Finite-size energy of non-interacting Fermi gases

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Abstract. We prove the asymptotics of the difference of the ground-state energies of two non-interacting \(N\)-particle Fermi gases on the half line of length \(L\) in the thermodynamic limit up to order \(1/L\), called finite-size energy. In the nineties Affleck and co-authors \cite{Aff97, ZA97, AL94} claimed that the finite-size energy equals the decay exponent occurring in Anderson’s orthogonality catastrophe. It turns out that the finite-size energy depends on the details of the thermodynamic limit and typically also includes a linear term in the scattering phase shift.

1. Introduction

Given two non-interacting \(N\)-particle Fermi gases confined to a finite volume \(\Lambda_L \subset (0, \infty)\) which differ by a local scattering potential, one can ask for two intimately connected asymptotics. The first one is the asymptotics of the ground-state overlap \(\langle \Phi^N_L, \Psi^N_L \rangle\) whereas the second related question is the asymptotics of the difference of the ground-state energies \(E^N_L - E^N_L\), both in the thermodynamic limit at some given Fermi energy \(E\), i.e. \(N/L \to \rho(E)\). Here, \(\rho\) is the integrated density of states of the unperturbed one-particle Schrödinger operator. Both asymptotics are related to effects where a sudden change by a scattering potential occurs, e.g. the Fermi edge singularity or the Kondo effect, see \cite{AL94}.

On the one hand the ground-state overlap vanishes as
\[
\langle \Phi^N_L, \Psi^N_L \rangle \sim L^{-\gamma(E)/2},
\]
where
\[
\gamma(E) := \frac{1}{\pi^2} \text{tr} \left( \arcsin \left| T(E)/2 \right|^2 \right)
\]
and \(T(E)\) equals the transition matrix of the unperturbed and perturbed one-particle Schrödinger operator. This relation was indicated by \cite{And67b} and \cite{And67a} and is now called Anderson’s orthogonality catastrophe. It was recently proved as an upper bound in \cite{GKMO14}, see also \cite{GKM14} and \cite{KOS13}.

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On the other hand, restricting ourselves to thermodynamic limits of the form \( \frac{N}{L} + O\left(\frac{1}{L}\right) = \rho(E) \) the difference of the ground-state energies admits the asymptotics

\[
E^N_L - E^N_N = \int_{-\infty}^{E} dx \xi(x) + \sqrt{\frac{E\pi}{L}} x_{FS}(E) + o\left(\frac{1}{L}\right)
\]

(1.3)
as \( N, L \to \infty \), where \( \xi \) is the spectral shift function for the unperturbed and perturbed one-particle Schrödinger operator. In the physics literature the first term is sometimes called the Fumi term and \( x_{FS} \) the finite-size correction or energy, see [Aff97]. For models on the half line with a local perturbation, the finite-size correction appearing in the energy difference was predicted to be the same as the exponent \( \gamma \) given in Anderson’s orthogonality catastrophe as well as the Fermi edge singularity exponent, see [Aff97, ZA97, AL94]. Afterwards there was some controversy about the correctness of the results given in the latter concerning the equality of these exponents, see [OF96, QFY96, QFY+97, OF97, EG97]. In this note we give a short and elementary proof of the correct asymptotics of the difference of the ground-state energies in the thermodynamic limit, see Theorem 2.2. It turns out that the finite-size energy heavily depends on the choice of the thermodynamic limit. Moreover, there is precisely one choice of the particle number and system size such that the finite-size energy equals the Anderson exponent. For other choices of the thermodynamic limit an additional linear term in the spectral shift function, or equivalently in the scattering phase shift occurs, see Corollary 2.4 below.

