Dynamical Analysis Predator-Prey Population with Holling Type II Functional Response

K Pusawidjayanti, Asmianto, V Kusumasari

Department of Mathematics, Faculty of Mathematics and sciences, Universitas Negeri Malang, Indonesia

E-mail: kridha.pusawidjayanti.fmipa@um.ac.id, asmianto.fmipa@um.ac.id, vita.kusumasari.fmipa@um.ac.id

Abstract. In this article, we investigate the dynamical analysis of predator prey model. Interaction among preys and predators use Holling type II functional response, and assuming prey refuge as well as harvesting in both populations. This study aims to study the predator prey model and to determine the effect of overharvesting which consequently will affect the ecosystem. In the model found three equilibrium points, i.e., \( E_1(0,0) \) is the extinction of predator and prey equilibrium, \( E_2(x_2,0) \) is the equilibrium with predatory populations extinct and the last equilibrium points \( E_3(x_3,y_3) \) is the coexist equilibrium. All equilibrium points are asymptotically stable (locally) under certain conditions. These analytical findings were confirmed by several numerical simulations.

1. Introduction

A model consisting of two different species with interaction behavior between them is called predator-prey model. In this research will be discussed predator prey system with functional response Holling type II, as well as refuge to the prey and harvesting in both populations. The predator prey model is a model that can be analyzed dynamically so that there is a possibility that could happen in the future. The growth rate is a logical model with carrying capacity in population growth [9]. Lotka-Volterra model is part of a modified predator prey model also uses a non-linear system equation that includes logistical growth of two species, carrying capacity prey and predator has been discussed by [10] and [6].

The interaction behavior between predators and prey in an ecosystem can cause the state of the population to change. These interactions can have a positive and negative impact, or even have no effect on the interacting species. So that what causes one species to become extinct is a very high predation rate of the prey and the low growth rate of prey [4]. Some researchers developed the Lotka-Volterra model with various assumptions, among others [12] assuming that the interactions between prey and predators using Holling type I, while [3], [5], [2], [7], [11] using Holling type II, and [4] using Holling type III and give consideration that this populations is a population that is useful for human life. In the model Holling II i.e. low prey population density, thus illustrating that the level of predation is high where the level of consumption reaches a saturation point, thus the authors provide this model with the Holling type II functional response.

Population in one ecosystem will provide many benefits for humans, from this situation there is an assumption that the population was harvested for human needs. So that [1] and [4] assumes the harvest in the prey and predator population. Refers to some of the research above then the authors assume refuge in prey, so that it is not easily preyed on by predators where refuge in prey is symbolized by \( n \) on the system. According to [8] such protection provides the prey with some degree of protection from predation and at the same time reduces the likelihood of predation due to predation.

In general, this study aims to study the predator prey model and to determine the effect of overharvesting which consequently will affect the ecosystem.
This paper is organized as follows. In part 2, we construct the model and derive its equilibrium points. In part 3, we will determine and analyze the stability at the equilibrium point. Numerical simulations by using program MATLAB R2013a with various conditions are performed in part 4. Finally, conclusions are presented in part 5.

2. Predator Prey model with functional respon holling type II, refuge prey and harvesting.

In this work, we consider a predator prey model with functional responses Holling type III and harvesting derived by [4]. If the functional responses become Holling II and provides refuge in the prey. Then the model is as follows:

\[
\begin{align*}
\frac{dx}{dt} &= r x \left(1 - \frac{x}{K}\right) - \frac{(1-n)mx y}{1+x} - a_1 Q_1 x \\
\frac{dy}{dt} &= \frac{(1-n)cx y}{1+x} - by - a_2 Q_2 y
\end{align*}
\]

Where \(x(t), y(t)\) represent the number of prey and predator population at time \(t\), respectively. Here \(r\) and \(c\) are growth rate of prey and predator, \(a_1\) and \(a_2\) are catch coefficient of prey and predator, \(Q_1\) and \(Q_2\) are harvest rate of prey and predator, \(K\) is carrying capacity, \(n\) is prey refuge rate, while \(n (0 \leq n \leq 1)\) is the indicates the prey ability to take cover, \(m\) is predation rate and \(b\) is the natural death rate of the population.

