Irregularity Strength of Circulant Graphs Using Algorithmic Approach

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ABSTRACT This paper deals with decomposition of complete graphs on \(n\) vertices into circulant graphs with reduced degree \(r < n - 1\). They are denoted as \(C_n(a_1, a_2, \ldots, a_m)\), where \(a_1\) to \(a_m\) are generators. Mathematical labeling for such bigger (higher order and huge size) and complex (strictly regular with so many triangles) graphs is very difficult. That is why after decomposition, an edge irregular \(k\)-labeling for these subgraphs is computed with the help of algorithmic approach. Results of \(k\) are computed by implementing this iterative algorithm in computer. Using the values of \(k\), an upper bound for edge irregularity strength is suggested for \(C_n(a_1, a_2, \ldots, a_m)\) that is \(|E|/2 \log_2 |V|\).

INDEX TERMS Edge irregular labeling, circulant graph, graph algorithm, computational complexity, Sidon sequence.

I. INTRODUCTION AND PRELIMINARY RESULTS

Let \(G\) be a connected, simple and undirected graph with vertex set \(V(G)\) and edge set \(E(G)\). In graph theory, degree of a graph is used as one of the most important graph invariant for comparison purpose. The degree of a vertex \(v\) is the number of edges incident to \(v\), it is denoted as \(\deg(v)\). The minimum and maximum degree of a graph \(G\) is denoted by \(\delta(G)\) and \(\Delta(G)\) respectively. If every vertex in a graph \(G\) has the same degree \(r\), that is \(\delta(G) = \Delta(G) = r\), then \(G\) is called an \(r\)-regular graph. A complete graph \(K_n\) is a circulant regular graph of order \(n\) and degree \(r = n - 1\), where every pair of distinct vertices is connected by a unique edge. That is why \(K_n\) is considered as super-circulant graph with biggest size that can be calculated using the property, \(|E(K_n)| = n(n - 1)/2\).

In graph theory new graphs are evolved from existing graphs by applying certain graph operations like by adding or deleting vertices or edges. Decomposition is one of the elementary graph operation in which a subgraph is extracted from a super-graph. Graph decomposition can be done by deleting \(2\)-factors or Hamiltonian cycles from a supergraph that results as reduction of two-degree.

In 2001 graph decomposition was introduced by Knopfmacher and Mays [13], they used enumerations to develop formulae for few families of graphs like paths, cycles, trees etc. In 2012 Cichacz et al. [8] decomposed the complete graphs into \((0, 2)\)-prisms. In 2015 Tichenor and Mays used decompositions for deleting edges from complete graphs and derived formulae for paths, cycles, star graphs and disjoint graphs using generating functions [21].

In computer science graphs play a vital role in computational linguistics, decision making software, coding theory and path determination in networks. In fifth-generation-computers, the interconnection network of the parallel processors are represented as complete graph \(K_n\), by considering vertices as \(n\) processors and edges as link between them [17]. Circulant graphs also have wide applications in the field of cloud computing and network topologies like ring topology and fully connected networks [5]. Sipper and Ruppin in [18] using the idea of [6] worked on cellular programing based on circulant graphs. In [14] Lu worked on fast methods for designing circulant network topology by decomposing complete graphs to construct new circulant graphs.

In real world applications temporal graphs are used more than complete graphs. A temporal graph is a dynamic...
A Sidon sequence $A = a_0, a_1, a_2, \ldots$ is a set of natural numbers in which all pairwise sums $a_i + a_j, i \leq j$ must be different. Sidon introduced this concept for Fourier series. In 2010 Cilleruelo et al. [9] solved the issue by giving a formula to calculate Sidon sequence terms. Table 1 shows the comparison between the algorithmic results of $k$ computed in [4], Sidon sequence terms from [9] and Fibonacci numbers $F_n$ as an upper bound from [1]. It can be seen clearly that algorithmic results are better than both sequences.

