SIBLINGS OF COUNTABLE COGRAPHS

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†Dedicated to the memory of Ivo G. Rosenberg

ABSTRACT. We show that every countable cograph has either one or infinitely many siblings. This answers, very partially, a conjecture of Thomassé. The main tools are the notion of well quasi ordering and the correspondence between cographs and some labelled ordered trees.

1. INTRODUCTION

1.1. Thomassé conjecture. A relation \( R \) is a pair \((V, \rho)\) where \( \rho \) is a subset of \( V^n \) for some non negative integer \( n \); this integer is the arity of \( \rho \) (which is also called an \( n \)-ary relation). A sibling of a relation \( R \) is any \( R' \) such that \( R \) and \( R' \) are embeddable in each other. In [10] (p. 2, Conjecture 2), Thomassé made the following:

Conjecture 1.1. Every countable relation \( R \) has 1, \( \aleph_0 \) or \( 2^{\aleph_0} \) siblings, these siblings counted up to isomorphy.

A positive answer was given for chains in [21]. It is not even known if \( R \) has one or infinitely many siblings, even if \( R \) is a countable loopless undirected graph; in fact, this is unsolved for countable trees as we will see later with a conjecture of Bonato and Tardif.

Graphs, and more generally binary structures, can be decomposed in simpler pieces via labelled trees, the bricks being the indecomposable structures. A binary relation \( R := (V, \rho) \) is indecomposable if it has no non-trivial module, alias ”interval”; a module of \( R \) is any subset \( A \) of \( V \) such that for every \( a, a' \in A, b \in V \setminus A \), the equalities \( \rho(a, b) = \rho(a', b) \) and \( \rho(b, a) = \rho(b, a') \) hold; \( A \) is trivial if \( A = \emptyset \), \( A = \{a\} \) for some \( a \in V \), or \( A = V \) (see [11] for the general theory, [15], [16], [17], [5] and [6] for infinite binary structures).

We conjecture that Thomassé’s conjecture reduces to the case of countable indecomposable structures. That is for graphs:

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Conjecture 1.2. A countable graph $G$ has $1$, $\aleph_0$ or $2^{\aleph_0}$ siblings if every induced indecomposable subgraph has $1$, $\aleph_0$ or $2^{\aleph_0}$ siblings.

This paper is a contribution in that direction.

If our conjecture holds, then Thomassé’s conjecture must hold for countable cographs. A cograph is a graph with no induced subgraph isomorphic to a $P_4$, a path on four vertices. As it is well known, no induced subgraph of a cograph with more than two vertices can be indecomposable (see [38] for finite graphs, [18] for infinite graphs).

We prove a weaker version of Thomassé’s conjecture:

Theorem 1.3. A countable cograph has either one or infinitely many siblings.

With the continuum hypothesis (CH), this says that a cograph with countably many vertices has one, $\aleph_0$ or $2^{\aleph_0}$ many siblings. We do hope to get rid of (CH) in a forthcoming publication.

1.2. Connected siblings and the conjectures of Bonato, Bruhn, Diestel, Sprüssel and Tardif. Indecomposable graphs with more than two vertices must be connected, hence a special consequence of our conjecture is the following fact, observed in [19].

Proposition 1.4. If Thomassé’s conjecture holds for connected graphs, it holds for all graphs.

Proof. Suppose that Thomassé conjecture holds for connected graphs. Let $G$ be a countable graph. If $G$ is connected, then it has one, $\aleph_0$ or $2^{\aleph_0}$ many siblings. If $G$ is not connected, then its complement $G^c$ is connected and thus has one, $\aleph_0$ or $2^{\aleph_0}$ many siblings. Since their complements are siblings of $G$, $G$ has one, $\aleph_0$ or $2^{\aleph_0}$ many siblings.

The study of the number of siblings of a direct sum of connected graphs relates to a conjecture of Bonato and Tardif about trees, not ordered trees but connected graphs with no cycles. The tree alternative property holds for a tree $T$, if either every tree equimorphic to $T$ is isomorphic to $T$ or there are infinitely many pairwise non-isomorphic trees which are equimorphic to $T$. Bonato and Tardif [3] conjectured that the tree alternative property holds for every tree and proved that it holds for rayless trees [3]. Laflamme, Pouzet, Sauer [22] proved that it holds for scattered trees (trees in which no subdivision of the binary tree can be embedded). But they could not conclude in the case of the complete ternary tree with some leaves attached. As it turns out, induced subgraphs equimorphic to these trees are connected; hence for such trees the tree alternative property amounts to the fact that every tree has one or infinitely many siblings.

The Bonato-Tardif conjecture was extended to (undirected loopless) graphs by Bonato et al, (2011) [4] in the following ways:

(1) For every connected graph $G$ the number $\text{sib}_{\text{conn}}(G)$ of connected graphs which are equimorphic to $G$ is 1 or is infinite.
(2) For every graph $G$ the number $sib(G)$ of graphs which are equimorphic to $G$ is 1 or is infinite.

The second conjecture is a weakening of Thomasse’s conjecture restricted to graphs. Both conjectures were proved true for rayless graphs by Bonato et al (2011) [4].

Note that the extension of the first conjecture of Bonato et al to binary relations is false. In fact it is false for undirected graphs with loops and for ordered sets.

There is a straightforward relationship between the extension of the Bonato-Tardif conjecture to connected graphs and the weakening of Thomasse’s conjecture for graphs.

To see this, first observe that:

**Lemma 1.5.** If some sibling of a connected graph $G$ is not connected, then $sib(G)$ is infinite.

Indeed, $G$ is equimorphic to the direct sum $G \oplus 1$. In this case, $G \oplus \overline{K}_n$, where $\overline{K}_n$ is an independent set of size $n$, and $n$ any positive integer, is equimorphic to $G$, hence $G$ has infinitely many siblings. Now, we have:

**Proposition 1.6.** If the extension of the Bonato-Tardif conjecture to connected graphs is true then the weakening of Thomasse’s conjecture for graphs is true.

**Proof.** According to Proposition 1.4, a graph has one or infinitely many siblings provided that all connected graphs have one of infinitely many siblings. Let $G$ be a connected graph. If all siblings of $G$ are connected, apply the extension of the Bonato-Tardif conjecture: $G$ has one or infinitely many siblings. If some sibling is not connected, apply Lemma 1.5. □

In November 2016, M.H. Shekarriz sent us a paper in which he proves the same result.

We prove that the extension of the Bonato-Tardif conjecture to countable connected cographs holds:
Theorem 1.7. A countable connected cograph has either one connected sibling or infinitely many connected siblings.

Theorem 1.3 follows from this and Proposition 1.6.

Apart from the one way infinite path, there are several countable connected graphs $G$ with $\text{sib}_{\text{conn}}(G) = 1$ and $\text{sib}(G)$ infinite. This is not the case with countable cographs. For a graph $G$ set $\text{sib}_c(G) = \text{sib}_{\text{conn}}(G)$ if $G$ is connected and $\text{sib}_c(G) = \text{sib}_{\text{conn}}(G^c)$ if $G$ is disconnected. We prove that:

Theorem 1.8. For a countable cograph, the following properties are equivalent

(i) $\text{sib}(G) = 1$;

(ii) $\text{sib}_c(G) = 1$;

(iii) $G$ is a finite lexicographic sum of cliques or independent sets.

The proofs of these two results have two ingredients. One is the tree decomposition of a cograph. Each cograph can be represented uniquely, up to isomorphism by a special type of (ordered) labelled tree. This fact is a special case of a very general result about tree decomposition of binary structures; a result which appears in [6], and based on [10, 11, 15] (a similar approach is in [16, 17, 5]). Due to the importance of the fact mentioned above, we give a detailed presentation and a proof (Theorem 6.19) in the appendix. With that fact, constructions of non isomorphic cographs reduce to constructions of labelled trees. We reduce our counting of siblings to the case of trees with no least element, for which we prove that there are $2^{\aleph_0}$ siblings (Theorem 2.5). This reduction is based on the second ingredient. This is the notion of well quasi ordering. A quasi-ordered set $Q$ is well quasi-ordered (w.q.o. for short) if every infinite sequence $q_0, \ldots, q_n, \ldots$ contains an infinite increasing subsequence $q_{n_0} \leq \cdots \leq q_{n_k} \leq \cdots$. In particular, $Q$ is well founded: every non-empty subset $A$ contains a minimal element (an element $a$ such that no $b < a$, i.e., $b \leq a$ and $a \not\leq b$) is in $A$. This property allows induction on the elements of $Q$. An example is the collection $\text{Cog}_{\leq \omega}$ of countable cographs quasi-ordered by embeddability. Indeed, according to a theorem of Thomassé [39], $\text{Cog}_{\leq \omega}$ is well quasi-ordered by embeddability. The reduction to the case of trees with no least element in Theorem 1.7 relies on a quasi order slightly different from embeddability which turns to be w.q.o. in virtue of Thomassé’s result. The counting of siblings in the case of these trees, which is the most difficult part of the paper, relies on properties of countable labelled chains. Induction can be done if the collection of our countable labelled chains is w.q.o. In general, the collection of countable chains labelled by a w.q.o. is not necessarily w.q.o. A strengthening of this notion is needed, this is the notion of better quasi ordering (in short b.q.o.), invented by C.St.J.A. Nash-Williams [28], as a tool for proving that some posets are w.q.o. No expertise about b.q.o. is needed in this paper, but the reader must be aware that this notion is unavoidable to prove that some posets are w.q.o. According to Laver [24], the collection $\mathcal{Q}^{D_{\leq \omega}}$ of countable chains labelled by a b.q.o $Q$ is b.q.o. (see p. 90 of [24]). The set $Q$ of labels we need is the direct product of the collection $\text{Cog}_{\leq \omega}$ of countable
cographs and the 2-element antichain. Since Thomassé proved in fact that
that \( \mathcal{C}og_{\omega} \) is b.q.o., \( Q \) is b.q.o. so our collection of labelled chains is w.q.o and
we can do induction (see Subsection 2.4). We conclude with some problems
(see Section 5).

Some results about the conjectures above were presented in [33] and also in
[19].

2. Ingredients

2.1. Elementary facts. We start with easy facts.

Lemma 2.1. If a disconnected graph \( G \) has finitely many non-trivial connected
components and each connected component has just one connected sibling then
either \( G \) has infinitely many disconnected siblings or just one sibling.

