New Hints from General Relativity

Olaf Dreyer

Perimeter Institute for Theoretical Physics, 35 King Street North, Waterloo, Ontario N2J 2W9, Canada

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Abstract

The search for a quantum theory of gravity has followed two parallel but different paths. One aims at arriving at the final theory starting from a priori assumptions as to its form and building it from the ground up. The other tries to infer as much as possible about the unknown theory from the existing ones and use our current knowledge to constrain the possibilities for the quantum theory of gravity.

Probably the biggest success of the second path has been the results of black hole thermodynamics. The subject of this essay is a new, highly promising such result, the application of quasinormal modes in quantum gravity.
A. Introduction

For almost sixty years, a major goal of theoretical physics has been to write an article on quantum gravity that does not start by recounting how long people have been trying with limited success. In their efforts, researchers in the field have followed two different paths. The first involves starting from basic principles and constructing a theory from the ground up. The second tries to accumulate as many hints as possible from the known and established theories, in the hope that this knowledge will constrain the possibilities for a quantum theory of gravity.

The best known examples of the first kind of approach are String Theory (see [1] for a recent introduction to the subject) and Loop Quantum Gravity (see [2] for a comprehensive overview). Both of these use conventional quantum theory. String theory is the quantum theory of string-like objects in space-time, whereas Loop Quantum Gravity is the attempt to quantize the gravitational field itself starting from the classical field equations.

By far the most important results of the second kind are Bekenstein’s and Hawking’s results on black hole thermodynamics. Starting from theorems in classical general relativity, Bekenstein [3] argued that a black hole has a temperature \( T \) which is proportional to its surface gravity \( \kappa \) and an entropy \( S \) that is proportional to its area \( A \). Using quantum field theory on curved space-time, Hawking [4] was then able to fix the proportionality constants and establish the relations

\[
T = \frac{\kappa \hbar}{2\pi} \quad (1)
\]

and

\[
S = \frac{A}{4\hbar G} \quad (2)
\]

In this way, well-established theories, namely, general relativity and quantum field theory, give us important insights into the problem of quantum gravity. The insight we should keep from this is that, given a classical black hole with area \( A \), there should be

\[
\exp\left(\frac{A}{4}\right) \quad (3)
\]

quantum mechanical states which, to a macroscopic observer, look like the given black hole. The important question then is: What are these states? It is arguably the most important success of String Theory and Loop Quantum Gravity so far that they can provide partial answers to this question.
In the next section we will discuss how Loop Quantum Gravity achieves this. In the process, we will encounter a new and surprising hint from General Relativity which provides additional insight into the problem, and which goes beyond the particular approach of Loop Quantum Gravity. We will close with a comparison of the two paths and an outlook.

B. Counting the Black Hole States... Almost

A basis for the Hilbert space of Loop Quantum Gravity is given by spin networks. These are graphs whose edges are labelled by representations of the gauge group of the theory. In the case of gravity this group is taken to be SU(2) and the representations are thus labelled by positive half-integers \( j = 0, 1/2, 1, 3/2, \ldots \). If a surface is intersected by an edge of the spin network carrying the label \( j \), the surface acquires the area

\[
A(j) = 8\pi l_p^2 \gamma \sqrt{j(j+1)},
\]

where \( l_p \) is the Planck length and \( \gamma \) is the so-called Immirzi parameter. It parameterizes an ambiguity in the choice of canonically conjugate variables that are used in the quantization. There is no a priori reason to fix this parameter to any particular value.

One can think of the area of a black hole as being a consequence of a large number of spin network edges puncturing its surface (see [8, 9, 10, 11, 12]). Each edge with spin \( j \) contributes the amount of area given by formula (4) to the black hole area. On the horizon, such a puncture also increases the dimensionality of the Hilbert space of the theory living on the boundary. Each puncture of an edge with spin \( j \) increases the dimension by a factor of \( 2j + 1 \), namely, by the dimension of the spin \( j \) representation. If there is a large number \( N \) of edges with spins \( j_i, i = 1, \ldots, N \), intersecting the horizon the dimension of the boundary Hilbert space is

\[
\prod_{i=1}^{N}(2j_i + 1).
\]

The entropy of a black hole with a given area \( A \) is simply given by the logarithm of the dimension of the Hilbert space of the boundary theory. It can be shown that the statistically most important contribution comes from those configurations in which the lowest possible spin dominates. Let us denote this spin by \( j_{\text{min}} \). The entropy then is

\[
S = N \ln(2j_{\text{min}} + 1),
\]
where \( N \) can be calculated from the area \( A \) of the black hole and from the amount of area \( A(j_{\text{min}}) \) contributed by every puncture. One obtains

\[
N = \frac{A}{8\pi l_P^2 \gamma \sqrt{j_{\text{min}}(j_{\text{min}} + 1)}}. \tag{7}
\]

We already see that the entropy of the black hole turns out to be proportional to the area, as we had hoped. The fly in the ointment is that we can not fix the constant \( \gamma \).

We will again turn to general relativity for help. It comes from the study of the so-called quasinormal modes of a black hole.

