Network, Popularity and Social Cohesion: A Game-Theoretic Approach

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Abstract
In studies of social dynamics, cohesion refers to a group’s tendency to stay in unity, which – as argued in sociometry – arises from the network topology of interpersonal ties. We follow this idea and propose a game-based model of cohesion that not only relies on the social network, but also reflects individuals’ social needs. In particular, our model is a type of cooperative games where players may gain popularity by strategically forming groups. A group is socially cohesive if the grand coalition is core stable. We study social cohesion in some special types of graphs and draw a link between social cohesion and the classical notion of structural cohesion (White and Harary 2001). We then focus on the problem of deciding whether a given social network is socially cohesive and show that this problem is CoNP-complete. Nevertheless, we give two efficient heuristics for coalition structures where players enjoy high popularity and experimentally evaluate their performances.

1 Introduction
Human has a natural desire to bind with others and needs to belong to groups. By understanding the basic instruments that generate coherent social groups, one can explain important phenomena such as the emergence of norms, group conformity, self-identity and social classes (Festinger 1950; Huisman and Bruggeman 2012; Hogg 1992). For example, studies reveal that on arrival to Western countries, immigrants tend to form cohesive groups within their ethnic communities, which may hamper their acculturation into the new society (Nee and Sanders 2001). Another study identifies cohesive groups of inhabitants in an Austrian village that correspond to stratified classes defined by succession to farmland ownership (Brudner and White 1997).

Most theories of group dynamics rely on two fundamental drives: cooperation and social needs. Indeed, every group exists to accomplish certain tasks. Cooperation is desirable because combining skills and resources leads to better outcomes. The theory of cooperative games studies distribution of collective gains among rational agents (Peleg and Sudhölter 2010). Social need is another important drive of group dynamics. A society contains complex interpersonal relations. The theory of self-categorization asserts that individuals mentally associate themselves into groups based on relations such as friendship and trust (Hogg 1992).

Cohesion denotes a tendency for a social group to stay in unity, which traditionally consists of two views. Firstly, cohesion refers to a “pulling force” that draws members together (Festinger 1950); Secondly, cohesion also means a type of “resistance” of the group to disruption (Gross and Martin 1952). In (2001), White and Harary propose a notion of structural cohesion, which unifies these two views.

However, we identify insufficiencies in the existing models for social cohesion: 1) Cooperative game theory often misses the crucial social network dimension. 2) The structural cohesion of a network refers to the minimum number of nodes whose removal results in network disintegration (White and Harary 2001); this is a property of the network on the whole, and does not embody individual needs. Since cohesion embodies both the micro-focus of psychology (fulfillment of personal objectives and needs) and the macro-focus of sociology (emergence of social classes) (Carron and Brawley 2000), the main challenge is to build a general but rigorous model to bridge the micro- and the macro-foci.

In this paper, we define a type of cooperative games on networked agents. Outcomes of the game not only rely on the network topology but also reflect individuals’ social needs. Our model is consistent with existing theories; firstly, we follow the network approach to study social phenomena, which is initiated by early pioneers such as Simmel and Durkheim. Secondly, our game-theoretic formulation is in line with group dynamics theories that focus on the independence among members (Lewin 1943). Thirdly, we verify that networks with high structural cohesion also tend to be socially cohesive according to our definition.

People prefer to be in a group where they are seen as valuable and popular members. Hence the payoff should reflect in some sense players’ social positions. Popularity – an important indicator of social position – arises from interpersonal ties such as liking or attraction (Lansu and Cillessen 2011). In particular, (Conti et al. 2013) uses the degrees of nodes as a measure of popularity and identify the economic benefits behind gaining popularity. Therefore, payoffs in our games are defined based on degrees of players.

We summarize our main contributions: (1) We propose popularity games on a social network and define social co-
hesion using core stability of the grand coalition. (2) We show that our notion is consistent with intuition for several s-
standard classes of networks and connect structural cohesion with our notion of social cohesion (Thm. 6). (3) We prove that deciding socially cohesion of a network is computation-
ally hard (Thm. 7). (4) Finally, we present two heuristics that decide social cohesion and compute group structures with high player payoffs and evaluate them by experiments.

Related works. The series of works (Narayanan et al. 2014; Michalak et al. 2013; Szczepański et al. 2015a; 2015b; 
Szczepański, Michalak, and Wooldridge 2014) investigates game-based network centrality. Their aim is to capture a 
player’s centrality using various instances of semivalues, which are based on the player’s expected payoff. In con-
trast, our study aims at games where the payoff of players are given a priori by degree centrality and focus on core sta-
bility. (Chen et al. 2011) uses non-cooperative games to exp-
plain community formation in a social network. Each player in their game decides among a fixed set of strategies (i.e.
a given set coalitions); the payoff is defined based on gain and loss which depend on the local graph structures, (Mc-
Sweeney, Mehrotra, and Oh 2014) studies community for-
mation through cooperative games. The payoffs of players are given by modularity and modularity-maximising parti-
tions correspond to Nash equilibria. The focus is on commu-
nity detection but not on social cohesion. Furthermore, our payoff function is not additively separable and hence does 
not extend from their model. Lastly, our work is different from community detection (Fortunato 2010). The notion of 
community structure originates from physics which focuses on a macro view of the network, while our work is motivat-
ed from group dynamics and focus on individual needs and preferences.

