RULES OF DISCRETIZATION FOR PAINLEVÉ EQUATIONS

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Abstract – The discrete Painlevé property is precisely defined, and basic discretization rules to preserve it are stated. The discrete Painlevé test is enriched with a new method which perturbs the continuum limit and generates infinitely many no-log conditions. A general, direct method is provided to search for discrete Lax pairs.
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1 Introduction

Given some continuous differential equation, we want to find its discretizations which preserve some global property, namely the explicit linearizability or more generally the discrete Painlevé property, a notion recalled section 2 together with its group of invariance. From this point of view, we will never have to address the question of finding a continuum limit, and consequently we will always write the discrete equations in a form as close as possible to the canonical form of the continuum limit, which is well established [33, 20].

Discrete equations are functional equations linking the values taken by some field variable $u$ at a finite number $N + 1$ of points, either arithmetically consecutive: $x + kh$, or geometrically consecutive: $x q^k$, $k = k_0, 1, \ldots, N$, where $h$ or $q$ is the lattice stepsize, assumed to lie in some neighborhood of, respectively, $0$ or $1$, and $k_0$ is just some convenient origin. The integer $N$ is called the order of the equation, and we denote, for brevity, d–(E) and q–(E) these “difference” and “q–difference” equations. Their study was initiated by Laguerre, mainly as three-term ($N = 2$) recurrence relations between coefficients of orthogonal polynomials. This remained for long a mathematical subject [52, 19], which then extended to topological field theory [4, 27]. Finally, the discrete equation

$$E \equiv -(\overline{u} - 2u + u)/h^2 + 2(\overline{u} + u + u)u + x = 0$$  (1)

already considered by the authors of last five references, was again encountered by statistical physicists in two-dimensional quantum gravity [6, 15, 25] who recognized it as a discrete analogue of the first Painlevé equation (P1)

$$E \equiv -u'' + 6u^2 + x = 0.$$  (2)

The same happened simultaneously with a discrete analogue of the second Painlevé equation (P2) [12, 36] (in the particular case $\alpha = 0$)

$$E \equiv -(\overline{u} - 2u + u)/h^2 + (\overline{u} + u)u^2 + xu + \alpha = 0$$  (3)

$$E \equiv -u'' + 2u^3 + xu + \alpha = 0.$$  (4)

The above short notation

$\begin{align*}
N \text{ even} : & \quad u = u(x), \overline{u} = u(x + h), \underline{u} = u(x - h), \overline{\underline{u}} = u(x + 2h), \\
N \text{ odd} : & \quad \overline{u} = u(x + h/2), \underline{u} = u(x - h/2), \overline{\underline{u}} = u(x + 3h/2),
\end{align*}$  (5-6)

is adopted throughout the article.

The reason why these two discrete equations, among many others with the same continuum limit, deserve the name of discrete Painlevé equations, in short d–(P1) and d–(P2), is that they possess the discrete Painlevé property. Indeed, both admit a discrete Lax pair.

Up to now, there exist two methods to find discrete Lax pairs: the discrete isomonodromic deformation method [3], and the discrete analogue [1] of the method of Zakharov–Shabat and Ablowitz–Kaup–Newell–Segur (AKNS). But both methods are inverse methods, i.e. they generate some discrete equation as a condition between the coefficients of two given linear operators. The drawback is that the obtained discrete equation may not be the one which was looked for. We propose here a direct method, based on discretization rules, to search for the Lax pair of a given discrete equation, and we obtain several new Lax pairs in this way.

Just like its continuous counterpart, the discrete Painlevé test is the set of all methods one can imagine to build necessary conditions for a given discrete equation to possess the discrete Painlevé property. Two such methods are known, the singularity confinement method [24] and the method of perturbation of the continuum...
The latter is based on the perturbation theorem of Poincaré, which is applicable to differential systems in an arbitrary number of independent variables, whether discrete, continuous or even mixed discrete–continuous.

The paper is organized as follows. The discrete Painlevé property (PP) and its group of invariance are defined in section 2, and a first set of basic rules of discretization is given in section 3. Section 4 recalls as an illustration the exact discretizations of the elliptic equations. In section 5, we define discrete Lax pairs from continuous Lax pairs. Section 6 states necessary discretization rules for Lax pairs and details the direct method to obtain the discrete Lax pair of a given equation, with an application to the d–(P1), the d–(P2) and a particular d–(P3) of degree one. In section 7, this method is extended to discrete equations with complementary terms which do not contribute to the continuum limit, and several new Lax pairs are obtained. Section 8 explains the method of perturbation of the continuum limit for the discrete Painlevé test, and applies it to a qualitative candidate d–(P1). Finally, we define in section 9 criteria for a systematic search for the fifty discrete Gambier equations.

2 The discrete Painlevé property and its group of invariance

The (continuous) Painlevé property is defined as the absence of movable critical singularities in the general solution of a differential equation

$$\forall x : E(x,u,u',\ldots,u^{(N)}) = 0,$$

where a singularity is said movable (as opposed to fixed) if its location in the complex plane of \(x\) depends on the initial conditions, and critical if the solution is multivalued around it. For shortness, following Bureau, we will use the terms “stability” for PP, “stable” or “unstable” for an equation with or without the PP.

The PP is invariant under the group of birational transformations

$$\begin{align*}
(u,x) \rightarrow (U,X) : & \quad u = r(x,U,dU/dX,dX^{-1}), x = \Xi(X), \\
(U,X) \rightarrow (u,x) : & \quad U = R(X,u,du/dx,dx^{-1}), X = \xi(x),
\end{align*}$$

\((r \text{ and } R \text{ rational in } U, u \text{ and their derivatives, analytic in } x,X). \) An easier to manage subgroup is made of the homographic transformations

$$\begin{align*}
(u,x) \rightarrow (U,X) : & \quad u = \frac{aU + b}{cU + d}, X = \xi(x), ad - bc \neq 0
\end{align*}$$

where \((a,b,c,d,\xi)\) are arbitrary analytic functions of \(x\). In his classification of second order first degree equations, Gambier has found respectively twenty-four and fifty equivalence classes for these two groups (with minor later corrections).

In the discrete case, let us consider equations

$$\begin{align*}
\forall x \forall h : & \quad E(x,h,u_x + kh_h) = 0, k = k_0 = 0, \ldots, N) = 0 \\
\forall x \forall q : & \quad E(x,q,u_x + q^k_h) = 0, k = k_0 = 0, \ldots, N) = 0
\end{align*}$$

algebraic in the values of the field variable, with coefficients analytic in \(x\) and the stepsize \(h\) or \(q\). It should be noted that \(u\) is a function of two variables, \(x\) and the stepsize. A natural definition for the discrete Painlevé property is the following.

