CLASSIFYING PERMUTATIONS UNDER CONTEXT-DIRECTED SWAPS AND THE CDS GAME

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Abstract. A special sorting operation called Context Directed Swap, and denoted cds, performs certain types of block interchanges on permutations. When a permutation is sortable by cds, then cds sorts it using the fewest possible block interchanges of any kind. This work introduces a classification of permutations based on their number of cds-eligible contexts. In prior work an object called the strategic pile of a permutation was discovered and shown to provide an efficient measure of the non-cds-sortability of a permutation. Focusing on the classification of permutations with maximal strategic pile, a complete characterization is given when the number of cds-eligible contexts is close to maximal as well as when the number of eligible contexts is minimal. A group action that preserves the number of cds-eligible contexts of a permutation provides, via the orbit-stabilizer theorem, enumerative results regarding the number of permutations with maximal strategic pile and a given number of cds-eligible contexts. Prior work introduced a natural two-person game on permutations that are not cds-sortable. The decision problem of which player has a winning strategy in a particular instance of the game appears to be of high computational complexity. Extending prior results, this work presents new conditions for player ONE to have a winning strategy in this combinatorial game.

1. Introduction

The sortability of permutations, non-repetitive arrays of integers, is of interest to a variety of fields including scientific computing and genetics. We study a a particular block-interchange sorting operation postulated to occur in the genomic sorting of single-celled organisms called ciliates [10]. This operation participates in decrypting the ciliate’s micronuclear genome to construct a new macronucleus. We refer to this block-interchange operation as cds, abbreviating “context directed swap”.

A permutation \( \pi \) is cds-sortable if successive applications of cds on \( \pi \) results in the identity permutation. Not all permutations are cds-sortable. The criteria for sortability were studied previously in [1] [8] [9] and others. In [3] Christie discovered that cds is a minimal block-interchange, sorting a cds-sortable permutation using the fewest possible block interchanges. If a permutation is not cds-sortable, successive applications of cds will result in a permutation where cds no longer applies. Define a permutation where cds does not apply as a cds fixed point. The set of cds fixed points reachable from a permutation \( \pi \), excluding the identity, is called the strategic pile of the permutation, and is denoted \( SP(\pi) \). An interesting phenomenon arises when a permutation is not cds-sortable; the fixed point of a permutation reached after applications of cds may depend on the order in which the cds operations were applied.

This phenomenon on non cds-sortable permutations gives rise to a combinatorial game previously investigated in [11] and [7]. In this game player ONE is assigned a subset of the strategic pile. The symbol \( CDS(\pi, A) \) denotes this game on permutation \( \pi \) where \( A \) is the subset of the strategic pile assigned to player ONE. Beginning with ONE, players ONE and TWO take turns successively applying cds to \( \pi \) until a fixed point is reached. If the fixed point is in \( A \), ONE wins; otherwise, TWO wins. Note that the number of moves in the game \( CDS(\pi, A) \) is bounded by the length of \( \pi \).
and thus the game is finite. By Zermelo’s Theorem \[13\], there exists a winning strategy for some player. Criteria for determining beforehand which of player ONE or player TWO has a winning strategy in a given instance of the game has yet to be discovered. For \textit{cds} non-sortable permutations at large \[1\] discovered a lowerbound on the size of \(A\) relative to the size of the strategic pile that ensures that ONE has a winning strategy in the game \(\text{CDS}(\pi, A)\). In \[7\] it was discovered for some class of permutations this bound is optimal. We broaden the exploration of tight bounds beyond the case explored in \[7\].

The basic definitions and notations as well as formal definitions for \textit{cds}, \textit{cds}-sortability, and strategic pile are introduced in section 2. Sections 3 through 5 will present our results. In section 3, we define a group action on permutations whose strategic pile is said to be \textit{maximal}. In section 4, we classify permutations with maximal strategic pile based on the number of possible applications of \textit{cds}. In section 5, we observe a tighter bound for \(A\) in the game \(\text{CDS}(\pi, A)\) for permutations \(\pi\) with maximal strategic pile and a certain number of valid pointer contexts. Finally, in section 6 we discuss directions for future work.

2. Preliminaries

In this paper the \textit{symmetric group} \(S_n\) is the group of bijections \(\pi: \mathbb{Z}_n \rightarrow \mathbb{Z}_n\), under function composition. An element of \(S_n\) is said to be a \textit{permutation} of length \(n\). All numbers appearing in permutations are always assumed to be coset representatives modulo \(n\). In general we choose the smallest \textit{strictly positive} member of the coset as representative, e.g. \(n\) is always the representative of the identity element. A permutation \(\pi \in S_n\) will be denoted

\[
[\pi(1) \pi(2) \pi(3) \ldots \pi(n)]
\]

For convenience the symbol \([k]\) denotes the set \{1, ..., \(k\)\}. A permutation \(\pi\) is \textit{sorted} if the entries appear in increasing order, or more precisely, if \(\pi\) is the identity function.

\textbf{Definition 2.1.} Fix a positive integer \(n\).

1. The set \(P_n = \{(1, 2), (2, 3), \ldots (n-1, n)\}\) is said to be the set of \(S_n\) \textit{pointers}.
2. For a positive integer \(p\), the symbol \(p^*\) denotes the pointer \((p, p+1)\).
3. For each positive integer \(k\), define \(L(k) = (k-1, k)\) and \(R(k) = (k, k+1)\). \(L(k)\) is said to be the \textit{left pointer} of \(k\), and \(R(k)\) is said to be the \textit{right pointer} of \(k\).
4. To each permutation \(\pi = [\pi(1) \pi(2) \cdots \pi(n-1) \pi(n)] \in S_n\) associate its \textit{pointer word} \(W(\pi) = [L(\pi(1)) R(\pi(1)) \cdots L(\pi(i)) R(\pi(i)) \cdots L(\pi(n)) R(\pi(n))]\) with entries \((0, 1)\) and \((n, n+1)\) removed.
5. For a pointer \(p = (j-1, j)\) of permutation \(\pi\), the \textit{pointer word, ignoring} \(p\) is denoted \(W(\pi) \setminus p\), and is the result of removing the segment \(L(\pi(j)) R(\pi(j))\) from \(W(\pi)\).

Note that the function

\[ W: S_n \rightarrow P_n^{2n-2} \]

is a one-to-one function. From an element \(\sigma\) in the range of \(W\) one can uniquely recover the permutation \(\pi \in S_n\) for which \(W(\pi) = \sigma\). Note that in \(W(\pi)\) every pointer appears exactly twice; the pointer \((k, k+1)\) is the right pointer of the element \(k\) and the left pointer of the element \(k+1\). Thus \(W(\pi)\) is an example of a \textit{double occurrence word} \(\text{II}\) over the alphabet \(P_n\).

\textbf{Example 2.2.} Let \(\pi\) be the permutation \([6 3 5 1 2 4]\). Then the pointer word of \(\pi\) is

\[ W(\pi) = [(5, 6) (2, 3) (3, 4) (4, 5) (5, 6) (1, 2) (1, 2) (2, 3) (3, 4) (4, 5)] \]

When convenient we write

\[
[ (5, 6)^6 (2, 3)^3 (3, 4) (4, 5)^5 (5, 6) (1, 2)^2 (1, 2)^2 (2, 3) (3, 4)^4 (4, 5) ]
\]
to display both \(\pi\) and its pointer word, or

\[
[ (5, 6)^6 (6, 7) (2, 3)^3 (3, 4) (4, 5)^5 (5, 6) (0, 1)^1 (1, 2)^1 (1, 2)^2 (2, 3) (3, 4)^4 (4, 5) ]
\]
to display both $\pi$ and its extended pointer word.

For the pointer $2^* = (2, 3)$, the pointer word of $\pi$, ignoring $2^*$, is

$$W(\pi) \setminus 2^* = [(5, 6) (4, 5) (5, 6) (1, 2) (1, 2) (2, 3) (3, 4) (4, 5)].$$

The notion of an adjacency in a permutation plays a fundamental role in the context directed swap operation $\text{cds}$ to be introduced shortly. In preparation for $\text{cds}$, we now define notions to facilitate the exposition.

**Definition 2.3.** Let $\pi \in S_n$ and $k \in \mathbb{Z}_n$ be such that $\pi(k) \neq n$. We say that $\pi(k)$ and $\pi(k + 1)$ form an adjacency if

$$\pi(k + 1) = \pi(k) + 1$$

When there is an adjacency between $\pi(k)$ and $\pi(k + 1)$, then in the pointer word of $\pi$ the right pointer of $\pi(k)$, i.e. $R(\pi(k)) = (\pi(k), \pi(k + 1)) = \pi(k)^*$, appears directly adjacent the left pointer of $\pi(k + 1)$, $L(\pi(k + 1))$, and $R(\pi(k)) = L(\pi(k + 1))$. Thus, when in $\pi$ there is an adjacency between $\pi(k)$ and $\pi(k + 1)$, there is a duplication in $W(\pi)$. We also say that there is an adjacency about the pointer $\pi(k)^* = (\pi(k), \pi(k) + 1)$ of $\pi$.

This leads us to the following definition:

**Definition 2.4.** Suppose $\pi \in S_n$ has an adjacency about a pointer $r^* = (r, r + 1)$.

1. The adjacency reduction of $\pi$ at $r^*$, denoted $R_{r^*}(\pi)$, is the following element of $S_{n-1}$: Fix $K$ for which $\pi(K) = r + 1$. Then define

$$\begin{cases} R_{r^*}(\pi)(k) = \pi(k) & \text{if } \pi(k) \leq r \text{ and } k < K \\ R_{r^*}(\pi)(k) = \pi(k) - 1 & \text{if } \pi(k) > r \text{ and } k < K \\ R_{r^*}(\pi)(k - 1) = \pi(k) & \text{if } \pi(k) \leq r \text{ and } k > K \\ R_{r^*}(\pi)(k - 1) = \pi(k) - 1 & \text{if } \pi(k) > r \text{ and } k > K \end{cases}$$

2. The reduced pointer word of $\pi$ at $r^*$, denoted $RW_{r^*}(\pi)$, is defined as follows: Fix $K$ for which $\pi(K) = r + 1$. Then for $0 \leq j \leq n - 2$ $RW_{r^*}(\pi)(2j + 1) = L(x_j)$ and $RW_{r^*}(\pi)(2j + 2) = R(x_j)$ where

$$x_j = \begin{cases} \pi(j + 1) & \text{if } j \neq K - 1 \text{ and } \pi(j + 1) < \pi(K) \\ \pi(j + 1) - 1 & \text{if } j \neq K - 1 \text{ and } \pi(j + 1) > \pi(K) \\ \pi(j + 1) & \text{otherwise.} \end{cases}$$

**Example 2.5.** For the permutation $\pi = [8 \ 5 \ 2 \ 4 \ 6 \ 7 \ 3 \ 1]$ consider the pointer $6^* = (6, 7)$. In the notation of Definition 2.4, $K = 6$. Now

$$W(\pi) = [(7, 8) (4, 5) (5, 6) (1, 2) (2, 3) (3, 4) (4, 5) (5, 6) (6, 7) (6, 7) (7, 8) (2, 3) (3, 4) (1, 2)]$$

Then we have

$$RW_{6^*}(\pi) = [(6, 7) (4, 5) (5, 6) (1, 2) (2, 3) (3, 4) (4, 5) (5, 6) (6, 7) (2, 3) (3, 4) (1, 2)]$$

The adjacency reduction operation induces a natural bijective mapping from the pointers of $\pi$, excluding pointer $p$, to the pointers of $R_p(\pi)$. This correspondence takes the left and right pointers of $\pi(k)$, $k < K - 1$ to the left and right pointers of $R_p(\pi)(k)$, and takes the left and right pointers of $\pi(k)$, $k > K$, to the left and right pointers of $R_p(\pi)(k - 1)$. The left pointer of $\pi(K - 1)$ is taken to the left pointer of $R_p(\pi)(K - 1)$, and the right pointer of $\pi(K)$ is taken to the right pointer of $R_p(\pi)(K - 1)$.