2 Popularity Games and Social Cohesion

A social network is an unweighted graph \( G = (V, E) \) where \( V \) is a set of nodes and \( E \) is a set of (undirected) edges. An 
edge \( \{u, v\} \subseteq V \) (where \( u \neq v \)), denoted by \( uv \), repre-
sents certain social relation between \( u, v \), such as attraction, 
interdependence and friendship. We do not allow loops of 
the form \( uu \). The reader is referred to (Peleg and Sudhölter 
2010) for more details on cooperative game theory.

Definition 1. A cooperative game (with non-transferable utility) is a pair \( G = (V, \rho) \) where \( V \) is a set of players, and 
\( \rho : V \times 2^V \rightarrow \mathbb{R} \) is a payoff function.

A coalition formation of \( G \) is a partition of \( V \) \( \mathcal{W} = \{V_1, \ldots, V_k\} \), i.e., \( \bigcup_{1 \leq i < j \leq k} V_i = V, \forall 1 \leq i < j \leq k: V_i \cap V_j = \emptyset \); each set \( V_i \) is called a coalition. The grand coalition formation is \( \mathcal{W}_G = \{V\} \) where \( V \) is called the grand coal-
tion. Cooperative games describe situations where players-
s strategically build coalitions based on individual payoffs.

Set \( \rho_{\mathcal{W}}(u) := \rho(u, S) \) where \( u \in S \) and \( S \subseteq \mathcal{W} \).

Definition 2. A non-empty set of players \( H \subseteq V \) is blocking for \( \mathcal{W} \) if \( \forall u \in H : \rho(u, H) > \rho_{\mathcal{W}}(u) \); In this case we say 
that \( \mathcal{W} \) is blocked by \( H \). A coalition formation \( \mathcal{W} \) of \( G \) is 
core stable w.r.t. \((V, \rho)\) if it is not blocked by any set \( H \subseteq V \).

Social positions, as argued in sociometric studies, arise 
from the network topology (Cillessen and Mayeux 2004). A 
long line of research studies how different centralities (e.g.
degree, closeness, betweenness, etc.) give rise to “position-
al advantage” of individuals. In particular, degree centrality 
refers to the number of edges attached to a node. Despite 
its conceptual simplicity, degree centrality naturally repre-
sents (sociometric) popularity, which plays a crucial role 
in a person’s self-efficacy and social needs (Zhang 2010; 
Conti et al. 2013). Popularity depends on the underlying 
group: a person may be very popular in one group while 
being unknown to another. Hence individuals may gain pop-
ularity by forming groups strategically. We thus make our 
next definition. The sub-network induced on a set \( S \subseteq V \) is 
\( G|S = (S, E|S) \) where \( E|S = E \cap S^2 \). \( \deg_S(u) \) denotes 
\( \{|v: uv \in E|S\} \) and we write \( \deg(u) \) for \( \deg_S(u) \).

Definition 3. The popularity of a node \( u \) in a subset \( S \subseteq V \) is 
\( p_S(u) := \deg_S(u)/|S| \).

Note that \( p_S(u) \) is not blocked. If \( u \in S \) has an edge to all other nodes in the graph \( G|S \), then \( u \) is the most 
popular node in \( S \) with \( p_S(u) = (|S| - 1)/|S| \).

Definition 4. The popularity game on \( G = (V, E) \) is a co-
operative game \( \Gamma(G) = (V, \rho) \) where \( \rho: V \times 2^V \rightarrow [0, 1] \) is 
defined by \( \rho(u, S) = p_S(u) \).

An outcome of the popularity game \( \Gamma(G) \) assigns any 
player \( u \) with a coalition \( S \ni u \). The sum of popular-
ity of members of \( S \) equals their average degree in \( S \), i.e.
\( \sum_{u \in S} p_S(u) = \sum_{u \in S} \deg_S(u)/|S| = 2|E|/|S| \).

The average degree measures the density of the set \( S \) which re-
fects the amount of interactions within \( S \), and thus can be 
regarded as a collective utility. In this sense, the popularity 
game is efficient in distributing such collective utility among 
players according to their popularity.

Social cohesion represents a group’s tendency to remain 
united (Cartwright and Zander 1953). We express cohesion 
through core stability w.r.t. the popularity game \( \Gamma(G) \): Sup-
pose a coalition formation \( \mathcal{W} \) is not core stable. Then there 
is a set \( S \subseteq V \) of each member of which would gain a high-
er popularity in \( S \) than in their own coalitions in \( \mathcal{W} \). Thus 
there is a latent incentive among members of \( S \) to disrupt 
\( \mathcal{W} \) and form a new coalition \( S \). On the contrary, a core sta-
ble \( \mathcal{W} \) represents a state of the network that is resilience to such “disruptions”. Thus, when the grand coalition formation \( \mathcal{W}_G = \{V\} \) is core stable, all members bind naturally and harmoniously into a single group and would remain so as 
long as the network topology does not change.