**Definition.** A discrete equation is said to possess the discrete Painlevé property if and only if there exists a neighborhood of \(h = 0\) (resp. \(q = 1\)) at every point of which the general solution \(x \rightarrow u(x,h)\) (resp. \(x \rightarrow u(x,q)\)) has no movable critical singularities.

**Remarks.**
1. The definition reduces to that of the continuous PP in the continuum limit.

2. The singularities in the definition belong to the complex plane of \( x \), not of the stepsize.

3. This definition immediately extends to equations in an arbitrary number of independent variables, discrete or continuous, the extension starting then from the definition of the PP suited to partial differential equations (PDEs), which we do not remind here since this is not our subject.

The discrete PP is invariant under the discrete analogue of (9), which is the group of nonlocal discrete birational transformations

\[
\begin{align*}
  u &= r(x, h \text{ or } q, U, \overline{U}, \ldots), \\
  U &= R(X, H \text{ or } Q, u, \overline{u}, \ldots), \\
  X &= \xi(x, h \text{ or } q), \\
  H &= \eta(h), \\
  Q &= \kappa(q),
\end{align*}
\]  

(13)

(r and \( R \) rational in \( U, \overline{U}, \ldots, u, \overline{u}, \ldots \), analytic in \( x \) and the stepsize, \( \xi, \eta, \kappa \) analytic). There exist two discrete analogues of the subgroup (10), and both may be useful to establish the discrete equivalent of the classification of Gambier. The first one is the group of transformations (13) which in the continuum limit reduce to the homographic transformations (10), where \( (a, b, c, d, \xi) \) are arbitrary analytic functions of \( x \) and of the stepsize. The second one is the group of local homographic transformations (\( r \) and \( R \) homographic in \( U \) and \( u \), independent of \( U, \overline{U}, \ldots, \overline{u}, \ldots \), analytic in \( x \) and the stepsize, \( \xi, \eta, \kappa \) analytic).

Remarks.

1. The first subgroup seems more useful, although it does not contain the transformation \( u = h^kU, \ k \in \mathbb{Z} \).

2. Just like in the continuous case, the birationality can only be proven by taking the discrete equation into account. For instance (18), given the equation

\[
(\overline{u} + u)(u + \overline{u})(4u + h^4x - 6) + 8 = 0, \tag{14}
\]

and the transformation

\[
U(X) = 2/(u(x - h/2) + u(x + h/2)), \tag{15}
\]

one first deduces the inverse transformation

\[
u(x) = (6 - 2U(X - H/2)U(X + H/2) - H^4X)/4, \ X = x, \ H = h(16)
\]

then one plugs it into the direct transformation to get the transformed equation

\[
(\overline{U} + U)U^2 + (H^4X - 6)U + 4 = 0. \tag{17}
\]

The fields which admit a continuum limit are \((u - 1)/h^2\) and \((1 - U)/H^2\), this limit being \((P1)\) for both fields.

3 Basic rules of discretization

Let us now give some basic rules for discretizing a given continuous equation (1) into either a difference equation (11) or a \( q \)-difference equation (12).

The question of discretization is well known in numerical analysis, where one looks for a scheme of discretization which maximizes the order, called scheme order, of the remainder of the expansion of the left-hand side of (11) in a Taylor series of \( h \).
around the center of the \( N + 1 \) points. A scheme of discretization is said \textit{exact} iff it has an infinite order, like that for the particular (P3) equation \( \alpha = \beta = \gamma = \delta = 0 \)

\[- uu'' + u'^2 - uu'/x = 0, \quad u = c_1 x^{2}, \quad u(xq)u(x/q) - u(x)^2 = 0. \quad (18)\]

Contrary to numerical analysis, only interested in a \textit{local} integration, we require the scheme of discretization to preserve the differential order \( N \), an essential element for a \textit{global} knowledge of the solution: every discretization must involve \( N + 1 \) consecutive points.

Like in the continuous case, there exists another important element concerning discretization rules.

\textit{Definition.} The \textit{degree} of a discrete equation is the highest of the two polynomial degrees of the LHS \( E \) of the equation in \( u(x) \) and \( u(x + Nh) \), or \( u(x) \) and \( u(xq^N) \), where \( E \) is assumed polynomial in the \( N + 1 \) variables \( u(\ldots) \).

The following conjecture has recently been made \[13\]: \textit{“Given an algebraic differential equation with the PP, there exists a discretization scheme of order two which conserves the degree.”}

This conjecture was supported by two examples, a second order first degree equation with the PP

\[ u = (c_1 x + c_2)^2, \quad uu'' - (1/2)u'^2 = 0, \] (19)

and a first order first degree equation without the PP

\[ u = (x - c_1)^{-1/2}, \quad u' + (1/2)u^3 = 0. \] (20)

Both admit an exact algebraic discretization, resulting from the elimination of \( c_1 \) or \((c_1, c_2)\) between the values of \( u \) taken at two or three contiguous points, and the resulting discrete equation has degree two. For the first example, we constructed a first degree discrete equation with the PP, while we showed the impossibility to do that for the second equation. Another example is displayed in section \ref{sec:example}.

This led us to state the additional rule, restricted to algebraic differential equations with the PP: every second order scheme must also conserve the degree.

4 The exact discrete elliptic equation

The (continuous) elliptic equation has two usual kinds of normalized forms, the one of Weierstrass and the twelve ones of Jacobi, defined by the first and second order equations \((g_2, g_3, a, b \text{ constants}, \ (p, q, r) \text{ arbitrary permutation of (c,d,n)})\)

\[
\begin{align*}
\wp' & = 4\wp^3 - g_2\wp - g_3 \\
\wp'' & = 6\wp^2 - g_2/2 \\
ps' & = qs^2 rs^2 = (ps^2 + a)(ps^2 + b) \\
ps'' & = ps(qs^2 + rs^2) = ps(2ps^2 + a + b).
\end{align*}
\] (21-24)

Elliptic functions possess an addition formula, i.e. an algebraic relation between the values of the function at the three points \((x_1, x_2, x_1 + x_2)\), with \((x_1, x_2)\) arbitrary. As noticed in the context of discretization by Baxter \[2\] and Potts \[44\], the choice \((x_1, x_2) = (x - h/2, h) \text{ \textit{ipso facto} defines an exact discretization scheme for the first order equations (21) and (23). A scheme for the second order equations (22) and (24) then results from the difference of the discrete first order equations taken for}

