Refined estimates of the blow-up profile for a strongly perturbed semilinear wave equations in one space dimension

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LR03ES04, 2092 Tunis, Tunisie

Abstract

We consider in this paper a class of strongly perturbed semilinear wave equations with a non-characteristic point in one space dimension, for general initial data. Working in the framework of similarity variables, in [13] Merle and Zaag constructed an explicit stationary solution of the unperturbed problem and proved an exponential convergence to this family of solutions. If we follow the same strategy under our strongly perturbed equation we just obtain a polynomial convergence which is a rough estimate compared to the one obtained in the unperturbed problem. In order to refine this approximation, we constructed an implicit solution to the perturbed problem which approaches the stationary solutions of the unperturbed problem and we prove the exponential convergence to this prescribed blow-up profile.

Keywords: Wave equation, stationary solutions, Blow-up, One-dimensional case, Perturbations.

MSC 2010 Classification: 35L05, 34K21, 35B44, 35L67, 35B20.

1 Introduction

1.1 Known results and motivation of the problem

In the current work, we consider the following one dimensionel semilinear wave equation:
\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial^2_t u = \partial^2_x u + |u|^{p-1}u + f(u) \\
(u(x,0), \partial_t u(x,0)) = (u_0(x), u_1(x))
\end{array} \right. 
\end{aligned}
\]  

(1.1)

where \( u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}, \ u_0(x) \in H^1_{loc,u}, \ u_1(x) \in L^2_{loc,u} \) and

\[

f(v) = \frac{|v|^{p-1}v}{\log^a(2 + |v|^2)} \quad \text{for all} \quad v \in \mathbb{R} \quad \text{with} \quad a > 1.
\]

(1.2)

The space \( L^2_{loc,u} \) is the set of all \( v \in L^2_{loc} \) such that

\[
\|v\|_{L^2_{loc,u}} \equiv \sup_{\alpha \in \mathbb{R}} \left( \int_{|x-\alpha|<1} |v(x)|^2 dx \right)^{\frac{1}{2}} < +\infty,
\]

the space \( H^1_{loc,u} = \{ v \mid v, |\partial_x v| \in L^2_{loc,u} \} \).

The Cauchy problem of equation (1.1) is wellposed in \( H^1_{loc,u} \times L^2_{loc,u} \). This is followed from the finite speed of propagation and the wellposedness in \( H^1 \times L^2 \), valid whenever \( 1 < p < p_S = 1 + \frac{4}{N-2} \). The existence of blow-up solutions \( u(t) \) of (1.1) follows from ODE techniques or the energy-based blow-up criterion of Levine [32] (see also Levine and Todorova [33] and Todorova [53]). More blow-up results can be found in Caffarelli and Friedman [6], [7], Kichenassamy and Littman [29], [30], Killip, Stovall and Visan [31].

Note that in this paper, we consider a class of perturbations of the idealized equation (when \( f \equiv 0 \)). This is quite meaningful since, physical models are sometimes damped and hardly come with a pure power source term (see Whitham [54]). For more application in general relativity, see Donninger, Shlag and Soffer [11].

If \( u(t) \) is a blow-up solution of (1.1), we define (see for example Alinhac [1] and [2]) \( \Gamma \) as the graph of a function \( x \mapsto T(x) \) such that the domain of definition of \( u \) (also called the maximal influence domain)

\[
D_u = \{ (x,t) | t < T(x) \}.
\]

Moreover, from the finite speed of propagation, \( T \) is a 1-Lipschitz function. Let us first introduce the following non degeneracy condition for \( \Gamma \). If we introduce for all \( x \in \mathbb{R}^N, \ t \leq T(x) \) and \( \delta > 0 \), the cone

\[
C_{x,t,\delta} = \{ (\xi, \tau) \neq (x,t) | 0 \leq \tau \leq t - \delta |\xi - x| \},
\]

(1.3)

then our non degeneracy condition is the following: \( x_0 \) is a non-characteristic point if

\[
\exists \ \delta_0 = \delta_0(x_0) \in (0, 1) \text{ such that } u \text{ is defined on } C_{x_0,T(x_0),\delta_0}.
\]

(1.4)

If condition (1.4) is not true, then we call \( x_0 \) a characteristic point. Already, we know from [10] and [42] that there exist blowup solutions for characteristic points. In our paper we are concerned to the non-characteristic case. We note by \( \mathcal{R} \) the set of non-characteristic points. In [13] and [14] Merle and Zaag have established the following:
• The set of non-characteristic points $\mathcal{R}$ is non empty and open.

• The function $T(x)$ is $C^1$ on $\mathcal{R}$ and for all $x_0 \in \mathbb{R}$, $T'(x_0) = d(x_0) \in (-1, 1)$.

Practically, we define for all $x_0 \in \mathbb{R}$, $0 < T_0 \leq T_0(x_0)$, the following self-similar transformation introduced in Antonini and Merle [3] and used in [22], [23], [39], [41], [40] and [20]:

$$y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t) \text{ and } w_{x_0, T_0}(y, s) = (T_0 - t)^\frac{2}{p-1} u(x, t). \quad (1.5)$$

The function $w_{x_0, T_0}$ (we write $w$ for simplicity) satisfies the following equation for all $y \in (-1, 1)$ and $s \geq -\log(T_0)$:

$$\partial_s^2 w = \mathcal{L}w - \frac{2(p + 1)}{(p - 1)^2} w + |w|^{p-1} w - \frac{p + 3}{p - 1} \partial_s w - 2y \partial_y^2 w + e^{-2ps} f(e^{\frac{2s}{p-1}}w) \quad (1.6)$$

where

$$\mathcal{L}w = \frac{1}{\rho} \partial_y (\rho (1 - y^2) \partial_y w) \quad \text{and} \quad \rho(y) = (1 - y^2)^\frac{2}{p-1}. \quad (1.7)$$

In the new set of variables $(y, s)$, the behavior of $u$ as $t \to T_0$ is equivalent to the behavior of $w$ as $s \to +\infty$. The equation (1.6) will be studied in the space $\mathcal{H}$ defined by

$$\mathcal{H} = \{ q = (q_1, q_2) | \int_{-1}^{1} \left( q_2^2 + (\partial_y q_1)^2 (1 - y^2) + q_1^2 \right) \rho dy < +\infty \}. \quad (1.8)$$

Note that, in one dimension case the energy space is equal to $\mathcal{H}_0 \times \mathcal{L}_\rho^2$, where

$$\mathcal{H}_0 = \{ r \in H^1_{\text{loc}}(-1, 1) | \| r \|_{\mathcal{H}_0}^2 = \int_{-1}^{1} \left( (\partial_y r)^2 (1 - y^2) + r^2 \right) \rho dy < +\infty \} \quad (1.9)$$

and $\mathcal{L}_\rho^2$ is the weighted $L^2$ space associated with the weight $\rho$ defined in (1.7). In the whole paper we denote

$$F(u) = \int_{0}^{u} f(v) dv. \quad (1.10)$$

Let us expose now some important properties and identities for our paper proved in some earlier works [39], [41], [40]. We start by recalling that

$$E_0(w(s)) = \int_{-1}^{1} \left( \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1 - y^2) + \frac{p + 1}{(p - 1)^2} w^2 - \frac{|w|^{p+1}}{p+1} \right) \rho dy \quad (1.11)$$

is a Lyapunov functional of equation (1.6) when $f \equiv 0$ which is defined in $\mathcal{H}$.

Then, we introduce the following functional:

$$E(w(s), s) = E_0(w(s)) - e^{-\frac{2(\rho+1)s}{p}} \int_{-1}^{1} F(e^{\frac{2s}{p-1}}w) \rho dy - \frac{1}{s^{\frac{2p}{p-1}}} \int_{-1}^{1} w \partial_sw \rho dy. \quad (1.12)$$
In our case and more generally under the assumptions

$$(H_f) \quad |f(v)| \leq M \left(1 + \frac{|v|^p}{\log^a(2 + v^2)}\right), \quad \text{for all } v \in \mathbb{R} \quad \text{with } (M > 0, \ a > 1)$$

and

$$(H_g) \quad |g(x, t, v, z)| \leq M(1 + |z|), \quad \text{for all } x, v \in \mathbb{R}^N, \ z \in \mathbb{R} \quad \text{with } (M > 0),$$

with $f$ and $g$ are perturbed terms added to equation (1.1), we have proved in [20], [21] and [19] that

$$H(w(s), s) = \exp \left(\frac{p + 3}{(a - 1)s^2 - 1}\right) E(w(s), s) + \theta e^{-\frac{(p+1)a}{p-1}},$$

(for a large constant $\theta$) is a Lyapunov functional of equation (1.6) which is defined in $\mathcal{H}$. Based on this functional and some energy estimates, we have established when $x_0$ is a non-characteristic point (in the sense (1.4)) that there exists $\hat{s}_0 = \hat{s}_0(x_0, p, a, T_0(x_0))$ such that for all $s \geq \hat{s}_0$, the following estimate holds:

$$0 < \varepsilon_0 \leq ||w_{x_0, T_0(x_0)}(s)||_{H^1(-1, 1)} + ||\partial_s w_{x_0, T_0(x_0)}(s)||_{L^2(-1, 1)} \leq M_1. \quad (1.13)$$

**Remark 1.1.** We present now some comments on the parameter $a$ related to the blow-up rate and the blow-up limit. Under the assumptions $(H_f)$ and $(H_g)$ and following our earlier work [20], we have proved the result (1.13) when the exponent $p$ is subconformal, i.e. $(1 < p < 1 + \frac{4}{N-2})$, when $N \geq 2$ for $a > 1$ and in [21] for $a > 2$ when the exponent $p$ is conformal, i.e. $(p \equiv 1 + \frac{4}{N-1})$. The method used in [20] breaks down when $a \in (0, 1]$, within some analysis we find an equality of type $\frac{d}{ds}(E(w(s), s)) \leq \frac{C}{s^a} E(w(s), s)$ with $E(w(s), s)$ is defined in (1.12) and this is a major reason preventing us from deriving the result in the case $a \in (0, 1]$ and explains the restriction of the parameter $a$ to be in $(1, +\infty)$ in this current paper. Except when the perturbed term $f$ satisfies the condition (1.2), the above estimate will be refined into $\frac{d}{ds}(E(w(s), s)) \leq \frac{C}{s^a} E(w(s), s)$ which implies that we can solve the question of the blow-up rate also when $a \in (0, 1]$. In this direction, we mention the work of Nguyen and Zaag [48] where the authors added the same perturbed term $f$ to the semilinear heat equation and they allowed values of $a$ in $(0, 1]$ at the expense of taking the particular form (1.2) of the perturbation $f$ via the derivation of a suitable Lyapunov functional. Going back to the problem of the blow-up limit, in this paper we solve the case when $a > 1$. It is very interesting to answer the question when $a$ in $(0, 1]$. For that purpose we may need some refinement to the polynomial decay and to the asymptotic behavior obtained in Claim 3.1. We felt that the study of this case can be our next challenge.

A natural question then is to know if $w_{x_0}$ has a limit or not, as $s \rightarrow +\infty$ (that is as $t \rightarrow T_0$). In order to make a simpler presentation, we start by the case $f \equiv 0$. This case
was treated by Merle and Zaag \cite{43} where the authors have proved the convergence of the solution \( w_{x_0} \) to the set of stationary solutions \( S \equiv \{ 0, \kappa(d, .), -\kappa(d, .) | |d| < 1 \} \) in one space dimension. In higher dimensions, \( N \geq 2 \) there is no classification of selfsimilar solutions of equation (1.6) when \((f \equiv 0)\). In other words, we already know that \( \kappa(d, \omega, y) \) defined in (1.14) is \( \mathcal{H}_0 \) stationary solution of equation (1.6) when \((f \equiv 0)\) for any \(|d| < 1\) and \( \omega \in \mathbb{R}^N \) with \(|\omega| = 1\), but we are unable to say whether there are other stationary solutions or not. Despite that, in higher dimensions Merle and Zaag extended in \cite{46} the openness of the set of non-characteristic points and regularity of the blow-up curve for the semilinear wave equation in one space dimension to the higher dimensions. In a companion paper \cite{47}, Merle and Zaag studied the dynamic of the solution of equation (1.6) when \((f \equiv 0)\) near explicit stationary solutions in similarity variables and they extended some properties of the one dimension to the higher dimension. The proof of the convergence in higher dimension is far from being a simple adaptation of the one dimensional case. Indeed, several difficulties arise when \( (N - 1) \) new degenerate directions in the linearized operator of equation (1.6) when \( f \equiv 0 \) around the stationary solution \( \kappa(d, y) \) defined in (1.14). These new directions naturally come from the derivative of \( \kappa(d, y) \) with respect to \((N - 1)\) angular directions of \( d \).

As announced above, we want to briefly recall the work of Merle and Zaag \cite{43} in the case when \( f \equiv 0 \) in a clear way. Let us first classify all the \( \mathcal{H}_0 \) stationary solutions of (1.6) when \( f \equiv 0 \) in one dimension. More precisely, we have:

If \( w \in \mathcal{H}_0 \) a stationary solution of (1.6) when \( f \equiv 0 \), then, either \( w = 0 \) or there exist \( d \in (-1, 1) \) and \( \omega \in \{-1, 1\} \) such that \( w(y) = \omega \kappa(d, y) \), where

\[
\forall (d, y) \in (-1, 1)^2 \quad \kappa(d, y) = \kappa_0 \left( \frac{1 - d^2}{1 + dy} \right)^{\frac{1}{p-1}} \quad \text{with} \quad \kappa_0 = \left( \frac{2(p + 1)}{(p - 1)^2} \right)^{\frac{1}{p-1}}. \tag{1.14}
\]

In a first time, (when \( f \equiv 0 \)) Merle and Zaag in \cite{43} show that \( w_{x_0}(y, s) \) defined in (1.5) strongly converges in \( H^1 \times L^2((-1, 1)) \) to a non-null connected component of the stationary solutions. More precisely, they have proved the following:

**Proposition. (Strong convergence related to the set of approximate stationary solution).** Consider \( w \in C([s^*, \infty), \mathcal{H}) \) for some \( s^* \in \mathbb{R} \) a solution of equation (1.6) (when \( f \equiv 0 \)). If \( x_0 \) is a non-characteristic point (in the sense (1.4)), then there exists \( \omega^*(x_0) \in \{-1, 1\} \) such that:

\[
\inf_{|d| < 1} \| w_{x_0}(., s) - \omega^*(x_0) \kappa(., .) \|_{H^1(-1, 1)} + \| \partial_s w_{x_0}(., s) \|_{L^2(-1, 1)} \longrightarrow 0 \quad \text{as} \quad s \longrightarrow \infty.
\]

**Remark 1.2.** We would like to emphasize that this proposition holds also when \( w_{x_0}(y, s) \) defined in (1.5) is a solution of (1.6) in the perturbed case. In other words, this proposition remains unchanged even under the strong perturbation \( f \) defined in (1.2) and can be generalized with the same arguments used in \cite{43} without any difficulty. Since the perturbed term \( f \) is polynomially small, it can be ignored in the proof.
Then, Merle and Zaag [43] derive the following result:

**Theorem. (Trapping near the set of non zero stationary solutions of (1.6) when \( f \equiv 0 \)).** There exist positive constants \( \epsilon_0, \mu_0 \) and \( C \) such that if \( w \in C([s^*, \infty), \mathcal{H}) \), for some \( s^* \in \mathbb{R} \) is a solution of equation (1.6) (when \( f \equiv 0 \)) such that:

\[
\forall \ s \geq s^* \ E_0(w(s)) \geq E_0(\kappa_0),
\]

and

\[
\left\| \left( \begin{array}{c} w(s^*) \\
\partial_s w(s^*) \end{array} \right) - \omega^* \left( \begin{array}{c} \kappa(d^*, \cdot) \\
0 \end{array} \right) \right\|_\mathcal{H} \leq \epsilon^*,
\]

for some \( d^* \in (-1, 1) \), \( \omega^* \in \{-1, 1\} \) and \( \epsilon^* \in (0, \epsilon_0] \), where \( \mathcal{H} \) and its norm are defined in (1.8) and \( \kappa(d, y) \) is defined in (1.14), then there exists \( d^*_\infty \in (-1, 1) \) such that

\[
|d^*_\infty - d^*| \leq C\epsilon^*(1 - d^*),
\]

and for all \( s \geq s^*, \)

\[
\left\| \left( \begin{array}{c} w(s) \\
\partial_s w(s) \end{array} \right) - \omega^* \left( \begin{array}{c} \kappa(d^*_\infty, \cdot) \\
0 \end{array} \right) \right\|_\mathcal{H} \leq C\epsilon^* e^{-\mu_0(s-s^*)}.
\]

The proof of (1.17) is performed in the framework of similarity variables defined in (1.5). To prove this result, Merle and Zaag [43], among other techniques, linearized (1.6) when \( f \equiv 0 \) around the solution \( \kappa(d, y) \) and got an exponential decay. Similarly to the problem of the blow-up rate, one can ask whether the result proved by Merle and Zaag in [43] holds also for the strongly perturbed semilinear wave equation. Before studying our class of perturbation we should recall that, in [24], Hamza and Zaag have proved a similar result as the one obtained in the case when \( f \equiv 0 \) under the assumption \((H_g)\) and the more restrictive assumption:

\[
|f(v)| \leq M(1 + |v|^q), \text{ for all } v \in \mathbb{R} \text{ with } (M > 0, \ q < p).
\]

As we said earlier, in this paper we are concerned with the non-characteristic case and the function \( f \) is defined in (1.2). Unlike the work of Merle and Zaag [43] and also Hamza and Zaag [24], we focus here on the assumption (1.2) which is much more complicated, so we need to invent a new idea in order to obtain the exponential decay. In fact, if we do the same as in [43] and [24] we may obtain some terms like \( \frac{1}{s^3} \) coming from the strong perturbation \( f \) defined in (1.2) and we may not able to control these terms. More precisely, we obtain the following

**Proposition 1.** Let \( x_0 \in \mathcal{R} \), there exist positive constants \( \epsilon_0, \mu \) and \( C \) such that if \( w \in C([s^*, \infty), \mathcal{H}) \), for some \( s^* \in \mathbb{R} \), is a solution of equation (1.6) such that:

\[
\forall \ s \geq s^* \ E(w(s)) \geq E_0(\kappa_0) - \frac{C}{s^{3/2}}.
\]
and
\[
\left\| \left( \begin{array}{c} w(s^*) \\ \partial_s w(s^*) \end{array} \right) - \omega^* \left( \begin{array}{c} \kappa(d^*, \cdot) \\ 0 \end{array} \right) \right\|_\mathcal{H} \leq \epsilon^*,
\]
for some \( d^* \in (-1, 1) \), \( \omega^* \in \{-1, 1\} \) and \( \epsilon^* \in (0, \epsilon_0] \), where \( \mathcal{H} \) and its norm are defined in (1.8) and \( \kappa(d, y) \) is defined in (1.14), then there exists \( d_\infty \in (-1, 1) \) such that
\[
|d_\infty - d^*| \leq C\epsilon^*(1 - d^2),
\]
and for all \( s \geq s^* \),
\[
\left\| \left( \begin{array}{c} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{array} \right) - \omega^* \left( \begin{array}{c} \kappa(d_\infty, \cdot) \\ 0 \end{array} \right) \right\|_\mathcal{H} \leq \frac{C}{s^\mu},
\]
(1.20)

**Remark 1.3.** Proposition 1 may be regarded as a rough version of our essential goal which is written for \( f \) defined in (1.2). In order to avoid unnecessary repetition, we kindly refer the reader to [43] for all the projections of the terms in (2.65) not involving \( f \) and we will explain briefly in this remark the terms with \( f \). Remarking that \( \kappa(d, y) \) is not a solution of (1.6), we can see that if we linearize (1.6) around \( \kappa(d, y) \), we get a remaining term of type \( |e^{-\frac{2p}{d}}f(e^{\frac{2p}{d}}(\kappa(d, y) + q_1))| \) (with \( q_1 \) will be defined later in (2.13)) provided from \( f \). If we control the projection of this remaining term we can see that the presence of the additional term \( \frac{C}{s^\mu} \) is natural which make the proof of this proposition very easy.

Our aim in this paper is to obtain a sharp estimate for the prescribed blow-up profile, more precisely an exponential convergence like the one obtained in the case when \( f \equiv 0 \), which is more advantageous than the polynomial decay. From the above remark the remaining term would act as a forcing term, preventing us from obtaining the exponential decay found in Merle and Zaag [43] in the case when \( f \equiv 0 \). In order to overcome this difficulty, instead of linearizing around the explicit function \( \kappa(d, y) \), which is not a solution of equation (1.6), (it is just an approximate solution) we will linearize around a new implicit profile function which happens to be an exact solution of equation (1.6) defined by:
\[
\overline{w}_1(d, y, s) = \kappa(d, y) \frac{\phi(s - \log(\frac{1 + dy}{\sqrt{1 - d^2}}))}{\kappa_0},
\]
with \( \phi \) is a solution of the associated ODE to the PDE (1.6). Moreover, we introduce
\[
\overline{w}(d, y, s) = \left( \begin{array}{c} \overline{w}_1(d, y, s) \\ \overline{w}_2(d, y, s) \end{array} \right) \text{ where } \overline{w}_2(d, y, s) = \partial_s \overline{w}_1(d, y, s).
\]
(1.22)

The most important property of our new solution is that
\[
\phi(s) - \kappa_0 \sim -\frac{\kappa_0}{p - 1} \left( \frac{p - 1}{4s} \right)^a \text{ as } s \to +\infty.
\]
(1.23)
This property is crucial in many steps in this paper. We will see later in Appendix B the proof of the equivalence (1.23) as well as some complementary results and in Appendix A we will see the details of the construction of \( w(d, y, s) \) defined in (1.21) and (1.22), where we can see this property in a clear way. In addition to that, from the proposition of the case when \( f \equiv 0 \) and the remark after, we can see also that \( w_{x_0}(y, s) \), defined in (1.5) is a solution of (1.6) in the perturbed case approaches \( w \) defined in (1.22) strongly in \( H^1 \times L^2(\mathbb{R}) \) norm. More precisely, we write, when \( x_0 \) is a non-characteristic point (in the sense (1.4)) and \( \omega^* = \omega^*(x_0) \in \{-1, 1\} \), the following:

\[
\inf_{|d| < 1} \| w_{x_0}(., s) - \omega^* w_1(d, ., s) \|_{H^1} + \| \partial_y w_{x_0}(., s) - w_2(d, ., s) \|_{L^2} \longrightarrow 0 \quad \text{as} \quad s \longrightarrow \infty. \quad (1.24)
\]

Our aim in this paper is to prove that \( w_1 \) is the blow-up profile such that if \( w \) is a solution of equation (1.6), we show the exponential convergence of \( w \) to this expected profile.

