Abstract. Frenkel-Reshetikhin introduced $q$-characters of finite dimensional representations of quantum affine algebras \cite{Frenkel-Reshetikhin}. We give a combinatorial algorithm to compute them for all simple modules. Our tool is $t$-analogue of the $q$-characters, which is similar to Kazhdan-Lusztig polynomials, and our algorithm has a resemblance with their definition.

We need the theory of quiver varieties for the definition of $t$-analogues and the proof. But it appear only in the last section. The rest of the paper is devoted to an explanation of the algorithm, which one can read without the knowledge about quiver varieties. A proof is given only in part. A full proof will appear elsewhere.

1. The quantum loop algebra

Let $\mathfrak{g}$ be a simple Lie algebra of type ADE over $\mathbb{C}$, $L_\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ be its loop algebra, and $U_q(L_\mathfrak{g})$ be its quantum universal enveloping algebra, or the quantum loop algebra for short. It is a subquotient of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$, i.e., without central extension and degree operator. Let $I$ be the set of simple roots, $P$ be the weight lattice, and $P^*$ be its dual lattice (all for $\mathfrak{g}$). The algebra has the so-called Drinfeld’s new realization: It is a $\mathbb{C}(q)$-algebra with generators $q^h$, $e_{k,r}$, $f_{k,r}$, $h_{k,n}$ ($h \in P^*$, $k \in I$, $r \in \mathbb{Z}$, $n \in \mathbb{Z} \setminus \{0\}$) with certain relations (see e.g., \cite{Frenkel-Reshetikhin} 12.2).

The algebra $U_q(L_\mathfrak{g})$ is a Hopf algebra, where the coproduct is defined using the Drinfeld-Jimbo realization of $U_q(L_\mathfrak{g})$. So a tensor product $M \otimes_{\mathbb{C}(q)} M'$ of $U_q(L_\mathfrak{g})$-modules $M$, $M'$ has a structure of a $U_q(L_\mathfrak{g})$-module.

Let $U_q(L_\mathfrak{g})$ be its specialization at $q = \varepsilon \in \mathbb{C}^*$. For precise definition of the specialization, we first introduce an integral form $U^\mathbb{Z}_q(L_\mathfrak{g})$ of $U_q(L_\mathfrak{g})$ and set $U_\varepsilon(L_\mathfrak{g}) = U^\mathbb{Z}_q(L_\mathfrak{g}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}$, where $\mathbb{Z}[q, q^{-1}] \to \mathbb{C}$ is given by $q^{\pm 1} \mapsto \varepsilon^{\pm 1}$. See \cite{Frenkel-Reshetikhin} for detail. But we assume $\varepsilon$ is not a root of unity in this paper. So we just replace $q$ by $\varepsilon$ in the definition of $U_\varepsilon(L_\mathfrak{g})$.

The quantum loop algebra $U_q(L_\mathfrak{g})$ contains the quantum enveloping algebra $U_q(\mathfrak{g})$ for the finite dimensional Lie algebra $\mathfrak{g}$ as a subalgebra. The specialization $U_\varepsilon(\mathfrak{g})$ of $U_q(\mathfrak{g})$. 

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1.1. Finite dimensional representations of \( \mathbf{U}_\varepsilon(\mathfrak{L}_g) \). The algebra \( \mathbf{U}_\varepsilon(\mathfrak{L}_g) \) contains a commutative subalgebra generated by \( q^h, h_{k,n} (h \in \mathbb{P}^*, k \in I, n \in \mathbb{Z} \setminus \{0\}) \). Let us introduce generating functions \( \psi^\pm_k(z) (k \in I) \) by

\[
\psi^\pm_k(z) \overset{\text{def.}}{=} q^{\pm h_k} \exp \left( \pm (q - q^{-1}) \sum_{m=1}^{\infty} h_{k, \pm m} z^{\pm m} \right).
\]

A \( \mathbf{U}_\varepsilon(\mathfrak{L}_g) \)-module \( M \) is called of type 1 if \( M \) has a weight space decomposition as a \( \mathbf{U}_\varepsilon(\mathfrak{g}) \)-module:

\[
M = \bigoplus_{\lambda \in P} M(\lambda), \quad M(\lambda) = \left\{ m \in M \middle| q^h \ast m = \varepsilon(h, \lambda) m \right\}.
\]

We will only consider type 1 modules in this paper.

A type 1 module \( M \) is an \( l \)-highest weight module ('l' stands for the loop) if there exists a vector \( m_0 \in M \) such that

\[
e_{k,r} \ast m_0 = 0, \quad \mathbf{U}_\varepsilon(\mathfrak{L}_g) \ast m_0 = M,
\]

\[
\psi^\pm_k(z) \ast m_0 = \Psi^\pm_k(z)m_0 \quad \text{for } k \in I
\]

for some \( \Psi^\pm_k(z) \in \mathbb{C}[z^\pm] \). The pair of the \( I \)-tuple \((\Psi^+(z), \Psi^-(z)) = (\Psi^+_k(z), \Psi^-_k(z))_{k \in I} \in (\mathbb{C}[z^\pm])^I \) is called the \( l \)-highest weight of \( M \), and \( m_0 \) is called the \( l \)-highest weight vector.

**Theorem 1.1.1** (Chari-Pressley [1]). (1) Every finite-dimensional simple \( \mathbf{U}_\varepsilon(\mathfrak{L}_g) \)-module of type 1 is an \( l \)-highest weight module, and its \( l \)-highest weight is given by

\[
(1.1.2) \quad \Psi^\pm_k(z) = \varepsilon^{\deg P_k} \left( \frac{P_k(\varepsilon^{-1}/z)}{P_k(\varepsilon/z)} \right)^{\pm}
\]

for some polynomials \( P_k(u) \in \mathbb{C}[u] \) with \( P_k(0) = 1 \). Here \( (\ )^\pm \in \mathbb{C}[z^\pm] \) denotes the expansion at \( z = \infty \) and 0 respectively.

(2) Conversely, for given \( P_k(u) \) as above, there exists a finite-dimensional simple \( l \)-highest weight \( \mathbf{U}_\varepsilon(\mathfrak{L}_g) \)-module \( M \) of type 1 such that the \( l \)-highest weight is given by the above formula.

Assigning to \( M \) the \( I \)-tuple \( P = (P_k)_{k \in I} \in \mathbb{C}[u]^I \) \( (P_k(0) = 1) \) defines a bijection between the set of all \( P \)'s and the set of isomorphism classes of finite-dimensional simple \( \mathbf{U}_\varepsilon(\mathfrak{L}_g) \)-modules of type 1.

We denote by \( L_P \) the simple \( \mathbf{U}_\varepsilon(\mathfrak{L}_g) \)-module associated to \( P \). We call \( P \) the Drinfeld polynomial. For the abuse of terminology, we also say \( ^\top P \) is the \( l \)-highest weight of \( L_P \).

Since \( \mathbb{C}[q^h, h_{k,n}] \) is a commutative subalgebra of \( \mathbf{U}_\varepsilon(\mathfrak{L}_g) \), any \( \mathbf{U}_\varepsilon(\mathfrak{L}_g) \)-module \( M \) decomposes into a direct sum \( M = \bigoplus M(\Psi^+, \Psi^-) \) of generalized eigenspaces, where

\[
M(\Psi^+, \Psi^-) \overset{\text{def.}}{=} \left\{ m \in M \middle| (\psi^\pm_k(z) - \Psi^\pm_k(z) \text{Id})^N \ast m = 0 \text{ for } k \in I \text{ and sufficiently large } N \right\},
\]

for \( \Psi^\pm_k(z) \in \mathbb{C}[z^\pm] \). The pair of the \( I \)-tuple \((\Psi^+, \Psi^-) = (\Psi^+_k, \Psi^-_k)_{k \in I} \) is called an \( l \)-weight, and \( M(\Psi^+, \Psi^-) \) is called an \( l \)-weight space of \( M \) if \( M(\Psi^+, \Psi^-) \neq 0 \).
THEOREM 1.1.3 (Frenkel-Reshetikhin [3]). Any $l$-weight of any finite dimensional $U_q(L_0)$-module $M$ of type 1 has the following form
\begin{equation}
\Psi^\pm_k(z) = z^{\deg R_k} \left( \frac{Q_k(z^{-1}) R_k(z)}{Q_k(z) R_k(z^{-1})} \right)^\pm
\end{equation}
for some polynomials
\[
Q_k(u) = \prod_{i=1}^{s_k} (1 - a_{ki} u), \quad R_k(u) = \prod_{j=1}^{r_k} (1 - b_{kj} u).
\]
Again for the abuse of terminology, we also say ‘$Q/R$ is an $l$-weight of $M$’. We denote the $l$-weight space $M(\Psi^+, \Psi^-)$ by $M(Q/R)$.