6
be discretized with two points, which we denote \((\overline{\tau} = u, 2)\) and \((\overline{\tau} = u, 3)\). The results are [14], [15], [16], with notation (3) for (25) and (27), (4) for (26) and (28).

\[
(\overline{\tau} - \overline{u})^2 \varphi(h) = 2 \overline{\tau} u(\overline{\tau} + \overline{u}) - (g_2/2)(\overline{\tau} + \overline{u}) - g_3 - [(\overline{\tau} u + g_2/4) + g_3(\overline{\tau} + \overline{u})] \varphi^{-1}(h) \tag{25}
\]

\[
(\overline{\tau} - 2u + \overline{u}) \varphi(h) = 2u(\overline{\tau} u + \overline{u}) - g_2/2 - [u^2(\overline{\tau} + \overline{u}) + (g_2/2)u + g_3] \varphi^{-1}(h) \tag{26}
\]

\[
(\overline{\tau} - \overline{u})^2 p\overline{s}^2(h) = (\overline{\tau} u)^2 - 2(p\overline{s}(h) + p\overline{s}(h)) \overline{\tau} u + ab \tag{27}
\]

\[
(\overline{\tau} - 2u + \overline{u}) p\overline{s}^2(h) = u^2(\overline{\tau} + \overline{u}) - 2(p\overline{s}(h) + p\overline{s}(h)) u \tag{28}
\]

**Remarks.**

1. The general solution of (23) and (27) is by construction \(\varphi(x - x_0, g_2, g_3)\) and \(p\overline{s}(x - x_0, k)\), where the step \(h\) is arbitrary, i.e. not necessarily small, with \(k\) the Jacobi modulus. These equations therefore possess the discrete PP. The equations (24) and (28) also possess the discrete PP since they admit a discrete Lax pair, see (67)–(68) and (62)–(63).

2. Order and degree are conserved by the four discretizations.

3. From these schemes of infinite order, the Laurent expansion of \(\varphi(h)\) around its double pole \(h = 0\), or of \(p\overline{s}(h)\) around its simple pole, defines the second order schemes

\[
(\overline{\tau} - 2u + \overline{u})h^{-2} = 2u(\overline{\tau} + \overline{u}) - g_2/2 \tag{29}
\]

\[
(\overline{\tau} - 2u + \overline{u})h^{-2} = u^2(\overline{\tau} + \overline{u}) - (a + b)u, \tag{30}
\]

which have the discrete PP, see (60)–(61) and (62)–(63).

4. Similar equations hold for the nine other Jacobi functions \(pq\), where the coefficients of the r.h.s. of (27) and (28) are polynomials in \(k^2, p\overline{s}(h), pq(h)\).

5. Equation (27) was obtained and integrated in 1973 by Baxter [2] in the eight-vertex model, as a commutation condition of the Yang–Baxter, or star–triangle relations. These Yang–Baxter relations [28], which are second order discrete tensorial equations, play in the discrete domain a role as central as the one played by the Yang–Mills equations in the continuous domain.

**5 Discrete Lax pairs**

In the two relations defining a Lax pair of a (continuous) ODE \(E(x, u) = 0\), e.g. in matrix form with \(t\) the spectral parameter

\[
\partial_x \psi = L \psi, \quad \partial_t \psi = M \psi, \quad (C \equiv \partial_t L - \partial_x M + LM - ML = 0) \leftrightarrow (E = 0) \tag{31}
\]

one can discretize either \(x\) alone or \(x\) and \(t\). Let us restrict here to the first case and to the difference type equations.

The rule of conservation of the differential order requires the column vector \(\psi\) to be discretized with two points, which we denote \(\overline{\psi} = \psi(x + h/2)\) and \(\overline{\psi} = \psi(x - h/2)\), and \(L\) to be discretized with as many points \(u\) as required by the differential order of the equation under consideration, points which we denote \(\overline{u} = u(x + h), u = u(x), \overline{u} = u(x - h)\) for a second order equation. In order to keep a linear correspondence between the continuous operators \((L, M)\) and their discrete counterparts, it is then convenient to discretize \(\partial_x \psi = L \psi\) in a dissymmetric-looking
way and to introduce [3] the linear operator $A$ linking $\psi$ to $\overline{\psi}$, thus defining the discrete Lax pair $(A, B, z, \psi, h)$ as [20]

$$
\overline{\psi} = A\psi, \quad \partial_z \psi = B\psi. \quad (K \equiv \partial_z A + AB - BA = 0) \Leftrightarrow (E = 0).
$$

(32)

The continuum limit is then

$$
\frac{A - 1}{h} \to L, \quad (dz/dt)B \to M,
$$

$$
(dz/dt)(\partial_z A + AB - BA)/h \to \partial_t L - \partial_x M + LM - ML,
$$

(33)

with some link $F(t, z, h) = 0$ between the spectral parameters $t$ and $z$.

For a second order equation $E(\pi, u, \underline{u}, x, h) = 0$, the operators $A$ and $B$ must have the $u$–dependences $A(\pi, u, \underline{u})$, $B(u, \underline{u})$.

Remark. The definition is invariant under the involution

$$
(E, A, B, x, h, \pi, u, \underline{u}) \to (E, A^{-1}, B, x, -h, u, u, \pi),
$$

(34)

which sometimes allows to suppress an undesired denominator in matrix $A$, like in the matricial Lax pair of d–(P1) given in Ref. [31].

The interest of a discrete Lax pair is to provide a constructive proof of the PP, just like in the continuous case.

6 A direct method towards matricial Lax pairs

The two methods recalled in the introduction to find discrete Lax pairs are inverse methods. From the point of view of discretization, a new, direct method emerges, which is as follows.

Let be given a continuous equation, its Lax pair, and some discretization of the continuous equation. We first state general rules for discretizing the Lax pair, involving some free functions; we then enforce the cross-derivative condition (32) to remove all the freedom on these functions, except the link between $t$ and $z$ which cannot be removed; finally, we choose the link between $t$ and $z$, so as to preserve the rational dependence of the matricial Lax pair on the spectral parameter and to ensure the existence of the continuum limit of the discrete Lax pair.

