Surfaces of revolution in the Heisenberg group and the spectral generalization of the Willmore functional

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1 Introduction and main results

In this paper we study the spectral generalization of the Willmore functional for the surfaces in the three-dimensional nilpotent Lie group Nil with left-invariant metric admitting four-dimensional isometry group, i.e. endowed by one of Thurston’s geometries.

The Weierstrass representation for surfaces in Nil was introduced by us in [3] where following the spectral point of view on the Willmore functional adopted in [10, 11] we proposed its generalization as

\[ E(M) = \int_M UV \frac{idz \wedge d\bar{z}}{2} \]

where \( U \) and \( V \) are the potentials of the Dirac operator coming into the representation. For the case of surfaces in \( \mathbb{R}^3 \) this formula gives the quarter of the Willmore functional \( W = \int H^2 d\mu \). However for surfaces in Nil the functional \( E \) is not proportional to the Willmore functional which in general is equal to \( \int (H^2 + \hat{K}) d\mu \) where \( \hat{K} \) is the sectional curvature of the ambient space along the tangent plane to the surface.

In this paper we demonstrate that for surfaces in Nil the functional \( E(M) \) resembles the Willmore functional for surfaces in \( \mathbb{R}^3 \) in many geometrical respects.

*The work was supported by RFBR (no. 06-01-00094a) and the complex integration project 1.1 of SB RAS. The second author (I.A.T.) was also supported by the Program of Basic Researches of Ministry of Education and Science of Kazakhstan (no. F03969-4).

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In particular, we prove that for closed surfaces of revolution $E$ is positive and for spheres of revolution the minima of $E$ are given by constant mean curvature spheres (see Theorem 2 and corollaries therein). Moreover these spheres are critical points of $E$ (see Theorem 3).

We observe the relation of the functionals $E$ and $W$ to the isoperimetric problem: in particular, both functionals $E$ and $\frac{1}{4}W$ attain the same value $\pi$ on the cmc spheres in Nil and $\mathbb{R}^3$ respectively (see Theorem 1). For $\mathbb{R}^3$ these spheres are isoperimetric surfaces and it is conjectured that the same is true for Nil. Therewith we also show in §6 how to derive some results from [4] and [5] by using the Weierstrass representation.

We find this relation of the theory of the Willmore functional to the isoperimetric problem interesting. We discuss this relation and some open questions in [6] and demonstrate it one more time for the case of surfaces in $S^2 \times \mathbb{R}$ in §6.2.

In §6.1 we also derive the Euler–Lagrange equation for the functional $E$.

2 The Weierstrass representation of surfaces in the Heisenberg group

The Heisenberg group Nil is the nilpotent Lie group formed by all matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R},$$

with the standard multiplication. It is assumed that the group is endowed by the left invariant metric

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$ 

The Lie algebra is spanned by three generators $e_1 = e_x, e_2 = e_y, e_3 = e_z$ which meet the commutativity relations

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0.$$ 

The scalar product at the Lie algebra to Nil induced by this left-invariant metric we denote by

$$\langle u, v \rangle = \sum_{i=1}^{3} u^i v^i, \quad u = \sum u^i e_i, \quad v = \sum v^k e_k.$$
As a smooth manifold this group is diffeomorphic to $\mathbb{R}^3$ and on Nil we also may introduce the cylindrical coordinates $(\rho, \phi, h)$ as follows:

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = \frac{\rho^2}{2} \cos \phi \sin \phi + h.$$  

Given a point $z = h$ on the $z$-axis we draw the geodesic $\gamma$ of the length $\rho$ orthogonally to the $z$-axis in the direction defined by $\phi$, the angle between $\gamma$ and the $x$-axis. The end-point of the geodesic has the coordinates $(\rho, \phi, h)$.

The metric in the cylindrical coordinates takes the form

$$ds^2 = d\rho^2 - \rho^2 dh d\phi + \frac{1}{4} \rho^2 (4 + \rho^2) d\phi^2 + dh^2$$

and we see that it is invariant under rotations $\phi \to \phi + \theta$ around the $z$-axis.

In fact from this formula for the metric it is easily derived that Nil has a four-dimensional isometry group generated by left translations: $g \to hg, h \in \text{Nil}$, and rotations around the $z$-axis.

Let us expose basic facts on the Weierstrass representation of surfaces in three-dimensional Lie groups $G$ introduced in [3].

The Weierstrass representation of a surface

$$f : M \to G$$

defines it in terms of a solution to a nonlinear equation

$$\mathcal{D}_\text{Nil} \psi = \left( \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right) \psi = 0,$$

where $z$ is a conformal parameter on the surface,

$$Z_1 = \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2), \quad Z_2 = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), \quad Z_3 = \psi_1 \bar{\psi}_2$$

and

$$f^{-1} f_z = \sum_{k=1}^{3} Z_k e_k$$

is the linear expansion of $f^{-1} f : M \to T_1 \text{Nil}$ in the generators $e_1, e_2, e_3$ of the Lie algebra, of Nil, identified with the tangent space to Nil at the unity.

The nonlinearity is hidden in the potentials $U$ and $V$ and for $G = \text{Nil}$ we have

$$U_{\text{Nil}} = V_{\text{Nil}} = \frac{H}{2} (|\psi_1|^2 + |\psi_2|^2) + \frac{i}{4} (|\psi_2|^2 - |\psi_1|^2),$$
where $H$ is the mean curvature.

