A mathematically interesting hyperbolic solution to the Einstein field equations is studied on an eight-dimensional pseudo-Riemannian manifold $X_{4,4}$ that is a space-time of four space dimensions and four time dimensions. [The signature and dimension of $X_{4,4}$ are chosen because its tangent spaces satisfy a triality principle (vectors and spinors are equivalent).]

This solution exhibits temporal hyperbolic inflation of three of the four space dimensions and temporal hyperbolic deflation of three of the four time dimensions. Comoving coordinates for the unscaled dimensions are chosen to be $(x^4 \leftrightarrow \text{time}, x^8 \leftrightarrow \text{space})$, where the $x^4$ coordinate corresponds to our universe’s observed physical time dimension and the $x^8$ coordinate corresponds to a predicted new physical spatial dimension. This solution of the field equations manifests temporal hyperbolic inflation $\cosh(\frac{1}{3}H x^4)$ of the scale factor associated with three of the four space dimensions, and temporal deflation $\text{sech}(\frac{1}{3}H x^4)$ of the scale factor associated with three of the four time dimensions. (Here $H$ is the Hubble parameter.) The scale factors possess a more complicated dependence on the spatial $x^8$ coordinate, which, however, turn out to be periodic in $x^8$ with period $\propto 1/H$. After “inflation” the observed physical macroscopic world has three space dimensions and one time dimension.

PACS numbers:

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1. INTRODUCTION

In general, inflationary cosmology [2–4] is a widely accepted and largely successful theory of the physics of the very early universe. Recent observational data from important experiments such as WMAP [5] and the Sloan Digital Sky Survey [6–8], and others support [9–12] the main predictions of inflation theory. Inflation employs a period of accelerated expansion at the beginning of the universe to solve the monopole, horizon and flatness problems. Here we discuss a new model of inflation/deflation that is based on the idea that our universe has as many time dimensions as space dimensions.

Let $X^{4,4}_4$ denote an eight-dimensional pseudo-Riemannian manifold that admits a spin structure, whose local tangent spaces are isomorphic to flat Minkowski spacetime $M^{4,4}_4$. $X^{4,4}_4$ is a spacetime of four space dimensions, with local comoving coordinates $(x^1, x^2, x^3, x^8)$, and four time dimensions, with local comoving coordinates $(x^4, x^5, x^6, x^7)$, employing the usual component notation in local charts. All coordinates have dimension of length [time coordinates are scaled with a normalized speed parameter $c = 1$ that represents the speed of gravitational waves in vacuum, so that they have correct units]. The domains of the comoving coordinates $x^\alpha$ (Greek indices run from 1 to 8) are all assumed to be $-\infty < x^\alpha < \infty$. If

\[
\begin{align*}
    f &= f(x^4, x^8) \text{ then } \\
    f^{(1,0)} &= \frac{\partial}{\partial x^4} f(x^4, x^8) \\
    f^{(0,1)} &= \frac{\partial}{\partial x^8} f(x^4, x^8) \\
    f^{(1,1)} &= \frac{\partial^2}{\partial x^4 \partial x^8} f(x^4, x^8) \\
    f^{(2,0)} &= \frac{\partial^2}{\partial x^4 \partial x^8} f(x^4, x^8) , \text{ etc.}
\end{align*}
\]

Let $g$ denote the pseudo-Riemannian metric tensor on $X^{4,4}_4$. The signature of the metric $g$ is $(+ + + - - - -)$. The covariant derivative with respect to the symmetric connection associated to the metric $g$ is denoted by a double-bar. $g \leftrightarrow g_{\alpha\beta} = g_{\alpha\beta}(x^\mu)$ is assumed to carry the Newton-Einstein gravitational degrees of freedom. It is moreover assumed that the ordinary Einstein field equations (on $X^{4,4}_4$)

\[
G^\mu_\nu = 8 \pi \mathbb{G} T^\mu_\nu
\]

are satisfied; here $\mathbb{G}$ denotes the Newtonian gravitational constant.
We study a solution to the Einstein field equations on $X_{4,4}$ that exhibits inflation/deflation: during “inflation”, the scale factor $a = a(x^4, x^8)$ for the three space dimensions $(x^1, x^2, x^3)$ exponentially inflates, and the scale factor $b = b(x^4, x^8)$ for the three time dimensions $(x^5, x^6, x^7)$ exponentially deflates; moreover, $x^4$ and $x^8$ do not scale. For brevity this phenomenon is sometimes called “inflation.”