If \(r_1 = r - a_1 Q_1\) and \(r_2 = b + a_2 Q_2\) so the model (1) will be

\[
\begin{align*}
\frac{dx}{dt} &= r_1 x - \frac{r x^2}{K} - \frac{(1-n)mx y}{1+x} \\
\frac{dy}{dt} &= \frac{(1-n)cx y}{1+x} - r_2 y
\end{align*}
\]

Equilibrium points is the point of motion of the constant state vector, or it can be said that equilibrium points is a solution that remains constant even when the time change [4]. Equilibrium points from model (2) get if \(\frac{dx}{dt} = 0, \frac{dy}{dt} = 0\)

From model (2) we have equilibrium points as follow:

(i). The trivial stage or the extinction of all population equilibrium point: \(E_1 (0,0)\)

(ii). The extinction of predator equilibrium point: \(E_2 (x_2, 0)\), where \(x_2 = \frac{r_2 K}{r}\)

(iii). The coexistent equilibrium point: \(E_3 (x_3, y_3)\), where \(x_3 = \frac{r_2}{(1-n)c-r_2}\) and \(y_3 = \frac{(1+x_3)(r_1-r_2)x_3}{(1-n)m}\)

3. Stability Analysis

In this Section, we will study stability analysis from the model (2), the Jacobian matrix of model (2) at equilibrium \(E^*(x^*, y^*)\) is given by:

\[
J(E^*) = \begin{bmatrix}
    r_1 - \frac{2r x^*}{K} - \frac{(1-n)my^*}{(1+x^*)^2} & -\frac{(1-n)mx^*}{1+x^*} \\
    \frac{(1-n)cy^*}{(1+x^*)^2} & \frac{(1-n)cx^*}{1+x^*} - r_2
\end{bmatrix}
\]
An equilibrium point $E^*$ is (locally) asymptotically stable if the real parts of all eigenvalues of $J(E^*)$ are negative. The properties of stability at equilibrium points are summarized in the theorem below.

**Theorem:**

(i). The extinction of all population equilibrium point: $E_1 (0,0)$ is stable if $r_1 < 0$  

$$r_2 > \frac{(1-r)cr_1K}{r+r_1K}$$

(ii). $E_2$ is the extinction of predator equilibrium point, $E_2$ is asymptotically stable if $r_2 > \frac{(1-r)cr_1K}{r+r_1K}$

(iii). $E_3$ is the coexistent equilibrium point, $E_3$ is asymptotically stable if

$$\frac{Kr_2 + rr_2 + r_1 K c (1-n) - r_2 - K c (1-n)}{2mr^2 + r^2} > 0$$

$$\frac{rr_2mc(1-n)(c(1-n) - r_2)(r_1 c K(1-n) - Kr_1 r_2 - rr_2)}{((1-n)^2mr^2 - mr^2 + r^2)(cK(1-n) - Kr_2)} > 0$$

**Proof:**

(i). The Jacobian matrix of model (2) at the extinction of all population equilibrium point $E_1 (0,0)$ is

$$J(E_1) = \begin{bmatrix} r_1 & 0 \\ 0 & -r_2 \end{bmatrix}$$

The eigen values of $J(E_1)$ are $\lambda_1 = r_1 > 0$ and $\lambda_2 = -r_2 < 0$.

Clearly that equilibrium point $E_1$ stable if $r_1 < 0$. Remember that $r_1 = r - a_1 Q_1$ it is mean that if harvesting is done on a large scale then the population of predator and prey are extinction.

If $r_1 > 0$ then prey will still survive but predator is extinction.

