MEAN-FIELD LIMIT FOR PARTICLE SYSTEMS WITH TOPOLOGICAL INTERACTIONS

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ABSTRACT. The mean-field limit for systems of self-propelled agents with “topological interaction” cannot be obtained by means of the usual Dobrushin approach. We get a result on this direction by adapting to the multidimensional case the techniques developed by Trocheris in 1986 to treat the Vlasov-Poisson equation in one dimension.

1. INTRODUCTION

Many interesting physical systems can be described at the microscopic level as particle dynamics and at the mesoscopic level with kinetic equations. In the wide field of two-body interactions, the link between these two regimes is mathematically well understood in the case of the mean-field limit, i.e. when the density of the particles diverges with their number $N$, the mean free path vanishes as $1/N$ and the interaction intensity scales with $1/N$. In this limit, each particle feels the interaction with the others as a mean.

A rigorous mathematical proof of this result can be done in the case of two-body interactions with sufficiently regular potentials. This classical achievement has been obtained independently by several authors in the ’70s (see [5, 14, 27]) and its explanation is particularly clear in the Dobrushin’s argument [14] where the result follows by noticing that the empirical measure associated with the particle system is a weak solution of the mean-field equation; the proof follows by showing the weak continuity, w.r.t the initial datum, of the weak solutions.

Although the theory for regular pairwise interactions is sufficiently well understood, going beyond it considering singular potentials, is instead a harder task. This is the case of the three-dimensional Vlasov-Poisson equation, which is the most important equation of plasma physics and of galactic dynamics, based on the choice of the Coulomb or Newton potential, respectively. In this equation, the potential $1/r$ is singular at the origin and does not belong to any $L^p$ space. Although the mean-field limit for the Vlasov-Poisson equation remains an open
problem, there has been important progress in recent years, see the works \[21, 22\] where the mean-field limit is proven for potentials with singularities “weaker than $1/r$” and also \[25, 26\]. However, in the case of the one-dimensional Vlasov-Poisson equation, the problem has been solved in \[30, 29\] and with a simpler proof in \[20\], being the force discontinuous, but not diverging.

The mean-field limit is a case of propagation of chaos, i.e. the $j$-particle distribution function factorize in the limit. This property is the key for obtaining a kinetic description of the particle dynamics (see for instance \[28\] and \[10, 16, 23\] for some reviews on this point of view).

In recent years, the conceptual and mathematical apparatus of kinetic equations has been used in the study of self-propelled particle systems of biological nature, in particular for the motion of swarms and other animals. Starting with the pioneering paper in \[31\], several models have been proposed to explain the evolution of these systems. In the simplest \[11, 12, 31\], a bird is modeled as a self-propelling particle that interacts with its neighbors. The interaction is such that neighboring birds tend to align their velocities. For many of these models, the mean-field limit has often been used to obtain a kinetic description of the dynamics (see, for instance, \[18, 7, 6, 8, 17, 3\]).

A few years ago, supported by observational data (\[2, 9, 1\]), “topological” models for interaction were introduced: an agent reacts to the presence of another not according to the distance, but according to the proximity ranking (see eqs (1.1), (1.2), (1.3) below for a rigorous formulation). These models come out of the case of two-body interaction, and present various problems in their kinetic treatment. In particular, the solutions of the kinetic equation are not weakly continuous w.r.t. the initial datum and there are also some difficulties in defining the particle motion.

In this paper we prove a result on the mean-field limit for topological models. We focus our attention on the topological Cucker-Smale model, but, with the same ideas, it is possible to consider more general cases. A first result in this direction has been proved in \[19\], for a smoothed version of the model in which the weak continuity in the initial datum is recovered. We also mention that a kinetic Boltzmann equation for a stochastic particle model with rank-based interaction has been obtained in \[13\], by using the BBGKY hierarchy.

We formulate the problem and summarize our results. A Cucker-Smale type model for the motion of $N$ agents, in the mean-field scaling, is the system

\[
\begin{align*}
\dot{X}_i(t) &= V_i(t) \\
\dot{V}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} p_{ij} (V_j(t) - V_i(t)),
\end{align*}
\]

(1.1)
where the “communication weights” \( \{p_{ij}\}_{i,j=1}^N \) are positive functions that take into account the interactions between agents. In classical models, \( p_{ij} \) depends only on the distance \( |X_i - X_j| \) between the agents. In topological models the weights depend on the positions of the agents by their rank

\[
p_{ij} := K(M(X_i, |X_i - X_j|)),
\]

where \( K: [0,1] \to \mathbb{R}^+ \) and, for \( r > 0 \), the function

\[
M(X_i, r) := \frac{1}{N} \sum_{k=1}^{N} \mathcal{X}(|X_k - X_i| \leq r)
\]

counts the number of agents at distance less than or equal to \( r \) from \( X_i \), normalized with \( N \). Note that in this case \( p_{ij} \) is a stepwise function of the positions of all the agents. In the sequel we assume that \( K \) is a positive decreasing function, Lipschitz continuous, and such that \( \int_0^\infty K(z) \, dz = \gamma \).

In the mean-field limit \( N \to +\infty \), the one-agent distribution function \( f_t = f(t, x, v) \) is expected to verify the equation

\[
\partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (W[Sf_t, f_t](x, v)f_t) = 0,
\]

where \( Sf_t(x) := \int f_t(x, v) \, dv \) is the spatial distribution and where, given a probability density \( f \) in \( \mathbb{R}^d \times \mathbb{R}^d \) and a probability density \( \rho \) in \( \mathbb{R}^d \),

\[
W[\rho, f](x, v) := \int K(M[\rho](x, |x - y|))(w - v)f(y, w) \, dy \, dw,
\]

with

\[
M[\rho](x, r) := \int_{|x' - x| \leq r} \rho(x') \, dx'.
\]

A weak formulation of this equation is given requiring that the solution \( f_t \) fulfills

\[
\int \alpha(x, v) \, df_t(x, v) = \int \alpha(X_t(x, v), V_t(x, v)) \, df_0(x, v)
\]

for any \( \alpha \in C_b(\mathbb{R}^d \times \mathbb{R}^d) \), where \( f_0 \) is the initial probability measure and \( (X_t(x, v), V_t(x, v)) \) is the flow defined by

\[
\begin{aligned}
\dot{X}_t(x, v) &= V_t(x, v) \\
\dot{V}_t(t, x, v) &= W[Sf_t, f_t](X_t(x, v), V_t(x, v)) \\
X_0(x, v) &= x, \quad V_0(x, v) = v.
\end{aligned}
\]

In other words, \( f_t \) is the push-forward of \( f_0 \) along the flow generated by the velocity field, determined by \( f_t \) itself.

