Radiation pressure approach to the repulsive Casimir force*

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We study the Casimir force between a perfectly conducting and an infinitely permeable plate with the radiation pressure approach. This method illustrates how a repulsive force arises as a consequence of the redistribution of the vacuum-field modes corresponding to specific boundary conditions. We discuss also how the method of the zero-point radiation pressure follows from QED.

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‘Understanding of signs is a sign of understanding.’ J. Sucher [1]

At the end of his work on a force between two perfectly conducting parallel plates in vacuum as a consequence of the change in zero-point energy [2], Casimir proposed that this force can be interpreted as radiation pressure from the vacuum field. Later this interpretation was reintroduced by several other authors [3 - 5] and used [4, 5] for calculation of the Casimir force in such a case. As was mentioned in [5], the radiation pressure approach can be systematically developed on the basis of QED. Actually this was already shown in 1969 by L. Brown and J. Maclay [6] (who also used this approach to compute the attractive Casimir force). We will discuss this question in some detail in Appendix.

P. Milonni, R. Cook and M. Goggin [5] noticed a puzzling character of the vacuum radiation pressure: In the case of two perfectly conducting plates the modes of the vacuum field in the space outside the plates form a continuum, corresponding to arbitrary wave vector $k$, whereas those inside are restricted to discrete values of $k_z$ (with the $z$-axis perpendicular to the plates). So there are ”more” modes outside to push the plates together by the radiation pressure than there are modes between the plates to push them apart. This results in the attractive Casimir force\(^2\). However, they concluded that this argument is superficial, since it cannot explain a radially outward Casimir force, known in the case of a spherically conducting shell, in spite of the fact that there also should be ”fewer” modes of the vacuum field inside the shell

\(^1\)Quotation marks since we compare infinite numbers!

\(^2\)This argument is also given in [7] and [8].
than there are outside.

In order to resolve the puzzle we should look more carefully to how boundaries affect the zero-point field. Doing this one can notice that, loosely speaking, as a whole there are no fewer normal modes inside confined volumes than there are in free space; they just are shifted to other frequencies. As it turns out this question was discussed in a rigorous way by G. Barton [9 - 10] and mentioned in [11]. Not knowing his results I came to the same conclusion from the following considerations. Consider, e. g., the space between two perfectly conducting plates separated by the distance \( l \). As was mentioned above, the vacuum electromagnetic field is here a sum of modes (standing waves) permitted by the boundary conditions. Each such a mode is mathematically equivalent to a harmonic oscillator of the same frequency. When the distance between plates is very large we actually have the free space case. If now one adiabatically moves the walls toward each other, the ”electromagnetic oscillators” will remain in the ground state; only their frequencies will be shifted to values corresponding to the changed boundary conditions [12]. Thus, the main effect of the boundaries is to redistribute normal modes: for some frequencies \( \omega \) there are more modes than in free space, for others there are fewer.

One can check this conclusion by comparison of the mode spectral densities. In free space the mode spectral density \( \rho(\omega) \) in a volume \( V \) is

\[
\rho_0(\omega) = \frac{V \omega^2}{\pi^2 c^3},
\]

while inside the cavity between two perfectly conducting plates [13], where
\( k_z = n\pi/l, \ n = 1, 2, ... \)

\[
\rho_1(\omega) = \rho_0(\omega) \frac{\omega_0}{\omega} [1/2 + \sum_{m=1}^{\infty} \theta(\frac{\omega}{\omega_0} - m)].
\]

(2)

Here \( c \) is velocity of light, \( \omega_0 = c\pi/l \), and \( \theta \) is the Heaviside step function.

In the case of a perfectly conducting plate parallel to an infinitely permeable plate, with the same separation \( l \), \( k_z = (n + 1/2)\pi/l, \ n = 0, 1, 2, ... \) [14] and therefore the mode spectral density is

\[
\rho_2(\omega) = \rho_0(\omega) \frac{\omega_0}{\omega} \sum_{m=0}^{\infty} \theta(\frac{\omega}{\omega_0} - m + 1/2).
\]

(3)

One can see that for about a half of the frequencies \( \rho_1(\omega) \) and \( \rho_2(\omega) \) are greater than \( \rho_0(\omega) \).

It is just a result of the redistribution of normal modes that the pressure from the inside vacuum field \( P_{\text{out}} \) is different than the oppositely directed pressure from a free space vacuum field \( P_{\text{in}} \). As noted above, for two conducting plates \( P_{\text{in}} > P_{\text{out}} \). However, since the redistribution of modes depends on boundary conditions, the relation between two pressures in other cases can be opposite.