Therewith the induced metric equals
\[ds^2 = (|\psi_1|^2 + |\psi_2|^2)^2 \, dzd\bar{z},\]
the Hopf differential $A = (\nabla_{f_z} f_z, n) (dz)^2$ takes the form
\[A = (\bar{\psi}_2 \partial \psi_1 - \psi_1 \partial \bar{\psi}_2) + i\psi_1^2 \bar{\psi}_2^2.\]

Let us denote by $n$ the normal vector, to the surface, translated to $T_1 \text{Nil}$ by the left multiplication by $f^{-1}$. It is equal to
\[n = e^{-\alpha} \left[ i(\psi_1 \psi_2 - \bar{\psi}_1 \bar{\psi}_2)e_1 - (\psi_1 \psi_2 + \bar{\psi}_1 \bar{\psi}_2)e_2 + (|\psi_2|^2 - |\psi_1|^2)e_3 \right], \quad (3)\]

The derivational equations express derivatives of $\psi$ in $z$ and $\bar{z}$ and this system is formed by (2) and the equations
\[\partial \psi_1 = \alpha_z \psi_1 + Ae^{-\alpha} \psi_2 - \frac{i}{2} \psi_1^2 \bar{\psi}_2,\]
\[\bar{\partial} \psi_2 = -\bar{A}e^{-\alpha} \psi_1 + \alpha_z \psi_2 - \frac{i}{2} \bar{\psi}_1 \psi_2^2.\]

The derivational equations are obtained by simple straightforward computations and one of the immediate consequences is as follows

- **a surface (in Nil) has constant mean curvature if and only if the quadratic differential**
\[\tilde{A}dz^2 = \left( A + \frac{Z_3^2}{2H+i} \right) dz^2 \quad (4)\]

is holomorphic. \footnote{Since it was proved earlier in $[2]$ that for constant mean curvature (cmc) surfaces in the products $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ some generalizations of the Hopf differential are holomorphic and the same was announced for surfaces Nil and other three-spaces with four-dimensional isometry group in $[1]$ and since these results by Abresch and Rosenberg motivated us to prove the same by our methods, in $[3]$ we attributed the statement on the holomorphicity of this differential for cmc surfaces to Abresch. However the detailed analysis of the formulas from $[1,2]$ shows that the Abresch–Rosenberg differential has the form
\[(H + i\tau)\tilde{A}dz^2.\]

Therewith Nil is locally considered as a one-dimensional fibration over the flat two-space with the bundle curvature $\tau$.}
For surfaces in $\mathbb{R}^3$ it is clear that Hopf differential is holomorphic if and only if the surface has constant mean curvature. We did manage to generalize this fact for surfaces in Nil and did not succeed for surfaces in $\widetilde{SL_2(\mathbb{R})}$. It appears to be impossible. Recently it was showed by Fernández and Mira that for surfaces in this group there are non-compact not cmc surfaces which the differential $\tilde{A}dz^2$ is holomorphic however all compact surfaces for which this differential is holomorphic are cmc surfaces (see this and similar results related to surfaces in other three-spaces with four-dimensional isometry group in [4]).

In [3] we also introduce the (spinor) energy of a compact surface $M$ without boundary in Nil (and also in $\widetilde{SL_2(\mathbb{R})}$) as

$$E(M) = \int_M UV \frac{idz \wedge d\bar{z}}{2}.$$  

For a surface in $\mathbb{R}^3$ $U = \bar{U} = V$ in its Weierstrass representation and $E(M) = \frac{1}{4} \mathcal{W}$ where $\mathcal{W}$ is the Willmore functional [10]. The point of view based on the spectral theory of Dirac operators $D$ coming in the Weierstrass representations and taken and demonstrated in [10] [11] assumes that the spectral properties of $D$ have to have important geometrical meanings. Thus we treat the functional $E$ as the spectral generalization of the Willmore functional. Although the product $UV$ is complex-valued it was showed in [3] that the integral taken over a compact surface without boundary equals

$$E(M) = \frac{1}{4} \int_M \left( H^2 + \frac{\hat{K}}{4} - \frac{1}{16} \right) d\mu$$  

where $\hat{K}$ is the sectional curvature of Nil along the tangent plane to a surface and $d\mu = e^{2\alpha} dx \wedge dy$ is the induced measure on $M$.

3 Constant mean curvature spheres in the Heisenberg group

3.1 The main equation for surfaces with $\tilde{A} = 0$

Let us first formulate some simple identities obtained from the derivational formulas and checked by straightforward computations:

$$\frac{\partial n_3}{\partial z} = \left( -H + \frac{i}{2} \right) Z_3 - 2e^{-2\alpha} A\bar{Z}_3$$  

(6)
where \( n_3 = \langle n, e_3 \rangle \),

\[ e^{2\alpha} = \frac{4|Z_3|^2}{1 - n_3^2} \]  

(7)

and

\[ \frac{\partial Z_3}{\partial z} = (2H - i)|Z_3|^2 \frac{n_3}{1 - n_3^2}. \]  

(8)

The formula (8) easily follows from the derivational equations written in terms of the immersion \( f \):

\[ \nabla_f n = -Hf - 2Ae^{-2\alpha}f. \]

Let us suppose that the differential \( \tilde{A}dz^2 \) vanishes which, in particular, implies that the mean curvature of a surface is constant [3]:

\[ A = -\frac{Z_3^2}{2H + i}, \quad H = \text{const}. \]  

(9)

Substituting (9) into (8) and expressing \( e^{2\alpha} \) via (7), we obtain

\[ \frac{\partial n_3}{\partial z} = \left( -\frac{i}{2} + \frac{1 - n_3^2}{4H + 2i} \right) Z_3 \]  

(10)

which together with (8) implies

\[ \Delta n_3 + \frac{2n_3(n_3x^2 + n_3y^2)}{1 - n_3^2} = 0. \]  

(11)

We have

**Proposition 1** For a surface in \( \text{Nil} \) with vanishing differential \( \tilde{A}dz^2 \) the equation (11) holds.

Moreover the metric \( e^{2\alpha} \) on this surface is uniquely reconstructed from the function \( n_3 \) and the constant \( H \).

The first statement is already derived. To prove the second part it is enough to reconstruct \( Z_3 \) from (10) and then by using (7) derive

\[ e^{2\alpha} = \frac{4}{1 - n_3^2} \frac{16H^2 + 4}{(4H^2 + n_3^2)^2} \left| \frac{\partial n_3}{\partial z} \right|^2. \]  

(12)
3.2 CMC spheres of revolution

Constant mean curvature of revolution and more general cmc surfaces with helicoidal symmetry were described in different terms in [5]. We demonstrate here how a description of cmc spheres is straightforwardly derived via the Weierstrass representation. Moreover these computations will be necessary for us for computing the values of different functionals (area, bounded volume, spinor energy) on these spheres.