2. Model and results

We consider a non-negative, continuous potential \( 0 \leq V \in C((0, \infty)) \) satisfying

\[
\int_0^\infty dx V(x) \left(1 + x^2\right) < \infty.
\]

(2.1)

Moreover, let \( L > 1 \) and \(-\Delta_L\) be the negative Laplacian on the interval \((0, L)\) with Dirichlet boundary conditions. Define the finite-volume one-particle Schrödinger operators

\[
H_L := -\Delta_L \quad H'_L := -\Delta_L + V.
\]

(2.2)

Here, \( V \) is understood as the canonical restriction of \( V \) to the interval \((0, L)\). These are densely defined self-adjoint operators on the Hilbert space \( L^2((0, L)) \) with compact resolvents. Thus, both operators admit an ONB of eigenfunctions and we denote the corresponding non-decreasing sequences of eigenvalues, counting multiplicities by \( \lambda^L_1 \leq \lambda^L_2 \leq \cdots \) and \( \mu^L_1 \leq \mu^L_2 \leq \cdots \). Note that \( \lambda^L_n = \left(\frac{n\pi}{L}\right)^2 \), \( n \in \mathbb{N} \), see e.g. [RS78]. Moreover, we write \( H := -\Delta \) and \( H' := -\Delta + V \) for the corresponding infinite volume operators on \( L^2((0, \infty)) \) with Dirichlet boundary condition at the origin.

Given \( N \in \mathbb{N} \), the induced (non-interacting) finite-volume fermionic \( N \)-particle Schrödinger operators \( H_L \) and \( H'_L \) act on the totally antisymmetric
subspace $\bigwedge_{j=1}^{N} L^2((0, L))$ of the $N$-fold tensor product space and are given by

$$\hat{H}_L^{(i)} := \sum_{j=1}^{N} I \otimes \cdots \otimes I \otimes \hat{H}_L^{(i)} \otimes I \otimes \cdots \otimes I,$$  \hspace{1cm} (2.3)$$

where the index $j$ determines the position of $H_L^{(i)}$ in the $N$-fold tensor product of operators. The corresponding ground-state energies are given by the sum of the $N$ smallest eigenvalues

$$E_L^N := \sum_{k=1}^{N} \lambda_k^L \quad \text{and} \quad E_L^{\prime N} := \sum_{j=1}^{N} \mu_j^L.$$  \hspace{1cm} (2.4)$$

We are interested in the difference of the ground state energies in the thermodynamic limit at a given Fermi energy $E > 0$. Thus, given $E > 0$ and the number of particles $N \in \mathbb{N}$, we choose the system length $L$ such that

$$\frac{N}{L} \to \frac{\sqrt{E}}{\pi} = \rho(E),$$  \hspace{1cm} (2.5)$$

where $\rho$ is the integrated density of states of the infinite-volume operator $H$.

For $k > 0$ we denote by $\delta(k)$ the phase shift corresponding to the pair of operators $H$ and $H'$ at the energy $k^2 > 0$. Since $V \geq 0$, the phase shift is non-positive, i.e. for $k > 0$

$$\delta(k) \leq 0.$$  \hspace{1cm} (2.6)$$

Then, the scattering matrix for the pair $H$ and $H'$ at the energy $E$ equals $S(E) = \exp\left(2i\delta(\sqrt{E})\right)$. Note that on the half line, the scattering matrix is just a complex number of modulus $1$. For a definition of the phase shift see e.g. Appendix A, [RS79, Chapter. XI.8] or [Cal67].

**Remark 2.1.**

(i) Let $\xi$ be the spectral shift function for the pair of operators $H, H'$. Then, we have the identity [BY92]

$$\frac{1}{\pi} \delta(\sqrt{E}) = -\xi(E),$$  \hspace{1cm} (2.7)$$

for every $E > 0$.

(ii) Let $T(E) := S(E) - 1$ be the transition matrix. Then, we define for $E > 0$

$$\gamma(E) := \frac{1}{\pi^2} \text{tr}(\text{arcsin} |T(E)/2|^2) = \frac{1}{\pi^2} \delta^2(\sqrt{E}).$$  \hspace{1cm} (2.8)$$

This constant equals the decay exponent in [GKMO14] which determines the asymptotics of the exponent in Anderson’s orthogonality catastrophe, i.e. the asymptotics (1.1) of the scalar product of the ground states of the pair of operators $\hat{H}_L$ and $\hat{H}_L^{(i)}$ in the thermodynamic limit.