In this section, we handle the d–(P1) (1) and the d–(P2) (3), whose matricial Lax pairs with a continuum limit have been found respectively by Refs. [31, 22], using an inverse method, and by Ref. [13], using the present method. We also handle the d–(P3) of Ref. [22] when its degree two reduces to one, i.e. for $(\alpha, \beta, \gamma, \delta) = (0, 0, 0, 0)$

$$
E \equiv -xu(\pi - 2u + \underline{u})/h^2 + x(\pi - u)(u - \underline{u})/h^2 - u(\pi - u)/(2h) = 0,
$$

(35)

with continuum limit either of the two canonical forms of the third Painlevé equation, (P3) or (P3′) (Ref. [10] p. 1115)

$$(\text{P3′}) \quad E \equiv -u'' + \frac{u^2}{u} - \frac{u'}{x} + \frac{\alpha u^2}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4u} = 0,
$$

(36)

$$(\text{P3}) \quad E \equiv -u'' + \frac{u^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \beta}{x} + \gamma u^3 + \frac{\delta}{u} = 0.
$$

(37)

One must first make a choice between three kinds of Lax pairs for the (Pn) equations : the second order scalar “Lax” pairs of Garnier [21], the second order matricial ones of Jimbo and Miwa [29], and the ones of Flaschka and Newell [17].
arising from the reduction of a PDE. In this paper, we restrict to the third type in matricial form when the matrix order is two. Using Pauli matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_j \sigma_k = \delta_{jk} + i \varepsilon_{jkl} \sigma_l, \] (38)

\[ \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

the continuous pairs are as follows.

For (P1) [29]:

\[ L = 2(u - t)\sigma^+ + \sigma^-, \]
\[ M = 2u'\sigma_3 + (-4u^2 - 2x + 8tu - 16t^2)\sigma^+ + (4u + 8t)\sigma^-, \]
\[ C = 2\sigma_3 E, \] (40)

For (P2) [17]:

\[ L = -t\sigma_3 + u\sigma_1, \]
\[ M = (4t^2 - 2u^2 - x)\sigma_3 + 2u'i\sigma_2 - (4tu + \alpha t^{-1})\sigma_1, \]
\[ C = 2i\sigma_2 E, \] (42)

For the (P3) [37]:

\[ L = \frac{1}{2}(u'/u + cu + d/u)\sigma_3 + t\sigma_1, \quad c^2 = \gamma, \quad d^2 = -\delta, \] (43)
\[ M = \left[ (x(2t))(u'/u + cu + d/u)\sigma_3 + x\sigma_1 + t^2cd^2x^2\sigma_1/2 - t^{-3}(\alpha c - \beta d)\sigma_3 \right. \\
\left. -t^{-2}(\alpha u + \gamma xu^2 + c(xu' + u))\sigma^+ /2 \\
- t^{-2}(\beta/u + \delta x/u^2 + d(x(1/u)' + 1/u))\sigma^- /2 \right] \frac{t^2}{t^2 + cd}, \]
\[ C = \left( (x/(2tu))(\sigma_3 - (cu/t)\sigma^+ + (d/(tu))\sigma^-) \right) \frac{t^2}{t^2 + cd} E. \] (44)

This Lax pair for (P3) is the extension to arbitrary \( \alpha, \beta, \gamma, \delta \) of the pair given by Milne [34] for \( \gamma \delta = 0. \)

The rules of discretization are the following.

1. conserve the matricial order. This is indeed the differential order of the scalar Lax pair, which must be conserved;
2. replace the continuous spectral parameter \( t \) by an unspecified function \( T(z,h) \);
3. discretize the operator \( L \) centered at the three points \( x - h, x, x + h \). If \( L \) is traceless, so is its discretization;
4. discretize the operator \( M \) centered at the two points \( x - h, x \). If \( M \) is traceless, so is its discretization;
5. for each matrix element, enforce conservation of order and degree;
6. replace each monomial \((du/dx)^k\) by its discretization obeying the general rules, multiplied by the \( k \)-th power of an unspecified function \( g(z,h) \). This function \( g \), whose continuum limit must be 1 for any \( z \), represents the ratio of the stepsize \( h \) to the differential element \( dx \);
7. take \( B \) as the product of the discretized \( M \) by an unspecified function \( J(z,h) \) (like Jacobian) representing a discretization of the derivative \( dT/dz \);
8. take $(A - 1)/h$ equal to the sum of the discretized operator $L$ and a diagonal matrix of unspecified functions of $(z, h)$ only, diag$(g_1, g_2)$; these functions, whose continuum limit must be zero, account for the dissymmetry of the formula defining $A$.

In the second order, first degree case of Painlevé equations (Pn), examples of discretizations obeying the above rules are

\[
\begin{align*}
  u^2 & \text{ in } L \rightarrow \nu_1 u (\nu + u)/2 + \nu_2 u^2 + \nu_3 \nu u, \quad \nu_1 + \nu_2 + \nu_3 = 1, \\
  u^2 & \text{ in } M \rightarrow u^2, \\
x \text{ in } M \rightarrow x - h/2. 
\end{align*}
\]  

(46)

Finally, after the cross-derivative condition has been enforced, one must perform the continuum limit on those of the functions $T, g, J, g_1, g_2$ which remain unspecified. One such free function is $T$, because the choice of $z$ is arbitrary. One chooses $T(z, h)$ as a rational function of $z$ (to conserve the rational dependence of the Lax pair on its spectral parameter) such that the inverse function $z$ of $(t, h)$ admits for every $t$ a finite nonzero limit when $h \to 0$.

The discretized matricial Lax pair of the d–(P1) is, with $\lambda_1 + \lambda_2 = 1$

\[
\begin{align*}
  A &= 1 + h \left( \frac{g_1}{1} 2\lambda_1 u + \lambda_2 \left( \mu + u \right) - 2T \right), \\
  B/J &= \left( \begin{array}{ccc} 2g(u - u)/h & 2(u + u) + 8T & -4u u - 2(x - h/2) + 4T(u + u) - 16T^2 \\
  2(u + u) + 8T & -2g(u - u)/h & \end{array} \right),
\end{align*}
\]

(47)

and that of the d–(P2) is

\[
\begin{align*}
  A &= 1 + h(-2\mu^2 - 2(\mu - h/2)) + g_1 \left( \begin{array}{cc} g_1 & 0 \\
  0 & g_2 \end{array} \right), \\
  B/J &= \left( \begin{array}{ccc} 2g(u - u)/h & -4u u - 2(x - h/2) + 4T(u + u) - 16T^2 \\
  -2g(u - u)/h & \end{array} \right),
\end{align*}
\]

(48)

depending on the yet unspecified functions $T, g, J, g_1, g_2$ of $(z, h)$. The cross-derivative condition is enforced by eliminating for instance $\mu$ (or here the variable $x$, which appears always at the first power) between the condition $K = 0$ and the discrete equation, and identifying to zero the resulting polynomial in the three variables $(x, u, \mu)$ (or here the three variables $(\mu, u, \mu)$). This results in the relations

\[
\begin{align*}
  \text{d–(P1)} & \quad \lambda_1 = 1, \quad g_1 = g_2 = (g - 1)/h, \quad 1 - g^2 - 2h^2 T^2 = 0, \quad J = -g'/h^2, \\
  \text{d–(P2)} & \quad \lambda_1 = 1, \quad g_1 = g_2 = (g - 1)/h, \quad 1 - g^2 + h^2 T^2 = 0, \quad J = g'/h^2 T, 
\end{align*}
\]

