Multi-resonance orbital model
applied to high-frequency quasi-periodic oscillations
observed in Sgr A*

Andrea Kotrlová, Zdeněk Stuchlík and Gabriel Török

Institute of Physics, Faculty of Philosophy and Science, Silesian University in Opava,
Bezručovo nám. 13, CZ-74601 Opava, Czech Republic
e-mail: andrea.kotrlova@fpf.slu.cz

ABSTRACT

The multi-resonance orbital model of high-frequency quasi-periodic oscillations (HF QPOs) en-
ables precise determination of the black hole dimensionless spin $a$ if observed set of oscillations
demonstrates three (or more) commensurable frequencies. The black hole spin $a$ is related to
the frequency ratio only, while its mass $M$ is related to the frequency magnitude. The model is
applied to the triple frequency set of HF QPOs observed in Sgr A* source with frequency ratio
$3:2:1$. Acceptable versions of the multi-resonance model are determined by the restrictions
on the Sgr A* supermassive black hole mass. Among the best candidates the version of strong
resonances related to the black hole “magic” spin $a = 0.983$ belongs. However, the version
demonstrating the best agreement with the mass restrictions predicts spin $a = 0.980$.

Keywords: Accretion, accretion disks — X-rays: general — Black hole physics — Sgr A*

1 Introduction

In the black hole systems observed in both Galactic and extragalactic sources, strong
gravity effects play a crucial role in three phenomena related to the accretion disk that
is the emitting source: the spectral continuum, spectral profiled lines, and oscillations
of the disk.

Quasi-periodic oscillations (QPOs) of X-ray brightness had been observed at low-
(Hz) and high-(kHz) frequencies in many Galactic low mass X-ray binaries (LMXBs)
containing neutron stars (see, e.g., van der Klis 2000, 2006; Barret et al. 2005; Belloni
et al. 2005, 2007) or black holes (see, e.g., McClintock & Remillard 2004; Remillard
2005; Remillard & McClintock 2006). Some of the HF QPOs are in the kHz range
and often come in pairs of the upper and lower frequencies ($\nu_U, \nu_L$) of twin peaks in
the Fourier power spectra. Since the peaks of high frequencies are close to the orbital
frequency of the marginally stable circular orbit, representing the inner edge of Kep-
lerian disks orbiting black holes (or neutron stars), the strong gravity effects must be
relevant in explaining HF QPOs (Abramowicz et al. 2004b).

The resonance orbital model of HF QPOs in black hole systems (Török et al. 2005)
is now partially supported by observations, in particular, when frequency ratio $3:2$
\(2\nu_3 = 3\nu_2\) is seen in twin peak QPOs in the LMXBs containing black holes (microquasars), namely GRO 1655−40, XTE 1550−564, GRS 1915+105 (Török et al. 2005). Nevertheless, there is a clear problem with explanation of the 3:2 frequency ratios in all three microquasars using a resonance model with unique variant of the twin oscillating modes, especially when the limit on the black hole spin given by the spectral continuum fitting (Remillard & McClintock 2006; McClintock et al. 2011) is used (Török et al. 2011; Aliev et al. 2013; Stefanov et al. 2013).

However, in the case of the GRS 1915+105 source the frequency set of HF QPOs is more complex— in fact, at least five HF QPOs were observed there (Remillard & McClintock 2006). Therefore, in this case we have to consider more complex models of HF QPOs. We are able to explain the complete observed frequency set in the framework of the extended resonant orbital model (Stuchlík et al. 2007b,a) based on the so-called Aschenbach effect (Aschenbach 2004; Stuchlík et al. 2005). Another possibility is related to the multi-resonance orbital model recently proposed in Stuchlík et al. (2013).

The multi-resonance orbital model can be profoundly tested in one of the most interesting and relevant cases related to the observations in the X-ray spectra of the Galaxy centre supermassive black hole in the source Sgr A* (Abramowicz et al. 2004a; Aschenbach 2004; Török 2005), where three frequency set with the 3 : 2 : 1 ratio has been observed. In the multi-resonance orbital model of HF QPOs, the resonance conditions on the frequency ratio of oscillations entering the resonances imply specific radii of accretion disks where the resonances can occur. For any specific version of the orbital resonance model, the resonant radii at thin, Keplerian accretion disks (Török et al. 2005) are given by the parameters of the central black hole, i.e., its mass \(M\) and dimensionless spin \(a\). The frequencies of the Keplerian disk oscillation modes can be expressed in terms of the geodetical orbital and epicyclic frequencies of the test particle motion in the Kerr spacetime (Török & Stuchlík 2005b). When HF QPOs are observed as a three frequency set, the multi-resonance orbital model can work in two ways. The first one, of strong resonant phenomena, assumes that all three oscillations in resonance occur at one fixed radius (Stuchlík et al. 2008). The second one, of duplex frequencies, assumes two radii where twin oscillations occur with one of the frequencies being common (see Stuchlík et al. 2013, for details).

In the framework of the multi-resonance model triple sets of frequency ratios determine the dimensionless black hole spin \(a\) precisely, quite independently of the black hole mass \(M\), but not uniquely as several versions of the model can enter the play (Stuchlík et al. 2013). The mass parameter \(M\) is then given by the magnitude of the observed frequencies. The precision of the frequency measurement thus determines precision of the mass parameter of the black hole. We expect the multi-resonance model will be applicable due to development of the observational techniques (e.g., the planned LOFT observatory, Feroci et al. 2012a,b).

Here we apply the multi-resonance orbital model to the observations of three HF QPOs reported by Aschenbach (Aschenbach 2004), using the precision of the frequency measurements.
quency measurements to give ranges of allowed values of the Sgr A* central black hole mass for each of the versions of the multi-resonance model allowing for the frequency ratio 3 : 2 : 1. These mass ranges are compared with the mass range given by the motion of the stars orbiting the central black hole as presented in Gillessen et al. (2009) and acceptable versions of the multi-resonance orbital model are thus found.

Of course, the spin (and mass) estimates of the Sgr A* black hole based on acceptable versions of the multi-resonance model have to be compared with the spin estimates based on the accretion disk spectral continuum fitting (McClintock et al. 2011; Done & Davis 2008; Done et al. 2007; McClintock et al. 2006; Middleton et al. 2006; Shafee et al. 2006) and profiled spectral lines (Laor 1991; Bao & Stuchlík 1992; Karas et al. 1992; Fabian & Miniutti 2005; Zakharov 2003; Čadež et al. 2003; Fanton et al. 1997; Čadež & Calvani 2005; Zakharov & Repin 2006; Schee & Stuchlík 2009b; Miller et al. 2009; Stuchlík & Schee 2012b) or by the measurements of the black hole (or super-spinar) shadow (Virbhadra & Ellis 2002; Schee & Stuchlík 2009a; Stuchlík & Schee 2010; Eiroa 2012). In the case of Sgr A* also the orbital precession of some stars moving in close vicinity of the central black hole could give interesting restriction on the black hole spin (Kraniotis 2005, 2007).