Successful results of $es(K_n)$ computed in [4] motivated us to extend the same approach for similar graph families, that is why edge irregular $k$-labeling of circulant graphs is computed in this article. Let $n, m$ and $a_1, a_2, \ldots, a_m$ be positive integers, $1 \leq a_i \leq \lfloor n/2 \rfloor$ and $a_i \neq a_j$, for all $1 \leq i, j \leq m$. An undirected graph with the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and the set of edges $E = \{v_iv_{i+1} : 1 \leq i \leq n, 1 \leq j \leq m\}$, the indices being taken modulo $n$, is called a circulant graph and denoted by $C_n(a_1, a_2, \ldots, a_m)$. The numbers $a_1, a_2, \ldots, a_m$ are called the generators and we say that the edge $v_iv_{i+1}$ is of type $a_i$. It is easy to see that the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ is a regular graph of degree $r$, where

$$r = \begin{cases} 2m - 1 & \text{if } \frac{n}{2} \in \{a_1, a_2, \ldots, a_m\} \\ 2m & \text{otherwise.} \end{cases}$$

Formation of circulant graphs leads us to the fact that degrees, generators and their decomposition depends on order of graph as even or odd. Both cases are explicitly explained as follows:

**Case 1**: $n \equiv 0 \pmod{2}$

If the order is even, then degrees will be odd, starting from $r = n - 1$ for $K_n$ up to $r = 3$ as maximum decomposed circulant graph. The value of $m = n/2$ and the graph can be represented as $C_n(1, 2, \ldots, n/2)$. The first decomposition can be done by deleting one 2-factor (outer cycle) that will result as $(n - 3)$-regular graph and graph can be represented as $C_n(2, 3, \ldots, n/2) = C_n(n - 3)$. The second decomposition can be done by deleting the second 2-factor that means all edges of type 2 will be deleted and the graph representation will be like $C_n(3, 4, \ldots, n/2) = C_{n - 5}$. This decomposition can continue until graph becomes 3-regular and can be represented as $C_n(n/2 + 1, n/2) = C_{n - 3}$.

**Case 2**: $n \equiv 1 \pmod{2}$

If the order is odd, then degrees will be even, starting from $r = n - 1$ for $K_n$ up to $r = 4$ as maximum decomposed circulant graph. Circulant graph with degree $r = 2$ is innermost cycle $C_n$ whose $es(C_n)$ has been proved already in [1]. The value of $m = (n - 1)/2$ and the graph can be represented as $C_n(1, 2, \ldots, (n - 1)/2)$. The first decomposition can be done by deleting a Hamiltonian cycle (outer cycle) that will result as $(n - 3)$-regular and the graph can be represented as $C_n(2, 3, \ldots, (n - 1)/2) = C_{n - 3}$. The second decomposition can be done by deleting another 2-factor that means all edges of type 2 will be deleted and the graph representation will be like
Cₙ(3, 4, ..., (n - 1)/2) = Cₙₙ₋₅. This decomposition can continue until graph becomes 4-regular and can be represented as Cₙ((n - 3)/2, (n - 1)/2) = Cₙ₄₄.

Phenomena of decomposition for both of the above cases is reflected in Figure 1.

III. ALGORITHMIC RESULTS

Using the decomposition process explained in Case 1 and 2, it is obvious that successive subgraphs have lesser size and according to handshaking lemma, size of each r-regular subgraph can be determined as nr/2. In [4] an upper bound for es(Kₙ) is computed as |E(Kₙ)| log₂ |V(Kₙ)|. The complete graph is super-circulant graph hence this result is considered as an upper bound, whereas a lower bound for any es(G) is presented as

$$\left\lceil \frac{n(r+2)}{4} \right\rceil.$$  

**Theorem 3:** Let Cₙₙ be an r-regular circulant graph of order n, n ≥ 5 and r ≥ 3. Then

$$\max \left\{ \left\lceil \frac{nr + 2}{4} \right\rceil, r \right\} \leq es(Cₙₙ) \leq \frac{nr}{2} \log_2 n.$$

**Proof 1:** The size of a circulant graph Cₙₙ is nr/2, therefore using Theorem 1 we obtained the (nr + 2)/4 as a lower bound for es(Cₙₙ). To prove the upper bound, an iterative algorithm is designed using back-tracking design strategy.