Before stating our result, we would like to mention that an extensive literature is devoted to the question of the construction of a solution to some PDE. We start by the work of Côte and Zaag [10] for the non-perturbed problem, where the authors for any integer \( k \geq 2 \) and \( \zeta_0 \in \mathbb{R} \) constructed a blow-up solution with a characteristic point \( a \), such that the asymptotic behavior of the solution near \( (a, T(a)) \) shows a decoupled sum of \( k \) solitons with alternative signs, whose centers (in the hyperbolic geometry) have \( \zeta_0 \) as a center of mass, for all times. Moreover, Hamza and Zaag [25] extended the result of Côte and Zaag [10] in the characteristic case where they added a perturbed terms satisfying the hypotheses \((H_f)\) and \((H_g)\) and they prescribed the center of mass of a multi-soliton solution for strongly perturbed semilinear wave equation. That question was investigated by Nguyen and Zaag in [49] where the authors construct an implicit profile function for a strongly perturbed semilinear heat equation and by Bressan in [4] and [5] for the semilinear heat equation with an exponential term source. Also we would like to mention the remarkable result of Ghoul, Nguyen and Zaag in [16] where the authors constructed blow-up solutions for non-variational semilinear parabolic system and the constructed solutions are stable under a small perturbation of initial data. Then in [17] the same authors extend their result to a higher order semilinear parabolic equation. Concerning the question of the construction of solutions to some PDE, we note that Mahmoudi, Nouaili and Zaag in [34] constructed a \( 2\pi \)-periodic solution to the nonlinear heat equation with power nonlinearity in one space dimension which blows up in finite time \( T \) only at one blow-up point. In the same sense Nouaili and Zaag in [50] constructed a solution for the complex nonlinear heat equation which blows up in finite time \( T \) only at a one blow-up point, the same type of solution is constructed for the complex Ginzburg-Landau Equation by Zaag in [55], by Masmoudi and Zaag in [36] and also by Nouaili and Zaag in [51] in the critical case. Willing to be as exhaustive as possible in our bibliography about the question of the construction, we would like to mention that this question was solved for \((\text{gKdV})\) by Côte in [8] and [9], for Shrödinger maps by Merle, Raphaël and Rodnianski in [38], the wave maps by Ghoul, Ibrahim and
Nguyen in [14] and for the Keller-Segel model by Raphaël and Schweyer in [52] and also Ghoul and Masmoudi in [15].

**Remark 1.4.** We believe that the present work can be a good start to understand the dynamics of many PDE’s without explicit solution. We can cite for example the PDE treated by Hamza and Zaag [26] in one space dimension and in [27] and [28] for the higher dimensions, where we don’t have any stationary solution to the semilinear wave equation and heat equation with nonlinearity including a logarithmic factor which is not scale invariant. In our opinion, we can construct an implicit solution adapted to the problem studied by the authors and prove a similar result as the one obtained in the present paper.

**1.2 Main results and strategy of the proof**

Let us give now our main result. After linearizing around $w_1$, which is the major novelty in our approach we derive, with more works, the following theorem:

**Theorem 2.** (Trapping near the set of the family of the implicit profile $w_1$). There exist $S_1 = S_1(x_0, p, a)$, $\epsilon_0$, $\mu_0$ and $C$ such that if $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \geq S_1$ a solution of equation (1.6) such that:

\[
\forall s \geq s^* \quad \text{and} \quad E(w(s), s) = E_0(w_1(d, y, s)) - \frac{C}{s^*},
\]

(1.25)

and

\[
\left\| \left( \frac{w(s^*)}{\partial s w(s^*)} \right) - \omega^* \left( \frac{w_1(d^*, \ldots, s^*)}{w_2(d^*, \ldots, s^*)} \right) \right\|_{\mathcal{H}} \leq \epsilon^*,
\]

(1.26)

for some $d^* \in (-1, 1)$, $\omega^* \in \{-1, 1\}$ and $\epsilon^* \in (0, \epsilon_0]$, where $\mathcal{H}$ and its norm are defined in (1.8), $w_1$ and $w_2$ are defined in (1.21) and (1.22), then there exists $d_\infty \in (-1, 1)$ such that

\[
|d_\infty - d^*| \leq C \epsilon^* (1 - d^*^2)
\]

and for all $s \geq s^*$,

\[
\left\| \left( \frac{w(s)}{\partial s w(s)} \right) - \omega^* \left( \frac{w_1(d_\infty, \ldots, s)}{w_2(d_\infty, \ldots, s)} \right) \right\|_{\mathcal{H}} \leq C \epsilon^* e^{-\mu_0(s - s^*)}.
\]

(1.27)

**Remark 1.5.** Thanks to (1.24), condition (1.26) is also guaranteed and the time $s^*$ is completely explicit and characterized by the fact that:

\[
s^* = \inf_{s \geq -\log(T(x_0))} \inf_{|d| < 1} \left\| \left( \frac{w(s)}{\partial s w(s)} \right) - \omega^* \left( \frac{w_1(d^*, \ldots, s)}{w_2(d^*, \ldots, s)} \right) \right\|_{\mathcal{H}} \leq \epsilon_0.
\]

(For more detail of this fact we can see Corollary 4 and the remark after that corollary in Merle and Zaag [43]).
Remark 1.6. We would like to compare (1.25) with (1.15) the condition imposed by Merle and Zaag in the unperturbed case, we can remark that we have an additional polynomially small term. This comes from the fact that the difference between $E(w(s),s)$ and $E_0(w(s))$ is polynomially small. In addition to that, we can replace $E_0(\overline{w}_1(d,y,s))$ in (1.25) by $E_0(\kappa_0)$ thanks to the monotonicity of $E_0(w(s))$, the fact that $\overline{w}_1(d,y,s) \sim \kappa(d,y)$ as $s \to \infty$ and the fact that $E_0(\kappa(d,y)) = E_0(\kappa_0)$ (for more details about the last equality, the interested reader may consult Merle and Zaag [43]).

Remark 1.7. Our proof under the assumption $(H_g)$ remains valid with exactly the same ideas and purely technical differences, that we omit to keep this paper in reasonable length.

Let us now comment on the method used to prove our results. Noting that the proof of Theorem 2 is far from being a simple adaptation of the case when $f \equiv 0$ treated by Merle and Zaag [43]. Indeed, there are additional difficulties arising from the perturbed term $f$ and the linearization around the new solution $\overline{w}_1$ defined in (1.21) which makes the technical details harder to elaborate. Accordingly, the exponential decay obtained in Theorem 2 requires more works than the case when $f \equiv 0$ and special arguments where we need to invent new idea to get it. However, in a first time, we follow the same strategy of the case when $f \equiv 0$ to get a rough estimate where the condition (1.25) will play an important role to obtain the polynomial decay. Based upon this estimate we can see that $\|q(s)\|_H \to 0$ as $s \to \infty$ with $q$ will be defined later in (2.13). Even better, thanks to the information asserted just before with some further refinement of the result obtained in the polynomial decay we conclude the exponential convergence at the end of this paper and we get our main Theorem 2.

Let us mention briefly the structure of the paper: The article is organized around three main results. The modulation theory which is a major step to obtain the polynomial decay and the exponential decay, we present the modulation theory in Section 2 the polynomial and the exponential decay in Section 3 and each of them in a separate subsection, where we conclude the proof of Theorem 2.

We mention that $C$ depends on $p$, $a$ and $x_0$ will be used in all this paper to denote a positive constant which may vary from line to line.

2 Modulation theory

In this part, we work in the space $H$ defined in (1.8), which is a natural choice (the energy space in $w$). We consider for some $s^*$ defined in Remark 1.5, the function $w \in C([s^*, \infty), H)$ a solution of equation (1.6), where $w$ may be equal to $w_{x_0}$ defined in (1.5) from a blow-up solution to equation (1.1), with no restriction to $x_0$.

In this section, we use modulation theory and introduce a parameter $d(s)$ adapted to the dispersive property of the equation (1.6) whenever (1.26) holds, in order to obtain the polynomial decay in the next section.
In the beginning, we are going to call back some tools introduced in Merle and Zaag [43] in the case when \( f \equiv 0 \). We briefly recall the crucial linear operator \( L_d \) and its spectral properties introduced in Merle and Zaag [43] defined in the energy space \( \mathcal{H} \) by:

\[
L_d \left( \begin{array}{c} q_1(y, s) \\ q_2(y, s) \end{array} \right) = \left( \begin{array}{c} L q_1 + \psi(d, y)q_1 - \frac{p+3}{p-1}q_2 - 2y\partial_y q_2 \\ \end{array} \right),
\]

(2.1)

where

\[
\psi(d, y) = p\kappa(d, y)^{p-1} - \frac{2(p+1)}{(p-1)^2} = \frac{2(p+1)}{(p-1)^2} \left( \frac{p(1-d^2)}{(1+dy)^2} - 1 \right).
\]

(2.2)

The linear operator \( L_d \) will play a fundamental role in our analysis, it is useful to recall some well known results link with the linear operator \( L_d \).

**Spectral properties related to \( L_d \).**

It is well known that \( \lambda = 1 \) and \( \lambda = 0 \) are the positive eigenvalues of the linear operator \( L_d \) and the rest of the eigenvalues are negative. It happens that the corresponding eigenfunctions of the positive eigenvalues are:

\[
F^d_1 = (1-d^2)^{\frac{p-1}{2}} \left( \frac{1+dy}{1+dy} \right)^{-\frac{d}{p-1}} \quad \text{and} \quad F^d_0 = (1-d^2)^{\frac{1}{p-1}} \left( \frac{y+d}{1+dy} \right)^{\frac{1}{p-1}+1}.
\]

(2.3)

Moreover, it holds for some \( C > 0 \) and any \( \lambda \in \{0,1\} \) that

\[
\forall \ |d| < 1, \quad \|F^d_\lambda\|_\mathcal{H} + (1-d^2)\|\partial_d F^d_\lambda\|_\mathcal{H} \leq C.
\]

(2.4)

In order to compute the projectors on the eigenfunctions of \( L_d \), we consider its conjugate with respect to the natural inner product \( \Upsilon \) of \( \mathcal{H} \) defined by:

\[
\Upsilon(q, r) = \int_{-1}^{1} (q_1(-yr_1 + r_1) + q_2r_2)\rho dy.
\]

(2.5)

The computation of \( L^*_d \) the conjugate operator of \( L_d \) with respect to \( \Upsilon \) is simple but lengthy that we omit, (for more details we kindly address the reader to see Lemma 4.1 page 81 in Merle and Zaag [43]). Furthermore, \( L^*_d \) has two nonnegative eigenvalues with eigenfunctions \( W^d_\lambda \) such that:

\[
W^d_{1,2}(y) = c_1 \frac{1-y^2}{(1+dy)^{\frac{p-1}{2}+1}}, \quad W^d_{0,2}(y) = c_0 \frac{y+d}{(1+dy)^{\frac{1}{p-1}+1}},
\]

(2.6)

\( W^d_{\lambda,1} \) is uniquely determined by

\[
-\mathcal{L} r + r = \left( \lambda - \frac{p+3}{p-1} \right) r_2 - 2y\partial_y r_2 + \frac{8}{p-1} \frac{r_2}{1-y^2}.
\]

(2.7)

with \( r_2 = W^d_{\lambda,2} \) and the \( C^1 \) function \( c_\lambda \) fixed by the relation

\[
\Upsilon(W^d_\lambda, F^d_\lambda) = 1.
\]

(2.8)
Finally, we also introduce for \( q \in \mathcal{H} \) and for \( \lambda = 1 \) and \( \lambda = 0 \) the following
\[
\pi_\lambda^d(q) = \Upsilon(W_\lambda^d, q) \quad \text{and} \quad q = \pi_1^d(q)F_1^d(y) + \pi_0^d(q)F_0^d(y) + \pi_-^d(q).
\]

**Remark 2.1.** Let us notice \( \pi_\lambda^d(q) \) is the projection of \( q \) on the eigenfunction of \( L_d \) associated to \( \lambda \) and that \( \pi_\lambda^d(q) \) is the negative part of \( q \) such that:
\[
\pi_-^d(q) \in \mathcal{H}_-^d = \{ r \in \mathcal{H} \mid \pi_1^d(r) = \pi_0^d(r) = 0 \}.
\]

Of course, we have the following orthogonality results:

1. **(Orthogonality)** For all \( |d| < 1 \) and \( \lambda \in \{0, 1\} \), we have \( \Upsilon(W_\lambda^d, F_{1-\lambda}^d) = 0 \).
2. **(Normalization)** There exists \( C > 0 \) such that for \( \lambda \in \{0, 1\} \) and \( |d| < 1 \),
\[
\|W_\lambda^d\|_\mathcal{H} + (1 - d^2)\|\partial_d W_\lambda^d\|_\mathcal{H} \leq C.
\]

After this reminder, we start the modulation theory and we claim the following:

**Proposition 2.2.** (Modulation of \( w \) with respect to \( \overline{w}_1(d, y, s) \)). There exist \( \epsilon_1 > 0 \) and \( K > 0 \) such that if \( (w, \partial_s w) \in \mathcal{C}([s^*, +\infty), \mathcal{H}) \) is a solution to equation (1.6) which satisfies (1.26) for some \( |d^*| < 1 \), \( \omega^* \in \{-1, 1\} \) and \( \epsilon^* \leq \epsilon_1 \), then the following is true:

(i) **(Choice of the modulation parameter)** There exists \( d(s) \in \mathcal{C}^1([s^*, +\infty), (-1, 1)) \) such that for all \( s \in [s^*, +\infty) \),
\[
\pi_0^{d(s)}(q(s)) = 0,
\]
where \( \pi_0^d \) is defined in (2.9), \( q = (q_1, q_2) \) is defined for all \( s \in [s^*, +\infty) \) by
\[
\begin{pmatrix}
  w(y, s) \\
  \partial_s w(y, s)
\end{pmatrix}
= \begin{pmatrix}
  \overline{w}_1(d(s), y, s) \\
  \overline{w}_2(d(s), y, s)
\end{pmatrix} + \begin{pmatrix}
  q_1(y, s) \\
  q_2(y, s)
\end{pmatrix}.
\]

Moreover, \( \|q(s^*)\|_\mathcal{H} \leq K\epsilon^* \).

(ii) **(Equation on \( q \))** For all \( s \in [s^*, +\infty) \)
\[
\frac{\partial}{\partial s} \begin{pmatrix}
  q_1(y, s) \\
  q_2(y, s)
\end{pmatrix}
= \mathcal{L}_{d(s)} \begin{pmatrix}
  q_1(y, s) \\
  q_2(y, s)
\end{pmatrix} - d^* \begin{pmatrix}
  \partial_d \overline{w}_1(d, y, s) \\
  \partial_d \overline{w}_2(d, y, s)
\end{pmatrix}
+ \begin{pmatrix}
  0 \\
  h(d, y, q_1)
\end{pmatrix}
+ \begin{pmatrix}
  0 \\
  \hat{f}(\overline{w}_1, q_1, s)
\end{pmatrix}.
\]
where

\[
\mathcal{T}_{d(s)} \left( \begin{array}{c} q_1 \\ q_2 \end{array} \right) = \left( \mathcal{L} q_1 + (\psi(d, y, s) + e^{-2s} f'(e^{\frac{2s}{p-1}}w_1))q_1 - \frac{p+3}{p-1}q_2 - 2y \partial_y q_2 \right),
\]

(2.16)

\[
h(d, y, q_1) = |\bar{w}_1(d, y, s) + q_1|^{p-1}(\bar{w}_1(d, y, s) + q_1) - \bar{w}_1(d, y, s)^p - p\bar{w}_1(d, y, s)^{p-1}q_1,
\]

(2.17)

\[
\bar{\psi}(d, y, s) = p\bar{w}_1^{p-1} - \frac{2(p + 1)}{(p - 1)^2} = \frac{2(p + 1)}{(p - 1)^2} \left( \frac{p(1 - d^2)\tilde{\phi}^{p-1}(d, y, s)}{(1 + dy)^2} - 1 \right),
\]

(2.18)

with

\[
\tilde{\phi}(d, y, s) = \frac{\phi(s - \log(\frac{1 + dy}{\sqrt{1 - d^2}}))}{\kappa_0}.
\]

(2.19)

We end with the definition of \( \tilde{f} \)

\[
\tilde{f}(\bar{w}_1, q_1, s) = e^{-2s/(p-1)} \left( f(e^{\frac{2s}{p-1}}(\bar{w}_1 + q_1)) - f(e^{\frac{2s}{p-1}}\bar{w}_1) - e^{\frac{2s}{p-1}}q_1 f'(e^{\frac{2s}{p-1}}\bar{w}_1) \right).
\]

(2.20)

**Remark 2.3.** As we said above, we are going to exploit the linear operator \( L_{d(s)} \) defined in (2.1). For that reason we express the linear term \( \mathcal{T}_{d(s)} \) differently:

\[
\mathcal{T}_{d(s)} \left( \begin{array}{c} q_1(y, s) \\ q_2(y, s) \end{array} \right) = L_{d(s)} \left( \begin{array}{c} q_1(y, s) \\ q_2(y, s) \end{array} \right) + \left( \begin{array}{c} 0 \\ \bar{V} q_1 \end{array} \right),
\]

(2.21)

with

\[
\bar{V} = \bar{\psi}(d, y, s) - \psi(d, y) + e^{-2s} f'(e^{\frac{2s}{p-1}}\bar{w}_1)
\]

\[
= pk^{p-1}(d, y) \left( \tilde{\phi}^{p-1}(d, y, s) - 1 \right) + e^{-2s} f'(e^{\frac{2s}{p-1}}\bar{w}_1),
\]

(2.22)

where \( \bar{\psi} \) and \( \psi \) are defined respectively in (2.2) and (2.18). We mention also that:

\[
\left( \begin{array}{c} \partial_d \bar{w}_1(d, y, s) \\ \partial_d \bar{w}_2(d, y, s) \end{array} \right) = \left( \begin{array}{cc} \partial_d \kappa(d, y) & 0 \\ 0 & \partial_d \left( \kappa(d, y)(\bar{\phi}(d, y, s) - 1) \right) \end{array} \right).
\]

(2.23)

This remark helps us to exploit the techniques used by Merle and Zaag [33] in a clear way.

**Proof.** The proof of the case when \( f \equiv 0 \) remains valid for our perturbation with exactly the same ideas and purely technical differences. We present it to convince the reader.

Up to replacing \( w(y, s) \) by \( -w(y, s) \), we can assume that \( \omega^* = 1 \) in (1.26).

(i) In [1.26], we see that there is a parameter \( d^* \in (-1, 1) \) which makes the distance between the solution \( (w(s^*), \partial_s w(s^*)) \) and a particular element \( (\bar{w}_1(d, y, s), \bar{w}_2(d, y, s)) \) small. Now, we would like to sharpen the decomposition and find for all \( s \in [s^*, \sigma^*] \)
for some \( \sigma^* > s^* \) a different parameter \( d(s) \) close to \( d^* \) which not only makes the difference between \( (w(s), \partial_s w(s)) \) and \( (\overline{w}_1(d, y, s), \overline{w}_2(d, y, s)) \) small but also satisfies the orthogonality condition \((2.12)\).

From \((2.9)\), we see that condition \((2.12)\) becomes \( \Phi((w(s), \partial_s w(s)), d, s) = 0 \) where \( \Phi \in \mathcal{C}(\mathcal{H} \times (-1, 1) \times [s^*, +\infty), \mathbb{R}) \) is defined by:

\[
\Phi(v, d, s) = \Upsilon \left( v - (\overline{w}_1(d, y, s), \overline{w}_2(d, y, s)), W_0^d \right)
\]

with \( \Upsilon \) and \( W_0^d \) are given in \((2.5)\) and \((2.6)\). The implicit function theorem allows us to conclude. Indeed,

- Note first that we have:
  \[
  \Phi((w_1(d^*, y, s), w_2(d^*, y, s)), d^*, s) = 0.
  \]  
  \((2.25)\)

- Then, we compute from \((2.24)\), the expression of \( F_0^d \) written in \((2.3)\) and the orthogonality relation \((2.8)\):

\[
D_v \Phi(v, d, s)(u) = \Upsilon(u, W_0^d) \text{ for all } u \in \mathcal{H},
\]

\[
\partial_d \Phi(v, d, s) = -\Upsilon \left( (\partial_d \overline{w}_1(d, y, s), \partial_d \overline{w}_2(d, y, s)), W_0^d \right) + \Upsilon \left( v - (\overline{w}_1(d, y, s), \overline{w}_2(d, y, s)), \partial_d W_0^d \right),
\]

\[
\partial_s \Phi(v, d, s) = -\Upsilon \left( (\partial_s \overline{w}_1(d, y, s), \partial_s \overline{w}_2(d, y, s)), W_0^d \right).
\]

According to the Cauchy-Shwarz inequality, the continuity of \( \Upsilon \) in \( \mathcal{H} \), the bounds \((2.11)\), \((B.22)\), \((B.37)\) and \((B.38)\), we see that if

\[
\left| \log \left( \frac{1 + d}{1 - d} \right) - \log \left( \frac{1 + d^*}{1 - d^*} \right) \right| + \| v - (\overline{w}_1(d^*, y, s), \overline{w}_2(d^*, y, s)) \|_{\mathcal{H}} \leq \epsilon_1,
\]

for some \( \epsilon_1 > 0 \) small enough independant of \( d^* \), then we have

\[
\| D_v \Phi(v, d, s) \| + | \partial_d \Phi(v, d, s) | \leq C \text{ and } \frac{1}{C(1 - d^2)} \leq \partial_d \Phi(v, d, s) \leq \frac{C}{1 - d^2}.
\]

\((2.26)\)

Now, if we introduce \( \Psi \in \mathcal{C}(\mathcal{H} \times \mathbb{R} \times [s^*, +\infty), \mathbb{R}) \) defined by

\[
\Psi(v, \theta, s) = \Phi(v, d, s) \text{ where } d = \tanh \theta,
\]

then, since \( \theta = \frac{1}{2} \log \left( \frac{1 + d}{1 - d} \right) \) and \( \tanh'(\theta) = 1 - \tanh^2(\theta) \), we see from \((2.25)\) and \((2.26)\) that the implicit function theorem applies to \( \Psi \) and we get the existence of
$d(s)$ for all $s \in [s^*, \sigma^*)$ for some $\sigma^* \leq \infty$.