(49)

and one of the five functions remains free, as expected. A convenient choice of $T$ is

\[
\begin{align*}
  \text{d–(P1)} & \quad T = z - h^2 z^2/2, \quad z = t + (t^2/2)h^2 + O(h^4), \\
  \text{d–(P2)} & \quad T = \frac{z - z}{2h}, \quad z = 1 + h t + O(h^2). 
\end{align*}
\]

(50)

(51)

(52)

Finally, the matricial Lax pair of the d–(P1) is

\[
\begin{align*}
  A &= \left( \begin{array}{cc} 1 - h^2 z & h(2u - 2t) \\
  h & 1 - h^2 z \end{array} \right), \quad t = z - h^2 z^2/2, \quad z = t + (t^2/2)h^2 + O(h^4) \\
  B &= \left( \begin{array}{cc} 2(1 - h^2 z)(u - u)/h & -4u u - 2(x - h/2) + 4T(u + u) - 16T^2 \\
  2(u + u) + 8t & -2(1 - h^2 z)(u - u)/h \end{array} \right), \quad h^{-1} K = 2\Sigma_3 E,
\end{align*}
\]

(53)
a result obtained \cite{3} with this method. This is the rewriting of eqns. (1.12)–(1.13) of Ref. \cite{18} which clearly shows the continuum limit, under the change of basis

\[
P = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) , \quad (\mathcal{P})^{-1} A P = h \left( \frac{2\zeta}{\sqrt{\omega}} - \frac{\sqrt{\omega}}{\omega} \right),
\]

\[
\zeta = (1 - h^2 z)/h, \quad \omega = (1 - 2h^2 u)/h^2, \quad f/f = \sqrt{1 - 2h^2 u}.
\]

As to the matricial Lax pair of the d–(P2) \cite{3}

\[
A = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} + h(gF(x)\sigma_3/2 + T^2 \sigma_1),
\]

\[
B/J = g(f(x - h/2) + f_0)\sigma_3/(2T) + (x - h/2)\sigma_1,
\]

with six unknown functions of \( z \) (\( T, g, J, G_1, G_2, f_0 \)) and one unknown function of \( x \) (\( F \)) representing the discretization of \( u'/u \). The condition that the discrete equation be a factor of the commutator \( K \) \cite{32} yields

\[
\text{d–(P3)} \quad G_1 = G_2, \quad g' = 0, \quad f_0' = 0, \quad J = G_1'(h^2 T), \quad (G_2^2 - h^2 T^2)' = 0,
\]

\[
F(x) = (f(x + h/2) + f(x - h/2) + 2K_0)/(2x), \quad K_0 \text{ constant},
\]

resulting in the same convenient choice of \( T(z) \) as in d–(P2), eq. \cite{52}.

The resulting Lax pair for this particular d–(P3) is finally

\[
f(x) = x \frac{u(x + h/2) - u(x - h/2)}{u(x + h/2) + u(x - h/2)} h, \quad F(x) = \frac{f(x + h/2) + f(x - h/2) + 2K_0}{2x},
\]

\[
A = (z + z^{-1})h/2 + h(F(x)\sigma_3/2 + ((z - z^{-1})/(2h))\sigma_1),
\]

\[
hzB = h/2 + h\sigma_3 + (x - h/2)\sigma_1,
\]

\[
zK = h/2 + h((z + z^{-1})\sigma_3/2 + hf(x)\sigma_1)E.
\]

\textit{Remarks.}
1. The scalar Lax pair satisfied by the second component $\psi_2$ of $\psi$ in (53)

\[
\frac{(\psi_2 - 2\psi_2 + \psi_2)/\hbar^2 - 2u\psi_2 + 2z\psi_2}{\hbar^2 - 2u\psi_2 + 2z\psi_2} = 0,
\]

\[
\psi_{2,z} + 2[4z(\psi_2 - \psi_2 + 2u\psi_2 - u\psi_2 - u\psi_2)/\hbar
+ 4z^2(\psi_2 - 3\psi_2) - zu\psi_2]h + 4z^3\psi_2h^3 = 0
\]

is of course the one obtained by Joshi et al. [31]; as to the scalar Lax pair
found in quantum gravity [27, 26], which contains algebraic coefficients, it
is the transformed of this one by $\psi_3 = G\psi_2$, with $G$ satisfying the discrete
equation $G/G = \sqrt{1 - 2h^2u}$.

2. The term $u'/u$ of operator $L$ for (P3) is discretized as the quotient of discr$(xuu')$
by discr$(xu^2)$, a result not easy to guess in advance.

3. The number of singular points in the complex plane of the spectral parameter
is not necessarily the same for the matrix $M(t)$ and for its discretized $B(z)$.

For (P2), $M$ has two singular points $t = 0, \infty$ while $B$ has four such points
$z = 0, \infty, 1, -1$.

One similarly obtains a Lax pair for the discrete Weierstrass equation without
complementary terms (29)

\[
t = (1 - \lambda^2)/(2h^2),
A = \begin{pmatrix} \lambda & 2h(u - t) \\ \hbar & \lambda \end{pmatrix},
B = 2\lambda((u - u)/h)\sigma_3 + (-16t^2 + 4t(u + u) + g_2 - 4uu)\sigma^+
+ (2(u + u) + 8t)\sigma^-
\]

\[
h^{-1}K = 2\sigma_3 E
\]

and one for the discrete Jacobi equation (30)

\[
t = (\lambda - 1/\lambda)/(2h),
A = \begin{pmatrix} 1/\lambda & hu \\ hu & \lambda \end{pmatrix},
B = (-2uu + 4t^2 - a - b)\sigma_3 + (\lambda + 1/\lambda)((u - u)/h)(i\sigma_2)
-2t(u + u)\sigma_1
\]

\[
h^{-1}K = 2i\sigma_2 E.
\]

These last two Lax pairs do not depend on the spectral parameter $z$, and $\lambda$ is
an arbitrary constant.

