Almost Morawetz estimates and global well-posedness for the defocusing $L^2$-critical nonlinear Schrödinger equation in higher dimensions

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Abstract: In this paper, we consider the global well-posedness of the defocusing, $L^2$ - critical nonlinear Schrödinger equation in dimensions $n \geq 3$. Using the I-method, we show the problem is globally well-posed in $n = 3$ when $s > \frac{2}{5}$, and when $n \geq 4$, for $s > \frac{n-2}{n}$. We combine energy increments for the I-method, interaction Morawetz estimates, and almost Morawetz estimates to prove the result.

1 Introduction

The defocusing, $L^2$ - critical nonlinear Schrödinger equation

$$iu_t + \Delta u = |u|^{4/n}u, \quad u(0, x) = u_0(x) \in H^s(\mathbb{R}^n),$$

has a local solution on some interval $[0, T]$, $T(\|u_0\|_{H^s(\mathbb{R}^n)}) > 0$ when $s > 0$. (See [4].) $\text{(1.1)}$ also has a local solution when $u_0 \in L^2(\mathbb{R}^n)$ on $[0, T)$, $T(u_0) > 0$, where $T$ depends on the profile of the initial data, not just its size. For global well-posedness to fail, and a solution to $\text{(2.2)}$ only exist on a maximal interval $[0, T_\ast)$, $T_\ast < \infty$, then

$$\lim_{t \to T_\ast} \|u(t)\|_{H^s(\mathbb{R}^n)} = \infty$$

for all $s > 0$. $\text{(1.1)}$ has the conserved quantities:

$$M(u(t)) = \int |u(t,x)|^2dx = M(u(0)),$$
\[ E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{n}{2n + 4} \int |u(t, x)|^{2+4/n} dx. \quad (1.4) \]

Combining the fact that \( E(u(t)) \) is positive definite, the Sobolev embedding theorem, and \([1,2]\); \([3]\) proved \((1.1)\) is globally well-posed when \( u_0 \in H^1(\mathbb{R}^n) \).

Furthermore, a solution to the equation
\[ iu_t + \Delta u = |u|^\alpha u, \quad u(0, x) = u_0(x), \quad (1.5) \]
can be rescaled in the following manner. If \( u(t, x) \) is a solution to \((1.5)\) on \([0, T_0]\), then
\[ u_\lambda(t, x) = \frac{1}{\lambda^{2/\alpha}} u\left( \frac{t}{\lambda^2}, \frac{x}{\lambda} \right) \]
is a solution to \((1.5)\) on \([0, \lambda^2 T_0]\) with initial data
\[ \frac{1}{\lambda^{2/\alpha}} u_0\left( \frac{x}{\lambda} \right). \]

\( \alpha = \frac{4}{n} \) is called the \( L^2 \)-critical exponent because a brief calculation will show that when \( \alpha = \frac{4}{n} \),
\[ \|u_\lambda\|_{L^\infty_t L^2_x(\mathbb{R}^n)} = \|u\|_{L^\infty_t L^2_x(\mathbb{R}^n)}. \quad (1.6) \]

Indeed, for any \( n \)-admissible pair \((p, q)\),
\[ \|u_\lambda\|_{L^p_t L^q_x(\mathbb{R}^n)} = \|u\|_{L^p_t L^q_x(\mathbb{R}^n)}. \quad (1.7) \]
(Admissible pairs will be discussed in greater detail in §2.)

Many have endeavored to prove global well-posedness for less regular data, \( u_0 \in H^s(\mathbb{R}^n), \ s < 1 \). The first progress was made in \([2]\), proving global well-posedness for \( s > \frac{3}{5} \) when \( n = 2 \) via the Fourier truncation method. In addition to proving global well-posedness, \([2]\) proved
\[ u(t, x) - e^{it\Delta} u_0 \in H^1(\mathbb{R}^2). \quad (1.8) \]
This method was modified in \([6]\) to produce the I-method, proving global well-posedness of \((1.1)\) when \( n = 2, \ s > \frac{4}{7} \). \([6]\) also discussed the cubic
nonlinear Schrödinger equation when \( n = 3 \), but that equation will not be discussed here, as it is \( \dot{H}^{1/2} \)-critical.

Since then, several improvements have been made when \( n = 2 \). In particular, improvements have utilized an almost Morawetz estimate. (See [5], [10].) Currently, the best known result for \( n = 2 \) is

**Theorem 1.1** \((1.1)\) is globally well-posed when \( n = 2 \) for \( s > \frac{1}{4} \).

*Proof:* See [13].

In [11], the I-method was extended to prove global well-posedness results for \((1.1)\) when \( n \geq 3 \). The chief difficulty with extending to \( n \geq 3 \) is that the nonlinearity \( |u|^{4/n} u \) is no longer "algebraic" when \( n > 2 \). That is, \( |u|^{4/n} u \) is no longer a polynomial of \( u \) and \( \bar{u} \) when \( n > 2 \). Nevertheless, it was proved that

**Theorem 1.2** \((1.1)\) is globally well-posed for \( s > \frac{\sqrt{7} - 1}{3} \) when \( n = 3 \), and \( s > \frac{-n+2+\sqrt{(n-2)^2+8(n-2)}}{4} \) when \( n \geq 4 \).

*Proof:* See [11].

In this paper, we will prove

**Theorem 1.3** When \( n \geq 4 \), \((1.1)\) is globally well-posed for \( u_0 \in H^s(\mathbb{R}^n), s > \frac{n-2}{n} \). Moreover,

\[
\sup_{t \in [0,T_0]} \|u(t)\|_{H^s(\mathbb{R}^n)} \leq C(\|u_0\|_{H^s(\mathbb{R}^n)})T_0^{\frac{(n-2)(1-s)^2}{2(n^2-(n-2))}}. \tag{1.9}
\]

**Theorem 1.4** When \( n = 3 \), \((1.1)\) is globally well-posed for \( u_0 \in H^s(\mathbb{R}^n), s > \frac{n-2}{n} \). Moreover,

\[
\sup_{t \in [0,T_0]} \|u(t)\|_{H^s(\mathbb{R}^3)} \leq C(\|u_0\|_{H^s(\mathbb{R}^3)})T_0^{\frac{(1-s)}{5s-2}}. \tag{1.10}
\]

**Description of Method:**
For \( u_0 \in H^s(\mathbb{R}^n) \), \( s < 1 \), the \( I \) - operator is defined to be the Fourier multiplier

\[
m(\xi) = \begin{cases} 
1, & |\xi| \leq N; \\
\left(\frac{N}{|\xi|}\right)^{1-s}, & |\xi| > N.
\end{cases}
\] (1.11)

Then if \( u(t, x) \) solves (1.1), \( I u(t, x) \) solves

\[
i I u_t + I \Delta u = I(|u|^{4/n}u),
\]

\( I u(0, x) \in H^1(\mathbb{R}^n) \).

\[
\|I u\|_{H^1(\mathbb{R}^n)} \lesssim N^{1-s} \|u\|_{H^s(\mathbb{R}^n)},
\]

\[
\|u\|_{H^s(\mathbb{R}^n)} \lesssim \|I u\|_{H^1(\mathbb{R}^n)},
\] (1.13)

so \( E(Iu(t)) \) very effectively controls \( \|u\|_{H^s(\mathbb{R}^n)} \). The chief difficulty is that, unlike (1.4), \( E(Iu(t)) \) is not a conserved quantity, rather,

\[
\frac{d}{dt} E(Iu(t)) = Re \int (Iu(t, x))\left(I(|u(t, x)|^{4/n} u(t, x)) - |I u(t, x)|^{4/n} I u(t, x)\right) dx
\] (1.14)

\[
= -Im \int I \nabla u(t, x)\left(I(|u(t, x)|^{4/n} u(t, x)) - |I u(t, x)|^{4/n} I u(t, x)\right) dx
\] (1.15)

\[
+ Im \int I(|u|^{4/n} u)\left(I(|u(t, x)|^{4/n} u(t, x)) - |I u(t, x)|^{4/n} I u(t, x)\right) dx.
\] (1.16)

To get around the fact that \( |u|^{4/n} u \) is not algebraic, we use the fact that \( |I u|^{4/n} \) can be very effectively approximated by \( I(|u|^{4/n}) \). Therefore, the analysis of (1.15) and (1.16) can be split into the analysis of a ”main term” and also a ”remainder term”, yielding a smaller energy increment than in (11). For the purposes of this paper, (1.15) will be called the linear term, and (1.16) will be called the nonlinear term.

In §2, some preliminary results from harmonic analysis will be discussed. In §3 some of the smoothness properties of \( |u|^{4/n} \) and \( |I u|^{4/n} \) that will be needed later will be proved. In §4, we will prove

\[
\left| \int_{t_1}^{t_2} (1.15) dt \right| \lesssim \frac{1}{N^{4/n}} \|\nabla I u\|_S^{2+4/n},
\] (1.17)
when $n \geq 4$. In §5, using a slightly different method, we will prove

$$\left| \int_{t_1}^{t_2} \left(1.15\right) dt \right| \lesssim \frac{1}{N^{1-}} \| \langle \nabla \rangle Iu \|_{S^0([t_1, t_2] \times \mathbb{R}^n)}^{10/3}, (1.18)$$

when $n = 3$. In §6 we will prove

$$\left| \int_{t_1}^{t_2} \left(1.16\right) dt \right| \lesssim \| \langle \nabla \rangle Iu \|_{S^0([t_1, t_2] \times \mathbb{R}^n)}^{2+4/n} \left\{ \begin{array}{ll} \frac{1}{N^{1-}} & \text{when } n = 3; \\ \frac{1}{N^{4/n}} & \text{when } n \geq 4. \end{array} \right. \ (1.19)$$

In §7 we will prove Theorem 1.3. In §8 we will prove an almost Morawetz estimate for $n = 3$, and in §9 we will prove Theorem 1.4.

Remark: (1.1) is globally well-posed when $\| u_0 \|_{L^2(\mathbb{R}^n)}$ is sufficiently small. [15] and [16] proved global well-posedness for all $u_0 \in L^2(\mathbb{R}^n)$, $u_0$ radial, based on an induction on mass method. This method will not be used here.

2 Some Harmonic Analysis

In this section, the harmonic analysis tools that will be needed later will be given. Let $\mathcal{F}$ be the Fourier transform,

$$\mathcal{F}(f)(\xi) = \int e^{-ix \cdot \xi} f(x) dx. \quad (2.1)$$

Definition 2.1 Suppose $\phi(\xi)$ is a $C^\infty_0$, decreasing, radial function. Also suppose,

$$\phi(\xi) = \begin{cases} 1, & |\xi| \leq 1/2; \\ 0, & |\xi| > 1. \end{cases} \quad (2.2)$$

Then define the frequency cutoff

$$\mathcal{F}(P_{\leq M} f) = \phi(\frac{\xi}{M}) \hat{f}(\xi). \quad (2.3)$$

$$P_{> M} f = f - P_{\leq M} f. \quad (2.4)$$

$$P_M f = P_{\leq M} f - P_{\leq \frac{M}{2}} f. \quad (2.5)$$
Lemma 2.1 When \( M > N \),
\[
\|P_{>M}u\|_{H^s(\mathbb{R}^n)} \lesssim \frac{1}{N^{1-s}} \|
abla Iu\|_{L^2(\mathbb{R}^n)}.
\] (2.6)

Proof: By definition of the I-operator,
\[
\|
abla IP_{>4N}P_{>M}u\|_{L^2(\mathbb{R}^n)} = N^{1-s} \|
abla |P_{>4N}P_{>M}u\|_{L^2(\mathbb{R}^n)}.
\]
so
\[
\|
abla |P_{>4N}P_{>M}u\|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{N^{1-s}} \|
abla Iu\|_{L^2(\mathbb{R}^n)}.
\]
Meanwhile,
\[
\|
abla |P_{\leq 4N}P_{>M}u\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{-5 \leq k \leq 5} (N2^k) \|
abla P_{2^kN}u\|_{L^2(\mathbb{R}^n)}
\]
\[
\lesssim \frac{1}{N^{1-s}} \sum_{-5 \leq k \leq 5} \|
abla P_{2^kN}u\|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{N^{1-s}} \|
abla Iu\|_{L^2(\mathbb{R}^n)}.
\]
This proves the lemma. □

Lemma 2.2 Suppose \( n \geq 3 \). A pair \((p, q)\) is called admissible if
\[
\frac{2}{p} = n(\frac{1}{2} - \frac{1}{q}), p \geq 2,
\] (2.7)

\[
\|e^{it\Delta} u_0\|_{L^p_t L^q_x(J \times \mathbb{R}^n)} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)},
\] (2.8)
for pairs \((p, q)\) that satisfy (2.7).