Frenkel-Reshetikhin [3] defined the $q$-character of $M$ by
\[
\chi_q(M) = \sum_{Q/R} \dim M(Q/R) \prod_{k \in I} \prod_{i=1}^{s_k} \prod_{j=1}^{r_k} Y_{k,a_{ki}} Y_{k,b_{kj}}^{-1}.
\]

THEOREM 1.1.5 (Frenkel-Reshetikhin [3]). (1) $\chi_q$ defines an injective ring homomorphism from the Grothendieck ring $\text{Rep} U_q(L_0)$ of finite dimensional $U_q(L_0)$-modules of type 1 to $\mathbb{Z}[Y^\pm_{k,a}]_{k \in I, a \in \mathbb{C}}$ (a ring of Laurent polynomials in infinitely many variables).

(2) If we compose a map $Y^\pm_{k,a} \mapsto y^\pm_{k,a}$ (forgetting ‘spectral parameters’), it gives the usual character of the restriction of $M$ to a $U_q(g)$-module.

DEFINITION 1.1.6. A monomial $\prod_{k \in I} \prod_{i=1}^{s_k} \prod_{j=1}^{r_k} Y_{k,a_{ki}} Y_{k,b_{kj}}^{-1}$ appearing in the $q$-character $\chi_q$ is called $l$-dominant if $r_k = 0$ for all $k$, i.e., a product of positive powers of $Y_{k,a}$’s or 1.

If $L_P$ is the simple $U_q(L_0)$-module with $l$-highest weight $P$, its $q$-character contains an $l$-dominant monomial corresponding to the $l$-highest weight. We denote it by $m_P$. Its coefficient in $\chi_q(L_P)$ is 1.

Since $\{L_P\}_P$ forms a basis of $\text{Rep} U_q(L_0)$, we have the following useful condition for the simplicity of a finite dimensional $U_q(L_0)$-module $M$ of type 1:
\begin{equation}
\text{If } \chi_q(M) \text{ contains only one } l \text{-dominant term, then } M \text{ is simple.}
\end{equation}

1.2. Example. We give examples of $q$-characters.

If $g = A_n$, we have an evaluation homomorphism $ev_a : U_q(L_0) \rightarrow U_q(g)$ corresponding to $L_0 \rightarrow g; z \mapsto a$ (Jimbo). Hence pullbacks of simple $U_q(g)$-modules are simple $U_q(L_0)$-modules.

EXAMPLE 1.2.1. Let $g = A_1 = sl_2$ and $V$ be the 2-dimensional simple $U_q(L_0)$-module. Then the $q$-character of $M_a = ev_a(M)$ is given by\footnote{This can be checked directly. But it also follows from Theorem 5.2.2 below.}
\[
\chi_q(M_a) = Y_{1,a} + Y_{1,b}^{-1}.
\]

Since $\chi_q$ is a ring homomorphism, we have
\[
\chi_q(M_a \otimes M_b) = \left( Y_{1,a} + Y_{1,b}^{-1} \right) \left( Y_{1,b} + Y_{1,a}^{-1} \right) = Y_{1,a} Y_{1,b} + Y_{1,a}^{-1} Y_{1,b} + Y_{1,a} Y_{1,b}^{-1} + Y_{1,a}^{-1} Y_{1,b}^{-1}.
\]
Thus we may denote the above module by $M$.

If $b \neq a \epsilon^2, a \epsilon^{-2}$, then $M_a \otimes M_b$ is simple by the criterion (1.1.7).

If $b = a \epsilon^2$ or $a \epsilon^{-2}$, then the second or third term becomes 1. In fact, it is known that $M_a \otimes M_{a \epsilon^2}$ decomposes (in $\text{Rep} \ U_q(Lg)$) to a sum $M'_a \oplus M''$, where $M'$ is the 3-dimensional simple $U_q(Lg)$-module, and $M''$ is the trivial module. Thus we have

$$\chi_q(M_a \otimes M_{a \epsilon^2}) = \chi_q(M'_a) + \chi_q(M'') = Y_{1,a}Y_{1,a \epsilon^2} + Y_{1,a}Y_{1,a \epsilon^4} + Y_{1,a \epsilon^2}Y_{1,a \epsilon^4} + 1.$$

See also Examples 4.1.6, 4.1.8, 37.2.9.

2. Standard modules

2.1. In [15] we defined a family of finite dimensional $U_q(Lg)$-modules of type 1 and called them standard modules. They are parametrized by the $I$-tuples $P = (P_k)_{k \in I} \in \mathbb{C}[u]^I$ exactly as simple modules. We denote by $M_P$ associated to $P$. The definition will be recalled in [18] but we give here their algebraic identification due to Varagnolo-Vasserot [16].

**Definition 2.1.1.** We say $L_P$ an $l$-fundamental representation if

$$P_k(u) = \begin{cases} 1 - su & \text{if } k = k_0, \\ 1 & \text{otherwise}, \end{cases}$$

for some $s \in \mathbb{C}^*$ and $k_0 \in I$. We denote $L_P$ by $L(\Lambda_{k_0})_s$. ($\Lambda_k$ is the $k$-th fundamental weight of $g$.)

For $s \in \mathbb{C}^*$ and a finite sequence $(k_\alpha)_\alpha = (k_1, k_2, \ldots)$ in $I$ and a sequence $(n_\alpha)_\alpha = (n_1 \geq n_2 \geq \ldots)$ of integers, we set

$$M(s; (k_\alpha)_\alpha, (n_\alpha)_\alpha) \overset{\text{def}}{=} L(\Lambda_{k_1})_{\epsilon^{n_1}} \otimes L(\Lambda_{k_2})_{\epsilon^{n_2}} \otimes \cdots.$$

Note that $U_q(Lg)$ is not cocommutative Hopf algebra, so the tensor product depends on the ordering of factors.

**Theorem 2.1.2 (Varagnolo-Vasserot [16]).** (1) A standard module $M$ is isomorphic to a module of the form

$$\bigotimes_i M(s^i; (k_\alpha^i)_{\alpha_i}, (n_\alpha^i)_{\alpha_i})$$

such that $s^i / s^j \notin \mathbb{Z}$ for $i \neq j$ and $n_1^i \geq n_2^i \geq \ldots$ for each $i$.

(2) The above tensor product is independent of the ordering of the factors $M(s^i; (k_\alpha^i)_{\alpha_i}, (n_\alpha^i)_{\alpha_i})$.

(3) The $I$-tuple of polynomials $P$ corresponding to $M$ is the product of Drinfeld polynomials of $l$-fundamental representations appearing as factors of $M$.

Note that if $P$ is given, we can define a module $M$ of the above form by decomposing $P$ into a product of Drinfeld polynomials of $l$-fundamental representations. Thus we may denote the above module by $M_P$.

The following properties of $M_P$ were shown in [15]:

(1) $\{M_P\}$ is a basis of $\text{Rep} \ U_q(Lg)$. 

(2) $M_P$ is an $l$-highest weight module with $l$-highest weight $P$ (i.e., given by \((1,1,2)\)).

(3) $L_P$ is the unique simple quotient of $M_P$.

(4) $M_P$ depends ‘continuously’ on $P$ in a certain sense. For example, $\dim M_P$ is independent of $P$.

