OPEN-CLOSED GROMOV-WITTEN INVARIANTS OF 3-DIMENSIONAL CALABI-YAU SMOOTH TORIC DM STACKS

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Abstract. We study open-closed orbifold Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric Deligne-Mumford stacks (with possibly non-trivial generic stabilizers K and semi-projective coarse moduli spaces) relative to Lagrangian branes of Aganagic-Vafa type. An Aganagic-Vafa brane in this paper is a possibly ineffective $C^\infty$ orbifold which admits a presentation $[(S^1 \times \mathbb{R}^2)/G_\tau]$, where $G_\tau$ is a finite abelian group containing $K$ and $G_\tau / K \cong \mu_m$ is cyclic of some order $m \in \mathbb{Z}_{>0}$.

1. Introduction

Open Gromov-Witten (GW) invariants of toric Calabi-Yau 3-folds have been studied extensively by both mathematicians and physicists. They correspond to “A-model topological open string amplitudes” in the physics literature and can be interpreted as intersection numbers of certain moduli spaces of holomorphic mathematicians and physicists. They correspond to “A-model topological open string amplitudes” in the A-model topological string theory of a Calabi-Yau 3-fold $X$ in full generality in [60].

1.1. Open GW invariants of smooth toric Calabi-Yau 3-folds. Aganagic-Vafa [6] introduce a class of Lagrangian submanifolds in smooth semi-projective toric Calabi-Yau 3-folds, which are diffeomorphic to $S^1 \times \mathbb{R}^2$. By mirror symmetry, Aganagic-Vafa and Aganagic-Klemm-Vafa [6, 5] relate genus-zero open GW invariants (disk invariants) of a smooth toric Calabi-Yau 3-fold $X$ to orbifold Riemann surfaces to 3-dimensional Calabi-Yau smooth toric DM stack $X$ with boundaries mapped into an Aganagic-Vafa brane $L$. All genus open-closed Gromov-Witten invariants of $X$ relative to $L$ are defined by torus localization and depend on the choice of a framing $f \in \mathbb{Z}$ of $L$.

(2) We provide another definition of all genus open-closed Gromov-Witten invariants in (1) based on algebraic relative orbifold Gromov-Witten theory. This generalizes the definition in [57] for smooth toric Calabi-Yau 3-folds and specifies an orientation on moduli of maps in (1) compatible with the canonical orientation determined by the complex structure on moduli of relative stable maps.

(3) When $X$ is a toric Calabi-Yau 3-orbifold (i.e. when the generic stabilizer $K$ is trivial), so that $G_\tau = \mu_m$, we define generating functions $F_{g,h}^{X,L,f}$ of open-closed Gromov-Witten invariants or arbitrary genus $g$ and number $h$ of boundary circles; it takes values in $H^*_{CR}(B\mu_m; C)^{\otimes h}$ where $H^*_ {CR}(B\mu_m; C) \cong \mathbb{C}^m$ is the Chen-Ruan orbifold cohomology of the classifying space $B\mu_m$ of $\mu_m$.

(4) We prove an open mirror theorem which relates the generating function $F_{0,1}^{X,L,f}$ of orbifold disk invariants to Abel-Jacobi maps of the mirror curve of $X$. This generalizes a conjecture by Aganagic-Vafa [3, 4] and Aganagic-Klemm-Vafa [5] (proved in full generality by the first and the second authors in [63]) on the disk potential of a smooth semi-projective toric Calabi-Yau 3-fold.

By the large-$N$ duality, Aganagic-Klemm-Mariño-Vafa propose the topological vertex [4], an algorithm of computing all genera generating functions $F_{\beta_1,\mu_1,\ldots,\mu_h}^{X,L}$ of open GW invariants of $(X,L)$ obtained by fixing a topological type of the map (determined by the degree $\beta' \in H_2(X,L;\mathbb{Z})$ and winding numbers $\mu_1, \ldots, \mu_h \in H_1(L;\mathbb{Z}) = \mathbb{Z}$) and summing over the genus of the domain. The algorithm of the topological vertex is proved in full generality in [60].
Bouchard-Klemm-Mariño-Pasquetti propose the Remodeling Conjecture [8], an algorithm of constructing the B-model topological open string amplitudes in all genera of \( \hat{X} \) following [58], using Eynard-Orantin’s topological recursion from the theory of matrix models [30]. Combined with the mirror symmetry prediction, this gives an algorithm of computing generating functions \( F_{g,h} \) of open GW invariants of \( (X,L) \) obtained by fixing a topological type of the domain (determined by the genus \( g \) and number \( h \) of boundary circles) and summing over the topological types of the map. Eynard-Orantin study the Remodeling Conjecture for any smooth symplectic toric Calabi-Yau threefolds [31].

1.2. Open GW invariants for 3-dimensional Calabi-Yau smooth DM stacks. There have been attempts to generalize some of the above results to 3-dimensional Calabi-Yau smooth toric Deligne-Mumford (DM) stacks. The closed GW theory of orbifolds has been studied for a long time. The physical literature dates back to early 1990s such as [17] [24], which study the quantum cohomology ring of orbifolds. The mathematical definition is given by Chen-Ruan [22] for symplectic orbifolds and by Abramovich-Graber-Vistoli [2] [3] for smooth DM stacks. Toric varieties are defined by a fan, while smooth toric DM stacks are defined by a stacky fan [7]. The coarse moduli of a smooth toric DM stack \( \mathcal{X} \) is a toric variety \( X_{\Sigma} \) defined by a simplicial fan \( \Sigma \). A toric orbifold is a smooth toric DM stack with trivial generic stabilizer. Any smooth toric DM stack \( \mathcal{X} \) is a \( K \)-gerbe over its rigidification \( \mathcal{X}^{rig} \), where \( K \) is the generic stabilizer (which is a finite abelian group) and \( \mathcal{X}^{rig} \) is a toric orbifold.

The definition of Aganagic-Vafa branes can be extended to the setting of 3-dimensional Calabi-Yau smooth toric DM stacks with semi-projective coarse moduli spaces. These branes are diffeomorphic to \( ([S^1 \times \mathbb{R}^2]/G_{\tau}) \) where \( G_{\tau} \) is a finite abelian group containing the generic stabilizer \( K \). The open GW invariants of 3-dimensional Calabi-Yau smooth toric DM stacks are defined via localization [60], generalizing the methods in [19]. By localization, open and closed GW invariants of a smooth toric Calabi-Yau 3-fold can be obtained by gluing the GW vertex, a generating function of open GW invariants of \( \mathcal{X} \), which can be reduced to a generating function of certain cubic Hodge integrals [20]. Similarly, open and closed orbifold GW invariants of a 3-dimensional Calabi-Yau smooth toric DM stack can be obtained by gluing the orbifold GW vertex, a generating function of open GW invariants of \( [C^3/G] \) (where \( G \) is a finite abelian subgroup of \( SL(3, \mathbb{C}) \)), which can be reduced to a generating function of certain cubic abelian Hurwitz-Hodge integrals [69]. The GW vertex has been evaluated in the general case [57] [60]. The orbifold GW vertex has been evaluated for \( [C^2/Z_n] \times \mathbb{C} \) where \( C^2/Z_n \) is the \( A_{n-1} \) surface singularity [77] [68] [69] [67], but not in the general case.

As for mirror symmetry, a mirror theorem for disk invariants of \( [C^3/Z_4] \) is proved in [11]. The Remodeling Conjecture is also expected to predict higher genus open GW invariants of toric Calabi-Yau 3-orbifolds via mirror symmetry [8] [9].

1.3. Summary of results. In this paper we study open-closed orbifold GW invariants of a 3-dimensional Calabi-Yau smooth toric DM stack \( \mathcal{X} \) relative to an Aganagic-Vafa A-brane \( \mathcal{L} \) at all genera.

Open GW invariants of the pair \( (\mathcal{X}, \mathcal{L}) \) count holomorphic maps from orbicurves to \( \mathcal{X} \) with boundaries mapped to \( \mathcal{L} \). Open-closed orbifold GW invariants of the pair \( (\mathcal{X}, \mathcal{L}) \) depend on the following discrete data:

(i) topological type \( (g,h) \) of the domain orbicurve \( (\Sigma, \partial \Sigma) \), where \( g \) is the genus and \( h \) is the number of boundary holes;

(ii) number of interior marked points \( n \);

(iii) topological type of the map \( u : (\Sigma, \partial \Sigma) \rightarrow (\mathcal{X}, \mathcal{L}) \) given by the degree \( \beta' = u_*[\Sigma] \in H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z}) \) and each \( [u_*(R_i)] \) is element of \( H_1(\mathcal{L}; \mathbb{Z}) \cong \mathbb{Z} \times G_{\tau} \), collectively denoted by \( \vec{\mu} = ([u_*(R_1)], \ldots, [u_*(R_h)]) \); and

(iv) framing \( f \in \mathbb{Z} \) of the Aganagic-Vafa A-brane \( \mathcal{L} \).

Let \( \mathcal{S}_{(g,h,n)}(\mathcal{X}) \) be the moduli space parametrizing holomorphic maps with discrete data (i)-(iii). We use the evaluation maps \( \text{ev}_i, \ i = 1, \ldots, n \) at interior points to pull back classes in the orbifold Chen-Ruan cohomology \( H^*_{CR}(\mathcal{X}) \) of \( \mathcal{X} \) to obtain open-closed GW invariants. More precisely, \( \mathcal{L} \) intersects a unique 1-dimensional orbit \( a_{\tau} \cong \mathbb{C}^* \times BG_{\tau} \). Given \( \gamma_1, \ldots, \gamma_n \in H^*_CR(\mathcal{X}; \mathbb{Q}) \), we define open-closed orbifold GW invariant \( \langle \gamma_1, \ldots, \gamma_n \rangle_{(g,h,n)}^{(\mathcal{X},\mathcal{L},f)} \) via localization using a circle action determined by the framing \( f \in \mathbb{Z} \); this is a rational number depending on \( f \) and can be viewed as an equivariant invariant. We also provide another definition based on algebraic relative Gromov-Witten theory, which generalizes the definition in [57] for smooth toric Calabi-Yau 3-folds.
When $X$ is a symplectic toric Calabi-Yau 3-orbifolds (i.e. when the generic stabilizer $K$ is trivial), $G_r \cong \mu_m$ is cyclic. In this case, for each topological type $(g, h)$ of the domain bordered Riemann surface we define a generating function $F_{g, h}^{X, (\mathcal{L}, f)}$ of open-closed GW invariants which takes value in $H^*_\text{CR}(BG; \mathbb{C})^\otimes h$, where $H^*_\text{CR}(BG; \mathbb{C}) = \oplus_{\lambda \in \mathbb{G}} \mathbb{C}\lambda$.

In particular, the disk potential $F_{0, 1}^{X, (\mathcal{L}, f)}$ takes values in $H^*_\text{CR}(BG; \mathbb{C})$. When $\mathcal{L}$ is an outer brand$^1$, the A-model disk potential is

$$F_{0, 1}^{X, (\mathcal{L}, f)}(\tau_2, X) = \sum_{\beta, n \geq 0} \sum_{(\mu, \lambda) \in H_1(\mathcal{L}, \mathbb{Z}) \otimes \mathbb{Z} \times G_r} \frac{\langle (\tau_2)^{\ast \lambda} X_{\mu, \lambda}(\beta, (\mu, \lambda)) \rangle}{n!} \cdot X^\mu (\xi_0)^\lambda \mathbf{1}_{\lambda^{-1}}$$

where $\tau_2$ is certain equivariant second Chen-Ruan cohomology class of $X$, $\xi_0$ is an $m$-th root of $-1$, and $\lambda \in \{0, 1, \ldots, m-1\}$ corresponds to $\lambda \in G$ under a group isomorphism $G_r \cong \mathbb{Z}/m\mathbb{Z}$. The precise definition of $\tau_2$ and $F_{0, 1}^{X, (\mathcal{L}, f)}$ will be given in Section 3.13. (Throughout the paper, $\beta'$ denotes a relative homology class in $H_2(X, \mathbb{Z})$ whereas $\beta$ denotes an absolute homology class in $H_2(X, \mathbb{Z})$.)

In this paper, we prove a mirror theorem regarding the disk potential $F_{0, 1}^{X, (\mathcal{L}, f)}$ when $X$ is a semi-projective toric Calabi-Yau 3-orbifold. Mirror symmetry relates the A-model topological string theory on a Calabi-Yau 3-fold to the B-model topological string theory on the mirror Calabi-Yau 3-fold. The mirror of a semi-projective toric Calabi-Yau 3-fold is a Calabi-Yau hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^2$ defined by an equation $uv = H(x, y, q)$, where $(u, v) \in \mathbb{C}^2$, $(x, y) \in (\mathbb{C}^*)^2$, and $q$ is the complex moduli parametrizing the B-model. The function $H(x, y, q)$ is determined by both the combinatorial toric data of $X$ and the framed brane $(\mathcal{L}, f)$. The affine curve $C_q = \{ H(x, y, q) = 0 \}$ in $(\mathbb{C}^*)^2$ is called the mirror curve. We can fix a labeling of the $m$ points with $x = 0$ on the mirror curve by the elements in $G_r \cong \text{Hom}(G, \mathbb{C}^*) \cong \mathbb{Z}/m\mathbb{Z}$. For each $\eta \in G_r$, there is a small open neighborhood $U^\circ_\eta$ of the compactified mirror curve of the $x = 0$ point labelled by $\eta$, and a branch $(\log y)_{U^\circ_\eta}$ of $\log y$ defined on $U^\circ_\eta$, where $y = y(x, q)$ is defined implicitly by the equation $H(x, y, q) = 0$. When $\mathcal{L}$ is an outer brane, the closure of the 1-dimensional orbit intersecting $\mathcal{L}$ contains a unique torus fixed (stacky) point $p_\sigma = BG_\sigma$, where $G_\sigma$ is the inertia group of $p_\sigma$. With the above convention, we state our open mirror theorem as follows.

**Theorem 1.1.** Under the closed mirror map $\tau_2 = \tau_2(q)$ and the open mirror map $X = xe^{A(q)}$ (the explicit formula of $\tau_2(q)$ and $A(q)$) will be given in Section 7.

$$x \frac{\partial}{\partial x} \left( \sum_{\eta \in \mu_m} (\log y)_{U^\circ_\eta} (q, x) \phi_\eta \right) = \frac{\partial^2}{\partial x^2} F_{0, 1}^{X, (\mathcal{L}, f)}(\tau_2, X)$$

where $\{\phi_\eta\}_{\eta \in G}$ is the canonical basis of $H^*_\text{CR}(BG_r; \mathbb{C})$.

**Remark 1.2.** The definition of the disk function $F_{0, 1}^{X, (\mathcal{L}, f)}$ and the formulation of the above Theorem 1.1 are slightly different from those in the first version of this paper in 2012 [34], but the above Theorem 1.1 implies [24 Theorem 1.1], which is used to prove an open version of Ruan’s Crepant Resolution Conjecture for disk invariants of toric Calabi-Yau 3-orbifold relative to an effective outer A-hategalagic-Vafa brane [30].

1.4. Similar results for compact Lagrangian tori. There are other open GW invariants relative to different types of Lagrangian submanifolds. C.-H. Cho [14] and J. Solomon [70] define disk invariants of a compact symplectic manifold in real dimension four and six relative to a Lagrangian submanifold which is the fixed locus of an anti-symplectic involution. The mirror theorem for disk invariants for the quintic 3-fold relative to the real quintic is conjectured in [72] and proved in [33]. It has been generalized to compact Calabi-Yau 3-folds which are projective complete intersections [64], where a mirror theorem for genus one open GW invariants (annulus invariants) is also proved.

Open orbifold GW invariants of compact toric orbifolds with respect to Lagrangian torus fibers of the moment map are defined in [24], which generalizes the work of [35] on compact toric manifolds. The third author and collaborators prove mirror theorems on disk invariants in this context [20, 18]. The third author and collaborators also prove mirror theorems on disk invariants of toric Calabi-Yau manifolds/orbifolds (which must be non-compact) with respect to Lagrangian torus fibers of the Gross fibration [21, 19].

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$^1$We work with both inner and outer branes. See Section 3.3 for the definition.
1.5. Applications. The main theorem (Theorem 1.1) of this paper has several applications. Here we mention two important applications.

- As mentioned in Remark 1.2 above, Theorem 1.1 has been applied to prove an open version of Ruan’s Crepant Resolution Conjecture for disk invariants of a toric Calabi-Yau 3-orbifold relative to an effective outer Aganagic-Vafa brane [50]. Using Theorem 1.1, S. Yu [73] proves an open version of Crepant Transformation Conjecture (and in particular Crepant Resolution Conjecture) for disk invariants of a semi-projective toric Calabi-Yau 3-orbifold relative to a general (effective or ineffective, inner or outer) Aganagic-Vafa brane defined in Section 3.3 of this paper. This generalizes Open Crepant Resolution Conjecture (OCRC) for disk invariants of $[C^2/Z_n] \times C$ relative to possibly ineffective Aganagic-Vafa branes proved in [12].

- Recently, the first two authors and Zong prove the BKMP Remodeling Conjecture for all semi-projective toric Calabi-Yau 3-orbifolds [36]. Theorem 1.1 is one of the key ingredients of this proof.

1.6. Overview of the paper. The rest of the paper is organized as follows. In Section 2 we review the necessary materials concerning smooth toric DM stacks. In Section 3 we apply localization to relate open-closed GW invariants and descendant GW invariants of 3-dimensional smooth Calabi-Yau toric DM stacks. In Section 4 we prove a mirror theorem for orbifold disk invariants.

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2. Smooth Toric DM Stacks

In this section, we follow the definitions in [47, Section 3.1], with slightly different notation. We work over $\mathbb{C}$.

2.1. Definition. Let $N$ be a finitely generated abelian group, and let $N_\mathbb{R} = N \otimes \mathbb{R}$. We have a short exact sequence of (additive) abelian groups:

$$0 \to N_{\text{tor}} \to N \to \tilde{N} = N/N_{\text{tor}} \to 0,$$

where $N_{\text{tor}}$ is the subgroup of torsion elements in $N$. Then $N_{\text{tor}}$ is a finite abelian group, and $\tilde{N} = \mathbb{Z}^n$, where $n = \dim \mathbb{R}N$. The natural projection $N \to \tilde{N}$ is denoted $b \mapsto \tilde{b}$. A smooth toric DM stack is an extension of toric varieties [39, 7]. A smooth toric DM stack is given by the following data:

- $b_1, \ldots, b_{r'} \in N$ which generate a subgroup of $N$ of finite index, and
- a simplicial fan $\Sigma$ in $N_{\mathbb{R}}$ such that the set of 1-cones is $\{\rho_1, \ldots, \rho_{r'}\}$,

where $\rho_i = \mathbb{R}_{\geq 0} \tilde{b}_i$, $i = 1, \ldots, r'$. The datum $\Sigma = (\Sigma, (b_1, \ldots, b_{r'}))$ is a stacky fan in the sense of [7]. The vectors $b_1, \ldots, b_{r'}$ may or may not generate $N$; if they do not, we choose additional vectors $b_{r'+1}, \ldots, b_r$ such that $b_1, \ldots, b_r$ generate $N$. There is a surjective group homomorphism

$$\phi: \tilde{N} := \bigoplus_{i=1}^{r'} \mathbb{Z}\tilde{b}_i \longrightarrow N,$$

$$\begin{array}{c}
\tilde{b}_i \\
\mapsto \\
b_i
\end{array}$$

Define $L := \ker(\phi) \cong \mathbb{Z}^k$, where $k := r - n$. Then we have the following short exact sequence of finitely generated abelian groups:

$$0 \to L \overset{\psi}{\longrightarrow} \tilde{N} \overset{\phi}{\longrightarrow} N \to 0.$$

Applying $- \otimes \mathbb{C}^*$ to (1), we obtain an exact sequence of abelian groups:

$$1 \to K \to G \to \mathbb{T} \to \mathbb{C} \to 1,$$
where
\[
T := N \otimes_{\mathbb{Z}} \mathbb{C}^* = \tilde{N} \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n,
\]
\[
\tilde{T} := \tilde{N} \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^r,
\]
\[
G := L \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^k,
\]
\[
K := \text{Tor}_1^T(N, \mathbb{C}^*) \cong N_{\text{tor}}.
\]

The action of \( \tilde{T} \) on itself extends to a \( \tilde{T} \)-action on \( \mathbb{C}^r = \text{Spec}\mathbb{C}[Z_1, \ldots, Z_r] \). The torus \( G \) acts on \( \mathbb{C}^r \) via the group homomorphism \( G \to \tilde{T} \) in (2), so \( K \subset G \) acts on \( \mathbb{C}^r \) trivially. The isomorphism \( K \cong N_{\text{tor}} \) is not canonical.

With the above preparation, we are now ready to define a smooth toric DM stack \( \mathcal{X} \). Let
\[
\mathcal{A} = \{ I \subset \{ 1, \ldots, r \} : \sum_{i \notin I} \mathbb{R}_{\geq 0} b_i \text{ is a cone of } \Sigma \}
\]
be the set of anti-cones; note that \( \{ r' + 1, \ldots, r \} \subset I \) for any anti-cone \( I \subset \mathcal{A} \). Given \( I \subset \{ 1, \ldots, r \} \), let \( \mathbb{C}^I \) be the subvariety of \( \mathbb{C}^r \) defined by the ideal in \( \mathbb{C}[Z_1, \ldots, Z_r] \) generated by \( \{ Z_i \mid i \notin I \} \). Define the smooth toric DM stack \( \mathcal{X} \) as the quotient stack
\[
\mathcal{X} := [U_\mathcal{A}/G],
\]
where
\[
U_\mathcal{A} := \mathbb{C}^r \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I = \bigcap_{I \notin \mathcal{A}} (\mathbb{C}^r \setminus \mathbb{C}^I).
\]

Note that for \( i = r' + 1, \ldots, r \), \( \mathbb{R}_{\geq 0} b_i \) is not a cone in \( \Sigma \), so \( \{ i \} : \{ 1, \ldots, r \} \setminus \{ i \} \notin \mathcal{A} \). Therefore,
\[
U_\mathcal{A} \subset \bigcap_{i = r'+1}^r \left( \mathbb{C}^r \setminus \mathbb{C}^{(i)'} \right) = \mathbb{C}^{r'} \times (\mathbb{C}^*)^{r-r'}.
\]

The stack \( \mathcal{X} \) contains the DM torus \( T := [\tilde{T}/G] \) as a dense open subset, and the \( \tilde{T} \)-action on \( U_\mathcal{A} \) descends to a \( T \)-action on \( \mathcal{X} \). The smooth toric DM stack \( \mathcal{X} \) is a toric orbifold if the \( G \)-action on \( \tilde{T} \) is free.

Let \( G_{\text{rig}} = G/K \). Then \( G_{\text{rig}} \) acts freely on \( \tilde{T} \) and \( \tilde{T}/G_{\text{rig}} = \mathbb{T} \). The rigidification of the smooth toric DM stack \( \mathcal{X} \) is the toric orbifold
\[
\mathcal{X}_{\text{rig}} = [U_\mathcal{A}/G_{\text{rig}}].
\]

The coarse moduli space of the stack \( \mathcal{X} \) is the simplicial toric variety \( X_\Sigma \) defined by the simplicial fan \( \Sigma \) in \( N_\mathbb{R} \cong \mathbb{R}^n \). By [37, Theorem 1], the morphism \( \mathcal{X} \to X_\Sigma \) factors canonically via toric morphisms
\[
\mathcal{X} \to \mathcal{X}_{\text{rig}} \to \mathcal{X}_{\text{can}} \to X_\Sigma
\]
where
\[
\begin{align*}
\mathcal{X} \to \mathcal{X}_{\text{rig}} & \text{ is a } K\text{-gerbe over } \mathcal{X}_{\text{rig}}; \\
\mathcal{X}_{\text{rig}} \to \mathcal{X}_{\text{can}} & \text{ is a fibered product of roots of toric divisors; } \\
\mathcal{X}_{\text{can}} \to X_\Sigma & \text{ is the minimal orbifold having } X_\Sigma \text{ as coarse moduli space.}
\end{align*}
\]

Restricting (3) to the open substack \( T \subset \mathcal{X} \), one obtains \( T \cong T \times BK \to T \to T \to T \), where \( T \times BK \to T \) is the projection to the first factor, and \( T \to T \) is the identity map.

**Remark 2.1.** The purpose of introducing additional vectors \( b_{r+1}, \ldots, b_r \) is to ensure \( G \) is connected. The stacky fan \( \Sigma \) together with the extra vectors \( b_{r+1}, \ldots, b_r \) is an extended stacky fan in the sense of Jiang [48]. It follows from the definition that \( \{ r' + 1, \ldots, r \} \subset I \) for any \( I \in \mathcal{A} \).

Let \( M, \tilde{M}, \) and \( L^\vee \) be the character lattices of the tori \( T, \tilde{T}, \) and \( G \), respectively:
\[
\begin{align*}
M &= \text{Hom}(N, \mathbb{Z}) = \text{Hom}(T, \mathbb{C}^*), \\
\tilde{M} &= \text{Hom}(\tilde{N}, \mathbb{Z}) = \text{Hom}(\tilde{T}, \mathbb{C}^*), \\
L^\vee &= \text{Hom}(L, \mathbb{Z}) = \text{Hom}(G, \mathbb{C}^*).
\end{align*}
\]

Applying \( \text{Hom}(-, \mathbb{Z}) \) to (1), we obtain the following exact sequence of (additive) abelian groups:
\[
0 \to M \overset{\phi^\vee}{\longrightarrow} \tilde{M} \overset{\psi^\vee}{\longrightarrow} L^\vee \to \text{Ext}^1(N, \mathbb{Z}) \to 0
\]
Therefore, the group homomorphism \( \psi^\vee : \tilde{M} \to L^\vee \) is surjective if and only if \( N_{\text{tor}} = 0 \).

We now consider a class of examples of 3-dimensional Calabi-Yau smooth toric DM stacks of the form \([\mathbb{C}^3/\mathbb{Z}_3]\). Let \( \omega = e^{2\pi i \sqrt{-1}/3} \) be the generator of \( \mathbb{Z}_3 \). Given \( i, j, k \in \{0, 1, 2\} \) such that \( i + j + k \in 3\mathbb{Z} \), we define \( \mathcal{X}_{i,j,k} \) to be the quotient stack of the following \( \mathbb{Z}_3 \)-action on \( \mathbb{C}^3 \):

\[
\omega \cdot (Z_1, Z_2, Z_3) = (\omega^i Z_1, \omega^j Z_2, \omega^k Z_3).
\]

In the following example, we consider \( \mathcal{X}_{1,1,1}, \mathcal{X}_{1,2,0} = [\mathbb{C}^2/\mathbb{Z}_3] \times \mathbb{C}, \mathcal{X}_{0,0,0} = \mathbb{C}^3 \times B\mathbb{Z}_3 \).