2 Orbital resonance model

The standard orbital resonance model (Abramowicz & Kluźniak 2001; Török et al. 2005; Aliev & Galtsov 1981) assumes non-linear resonance between oscillation modes of an accretion disk orbiting a central object, here considered to be a rotating Kerr black hole.²

The accretion disk can be a thin disk with Keplerian angular velocity profile (Novikov & Thorne 1973), or a thick toroidal disk with angular velocity profile given by the distribution of the specific angular momentum of the fluid (Kozłowski et al. 1978; Abramowicz et al. 1978; Stuchlík et al. 2000; Stuchlík & Kovář 2008; Stuchlík et al. 2009).

The frequency of the oscillations is related to the Keplerian frequency (orbital frequency of tori) and to the radial and vertical epicyclic frequencies of the circular test particle motion. The epicyclic frequencies can be relevant both for the thin, Keplerian disks with quasicircular geodetical motion (Kato et al. 1998; Kluźniak et al. 2007; Kato 2001a,b; 2004; Török et al. 2005; Török & Stuchlík 2005b) and for slender toroidal disks (Schnittman & Rezzolla 2006; Rezzolla et al. 2003). However, with the thickness of an oscillating toroidal growing, the eigenfrequencies of its radial and vertical oscillations deviate from the epicyclic test particle frequencies (Blaes et al. 2007; Straub & Šrámková 2009). Here we focus our attention on the Keplerian thin disks.

A variety of different versions of the orbital resonance model exists. They can be classified due to the three following criteria:

a) the type of the resonance (parametric or forced),

²We consider here only the Kerr spacetime as the standard description for rotating black holes, although alternatives have been discussed in the same context (see Kotrlová et al. 2008; Stuchlík & Kotrlová 2009; Aliev & Talazan 2009; Rahimov et al. 2011; Abdjabbarov et al. 2011; Abdjabbarov & Ahmedov 2010; Aliev et al. 2013; Horváth & Gergely 2012; Stefanov et al. 2013).
b) the presence of beat, combinational frequencies,
c) the type of oscillations entering the resonance.

According to the basical criterion (a), two main groups of orbital resonance model versions exist, differing by the type of the resonance. In both of them, the epicyclic frequencies of the equatorial test particle circular motion play a crucial role (Török et al. 2005).

### 2.1 Parametric internal resonance

The internal resonance model assumes parametric resonance between vertical and radial epicyclic oscillations with the frequencies \( \nu_\theta = \omega_\theta / 2\pi \) and \( \nu_r = \omega_r / 2\pi \). The parametric resonance is described by the Mathieu equation (Landau & Lifshitz 1976)

\[
\delta \ddot{\theta} + \omega_\theta^2 [1 + h \cos(\omega_r t)] \delta \theta = 0.
\]  
(1)

The theory behind the Mathieu equation implies that a parametric resonance is excited when

\[
\frac{\omega_r}{\omega_\theta} = \frac{\nu_r}{\nu_\theta} = \frac{2}{n}, \quad n = 1, 2, 3, \ldots
\]  
(2)

and is strongest for the lowest possible value of \( n \) (Landau & Lifshitz 1976). Because there is \( \nu_r < \nu_\theta \) near black holes, the lowest possible value for the parametric resonance in the so-called epicyclic resonance model is \( n = 3 \) implying \( 2\nu_\theta = 3\nu_r \). This explains the 3 : 2 ratio observed in the microquasars, if \( \nu_U = \nu_\theta \) and \( \nu_L = \nu_r \). The internal resonance corresponds to a system with conserved energy, as shown in Horák et al. (2009).

### 2.2 Forced resonance

The forced resonance model comes from the idea of a forced non-linear oscillator, when oscillations are governed by

\[
\delta \ddot{\theta} + \omega_\theta^2 \delta \theta + [\text{non linear terms in } \delta \theta] = g(r) \cos(\omega_0 t),
\]  
(3)

\[
\delta \ddot{r} + \omega_r^2 \delta r + [\text{non linear terms in } \delta \theta, \delta r] = h(r) \cos(\omega_0 t),
\]  
(4)

with

\[
\omega_0 = \left( \frac{k}{l} \right) \omega_r,
\]  
(5)

where \( k, l \) are small natural numbers and \( \omega_0 \) is the frequency of the external force. The non-linear terms allow combination (beat) frequencies in resonant solutions for \( \delta \theta(t) \) and \( \delta r(t) \) (see, e.g., Landau & Lifshitz 1976), which in the simplest case give

\[
\omega_- = \omega_0 - \omega_r, \quad \omega_+ = \omega_0 + \omega_r.
\]  
(6)

Such resonances can produce the observable frequencies in the 3 : 2 ratio, as well as in other rational ratios. (One of the cases that give 3 : 2 observed ratio is also the “direct”
case of \( k:l = 3:2 \) corresponding to the same frequencies and radius as in the case of \( 3:2 \) parametric resonance.

The “Keplerian” resonance model assumes parametric or forced resonances between oscillations with radial epicyclic frequency \( \nu_r \) and Keplerian orbital frequency \( \nu_K \). Of course, there are many additional possibilities for composing the resonance conditions using the Keplerian orbital and epicyclic frequencies in the framework of the multi-resonance model (Stuchlík et al. 2013).

The resonance conditions related to the frequency ratio of oscillations are common, however, physical details, such as the time evolution of the resonance, the dependence of the resonance strength and the resonant frequency width on the order of the resonance, are different (see Landau & Lifshitz 1976; Nayfeh & Mook 1979; Stuchlík et al. 2008). Here we concentrate on the frequency-ratio resonant conditions only.

### 2.3 Orbital and epicyclic frequencies of the Keplerian motion

The formulae for the vertical epicyclic frequency \( \nu_\theta \) and the radial epicyclic frequency \( \nu_r \) take in the Kerr spacetime (with the mass \( M \) and dimensionless spin \( a \)) the form

\[
\nu_\theta^2 = \alpha_\theta \nu_K^2, \quad \nu_r^2 = \alpha_r \nu_K^2
\]

(e.g., Aliev & Galtsov 1981; Kato et al. 1998; Stella & Vietri 1998; Török & Stuchlík 2005b) where the Keplerian frequency \( \nu_K \) and related epicyclic frequencies are given by the formulae

\[
\nu_K = \frac{1}{2\pi} \left( \frac{GM}{r_\odot^3} \right)^{1/2} \frac{1}{x^{3/2} + a} = \frac{1}{2\pi} \left( \frac{c^3}{GM} \right) \frac{1}{x^{3/2} + a},
\]

\[
\alpha_\theta = 1 - \frac{4a}{x^{3/2} + \frac{3a^2}{x^2}},
\]

\[
\alpha_r = 1 - \frac{6}{x} + \frac{8a}{x^{3/2} - \frac{3a^2}{x^2}}.
\]

Here \( x = r/(GM/c^2) \) is the dimensionless radius, expressed in terms of the gravitational radius of the black hole, and \( a \) is the dimensionless spin.

