Research article

On generalizations of quantum Simpson’s and quantum Newton’s inequalities with some parameters

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Abstract: In this paper, we prove two identities concerning quantum derivatives, quantum integrals, and some parameters. Using the newly proved identities, we prove new Simpson’s and Newton’s type inequalities for quantum differentiable convex functions with two and three parameters, respectively. We also look at the special cases of our key findings and find some new and old Simpson’s type inequalities, Newton’s type inequalities, midpoint type inequalities, and trapezoidal type inequalities.

Keywords: Simpson’s inequalities; Newton’s inequalities; quantum calculus; convex functions

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1. Introduction

Simpson’s rules are well-known methods for numerical integration and numerical estimation of definite integral. Thomas Simpson is credited with inventing this process (1710–1761). However, about 100 years earlier, Johannes Kepler used the same approximation, so this form is also known as Kepler’s law. The three-point Newton-Cotes quadrature rule is included in Simpson’s rule, so estimation based on three steps quadratic kernel is often referred to as Newton type results.
1) Simpson’s quadrature formula (Simpson’s 1/3 rule)
\[
\int_{\pi_1}^{\pi_2} \Pi(x) dx \approx \frac{\pi_2 - \pi_1}{6} \left[ \Pi(\pi_1) + 4\Pi \left( \frac{\pi_1 + \pi_2}{2} \right) + \Pi(\pi_2) \right].
\]

2) Simpson’s second formula or Newton-Cotes quadrature formula (Simpson’s 3/8 rule).
\[
\int_{\pi_1}^{\pi_2} \Pi(x) dx \approx \frac{\pi_2 - \pi_1}{8} \left[ \Pi(\pi_1) + 3\Pi \left( \frac{2\pi_1 + \pi_2}{3} \right) + 3\Pi \left( \frac{\pi_1 + 2\pi_2}{3} \right) + \Pi(\pi_2) \right].
\]

In the literature, there are several estimations linked to these quadrature laws, one of which is known as Simpson’s inequality:

**Theorem 1.1.** Suppose that \( \Pi : [\pi_1, \pi_2] \to \mathbb{R} \) is a four times continuously differentiable mapping on \( (\pi_1, \pi_2) \), and let \( \|\Pi^{(4)}\|_{\infty} = \sup_{x \in (\pi_1, \pi_2)} |\Pi^{(4)}(x)| < \infty \). Then, one has the inequality
\[
\left| \frac{1}{3} \left[ \Pi(\pi_1) + \Pi(\pi_2) \right] + 2\Pi \left( \frac{\pi_1 + \pi_2}{2} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) dx \right| \leq \frac{1}{2880} \|\Pi^{(4)}\|_{\infty} (\pi_2 - \pi_1)^4.
\]

Many authors have concentrated on Simpson’s type inequalities for different classes of functions in recent years. Since convexity theory is an effective and efficient method for solving a large number of problems that exist within various branches of pure and applied mathematics, some mathematicians have worked on Simpson’s and Newton’s type results for convex mappings. Dragomir et al. [1], presented new Simpson’s type inequalities and their applications to numerical integration quadrature formulas. Furthermore, Alomari et al. in [2] derive some Simpson’s type inequalities for s-convex functions. Following that, in [3], Sarikaya et al. discovered variants of Simpson’s type inequalities dependent on convexity. The authors given some Newton’s type inequalities for harmonic and \( p \)-harmonic convex functions in [4, 5]. Iftekhar et al. also have new Newton’s type inequalities for functions whose local fractional derivatives are generalized convex in [6].

On the other hand, in the domain of \( q \) analysis, many works are being carried out as initiated by Euler in order to attain adeptness in mathematics that constructs quantum computing \( q \) calculus considered as a relationship between physics and mathematics. In different areas of mathematics, it has numerous applications such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, as well as mechanics, the theory of relativity, and quantum theory [7, 8]. Quantum calculus also has many applications in quantum information theory, which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [9, 10]. Apparently, Euler invented this important branch of mathematics. He used the \( q \) parameter in Newton’s work on infinite series. Later, in a methodical manner, the \( q \)-calculus, calculus without limits, was firstly given by Jackson [11, 12]. In 1966, Al-Salam [13] introduced a \( q \)-analogue of the \( q \)-fractional integral and \( q \)-Riemann–Liouville fractional. Since then, related research has gradually increased. In particular, in 2013, Tariboon [14] introduced the \( \pi_1D_q \)-difference operator and \( q_{\pi_1} \)-integral. In 2020, Bermudo et al. [15] introduced the notion of \( \pi D_q \)-derivative and \( q^{\pi}\)-integral. Sadjaq [16] generalized to quantum calculus and introduced the notions of post-quantum calculus, or briefly \((p, q)\)-calculus. Soontharanon et al. [17] introduced the fractional \((p, q)\)-calculus later on. In [18], Tunç and Göv gave the post-quantum variant of \( \pi D_q \)-difference operator and \( q_{\pi_1} \)-integral. Recently, in 2021, Chu et al. [19] introduced the notions of \( \pi D_{p,q} \) derivative and \((p, q)^{\pi}\)-integral.
Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [15, 20–27], the authors used $\pi_1 D_q^{\pi_2} D_q$-derivatives and $q_{\pi_1}, q^{\pi_2}$-integrals to prove Hermite–Hadamard integral inequalities and their left–right estimates for convex and coordinated convex functions. In [28], Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, Nwaeze et al. proved certain parameterized quantum integral inequalities in [29]. Khan et al. proved quantum Hermite–Hadamard inequality using the green function in [30]. Budak et al. [31], Ali et al. [32, 33], and Vivas-Cortez et al. [34] developed new quantum Simpson’s and quantum Newton’s type inequalities for convex and coordinated convex functions. For quantum Ostrowski’s inequalities for convex and co-ordinated convex functions, one can consult [35–38]. Kunt et al. [39] generalized the results of [22] and proved Hermite–Hadamard-type inequalities and their left estimates using $\pi_1 D_p^{\pi_2} D_q$-difference operator and $(p, q)_{\pi_1}$ integral. Recently, Latif et al. [40] found the right estimates of Hermite–Hadamard type inequalities proved by Kunt et al. [39]. To prove Ostrowski’s inequalities, Chu et al. [19] used the concepts of $\pi_2 D_p^{\pi_2}$-difference operator and $(p, q)_{\pi_2}$ integral.

Inspired by this ongoing studies, we offer some new quantum parameterized Simpson’s and Newton’s type inequalities for convex functions using the notions of quantum derivatives and integrals.

The structure of this paper is as follows: Section 2 provides a quick review of the ideas of $q$-calculus, as well as some related works. In Section 3, we present two integral identities that aid in the proof of the key conclusions. We prove quantum Simpson’s and quantum Newton’s inequalities in sections 4 and 5, respectively. Section 6 finishes with a few suggestions for future research.

2. Preliminaries of $q$-calculus and some inequalities

In this section, we first present some known definitions and related inequalities in $q$-calculus. Set the following notation (see, [8]):

$$[n]_q = \frac{1 - q^n}{1 - q} = \sum_{k=0}^{n-1} q^k, \quad q \in (0, 1).$$

Jackson [11] defined the $q$-integral of a given function $\Pi$ from 0 to $\pi_2$ as follows:

$$\int_0^{\pi_2} \Pi(x) \, dq x = (1 - q) \pi_2 \sum_{n=0}^{\infty} q^n \Pi(\pi_2 q^n), \quad \text{where } 0 < q < 1 \quad (2.1)$$

provided that the sum converges absolutely. Moreover, he defined the $q$-integral of a given function over the interval $[\pi_1, \pi_2]$ as follows:

$$\int_{\pi_1}^{\pi_2} \Pi(x) \, dq x = \int_0^{\pi_2} \Pi(x) \, dq x - \int_0^{\pi_1} \Pi(x) \, dq x.$$

**Definition 2.1.** [14] We consider the mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$. Then, the $q_{\pi_1}$-derivative of $\Pi$ at $x \in [\pi_1, \pi_2]$ is defined by the following expression

$$\pi_1 D_q \Pi(x) = \frac{\Pi(x) - \Pi(qx + (1 - q) \pi_1)}{(1 - q)(x - \pi_1)}, \quad x \neq \pi_1. \quad (2.2)$$
If \( x = \pi_1 \), we define \( \pi_1 D_q \Pi (\pi_1) = \lim_{x \to \pi_1} \pi_1 D_q \Pi (x) \) if it exists and it is finite.