**Example 2.2.**

1. \( \mathcal{X} = \mathcal{X}_{1,1,1} \). The toric data are given as follows.

\[
N = \mathbb{Z}^3, \quad N_{\text{tor}} = 0;
\]

\[
b_1 = (1, 0, 1), b_2 = (0, 1, 1), b_3 = (-1, -1, 1), b_4 = (0, 0, 1);
\]

\[
r = 4, r' = 3, k = 1;
\]

\( \Sigma = \{ \) the 3-cone spanned by \( \{b_1, b_2, b_3\}, \) and its faces, and faces of faces, etc.\( \} \);

\( \mathcal{A} = \{ I \subset \{ 1, 2, 3, 4 \} : 4 \in I \}; \)

\( \mathbb{L} \cong \mathbb{Z}, \quad \mathbb{L}^\vee \cong \mathbb{Z}; \)

![Figure 1. \( \mathcal{X}_{1,1,1} \) and its crepant resolution \( O_{\mathbb{P}^2}(-3) \)]

2. \( \mathcal{X} = \mathcal{X}_{1,2,0} \), transversal \( A_2 \)-singularity. The toric data are given as follows.

\[
N = \mathbb{Z}^3, \quad N_{\text{tor}} = 0;
\]

\[
b_1 = (1, 0, 1), b_2 = (0, 3, 1), b_3 = (0, 0, 1), b_4 = (0, 1, 1), b_5 = (0, 2, 1);
\]

\[
r = 5, r' = 3, k = 2;
\]

\( \Sigma = \{ \) the 3-cone spanned by \( \{b_1, b_2, b_3\}, \) and its faces, and faces of faces, etc.\( \} \);

\( \mathcal{A} = \{ I \subset \{ 1, 2, 3, 4, 5 \} : \{ 4, 5 \} \subset I \}; \)

\( \mathbb{L} \cong \mathbb{Z}^2, \quad \mathbb{L}^\vee \cong \mathbb{Z}^2. \)

![Figure 2. \( \mathcal{X}_{1,2,0} \) and its (partial) crepant resolutions]
(3) \( X = X_{0,0,0} \). The toric data is given as follows.
\[
N = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \quad N_{\text{tor}} = \mathbb{Z}_3;
\]
\[
b_1 = (1, 0, 0, 0), b_2 = (0, 1, 0, 0), b_3 = (0, 0, 1, 0), b_4 = (1, 0, 0, 1);
\]
\[
r = 4, r' = 3, k = 1;
\]
\[
\Sigma = \{ \text{the 3-cone spanned by } \{ b_1, b_2, b_3 \}, \text{and its faces, and faces of faces, etc.} \};
\]
\[
\mathcal{A} = \{ I \subset \{ 1, 2, 3, 4 \} : 4 \in I \};
\]
\[
\mathbb{L} \cong \mathbb{Z}, \quad \mathbb{L}^{r'} \cong \mathbb{Z}.
\]

2.2. Equivariant line bundles and torus-invariant Cartier divisors. A character \( \chi \in \hat{M} \) gives a \( \mathbb{T} \)-action on \( \mathbb{C}^r \times \mathbb{C} \) by
\[
(\tilde{t}_1, \ldots, \tilde{t}_r) \cdot (Z_1, \ldots, Z_r, u) = (\tilde{t}_1 Z_1, \ldots, \tilde{t}_r Z_r, \chi(\tilde{t}_1, \ldots, \tilde{t}_r) u),
\]
where
\[
(\tilde{t}_1, \ldots, \tilde{t}_r) \in \hat{\mathbb{T}} \cong (\mathbb{C}^*)^r, \quad (Z_1, \ldots, Z_r) \in \mathbb{C}^r, \quad u \in \mathbb{C}.
\]
Therefore \( \mathbb{C}^r \times \mathbb{C} \) can be viewed as the total space of a \( \hat{\mathbb{T}} \)-equivariant line bundle \( \tilde{L}_X \) over \( \mathbb{C}^r \). If
\[
\chi(\tilde{t}_1, \ldots, \tilde{t}_r) = \prod_{i=1}^r \tilde{t}_i^{c_i},
\]
where \( c_1, \ldots, c_r \in \mathbb{Z} \), then
\[
\tilde{L}_X = \mathcal{O}_{\mathbb{T}}(\sum_{i=1}^r c_i \tilde{D}_i),
\]
where \( \tilde{D}_i \) is the \( \hat{\mathbb{T}} \)-divisor in \( \mathbb{C}^r \) defined by \( Z_i = 0 \). We have
\[
\hat{M} \cong \text{Pic}_\mathbb{C}(\mathbb{C}^r) \cong H^2_\mathbb{T}(\mathbb{C}^r; \mathbb{Z}),
\]
where the first isomorphism is given by \( \chi \mapsto \tilde{L}_X \) and the second isomorphism is given by the \( \hat{\mathbb{T}} \)-equivariant first Chern class \((c_1)_{\hat{\mathbb{T}}}\). Define
\[
D^T_i := (c_1)_{\hat{\mathbb{T}}} \mathcal{O}_{\mathbb{T}}(\tilde{D}_i) \in H^2_\mathbb{T}(\mathbb{C}^r; \mathbb{Z}) \cong H^2_\mathbb{T}(\mathbb{C}^r/G; \mathbb{Z}).
\]
Then \( \{ D^T_1, \ldots, D^T_r \} \) is a \( \mathbb{Z} \)-basis of \( H^2_\mathbb{T}(\mathbb{C}^r; \mathbb{Z}) \cong \hat{M} \) dual to the \( \mathbb{Z} \)-basis \( \{ \tilde{b}_1, \ldots, \tilde{b}_r \} \) of \( \hat{N} \). We have a commutative diagram
\[
\begin{array}{ccc}
\text{Pic}_\mathbb{C}(\mathbb{C}^r) & \xrightarrow{\iota_{\mathbb{T}}} & \text{Pic}_\mathbb{T}(\mathcal{U}_\mathcal{A}) \cong \text{Pic}_\mathbb{T}(X) \\
(c_1)_{\hat{\mathbb{T}}} \downarrow & & (c_1)_{\mathbb{T}} \downarrow \\
H^2_\mathbb{T}(\mathbb{C}^r; \mathbb{Z}) & \xrightarrow{\iota_{\mathbb{T}}} & H^2_\mathbb{T}(\mathcal{U}_\mathcal{A}; \mathbb{Z}) \cong H^2_\mathbb{T}(X; \mathbb{Z}),
\end{array}
\]
where \( \iota_{\mathbb{T}} \) is a surjective group homomorphism induced by the inclusion \( \iota : \mathcal{U}_\mathcal{A} \hookrightarrow \mathbb{C}^r \), and
\[
\text{Ker}(\iota_{\mathbb{T}}) = \bigoplus_{i=r'+1}^r \mathbb{Z}D^T_i.
\]
Therefore,
\[
\text{Pic}_\mathbb{T}(X) \cong H^2_\mathbb{T}(X; \mathbb{Z}) \cong \hat{M} / \bigoplus_{i=r'+1}^r \mathbb{Z}D^T_i
\]
Let \( \tilde{D}^T_i := \iota_{\mathbb{T}} D^T_i \). Then
\[
\tilde{D}^T_i = 0, \quad i = r'+1, \ldots, r,
\]
and
\[
H^2_\mathbb{T}(X; \mathbb{Z}) = \bigoplus_{i=1}^{r'} \mathbb{Z}\tilde{D}^T_i \cong \mathbb{Z}^{r'}.
\]
For \( i = 1, \ldots, r' \), \( \tilde{D}_i \cap \mathcal{U}_\mathcal{A} \) is a \( \mathbb{T} \)-divisor in \( \mathcal{U}_\mathcal{A} \), and it descends to a \( \mathbb{T} \)-divisor \( D_i \) in \( X \). We have
\[
\tilde{D}^T_i = (c_1)_{\mathbb{T}}(\mathcal{O}_X(\tilde{D}_i)), \quad i = 1, \ldots, r'.
\]
For \( i = r' + 1, \ldots, r \), \( \bar{D}_i \cap U_A \) is empty, so it is the zero \( \bar{T} \)-divisor.

2.3. Line bundles and Cartier divisors. We have group isomorphisms

\[ L^\vee \cong \text{Pic}_G(C^r) \cong H^2_G(C^r; \mathbb{Z}), \]

where the first isomorphism is given by \( \chi \in L^\vee = \text{Hom}(G, C^r) \to \bar{L}_\chi \), and the second isomorphism is given by the \( G \)-equivariant first Chern class \((c_1)_G\). We have a commutative diagram

\[
\begin{array}{ccc}
\text{Pic}_G(C^r) & \xrightarrow{\iota^*} & \text{Pic}_G(U_A) \\
(c_1)_G & \downarrow & (c_1)_G \\
H^2_G(C^r; \mathbb{Z}) & \xrightarrow{\iota^*} & H^2_G(U_A; \mathbb{Z}) \\
\end{array}
\]

where \( \iota^* \) is a surjective group homomorphism induced by the inclusion \( \iota : U_A \hookrightarrow C^r \). The surjective map \( H^2_G(C^r; \mathbb{Z}) \to H^2(\mathcal{X}; \mathbb{Z}) \) is the restriction of the Kirwan map

\[ \kappa : H^2_G(C^r; \mathbb{Z}) \to H^2(\mathcal{X}; \mathbb{Z}). \]

Define

\[ D_i := (c_1)_G(O_{C^r}((\bar{D}_i))) \in H^2_G(C^r; \mathbb{Z}) \cong H^2([C^r/G]; \mathbb{Z}). \]

Then

\[ \text{Ker}(\iota^*) = \bigoplus_{i=r' + 1}^r \mathbb{Z}D_i. \]

Therefore,

\[ \text{Pic}(\mathcal{X}) \cong H^2(\mathcal{X}; \mathbb{Z}) \cong L^\vee / \bigoplus_{i=r' + 1}^r \mathbb{Z}D_i. \]

Recall that \( \psi^\vee : \bar{M} \to L^\vee \) is surjective if and only if \( N_{\text{tor}} = 0 \). Let

\[ \bar{D}_i = c_1(O_{\mathcal{X}}(\bar{D}_i)) \in H^2(\mathcal{X}; \mathbb{Z}), \quad i = 1, \ldots, r. \]

The map

\[ \psi^\vee : \text{Pic}_G(\mathcal{X}) \cong H^2_G(C^r; \mathbb{Z}) \to \text{Pic}(\mathcal{X}) \cong H^2(\mathcal{X}; \mathbb{Z}), \]

given by

\[ \bar{D}_i \mapsto D_i, \quad i = 1, \ldots, r', \]

is surjective if and only if \( N_{\text{tor}} = 0 \). In general, \( \text{Coker}(\psi^\vee) \cong \text{Coker}(\psi^\vee) \) is a finite abelian group.

Pick a \( \mathbb{Z} \)-basis \( \{e_1, \ldots, e_k\} \) of \( L \cong \mathbb{Z}^k \), and let \( \{e_1^\vee, \ldots, e_k^\vee\} \) be the dual \( \mathbb{Z} \)-basis of \( L^\vee \). For each \( a \in \{1, \ldots, k\} \), we define a charge vector

\[ \ell(a) = (\ell_1^{(a)}, \ldots, \ell_r^{(a)}) \in \mathbb{Z}^r \]

by

\[ \psi(e_a) = \sum_{i=1}^r \ell_i^{(a)} b_i, \]

where \( \psi : L \to \bar{N} \) is the inclusion map. Then

\[ D_i = \psi^\vee(D_i^T) = \sum_{a=1}^k \ell_i^{(a)} e_a^\vee, \quad i = 1, \ldots, r, \]

and

\[ \sum_{i=1}^r \ell_i^{(a)} b_i = \phi \circ \psi(e_a) = 0, \quad a = 1, \ldots, k. \]

Example 2.3. We use the notation in Example 2.2

(1) \( \mathcal{X} = X_{1,1,1} \).

\[
\begin{align*}
D_1 &= D_2 = D_3 = 1, \quad D_4 = -3; \\
\ell^{(1)} &= (1, 1, 1, -3); \\
\text{Pic}_G(\mathcal{X}) &\cong \mathbb{Z}^3, \quad \text{Pic}(\mathcal{X}) \cong \mathbb{Z}/3\mathbb{Z};
\end{align*}
\]
Example 2.4. Let $\rho \in \Sigma$ be an $r$-th root of unity. Then $\rho G$ is contained in $I_\rho \subset \tilde{\kappa}$. The representation of $G$ on $\tilde{\kappa} G$ is given by $g \mapsto (\chi_1(g), \ldots, \chi_r(g))$.

Let $G_\sigma := \{ g \in G \mid g \cdot z = z \text{ for all } z \in \tilde{\kappa}(\sigma) \} = \bigcap_{i \in I_\sigma} \ker(\chi_i)$.

Then $G_\sigma$ is the generic stabilizer of $\tilde{\kappa}(\sigma)$. It is a finite subgroup of $G$. If $\tau \subset \sigma$ then $I_\sigma \subset I_\tau$, so $G_\tau \subset G_\sigma$.

There are two special cases:

- Let $\langle 0 \rangle$ be the unique 0-dimensional cone. Then $G_{\langle 0 \rangle} = K$ is the generic stabilizer of $\tilde{\kappa}(\langle 0 \rangle) = X$.
- If $\sigma \in \Sigma(n)$ where $n = \dim \mathcal{X}$, then $p_\sigma := \tilde{\kappa}(\sigma)$ is a $T$-fixed point in $\mathcal{X}$, and $p_\sigma = B G_\sigma$.

Example 2.4. We use the notation in Example 2.2. Let $\sigma \in N_\mathbb{R} \cong \mathbb{R}^3$ denote the 3-dimensional cone spanned by $b_1, b_2, b_3$. For $j = 1, 2, 3$, let $\tau_j$ denote the 2-dimensional cone in $N_\mathbb{R}$ spanned by $\{b_i : i \in \{1, 2, 3\} \setminus \{j\}\}$.

(1) $\mathcal{X} = X_{1,1,1}; \ G_\sigma = \mathbb{Z}_3, \ G_{\tau_1} = G_{\tau_2} = G_{\tau_3} = \{1\}$.

(2) $\mathcal{X} = X_{1,2,0}; \ G_\sigma = \mathbb{Z}_4, \ G_{\tau_1} = G_{\tau_2} = G_{\tau_3} = \{1\}$.

(3) $\mathcal{X} = X_{0,0,0}; \ G_\sigma = \mathbb{Z}_3, \ G_{\tau_1} = G_{\tau_2} = G_{\tau_3}$.

Define the set of flags in $\Sigma$ to be $F(\Sigma) = \{ (\tau, \sigma) \in \Sigma(n-1) \times \Sigma(n) : \tau \subset \sigma \}$.

Given $(\tau, \sigma) \in F(\Sigma)$, let $I_\tau := \tilde{\kappa}(\tau)$ be the 1-dimensional $T$-invariant subvariety of $\mathcal{X}$. Then $p_\sigma$ is contained in $I_\tau$. There is a unique $i \in \{1, \ldots, r\}$ such that $j \in I_\rho \setminus I_\tau$. The representation of $G_\sigma$ on the tangent line $T_{p_\tau} I_\tau$ at the stacky point $p_\tau$ is given by $\chi((\tau, \sigma)) := \chi_i | G_\sigma : G_\sigma \rightarrow \mathbb{C}^*$. The image $\chi_\sigma(G_\sigma) \subset \mathbb{C}^*$ is a cyclic subgroup of $\mathbb{C}^*$; we define the order of this group to be $r(\tau, \sigma)$. Then there is a short exact sequence of finite abelian groups:

$$1 \rightarrow G_\tau \rightarrow G_\sigma \xrightarrow{\chi_{(\tau, \sigma)}} \mu_{r(\tau, \sigma)} \rightarrow 1,$$

where $\mu_a = \{ z \in \mathbb{C}^* \mid z^a = 1 \}$ is the group of $a$-th roots of unity.
2.5. The extended nef cone and the extended Mori cone. In this paragraph, \( F = \mathbb{Q}, \mathbb{R}, \) or \( \mathbb{C} \). Given a finitely generated abelian group \( \Lambda \) with \( \Lambda / \Lambda_{\text{tor}} \cong \mathbb{Z}^m \), define \( \Lambda_F = \Lambda \otimes \mathbb{Z} F \cong F^m \). We have the following short exact sequences of vector spaces:

\[
0 \to L_F \to \tilde{N}_F \to N_F \to 0, \\
0 \to M_F \to \tilde{M}_F \to \mathbb{L}_F^\vee \to 0.
\]

We also have the following isomorphisms of vector spaces over \( F \):

\[
H^2(X; F) \cong H^2(X; \mathbb{F}) \cong \mathbb{L}_F^\vee / \oplus_{i=r+1}^r FD_i, \\
H^2(X; \mathbb{F}) \cong H^2(X; \mathbb{F}) \cong \tilde{M}_F / \oplus_{i=r+1}^r FD_i^T,
\]

where \( X \) is the coarse moduli space of \( \mathcal{X} \).

From now on, we assume all the maximal cones in \( \Sigma \) are \( n \)-dimensional, where \( n = \dim_{\mathbb{C}} \mathcal{X} \). Given a maximal cone \( \sigma \in \Sigma(n) \), we define

\[
\mathbb{K}_\sigma^\vee := \bigoplus_{i \in I_\sigma} \mathbb{Z}D_i.
\]

Then \( \mathbb{K}_\sigma^\vee \) is a sublattice of \( \mathbb{L}_\sigma^\vee \) of finite index. We define the extended \( \sigma \)-nef cone to be

\[
\widetilde{\text{Nef}}_\sigma = \sum_{i \in I_\sigma} \mathbb{R}_{\geq 0} D_i,
\]

which is a \( k \)-dimensional cone in \( \mathbb{L}_\sigma^\vee \cong \mathbb{R}^k \). The extended nef cone of the extended stacky fan \( (\Sigma, b_1, \ldots, b_r) \) is

\[
\widetilde{\text{Nef}}_{\mathcal{X}} := \bigcap_{\sigma \in \Sigma(n)} \widetilde{\text{Nef}}_\sigma.
\]

The extended \( \sigma \)-Kähler cone \( \tilde{C}_\sigma \) is defined to be the interior of \( \tilde{\text{Nef}}_\sigma \); the extended Kähler cone of \( \mathcal{X}, \tilde{C}_{\mathcal{X}} \), is defined to be the interior of the extended nef cone \( \tilde{\text{Nef}}_{\mathcal{X}} \).

Let \( \mathbb{K}_\sigma \) be the dual lattice of \( \mathbb{K}_\sigma^\vee \); it can be viewed as an additive subgroup of \( \mathbb{L}_\mathbb{Q} \):

\[
\mathbb{K}_\sigma = \{ \beta \in \mathbb{L}_\mathbb{Q} \mid (D, \beta) \in \mathbb{Z} \forall D \in \mathbb{K}_\sigma^\vee \},
\]

where \( (\cdot, \cdot) \) is the natural pairing between \( \mathbb{L}_\mathbb{Q}^\vee \) and \( \mathbb{L}_\mathbb{Q} \). Define

\[
\mathbb{K} := \bigcup_{\sigma \in \Sigma(n)} \mathbb{K}_\sigma.
\]

Then \( \mathbb{K} \) is a subset (which is not necessarily a subgroup) of \( \mathbb{L}_\mathbb{Q} \), and \( \mathbb{L} \subseteq \mathbb{K} \).

We define the extended \( \sigma \)-Mori cone \( \tilde{\text{NE}}_\sigma \subset \mathbb{L}_\mathbb{R} \) to be the dual cone of \( \tilde{\text{Nef}}_\sigma \subset \mathbb{L}_\mathbb{R}^\vee \):

\[
\tilde{\text{NE}}_\sigma = \{ \beta \in \mathbb{L}_\mathbb{R} \mid (D, \beta) \geq 0 \forall D \in \tilde{\text{Nef}}_\sigma \}.
\]

It is a \( k \)-dimensional cone in \( \mathbb{L}_\mathbb{R} \). The extended Mori cone of the extended stacky fan \( (\Sigma, b_1, \ldots, b_r) \) is

\[
\tilde{\text{NE}}_{\mathcal{X}} := \bigcup_{\sigma \in \Sigma(n)} \tilde{\text{NE}}_\sigma.
\]

Finally, we define

\[
\mathbb{K}_{\text{eff}, \sigma} := \mathbb{K}_\sigma \cap \tilde{\text{NE}}_\sigma, \quad \mathbb{K}_{\text{eff}} := \mathbb{K} \cap \tilde{\text{NE}}(\mathcal{X}) = \bigcup_{\sigma \in \Sigma(n)} \mathbb{K}_{\text{eff}, \sigma}.
\]

Example 2.5. (1) \( \mathcal{X} = \mathcal{X}_{1,1,1} \).

\[
\mathbb{K}_{\text{eff}, \sigma} = \frac{1}{3} \mathbb{Z}, \quad \tilde{\text{NE}}_{\mathcal{X}} = \mathbb{R}_{\leq 0};
\]

\[
\mathbb{K} = \frac{1}{3} \mathbb{Z}, \quad \tilde{\text{NE}}_{\mathcal{X}} = \mathbb{R}_{\leq 0}, \quad \mathbb{K}_{\text{eff}} = \frac{1}{3} \mathbb{Z}_{\leq 0}.
\]
Figure 3. $K_{\text{eff}}$ of $X_{1,1,1}$ and its crepant resolution $O_{\mathbb{P}^2}(-3)$

(2) $\mathcal{X} = X_{1,2,0}$.

$K^\vee \cong \mathbb{Z}(-2,1) \oplus \mathbb{Z}(1,-2)$, \quad $\widetilde{\text{Nef}}_{\mathcal{X}} = \mathbb{R}_{\geq 0}(-2,1) + \mathbb{R}_{\geq 0}(1,-2)$;

$K \cong \mathbb{Z}(-\frac{2}{3},-\frac{1}{3}) \oplus \mathbb{Z}(-\frac{1}{3},-\frac{2}{3})$, \quad $\widetilde{\text{NE}}_{\mathcal{X}} = \mathbb{R}_{\geq 0}(-\frac{2}{3},-\frac{1}{3}) + \mathbb{R}_{\geq 0}(-\frac{1}{3},-\frac{2}{3})$,

$K_{\text{eff}} = \mathbb{Z}_{\geq 0}(-\frac{2}{3},-\frac{1}{3}) + \mathbb{Z}_{\geq 0}(-\frac{1}{3},-\frac{2}{3})$.

Figure 4. The secondary fan of the crepant resolution of $X_{1,2,0}$

(3) $\mathcal{X} = X_{0,0,0}$.

$K^\vee \cong 3\mathbb{Z}$, \quad $\widetilde{\text{Nef}}_{\mathcal{X}} = \mathbb{R}_{\leq 0}$;

$K \cong \frac{1}{3}\mathbb{Z}$, \quad $\widetilde{\text{NE}}_{\mathcal{X}} = \mathbb{R}_{\leq 0}$, \quad $K_{\text{eff}} = \frac{1}{3}\mathbb{Z}_{\leq 0}$.

Assumption 2.6. From now on, we make the following assumptions on $\mathcal{X}$.

(a) The coarse moduli space $X_{\Sigma}$ of $\mathcal{X}$ is semi-projective.

(b) We may choose $b_{r+1}, \ldots, b_r$ such that $\hat{\rho} := D_1 + \cdots + D_r$ is contained in the closure of the extended Kähler cone $\overline{C}_{\mathcal{X}}$.

Remark 2.7. (1) We make the above assumptions (a) and (b) so that the equivariant mirror theorem [27, Theorem 31] takes a particularly simple form. See Section 4.1 in this paper for the precise statement.

(2) By [28, Proposition 14.4.1], $X_{\Sigma}$ is semi-projective if and only if $|\Sigma|$ is equal to the cone spanned by $b_1, \ldots, b_r$. For example, the total space of $O_{\mathbb{P}^1}(-3) \oplus O_{\mathbb{P}^1}(1)$ is a smooth toric Calabi-Yau 3-fold which is not semi-projective.

(3) When $\mathcal{X}$ is a Calabi-Yau smooth toric DM stack, Assumption (b) holds if its coarse moduli space $X_{\Sigma}$ has a toric crepant resolution of singularities; see [47, Remark 3.4]. By [28, Proposition 11.4.19], any 3-dimensional Gorenstein toric variety $X_{\Sigma}$ has a resolution of singularities $\phi : X_{\Sigma'} \to X_{\Sigma}$ such that $\phi$ is projective and crepant. So Assumption 2.6 (b) holds for any 3-dimensional Calabi-Yau smooth toric DM stacks.
2.6. Smooth toric DM stacks as symplectic quotients. Let $G_R \cong U(1)^k$ be the maximal compact subgroup of $G \cong (\mathbb{C}^*)^k$. Then the Lie algebra of $G_R$ is $\mathbb{L}_R$. Let

$$\tilde{\mu} : \mathbb{C}^r \to L_R^\vee = \bigoplus_{a=1}^{k} \mathbb{R} e_a^\vee$$

be the moment map of the Hamiltonian $G_R$-action on $\mathbb{C}^r$, equipped with the Kähler form

$$\sqrt{-1} \sum_{i=1}^r dZ_i \wedge d\bar{Z}_i.$$ 

Then

$$\tilde{\mu}(Z_1, \ldots, Z_r) = \sum_{i=1}^r \sum_{a=1}^k l^{(a)}_i |Z_i|^2 e_a^\vee.$$ 

If $r = \sum_{a=1}^k r_a e_a^\vee$ is in the extended Kähler cone of $\mathcal{X}$, then

$$\mathcal{X} = \left[ \tilde{\mu}^{-1}(r)/G_R \right].$$

The generic stabilizer $K$ (which is a finite subgroup of $G \cong (\mathbb{C}^*)^k$) is contained in the maximal compact subgroup $G_R$ of $G$. The quotient $G_{\text{rig}} := G_R/K \cong U(1)^k$ is the maximal compact subgroup of $G_{\text{rig}} = G/K \cong (\mathbb{C}^*)^k$, and

$$\mathcal{X}_{\text{rig}} = \left[ \tilde{\mu}^{-1}(r)/G_{\text{rig}} \right]$$

as a symplectic quotient.
The real numbers $r_1, \ldots, r_k$ are extended Kähler parameters. The symplectic structure $\omega(r)$ depends on $r$. The map $r \mapsto |\omega(r)|$ is given by $L^2_{\mathbb{R}} \rightarrow H^2(\mathcal{X}; \mathbb{R})$. Let $T_a = -r_a + \sqrt{-1} \theta_a$ be complexified extended Kähler parameters of $\mathcal{X}$.

2.7. The inertia stack and the Chen-Ruan orbifold cohomology. Given $\sigma \in \Sigma$, define

$$\Box(\sigma) := \left\{ v \in N : \bar{v} = \sum_{i \in I'_\sigma} c_i \bar{b_i}, \ 0 \leq c_i < 1 \right\}.$$ 

Then $N_{\text{tor}} \subset \Box(\sigma) \subset N$. If $\sigma$ is a $d$-dimensional cone, then the set $\{ \sum_{i \in I'_\sigma} c_i \bar{b_i} : c_i \in \mathbb{R}, 0 \leq c_i < 1 \}$ is a fundamental domain of the action of $N_\sigma = \oplus_{i \in I'_\sigma} \mathbb{Z} b_i \cong \mathbb{Z}^d$ on $N_\sigma \otimes \mathbb{R} = \oplus_{i \in I'_\sigma} \mathbb{R} \bar{b_i} \cong \mathbb{R}^d$. If $\tau \subset \sigma$ then $I'_\tau \subset I'_\sigma$, so $\Box(\tau) \subset \Box(\sigma)$.

Let $\sigma \in \Sigma(n)$ be a maximal cone in $\Sigma$. We have a short exact sequence of abelian groups

$$0 \rightarrow \mathbb{K}_\sigma/\mathbb{L} \rightarrow L_\mathbb{R}/\mathbb{L} \rightarrow \mathbb{L}_\mathbb{R}/\mathbb{K}_\sigma \rightarrow 0,$$

which can be identified with the following short exact sequence of multiplicative abelian groups

$$1 \rightarrow G_\sigma \rightarrow G_\mathbb{R} \rightarrow (G/G_\sigma)_\mathbb{R} \rightarrow 1$$

where $G_\mathbb{R} \cong U(1)^k$ is the maximal compact subgroup of $G \cong (\mathbb{C}^*)^k$, and $(G/G_\sigma)_\mathbb{R} \cong U(1)^k$ is the maximal compact subgroup of $(G/G_\sigma) \cong (\mathbb{C}^*)^k$.