Definition 5. A network \( G = (V, E) \) is socially cohesive 
or simply cohesive if the grand coalition formation \( \mathcal{W}_G \) is 
core stable w.r.t. the popularity game \( \Gamma(G) \).

Example 1. Fig. 1(a) displays \( G_1 = (V_1, E_1) \). The popu-
ularity \( p_{V_1}(i) \) is \( 1/3 \) for \( i = b, f \), and is \( 1/2 \) for \( i = a, c, d, e \). 
The set \( \{a, b, c\} \) blocks \( W_{V_1} \), as each member has popularity 
\( 2/3 \). The only core stable formation is \( \{(a, b, e), (d, e, f)\} \).

Adding the edge \( ad \) makes \( G_1 \) cohesive as the popularity of 
both \( a \) and \( d \) in \( V_1 \) reaches \( 2/3 \). Fig. 1(b) displays \( G_2 = 
(V_2, E_2) \) where \( p_{V_2}(a) = 4/5 \) and \( p_{V_2}(i) = 1/5 \) for all 
\( i = b, \ldots, e \). This graph is cohesive as \( W_{V_2} \) is not blocked.
Figure 1: The graphs considered in Example 1 are in black. The added edges are highlighted in red.

However, adding the edge $bc$ will destroy social cohesion as then \{b, c\} blocks $W_{G_2}$. Social psychological studies often presume that more ties leads to higher cohesion; this example displays a more complicated picture: Adding an edge may establish cohesion, but may also sabotage cohesion.

**Theorem 1 (Connectivity).** If a coalition formation $W$ of $G$ is core stable then any $S \subseteq W$ induces a set of isolated nodes or a connected subgraph.

Proof. Suppose $S = V_1 \cup V_2$ where $V_1, V_2$ are non-empty and no edge exists between any pair in $V_1 \times V_2$. Then for every $u \in V_1$, $p_{V_1}(u) = \frac{\deg v(u)}{|V_1|} > \frac{\deg_S(u)}{|S|} = p_S(u)$. Hence $W$ does not contain $S$. □

By Thm. 1, it is sufficient to consider only coalitions that induce connected sub-networks of a social network.

**Definition 6.** A set $S \subseteq V$ is called a social group of $G$ if $S$ induces a connected sub-network. A group structure is a coalition formation containing only social groups.

The next theorem shows that socially cohesive networks have bounded size.

**Theorem 2.** Let $\delta(G)$ be the maximum degree of nodes in $G = (V, E)$. Then $G$ is socially cohesive only when $|V| \leq 2\delta(G)$ unless $|V| = 1$.

Proof. Suppose $|V| > 2\delta(G)$ and $|V| > 1$. If $E = \emptyset$, $G$ is not socially cohesive by Thm. 1. Otherwise, pick an edge $uv$. Then $\max\{\deg(u), \deg(v)\} \leq \delta(G) < |V|/2$. This means that $\max\{p_V(u), p_V(v)\} < 1/2$, and the edge $\{u, v\}$ forms a blocking set. Thus $G$ is not socially cohesive. □

We now investigate our games on some standard classes of graphs and characterize core stable group structures.

**Complete networks.** A graph $G = (V, E)$ is complete if an edge exists between any pair of nodes. Naturally, one would expect a complete network to be socially cohesive.

**Theorem 3.** Let $G = (V, E)$ be a complete network. The grand coalition is the only core stable group structure.

Proof. Any induced sub-network $G[S]$ of a complete network $G = (V, E)$ (where $S \subseteq V$) is also complete. Thus $p_{S}(u) = \frac{|S|-1}{|S|} < \frac{|V|-1}{|V|} = p_{V}(u)$. Therefore any player’s popularity is maximised in the grand coalition $V$. □

**Star networks.** A star network contains a node $c$ (centre), a number of other nodes $u_1, \ldots, u_m$ (tails) where $m > 1$, and edges $\{cu_1, \ldots, cu_m\}$. Intuitively, the centre $c$ would like to be in a social group with as many players as possible, while a tail would like to be with as few others as possible.

**Theorem 4.** A group structure $W$ of a star network is core stable iff the centre is in the same social group with at least half of the tails. Thus, any star network is socially cohesive.

Proof. Take any group structure $W$ and suppose the centre $c$ is in a social group $S$ with $\ell$ tails. Then $p_{S}(c) = \ell/(\ell+1)$ and for any tail $u_i \in S$, $p_{S}(u_i) = 1/(\ell+1)$. All players not in $S$ has popularity 0.

Suppose $\ell \geq m/2$. Take any set $S' \neq S$ that contains $c$. If $|S'| \leq |S|$, then $p_{S'}(c) \leq p_{S}(c)$. If $|S'| > |S|$, then $p_{S'}(v) < p_{S}(v)$ for some tail $v$. In either case $S'$ does not block $W$. Hence $W$ is core stable. Suppose $\ell < m/2$. Then let $N'$ be the set of all tails not in $S$. Then $N \cup \{c\}$ blocks $W$. Thus $W$ is core stable iff $\ell \geq m/2$.