**Definition 2.2.** [15] We consider the mapping \( \Pi : [\pi_1, \pi_2] \to \mathbb{R} \). Then, the \( q^{\pi_1} \)-derivative of \( \Pi \) at \( x \in [\pi_1, \pi_2] \) is defined by

\[
\pi_1 D_q \Pi (x) = \frac{\Pi (q x + (1 - q) \pi_2) - \Pi (x)}{(1 - q) (\pi_2 - x)}, \quad x \neq \pi_2. \tag{2.3}
\]

If \( x = \pi_2 \), we define \( \pi_2 D_q \Pi (\pi_2) = \lim_{x \to \pi_2} \pi_2 D_q \Pi (x) \) if it exists and it is finite.

**Definition 2.3.** [14] We consider the mapping \( \Pi : [\pi_1, \pi_2] \to \mathbb{R} \). Then, the \( q^{\pi_2} \)-definite integral on \([\pi_1, \pi_2]\) is defined by

\[
\pi_2 \int_{\pi_1}^{\pi_2} \Pi (x) \pi_2 d_q x = (1 - q) (\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n \Pi (q^n \pi_2 + (1 - q^n) \pi_1) \tag{2.4}
\]

\[= (\pi_2 - \pi_1) \int_{0}^{1} \Pi ((1 - \tau) \pi_1 + \tau \pi_2) \, d_q \tau.
\]

**Remark 2.1.** If we set \( \pi_1 = 0 \) in Definition 2.3, then we obtain \( q \)-Jackson integral, which is given in expression (2.1).

In [22, 27], the authors proved quantum Hermite-Hadamard type inequalities and their estimations by using the notions of \( q^{\pi_1} \)-derivative and \( q^{\pi_2} \)-integral.

On the other hand, in [15], Bermudo et al. gave the following definition and obtained the related Hermite-Hadamard type inequalities:

**Definition 2.4.** [15] We consider the mapping \( \Pi : [\pi_1, \pi_2] \to \mathbb{R} \). Then, the \( q^{\pi_2} \)-definite integral on \([\pi_1, \pi_2]\) is defined by

\[
\pi_2 \int_{\pi_1}^{\pi_2} \Pi (x) \pi_2 d_q x = (1 - q) (\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n \Pi (q^n \pi_1 + (1 - q^n) \pi_2) \tag{2.4}
\]

\[= (\pi_2 - \pi_1) \int_{0}^{1} \Pi (\tau \pi_1 + (1 - \tau) \pi_2) \, d_q \tau.
\]

**Theorem 2.1.** [15] Let \( \Pi : [\pi_1, \pi_2] \to \mathbb{R} \) be a convex function on \([\pi_1, \pi_2]\) and \( 0 < q < 1 \). Then, \( q^{\pi_2} \)-Hermite-Hadamard inequalities are given as follows:

\[
\Pi \left( \frac{\pi_1 + q \pi_2}{2} \right) \leq \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) \pi_2 d_q x \leq \frac{\Pi (\pi_1) + q \Pi (\pi_2)}{2}. \tag{2.5}
\]

In [24], Budak proved the left and right bounds of the inequality (2.5).
3. Crucial identities

To obtain the key results of this paper, we prove three separate identities in this section. Let’s begin with the following crucial Lemma.

**Lemma 3.1.** If $\Pi : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R}$ is a $q_s$-differentiable function on $(\pi_1, \pi_2)$ such that $\pi_1 D_q \Pi$ is continuous and integrable on $[\pi_1, \pi_2]$, then we have the following identity:

$$q \Pi (\pi_1) + (1 - \mu q) \Pi (\pi_2) + q (\mu - \lambda) \Pi \left( \frac{\pi_1 q + \pi_2}{2} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) \pi_2 dq x \quad (3.1)$$

where $q \in (0, 1)$.

**Proof.** From Definition 2.1, we have

$$\pi_1 D_q \Pi (t \pi_2 + (1 - t) \pi_1) = \frac{\Pi (t \pi_2 + (1 - t) \pi_1) - \Pi (t q \pi_2 + (1 - q t) \pi_1)}{(1 - q) (\pi_2 - \pi_1) t} \quad (3.2)$$

By utilizing the properties of quantum integrals, we obtain

$$\int_0^{1/q} (t - \lambda) \pi_1 D_q \Pi (t \pi_2 + (1 - t) \pi_1) dq t + \int_0^{1/q} (t - \mu) \pi_1 D_q \Pi (t \pi_2 + (1 - t) \pi_1) dq t \quad (3.3)$$

$$= \int_0^{1/q} (\mu - \lambda) \pi_1 D_q \Pi (t \pi_2 + (1 - t) \pi_1) dq t + \int_0^{1/q} (t - \mu) \pi_1 D_q \Pi (t \pi_2 + (1 - t) \pi_1) dq t$$

$$= (\mu - \lambda) \int_0^{1/q} \frac{\Pi (t \pi_2 + (1 - t) \pi_1) - \Pi (t q \pi_2 + (1 - q t) \pi_1)}{(1 - q) (\pi_2 - \pi_1) t} dq t$$

$$+ \int_0^{1/q} \frac{\Pi (t \pi_2 + (1 - t) \pi_1) - \Pi (t q \pi_2 + (1 - q t) \pi_1)}{(1 - q) (\pi_2 - \pi_1) t} dq t$$

$$- \mu \int_0^{1/q} \frac{\Pi (t \pi_2 + (1 - t) \pi_1) - \Pi (t q \pi_2 + (1 - q t) \pi_1)}{(1 - q) (\pi_2 - \pi_1) t} dq t.$$

By Definition 2.3, we have the following equalities

$$\int_0^{1/q} \frac{\Pi (t \pi_2 + (1 - t) \pi_1) - \Pi (t q \pi_2 + (1 - q t) \pi_1)}{(1 - q) (\pi_2 - \pi_1) t} dq t \quad (3.4)$$

$$= \frac{1}{\pi_2 - \pi_1} \left[ \sum_{n=0}^{\infty} \Pi \left( \frac{q^n}{2q} \pi_2 + \frac{1 - q^n}{2} \pi_1 \right) - \sum_{n=0}^{\infty} \Pi \left( \frac{q^{n+1}}{2q} \pi_2 + \frac{1 - q^{n+1}}{2} \pi_1 \right) \right]$$

$$= \frac{1}{\pi_2 - \pi_1} \left[ \Pi \left( \frac{\pi_1 q + \pi_2}{2} \right) - \Pi (\pi_1) \right],$$

$$\int_0^{1/q} \frac{\Pi (t \pi_2 + (1 - t) \pi_1) - \Pi (t q \pi_2 + (1 - q t) \pi_1)}{(1 - q) (\pi_2 - \pi_1) t} dq t \quad (3.5)$$
and
\[
\int_0^1 \frac{\Pi (t \pi_2 + (1 - t) \pi_1) - \Pi (q t \pi_2 + (1 - q t) \pi_1)}{(1 - q) (\pi_2 - \pi_1)} \, dq \, dt = \frac{1}{\pi_2 - \pi_1} [\Pi (\pi_2) - \Pi (\pi_1)]
\]

Remark 3.2. In Lemma 3.1, if we take the limit \( q \to 1^- \), then we have [43, Lemma 2.1 for \( m = 1 \)].