Now, let’s prove that $\sigma^* = +\infty$. We argue by contradiction and assume that $\sigma^* < \infty$, we apply the implicit function theorem around $(v, d) = ((w(s_n), \partial_s w(s_n)), d(s_n))$ where $s_n = \sigma^* - \frac{1}{n}$ and the uniform continuity of $(w(s_n), \partial_s w(s_n))$ from $[\sigma^* - \eta_0, \sigma^* + \eta_0]$ to $\mathcal{H}$ for some $\eta_0 > 0$, we see that for $n$ large enough, we can define $d(s)$ for all $s \in [s_n, s_n + \epsilon_0]$ for some $\epsilon_0 > 0$ independent of $n$. Therefore, for $n$ large enough, $d(s)$ exist beyond $\sigma^*$, which is a contradiction. Thus, $\sigma^* = \infty$ which ends the proof of (i) of Proposition 2.2.

(ii) is a direct consequence of the equation (1.6) satisfied by $w$ put in the vectorial form:

$$\partial_s w = v,$$

$$\partial_s v = \mathcal{L}w - \frac{2(p + 1)}{(p - 1)^2} \partial_s \eta w + |w|^{p-1}w - \frac{p + 3}{p - 1} v - 2y \partial_y v + e^{\frac{2p\sigma}{p-1}} f(e^{\frac{2p\sigma}{p-1}} w),$$

and the fact that $(\eta_1(d, y, s), \eta_2(d, y, s))$ is a solution of (2.27)-(2.28), that is $\eta_1(d, y, s)$ is a solution of

$$\partial_s \eta_2 = \mathcal{L}\eta_1 - \frac{2(p + 1)}{(p - 1)^2} \eta_1 \eta_1 + \eta_1^{p-1} \eta_1 - \frac{p + 3}{p - 1} \eta_1 = -2y \partial_y \eta_1 + e^{\frac{2p\sigma}{p-1}} f(e^{\frac{2p\sigma}{p-1}} \eta_1),$$

(see (1.21)).

Indeed, since we have from (2.13), the definition of $\mathcal{L}$ written in (1.7) and the expression of $h(d, y, q_1)$ written in (2.17)

$$w(y, s) = \eta_1(d, y, s) + q_1(y, s),$$

$$\partial_s w(y, s) = \partial_s \eta_1(d, y, s) + d' \partial_y \eta_1(d, y, s) + \partial_s q_1(y, s),$$

$$\mathcal{L}w(y, s) = \mathcal{L}\eta_1(d, y, s) + \mathcal{L}q_1(y, s),$$

$$|w|^{p-1}w = h(d, y, q_1(y, s)) + \eta_1^{p-1}(d, y, s) + \eta_1^{p-1}(d, y, s)q_1(y, s),$$

$$-2y \partial_y \eta_1 = -2y \partial_y \eta_1(d, y, s) - 2y \partial_y q_2(y, s),$$

$$e^{\frac{2p\sigma}{p-1}} f(e^{\frac{2p\sigma}{p-1}} w(y, s)) = e^{\frac{2p\sigma}{p-1}} f(e^{\frac{2p\sigma}{p-1}} \eta_1(d, y, s) + q_1(y, s))).$$

We see that equation (2.13) follows immediately from (2.27)-(2.29). This conclude the proof of Proposition 2.2.

2.1 Projection on the eigenspaces of the operator $L_d$

For the proof of the main Theorem 2 we need to prove in some sense dispersive estimates on $q_- = \pi^d(q)$ when $q$ is a solution to (2.15). In order to achieve this, we need to
manipulate a function of $\mathbf{q}$ (equivalent to the norm $\|\mathbf{q}\|_{\mathcal{H}} = \Upsilon(\mathbf{q_-}, \mathbf{q_-})^{\frac{1}{2}}$ in $\mathcal{H}^d$) which will capture the dispersive character of the equation (2.15). Such a quantity will be

$$
\varphi_d(q, r) = \int_{-1}^{1} (-\psi(d, y)q_1 r_1 + \partial_y q_1 \partial_y r_1 (1 - y^2) + q_2 r_2) \rho dy
$$

$$
= \int_{-1}^{1} (-q_1 (\mathcal{L} r_1 + \psi(d, y) r_1) + q_2 r_2) \rho dy,
$$

(2.30)

with $\psi(d, y)$ is defined in (2.2).

**Remark 2.4.** We remark that $\varphi_d(q, r)$ is the same bilinear form introduced in Merle and Zaag [43] in the case when $\mathbf{f} \equiv 0$.

It is worth mentioning that this bilinear form $\varphi_d(q, r)$ is in fact the second variation of $E_0(w(s))$ defined in (1.11) around $\kappa(d, y)$ defined in (1.14), which is a stationary solution of (1.6) when $(f \equiv 0)$ and can be seen as the energy norm in $\mathcal{H}^d$ (space where it will be definite positive). We recall briefly from Merle and Zaag [43] in the following the continuity of $\varphi_d(q, r)$:

For all $(q, r) \in \mathcal{H}^2$ and $s \in [s^*, +\infty)$, we have:

$$
|\varphi_d(q, r)| \leq C \|q\|_{\mathcal{H}} \|r\|_{\mathcal{H}}.
$$

(2.31)

As a matter of fact, it is reasonable to recall the following a priori estimate:

$$
\|q\|_{\mathcal{H}} \leq \epsilon,
$$

(2.32)

for some $s \geq s^*$ and some $\epsilon > 0$. In addition to that, we would like to expand $q$ from (2.12) according to the linear operator $\mathcal{L}_d$:

$$
q(y, s) = \alpha_1(s) F^d_1(y) + q_-(y, s),
$$

(2.33)

where

$$
\alpha_1(s) = \pi^d_1(q), \quad \alpha_0(s) = \pi^d_0(q) = 0, \quad \alpha_-(s) = \sqrt{\varphi_d(q_-, q_-)}
$$

(2.34)

and

$$
\mathbf{q}_- = \begin{pmatrix} q_{-, 1} \\ q_{-, 2} \end{pmatrix} = \pi^d(q) = \pi^d_0 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.
$$

(2.35)

Beside that, from Proposition 4.7 Page 90 in Merle and Zaag [43] and (2.33), we see that for all $s \geq s^*$,

(i) (**Equivalence of norms in $\mathcal{H}^d$**) For all $q_- \in \mathcal{H}^d$,

$$
\frac{1}{C} \alpha_-(s) \leq \|q_-(s)\|_{\mathcal{H}} \leq C \alpha_-(s).
$$

(2.36)
(ii) (Equivalence of norms in $\mathcal{H}$) For all $q \in \mathcal{H}$,
\[
\frac{1}{C}(|\alpha_1(s)| + \alpha_-(s)) \leq \|q(s)\|_\mathcal{H} \leq C(|\alpha_1(s)| + \alpha_-(s)).
\] (2.37)

for some $C > 0$. We present now the very heart of our argument. Here, we derive from (2.15) a differential inequalities satisfied by $\alpha_1(s)$, $\alpha_-(s)$ and $d(s)$:

**Proposition 2.5.** There exists $\epsilon_2 > 0$ such that if $w$ is a solution to equation (1.6) satisfying (2.12) and (2.32) at some time $s$ for some $\epsilon < \epsilon_2$, where $q$ is defined in (2.13), then

(i) (Control of the modulation parameter) For all $s \geq s^*$, we have
\[
\frac{|d'|}{1 - d^2} \leq C(\alpha^2_1 + \alpha^2_-) + \frac{C}{s^a}(\alpha^2_1 + \alpha^2_-)^{\frac{1}{2}}.
\] (2.38)

(ii) (Projection of equation (2.15) on the mode $\lambda = 1$ and the negative part) For all $s \geq s^*$, we have
\[
|\alpha'_1 - \alpha_1| \leq C(\alpha^2_1 + \alpha^2_-) + \frac{C}{s^a}(\alpha^2_1 + \alpha^2_-)^{\frac{1}{2}},
\] (2.39)

\[
\left( R_- + \frac{1}{2} \alpha_-^2 \right)' \leq -\frac{4}{p - 1} \int_{-1}^{1} q^2 \rho \frac{\rho}{1 - y^2} dy + C(\alpha^2_1 + \alpha^2_-)^{\frac{1}{2+p}} + \frac{C}{s^a}(\alpha^2_1 + \alpha^2_-)
\] (2.40)

for some $R_-(s)$ satisfying
\[
|R_-(s)| \leq C(\alpha^2_1 + \alpha^2_-)^{\frac{1}{2+p}}, \text{ where } p = \min(p, 2) > 1.
\] (2.41)

(iii) (Additional relation) For all $s \geq s^*$, we have
\[
\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho dy \leq -\frac{1}{5} \alpha_-^2 + C \int_{-1}^{1} q^2 \frac{\rho}{1 - y^2} dy + C \alpha^2_1.
\] (2.42)

(iv) (Energy barrier) If moreover (1.25) holds, then
\[
|\alpha_1(s)|^2 \leq c_2 \alpha_-(s)^2 + \frac{C}{s^{p+a}}.
\] (2.43)

**Remark 2.6.** The proof of Proposition 2.5 is the major step of this paper. Indeed this proposition allows us to derive in the next section the polynomial and the exponential decay. From the polynomial decay, we show that for $d(s)$ introduced in (2.12), we have $\|q(s)\|_\mathcal{H} \to 0$ as $s \to +\infty$. Thanks to this special information combined with some refinement of some results obtained in the polynomial decay, we get the exponential decay. Here lays a major difference between our approach and the case when $f \equiv 0$. 
Proof of Proposition 2.5: Before going into the proof of Proposition 2.5, let us first derive some nonlinear estimates which will be useful for getting Proposition 2.5. Following the method used in Merle and Zaag [43], the proof of Proposition 2.5 requests the following nonlinear estimates.

Lemma 2.7. (Nonlinear estimates) For all \( y \in (-1, 1) \), we have

\[
|h(d, y, q_1)| \leq C \delta_{p \geq 2} |\kappa(d(s), y)||p-2|q_1(y, s)|^2 + C|q_1(y, s)|^p, \tag{2.44}
\]

\[
|H(d, y, q_1)| \leq C \delta_{p \geq 2} |\kappa(d(s), y)||p-2|q_1(y, s)|^3 + C|q_1(y, s)|^{p+1}, \tag{2.45}
\]

where \( \delta_{p \geq 2} \) is 0 if \( 1 < p < 2 \) and 1 otherwise, the function \( h \) is defined in (2.17) and

\[
H(d, y, q_1) = \int_0^{q_1} h(d, y, q') dq' = \frac{|\bar{w}_1 + q_1|^{p+1}}{p+1} - \frac{|\bar{w}_1|^{p+1}}{p+1} - \frac{p}{2} \bar{w}_1 q_1. \tag{2.46}
\]

Proof: The proof of (2.44) and (2.45) is exactly the same as the one written in Claim 5.3 page 104 in Merle and Zaag [43]: just replace \( \kappa(d, y) \) by \( \bar{w}_1(d, y, s) \) and use the fact that \( \frac{\bar{w}_1(d, y, s)}{\kappa(d, y)} \) is a bounded function from (A.5).

Let us now introduce the following lemma, where we give some nonlinear estimates related to our perturbation \( f \) defined in (1.2) and the new solution \( \bar{w}_1 \) defined in (1.21) which will play a central role in our analysis.

Lemma 2.8. (Nonlinear estimates related to \( f \) and \( \bar{w}_1 \)) For all \( y \in (-1, 1) \), we have

\[
|V| \leq Ce^{-s} + \frac{C}{s a} |p-1(d, y)|, \tag{2.47}
\]

\[
|\hat{f}| \leq C \delta_{p \geq 2} |\kappa(d(s), y)||p-2|q_1(y, s)|^2 + C|q_1(y, s)|^p, \tag{2.48}
\]

\[
|\hat{F}| \leq C \delta_{p \geq 2} |\kappa(d(s), y)||p-2|q_1(y, s)|^3 + C|q_1(y, s)|^{p+1}, \tag{2.49}
\]

where \( \delta_{p \geq 2} \) is 0 if \( 1 < p < 2 \) and 1 otherwise and

\[
\hat{F}(\bar{w}_1, q_1, s) = \int_0^{q_1} \hat{f}(\bar{w}_1, q', s) dq',
\]

with \( \hat{f}(\bar{w}_1, q_1, s) \) and \( V \) defined in (2.20).

Proof: We start by the proof of (2.47). Inspired by the proof of Lemma 2.1 page 1121 in our paper [20], the following holds

\[
e^{-2s}|f'(e^{\frac{q_1}{n}} \bar{w}_1)| \leq Ce^{-s} + \frac{C}{s a} |\bar{w}_1|^p. \tag{2.50}
\]
In the case when \( |p| < \frac{1}{2} \), we apply the mean value theorem to derive the following

\[
|\hat{f}_1(\overline{w}, q_1, s)| \leq C \frac{e_{\frac{4s}{p-1}}|\overline{w}| |q_1|}{2 + e_{\frac{4s}{p-1}}(\overline{w} + \theta_1 q_1)^2}.
\]  

In this case (when \( |p| < \frac{1}{2} \)), we separate the cases \( p \geq 2 \) and \( 1 < p < 2 \). In the case when \( p \geq 2 \), by virtue of (2.54) and (2.56) entails

\[
|\hat{f}_1(\overline{w}, q_1, s)| \leq C|\overline{w}|^{p-2}|q_1|^2.
\]  

19
For the case when $1 < p < 2$, we remark that $|q_1|^{p-2} \geq \frac{|w_1|^{p-2}}{2^{p-2}}$, which imply from inequalities (2.54) and (2.56) that

$$|\hat{f}_1(\overline{w}_1, q_1, s)| \leq C|q_1|^p. \quad (2.58)$$

We combine now (2.55), (2.57) and (2.58) to deduce that for all $q_1$ and $\overline{w}_1$

$$|\hat{f}_1(\overline{w}_1, q_1, s)| \leq C\delta_{p \geq 2}|w_1|^{p-2}|q_1|^2 + C|q_1|^p, \quad (2.59)$$

where $\delta_{p \geq 2}$ is 0 if $1 < p < 2$ and 1 otherwise. We treat now the term $\hat{f}_2(\overline{w}_1, q_1, s)$. We write

$$|\hat{f}_2(\overline{w}_1, q_1, s)| \leq C|\overline{w}_1|^p + C|\overline{w}_1|^{p-1}|q_1|. \quad (2.60)$$

Directly, we can write from (2.60) if $\frac{|q_1|}{|\overline{w}_1|} \geq \frac{1}{2}$

$$|\hat{f}_2(\overline{w}_1, q_1, s)| \leq C|q_1|^p. \quad (2.61)$$

Now, if $\frac{|q_1|}{|\overline{w}_1|} < \frac{1}{2}$, similarly to (2.57) and (2.58), we apply again the mean value theorem to write the following when $p \geq 2$

$$|\hat{f}_2(\overline{w}_1, q_1, s)| \leq C|\overline{w}_1|^{p-2}|q_1|^2. \quad (2.62)$$

When $1 < p < 2$, we write

$$|\hat{f}_2(\overline{w}_1, q_1, s)| \leq C|\overline{w}_1|^p. \quad (2.63)$$

We combine (2.61), (2.62) and (2.63) to deduce that for all $q_1$ and $\overline{w}_1$

$$|\hat{f}_2(\overline{w}_1, q_1, s)| \leq C\delta_{p \geq 2}|w_1|^{p-2}|q_1|^2 + C|q_1|^p. \quad (2.64)$$

Thanks to (2.44) of Lemma 2.7 combined with (2.52), (2.59) and (2.64), we conclude the proof of (2.48). Then, we exploite the expression of $\hat{F}$ to conclude the proof (2.49). This conclude the proof of Lemma 2.8.

We give now the strategy of the proof of (i)-(ii) of Proposition 2.5. We proceed in two steps:

- In Step 1, we project equation (2.15) with the projector $\pi^d$ defined in (2.9) for $\lambda = 0$ and $\lambda = 1$ and derive the smallness condition on $d'$ in (2.38) and the equation satisfied by $\alpha_1$ in (2.39).

- In Step 2, we write an equation satisfied by $(q_{-1}, q_{-2})$ which is the difficult part in this non self-adjoint framework. We claim that inequality (2.40) follows from the existence of the Lyapunov functional (1.12) for equation (1.11). Here, the Lyapunov functional structure will be revealed by the quadratic form $\varphi_d$ (2.30).
Step1: Projection of equation (2.15) on the modes $\lambda = 0$ and $\lambda = 1$. Projecting equation (2.15) with the projector $\pi^d_\lambda$ defined in (2.9) for $\lambda = 0$ and $\lambda = 1$, we write

$$\pi^d_\lambda(\partial_s q) = \pi^d_\lambda(L_d q) + \pi^d_\lambda \left( \begin{array}{c} 0 \\ h(d, y, q_1) \end{array} \right) - d'\pi^d_\lambda \left( \begin{array}{c} \partial_d\kappa(d, y) \\ 0 \end{array} \right) + \Sigma^1_{\lambda, d}(s) - d'\Sigma^2_{\lambda, d}(s) + \Sigma^3_{\lambda, d}(s), \quad (2.65)$$

where

- $\Sigma^1_{\lambda, d}(s) = \pi^d_\lambda \left( \begin{array}{c} 0 \\ \nabla q_1 \end{array} \right)$,
- $\Sigma^2_{\lambda, d}(s) = \pi^d_\lambda \left( \begin{array}{c} \partial_d(\kappa(d, y)(\hat{\phi}(d, y, s) - 1)) \\ \partial_d\hat{w}_2(d, y, s) \end{array} \right)$,
- $\Sigma^3_{\lambda, d}(s) = \pi^d_\lambda \left( \begin{array}{c} 0 \\ \hat{f}(\hat{w}_1, q_1, s) \end{array} \right)$,

with $L_d, \kappa(d, y), \nabla$ and $\hat{f}(\hat{w}_1, q_1, s)$ defined respectively in (2.1), (1.14), (2.22) and (2.20). According to the first step of the proof of Proposition 5.2 page 105 in Merle and Zaag [43], we write directly when $\lambda = 0$,

$$\frac{2\kappa_0 |d'|}{(p - 1)(1 - d^2)} \leq \frac{C|d'|(|\alpha_1(s)| + \alpha_-(s))}{1 - d^2} + C\|q\|_{\mathcal{H}}^2 + |\Sigma^1_{0, d}(s)| + |d'\Sigma^2_{0, d}(s)| + |\Sigma^3_{0, d}(s)| \quad (2.66)$$

and when $\lambda = 1$,

$$|\alpha'_1(s) - \alpha_1(s)| \leq \frac{C|d'|(|\alpha_1(s)| + \alpha_-(s))}{1 - d^2} + C\|q\|_{\mathcal{H}}^2 + |\Sigma^1_{1, d}(s)| + |d'\Sigma^2_{1, d}(s)| + |\Sigma^3_{1, d}(s)| . \quad (2.67)$$

Our focal interest now is to treat the new terms $\Sigma^1_{\lambda, d}(s), \Sigma^2_{\lambda, d}(s)$ and $\Sigma^3_{\lambda, d}(s)$ with $\lambda \in \{0, 1\}$. With Claim $[B.1]$ we are in position to give an estimation to $\Sigma^1_{\lambda, d}(s)$. We use the definition (2.9) of $\pi^d_\lambda$, the expression of $T$ given in (2.5) and the inequality (2.47), we see that

$$|\Sigma^1_{\lambda, d}(s)| \leq Ce^{-s} \int_{-1}^1 |q_1||W^d_{\lambda, 2}||\rho dy + \frac{C}{s^a} \int_{-1}^1 |\kappa^{p-1}(d, y)||q_1||W^d_{\lambda, 2}||\rho dy . \quad (2.68)$$

