Finiteness of fixed equilibrium configurations of point vortices in the plane with a background flow

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Abstract
For a dynamic system consisting of \( n \) point vortices in an ideal plane fluid with a steady, incompressible and irrotational background flow, a more physically significant definition of a fixed equilibrium configuration is suggested. Under this new definition, if the complex polynomial \( w \) that determines the aforesaid background flow is non-constant, we have found an attainable generic upper bound \( (m+n-1)!/\prod_{i=0}^{n-1} (m-i)!n_i! \) for the number of fixed equilibrium configurations. Here, \( m = \deg w \), \( i_0 \) is the number of species, and each \( n_i \) is the number of vortices in a species. We transform the rational function system arising from fixed equilibria into a polynomial system, whose form is good enough to apply the BKK theory (named after Bernshtein (1975 Funct. Anal. Appl. 9 183–5), Khovanskii (1978 Funct. Anal. Appl. 12 38–46) and Kushnirenko (1976 Funct. Anal. Appl. 10 233–5)) to show the finiteness of its number of solutions. Having this finiteness, the required bound follows from Bézout’s theorem or the BKK root count by Li and Wang (1996 Math. Comput. 65 1477–84).

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1. Introduction
Every polynomial \( w \) in one complex variable generates a steady (i.e. independent of time) fluid flow in the complex plane \( \mathbb{C} \) by the map \( \zeta \mapsto (\Re w(\zeta), -\Im w(\zeta)) \), which is identified with \( w(\zeta) \). The polynomial \( w \) is called the complex velocity of this flow, and since \( w \) satisfies the Cauchy–Riemann equations, the flow is incompressible and irrotational. Assume that the fluid
in \( \mathbb{C} \) is ideal (in the sense of [4, section 1.3, p 6]). If \( n \geq 2 \) (point) vortices \( z_1(t), \ldots, z_n(t) \in \mathbb{C} \) with their respective circulations
\[
\Gamma_1, \ldots, \Gamma_n \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}
\] (1)

(which are constants, due to Helmholtz’s theorems; see [5, section 2.2]) are situated in the ideal fluid with the above flow \( \overrightarrow{w} \) in the background, then the dynamics of these vortices will be governed by
\[
\frac{dz_j(t)}{dt} = -\frac{1}{2\pi i} \sum_{k=1}^{n} \frac{\Gamma_k}{z_j(t) - z_k(t)} + w(z_j(t)), \quad j = 1, \ldots, n
\] (2)

(see [6, equation (2)] (caution: typo on the left-hand side) and [7, equation (34)]).

The case of no background flow is when \( w \equiv 0 \). Since the early 1980s, there have been many studies on all types of stationary configurations (in the sense of [8, definitions 0.2 and 1.1.2], see also [8, definitions 0.3 and 1.1.3 and proposition 1.1.4]), as well as other aspects of the vortex dynamics (2). So far, researchers have only found configurations with special patterns or for small \( n \) (see [6, 7, 9], and so on, for these results). Given the difficulty of determining (analytically or numerically) the stationary configurations, inquiry into their number becomes a natural alternative. O’Neil [8, theorems 5.1.1, 5.2.1 and 6.5.1] (see [10, propositions 1 and 2]; beware of different terminologies) gave such results for three types.

(\textit{Let } w \equiv 0. \textit{ For almost every choice of circulations (1) that satisfies:})

- \( \sum_{i < j} \Gamma_i \Gamma_j = 0 \), there are exactly \((n - 2)!\) equilibrium configurations. (This relation among the circulations (1) is necessary for the existence of such configurations.)
- \( \sum_j \Gamma_j = 0 \), there are exactly \((n - 1)!\) rigidly translating configurations. (This relation among the circulations (1) is necessary for the existence of such configurations.)
- \( \sum_{i < j} \Gamma_i \Gamma_j \neq 0 \) and \( \sum_j \Gamma_j \neq 0 \), there are no more than \( n!/2 \) collinear relative equilibrium configurations.

See also [11, 12] for more results about the number of configurations when \( n = 4, 5 \).

As for the case when a background flow is present (i.e. \( w \neq 0 \)), there appears to be no corresponding knowledge so far apart from the few cases in [6, sections III.B.2 and III.D] and [7, section 3.3]. Our main result (theorem 7) concerning the finiteness of the number of fixed equilibrium configurations (to be defined in definition 4; synonymous with ‘equilibrium configurations’ in [8] and ‘stationary equilibrium configurations’ in [10], but the current terminology seems more common in the literature) will fill this deficiency.

Fixed equilibria are the distinct solutions of a polynomial system in the \( n \) unknowns \( z_1, \ldots, z_n \) (by clearing denominators in (4) below). Instead of that, we have found an alternative one (6) (lemma 1) whose form is good enough for applying a test (lemma 3) in the BKK theory ([1–3], as detailed in [14, section 3]) to discuss the finiteness of the number of solutions of this new polynomial system (proposition 2). These finitely many solutions \((z_1, \ldots, z_n)\) are then reduced to the required configurations by the notion of equivalent solutions (definition 4, which is different from those in [8, 10, 12]). We will consider a natural situation (definition 6) where equivalent solutions arise, and arrive at the main result.
2. Main result

Before proceeding, two conventions are to be understood throughout this paper: (i) ‘number of solutions’ of any single polynomial or rational function equation, or any such system, counts multiplicity; (ii) ‘finitely many’ includes ‘none’. 

For fixed equilibria, set \( z_j(t) \equiv z_j \) (so that \( \frac{dz_j(t)}{dt} \equiv 0 \)) in the system (2) and hence we have

\[
- w(z_j) = \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{\Gamma_k}{z_j - z_k}, \quad j = 1, \ldots, n. \tag{3}
\]

Note that the case \( w \equiv c \in \mathbb{C}^* \) goes back to O’Neil’s results in [8, 10], therefore we assume that the degree \( m := \deg w \) of the background flow is positive in what follows. Then, by complex conjugation and an appropriate rescaling, it only suffices to consider the normalized system

\[
z_j^m + W(z_j) = \sum_{k=1}^{n} \frac{\Gamma_k}{z_j - z_k} =: L_j(z_1, \ldots, z_n), \quad j = 1, \ldots, n, \tag{4}
\]

where \( W \) is a polynomial of degree at most \( m - 1 \) with coefficients determined by \( w \). One might clear denominators to obtain a polynomial system \( \mathcal{S} \) of \( n \) equations, where each equation is of degree \( m + n - 1 \) and is in the \( n \) unknowns \( z_1, \ldots, z_n \). Polynomial systems could have infinitely or finitely many solutions. If \( \mathcal{S} \) falls into the latter case, then Bézout’s theorem [13, theorem 2.3.1] would provide the upper bound

\[
(m + n - 1)^n \tag{5}
\]

for the number of solutions.

However, it seems not obvious whether the complicated system \( \mathcal{S} \) would have finitely many solutions. This difficulty has motivated us to transform (4) into the following better polynomial system:

**Lemma 1. (An equivalent polynomial system.)** The system (4) is equivalent to

\[
\begin{aligned}
\sum_j \Gamma_j z_j^m + \sum_j \Gamma_j W(z_j) &= 0 \\
\sum_j \Gamma_j z_j^{m+1} + \sum_{i < j} \Gamma_j z_j W(z_j) &= \sum_i \Gamma_i \Gamma_j \\
\sum_j \Gamma_j z_j^{m+2} + \sum_j \Gamma_j z_j^2 W(z_j) &= \sum_j \Gamma_j \Gamma_j z_j \\
&\vdots \\
\sum_j \Gamma_j z_j^{m+k-1} + \sum_j \Gamma_j z_j^{k-1} W(z_j) &= \sum_j \Gamma_j \Gamma_j z_j^{k-2} + \sum_{r+s=k-2 \atop r, s \neq 0, i < j} \Gamma_i \Gamma_j z_i r z_j s \\
&\vdots \\
\sum_j \Gamma_j z_j^{m+n-1} + \sum_j \Gamma_j z_j^{n-1} W(z_j) &= \sum_j \Gamma_j \Gamma_j z_j^{n-2} + \sum_{r+s=n-2 \atop r, s \neq 0, i < j} \Gamma_i \Gamma_j z_i r z_j s \\
\end{aligned} \tag{6}
\]

with constraint

\[
z_i \neq z_j, \quad i \neq j, \tag{7}
\]

where \( \Gamma_i, j := \Gamma_i \Gamma_j \) and \( \Gamma^j := \sum_{i \neq j} \Gamma_i. \)
Remark. In this new system (6):

(i) The \( n \) equations are of degrees \( m, m+1, \ldots \) and \( m+n-1 \), respectively, and the leftmost sums are the only sources of these degrees.