2. PROBLEM FORMULATION AND SOLUTION

The line element for inflation/deflation is assumed to be given by

$$\{ds\}^2 = \{a(x^4, x^8)\}^2 [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2$$

$$- \{b(x^4, x^8)\}^2 [(dx^5)^2 + (dx^6)^2 + (dx^7)^2] + (dx^8)^2$$

$$= a^2 [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - b^2 [(dx^5)^2 + (dx^6)^2 + (dx^7)^2]$$

$$- (dx^4)^2 + (dx^8)^2 ;$$

(3)

where $a = a(x^4, x^8)$ and $b = b(x^4, x^8)$ carry the metric degrees of freedom in this model. The dark energy density is denoted $\varrho = \varrho(x^4, x^8)$, the dark matter density is denoted $\Psi = \Psi(x^4, x^8)$ and the scalar inflaton field is $\varphi = \varphi(x^4, x^8)$. In this paper, in order to approximate the familiar “slow roll” conditions that are usually assumed to prevail during inflation, the inflaton field $\varphi$ is approximated by 0. The dark energy density $\varrho$ and the dark matter density $\Psi$ will be seen (below) to be dynamically determined by the field equations.

The action for the metric and inflaton degrees of freedom is assumed to be given by

$$S = \int d^8x \sqrt{det(g_{\alpha \beta})} \left[ \frac{1}{16 \pi G} R - \frac{1}{2} g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right],$$

(4)

where $V(0) = V_0 \neq 0$. The dark matter and dark energy degrees of freedom are modeled phenomenologically, through a postulated effective stress-energy tensor $T^\mu_\nu$.

$T^\mu_\nu$ is constructed from the canonical contribution $-g^{\mu \alpha} \frac{2}{\sqrt{det(g_{\alpha \beta})}} \frac{\partial}{\partial g_{\alpha \nu}} L_\varphi$, $L_\varphi = \sqrt{det(g_{\alpha \beta})} \left[ -\frac{1}{2} g^{\alpha \nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right]$, plus a contribution that is diagonal in the dark energy fields, plus a contribution whose non-zero off-diagonal components are assumed to be $T^4_8 = -T^8_4 = \Psi(x^4, x^8)$. The dark energy components of $T^\mu_\nu$ are parameterized by

$$^\text{DE}T^\mu_\nu = \text{diag}(p, p, p, -\rho, -\Omega, -\Omega, -\Omega, P).$$

(5)
Note that $T_{\mu\nu}$ is symmetric. The phenomenological relations connecting the fields \{\(p(x^4, x^8), \rho(x^4, x^8), \Omega(x^4, x^8), P(x^4, x^8)\)\} and the dark energy density \(\varrho(x^4, x^8)\) are assumed to be

\[
\begin{align*}
    p &= w \varrho \\
    P &= W \varrho \\
    \Omega &= \Lambda \varrho \\
    \rho &= \lambda \varrho.
\end{align*}
\]  

(6)

The parameter \(\lambda\) is included for bookkeeping purposes, and may be set equal to one. In this model the remaining parameters \{\(w, W, \Lambda, V_0\)\} are dynamically determined by the field equations, at least for the epoch of inflation.

The distinct field equation components may be written as

\[
\begin{align*}
    G_{48} = G_{84} &= -\frac{3a^{(1,1)}}{a} - \frac{3b^{(1,1)}}{b} = 8\pi \mathbb{G} \left(\varphi^{(0,1)}\varphi^{(1,0)} + \Psi\right) \\
    G_{33} &= \frac{1}{b^2} \left[ 3ab \left(2a^{(0,1)}b^{(0,1)} - 2a^{(1,0)}b^{(1,0)} + a \left(b^{(0,2)} - b^{(2,0)}\right)\right) \\
    &\quad + \left(a^{(0,1)} - a^{(1,0)}\right)^2 + 2a \left(a^{(0,2)} - a^{(2,0)}\right) \right] b^2 + 3a^2 \left(b^{(0,1)} - b^{(1,0)}\right) \\
    &= 4\pi \mathbb{G} a^2 \left(2p - 2V(\varphi) - \varphi^{(0,1)^2} + \varphi^{(1,0)^2}\right) \\
    G_{55} &= \frac{1}{a^2} \left[ 2ab \left(-3a^{(0,1)}b^{(0,1)} + 3a^{(1,0)}b^{(1,0)} + a \left(b^{(2,0)} - b^{(0,2)}\right)\right) \\
    &\quad + 3 \left(-a^{(0,1)} + a^{(1,0)}\right)^2 + a \left(a^{(2,0)} - a^{(0,2)}\right) \right] b^2 + a^2 \left(b^{(1,0)} - b^{(0,1)}\right) \\
    &= 4\pi \mathbb{G} b^2 \left(2V(\varphi) + \varphi^{(0,1)^2} - \varphi^{(1,0)^2} + 2\Omega\right) \\
    G_{44} &= -\frac{3}{a^2b^2} \left[ b^2 \left(a^{(0,1)^2} - a^{(1,0)^2} + aa^{(0,2)}\right) + ab \left(3a^{(0,1)}b^{(0,1)} - 3a^{(1,0)}b^{(1,0)} + ab^{(0,2)}\right) \\
    &\quad + a^2 \left(b^{(0,1)} - b^{(1,0)}\right) \right] \\
    &= 4\pi \mathbb{G} \left(2\varrho + 2V(\varphi) + \varphi^{(0,1)^2} + \varphi^{(1,0)^2}\right)
\end{align*}
\]  