Using the notation of Remark 2.1 our main result is the following:
Theorem 2.2. For all Fermi energies $E > 0$ the difference of the ground-state energies admits the asymptotics

$$E_N^N - E_N^N = -\frac{1}{\pi} \int_0^{(\frac{N\pi}{L})^2} dx \delta(\sqrt{x}) + \frac{\sqrt{E}}{L} \left( -\delta(\sqrt{E}) + \frac{1}{\pi} \delta^2(\sqrt{E}) \right) + o \left( \frac{1}{L} \right)$$

(2.9)

$$= \int_0^E dx \xi(x) + \int_E^{(\frac{N\pi}{L})^2} dx \xi(x) + \frac{\sqrt{E} \pi}{L} (\xi(E) + \gamma(E)) + o \left( \frac{1}{L} \right)$$

(2.10)

as $N, L \to \infty$, and $\frac{N}{L} \to \frac{\sqrt{E}}{\pi}$.

Remark 2.3. Since $\xi$ is continuous, see Lemma 3.2 below,

$$\int_E^{(\frac{N\pi}{L})^2} dx \xi(x) = \left( \left( \frac{N\pi}{L} \right)^2 - E \right) \xi(E) + o \left( \left( \frac{N\pi}{L} \right)^2 - E \right)$$

(2.11)

as $\frac{N}{L} \to \frac{\sqrt{E}}{\pi} > 0$. This immediately implies that the asymptotics depend on the rate of convergence of the thermodynamic limit and that the finite-size energy defined in (1.3) is non-universal. In general, it may even be $L$ dependent.

Remark 2.4 implies for the particular family of thermodynamic limits considered in the introduction the following:

Corollary 2.4 (Finite-size energy). For a given Fermi energy $E > 0$, some particle number $N \in \mathbb{N}$ and $a \in \mathbb{R}$ we choose the system length $L$ such that

$$\frac{N + a}{L} := \frac{\sqrt{E}}{\pi}.$$  

(2.12)

Then the finite-size energy $x_{FS}$ defined in (1.3) equals

$$x_{FS}(E) = (1 - 2a)\xi(E) + \gamma(E).$$

(2.13)

Thus,

(i) for the particular choice $a = \frac{1}{2}$ the finite-size energy is

$$x_{FS}(E) = \gamma(E),$$

(2.14)

(ii) whereas for the choice $a = 0$ the finite-size energy equals

$$x_{FS}(E) = \xi(E) + \gamma(E).$$

(2.15)

Remark 2.5. (i) In our case of $V \geq 0$ the integrals in Theorem 2.2 may start from 0, since $\delta(x) = 0$ for $x \leq 0$.

(ii) The first term in the expansion is not surprising since

$$E_N^N - E_N^N = \int_{-\infty}^E dx \xi_L(x) + o(1),$$

(2.16)

where $\xi_L$ is the finite-dimensional spectral shift function. And $\xi_L \to \xi$ weakly as $L \to \infty$, see [HM10] for definitions and details.
(iii) The same result with an analogous proof holds also for a Dirac $\delta$-perturbation or s-wave scattering in three dimensions. In the special case of the Neumann and Dirichlet Laplacian $H := -\Delta^N$ and $H' := -\Delta^D$ on $L^2((0, \infty))$ the proof is even simpler since the phase shift is constant $\delta(\sqrt{E}) = \frac{\pi}{2}$.

(iv) We chose $V \geq 0$ since we want to avoid bound states or zero-energy resonances. Moreover, the integrability assumption (2.1) on $V$ ensures sufficient regularity of the phase shift $\delta$. In contrast, the continuity condition on $V$ is only technical and due to the references we use.

(v) Our result allows also a conclusion for the same problem on $\mathbb{R}$ with a symmetric perturbation $V$ because in this case the problem is reduced to two problems on the half axis with either Neumann or Dirichlet boundary condition at the origin.

