Asymptotic lower bounds for Gallai-Ramsey functions and numbers

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Abstract

For two graphs $G, H$ and a positive integer $k$, the Gallai-Ramsey number $gr_k(G, H)$ is defined as the minimum number of vertices $n$ such that any $k$-edge-coloring of $K_n$ contains either a rainbow (all different colored) copy of $G$ or a monochromatic copy of $H$. If $G$ and $H$ are both complete graphs, then we call it Gallai-Ramsey function $GR_k(s, t)$, which is the minimum number of vertices $n$ such that any $k$-edge-coloring of $K_n$ contains either a rainbow copy of $K_s$ or a monochromatic copy of $K_t$. In this paper, we derive some lower bounds for Gallai-Ramsey functions and numbers by Lovász Local Lemma.

Keywords: Ramsey theory; Gallai-Ramsey function; Gallai-Ramsey number; Lovász Local Lemma.

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1 Research Background

Ramsey theory, named from the British mathematician Frank P. Ramsey, is a branch of mathematics that studies the conditions under which order must appear. Problems in Ramsey theory typically ask a question of the form: “how many elements of some structure must there be to guarantee that a particular property will hold?” More specifically, Ron Graham describes Ramsey theory as a “branch of combinatorics”. We refer the readers to [10] for a classical book of Ramsey theory.

1.1 Ramsey theorem

In this work, we consider only edge-colorings of graphs. A coloring of a graph is called rainbow if no two edges have the same color. Let $[n] = \{1, 2, \ldots, n\}$ and $[n]^2 = \{Y : Y \subset \{1, 2, \ldots, n\}, |Y| = 2\}$.

We write $n \rightarrow (\ell_1, \ell_2, \ldots, \ell_r)$ if, for every $r$-coloring of $[n]^2$, there exists $i$, $1 \leq i \leq r$, and a set $T$, $|T| = \ell_i$ so that $[T]^2$ is colored $i$. The Ramsey function $R(\ell_1, \ell_2, \ldots, \ell_r)$ denotes the minimal $n$ such that

\[ n \rightarrow (\ell_1, \ell_2, \ldots, \ell_r). \]
Theorem 1.1. (Ramsey’s theorem) \cite{10} The function $R$ is well defined, that is, for all $\ell_1, \ell_2, \ldots, \ell_r$ there exists $n$ such that

$$n \rightarrow (\ell_1, \ell_2, \ldots, \ell_r).$$

The Ramsey function only consider complete graphs. But later, the Ramsey number are considered for general graphs. Given $k$ graphs $H_1, H_2, \ldots, H_k$, let $R(H_1, H_2, \ldots, H_k)$ denote the minimum number of vertices $n$ needed so that every $k$-edge-coloring of $K_n$ contains a monochromatic $H_i$, where $1 \leq i \leq n$. If $H_1 = H_2 = \cdots = H_k$, then we write the number as $R_k(H)$.

We refer the readers to \cite{16} for a dynamic survey of small Ramsey numbers.

1.2 Gallai-Ramsey number and function

Colorings of complete graphs that contain no rainbow triangle have very interesting and somewhat surprising structure. In 1967, Gallai \cite{9} first examined this structure under the guise of transitive orientations. The result was reproven in \cite{14} in the terminology of graphs and can also be traced to \cite{1}. For the following statement, a trivial partition is a partition into only one part.

Theorem 1.2 (\cite{1, 9, 14}). In any coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition of the vertices (that is, with at least two parts) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.

For ease of notation, we refer to a colored complete graph with no rainbow triangle as a *Gallai-coloring* and the partition provided by Theorem 1.2 as a *Gallai-partition*. The induced subgraph of a Gallai colored complete graph constructed by selecting a single vertex from each part of a Gallai partition is called the *reduced graph* of that partition. By Theorem 1.2, the reduced graph is a 2-colored complete graph.

Although the reduced graph of a Gallai partition uses only two colors, the original Gallai-colored complete graph could certainly use more colors. With this in mind, we consider the following generalization of the Ramsey numbers. Given two graphs $G$ and $H$, the *$k$-colored Gallai-Ramsey number* $g_k(G : H)$ is defined to be the minimum integer $m$ such that every $k$-edge-coloring of the complete graph on $m$ vertices contains either a rainbow copy of $G$ or a monochromatic copy of $H$. With the additional restriction of forbidding the rainbow copy of $G$, it is clear that $g_k(G : H) \leq R_k(H)$ for any graph $G$.

In \cite{13}, Gyárfás et al. obtained the following nice result.