(7), (8), (9), (10)
\[ G_{88} = -\frac{3}{a^2 b^2} \left[ a^2 \left( b^{(1,0)} - b^{(0,1)} \right)^2 + \left( -a^{(0,1)} + a^{(1,0)} + a a^{(2,0)} \right) b^2 \right. \\
+ \left. a b \left( -3 a^{(0,1)} b^{(0,1)} + 3 a^{(1,0)} b^{(1,0)} + a b^{(2,0)} \right) \right] \]
\[ = 4 \pi G \left( 2P - 2V(\varphi) + \varphi^{(0,1)^2} + \varphi^{(1,0)^2} \right) \]  
(11)

The components of \( T_{\mu \alpha}^{\mu} \) that are not identically zero must satisfy
\[ a b T_{\mu 4}^{\mu} = 0 = 3a \left( -b^{(1,0)} \left( r + \varphi^{(1,0)^2} - \Omega \right) + b^{(0,1)} \varphi^{(0,1)} \varphi^{(1,0)} + b^{(0,1)} \Psi \right) \\
+ b \left( -3 a^{(1,0)} \left( P + r + \varphi^{(0,2)} \right) + 3 a^{(0,1)} \varphi^{(0,1)} \varphi^{(1,0)} \right) \\
+ 3 a^{(0,1)} \Psi + a \left( -\varphi^{(1,0)} - \varphi^{(0,0)} \left( V'(\varphi) - \varphi^{(0,2)} + \varphi^{(2,0)} \right) + \varphi^{(0,1)} \right) \]  
\[ a b T_{\mu 8}^{\mu} = 0 = 3a \left( b^{(0,1)} \left( P + \varphi^{(0,1)^2} + \Omega \right) - b^{(1,0)} \varphi^{(0,1)} \varphi^{(1,0)} - b^{(1,0)} \Psi \right) \\
- b \left( 3 a^{(0,1)} P - 3 a^{(0,1)} P - 3 a^{(0,1)} \varphi^{(0,1)^2} + 3 a^{(1,0)} \varphi^{(1,0)} \varphi^{(0,1)} \right) \\
+ 3 a^{(1,0)} \Psi + a \left( \varphi^{(0,1)} V'(\varphi) - P^{(0,1)} \right) - a \varphi^{(0,2)} \varphi^{(0,1)} + a \varphi^{(2,0)} \varphi^{(0,1)} + a \varphi^{(1,0)} \Psi \]  
(12)

This system of equations admits a hyperbolic inflationary solution to the field equations.

With the assumption that \( \varphi = 0 \), a separable solution to the field equations of the form
\[ a = a(x^4, x^8) = \cosh \left( \frac{H}{3} x^4 \right) a_8(x^8) \]  
\[ b = b(x^4, x^8) = \sech \left( \frac{H}{3} x^4 \right) b_8(x^8) \]  
\[ \varrho = \varrho(x^4, x^8) = \rho_4(x^4) \rho_8(x^8) \]  
\[ \Psi = \Psi(x^4, x^8) = \psi_4(x^4) \psi_8(x^8) \]  
(13)

has the following solution:

\[ V_0 = \frac{H^2}{4 \pi G} \]  
\[ \varrho = \frac{H^2}{24 \pi G \lambda} \sech^2 \left( \frac{H x^4}{3} \right) \]  
\[ \Psi = -\frac{5 H^2 \tanh \left( \frac{H x^4}{3} \right)}{12 \pi G \sin(2H x^8) + 24 \pi G \cos(2H x^8)} \]  
\[ w = \frac{4 \lambda}{3} > 0 \text{ when } \lambda > 0 \]  
\[ \mathbb{W} = \lambda \]  
\[ \Lambda = -\frac{2 \lambda}{3}. \]  
(14)
The scale factors may be expressed in terms of the quadratic polynomial