3. Proof of Theorem 2.2

We start with a lemma relating the eigenvalues of the pair of finite-volume operators.

**Lemma 3.1.** Let $\delta$ be the phase shift for the pair of operators $H$ and $H'$ then the $n$th eigenvalues of $H_L$ and $H'_L$ satisfy

$$\sqrt{\mu_n} = \sqrt{\lambda_n} - \frac{\delta(\sqrt{\mu_n})}{L} + o\left(\frac{1}{L^2}\right),$$

where the error depends only on the potential $V$.

This follows directly from introducing Prüfer variables in the theory of Sturm-Liouville operators.

We have to investigate the behaviour of $\delta$ at $k = 0$ to obtain suitable error estimates.

**Lemma 3.2.** Let $\delta$ be the phase shift corresponding to the operators $H$ and $H'$. Then, $\delta \in C^2((0, \infty))$ and there exists a constant $c$, depending on the potential $V$, such that for all $k > 0$

(i) $|\delta(k)| \leq c \min\{k, \frac{1}{k}\}$, in particular $\delta \in L^\infty((0, \infty))$.

(ii) $\delta' \in L^\infty((0, \infty))$.

(iii) $|\delta''(k)| \leq \frac{c}{k}$.

Moreover,

(iv) we have the following expansion of the phase shift

$$\delta(\sqrt{\mu_n}) = \delta(\sqrt{\lambda_n}) - \frac{\delta'(\sqrt{\lambda_n})\delta(\sqrt{\lambda_n})}{L} + \frac{F(\sqrt{\lambda_n})}{L^2},$$

where the remainder term obeys for $x > 0$

$$|F(x)| \leq c \left(\frac{1}{x} + 1\right)$$

(3.3)
for some constant $c$ depending on the potential $V$.

Lemma 3.1 and 3.2 are well known to experts in the theory of Sturm-Liouville operators. Unfortunately, we did not find a precise reference. For convenience, we prove both results in Appendix A.

The third ingredient to the proof of Theorem 2.2 is the following:

**Lemma 3.3. (Euler-MacLaurin)**

(i) Let $f \in C^1((0, \infty))$ then
\[
\frac{1}{L} \sum_{n=1}^{N} f \left( \frac{n}{L} \right) = \int_{0}^{\frac{N}{L}} dx f(x) + O \left( \frac{N}{L^2} \right).
\] (3.4)

(ii) Let $f \in C^2((0, \infty))$ with $f'' \in L^\infty((0, \infty))$ then
\[
\frac{1}{L} \sum_{n=1}^{N} f \left( \frac{n}{L} \right) = \int_{0}^{\frac{N}{L}} dx f(x) + \frac{1}{2L} \int_{0}^{\frac{N}{L}} dx f'(x) + O \left( \frac{N}{L^3} \right).
\] (3.5)

The proof of this lemma is elementary, see also [Kno96, Chapter XIV].

**Proof of Theorem 2.2** Using Lemma 3.1, we obtain
\[
\sum_{n=1}^{N} (\mu_n - \lambda_n) = \sum_{n=1}^{N} \left( -2\sqrt{\lambda_n} \delta(\sqrt{\mu_n}) + \frac{\delta^2(\sqrt{\mu_n})}{L^2} \right) + o \left( \frac{N}{L^2} \right)
\] (3.6)

On the other hand Lemma 3.2 (iv) provides
\[
\sum_{n=1}^{N} \left( -2\delta(\sqrt{\lambda_n}) \sqrt{\lambda_n} \delta(\sqrt{\mu_n}) + \frac{2\delta'(\sqrt{\lambda_n}) \delta(\sqrt{\mu_n}) \sqrt{\lambda_n}}{L^2} + \frac{\delta^2(\sqrt{\lambda_n})}{L^2} \right)
\] (3.7)
\[
+ \frac{1}{L^3} \sum_{n=1}^{N} G(\sqrt{\lambda_n}) + o \left( \frac{N}{L^2} \right),
\] (3.8)

where
\[
G(x) = \left( -2\delta'(x) \delta^2(x) - 2xF(x) + \frac{1}{L} (\delta'(x) \delta(x))^2 + 2\delta(x)F(x) \right)
\] (3.9)
\[
+ \frac{2}{L^2} F(x) \delta'(x) \delta(x) + \frac{1}{L^3} F^2(x) \). (3.10)