Lemma 3.2. If \( \Pi : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R} \) is a \( q_{\pi_1} \)-differentiable function on \( (\pi_1, \pi_2) \) such that \( \pi_1 D_q \Pi \) is continuous and integrable on \( [\pi_1, \pi_2] \), then we have the following identity:
\[
q \lambda \Pi (\pi_1) + q (\mu - \lambda) \Pi \left( \frac{\pi_1 q [2]_q + \pi_2}{[3]_q} \right) \\
+ q (\nu - \mu) \Pi \left( \frac{\pi_1 q^2 + \pi_2 [2]_q}{[3]_q} \right) + (1 - q (\nu - 1)) \Pi (\pi_2) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) \, \pi_1 d_q x
\]
By applying the same steps in the proof of Lemma 3.1 for rest of this proof, one can obtain the desired identity

\[ q \in (0, 1) \]

**Proof.** By the fundamental properties of quantum integrals, we have

\[
\begin{align*}
& \int_{\pi}^{1} (t - \lambda) \pi_{1} D_{q} \Pi (t \pi_{2} + (1 - t) \pi_{1}) d_{q} t \\
& + \int_{\pi}^{1} (t - \nu) \pi_{1} D_{q} \Pi (t \pi_{2} + (1 - t) \pi_{1}) d_{q} t \\
& \quad = \int_{\pi}^{1} (t - \lambda) \pi_{1} D_{q} \Pi (t \pi_{2} + (1 - t) \pi_{1}) d_{q} t \\
& + \int_{\pi}^{1} (t - \nu) \pi_{1} D_{q} \Pi (t \pi_{2} + (1 - t) \pi_{1}) d_{q} t.
\end{align*}
\]

By applying the same steps in the proof of Lemma 3.1 for rest of this proof, one can obtain the desired identity (3.7).

**Remark 3.5.** If we take \( \lambda = \frac{1}{[8]_{q}}, \mu = \frac{1}{[2]_{q}}, \) and \( \nu = \frac{[7]_{q}}{[8]_{q}} \) in Lemma 3.2, then we obtain the following identity

\[
\begin{align*}
& \frac{1}{[8]_{q}} \left[ q \Pi (\pi_{1}) + \frac{q^{3} [6]_{q}}{[2]_{q}} \Pi \left( \pi_{1} q [2]_{q} + \pi_{2} \right) + \frac{q^{2} [6]_{q}}{[2]_{q}} \Pi \left( \pi_{1} q^{2} + \pi_{2} [2]_{q} \right) + \Pi (\pi_{2}) \right] \\
& - \frac{1}{\pi_{2} - \pi_{1}} \int_{\pi_{1}}^{\pi_{2}} \Pi (x) \pi_{1} d_{q} x \\
& \quad = q (\pi_{2} - \pi_{1}) \left[ \int_{\pi_{1}}^{\pi_{2}} \left( t - \frac{1}{[8]_{q}} \right) \pi_{1} D_{q} \Pi (t \pi_{2} + (1 - t) \pi_{1}) d_{q} t \\
& + \int_{\pi_{1}}^{1} \left( t - \frac{1}{[2]_{q}} \right) \pi_{1} D_{q} \Pi (t \pi_{2} + (1 - t) \pi_{1}) d_{q} t \\
& + \int_{\pi_{1}}^{1} \left( t - \frac{[7]_{q}}{[8]_{q}} \right) \pi_{1} D_{q} \Pi (t \pi_{2} + (1 - t) \pi_{1}) d_{q} t \right]
\end{align*}
\]

which is proved by Erden et al. in [44].

**Remark 3.6.** If we take \( \lambda = \mu = \nu = \frac{1}{[2]_{q}} \), in Lemma 3.2, then we obtain [42, Lemma 3.1].

**Corollary 3.1.** If we take the limit \( q \rightarrow 1^{-} \) in Lemma 3.2, then we obtain the following new identity

\[
\lambda \Pi (\pi_{1}) + (\mu - \lambda) \Pi \left( \frac{2 \pi_{1} + \pi_{2}}{3} \right) + (\nu - \mu) \Pi \left( \frac{\pi_{1} + 2 \pi_{2}}{3} \right) + (1 - \nu) \Pi (\pi_{2})
\]
\[
\frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \, dx
\]

\[
= (\pi_2 - \pi_1) \left[ \int_0^{\frac{1}{2}} (t - \lambda) \Pi' (t \pi_2 + (1 - t) \pi_1) \, dt + \int_{\frac{1}{2}}^{1} (t - \mu) \Pi' (t \pi_2 + (1 - t) \pi_1) \, dt \right.
\]

\[
+ \int_{\frac{1}{2}}^{1} (t - \nu) \Pi' (t \pi_2 + (1 - t) \pi_1) \, dt \right]
\]

For brevity, let us prove another lemma that will be used frequently in the main results.

**Lemma 3.3.** The following quantum integrals holds for \( \lambda, \mu, \nu \geq 0 \):

\[
\Omega_{11} = \int_{\frac{1}{3} [t - \lambda]}^{\frac{1}{2} [t]} |t - \lambda| \, d_q t = \frac{2\lambda^2 q}{[2]_q} + \frac{1}{([2]_q)^3} - \frac{\lambda}{[2]_q} \quad (3.8)
\]

\[
\Omega_{12} = \int_{\frac{1}{3} [t - \mu]}^{\frac{1}{2} [t]} |t - \mu| \, d_q t = \frac{2\mu^2 q}{[2]_q} + \frac{[2]_q + 1}{([2]_q)^3} - \frac{\mu([2]_q + 1)}{[2]_q} \quad (3.9)
\]

\[
\Omega_{13} = \int_{\frac{1}{3} [t - \lambda]}^{\frac{1}{2} [t]} |t - \nu| \, d_q t = \frac{2\lambda^2 q}{[2]_q} + \frac{1}{([2]_q)([3]_q)^2} - \frac{\lambda}{[3]_q} \quad (3.10)
\]

\[
\Omega_{14} = \int_{\frac{1}{3} [t - \mu]}^{\frac{1}{2} [t]} |t - \nu| \, d_q t = \frac{2\mu^2 q}{[2]_q} - \frac{\mu([2]_q + 1)}{[3]_q} + \frac{[2]_q + 1}{([2]_q)([3]_q)^2} \quad (3.11)
\]

\[
\Omega_{15} = \int_{\frac{1}{3} [t - \lambda]}^{\frac{1}{2} [t]} |t - \nu| \, d_q t = \frac{2\nu^2 q}{[2]_q} - \frac{\nu([2]_q + [3]_q)}{([3]_q)^2} + \frac{[2]_q}{([3]_q)^2} + \frac{1}{[2]_q} \quad (3.12)
\]

\[
\Omega_1 = \int_{\frac{1}{3} [t - \lambda]}^{\frac{1}{2} [t]} t |t - \lambda| \, d_q t = \frac{2\lambda^2 q^2}{[2]_q [3]_q} + \frac{1}{([2]_q)^3 [3]_q} - \frac{\lambda}{([2]_q)^3} \quad (3.13)
\]

\[
\Omega_2 = \int_{\frac{1}{3} [t - \lambda]}^{\frac{1}{2} [t]} (1 - t) |t - \lambda| \, d_q t = \Omega_{11} - \Omega_1 \quad (3.14)
\]