Proof. Let \( G \) satisfy the conditions of the lemma. Either every component
is trivial, in that case \( G \) is an independent set and \( \text{sib}(G) = 1 \), or some com-
ponent is not trivial. In that case, let \( H_1, \ldots, H_k \) enumerate the non-trivial
components.

Case 1 There is a component \( H \) and a non-trivial and connected graph \( L \)
such that \( H \oplus L \leq H \). In that case, for each \( n \geq 1 \), the direct sum of \( G \) with
the direct sum of \( n \) copies of \( L \) is a disconnected sibling of \( G \), hence \( G \) has
infinitely many disconnected siblings.

Case 2 Otherwise, \( H \oplus L \leq H \) implies that \( L \) cannot be connected. In this
case, if the set \( T \) of trivial components is non empty and for some connected
component \( H, H \oplus K_\kappa \leq H \), where \( \kappa := |T| \) then \( G \) is equimorphic to the
set \( G' \) of non-trivial components, and also to \( G' \oplus 1 \) thus \( G \) has infinitely
many disconnected siblings. If \( T \) is infinite or \( H \oplus 1 \not\leq H \) for each connected
component \( H \) then \( \text{sib}(G) = 1 \). Indeed, suppose that \( G' \subseteq G \) induces a sibling
of \( G \) via an embedding \( \phi \). Since each \( H_i \) is connected the image under \( \phi \) must
be a contained in some component \( H_{\varphi(i)} \) of \( G \). Furthermore, the map \( \varphi \) must
be one to one, hence bijective (otherwise, by iterating it we will find \( i \neq j \) such
that \( H_i \oplus H_j \) embeds into \( H_j \) contradicting our hypothesis). We may suppose
that \( \varphi \) is the identity, so, if \( T \) is infinite, infinitely many elements will stay in
\( T \) and if \( T \) is finite, none will be mapped into some non-trivial component,
hence \( G' \) is isomorphic to \( G \).

\[ \square \]

Lemma 2.2. Let \( G \) be a countable graph. Suppose that some connected com-
ponent has infinitely many connected siblings. Then \( G \) has infinitely many
siblings and if it has at least two connected components, infinitely many are
disconnected.

Proof. Let \( G \) satisfy the conditions of the lemma. Let \( H \) be a con-
nected component with infinitely many non-isomorphic connected siblings
\( H_0 = H, H_1, \ldots, H_k, \ldots \) Let \( G_i \) result from \( G \) by replacing every component
that is a sibling of \( H \) by \( H_i \). Then \( G_i \) is a sibling of \( G \) and the \( G_i \)'s are
pairwise non isomorphic.
If $G$ has has at least two components, the construction yields disconnected siblings.

**Lemma 2.3.** Let $G$ be a countable graph. Suppose that $G$ is disconnected and that there is an infinite sequence $(H_n)_{n<\omega}$ of non-trivial components which is increasing w.r.t. embeddability. Then $G$ has infinitely many disconnected siblings.

**Proof.** First, $G \oplus \overline{K}_\omega$ embeds into $G$. Indeed, if $K$ is the union of the connected components distinct from the $H_n$'s, then $G = \bigoplus_{n \in \mathbb{N}} H_n \oplus K$. Since $\bigoplus_{n \in \mathbb{N}} H_n$ embeds into $\bigoplus_{n \in \mathbb{N}} H_{2n}$, $G \oplus \overline{K}_\omega$ embeds into $G$. Let $G'$ be the restriction of $G$ to the nontrivial components of $G$. Then $G'$ is equimorphic to $G$. And also $G' \oplus \overline{K}_n$. Thus, $G$ has infinitely many siblings and infinitely many are disconnected.

One can obtain a bit more under the stronger hypothesis that $H_n$ embeds in $H_m$ just in case $n < m$. In this case, we can obtain a continuum of non-isomorphic siblings. Let $\mathcal{K}$ be the family of components $K$ for which there is $n$ such that $K$ embeds in $H_n$. Enumerate $\mathcal{K}$ as $\{K_1, \ldots, K_n, \ldots\}$. Let the union of the components not contained in $\mathcal{K}$ be denoted $L$. Let $J$ be an infinite subset of $\{1, 2, \ldots\}$. Set $G_J := L \oplus \bigoplus_{n \in J} H_n$. Since $K_m$ embeds in infinitely many $H_k$, and the same is true for $H_n$ for $n \notin J$, by interleaving we see that $G_J$ is a sibling of $G$.

Now let $J_1 \neq J_2$, and let $i$ be the least element in the symmetric difference. We may suppose that $i \in J_2 \setminus J_1$. Then $G_{J_2}$ has a component isomorphic to $H_i$ while $G_{J_1}$ does not.

In summary, with Lemma 1.5, 2.2 and 2.3 we have:

**Proposition 2.4.** A disconnected graph $G$ has infinitely many disconnected siblings provided that either $G$ is equimorphic to some connected component, or some connected component has infinitely many connected siblings, or there is an infinite sequence $(H_n)_{n<\omega}$ of non-trivial components which is increasing w.r.t. embeddability.

**2.2. Tree decomposition of a cograph.** The crucial result we will use for the proof of Theorems 1.7 and 1.8 is the following.

**Theorem 2.5.** If a countably infinite cograph is connected and its complement too then $G$ has $2^{\aleph_0}$ siblings with the same property.

This relies on properties of the tree decomposition of a cograph. Cographs have a simple structure. They can be obtained from the one vertex graph by iteration of three operations: direct sum, complete sum and sum over a labelled chain. If $(G_i)_{i \in I}$ is a family of at least two non-empty graphs, their direct sum $\bigoplus_{i \in I} G_i$ is the disjoint union of the $G_i$'s with no edge between distinct $G_i$'s; their complete sum $\bigoplus_{i \in I} G_i$ is the disjoint union of the $G_i$'s with any pair of vertices between distinct $G_i$'s connected by an edge. If there is a linear order $\leq$ on $I$ and a labelling $r$ of $I$ by 0 and 1, this structure being denoted $C := (I, \leq, r)$, then the labelled sum denoted by $\sum_{i \in C} G_i$ is the disjoint union of the $G_i$'s, two vertices
SIBLINGS 7

$x \in G_i$, $y \in G_j$, with $i < j$, being linked by an edge if and only $r(i) = 1$. We may view such a sum as the graph associated to the labelled chain $C := (I, \leq, \ell)$ where $\ell(i) := (G_i, r(i))$ and denote it by $\Sigma C$. Due to the associativity of these operations, all possible sums do not need to be considered. The direct or complete sums we need to consider are the direct sums of connected graphs, and the complete sums of graphs whose complements are connected. Since the complement of a finite connected cograph is disconnected, finite cographs are obtained by means of direct or complete sums. Infinite cographs may require labelled sums, but it suffices to consider those indexed by infinite densely labelled chains (that is chains in which the two labels occur in every interval having at least two elements) such that if the label of $i$ is 0, resp. 1, then $G_i$ is not a complete sum, resp. a direct sum, of at least two nonempty cographs, and if $i$ is the largest element of $C$, then $G_i$ is either a direct sum or a complete sum of at least two singletons. We will say that the labelled chain $C$ and its sum are reduced.

With these operations, one has (see Boudabbous and Delhomme [2], Lemma 2.2 page 1747):

**Theorem 2.6.** Let $G$ be a cograph with more than one vertex. Then either

1. $G$ is a direct sum of at least two nonempty connected cographs, or
2. $G$ is a complete sum of at least two nonempty cographs whose complements are connected, or
3. $G$ is the sum $\Sigma C$ of a reduced labelled chain $C := (I, \leq, \ell)$, where $(I, \leq)$ is an infinite chain with no first element, each label $\ell(i)$ is the pair $(G_i, r(i))$ made of a non-empty cograph $G_i$ and an element $r(i) \in \{0, 1\}$ in such a way that $r$ is a dense labelling of the chain $(I, \leq)$.

With some effort, one can (partially) evaluate the number of siblings of a direct sum or a complete sum in relation with the number of siblings of its components. For the case of labelled chains, more substantial information is needed. This information comes from the tree decomposition of a graph, that we consider here only for cographs. The tree decomposition of a cograph $G$ is a labelled tree $T(G) := (R(G), v)$ defined as follows: The nodes of the tree $R(G)$ are the robust modules of $G$; a module $A$ is robust if it is the least strong module containing two vertices $a, b$ of $G$ (a module is strong if it is either comparable to or disjoint from every module). Non-trivial robust modules are labelled with one of two symbols 0 and 1. If $A$ is the least robust module containing two distinct vertices $a$ and $b$ of $G$, the label $v(A)$ is 1 if $\{a, b\}$ forms an edge, whereas it is 0 if $\{a, b\}$ does not form an edge (it turns out that the label does not depend upon the choice of $a$ and $b$). The order on the nodes of the tree is reverse inclusion. The graph $G$ can be recovered from the labelled tree. A description of these labelled trees is given in the Appendix. An example made of a clique and an independent set (with some extra edges) is given in Figure 2.2.
The first two cases of Theorem 2.6 correspond to the case of a cograph whose decomposition tree has a least element; the last case to the non-existence of a least element.

The stated condition in Theorem 2.5 amounts to the fact that the decomposition tree has no least element. The conclusion follows from the next two lemmas.

**Lemma 2.7.** Let $G$, $G'$ be two isomorphic cographs such that their tree decomposition has no least element. If $G = \Sigma C$ and $G' = \Sigma C'$ where $C := (I, \leq, \ell)$ and $C' := (I', \leq', \ell')$ are two reduced labelled chains then there are two infinite initial segments $W$ of $I$ and $W'$ of $I'$ and an isomorphism $h$ of the induced labelled chains $C|_W$ and $C'|_{W'}$.

**Proof.** Since $G$ and $G'$ are isomorphic, their decomposition trees are isomorphic. We may suppose that they are identical. Let $T$ be such a tree. Then
(I, ≤) and (I', ≤) correspond to maximal chains in T. If these chains are identical, there is nothing to prove (W = W' = I). If not, then since T has no least element these chains meet in some element, say a. Set W = W' := ↓ a and h be the identity on the initial segment ↓ a.