(5) For a generic $P$, $M_P \cong L_P$.

Conjecturally $M_P$ is isomorphic to the specialization of the module $V^{\max}(\lambda)$, introduced by Kashiwara [8], and further studied by Chari-Pressley [4].

3. $t$-analogues of $q$-characters

A main tool in this paper is a $t$-analogue of the $q$-character:

$$\chi_{q,t}: \text{Rep} U_t(L_q) \to \mathbb{Z}[t,t^{-1}][Y_{k,a}^{\pm}]_{k \in I, a \in C^*}.$$ 

This is a homomorphism of additive groups, not of rings, and has the property $\chi_{q,t} = \chi_q$. We define $\chi_{q,t}$ for all standard modules $M_P$. Since $\{M_P\}_P$ is a basis of $\text{Rep} U_t(L_q)$, we can extend it linearly to any finite dimensional $U_t(L_q)$-modules.

For the definition we need geometric constructions of standard modules, so we will postpone it to $§8.3$. We give an alternative definition, which is conjecturally the same as the geometric definition.

3.1. A conjectural definition. Let $M = M_P$ be a standard module, $Q/R$ be an $l$-weight of $M$, $M(Q/R)$ be the corresponding $l$-weight space. Define a filtration on $M(Q/R)$ by

$$0 = M^{-1}(Q/R) \subset M^0(Q/R) \subset M^1(Q/R) \subset \cdots \subset M^n(Q/R) \overset{\text{def.}}{=} \bigcap_k \ker(\psi_k^\pm(z) - \Psi_k^\pm(z) id)^{n+1}.$$ 

**Conjecture 3.1.1.** The $t$-analogue $\chi_{q,t}(M_P)$, defined geometrically in $§8.3$, is equal to

$$\chi_{q,t}(M_P) = \sum_{Q/R} \sum_n t^{2n-d(Q/R,P)} \dim \left( \frac{M^n(Q/R)}{M^{n-1}(Q/R)} \right) m_{Q/R},$$ 

where $d(Q/R,P)$ is an integer (determined explicitly from $Q/R$, $P$ by \((5.1.2)\) below), and $m_{Q/R}$ is a monomial in $Y_{k,a}^{\pm}$ corresponding to the $l$-weight space $M(Q/R)$.

This definition makes sense for any finite dimensional modules, but is not well-defined on the Grothendieck group $\text{Rep} U_t(L_q)$. Thus the above does not hold for simple modules.

3.2. A main result of this paper is a combinatorial algorithm for computing $\chi_{q,t}(M_P)$ and $[M_P : L_Q]$. It is divided into three steps:

**Step 1:** Compute $\chi_{q,t}$ for all $l$-fundamental representations.

**Step 2:** Compute $\chi_{q,t}(M_P)$ for all standard modules $M_P$.

**Step 3:** Express the multiplicity $[M_P : L_Q]$ in terms of $\chi_{q,t}(M_R)$ for various $R$.

Step 1 is a modification of Frenkel-Mukhin’s algorithm [8] for computing $\chi_q$ of $l$-fundamental representations. Step 2 is nothing but a study of $\chi_{q,t}$ of tensor products of $l$-fundamental representations. Although $\chi_{q,t}$ is not a ring homomorphism, $\chi_{q,t}$ of tensor products is given by a simply modified multiplication. For the proof we use an idea in [13]. Step 3 was essentially done in [15].
4. Step 3

We start with Step 3. The algorithm is similar to the definition of Kazhdan-Lusztig polynomials [4]. It is also similar to the algorithm for computing the transition matrix between the canonical basis and the PBW basis of type ADE [10].

4.1. Let

\[ A_{k,a} \overset{\text{def}}{=} Y_{k,a}Y_{k,a}^{-1} \prod_{l \neq k} Y_{l,a}^{c_{kl}}, \]

where \( c_{kl} \) is the \((k,l)\)-entry of the Cartan matrix.

**Definition 4.1.1.** (1) Let \( m, m' \) be monomials in \( Y_{k,a}^\pm \) \((k \in I, a \in C^*)\). We define an ordering \( \leq \) among monomials by

\[ m \leq m' \iff \frac{m'}{m} \text{ is a monomial in } A_{k,a}^{-1} \]

Here a monomial in \( A_{k,a}^{-1} \) means a product of nonnegative powers of \( A_{k,a}^{-1} \). It does not contain any factors \( A_{k,a} \).

(2) If \( \Psi^\pm, \Psi'^\pm \) are \( l \)-weights of finite dimensional \( U_q(L_g) \) modules, or \( Q/R, Q'/R' \) are related to \( l \)-weights by (1.1.4), we write \( \Psi^\pm \leq \Psi'^\pm, Q/R \leq Q'/R' \) if the corresponding monomials \( m, m' \) satisfy \( m \leq m' \).

Recall that \( \chi_q(L_P) \) contains an \( l \)-dominant monomial \( m_P \) corresponding to the highest weight vector. It is known that any monomial \( m \) appearing \( \chi_q(L_P) \) satisfies \( m \leq m_P \) ([5], 4.1), ([15], 13.5.2)].

Let

\[ c_{QP}(t) \overset{\text{def}}{=} \text{the coefficient of } m_Q \text{ in } \chi_q(t)(M_P). \]

Then \( (c_{QP}(t))_{P,Q} \) is upper-triangular and \( c_{PP}(t) = 1 \) by the above mentioned result.

Let \((c^{QP}(t))\) be the inverse matrix \((c_{QP}(t))^{-1}\). Let

\[ u_{RP}(t) \overset{\text{def}}{=} \sum_Q c^{RQ}(t^{-1})c_{QP}(t), \]

Let \(-\) be the involution on \( Z[t,t^{-1}] \) given by \( t^\pm \mapsto t^{\mp} \).

**Lemma 4.1.2 (Lusztig [10], 7.10).** There exists a unique solution \( Z_{QP}(t) \in Z[t^{-1}] \) \((Q \leq P)\) of

\[ Z_{RP}(t) = \sum_{Q : R \leq Q \leq P} Z_{RQ}(t)u_{QP}(t), \]

\[ Z_{PP}(t) = 1, \quad Z_{QP}(t) \in t^{-1}Z[t^{-1}] \text{ for } Q < P. \]

This lemma is proved by induction, and holds in a general setting. Lusztig has been using this (or its variant) in many places.

**Theorem 4.1.5.** The multiplicity \([M_P : L_Q]\) of a simple module \( L_Q \) in a standard module \( M_P \) is equal to \( Z_{QP}(1) \).

The proof will be given in \( \S 8.4. \)
We denote \( d \) where
\[
5.1.2 \quad M = m \text{ where } w \text{ is contained in space as before. We denote by } m \text{ where } l = \text{the corresponding to the simple } U. 
\]
Let \( k,a \) We also define \( P \) We need the following modification of \( 5.2. \) Frenkel-Mukhin \( 5.1.5.2 \) proved that the image of the \( q \)-character \( \chi_q \) is contained in
\[
5.2. \quad \chi_{q,t}(M_Q) = Y_{1,a} Y_{1,a}^{-1} + Y_{1,a}^{-1} Y_{1,a} + t^{-1} Y_{1,a}^{-1} Y_{1,a} + t Y_{1,a}^{-1} Y_{1,a} + t^{-1} Y_{1,a}^{-1} Y_{1,a} + t Y_{1,a}^{-1} Y_{1,a}.
\]
Let \( P(u) = (1 - au)(1 - a^{-2}u) \) (i.e., \( M_P = M_{a^2} \otimes M_a \), \( Q(u) = 1 \) (i.e. \( M_Q \) is trivial module). Then the above algorithm gives us \( Z_{QP}(t) = t^{-1} \).