Given a real number $x$, we recall some standard notation: $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$, $\lceil x \rceil$ is the least integer greater or equal to $x$, and $\{ x \} = x - \lfloor x \rfloor$ is the fractional part of $x$. Define $v : \mathbb{K}_\sigma \rightarrow N$ by

$$v(\beta) = \sum_{i=1}^r [\langle D_i, \beta \rangle] b_i.$$

Then

$$\overline{v(\beta)} = \sum_{i \in I'_\sigma} \{ -(\langle D_i, \beta \rangle) \} \bar{b_i},$$

so $v(\beta) \in \Box(\sigma)$. Indeed, $v$ induces a bijection $K_\sigma/\mathbb{L} \cong \Box(\sigma)$.

For any $\tau \in \Sigma$ there exists $\sigma \in \Sigma(n)$ such that $\tau \subset \sigma$. The bijection $G_\sigma \rightarrow \Box(\sigma)$ restricts to a bijection $G_\tau \rightarrow \Box(\tau)$.

Define

$$\Box(\Sigma) := \bigcup_{\sigma \in \Sigma} \Box(\sigma) = \bigcup_{\sigma \in \Sigma(n)} \Box(\sigma).$$

Then $N_{\text{tor}} \subset \Box(\Sigma) \subset N$. There is a bijection $\mathbb{K}/\mathbb{L} \rightarrow \Box(\Sigma)$.

Given $v \in \Box(\sigma)$, where $\sigma \in \Sigma(d)$, define $c_i(v) \in [0, 1) \cap \mathbb{Q}$ by

$$\bar{v} = \sum_{i \in I'_\sigma} c_i(v) \bar{b_i}.$$

Suppose that $k \in G_\sigma$ corresponds to $v \in \Box(\sigma)$ under the bijection $G_\sigma \cong \Box(\sigma)$, then

$$\chi_i(k) = \begin{cases} 1, & i \in I_\sigma, \\ e^{2\pi \sqrt{-1} c_i(v)}, & i \in I'_\sigma. \end{cases}$$

Define

$$\text{age}(k) = \text{age}(v) = \sum_{i \notin I_\sigma} c_i(v).$$

Let $IU = \{ (z, k) \in U_\mathcal{A} \times G \mid k \cdot z = z \}$, and let $G$ act on $IU$ by $h \cdot (z, k) = (h \cdot z, k)$. The inertia stack $\mathcal{I}\mathcal{X}$ of $\mathcal{X}$ is defined to be the quotient stack

$$\mathcal{I}\mathcal{X} := [IU/G].$$

Note that $z = (Z_1, \ldots, Z_r), k \in IU$ if and only if $k \in \bigcup_{\sigma \in \Sigma} G_\sigma$ and $Z_i = 0$ whenever $\chi_i(k) \neq 1$. 

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So

\[ IU = \bigcup_{v \in \text{Box}(\Sigma)} U_v, \]

where

\[ U_v := \{(Z_1, \ldots, Z_m) \in U_A : Z_i = 0 \text{ if } c_i(v) \neq 0\}. \]

The connected components of \( \mathcal{I} \mathcal{X} \) are

\[ \{\mathcal{X}_v := [U_v/G] : v \in \text{Box}(\Sigma)\}. \]

The involution \( IU \to IU, (z, k) \mapsto (z, k^{-1}) \) induces involutions \( \text{inv} : \mathcal{I} \mathcal{X} \to \mathcal{I} \mathcal{X} \) and \( \text{inv} : \text{Box}(\Sigma) \to \text{Box}(\Sigma) \) such that \( \text{inv}(\mathcal{X}_v) = \mathcal{X}_{\text{inv}(v)} \).

In the remainder of this subsection, we consider rational cohomology, and write \( H^*(-) \) instead of \( H^*(-; \mathbb{Q}) \).

As a graded vector space over \( \mathbb{Q} \) (and as the state-space of the relevant quantum theory in physics \([74]\)), the Chen-Ruan orbifold cohomology \([23]\) is defined to be

\[ H^*_{\text{CR}}(\mathcal{X}) = \bigoplus_{v \in \text{Box}(\Sigma)} H^*_{\text{CR}}(\mathcal{X}_v)[\text{2age}(v)]. \]

Let \( 1_v \) be the unit in \( H^*(\mathcal{X}_v) \). Then \( 1_v \in H^2_{\text{CR}}(\mathcal{X}_v)[\mathcal{X}] \). In particular,

\[ H^0_{\text{CR}}(\mathcal{X}) = \bigoplus_{v \in N_{\text{tor}}} \mathbb{Q} 1_v. \]

Suppose that \( \mathcal{X} \) is a proper toric DM stack. Then the orbifold Poincaré pairing on \( H^*_{\text{CR}}(\mathcal{X}) \) is defined as

\[ (\alpha, \beta) := \int_{\mathcal{I} \mathcal{X}} \alpha \cup \text{inv}^* (\beta), \]

We also have an equivariant pairing on \( H^*_{\text{CR, T}}(\mathcal{X}) \):

\[ (\alpha, \beta)_T := \int_{\mathcal{I} \mathcal{X}_T} \alpha \cup \text{inv}^* (\beta), \]

where

\[ \int_{\mathcal{I} \mathcal{X}_T} : H^*_{\text{CR, T}}(\mathcal{X}) \to H^*_T(\text{point}) = H^*(B\mathbb{T}) \]

is the equivariant pushforward to a point. When \( \mathcal{X} \) is not proper, \([5]\) is not defined, but we can still define via \([6]\) an equivariant pairing \( H^*_{\text{CR, T}}(\mathcal{X}) \otimes H^*_{\text{CR, T}}(\mathcal{X}) \to Q_T \), where \( Q_T \) is the fractional field of the ring \( H^*(B\mathbb{T}) \).

**Example 2.8.**

1. \( \mathcal{X} = X_{1,1,1} \).

\[ N = \mathbb{Z}^3, \quad \text{Box}(\Sigma) = \{(0, 0, 0), (0, 0, 1), (0, 0, 2)\}; \]

\[ H^0_{\text{CR}}(\mathcal{X}) = \mathbb{Q} 1_{(0,0,0)}, \quad H^2_{\text{CR}}(\mathcal{X}) = \mathbb{Q} 1_{(0,0,1)}, \quad H^4_{\text{CR}}(\mathcal{X}) = \mathbb{Q} 1_{(0,0,2)}. \]

2. \( \mathcal{X} = X_{1,2,0} \).

\[ N = \mathbb{Z}^3, \quad \text{Box}(\Sigma) = \{(0, 0, 0), (0, 2, 1), (0, 1, 1)\}; \]

\[ H^0_{\text{CR}}(\mathcal{X}) = \mathbb{Q} 1_{(0,0,0)}, \quad H^2_{\text{CR}}(\mathcal{X}) = \mathbb{Q} 1_{(0,2,1)} \oplus \mathbb{Q} 1_{(1,0,1)}. \]

3. \( \mathcal{X} = X_{0,0,0} \).

\[ N = \mathbb{Z}^3 \oplus \mathbb{Z}_3, \quad \text{Box}(\Sigma) = N_{\text{tor}} = \mathbb{Z}_3 = \{0, 1, 2\}; \]

\[ H^0_{\text{CR}}(\mathcal{X}) = \mathbb{Q} 1_0 \oplus \mathbb{Q} 1_1 \oplus \mathbb{Q} 1_2. \]

3. **All Genus Open-Closed Gromov Witten invariants**

In this section, \( \mathcal{X} \) is a 3-dimensional Calabi-Yau smooth Deligne-Mumford stack.
3.1. Rigidification. The rigidification $\mathcal{X}^\text{rig}$ of $\mathcal{X}$ is a toric Calabi-Yau 3-orbifold. The Calabi-Yau condition implies $\mathcal{X}^\text{rig} = \mathcal{X}^\text{can}$, where $\mathcal{X}^\text{can}$ is determined by the simplicial fan $\Sigma$ and then by choosing each $b_i$ to be the primitive generator of each 1-cone (cf. Equation (3) in Section 2.1). Let $T'$ (resp. $T$) be the subtorus of $T \cong (\mathbb{C}^*)^3 \times BK$ (resp. $T \cong (\mathbb{C}^*)^3$) preserving the Calabi-Yau 3-form on $\mathcal{X}$ (resp. $\mathcal{X}^\text{rig}$). Then $T' \cong (\mathbb{C}^*)^2$ and $T'' \cong (\mathbb{C}^*)^2 \times BK$. There is a primitive $u_3 \in M = \text{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^3$ such that $\text{Ker}(u_3) = T'$. Define $M' := M/\langle u_3 \rangle \cong \mathbb{Z}^2$. Then $N' := u_3^T = \{v \in N : \langle u_3, v \rangle = 0\}$ is the dual lattice of $M' = \text{Hom}(T, \mathbb{C}^*)$.

The simplicial fan $\Sigma$ is determined by a convex polytope $\Delta$ where all the vertices are in the lattice $\bar{N}$. The fan $\Sigma$ is a cone over this triangulation in $N'_{1,R} \subset N_R$, where $N'_{1,R} = \{v \in N_R : \langle u_3, v \rangle = 1\}$.

3.2. Toric graph. Let $T_R \cong U(1)^3$ (resp. $T'_R \cong U(1)^2$) be the maximal compact subgroup of $T \cong (\mathbb{C}^*)^3$ (resp. $T' \cong (\mathbb{C}^*)^2$), and we choose an $r$ in the extended Kähler cone. The $T_R$-action on $\mathcal{X}^\text{rig}$ restricts to a Hamiltonian $T_R$-action on the Kähler orbifold $(\mathcal{X}^\text{rig}, \omega(r))$. Since $M_R$ (resp. $M'_R$) is canonically identified with the dual of the Lie algebra of $T_R$ (resp. $T'_R$), the Kähler form $\omega(r)$ determines a moment map $\mu_{T_R} : \mathcal{X}^\text{rig} \rightarrow M_R$ up to translation by a vector in $M_R$. The image $\mu_{T_R}(\mathcal{X}^\text{rig})$ is a convex polyhedron. The moment map $\mu_{T_R} : \mathcal{X}^\text{rig} \rightarrow M_R$ is the composition $\pi \circ \mu_{T_R}$, where $\pi : M_R \cong \mathbb{R}^3 \rightarrow M'_{R} \cong \mathbb{R}^2$ is the projection. The map $\mu_{T_R}$ is surjective. Let $\mathcal{X}_R^\text{rig} \subset \mathcal{X}^\text{rig}$ be the union of 0-dimensional and 1-dimensional $T$-orbits in $\mathcal{X}^\text{rig}$. The toric graph is defined by $\Gamma := \mu_{T_R}(\mathcal{X}_1^\text{rig}) \subset M'_{R} \cong \mathbb{R}^2$. It is determined by the Kähler class $[\omega(r)] \in H^2(\mathcal{X}^\text{rig}; \mathbb{R}) = H^2(\mathcal{X}; \mathbb{R})$ up to translation by a vector in $M'_{R}$. The vertices (resp. edges) of $\Gamma$ are in one-to-one correspondence to 1-dimensional (resp. 2-dimensional) cones in $\Sigma$. Conversely, the Kähler class $[\omega(r)] \in H^2(\mathcal{X}^\text{rig}; \mathbb{R})$ is determined by the toric graph.

Pulling back under the map $\mathcal{X} \rightarrow \mathcal{X}^\text{rig}$ defines a one-to-one correspondence between Kähler forms/classes on $\mathcal{X}$ and on its rigidification $\mathcal{X}^\text{rig}$.

3.3. Aganagic-Vafa A-branes. In [6], Aganagic-Vafa introduced a class of Lagrangian submanifolds of semi-projective smooth toric Calabi-Yau 3-folds. In this section, we generalize this construction and define Aganagic-Vafa A-branes in a general 3-dimensional Calabi-Yau smooth toric DM stack with semi-projective coarse moduli space.

Let $\mathcal{X} = [\tilde{\mu}^{-1}(r)/\text{G}_R]$ be a 3-dimensional Calabi-Yau smooth toric DM stack, where

$$\mathbf{r} = \sum_{a=1}^{k} r_a e_a^\mathcal{X} \in \bar{C}(\mathcal{X}) \subset L^\mathcal{X}_R,$$

and $\tilde{\mu}^{-1}(r) \subset \mathbb{C}^{k+3}$ is defined by the following equations:

$$\sum_{i=1}^{k+3} l_i^{(a)} |X_i|^2 = r_a, \quad a = 1, \ldots, k.$$

Write $X_i = \rho_i e^{-\sqrt{-1} \phi_i}$, where $\rho_i = |X_i|$. An Aganagic-Vafa brane is a Lagrangian sub-orbifold of $\mathcal{X}$ of the form

$$\mathcal{L} = [\tilde{L}/\text{G}_R]$$

where

$$\tilde{L} = \{(X_1, \ldots, X_{k+3}) \in \tilde{\mu}^{-1}(r) : \sum_{i=1}^{k+3} \hat{l}_i |X_i|^2 = c_1, \sum_{i=1}^{k+3} \hat{\ell}_i |X_i|^2 = c_2, \sum_{i=1}^{k+3} \phi_i = \text{const}\}$$

for some $\hat{l}_i \in \mathbb{Z}, \sum_{i=1}^{k+3} \hat{l}_i = 0, \alpha = 1, 2$. Note that the action of $\text{G}_R$ on $\mathbb{C}^{k+3}$ preserves the subsets $\tilde{\mu}^{-1}(r)$ and $\tilde{L}$. If we view $\mathcal{X} = [\tilde{\mu}^{-1}(r)/\text{G}_R]$ as a Lie groupoid (and in particular a category) then $\mathcal{L} = [\tilde{L}/\text{G}_R]$ is a full subcategory.

An Aganagic-Vafa brane $\mathcal{L}$ intersects a unique 1-dimensional orbit $\mathfrak{o}_r \cong \mathbb{C}^* \times BG_r$ along $S_r := \mathcal{L} \cap \mathfrak{o}_r \cong S^1 \times BG_r$. The inclusion $S_r \subset \mathcal{L}$ is a homotopic equivalence, so the fundamental group of $\mathcal{L}$ is

$$\pi_1(\mathcal{L}) \cong \pi_1(S^1 \times BG_r) \cong \mathbb{Z} \times G_r.$$

In particular, it is abelian, so it is isomorphic to its abelianization $H_1(\mathcal{L}, \mathbb{Z})$. 
If \((\tau, \sigma) \in F(\Sigma)\) then there is an inclusion \(i^{(\tau, \sigma)} : S_{\tau} \hookrightarrow X_{\sigma} = [\mathbb{C}^3/G_{\sigma}]\) which induces
\[
i^{(\tau, \sigma)} : \pi_1(S_{\tau}) \cong \mathbb{Z} \times G_{\tau} \to \pi_1(X_{\sigma}) \cong G_{\sigma}.
\]

3.4. Moduli spaces of stable maps to \((X, L)\). In [49], Katz-Liu introduced stable maps to a symplectic manifold with Lagrangian boundary conditions at all genera; the domain of such a map is a prestable bordered Riemann surface, i.e. a smooth or nodal bordered Riemann surface. (See also [55], [38].) In [24, Section 2], Cho-Poddar define stable maps to a symplectic orbifold \(X\) with Lagrangian boundary conditions, under the assumption that the Lagrangian sub-orbifold \(L\) does not contain any stacky points (so that \(L\) is indeed a smooth manifold); the domain of such a map is a prestable bordered orbifold Riemann surface in the sense of [24, Section 2], i.e. a smooth or nodal bordered orbifold Riemann surface, where a stacky point is either an interior marked point or an interior node.

In general, \(L\) is a sub-orbifold which contains stacky points. To obtain compactness of the moduli spaces when \(X\) and \(L\) are compact, one needs to allow orbifold structures at boundary marked points and boundary nodes. In the present paper \(L\) may contain stacky points, but we do not need to allow orbifold structures on the boundary of the domain, for the following two reasons.

(i) Our enumerative problem only requires interior insertions, so we do not need to introduce any boundary marked points.

(ii) In our case \(X\) and \(L\) are non-compact and we will define and compute open GW invariants by torus localization on moduli space of stable maps \(X\) with boundaries in \(L\). If a stable map represents a torus fixed point in the moduli space then any node in the domain must be mapped to a torus fixed (scheme or stacky) point in \(X\), but \(L\) does not contain any torus fixed point, so the domain does not contain any boundary nodes.

Let \((\Sigma, x_1, \ldots, x_n)\) be a prestable bordered orbifold Riemann surface with \(n\) interior marked point. Then the coarse moduli space \((\Sigma, \bar{x}_1, \ldots, \bar{x}_n)\) is a prestable bordered Riemann surface with \(n\) interior marked points, defined in [49, Section 3.6] and [55, Section 3.2]. We define the topological type \((g, h)\) of \(\Sigma\) to be the topological type of \(\Sigma\) (see [55, Section 3.2]).

Let \((\Sigma, \partial \Sigma)\) be a prestable bordered orbifold Riemann surface of type \((g, h)\), and let \(\partial \Sigma = R_1 \cup \cdots \cup R_h\) be union of connected components. Each connected component is a circle which contains no orbifold points. A (bordered) prestable map to the pair \((X, L)\) is a map \(u : (\Sigma, \partial \Sigma) \to (X, L)\) where \(\Sigma\) is a prestable bordered orbifold Riemann surface, such that \(u \circ \nu : \Sigma \to X\) is holomorphic, where \(\nu : \Sigma \to \Sigma\) is the normalization (so \(\Sigma\) is a possibly disconnected smooth bordered orbifold Riemann surface); a prestable map to \((X, L)\) is stable if its automorphism group is finite. The topological type of a stable map \(u\) is given by the degree \(\beta' = u_*[\Sigma] \in H_2(X, L; \mathbb{Z})\) (where \(X\) and \(L\) are the coarse moduli spaces of \(X\) and \(L\) respectively, and \(\hat{u} : \hat{\Sigma} \to X\) is the map between coarse moduli spaces) and \(\check{\mu}_i = u_*[R_i] = (\mu_i, \lambda_i) \in H_1(L; \mathbb{Z}) \cong \mathbb{Z} \times G_{\tau}\) (where \(\mu_i \in \mathbb{Z}\) is the winding number and \(\lambda_i \in G_{\tau}\) is the monodromy). Given \(\beta' \in H_2(X, L; \mathbb{Z})\) and \(\check{\mu} = ((\mu_1, \lambda_1), \ldots, (\mu_h, \lambda_h)) \in H_1(L; \mathbb{Z})^h\).

Let \(\overline{\mathcal{M}}_{(g,h),n}(X, L | \beta', \check{\mu})\) be the moduli space of stable maps of type \((g, h)\), degree \(\beta'\), winding numbers \(\mu_i \in \mathbb{Z}\) and monodromies \(\lambda_i \in G_{\tau}\), with \(n\) interior marked points.

3.5. The tangent-obstruction complex and the virtual dimension. Similar to [49, Section 4.2], the tangent space \(T^1_{\xi}\) and the obstruction space \(T^2_{\xi}\) at a moduli point
\[
\xi = [u : ((\Sigma, x_1, \ldots, x_n), \partial \Sigma) \to (X, L)] \in \overline{\mathcal{M}}_{(g,h),n}(X, L | \beta', \check{\mu})
\]
fit into the following exact sequence of real vector spaces:
\[
0 \to \text{Aut}((\Sigma, x_1, \ldots, x_n), \partial \Sigma) \to H^0(\Sigma, \partial \Sigma, u^*T_X, (u|_{\partial \Sigma})^*T_L) \to T^1_{\xi} \to 0,
\]
\[
de(\Sigma, x_1, \ldots, x_n, \partial \Sigma) \to H^1(\Sigma, \partial \Sigma, u^*T_X, (u|_{\partial \Sigma})^*T_L) \to T^2_{\xi},
\]
where
- \(\text{Aut}((\Sigma, x_1, \ldots, x_n), \partial \Sigma)\) is the space of infinitesimal automorphism of the domain \(((\Sigma, x_1, \ldots, x_n), \partial \Sigma)\) and is equal to \(H^0(\Sigma, \partial \Sigma, T_\Sigma(- \sum_{j=1}^n x_j)T_{\partial \Sigma})\) when \(\Sigma\) is a smooth bordered orbifold Riemann surface;
\textbullet{} \text{Def}(\Sigma, x_1, \ldots, x_n, \partial \Sigma) is the space of infinitesimal deformations of the domain, and is equal to 
\[ H^1(\Sigma, \partial \Sigma, T_\Sigma(- \sum_{j=1}^n x_j), T_{\partial \Sigma}) \] 
when \( \Sigma \) is a smooth bordered orbifold Riemann surface;
\textbullet{} \[ H^0(\Sigma, \partial \Sigma, u^* T_X, (u|_{\partial \Sigma})^* T_L) \] is the space of infinitesimal deformation of the map for a fixed domain;
\textbullet{} \[ H^1(\Sigma, \partial \Sigma, u^* T_X, (u|_{\partial \Sigma})^* T_L) \] is the space of obstructions to deforming the map for a fixed domain.

Globally on the moduli space \( \overline{M}_{(g,h)}(\mathcal{X}, \mathcal{L}, \beta', \bar{\mu}) \), there is an exact sequence of sheaves
\[ 0 \to B_1 \to B_2 \to T^1 \to B_4 \to B_5 \to T^2 \to 0 \] 
whose fiber at the moduli point \( \xi \) is \( 7 \).

Let \( \mathcal{M}_{(g,h),n} \) be the moduli of prestable bordered orbifold Riemann surfaces of type \((g,h)\) with \(n\) interior marked point. Then \( \mathcal{M}_{(g,h),n} \) is a differentiable stack (with corners) of real dimension
\[ 3(2g - 2 + h) + 2n = \dim \text{Def}(\Sigma, x_1, \ldots, x_n, \partial \Sigma) - \dim \text{Aut}(\Sigma, x_1, \ldots, x_n, \partial \Sigma). \]

There are evaluation maps (at interior marked points)
\[ \text{ev}_j : \overline{M}_{(g,h),n}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) \to \mathcal{I} \mathcal{X}, \quad j = 1, \ldots, n. \]
Given \( \vec{v} = (v_1, \ldots, v_n) \), where \( v_1, \ldots, v_n \in \text{Box}(\Sigma) \), define
\[ \overline{M}_{(g,h),\vec{v}}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) := \bigcap_{j=1}^n \text{ev}_j^{-1}(\mathcal{X}_{v_j}). \]

Suppose that \( \xi \in \overline{M}_{(g,h),\vec{v}}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) \). By the Riemann-Roch theorem for prestable bordered orbifold Riemann surface (which can be derived by combining the proof of the Riemann-Roch theorem for prestable bordered Riemann surfaces and prestable orbifold closed Riemann surfaces),
\[ \dim \mathbb{R} H^0(\Sigma, \partial \Sigma, u^* T_X, (u|_{\partial \Sigma})^* T_L) - \dim \mathbb{R} H^1(\Sigma, \partial \Sigma, u^* T_X, (u|_{\partial \Sigma})^* T_L) = 3(2 - 2g - h) - 2 \sum_{j=1}^n \text{age}(v_j). \]

The above Equation \( 10 \) is the relative virtual dimension of \( \overline{M}_{(g,h),\vec{v}}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) \to \mathcal{M}_{(g,h),n} \) which sends a stable map to its domain. The virtual (real) dimension of \( \overline{M}_{(g,h),\vec{v}}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) \) is equal to
\[ \dim \mathbb{R} T^1_{\xi} - \dim \mathbb{R} T^2_{\xi} = 2 \sum_{j=1}^n (1 - \text{age}(v_j)), \]
where \( \text{age}(v_j) \in \{0, 1, 2\} \).

3.6. Torus action and equivariant invariants. Let \( T_{\mathbb{R}}^n \cong U(1)^2 \) be the maximal compact subgroup of \( T' \cong (\mathbb{C}^*)^2 \). For any \( t \in T_{\mathbb{R}}^n \), the map \( \phi_t : \mathcal{X} \to \mathcal{X} \) given by \( x \mapsto t \cdot x \) is an automorphism of the smooth toric DM stack \( \mathcal{X} \), and \( \phi_t(\mathcal{L}) = \mathcal{L} \), so \( T_{\mathbb{R}}^n \) acts on the moduli spaces \( \overline{M}_{(g,h),n}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) \); here we use the notion of group actions on stacks in \([65]\). Let \( F \subset \overline{M}_{(g,h),n}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) \) be the substack of \( T_{\mathbb{R}}^n \) fixed points. The restriction of the exact sequence \( 3 \) to the substack \( F \) is the direct sum of two exact sequences
\[ 0 \to B_1^f \to B_2^f \to T^{1,f} \to B_4^f \to B_5^f \to T^{2,f} \to 0, \]
\[ 0 \to B_1^m \to B_2^m \to T^{1,m} \to B_4^m \to B_5^m \to T^{2,m} \to 0, \]
where \( 11 \) is the subcomplex fixed by the torus action. The virtual tangent bundle \( T^\text{vir}_F \) of \( F \) is
\[ T^\text{vir}_F = T^{1,f} - T^{2,f} \]
whose ranks can be different on different connected components of \( F \). We will see that each connected component of \( F \) is a compact orbifold, and that \( T^\text{vir}_F \) is equal to the tangent bundle \( T_F \) of \( F \). So
\[ [F]^\text{vir} = [F]. \]

The virtual normal bundle \( N^\text{vir} \) of \( F \) in \( \overline{M}_{(g,h),n}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) \) is
\[ N^\text{vir} = T^{1,m} - T^{2,m}. \]
Given $\gamma_1, \ldots, \gamma_n \in H^*_\mathfrak{T}_k; \mathbb{C}R(X; \mathbb{Q}) = H^*_{\mathfrak{T}_k; \mathbb{C}R}(X; \mathbb{Q})$, we define
\begin{equation}
(\gamma_1, \ldots, \gamma_n)_{\alpha, \beta, \gamma} := \int_{[F]_{\text{vir}}} \prod_{i=1}^{n} \frac{(ev_i^* \gamma_i)|_F}{e_{(k)}(N^\text{vir})} \in \mathcal{Q}_{\mathfrak{T}_k}
\end{equation}

where $\mathcal{Q}_{\mathfrak{T}_k}$ is the fractional field of $H^*_k$ (point; $\mathbb{Q}$), and
\[
\frac{1}{e_{(k)}(N^\text{vir})} = \frac{e_{(k)}(T^2, m)}{e_{(k)}(T^1, m)} = \frac{e_{(k)}(B^1_1) e_{(k)}(B^m_1)}{e_{(k)}(B^1_1) e_{(k)}(B^m_1)}.
\]

More precisely, the definition \((13)\) also requires an orientation on the virtual tangent bundle $T^1 - T^2$, which we will specify later.

### 3.7. Tangent weights: the 3-torus, the Calabi-Yau 2-torus, and the framing 1-torus

Let $\mathfrak{c}_\tau \cong \mathbb{C}^* \times \mathbb{B}_{\mathfrak{c}}$ be the unique 1-dimensional $\mathbb{T}$-orbit which intersects the Aganagic-Vafa A-brane $\mathcal{L}$, where $\tau \in \Sigma(2)$, as before. Let $\mathfrak{l}_\tau$ be the closure of $\mathfrak{c}_\tau$, and let $\ell_\tau$ be the coarse moduli of $\mathfrak{l}_\tau$. Then $\ell_\tau$ is either $\mathbb{P}^1$ or $\mathbb{C}$.

**Definition 3.1.** We say $\mathcal{L}$ is an inner brane if $\ell_\tau \cong \mathbb{P}^1$; we say $\mathcal{L}$ is an outer brane if $\ell_\tau \cong \mathbb{C}$.