(ii). At the predator extinction equilibrium point $E_2$, the Jacobian matrix is given by

$$J(E_2) = \begin{bmatrix} r_1 - \frac{(1-n)cr_1K}{r+r_1K} & 0 \\ 0 & r_2 \end{bmatrix}$$

Which its eigenvalues are $\lambda_1 = -r_1$ and $\lambda_2 = \frac{(1-n)cr_1K}{r+r_1K} - r_2$. It is clear that $\lambda_{1,2} < 0$ only if $r_2 > \frac{(1-n)cr_1K}{r+r_1K}$.

Thus, $E_2$ is stable if $r_2 > \frac{(1-n)cr_1K}{r+r_1K}$.

(iii). The Jacobian matrix at the coexistent equilibrium $E_3$ is

$$J(E_3) = \begin{bmatrix} r_1 - \frac{2rx_5 - (1-n)my_5}{K} & 0 & -(1-n)mx_5 \\ 0 & (1-n)cy_5 & 1 + x_5 \\ (1-n)x_5 & (1-n)cx_5 & 1 + x_5 \end{bmatrix}$$

The characteristic equation of $J(E_3)$ is given by $\lambda^2 + A\lambda + B = 0$,

$$A = \frac{Kr_2 + rr_2 + r_1 K c (1-n) - r_2 - K c (1-n)}{Kr_2 + rr_2 + r_1 K c (1-n) - r_2 - K c (1-n)}$$

$$B = \frac{rr_2mc(1-n)(c(1-n) - r_2)(r_1 c K(1-n) - Kr_1 r_2 - rr_2)}{((1-n)^2mr^2 - mr^2 + r^2)(cK(1-n) - Kr_2)}$$

In Accordance with the Routh-Hurwitz criterion, the equilibrium point $E_3$ is the coexistent equilibrium will be asymptotically stable if $A, B > 0$ which are equivalent to conditions (5) and (6). □
4. Numerical Simulation

To support our theoretical result, we perform some numerical simulations of system (2). We divide these simulations into three cases as follows.

(i). \( r_1 = -0.01; \ r = 0.7; \ n = 0.5; \ c = 0.3; \ m = 0.7; \ r_2 = 0.1; \ K = 2 \)

(ii). \( r_1 = 0.5; \ r = 0.7; \ n = 0.5; \ c = 0.3; \ m = 0.7; \ r_2 = 0.1; \ K = 2 \)

(iii). \( r_1 = 0.5; \ r = 0.7; \ n = 0.3; \ c = 0.5; \ m = 0.7; \ r_2 = 0.1; \ K = 2 \)

**Case 1.** In this case model (2) has one equilibrium point, i.e., \( E_1(0,0) \) and the other equilibrium points do not exist. In figure 1 we plot the solution of model (2) using four different initial values. It is shown that all solutions converge to \( E_1 \), as predicted by our theoretical result. This is the condition that all population is extinction the result of large-scale harvesting.

**Figure 1.** Numerical solution of system (2) with parameters \( r_1 = -0.01; \ r = 0.7; \ n = 0.5; \ c = 0.3; \ m = 0.7; \ r_2 = 0.1; \ K = 2 \). All solution converges to \( E_1 \). 

**Case 2.** In this case model (2) has two equilibrium points are exist, i.e., \( E_1(0,0) \) and \( E_2(1.43; 0) \), while the \( E_3 \) is not exist. In the \( E_2 \) is asymptotically stable, but in \( E_1 \) is unstable. In Figure 2, by giving four different initial values it is shown that all solution converges to \( E_2 \) this is conditioning that population of predator is extinction.
Figure 2. Numerical solution of system (2) with parameters \( r_1 = 0.5; r = 0.7; n = 0.5; c = 0.3; m = 0.7; r_2 = 0.1; K = 2 \). All solution converges to \( E_2 \).

