Pairwise Covariates-Adjusted Block Model for Community Detection*

Sihan Huang, Jiajin Sun
Department of Statistics, Columbia University
Yang Feng
Department of Biostatistics, New York University

Abstract

One of the most fundamental problems in network study is community detection. The stochastic block model (SBM) is one popular model with different estimation methods developed with their community detection consistency results unveiled. However, the SBM is restricted by the strong assumption that all nodes in the same community are stochastically equivalent, which may not be suitable for practical applications. We introduce a pairwise covariates-adjusted stochastic block model (PCABM), a generalization of SBM that incorporates pairwise covariate information. We study the maximum likelihood estimates of the coefficients for the covariates as well as the community assignments. It is shown that both the coefficient estimates of the covariates and the community assignments are consistent under suitable sparsity conditions. Spectral clustering with adjustment (SCWA) is introduced to efficiently solve PCABM. Under certain conditions, we derive the error bound of community detection under SCWA and show that it is community detection consistent. In addition, the model selection in terms of the number of communities and the feature selection for the pairwise covariates are investigated, and two corresponding algorithms are proposed. PCABM compares favorably with the SBM or degree-corrected stochastic block model (DCBM) under a wide range of simulated and real networks when covariate information is accessible.

Keywords: Covariates-adjusted; Network; Consistency; Community Detection; Spectral Clustering with Adjustment

*Huang and Sun contribute equally to this work. Corresponding Author: Yang Feng (yang.feng@nyu.edu)
1 Introduction

Networks describe the connections among subjects in a population of interest. Its wide applications have attracted researchers from different fields. In social media, people’s behaviors and interests can be unveiled by network analysis (Facebook friends and Twitter followers). In ecology, a food web of predator-prey interactions can provide valuable information about the habits of individuals and the structure of biocoenosis. There are also wide applications in computer science, biology, physics, and economics (Getoor & Diehl, 2005; Goldenberg et al., 2010; Newman, 1963; Graham, 2014).

Community detection is one of the most studied problems for network data. Communities can be intuitively understood as groups of nodes that are densely connected within groups while sparsely connected between groups. Identifying network communities not only help better understand the structural features of the network but also offers practical benefits. For example, communities in social networks tend to share similar interests, which could provide useful information to build recommendation systems. Most community detection methods fall into two categories: algorithm-based and model-based. For algorithm-based methods (Bickel & Chen, 2009; Newman, 2006; Zhao et al., 2011; Wilson et al., 2014, 2017), we come up with an objective function (e.g., modularity) and then optimize it to conduct community detection. For model-based methods, we assume that the edges are generated from a probabilistic model. Some popular models include the stochastic block model (Holland et al., 1983), mixture model (Newman & Leicht, 2007), degree-corrected stochastic block model (Karrer & Newman, 2011), latent space models (Hoff et al., 2002; Handcock et al., 2007; Hoff, 2008), and so on. For a systematic review of statistical network models, see Goldenberg et al. (2010) and Fortunato (2010).

The classical stochastic block model (SBM) assumes that the connection between each pair of nodes only depends on their community labels. For SBM, community detection consistency has been established for various methods, including modularity maximization (Newman, 2006),
profile likelihood (Bickel & Chen, 2009; Choi et al., 2012), spectral clustering (Rohe et al., 2011; Lei & Rinaldo, 2015), variational inference (Bickel et al., 2013), penalized local maximum likelihood estimation (Gao et al., 2017), among others. In the real world, the connection of nodes may depend not only on community structure but also on nodal or pairwise covariates. For example, in an ecological network, the predator-prey link between species may depend on their prey types as well as their habits, body sizes, and living environment. Incorporating nodal and pairwise information into the network model could help us recover a more accurate community structure.

Depending on the relationship between communities and covariates, there are, in general, two classes of models, as shown in Figure 1: covariates-adjusted and covariates-confounding. \( c, Z, \) and \( A \) stand for latent community label, pairwise covariates, and adjacency matrix, respectively. In Figure 1a, the latent community and the covariates jointly determine the network structure. One typical example of this model is the friendship network between students. Students become friends for various reasons: they are in the same class; they have the same hobbies; they are of the same ethnic group. Without adjusting those covariates, it is hard to believe \( A \) represents any
single community membership. We will analyze one such example in detail in Section 8. On the other hand, covariates sometimes carry the same community information as the adjacency matrix, which is shown in Figure 1b. The name “confounding” comes from the graph model (Greenland et al., 1999). The citation network is a perfect example of this model (Tan et al., 2016). When the research topic is treated as the community label for each article, the citation links would largely depend on the research topics of the article pair. Meanwhile, the distribution of the keywords is also likely to be driven by the specific topic the article is about.

Most researchers modify SBM in the above two ways to incorporate covariates’ information. For the covariates-confounding model, Newman & Clauset (2016) uses covariates to construct the prior for community label and then generates edges by degree-corrected model. Weng & Feng (2016) uses a logistic model as the prior for community labels. Zhang et al. (2016) proposes a joint community detection criterion, which is an analog of modularity, to incorporate node features. For the covariates-adjusted model, Yan et al. (2019) proposes a directed network model with logistic function, but it does not consider possible community structure. Wu et al. (2017) proposes a generalized linear model with low-rank effects to model network edges, which could imply a community structure or a latent space structure though not mentioned explicitly; Ma et al. (2020) presents algorithms for a latent space model that incorporates edge covariates; both of those two works consider penalized MLE with convex relaxation and gradient-based algorithms. For real networks, the true model could be a mix of those two ingredients. We will use an example in Section 6.2 to show some connections between those two types of models.

In this work, we propose a simple yet effective model called **Pairwise Covariates-Adjusted Stochastic Block Model** (PCABM), which extends the SBM by adjusting the likelihood of connections by the contribution of pairwise covariates. Through this model, we can learn how each covariate affects the connections by looking at its corresponding regression coefficient. In addition, we show the consistency and asymptotic normality for MLE of the regression coefficients. Besides likelihood methods, we also propose a novel spectral clustering method called **spectral**
clustering with adjustment (SCWA), which works for sparse settings. Note that Binkiewicz et al. (2017) also uses a modified version of spectral clustering to incorporate nodal covariates, but it is not based on a specific model. We prove desirable theoretical properties for SCWA applied to PCABM, and show that as a fast algorithm, using it as an initial estimator for the likelihood method usually leads to more accurate community detection than random initialization. We also consider the model selection problems of estimating the number of communities and selecting the important confounding covariates, and provide algorithms to address those two problems based on the edge cross-validation framework introduced by Li et al. (2020).

The rest of the paper is organized as follows. In Section 2, we introduce the PCABM. We then show the asymptotic properties of the coefficient estimates in Section 3. Following that, we introduce two methods for community detection, namely a likelihood approach in Section 4 and a spectral approach in Section 5. In addition, we present two algorithms for model selection in Section 6. Simulations and applications on real networks are discussed in Sections 7 and 8, respectively. We conclude the paper with a short discussion in Section 9. All proofs are relegated to the Supplementary Materials.

Here, we introduce some notations to facilitate the discussion. For a square matrix $M \in \mathbb{R}^{n \times n}$, let $\|M\|$ be the operator norm of $M$, $\|M\|_F = \sqrt{\text{trace}(M^T M)}$, $\|M\|_\infty = \max_i \sum_{j=1}^n |M_{ij}|$, $\|M\|_0 = \#\{(i,j) | M_{ij} \neq 0\}$, and $\|M\|_{\max} = \max_{ij} |M_{ij}|$. $\lambda_{\min}(M)$ is the minimum eigenvalue of $M$. For index sets $I, J \subset [n] := \{1, 2, \cdots, n\}$, $M_I$ and $M_J$ are the sub-matrices of $M$ consisting the corresponding rows and columns, respectively. For a vector $x \in \mathbb{R}^n$, let $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ and $\|x\|_\infty = \max_i |x_i|$. We define the Kronecker power by $x \otimes (k+1) = x \otimes k \otimes x$, where $\otimes$ is the Kronecker product.

For any positive integer $K$, we define $I_K \in \mathbb{R}^{K \times K}$ to be the identity matrix, $J_K \in \mathbb{R}^{K \times K}$ to be the all-one matrix, $1_K$ to be the all-one vector. When there is no confusion, we will sometimes omit the subscript $K$. For a vector $x \in \mathbb{R}^K$, $D(x) \in \mathbb{R}^{K \times K}$ represents the diagonal matrix whose diagonal elements take the value of $x$. For an event $A$, its indicator function is written
For two real number sequences \( x_n \) and \( y_n \), we say \( x_n = o(y_n) \) or \( y_n = \omega(x_n) \) if \( \lim_{n \to \infty} x_n/y_n = 0 \), \( x_n = O(y_n) \) or \( y_n = \Omega(x_n) \) if \( \limsup_{n \to \infty} |x_n/y_n| \leq \infty \).

2 Pairwise Covariates-Adjusted Stochastic Block Model

We consider a graph with \( n \) nodes and \( K \) communities, where \( K \) could be fixed or increase with \( n \). In this paper, we focus on undirected weighted graphs without self-loops. All edge information is incorporated into a symmetric adjacency matrix \( A = [A_{ij}] \in \mathbb{N}^{n \times n} \) with diagonal elements being zero, where \( \mathbb{N} \) represents the set of nonnegative integers. The total number of possible edges is denoted by \( N_n = n(n - 1)/2 \). The true node labels \( c = \{c_1, \ldots, c_n\} \in \{1, \ldots, K\}^n \) are drawn independently from a multinomial distribution with parameter vector \( \pi = (\pi_1, \ldots, \pi_K)^T \), where \( \sum_{k=1}^{K} \pi_k = 1 \) and \( \pi_k > 0 \) for all \( k \). The community detection problem aims to find a disjoint partition of the nodes, or equivalently, estimated node labels \( e = \{e_1, \ldots, e_n\} \in \{1, \ldots, K\}^n \) that is close to \( c \), where \( e_i \in \{1, \ldots, K\} \) is the label for node \( i \).

In classical SBM, we assume \( \Pr(A_{ij} = 1|c) = B_{c_ic_j} \), where \( B = [B_{ab}] \in [0, 1]^{K \times K} \) is a symmetric matrix with no identical rows. In practice, the connection between two nodes may depend not only on the communities they belong to, but also on the nodal covariates (e.g., gender, age, religion). To fix idea, assume in addition to \( A \), we have observed a pairwise \( p \)-dimensional vector \( z_{ij} \) between nodes \( i \) and \( j \). Denote the collection of the pairwise covariates among nodes as \( Z = [z_{ij}^T] \in \mathbb{R}^{n^2 \times p} \). Here, we assume \( z_{ij} = z_{ji} \) and \( z_{ii} = 0 \), for all \( i \) and \( j \).

Now, we are ready to introduce the Pairwise Covariates-Adjusted Stochastic Block Model (PCABM). For \( i < j \), conditional on the community label \( c \) and the pairwise covariate matrix \( Z \), \( A_{ij} \)'s are independent and

\[
A_{ij} \sim \text{Poisson}(\lambda_{ij}), \quad \lambda_{ij} = B_{c_ic_j} e^{z_{ij}^T \gamma_0},
\]
where $\gamma^0$ is the true coefficient vector for the pairwise covariates. In addition to the goal of recovering the community membership vector $c$, we would also like to accurately estimate $\gamma^0$.

The specific term $\exp(z^T_{ij} \gamma^0)$ is introduced here to adjust the connectivity between nodes $i$ and $j$. Here, as in the vanilla SBM, we assume a sparse setting for $B = \rho_n \bar{B}$, with $\bar{B}$ fixed and $\rho_n \to 0$ as $n \to \infty$. Note that due to the contribution of $Z$, $\varphi_n = n\rho_n$ is no longer the expected degree as in the vanilla SBM (Zhao et al., 2012), but it is still useful as a measure of the network sparsity. It is easy to observe that when $\gamma^0 = 0$, PCABM reduces into the vanilla Poisson SBM.

Under PCABM, the likelihood function is

$$L(e, \gamma, B, \pi | A, Z) \propto \prod_{i=1}^{n} \pi_{e_i} \prod_{i<j} B_{e_i e_j}^{A_{ij}} e^{A_{ij} z^T_{ij} \gamma} \exp \left( -B_{e_i e_j} e^{z^T_{ij} \gamma} \right).$$

Define

$$n_k(e) = \sum_{i=1}^{n} 1(e_i = k), \quad O_{kl}(e) = \sum_{i,j} A_{ij} 1(e_i = k, e_j = l),$$

$$E_{kl}(e, \gamma) = \sum_{i \neq j} e^{z^T_{ij} \gamma} 1(e_i = k, e_j = l) = \sum_{(i,j) \in s_e(k,l)} e^{z^T_{ij} \gamma},$$

where $s_e(k, l) = \{(i, j) | e_i = k, e_j = l, i \neq j\}$. Under assignment $e$, $n_k(e)$ represents the number of nodes estimated to be in community $k$. For $k \neq l$, $O_{kl}$ is the total number of edges between estimated communities $k$ and $l$; for $k = l$, $O_{kk}$ is twice of the number of edges within estimated community $k$. $E_{kl}$ is the summation of all pair-level factors between estimated communities $k$ and $l$. Up to a constant term, we can write the log-likelihood function as

$$\log L(e, \gamma, B, \pi | A, Z) = \sum_k n_k(e) \log \pi_k + \frac{1}{2} \sum_{kl} O_{kl}(e) \log B_{kl}$$

$$- \frac{1}{2} \sum_{kl} B_{kl} E_{kl}(e, \gamma) + \sum_{i<j} A_{ij} Z^T_{ij} \gamma.$$

Given $e$ and $\gamma$, we derive the MLE $\hat{\pi}_k(e) = \frac{n_k(e)}{n}$ and $\hat{B}_{kl}(e, \gamma) = \frac{O_{kl}(e)}{E_{kl}(e, \gamma)}$. Plugging $\hat{B}(e, \gamma)$
and $\hat{\pi}(e)$ into the original log-likelihood and discarding the constant terms, we have

$$\log L(e, \gamma, \hat{B}, \hat{\pi}|A, Z)$$

$$\propto \frac{1}{2} \sum_{kl} O_{kl}(e) \log \frac{O_{kl}(e)}{E_{kl}(e, \gamma)} + \sum_{i<j} A_{ij} z_{ij}^T \gamma + \sum_k n_k(e) \log \frac{n_k(e)}{n_k(e)}.$$ 

(1)

Out target is to maximize (1) w.r.t. $e$ and $\gamma$. We consider a two-step sequential estimation procedure by first studying the estimation of $\gamma^0$ in Section 3 and then the estimation of $c$ in Section 4 (likelihood method) and Section 5 (spectral method).

It is worth mentioning that the proposed model includes DCBM in the following sense: by choosing $p = 1$, $z_{ij} = \log(d_i d_j)$ and $\gamma = 1$ where $d_i$ is the degree of node $i$, (1) becomes

$$\log L(e, \gamma = 1, \hat{B}, \hat{\pi}|A, Z = (\log(d_i d_j))_{n \times 1}) \propto \frac{1}{2} \sum_{kl} O_{kl}(e) \log \frac{O_{kl}(e)}{n_k(e)n_l(e)},$$

which is exactly the profile log-likelihood under DCBM derived by maximizing over “$\theta$ and $P$” (degree parameter and block connection probability) in DCBM. From this perspective, one can view PCABM as a generalization of DCBM.

3 Estimation of Coefficients for Pairwise Covariates

As the first step to maximize the log-likelihood, we consider the estimation of coefficients $\gamma^0$ for pairwise covariates. To this end, we impose the following conditions on $Z$.

**Condition 1.** $\{z_{ij}, i < j\}$ are i.i.d. and uniformly bounded, i.e., for $\forall i < j$, $\|z_{ij}\|_\infty \leq \zeta$, where $\zeta > 0$ is some constant. Denote $\xi = \exp(\zeta \|\gamma^0\|_1)$.

**Remark 1.** The bounded support condition for $z_{ij}$ is introduced to simplify the proof. It could be relaxed to $z_{ij}$ having a light tail or to allow the upper bound to grow slowly with network size $n$. 

8
By Condition 1, we know that $e^{z_{ij}^T \gamma^0}$ is also uniformly bounded. We denote the bound as $\beta_l \leq e^{z_{ij}^T \gamma^0} \leq \beta_u$. Thus, the following expectations exist: $\theta(\gamma^0) \equiv \mathbb{E}e^{z_{ij}^T \gamma^0} \in \mathbb{R}^+$, $\mu(\gamma^0) \equiv \mathbb{E}z_{ij}e^{z_{ij}^T \gamma^0} \in \mathbb{R}^p$, and $\Sigma(\gamma^0) \equiv \mathbb{E}z_{ij}z_{ij}^Te^{z_{ij}^T \gamma^0} \in \mathbb{R}^{p \times p}$.

To ensure that $\gamma^0$ is the unique solution to maximize the likelihood in the population version, we impose the following regularity condition at the true $\gamma^0$.

**Condition 2.** $\Sigma(\gamma^0) - \theta(\gamma^0)^{-1} \mu(\gamma^0)\otimes 2$ is positive definite.

**Remark 2.** To understand the implication of Condition 2, consider the function $g(\gamma) = \theta(\gamma)\Sigma(\gamma) - \mu(\gamma)\otimes 2$. In the special case of SBM where $\gamma^0 = 0$, we have $g(0) = \mathbb{E}[z\otimes 2] - \mathbb{E}[z] \otimes z = \text{cov}(z)$. To avoid multicollinearity, it’s natural for us to require $\text{cov}(z)$ to be positive definite. For a general PCABM, we require $g(\gamma)$ to be positive definite at the true value $\gamma^0$.

For a given initial community assignment $e_0$, denote by $\ell_{e_0}$ the log-likelihood terms in (1) containing $\gamma$, which is

$$
\ell_{e_0}(\gamma) \equiv \sum_{i<j} A_{ij}z_{ij}^T \gamma - \frac{1}{2} \sum_{kl} O_{kl}(e_0) \log E_{kl}(e_0, \gamma).
$$

We consider the following estimate:

$$
\hat{\gamma}(e_0) = \arg \max_{\gamma} \ell_{e_0}(\gamma). \tag{2}
$$

Note that since $\ell_{e_0}(\gamma)$ is concave in $\gamma$, the global optimizer in (2) can be efficiently solved by a BFGS algorithm. When there is no ambiguity, we will just write it as $\hat{\gamma}$, as we will see in the theory that under some mild conditions, the asymptotic result does not depend on the choice of $e_0$. In fact, one could simply choose $e_0 = 1$, the all-one vector, when estimating $\gamma^0$. To accommodate the “$K$ growing with $n$” case, we also need the following stability condition.

**Condition 3.** $\bar{B}_{\lim} = \lim_{n \to \infty} \sum_{a,b}^K \pi_a \pi_b \bar{B}_{ab}$ exists, where the number of communities $K$ could either be fixed or grow to $\infty$ at an arbitrary rate.
Note that when $K$ is fixed, Condition 3 is automatically satisfied. When $K$ grows with $n$, we need the $\pi$-weighted average of matrix $\bar{B}$ to have a limit. This is a mild condition since otherwise, the sequence of observed graphs indexed by $n$ is not coming from a consistent data generating process. Now, we are ready to present the consistency and asymptotic normality of $\hat{\gamma}$ with the following theorem.