As an example of such a case let us consider a perfectly conducting plate parallel to an infinitely permeable plate, mentioned above. If a plane wave has an angle of incidence \( \theta \) the radiation pressure exerted by such a wave on a plane, \( P = 2w \cos^2(\theta) \), where \( w \) is the energy density. So a vacuum field mode of frequency \( \omega \), which has an angle of incidence \( \theta \), makes a contribution to the pressure

\[
P(\omega) = 2 \frac{1}{2^2} \frac{h\omega}{2V^2} \cos^2(\theta) = \frac{h\omega}{2V} \left( \frac{k_z}{k} \right)^2,
\]

(4)
where \( k = wc \) and \( V \) is a quantization volume. A factor \( 1/2 \) has been inserted because the zero-point energy of each mode is divided equally between waves propagating toward or away from each plate [5].

Therefore we find for the net pressure \( P = P_{\text{out}} - P_{\text{in}} \), where \( P_{\text{out}} \) and \( P_{\text{in}} \) are vacuum radiation pressures directed outward and inward, correspondingly, the expression

\[
P = \left( \frac{\hbar c}{\pi^2 l^2} \right) \sum_{n=0}^{\infty} \frac{1}{2} \int_0^\infty dk_x \int_0^\infty dk_y \frac{[n + 1/2 \pi/l]^2}{(k_x^2 + k_y^2 + [n + 1/2 \pi/l]^2)^{1/2}}
\]

\[
- \left( \frac{\hbar c}{\pi^3} \right) \int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z \frac{k_z^2}{(k_x^2 + k_y^2 + k_z^2)^{3/2}},
\]

(5)

which can be transformed using variables \( s \equiv (l/\pi)^2(k_z^2 + k_y^2) \) and \( u \equiv (l/\pi)k_z \) into the form

\[
P = \left( \frac{\hbar c \pi}{4 l^4} \right) \sum_{n=0}^{\infty} (n + 1/2)^2 \int_0^\infty ds \frac{1}{(s + (n + 1/2)^2)^{1/2}}
\]

\[
- \int_0^\infty du u^2 \int_0^\infty ds \frac{1}{(s + u^2)^{1/2}}.
\]

(6)

In order to regularize the divergent integrals in (6) a cutoff function \( f_a([s + u^2]^{1/2}) \) must be introduced, with \( f_a \to 1 \) when the parameter \( a \) tends to, say, zero:

\[
\lim_{a \to 0} f_a(p) \to 1,
\]

(7)

here \( p \equiv ([s + u^2]^{1/2}) \). Further, one requires that \( f_a(p) \) vanish rapidly enough for \( y \to \infty \),

\[
\lim_{p \to \infty} f_a(p) \to 0.
\]

(8)

so that the function

\[
G_a(u) \equiv u^2 \int_0^\infty ds \frac{f_a([s + u^2]^{1/2})}{(s + u^2)^{1/2}}
\]

(9)
is finite and \( G_a(\infty) = 0 \). We can thus rewrite (6) as

\[
P = \lim_{a \to 0} \left( \frac{\hbar c \pi}{4l^4} \sum_{n=0}^{\infty} G_a(n + 1/2) - \int_{0}^{\infty} du G_a(u) \right).
\]  

(10)

Using the Euler-Maclaurin formula [15] one can find [12] that in the limit \( a \to 0 \) the difference in pressure is finite, independent of cutoff, and reduces to

\[
P = \frac{7}{8} \frac{\hbar c \pi}{240l^4}.
\]  

(11)

This coincides with the expression for the Casimir force for such a system obtained by Boyer [14], who used the energy difference method and a special (exponential) form of cutoff function.

Let us discuss shortly as to why the net vacuum radiation pressure has positive sign in the case under consideration. Let \( H_a(u) \equiv G_a(n + 1/2) \) for \( n < u < n + 1 \), where \( n \) is an integer. It follows from (10) that the net sign of the vacuum radiation pressure depends on whether the area under the step function \( H_a(u) \) is larger or smaller than the area under \( G_a(u) \).

It is not difficult to see that if the curvature of \( G_a(u) \) is zero for \( n < u < n + 1 \), the area difference between \( H_a(u) \) and \( G_a(u) \) in this region is equal to zero. If the curvature is not zero the area difference will be bigger or smaller than these values depending on the sign of \( d^2G_a/du^2 \), i.e. on whether in the considered region \( G_a(u) \) is concave or convex.

One can check that \( G_a(u) \) is primarily convex for any acceptable cutoff function \( f_a \). So the net area beneath \( H_a(u) \) is larger than that beneath \( G_a(u) \) and, therefore, in the case of one conducting and one permeable plate \( P_{out} - P_{in} > 0 \) and we have repulsion.
So we showed in a very simple case how zero-point radiation pressure can lead to a repulsive rather than attractive Casimir force. It is the precise distribution of normal mode frequencies associated with specific boundary conditions, together with the fact that the function $F_a(u)$ is primarily convex, that determines the sign of the force.

