A Generalization of the Passivity Theorem and the Small Gain Theorem Based on $\rho$-Stability, with Application to a Parameter Adaptation Algorithm for Recursive Identification

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Abstract

The usual passivity theorem considers a closed-loop, the direct chain of which consists of a strictly passive stable operator $H_1$, and the feedback chain of which consists of a passive operator $H_2$. Then the closed-loop is stable. Let $\rho > 1$ and let us adopt the terminology introduced in [4]. We show here that the closed-loop is still stable when the direct chain consists of a strictly $\rho^{-1}$-passive $\rho^{-1}$-stable operator (a weaker condition than above) and the feedback chain consists of a $\rho$-passive operator (a stronger condition than above). Variations on the theme of the small gain theorem (incremental or not) can be made similarly. This approach explains the results obtained in a paper on identification which was recently published [6].

1 Introduction and preliminaries

Many stability theorems were derived for a standard closed-loop system (as depicted in, e.g., Figure III.1 of [5], p. 37), the direct chain of which consists of an operator $H_1$, with input $e_1$ and output $y_1$, and the feedback chain of which consists of an operator $H_2$, with input $e_2$ and output $y_2$. The interconnection equations are $e_1 = u_1 - y_2$, $e_2 = u_2 + y_1$ where $u_1, u_2$ are external signals.

Let $T = \mathbb{Z}$ in the discrete-time case and $T = \mathbb{R}$ in the continuous-time case. In addition, let $S^n$ be the subspace of $(l^2)^n$ in the former case, of $(L^2)^n$ in the latter, consisting of those signals which have a left-bounded support; $S^n$ is a Hilbert space. Let $T \in T$, let $P_T$ be the truncation operator, such that $(P_T x)(t) = x(t)$ if $t \leq T$ and $(P_T x)(t) = 0$ otherwise.

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(Sect. 2.3), and let $S^n_T$ be the extended space consisting of all signals $x \in (\mathbb{R}^n)^T$ such that $P_T x \in S$ for all $T \in \mathbb{T}$. If $x, y \in S^n_T$, the inner product $(P_T x, P_T y)_{S^n_T}$ is denoted by $(x, y)_T$, and $\|x\|_T := \sqrt{(x, x)_T}$. Let $H : S^n_T \to S^n_T$ be an operator. Its gain $\gamma(H) \leq +\infty$ is defined to be ([5], Sect. 3.1)

$$\gamma(H) = \inf \{ \delta \geq 0 : \exists \beta \in \mathbb{R}, \|Hx\|_T \leq \delta \|x\|_T + \beta \text{ for all } T \in \mathbb{T} \}.$$ 

We put

$$\gamma^0(H) = \inf \{ \delta \geq 0 : \|Hx\|_T \leq \delta \|x\|_T \text{ for all } T \in \mathbb{T} \}$$

and $H$ is said to be $S$-stable if $\gamma^0(H) < +\infty$ ([5], Sect. 3.7).

Let $G$ be the multiplicative Abelian group of all positive real numbers. This group acts on $S^n_T$ as follows: if $\rho \in G$, $x \in S^n_T$, then $(\rho \circ x)(t) = \rho t x(t)$. In the continuous-time case, let $\alpha = \ln(\rho)$; then the concept of $\rho$-stability as defined in [4] is equivalent to $\alpha$-stability as introduced in [1] and developed in [2], [3].

The following is assumed in this paper (with the above notation): if $u_1, u_2 \in S^n_T$, then there are solutions $e_1, e_2 \in S^n_T$ ("well-posedness" of the closed-loop).

The classical passivity (resp. small gain) theorem states that if $H_1$ is strictly passive and such that $\gamma^0(H_1) < +\infty$ (resp. is such that $\gamma^0(H_1) < +\infty$) and $H_2$ is passive (resp. is such that $\gamma^0(H_2) < +\infty$ and $\gamma^0(H_1), \gamma^0(H_2) < 1$) then the operator $(u_1, u_2) \mapsto (e_1, e_2, y_1, y_2)$ is $S$-stable. This result has variants which will be mentioned below.

In what follows, using the action of $G$, we relax the assumption on $H_1$ and strengthen the assumption on $H_2$, or vice-versa.

## 2 Extension of stability results

### 2.1 Extended passivity stability theorem

Consider again the closed-loop system as specified in Section [1] assumed to be well-posed, with $H_1$ replaced by $\rho^{-1} \circ H_1 \circ \rho$ and $H_2$ replaced by $\rho \circ H_2 \circ \rho^{-1}$, so that

$$e_1 = u_1 - y_2 = u_1 - \rho \circ H_2 \circ \rho^{-1} \circ e_2 \quad (1)$$

$$e_2 = u_2 + y_1 = u_2 + \rho^{-1} \circ H_1 \circ \rho \circ e_1 \quad (2)$$

One passes from the original closed-loop to the new one by introducing multipliers $\rho \circ$ and $\rho^{-1} \circ$. 