Theorem 1.3. \cite{13} Let $H$ be a fixed graph with no isolated vertices. If $H$ is not bipartite, then $g_k(K_3 : H)$ is exponential in $k$. If $H$ is bipartite, then $g_k(K_3 : H)$ is linear in $k$.

We refer the interested reader to \cite{8} for a dynamic survey of rainbow generalizations of Ramsey theory, including topics like Gallai-Ramsey numbers.

We write $n \rightarrow^\text{GR}_k (s, t)$ if, for every $k$-coloring of $[n]^2$, there exists a set $S$ such that $|S|^2$ is rainbow or there exists a set $T$ so that $|T|^2$ is monochromatic, where $|S| = s$ and $|T| = t$. The *Gallai-Ramsey function* $\text{GR}_k(s, t, t, \ldots, t)$ ($k$ times of $t$) or $\text{GR}_k(s, t)$ denotes the minimal $n$ such that

$$n \rightarrow^\text{GR}_k (s, t).$$

Note that $\text{GR}_k(s, t) = g_k(K_s : K_t)$. If $k = 2$, then $\text{GR}_2(s, t) = g_2(K_s : K_t) = R(s, t)$.

Since $g_k(G : H) \leq R_k(H)$ for any graph $G$, it follows that the following corollary is immediate from Theorem 1.1.
Corollary 1.4. The function $GR_k$ is well defined, that is, for all $s, t$ there exists $n$ such that

$$n \xrightarrow{GR_k} (s, t).$$

1.3 Main results

The probabilistic method is a powerful technique for approaching asymptotic combinatorial problems. The following probability result, due to L. Lovász, fundamentally improves the Existence argument in many instances. Let $A_1, \ldots, A_n$ be events in a probability space $\Omega$.

A graph $G$ on $[n]$ is said to be a dependency graph of $\{A_i\}$ if, for all $i$, the event $A_i$ is mutually independent of $\{A_j : \{i, j\} \notin E(G)\}$. $A_i$ must be not only independent of each $A_j$ but of any combination of the $A_j$.

**Theorem 1.5.** (Lovász Local Lemma [6]) Let $A_1, \ldots, A_n$ be events with a dependency graph $G$. Suppose that there exists $x_1, \ldots, x_n$, $0 < x_i < 1$, so that, for all $i$,

$$\Pr[A_i] < x_i \prod_{\{i,j\} \in E(G)} (1 - x_j).$$

Then $\Pr[\bigwedge A_i] > 0$.

In [18], Spencer studied the some asymptotic lower bounds for Ramsey functions. Li et al. [15] investigated the asymptotic upper bounds for Ramsey functions. Chen et al. [3] got the asymptotic bounds for irredundant and mixed Ramsey numbers. Caro et al. [2] obtained the asymptotic bounds for some bipartite graphs. In [12], Godbole et al. studied the asymptotic lower bound on the diagonal Ramsey numbers. Erdős and Hattingh [5] investigated the asymptotic bounds for irredundant Ramsey numbers.

In Subsection 2.1, we obtain a lower bound of Gallai-Ramsey function for a fixed probability of receiving colors for each edge. By Lovász Local Lemma, we derive another lower bound of Gallai-Ramsey function for a flexible probability of receiving colors for each edge. In Subsection 2.2, we got some lower bounds for Gallai-Ramsey numbers by the same method.

2 Main results

The following lemma, which is a consequence of Theorem [1.5] will be used later.

**Lemma 2.1.** [10] Under the assumption of Theorem [1.5], if there exist positive $y_1, \ldots, y_n$, with $y_i \Pr(A_i) < 1$ such that

$$\ln y_i > \sum_{\{i,j\} \in E(G)} y_j \Pr(A_j),$$

then $\Pr[\bigwedge \overline{A_i}] > 0$.

2.1 Results for general Gallai-Ramsey function

For a fixed probability of receiving colors for each edge, we can derive the following lower bound of $GR_k(s, t)$. 