5. Step 1

5.1. Some definitions. Let \( M_P \) be a standard module. Let \( m_P \) be the monomial corresponding to the \( l \)-highest weight vector. Let \( M_P(Q/R) \) be an \( l \)-weight space as before. We denote by \( m_{Q/R} \) the corresponding monomial. We define \( w_{k,a}(P), v_{k,a}(Q/R, P) \in \mathbb{Z}_{\geq 0}, u_{k,a}(Q/R) \in \mathbb{Z} \) by
\[
5.1.6 \quad m_P = \prod_{k \in I, a \in \mathbb{C}^*} Y_{k,a}(P), \quad m_{Q/R} = m_P \prod_{k \in I, a \in \mathbb{C}^*} A_{k,a}(Q/R, P) = \prod_{k \in I, a \in \mathbb{C}^*} Y_{k,a}(Q/R).
\]
Suppose two standard modules \( M_{P1}, M_{P2} \) and \( l \)-weight spaces \( M_{P1}(Q^1/R^1) \subset M_{P1}, M_{P2}(Q^2/R^2) \subset M_{P2} \) are given. We define
\[
5.1.7 \quad d(Q^1/R^1, P^1; Q^2/R^2, P^2) \overset{\text{def.}}{=} \sum_{k,a} \left( v_{k,a}(Q^1/R^1, P^1) u_{k,a}^{-1}(Q^2/R^2) + v_{k,a}(P^1) u_{k,a}(Q^2/R^2, P^2) \right).
\]
We also define
\[
5.1.8 \quad d(Q/R, P) \overset{\text{def.}}{=} d(Q/R, P; Q/R, P).
\]
We denote \( d(Q^1/R^1, P^1; Q^2/R^2, P^2) \) also by \( d(m_{Q^1/R^1}, m_{P^1}; m_{Q^2/R^2}, m_{P^2}) \).
We need the following modification of \( \chi_{q,t} \). Write \( \chi_{q,t}(M_P) = \sum_m a_m(t) m, \) where \( m \) is a monomial and \( a_m(t) \) is its coefficient. Let
\[
5.1.9 \quad \tilde{\chi}_{q,t}(M_P) \overset{\text{def.}}{=} \sum_m t^{d(m, m_P)} a_m(t) m,
\]
where \( d(m, m_P) \) is defined in \( 5.1.2 \) \quad 5

5.2. Frenkel-Mukhin \( 5.1.5.2 \) proved that the image of the \( q \)-character \( \chi_q \) is contained in
\[
\begin{align*}
\bigcap_{k \in I} \left( \mathbb{Z}[Y_{l,a}]_{l \neq k, a \in \mathbb{C}^*} \otimes \mathbb{Z}[Y_{l,b}(1 + A_{k,b}^{-1})_{b \in \mathbb{C}^*}] \right).
\end{align*}
\]
We have the \( t \)-analogue of this result, replacing \( (1 + A_{k,b}^{-1})^n \) by
\[
\begin{align*}
(1 + A_{k,b}^{-1})^n_t \overset{\text{def.}}{=} \sum_{r=0}^{n} t^{(n-r)} \begin{pmatrix} n \cr r \end{pmatrix} A_{k,b}^{-r},
\end{align*}
\]
where \( \begin{pmatrix} n \cr r \end{pmatrix}_t \) is the \( t \)-binomial coefficient. More precisely, we have
\[\text{or direct calculation for the definition [8.3.1]}\]
\[\text{In fact, } d(m, m_P) \text{ is determined from } a_m(t) \text{ so that } t^{d(m, m_P)} a_m(t) \text{ is a polynomial in } t \text{ with nonzero constant term.}\]
THEOREM 5.2.1. (1) For each $k \in I$, $\widetilde{\chi}_{q,t}(MP)$ is expressed as a linear combination of
\[
\prod_i Y_{k,b_i}^{n_i} \left(1 + A_{k,b_i}^{-1}\right)_{t}^{n_i} = Y_{k,b_1}^{n_1} \left(1 + A_{k,b_1}^{-1}\right)_{t}^{n_1} Y_{k,b_2}^{n_2} \left(1 + A_{k,b_2}^{-1}\right)_{t}^{n_2} \cdots
\]
with coefficients in $\mathbb{Z}[t][Y_{i,a}^{\pm}]_{i \neq k,a \in C^*}$, where $b_i \in \mathbb{C}^*$, $n_i \in \mathbb{Z}_{\geq 0}$ with $b_i \neq b_j$ for $i \neq j$.

(2) If $L_P$ is an $l$-fundamental representation (and hence $M_P = L_P$), then $\chi_{q,t}(M_P)$ contains no $l$-dominant monomials other than $m_P$ and the condition above uniquely determines $\chi_{q,t}(M_P)$.

REMARK 5.2.2. The statement (1) for $t = 1$ was proved by Frenkel-Mukhin [5]. And the proof of (2) is the same for $t = 1$ and the general case, as illustrated in the following examples. In this sense, (2) should also be credited to them.

5.3. Graph. We give few examples of $\chi_{q,t}$ of $l$-fundamental representations determined by the above theorem.

We attach to each standard module $M_P$, an oriented colored graph $\Gamma_P$. (It is a slight modification of the graph in [3 5.3].) The vertices are monomials in $\chi_{q,t}(M_P)$. We draw an colored edge $\frac{k,a}{\epsilon}$ from $m_1$ to $m_2$ if $m_2 = m_1A_{k,a}^{-1}$. We also write the multiplicity of the monomials in $\chi_{q,t}(M_P)$.

EXAMPLE 5.3.1. Let $g = A_3 = \mathfrak{sl}_4$ and $M_P = L(\Lambda_2)$. Then the corresponding graph $\Gamma_P$ is
\[
\begin{array}{c}
Y_{2,1} \xrightarrow{2,\epsilon} Y_{1,\epsilon}Y_{2,1}\overline{Y}_{3,\epsilon} \xrightarrow{1,\epsilon^2} Y_{1,\epsilon}Y_{3,\epsilon} \xrightarrow{3,\epsilon^2} Y_{1,\epsilon}Y_{3,\epsilon} \xrightarrow{1,\epsilon^2} Y_{1,\epsilon}Y_{2,\epsilon^2}Y_{3,\epsilon} \xrightarrow{2,\epsilon^3} Y_{2,\epsilon^4}.
\end{array}
\]
Let us explain how we determine this graph inductively. We start with the $l$-highest weight $Y_{2,1}$. We know that its coefficient is 1. Applying Theorem 5.2.1(1) with $k = 2$, we get $Y_{1,\epsilon}Y_{2,1}\overline{Y}_{3,\epsilon}$ with coefficient 1. Then we apply Theorem 5.2.1(1) with $k = 1$ to get $Y_{1,\epsilon}Y_{3,\epsilon}$. And so on. All multiplicities are 1 in this case.

For $g = A_n$, it is known that the coefficients of $\chi_{q,t}(L(\Lambda_k))$ are all 1.\footnote{More generally, if the coefficients of $\alpha_k$ in the highest root is 1, then the same holds. This result easily follows from the theory of quiver varieties. Exercise: Check this using the above algorithm.}

EXAMPLE 5.3.2. Let $g = D_4$ and $M_P = L(\Lambda_2)$. The graph $\Gamma_P$ is Figure 1. It is known that the restriction of $M_P$ to a $U_\epsilon(g)$-module is a direct sum of the adjoint representation and the trivial representation. This fact is reflected in $\chi_{q,t}(M_P)$ where $Y_{2,\epsilon}Y_{2,\epsilon}^{-1}$ has the coefficient $[2]_1$ and all others has 1. Note that the number of monomials is 28, which is the dimension of the adjoint representation. See also Example 7.2.3 below.

Let us give a more complicated example.

EXAMPLE 5.3.3. Let $g = A_2$ and $M_P = L(\Lambda_2)^{\otimes 2} \otimes L(\Lambda_1)$. Although this is not an $l$-fundamental representation, $\chi_{q,t}(M_P)$ has no $l$-dominant terms other than $m_P$, so the condition Theorem 5.2.1(1) gives us $\chi_{q,t}$. The graph is Figure 2.
The graph for Figure 1.

Figure 1. The graph for $L(A_2)_1$
**Remark 5.3.4.** As we can see in above examples, the crystal graphs are subgraphs of $\Gamma_P$. The set of vertices is the same, but the set of arrows is smaller. We would like to discuss this further elsewhere.