Since $\lambda_n = \left( \frac{n\pi}{L} \right)^2$, $\frac{N}{L} \to \sqrt{E}$, using Lemma 3.2 (i)-(iii) and (3.3), we obtain for the error
\[
\frac{1}{L^3} \sum_{n=1}^{N} G(\sqrt{\lambda_n}) = O \left( \frac{1}{L^2} \right).
\] (3.11)

Note that by Lemma 3.2 the function $f : x \mapsto x\delta(x)$ fulfills the assumptions of Lemma 3.3 (ii). Thus, we compute
\begin{align*}
\sum_{n=1}^{N} \frac{2\delta(\sqrt{\lambda_n})\sqrt{\lambda_n}}{L} &= -\frac{1}{L} \sum_{n=1}^{N} 2\delta \left( \frac{n\pi}{L} \right) \frac{n\pi}{L} \\
&= -\int_{0}^{N/L} dx 2\delta(x\pi)(x\pi) - \frac{1}{L} \int_{0}^{N/L} dx (\delta(x\pi)(x\pi))' + O \left( \frac{N}{L^3} \right) \\
&= -\frac{1}{\pi} \int_{0}^{(N/L)^2} dx \delta(\sqrt{x}) - \frac{1}{L} \delta(\sqrt{E})\sqrt{E} + o \left( \frac{1}{L} \right),
\end{align*}

where we used in the last equality the convergence \( \frac{N}{L} \to \frac{\sqrt{E}}{\pi} \) and the continuity of \( \delta \). Using Lemma 3.2 we see that \( g : x \mapsto x\delta(x)\delta'(x) \) satisfies the assumptions of Lemma 3.3 (i) with \( \|g'\|_{L^\infty((0,\infty))} \leq c(1 + \frac{N}{L}) \). Therefore,

\begin{align*}
\sum_{n=1}^{N} \frac{2\delta'(\sqrt{\lambda_n})\sqrt{\lambda_n}}{L^2} &= \frac{1}{L} \left( \frac{1}{L} \sum_{n=1}^{N} 2\delta' \left( \frac{n\pi}{L} \right) \frac{n\pi}{L} \right) \\
&= \frac{1}{L} \int_{0}^{N/L} dx 2\delta'(x\pi)(x\pi) + O \left( \frac{N}{L^3} \right) \left( 1 + \frac{N}{L} \right) \\
&= \frac{1}{L\pi} \left( \delta'(\sqrt{E})\sqrt{E} - \int_{0}^{(N/L)^2} dx \delta'(x\pi) \right) + o \left( \frac{1}{L} \right),
\end{align*}

where we used integration by parts, the convergence \( \frac{N}{L} \to \frac{\sqrt{E}}{\pi} \) and the continuity of \( \delta \) in the last line. Lemma 3.2 yields the assumptions of Lemma 3.3 (i) for \( h : x \mapsto \delta^2(x) \) with \( h' \in L^\infty((0,\infty)) \). Thus,

\begin{align*}
\sum_{n=1}^{N} \frac{\delta^2(\sqrt{\lambda_n})}{L^2} &= \frac{1}{L} \left( \frac{1}{L} \sum_{n=1}^{N} \delta^2 \left( \frac{n\pi}{L} \right) \right) \\
&= \frac{1}{L} \int_{0}^{(N/L)^2} dx \delta^2(x\pi) + o \left( \frac{1}{L^2} \right).
\end{align*}