**Complete Bipartite Graph.** A complete bipartite graph $K_{n,n}$ consists of disjoint sets of nodes $V_1, V_2$ with $n$ nodes each and $E = \{uv \mid u \in V_1, v \in V_2\}$. Let $W$ be a group structure.

For every $S \subseteq W$, we use $\ell(S)$ and $r(S)$ to denote $|\{v \mid v \in S \cap V_1\}|$ and $|\{v \mid v \in S \cap V_2\}|$, respectively.

**Lemma 1.** $W$ is core stable only if $\forall S \in W$: $\ell(S) \geq r(S)$.

Proof. Suppose there is $S \subseteq W$ with $\ell(S) < r(S)$. Since $m \geq n$, there is $H \in W$ with $\ell(H) > r(H)$. Take any $u \in S \cap V_2$ and $v \in H \cap V_1$. Then we have $p_{S}(u) = \frac{\ell(S)}{\ell(S)+r(S)} < \frac{1}{2}$ and $p_{H}(v) = \frac{\ell(H)}{\ell(H)+r(H)} < \frac{1}{2}$. Hence, the set $\{u, v\}$ blocks $W$ as $p(u,v) = p_{\{u,v\}}(v) = 1/2$. □

We next characterize core stable group structure in $K_{n,n}$. In particular, perfect matchings, i.e., situations when every $v \in V_1$ is matched with a unique player in $V_2$, are core stable.

**Theorem 5.** A group structure $W$ of $K_{n,n}$ is core stable iff $\forall S \in W$: $\ell(S) = r(S)$.

Proof. By Lem. 1, if $W$ is core stable then $\forall S \in W$: $\ell(S) = r(S)$. Conversely, if $\forall S \in W$: $\ell(S) = r(S)$, then any $v$ has payoff $1/2$. Thus $W$ is core stable as every $H \subseteq V$ contains some player with payoff at most $1/2$. □

### 3 Structural Cohesion and Social Cohesion

**Definition 7** (White and Harary 2001). The structural cohesion $\kappa$ of a connected graph $G$ is the minimal number of nodes upon removal of which $G$ becomes disconnected.

As stated in (White and Harary 2001), a larger $\kappa$ implies that $G$ is more resilient to conflicts or the departure of group members, and is thus more cohesive. Moreover, Menger’s theorem states that $\kappa$ is the greatest lower bound on the number of paths between any pairs of nodes. Hence $\kappa$ is a reasonable measure of cohesion. We next link $\kappa$ with our notion of social cohesion. In (Granovetter 1973), a pair $uv \notin E$ is seen as a type of “structural hole” that forbids communication and is thus referred to as an *absent tie*. For each $S \subseteq V$ and $u \in S$ we define the following:

- $f_{\text{in}}(u, S) := \deg_{S}(u)$ and $f_{\text{out}}(u, S) := \{|v \notin S \mid uv \in E\}$ are the numbers of actual ties of $u$ within the group $S$ and outside $S$, resp.
- $s_{\text{in}}(u, S) := |S| - f_{\text{in}}(u, S)$ and $s_{\text{out}}(u, S) := \{|v \notin S \mid uv \notin E\}$ are the number of absent ties in $S$ (including $u$ itself) and outside $S$, resp.
Intuitively, if $S \subseteq V$ is a blocking set, each member $u$ tends to have many actual ties within $S$ and absent ties outside $S$, i.e., high $f_{in}(u, S)$ and $s_{out}(u, S)$, and $u$ tends to have few absent ties in $S$ and actual ties outside $S$, i.e., low $f_{out}(u, S)$ and $s_{in}(u, S)$. Thus, we define for all $S \subseteq V$, $u \in S$,

$$\gamma(u, S) := f_{in}(u, S)s_{out}(u, S) - f_{out}(u, S)s_{in}(u, S)$$

(1)

**Lemma 2.** For all $S \subseteq V$, $S$ blocks $W_G = \{V\}$ iff $\forall u \in S: \gamma(u, S) > 0$.

**Proof.** For each $u \in S$, $p_S(u) = \frac{f_{in}(u, S)}{f_{in}(u, S) + s_{in}(u, S)}$ and $p_V(u) = \frac{f_{in}(u, S) + f_{out}(u, S) + s_{in}(u, S)}{f_{in}(u, S) + f_{out}(u, S) + s_{in}(u, S) + s_{out}(u, S)}$.

The set $S$ blocks $W_G$ iff $\forall u \in S: p_S(u) > p_V(u)$, which can be shown to be equivalent to $f_{in}(u, S)s_{out}(u, S) > f_{out}(u, S)s_{in}(u, S)$ using the above equalities. $\square$

A network $G$ contains a minimal cut $A_0 \subseteq V$ of size $\kappa$, i.e., removing $A_0$ from $G$ decomposes the graph into $m$ distinct connected components $A_1, \ldots, A_m \subseteq V$ where $m \geq 2$. We further assume that $|A_1| \leq \cdots \leq |A_m|$ and $A_0$ is chosen in a way where $|A_1|$ is as small as possible. Let $\chi$ be the size $|A_1|$, and let $\mu$ be the largest possible length $m$ of the sequence of $A_i$’s, first look at the case where $\kappa = 1$.