**Theorem 1** (Consistency and asymptotic normality of MLE of $\hat{\gamma}$). Under PCABM, assume Conditions 1, 2 and 3 hold, where the number of communities $K$ could either be fixed or grow to $\infty$ at an arbitrary rate. Then fixing $e_0 = 1$, as $n \to \infty$, if $N_n \rho_n \to \infty$ and $\rho_n \to 0$, we have $\hat{\gamma}(e_0) \xrightarrow{P} \gamma^0$ and
\[
\sqrt{N_n \rho_n} \left[ \hat{\gamma}(e_0) - \gamma^0 \right] \xrightarrow{d} N(0, \Sigma^{-1}_\infty(\gamma^0)),
\]
where $\Sigma_\infty(\gamma^0) = \bar{B} \lim [\Sigma(\gamma^0) - \theta(\gamma^0)^{-1} \mu(\gamma^0) \otimes 2]$.

Different from Yan et al. (2019) in which the network is dense, the convergence rate is $\sqrt{N_n \rho_n}$ rather than $\sqrt{N_n}$ since the effective number of edges is reduced from $N_n$ to $N_n \rho_n$. The asymptotic covariance matrix $\Sigma^{-1}_\infty(\gamma^0)$ depends on $\theta(\gamma^0)$, $\mu(\gamma^0)$, and $\Sigma(\gamma^0)$, which can be estimated empirically by the plug-in method.

Now, with a consistent estimate of $\gamma^0$, we are ready to study the estimation of $c$. In the next two sections, we will present two different methods for estimating $c$, namely the likelihood-based estimate in Section 4 and the spectral method in Section 5.

## 4 Likelihood Based Estimate for Community Labels

This section presents a likelihood based estimate for community labels by maximizing $\log L$ regarding $e_0$ with $\hat{\gamma}$ from Section 3. We only present the fixed $K$ setting here, and the results for the growing $K$ scenario are relegated into the supplementary material, partly because in the class label MLE for growing $K$, we consider a slightly different regime from $B = \rho_n \bar{B}$: we need the
signal-noise-ratio, or approximately in-class probability over between-class probability, to also grow with \( n \) and \( K \), and the conditions are imposed on \( \sum_{i<j} B_{c_i c_j} \) rather than \( \rho_n \). See Section A.4 in the supplementary material for details.

We will show that as long as \( \hat{\gamma}(e_0) \) is consistent, the consistency of \( \hat{c}(\hat{\gamma}) \) is guaranteed. Plugging \( \hat{\gamma} \) into (1), the log-likelihood function can be rewritten as

\[
\ell_{\hat{\gamma}}(e) = \frac{1}{2} \sum_{kl} O_{kl}(e) \log \frac{O_{kl}(e)}{E_{kl}(e, \hat{\gamma})} + \sum_k n_k(e) \log \frac{n_k(e)}{n}.
\]

Then, our maximum likelihood estimate for the community label is

\[
\hat{c} = \hat{c}(\hat{\gamma}) := \arg \max_e \ell_{\hat{\gamma}}(e).
\] (4)

Note that here we omit \( e_0 \) to avoid confusion. Following Zhao et al. (2012), we consider two versions of community detection consistency. Note that the consistency in community detection is understood under any permutation of the labels. To be more precise, let \( \mathcal{P}_K \) be the collection of all permutation functions of \([K]\). (1) We say the label estimate \( \hat{c} \) is weakly consistent if \( \Pr[n^{-1} \sum_{i=1}^n \mathbb{1}(\sigma(\hat{c}_i) \neq c_i) < \varepsilon] \to 1 \) for any \( \varepsilon > 0 \) as \( n \to \infty \). (2) We say \( \hat{c} \) is strongly consistent if \( \Pr[\min_{\sigma \in \mathcal{P}_K} \sum_{i=1}^n \mathbb{1}(\sigma(\hat{c}_i) \neq c_i) = 0] \to 1 \), as \( n \to \infty \). We establish both versions of consistency for MLE \( \hat{c} \) in the following theorem.

**Theorem 2.** Under PCABM that satisfies the Conditions 1 and 2, when \( K \) is fixed, the community label estimate \( \hat{c} \) defined in (4) is weakly consistent if \( \varphi_n \to \infty \) and strongly consistent if \( \varphi_n / \log n \to \infty \), where \( \varphi_n = n \rho_n \).

In addition to the fixed case shown in Theorem 2, we have also shown the consistency of maximum likelihood label estimate in the case when \( K \) grows as fast as \( K = O(\sqrt{n}) \), where we require a slightly stronger condition on the sparsity, \( \varphi_n / (\log n)^{3+\delta} \to \infty \). Details are presented in the supplementary material, Section A.4.

Since finding \( \hat{c} \) is a non-convex problem, we use tabu search (Beasley, 1998; Glover & Laguna, 2013). The detailed algorithm is described in Algorithm 1. The idea of tabu search is to
switch the class labels for a randomly-chosen pair of nodes. If the value of the log-likelihood function increases after switching, we proceed with the switch. Otherwise, we ignore the switch by sticking with old labels. Because this algorithm is greedy, to avoid being trapped in local maximum, we “tabu” those nodes whose labels have been switched recently, i.e., we don’t consider switching the label of a node if it is in the tabu set. Though the global maximum is not guaranteed, tabu search usually gives satisfactory results from our limited numerical experience.

**Algorithm 1: PCABM.MLE0**

| Input: | Adjacency matrix $A$; pairwise covariates $Z$; initial community assignment $e$; the number of communities $K$. |
|--------|--------------------------------------------------------------------------------------------------|
| Output:| Coefficient estimate $\hat{\gamma}$ and community label estimate $\hat{c}$. |

1: Maximize $\ell(\gamma)$ in (2) by some optimization algorithm (e.g., BFGS) to get $\hat{\gamma}$.

2: Use tabu search to maximize $\ell_\gamma(e)$ in (4) to get $\hat{c}$.

5 Spectral Clustering with Adjustment

Though the likelihood-based method has appealing theoretical properties, tabu search can sometimes be slow when the network size is large. In addition, the community detection results can be sensitive to the initial label assignments $e$. As a result, we propose a computationally efficient algorithm in the flavor of spectral clustering (Rohe et al., 2011), which can also be used as the initial community label assignments for PCABM.

5.1 A Brief Review on Spectral Clustering

First, we introduce some notations and briefly review the classical spectral clustering with $K$-means for SBM. Let $\mathbb{M}_{n,K}$ be the space of all $n \times K$ matrices where each row has exactly one 1 and $(K - 1)$ 0’s. We usually call $M \in \mathbb{M}_{n,K}$ a *membership matrix* with $M_{ic_i} = 1$ for node $i$ with community label $c_i$. Note that $M$ contains the same information as $c$, and is only introduced to
facilitate the discussion.

From now on, we use PCABM($M, B, Z, \gamma^0$) to represent PCABM generated with parameters in the parentheses. Let $G_k = G_k(M) = \{1 \leq i \leq n : c_i = k\}$ and $n_k = |G_k|$ for $k = 1, \cdots, K$. Let $n_{\min} = \min_{1 \leq k \leq K} n_k$, $n_{\max} = \max_{1 \leq k \leq K} n_k$ and $n'_{\max}$ is the second largest community size.

For convenience, we define matrix $P = [P_{ij}] \in [0, \infty)^{n \times n}$, where $P_{ij} = B_{c_i, c_j}$, then it is easy to observe $P = MBM^T$. When $A$ is generated from a SBM with $(M, B)$, the $K$-dimensional eigen-decomposition of $P = UDU^T$ and $A = \hat{U} \hat{D} \hat{U}^T$ are expected to be close, where $\hat{U}^T \hat{U} = I_K$ and $\hat{D}, \hat{D} \in \mathbb{R}^{K \times K}$. Since $U$ has only $K$ unique rows, which represent the community labels, the $K$-means clustering on the rows of $\hat{U}$ usually leads to a good estimate of $M$. While finding a global minimizer for the $K$-means problem is NP-hard (Aloise et al., 2009), for any positive constant $\epsilon$, we have efficient algorithms to find an $(1 + \epsilon)$-approximate solution (Kumar et al., 2004; Lu & Zhou, 2016):

$$(\hat{M}, \hat{X}) \in \mathbb{M}_{n,K} \times \mathbb{R}^{K \times K} \quad \text{s.t.} \quad \|\hat{M} \hat{X} - \hat{U}\|_F^2 \leq (1 + \epsilon) \min_{M \in \mathbb{M}_{n,K}, X \in \mathbb{R}^{K \times K}} \|M X - \hat{U}\|_F^2.$$ 

The goal of community detection is to find $\hat{M}$ that is close to $M$. To define a loss function, we need to take permutation into account. Let $S_K$ be the collection of all $K \times K$ permutation matrices. Following Lei & Rinaldo (2015), we define two measures of estimation error: the overall error and the worst case relative error:

$$L_1(\hat{M}, M) = n^{-1} \min_{S \in S_K} \|\hat{M} S - M\|_0, \quad L_2(\hat{M}, M) = \min_{S \in S_K} \max_{1 \leq k \leq K} n_k^{-1} \|\hat{M} S_{G_k} - M_{G_k}\|_0.$$ 

It can be seen that $0 \leq L_1(\hat{M}, M) \leq L_2(\hat{M}, M) \leq 2$. While $L_1$ measures the overall proportion of mis-clustered nodes, $L_2$ measures the worst performance across all communities.

Vanilla spectral clustering on SBM requires the average degree of the network to be of the order $\Omega(\log n)$ (Lei & Rinaldo, 2015), mainly because sparser networks do not have desired concentration properties like $\|A - E_A\| = O(\sqrt{\varphi n})$. In particular, because the true $E_A$ has elements
of the same scale, one can imagine a node with very large degree will harm the closeness between $A$ and $E A$, which is the basis that spectral clustering lies on. Recent works (Le et al., 2017; Gao et al., 2017; Joseph & Yu, 2016) have shown that regularized versions of spectral clustering (Amini et al., 2013), which basically means performing spectral clustering on a regularized adjacency matrix, could enable the concentration of the adjacency matrix under sparser settings and thus relax the average degree assumption required in vanilla spectral clustering. In our algorithms, we adopt the “reduce weight of edges proportionally to the excess of degrees” version of regularization (Le et al., 2017), i.e. assigning weight $\sqrt{\lambda_i \lambda_j}$ to $A_{ij}$, where $\lambda_i := \min\{2d/d_i, 1\}$, $d = \max_{ij} nP_{ij}$, and $d_i$ is the degree of node $i$. \(^1\)

5.2 Regularized Spectral Clustering with Adjustment

The existence of covariates in PCABM prevents us from applying (regularized) spectral clustering directly on $A$. Unlike SBM where $A$ is generated from a low rank matrix $P$, $A$ consists of both community and covariate information in PCABM. Since $P_{ij} = \mathbb{E}[A_{ij}/e^{Z_i^T z_i \gamma_0}]$, an intuitive idea to take advantage of the low rank structure is to remove the covariate effects, i.e. using the adjusted adjacency matrix $[A_{ij}/e^{z_i^T z_j \hat{\gamma}}]$ for spectral clustering.

However, in practice, we don’t know the true value of the parameter $\gamma_0$ is, and we naturally replace $\gamma_0$ with the empirical estimate $\hat{\gamma}$ from (2). To this end, define the adjusted adjacency matrix as $A' = [A'_{ij}]$ where $A'_{ij} = A_{ij} \exp(-z_i^T z_j \hat{\gamma})$. Furthermore, for regularized spectral clustering, define the weighted version of $A'$ to be $A'^R$, called weighted adjusted adjacency matrix. By the asymptotic properties of $\hat{\gamma}$ proved in Theorem 1, we can show $\|A'^R - P\|$ achieves the desirable spectral bound of order $O_p(\sqrt{n})$ as in Le et al. (2017), and the proof is available in Section A.5 of the Supplementary Materials.

Based on this bound, we could then apply the regularized spectral clustering algorithm on

\(^1\)Our algorithms set $d$ as the max degree for simplicity.
matrix $A'$ to detect the communities. We call this adjustment scheme the Spectral Clustering with Adjustment (SCWA) algorithm, which is elaborated in Algorithm 2.

**Algorithm 2: PCABM.SCWA**

**Input:** Adjacency matrix $A$; pairwise covariates $Z$; initial community assignment $e$; the number of communities $K$; approximation parameter $\epsilon$.

**Output:** Coefficient estimate $\hat{\gamma}$, community estimate $\hat{c}$.

1: Maximize $\ell(\gamma)$ as in (2) by some optimization algorithm (e.g., BFGS) to derive $\hat{\gamma}$.
2: Compute the adjusted adjacency matrix $A' = [A'_{ij}]$ where $A'_{ij} = A_{ij} \exp(-Z_{ij}^T \hat{\gamma})$.
3: Compute the weighted adjusted adjacency matrix $A'^R = [A'^R_{ij}]$, where $A'^R_{ij} = A'_{ij} \sqrt{\lambda_i \lambda_j}$, where $\lambda_i = \min\{2d'/d'_i, 1\}$, $d'_i$ is the degree of node $i$ in $A'$ and $d' = \max_i d'_i$.
4: Calculate $\hat{U} \in \mathbb{R}^{n \times K}$ consisting of the leading $K$ eigenvectors (ordered in absolute eigenvalue) of $A'^R$.
5: Calculate the $(1 + \epsilon)$-approximate solution $\hat{M}$ to the $K$-means problem with $K$ clusters and input matrix $\hat{U}$.
6: Output $\hat{c}$ according to $\hat{M}$.

To show the consistency of Algorithm 2, one natural requirement is that $A'^R$ and $P$ are close enough, which is stated rigorously in the following theorem.

**Theorem 3** (Spectral bound of adjusted, regularized Poisson random matrices). Let $A$ be the adjacency matrix generated by the undirected PCABM $(M, B, Z, \gamma^0)$. Assume Conditions 1, 2, 3 hold. Let $d = \max_{ij} nP_{ij}$. Further assume each element of $\bar{B}$ is bounded from above by a constant $C_B$, i.e. $\|\bar{B}\|_{\max} \leq C_B$. For any $r > 1$, the following holds with probability at least $1 - 3n^{-r} - C_\eta \exp(-v_\eta n)$ (where $\eta = (p\zeta)^{-1}$, $C_\eta$, and $v_\eta$ are constants in Lemma A.11): the regularized adjusted adjacency matrix $A'^R$ in Algorithm 2 satisfies

$$\|A'^R - P\| \leq C\sqrt{\varphi_n}$$

where $C$ is a constant that depends on $\xi$, $r$ and $C_B$.
With similar proof of Theorem 3.1 in Lei & Rinaldo (2015), we can prove the following
Theorem 4 by combining Lemmas 5.1 and 5.3 in Lei & Rinaldo (2015), and Theorem 3. Without
loss of generality, we now assume \( \| B_{\max} \| \leq 1 \), making the theorem statement simpler.

**Theorem 4.** In addition to the conditions of Theorem 3, assume that \( P = MBM^T \) is of rank
\( K \) with the smallest absolute non-zero eigenvalue at least \( \xi_n \). Let \( \hat{M} \) be the output of spectral
clustering using \( (1 + \epsilon) \) approximate \( K \)-means on \( A^R \) (defined in Algorithm 2, step 3). For any
constant \( r > 0 \), there exists an absolute constant \( C > 0 \), such that, if
\[
(2 + \epsilon) \frac{Kn\rho_n}{\xi_n^2} < C,
\]
then, with probability at least \( 1 - 3n^{-r} - C\eta \exp(-v_\eta n) \), there exist subsets \( H_k \subset G_k \) for \( k = 1, \ldots, K \), and a \( K \times K \) permutation matrix \( J \) such that \( \hat{M}_GJ = M_G \), where \( G = \bigcup_{k=1}^K (G_k \setminus H_k) \), and
\[
\sum_{k=1}^K \frac{|H_k|}{n_k} \leq C^{-1}(2 + \epsilon) \frac{Kn\rho_n}{\xi_n^2}.
\]

Inequality (7) provides an error bound for the overall relative error. Theorem 4 doesn’t pro-
vide us with an error bound in a straightforward form since \( \xi_n \) contains \( \rho_n \). The following corol-
Jary gives us a clearer view of the error bound in terms of model parameters. The condition that
the maximum normalized probability equals 1 can be replaced by any constant, but we just use 1
here for simplicity, since any constant can always be absorbed into the sparsity parameter \( \rho_n \).

**Corollary 1.** In addition to the conditions of Theorem 3, assume that \( \bar{B}' \)’s minimum absolute
eigenvalue bounded below is by \( \tau > 0 \) and \( \max_{kl} \bar{B}(k, l) = 1 \). Let \( \hat{M} \) be the output of spectral
clustering using \( (1 + \epsilon) \) approximate \( K \)-means. For any constant \( r > 0 \), there exists an absolute
constant \( C \) such that if
\[
(2 + \epsilon) \frac{Kn}{n_{\min}^2 \tau^2 \rho_n} < C,
\]
then with probability at least \( 1 - 3n^{-r} - C\eta \exp(-v_\eta n) \),
\[
L_2(\hat{M}, M) \leq C^{-1}(2 + \epsilon) \frac{Kn}{n_{\min}^2 \tau^2 \rho_n}, \quad L_1(\hat{M}, M) \leq C^{-1}(2 + \epsilon) \frac{K\max_{kl} n}{n_{\min}^2 \tau^2 \rho_n}.
\]
It is worth mentioning that Theorem 3, Theorem 4, and Corollary 1 all allow \( K \) to go to infinity with \( n \).

Compared to SCWA, the likelihood tabu search could lead to more precise results but takes a longer time. Also, the likelihood tabu search results could be sensitive to the initial labels \( e \) in some settings. To combine the advantages of those two methods, we propose to use the results of SCWA as the initial solution for tabu search (PCABM.MLE as described in Algorithm 3). We will conduct extensive simulation studies in Section 7 to study the performances of PCABM.SCWA and PCABM.MLE.

### Algorithm 3: PCABM.MLE

**Input:** Adjacency matrix \( A \); pairwise covariates \( Z \); initial community assignment \( e \); the number of communities \( K \); approximation parameter \( \epsilon \).

**Output:** Community estimate \( \hat{c} \).

1. Use Algorithm 2 to get an initial community estimate \( \tilde{M} \).
2. Use \( \tilde{M} \) as the initial communities for tabu search to find the MLE \( \hat{c} \).