Summing up (3.14), (3.17), (3.19) and equations (3.6), (3.11) give the claim. \( \square \)

### Appendix A. Prüfer variables and the phase shift

Our approach to the phase shift uses a non-linear ODE called the variable-phase equation, see e.g. [Cal67] or [RS79, Thm. XI.54]. Let \( k > 0 \). First note that there is a unique solution \( \delta_k \) of

\begin{equation}
\delta_k'(x) = -\frac{1}{k} V(x) \sin^2(kx + \delta_k(x)), \quad x > 0
\end{equation}

with the boundary condition \( \limsup_{x \to 0} \frac{1}{2} |\delta_k(x)| < \infty \). We call this solution the phase-shift function. Moreover,

\begin{equation}
\lim_{x \to \infty} \delta_k(x) = \delta(k)
\end{equation}

is the phase shift for \( H \) and \( H' \).
On the other hand consider the eigenvalue problem on $(0, \infty)$

$$-u'' + Vu = k^2 u, \quad u(0) = 0.$$  
(A.3)

Introducing Prüfer variables

$$u(x) = \rho_u(x) \sin(\theta_k(x)) \quad u'(x) = k \rho_u(x) \cos(\theta_k(x)),$$
(A.4)

is equivalent to the system

$$\theta'_k = k - \frac{1}{k} V \sin^2(\theta_k), \quad \theta_k(0) = 0,$$
(A.5)

$$\rho'_u = \frac{V}{2k} \rho_u,$$
(A.6)

see e.g. [Tes12, Section 5.5]. We call $\theta_k$ the Prüfer angle. Note that $\rho_u(x) \neq 0$ for all $x \geq 0$. We did not choose the standard Prüfer variables since we want to compare the Prüfer angle with the phase-shift function. Given the phase-shift function in (A.1) we obtain a solution $\theta_k$ to (A.5) by setting

$$\theta_k(x) := \delta_k(x) + kx,$$
(A.7)

Since any solution of (A.5) fulfills $|\theta_k(x)| \leq kx$, see (A.9) below, we obtain that $\delta_k(x) := \theta_k(x) - kx$ is the unique solution of (A.1). This implies uniqueness of $\theta_k$ and

$$\delta(k) = -\frac{1}{k} \int_0^\infty dt V(t) \sin^2(\theta_k(t)).$$
(A.8)

We state some properties of the Prüfer angle, respectively of the phase-shift function, which we use in the sequel.

**Proposition A.1.** Given $k > 0$, let $\delta_k$ and $\theta_k$ be the solution of (A.1), respectively (A.3). Fix $x > 0$. Then,

(i) $\theta_k(x)$ is non-negative, moreover,

$$0 \leq \theta_k(x) \leq kx.$$  
(A.9)

(ii) we have

$$\lim_{k \to 0} \theta_k(x) = 0, \quad \lim_{k \to \infty} \theta_k(x) = \infty.$$  
(A.10)

(iii) the functions $k \mapsto \theta_k(x)$ and $k \mapsto \delta_k(x)$ are smooth, i.e.

$$\theta_{(\cdot)}(x), \quad \delta_{(\cdot)}(x) \in C^\infty((0, \infty)).$$  
(A.11)

(iv) the derivative of the Prüfer angle with respect to the energy is strictly positive, i.e.

$$\frac{\partial}{\partial k} \theta_k(x) > 0.$$  
(A.12)

**Proof of Proposition A.1.** For (i) first note that $\lim_{x \to \infty} \theta'_k(x) = k > 0$ and $\theta'_k(x) > 0$ for all $x > 0$ such that $\theta_k(x) = 0$. Since $\theta_k(0) = 0$ and $\theta_k \in C^1((0, \infty))$ we have $\theta_k > 0$. One the other hand $k - \frac{1}{k} V \sin^2(y) \leq k$, $y \in \mathbb{R}$, since $V \geq 0$. This yields $\theta_k(x) \leq kx$, see e.g. [Har64, Chapter III, 4.2].