Case 3. Finally, in this case model (2) has all equilibrium points are exist, i.e., \( E_1 (0,0), E_2 (1,43; 0) \) and \( E_3 (0.4; 1.03) \). In the \( E_1 \) and \( E_2 \) is unstable but \( E_3 \) is asymptotically stable, with four different initial value it is shown that all solution converges to \( E_3 \), showing that all population is co-exist.

Figure 3. Numerical solution system (2) with parameters \( r_1 = 0.5; r = 0.7; n = 0.3; c = 0.5; m = 0.7; r_2 = 0.1; K = 2 \). All solution converges to \( E_3 \).

5. Conclusion
In this paper, a predator-prey model with Holling type II functional response, refuge in prey and harvesting is discussed and analyzed. We have found a possible equilibrium point and have also found the stability properties at this equilibrium points. There are three equilibrium points, i.e., the extinction of prey and predator, the extinction of predator and the co-existent equilibrium point. It is found that all equilibrium points are conditionally asymptotically stable. For further research, we can add consideration of
assumptions, for example by giving a time delay to see the dynamic changes. In addition, it is necessary to add the maximum benefit from harvesting the population.

References
[1] Al-Omari J 2007 Stability and Optimal Harvesting in Lotka-Volterra Competition Model for Two-Species with Stage Structure. Kyungpook Mathematical Journal 47 31-56
[2] Banshidhar S 2012 Predator-Prey Model with Different Growth Rates and Different Functional Responses: A Comparative Study with Additional Food International Journal of Allied Mathematical Research 1 117-129
[3] Chakraborty S, Pal S, Bairaig N 2012 Predator-Prey Interaction with Harvesting: Mathematical Study with Biological ramifications Applied Mathematical Modelling 36 4044-4059
[4] Didiharyono 2016 Analisis Kestabilan dan Keuntungan Maksimum Model Predator-Prey Fungsi Respon tipe Holling III dengan Usaha Pemanenan Jurnal Masagena Vol. 11, nomor: 2
[5] Hafizul M, Sabiar R Md and Sahabuddin S 2018 Dynamics of a Predator-Prey Model with Holling Type II Functional Response Incorporating a Prey Refuge Depending On Both Species International Journal of Nonlinear Sciences and Numerical Simulation doi:10.1515/ijnsns-2017-0224
[6] Hang D, Fengde C, Zhenliang Z and Zhong L 2019 Dynamic behaviors of Lotka-Volterra Predator-Prey Model Incorporating Predator Cannibalism Advances in Difference Equations 359 https://doi.org/10.1186/s13662-019-2289-8
[7] Jean J T, Valaire Y D and Samuel B 2013 Predator-Prey model with Holling response function of Type II and SIS Infectious Disease Applied Mathematical Modelling 37 4825-4841
[8] Pusawidjayanti K, Suryanto A and Wibowo R B E 2015 Dynamics of a Predators-Prey Model Incorporating Prey Refuge, Predator Infection and Harvesting International Journal of Applied Mathematical Sciences. Vol 9 no. 76 3751-3760 HIKARI Ltd
[9] Safuan H M, Sidhu H S, Jovanoski Z and Towers I N 2014 A Two-Species Predator-Prey Model In An Environment Enriched By A Biotic Resource Anziam J 54 pp C768-C787
[10] Vahidin H, Midhat M and Jasmin B 2017 Lotka-Volterra Model with Two Predators and Their Prey TEM Journal Vol 6 Issue 1 Page 132-136 DOI: 10.18421/TEM61-19
[11] Xiao-Ke S, Hai-Feng H, and Xiao-Bing Z 2012 A Predator-Prey Model with Functional Response and Stage Structure for Prey Hindawi Publishing Corporation Abstract and Applied Analysis Vol 2012 article ID 628103 19 pages doi:10.1155/2012/628103
[12] Yong Y, Hua L, Yumei W, Kai Z, Ming M and Jianhua Y 2019 Dynamic Study of a Predator-Prey Model with Alle effect and Holling type I Functional Response Advances in Difference Equations 369 https://doi.org/10.1186/s13662-019-2311-1