Let us consider the following solutions to (11):

\[ n_3 = \frac{r^2 - 1}{r^2 + 1}, \]  

where \( r^2 = x^2 + y^2, z = x + iy. \)

If such a solution corresponds to a surface with \( \tilde{A} = 0 \) then, by (12), the induced metric \( e^{2\alpha} dzd\bar{z} \) on the surface takes the form

\[ e^{2\alpha} = \frac{16(1 + 4H^2)(1 + r^2)^2}{(r^2 - 1)^2 + 4H^2(1 + r^2)^2} \]  

We look for a surface which is obtained by a revolution of curve \( \gamma(r) = (\rho(r), \psi(r), h(r)) \) and for which the induced metric takes the form (14) where \( x = r \cos \theta, y = r \sin \theta \) are the conformal coordinates on the surface with \( \theta \) the angle of rotation.

The induced metric on a surface of revolution in the coordinates \( r \) and \( \theta \) is equal to

\[
\left( \rho^2 + \frac{\rho^4}{4} \right) d\theta^2 + \left( 2\rho^2 + \frac{\rho^4}{2} \right) \psi' - \rho^2 h' \right) dr d\theta + \\
\left( h'^2 + \rho^2 \psi'^2 + \frac{1}{4} \rho^4 \psi'^2 - \rho^2 h' \psi' + \rho^2 \right) dr^2 .
\]

Such a metric takes the form \( e^{2\alpha} dzd\bar{z} \) if and only if \( \rho, h \) and \( \psi \) satisfy the following equations:

\[
\rho^2 + \frac{\rho^4}{4} = r^2 e^{2\alpha}, \quad \left( 2\rho^2 + \frac{\rho^4}{2} \right) \psi' - \rho^2 h' = 0, \\
h'^2 + \rho^2 \psi'^2 + \frac{1}{4} \rho^4 \psi'^2 - \rho^2 h' \psi' = e^{2\alpha} .
\]

This system is rewritten as follows:

\[
\rho = \sqrt{\sigma}, \quad h' = \sqrt{1 + \frac{\sigma}{4}} \sqrt{e^{2\alpha} - \frac{\sigma^2}{4\sigma}}, \quad \psi' = \frac{1}{2} \sqrt{\frac{e^{2\alpha} - \frac{\sigma^2}{4\sigma}}{1 + \frac{\sigma}{4}}}
\]
where \( \sigma = 2 \sqrt{1 + r^2 e^{2\alpha}} - 2 \).

If \( e^{2\alpha} \) takes the form (14) then

\[
\sigma = \frac{16r^2}{(r^2 - 1)^2 + 4H^2(r^2 + 1)^2}
\]

and we have

\[
\rho = \sqrt{\frac{16r^2}{(r^2 - 1)^2 + 4H^2(r^2 + 1)^2}},
\]

\[
h' = \frac{16H(1 + 4H^2)r(1 + r^2)^2}{((r^2 - 1)^2 + 4H^2(1 + r^2)^2)^2},
\]

\[
\psi' = \frac{8Hr}{(r^2 - 1)^2 + 4H^2(1 + r^2)^2}.
\]

The final formulas for the generating curve \( \gamma(r) \) of the surface of revolution are as follows:

\[
\rho = \frac{4r}{\sqrt{(r^2 - 1)^2 + 4H^2(r^2 + 1)^2}},
\]

\[
h = \frac{1 + 4H^2}{4H^2} \left( - \frac{4H(1 - r^2 + 4H^2(1 + r^2))}{(r^2 - 1)^2 + 16H^4(1 + r^2)^2 + 8H^2(1 + r^4)} + \arctan \left[ \frac{1}{4H(r^2 - 1 + 4H^2(r^2 + 1))} \right] \right),
\]

\[
\psi = \arctan \left[ \frac{1}{4H(4H^2 - 1 + (1 + 4H^2)r^2)} \right],
\]

where \( r \in [0, \infty] \).

The following proposition is checked by straightforward computations.

**Proposition 2** For any \( H, 0 < H < \infty \), the curve (15) generates by revolution a sphere with constant mean curvature \( H \).

Let \( T_1\text{Nil} \) be the \( S^1 \)-fiber bundle over \( \text{Nil} \) formed by all unit vectors. We denote by \( \hat{\mathcal{f}} : M \to T_1\text{Nil} \) the Gauss map which corresponds to a point \( p \in M \) the unit normal vector at \( p \).

**Proposition 3** Given \( H, 0 < H < \infty \), for any point \( q \in T_1\text{Nil} \) there exists a sphere of revolution \( M \) with constant mean curvature \( H \) such that \( q \in \hat{\mathcal{f}}(M) \).
Of course, here and in the sequel we mean by spheres of revolution not only spheres given by \((15)\) but also their left translates in the group \(\text{Nil}\).

**Proof of Proposition 3** Let \(q = (p, \xi)\) with \(p \in \text{Nil}\) and \(\xi \in T_p\text{Nil}\). Given the sphere \(S_H\), it follows from \((15)\) that \(n_3\) takes all values from \(-1\) till 1. Therefore let us take \(p_1 \in S_H\) such that \(\xi_3 = n_3(p_1)\) and translate \(S_H\) into a cmc sphere \(S_1\) by left-translation \(g \rightarrow hg\) such that \(hp_1 = p\). The normal to \(S_1\) at \(p_1\) equals \((\xi_1, \xi_2, \xi_3)\). Then we rotate \(S_1\) around the \(z\)-axis coming through \(p\) to achieve a sphere \(S_2\) for which the normal at \(p\) is equal to \(\xi\). This proves the proposition.

### 3.3 CMC spheres

To finish the description of all cmc spheres in \(\text{Nil}\) we are left to show all such spheres are just spheres of revolution: this fact was proved in [2] for surfaces in \(S^2 \times \mathbb{R}\) and \(H^2 \times \mathbb{R}\) and was stated in [1] for other three-manifolds with four-dimensional isometry group. Moreover in [1] it is explained that the proof for the latter case is almost the same as for the cases of products [2]. Here we expose such a proof for in the particular case of \(\text{Nil}\).