\[
= \frac{2\lambda^2 q}{[2]_q} - \frac{2\lambda^3 q^2}{[2]_q [3]_q} - \frac{\lambda([2]_q)^2 - 1}{([2]_q)^3} + \frac{[3]_q - 1}{([2]_q)^3 [3]_q}
\]

\[
\Omega_3 = \int_{\frac{1}{3} [t - \mu]}^{\frac{1}{2} [t]} t |t - \mu| \, d_q t \quad (3.15)
\]
\[= \frac{2\mu^3 q^2}{[2]_q [3]_q} + \frac{1 + ([2]_q)^3}{([2]_q)^3 [3]_q} - \frac{\mu\left(([2]_q)^2 + 1\right)}{([2]_q)^3}\]

\[\Omega_4 = \int_{\frac{1}{\sqrt[3]{2}q}}^1 (1 - t) |t - \mu| d_q t = \]

\[= \Omega_{12} - \Omega_3\]

\[= \frac{2\mu^2 q}{[2]_q} - \frac{2\mu^3 q^2}{[2]_q [3]_q} - \frac{\mu\left(([2]_q)^3 - 1\right)}{([2]_q)^3 + \frac{[3]_q\left(1 + ([2]_q)^2\right) - ([2]_q)^3 - 1}{([2]_q)^3 [3]_q}\]

\[\Omega_5 = \int_{0}^{\frac{1}{\sqrt[3]{2}q}} |t - \lambda| d_q t = \frac{2\lambda^3 q^2}{[2]_q [3]_q} + \frac{1}{([3]_q)^4} - \frac{\lambda}{([3]_q)^2 [2]_q}\]

\[\Omega_6 = \int_{0}^{\frac{1}{\sqrt[3]{2}q}} (1 - t) |t - \lambda| d_q t = \]

\[= \Omega_{13} - \Omega_5\]

\[= \frac{2\lambda^2 q}{[2]_q} - \frac{2\lambda^3 q^2}{[2]_q [3]_q} + \frac{\lambda\left(1 - [2]_q [3]_q\right)}{([3]_q)^2 [2]_q} + \frac{([3]_q)^2 - [2]_q}{([3]_q)^2 [2]_q}\]

\[\Omega_7 = \int_{\frac{1}{\sqrt[3]{2}q}}^{\frac{1}{\sqrt[3]{3}q}} |t - \mu| d_q t = \frac{2\mu^3 q^2}{[2]_q [3]_q} + \frac{1 + ([2]_q)^3}{([3]_q)^4} - \frac{\mu\left(([2]_q)^2 + 1\right)}{([3]_q)^2 [2]_q}\]

\[\Omega_8 = \int_{\frac{1}{\sqrt[3]{3}q}}^{\frac{1}{\sqrt[3]{2}q}} (1 - t) |t - \mu| d_q t = \]

\[= \Omega_{14} - \Omega_7\]

\[= \frac{2\mu^2 q}{[2]_q} - \frac{2\mu^3 q^2}{[2]_q [3]_q} - \frac{\mu\left(([2]_q)^2 ([3]_q - 1) + [2]_q [3]_q\right)}{([3]_q)^2 [2]_q}

\[+ \frac{\left(([2]_q)^2 + 1\right)([3]_q)^3 - [2]_q - ([2]_q)^4}{([3]_q)^4 [2]_q}\]
$$\Omega_9 = \int_{\frac{\pi_1}{n_q}}^{\frac{\pi_2}{n_q}} t |t - \nu| d_q t = \frac{2 \nu^3 q^2}{[2]_q [3]_q} - \frac{\nu \left( ([2]_q)^2 + ([3]_q)^2 \right)}{[2]_q ([3]_q)^2} + \frac{([2]_q)^3 + ([3]_q)^3}{([3]_q)^4}$$  \hspace{1cm} (3.21)

$$\Omega_{10} = \int_{\frac{\pi_1}{n_q}}^{\frac{\pi_2}{n_q}} (1 - t) |t - \nu| d_q t $$  \hspace{1cm} (3.22)

$$= \Omega_{15} - \Omega_9$$

$$= \frac{2 \nu^3 q^2}{[2]_q} - \frac{2 \nu^3 q^2}{[2]_q [3]_q} - \frac{\nu \left( ([3]_q)^2 ([2]_q - 1) + ([2]_q)^2 ([3]_q - 1) \right)}{([3]_q)^2 [2]_q}$$

$$+ \frac{([3]_q)^2 ([2]_q - [3]_q) - ([2]_q)^3}{([3]_q)^4}$$  \hspace{1cm} (3.23)

$$+ \frac{([3]_q)^2 ([2]_q - [3]_q) - ([2]_q)^3}{([3]_q)^4}$$  \hspace{1cm} (3.24)

**Proof.** By the definition of $q$-integral, we have

$$\Omega_1 = \int_{0}^{\frac{\pi_1}{n_q}} t |t - \lambda| d_q t$$

$$= \int_{0}^{1} t (\lambda - t) d_q t + \int_{1}^{\frac{\pi_1}{n_q}} t (t - \lambda) d_q t$$

$$= 2 \int_{0}^{1} t (\lambda - t) d_q t + \int_{0}^{\frac{\pi_1}{n_q}} t (t - \lambda) d_q t$$

$$= 2 \frac{\lambda^3 q^2}{[2]_q [3]_q} + \frac{1}{([2]_q)^3 [3]_q} - \frac{\lambda}{([2]_q)^3}$$

and so

$$\Omega_1 = \frac{2 \lambda^3 q^2}{[2]_q [3]_q} + \frac{1}{([2]_q)^3 [3]_q} - \frac{\lambda}{([2]_q)^3}.$$ 

This gives the proof of the equality (3.13). The others can be calculated in similar way. \hspace{1cm} \Box

### 4. Generalizations of Simpson’s type inequalities for quantum integrals with two parameters

In this section, we prove a new generalization of quantum Simpson’s, Midpoint and Trapezoid type inequalities for quantum differentiable convex functions.

**Theorem 4.1.** We assume that the given conditions of Lemma 3.1 hold. If the mapping $|\pi_1 D_q \Pi|$ is convex on $[\pi_1, \pi_2]$, then the following inequality holds:

$$\left| q \lambda \Pi (\pi_1) + (1 - \mu q) \Pi (\pi_2) + q (\mu - \lambda) \Pi \left( \frac{\pi_1 q + \pi_2}{[2]_q} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) |x| d_q x \right|$$  \hspace{1cm} (4.1)
where \( \Omega_1 \cdot \Omega_4 \) are given in (3.13)-(3.16), respectively.

**Proof.** By taking the modulus in Lemma 3.1 and using the convexity of \( |\pi_1 D_q \Pi| \), we obtain

\[
\begin{align*}
q \lambda \Pi (\pi_1) + (1 - \mu q) \Pi (\pi_2) + q (\mu - \lambda) \Pi \left( \frac{\pi_1 q + \pi_2}{[2]_q} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) \, \pi_1 d_q x \\
\leq q (\pi_2 - \pi_1)
\times \left[ \int_0^\frac{1}{2q} |t - \lambda| \, |\pi_1 D_q \Pi (t\pi_2 + (1 - t) \pi_1)| \, dq \int_0^1 t \, dq \int_0^1 (1 - t) \, dq \right]

\leq (\pi_2 - \pi_1) q \left[ |\pi_1 D_q \Pi (\pi_2)| \left\{ \int_0^\frac{1}{2q} t \, dq \int_0^1 t \, dq \int_0^1 (1 - t) \, dq \right\} + |\pi_1 D_q \Pi (\pi_1)| \left\{ \int_0^\frac{1}{2q} (1 - t) \, dq \int_0^1 t \, dq \int_0^1 (1 - t) \, dq \right\} \right]

\leq (\pi_2 - \pi_1) q \left[ \Omega_1 + \Omega_3 \right] |\pi_1 D_q \Pi (\pi_2)| + \Omega_2 \left| \pi_1 D_q \Pi (\pi_1) \right|

\right]

which is the desired inequality. \( \Box \)

**Remark 4.1.** If we take the limit \( q \rightarrow 1^- \) in Theorem 4.1, then we have [43, Theorem 2.1 for \( s = m = 1 \)].