**Example 2.2 continued.** Taking $\pi'$ as in example 2.2 we have adjacencies around the pointers $5^* = (5, 6)$ and $3^* = (3, 4)$. If we reduce the permutation $\pi'$ at $3^* = (3, 4)$, the resulting permutation, $R_{(3,4)}(\pi')$, is:

$$[1_{(1,2)} (1,2) 2_{(2,3)} (3,4) 4_{(4,5)} (4,5) 5_{(2,3)} 3_{(3,4)}]$$
Of the pointers that are not preserved under the induced correspondence, pointer $4^* = (4, 5)$ of $\pi'$ is sent to pointer $3^* = (3, 4)$ of $R_{(3,4)}(\pi')$, and pointer $5^* = (5, 6)$ of $\pi'$ is sent to pointer $4^* = (4, 5)$ of $R_{(3,4)}(\pi')$.

**Definition 2.6.** Let $\pi \in S_n$ be given. Let $p$ and $q$ be distinct pointers in $\pi$. We say that $(p, q)$ is a valid pointer context if the pointers $p$ and $q$ appear in the order $p \ldots q \ldots p \ldots q$, or in the order $q \ldots p \ldots q \ldots p$, in the pointer word $W(\pi)$ of $\pi$.

**Example 2.2 continued.** Taking $\pi$ as in Example 2.2, the pointers $5^* = (5, 6)$ and $3^* = (3, 4)$ appear in the desired order, so $(5^* = (5, 6), 3^* = (3, 4))$ is a valid pointer context.

**Definition 2.7.** Let $\pi \in S_n$ be given. Let $p$ and $q$ be distinct pointers in $\pi$ that form a valid pointer context $(p, q)$. Define the context-directed swap (or cds) on the permutation $\pi$ at pointers $p, q$ to be the permutation resulting from swapping the two individual blocks of entries of $\pi$ appearing between the pointers $p$ and $q$. Let $\text{cds}_{p,q}(\pi) \in S_n$ be the permutation resulting from applying $\text{cds}$ for valid pointer context $(p, q)$ to $\pi$.

**Example 2.2 continued.** Take permutation $\pi$ again as in Example 2.2 and take the pointers $5^* = (5, 6)$ and $3^* = (3, 4)$ that form a valid pointer context. The block “6 3” appears between the first $(5, 6) (3, 4)$ pointer pair and the block “1 2” appears between the second occurrence. Applying $\text{cds}$ for the valid pointer context $(5^*, 3^*)$ swaps the positions of these two blocks, yielding:

$$\pi' = \left[ 1_{(1,2)} \ (1,2)^2_{(2,3)} \ (4,5)^5_{(5,6)} \ (5,6)^6 \ (2,3)^3_{(3,4)} \ (3,4)^4_{(4,5)} \right]$$

Observe that $\text{cds}$ always introduces adjacencies about the pointers $p$ and $q$; in Example 2.2, $\pi'(3) = 5$, $\pi'(3+1) = 6$, so 5 and 6 form an adjacency, and similarly, $\pi'(5) = 3$, $\pi'(5+1) = 4$, so 3 and 4 form an adjacency.

Since an adjacency about a pointer $p$ guarantees that $p$ will not subsequently appear in a valid pointer context, the adjacency will persist regardless of any subsequent $\text{cds}$ operation performed on the permutation.

A $\text{cds}$ fixed point is a permutation with no valid pointer contexts. Besides the identity permutation $[1 \ 2 \ \cdots \ n]$, all other $\text{cds}$ fixed points in $S_n$ are of the form

$$[ \ (k + 1) \ (k + 2) \ (k + 3) \ \ldots \ n \ 1 \ 2 \ 3 \ \ldots \ k ]$$

for some $k \in \mathbb{Z}_n$ [1].

**Strategic Pile.** We now introduce a set associated to each permutation that characterizes its sortability and fixed points under $\text{cds}$.

Fix arbitrary $\pi \in S_n$, say $\pi = [ \ a_1 \ a_2 \ a_3 \ \ldots \ a_n \ ]$. For the purposes of the upcoming definition of the strategic pile, view all numbers as elements of $\mathbb{Z}$, and let parentheses denote the usual cycle notation for permutations. Define

$$X_n := (0 \ 1 \ 2 \ \ldots \ n)$$
$$Y_\pi := (a_n \ a_{n-1} \ a_{n-2} \ \ldots \ a_1 \ 0)$$
$$C_\pi := Y_\pi \circ X_n,$$

the function composition of permutations $Y_\pi$ and $X_n$.

**Example 2.8.** For $\pi = [8 \ 1 \ 5 \ 2 \ 4 \ 3 \ 7 \ 6]$ we find:

$$X_8 := (0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8)$$
$$Y_\pi := (6 \ 7 \ 3 \ 4 \ 2 \ 5 \ 1 \ 8 \ 0)$$
$$C_\pi := Y_\pi \circ X_8$$
$$= (0 \ 8 \ 6 \ 3 \ 2 \ 4 \ 1 \ 5 \ 7)$$
Definition 2.9. Consider a permutation $\pi \in S_n$. The strategic pile of $\pi$, denoted $SP(\pi)$, is the set of numbers appearing after $n$ and before 0 in the cycle of $C_\pi$ containing 0 and $n$. If 0 and $n$ do not appear in the same cycle, then $SP(\pi) = \emptyset$.

Example 2.8 continued. For $\pi = [8 \ 1 \ 5 \ 2 \ 4 \ 3 \ 7 \ 6]$ the strategic pile is $SP(\pi) = \{6, \ 3, \ 2, \ 4, \ 1, \ 5, \ 7\}$.

The strategic pile completely characterizes when a permutation is sortable under cds.

Theorem 2.10 ([1], Theorem 2.18). A permutation $\pi \in S_n$ is cds sortable if and only if $SP(\pi) = \emptyset$.

In the case that a permutation is not sortable, the strategic pile computes precisely which fixed points are reachable under cds.

Theorem 2.11 ([4] Lemma 3.1, Corollary 3.2). Let $\pi \in S_n$ be given. Then

$$[k \ k+1 \ ... \ n \ 1 \ 2 \ ... \ k-1]$$

is a fixed point of $\pi$ if and only if $k-1 \in SP(\pi)$.

We now have a way to make precise what we mean by a permutation having the largest possible strategic pile.

Definition 2.13. Let $\pi \in S_n$. We say that $\pi$ has a maximal strategic pile if $n$ is even and $|SP(\pi)| = n-1$, or $n$ is odd and $|SP(\pi)| = n-2$.

Example 2.8 continued. $\pi = [8 \ 1 \ 5 \ 2 \ 4 \ 3 \ 7 \ 6]$ has a maximal strategic pile. The fixed points reachable by repeated applications of cds are $[8 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7], [7 \ 8 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6], [6 \ 7 \ 8 \ 1 \ 2 \ 3 \ 4], [5 \ 6 \ 7 \ 8 \ 1 \ 2], [4 \ 5 \ 6 \ 7 \ 8 \ 1], [3 \ 4 \ 5 \ 6 \ 7 \ 8], \text{ and } [2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8]$.

The following proposition collects several properties of permutations with maximal strategic pile.

Proposition 2.14. If $\pi \in S_n$ have maximal strategic pile, then the following are true.

1. Let $n$ be even. If $\pi(k) = n$, then $\pi(k+1) = 1$.
2. If $n$ is even, then $\pi$ has no adjacencies.
3. If $n$ is odd, $\pi$ has precisely one adjacency.
4. Let $n$ be even. An application of cds to $\pi$ introduces exactly two adjacencies, and removes exactly two elements from the strategic pile.
5. Let $n$ be even. If one performs cds on $\pi$, after reducing the resulting adjacencies we have a new permutation of maximal strategic pile in $S_{n-2}$.

Proof. We first prove item (1). If $\pi \in S_n$ has a strategic pile of size $n-1$, then each number in $[n-1]$ must appear between $n$ and 0 in a cycle of $C_\pi$. Thus, $C_\pi(0) = n$. By the definition of $C_\pi$, we must have that $\pi^{-1}(0) = \pi^{-1}(n) - 1$, as desired. Items (2) and (3) are from [4], Lemma 3.11. Moreover, (5) is a corollary of (4), so we prove (4). Each application of cds introduces at least two adjacencies, and thus removes at least two elements from the strategic pile ([1], Lemma 2.7). However, no more than two elements can be removed from the strategic pile with a single cds move ([1], Corollary 2.16).

By the preceding proposition, to study permutations with maximal strategic pile, it suffices to study such permutations with even length: Permutations of odd length with maximal strategic pile can be reduced to permutations with even length and maximal strategic pile.

Furthermore, for a permutation $\pi \in S_{2n}$ with maximal strategic pile, entry $2n$ always occurs directly to the left of 1. Thus we can contract these entries in a way analogous to reductions of adjacencies.
Definition 2.15. To each \( \pi \in S_{2n} \) with maximal strategic pile, associate a permutation \( \pi' \in S_{2n-1} \), defined as follows: If \( m \) is such that \( \pi(m) = 2n \), then for all \( k < m \) define
\[
\pi'(k) = \pi(k)
\]
and for all \( m \leq k < 2n-1 \) define
\[
\pi'(k) = \pi(k+1)
\]
and assign \( \pi' \) the same pointers as the corresponding elements in \( \pi \), except 1 has the left pointer \((2n-1, 1)\), and \( 2n-1 \) has the right pointer \((2n-1, 1)\).

\( M_{2n,k} \) is the set of all such contractions on permutations of length \( 2n \) with maximal strategic pile and \( k \) valid pointer contexts.

Example 2.16. The permutation \( \pi = [2 4 6 1 3 5] \) in \( S_6 \) has a maximum strategic pile. Also, \( \pi \) has \( k = \left(\frac{2 \cdot 5 - 1}{2}\right) \) pointer pairs that constitute valid pointer contexts. Its contraction to a permutation in \( M_{6,5} \) is \([2 4 1 3 5]\).

3. A Group Action on \( M_{2n,k} \)

A goal of this section is to count, fixing \( k \) and \( n \), the number of permutations that are of length \( 2n \) have maximum strategic pile, and exactly \( k \) valid pointer contexts. Note that as a consequence of 1) in Proposition 2.14, \(|M_{2n,k}|\) is the number of permutations \( S_{2n} \) with maximum strategic pile and \( k \) pairs of pointers that constitute valid pointer contexts. The enumeration method relies on a group action on permutations which preserves membership to the set \( M_{2n,k} \), and an application of the orbit-stabilizer theorem.

Definition 3.1. Define the map \( \phi : \mathbb{Z}_n \times \mathbb{Z}_n \times S_n \rightarrow S_n \) by
\[
\phi(a, b, \pi)(x) = \pi(x - a) + b
\]
where \( a, b \in \mathbb{Z}_n \) and \( \pi \in S_n \).

For \( \pi \in S_n \), this operation has the effect of applying a cyclic shift by \( a \) to \( \pi \), followed by adding \( b \) to each element in \( \pi \).

Example 3.2. If we consider \( \pi = [5 4 1 3 2] \) in \( S_5 \), then
\[
\phi(2, 3, \pi) = [3 + 3 2 + 3 5 + 3 4 + 3 1 + 3] = [1 5 3 2 4]
\]

As is exhibited in Example 3.2, the function \( \phi \) is composed of two elementary functions: (a) cyclic shift by \( a \in \mathbb{Z}_n \) which is defined by
\[
C_a : S_n \rightarrow S_n \\
\pi \mapsto \pi(x - a)
\]
and (b) translation by \( b \in \mathbb{Z}_n \) which is defined by
\[
T_b : S_n \rightarrow S_n \\
\pi \mapsto \pi(x) + b
\]
Then we have that
\[
\phi(a, b, \pi) = C_a(T_b(\pi)) = T_b(C_a(\pi))
\]
for all \( a, b \in \mathbb{Z}_n \), \( \pi \in S_n \). Furthermore, note that
\[
C_a \circ C_b = C_{a+b} \\
T_a \circ T_b = T_{a+b}
\]
for all \( a, b \in \mathbb{Z}_n \).

We now show that the action of \( \mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1} \) on any element of \( M_{2n,k} \) gives an element of \( M_{2n,k} \).
Theorem 3.3. \( \phi \) restricted to \( \mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1} \times M_{2n,k} \) has image a subset of \( M_{2n,k} \).