**Proposition 4 (Abresch–Rosenberg [1])** Given \(H, 0 < H < \infty\), any complete surface with \(\tilde{A} = 0\) is a sphere of revolution.

**Proof.** One of the Gauss–Weingarten equations reads

\[
\nabla f_z n = -H f_z - 2A e^{-2\alpha} f_z.
\]

(16)

Since \(\tilde{A} = 0\), we have \(A = -\frac{Z_2^2}{2H+1}\) and therefore at any point \(p\) the vectors \(\nabla f_z n\) are \(\nabla f_y n\) uniquely defined by \(f_z, f_z\), and the point \(p\). Moreover we have

\[
(\nabla f_z n)^i = \frac{\partial n_i}{\partial x} + \Gamma^i_{jk}(p)f_j n^k.
\]

The equation (16) takes the same form for any conformal coordinate \(w = w(z)\) on the surface. In fact it defines a two-plane \(\Pi_q\) in \(T_q(T_1\text{Nil})\) with \(q = (p, n)\) such that \(\Pi_q\) is tangent to the image of the Gauss mapping of any surface with \(\tilde{A} = 0\). Thus we have a two-dimensional distribution \(\Pi\) on \(T_1\text{Nil}\). Any integral surface of this distribution is uniquely determined by any its point and, since through any point of \(T_1\text{Nil}\) goes the image of the Gauss map of a cmc sphere, we conclude that all complete surfaces with \(\tilde{A} = 0\) are cmc spheres. Proposition is proved.
3.4 Remark on the isoperimetric problem for Nil

Since the metric on the sphere equals
\[ e^{2\alpha}(dr^2 + r^2d\theta^2), \]
the area element is equal
\[ d\mu = re^{2\alpha}drd\theta. \]

Substituting (14) into this formula we compute the area \( A(H) \) of the sphere \( S_H \) with constant mean curvature \( H \):
\[
A(H) = \int_{S_H} d\mu = 2\pi \int_0^\infty re^{2\alpha}dr = 2\pi \int_0^\infty \frac{16(1 + 4H^2)r(1 + r^2)^2}{((r^2 - 1)^2 + 4H^2(r^2 + 1)^2)^2} dr = 2\pi \left( \frac{1}{H^2} + \frac{1 + 4H^2}{4H^3} \left( \frac{\pi}{2} - \arctan \left[ \frac{4H^2 - 1}{4H} \right] \right) \right).
\]

In the domain \( D_H \) bounded by the sphere \( S_H \) we take for coordinates the parameters \( \delta \in [0, 1], r \in (0, \infty) \) and \( \theta \in [0, 2\pi] \) such that the cylindrical coordinates of the point \( (\delta, r, \theta) \) are equal to \( \rho = \delta\rho(r), \phi = \psi(r) + \theta \) and \( h = h(r) \) where the functions \( \rho(r), h(r), \) and \( \psi(r) \) from (15) define a sphere of revolution.

We have
\[ d\rho = \delta\rho'(r)dr + \rho(r)d\delta, \quad d\phi = \psi'(r)dr + d\theta, \quad dh = h'(r)dr \]
and, substituting these formulas to (11), we compute the induced metric:
\[
ds^2 = \rho^2d\delta^2 + 2\delta\rho\rho'dr d\delta + (\delta^2\rho^2 - \delta^2\rho^2h'\psi' + \delta^2\rho^2\psi'^2 + \frac{1}{4}\delta^4\rho^4\psi'^2 + h'^2)dr^2 + \frac{1}{4}\delta^2\rho^2(4 + \delta^2\rho^2)d\theta^2 + (-\delta^2\rho^2h' + 2\delta^2\rho^2\psi' + \frac{1}{2}\delta^4\rho^4\psi')dr d\theta,
\]
and the volume form \( d\nu \):
\[
d\nu = (\rho^2h'\delta) d\delta dr d\theta = \frac{256H(1 + 4H^2)r^3(1 + r^2)^2}{(r^2 - 1)^2 + 4H^2(1 + r^2)^2} dr d\delta d\theta.
\]

Therefore the volume \( V(H) \) of \( D_H \) equals
\[
V(H) = \int_{D_H} d\nu = \pi \int_0^\infty \frac{256H(1 + 4H^2)r^3(1 + r^2)^2}{(r^2 - 1)^2 + 4H^2(1 + r^2)^2} dr = \]
\[ \frac{\pi}{16H^4} \left( 4H(4H^2 + 3) - (4H^2 + 1)(4H^2 - 3) \left( \frac{\pi}{2} - \arctan \left[ \frac{4H^2 - 1}{4H} \right] \right) \right). \]

Finally we obtain the relation between the area \( V(H) \) of a cmc sphere and the volume \( V(H) \) of the domain bounded by this sphere:

\[ V(H) = \frac{4\pi}{H} - \frac{4H^2 - 3}{8H} A(H). \] (17)

Conjecturally the relation between \( A(H) \) and \( S(H) \) gives a solution to the isoperimetric problem for \( \text{Nil} \).

For a general \( n \)-dimensional Riemannian manifold this problem consists in finding a hypersurface \( S \) which minimizes the \((n-1)\)-volume \( V_{n-1} \) among all surfaces bounding domains of \( n \)-volume \( d \). This surface has constant mean curvature and is called isoperimetric and we denote its volume by \( V_{n-1}(d) \). From the geometric measure theory it is known that for \( n \leq 7 \) an isoperimetric hypersurface is smooth [6].

For \( \mathbb{R}^3 \) the isoperimetric surfaces are the round spheres and \( V_2(d) = (24\pi d^2)^{1/3} \). This was originally proved by Schmidt in 1930s by the symmetrization method [8] however now it also can be derived from the Alexandrov theorem that all embedded compact cmc surfaces without boundary in \( \mathbb{R}^3 \) are homeomorphic to spheres and the Hopf theorem that all cmc spheres in \( \mathbb{R}^3 \) are the round spheres.