The Keplerian orbital frequency \( \nu_K(x, a) \) is a monotonically decreasing function of the radial coordinate for any value of the black hole spin. The radial epicyclic frequency has a global maximum for any Kerr black hole. However, the vertical epicyclic frequency is also not monotonic if the spin is sufficiently high (see, e.g., Kato et al. 1998; Török & Stuchlík 2005b). For the Kerr spacetimes, the locations \( R_r(a) \), \( R_\theta(a) \) of maxima of the epicyclic frequencies \( \nu_r \), \( \nu_\theta \) are implicitly given by the conditions (Török & Stuchlík 2005b)

\[
\beta_j(x, a) = \frac{1}{2} \frac{\sqrt{x}}{x^{3/2} + a} \alpha_j(x, a), \quad \text{where } j \in \{r, \theta\},
\]

\[
\beta_r(x, a) = \frac{1}{x^2} - \frac{2a}{x^{3/2} + \frac{a^2}{x^3}},
\]

\[
\beta_\theta(x, a) = \frac{a}{x^{3/2} - \frac{a^2}{x^3}}.
\]
For any black hole spin \( a \), the extrema of the radial epicyclic frequency \( R_r(a) \) must be located above the marginally stable orbit. On the other hand, the latitudinal extrema \( R_\theta(a) \) are located above the photon (marginally bound or marginally stable) circular orbit only if the limits on the black hole spin \( a > 0.748 \) (0.852, 0.952) are satisfied (Török & Stuchlík 2005b). In the Keplerian disks, with the inner boundary \( x_{in} \approx x_{ms} \), the limiting value \( a = 0.952 \) is relevant. In the Kerr black hole spacetimes (0 < \( a < 1 \)) there is always \( \nu_K(x) > \nu_\theta(x) > \nu_r(x) \), while in the Kerr naked singularity spacetimes (\( a > 1 \)), the situation is more complex and spin dependent (Török & Stuchlík 2005b; Stuchlík & Schee 2012b).

For a particular resonance \( n : m \), the equation
\[
np_r = m\nu_v; \quad \nu_v \in \{\nu_\theta, \nu_K\}
\]
determines the dimensionless resonance radius \( x_{nm} \) as a function of the spin \( a \) in the case of direct resonances. This can be easily extended to the resonances with combinational frequencies (Stuchlík et al. 2013).

The known mass of the central black hole, the observed twin-peak frequencies (\( \nu_U, \nu_L \)), the equations (7) – (14) and a concrete type of resonance, assumed to be direct or to have the combinational frequencies, enable to determine the black hole spin. This procedure was first applied to the microquasar GRO 1655−40 by Abramowicz & Kluzniak (2001), to the other three microquasars (Török et al. 2005), and also to the Galaxy centre black hole Sgr A* (Török 2005).

More complex resonant phenomena in HF QPOs have to be expected in the field of Kerr naked singularities (Török & Stuchlík 2005b; Stuchlík & Schee 2012b). In the naked singularity backgrounds, the optical effects also demonstrate considerable differences as compared with those generated in the field of black holes (Stuchlík & Schee 2010, 2012a,b; Virbhadra & Ellis 2002; Virbhadra & Keeton 2008; Takahashi & Harada 2010; Eiroa 2012). Here we restrict our attention to the black hole spacetimes.

3 Multi-resonance model

The very probable interpretation of twin peak frequencies observed in microquasars is the 3:2 parametric resonance of the epicyclic oscillations; however, we expect that more than one resonance could be excited in the disk simultaneously (or at different times) under different internal conditions. Indeed, observations of the HF QPOs in the microquasar GRS 1915+105 (Remillard 2005), in the extragalactic sources NGC 4051, MCG-6-30-15 (Lachowicz et al. 2006), NGC 5408 X-1 (Strohmayer et al. 2007), and the Galaxy centre Sgr A* (Aschenbach et al. 2004) show a variety of QPOs with frequency ratios differing from (or additional to) the 3:2 ratio.

In the framework of the multi-resonance model we consider two different resonances determined by a doubled ratio of natural numbers \( n : m \) and \( n' : m' \). They occur at corresponding radii of the disk \( x_{nm}, x_{n'm'} \) that can be determined from the observed set of frequencies using the relevant versions of the orbital resonance model. In a degenerated case when only triple frequency set is observed, we can distinguish in the multi-resonance model two relevant cases. First, the three frequencies could be related to three oscillating modes occurring at a common radius, if the black hole has
a specific, “magic”, spin given by the frequency ratio of the set – this case corresponds to the strong resonance model as we can assume cooperative resonant phenomena occurring at the common radius (Stuchlík et al. 2008, 2013). Second, three frequency sets with a duplex frequency could occur accidentally for proper values of the black hole dimensionless spin. In such a case, two twin peak QPOs observed at the radii \(x_{n,m}\) and \(x_{n',m'}\) have the bottom, top, or mixed (the bottom at the inner radius and the top in the outer radius, or vice versa) frequencies identical. Such situations can be characterized by sets of three frequencies (upper \(\nu_U\), middle \(\nu_M\) and lower \(\nu_L\)) with ratio \(\nu_U : \nu_M : \nu_L = s : t : u\), given by the \(n : m\) and \(n' : m'\) ratios, the relevant versions of the resonance, and the type of the duplex (common) frequency. The guide book of the possible combinations generated by the orbital and epicyclic frequencies and their combinations is presented in Stuchlík et al. (2013). When only direct resonances of the epicyclic oscillations are allowed, the first case with “bottom identity” can be realized by the situation with two resonances having common radial epicyclic frequency, while the second case with “top identity” can be realized by the situation with two resonances having common vertical epicyclic frequency. These two possibilities only are in principle allowed by the non-monotonocity of the epicyclic frequencies (7) discussed in detail by Török & Stuchlík (2005a,b). When the Keplerian oscillations and the combinational frequencies are allowed, all the mixed, bottom, and top identities are possible. We consider frequency ratios of small integers, with the order of the resonances \(n + m \leq 9\) (\(n' + m' \leq 9\)), since the resonant phenomena are realistic only for \(n \leq 5\) (see Landau & Lifshitz 1976; Nayfeh & Mook 1979, for details).

### 3.1 Resonance conditions

For all possible resonances of the epicyclic and Keplerian frequencies, the resonance condition for the ratio \(\nu_U : \nu_L = n : m\) is given in terms of the frequency ratio parameter

\[
p = \left(\frac{m}{n}\right)^2.
\]

The resonant conditions that implicitly determine the resonant radius \(x_{n,m}(a, p)\) have to be related to the radius of the innermost stable circular geodesic \(x_{ms}(a)\) giving the inner edge of Keplerian disks. We require \(x_{n,m}(a, p) \geq x_{ms}(a)\), where \(x_{ms}(a)\) is implicitly given by

\[
a = a_{ms} \equiv \frac{\sqrt{x}}{3} \left(4 - \sqrt{3x - 2}\right).
\]

The resonance functions are denoted as \(a^{\nu_U/\nu_L}(x, p)\). As an example we give the resonance function (and the resonance condition) for the radial and vertical epicyclic frequencies \((\nu_U = \nu_\theta, \nu_L = \nu_r)\) that read

\[
a = a^{\theta/r}(x, p) = \frac{\sqrt{x}}{3(p + 1)} \left(2(p + 2) - \sqrt{(1 - p) \left[3x(p + 1) - 2(2p + 1)\right]}\right),
\]