We are going now to estimate one by one the terms of the right-hand side of inequality (2.68). According to (B.36), the Hölder inequality and the Hardy-Sobolev’s inequality in Lemma [B.6] we can see the following estimate:

$$\int_{-1}^1 |\kappa^{p-1}(d, y)||q_1||W^d_{\lambda, 2}||\rho dy \leq C\int_{-1}^1 |\kappa^{p}(d, y)||q_1||\rho dy \leq C\|\kappa\|_{\mathcal{H}_0}^p \|q\|_{\mathcal{H}} \leq C\|q\|_{\mathcal{H}} . \quad (2.69)$$
Again via (B.36), the Hölder inequality and the Hardy-Sobolev's inequality in Lemma B.6 to obtain
\[ \int_{-1}^{1} |q_1| |W_{\lambda,1}^d| \rho dy \leq C \|q\|_\mathcal{H}. \quad (2.70) \]
Collecting (2.68), (2.69) and (2.70) together, we deduce
\[ \left| \Sigma_{\lambda,d}^1(s) \right| \leq C \frac{s}{a} \|q\|_\mathcal{H}. \quad (2.71) \]
We would like now to estimate \( \Sigma_{\lambda,d}^2(s) \). From the definition (2.9) of \( \pi^d_\lambda \), the expression of \( \Upsilon \) given in (2.5) and inequality (B.36), it holds that:
\[ \left| \Sigma_{\lambda,d}^2(s) \right| \leq \int_{-1}^{1} |\partial_d \left( \kappa(d, y)(\tilde{\phi}(d, y, s) - 1) \right) \| - \mathcal{L} W_{\lambda,1}^d + W_{\lambda,1}^d | \rho dy \]
\[ \quad + C \int_{-1}^{1} |\kappa(d, y)||\partial_d \psi_2(d, y, s)| |\rho dy. \quad (2.72) \]
By using equation (2.7) satisfied by \( W_{\lambda,1}^d \) and (B.36), we can see that
\[ | - \mathcal{L} W_{\lambda,1}^d + W_{\lambda,1}^d | \leq C \frac{\kappa(d, y)}{1 - y^2}. \quad (2.73) \]
Using Claim B.1 inequalities (B.24), (B.35) and (ii) of Claim B.6 to deduce that
\[ \int_{-1}^{1} |\partial_d \left( \kappa(d, y)(\tilde{\phi}(d, y, s) - 1) \right) \| - \mathcal{L} W_{\lambda,1}^d + W_{\lambda,1}^d | \rho dy \leq C \frac{s}{a(1 - d^2)}. \quad (2.74) \]
We get directly from inequality (B.21) and (ii) of Claim B.6 the following
\[ \int_{-1}^{1} \kappa(d, y)||\partial_d \psi_2(d, y, s)|| \rho dy \leq \frac{C}{s_a(1 - d^2)}. \quad (2.75) \]
Plugging (2.74) and (2.75) into (2.72), we observe that
\[ \left| \Sigma_{\lambda,d}^2(s) \right| \leq \frac{C}{s_a(1 - d^2)}. \quad (2.76) \]
We treat now the term \( \Sigma_{\lambda,d}^3(s) \) which is provided from the perturbation \( f \) defined in (1.2). We use the definition (2.9) of \( \pi^d_\lambda \), the expression of \( \Upsilon \) given in (2.5), inequality (2.48) of Lemma 2.8 and (B.36), we write
\[ \left| \Sigma_{\lambda,d}^3(s) \right| \leq C \int_{-1}^{1} |\kappa(d, y)|^{p-1} |q_1|^2 |\rho dy + C \delta_{p \geq 2} \int_{-1}^{1} |\kappa(d, y)||q_1|^p |\rho dy, \quad (2.77) \]
where \( \delta_{p \geq 2} \) is 0 if \( 1 < p < 2 \) and 1 otherwise. From the Hölder inequality and the Hardy-Sobolev’s inequality in Lemma B.6 we write
\[
\int_{-1}^{1} |\kappa(d, y)|^{p-1}|q_1|^2 \rho dy \leq \|\kappa\|_{\mathcal{H}_0}^{p-1}\|q\|_{\mathcal{H}}^2 \leq C\|q\|_{\mathcal{H}}^2, \tag{2.78}
\]
for the same reason, we write also the following
\[
\int_{-1}^{1} |\kappa(d, y)| |q_1|^p \rho dy \leq \|\kappa\|_{\mathcal{H}_0}\|q\|_{\mathcal{H}}^p \leq C\|q\|_{\mathcal{H}}^p. \tag{2.79}
\]
Observing inequalities (2.77), (2.78), (2.79) along with the a priori estimate (2.32) yield
\[
\left|\Sigma_{\lambda, d}^3(s)\right| \leq C\|q\|_{\mathcal{H}}^2. \tag{2.80}
\]
According to (2.65), (2.66), (2.67), (2.71), (2.76) and (2.80) to obtain when \( \lambda = 0 \)
\[
\frac{2\kappa_0 | d' |}{(p - 1)(1 - d^2)} \leq \frac{C|d'|}{1 - d^2}(\|\alpha_1(s)\| + \alpha_-(s)) + C\|q\|_{\mathcal{H}}^2 + \frac{C|d'|}{s^a\|q\|_{\mathcal{H}}} + \frac{C|d'|}{s^a(1 - d^2)}, \tag{2.81}
\]
and when \( \lambda = 1 \)
\[
|\alpha_1'(s) - \alpha_1(s)| \leq \frac{C|d'|}{1 - d^2}(\|\alpha_1(s)\| + \alpha_-(s)) + C\|q\|_{\mathcal{H}}^2 + \frac{C|d'|}{s^a\|q\|_{\mathcal{H}}} + \frac{C|d'|}{s^a(1 - d^2)}. \tag{2.82}
\]
Using the smallness condition (2.32), the equivalence of the norms in (2.37), the fact that
\[
C\frac{|d'|}{1 - d^2}\|q\|_{\mathcal{H}} \leq \frac{\kappa_0}{4(p-1)} \frac{|d'|}{1 - d^2}
\]
and the fact that for some \( s^* \) large enough and for all \( s \geq s^* \), we have \( C\frac{|d'|}{s^a(1 - d^2)} \leq \frac{\kappa_0}{4(p-1)} \frac{|d'|}{1 - d^2} \), we get (2.38) and (2.39) for \( \epsilon \) small enough.

**Step 2: Differential inequality on \( \alpha_- \).** In the following, we project equation (2.15) on the negative modes, which gives a partial differential inequality satisfied by \( q_- \). In order to simplify the presentation, let us introduce the following terms which will be useful in many steps of this part.
\[
\Sigma_{\lambda, d}^1(s) = \pi_d^\bot \begin{pmatrix} 0 \\ -\nabla q_1 \end{pmatrix}, \tag{2.83}
\]
\[
\Sigma_{\lambda, d}^2(s) = d'\pi_d^\bot \begin{pmatrix} -\partial_d \left( \kappa(d, y) \tilde{\phi}(d, y, s) - 1 \right) \\ \partial_d \tilde{w}_2(d, y, s) \end{pmatrix}, \tag{2.84}
\]
\[
\Sigma_{\lambda, d}^3(s) = \pi_d^\bot \begin{pmatrix} 0 \\ h(d, y, q_1) + \tilde{f}(\tilde{w}_1, q_1, s) \end{pmatrix}. \tag{2.85}
\]
\[ \Sigma^3_{-,d}(s) = \pi_d^d \begin{pmatrix} 0 & \hat{f}(\bar{w}_1, q_1, s) \\ 0 & h(d, y, q_1) \end{pmatrix}, \]
\[ \Sigma^3_{-,d}(s) = \pi_d^d \begin{pmatrix} 0 & h(d, y, q_1) \\ 0 & \hat{f}(\bar{w}_1, q_1, s) \end{pmatrix}. \]

We now claim the following:

**Claim 2.9. (Preliminary estimates)** There exists \( \epsilon_3 > 0 \) such that if \( \epsilon < \epsilon_3 \) in the hypotheses of Proposition 2.5, then

\[ \| \partial_s q_- - L_d q_- - \Sigma^1_{-,d}(s) - \Sigma^2_{-,d}(s) - \Sigma^3_{-,d}(s) \|_H \leq C(\alpha^2 + \alpha^-)^{3/2}, \]  
(2.86)

\[ \left| \varphi_d(q_-, \Sigma^1_{-,d}(s)) \right| \leq \frac{C}{s^a}(\alpha^2 + \alpha^-), \]  
(2.87)

\[ \left| \varphi_d(q_-, \Sigma^2_{-,d}(s)) \right| \leq C(\alpha^2 + \alpha^-)^{3/2} + \frac{C}{s^a}(\alpha^2 + \alpha^-), \]  
(2.88)

\[ \left| \varphi_d(q_-, \Sigma^3_{-,d}(s)) - \int_{-1}^1 q_2 h(d, y, q_1) \rho dy \right| \leq C(\alpha^2 + \alpha^-)^{3/2}, \]  
(2.89)

\[ \left| \varphi_d(q_-, \Sigma^3_{-,d}(s)) - \int_{-1}^1 q_2 \hat{f}(\bar{w}_1, q_1, s) \rho dy \right| \leq C(\alpha^2 + \alpha^-)^{3/2}, \]  
(2.90)

\[ \left| \int_{-1}^1 q_2 h(d, y, q_1) \rho dy - \frac{d}{ds} \int_{-1}^1 H(d, y, q_1) \rho dy \right| \leq C(\alpha^2 + \alpha^-)^2 + \frac{C}{s^a}(\alpha^2 + \alpha^-), \]  
(2.91)

\[ \left| \int_{-1}^1 q_2 \hat{f}(\bar{w}_1, q_1, s) \rho dy - \frac{d}{ds} \int_{-1}^1 \hat{F}(\bar{w}_1, q_1, s) \rho dy \right| \leq C(\alpha^2 + \alpha^-)^{11/2} + \frac{C}{s^a}(\alpha^2 + \alpha^-), \]  
(2.92)

where \( h(d, y, q_1), H(d, y, q_1) \) and \( \hat{F}(\bar{w}_1, q_1, s) \) are defined respectively in (2.17), (2.46) and (2.110).

**Remark 2.10.** Note that the terms in (2.91) and (2.92) cannot be controlled directly and have to be seen as time derivatives.

Let us now use Claim 2.9 to derive the proof of the differential inequality (2.40) satisfied by \( \alpha_- \), then we will prove it later.

**Proof of (2.40) admitting Claim 2.9** In order to avoid unnecessary repetition, we kindly refer the interested reader to Merle and Zaag [43] (the proof of inequality (186) page 107). Only we focus here on the new terms coming from \( f \) defined in (1.2) and the new solution \( \bar{w} \) defined in (1.21) and (1.22).

In fact, the whole proof of (2.40) is based on the fact that the derivative of \( \alpha_- \) is related to the quadratic form \( \varphi_d(q_-, L_d(q_-)) \), defined in (2.30), which inherits the properties of the Lyapunov functional defined in (1.12) (and give an almost self-adjoint behavior).
We use here the same bilinear form $\varphi_d$ introduced in Merle and Zaag [43], therefore using the bound (2.38) on $|d'|$, we get
\[ |\alpha - \alpha' - \varphi_d(q_-, \partial_s q_-) - \frac{d}{ds} \int_{-1}^{1} H(d, y, q_1) \rho dy | \leq C \|q\|_{H}^4 + \frac{C}{s^a} \|q\|_{H}^3. \tag{2.93} \]

For the reader’s interest, we mention that the proof of inequality (2.93) is written in inequality (206) page 107 in Merle and Zaag [43].

From Claim 2.9, the continuity of $\varphi_d$ (2.31), the equivalence norms (2.36) and inequality (2.93), we write
\[ \left| \alpha - \alpha' - \varphi_d(q_-, L_d(q_-)) \right| \leq C \|q\|_{H}^2 + \frac{C}{s^a} \|q\|_{H}^2, \tag{2.94} \]

Using (2.93), (2.94) and (2.95), we see that the estimate (2.40) holds with
\[ R_-(s) = - \int_{-1}^{1} H(d, y, q_1) \rho dy - \int_{-1}^{1} \tilde{F}(\overline{w}_1, q_1, s) \rho dy. \tag{2.96} \]

Let us recall that $\varphi_d(q, r)$ is the same bilinear form used by Merle and Zaag [43]. For that reason, we refer the reader to page 107 and the beginning of page 108 in Merle and Zaag [43] to see the details of the proof of equality (2.95).

Using (2.93), (2.94) and (2.95), we see that (2.41) holds. It remains to prove Claim 2.9 in order to conclude the proof of (i) – (ii) of Proposition 2.5.

**Proof of Claim 2.9**

Proof of (2.86): We first project equation (2.15) and using the negative projector $\pi_-^d$ introduced in (2.10):
\[ \pi_-^d(\partial_s q) = \pi_-^d(L_d q) - d' \pi_-' \left( \frac{\partial_d \kappa(d, y)}{0} \right) + \Sigma_{-d}(s) + \Sigma_{-d}(s), \tag{2.97} \]
where $\Sigma^2_{-d}$ and $\Sigma^3_{-d}$ are defined respectively in (2.84) and (2.85). According to the proof of Claim 5.4 page 108 in [13], the Remark 2.3 equations (2.83) and (2.97), we can directly write the following

$$\|\partial q_+ - L_d q_+ - \Sigma^1_{-d}(s) - \Sigma^2_{-d}(s) - \Sigma^3_{-d}(s)\| \leq C(\alpha_1^2 + \alpha_2^2)^\frac{1}{2},$$

(2.98)

which end the proof of (2.86).

**Proof of (2.87):** This inequality is a direct consequence of the continuity of $\varphi_d$ (2.31), from the equivalence norms in (2.36) and (2.37) and inequality (2.47) in Lemma 2.8.

**Proof of (2.88):** This inequality is a direct consequence of the continuity of $\varphi_d$, (2.31), inequality (2.38) on $d'$ and the a priori estimate (2.32).

**Proof of (2.89) and (2.90):** Recall from (2.33) that we have

$$q(y, s) = \alpha_1 F_1^d(y) + q_+(y, s),$$

$$\left( \begin{array}{c} \alpha_1 F_1^d(y) + \beta_0(s) F_0^d(y) + \pi_d \left( \begin{array}{c} 0 \\ h(d, y, q_1) \end{array} \right) \\ \tilde{\beta}_1(s) F_1^d(y) + \tilde{\beta}_0(s) F_0^d(y) + \pi_d \left( \begin{array}{c} 0 \\ \tilde{f}(w_1, q_1, s) \end{array} \right) \end{array} \right),$$

where $\beta_\lambda(s) = \pi^d_\lambda \left( \begin{array}{c} 0 \\ h(d, y, q_1) \end{array} \right)$ and $\tilde{\beta}_\lambda(s) = \pi^d_\lambda \left( \begin{array}{c} 0 \\ \tilde{f}(w_1, q_1, s) \end{array} \right).$ Note from the definition (2.30) of $\varphi_d$, we have

$$\int_{-1}^1 q_2 h(d, y, q_1) \rho dy = \varphi_d \left( \begin{array}{c} 0 \\ h(d, y, q_1) \end{array} \right).$$

(2.99)

In addition by virtue of (2.99), the bilinearity of $\varphi_d$, the bound (2.4) on the norm of $F_\chi$ and the equivalence of norms in (2.36), we write

$$|\varphi_d \left( \begin{array}{c} 0 \\ h(d, y, q_1) \end{array} \right) + \int_{-1}^1 q_2 h(d, y, q_1) \rho dy| \leq C(|\alpha_1| + |\alpha_-|) (|\beta_1| + |\beta_0|) + |\alpha_1| \left| \varphi_d \left( \begin{array}{c} 0 \\ h(d, y, q_1) \end{array} \right) \right|.$$ (2.100)

Since, we have from the expression (2.30) of $\varphi_d$, the fact that $|F_{1,2}^d(y)| \leq C\kappa(d, y)$ and Lemma 2.7,

$$\left| \varphi_d \left( \begin{array}{c} 0 \\ h(d, y, q_1) \end{array} \right) \right| = \left| \int_{-1}^1 F_{1,2}^d(y) h(d, y, q_1) \rho dy \right| \leq C(\alpha_1^2 + \alpha_2^2),$$

(2.101)

$$|\beta_1| + |\beta_0| \leq C \int_{-1}^1 \kappa(d, y) h(d, y, q_1) \rho dy \leq C(\alpha_1^2 + \alpha_2^2),$$

(2.102)
we combine (2.100), (2.101) and (2.102) to conclude (2.89).

Note from the definition (2.30), the bilinearity of \( \varphi_d \), the bound (2.4) on the norm of \( F^d \) and the equivalence norm (2.36), we derive that
\[
\left| \varphi_d(q_1, \pi_d \left( \frac{0}{\hat{f}(\bar{w}_1, q_1, s)} \right) \right) - \int_{-1}^{1} q_2 \hat{f}(\bar{w}_1, q_1, s) \rho dy \right| 
= \left| \varphi_d(q_1, \pi_d \left( \frac{0}{\hat{f}(\bar{w}_1, q_1, s)} \right) \right) - \varphi_d(q_1, \left( \frac{0}{\hat{f}(\bar{w}_1, q_1, s)} \right) \right| 
\leq C(|\alpha_1| + |\alpha_2|)(|\tilde{\beta}_1| + |\tilde{\beta}_0|) + |\alpha_1| \left| \varphi_d(F^d_1, \left( \frac{0}{\hat{f}(\bar{w}_1, q_1, s)} \right) \right|.
\]

Since, we see from the Hardy-Sobolev inequality of Lemma B.2, the equivalence norm (2.36), we derive that
\[
\left| \varphi_d(F^d_1, \left( \frac{0}{\hat{f}(\bar{w}_1, q_1, s)} \right) \right) \right| = \left| \int_{-1}^{1} F^d_{1,2}(y) \hat{f}(\bar{w}_1, q_1, s) \rho dy \right| \leq C(\alpha_1^2 + \alpha_2^2)
\]
and from (2.80), we obtain
\[
|\tilde{\beta}_1| + |\tilde{\beta}_0| \leq C \int_{-1}^{1} \kappa(d, y) \hat{f}(\bar{w}_1, q_1, s) \rho dy \leq C(\alpha_1^2 + \alpha_2^2),
\]
this gives (2.90).

**Proof of (2.91):** Since \( q_2 = \partial_s q_1 + d' \partial_q \bar{w}_1 \) by (2.13) and (1.22), we use the expression (2.46) of \( H \) to write
\[
\int_{-1}^{1} q_2 h(d, y, q_1) \rho dy = \frac{d}{ds} \int_{-1}^{1} H(d, y, q_1) \rho dy + d' \int_{-1}^{1} \partial_q \bar{w}_1 \left( h(d, y, q_1) - \partial_d H(d, y, q_1) \right) \rho dy 
- \int_{-1}^{1} \frac{\partial H(d, y, q_1)}{\partial \bar{w}_1} \frac{\partial \bar{w}_1}{\partial s} \rho dy
= \frac{d}{ds} \int_{-1}^{1} H(d, y, q_1) \rho dy + d' \frac{p(p-1)}{2} \int_{-1}^{1} \partial_q \bar{w}_1 \bar{w}_1^{p-2} q_1^2 \rho dy 
- \int_{-1}^{1} \left( h(d, y, q_1) + \frac{p(p-1)}{2} \bar{w}_1^{p-2} q_1^2 \right) \frac{\partial \bar{w}_1}{\partial s} \rho dy.
\]

Since, we see from the Hardy-Sobolev inequality of Lemma [B.6] inequality [B.24] and (A.4) that the function \( \phi \) defined in (iii) of Lemma [A.4] is bounded, it follows that
\[
\| \partial_q \bar{w}_1 \bar{w}_1^{p-2} \|_{L_p^{\frac{d}{d-1}}} \leq C \| q_1 \|_{L_p^{\frac{d}{d-1}}} \leq C(\alpha_1^2 + \alpha_2^2)
\]
From this fact, we derive the following:
\[
\int_{-1}^{1} \partial_q \bar{w}_1 \bar{w}_1^{p-2} q_1^2 \rho dy \leq \frac{C}{1 - d^2} \| q_1 \|_{L_p^{\frac{d}{d-1}}} \leq \frac{C(\alpha_1^2 + \alpha_2^2)}{1 - d^2}.
\]
From Lemma 2.7 and the Hölder inequality, it follows that

$$\int_{-1}^{1} \left( h(d, y, q_1) + \frac{p(p-1)}{2} \bar{w}_1^{p-2} q_1^2 \right) \frac{\partial \bar{w}_1}{\partial s} \rho dy \leq \frac{C}{s^a} \|q_1\|_{L_p}^{2} \|\kappa\|_{L^p}^{p-1}$$

where \( \delta_{p\geq 2} \) is 0 if \( 1 < p < 2 \) and 1 otherwise. Therefore, using \((ii)\) of Lemma B.6, the a priori estimate (2.32), the equivalence of norms (2.37) and (2.107), we write

$$\int_{-1}^{1} \left( h(d, y, q_1) + \frac{p(p-1)}{2} \bar{w}_1^{p-2} q_1^2 \right) \frac{\partial \bar{w}_1}{\partial s} \rho dy \leq \frac{C}{s^a} \|q_1\|_{L_p}^{2} \|\kappa\|_{L^p}^{p-1}, \quad (2.107)$$

Finally from the bound (2.38) on \( d' \), the a priori estimate (2.32), identity (2.105), inequalities (2.106) and (2.108), we deduce that

$$\left| \int_{-1}^{1} q_2 h(d, y, q_1) \rho dy - \frac{d}{ds} \int_{-1}^{1} H(d, y, q_1) \rho dy \right| \leq C(\alpha_1^2 + \alpha_2^2) + \frac{C}{s^a}(\alpha_1^2 + \alpha_2^2). \quad (2.109)$$