7 More on discrete Lax pairs

Some discrete equations have complementary terms which do not contribute to
the continuum limit, such as the discrete Weierstrass equation with its terms $\psi^{-2}(h)$,
or the following d–(P1) (Ref. [18] eq. (2.8)), which only differs from the d–(P1) [1]
by terms homogeneous to $h^2 xu$ and $h^2 u^3$

\[
E \equiv -(\bar{u} - 2u + u)/\hbar^2 + 2(\bar{u} + u + u)u + x
- h^2 xu - h^2 u^2(\bar{u} + u + u) = 0.
\]
A Lax pair has already been obtained [18], not for this d–(P1) exactly, but only for the discrete derivative of a birational transform of it, which makes it not simple at all. Let us obtain the natural Lax pair.

One assumes the same qualitative form (47) than for the d–(P1) without complementary terms, and one adds to \(A\) and \(B\) as many matrices of order at least \(h\) as there exist divisors of these complementary terms, each matrix being the product of such a divisor by a matrix of free functions of \(z\). The result for (64) is

\[
t = \frac{(1-z)}{h^2}, \quad z = 1 - h^2t,
\]

\[
A = \begin{pmatrix} \frac{z}{(h/2)(z+1-h^2u)} & \frac{(2/h)(z-1+h^2u)}{z} \\ \end{pmatrix},
\]

\[
-h^2zB = 2z((u - \bar{u})/h)\sigma_3 - h(2u\bar{u} + x - h/2) \begin{pmatrix} 0 & 2/h \\ h/2 & 0 \\ \end{pmatrix}
\]

\[
+ (u + \bar{u}) \begin{pmatrix} 0 & 4t \\ 1 + z & 0 \\ \end{pmatrix} - 2t(1+z)/z \begin{pmatrix} 0 & 4t \\ -1 - z & 0 \\ \end{pmatrix}
\]

\[
h^{-1}K = 2\sigma_3E.
\]

This Lax pair admits by construction (39)–(40) as its continuum limit.

One similarly obtains a Lax pair for the discrete Weierstrass equation with complementary terms (26)

\[
t = \frac{(1-\lambda)}{H^2}, \quad H^{-2} = \varphi(h),
\]

\[
A = \begin{pmatrix} \frac{\lambda}{(H/2)(\lambda + 1 - H^2u)} & \frac{(2/H)(\lambda - 1 + H^2u)}{\lambda} \\ \end{pmatrix},
\]

\[
B = 2\lambda((u - \bar{u})/H)\sigma_3 - H(2u\bar{u} - g_2 - 2 - H^2g_3) \begin{pmatrix} 0 & 2/H \\ H/2 & 0 \\ \end{pmatrix}
\]

\[
+ (u + \bar{u}) \begin{pmatrix} 0 & 4t \\ 1 + \lambda & 0 \\ \end{pmatrix} - 2t((1 + \lambda)/\lambda - (H^4g_2 + H^6g_3)/4) \begin{pmatrix} 0 & 4t \\ -1 - \lambda & 0 \\ \end{pmatrix}
\]

\[
H^{-1}K = 2\sigma_3E.
\]

8 The discrete Painlevé test

In the continuous case, all the methods of the Painlevé test, without exception, are based on two theorems and only two, namely the existence theorem of Cauchy and the theorem of perturbations of Poincaré [43]. This is explained in detail in the lecture notes of a Chamonix school [3], an updated version of which is in preparation [11].

For discrete equations, this is also the case for all methods but one, the singularity confinement method, which really seems outside the scope of the theorem of Poincaré.

Consider an arbitrary discrete equation (P1), also depending on some parameters \(a\), and let \((x, h, u, a) \rightarrow (X, H, U, A, \varepsilon)\) be an arbitrary perturbation admissible by the theorem of Poincaré (which excludes any nonanalyticity, like \(\varepsilon^{1/5}\), for the perturbed variables). A necessary condition is that the limit \(\varepsilon = 0\) possesses the PP (discrete or continuous, this does not matter).

To illustrate the different methods, let us discretize the equation (P1) by a second order scheme, using the rules previously stated

\[
E \equiv -(\bar{u} - 2u + \bar{u})h^{-2} + 3\lambda_1(\bar{u} - 2u + \bar{u})u + 6\lambda_2u^2 + 6\lambda_3\bar{u}u + g = 0.
\]
with $\sum \lambda_k = 1$, and $g$ an unspecified function of $x$.

The test will generate necessary conditions on $(\lambda_k, g)$. One must find at least the following solution, where the equation has a Lax pair: $g = x$ and $\lambda = (2/3, 1/3, 0)$; one may also find the following solutions, isolated by the singularity confinement method: $g = x$ and $\lambda = (1, 0, 0)$ (Ref. [24]), $\lambda = (1/2, 1/4, 1/4)$ (Ref. [17], eq. (5.5)).

8.1 Singularity confinement method

If the field $u$ admits a pole at some point $x_0$ in the complex plane

$$u(x) \sim u_0 \chi^p, \quad \chi = x - x_0 \to 0, \quad u_0 \neq 0, \quad -p \in \mathcal{N},$$

it is generically regular at any point $x_0 + x_1$ where $x_1$ is not infinitesimal

$$\forall x_1, \quad |x_1| >> 0 : u(x_0 + x_1) \neq \infty.$$ (71)

When $u$ satisfies a discrete equation of order $N$, the implementation of this “confinement condition” consists in requiring the property (71) for $N + 1$ consecutive iterates, which generically ensures the property for the next iterates. The polar behaviour is then only sensitive during a finite number of iterations.

8.2 Method of perturbation of the continuum limit

Defined by an expansion of $u$ as a Taylor series in the lattice stepsize $\varepsilon$

$$x \text{ unchanged, } h = \varepsilon, \quad q = e^\varepsilon, \quad u = \sum_{n=0}^{+\infty} \varepsilon^n u^{(n)}, \quad a = \text{ analytic } (A, \varepsilon),$$ (72)

this perturbation generates an infinite sequence of differential equations $E^{(n)} = 0$

$$E = \sum_{n=0}^{+\infty} \varepsilon^n E^{(n)}$$

$$E^{(n)}(x, u^{(0)}, \ldots, u^{(n)}) \equiv E^{(0)}(x, u^{(0)'}) u^{(n)} + R^{(n)}(x, u^{(0)}, \ldots, u^{(n-1)}) = 0, \quad n \geq 1,$$ (74)

whose first one $n = 0$ is the “continuum limit”. The next ones $n \geq 1$, which are linear inhomogeneous, have the same homogeneous part $E^{(0)} u^{(n)} = 0$ independent of $n$, defined by the derivative of the equation of the continuum limit, while their inhomogeneous part $R^{(n)}$ (“right-hand side”) comes at the same time from the nonlinearities and the discretization.