Proof. Let \( \pi \in M_{2n,k} \). It suffices to show that
\[
\phi(1,0,\pi) \in M_{2n,k} \\
\phi(0,1,\pi) \in M_{2n,k}
\]
Let \( \pi' \in S_{2n} \) be the permutation for which \( \pi \) is a reduction.

First \( \phi(1,0,\pi) \) is a reduction of a permutation with the same number of moves as \( \pi' \). This is seen by noting if \( \sigma \) is the permutation for which \( \phi(1,0,\pi) \) is a reduction, then the order of the pointers is the same in \( \sigma \) as it is in \( \pi' \), except the second to last and last pointers in \( \pi' \) appear first and second respectively in \( \sigma \). Suppose \((p,q)\) are a valid pointer context in \( \pi' \). Then they appear in the order \( \ldots p \ldots q \ldots p \ldots q \) in \( \pi' \). Suppose then \( p \) nor \( q \) was one of the last two pointers in \( \pi' \), in which case the pointers appear in the order \( \ldots p \ldots q \ldots p \ldots q \) in \( \sigma \), and thus form a valid pointer context. Otherwise, if \( q \) was one of the last two pointers, then the list of pointers is of the form \( \ldots q \ldots p \ldots q \ldots p \) in \( \sigma \) and thus \((p,q)\) is a valid pointer context. Once again if \( p \) and \( q \) were the last two pointers in \( \pi' \) then they appear as \( pq \ldots p \ldots q \) in \( \sigma \) and once again form a valid pointer context.

The same is true for \( \phi(0,1,\pi) \) since if \( \ldots p \ldots q \ldots p \ldots q \ldots \) is a valid pointer context in \( \pi' \), then the pointer word after translation has the form \( \ldots (p+1) \ldots (q+1) \ldots (p+1) \ldots (q+1) \ldots \), containing a valid pointer context.

For permutations with even length and maximal strategic pile we have that \( C_{\pi}(0) = 2n \) and \( C_{\pi}(2n) = C_{\pi'(2n)} \), since \( 2n \) appears directly before 1 in such a permutation. We have that the strategic pile of the permutation that reduces to \( \phi(1,0,\pi) \) is a cyclic shift by one of the original permutation, and likewise the strategic pile of the permutation that reduces to \( \phi(0,1,\pi) \) is a translation by 1 of the original permutation. This implies the theorem. \( \square \)

From now on, let \( \phi' \) denote the restriction of \( \phi \) to \( \mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1} \times M_{2n,k} \).

Corollary 3.3.1. \( \phi' \) is a \( \mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1} \)-action on \( M_{2n,k} \).

Proof. Firstly, the identity \( (0,0) \in \mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1} \) preserves any element in \( M_{2n,k} \). Let \( \pi \in M_{2n,k} \).
\[
\phi(0,0,\pi)(x) = \pi(x) - 0 = \pi(x)
\]
Thus, \( \phi(0,0,\pi) = \pi \). Secondly, if \( (a,b), (c,d) \in \mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1} \)
\[
\phi(a + c, b + d, \pi)(x) = \pi((a + c) - (a + c)) + b + d = [\pi((x - c) - a) + b] + d = \phi(c, d, \phi(a, b, \pi))(x)
\]
which proves the statement. \( \square \)

Notice that
\[
\pi(x + 1) - \pi(x) = T_b(\pi)(x + 1) - T_b(\pi)(x)
\]
and so, intuitively, two permutations are translations of one another if the differences between their elements are the same. This motivates the following definition.

Definition 3.4. Let permutation \( \pi \) be an element of \( S_n \). The difference sequence of \( \pi \) is the \( n \)-tuple \( D_\pi \in \mathbb{Z}_n := \prod_{i=1}^{n} \mathbb{Z}_n \) where for \( 1 \leq k \leq n - 1 \), the \( k \)th component of \( D_\pi \) is defined as follows:
\[
D_\pi(k) = \begin{cases} 
\pi(k + 1) - \pi(k) & \text{if } 1 \leq k < n \\
\pi(1) - \pi(n) & \text{otherwise}
\end{cases}
\]
In the case that there exists a \( 0 < p < n \) such that
\[
D_\pi(k + p) = D_\pi(k)
\]
Corollary 3.8.2. The orbit of $\pi$ under $\phi'$, denoted $O_\pi$, has order $(2n-1)p$ if $D_\pi$ is periodic with period $p$. $O_\pi$ is of order $(2n-1)^2$ otherwise.
Classifying permutations under context-directed swaps and the CDS game

Proof. This is a consequence of the orbit-stabilizer theorem. First let $D_\pi$ be periodic with period $p$. The order of $(p, \pi(p) - \pi(0)) \in \mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1}$ is $\frac{2n-1}{p}$ since $p$ has order $\frac{2n-1}{p}$ in $\mathbb{Z}_{2n-1}$, and

$$\frac{2n-1}{p} (\pi(p) - \pi(0)) = \sum_{i=1}^{(2n-1)/p} [\pi(ip) - \pi((i-1)p)] = \pi \left( \frac{2n-1}{p} \right) - \pi(0) = 0$$

Thus $\pi(p) - \pi(0)$ has order dividing $\frac{2n-1}{p}$ in $\mathbb{Z}_{2n-1}$. Hence $|\text{Stab}(\pi)| = \frac{2n-1}{p}$. Therefore by the orbit-stabilizer theorem

$$|O_\pi| = \frac{|\mathbb{Z}_{2n-1} \times \mathbb{Z}_{2n-1}|}{|\text{Stab}(\pi)|} = \frac{(2n-1)^2}{\frac{2n-1}{p}} = (2n-1)p$$

as desired. For the case when $D_\pi$ is not periodic, $|\text{Stab}(\pi)| = 1$ and thus

$$|O_\pi| = \frac{(2n-1)^2}{1}$$

as desired. □

Periodic Difference Sequence Characterization. We return to difference sequences to characterize permutations permutations with nontrivial stabilizer. In particular we will characterize permutations with periodic difference sequences and further characterize permutations with both periodic difference sequence and maximal strategic pile. These results aid in counting the number of permutations with maximal strategic pile and periodic difference sequence with specified period.

Lemma 3.9. For a permutation $\pi \in S_n$ with difference sequence $D_\pi$, for all $i, j \in [n]$

$$\pi(i + j) - \pi(i) = \sum_{k=0}^{j-1} D_\pi(i + k)$$

Proof.

$$\pi(i + j) - \pi(i) = \pi(i + j) - \pi(i) + \sum_{k=1}^{j-1} \pi(i + k) - \sum_{k=1}^{j-1} \pi(i + k)$$

$$= \sum_{k=1}^{j} \pi(i + k) - \sum_{k=0}^{j-1} \pi(i + k)$$

$$= \sum_{k=0}^{j-1} \pi(i + k + 1) - \pi(i + k)$$

$$= \sum_{k=0}^{j-1} D_\pi(i + k)$$

□

Lemma 3.10. An $n$-tuple $E \in \mathbb{Z}_n^n$ is the difference sequence of a permutation in $S_n$ if and only if $\sum_{i=1}^{n} E(i) = 0$ and there do not exist $i, j \in [n], j < n$ such that $\sum_{k=0}^{j-1} E(i + k) = 0$

Proof. Suppose $E$ is the difference sequence of some permutation $\pi \in S_n$. Fix arbitrary $i, j \in [n]$. By Lemma 3.9,

$$\sum_{k=1}^{n} E(k) = \sum_{k=0}^{n-1} E(1 + k) = \pi(1) - \pi(1 + n) = 0$$
Suppose towards a contradiction that there are \( i, j \in [n], j < n \) such that

\[
\sum_{k=0}^{j-1} E(i + k) = 0
\]

Then by Lemma 3.9, \( \pi(i + j) - \pi(i) = 0 \), so \( \pi(i) = \pi(i + j) \) but because \( j \neq n, i \neq i + j \), contradicting the injectivity of \( \pi \).

Now let \( E \in \mathbb{Z}_n^\pi \) such that \( \sum_{i=1}^{n} E(i) = 0 \) and assume that there are no \( i, j \in [n], j < n \) such that \( \sum_{k=0}^{j-1} E(i + k) = 0 \). Fix arbitrary \( a_1 \in \mathbb{Z}_n \) and define \( \alpha : \mathbb{Z}_n \rightarrow \mathbb{Z}_n \) by:

\[
\alpha(1) \mapsto a_1 \\
\alpha(i) \mapsto \alpha(i - 1) + E(i - 1)
\]

for \( i \in [n], i > 1 \)

To show that \( \alpha \) is a permutation, it is enough to show that \( \alpha \) is injective. By construction, we have that for all \( i, j \in [n] \),

\[
\alpha(i) + \sum_{k=0}^{j-1} E(i + k) = \alpha(i + j)
\]

Assume for a contradiction that there exist \( i, j \in \mathbb{Z}_n, i \neq j \) such that \( \alpha(i) = \alpha(j) \). Then

\[
\alpha(i) - \alpha(j) = \sum_{k=i}^{j-1} E(k) = 0
\]

contradicting the choice of \( E \). \( \Box \)

Remark 3.11. The permutation \( \alpha \) constructed above is not unique, as any element of \( \mathbb{Z}_n \) can be chosen to be \( a_1 \). In particular, \( \alpha \) constructed by a given choice of \( a_1 \) is some translation away from every other permutation with the same difference sequence.

For a permutation \( \pi \in S_n \), let \( \pi_k \) denote the function \( \pi_k : \mathbb{Z}_k \rightarrow \mathbb{Z}_k \) satisfying \( \pi_k(i) \equiv \pi(i) \mod k \) for all \( i \in \mathbb{Z}_k \).

Lemma 3.12. Let \( \pi \in S_n \) and \( p \mid n \), and suppose the difference sequence of \( \pi \) is periodic with a period dividing \( p \). Then for all \( 1 \leq i \leq n \), \( \pi(i) \equiv \pi(i + p) \mod p \).

Proof. Recall that \( \sum_{i=1}^{n} D_{\pi}(i) = 0 \). By periodicity of the \( D_{\pi} \), for all \( i, j \in [n] \), we have

\[
\sum_{k=i}^{i+p-1} D_{\pi}(k) = \sum_{k=j}^{j+p-1} D_{\pi}(k)
\]

Let \( s \) denote this sum of any \( p \) consecutive elements of \( D_{\pi} \). Necessarily,

\[
\sum_{k=1}^{n} D_{\pi}(k) = \frac{n}{p} \cdot s \equiv 0 \mod n
\]

Therefore, \( p \mid s \), so for all \( i \in \mathbb{Z}_n \), \( \pi(i) \equiv \pi(i + p) \mod p \). \( \Box \)

Lemma 3.13. Let \( \pi, p \) be defined as in Lemma 3.12. Then \( \pi_0 \) is a permutation in \( S_p \).

Proof. We show \( D_{\pi_p} \) satisfies the criteria of Lemma 3.10. Let \( s \) be defined as in Lemma 3.12. \( \sum D_{\pi_p} \equiv s \mod p \), and as seen in Lemma 3.12, \( p \mid s \), so \( \sum D_{\pi_p} \equiv 0 \mod p \).

Assume that on the contrary there exist \( i, t \in [p], t < p \) such that \( \sum_{k=i}^{i+t-1} D_{\pi_p}(k) \equiv 0 \mod p \). Let \( j = i + t \). Note that because \( t \neq p, i \neq j \). The assumption is equivalently, \( \pi_p(i) \equiv \pi_p(j) \mod p \). For all \( i \in \mathbb{Z}_n \), there are \( \frac{n}{p} \) elements of \( \mathbb{Z}_n \) (including \( i \) itself) equivalent \( \mod p \) to \( i \). By Lemma 3.12. For each \( f \in \mathbb{Z}_p \), \( \pi(i) \equiv \pi(i + fp) \mod p \), so because \( \pi \) is a bijection, there must exist \( f \in \mathbb{Z}_p \) such that \( j = i + fp \). But then \( j \equiv i + fp \), a contradiction. \( \Box \)
Example 3.5 continued. Consider the permutation \( \pi \in S_9 \) with periodic difference sequence:

\[
\pi = [2 \ 4 \ 3 \ 8 \ 1 \ 9 \ 5 \ 7 \ 6]
\]

\( D_\pi = (2, 8, 5, 2, 8, 5, 2, 8, 5) \)

\( D_\pi \) has period 3. \( \pi_3 = [2 \ 1 \ 3 \ 2 \ 1 \ 3 \ 2 \ 1 \ 3] \) which is \( \frac{9}{3} = 3 \) copies of \([2 \ 1 \ 3] \).