If $\mathcal{L}$ is an outer brane, let $\sigma \in \Sigma(3)$ be the unique 3-cone such that $(\tau, \sigma) \in F(\Sigma)$; if $\mathcal{L}$ is an inner brane, we choose $\sigma \in \Sigma(3)$ such that $(\tau, \sigma) \in F(\Sigma)$ and let $\sigma_{-} \in \Sigma(3)$ denote the other choice, so that $p_\sigma$ and $p_{\sigma'}$ are the two torus fixed points in $\mathfrak{l}_\tau$. In both cases, we also denote $\sigma = \sigma_{+}.$

By permuting $b_1, \ldots, b_3$, if necessary, we may assume that $I'_\sigma = \{1, 2, 3\}$, and $(\tau_1, \sigma) = (\tau, \sigma)$, $(\tau_2, \sigma)$ and $(\tau_3, \sigma)$ are three flags in the toric graph in the counterclockwise direction such that
\[
I'_{\tau_1} = \{2, 3\}, \quad I'_{\tau_2} = \{3, 1\}, \quad I'_{\tau_3} = \{1, 2\}.
\]

Here we fixed an orientation of $\mathbb{R}^2$. If $\mathcal{L}$ is an inner brane, we assume in addition $I'_{-\sigma} = \{2, 3, 4\}$.

Recall from Section 2.3 that for any flag $(\tau, \sigma) \in F(\Sigma)$, $\chi_{(\tau, \sigma)}$ is the character of the 1-dimensional $G_{\sigma}$ representation $T_{p_{\sigma}} \mathfrak{l}_\tau$. Let
\[
\tau := r(\tau, \sigma) = |G_{\sigma}/G_{\ell_\tau}|, \quad m := |G_{\tau}/K|.
\]

Then we have the following two short exact sequences of finite abelian groups
\[
1 \rightarrow G_{\tau} \rightarrow G_{\sigma} \xrightarrow{\chi_{(\tau, \sigma)}} \mu_{m} \rightarrow 1, \quad 1 \rightarrow K \rightarrow G_{\tau} \xrightarrow{\chi_{(\tau, \sigma)}} \mu_{m} \rightarrow 1.
\]

Note that for any $\lambda \in G_{\tau}$, $\chi_{(\tau, \sigma)}(\lambda) = 1$, and $\chi_{(\tau_2, \sigma)}(\lambda)\chi_{(\tau_3, \sigma)}(\lambda) = 1$. Let $\lambda$ denote the unique element in $\{0, 1, \ldots, m-1\}$ such that
\[
\chi_{\lambda}(\lambda) = e^{2\pi \sqrt{-1}/m}.
\]

Let $u_3 \in M$ be defined as in Section 3.1 so that $\langle u_3, b_i \rangle = 1$. We may choose a $\mathbb{Z}$-basis $\{v_1, v_2, v_3\}$ of $\tilde{N}$ such that $\langle u_3, v_i \rangle = \delta_{i, 3}$, and
\[
\tilde{b}_1 = v_1 - \alpha v_2 + v_3, \quad \tilde{b}_2 = \beta v_2 + v_3, \quad \tilde{b}_3 = v_3.
\]

Moreover, the choice $\{v_1, v_2, v_3\}$ is unique if we require $s \in \{0, 1, \ldots, r-1\}$. Let $\{u_1, u_2, u_3\}$ be the $\mathbb{Z}$-basis of $M$ which is dual to the $\mathbb{Z}$-basis $\{v_1, v_2, v_3\}$ of $\tilde{N}$. Let $\{w_1, w_2, w_3\}$ be the $\mathbb{Q}$-basis of $M_Q$ which is dual to the $\mathbb{Q}$-basis $\{\tilde{b}_1, \tilde{b}_2, \tilde{b}_3\}$ of $\tilde{N}_Q = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Then
\[
w_1 = \frac{1}{r} u_1, \quad w_2 = \frac{s}{rm} u_1 + \frac{1}{m} u_2, \quad w_3 = -\frac{s + m}{rm} u_1 - \frac{1}{m} u_2 + u_3.
\]

Moreover, for $i \in \{1, 2, 3\}$,
\[
w_i = e_{\tau}(T_{p_{\sigma} \mathfrak{l}_\tau}) = e_{\tau}(O_{\chi}(\mathcal{D}_i))|_{p_{\sigma}}.
\]

The inclusion $T' \subset T$ induces the following surjective ring homomorphism
\begin{equation}
(14) \quad H^*(\mathbb{B}^{T}; R) = R[u_1, u_2, u_3] \longrightarrow H^*(\mathbb{B}^{T'}; R) = R[u'_1, u'_2], \quad u_1 \mapsto u'_1, \quad u_2 \mapsto u'_2, \quad u_3 \mapsto 0
\end{equation}

where $R = \mathbb{Z}$ or $\mathbb{Q}$.

Given a framing which is an integer $f \in \mathbb{Z}$, let $\mathbb{T}_f \subset \mathbb{T}'$ be the kernel of the character $u'_2 - fu'_1 \in \text{Hom}(\mathbb{T}'; \mathbb{C}^*)$. Then $\mathbb{T}_f \cong \mathbb{C}^*$ is a 1-dimensional subtorus of the Calabi-Yau torus $\mathbb{T}'$. The inclusion $\mathbb{T}_f \subset \mathbb{T}'$ induces a surjective ring homomorphism
\begin{equation}
(15) \quad H^*(\mathbb{B}^{T'}; R) = R[u'_1, u'_2] \longrightarrow H^*(\mathbb{B}^{T_f}; R) = R[u], \quad u'_1 \mapsto u, \quad u'_2 \mapsto fu
\end{equation}
where \( R = \mathbb{Z} \) or \( \mathbb{Q} \). For \( i = 1, 2, 3 \), let \( w'_i \) denote the image of \( w_i \) under the ring homomorphism \( \mathbb{Q} \), and let \( w'_i \) denote the image of \( w'_i \) under the ring homomorphism \( \mathbb{Q} \). Then

\[
w'_1 = \frac{1}{r} u'_1, \quad w'_2 = \frac{s}{rm} u'_1 + \frac{1}{m} u'_2, \quad w'_3 = -\frac{s + m}{rm} u'_1 - \frac{1}{m} u'_2 = -w'_1 - w'_2 \in H^2(B^T) = \mathbb{Q}u'_1 \oplus \mathbb{Q}u'_2,
\]

and \( w'_i = w_i u \), where \( w_i \in \mathbb{Q} \) are given by

\[
w_1 = \frac{1}{r}, \quad w_2 = \frac{s + rf}{rm}, \quad w_3 = -w_1 - w_2 = -\frac{m - s - rf}{rm}.
\]

3.8. Disk factor as equivariant open GW invariants. A framed Aganagic-Vafa Lagrangian brane is a pair \((\mathcal{L}, f)\) where \( \mathcal{L} \) is a Aganagic-Vafa brane together with a choice of a flag \((\tau, \sigma) \in F(\Sigma)\) such that \( \sigma \) is the unique 1-dimensional orbit intersecting \( \mathcal{L} \) and a choice of framing \( f \in \mathbb{Z} \). Given a framed Aganagic-Vafa Lagrangian brane \((\mathcal{L}, f)\), we choose an isomorphism \( \pi_1(\mathcal{L}) \cong \mathbb{Z} \times G_{\tau} \) such that if \( h = \iota_{(\tau, \sigma)}(d_0, \lambda) \) (where \( \iota_{(\tau, \sigma)} \) is defined in Section 3.3) then

\[
\chi_{(\tau_1, \sigma)}(h) = e^{2\pi \sqrt{-1} d_0 w_1}, \quad \chi_{(\tau_2, \sigma)}(h) = e^{2\pi \sqrt{-1} d_0 (w_2 - \frac{1}{m})}, \quad \chi_{(\tau_3, \sigma)}(h) = e^{2\pi \sqrt{-1} d_0 (w_3 + \frac{1}{m})}.
\]

Let \( \ell_\tau \) be the coarse moduli of \( \mathcal{L}_\tau \), as before. Let \( p_\sigma \in \ell_\tau \) be the coarse moduli of \( p_\sigma \cong B G_{\sigma} \), and let \( S_\tau := L \cap \ell_\tau \cong S^1 \) be the coarse moduli of \( S_\tau = \mathcal{L} \cap \ell_\tau \cong S^1 \times B G_{\tau} \).

3.8.1. \((\mathcal{L}, f)\) is a framed outer brane. In this case \( \ell_\tau = \mathbb{C} \). Let \( D \subset \ell_\tau \) be the disk which contains \( p_\sigma \) with boundary \( S_\tau \), oriented by the complex structure on \( \ell_\tau \), and let \( b := |D| \in H_2(X; \mathbb{L}; \mathbb{Z}) \). Given \( (d_0, \lambda) \in H_1(\mathcal{L}; \mathbb{Z}) \cong \mathbb{Z} \times G_{\tau} \), where \( d_0 > 0 \), define

\[
\overline{\mathcal{M}}(d_0, \lambda) := \overline{\mathcal{M}}(0, 1, 1)(X, \mathcal{L} \mid d_0 b, (d_0, \lambda)).
\]

The virtual real dimension of \( \overline{\mathcal{M}}(d_0, \lambda) \) is \( 2(1 - \text{age}(h(d_0, \lambda))) \), where \( h(d_0, \lambda) := \iota_{(\tau, \sigma)}^*(d_0, \lambda) \in G_{\tau} \). Define the disk factor

\[
D_{d_0, \lambda} := \frac{\text{dim}_{d^* (\mathbb{C}, \mathcal{L}, \mathbb{Q})}^{\mathcal{L}, C^*}}{d_0 \partial(d_0, \lambda)}
\]

which is a rational function in \( w'_1, w'_2 \), homogeneous of degree \( \text{age}(h(d_0, \lambda)) - 1 \). The disk factor is computed in [11] when \( G_{\sigma} \) is cyclic, and in [66] Section 3.3] for general \( G_{\sigma} \). In our notation, the formula in [66] Section 3.3] says \(^2\)

\[
D_{d_0, \lambda} = \frac{\text{dim}_{d^* (\mathbb{C}, \mathcal{L}, \mathbb{Q})}^{\mathcal{L}, C^*}}{d_0 \partial(d_0, \lambda)} \cdot \prod_{a_{d^* (\mathbb{C}, \mathcal{L}, \mathbb{Q})}^{\mathcal{L}, C^*}} \left( \text{dim}_{d^* (\mathbb{C}, \mathcal{L}, \mathbb{Q})}^{\mathcal{L}, C^*} \right)^{-1} \left( \text{dim}_{d^* (\mathbb{C}, \mathcal{L}, \mathbb{Q})}^{\mathcal{L}, C^*} \right) - a \left( \text{dim}_{d^* (\mathbb{C}, \mathcal{L}, \mathbb{Q})}^{\mathcal{L}, C^*} \right) \right)
\]

where \( c_i(\cdot) \in \mathbb{Q} \cap [0, 1) \) is defined in Section 2.7. More explicitly,

\[
c_2(h(d_0, \lambda)) = \langle d_0 w_2 - \frac{\lambda}{m} \rangle.
\]

3.8.2. \((\mathcal{L}, f)\) is a framed inner brane. In this case \( \ell_\tau \cong \mathbb{P}^1 \). It contains two torus fixed points \( p_+ = p_\sigma \) and \( p_- = p_{\sigma_-} \), where \( \sigma_- \in \Sigma(3) \). The circle \( S_\tau \) is the intersection of two disks \( D_+ \) and \( D_- \) which contain \( p_+ \) and \( p_- \), respectively. Let \( b := |D| \in H_2(X, \mathbb{L}; \mathbb{Z}) \), \( \alpha = [\ell_\tau] \in H_2(X; \mathbb{Z}) \).

Then \( |D_-| = \alpha - b \in H_2(X, \mathbb{L}; \mathbb{Z}) \). Given \( (d_0, \lambda) \in H_1(\mathcal{L}; \mathbb{Z}) \cong \mathbb{Z} \times G_{\tau} \), where \( d_0 \neq 0 \), we define

\[
\overline{\mathcal{M}}(d_0, \lambda) := \begin{cases} \overline{\mathcal{M}}(0, 1, 1)(X, \mathcal{L} \mid d_0 b, (d_0, \lambda)), & d_0 > 0, \\ \overline{\mathcal{M}}(0, 1, 1)(X, \mathcal{L} \mid -d_0 \alpha + b, (d_0, \lambda)), & d_0 < 0. \end{cases}
\]

Then

\[
\text{virtual dimension of } \overline{\mathcal{M}}(d_0, \lambda) = \begin{cases} 1 - \text{age}(h^+(d_0, \lambda)), & d_0 > 0, \\ 1 - \text{age}(h^-(d_0, \lambda)), & d_0 < 0, \end{cases}
\]

\(^2\)The disk function in [66] Section 3.3] and our disk factor are the same when \( h(d_0, \lambda) \neq 0 \). When \( h(d_0, \lambda) = 0 \), the disk function is \( (\cdot)^X \cap \mathcal{L} \) (no insertion), while the disk factor is \( (1)^X \cap \mathcal{L} \) (one insertion of 1), so there is an additional factor of \( \left( \frac{1}{w_1} \right)^{h(b, h(d_0, \lambda))} \) in the disk function in [66] Section 3.3].
where $h^\pm(d_0, \lambda) = \ell_\tau^{(\tau, \sigma, \pm)}(d_0, \lambda) \in G_{\sigma, \pm}$. Define

$$D_{d_0, \lambda} := \begin{cases} 
(1_{h^+(d_0, \lambda)})_{0, d_0, b, c}(d_0, \lambda), & d_0 > 0, \\
(1_{h^-(d_0, \lambda)})_{0, d_0, a, b, c}(d_0, \lambda), & d_0 < 0.
\end{cases}$$

Then $D_{d_0, \lambda}$ is a rational function in $u'^1_1, u'^1_2$, homogeneous of degree $\text{deg}(h^\pm(d_0, \lambda)) - 1$ if $\pm d_0 > 0$.

More precisely, the disk factor $D_{d_0, \lambda}$ is defined up to a sign depending on choice of orientation of $\mathcal{M}(d_0, \lambda)$, which will be clarified in Section 3.11 by relative GW invariants.

3.9. Normal bundle to $I_\tau$. Let $L$ be an inner brane, so that $I_\tau$ is a proper smooth toric DM curve. Let $\hat{I}_\tau$ be the image of $I_\tau$ under the morphism $X \to X^{rig}$. We have

$$I_\tau \longrightarrow \hat{I}_\tau \longrightarrow \hat{I}^{rig}_\tau \longrightarrow \ell_\tau \cong \mathbb{P}^1,$$

where $I_\tau \to \hat{I}_\tau$ is a $K$-banded gerbe, $\hat{I}_\tau \to \hat{I}^{rig}_\tau$ is a $\mu_\tau$-banded gerbe, and $I_\tau \to \ell_\tau$ is a $G_\tau$-banded gerbe. The normal bundle $I_\tau$ in $X$ is a direct sum of two $T$-equivariant line bundles over $I_\tau$:

$$N_{I_\tau/X} = L_2 \oplus L_3,$$

where $L_2 = \mathcal{O}_X(D_2)|_{I_\tau}$ and $L_3 = \mathcal{O}_X(D_3)|_{I_\tau}$. The total space of $N_{I_\tau/X}$ is a smooth toric DM stack which is isomorphic to the open substack $\mathcal{Y} := X_\sigma \times \mathcal{X}_\tau$ of $X$. Let $\hat{D}_\tau$ be the image of $D_\tau$ under $X \to X^{rig}$. Then

$$N_{I_\tau/X^{rig}} = \hat{L}_2 \oplus \hat{L}_3,$$

where $\hat{L}_2 = \mathcal{O}_{X^{rig}}(\hat{D}_2)|_{I_\tau}$ and $\hat{L}_3 = \mathcal{O}_{X^{rig}}(\hat{D}_3)|_{I_\tau}$. The total space of $N_{I_\tau/X^{rig}}$ is a toric orbifold which is isomorphic to the open substack $\mathcal{Y}^{rig} = X^{rig}_\sigma \times X^{rig}_\tau$ of the toric orbifold $X^{rig}$.

Let $\Sigma_0$ be the simplicial fan in $N_\mathbb{R}$ consisting of $\sigma, \sigma_-$ and their subcones. The stacky fan of $\mathcal{Y}^{rig}$ is given by $(\Sigma_0, (b_1, b_2, b_3, b_4))$, where

$$\delta_1 = qv_1 - sv_2 + v_3, \quad \delta_2 = mv_2 + v_3, \quad \delta_3 = v_3, \quad \delta_4 = -v_1 + cv_2 + v_3,$$

where $c$ is some integer. For inner branes, we also denote $\tau_+ = \tau$. We have $Y = [U/G_0]$, where

$$U = \{(Z_1, Z_2, Z_3, Z_4) \in \mathbb{C}^4 : (Z_1, Z_4) \neq (0, 0), \}$$

$$G_0 = \{(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4 : t_1^i t_2^j t_3^k t_4^l = (t_1)^{-i} (t_2)^{-j} (t_3)^{-k} (t_4)^{-l} = 1\}.$$}

We have a short exact sequence of abelian groups:

$$1 \to \mu_m \longrightarrow G_0 \xrightarrow{\chi \times \chi \times \chi} G_{\tau, \tau} \longrightarrow 1,$$

where $G_{\tau, \tau} = \{(1_1, 1_4) \in (\mathbb{C}^*)^2 : t_1^i t_4^{-i} = 1\}$, and $\chi(t_1, t_2, t_3, t_4) = t_1$. The subgroup $\mu_m$ of $G_0$ acts trivially on $V = \{(Z_1, Z_2, Z_3, Z_4) \in U_A : Z_2 = Z_3 = 0\}$, so the $G_0$-action on $V$ factors through a $G_{\tau, \tau}$-action on $V$, and

$$\hat{I}_\tau = [V/G_0], \quad \mathcal{I}^{rig}_\tau = [V/G_{\tau, \tau} \cong \mathcal{F}_{\tau, \tau}.$$}

where $\mathcal{F}_{\tau, \tau}$ denotes the football obtained by gluing $[\mathbb{C}^*/\mu_\tau]$ and $[\mathbb{C}^*/\mu_{-\tau}]$ along $[\mathbb{C}^*/\mu_{\tau}] \cong [\mathbb{C}^*/\mu_{-\tau}] \cong \mathbb{C}^*$. The two torus fixed points in $\mathcal{I}^{rig}_\tau$ are

$$p_x = [(0, 0, 0) \times \mathbb{C}^*)/G_{\tau, \tau}] \cong B_{\mu_\tau}, \quad p_y = [((0, 0, 0)) \times \mathbb{C}^*)/G_{\tau, \tau}] \cong B_{\mu_{-\tau}},$$

and $\mathcal{I}^{rig}_\tau \sim p_x, p_y \cong \mathbb{C}^*$. We have a surjective group homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \text{Pic}(\mathcal{I}^{rig}_\tau)$ sending $(n_x, n_y)$ to $O^{rig}_{\mathcal{I}^{rig}_\tau}(n_x p_x + n_y p_y)$; the kernel is $\mathbb{Z}(t - \tau)$.

Let $\mathcal{O}(-1)$ denote the tautological line bundle over $\mathcal{B}_C^* \cong GL(1, \mathbb{C}), t \mapsto t$. Given a line bundle $L$ over a DM stack $Z$ and a positive integer $m$, let $\sqrt[m]{L/Z}$ denote the following fiber product (cf. [2, Definition 2.2.6]):

$$\sqrt[m]{L/Z} = Z \times_{\mathcal{B}_C^*} \mathcal{B}_C^* \longrightarrow \mathcal{B}_C^* \quad \text{by } \phi_L \quad \bigg|_{\mathcal{O}(m)}.$$
where the morphism $\phi_L : Z \to BC^*$ is defined by $L$ (so that $\phi_L^* O(-1) = L$), and $BC^* \to BC^*$ is induced by the $m$-th power map from $C^*$ to itself. Then $p_1 : \sqrt{L}/Z \to Z$ is a $\mu_m$-banded gerbe. Let $\sqrt{L} := p_2^* O(-1) \in$ Pic($\sqrt{L}/Z$). Then ($\sqrt{L})^\otimes_m = p_1^* L$, i.e., $\sqrt{L}$ is an $m$-th root of $p_1^* L$.

It is straightforward to check that

- $\hat{1}_r$ is isomorphic to $\sqrt{O_{\mu_m}(sp_x - cp_y)/\hat{1}_r^\circ}$ as a $\mu_m$-banded gerbe over $\hat{1}_r^\circ \cong F_{r,-}$, and
- $\hat{L}_2 \cong \sqrt{O_{\mu_m}(sp_x - cp_y)}, \hat{L}_3 \cong \hat{L}_2^{-1} \otimes p_1^* O_{\mu_m}(-sp_x - cp_y)$, where $p_1 : \hat{1}_r \to \hat{1}_r^\circ$ and $O_{\mu_m}(-sp_x - cp_y)$ is the cotangent bundle of $\hat{1}_r^\circ$.

3.10. **Degeneration.** Let $\Sigma_1 = \{(0, R_{\geq 0} v_1, R_{< 0} (-v_1)) \}$ be the complete 1-dimensional fan in $R v_1 \cong \mathbb{R}$, and let $\Sigma_2 = \{(0, R_{\geq 0} v_4, R_{< 0} (-v_4)) \}$ be the complete 1-dimensional fan in $R v_4 \cong \mathbb{R}$. Then $X_{\Sigma_1} = X_{\Sigma_2} \cong \mathbb{P}^1$. The stacky fan $(\Sigma_1, (v_1, -v_4))$ defines the 1-dimensional toric orbifold $F_{r,-}$, and the stacky fan

$$\Sigma_2 = (\Sigma_1 \times \Sigma_2, (b_1^0 = v_1, b_3 = -v_4, b_4 = -v_4))$$

defines the 2-dimensional toric orbifold $F_{r,-} \times \mathbb{P}^1$. The 1-dimensional cones in the fan $\Sigma_1 \times \Sigma_2$ are \{\(\rho_i \in R_{\geq 0} b_i^0 : 1 \leq i \leq 4\). Let $S'$ be the fan obtained by adding a 1-dimensional cone $\rho_5 = R_{\geq 0} b_5^0$ where $b_5^0 = -v_1 - v_4$. Let $S'$ be the 2-dimensional toric orbifold defined by the stacky fan $\Sigma' = (\Sigma_1 \times \Sigma_2, (b_1^0, b_2^0, b_4^0, b_5^0))$, and let $\hat{t}_r' = V(\rho_1) \subset S'$ be 1-dimensional closed toric substack associated with the ray $\rho_1$. The morphism $\Sigma' \to \Sigma_2$ of stacky fans induces a morphism $\nu : S' \to F_{r,-} \times \mathbb{P}^1$ of toric orbifolds; $\nu$ contracts the divisor $\hat{t}_r'$ to the torus fixed point $[0,1] \times [0,1] \cong B \mu_{\tau, \nu}$ in $F_{r,-} \times \mathbb{P}^1$. Let $p : F_{r,-} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection to the second factor. The composition $\pi' := p \circ \nu$ is a flat morphism, and

$$\pi^{-1}((0,1) = \hat{t}_r', \pi^{-1}([1,0]) = \hat{t}_r' \cup \hat{t}_5,$$

where $\hat{t}_r' \cong F_{r,-}$, and $\hat{t}_5 \cong F_{r,1}$. The torus fixed points in $S'$ are

$$p_0^0 = \hat{t}_r'^0 \cap \hat{t}_5^0 \cong B \mu_{\nu}, \quad p_y^0 = \hat{t}_r'^0 \cup \hat{t}_5^0 \cong B \mu_{\tau, \nu}, \quad p_0^\infty = \hat{t}_r'^0 \cap \hat{t}_5^0 \cong B \mu_{\tau, \nu}, \quad p_y^\infty = \hat{t}_r'^0 \cup \hat{t}_5^0 \cong B \mu_{\tau, \nu}, \quad p_z = \hat{t}_r'^0 \cap \hat{t}_5^0,$$

where $p_z$ is a scheme point.

Given any $f \in \mathbb{Z}$, define

$$\hat{S} = \sqrt{O_{S'}(sp_x^0 - cp_y^0)/S'}$$

which is a $\mu_m$-banded gerbe over $S'$, and let $\hat{q} : \hat{S} \to S' = \hat{S}^{rig}$ be the morphism to the rigidification. Define $\pi := \hat{q} \circ \pi' : S' \to \mathbb{P}^1$, and let $\hat{t}_i \subset \hat{S}$ be the divisor which corresponds to $\hat{t}_i' \subset S'$ under $\hat{q} : \hat{S} \to S'$. Then $\hat{q}_i := \hat{q}|_{\hat{t}_i} : \hat{t}_i \to \hat{t}_i' = \hat{t}_i^{rig}$ is a $\mu_m$-banded gerbe. We have

$$\pi^{-1}((0,1) = \hat{t}_4 \cong \hat{t}_r', \pi^{-1}([1,0]) = \hat{t}_4 \cup \hat{t}_5.$$ 

Define $\hat{L}_2, \hat{L}_3 \in$ Pic($\hat{S}$) by

$$\hat{L}_2 := \sqrt{O_{\hat{S}'}(sp_x^0 - cp_y^0)/\hat{S}} \quad \hat{L}_3 := \hat{L}_2^{-1} \otimes q^* O_{\hat{S}'}(-t_1 - t_2).$$

Then

$$\hat{L}_2|_{\hat{t}_4} = \sqrt{O_{\hat{t}_4'}(sp_x^0 - cp_y^0)} \cong \hat{L}_2, \quad \hat{L}_3|_{\hat{t}_4} = \sqrt{O_{\hat{t}_4'}(-sp_x^0 + cp_y^0) \otimes \hat{q}_4^* O_{\hat{t}_4'}(-p_x - p_y)} \cong \hat{L}_3.$$

For $i \in \{2,3\}$, define $\hat{L}_i^+ = \hat{L}_i|_{\hat{t}_4}$ and $\hat{L}_i^- = \hat{L}_i|_{\hat{t}_5}$. Then

$$\hat{L}_2^+ = \sqrt{O_{\hat{t}_4'}(sp_x^0 + f p_z)}, \quad \hat{L}_2^- = \sqrt{O_{\hat{t}_5'}(-cp_y^0 - f p_z)}, \quad \hat{L}_3^+ = \sqrt{O_{\hat{t}_4'}(-sp_x^0 - f p_z) \otimes \hat{q}_4^* O_{\hat{t}_4'}(-p_x - p_y)}, \quad \hat{L}_3^- = \sqrt{O_{\hat{t}_5'}(cp_y^0 + f p_z) \otimes \hat{q}_5^* O_{\hat{t}_5'}(-p_y^\infty)}.$$

To summarize:

- $\hat{S}$ is a degeneration from $\hat{t}_r$ to a nodal DM curve $\hat{t}_4 \cup \hat{t}_5$, and $S' = \hat{S}^{rig}$ is a degeneration from the football $\hat{t}_r^{rig} \cong F_{r,-}$ to the nodal DM curve $\hat{t}_4 \cup \hat{t}_5$.
- For $i = 2,3$, the line bundle $\hat{L}_i$ on $\hat{S}$ defines a degeneration of the line bundle $\hat{L}_i \to \hat{t}_r$ to a line bundle on $\hat{t}_4 \cup \hat{t}_5$ which restricts to $\hat{L}_i^+$ on $\hat{t}_4$ and $\hat{L}_i^-$ on $\hat{t}_5$.
- $\hat{S}, \hat{L}_i, \hat{t}_4, \hat{t}_5, \hat{L}_i^+$ depend on $f$, while $S', \hat{t}_4, \hat{t}_5$ do not.
Moreover, the total space of $\tilde{L}_2 \oplus \tilde{L}_3 \to \tilde{S}$ is a four dimensional toric orbifold $\tilde{W}$ defined by a stacky fan
\begin{equation}
(\Sigma_f, (b_1, b_2, b_3, b_4, v_4, -v_4, -v_1 - f v_2 - v_4))
\end{equation}
where $\Sigma_f$ is a simplicial fan in $\bigoplus_{i=1}^4 \mathbb{R} v_i$, and $(\tilde{b}_1, \ldots, -v_1 - f v_2 - v_4)$ is a 7-tuple of vectors in $\tilde{N} \oplus \mathbb{Z} v_4 \cong \mathbb{Z}^4$.