\[ Q(z) = \left[ z - \frac{1}{2} \left( 1 - \sqrt{5} \right) \right] \left[ z - \frac{1}{2} \left( 1 + \sqrt{5} \right) \right] \equiv (z - z_-)(z - z_+) \]

\[ = = z^2 - z - 1 \quad (15) \]

that is often introduced to calculate the *golden ratio* \( z_+ \). Let

\[ z = \tan(Hx^8); \quad (16) \]

then

\[
\begin{align*}
    a &= a(x^4, x^8) = a_0 \cosh \left( \frac{H}{3} x^4 \right) \frac{12}{\cos^4(Hx^8)[Q(z)]^2} \left( \frac{z - z_-}{z - z_+} \right)^2 \frac{z_+ - z_-}{z - z_+} \\
    b &= b(x^4, x^8) = b_0 \text{sech} \left( \frac{H}{3} x^4 \right) \frac{12}{\cos^4(Hx^8)[Q(z)]^2} \left( \frac{z - z_+}{z - z_-} \right)^2 \frac{z_+ - z_-}{z - z_-}. \quad (17)
\end{align*}
\]

Here \( a_0 \) and \( b_0 \) are constants.

The dark matter density, in this notation, is

\[
\Psi = -\frac{5H^2 \sec^2(z) \tanh \left( \frac{Hx^4}{3} \right)}{24\pi G \left( -\tan^2(z) + \tan(z) + 1 \right)}. \quad (18)
\]

3. CONCLUSION

This model admits an inflation model for which time may flow in either the positive or negative sense. This solution may correspond to a pair of universes created simultaneously at \( x^4 = 0 \), and which are physically separated further in time as they evolve. Note that the dark matter densities in these two universes, for identical values of \( x^8 \), have different algebraic signs, but that the dark energy densities have the same sign. The interpretation of \( \Psi \) as the dark matter density comes from an unpublished calculation in which the dark energy density is constant, \( a \propto e^{Hx^4} \) and \( b \propto e^{-Hx^4} \). The field equations, for this case, predict that \( \Psi = \text{const} / (ab)^3 = \text{const} / \text{volume element} \), which mimics the behavior of the ordinary matter density in a flat Friedmann cosmological model which is matter-dominated.
Functions of the extra space dimension, coordinatized by $x^8$, have spatial period $\pi/H$, where from Eq.[14], $H^2 = 24\pi G \lambda \varrho(0,0) = 4\pi G V_0$. From Figure[1] it is clear that the determinant of the metric vanishes twice over this interval, at the singular points at which one of $\{a, b\}$ is zero and the other infinite, but $a \times b \to 0$. This suggests that the extra space dimension is either compact, or that the model actually represents an infinity of classical (i.e., pre-quantized) distinguishable universes that are enumerated by $x^8$ intervals. Quantum effects may average out the singular effects, so one cannot assert that these singularities delimit classical multiverse members without further study. Certainly the classical minimum $V_0$ of the inflaton potential will be shifted by renormalization terms once $\varphi$ is quantized. The effective inflaton potential is expected to be temperature dependent, and perhaps this may drive a phase transition that ends inflation. From a physics perspective, the model cannot be taken seriously until a way to end inflation is found.

Both the fourth space dimension, whose associated co-moving coordinate is $x^8$, as well as the extra time dimensions, evidently pose a challenge to observe, if they exist. Relative to the first three space dimensions, whose co-moving coordinates are $(x^1, x^2, x^3) \in \mathbb{R}^3$, the distance $\Delta X$ between two points $(x_0, x_0 + \Delta X)$ on the $x^8$-axis is expected to be exponentially smaller by about 60 e-folds than the distance between two points in $\mathbb{R}^3$ separated by the same coordinate difference $\Delta X$, but lying on, say, the $x^3$-axis; the distance between two points $(x_0, x_0 + \Delta X)$ lying on, say, the $x^7$-axis is even smaller, since the extra time dimensions experience deflation.

Acknowledgments

I would like to acknowledge main author José M. Martn-Garca of the software package “xAct” URL= [http://www.xact.es/index.html](http://www.xact.es/index.html) which was employed to calculate the Einstein tensor for this paper.

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FIG. 1: \(a(0, x^8)\) (red), \(b(0, x^8)\) (blue, dashed), \(a(0, x^8) \times b(0, x^8)\) (green, blocks), \(\Psi(10, x^8)\) (black, dot-dashed). [Color online]