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Theorem 2.1. Let $s, t$ be two positive integers with $r, s \geq 3$. For $k \geq \binom{s}{2}$, we have

$$\text{GR}_k(s, t) > \frac{1}{e} \min \left\{ \frac{s}{\sqrt{L + k^1 - \binom{i}{2}}}, \frac{t}{\sqrt{L + k^1 - \binom{i}{2}}} \right\}.$$ 

Proof. More precisely, we show that if

$$\max \left\{ \left( \binom{ne}{s} \right)^s \left( L + k^1 - \binom{i}{2} \right), \left( \binom{ne}{t} \right)^t \left( L + k^1 - \binom{i}{2} \right) \right\} < 1,$$

then $\text{GR}_k(s, t) > n$, that is, there exists a $k$-coloring of $K_n$ with vertex set $\{u_1, u_2, \ldots, u_n\}$ containing neither a rainbow $K_s$ nor a monochromatic $K_t$. Consider a random $k$-coloring of $K_n$, where the color of each edge is determined by the toss of a fair coin. More precisely, we have a probability space whose elements are the $k$-colorings of $K_n$, and whose probabilities are determined by setting

$$\Pr\{\{u_i, u_j\} \text{ is } c_x\} = k^{-1},$$

where $u_i, u_j \in V(K_n)$, $c_1, c_2, \ldots, c_k$ are the all colors and $1 \leq x \leq k$, for all $i, j$ and making these probabilities mutually independent.

Thus there are $k^{\binom{n}{2}}$ colorings, each with probability $k^{-\binom{n}{2}}$. For any set of vertices $S$, $|S| = s$, let $A_S$ denote the event “$S$ is rainbow.” Then

$$\Pr[A_S] = \frac{k(k-1)(k-2) \cdots (k-\binom{s}{2} + 1)}{k^{\binom{s}{2}}}.$$

For any set of vertices $T$, $|T| = t$, let $B_T$ denote the event “$T$ is monochromatic.” Then

$$\Pr[B_T] = k^{1 - \binom{t}{2}},$$

as the $\binom{t}{2}$ “coin flips” to determine the colors of $|S|^2$ must be the same.

The event “some $s$-element set of vertices $S$ is rainbow” is represented by $\bigvee_{|S|=s} A_S$, and “some $t$-element set of vertices $T$ is monochromatic” is represented by $\bigvee_{|T|=t} B_T$. Then

$$\Pr \left[ \left( \bigvee_{|S|=s} A_S \right) \bigvee \left( \bigvee_{|T|=t} B_T \right) \right] \leq \Pr \left[ \left( \bigvee_{|S|=s} A_S \right) \right] + \Pr \left[ \left( \bigvee_{|T|=t} B_T \right) \right]$$

$$\leq \sum_{|S|=s} \Pr[A_S] + \sum_{|T|=t} \Pr[B_T]$$

$$= \binom{n}{s} k(k-1)(k-2) \cdots (k-\binom{s}{2} + 1) \frac{1}{k^{\binom{s}{2}}} + \binom{n}{t} k^{1 - \binom{t}{2}}$$

$$< \binom{n}{s} \binom{k}{\binom{s}{2}} \binom{\binom{s}{2}}{k^{\binom{s}{2}}} + \binom{n}{t} k^{1 - \binom{t}{2}}$$

$$< \binom{n}{s} \left( \frac{ke}{\binom{s}{2}} \right)^{\binom{s}{2}} \binom{\binom{s}{2}}{k^{\binom{s}{2}}} + \binom{n}{t} k^{1 - \binom{t}{2}}$$

$$< \binom{n}{s} \left( \frac{e}{\binom{s}{2}} \right)^{\binom{s}{2}} \binom{\binom{s}{2}}{2} + \binom{n}{t} k^{1 - \binom{t}{2}}$$

$$= \binom{n}{s} L + \binom{n}{t} k^{1 - \binom{t}{2}}.$$
Let \( N = \binom{n}{s}L + \binom{n}{t}k^{1-\left(\frac{1}{2}\right)} \). If \( s \leq t \), then

\[
N < \binom{n}{t} \left( L + k^{1-\left(\frac{1}{2}\right)} \right) \leq \left( \frac{ne}{t} \right)^t \left( L + k^{1-\left(\frac{1}{2}\right)} \right),
\]

and hence \( n < \frac{t}{e \sqrt{L + k^{1-\left(\frac{1}{2}\right)}}} \).

If \( s > t \), then

\[
N < \binom{n}{s} \left( L + k^{1-\left(\frac{1}{2}\right)} \right) \leq \left( \frac{ne}{s} \right)^s \left( L + k^{1-\left(\frac{1}{2}\right)} \right),
\]

and hence \( n < \frac{s}{e \sqrt{L + k^{1-\left(\frac{1}{2}\right)}}} \).

From the above argument, we have

\[
\text{GR}_k(s, t) > \frac{1}{e} \cdot \min \left\{ \frac{s}{\sqrt{L + k^{1-\left(\frac{1}{2}\right)}}}, \frac{t}{\sqrt{L + k^{1-\left(\frac{1}{2}\right)}}} \right\}.
\]

For a flexible probability of receiving colors for each edge, we have the following two results.

