Green Function on the Quantum Plane

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Abstract: Green function (which can be called the q-analogous of the Hankel function) on the quantum plane $E_q^2 = E_q(2)/U(1)$ is constructed.

1 Introduction

Green functions play important roles in physics. Field theoretical problems involving boundaries, such as the Casimir interactions, particle pair productions i.e., all employ Green functions. Therefore if one is interested in the investigation of some physical effects on the non–commutative spaces construction of the Green functions in these media is useful. Motivated by these considerations we think it is of interest to study the Green functions on the quantum group spaces which are the natural examples of the non–commutative geometries.

Previously we have constructed the Green function on the quantum sphere $S^2_q$ \cite{1}. In this paper we study the same problem for the quantum plane $E_q^2$ which may be more relevant to physics.

In Section 2 we recall main result \cite{2, 3, 4, 5, 6, 7} concerning the quantum group $E_q(2)$ and its homogeneous spaces.

In Section 3 we construct the Green function on the quantum plane. The Green function we obtain, provides the possibility of the future studies on the new q-functions which are the deformations of the Neumann and Hankel functions.

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2 Quantum Group $E_q(2)$ and its Homogeneous Spaces

Let $A$ be the set of linear operators in the Hilbert space $l^2(Z)$ subject to the condition
\[
\sum_{j=-\infty}^{\infty} q^{2j}(e_j, F^*F e_j) < \infty; \quad F \in A.
\] (1)

Here $0 < q < 1$ and $\{e_j\}$ is the orthonormal basis in $l^2(Z)$. Explicit form of $e_j$ is
\[
e_j = (0, \cdots, 0, 1, 0, \cdots),
\] (2)
where either $j^{th}$ (for $j > 0$) or $(-j)^{th}$ (for $j < 0$) component is one, all others are zero. Any vector $x = (x_0, x_1, x_{-1}, \cdots, x_n, x_{-n}, \cdots)$ of $l^2(Z)$ has representation
\[
x = \sum_{j=-\infty}^{\infty} x_j e_j.
\] (3)

$(\cdot, \cdot)$ in (1) is the scalar product in $l^2(Z)$:
\[
(x, y) = \sum_{j=-\infty}^{\infty} \overline{x_j} y_j.
\] (4)

$A$ is the Hilbert space with the scalar product
\[
(F, G)_A = (1 - q^2) \sum_{j=-\infty}^{\infty} q^{2j}(e_j, F^*G e_j); \quad F, G \in A.
\] (5)

Let us introduce the linear operators acting in $l^2(Z)$
\[
ze_j = e^{i\psi}q^j e_j, \quad ve_j = e^{i\phi} e_{j+1},
\] (6)
where $\psi$ and $\phi$ are the classical phase variables. $n$ is normal and $v$ is unitary operator in $l^2(Z)$. It is easy to show that they satisfy the relations:
\[
zv = qvz, \quad z^* v = qvn^*, \quad zz^* = z^* z.
\] (7)

Any element $F \in A$ can be represented as
\[
F = \sum_{j=-\infty}^{\infty} f_j(z, z^*) v^j
\] (8)
by suitable choice of the functions \( f_j \).

The linear operators \( Z \) and \( V \) given by
\[
Z = z \otimes v^{-1} + v \otimes z, \quad V = v \otimes v
\]
are normal and unitary in \( l^2(Z \times Z) \) They satisfy the relations
\[
ZV = qVZ, \quad Z^*V = qVZ^*, \quad ZZ^* = Z^*Z. \tag{10}
\]
Note that the operators \( N \) and \( V \) have the same properties as \( n \) and \( v \).

Therefore there exists the linear map \( \Delta : A \to A \otimes_A A \) \tag{11}
defined as
\[
\Delta(f(z, z^*)v^j) = f(Z, Z^*)V^j. \tag{12}
\]
Here \( \otimes_A \) is the completed tensor product \( \otimes \) with respect to the scalar product
\[
(F_1 \otimes F_2, F_3 \otimes F_4)_A = (F_1, F_3)_A(F_2, F_4)_A; \quad F_n \in A \tag{13}
\]
in \( A \otimes_A A \). \( A \) is the space of square integrable functions on the quantum group \( E_q(2) \) and \( \Delta \) is the quantum analog of the group multiplication.