It is easy to verify that the empirical measure

\[
\mu_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^N(i)} \delta_{V_t^N(i)}
\]
associated with the solution of (1.1), (1.2) and (1.3) is a weak solution of (1.4). Namely, $M[S\mu^N_t](X, r)$ is exactly $M(X, r)$ defined in (1.3) (from now on we use the more complete notation $M[S\mu^N_t](X, r)$). Thus, we can rewrite the agent evolution in (1.1) as

$$
\begin{align*}
\dot{X}^N_t(t) &= V^N_t(t) \\
\dot{V}^N_t(t) &= W[S\mu^N_t, \mu^N_t](X^N_t(t), V^N_t(t)).
\end{align*}
$$

(1.8)

In the Dobrushin approach to the mean-field limit, the result is achieved from this fact and from the weak continuity, w.r.t the initial datum, of the weak solutions of (1.4). We cannot use this approach in presence of topological interaction, since in general the solutions of (1.7) are not weakly continuous w.r.t the initial datum (see Section 3). We can overcome this difficulty if the solution of (1.4) has a bounded density. To obtain our result, we adapt the ideas used in [30] for the derivation of the one-dimensional Vlasov equation in presence of discontinuity of the force. In particular we prove that

[Thm. 5.1] the $N$-particle dynamics is well defined, except for a set of measure zero;

[Thm. 6.1] if $f_0$ is bounded, there exists a unique weak solution $f_t$ of the topological Cucker-Smale equation, which is bounded;

[Thm. 7.1] if $\mu^N_t$ solves (1.8) and $\mu^N_0 \rightharpoonup \mu_0$, then $\mu^N_t \rightharpoonup f_t$.

We divide the work as follows: in Section 2 we discuss some properties of the “discrepancy distance”, the main tool for dealing with topological interactions. In Section 3 we discuss existence, uniqueness and regularity of the agent dynamics (1.8), proving Thm. 6.1. In Section 4 we discuss existence, uniqueness and regularity of the weak solutions of the mean-field equation (1.4) with bounded initial datum, proving Thm. 7.1. In Section 5 we prove Thm. 5.1.

2. DISTANCES AND WEAK CONVERGENCE

We recall that the 1-Wasserstein distance $\mathcal{W}$ of two probability measures $\rho_1$ and $\rho_2$ on $\mathbb{R}^d$ can be defined by duality with Lipschitz functions:

$$
\mathcal{W}(\rho_1, \rho_2) = \sup_{\phi \in C_b(\mathbb{R}^d), \text{Lip}(\phi) \leq 1} \int \phi(d\rho_1 - d\rho_2)
$$

$$
= \sup_{\phi \in C^1_b(\mathbb{R}^d), \|\nabla \phi\|_{\infty} \leq 1} \int \phi(d\rho_1 - d\rho_2),
$$

where $\text{Lip}(\phi)$ is the Lipschitz constant of $\phi$.

The counter of the number of particles in (1.9) is not continuous w.r.t. $\mathcal{W}$, so we work with the weaker topology induced by another distance, the discrepancy, defined as

$$
\mathcal{D}(\rho_1, \rho_2) := \sup_{x, r > 0} \left| \int_{B_r(x)} d\rho_1 - \int_{B_r(x)} d\rho_2 \right|.
$$
Here and after, we denote by $B_r(x)$ the closed ball of center $x$ and radius $r$ in $\mathbb{R}^d$. In the sequel, we also indicate by $B_r$ the closed ball $B_R(0)$. The discrepancy distance is mostly used in one dimension to quantify the uniformity of sequence of points (see [24, 15]), but its multidimensional version is cited in [27], in the context of kinetic limits. By definition, it holds the following proposition.

**Proposition 2.1** (Lipschitzianity of $M$ w.r.t. $D$). Let $\rho_1$ and $\rho_2$ be two probability measures on $\mathbb{R}^d$. Then, for any $x \in \mathbb{R}^d$ and $r > 0$,

$$|M[\rho_1](x, r) - M[\rho_2](x, r)| \leq D(\rho_1, \rho_2).$$

We can also define $D$ in terms of regular functions. Let $X$ be the subset of $C^1_b([0, +\infty); \mathbb{R})$, and define

$$\|\phi\|_X := \int_{0}^{+\infty} |\phi'(r)| \, dr.$$  

Then

$$D(\rho_1, \rho_2) = \sup_{\phi \in X: \|\phi\|_X \leq 1} \sup_x \int x \phi(|x - y|) (d\rho_1(y) - d\rho_2(y)).$$

This assertion is an easy consequence of the following lemma.

**Lemma 2.1.** Let $g_1$ and $g_2$ be two probability measures on $[0, +\infty)$. Then

$$\sup_{r \geq 0} \left| \int_{[0,r]} dg_1 - \int_{[0,r]} dg_2 \right| = \sup_{\phi \in X: \|\phi\|_X \leq 1} \int_{0}^{+\infty} \phi(dg_1 - dg_2). \quad (2.1)$$

**Proof.** Fix $r > 0$, there exists $\phi_{r,\varepsilon} \in X$ with $\|\phi_{r,\varepsilon}\|_X = 1$ and such that $\phi_{r,\varepsilon}(s) = 1$ if $0 \leq s \leq r$ and $\phi_{r,\varepsilon}(s) = 0$ if $s \geq r + \varepsilon$. For any measure $g$,

$$\lim_{\varepsilon \to 0} \int_{0}^{+\infty} \left( \phi_{r,\varepsilon}(s) - X\{s \in [0, r]\} \right) dg(s) = 0,$$

then

$$\int_{[0,r]} (dg_1 - dg_2) = \lim_{\varepsilon \to 0} \int_{0}^{+\infty} \phi_{r,\varepsilon}(dg_1 - dg_2) \leq \sup_{\phi \in X: \|\phi\|_X \leq 1} \int_{0}^{+\infty} \phi(dg_1 - dg_2).$$

To prove the opposite inequality, we denote by $G_1$ and $G_2$ the distribution functions of $g_1$ and $g_2$:

$$G_i(r) := \int_{[0,r]} dg_i.$$

Then, integrating by parts,

$$\int_{0}^{+\infty} \phi(dg_1 - dg_2) = - \int_{0}^{+\infty} \phi'(r) (G_1(r) - G_2(r)) \, dr \leq \|\phi\|_X \|G_1 - G_2\|_X.$$  

We conclude the proof by noticing that $\|G_1 - G_2\|_X$ is exactly the left-hand-side of (2.1). \qed
For our purposes, we need the equivalence of $D$ and $W$ in the case in which one of the two measures has bounded density. We note that in the general case the equivalence is false, as can be easily checked by considering two Dirac measures $\delta_{x_1}$ and $\delta_{x_2}$: $W$ vanishes when $|x_1 - x_2| \to 0$, while $D$ is one whenever $x_1 \neq x_2$. Nevertheless, using the covering principles as in [4], for measures on a compact set, it can be proved the continuity of the Wasserstein distance $W$ w.r.t. the discrepancy distance $D$. For the sake of completeness, we give a proof in the appendix, although this property is not really necessary for our results.