All the other resonance conditions and functions for direct and combinational resonances can be found in Stuchlík et al. (2013).
3.2 Frequency sets with a duplex frequency

From the point of view of the observational consequences, it is important to know for which frequency ratios $n : m$ the resonant frequency $\nu_0(a, n : m)$, which is considered as a function of the black hole spin $a$ for a given frequency ratio $n : m$, has a non-monotonic character. A detailed analysis (Török & Stuchlík 2005b) shows that $\nu_0(a, n : m)$ has a local maximum for $n : m > 11 : 5$; i.e., in physically relevant situations ($n, m$ small enough for the resonance), it occurs for the ratios $\nu_0 : \nu_r = 5 : 2, 3 : 1, 4 : 1, 5 : 1$. This means that while the “bottom identity” could happen for any black hole spin $a$, the “top identity” can only arise for $a \sim 1$ if only the epicyclic oscillations are considered. For details see Stuchlík et al. (2013) – we adopt here the system of notation introduced in this paper: $T(X)(Y)$, $B(X)(Y)$, and $M(X)(Y)$ where $T$ stands for top, $B$ for bottom, and $M$ for mixed identity. The $(X)(Y)$ corresponds to the given types of the doubled resonances ($X$) and ($Y$) with identical top, bottom, or mixed frequencies (for the case of mixed identity ($X$) always denotes the resonance with the top frequency and ($Y$) the resonance with the identical bottom frequency). For the triple frequency sets with the ratio $3 : 2 : 1$, the concrete types of the resonances ($X)(Y)$ are presented in Table 1.

3.3 Triple frequency sets and black hole spin

The multi-resonance model gives strong restrictions on the black hole spin $a$ when the triple frequency sets are considered (Stuchlík et al. 2013). For the simple case of the “top identity” of the upper frequencies in two resonances between the radial and vertical epicyclic oscillations at the radii $x_p, x_p'$ with $p^{1/2} = m : n, p'^{1/2} = m' : n'$, the condition $\nu_0(a, x_p) = \nu_0(a, x_p')$ can be transformed to the relation

$$\frac{a_0^{1/2}(a, x_p)}{x_p^{3/2} + a} = \frac{a_0^{1/2}(a, x_p')}{x_p'^{3/2} + a},$$

which uniquely determines the black hole spin $a$. When two different resonances are combined, we proceed in the same manner. For example, the case of “bottom identity” in the resonance between the radial and vertical epicyclic oscillations at $x_p$ and the resonance between the oscillations with the Keplerian frequency $\nu_K$ and total precession oscillations with $\nu_r = \nu_0 - \nu_r$ at $x_p$ implies the condition $\nu_r(a, x_p) = (\nu_0 - \nu_r)(a, x_p')$ that leads to the relation

$$\frac{a_r^{1/2}(a, x_p)}{x_p^{3/2} + a} = \frac{(a_0 - a_r)(a, x_p')}{x_p'^{3/2} + a},$$

which uniquely determines the dimensionless spin $a$, since the radii $x_p$ and $x_p'$ are related to the spin $a$ by the resonance conditions for $a^{0/1/2}(x, p)$ (see Eq. (17)) and $a^{K/(0-\tau)}(x, p')$, respectively. Therefore, for given types of the doubled resonances, the ratios $n : m$ and $n' : m'$ determine the ratio in the triple frequency set $s : t : u$. The black hole spin $a$ is given by the types of the two resonances and the ratios $p, p'$, quite independently of the black hole mass $M$.

Since the radial and vertical epicyclic frequencies and the orbital frequency have the same dependence on the black hole mass $M$, the above arguments hold for any
For a given frequency ratio set, several values of the black hole spin and the triple frequency ratios is not unique in general. However, the relation between the black hole spin and the triple frequency ratios can be very high (Stuchlýk et al. 2013). The mass parameter can be addressed by the magnitude of the measured frequencies. Therefore, the triple frequency sets with the "duplex" frequencies can be used to determine the black hole spin with very high precision given by precision of the frequency measurements that can be very high (Stuchlýk et al. 2013). The mass parameter can be addressed by the magnitude of the measured frequencies. However, the relation between the black hole spin and the triple frequency ratios is not unique in general. For a given frequency ratio set, several values of \( a \) are allowed, and some other meth-

| Set  | \( v_U \) | \( v_M \) | \( v_L \) |
|------|-----------|-----------|-----------|
| T23 (“magic”) | \( v_K \) | \( v^3:2_\theta \) | \( v^3:1_r \) |
| T45 | \( v^3:1_K = v^3:2_\theta \) | \( (v_K - v_r)^{3:2} \) | \( (v_\theta - v_r)^3:1 \) |
| B15 | \( v^{3:1}_\theta \) | \( v^{2:1}_\theta \) | \( v^{3:1}_r = (v_K - v_r)^{2:1} \) |
| B22 | \( v^{2:1}_K \) | \( v^3:1_\theta \) | \( v_r \) |
| B24 | \( v^3:1_K \) | \( v^{2:1}_K \) | \( v_r = (v_\theta - v_r)^{3:1} \) |
| B25 | \( v^3:1_K \) | \( v^{2:1}_\theta \) | \( v^{3:1}_r = (v_K - v_r)^{2:1} \) |
| B45 | \( v^3:1_K \) | \( v^{2:1}_\theta \) | \( (v_\theta - v_r)^{3:1} = (v_K - v_r)^{2:1} \) |
| M14 | \( v^{3:2}_K \) | \( v^{2:1}_\theta \) | \( (v_\theta - v_r)^{3:2} \) |
| M24 | \( v^{3:2}_K \) | \( v^{2:1}_K \) | \( (v_\theta - v_r)^{3:2} \) |
| M54 | \( v^{3:2}_K \) | \( v^{2:1}_\theta \) | \( (v_\theta - v_r)^{3:2} \) |
| T1(DC1) | \( (v_K - v_r)^{3:2} = v^{3:1}_\theta \) | \( (v_\theta - v_r)^{3:2} \) | \( v^{3:1}_r \) |
| B1(DC11) | \( v^{3:1}_\theta \) | \( (v_\theta - v_r)^{2:1} \) | \( v^{3:1}_r = (v_K - v_\theta)^{2:1} \) |
| T1(DC3) | \( (v_K - v_\theta)^{3:2} = v^{3:1}_\theta \) | \( (v_\theta - v_r)^{3:2} \) | \( v^{3:1}_r \) |
| T1(DC9) | \( (v_K - v_r)^{3:1} = v^{3:2}_\theta \) | \( v^{3:2}_r \) | \( (v_K - v_\theta)^{3:1} \) |
| M(DC11) | \( v^{3:2}_\theta \) | \( v^{2:1}_r = (v_\theta - v_r)^{2:1} \) | \( (v_K - v_\theta)^{2:1} \) |
| T1(DC11)$^a$ | \( (v_\theta - v_r)^{3:1} = v^{3:2}_\theta \) | \( v^{3:2}_r \) | \( (v_K - v_\theta)^{3:1} \) |
| T1(DC11)$^b$ | \( (v_\theta - v_r)^{3:2} = v^{3:1}_\theta \) | \( (v_K - v_\theta)^{3:2} \) | \( v^{3:1}_r \) |
| B1(DC9) | \( (v_K - v_r)^{3:1} = v^{3:1}_\theta \) | \( v^{3:1}_r \) | \( (v_K - v_\theta)^{3:2} \) |
| M1(DC1) | \( (v_K - v_r)^{3:2} = v^{2:1}_\theta \) | \( (v_\theta - v_r)^{3:2} \) | \( v^{2:1}_r \) |
| B1(DC12) | \( (v_\theta + v_r)^{3:1} \) | \( v^{2:1}_\theta \) | \( (v_\theta - v_r)^{3:2} \) |
| M1(DC7) | \( (v_K - v_\theta)^{3:2} \) | \( v^{2:1}_\theta \) | \( (v_K - v_r)^{3:2} \) |