The one parameter groups \( \{\sigma_1\} \) and \( \{\sigma_2\} \) of automorphism of \( A \) given by
\[
\sigma_1(v) = e^{-it}v, \quad \sigma_1(z) = e^{it}z \tag{14}
\]
and
\[
\sigma_2(v) = v, \quad \sigma_2(z) = e^{it}z \tag{15}
\]
with \( t \in \mathbb{R} \) are isomorphic to \( U(1) \). The subspaces
\[
B = \{ F \in A : \sigma_1(F) = F, \text{ for all } t \in \mathbb{R} \} \tag{16}
\]
and
\[
H = \{ F \in B : \sigma_2(F) = F \text{ for all } t \in \mathbb{R} \} \tag{17}
\]
are the space of square integrable functions on the quantum plane \( E_q^2 \) and two sided coset space \( U(1)\backslash E_q(2)/U(1) \). Any element of \( H \) is the function of \( \rho = zz^* \). Note that the scalar product \( \mathbb{B} \) on \( H \) becomes a q-integration
\[
(f(\rho), g(\rho))_A = (1 - q^2) \sum_{j=-\infty}^{\infty} q^{j^2} f(q^{2j})g(q^{2j}) = \int_0^{\infty} \overline{f(\rho)} g(\rho) d_{q^2}\rho \tag{18}
\]
Let $U_q(e(2))$ be the $*$-algebra generated by $p$ and $\kappa^{\pm 1}$ such that

$$p^* p = q^2 p^* p, \quad \kappa^* = \kappa, \quad \kappa p = q^2 p \kappa. \quad (19)$$

The representation $\mathcal{L}$ of $U_q(e(2))$ in $A$ is given by

$$\mathcal{L}(p) f(z, z^*) \nu^j = i q^{j+1} D^z_+ f(z, z^*) \nu^{j+1} \quad (20)$$

$$\mathcal{L}(p^*) f(z, z^*) \nu^j = i q^j D_-^z f(z, z^*) \nu^{j-1} \quad (21)$$

$$\mathcal{L}(\kappa) f(z, z^*) \nu^j = q^j f(q^{-1} z, q z^*) \nu^j, \quad (22)$$

where

$$D^x_\pm f(x) = \frac{f(x) - f(q^{\pm 2} x)}{(1 - q^{\pm 2}) x}. \quad (23)$$

For the Casimir element $C = -q^{-1} \kappa^{-1} p p^*$ we have

$$\mathcal{L}(C) f(z, z^*) \nu^j = q^j D_-^z D_+^z f(q z, q^{-1} z^*) \nu^j. \quad (24)$$

The restriction $\Box$ of $\mathcal{L}(C)$ on $H$ is

$$\Box = D_-^\rho \rho D_+^\rho \quad (25)$$

which we call the radial part of $\mathcal{L}(C)$.
3 Green Function on the Quantum Plane

(i) Green Function on $U(1) \backslash E_{q}(2) / U(1)$

The Green function $\mathcal{G}^{p}(\rho)$ on the two sided coset space is defined as

$$(\Box + p)\mathcal{G}^{p}(\rho) = \delta(\rho),$$

(26)

where $\delta$ is the delta function which defined with respect to the scalar product $\delta, f)_{A} = f(0)$

(27)

for any $f \in H_{0}$. The equation (26) is understood as

$$(\mathcal{G}^{p}, f)_{A} = \lim_{\epsilon \to 0} ((\delta(\rho), 1_{\Box + p + i\epsilon}f(\rho))_{A}$$

(28)

For $\rho \neq 0$ the equation (26) is solved by

$$\mathcal{J}(\sqrt{p\rho}) = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{([k]!^{2})(pp)^{k}}$$

(29)

and

$$\mathcal{N}(\sqrt{pp}) = \frac{q - q^{-1}}{2q \log(q)} \mathcal{J}(\sqrt{pp})(\log(pp) + 2C_{q}) \frac{1}{\sum_{k=1}^{\infty} \frac{(-1)^{k}}{([k]!^{2})(pp)^{k}} \sum_{m=1}^{k} \frac{q^{m} + q^{-m}}{[m]}}$$

(30)

where

$$[m] = \frac{q^{m} - q^{-m}}{q - q^{-1}}, \quad [m]! = [1][2] \cdots [m].$$

(31)

The Hahn-Exton q-Bessel function $\mathcal{J}$ is regular at $\rho = 0$. It is the zonal spherical function of the unitary irreducible representations of $E_{q}(2)$. $\mathcal{N}$ can be called q-Neuman function which indeed is reduced to the usual Neuman function in $q \to 1$ limit. Here $C_{q}$ is some constant, which in $q \to 1$ limit should become the Euler constant $[8]$.