In the sequel, in the definition of $D$ we choose functions in $\phi \in C([0, +\infty), \mathbb{R})$, with first derivative continuous up to a finite number of jumps. With abuse of notation, we keep calling this set of functions $X$. Let us expose some technical properties.

Given $\phi \in X$, we define some useful regularizations, $\phi^\pm$, $\phi_\varepsilon$ and $\psi_\varepsilon$, with $\varepsilon > 0$, as follows. Denoting by $\tilde{\phi}$ the function

$$
\tilde{\phi}(r) := \int_0^r |\phi'(s)| \, ds,
$$

we define

$$
\phi^\pm(r) := \begin{cases} 
\frac{1}{2}(\tilde{\phi}(r) \pm \phi(r)), & \text{if } r \geq 0, \\
\pm \frac{1}{2}\phi(0), & \text{if } r < 0,
\end{cases}
$$

and

$$
\phi_\varepsilon(r) := \phi^+(r + \varepsilon) - \phi^-(r - \varepsilon).
$$

Finally, fixed a regular mollifier $\eta$ supported in $(0,1)$, we define

$$
\psi_\varepsilon(r) := \int_0^\varepsilon \eta_\varepsilon(s)\phi^+(r + s) \, ds - \int_0^\varepsilon \eta_\varepsilon(s)\phi^-(r - s) \, ds.
$$

We summarize the properties of these regularizations in the following lemma, where we indicate with $c$ any constant which does not depend on $\phi$ and $\varepsilon$.

**Lemma 2.2.**

i) $\phi^\pm$ are not decreasing. Moreover

$$
\int_0^{+\infty} (\phi^\pm)'(r) \, dr \leq \|\phi\|_X
$$

and $\phi(r) = \phi^+(r) - \phi^-(r)$ for $r \geq 0$.

ii) $\phi_\varepsilon \in X$, $\phi(r) \leq \phi_\varepsilon(r)$ and

$$
\int_0^{+\infty} (\phi_\varepsilon(r) - \phi(r)) \, dr \leq 2\varepsilon\|\phi\|_X.
$$
iii) \( \psi_\varepsilon(r) \geq \phi(r) \). Moreover \( \psi_\varepsilon \) is a \( C^1 \) function in \( X \),

\[
\| (\psi_\varepsilon)' \|_X \leq \frac{2}{\varepsilon} \| \eta \|_X \| \phi \|_X \tag{2.6}
\]

and

\[
\int_0^{+\infty} |\psi_\varepsilon(r) - \phi(r)| \, dr \leq c \varepsilon \| \phi \|_X. \tag{2.7}
\]

**Proof.** The proof is elementary, we only describe how to get the bounds in ii) and iii). Since \( \phi = \phi^+ - \phi^- \), we rewrite the l.h.s. of (2.5) as

\[
\int_0^{+\infty} (\phi^+(r + \varepsilon) - \phi^+(r)) + (\phi^-(r) - \phi^-(r - \varepsilon)) \, dr
\]

\[
= \int_0^{+\infty} \left( \int_0^\varepsilon \left( (\phi^+)'(r + \xi) + (\phi^-)'(r - \xi) \right) \, d\xi \right) \, dr \leq 2 \varepsilon \| \phi \|_X.
\]

The estimate in (2.6) is immediate while, regarding (2.7), we rewrite \( \psi_\varepsilon(r) - \phi(r) \) as

\[
\int_0^1 \eta(s) (\phi^+(r + \varepsilon s) - \phi^+(r) + \phi^-(r) - \phi^-(r - \varepsilon s)) \, ds
\]

\[
= \varepsilon \int_0^1 s \eta(s) \left( \int_0^\varepsilon (\phi^+)'(r + \varepsilon s \xi) \, d\xi + \int_0^\varepsilon (\phi^-)'(r - \varepsilon s \xi) \, d\xi \right) \, ds.
\]

We conclude by integrating in \( r \), switching the order of integration and using (2.4). \( \square \)

Now we can prove the following proposition.

**Proposition 2.2.** Let \( \rho \) and \( \nu \) be two probability measures on \( \mathbb{R}^d \) with support in a ball \( B_R \) and such that \( \rho \in L^\infty(\mathbb{R}^d) \). Then

\[
\mathcal{D}(\nu, \rho) \leq C \| \rho \|_X \| \nabla \phi \|_X \mathcal{W}(\nu, \rho),
\]

where \( C \) is a constant that depends on the dimension \( d \), as well as on \( \| \rho \|_X \) and on \( R \).

**Proof.** Let \( \phi \) be in \( X \) and consider \( \psi_\varepsilon \) as in (2.3). Fixed \( x \in \mathbb{R}^d \), let \( \Phi \) and \( \Psi_\varepsilon \) be the functions

\( \Phi(y) := \phi(|x - y|) \) and \( \Psi_\varepsilon(y) := \psi_\varepsilon(|x - y|) \).

Then, from iii) of Lemma 2.2,

\[
\int \Phi \, d\nu - \int \Phi \, d\rho \leq \int \Psi_\varepsilon \, d\nu - \int \Phi \, d\rho = \int \Psi_\varepsilon \, d(\nu - \rho) + \int (\Psi_\varepsilon - \Phi) \, d\rho.
\]

From (2.6) of Lemma 2.2 the first term is bounded by \( \varepsilon \| \phi \|_X \mathcal{W}(\nu, \rho) \). Regarding the second term, denoting by \( \sigma_r \) the uniform measure on
\[ \partial B_r(x), \] we have
\[
\int (\Psi_\varepsilon - \Phi) \, d\rho \leq \|\rho\|_\infty \int_0^{\infty} dr (\psi_\varepsilon(r) - \phi(r)) \int_{\partial B_r(x)} X\{z \in B_R\} \sigma(dz)
\]
\[
\leq c\varepsilon R^{d-1}\|\phi\|_X, \tag{2.8}
\]
where in the last inequality we have used (2.7). Optimizing on \(\varepsilon\) and passing to the supremum in \(\phi\), we get the proof. \(\square\)

Note that if \(\mu^N\) is an empirical measure and \(\nu\) a probability measure that does not give mass to the atoms of \(\mu^N\), \(\mathcal{D}(\mu^N, \rho) \geq 1/N\). With this constraint, the discrepancy between two empirical measures is “small” if the measures are close in the sense specified in the following proposition.