### 6 Model Selection

#### 6.1 Estimating the Number of Communities \( K \)

So far we have been treating the number of communities \( K \) as given, while in practice, the true value of \( K \) may be unknown to us. In those cases, we would be interested in estimating \( K \). To provide a systematic approach, we propose to adapt the edge-sampling cross-validation (ECV) method (Li et al., 2020) to the PCABM. The main idea of the ECV procedure is reviewed as follows: in each iteration, we randomly sample some edges in the network, and predict the rest of the edges under certain models based on matrix completion on the selected edges’ adjacency matrix; after all iterations, we compare the average prediction performances or held-out losses under different models and choose the best model accordingly. A detailed
algorithm of applying this idea for estimating $K$ in PCABM is presented in Algorithm 4. The notation $P_\Omega A$ stands for the matrix that keeps all elements of $A$ in the index set $\Omega$ and sets all the other elements to be 0. As recommended in Li et al. (2020), in the rank-$K$ matrix completion step, we adopt the SVD truncation approach $\hat{A}'_K = S_H \left( \frac{1}{p} P_\Omega A', K \right)$, which means in SVD of $P_\Omega A' = UDV^\top$, we keep the $K$ largest elements of diagonal $D$ so that $\hat{A}'_K = \frac{1}{p} UD_K V^\top$. For the loss evaluated in step 7 of Algorithm 4, two common choices are the negative log-likelihood (nll) 
\[
\sum_{(i,j) \in \Omega^c} \left[ \hat{B}_{\hat{e}_i \hat{e}_j} \exp (z_{ij}\hat{\gamma}) - A_{ij} \log \hat{B}_{\hat{e}_i \hat{e}_j} \right]
\]
and the $L_2$ loss 
\[
\sum_{(i,j) \in \Omega^c} \left[ A_{ij} \exp (-z_{ij}\hat{\gamma}) - \hat{B}_{\hat{e}_i \hat{e}_j} \right]^2.
\]

**Algorithm 4: ECV for selecting $K$ in PCABM**

**Input**: Adjacency matrix $A$, covariates $Z$, the maximum number of communities to consider $K_{max}$, training proportion $p$, number of replications $N_{rep}$.

**Output**: Estimated number of communities $\hat{K}$.

1. Calculate MLE $\hat{\gamma}$ with $A$, $Z$ with $e$ being all 1 vector.
2. For $m = 1$ to $N_{rep}$ do
3.   Randomly choose a subset of node pairs $\Omega$: selecting each pair $(i, j), i < j$ independently with probability $p$, and adding $(j, i)$ if $(i, j)$ is selected.
4.   For $K = 1$ to $K_{max}$ do
5.     Apply matrix completion to $P_\Omega A'$ with rank constraint $K$ to obtain $\hat{A}'_K$, where $A'$ denotes the adjusted adjacency matrix $A'_{ij} = A_{ij} / \exp(z_{ij}\hat{\gamma})$.
6.     Run spectral clustering on $\hat{A}'_K$ to obtain the estimated membership vector $e_{K}^{(m)}$.
7.     Estimate the probability matrix $\hat{B}_{K}^{(m)}$ with $\hat{B}_{kl}(\hat{e}, \hat{\gamma}) = \hat{O}_{kl}(\hat{e}) / E_{kl}(\hat{e}, \hat{\gamma})$, and evaluate the corresponding losses $L_{K}^{(m)}$, by applying the loss function $L$ with the estimated parameters to $\{A_{ij}, (i, j) \in \Omega^c\}$.
8.   End
9. End
10. Let $L_K = \sum_{m=1}^{N_{rep}} L_{K}^{(m)} / N_{rep}$. Return $\hat{K} = \arg\min_{K=1, \ldots, K_{max}} L_K$ as the best model.
6.2 Feature Selection

In the covariate-adjusted model, pairwise covariates \( Z \) and the class labels \( c \) are independent. On the contrary, in the covariates-confounding model, the covariates\( \epsilon^\text{TM} \) distribution is governed by the community labels. An interesting question to ask is what will happen if the covariates used in fitting the covariates-adjusted model are correlated with the community information.

Consider the following example: \( n = 500, \bar{B} = (2 \quad \frac{1}{2}) \), \( \rho_n = 4 \log n/n \). A PCABM network is generated from \( A_{ij} \sim \text{Poisson}(B_{c_i c_j} \exp(z_{ij} \gamma^0)) \), where we have one pairwise covariate \( z_{ij} \sim \text{Poisson}(0.09) \) and \( \gamma^0 = 2 \). However, when fitting the model, a “false” covariate \( Z' \) is also included where \( z'_{ij} \sim \text{Poisson}(0.09) + 0.6(\bar{B}_{c_i c_j} - 1.5)r(1 - r^2)^{-1/2} \), which makes the Pearson correlation between \([z'_{ij}, i < j]\) and \([P_{ij}, i < j]\) to be \( r \). We consider evaluating the community detection performance of PCABM under three scenarios: (1) using only the true covariate \( z_{ij} \), (2) using only the false covariate \( z'_{ij} \), (3) using both covariates. We vary the correlation \( r \) from 0 to 1 to see how it will impact the community detection accuracy. As shown in Figure 2, as the correlation increases, when fitting model with \( Z' \) or with \( \{Z, Z'\} \), community detection performance becomes worse. Why does this happen? As far as we could understand, the reason is that when the false covariate is correlated with the matrix \( P \), it will contribute substantially to fitting the model. More specifically, when estimating \( \gamma \), the MLE mistakenly recognized the effect of \( B_{c_i c_j} \) as the effect of \( \exp(z'_{ij} \gamma) \), so that the \( \gamma \) estimate is very biased, and as a result what we get after adjusting for such a covariate will contain less community information. This is demonstrated in Table 1: when fitting PCABM with both covariates, the coefficient of \( Z' \) is more and more biased as the correlation between \( Z' \) and \( B \) grows. In another word, the MLE cannot distinguish the effects of \( B_{c_i c_j} \) and \( \exp(z'_{ij} \gamma) \).

However, one can imagine the prediction of an inaccurate \((\gamma, e)\) estimate on a test set will not be as satisfactory. In particular, the out-of-sample likelihood of \((\gamma, e)\) estimation under the \( \{Z, Z'\} \) model should be smaller than its counterpart of the model that only has the true covariate.
Algorithm 5: ECV for stepwise covariate selection in PCABM

**input**: Adjacency matrix $A$, covariates $z_{ij} \in \mathbb{R}^d, i, j = 1, ..., n$, the number of communities $K$, training proportion $p$, number of replications $N_{rep}$, constant $\epsilon_L$.

**output**: Selected covariate index set $S$.

1. Initialize selected covariate index set $S = \emptyset$, and the best loss $L_{old} = $ some large number.

2. while $L_{old}$ has not converged do

3. for $m = 1$ to $N_{rep}$ do

4. Randomly choose a subset of node pairs $\Omega$: selecting each pair $(i, j)$, $i < j$ independently with probability $p$, and adding $(j, i)$ if $(i, j)$ is selected.

5. for $d_1$ in $[d] \setminus S$ do

6. Let $S' = S \cup \{d_1\}$. Consider the model with covariates $Z_{S'}$.

7. Calculate MLE $\hat{\gamma}_S^{(\Omega)}$ with $P_{\Omega}A$, $P_{\Omega}Z$: Optimize

8. $l_e^{(\Omega)}(\hat{\gamma}) = \sum_{(i,j) \in \Omega} A_{ij}z_{ij}^\top \hat{\gamma} - \frac{1}{2} \sum_{kl} O_{kl}^{(\Omega)}(e) \log E_{kl}^{(\Omega)}(e, \hat{\gamma})$ with $z_{ij}$ restricted on $S'$.

9. Apply matrix completion to $P_{\Omega}A'$ with rank constraint $K$ to obtain $\hat{A}'_{d_1}$, where $A'$ denotes the adjacency matrix adjusted by $\hat{\gamma}$.

10. Run spectral clustering on $\hat{A}'_{d_1}$ to obtain the estimated membership vector $\hat{e}_{d_1}^{(m)}$. Estimate the probability matrix $\hat{B}_{d_1}^{(m)}$: $\hat{B}_{kl}(\hat{e}, \hat{\gamma}) = O_{kl}^{(\Omega)}(\hat{e})/E_{kl}^{(\Omega)}(\hat{e}, \hat{\gamma})$.

11. Evaluate the corresponding losses $L_{d_1}^{(m)}$, by applying the loss function $L$ with the estimated parameters to $A_{ij}, (i, j) \in \Omega^e$.

12. end

13. Let $L_{d_1} = \sum_{m=1}^{N_{rep}} L_{d_1}^{(m)}/N_{rep}$ for all $d_1 \in [d] \setminus S$. If $L_{old} - \min_{d_1 \in [d] \setminus S} L_{d_1} > \epsilon_L |L_{old}|$, add $d^* = \arg\min_{d_1 \in [d] \setminus S} L_{d_1}$ into the selected covariate index set $S$ and set $L_{old} = L_{d^*}$. Otherwise, we claim $L_{old}$ as converged and stop the algorithm.

14. end
Figure 2: Simulation results for adding covariates of different correlations with community structure.

Table 1: Estimation of $\hat{\gamma}$ when including both covariates $\{Z, Z'\}$. Averaged over 100 replicates.

| $r$ | 0   | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\gamma(Z)$ | 2.001 | 2.001 | 2.000 | 2.002 | 2.002 | 2.001 | 2.000 | 1.999 | 2.001 | 2.003 |
| $\gamma(Z')$ | 0.000 | 0.108 | 0.198 | 0.282 | 0.351 | 0.412 | 0.456 | 0.485 | 0.489 | 0.424 |

Z. The edge cross-validation model selection procedure introduced above is a perfect framework to compare out-of-sample likelihoods when fitting different subsets of covariates. Therefore, we expect the ECV approach could select the best subset of covariates, and pick out the confounding $Z'$ in this case. An ECV algorithm for selecting covariates is given in Algorithm 5. It basically uses a forward stepwise selection method, with a stopping criterion defined by the convergence of out-of-sample likelihood. Applying this variable selection procedure on the example introduced in the beginning of this subsection, where we used $p = 0.9, N_{rep} = 5$ and $\epsilon_L = 0.1$, the frequencies of selecting $Z$ or $Z'$ in 100 replicates when $Z'$ has different correlation $r$ with $B$
are presented in Table 2. We can see that the algorithm almost perfectly selects the true covariate \( Z \) and screens out the false covariate \( Z' \). As a result, the performance of spectral clustering using selected covariates is very close to using only true covariates (oracle), and is much better than the result when both covariates are included, as is shown in Figure 2. Thus, we conclude that while including false correlated covariates could be very harmful to clustering performance under PCABM, Algorithm 5 could screen out these false covariates, which makes the whole clustering procedure more robust to confounding variables.

| \( r \) | 0   | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( Z \) | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |    |
| \( Z' \) |    |    |    | 1   | 0   | 0   | 0   | 0   | 0   | 2   |

7 Simulation Studies

For all simulations, we consider \( K \) communities with prior probabilities \( \pi_i = 1/K, i = 1, \ldots, K \). In addition, we fix \( \bar{B} \) to have all diagonal elements equaling 2 and off-diagonal elements 1; and we fix \( K = 2 \) except in subsection 7.6 where \( K \) varies. We generate data by applying the following procedure:

S1. Determine parameters \( \rho_n \) and \( \gamma^0 \). Generate \( z_{ij} \) from certain distributions to be specified.

S2. Generate adjacency matrix \( A = [A_{ij}] \) from the Poisson distribution with parameters calculated using PCABM with parameters in S1.
7.1 $\gamma$ Estimation

For PCABM, estimating $\gamma$ would be the first step, so we check the consistency and asymptotic normality of $\hat{\gamma}$ claimed in our theory section.

The pairwise covariate vector $z_{ij}$ has 5 variables, generated independently from Bernoulli(0.1), Poisson(0.1), Uniform[0, 1], Exponential(0.3), and $N(0, 0.3)$, respectively. The parameters for each distribution are chosen to make the variances of covariates similar.

We ran 100 simulations respectively for $n = 100, 300, 500$. The parameters are set as $\rho_n = 2 \log n/n$, $\gamma^0 = (0.4, 0.8, 1.2, 1.6, 2)^T$. We obtained $\hat{\gamma}$ by using BFGS to optimize likelihood function under initial community assignment $e_0 = 1$. We list the mean and standard deviation of $\hat{\gamma}$ in Table 3. It is clear that $\hat{\gamma}$ is very close to $\gamma^0$ even for a small network. The shrinkage of standard deviation implies consistency of $\hat{\gamma}$. We also repeated the experiment by initializing with random community assignments, which leads to very similar results; the corresponding results are in Table A.1 of Supplementary Materials. This validates the observation that estimating $\gamma$ and communities are decoupled.

![Figure 3: Simulation results for $\hat{\gamma}$ compared with theoretical values.](image)

By taking a closer look at the network of size $n = 500$, we compare the distribution of $\hat{\gamma}$ with the theoretical asymptotic normal distribution derived in Theorem 1. We show the histogram for the first three coefficients in Figure 3. We can see that the empirical distribution matches well with the theoretical counterpart.
Table 3: Simulated results of $\hat{\gamma}$ over 100 repetitions, displayed as mean (standard deviation).

| $n$ | $\gamma_1^0 = 0.4$ | $\gamma_2^0 = 0.8$ | $\gamma_3^0 = 1.2$ | $\gamma_4^0 = 1.6$ | $\gamma_5^0 = 2$ |
|-----|--------------------|--------------------|--------------------|--------------------|--------------------|
| 100 | 0.393(0.0471)      | 0.796(0.0345)      | 1.206(0.0560)      | 1.596(0.0410)      | 2.005(0.0454)      |
| 300 | 0.399(0.0198)      | 0.801(0.0160)      | 1.198(0.0256)      | 1.603(0.0180)      | 2.003(0.0213)      |
| 500 | 0.399(0.0147)      | 0.800(0.0117)      | 1.197(0.0162)      | 1.599(0.0148)      | 2.002(0.0155)      |

7.2 Community Detection

After obtaining $\hat{\gamma}$, we now move on to the estimation of community labels. Under PCABM, there are three parameters that we could tune to change the property of the network: $\gamma^0$, $\rho_n$, and $n$. To illustrate the impact of these parameters on the performance of community detection, we vary one parameter while fixing the remaining two in each experiment. More specifically, we consider the form $\rho_n = c_\rho \log n/n$ and $\gamma^0 = c_\gamma (0.4, 0.8, 1.2, 1.6, 2)$ in which we will vary the multipliers $c_\rho$ and $c_\gamma$. The detailed parameter settings for the three experiments are as follows.

(a) $n \in \{200, 400, 600, 800, 1000\}$, with $c_\rho = 5$ and $c_\gamma = 1.2$.

(b) $c_\rho \in \{2, 3, 4, 5, 6\}$, with $n = 200$ and $c_\gamma = 1.2$.

(c) $c_\gamma \in \{0, 0.4, 0.8, 1.2, 1.6, 2.0\}$, with $n = 200$ and $c_\rho = 5$.

The results for the three experiments are presented in Figure 4. Each experiment is carried out 100 times. The error rate is reported in terms of the average Adjusted Rand Index (ARI) (Hubert & Arabie, 1985), which is a measure of the similarity between two data clusterings. SBM.MLE and SBM.SC refer to the likelihood and spectral clustering methods under SBM, respectively; DCBM.MLE is the maximum likelihood method based on DCBM (Zhao et al., 2012); PCABM.SCWA and PCABM.MLE refer to Algorithms 2 and 3, respectively.

When the number of nodes increases, it is clear from the panel 4a in Figure 4 that both PCABM-based algorithms perform exceptionally well with PCABM.MLE having nearly perfect community detection performance throughout all $n$. Spectral clustering under SBM leads to
a result like random guesses. DCBM and MLE under SBM are better when \( n \) is large, but still worse than the PCABM-based algorithms. As we increase the density of the network, the performance doesn’t change much. When the scale of \( \gamma^0 \) is changed, both algorithms under PCABM still yield great results. As we know, when \( \gamma^0 = 0 \), our model reduces to SBM, so it is not surprising that SBM.MLE and SBM.SC both perform well when the magnitude of \( \gamma^0 \) is relatively small and fails when the magnitude increases.
7.3 Impact of Inaccurate $\gamma$ Estimate

In this section, we investigate how the accuracy of $\hat{\gamma}$ affects the community label estimate. Here we use one dimensional covariate following Poisson(0.1) and set the true coefficient $\gamma^0 = 2.5$, $n = 500$ and $\rho_n = 2.5 \log n/n$. We vary the coefficient estimates $\hat{\gamma} \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5\}$.

The community detection results are shown in the last plot of Figure 4 under different $\hat{\gamma}$ values. We can see that using a more accurate $\hat{\gamma}$ would yield more accurate community detection results. The maximum ARI is reached by PCABM.MLE when $\hat{\gamma}$ equals the true $\gamma^0$ value. As the bias of $\hat{\gamma}$ increases, the community detection performance deteriorates. In particular, we see the choice of $\hat{\gamma} = 0$ (i.e. SBM) leads to a result like random guesses. This shows the importance of considering the pairwise covariates in the community detection procedure.

7.4 Impact of Initial Assignments Accuracy

As we know, the tabu search is sensitive to the initial assignments. To further understand how influential it is in our setting, we simulate initial community assignments with different accuracy and check how that affects the prediction accuracy. The parameters are fixed to be $n = 200$, $c_\rho = 2$, $c_\gamma = 1.5$. We change the accuracy of initial assignments from 0.5 to 1. To make the results easier to interpret, we use accuracy rather than ARI to evaluate the performance. Note that SCWA does not use a class initialization, and we plot its accuracy as a reference flat line in Figure 5. As the accuracy of initial assignments increases, the prediction accuracy of the MLE method increases as well. As long as the prediction accuracy of SCWA is above 0.8, if we use it as the initial assignments for the MLE method, we could improve the prediction accuracy from 0.75 (random initial) to almost 1. This shows why it is always preferable to have some good initial assignments for MLE methods.
7.5 DCBM

In view of the observation that PCABM includes DCBM as a special case in the sense of having the same profile likelihood, we want to see the performance of Algorithms 2 and 3 on networks generated by DCBM. The degree parameter for each node is chosen from $\{0.5, 1.5\}$ with equal probability, $\bar{B} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and $\rho_n = c_\rho \log n/n$. For covariates construction, we take $z_{ij} = \log d_i + \log d_j$, where $d_i$ is the degree of node $i$. As a comparison, we also implemented the likelihood method in Zhao et al. (2012) (DCBM.MLE) and the SCORE method in Jin (2015). As in Section 7.2, we vary one parameter while fixing the remaining one in each experiment. The detailed parameter settings for the two experiments are as follows with results presented in Figure 6.

(a) $n \in \{200, 400, 600, 800, 1000\}$, with $c_\rho = 2$.

(b) $c_\rho \in \{2, 3, 4, 5, 6\}$, with $n = 200$.

From the results, we observe that except for SBM.MLE and SBM.SC, all the other methods work well, with the ARI being almost one when $n$ or $c_\rho$ is large. It is interesting to observe that the likelihood-based methods (PCABM.MLE and DCBM.MLE) perform a bit better than the spectral counterparts PCABM.SCWA and SCORE. The flexibility of PCABM allows us to model any factors that may contribute to the structure of network in addition to the underlying
communities.

7.6 Estimation of the Number of Communities $K$

In this subsection, we study the performance of our approach for selecting the number of communities $K$, i.e. Algorithm 4. We set $\bar{B}$ to have diagonal elements 2 and off-diagonal elements 1; and we set $n = 1000$, $\rho_n = 5 \log n / n$. $\gamma^0$ and covariates $Z$ are generated in the same way as in section 7.1. We consider the cases of true underlying $K$ being 2, 3 or 4, and let $K_{\text{max}} = 6$, i.e. select $\hat{K}$ from $\{1, 2, ..., 6\}$. The simulation results are presented in Table 4. The results show that Algorithm 4 chooses the correct $K$ with a high probability. Besides, $L_2$ loss is more conservative than negative log-likelihood (nll) loss, in the sense that it sometimes leads to a larger $\hat{K}$.