The Green function on $U(1) \backslash E_{q}(2) / U(1)$ is then

$$\mathcal{G}^{p}(\rho) = \mathcal{N}(\sqrt{pp}) - i\mathcal{J}(\sqrt{pp}),$$

(32)
which in classical limit becomes the Hankel function. Using the Fourier-Bessel integral \[9\]

\[
\int_0^\infty dq^2 \rho J(q^n \sqrt{\rho}) J(q^m \sqrt{\rho}) = \frac{q^{2m+2}}{1-q^2} \delta_{mn}
\]

(33)

we arrive at the following representation for the Green function

\[
\mathcal{G}_p(\rho) = \lim_{\epsilon \to 0} q^{-2} \int_0^\infty dq^2 \lambda \frac{J(\sqrt{\lambda q})}{\lambda - \lambda + i\epsilon}
\]

(34)

from which one can derive the constant \(C_q\).

To prove that \(\mathcal{G}\) solves (26) we first have to show that

\[
\square \log \rho = \frac{2q \log(q)}{q - q^{-1}} \delta(\rho).
\]

(35)

For \(\rho \neq 0\) we have

\[
\square \log \rho = 0.
\]

(36)

Since the operator \(\square\) is symmetric in \(H\) we have

\[
(\square \log \rho, f)_A = (\log \rho, \square f)_A
\]

\[
= \frac{2q \log(q)}{q - q^{-1}} \lim_{n \to \infty} \sum_{j=-\infty}^n j[2f(q^{2j}) - f(q^{2(j+1)}) - f(q^{2(j-1)})]
\]

\[
= \frac{2q \log(q)}{q - q^{-1}} \lim_{n \to \infty} [f(q^{2n}) - n(f(q^{2n+2}) - f(q^{2n}))].
\]

(37)

We then employ the q–Taylor expansion at the neighborhood of \(\rho = 0\)

\[
f(q^2 \rho) - f(\rho) \sim D_+^q f(0)(q^2 - 1) \rho.
\]

(38)

For \(n \gg 1\) we get

\[
n(f(q^{2n+2}) - f(q^{2n})) \sim nD_+^q f(0)(q^2 - 1)q^{2n}.
\]

(39)

Since \(nq^{2n}\) vanishes as \(n \to \infty\), we arrive at

\[
(\square \log \rho, f)_A = \frac{2q \log(q)}{q - q^{-1}} \lim_{n \to \infty} f(q^{2n}) = \frac{2q \log(q)}{q - q^{-1}} f(0).
\]

(40)
In a similar fashion one can show that

\[(\Box + p)G^p(\rho), f)_A = f(0). \] (41)

(ii) Green Function on \(E^2_q\)

We obtain the Green function \(G^p(R)\) on the quantum plane \(E^2_q\) from the one \(G^p(\rho)\) on the two sided coset space by the group multiplication [1] :

\[G^p(R) = \Delta G^p(\rho). \] (42)

Here

\[R = \Delta(\rho) = \rho \otimes 1 + 1 \otimes \rho + vz^* \otimes zv + zv^* \otimes v^*z^* \] (43)

is self-adjoint operator in \(l^2(\mathbb{Z} \times \mathbb{Z})\) and

\[R e_{ts} = q^{2t} e_{ts} \] (44)

where the eigenfunctions \(e_{ts}\) are given by [3]

\[e_{ts} = \sum_{j=-\infty}^{\infty} (-1)^j q^{t-j} \mathcal{J}_s(q^{t-j}) e_{s+j} \otimes e_j. \] (45)

They satisfy the orthogonality condition

\[(e_{ts}, e_{ij}) = \delta_{ti} \delta_{sj}. \] (46)

We also have

\[e_{s+j} \otimes e_j = \sum_{t=-\infty}^{\infty} (-1)^j q^{t-j} \mathcal{J}_s(q^{t-j}) e_{ts}. \] (47)

Therefore the basis elements \(e_{ts} \); \(t, s \in (-\infty, \infty)\) form the complete set in \(l^2(\mathbb{Z} \times \mathbb{Z})\). The Green function on the quantum plane is the linear operator in this space defined as

\[G^p(R)e_{ts} = G^p(q^{2t}) e_{ts}. \] (48)
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