Let $\mathcal{W}$ be the four dimensional smooth toric DM stack defined by the the stacky fan
\begin{equation}
(\Sigma_f, (b_1, b_2, b_3, b_4, v_4, -v_4, -v_1 - f \tilde{v}_2 - v_4)),
\end{equation}
where $\tilde{v}_1, \tilde{v}_2 \in N$ are lifts of $v_1, v_2 \in \tilde{N}$, so that $(b_1, \ldots, -\tilde{v}_1 - f \tilde{v}_2 - v_4)$ is a 7-tuple of elements in $N \oplus \mathbb{Z} v_4$.

Then $\mathcal{W}$ is a $K$-banded gerbe over $\tilde{W} = \mathcal{W}_{\text{rig}}$, and is a degeneration from the total space $\mathcal{Y}_\infty$ of $N_{l_+/\lambda}$ to the total space $\mathcal{Y}_\infty$ of a direct sum $L_2^\infty \oplus L_3^\infty$ of line bundles over a nodal DM curve $l_+ \cup l_-$. $l_+ \cup l_-$ is a $K$-banded gerbe over $l_4 \cup \bar{l}_5$ and a $G_r$-banded gerbe over $l'_4 \cup l'_5$. For $i = 2, 3$, let $L_i^\pm = L_i^\infty|_{l_i}$. Then $L^+_i$ is the pullback of $\tilde{L}^+_i$. Let $p_0 \equiv BG_\tau$ be the node which is the intersection of $l_+$ and $l_-$ and let $p_\pm$ be the unique torus fixed point in $l_\pm - \{p_0\}$. Then $p_\pm \equiv BG_\tau$ and $p_- \equiv BG_\tau$. Define $T'$ weights
\begin{align*}
w^+_1 &= : (c_1)_{T^2}^0 (T_{p_0}^+ 1^\pm), & w^+_2 &= : (c_1)_{T^2}^0 (L_2^0 1^\pm), & w^+_3 &= : (c_1)_{T^2}^0 (L_3^0 1^\pm) \in H^2(BT') = \mathbb{Q} u_1' \oplus \mathbb{Q} u_2',
\end{align*}
Then $w^+_i = w'_i$ is given by Equation (19), $w^+_1 + w^+_2 + w^+_3 = 0$, and
\begin{equation}
w^-_1 = \frac{1}{r_-} u'_1, \quad w^-_2 = \frac{c}{r_- m} u'_1 + \frac{1}{m} u'_2, \quad w^-_3 = \frac{-c + m}{r_- m} u'_1 - \frac{1}{m} u'_2.
\end{equation}
We also have
\begin{align*}
(c_1)_{T^2}^0 (T_{p_0}^- 1^\pm) &= \mp u'_1, & (c_1)_{T^2}^0 (L_2^0 1^-) &= \mp \frac{u'_2 - f u'_1}{m} = -(c_1)_{T^2}^0 (L_3^0 1^-) p_0.
\end{align*}
The above weights are summarized in Figure 7 below.

![Figure 7](image)

**Figure 7.** Degenerated $N_{l_+/\lambda}$ and the $T'$-weights

### 3.11. Disk factor as equivariant relative GW invariants.

When $d_0 > 0$, let $\mathcal{M}_{0,1}(l_+/p_0,(d_0, \lambda))$ be the moduli space of relative maps to $(l_+,p_0)$ with the relative condition $(d_0, \lambda)$, where $\lambda \in G_r$ [1]. A relative stable map to $(l_+,p_0)$ is a morphism to $l_+[m]$ which is the union of $l_+$ and a chain of $m$ copies of $\mathbb{P}^1 \times BG_\lambda$. Let $\mathcal{M}_{0,1}(l_+/p_0,(d_0, \lambda)) \subset \mathcal{M}_{0,1}(l_+/p_0,(d_0, \lambda))$ be the open substack where the target is $l_+[0] = l_+$. The tangent space $T^*_\xi$ and the obstruction space $T^*_\xi$ at a moduli point $\xi = [u : (C, x, y) \to (l_+,p_0)]$ in $\mathcal{M}_{0,1}(l_+/p_0,(d_0, \lambda))$ (where $u^{-1}(p_0) = d_0 y$ as Cartier divisors) fit into the following exact sequence of complex vector spaces:
\begin{align}
0 &\to \text{Ext}^0(\Omega_C(x + y), \mathcal{O}_C) \to H^0(C, u^* (T_{l_+}(-p_0))) \to T^*_\xi^1 \\
&\to \text{Ext}^1(\Omega_C(x + y), \mathcal{O}_C) \to H^1(C, u^* (T_{l_+}(-p_0))) \to T^*_\xi^2.
\end{align}

Globally on $\mathcal{M}_{0,1}(l_+/p_0,(d_0, \lambda))$, there is an exact sequence of sheaves
\begin{equation}
0 \to B_1 \to B_2 \to T^1 \to B_4 \to B_5 \to T^2 \to 0
\end{equation}
whose fiber at the moduli point $\xi$ is [21].

Let $\pi : \mathcal{U}_+ \to \mathcal{M}_{0,1}(l_+/p_0,(d_0, \lambda))$ be the universal domain curve and let $F_+ : \mathcal{U}_+ \to l_+$ be the evaluation map. We define
\begin{align*}
V_{\mathcal{M}_{0,1}}^+ &= R^* \pi_* F_+^* (L_{l_+}^+ \oplus L_{l_+}^-) \in K_{T^*}^*(\mathcal{M}_{0,1}(l_+/p_0,(d_0, \lambda)))
\end{align*}
where $R^* \pi_*$ is the K-theoretic push-forward.
For $d_0 > 0$, we define
\[
D_{d_0, \lambda} = (1_{h^+(d_0, \lambda)})_{\mathcal{X}, \mathcal{L}, T} \mathcal{M}_{0,1}(\mathcal{L}_- / \mathcal{L}_+),
\]
and define $\pi$ and the following identities relative to condition $(\mathcal{L}_- / \mathcal{L}_+)$.

When $d_0 < 0$, let $\mathcal{M}_{0,1}(\mathcal{L}_- / \mathcal{L}_+(d_0, \lambda^{-1}))$ be the moduli space of relative stable maps to $(\mathcal{L}_- / \mathcal{L}_+)$ with relative condition $(-d_0, \lambda^{-1})$, and let $\mathcal{M}_{0,1}(\mathcal{L}_- / \mathcal{L}_+(d_0, \lambda^{-1}))$ be the open substack where the target is $\mathcal{L}_-$. Let $\pi : \mathcal{M}_{0,1}(\mathcal{L}_- / \mathcal{L}_+(d_0, \lambda^{-1}))$ be the universal domain curve and let $F_+ : \mathcal{U}_{\mathcal{L}_+} \to \mathcal{L}_+$ be the evaluation map. We define
\[
V_{0,1} = R^\bullet \pi_* F^+_{\mathcal{L}_2} \mathcal{L}_3 \in K_{\mathcal{U}_{\mathcal{L}_+}} \mathcal{M}_{0,1}(\mathcal{L}_- / \mathcal{L}_+(d_0, \lambda^{-1}))
\]
and define
\[
D_{d_0, \lambda} = (1_{h^+(d_0, \lambda)})_{\mathcal{X}, \mathcal{L}, T} \mathcal{M}_{0,1}(\mathcal{L}_- / \mathcal{L}_+(d_0, \lambda^{-1}))
\]
and define $\pi$ and the following identities relative to condition $(\mathcal{L}_- / \mathcal{L}_+)$.

Let $u : (C, x, y) \to \mathcal{L}_+$ be a relative stable map which represents a point in $\mathcal{M}$. Suppose that $u$ is fixed by the torus action. Recall that $c_1 : G_{\mathcal{L}_+} \to [0, 1] \cap \mathbb{Q}$ is defined by $c_1(k) = \exp(2\pi \sqrt{-1} c_1(k))$. In the computation below, let $k^\pm = h^\pm(d_0, \lambda)$. For $j = 1, 2, 3$, let $\epsilon_j = c_j(k^+)$. Then $\epsilon_1 = \langle \frac{d_0 w_1}{s_1^+} \rangle$. We have the following weights
\[
\text{ch}_{\mathcal{U}_{\mathcal{L}_+}}(H^0(C, u^* L_1^+)) = \sum_{a=0}^{d_0 w_1} e^{\frac{a}{w_1^+}}, \quad \text{ch}_{\mathcal{U}_{\mathcal{L}_+}}(H^1(C, u^* L_1^+)) = 0,
\]
\[
\text{ch}_{\mathcal{U}_{\mathcal{L}_+}}(H^0(C, u^* L_2^+ \otimes \mathcal{O}_y)) = \delta_{\langle d_0 w_2 - \epsilon_2 \rangle} \frac{e^{\frac{w_1^+ - f_1^+}{m}}}{s_1^+ m}, \quad \text{ch}_{\mathcal{U}_{\mathcal{L}_+}}(H^1(C, u^* L_2^+ \otimes \mathcal{O}_y)) = 0,
\]
and the following identities
\[
\sum_{a=-\langle w_3 d_0 - \epsilon_3 \rangle}^{0} e^{\frac{w_3^+ + \epsilon_3}{w_3^+}} = \sum_{a=d_0 w_1 + \epsilon_2 + \epsilon_3}^{d_0 w_2 - \epsilon_2 - \delta_{\langle d_0 w_2 - \epsilon_2 \rangle}} e^{\frac{w_2^+ + \epsilon_2 - a}{w_2^+}}
\]
and
\[
\sum_{a=-\langle w_3 d_0 + 1 - \epsilon_3 \rangle}^{0} e^{\frac{w_3^+ + \epsilon_3}{w_3^+}} = \sum_{a=-\langle d_0 w_2 - \epsilon_2 \rangle + \delta_{\langle d_0 w_2 - \epsilon_2 \rangle}}^{d_0 w_1 + \epsilon_2 + \epsilon_3 - 1} e^{\frac{w_2^+ + \epsilon_2 - a}{w_2^+}}
\]
and
\[
d_0 w_1 + \epsilon_2 + \epsilon_3 = \frac{d_0}{s_1^+} + \text{age}(k^+).
\]
The $T'$-equivariant Euler classes are
\[ e_{T'}(B_2^m) = 1, \]
\[ e_{T'}(B_3^m) = |d_0 w_1| \left( \frac{u'_1}{d_0} \right)^{|d_0 w_1|}. \]
\[ \frac{e_{T'}(B_3^m)}{e_{T'}(B_2^m)} = (-1)^{|d_0 w_2 - \epsilon_2| + |d_0 w_1| + \text{age}(k^+) - 1} \prod_{a=1}^{d_0 w_1} (w_2 + (a - \epsilon_2) \frac{u'_1}{d_0}). \]

Since $|\text{Aut}(f)| = d_0 |G_T|$, by localization
\[ D_{d_0, \lambda} = D(d_0, k^+, k^-) = \frac{1}{|\text{Aut}(f)|} \frac{e_{T'}(B_3^m) e_{T'}(B_2^m)}{e_{T'}(B_1^m) e_{T'}(B_3^m)} \]
\[ = \frac{1}{d_0 |G_T|} \prod_{a=1}^{d_0 w_1} |d_0 w_1| \left( \frac{u'_1}{d_0} \right)^{|d_0 w_1|} \frac{1}{|d_0 w_1|} \prod_{a=1}^{d_0 w_1} (w_2 + (a - \epsilon_2) \frac{u'_1}{d_0}). \]
\[ = (-1)^{|d_0 w_2 - \epsilon_2| + |d_0 w_1| + \text{age}(k^+) - 1} (\frac{u'_1}{d_0})^{\text{age}(k^+) - 1} \frac{1}{d_0 |G_T|} \prod_{a=1}^{d_0 w_1} (w_2 + (a - \epsilon_2) \frac{u'_1}{d_0}). \]

Let $u := \iota_f^* u'_1 \in H^2(T_f; \mathbb{Z})$. Then $H^*(T_f; \mathbb{Q}) = \mathbb{Q}[u]$ and $\iota_f^* u'_2 = fu$. Define $D_{d_0, \lambda, f} = \iota_f^* D_{d_0, \lambda}$. Hence when $d_0 > 0$
\[ D_{d_0, \lambda, f} = (-1)^{|d_0 w_2 + \frac{1}{2}|} (\frac{u}{d_0})^{\text{age}(h^+(d_0, \lambda)) - 1} \frac{1}{-d_0 |G_T|} \prod_{a=1}^{d_0 w_1} (w_2 + c_2 h^+(d_0, \lambda)). \]
If $d_0 < 0$, similar computation shows (notice $\lambda^{-1} = (1 - \delta_\lambda, 0)(m - \lambda) \in \{0, \ldots, m - 1\}$)
\[ D_{d_0, \lambda, f} = (-1)^{|d_0 w_2 - \frac{1}{2}|} (\frac{u}{d_0})^{\text{age}(h^-(d_0, \lambda)) - 1} \frac{1}{d_0 |G_T|} \prod_{a=1}^{d_0 w_1} (w_2 + c_2 h^-(d_0, \lambda) + a). \]
If $L$ is an outer brane, it is the same as $d_0 > 0$. Define
\[ D_{d_0, \lambda, f} = (-1)^{|d_0 w_2 + \frac{1}{2}|} (\frac{u}{d_0})^{\text{age}(h(d_0, \lambda)) - 1} \frac{1}{d_0 |G_T|} \prod_{a=1}^{d_0 w_1} (w_2 + c_2 h(d_0, \lambda)). \]

3.12. **Open-closed GW invariants and descendant GW invariants.** For any torus fixed point $p_\sigma$ of $X$, where $\sigma \in \Sigma(3)$, we have
\[ H^*_{\text{CR}}(p_\sigma) = \bigoplus_{k \in G_\sigma} \mathbb{Q}[1_k], \quad H^*_{\text{CR}, T'}(p_\sigma) = \bigoplus_{k \in G_\sigma} \mathbb{Q}[w'_1, w'_2, 1_k]. \]
The inclusion $\iota_\sigma : p_\sigma \to X$ induces
\[ \iota_\sigma^* : H^*_{\text{CR}, T'}(p_\sigma) = H^*_{\text{CR}, T'}(I p_\sigma) \to H^*_{\text{CR}, T'}(X) = H^*_{\text{CR}, T'}(X). \]
Define
\[ \phi_{\sigma, k} = \iota_\sigma^* 1_k \in H^*_{\text{CR}, T'}(X), \quad \phi_{\sigma, k}^f = \iota_f^* \phi_{\sigma, k} \in H^*_{\text{CR}, T'}(X). \]

**Proposition 3.2** (framed inner brane). Suppose that $(L, f)$ is a framed inner brane, and $\bar{\mu} = ((\mu_1, \lambda_1), \ldots, (\mu_h, \lambda_h))$,
where $(\mu_j, \lambda_j) \in H_1(L; \mathbb{Z}) \cong \mathbb{Z} \times BG_T$. Let $J_\pm = \{ j \in \{1, \ldots, h \} : \pm \mu_j > 0 \}$, and let $k_+^f = h^+(\mu_j, \lambda_j)$. Then
\[ D_{\mu_j^+, \lambda_j, f} \left( \Pi_{j=1}^h \text{ev}^*_{\gamma_i} \Pi_{j \in J_+} \text{ev}^*_{n+j} \phi_{\sigma, (h^+(\lambda_j), \lambda_j) - 1} \Pi_{j \in J_-} \text{ev}^*_{n+j} \phi_{\sigma, (h^-(\lambda_j), \lambda_j) - 1} \right) \]
\[ = \frac{1}{\mu_j} \left( \frac{u}{\mu_j} - \psi_{n+j} \right). \]
where
\[ \beta \in H_2(\mathcal{X}), \quad \beta' = \beta + \left( \sum_{j \in J_+} \mu_j \right) b - \left( \sum_{j \in J_-} \mu_j \right) (\alpha - b) \in H_2(\mathcal{X}, \mathcal{L}). \]

**Proof.** There exists \( \beta \in H_2(\mathcal{X}) \) such that
\[ \beta' = \beta + \left( \sum_{j \in J_+} \mu_j \right) b + \sum_{j \in J_-} (-\mu_j)(\alpha - b). \]

Let \( \langle k_j^+ \rangle \) be the cyclic subgroup generated by \( k_j^+ \), and let \( r_j \) be the cardinality of \( \langle k_j^+ \rangle \) for \( j \in J_+ \) and \( r_j \) be the cardinality of \( \langle k_j^- \rangle \) for \( j \in J_- \).

We have
\[\begin{align*}
\overline{\mathcal{M}}_{(g,n),\mathcal{L}}(\mathcal{X} | \beta', \tilde{\mu})^{\mathcal{T}\mathcal{R}}_k &= \bigcup_{r \in G_{g,n}(\mathcal{X}, \mathcal{L} | \beta', \tilde{\mu})} \mathcal{V}_{\mathcal{T}\mathcal{R}}_k \\
\overline{\mathcal{M}}_{g,n+h}(\mathcal{X}, \beta)^{\mathcal{T}\mathcal{R}}_k &= \overline{\mathcal{M}}_{g,n+h}(\mathcal{X}, \beta)^{\mathcal{T}\mathcal{R}}_k = \bigcup_{\tilde{r} \in G_{g,n+h}(\mathcal{X}, \beta)} \mathcal{V}_{\mathcal{T}\mathcal{R}}_k
\end{align*}\]

In the remaining part of this subsection, we use the following abbreviations:
\[ \mathcal{M} = \overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta), \quad \mathcal{N} = \overline{\mathcal{M}}_{g,n+h}(\mathcal{X}, \beta), \]
\[ \mathcal{M}_j = \begin{cases} \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X}, \beta), & j \in J_+ \\ \overline{\mathcal{M}}_{(0,1),1}(\mathcal{X}, \beta), & j \in J_- \end{cases} \]
\[ \mathcal{G} = G_{g,n}(\mathcal{X}, \beta), \quad \mathcal{G} = G_{g,n+h}(\mathcal{X}, \beta) \]
\[ \mathcal{x} = (x_1, \ldots, x_h), \quad \mathbf{y} = (y_1, \ldots, y_h). \]

Given \( u : (\Sigma, \mathbf{x}, \partial \Sigma) \to (\mathcal{X}, \mathcal{L}) \) which represents a point \( \xi \in \mathcal{N}^{\mathcal{T}\mathcal{R}}_k \), we have
\[ \Sigma = \mathcal{C} \cup \left( \bigcup_{j=1}^h D_j \right), \]

where \( \mathcal{C} \) is an orbicurve of genus \( g \), \( x_1, \ldots, x_n \in \mathcal{C} \), \( D_j = \{ z \in \mathbb{C} \mid |z| \leq 1 \}/\mathbb{Z}_{r_j} \}, \mathcal{C} \) and \( D_j \) intersect at \( y_j = B \mathbb{Z}_{r_j} \). Let \( u_j = u_{D_j} \) and \( \tilde{u} = u_{|\mathcal{C}}. \) Then

1. For \( j = 1, \ldots, h \), \( u_j : (D_j, \partial D_j) \to (\mathcal{X}, \mathcal{L}) \) represents a point in \( \mathcal{M}_j^{\mathcal{T}\mathcal{R}}_k \).
2. \( \tilde{u} : (\mathcal{C}, \mathbf{x}, \mathbf{y}) \to \mathcal{X} \) represents a point \( \tilde{\xi} \in \mathcal{M}^{\mathcal{T}\mathcal{R}}_k \), and \( \tilde{u}(y_j) = [p_{\pm}, (k_j^\pm)^{-1}] \in \mathcal{L}p_{\pm} \subset \mathcal{L}\mathcal{X} \) if \( j \in J_{\pm} \).

Let \( x_{n+j} = y_j \). Let \( F_\Gamma \) be the connected component of \( \mathcal{M}_\Gamma^{\mathcal{T}\mathcal{R}}_k \) associated to the decorated graph \( \Gamma \in \mathcal{G} \), and let \( \hat{F}_\Gamma \) be the connected component of \( \hat{\mathcal{M}}_\Gamma^{\mathcal{T}\mathcal{R}}_k \) associated to the decorated graph \( \hat{\Gamma} \in \hat{\mathcal{G}} \). Then for any \( \Gamma \in \mathcal{G} \) there exists \( \hat{\Gamma} \in \hat{\mathcal{G}} \) such that
\[ ev_{n+j}(F_\Gamma) = (p_{\pm}, (k_j^\pm)^{-1}) \in \mathcal{L}p_{\pm} \subset \mathcal{L}\mathcal{X} \]
if \( j \in J_{\pm} \), and \( F_\Gamma \) can be identified with \( \hat{F}_\Gamma \) up to a finite morphism. More precisely,
\[ [F_\Gamma]^{\mathcal{T}\mathcal{R}}_k = \prod_{j \in J_+} \frac{|G_{\Sigma_+}|}{r_j |\text{Aut}(u_j)|} \prod_{j \in J_-} \frac{|G_{\Sigma_-}|}{r_j |\text{Aut}(u_j)|} [F_\Gamma]^{\mathcal{T}\mathcal{R}}_k \]
\[ = \prod_{j \in J_+} \frac{s_j^+}{r_j \mu_j} \prod_{j \in J_-} \frac{s_j^-}{r_j \mu_j} [F_\Gamma]^{\mathcal{T}\mathcal{R}}_k. \]

We have
\[ \frac{1}{ev_{\mathcal{X}}(N^{\mathcal{T}\mathcal{R}}_k)} = ev_{\mathcal{X}}(B_1^m)ev_{\mathcal{X}}(B_2^m)ev_{\mathcal{X}}(B_3^m), \quad \frac{1}{ev_{\mathcal{X}}(N^{\mathcal{T}\mathcal{R}}_\Gamma)} = ev_{\mathcal{X}}(\hat{B}_1^m)ev_{\mathcal{X}}(\hat{B}_2^m)ev_{\mathcal{X}}(\hat{B}_3^m), \]
where

\[
\begin{align*}
\text{er}_k(B^m) &= \text{er}_k(\hat{B}^m), \\
\text{er}_k(B^m_\pm) &= \text{er}_k(\hat{B}^m_\pm) \prod_{j \in J_+} \left( \frac{u'_j}{r_j \mu_j} - \frac{\hat{\psi}_j}{r_j} \right) \prod_{j \in J_-} \left( \frac{u'_j}{\sigma_j \mu_j} - \frac{\hat{\psi}_j}{r_j} \right)
\end{align*}
\]

For \( k = 0, 1 \) and \( j = 1, \ldots, h \), let

\[
H^k(D_j) = H^k(D_j, \partial D_j, u_j^* T \chi, (u_j | \partial D_j)^* T \mathcal{L}).
\]

Then there is a long exact sequence

\[
0 \to B_2 \to \hat{B}_2 \oplus \bigoplus_{j=1}^{h} H^0(D_j) \to \bigoplus_{j \in J_+} (T_{p+} \mathcal{X})^{k^+_j} \oplus \bigoplus_{j \in J_-} (T_{p-} \mathcal{X})^{k^-_j} \to B_5 \to \hat{B}_5 \oplus \bigoplus_{j=1}^{h} H^1(D_j) \to 0,
\]

where \((T_{p \pm} \mathcal{X})^{k^\pm_j}\) denote the \( k^\pm_j \)-invariant part of \( T_{p \pm} \mathcal{X} \). Note that

\[
(T_{p \pm} \mathcal{X})^{k^\pm_j} = T^{(p \pm, k^\pm_j)} \mathcal{L} \mathcal{X} = T^{(p \pm, k^\pm_j)} \mathcal{L} \mathcal{X}.
\]

\[
\frac{\text{er}_k(H^1(D_j)^m)}{\text{er}_k(H^0(D_j)^m)} = |\mu_j| ||G_r|| D_{\mu_j, \lambda_j}
\]

Let

\[
\text{er}_f(N_{\Gamma}^\text{vir}) := \tau^*_f \text{er}_k(N_{\Gamma}^\text{vir}) = \left. \text{er}_k(N_{\Gamma}^\text{vir}) \right|_{u'_i = u, u'_j = f u},
\]

\[
\text{er}_j(N_{\Gamma}^\text{vir}) := \tau^*_j \text{er}_k(N_{\Gamma}^\text{vir}) = \left. \text{er}_k(N_{\Gamma}^\text{vir}) \right|_{u'_i = u, u'_j = f u}.
\]

Given \( \gamma_1, \ldots, \gamma_n \in H^*_\text{CR,T}_f(\mathcal{X}) \), we define

\[
\langle \gamma_1, \ldots, \gamma_h \rangle_{\mathcal{X},(\mathcal{L},f)}^{\mathcal{X},(\mathcal{L},f)} := \int_{[F_{\Gamma}]^\text{vir}} \left( \prod_{i=1}^{n} \text{ev}_i^* \gamma_i \right) |F_{\Gamma}|^{-1} \frac{\text{er}_f(N_{\Gamma}^\text{vir})}{\text{er}_f(N_{\Gamma}^\text{vir})} \]

\[
= \prod_{j \in J_+} \frac{r_j}{r_j \mu_j} \prod_{j \in J_-} \frac{-r_j \mu_j}{r_j} \prod_{j=1}^{h} (|\mu_j| ||G_r|| D_{\mu_j, \lambda_j, f}) \cdot \int_{[F_{\Gamma}]^\text{vir}} \prod_{j \in J_+} \left( \frac{u'_j}{r_j \mu_j} - \frac{\hat{\psi}_j}{r_j} \right) \prod_{j \in J_-} \left( \frac{u'_j}{r_j \mu_j} - \frac{\hat{\psi}_j}{r_j} \right) \text{er}_f(N_{\Gamma}^\text{vir}) \]

\[
= \prod_{j=1}^{h} D'_{\mu_j, \lambda_j, f} \cdot \int_{[F_{\Gamma}]^\text{vir}} \left( \prod_{i=1}^{n} \text{ev}_i^* \gamma_i \prod_{j \in J_+} \text{ev}_n^* j^f_{\sigma_+(k^+_j)-1} \prod_{j \in J_-} \text{ev}_n^* j^f_{\sigma_-(k^-_j)-1} \right) |F_{\Gamma}|^{-1} \frac{\text{er}_f(N_{\Gamma}^\text{vir})}{\text{er}_f(N_{\Gamma}^\text{vir})} \]

where

\[
D'_{\mu_0, \lambda_0, f} = \begin{cases} 
-(-1)^{ \lceil d_0 w_1^+ + \hat{k}_1 \rceil / d_0 \cdot u^{\text{age}(k^+)} / d_0 \cdot \prod_{\alpha=1}^{d_0 w_1^+ - 1} (d_0 w_1^+ - a - c_2(k^+)) / |d_0 w_1^+|! } & d_0 > 0, \\
-(-1)^{ \lceil d_0 w_3^+ + (1 - k - \delta_{3,0}) \rceil / d_0 \cdot u^{\text{age}(k^-)} / d_0 \cdot \prod_{\alpha=1}^{d_0 w_3^- - 1} (d_0 w_3^- - c_3(k^-) - a) / |d_0 w_3^-|! } & d_0 < 0.
\end{cases}
\]

\[
\square
\]
Suppose that \((L, f)\) is a framed outer brane. Define
\[
D'_{d_0, \lambda, f} = -(-1)^{[d_0 w_1 + \frac{1}{2}]} \left( \frac{u}{d_0} \right)^{\text{age}(k)} \cdot \prod_{n=1}^{[d_0 w_1]} \frac{\prod_{j=1}^{\max(0, n)} (d_0 w_2 + a - c_2(k))}{[d_0 w_1]!}
\]
where \(k = h(d_0, \lambda)\). By the \(d_0 > 0\) part of the proof of Proposition 3.2, we obtain:

**Proposition 3.3** (framed outer brane). Suppose that \((L, f)\) is a framed inner brane, and \(\bar{\mu} = ((\mu_1, \lambda_1), \ldots, (\mu_h, \lambda_h))\), where \((\mu_j, \lambda_j) \in H_1(L; \mathbb{Z})\). Then
\[
\langle \gamma_1, \ldots, \gamma_n \rangle^{X, (L, f)}_{g, \beta', \bar{\mu}} = \prod_{j=1}^h D'_{\mu_j, \lambda_j, f} \cdot \int_{(\chi_{g, n+h}(X, \beta))^\text{vir}} \frac{\left( \prod_{j=1}^h \psi_{\nu_j} \prod_{j=1}^h \psi_{\nu_j} \right)}{\prod_{j=1}^h \psi_{\nu_j} (\psi_{\nu_j} - \psi_{n+j})}
\]
where
\[
\beta \in H_2(X), \quad \beta' = \beta + \left( \sum_{j=1}^h \mu_j \right) b.
\]

### 3.13. Generating functions of open-closed GW invariants

From now on, we assume the generic stabilizer \(K\) is trivial, so that \(X = X^{\text{rig}}\) is a toric Calabi-Yau 3-orbifold. Then
\[
\chi_3 : G_\tau \rightarrow \mu_m, \quad \lambda \mapsto e^{2\pi \sqrt{-1} \lambda/m}
\]
is a group isomorphism. We have \(N = N\) and \(\bar{b}_i = \bar{b}_i\). In particular,
\[
b_i = v_1 - s v_2 + v_3, \quad b_2 = m v_2 + v_3, \quad b_3 = v_3.
\]
There exists \(m_a, n_a \in \mathbb{Z}\), such that
\[
b_{j+a} = m_a v_1 + n_a v_2 + v_3, \quad a = 1, \ldots, k.
\]
Introduce variables \(\{X_j \mid j = 1, \ldots, h\}\) and let
\[
\tau_2 = \sum_{i=1}^m \tau_i u_i
\]
where \(u_1, \ldots, u_m\) form a basis of \(H^2_{\text{CR}}(X; \mathbb{Q})\). We choose \(T^*\)-equivariant lifting of \(\tau_2\) as follows: for each \(u_i \in H^2_{\text{CR}}(X; \mathbb{Q})\), we choose the unique \(T^*\)-equivariant lifting \(u_i^\tau \in H^2_{\text{CR}, T^*}(X; \mathbb{Q})\) such that \(\iota_\sigma^* u_i^\tau = 0 \in H^2_{\text{CR}, T^*}(p_\sigma; \mathbb{Q})\), where \(\iota_\sigma : H^2_{\text{CR}, T^*}(X; \mathbb{Q}) \rightarrow H^2_{\text{CR}, T^*}(p_\sigma; \mathbb{Q})\) is induced by the inclusion map \(\iota_\sigma : p_\sigma \rightarrow X\).