Figure 6: Simulation results under DCBM for different parameter settings

(a) Different $n$ with $c_\rho = 2$

(b) Different $c_\rho$ with $n = 200$
Table 4: Counts of ECV estimation of community number $K$ in 100 realizations under negative log-likelihood (nll) and $L_2$ loss.

| Loss | $\mathbb{P}(\hat{K} = K)$ | $\mathbb{P}(\hat{K} \geq K)$ | $\mathbb{P}(\hat{K} = K)$ | $\mathbb{P}(\hat{K} \geq K)$ |
|------|-------------------|-----------------|-------------------|-----------------|
| $K = 2$ | 100% | 100% | 91% | 100% |
| $K = 3$ | 99% | 99% | 91% | 99% |
| $K = 4$ | 95% | 95% | 74% | 100% |

8 Real Data Examples

8.1 Example 1: Political Blogs

The first real-world dataset we used is the network of political blogs created by Adamic & Glance (2005). The nodes are blogs about US politics, and edges represent hyperlinks between them. We treated the network as undirected and only focused on the largest connected component of the network, resulting in a subnetwork with 1,222 nodes and 16,714 edges.

Because there are no other nodal covariates available in this dataset, we created one pairwise covariate by aggregating the degree information. We let $z_{ij} = \log(d_i \times d_j)$, where $d_i$ is the degree for the $i$-th node. The coefficient estimate for the covariate $\hat{\gamma}$ is 1.0005 with the 95% confidence interval being $(0.9898, 1.0111)$. Table 5 summarizes the performance comparison of PCABM with some existing results on this dataset. Besides ARI, we also evaluated normalized mutual information (NMI) (Danon et al., 2005), which is a measure of the mutual dependence between the two variables. It is observed that our model slightly outperforms all previous results, and the error rate is very close to the ideal results mentioned in Jin (2015), which is 55/1222. This shows that PCABM provides an alternative way to the DCBM by incorporating the degree information into a specific pairwise covariate. As a more flexible model, it shows that DCBM is indeed a
suitable model for this dataset since the coefficient estimate is close to 1. PCABM also provides a significant improvement over the vanilla SBM, whose NMI is only 0.0001 as reported in Karrer & Newman (2011).

|              | DCBM.MLE | DCBM.RSC | DCBM.CMM | SCORE  | PCABM.MLE |
|--------------|----------|----------|----------|--------|-----------|
| ARI          | 0.819    | –        | –        | 0.819  | **0.825** |
| NMI          | 0.72     | –        | –        | 0.725  | **0.737** |
| Errors       | –        | –        | 62       | 58     | **56**    |
| Accuracy     | –        | 95%      | 94.9%    | 95.3%  | **95.4%** |

Table 5: Performance comparison on political blogs data. The performance of DCBM.MLE is cited from Karrer & Newman (2011); Zhao et al. (2012); the performance of SCORE is from Jin (2015); the performance of regularized spectral clustering (RSC) based on DCBM is reported in Joseph & Yu (2016); the performance of convexified modularity maximization (CMM) for DCBM is from Chen et al. (2018).

8.2 Example 2: School Friendship

For real networks, people often use specific nodal covariates as the ground “truth” for community labels to evaluate the performance of various community detection methods. However, there could be different “true” community assignments based on different nodal covariates (e.g., gender, job, and age). Peel et al. (2017) mentioned that communities and the covariates might capture various aspects of the network, which is in line with the idea presented in this paper. To examine whether PCABM can discover different community structures, in our second example, we treat one covariate as the indicator for the unknown “true” community assignments while using the remaining covariates as the known covariates in our PCABM model.

The dataset is a friendship network of school students from the National Longitudinal Study of Adolescent to Adult Health (Add Health). It contains 795 students from a high school (Grades
9-12) and its feeder middle school (Grade 7-8). The nodal covariates include grade, gender, ethnicity, and the number of friends nominated (up to 10). We focused on the largest connected component with at least one covariate non-missing and treated the network as undirected, resulting in a network with 777 nodes and 4124 edges. For the nodes without gender, we let them be female, which is the smaller group. For those without grades, we generated a random grade according to their schools.

Unlike traditional community detection algorithms that can only detect one underlying community structure, PCABM provides us with more flexibility to uncover different community structures by controlling different covariates. Our intuition is that social network is usually determined by multiple underlying structures and cannot be simply explained by one covariate. Sometimes one community structure seems to dominate the network, but if we adjust the covariate associated with that structure, we may discover other interesting community structures.

In this example, we conducted two community detection experiments. In each experiment, out of the two nodal covariates, school and ethnicity, one was viewed as the proxy for the “true” underlying community, and community detection was carried out by using the pairwise covariates constructed using the other covariates. For school and ethnicity, we created indicator variables to represent whether the corresponding covariate values were the same for the pair of nodes. For example, if two students come from the same school, the corresponding pairwise covariate equals 1; if they have different genders, the corresponding pairwise covariate equals 0. Also, we considered the number of nominated friends in all experiments and grade for predicting ethnicity and gender. For number of nominated friends, we used \( \log(n_i + 1) + \log(n_j + 1) \) as one pairwise covariate, where \( n_i \) is the number of nominated friends for the \( i \)-th student. “+1” was used here because some students didn’t nominate anyone. For grades, we used the absolute difference to form a pairwise covariate. Using random initial community labels, we derived the estimates \( \hat{\gamma} \) in each experiment. In Tables 6 and 7, we show respectively the estimates when school and ethnicity are taken as the targeted community.
Table 6: Inference results when school is targeted community.

| Covariate | Estimate | $t$ value | $P(>|t|)$ |
|-----------|----------|-----------|-----------|
| White     | 1.251    | 29.002    | $<0.001^{***}$ |
| Black     | 1.999    | 38.886    | $<0.001^{***}$ |
| Hispanic  | 0.048    | 0.091     | 0.927     |
| Others    | 0.019    | 0.035     | 0.972     |
| Gender    | 0.192    | 5.620     | $<0.001^{***}$ |
| Nomination| 0.438    | 18.584    | $<0.001^{***}$ |

Table 7: Inference results when ethnicity is targeted community.

| Covariate | Estimate | $t$ value | $P(>|t|)$ |
|-----------|----------|-----------|-----------|
| School    | 1.005    | 13.168    | $<0.001^{***}$ |
| Grade     | -1.100   | -39.182   | $<0.001^{***}$ |
| Gender    | 0.198    | 5.813     | $<0.001^{***}$ |
| Nomination| 0.498    | 21.679    | $<0.001^{***}$ |

In both tables, the standard error is calculated by Theorem 1, with the theoretical values replaced by the estimated counterparts. Thus, we can calculate the $t$ value for each coefficient and perform statistical tests. We can see that in both experiments, the coefficients for gender and the number of nominations are positive and significant in the creation of the friendship network. The significant positive coefficient of nominations shows that students with a large number of nominations tend to be friends with each other, which is intuitive. The positive coefficients of gender and school show students of the same gender and school are more likely to be friends with each other, which is in line with our expectations. The negative coefficient of grade means that students with closer grades are more likely to be friends. If we take a closer look at the coefficients of different ethnic groups in Table 6, we find that only those corresponding to white and black are significant. This is understandable as we observe that among 777 students, 476 are white, and 221 are black. As for school and grade, students in the same school or grade tend to be friends with each other, as expected.

The network is divided into two communities each time (we only look at white and black students in the second experiment because the sizes for other ethnicities are very small). We apply our algorithm PCABM.MLE as well as some classic methods on SBM and DCBM to
cluster network in both experiments. The results in terms of ARI are shown in Table 8. It could be seen that while DCBM.MLE could capture one main structure of the network “School” which is probably the dominating structure, our method could not only capture “School” but also capture “Race” when having adjusted for the covariate “School”. Note that, for all methods other than ours, we would get only one community structure, whose performance is doomed to be bad for capturing different community structures. Also, to test the robustness of our method, in the experiment of detecting the ethnicity community, we tried to use the square of the grade difference, which led to almost the same ARI.

Table 8: ARI comparison on school friendship data.

|          | PCABM.MLE | SBM.MLE | SBM.SC | DCBM.MLE | SCORE |
|----------|-----------|---------|--------|----------|-------|
| School   | 0.909     | 0.048   | 0.043  | 0.909    | 0.799 |
| Race     | 0.914     | 0.138   | -0.024 | 0.001    | 0.012 |

9 Discussion

In this paper, we extend the classical stochastic block model to allow the connection rate between nodes to not only depend on the community memberships but also on the pairwise covariates. We prove consistency in terms of both coefficient estimates and community label assignments for MLE under PCABM. Also, we introduce a fast spectral method SCWA with theoretical justification, which may serve as a good initial solution for the likelihood method. Lastly, we propose cross-validation-based algorithms for estimating the number of communities and feature selection.

There are many interesting future research directions on PCABM. In our paper, we assumed the entries in the adjacency matrix are non-negative integers. However, it can be relaxed to be any non-negative numbers, and we expect similar theoretical results to hold. It would also be
interesting to consider highly imbalanced community sizes $n_{\text{min}}/n_{\text{max}} = o(1)$. Besides, when we have high-dimensional pairwise covariates, adding a penalty term to conduct variable selection is also worth investigating. For instance, in the estimation of $\gamma$ we can regularize (2) by adding an $L_1$ penalty $\hat{\gamma}_{\lambda}(e_0) = \arg \max_{\gamma} (l_{e_0}(\gamma) - \lambda\|\gamma\|_1)$ to estimate a sparse high-dimensional $\gamma$.

One model assumption in PCABM is the independence among edges conditional on observed covariates. However, the independence might be inappropriate if there are unobserved covariates. To address this, one possible extension is a degree corrected pairwise covariate adjusted block model, which can incorporate unobserved nodal covariates. The adjacency matrix could be modeled as, say, $A_{ij}|c, Z, \theta \sim \text{Poisson}(B_{c_{ij}}\theta_i\theta_j \exp(z_{ij}^\top\gamma))$, where $\theta$ represents degree correction parameters. From a modeling perspective, the $\theta$ term could be one way of modeling unobserved nodal covariates or random effects. From a fitting point of view, a first question to ask about this model is whether it is in some sense equivalent to PCABM, by adding the covariate $\log(d_id_j)$, where $d_i$ is the degree of the $i$th node.

Last but not least, while the feature selection algorithm given in section 6.2 is an interesting first-step attempt on dealing with the confounding covariates, it is still based on the PCABM which only models covariate adjusting. It would be desirable to propose an exhaustive covariate block model as well as corresponding community detection methods that could integrate both covariate-adjusting and covariate-confounding.

The code for implementing the proposed algorithms is available on GitHub.

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Supplementary Materials for “Pairwise Covariates-Adjusted Block Model for Community Detection”

Sihan Huang, Jiajin Sun
Department of Statistics, Columbia University
Yang Feng
Department of Biostatistics, New York University

The Supplementary Material contains proofs and additional theoretical results in Section A and presents additional simulation and real data analysis results in Section B.

A Additional Theoretical Results

A.1 Proof of Theorem 1

Proof. In the following proof, we will use $\hat{\gamma}$ instead of $\hat{\gamma}(e_0)$ for simplicity. Since

$$-l'_{e_0}(\gamma^0) = l'_{e_0}(\hat{\gamma}) - l'_{e_0}(\gamma^0) = l''_{e_0}(\hat{\gamma})(\hat{\gamma} - \gamma^0)$$

(A.1)

*Huang and Sun contribute equally to this work. Corresponding Author: Yang Feng (yang.feng@nyu.edu)
where \( \hat{\gamma} = q\gamma + (1 - q)\gamma^0 \) for some \( q \in [0, 1] \), we want to analyze the asymptotic behavior of \( l'_{e_0}(\gamma^0) \). Define the empirical version of \( \theta(\gamma) \), \( \mu(\gamma) \) and \( \Sigma(\gamma) \) as

\[
\hat{\theta}(\gamma) = \sum_{u,v \in [n], u \neq v} e^{z_{uv}^T \gamma} / (n(n - 1)), \\
\hat{\mu}(\gamma) = \sum_{u,v \in [n], u \neq v} z_{uv} e^{z_{uv}^T \gamma} / (n(n - 1)), \\
\hat{\Sigma}(\gamma) = \sum_{u,v \in [n], u \neq v} z_{uv} z_{uv}^T e^{z_{uv}^T \gamma} / (n(n - 1)).
\]

For fixed \( \gamma \), by Chebyshev’s inequality, we know the weak law of large numbers holds, i.e.,

\[
\hat{\theta}(\gamma) \xrightarrow{p} \theta(\gamma), \quad \hat{\mu}(\gamma) \xrightarrow{p} \mu(\gamma) \quad \text{and} \quad \hat{\Sigma}(\gamma) \xrightarrow{p} \Sigma(\gamma).
\]

For the given cluster assignment \( e_0 \), the log-likelihood for covariate coefficient \( \gamma \) is

\[
l_{e_0}(\gamma) = \sum_{i < j} A_{ij} z_{ij}^T \gamma - \frac{1}{2} \sum_{kl} O_{kl}(e_0) \log E_{kl}(e_0, \gamma). \quad (A.2)
\]

Note that the likelihood is a concave random function of \( \gamma \). Thus a direct application of Theorem II.1 and Corollary II.2 of Andersen & Gill (1982) gives the consistency of the MLE \( \hat{\gamma} \).

Next we show the asymptotic normality of \( \hat{\gamma} \). The score function for \( \gamma \) is given by

\[
l'_{e_0}(\gamma) = \sum_{i < j} A_{ij} \left[ z_{ij} - \frac{\mu(\gamma)}{\hat{\theta}(\gamma)} \right] \quad (A.3)
\]

which could be decomposed into two parts

\[
l'_{e_0}(\gamma) = \sum_{i < j} A_{ij} \left[ z_{ij} - \frac{\mu(\gamma)}{\theta(\gamma)} \right] + \sum_{i < j} A_{ij} \left[ \frac{\mu(\gamma)}{\theta(\gamma)} - \frac{\hat{\mu}(\gamma)}{\hat{\theta}(\gamma)} \right] =: I(\gamma) + II(\gamma). \quad (A.4)
\]

In the decomposition (A.4), conditioning on \( c \), \( I(\gamma^0) \) is a sum of independent random variables. The mean and variance of each summand in \( I(\gamma^0) \) (scaled by \( \rho_n \)) are given by

\[
\mathbb{E} \left[ A_{ij} \left( z_{ij} - \frac{\mu(\gamma^0)}{\theta(\gamma^0)} \right) / \rho_n \right] = B_{c_i c_j} e^{z_{ij}^T \gamma^0} \left( z_{ij} - \frac{\mu(\gamma^0)}{\theta(\gamma^0)} \right), \\
\mathbb{E} \left[ A_{ij} \left( z_{ij} - \frac{\mu(\gamma^0)}{\theta(\gamma^0)} \right) / \rho_n \right] = 0,
\]
\[ Var \left[ A_{ij} \left( z_{ij} - \frac{\mu(\gamma^0)}{\theta(\gamma^0)} \right) / \rho_n \right] = E \left\{ Var \left[ A_{ij} \left( z_{ij} - \frac{\mu(\gamma^0)}{\theta(\gamma^0)} \right) / \rho_n \mid Z \right] \right\} + Var \left\{ E \left[ A_{ij} \left( z_{ij} - \frac{\mu(\gamma^0)}{\theta(\gamma^0)} \right) / \rho_n \mid Z \right] \right\} = E \left[ \frac{\bar{B}_{c_i c_j} e^{z_{ij}^\top \gamma^0}}{\rho_n} \left( z_{ij} - \frac{\mu(\gamma^0)}{\theta(\gamma^0)} \right) \otimes^2 \right] + E \left[ \frac{\bar{B}_{c_i c_j} e^{z_{ij}^\top \gamma^0} \left( z_{ij} - \frac{\mu(\gamma^0)}{\theta(\gamma^0)} \right)}{\rho_n} \right] \otimes^2 \right] = \frac{\bar{B}_{c_i c_j}}{\rho_n} \left[ \Sigma(\gamma^0) - \frac{\mu(\gamma^0) \otimes^2}{\theta(\gamma^0)} \right] (1 + o(1)). \]

Thus, by Lyapunov CLT, (noting that the third central moment of Poisson(\( \lambda \)) is \( \lambda \),) we have

\[
\frac{I(\gamma^0)}{\sqrt{\rho_n}} \overset{d}{\to} N \left( 0, \sum_{i<j} \bar{B}_{c_i c_j} \left[ \Sigma(\gamma^0) - \frac{\mu(\gamma^0) \otimes^2}{\theta(\gamma^0)} \right] \right); \quad (A.5)
\]

and unconditioning on \( c \), we obtain

\[
\frac{I(\gamma^0)}{\sqrt{N_n \rho_n}} \overset{d}{\to} N \left( 0, \Sigma(\gamma^0) \right) \quad (A.6)
\]

from the U-statistic type LLN \( \sum_{i<j} \bar{B}_{c_i c_j} / N_n \overset{d}{\to} \bar{B}_{\text{lim}} \).

Now we analyze part II(\( \gamma^0 \)) in \( l'_e(\gamma^0) \). \( \sum_{i<j} A_{ij} \) is a sum of independent random variables conditioning on \( c \) and by triangular array WLLN

\[
\frac{\sum_{i<j} A_{ij}}{N_n \rho_n} \overset{d}{\to} \frac{\sum_{i<j} \bar{B}_{c_i c_j} E e^{z_{ij}^\top \gamma^0}}{N_n}; \quad (A.7)
\]

and unconditioning on \( c \) we obtain

\[
\frac{\sum_{i<j} A_{ij}}{N_n \rho_n} \overset{d}{\to} \bar{B}_{\text{lim}} \theta(\gamma^0). \quad (A.8)
\]

\( \hat{\mu}(\gamma^0) \) and \( \hat{\theta}(\gamma^0) \) are both averages of independent random variables so by CLT we have

\[
\sqrt{N_n} \begin{pmatrix} \hat{\mu}(\gamma^0) - \mu(\gamma^0) \\ \hat{\theta}(\gamma^0) - \theta(\gamma^0) \end{pmatrix} \overset{d}{\to} N \left( 0, \begin{pmatrix} Var(z_{ij} e^{z_{ij}^\top \gamma^0}) & Cov(z_{ij} e^{z_{ij}^\top \gamma^0}, e^{z_{ij}^\top \gamma^0}) \\ Cov(z_{ij} e^{z_{ij}^\top \gamma^0}, e^{z_{ij}^\top \gamma^0}) & Var(e^{z_{ij}^\top \gamma^0}) \end{pmatrix} \right). \]

3
Since $Z$ is bounded as is assumed in Condition 1, by delta method we could see $\sqrt{N_n} \left( \frac{\hat{\mu}(\gamma) - \mu(\gamma)}{\hat{\theta}(\gamma)} - \frac{\mu(\gamma) - \theta(\gamma)}{\theta(\gamma)} \right)$ converges to a certain normal distribution with bounded variance. Thus, $II(\gamma^0) = \sum_{i<j} A_{ij} \left[ \frac{\hat{\mu}(\gamma_{0})}{\hat{\theta}(\gamma_{0})} - \frac{\mu(\gamma_{0})}{\theta(\gamma_{0})} \right]$ is of the order $O_p(\rho_n \sqrt{N_n})$ while $I(\gamma^0)$ is of the order $O_p(\sqrt{N_n \rho_n})$. We could now conclude that $I_{e_0}'(\gamma^0) = I(\gamma^0)(1 + o_p(1))$, and hence $l_{e_0}'(\gamma^0)/\sqrt{N_n \rho_n} \overset{d}{\to} N(0, \Sigma_{\infty}(\gamma^0))$.