This perturbation of the continuum limit is entirely analogous to the perturbative method of the continuous case, either in its Fuchsian version [12] or in its non-Fuchsian one [33], depending on the nature, Fuchsian or non-Fuchsian, of the linearized equation $E^{(1)} = 0$ at a singular point of $u^{(0)}$.

The simplicity of the method is best seen on the Euler scheme for the Bernoulli equation [13]

$$E \equiv (u(x + h) - u(x))/h + u(x)^2 = 0,$$ (75)

i.e. the logistic map of Verhulst, a paradigm of chaotic behaviour which should therefore fail the test. Let us expand the terms of (73) according to the perturbation
up to an order in $\varepsilon$ sufficient to build the first equation $E^{(1)} = 0$ beyond the continuum limit $E^{(0)} = 0$

$$u = u^{(0)} + u^{(1)}\varepsilon + O(\varepsilon^2)$$  \hfill (76)

$$u^2 = u^{(0)^2} + 2u^{(0)}u^{(1)}\varepsilon + O(\varepsilon^2)$$  \hfill (77)

$$u(x + h) = u(x) + u'(x)h + (1/2)u''(x)h^2 + O(h^3)$$  \hfill (78)

$$\frac{u(x + h) - u(x)}{h} = u^{(0)'} + (u^{(1)})' + (1/2)u^{(0)''}\varepsilon + O(\varepsilon^2).$$  \hfill (79)

The equations of orders $n = 0$ and $n = 1$

$$E^{(0)} = u^{(0)'} + u^{(0)^2} = 0$$  \hfill (80)

$$E^{(1)} = E^{(0)'}u^{(1)} + (1/2)u^{(0)''} = 0, \ E^{(0)'} = \partial_x + 2u^{(0)}.$$  \hfill (81)

have the general solution

$$u^{(0)} = \chi^{-1}, \chi = x - x_0, \ x_0 \text{ arbitrary}$$  \hfill (82)

$$u^{(1)} = u^{(1)}(-1)\chi^{-2} - \chi^{-2}\log \psi, \ \psi = x - x_0, \ u^{(1)}(-1) \text{ arbitrary},$$  \hfill (83)

and the movable logarithm proves the instability as soon as order $n = 1$, at the Fuchs index $i = -1$.

Remark. The only restriction on $u^{(0)}$ is not to be what is called a singular solution (not obtainable from the general solution by assigning values to the arbitrary data), i.e. it can be either the general solution (as above) or a particular one, it can also be either global (as above) or local (Laurent series).

The processing of the example (69) \[13\] isolates three values of $\lambda$, with $g = x$.

The first value $\lambda = (2/3, 1/3, 0)$ (case $a = 1$ in Ref. \[24\]) corresponds to the d–(P1) \[1\] with a Lax pair found in quantum gravity, the condition is then sufficient. The second value $(1, 0, 0)$ (case $a = 0$ in Ref. \[24\]) corresponds to a candidate d–(P1) with a second order Lax pair \[13\]. The third value $(1/2, 1/4, 1/4)$ defines an equation equivalent to that for $(1, 0, 0)$ under a discrete birational transformation \[13\] \[47\].

Remark. Following Painlevé \[38\], one should in fact search for the “complete equation”, i.e. for all the admissible nondominant terms which can be added to the candidate d–(P1) \[33\] without destroying the PP. According to the already known difference or $q$–difference d–(P1) candidates (fifteen to our knowledge), the only admissible complementary terms seem to be

$$h^2X \ \text{discr}(u), \ h^2u \ \text{discr}(u^2), \ h^3(u - \bar{u})/(2h),$$

$$h^4X^2, \ h^4X \ \text{discr}(u^2), \ h^4u^2 \ \text{discr}(u^2),$$  \hfill (84)

in which $X$ is $x$ for difference equations or the suitable exponential function of $x$ for $q$–difference equations.

### 8.3 Comparison of the two main methods

The two main methods which define the discrete Painlevé test, namely the singularity confinement method and the perturbation of the continuum, happen to find the same necessary conditions when applied to a sample of equations : the candidate d–(P1) \[33\], a candidate discrete Chazy equation of class III not yet fully integrated \[33\], discrete nonlinear Schrödinger equations or various discrete Korteweg-de Vries equations \[32\]. Other examples are currently under investigation to detect situations where the two methods would produce complementary, not identical results.
Since it only relies on the existence of a continuum limit, our method can be extended without difficulty to equations in an arbitrary number of dependent or independent variables, whether the equations be discrete or mixed continuous and discrete.

The singularity confinement method has also been extended to such situations [50, 51]; however, in the case of \( m \) discrete independent variables, one must check in addition that the result of the iteration is independent of the path followed on the \( m - \)th dimensional lattice.

In the case of equations of second degree and higher, our method is unchanged since, again, it relies on the continuum limit, for which this technical question is settled. In such a case, the confinement method must make at each step a coherent choice of determination then compute the confinement condition.

9 Towards the discrete Painlevé and Gambier equations

The task of finding discrete analogues of the fifty canonical equations of Gambier d–(Gn) is, at present time, far from being achieved. Let us give here a brief summary of the situation and a few lines of conduct to improve it.

In the continuous case, the PP has been proved either by explicitly linearizing, or by integrating with elliptic functions, or by proving the irreducibility and the absence of movable critical singularities. This is equivalent to either linearize or find a Lax pair.

In the discrete case, after having performed the discrete Painlevé test in order to isolate candidates d–(Gn), one must do the same: either linearize or find a discrete Lax pair.

The only sure informations at our disposal are: the fifty continuous (Gn) equations of course, the exact discrete elliptic equations.

To be precise, according to the conjecture of Section 3, one should look for d–(Gn) equations satisfying the following criteria.

1. Each d–(Gn) passes the discrete Painlevé test.
2. Each d–(Gn) has, like (Gn), order two and degree one.
3. Each d–(Gn) must either be explicitly linearizable, or possess a first integral identical to an elliptic equation, or possess a matricial Lax pair \((A, B, z)\) admitting as continuum limit a Lax pair \((L, M, t)\) of (Gn). This implies an order two for these discrete Lax pairs.
4. The d–(Pn) equations satisfy a confluence cascade admitting as continuum limit the cascade of Painlevé and Gambier [40, 20] down to the Weierstrass level included, see formulae below.