**Lemma 2.8.** Let C := (I, ≤, ℓ) be a countable reduced labelled chain with no least element. Then there are 2^{80} reduced labelled chains with no least element C_α := (I_α, ≤_α, ℓ_α) such that:

1. all their labelled sums are siblings of the labelled sum of C;
2. there are no distinct α, α' and nonempty initial segment W of I_α and W' of I_α' such that the induced labelled chains on W and W' are isomorphic.

### 2.3. Sketch of the proof of Lemma 2.8.

We start with a countable reduced labelled chain C := (I, ≤, ℓ) with ℓ(i) := (G_i, v(i)) for each i ∈ I. We suppose that (I, ≤) has no least element.

We select an initial segment J of I such that the restriction C_J := (J, ≤_J, ℓ_J) is indecomposable on the left (see Subsection 2.4 below for the definition and existence). Let E_v(J) be the set of i ∈ J such that G_i is a clique or an independent set of even size. To each map f : N → {0, 1} we associate a reduced labelled chain C_f as follows. We select an infinite descending sequence (a_n)_{n∈N} of elements of J coinitial in J and such that v(a_n) = 1 for each n ∈ N. Now we insert infinitely many elements b_n, c_n such that b_n covers a_n and c_n covers b_n. Let J := J ∪ {b_n, c_n : n ∈ N}, K := I \ J and T = J ∪ K. Let f : N → {0, 1}. We define a labelling ℓ_f on T, with ℓ_f(i) := (G_i, v_f(i)) as follows. If i ∈ I \ E_v(J), then v_f(i) = v(i) and ℓ_f extends G_i to an extra element in such a way that the new graph is a clique if G_i is a clique and an independent set, otherwise. If i ∈ {b_n, c_n}, recalling that v(a_n) = 1, we set v_f(b_n) = 0, v_f(c_n) = 1, G_i being an independent set, resp., a clique, if i = b_n, resp. i = c_n, of size 2f(n) + 2. Due to the labelling, the labelled chain C_f := (T, ≤, ℓ_f) is reduced. If f and f' are two maps from N to N such that the sums ΣC_f of C_f := (T, ≤, ℓ_f) and ΣC_{f'} of C_{f'} := (T, ≤, ℓ_{f'}) are isomorphic then according to Lemma 2.7 there are two initial segment W and W' of T and an isomorphism h from W onto W' preserving the labels, that is the labels ℓ_f(i) := (G_i, v(i)) and ℓ_{f'}(h(i)) := (G_{h(i)}, v(h(i))) are isomorphic, meaning that G_i and G_{h(i)} are isomorphic and v_f(i) = v'_{f'}(h(i)) for each i ∈ W. If i = b_n then necessarily h(b_n) = b_m for some m. Indeed, G_{h(i)} must be an independent set of even size, thus h(b_n) ∉ E_v(J) hence h(b_n) must be either some b_m or some c_m. Since v(h(b_n)) = v(b_n) = 0 this is some b_m. Consequently, there are two final segments D and D' of N and an isomorphism h from D to D' such that f'(h(n)) = f(n). Thus, in order to obtain 2^{80} non isomorphic reduced labelled chains, we may apply the following lemma.

**Lemma 2.9.** There is a set of 2^{80} maps f from N into {0, 1}^N such that for every pair f, f' of distinct maps, and every isomorphism h from a final segment D onto another D' there is some n such that f(n) ≠ f'(h(n)).
This lemma appears as Lemma 4 p. 39 of [20]. For the reader’s convenience we reproduce the short proof given there.

**Proof.** We start with a subset \( X := \{ x_n : n \in \mathbb{N} \} \) where \( x_0 = 0 \) and \( x_{n+1} = x_n + n \) (one just need the gaps increasing). Now, let \( A \) be an almost disjoint family of \( 2^{\aleph_0} \) infinite subsets of \( X \). For any \( A \in \mathcal{A} \), and \( n > 0 \), \( A + n \) is almost disjoint from \( X \), and thus almost disjoint from any other \( A' \). The characteristic functions of members of \( \mathcal{A} \) have the required property.

In order to complete the proof of Lemma [2.8] it suffices to prove that each sum \( \Sigma C_f \) of \( C_f \) is equimorphic to the sum \( \Sigma C \) of \( C \). This is quite easy if in \( C \) no \( G_i \) has an even size (e.g. each \( G_i \) is finite but of odd size or is infinite). Some difficulties arise in general. Our proof, given in Lemma 2.12 of the next Subsection, is based on properties of chains labelled by a quasi order \( Q \).

### 2.4. Chains labelled by a better quasi order

We consider below chains labelled by a quasi-ordered-set. There is a strong similarity between properties of countable chains and properties of labelled countable chains, provided that the quasi-ordered-set is a better quasi ordering (in short b.q.o.). We present the facts we need. We refer to chapter 10 of [35] and to chapter 6 and 7 of [13] for properties of chains and an exposition of the solution of Fraïssé’s conjectures by Laver. We refer to the papers of Laver [24, 25] and also to [31] for properties of labelled chains.

A labelled chain is a pair \( C := (I, \ell) \) where \( \ell \) is a map from \( I \) to a quasi-ordered-set \( Q \). A \( Q \)-embedding of a labelled chain \( C := (I, \ell) \) into another \( C' := (I', \ell') \) is an order embedding \( f : I \to I' \) such that \( \ell(x) \leq \ell'(f(x)) \) for all \( x \in I \), a fact that we denote by \( C \leq C' \). These two labelled chains are equimorphic, and we set \( C \equiv C' \), if they are \( Q \)-embeddable in each other. They are isomorphic if there is an isomorphism \( \phi \) from \( I \) to \( I' \) such that \( \ell' \circ \phi = \ell \).

A labelled chain \( C := (I, \ell) \) is additively indecomposable, or briefly indecomposable, if for every partition of \( I \) into an initial interval \( I \) and a final interval \( F \), \( C \) embeds into \( C_{iF} \) or into \( C_{iF} \). We say that \( C \) is left-indecomposable, resp., strictly left-indecomposable, if \( C \) embeds in every non-empty initial segment, resp. if \( C \) is left-indecomposable and embeds in no proper final segment. The right-indecomposability and the strict right-indecomposability are defined in the same way. The notions of indecomposability and strict-right or left-indecomposability are preserved under equimorphy (but not the right or the left-indecomposability).

The sum \( \Sigma_{n \in \mathbb{N}} C_i \) of labelled chains over a chain (not over a labelled chain) is defined as in the case of chains. If \( I \) is the \( n \)-element chain \( n := \{ 0, 1, \ldots, n-1 \} \) with \( 0 < 1 < \cdots < n-1 \), the sum is rather denoted by \( C_0 + C_1 + \cdots + C_{n-1} \). With the notion of sum, a labelled chain \( C \) is indecomposable if \( C \leq C_0 + C_1 \) implies \( C \leq C_0 \) or \( C \leq C_1 \). A sequence \( (C_n)_{n \in \mathbb{N}} \) of labelled chains is quasi-monotonic if \( \{ m : C_n \leq C_m \} \) is infinite for each \( n \), its sum \( C := \Sigma_{n \in \mathbb{N}} C_n \) is right-indecomposable or equivalently \( C^* = \Sigma_{n \in \mathbb{N}} C_n \) is left-indecomposable.

Let \( C := (I, \leq, \ell) \) be a labelled chain. Two elements \( x, y \) of \( I \) are equivalent and we set \( x \equiv_C y \) if \( C \) does not embed in the restriction of \( C \) to the interval
The following lemma lists the properties we need. The first three are due to Laver, see [24] p. 109 for the first two and [25] p. 179 for the third. For the reader’s convenience, we give a (short) proof. For this purpose, let $D$ be the collection of countable chains quasi-ordered by embeddability and let $Q^{D_{\omega}}$ be the collection of countable chains, labelled by a quasi-ordered set $Q$. According to Laver [24], $D_{\omega}$ quasi-ordered by embeddability is better- quasi-ordered (b.q.o.) and more strongly, the collection $Q^{D_{\omega}}$ of countable chains labelled by a b.q.o $Q$ is b.q.o. (see p. 90 of [24]).

**Lemma 2.10.** Let $C := (I, \ell)$ be a countable chain labelled over a b.q.o. $Q$. Then

1. $C$ is a finite sum of indecomposable labelled chains;
2. If $I$ has no least element, then there is some initial interval $J$ such that $C_{I,\ell}$ is left-indecomposable;
3. If $C$ is left-indecomposable then $C$ is an $\omega^*$ sum $\Sigma_{n<\omega} C_n$ where each $C_n$ is indecomposable and the set of $m$ such that $C_n$ embeds into $C_m$ is infinite;
4. If $C$ is indecomposable and the quotient $I/\equiv_C$ is dense then $C$ is equimorphic to a sum $\Sigma_{q\in Q} C_q$ such that for every $p < q$ and $r$ in $Q$ there are $s_0, \ldots, s_{n-1}$ with $p < s_0 < \cdots < s_{n-1} < q$ and $C_r \leq C_{s_0} + \cdots + C_{s_{n-1}} < C$.

**Proof.** (1) Since $Q^{D_{\omega}}$ is b.q.o. it is w.q.o. hence every non-empty subset has a minimal element. If (i) fails, it fails for some minimal $C$. We claim that $C$ is indecomposable. Indeed, if $C = C_0 + C_1$ with $C_0 < C$ and $C_1 < C$ then $C_0$ and $C_1$ are finite sums of indecomposables thus $C$ is such, contradicting our hypothesis. Hence $C$ is indecomposable, contradicting our hypothesis too. Hence (2) holds.

(2) For each $a \in I$, let $C_a := (I_a, \ell_{I_a})$ where $I_a := (a \mapsto a) := \{x \in I : x \leq a\}$, a set that we denote also by $a).$ As a subset of a w.q.o. the set of these $C_a$ has a minimal element. This minimal element is left-indecomposable.

(3) Pick a coinitial sequence in $I$, say $(x_n)_{n<\omega}$. Write $I$ as $\bigcup_{n<\omega} I_n$ where $I_0 := \{x_0, \to\}$, $I_{n+1} := [x_{n+1}, x_n]$. Write $C$ as $\Sigma_{n<\omega} X_n$, where $X_n := (I_n, \ell_{I_n})$. Since, by (i), each $X_n$ is a finite sum of indecomposables, we may rewrite $C$ as a sum $\Sigma_{n<\omega} C_n$ where each $C_n$ is indecomposable. Pick $n < \omega$ and any $k > n$. Since $C$ is left-indecomposable, $C$ embeds into $\Sigma_{k<\omega} C_m$, hence $C_{n+1} + C_n$ embeds in that sum, thus $C_n$ embeds in a finite sum of $C_m$, $m > k$. This $C_n$ being indecomposable, it embeds into some $C_m$.