(ii) The right-hand side of the \( k \)th \( (k = 2, \ldots, n) \) equation is either a homogeneous polynomial of degree \( k-2 \) or, in the degenerate case, identically zero.

(iii) Each equation is invariant under any permutation of the \( n \) circulation–vortex pairs \( \{ (\Gamma_j, z_j) : j = 1, \ldots, n \} \), while in the original system (4), there will be a permutation of the \( n \) equations.

(iv) Admissible solutions of (6) are those that satisfy constraint (7). This terminology will not appear until section 5.

(v) Lemma 1 will be proved in section 3 by transforming (4) into (6) via an explicit matrix which is invertible under (7). This new system (6) can also be obtained by replacing \( 2\pi i \bar{\nu}_j \) by \( z_j^m + W(z_j) \) in O’Neil’s [10, equation (4.1)]. Our and O’Neil’s methods are different. In particular, the involvement of the Vandermonde determinant in (17) in our method seems surprising.

The following proposition shows that the condition

\[
\sum_{j \in I} \Gamma_j \neq 0 \quad \text{for all} \quad I \subset \{1, \ldots, n\} \quad \text{and} \quad \sum_{i<j} \Gamma_i \Gamma_j \neq 0
\]

guarantees that (6) alone (i.e. without the constraint (7)) already has finitely many solutions.

**Proposition 2. (Upper bound for the number of solutions of the equivalent polynomial system.)** Assume that \( m \geq 1 \). Then, for any choice of the circulations (1) that satisfies (8), the system (6) has at most \( \frac{(m+n-1)!}{(m-1)!} \) solutions, and so does (4). This bound can be attained.

**Remark.** There are cases where (4) has infinitely many solutions, such as in [6, sections III.B and III.C] and [7, section 3.2]. But these existing cases either have no background flow or a background flow of degree 0 (i.e. \( m < 1 \)), thus are not covered by proposition 2.

The finiteness of the number of solutions of (6) under assumption (8) will be shown in section 4. Then by remark (i) above, Bézout’s theorem provides the required upper bound \( \frac{(m+n-1)!}{(m-1)!} \) for the number of solutions which is better than the bound (5). As we shall see in the proof of lemma 1 in section 3, every solution of (4) also satisfies (6), so (4) inherits the upper bound for its number of solutions. Finally, section 5 will provide examples where this bound is attained.

As the reader will see in section 4, the advantage of the transformation of lemma 1 is that (6) is in a special form that facilitates using the following test in the BKK theory (for details, the reader is referred to [14, section 3], and also [12] for an application):

**Lemma 3. (Test for finiteness of the number of solutions of a polynomial system in \( \mathbb{C}^n_{\alpha_0} \))** [14, propositions 2 and 3] Consider a system of \( m \) polynomial equations in \( n \) unknowns:

\[
P_k(z_1, \ldots, z_n) = \sum_{r = (r_1, \ldots, r_n) \in \mathcal{A}_k} c_r z_1^{r_1} \cdots z_n^{r_n} = 0, \quad c_r \in \mathbb{C}^*, \quad k = 1, \ldots, m,
\]

where each finite subset \( \mathcal{A}_k \) of \( (\mathbb{N} \cup \{0\})^n \) is called the support of \( P_k \). For each \( \alpha \in \mathbb{R}^n \), the system

\[
P_{k, \alpha}(z_1, \ldots, z_n) := \sum_{r \in \mathcal{A}_k} c_r z_1^{r_1} \cdots z_n^{r_n} = 0, \quad k = 1, \ldots, m,
\]

is called the reduced system (of (9)) determined by \( \alpha \). Suppose that there exists an \( \alpha_0 \in \mathbb{Z}^n \) such that every reduced system (10) with \( \alpha_0 \cdot \alpha \leq 0 \) has no solution in \( \mathbb{C}^n_{\alpha_0} \). Then, the original system (9) has only finitely many solutions in \( \mathbb{C}^n_{\alpha_0} \).
The seemingly weird condition ‘$\alpha \cdot r = \min_{r' \in A_k} \alpha \cdot r'$’ in (10) actually admits a beautiful geometric interpretation in terms of a supporting hyperplane. We refer the reader to the paragraph preceding [14, proposition 3]. This interpretation will be used when proving proposition 2 in section 4.

Next, we suggest a, perhaps, new definition of the fixed equilibrium configuration in order to state the main result, theorem 7, of this paper. Fix any polynomial $w$ of degree $m \geq 1$, which provides the background flow in $\mathbb{C}$ as in section 1. Let $z = (z_1, \ldots, z_n)$ be a solution of (3) (necessarily, $z_i \neq z_j$ for $i \neq j$). Recall that these vortices in the background flow $w$ generate the flow

\[ V_z(\zeta) := -\frac{1}{2\pi i} \sum_j \frac{\Gamma_j}{\zeta - z_j} + w(\zeta) \]

or the complex velocity

\[ V_z(\zeta) := \overline{V_z}(\zeta) = \frac{1}{2\pi i} \sum_j \frac{\Gamma_j}{\zeta - z_j} + w(\zeta) \]

on $\mathbb{C}_z := \mathbb{C} \setminus \{z_1, \ldots, z_n\}$. Note that $V_z$ is a rational function on $\mathbb{C}$ with simple poles at $z_1, \ldots, z_n$.

**Definition 4.** (Equivalent solutions. Fixed equilibrium configuration.) Two solutions $z = (z_1, \ldots, z_n)$ and $z' = (z_1', \ldots, z_n')$ of (3) are said to be equivalent (denoted by $z \sim z'$) if they generate two non-reflectively similar flows with $w$, in a plane-geometrical sense. More precisely, $z \sim z'$ if there exists $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ such that $aV_{z'}(a\zeta + b) = V_z(\zeta)$ for $\zeta \in \mathbb{C}_z$, or, equivalently (in terms of complex velocity),

\[ aV_{z'}(a\zeta + b) = V_z(\zeta) \quad \text{for} \quad \zeta \in \mathbb{C}_z. \]  

Then, each equivalence class $[z]$ is called a fixed equilibrium configuration.

In O’Neil’s definition [8, definitions 0.4 and 1.1.5], two solutions $z$ and $z'$ are regarded as equivalent if there exists $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ such that $z_j' = az_j + b$ for all $j$, so only the shapes of vortex sets are involved. Our definition of equivalent solutions of (3) has an extra physical significance: besides the shapes of vortex sets, their effects on the rest of the plane are also considered. Here we illustrate the difference between our and O’Neil’s definitions.

**Example 5.** Consider the two-vortex case $n = 2$, both with circulation $\Gamma_1 = \Gamma_2 = 1$, and with background flow $w(\zeta) = -\zeta^2 + \frac{1}{2\pi i}$ (thus $m = 2$). By lemma 1, we are to solve

\[
\begin{cases}
  z_1^2 + z_2^2 + 2 = 0, \\
  z_1^3 + z_2^3 + z_1 + z_2 = 1
\end{cases}
\]

with constraint $z_1 \neq z_2$, and the solutions are

\[
(z_1, z_2) \approx (-0.250 \pm 1.349 \, i, 0.487 \pm 0.693 \, i), \\
(0.487 \pm 0.693 \, i, -0.250 \pm 1.349 \, i), \\
(-0.237 \pm 1.028 \, i, -0.237 \pm 1.028 \, i).
\]  

Any pair of two-point sets in the plane differs by translation, rotation and/or scaling, so all the six solutions in (12) are equivalent in O’Neil’s sense, thereby constituting exactly one fixed equilibrium configuration. But by definition 4, they constitute three fixed equilibrium configurations because they generate three flows with $w$ that are not non-reflectively similar, as shown in figure 1. The streamlines in figure 1 are actually formed by superimpositions in figure 2. (Figures 1 and 2 are generated by Mathematica.)
Figure 1. The streamlines of the flows $V(z_1, z_2)$ generated by the vortex sets (12) with the background flow $w(\zeta) = -\frac{\zeta^2 + 1}{2\pi i}$ in example 5.

Our definition introduces a comparison (11) of rational functions to distinguish between different configurations. We are not going to elaborate this definition to the full extent, but just observe a natural situation where equivalent solutions, in our sense, of (3) arise.