A direct calculation gives us

$$I_{e_0}''(\gamma) = \sum_{i<j} A_{ij} \left[ \frac{\hat{\mu}(\gamma) \otimes^2 \hat{\theta}(\gamma)}{\theta(\gamma)} - \hat{\Sigma}(\gamma) \right].$$

(A.9)

By a typical argument of the uniform weak law of large numbers followed by continuous mapping theorem, we get

$$\frac{\hat{\mu}(\gamma) \otimes^2 \hat{\theta}(\gamma) - \hat{\Sigma}(\gamma)}{\theta(\gamma)} \overset{d}{\to} \frac{\mu(\gamma^0) \otimes^2 \theta(\gamma)}{\theta(\gamma)} - \Sigma(\gamma^0)$$

where $\gamma$ is a mean value of $\hat{\gamma}$ and $\gamma^0$. Thus

$$\frac{l_{e_0}''(\gamma)}{N_n \rho_n} \overset{d}{\to} \Sigma_{\infty}(\gamma^0).$$

(A.10)

Substituting the above result and the asymptotic normality of $l_{e_0}'(\gamma^0)$ back into equation (A.1) finishes the proof.

\[\square\]

### A.2 Some Concentration Inequalities and Notations

To prepare later proofs, we introduce some concentration inequalities and additional notations in this part.

One inequality that we will apply repeatedly is an extended version of Bernstein inequality for unbounded random variables introduced in Wellner (2005).

**Lemma A.1 (Bernstein inequality).** Suppose $X_1, \ldots, X_n$ are independent random variables with $\mathbb{E}X_i = 0$ and $\mathbb{E}|X_i|^k \leq \frac{1}{2} \mathbb{E}X_i^2 L^{k-2} k!$ for $k \geq 2$. For $M \geq \sum_{i \leq n} \mathbb{E}X_i^2$ and $x \geq 0$,

$$\Pr\left( \sum_{i \leq n} X_i \geq x \right) \leq \exp\left( -\frac{x^2}{2(M + xL)} \right).$$
To show that all Poisson distributions satisfy the above Bernstein condition uniformly under some constant $\bar{L}$, we give the following lemma.

**Lemma A.2** (Bernstein condition). Assume $A \sim \text{Pois}(\lambda)$, let $X = A - \lambda$, then for any $0 < \lambda < 1/2$, there exists a constant $\bar{L} > 0$ s.t. for any integer $k > 2$, $\mathbb{E}[|X|^k] \leq \mathbb{E}[X^2]\bar{L}^{k-2}k!/2$.

**Proof.**

\[
\frac{2\mathbb{E}[|A - \lambda|^k]}{\lambda k!} = \frac{2}{\lambda k!} \mathbb{E}[(A - \lambda)^k | A \geq 1] \Pr(A \geq 1) + \frac{2\lambda^{k-1}e^{-\lambda}}{k!} \\
\leq \frac{2}{\lambda k!} \mathbb{E}[A^k | A \geq 1] \Pr(A \geq 1) + e^{-\lambda} = \frac{2}{\lambda k!} \mathbb{E}[A^k] + e^{-\lambda} \\
= \frac{2}{k!} \sum_{i=1}^{k} \binom{k}{i} \lambda^{i-1} + e^{-\lambda} \leq \frac{1}{k!} \sum_{i=1}^{k} \binom{k}{i} i^{-1} \lambda^{i-1} + e^{-\lambda} \\
\leq e^{k-1} \frac{k}{k!} \sum_{i=1}^{k} \binom{k}{i} i^{-1} \lambda^{i-1} + e^{-\lambda} = \sum_{i=1}^{k} e^{i+k-1} i^{-1} \lambda^{i-1} + e^{-\lambda} \\
< \sum_{i=1}^{k} e^{i+k-1} e^{-i} \lambda^{i-1} + e^{-\lambda} = e^{k-1} \frac{1 - \lambda^k}{1 - \lambda} + e^{-\lambda} \leq e^{k-1} \frac{1}{1 - \lambda} + 1 \\
\leq \left( \frac{e^2 + 1}{1 - \lambda} \right)^{k-2}.
\]

Notice that when $\lambda$ is bounded away from 1, say $\lambda < 1/2$, we can simply set $\bar{L} = 2(e^2 + 1)$, then Bernstein condition is satisfied uniformly for all $\lambda$. \(\square\)

We introduce some notations. Let $|e - c| = \sum_{i=1}^{n} \mathbb{1}(e_i \neq c_i)$. Given a community assignment $e \in \mathbb{K}^n$, we define $V(e) \in \mathbb{R}^{K \times K}$ with their elements being

\[
V_{ka}(e) = \frac{\sum_{i=1}^{n} \mathbb{1}(e_i = k, c_i = a)}{\sum_{i=1}^{n} \mathbb{1}(c_i = a)} = \frac{R_{ka}(e)}{\pi_a(e)}.
\]

One can view $R$ as the empirical joint distribution of $e$ and $c$, and $V$ as the empirical conditional distribution of $e$ given $c$. We can see that $V(e) = R(e)(D(c))^{-1}$, where $D(c) =$
also, note that \( V(e)^T 1 = 1 \), \( V(e)\pi(c) = \pi(e) \) and \( V(c) = I_K \). For the convenience of later proof, we also define \( W(c) = D(c)\overline{B}D(c) \) and

\[
\hat{T}(e) \triangleq R(e)\overline{B}R(e)^T = V(e)W(c)V(e)^T, \\
\hat{S}(e) \triangleq V(e)\pi(c)\pi(c)^TV(e)^T.
\]

Replacing the empirical distribution \( \pi(c) \) by the true distribution \( \pi_0 \), we define \( W_0 = D(\pi_0)\overline{B}D(\pi_0) \), where \( D(\pi_0) = \text{diag}(\pi_0) \), and \( T(e), S(e) \in \mathbb{R}^{K \times K} \) as

\[
T(e) \triangleq V(e)W_0V(e)^T, \\
S(e) \triangleq V(e)\pi_0\pi_0^TV(e)^T.
\]

The population version of \( F(O_{2N_n\rho_n}, E_{2N_n}) \) is

\[
F(\theta(\gamma_0)T(e), \theta(\hat{\gamma})S(e)).
\]

To measure the discrepancy between empirical and population version of \( F \), we define \( X(e), Y(e, \hat{\gamma}) \in \mathbb{R}^{K \times K} \) to be the rescaled difference between \( O, E \) and their expectations

\[
X(e) \triangleq \frac{O(e)}{2N_n\rho_n} - \theta(\gamma_0)\hat{T}(e), \\
Y(e, \hat{\gamma}) \triangleq \frac{E(e, \hat{\gamma})}{2N_n} - \theta(\hat{\gamma})\hat{S}(e).
\]

Before we establish bound for \( Y(e, \hat{\gamma}) \), we present the following lemma for \( \hat{\gamma} \).

**Lemma A.3.** For any constant \( \phi > 0 \), there exist positive constants \( C_\phi \) and \( v_\phi \) s.t., \( \Pr(||\hat{\gamma} - \gamma_0||_{\infty} > \phi) < C_\phi \exp(-v_\phi N_n\rho_n) \).

We omit the proof of the Lemma as it follows easily from the proof of Theorem 1. Conditioned on \( ||\hat{\gamma} - \gamma_0||_{\infty} \leq \phi \), we have \( |e^{\hat{x}_{ij}} - \mathbb{E}e^{\hat{x}_{ij}}| \leq \exp\{p\alpha(\phi + ||\gamma_0||_{\infty})\} \equiv \chi \) uniformly for any \( i, j \in [n] \) and \( \hat{\gamma} \). Under this condition, we establish Lemma A.4 using Bernstein inequality.
Lemma A.4.

\[
\Pr(\max_e \|X(e)\|_\infty \geq \epsilon) \leq 2K^{n+2} \exp\left(-C_1\epsilon^2 N_n \rho_n\right) \tag{A.11}
\]

for \( \epsilon < \beta_u \|\tilde{B}\|_{\text{max}/L}\).

\[
\Pr\left( \max_{|e-c| \leq m} \|X(e) - X(c)\|_\infty \geq \epsilon \right) \leq 2 \left(\frac{n}{m}\right) K^{m+2} \exp\left(-\frac{C_3n}{m} \epsilon^2 N_n \rho_n\right) \tag{A.12}
\]

for \( \epsilon < \eta m/n \), where \( \eta = 2 \beta_u \|\tilde{B}\|_{\text{max}/L}\).

\[
\Pr(\max_e \|Y(e, \hat{\gamma})\|_\infty \geq \epsilon) \leq 2K^{n+2} \exp\left(-C_2\epsilon^2 N_n \rho_n\right) \tag{A.13}
\]

for \( \epsilon \leq \eta m/n \).

\[
\Pr(\max_e \|Y(e, \hat{\gamma}) - Y(c, \hat{\gamma})\|_\infty \geq \epsilon) \leq 2 \left(\frac{n}{m}\right) K^{m+2} \exp\left(-\frac{C_5n}{m} \epsilon^2 N_n \rho_n\right) \tag{A.14}
\]

for \( \epsilon < \chi \kappa^2_2 \).

\[
\Pr(\max_e \|Y(e, \hat{\gamma}) - Y(c, \hat{\gamma})\|_\infty \geq \epsilon) \leq 2 \left(\frac{n}{m}\right) K^{m+2} \exp\left(-\frac{C_6n}{m} \epsilon^2 N_n \rho_n\right) \tag{A.15}
\]

for \( \epsilon < \frac{2\chi m}{n} \).

\[
\Pr(\max_e \|Y(e, \hat{\gamma}) - Y(c, \hat{\gamma})\|_\infty \geq \epsilon) \leq 2 \left(\frac{n}{m}\right) K^{m+2} \exp\left(-\frac{C_6n}{m} \epsilon^2 N_n \rho_n\right) \tag{A.16}
\]

for \( \epsilon \geq \frac{2\chi m}{n} \).

**Proof.** The proofs are all given conditioned on \( |e^{x_{ij} \hat{\gamma}} - \mathbb{E}e^{x_{ij} \hat{\gamma}}| \leq \chi \). By combining Lemma A.3, we could have the conclusion directly. For any \( \hat{\gamma} \), by Bernstein inequality, when \( \epsilon < \chi \kappa^2_2 \),

\[
\Pr(|Y_{kl}(e, \hat{\gamma})| \geq \epsilon) \leq 2 \exp\left(-\frac{1}{2} \left(\frac{2N_n \epsilon}{|s_e(k, l)| \chi^2 + \frac{2}{3} \chi N_n \epsilon}\right)^2\right)
\]

\[
\leq 2 \exp\left(-\frac{6N_n \epsilon^2}{3 \kappa^2_2 \chi^2 + 2 \chi \epsilon}\right) \leq 2 \exp\left(-\frac{6}{5 \kappa^2_2 \chi^2} \epsilon^2 N_n\right).
\]
Let \(X^1(e) = \frac{O(e) - \mathbb{E}[O(e)]}{2N_n \rho_n}\) and \(X^2(e) = X(e) - X^1(e)\), and we establish bound for \(X^1(e)\) and \(X^2(e)\) respectively. By Lemma A.1, for any \(k, l \in [K]\), let \(M = \beta_u \|\bar{B}\|_{\max}|s_e(k, l)|\rho_n\), \(L = \bar{L}\), and \(x = 2N_n \rho_n \epsilon\), then for \(\epsilon < \beta_u \|\bar{B}\|_{\max}/L\),

\[
\Pr(X^1_{kl}(e) \geq \epsilon) \leq \exp\left(-\frac{4N^2_n \rho_n^2 \epsilon^2}{2(\beta_u \|\bar{B}\|_{\max}|s_e(k, l)|\rho_n + 2N_n \rho_n \epsilon L)}\right)
\]

\[
\leq \exp\left(-\frac{N_n \rho_n \epsilon^2}{\beta_u \|\bar{B}\|_{\max} + \epsilon L}\right) \leq \exp\left(-\frac{\epsilon^2 N_n \rho_n}{2 \beta_u \|\bar{B}\|_{\max}}\right).
\]

Notice that \(|X^2_{kl}(e)|/\|\bar{B}\|_{\max} \leq |Y_{kl}(e, \gamma^0)|\). Thus, for \(\epsilon < \chi^2_k \|\bar{B}\|_{\max}\),

\[
\Pr(|X^2_{kl}(e)| \geq \epsilon) \leq \Pr\left(|Y_{kl}(e, \gamma^0)| \geq \frac{\epsilon}{\|\bar{B}\|_{\max}}\right)
\]

\[
\leq 2 \exp\left(-\frac{6}{5 \chi^2_k k \rho_n \epsilon^2 N_n}\right).
\]

Thus, the bound of \(X(e)\) will be dominated by \(X^1(e)\), and we will ignore the second term in the bound because it is just a small order and can be absorbed into the first one.

Similar to the arguments in Zhao et al. (2012), for \(|e - c| \leq m, \epsilon < \frac{2\chi m}{n}\),

\[
\Pr(|Y_{kl}(e, \hat{\gamma}) - Y_{kl}(c, \hat{\gamma})| \geq \epsilon) \leq 2 \exp\left(-\frac{6N_n \epsilon^2}{6\chi^2 mn/N_n + 2\chi \epsilon}\right)
\]

\[
\leq 2 \exp\left(-\frac{3(n-1)}{8\chi^2 m} \epsilon^2 N_n\right) \leq 2 \exp\left(-\frac{n}{4\chi^2 m} \epsilon^2 N_n\right).
\]

For \(\epsilon \geq \frac{2\chi m}{n}\),

\[
\Pr(|Y_{kl}(e, \hat{\gamma}) - Y_{kl}(c, \hat{\gamma})| \geq \epsilon) \leq 2 \exp\left(-\frac{6N_n \epsilon^2}{6\chi^2 mn/N_n + 2\chi \epsilon}\right)
\]

\[
\leq 2 \exp\left(-\frac{3}{4\chi} N_n\right).
\]

Also, for \(\epsilon < \frac{2\beta_u \|\bar{B}\|_{\max} m}{nL}\),

\[
\Pr(|X^1_{kl}(e) - X^1_{kl}(c)| \geq \epsilon) \leq \exp\left(-\frac{N_n \rho_n \epsilon^2}{\beta_u \|\bar{B}\|_{\max} mn/N_n + \epsilon L}\right)
\]

\[
\leq \exp\left(-\frac{n - 1}{2 \beta_u \|\bar{B}\|_{\max} m} \epsilon^2 N_n \rho_n\right) \leq \exp\left(-\frac{n}{4 \beta_u \|\bar{B}\|_{\max} m} \epsilon^2 N_n \rho_n\right).
\]
For $\epsilon \geq \frac{2\beta_u \| B \|_{\text{max} m}}{nL}$,

$$
\Pr(|X_{kl}^1(e) - X_{kl}^1(c)| \geq \epsilon) \leq \exp \left( -\frac{N_n \rho_n \epsilon^2}{\beta_u \| B \|_{\text{max} m n / N_n + \epsilon L}} \right) \\
\leq \exp \left( -\frac{1}{3L} \epsilon N_n \rho_n \right).
$$

We will omit the bound for $|X_{kl}^2(e) - X_{kl}^2(c)|$ since it’s a smaller order.

\square

A.3 Proof of Theorem 2

A.3.1 Consistency of a General Class of Criteria

Instead of directly analyzing $\ell_{q_i}(e)$, similar to Zhao et al. (2012), we first investigate the maximizer of a general class of criteria defined as

$$
Q(e, \hat{\gamma}) := F \left( \frac{O(e)}{2N_n \rho_n}, \frac{E(e, \hat{\gamma})}{2N_n} \right),
$$

(A.17)

where $O(e) = [O_{kl}(e), k, l \in [K]]$ and $E(e, \hat{\gamma}) = [E_{kl}(e, \hat{\gamma}), k, l \in [K]]$. Then, we show our log-likelihood function falls in this class of criteria, implying the consistency of label estimation. We say the criterion $Q$ is consistent if the estimated labels, obtained by maximizing the criterion, $\hat{c} = \arg \max_e Q(e, \hat{\gamma})$ is consistent.

One key condition of $Q$ for implying consistent community detection is that it reaches the maximum at $c$ under the true parameter $\gamma^0$ in the “population version”, which is $F \left( \frac{E[O(e)]}{2N_n \rho_n}, \frac{E[E(e, \gamma^0)]}{2N_n} \right)$. To further demonstrate what the “population version” is, we introduce some notations. Given a community assignment $e \in [K]^n$, we define $R(e) \in \mathbb{R}^{K \times K}$ with its elements being $R_{ka}(e) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(e_i = k, c_i = a)$. One can view $R$ as the empirical joint distribution of $e$ and $c$. Next, we introduce the key condition for the function $F$ in terms of $R$ as follows.
**Condition 4.** $F(R\bar{B}R^T, RJR^T)$ is uniquely\(^1\) maximized over $\mathcal{R} = \{R : R \geq 0, R^T 1 = \pi\}$ by $R = D(\pi)$, where $J$ is the matrix of ones and $D(\pi)$ is the diagonal matrix with diagonal entries $\pi$.

Besides the common factor $\mathbb{E}[\exp(Z^T \gamma)^0]$, the first term is $\bar{B}$ weighted by pairwise community proportions, the second term is the normalized pairwise count between two communities. This reduces the criteria to the form described in Zhao et al. (2012), thus similar methods can be applied to show the consistency of community detection. In addition, we need more regularity conditions for $F$, analogous to those in Zhao et al. (2012).

**Condition 5.** Some regularity conditions hold for $F$.

1. $F$ is Lipschitz in its arguments and $F(cX_0, cY_0) = cF(X_0, Y_0)$ for constant $c \neq 0$.

2. The directional derivatives $\frac{\partial^2 F}{\partial \epsilon^2}(X_0 + \epsilon(X_1 - X_0), Y_0 + \epsilon(Y_1 - Y_0))|_{\epsilon=0^+}$ are continuous in $(X_1, Y_1)$ for all $(X_0, Y_0)$ that is in a neighborhood of $(D(\pi)\bar{B}D(\pi)^T, \pi \pi^T)$.

3. Let $G(R, \bar{B}) = F(R\bar{B}R^T, RJR^T)$. On $\mathcal{R}$, for all $\pi$, $\bar{B}$ and some constant $C > 0$, the gradient satisfies $\frac{\partial G((1-\epsilon)D(\pi) + \epsilon R, \bar{B})}{\partial \epsilon}|_{\epsilon=0^+} < -C$.

Notice that the first condition in Condition 5 ensures that we could extract the common exponential factor. Thus we can ignore that term when we consider the population maximum in Condition 4. Naturally, the consistency of $\hat{\gamma}$ is also required to ensure that the “sample version” is close to the “population version”. Now the main theorem is stated as follows.

**Theorem 5.** Under PCABM, if Conditions 1 and 2 hold for $Z$, then the criteria function $Q$ of the form (A.17), which satisfies Conditions 4, 5, is weakly consistent if $\varphi_n \to \infty$ and strongly consistent if $\varphi_n / \log n \to \infty$.