We now describe what information is needed to construct a permutation with periodic difference sequence.

**Theorem 3.14.** Let \( n \in \mathbb{N} \), \( p \mid n \). A triple

\[
(\varphi, R, k) \in S_p \times \left( \prod_{i=1}^{p} \{0\} \cup \left[ \frac{n}{p} - 1 \right] \right) \times \mathbb{Z}_p^2
\]

defines a permutation \( \pi \in S_n \) with periodic difference sequence (having period dividing \( p \)) by

\[
\pi(i) = R_{\varphi \bmod p} \cdot (\varphi(i \bmod p)) + kp \left( \left[ \frac{i - 1}{p} \right] - \left[ \frac{j - 1}{p} \right] \right)
\]

if \( \gcd(k, \frac{n}{p}) = 1 \).

**Proof.** Let \( i, j \in \mathbb{Z}_n, i \neq j \bmod n \). Without loss of generality, let \( i > j \). We have

\[
\pi(i) - \pi(j) = p \cdot \left( R_{\varphi \bmod p} \cdot (\varphi(i \bmod p)) + kp \left( \left[ \frac{i - 1}{p} \right] - \left[ \frac{j - 1}{p} \right] \right) \right)
\]

In the case that \( i \neq j \bmod p \), \( 0 < \varphi(i \bmod p) - \varphi(j \bmod p) < p \) and so \( \pi(i) - \pi(j) \equiv 0 \bmod p \). Since \( p \mid n \), \( \pi(i) - \pi(j) \neq 0 \bmod n \). Thus \( \pi \) is a permutation.

In the case that \( i \equiv j \bmod p \), we have

\[
\pi(i) - \pi(j) = kp \cdot \left( \left[ \frac{i - 1}{p} \right] - \left[ \frac{j - 1}{p} \right] \right)
\]

Since \( i \neq j \) and \( i \equiv j \bmod p \), \( \left\lfloor \frac{i - 1}{p} \right\rfloor \neq \left\lfloor \frac{j - 1}{p} \right\rfloor \). Therefore, \( 0 < \left\lfloor \frac{i - 1}{p} \right\rfloor - \left\lfloor \frac{j - 1}{p} \right\rfloor < \frac{n}{p} < n \) and since \( \gcd(k, \frac{n}{p}) = 1 \) as well, \( \pi(i) - \pi(j) \neq 0 \bmod n \).

Now we show \( \pi \) has periodic difference sequence. Let \( i \in \mathbb{Z}_n \).

\[
[\pi(i + 1) - \pi(i)] - [\pi(i + 1 + p) - \pi(i + p)]
\]

\[
= [\pi(i + p) - \pi(i)] - [\pi(i + 1) - \pi(i + p + 1)]
\]

\[
= (R_{(i+p) \bmod p} \cdot p + \varphi((i + p) \bmod p)) - \varphi(i \bmod p) + kp \left( \left[ \frac{i + p - 1}{p} \right] - \left[ \frac{i - 1}{p} \right] \right)
\]

\[
- \left( (R_{(i+1) \bmod p} - R_{(i+p+1) \bmod p}) \cdot p + \varphi((i + 1) \bmod p) - \varphi((i + 1) \bmod p) + kp \left( \left[ \frac{i}{p} \right] - \left[ \frac{i + p}{p} \right] \right) \right)
\]

\[
= kp \left( \left[ \frac{i + p - 1}{p} \right] - \left[ \frac{i - 1}{p} \right] + \left[ \frac{i}{p} \right] - \left[ \frac{i + p}{p} \right] \right)
\]

\[
= kp \left( \left[ \frac{i - 1}{p} \right] - \frac{i}{p} + 1 + \left[ \frac{i}{p} \right] - \left[ \frac{i}{p} \right] - 1 \right)
\]

\[
= 0 \quad \square
\]

The converse is actually true. That is, the information in 3.14 is precisely the amount of information needed to construct a permutation with periodic difference sequence.
Theorem 3.15. \( \pi \in S_n \) has a periodic difference sequence with period dividing \( p \) (for \( p \mid n \)) if and only if there exists a triple:

\[
(\varphi, R, k) \in S_p \times \left( \prod_{i=1}^{p} \{0\} \cup \left[ \frac{n}{p} - 1 \right] \right) \times \mathbb{Z}_{n/p}
\]

such that for all \( i \in \mathbb{Z}_n \), \( \pi(i) = R_i \mod p + \varphi(i \mod p) + kp \left\lfloor \frac{i-1}{p} \right\rfloor \) and \( \gcd(k, \frac{n}{p}) = 1 \). In fact \( \pi \) is uniquely defined by such a triple.

Proof. The 'if' portion of this statement was shown in Theorem 3.14. We now prove the 'only if' portion. Let \( \pi \in S_n \) have periodic difference sequence with period dividing \( p \). We recover \( \varphi \in S_p \) from the first \( p \) elements of \( \pi \) taken mod \( p \): \( \varphi(i) := \pi(i \mod p) \). By Lemma 3.13 \( \varphi \) defined in this way is indeed a permutation in \( S_p \).

Define \( R \) for \( 1 \leq i \leq p \) by

\[
R_i := \frac{\pi(i) - (\pi(i \mod p))}{p}
\]

Let \( k \in \mathbb{Z}_{n/p} \) be defined by

\[
k := \frac{\pi(p+1) - \pi(1)}{p}
\]

By Lemma 3.12 \( k \) defined in this manner is indeed an integer. Also we have that

\[
\pi(1+mp) - \pi(1) = kpn
\]

for any integer \( 0 \leq m < \frac{n}{p} \). If \( k \) and \( \frac{n}{p} \) were not coprime, then we’d have that \( kp \) and \( n \) are not coprime, in which case there is an \( m \) in the specified range so that

\[
= 0 \mod n
\]

which contradicts the fact that \( \pi \) is a permutation. We now show that we can recover \( \pi \) with the formula in the statement of the theorem.

\[
\begin{align*}
R_{i \mod p} \cdot p + \varphi(i \mod p) + kp \left\lfloor \frac{i-1}{p} \right\rfloor \\
= \pi(i \mod p) + (\pi(p+1) - \pi(1)) \left\lfloor \frac{i-1}{p} \right\rfloor \\
= \pi(i \mod p) + (\pi(i) - \pi(i-p)) \left\lfloor \frac{i-1}{p} \right\rfloor \\
= \pi(i \mod p) + \left( \sum_{\ell=i-p}^{i-1} \pi(\ell+1) - \pi(\ell) \right) \left\lfloor \frac{i-1}{p} \right\rfloor \\
= \pi(i \mod p) + \sum_{j=1}^{\left\lfloor \frac{i-1}{p} \right\rfloor} \left( \sum_{\ell=i \mod p+jp}^{i \mod p+(j+1)p-1} \pi(\ell+1) - \pi(\ell) \right) \quad \text{by Lemma 3.12} \\
= \pi(i \mod p) + \sum_{\ell=i \mod p}^{i-1} \pi(\ell+1) - \pi(\ell) \\
= \pi(i \mod p) + \pi(i) - \pi(i \mod p) \\
= \pi(i)
\end{align*}
\]

\[\square\]
The Strategic Pile of Permutations with Periodic Difference Sequences.

Theorem 3.16. A permutation \( \pi \in S_{2n-1} \) with difference sequence having period \( p \) has maximal strategic pile if and only if the following conditions hold:

1. Let \( \varphi \) be the permutation in \( S_p \) gotten from reducing the first \( p \) elements of \( \pi \) mod \( p \). The unreduced counterpart of \( \varphi \) in \( S_{p+1} \) (relinquishing the identification of 1 and \( p+1 \)) has maximal strategic pile.

2. If \( K = \frac{1}{p}(\pi(p+1) - \pi(1)) \), then the order of \( K - 1 \mod \frac{2n-1}{p} \) is \( \frac{2n-1}{p} \).

Proof. Let \( \rho \) be the permutation achieved by reducing the first \( p \) elements of \( \pi \mod p \). We will first show that \( \rho \) must have maximal strategic pile. Let \( S \) be the size of the strategic pile of \( \rho \). By the definition of \( C_\pi \) and the periodicity of \( \pi \mod p \), if one reduces the numbers listed in \( C_\pi \mod p \), the resulting list of numbers after \( 2n \) and before 0 will have period \( S \). Thus, for this list of numbers to have size \( 2n-1 \), we must have that \( S = p \). Now, assume that \( \rho \) has maximal strategic pile. Let \( O \) denote the order of \( p(K-1) \mod 2n-1 \).

We will show that the strategic pile of \( \pi \) has size \( p \cdot O \). Since \( \rho \) has max strategic pile, the first \( p \) elements in the orbit of \( x \) in \( C_\pi \) have one element of each coset \( \mod p \); \( C_\pi^k(x) = x \mod p \) iff \( k = 0 \mod p \). Let \( d_k = \varphi(c_k - c_{k+1}) \). There exists a permutation \( \sigma \in S_p \) such that for each \( k \in [p] \), \( C_\pi^{k-1}(x) + 1 - d_{\sigma(k)} = C_\pi^k(x) \). Thus, \( C_\pi^p(x) = \pi(x) + p - \sum_{k=1}^{p} d_{\sigma(k)} = \pi(x) + p(K - 1) \). Therefore, the smallest \( kp \) such that \( C_\pi^{kp}(\pi(n)) = \pi(n) \) is the smallest \( kp \) such that \( kp(K - 1) = 0 \mod n \); by the definition of the strategic pile, \( kp \) is also the size of the strategic pile. But \( k = O \), as desired.

Since the case of odd length permutations with maximal strategic pile reduces to the even length case, we are specifically interested in permutation in \( M_{2n} \). Using the above characterizations, we can count the members of \( M_{2n} \) having periodic difference sequence with period dividing \( p|2n-1 \). Since reduced permutations in \( M_{2n} \) have length \( 2n-1 \), \( p \) must be odd. To count such permutations we must count \( \varphi, R, K \) subject to the conditions in Theorem 3.16 and Theorem 3.17. Counting \( \varphi \) is counting permutations of (even) length \( p+1 \) having maximal strategic pile, which is \( \frac{2^{2n-1}}{p+1} \) by Theorem 3.3 in \([4]\). There are \( \left( \frac{2n-1}{p} \right)^p \) choices for \( R \). Finally, to count the choices of \( k \) that yield a permutation that has both periodic difference sequence and maximal strategic pile, we need to define a variant of Euler’s totient function. Let

\[
\psi(m) := \{0 < c <= m : \gcd(c, m) = \gcd(c-1, m) = 1\}
\]

It turns out that this function has closed form:

\[
\psi(m) = m \prod_{\substack{q \mid m \; q \; \text{prime}}} (1 - \frac{2}{q}).
\]

We have then shown the following:

Corollary 3.16.1. Let \( n \) be a positive integer and let \( p \) be a divisor of \( 2n-1 \). The number of permutations in \( M_{2n} \) having periodic difference sequence with period dividing \( p \) is

\[
2 \cdot \frac{p!}{p+1} \cdot \left( \frac{2n-1}{p} \right)^p \cdot \psi(n).
\]

Proof. By Theorem 3.16, \( K \) must satisfy \( \gcd \left( K, \frac{2n-1}{p} \right) = 1 \). By Theorem 3.17, in order for the resulting permutation to have maximal strategic pile, we must also have that \( K - 1 \) has order \( \frac{2n-1}{p} \) in \( \mathbb{Z}_{2n-1} \), which is equivalent to \( \gcd \left( K - 1, \frac{2n-1}{p} \right) = 1 \). Thus \( \psi \left( \frac{2n-1}{p} \right) \) is the number of valid choices for \( K \). The result then follows from the preceding exposition.
4. Taxonomy

The techniques developed in the previous section will now be used to analyze \( M_{2n,k} \) for certain values of \( n \) and \( k \). As an outline of the upcoming work, difference sequences will be used to compute some values \( |M_{2n,k}| \mod (2n-1)^2 \). Then, elements of \( M_{2n,k} \) are characterized for the following specific values of \( k \): \( k = \binom{2n-1}{2}, k = \binom{2n-1}{2} - 4 \), and \( k = 2n-1 \).