**Theorem 2.2.** If, for some \( p_1, p_2, \ldots, p_k \), \( 0 \leq p_i \leq 1 \) (\( 1 \leq i \leq k \)),

\[
\binom{n}{s} \left[ \left( \frac{s(s-1)}{2} \right)! \right] \sum_{x_1, x_2, \ldots, x_\binom{s}{2} \in \{1, 2, \ldots, k\}} \frac{p_{x_1}p_{x_2} \cdots p_{x_\binom{s}{2}}}{\left( n \right)^{\binom{s}{2}}} + \binom{n}{t} \sum_{x_i \in \{1, 2, \ldots, k\}} p_{x_i} < 1,
\]

then \( \text{GR}_k(s, t) > n \), where \( k \geq \binom{s}{2} \) and \( c_1, c_2, \ldots, c_k \) are all \( k \) colors.

**Proof.** We use the existence argument of Theorem 2.1, replacing by \( 1 \),

\[
\Pr\{u_i, u_j\} \text{ is } c_x\} = p_x,
\]

where \( u_i, u_j \in V(K_n) \), \( c_1, c_2, \ldots, c_k \) are the all colors and \( 1 \leq x \leq k \), for all \( i, j \) and making these probabilities mutually independent.

For \( S, |S| = s \) let \( A_S \) be the event \( "[S]^2 \text{ is rainbow,}" \) and for \( T, |T| = t \), let \( B_T \) be the event \( "[T]^2 \text{ is monochromatic.}" \) Then

\[
\Pr\left( \bigvee_{|S|=s} A_S \bigvee \left( \bigvee_{|T|=t} B_T \right) \right) < 1,
\]

so the desired coloring of \( K_n \) exists.

**Theorem 2.3.** Let \( k, s, t \) be two positive integers with \( r, s \geq 6 \) and \( k \geq \binom{s}{2} \). Then

\[
\text{GR}_k(s, t) > \frac{1}{\beta c_2} \left( \frac{\lambda N^{-1/\gamma}}{\ln((t-1)N^{-1/\gamma})} \right)^\beta,
\]

where

\[
\beta = \left( \frac{s}{2} \right) / (s + 1), \quad \gamma = \left( \frac{s}{2} \right) - 1, \quad N = \left( \frac{s}{2} \right) (k - 1)^{2-\left(\frac{1}{2}\right)} (k - 2)(k - 3) \cdots \left( k - \left( \frac{s}{2} \right) + 1 \right).
\]
Proof. Let the edges of $K_n$ be independently $k$-colored with the probability that an edge is colored $c_i$ ($1 \leq i \leq k-1$) always being $\frac{p}{k-1}$, and $c_k$ being $1-p$. To each $s$-element subset of vertices $S$ associate the event $A_S$ that all the edges spanned by $S$ have colored rainbow. To each $t$-element subset of vertices $T$ associate the event $B_T$ that all the edges spanned by $T$ have colored monochromatic. Observe that $\text{GR}_k(s,t) > n$ if

$$\Pr\left[\left(\bigwedge_S A_S\right) \land \left(\bigwedge_T B_T\right)\right] > 0.$$  

Let $\Gamma$ denote the graph $\binom{n}{s}$ vertices corresponding to all possible $A_S$ and $B_T$, where $\{A_S, B_T\}$ is an edge of $\Gamma$ if and only if $|S \cap T| \geq 2$ (i.e., the events $A_S$ and $B_T$ are dependent), the same applies to pairs of the form $\{A_S, A_S^\prime\}$ and $\{B_T, B_T^\prime\}$. Let $N_{AA}$ denote the number of vertices of the form $A_S$ for some $S$ joined to some other vertex of this form, and let $N_{AB}, N_{BA}$ and $N_{BB}$ be defined analogously. If there exist positive $p, y, z$ such that

$$\log y > y \Pr[A_S](N_{AA} + 1) + z \Pr[B_T]N_{AB}, \quad \log z > y \Pr[A_S]N_{BA} + z \Pr[B_T](N_{BB} + 1),$$  

then $\text{GR}_k(s,t) > n$. Since

$$\Pr[A_S] = (k-1)(k-2) \cdots \left(k - \binom{s}{2}\right) \left(\frac{p}{k-1}\right)^{\binom{s}{2}}$$

$$+ \binom{s}{2}(1-p)(k-1)(k-2) \cdots \left(k - \binom{s}{2} + 1\right) \left(\frac{p}{k-1}\right)^{\binom{s}{2}-1}$$