The analog of the Alexandrov theorem is not known for \( \text{Nil} \). However it is very unlikely that isoperimetric surfaces in \( \text{Nil} \) are non-spherical and it is a reasonable and known conjecture that isoperimetric surfaces in \( \text{Nil} \) are homeomorphic to spheres. If it is true the cmc spheres \( S_H, 0 < H < \infty \), give isoperimetric surfaces for all \( d, 0 < d < \infty \). We remark that for a compact Riemannian manifold and for small volumes the isoperimetric hypersurfaces are homeomorphic to a sphere [7].

\footnote{After the posting of the first version of this paper in the internet F. Morgan pointed out to us the paper [12] where the constant mean curvature spheres of revolution in \( \text{Nil} \) are described and it is proved that for small volumes, i.e. for \( H \gg 0 \), they are solutions to the isoperimetric problem.}
4 The spectral generalization of the Willmore functional

For closed oriented surfaces in Nil the spinor energy functional introduced in [3] is equal to

$$E(M) = \int_M UV \frac{dz \wedge d\bar{z}}{2} =$$

$$= \frac{1}{4} \int_M \left( H^2 - \frac{n_3^2}{4} \right) d\mu = \frac{1}{4} \int_M \left( H^2 + \frac{\tilde{K}}{4} - \frac{1}{16} \right) d\mu.$$  \hfill (18)

For cmc spheres $S_H$ the spinor energy takes the form

$$E(S_H) = \frac{\pi}{2} \int_0^\infty \left( H^2 - \frac{1}{4} n_3^2 \right) e^{2\alpha r} dr$$

where $n_3$ and $e^{2\alpha}$ are given by (13) and (14). Substituting these formulas for $n_3$ and the metric into (19), by straightforward computations we prove

**Theorem 1** *For all cmc spheres in Nil the spinor energy is equal to $\pi$:*

$$E(S_H) = \pi.$$  \hfill (20)

Let us compute the classical Willmore functional

$$W(M) = \int_M (H^2 + \tilde{K}) d\mu.$$  \hfill (21)

for these spheres. Since for surfaces in Nil we have $\tilde{K} = \frac{1}{4} - e^{-2\alpha} (|\psi_2|^2 - |\psi_1|^2) = \frac{1}{4} - n_3^2$, it follows from (13) and (21) that $\int_{S_H} \tilde{K} d\mu = 16\pi - (4H^2 - \frac{1}{4}) A(H)$ and finally we derive that

$$W(S_H) = 10\pi + \frac{\pi}{2H^2} - \frac{\pi}{2} \left( \frac{1 + 4H^2}{2} \right) A(H)^2.$$  \hfill (22)

Therefore we see that the Willmore functional does not take a constant value on constant mean curvature spheres.

Let us compute the spinor energy for closed surfaces of revolution.

We have the $SO(2)$-action on Nil by rotations around the $z$-axis and the quotient space $\text{Nil} / SO(2)$ is the half-plane $u \geq 0$ with the local coordinates $u = \rho$ and $v = z$ where $\rho, \phi$, and $z$ are the cylindrical coordinates. By (11), there is a submersion $\text{Nil} \to B = \text{Nil} / SO(2)$

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where \( \text{Nil}/SO(2) \) is endowed with the metric
\[
du^2 + \frac{4u^2}{4u^2 + u^4}dv^2.
\]

Let \( \gamma(s) = (u(s), v(s)) \) be a smooth curve in \( B \) which generates by revolution a smooth surface in \( \text{Nil} \). Here we denote by \( s \) the natural parameter on \( \gamma \). Let \( \sigma \) be the angle between \( \gamma \) and the direction \( \partial / \partial u \). We have the following formulas for the tangent vector \( t \) and the normal vector \( n \):
\[
\begin{align*}
t &= (\cos \sigma, (2u)^{-1}\sqrt{4u^2 + u^4}\sin \sigma), \\
n &= (-\sin \sigma, (2u)^{-1}\sqrt{4u^2 + u^4}\cos \sigma).
\end{align*}
\]

Moreover \( u, v, \) and \( \sigma \) satisfy the following ordinary differential equations:
\[
\begin{align*}
\dot{u} &= \cos \sigma, \\
\dot{v} &= (2u)^{-1}\sqrt{4u^2 + u^4}\sin \sigma, \\
\dot{\sigma} &= 2H - u^{-1}\sin \sigma,
\end{align*}
\]
where the dot denotes the derivation in \( s \). It follows from (22) that
\[
H = \frac{1}{2}(\dot{\sigma} + u^{-1}\sin \sigma).
\]

These formulas for \( t, n, \) and \( H \) were derived in [5].

It is easy to compute
\[
n_3 = \left\langle n, \frac{\partial}{\partial z} \right\rangle = \frac{2u}{\sqrt{4u^2 + u^4}}\cos \sigma, \quad d\mu = \frac{1}{2}\sqrt{4u^2 + u^4}d\theta ds.
\]

Rewriting the functional \( E \) in terms of \( u \) and \( \sigma \) we compute
\[
E(M) = \frac{\pi}{4} \int_{\gamma} \left[ \frac{(\dot{\sigma} - \frac{\sin \sigma}{u})^2}{4} \sqrt{4u^2 + u^4} + \frac{\dot{\sigma} \sin \sigma}{u} \sqrt{4u^2 + u^4} - \frac{u^2 \cos^2 \sigma}{\sqrt{4u^2 + u^4}} \right] ds =
\]
\[
= \frac{\pi}{4} \int_{\gamma} \left[ \frac{(\dot{\sigma} - \frac{\sin \sigma}{u})^2}{4} \sqrt{4u^2 + u^4} + \frac{\dot{\sigma} \sin \sigma}{u} \sqrt{4u^2 + u^4} - \frac{u^2 \cos^2 \sigma}{\sqrt{4u^2 + u^4}} \right] ds.
\]