**Lemma 3.** If $\kappa = 1$ and $G$ is socially cohesive, then $\chi < 2$.

**Proof.** Suppose $\kappa = 1$ and $\chi \geq 2$. Let $(A_1, \ldots, A_m)$ be an optimal cut sequence. Take $u \in A_1$. As $G$ contains a cut node, $f_{out}(u, A_1) \leq 1$ and $s_{out}(u, A_1) \geq |V| - \chi - 1$. Then $\gamma(u, A_1) \geq f_{in}(u, A_1) \cdot (|V| - \chi - 1) - s_{in}(u, A_1)$. Since $f_{in}(u, A_1) + s_{in}(u, A_1) = \chi$,

$$\begin{align*}
\gamma(u, A_1) &\geq f_{in}(u, A_1)(|V| - \chi - 1) - (\chi - f_{in}(u, A_1)) \\
&= f_{in}(u, A_1)(|V| - \chi - 1) - \chi.
\end{align*}$$

Since $|V| - \chi > \chi$, $\gamma(u, A_1) > 0$. Thus by Lem. 2, $A_1$ blocks the grand coalition $W_G$. $\square$

**Lemma 4.** Suppose $\mu > 2$. Then any network $G$ is socially cohesive only if $\chi < \frac{\kappa}{\mu - 2}$.

**Proof.** Suppose $\mu > 2$. Take an optimal cut sequence $(A_1, \ldots, A_m)$ and $u \in V_1$. Since $\deg(u) < \chi + \kappa$ and $|V| \geq \mu \chi + \kappa$, we have $p_V(u) < \frac{\chi + \kappa}{\mu \chi + \kappa}$. Suppose $\chi \geq \frac{\kappa}{\mu - 2}$. Then $\mu \chi - 2\chi > \kappa$. One can then derive that $p_V(u) < \frac{\chi + \kappa}{\mu \chi + \kappa} \leq \frac{2}{2}$. Thus any edge $\{u, v\}$ in $G$ forms a blocking set of the grand coalition formation $W_G$. $\square$

**Theorem 7.** The network $G$ belongs to NDU2 iff its nodes $V$ can be partitioned into two non-empty set $V_1$ and $V_2$, where $V_1 = \{u \mid vu \in E \text{ for all } v \neq u\}$.

**Proof.** Since $G$ has diameter 2, and is not diametrically uniform, there is a non-empty set $V_1$ of nodes with eccentricity 1, and the other nodes (call them $V_2$) have eccentricity 2. The sets $V_1, V_2$ satisfy the condition in the theorem. $\square$

Let $G$ be a graph in NDU2. We call $\{V_1, V_2\}$ as described in Thm. 7 the eccentricity partition of $G$. We first present some simple properties of NDU2.

**Lemma 5.** The network $G$ in NDU2 is socially cohesive iff no set $S \subseteq V_2$ blocks $W_G$.

**Proof.** One direction (left to right) is clear. Conversely, suppose the network is not socially cohesive. Let $S \subseteq V_2$ be a blocking set of the grand coalition formation, i.e., $\forall u \in S: p_S(u) > p_V(u)$. If $S \cap V_1 \neq \emptyset$, then $\forall u \in S \cap V_1: p_V(u) = \frac{|V_1| - 1}{|V|}$. However, $p_V(u) > \frac{|S| - 1}{|V|} \geq p_S(u)$ which contradicts that fact that $S$ is a blocking set.

By the lemma above, the structure of $G$ is crucial in determining social cohesion of $S$. For any $S \subseteq V_2$ and $u \in S$, we recall the notions $f_{in}(u, S)$, $f_{out}(u, S)$, $s_{in}(u, S)$, and $s_{out}(u, S)$ from Section 3, but re-interpret these values within the sub-network $G[V_2]$. Hence, we now set $f_{in}(u, S) = \{v \in V_2 \mid S \subseteq V \text{ where } uv \in E\}$, i.e., the number of ties that $u$ has within $V_2$ but not in $S$, the other variables remain as originally defined. Thus

$$|V_2| = f_{in}(u, S) + f_{out}(u, S) + s_{in}(u, S) + s_{out}(u, S) \tag{2}$$

We then define the value

$$\lambda(u, S) = \frac{f_{in}(u, S) \cdot s_{out}(u, S)}{s_{in}(u, S)} - f_{out}(u, S)$$

**Theorem 8.** A network $G$ in NDU2 is socially cohesive iff for all $S \subseteq V_2$ there exists $u \in S$ such that $|V_1| \geq \lambda(u, S)$.