**Proposition 2.3.** Let
\[
\mu^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \quad \text{and} \quad \nu^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{y_i}
\]
be two empirical measures on \(\mathbb{R}^d\) and take \(\delta > 0\) such that \(|x_i - y_i| \leq \delta\) for all \(i = 1, \ldots, N\). Then, for any probability measure \(\rho \in L^\infty(\mathbb{R}^d)\) supported on a ball \(B_R\),
\[
\mathcal{D}(\mu^N, \nu^N) \leq cR^{d-1}\|\rho\|_\infty + c\mathcal{D}(\mu^N, \rho).
\]

**Proof.** Given \(\phi \in X\) with \(\|\phi\|_X \leq 1\), we construct \(\phi_\delta\) as in (2.2) and, fixed \(x \in \mathbb{R}^d\), we consider \(\Phi(y) := \phi(|x - y|), \Phi_\delta(y) := \phi_\delta(|x - y|)\).

Since \(|x - x_i| - \delta \leq |x - y_i| \leq |x - x_i| + \delta\), we have that
\[
\Phi(y_i) = \phi^+(|x - y_i|) - \phi^-(|x - y_i|) \leq \Phi_\delta(x_i).
\]
Then
\[
\int \Phi \, d(\nu^N - \mu^N) \leq \int (\Phi_\delta - \Phi) \, d\mu^N = \int (\Phi_\delta - \Phi) \, d(\mu^N - \rho) + \int (\Phi_\delta - \Phi) \, d\rho.
\]
Since \((\phi_\delta - \phi) \in X\), the first term is bounded by \(c\mathcal{D}(\mu^N, \rho)\). Using (2.5) and reasoning as in (2.8) we estimate the second term with \(c\delta R^{d-1}\|\rho\|_\infty\). \(\square\)

### 3. Agent Dynamics

One of the difficulties in handling (1.8) is that the dynamic is not continuous w.r.t the initial datum. For instance, consider three agents \(\{X_i\}_{i=1}^3\) on a line, such that
\[
X_1(0) = -1, \quad X_2(0) = \varepsilon, \quad X_3(0) = 1,
\]
\[
V_1(0) = -1, \quad V_2(0) = 0, \quad V_3(0) = 1, \quad \tag{3.1}
\]
with \( \varepsilon \in (-1, 1) \setminus \{0\} \). Then \( p_{i,j} = M(X_i, |X_i - X_j|) \) takes the values 1/3, 2/3, 1. Suppose for simplicity that \( K(2/3) = 3 \) and \( K(1) = 0 \), then the equations for \( V_1 \) and \( V_3 \) read as

\[
\begin{align*}
\dot{V}_1(t) &= V_2(t) - V_1(t) \\
\dot{V}_3(t) &= V_2(t) - V_3(t),
\end{align*}
\]

while

\[
\dot{V}_2(t) = \begin{cases} 
V_3(t) - V_2(t) & \text{if } \varepsilon \in (0, 1) \\
V_1(t) - V_2(t) & \text{if } \varepsilon \in (-1, 0).
\end{cases}
\]

It follows that

\[
\begin{align*}
\dot{V}_1(t) &= -(1 + e^{-2t})/2 \\
\dot{V}_2(t) &= -(1 - e^{-2t})/2 \\
\dot{V}_3(t) &= (-1 + 4e^{-t} - e^{-2t})/2
\end{align*}
\]

if \( \varepsilon \in (-1, 0) \), while

\[
\begin{align*}
\dot{V}_1(t) &= -(1 + 4e^{-t} - e^{-2t})/2 \\
\dot{V}_2(t) &= (1 - e^{-2t})/2 \\
\dot{V}_3(t) &= (1 + e^{-2t})/2
\end{align*}
\]

if \( \varepsilon \in (0, 1) \), so that \( \{X_i(t), V_i(t)\}_{i=1}^3 \) is discontinuous in \( \varepsilon = 0 \). Note that the discontinuity of the trajectories in the phase space is easily translated in the weak discontinuity of the empirical measure at time \( t \), w.r.t the initial measure.

This discontinuity reflects the fact that, for data as in (3.1) with \( \varepsilon = 0 \), there is not a unique way to define the dynamics. Nevertheless, we can prove that the system (1.8) is well-posed for almost all initial data. To do so, let us define some subsets of the phase space

\[
\{(X, V) = (x_1, \ldots, x_N, v_1, \ldots, v_N) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}\},
\]

where \( d \geq 1 \) is the dimension of the configuration space of the agents.

\textbf{Definition 3.1.}

\( \mathcal{R} \) is the set of “the regular points”, i.e. the set of points \( (X, V) \) such that for each triad of different indices it holds that \( |x_i - x_k| \neq |x_j - x_k| \).

\( \mathcal{S} \) is the “iso-rank” manifold, i.e. the set of points \( (X, V) \) such that there exists a triad of different indices \( i, j, k \) for which \( |x_i - x_k| = |x_j - x_k| \), i.e. the agents \( i \) and \( j \) have the same rank with respect to the agent \( k \).

\( \mathcal{S}_r \) is the set of the “regular points” of the iso-rank manifold, i.e. the subset of points \( (X, V) \in \mathcal{S} \) such that if \( |x_i - x_k| = |x_j - x_k| \) then \( x_i, x_j, x_k \) are different and \( (v_i - v_k) \cdot \hat{n}_{ik} \neq (v_j - v_k) \cdot \hat{n}_{jk} \), where \( \hat{n}_{ab} := (x_a - x_b)/|x_a - x_b| \).
We can define the dynamics locally in time, not only for initial data in \( \mathcal{R} \), but also in \( \mathcal{S}_r \). Namely, if initially the agents \( i \) and \( j \) have the same rank with respect to the agent \( k \), we can redefine the force exerted on the agent \( k \) accordingly to the velocities: if \( (v_i - v_k) \cdot \hat{n}_{ik} > (v_j - v_k) \cdot \hat{n}_{jk} \) we evaluate the rank as if \( |x_i - x_k| > |x_j - x_k| \) for \( t > 0 \) and as if \( |x_i - x_k| < |x_j - x_k| \) for \( t < 0 \). In other words, the different speeds of change of the distances among the agents allow the dynamics to leave \( \mathcal{S} \) instantaneously.