### 6. Step 2

**6.1.** Let $M_P = M(s^1; (k_{\alpha_1}^1), (n_{\alpha_1}^1)) \otimes M(s^2; (k_{\alpha_2}^2), (n_{\alpha_2}^2)) \otimes \cdots$ be a standard module with $s^i/s^j \notin \varepsilon\mathbb{Z}$ as in Theorem 2.1.2.

**Proposition 6.1.1.** We have

$$\chi_{q,t}(M_P) = \chi_{q,t}(M(s^1; (k_{\alpha_1}^1), (n_{\alpha_1}^1)) \otimes M(s^2; (k_{\alpha_2}^2), (n_{\alpha_2}^2)) \otimes \cdots$$

if $s^i/s^j \notin \varepsilon\mathbb{Z}$ for $i \neq j$.

Thus it is enough to study

$$\chi_{q,t}(M(s; (k_{\alpha}), (n_{\alpha}))) = \chi_{q,t}(L(\Lambda_{k_1})_{\varepsilon_{1}} \otimes L(\Lambda_{k_2})_{\varepsilon_{2}} \otimes \cdots).$$
Let 
\[ \chi_{q,t}(L(\Lambda_{k_a})_{x=\alpha}) = \sum_{r_\alpha} a_{m_\alpha,r_\alpha}(t)m_{\alpha,r_\alpha}, \]
where \(m_{\alpha,r_\alpha}\) is a monomial in \(Y_{k,a}^2\) and \(a_{m_\alpha,r_\alpha}(t) \in \mathbb{Z}[t,t^{-1}]\) is its coefficient.

If \(t = 1\), \(\chi_{q,1}\) is a ring homomorphism, hence we have
\[ \chi_{q,1}(M(a; (k_\alpha)_\alpha, (n_\alpha)_\alpha)) = \sum_{r_1, r_2, \ldots} \prod_{\alpha} a_{m_\alpha, r_\alpha}(t)m_{\alpha, r_\alpha}. \]

**Theorem 6.1.2.** Let \(P^n\) be the Drinfeld polynomial of \(L(\Lambda_{k_a})_{x=\alpha}s\). Then we have
\[ \chi_{q,t}(M(a; (k_\alpha)_\alpha, (n_\alpha)_\alpha)) = \sum_{r_1, r_2, \ldots} t^{\sum_{\alpha, \beta} \pm d(m_\alpha, r_\alpha, m_\rho; m_{\beta, r_\beta, \rho_{\beta}})} \prod_{\alpha} a_{m_\alpha, r_\alpha}(t)m_{\alpha, r_\alpha}, \]
where the sign for \(d(m_\alpha, r_\alpha, m_\rho; m_{\beta, r_\beta, \rho_{\beta}})\) is \(-\) if \(\alpha \leq \beta\) and \(+\) otherwise.

**Example 6.1.3.** For \(g = A_1\), we have
\[ d(Y_{1,az^2}, Y_{1,a}; Y_{1,az}, Y_{1,a}) = 1, \quad d(Y_{1,a}, Y_{1,a}; Y_{1,az}, Y_{1,az^2}) = 1 \]
and all others are 0. Then we get (6.1.7).

If \(P = (1 - au)^n\), we get
\[ \chi_{q,t}(M_P) = \sum_{r=0}^n \binom{n}{r} t^{n-r} Y_{1,a}^r Y_{1,az^2}^{-r} \]
from \(\chi_{q,t}(L(\Lambda_1)_a) = Y_{1,a} + Y_{1,az^2}^{-1}\). This also follows directly from the definition (8.3.1) below. The \(t\)-binomial coefficients appear as Poincaré polynomials of Grassmann manifolds.

### 7. Restriction to \(U_\mathfrak{g}(g)\)

Finite dimensional simple \(U_\mathfrak{g}(g)\)-modules are classified by highest weights. Let \(\text{Res} M_P\) be the restriction of a standard module \(M_P\) to a \(U_\mathfrak{g}(g)\)-module. It decomposes into a sum of various simple modules. Once \(\chi_q(M_P)\) is computed, the character of \(\text{Res} M_P\) is given by replacing \(Y_{r,a}^\pm\) by \(y_{r,a}^\pm\) (Theorem 1.1.3(2)). Combining with the knowledge of characters of simple finite dimensional \(U_\mathfrak{g}(g)\)-modules, we can determine the multiplicity of simple modules in \(\text{Res} M_P\).

Characters of simple finite dimensional \(U_\mathfrak{g}(g)\)-modules are the same as that of simple \(g\)-modules, hence are known. However, we express them in terms of \(\chi_{q,t}\) in this section.

#### 7.1. For a dominant weight \(w = \sum w_k \Lambda_k\) we denote by \(L_w\) the simple highest weight \(U_\mathfrak{g}(g)\)-module with the highest weight \(w\).

We consider a standard module \(M_P\) with \(\deg P_k = w_k\). By the ‘continuity’ of \(M_P\) on \(P\), \(\text{Res} M_P\) depends only on \(w_k = \deg P_k\), and not on \(P\) itself. Let us denote the multiplicity of \(L_{w'}\) in \(\text{Res} M_P\) by \(Z_{w',w}\), i.e.,
\[ \text{Res} M_P = \bigoplus_{w'} L_{w'}^\oplus Z_{w',w}. \]
We will give a formula expressing $Z_{w',w}$ in terms of $\chi_{q,t}(M_P)$. Although we can give algorithm for arbitrary $P$ in principle, the following choice will make the formula simple.

Choose and fix orientations of edges in the Dynkin diagram. We define integer $m(k)$ for each vertex $k$ so that $m(k) - m(l) = 1$ if we have an oriented edge from $k$ to $l$, i.e., $k \rightarrow l$. Then we define $P_k(u) = (1 - u^{m(k)})^{\text{wk}}$.

Let $\bar{\chi}_{q,t}(M_P)$ as in \[1.1.3\]. Let $\tilde{\chi}_t(M_P) \in \mathbb{Z}[t] \otimes \mathbb{Z}[t^\pm_{k}]_{k \in I}$ be a $t$-analogue of the ordinary character which is obtained from $\tilde{\chi}_{q,t}(M_P)$ by sending $Y_{k,a}$ to $y_k^\pm$.

For another dominant weight $w'$, let $Z_{w',w} = \sum_{M} M$ be the multiplicities of the restriction of $M$ to $w$. Then we define $P_k(u) = (1 - u^{m(k)})^{\text{wk}}$. By Example 6.1.3, we have

$$ Z_{w',w} = \sum_{M} w_{k}' \Lambda_k, $$

so that

$$ Z_{w',w} = \prod_{k} y_k^{w_{k}'}. $$

The matrix $(Z_{w',w}(t))_{w',w}$ is upper-triangular with respect to the usual order on weights, and diagonal entries are all 1.

**Theorem 7.1.1.** $c_{w',w}(0)$ is the weight multiplicity of $w'$ in the highest weight module $L_w$ with the highest weight $w$.

This is just a simple rephrasing of a main result in \[12, 14\]. The proof will be given in \[8.5\].

Note that $c_{w',w}(1)$ gives the weight multiplicity of $w'$ in $\text{Res} M_P$ since $\tilde{\chi}_{1=1}$ is the ordinary character. Thus we have

$$ c_{w'',w}(1) = \sum_{w'} c_{w'',w}(0) Z_{w',w}. $$

This equation determines the multiplicity $Z_{w',w}$ only from the knowledge of $\chi_{q,t}$.

According to a conjecture of Lusztig \[11\] together with a formula \[8.5.1\] below, $c_{w',w}(t)$ should be written by ferminonic form of Hatayama el al. \[7\]. More precisely, we should have $\sum_{w''} c_{w'',w}(0) c_{w',w}(t) = M(w,w'' ,t^2)$, where $(c_{w'',w}(0))$ is the inverse matrix of $(c_{w',w}(0))$. See \[11\] for the definition of $M(w,w',q)$. Although this formula can be checked in many examples, the complexity of the combinatorics prevent us from proving it in full generality. Conjecturally $M(w,w',q = 1)$ gives us the multiplicities of the restriction of $M_P$ (Kirillov-Reshetikhin). Thus the conjecture is compatible with our result in this section.