[2]
**Theorem 1** Assume that $\gamma (\rho^{-1} \circ H_1 \circ \rho) < +\infty$ and that there are constants $\delta_1, \beta'_1, \varepsilon_2, \beta'_2$ such that

\[
\langle x, \rho^{-1} \circ H_1 \circ \rho \circ x \rangle \geq \delta_1 \|x\|_T^2 + \beta'_1 \\
\langle x, \rho \circ H_2 \circ \rho^{-1} \circ x \rangle \geq \delta_2 \|\rho \circ H_2 \circ \rho^{-1} \circ x\|_T^2 + \beta'_2
\]

for all $x \in S^n_e$ and all $T \in \mathbb{T}$. If

\[
\delta_1 + \delta_2 > 0
\]

then $e_1, e_2, y_1, y_2 \in S^n$ whenever $u_1, u_2 \in S^n$.

**Proof.** For any $T \in \mathbb{T}$, we have that

\[
\langle e_1, y_1 \rangle_T + \langle e_2, y_2 \rangle_T = \langle u_1 - y_2, y_1 \rangle_T + \langle u_2 + y_1, y_2 \rangle_T = \langle u_1, y_1 \rangle_T + \langle u_2, y_2 \rangle_T.
\]

In addition,

\[
\langle e_1, y_1 \rangle_T = \langle e_1, \rho^{-1} \circ H_1 \circ \rho \circ e_1 \rangle_T \geq \delta_1 \|e_1\|_T^2 + \beta'_1, \\
\langle e_2, y_2 \rangle_T = \langle e_2, \rho \circ H_2 \circ \rho^{-1} \circ e_2 \rangle_T \geq \delta_2 \left\| \rho \circ H_2 \circ \rho^{-1} \circ e_2 \right\|_T^2 + \beta'_2
\]

\[
\geq \delta_2 \left( \|u_1\|_T^2 - 2 \|u_1\|_T \|e_1\|_T + \|e_1\|_T^2 \right) + \beta'_2.
\]

Therefore, setting $\gamma_1 = \gamma (\rho^{-1} \circ H_1 \circ \rho)$,

\[
\delta_1 \|e_1\|_T^2 + \beta'_1 + \delta_2 \left( \|u_1\|_T^2 - 2 \|u_1\|_T \|e_1\|_T + \|e_1\|_T^2 \right) + \beta'_2
\]

\[
\leq \langle u_1, y_1 \rangle_T + \langle u_2, y_2 \rangle_T \leq \|u_1\|_T \|y_1\|_T + \|u_2\|_T \|y_2\|_T
\]

\[
\leq \|u_1\|_T (\gamma_1 \|e_1\|_T + \beta_1) + \|u_2\|_T (\|u_1\|_T + \|e_1\|_T)
\]

which implies

\[
(\delta_1 + \delta_2) \|e_1\|_T^2 \leq \|e_1\|_T \left[ (2 \|u_1\|_T + \gamma_1 \|u_1\|_T) + \|u_2\|_T \right]
\]

\[
+ \|u_1\|_T \|u_2\|_T + \beta_1 \|u_1\|_T + \|u_1\|_T^2 - \beta'_1 - \beta'_2.
\]

This is the same equality as in ([3], section 6.5, (23)) (correcting an obvious misprint) and the result follows as in this reference. ■

**Corollary 2** (Extended passivity theorem) Assume that $\gamma^0 (\rho^{-1} \circ H_1 \circ \rho) < +\infty$ and that there exists $\delta_1 > 0$ such that

\[
\langle x, \rho^{-1} \circ H_1 \circ \rho \circ x \rangle \geq \delta_1 \|x\|_T^2 \tag{3}
\]

\[
\langle x, \rho \circ H_2 \circ \rho^{-1} \circ x \rangle \geq 0 \tag{4}
\]

for all $x \in S^n_e$ and all $T \in \mathbb{T}$. (In this case, we will say that $H_1$ is $\rho^{-1}$-
stable and is $\rho^{-1}$-passive, and that $H_2$ is $\rho$-passive.) Then the operator $(u_1, u_2) \mapsto (e_1, e_2, y_1, y_2)$ is $S$-stable.
Proof. This follows from Theorem \[ \| \] by the same rationale as in the proof of Corollary 27 of (\[5\], Sect. 6.5). □

The proof of the following is elementary:

**Proposition 3** In the discrete-time case, let $H_1$ be the linear operator such that for any $x \in S^n$

$$(H_1 x) (t) = \sum_{\tau \in \mathbb{Z}} h_1 (t, \tau) x (\tau).$$

Then

$$(\rho^{-1} \circ H_1 \circ \rho \circ x) (t) = \sum_{\tau \in \mathbb{Z}} h_1 (t, \tau) \rho^{\tau-t} x (\tau).$$

In particular, if $H_1$ is LTI, then $h_1 (t, \tau) = h_1 (t - \tau)$ (where $h_1$ is the impulse response), so that the transfer matrix of $\rho^{-1} \circ H_1 \circ \rho$ is $z \mapsto \hat{h}_1 (\rho z)$ where $z \mapsto \hat{h}_1 (z)$ is the transfer matrix of $H_1$.