kind of the three frequency sets, for any of the bottom, top, or mixed frequency identity with any two resonances containing any combination of the frequencies \( v_K, v_\theta, v_r \). Therefore, the triple frequency sets with the “duplex” frequencies can be used to determine the black hole spin with very high precision given by precision of the frequency measurements that can be very high (Stuchlýk et al. 2013). The mass parameter \( M \) can be addressed by the magnitude of the measured frequencies. However, the relation between the black hole spin and the triple frequency ratios is not unique in general. For a given frequency ratio set, several values of \( a \) are allowed, and some other meth-

Table 1: Relevant versions of the multi-resonant model with assumed observed characteristic frequency ratio set \( v_U : v_M : v_L = 3 : 2 : 1 \).
ods of the spin measurement (spectral continuum fitting, profiled spectral lines) must be involved. It is then important that the spin measurements can be considered quite independently of the mass measurements based on different methods.

3.4 Strong resonant radii and related black hole spin

In situations discussed above, the triple frequency sets fixing the black hole spin $a$ occur at two different radii. However, there are important cases when the triple frequencies occur at the same (shared) radius (Stuchlík et al. 2008). Then we expect higher probability that the resonant phenomena will arise, especially in the cases of ratios of very low integers because a causally related cooperation of the resonances at the given radius can be relevant. A crucial role is expected for direct resonances of oscillations with all three orbital frequencies characterized by a triple frequency ratio set $(s, t, u)$ being small natural numbers

$$\nu_K : \nu_\theta : \nu_r = s : t : u$$  \hspace{1cm} (20)

when strong resonant phenomena are possible. The radius giving strong resonance $s : t : u$ ratio is given by (Stuchlík et al. 2008)

$$x(s/u, t/u) \equiv 6(s/u)^2 X^{-1},$$
$$X = 6(s/u)^2 \pm 2 \sqrt{2} \sqrt{(t/u - 1)(t/u + 1)[3(s/u)^2 - (t/u)^2 - 2] - (t/u)^2 + 5},$$  \hspace{1cm} (21)

and the related black hole “magic” spin is given by

$$a(x(s/u, t/u), u/s) \equiv \frac{\sqrt{x}}{3} \left(4 \pm \sqrt{-2 + 3x \left[1 - (u/s)^2\right]}\right).$$  \hspace{1cm} (22)

A detailed discussion of the black holes that admit strong resonant phenomena can be found for small integers $(s \leq 5)$ in Stuchlík et al. (2008).

Of special interest is the case of the “magic” spin $a = 0.983$, when the Keplerian and epicyclic frequencies are in the ratio $\nu_K : \nu_\theta : \nu_r = 3 : 2 : 1$ at the common radius $x_{3:2:1} = 2.395$. In fact, this case involves rather extended structure of resonances (see Stuchlík et al. 2008). It should be stressed that beside the case of strong resonances between oscillations with $\nu_K, \nu_\theta, \nu_r$ sharing the same radius, the characteristic set $3 : 2 : 1$ could appear also due to resonances at different radii (see Fig. 5 in Stuchlík et al. 2013). All the relevant versions of the multi-resonant model with $3 : 2 : 1$ frequency ratio set are given in Table 1.

4 Acceptable variants of the multi-resonance orbital model and possible spin of the supermassive black hole at Sgr A*

4.1 HF QPOs observed in Sgr A*

The Galaxy centre supermassive black hole source Sgr A* serves as an appropriate astrophysical test system for the multi-resonance orbital model. The HF QPOs with
frequency ratio $\sim 3 : 2 : 1$ were reported for Sgr A* (Aschenbach 2004; Aschenbach et al. 2004; Török 2005) and represent therefore an appropriately simple sample to test the multi-resonant model. Although there are doubts on validity of the data that are not fully accepted by the astrophysical community, we feel it could be important, illustrative and interesting to test possible implications of the observations, assuming their relevance.

The observed frequency ratio is

$$\frac{1}{692} : \frac{1}{1130} : \frac{1}{2178} \sim 3 : 2 : 1.$$ \hspace{1cm} (23)

The upper frequency was observed with a rather high error (Aschenbach 2004)

$$\nu_U = (1.445 \pm 0.16) \text{ mHz}.$$ \hspace{1cm} (24)

Although the reported HF QPOs are considered as controversial, it is still worth to consider them seriously and take them as a starting point to demonstrate the effectiveness of the multi-resonance model in the most interesting case of triple frequency set with the ratio $3 : 2 : 1$. The frequency measurement precision is very poor in the case of data given in Aschenbach (2004), it is much worse in comparison with data obtained by measurements in the LMXBs containing microquasars, where the precision is of the order of 1% (Remillard & McClintock 2006; McClintock et al. 2011) and can be expected by one order better in the planned X-ray satellite observatory LOFT (Feroci et al. 2012b). Here we assume for simplicity the frequency ratio condition $3 : 2 : 1$ to be exactly fulfilled, giving thus the spin precisely, and shifting all the uncertainties of the frequency measurement to the estimates of the black hole mass that is given by the magnitude of the measured frequency.$^3$

Comparing the mass uncertainty related to the measurement error of the upper frequency to the range of mass implied by the motion of the stars observed in vicinity of the Sgr A* (Ghez et al. 2008), we are able to find acceptable variants of the multi-resonance model predicting the related spin of the supermassive black hole at Sgr A*.

### 4.2 Sgr A* black hole parameters

It has been shown that if the strong resonant model with $\nu_K : \nu_\theta : \nu_r = 3 : 2 : 1$ is applied (Stuchlík et al. 2008), the observed data imply spin $a$ and mass $M$ of Sgr A* black hole not contradicting the black hole mass estimate given by the star orbital motion that reads $M_{\text{Sgr}} \sim 4.3 \times 10^6 M_\odot$ (Gillessen et al. 2009) with the interval of allowed values given by

$$3.9 \times 10^6 M_\odot < M < 4.7 \times 10^6 M_\odot,$$ \hspace{1cm} (25)

considering also the error given by uncertainty in distance measurement to Sgr A*.