Definition 6. (Species.) Partition the circulations (1) into their $t_0$ distinct values:

$$
\Gamma_{k,1} = \cdots = \Gamma_{k,t_0} =: [\Gamma_i], \quad i = 1, \ldots, t_0, \quad [\Gamma_i] \neq [\Gamma_{i'}], \quad i \neq i', \quad \sum_i n_i = n. \quad (13)
$$

For each solution $(z_1, \ldots, z_n)$ of (3), the set of all the vortices

$$
[z_i] := \{z_{k,1}, \ldots, z_{k,n_i}\},
$$

which possess the common circulation $[\Gamma_i]$, is called a species (see [6]).
Note that the concept of species was not involved in defining equivalent solutions in [8]. Here, it acts as follows: if $\Gamma_i = \Gamma_j$ for some $i < j$, then with every solution $z = (z_1, \ldots, z_i, \ldots, z_j, \ldots, z_n)$ of (3) there associates another solution $z' = (z_1, \ldots, z_j, \ldots, z_i, \ldots, z_n)$, and then $z \sim z'$ because (11) holds with $(a, b) = (1, 0)$. Consequently, a standard combinatorial argument bridges proposition 2 to our main theorem.

Theorem 7. (Upper bound for the number of fixed equilibrium configurations in a background flow.) Assume that $m \geq 1$. Then, for any choice of the circulations (1) that satisfies (8), there are at most $\frac{(m + n - 1)!}{(m - 1)! n_1 \cdot \cdots \cdot n_m}$ fixed equilibrium configurations, where $n_i$ are the sizes of the species as in (13). This bound can be attained.
Section 5 will provide examples where this bound is and is not attained. It will also suggest another factor in reducing the solutions of (3), (4) or (6) to fixed equilibrium configurations via definition 4.

3. Proof of lemma 1

Write (4) in vector form

$$Z_n + W_n = L_n,$$

where

$$Z_n := (z_i^m)_{i=1,\ldots,n}, \quad W_n := (W(z_i))_{i=1,\ldots,n} \quad \text{and} \quad L_n := (L_i(z_1, \ldots, z_n))_{i=1,\ldots,n}.$$  

By left-multiplying each side by the square matrix

$$T_n := \left( \frac{\Gamma_j}{\Gamma_i} \right)_{i,j=1,\ldots,n},$$

the left-hand side simply becomes

$$T_n(Z_n + W_n) = \left( \frac{\Gamma_1}{\Gamma_i} \frac{\Gamma_2}{\Gamma_i} \cdots \frac{\Gamma_n}{\Gamma_i} \right) \left( \begin{array}{c} \left( \sum_{i \neq 1} \frac{\Gamma_i}{\Gamma_i} \frac{z_i - z_1}{z_1 - z_i} \right) \\ \left( \sum_{i \neq 2} \frac{\Gamma_i}{\Gamma_i} \frac{z_i - z_2}{z_2 - z_i} \right) \\ \vdots \\ \left( \sum_{i \neq n} \frac{\Gamma_i}{\Gamma_i} \frac{z_i - z_n}{z_n - z_i} \right) \end{array} \right) + \left( \begin{array}{c} \left( \sum_{j=1}^{n-1} \frac{\Gamma_j}{\Gamma_j} \frac{z_j}{z_j - z_{j+1}} \right) \\ \left( \sum_{j=2}^{n} \frac{\Gamma_j}{\Gamma_j} \frac{\sum_j \Gamma_j W(z_j)}{z_j - z_{j+1}} \right) \\ \vdots \\ \left( \sum_{j=n}^{1} \frac{\Gamma_j}{\Gamma_j} \frac{\sum_j \Gamma_j z_j W(z_j)}{z_j - z_{j+1}} \right) \end{array} \right).$$

On the right-hand side, the first entry of $T_n L_n$ is

$$\left( \frac{\Gamma_1}{\Gamma_i} \right) \left( \sum_{i \neq 1} \frac{\Gamma_i}{\Gamma_i} \frac{z_i - z_1}{z_1 - z_i} \right) = \sum_j \Gamma_j \sum_{i \neq j} \frac{\Gamma_i}{\Gamma_i} \frac{z_j - z_i}{z_j - z_i} = \sum_{i < j} \left( \frac{\Gamma_i \Gamma_j}{z_j - z_i} + \frac{\Gamma_i \Gamma_j}{z_i - z_j} \right) = 0$$

by cancellations. The second entry is

$$\left( \frac{\Gamma_1}{\Gamma_i} \frac{\Gamma_2}{\Gamma_i} \cdots \frac{\Gamma_n}{\Gamma_i} \right) \left( \sum_{i \neq 1} \frac{\Gamma_i}{\Gamma_i} \frac{z_i - z_1}{z_1 - z_i} \right) = \sum_j \Gamma_j \sum_{i \neq j} \frac{\Gamma_i}{\Gamma_i} \frac{z_j - z_i}{z_j - z_i}$$

$$= \sum_{i < j} \left( \frac{\Gamma_i \Gamma_j}{z_j - z_i} + \frac{\Gamma_i \Gamma_j}{z_i - z_j} \right) = \sum_{i < j} \Gamma_i \Gamma_j.$$
The $k$th entry, $k = 3, \ldots, n$, is
\[
\left( \frac{\Gamma_i}{z_i - \bar{z}_i} \frac{\Gamma_i}{z_i - \bar{z}_2} \cdots \frac{\Gamma_i}{z_i - \bar{z}_n} \right) = \sum_j \frac{\Gamma_j}{z_j - \bar{z}_i} \sum_{j \neq i} \frac{\Gamma_j}{z_j - \bar{z}_i}
\]
\[
= \sum_{i < j} \left( \frac{\Gamma_j \Gamma_i z_{j-1}}{z_j - z_i} + \frac{\Gamma_i \Gamma_j z_{i-1}}{z_i - z_j} \right)
\]
\[
= \sum_{i < j} \Gamma_i \Gamma_j (z_i^{k-2} + z_i^{k-3} z_j + \cdots + z_i z_j^{k-3} + z_j^{k-2}).
\]

We have left-multiplied each side of (14) by $T_n$ to obtain (6), thus the claimed equivalence between (4) and (6) and (7) would follow if $T_n$ is invertible. Indeed,
\[
\det T_n = \begin{vmatrix}
\Gamma_1 & \Gamma_2 & \cdots & \Gamma_n \\
\Gamma_1 z_1 & \Gamma_2 z_2 & \cdots & \Gamma_n z_n \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma_1 z_1 n^{-1} & \Gamma_2 z_2 n^{-1} & \cdots & \Gamma_n z_n n^{-1}
\end{vmatrix} = \prod_j \Gamma_j \cdot \prod_{i < j} (z_j - z_i) \quad (17)
\]
is just $\prod_j \Gamma_j$ times of the Vandermonde determinant $\prod_{i < j} (z_j - z_i)$, where (1) and (7) guarantee that their product is non-zero.

4. Proof of proposition 2

4.1. A slightly more general system

Consider the following more general system than (6):
\[
P_1(z_1, \ldots, z_n) := \sum_j \Gamma_j z_j^m + \sum_{j,r} A_{j,r} z_j^r = 0
\]
\[
P_2(z_1, \ldots, z_n) := \sum_j \Gamma_j z_j^{m+1} + \sum_{j,r} A_{j,r} z_j^{r+1} + C_0 = 0
\]
\[
P_3(z_1, \ldots, z_n) := \sum_j \Gamma_j z_j^{m+2} + \sum_{j,r} A_{j,r} z_j^{r+2} + \sum_j C_3 z_j = 0
\]
\[
\vdots
\]
\[
P_k(z_1, \ldots, z_n) := \sum_j \Gamma_j z_j^{m+k-1} + \sum_{j,r} A_{j,r} z_j^{r+k-1} + \sum_j C_{j} z_j^{k-2} + \sum_{r+s=k-2, r,s \neq 0} C_r z_j^r z_j^s = 0
\]
\[
\vdots
\]
\[
P_n(z_1, \ldots, z_n) := \sum_j \Gamma_j z_j^{m+n-1} + \sum_{j,r} A_{j,r} z_j^{r+n-1} + \sum_j C_{j} z_j^{n-2} + \sum_{r+s=n-2, r,s \neq 0, r,s \neq 0} C_r z_j^r z_j^s = 0
\]

with coefficients $\Gamma_j, C_0 \in \mathbb{C}$ and $A_{j,r}, C_j, C_{i,j,r,s} \in \mathbb{C}$, where $\sum_r = \sum_{r=0}^{m-1}$. We will prove the following finiteness result for (18) via sections 4.2 and 4.3:

Proposition 8. (Finiteness of the number of solutions.) The system (18) has finitely many solutions if $\sum_{j \in I} \Gamma_j \neq 0$ for all $I \subseteq \{1, \ldots, n\}$ and $C_0 \neq 0$. 