This algorithm computes an edge irregular k-labeling for a circulant graph Cₙᵣ with order n ≥ 5 and any positive integer 4 ≤ r ≤ n - 1 as degree. Algorithm returns output in the form of an array containing label of vertices where nⁿ location vertex is k. Values of edge weights are computed, then compared their uniqueness and finally stored in a 2-D array.

**Input:** A positive integer n ≥ 5 as the order of the graph and r as degree of the graph.

**Output:** Labels of vertices V[n] → {1, 2, ..., k}

**Algorithm 1 CR-Labeling(n, r)**

1: V[n] ← {1, 2, 3, 4}
2: Diff ← n - r
3: dfact ← ⌈}\frac{\text{Diff}}{r}\rceil \rceil
4: for each edge w₁(x, y) ← 1 where x ≠ y
5: t ← 5
6: m ← 4
7: repeat
8: V[t] ← m + 1
9: Edge-Calculate(G, t)
10: if (Edge-Duplicate(G, t) ≠ TRUE)
11: t ← t + 1
12: until t ≤ n
13: return V

**Description of the Algorithm:** CR-Labeling(n, r) computes the label of vertices {1, 2, ..., k} and store them in an array V[n], where {1, 2, 3, 4} are four initial labels used as seed values to initialize the back-tracking algorithm. Algorithm computes a variable “dfact” (difference factor), using the given inputs n and r that actually identi-
TABLE 2. Algorithmic results for circulant graphs \((n - 3)\) to \((n - 9)\) and suggested upper-bound.

| \(n\) | \(r = n - 3\) | \(r = n - 5\) | \(r = n - 7\) | \(r = n - 9\) |
|-------|---------------|---------------|---------------|---------------|
| \(|E|\)  | \(k\) bound | \(|E|\)  | \(k\) bound | \(|E|\)  | \(k\) bound | \(|E|\)  | \(k\) bound |
| 7     | 14           | 13           | 20           | 12           | 14           | 18           | 15           | 20           | 25           |
| 8     | 20           | 19           | 30           | 12           | 14           | 18           | 15           | 20           | 25           |
| 9     | 27           | 28           | 43           | 18           | 19           | 29           | 22           | 26           | 38           |
| 10    | 35           | 41           | 58           | 25           | 26           | 42           | 15           | 20           | 25           |
| 11    | 44           | 55           | 76           | 33           | 36           | 57           | 22           | 26           | 38           |
| 12    | 54           | 69           | 97           | 42           | 50           | 75           | 30           | 34           | 54           | 18           | 27           | 32           |
| 13    | 65           | 83           | 120          | 52           | 69           | 96           | 39           | 45           | 72           | 26           | 34           | 48           |
| 14    | 77           | 103          | 147          | 63           | 89           | 120          | 49           | 60           | 93           | 35           | 43           | 67           |
| 15    | 90           | 124          | 176          | 75           | 109          | 147          | 60           | 80           | 117          | 45           | 55           | 88           |
| 16    | 104          | 147          | 208          | 88           | 129          | 176          | 72           | 106          | 144          | 56           | 71           | 112          |
| 17    | 119          | 208          | 245          | 102          | 159          | 208          | 85           | 133          | 174          | 68           | 92           | 139          |
| 18    | 135          | 238          | 281          | 117          | 186          | 244          | 99           | 160          | 206          | 81           | 119          | 169          |
| 19    | 152          | 267          | 325          | 133          | 213          | 282          | 114          | 187          | 242          | 95           | 153          | 202          |
| 20    | 170          | 296          | 367          | 150          | 240          | 324          | 130          | 223          | 281          | 110          | 188          | 238          |
| 21    | 189          | 325          | 415          | 168          | 267          | 369          | 147          | 250          | 323          | 126          | 223          | 277          |
| 22    | 209          | 423          | 466          | 187          | 309          | 417          | 165          | 286          | 368          | 143          | 258          | 319          |
| 23    | 230          | 454          | 520          | 207          | 361          | 468          | 184          | 333          | 416          | 161          | 303          | 364          |
| 24    | 252          | 499          | 578          | 228          | 405          | 523          | 204          | 369          | 468          | 180          | 350          | 413          |
| 25    | 275          | 570          | 639          | 250          | 457          | 580          | 225          | 439          | 522          | 200          | 385          | 464          |
| 26    | 299          | 641          | 703          | 273          | 535          | 642          | 247          | 506          | 581          | 221          | 445          | 519          |
| 27    | 324          | 674          | 770          | 297          | 627          | 706          | 270          | 576          | 642          | 243          | 478          | 578          |
| 28    | 350          | 781          | 841          | 322          | 664          | 774          | 294          | 614          | 707          | 266          | 519          | 639          |
| 29    | 377          | 874          | 916          | 348          | 699          | 845          | 319          | 665          | 775          | 290          | 538          | 704          |
| 30    | 405          | 946          | 994          | 375          | 791          | 920          | 345          | 707          | 846          | 315          | 579          | 773          |