**Proof of (2.92):** We write first

$$\hat{F}(\bar{w}_1, q_1, s) = \int_{0}^{q_1} \hat{f}(\bar{w}_1, q', s) dq' = e^{-\frac{2(p+1)}{p-1} s} \left( F\left( e^{\frac{2s}{p-1}} (\bar{w}_1(d, y, s) + q_1) \right) \right)$$

$$- F\left( e^{\frac{2s}{p-1}} (\bar{w}_1(d, y, s)) \right) - e^{\frac{2s}{p-1}} q_1 f\left( e^{\frac{2s}{p-1}} (\bar{w}_1(d, y, s)) \right) - e^{\frac{4s}{p-1}} q_1^2 f'\left( e^{\frac{2s}{p-1}} (\bar{w}_1(d, y, s)) \right),$$

with \( F \) and \( \hat{f} \) defined respectively in (1.10) and (2.20). Since \( q_2 = \partial_s q_1 + d' \partial_d \bar{w}_1 \) by (2.13) and (1.22), thanks to the expression (2.110) of \( \hat{F} \), we write

$$\int_{-1}^{1} q_2 \hat{f}(\bar{w}_1, q_1, s) \rho dy = \int_{-1}^{1} \partial_s q_1 \hat{f}(\bar{w}_1, q_1, s) \rho dy + d' \int_{-1}^{1} \partial_d \bar{w}_1 \hat{f}(\bar{w}_1, q_1, s) \rho dy$$

$$= \frac{d}{ds} \int_{-1}^{1} \hat{F}(\bar{w}_1, q_1, s) \rho dy + d' \int_{-1}^{1} \partial_d \bar{w}_1 \left( \hat{f}(\bar{w}_1, q_1, s) - \frac{\partial \hat{F}(\bar{w}_1, q_1, s)}{\partial \bar{w}_1} \right) \rho dy$$

$$- \int_{-1}^{1} \frac{\partial \hat{F}(\bar{w}_1, q_1, s)}{\partial \bar{w}_1} \frac{\partial \bar{w}_1}{\partial s} \rho dy - \int_{-1}^{1} \frac{\partial \hat{F}(\bar{w}_1, q_1, s)}{\partial s} \rho dy. \quad (2.111)$$

By a careful calculation, we can see that

$$\partial_d \bar{w}_1 \left( \hat{f}(\bar{w}_1, q_1, s) - \frac{\partial \hat{F}(\bar{w}_1, q_1, s)}{\partial \bar{w}_1} \right) = e^{-\frac{2(p+1)}{p-1} s} q_1 \frac{2p}{p-1} \partial_d \bar{w}_1 f''\left( e^{\frac{2s}{p-1}} \bar{w}_1 \right). \quad (2.112)$$

We derive from (2.110)

$$\frac{\partial \hat{F}(\bar{w}_1, q_1, s)}{\partial s} = -\frac{2(p+1)}{p-1} \hat{F}(\bar{w}_1, q_1, s) + g_p(A + X) - g_p(A)$$

28
where $A(y, s) = e^{2s - pW_1}$, $X(y, s) = e^{2s - q_1}$ and the function $g_p(X) = \frac{2}{(p-1)s}e^{-2(p+1)Xf(X)}$. We apply the mean value theorem to say that there exists $\theta \in (0, 1)$ such that:

$$g_p(A + X) - g_p(A) - Xg'_p(A) - \frac{X^2}{2}g''_p(A) = \frac{X^3}{6}g^{(3)}_p(A + \theta X).$$  \hfill (2.113)

Remark that

$$g^{(3)}_p(A + \theta X) = \frac{2}{p-1}e^{-2(p+1)s}3f''(A + \theta X) + (A + \theta X)f^{(3)}(A + \theta X)).$$  \hfill (2.114)

After straightforward computations of the expressions of $f'$, $f''$ and $f^{(3)}$, we write the following

$$\frac{|X|^3}{6}|g^{(3)}_p(A + \theta X)| \leq C|q_1|^3|w_1| + \theta|q_1|^{p-2}.$$  \hfill (2.115)

Hence, if $\frac{|q_1|}{|w_1|} \geq \frac{1}{2}$, we obtain

$$\frac{|X|^3}{6}|g^{(3)}_p(A + \theta X)| \leq C|q_1|^{p+1}.  \hfill (2.116)$$

For the case when $\frac{|q_1|}{|w_1|} < \frac{1}{2}$, similarly to the proof of (2.57) and (2.58), we derive if $p \geq 2$

$$\frac{|X|^3}{6}|g^{(3)}_p(A + \theta X)| \leq C|q_1|^3|w_1|^{p-2}$$  \hfill (2.117)

and if $1 < p < 2$, we obtain

$$\frac{|X|^3}{6}|g^{(3)}_p(A + \theta X)| \leq C|q_1|^{p+1}. \hfill (2.118)$$

Adding estimates (2.117), (2.118) and (2.119), we write

$$\frac{|X|^3}{6}|g^{(3)}_p(A + \theta X)| \leq C\delta_{p \geq 2}|w_1|^{p-2}|q_1|^3 + C|q_1|^{p+1},  \hfill (2.119)$$

where $\delta_{p \geq 2}$ is 0 if $1 < p < 2$ and 1 otherwise. Collecting (2.113), (2.49) in Lemma 2.8 inequality (2.120) and the a priori estimate (2.32) together combined with the Hardy-Sobolev inequality, we deduce that

$$|\int_{-1}^{1} \frac{\partial F(w_1, q_1, s)}{\partial s} \rho dy| \leq C||q||^{1+\frac{1}{n}}; \hfill (2.121)$$

29
where \( \overline{p} \) is defined in (2.41). Now we deal with the term written in (2.112). From (B.22), (2.38) and the a priori estimate (2.32) combined with the Hardy-Sobolev inequality, we obtain

\[
|d' \int_{-1}^{1} \partial_d \overline{w}_1 \left( \hat{f}(\overline{w}_1, q_1, s) - \frac{\partial \hat{F}(\overline{w}_1, q_1, s)}{\partial \overline{w}_1} \right) \rho dy| \leq C \| q \|_H^4 + C \frac{s^a}{s^a} \| q \|_H^3. \quad (2.122)
\]

We apply the mean value theorem to say that there exists \( \theta \in (0, 1) \) such that:

\[
|\partial \hat{F}(\overline{w}_1, q_1, s) \partial \overline{w}_1| \leq C |q_1|^2 |\overline{w}_1| + \theta |q_1|^p - 2 + C |q_1|^2 |\overline{w}_1|^{p-2}. \quad (2.123)
\]

According to inequality (2.123) and (B.23) combined with the Hardy-Sobolev inequality, we can see that

\[
\left| \int_{-1}^{1} q_2 \hat{f}(\overline{w}_1, q_1, s) \rho dy - \frac{d}{ds} \int_{-1}^{1} \hat{F}(\overline{w}_1, q_1, s) \rho dy \right| \leq C \| q \|_H^{1+\overline{p}} + C \frac{s^a}{s^a} \| q \|_H^3. \quad (2.124)
\]

Gathering the bounds (2.111), (2.121), (2.122), (2.124) and the a priori estimate (2.32), it follows that

\[
\left| \int_{-1}^{1} q_2 \hat{f}(\overline{w}_1, q_1, s) \rho dy - \frac{d}{ds} \int_{-1}^{1} \hat{F}(\overline{w}_1, q_1, s) \rho dy \right| \leq C \| q \|_H^{1+\overline{p}} + C \frac{s^a}{s^a} \| q \|_H^3. \quad (2.125)
\]

Finally, (2.109) and (2.125) ends the proof of Claim 2.9 as well as (i) and (ii) of Proposition 2.5.

(iii) This inequality is a consequence of the coercivity of the quadratic form \( \varphi_d \) on the space \( H^d \) stated in (2.36) and (2.37).

From equation (2.15), the fact that \( q_2 = \partial_s q_1 + d' \partial_d \overline{w}_1 \) and the definition (2.1) of \( L_d \), we write

\[
\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho dy = \int_{-1}^{1} q_2 \partial_s q_1 \rho dy + \int_{-1}^{1} q_1 \partial_s q_2 \rho dy
\]

\[
= \int_{-1}^{1} q_2 (q_2 - d' \partial_d \overline{w}_1) \rho dy \quad (2.126)
\]

\[
+ \int_{-1}^{1} \sum q_1 \left( \mathcal{L} q_1 + \psi(d, y) q_1 - \frac{p+3}{p-1} q_2 - 2y \partial_y q_2 + h(d, y, q_1) \right) \rho dy
\]

\[
+ \int_{-1}^{1} \nabla q_1^2 \rho dy - d' \int_{-1}^{1} q_1 \partial_d \overline{w}_2 \rho dy + \int_{-1}^{1} q_1 \hat{f}(\overline{w}_1, q_1, s) \rho dy.
\]

According to (2.126) and the proof of item (iii) of Proposition 5.2 page 110 in Merle and Zaag [43], we write directly that

\[
\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho dy \leq - \frac{4}{5} \alpha_2 + C \int_{-1}^{1} q_1^2 (\partial_s^2 - \frac{\rho}{1-y^2}) dy + C \alpha_2^2
\]

30
with  

\[ + \int_{-1}^{1} \nabla q_{1}^{2} \rho dy - d' \int_{-1}^{1} q_{1} \partial_{d} \psi_{2} \rho dy + \int_{-1}^{1} q_{1} \hat{f}(\psi_{1}, q_{1}, s) \rho dy. \]

We see from (2.47), (2.104) and the fact that \( s^{*} \) is large enough such that for all \( s \geq s^{*} \), the following estimate holds

\[ \left| \int_{-1}^{1} \nabla q_{1}^{2} \rho dy \right| \leq C e^{-s} \int_{-1}^{1} q_{1}^{2} \rho dy + C \frac{s}{s} \int_{-1}^{1} |\kappa(d, y)|^{p-1} q_{1}^{2} \rho dy \leq \frac{C}{s} \|q\|_{H}^{2} \leq \frac{1}{5} \|q\|_{H}^{2}. \]  

(2.127)

We can see from (2.48), the smallness condition (2.32) and Lemma (2.6) we have

\[ \left| \int_{-1}^{1} q_{1} \hat{f}(\psi_{1}, q_{1}, s) \rho dy \right| \leq C \delta_{p \geq 2} \int_{-1}^{1} \kappa^{p-2}(d, y)|q_{1}|^{3} \rho dy + C \int_{-1}^{1} |q_{1}|^{p+1} \rho dy \]

\[ \leq C \|\kappa\|_{L_{p+1}}^{p-2} \|q_{1}\|_{L_{p+1}}^{3} + C \delta_{p \geq 2} \|q_{1}\|_{L_{p+1}}^{p+1} \]

\[ \leq C \|q\|_{H}^{p+1} \leq \frac{1}{5} \|q\|_{H}^{2}. \]

(2.128)

Using (B.21) to write

\[ \left| \int_{-1}^{1} q_{1} \partial_{d} \psi_{2} \rho dy \right| \leq \frac{C}{s} \int_{-1}^{1} |\kappa(d, y)| \rho dy. \]

(2.129)

According to (2.129), Lemma (B.6) the smallness condition (2.32), (i) of Proposition 2.5 and the fact that \( s^{*} \) is large enough, we deduce that for all \( s \geq s^{*} \), we have

\[ \left| d' \int_{-1}^{1} q_{1} \partial_{d} \psi_{2} \rho dy \right| \leq \frac{C |d'|}{s} \|q\|_{H} \leq \frac{C}{s} \|q\|_{H}^{2} + \frac{C'}{s} \|q\|_{H}^{2} \leq \frac{1}{5} \|q\|_{H}^{2}. \]

(2.130)

Combining (2.126) with (2.127), (2.128) and (2.130) to deduce the proof of (iii) of Proposition 2.5

(iv) We use first the hypothesis (1.25) of Theorem 2 and then we search to an upper bound to the functional \( E(w, s) \). Recalling that

\[ E(w, s) = E_{0}(w(s)) + I(w, s) + J(w, s), \]

with

\[ I(w, s) = -e^{-\frac{2(p+1)s}{p-1}} \int_{-1}^{1} F(e^{\frac{2s}{p-1}} w) \rho dy \text{ and } J(w, s) = -\frac{1}{s} \int_{-1}^{1} w \partial_{s} w \rho dy. \]

(2.131)

Using the definition of \( q(y, s) \) written in (2.13), we can make an expansion of \( E_{0}(w(s)) \) defined in (1.11) for \( q \rightarrow 0 \) in \( H \) and get after some straightforward computations

\[ E_{0}(w(s)) = E_{0}(\bar{w}(d, y, s)) + \frac{1}{2} \varphi_{d}(q, q) - \int_{-1}^{1} H(d, y, q_{1}) \rho dy \]

\[ + R_{1}(s) + R_{2}(s) + R_{3}(s), \]

(2.132)

with
\[
\begin{align*}
R_1(s) &= \int_{-1}^{1} \left( -\mathcal{L}w_1 + 2 \frac{p+1}{(p-1)^2} w_1 - |w_1|^p \right) q_1 \rho dy, \\
R_2(s) &= \frac{\xi}{2} \int_{-1}^{1} \left( \kappa^{p-1} - \overline{w}_1^{p-1} \right) q_1^2 \rho dy, \\
R_3(s) &= \frac{1}{2} d'' \int_{-1}^{1} (\partial \overline{w}_1)^2 \rho dy + d' \int_{-1}^{1} \partial_y \overline{w}_1 \partial_y w_1 \rho dy \quad - d' \int_{-1}^{1} \partial_y \overline{w}_1 q_1 \rho dy + \int_{-1}^{1} \overline{w}_2 q_2 \rho dy.
\end{align*}
\]

Recalling that \( \varphi_d \) (the same bilinear form used by Merle and Zaag \[43\]) satisfies the following
\[
\varphi_d(q, q) \leq c_2 \alpha_+^2 - c_3 \alpha_+^2,
\]
for some \( c_2 > 0 \) and \( c_3 > 0 \). Using (2.32), (2.37), (2.41) and (2.96), to obtain
\[
- \int_{-1}^{1} H(d, y, q_1) \rho dy \leq C \|q\|_{\mathcal{H}_0} \leq C \epsilon (\alpha_2(s) + \alpha_2^- (s)),
\]
where \( \overline{\epsilon} = \min(p, 2) \). Recalling that \( \kappa(d, y) \) defined in (1.14) is a stationary solution of the unperturbed case, so we can write the following
\[
\mathcal{L} \kappa(d, y) - 2 \frac{p+1}{(p-1)^2} \kappa(d, y) + |\kappa(d, y)|^p = 0,
\]
for more detail of equality (2.135) we can see the proof of equality (47) page 58 in Merle and Zaag \[43\]. It is easy to check
\[
R_1(s) = \int_{-1}^{1} \left( \mathcal{L}(\kappa(d, y) - \overline{w}_1) + 2 \frac{p+1}{(p-1)^2} (\overline{w}_1 - \kappa(d, y)) + (|\kappa(d, y)|^p - |\overline{w}_1|^p) \right) q_1 \rho dy.
\]

After an integration by parts, using the fact that \( |a^p - 1| \leq C |a - 1| \) where \( a \) is bounded, together with Claim \[B.1\] the invariance of equation (1.6) and the norm in \( \mathcal{H}_0 \) under the Lorentz transform written in \[B.3\] we get
\[
R_1(s) \leq \frac{C}{s^a}.
\]

We use again Claim \[B.1\] and the fact that \( |a^p - 1| \leq C |a - 1| \) where \( a \) is bounded to write
\[
R_2(s) \leq \frac{C}{s^a} \|q\|_{\mathcal{H}_0}^2.
\]

According to the classical inequality \( ab \leq \frac{1}{2} (a^2 + b^2) \), inequalities \([B.22]\), \([B.23]\) and \([2.38]\) on \( d' \), we can derive
\[
R_3(s) \leq \frac{C}{s^a}.
\]

We use the expression \([2.13]\) of \( w \) and exactly the same techniques used in the proof of Lemma 2.1 page 1121 in our paper \[20\], we prove
\[
I(w(s), s) \leq \frac{C}{s^a}.
\]
From the expression (2.13) of $w$ and (2.131) of $J(w(s), s)$, we can write

$$J(w(s), s) = \frac{1}{s^{a+1/2}} \int_{-1}^{1} w_1 \partial_s w_1 \rho dy - \frac{1}{s^{a+1/2}} \int_{-1}^{1} w_1 \partial_s q_1 \rho dy$$

$$- \frac{1}{s^{a+1/2}} \int_{-1}^{1} q_1 \partial_s w_1 \rho dy - \frac{1}{s^{a+1/2}} \int_{-1}^{1} q_1 \partial_s q_1 \rho dy. \tag{2.141}$$

Using the fact that $q_2 = \partial_s q_1 + d' \partial_d w_1$, the classical inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, inequalities (B.22), (B.23), (2.141) and (2.38) on $d'$, we can deduce

$$J(w(s), s) \leq \frac{C}{s^a}. \tag{2.142}$$

To conclude we need to combine the expression of the functional $E(w(s), s)$ with condition (1.25), identity (2.132), inequalities (2.133), (2.134), (2.137), (2.138), (2.139), (2.140), (2.142) together with the smallness condition (2.32) and taking $\epsilon$ small enough so that $C\epsilon^{-1} \leq \frac{\epsilon}{4}$ to deduce that

$$- \frac{C}{s^{a+1/2}} \leq E(w(s)) - E_0(\bar{w}_1(d, y, s)) \leq \left(\frac{c_2}{2} + \frac{c_3}{4}\right)\alpha_2^2 - \frac{c_3}{4}\alpha_1^2 + \frac{C}{s^a},$$

which yields (2.43) and concludes the proof of Proposition 2.5.

3 Polynomial decay of the different components

We are here mainly interested in proving the polynomial decay which may appear as a rough estimate compared to (1.27). However it is a key result to get Theorem 2. This estimate guarantee that $\|q(s)\|_H \to 0$ as $s \to \infty$ where $q$ is defined in (2.13). More precisely we prove the following proposition:

**Proposition 3.1. (Polynomial decay).** Assume that for $w \in C([s^*, \infty), H)$ a solution of equation (1.6) the following conditions hold:

$$\forall \ s \geq s^* \ E(w(s), s) \geq E_0(\bar{w}_1(d, y, s)) - \frac{C}{s^{a+1/2}}$$

and

$$\left\| \left( \begin{array}{c} w(s^*) \\ \partial_s w(s^*) \end{array} \right) - \omega^* \left( \begin{array}{c} \bar{w}_1(d^*, s^*) \\ \bar{w}_2(d^*, s^*) \end{array} \right) \right\|_H \leq \epsilon^*,$$

for some $d^* \in (-1, 1)$, $\omega^* \in \{-1, 1\}$ and $\epsilon^* \in (0, \epsilon_0]$, where $H$ and its norm are defined in (1.8), $\bar{w}_1$ and $\bar{w}_2$ are defined respectively in (1.21) and (1.22), there exists $d_\infty \in (-1, 1)$ such that

$$|d_\infty - d^*| \leq C\epsilon^*(1 - d^*^2).$$
Then there exists positive constant $C$ such that we have for all $s \geq s^*$,

$$\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \omega^* \begin{pmatrix} w_1(d_\infty, \ldots, s) \\ w_2(d_\infty, \ldots, s) \end{pmatrix} \right\|_H \leq \frac{C}{s^{a+1}},$$

(3.1)

**Remark 3.2.** Admitting Proposition 3.1, we can prove that Proposition 11 follows trivially from Proposition 3.1 by the classical triangular inequality and we use the fact that the solution $\bar{w}_1$ of equation (1.6) approaches the stationary solutions of equation (1.6) when ($f \equiv 0$) namely $\kappa(d, y)$ defined in (1.14).

A good understanding of Proposition 3.1 gives us the permission to introduce a parameter $d(s) \in (-1, 1)$ such that $\|q(s)\|_H \rightarrow 0$ as $s \rightarrow \infty$ with $q$ is defined in (2.13). Following this crucial information, we obtain the exponential decay and conclude our main Theorem 2. In the following, we showed that if $w(s^*)$ is close enough to some class of solution $w_1(d^*, \ldots, s)$ and satisfies an energy barrier, then $w(s)$ converges to a neighboring class of solution as $s \rightarrow \infty$. Our aim is to show the convergence of $\begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix}$ as $s \rightarrow \infty$ to some $\begin{pmatrix} \bar{w}_1(d_\infty, y, s) \\ \bar{w}_2(d_\infty, y, s) \end{pmatrix}$ for some $d_\infty$ close to $d^* \in (-1, 1)$ in order to obtain Proposition 3.1.

After the proof of the Proposition 2.5, we can adapt with no difficulty the proof given in the case when $f \equiv 0$ treated by Merle and Zaag [43]. It happens that the same adaptation pattern works in the present case to obtain the polynomial decay. To be accurate and concise in our result, we are going to give the detail of the proof. Let us first introduce a more adapted notation and rewrite Proposition 2.5.