**Remark 4.2.** If we assume \( \lambda = \mu = \frac{1}{[6]_q} \) in Theorem 4.1, then we obtain [42, Theorem 4.1].

**Remark 4.3.** In Theorem 4.1, if we choose \( \lambda = 0 \) and \( \mu = \frac{1}{[2]_q} \), then Theorem 4.1 reduces to [22, Theorem 13].

**Remark 4.4.** If we assume \( \lambda = \frac{1}{[6]_q} \) and \( \mu = \frac{[5]_q}{[6]_q} \) in Theorem 4.1, then we obtain the following inequality

\[
\begin{align*}
&\left| \frac{1}{[6]_q} \left( q \Pi (\alpha) + q^2 [4]_q \Pi \left( \frac{q \pi_1 + \pi_2}{[2]_q} \right) + \Pi (\pi_2) \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) \, \pi_1 d_q x \right|

\leq q (\pi_2 - \pi_1) \left[ \left| \pi_1 D_q \Pi (\pi_2) \right| A_1 (q) + A_2 (q) + \left| \pi_1 D_q \Pi (\pi_1) \right| B_1 (q) + B_2 (q) \right],
\end{align*}
\]

where

\[
\begin{align*}
A_1 (q) &= \frac{2 q^2 [2]_q^3 + [6]_q^3 \left( [6]_q - [3]_q \right)}{[2]_q [3]_q [6]_q [6]_q}, \\
B_1 (q) &= 2 q \left[ [3]_q [6]_q - q^2 \right] \left( \frac{1}{[2]_q [3]_q [6]_q^3} \right), \\
A_2 (q) &= 2 q^2 \left( \frac{5}[[6]_q^3] \right) + \left[ 6]_q \left( [1 + [2]_q^3] - [3]_q [5]_q (1 + [2]_q^2) \right) \right] \left( \frac{[2]_q [3]_q [6]_q}{[2]_q [3]_q [6]_q} \right), \\
B_2 (q) &= 2 q \left[ [5]_q^2 [6]_q [3]_q - q^2 [5]_q^3 \right] + \left( \frac{q^2 [5]_q}{[2]_q [3]_q [6]_q} - \frac{[5]_q}{[2]_q [3]_q [6]_q} \right).
\end{align*}
\]
Theorem 4.2. We assume that the given conditions of Lemma 3.1 hold. If the mapping \( |\pi_1 D_q \Pi|^{\rho_1} \), \( p_1 \geq 1 \) is convex on \([\pi_1, \pi_2]\), then the following inequality holds:

\[
\left| \lambda q \Pi(\pi_1) + (1 - \mu q) \Pi(\pi_2) + q(\mu - \lambda) \Pi \left( \frac{\pi_1 q + \pi_2}{2} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \, |\pi_1 D_q x| \right|^{\rho_1} 
\]

where \( \Omega_{11}, \Omega_{12} \) and \( \Omega_{1} - \Omega_{4} \) are given in (3.8), (3.9), and (3.13)–(3.16), respectively.

Proof. By taking the modulus in Lemma 3.1 and using the power mean inequality, we have

\[
\left| \lambda q \Pi(\pi_1) + (1 - \mu q) \Pi(\pi_2) + q(\mu - \lambda) \Pi \left( \frac{\pi_1 q + \pi_2}{2} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \, |\pi_1 D_q x| \right|^{\rho_1} 
\]

By using the convexity of \( |\pi_1 D_q \Pi|^{\rho_1} \), we have

\[
\left| \lambda q \Pi(\pi_1) + (1 - \mu q) \Pi(\pi_2) + q(\mu - \lambda) \Pi \left( \frac{\pi_1 q + \pi_2}{2} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \, |\pi_1 D_q x| \right|^{\rho_1} 
\]
Theorem 4.3. We assume that the given conditions of Lemma 3.1 hold. If the mapping \( |x|D_4 \Pi (\pi_2) |^p_1 \), \( p_1 > 1 \) is convex on \([\pi_1, \pi_2] \), then the following inequality holds:

\[
\left| \lambda q \Pi (\pi_1) + (1 - \mu q) \Pi (\pi_2) + q (\mu - \lambda) \Pi \left( \frac{\pi_1 q + \pi_2}{[2]_q} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) |x| d_q x \right| \quad (4.3)
\]

\[
\leq (\pi_2 - \pi_1) q \left[ \Omega_{16}^{\frac{1}{3}} \left( \frac{|x_1|D_4 \Pi (\pi_2) |^p_1}{[2]_q} \right)^{\frac{3}{2}} + \frac{\left( [2]_q ^2 - 1 \right)}{[2]_q ^3} \right]^{\frac{1}{p_1}}
\]

and the proof is completed.

Remark 4.5. If we take the limit \( q \rightarrow 1^- \) in Theorem 4.2, then we have [43, Theorem 2.3 for \( s = m = 1 \)].

Remark 4.6. If we assume \( \lambda = \mu = \frac{1}{[2]_q} \) in Theorem 4.2, then we obtain [42, Theorem 4.2].

Remark 4.7. If we assume \( \lambda = \frac{1}{[6]_q} \) and \( \mu = \frac{[5]_q}{[6]_q} \) in Theorem 4.2, then we obtain the following inequality

\[
\left| \frac{1}{[6]_q} \left[ q \Pi (\alpha) + q^2 [4]_q \Pi \left( \frac{q \pi_1 + \pi_2}{[2]_q} \right) + \Pi (\pi_2) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (s) |s| d_q s \right|
\]

\[
\leq q (\pi_2 - \pi_1) \left[ \frac{2q}{[2]_q \cdot [6]_q} + \frac{q^3 [3]_q - q}{[6]_q \cdot [2]_q^2} \right]^{\frac{1}{p_1}}
\]

\[
\times \left( A_1 (q) \left| \pi_1 |D_4 \Pi (\pi_2) |^p_1 + B_1 (q) \left| \pi_1 |D_4 \Pi (\pi_1) |^p_1 \right| \right. \right)^{\frac{1}{p_1}}
\]

\[
+ \left( 2q \frac{[5]_q^2}{[2]_q \cdot [6]_q^2} + \frac{1}{[2]_q} - \frac{[5]_q}{[6]_q} - \frac{[5]_q \cdot [2]_q - [6]_q}{[6]_q \cdot [2]_q^3} \right)^{\frac{1}{p_1}}
\]

\[
\times \left( A_2 (q) \left| \pi_1 |D_4 \Pi (\pi_2) |^p_1 + B_2 (q) \left| \pi_1 |D_4 \Pi (\pi_1) |^p_1 \right| \right. \right)^{\frac{1}{p_1}}
\]

where \( A_1 (q) \), \( A_2 (q) \), \( B_1 (q) \) and \( B_2 (q) \) are defined in Remark 4.4. The above inequality is proved by Ifitikhar et al. [41].