We define \(\xi_0 := e^{-\pi \sqrt{-1}/m}\). If \((L, f)\) is a framed outer brane, define
\[
F^\tau_{g, h, f}(\tau_2, Q, X_1, \ldots, X_h) = \sum_{\beta', n \geq 0 \in \mathbb{Z}} \sum_{\lambda \in G_\tau} \langle \iota_\sigma^* \tau_2 \rangle^{n}_{g, \beta', (\mu_1, \lambda_1), \ldots, (\mu_h, \lambda_h)} \prod_{j=1}^h (Q^b X_j)^{\mu_j} \cdot \xi_0^{1/\lambda_1} \cdots \xi_0^{1/\lambda_h}
\]
which is a function which takes values in \(H^*_{\text{CR}}(BG_{\tau}; \mathbb{C})^\otimes h\), where
\[
H^*_{\text{CR}}(BG_{\tau}; \mathbb{C}) = \bigoplus_{\lambda \in G_\tau} \mathbb{C} \mathbf{1}_\lambda.
\]
When \(\lambda = 1\) is the identity element of \(G_\tau\), \(1 = 1\) is the unit of \(H^*_{\text{CR}}(BG_{\tau}; \mathbb{C})\).

If \((L, f)\) is a framed inner brane, define
\[
F^\tau_{g, h, f}(\tau_2, Q, X_1, \ldots, X_h) = \sum_{\beta', n \geq 0 \in \mathbb{Z}} \sum_{\lambda \in G_\tau} \langle \iota_\sigma^* \tau_2 \rangle^{n}_{g, \beta', (\mu_1, \lambda_1), \ldots, (\mu_h, \lambda_h)} \prod_{j=1}^h (Q^b X_j)^{\mu_j} \cdot \xi_0^{1/\lambda_1} \cdots \xi_0^{1/\lambda_h}
\]
which is a function which takes values in \(H^*_{\text{CR}}(BG_{\tau}; \mathbb{C})^\otimes h\).
3.14. The equivariant $J$-function and the disk potential. Let $\{u_i\}_{i=1}^N$ be a homogeneous basis of $H^*_{T,CR}(X;\mathbb{Q})$, and $\{u^*_i\}_{i=1}^N$ be its dual basis. Define

$$\tau = \sum_{i=1}^N \tau_i u_i = \tau_0 + \tau_2 + \tau_{>2}$$

where

$$\tau_0 \in H^0_{T,CR}(X;\mathbb{C}), \quad \tau_2 \in H^2_{T,CR}(X;\mathbb{C}), \quad \tau_{>2} \in H^{>2}_{T,CR}(X;\mathbb{C}).$$

The $J$-function ([11][26][40]) is a $H^*_{T,CR}(X)$-valued function:

$$J(\tau, z) := 1 + \sum_{\beta \geq 0, n \geq 0} \frac{1}{n!} \sum_{i=1}^N \langle 1, \tau^n, \frac{u_i}{z - \psi} \rangle_{X,T} u^i.$$ 

Then

$$i^*_\sigma J(\tau, z) \big|_{u_i = u, u_2 = f, u_3 = 0} = \sum_{k \in G_\sigma} J^f_{\sigma, k}(\tau, z) 1_k,$$

where

$$J^f_{\sigma, k}(\tau, z) = \delta_{k,1} + \sum_{\beta \geq 0, n \geq 0} \frac{1}{n!} \sum_{i=1}^N \langle 1, \tau^n, \frac{[G_\sigma \phi^{f}_{\sigma, k-1}]_{X,T}}{z - \psi} \rangle_{0,\beta}.$$ 

As a special case of Proposition [363]

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{X, (\mathcal{L}, f)} = D'_{d_0, \lambda, f} \frac{\left( \prod_{i=1}^n \frac{\nu_{\lambda} \gamma_i + \nu_{\lambda} \psi_{n+1} \phi^{f}_{h^+(d_0,\lambda)} - 1}{\nu_{d_0} - \nu_{n+1}} \right)}{\nu_{d_0}} = D'_{d_0, \lambda, f} (1, \gamma_1, \ldots, \gamma_n, \frac{\phi^{f}_{h^+(d_0,\lambda)} - 1}{\nu_{d_0} - \nu_{n+1}}).$$ 

$$F^X_{0,1}(\mathcal{L}, f)(\tau_2, Q^b, X) = \sum_{\beta \geq 0, n \geq 0} \sum_{(d_0,\lambda) \in H_1(X,\mathbb{Z})} (\langle 1, \gamma \rangle_{X, (\mathcal{L}, f)})_{0,\beta + d_0, \lambda} (Q^b X)^{d_0} \sum_{(d_0,\lambda) \in H_1(X,\mathbb{Z})} \langle \gamma_1, \ldots, \gamma_n \rangle_{X, (\mathcal{L}, f)} \frac{\nu_{d_0}}{\nu_{d_0} - \nu_{n+1}}.$$ 

**Proposition 3.4.** Let $X = Q^b X_1$. If $(\mathcal{L}, f)$ is a framed outer brane, then

$$F^X_{0,1}(\mathcal{L}, f)(\tau_2, X) = \frac{1}{|G_\sigma|} \sum_{(d_0,\lambda) \in H_1(X,\mathbb{Z})} X^{d_0} D'_{d_0, \lambda, f} J_{\sigma, h^+(d_0,\lambda), \lambda}^f (\tau_2, \frac{u}{d_0}) \xi_0^d 1_{\lambda^{-1}}.$$ 

If $(\mathcal{L}, f)$ is a framed inner brane, then

$$F^X_{0,1}(\mathcal{L}, f)(\tau_2, Q, X) = \frac{1}{|G_\sigma|} \sum_{(d_0,\lambda) \in H_1(X,\mathbb{Z}), d_0 > 0} X^{d_0} D'_{d_0, \lambda, f} J_{\sigma, h^+(d_0,\lambda), \lambda}^f (\tau_2, \frac{u}{d_0}) \xi_0^d 1_{\lambda^{-1}}.$$ 

4. Mirror symmetry for the disk amplitudes

4.1. The equivariant $J$-function and the equivariant mirror theorem. We choose $p_1, \ldots, p_k \in L^\vee \cap \text{Nef}_X$ such that

- $\{p_1, \ldots, p_k\}$ is a $\mathbb{Q}$-basis of $L^\vee$.
- $\{\bar{p}_1, \ldots, \bar{p}_k\}$ is a $\mathbb{Q}$-basis of $H^2(X;\mathbb{Q})$.
- $p_a = D_{3+a}$ for $a = k' + 1, \ldots, k$.

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We define charges $m_i^{(a)} \in \mathbb{Q}$ by $D_i = \sum_{a=1}^{k} m_i^{(a)} p_a$.

Let $q_0, q_1, \ldots, q_k$ be $k + 1$ formal variables, and define $q^\beta = q_1^{\langle p_1, \beta \rangle} \cdots q_k^{\langle p_k, \beta \rangle}$ for $\beta \in K$. We take the equivariant lifting $\tilde{\rho}^T_a \in H^2_\ast(X; \mathbb{Q})$ of $\rho_a \in H^2(X; \mathbb{Q})$. The equivariant I-function is an $H^\ast_{CR,T}(X)$-valued power series defined as follows [17]:

$$I(q_0', q, z) = e^{\frac{\log q_0' + \sum_{a=1}^{k} q^\beta}{2} \log q_a} \sum_{\beta \in K_{\text{eff}}} q^\beta \prod_{i=1}^{r'} \frac{\prod_{m=0}^{\infty} ((D_i, \beta) - m) z^m}{\prod_{m=0}^{\infty} ((D_i, \beta) - m)} 1_{v(\beta)}$$

where $q^\beta = \prod_{a=1}^{k} q_a^{\langle p_a, \beta \rangle}$. Note that $\langle p_a, \beta \rangle \geq 0$ for $\beta \in K_{\text{eff}}$. The equivariant I-function can be rewritten as

$$I(q_0', q, z) = e^{\frac{\log q_0' + \sum_{a=1}^{k} q^\beta}{2} \log q_a} \sum_{\beta \in K_{\text{eff}}} q^\beta \prod_{i=1}^{r} \frac{\prod_{m=0}^{\infty} ((D_i, \beta) - m) z^m}{\prod_{m=0}^{\infty} ((D_i, \beta) - m)} 1_{v(\beta)}$$

where $\hat{\rho} = D_1 + \cdots + D_r \in C_X$.

Since $X$ is a Calabi-Yau orbifold, $\text{age}(v)$ is an integer for any $v \in \text{Box}(\Sigma)$. Then

$$H^2_{CR,T}(X; \mathbb{Q}) = H^0_{CR,T}(X; \mathbb{Q}) \oplus H^2_{CR,T}(X; \mathbb{Q})$$

Let $Q = Q(u_1, u_2, u_3)$ be the fractional field of $H^2_\ast(\text{point}; \mathbb{Q})$.

$$H^0_{CR,T}(X; Q) = \mathbb{Q}1,

H^2_{CR,T}(X; Q) = \bigoplus_{a=1}^{k} Q\tilde{\rho}^T_a \oplus \bigoplus_{v \in \text{Box}(\Sigma), \text{age}(v) = 1} \mathbb{Q}v.$$
where \(c_j(b_i) \in (0, 1)\) and \(\sum_{j \in I_\sigma} c_j(b_i) = 1\). There exists a unique \(D_i^\vee \in \mathbb{L}_\sigma\) such that

\[
\langle D_j, D_i^\vee \rangle = \begin{cases} 
1, & j = i, \\
-c_j(b_i), & j \in I_\sigma', \\
0, & j \in I_\sigma - \{i\}.
\end{cases}
\]

Then

\[
A_i(q) = q^{D_i^\vee} + \text{higher order terms}
\]

\[
I(q_0', q, z) = 1 + \frac{1}{z} \left( \log q_0' + \sum_{a=1}^{k'} \log(q_a) \tilde{p}_a^T + \sum_{i=1}^{r'} A_i(q) \tilde{D}_i^T + \sum_{i=r'+1}^r A_i(q) \mathbf{1}_b \right) + o(z^{-1}).
\]

For \(i = 1, \ldots, r'\),

\[
\tilde{D}_i^T = \sum_{a=1}^{k'} m_i^{(a)} \tilde{p}_a^T + \lambda_i
\]

where \(\lambda_i \in H^2(BT; \mathbb{Q})\). Let \(S_a(q) := \sum_{i=1}^r m_i^{(a)} A_i(q)\). Then

\[
I(q_0', q, z) = 1 + \frac{1}{z} \left( (\log q_0' + \sum_{a=1}^{k'} \lambda_i A_i(q)) \mathbf{1} + \sum_{a=1}^{k'} (\log(q_a) + S_a(q)) \tilde{p}_a^T + \sum_{i=r'+1}^r A_i(q) \mathbf{1}_b \right) + o(z^{-1}).
\]

Recall that the \(T\)-equivariant \(J\)-function for \(X\) is

\[
J(\tau, z) = 1 + \sum_{\beta \geq 0, n \geq 0} \frac{1}{n!} \left( 1, \tau^n, \sum_{j \in \mathbb{N}} u_j \right) T^{(n, \tau^j)} u^j,
\]

where \(\{u_j\}_{j=1}^N\) is an \(H^*(BT; \mathbb{Q})\)-basis of \(H^*_T(X; \mathbb{Q})\) and \(\{u^j\}_{j=1}^N\) is the dual basis. Assume \(u_0 = 1, u_a = \tilde{p}_a^T\) for \(a = 1, \ldots, k'\) and \(u_a = \mathbf{1}_{b_{a-3}}\) for \(a = k'+1, \ldots, k\). The mirror theorem for toric orbifolds [27] implies following theorem.

**Theorem 4.1** (Coates-Corti-Iritani-Tseng [27]). If the toric orbifold \(X\) satisfies Assumption [27], then

\[
e^{\frac{\tau_0(q_0', q)}{z} + \tau_2(q)} J(\tau_2(q), z) = I(q_0', q, z),
\]

where the equivariant closed mirror map \((q_0', q) \mapsto \tau_0(q_0', q) + \tau_2(q)\) is determined by the first-order term in the asymptotic expansion of the \(J\)-function

\[
I(q_0', q, z) = 1 + \frac{\tau_0(q_0', q) + \tau_2(q)}{z} + o(z^{-1}).
\]

More explicitly, the equivariant closed mirror map is given by

\[
\tau_0 = \log(q_0') + \sum_{a=1}^{r'} \lambda_i A_i(q),
\]

\[
\tau_a = \begin{cases} 
\log(q_a) + S_a(q), & 1 \leq a \leq k', \\
A_{a-3}(q), & k'+1 \leq a \leq k.
\end{cases}
\]

4.2. The pullback of the disk potential under the mirror map. By Proposition 3.4 if \((L, f)\) is a framed outer brane, then

\[
F_{0,1}^{X,(L,f)}(\tau_2, Q, X) = \frac{1}{|G_\sigma|} \sum_{(d_0, \lambda) \in H_1(L; \mathbb{Z})} X^{d_0} D_0^{d_0, \lambda, f} J^{f}_{d_0, \lambda, f} (\tau_2, u_0) \mathbf{1}_{d_0} \mathbf{1}_{\lambda - 1}.
\]

Let \(F^{X,(L,f)}(Q, X)\) be the pullback of \(F_{0,1}^{X,(L,f)}(\tau_2, Q, X)\) under the closed mirror map.
By Proposition 3.3, if \((L, f)\) is a framed inner brane, then

\[
F_{0,1}^{X, (L, f)}(\tau_2, Q, X) = \frac{1}{|G_\sigma|} \sum_{(d_0, \lambda) \in H_1(L; \mathbb{Z}), d_0 > 0} X^{d_0} D'_{d_0, \lambda, f} J_{\sigma, h(d_0, \lambda)}(\tau_2, \frac{u}{d_0}) \xi_0^1 \lambda^{-1} + \sum_{(d_0, \lambda, d_0, \lambda) \in H_1(L; \mathbb{Z}), d_0 > 0} X^{d_0} Q^{-d_0 \alpha} D'_{d_0, \lambda, f} J_{\sigma, h(d_0, \lambda)}(\tau_2, \frac{u}{d_0}) \xi_0^1 \lambda^{-1}.
\]

Given \(\sigma \in \Sigma(3)\), \(k \in G_\sigma\), and \(f \in \mathbb{Z}\), define \(I_{\sigma, k}^I(q, z)\) by

\[
t'_{\sigma} I(q, z) \bigg|_{u_1 = u_2 = f, u_3 = 0} = \sum_{k \in G_\sigma} I_{\sigma, k}^I(q, z) 1_k.
\]

Since a toric Calabi-Yau orbifold satisfies the weak Fano condition, by the equivariant mirror theorem (Theorem 4.1), we may write \(F_{X, (L, f)}(q, X)\) in terms of \(I_{\sigma, k}^I(q, z)\) in case of an outer brane, and in terms of \(I_{\sigma, k^+}^I(q, z)\) and \(I_{\sigma, k^-}^I(q, z)\) in case of an inner brane.

**Lemma 4.2.** If \((L, f)\) is a framed outer brane, then

\[
F_{0,1}^{X, (L, f)}(q, X) = \frac{1}{|G_\sigma|} \sum_{(d_0, \lambda) \in H_1(L; \mathbb{Z})} X^{d_0} D'_{d_0, \lambda, f} e^{-d_0 \gamma_0(q)} I_{\sigma, h(d_0, \lambda)}(q, \frac{u}{d_0}) \xi_0^1 \lambda^{-1}.
\]

If \((L, f)\) is a framed inner brane, then

\[
F_{0,1}^{X, (L, f)}(q, X) = \frac{1}{|G_\sigma|} \sum_{(d_0, \lambda) \in H_1(L; \mathbb{Z}), d_0 > 0} X^{d_0} D'_{d_0, \lambda, f} e^{-d_0 \gamma_0(q)} I_{\sigma, h(d_0, \lambda)}(q, \frac{u}{d_0}) \xi_0^1 \lambda^{-1} + \frac{1}{|G_\sigma|} \sum_{(d_0, \lambda) \in H_1(L; \mathbb{Z}), d_0 > 0} X^{d_0} Q^{-d_0 \alpha} D'_{d_0, \lambda, f} e^{-d_0 \gamma_0(q)} I_{\sigma, h(d_0, \lambda)}(q, \frac{u}{d_0}) \xi_0^1 \lambda^{-1}.
\]

Let \((L, f)\) be a framed brane, and let \(\tau, \sigma = \sigma_+, \sigma_-\) be defined as in Section 3.3. Recall that

\[
I_{\sigma}^+ = \{ i \in \{1, \ldots, r\} : \rho_i \subset \sigma\} = \{1, 2, 3\}, \quad I_{\tau} = \{1, \ldots, r\} \setminus I_{\sigma}^+, \quad I_{\sigma}^- = \{ i \in \{1, \ldots, r\} : \rho_i \subset \tau\} = \{2, 3\}, \quad I_{\tau} = \{1, \ldots, r\} \setminus I_{\sigma}^-.
\]

In case that \(L\) is inner,

\[
I_{\sigma}^- = \{2, 3, 4\}, \quad I_{\sigma}^- = \{1, \ldots, r\} \setminus I_{\sigma}^-, \quad I_{\sigma}^+ = \{ i \in \{1, \ldots, r\} \} \setminus I_{\sigma}^-.
\]

Let \(b_{\sigma, i} = \tau'_{\sigma} D_i^T \in H^2(\mathbb{Z}, \mathbb{Z}) = H^2(BT; \mathbb{Q})\) for \(1 \leq i \leq r\), and then \(b_i = 0\) for \(r' + 1 \leq i \leq r\). For \(\beta \in \mathbb{K}_\text{eff, } \sigma,\) define an \(H^*(BT; \mathbb{Q})\)-valued

\[
I(\sigma, \beta) := \prod_{i=1}^r \frac{\prod_{m=0}^{1} (b_{\sigma, i} + (D_i, \beta) - m) \psi_i}{\prod_{m=0}^{1} (b_{\sigma, i} + (D_i, \beta) - m) \psi_i}.
\]

Recall that \(\tau'_{\sigma} b_i^T = 0\), so

\[
t'_{\sigma} I(q, z) \bigg|_{u_3 = \frac{1}{\psi_i}} = \sum_{\beta \in \mathbb{K}_\text{eff, } \sigma} e^{\frac{1}{\psi_i}} \log q^\beta I(\sigma, \beta) 1_{\psi_i}. \]

With the above notation, if \(L\) is an outer brane we can rewrite \(F_{0,1}^{X, (L, f)}(q, X)\) as

\[
F_{0,1}^{X, (L, f)}(q, X) = \frac{1}{|G_\sigma|} \sum_{(d_0, \lambda) \in H_1(L; \mathbb{Z})} \sum_{\beta \in \mathbb{K}_\text{eff, } \sigma, \psi(\beta) = h(d_0, \lambda)} X^{d_0} q^3 D'_{d_0, \lambda, f} J_{\sigma, \beta}(q, \frac{u}{d_0}) \xi_0^1 \lambda^{-1}.
\]
where $I^f(\sigma, \beta) = I(\sigma, \beta)|_{u_1 = u_2 = f_0, u_3 = 0}$, and

$$x = X \exp \left( \frac{\log q_0' - \tau_0(q)}{u_1} \right)$$

is the B-brane moduli parameter.

Following [51, 59], we define extended charge vectors

$$\{m_i^{(a)}\}_{a = 1, \ldots, r} = \left( \begin{array}{c} w_1 \\ w_2 \\ w_3 \\ \vdots \end{array} \right),$$

such that $m_i^{(0)} = w_i$ for $i = 1, 2, 3$ and $m_i^{(0)} = 0$ for $i = 4, \ldots, r$. Recall that

$$\tau_0 + \sum_{a=1}^{k'} \tau_a \bar{p}_a^T + \sum_{a=k'+1}^{k} \tau_a 1_{b_{a+3}} = \log q_0 + \sum_{a=1}^{k'} \log q_a \bar{p}_a^T + \sum_{i=1}^{r'} A_i(q) \bar{D}_i^T + \sum_{i=r'+1}^{r} A_i(q) 1_{b_i}.$$

We pull back the above identity under $t_\sigma^*$. Since

$$t_\sigma^* \bar{p}_a^T = 0, \ t_\sigma^* \bar{D}_i^T \big|_{u_1 = u_2 = f_0, u_3 = 0} = m_i^{(0)} u,$$

we get

$$\tau_0(q_0', q) = \log q_0' + \sum_{i=1}^{r'} m_i^{(0)} A_i(q).$$

So the open mirror map is given by

$$\log X = \log x + \sum_{i=1}^{r'} m_i^{(0)} A_i(q). \quad (28)$$

If $L$ is inner, we further set $Q = q$. Denote the pullback of the disk potential $W^X(\mathcal{L}, f)(q, x)$ to be the pullback of $F_{0,1}^{X, (\mathcal{L}, f)}(q, X)$ under this open mirror map. Then by Lemma [4.2]

$$|G_\sigma| W^X(\mathcal{L}, f)(q, x) = \begin{cases} \sum_{d_0 > 0, \beta \in \mathbb{K}_{\text{eff}, \sigma} \atop v(\beta) = h(d_0, \lambda)} x^{d_0} q^\beta D'_{d_0, \lambda, f} I^f(\sigma, \beta) \xi \xi_0 1_{\lambda = 1}, & \mathcal{L} \text{ is outer,} \\ \sum_{d_0 > 0, \beta \in \mathbb{K}_{\text{eff}, \sigma} \atop v(\beta) = h^+(d_0, \lambda)} x^{d_0} q^\beta D'_{d_0, \lambda, f} I^f(\sigma^+, \beta) \xi_0^\lambda 1_{\lambda = 1} \\ + \sum_{d_0 < 0, \beta \in \mathbb{K}_{\text{eff}, \sigma} \atop v(\beta) = h^-(d_0, \lambda)} x^{d_0} q^\beta D'_{d_0, \lambda, f} I^f(\sigma^-, \beta) \xi_0^\lambda 1_{\lambda = 1}, & \mathcal{L} \text{ is inner.} \end{cases} \quad (29)$$

Given $\bar{\beta} = (d_0, \beta) \in \mathbb{Z} \times \mathbb{K}_\sigma$, define the extended or open sector pairing to be

$$\langle D_1, \bar{\beta} \rangle = m_i^{(0)} d_0 + \langle D_1, \beta \rangle.$$

Recall that $\{D_i : i \in I_{\sigma}\}$ is a $\mathbb{Q}$-basis of $\mathbb{L}_{\sigma}^\vee \cong \mathbb{Q}^k$ and a $\mathbb{Z}$-basis of $\mathbb{K}_{\sigma}^\vee \cong \mathbb{Z}^k$. Let $v_a = D_{a+3}$ for $a = 1, \ldots, k$, and let $\{h_a\}_{a=1, \ldots, k}$ be the dual $\mathbb{Q}$-basis of $\mathbb{L}_{\sigma}$. Then $\{h_a\}_{a=1, \ldots, k}$ is a $\mathbb{Z}$-basis of $\mathbb{K}_\sigma \cong \mathbb{Z}^k$, and

$$\mathbb{K}_{\text{eff}, \sigma} = \sum_{a=0}^{k} \mathbb{Z}_{\geq 0} h_a.$$

Given any $(d_0, \beta) \in \mathbb{Z} \times \mathbb{K}_\sigma$, define

$$q^{\bar{\beta}} = x^{d_0} q^{\beta} = x^{d_0} \prod_{a=1}^{k} q^{(p_a^*, \beta)}.$$

Define

$$\mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) = \{ \bar{\beta} = (d_0, \beta) \in \mathbb{Z} \times \mathbb{K}_{\text{eff}, \sigma} : \langle D_1, \bar{\beta} \rangle \in \mathbb{Z}_{\geq 0}, d_0 \neq 0 \}.$$
Theorem 4.3. Assuming the Aganagic-Vafa brane $(\mathcal{L}, f)$ is either inner or outer,

\begin{equation}
W^{X, (\mathcal{L}, f)}(q, x) = \sum_{\substack{\beta \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \\ v(\beta) = h(d_0, \lambda)}} q^\beta A^{X, (\mathcal{L}, f)}_\beta \xi_0^\lambda \mathbf{1}_{\lambda-1},
\end{equation}

where

\[ A^{X, (\mathcal{L}, f)}_{\beta = (d_0, \beta)} = \frac{(-1)^{(\langle D_3, \bar{\beta} \rangle)}}{md_0 \prod_{i \in I_\sigma} \Gamma((D_i, \bar{\beta}) + 1)} \cdot \frac{\Gamma(\langle \bar{D}_3, \bar{\beta} \rangle)}{\Gamma((D_2, \bar{\beta}) + 1)}.
\]

Proof. Assume $\mathcal{L}$ is an outer brane. Let $\bar{\beta} = (d_0, \beta)$, and let $\epsilon_j = c_j(v)$ for $j = 1, 2, 3$. By (29) (and real that $|G_\sigma| = \text{rm}$)

\[ W^{X, (\mathcal{L}, f)}(q, x) = \frac{1}{\text{rm}} \sum_{d_0 > 0 \atop \beta \in \mathbb{K}_{\text{eff}, \sigma}} x^{d_0 q^\beta} D^{d_0, \lambda, f} I^f(\sigma, \beta) \xi_0^\lambda \mathbf{1}_{\lambda-1}.
\]

Given any $\beta \in \mathbb{K}_{\text{eff}, \sigma}$, we have $\langle (D_j, \beta) \rangle - \epsilon_j = \langle (D_j, \beta) \rangle$ for $j = 1, 2, 3$, and $\langle (D_i, \beta) \rangle = \langle (D_i, \beta) \rangle$ for $i \in I_\sigma$.