We discuss the existence of the dynamics, so redefined.

**Lemma 3.1.** If \((X, V) \in \mathcal{R} \cup \mathcal{S}_r\), there exists \( \tau > 0 \) such that the system (1.8) has a unique solution for \( t \in (-\tau, \tau) \), with initial datum \((X, V)\). Moreover the solution is locally Lipschitz in \( t \) and in \((X, V)\).

We omit the proof.

In \( \mathcal{R} \) the solution is regular, so we can compute the determinant of the Jacobian of the flow \( J(t) \equiv J(X, V, t) \). It verifies the equation

\[
\frac{d}{dt} J(t) = -\left( \frac{d}{N} \sum_{i,j \neq j} p_{ij} \right) J(t) = -dN \gamma_N J(t),
\]

where

\[
\gamma_N := \frac{1}{N} \sum_{n=2}^{N} K(n/N).
\]

Thus, volumes of the phase space are shrunk in time at a constant rate, therefore their measure cannot vanish in finite time. This implies the following fact, of which we omit the proof.

**Lemma 3.2.** The subset of initial data \((X, V) \in \mathcal{R} \) such that the trajectory, at a first time in the future or in the past, intersects \( \mathcal{S} \setminus \mathcal{S}_r \), has Lebesgue measure zero. Namely, \( \mathcal{S} \setminus \mathcal{S}_r \) has dimension \( 2Nd - 2 \).

This lemma guarantees that, except for a subset of Lebesgue measure zero, we can prolong the dynamics with initial data in \( \mathcal{R} \) also after a crossing in \( \mathcal{S} \). To define the dynamics for all times, we need to control the number of crossings.

**Lemma 3.3.** The subset of initial data \((X, V) \in \mathcal{R} \) such that the trajectory intersect \( \mathcal{S}_r \) infinitely many times in finite time, has Lebesgue measure zero.

**Proof.** Fix \( T > 0 \) and suppose to take \((X, V) \in \mathcal{R} \) such that the solution \((X^N(t), V^N(t)) = (X_1(t), \ldots, X_N(t), V_1(t), \ldots, V_N(t)) \) with initial data \((X, V)\) intersects \( \mathcal{S}_r \) a finite number of times in \([0, T - \varepsilon]\) and infinitely many times in \([0, T)\). The number of particles is finite, so we can assume that there exists a triad of indices such that \(|X_i - X_k| = |X_j - X_k| \) infinitely many times. Since the velocities \( V_i \) are bounded by a constant, as follows by simple considerations (see also Thm. 3.1), from
the equation we have that \(|X_i - X_k|\) and \(|X_j - X_k|\) are \(C^1\) functions, with time derivatives uniformly Lipschitz, if \(|X_i - X_k|\) and \(|X_j - X_k|\) remain far from 0. Then, as \(t \to T\), either \(|X_i - X_k| \to 0\) or \((V_i - V_k) \cdot \hat{n}_{ik}\) and \((V_j - V_k) \cdot \hat{n}_{jk}\) converge to the same limit. In both the cases, the trajectory reaches \(S\) at a point that is not in \(S_r\). We conclude the proof observing that the intial point with these properties lives in a subset of dimension \(2Nd - 1\).

From these lemmas and other few considerations, we obtain the following theorem.

**Theorem 3.1.** Except for a set of measure zero, given \((X, V) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}\), there exists a unique global solution

\[
(X^N(t, X, V), V^N(t, X, V)) \in C^1(\mathbb{R}^+, \mathbb{R}^{2Nd}) \times C(\mathbb{R}^+, \mathbb{R}^{2Nd})
\]

with initial datum \((X, V)\).

Moreover, given \(R_x > 0\) and \(R_v > 0\), we have that

\[|X_i(t)| \leq R_x + tR_v, \quad |V_i(t)| \leq R_v\]

for any \(i\), if \(|x_i| \leq R_x\) and \(|v_i| \leq R_v\). Therefore \(V_i(t, X, V)\) has Lipschitz constant bounded by \(2R_vK(0)\).

**Proof.** The proof follows easily from Lemma 3.1, Lemma 3.2 and Lemma 3.3.

The a-priori bound on the support follows from (3.2) and by noticing that

\[
\frac{d}{dt}|V_i(t)|^2 = -2 \sum_{j \neq i} p_{ij} \left(|V_i(t)|^2 - V_i(t) \cdot V_j(t)\right)
\]

is null or negative if \(|V_i|^2\) is maximum in \(i\).

\[\square\]

### 4. The Mean-Field Equation in \(L^\infty\)

In this section we show how to get an existence and uniqueness result for bounded weak solutions of equation (1.4). We start by stating some elementary facts.

**Lemma 4.1.** Let \(\rho \in L^\infty(\mathbb{R}^d)\) be a probability density.

i) Given \(r_1, r_2 > 0\),

\[|M[\rho](x, r_1) - M[\rho](x, r_2)| \leq c\|\rho\|_\infty |r_1^d - r_2^d|.
\]

ii) Given \(x_1, x_2 \in \mathbb{R}^d\) and \(r > 0\),

\[|M[\rho](x_1, r) - M[\rho](x_2, r)| \leq c\|\rho\|_\infty r^{d-1}|x_1 - x_2|.
\]

**Proof.** The proof of the first assertion is immediate. For the second, we use the following splitting

\[
\mathcal{X}\{|x_1 - y| < r\} - \mathcal{X}\{|x_2 - y| < r\} = \mathcal{X}\{|x_1 - y| < r\}\mathcal{X}\{|x_2 - y| \geq r\} - \mathcal{X}\{|x_2 - y| < r\}\mathcal{X}\{|x_1 - y| \geq r\}
\]
and we note that, if $|x_1 - x_2| \geq r$,

$$
\int_{|x_1 - y| < r} \mathcal{X} \{ |x_2 - y| \geq r \} \, dy \leq cr^d \leq cr^{d-1}|x_1 - x_2|,
$$

while, if $|x_1 - x_2| < r$,

$$
\int_{|x_1 - y| < r} \mathcal{X} \{ |x_2 - y| \geq r \} \, dy \leq \int \mathcal{X} \{ r - |x_1 - x_2| < |x_1 - y| < r \} \, dy
$$

$$
= cr^d \left( 1 - (1 - |x_1 - x_2|/r)^d \right) \leq cd r^{d-1}|x_1 - x_2|.
$$

\[ \square \]

In the following, we denote by $B_r$ the closed ball of center 0 and radius $r$ in $L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and by $C_w \left( [0, +\infty); L^\infty(\mathbb{R}^d \times \mathbb{R}^d) \right)$ the set of families of bounded probability densities $\{f_t\}_{t \geq 0}$ which are weakly continuous in time in the sense of measures.