**Definition 4.1** (Compatible Pointers). Let \( \pi \) be a permutation, and let \( p^* = (p, p+1) \), \( q^* = (q, q+1) \) be a pair of pointers. Pointers \( p^* \) and \( q^* \) are compatible if they constitute a valid pointer context in \( \pi \).

**Definition 4.2** (Universal Pointer). Let \( \pi \) be a permutation. A pointer is \( \pi \)-universal if it is compatible with each other pointer.

**Counting Classes mod \((2n-1)^2\).**

In the following theorem and proof, for \( d \in \mathbb{Z}_{2n-1}^* \), let \( d^{-1} \) denote the multiplicative inverse of \( d \) in \( \mathbb{Z}_{2n-1} \). Besides this, all arithmetic will be done in \( \mathbb{Z} \).

**Theorem 4.3.** Let \( \pi \in S_{2n} \) have constant difference sequence with difference value \( d \). If \( \gcd(2n-1,d) = 1 \), then the number of valid pointer contexts for \( \pi \) is \( (2n-1) \cdot \min(d-1, 2n-1 - d^{-1}) \).

**Proof.** Let \( p = \pi(1) \). We have \( \pi^{-1}(p+1) - 1 = d^{-1} \) and \( \pi^{-1}(p-1) = -d^{-1} \).

First consider the case where \( \pi^{-1}(p+1) < \pi^{-1}(p-1) \). Each pointer appearing between \( p \) and \( p+1 \) occurs exactly once, so \( p^* = (p, p+1) \) is compatible with \( 2(d^{-1} - 1) \) pointers. If \( q \in \mathbb{Z}_{2n-1} \), then we can cyclically shift so that \( \pi(1) = q \) without changing the difference sequence. Thus, \( q^* = (q, q+1) \) is also compatible with \( 2(d^{-1} - 1) \) pointers. Thus, the total number of available valid pointer contexts is

\[
\frac{1}{2}(2n-1) \cdot 2(d^{-1} - 1) = (2n-1)(d^{-1} - 1).
\]

Now, consider the case where \( \pi^{-1}(p-1) < \pi^{-1}(p+1) \). Each pointer appearing between \( p \) and \( p-1 \) occurs exactly once, so \( (p-1, p) \) is compatible with \( 2(2n-1 - d^{-1}) \) other pointers. Using an appropriate cyclic shift, we can show that for all \( q \in \mathbb{Z}_{2n-1} \), \((q-1, q) \) is compatible with \( 2(2n-1 - d^{-1}) \) other pointers. Thus, the total number of available valid pointer contexts is

\[
\frac{1}{2}(2n-1) \cdot 2(2n-1 - d^{-1})) = (2n-1)(2n-1 - d^{-1}).
\]

**Corollary 4.3.1.** Let \( \pi \) be a member of \( S_{2n} \), where \( p = 2n-1 \) is an odd prime number.

1. For \( k \in \lfloor (2n-4)/2 \rfloor \) there are exactly \( 2p \) permutations with constant difference sequence, maximal strategic pile, and \( kp \) valid pointer contexts.
2. There are exactly \( p \) permutations with constant difference sequence, maximal strategic pile, and \( \binom{p}{2} \) valid pointer contexts.
3. All other permutations in \( S_{2n} \) with max strategic pile have orbit size \( p^2 \).

**Proof.** When \( \pi \in S_{2n} \) has nonperiodic difference sequence, its orbit under \( \phi' \) is of order \((2n-1)^2\). The equation \( k = \min(d^{-1} - 1, 2n-1 - d^{-1}) \) has two solutions for \( k \in \lfloor (2n-1)/4 \rfloor \) and one solution for \( k = \binom{2n-1}{2} \).

**Theorem 4.4.** Let \( \pi \in M_{2n} \) have a difference sequence with period \( p \). Then, the number of available valid pointer contexts is a multiple of \( \frac{2n-1}{p} \).

**Proof.** Let \( K = \pi(p+1) - \pi(1) \). The pointers \((p, p+1)\) and \((q, q+1)\) are compatible if and only if \((p+K, p+K+1)\) is compatible with \((q+K, q+K+1)\). Since the order of \( K \mod 2n-1 \) is \( \frac{2n-1}{p} \), the result follows.
Corollary 4.4.1. If $k$ is relatively prime to $2n - 1$, then $|M_{2n,k}| = 0 \mod (2n - 1)^2$.

Proof. By the previous theorem, no element of $M_{2n,k}$ can have a periodic difference sequence, so every element of $M_{2n,k}$ has orbit size $(2n - 1)^2$. □

Characterizing $M_{2n,k}$ for $k = \binom{2n-1}{2}$.

Proposition 4.5. The permutation $[2 4 \ldots 2n - 2 2n 1 3 \ldots 2n - 1]$ has maximal strategic pile and $\binom{2n-1}{2}$ valid pointer contexts.

Lemma 4.6. If $\pi \in S_{2n}$ has a universal pointer $(p,p+1)$, then exactly $2n - 2$ pointers appear to the right of the leftmost instance of $(p,p+1)$ and to the left of the rightmost instance of $(p,p+1)$.

Proof. Each pointer other than $(p,p+1)$ must appear exactly once the right of the leftmost instance of $(p,p+1)$ and to the left of the rightmost instance of $(p,p+1)$, since each other pointer is compatible for CDS with $\pi$. There are $2n - 2$ such pointers. □

Theorem 4.7. Every $\pi \in M_{2n,\binom{2n-1}{2}}$, is a cyclic shift of $[2 4 6 \ldots 2n - 2 1 3 5 \ldots 2n - 1]$.

Proof. We will show that $\pi$ has constant difference sequence 2. Let $k \in [2n - 1]$, and cyclically shift $\pi$ so that $\pi(1) = k$. Then, since exactly $2n - 2$ pointers must appear to the right of the leftmost instance of $(k,k+1)$ and to the left of the rightmost instance of $(k,k+1)$, we have $\pi(n+1) = k + 1$. Since exactly $2n - 2$ pointers must appear to the right of the leftmost instance of $(k+1,k+2)$ and to the left of the rightmost instance of $(k+1,k+2)$, we have $\pi(2) = k + 2$, as desired. □

Corollary 4.7.1. $|M_{2n,\binom{2n-1}{2}}| = 2n - 1$.

Characterizing $M_{2n,k}$ for $k = \binom{2n-1}{2} - 4$.

Proposition 4.8. For $n \geq 3$, the permutation $[4 2 6 \ldots 2n - 2 2n 1 3 5 \ldots 2n - 1]$ has maximal strategic pile and $\binom{2n-1}{2} - 4$ valid pointer contexts and a non-periodic difference sequence. Thus, there are at least $(2n - 1)^2$ permutations in $M_{2n,\binom{2n-1}{2}-4}$.

Lemma 4.9. For $\pi \in M_{2n}$, each pointer $(p,p+1)$ is compatible with an even number of pointers.

Proof. For $p \in [2n - 1]$, cyclically shift $\pi$ so that $\pi(1) = p$. Then, since each number occurring after $p$ and before $p+1$ in $\pi$ has two associated pointers, there are an even number of pointers to the right of the leftmost instance of $(p,p+1)$ and to the left of the rightmost instance of $(p,p+1)$. Let $\alpha$ be the multiset of pointers that appear to the right of the leftmost instance of $(p,p+1)$ and to the left of the rightmost instance of $(p,p+1)$. Let $\beta \subset \alpha$ be the set of elements that appear exactly once in $\alpha$. Then, $|\beta|$ is the number of pointers compatible with $(p,p+1)$, and since every other element of $\alpha$ appears twice, $|\beta|$ is even. □

Corollary 4.9.1. For $\pi \in M_{2n}$, each pointer is incompatible with an even number of pointers.

Lemma 4.10. If $\pi \in M_{2n}$ has $\binom{2n-1}{2} - c$ valid pointer contexts for $c > 0$ and a non-universal pointers, then $\binom{c}{2} \geq c \geq a$.

Proof. Let $\pi \in M_{2n}$ have $\binom{2n-1}{2} - c$ valid pointer contexts and a non-universal pointers. Define the “incompatibility” graph $G_\pi$ as follows: the vertices are the set of non-universal pointers, and draw an edge between two non-universal pointers if they are incompatible. Then, $G_\pi$ has $a$ vertices and $c$ edges. Moreover, each vertex has degree at least 2. Thus, we have $c \leq \binom{a}{2}$ and $c \geq a$. □

Lemma 4.11. If $\pi \in M_{2n}$ has $\binom{2n-1}{2} - c$ valid pointer contexts for $c > 0$, then $c \geq 3$.

Proof. Let $a$ be the number of non-universal pointers of $\pi$. If $c = 1$ or $c = 2$, then $a \leq c \leq \binom{a}{2} < a$, which is impossible. □
Lemma 4.12. Let \( \pi \in M_{2^n} \), and let \((p, p + 1)\) be a non-universal pointer. Then, either \((p - 1, p)\) or \((p + 1, p + 2)\) is non-universal.

Proof. Assume for a contradiction that \((p, p + 1)\) is non-universal but \((p - 1, p)\) and \((p + 1, p + 2)\) are both universal. Cyclically shift \(\pi\) so that \(\pi(1) = p\). Then, we have \(\pi(n) = p - 1\). Since \((p, p + 1)\) is compatible with \((p - 1, p)\), we must have \(\pi(n + k) = p + 1\) for \(1 \leq k \leq n - 1\). Then, since \((p + 1, p + 2)\) is universal, we have \(\pi(k + 1) = p + 2\). The sequence of pointers of \(\pi\) is as follows: 
\[ [(p - 1, p), (p, p + 1), \alpha, (p + 1, p + 2), (p + 2, p + 3), \beta, (p - 2, p - 1), (p - 1, p), \gamma, (p, p + 1), (p + 1, p + 2), \delta].\]
Thus, each pointer appears exactly once in the sequence 
\[ [(p, p + 1), \alpha, (p + 1, p + 2), (p + 2, p + 3), \beta, (p - 2, p - 1), (p - 1, p)], \]
and each pointer appears exactly once in the sequence 
\[ [(p + 2, p + 3), \beta, (p - 2, p - 1), (p - 1, p), \gamma, (p, p + 1), (p + 1, p + 2)].\]
Thus, \(\alpha\) and \(\gamma\) contain exactly the same pointers. Let \(A\) and \(C\) be the sequences of elements of \(\mathbb{Z}_{2n-1}\) with pointer sequences \(\alpha\) and \(\gamma\), respectively. If \(q \in A\), then \(q - 1\) and \(q + 1\) must be in \(C\), so \(q - 2\) and \(q + 2\) must be in \(A\), and so on. But then, depending on the parity of \(p - q\), either \(p\) or \(p - 1\) is in \(A\), a contradiction. \(\square\)

Lemma 4.13. A permutation \(\pi \in M_{2^n}\) cannot have a collection of pointers \(\{(p, p + 1), (p + 1, p + 2), (p + 2, p + 3)\}\) which are pairwise incompatible but every other pointer is universal.

