CONTROLLED TOPOLOGY IN GEOMETRY

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The purpose of the present note is to announce some finiteness theorems for classes of Riemannian manifolds (cf. A, B and D below).

Let \( \mathcal{M}_{k,d,v}^{K,D,v}(n) \) denote the class of closed Riemannian \( n \)-manifolds with sectional curvatures between \( k \) and \( K \), diameter between \( d \) and \( D \), and volume between \( v \) and \( V \). Here \( k \leq K \) are arbitrary, \( 0 < d < D \), and \( 0 < v < V \).

**Theorem A.** For \( n \neq 3, 4 \) the class \( \mathcal{M}_{k,0,v}^{\infty,D,\infty}(n) \) contains at most finitely many diffeomorphism types.

This unifies and generalizes the following two theorems in high dimensions.

**Theorem (J. Cheeger [C, P]).** The class \( \mathcal{M}_{k,0,v}^{\infty,K,\infty}(n) \) contains at most finitely many diffeomorphism types.

**Theorem (K. Grove, P. Petersen [GP]).** The class \( \mathcal{M}_{k,0,v}^{\infty,D,\infty} \) contains at most finitely many homotopy types.

For \( k > 0 \) and \( n = 3 \), the conclusion in Theorem A follows by Hamilton’s theorem in [H]. For \( k > 0 \) and \( n = 4 \) the fundamental group is either trivial or \( \mathbb{Z}_2 \) by Synge’s theorem. Using Freedman’s classification of simply connected topological 4-manifolds together with the above theorem and standard surgery theory then yields (cf. also [HK]).

**Corollary B.** For \( k > 0 \) the class \( \mathcal{M}_{k,0,v}^{\infty,D,\infty}(n) \) contains at most finitely many diffeomorphism (resp. homeomorphism) types when \( n \neq 4 \) (resp. \( n = 4 \)).
for large $k$, where $\text{diam } f_k = \sup\{\text{diam } f_k^{-1}(x) | x \in X\} \rightarrow 0$ and $\text{diam } g_k \rightarrow 0$ as $k \rightarrow \infty$, cf. [PV]. In particular, $X$ is $n$-dimensional.

It is a fairly easy consequence of results due to Begle [B] that $X = \lim M_k$ must be a homology manifold.

Combining all these properties of $X$ allows us to apply a result of F. Quinn [Q] to conclude that $X$ admits a resolution for $n \geq 4$. If in addition $n \geq 5$, and $X$ satisfies the disjoint disc property (DDP), it must be a topological manifold according to a theorem of R. D. Edwards, cf. [E, D]. To see that $X = \lim M_k$ indeed satisfies the DDP, one uses the deformations associated with $M_k$ together with the homotopy equivalences $f_k, g_k$. Hence

**Theorem C.** For $n \geq 5$ any compact metric space in the Gromov-Hausdorff closure of $\mathcal{M}_{k,0,v}^{\infty, D, \infty}(n)$ is a topological $n$-manifold.

Having shown that $X$ is a topological manifold of dimension $\geq 5$ a result of T. A. Chapman and S. Ferry [CF, F] implies that for $k$ sufficiently large $g_k$ can be deformed to a homeomorphism. Since the closure of $\mathcal{M}_{k,0,v}^{\infty, D, \infty}(n)$ is compact, cf. [G], we conclude that this class contains at most finitely many homeomorphism types. By a general result of R. Kirby and L. Siebenmann [KS], Theorem A follows.

The same argument as outlined above yields finiteness for diffeomorphism types rather than homotopy types in a finiteness theorem by T. Yamaguchi, cf. [Y]. In particular,

**Theorem D.** For $n \geq 5$ the class of closed $n$-manifolds with injectivity radius bounded from below and volume from above, contains at most finitely many diffeomorphism types.

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