**Lemma 4.2.** Let $\{f_t\}_{t \geq 0}$ be a family of probability densities such that $\{f_t\} \in C_w \left( [0, +\infty); B_{r(t)} \right)$, with $r(t)$ a continuous nondecreasing function. Suppose that

$$
supp(f_t) \subset B_{R_v(t)} \times B_{R_v(t)}, \tag{4.1}
$$

where $R_v(t)$ and $R_x(t)$ are two continuous nondecreasing functions. Then, for any initial datum $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, there exists a unique global solution of (1.7).

**Proof.** From the classical Cauchy-Lipschitz theory, we only have to verify that $W[Sf_t, f_t](x, v)$ is bounded on compact sets, locally Lipschitz and continuous in $t$.

Recalling 1.3, the boundness on compact sets follows from

$$
|W[Sf_t, f_t](x, v)| \leq \|K\|_{\infty} (R_v(t) + |v|).
$$

Since from $i)$ and $ii)$ of Lemma 3.1

$$
|M[Sf_t](x_1, |x_1 - y|) - M[Sf_t](x_2, |x_2 - y|)|
$$

$$
\leq c\|Sf_t\|_{\infty} (|x_1| + |x_2| + |y|)^{d-1}|x_1 - x_2|,
$$

we have that, if $(x_1, v_1)$ and $(x_2, v_2)$ belong to a compact subset of $\mathbb{R}^d \times \mathbb{R}^d$,

$$
|W[Sf_t, f_t](x_1, v_1) - W[Sf_t, f_t](x_2, v_2)| \leq C(|x_1 - x_2| + |v_1 - v_2|),
$$

where $C$ depends on $R_x, R_v$ and on the diameter of the compact set.

In order to prove that $W[Sf_t, f_t](x, v)$ is continuous in $t$, we first observe that $\mathcal{W}(Sf_t, Sf_s) \leq \mathcal{W}(f_t, f_s)$ and that, from the Lipschitzianity of $K$ and Propositions 2.1 and 2.2 $K(M([Sf_t](x, |x - y|)))$ is continuous in $t$. Since $K(M([Sf_t](x, |x - y|)))$ is Lipschitz in $y$, also

$$
\int K(M([Sf_t](x, |x - y|))) (v - w) (f_t(y, w) - f_s(y, w)) \, dy \, dw
$$

vanishes when $\mathcal{W}(f_t, f_s) \to 0$. \[ \square \]
Now we can prove the main theorem of this section.

**Theorem 4.1.** Let $f_0(x,v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a probability density such that $\text{supp}(f_0) \subset B_{R_x} \times B_{R_v}$. Given $T > 0$, there exists a unique weak solution $f \in C_w([0,T]; L^\infty(\mathbb{R}^d \times \mathbb{R}^d))$ of the topological Cucker-Smale equation. Moreover

$$\text{supp}(f_t) \subset B_{R_x+tR_x} \times B_{R_v}. \quad (4.2)$$

**Proof.** We first note that, if the solution exists, (4.2) follows from an argument similar to the one used in the discrete case (see Theorem 3.1).

We now prove the existence. As in Lemma 4.2, consider a family of probability densities $\{g_t\}_{t \geq 0} \in C_w([0,T]; B_M)$, with $M := \|f_0\|_{L^\infty} e^{dR_xT}$ and such that (4.1) holds with $R_x(t) = R_x + tR_x$ and $R_v(t) = R_v$. The push-forward of $f_0$ along the flow generated by $g_t$, denoted by $\tilde{g}_t$, is weakly continuous in $t$, uniformly in $g_t$, with $t \in [0,T]$. Moreover, the determinant of the Jacobian of the flow $J(t) = J(t,x,v)$ verifies

$$\frac{d}{dt} J(t) = -J(t) d\gamma.$$

So the push-forward $\tilde{g}_t$ is bounded by $\|f_0\|_{L^\infty} e^{dR_x t}$.

With a standard construction we can prove that, for $T$ sufficiently small, the map $\{g_t\} \mapsto \{\tilde{g}_t\}$ is a contraction in $C_w([0,T]; B_M)$, with the distance defined by the supremum on time of the Wasserstein distance; in this way we prove local existence and uniqueness. Using the a-priori estimate on the supremum and on the support, we get the global result. □

5. **The mean-field limit**

In this section we prove the main result regarding the mean-field limit for the topological Cucker-Smale equation. In the sequel, $f_t$ is the fixed global solution of eq. (1.7) as in Theorem 4.1 with initial datum $f_0$, and $\mu^N_t$ is the global solution of eq. (1.8) in the sense of Theorem 3.1 with initial datum

$$\mu^N_0 = \frac{1}{N} \sum_{i=0}^N \delta_{x_i} \delta_{v_i}.$$  

We assume that $f_0$ and $\mu^N_0$ are supported in $B_{R_x} \times B_{R_v}$. Fixed $T$, we indicate by $C(T)$ any constant that depends only on $T$, $R_x$, $R_v$ and $\|f_0\|_{L^\infty}$.

In order to get the result, we compare the $N$-agent dynamics with the “intermediate” dynamics given by

$$\begin{cases}
\dot{X}^f_i(t) = V^f_i(t) \\
\dot{V}^f_i(t) = W[Sf_t, \nu^N_t](X^f_i, V^f_i),
\end{cases} \quad (5.1)$$
Proposition 5.1. Given $\nu^N_t$ is the empirical measure. The initial datum is $\nu^N_0 = \mu^N_0$, i.e.

$$\{(X^f_i(0), V^f_i(0))\}_{i=1}^N = \{(x_i, v_i)\}_{i=1}^N.$$ 

**Proposition 5.1.** Given $T > 0$, it holds that

1. For $t \in [0, T]$,
   $$\mathcal{W}(f_t, \nu^N_t) \leq C(T)\mathcal{W}(f_0, \mu^N_0).$$  
   
2. For $t \in [0, T]$, the distance
   $$\delta(t) := \max_{i=1, \ldots, N} \left( |X^f_i(t) - X^N_i(t)| + |V^f_i(t) - V^N_i(t)| \right)$$ 
   verifies
   $$\delta(t) \leq C(T)\sqrt{\mathcal{W}(f_0, \mu^N_0)}.$$  

**Proof.** Since $f_t$ is bounded, $K(M[Sf_t](x, |x-y|))$ is locally Lipschitz in $x$ and $y$ (see i) and ii) of Lemma 4.1 and then $W[f_t, \nu](x, v)$ is weakly continuous in $\nu$ in the sense that

$$\sup_{x,v} |W[f_t, \nu_1](x, v) - W[f_t, \nu_2](x, v)| \leq C(T)\mathcal{W}(\nu_1, \nu_2).$$

It is straightforward to prove that the solution $\nu_t$ of the system

$$\begin{cases} 
\dot{X}_t = V_t \\
\dot{V}_t = W[Sf_t, \nu_t](X_t, V_t) \\
\nu_t = \text{push-forward of } \nu_0 \text{ along the flow } (X_t, V_t)
\end{cases}$$

is continuous in $\mathcal{W}$ w.r.t. the initial datum $\nu_0$. Taking $\nu_0 = f_0$ and $\nu_0 = \mu^N_0$ we get the proof of i).