\(^1\)The uniqueness is interpreted up to a permutation of the labels.
Proof. We divide the proof into three steps.

**Step 1**: sample and population version comparison. We prove \( \exists \epsilon_n \to 0 \), such that

\[
\Pr \left( \max_e \left| F \left( \frac{O(e)}{2N_n \rho_n}, \frac{E(e, \hat{\gamma})}{2N_n} \right) - F(\theta(\gamma^0)T(e), \theta(\gamma^0)S(e)) \right| \leq \epsilon_n \right) \to 1, \tag{A.18}
\]

if \( \varphi_n \to \infty \) and \( \hat{\gamma} \overset{p}{\to} \gamma^0 \).

Since

\[
\left| F \left( \frac{O(e)}{2N_n \rho_n}, \frac{E(e, \hat{\gamma})}{2N_n} \right) - \theta(\gamma^0)F(T(e), S(e)) \right| \\
\leq \left| F \left( \frac{O(e)}{2N_n \rho_n}, \frac{E(e, \hat{\gamma})}{2N_n} \right) - F(\theta(\gamma^0)\hat{T}(e), \theta(\hat{\gamma})\hat{S}(e)) \right| \\
+ \left| F(\theta(\gamma^0)\hat{T}(e), \theta(\hat{\gamma})\hat{S}(e)) - \theta(\gamma^0)F(\hat{T}(e), \hat{S}(e)) \right| \\
+ \theta(\gamma^0) \left| F(\hat{T}(e), \hat{S}(e)) - F(T(e), S(e)) \right|,
\]

it is sufficient to bound these three terms uniformly. By Lipschitz continuity,

\[
\left| F \left( \frac{O(e)}{2N_n \rho_n}, \frac{E(e, \hat{\gamma})}{2N_n} \right) - \theta(\gamma^0)F \left( \hat{T}(e), \hat{S}(e) \right) \right| \\
\leq M_1 \|X(e)\|_\infty + M_2 \|Y(e, \hat{\gamma})\|_\infty, \tag{A.19}
\]

\[
\left| F(\theta(\gamma^0)\hat{T}(e), \theta(\hat{\gamma})\hat{S}(e)) - \theta(\gamma^0)F(\hat{T}(e), \hat{S}(e)) \right| \\
\leq M_2 \|\theta(\hat{\gamma}) - \theta(\gamma^0)\| \|\hat{S}(e)\|_\infty. \tag{A.20}
\]

By (A.11) and (A.14), (A.19) converges to 0 uniformly if \( \varphi_n \to \infty \). Since \( \|\hat{S}(e)\|_\infty \) is uniformly bounded by 1, (A.20) also converges to 0 uniformly.

\[
\left| F \left( \hat{T}(e), \hat{S}(e) \right) - F(\hat{T}(e), \hat{S}(e)) \right| \\
\leq M_1 \|\hat{T}(e) - T(e)\|_\infty + M_2 \|\hat{S}(e) - S(e)\|_\infty. \tag{A.21}
\]
Since \( \pi(c) \overset{p}{\to} \pi_0 \), (A.21) converges to 0 uniformly. So we prove (A.18).

**Step 2**: proof of weak consistency. We prove that there exists \( \delta_n \to 0 \), such that

\[
\Pr\left( \max_{\{e: \|V(e) - I_K\| \geq \delta_n\}} F\left( \frac{O(e)}{2Nn\rho_n}, \frac{E(e, \gamma)}{2N_n} \right) < F\left( \frac{O(c)}{2Nn\rho_n}, \frac{E(c, \gamma)}{2N_n} \right) \right) \to 1. \tag{A.22}
\]

By continuity property of \( F \) and Condition 4, there exists \( \delta_n \to 0 \), such that

\[
\theta(\gamma^0)F(T(c), S(c)) - \theta(\gamma^0)F(T(e), S(e)) > 2\epsilon_n
\]

if \( \|V(e) - I_K\| \geq \delta_n \), where \( I_K = V(c) \). Thus, following (A.18),

\[
\Pr\left( \max_{\{e: \|V(e) - I_K\| \geq \delta_n\}} F\left( \frac{O(e)}{2Nn\rho_n}, \frac{E(e, \gamma)}{2N_n} \right) < F\left( \frac{O(c)}{2Nn\rho_n}, \frac{E(c, \gamma)}{2N_n} \right) \right)
\]

\[
\geq \Pr\left( \max_{e: \|V(e) - I_K\| \geq \delta_n} \theta(\gamma^0)F(T(e), S(e)) - \theta(\gamma^0)F(T(c), S(c)) - F\left( \frac{O(c)}{2Nn\rho_n}, \frac{E(c, \gamma)}{2N_n} \right) \leq \epsilon_n \right)
\]

\[
\to 1.
\]

(A.22) implies \( \Pr(\|V(e) - I_K\| < \delta_n) \to 1 \). Since

\[
\frac{1}{n}|e - c| = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(c_i \neq e_i) = \sum_k \pi_k(1 - V_{kk}(e))
\]

\[
\leq \sum_k (1 - V_{kk}(e)) = \|V(e) - I_K\|_1/2,
\]

weak consistency follows.

**Step 3**: proof of strong consistency.

To prove strong consistency, we need to show

\[
\Pr\left( \max_{\{e: \|V(e) - I_K\|_1 < \delta_n\}} F\left( \frac{O(e)}{2Nn\rho_n}, \frac{E(e, \gamma)}{2N_n} \right) < F\left( \frac{O(c)}{2Nn\rho_n}, \frac{E(c, \gamma)}{2N_n} \right) \right) \to 1. \tag{A.23}
\]

12
Combining (A.22) and (A.23), we have

\[
\Pr \left( \max_{\{e \neq e' \}} F \left( \frac{O(e)}{2N_n\rho_n}, \frac{E(e, \hat{\gamma})}{2N_n} \right) < F \left( \frac{O(e')}{2N_n\rho_n}, \frac{E(e', \hat{\gamma})}{2N_n} \right) \right) \to 1,
\]

which implies strong consistency.

By Lipschitz continuity and the continuity of derivative of \( F \) w.r.t. \( V(e) \) in the neighborhood of \( I_K \), we have

\[
F \left( \frac{O(e)}{2N_n\rho_n}, \frac{E(e, \hat{\gamma})}{2N_n} \right) - F \left( \frac{O(e')}{2N_n\rho_n}, \frac{E(e', \hat{\gamma})}{2N_n} \right) = \theta(\gamma^0)F(\hat{T}(e), \hat{S}(e)) - \theta(\gamma^0)F(\hat{T}(e), \hat{S}(e)) + \Delta(e, e'),
\]

where \( |\Delta(e, e')| \leq M_3(\|X(e) - X(e')\|_\infty) + M_4\|Y(e, \hat{\gamma}) - Y(e', \hat{\gamma})\|_\infty \), and

\[
F(T(e), S(e)) - F(T(c), S(c)) \leq -\bar{C}\|V(e) - I\|_1 + o(\|V(e) - I_K\|_1).
\]

Since the derivative of \( F \) is continuous w.r.t. \( V(e) \) in the neighborhood of \( I_K \), there exists a \( \delta' \) such that

\[
F(\hat{T}(e), \hat{S}(e)) - F(\hat{T}(c), \hat{S}(c)) \leq -(C'/2)\|V(e) - I\|_1 + o(\|V(e) - I_K\|_1)
\]

(A.25)

holds when \( \|\pi(c) - \pi_0\|_\infty \leq \delta' \). Since \( \pi(c) \to \pi_0 \), (A.25) holds with probability approaching 1. Combining (A.24) and (A.25), it is easy to see strong consistency follows if we can show

\[
\Pr(\max_{\{e \neq e' \}} |\Delta(e, e)| \leq C'\|V(e) - I_K\|_1/4) \to 1.
\]

Note \( \frac{1}{n}|e - e'| \leq \frac{1}{2}\|V(e) - I_K\|_1 \). So for each \( m \geq 1 \),

\[
\Pr \left( \max_{|e - e'| = m} |\Delta(e, e)| > C'\|V(e - I_K)\|_1/4 \right) \\
\leq \Pr \left( \max_{|e - e'| \leq m} \|X(e) - X(e')\|_\infty \leq \frac{C'm}{4M_4n} \right) \equiv I_1
\]

and

\[
\Pr \left( \max_{|e - e'| \leq m} \|Y(e, \hat{\gamma}) - Y(e', \hat{\gamma})\|_\infty > \frac{C'm}{4M_4n} \right) \equiv I_2.
\]

(A.26)
Let $\eta_1 = C'/4N_3$, if $\eta_1 < \eta$, by (A.12),

$$I_1 \leq 2K^{m+2}n^m \exp(-\eta_1^2 \frac{C_3m}{n} N_n \rho_n) = 2K^2 [K \exp(\log n - \eta_1^2 C_3 N_n \rho_n / n)]^m.$$ 

If $\eta_1 > \eta$, by (A.13),

$$I_1 \leq 2K^{m+2}n^m \exp(-\eta_1 \frac{C_4m}{n} N_n \rho_n) = 2K^2 [K \exp(\log n - \eta_1 C_4 N_n \rho_n / n)]^m.$$ 

Similar arguments hold for $I_2$ by using (A.15) and (A.16). In all cases, since $\varphi_n / \log n \to \infty$,

$$\Pr(\max_{c \neq c} |\Delta(e, c)| > C' \|V(e) - I_K\|_1 / 4)$$

$$= \sum_{m=1}^{\infty} \Pr(\max_{|c-c|=m} |\Delta(e, c)| > C' \|V(e) - I_K\|_1 / 4) \to 0.$$ 

as $n \to \infty$. The proof is completed.

\[ \Box \]

### A.3.2 Proof of Theorem 2

By Theorem 5, it suffices to show that the log-likelihood satisfies the above conditions 4 and 5. By scaling $\ell_\gamma(e)$, we have

$$\frac{1}{N_n} \ell_\gamma(e) = \rho_n F \left( \frac{O}{2N_n \rho_n}, \frac{E}{2N_n} \right) + (\rho_n \log \rho_n) \sum_{kl} \frac{O_{kl}(e)}{2N_n \rho_n} + O(n^{-1}),$$

where $F(X, Y) = \sum_{kl} X_{kl} \log \left( \frac{X_{kl}}{\hat{X}_{kl}} \right)$, $X, Y \in \mathbb{R}^{K \times K}$. Note that $F$ is closely related to the likelihood criterion used in Zhao et al. (2012). In our case,

$$F(\theta(\gamma^0) R \bar{B} R^T, \theta(\gamma^0) \pi \pi^T) - F(\theta(\gamma^0) R \bar{B} R^T, \hat{\theta}(\gamma) \pi \pi^T)$$

$$= \theta(\gamma^0) \log \frac{\theta(\gamma^0)}{\hat{\theta}(\gamma)} (1^T R \bar{B} R^T 1),$$
which is basically the population degree up to a constant. This fact shows that the consistency of $\hat{\gamma}$ is unnecessary to ensure the consistency of community detection. Thus, we can plug in any random fixed $\hat{\gamma}$, which is different from our general theorem. For simplicity, just assume we use the true value $\gamma^0$ here. Observe that

$$F(\theta(\gamma^0)R\bar{B}R^T, \theta(\gamma^0)\pi\pi^T) = \theta(\gamma^0)F(R\bar{B}R^T, \pi\pi^T),$$

then the form of $F$ is exactly the same as $F$ defined in Bickel & Chen (2009), which automatically satisfies all conditions for $F$.

A.4 Consistency of Maximum Likelihood Community Detection when the Number of Communities $K$ Grows with $n$

In this section we consider the MLE for cluster assignment $c$ when the number of communities $K$ grows with $n$. We start with some notations and definitions. In our model $A_{ij} | Z \sim \text{Poisson}(B_{e_i e_j} \exp(z_{ij}^\top \gamma))$ we denote the true value of parameters $e, B$ and $\gamma$ by $c, B^0$ and $\gamma^0$, respectively. Let $P_{ij} := B_{e_i e_j}^0$. Recall the log-likelihood of our model is

$$\log L(e, \gamma, B, \pi | A, Z) \propto \sum_{i=1}^{n} \pi_i + \sum_{i<j} A_{ij} \log B_{e_i e_j} + \sum_{i<j} A_{ij} z_{ij}^\top \gamma - \sum_{i<j} B_{e_i e_j} \exp(z_{ij}^\top \gamma). \quad (A.27)$$

Since we already have an estimate of $\gamma$, the $\sum_{i<j} A_{ij} z_{ij}^\top \gamma$ term in (A.27) does not contribute to the estimation of $e$. Besides, under the average degree $= \omega(\text{Poly}(\log n))$ regime, the $\sum_{i=1}^{n} \pi_i$ term is smaller order of the other terms. Thus, for the MLE of $e$ we are (asymptotically) actually optimizing the following loss function:

$$L(A, Z; e, B, \gamma) := \sum_{i<j} A_{ij} \log B_{e_i e_j} - \sum_{i<j} B_{e_i e_j} \exp(z_{ij}^\top \gamma). \quad (A.28)$$

We define a “population” version of the above loss as

$$L_p(Z; e, B, \gamma) := \sum_{i<j} P_{ij} \exp(z_{ij}^\top \gamma) \log B_{e_i e_j} - \sum_{i<j} B_{e_i e_j} \exp(z_{ij}^\top \gamma) \quad (A.29)$$
which is the expectation of \( L(A, Z; e, B, \gamma) \) given \( e \) and \( Z \). \( L(A, Z; e, B, \gamma) \) and \( L_p(Z; e, B, \gamma) \) could be optimized with respect to \( B \) with \( \hat{B}_a^b := \frac{O_{ab}(e)}{E_{ab}(e)} \) and \( \tilde{B}_a^b := \frac{\sum_{(i,j) \in \pi(a, b)} P_{ij} \exp(x_j^i \gamma)}{E_{ab}(e)} \).

Thus, profiling \( B \) we define

\[
L(A, Z; e, \hat{B}^e, \gamma) = \frac{1}{2} \sum_{a,b} [O_{ab} \log(\hat{B}_{ab}) - E_{ab}\hat{B}_{ab}] =: PL(A, Z; e, \gamma),
\]

\[
L_p(Z; e, \hat{B}^e, \gamma) = \frac{1}{2} \sum_{a,b} [E_{ab}\tilde{B}_{ab} \log(\tilde{B}_{ab}) - E_{ab}\tilde{B}_{ab}] =: PL_p(Z; e, \gamma),
\]

(A.30)

where for simplicity we omit the \((e)\) for \( \hat{B} \) and \( \tilde{B} \) when there is no confusion.

The goal is to prove consistency of the MLE of cluster assignment \( \hat{e} = \arg\max_e PL(A, Z; e, \gamma^0) \). In fact, the main idea is to adopt the very classical approach of first showing a “uniform weak law of large numbers” type result (Theorem 6), and then establish some identifiability conditions such that the expected likelihood \( PL_p(Z; e, \gamma^0) \) is large and close to \( PL_p(Z; c, \gamma^0) \) only if \( e \) is close enough to the true \( c \) (Theorem 8).

First, we introduce some important conditions. We want to point that in this section (class assignment MLE when \( K \) grows) we are not using the \( B = \rho_n\bar{B} \) setting, and are imposing assumptions directly on \( P_{ij} \)'s. This is mainly because when \( K \to \infty \) we need stronger signal to noise ratio, which can be approximately understood as in-class probability over between-class probability, to identify the communities (see the remark under Theorem 8 for more detailed discussions).

**Condition 6.** \( M = \sum_{i<j} P_{ij} \) satisfies \( M = \omega(n(\log n)^{3+\delta}) \) for some positive constant \( \delta \).

**Condition 7.** The number of communities \( K \) satisfies \( K = O(n^{1/2}) \).

**Condition 8.** There exist some constants \( 0 < c_B < C_B \) such that \( c_B/((n^2) \leq P_{ij} \leq n^2C_B \).

Condition 6 requires the average degree to grow in at least Poly(\( \log n \)) rate, which is still a sparse network setting. Condition 7 allows the number of communities \( K \) to grow at a rate
as fast as $n^{1/2}$, which matches the growth rate allowed in SBM’s MLE consistency (Choi et al., 2012). Condition 8 is a mild condition: note that roughly speaking $B_{c_i, c_j}$ is of the order $\rho_n = \omega((\log n)^{3+\delta}/n)$, so the required range from $O(n^{-2})$ to $O(n^2)$ is very loose for $B$. The sparsity of the network is already controlled by Condition 6; and Condition 8 is not a sparsity condition, but just some technical requirement so that $\log B_{c_i, c_j}$ does not blow up too much.

Now we present our main results.

**Theorem 6.** Under Conditions 1, 6, 7 and 8,

$$
\max_e \{|PL(A, Z; e, \gamma^0) - PL_p(Z; e, \gamma^0)|\} = o_p(M), \quad (A.31)
$$

where $M = \sum_{i<j} P_{ij}$.

Theorem 6 states a uniform concentration result of the profile likelihood around its population version, which plays the role of “uniform weak law of large numbers” in the classical MLE consistency proof, where $M$ is the actual “sample size”. Our next step is to show $\hat{e}$ is close enough to $c$ in their expected likelihood.

**Theorem 7.** Let $\hat{e} = \arg \max_e PL(A, Z; e, \hat{\gamma})$ to be the MLE for the true community assignment $c$. Then under Conditions 1, 6, 7 and 8, we have

$$
PL_p(Z; c, \gamma^0) - PL_p(Z; \hat{e}, \gamma^0) = o_p(M). \quad (A.32)
$$

Theorem 7 already shows that $\hat{e}$ is close to the true $c$ in some sense. With some identifiability conditions, we will be able to translate this closeness into consistency. In particular, this consistency is defined in terms of a notion of classification error used in Choi et al. (2012): $N(\hat{e})$, the number of incorrect class assignments under $\hat{e}$, is counted for every node whose true class under $c$ is not in the majority within its estimated class $\hat{e}$; and $\hat{e}$ achieves weak consistency when $N(\hat{e}) = o_p(n)$.

**Theorem 8.** Suppose the following two conditions hold for $\forall a, b, c \in [K]$:
1. \( \min_an_a(c) = \Omega(n/K) \), i.e. all cluster sizes are of the same scale;

2. \( \min_{a \neq b} \max_c D'(B^0_{ac}; B^0_{bc}) = \Omega\left(\frac{MK}{n^2}\right) \) where \( D'(a, b) := a \log a + b \log b - (a + b) \log \left(\frac{a + b}{2}\right) \).

Then (A.32) implies \( N(\hat{e}) = o_p(n) \), where \( \hat{e} \) denotes the MLE.

The second condition is basically saying any two classes \( a \) and \( b \) are “well-separated” in the sense that there exists a class \( c \) that connects with \( a \) and \( b \) very differently so that one can distinguish \( a \) and \( b \) through their connections with \( c \). This condition may not be trivially satisfied. A simple calculation could show that the left hand side of the condition is of order \( \min_a \max_c B^0_{ac} \), while the right hand side is \( \Omega(\text{Poly}(\log n)K/n) \). When \( K \) is fixed, this is satisfied in the usual \( \rho_n = O(\text{Poly}(\log n)/n) \) scenario. When \( K \) is growing at a rate no faster than \( O(\sqrt{n}) \), an example scenario for this to hold would be \( B^0_{aa} = O(\text{Poly}(\log n)K/n) \) and \( B^0_{ab} = O(\text{Poly}(\log n)/n) \) for \( a \neq b \). Note that in the second scenario, though we have a stronger requirement for in-class edge probability, the average degree of each node is still of the order \( O(\text{Poly}(\log n)) \), which is not beyond the sparse setting satisfying Condition 6.