If we introduce

$$\theta(s) = \frac{1}{2} \log \left( \frac{1 + d(s)}{1 - d(s)} \right), \quad A(s) = \alpha^2_1(s) \quad \text{and} \quad B(s) = \alpha^2_2(s) + 2R_-(s),$$

(note that $d(s) = \tanh(\theta(s))$), then we see from (2.37) and (2.41) that if (2.32) holds, then $|B(s) - \alpha^2_-| \leq Ce^{\bar{p}} \epsilon^{-1}(\alpha^2_1 + \alpha^2_2)$, hence

$$\frac{99}{100} \alpha^2_- - \frac{1}{100} A(s) \leq B(s) \leq \frac{101}{100} \alpha^2_- + \frac{1}{100} A(s)$$

(3.3)

for $\epsilon$ small enough. Therefore, using Proposition 2.5 estimates (2.32), (2.37) and the fact that $\theta'(s) = \frac{d'(s)}{1-d^2(s)}$, we derive the following:

**Corollary 3.3. (Equations in the new framework)** **There exist positive $\epsilon_4$, $K_0$, $K_1$ and $C_i$ for $i \in \{0, 1, 2, 3\}$ such that if $w$ is a solution to equation (1.6) such that (2.12) and (2.32) hold for some $\epsilon \leq \epsilon_4$, where $q$ is defined in (2.13), then using the notation (3.2), we have for all $s \geq s^*$,
(i) (Size of the solution)

\[ \frac{1}{K_0}(A(s) + B(s)) \leq \|q\|_H^2 \leq K_0(A(s) + B(s)) \leq K_0^2 \epsilon^2, \]  
\[ \theta'(s) \leq K_0(A(s) + B(s)) + \frac{C_0}{s^a} \leq K_0^2 \|q\|_H^2 + \frac{C_0}{s^a}, \]
\[ \left| \int_{-1}^{1} q_1 q_2 \rho dy \right| \leq K_0(A(s) + B(s)). \]  
\[ |\theta'(s)| \leq K_0(A(s) + B(s)) + \frac{C_0}{s^a} \leq K_0^2 \|q\|_H^2 + \frac{C_0}{s^a}, \]
\[ \left| \int_{-1}^{1} q_1 q_2 \rho dy \right| \leq K_0(A(s) + B(s)). \]

(ii) (Equations)

\[ \frac{3}{2} A(s) - K_0 \epsilon B(s) - \frac{C_1}{s^a} \leq A'(s) \leq \frac{5}{2} A(s) + K_0 \epsilon B(s) + \frac{C_1}{s^a}, \]
\[ B'(s) \leq -\frac{8}{p-1} \int_{-1}^{1} q_{1,2}^2 \frac{\rho}{1-y^2} dy + K_0 \epsilon (A(s) + B(s)) + \frac{C_2}{s^a}, \]
\[ \frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho dy \leq -\frac{1}{10} B(s) + K_0 \int_{-1}^{1} q_{1,2}^2 \frac{\rho}{1-y^2} dy + K_0 A(s). \]

(iii) (Energy barrier) If (1.25) holds, then

\[ A(s) \leq K_1 B(s) + \frac{C_3}{s^{\frac{a+1}{2}}}. \]

Coming at this level, we are able to establish the polynomial decay and conclude the proof of Proposition 3.1.

**Proof of Proposition 3.1** Consider \( w \in C([s^*, +\infty), H) \) for some \( s^* \) large enough a solution of equation (1.6) such that (1.25) and (1.26) hold for some \( d^* \in (-1, 1) \). Up to replacing \( w(y,s) \) by \(-w(y,s)\) we may assume that \( \omega^* = 1 \) in (3.1). Consider then \( \epsilon = 2KK_2 \epsilon^* \) where \( K \) is given in Proposition 2.2 and \( K_2 \) will be fixed later. If

\[ \epsilon^* \leq \epsilon_1 \text{ and } \epsilon \leq \epsilon_4, \]  

then we see that Proposition 2.2, Corollary 3.3 and (3.3) apply respectively with \( \epsilon^* \) and \( \epsilon \). In particular, there is a maximal solution \( d(s) \in C^1([s^*, +\infty), (-1, 1)) \) such that (2.12) holds for all \( s \in [s^*, +\infty) \) where \( q(y,s) \) is defined in (2.13) and

\[ |\theta(s^*) - \theta^*| + \|q(s^*)\|_H \leq K \epsilon^* \text{ with } \theta^* = \frac{1}{2} \log \left( \frac{1 + d^*}{1 - d^*} \right). \]  

If in addition we have

\[ K_2 \geq 1 \text{ hence, } \epsilon \geq 2K \epsilon^*, \]  

then, we can give two definitions:
• We define first from (3.12) and (3.13) \( s^*_1 \in (s^*, +\infty) \) such that for all \( s \in [s^*, s_1] \),

\[
\|q(s)\|_H < \epsilon
\]  

and if \( s^*_1 < +\infty \), then \( \|q(s^*_1)\|_H = \epsilon \).

• Then, we define \( s^*_2 \in [s^*, s^*_1] \) as the first \( s \in [s^*, s^*_1] \) such that

\[
A(s) \geq \frac{B(s)}{20K_0} + \frac{C_4}{s^{a+1}}
\]  

for some positive \( C_4 \) and \( K_0 \) is introduced in Corollary 3.3, or \( s^*_2 = s^*_1 \) if (3.15) is never satisfied on \([s^*, s^*_1]\).

Here we outline our formal approach into three steps:

• In Step 1, using (3.15), we integrate equations (3.8) and (3.9) on the time interval \([s^*, s^*_2]\) and obtain for some positive \( K_3, \eta_1, C_5, C_6 \) and some \( f_0(s) \)

\[
\forall s \in [s^*, s^*_1] \quad \frac{1}{K_3} \|q\|_H^2 - \frac{C_5}{s^{a+1}} \leq f_0(s) \leq K_3 \|q\|_H^2 + \frac{C_5}{s^{a+1}} \quad \text{and} \quad f_0'(s) \leq -\frac{\eta_1}{80} f_0(s) + \frac{C_6}{s^{a+1}}.
\]  

• In Step 2, integrating equation (3.7) satisfied by \( A(s) \) on the time interval \([s^*, s^*_2]\), we obtain some polynomial estimate.

• In Step 3, we conclude the proof by showing first that \( s^*_1 - s^*_2 \leq \sigma_0 \) for some \( \sigma_0 \), then \( s^*_1 = +\infty \). Finally, integrating the equation obtained in Step 1, we conclude the proof of Proposition 3.1.

**Step 1:** Integration of the equations on \([s^*, s^*_2]\). We claim the following.

**Claim 3.4.** There exist positive \( \epsilon_5, K_3, \eta_1, C_5, C_6, C_7 \) and \( f_0(s) \in C^1([s^*, s^*_2], \mathbb{R}^+) \) such that if \( \epsilon \leq \epsilon_5 \), then for all \( s \in [s^*, s^*_2] \),

(i) \[
\frac{1}{2} f_0(s) - \frac{C_5}{s^{a+1}} \leq B(s) \leq 2 f_0(s) + \frac{C_5}{s^{a+1}}
\]  

and

\[
f_0'(s) \leq -\frac{\eta_1}{80} f_0(s) + \frac{C_6}{s^{a+1}}.
\]  

(ii) \[
\|q(s)\|_H \leq K_3 ||q(s^*)||_H e^{-\frac{\eta_1}{160}(s-s^*)} + \frac{C_7}{s^{a+1}}.
\]
Proof: (i) By definition of $s_2^*$, we see that

$$\forall \ s \in [s^*, s_2^*], \ A(s) \leq \frac{B(s)}{20K_0} + \frac{C_4}{s^{a+1}}, \quad (3.18)$$

where $A(s)$ and $B(s)$ are defined in (3.2). Since $[s^*, s_2^*] \subset [s^*, s_1^*]$, the interval where (3.14) is satisfied, we can apply Corollary 3.3. Therefore, using equations (3.8) and (3.9), we write for all $s \in [s^*, s_2^*]$:

$$B'(s) \leq - \frac{8}{p-1} \int_{-1}^{1} q_{-2}^2 \frac{\rho}{1-y^2} dy + C\epsilon B(s) + \frac{C_2}{s^{a+1}}, \quad (3.19)$$

$$\frac{d}{ds} \int_{-1}^{1} q_1 q_2 \rho dy \leq - \frac{1}{20} B(s) + K_0 \int_{-1}^{1} q_{-2}^2 \frac{\rho}{1-y^2} dy + \frac{C_4K_0}{s^{a+1}}, \quad (3.20)$$

for some $C > 0$ depends on $K_0$ and $\epsilon$ small enough. We claim that

$$f_0(s) = B(s) + \eta_1 \int_{-1}^{1} q_1 q_2 \rho dy \quad (3.21)$$

satisfies all the desired property, where $\eta_1 > 0$ will be fixed small independant of $\epsilon$. Using (3.6) and (3.18), we see that if $\eta_1$ is small enough, then we get for all $s \in [s^*, s_2^*]$,

$$\frac{1}{2} B(s) - \frac{C_4\eta_1 K_0}{s^{a+1} \frac{s}{s^{a+1}}} \leq f_0(s) \leq 2B(s) + \frac{C_4\eta_1 K_0}{s^{a+1} \frac{s}{s^{a+1}}}. \quad (3.22)$$

Using (3.18) and the equivalence of norms (3.4), we obtain for some $C > 0$

$$\frac{1}{C} \|q(s)\|_H^2 - \frac{C_4\eta_1 K_0}{s^{a+1} \frac{s}{s^{a+1}}} \leq f_0(s) \leq C \|q(s)\|_H^2 + \frac{C_4\eta_1 K_0}{s^{a+1} \frac{s}{s^{a+1}}}. \quad (3.23)$$

We combine (3.22) and (3.23) to conclude the proof of (3.16). Then using (3.19), (3.20) and (3.21), we have for all $s \in [s^*, s_2^*]$,

$$f_0'(s) \leq - \left( \frac{1}{20} \eta_1 - C\epsilon \right) B(s) - \left( \frac{8}{p-1} - K_0 \eta_1 \right) \int_{-1}^{1} q_{-2}^2 \frac{\rho}{1-y^2} dy + \frac{C_4K_0}{s^{a+1} \frac{s}{s^{a+1}}}$$

$$\leq - \frac{\eta_1}{40} B(s) + \frac{C_4K_0}{s^{a+1} \frac{s}{s^{a+1}}}, \quad (3.24)$$

where $\eta_1$ is small enough independant of $\epsilon$ and $\epsilon$ is choosen small enough. Using (3.12), (3.22) and (3.24), to write

$$f_0'(s) \leq - \frac{\eta_1}{80} f_0(s) + \frac{C_4K_0}{s^{a+1} \frac{s}{s^{a+1}}} \quad (3.25)$$

37
We integrate \((3.31)\) between \(s^*\) and \(s\) to obtain

\[
\left( e^{\frac{n_1}{\alpha_0}(s-s^*)} f_0(s) \right) = \frac{C_4 K_0 e^{\frac{n_1}{\alpha_0}(s-s^*)}}{s^{a+1}}. \tag{3.26}
\]

That’s imply that there exists \(C_7 > 0\) depends on \(C_4, K_0\) and \(\eta_1\) such that

\[
f_0(s) \leq e^{-\frac{n_1}{\alpha_0}(s-s^*)} f_0(s^*) + C_7 K_0 e^{-\frac{n_1}{\alpha_0}s} \int_{s^*}^{s} e^{\frac{n_1}{\alpha_0}t} dt. \tag{3.27}
\]

(ii) Multiplying inequality \((3.25)\) by \(e^{\frac{n_1}{\alpha_0}(s-s^*)}\), we obtain the following:

\[
\left( e^{\frac{n_1}{\alpha_0}(s-s^*)} f_0(s) \right) \leq \frac{C_4 K_0 e^{\frac{n_1}{\alpha_0}(s-s^*)}}{s^{a+1}}. \tag{3.26}
\]

We integrate \((3.26)\) between \(s^*\) and \(s\) to obtain

\[
f_0(s) \leq e^{-\frac{n_1}{\alpha_0}(s-s^*)} f_0(s^*) + C_7 K_0 e^{-\frac{n_1}{\alpha_0}s} \int_{s^*}^{s} e^{\frac{n_1}{\alpha_0}t} dt. \tag{3.27}
\]

Using \((3.23)\) to conclude the proof of Claim \(3.4\).

\begin{flushright}
\textbf{Claim 3.5:} Integration of the equations on \([s_2^*, s_1^*]\). We claim the following.
\end{flushright}

\begin{itemize}
\item[(i)] There exists \(\epsilon_6 > 0\) such that for all \(\sigma > 0\), there exist \(K_4(\sigma) > 0\) and \(C_8(\sigma) > 0\) such that if \(\epsilon \leq \epsilon_6\), then

\[
\forall \ s \in [s_2^*, \min(s_2^* + \sigma, s_1^*)], \quad \|q(s)\|_H \leq K_4 e^{-\frac{n_1}{\alpha_0}(s-s^*)} \|q(s^*)\|_H + \frac{C_8}{(s_2^*)^{a+1}}.
\]

\item[(ii)] There exist positive \(\epsilon_7, C_{12}, C_{13}\) and \(C\) such that if \(\epsilon \leq \epsilon_7\) and \(s \geq s^*\), then

\[
\forall \ s \in [s_2^*, s_1^*], \quad B(s) \leq (A(s) + \frac{C_{12}}{s^a}) \left( 20 K_0 e^{\frac{(s-s_2^*)^2}{2}} + 2 C_1 + \frac{C_{13}}{s^a} \right), \tag{3.29}
\]

where \(K_0\) is introduced in Corollary \(3.33\).
\end{itemize}

\textbf{Proof:} (i) Using equations \((3.7)\) and \((3.8)\), we see that for all \(s \in [s_2^*, \min(s_2^* + \sigma, s_1^*)]\),

\[
(A(s) + B(s))' \leq 3 (A(s) + B(s)) + \frac{C}{s^a}, \tag{3.30}
\]

with \(C = \max(C_1, C_2)\). Multiplying inequality \((3.30)\) by \(e^{-3(s-s_2^*)}\), we obtain

\[
\left( e^{-3(s-s_2^*)}(A(s) + B(s)) \right)' \leq \frac{C e^{-3(s-s_2^*)}}{s^a}. \tag{3.31}
\]

We integrate \((3.31)\) between \(s_2^*\) and \(s\), we write

\[
A(s) + B(s) \leq e^{3(s-s_2^*)} \left( A(s_2^*) + B(s_1^*) \right) + C e^{3(s-s_2^*)} \int_{s_2^*}^{s} \frac{e^{-3(t-s_2^*)}}{t^a} dt. \tag{3.32}
\]
Thanks to (3.4) and we use the fact that $\frac{1}{2} \leq \frac{s}{s_2^*} \leq 2$, we can see that there exist $C > 0$ such that

$$\|q(s)\|_H \leq K_0 e^{\frac{2\sigma}{2}} \|q(s_2^*)\|_H + \frac{C e^{\frac{2\sigma}{2}}}{(s_2^*)^a}.$$  (3.33)

Using (ii) in Claim 3.3 with $s = s_2^*$ gives the conclusion of (i).

(ii) By definition of $s_1^*$, (3.14) is satisfied for all $s \in [s_2^*, s_1^*]$, hence, Corollary 3.3 applies and equations (3.7) and (3.8) holds.

Let us first prove that

$$\forall \ s \in (s_2^*, s_1^*], \ A(s) \geq \frac{B(s)}{20K_0} + \frac{C_4}{s^a},$$  (3.34)

where $K_0$ is introduced in Corollary 3.3. We need to assume that $s_2^* < s_1^*$, otherwise the set $(s_2^*, s_1^*)$ is empty. Let $g_0(s) = A(s) - \frac{B(s)}{20K_0} - \frac{C_4}{s^a}$, where $A(s)$ and $B(s)$ are defined in (3.2). From equations (3.7) and (3.8), we write for some $C > 0$ and for all $s \in [s_2^*, s_1^*]$

$$A'(s) \geq \frac{3}{2} A(s) - C\epsilon B(s) - \frac{C_1}{s^a}, \quad B'(s) \leq C\epsilon (A(s) + B(s)) + \frac{C_2}{s^a},$$  (3.35)

then, we derive the function $g_0$, we obtain

$$g_0'(s) = \left( A(s) - \frac{B(s)}{20K_0} - \frac{C_4}{s^a} \right)'$$

$$\geq \frac{3}{2} A(s) - C\epsilon B(s) - \frac{C_1}{s^a} - \frac{C_9}{20K_0} (A(s) + B(s)) + \frac{C_4(a+1)}{2s^{\frac{a+3}{2}}}$$

$$\geq A(s) - \frac{B(s)}{20K_0} - \frac{C_4}{s^{\frac{a+1}{2}}} + \frac{C_4(a+1)}{2s^{\frac{a+1}{2}}} + \frac{C_4}{s^{\frac{a+1}{2}}} - \frac{C_9}{s^a},$$  (3.36)

for $\epsilon$ small enough. Note that $s^*$ is large enough, so for all $s \geq s^*$, we have:

$$\frac{C_4}{s^{\frac{a+1}{2}}} - \frac{C_9}{s^a} = \frac{C_4}{s^{\frac{a+1}{2}}} \left( 1 - \frac{C_9}{C_4 s^{\frac{a+1}{2}}} \right) \geq \frac{C_4}{2s^{\frac{a+3}{2}}} \geq \frac{C_4}{2s^{\frac{a+3}{2}}}.$$  (3.37)

In view of (3.36) and (3.37), we write

$$g_0'(s) \geq A(s) - \frac{B(s)}{20K_0} - \frac{C_4}{s^{\frac{a+1}{2}}} + \frac{C_4(a+2)}{2s^{\frac{a+3}{2}}} = g_0(s) + \frac{C_4(a+2)}{2s^{\frac{a+3}{2}}}.$$  (3.36)

Since, by definition of $s_2^*$, we have $g_0(s_2^*) \geq 0$ and (3.34) follows.

Using (3.34) and (3.35), we obtain for $\epsilon$ small enough and for all $s \in (s_2^*, s_1^*)$ the following

$$A'(s) \geq \frac{3}{2} A(s) - 20K_0 C\epsilon (A(s) - \frac{C_4}{s^{\frac{a+1}{2}}}) - \frac{C_1}{s^a} \geq A(s) - \frac{C_1}{s^a},$$  (3.38)
The same reasoning in (3.33) can be applied to write the following

\[ A(s) \geq e^{s-s_2}A(s_2) - \frac{C_{10}}{s^a}, \]  

(3.39)

If \( q(s_2^*) \equiv 0 \), then \( w(y, s_2^*) \equiv w_1(d(s_2^*), y, s_2^*) \) by (2.13) and from the uniqueness of solutions to equation (1.6), we have \( w(y, s) \equiv w_1(d(s_2^*), y, s_2^*) \) and \( q(y, s) \equiv 0 \) for all \( s \geq s_2^* \), hence \( A(s) = B(s) = 0 \) by (3.4) and (3.29) follows trivially.

Now, if \( q(s_2^*) \neq 0 \), we can define \( h(s) = \frac{B(s)}{A(s) + \frac{C_{11}}{s^a}} \) for all \( s \in (s_2^*, s_1^*] \) and derive from (3.35) and (3.38) for all \( s \in (s_2^*, s_1^*] \)

\[
h'(s) = \frac{B'(s)(A(s) + \frac{C_{11}}{s^a}) - B(s)(A'(s) - \frac{aC_{11}}{s^{a+1}})}{(A(s) + \frac{C_{11}}{s^a})^2}
\leq \frac{(C\epsilon(A(s) + B(s)) + \frac{C_{11}}{s^a})(A(s) + \frac{C_{11}}{s^a}) - B(s)A'(s) + \frac{aC_{11}B(s)}{s^{a+1}}}{(A(s) + \frac{C_{11}}{s^a})^2}
\leq \left(C\epsilon + \frac{aC}{s^{a+1}e^{s-s_2^*}A(s_2^*)} + \frac{2C}{s^a e^{s-s_2^*}A(s_2^*) - 1}\right)h(s) + C\epsilon + \frac{C - C\epsilon}{s^a e^{s-s_2^*}A(s_2^*)}
\]

for \( \epsilon \) small enough. We know that \( s^* \) is large enough and \( \epsilon \) is small enough, one can check that for all \( s \geq s^* \), we have:

\[
h'(s) \leq -\frac{h(s)}{2} + C\epsilon + \frac{C}{2s^a e^{s-s_2^*}A(s_2^*)}. \]  

(3.40)

Integrating (3.40) between \( s_2^* \) and \( s \) to obtain

\[
h(s) \leq e^{-\frac{s-s_2^*}{2}}h(s_2^*) + 2C\epsilon + \frac{C}{s^a}, \]  

(3.41)

where \( C \) depends on \( a, s_2^* \) and \( A(s_2^*) \), we can write now

\[
B(s) \leq (A(s) + \frac{C_{11}}{s^a}) \left(e^{-\frac{s-s_2^*}{2}} \frac{B(s_2^*)}{A(s_2^*)} + \frac{C - C\epsilon}{s^a}\right).
\]

Using (3.41) and taking \( \epsilon \) small enough gives (3.29) and concludes the proof of Claim 3.5

**Step 3:** Conclusion of the proof. We use Step 1 and Step 2 to conclude the proof of Proposition 3.1 here. Let us first fix \( \sigma_0 \) such that

\[
20K_0e^{-\sigma_0} + 2C\epsilon + \frac{C_{13}}{\sigma_0^6} \leq \frac{1}{2K_1}, \]  

(3.42)
where \( K_0, K_1 \) are introduced in Corollary \( \text{3.3} \). Then, we impose the condition
\[
\epsilon = 2K_2K\epsilon^*, \quad \text{where} \quad K_2 = \max(2, K_3, K_4).
\]

Finally, we fix
\[
\epsilon_0 = \min \left( 1, \epsilon_i, \quad \text{for} \quad i \in \{1, 4, 5, 6, 7\} \right), \quad C_{14} = \max(C_0, C_7, C_8)
\]
and the constants are defined in Proposition \( \text{2.2} \) Corollary \( \text{3.3} \) and Claims \( \text{3.4} \) and \( \text{3.5} \).

Now if \( \epsilon^* \leq \epsilon_0 \), then Corollary \( \text{3.3} \) and Steps 1 and 2 apply. We claim that for all \( s \in [s^*, s_1^*] \),
\[
\|q(s)\|_{\mathcal{H}} \leq K_2\|q(s^*)\|_{\mathcal{H}}e^{-\frac{\theta_0}{2}} + \frac{C_{14}}{s} \leq K_2\epsilon^*e^{-\frac{\theta_0}{2}} + \frac{C_{14}}{s}.
\]
Indeed, if \( s \in \left[ s^*, \min(s^*_2 + \sigma_0, s_1^*) \right] \), then this comes from (ii) of Claim \( \text{3.3} \) or (i) of Claim \( \text{3.5} \) and the definition \( \text{(3.43)} \) of \( K_2 \).