$$\leq \binom{s}{2}(k-1)(k-2) \cdots \left(k - \binom{s}{2} + 1\right) \left(\frac{p}{k-1}\right)^{\binom{s}{2}-1} \left(\binom{s}{2} \left(\frac{p}{k-1}\right) + (1-p)\right)$$

$$\leq N_p^{\binom{s}{2}-1},$$

and

$$\Pr[B_T] = (1-p)^{\binom{t}{2}} + (k-1) \left(\frac{p}{k-1}\right)^{\binom{t}{2}} \leq (1-p)^{\binom{t}{2}} + (k-1) \left(\frac{p}{k-1}\right)^{\binom{t}{2}}$$

$$\leq (1-p)^{\binom{t}{2}} + p \left(\frac{p}{k-1}\right)^{\binom{t}{2}-1} \leq (1-p)^{\binom{t}{2}} + p(1-p)^{\binom{t}{2}-1} = (1-p)^{\binom{t}{2}-1},$$

it follows that

$$N(A, B) = \binom{n}{t} - \binom{n-s}{t} - s \binom{n-s-1}{t-1} \leq \binom{n}{t}, \quad N(A, A) + 1 \leq \binom{n}{s},$$

$$N(B, B) + 1 = \binom{n}{t} - \binom{n-t}{t} - t \binom{n-t}{t-1} < \binom{n}{t}, \quad N(B, A) \leq \binom{n}{s}.$$

Set

$$p = c_1 n^{-1/\beta} \cdot N^{(1-1)/\gamma}, \quad t - 1 = c_2 n^{1/\beta} (\ln n) N^{1/\gamma},$$

and

$$z = \exp \left[c_3 n^{1/\beta} (\ln n)^2 N^{1/\gamma}\right], \quad y = 1 + \epsilon,$$
Theorem 2.4. Let \( gr \)

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where

\[
\ln(1 + \epsilon) > \frac{(1 + \epsilon)c_{2}^{(x)} - 1}{n} + \exp(n^{1/\beta}(\ln n)^{2}N^{1/\gamma})(c_{3} + c_{2} - \frac{c_{1}c_{2}^{2}}{4}).
\]

Observe that

\[
\ln y > y \cdot N p^{(x)} - 1 \cdot \left(\binom{n}{s}\right) + z \left((1 - p)^{(x)} - 1\right) \cdot \left(\binom{n}{t}\right)
\]

and

\[
\ln z > y \cdot N p^{(x)} - 1 \cdot \left(\binom{n}{s}\right) + z \left((1 - p)^{(x)} - 1\right) \cdot \left(\binom{n}{t}\right).
\]

If \( c_{3} + c_{2} - \frac{c_{1}c_{2}^{2}}{4} < 0 \) and \( t \) is large, then the equations (2) hold. Since

\[
t - 1 = c_{2}n^{1/\beta}(\ln n)N^{1/\gamma} = (\beta c_{2})n^{1/\beta}(\ln n)^{1/\gamma}N^{1/\gamma} < (\beta c_{2})n^{1/\beta} \ln(tN^{-1/\gamma}),
\]

it follows that

\[
n^{1/\beta} > \frac{(t - 1) \cdot N^{1/\gamma}}{(\beta c_{2}) \ln((t - 1)N^{-1/\gamma})},
\]

and hence the result follows.

\[\Box\]

2.2 Results for general Gallai-Ramsey number

For a fixed probability of receiving colors for each edge, we can derive the following lower bound of \( gr_{k}(G, H) \).

Theorem 2.4. Let \( G, H \) be two graphs of order \( r, s \geq 4 \) and size \( m_{s}, m_{t} \), respectively. Let \( x^{*} = \min\{x | (\frac{x}{k}) \geq m_{s}\} \) and \( y^{*} = \min\{y | (\frac{y}{k}) \geq m_{t}\} \). For \( k \geq 2 \),

\[
gr_{k}(G, H) > \frac{\ell}{e} \cdot (k(k - 1)(k - 2) \cdots (k - m_{s} + 1)X \left(\frac{1}{k}\right) m_{s} + k \left(\frac{1}{k}\right) m_{t} Y)^{(-1)/\ell},
\]

where

\[
X = \left(\binom{s}{m_{s}} - \sum_{i=1}^{s-x^{*}} \binom{s}{i} \binom{m_{s}}{2}\right), \quad Y = \left(\binom{t}{m_{t}} - \sum_{i=1}^{t-y^{*}} \binom{t}{i} \binom{m_{t}}{2}\right).
\]