**Proof.** By Lem. 5, we only need to examine subsets $S \subseteq V_2$. Every $u \in S$ has $p_S(u) = \frac{f_{in}(u, S)}{f_{in}(u, S) + s_{in}(u, S)}$ and

$$p_V(u) = \frac{f_{in}(u, S) + f_{out}(u, S) + |V_1|}{|V_2| + |V_1|}$$

Applying (2), $S$ blocks $W_G$ iff $\forall u \in S: p_S(u) > p_V(u)$, iff $\forall u \in S: |V_1| < \frac{f_{in}(u, S)s_{out}(u, S)}{s_{in}(u, S)} - f_{out}(u, S) = \lambda(u, S)$. $\square$
We now give a sufficient condition for social cohesion of an NDU₂ network. The size of a network is its number of nodes. A clique is a complete subgraph. The clique number of $G$, denoted by $\omega(G)$, is the size of the largest clique. Turán's theorem relates $\omega(G)$ with the number of edges in $G$:

**Theorem 9** (Turán 1941). For any $p \geq 2$, if a graph $G$ with $n$ nodes has more than \( \frac{p-2}{p-1} \) \( n^2 \) edges, then $\omega(G) \geq p$.

**Lemma 6.** For any social group $S \subseteq V$, there exists $u \in S$ with $\frac{f_{in}(u, S)}{s_{in}(u, S)} \geq \omega(G\setminus S) - 1$.

**Proof.** Let $k = \omega(G\setminus S)$ and suppose for all $u \in S, \frac{f_{in}(u, S)}{s_{in}(u, S)} > k - 1$. Since $|S| = f_{in}(u, S) + s_{in}(u, S)$,

\[
k - 1 < \frac{f_{in}(u, S)}{|S| - f_{in}(u, S)}\]

\[
f_{in}(u, S) > k|S| - k f_{in}(u, S) - |S| + f_{in}(u, S)\]

\[
f_{in}(u, S) > (k-1)|S|/k\]

Thus $E\setminus S$ contains $\frac{k-1}{2k} |S|^2$ edges. By Thm. 9, $G\setminus S$ contains a size-$(k+1)$ clique, contradicting $k$'s definition. \( \square \)

The set $V_1$ contains the most socially active members—who those interact with everyone else. Hence the larger $V_1$ gets, the more likely $G$ will be socially cohesive. There is a bound such that once $|V_1|$ exceeds it, the network $G$ is guaranteed to be socially cohesive.

**Lemma 7.** Suppose $c = \omega(G\setminus V_2)$ and $|V_2| > c(e^{-1})$. Then $G$ is socially cohesive whenever $|V_1| \geq (c-1)(|V_2| - c)$.

**Proof.** Suppose $|V_2| > c(e^{-1})$, $|V_1| \geq (c-1)(|V_2| - c)$. Take any $S \subseteq V_2$. If $S$ has a size-$c$ clique, by Lem. 6 there exists $u \in V_2$ with $\frac{f_{in}(u, S)}{s_{in}(u, S)} \leq c - 1$. Since $s_{out}(u, S) \leq |V_2| - c$, $|V_1| \geq (c-1)(|V_2| - c) \geq \frac{f_{in}(u, S)}{s_{in}(u, S)} s_{out}(u, S) > \lambda(u, S)$ and thus $G$ is socially cohesive by Thm. 8.

If $S$ contains no clique of size $c$, then $\omega(G\setminus S) \leq c - 1$. Let $k = \omega(G\setminus S)$. By Lem. 6, $\frac{f_{in}(u, S)}{s_{in}(u, S)} \leq k - 1 < c - 1$. Thus $c - \frac{|V_2|}{s_{in}(u, S)} \geq 1$. Since $s_{out}(u, S) = |V_2| - f_{in}(u, V_2) - s_{in}(u, S)$,

\[
\lambda(u, S) = \frac{f_{in}(u, S)}{s_{in}(u, S)} \geq \frac{|V_2| - f_{in}(u, V_2) - f_{in}(u, V_2)}{|V_2| - f_{in}(u, V_2)} = \frac{|V_2| - f_{in}(u, V_2)}{|V_2| - f_{in}(u, V_2)}\]

Furthermore, since $f_{in}(u, S) = |S| - s_{in}(u, S)$,

\[
\lambda(u, S) = \frac{|S| - |V_2|}{s_{in}(u, S)} - \frac{|S| f_{in}(u, V_2)}{s_{in}(u, S)} - |V_2|\]

Hence, $|V_1| - \lambda(u, S)$ is at least

\[
(c - 1)(|V_2| - c) - \left( \frac{|S| - |V_2|}{s_{in}(u, S)} - \frac{|S| f_{in}(u, V_2)}{s_{in}(u, S)} - |V_2| \right)\]

\[
= |V_2| \left( c - \frac{|S|}{s_{in}(u, S)} \right) - c(c-1) + \frac{|S| f_{in}(u, V_2)}{s_{in}(u, S)} \geq 0\]

The last step is by $c - \frac{|S|}{s_{in}(u, S)} \geq 1$ and $|V_2| > c(e^{-1})$. By Thm. 8, $G$ is socially cohesive. \( \square \)

**Theorem 10.** The problem COH is CoNP-complete. Furthermore CoNP-hardness holds already for the class NDU₂.

**Proof.** The complement of COH, \( \overline{\text{COH}} \), asks whether a set $S$ blocks the grand coalition \( \mathcal{W}_G \) of a given network $G$; this problem is clearly in NP and thus \( \overline{\text{COH}} \) is in CoNP. For hardness, we reduce MaxClique (asking whether a graph contains a clique of a given size $k$) to \( \overline{\text{COH}} \). CoNP-hardness of \( \overline{\text{COH}} \) then follows from the NP-hardness of MaxClique (Garey and Johnson 1979).