**7.2.**

**Example 7.2.1.** Let $g = A_1$ and $w = 2\Lambda_1$. We take $P = (1 - u)^2$ by the above choice. By Example 6.1.3, we have

$$ \tilde{\chi}_t(M_P) = y_1^2 + (1 + t^2) + y_1^{-2}. $$

Thus $Z_{0,w} = 1$. Since $\text{Res}(M_P) = L_{\Lambda_1} \otimes L_{\Lambda_1} = L_{2\Lambda_1} \oplus L_0$, this is the correct answer !

**Example 7.2.2.** Let $g = A_3$ and $w = \Lambda_2$. By Example 5.3.1 all the coefficients of $\chi_{q,t}(L(\Lambda_2)_{1})$ are 1. Hence $\text{Res} M_P = \text{Res} L(\Lambda_2)_{1}$ is simple as a $U_v(g)$-module.

**Example 7.2.3.** Let $g = D_4$, $w = \Lambda_2$, $w' = 0$. By Example 5.3.2 we have $c_{w,w}(t) = 4 + t^2$. Thus $Z_{w',w} = 1$, i.e. $\text{Res}(L(\Lambda_2)_{1}) = L_{\Lambda_2} \oplus L_0$.\footnote{In fact, they consider more general modules, not necessarily standard modules.}
8. Quiver varieties

In this section, we give the definition of $\chi_{q,t}$ and prove Theorems 4.1.3, 12.1.1. As we mentioned, those proofs are essentially given in [12, 14] respectively. The only things we do here are translation of results into the language of $\chi_{q,t}$. We believe that this section gives good introductions to [12, 14, 15].

8.1. Let $w = \sum w_k \Lambda_k$ ($w_k \in \mathbb{Z}_{\geq 0}$) be a dominant weight of the finite dimensional Lie algebra $\mathfrak{g}$. In [12, 14, 15], we have attached to each $w$, a map $\pi: \mathcal{M}(\mathfrak{g}) \to \mathcal{M}_0(\infty, w)$ with the following properties:

1. $\mathcal{M}(\mathfrak{g})$ is a finite disjoint union of nonsingular quasi-projective varieties of various dimensions.
2. $\mathcal{M}_0(\infty, w)$ is an affine algebraic variety.
3. $\pi$ is a projective morphism.
4. There exist actions of $G_w \times \mathbb{C}^*$ on $\mathcal{M}(\mathfrak{g})$ and $\mathcal{M}_0(\infty, w)$ such that $\pi$ is equivariant.
5. $\mathcal{M}_0(\infty, w)$ is a cone, and the vertex (denoted by 0) is the unique fixed point of the $\mathbb{C}^*$-action (restriction of $G_w \times \mathbb{C}^*$-action to the second factor).

Here $G_w = \prod_{k \in I} \text{GL}(w_k, \mathbb{C})$.

We consider the fiber product

$$Z(w) \overset{\text{def}}{=} \mathcal{M}(w) \times_{\mathcal{M}_0(\infty, w)} \mathcal{M}(w).$$

The convolution product makes the (Borel-Moore) homology group $H_*(Z(w), \mathbb{C})$ into an associative (noncommutative) algebra. One of main results in [14] is a construction of a surjective algebra homomorphism

$$\mathfrak{U}(\mathfrak{g}) \to H_{\text{top}}(Z(w), \mathbb{C}),$$

where $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ (NB: not a ‘quantum’ version). Here $H_{\text{top}}(\mathfrak{g})$ means the degree = dimension part of the homology group. More precisely, we take degree = dimension part on each connected components of $Z(w)$, and then make the direct sum. Note that the the dimension differs on various components.

Let $\mathcal{L}(w) = \pi^{-1}(0)$. It is known that $\mathcal{M}(w)$ has a holomorphic symplectic form such that $\mathcal{L}(w)$ is a lagrangian subvariety. The convolution makes $H_{\text{top}}(\mathcal{L}(w), \mathbb{C})$ (the top degree part of the Borel-Moore homology group, in the same sense as above) into an $H_{\text{top}}(Z(w), \mathbb{C})$-module. It is a $\mathfrak{U}(\mathfrak{g})$-module by the above homomorphism. By [14, 10.2] it is the simple finite dimensional $\mathfrak{U}(\mathfrak{g})$-module $L_w$ with highest weight $w$. And connected components $\mathcal{M}(v, w)$ of $\mathcal{M}(w)$ are parametrized by vectors $v = \sum v_k \alpha_k$ ($\alpha_k$ is the $k$th simple root of $\mathfrak{g}$) so that

$$H_{\text{top}}(\mathcal{L}(w), \mathbb{C}) = \bigoplus_v H_{\text{top}}(\mathcal{M}(v, w) \cap \mathcal{L}(w), \mathbb{C})$$

is the weight space decomposition of the simple highest weight module $L_w$, where $H_{\text{top}}(\mathcal{M}(v, w) \cap \mathcal{L}(w), \mathbb{C})$ has weight $w - v$. In particular, $v = 0$ corresponds to the highest weight vector. In fact, $\mathcal{M}(0, w)$ is consisting of a single point.

The space $\mathcal{M}_0(\infty, w)$ has a stratification

$$\mathcal{M}_0(\infty, w) = \bigcup v \in \mathcal{M}_0^{\infty}(v, w),$$

where $v$ runs over the set of vectors such that $w - v$ is a weight of $L_w$ which is dominant [12, §3].
Let us give the \( U_q(L_\mathfrak{q}) \)-version of the construction of the previous subsection.

We use the following notation: Let \( R(G) \) denote the representation ring of a linear algebraic group \( G \). If \( G \) acts a quasi-projective variety \( X \), \( K^G(X) \) denotes the Grothendieck group of \( G \)-equivariant coherent sheaves on \( X \).

The representation ring \( R(G_\mathfrak{w} \times \mathbb{C}^*) \) of \( G_\mathfrak{w} \times \mathbb{C}^* \) is isomorphic to the tensor product \( R(G_\mathfrak{w}) \otimes_\mathbb{Z} \mathbb{C}(\mathbb{C}^*) \). Moreover, \( R(\mathbb{C}(\mathbb{C}^*)) \) is isomorphic to \( \mathbb{Z}[q,q^{-1}] \), where \( q \) is the canonical 1-dimensional representation of \( \mathbb{C}^* \).

The convolution makes the Grothendieck group \( K^{G_\mathfrak{w} \times \mathbb{C}^*}(Z(\mathfrak{w})) \) into a \( R(G_\mathfrak{w} \times \mathbb{C}^*) \)-algebra. One of main results in \( \text{[13]} \) is a construction of an algebra homomorphism

\[
U^\mathfrak{w}_q(L_\mathfrak{q}) \otimes_\mathbb{Z} R(G_\mathfrak{w}) \to K^{G_\mathfrak{w} \times \mathbb{C}^*}(Z(\mathfrak{w})) / \text{tor}.
\]

By the equivariance of \( \pi \), \( \mathcal{L}(\mathfrak{w}) = \pi^{-1}(0) \) is invariant under \( G_\mathfrak{w} \times \mathbb{C}^* \). The convolution makes \( K^{G_\mathfrak{w} \times \mathbb{C}^*}(\mathcal{L}(\mathfrak{w})) \) into a \( K^{G_\mathfrak{w} \times \mathbb{C}^*}(Z(\mathfrak{w})) \)-module. Moreover, it is free of finite rank over \( R(G_\mathfrak{w} \times \mathbb{C}^*) \) \( \text{[15]} \) \( \S 7 \). It is a \( U^\mathfrak{w}_q(L_\mathfrak{q}) \otimes_\mathbb{Z} R(G_\mathfrak{w}) \)-module by the above homomorphism. By \( \text{[13]} \) \( \S 13 \), it contains a vector \( m_0 \) such that

\[
e_{\epsilon,k,r} * m_0 = 0, \quad (U^\mathfrak{w}_q(L_\mathfrak{q}) \otimes_\mathbb{Z} R(G_\mathfrak{w})) * m_0 = K^{G_\mathfrak{w} \times \mathbb{C}^*}(\mathcal{L}(\mathfrak{w})) \tag{8.2.1}
\]