Proof. More precisely, we show that if

\[
\left(\frac{m_{e}}{\ell}\right) N < 1,
\]

then \( gr_{k}(s, t) > n \), that is, there exists a \( k \)-coloring of \( K_{n} \) with vertex set \( \{u_{1}, u_{2}, \ldots, u_{n}\} \) containing neither a rainbow \( G \) nor a monochromatic \( H \). Consider a random \( k \)-coloring of \( K_{n} \), where the color of each edge is determined by the toss of a fair coin. More precisely, we have a probability space whose elements are the \( k \)-colorings of \( K_{n} \), and whose probabilities are determined by setting

\[
\Pr\{\{u_{i}, u_{j}\} is c_{x}\} = k^{-1},
\]

where \( c_{1}, c_{2}, \ldots, c_{k} \) are the all colors and \( 1 \leq x \leq k \), for all \( i, j \) and making these probabilities mutually independent.

Thus there are \( k^{(s)} \) colorings, each with probability \( k^{-1} \). For any set of vertices \( S \), \( |S| = s \), let \( A_{S} \) denote the event that all the induced graphs spanned by \( S \) contains a rainbow \( G \). Then

\[
\Pr[A_{S}] \leq k(k - 1)(k - 2) \cdots (k - m_{s} + 1) X \left(\frac{1}{k}\right) m_{s}.
\]
For any set of vertices \( T \), let \( B_T \) denote the event that all the induced graphs spanned by \( T \) contains a monochromatic \( H \). Then
\[
\Pr[B_T] \leq k \left( \frac{1}{k} \right)^{mt} Y.
\]
as the \( \binom{n}{2} \) “coin flips” to determine the colors of \([S]^2\) must be the same.

The event “some \( S \) of vertices that \( G \) is rainbow” is represented by \( \bigvee_{|S|=s} A_S \), and “some \( T \) of vertices that \( H \) is monochromatic” is represented by \( \bigvee_{|T|=t} B_T \). Then
\[
\Pr \left[ \bigvee_{|S|=s} A_S \bigvee \bigvee_{|T|=t} B_T \right] \leq \Pr \left[ \bigvee_{|S|=s} A_S \right] + \Pr \left[ \bigvee_{|T|=t} B_T \right]
\]
\[
\leq \sum_{|S|=s} \Pr[A_S] + \sum_{|T|=t} \Pr[B_T]
\]
\[
= \binom{n}{s} k(k-1)(k-2) \cdots (k-m_s+1) X \cdot \left( \frac{1}{k} \right)^{m_s}
\]
\[
+ \binom{n}{t} k \left( \frac{1}{k} \right)^{mt} Y
\]
\[
\leq \binom{n}{\ell} N,
\]
where \( \ell = \max\{\min\{s, n-s\}, \min\{t, n-t\}\} \) and
\[
N = k(k-1)(k-2) \cdots (k-m_s+1) X \left( \frac{1}{k} \right)^{m_s} + k \left( \frac{1}{k} \right)^{m_t} Y.
\]

Furthermore, if
\[
\binom{n}{\ell} \cdot \frac{ne}{\ell} N < \binom{n}{\ell} \cdot N < 1,
\]
then
\[
n < \frac{\ell}{e N^{1/\ell}}.
\]

From the above argument, we have
\[
gr_k(G, H) > \frac{\ell}{e} \cdot \left( k(k-1)(k-2) \cdots (k-m_s+1) X \left( \frac{1}{k} \right)^{m_s} + k \left( \frac{1}{k} \right)^{m_t} Y \right)^{(-1)/\ell}.
\]

\[\square\]

For two general graphs, we can give lower bound for Gallai-Ramsey number.

**Theorem 2.5.** Let \( G \) be a graph of order \( s \geq 4 \) and size \( m_s \), respectively, and \( H \) is a complete graph of order \( t \). Let \( c_1, c_2, c_3 \) be three numbers with \( c_3 + \frac{c_2}{2} - \frac{c_1}{2} < 0 \). For \( k \geq 2 \), if and only if \( m_s \geq 2s \), then
\[
gr_k(G, H) > \left( \frac{(t-1)(s+1)L^{(s+1)/(m_s-1)}}{c_2(m_s-1) \ln [(t-1)L^{(s+1)/(m_s-1)}]} \right)^{(m_s-1)/(s+1)},
\]
where
\[
L = m_s(k-1)^{2-m_s}(k-2)(k-3) \cdots (k-m_s+1) X,
\]
and

\[ X = \binom{s}{2} - \sum_{i=1}^{s} \binom{s}{i} \binom{s-i}{2}. \]