(4) Let $D := (I/\equiv_C, \ell_{\equiv_C})$ be the labelled chain where $\ell_{\equiv_C} (F) := (F, \ell_{I_F})$ where $F \in D$. The set of labels belongs to $Q^{D_{\omega}}$ hence is b.q.o. As a labelled chain over a b.q.o. $D$ is a finite sum of indecomposable labelled chains. The
decomposition of $I/\equiv_C$ in finitely many intervals on which the labelled chains are indecomposable induces a decomposition of $I$ into finitely many intervals. Since $C$ is indecomposable, $C$ is equimorphic to its restriction to some interval. Since $C_{IF} \leq C$ for each $F \in I/\equiv_C$ we may throw out the extremities of this interval if any, hence we may suppose that this interval is isomorphic to $\mathbb{Q}$. Since each $C_{IF}$ is a finite sum of indecomposable labelled chains and $C$ embeds in $C_{[x,y]}$ for every $x < y$ with $x \neq_C y$, the conclusion follows.

Let $C := (I, \ell)$ be a chain labelled by a poset $Q$ and $n$ be a positive integer, the ordinal product $n.C$ is the labelled chain $(n.I, \ell_n)$ in which $n.I$ is the ordinal product of the $n$-element chain $n := \{0, \ldots, n-1\}$ and the chain $I$, that is the ordinal sum of $C$ copies of $n.$, and $\ell_n(m,i) = \ell(i)$ for every $m \in \{0,n-1\}, i \in I.$

**Lemma 2.11.** Let $C := (I, \ell)$ be a countably infinite labelled chain. If the labels belong to a b.q.o. and $C$ is indecomposable then for every positive integer $n$, the ordinal product $n.C$ embeds in $C$.

**Proof.** It suffices to prove that the property holds for $n = 2$. Indeed, suppose that the property holds for $n$. Let $C$ be an indecomposable chain, then trivially $(n + 1).C$ embeds into $2n.C$. Since $2n.C = n.(2.C)$ and, as it can be checked easily, $2.C$ is indecomposable, induction ensures that $n.(2.C)$ embeds into $2.C$. With the fact that $2.C$ embeds into $C$ we obtain that $(n + 1).C$ embeds into $C$.

To prove that this property holds, we use induction. Since the collection of labelled countable chains over a b.q.o. is b.q.o. it is well founded, hence we may suppose that $C$ is a chain such that every countably infinite indecomposable labelled chain $D$ satisfying $D < C$ satisfies the property.

We consider two cases.

**Case 1.** The equivalence relation $\equiv_C$ has just one class. Then $C$ is either strictly left-indecomposable or strictly right-indecomposable. We may suppose that $C$ is strictly left-indecomposable. We apply Item $(iii)$ of Lemma 2.10. $C$ is equimorphic in an $\omega^*$ sum $\sum_{n<\omega} C_n$ where each $C_n$ is indecomposable and the set of $m$ such that $C_n$ embeds into $C_m$ is infinite. Hence $2.C = \sum_{n<\omega} 2.C_n$. We define an embedding of $2.C$ into $C$ as follows. Suppose that we have embedded $\sum_{n<\omega} 2.C_n$ in $\sum_{n<\varphi(m)} C_n$. We extend this embedding to $2.C_m$ as follows. If $C_m$ is infinite, we embed it into some $C_k$ with $k \geq \varphi(m)$. According to the induction hypothesis, $2.C_k$ embeds into $C_k$, and we may set $\varphi(m + 1) = k$. If $C_m$ is a one element labelled chain, $2.C_m = C_m + C_m$, we send the first copy of $C_m$ into some $C_k$ and the second copy into another $C_k'$ for $k' > k \geq \varphi(m)$.

**Case 2.** The equivalence relation has at least two classes, and in fact a dense set of classes. According to $(iv)$ of Lemma 2.10, $C$ is equimorphic to a sum $\Sigma_{q \in \mathbb{Q}} C_q$ such that for every $p < q$ and $r$ in $\mathbb{Q}$ there are $s_0, \ldots, s_{n_r-1}$ with $p < s_0 < \cdots < s_{n_r-1} < q$ and $C_r \leq C_{s_0} + \cdots + C_{s_{n_r-1}} < C$. Let $p_0, \ldots, p_n, \ldots$ be an enumeration of $\mathbb{Q}$. Suppose that we have defined an embedding $\varphi_m$ of $\Sigma_{p_n \leq q} 2.C_{p_n}$ in a sum $\Sigma_{q \in A} C_q$ where $A$ is a finite subset of $\mathbb{Q}$ and in such a way that the projections on $\mathbb{Q}$ of the images of the $2.C_{p_n}$’s do not intersect.
We extend it to $2C_p$ as follows. Let $A^-$ be the initial, resp. $A^+$ be the final, segment of $\mathbb{Q}$ generated by the projections on $\mathbb{Q}$ of the images via $\varphi_m$ of the $2C_p$’s for $p_n < p_m$, resp., for $p_m < p_n$. The complement is a dense interval of $\mathbb{Q}$. The labelled chain $2C_{p_n}$ is a finite sum $2C_{s_0} + \cdots + 2C_{s_{m-1}}$, the $C_{s_i}$ being indecomposable. If $C_{s_i}$ is infinite, we may send $2C_{s_i}$ into some $C_j$ and if $C_i$ is a one element labelled chain, we may send $2C_{s_i}$ into a sum $C_i + C_j$. 

Let $\text{Cog}_{\omega}$ be the collection of countable cographs, quasi-ordered by embeddability. According to Thomassé [39], $\text{Cog}_{\omega}$ is b.q.o. Let $Q := \text{Cog}_{\omega} \times \{0, 1\}$ be the direct product of $\text{Cog}_{\omega}$ and the two element antichain $\{0, 1\}$. This poset is b.q.o. as a union of two b.q.o.’s. And thus, from Laver’s theorem, $Q^{\omega}$ is b.q.o.

**Lemma 2.12.** Let $C := (I, \preceq, \ell)$ be a countable reduced labelled chain such that $I$ has no first element and the labels belong to $\text{Cog}_{\omega} \times \{0, 1\}$. Then, there is an initial segment $J$ of $I$ such that the restriction $C_J := (J, \preceq, \ell_J)$ is left-indecomposable. If $J$ is such an initial segment then, for every map $f : \mathbb{N} \to \{0, 1\}$, the sum $\sum C_f$ of $C_f$ is embeddable into the sum $\sum C$ of $C$.

**Proof.** The existence of $J$ follows from the fact that the set of countable chains labelled by $\text{Cog}_{\omega} \times \{0, 1\}$ is b.q.o. and (ii) of Lemma 2.10. Let $f : \mathbb{N} \to \{0, 1\}$ and $C_f := (I, \preceq, \ell_f)$ be the labelled chain defined in Subsection 2.3. We prove that $\sum C_f \preceq \sum C$. Let $C_f|J$, resp., $C_f|K$ the restriction of $C_f$ to $K$. We have $C_f = C_f|J + C_f|K$. As it is easy to see, $\sum C_f$ extends both $\sum C_f|J$ and $\sum C_f|K$ in a simple way: If $a \in \sum C_f|J$ and $b \in \sum C_f|K$, we link $a$ and $b$ by an edge if $v_f(i) = 1$ where $i \in J$ and $a \in G_i$. We denote by $\bar{+}$ this operation, hence $\sum C_f = \sum C_f|J \bar{+} \sum C_f|K$.

To conclude, it suffices to prove that $\sum C_f|J$ embeds into $\sum C_J$ and $\sum C_f|K$ embeds into $\sum C_K$. Only the first statement needs a proof. In order to prove it, we define an auxiliary labelled chain $D := (L, \preceq, d)$ as follows. The domain $L$ is $2\cdot J \cup X$ where $X := \{b_n, c_n : n < \omega\}$, $b_n$ covers $(1, a_n)$, $c_n$ covers $b_n$ and $(a_n)$ is the sequence coinitial in $J$ defined in Subsection 2.3. The labelling $d$ is defined by $d(i) := \ell(r)$ if $i := (j, r) \in \{0, 1\} \cdot J$, $d(i) := \ell_f(i)$ if $i \in \{a_n, b_n\}$ (so $D$ is not densely labelled, but this does not matter). We prove that the following inequalities hold:

$$\sum C_f|J \preceq \sum D \preceq \sum C_J \preceq \sum C_f|J.$$  

For the first inequality, note that by definition $\sum C_f|J = \sum_{i \in (J, \ell_f)} \bar{G}_i$ and $\sum D = \sum_{j \in L} \bar{D}_j$ such that $D_j$ is the first component of $d(j)$. We have $\sum D = \sum_{i \in (J, \ell_f)} H_i$ with $H_i = \bar{G}_i$ if $i \in \{a_n, b_n\}$ and $H_i = G_0,i \oplus G_1,i$, resp. $G_0,i + G_1,i$ if $v_f(i) = 0$, resp. $v_f(i) = 1$ if $i \in I$. It follows that $\bar{G}_i \preceq H_i$ for every $i \in J$. Thus $\sum C_f|J \preceq \sum D$.

For the second inequality, we observe that $C_J$ being left-indecomposable, we may write it as an $\omega^*$ sum $\sum_{n \in \omega} C_n$ where each $C_n$ is indecomposable and the set of $m$ such that $C_n$ embeds into $C_m$ is infinite. For each $n$, let $J_n$
be the domain of $C_n$ and let $L_n := 2J_n \cup \{b_m, c_m : (1, a_m) \in 2J_n\}$. Set $D_n := D_{|L_n}$. Then $D = \Sigma_{n<\omega} D_n$. Define an embedding from $\Sigma D$ in $\Sigma_2 C_{1J}$ by induction. Let $n < \omega$. Suppose that $\Sigma(D_{n-1} + \cdots + D_0)$ has been embedded in $\Sigma(2C_{\varphi(n-1)} + \cdots + 2C_0)$. The labelled chain $D_n$ consists of $2C_n$ plus finitely many elements, each labelled by a finite graph. Thus $D_n$ can be written $D_{n,0} + \alpha(0,0) + \alpha(0,1) + D_{n,1} + \alpha(1,0) + \alpha(1,1) + \cdots + D_{k-2} + \alpha(k-2,0) + \alpha(k-2,1) + D_{k-1}$ with $2C_n = D_{n,0} + \cdots + D_{n,k-1}$ and the $\alpha_i$ are singletons belonging to $X$ labelled by a 2 or 4-element cograph. Since the set of $m$ such that $C_n$ embeds in $C_m$ is infinite, of the $C_m$ we may find $m \geq \varphi(n-1)$ and $k$ such that $\Sigma D_n$ embeds in $\Sigma(D_{m+k} + \cdots + D_m)$. Then set $\varphi(n) = m + k$.