Proof: See [19] for the case \( p > 2 \) and [14] for \( p = 2 \). □

There is a very useful space of functions, called the Strichartz space.
\[
\|u\|_{S^0(J \times \mathbb{R}^n)} = \sup_{(p, q) \text{ admissible}} \|u\|_{L^p_t L^q_x(J \times \mathbb{R}^n)}.
\] (2.9)

Let \( p' \) denote the dual exponent to \( p \), \( \frac{1}{p'} = 1 - \frac{1}{p} \). The dual space to (2.9) is
\[
\|F\|_{N^0(J \times \mathbb{R}^n)} = \inf_{(p, q) \text{ admissible}} \|F\|_{L^{p'}_t L^{q'}_x(J \times \mathbb{R}^n)}.
\] (2.10)
\[ iu_t + \Delta u = F, \]
\[ u(a) = u_0, \]  
(2.11)

then

\[ \|u\|_{S^0([a,b] \times \mathbb{R}^n)} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)} + \|F\|_{N^0([a,b] \times \mathbb{R}^n)}. \]  
(2.12)

See [19] for more information on this space.

Finally, the bilinear Strichartz estimate when \( n = 3 \) will be used to resolve one technical issue.

**Theorem 2.3** For any spacetime slab \( I \times \mathbb{R}^3 \) and any \( t_0 \in I \), and for any \( \delta > 0 \), suppose \( M \ll N \), \( u \) is supported on frequency \( |\xi| \geq N \), \( v \) on frequency \( |\xi| \leq M \),

\[ \|uv\|_{L^2_{t,x}(I \times \mathbb{R}^3)} \leq C(\delta) \frac{M^{1-\delta}}{N^{1/2-\delta}} \left( \|u(t_0)\|_{L^2(\mathbb{R}^3)} + \|(i\partial_t + \Delta)u\|_{L^1_t L^2_x(I \times \mathbb{R}^3)} \right) \times \left( \|v(t_0)\|_{L^2(\mathbb{R}^3)} + \|(i\partial_t + \Delta)v\|_{L^1_t L^2_x(I \times \mathbb{R}^3)} \right). \]  
(2.13)

**Proof:** See [9].

### 3 Smoothness Estimates

The principle difficulty that arises for \( n \geq 3 \) is that the nonlinearity \( |u|^{4/n}u \) is no longer algebraic. To circumvent this problem, it is necessary to understand how the smoothness of \( IU \) affects the smoothness of \( I(|u|^{4/n}) \).

**Theorem 3.1** Suppose \( n = 3, 4 \). If \( M \geq N \),

\[ \|P_{>M}|u|^{4/n}\|_{L^{n/2}(\mathbb{R}^n)} \lesssim \frac{1}{M^{sN^{1-s}}} \|\nabla Iu\|_{L^2(\mathbb{R}^n)}^{4/n}, \]
\[ \|P_{>M}\frac{u^2}{|u|^{2-4/n}}\|_{L^{n/2}(\mathbb{R}^n)} \lesssim \frac{1}{M^{sN^{1-s}}} \|\nabla Iu\|_{L^2(\mathbb{R}^n)}^{4/n}. \]  
(3.1)

If \( M < N \),

\[ \|P_{>M}|u|^{4/n}\|_{L^{n/2}(\mathbb{R}^n)} \lesssim \frac{1}{M} \|\nabla Iu\|_{L^2(\mathbb{R}^n)}^{4/n}, \]
\[ \|P_{>M}\frac{u^2}{|u|^{2-4/n}}\|_{L^{n/2}(\mathbb{R}^n)} \lesssim \frac{1}{M} \|\nabla Iu\|_{L^2(\mathbb{R}^n)}^{4/n}. \]  
(3.2)
Prove:

Case 1, \( M \geq N \): Split the data, \( u = u_l + u_h \) with \( u_l = P_{\leq M} u, u_h = P_{> M} u \).

\[
\| \nabla u_l \|_{L^2(\mathbb{R}^n)} \lesssim \frac{M^{1-s}}{N^{1-s}} \| \nabla Iu \|_{L^2(\mathbb{R}^n)}.
\]

By elementary calculation and the Leibniz rule

\[
\| \nabla |u|^{4/n} \|_{L^{n/2}(\mathbb{R}^n)} \lesssim \frac{M^{1-s}}{N^{1-s}} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}, \quad (3.3)
\]

\[
\| \nabla \frac{u_l^2}{|u_l|^{2-4/n}} \|_{L^{n/2}(\mathbb{R}^n)} \lesssim \frac{M^{1-s}}{N^{1-s}} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)} \quad (3.4)
\]

Therefore,

\[
\| P_{> M} \frac{u_l^2}{|u_l|^{2-4/n}} \|_{L^{n/2}(\mathbb{R}^n)} + \| P_{> M} |u|^{4/n} \|_{L^{n/2}(\mathbb{R}^n)} \lesssim \frac{1}{M} \frac{M^{1-s} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}}{N^{1-s}}. \quad (3.5)
\]

On the other hand, by lemma 2.1 \( M > N \),

\[
\| u_h \|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{M^s} \| \nabla |u| \|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{M^s} \frac{1}{N^{1-s}} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}.
\]

The functions \( F(x) = |x| \) and \( G(x) = \frac{x^2}{|x|} \) are Lipschitz functions, so

\[
\| F(u_l + u_h) - F(u_l) \|_{L^2(\mathbb{R}^n)} + \| G(u_l + u_h) - G(u_l) \|_{L^2(\mathbb{R}^n)}
\]

\[
\lesssim \| u_h \|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{M^s N^{1-s}} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^2)}. \quad (3.6)
\]

This takes care of \( n = 4 \). For \( n = 3 \) let \( F(x) = |x|^{4/3} \) and \( G(x) = \frac{x^2}{|x|^{2/3}} \).

\[
|F(x + y) - F(x)| + |G(x + y) - G(x)| \lesssim |y|(|x|^{1/3} + |y|^{1/3}). \quad (3.7)
\]

Therefore,

\[
\| F(u_l + u_h) - F(u_l) \|_{L^{3/2}(\mathbb{R}^3)} + \| G(u_l + u_h) - G(u_l) \|_{L^{3/2}(\mathbb{R}^3)}
\]

\[
\lesssim \| u_h \|_{L^2(\mathbb{R}^3)} \| u_l \|_{L^{3/2}(\mathbb{R}^3)}^{1/3} + \| u_h \|_{L^{3/2}(\mathbb{R}^3)}^{1/3} \lesssim \frac{1}{N^{1-s} M^s} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^2)}. \quad (3.8)
\]

Case 2, \( M \leq N \): In this case let \( u_l = P_{\leq N} u \) and \( u_h = P_{> N} u \). By (3.3) and (3.4).
\[
\|P > M \frac{u^2}{|u|^{2-4/n}}\|_{L^{n/2}(R^n)} + \|P > M |u|^{4/n}\|_{L^{n/2}(R^n)} \lesssim \frac{1}{M} \|\langle \nabla \rangle Iu\|_{L^2(R^n)}^{4/n}.
\]

Using (3.7) when \(n = 3\), \(F, G\) Lipschitz when \(n = 4\),

\[
\|u|^{4/n} - |u|^{4/n}\|_{L^{n/2}(R^n)} + \|\frac{u^2}{|u|^{2-4/n}} - \frac{u^2}{|u|^{2-4/n}}\|_{L^{n/2}(R^n)} \lesssim \frac{1}{N} \|\langle \nabla \rangle Iu\|_{L^2(R^n)}^{4/n}.
\]

This completes the proof of the theorem. \(\square\)

When \(n > 4\), \(u \in H^1(R^n)\) no longer implies \(|u|^{4/n} \in H^{1,p}(R^n)\) for any \(p\). Instead, it is necessary to rely on a proposition from [24].

**Proposition 3.2** Let \(F\) be a Hölder continuous function of order \(0 < \alpha < 1\).

For every \(0 < \sigma < \alpha\), \(1 < p < \infty\), \(\sigma \rho < \rho < 1\),

\[
\|\nabla^{\sigma} F(u)\|_{L^p(R^n)} \lesssim \|u|^{\alpha - \sigma/\rho}\|_{L^{p_1}(R^n)} \|\nabla^{\rho} u\|_{L^{\frac{\sigma \rho}{\rho}}(R^n)},
\]

provided \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \ (1 - \frac{\sigma}{\alpha})p_1 > 1\).

**Proof:** See [24].

**Theorem 3.3** Suppose \(n > 4\). If \(M > N\),

\[
\|P > M \frac{u^2}{|u|^{2-4/n}}\|_{L^{n/2}(R^n)} + \|P > M |u|^{4/n}\|_{L^{n/2}(R^n)} \lesssim \frac{1}{M^{\frac{4}{n}}} - \frac{1}{N^{\frac{4}{n}}(1-s)} \|\langle \nabla \rangle Iu\|_{L^2(R^n)}^{4/n}.
\]

If \(M \leq N\),

\[
\|P > M \frac{u^2}{|u|^{2-4/n}}\|_{L^2(R^n)} + \|P > M |u|^{4/n}\|_{L^{n/2}(R^n)} \lesssim \frac{1}{M^{\frac{4}{n}}} \|\langle \nabla \rangle Iu\|_{L^2(R^n)}^{4/n}.
\]

**Proof:** Let \(F(x) = |x|^{4/n}\) and \(G(x) = \frac{x^2}{|x|^{2-4/n}}\), then \(F, G \in C^{0,4/n}(C)\). Choose \(\frac{\sigma}{\rho} = \frac{4}{n} - \epsilon, \rho = 1 - \delta\).