Here, we test all the relevant versions of the multi-resonant orbital model. The results are summarized in Fig. 1 and related Table 2, including the case of the strong resonant phenomena.

$^3$Error in determination of the dimensionless spin of the black hole is discussed in Stuchlík et al. (2008, 2013).
We can see in Table 2 that from all of the theoretically allowed (19) versions, only six versions are compatible with observational restrictions from the orbital motion of the stars in vicinity of Sgr A∗. In all of the six cases, the black hole spin $a > 0.85$, in agreement with the assumption that the Galactic centre black hole should be fast rotating. The best fit (B24) is obtained for the spin $a = 0.980$, and mass $M \in (3.983 - 4.975) \times 10^6 M_\odot$, with resonances $\nu_K : (\nu_\theta - \nu_r) = 3 : 1, \nu_K : \nu_r = 2 : 1$, having a common bottom frequency. As we can see from Fig. 1, for the case B24 with $a = 0.980$, the mean value of estimated mass $M = 4.42 \times 10^6 M_\odot$ is close to the orbital motion estimate, while in the other five acceptable cases, the difference is much greater, however, two of them are still well acceptable. The good fits are obtained also for the cases T23 and B22, where the strong resonance for the “magic” spin $a = 0.983$ (T23) implies $M \in (4.293 - 5.362) \times 10^6 M_\odot$, and the case B22 with $a = 0.914$ gives the black hole mass $M \in (3.463 - 4.325) \times 10^6 M_\odot$. The frequency relations corresponding to the cases T23 and B22 are illustrated in Fig. 5 in Stuchlík et al. (2013).

The multi-resonance orbital model should be further tested and more precise frequency measurements are very important. We have to compare the results of our model to the other methods of black hole spin measurements, namely those related to the optical phenomena in close vicinity of the supermassive black hole at Sgr A∗ enabling
Table 2: Black hole spin and mass of Sgr A* calculated for all the relevant versions of the multi-
resonant model with assumed observed characteristic frequency ratio set $\nu_U : \nu_M : \nu_L = 3:2:1$;
$\nu_U = (1.445 \pm 0.16)$ mHz is used to determine the black hole mass. The radius of marginally stable
orbit $x_{ms}$ and corresponding resonant radii $x_1$ and $x_2$ are given. An asterisk denotes the special
values of the black hole spin when the resonance points share the same radius ($x_1 = x_2 \equiv x_{3:2:1}$).
Gray shaded rows represent the resonant model versions that are compatible with mass estimates
given by the star orbital motion (Gillessen et al. 2009). Dark grey shaded row corresponds to
the best fit, see also Fig. 1.

| Set          | $a$   | $x_{ms}$ | $x_1$ | $x_2$ | $M \times 10^6 M_\odot$ |
|--------------|-------|----------|-------|-------|-------------------------|
| T23 (“magic”)| 0.983043* | 1.571    | 2.395 | 4.293 | 4.932 – 5.362            |
| T45          | 0.885010 | 2.419    | 3.720 | 4.299 | 2.054 – 2.566            |
| B15          | 0.616894 | 3.758    | 4.241 | 5.833 | 1.903 – 2.376            |
| B15          | 0.999667 | 1.121    | 1.411 | 4.250 | 2.606 – 2.355            |
| B22          | 0.913806 | 2.225    | 2.885 | 3.935 | 3.463 – 4.325            |
| B24          | 0.980124 | 1.612    | 2.551 | 3.519 | 3.983 – 4.975            |
| B25          | 0.475159 | 4.330    | 4.988 | 6.359 | 1.733 – 2.165            |
| B45          | 0.922985 | 2.158    | 3.794 | 4.594 | 2.422 – 3.025            |
| M14          | 0.544870 | 4.055    | 4.347 | 5.477 | 2.095 – 2.617            |
| M24          | 0.535413 | 4.093    | 4.394 | 5.832 | 2.065 – 2.580            |
| M54          | 0.336030 | 4.849    | 5.327 | 6.857 | 1.594 – 1.991            |
| T1(DC1) B1(DC1) | 0.865670* | 2.539    | 2.880 | 2.625 | 3.278                   |
| T1(DC3) M(DC11) | 0.986666* | 1.514    | 1.753 | 3.043 | 3.800                   |
| T1(DC9) M(DC11) | 0.892290 | 2.372    | 3.601 | 5.034 | 1.457 – 1.820            |
| T1(DC11)$^a$ | 0.772687 | 3.046    | 3.792 | 6.036 | 1.183 – 1.477            |
| T1(DC11)$^b$ | 0.927324 | 2.125    | 2.131 | 2.419 | 2.895 – 3.616            |
| B1(DC9) M1(DC1) | 0.868917 | 2.520    | 2.648 | 3.510 | 3.283 – 4.101            |
| B1(DC12) M1(DC7) | 0.851581 | 2.623    | 2.675 | 3.640 | 3.179 – 3.971            |
| M1(DC7)      | 0.987594 | 1.498    | 1.502 | 2.326 | 4.352 – 5.435            |

* to observe even silhouette of the black hole (Schae & Stuchlik 2009a; Bin-Nun 2010; Virbhadra & Ellis 2002; Virbhadra & Keeton 2008; Stuchlik & Schae 2010; Amarilla
& Eiroa 2012; Eiroa 2012). For Sgr A*, the relativistic precession of the nearby star orbits is also very promising (Kraniotis 2005, 2007).

5 Conclusions

We have demonstrated that the multi-resonant model of HF QPOs based on the orbital motion is capable to explain the HF QPO data observed in the supermassive Galactic centre (Sgr A*) black hole (Aschenbach 2004).

There are three versions of the multi-resonance orbital model predicting the mass parameter $M$ of the central supermassive black hole at Sgr A* in agreement with the restrictions determined from the measurements of the motion of the stars orbiting the black hole. All the versions also predict relatively large spin ($a > 0.9$) in accord with expectation of large spin of the central black hole. Among the acceptable versions the strong resonance model with the magic spin $a = 0.983$ belongs, but the mean value of the Sgr A* black hole mass predicted by this version does not meet the range of the mass predicted by the star motion. On the other hand, the best version (B24) predicts $a = 0.980$ and the mean value of the mass parameter very close to the mean value determined by the orbits of stars. The strong limit on the supermassive black hole predicted by the acceptable versions of the multi-resonance model can be useful in studies of the optical phenomena expected to be observed in very close vicinity of the black hole horizon because of development of the recent VLBI observational technique (Doeleman et al. 2009).

Despite the fact that there are doubts on the observational data used for our study of the HF QPOs in the Sgr A* source, we can conclude that the multi-resonance model can work efficiently in restricting the black hole mass parameters, especially when more precise observations of the HF QPOs will be obtained in the measurements of HF QPOs in microquasars and in galaxy centers. We expect very precise measurements obtained by the planned X-ray satellite observatory LOFT (Feroci et al. 2012a,b). In the case of Sgr A* some active accretion periods are necessary in order to have possibility of creating observable HF QPOs.