The last inequality is because $2C_{1J} \leq C_{1J}$, a fact which follows from Lemma 2.11 since $C_{1J}$ is indecomposable.

With that the proof of the lemma is complete. 

3. Proof of Theorem 1.7

We use induction on the collection $Cog_{\leq \omega}$ of countable cographs with a quasi-order slightly different from embeddability. If countable $G$ and $G'$ are two connected cographs, we set $G \leq G'$ if $G \leq G'$ or $G \leq G^{ce}$. This is a quasi ordering; since it extends the embeddability, which is w.q.o., it is w.q.o. hence well-founded. So, in order to prove that a connected countable cograph $G$ has one or infinitely many connected siblings, we may suppose that this property holds for all connected countable cographs $G'$ such that $G' < G$, that is either $G'$ strictly embeds into $G$ or into $G^{ce}$. There are two cases to consider:

Case 1. The complement $G^c$ of $G$ is not connected. We apply Proposition 2.4. We decompose $G^c$ into connected components, say $G^c := \bigoplus_i H_i$, where each $H_i$ is connected.

We have $H_i \leq G^{ce}$, hence $H_i \leq G$.

Subcase 1 $G \leq H_i$ for some $i$. This means $G \leq H_i$ or $G^c \leq H_i$. This implies that $G^c$ is equimorphic to some connected graph $K$ ($K = H_i$ in the second case, $K = G$ in the first case). It follows from Lemma 1.5 that $G^c$ has infinitely many disconnected siblings, hence $G$ has infinitely many connected siblings.

Subcase 2 $H_i < G$ for all $i$.

According to the induction hypothesis, each $H_i$ has either one or infinitely many connected siblings.

Subcase 2.1 Some connected component of $G^c$ has infinitely many connected siblings. According to Lemma 2.2, $G^c$ has infinitely many disconnected siblings. It follows that $G$ has infinitely many connected siblings.

Subcase 2.2 Every connected component $H_i$ of $G^c$ has only one connected sibling.

Subcase 2.2.1 The number of non-trivial connected components of $G^c$ is infinite. Since the collection of countable cographs is w.q.o. under embeddability,
there is an increasing sequence among these connected components. According to Lemma 2.3, $G^c$ has infinitely many disconnected siblings, hence $G$ has infinitely many connected siblings.

**Subcase 2.2.2.** $G^c$ has only finitely many non-trivial connected components. We apply Lemma 2.1. Then $G^c$ has either one sibling or infinitely many disconnected siblings. Hence $G$ has either one sibling or infinitely many connected siblings.

**Case 2.** The complement $G^c$ of $G$ is connected. In this case, the decomposition tree $T(G)$ has no least element.

If $T(G)$ has no least element, then $G$ is a sum $\sum_{i \in (C, \ell)} G_i$ of non-empty graphs, where $C$ is a chain with no first element, $v$ is a dense labelling of $C$ by 0 and 1, $G$ extends each $G_i$, and for $x \in G_i$, $y \in G_j$ with $i < j$, $\{x, y\}$ forms an edge iff $v(i) = 1$. According to Theorem 2.5, $G$ has $2^{\aleph_0}$ connected siblings.

4. **Proof of Theorem 1.8**

We prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

The implication $(i) \Rightarrow (ii)$ is trivial. Implication $(iii) \Rightarrow (i)$ is an immediate consequence of the following result.

**Theorem 4.1.** If a graph $G$ is a finite lexicographic sum of cliques or independent sets then it has just one sibling: itself.

This is a consequence of properties of monomorphic decompositions of relational structures, a notion introduced in [34] and developed in [29, 32, 30, 23].

Let $R := (V, (\rho_i)_{i \in I})$ be a relational structure. A **monomorphic decomposition** of $R$ is any partition $(V_j)_{j \in J}$ of $V$ such that for every pair of finite subsets $F, F'$ of $V$, the restrictions $R_{\upharpoonright F}$ and $R_{\upharpoonright F'}$ are isomorphic whenever $|F \cap V_j| = |F' \cap V_j|$ for every $j \in J$. Among the monomorphic decompositions of $R$ there is a largest one: every other is included in it (see [34], Proposition 2.12). We call it the **canonical decomposition** of $R$ and denote it by $M(R)$. Its existence is a consequence of the following notion: say that two elements $x, y$ of $V$ are **equivalent** and set $x \equiv_R y$ if for every finite set $F$ of $V \setminus \{x, y\}$ the restrictions $R_{\upharpoonright F \cup \{x\}}$ and $R_{\upharpoonright F \cup \{y\}}$ are isomorphic.

Oudrar and Pouzet showed (see Lemma 7.48 and Lemma 7.49 in Section 7.2.5 of [30]):

**Theorem 4.2.** The partition of the domain $V$ of a relational structure $R$ into equivalence classes forms a monomorphic decomposition of $R$ and every other monomorphic decomposition of $R$ is a refinement of it.

We will need the following fact.

**Proposition 4.3.** Let $R := (V, (\rho_i)_{i \in I})$ be a relational structure and $A$ a subset of $V$. Then

(1) Every monomorphic decomposition of $R$ induces a monomorphic decomposition of $R_{\upharpoonright A}$;
(2) If $R$ has a monomorphic decomposition into finitely many classes and $R$ embeds into $R_{1A}$ then for each class $C$ of $M(R)$, $R_{1AC}$ is a class of the canonical decomposition of $M(R_{1A})$ and $|A \cap C| = |C|$.

**Proof.** Item (1) is obvious.

(2) Let $f$ be an embedding of $R$ into $R_{1A}$, $A'$ be the range of $f$ and $R' := R_{1A'}$. According to (1), $(A' \cap C)_{C \in M(R)}$ is a monomorphic decomposition of $R'$, hence it is finer than $M(R')$. Thus, it has as many classes as $M(R)$. Since $R$ and $R'$ are isomorphic, their canonical decompositions have the same number of classes, hence $(A' \cap C)_{C \in M(R)}$ has the same number of classes as $M(R)$. These number being finite, the partition $(A' \cap C)_{C \in M(R)}$ coincides with $M(R')$, hence the frequency sequences $(|A' \cap C|)_{C \in M(R)}$ and $(|C|)_{C \in M(R)}$ are equal up to a permutation. Since $|A' \cap C| \leq |C|$ for each $C \in M(R)$, these sequences must be equal. By the same token, we obtain that the frequency sequences $(|A \cap C|)_{C \in M(R)}$ and $(|C|)_{C \in M(R)}$ coincide, proving that (2) holds.

The case of symmetric graphs is particularly simple:

**Lemma 4.4.** Let $G := (V, E)$ be a symmetric graph. A partition $(V_j)_{j \in J}$ of $V$ is a monomorphic decomposition of $G$ if and only if each $G_{V_j}$ is a clique or an independent set and $G$ is the lexicographic sum of the $G_{V_j}$’s indexed by a graph $H$ on $J$. In particular, $G$ has a finite monomorphic decomposition if and only if it is a lexicographic sum of cliques or independent sets indexed by a finite graph.

**Proof.** Since $G$ is symmetric, if $(V_j)_{j \in J}$ is a monomorphic decomposition of $G$ then each $G_{V_j}$ is either a clique or an independent set and a module of $G$. Hence, $G$ is the lexicographic sum of the $G_{V_j}$ indexed by a graph $H$ on $J$. Conversely, if $G$ is a lexicographic sum $\sum_{j \in H} L_j$ of cliques or independent sets $L_j$ indexed by a graph $H$ then the family of $L_j$ forms a monomorphic decomposition of $G$. Indeed, let $F$, $F'$ be two finite subsets of $V$ such that $|F \cap L_j| = |F' \cap L_j|$ for every $j \in H$. For $j \in J$, let $f_j$ be any bijective map from $F \cap L_j$ onto $F' \cap L_j$, then $f := \cup_{j \in J} f_j$ is an isomorphism of $G_{1F}$ onto $G_{1F'}$.

**Proof of Theorem 4.1** Let $A \subseteq V$ be such that $G$ embeds into $G \upharpoonright A$. Our aim is to show that $G_{1A}$ is isomorphic to $G$. According to (2) of Proposition 4.3, $|C \cap A| = |C|$ for each equivalence class $C$ of $\approx_G$. For each equivalence class $C$, let $f_C$ be any bijective map from $C$ onto $C \cap A$. Then $f := \cup_C f_C$ is an isomorphism from $G$ onto $G_{1A}$.

**Proof of implication (ii) \Rightarrow (iii) of Theorem 1.8.** We argue by induction. Since the quasi order $\leq$ defined in the proof of Theorem 1.7 is a well quasi order, in order to prove that $\text{sib}_K(G) = 1$ implies that $G$ is a lexicographical sum of cliques or independent sets, we may suppose that for every graph $H$ such that $H < G$ and $\text{sib}_K(H) = 1$ is such a lexicographical sum. According to Theorem 2.5 $G$ or $G^c$ is disconnected. Without loss of generality, we may suppose
that $G$ is disconnected. If $H$ is a connected component, $H \prec G^c$. Otherwise, as in the proof of Theorem 1.7, $G$ has infinitely many disconnected siblings contradicting $sibc(G) \neq 1$. We apply Proposition 2.4. For each connected component $sibc(H) = 1$ and, due to the w.q.o. of embeddability, there are only finitely many connected components. Let $\{H_i : i < m\}$ be the set of non-trivial connected components of $G$. Due to the induction hypothesis, each one is of the form $\sum_{j \in K_i} L_{ij}$, where each $L_{ij}$ is a finite cograph. Let $K = \bigoplus_{i < m} K_i \oplus \{a\}$ and $L_a$ be the independent set made of the trivial components. Then, $G$ is the lexicographical sum of the $L_{ij}$ and $L_a$ over $K$.