**Case 1, \(M \geq N\):** Let \(u_l = P_{\leq M} u\).
\[ \| \nabla^\rho u \|_{L^2(\mathbb{R}^n)} \lesssim \frac{M^{1-s-\delta}}{N^{1-s-\delta}} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}. \]  

Let \( p = \frac{n}{2} \), \( p_1 = \frac{2}{\epsilon} \), \( p_2 = \frac{2n}{4-n\epsilon} \), \( p_2 - \frac{2}{\rho} = 2 \).

\[ \| \nabla^\sigma F(u) \|_{L^{n/2}(\mathbb{R}^n)} \lesssim \| u_1 \|_{L^{2/(1-s)}(\mathbb{R}^n)} \| \nabla^\rho u \|_{L^2(\mathbb{R}^n)}^{\sigma/\rho} \lesssim \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)} \left( \frac{M^{1-s-\delta}}{N^{1-s-\delta}} \right)^{\sigma/\rho} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}^{4/n-\epsilon}. \]  

\[ \| P_{>M} F(u) \|_{L^{n/2}(\mathbb{R}^n)} \lesssim M^{-s(4/n-\epsilon)} N^{(4/n-\epsilon)(1-s-\delta)} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}^{4/n}. \]  

Make a similar calculation for G. Since F and G are both Hölder continuous,

\[ |F(u_l + u_h) - F(u_l)| \lesssim |u_h|^{4/n}, \]

\[ |G(u_l + u_h) - G(u_l)| \lesssim |u_h|^{4/n}. \]

This implies,

\[ \| F(u_l + u_h) - F(u_l) \|_{L^{n/2}(\mathbb{R}^n)} + \| G(u_l + u_h) - G(u_l) \|_{L^{n/2}(\mathbb{R}^n)} \lesssim \left( \frac{1}{M^{s} N^{1-s}} \right)^{4/n} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}^{4/n}. \]  

**Case 2, \( M \leq N \):** In this case let \( u_l = P_{\leq N} u \). In this case

\[ \| F(u_l + u_h) - F(u_l) \|_{L^{n/2}(\mathbb{R}^n)} + \| G(u_l + u_h) - G(u_l) \|_{L^{n/2}(\mathbb{R}^n)} \lesssim \frac{1}{N^{4/n}} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}^{4/n}. \]  

Also, for \( \sigma = \left( \frac{4}{n} - \epsilon \right)(1 - \delta) \),

\[ \| \nabla^\sigma G(u) \|_{L^{n/2}(\mathbb{R}^n)} \lesssim \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}^{4/n}. \]  

So in this case,

\[ \| P_{>M} G(u) \|_{L^{n/2}(\mathbb{R}^n)} \lesssim \frac{1}{M^{(4/n-\epsilon)(1-\delta)}} \| \langle \nabla \rangle Iu \|_{L^2(\mathbb{R}^n)}^{4/n}. \]  

Taking \( \epsilon, \delta > 0 \) arbitrarily small proves the theorem. \( \Box \)
4 Linear Term for $n \geq 4$

In this section the linear term (1.15) for $n \geq 4$. The $n = 3$ case is put off until the next section, due to a technical complication.

**Theorem 4.1**  

$$ |\text{Im} \int_{t_1}^{t_2} (\mathcal{I}u)[|Iu|^{4/n}(Iu) - I(|u|^{4/n}u)]dxdt| \lesssim \frac{1}{N^{4/n-}} \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}. $$

(4.1)

To prove this integrate by parts,

$$ |\text{Im} \int_{t_1}^{t_2} (\mathcal{I}u)[|Iu|^{4/n}(Iu) - I(|u|^{4/n}u)]dxdt| \lesssim \frac{1}{N^{4/n-}} \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}. $$

(4.2)

$$ = \left(-\frac{n+2}{n}\right) \text{Im} \int_{t_1}^{t_2} (\mathcal{I}u)(I(|u|^{4/n}\nabla u) - |Iu|^{4/n}(\nabla Iu))dxdt $$

(4.3)

$$ - \frac{2}{n} \text{Im} \int_{t_1}^{t_2} (\mathcal{I}u)(I\left(\frac{u^2}{|u|^{2-4/n}}\nabla u\right) - \frac{(Iu)^2}{|Iu|^{2-4/n}}(\nabla Iu))dxdt. $$

(4.4)

First estimate a slightly modified version of (4.3) and (4.4), approximating $|Iu|^{4/n}$ by $I(|u|^{4/n})$.

**Lemma 4.2**  

Suppose $J$ is an interval.

$$ |\int_{t_1}^{t_2} (\nabla Iu)(I(|u|^{4/n}\nabla u) - I(|u|^{4/n})(\nabla Iu))dxdt| \lesssim \frac{\|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}}{N^{4/n-}}. $$

(4.5)

$$ |\int_{t_1}^{t_2} (\nabla Iu)(I\left(\frac{u^2}{|u|^{2-4/n}}\nabla u\right) - I\left(\frac{u^2}{|u|^{2-4/n}}\right)(\nabla Iu))dxdt| \lesssim \frac{\|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}}{N^{4/n-}}. $$

(4.6)

**Proof:** The proof of (4.6) is virtually identical to the proof of (4.5), so only (4.5) will be proved. Recall that $F(u) = |u|^{4/n}$ and $G(u) = \frac{u^2}{|u|^{2-4/n}}$. All that will be used to prove (4.5) is $F \in C^{0,4/n}(\mathbb{C})$, which is also true for $G$. 

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\[
\int_{t_1}^{t_2} \int (\nabla Iu)(|u|^4/n u) - I(|u|^4/n) (\nabla Iu) dx dt
\]

\[
= \int_{t_1}^{t_2} \int_{\Sigma} (\xi_1 \tilde{I}u(t, \xi_1))(1 - \frac{m(\xi_2 + \xi_3)}{m(\xi_2)m(\xi_3)}) \cdot (\xi_2 \tilde{I}u(t, \xi_2))(\tilde{IF}(u)(t, \xi_3)) d\xi dt,
\]

where \( \Sigma \) is the hyperplane \( \{ \xi_1 + \xi_2 + \xi_3 = 0 \} \) and \( d\xi \) is the Lebesgue measure on \( \Sigma \). Make a Littlewood - Paley decomposition and consider several cases separately.

**Case 1, \( N_2, N_3 << N \):** In this case the multiplier is \( \equiv 0 \).

**Case 2(a), \( N_2 \geq N >> N_3 \):** Here \( N_1 \sim N_2 \). Using the fundamental theorem of calculus,

\[
|1 - \frac{m(\xi_2 + \xi_3)}{m(\xi_2)m(\xi_3)}| = |1 - \frac{\nabla m(\xi_2)}{m(\xi_2)}| \lesssim \frac{N_3}{N_2}.
\]

\[
\sum_{N \lesssim N_1 \sim N_2} \|P_N \nabla Iu\|_{L_t^2 L_x^{2n/(n+2)}(J \times \mathbb{R}^n)} \|P_{N_2} \nabla Iu\|_{L_t^2 L_x^{2n/(n+2)}(J \times \mathbb{R}^n)} \times \sum_{N_3 << N} \frac{N_3}{N_2} \|P_{N_3} IF(u)\|_{L_t^\infty L_x^{n/(n+2)}(J \times \mathbb{R}^n)},
\]

Applying Theorems 3.1 and 3.3

\[
\lesssim \sum_{N \lesssim N_1 \sim N_2} \frac{1}{N_2} \|P_{N_1} \nabla Iu\|_{S^0(J \times \mathbb{R}^n)} \|P_{N_2} \nabla Iu\|_{S^0(J \times \mathbb{R}^n)} \times \sum_{N_3 << N} N_3^{(1-4/n)+} \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{4/n} \lesssim \frac{1}{N_4^{4/n-}} \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}.
\]

**Case 2(b), \( N_3 \geq N >> N_2 \):** Making a similar calculation, it suffices to estimate

\[
\sum_{N \lesssim N_1 \sim N_3} \|P_{N_1} \nabla Iu\|_{L_t^2 L_x^{2n/(n+2)}(J \times \mathbb{R}^n)} \|P_{N_3} IF(u)\|_{L_t^\infty L_x^{n/(n+2)}(J \times \mathbb{R}^n)}
\]

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\[
\times \sum_{N_2 \ll N} \frac{N_2}{N_3} \| P_{N_2} \nabla Iu \|_{L^2_2 L^{2n/(n-2)}_{L^2}(J \times \mathbb{R}^n)}.
\]

Again applying Theorems 3.1 and 3.3

\[
\lesssim \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}^{2 + 4/n} \sum_{N \lesssim N_1 \sim N_2} N \lesssim \frac{1}{N^{4/n}} \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}^{2 + 4/n}.
\]

Case 3, \( N_2 \gtrsim N \) and \( N_3 \gtrsim N \) There are three subcases to consider.

Case 3(a), \( N_1 \sim N_2, N_2 \gg N_3 \): In this case \( |1 - \frac{m(\xi_2 + \xi_3)}{m(\xi_2 + \xi_3)}| \lesssim \frac{1}{m(\xi_3)} \).

\[
\sum_{N \lesssim N_1 \sim N_2} \| P_{N_1} \nabla Iu \|_{L^2_2 L^{2n/(n-2)}(J \times \mathbb{R}^n)} \| P_{N_2} \nabla Iu \|_{L^2_2 L^{2n/(n-2)}(J \times \mathbb{R}^n)}
\]

\[
\times \sum_{N \lesssim N_3 \ll N_2} \frac{1}{m(N_3)} \| P_{N_3} \mathcal{F}(u) \|_{L^2_2 L^{n/2}(J \times \mathbb{R}^n)}
\]

\[
\lesssim \sum_{N \lesssim N_1 \sim N_2} \| P_{N_1} \nabla Iu \|_{S^0(J \times \mathbb{R}^n)} \| P_{N_2} \nabla Iu \|_{S^0(J \times \mathbb{R}^n)}
\]

\[
\times \sum_{N \lesssim N_3 \ll N_2} \frac{1}{N N_1^{4(1-s)/n} N_1^{4/s}} \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}^{4/n}
\]

\[
\lesssim \frac{1}{N^{4/n}} \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}^{2 + 4/n}.
\]

Remark: The last sum follows by Cauchy Schwartz.

Case 3(b), \( N_1 \sim N_3, N_3 \gg N_2 \): In a similar manner,

\[
\sum_{N_1 \sim N_3} \| P_{N_1} \nabla Iu \|_{L^2_2 L^{2n/(n-2)}(J \times \mathbb{R}^n)} \| P_{N_3} \mathcal{F}(u) \|_{L^2_2 L^{n/2}(J \times \mathbb{R}^n)}
\]

\[
\times \sum_{N \lesssim N_2 \ll N_3} \frac{1}{m(N_2)} \| P_{N_2} \nabla Iu \|_{L^2_2 L^{2n/(n-2)}(J \times \mathbb{R}^n)}
\]
\[ \lesssim \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{2+4/n} \sum_{N \lesssim N_1 \sim N_3} \frac{1}{N_3} \frac{N_1^{1-s}}{N_1^{1-\frac{2}{n}}} \]

\[ \lesssim \frac{1}{N_4^{4/n-}} \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}. \]

**Remark:** This proof utilizes the fact that \(1 - s < \frac{4}{n}\) in Theorem 3.3.

**Case 3(c),** \(N_2 \sim N_3, N_2 \gtrsim N_1\): In this case \(|1 - \frac{m(\xi_2 + \xi_3)}{m(\xi_2)m(\xi_3)}| \lesssim \frac{1}{m(\xi_2)m(\xi_3)}\).

\[ \sum_{N \lesssim N_2 \sim N_3} \|P_{N_2} \nabla Iu\|_{L^2_{tL^2_x} (J \times \mathbb{R}^n)} \|P_{N_3} IF(u)\|_{L^\infty_{t} L^{n/2}_{x}(J \times \mathbb{R}^n)} \]

\[ \times \sum_{N \lesssim N_1 \lesssim N_2} \frac{m(N_1)}{m(N_2)^2} \|P_{N_1} \nabla Iu\|_{L^2_{tL^2_x} (J \times \mathbb{R}^n)} \]

\[ \lesssim \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{2+4/n} \sum_{N \lesssim N_2} \frac{\ln(N)}{m(N_2)^2 N_3^{4/n-}} \]

\[ \lesssim \frac{1}{N_4^{4/n-}} \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}. \]

When \(n \geq 4, 2(1 - s) < \frac{4}{n}\) since \(s > \frac{n-2}{n}\). This takes care of the lemma. \(\square\)

**Remark:** It is in this particular case where the above argument would break down when \(n = 3\). Therefore, \(n = 3\) requires a different method.