**Remark 4** Likewise, in the continuous LTI case, the transfer matrix of $\rho^{-1} \circ H_1 \circ \rho$, with $\rho = e^{\alpha}$, is $s \mapsto \hat{h}_1 (s + \alpha)$ where $s \mapsto \hat{h}_1 (s)$ is the transfer matrix of $H_1$.

The proof of the following is easy:

**Corollary 5** Let $H_1$ be an LTI operator with rational transfer matrix $\hat{h}_1$. Then the conditions on $H_1$ in Corollary \[ \| \] are satisfied provided that:

- In the discrete-time case, the transfer matrix $\hat{h}_1$ is analytic and bounded in $|z| > \rho$, and for all $\theta \in [0, \pi]$

$$\lambda_{\min} \left\{ \frac{\hat{h}^T (\rho e^{-i\theta}) + \hat{h} (\rho e^{i\theta})}{2} \right\} \geq \delta_1$$

- In the continuous-time case, the transfer matrix $\hat{h}_1$ is analytic and bounded in $\Re (s) > \alpha$, and for all $\omega \geq 0$

$$\lambda_{\min} \left\{ \frac{\hat{h}^T (\alpha - i\omega) + \hat{h} (\alpha + i\omega)}{2} \right\} \geq \delta_1$$

### 2.2 Extended small gain theorem

Consider again the closed-loop system, assumed to be well-posed and defined by equations \[ \| \] , \(2\).
Theorem 6  (i) Assume that \( \gamma_1 := \gamma (\rho^{-1} \circ H_1 \circ \rho) < +\infty, \gamma_2 := \gamma (\rho \circ H_2 \circ \rho^{-1}) < +\infty, \) and that \( \gamma_1 \gamma_2 < 1. \) Then, the operator \((u_1, u_2) \mapsto (e_1, e_2)\) has finite gain.

(ii) Assume that \( \gamma_0^1 := \gamma_0 (\rho^{-1} \circ H_1 \circ \rho) < +\infty, \gamma_0^2 := \gamma_0 (\rho \circ H_2 \circ \rho^{-1}) < +\infty, \) and that \( \gamma_0^1 \gamma_0^2 < 1. \) Then, the operator \((u_1, u_2) \mapsto (e_1, e_2)\) is \( S \)-stable.

Proof. The proof is similar to that of ([5], section 3.2, Theorem 1).

Remark 7  (1) As in the usual case \( \rho = 1 \), the extended small gain theorem and the extended passivity theorem are closely related. Indeed, let \( H : S_n^e \rightarrow S_n^e \) be such that \( (I + H)^{-1} \) is a well-defined operator \( S_n^e \rightarrow S_n^e. \) As easily seen, \( \rho^{-1} \circ (I + H) \rho = (I + \rho^{-1} \circ H \circ \rho)^{-1}, \) thus \( (I + \rho^{-1} \circ H \circ \rho)^{-1} \) is well-defined. In addition,

\[
(\rho^{-1} \circ H \circ \rho - I)^{-1} (I + \rho^{-1} \circ H \circ \rho)^{-1} = \rho^{-1} (H - I) (I + H) \rho.
\]

Therefore, putting \( S := (H - I) (I + H)^{-1}, \)

(a) Condition (4) is satisfied if and only if \( \gamma_0^0 (S) \leq 1. \)

(b) The following conditions (i), (ii) are equivalent: (i) there exists \( \delta_1 > 0 \) such that Condition (3) is satisfied and \( \gamma_0^0 (\rho^{-1} \circ H \circ \rho) < +\infty; \)

(ii) \( \gamma_0^0 (S) < 1 \) (see [5], section 6.10, lemma 7 for the details). Thus, one passes from Corollary 2 to statement (ii) of Theorem 6 via the usual loop transformation described in ([5], section 6.10).

(2) A generalized version of the incremental small gain theorem ([5], section 3.3) can be obtained following the same line, and its statement is left to the reader. The pattern of noncausal multiplier technique, as described in ([5], section 9.2), can also be extended in a similar way.

3 Application to a parameter adaptation algorithm

We consider now the parameter adaptation algorithm (PAA) in [6]. The aim of the algorithm is to identify a discrete-time system with poles on or outside the unit circle. The simulations in Section 4 of [6] show that this is indeed possible since the PAA is \( \rho \)-stable with \( \rho > 1. \) However, although the theorems of [6] are correct mathematically, they do not explain this result. In the two theorems of [6], the condition that \( H (z/\rho) - \frac{\lambda_2}{2} (\rho > 1) \) be strictly positive real is indeed more restrictive than the condition that \( H (z) - \frac{\lambda_2}{2} \) be strictly positive real. Thus, this condition must be replaced by: \( H (\rho z) - \frac{\lambda_2}{2} \) is strictly positive real. By Corollary 2 and Proposition 3 here above, with this change the identification algorithm of [6] converges (with degree of stability 1, not \( \rho \)). This observation was the first motivation of this paper.
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