Proof. Assume for a contradiction that \(\pi \in M_{2^n}\) has this property. Cyclically shift \(\pi\) so that \(\pi(1) = p\). Since \((p - 1, p)\) is universal, we have \(\pi(n) = p - 1\). Since \((p, p + 1)\) is compatible with \(2n - 4\) pointers, either \(\pi(n) = p + 1\) or \(\pi(n + 2) = p + 1\); since \(\pi(n) = p - 1\), we have \(\pi(n + 2) = p + 1\). Since \(2n\) pointers appear to the right of the leftmost instance of \((p, p + 1)\) and to the left of the rightmost instance of \((p, p + 1)\), \((p + 1, p + 2)\) and \((p + 2, p + 3)\) must both appear twice in the sequence of pointers to the right of the leftmost instance of \((p, p + 1)\) and to the left of the rightmost instance of \((p, p + 1)\). We then have
\[ \pi = [p A p - 1 b p + 1 C], \]
where \(A\) is a sequence of \(n - 2\) elements of \(\mathbb{Z}_{2n-1}\), \(b \in \mathbb{Z}_{2n-1}\), and \(C\) is a sequence of \(n - 3\) elements of \(\mathbb{Z}_{2n-1}\). Let \(\alpha\) be the multiset of pointers of elements of \(A\), then since \((p - 1, p)\) is universal, \(\{(p - 1, p), (p, p + 1), (p - 2, p - 1)\} \cup \alpha\) contains each pointer exactly once. Then, since \((p, p + 1)\) is incompatible with \((p + 1, p + 2)\) and \((p + 2, p + 3)\), we have \(b = p + 2\). But then, \((p - 1, p)\) is not compatible with \((p + 1, p + 2)\), a contradiction. \(\square\)

Corollary 4.13.1. \(\pi \in M_{2^n}\) cannot have exactly \(\binom{2n-1}{2} - 3\) valid pointer contexts.

Theorem 4.14. A permutation \(\pi \in M_{2^n}\) with \(\binom{2n-1}{2} - 4\) valid pointer contexts must be in the orbit of \([4 \ 2 \ 6 \ \ldots \ 2n - 2 \ 1 \ 3 \ \ldots \ 2n - 1]\).

Proof. By lemma 4.12, the non-universal pointers of \(\pi\) are of the form 
\(\{(p - 1, p), (p, p + 1), (q - 1, q), (q, q + 1)\}\), where without loss of generality \(q \neq p + 2\). Up to a cyclic shift and a translation, we have \(p = 4\) and \(\pi(1) = 4\). Since each non-universal pointer is incompatible with at least two other pointers, and since the number of available valid pointer contexts is \(\binom{2n-1}{2} - 4\), each non-universal pointer is incompatible with exactly two other pointers. We thus have that either \(\pi(n) = 5\) or \(\pi(n + 2) = 5\). If \(\pi(k) = 6\), then since \((5, 6)\) is universal, we have \(\pi(k + n - 1) = 5\). Since \((5, 6)\) is compatible with \((4, 5)\), we have \(\pi^{-1}(6) < \pi^{-1}(5)\), so \(\pi(n + 2) = 5\) and \(\pi(3) = 6\). Since \((3, 4)\) is compatible with \(2n - 4\) pointers, we either have \(\pi(n - 1) = 3\) or \(\pi(n + 1) = 3\), so \((3, 4)\) is compatible with \((4, 5)\), and both are not compatible with \((q - 1, q)\) and \((q, q + 1)\). We have that
\[ \pi = [4 a 6 B 5 C], \]
where \(a \in \mathbb{Z}_{2n-1}\), \(B\) is a sequence of \(n-2\) elements of \(\mathbb{Z}_{2n-1}\), and \(C\) is another sequence of elements of \(\mathbb{Z}_{2n-1}\). Let \(\beta\) be the multiset of pointers of \(B\). Since \((5,6)\) is universal, every pointer appears exactly once in the multiset \(\{(5,6),(6,7),(4,5)\} \cup \beta\). Then, \((4,5)\) is incompatible with \((a-1,a)\) and \((a,a+1)\), so \(q = a = \pi(2)\). Since \(q\) appears between 4 and 3, the sequence of pointers to the right of the leftmost instance of \((3,4)\) and to the left of the rightmost instance of \((3,4)\) contains two copies of each of \((q-1,q),(q,q+1)\), and contains all pointers other than \((3,4)\), \((q-1,q)\), and \((q,q+1)\) exactly once. Thus, \(\pi(n+1) = 3\), and \(\pi^{-1}(q+1) \leq \pi^{-1}(3)\). Since \((q,q+1)\) is compatible with \(2n-4\) other pointers, we have \(\pi(2+n-1) = q+1\) or \(\pi(2+n+1) = q+1\). Since \(\pi^{-1}(q+1) \leq \pi^{-1}(3) = n+1\), we have \(\pi(n+1) = q+1\) and thus \(q+1 = 3\), so \(q = 2\). For all numbers \(c \in \{6,\ldots,2n-2\}\), since the pointer \((c,c+1)\) is universal, we have that if \(\pi(k) = c\), then \(\pi(k+n) = c+1\). Since we know already that \(\pi(3) = 6\), the rest of the permutation is determined, and is equal to \([4 \ 2 \ 6 \ldots \ 2n-2 \ 1 \ 3 \ldots \ 2n-1]\].

**Corollary 4.14.1.** For \(n \geq 3\), \(|M_{2n,(2n-1)-4}| = (2n-1)^2\).

**Characterizing \(M_{2n,k}\) for \(k = 2n-1\).**

**Proposition 4.15.** For \(n \geq 3\), the permutations \([2n \ 1 \ n+1 \ 2n \ 2 \ldots \ n-1 \ 2n-1 \ n]\) and \([2n-1 \ 2n-2 \ldots \ 2n \ 1]\) have maximal strategic pile, \((2n-1)\) valid pointer contexts, and constant difference sequences. Thus, there are at least \(2(2n-1)\) permutations in \(M_{2n,2n-1}\).

**Lemma 4.16.** If \(\pi \in M_{2n,2n-1}\), then each pointer is compatible for \(\text{cds}\) with exactly two other pointers.

**Proof.** By [1], Theorem 2.19, each pointer must be compatible with some other pointer. By Lemma 4.13, each pointer must be compatible with an even number of pointers. Since there are \(2n-1\) pointers, the result follows.

**Lemma 4.17.** Let \(\pi \in M_{2n}\). Assume \((p, p+1)\) is compatible with two other pointers. Let \(a = \pi^{-1}(p) < \pi^{-1}(p+1) = b\). Then, \(\pi((a+1,\ldots,b-1)) = \{k,k+1,\ldots,k+l\}\) for some \(k,l\).

**Proof.** Assume not. Then, there are \(q,r \in \pi((a+1,\ldots,b-1))\) with \(q+1,r+1 \notin \pi((a+1,\ldots,b-1))\), and there is an \(s \in \pi((a+1,\ldots,b-1))\) with \(s-1 \notin \pi((a+1,\ldots,b-1))\). But then, \((p,p+1)\) is compatible with \((r,r+1), (q,q+1)\), and \((s-1,s)\), a contradiction.

**Lemma 4.18.** Let \(\pi \in M_{2n}\). Assume \((p,p+1)\) is compatible with two other pointers. Let \(a = \pi^{-1}(p+1) < \pi^{-1}(p) = b\). Then, \(\pi((a,\ldots,b)) = \{k,k+1,\ldots,k+l\}\) for some \(k,l\).

**Proof.** If we cyclically shift \(\pi\) to a permutation \(\phi\) so that \(\pi^{-1}(p) < \pi^{-1}(p-1)\), applying Lemma 4.17 gives that \(\mathbb{Z}_{2n-1} \setminus \pi((a,\ldots,b)) = \{k',k'+1,\ldots,k'+l'\}\) for some \(k',l'\). Thus, there are \(k,l\) such that \(\pi((a,\ldots,b)) = \{k,k+1,\ldots,k+l\}\), as desired.

Let \(\pi\) be a permutation. For \(m \geq 1\), call a proper subsequence \([\pi(k),\ldots,\pi(k+m)]\) violating if \(\pi((k,k+1,\ldots,k+m)) = \{a,a+1,\ldots,a+m\}\) for some \(a\), if \(\pi(k) = a\) and if \(\pi(k+m) = a+m\).

**Lemma 4.19.** If \(\pi\) is a permutation with a violating subsequence, then \(\pi\) cannot be in \(M_{2n}\).

**Proof.** Let \([\pi(k),\ldots,\pi(k+m)]\) be the violating subsequence. Assume for a contradiction that \(\pi \in M_{2n}\), and let \(\pi'\) be the permutation in \(S_{2n}\) with maximal strategic pile such that \(\pi\) is the reduction of \(\pi'\). Then, if \(c \in \{\pi(k),\ldots,\pi(k+m-1)\}\), we have \(C_{\pi'}(c) \in \{\pi(k),\ldots,\pi(k+m-1)\}\). Thus, \(\pi(k) \notin \text{SP}(\pi')\), so \(\pi'\) does not have maximal strategic pile.

**Theorem 4.20.** Let \(\pi \in M_{2n,2n-1}\) have \(\pi(2n-1) = 1\). Then, \(\pi = [2n-1 \ 2n-2 \ldots \ 2 \ 1]\) or \(\pi = [n+1 \ 2n+2 \ 3 \ldots \ n \ 1]\).

**Proof.** **Case 1:** Assume \(\pi^{-1}(3) < \pi^{-1}(2)\). We will show that for all \(k\), \(\pi^{-1}(k) = 2n-k\).
First, we will show that $\pi(2n - 2) = 2$. By Lemma 4.18 we have

$$\pi = [\alpha \ 2 \ \beta \ 1],$$

where for some $a$, $\alpha$ is a permutation of $\{3, \ldots, a\}$, and $\beta$ is a permutation of $\{a + 1, \ldots, 2n - 1\}$. Note that we retain the possibility that either $\alpha$ or $\beta$ is empty, and in fact seek to prove that $\beta$ is empty. Assume $\beta$ is nonempty. By applying Lemma 4.17 with the pointer $(a, a + 1)$, we have $\pi(\pi^{-1}(2) + 1) = a + 1$. In other words, $a + 1$ is the “first” element of $\beta$. Then, the subsequence $[\beta \ 1]$ is violating, so $\beta$ must be empty by Lemma 4.19.

Now, let $m$ be such that for all $l < m$, $\pi(2n - l) = l$. We will show that $\pi(2n - m) = m$. By induction, this will complete the proof of Case 1. By Lemma 4.18 we have

$$\pi = [\gamma \ m \ \delta \ m - 1 \ \ldots \ 1],$$

where $\gamma$ is a permutation of $\{l + 1, \ldots, 2n - 1\}$ and $\delta$ is a permutation of $\{m + 1, \ldots, l\}$. Again, we retain the possibility that either $\gamma$ or $\delta$ is empty, and in fact want to prove that $\delta$ is empty.

Assume $\delta$ is nonempty. Cyclically shift $\pi$ to a $\phi$ such that $\phi(2n - 1) = m$. Then, $a = \phi^{-1}(l) < \phi^{-1}(l + 1) = b$, and by Lemma 4.17 no element of $\{m + 1, \ldots, l - 1\}$ can be in $\phi(\{a + 1, \ldots, b - 1\})$. Thus, $l = \phi(\phi^{-1}(m - 1) - 1)$, and thus $l = \pi(2n - m)$. In other words, $l$ is the “last” element of $\delta$.

Thus, $[m \ \delta]$ is a violating subsequence, which is impossible by Lemma 4.19.

Case 2: Assume that $\pi$ is not in the orbit of $[2n - 1 \ 2n - 2 \ \ldots \ 2 \ 1]$. We will show that $\pi = [n + 1 \ 2 \ n + 2 \ 3 \ \ldots \ n \ 1]$. Let $x \in \mathbb{Z}_{2n-1}$. If we cyclically shift $\pi$ to a permutation $\psi$ such that $\psi(2n - 1) = x$, then $\psi^{-1}(x + 2) > \psi^{-1}(x + 1)$. To see this, if $\psi^{-1}(x + 2) < \psi^{-1}(x + 1)$, then we could translate $\psi$ by subtracting $x - 1$ from every element to get $\psi'$ with the property that $\psi'(3) < \psi'(2)$ and $\psi'(2n - 1) = 1$. Then, by the proof of case 1, $\psi' = [2n - 1 \ 2n - 2 \ \ldots \ 2 \ 1]$, but $\pi$ is in the orbit of $\psi'$, so this is impossible.

Let $x \in \mathbb{Z}_{2n-1}$. We will show that $\pi^{-1}(x + 1) - \pi^{-1}(x) = 2$. From this, the result will follow. In fact, it suffices to show that $\pi(2) = 2$. To see this, cyclically shift $\pi$ to a permutation $\rho$ such that $\rho(2n - 1) = x$. Then translate $\rho$ to a permutation $\rho'$ by subtracting every element by $x - 1$. Then, $\rho'$ is a permutation in $M_{2n, 2n-1}$ with $\rho'(2n - 1) = 1$ and such that $\rho'$ is not in the orbit of $[2n - 1 \ 2n - 2 \ \ldots \ 2 \ 1]$. Thus, $\rho'(2) = 2$, so $\pi^{-1}(x + 1) - \pi^{-1}(x) = 2$.