Now, if \( s^*_2 + \sigma_0 < s_1^* \) and \( s \in \left[ s^*_2 + \sigma_0, s_1^* \right] \), then we have from \( \text{(3.29)} \) and the definition of \( \sigma_0 \) that \( B(s) \leq \frac{1}{2K_1} \left( A(s) + \frac{C_{11}}{s} \right) \) on the one hand. On the other hand, from (iii) of Corollary \( \text{3.3} \) we have \( A(s) \leq K_1B(s) + \frac{C_{12}}{s} \), we use \( \text{(3.4)} \) to deduce that \( \text{(3.44)} \) is satisfied.

In particular, we have for all \( s \in [s^*, s_1^*] \), \( \|q(s)\|_{\mathcal{H}} \leq \frac{\epsilon}{2} + \frac{C_{14}}{s} \). Note that \( s^* \) is large enough, so we can say that for all \( s \in [s^*, s_1^*] \), we have:
\[
\|q(s)\|_{\mathcal{H}} \leq \frac{3\epsilon}{4},
\]
hence, by definition of \( s_1^* \), this means that \( s_1^* = +\infty \). Therefore, from \( \text{(3.44)} \) and \( \text{(3.5)} \), we have
\[
\forall \ s \geq s^*, \quad \|q(s)\|_{\mathcal{H}} \leq \frac{C_{15}}{s} \quad \text{and} \quad |\theta'(s)| \leq \frac{C_{16}}{s},
\]
Hence, there is \( \theta_\infty \in \mathbb{R} \) such that \( \theta(s) \to \theta_\infty \) as \( s \to \infty \) and
\[
\forall \ s \geq s^*, \quad |\theta(s) - \theta_\infty| \leq \frac{C_{17}}{s^\frac{1}{2}}.
\]
Taking \( s = s^* \) here and using \( \text{(3.12)} \), we see that \( |\theta_\infty - \theta^*| \leq C\epsilon^* \).
If \( d_\infty = \tanh \theta_\infty \), then we see that \( |d_\infty - d^*| \leq C(1 - d^2)\epsilon^* \).

Using the definition of \( q \) given in \( \text{2.13} \), inequalities \( \text{(3.37)}, \text{(3.45)} \) and \( \text{(3.46)} \), we write
\[
\left\| \left( \begin{array}{c} w(s) \\ \partial_x w(s) \end{array} \right) \right\|_{\mathcal{H}} - \omega^* \left( \begin{array}{c} \underline{w}_1(d_\infty, y, s) \\ \underline{w}_2(d_\infty, y, s) \end{array} \right)
\]
\[
\leq \left\| \left( \frac{w(s)}{\partial_s w(s)} \right) - \omega^* \left( \frac{w_1(d(s), y, s)}{w_2(d(s), y, s)} \right) \right\|_H + \left\| \left( \frac{w_1(d(s), y, s)}{w_2(d(s), y, s)} \right) - \omega^* \left( \frac{w_1(d_\infty, y, s)}{w_2(d_\infty, y, s)} \right) \right\|_H
\leq \| q(s) \|_H + C|\theta(s) - \theta_\infty| \leq \frac{C_{18}}{s^4}.
\]

This concludes the proof of Proposition 3.1.

As a consequence of our polynomial decay obtained in Proposition 3.1 we can deduce that for \( d(s) \in C^1([s^*, +\infty), (-1, 1)) \) introduced in (2.12), we have
\[
\| q(s) \|_H \to 0 \text{ as } s \to +\infty.
\]

This crucial information helps us to obtain in the next section the desired exponential decay.

3.1 Proof of Theorem 2

In this subsection we prove Theorem 2. We should keep in mind that from Proposition 3.1 we have \( \| q(s) \|_H \to 0 \text{ as } s \to +\infty \) for the same \( d(s) \) introduced in (2.12). Following this information, we can see that
\[
A(s) + B(s) \to 0 \text{ as } s \to \infty,
\]
where \( A(s) \) and \( B(s) \) are defined in (3.2). We first show that \( A(s) \) is controlled by \( B(s) \), which is not our goal but this control and suitable refinements of some results obtained in the previous part allow us to find the exponential decay and conclude the proof of our Theorem 2. We start by the following lemma.

Lemma 3.6. For all \( s \geq s^* \), we have:
\[
A(s) \leq \frac{1}{4}B(s).
\]

Proof: We define for all \( s \geq s^* \) the function:
\[
\gamma(s) = A(s) - \frac{1}{4}B(s).
\]

A direct consequence of (2.39) and (2.40), we can choose \( \epsilon \) small enough in Corollary 3.3 and \( s^* \) large such that for all \( s \geq s^* \), the following estimates hold:
\[
A'(s) \geq \frac{1}{2}A(s) - \frac{3}{64}B(s),
\]
\[
B'(s) \leq A(s) + \frac{1}{16}B(s).
\]

42
We can see that
\[
\gamma'(s) = A'(s) - \frac{1}{4} B'(s) \geq \frac{1}{4} \gamma(s),
\]
since \( \gamma(s) \to 0 \) as \( s \to \infty \) (see (3.47)), implies \( \gamma(s) \leq 0 \), which conclude the proof of Lemma 3.6.

This way, we are in a position to perform some refinements of the estimates on the function \( f_0 \) defined in (3.21) and some other estimates obtained in the previous subsection. More precisely, we have the following:

**Claim 3.7.** For all \( s \geq s^* \), we have

\[
\frac{1}{2} f_0(s) \leq B(s) \leq 2 f_0(s),
\]

(3.48)

\[
f'_0(s) \leq -\frac{\eta_1}{8} f_0(s),
\]

(3.49)

\[
\|q(s)\|_\mathcal{H} \leq C \|q(s^*)\|_\mathcal{H} e^{-\frac{\eta_1}{16} (s-s^*)} \leq \tilde{K} e^{-\frac{\eta_1}{16} (s-s^*)},
\]

(3.50)

\[
|\theta'(s)| \leq \tilde{K}^2 e^{-\frac{\eta_1}{16} (s-s^*)},
\]

(3.51)

such that \( \tilde{K} \) depends on \( \epsilon \), the constant \( K_0 \) and the function \( f_0 \) defined respectively in (3.21) and Corollary 3.3.

**Proof:** Using (3.6), (3.21) and Lemma 3.6 we see that if \( \eta_1 \) is small enough, then we get

\[
\frac{1}{2} B(s) \leq f_0(s) \leq 2 B(s).
\]

(3.52)

We use again Lemma 3.6 estimate (3.52) and the equivalence of norms (3.4), we obtain for some \( C > 0 \)

\[
\frac{1}{C} \|q(s)\|_\mathcal{H}^2 \leq f_0(s) \leq C \|q(s)\|_\mathcal{H}^2.
\]

(3.53)

Using (2.40), Corollary 3.3 and Lemma 3.6 we get for some \( C > 0 \) and for all \( s \geq s^* \)

\[
B'(s) \leq -\frac{8}{p-1} \int_{-1}^{1} q_{-2}^2 \frac{\rho}{1-y^2} dy + \frac{5K_0 \epsilon}{4} B(s) + \frac{C}{s^a} B(s),
\]

(3.54)

\[
\frac{d}{ds} \int_{-1}^{1} q_{1} q_{2} \rho dy \leq -\frac{1}{20} B(s) + K_0 \int_{-1}^{1} q_{-2}^2 \frac{\rho}{1-y^2} dy.
\]

(3.55)

Then using (3.54), (3.55), taking \( \epsilon \) small enough and \( s^* \) large enough, we obtain for all \( s \geq s^* \)

\[
f'_0(s) \leq -\frac{1}{20} \eta_1 - \frac{5K_0 \epsilon}{4} - \frac{C}{s^a} B(s) - (\frac{8}{p-1} - K_0 \eta_1) \int_{-1}^{1} q_{-2}^2 \frac{\rho}{1-y^2} dy \leq -\frac{\eta_1}{4} B(s)
\]

43
We conclude the proof of (3.49). Integrating (3.56), we get for all \( s \geq s^* \), \( f_0(s) \leq f_0(s^*) e^{-\eta_{18}(s-s^*)} \). Using (3.53) and the a priori estimate (2.32), we can deduce that (3.50) follows. The proof of (3.51) follows directly from (3.50). Which conclude the proof of Claim 3.7.

At this level, we are ready to adapt the proof of the case when \( f \equiv 0 \) and deduce our main result in Theorem 2. We are going to give the deduction of the proof:

From (3.51), we can see that there exists \( \theta_\infty \in \mathbb{R} \) such that \( \theta(s) \to \theta_\infty \) as \( s \to \infty \) and

\[
\forall \ s \geq s^*, \quad |\theta(s) - \theta_\infty| \leq \tilde{K} e^{-\eta_{18}(s-s^*)},
\]

Using the definition of \( q \) given in (2.13), inequalities (B.37), (B.38), (3.50) and (3.51), we write:

\[
\left\| \left( w(s) \overline{\omega}(d_\infty, y, s) \right) - \omega^* \left( \frac{\overline{w}_1(d_\infty, y, s)}{\overline{w}_2(d_\infty, y, s)} \right) \right\|_{\mathcal{H}} \\
\leq \left\| \left( \frac{w(s)}{\partial_s w(s)} \right) - \omega^* \left( \frac{\overline{w}_1(d(s), y, s)}{\overline{w}_2(d(s), y, s)} \right) \right\|_{\mathcal{H}} + \left\| \omega^* \left( \frac{\overline{w}_1(d_\infty, y, s)}{\overline{w}_2(d_\infty, y, s)} \right) \right\|_{\mathcal{H}} \\
\leq \|q(s)\|_{\mathcal{H}} + C|\theta(s) - \theta_\infty| \leq \tilde{K}_0 e^{-\eta_{18}(s-s^*)},
\]

where \( \tilde{K}_0 \) depends on \( \tilde{K} \). This concludes the proof of Theorem 2.

**Acknowledgment:** The authors wish to thank Professor Hatem ZAAG for many fruitful discussions, valuable suggestions and guidance in this work. Part of this work was done when the second author was visiting the Laboratoire Analyse Géométrie et Applications (LAGA) of university Sorbonne Paris Nord. He is grateful to LAGA for the hospitality and the stimulating atmosphere.

### A Construction of a particular solution of equation (1.6) in similarity variables

As already written in the introduction, our aim in this section is to show the detail of the construction of the solution defined in (1.21) and (1.22). We start by the case where \( d = 0 \) and the deduction of the case where \( d \neq 0 \) will follow from the fact that (1.1) is invariant under the Lorentz transform.

**Case** \( d = 0 \). Note that \( \kappa(d, y) = \kappa_0 \) is not a solution of equation (1.6). In order to find \( \phi(s) \) a solution of equation (1.6) such that \( \phi(s) \to \kappa_0 \) as \( s \to +\infty \), it is equivalent to show a solution \( \varphi_T(t) \) of equation (1.1) such that \( \varphi_T(t) \) blows-up at time \( T \) with \( \varphi_T(t) \sim \kappa_0 (T - t)^{-\frac{2}{p-1}} \) as \( t \to T \).

**Case** \( d \neq 0 \). We know that equation (1.1) is invariant under the Lorentz transform. So,
for any $d \in (-1, 1)$, if we define $\varphi(d, x, t) = \varphi_0(\frac{t-x}{\sqrt{1-d^2}})$, then $\varphi$ is a solution of equation (1.1) which blows-up at the line $\{t = dx\}$. Let us transform it with the definition of $w$:

$$y = -\frac{x}{t}, \quad s = -\log(-t) \text{ and } w_1(d, y, s) = (-t)^{\frac{2}{p-1}} \varphi(d, x, t). \tag{A.1}$$

Some simple computation gives:

$$w_1(d, y, s) = e^{\frac{2s}{p-1}} \varphi_0 \left( -\frac{1}{\sqrt{1-d^2}} \right).$$

In other words, we look in the following lemma for $\varphi_T(t) \in C^2([-t_0, T), \mathbb{R})$ a solution of $\varphi'' = \varphi^p + f(\varphi)$ with $\varphi_T(t) \sim \kappa_0(T-t)^{-\frac{p}{p-1}}$ as $t \to T$ and $f$ defined in (1.2).

**Lemma A.1.** The following items hold,

(i) There exists $\varphi_0(t) \in C^2([-t_0, 0), \mathbb{R})$ solution of $\varphi'' = \varphi^p(t) + f(\varphi(t))$, with $\varphi_0(t) \sim \kappa_0(-t)^{-\frac{p}{p-1}}$ as $t \to 0$.

(ii) If $\varphi_T(t) = \varphi_0(t-T)$, then $\varphi_T''(t) = \varphi_T^p(t) + f(\varphi_T(t))$, with $\varphi_T(t) \sim \kappa_0(T-t)^{-\frac{p}{p-1}}$ as $t \to T$.

(iii) $\phi(s) = e^{\frac{2s}{p-1}} \varphi_T(T-e^s) = e^{\frac{2s}{p-1}} \varphi_0(-e^s)$.

**Proof:** We have the following associated ODE to the PDE (1.1).

$$\begin{cases}
\varphi''(t) = \varphi^p(t) + f(\varphi(t)), \\
(\varphi(0), \varphi'(0)) = (A, B),
\end{cases} \tag{A.2}$$

with $A > 0$ and $B = B(A) > 0$ such that $B^2 - \frac{Ap}{p+1} \geq 0$. By the Cauchy theory, there exists a maximal solution $\varphi_A(t)$ defined in $[0, T_A)$, with $T_A \leq +\infty$. In order to conclude the proof of Lemma A.1 we proceed in 3 steps.

**Step 1:** In this step we prove that for $A$ large enough and for all $t \in [0, T_A)$, we have $\varphi'_A(t) \geq 0$. Let $A_0$ large enough, such that

$$\forall \xi \geq A_0, \quad |f(\xi)| \leq \frac{\xi^p}{2},$$

we consider $A \geq A_0$. Since, we have $\varphi_A''(t) \geq \frac{\varphi_A^p(t)}{2}$, whenever $\varphi_A(t) \geq A$, using a contradiction argument, the result follows.

**Step 2:** The solution $\varphi_A(t)$ is an increasing function, so for all $t \in [0, T_A)$, we have $\varphi_A(t) \geq \varphi_A(0) = A > 0$.

Recalling that $\varphi'_A(t) \geq 0$, so, by multiplying inequality $\varphi_A''(t) \geq \frac{\varphi_A^p(t)}{2}$ by $\varphi'(t)$, we obtain

$$\frac{d}{dt}(\varphi_A(t)) \geq 0,$$

45
with \( \Xi(\varphi_A(t)) = (\varphi_A(t))^2 - \frac{\varphi_A^{p+1}(t)}{p+1} \). From the fact that \( B^2 \geq \frac{A p + 1}{p+1} \), we obtain
\[
\Xi(\varphi_A(t)) \geq \Xi(\varphi_A(0)) \geq 0.
\]
As mentioned above, \( \varphi_A(t) > 0 \) for all \( t \in [0, T_A) \), so
\[
\frac{\varphi_A'(t)}{\varphi_A^p(t)} \geq \frac{1}{\sqrt{p+1}}.
\]
After integration between 0 and \( T_A \), we deduce that \( \varphi_A(t) \) blow’s up in finite time \( T_A \).

**Step 3:** If \( \varphi(t) \) is a solution of equation (A.2) which blow’s up in finite time \( T \) (for example \( \varphi(t) = \varphi_A(t) \) constructed in Step 2). If \( \epsilon > 0 \), then there exists \( t_0(\epsilon) \) such that \( \forall \ t \in [t_0(\epsilon), T) \), we have:
\[
|f(\varphi_A(t))| \leq \epsilon \varphi_A^p(t).
\]
Next, we multiply (A.2) by \( \varphi' \) and we integrate between 0 and \( t \) in order to get the following:
\[
2\frac{\varphi_A^{p+1}(t)(1 - \epsilon)}{p+1} - C \leq \varphi_A^2(t) \leq 2\frac{\varphi_A^{p+1}(t)(1 + \epsilon)}{p+1} + C,
\]
with \( C \leq \epsilon \varphi_A^{p+1}(t)/(p+1) \). Solving (A.3), yields to
\[
\kappa_0(T_A - t)^{-2/(p-1)} (1 - 2\epsilon)^{-\frac{1}{p-1}} \leq \varphi_A(t) \leq \kappa_0(T_A - t)^{-2/(p-1)} (1 + 2\epsilon)^{-\frac{1}{p-1}},
\]
where \( \kappa_0 \) is introduced in (1.14). Since \( \epsilon > 0 \) is arbitrary, we can find that
\[
\varphi_A(t) \sim \kappa_0(T - t)^{-\frac{2}{p-1}},
\]
which concludes the proof of Lemma A.1.

**Remark A.2.** Combining (i) and (iii) of Lemma A.1 we see that \( \phi(s) \) is a solution of equation (A.6) with
\[
\phi(s) \rightarrow \kappa_0 \text{ as } s \rightarrow +\infty.
\]

**Remark A.3.** In particular, when \( d = 0 \), we show the following equivalence
\[
\phi(s) = e^{-\frac{2s}{p-1}} \varphi_0(-e^{-s}) \iff \varphi_0(t) = \phi(s)(-t)^{-\frac{2}{p-1}}.
\]
We now focus on the case when \( d \neq 0 \), we exploite the expression (1.4) of \( \kappa(d, y) \) to write the following
\[
\overline{w}_1(d, y, s) = e^{-\frac{2s}{p-1}} \varphi_0(-e^{-s} \frac{1 + dy}{\sqrt{1 - d^2}}) = e^{-\frac{2s}{p-1}}(e^{-s} \frac{1 + dy}{\sqrt{1 - d^2}})^{\frac{2}{p-1}} \phi(-\log(e^{-s} \frac{1 + dy}{\sqrt{1 - d^2}}))
\]
\[
= \frac{1}{(1 + dy)^{\frac{1}{p-1}}} \phi(s) - \log(\frac{1 + dy}{\sqrt{1 - d^2}}) = \kappa(d, y) \frac{\phi(s) - \log(\frac{1 + dy}{\sqrt{1 - d^2}})}{\kappa_0}.
\]
Finally, we find our solution defined in (1.21) and (1.22).
Remark A.4. In the case when $f \equiv 0$, $\varphi_0 = \kappa_0(-t)^{-\frac{2}{p-1}}$ and $\varphi(d, x, t) = \kappa_0 \left( \frac{-t+dx}{\sqrt{1-d^2}} \right)^{-\frac{2}{p-1}}$, we can see that

$$\varpi_1(d, y, s) = \kappa(d, y).$$

In this case, $\varphi_0 \sim \kappa_0(-t)^{-\frac{2}{p-1}}$, therefore

$$\varpi_1(d, y, s) \sim \kappa(d, y) \text{ as } s \to \infty. \quad (A.5)$$

B Property of the particular solution $\phi(s)$

The following claim shows the asymptotic behavior of the particular solution $\phi(s)$ of equation (1.6), which is crucial in many steps in this paper.

Claim B.1. (Equivalent to $\phi(s) - \kappa_0$) For all $p > 1$ and $a > 1$, we have,

(i) $\phi(s) - \kappa_0 \sim -\kappa_0 \left( \frac{p-1}{4s} \right)^a$ as $s \to +\infty$,

(ii) $|\phi'(s)| \leq \frac{C}{s^2}$,

where $\phi(s)$ satisfies equation (1.6), the constant $\kappa_0$ is defined in (1.14) and $C > 0$.

Proof: The proof of item (ii) is a direct consequence from the proof of (i) and we use the fact that $\phi(s)$ satisfies equation (1.6). We are going now to give the proof of item (i).

We proceed in three steps:

Step 1: Let us recall the following equation:

$$\varphi_0''(t) = \varphi_0'(t) + f(\varphi_0(t)), \quad (B.1)$$

and also from the previous part $\varphi_0(t) \sim \kappa_0(-t)^{-\frac{2}{p-1}}$, $\varphi_0'(t) > 0$ and $\varphi_0''(t) > 0$.

Multiplying equation (B.1) by $\varphi_0(t)$ and integrating between 0 and $t$, we can see that

$$\varphi_0(t) = \sqrt{\frac{2\varphi_0^{p+1}(t)}{p+1}} + 2F(\varphi_0(t)) + M_0, \quad (B.2)$$

where $M_0 = M_0 \left( \varphi_0'(0), \varphi_0(0) \right)$ and $F$ is defined in (1.10).