The first equality in (ii) follows by (i). For the second equality observe $\theta_k(x) \geq kx - \frac{1}{k} \|V\|_1$, where $x, k > 0$ and $\|\cdot\|_1$ denotes the $L^1((0, \infty))$ norm.
For (iii) note that \( k \mapsto k - \frac{1}{k} V(x) \sin^2(y) \in C^\infty((0, \infty)) \) for fixed \( x > 0, y \in \mathbb{R} \). Then, standard results imply that the solution \( \theta_{\nu}(x) \in C^\infty((0, \infty)) \) for fixed \( x > 0 \), see e.g. [Har64], Chapter V, 4.1.

For (iv) note that \( k - \frac{1}{k} V \sin^2(y) \leq k' - \frac{1}{k'} V \sin^2(y) \) for all \( k \leq k', y \in \mathbb{R} \) since \( V \geq 0 \) and use [Har64], Chapter III, 4.2.

**Proof of Lemma 3.1.** Let \( \mu > 0 \). Consider the eigenvalue equation on \([0, L]\)

\[
- u'' + Vu = \mu u, \quad u(0) = 0.
\]  

(A.13)

We introduce Prüfer variables according to (A.4). Note that any eigenfunction \( u \) of \( h'_{L} \) corresponding to an eigenvalue \( \mu \) has to fulfill \( u(L) = 0 \) due to the Dirichlet boundary condition at \( L \). Thus, using \( \rho_u(x) \neq 0 \) for all \( x \geq 0 \), we obtain \( \sin \left( \theta_{\sqrt{\mu}}(L) \right) = 0 \). With (A.10) and (A.12) this implies for the \( n \)th eigenvalue \( \mu_n \) of \( h'_{L} \)

\[
\theta_{\sqrt{\mu_n}}(L) = n\pi.
\]  

(A.14)

Therefore, integrating (A.5) yields

\[
\sqrt{\mu_n} = \frac{n\pi}{L} + \frac{1}{L\sqrt{\mu_n}} \int_{0}^{L} dt V(t) \sin^2(\theta_{\sqrt{\mu_n}}(t)).
\]  

(A.15)

Now, using \(|\sin(x)| \leq |x|\), (A.9), \(|\sin(x)| \leq 1\) and (2.1) we obtain

\[
\frac{1}{\sqrt{\mu_n}} \int_{L}^{\infty} dt V(t) \sin^2(\theta_{\sqrt{\mu_n}}(t)) \leq \int_{L}^{\infty} dt V(t) t
\]

\[
\leq \frac{1}{L} \int_{L}^{\infty} dt t^2 V(t) = o \left( \frac{1}{L} \right).
\]  

(A.16)

(A.17)

Then, (A.8) and \( \sqrt{\lambda_n} = \frac{n\pi}{L} \) give the claim.

**Proof of Lemma 3.2.** Part (i) follows from (A.8), (A.9) and (2.1).

Concerning (ii), we first note that \( \theta_{k} \in C^1((0, \infty)) \) for fixed \( k > 0 \) because \( V \) is assumed to be continuous and \( \theta_{\nu}(x) \in C^\infty((0, \infty)) \) for fixed \( x > 0 \) by (A.11). From now on we consider \( \theta \) as a function of two variables and write, in abuse of notation, the abbreviation \( f_{x} \) for the partial derivative \( \frac{\partial}{\partial x} f \) of a function \( f \in C^1(\mathbb{R}^2) \). Also we drop the \( u \) index of \( \rho \). Then the ODEs (A.5) and (A.6) imply

\[
\left( \rho^2 \frac{\partial}{\partial k} \theta \right)_{x} = 2\rho \frac{\partial}{\partial k} \theta + \rho^2 \frac{\partial}{\partial k} \theta_x
\]

(A.18)

\[
= 2\rho \frac{\partial}{\partial k} \theta + \rho^2 \left( k - \frac{V \sin^2(\theta)}{k} \right)
\]