In order to estimate $\delta(t)$, we need to evaluate, for $0 \leq s \leq t$ and for $i = 1, \ldots, N$, the difference $|V^f_i(s) - V^N_i(s)|$ given by

$$W[Sf_s, \nu^N_s](X^f_i(s), V^f_i(s)) - W[Sf_s, \nu^N_s](X^N_i(s), V^N_i(s)).$$

We estimate this quantity with the sum of three terms:

1. $|W[Sf_s, \nu^N_s](X^f_i(s), V^f_i(s)) - W[Sf_s, \nu^N_s](X^N_i(s), V^N_i(s))|$, 
2. $|W[Sf_s, \nu^N_s](X^N_i(s), V^N_i(s)) - W[Sf_s, \mu^N_s](X^N_i(s), V^N_i(s))|$, 
3. $|W[Sf_s, \mu^N_s](X^N_i(s), V^N_i(s)) - W[Sf_s, \mu^N_s](X^N_i(s), V^N_i(s))|$. 

Since $K(M[Sf_s](x, |x-y|))$ is Lipschitz in $x$, from the definition of $W$ it is easy to prove that (a) is bounded by

$$c\text{Lip}(K)\|Sf_s\|_x R_x^{-1}(s) R_v + c\|K\|_x \delta(s)$$

and that (b) is estimated by

$$c\text{Lip}(K)\|Sf_s\|_x R_x^{-1}(s) R_v \delta(s).$$
Note that $\|Sf_s\|_\infty \leq cR_v^d f_\infty$. From Proposition 2.1 we have that (c) is bounded by
\[ c\text{Lip}(K)R_v \mathcal{D}(Sf_s, S\mu_s^N). \]
Since
\[ \mathcal{D}(Sf_s, S\mu_s^N) \leq \mathcal{D}(Sf_s, S\nu_s^N) + \mathcal{D}(S\nu_s^N, S\mu_s^N), \]
by Proposition 2.3 with $\rho = Sf_s$, $\mu^N = S\nu_s^N$ and $\nu^N = S\mu_s^N$, we get
\[ \mathcal{D}(S\nu_s^N, S\mu_s^N) \leq c\delta(s) + c\mathcal{D}(Sf_s, S\nu_s^N). \]
Writing $\delta(t)$ in terms of the time integral of $\delta(s)$ and the difference of the interaction terms and using the Gronwall lemma, we readily get the estimate
\[ \delta(t) \leq C(T) \int_0^t \mathcal{D}(Sf_s, S\nu_s^N) \, ds, \]
valid for $0 \leq t \leq T$. We conclude the proof by using Proposition 2.2, eq. (5.2) and the fact that $\mathcal{W}(Sf_s, S\nu_s^N) \leq \mathcal{W}(f_s, \nu_s^N)$.

**Theorem 5.1.** Fixed $T > 0$, let $f_t$ be a solution of eq. (1.7) as in Theorem 4.1 with initial datum $f_0$ and let $\mu_t^N$ be a solution of eq. (1.8) in the sense of Theorem 3.1 with initial datum $\mu_0^N$. Then, for $0 \leq t \leq T$,
\[ \mathcal{W}(f_t, \mu_t^N) \leq C(T) \max \left\{ \mathcal{W}(f_0, \mu_0^N), \sqrt{\mathcal{W}(f_0, \mu_0^N)} \right\}. \]

**Proof.** By the triangular inequality,
\[ \mathcal{W}(f_t, \mu_t^N) \leq \mathcal{W}(f_t, \nu_t^N) + \mathcal{W}(\nu_t^N, \mu_t^N). \]
From (5.2), using that $\mathcal{W}(\nu_t^N, \mu_t^N) \leq \delta(t)$ and (5.3), we get the thesis. \hfill \square

**APPENDIX: CONTINUITY OF $\mathcal{W}$ W.R.T. $\mathcal{D}$**

In this appendix we prove the continuity of the Wasserstein distance $\mathcal{W}$ w.r.t. the discrepancy distance $\mathcal{D}$ for compactly supported measures.

Consider two probability measures $\mu$ and $\nu$, both with support in the ball $B_R$ of $\mathbb{R}^d$. Fix $\varepsilon > 0$ and consider a Lipschitz test function $\phi$; it is sufficient to consider $\phi$ with support of diameter less than $cR$, so that $\|\phi\|_\infty \leq cR$. Given such $\phi$, take $\delta_1 > 0$ such that $\text{Lip}(\phi)\delta_1 < \varepsilon$.

By the Besicovitch covering principle (see [41]), there exist $N_{\varepsilon}$ disjoint closed balls $\{B_i\}_{i=1}^{N_{\varepsilon}}$ of radius at most $\delta_1$ such that
\[ \mu \left( \bigcup_{i=1}^{N_{\varepsilon}} B_i \right) \geq 1 - \varepsilon. \]
We estimate
\[ \int \phi \, d(\mu - \nu) = \int_{\mathbb{R}^d \setminus \bigcup_{i} B_i} \phi \, d(\mu - \nu) + \int_{\bigcup_{i} B_i} \phi \, d(\mu - \nu) \equiv A + B. \]
We have that
\[
A \leq \|\phi\|_\infty \left( \mu \left( \mathbb{R}^d \setminus \bigcup_{i=1}^{N_\epsilon} B_i \right) + \nu \left( \mathbb{R}^d \setminus \bigcup_{i=1}^{N_\epsilon} B_i \right) \right) \\
\leq \|\phi\|_\infty (2\varepsilon + N_\epsilon \mathcal{D}(\mu, \nu)),
\]
while
\[
B \leq \sum_{i=1}^{N_\epsilon} \int_{B_i} \left( \sup \phi - \inf \phi \right) \, d\nu + \sum_{i=1}^{N_\epsilon} \int_{B_i} \sup \phi \, d(\mu - \nu) \\
\leq 2\text{Lip}(\phi)\delta_1 + N_\epsilon \|\phi\|_\infty \mathcal{D}(\mu, \nu).
\]
Hence we obtain
\[
\int \phi \, d(\mu - \nu) \leq cR\varepsilon + cRN_\epsilon \mathcal{D}(\mu, \nu).
\]
Taking \(\mathcal{D}(\mu, \nu) < \delta_2\) such that \(N_\epsilon \delta_2 < \varepsilon\), we get the thesis.