fies how many generators are deleted from \(K_n\). This variable controls the starting and termination of loops. Algorithm is designed in a way that it can handle the even and odd value of \(n\). Sub-procedure “Edge-Weights\((G, t)\)” is based on two nested loops, that are executed in style of Gaussian arithmetic series to compute weights of all edges and then store in two dimensional array. Whereas “Weight-Duplicate\((G, t)\)” ensures that all the edge weights are unique to verify the condition \(w(u, v) \neq w(u', v')\) for all edges \((u, v)\) and \((u', v')\). CR-Labeling\((n, r)\) works in a fashion of backtracking because any of the \(i^{th}\) label for \(C_i(a_1, a_2, \ldots, a_m)\) is computed using the labels of \(C_i-1(a_1, a_2, \ldots, a_m)\).

**FIGURE 2.** Labeling of circulant graph \(C_{10}\) from \((n - 1)\) to \((n - 7)\) regular.

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Weight-Duplicate\((G, t)\)
1: for \(i \leftarrow 2\) to \(t - dfac - 2\)
2: for \(j \leftarrow dfac + i + 1\) to \(t - 1\)
3: for \(l \leftarrow 1\) to \(i - 1\)
4: for \(w \leftarrow j + 1\) to \(t\)
5: if \(E[l][j] = E[l][w] AND E[l][w] \neq NULL\)
6: return TRUE
7: break
8: return FALSE
```

Outcomes of the algorithm is shown pictorially in Figure 2 as labeled graphs and their representation in 2-D matrices.
It can be observed clearly that by deleting one generator sequentially, the graph is decomposed from 9-regular to 3-regular and the value of the largest label is also reduced from $k = 47$ to $k = 14$.

Examples of circulant graphs given in Figure 2, in Table 2 and the chart given in Figure 3 for higher order circulant graphs, prove the claim of this study. It can be observed as mathematical inequality that

$$
\max \left\{ \frac{nr + 2}{4}, r \right\} \leq \frac{nr}{4} \log_2(n) \leq \frac{nr}{2} \log_2(n),
$$

that completes the proof.

IV. CONCLUDING REMARKS

Using the values of $k$ for circulant graphs calculated with the help of algorithm, and suggested upper bound that is definitely smaller than upper bound of complete graph because decomposed circulant graphs are subgraphs of $K_n$. Line chart given in Figure 3 shows the comparison between the curves of a lower bound for any graph, algorithmic results of circulant graphs as $k$, suggested upper-bound of $es(C_n(a_2), a_3, \ldots, a_m)) \leq |E|/2\log_2(|V|) \leq es(K_n) = |E| \log_2 |V|.$

REFERENCES

[1] A. Ahmad, O. B. S. Al-Mushayt, and M. Bača, “On edge irregularity strength of graphs,” Appl. Math. Comput., vol. 243, pp. 607–610, Sep. 2014.

[2] A. Ahmad, M. A. Asim, M. Bača, and R. Hasni, “Computing edge irregularity strength of complete m-ary trees using algorithmic approach,” U.P.B. Sci. Bull., Sec. A, vol. 80, no. 3, pp. 145–152, 2018.

[3] M. A. Asim, A. Ahmad, and R. Hasni, “Edge irregular k-labeling for several classes of trees,” Utilitas Math., vol. 111, pp. 75–83, Jun. 2019.