As an example of some difficulties, consider the (G27) equation in the particular case of Ermakov and Pinney

\[-uu'' + (1/2)u'^2 + f(x)u^2 - c^2/2 = 0, \quad c \neq 0, \quad c \text{ constant},\]  

\[85\]

which is linearizable either by derivation

\[-u''' + 2fu' + f'u = 0, \]  

\[86\]

or by the singular part transformation

\[u^{-1} = c(\Log(\psi_2/\psi_1))', \quad \psi_k'' - (f/2)\psi_k = 0, \quad k = 1, 2.\]  

\[87\]
Three discrete candidates for \( \mathcal{M} \) have been proposed, the first one of degree one \[22\]
\[-u(\mp - 2u + \mp)h^{-2} + (1/2)(\mp - u)(u - \mp)h^{-2} + f(\mp + u)(u + \mp)/4 - c^2/2 = (88)\]
the second one also of degree one \[23\]
\[-u(\mp - 2u + \mp)h^{-2} + (1/2)(\mp - u)(u - \mp)h^{-2} + f\mp\mp - c^2/2 + (h/2)c(\mp - 2u + \mp)h^{-2} = 0, \quad (89)\]
the third one of degree two \[3, 10\]
\[(\mp - u - (2 + \h f/2)^2)u^2 - (2 + \h f/2)^2(4\mp\mp + \h^2c^2) / (8\h^2) = 0. \quad (90)\]

The third one has been linearized by discrete analogues of both transformations \( \mathcal{N} \) and \( \mathcal{M} \) \[10\] but it has degree two. The first one has been linearized by a discrete analogue of \( \mathcal{N} \) only, but it has two nice features: it obeys the rules and has no complementary terms. The second one, to our knowledge, has not yet been integrated.

Before proceeding, let us recall for later use the definition of \( \mathcal{P} \)
\[u'' = \frac{u'^2}{2u} + \frac{3}{2}u^3 + 4ux^2 + 2(x^2 - \alpha)u + \beta \quad (91)\]
and the restriction to the subset \( (\mathcal{P}_m)(x, u, \alpha, \beta, \gamma, \delta) \) of the confine cascade from \( (\mathcal{P}_1)(x, u, \alpha, \beta, \gamma, \delta) \) to \( (\mathcal{P}_m)(X, U, A, B, C, D) \), \( m < n \),
\[
4 \to 2: \quad (x, u, \alpha, \beta) = (\varepsilon X - \varepsilon^{-3}/4, \varepsilon^{-3}/4 + \varepsilon^{-1}U, -\varepsilon^{-6}/32 - 2A, -\varepsilon^{-12}/512) \quad (92)\\
2 \to 1: \quad (x, u, \alpha) = (-6\varepsilon^{-10} + \varepsilon^2X, \varepsilon^{-5} + \varepsilon U, 4\varepsilon^{-15}) \quad (93)\\
1 \to \varnothing: \quad (x, u) = (-\varepsilon^{-4}g_2/2 + \varepsilon X, \varepsilon^{-2}U) \quad (94)\\
\]
with \( \varepsilon \to 0 \).

Let us from now on restrict to discretizations without complementary terms, such as, for \( \mathcal{P}_1 \), equation \[23\] but not equation \[24\].

For \( \mathcal{P}_1 \), three candidates \[23\] pass the test:
\[
\begin{align*}
\lambda \mp &= (2/3, 1/3, 0), (1, 0, 0), (1/2, 1/4, 1/4). \text{ Among these, the first one is the only one to} \\
\text{admit a confluence to the discrete equation of Weierstrass and, since it satisfies} \\
\text{the other above criteria, it definitely can be called “the (unique) \mathcal{P}_1 \text{ without} \\
\text{complementary terms”}}.
\end{align*}
\]

For \( \mathcal{P}_2 \), this is \[3\] the good equation, because of both its Lax pair and the confine from \( \mathcal{P}_2 \)(\( u, x, h, \alpha \)) to \( \mathcal{P}_1(U, X, H) \) \[48\]; this confine is better written as
\[
(x, h, u, \alpha) = (\lambda^2(X - 6\varepsilon^{-12}), \lambda^2 H, \lambda(\varepsilon^{-6} + U), 4\lambda^3\varepsilon^{-18}), \quad \lambda^{-6} = \varepsilon^{-6} + H^2\varepsilon^{-12}, \quad (95)
\]
which proves that its continuum limit is the continuous confine \[93\].

For the four other equations \( \mathcal{P}_3 \), \( \mathcal{P}_4 \), \( \mathcal{P}_5 \), \( \mathcal{P}_6 \), there is not yet a fully satisfactory result, i.e. for each of them at least one of the enumerated points is not satisfied.

Two discrete \( \mathcal{P}_3 \) candidates have been proposed. The \( q-\mathcal{P}_3 \) candidate \[18\] has a matricial Lax pair \[11\] of order four, not two, without clear continuum limit. The \( d-\mathcal{P}_3 \) candidate, system \( (25a) \) of two equations in two fields in Ref. \[22\], has a fourth order Lax pair \[24, 31\], but it also has a subtle drawback. Indeed, after elimination of the second field, the discrete equation has degree two, although its
The continuum limit (P3) has degree one. The second order Lax pair (58) is a starting point to remedy this situation.

The d–(P4) candidate

\[
E \equiv -u(\pi - 2u + w)h^{-2} + (1/2)(\pi - w)(u - w)h^{-2} \\
+ u^2(w\pi + wu + uw)/2 + (xu + x^2/2)(w + u)(u + w) - 2\alpha u^2 + \beta = 0
\]  

admits a confluence to the d–(P2) [18], and one can even check that this confluence is independent of the stepsize and given by (92). Although this is evidently the good d–(P4), its Lax pair is still unknown.

No d–(P5) candidate is known. The q–(P5) candidate [18] has the correct degree (one), it admits confluenes [18] to both the q–(P3) candidate and the d–(P2), but its Lax pair is unknown.

Two discrete (P6) candidates have also been proposed. Both the q–(P6) candidate [30] and the d–(P6) candidate [49] are defined as a system of two equations in two fields and, just like for the d–(P3) candidate, the discrete equation in a single field has second order but also second degree (as remarked in Ref. [30]), although its continuum limit is (P6) itself. The d–(P6) candidate has not yet a Lax pair, but the q–(P6) candidate has one by construction, since Jimbo and Sakai started in fact from a q–difference isomonodromic deformation problem in order to obtain their system.

10 Conclusion

These precise definitions and guidelines may render more systematic the search for discrete analogues of differential equations integrable in the sense of Painlevé.

The most fundamental anchor point consists of the discrete elliptic equations because they are exact.

As to the new perturbative method for the discrete Painlevé test, its full applicability (to any number of independent variables, whether discrete or mixed discrete-continuous) should make it a quite efficient tool to investigate which discretizations of partial differential equations preserve the Painlevé property.

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