Step 2: We use now the following self-similar change of variables:

$$\phi(s) = (-t)^{\frac{2}{p-1}} \varphi_0(t) \text{ and } s = -\log(-t). \quad (B.3)$$

Some computation gives

$$\phi'(s) = -\frac{2}{p-1} \phi(s) + \sqrt{\frac{2\varphi_0^{p+1}(s)}{p+1}} + 2e^{-\frac{2(p+1)s}{p-1}} F(e^{-\frac{2s}{p-1}} \phi(s)) + e^{-\frac{2(p+1)s}{p-1}} M_0. \quad (B.4)$$
**Step 3:** According to Step 1 and Step 2, we may try to find an equivalent to $\phi(s) - \kappa_0$. If we note by $q = \phi(s) - \kappa_0$, according to equation (B.4), we can see that $q(s)$ satisfies the following equation

$$q'(s) = -\frac{2q(s) + \kappa_0}{p-1} + \sqrt{\frac{2(q(s) + \kappa_0)(p+1)}{p+1} + e^{-2(q(s) + \kappa_0)(p+1)}(2F(e^{-q(s) + \kappa_0}) + M_0) + e^{-q(s) + \kappa_0}(2F(e^{-q(s) + \kappa_0}) + M_0)}.$$  \hspace{1cm} (B.5)

Using Taylor expansion to derive formally from (A.4) the following

$$(q(s) + \kappa_0)^{p+1} = \kappa_0^{p+1} + (p+1)\kappa_0^p q(s) + O(q^2(s)).$$ \hspace{1cm} (B.6)

An integration by part gives when $\theta \to +\infty$

$$F(\theta) = \frac{\theta^{p+1}}{(p+1)\log^a(2 + \theta^2)} + O\left(\frac{\theta^{p+1}}{\log^{a+1}(2 + \theta^2)}\right).$$ \hspace{1cm} (B.7)

According to (B.6) and (B.7), we can write

$$F(e^{-q(s) + \kappa_0}) = \frac{e^{2q(s) + \kappa_0}}{(p+1)\left(1 + \frac{4s}{p-1}\right)^{a}} + o\left(\frac{2^{(p+1)a}}{s^a}\right).$$ \hspace{1cm} (B.8)

Combining equation (B.5), (B.6) and (B.8), we write

$$q'(s) = -\frac{2\kappa_0}{p-1} - \frac{2q(s)}{p-1} + \sqrt{\frac{2}{p+1}\left[\kappa_0^{p+1} + (p+1)\kappa_0^p q(s) + \frac{\kappa_0^{p+1}}{2}\right] + O(q^2(s)) + o\left(\frac{1}{s^a}\right).} $$ \hspace{1cm} (B.9)

From (B.9) and the fact that $\sqrt{\frac{2\kappa_0^{p+1}}{p+1}} = \frac{2\kappa_0}{p-1}$, it is simple to write the following

$$q'(s) = -\frac{2\kappa_0}{p-1} - \frac{2q(s)}{p-1} + \frac{2\kappa_0}{p-1}\sqrt{1 + \frac{p+1}{\kappa_0} q(s) + \left(\frac{p-1}{4s}\right)^a} + O(q^2(s)) + o\left(\frac{1}{s^a}\right).$$

Using Taylor expansion of the function $Z \to \sqrt{1+Z}$ as $Z \to 0$, applying at $Z = \frac{p+1}{\kappa_0} q(s) + \left(\frac{p-1}{4s}\right)^a + O(q^2(s)) + o\left(\frac{1}{s^a}\right)$, we derive

$$q'(s) = -\frac{2\kappa_0}{p-1} - \frac{2q(s)}{p-1} + \frac{2\kappa_0}{p-1}\left(1 + \frac{p+1}{2\kappa_0} q(s) + \frac{1}{2}\left(\frac{p-1}{4s}\right)^a + O(q^2(s)) + o\left(\frac{1}{s^a}\right)\right).$$ \hspace{1cm} (B.10)

Finally, by (B.10) it is obvious to write

$$q'(s) = q(s) + \frac{\kappa_0}{p-1}\left(\frac{p-1}{4s}\right)^a + O(q^2(s)) + o\left(\frac{1}{s^a}\right).$$ \hspace{1cm} (B.11)
We would like now to prove that \( q(s) = \phi(s) - \kappa_0 \sim -\frac{c_p}{s^a} \), where \( c_p = \frac{\kappa_0(p-1)a^{-1}}{4a} \). To do that let \( g_-(s) = -\frac{1}{s^{a+\frac{1}{2}}} \), \( g_+(s) = \frac{1}{s^{a+\frac{1}{2}}} \) and \( g(s) = -\frac{c_p}{s^a} \). We start by remarking that the flow is transverse outgoing in the curve of \( g_+ \) and \( g_- \). We explain in the following this fact. Firstly, we start by the case when \( q(s) = g_-(s) \). A simple derivation gives \( g'_-(s) = -\frac{1}{s^{a+\frac{1}{2}}} > 0 \). We exploite \( (B.11) \) to write when \( s \) is large enough
\[
q'(s) = -\frac{1}{s^{a+\frac{1}{2}}} + \frac{c_p}{s^a} + O\left(\frac{1}{s^{2a-1}}\right) + o\left(\frac{1}{s^a}\right) < 0.
\] (B.12)
We conclude that \( q'(s) < g'_-(s) \) for \( s \) large enough and the flow is transverse outgoing in the curve of \( g_- \). Now, we treat the case when \( q(s) = g_+(s) \). A simple derivation gives \( g'_+(s) = -\frac{1}{2s^2} < 0 \). We exploite \( (B.11) \) to write when \( s \) is large enough
\[
q'(s) = \frac{1}{s^a} + \frac{c_p}{s^a} + O\left(\frac{1}{s}\right) + o\left(\frac{1}{s^a}\right) > 0.
\] (B.13)
We conclude that \( q'(s) > g'_+(s) \) for \( s \) large enough and the flow is transverse outgoing in the curve of \( g_+ \). The conclusion of this part is that the flow is transverse outgoing in the curve of \( g_+ \) and \( g_- \). Three cases then arise:

**The first case:** If for all \( s \) large enough, we have \( g_-(s) \leq q(s) \leq g_+(s) \).

Since, \( q(s) \rightarrow 0 \) as \( s \rightarrow +\infty \), we rewrite \( (B.11) \) as follow
\[
q'(s) = q(s) + \frac{c_p}{s^a} + O(q^2(s)) + o\left(\frac{1}{s^a}\right).
\] (B.14)
In this case we have \( O(q^2(s)) + o\left(\frac{1}{s^a}\right) = o\left(\frac{1}{s^a}\right) \), which imply that we can write
\[
O(q^2(s)) + o\left(\frac{1}{s^a}\right) = \frac{\varepsilon(s)}{s^a} \quad \text{with} \quad \varepsilon(s) \rightarrow 0 \quad \text{as} \quad s \rightarrow +\infty.
\]
Multyplying equation \( (B.14) \) by \( e^{-\tau} \) and we integrate between \( s \) and \( +\infty \), we obtain the following
\[
-e^{-s}q(s) = \int_s^{+\infty} \frac{c_p e^{-\tau}}{\tau^a} d\tau + \int_s^{+\infty} \varepsilon(\tau)e^{-\tau} d\tau, \quad \text{with} \quad \varepsilon(s) \rightarrow 0 \quad \text{as} \quad s \rightarrow +\infty, \quad (B.15)
\]
with equation \( (B.15) \), we can deduce that
\[
q(s) \sim -\frac{c_p}{s^a}, \quad \text{as} \quad s \rightarrow +\infty.
\]

**The second case:** If there exist \( \tilde{s} \) large enough such that \( q(\tilde{s}) < g_-(\tilde{s}) \).
In this case, for all \( s \geq \tilde{s} \), \( q(s) < g_-(s) = -\frac{1}{s^{\frac{p}{2}}} \). From this fact, we obtain for some \( C > 0 \) the following
\[
\frac{c_p}{s^a} + o\left(\frac{1}{s^a}\right) \leq \frac{C}{s^a} < -\frac{Cq(s)}{s^{\frac{p}{2}}}
\]
and deduce that
\[
\frac{c_p}{s^a} + o\left(\frac{1}{s^a}\right) = \mathcal{O}\left(-\frac{q(s)}{s^{\frac{p}{2}}}\right).
\] (B.16)

Combining (B.14) and (B.16), we get
\[
q'(s) = q(s) + \mathcal{O}\left(-\frac{q(s)}{s^{\frac{p}{2}}}ight) + \mathcal{O}(q^2(s)) = q(s)\left(1 - \mathcal{O}\left(\frac{1}{s^{\frac{p}{2}}}ight) + \mathcal{O}(q(s))\right).
\] (B.17)

We use the fact that in this case \( q(s) < 0 \) and \( 1 - \mathcal{O}\left(\frac{1}{s^{\frac{p}{2}}}ight) + \mathcal{O}(q(s)) > \frac{1}{2} \), therefore
\[
q'(s) \leq \frac{q(s)}{2}.
\] (B.18)

Solving (B.18) and using the fact that \( q(\tilde{s}) < 0 \), we derive the following
\[
q(s) \leq e^{s-\tilde{s}}q(\tilde{s}) \to -\infty, \text{ as } s \to +\infty.
\]

Recalling that \( q \) is small, we get a contradiction.

**The third case:** If there exist \( \tilde{s} \) large enough such that \( q(\tilde{s}) > g_+(\tilde{s}) \).

In this case, for all \( s \geq \tilde{s} \), \( g_+(s) = \frac{1}{s^{\frac{p}{2}}} < q(s) \), which imply that
\[
\frac{c_p}{s^a} + o\left(\frac{1}{s^a}\right) = \mathcal{O}(q^2(s)).
\]

We use again (B.14), to write
\[
q'(s) = q(s) + \mathcal{O}(q^2(s)) = q(s)(1 + \mathcal{O}(q(s))) \geq q(s).
\] (B.19)

We solve (B.19) and we use the fact that \( q(\tilde{s}) > 0 \), we derive the following
\[
q(s) \geq e^{s-\tilde{s}}q(\tilde{s}) \to +\infty, \text{ as } s \to +\infty,
\]
which is a contradiction because \( q \) is small. Consequently, we get
\[
q(s) = \phi(s) - \kappa_0 \sim -\frac{\kappa_0}{p - 1} \left(\frac{p - 1}{4s}\right)^a, \text{ as } s \to +\infty.
\] (B.20)

This conclude the proof of Claim \( B.1 \)

\[\blacksquare\]

In the rest of this section, we are going to give the proof of some identities used in many steps of this paper. We claim the following:
Corollary B.2. For all \((d, y) \in (-1, 1)^2\) and \(s \in [s^*, +\infty)\), we have the following:

\[
\left| \partial_d \overline{w}_2(d, y, s) \right| \leq \frac{C \kappa(d, y)}{s^a(1 - d^2)} \tag{B.21}
\]

\[
\left| \partial_d \overline{w}_1(d, y, s) \right| + \left| \partial_d \overline{w}_2(d, y, s) \right| + \| \partial_d \psi(d, y, s) \|_{L^p} \leq \frac{C}{1 - d^2} \tag{B.22}
\]

\[
\left| \partial_s \overline{w}_1(d, y, s) \right| \leq \frac{C}{s^a} \kappa(d, y) \tag{B.23}
\]

\[
\left| \partial(d \dot{\phi}(d, y, s)) \right| + \| \partial_s \psi(d, y, s) \|_{L^p} \leq \frac{C}{s^a(1 - d^2)} \tag{B.24}
\]

Proof:

Proof of (B.21): Some simple calculations give the following

\[
\partial_d \overline{w}_2(d, y, s) = \frac{1}{\kappa_0} \partial_d (\kappa(d, y)) \phi'(s - \log \left( \frac{1 + dy}{\sqrt{1 - d^2}} \right)) - \kappa(d, y) \phi''(s - \log \left( \frac{1 + dy}{\sqrt{1 - d^2}} \right)) \left( \frac{y + d}{1 + dy} \right) \tag{B.25}
\]

From (ii) of Claim B.1, the expression (B.25) of \(\partial_d \overline{w}_2(d, y, s)\) and inequality (B.35), we conclude the proof of (B.21).

Proof of (B.22): By a careful calculation, we write

\[
\partial_d \overline{w}_1(d, y, s) = -\frac{\kappa(d, y)}{\kappa_0(1 + dy)} \left[ \frac{2d}{(p - 1)} \phi(s - \log \left( \frac{1 + dy}{\sqrt{1 - d^2}} \right)) - \frac{y + d}{1 - d^2} \phi'(s - \log \left( \frac{1 + dy}{\sqrt{1 - d^2}} \right)) \right] \tag{B.26}
\]

From the expression (B.26) of \(\partial_d \overline{w}_1(d, y, s)\) and Claim B.1, we can see that:

\[
\left| \partial_d \overline{w}_1(d, y, s) \right| \leq \frac{C}{s^a(1 - d^2)} \left[ \frac{1 - d^2}{1 + dy} + \frac{|y + d|}{1 + dy} \right] \tag{B.27}
\]

Using (B.27), (B.34), the expression (B.25) of \(\partial_d \overline{w}_2(d, y, s)\), the fact that \(\partial_d \kappa(d, y) \leq \frac{C}{1 - d^2}\); Claim B.1 and the fact that \(s^*\) is large enough such that for all \(s \geq s^*\), we have:

\[
\left| \partial_d \overline{w}_1(d, y, s) \right| + \left| \partial_d \overline{w}_2(d, y, s) \right| \leq \frac{C}{1 - d^2} \tag{B.28}
\]

According to the expression (2.18) of \(\overline{\psi}(d, y, s)\), we write for all \((d, y, s) \in (-1, 1)^2 \times [s^*, +\infty)\), we obtain

\[
\left| \partial_d \overline{\psi}(d, y, s) \right| \leq \frac{C}{(1 + dy)^2} \tag{B.29}
\]
We apply \( (iii) \) of Claim \( B.4 \) and \( (B.29) \) to conclude that
\[
\|\partial_d \psi(d, y, s)\|_{L^{p+1}} \leq \frac{C}{(1 + dy)^2}.
\]

Combining \( (B.28) \) and \( (B.30) \) to conclude the proof of \( (B.22) \).

**Proof of \( (B.23) \):** Inequality \( (B.23) \) is a direct consequence of item \( (ii) \) of Claim \( B.1 \).

**Proof of \( (B.24) \):** We write in the following the expression of \( \partial_d (\tilde{\phi}(d, y, s)) \)
\[
\partial_d (\tilde{\phi}(d, y, s)) = -\frac{1}{\kappa_0} \phi'(s - \log \left( \frac{1 + dy}{\sqrt{1 - d^2}} \right)) \frac{y + d}{(1 + dy)(1 - d^2)}.
\] \hspace{1cm} (B.31)

From \( (B.31) \), we have:
\[
|\partial_d (\tilde{\phi}(d, y, s))| \leq \frac{1}{\kappa_0} |\phi'(s - \log \left( \frac{1 + dy}{\sqrt{1 - d^2}} \right))| \frac{y + d}{(1 + dy)(1 - d^2)^{\frac{3}{2}}} \leq \frac{C}{s^a(1 - d^2)}. \] \hspace{1cm} (B.32)

From the expression \( (2.18) \) of \( \tilde{\psi}(d, y, s) \) and \( (ii) \) of Claim \( B.1 \) we can see that
\[
|\partial_s \tilde{\psi}(d, y, s)| \leq \frac{C}{s^a(1 + dy)^2}. \] \hspace{1cm} (B.33)

Then, we apply \( (iii) \) of Claim \( B.4 \) combined with \( (B.32) \) and \( (B.33) \) to deduce \( (B.24) \). 

We recall in the following claim some basic bounds and properties used in many steps in our paper and exceptionally in the proof of the Proposition 2.2.

**Claim B.3. (Useful properties):** For all \( (d, d_1, d_2, y) \in (-1, 1)^4 \), we have the following:
\[
|y + d| + (1 - d^2) + (1 - y^2) \leq C|1 + dy|, \] \hspace{1cm} (B.34)
\[
|\partial_d (\kappa(d, y))| \leq \frac{C \kappa(d, y)}{1 - d^2}, \] \hspace{1cm} (B.35)
\[
|W_{d, 2}(y)| + (1 - y^2) |\partial_y W_{d, 2}(y)| \leq C \kappa(d, y), \] \hspace{1cm} (B.36)
\[
|\bar{w}_1(d_1, y, s) - \bar{w}_1(d_2, y, s)|_{\mathcal{H}_0} \leq C|\theta_1 - \theta_2|, \] \hspace{1cm} (B.37)
\[
|\bar{w}_2(d_1, y, s) - \bar{w}_2(d_2, y, s)|_{L^2} \leq C|\theta_1 - \theta_2|, \] \hspace{1cm} (B.38)

where \( W_{d, 2} \) is defined in \( (2.6) \) and \( \theta_i = \frac{1}{2} \log \left( \frac{1 + dy}{1 - d^2} \right) \), for \( i \in \{1, 2\} \).

**Proof:**

**Proof of \( (B.35) \):** The proof is omitted since it is classical and known from [43].

**Proof of \( (B.36) \):** The proof is omitted since it is the same as the proof of \( (iii) \) of Lemma 4.4 page 85 in [43].

52
Proof of \((B.37)\): We exploit here the proof of inequality (174) in Merle and Zaag \([43]\) page 102. Since, we have from item \((i)\) of Claim \(B.1\) to deduce:

\[
\overline{w}_1(d_1, y, s) - \kappa(d_1, y) \leq -\frac{\kappa_0}{p-1} \left( \frac{p-1}{4s} \right)^a \kappa(d_1, y) \quad (B.39)
\]

and

\[
\kappa(d_2, y) - \overline{w}_1(d_2, y, s) \leq -\frac{\kappa_0}{p-1} \left( \frac{p-1}{4s} \right)^a \kappa(d_2, y). \quad (B.40)
\]

Combining \((B.39)\), \((B.40)\) and the triangular inequality to conclude that

\[
\|\overline{w}_1(d_1, y, s) - \overline{w}_1(d_2, y, s)\|_{H_0} \leq C_s a \|\kappa(d_1, y) - \kappa(d_2, y)\|_{H_0}. \quad (B.41)
\]

Coming at this level, we apply the proof of inequality (174) in Merle and Zaag \([43]\) page 102 to obtain \((B.37)\).

Proof of \((B.38)\): The proof is similar as the proof of \((B.37)\). According to the expression of \(\overline{w}_2(d, y, s)\) and item \((ii)\) of Claim \(B.1\), we obtain:

\[
\|\overline{w}_2(d_1, y, s) - \overline{w}_2(d_2, y, s)\|_{L^2} \leq C_s a \|\kappa(d_1, y) - \kappa(d_2, y)\|_{L^2}. \quad (B.42)
\]

It is clear that we have

\[
\|\kappa(d_1, y) - \kappa(d_2, y)\|_{L^2} \leq \|\kappa(d_1, y) - \kappa(d_2, y)\|_{H_0}. \quad (B.43)
\]

Now, we are in position to apply the proof of inequality (174) in Merle and Zaag \([43]\) page 102 to obtain \((B.38)\).

---

We recall now from \([43]\) the following estimate:

Claim B.4. (Integral computation table ). Consider for some \(\alpha > -1\) and \(\beta \in \mathbb{R}\) the following integral,

\[
I(d) = \int_{-1}^{1} \frac{(1-y^2)^\alpha}{(1+dy)^\beta} dy,
\]

then there exists \(K(\alpha, \beta)\) such that the following holds for all \(d \in (-1, 1)\),

(i) if \(\alpha + 1 - \beta > 0\), then \(\frac{1}{K} \leq I(d) \leq K\),

(ii) if \(\alpha + 1 - \beta = 0\), then \(\frac{1}{K} \leq I(d) / \log(1-d^2) \leq K\),

(iii) if \(\alpha + 1 - \beta < 0\), then \(\frac{1}{K} \leq I(d)(1-d^2)^{-\alpha+1+\beta} \leq K\).

Proof: See the proof of Claim 4.3 page 84 in Merle and Zaag \([43]\).

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We would like now to recall that the Lorentz transform keeps equation (1.6) and the norms in \(\mathcal{H}_0\) invariant. More precisely, we write from Merle and Zaag \([43]\) the following lemma:

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53
Lemma B.5. (The invariance of the Lorentz transform in similarity variables) Consider 
\( w(y, s) \) a solution of equation (1.6) defined for all \(|y| < 1\) and \( s \in (s_0, s_1) \) for some \( s_0 \) and \( s_1 \) in \( \mathbb{R} \) and introduce for any \( d \in (-1, 1) \), the function \( W \equiv \tau_d(w) \) defined by,

\[
W(Y, S) = \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dY)^{\frac{1}{p-1}}} w(y, s), \quad \text{where}
\]

\[
y = \frac{Y + d}{1 + dY} \quad \text{and} \quad s = S - \log \left( \frac{1 + dY}{\sqrt{1 - d^2}} \right).
\]

Then \( W(Y, S) = \tau_d(w) \) is also a solution of (1.6) defined for all \(|y| < 1\) and

\[
S \in \left( s_0 + \frac{1}{2} \log \frac{1 + |d|}{1 - |d|}, s_1 - \frac{1}{2} \log \frac{1 + |d|}{1 - |d|} \right).
\]

Also, we have the continuity of \( \tau_d \) in \( \mathcal{H}_0 \),

\[
\|\tau_d(w)\|_{\mathcal{H}_0} \leq C\|w\|_{\mathcal{H}_0}.
\]

Proof: See the proof of Lemma 2.6 page 54 and Lemma 2.8 page 57 in Merle and Zaag [43].

We end this section by the Hardy-Sobolev identity in the space \( \mathcal{H}_0 \) defined in (1.9) combined with some basic bounds on the stationary solution of (1.6) when \( f \equiv 0 \), namely \( \kappa(d, y) \) defined in (1.14):

Lemma B.6. We have the following identities,

(i) (A Hardy-Sobolev type identity) For all \( h \in \mathcal{H}_0 \), we have

\[
\|h\|_{L^2_{\rho} \cdot \frac{1}{1 - y^2} (-1,1)} + \|h\|_{L^{p+1} \cdot \frac{1}{1 - y^2} (-1,1)} + \|h(1 - y^2)^{\frac{1}{p-1}}\|_{L^\infty (-1,1)} \leq C\|h\|_{\mathcal{H}_0}.
\]

(ii) (Boundedness of \( \kappa(d, y) \) in several norms) For all \( d \in (-1, 1) \), we have

\[
\|\kappa(d, y)\|_{L^2_{\rho} \cdot \frac{1}{1 - y^2} (-1,1)} + \|\kappa(d, y)\|_{L^{p+1} \cdot \frac{1}{1 - y^2} (-1,1)} + \|\kappa(d, y)(1 - y^2)^{\frac{1}{p-1}}\|_{L^\infty (-1,1)} + \|\kappa(d, y)\|_{\mathcal{H}_0} \leq C.
\]

Proof: For the proof of (i), we can see the proof of Lemma 2.2 page 51 in Merle and Zaag [43]. For the proof of (ii), we need to use (i) and identity (49) page (59) in [43].
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