2453
The system (18) has Bézout bound \( \frac{(m+n-1)!}{(m-1)!} \). This coincides with the generally finer Li and Wang’s BKK root count [15], due to the somewhat special structure of (18) (to be seen in (20)).

**Proposition 9. (BKK root count in \( \mathbb{C}^n \).)** Assume that \( \Gamma_j, C_0 \in \mathbb{C}^* \). Let \( \mathcal{M}(N_1, \ldots, N_n) \) denote the mixed volume of the Newton polytopes \( N_k \) of \( A_k \cup \{0\} \), \( k = 1, \ldots, n \), where each \( A_k \) is the support of the polynomial \( P_k \) in (18). Then,

\[
\mathcal{M}(N_1, \ldots, N_n) = \frac{(m+n-1)!}{(m-1)!}.
\]

**Proof of proposition 9.** The supports of \( P_k \) are respectively

\[
\begin{align*}
\mathcal{A}_1 &= m \mathcal{E}_n \cup \bigcup_{j,r} S_{j,r}^1, \\
\mathcal{A}_2 &= (m+1) \mathcal{E}_n \cup \bigcup_{j,r} S_{j,r}^2 \cup \{0\}, \\
\mathcal{A}_3 &= (m+2) \mathcal{E}_n \cup \bigcup_{j,r} S_{j,r}^3 \cup \bigcup_j S_j^3, \\
& \vdots \\
\mathcal{A}_k &= (m+k-1) \mathcal{E}_n \cup \bigcup_{j,r} S_{j,r}^k \cup \bigcup_j S_j^k \cup \bigcup_{r+s = k-2} S_{i,j,r,s}^{k,r,s} \\
& \vdots \\
\mathcal{A}_n &= (m+n-1) \mathcal{E}_n \cup \bigcup_{j,r} S_{j,r}^n \cup \bigcup_j S_j^n \cup \bigcup_{r+s = n-2} S_{i,j,r,s}^{n,r,s},
\end{align*}
\]

where \( \mathcal{E}_n := \{e_1, \ldots, e_n\} \) is the standard basis of \( \mathbb{R}^n \), \( \bigcup_{r=0}^{m-1} \) and

\[
S_{j,r}^k := \begin{cases} \{(r+k-1)e_j\} & \text{if } A_{k,j,r} \neq 0, \\ \emptyset & \text{if } A_{k,j,r} = 0, \end{cases} \quad S_j^k := \begin{cases} \{(k-2)e_j\} & \text{if } C_{k,j} \neq 0, \\ \emptyset & \text{if } C_{k,j} = 0, \end{cases}
\]

and

\[
S_{i,j,r,s}^{k,r,s} := \begin{cases} \{re_i + se_j\} & \text{if } C_{i,j,r,s}^{k,r,s} \neq 0, \\ \emptyset & \text{if } C_{i,j,r,s}^{k,r,s} = 0. \end{cases}
\]

No matter what value \( S_{j,r}^k, S_j^k \) and \( S_{i,j,r,s}^{k,r,s} \) take in (19), the Newton polytopes \( N_k \) of \( A_k \cup \{0\} \) are

\[
\begin{align*}
\mathcal{N}_1 &= \text{Conv}(m \mathcal{E}_n \cup \{0\}) = m \Delta_n, \\
\mathcal{N}_2 &= \text{Conv}((m+1) \mathcal{E}_n \cup \{0\}) = (m+1) \Delta_n, \\
\mathcal{N}_3 &= \text{Conv}((m+2) \mathcal{E}_n \cup \{0\}) = (m+2) \Delta_n, \\
& \vdots \\
\mathcal{N}_k &= \text{Conv}((m+k-1) \mathcal{E}_n \cup \{0\}) = (m+k-1) \Delta_n \\
& \vdots \\
\mathcal{N}_n &= \text{Conv}((m+n-1) \mathcal{E}_n \cup \{0\}) = (m+n-1) \Delta_n,
\end{align*}
\]

where

\[
\Delta_n := \text{Conv}(\mathcal{E}_n \cup \{0\})
\]

is the unit simplex in \( \mathbb{R}^n \). Such a simplification is due to the fact that \( (r+k-1)e_j \) and \( re_i + se_j \) \((k = 3, \ldots, n, r + s = k - 2 \) and \( i < j \)) actually lie in \( \text{Conv}((m+k-1) \mathcal{E}_n \cup \{0\}) \). By the multilinearity [16, theorem 7.4.12.b, p 338] of mixed volume and [16, exercise 7.3.b, p 306, and exercise 7.7.b, p 338], the required mixed volume is

\[
\mathcal{M}(N_1, \ldots, N_n) = \mathcal{M}(m \Delta_n, (m+1) \Delta_n, \ldots, (m+n-1) \Delta_n) = m \cdot (m+1) \cdot \ldots \cdot (m+n-1) \cdot \mathcal{M}((m+n-2) \Delta_n, \ldots, \Delta_n) = \frac{(m+n-1)!}{(m-1)!} \cdot n! \text{Vol}_n(\Delta_n) = \frac{(m+n-1)!}{(m-1)!}.
\]

\[\square\]
Equation (20) has just revealed that the elements of the supports $A_k$ of $P_k$ corresponding to the terms with coefficients $A$ and $C$ can simply be ignored in the formation of the Newton polytopes $N_k$. This will be useful in the upcoming sections.

4.2. The initial case

To prove proposition 8, we shall apply strong induction on the number of vortices $n$; it contains proposition 2 as a case, and then leads to theorem 7 as analysed in section 2. Lemma 3 will enter to test the finiteness of the number of solutions of polynomial systems in $\mathbb{C}^n$. Taking the geometric interpretation of the reduced system in lemma 3 as detailed in the paragraph preceding [14, proposition 3] for granted, we will need the following notation for brevity of the upcoming discussions: let $\ell \ N \ \alpha$ denote the supporting hyperplane of a polytope $N$ with inward normal vector $\alpha$.

Here we start with the initial case $n = 2$:

\[
\begin{align*}
\left\{ \begin{array}{l}
P_1(z_1, z_2) &= \Gamma_1 z_1^m + \Gamma_2 z_2^m + A_1 z_1^{m-1} z_2^{m-1} + \cdots + (A_1^{1,0} + A_1^{1,0}) = 0, \\
P_2(z_1, z_2) &= \Gamma_1 z_1^{m+1} + \Gamma_2 z_2^{m+1} + A_2 z_1^{m-1} z_2^{m-1} + \cdots + C_0 = 0
\end{array} \right.
\]

(22)

Case 1: $(z_1, z_2) \in \mathbb{C}_2^2$. Lemma 3 will show that (22) has only finitely many solutions in $\mathbb{C}_2^2$. To this end, consider, by (19), the Newton polytopes $N_1 = \text{Conv}(\{m e_1, m e_2\} \cup \cdots)$ and $N_2 = \text{Conv}(\{(m+1) e_1, (m+1) e_2, 0\} \cup \cdots)$ of the supports $A_1$ and $A_2$ of $P_1$ and $P_2$, and their Minkowski sum

\[
N = N_1 + N_2 = \text{Conv}(\{a_2 e_1, a_2 e_2\} \cup \cdots),
\]

(23)

where and hereafter,

\[
a_n := m + (m + 1) + \cdots + (m + n - 1) = nm + \binom{n}{2}.
\]

(24)

Note that this ‘· · ·’ in (23) does not alter the fact that

\[
N \subset a_2 \Delta_2,
\]

(25)

where $\Delta_2$ is as in (21). Now, consider the reduced systems of (22) determined by all the $\alpha$ or $\ell \ N \ \alpha$ satisfying $(e_1 + e_2) \cdot \alpha \leq 0$. Because of (25) and such a choice of $\alpha$, each $\ell \ N \ \alpha$ actually supports $N$ at a face of the facet $\text{Conv}(\{a_2 e_1, a_2 e_2\}) = a_2 \Delta_2$.