(A.19)

\[
= 2\rho \frac{\partial}{\partial k} \theta + \rho^2 \left( 1 + \frac{V \sin^2(\theta)}{k^2} - \frac{V \sin(2\theta)}{k} \frac{\partial}{\partial k} \theta \right)
\]

(A.20)

\[
= \rho^2 \left( 1 + \frac{V \sin^2(\theta)}{k^2} \right).
\]  

(A.21)
Integrating the latter yields
\[
\frac{\partial}{\partial k} \theta(x) = \int_0^x dt \frac{\rho^2(t)}{\rho^2(x)} \left( 1 + \frac{V(t) \sin^2(\theta_k(t))}{k^2} \right). \tag{A.22}
\]
The ODE (A.6), (A.9), the elementary inequality $|\sin x| \leq |x|$ and (2.4) imply
\[
\left| \frac{\rho(t)}{\rho(x)} \right| \leq \exp \left( \int_t^x ds V(s) \right) \leq \exp (\|V\|_1) < \infty. \tag{A.23}
\]
From this, (A.9) and $|\sin x| \leq |x|$ we infer the existence of a constant $c$ depending on the potential $V$ such that
\[
\left| \frac{\partial}{\partial k} \theta_k(x) \right| \leq c(1 + x). \tag{A.24}
\]
Then, the above, (A.9) and dominated convergence provide $\delta \in C^1((0, \infty))$ and
\[
|\delta'(k)| \leq c \int_0^\infty dt V(t)(1 + t + t^2). \tag{A.25}
\]
The assumptions on the potential give the claim.

For (iii) we compute as above
\[
\left( \frac{\partial^2}{\partial k^2} \theta \right)_{x} = 2 \rho^2 V \left( \frac{- \sin^2(\theta)}{k} + \frac{\sin(2\theta) \frac{\partial}{\partial k} \theta}{k} - \frac{\cos(2\theta) \left( \frac{\partial}{\partial k} \theta \right)^2}{k} \right). \tag{A.26}
\]
Using (A.9), $|\sin x| \leq |x|$, (A.23) and (A.24), we see
\[
\left| \frac{\partial^2}{\partial k^2} \theta_k(x) \right| \leq \tilde{c} \frac{1}{k}, \tag{A.27}
\]
where $\tilde{c}$ depends on $V$. Then dominated convergence yields $\delta \in C^2((0, \infty))$ and (A.9) and (A.27) provide
\[
|\delta''(k)| \leq \frac{C}{k} \int_0^\infty dt V(t)(1 + t + t^2) \tag{A.28}
\]
for some $C$ depending on the potential $V$.

To prove (iv) we use Lemma 3.1. Thus,
\[
\sqrt{\mu_n} = \sqrt{\lambda_n} + \frac{\delta(\sqrt{\mu_n})}{L} + o\left( \frac{1}{L} \right), \tag{A.29}
\]
Since $\delta \in C^2((0, \infty))$ we compute for $x, y \in (0, \infty)$ with $y > x$ and $y = x + \frac{\delta(y)}{L} + o\left( \frac{1}{L} \right)$
\[
\left| \delta(y) - \delta(x) + \frac{\delta'(x) \delta(x)}{L} \right| \leq \left| \int_x^y dt \int_x^t ds \delta''(s) \right| + \left| \delta'(x) \right| \left| y - x + \frac{\delta(x)}{L} \right| \leq \frac{1}{x} |y - x|^2 + \frac{\|\delta\|_\infty}{L} \left| \int_x^y dt \delta'(t) + o\left( \frac{1}{L} \right) \right|, \tag{A.30}
\]
Using Lemma 3.2 (ii) and once again the recursion relation we obtain
\[
\left| \delta(y) - \delta(x) + \frac{\delta'(x)\delta(x)}{L} \right| \leq \left( \frac{1}{x} + 1 \right) O \left( \frac{1}{L^2} \right).
\] (A.31)

The claim follows from setting \( x := \lambda_n \) and \( y := \mu_n \). \( \square \)

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