Remark 4.8. In Theorem 4.2, if we choose \( \lambda = 0 \) and \( \mu = \frac{1}{q} \), then Theorem 4.2 reduces to [22, Theorem 16].
where \( p_1^{-1} + r_1^{-1} = 1 \) and
\[
\Omega_{16} = \int_0^{1/p_1} |t - \lambda|^{\pi_1} d_q t, \quad \Omega_{17} = \int_0^{1} |t - \mu|^{\pi_1} d_q t
\]

Proof. By taking the modulus in Lemma 3.1 and using the Hölder inequality, we have
\[
\left| \lambda q \Pi (\pi_1) + (1 - \mu q) \Pi (\pi_2) + q (\mu - \lambda) \Pi \left( \frac{\pi_1 q + \pi_2}{2} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) |_{\pi_1} d_q x \right|
\leq (\pi_2 - \pi_1) q \left[ \left( \int_0^{1/p_1} |t - \lambda|^{\pi_1} d_q t \right)^{\frac{1}{\pi_1}} \left( \int_0^{1/p_2} |\pi_1 D_q \Pi (t \pi_2 + (1 - t) \pi_1)|^{\pi_1} d_q t \right)^{\frac{1}{\pi_1}} \right]
+ \frac{1}{\pi_1} \left( \int_0^{1} |t - \mu|^{\pi_1} d_q t \right)^{\frac{1}{\pi_1}} \left( \int_0^{1} |\pi_1 D_q \Pi (t \pi_2 + (1 - t) \pi_1)|^{\pi_1} d_q t \right)^{\frac{1}{\pi_1}}.
\]
Since \( |\pi_1 D_q \Pi|^{\pi_1} \) is convex on \([\pi_1, \pi_2]\), we have
\[
\left| \lambda q \Pi (\pi_1) + (1 - \mu q) \Pi (\pi_2) + q (\mu - \lambda) \Pi \left( \frac{\pi_1 q + \pi_2}{2} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) |_{\pi_1} d_q x \right|
\leq (\pi_2 - \pi_1)
\times \left[ \left( \int_0^{1/p_1} |t - \lambda|^{\pi_1} d_q t \right)^{\frac{1}{\pi_1}} \left( \int_0^{1/p_2} |\pi_1 D_q \Pi (\pi_2)|^{\pi_1} d_q t + \int_0^{1/p_1} \int_0^{1/p_2} |\pi_1 D_q \Pi (\pi_1)|^{\pi_1} d_q t \right)^{\frac{1}{\pi_1}} \right]
+ \frac{1}{\pi_1} \left( \int_0^{1} |t - \mu|^{\pi_1} d_q t \right)^{\frac{1}{\pi_1}} \left( \int_0^{1} |\pi_1 D_q \Pi (\pi_2)|^{\pi_1} d_q t + \int_0^{1} \int_0^{1} |\pi_1 D_q \Pi (\pi_1)|^{\pi_1} d_q t \right)^{\frac{1}{\pi_1}}
\]
\[
= (\pi_2 - \pi_1) q \left[ \Omega_{16}^{\frac{1}{\pi_1}} \left( \frac{|\pi_1 D_q \Pi (\pi_2)|^{\pi_1}}{(2 \lambda)^3} + \frac{((2 \lambda)^2 - 1) |\pi_1 D_q \Pi (\pi_1)|^{\pi_1}}{(2 \lambda)^3} \right)^{\frac{1}{\pi_1}} \right]
+ \Omega_{17}^{\frac{1}{\pi_1}} \left( \frac{|\pi_1 D_q \Pi (\pi_2)|^{\pi_1}}{(2 \lambda)^3} + \frac{((2 \lambda)^2 - 2 (2 \lambda)^2 + 1) |\pi_1 D_q \Pi (\pi_1)|^{\pi_1}}{(2 \lambda)^3} \right)^{\frac{1}{\pi_1}} \right] \] .

This completes the proof. \( \square \)

Remark 4.9. If we take the limit \( q \to 1^- \) in Theorem 4.3, then Theorem 4.3 becomes [43, Theorem 2.2 for \( s = m = 1 \)].

Remark 4.10. If we assume \( \lambda = \mu = \frac{1}{2} \) in Theorem 4.3, then we obtain [27, Theorem 3.3].

Remark 4.11. If we assume \( \lambda = \frac{1}{6} \) and \( \mu = \frac{5}{6} \) in Theorem 4.3, then we obtain the following inequality
\[
\left| \frac{1}{6} q \Pi (\alpha) + q^2 [4]_q \Pi \left( \frac{q \pi_1 + \pi_2}{2} \right) + \Pi (\pi_2) \right| - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) |_{\pi_1} d_q x
\]

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We assume that the given conditions of Lemma 3.2 hold. If the mapping £1 \rightarrow £2

\text{Theorem 5.1.}

Remark 4.12. In Theorem 4.2, if we choose \lambda = 0 and \mu = \frac{1}{q}, then Theorem 4.3 reduces to [22, Theorem 18].

5. Generalizations of Newton’s type inequalities for quantum integrals with three parameters

Some new generalized versions of quantum Newton’s and Trapezoid type inequalities for quantum differentiable convex functions are offered in this section.

\text{Theorem 5.1.} We assume that the given conditions of Lemma 3.2 hold. If the mapping \[|\pi_1 D_q \Pi|\] is convex on [\pi_1, \pi_2], then the following inequality holds:

\begin{align*}
&\leq q (\pi_2 - \pi_1) \left[ \left( \frac{q^{2 \pi_1} [4]_q^{\pi_1}}{[2]_q^{\pi_1 + 1} [6]_q^{\pi_1}} \right)^{\frac{1}{2}} \right. \\
&+ \left. \left( \frac{[2]_q^{\pi_1 + 1} [5]_q - [4]_q^{\pi_1}}{[2]_q^{\pi_1 + 1} [6]_q} \right) \right] ^{\frac{1}{2}} \\
&\times \left( q^2 + 2q \right) \left[ \pi_1 D_q \Pi (\pi_2) \right]^{p_1} + \left( q^3 + q^2 - q \right) \left[ \pi_1 D_q \Pi (\pi_1) \right]^{p_1}
\end{align*}

which is established by Iftikhar et al. in [41].

Remark 5.1. If we take the limit \( q \rightarrow 1^- \) in Theorem 5.1, then we obtain the desired inequality (5.1).

Proof. By considering Lemma 3.2 and applying the same method that used in the proof of Theorem 4.1, then we can obtain the desired inequality (5.1).

\text{Remark 5.1.} If we assume \lambda = \mu = \nu = \frac{1}{\Omega_1} in Theorem 5.1, then we obtain [42, Theorem 4.1].

\text{Corollary 5.1.} If we take the limit \( q \rightarrow 1^- \) in Theorem 5.1, then we obtain the following inequality

\begin{align*}
&\left| \lambda \Pi (\pi_1) + (\mu - \lambda) \Pi \left( \frac{2\pi_1 + \pi_2}{3} \right) + (\nu - \mu) \Pi \left( \frac{\pi_1 + 2\pi_2}{3} \right) \right. \\
&+ (1 - \nu) \Pi (\pi_2) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) \pi_1 d_q x \right| \\
\leq & \left( \pi_2 - \pi_1 \right) q \left[ \left( \Omega_5 + \Omega_7 + \Omega_9 \right) \left| \pi_1 D_q \Pi (\pi_2) \right| + \left( \Omega_6 + \Omega_8 + \Omega_{10} \right) \left| \pi_1 D_q \Pi (\pi_1) \right| \right]
\end{align*}

where

\[ \Omega_5 = \int_0^1 t |t - \lambda| dt = \frac{\lambda^3}{3} + \frac{1}{81} - \frac{\lambda}{18}. \]
Remark 5.2. If we take \( \lambda = \frac{1}{[8]q} \), \( \mu = \frac{1}{[2]q} \), and \( \nu = \frac{[7]q}{[8]q} \) in Theorem 5.1, then we obtain the following inequality