- Case I. $\ell \ N \ \alpha$ supports $N$ at the 0-face

\[
a_2 e_j, \quad j = 1, 2,
\]

then $\ell \ N_1$ and $\ell \ N_2$ support $N_1$ and $N_2$ at the 0-faces

\[
me_j \quad \text{and} \quad (m+1)e_j,
\]

respectively, giving the reduced system

\[
\begin{align*}
\Gamma_j z_j^m &= 0, \\
\Gamma_j z_j^{m+1} &= 0, \\
\Gamma_j &\neq 0,
\end{align*}
\]

which has no solution in $\mathbb{C}_2^2$.
• **Case II.** $\ell^N_\alpha$ supports $\mathcal{N}$ at the 1-face

\[ \text{Conv}(\{ae_1, ae_2\}) \]

then $\ell^N_1$ and $\ell^N_2$ support $\mathcal{N}_1$ and $\mathcal{N}_2$ at the 1-faces

\[ \text{Conv}(\{me_1, me_2\}) \quad \text{and} \quad \text{Conv}(\{(m + 1)e_1, (m + 1)e_2\}), \]

respectively, giving the reduced system

\[
\begin{align*}
\tilde{P}_1(z_1, z_2) & : = \Gamma_1 z_1^m + \Gamma_2 z_2^m = 0, \\
\tilde{P}_2(z_1, z_2) & : = \Gamma_1 z_1^{m+1} + \Gamma_2 z_2^{m+1} = 0, \\
\Gamma_1, \Gamma_2, \Gamma_1 + \Gamma_2 & \neq 0.
\end{align*}
\]

(26)

(i) If $z_1 \neq z_2$, then $\det T_2 \neq 0$ by (17), so that $T_2$ is invertible. By left-multiplying both sides of (26) by $T_2^{-1}$, it is transformed to

\[ Z_2 = 0 \]

(see (16)), where $Z_2$ is as in (15). But this contradicts that $z_1 \neq z_2$.

(ii) If $z_1 = z_2$, then the sub-system consisting of the first equation

\[
\begin{align*}
(\Gamma_1 + \Gamma_2)z_1^m & = 0, \\
\Gamma_1 + \Gamma_2 & \neq 0
\end{align*}
\]

already has no solution in $\mathbb{C}_+^*$. Therefore, in any case, (26) has no solution in $\mathbb{C}_+^2$.

What we have established so far is that for every $\alpha$ or $\ell^N_\alpha$ with $(e_1 + e_2) \cdot \alpha \leq 0$, the corresponding reduced system has no solution in $\mathbb{C}_+^2$. Hence, it follows from lemma 3 that (22) has only finitely many solutions in $\mathbb{C}_+^2$.

**Case 2:** $(z_1, z_2) \in \mathbb{C}_+ \times \{0\}$. The system (22) degrades to

\[
\begin{align*}
P_1(z_1, 0) & = \Gamma_1 z_1^m + A_1^{1,0} z_1^{m-1} + \ldots + (A_1^{1,0} + A_2^{1,0}) = 0, \\
P_2(z_1, 0) & = \Gamma_1 z_1^{m+1} + A_1^{2,0} z_1^{m-1} + \ldots + C_0 = 0, \\
\Gamma_1, C_0 & \neq 0.
\end{align*}
\]

(27)

where the first equation already has only finitely many (at most $m$) solutions, thus so does (27). Similar for the case where $(z_1, z_2) \in \{0\} \times \mathbb{C}_+^*$. 

**Case 3:** $(z_1, z_2) \in \{0\}^2$. The system (22) degrades to

\[
\begin{align*}
P_1(0, 0) & = A_1^{1,0} + A_2^{1,0} = 0, \\
P_2(0, 0) & = C_0 = 0, \\
C_0 & \neq 0,
\end{align*}
\]

which is simply inconsistent.

Combining all the above three cases, (22) has only finitely many solutions in $\mathbb{C}_+^2$, and proposition 8 with $n = 2$ is proved.

**4.3. Strong induction**

As the reader may have noticed or will see soon, everything actually lies in the non-solvability of the following special reduced system in $\mathbb{C}_+^2$ under the assumption of proposition 8:

\[
\begin{align*}
P_1(z_1, z_2) & : = \Gamma_1 z_1^m + \Gamma_2 z_2^m = 0, \\
P_2(z_1, z_2) & : = \Gamma_1 z_1^{m+1} + \Gamma_2 z_2^{m+1} = 0, \\
\Gamma_1, \Gamma_2, \Gamma_1 + \Gamma_2 & \neq 0.
\end{align*}
\]
Lemma 10. *(Special reduced system.* The system
\[
\begin{align*}
\tilde{P}_1(z_1, \ldots, z_k) & := \Gamma_1 z_1^m + \cdots + \Gamma_k z_k^m = 0 \\
\tilde{P}_2(z_1, \ldots, z_k) & := \Gamma_1 z_1^{m+1} + \cdots + \Gamma_k z_k^{m+1} = 0 \\
& \vdots \\
\tilde{P}_k(z_1, \ldots, z_k) & := \Gamma_1 z_1^{m+k-1} + \cdots + \Gamma_k z_k^{m+k-1} = 0 \\
\sum_{j \in I} \Gamma_j & \neq 0, \ I \subset \{1, \ldots, k\}
\end{align*}
\]
(28)

has no solution in \( \mathbb{C}^k \).

**Proof of lemma 10.** The case \( k = 2 \) is just (26). Assume that the lemma holds when \( k = K \), then consider (28) with \( k = K + 1 \):
\[
\begin{align*}
\tilde{P}_1(z_1, z_K, z_{K+1}) & := \Gamma_1 z_1^m + \cdots + \Gamma_K z_K^m + \Gamma_{K+1} z_{K+1}^m = 0 \\
\tilde{P}_2(z_1, z_K, z_{K+1}) & := \Gamma_1 z_1^{m+1} + \cdots + \Gamma_K z_K^{m+1} + \Gamma_{K+1} z_{K+1}^{m+1} = 0 \\
& \vdots \\
\tilde{P}_{K+1}(z_1, z_K, z_{K+1}) & := \Gamma_1 z_1^{mK-1} + \cdots + \Gamma_K z_K^{mK-1} + \Gamma_{K+1} z_{K+1}^{mK-1} = 0 \\
\sum_{j \in I} \Gamma_j & \neq 0, \ I \subset \{1, \ldots, K + 1\}
\end{align*}
\]
(29)

(i) If all \( z_j \) are distinct, then \( \det T_n \neq 0 \) by (17), so that \( T_n \) is invertible. By left-multiplying both sides of (29) by \( T_n^{-1} \), it is transformed to
\[
Z_0 = 0
\]
(see (16), where \( Z_0 \) is as in (15). But this contradicts that all \( z_j \) are distinct.

(ii) If some \( z_j \) are equal, say, \( z_K = z_{K+1} \), then the sub-system consisting of the first \( K \) equations
\[
\begin{align*}
\tilde{P}_1(z_1, \ldots, z_K, z_K) & := \Gamma_1 z_1^m + \cdots + \Gamma_{K-1} z_{K-1}^m + (\Gamma_K + \Gamma_{K+1}) z_K^m = 0 \\
\tilde{P}_2(z_1, \ldots, z_K, z_K) & := \Gamma_1 z_1^{m+1} + \cdots + \Gamma_{K-1} z_{K-1}^{m+1} + (\Gamma_K + \Gamma_{K+1}) z_K^{m+1} = 0 \\
& \vdots \\
\tilde{P}_{K}(z_1, \ldots, z_K, z_K) & := \Gamma_1 z_1^{mK-1} + \cdots + \Gamma_{K-1} z_{K-1}^{mK-1} + (\Gamma_K + \Gamma_{K+1}) z_K^{mK-1} = 0 \\
\sum_{j \in I} \Gamma_j & \neq 0, \ I \subset \{1, \ldots, K + 1\}
\end{align*}
\]

already has no solution in \( \mathbb{C}^K \) due to the induction hypothesis (with \( \Gamma_K \) replaced by \( \Gamma_K + \Gamma_{K+1} \)), thus so does (29) in \( \mathbb{C}^{K+1} \). Similar for the other cases.