Now we are left to notice that
\[
\frac{\pi}{4} \int_{\gamma} \left( \frac{\dot{\sigma} \sin \sigma}{u} \sqrt{4u^2 + u^4} - \frac{u^2 \cos^2 \sigma}{\sqrt{4u^2 + u^4}} \right) ds =
\]
\[
= -\frac{\pi}{4} \int_{\gamma} \left( \dot{u} \sqrt{4u^2 + u^4} + \frac{u}{\sqrt{4u^2 + u^4}} \right) ds = -\frac{\pi}{4} \int_{\gamma} \frac{\partial}{\partial s} (\dot{u} \sqrt{4u^2 + u^4}) ds.
\]
Thus we prove the following theorem.
**Theorem 2**  Given a closed surface $M$ in $\text{Nil}$ obtained by revolving a curve $\gamma \subset B$ around the $z$-axis, the spinor energy of $M$ equals

$$E(M) = \frac{1}{4} \int_{\gamma} \left( H^2 - \frac{1}{4} n_3^2 \right) d\mu =$$

$$\frac{\pi}{8} \int_{\gamma} \left( \dot{\sigma} - \frac{\sin \sigma}{u} \right)^2 \sqrt{4u^2 + u^4} ds - \frac{\pi}{4} \int_{\gamma} \frac{\partial u \sqrt{4 + u^2}}{\partial s} ds =$$

$$\frac{\pi}{8} \int_{\gamma} \left( \dot{\sigma} - \frac{\sin \sigma}{u} \right)^2 \sqrt{4u^2 + u^4} ds + \frac{\pi \chi(M)}{2}$$

where $\chi(M)$ is the Euler characteristic of $M$.

Moreover if $\dot{\sigma} = \frac{\sin \sigma}{u}$ everywhere on the surface then it is a cmc sphere.

**Corollary 1**  For spheres of revolution $E(M) \geq \pi$ and the equality is attained exactly at cmc spheres.

**Corollary 2**  For tori of revolution $E(M) > 0$.

We also have

**Theorem 3**  The cmc spheres in $\text{Nil}$ are the critical points of the spinor energy functional $E$.

**Proof.**  The Euler–Lagrange equation for $E$ takes the form

$$\Delta H + 2H(H^2 - K) + 2e^{-4\alpha}(AZ_3^2 + \bar{A}Z_3^2) = 0$$

(see Theorem 4 in §6). The cmc spheres meets the equations $H = \text{const}$ and $A = -\frac{Z_3^2}{2H + 1}$, which imply

$$\Delta H = 0, \quad 2H(H^2 - K) = 8e^{-4\alpha} H |A|^2 = 8e^{-4\alpha} H \frac{|Z_3|^4}{4H^2 + 1},$$

$$2e^{-4\alpha} (AZ_3^2 + \bar{A}Z_3^2) = -8e^{-4\alpha} H \frac{|Z_3|^4}{4H^2 + 1}.$$}

It follows from these formulas that the cmc spheres in $\text{Nil}$ meet the Euler–Lagrange equation for $E$. Theorem is proved.
5 Final remarks and open questions

We see that the energy functional in many geometrical respects behaves similarly to the functional
\[ \frac{\mathcal{W}(M)}{4} = \frac{1}{4} \int_M H^2 d\mu \]
for closed oriented surfaces in \( \mathbb{R}^3 \).

Indeed

1. as in (20) we have
\[ \frac{\mathcal{W}}{4} = \pi \]
for all isoperimetric surfaces, i.e. the round spheres, in \( \mathbb{R}^3 \);

2. we have
\[ \frac{\mathcal{W}(M)}{4} = \frac{1}{4} \int_M \left( \left( \frac{\kappa_1 - \kappa_2}{2} \right)^2 + \kappa_1 \kappa_2 \right) d\mu = \frac{1}{4} \int_M \left( \frac{\kappa_1 - \kappa_2}{2} \right)^2 d\mu + \frac{\pi}{2} \chi(M) \]
where \( \kappa_1 \) and \( \kappa_2 \) are the principal curvatures. The latter formula is similar to (23) however the quantities \( \dot{\sigma} \) and \( \sin \sigma u \) are not the principal curvatures of a surface of revolution;

3. the condition \( A = 0 \) distinguishes among complete compact surfaces in \( \mathbb{R}^3 \) exactly thecmc spheres which are the minima of the Willmore functional \( \mathcal{W} \) for surfaces. Among closed surfaces of revolution in Nil the similar condition \( \tilde{A} = 0 \) distinguishes exactly thecmc spheres which are the minima of the functional \( E \) (among spheres of revolution).

These geometrical observations confirm that the functional \( E \) coming from the spectral theory of the Weierstrass representation sounds to be the right generalization of the Willmore functional to surfaces in Nil.

We also have to consider the equation \( \tilde{A} = 0 \) as distinguishing generalized umbilic surfaces: both in \( \mathbb{R}^3 \) and in Nil the complete compact “umbilic” surfaces are the cmc spheres. We remark that for cmc spheres in Nil only the poles, i.e points invariant under rotation symmetry, are umbilics in the classical sense.

From the point of view of this generalization the following open problems are interesting for study:
1. to prove that \( E \) is bounded from below for each topological type of closed oriented surfaces or even to prove that \( E \) is positive;

2. to prove that the cmc spheres are the global minima of \( E \) among spheres;

3. to generalize the formula (23) for general surfaces;

4. to find the minima of \( E \) among surfaces of fixed topological type and, in particular, to find the substitution of the Willmore conjecture.

Of course it sounds interesting to consider the same questions for surfaces in \( \tilde{SL}_2(\mathbb{R}) \) for which case the Weierstrass representation and the energy functional were also derived in [3].

### Appendices

#### 6.1 The Euler–Lagrange equations for the functional \( E \)

**Theorem 4** Let \( f : M \to \text{Nil} \) be a regular surface and \( r : M \times [0, 1] \to \text{Nil} \) be its smooth variation: \( r_0 = f \). We assume that \( r \) is constant on the boundary of \( M \) if it exists: \( r(p,t) = f(p) \). Let \( \frac{\partial r(p,t)}{\partial t} \big|_{t=0} = \varphi n \) where \( n \) is the unit normal field to \( M \). Then the variation of \( E \) at \( t = 0 \) equals

\[
\delta E(M) = \frac{1}{4} \int_M \left( \Delta H + 2H(H^2 - K) + 2e^{-4\alpha}(AZ_3^2 + \tilde{A}Z_3^2) \right) \varphi d\mu. \tag{24}
\]

**Proof.** By [3], we have to compute

\[
\delta E = \frac{1}{4} \left( \delta \int_M H^2 d\mu - \delta \int_M \frac{1}{4} n_3^2 \right) d\mu. \tag{25}
\]

Given parameters \( x \) and \( y \) on \( M \) we mean by \( r_1 \) and \( r_2 \) the derivatives \( \frac{\partial r}{\partial x} \) and \( \frac{\partial r}{\partial y} \).