The disk factors

\[ D^{d_0, \lambda, f} = (-1)^{(\langle D_0, d_0 \rangle + \lambda)} \frac{\left( \begin{array}{c} u \\ d_0 \end{array} \right)^{\text{deg}(h(d_0, \lambda))} \prod_{a=1}^{d_0 w_1 + \text{deg}(h(d_0, \lambda)) - 1} (d_0 w_1 + a - \epsilon_2)}{(d_0 w_1)!} \cdot \frac{1}{\Gamma(w_1 d_0 - \epsilon_1 + 1)} \cdot \frac{\Gamma(-w_3 d_0 + \epsilon_3)}{\Gamma(w_2 d_0 - \epsilon_2 + 1)}.
\]

The pullback of the coefficients of the $I$-function is

\[ I^f(\sigma, \beta) = \frac{1}{\prod_{i \in I_\sigma} \Gamma((D_i, \bar{\beta}) + 1)} \frac{\Gamma(w_1 d_0 - \epsilon_1 + 1)}{\Gamma((D_1, \bar{\beta}) + 1)} \cdot \frac{\Gamma(w_2 d_0 - \epsilon_2 + 1)}{\Gamma((D_2, \bar{\beta}) + 1)} \cdot \frac{\Gamma(w_3 d_0 - \epsilon_3 + 1)}{\Gamma((D_3, \bar{\beta}) + 1)}.
\]

Hence

\[ W^{X, (\mathcal{L}, f)}(q, x) = \sum_{\beta \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \atop v(\beta) = h(d_0, \lambda)} x^{d_0 q^\beta} A^{X, (\mathcal{L}, f)}_\beta \xi_0^\lambda \mathbf{1}_{\lambda-1},
\]

where

\[ A^{X, (\mathcal{L}, f)}_{\beta} = (-1)^{(\langle D_3, \bar{\beta} \rangle)} \frac{1}{md_0 \prod_{i \in I_\sigma} \Gamma((D_i, \bar{\beta}) + 1)} \cdot \frac{\Gamma(\langle \bar{D}_3, \bar{\beta} \rangle)}{\Gamma((D_2, \bar{\beta}) + 1)}.
\]

Note that $\mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \subset \mathbb{Z}_{>0} \times \mathbb{K}_{\text{eff}, \sigma}$, and for any $(d_0, \beta) \in (\mathbb{Z}_{>0} \times \mathbb{K}_{\text{eff}, \sigma}) \setminus \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L})$, $A^{X, (\mathcal{L}, f)}_{(d_0, \beta)} = 0$.

In case that $\mathcal{L}$ is inner, by (29)

\[ W^{X, (\mathcal{L}, f)}(q, x) = I_+ + I_-,
\]

where

\begin{align*}
I_+ &= \sum_{d_0 > 0, \beta \in \mathbb{K}_{\text{eff}, \sigma} \atop v(\beta) = h^+(d_0, \lambda)} x^{d_0 q^\beta} \frac{-1}{md_0 \prod_{i \in I_+} \Gamma((D_i, \bar{\beta}) + 1)} \cdot \frac{\Gamma(-w_2^{+} d_0 - \langle D_3, \bar{\beta} \rangle)}{\Gamma(w_2^{+} d_0 + (D_2, \bar{\beta}) + 1) \Gamma(w_1^{+} d_0 + (D_1, \bar{\beta}) + 1)} \xi_0^\lambda \mathbf{1}_{\lambda-1},
\end{align*}

\begin{align*}
I_- &= \sum_{d_0 < 0, \beta \in \mathbb{K}_{\text{eff}, \sigma} \atop v(\beta) = h^-(d_0, \lambda)} x^{d_0 q^\beta} \frac{-1}{md_0 \prod_{i \in I_-} \Gamma((D_i, \bar{\beta}) + 1)} \cdot \frac{\Gamma(-w_2^{-} d_0 - \langle D_2, \bar{\beta} \rangle)}{\Gamma(w_2^{-} d_0 + (D_3, \bar{\beta}) + 1) \Gamma(w_1^{-} d_0 + (D_1, \bar{\beta}) + 1)} \xi_0^\lambda \mathbf{1}_{\lambda-1},
\end{align*}

\begin{align*}
I_+ &= \sum_{\beta \in \mathbb{K}_{\text{eff}, \sigma} \atop (\beta, D_4) + w^{-} \epsilon_0 d_0 \in \mathbb{Z}_{\geq 0} \atop d_0 < 0, v(\beta) = h^-(d_0, \lambda)} x^{d_0 q^\beta} \frac{-1}{md_0 \prod_{i \in I_-} \Gamma((D_i, \bar{\beta}) + 1)} \cdot \frac{\Gamma(-w_2^{-} d_0 - \langle D_2, \bar{\beta} \rangle)}{\Gamma(w_2^{-} d_0 + (D_3, \bar{\beta}) + 1) \Gamma(w_1^{-} d_0 + (D_1, \bar{\beta}) + 1)} \xi_0^\lambda \mathbf{1}_{\lambda-1}.
\end{align*}

We have

\[ \langle D_1, \alpha \rangle = w_1^+, \quad \langle D_2, \alpha \rangle = w_2^+ - w_2^- , \quad \langle D_3, \alpha \rangle = w_3^+ - w_3^- , \quad \langle D_4, \alpha \rangle = -w_1^-.
\]
and $\langle D_i, \alpha \rangle = 0$ for $i \in I \setminus \{1, 2, 3, 4\}$. So for $\beta \in \mathbb{K}_{\text{eff,}s_-}$,

$$
\langle D_1, \beta \rangle = (D_1, \beta - d_0 \alpha) + d_0 w_1^+,
$$

$$
\langle D_2, \beta \rangle = w_2^+ d_0 + \langle D_2, \beta - d_0 \alpha \rangle,
$$

$$
\langle D_3, \beta \rangle = w_3^+ d_0 + \langle D_3, \beta - d_0 \alpha \rangle,
$$

$$
\langle D_4, \beta \rangle = \langle D_4, \beta - d_0 \alpha \rangle.
$$

Since the conditions $\langle \beta, D_4 \rangle + w_1 d_0 \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{K}_{\text{eff,}s_-}$ implies $(d_0, \beta - d_0 \alpha) \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L})$,

$$
I_- = \sum_{(d_0, \beta - d_0 \alpha) \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \atop d_0 < 0} x^{d_0} q^{\beta - d_0 \alpha} \frac{1}{m(d_0)} \frac{(-1)^{\lfloor (D_2, \beta - d_0 \alpha) \rfloor}}{\Gamma(\langle D_1, \beta - d_0 \alpha \rangle + 1)} \Gamma(w_2^+ d_0 + \langle D_2, \beta - d_0 \alpha \rangle + 1) \Gamma(w_1 d_0 + \langle D_1, \beta - d_0 \alpha \rangle + 1) \xi_0^3 \lambda^{-1}.
$$

So

$$
I_+ + I_- = \sum_{(d_0, \beta - d_0 \alpha) \in \mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \atop \nu(\beta) = h^+(d_0, \alpha)} x^{d_0} q^{\beta} \frac{1}{m(d_0)} \frac{(-1)^{\lfloor (D_2, \beta) \rfloor}}{\Gamma(\langle D_1, \beta \rangle + 1)} \Gamma(\langle D_2, \beta \rangle + 1) \Gamma(\langle D_1, \beta \rangle + 1) \xi_0^3 \lambda^{-1}.
$$

\[ \square \]

**Remark 4.4.** When $\mathcal{L}$ is an outer brane, the condition $\bar{\beta} = (d_0, \beta) \in \mathbb{K}(\mathcal{X}, \mathcal{L})$ implies $d_0 > 0$. One has

$$
\sum_{a=1}^k \langle D_{a+3}, \beta \rangle b_{a+3} = -\sum_{i=1}^3 \langle D_i, \beta \rangle b_i.
$$

Since $\langle D_{a+3}, \beta \rangle \geq 0$, the fan $\Sigma$ is convex, and this is an outer brane, every $1$-cone $b_i$ is on the same side of the plane spanned by $b_2, b_3$. Therefore, $\langle \beta, D_1 \rangle \leq 0$. From $w_1 d_0 + \langle D_1, \beta \rangle \in \mathbb{Z}_{\geq 0}$ we see that $d_0 > 0$.

4.3. **The B-model and the framed mirror curve.** The mirror B-model to the toric Calabi-Yau threefold $\mathcal{X}$ is another non-compact Calabi-Yau hypersurface $Y \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2$, constructed as the Hori-Vafa mirror [45]. It is equivalent to an affine mirror curve $C_q \subset (\mathbb{C}^*)^2$. We state the relevant mirror prediction for disk amplitudes from [0 5].

4.3.1. **Toric degeneration.** The main reference of this subsection is [61, Section 3].

The set

$$
\Theta_0 = \bigcap_{i \in A} \sum_{i \in I} \mathbb{Q}_{\geq 0} D_i \subset \mathbb{L}_{\mathbb{Q}}^\vee
$$

is a top dimensional convex cone in $\mathbb{L}_{\mathbb{Q}}^\vee \cong \mathbb{Q}^k$. The cone $\Theta_0$ together with its faces is a fan in $\mathbb{L}_{\mathbb{R}}^\vee$ denoted by $\Theta$. This fan determines a $k$-dimensional affine toric variety $X_{\Theta}$.

Consider the exact sequence induced from (4) (notice $N_{\text{tor}} = 0$)

$$
0 \rightarrow M' \overset{\phi'}{\rightarrow} \tilde{M}' \overset{\psi'}{\rightarrow} \mathbb{L}^\vee \rightarrow 0
$$

where $M' = M/\langle v_3 \rangle$ and $\tilde{M}' = \tilde{M}/\langle \phi'(v_3) \rangle$. Let $D_i^{\tilde{T}}$ be the image of $D_i^T$ when passing to $\tilde{M}'$. For any proper subset $I \subset \{1, \ldots, r\}$ and a cone $\nu \in \Theta$, define

$$
\Xi_I = \sum_{i \in I} \mathbb{Q}_{\geq 0} D_i^{\tilde{T}}, \quad \Theta_{I, \nu} = (\psi')^{-1}(\nu) \cap \Xi_I.
$$

Define a fan

$$
\tilde{\Theta} = \{ \tilde{\Theta}_{I, \nu} \mid I \subset \{1, \ldots, r\}, \nu \in \Theta \} \sqcup \{0\}.
$$

This fan determines a toric variety $X_{\tilde{\Theta}}$. There is a fan morphism $\rho : X_{\tilde{\Theta}} \rightarrow X_{\Theta}$, which induces a flat family of toric surfaces $\rho : X_{\tilde{\Theta}} \rightarrow X_{\Theta}$.
Let \( \Theta'_0 \subset L'_\Omega \) be the cone spanned by \( p_1, \ldots, p_k \). Let \( L^{\vee} := \bigoplus_{i=1}^{k} \mathbb{Z}p_i \) and let \( \mathbb{L}' \) be the dual lattice. Then \( L^{\vee} \) is a sublattice of \( L^{\vee} \) of finite index, and \( \mathbb{L} \) is a sublattice of \( \mathbb{L}' \) of finite index. Let \( \Theta'_0 \) and \( \Theta'_0 \) be the dual cones of \( \Theta_0 \) and \( \Theta_0 \), respectively. We have inclusions

\[
\Theta'_0 \subset \Theta_0, \quad \Theta'_0 \subset \Theta'_0.
\]

Since that \( \Theta'_0 \cap \mathbb{L} \) is a subset of \( \Theta'_0 \cap \mathbb{L}' \), we have an injective ring homomorphism

\[
\mathbb{C}[\Theta'_0 \cap \mathbb{L}] \rightarrow \mathbb{C}[\Theta'_0 \cap \mathbb{L}'] = \mathbb{C}[q_1, \ldots, q_k]
\]

where \( q_1, \ldots, q_k \) are the variables in Section 4.1. Taking the spectrum, we obtain a morphism

\[
A^k = \operatorname{Spec} \mathbb{C}[q_1, \ldots, q_k] \longrightarrow X_\Theta = \operatorname{Spec} \mathbb{C}[\Theta'_0 \cap \mathbb{L}].
\]

and a cartesian diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\tilde{\nu}} & X_\Theta \\
\downarrow{\tilde{\rho}} & & \downarrow{\rho} \\
A^k & \xrightarrow{\nu} & X_\Theta
\end{array}
\]

where \( \tilde{\rho} : X \rightarrow A^k \) is a flat family of toric surfaces.

We choose a Kähler class \( [\omega(\eta)] \in H^2(X_S; \mathbb{Z}) \) associated to a lattice point \( \eta \in L^{\vee} \); \( [\omega(\eta)] \) is the first Chern class of some ample line bundle over \( X_S \). Then it determines a toric graph \( \Gamma \in M^\vee_k \cong \mathbb{R}^2 \) up to translation by an element in \( M' \cong \mathbb{Z}^2 \) (see Section 5.2). The toric graph gives a polyhedral decomposition of \( M_\Omega \) in the sense of [61, Section 3]. It is a covering \( \mathcal{P} \) of \( M_\Omega \) by strongly convex lattice polyhedra. The asymptotic fan of \( \mathcal{P} \) is defined to be

\[
\Sigma_\mathcal{P} := \{ \lim_{\alpha \to 0} \alpha \Xi \subset M'_\Omega : \Xi \in \mathcal{P} \}.
\]

The fan \( \Sigma_\mathcal{P} = \tilde{\Theta} \cap \rho'^{-1}(0) \) is a toric surface \( \mathbb{S} \), which is the same as the toric surface given by the defining polytope \( \Delta \). For each \( \Pi \in \mathcal{P} \), let \( C(\Pi) \subset M'_\Omega \times \mathbb{Q}_{\geq 0} \) be the closure of the cone over \( \Xi \times \{1\} \) in \( M'_\Omega \times \mathbb{Q} \). Then

\[
\Sigma_\mathcal{P} := \{ \sigma \text{ is a face of } C(\Pi) : \Pi \in \mathcal{P} \} = \tilde{\Theta} \cap \rho'^{-1}(\mathbb{Q}_{\geq 0} \eta)
\]

is a fan in \( M'_\Omega \times \mathbb{Q} \) with support \( \Sigma_\mathcal{P} = M'_\Omega \times \mathbb{Q}_{\geq 0} \). The projection \( \pi' : M'_\Omega \times \mathbb{Q} \twoheadrightarrow \mathbb{Q} \) to the second factor defines a map from the fan \( \Sigma_\mathcal{P} \) to the fan \( \{0, \mathbb{Q}_{\geq 0}\} \). This map of fans determines a flat toric morphism \( \pi : X_{\Sigma_\mathcal{P}} \rightarrow \mathbb{A}^1 \), where \( X_{\Sigma_\mathcal{P}} \) is the toric 3-fold defined by the fan \( \Sigma_\mathcal{P} \). Let \( t \) be a closed point in \( \mathbb{A}^1 \), and let \( X_t \) denote the fiber of \( \pi \) over \( t \). Then \( X_t \cong \mathbb{S} \) for \( t \neq 0 \). As shown in [61], when \( t = 0 \), we have a union of irreducible components, where each \( S_v \) is the toric surface defined by the polytope \( \Delta_v \) (recall each 3-cone is a cone over a triangle \( \Delta_v \subset \Delta \) in \( \mathbb{N}^2 \)).

\[
X_0 = \bigcup_{v \in \Sigma(3)} S_v.
\]

If \( v' \in \Sigma(2), v \in \Sigma(3) \) and \( v' \subset v \), \( v' \) corresponds to a torus invariant divisor \( D_{v'} \subset S_v \). We have the following commutative diagram

\[
\begin{array}{ccc}
X_0 & \xleftarrow{\pi} & X_{\Sigma_\mathcal{P}} & \xleftarrow{\pi} & X \xleftarrow{\tilde{\nu}} X_\Theta \\
\downarrow{\pi} & & \downarrow{\tilde{\rho}} & & \downarrow{\rho} \\
\{0\} & \xrightarrow{\nu} & \mathbb{A}^1 & \xrightarrow{\nu} & A^k & \xrightarrow{\nu} & X_\Theta.
\end{array}
\]

The polytope \( \operatorname{Hull}(\hat{b}_1, \ldots, \hat{b}_r) \subset \hat{N} \) lies on the hyperplane \( \langle \hat{\phi}(u_3), \bullet \rangle = 1 \). It determines a polytope on \( \hat{N}' = \{ \langle \hat{\phi}(u_3), \bullet \rangle = 0 \} \) up to a translation. The associated line bundle \( \mathcal{L} \) on \( X_\Theta \) has sections \( u_i, i = 1, \ldots, r \) associated to each integer point in this polytope. Define

\[
u = \sum_{i=1}^{r} u_i, \quad \mathcal{E} = \nu^{-1}(0).
\]

The divisor \( \mathcal{E} \subset X_\Theta \). Let \( \mathcal{E} := \nu^{-1}(\mathcal{C}) \subset X \) be the pullback divisor under the morphism \( \tilde{\nu} : X \rightarrow X_\Theta \). Then \( \mathcal{E} \rightarrow A^k \) is a flat family of curves over \( A^k \).
For $q \neq 0$, $\mathcal{C}_q = \overline{\rho}^{-1}(q) \cap \mathbb{C}$ can be identified with the zero locus of

$$H_q(x, y) = x^k y^{s-r}y + y^m + \sum_{a=1}^k s_a(q)x^{m_a}y^{n_a-f ma},$$

where $x^k y^{s-r}y = u_1 u_3^{-1}$, $y^m = u_3 u_3^{-1}$ while $s_a(q)x^{m_a}y^{n_a-f ma} = u_3 u_3^{-1}$ for $a = 1, \ldots, k$. Here $x, y$ are affine coordinates of the toric surface $S$.

For any $\beta \in \mathbb{K}_\text{eff}(X, L)$, let $s^\beta = \prod_{a=1}^k s_a^\beta$. If we write $p_a = \sum_{b=1}^k \rho_3^b D_{b+3}$, we have

$$s_a = \prod_{b=1}^k q_a^b, \quad s^\beta = q^\beta.$$

Denote $s^\beta = x^d_0 s^\beta$ for $\overline{\beta} = \langle d_0, \beta \rangle$.

When $q = 0$, we have a union of irreducible components

$$\mathcal{C}_0 = \bigcup_{v \in \Sigma(3)} \overline{C}_{0,v}.$$

Each irreducible component $\mathbb{S}_v$ of the central fiber $X_0$ is given by the equation $\{u_i = 0, \beta_i \notin v\}$. On $\mathbb{S}_\{x\}$, the coordinates in the affine chart $u_3 \neq 0$ are

$$x^k y^{s-r}y = u_1 u_3^{-1}, \quad y^m = u_2 u_3^{-1},$$

while on $\mathbb{S}_\{-x\}$, the coordinates are

$$u_4 u_3^{-1} = (q^a x^s y) \pi y^{n_1-s} = u_2 u_3^{-1}.$$ 

Here $b_1 = m_1 v_1 + n_1 v_2 + v_3$, $m_1 = -s$ and $\alpha = [t_\tau]$. Define

$$U = \{(q_1, \ldots, q_k) \in (\mathbb{C}^*)^k \times \mathbb{C}^{k-1} : \mathbb{C}_q \text{ is smooth and intersects } \partial S \text{ transversally at distinct points}\}.$$ 

Then $U$ is a dense open subset of $\mathbb{A}^k$.

4.3.2. Mirror curve and the mirror conjecture for disk amplitudes. When $q \neq 0$, we denote $\mathbb{C}_q = \mathbb{C}_q \setminus (\partial S)$. Thus the mirror curve $\mathbb{C}_q \subset (\mathbb{C}^*)^2$ is given by Equation (33). On $\overline{C}_{0,v}$, when $x = 0$, there are $m$ points, called large radius limit (LRL) points. They are given by

$$x = 0, \quad y^m = -1.$$ 

If $L$ is outer, these points are smooth points in $\overline{C}_{0,v} \subset \mathbb{C}_0$; if $L$ is inner, they are the nodal points $\overline{C}_{0,v} \cap \overline{C}_{0,v}$. The group $G^*_a = \{(t_1, t_2) \in (\mathbb{C}^*)^2 | t_1^a = t_2^a, t_2^a = 1\}$ fits into the short exact sequence

$$1 \to \mu_2^* \to G^*_a \to \mu^*_m \to 1,$$

where $G^*_a \to \mu^*_m$ is given by $(t_1, t_2) \mapsto t_2$. Let

$$\chi_1 = (e^{2\pi \sqrt{-1} t_1^a}, 1), \quad \chi_2 = (e^{2\pi \sqrt{-1} t_2^a}, 1).$$

Then $G^*_a = \{\chi_1^j \chi_2^l | j \in \{0, \ldots, r-1\}, l \in \{0, \ldots, m-1\}\}$. It pairs with $G_a$ by

$$\chi_1(h) = e^{2\pi \sqrt{-1} \tau_1(h)}, \quad \chi_2(h) = e^{2\pi \sqrt{-1} \tau_2(h)}, \quad h \in G;$$

and acts on the family of compactified mirror curves $\overline{C}$ by

$$\chi_1 \cdot (x, y, s_a) = (e^{2\pi \sqrt{-1} \tau_1 x}, y, e^{-2\pi \sqrt{-1} \tau_1 (b+3)} s_a), \quad \chi_2 \cdot (x, y, s_a) = (e^{2\pi \sqrt{-1} \tau_2 t_1 x}, e^{2\pi \sqrt{-1} \tau_1 y}, e^{-2\pi \sqrt{-1} \tau_1 (b+3)} s_a).$$

Here $c_i^*(b_{a+3})$ is defined as $h_{a+3} = \sum_{i=1}^3 c_i^*(b_{a+3}) b_i$. The group $\mu^*_m$ acts freely and transitively on the set of LRL points ($x = 0$ on $\overline{C}_{0,v}$).

Given $\bar{\eta} \in \{0, 1, \ldots, m-1\}$, let $\eta \in G^*_a$ be the element associated to the character

$$\chi_\eta : G \to \mathbb{C}^*, \quad \chi_\eta(\lambda) = \exp\left(\frac{2\pi \sqrt{-1}}{m} \bar{\eta} \lambda\right).$$
Then \( \tilde{\eta} \mapsto \eta \) is a bijection from \( \{ 0, 1, \ldots, m - 1 \} \) to \( G_2^* \). Given \( \eta \in G_2^* \), define \( u_\eta \in \xi_0 \) by
\[
u_\eta = (0, e^{\pi \sqrt{(-1 + 2i)/m}}).
\]

For a small \( \epsilon \), one can always find small \( \epsilon'(\epsilon) < \epsilon \) such that when \( \|q\| < \epsilon'(\epsilon) \) the following set
\[
U^\epsilon,\epsilon' = \begin{cases}
\{(x, q), |x| < \epsilon, U \}, & \text{if } \mathcal{L} \text{ is outer};
\{(x, q), |x| < \epsilon, |q|^a - 1 | < \epsilon \text{ whenever } \|q\| < \epsilon' \}, & \text{if } \mathcal{L} \text{ is inner};
\end{cases}
\]
is not empty. Let \( U^\epsilon = U^\epsilon,\epsilon'(\epsilon) \times \{ \|q\| < \epsilon'(\epsilon), q \in U \} \subset \mathcal{C} \). When \( \epsilon \) is sufficiently small, \( U^\epsilon,\epsilon'(\epsilon) \) is a disjoint union of \( m \) small contractible regions when \( \mathcal{L} \) is outer, or is a disjoint union of \( m \) annuli when \( \mathcal{L} \) is inner. Let \( U^\epsilon_\eta \) be the unique connected component of \( U^\epsilon \) containing \( u_\eta \). So \( \log y \) is well defined on \( U^\epsilon_\eta \) up to an integral multiple of \( 2\pi \sqrt{-1} \), and it could be written as a power series in \( x \) when \( \mathcal{L} \) is outer, and a Laurent series in \( x \) when \( \mathcal{L} \) is inner.

Define
\[
\phi_\eta := \frac{1}{m} \sum_{\lambda \in G_2^*} \chi_\eta (\lambda^{-1}) 1_\lambda.
\]

Then \( \{ \phi_\eta : \eta \in G_2^* \} \) is the canonical basis of \( H_{\text{CR}}^*(B\mu_m) \), and
\[
1_\lambda = \sum_{\eta \in \mu_m^*} \chi_\eta (\lambda) \phi_\eta.
\]

We prove the following mirror theorem for disk amplitudes. Note that the ambiguity in \( \log y \) does not play any role in the statement.

**Theorem 4.5.**
\[
\left( \frac{\partial}{\partial x} \sum_{\eta \in \mu_m^*} (\log y|_{U^\epsilon_\eta}) \phi_\eta \right)^2 W^{X,(\mathcal{L},f)}(q, x) = \left( \frac{\partial}{\partial x} \right)^2 F_{0,1}^{X,(\mathcal{L},f)}(\tau_2, X).
\]

*Here s and q are related by (33). The A-model flat coordinates \( \tau_2, X \) and the B-model coordinates \( q, x \) are related by the mirror maps (25) and (28).*

**Remark 4.6.**
(i) Theorem 4.5 for smooth toric Calabi-Yau 3-folds was conjectured in [6, 5] and proved in [53]. Theorem 4.5 can also be written as
\[
\int \sum_{\eta \in \mu_m^*} (\log y |_{U^\epsilon_\eta}) \phi_\eta W^{X,(\mathcal{L},f)}(q, x),
\]
where the integral is indefinite and \( \sim \) means their instanton parts are equal in the following sense. The left side is the sum of a power series with no constant term in \( x \) and an extra term in the form of \( f(q) \log x + c \). The power series part is equal to the right side. Note that the constant ambiguity in the indefinite integral is irrelevant here. If \( X \) is a smooth variety, then \( \mu_m^* \) is trivial, and we revert to the original form of the conjecture in [6, 5]. We will prove this conjecture in the next subsection.

**4.4. Open mirror theorem for disk amplitudes.**

**Lemma 4.7.** The solution \( v \) to the exponential polynomial equation
\[
(35) \quad \sum_{a=0}^{k} t_a e^{r_a v} - e^v + 1 = 0,
\]
around \( t_0 = \cdots = t_k = 0, v = 0 \) is in the following power series form
\[
(36) \quad v = \sum_{l_0, \ldots, l_k \geq 0} \frac{(r_0 l_0 + \cdots + r_k l_k - 1)(l_0 + \cdots + l_k - 1)}{l_0! \cdots l_k!} t_0^{l_0} \cdots t_k^{l_k}.
\]
Here we adopt the Pochhammer symbol

\[
(a)_n = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} = \begin{cases} 
  a(a-1) \cdots (a-n+1), & n > 0; \\
  1, & n = 0; \\
  (a+1) \cdots (a-n), & n < 0;
\end{cases}
\]

where \(a \in \mathbb{C}\) and \(n \in \mathbb{Z}\).