We will show that $\pi(2) = 2$. Since $1, 2$ do not form an adjacency, we do not have $\pi(1) = 2$. Moreover, since $\pi(3) > \pi(2)$, the set of elements $y$ with $\pi^{-1}(2) < \pi^{-1}(y) < \pi^{-1}(1)$ is nonempty.

Let $\pi(k + 1) = 2$. By applying Lemma 4.17 to an appropriate cyclic shift of $\pi$, we see that $\pi(\{1, \ldots, k\}) = \{a, a + 1, \ldots, a + k - 1\}$ for some $a$. We will show that $a = a + k - 1$, establishing $k = 1$, as desired.

Assume for a contradiction that $\pi^{-1}(a) < \pi^{-1}(a + k - 1)$. Since

$$\pi^{-1}(a) < \pi^{-1}(a + k - 1) < \pi^{-1}(2) < \pi^{-1}(a + k),$$

by Lemma 4.17, we have $\pi^{-1}(a + b) < \pi^{-1}(a + k - 1)$ for all $0 \leq b < k - 1$. Thus, $\pi(k) = a + k - 1$. We also have

$$\pi^{-1}(a) < \pi^{-1}(a + k - 1) < \pi^{-1}(2) < \pi^{-1}(a - 1) < \pi^{-1}(1).$$

By Lemma 4.18, we must have $\pi^{-1}(a) < \pi^{-1}(a + b)$ for every $1 \leq b \leq k - 1$. Thus, $\pi(1) = a$. Then, $[\pi(1) \ \pi(2) \ \ldots \ \pi(k)]$ is a violating subsequence, which is impossible by Lemma 4.19.

Instead, assume for a contradiction that $\pi^{-1}(a + k - 1) < \pi^{-1}(a)$. We then have

$$\pi^{-1}(a + k - 1) < \pi^{-1}(a) < \pi^{-1}(2) < \pi^{-1}(a + k) \leq \pi^{-1}(1).$$

By Lemma 4.17, all elements $c \in \{2, 3, \ldots, a - 1\}$ have $\pi^{-1}(c) < \pi^{-1}(a + k)$. Moreover, any element $d \in \{a + k + 1, \ldots, 2n - 1\}$ must have $\pi^{-1}(d) > \pi^{-1}(a + k)$. Then, $[a + k \ \ldots \ 1]$ is a violating subsequence, which is impossible by Lemma 4.19.

Thus, $\pi^{-1}(a) = \pi^{-1}(a + k - 1)$, so $a = a + k - 1$, $k = 1$, and $\pi(2) = 2$. □
Corollary 4.20.1. For \( n \geq 3 \), \(|M_{2n,2n-1}| = 2(2n - 1)\).

5. The \textbf{cds} Game

The following combinatorial game associated with \textbf{cds} was introduced in [1]:

**Definition 5.1.** [cds game] For permutation \( \pi \) and set \( A \subseteq SP(\pi) \) the two-player game \textbf{CDS}(\( \pi, A \)), called the \textbf{cds} game, is played as follows:

Player ONE selects a \( \pi \)-valid pointer context and performs \textbf{cds} on \( \pi \) for that pointer context. Let \( \pi_1 \) be the resulting permutation. Then player TWO selects a \( \pi_1 \)-valid pointer context and performs \textbf{cds} on \( \pi_1 \) for that pointer context. Let \( \pi_2 \) be the resulting permutation. ONE and TWO alternate making such moves until a \textbf{cds} fixed point \( \phi \) is reached. If \( \phi \in A \), ONE wins. Otherwise, TWO wins.

The number of moves in this game can be pre-computed efficiently from a given permutation \( \pi \): For let \( c(\pi) \) denote the number of cycles (including length 1 cycles) in the disjoint cycle decomposition of \( C_\pi \). The Duration Theorem, Theorem 3.2 of [1] (see also Theorem 4 of [3]), states

**Theorem 5.2.** For each \( \pi \in S_n \) that is not a \textbf{cds} fixed point, the number of consecutive applications of \textbf{cds} that results in a \textbf{cds} fixed point is

\[
\begin{align*}
\frac{n+1-c(\pi)}{2} & - 1 \\
\frac{n+1-c(\pi)}{2} & - 1
\end{align*}
\]

if \( \pi \) is \textbf{cds} sortable

otherwise

In the case when \( n \) is even and \( \pi \) has a maximum sized strategic pile, \( c(\pi) = 1 \), and thus the duration of any instance of the game \textbf{CDS}(\( \pi, A \)) is \( \frac{n}{2} - 1 \).

As the game \textbf{CDS}(\( \pi, A \)) is in the category of finite two-person perfect information combinatorial games, by a classical theorem of Zermelo [13] one of the players has a winning strategy. Determining which player has a winning strategy in a generic instance of the game appears to be of high computational complexity, and no simple criterion is known at this time. A number of sufficient conditions for ONE to have a winning strategy, or for TWO to have a winning strategy, have been obtained in prior work [1] [7]. In this section of the paper additional sufficient conditions for player ONE to have a winning strategy in this game are developed, and a connection with classical Sprague-Grundy numbering [12] [6] in combinatorial games is adapted to this game.

The following notational conventions will be followed in this section: \( M_{2n} \) denotes the permutations in \( S_{2n} \) with maximal strategic pile. Thus, \( M_{2n} \) is the set of contracted versions of permutations in \( M_{2n} \). For positive integer \( p \) the symbol \( p^* \) denotes the pointer \( (p,p+1) \), and the symbol \( \sigma_{2n,p} \) denotes the \textbf{cds} fixed point of length \( 2n \) of the form \([p+1 \cdot p+2 \cdot \cdots \cdot 2n \cdot 2 \cdot \cdots \cdot p]\).

**Sprague-Grundy Numbering.** In this subsection the Strategic Pile Retention Theorem, Theorem 2.21 of [1], will be useful:

**Theorem 5.3.** Let \( \pi \in S_n \) be a permutation for which \( SP(\pi) \) has more than one element. Then for any \( x \in SP(\pi) \) there is a \( \pi \)-compatible pair \( (p,q) \) of pointers such that \( x \in SP(\text{cds}_{p,q}(\pi)) \).

Moreover, as a direct consequence of the Strategic Pile Removal Theorem, Theorem 2.19 of [1],

**Lemma 5.4.** For \( \pi \in M_{2n} \) and \( \pi \)-compatible pointers \( p \) and \( q \), \( SP(\text{cds}_{p^*,q^*}(\pi)) = SP(\pi) \setminus \{\sigma_{2n,p}(n), \sigma_{2n,q}(n)\} \).

Thus, when player ONE executes \textbf{cds} on permutation \( \pi \in M_{2n} \) with a \( \pi \)-valid pointer context \((p^*,q^*)\), then by Lemma 5.3 the resulting permutation has strategic pile \( SP(\pi) \setminus \{\sigma_{2n,p}(n), \sigma_{2n,q}(n)\} \). Now player TWO is confronted with executing \textbf{cds} on the permutation \textbf{cds}_{p^*,q^*}(\pi). In this position the subset of the strategic pile corresponding to a win for TWO is \( (SP(\pi) \setminus A) \setminus \{\sigma_{2n,p}(n), \sigma_{2n,q}(n)\}) \).

**Definition 5.5.** Let a permutation \( \pi \in S_n \), a set \( A \subseteq SP(\pi) \), and a valid pointer context \((p^*, q^*)\) be given.
(1) The game \( \text{CDS}(p^*, q^*)(\pi), (\text{SP}(\pi) \setminus A) \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n)\} \) is said to be a child of the game \( \text{CDS}(\pi, A) \).

(2) The set of children of \( \text{CDS}(\pi, A) \) is the set \( \{\text{CDS}(p^*, q^*)(\pi), (\text{SP}(\pi) \setminus A) \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n)\} : (p^*, q^*) \text{ a valid pointer context on } \pi \} \).

With the notion of a child of a game defined, an analogue of the notion of the Sprague-Grundy Number \([6, 12]\) for a combinatorial game can now be defined for games of the form \( \text{CDS}(\pi, A) \).

**Definition 5.6.** Let \( \pi \in S_n \) and a subset \( A \) of \( \text{SP}(\pi) \) be given. Define \( \text{SG}(\pi, A) \), the Sprague-Grundy number of \( \text{CDS}(\pi, A) \) as follows:

1. If \( \pi \) is a \( \text{cds} \) fixed point,
   \[
   \text{SG}(\pi, A) = \begin{cases} 
   1 & \text{if } \pi \in A \\
   0 & \text{otherwise} 
   \end{cases}
   \]

2. If \( \pi \) is not a \( \text{cds} \) fixed point,
   \[
   \text{SG}(\pi, A) = \min\{n \geq 0 : n \notin \{\text{SG}(\sigma, B) : \text{CDS}(\sigma, B) \text{ a child of } \text{CDS}(\pi, A)\}\}
   \]

**Lemma 5.7.** Let \( \pi \) be a permutation. Let \( p, q \) be \( \pi \)-compatible pointers such that the set of pointers other than \( q \) \( \pi \)-compatible with \( p \) is the same as the set of pointers other than \( p \) \( \pi \)-compatible with \( q \). Let \( r, s \) be \( \pi \)-compatible pointers. Then,

1. The set of pointers other than \( q \) that are \( \text{cds}_{r,s}(\pi) \)-compatible with \( p \) is the same as the set of pointers other than \( p \) that are \( \text{cds}_{r,s}(\pi) \)-compatible with \( q \),

2. \( p \) and \( q \) are \( \text{cds}_{r,s}(\pi) \)-compatible.

**Proof.** By Theorem 2 of \([7]\). In particular, \( p \) and \( q \) appear in exactly the same columns of the “master list” \( M(r, s) \), so a pointer \( t \) is \( \text{cds}_{r,s}(\pi) \)-compatible with \( p \) if and only if it is \( \text{cds}_{r,s}(\pi) \) compatible with \( q \). This proves (1). Moreover, \( p \) and \( q \) appear an even number of times \((0 \text{ or } 2)\) in each column of the “master list”, so they remain compatible in \( \text{cds}_{r,s}(\pi) \), proving (2).

**Definition 5.8.** Let \( P \subseteq P_n \) be a set of pointers. A permutation \( \pi \in S_n \) is \( P \)-excellent if

1. Any pair of pointers in \( P \) is \( \pi \)-compatible,
2. Pointers \( p \in P_n \setminus P \) are not \( \pi \)-compatible with any other pointers,
3. \( \text{SP}(\pi) = P \).

The \( P \)-excellent permutations are exactly those which, after reduction of adjacencies, have maximal strategic pile and a maximal number of valid pointer contexts.

**Lemma 5.9.** If \( \pi \in S_n \) is \( P \)-excellent, then \( \text{cds}_{p,q}(\pi) \) is \( P \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n)\} \)-excellent.

**Proof.** Let \( \sigma_{n,r}(n), \sigma_{n,s}(n) \in P \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n)\} \). The set of pointers other than \( r \) \( \pi \)-compatible with \( s \) is the same as the set of pointers other than \( s \) \( \pi \)-compatible with \( r \), and \( r \) is \( \pi \)-compatible with \( s \), so by part 2 of Lemma 5.7 \( r \) and \( s \) are \( \text{cds}_{p,q}(\pi) \)-compatible. Thus, condition (1) is satisfied.

Applying \( \text{cds} \) with valid pointer context \( p, q \) creates adjacencies at both \( p \) and \( q \). Moreover, any pointer \( r \) that is part of an adjacency in \( \pi \) is part of an adjacency in \( \text{cds}_{p,q}(\pi) \), so every pointer not in \( P \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n)\} \) is part of an adjacency in \( \text{cds}_{p,q}(\pi) \), proving condition 2.