The right hand side of the third equation needs an explanation: First \( W_k \) is the vector representation of \( \text{GL}(w_k, \mathbb{C}) \), considered as a \( G_\mathfrak{w} \times \mathbb{C}^* \)-module. Then \( \sum u^i \Lambda^i V \). Since \( \Lambda^{-q/z} = 1 - (1/z)W_k + \cdots \) (1 is the trivial module), we can define \( \Lambda_{-q/z}^{-1} \) as a formal power series in \( 1/z \). This gives us the case \( ( )^+ \) of the above formula. In the case \( ( )^- \), we expand as \( \Lambda_{-q/z}^{-1} = (-1/z)^{W_k} (\Lambda^{w_k} W_k - z \Lambda^{w_k-1} W_k + \cdots) \). Then \( \Lambda^{W_k} W_k \) is an invertible element, we can also define \( (\Lambda_{-q/z}^{-1})^{-1} \). The vector \( m_0 \) is the canonical generator of \( K^{G_\mathfrak{w} \times \mathbb{C}^*}(\mathfrak{M}(0, \mathfrak{w})) \). (Recall \( \mathfrak{M}(0, \mathfrak{w}) \) is a point.)

The module \( K^{G_\mathfrak{w} \times \mathbb{C}^*}(\mathcal{L}(\mathfrak{w})) \) should be considered as a ‘universal’ standard module since standard modules are obtained from it by specializations as we explain now.

Let \( a = (s,z) \in G_\mathfrak{w} \times \mathbb{C}^* \) be a semisimple element. It defines a homomorphism \( \chi_a : R(G_\mathfrak{w} \times \mathbb{C}^*) \to \mathbb{C} \) by sending a representation to the value of the character at \( a \). Then

\[
K^{G_\mathfrak{w} \times \mathbb{C}^*}(\mathcal{L}(\mathfrak{w})) \otimes_{R(G_\mathfrak{w} \times \mathbb{C}^*)} \mathbb{C}
\]

is a module over \( U^\mathfrak{w}_z(L_\mathfrak{q}) = U^\mathfrak{w}_q(L_\mathfrak{q}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} \). By \( \text{[8.2.1]} \) it is a finite-dimensional \( l \)-highest weight module. This is the standard module \( M_P \), where \( P_k(u) = \chi_a(\Lambda_{-u}^{-1} W_k) \). Note that the set of conjugacy classes of \( a = (s,z) \) bijectively corresponds to the set of \( I \)-tuple of polynomials \( P \) with \( \deg P_k = w_k \).

Let \( A \) be the Zariski closure of \( a^Z \) in \( G_\mathfrak{w} \times \mathbb{C}^* \). It is an abelian reductive group. We have \( K^{G_\mathfrak{w} \times \mathbb{C}^*}(\mathcal{L}(\mathfrak{w})) \otimes_{R(G_\mathfrak{w} \times \mathbb{C}^*)} R(A) \cong K^A(\mathcal{L}(\mathfrak{w})) \) \( \text{[15]} \) \( \S 7 \). Since \( \chi_a \) factors through \( R(A) \), the standard module \( M_P \) is isomorphic to \( K^A(\mathcal{L}(\mathfrak{w})) \otimes_{R(A)} \mathbb{C} \). By Thomason’s localization theorem, it is isomorphic to \( K(\mathcal{L}(\mathfrak{w})^A) \otimes_{\mathbb{Z}} \mathbb{C} \), where
\( \mathfrak{L}(w)^A \) is the fixed point set. Furthermore, it is isomorphic to \( H_\ast(\mathfrak{L}(w)^A, \mathbb{C}) \) via the Chern character homomorphism [15, §7].

Let \( \pi^A : \mathfrak{M}(w)^A \to \mathfrak{M}(\infty, w)^A \) be the restriction of the map \( \pi : \mathfrak{M}(w) \to \mathfrak{M}(\infty, w) \) to the fixed point set. Let \( \mathfrak{M}(w)^A = \bigsqcup \mathfrak{M}(\rho) \) be the decomposition into connected components. Each \( \mathfrak{M}(\rho) \) is a nonsingular quasi-projective variety. Then we have the direct sum decomposition

\[
M_P \cong H_\ast(\mathfrak{L}(w)^A, \mathbb{C}) \cong \bigoplus_{\rho} H_\ast(\mathfrak{M}(\rho) \cap \mathfrak{L}(w), \mathbb{C}).
\]

In [15, §13, §14] we have shown that this is the \( l \)-weight space decomposition of \( M_P \). In particular, the index \( \rho \) can be considered as an \( l \)-weight of \( M_P \). Thus we have arrived at a geometric interpretation of \( \chi_q \):

\[
\chi_q(M_P) = \sum_{\rho} \dim H_\ast(\mathfrak{M}(\rho) \cap \mathfrak{L}(w), \mathbb{C}) m_{\rho},
\]

where \( m_{\rho} \) is the monomial corresponding to the \( l \)-weight \( \rho \).

Now we define the \( l \)-analogue \( \chi_{q,t} \) by

\[
(8.3.1) \quad \chi_{q,t}(M_P) \overset{\text{def}}{=} \sum_{\rho} \sum_k \dim H_k(\mathfrak{M}(\rho) \cap \mathfrak{L}(w), \mathbb{C}) t^{k - \dim \mathfrak{M}(\rho)} m_{\rho}.
\]

By [15, §14] we have a stratification

\[
\mathfrak{M}_0(\infty, w)^A = \bigsqcup_{\rho} \mathfrak{M}_0^{\text{reg}}(\rho),
\]

consisting of nonsingular locally closed subvarieties. Here the index set \( \{ \rho \} \) is the subset of the above index set consisting of \( l \)-dominant \( l \)-weights.

**8.4. Proof of Theorem 4.1.3.** The \( l \)-highest weight \( P \) is fixed throughout the proof. Thus the dominant weight vector \( w \) and the element \( a = (s, \varepsilon) \in G_w \) are fixed.

We change the notation now. If \( \rho \) corresponds to an \( l \)-weight space \( M_P(Q/R) \), we denote above \( \mathfrak{M}(\rho) \) by \( \mathfrak{M}(Q/R, P) \). We also denote by \( \mathfrak{M}_0^{\text{reg}}(Q, P) \) for above \( \mathfrak{M}_0^{\text{reg}}(\rho) \) if \( \rho \) corresponds to an \( l \)-dominant \( l \)-weight \( Q \). Thus we have

\[
\mathfrak{M}(w)^A = \bigsqcup_{Q/R} \mathfrak{M}(Q/R, P), \quad \mathfrak{M}_0(\infty, w)^A = \bigsqcup_Q \mathfrak{M}_0^{\text{reg}}(Q, P).
\]

In this notation \( H_\ast(\mathfrak{M}(P, P) \cap \mathfrak{L}(w), \mathbb{C}) \) is the \( l \)-highest weight space. Since \( \mathfrak{M}(0, w) \) is a single point as we explained, we have \( \mathfrak{M}(P, P) = \mathfrak{M}(0, w) \). We also have \( \mathfrak{M}_0^{\text{reg}}(P, P) = \{0\} \).

**Lemma 8.4.1.** (1) \( \dim_{\mathbb{C}} \mathfrak{M}(Q/R, P) = d(Q/R, P) \), \( \dim_{\mathbb{C}} \mathfrak{M}_0^{\text{reg}}(Q, P) = d(Q, P) \).

(2) If \( \mathfrak{M}_0^{\text{reg}}(Q, P) \subseteq \mathfrak{M}_0^{\text{reg}}(R, P) \), then \( R \leq Q \).