To finish the proof of theorem 4.1, it remains to prove

\[ IF(u) + IG(u) \]

is a good approximation of

\[ F(Iu) + G(Iu). \]

\[ (\overline{\nabla Iu})I(|u|^{4/n} \nabla u) - |Iu|^{4/n}(\nabla Iu)(\overline{\nabla Iu}) \]

\[ = (\overline{\nabla Iu})I(|u|^{4/n} \nabla u) - (\overline{\nabla Iu})(\nabla Iu)I(|u|^{4/n}) \]

\[ + (\overline{\nabla Iu})(\nabla Iu)I(|u|^{4/n}) - |Iu|^{4/n}(\nabla Iu)(\overline{\nabla Iu}), \]

and similarly for \(G(u)\).
Lemma 4.3

\[ | \int_{t_1}^{t_2} \left| \nabla Iu \right|^2 [I(\left| u \right|^{4/n}) - |u|^{4/n}] dx \, dt | \lesssim \frac{1}{N^{4/n}} \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}, \quad (4.7) \]

\[ | \int_{t_1}^{t_2} \left( \nabla Iu \right)^2 \left[ I\left( \frac{u^2}{|u|^{2-4/n}} \right) - \frac{(Iu)^2}{|Iu|^{2-4/n}} \right] dx \, dt | \lesssim \frac{1}{N^{4/n}} \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}. \quad (4.8) \]

Proof: Split the data \( u = u_t + u_h \), with \( u_t = P_{\leq \frac{4}{n}} u \), in particular \( Iu_t = u_t \), and

\[ \| u_h \|_{L^\infty L^{n/2}_x(J \times \mathbb{R}^n)}^{4/n} \lesssim \frac{1}{N^{4/n}} \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}. \quad (4.9) \]

\[ IF(u) - F(Iu) = [IF(u) - IF(u_t)] + [IF(u_t) - F(Iu_t)] + [F(Iu_t) - F(Iu)], \]

\[ IG(u) - G(Iu) = [IG(u) - IG(u_t)] + [IG(u_t) - G(Iu_t)] + [G(Iu_t) - G(Iu)]. \]

Since \( F, G \in C^{0,4/n} \),

\[ \| F(u) - F(u_t) \|_{L^\infty L^{n/2}_x(J \times \mathbb{R}^n)} + \| G(u) - G(u_t) \|_{L^\infty L^{n/2}_x(J \times \mathbb{R}^n)} \lesssim \| u_h \|_{L^\infty L^2_x(J \times \mathbb{R}^n)}^{4/n} \lesssim \frac{1}{N^{4/n}} \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}. \quad (4.10) \]

Similarly, since \( Iu_t = u_t \);

\[ \| F(Iu) - F(u_t) \|_{L^\infty L^{n/2}_x(J \times \mathbb{R}^n)} + \| G(Iu) - G(u_t) \|_{L^\infty L^{n/2}_x(J \times \mathbb{R}^n)} \lesssim \frac{1}{N^{4/n}} \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}^{2+4/n}. \quad (4.11) \]

Finally,

\[ \| I(\left| u \right|^{4/n}) - |u|^{4/n} \|_{L^\infty L^{n/2}_x(J \times \mathbb{R}^n)} + \| I(\frac{u^2}{|u|^{2-4/n}}) - \frac{u^2}{|u|^{2-4/n}} \|_{L^\infty L^{n/2}_x(J \times \mathbb{R}^n)} \lesssim \frac{1}{N^{4/n}} \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^n)}, \quad (4.12) \]

by \( m(\xi) \equiv 1 \) on \( |\xi| \leq N \), theorems 3.1 and 3.3. This proves the lemma. \( \square \)

Combining Lemma 4.2 and Lemma 4.3 proves Theorem 4.1. \( \square \)
5 Linear Term for $n = 3$

When $n = 3$, it is necessary to use a different method than was used for $n \geq 4$.

**Theorem 5.1**

\[
|\text{Im} \int_{t_1}^{t_2} (I \Delta u) | |u|^{4/3}(Iu) - I(|u|^{4/3}u)| dxdt| \lesssim \frac{1}{N_1^2} \| \langle \nabla \rangle Iu \|_{S^0(J \times R^n)}^{7/3}. \tag{5.1}
\]

**Proof:** Let $u = u_l + u_h$, $u_l = P_{\leq N/4} u$. Then $Iu_l = u_l$. Integrating by parts,

\[
\int_{t_1}^{t_2} (I \Delta u) [|u_l|^{4/n} - I(|u_l|^{4/n}u_l)] dx dt,
\]

\[
= \int_{t_1}^{t_2} (\nabla Iu) [I \left( \frac{5}{3} |u_l|^{4/3} \nabla u_l + \frac{2}{3} \frac{u_l^2}{|u_l|^{2/3}} \nabla \bar{u_l}} + \frac{2}{3} \frac{u_l^2}{|u_l|^{2/3}} \nabla \bar{u_l}} \right) dx dt. \tag{5.2}
\]

When $N_1 \leq N$,

\[
P_{N_1} [I \left( \frac{5}{3} |u_l|^{4/3} \nabla u_l + \frac{2}{3} \frac{u_l^2}{|u_l|^{2/3}} \nabla \bar{u_l}} + \frac{2}{3} \frac{u_l^2}{|u_l|^{2/3}} \nabla \bar{u_l}} \right) \equiv 0.
\]

For $N_1 \geq N$, since $\nabla u_l$ is supported on $|\xi| \leq \frac{N}{4}$ it suffices to estimate

\[
\frac{5}{3} \| P_{|\xi| \sim N_1} |u_l|^{4/3} \|_{L_{t,x}^{\infty,3/2}(J \times R^n)} + \frac{2}{3} \| P_{|\xi| \sim N_1} \frac{u_l^2}{|u_l|^{2/3}} \|_{L_{t,x}^{\infty,3/2}(J \times R^n)}
\]

\[
\lesssim \frac{1}{N_1} \| \nabla P_{|\xi| \sim N_1} |u_l|^{4/3} \|_{L_{t,x}^{\infty,3/2}(J \times R^n)} + \frac{1}{N_1} \| \nabla P_{|\xi| \sim N_1} \frac{u_l^2}{|u_l|^{2/3}} \|_{L_{t,x}^{\infty,3/2}(J \times R^n)}
\]

\[
\lesssim \frac{1}{N_1} \| \langle \nabla \rangle Iu \|_{S^0(J \times R^n)}^{4/3},
\]

so,

\[
\int_{t_1}^{t_2} (\nabla Iu) | |u|^{4/3}(Iu) - I(|u|^{4/3}u)| dxdt| \lesssim \sum_{N \leq N_1} \frac{1}{N_1^2} \| \langle \nabla \rangle Iu \|_{S^0(J \times R^n)}^{10/3} \lesssim \frac{1}{N} \| \langle \nabla \rangle Iu \|_{S^0(J \times R^n)}^{10/3}. \tag{5.3}
\]
Next, use the Taylor expansion,

\[ f(x + y) = f(x) + \int_0^1 y f'(x + \tau y) d\tau. \]

\[ |u|^{4/3}(I u) = |u|^{4/3} u + \int_0^1 \frac{5}{3} |u + \tau u h|^{4/3} (I u h) + \frac{2}{3} \left( \frac{u_l + \tau I u h}{|u_l + \tau I u h|^{2/3}} \right) (I u h) d\tau. \]

(5.4)

\[ I(|u|^{4/3} u) = I(|u|^{4/3} u) + \int_0^1 \frac{5}{3} I(|u_l + \tau u h|^{4/3} (u_h)) + \frac{2}{3} I \left( \frac{(u_l + \tau u h)^2}{|u_l + \tau u h|^{2/3}} \right) (u_h) d\tau. \]

(5.5)

\[ \int_{t_1}^{t_2} \int (\Delta I u) \left[ I(F(u_l + \tau u h) u_h) - F(u_l + \tau I u h) I u h \right] dx dt \]

\[ = \int_{t_1}^{t_2} \int \left( |\xi_1|^2 \hat{I} u(t, \xi_1) \right) m(\xi_2 + \xi_3) \hat{F}(u_l + \tau u h)(t, \xi_2) \hat{u}_h(t, \xi_3) \]

\[ - m(\xi_3) \hat{F}(u_l + \tau u h)(t, \xi_2) \hat{I} u_h(t, \xi_3) \]  \[ \Sigma \] \[ d\xi dt. \]

(5.6)

As usual, make a Littlewood - Paley decomposition and consider several cases separately. It suffices to consider only \( N_3 \geq N \) because of the support of \( u_h \).

**Case 1,** \( N_1 \sim N_3 \geq N, \ N_2 \ll N \):

(5.7)

\[ \int_{t_1}^{t_2} \int \left( |\xi_1|^2 \hat{I} u(t, \xi_1) \right) m(\xi_2 + \xi_3) \hat{F}(u_l + \tau u h)(t, \xi_2) \hat{u}_h(t, \xi_3) \]

\[ - m(\xi_3) \hat{F}(u_l + \tau u h)(t, \xi_2) \hat{I} u_h(t, \xi_3) \]  \[ \Sigma \] \[ d\xi dt \]

\[ + \int_{t_1}^{t_2} \int \left( |\xi_1|^2 \hat{I} u(t, \xi_1) \right) m(\xi_3) \hat{F}(u_l + \tau u h)(t, \xi_2) \]

\[ - \hat{F}(u_l + \tau I u h)(t, \xi_2) \hat{I} u_h(t, \xi_3) \]  \[ \Sigma \] \[ d\xi dt \]  \[ \hat{u}_h(t, \xi_3) \]  \[ \hat{I} u_h(t, \xi_3) \]  \[ d\xi dt \]  \[ \hat{I} u_h(t, \xi_3) \]  \[ d\xi dt \]

(5.8)

For (5.7), using the fundamental theorem of calculus,

\[ |m(N_2 + N_3) - m(N_3)| \lesssim \frac{N_2 m(N_3)}{N_3}. \]

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\[
\sum_{N \lesssim N_1 \sim N_3} \| P_{N_1} \Delta Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \frac{m(N_3)}{N_3} \| P_{N_3} u_h \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \\
\times \sum_{N_2 \ll N} \frac{N}{N_2} \| P_{N_2} F(u_t + \tau u_h) \|_{L_t^\infty L_x^{3/2}(J \times \mathbb{R}^3)} \\
\lesssim \| \langle \nabla \rangle Iu \|^{10/3}_{S^0(J \times \mathbb{R}^3)} \sum_{N \lesssim N_1 \sim N_3} \frac{N_1 \ln(N)}{N_3^2} \lesssim \frac{1}{N^2} \| \langle \nabla \rangle Iu \|^{10/3}_{S^0(J \times \mathbb{R}^3)},
\]

by theorem 3.1. To estimate (5.8),

\[ |F(u_t + \tau u_h) - F(u_t + \tau Iu_h)| \lesssim |(1 - I)u_h| |u_t|^{1/3} + |u_h|^{1/3}, \]

therefore,

\[ \| P_{N_2} [F(u_t + \tau u_h) - F(u_t + \tau Iu_h)] \|_{L_t^\infty L_x^{3/2}(J \times \mathbb{R}^3)} \lesssim \frac{1}{N} \| \langle \nabla \rangle Iu \|^{4/3}_{S^0(J \times \mathbb{R}^3)}. \tag{5.9} \]

\[ \sum_{N \lesssim N_1 \sim N_3} \frac{N_1 \ln(N)}{N_3^2} \sum_{N_2 \ll N} \frac{N_1 \ln(N)}{N_3^2} \| P_{N_1} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \| P_{N_3} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \]

\[ \lesssim \frac{1}{N^2} \| \langle \nabla \rangle Iu \|^{10/3}_{S^0(J \times \mathbb{R}^3)}. \]