The Gauss–Weingarten equations reads

\[
\nabla_j r_i = \Gamma^k_{ij} r_k + h_{ij} n, \quad \nabla_i n = -h^j_i r_j = -g^{kj} h_{ki} r_j,
\]

where \( g_{ij} \) and \( h_{ij} \) are the first and second fundamental forms of \( M \). By the definition of the mean curvature, we have

\[
\delta d\mu = -2H \varphi d\mu.
\]
a) Let us compute
\[
\delta \int H^2 d\mu = 2 \int H \delta H d\mu + \int H^2 \delta d\mu = 2 \int H(\delta H) d\mu - 2 \int H^3 \varphi d\mu.
\]
We have
\[
2\delta H = \delta 2H = \delta(g^{ij} h_{ij}) = (\delta g^{ij}) h_{ij} + g^{ij} \delta h_{ij}
\]
and the Gauss–Weingarten equations imply that \(\delta g_{ij} = \delta(r_i, r_j) = -2\varphi h_{ij}\).
Since \(0 = \delta (g_{ij} g^{jk}) = g_{ij} \delta g^{jk} + g^{jk} \delta g_{ij} = g_{ij} \delta g^{jk} - 2\varphi h_{ij} g^{jk}\), we have
\[
\delta g^{ij} = 2\varphi g^{ik} h^j_k.
\]

Let us compute \(\delta h_{ij}\) which equals \(\delta h_{ij} = \delta(\nabla_j r_i, n) = \langle \nabla_j r_i, \delta n \rangle + \langle \delta \nabla_j r_i, n \rangle\). By the Gauss–Weingarten equations, we have \(\delta n = -g^{ij} \varphi j r_i\)
which implies \(\langle \nabla_j r_i, \delta n \rangle = -\Gamma^k_{ij} \varphi_k\). We also have \(\delta \nabla_j r_i = \nabla_{\partial t} \nabla_j r_i = \nabla_j \nabla_{\partial t} r_i + (\nabla_{\partial t} \nabla_j r_i - \nabla_j \nabla_{\partial t} r_i) = \nabla_j \nabla_i (\varphi) n + \varphi R(r_j, n) r_i\) from which by straightforward computations we derive that \(\langle \nabla_j \nabla_i \varphi n, n \rangle = \varphi_{ij} - \varphi h^k_k h_{kj}\).
Combining the previous computations we obtain
\[
\delta h_{ij} = -\Gamma^k_{ij} \varphi_k + \varphi_{ij} - \varphi h^k_k h_{kj} + \varphi (R(r_j, n) r_i, n).
\]
Substituting the derived formulas for \(\delta g^{ij}\) and \(\delta h_{ij}\) into (26) we conclude that
\[
2\delta H = g^{ij} (\varphi_{ij} - \Gamma^k_{ij} \varphi_k) + \varphi g^{jk} h^i_k h_{ij} + \varphi g^{ij} (R(r_j, n) r_i, n) = \Delta \varphi + \varphi h^k_k h^i_i + \varphi g^{ij} (R(r_j, n) r_i, n)
\]
where \(\Delta\) is the Laplace–Beltrami operator on the surface. Since \(h^k_k h^i_i = \text{Tr} h^2 = k_1^2 + k_2^2 = (k_1 + k_2)^2 - 2k_1 k_2 = 4H^2 - 2K\), we rewrite the previous formula as
\[
2\delta H = \Delta \varphi + (4H^2 - 2K) \varphi + g^{ij} (R(r_j, n) r_i, n) \varphi.
\]
Therefore, by using the equality \(\int_M (\Delta \varphi) H d\mu = \int_M (\Delta H) \varphi d\mu\) we derive
\[
\delta \int_M H^2 d\mu = \int_M (\Delta H + 2H(\Delta^2 H - 2K) \varphi + H g^{ij} (R(r_j, n) r_i, n)) \varphi d\mu.
\]
Assuming that the coordinates \(x, y\) are curvilinear orthogonal: \(g_{12} = 0\), we have
\[
g^{ij} (R(r_j, n) r_i, n) = g^{11} (R(r_1, n) r_1, n) + g^{22} (R(r_2, n) r_2, n) =
\]
= \tilde{K}(r_1, n) + \tilde{K}(r_2, n)

where \(\tilde{K}(u, v)\) is the sectional curvature of the ambient space along the plane spanned by \(u\) and \(v\).

Let us specialize the formula for \(\delta \int H^2 d\mu\) for the case of surfaces in Nil. In this case the sectional curvature depends only on \(n_3\) and equals \(\frac{1}{4} - n_3^2\) (see, for instance, [3]) and we have \(\tilde{K}(r_1, n) + \tilde{K}(r_2, n) = n_3^2 - \frac{1}{4}\) which implies

\[
\delta \int_M H^2 d\mu = \int_M (\Delta H + 2H(H^2 - K) + H(n_3^2 - \frac{1}{4})\varphi d\mu. \tag{27}
\]

b) Let us compute

\[
\delta \int_M n_3^2 d\mu = \int 2n_3 \delta n_3 d\mu + \int n_3^2 \delta d\mu.
\]

Therefore we assume that \(z = x + iy\) is the conformal parameter on the surface and the metric takes the form \(e^{2\alpha} dz d\bar{z}\).