**Appendix**

Let us show how the method of the zero-point radiation pressure follows from QED and that it is equivalent to the method of the change in the energy of quantum fluctuations of the electromagnetic field.

It follows from the operator Maxwell’s equations that $g$, the operator for the momentum density of the electromagnetic field is

$$g = \frac{1}{8\pi}(E \times B - B \times E),$$

where $E = E(r, t)$ and $B = B(r, t)$ are the Heisenberg operators of the electric and magnetic fields. $g$ obeys the equation of continuity

$$\frac{\partial g^i}{\partial t} + \frac{\partial T^{ij}}{\partial x^j} = 0,$$

where $T^{ij}$, $i, j = 1, 2, 3$ is the spatial part of the 4-dim. energy-momentum tensor operator $T^{\mu\lambda}$, $\mu, \lambda = 0, 1, 2, 3$ of the electromagnetic field, $x^0 = ct$, $x^1 = x, x^2 = y, x^3 = z$. $T^{\mu\lambda}$ is a certain combination of electric and magnetic fields components and, therefore, is an operator function of $x^\mu$, the form of which depends on the problem. We will consider below $T^{\mu\lambda}$ renormalized in
such a way, that its expectation value in the ground state of the electromagnetic field, $< T^\mu_\lambda >$, is zero in free space.\footnote{The following is basically a slightly modified derivation given in [6].}

Eq. (13) describes the local conservation of the field momentum. Integrating $\partial g^i / \partial t$ over a volume $V$ and using the divergence theorem one can find the force exerted on this volume from the internal electromagnetic field. The expectation value for a force component, $F^i$ is

$$ F^i = \int dv < \frac{\partial g^i}{\partial t} > = - \int dv < \frac{\partial T^{ij}}{\partial x^j} > = \int_\Sigma < T^{ij} > n^i da, $$

where $n$ is a unit outward normal to the surface which surrounds volume $V$ and $da$ is an element of the surface area.

Let us show that $F^i$ determined by (14) coincides with the force determined by the zero-point energy method:

$$ F^i = - \frac{\delta \varepsilon}{\delta x^i}, $$

where $\delta \varepsilon$ is a infinitesimal change in the zero-point energy of the electromagnetic field in the volume $V$ under the influence of a virtual displacement $\delta x^i$ of its surface.

Consider for simplicity the case of two parallel infinite plates separated by a distance $l$, with the $z$-axis perpendicular to them. Each plate is either a perfect conductor or is infinitely permeable. As follows from dimensional considerations, the energy per unite area of the electromagnetic field between the plates, $\varepsilon$, is

$$ \varepsilon \propto l^{-3}. $$

\footnote{The following is basically a slightly modified derivation given in [6].}
For a displacement of one of the plates along the $z$-axis $\delta x^3 \equiv \delta z = \delta l$. That is why the force per unit area acting on a plate in terms of the energy density $w = \varepsilon/l$ is

$$F_z = -\frac{\delta \varepsilon}{\delta l} = 3w. \quad (17)$$

At the same time as follows from (14)

$$F_z = \int_\Sigma < T^{33} > n^3 \, da, \quad (18)$$

where $\Sigma$ is a surface of a plate, which has a unit area. It follows from the requirements $T^\mu_\mu = 0$, $\partial_\mu T^{\mu\lambda} = 0$ and the symmetry of the problem that for the case under consideration [6]

$$< T^{\mu\lambda} >= C \left( \frac{1}{4} g^{\mu\lambda} - \hat{x}^\mu \hat{x}^\lambda \right) \quad (19)$$

where $C$ is a constant, the metric $g^{\mu\lambda}$ has the signature $(-1,1,1,1)$, and $\hat{x}^\mu = (0,0,0,1)$ is a unit vector along the $z$ axis. Thus, since $g^{00} = -1$ and $g^{33} = 1$,

$$w \equiv < T^{00} >= -\frac{1}{4} C, \quad (20)$$

and

$$T^{33} = \left( \frac{1}{4} g^{33} - 1 \right) C = -\frac{3}{4} C. \quad (21)$$

So, taking into account that $< T^{33} >$ is a constant we have from (19)

$$F_z = < T^{33} >= -\frac{3}{4} C. \quad (22)$$

Finally, from the comparison of this expression with (20) it follows that

$$F_z = 3w, \quad (23)$$
which coincides with (17).

In the case of two perfectly conducting or two infinitely permeable plates the renormalized energy density $w$ is negative. So, as follows from (23), the force is attractive. In the case of a perfectly conducting plate parallel to an infinitely permeable plate $w$ is positive and therefore the force is repulsive.

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