A weak form of beyond endoscopic decomposition for the stable trace formula of odd orthogonal groups

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Received February 6, 2017; accepted April 20, 2017; published online December 29, 2017

Abstract We show that the cuspidal component of the stable trace formula of a split special odd orthogonal group over a number field, satisfies a weak form of beyond endoscopic decomposition. We also study the r-stable trace formula, when r is the standard or the second fundamental representation of the dual group, and show that they satisfy a similar kind of beyond endoscopic decomposition. The results are consequences of Arthur’s works (2013) on endoscopic classification of automorphic representations, together with known results concerning a class of Langlands L-functions for special odd orthogonal groups.

Keywords cuspidal component, stable trace formula, beyond endoscopic decomposition, r-stable trace formula

MSC(2010) 11F66, 11F70

Citation: Mok C P. A weak form of beyond endoscopic decomposition for the stable trace formula of odd orthogonal groups. Sci China Math, 2018, 61: 993–1012, https://doi.org/10.1007/s11425-017-9089-8

1 Introduction

In the paper [22] (and later expanded in [13]), Langlands suggested an approach to establish the Principle of Functoriality for automorphic representations of reductive groups over number fields, an approach that he termed “beyond endoscopy”. In this approach, the stable trace formula plays a crucial role; more precisely, one of the main innovation is the insertion of Langlands L-functions, or their logarithmic derivatives, into the spectral side of the stable trace formula, via the use of modified form of test functions. An important step in the beyond endoscopy approach is to establish a certain decomposition for the cuspidal component of the stable trace formula, known as beyond endoscopic decomposition.

In this paper, we study the case of a split special odd orthogonal group $G = SO(2N + 1)$ over a number field $F$, and establish a weak form of the beyond endoscopic decomposition for the cuspidal component of the stable trace formula, by using Arthur’s work [8] on endoscopic classification of automorphic representations for these groups. Equivalently, we would like to put the endoscopic classification framework in [8] in the setting of the beyond endoscopy framework as in [9]. In addition, by using results on Langlands L-functions $L(s, \pi, r)$, attached to automorphic representation $\pi$ of $G$, with $r$ being either the standard or the second fundamental representation of the dual group $\hat{G} = Sp(2N, \mathbb{C})$, we show that the r-stable trace formula for $G$ (in the sense of [9]) exists, and that it also satisfies a similar weak form of beyond endoscopic decomposition; in the case when $r$ is the standard representation, the r-stable trace formula in fact vanishes identically.
To state these results, we must recall some formulation of stable trace formula of Arthur [3–5] and his work [8] on endoscopic classification, which we will turn to in the next section. The weak beyond endoscopic decomposition of the cuspidal component of the stable trace formula will be established in Section 3. In Section 4, we study the $r$-stable trace formula for $r$ being the standard of the second fundamental representations of $G$. The main results are established as Theorems 3.6, 4.1 and 4.4. In Section 5, we will pose some closely related questions. We present another construction of the stable transfer in Appendix A.

2 Résumé on stable trace formula and endoscopic classification

In the general context of the Arthur trace formula, one has a connected reductive group $G$ over a number field $F$. In the present paper, we will need the case where $G$ is the split form of the special odd orthogonal group $SO(2N + 1)$ over $F$ (with $N \geq 1$), or finite products of such groups. The Hecke space of test functions on the adelic group $G(A_F)$ will be denoted as

$$\mathcal{H}(G(A_F)) = \bigotimes_v \mathcal{H}(G(F_v)),$$

which is a restricted direct product of the local Hecke space $\mathcal{H}(G(F_v))$. Here, the restricted direct product is taken with respect to the unit element of the spherical Hecke algebra $\mathcal{H}^{\text{sp}}(G(F_v))$ at places $v$ of $F$, with respect to the standard maximal compact subgroup $G(O_{F_v})$ of $G(F_v)$. Then for each non-negative real number $\tau$, one has the linear form $I^G_{\text{disc}, \tau}$ on $\mathcal{H}(G(A_F))$, known as the discrete component of the invariant trace formula for $G$; here and thereafter, the parameter $\tau$ controls the norm of the imaginary part of the archimedean infinitesimal characters of representations. We refer the reader [8, Subsection 3.1] for more detailed discussions. The value of the linear form $I^G_{\text{disc}, \tau}$ on a test function $f \in \mathcal{H}(G(A_F))$ depends only on the invariant orbital integral $f_G$, with $f_G$ being regarded as a function on the set of regular semi-simple conjugacy classes of $G(A_F)$.

For questions related to the Principle of Functoriality, it is the stable trace formula that plays the crucial role (see [3–5]). More precisely, we have the linear form $S^G_{\text{disc}, \tau}$ on $\mathcal{H}(G(A_F))$, known as the discrete component of the stable trace formula for $G$. The linear form satisfies the important condition of being stable, namely that its value on a test function $f \in \mathcal{H}(G(A_F))$ depends only on the stable orbital integral $f^G$, with $f^G$ being regarded as a function on the set of regular semi-simple stable conjugacy classes of $G(A_F)$; for these notions we refer for example to [8, Subsections 2.1 and 3.2]. We denote by $S(G(A_F))$ the space spanned by $f^G$ for $f \in \mathcal{H}(G(A_F))$. One has a corresponding local definition, and we have

$$S(G(A_F)) = \bigotimes_v S(G(F_v)),$$

where the restricted tensor products are taken with respect to the stable orbital integral of the unit element of the spherical Hecke algebra at places $v$ of $F$.

Following [8], given a linear form $S$ on $\mathcal{H}(G(A_F))$ that is stable, we will denote by $\tilde{S}$ the corresponding linear form on the space of stable orbital integrals $S(G(A_F))$; in other words, for $f \in \mathcal{H}(G(A_F))$,

$$S(f) = \tilde{S}(f^G).$$

The stable linear form $S^G_{\text{disc}, \tau}$ is constructed from $I^G_{\text{disc}, \tau}$ via the inductive procedure as described for example in [8, Subsection 3.2]. This inductive procedure critically relies on the existence of the Langlands-Shelstad transfer and the fundamental lemma. As is well-known, these are now all established theorems; for the precise reference we refer to [8, Subsection 2.1]. Finally, the linear forms $I^G_{\text{disc}, \tau}$ and $S^G_{\text{disc}, \tau}$ satisfy an admissible condition (see [8, Subsection 3.1]).

We now specialize to the case where $G$ is the split $SO(2N + 1)$. In the work [8], Arthur gave an explicit formula for $S^G_{\text{disc}, \tau}$ in terms of the set of (formal) Arthur parameters $\Psi(G)$ for the group $G$. More precisely, since we have fixed the parameter $\tau$, it suffices to work with the subset

$$\Psi^\tau(G) \subset \Psi(G)$$