5. Extensions

Our result is crude in several aspects.
First, we think that one can prove without (CH) that a countable cograph has one, $\aleph_0$ or $2^{\aleph_0}$ siblings.

We think that the following holds:

Let $G$ be a countable cograph and $T(G) := (R(G), v)$ its decomposition tree.

1. $sib(G) = 2^{\aleph_0}$ if $T(G)$ contains an infinite set $A$ such that for every integer $n$ the set of $a \in A$ such that the subtree $T_{\uparrow n}$ has cardinality at most $n$ is finite.

2. $sib(G) = \aleph_0$ if $T(G)$ has only finitely many levels and for each $a \in T$ with infinitely many successors, there is an integer $n$ which bounds the cardinality of almost all $T_{\uparrow b}$ (where $b$ is a successor of $a$), and there is some $a \in T$ with infinitely many successors $b$ such that all $T_{\uparrow b}$ have at least two elements.

3. $sib(G) = 1$ if $T(G)$ has only finitely many levels and if some element $a$ has infinitely many successors then almost all are maximal in $T(G)$.

What is needed?

- A countable connected cograph $G$ embedding $G \oplus 1$ has $2^{\aleph_0}$ siblings.
This will be true if we can prove that
- If $T(G)$ is well founded with an infinite chain then $sib(G) = 2^{\aleph_0}$.

**Problem 5.1.** If a a countable connected graph $G$ embeds $G \oplus 1$, is $sib(G) = \aleph_0$ or $sib(G) = 2^{\aleph_0}$?

We guess that the alternative "one" or "infinite" may hold for arbitrary cographs, possibly uncountable. But, we may note that then the well quasi ordering arguments cannot be used. The collection of uncountable cographs is not w.q.o. under embeddability. Simple examples can be made with rigid chains and the comb construction. Also, there are uncountable cographs with no proper embedding (in particular, they have just one sibling) while countable cographs with one sibling have plenty of proper embeddings. To illustrate, say that a comb is the sum $G$ of a dense labelled chain $C := (I, \leq, \ell)$ such that for each $i$, the label $\ell(i) := (G_i, v(i))$ is made of a one vertex graph if $i$ is not the largest element, otherwise $G_i$ has two vertices, and $v(i) \in \{0, 1\}$. The decomposition tree of $G$ is the pair $(T, w)$ where $T$ is the tree on $I \cup I'$ with an
extra element \(a\) if \(I\) has a largest element. The order on \(I \cup I'\) extend the order on \(I\), the set \(I'\) is an antichain, every element \(i\) of \(I\) has a unique successor \(i' \in I'\), except if \(i\) is maximal in \(I\), in which case it has two, namely \(i'\) and \(a\).

The label function \(w\) is \(v\) (see the example given in Figure 2.2).

It is easy to construct uncountable combs with no proper sibling. A rigid chain will do, but this is unnecessary. Indeed, Dushik and Miller [9] (see also Chapter 9 of Rosenstein [35], Theorem 9.6 page 151) showed that the real line \(\mathbb{R}\) can be decomposed into two disjoint dense subsets \(E\) and \(F\) such that \(g(E) \cap F \neq \emptyset\) and \(g(F) \cap E \neq \emptyset\) for any non-identity order preserving map \(g : \mathbb{R} \to \mathbb{R}\). Thus, let \(C := (\mathbb{R}, \leq, \ell)\) where \(\ell(i) := (G_i, \chi_F(i))\) is made of the one vertex vertex graph and \(\chi_F(i) = 1\) if \(i \in F\), and 0 if \(i \notin F\). Then the corresponding comb has no proper embedding.

Instead of cographs, one could consider series-parallel posets, that is posets not embedding an "N" or equivalently posets whose comparability graph is a cograph. More generally, let \(\mathcal{C}\) be a hereditary class of finite binary structures containing only finitely many indecomposable structures. According to unpublished results of Delhommé [7] and G.McKay [26], the collection of countable binary structures \(R\) such that \(Age(R) \subseteq \mathcal{C}\) is b.q.o. (even if we add labels). There is not much difference with the case of cographs; we have just to add the case of a lexicographical sum indexed by a finite indecomposable structure.

6. Appendix: cographs and labelled trees

In this section, we prove the existence of a one-to-one correspondence between cographs and ramified meet-trees densely valued by \(\{0, 1\}\) (cf. Theorem 6.19). This result follows from Lemma 5.1 and Proposition 5.4 of [6] as follows. In [6] a labelled tree \(\text{mdec}(G)\) is constructed which consists of the one element subsets of \(G\), and the robust modules of \(G\) together with strong modules which are limit modules that are maximal proper strong submodules of a robust module. If \(A\) is a robust module which is not a singleton, a label is assigned according to the structure of the quotient of \(A\) by the family of maximal proper strong submodules of \(A\). For cographs only two labels arise according to whether the quotient is a complete graph, or an independent set. From this tree one defines a graph on the "leaves", by assigning an edge between distinct elements \(x\) and \(y\) according to the label of the robust module that \(x\) and \(y\) determine. Lemma 5.1 asserts that the original graph \(G\) is recovered. Proposition 5.4 of [6] asserts that the labeled tree \(\text{mdec}(G)\) is uniquely determined by \(\text{rdec}(G)\), the labelled tree of robust modules, by a process of completion, essentially adding the limit strong modules that are maximal proper submodules of a non-singleton robust module. Since their tree \(\text{rdec}(G)\) is obtained from our tree of robust submodules ordered by reversing inclusion, changing join to meet, their results apply. This establishes the result. We think that correspondence is simple and important enough to justify a detailed presentation.
We first put together some general properties of modules, and the modular decomposition of a graph. In order that the Appendix may be used for other work, we present results in more generality, rather than provide statements and proofs strictly in the context of cographs.

6.1. Modules. We recall some basic ingredients of binary structures, alias 2-structures. Most of it can be found in [11]. Let $W$ be a set. A binary structure over $W$ is a pair $M := (V, d)$ where $d$ is a map from $V \times V$ into $W$; its restriction to a subset $A$ of $V$ is $M\upharpoonright_A := (A, d\upharpoonright_{A\times A})$. The value of $d$ on $\Delta_V := \{(x, x) \in V \times V : x \in V\}$, the diagonal of $V$, plays no role in the notions involved below, and the reader could suppose that it is constant.

A subset $A$ of $V$ is a module of $M$ if $d(x, y) = d(x, y')$ and $d(y, x) = d(y', x)$ for every $x \in V \setminus A$, $y, y' \in A$. (other names are autonomous sets, or intervals).

The whole set, the empty set and the singletons are modules. These are the trivial modules. A binary structure whose modules are trivial is indecomposable. If moreover it has more than two vertices it is prime.

We recall the basic and well-known properties of modules, under the form given in [6].

**Lemma 6.1.** Let $M := (V, d)$ be a binary structure. Then:

1. The intersection of a non-empty set of modules is a module (possibly empty).
2. The union of two modules that meet is a module, and more generally, the union of a set of modules is a module as soon as the meeting relation on that set is connected.
3. For two modules $A$ and $B$, if $B \setminus A$ is non-empty, then $A \setminus B$ is a module.

A module is strong if it is either comparable w.r.t. inclusion or disjoint from every other module.

**Lemma 6.2.** (1) The intersection of any set of strong modules is empty or is a strong module.

2. The union of any non-empty directed set of strong modules is a strong module.

Let $A$ be a subset of $V$. Let $S_M(A)$ be the intersection of strong modules of $M$ which contain $A$. We write $S_M(x, y)$ instead of $S_M(\{x, y\})$. According to (1) of Lemma 6.2, $S_M(A)$ is a strong module.

According to Courcelle, Delhommé 2008 [6], a module $A$ is robust if it is either a singleton or the least strong module containing two distinct vertices; that is there are $x, y \in A$ such that $A$ is strong and every strong module containing $x$ and $y$ contains $A$. Alternatively, $A = S_M(x, y)$ for some $x, y \in V$.

**Example 6.3.** If $A$ is a module and $M\upharpoonright_A$ is prime then $A$ is robust.

**Definition 6.4.** Let $A$ be a strong module. Let $x, y \in A$. We set $x \equiv_A y$ if either $x = y$ or there is a strong module containing $x$ and $y$ and properly contained in $A$. 
Lemma 6.5. The relation $\equiv_A$ is an equivalence relation on $A$ whose equivalence classes are strong modules. If $A$ has more than one element then there are at least two classes iff $A$ is robust. Furthermore, these classes are the maximal strong modules properly included in $A$.

Proof. Since strong modules are disjoint or comparable, $\equiv_A$ is an equivalence relation. Suppose that $A$ has at least two elements. If $A$ is robust then there are two distinct elements such that $A = S_M(x, y)$, hence $x \not\equiv_A y$ proving that there are at least two equivalence classes. Conversely, if there are at least two classes, then pick $x$ in one and $y$ in another; since $x \not\equiv_A y$ we have $S_M(x, y) = A$.

Let $I$ be an equivalence class. Suppose that there is some strong module $F$ such that $I \subseteq F \subset A$ with $F$ strong. Then all members of $F$ are $A$-equivalent. Since $I$ is an equivalence class, $I = F$. This shows that if $I$ is strong then it is maximal. The fact that $I$ is strong follows from (2) of Lemma 6.1, but can be obtained directly as follows. Let $J$ be a module which meets $I$. We claim $J$ is comparable to $I$. Since $A$ is strong, we may suppose that $J \subseteq A$ and $J$ is incomparable to $I$. Let $a \in I \cap J$ et $b \notin I \setminus J$. Then $S_M\{a, b\}$, the least strong module containing $a$ and $b$, which is necessarily contained in $I$, intersects $J$ properly, contradicting the fact that it is strong.

We call components of $A$ the equivalence classes of the relation $\equiv_A$. Except if $A$ is finite, the components need not be robust.

A notion equivalent to the notion of robust module was previously introduced by Kelly [18] see also [5]. A module $I$ is non-limit if it is strong and contains a non-empty strong module $J$ which is maximal among those contained in $I$ and distinct from $I$.

Proposition 6.6. Let $A$ be a subset of $V$. Then $A$ is a robust module with at least two elements iff $A$ is a non-limit module.