**Remark:** Summing in $N_1 \sim N_3$ follows by Cauchy-Schwartz.

**Case 2:** $N_1, N_2, N_3 \gtrsim N$. In this case consider $I(F(u)u_h)$ and $F(Iu)(Iu_h)$ separately.

**Case 2(a):** $N_1 \sim N_3 \gg N_2 \gtrsim N$. For the $I(F(u)u_h)$ term, $m(\xi_2 + \xi_3) \sim m(\xi_3)$. Using theorem 3.1 again,

\[ \sum_{N \lesssim N_1 \sim N_3} \| P_{N_1} \Delta Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \| P_{N_3} Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \sum_{N \lesssim N_2 \ll N_3} \| P_{N_2} F(u) \|_{L_t^\infty L_x^{3/2}(J \times \mathbb{R}^3)} \]

\[ \lesssim \| \langle \nabla \rangle Iu \|^{4/3}_{S^0(J \times \mathbb{R}^3)} \sum_{N \lesssim N_1 \sim N_3} \frac{N_1}{N_3^2} \| P_{N_1} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \]
\[ \times \left\| P_{N_3} \nabla Iu \right\|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \sum_{N \lesssim N_2 < N_3} \frac{1}{N^{1-s} N_2^s} \lesssim \frac{1}{N^{1-s}} \left\| \langle \nabla \rangle Iu \right\|_{S^0(J \times \mathbb{R}^3)}. \]

For the \( F(Iu)(Iu) \) term,

\[ \sum_{N \lesssim N_1 \sim N_3} \| P_{N_1} \Delta Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \| P_{N_3} Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \sum_{N \lesssim N_2 < N_3} \| P_{N_3} F(Iu) \|_{L_t^8 L_x^{3/2}(J \times \mathbb{R}^3)} \]

\[ \lesssim \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^3)}^{4/3} \sum_{N \lesssim N_1 \sim N_3} \frac{N_1}{N_3} \| P_{N_1} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \]

\[ \times \left\| P_{N_3} \nabla Iu \right\|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \sum_{N \lesssim N_2 < N_3} \frac{1}{N_2} \lesssim \frac{1}{N^{1-s}} \left\| \langle \nabla \rangle Iu \right\|_{S^0(J \times \mathbb{R}^3)}. \]

**Case 2(b):** \( N_1 \sim N_2 \gg N_3 \sim N \) In this case \( m(\xi_2 + \xi_3) \sim m(\xi_2) \). To estimate

\[ \int_{t_1}^{t_2} \int \sum_{N \lesssim N_1 \sim N_2} (P_{N_1} \Delta Iu)(P_{N_3} Iu) \sum_{N \lesssim N_3 < N_1} (P_{N_3} u_h), \quad (5.10) \]

there is a slight technical complication due to the fact that Cauchy - Schwartz is not available for \( P_{N_2} IF(u) \) (§3 only proved an estimate on the decay of \( P_{N_2} IF(u), \) it did not prove \( IF(u) \in H^{1,3/2}(\mathbb{R}^3) \)). Therefore, it is necessary to utilize the bilinear estimates of theorem 2.3. Interpolating

\[ \sum_{N \lesssim N_1 \sim N_2} \sum_{N \lesssim N_3 < N} \| (P_{N_1} \Delta Iu)(P_{N_3} u_h) \|_{L_t^{4/3} L_x^2(J \times \mathbb{R}^3)} \| P_{N_3} IF(u) \|_{L_t^4 L_x^2(J \times \mathbb{R}^3)}. \]

with the bilinear Strichartz estimate

\[ \| (P_{N_1} \Delta Iu)(P_{N_3} u_h) \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \leq C(\delta) \frac{N_3^{1-\delta}}{N_1^{1/2-\delta} N_3^s N_1^{1-s}} \]

\[ \times \left( \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^3)}^2 + \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R}^3)}^{14/3} \right). \quad (5.11) \]
Let $\frac{1}{p} = \frac{3-\epsilon}{4}$, 

\[
\| (P_{N_1} \Delta Iu)(P_{N_3} u_h) \|_{L_t^p L_x^2(J \times \mathbb{R}^3)} \lesssim \frac{N_3^{\epsilon-\delta \epsilon}}{N_1^{\epsilon/2-\epsilon \delta}} \frac{N_1}{N_3^3 N_1^{1-s}} (\| \nabla \|_{S^0(J \times \mathbb{R}^3)}^2 + \| \nabla \|_{S^0(J \times \mathbb{R}^3)}^{2+8 \epsilon/3}).
\]

Suppose $|J| \lesssim N^\alpha$ for some $\alpha$, 

\[
\| (P_{N_1} \Delta Iu)(P_{N_3} u_h) \|_{L_t^{4/3} L_x^2(J \times \mathbb{R}^3)} \lesssim \frac{N_3^{\epsilon-\delta \epsilon}}{N_1^{\epsilon/2-\epsilon \delta}} \frac{N_1}{N_3^3 N_1^{1-s}} (\| \nabla \|_{S^0(J \times \mathbb{R}^3)}^2 + \| \nabla \|_{S^0(J \times \mathbb{R}^3)}^{2+14 \epsilon/3}).
\]

\[
\| \nabla \|_{S^0(J \times \mathbb{R}^3)}^{10/3} \lesssim \frac{N_3^{\epsilon-\delta \epsilon}}{N_1^{\epsilon/2-\epsilon \delta}} \frac{N_1}{N_3^3 N_1^{1-s}} \sum_{N \ll N_1} \frac{N_1}{N_2} \sum_{N \ll N_3} \frac{N_3^{\epsilon-\delta \epsilon}}{N_2^2 N_1^{1-s}}
\]

\[
\lesssim \| \nabla \|_{S^0(J \times \mathbb{R}^3)}^{10/3} \frac{N_3^{\epsilon-\delta \epsilon} N_1^\epsilon/2}{N}.
\]

Letting $\epsilon \searrow 0$ proves the claim.

Remark: In §7 we will rescale to make $\| \nabla \|_{S^0(J \times \mathbb{R}^3)} \lesssim 1$, so it will not be necessary to worry about the $\| \nabla \|_{S^0(J \times \mathbb{R}^3)}^{2+8 \epsilon/3}$ term here, since it will be $\lesssim \| \nabla \|_{S^0(J \times \mathbb{R}^3)}^2$. This rescaling will rescale the interval $[0, T_0]$ to $[0, N^{2(1-\alpha)} T_0]$, so $|J| \lesssim N^\alpha$.

For the $F(Iu)(Iu_h)$ term,

\[
\| \nabla F(Iu) \|_{L_t^\infty L_x^{3/2}(J \times \mathbb{R}^3)} \lesssim \| \nabla \|_{L_t^\infty L_x^{3/2}(J \times \mathbb{R}^3)}.
\]

\[
\sum_{N \ll N_1} \frac{N_1}{N_2} \sum_{N \ll N_1} \frac{N_3^{\epsilon-\delta \epsilon}}{N_2^2 N_1^{1-s}}
\]

\[
\lesssim \frac{1}{N} \| \nabla \|_{S^0(J \times \mathbb{R}^3)} \sum_{N \ll N_1} \frac{N_1}{N_2} \| P_{N_1} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \| P_{N_2} \nabla F(Iu) \|_{L_t^\infty L_x^{3/2}}
\]

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In this case $F(Iu) \in H^{1,3/2}(\mathbb{R}^3)$, so it is possible to use Cauchy-Schwartz.

**Case 2(c),** $N_2 \sim N_3 \gtrsim N_1 \gtrsim N$: For the $F(I(u)u_h)$ term, use the fact that $m(\xi_2 + \xi_3) = m(\xi_1)$.

\[
\sum_{N \leq N_2 \sim N_3} \|P_{N_3} u_h\|_{L^2_t L^6_x(J \times \mathbb{R}^3)} \|P_{N_2} F(u)\|_{L^\infty_t L^{3/2}_x(J \times \mathbb{R}^3)} \sum_{N \leq N_1 \leq N_2} \|P_{N_1} \Delta I^2 u\|_{L^2_t L^6_x(J \times \mathbb{R}^3)} \lesssim \|F(Iu)\|_{S^0(J \times \mathbb{R}^3)}^{10/3} \sum_{N \leq N_2 \sim N_3} \frac{1}{N_3^s N_1^2 N^{2(1-s)}} \sum_{N \leq N_1 \leq N_2} N_1^s N_1^{1-s} \lesssim \frac{1}{N_1^{1-s}} \|F(Iu)\|_{S^0(J \times \mathbb{R}^3)}^{10/3}.
\]

For the $F(Iu)(Iu)$ term,

\[
\sum_{N \leq N_2 \sim N_3} \|P_{N_3} Iu_h\|_{L^2_t L^6_x(J \times \mathbb{R}^3)} \|P_{N_2} F(Iu)\|_{L^\infty_t L^{3/2}_x(J \times \mathbb{R}^3)} \sum_{N \leq N_1 \leq N_2} \|P_{N_1} \Delta I^2 u\|_{L^2_t L^6_x(J \times \mathbb{R}^3)} \lesssim \|F(Iu)\|_{S^0(J \times \mathbb{R}^3)}^{7/3} \sum_{N \leq N_2 \sim N_3} \frac{1}{N_2 N_3} \sum_{N \leq N_1 \leq N_2} N_1 \lesssim \frac{1}{N_1^{1-s}} \|F(Iu)\|_{S^0(J \times \mathbb{R}^3)}^{7/3}.
\]

A similar calculation can be made for the $G$ term. This proves Theorem 5.1. □
\section{Nonlinear Estimate}

Having dealt with (1.14), we turn our attention to (1.15).

\textbf{Theorem 6.1}

\begin{equation}
\left\| \int_{t_1}^{t_2} I(|u|^{4/n}u)(I(|u|^{4/n}u) - |Iu|^{4/n}(Iu))dxdt \right\| \lesssim \begin{cases} 
\frac{1}{N^{\sigma(n)}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n} & \text{when } n = 3; \\
\frac{1}{N^{\sigma(n)}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^n)}^{2+8/n} & \text{when } n \geq 4.
\end{cases}
\end{equation}

\textbf{Proof:} For simplicity of notation let \( \sigma(n) \) be the exponent for \( N \) in (6.1). It suffices to prove

\begin{equation}
\| I(|u|^{4/n}u) - |Iu|^{4/n}(Iu) \|_{L^2_t L^\infty_x (J \times \mathbb{R}^n)} \lesssim \frac{1}{N^{\sigma(n)}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n},
\end{equation}

since

\begin{equation}
\| I(|u|^{4/n}u) \|_{L^2_t L^\infty_x (J \times \mathbb{R}^n)} \lesssim \| \nabla I(|u|^{4/n}u) \|_{L^2_t L^{2n/(n-4)}(J \times \mathbb{R}^n)} \lesssim \| \nabla Iu \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n}.
\end{equation}

As in the linear case, the quantity in (6.2) will be split into a main term and a remainder term. This time, we will deal with the remainder term first.