Proof. See Appendix \(\Delta\) \(\square\)

Starting from the above observation, we prove Theorem 4.3 in this section. In order to find the expansion of \(\log y\) on \(U^r_{\eta}\), we assume

\[
\log y = \log \xi_0 + \frac{2\pi \sqrt{-1}}{m} \bar{\eta} + \frac{v(q,x)}{m} = \frac{\pi \sqrt{-1}}{m}(-1 + 2\bar{\eta}) + \frac{v(q,x)}{m}
\]

where \(v\) is a power series in \(q\) and \(x\). Setting

\[
\xi_{\bar{\eta}} = e^{\frac{2\pi \sqrt{-1}}{m} \eta}, \quad t_0 = x^r(\xi_0\bar{\xi}_{\bar{\eta}})^{-s-rf}, \quad r_0 = -w_2 \tau,
\]

\[
t_a = s_a x^{m_a}(\xi_0\bar{\xi}_{\bar{\eta}})^{n_a - fm_a}, \quad r_a = \frac{n_a - fm_a}{m},
\]

the mirror curve is

\[
H(x,y) = \sum_{a=0}^{k} t_a e^{r_a v} - e^v + 1 = 0.
\]

Let \(\mathbb{D}_{\text{eff}}(\mathcal{X}, \mathcal{L}) = \{\bar{\beta} = (d_0, \beta) \in \mathbb{Z} \times \mathbb{L} | \langle \bar{\beta}, D_i \rangle \in \mathbb{Z}_{\geq 0}, i \neq 2,3\}\). So \(\mathbb{K}_{\text{eff}}(\mathcal{X}, \mathcal{L}) \subset \mathbb{D}_{\text{eff}}(\mathcal{X}, \mathcal{L})\). By Lemma 4.7

\[
v = \sum_{l_0, \ldots, l_k \neq 0, (l_0, \ldots, l_k) \neq 0} \frac{(r_0 l_0 + \ldots r_k l_k - 1)!}{l_0! \ldots l_k!} \prod_{a=0}^{k} t_a^{l_a}, \quad \mathcal{P}_{\text{eff}}(\mathcal{X}, \mathcal{L}, \bar{\beta}) = \prod_{i \in I_{\mathcal{L}}(D_i, \beta)} \prod_{a=1}^{k} s_a^{(D_{a+3}, \beta)} \prod_{i \in I_{\mathcal{L}}(D_i, \beta)} \prod_{a=1}^{k} s_a^{(D_{a+3}, \beta)}.
\]

Suppose that \(\mathcal{L}\) is an outer brane. For any \(\bar{\beta} = (d_0, \beta) \in \mathbb{D}_{\text{eff}}(\mathcal{X}, \mathcal{L})\), we have

\[
d_0(v_1 + f v_2) - v(\beta) + N_\sigma \in G_\tau \subset G_\sigma = \mathbb{N}/N_\sigma.
\]

Let \(\lambda = d_0(v_1 + f v_2) - v(\beta) + N_\sigma \in G_\tau\). Then \(h(d_0, \lambda) = v(\beta) \in G_\sigma\). If \(\mathcal{L}\) is an inner brane, we replace \(\sigma\) by \(\sigma_+\) in the above discussion and define \(\lambda \in G_1\) similarly. Then

\[
\langle D_2, \bar{\beta} \rangle \in \frac{\lambda}{m} + \mathbb{Z}, \quad \langle D_3, \bar{\beta} \rangle \in -\frac{\lambda}{m} + \mathbb{Z}.
\]

So

\[
\xi_{\bar{\eta}}^{-\langle D_2, \bar{\beta} \rangle} = \exp(-\frac{2\pi \sqrt{-1}}{m} \bar{\eta} \lambda) = \chi_\eta(\lambda^{-1})
\]
Let \( X \) be a set of variables around \( t \). This is done by elementary recursive calculation. We illustrate the expansion at 

\[
\begin{align*}
  \text{Proof.} \\
  \text{Implicit function theorem (applying to } Y \text{ and the sum in Equation (35) in Lemma 4.7 starts from } r \text{ rational function of } \beta \text{).} \\
  \text{For Equation } s \text{ and } a \text{, one can expand with each coefficient rational in } L \text{.)}
\end{align*}
\]

Thus Theorem 4.5 follows.

**Appendix A. Proof of Lemma 4.7**

In this appendix, we obtain a power series solution to the following exponential polynomial where \( r_a \in \mathbb{R} \)

\[
1 - e^x + \sum_{a=1}^{k} t_a e^{r_a y} = 0
\]

around \( t_1 = \cdots = t_k = 0 \) by oscillatory integral and inverse Laplace transform. Note that the notation here is slightly different from that in Lemma 4.7, the sum in the above Equation starts from \( a = 1 \), whereas the sum in Equation (35) in Lemma 4.7 starts from \( a = 0 \).

We also consider the following equation where \( f, r_a \in \mathbb{Z}_{>0} \)

\[
L(X, Y) = 1 + XY^{-f} + Y + \sum_{a=1}^{k} s_a Y^{r_a} = 0.
\]

Let \( X = e^{-x} \) and \( Y = e^{-y} \). This equation identifies with Equation (37) after setting \( X = 0 \) and a change of variables \( v = \sqrt{-1} \pi - y, t_a = (-1)^{r_a} s_a \).

**Lemma A.1.** For Equation (37), one can expand \( v \) as a power series in \( t_a \), where each coefficient is a rational function of \( r_a \). For Equation (38), the variable \( Y \) can be expanded as a power series of \((-1)^{r_a} s_a\) and \((-1)^{f} X \) around \( Y = -1 \) with each coefficient rational in \( r_a \) and \( f \). One can also expand \( Y \) as a power series of \( s_a \) and \((-X)^{f} \) around \( Y = 0 \) with each coefficient rational in \( r_a \) and \( f \).

**Proof.** This is done by elementary recursive calculation. We illustrate the expansion at \( Y = 0 \) for Equation (38). The equation can be written as

\[
Y^f + Y^{f+1} + \sum_{a=1}^{k} s_a Y^{r_a + f} = ((-X)^f).
\]

Implicit function theorem (applying to \( Y(Y + 1 + \sum_{a=1}^{k} s_a Y^{r_a})^f = ((-X)^f) \) says \( Y \) is analytic in \((-X)^f\) and \( s_a \) around \((X, Y, s) = (0, 0, 0)\), and recursive calculation shows each coefficient is a rational function of lower degree coefficients. \( \square \)
We consider an affine curve
\[ C := \{(X, Y) \in (\mathbb{C}^*)^2 \mid L(X, Y) = 0\} \]
and its partial compactification \( \tilde{C} \subset \mathbb{C}^2 \) with two points \((X, Y) = (0, 0)\) and \((X, Y) = (0, -1 + O(s))\) added.
Let \( e^{-x_0} \) be the branch point of the map \((X, Y) \mapsto X\) such that \( e^{-x_0} = -\frac{1}{2} + O(s) \). Let \( \gamma_s \) be the oriented Lefschetz thimble which passes through the ramification point \((e^{-x_0}, e^{-y_0}) = (-\frac{1}{2} + O(s), -\frac{1}{2} + O(s))\) and goes from \((X, Y) = (0, -1 + O(s))\) to \((X, Y) = (0, 0)\). So the coordinate on \( \gamma_s \) is \( z \) such that \( x - x_0 = z^2 \).
We choose the sign of \( z \) such that \((X, Y) = (0, 0)\) is at \( z = +\infty \).

**Lemma A.2.**
\[
\int_{\gamma_s} e^{-ux} ydx = \sum_{t_1, \ldots, t_k \geq 0} \frac{\Gamma(u) \Gamma(f u + \sum_{a=1}^k r_a l_a)}{\Gamma((f + 1)u + \sum_{a=1}^k (r_a - 1) l_a + 1) \prod_{a=1}^k l_a!} \prod_{a=1}^k s_a^{l_a}.
\]

**Proof.** Consider a Landau-Ginzburg model \( W_s : (\mathbb{C}^*)^3 \to \mathbb{C} \), where
\[
W_s = X_1 X_2^{-f} X_3 + X_2 X_3 + X_3 + \sum_{a=1}^k s_a X_2^{r_a} X_3 - u \log X_1.
\]
Define \( \tilde{t}_1 = X_1 X_2^{-f} X_3, \tilde{t}_2 = X_2 X_3, \tilde{t}_1 = X_1 X_2^{-f}, \tilde{t}_2 = X_2 \).
Let \( X_3 \in \Gamma_3 \) be a cycle that counter-clockwise encircles the positive real axis, starting and ending on the positive real infinity. We require the argument of each \( X_3 \in \Gamma_3 \) takes every value in \((0, 2\pi)\) once. Define the relative connected cycle \( \Gamma_s \) to be
\[
\Gamma_s = \{(X_1, X_2, X_3) \in (\mathbb{C}^*)^3 \mid \tilde{t}_1 > 0, \tilde{t}_2 > 0, X_3 \in \Gamma_3, \text{when } X_3 < 0 \text{ and } s = 0, X_2 \in \mathbb{R}^-\}.
\]
When \(|s| < \epsilon\) for small \( \epsilon \), the superpotential \( \text{Re}(W) \to \infty \) in the non-compact direction of \( \Gamma_s \). On \( \Gamma_s \) the logarithm is taken in the following way: when \( X_3 < 0 \) and \( s = 0 \),
\[
\arg(X_1) = -(f + 1)\pi, \quad \arg(X_2) = -\pi, \quad \arg(X_3) = \pi.
\]
Since the cycle \( \Gamma_s \) is simply-connected and deforms continuously with respect to \( s \), this choice is fixed. Evaluate the following oscillatory integral of \( W_s \)
\[
I(u) = \int_{\Gamma_s} e^{-W} \frac{dX_1}{X_1} \frac{dX_2}{X_2} \frac{dX_3}{X_3}
= \int_{\Gamma_s} \exp\left(-\sum_{a=1}^k s_a \log(X_3^{-r_a} - \tilde{t}_1 - \tilde{t}_2 - X_3 + u \log \tilde{t}_1 + f u \log \tilde{t}_2 - (f + 1) u \log X_3\right) \frac{d\tilde{t}_1}{\tilde{t}_1} \frac{d\tilde{t}_2}{\tilde{t}_2} \frac{dX_3}{X_3}
= \int_{\Gamma_s} e^{-\sum_{a=1}^k s_a \log(X_3^{-r_a} - \tilde{t}_1 - \tilde{t}_2 - X_3 + f u \log X_3 - \sqrt{-1} \pi) \frac{d\tilde{t}_1}{\tilde{t}_1} \frac{d\tilde{t}_2}{\tilde{t}_2} \frac{dX_3}{X_3}
= -e^{-f-1}(\sqrt{-1})^u \sum_{t_1, \ldots, t_k \geq 0} (-1)^{\sum_{a=1}^k (r_a - 1) l_a} \prod_{a=1}^k \frac{(-s_a)^{l_a}}{l_a!} \prod_{t_1 > 0} e^{-\tilde{t}_1 \tilde{t}^{u-1} d\tilde{t}_1}
\cdot \left( \int_{t_2 > 0} e^{-2t_2 \sum_{a=0}^k r_a l_a f u - 1} d\tilde{t}_2 \right) \left( \int_{X_3 \in \Gamma_3} e^{-X_3 \log X_3 - \sqrt{-1} \pi) (\sum_{a=1}^k (r_a - 1) l_a)(f + 1) u \log X_3 dX_3\right)
= 2\pi \sqrt{-1} e^{-f - 1} \sqrt{-1} \pi u \sum_{t_1, \ldots, t_k \geq 0} (-1)^{\sum_{a=1}^k r_a l_a} \prod_{a=1}^k \frac{s_a^{l_a}}{l_a!} \Gamma(f u + \sum_{a=1}^k (r_a - 1) l_a + 1)
\]
Here we use the Hankel’s formula
\[
\frac{\sqrt{-1}}{2\pi} \int_{\Gamma_3} e^{-z(\log(z) - \sqrt{-1} \pi)} e^{-t dt} \frac{1}{\Gamma(z)}
\]
By Hori-Iqbal-Vafa [44], this oscillatory integral could be reduced to a Laplace transform on the curve \( C \).
Introduce two variables \( v^+, v^- \in \mathbb{C} \), and the extended cycle
\[
\tilde{\Gamma}_s = \Gamma_s \times \{v^+ = -v^-\}.
\]
Define $H = \frac{W}{X^3}$. Define the holomorphic volume form

$$\Omega = \frac{dX_1}{X_1} \frac{dX_2}{X_2} \frac{dv^-}{v^-} = dx dy \frac{dv^-}{v^-}.$$

We reduce the oscillatory integral to the curve $C$ as follows. Let $\hat{\Gamma}_{\text{red}} = \{(\hat{t}_1, \hat{t}_2) | \arg \hat{t}_1 = \arg \hat{t}_2 \} \times \{v^+ = -v^- \}$. Further reduce the integral to the curve $C$ as follows.

$$I(u) = \frac{1}{2\sqrt{-1}\pi} \int_{C} e^{-X_3(H-v^+ v^-)} e^{-u X_3} \frac{dX_1}{X_1} \frac{dX_2}{X_2} dX_3 dv^+ dv^-$$

This integration is further reduced to the curve $C = \{ H(e^{-x}, e^{-y}) = 0 \}$ as follows.

$$I(u) = -\int_{\hat{\Gamma}_{\text{red}} \cap \{ H-v^+ v^- = 0 \}} e^{-u x} dx dy \frac{dv^-}{v^-} = 2\sqrt{-1}\pi \int_{\gamma_a} e^{-u x} y dx.$$

Notice that we use the fact $d(e^{-u x} y dx \frac{dv^-}{v^-}) = -e^{-u x} \Omega$ near $\hat{\Gamma}_{\text{red}} \cap \{ H-v^+ v^- = 0 \}$. □

The function $\frac{du}{dx}$ is a meromorphic function on the partially compactified curve $\bar{C}$ with the only pole at $x_0$. Its expansion at $(X, Y) = (0, -1 + O(s))$ is a power series in $X$, while its expansion at $(X, Y) = (0, 0)$ is a series in $X^\pm$. Denote $g^\pm(x) = \frac{du}{dx} \bigg|_{x = \pm \sqrt{x_0}}$. Then $g^+$ is a power series in $X$ and $s_a$, while $g^+$ is a power series in $X^+$ and $s_a$. Since they are expansions of $\frac{du}{dx} = \frac{dX}{dY}$ regarding to the curve equation (A.35), as series of $(-1)^l X$, $(-1)^s s_a$ and $(-X)^l$, $s_a$ respectively, their coefficients are rational in $r_a$ and $f$ by Lemma A.1.

By Lemma A.2, the “classical Laplace transform” is

$$\mathcal{G}(u) = \int_{x-x_0 \in \mathbb{R}^+} e^{-u(x-x_0)} \left( g^+(x) - g^-(x) \right) dx$$

$$= \int_{x-x_0 \in \mathbb{R}^+} e^{-u(x-x_0)} \left( \frac{dy}{dx} \right) dx(x-x_0) = \int_{\gamma_a} e^{-u x} y dx$$

$$= \sum_{l_1, \ldots, l_k \geq 0} \frac{u e^{ux_0}}{\sqrt{-1\pi}} \frac{1}{\Gamma((-f+1)u + \sum_{a=1}^{k} r_a l_a)} \frac{\Gamma(u) \Gamma(f u + \sum_{a=1}^{k} r_a l_a) \prod_{a=1}^{k} s_a^{l_a}}{\Gamma((f+1)u + \sum_{a=1}^{k} (r_a - 1) l_a + 1) l_1! \ldots l_k!}.$$

By the inverse Laplace transform formula,

$$(g^+ - g^-) = \int_{u=-\infty}^{u=+\infty} \mathcal{G}(u) e^{u(x-x_0)} du,$$

where $T$ is large enough such that all poles of $\mathcal{G}(u)$ is on the left of the integration contour. Here the inverse Laplace transform takes resides around poles of $\Gamma(u)$ and $\Gamma(f u + \sum_{a=1}^{k} r_a l_a)$. Taking the residues around all poles (other than the possible pole at $u = 0$) of $\Gamma(f u + \sum_{a=1}^{k} r_a l_a)$ gives a series of $(-X)^l$ with coefficients rational in $r_a$ and $f$, denoted by $h^+.

$$h^+ = \sum_{l > 0, l_1, \ldots, l_k \geq 0} \frac{k}{l_1! \ldots l_k!} \prod_{a=1}^{k} s_a^{l_a} ((-X)^l).$$

$$= \sum_{l > 0, l_1, \ldots, l_k \geq 0} \frac{(-l/f) e^{-\sqrt{-1\pi} (-\frac{l}{2} + \sum_{a=1}^{k} r_a l_a) (X^+)^l}}{\prod_{a=1}^{k} s_a^{l_a}} \frac{\Gamma(-\frac{l}{2}) \Gamma((-\frac{l}{2} + 1) + \sum_{a=1}^{k} l_a)}{\Gamma((-\frac{l}{2} + 1) + \sum_{a=1}^{k} (r_a - 1) l_a + 1) l_1! \ldots l_k!}.$$
while taking residues around the poles of $\Gamma(u)$ (other than the possible pole at $u = 0$) we get a power series in $X$

$$h^{-} = \sum_{l>0,k_1,\ldots,k_r \geq 0} (-l)^{l} \Gamma(-f(-1)^{(f+l)}+\sum_{a=1}^{k} r_{a} l_{a}) \frac{\text{Res}_{u=-l} (\Gamma(u)) \Gamma(-f(-1)^{(f+l)}+\sum_{a=1}^{k} r_{a} l_{a}) \prod_{a=1}^{k} s_{a}^{l_{a}}}{\Gamma(-f(-1)^{(f+l)}+\sum_{a=1}^{k} r_{a} l_{a} + 1) l_{1}! \ldots l_{k}!}$$

$$= \sum_{l>0,k_1,\ldots,k_r \geq 0} (-l)^{l} ((-1)^{l} f)^{l} \frac{\Gamma(-f(-1)^{(f+l)}+\sum_{a=1}^{k} r_{a} l_{a}) \prod_{a=1}^{k} ((-1)^{l_{a}} s_{a})^{l_{a}}}{\Gamma(-f(-1)^{(f+l)}+\sum_{a=1}^{k} r_{a} l_{a} + 1) l_{1}! \ldots l_{k}!}.$$ 

So $g^{+} - g^{-} = h^{+} + h^{-} + \text{const.}$, where the constant difference (in $X$) arises since we don’t consider the residue around $u = 0$. For any degree $l \geq 1$, choose $f > l$ such that the term $((-X)^{f})^{l} \prod_{a=1}^{k} s_{a}^{l_{a}}$ is not a monomial in $X$, and thus can only come from $h^{+}$. Since the coefficient of the term is rational in $f$, it has to be equal to the corresponding term $h^{+}_{i_{1},\ldots,i_{k}}$ for all $f > 0$. Therefore $g^{+} = h^{+} + \text{const.}$ and $-g^{-} = h^{-}$. Here note that $g^{-}$ is the expansion of $\frac{dy}{dx}$, and since $y$ is analytic in $X$ at $(X,Y) = (0, -1 + O(s))$, $-g^{-}$ has no degree 0 term and does not differ from $h^{-}$ by a degree 0 term in $X$.

Suppose the expansion of $y$ at $(X,Y) = (0, -1 + O(s))$ is $y = A_{0} + \sum_{l>0} A_{l} X^{l}$, then the expansion of $\frac{dy}{dx}$ at this point is

$$g^{-} = \frac{dy}{dx} = -\sum_{l>0} l A_{l} X^{l}.$$

Therefore for $l \geq 1$,

$$A_{l} = -\sum_{l_{1},\ldots,l_{k} \geq 0} e^{-\pi u} (-1)^{(f+l)} \sum_{a=1}^{k} r_{a} l_{a} \frac{\Gamma(-f(-1)^{(f+l)}+\sum_{a=1}^{k} r_{a} l_{a}) \prod_{a=1}^{k} s_{a}^{l_{a}}}{\Gamma(-f(-1)^{(f+l)}+\sum_{a=1}^{k} r_{a} l_{a} + 1) l_{1}! \ldots l_{k}!}.$$

We prove Lemma 4.7 by induction. The statement is true for $k = 0$ trivially. Assume it is true for $k = m - 1$ ($m \geq 1$). For $k = m$ we first assume $r_{m} \in \mathbb{Z}_{<0}$ and all other $r_{a}$ ($a = 1, \ldots, m - 1$) are positive integers. By the induction assumption we know the expansion is given as in Lemma 4.7 for terms of degree 0 in $t_{m}$. Let $f = -r_{m}$. After a change of variables $v = \sqrt{-\pi} u - y$, $(-1)^{l_{a}} s_{a} = t_{a}$ for $a = 1, \ldots, m - 1$ and $X = (-1)^{l} t_{m}$ we obtain Equation 4. Then from Equation 39 we know the expansion of $y$ for positive degree terms in $X$, thus conclude that for positive degree terms in $t_{m}$ the lemma also holds. Then for all degrees the lemma holds

$$v = \sum_{l_{1},\ldots,l_{k} \neq 0} \frac{r_{1} l_{1} + \ldots + r_{k} l_{k} - 1)(l_{1} + \ldots + l_{k} - 1)}{l_{1}! \ldots l_{k}!} t_{1}^{l_{1}} \ldots t_{k}^{l_{k}}.$$ 

Each coefficient is a rational function of $r_{1},\ldots,r_{m}$. The above equation holds for $r_{1},\ldots,r_{m-1} \in \mathbb{Z}_{>0}$ and $r_{m} \in \mathbb{Z}_{<0}$, so it is true for all $r_{a} \in \mathbb{R}$.

References

[1] D. Abramovich, B. Fantechi, “Orbifold techniques in degeneration formulas,” Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 16 (2016), no. 2, 519–579.
[2] D. Abramovich, T. Graber, A. Vistoli, “Algebraic orbifold quantum products,” Orbifolds in mathematics and physics (Madison, WI, 2001), 1–24, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.
[3] D. Abramovich, T. Graber, A. Vistoli, “Gromov-Witten theory of Deligne-Mumford stacks,” Amer. J. Math. 130 (2008), no. 5, 1337–1398.
[4] M. Aganagic, A. Klemm, M. Mariño, C. Vafa, “The topological vertex,” Comm. Math. Phys. 254 (2005), no. 2, 425–478.
[5] M. Aganagic, A. Klemm, C. Vafa, “Disk instantons, mirror symmetry and the duality web,” Z. Naturforsch. A 57 (2002), no. 1-2, 1–28.
[6] M. Aganagic, C. Vafa, “Mirror Symmetry, D-Branes and Counting Holomorphic Discs,” hep-th/0012041
[7] L. Borisov, L. Chen, G. Smith, “The orbifold Chow ring of toric Deligne-Mumford stacks,” J. Amer. Math. Soc. 18 (2005), no. 1, 193–215.
[8] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, “Remodeling the B-model,” Comm. Math. Phys. 287 (2009), no. 1, 117–178.
[9] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, “Topological open strings on orbifolds,” Comm. Math. Phys. 296 (2010), 589–623.
[10] A. Brini, “Open topological strings and integrable hierarchies: remodeling the A-model,” Comm. Math. Phys. 312 (2012), no. 3, 735–780.
H. Iritani, “An integral structure in quantum cohomology and mirror symmetry for toric orbifolds,” Adv. Math. 222 (2009), no. 3, 1016–1079.

Y. Jiang, “The orbifold cohomology ring of simplicial toric stack bundles,” Illinois J. Math. 52 (2008), no. 2, 493–514.

S. Katz, C.C. Liu, “Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc,” Adv. Theor. Math. Phys. 5 (2001), no. 1, 1–49.

H.-Z. Ke, J. Zhou, “Open-Closed Gromov-Witten Invariants of 3-dimensional Calabi-Yau Smooth Toric DM Stacks,” Lett. Math. Phys. 105 (2015), no. 1, 63–88.

B. H. Lian, K. Liu, S.-T. Yau, “Mirror principle. I,” Asian J. Math. 1 (1997), no. 4, 729–763.

B. H. Lian, K. Liu, S.-T. Yau, “Mirror principle. II,” Asian J. Math. 3 (1997), no. 1, 109–146.

C. Lin, “Bouchard-Klemm-Marino-Pasquetti Conjecture for C^3,” Topological recursion and its influence in analysis, geometry, and topology, 83–102, Proc. Sympos. Pure Math., 100, Amer. Math. Soc., Providence, RI, 2018.

C.-C. M. Liu, “Moduli of J-holomorphic curves with Lagrangian boundary conditions and open Gromov-Witten invariants for an S^1-equivariant pair,” arXiv:math/0211388.

C.-C. M. Liu, “Localization in Gromov-Witten theory and orbifold Gromov-Witten theory,” Handbook of moduli. Vol. II, 353–425, Adv. Lect. Math. (ALM), 25, Int. Press, Somerville, MA, 2013.

J. Li, C.-C. M. Liu, K. Liu, J. Zhou, “A mathematical theory of the topological vertex,” Geom. Topol. 13 (2009), no. 1, 527–621.

M. Mariño, “Open string amplitudes and large order behavior in topological string theory,” J. High Energy Phys. (2008), no. 3, 060, 34 pp.

P. Mayr, “A = 1 mirror symmetry and open/closed string duality,” Adv. Theor. Math. Phys. 5 (2001), no. 2, 213–242.

D. Maulik, A. Oblomkov, A. Okounkov, “Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds,” Invent. Math. 186 (2011), no. 2, 435–479.

T. Nishinou, B. Siebert, “Toric degeneration of toric varieties and tropical curves,” Duke Math. J. 135 (2006), no. 1, 1–51.

A. Okounkov, R. Pandharipande, “Hodge integrals and invariants of the unknot,” Geom. Topol. 8 (2004), 675–699.

R. Pandharipande, J. Solomon, J. Walcher, “Disk enumeration on the quintic 3-fold,” J. Amer. Math. Soc. 21 (2008), no. 4, 1169–1209.

A. Popa, A. Zinger, “Mirror symmetry for closed, open, and unoriented Gromov-Witten Invariants,” Adv. Math. 259 (2014), 448–510.

M. Romagny, “Group actions on stacks and applications,” Michigan Math. J. 53 (2005), no. 1, 209–236.

D. Ross, “Localization and gluing of orbifold amplitudes: the Gromov-Witten orbifold vertex,” Trans. Amer. Math. Soc. 366 (2014), no. 3, 1587–1620.

D. Ross, “On GW/DT and Ruan’s Conjecture for Calabi-Yau 3-Orbifolds,” Comm. in Math. Phys. 340 (2015), 851–864.

D. Ross, Z. Zong, “The gerby Gopakumar-Mariño-Vafa formula,” Geom. Topol. 17 (2013), no. 5, 2935–2976.

D. Ross, Z. Zong, “Cyclic Hodge integrals and loop Schur functions,” Adv. Math. 285 (2015), 1448–1486.

J. Solomon, “Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions,” arXiv:math/0606429.

H.-H. Tseng, “Orbifold quantum Riemann-Roch, Lefschetz and Serre,” Geom. Topol. 14 (2010), no. 1, 1–81.

J. Walcher, “Opening mirror symmetry on the quintic,” Comm. Math. Phys. 276 (2007), no. 3, 671–689.

S. Yu, “The Open Crepant Transformation Conjecture for Toric Calabi-Yau 3-Orbifolds,” arXiv:2002.08524.

E. Zaslow, “Topological orbifold models and quantum cohomology rings,” Comm. Math. Phys. 156 (1993), no. 2, 301–331.

J. Zhou, “Local mirror symmetry for one-legged topological vertex,” arXiv:0910.4320.

J. Zhou, “Open string invariants and mirror curve of the resolved conifold,” arXiv:1001.0447.

Z. Zong, “Generalized Mariño-Vafa formula and local Gromov-Witten theory of orbicurves,” J. Differential Geom. 100 (2015), no. 1, 161–190.

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