REFERENCES

[1] A. Attanasi, A. Cavagna, L. Del Castello, I. Giardina, T.S. Grigera, A. Jelić, S. Melillo, L. Parisi, O. Pohl, E. Shen, M. Viale Information transfer and behavioural inertia in starling flocks Nat. Phys. 10 (2014) 691–696.
[2] M. Ballerini, N. Cabibbo, R. Andelier, A. Cavagna, A. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study Proc. Nat. Acad. Sci. USA 105 (2008) 1232–1237.
[3] D. Benedetto, P. Buttà, E. Caglioti Some aspects of the inertial spin model for flocks and related kinetic equations Math. Mod. Meth. Appl. Sci. 30 (10) (2020) 1987–2022.
[4] A. S. Besicovitch A general form of the covering principle and relative differentiation of additive functions I Proc. Cambridge Philos. Soc, 41 (1945), 103-110; II, Proc. Cambridge Philos. Soc, 42 (1946), 1-10, with corrections in Proc. Cambridge Philos. Soc, 43 (1947), 590.
[5] W. Braun, W.K. Hepp The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles Commun. Math. Phys 56 (1977) 101–113.
[6] J.A. Cañizo, J.A. Carrillo, J. Rosado A well-posedness theory in measures for some kinetic models of collective motion Math. Mod. Meth. Appl. Sci. 21 3 (2011) 515–539.
[7] J.A. Carrillo, M. Fornasier, G. Toscani, F. Vecil Particle, kinetic, and hydrodynamic models of swarming in: Naldi G., Pareschi L., Toscani G. (eds) Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser, Boston (2010)
[8] J.A. Carrillo, Y.-P. Choi, M. Hauray The derivation of swarming models: mean-field limit and Wasserstein distances in: Muntean A., Toschi F. (eds) Collective Dynamics from Bacteria to Crowds. CISM International Centre for Mechanical Sciences, vol. 553, Springer, Vienna (2014)
[9] A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini, R. Tavarone From empirical data to inter-individual interactions: Unveiling the rules of collective animal behavior Math. Mod. Meth. App. Sci. 20 (2010) 1491–1510.
[10] L.-P. Chaintron, A. Diez Propagation of chaos: a review of models, methods and applications arXiv:2106.14812v1 (2021)
[11] F. Cucker, S. Smale *On the mathematics of emergence* Japan. J. Math. 2 (2007) 197–227.

[12] F. Cucker, S. Smale *Emergence behavior in flocks* IEEE Trans. Automat. Control 52 (2007) 852–862.

[13] P. Degond, M. Pulvirenti *Propagation of chaos for topological interaction* Ann. Appl. Prob. 29 (2019) 2594–2612.

[14] R. Dobrushin *Vlasov equations* Funct. Anal. Appl. (1979) 13 115–123

[15] A.L. Gibbs, F.E. Su *On choosing and bounding probability metrics* International Statistical Review / Revue Internationale De Statistique, vol. 70, no. 3 (2002) 419–435.

[16] F. Golse *On the Dynamics of Large Particle Systems in the Mean Field Limit* in: Muntean A., Rademacher J., Zagaris A. (eds) Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity. Lecture Notes in Applied Mathematics and Mechanics, vol 3. Springer, (2016).

[17] F. Golse, S.-Y. Ha *A mean-field limit of the Lohe matrix model and emergent dynamics* Arch. Rational Mech. Anal. 234 (2019) 1445–1491.

[18] S.-Y. Ha, J.-G. Liu *A simple proof of the Cucker-Smale flocking dynamics and mean-field limit* Comm. Math. Sci. 7 (2009) 297–325.

[19] J. Haskovec *Flocking dynamics and mean-field limit in the Cucker-Smale-type model with topological interactions* Phys. D 261 (2013) 42–51.

[20] M. Hauray *Mean field limit for the one dimensional Vlasov-Poisson equation* Séminaire Laurent Schwartz — EDP et applications, Exposé no. 21 (2012-2013).

[21] M. Hauray, P.-E. Jabin *N-particle approximation of the Vlasov equations with singular potential* Arch. Ration. Mech. Anal. 183 (2007) 489–524.

[22] M. Hauray, P.-E. Jabin *Particle approximation of Vlasov equations with singular forces* Ann. Sci. Ecol. Norm. Sup. 48 (2015) 891–940.

[23] P.-E. Jabin *A review of the mean field limits for Vlasov equations* Kinet. Relat. Models 7 (2014) 661–711.

[24] L. Kuipers, H. Niederreiter *Uniform Distribution of Sequences* Wiley, New York (1974).

[25] D. Lazarovici *The Vlasov–Poisson dynamics as the mean-field limit of rigid charges* Commun. Math. Phys. 347 (2016) 271–289.

[26] D. Lazarovici, P. Pickl *A mean-field limit for the Vlasov–Poisson system* Arch. Ration. Mech. Anal. 225 (2017) 1201–1231.

[27] H. Neunzert *An introduction to the nonlinear Boltzmann Vlasov equation in: Kinetic Theories and the Boltzmann Equation (C. Cercignani ed.), Lecture Notes in Math., vol. 1048, Springer, Berlin, Heidelberg, (1984) 60–110.

[28] A.-S. Sznitman *Topics in propagation of chaos* in: Ecole d’Eté de Probabilités de Saint-Flour XIX (P.-L. Hennequin ed.), Lecture Notes in Math., vol. 1464, Springer-Verlag, Berlin, (1991).

[29] M. Trocheris *Continuite entre une solution de l’équation de Vlasov a une dimension et le mouvement d’un systeme de points* (EUR-CEA-FC–1222) France (1984).

[30] M. Trocheris *On the derivation of the one dimensional Vlasov equation* Transport Theory and Statistical Physics 15, 5 (1986) 597–628.

[31] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Shochet *Novel type of phase transition in a system of self-driven particles* Phys. Rev. Lett. 75 (1995) 1226.
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