\textbf{Lemma 6.2}

\begin{equation}
\| |Iu|^{4/n} - |u|^{4/n} \|_{L^2_t L^\infty_x (J \times \mathbb{R}^n)} \lesssim \frac{1}{N^{\sigma(n)}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n}.
\end{equation}

\textbf{Proof:} First consider \( n > 4 \).

\begin{equation}
\| |Iu|^{4/n} - |u|^{4/n} \|_{L^2_t L^\infty_x (J \times \mathbb{R}^n)} \lesssim \| Iu \|_{L^2_t L^{2n/(n-4)}(J \times \mathbb{R}^n)} \| |Iu|^{4/n} - |u|^{4/n} \|_{L^\infty_t L^{n/2}(J \times \mathbb{R}^n)}
\end{equation}

\begin{equation}
\lesssim \| \nabla Iu \|_{S^0(J \times \mathbb{R}^n)} \| |Iu|^{4/n} - |u|^{4/n} \|_{L^\infty_t L^{n/2}(J \times \mathbb{R}^n)}.
\end{equation}

Let \( u_l = P_{\leq N} u \), since \( F(x) \in C^{0,4/n} \),

\begin{equation}
\lesssim \| \nabla Iu \|_{S^0(J \times \mathbb{R}^n)} \| |Iu|^{4/n} + |u_l|^{4/n} \|_{L^\infty_t L^{n/2}(J \times \mathbb{R}^n)}
\end{equation}
\[
\lesssim \frac{1}{N^{\frac{4}{n}}} \| (\nabla) Iu \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n}.
\]

For \( n = 3, 4 \),

\[
\| [Iu]^{4/n} - |u|^{4/n} (Iu) \|_{L^2_{t,x}(J \times \mathbb{R}^n)} \lesssim \| Iu \|_{L^\infty_t L^2_x(J \times \mathbb{R}^n)} \| Iu \|_{L^1_t L^{4/n}_x(J \times \mathbb{R}^n)}^{2/n} + \| u \|_{L^2_t L^4(J \times \mathbb{R}^n)}^{2/n} \lesssim \frac{1}{N^{1-}} \| (\nabla) Iu \|_{S^0(J \times \mathbb{R}^n)}^{2}. \tag{6.4}
\]

When \( n = 4 \),

\[
\| [Iu]^{4/n} - |u|^{4/n} (Iu) \|_{L^2_{t,x}(J \times \mathbb{R}^n)} \lesssim \| Iu \|_{L^\infty_t L^2_x(J \times \mathbb{R}^n)} \| Iu \|_{L^1_t L^{4/n}_x(J \times \mathbb{R}^n)}^{2/n} + \| u \|_{L^2_t L^4(J \times \mathbb{R}^n)}^{2/n} \lesssim \frac{1}{N^{1-}} \| (\nabla) Iu \|_{S^0(J \times \mathbb{R}^n)}^{2}. \tag{6.4}
\]

When \( n = 3 \),

\[
\| [Iu]^{4/3} - |u|^{4/3} \|_{L^2_{t,x}(J \times \mathbb{R}^n)} \lesssim \| Iu_h \|_{L^2_{t,x}(J \times \mathbb{R}^n)} + \| u_h \|_{L^2_{t,x}(J \times \mathbb{R}^n)}^{1/3} \lesssim \frac{1}{N^{1-}} \| (\nabla) Iu \|_{S^0(J \times \mathbb{R}^n)}^{4/3}. \] □

Now we tackle the main term.

**Lemma 6.3**

\[
\| I(|u|^{4/n} u) - (|u|^{4/n}) (Iu) \|_{L^2_{t,x}(J \times \mathbb{R}^n)} \lesssim \frac{1}{N^{\sigma}} \| (\nabla) Iu \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n}. \tag{6.4}
\]

**Proof:** Let

\[
f(t, \xi) = \int_{\xi_2 + \xi_3 = \xi} \left[ m(\xi_2 + \xi_3) - m(\xi_3) \right] \hat{F}(\xi_2) \hat{u}(t, \xi) d\xi_2. \tag{6.5}
\]

As usual, make a Littlewood - Paley decomposition.

**Case 1, \( N_2, N_3 << N \):** In this case \( m(\xi_2 + \xi_3) - m(\xi_3) \equiv 0 \).

For the remaining cases, to simplify notation, let \( p_1 = 2, q_1 = \frac{2n}{n-1}, p_2 = \infty, q_2 = \frac{n}{2} \) when \( n > 4 \) and \( p_1 = \infty, q_1 = \frac{2n}{n-2}, p_2 = 2, q_2 = n \) when \( n = 3, 4 \).
Case 2, $N_2 \gtrsim N, N_3 \ll N$: In this case $|m(N_2 + N_3) - m(N_3)| \lesssim 1$. By Theorem 3.1 and the Sobolev embedding theorem,

$$
\sum_{N_2 \gtrsim N} \|P_{N_2}F(u)\|_{L_t^{p_2}L_x^{q_2}(J \times \mathbb{R}^n)} \lesssim \ln(N) \sum_{N_3 \ll N} \|P_{N_3}u\|_{L_t^{p_1}L_x^{q_1}(J \times \mathbb{R}^n)}
$$

$$
\lesssim \|\langle \nabla \rangle Iu\|^{1+4/n}_{S^0(J \times \mathbb{R}^n)}
$$

$$
\lesssim \frac{1}{N^\sigma} \|\langle \nabla \rangle Iu\|^{1+4/n}_{S^0(J \times \mathbb{R}^n)}.
$$

Case 3, $N_2 \ll N, N_3 \gtrsim N$: In this case use the fundamental theorem of calculus,

$$
|m(N_2 + N_3) - m(N_3)| \lesssim \frac{N_2 m(N_3)}{N_3}.
$$

Again by Theorem 3.1 Sobolev embedding,

$$
\sum_{N_3 \lesssim N_3} \frac{m(N_3)}{N_3} \|P_{N_3}u\|_{L_t^{p_1}L_x^{q_1}(J \times \mathbb{R}^n)} \sum_{N_2 \ll N} \|P_{N_2}F(u)\|_{L_t^{p_2}L_x^{q_2}(J \times \mathbb{R}^n)}
$$

$$
\lesssim \|\langle \nabla \rangle Iu\|^{1+4/n}_{S^0(J \times \mathbb{R}^n)} \sum_{N_3 \gtrsim N} \sum_{N_2 \ll N} \frac{N_2}{N_2^\sigma}
$$

$$
\lesssim \frac{1}{N^\sigma} \|\langle \nabla \rangle Iu\|^{1+4/n}_{S^0(J \times \mathbb{R}^n)}.
$$

Case 4, $N_2, N_3 \gtrsim N$: In this case,

$$
\|P_{\geq N}Iu\|_{L_t^{p_1}L_x^{q_1}(J \times \mathbb{R}^n)} \|P_{\geq N}F(u)\|_{L_t^{p_2}L_x^{q_2}(J \times \mathbb{R}^n)} \lesssim \|\nabla Iu\|_{S^0(J \times \mathbb{R}^n)} \|P_{\geq N}F(u)\|_{L_t^{p_2}L_x^{q_2}(J \times \mathbb{R}^n)}
$$

$$
\lesssim \frac{1}{N^\sigma} \|\langle \nabla \rangle Iu\|^{1+4/n}_{S^0(J \times \mathbb{R}^n)}.
$$

Therefore, it remains to tackle $\|I(|u|^{4/n}u)\|_{L_t^{2}L_x(J \times \mathbb{R}^n)}$. 

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Case 4(a), $N_2 \gg N_3$: In this case $m(N_2 + N_3) \sim m(N_2)$.

$$
\sum_{N_2 \geq N} m(N_2) \| P_{N_2} F(u) \|_{L_t^{p_2} L_x^{q_3}(J \times \mathbb{R}^n)} \sum_{N \lesssim N_3 \lesssim N_2} \| P_{N_3} u \|_{L_t^{p_1} L_x^{q_3}(J \times \mathbb{R}^n)}
\lesssim \| \langle \nabla \rangle I u \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n} \sum_{N \lesssim N_3 < N_2} \frac{m(N_2)}{N_2^{\sigma} N_3^{\sigma(1-s)}} \sum_{N \lesssim N_3 < N_2} \frac{N_3^{1-s}}{N^{1-s}}
\lesssim \frac{1}{N^\sigma} \| \langle \nabla \rangle I u \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n},
$$

Case 4(b), $N_2 \ll N_3$: In this case $m(N_2 + N_3) \sim m(N_3)$. Choose $g(t, x)$ such that $\| g(t, x) \|_{L_{t,x}^{2}(J \times \mathbb{R}^n)} = 1$. Then decompose

$$
\int \hat{g}(t, \xi) \hat{f}(t, \xi) d\xi,
$$

using Cauchy-Schwartz.

Case 4(c), $N_2 \sim N_3$: In this case use the Sobolev estimate

$$
\| I(\| u^{1/n} \|_{L_{t,x}^{2}(J \times \mathbb{R}^n)}) \|_{L_t^{\infty} L_x^{2/(1-s)}(J \times \mathbb{R}^n)} \lesssim \nabla I(\| u^{1/n} \|_{L_{t,x}^{2}(J \times \mathbb{R}^n)})
$$

In this case $|N_2 + N_3|m(N_2 + N_3) \lesssim (N_3)|N_3|$.

$$
\sum_{N \lesssim N_2 \sim N_3} \| P_{N_2} F(u) \|_{L_t^{\infty} L_x^{n/2}(J \times \mathbb{R}^n)} \| P_{N_3} I u \|_{L_t^{\infty} L_x^{2/(1-s)}(J \times \mathbb{R}^n)}
\lesssim \| \langle \nabla \rangle I u \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n} \sum_{N \lesssim N_2 \sim N_3} \frac{1}{N_2^{\sigma} N_3^{\sigma(1-s)}} \lesssim \frac{1}{N^\sigma} \| \langle \nabla \rangle I u \|_{S^0(J \times \mathbb{R}^n)}^{1+4/n}.
$$

This completes the proof of Lemma 6.3 and consequently Theorem 6.1.
7 Proof for $n \geq 4$

The interaction Morawetz estimates will be stated without proof.

**Theorem 7.1** Suppose $u$ solves \([1, 2]\), then

$$
\|u\|_{L_t^{2(n-1)} L_x^{2(n-1)/n} (J \times \mathbb{R}^n)} \lesssim \|u_0\|_{L^{2(n-2)}(\mathbb{R}^n)}^{1/2} \|u\|_{L_t^{\infty} \dot{H}_x^{1/2}(J \times \mathbb{R}^n)}^{n-2}. \tag{7.1}
$$

In addition, suppose $J = [0, T]$,

$$
\|u\|_{L_t^{4(n-1)} L_x^{2(n-1)/n} (J \times \mathbb{R}^n)} \lesssim \|u_0\|_{L^{2(n-2)}(\mathbb{R}^n)}^{1/2} \|u\|_{L_t^{\infty} \dot{H}_x^{1/2}(J \times \mathbb{R}^n)}^{n-2}. \tag{7.2}
$$

**Proof:** See [7] for $n = 3$, [20] for $n \geq 4$.

A local well-posedness result is also needed.

**Theorem 7.2** There exists $\epsilon > 0$ such that if

$$
\|u\|_{L_t^{4(n-1)} L_x^{2(n-1)/n} (J \times \mathbb{R}^n)} < \epsilon, \tag{7.3}
$$

and $\|\nabla Iu_0\|_{L^2(\mathbb{R}^n)} \leq 1$, then

$$
\|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)} \lesssim 1. \tag{7.4}
$$

**Proof:** Let $J = [a, b]$. A solution to \([1.1]\) satisfies Duhamel’s formula,

$$
Iu(t, x) = Ie^{i(t-a)\Delta} u(a) + \int_a^t e^{i(t-\tau)\Delta} I(|u(\tau)|^{4/n} u(\tau)) d\tau. \tag{7.5}
$$

Make the Strichartz estimates,

$$
\|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)} \lesssim \|\langle \nabla \rangle Iu_0\|_{L^2(\mathbb{R}^n)} + \|\langle \nabla \rangle Iu\|_{L_t^{n-1} L_x^{2(n-1)/n} (J \times \mathbb{R}^n)} \|u\|_{L^4(\mathbb{R}^n)}^{4/n} \|u\|_{L_t^{\infty} \dot{H}_x^{1/2}(J \times \mathbb{R}^n)}^{n-2}.
$$

So by the continuity method, for $\epsilon > 0$ sufficiently small,

$$
\|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^n)} \lesssim 1. \tag{7.6}
$$
Proof of Theorem 1.3:

\[ \int |\nabla Iu_0(x)|^2 \, dx \leq N^{2(1-s)} \|u_0\|_{H^s(\mathbb{R}^n)}^2. \]

By the Sobolev embedding theorem, \( H^\frac{n}{4+2}(\mathbb{R}^n) \subset L^{2+4/n}(\mathbb{R}^n) \), so for \( n \geq 4 \),

\[ \int |Iu_0(x)|^{2+4/n} \, dx \leq C \|u_0\|_{H^s(\mathbb{R}^n)}^{2+4/n}. \]

Next, fix an interval \([0, T_0]\). Rescaling,

\[ \|u_{0,\lambda}(x)\|_{H^s(\mathbb{R}^n)} = \lambda^{-s} \|u_0\|_{H^s(\mathbb{R}^n)}. \tag{7.7} \]

Therefore, choose \( \lambda = C(\|u_0\|_{H^s(\mathbb{R}^n)}) N^{\frac{1}{2s}} \) such that \( E(Iu_{0,\lambda}) \leq \frac{1}{2} \). This also proves \( |\lambda^2 T_0| \lesssim N^\alpha \).