Let \( r \) be a pointer not part of an adjacency in \( \text{cds}_{p,q}(\pi) \). Then \( r \) was not part of an adjacency in \( \pi \), so \( \sigma_{n,r}(n) \in \text{SP}(\pi) \). By [1], Lemma 2.15, \( \text{SP}(\text{cds}_{p,q}(\pi)) = \text{SP}(\pi) \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n)\} \), so \( \sigma_{n,r}(n) \in \text{SP}(\text{cds}_{p,q}(\pi)) \). Thus, condition (3) is satisfied.

**Remark 5.10.** By Lemma 4.9, if \( \pi \) is \( P \)-excellent, then \(|P|\) is odd.

**Theorem 5.11.** Let \( g_m(a) \) be the Grundy number of the game \( \text{CDS}(\pi, A) \), where \( \pi \in S_n \) is a permutation which is \( P \)-excellent for a set of pointers \( P \) with \(|P| = 2m - 1\) and \(|A| = a\).
Theorem 5.12

Let \( g \) be a winning strategy.

Proof. By Lemma 5.9, the children of a \( P \)-excellent permutation with \(|P| = 2m - 1 \) are \( P \)-excellent permutations with \(|P| = 2m - 3 \). We thus proceed by induction.

The base case \( m = 1 \) is clear, since \( \pi \) would already be \( P \)-excellent. Let \( m \geq 2 \). If \( a = 2m - 1 \), then \( g_{2m}(a) \) is the minimal excludant of \( \{g_{2m-2}(0)\} \). If \( a = 2m - 2 \), then \( g_{2m}(a) \) is the minimal excludant of \( \{g_{2m-2}(0), g_{2m-2}(1)\} \). If \( a = 0 \), then \( g_{2m}(a) \) is the minimal excludant of \( \{g_{2m-2}(2m - 3)\} \). If \( a = 1 \), then \( g_{2m}(a) \) is the minimal excludant of \( \{g_{2m-2}(2m - 3), g_{2m-2}(2m - 4)\} \). Otherwise, \( g_{2m}(a) \) is the minimal excludant of \( \{g_{2m-2}(2m - 3 - a), g_{2m-2}(2m - 2 - a), g_{2m-2}(2m - 1 - a)\} \).

The formula then follows by induction.

\[ g_{2m}(a) = \begin{cases} 0, & a \leq m - 2 \text{ or } a = m - 1 \text{ and } m \text{ is odd} \\ 1, & a \geq m + 1 \text{ or } a = m \text{ and } m \text{ is odd} \\ 2, & a = m \text{ or } m - 1 \text{ and } m \text{ is even} \end{cases} \]

Corollary 5.11.1. If \( \pi \) is \( P \)-excellent and \(|A| > SP(\pi) - |A|\), then ONE has a winning strategy in \( CDS(\pi, A) \).

Winning Strategies for ONE.

There exists a bound on the size of \( A \) relative to the strategic pile necessary for a player to have a winning strategy.

Theorem 5.12 (\([\pi]\), Theorem 4.3). Let \( \pi \in S_n \) and \( A \subseteq SP(\pi) \).

1. If \(|A| \geq \frac{3}{4}|SP(\pi)|\), then ONE has a winning strategy in \( CDS(\pi, A) \).
2. If \(|A| < \frac{3}{4}|SP(\pi)| - 2\), then TWO has a winning strategy in \( CDS(\pi, A) \).

Intuitively, increasing the size of \( A \) should provide an advantage to ONE. Despite this, examples given in \([\pi]\) show that if \( \epsilon > 0 \), there are permutations \( \pi \) and subsets \( A \subseteq SP(\pi) \) with \(|A| \geq (3/4 - \epsilon)|SP(\pi)|\) for which ONE does not have a winning strategy in \( CDS(\pi, A) \). However, if a certain kind of “symmetry” is imposed on a large subset of the elements of \( SP(\pi) \), the constant \( 3/4 \) can be improved to be arbitrarily close to \( 1/2 \). We formalize this as follows.

Definition 5.13 (\( P \)-good). Let \( \pi \) be a permutation and \( P \) be a set of pointers. Call \( \pi \) \( P \)-good if

1. Every pair of pointers in \( P \) is compatible.
2. For all pointers \( p, q \in P \), if \( r \) is another pointer, \( r \) and \( p \) are compatible if and only if \( r \) and \( q \) are compatible.
3. Given any pointer \( p \) not part of an adjacency, \( \sigma_{n,p}(n) \in SP(\pi) \).

Lemma 5.14. Let \( \pi \in \mathcal{M}_{2n} \) and let \( P \) be the set of universal pointers in \( \pi \). Then, \( \pi \) is \( P \)-good.

Proof. Conditions (1) and (2) are satisfied, as each pointer in \( P \) is compatible with every other pointer. Condition (3) is satisfied, since the strategic pile is maximal.

Lemma 5.15. Let \( \pi \in S_n \) be \( P \)-good. Then \( \text{cds}_{p,q}(\pi) \) is \( P \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n)\} \)-good.

Proof. Let \( r, s \in P \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n)\} \). The set of pointers other than \( s^* \) compatible with \( r^* \) in \( \pi \) is the same as the set of pointers other than \( r^* \) compatible with \( s^* \) in \( \pi \). Moreover, \( r^* \) and \( s^* \) are \( \pi \)-compatible. Thus, part 2 of Lemma 5.7 gives that \( r^* \) and \( s^* \) are \( \text{cds}_{p,q}(\pi) \)-compatible, proving condition (1). Moreover, by part 1 of Lemma 5.7, the set of pointers other than \( s^* \) \( \text{cds}_{p,q}(\pi) \)-compatible with \( r^* \) is the same as the set of pointers other than \( r^* \) \( \text{cds}_{p,q}(\pi) \)-compatible with \( s^* \), so condition (2) is satisfied.

Let \( r \) be a pointer not part of an adjacency in \( \text{cds}_{p,q}(\pi) \). Then \( r \) was not part of an adjacency in \( \pi \), so \( \sigma_{n,r}(n) \in SP(\pi) \). By \([\pi]\), Lemma 2.15, \( SP(\text{cds}_{p,q}(\pi)) = SP(\pi) \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n)\} \), so \( \sigma_{n,r}(n) \in SP(\text{cds}_{p,q}(\pi)) \). Thus, condition (3) is satisfied.

Lemma 5.16. Let \( b > 0 \) and let \( n \) be such that \( 2n - 1 > b \). If \( \pi \in \mathcal{M}_{2n} \) has at least \( \binom{2n-1}{2} - b \) valid pointer contexts, then there are at least \( 2n - 1 - b \) universal pointers.
Let \( \pi \in S_n \) be \( P \)-good, where \( |SP(\pi)| = k \). Write \( c = k - |P| \). Assume further that \( |P| > 3c \), and let \( A \subset SP(\pi) \) satisfy \( |A| \geq \left\lceil \frac{k}{2} \right\rceil + 1 + 2c \). Then, \( \text{ONE} \) has a winning strategy in \( \text{CDS}(\pi, A) \).

**Proof.** By Corollary 5.11.1 each non-universal pointer is incompatible with at least two other pointers. Since there are at most \( b/2 \) pairs of incompatible pointers, there are at most \( b \) non-universal pointers, and therefore there are at least \( 2n - 1 - b \) universal pointers.

**Theorem 5.17.** Let \( \pi \in S_n \) be \( P \)-good, where \( |SP(\pi)| = k \). Write \( c = k - |P| \). Assume further that \( |P| > 3c \), and let \( A \subset SP(\pi) \) satisfy \( |A| \geq \left\lceil \frac{k}{2} \right\rceil + 1 + 2c \). Then, \( \text{ONE} \) has a winning strategy in \( \text{CDS}(\pi, A) \).

**Proof.** If \( c = 0 \), then every pointer is either in \( P \) or part of an adjacency, so \( \pi \) is \( P \)-excellent. We have \( 2|A| > k \), so \( |A| > k - |A| \), and \( \text{ONE} \) has a winning strategy by Corollary 5.11.1.

Now, assume \( c \geq 1 \). Let \( p^* \) be a pointer that is not part of an adjacency and is also not in \( P \). Since \( \sigma_{n,p}(u) \in SP(\pi) \), the Strategic Pile Removal Theorem (II, Theorem 2.19), there is a pointer \( q^* \) such that \( p^* \) and \( q^* \) are compatible. Thus, let \( \text{ONE} \) perform \( \text{CDS} \) with valid pointer context \( p^*, q^* \).

Let \( \text{TWO} \) perform \( \text{CDS} \) with valid pointer context \( r^*, s^* \), and let \( \pi' = \text{CDS}_{r^*, s^*}(\text{CDS}_{p^*, q^*}(\pi)) \). Let \( A' = A \cap SP(\pi') \), let \( k' = |SP(\pi')| \), let \( P' = P \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n), \sigma_{n,r}(n), \sigma_{n,s}(n)\} \), and let \( c' = k' - |P'| \). It suffices to show that \( \text{ONE} \) has a winning strategy in \( \text{CDS}(\pi', A') \). By Lemma 5.15, \( \pi' \) is \( P' \)-good. Moreover, since \( p \notin P \), \( |P| - |P'| \leq 3 \). We have \( \{\sigma_{n,p}(n), \sigma_{n,q}(n), \sigma_{n,r}(n), \sigma_{n,s}(n)\} \subset SP(\pi) \), and \( SP(\pi') = SP(\pi) \setminus \{\sigma_{n,p}(n), \sigma_{n,q}(n), \sigma_{n,r}(n), \sigma_{n,s}(n)\} \) by [I], Lemma 2.15. Thus \( k - k' = 4 \). Then, \( c' = k' - |P'| \leq k - 4 - |P| + 3 = c - 1 \).

Moreover, since \( A \setminus A' = A \setminus (SP(\pi) \setminus SP(\pi')) \), we have \( |A| - |A'| \leq 4 \). Thus, \( |P'| \geq |P| - 3 > 3c - 3 \geq 3c' \), and

\[
|A'| \geq |A| - 4 \geq \left\lceil \frac{k}{2} \right\rceil + 1 + 2c - 4 \geq \left\lceil \frac{k' + 4}{2} \right\rceil + 1 + 2(c' + 1) - 4 = \left\lceil \frac{k'}{2} \right\rceil + 1 + 2c'.
\]

By induction, \( \text{ONE} \) has a winning strategy in \( \text{CDS}(\pi', A') \).

**Corollary 5.17.1.** Let \( b > 0 \). Choose \( n \) such that \( 2n - 1 - b > 3b \), and let \( \pi \in M_{2n} \) have at least \( \binom{2n - 1}{b} - b \) valid pointer contexts. If \( A \subset SP(\pi) \) satisfies \( |A| \geq n + 2b \), then \( \text{ONE} \) has a winning strategy in \( \text{CDS}(\pi, A) \).

**Proof.** Let \( P \) be the set of universal pointers of \( \pi \). By Lemma 5.14, \( \pi \) is \( P \)-good. By Lemma 5.16, \( |P| \geq 2n - 1 - b \). Let \( c = b \), and let \( k = 2n - 1 - |SP(\pi)| \). We have \( |P| \geq 2n - 1 - b > 3b = 3c \), and \( |A| \geq n + 2b = \left\lceil \frac{2n - 1}{2} \right\rceil + 1 + 2e \). Thus, by Theorem 5.17 \( \text{ONE} \) has a winning strategy.

**Corollary 5.17.2.** Let \( b > 0 \). For all \( r > 1/2 \), there exists an \( N \) such that for all \( n \geq N \), if \( \pi \in M_{2n} \) has at least \( \binom{2n - 1}{b} - b \) available valid pointer contexts and \( A \subset SP(\pi) \) has \( |A| \geq r|SP(\pi)| \), then \( \text{ONE} \) has a winning strategy in \( \text{CDS}(\pi, A) \).

**6. Future Work**

1. Compute the number of valid pointer contexts from a periodic difference sequence. Given this result, it would be possible to compute the number of permutations in \( S_{2n} \) with maximal strategic pile and a given number of valid pointer contexts \( \text{mod}(2n - 1)^2 \).
2. Classify permutations with other strategic pile sizes.
3. Prove the boundedness or exhibit the unboundedness of Grundy numbers of \( \text{CDS} \) games on permutations with maximal strategic pile.

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