Since \(\langle n, \partial e_3 \rangle = \langle n, \nabla \phi n e_3 \rangle = \langle n, \phi(\frac{1}{2}n_2 e_1 - \frac{1}{2}n_1 e_2) \rangle = \frac{1}{2}\phi(n_2 n_1 - n_1 n_2) = 0\), \(^3\) we have \(\vartheta n_3 = \delta \langle n, e_3 \rangle = \langle \delta n, e_3 \rangle\) which is equal to \(\langle \delta n, e_3 \rangle = \langle -e^{2\alpha} \varphi_3 r^*_z + \varphi_3 r z, e_3 \rangle\).

Thus we compute

\[
2 \int_M n_3 \delta n_3 d\mu = -4 \int_M n_3 \langle \varphi_3 r^*_z + \varphi_3 r z, e_3 \rangle dx \wedge dy =
\]

\[
= 4 \int_M ((n_3 \langle r^*_z, e_3 \rangle)_z + (n_3 \langle r z, e_3 \rangle)_z) \varphi dx \wedge dy.
\]

By the Weierstrass representation formulas (see [2]), we have \(\langle r^{-1} r_z, e_3 \rangle = Z_3\) and, since the metric is left-invariant, we conclude that

\[
\int_M n_3 \delta n_3 d\mu = 2 \int_M (n_3 (\partial Z_3 + \overline{\partial} Z_3) + (\langle \nabla \partial_z n, e_3 \rangle + \langle n, \nabla \partial_z e_3 \rangle) Z_3 +
\]

\[
+ (\langle \nabla \partial \overline{\partial} n, e_3 \rangle + \langle n, \nabla \partial \overline{\partial} e_3 \rangle) \overline{Z}_3) \varphi dx \wedge dy.
\]

Since \(\partial Z_3 + \overline{\partial} Z_3 = H n_3 e^{2\alpha}\) (see [3]), it follows from [16] that \(\langle \nabla \partial_z n, e_3 \rangle = -HZ_3 - 2A e^{-2\alpha} Z_3\). From the formulas for the Levi-Civita connection on Nil it also follows that \(\langle n, \nabla \partial_z e_3 \rangle = \langle n, \nabla Z_1 e_1 + Z_2 e_2 + Z_3 e_3 \rangle = \langle n_1 e_1 + n_2 e_2 +
\]

Here we use the formulas for the Levi-Civita connection on Nil exposed, for instance, in [4].
\[ n_3 e_3, \frac{1}{2} Z_2 e_1 - \frac{1}{2} Z_4 e_2 \rangle = \frac{1}{2} (n_1 Z_2 - n_2 Z_4) = \frac{i}{2} Z_3. \]

Substituting these formulas into the formula for \( \int_M n_3 \delta n_3 d\mu \) we obtain

\[ \int_M n_3 \langle \delta n, e_3 \rangle d\mu = 2 \int_M H n_3^2 \varphi d\mu + 2 \int_M (-2H |Z_3|^2 - 2A Z_3^2 e^{-2\alpha} - 2 \overline{A} Z_3^2 e^{-2\alpha}) \varphi dx \wedge dy \]

Since \( 4|Z_3|^2 = e^{2\alpha}(1 - n_3^2) \) (see (7)), we have

\[ \int_M n_3 \langle \delta n, e_3 \rangle d\mu = \frac{1}{2} \int_M \left( 6H n_3^2 - 2H - 8e^{-4\alpha}(A Z_3^2 + \overline{A} Z_3^2) \right) \varphi d\mu, \]

and finally derive

\[ \delta \int_M n_3^2 d\mu = \int_M \left( 4H n_3^2 - 2H - 8e^{-4\alpha}(A Z_3^2 + \overline{A} Z_3^2) \right) \varphi d\mu. \quad (28) \]

Now by substituting (27) and (28) into (25) we prove the theorem.

6.2 The isoperimetric problem in \( S^2 \times \mathbb{R} \) and a certain Willmore-type functional

Proposition 5 For a nonminimal constant mean curvature sphere \( M \) in \( S^2 \times \mathbb{R} \) we have

\[ \int_M (H^2 + \widehat{K} + 1) d\mu = 16\pi. \]

Proof. By [2] we know that each cmc sphere is a sphere of revolution for \( H \neq 0 \) and a spherical section for \( H = 0 \). By [8] every constant mean curvature sphere of revolution \( S_H \), with \( H \neq 0 \), is generated by the curve \( \gamma_H = S_H / O(2) \) meeting the following equations \( B = S^2 \times \mathbb{R} / O(2) = \{(x, y) : x \in \mathbb{R}, y \in [0, \pi]\} \):

\[
\frac{dx}{ds} = \cos \sigma, \quad \frac{dy}{ds} = \sin \sigma, \quad \frac{d\sigma}{ds} = h + \cot y \cos \sigma,
\]

where \( \sigma \) is the angle between \( \gamma \) and the \( x \)-axis. Thus \( \widehat{K} = \sin^2 \sigma \) and we have

\[ \int_{S_H} \widehat{K} d\mu = 4\pi \left( 2 - \frac{h^2}{\sqrt{1 + h^2}} \ln \frac{\sqrt{1 + h^2} + 1}{\sqrt{1 + h^2} - 1} \right). \quad (29) \]
By $[8]$, the area of $S_H$ equals

$$A(S_H) = \int_M d\mu = 4\pi \left( \frac{2}{1 + h^2} + \frac{h^2}{(1 + h^2)^{3/2}} \ln \frac{\sqrt{1 + h^2} + 1}{\sqrt{1 + h^2} - 1} \right). \quad (30)$$

Combining (29) and (30) we obtain the proof of the proposition.

As it was proved by Pedrosa $[8]$ the isoperimetric problem for $S^2 \times \mathbb{R}$ is solved by domains bounded by cmc spheres for small volumes $d$ and by products cylinders $S^2 \times [0, \frac{d}{4\pi}]$ for large volumes $d$ with one point $d_0$ of transition from one topological class of solutions to another.

Proposition $[5]$ one more time demonstrates that a certain generalization of the Willmore functional of the type $(H^2 + \alpha \tilde{K} + \beta) d\mu$ respects the isoperimetric surfaces of spherical topology by attaining on them some constant value which is even probably the minimum of the functional.

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