Define a set

\[ W = \{ t \in [0, \lambda^2 T_0] : E(Iu_\lambda(t)) \leq \frac{3}{4} \}. \tag{7.8} \]

We aim to prove \( W = [0, \lambda^2 T_0] \) for \( s > \frac{n-2}{2} \). Now \( 0 \in W \) and \( W \) is closed, so it suffices to show \( W \) is open in \([0, T_0]^2\). Suppose \( W = [0, T] \subset [0, \lambda^2 T_0] \), then there exists \( \delta > 0 \) such that \( E(Iu_\lambda(t)) \leq 1 \) on \([0, T+\delta]\).

Next, apply the Morawetz estimates.

\[ \|P_{\leq N} u(t)\|_{H^{1/2}(\mathbb{R}^n)} \leq \|P_{\leq N} u(t)\|_{H^1(\mathbb{R}^n)}^{1/2} \|u_0\|_{L^2(\mathbb{R}^n)}^{1/2}. \tag{7.9} \]

\[ \|P_{> N} u(t)\|_{H^{1/2}(\mathbb{R}^n)} \leq \|P_{> N} u(t)\|_{H^s(\mathbb{R}^n)}^{1/2s} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-1/2s}. \tag{7.10} \]

So if \( m_0 = \|u_0\|_{L^2(\mathbb{R}^n)} \), then combining Theorem 7.1 properties of the I-operator, and \( E(Iu_\lambda(t)) \leq 1 \) on \([0, T+\delta]\):

\[ \|u(t)\|_{L^{2n/(n-1)}_t L^{(2n-1)/(n-1)}_x ([0,T+\delta] \times \mathbb{R}^n)} \leq C(m_0). \tag{7.11} \]

Then by (7.12),

\[ \|u(t)\|_{L^{2n/(n-1)}_t L^{(2n-1)/(n-1)}_x ([J \times \mathbb{R}^n)}} \leq \lambda^{\frac{n-2}{2(n-1)}} T_0^{\frac{n-2}{2(n-1)}} C(m_0). \tag{7.12} \]
Partition $[0, T + \delta]$ into
\[
\left[ C \lambda^{\frac{n-2}{n}} T_0^{\frac{n-2}{2(n-1)}} \delta \right] \sim N^{\frac{2(n-2)(1-s)}{ns}} T_0^{\frac{(n-2)(1-s)}{ns}}
\]
subintervals with $\|u\|_{L_t^{\frac{4(n-1)}{2n}} L_x^{\frac{2(n-1)}{n-2}}} \leq \epsilon$ on each subinterval. Combining this with the estimate for the energy increment,
\[
|E(Iu_\lambda(t))| \leq \frac{1}{2} + CN^{\frac{2(n-2)(1-s)}{ns}} - N^{-\frac{4}{n-2}} + T_0^{\frac{(n-2)(1-s)}{ns}}.
\]
When $s > \frac{n-2}{n}$, choosing $N$ sufficiently large,
\[
E(Iu_\lambda(t)) \leq \frac{3}{4}.
\]
Therefore, $W$ is both open and closed in $[0, \lambda^2 T_0]$, and $W = [0, \lambda^2 T_0]$.

**Remark:** It suffices to choose
\[
N \geq (4C)^{\frac{ns}{2(n^s-(n-2)^s)}} + (T_0^{\frac{(n-2)(1-s)}{ns}})^{\frac{2(n^s-(n-2)^s)}{ns}}.
\]
Since $\lambda = C_0 N^{\frac{1-s}{n}}$ and
\[
\|u(t)\|_{\dot{H}^s(\mathbb{R}^n)} = \lambda^s \|u_\lambda(t)\|_{\dot{H}^s(\mathbb{R}^n)},
\]
\[
\sup_{[0,T_0]} \|u(t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq C T_0^{\frac{(n-2)(1-s)^2}{2(n^s-(n-2)^s)}},
\] (7.13)
and the proof is complete. □

**8 Almost Morawetz Estimate for $n = 3$**

Since we wish to prove global well-posedness for $s > \frac{2}{5}$, it is not enough to use
\[
\|u\|_{L_{t,x}^1(J \times \mathbb{R}^3)} \lesssim \|u\|_{L_{t,x}^{\infty} \dot{H}^{1/2} (J \times \mathbb{R}^3)}^2 \|u_0\|_{L^2(\mathbb{R}^3)}^2.
\]
Instead it is necessary to use almost Morawetz estimates in $n = 3$. (See [5], [10], [13] for a discussion of the $n = 2$ case, [12] for the $n = 1$ case.)
**Theorem 8.1** Suppose $u$ solves the equation

$$iu_t + \Delta u = |u|^{4/3}u,$$  \hspace{1cm} (8.1)

Suppose also that $[0, T] = \bigcup_{k=1}^K J_k$.

$$\|Iu\|_{L^4_t([0,T] \times \mathbb{R}^3)} \lesssim \|u_0\|_{L^2(\mathbb{R}^3)}^2 \|Iu\|_{L^\infty_t H^1(\mathbb{R}^3)} + \frac{1}{N^1} \sum_{k=1}^K \|\nabla Iu\|_{S^{16/3}_0(J_k \times \mathbb{R}^3)}^{16/3}.\hspace{1cm} (8.2)$$

**Proof:** Suppose $v$ satisfies the partial differential equation

$$iv_t + \Delta v = F.$$  \hspace{1cm} (8.3)

Let

$$T_{0j} = 2\text{Im}(v(t, z)\partial_j v(t, z)),$$  \hspace{1cm} (8.4)

$$L_{jk} = -\partial_j \partial_k (|v(t, z)|^2) + 4\text{Re}(\partial_j v(t, z)\partial_k v(t, z)),$$  \hspace{1cm} (8.5)

$$\partial_t T_{0j} + \partial_k L_{jk} = 2(F(t, z)\partial_j v(t, z) - \overline{v(t, z)}\partial_j F(t, z) + \partial_t F(t, z)) - v(t, z)\partial_j F(t, z),$$  \hspace{1cm} (8.6)

Let $v(t, z)$ be the solution on $\mathbb{R}^3 \times \mathbb{R}^3$ given by the coordinates $(x, y) = z$,

$$v(t, z) = Iu(t, x)Iu(t, y).$$

Then $v(t, z)$ solves the equation

$$i\partial_t v(t, z) + \Delta_z v(t, z) = I(|u(t, x)|^{4/3}u(t, x))Iu(t, y) + Iu(t, x)I(|u(t, y)|^{4/3}u(t, y)).$$  \hspace{1cm} (8.7)

Next define the Morawetz action,

$$M^\otimes_a(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \partial_j a(z) \cdot \text{Im}(v(t, z)\partial_j v(t, z))dz,$$  \hspace{1cm} (8.8)

with $a(z) = |x - y|$.

$$\partial_t M^\otimes_a(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \partial_j a(z)\partial_t T_{0j}(t, z)dz$$
\[= - \int [\partial_k L_{jk}(t, z)] \partial_j a(z) dz \]

\[+ 2 \int [F(t, z) \partial_j v(t, z) - v(t, z) \partial_j F(t, z) + F(t, z) \partial_j v(t, z) - v(t, z) \partial_j F(t, z)] \partial_j a(z) dz.\]

\[\partial_t M_a^{\otimes 2}(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \partial_{jkk} (|v(t, z)|^2) \partial_j a(z) dz \quad (8.9)\]

\[- 4 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \partial_k \text{Re}(\partial_j v(t, z) \partial_k v(t, z)) \partial_j a(z) dz \quad (8.10)\]

\[+ 2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} [F(t, z) \partial_j v(t, z) - v(t, z) \partial_j F(t, z)] \partial_j a(z) dz. \quad (8.11)\]

Integrating by parts three times in (8.9), and using the identity \(\Delta \Delta |x - y| = -\delta(|x - y|)\) in \(\mathbb{R}^3\),

\[\int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \partial_{jkk} (|v(t, z)|^2) \partial_j a(z) dz = \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(|x - y|) |Iu(t, x)|^2 |Iu(t, y)|^2 dxdydt \quad (8.12)\]

\[= \int_0^T \int_{\mathbb{R}^3} |Iu(t, x)|^4 dxdt.\]

Integrating (8.10) by parts once,

\[- 4 \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \partial_k \text{Re}(\partial_j v(t, z) \partial_k v(t, z)) \partial_j a(z) dz = 4 \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \text{Re}(\partial_j v(t, z) \partial_k v(t, z)) \partial_{jk} a(z) dz. \quad (8.13)\]

This quantity is a positive definite matrix.

\[\frac{\delta_{jk}}{|x - y|} - \frac{(x - y)_j (x - y)_k}{|x - y|^3} v_j v_k = \frac{|v|^2}{|x - y|} - \frac{(v \cdot (x - y))^2}{|x - y|^3} \geq 0.\]

This proves in particular that the quantity (8.13) is \(\geq 0\).
Split $F = F_g + F_b$, with

\[ F_g = |Iu(t,x)|^{4/3}Iu(t,x)Iu(t,y) + |Iu(t,y)|^{4/3}Iu(t,x)Iu(t,y), \]

\[ F_b = F - F_g. \]  

(8.14)

Now for (8.11), without loss of generality let $j = 1, 2, 3$.

\[ |Iu(t,y)|^{4/3}Iu(t,y)Iu(t,x)\partial_j(Iu(t,x)Iu(t,y)) \]

\[ + |Iu(t,y)|^{4/3}Iu(t,y)Iu(t,x)\partial_j(Iu(t,x)Iu(t,y)) = \partial_j(|Iu(t,y)|^{10/3}|Iu(t,x)|^2). \]

This cancels with the term

\[ -Iu(t,x)Iu(t,y)\partial_j(|Iu(t,y)|^{4/3}Iu(t,y)Iu(t,x)) \]

\[ -Iu(t,x)Iu(t,y)\partial_j(|Iu(t,y)|^{4/3}Iu(t,y)Iu(t,x)) = -\partial_j(|Iu(t,y)|^{10/3}|Iu(t,x)|^2). \]

On the other hand,

\[ |Iu(t,x)|^{4/3}Iu(t,x)Iu(t,y)\partial_j(Iu(t,x)Iu(t,y)) \]

\[ + |Iu(t,x)|^{4/3}(Iu(t,x)Iu(t,y))\partial_jIu(t,x)Iu(t,y) \]

\[ -Iu(t,x)Iu(t,y)\partial_j(|Iu(t,x)|^{4/3}Iu(t,x)Iu(t,y)) \]

\[ -Iu(t,x)Iu(t,y)\partial_j(|Iu(t,x)|^{4/3}Iu(t,x)Iu(t,y)) \]

\[ = -2|Iu(t,x)|^2|Iu(t,y)|^2\partial_j|Iu(t,x)|^{4/3} = -\frac{4}{5}\partial_j(|Iu(t,y)|^2|Iu(t,x)|^{10/3}). \]

Integrating by parts,

\[ -\int_0^T \int \partial_j a(z)\partial_j(|Iu(t,y)|^2|Iu(t,x)|^{10/3})dxdydt \]

\[ = \int_0^T \int (\partial_j a(z))|Iu(t,y)|^2|Iu(t,x)|^{10/3}dxdydt \geq 0. \]

All this together proves (8.11) with $F$ replaced by $F_g$ is $\geq 0$.