Therefore, in any case, (29) has no solution in \( \mathbb{C}^{K+1} \), and the lemma follows from induction. □

Finally, we are in a position to prove proposition 8. Recall that the case \( n = 2 \) has already been proved above. Now, assume that it holds for \( n = 2, \ldots, N - 1 \), then consider the case \( n = N \):
\[
\begin{align*}
P_1(z_1, \ldots, z_N) & = \sum_j \Gamma_j z_j^m + \sum_{j,r} A_j^{r} z_j^r = 0 \\
P_2(z_1, \ldots, z_N) & = \sum_j \Gamma_j z_j^{m+1} + \sum_{j,r} A_j^{2r} z_j^{r+1} + C_0 = 0 \\
P_k(z_1, \ldots, z_N) & = \sum_j \Gamma_j z_j^{m+k-1} + \sum_{j,r} A_j^{k} z_j^{r+k-1} \\
& + \sum_{r,s} C_{j,r} z_j^{k-2} + \sum_{r,s \neq 0, i < j} C_{i,j} z_i^r z_j^s = 0,
\end{align*}
\]
(30)

\( k = 3, \ldots, N; \sum_{j \in I} \Gamma_j \neq 0, \ I \subset \{1, \ldots, N\}, \ C_0 \neq 0. \)
Case 1: \((z_1, \ldots, z_N) \in \mathbb{C}_a^N\). Lemma 3 will show that \((30)\) has only finitely many solutions in \(\mathbb{C}_a^N\). To this end, consider, by \((19)\), the Newton polytopes
\[
\mathcal{N}_1 = \text{Conv}(\{me_1, \ldots, me_N\} \cup \cdots)
\]
\[
\mathcal{N}_2 = \text{Conv}(\{(m+1)e_1, \ldots, (m+1)e_N, 0\} \cup \cdots)
\]
\[
\mathcal{N}_3 = \text{Conv}(\{(m+2)e_1, \ldots, (m+2)e_N\} \cup \cdots)
\]
\[
\vdots
\]
\[
\mathcal{N}_N = \text{Conv}(\{(m+N-1)e_1, \ldots, (m+N-1)e_N\} \cup \cdots)
\]
of the supports \(A_1, \ldots, A_N\) of \(P_1, \ldots, P_N\), and their Minkowski sum of \(\mathcal{N} = \mathcal{N}_1 + \cdots + \mathcal{N}_N = \text{Conv}(\{a_N e_1, \ldots, a_N e_N\} \cup \cdots)\), where \(a_N\) is as in \((24)\). Note that this ‘\(\cdots\)’ does not alter the fact that
\[
\mathcal{N} \subset a_N \Delta_N.
\]

Now, consider the reduced systems of \((30)\) determined by all the \(\alpha\) or \(\ell_\alpha^N\) satisfying \((e_1 + \cdots + e_N) \cdot \alpha \leq 0\). Because of \((25)\) and such a choice of \(\alpha\), each \(\ell_\alpha^N\) actually supports \(\mathcal{N}\) at a face of the facet \(\text{Conv}(\{a_N e_1, \ldots, a_N e_N\}) = a_N \Delta_N\).

- **Case I.** \(\ell_\alpha^N\) supports \(\mathcal{N}\) at the 0-face
  \[
a_N e_j, \quad j = 1, \ldots, N,
  \]
  then \(\ell_\alpha^N, \ldots, \ell_\alpha^N\) support \(\mathcal{N}_1, \ldots, \mathcal{N}_N\) at the 0-faces
  \[
  me_j, \ldots, (m+N-1)e_j,
  \]
  respectively, giving the reduced system
  \[
  \begin{align*}
  \Gamma_j z_j^m &= 0 \\
  \Gamma_j z_j^{m+1} &= 0 \\
  \vdots \\
  \Gamma_j z_j^{m+N-1} &= 0 \\
  \Gamma_j &\neq 0,
  \end{align*}
  \]
  which has no solution in \(\mathbb{C}_a^N\).

- **Case II.** \(\ell_\alpha^N\) supports \(\mathcal{N}\) at the \((k-1)\)-face \((k = 2, \ldots, N)\)
  \[
  \text{Conv}(\{a_N e_j, \ldots, a_N e_j\}), \quad j_1 < \cdots < j_k,
  \]
  then \(\ell_\alpha^N, \ldots, \ell_\alpha^N\) support \(\mathcal{N}_1, \ldots, \mathcal{N}_N\) at the \((k-1)\)-faces
  \[
  \text{Conv}(\{me_j, \ldots, me_j\}), \ldots, \text{Conv}(\{(m+N-1)e_j, \ldots, (m+N-1)e_j\}),
  \]
  respectively, giving the reduced system
  \[
  \begin{align*}
  \tilde{P}_1(z_1, \ldots, z_N) &= \Gamma_{j_1} z_{j_1}^m + \cdots + \Gamma_{j_k} z_{j_k}^{m+k-1} = 0 \\
  \tilde{P}_2(z_1, \ldots, z_N) &= \Gamma_{j_1} z_{j_1}^{m+1} + \cdots + \Gamma_{j_k} z_{j_k}^{m+k} = 0 \\
  &\vdots \\
  \tilde{P}_k(z_1, \ldots, z_N) &= \Gamma_{j_1} z_{j_1}^{m+k-1} + \cdots + \Gamma_{j_k} z_{j_k}^{m+k-1} = 0 \\
  \vdots \\
  \tilde{P}_N(z_1, \ldots, z_N) &= \Gamma_{j_1} z_{j_1}^{m+N-1} + \cdots + \Gamma_{j_k} z_{j_k}^{m+N-1} = 0,
  \end{align*}
  \]
  where the sub-system consisting of the first \(k\) equations already has no solution in \(\mathbb{C}_a^k\) by lemma 10.
What we have established so far is that for every $\alpha$ or $\hat{\alpha}$ with $(e_1 + \cdots + e_N) \cdot \alpha \leq 0$, the corresponding reduced system has no solution in $\mathbb{C}_\alpha^N$. Hence, it follows from lemma 3 that (30) has only finitely many solutions in $\mathbb{C}_\alpha^N$.

**Case 2:** $(z_1, \ldots, z_N) \in \mathbb{C}_\alpha^N \times [0]^{N-k} (k = 2, \ldots, N - 1)$. The system (30) degrades to

$$
\begin{align*}
P_1(z_1, \ldots, z_k, 0, \ldots, 0) &= \sum_{j=1}^k \Gamma_j z_j^{n} + \sum_{j=1}^k \sum_{r} A_j^{1r} z_j^r = 0 \\
P_2(z_1, \ldots, z_k, 0, \ldots, 0) &= \sum_{j=1}^k \Gamma_j z_j^{m+1} + \sum_{j=1}^k \sum_{r} A_j^{2r} z_j^r + C_0 = 0 \\
P_k(z_1, \ldots, z_k, 0, \ldots, 0) &= \sum_{j=1}^k \Gamma_j z_j^{m+k-1} + \sum_{j=1}^k \sum_{r} A_j^{k} z_j^r + C_0 = 0 \\
&\quad + \sum_{j=1}^k C_j k - 2 + \sum_{i,j} \sum_a r \neq 0, \ i < j \\
&\quad C_{i,j} z_i z_j^r = 0,
\end{align*}
$$

where the sub-system consisting of the first $k$ equations already has only finitely many solutions in $\mathbb{C}^k$ by induction hypothesis, thus so does (31). Similar for the other cases where in $(z_1, \ldots, z_N)$ exactly $N - k$ coordinates equal 0.

**Case 3:** $(z_1, \ldots, z_N) \in \mathbb{C}_\alpha \times [0]^{N-1}$. The system (30) degrades to

$$
\begin{align*}
P_1(z_1, 0, \ldots, 0) &= \Gamma_1 z_1^{m} + \sum_{r} A_1^{1r} z_1^r = 0 \\
P_2(z_1, 0, \ldots, 0) &= \Gamma_1 z_1^{m+1} + \sum_{r} A_1^{2r} z_1^r + C_0 = 0 \\
P_k(z_1, 0, \ldots, 0) &= \Gamma_1 z_1^{m+k-1} + \sum_{r} A_1^{k} z_1^r + C_0 = 0 \\
&\quad + \sum_{j=1}^k C_j z_1^k - 2 + \sum_{i,j} \sum_a r \neq 0, \ i < j \\
&\quad C_{i,j} z_i z_j^r = 0,
\end{align*}
$$

where the first equation already has only finitely many (at most $m$) solutions, thus so does (32). Similar for the other cases where in $(z_1, \ldots, z_N)$ exactly $N - 1$ coordinates equal 0.