**Algorithm 1** Construction of $H$ given $G = (V, E)$ and $k > 2$

1. Set $d := k \cdot \max\{\deg(u) \mid u \in V\}$
2. Create $G'$ by adding $k(k-1)+d$ isolated nodes (and no new edges) to $G$
3. Let $V_2$ be the set of nodes in $G'$ (which contains all nodes in $V$ and new nodes)
4. Create a complete graph with $(k-1)(|V_2| - k) - d$ nodes; Let $V_1$ be the set of these nodes
5. Create edges \{$(u, v) \mid u \in V_1, v \in V_2$\} to connect $V_1, V_2$.

The resulting graph is $H$.

To this end, we construct, for a given $G = (V, E)$ and $k > 2$, a graph $H \in \text{NDU}_2$ as in Alg. 1. Our goal is to show that $H$ is not socially cohesive if $G$ contains a clique of size $k$. It is clear that $H$ is a NDU₂ network with eccentricity partition $\{V_1, V_2\}$. Let $c = \omega(G)$. Suppose $c < k$. By definition of $V_1$ and $V_2$, we have $|V_1| - (c-1)(|V_2| - c) = |V_2|(|k-1| - c) - (c-1)(|V_2| - c)$. Since $k \geq c$ and $|V_2| > k(k-1)+d$, $|V_1| > (c-1)(|V_2| - c)$. By Thm. 7, $H$ is cohesive.

Conversely, suppose $\omega(G) \geq k$. Let $C \subseteq V_2$ be a clique of size $k$. Take $u \in C$. Since $f_{in}(u, C) = k - 1, s_{out}(u, C) = |V_2| - k - f_{out}(u, C)$ and $s_{in}(u, C) = 1$, $\lambda(u, C) = (k-1)(|V_2| - k - f_{out}(u, C)) - f_{out}(u, C) = (k-1)(|V_2| - k) = k \cdot f_{out}(u, C)$.

Hence, $\lambda(u, C) - |V_1| = d - k f_{out}(u, C)$. Since $d \geq k \cdot \deg(u)$, $\lambda(u, C) - |V_1| > 0$. By Lem. 8, $G$ is not socially cohesive. Therefore, $G$ contains a clique of size $k$ iff $H$ is not socially cohesive and the reduction is complete. \( \square \)

### 5 Efficient Heuristics

We propose two heuristics that construct group structures of a given network where players enjoy high popularity. These heuristics (partially) solve COH despite COH’s inherent complexity: Each heuristic builds a group structure $W$ and checks if any set $S \subseteq \mathcal{W}$ blocks $\mathcal{W}_G$. If $G$ is socially cohesive, then no such $S$ will be found; On the other hand, if a blocking set $S$ is found, $G$ is surely not socially cohesive.

**Heuristic 1: Louvain’s method (LM)** We observe that blocking sets of $\mathcal{W}_G$ are usually tightly connected within, but are sparsely connected with nodes outside. This property corresponds to the well-studied notion of **communities** (Fortunato 2010). Therefore, the first heuristic uses a well-known community detection algorithm, Louvain’s method (Blondel et al. 2008), to compute a group structure in $G$.

**Heuristic 2: Average payoff (AP)** The second heuristic aims to optimize the average payoffs of members of a coalition. Socially cohesive networks usually have small diameters ($\leq 2$). Thus we consider neighborhood $N(v) := \{v\} \cup$
We further evaluate the heuristics on real-world networks. AP sampled cases when ZA, LM nodes get a higher payoff in the coalitions identified by that, compared to the payoffs in the grand coalition, more ble. 1. Expectedly, none of these networks are socially co- the largest components in each network; see details in Ta- works (Leskovec et al. 2009), and three physics collaboration net- beaches. The box-and-whisker diagrams in Fig. 3 show the distribution of payoffs of players in the grand coalition as well as in the coalitions output by each heuristic (outliers omitted). In all cases, the heuristics improve players’ payoffs considerably compared to the grand coalition, while AP in particular achieves higher payoffs. Furthermore, Fig. 4 shows that all nodes get higher payoffs through LM. In summary, both of the heuristics are useful in computing coalitions; while LM may benefit a larger portion of players, AP tends to obtain higher payoffs.

6 Conclusion and future work

We aim to investigate natural game-theoretical and computational questions as future works: Does a core stable group structure exists for every network? What about other stability concepts? What would be strategies of players to improve popularity? The proposed games are instances of a more general framework for network-based cooperative games, where payoffs of players are given by various centrality indices. It is interesting to extend the work by considering other centralities, and different forms of social networks (e.g. directed, weighted, signed). Furthermore, one could also explore the evolution of social groups in a dynamic setting.
References
Blondel, V. D.; Guillaume, J. L.; Lambiotte, R.; and Lefebvre, E. 2008. Fast unfolding of communities in large networks. *J. Stat. Mech.-Theory E.* 10.

Brudner, L., and White, D. 1997. Class, property, and structural endogamy: Visualizing networked histories. *Theor. Soc.* 26:161–208.