Proof. Suppose that $A$ is a non-limit module. Let $x \in I \subset A$ with $I$ maximal among the strong modules contained in $A$ and distinct from $A$. Let $y \in A \setminus I$. Let $A' := S_M(x, y)$. Since $I$ is strong, $A'$ is a strong module properly containing $I$. Due to the choice of $I$, it is equal to $A$ hence $A$ is robust. Conversely, suppose that $A$ is robust with at least two elements. Then the components of $A$, as defined above, are the maximal non-empty strong modules properly contained in $A$ and in particular $A$ is non-limit.

If $A$ is a robust module with at least two elements of a binary structure $M := (V, d)$ then for two distinct components $I, J$ of $A$, the values $d(x, y)$ for $x \in I$ and $y \in J$ depends only upon $I$ and $J$. Hence, the binary structure on $A$ induces a binary structure $M/\equiv_A$ on the set $A/\equiv_A$ of components of $A$, called the Gallai quotient of $A$, and $M_{1/A}$ is the lexicographical sum of the $M_{1/I}$'s indexed by $M/\equiv_A$. This quotient $M_{1/A}/\equiv_A$ has a special structure: its strong modules are trivial, there are only the empty set, the whole set $A/\equiv$ and the singletons.

The central result of the decomposition theory of binary structures describes the structure of the Gallai quotient. It is due to Gallai [14] for finite graphs,
Theorem 6.7. The strong modules of a binary structure $\mathbb{M}$ are trivial iff either $\mathbb{M}$ is prime, or constant, that is $d(x,y) = \alpha$ for all $x \neq y \in V$, or linear, that is there are $\alpha \neq \beta$ such that $\{(x,y) \in V \times V : x \neq y, d(x,y) = \alpha\}$ is a linear (strict) order and $\{(x,y) \in V \times V : x \neq y, d(x,y) = \beta\}$ is the opposite order.

We say that the type of $X$, $t(A)$, is prime if $\mathbb{M}_{1|A/\equiv}$ is prime, otherwise its type is $\alpha$ if $\mathbb{M}_{1|A/\equiv}$ is constant, and $\{\alpha, \beta\}$ if it is linear.

In the case of directed graphs, this yields:

Theorem 6.8. If $\mathbb{M}$ is a directed graph, every robust module with at least two elements is the lexicographic sum of its components and the quotient is either a clique or an independent set or a chain or a prime graph.

Let $X, Y$ be two disjoint non-empty subsets of $\mathbb{M}$. If $X, Y$ are two modules, the value $d(x,y)$ where $x \in X$ and $y \in Y$ is independent of $X$ and $Y$, we will denote it by $d(X,Y)$.

Lemma 6.9. Let $X, Y$ be two disjoint non-empty modules of $\mathbb{M}$. If $Y$ is robust, non-trivial and $\{d(X,Y),d(Y,X)\} = t(Y)$ then $X \cup Y$ is not a module.

Proof. Let $\alpha := d(X,Y)$ and $\beta := d(Y,X)$. By hypothesis, we have $t(Y) = \{\alpha, \beta\}$. Let $L := \{(p,q) \in Y/\equiv_Y : p \neq q$ and $d(p,q) = \alpha\}$. If $\alpha \neq \beta$ this is a linear order, otherwise this is a complete graph or an independent set. Since the quotient has at least two elements, we may divide it into two non-empty subsets, which in the case $\alpha \neq \beta$ are an initial segment $I$ and a final segment $F$ w.r.t to this order. Let $Y'$ be the union of components of $Y$ which belong to $I$. We claim that $X \cup Y'$ is a module whenever $X \cup Y$ is a module.

Indeed, suppose that $X \cup Y$ is a module. Let $x, x' \in X \cup Y'$ and $y \in V \setminus (X \cup Y')$. We check that $d(x,y) = d(x',y)$ and $d(y,x) = d(y,x')$. If $y \notin X \cup Y$ this holds since $X \cup Y$ is a module. Thus, we may suppose $y \in Y \setminus Y'$. If $x, x' \in Y'$ this holds because due to our choice, $Y'$ is a module of $Y$. If $x, x' \in X$ this holds because $X$ is a module. Hence, we may suppose $x \in X, x' \in Y'$. In this case, we have $d(x,y) = d(X,Y) = \alpha$ and $d(x',y) = \alpha$, hence $d(x,y) = d(x',y)$; similarly $d(y,x) = d(y,x')$, proving that $X \cup Y'$ is a module. But, this is impossible since it meets $Y$ properly and $Y$ is strong.

Proposition 6.10. If two robust modules $A$ and $B$ with at least two elements and such that $B \subset A$ have the same non prime type $\{\alpha, \beta\}$ then there is a robust module $C$ with $B \subset C \subset A$ whose type is distinct from the type of $\{\alpha, \beta\}$.

Proof. Suppose that this is not the case. That is $t(C) = \{\alpha, \beta\}$ for every robust module $C$ with $B \subset C \subset A$.

Claim 6.11. Let $x \in A \setminus B$ and $y, y' \in B$. Then $d(x,y) = d(x,y') \in \{\alpha, \beta\}$.

Proof of Claim 6.11. Let $C$ be the least strong module containing $x, y$ and $y'$. We have $C = S_C(x,y) = S_C(x,y')$, hence $B \subseteq C \subseteq A$ thus $t(C) = \{\alpha, \beta\}$. 

to Ehrenfeucht and Rozenberg [10] for finite binary structures and to Harju and Rozenberg [15] for infinite binary structures (see also [6] Corollary 4.4.)
Since \( x \) and \( y \) belong to two different components of \( C \), \( d(x,y) \in t(C) \) and \( d(y,x) \in t(C) \); similarly \( d(x,y') \in t(C) \). Since \( B \) is a module, \( d(x,y) = d(x,y') \) and \( d(y,x) = d(y',x) \). The claim follows.

Let \( d(x,B) := d(x,y) \) where \( y \in B \). Let \( \gamma \in t(A) \) and 
\[
X_\gamma := \{ x \in A \setminus B : d(x,B) = \gamma \}
\]
Then, according to Claim 6.11 \( A = X_\alpha \cup X_\beta \cup B \).

Claim 6.12. \( X_\alpha \) and \( X_\beta \) are modules of \( M \).

Proof of Claim 6.12. Let \( x,x' \in X_\alpha \) and \( y \in V \setminus X_\alpha \). If \( y \notin A \) then since \( A \) is a module of \( M \) we have \( d(x,y) = d(x',y) \) and \( d(y,x) = d(y,x') \) as required.

If \( y \in A \) then since \( A = X_\alpha \cup X_\beta \cup B \), either \( y \in B \) or \( y \in X_\beta \) in which case \( \alpha \neq \beta \). If \( y \in B \) then by definition of \( X_\alpha \) we have \( d(x,y) = d(x',y) = \alpha \) and hence \( d(y,x) = d(y,x') = \beta \). If \( y \in X_\beta \), pick \( z \in B \). Since \( x \in X_\alpha \) and \( y \in X_\beta \) we have \( d(x,z) = \alpha \) and \( d(z,y) = \alpha \); since \( \alpha \neq \beta \), the Gallai quotient of \( A \) is linear, hence \( d(x,y) = \alpha \); similarly, \( d(x',y) = \alpha \). Thus \( X_\alpha \) is a module. The same holds for \( X_\beta \).

Since \( A = X_\alpha \cup X_\beta \cup B \) one of the sets \( X_\alpha, X_\beta \) is non-empty. Suppose that this is \( X_\alpha \).

Claim 6.13. \( X_\alpha \cup B \) is a module.

Proof of Claim 6.13. If \( \alpha = \beta \), \( X_\alpha \cup B = A \) and there is nothing to prove. Suppose \( \alpha \neq \beta \). Let \( x,x' \in X_\alpha \cup B \) and \( y \in V \setminus (X_\alpha \cup B) \). We check that \( d(x,y) = d(x',y) \) and \( d(y,x) = d(y,x') \). If \( y \notin A \) this holds since \( A \) is a module.

Thus, we may suppose \( y \in A \setminus (X_\alpha \cup B) \), that is \( y \in X_\beta \). If \( x,x' \in X_\alpha \) or \( x,x' \in B \) these equalities hold since \( X_\alpha \) and \( B \) are modules. Thus we may suppose \( x \in X_\alpha \) and \( x' \in B \). In this case, we have \( d(x,x') = d(x',y) = \alpha \) and since \( L \) is linear, \( d(x,y) = \alpha = d(x',y) \); by the same argument we also have \( d(y,x) = d(y,x') = \beta \). This proves our claim.

Claim 6.14. There is some non-empty proper subset \( D \) of \( B \) such that \( X := X_\alpha \cup D \) is a module.

Proof of Claim 6.14. Case 1. \( \alpha = \beta \). Let \( D \) be a component of \( B \). Since \( B \) is non-trivial, \( D \) is a proper subset of \( B \). It is easy to check that \( X := X_\alpha \cup D \) is a module. For that, pick \( x,x' \in X \) and \( y \in V \setminus X \). If \( x \in X_\alpha \), \( x' \in D \) we have \( d(x,x') = \alpha \); hence, for every \( y \in A \) we have \( d(x,y) = d(x',y) = \alpha \); the equality \( d(x,y) = d(x',y) \) holds trivially in all other cases.

Case 2. \( \alpha \neq \beta \). In this case, the Gallai quotient of \( B \) is linear. The set \( L := \{(p,q) \in B \mid \exists_B : p \neq q \text{ and } d(p,q) = \alpha \} \) is a linear order. Since the quotient has at least two elements, we may divide it into a non-empty initial segment and a non-empty final segment w.r.t. to this order. Let \( D \) be the union of components of \( B \) which belong to such an initial segment. As above one, can check that \( X \) is a module.

This claim contradicts Lemma 6.9.

6.2. Decomposition tree of cographs. The presentation followed below is equivalent to that employed by Courcelle and Delhomme [6], except they use
modules. The collection of robust modules of a binary structure $M$ forms a tree. We prefer to consider a refinement of this tree made of robust lattice as we do.

Once ordered by the reverse of inclusion, the collection of strong modules forms a tree. We prefer to consider a refinement of this tree made of robust lattice as we do.

Inclusion instead of reverse inclusion and find a join-lattice rather than a meet-lattice as we do.