\[
\left| \frac{1}{[8]q} \left[ q \Pi(\pi_1) + \frac{q^2 [6]_q}{[2]_q} \Pi \left( \frac{\pi_1 q [2]_q + \pi_2}{[3]_q} \right) + \frac{q^2 [6]_q}{[2]_q} \Pi \left( \frac{\pi_1 q^2 + \pi_2 [2]_q}{[3]_q} \right) \right] \right|
\]

\[
\leq q (\pi_2 - \pi_1) \left[ \left| \int_{\pi_1}^{\pi_2} \Pi (x) \, q \, d_q x \right| \right]
\]

where

\[
A_3 (q) = \frac{2q^2 [3]_q^2 + [8]_q [18]_q [2]_q - [3]_q^2}{[8]_q [3]_q^2 [2]_q},
\]

\[
B_3 (q) = 2q [8]_q [2]_q - q^2 \frac{[3]_q - [2]_q}{[3]_q^2 [2]_q} + \frac{1 - [3]_q [2]_q}{[8]_q [3]_q^2},
\]

\[
A_4 (q) = \frac{2q^2}{[2]_q [3]_q} + \frac{[2]_q^2 (1 + [2]_q^2) - [3]_q^2 (1 + [2]_q^2)}{[3]_q^2 [2]_q^2},
\]

\[
B_4 (q) = \frac{2q}{[2]_q^3} - q \frac{[3]_q}{[3]_q^2} - A_4 (q),
\]

\[
A_5 (q) = \frac{2q^2 [7]_q^2}{[8]_q [2]_q [3]_q} + \frac{[2]_q [8]_q [2]_q [3]_q - [7]_q [3]_q [2]_q + [3]_q^2 [2]_q}{[3]_q [8]_q [2]_q}
\]

and

\[
B_5 (q) = 2q [7]_q^2 [8]_q [3]_q - q^2 [7]_q^3 + \frac{q^2}{[2]_q [3]_q} - \frac{q [7]_q}{[2]_q [8]_q}
\]
\begin{align*}
+ \frac{[2]_q \left( [3]_q^2 - [2]_q^2 \right)}{[3]_q^4} - \frac{(q + q^2) [7]_q [2]_q}{[3]_q^2 [8]_q}.
\end{align*}

**Theorem 5.2.** We assume that the given conditions of Lemma 3.2 hold. If the mapping \([\pi_1 D_q \Pi]^{p_1}\), \(p_1 \geq 1\) is convex on \([\pi_1, \pi_2]\), then the following inequality holds:

\begin{equation}
q \lambda \Pi (\pi_1) + q (\mu - \lambda) \Pi \left( \frac{\pi_1 q [2]_q + \pi_2}{[3]_q} \right) + q (\nu - \mu) \Pi \left( \frac{\pi_1 q^2 + \pi_2 [2]_q}{[3]_q} \right) + (1 - \nu q) \Pi (\pi_2) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) \left[ \prod_{\pi_1} (x) \right] d\pi
\end{equation}

\begin{align*}
\leq \frac{(\pi_2 - \pi_1) q}{\Omega_{13} \left( \Omega_5 \left| \pi_1 D_q \Pi (\pi_2) \right|^{p_1} + \Omega_6 \left| \pi_1 D_q \Pi (\pi_1) \right|^{p_1} \right)^{1/\mu}} + \frac{\Omega_{14} (\pi_2 - \pi_1) q}{\Omega_7 \left| \pi_1 D_q \Pi (\pi_2) \right|^{p_1} + \Omega_8 \left| \pi_1 D_q \Pi (\pi_1) \right|^{p_1}} + \frac{\Omega_{15} (\pi_2 - \pi_1) q}{\Omega_9 \left| \pi_1 D_q \Pi (\pi_2) \right|^{p_1} + \Omega_{10} \left| \pi_1 D_q \Pi (\pi_1) \right|^{p_1}}
\end{align*}

where \(\Omega_5 - \Omega_{10}\) and \(\Omega_{13} - \Omega_{15}\) are given in (3.17)–(3.22) and (3.10)–(3.12), respectively. The above inequality established by Erden et al. in [44].

**Proof.** By applying the steps used in the proof of Theorem 4.2 and taking into account Lemma 3.2, we can obtain the required inequality (5.2). \(\square\)

**Corollary 5.2.** If we take the limit \(q \to -1^\mu\) in Theorem 5.2, then we obtain the following inequality

\begin{equation}
\lambda \Pi (\pi_1) + (\mu - \lambda) \Pi \left( \frac{2\pi_1 + \pi_2}{3} \right) + (\nu - \mu) \Pi \left( \frac{2\pi_2 + \pi_1}{3} \right) + (1 - \nu) \Pi (\pi_2)
\end{equation}

\begin{align*}
\leq \frac{(\pi_2 - \pi_1) q}{\Theta_{11} \left( \Omega_5^* \left| \pi_1 D_q \Pi (\pi_2) \right|^{p_1} + \Omega_6^* \left| \pi_1 D_q \Pi (\pi_1) \right|^{p_1} \right)^{1/\mu}} + \frac{\Theta_{12} (\pi_2 - \pi_1) q}{\Omega_7^* \left| \pi_1 D_q \Pi (\pi_2) \right|^{p_1} + \Omega_8^* \left| \pi_1 D_q \Pi (\pi_1) \right|^{p_1}} + \frac{\Theta_{13} (\pi_2 - \pi_1) q}{\Omega_9^* \left| \pi_1 D_q \Pi (\pi_2) \right|^{p_1} + \Omega_{10}^* \left| \pi_1 D_q \Pi (\pi_1) \right|^{p_1}}
\end{align*}

where \(\Omega_5^* - \Omega_{10}^*\) are defined in Corollary 5.1 and

\begin{align*}
\Theta_{11} &= \int_0^1 |t - \lambda| dt = \lambda^2 + \frac{1}{9} [2]_q - \frac{\lambda}{3}, \\
\Theta_{12} &= \int_0^1 |t - \mu| dt = \frac{18 \mu^2 + 5}{18} - \mu, \\
\Theta_{13} &= \int_0^1 |t - \nu| dt = \frac{18 \nu^2 + 13}{18} - \frac{5 \nu}{3}.
\end{align*}
Remark 5.3. If we take $\lambda = \frac{1}{[8]_q}$, $\mu = \frac{1}{[2]_q}$, and $\nu = \frac{[7]_q}{[8]_q}$ in Theorem 5.2, then we obtain the following inequality

\[
\left| \frac{1}{[8]_q} \left[ q\Pi(\pi_1) + \frac{q^3 [6]_q}{[2]_q} \Pi\left( \frac{\pi_1 q [2]_{2q} + \pi_2}{[3]_q} \right) + \frac{q^2 [6]_q}{[2]_q} \Pi\left( \frac{\pi_1 q^2 + \pi_2 [2]_q}{[3]_q} \right) + \Pi(\pi_2) \right] \right| \\
- \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \delta_q x \right|
\]

\[
\leq q(\pi_2 - \pi_1) \left( \frac{2q}{[8]_q [2]_q} + \frac{[8]_q - [3]_q [2]_q}{[3]_q^2 [2]_q [8]_q} \right)^{1-\frac{1}{\pi}}
\]

\[
\times \left( A_3(q) \left| \delta_q \left( \pi_1 \Pi(\pi_2) \right)^{p_1} + B_3(q) \left| \delta_q \left( \pi_1 \Pi(\pi_1) \right)^{p_1} \right| \right| \right)^{\frac{1}{\pi}}
\]