Carron, A. V., and Brawley, L. R. 2000. Cohesion conceptual and measurement issues. *Small Gr. Res.* 31(1).

Cartwright, D., and Zander, A. 1953. *Group Dynamics: Research and Theory.* Evanston, IL: Row, Peterson.

Chen, W.; Liu, Z.; Sun, X.; and Wang, Y. 2011. Community detection in social networks through community detection games. In *Proceedings of International Joint Conference on Artificial Intelligence (IJCAI’11)*, volume 22, No. 3.

Cillessen, A., and Mayeux, L. 2004. Sociometric status and peer group behavior: Previous findings and current directions. In Kupersmidt, J., and Dodge, K., eds., *Children’s peer relations: From development to intervention.* Decade of behavior. American Psychological Association. 3–20.

Conti, G.; Galeotti, A.; Mueller, G.; and Pudney, S. 2013. Popularity. *J. Hum. Resour.* 48(4).

Festinger, L. 1950. The spatial ecology of group formation. (L. festinger, s. schachter, & k. back, eds.). *Social Pressure in Information Groups* 33–59.

Fortunato, S. 2010. Community detection in graphs. *Physics reports* 486(3):75–174.

Garey, M. R., and Johnson, D. S. 1979. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W.H.Freeman.

Girvan, M., and Newman, M. E. 2002. Community structure in social and biological networks. In *P. Natl. Acad. Sci. USA*, volume 99(12).

Granovetter, M. S. 1973. The strength of weak ties. *Am J. Sociol.* 78(2):1360–1380.

Gross, N., and Martin, W. E. 1952. On group cohesiveness. *Am. J. Sociol.* 57:546–564.

Hogg, M. A. 1992. *The social psychology of group cohesiveness: From attraction to social identity.* Harvester Wheatsheaf.

Huisman, C., and Bruggeman, J. 2012. The social network, socioeconomic background, and school type of adolescent smokers. *Int. J. Behav. Dev.* 36(5).

Lansu, T. A., and Cillessen, A. H. 2011. Peer status in emerging adulthood: Associations of popularity and preference with social roles and behavior. *J. Adolescent. Res.* 0743558411402341.

Leskovec, J.; Kleinberg, J.; ; and Faloutsos, C. 2007. Graph evolution: Densification and shrinking diameters. *ACM Transactions on Knowledge Discovery from Data (TKDD)* 1(1).

Leskovec, J.; Lang, K. J.; Dasgupta, A.; and Mahoney, M. W. 2009. Community structure in large networks: Natural cluster sizes and the absence of large well-defined clusters. *Internet Mathematics* 6(1).

Lewin, K. 1943. Psychology and the process of group living. *J. Soc. Psychol.* 17:113–131.

Lusseau, D.; Schneider, K.; Boisseau, O. J.; Haase, P.; Slooten, E.; and Dawson, S. M. 2003. The bottlenose dolphin community of doubtful sound features a large proportion of long-lasting associations. *Behav Ecol Sociobiol* 54(4).

McSweeney, P. J.; Mehrrota, K.; and Oh, J. C. 2014. Game-theoretic framework for community detection. In *Encyclopedia of Social Network Analysis and Mining.* Springer New York.

Michalak, T. P.; Aadithya, K. V.; Szczepanski, P. L.; Ravindran, B.; and Jennings, N. R. 2013. Efficient computation of the shapley value for game-theoretic network centrality. *Journal of Artificial Intelligence Research.*

Narayanan, R.; Skibski, O.; Lamba, H.; and Michalak, T. 2014. A shapley value-based approach to determine gatekeepers in social networks with applications. In *ECAI’14*, 651–656. IOS Press.

Nee, V., and Sanders, J. 2001. Trust in ethnic ties: Social capital and immigrants. *Trust in Society.* 2.

Peleg, B., and Sudhölter, P. 2010. *Introduction to the Theory of Cooperative Games (2nd Ed.).* Springer.

Szczepański, P. L.; Barcz, A.; Michalak, T. P.; and Rahwan, T. 2015a. The game-theoretic interaction index on social networks with applications to link prediction and community detection. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI 2015)*, volume 22, No. 3. AAAI Press.

Szczepański, P.; Tarkowski, M.; Michalak, T.; Harrenstein, P.; and Wooldridge, M. 2015b. Efficient computation of semivalues for game-theoretic network centrality. In *Proceedings of the 29th AAAI Conference on AI (AAAI’15).* AAAI Press.

Szczepański, P.; Michalak, T.; and Wooldridge, M. 2014. A centrality measure for networks with community structure based on a generalization of the Owen value. In *Proceedings of the 21th European Conference on Artificial Intelligence (ECAI’14)*, 867–872.

Turán, P. 1941. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok* (in Hungarian) 48:436–452.

White, D. R., and Harary, F. 2001. The cohesiveness of blocks in social networks: Node connectivity and conditional density. *Social. Methodol.*

Zachary, W. 1977. An information flow model for conflict and fission in small groups. *J. Anthropol. Res.* 48:75–174.

Zhang, M. 2010. Social network analysis: History, concepts and research. *Handbook of Social Network Technologies and Applications.*