Acknowledgements. We would like to express our gratitude to the Czech grant GAČR 202/09/0772 and the internal grants of the Silesian University in Opava SGS/11/2013 and SGS/23/2013. The authors further acknowledge the project Supporting Integration with the International Theoretical and Observational Research Network in Relativistic Astrophysics of Compact Objects, CZ.1.07/2.3.00/20.0071, supported by Operational Programme Education for Competitiveness funded by Structural Funds of the European Union and the state budget of the Czech Republic.

References

Abdujabbarov, A. & Ahmedov, B. 2010, Phys. Rev. D, 81, 044022, arXiv:0905.2730 [gr-qc]
Abdujabbarov, A., Ahmedov, B., & Hakimov, A. 2011, Phys. Rev. D, 83, 044053, arXiv:1101.4741 [gr-qc]

Abramowicz, M. A., Jaroszyński, M., & Sikora, M. 1978, A&A, 63, 221

Abramowicz, M. A. & Kluźniak, W. 2001, A&A, 374, L19, arXiv:astro-ph/0105077

Abramowicz, M. A., Kluźniak, W., McClintock, J. E., & Remillard, R. A. 2004a, ApJ, 609, L63

Abramowicz, M. A., Kluźniak, W., Stuchlík, Z., & Török, G. 2004b, in Proceedings of RAGtime 4/5: Workshops on black holes and neutron stars, Opava, 14–16/13–15 October 2002/2003, ed. S. Hledík & Z. Stuchlík, Opava: Silesian University in Opava, 1–23

Aliev, A. N., Esmer, G. D., & Talazan, P. 2013, Classical Quant. Grav., 30, 045010, arXiv:1205.2838 [gr-qc]

Aliev, A. N. & Galtsov, D. V. 1981, Gen. Relativity Gravitation, 13, 899

Aliev, A. N. & Talazan, P. 2009, Phys. Rev. D, 80, 044023, arXiv:0906.1465 [gr-qc]

Amarilla, L. & Eiroa, E. F. 2012, Phys. Rev. D, 85, 064019, arXiv:1112.6349 [gr-qc]

Aschenbach, B. 2004, A&A, 425, 1075, arXiv:astro-ph/0406545

Aschenbach, B., Grosso, N., Porquet, D., & Predehl, P. 2004, A&A, 417, 71, arXiv:astro-ph/0401589

Bakala, P., Šrámková, E., Stuchlík, Z., & Török, G. 2010, Classical Quant. Grav., 27, 045001

Bakala, P., Urbanec, M., Šrámková, E., Stuchlík, Z., & Török, G. 2012, Classical Quant. Grav., 29, 065012

Bao, G. & Stuchlík, Z. 1992, ApJ, 400, 163

Barret, D., Olive, J.-F., & Miller, M. C. 2005, MNRAS, 361, 855, arXiv:astro-ph/0505402

Belloni, T., Méndez, M., & Homan, J. 2005, A&A, 437, 209, arXiv:astro-ph/0501186

Belloni, T., Méndez, M., & Homan, J. 2007, MNRAS, 376, 1133, arXiv:astro-ph/0702157

Bin-Nun, A. Y. 2010, Phys. Rev. D, 82, 064009, arXiv:1004.0379 [gr-qc]

Blaes, O. M., Šrámková, E., Abramowicz, M. A., Kluźniak, W., & Torkelsson, U. 2007, ApJ, 665, 642, arXiv:0706.4483 [astro-ph]

Čadež, A. & Calvani, M. 2005, MNRAS, 363, 177

Čadež, A., Calvani, M., & Fanton, C. 2003, Mem. Soc. Astron. Italiana, 74, 446
Doeleman, S. S., Fish, V. L., Broderick, A. E., Loeb, A., & Rogers, A. E. E. 2009, ApJ, 695, 59, arXiv:0809.3424 [astro-ph]

Done, C. & Davis, S. W. 2008, ApJ, 683, 89, arXiv:0803.0584v2 [astro-ph]

Done, C., Gierliński, M., & Kubota, A. 2007, A&A Rev., 15, 1, arXiv:0708.0148 [astro-ph]

Eiroa, E. F. 2012, ArXiv e-prints, arXiv:1212.4535 [gr-qc]

Fabian, A. C. & Miniutti, G. 2005, Kerr Spacetime: Rotating Black Holes in General Relativity, Cambridge: Cambridge University Press, arXiv:astro-ph/0507409

Fanton, C., Calvani, M., de Felice, F., & Čadež, A. 1997, PASJ, 49, 159

Feroci, M., den Herder, J. W., Bozzo, E., et al. 2012a, in Society of Photo-Optical Instrumentation Engineers (SPIE) Conference Series, Vol. 8443, arXiv:1209.1497 [astro-ph.IM]

Feroci, M., Stella, L., van der Klis, M., et al. 2012b, Experimental Astronomy, 34, 415, arXiv:1107.0436 [astro-ph.IM]

Ghez, A. M., Salim, S., Weinberg, N. N., et al. 2008, ApJ, 689, 1044, arXiv:0808.2870 [astro-ph]

Gillessen, S., Eisenhauer, F., Fritz, T. K., et al. 2009, ApJ, 707, L114, arXiv:0910.3069 [astro-ph.GA]

Horáček, J., Abramowicz, M. A., Kluzniak, W., Rebusco, P., & Török, G. 2009, A&A, 499, 535, arXiv:0901.3076 [astro-ph.CO]

Horváth, Z. & Gergely, L. Á. 2012, ArXiv e-prints, arXiv:1203.6576 [gr-qc]

Karas, V., Vokrouhlický, D., & Polnarev, A. G. 1992, MNRAS, 259, 569

Kato, S. 2001a, PASJ, 53, 1

Kato, S. 2001b, PASJ, 53, L37

Kato, S. 2004, PASJ, 56, 905, arXiv:astro-ph/0409051

Kato, S., Fukue, J., & Mineshige, S. 1998, in Black-hole accretion disks, ed. S. Kato, J. Fukue, & S. Mineshige, Kyoto, Japan: Kyoto University Press

Kluzniak, W., Abramowicz, M. A., Bursa, M., & Török, G. 2007, in Revista Mexicana de Astronomia y Astrofisica (Serie de Conferencias), Vol. 27, 18–25

Kotrlová, A., Stuchlík, Z., & Török, G. 2008, Classical Quant. Grav., 25, 225016, arXiv:0812.0720 [astro-ph]