\[
+ \frac{2q}{[2]_q^3} + \frac{q}{[3]_q^2 [2]_q} + \frac{[2]_q - [3]_q [2]_q}{[3]_q^2 [2]_q} \right)^{1-\frac{1}{\pi}}
\]

\[
\times \left( A_4(q) \left| \delta_q \left( \pi_1 \Pi(\pi_2) \right)^{p_1} + B_4(q) \left| \delta_q \left( \pi_1 \Pi(\pi_1) \right)^{p_1} \right| \right| \right)^{\frac{1}{\pi}}
\]

\[
+ \frac{2q [7]_q^2}{[8]_q [2]_q} + \frac{[3]_q^2 + [2]_q^2}{[2]_q [3]_q^2 - [8]_q [3]_q} \left( \frac{7}{[3]_q} \left[ [3]_q + [2]_q \right] \right)^{1-\frac{1}{\pi}}
\]

\[
\times \left( A_5(q) \left| \delta_q \left( \pi_1 \Pi(\pi_2) \right)^{p_1} + B_5(q) \left| \delta_q \left( \pi_1 \Pi(\pi_1) \right)^{p_1} \right| \right| \right)^{\frac{1}{\pi}}
\]

where $A_3(q) - A_5(q)$ and $B_3(q) - B_5(q)$ are given in Remark 5.2. The above inequality established by Erden et al. in [44].

Remark 5.4. If we assume $\lambda = \mu = \nu = \frac{1}{[2]_q}$ in Theorem 5.2, then we obtain [42, Theorem 4.2].

Theorem 5.3. We assume that the given conditions of Lemma 3.2 hold. If the mapping $\left| \delta_q \Pi(\pi_1) \right|^{p_1}$, $p_1 > 1$ is convex on $[\pi_1, \pi_2]$, then the following inequality holds:

\[
\left| q\Pi(\pi_1) + q(\mu - \lambda) \Pi\left( \frac{\pi_1 q [2]_{2q} + \pi_2}{[3]_q} \right) + q(\nu - \mu) \Pi\left( \frac{\pi_1 q^2 + \pi_2 [2]_q}{[3]_q} \right) \right| \leq (\pi_2 - \pi_1) q \left( \frac{\left| \delta_q \Pi(\pi_2) \right|^{p_1}}{\left| \delta_q \Pi(\pi_1) \right|^{p_1}} \right) \left( \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \delta_q x \right) \right|
\]

\[
\leq \left( \pi_2 - \pi_1 \right) q \left[ \frac{1}{\left| \delta_q \Pi(\pi_2) \right|^{p_1}} \right] \left( \frac{1}{\left| \delta_q \Pi(\pi_1) \right|^{p_1}} \right) \right] \left( \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \delta_q x \right) \right|
\]

\[
\leq \left( \pi_2 - \pi_1 \right) q \left[ \frac{1}{\left| \delta_q \Pi(\pi_2) \right|^{p_1}} \right] \left( \frac{1}{\left| \delta_q \Pi(\pi_1) \right|^{p_1}} \right) \right] \left( \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \delta_q x \right) \right|
\]

\[
+ \left( \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \delta_q x \right) \right|
\]

\[
\leq \left( \pi_2 - \pi_1 \right) q \left[ \frac{1}{\left| \delta_q \Pi(\pi_2) \right|^{p_1}} \right] \left( \frac{1}{\left| \delta_q \Pi(\pi_1) \right|^{p_1}} \right) \right] \left( \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \delta_q x \right) \right|
\]

\[
+ \left( \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \delta_q x \right) \right|
\]

\[
+ \left( \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \delta_q x \right) \right|
\]

\[
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where \( p_1^{-1} + r_1^{-1} = 1 \) and
\[
\Omega_{18} = \int_{0}^{\pi_{q}} |t - \lambda_1|^{\nu_1} d_{q}t, \; \Omega_{19} = \int_{0}^{\pi_{q}} |t - \mu_1|^{\nu_1} d_{q}t, \; \Omega_{20} = \int_{0}^{\pi_{q}} |t - \nu_1|^{\nu_1} d_{q}t.
\]

**Proof.** By applying the steps used in the proof of Theorem 4.3 and taking into account Lemma 3.2, we can obtain the required inequality (5.3).

**Remark 5.5.** If we assume \( \lambda = \mu = \frac{1}{2 [\lambda_q]} \) in Theorem 5.3, then we obtain [27, Theorem 3.3].

**Remark 5.6.** If we take \( \lambda = \frac{1}{[8]_q}, \; \mu = \frac{1}{[2]_q}, \) and \( \nu = \frac{[7]_q}{[8]_q} \) in Theorem 5.3, then we obtain the following inequality
\[
\frac{1}{[8]_q} \left[ q \Pi (\pi_1) + \frac{q^3 [6]_q}{[2]_q} \Pi \left( \frac{[\pi_1] [2]_q + \pi_2}{[3]_q} \right) + \frac{q^2 [6]_q}{[2]_q} \Pi \left( \frac{[\pi_1] [2]_q + \pi_2}{[3]_q} \right) + \Pi (\pi_2) \right]
\]
\[
- \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi (x) \pi_1 d_{q} x
\]
\[
\leq q (\pi_2 - \pi_1) \left( \frac{q^{\pi_1} [5]_q}{[3]_q^{\pi_1 + 1} [8]_q} \right)^{\frac{1}{\pi_1}}
\]
\[
\times \left( \frac{1}{[3]_q^2 [2]_q} \left| \pi_1 D_q \Pi (\pi_1) \right|^{\pi_1} + \frac{[3]_q^2 [2]_q - 1}{[3]_q^2 [2]_q} \left| \pi_1 D_q \Pi (\pi_2) \right|^{\pi_1} \right)^{\frac{1}{\pi_1}}
\]
\[
+ \left( \frac{q^{\pi_1} [2]_q - q^{2 \pi_1}}{[3]_q^{\pi_1 + 1} [2]_q} \right)^{\frac{1}{\pi_1}}
\]
\[
\times \left( \frac{q^2 + 2}{[3]_q^2 [2]_q} \left| \pi_1 D_q \Pi (\pi_1) \right|^{\pi_1} + \frac{q [3]_q [2]_q - (q^2 + 2q)}{[3]_q^2 [2]_q} \left| \pi_1 D_q \Pi (\pi_2) \right|^{\pi_1} \right)^{\frac{1}{\pi_1}}
\]
\[
+ \left( \frac{2 q^{\pi_1} [7]_q [3]_q - [8]_q [2]_q}{[8]_q [3]_q^{\pi_1 + 1}} \right)^{\frac{1}{\pi_1}}
\]
\[
\times \left( \frac{[3]_q^2 - [2]_q^2}{[3]_q^2 [2]_q} \left| \pi_1 D_q \Pi (\pi_1) \right|^{\pi_1} + \frac{q^2 [3]_q [2]_q + [2]_q^2 - [3]_q^2}{[3]_q^2 [2]_q} \left| \pi_1 D_q \Pi (\pi_2) \right|^{\pi_1} \right)^{\frac{1}{\pi_1}}
\]
which is proved by Iftikhar et al. in [41].

6. Conclusions

To sum up, we provided some generalisations of quantum Simpson’s and quantum Newton’s inequalities for quantum differentiable convex functions with two and three parameters, respectively. It is important to note that by considering the limit \( q \rightarrow 1^- \) and different special choices of the involved parameters in our key results, our results transformed into some new and well-known results. We believe that it is an interesting and innovative problem for future researchers who can obtain similar inequalities for different types of convexity and quantum integrals.
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Conflict of interest

The authors declare no conflict of interest.

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