C. A New Hint: Quasinormal Modes in Quantum Gravity

The reaction of a black hole to perturbations is dominated by a discrete set of damped oscillations called quasinormal modes (see \cite{13, 14} for more details.) A remarkable property of these modes is that in the large damping limit the real part of their frequencies \( \omega \) becomes a non-zero constant. It was shown in \cite{15, 16} that this real part is equal to

\[
\frac{\ln 3}{8\pi M}. \tag{8}
\]

Thus, a particular frequency is associated to any black hole. If we had a quantum theory of black holes, we would expect that there is a transition between two black hole states that gives rise to exactly this classical frequency. In the picture of a black hole described in the previous section there is a natural candidate for such a transition. It is the appearance or disappearance of a puncture with spin \( j_{\text{min}} \). This causes a change in the area of the black hole given by equation (4):

\[
\Delta A = A(j_{\text{min}}) = \frac{8\pi l_P^2 \gamma \sqrt{j_{\text{min}}(j_{\text{min}} + 1)}}{\ln 3}. \tag{9}
\]

We can now fix the Immirzi parameter \( \gamma \) appearing in this equation simply by requiring that the change \( \Delta M \) in the mass, corresponding to this change in area, equals the energy of a quantum with the frequency of equation (8). That is, we fix \( \gamma \) by setting

\[
\Delta M = \frac{\hbar \ln 3}{8\pi M}. \tag{10}
\]

Since the area \( A \) and the mass \( M \) of a Schwarzschild black hole are related by \( A = 16\pi M^2 \) the mass change of equation (10) translates into the area change

\[
\Delta A = 4\ln 3 l_P^2. \tag{11}
\]
Comparing this with equation (9) gives the desired expression for the Immirzi parameter $\gamma$:

$$\gamma = \frac{\ln 3}{2\pi \sqrt{j_{\min}(j_{\min} + 1)}}.$$  \hspace{1cm} (12)

Using this value for the Immirzi parameter in formulae (6) and (7) gives

$$S = \frac{A}{4l_p^2} \frac{\ln(2j_{\min} + 1)}{\ln 3}.$$ \hspace{1cm} (13)

We thus see that we get an exact agreement with the Bekenstein-Hawking result if we set $j_{\min} = 1$.

D. Discussion

The classical theory of general relativity is still able to give us hints about the so far elusive quantum theory of gravity. Using the unique feature of the quasinormal mode spectrum from the classical theory, we were able to argue that a free parameter appearing in a quantum theory should be fixed to a certain value. What is even more amazing is that this gives the right value for the entropy of a black hole.

But it does not stop here. To get the right value for the black hole entropy we had to choose $j_{\min} = 1$. It was argued in [17] that this implies that the appropriate gauge group to be used in quantum gravity is $\text{SO}(3)$ instead of $\text{SU}(2)$. Another surprising insight on the quantum theory, resulting from a closer look at the classical theory.

The results mentioned in the previous section are just one example of how useful the classical theory can be in our search for a quantum theory of gravity. We wish to mention one more recent such example. Using an extended notion of symmetry, one can argue that a generic black hole spacetime possesses in the near horizon region a set of symmetry vector fields $\xi_n, n \in \mathbb{N}$, that form a $\text{Diff}(S^1)$ (see [18] and [19] for more details). When represented on phase space, the $\text{Diff}(S^1)$ acquires a central charge and we thus obtain a Virasoro algebra. Knowing the representation theory of the Virasoro algebra then gives the dimensionality of the corresponding representation space. The logarithm of this dimension turns out to be equal to the Bekenstein-Hawking entropy of a black hole.

This leads to another hint from general relativity. The symmetry vector fields found in the classical black hole space-time tell us about the space of states of the quantum theory.
Given a quantum theory of black holes we should look for representations of a Virasoro algebra in it.

One might ask: Why bother? Since we have the final theory already, string theory or loop quantum gravity, or at least we are very close to completing it, aren’t these kinds of investigations just distractions from the real work?

I would argue that this is not so. To this day, it is far from clear whether string theory or loop quantum gravity is the right quantum theory of gravity. There are plenty of open questions. Both in the way they treat black holes, and in their general status.

String theory is not yet able to deal with non-extremal black holes since all the tools so far rely heavily on supersymmetry. The loop quantum gravity treatment of black hole entropy requires the a priori assumption of a horizon surface. One should instead be able to infer the existence of such a surface from the theory itself.

On a more general level, string theory holds the promise of a beautiful unified theory but, for the moment, suffers from an embarrassment of riches. The number of possible vacua seems to be so large ([20]) that is is not clear what the predictive content of the theory is. Loop quantum gravity has shown us how to quantize a background independent theory, but has yet to convincingly argue how general relativity is recovered in the large scale limit.

The results we have been discussing in this essay are of a different nature. They are likely to be true in any theory. The example we have been looking at in some detail might be interpreted as saying that a quantum theory describing a black hole should allow for spin one excitations. This can be true even outside the framework of loop quantum gravity.

As long as we do not have the final theory, results of the kind discussed here are still very important and we can only hope that general relativity has some more hints in store for us.

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