^l. \quad (17)$$

Using suitable identities on inverse trigonometric functions and a bit of algebra, we can write the following probabilities,

$$p(\theta, l) = \cos^2 \left(\frac{\Theta}{2}\right) = \frac{1}{2} \left(1 + \sqrt{\frac{1+c^l}{2}}\right), \quad (18)$$

$$q(\theta, l) = \cos^2 \left(\frac{3\Theta}{2}\right) = \frac{1}{2} \left(1 + \sqrt{\frac{1+c^l}{2}}\right) \times$$

$$\times \left(1 - 2\sqrt{\frac{1+c^l}{2}}\right)^2. \quad (19)$$

Given a fixed $N$, $l$ and $\theta$, as well as error constraints $\alpha$ and $\beta$ to meet we can obtain an expression for $k$. We also have the obvious constraint

$$k \leq N/l \quad (20)$$

which basically says that we cannot measure copies that are not available. For small values of $\alpha$ and $\beta$ we have that the constraint [20] becomes nontrivial and makes a hard cut in the values of $k$, which means that there is an $l$ such that taking larger chunks would be detrimental.
because there is no more resources (chunks of states) to make measurements such that the error requirements are fulfilled. This can be seen in Fig. 2.

The \( k(N_0) \) curve decreases with \( l \), which means that larger chunks yield lower error, as expected [13]. However, there is a point where it is no longer possible to have larger chunks because the available resources are finite. The optimal \( l \) is the maximum one within available resources, that corresponds when the \( k(N_0) \) curve crosses the \( N/l \) curve, i.e. when \( k = N/l \). Such condition yields the equality

\[
e^{-\frac{A}{2l}} = (1 - \sqrt{2\sqrt{1 + c^2}})^{-2}.
\]

Observe that \( (1 - \sqrt{2\sqrt{1 + c^2}})^{-2} \) grows fast until it is very near a constant \( (1 - \sqrt{2})^{-2} \) and \( e^{-\frac{A}{2l}} \) grows exponentially with \( l \) therefore, the equality [21] is reached normally when the right hand is near its constant value, we can thus make the approximation \( c^2 \approx 0 \) then we get

\[
l_0 \approx -\frac{2N}{\log A} \log((1 - \sqrt{2})).
\]

This is a value for when \( \rho = \sigma_0 \) the other case, \( \rho = \sigma_1 \) yields a different \( l \) calculated with the same techniques, just doing the change \( A \rightarrow B \) and thus obtaining a \( l_1 \). The \( l \) that saturates inequality [20] is the maximum possible \( l \) such that the SPRT fulfills the error constraints.

In the scenario that we don’t know a priori which state we have been given we note that a priori that perhaps it can be detrimental to make the chunks too large if the other hypothesis is true. Therefore, a chunk size that takes this into account both possible hypotheses is

\[
l = \min\{l_0, l_1\}.
\]

IV. CONCLUSIONS

One of the departures of quantum physics from classical physics is that a composite Hilbert space admits entangled operators. These operators can yield more powerful measurements than those classically accessible. However, these entangled measurements imply a simultaneous and instant operation. We investigate if an information-processing task implies an abstract notion of time. First, we notice the existence of unambiguous protocols where there is no difference between global and sequential strategies. Afterwards, we find an instance were entanglement on subsets is useful if the subsets are not too large. The point of not considering larger sets is to have enough copies so that the statistical processing becomes trustworthy. We thus find the optimal size of the sets, or chunks. The statistical process reveals to us that we have been using implicitly an ordering in “time” of a set of copies.

Being more explicit, we fix the error bounds we want and use a statistical test that minimizes the average number of samples. The treatment here was with the most simple quantum strategy that involves only pure states and fixed measurements. Perhaps an adaptive strategy in the measurement apparatus gives more insight into when sequential information processing is necessary [20].

The possible relationship between the necessary subjective notion of time ordering and other abstract notions of time is a topic for future research. Similar to other approaches here we are adopting an operationalist perspective [2]. The extra key ingredient that we consider is the processing of copies of states by Bob.

The possible metaphysical implications of this work ask for some clarifications. We should stress that the results from this article are not related to any issue of consciousness in quantum mechanics or the like. The main point here is that if information processing is related to physical interactions then an implicit ordering would be useful. A physical interaction would then imply “temporalization” or the arise of the necessity of time.

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[1] Carlo Rovelli. *Quantum Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2004.