**Proof.** Let the edges of \( K_n \) be independently \( k \)-colored with the probability that an edge is colored \( c_i \) (\( 1 \leq i \leq k - 1 \)) always being \( \frac{p}{k-1} \), and \( c_k \) being \( 1 - p \). For a vertex subset \( S \) with exactly \( s \) vertices, the event \( A_S \) that all the induced graphs spanned by \( S \) contains a rainbow \( G \). For a vertex subset \( T \) with exactly \( t \) vertices, the event \( B_T \) that all the induced graphs spanned by \( T \) contains a rainbow \( H \). Observe that \( gr_k(G, H) > n \) if

\[ \Pr \left[ \left( \bigwedge_{S} \overline{A_S} \right) \land \left( \bigwedge_{T} B_T \right) \right] > 0. \]

Let \( \Gamma \) denote the graph \( \binom{n}{s} + \binom{n}{t} \) vertices corresponding to all possible \( A_S \) and \( B_T \), where \( \{A_S, B_T\} \) is an edge of \( \Gamma \) if and only if \( |S \cap T| \geq 2 \) (i.e., the events \( A_S \) and \( B_T \) are dependent), the same applies to pairs of the form \( \{A_S, A_S\} \) and \( \{B_T, B_T\} \). Let \( N_{AA} \) denote the number of vertices of the form \( A_S \) for some \( S \) joined to some other vertex of this form, and let \( N_{AB}, N_{BA} \) and \( N_{BB} \) be defined analogously.

If there exist positive \( p, y, z \) such that

\[ \log y > y Pr[A_S](N_{AA} + 1) + z Pr[B_T]N_{AB}, \quad \log z > y Pr[A_S]N_{BA} + z Pr[B_T](N_{BB} + 1), \quad \text{(4)} \]

then \( gr_k(G, H) > n \). Since

\[ \Pr[A_S] \leq (k-1)(k-2) \cdots (k-m_s) \frac{p}{k-1}^{m_s} \]

\[ + m_s(1-p)(k-1)(k-2) \cdots (k-m_s + 1) \frac{p}{k-1}^{m_s-1} \]

\[ = m_s(k-1)(k-2) \cdots (k-m_s + 1) \frac{p}{k-1}^{m_s-1} \cdot \left[ (k-m_s) \left( \frac{p}{k-1} \right) + 1 - p \right] \]

\[ \leq m_s(k-1)(k-2) \cdots (k-m_s + 1) \frac{p}{k-1}^{m_s-1} \]

\[ = m_s(k-1)^{2-m_s}(k-2)(k-3) \cdots (k-m_s + 1) Xp^{m_s-1} = Lp^{m_s-1} \]

and

\[ \Pr[B_T] \leq \left[ (1-p)^{m_t} + (k-1) \frac{p}{k-1}^{m_t} \right] = \left[ (1-p)^{m_t} + p \frac{p}{k-1}^{m_t-1} \right] \]

\[ \leq \left[ (1-p)^{m_t} + p (1-p)^{m_t-1} \right] = (1-p)^{m_t-1} \]

it follows that

\[ N(A, B) = \binom{n}{t} - \binom{n-s}{t} - s \binom{n-s-1}{t-1} \leq \binom{n}{t}, \quad N(A, A) + 1 \leq \binom{n}{s}, \]

\[ N(B, B) + 1 = \binom{n}{t} - \binom{n-t}{t} - t \binom{n-t-1}{t-1} < \binom{n}{t}, \quad N(B, A) \leq \binom{n}{s}. \]
Set
\[ p = c_1 n^{-(s+1)/(m_s-1)} \cdot L^{(-1)/(m_s-1)}, \quad t - 1 = c_2 n^{(s+1)/(m_s-1)} (\ln n) L^{1/(m_s-1)}, \]
and
\[ z = \exp \left[ c_3 n^{(s+1)/(m_s-1)} (\ln n)^2 L^{1/(m_s-1)} \right], \quad y = 1 + \epsilon. \]
Observe that
\[ \ln y > y \cdot L p^{m_s-1} \cdot \left( \begin{pmatrix} n \\ s \end{pmatrix} + z \cdot (1 - p)^{m_t-1} \cdot \left( \begin{pmatrix} n \\ t \end{pmatrix} \right) \right) \]
and
\[ \ln z > y \cdot L p^{m_s-1} \cdot \left( \begin{pmatrix} n \\ s \end{pmatrix} + z \cdot (1 - p)^{m_t-1} \cdot \left( \begin{pmatrix} n \\ t \end{pmatrix} \right) \right). \]
Note that \( m_t = \left( \frac{t}{2} \right) \). If \( c_3 + \frac{c_2}{2} - \frac{c_3 c_2^2}{2} < 0 \) and \( t \) is large, then the equations \([1]\) hold. Since
\[ t = c_2 n^{(s+1)/(m_s-1)} (\ln n) L^{1/(m_s-1)} + 1 \]
\[ = c_2 n^{(s+1)/(m_s-1)} \frac{m_s-1}{s+1} (\ln n)^{s+1/(m_s-1)} L^{1/(m_s-1)} + 1 \]
\[ \leq c_2 n^{(s+1)/(m_s-1)} \frac{m_s-1}{s+1} (\ln n)^{s+1/(m_s-1)} L^{1/(m_s-1)} + 1 \]
\[ \leq c_2 n^{(s+1)/(m_s-1)} \frac{m_s-1}{s+1} \cdot \ln \left[ \frac{(t-1) L^{(-1)/(m_s-1)}}{c_2 (m_s-1) \ln \left[ (t-1) L^{(-1)/(m_s-1)} \right]} \right], \]
it follows that
\[ n^{(s+1)/(m_s-1)} > \frac{(t-1)(s+1) L^{(-1)/(m_s-1)}}{c_2 (m_s-1) \ln \left[ (t-1) L^{(-1)/(m_s-1)} \right]}, \]
and hence the result follows. \( \square \)