The proofs of our main theorems are basically divided into several lemmas.

**Lemma A.5.** For any \( e, B \),

\[
L_p(Z; e, B^0; \gamma^0) - L_p(Z; e, B; \gamma^0) = \sum_{i<j} e^{\gamma^0_{ij}} D(P_{ij} || B_{e_i e_j}) \geq 0,
\]

where \( D(\lambda||\mu) := \lambda \log(\lambda/\mu) - \lambda + \mu \geq 0 \) is the KL divergence from a Poisson(\( \lambda \)) distribution to a Poisson(\( \mu \)) one.

Lemma A.5 basically says the population version of likelihood \( L_p(Z; \cdot, \cdot, \gamma^0) \) achieves its maximum at true \((e, B^0)\). The next few lemmas establish concentration results of the profile likelihood around its population version.
Lemma A.6. For any \( e \),

\[
PL(A, Z; e, \gamma^0) - PL_p(Z; e, \gamma^0) = \frac{1}{2} \sum_{ab} \left[ E_{ab} D(\tilde{B}_{ab}||\hat{B}_{ab}) + E_{ab}(\hat{B}_{ab} - \tilde{B}_{ab}) \log \tilde{B}_{ab} \right]
\]

where \( E_{ab} = E_{ab}(e, \gamma^0) \).

Lemma A.7. Under Conditions 1 and 6, 7,

\[
\max_e \left\{ \sum_{a \leq b} E_{ab} D(\tilde{B}_{ab}||\hat{B}_{ab}) \right\} = o_p(M)
\]

where \( E_{ab} = E_{ab}(e, \gamma^0) \).

Proof of Lemma A.7. Given covariates \( Z \) and class assignment \( e \), let \( \hat{\Theta}_\epsilon := \{ \hat{B} : \sum_{a \leq b} E_{ab} D(\hat{B}_{ab}||\tilde{B}_{ab}) \geq \epsilon \} \) and \( \hat{\Theta}_1 := \{ \hat{B} : \exists a, b \in [K] \text{ s.t. } \hat{B}_{ab} \geq 2M/E_{ab} \} \).

We first bound the probability of event \( \hat{\Theta}_1 \). Under the conditions \( ||z_{ij}||_\infty \leq \zeta \) and \( \xi = \exp(\zeta||\gamma^0||_1) \), by Bernstein’s inequality for Poisson variables (Lemma A.1 and A.2),

\[
\Pr \left( \hat{B}_{ab} \geq \frac{2M}{E_{ab}} \right) = \Pr \left( \sum_{(i,j) \in S_{(a,b)}} A_{ij} \geq M + \sum_{(i,j) \in S_{(a,b)}} P_{ij} \right) \leq \exp \left( -\frac{M^2}{2(M\xi + M^2L)} \right) \leq \exp(-C_1M)
\]

where \( C_1, C_2, \ldots \) are constants. Thus,

\[
\Pr(\hat{\Theta}_1) \leq \sum_{a \leq b} \Pr \left( \hat{B}_{ab} \geq \frac{2M}{E_{ab}} \right) \leq K^2 \exp(-C_1M).
\]

Next, we bound the probability of event \( \hat{\Theta}_\epsilon \). Applying a Poisson variable’s Chernoff inequal-
ity (Vershynin (2018), p20) we have
\[
\Pr(\hat{B}_{ab} \geq \tilde{B}_{ab} + t) = \Pr \left( O_{ab} \geq \sum_{(i,j) \in S_{a}(a,b)} P_{ij} e^{\sum_{i,j} e_{ij} \gamma_{ij}^{0} + E_{ab} t} \right)
\]
\[
\leq \left( \frac{e^{\sum_{(i,j) \in S_{a}(a,b)} P_{ij} e^{\sum_{i,j} e_{ij} \gamma_{ij}^{0} + E_{ab} t}}}{\sum_{(i,j) \in S_{a}(a,b)} P_{ij} e^{\sum_{i,j} e_{ij} \gamma_{ij}^{0} + E_{ab} t}} \sum_{(i,j) \in S_{a}(a,b)} P_{ij} e^{\sum_{i,j} e_{ij} \gamma_{ij}^{0} + E_{ab} t}} \exp \left( - \sum_{(i,j) \in S_{a}(a,b)} P_{ij} e^{\sum_{i,j} e_{ij} \gamma_{ij}^{0} + E_{ab} t} \right) \right)
\]
\[
= \frac{e^{E_{ab} t}}{(1 + \frac{t}{B_{ab}})^{E_{ab} \tilde{B}_{ab} + E_{ab} t}}.
\]
Since \( D(\tilde{B}_{ab} + t||\tilde{B}_{ab}) = (\tilde{B}_{ab} + t) \log(1 + \frac{t}{B_{ab}}) - t \), we have
\[
\exp(-E_{ab} D(\tilde{B}_{ab} + t||\tilde{B}_{ab})) \geq \Pr(\hat{B}_{ab} \geq \tilde{B}_{ab} + t),
\]
which indicates
\[
\Pr(\hat{B}_{ab} = v) \leq \exp(-E_{ab} D(v||\tilde{B}_{ab})).
\]

And by the independence between \( \{A_{ij}\}_{i<j} \) we have
\[
\Pr(\hat{B}) \leq \exp\left(- \sum_{a \leq b} E_{ab} D(\hat{B}_{ab}||\tilde{B}_{ab})\right).
\]

Thus, for any \( \hat{B} \in \hat{\Theta}_{e} \), we have \( \Pr(\hat{B}) \leq \exp(-c) \). Now we bound \( \Pr(\hat{B} \in \hat{\Theta}_{e}) \) by
\[
\Pr(\hat{\Theta}_{e}) \leq \Pr(\hat{\Theta}_{1}) + \Pr(\hat{\Theta}_{e} \setminus \hat{\Theta}_{1})
\]
\[
\leq K^{2} \exp(-C_{1} M) + |\hat{\Theta}_{1}^{c}| e^{-\epsilon}
\]
\[
\leq K^{2} \exp(-C_{1} M) + (2M + 1) \frac{K^{2} + K}{2} e^{-\epsilon},
\]
where the bound on cardinality of complement of \( \hat{\Theta}_{1} \) comes from the fact that \( \hat{B}_{ab} = O_{ab}/E_{ab} \) and \( O_{ab} \) only takes integer values. Therefore, for any \( \epsilon' > 0 \), a union bound over all \( [K]^{n} \) possible
e’s gives

\[
\Pr \left( \max_e \left\{ \sum_{a \leq b} E_{ab} D(\hat{B}_{ab}||\tilde{B}_{ab}) \right\} > \epsilon' M \right)
\]

\[
\leq \exp \left( (n + 2) \log K - C_1 M \right) + \exp \left( n \log K + \frac{K^2 + K}{2} \log(2M + 1) - \epsilon' M \right).
\]

(A.33)

Under the conditions \( M = \omega(n(\log n)^{3+\delta}) \) and \( K = O(n^{1/2}) \), the probability bound in (A.33) goes to 0 as \( n \to \infty \), which proves the desired result.

**Lemma A.8.** Under Conditions 1 and 6, 7 and 8,

\[
\max_e \left\{ \left| \sum_{ab} E_{ab}(\hat{B}_{ab} - \tilde{b}_{ab}) \log \tilde{B}_{ab} \right| \right\} = o_p(M)
\]

where \( E_{ab} = E_{ab}(e, \gamma^0) \).

**Proof of Lemma A.8.** Given \( Z \) and \( e \), let \( X_{ij} := A_{ij} \log \tilde{B}_{e_i e_j} \), then \( \mathbb{E}X_{ij} = P_{ij} e^{x_i^* \gamma^0} \log \tilde{B}_{e_i e_j} \). The term we are considering could be expressed as \( \sum_{ab} E_{ab}(\hat{B}_{ab} - \tilde{b}_{ab}) \log \tilde{B}_{ab} = 2 \sum_{i < j}(X_{ij} - \mathbb{E}X_{ij}) =: 2(X - \mathbb{E}X) \). \( \{X_{ij}\}_{i < j} \) follow independent scaled Poisson distributions, so they satisfy the Bernstein condition in Lemma A.2 with \( L = 2\bar{L} \log n \geq \bar{L} \log |\tilde{B}_{e_i e_j}| \). Thus, by Bernstein’s inequality, for any \( \epsilon > 0 \)

\[
\Pr(|X - \mathbb{E}X| \geq \epsilon) \leq 2 \exp \left( -\frac{\epsilon^2}{2(M\xi(2\log n)^2 + 2\epsilon\bar{L} \log n)} \right).
\]

And a union bound over all possible \( e \) gives

\[
\Pr(\max_e |X - \mathbb{E}X| \geq \epsilon M) \leq 2K^n \exp \left( -\frac{\epsilon^2 M^2}{4(2M\xi(\log n)^2 + \epsilon M \bar{L} \log n)} \right)
\]

\[
\leq 2 \exp \left( n \log K - \frac{C_2 \epsilon^2 M}{(\log n)^2} \right).
\]

(A.34)

Under the conditions \( M = \omega(n(\log n)^{3+\delta}) \) and \( K = O(n^{1/2}) \), the probability bound in (A.34) goes to 0 as \( n \to \infty \), which proves the desired result.
Combining Lemma A.6, A.7 and A.8 together we immediately derive Theorem 6. Our next step is to show \( \hat{e} \) is close enough to \( c \) in their expected likelihood (Theorem 7). First, we show a lemma that bridges likelihood with true parameter \( \gamma^0 \) to that with the MLE \( \hat{\gamma} \).

**Lemma A.9.** Assume MLE \( \hat{\gamma} \) is consistent. Then under Conditions 1 and 6, for any \( e \),

\[
|PL(A, Z; e, \hat{\gamma}) - PL(A, Z; e, \gamma^0)| = o_p(M).
\]

**Proof of Lemma A.9.** Since \( PL(A, Z; e, \gamma) = \sum_{i<j} (A_{ij} \log \hat{B}_{ei,ej} - A_{ij}) \), we could see that

\[
PL(A, Z; e, \hat{\gamma}) - PL(A, Z; e, \gamma^0) = \sum_{i<j} A_{ij} \log \frac{E_{ei,ej}(e, \gamma^0)}{E_{ei,ej}(e, \hat{\gamma})}.
\]

(A.35)

Under the assumption \(||z_{ij}||_\infty \leq \zeta \), we have

\[
\left| \log \frac{E_{ei,ej}(e, \gamma^0)}{E_{ei,ej}(e, \hat{\gamma})} \right| \leq \zeta ||\hat{\gamma} - \gamma^0||_1 = o_p(1).
\]

(A.36)

Since \( \{A_{ij}\}_{i<j} \) are independent, by Bernstein’s inequality for Poisson variables (Lemma A.1 and A.2), for any constant \( a > 0 \)

\[
\Pr(\sum_{i<j} A_{ij} \geq (a + 1)M) \leq \exp \left( - \frac{a^2 M^2}{2(M\xi + aML)} \right) \leq \exp(-C_3aM),
\]

i.e. \( \sum_{i<j} A_{ij} = O_p(M) \). Combining that result with (A.35) and (A.36) finishes the proof. \( \square \)

With Theorem 6 and Lemma A.9 we are ready to show Theorem 7 characterizing the MLE \( \hat{e} \) being close to the true \( c \) in their population version profile likelihood, and furthermore, Theorem 8 stating the consistency of \( \hat{e} \) to \( c \).

**Proof of Theorem 7.** First note that by Lemma A.5 \( PL_p(Z; c, \gamma^0) = L_p(Z; c, B^0, \gamma^0) \geq PL_p(Z; \hat{e}, \gamma^0) \). Hence it suffices to upper bound \( PL_p(Z; c, \gamma^0) - PL_p(Z; \hat{e}, \gamma^0) \). By the definition of \( \hat{e} \), Lemma
and define
\[ \Pi = \text{one induced by class assignments.} \]
For any partition \( T \) disjoint subsets of \( V \), let \( \Pi \) be a refinement of \( \Pi \).

**Proof of Theorem 8.** We first define a partition \( \Pi \) of the edge set \( \{(i, j)\}_{i < j} \) to be a collection of disjoint subsets \( T_1^\Pi, \ldots, T_R^\Pi \) such that \( \cup_{r=1}^R T_r^\Pi = \{(i, j)\}_{i < j} \). We denote by \( \Pi_{ij} = T_r^\Pi \) if \((i, j) \in T_r^\Pi \). A example is that node class assignments \( \Pi(e) \) naturally induces a partition of the edge set \( \{T_{kl}^\Pi(e)\}_{1 \leq k \leq l \leq K} \), in which case \( \Pi_{ij} = T_{(e_i, e_j)}^\Pi(e) \). In general a partition could be more flexible than one induced by class assignments. For any partition \( \Pi = \{T_1^\Pi, \ldots, T_R^\Pi\} \) of \( \{(i, j)\}_{1 \leq i < j \leq n} \), define \( \tilde{P}_r := \frac{\sum_{(i, j) \in T_r^\Pi} P_{ij} e^{x_i^j} \gamma^0}{\sum_{(i, j) \in T_r^\Pi} e^{x_i^j} \gamma^0} \) which corresponds to the \( \tilde{B} \) defined previously; and define
\[ PL_p(Z; \Pi) := \sum_{i < j} (P_{ij} e^{x_i^j} \gamma^0 \log \tilde{P}_{i,j} - \tilde{P}_{i,j} e^{x_i^j} \gamma^0) = \sum_{i < j} P_{ij} e^{x_i^j} \gamma^0 (\log \tilde{P}_{i,j} - 1) \tag{A.37} \]
which corresponds to the \( PL_p(Z; e, \gamma) \) previously (we omitted the argument \( \gamma \) in (A.37) since we are only dealing with \( \gamma^0 \) now by Theorem 7). When the partition is induced by a class assignment \( e \), it is easy to see that \( PL_p(Z; e, \gamma^0) = PL_p^\gamma(Z; \Pi(e)) \). Besides, by noting that \( PL_p^\gamma(Z; \Pi) \) is the optimal value of the problem \( \max_Q L_p^\gamma(Z; \Pi, Q) := \sum_{i < j} (P_{ij} e^{x_i^j} \gamma^0 \log Q_{ij} - Q_{ij} e^{x_i^j} \gamma^0) \) subject to the constraint \( Q_{ii j_1} = Q_{i j_2 j} \) if \( \Pi_{i i j_1} = \Pi_{i j_2 j} \), we have the following property:

**Lemma A.10.** Let \( \Pi' \) be a refinement of partition \( \Pi \) of set \( \{(i, j)\}_{i < j} \), then \( PL_p^\gamma(Z; \Pi) \leq PL_p^\gamma(Z; \Pi') \).

Next, we want to construct a refinement \( \Pi^* \) of \( \Pi(\hat{e}) \) such that
\[ PL_p(Z; c, \gamma^0) - PL_p^\gamma(Z; \Pi^*) = N(\hat{e}) \Omega(M/n). \tag{A.38} \]
Combining (A.38) with (A.32) and Lemma A.10 we have
\[N(\hat{e})\Omega(M/n) = PL_p(Z; c, \gamma^0) - PL^*_p(Z; \Pi^*) \leq PL_p(Z; c, \gamma^0) - PL_p(Z; \hat{e}, \gamma^0) = o_p(M).\]
Hence it suffices to show there exists a refinement such that (A.38) holds. We construct \(\Pi^*\) as follows. In each class \(k\) of \(\hat{e}\), we take out pairs \((i, j)\) such that \(\hat{e}_i = \hat{e}_j = k\) but \(c_i \neq c_j\). Continue this process in this cluster of \(\hat{e}_i = k\) until all nodes remaining in it have the same true class membership under \(c\). Denote the total number of pairs we have taken out by \(N_1\). Since by definition we see each node whose true class under \(c\) is in the majority within its estimated class \(\hat{e}\) as being correctly classified, the total number of nodes remaining after the pairing process must be smaller than or equal to the number of correctly classified nodes, and hence \(2N_1 \geq N(\hat{e})\). Next, for each picked out pair \((i, j)\), select all nodes \(h\) such that \(D'(P_{ih}, P_{jh}) \geq CMK/(n^2)\) where \(C\) is the constant from the second condition of the theorem. In \(\Pi(\hat{e})\) we know that \(\Pi(\hat{e})_{ih} = \Pi(\hat{e})_{jh} = T_{(\hat{e}_i, \hat{e}_h)}\); now we separate \(T_{(\hat{e}_i, \hat{e}_h)}\) into \(\{(i, h), (j, h)\}\) and \(T_{(\hat{e}_i, \hat{e}_h)} \setminus \{(i, h), (j, h)\}\). Perform this separation for all \(((i, j), h)\) such that \((i, j)\) pair is picked out in the first step and \(D'(P_{ih}, P_{jh}) \geq CMK/(n^2)\), and the resulted refinement of \(\Pi(\hat{e})\) is the \(\Pi^*\) we wanted. To see this, denote the number of triples \(((i, j), h)\) selected in the second step by \(N_2\). By condition 2 in the theorem, for each pair \((i, j)\) there is at least one true class such that all nodes in it could form a selected triple with \((i, j)\). Hence \(N_2 \geq N_1 \min_a n_a(c) \geq N_1\Omega(n/K)\). From (A.37) we could calculate
\[
PL_p(Z; c, \gamma^0) - PL^*_p(Z; \Pi^*) = \sum_{i<j} P_{ij} e^{\sum_j \gamma^0_j \log \frac{P_{ij}}{\hat{P}_{ij}}} \geq \sum_{((i, j), h)\text{picked out}} P_{ih} \log \frac{P_{ih}}{\hat{P}_{ih}} + P_{jh} \log \frac{P_{jh}}{\hat{P}_{jh}} = D'(P_{ih}, P_{jh}) \geq N_2\Omega\left(\frac{MK}{n^2}\right) \geq N_1\Omega\left(\frac{n}{K}\right)\Omega\left(\frac{MK}{n^2}\right) \geq N(\hat{e})\Omega\left(\frac{M}{n}\right)
\]
which shows (A.38) and finishes the proof. \(\square\)
A.5 Proof of Theorem 3

From Theorem 1, it is not hard to show the following error bound for \( \hat{\gamma} \).

**Lemma A.11.** For any constant \( \eta > 0 \), \( \exists \) positive constants \( C_\eta \) and \( v_\eta \) s.t., \( \Pr(\sqrt{n}p_n\|\gamma_0 - \hat{\gamma}\|_\infty > \eta) < C_\eta \exp(-v_\eta n) \).

To prove Theorem 3, we first state a concentration result for directed, \( \gamma_0 \) adjusted adjacency matrix (Theorem 9), then derive Theorem 3 based on Theorem 9, and finally give a proof of Theorem 9.