**Case 4:** $(z_1, \ldots, z_N) \in [0]^N$. The system (30) degrades to

$$
\begin{align*}
P_1(0, 0, \ldots, 0) &= \sum_{j} A_j^{10} = 0 \\
P_2(0, 0, \ldots, 0) &= C_0 = 0 \\
P_k(0, 0, \ldots, 0) &= 0, \\
&\quad k = 3, \ldots, N, \ C_0 \neq 0
\end{align*}
$$

and is simply inconsistent.

Combining all the above four cases, (30) has only finitely many solutions in $\mathbb{C}^N$, and proposition 8 follows from strong induction.

### 5. Background flow of degree one

By a background flow of degree one, we mean, without any loss of generality, $w(\zeta) = -\frac{\zeta c}{2\pi i}$ for some constant $c \in \mathbb{C}$ in (3) or $m = 1$ and $W \equiv c$ in (4) or (6) (the case with $c = 0$ is commonly called a quadrupole background flow). For whichever two distinct solutions $z = (z_1, \ldots, z_n)$ and $z' = (z'_1, \ldots, z'_n)$ of (3) with such $w$ to be equivalent, by (11), one looks for $(a, b) \in \mathbb{C}_\alpha \times \mathbb{C}$ such that

$$
aV_{z'}(a\zeta + b) = V_{z}(\zeta),
$$

$$
\sum_j \frac{\Gamma_j}{\zeta - z_j} - a^2 \zeta - ab - ac = \sum_j \frac{\Gamma_j}{\zeta - z_j} - \zeta - c.
$$

The equality between the two analytic parts already forces

$$(a, b) = (1, 0) \quad \text{or} \quad (-1, -2c).$$
Table 1. This table summarizes section 5.1, i.e. the case \( n = 2, m = 1 \) and \( W \equiv \) constant in (6). \( 'a / b / c' \) means that \( ' \) the system (6) has \( c \) solutions (counting multiplicity), \( b \) of which are distinct and admissible, and that these solutions constitute a fixed equilibrium configurations (in the sense of definition 4)' \( . '—' \) means non-existence of the case.

| \( \Gamma_1 + \Gamma_2 = 0 \) | \( \Gamma_1 + \Gamma_2 \neq 0 \) |
|-----------------------------|-----------------------------|
| \( \Gamma_1 = \Gamma_2 \) (1 species) | 1/2/2 |
| \( \Gamma_1 \neq \Gamma_2 \) (2 species) | 0/0/0 1/2/2 |

Table 2. This table summarizes section 5.2, i.e. the case \( n = 3, m = 1 \) and \( W \equiv 0 \) with \( \Gamma_1 = \Gamma_2 \) in (6). \( 'a / b / c' \) means that \( ' \) the system (6) has \( c \) solutions (counting multiplicity), \( b \) of which are distinct and admissible, and that these solutions constitute a fixed equilibrium configurations (in the sense of definition 4)' \( . '—' \) means non-existence of the case.

| \( \Gamma_1 = \Gamma_2 \) \( (\Gamma_1 + \Gamma_3)(2 \Gamma_1 + \Gamma_3) = 0 \) \( 5 \Gamma_1 + 4 \Gamma_3 = 0 \) (necessarily, \( (\Gamma_1 + \Gamma_3)(2 \Gamma_1 + \Gamma_3) \neq 0 \)) |
|-----------------------------|-------------|-----------------------------|
| \( \Gamma_1 = \Gamma_3 \) (1 species) | — | — |
| \( \Gamma_1 \neq \Gamma_3 \) (2 species) | 1/2/2 | 1/2/6 |

In section 5.1, through the simplest example of two vortices, we will illustrate that the bounds in proposition 2 and theorem 7 may or may not be attained. In this example, in addition, all the fixed equilibrium configurations come from the reduction, via definition 4, of the solutions of (3), (4) or (6) by symmetries of the vortex sets and/or the given background flow only. Section 5.2 will provide an example that a repeated solution of (3), (4) or (6) exists, thus suggesting another factor in the reduction of the solutions to fixed equilibrium configurations via definition 4. All these results will be summarized in tables 1 and 2.

5.1. Two vortices

For two vortices \( i.e. n = 2 \), (6) reads

\[
\begin{align*}
P_1(z_1, z_2) &= \Gamma_1 z_1 + \Gamma_2 z_2 + c(\Gamma_1 + \Gamma_2) = 0 \\
P_2(z_1, z_2) &= \Gamma_1 z_1^2 + \Gamma_2 z_2^2 + c \Gamma_1 z_1 + c \Gamma_2 z_2 - \Gamma_1 \Gamma_2 = 0 \\
&\quad \Gamma_1, \Gamma_2 \neq 0. \tag{35}
\end{align*}
\]

Computing by (the improved) Buchberger’s algorithm ( [17, algorithm GRÖBNERNEW2, p 232, and subalgorithm UPDATE, p 230]) with respect to the variable ordering \( z_1 > z_2 \), we obtain a Gröbner basis

\[ \mathcal{G} = \{P_1, Q_3\} , \quad \text{where } Q_3 := -\frac{\Gamma_1 + \Gamma_2}{\Gamma_1} z_2^2 - \frac{2c(\Gamma_1 + \Gamma_2)}{\Gamma_1} z_2 + \Gamma_1 - \frac{\Gamma_2(\Gamma_1 + \Gamma_2)}{\Gamma_1} \neq 0, \]

of \( \mathcal{F} = \{P_1, P_2\} \). Thus, (35) is equivalent to \( P_1 = Q_3 = 0, i.e. \)

\[
\begin{align*}
\Gamma_1 z_1 + \Gamma_2 z_2 + c(\Gamma_1 + \Gamma_2) &= 0 \\
-\frac{\Gamma_1 + \Gamma_2}{\Gamma_1} z_2^2 - \frac{2c(\Gamma_1 + \Gamma_2)}{\Gamma_1} z_2 + \Gamma_1 - \frac{\Gamma_2(\Gamma_1 + \Gamma_2)}{\Gamma_1} &= 0 \\
\Gamma_1, \Gamma_2 \neq 0. \tag{36}
\end{align*}
\]

Case 1. \( \Gamma_1 + \Gamma_2 = 0 \), then \( \Gamma_1 \neq \Gamma_2 \) because \( \Gamma_1, \Gamma_2 \neq 0 \), and then the second equation in (36) reads \( \Gamma_1 = 0 \) which already has no solution. But this case is beyond the scope of proposition 2 and theorem 7.
Case 2. \( \Gamma_1 + \Gamma_2 \neq 0 \), then the two solutions are
\[
(z_1, z_2) = \left( -c \pm \frac{\Gamma_2}{\sqrt{(\Gamma_1 + \Gamma_2)}}, -c \pm \frac{\Gamma_1}{\sqrt{(\Gamma_1 + \Gamma_2)}} \right) =: z^\pm,
\]
and the bound \( \frac{(1+2-1)!}{(1-1)!} = 2 \) in proposition 2 is attained. Moreover, \( z^\pm \) are admissible (see remark (iv) following definition 4) since \( \Gamma_1, \Gamma_2 \neq 0 \), and are distinct since \( \Gamma_1, \Gamma_2 \neq 0 \).

- Case I. \( \Gamma_1 = \Gamma_2 \) (one species), then
\[
2\pi i V_x(\xi) = \frac{\Gamma_1}{\xi + c - \sqrt{\frac{\Gamma_1}{\Gamma_1 + \Gamma_2}}} + \frac{\Gamma_1}{\xi + c + \sqrt{\frac{\Gamma_1}{\Gamma_1 + \Gamma_2}}} - \xi - c
\]
so that (33) with \( (a, b) = (1, 0) \) in (34) is satisfied, and \( z^\pm \) constitute only one fixed equilibrium configuration. In this case, the bound \( \frac{(1+2-1)!}{(1-1)!} = 1 \) in theorem 7 is attained.