References

[1] K. Cameron and J. Edmonds, Lambda composition, *J. Graph Theory* 26(1) (1997), 9–16.
[2] Y. Caro, Y. Li, C.C. Rousseau, Y. Zhang, Asymptotic bounds for some bipartite graph: complete graph Ramsey numbers, *Discrete Math.* 220 (1-3) (2000), 51–56.
[3] G. Chen, J.H. Hattingh, C.C. Rousseau, Asymptotic bounds for irredundant and mixed Ramsey numbers, *J. Graph Theory* 17 (2) (1993), 193–206.
[4] G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* (3)2 (1993), 69–81.
[5] P. Erdős, J.H. Hattingh, Asymptotic bounds for irredundant Ramsey numbers, *Quaest. Math.* 16(3) (1993), 319–331.
[6] P. Erdős, L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions. In: A. Hajnal, R. Rado, V.T. Sós (eds.) Infinite and Finite Sets (to Paul Erdős on his 60th birthday), (1975), 609–627.
[7] R. J. Faudree and R. H. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Math.* 8 (1974), 313–329.
[8] S. Fujita, C. Magnant, and K. Ozeki. Rainbow generalizations of Ramsey theory - a dynamic survey. *Theo. Appl. Graphs*, 0(1), 2014.

[9] T. Gallai, Transitiv orientierbare Graphen, *Acta Math. Acad. Sci. Hungar* 18 (1967), 25–66.

[10] R.L. Graham, B.L. Rothschild, J.H. Spencer, *Ramsey Theory*, JOHN WILEY & SONS, 1990.

[11] R.L. Graham, V. Rödl, *Numbers in Ramsey theory*, Surveys in Combinatorics 1987, London Math. Soc. Lecture Notes 123 (1987), 111–153.

[12] A.P. Godbole, D.E. Skipper, R.A. Sunley (1995), *The Asymptotic Lower Bound on the Diagonal Ramsey Numbers: A Closer Look*, In: D. Aldous, P. Diaconis, J. Spencer, J.M. Steele (eds) Discrete Probability and Algorithms. The IMA Volumes in Mathematics and its Applications, vol 72. Springer, New York, NY.

[13] A. Gyárfás, G. Sárközy, A. Sebő, and S. Selkow, Ramsey-type results for gallai colorings, *J. Graph Theory* 64(3) (2010), 233–243.

[14] A. Gyárfás, G. Simonyi, Edge colorings of complete graphs without tricolored triangles, *J. Graph Theory* 46(3) (2004), 211–216.

[15] Y. Li, C. Rousseau, W. Zang, Asymptotic Upper Bounds for Ramsey Functions, *Graphs Comb.* 17 (2001), 123–128.

[16] S. P. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.*, Dynamic Survey 1, 30 pp. (electronic), 1994.

[17] V. Rosta, On a Ramsey-type problem of J. A. Bondy and P. Erdős. I, II, *J. Combinatorial Theory Ser. B*, 15 (1973), 94–104; 15 (1973), 105–120.

[18] J. Spencer, Asymptotic lower bounds for Ramsey functions, *Discrete Math.* 20 (1977), 69–76.

[19] E. Szemerédi, Regular partitions of graphs, in “Proc. Colloque Inter. CNRS” (J.C. Bermond, J.C. Fournier. M. das Vergnas, and D. Sotteau, Eds.), 1978.