**Theorem 9** (Concentraion for directed, \( \gamma_0 \) adjusted adjacency matrix; A covariate adjusted, Poisson variant of Theorem 2.1 of Le et al. (2017)). Let \( A \) be the adjacency matrix generated by the directed PCABM \((M, B, Z, \gamma_0)\). Assume Condition 1 holds; and \( \|B\|_{\text{max}} \leq C\bar{B} \). Also, let

\[
d = \max_{i,j}nP_{ij} = \max_{i,j}nB_{c_ic_j}. \tag{A.39}
\]

Consider the adjusted adjacency matrix \( A_0 \) derived from \( \gamma_0 \), i.e. \( A^0_{ij} = A_{ij}/\exp(z_{ij}^\top \gamma_0) \). For any \( r > 1 \), the following holds with probability at least \( 1 - 3n^{-r} \): Consider any subset consisting of at most \( 10n/d \) vertices, and reduce the weights of the edges incident to those vertices in an arbitrary way. Denote the adjacency matrix of the new (weighted) graph by \( A_0^{0R} \), and let \( d^{(R)} \) be the maximal row and column \( l_1 \) norm of \( A_0^{0R} \). Then \( A_0^{0R} \) satisfies

\[
\|A_0^{0R} - P\| \leq C_\varepsilon \sqrt{\varepsilon^3} (\sqrt{d} + \sqrt{d^{(R)}}) \tag{A.40}
\]

where \( C \) is a constant that does not depend on \( \varepsilon \). Moreover, the bound (A.40) still holds when \( d^{(R)} \) is the maximal row and column \( l_2 \) norm of \( A_0^{0R} \).

In this result and in the rest of this section, \( C \) denotes absolute constant whose value may be different from line to line.
Proof of Theorem 3. We first prove the result for directed case, i.e. $A_{ij} \sim \text{Poisson}(B_{ci}c_e \exp(z_{ij}^T \gamma^0))$ independent for all $i \neq j$, and then use a coupling argument to extend the result to undirected case. In the directed setting, by Theorem 9 it suffices to bound $\|A^r - A^{0r}\|$. Let $w_{ij} \in [0, 1]$ be the weight imposed on edge $A_{ij}$. By Lemma A.11,

$$\|\gamma^0 - \hat{\gamma}\|_{\infty} \leq \eta/\sqrt{n \rho_n}$$  \hspace{1cm} (A.41)

with probability at least $1 - c_n \exp(-v_n n)$. All of our arguments in this proof are conditioned on event (A.41). For any $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$,

$$|x^T (A^r - A^{0r}) x| = \left| \sum_{i,j} x_i x_j \left( A_{ij}/\exp(z_{ij}^T \gamma) - A_{ij}/\exp(z_{ij}^T \gamma^0) \right) w_{ij} \right|
\leq \sum_{i,j} |x_i||x_j||A^{0r}_{ij} \left[ \exp\{z_{ij}^T(\gamma^0 - \hat{\gamma})\} - 1 \right]|
\leq \sum_{i,j} 2|x_i||x_j||A^{0r}_{ij} |z_{ij}^T(\gamma^0 - \hat{\gamma})|
\leq \frac{2}{\sqrt{n \rho_n}} \sum_{i,j} |x_i||x_j||A^{0r}_{ij}|
\leq \frac{2}{\sqrt{n \rho_n}} \|A^{0r}\|$$  \hspace{1cm} (A.42)

in which the third line is due to $|e^t - 1| < 2|t|$ when $|t| < 1$, and the fourth line is due to (A.41) and $\eta = (p \zeta)^{-1}$. From (A.42) we get $\|A^r - A^{0r}\| \leq 2 \|A^{0r}\|/\sqrt{n \rho_n}$. Furthermore, we have

$$\|P\| \leq \|P\|_F \leq \sqrt{n^2 \rho_n^2 \|B\|_\infty^2} \leq n \rho_n C_B.$$  \hspace{1cm} (A.43)

Combining (A.43) with Theorem 9, we could bound $\|A^{0r}\|$ by $\|A^{0r}\| \leq \|A^r - P\| + \|P\| \leq C r \sqrt{\xi^3(\sqrt{d} + \sqrt{d(R)})} + n \rho_n C_B$. Thus,

$$\|A^r - P\| \leq \|A^r - A^{0r}\| + \|A^{0r} - P\|
\leq 2 \|A^{0r}\|/\sqrt{n \rho_n} + \|A^{0r} - P\|
\leq 2C_B \sqrt{n \rho_n} + C r \sqrt{\xi^3(1 + \frac{1}{\sqrt{n \rho_n}})(\sqrt{d} + \sqrt{d(R)})}.$$  \hspace{1cm} (A.44)
It is not hard to see that after performing the previously stated “reduce weight” regularization on $A^0$, the resulting $A^{0R}$ has row and column $l_2$ norms bounded by $d$ up to a constant:

$$\sum_j A_{ij}^0 \leq 4 \sum_j A_{ij}^0 \cdot \frac{d'}{d_i} \frac{d'}{d_j} \leq 4 \sqrt{\phi n} d_{(0)}^2 \sum_j A_{ij}^0 / d_{(0)}^i \leq 5d^2$$

where $d_{(i)}^0$ and $d_{(0)}^0$ are $i$th node degree and maximum degree of $A^0$. Hence the concentration (A.44) reads $\|A^r - P\| \lesssim \sqrt{\phi n}$.

Now we bridge our result for directed case to the undirected case with a coupling approach (Amini et al., 2013). Consider the directed model

$$\tilde{A}_{ij} | Z \overset{indpt}{\sim} \text{Poisson}(\frac{1}{2} B_{c_i c_j} \exp(z_{ij}^T \gamma^0)) \quad \text{for} \quad \forall i \neq j.$$ 

Now let $A_{ij} = \tilde{A}_{ij} + \tilde{A}_{ji}$. Then the resulting adjacency matrix $A = \{A_{ij}\}$ satisfies: (1) $A_{ij} = A_{ji}$ for all $i < j$; (2) $A_{ij}, i < j$ are all independent; and (3) $A_{ij} | Z \sim \text{Poisson}(B_{c_i c_j} \exp(z_{ij}^T \gamma^0))$. Thus, the $\{A_{ij}\}$ defined this way follows our original undirected PCABM. Denote by $\tilde{A}^r$ the adjusted version of $\tilde{A}$ on the new weighted graph. The result for directed case gives $\|\tilde{A}^r - P/2\| \lesssim \sqrt{\phi n}$, so that a triangle inequality $\|A^r - P\| \leq \|\tilde{A}^r - P/2\| + \|\tilde{A}^r - P/2\|$ proves the statement of Theorem 3 for the undirected case.

**A.5.1 Proof of Theorem 9**

**Theorem 10** (Graph decomposition for the covariate adjusted Poisson adjacency matrix; counterpart of Theorem 2.6 in Le et al. (2017)). *Consider the adjusted adjacency matrix $A^0$ derived*
from $\gamma^0$ in Theorem 9. Under the same assumptions of Theorem 9, for any $r > 1$, the following holds with probability at least $1 - 3n^{-r}$: One can decompose the set of edges $[n] \times [n]$ into three classes $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{C}$ so that the following properties are satisfied for the adjusted adjacency matrix $A^0$:

1. The graph concentrates on $\mathcal{N}$, namely $\| (A^0 - E A^0)_{\mathcal{N}} \| \leq C r \sqrt{\xi^3 d}$.

2. Each row of $A^0_{\mathcal{R}}$ and each column of $A^0_{\mathcal{C}}$ has $l_1$ norm $\leq 32 r \xi^2$.

3. $\mathcal{R}$ intersects at most $\sqrt{2n}/(\sqrt{2} - 1) d$ columns, and $\mathcal{C}$ intersects at most $\sqrt{2n}/(\sqrt{2} - 1) d$ rows of $[n] \times [n]$.

Moreover, the same result also holds for the second property being replaced by “each row of $A^0_{\mathcal{R}}$, each column of $A^0_{\mathcal{C}}$ has $l_0$ norm $\leq 32 r \xi$.”

An illustration of the graph decomposition in Theorem 10 is given in Figure 7 (the picture comes from Le et al. (2017)). We put those well-concentrated edges into $\mathcal{N}$, while the number of the not well-behaved, high degree node attached edges are bounded as $\mathcal{R}$ and $\mathcal{C}$ have bounded column and row norms.

![Figure 7: An illustration of the graph decomposition in Theorem 10.](image)

Several lemmas are established as steps for the proof of the decomposition Theorem 10.
**Lemma A.12** (Concentration in $l_\infty \to l_2$ norm). Let $1 \leq m \leq n$ and $\alpha \geq m/n$. Then for $r \geq 1$ the following holds with probability at least $1 - n^{-r}$. Consider a block $I \times J$ of size $m \times m$. Let $I'$ be the set of indices of the rows of $A^0_{I \times J}$ whose $l_1$ norm $\leq \alpha d$. Then

$$\| (A^0 - \mathbb{E}A^0)_{I' \times J} \|_{\infty \to 2} \leq C g(\xi) \sqrt{\alpha dm r \log(en/m)}$$

where $g(\xi) = \sqrt{(1 + 2\bar{L})\xi}$ is a constant that is a function of $\xi$ with $\bar{L} = 2(e^2 + 1)$.

**Proof of Lemma A.12.** Note that centralized Poisson random variables satisfy the Bernstein condition $\mathbb{E}|A_{ij} - \mathbb{E}A_{ij}|^k \leq \frac{1}{2}\mathbb{E}(A_{ij} - \mathbb{E}A_{ij})^2 L^{k-2}k$! (Lemma A.2). By the scale invariance of the Bernstein condition, the adjusted adjacency matrix elements also satisfy the Bernstein condition with the constant $L$ replaced by $L\xi$. Thus, defining

$$x \in \{1, -1\}^m, \quad X_i := \sum_{j \in J} (A^0_{ij} - \mathbb{E}A^0_{ij})x_j, \quad \eta_i := \mathbb{1}_{\{\sum_{j \in J} A^0_{ij} \leq \alpha d\}}$$

one can recover equation (3.5) and (3.6) in the proof of Lemma 3.3 of Le et al. (2017) by

$$\Pr(|X_i\eta_i| > tm) \leq 2 \exp\left(\frac{-mt^2/2}{\xi d/n + L\xi t}\right) \leq 2 \exp\left(\frac{-m^2t^2/2}{\xi d/n + 2L\xi d}\right)$$

where the last inequality is because $\alpha \geq m/n$ and

$$|X_i\eta_i| \leq \sum_{j \in J} A^0_{ij} + \mathbb{E}A^0_{ij} \leq \alpha d + m \cdot \frac{d}{n} \leq 2\alpha d.$$

Thus, $|X_i\eta_i|$ has sub-gaussian norm at most $\sqrt{(1 + 2L)\xi d}$, and the rest of the proof follows from the proof of Lemma 3.3 in Le et al. (2017). \qed

Combining Lemma A.12 with Theorem 3.2 (Grothendieck-Pietsch factorization, sub-matrix version) of Le et al. (2017), we immediately get the following Lemma A.13.

**Lemma A.13** (Concentration in spectral norm). Let $1 \leq m \leq n$ and $\alpha \geq m/n$. Then for $r \geq 1$ the following holds with probability at least $1 - n^{-r}$. Consider a block $I \times J$ of size $m \times m$.
Let $I'$ be the set of indices of the rows of $A^0_{I \times J}$ whose $l_1$ norm $\leq \alpha d$. Then there exists a subset $J' \subseteq J$ of at least $3m/4$ columns such that

$$\| (A^0 - EA^0)_{I' \times J'} \| \leq C \sqrt{\xi a d r \log(en/m)}.$$

The following Lemma A.14 shows that most rows satisfy the condition for $I'$ in Lemma A.13. The proof of this Lemma involves three steps: first bound the probability of each row having large $l_1$ norm; then bound the number of high $l_1$ norm rows by seeing it as sum of independent Bernoulli variables; finally apply a union bound for $m, I$, and $J$. In the first step of the proof, we need to deal with the covariates as well as the Poisson edges, which is different from the Erdos-Renyi scenario in Le et al. (2017).

**Lemma A.14** (Most rows have $l_1$ norm $\leq O(\alpha d)$). Let $1 \leq m \leq n$ and $\alpha \geq \sqrt{m/n}$. Then for $r \geq 1$ the following holds with probability at least $1 - n^{-r}$. Consider a block $I \times J$ of size $m \times m$. Then all but $m/\alpha d$ rows of $A^0_{I \times J}$ have $l_1$ norm $\leq 8\xi^2 r d$.

**Proof of Lemma A.14.** Fix a block $I \times J$, and denote by $d_i$ the $l_1$ norm of the $i$-th row of $A^0_{I \times J}$, i.e. $d_i = \sum_{j \in J} A_{ij}$. We apply a Poisson variable’s Chernoff inequality (Vershynin (2018), p20) to bound $d_i$:

$$\Pr(d_i \geq 8r \alpha d \xi^2) = \Pr\left( \sum_{j \in J} \frac{A_{ij}}{\exp(z_{ij}^0)} \geq 8r \alpha d \xi \right)$$

$$\leq \Pr\left( \sum_{j \in J} A_{ij} \geq 8r \alpha d \xi \right)$$

$$\leq \left( \frac{8r \alpha d \xi}{e \cdot \frac{md}{n} \cdot \sum_{j \in J} \frac{\exp(z_{ij}^0)}{m} / m} \right)^{-8r \alpha d \xi}$$

$$\leq \left( \frac{2\alpha n}{m} \right)^{-8r \alpha d},$$

30
in which we used $\sum_{j \in J} \exp(z_{ij}^T \gamma^0)/m \leq \xi$ and $\xi \geq 1$, and all the inequalities should be understood as first conditioning on and then averaging out the covariates $Z$. The rest of the proof follows from proof of Lemma 3.5 in Le et al. (2017). \hfill \Box

**Lemma A.15** (For block of large row $l_1$ norm, most columns have $O(1)$ $l_1$ norm; and most columns have $O(1)$ $l_0$ norm). Let $1 \leq m \leq n$ and $\alpha \geq \sqrt{m/n}$. Then for $r \geq 1$ the following holds with probability at least $1 - n^{-r}$. Consider a block $I \times J$ of size $k \times m$ with some $k \leq m/\alpha d$. Then all but $m/4$ columns of $A_{I \times J}^0$ have $l_1$ norm $\leq 32r\xi^2$. Moreover, all but $m/4$ columns of $A_{I \times J}^0$ have $l_0$ norm $\leq 32r\xi$.

**Proof of Lemma A.15.** Fix a block $I \times J$, and denote by $d_j$ the $l_1$ norm of the $j$-th column of $A_{I \times J}^0$, i.e. $d_j = \sum_{i \in I} A_{ij}^0$. We apply a Poisson variable’s Chernoff inequality (Vershynin (2018), p20) to bound $d_j$:

$$\Pr(d_j \geq 32r\xi^2) = \Pr \left( \sum_{i \in I} \frac{A_{ij}^0}{\exp(z_{ij}^T \gamma^0)\xi} \geq 32r\xi \right)$$

$$\leq \Pr \left( \sum_{j \in J} A_{ij} \geq 32r\xi \right)$$

$$\leq \left( \frac{32r\xi}{e \cdot \frac{kd}{n} \cdot \sum_{i \in I} \frac{\exp(z_{ij}^T \gamma^0)}{k} } \right)^{-32r\xi}$$

$$\leq \left( \frac{10\alpha n}{m} \right)^{-32r} .$$

For the “moreover” part, denote by $d'_j$ the $l_0$ norm, i.e. the number of none zero elements, of the $j$-th column of $A_{I \times J}^0$. By a similar argument as above, we could bound $d'_j$ by

$$\Pr(d'_j \geq 32r\xi) \leq \Pr \left( \sum_{j \in J} A_{ij} \geq 32r\xi \right) \leq \left( \frac{10\alpha n}{m} \right)^{-32r} .$$

Again all the inequalities should be understood as first conditioning on and then averaging out the covariates $Z$. The rest of the proof follows from proof of Lemma 3.6 in Le et al. (2017). \hfill \Box

31
Combining Lemma A.13, A.14 and A.15 we could get the following Lemma A.16, which gives the decomposition of one block.

**Lemma A.16 (Decomposition of one block).** Let $1 \leq m \leq n$ and $\alpha \geq \sqrt{m/n}$. Then for $r \geq 1$ the following holds with probability at least $1 - 3n^{-r}$. Consider a block $I \times J$ of size $m \times m$. Then there exists an exceptional sub-block $I_1 \times J_1$ with dimensions at most $m/2 \times m/2$ such that the remaining part of the block, that is $(I \times J) \setminus (I_1 \times J_1)$, can be decomposed into three classes $\mathcal{N}, \mathcal{R} \subset (I \setminus I_1) \times J$ and $\mathcal{C} \subset I \times (J \setminus J_1)$ so that the following hold:

1. The graph concentrates on $\mathcal{N}$, i.e. $\| (A_0^0 - \mathbb{E} A_0^0)_{\mathcal{N}} \| \leq C r \sqrt{\xi^4 \alpha d \log(en/m)}$.

2. Each row of $A_0^0 \mathcal{R}$, each column of $A_0^0 \mathcal{C}$ has $l_1$ norm $\leq 32 r \xi^2$.

3. $\mathcal{R}$ intersects at most $m/\alpha d$ columns, and $\mathcal{C}$ intersects at most $m/\alpha d$ rows of $I \times J$.

Moreover, the same result also holds for the second property being replaced by “each row of $A_0^0 \mathcal{R}$, each column of $A_0^0 \mathcal{C}$ has $l_0$ norm $\leq 32 r \xi$.”

Repeatedly applying Lemma A.16 to the ‘exceptional’ block in each iteration, we would finally arrive at Theorem 10. And with the decomposition in Theorem 10 we could prove Theorem 9 by bounding the spectral norm separately on $\mathcal{N}, \mathcal{R}$ and $\mathcal{C}$. The proof of Lemma A.16, Theorem 10 and Theorem 9 are all same as in Le et al. (2017), except for some changes in the constants. Thus, we omit the proof for these three results.
B  Additional Numerical Results

B.1  Estimating $\gamma$ with Random Initial $e$

Here, we present in Table A.1 the simulation results on the estimation of $\gamma$ for Section 7.1 under random initial community assignments. It is very similar to Table 3 when we ignore the community structure.

Table A.1: Simulated results over 100 replicates of $\hat{\gamma}$, displayed as mean (standard deviation).

| $n$ | $\gamma_0^0 = 0.4$ | $\gamma_2^0 = 0.8$ | $\gamma_3^0 = 1.2$ | $\gamma_4^0 = 1.6$ | $\gamma_5^0 = 2$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|
| 100 | 0.399 (0.0414)    | 0.797 (0.0354)    | 1.197 (0.0455)    | 1.596 (0.0467)    | 1.995 (0.0484)    |
| 300 | 0.399 (0.0205)    | 0.801 (0.0151)    | 1.199 (0.0227)    | 1.603 (0.0217)    | 2.000 (0.0234)    |
| 500 | 0.395 (0.0131)    | 0.799 (0.0118)    | 1.197 (0.0173)    | 1.599 (0.0140)    | 2.002 (0.0147)    |

B.2  PCABM Clustering Visualization in the School Friendship Data

In Figure 8, school and ethnicity are targeted communities, respectively. We use different shades to distinguish true communities. Predicted communities are separated by the middle dash line so that the ideal split would be shades vs. tints on two sides. By these criteria, our model performs pretty well in both cases.
Figure 8: Community detection with different pairwise covariates. From top to bottom, we present community prediction results for school and ethnicity.