(3) Choose \( x \in \mathfrak{M}_0^{\text{reg}}(Q, P) \). Then \( (\pi^A)^{-1}(x) \cap \mathfrak{M}(S/T, P) \) is isomorphic to \( \mathfrak{M}(S/T, Q) \cap \mathfrak{L}(w) \).

**Proof.** (1) The first equation is the dimension formula [15, 4.1.6]. The second equation follows from \( \dim_{\mathbb{C}} \mathfrak{M}_0^{\text{reg}}(Q, P) = \dim_{\mathbb{C}} \mathfrak{M}(Q, P) \), which is clear from the definition [15, §4].

(2),(3) The results are known or trivial for \( Q = P \). Now use the transversal slice at \( x \in \mathfrak{M}_0^{\text{reg}}(Q, P) \) [15, §3] to reduce a general case to this case. \( \square \)
Let $D^b(\mathcal{M}_0(\infty, w)^A)$ be the bounded derived category of complexes of sheaves such that cohomology sheaves are constant along each stratum $\mathcal{M}_0^{reg}(Q, P)$. Let $IC(\mathcal{M}_0^{reg}(Q, P))$ be the intersection homology complex associated with the constant local system $\mathcal{C}_{\mathcal{M}_0^{reg}(Q, P)}$ on $\mathcal{M}_0^{reg}(Q, P)$. By using the transversal slice [15, §3], one can check that it is an object in $D^b(\mathcal{M}_0(\infty, w)^A)$. Let $\mathcal{C}_{\mathcal{M}(Q/R, P)}$ be the constant local system on $\mathcal{M}(Q/R, P)$. Then $\pi_*^A(\mathcal{C}_{\mathcal{M}(Q/R, P)})$ is an object of $D^b(\mathcal{M}_0(\infty, w)^A)$ again by the transversal slice argument. Using the decomposition theorem of Beilinson-Bernstein-Deligne, we have shown that there exists an isomorphism in $D^b(\mathcal{M}_0(\infty, w)^A)$:

\[(8.4.2) \quad \pi_*^A(\mathcal{C}_{\mathcal{M}(R, P)}[\dim \mathcal{M}(R, P)]) \cong \bigoplus_{Q, k} L_{Q, k}(R, P) \otimes IC(\mathcal{M}_0^{reg}(Q, P))[k]\]

for some vector space $L_{Q, k}(R, P)$ [15, 14.3.2]. Since $\pi_*^A(\mathcal{M}(R, P)) \subset \mathcal{M}_0^{reg}(Q, P)$ by definition [15, §4], the summation runs over $Q \geq R$ by Lemma 8.4.1. Let $L_{RQ}(t) \overset{\text{def}}{=} \sum_k \dim L_{Q, k}(R, P) t^{-k}$.

Applying the Verdier duality to the both hand side of (8.4.2) and using the self-duality of $\pi_*^A(\mathcal{C}_{\mathcal{M}(R, P)}[\dim \mathcal{M}(R, P)])$ and $IC(\mathcal{M}_0^{reg}(Q, P))$, we find $L_{RQ}(t) = L_{RQ}(t)$.

Choose a point $x_Q$ from $\mathcal{M}_0^{reg}(Q, P)$ for each stratum. Let $i_{x_Q} : \{x_Q\} \to \mathcal{M}_0(\infty, w)^A$ denote the inclusion. Consider

\[H^k(i_{x_Q}^! \pi_*^A \mathcal{C}_{\mathcal{M}(R, P)}[\dim \mathcal{M}(R, P)]) = H_{\dim \mathcal{M}(R, P) - k}((\pi^A)^{-1}(x_Q)) \cap \mathcal{M}(R, P, \mathbb{C}).\]

By Lemma 8.4.1(3) this is isomorphic to $H_{\dim \mathcal{M}(R, P) - k}(\mathcal{M}(R, Q) \cap \mathcal{L}(w), \mathbb{C})$. Therefore we have

\[(8.4.3) \quad \sum_k \dim H^k(i_{x_Q}^! \pi_*^A \mathcal{C}_{\mathcal{M}(R, P)}[\dim \mathcal{M}(R, P)]) t^{\dim \mathcal{M}(Q, P) - k} = \sum_d \dim H_d(\mathcal{M}(R, Q) \cap \mathcal{L}(w), \mathbb{C}) t^{d + \dim \mathcal{M}(Q, P) - \dim \mathcal{M}(R, P)} = c_{RQ}(t),\]

where we used $\dim \mathcal{M}(R, P) - \dim \mathcal{M}(R, Q) = \dim \mathcal{M}(R, Q)$ in the last equality.

By [15, 14.3.10], we have

\[M_Q : L_R = \dim H^*(i_{x_Q}^! IC(\mathcal{M}_0^{reg}(R, P))).\]

(In fact, we defined the standard module $M_Q$ as $H^*((\pi^A)^{-1}(x_Q), \mathbb{C})$ in [15, §13], which a priori depends on $P$. Thus the definition coincides only when $Q = P$. However, by using the transversal slice, we can show that the right hand side is the same for both definitions. cf. Lemma 8.4.1.)

Let $Z_{RQ}(t) \overset{\text{def}}{=} \sum_k \dim H^k(i_{x_Q}^! IC(\mathcal{M}_0^{reg}(R, P))) t^{\dim \mathcal{M}_0^{reg}(Q, P) - k}$.

We have $[M_Q : L_R] = Z_{RQ}(1)$. By the defining property of the intersection homology, $Z_{RQ}(t)$ satisfies (11.1).

Substituting (8.4.2) into (8.4.3), we get

\[c_{SQ}(t) = \sum_R L_{SR}(t) Z_{RQ}(t).\]
Now $L_{SR}(t) = \overline{L_{SR}(t)}$ implies (4.1.3). This completes the proof of Theorem 4.1.3.

8.5. Proof of Theorem 7.1.1. By the result explained in §8.1, the weight multiplicity of $w'$ in $L_w$ is equal to
\[
\dim H_{\text{top}}(M(w - w', w) \cap L(w), \mathbb{C}).
\]
The assertion follows from more general formula
\[
\chi_i(M_P) = \sum_{w'} \sum_d H_d(M(w - w', w) \cap L(w), \mathbb{C}) \cdot \dim M(w - w', w) - d \prod_k y_k^{w'_k}.
\]
Note that $M(w - w', w) \cap L(w)$ is a lagrangian subvariety in $M(w - w', w)$, so we have $\text{top} = \dim M(w - w', w)$.

In order to prove (8.5.1), we use [12, 5.7], where the Betti numbers are given in terms of those of fixed point components. It looks almost the same as above. However, there is one significant difference. The $\mathbb{C}^*$-action used there is different from our $\mathbb{C}^*$-action used here, defined in [15, §2]. This is the reason why we choose $P$ and corresponding $a = (s, \varepsilon)$ as explained in §4. Then $A = \mathbb{C}^*$ is isomorphic to $\mathbb{C}^*$ and the action is the same as the $\mathbb{C}^*$-action considered in [12, §5].

We decompose $M(w)^A = \bigsqcup M(\rho)$ into connected components as before. By [12, 5.7] we have:
\[
\dim H_d(M(w - w', w) \cap L(w), \mathbb{C}) = \sum_{\rho} \dim H_{\text{dim } M(w - w', w) - d}(M(\rho), \mathbb{C}),
\]
where the summation runs over the set of $\rho$ such that the corresponding monomial $m_{\rho}$ is sent to $\prod_k y_k^{w'_k}$ after $Y_{k,a} \to y_k$. The $\mathbb{C}^*$-action makes $M(\infty, w)^\mathbb{C}^* = \{0\}$, so $M(\rho) = M(\rho) \cap L(w)$. Hence the above expression coincides with the definition of the coefficients $\chi_i$.

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In fact, this formula even holds for general $P$ if we replace $\dim M(w - w', w) - d$ by a suitable degree. However, this degree shift is given by a complicated expression in $\rho$. So our choice of $P$ is most economical.
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