Let $P$ be a poset. We recall that $P$ is a forest if for every element $x \in P$ the initial segment $\downarrow x := \{y \in P : y \leq x\}$ is a chain; this is a tree if in addition every pair of elements has a lower bound. We say that $P$ is a meet-lattice if every pair of elements $x, y \in P$ has a meet that we denote by $x \land y$ (and which is the largest lower bound of $x$ and $y$).

Let $T$ be a meet-tree. We observe that if an element $x$ of $T$ is the meet of a finite set $X$ of the maximal elements of $T$, denoted $\text{Max}(T)$, then $x$ is the meet of a subset $X'$ of $X$ with at most two elements.

We say that a meet-tree $T$ is ramified if every element of $T$ is the meet of a finite set of maximal elements of $T$.

Let $T$ be a ramified meet-tree and $T' := T \setminus \text{Max}(T)$. A $\{0, 1\}$-valuation is a map $v : T' \rightarrow \{0, 1\}$. The valuation is dense if for every $a < b$ in $T'$ there is some $c$ with $a < c \leq b$ such that $v(c) \neq v(a)$.

**Lemma 6.15.** Let $(T, v)$ be a densely valued ramified meet tree. Let $G := G(T)$ be the graph with vertex set $V := \text{Max}(T)$, two vertices $x$ and $y$ being joined by an edge if $v(x \land y) = 1$. Then $G$ is a cograph and $T(G) := (R(G), v_G)$ is isomorphic to $(T, v)$.

**Proof.** Let $x, y \in V$. Let $S_G(x, y)$ be the least strong module of $G$ containing $x$ and $y$, let $a := x \land y$ and $B(x, y) := (\uparrow a) \cap V := \{z \in V : z \geq x \land y\}$.

We prove that the following equality holds.

\begin{equation}
S_G(x, y) = B(x, y).
\end{equation}

\begin{equation}
v_G(S_G(x, y)) = v(x \land y).
\end{equation}

The fact that $(R(G), v_G)$ and $(T, v)$ are isomorphic follows.

**Claim 6.16.** $Z := B(x, y)$ is a strong module of $G$.

**Proof of Claim 6.16.** We prove first that $Z$ is a module containing $x$ and $y$. Let $t \in V \setminus Z$. Let $a' := x \land y \land t$. We claim that $z \land t = a'$ for every $z \in Z$. Since $G(z, t) = c(z \land t) = c(a')$, if this equality holds, $G(z, t)$ is independent of $z$, hence $Z$ is a module of $G$. To prove that this equality holds, let $z \in Z$. By definition, we have $x \land y \leq z$. From this inequality, we get $a' = x \land y \land t \leq z \land t$. Since $x \land y \leq z$ and $z \land t \leq z$ and $T$ is a tree, $x \land y$ and $z \land t$ are comparable. We cannot have $x \land y \leq z \land t$ otherwise we would have $x \land y \leq t$ contradicting...
Then $t \notin Z$. Thus, we have $z \land t \leq x \land y$. Since $z \land t \leq t$ and $a' = x \land y \land t$ it follows that $z \land t \leq a'$. With the inequality $a' \leq z \land t$ obtained above, this gives $z \land t = a'$ as claimed.

Now we prove that $Z$ is a strong module. Suppose not. Let $I$ be a module which intersects $Z$ properly. Let $x' \in Z \setminus I$, $y' \in Z \cap I$ and $t \in I \setminus Z$. Let $Z' := B(x', y')$, $b := x' \land y'$ and $b' := b \land t$. We have $Z' \subseteq Z$, hence $t \notin Z'$. The set $Z'$ is a module and $x' \land t = y' \land t$. Since $I$ is a module, $G(x', y') = G(x', t)$ hence $v(b) = v(b')$. We have $b' < b$ hence according to Lemma 6.10 there is some element $c$ with $b' < c < b$ such that $v(b') \neq v(c)$. Since $T$ is ramified, there are two elements $x''$, $y'' \in V$ such that $x'' \land y'' = c$. Let $Z'' := B(x'', y'')$. Then $Z \subseteq Z''$. Necessarily, $x''$ or $y''$ is not in $Z'$. Suppose that this is $y''$. In this case, we have $x' \land y'' = y' \land y'' = x'' \land y'' = c$. If $y'' \notin I$ then, since $I$ is a module, we must have $v(y' \land y'') = v(t \land y'')$. This is impossible since $y' \land y'' = c$, $t \land y'' = b'$ and $v(b') \neq v(c)$. Suppose that $y'' \in I$ then since $I$ is a module we must have $v(x' \land y') = v(x' \land y'')$ which is impossible since $x' \land y' = b$, $x' \land y'' = c$ and $v(b) = v(b') \neq v(c)$. Consequently, $I$ cannot intersect $Z$ properly, proving that $Z$ is strong.

On $Z$ define the following binary relation $\equiv_Z$:

$$
(3) \quad p \equiv_Z q \text{ if } p, q \in Z \text{ and } p = q \text{ or } p \land q \neq a.
$$

**Claim 6.17.** The relation $\equiv_Z$ is an equivalence relation whose blocks are strong modules. If $Z$ is not a singleton then there are at least two blocks and $G_{1Z}$ is a lexicographical sum on these blocks indexed by a clique or an independent set.

**Proof of Claim 6.17.** The relation $\equiv_Z$ is clearly reflexive and symmetric. We check that it is transitive. Let $p, q, r \in Z$ such that $p \equiv q$ and $q \equiv r$. We may suppose that $p, q, r$ are pairwise distinct, hence $p \land q > a$ and $q \land r > a$. Since $p \land q \leq q$ and $q \land r \leq q$ and $T$ is a tree, $p \land q \leq q$ and $q \land r \leq q$ are comparable, hence $a < \text{Min}\{p \land q, q \land r\} \leq p \land r$ proving that $p \equiv_Z r$ is transitive. If $Z$ is not a singleton, then $x \neq y$ hence $x \neq_Z y$ and $x \land y = a$, hence, there are at least two blocks. Let $I$ be a block. Pick $p \in I$, then $I = \bigcup_{q \in I} B(p, q)$. Since, according to Claim 6.16, each $B(p, q)$ is a strong module, $I$ is a strong module (as a union of strong modules with a common vertex, see Lemma 6.2). If $I$ and $J$ are two distinct blocks, let $p \in I$ and $q \in J$. Since $p \land q = a$, $G(p, q) = v(a)$ hence $G_{1Z}$ is a lexicographical sum on the blocks indexed by a clique if $v(a) = 1$ and indexed by an independent set if $v(a) = 0$. □

**Proof of Equation (1).** If $x = y$, $B(x, y) = S_G(x, y) = \{x\}$. Suppose $x \neq y$. We have $S_G(x, y) \subseteq B(x, y)$. Indeed, by definition $S_G(x, y)$ is the least strong module containing $x$ and $y$. According to Claim 6.16, $B(x, y)$ is a strong module. Since it contains $x$ and $y$ it contains $S_G(x, y)$. Let us prove the converse. Set $Z := B(x, y)$. Since each block of $\equiv_Z$ is a module, $S_G(x, y)$, which is a strong module, must be comparable to every block that it meets. Since it meets the block containing $x$ and the block containing $y$, it contains
these two blocks. Due to that fact, it contains every other block that it meets. Hence $S_G(x,y)$ is a union of at least two blocks. Since this is a strong module, it induces a strong module on the quotient. This quotient being a clique or an independent set, this strong module must be either a singleton, that is $S_G(x,y)$ is a block, which is not the case, or the the whole set, in which case $S_G(x,y) = Z$ as claimed.

\[ \square \]

**Proof of Equation \([2]\).** According to Equation \([1]\), we have $S_G(x,y) = B(x,y)$. We claim that the equivalence relations $\equiv_Z$ and $\equiv_a$ coincide. Indeed, let $p,q \in Z$. We have $S_G(p,q) = B(p,q)$. Hence, $p \equiv_Z q$ amounting to $S_G(p,q) \neq Z$ is equivalent to $B(p,q) \neq Z$ amounting to $p \equiv_a q$. It follows that $v_G(S_G(x,y)) = v(a)$ as claimed.

\[ \square \]

**Lemma 6.18.** Let $G$ be a cograph, $R(G)$ be the set of robust modules of $G$ ordered by reverse inclusion, $R_{22}(G) := \{A \in R(G) : |A| \geq 2\}$ and $v_G : R_{22}(G) \rightarrow \{0,1\}$ defined by setting $v_G(A) := 0$ if the Gallai quotient of $A$ is an independent set and $v_G(A) := 1$ otherwise. Then $T(G) := (R(G),v_G)$ is a dense valued meet tree and the graph $G$ associated with $T(G)$ is $G$.

**Proof.**

We prove successively:

1. The set $R(G)$ of robust modules of $G$ ordered by reverse inclusion is a meet-tree; the maximal elements of $R(G)$ are the singletons of $V(G)$, and thus $R(G)$ is ramified.

Since robust modules are strong, they are disjoint or comparable hence $R(G)$ is a forest. Let $A,B \in T(G)$. Pick $x \in A$ and $y \in B$. Then $S_G(x,y)$ is the meet of $A$ and $B$ in $R(G)$ hence $R(G)$ is a meet-tree. For each $x \in V := V(G)$, $\{x\} \in R(G)$. Let $A \in R(G)$; since $A$ contains $x,y$ such that $A = S_G(x,y)$, we have $A = \{x\} \land \{y\}$ in $R(G)$, hence $R(G)$ is ramified.

2. $v_G$ is a dense valuation.

Since $G$ is a cograph, no strong module can be prime; since $G$ is undirected the type of a robust module is $\{0\}$ or $\{1\}$. Hence the valuation of a robust module is essentially its type. The density property follows from Lemma 6.10.

3. The graph $G$ associated with $T(G)$ is $G$.

Let $x,y$ be two distinct vertices of $G$ and $A$ be the least robust module containing $x$ and $y$. Since $G$ is a cograph, Theorem 6.8 asserts that $G_{1A}$ is a lexicographic sum of its components and the quotient is a clique or an independent. Thus, $\{x,y\}$ is an edge iff $v_G(A) = 1$.

From these two lemmas we deduce:

**Theorem 6.19.** There is a one-to-one correspondence between cographs and ramified meet-trees densely valued by $\{0,1\}$.

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