Kovář, J., Stuchlík, Z., & Karas, V. 2008, Classical Quant. Grav., 25, 095011, arXiv:0803.3155 [astro-ph]
Kozłowski, M., Jaroszyński, M., & Abramowicz, M. A. 1978, A&A, 63, 209
Kraniotis, G. V. 2005, Classical Quant. Grav., 22, 4391, arXiv:astro-ph/0503386
Kraniotis, G. V. 2007, Classical Quant. Grav., 24, 1775, arXiv:gr-qc/0602056
Lachowicz, P., Czerny, B., & Abramowicz, M. A. 2006, submitted to MNRAS, arXiv:astro-ph/0607594
Landau, L. D. & Lifshitz, E. M. 1976, Course of Theoretical Physics, Vol. I, Mechanics, 3rd edn., Oxford: Elsevier Butterworth-Heinemann
Laor, A. 1991, ApJ, 376, 90
McClintock, J. E., Narayan, R., Davis, S. W., et al. 2011, Classical Quant. Grav., 28, 114009, arXiv:1101.0811 [astro-ph.HE]
McClintock, J. E. & Remillard, R. A. 2004, in Compact Stellar X-Ray Sources, ed. W.H.G. Lewin & M. van der Klis, Cambridge: Cambridge University Press, arXiv:astro-ph/0306213
McClintock, J. E., Shafee, R., Narayan, R., et al. 2006, ApJ, 652, 518, arXiv:astro-ph/0606076
Middleton, M., Done, C., Gierliński, M., & Davis, S. W. 2006, MNRAS, 373, 1004, arXiv:astro-ph/0601540
Miller, J. M., Reynolds, C. S., Fabian, A. C., Miniutti, G., & Gallo, L. C. 2009, ApJ, 697, 900, arXiv:0902.2840 [astro-ph.HE]
Montero, P. J. & Zanotti, O. 2012, MNRAS, 419, 1507, arXiv:1107.2546 [astro-ph.SR]
Nayfeh, A. H. & Mook, D. T. 1979, Nonlinear Oscillations, New York: Wiley-interscience
Novikov, I. D. & Thorne, K. S. 1973, in Black Holes, ed. C. D. Witt & B.S.D. Witt, New York–London–Paris: Gordon and Breach, 343
Rahimov, O. G., Abdujabbarov, A. A., & Ahmedov, B. J. 2011, Ap&SS, 335, 499, arXiv:1105.4543 [astro-ph.SR]
Remillard, R. A. 2005, Astronom. Nachr., 326, 804, arXiv:astro-ph/0510699
Remillard, R. A. & McClintock, J. E. 2006, ARA&A, 44, 49, arXiv:astro-ph/0606352
Rezzolla, L., Ahmedov, B. J., & Miller, J. C. 2001, MNRAS, 322, 723, ibid. (2003), 338, 816, arXiv:astro-ph/0011316
Rezzolla, L., Yoshida, S., & Zanotti, O. 2003, MNRAS, 344, 978, arXiv:astro-ph/0307488
Schell, J. & Stuchlik, Z. 2009a, Internat. J. Modern Phys. D, 18, 983, arXiv:0810.4445 [astro-ph]
Schee, J. & Stuchlík, Z. 2009b, *Gen. Relativity Gravitation*, 41, 1795, arXiv:0812.3017 [astro-ph]

Schnittman, J. D. & Rezzolla, L. 2006, *ApJ*, 637, L113, arXiv:astro-ph/0506702

Shafee, R., McClintock, J. E., Narayan, R., et al. 2006, *ApJ*, 636, L113, arXiv:astro-ph/0508302

Stefanov, I. Z., Gyulchev, G. G., & Yazadjiev, S. S. 2013, *Phys. Rev. D*, 87, 083005, arXiv:1212.2878 [astro-ph.HE]

Stella, L. & Vietri, M. 1998, *ApJ*, 492, L59, arXiv:astro-ph/9709085

Straub, O. & Šrámková, E. 2009, *Classical Quant. Grav.*, 26, 055011

Strohmayer, T. E., Mushotzky, R. F., Winter, L., et al. 2007, *ApJ*, 660, 580, arXiv:astro-ph/0701390

Stuchlík, Z. & Kološ, M. 2012a, *Phys. Rev. D*, 85, 065022, arXiv:1206.5658 [gr-qc]

Stuchlík, Z. & Kološ, M. 2012b, *J. Cosmology Astropart. Phys.*, 10, 8

Stuchlík, Z. & Kotrlová, A. 2009, *Gen. Relativity Gravitation*, 41, 1305, arXiv:0812.5066 [astro-ph]

Stuchlík, Z., Kotrlová, A., & Török, G. 2008, *Acta Astron.*, 58, 441, arXiv:0812.4418 [astro-ph]

Stuchlík, Z., Kotrlová, A., & Török, G. 2013, *A&A*, 552, arXiv:1305.3552 [astro-ph.HE]

Stuchlík, Z. & Kovář, *Int. J. Modern Phys. D*, 2008, 17, 2089, arXiv:0803.3641 [gr-qc]

Stuchlík, Z. & Schee, J. 2010, *Classical Quant. Grav.*, 27, 215017, arXiv:1101.3569 [gr-qc]

Stuchlík, Z. & Schee, J. 2012a, *Classical Quant. Grav.*, 29, 025008

Stuchlík, Z. & Schee, J. 2012b, *Classical Quant. Grav.*, 29, 065002

Stuchlík, Z., Slaný, P., & Hledík, S. 2000, *A&A*, 363, 425

Stuchlík, Z., Slaný, P., & Kovář, J. 2009, *Classical Quant. Grav.*, 26, 215013, arXiv:0910.3184 [gr-qc]

Stuchlík, Z., Slaný, P., & Török, G. 2007a, *A&A*, 463, 807, arXiv:astro-ph/0612439

Stuchlík, Z., Slaný, P., & Török, G. 2007b, *A&A*, 470, 401, arXiv:0704.1252 [astro-ph]

Stuchlík, Z., Slaný, P., Török, G., & Abramowicz, M. A. 2005, *Phys. Rev. D*, 71, 024037, arXiv:gr-qc/0411091

Takahashi, R. & Harada, T. 2010, *Classical Quant. Grav.*, 27, 075003
Török, G. 2005, *A&A*, **440**, 1, arXiv:astro-ph/0412500

Török, G., Abramowicz, M. A., Kluźniak, W., & Stuchlík, Z. 2005, *A&A*, **436**, 1, arXiv:astro-ph/0401464

Török, G., Kotrllová, A., Šrámková, E., & Stuchlík, Z. 2011, *A&A*, **531**, A59, arXiv:1103.2438 [astro-ph.HE]

Török, G. & Stuchlík, Z. 2005a, in Proceedings of RAGtime 6/7: Workshops on black holes and neutron stars, Opava, 16–18/18–20 September 2004/2005, ed. S. Hledík & Z. Stuchlík, Opava: Silesian University in Opava, 315–338

Török, G. & Stuchlík, Z. 2005b, *A&A*, **437**, 775, arXiv:astro-ph/0502127

van der Klis, M. 2000, *ARA&A*, **38**, 717, arXiv:astro-ph/0001167

van der Klis, M. 2006, in Compact Stellar X-Ray Sources, ed. W.H.G. Lewin & M. van der Klis, Cambridge: Cambridge University Press, 39–112, arXiv:astro-ph/0410551

Virbhadra, K. S. & Ellis, G. F. 2002, *Phys. Rev. D*, **65**, 103004

Virbhadra, K. S. & Keeton, C. R. 2008, *Phys. Rev. D*, **77**, 124014, arXiv:0710.2333 [gr-qc]

Zakharov, A. F. 2003, Publications of the Astronomical Observatory of Belgrade, **76**, 147, arXiv:astro-ph/0411611

Zakharov, A. F. & Repin, S. V. 2006, *New A*, **11**, 405, arXiv:astro-ph/0510548