To evaluate (8.11) with $F$ replaced by $F_b$, there are terms of the form
\[ \int_0^T \int_0^T \partial_j a(x - y) |Iu(t, x)|^2 \partial_j |Iu(t, x)|^2 \, dx \, dy \, dt, \]  
(8.15) 
terms of the form

\[ \int_0^T \int_0^T \partial_j a(x - y) |Iu(t, x)|^2 |Iu(t, x)|^2 \partial_j |Iu(t, x)|^2 \, dx \, dy \, dt, \]  
(8.16) 
and also terms of the form

\[ \int_0^T \int_0^T \partial_j a(x - y) |Iu(t, x)|^2 |Iu(t, x)|^2 \partial_j |Iu(t, x)|^2 \, dx \, dy \, dt. \]  
(8.17) 

To evaluate a term of the form (8.15), let

\[ \|u\|_{L^4} \leq 1/10 \|\nabla u\|_{L^2} \] 

and also terms of the form

\[ \int_0^T \int_0^T \partial_j a(x - y) |Iu(t, x)|^2 |Iu(t, x)|^2 \partial_j |Iu(t, x)|^2 \, dx \, dy \, dt. \]  
(8.18) 

For a term of the form (8.16), integrate by parts. Then (8.16) is equal to an integral of the form (8.15), as well as a term of the form

\[ \int_0^T \int_0^T \partial_j a(x - y)|Iu(t, x)|^2 |Iu(t, x)|^2 \partial_j |Iu(t, x)|^2 \, dx \, dy \, dt. \]  
(8.19)
\[ \int |Iu(t,y)|^2 \partial_j a(x-y) \, dy \]

is controlled by a term of the form
\[ \int \frac{1}{|x-y|} |Iu(t,y)|^2 \, dy \lesssim \|\langle \nabla \rangle Iu(t,y)\|_{L_\infty L_2}^2. \]

When \(|x-y| > 1, |\Delta a(x-y)| \lesssim 1\), so
\[ \sup_{t,x} \int_{|x-y| > 1} |Iu(t,y)|^2 \Delta a(x-y) \, dy \lesssim \|Iu\|_{L_\infty L_2(J \times R^3)}^2. \quad (8.20) \]

On \(|x-y| \leq 1, \)
\[ \sup_{t,x} \int_{|x-y| \leq 1} \frac{|Iu(t,y)|^2}{|x-y|} \, dy \lesssim \|Iu\|_{L_\infty L_6(J \times R^3)} \|\langle \nabla \rangle Iu\|_{S_0}^2. \quad (8.21) \]

Therefore,
\[ \text{[8.19]} \lesssim \|Iu(t,x)^{4/3} Iu(t,x) - I(|u(t,x)|^{4/3} u(t,x))\|_{L_2^{6/5}(J \times R^3)} \times \|Iu(t,x)\|_{L_6 L_5(J \times R^3)} \|\langle \nabla \rangle Iu\|_{S_0}^2 \]
\[ \lesssim \frac{1}{N} \|\nabla Iu\|_{S_0(J \times R^3)}^{16/3}. \]

Finally,
\[ \int \int \partial_j a(x,y)Iu(t,x)(\partial_j Iu(t,x))Iu(t,y)\left[|Iu(t,y)|^{4/3}(Iu(t,y)) - I(|u(t,y)|^{4/3} u(t,y))\right] \]
\[ \lesssim \|\langle \nabla \rangle Iu\|_{L_\infty L_2(J \times R^3)}^2 \|Iu\|_{L_\infty L_6(J \times R^3)} \|I(|u|^{4/3} u) - |Iu|^{4/3} Iu\|_{L_2^{6/5}(J \times R^3)} \]
\[ \lesssim \frac{1}{N} \|\langle \nabla \rangle Iu\|_{S_0(J \times R^3)}^{16/3}. \]

Putting this all together,
\[ \int_0^T \int_{\mathbb{R}^3} |Iu(t,x)|^4 \, dx \, dt \leq |M_0^{\otimes 2}(T) - M_0^{\otimes 2}(0)| + \sum_{J_k} \frac{\|\langle \nabla \rangle Iu\|_{S^0(J_k \times \mathbb{R}^3)}}{N^1}. \] 

(8.22)

\[ M_0^{\otimes 2}(t) = \left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} Iu(t,x) Iu(t,y) \partial_j (Iu(t,x) Iu(t,y)) \, dx \, dy \right| \lesssim \|\nabla Iu\|_{L_\infty L_2} \|Iu\|_{L_\infty L_2}^3. \] 

(8.23)

and Theorem 8.1 is proved. □

9 Proof for \( n = 3 \)

First prove a local well-posedness result.

**Theorem 9.1** There exists \( \epsilon > 0 \) such that if \( \|Iu\|_{L_8^{3/4} L_2^2(J \times \mathbb{R}^3)} \leq \epsilon, \|\nabla Iu_0\|_{L^2(\mathbb{R}^3)} \leq 1 \), then (1.1) has a local solution with

\[ \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^3)} \lesssim 1. \] 

(9.1)

**Proof:**

\[ \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^3)} \lesssim \|\langle \nabla \rangle Iu_0\|_{L^2(\mathbb{R}^3)} + \|\langle \nabla \rangle Iu\|_{L_1^2 L_{\infty}^2(J \times \mathbb{R}^3)} \|u\|_{L_1^4 L_2^2(J \times \mathbb{R}^3)}^{4/3}. \]

\[ \|u\|_{L_1^4 L_2^2(J \times \mathbb{R}^3)} \leq \|P_{\leq N} u\|_{L_1^4 L_2^2(J \times \mathbb{R}^3)} + \|P_{> N} u\|_{L_1^4 L_2^2(J \times \mathbb{R}^3)} \]

\[ \leq \|Iu\|_{L_1^4 L_2^2(J \times \mathbb{R}^3)} + \frac{1}{N} \|\langle \nabla \rangle Iu\|_{L_1^4 L_2^2(J \times \mathbb{R}^3)}^{4/3}. \]

Therefore, by the continuity method, \( \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R}^3)} \lesssim 1. \) □

**Proof of Theorem 1.4:** In this case,

\[ \int |\nabla Iu_0(x)|^2 \, dx \leq N^{2(1-s)} \|u_0(x)\|_{H^s(\mathbb{R}^3)}^2. \]
\[
\int |Iu(x)|^{10/3} \, dx \leq \|Iu(x)\|^{10/3}_{H^{4/5}(\mathbb{R}^3)} \leq N^{2-\frac{10s}{3}} \|u(x)\|^{10/3}_{H^s(\mathbb{R}^n)}.
\]

Once again, choose \( \lambda = C(\|u_0\|_{H^s(\mathbb{R}^n)}) N^{\frac{1-s}{2}} \) such that \( E(Iu_{0,\lambda}(x)) \leq \frac{1}{2} \).

Define the set,

\[
W = \{ t : E(Iu_{\lambda}(t)) \leq \frac{3}{4} \} \subset [0, \lambda^2 T_0].
\] (9.3)

0 \in W, W is closed. Suppose \( W = [0, T] \), there exists \( \delta > 0 \) with \( E(Iu_{\lambda}(t)) \leq 1 \) on \([0, T + \delta]\).

Lemma 9.2 If \( E(Iu_{\lambda}(t)) \leq 1 \) on \([0, T + \delta]\),

\[
\|Iu_{\lambda}(t)\|_{L_{t,x}^4([0,T+\delta])}^4 \leq \frac{3C}{2} m_0^3.
\] (9.4)

**Proof:** Let

\[
\tau = \sup \{ \bar{T} : \|Iu_{\lambda}\|_{L_{t,x}^4([0,\bar{T}] \times \mathbb{R}^3)} \leq \frac{3Cm_0^3}{2} \}.
\]

If \( \tau < T + \delta \), then

\[
\|Iu_{\lambda}(t)\|_{L_{t,x}^4([0,\tau])}^4 \leq \frac{3C}{2} m_0^3,
\] (9.5)

and there exists some \( \delta' > 0 \) such that

\[
\|Iu_{\lambda}(t)\|_{L_{t,x}^4([0,\tau+\delta'])}^4 \leq 2Cm_0^3.
\] (9.6)

Then \([0, \tau + \delta']\) can be partitioned into

\[
\lesssim (\tau + \delta')^{1/3} \|Iu_{\lambda}\|_{L_{t,x}^4([0,\tau+\delta'] \times \mathbb{R}^3)} \leq (2C)^{8/3} \lambda^{2/3} T_0^{1/3} m_0^8
\]

subintervals, and on each subinterval \( J_k \), \( \|Iu_{\lambda}\|_{L_{t,x}^s L_{x}^4(J_k \times \mathbb{R}^3)} \leq \epsilon, \)

\[
\|\langle \nabla \rangle Iu\|_{S^0(J_k \times \mathbb{R}^3)} \leq C'.
\] (9.7)

Applying the almost Morawetz estimate of the previous section,

\[
\|Iu_{\lambda}\|_{L_{t,x}^4([0,\tau+\delta])}^4 \leq Cm_0^3 + C' \frac{\lambda^{2/3} T_0^{1/3} (2C)^{8/3} m_0^8}{N^{1-}},
\] (9.8)

so for \( s > \frac{2}{3} \), take \( N \geq T_0^{s-\frac{2}{3}} \),

\[
\|Iu_{\lambda}\|_{L_{t,x}^4([0,\tau+\delta'])}^4 \leq \frac{3C}{2} m_0^3.
\] (9.9)

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Therefore, \( \tau = T + \delta. \) □

Returning to the proof of the theorem,

\[
\|Iu_\lambda\|_{L^{2/3}_t L^{2/3}_x ([0,T+\delta] \times \mathbb{R}^3)} \lesssim (\lambda^2 T_0)^{1/8} \|Iu_\lambda\|_{L^{12/5}_t L^{4/5}_x ([0,T+\delta] \times \mathbb{R}^3)}. \tag{9.10}
\]

Therefore, \([0,T+\delta]\) can be partitioned into \( \lesssim \lambda^{2/3} T_0^{1/3} \) subintervals \( J_k \) with

\[
\|Iu_\lambda\|_{L^{12/5}_t L^{4/5}_x (J_k \times \mathbb{R}^3)} \leq \epsilon.
\]

\[
E(Iu_\lambda(t)) \leq \frac{1}{2} + \frac{C \lambda^{2/3} T_0^{1/3}}{N^{1-}}.
\]

Again, choosing some \( N \gtrsim T_0^{5s-2} \) with a possibly bigger constant,

\[
E(Iu_\lambda(t)) \leq \frac{3}{4}.
\]

This implies \( W = [0, \lambda^2 T_0] \). It suffices to take \( N \gtrsim T_0^{5s-2} \), so

\[
\|u(t)\|_{H^s(\mathbb{R}^3)} \lesssim T_0^{\frac{(1-s)}{5s-2}+}. \tag{9.11}
\]

This concludes the proof of Theorem 1.4 □
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