- Case II. \( \Gamma_1 \neq \Gamma_2 \) (two species), then
\[
2\pi i V_x(\xi) = \frac{\Gamma_1}{\xi + c - \sqrt{\frac{\Gamma_1}{\Gamma_1 + \Gamma_2}}} + \frac{\Gamma_1}{\xi + c + \sqrt{\frac{\Gamma_1}{\Gamma_1 + \Gamma_2}}} - \xi - c
\]
so that (33) with \( (a, b) = (1, 0) \) in (34) is not satisfied, but
\[
-2\pi i V_x(-\xi - 2c) = -\frac{\Gamma_1}{\xi + c - \sqrt{\frac{\Gamma_1}{\Gamma_1 + \Gamma_2}}} = -\frac{\Gamma_2}{\xi + c + \sqrt{\frac{\Gamma_2}{\Gamma_1 + \Gamma_2}}}
\]
so that (33) with \( (a, b) = (-1, -2c) \) in (34) is satisfied, hence \( z^\pm \) still constitute only one fixed equilibrium configuration. In this case, the bound \( \frac{(1+2-1)!}{(1-1)!} = 2 \) in theorem 7 is not attained.

5.2. Repeated solution

There is a case of three vortices in a quadrupole background flow (i.e. \( n = 3 \) and \( w(\xi) = -\frac{\xi}{2\pi} \) in (3) or \( m = 1 \) and \( W \equiv 0 \) in (4) or (6)) where (3), (4) or (6) has a repeated solution. In such a case, the bound in theorem 7 must not be attained. Consider (6) with the extra assumption that \( \Gamma_1 = \Gamma_2 \) (at most two species):
\[
\begin{align*}
P_1(z_1, z_2, z_3) &= \Gamma_1 z_1 + \Gamma_1 z_2 + \Gamma_3 z_3 = 0 \\
P_2(z_1, z_2, z_3) &= \Gamma_1 z_1^2 + \Gamma_1 z_2^2 + \Gamma_3 z_3^2 - \Gamma_1^2 - 2 \Gamma_1 \Gamma_3 = 0 \\
P_3(z_1, z_2, z_3) &= \Gamma_1 z_1^3 + \Gamma_1 z_2^3 + \Gamma_3 z_3^3 - \Gamma_1 (\Gamma_1 + \Gamma_3) z_1 \\
-\Gamma_1 (\Gamma_1 + \Gamma_3) z_2 - 2 \Gamma_1 \Gamma_3 z_3 = 0
\end{align*}
\]

(37)

Computing by the aforesaid Buchberger’s algorithm with respect to the variable ordering \( z_1 > z_2 > z_3 \), we obtain a Gröbner basis
\[
G = \{ P_1, Q_2, Q_3 \}.
\]
where
\[ Q_4 := -2 \Gamma_1 z_2^2 - 2 \Gamma_3 z_2 z_3 - \frac{\Gamma_3 (\Gamma_1 + \Gamma_3)}{\Gamma_1} z_3^2 + \Gamma_1 (\Gamma_1 + 2 \Gamma_3) \neq 0 \quad \text{and} \]
\[ Q_5 := -\frac{\Gamma_3 (\Gamma_1 + \Gamma_3)(2 \Gamma_1 + 3 \Gamma_3)}{2 \Gamma_1^2} z_3^3 + \frac{\Gamma_3 (5 \Gamma_1 + 4 \Gamma_3)}{2} z_3 \neq 0, \]
of \( \mathcal{F} = \{ P_1, P_2, P_3 \} \). Thus, (37) is equivalent to \( P_1 = Q_4 = Q_5 = 0 \), i.e.
\[
\begin{align*}
\Gamma_1 z_1 + \Gamma_1 z_2 + \Gamma_3 z_3 &= 0 \\
-2 \Gamma_1 z_2^2 - 2 \Gamma_3 z_2 z_3 - \frac{\Gamma_3 (\Gamma_1 + \Gamma_3)}{\Gamma_1} z_3^2 + \Gamma_1 (\Gamma_1 + 2 \Gamma_3) &= 0 \\
-\frac{\Gamma_3 (\Gamma_1 + \Gamma_3)(2 \Gamma_1 + 3 \Gamma_3)}{2 \Gamma_1^2} z_3^3 + \frac{\Gamma_3 (5 \Gamma_1 + 4 \Gamma_3)}{2} z_3 &= 0
\end{align*}
\]
(38)

**Case 1.** \( (\Gamma_1 + \Gamma_3)(2 \Gamma_1 + \Gamma_3) = 0 \), then \( \Gamma_1 \neq \Gamma_3 \) because \( \Gamma_1, \Gamma_3 \neq 0 \), so that there are two species. And then writing \( \Gamma_3 = -\alpha \Gamma_1 \), where \( \alpha = 1 \) or 2, (38) reads
\[
\begin{align*}
\Gamma_1 z_1 + \Gamma_1 z_2 - \alpha \Gamma_1 z_3 &= 0 \\
-2 \Gamma_1 z_2^2 + 2 \alpha \Gamma_1 z_2 z_3 + \alpha (1 - \alpha) \Gamma_1 z_3^2 + (1 - 2\alpha) \Gamma_1^2 &= 0 \\
-\alpha (5 - 4\alpha) \Gamma_1^2 z_3 &= 0
\end{align*}
\]
(38)

The two solutions
\[
(z_1, z_2, z_3) = \left( \pm \sqrt{\frac{(1 - 2\alpha)\Gamma_1}{2}}, \pm \sqrt{\frac{(1 - 2\alpha)\Gamma_1}{2}}, 0 \right) =: z^\pm
\]
are admissible and distinct, both because of \( \Gamma_1 \neq 0 \) and \( \alpha \neq \frac{1}{2} \). Moreover,
\[
2\pi i V^-_z(\zeta) = \frac{\Gamma_1}{\zeta - \sqrt{\frac{(1-2\alpha)\Gamma_1}{2}}} + \frac{\Gamma_1}{\zeta + \sqrt{\frac{(1-2\alpha)\Gamma_1}{2}}} - \frac{-\alpha \Gamma_1}{\zeta} - \zeta
\]
\[
= \frac{\Gamma_1}{\zeta + \sqrt{\frac{(1-2\alpha)\Gamma_1}{2}}} + \frac{\Gamma_1}{\zeta - \sqrt{\frac{(1-2\alpha)\Gamma_1}{2}}} - \frac{-\alpha \Gamma_1}{\zeta} - \zeta = 2\pi i V^-_z(\zeta)
\]
so that (33) with \( (a, b) = (1, 0) \) in (34) is satisfied, and they constitute only one fixed equilibrium configuration. But this case is beyond the scope of proposition 2 and theorem 7.

**Case 2.** \( (\Gamma_1 + \Gamma_3)(2 \Gamma_1 + \Gamma_3) \neq 0 \), then, first of all, the third equation of (38) has three solutions
\[
z_3 = 0, \quad \pm \sqrt{\frac{5 \Gamma_1 + 4 \Gamma_3}{(\Gamma_1 + \Gamma_3)(2 \Gamma_1 + \Gamma_3)}}
\]
Each leads to two \( z_2 \) via the second equation in (38) and then one \( z_1 \) via the first, resulting in six solutions of (38), thus the bound \( \frac{(1+1)(3+1)(3+1)!}{(1-1)(3+1)} = 6 \) in proposition 2 is attained. Now, one could observe that if
- \( 5 \Gamma_1 + 4 \Gamma_3 = 0 \), then \( \Gamma_1 \neq \Gamma_3 \) because \( \Gamma_1, \Gamma_3 \neq 0 \), so that there are two species. Now, \( z_3 = 0 \) is actually a triple zero, and the six solutions of (38) are \( (z_1, z_2, z_3) = z^\pm := \left( \pm \sqrt{\frac{\Gamma_1 + 2 \Gamma_3}{2}}, \pm \sqrt{\frac{\Gamma_1 + 2 \Gamma_3}{2}}, 0 \right) \) (each repeated thrice).
Note that $5 \Gamma_1 + 4 \Gamma_3 = 0$ with $\Gamma_1, \Gamma_3 \neq 0$ implies that $\Gamma_1 + 2 \Gamma_3 \neq 0$, so $z^\pm$ are admissible and distinct. Moreover,

$$2\pi i V_z(\xi) = \frac{\Gamma_1}{\xi - \sqrt{\Gamma_1 + 2 \Gamma_3}} + \frac{\Gamma_1}{\xi + \sqrt{\Gamma_1 + 2 \Gamma_3}} - \frac{-\frac{5}{4} \Gamma_1}{\xi} - \xi$$

so that (33) with $(a, b) = (1, 0)$ in (34) is satisfied, and they constitute only one fixed equilibrium configuration. Thus, the bound $\frac{1 + 3 - 1}{4} \Gamma_2 \frac{1 + 3 - 1}{4} = 3$ in theorem 7 is not attained.

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