[4] M. A. Asim, A. Ahmad, and R. Hasni, “Iterative algorithm for computing irregularity strength of complete graph,” Ars Combinatoria, vol. 138, pp. 17–24, Apr. 2018.

[5] J. C. Bermond, F. Comellas, and D. F. Hu, “Distributed loop computer-networks: A survey,” J. Parallel Distrib. Comput., vol. 24, no. 1, pp. 2–10, Jan. 1995.

[6] F. Buckley and F. Harary, Distance in Graphs. Redwood City, CA, USA: Addison-Wesley, 1990.

[7] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, and F. Saba, “Irregular networks,” Congr. Numer., vol. 64, pp. 187–192, Jan. 1988.

[8] S. Cichacz, S. Dib, and D. Froncek, “Decomposition of complete graphs into (0, 2)-prisms,” Czech. Math. J., vol. 64, no. 1, pp. 37–43, 2014.

[9] J. Cilleruelo, I. Ruzsa, and C. Viseuca, “Generalized sidon sets,” Adv. Math., vol. 225, no. 5, pp. 2786–2807, Dec. 2010.

[10] R. J. Faudree, R. H. Schelp, M. S. Jacobson, and J. Lehel, “Irregular networks, regular graphs and integer matrices with distinct row and column sums,” Discrete Math., vol. 76, no. 3, pp. 223–240, 1989.

[11] A. Frieze, R. J. Gould, M. Karonski, and F. Pfender, “On graph irregularity strength,” J. Graph Theory, vol. 41, no. 2, pp. 120–137, 2002.

[12] M. Kalkowski, M. Karonski, and F. Pfender. “A new upper bound for the irregularity strength of graphs,” SIAM J. Discrete Math., vol. 25, no. 3, pp. 1319–1321, Jan. 2011.

[13] A. Knopfmacher and M. E. Mays, “Graph compositions I: Basic enumeration,” Integers, Electron. J. Combin. Numb. Theory Theory, vol. 1, no. 4, p. 11, 2001.

[14] R. Lu, “Fast methods for designing circulant network topology with high connectivity and survivability,” J. Cloud. Comp., vol. 5, no. 1, pp. 1–13, 2016.

[15] P. Majerski and J. Przybyło, “On the irregularity strength of dense graphs,” SIAM J. Discrete Math., vol. 28, no. 1, pp. 197–205, Jan. 2014.

[16] O. Michail, “An introduction to temporal graphs: An algorithmic perspective,” Internet Math., vol. 12, no. 4, pp. 239–280, Jul. 2016.

[17] K. H. Rosen, Discrete Mathematics and Its Applications, 7th ed New York, NY, USA: McGraw-Hill, 2012.

[18] M. Sipper and E. Ruppin, “Co-evolving cellular architectures by cellular programming,” in Proc. IEEE Int. Conf. Evol. Comput., May 1996, pp. 306–311.

[19] I. Tarawneh, R. Hasni, M. K. Siddiqui, and M. A. Asim, “On the edge irregularity strength of disjoint union of graphs,” Ars Combinatoria, vol. 142, pp. 239–249, Jan. 2019.

[20] I. Tarawneh, R. Hasni, and M. A. Asim, “On the edge irregularity strength of disjoint union of star graph and subdivision of star graph,” Ars Combinatoria, vol. 141, pp. 93–100, Oct. 2018.

[21] T. Tichenor and M. E. Mays, “Graph compositions: Deleting edges from complete graphs,” Integers, vol. 15, no. 35, p. 12, 2015.
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