A FAMILY OF EXPONENTIAL INTEGRALS
SUGGESTED BY STELLAR DYNAMICS

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Abstract

While investigating the generalization of the Chandrasekhar (1943) dynamical friction to the case of field stars with a power-law mass spectrum and equipartition Maxwell-Boltzmann velocity distribution, a pair of 2-dimensional integrals involving the Error function occurred, with closed form solution in terms of Exponential Integrals (Ciotti 2010). Here we show that both the integrals are very special cases of the family of (real) functions

\[ I(\lambda, \mu, \nu; z) := \int_0^z x^\lambda E_\nu(x^\mu) \, dx = \frac{\gamma\left(1 + \frac{1 + \lambda}{\mu}, z^\mu\right) + z^{1 + \lambda}E_\nu(z^\mu)}{1 + \lambda + \mu(\nu - 1)}, \quad \mu > 0, \quad z \geq 0, \quad (1) \]

where \( E_\nu \) is the Exponential Integral, \( \gamma \) is the incomplete Euler gamma function, and for existence \( \lambda > \max\{-1, -1 - \mu(\nu - 1)\} \). Only in one of the consulted tables a related integral appears, that with some work can be reduced to eq. (1), while computer algebra systems seem to be able to evaluate the integral in closed (and more complicated) form only provided numerical values for some of the parameters are assigned. Here we show how eq. (1) can in fact be established by elementary methods.

1. Introduction

Two interesting integrals, that can be expressed in closed form in terms of the Error Function and of the Exponential Integral, were encountered while generalizing the Chandrasekhar (1943) dynamical friction formula to the case of a test mass moving in a field of stars with a power-law mass spectrum, and equipartition Maxwell-Boltzmann velocity distribution (eqs. [30]-[31] in Ciotti 2010). They both belong to the family of functions in eq. (1): quite surprisingly, this simple-looking identity is not found in the most important tables of integrals (e.g., Erdélyi et al. 1953, Gradshteyn and Ryzhik 2007, Prudnikov et al. 1990), and neither the latest releases of Mathematica and Maple seem to be able to recover the general result, but only particular cases for numerical values of some of the parameters. In the following I show how the identity in eq. (1) can be established with elementary methods.
2. Some preliminary material

For succesive use, we report the relevant identities obeyed by the Exponential Integrals. They are defined for $\Re(z) > 0$ as

$$E_\nu(z) := \int_1^\infty t^{-\nu} e^{-tz} dt = z^{\nu-1} \Gamma(1 - \nu, z),$$

(2)
e.g., Abramowitz & Stegun, Chapter 5; Arfken & Weber 2005, Exercise 8.5.8; Erdélyi et al. 1953, Vol.2, Chapter 9; see also https://functions.wolfram.com, https://dlmf.nist.gov).

The last expression above, where $\Gamma(1 - \nu, z)$ is the incomplete right Euler Gamma function, is obtained with an obvious change of integration variable. The Euler incomplete left and right Gamma functions (over the reals) can be expressed in integral form as

$$\gamma(a, x) := \int_0^x t^{a-1} e^{-t} dt, \quad \Gamma(a, x) := \int_x^\infty t^{a-1} e^{-t} dt,$$

(3)
where $\Re(a) > 0$ for convergence of the $\gamma$ function\(^1\), therefore they obey the relations of easy proof:

$$\gamma(a + 1, x) = a\gamma(a, x) - x^a e^{-x}, \quad \Gamma(a + 1, x) = a\Gamma(a, x) + x^a e^{-x}. \quad (4)$$

and

$$\gamma(a, x) + \Gamma(a, x) = \Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt = \gamma(a, \infty) = \Gamma(a, 0), \quad (5)$$

where $\Gamma(a)$ is the complete Gamma function.

About the Exponential Integrals $E_\nu$, from their integral expression in eq. (2) it is a simple exercise to show that

$$E_0(z) = \frac{e^{-z}}{z}, \quad \frac{dE_\nu(z)}{dz} = -E_{\nu-1}(z). \quad (6)$$

Moreover, from integration by parts of eq. (2), by using the first and the second function in the integrand as differential factor, for $z \neq 0$ one obtain respectively

$$E_\nu(z) = \frac{e^{-z} - z E_{\nu-1}(z)}{\nu - 1} = \frac{e^{-z} - \nu E_{\nu+1}(z)}{z}, \quad (7)$$

where of course $\nu \neq 1$ in the first identity. Finally, from standard asymptotic expansion it follows that, at the leading order for $z \to 0$,

$$E_\nu(z) \sim \begin{cases} \frac{1}{\nu - 1}, & \nu > 1, \\ -\ln z, & \nu = 1, \\ \frac{\Gamma(1 - \nu)}{z^{1-\nu}}, & \nu < 1, \end{cases} \quad (8)$$

\(^1\) As we do not use the continuation of the functions to the Complex plane, from now on all the quantities are intended reals.
and in particular it follows that the divergence of the Exponential Integrals near the origin gets worse for decreasing $\nu \leq 1$, an obvious consequence of eq. (2). The leading-order expansions in eq. (8) will be used in the next Section to determine the limitations on the values of the parameters $(\lambda, \mu, \nu)$ required for existence of the function $I(\lambda, \mu, \nu; z)$; no special difficulties are encountered to evaluate higher order terms, and they can be also used to check the consistency of the recursion identities in eq. (7) for $z \to 0$.

3. The parameter space

Before proceeding to prove eq. (1), it is convenient to determine the restrictions on the values of the parameters $(\lambda, \mu, \nu)$ to assure existence of the function $I$. Equation (8) shows that we must consider three different cases as a function of the value of $\nu$ (a generic real number), and in fact elementary integration shows that at the leading order for $z \to 0^{+}$ and $\mu > 0$

$$I(\lambda, \mu, \nu; z) \sim \begin{cases} \frac{z^{\lambda+1}}{(\lambda + 1)(\nu - 1)}, & \nu > 1, \\ -\frac{\mu}{\lambda + 1}z^{\lambda+1}\ln z, & \nu = 1, \\ \frac{\Gamma(1-\nu)}{1 + \lambda + \mu(\nu - 1)}z^{1+\lambda+\mu(\nu-1)}, & \nu < 1, \end{cases}$$

provided the conditions

$$\lambda > \begin{cases} -1, & \nu \geq 1, \\ -1 - \mu(\nu - 1), & \nu \leq 1 \end{cases} = \max\{-1, -1 - \mu(\nu - 1)\},$$

are satisfied (see Figure 1).

We finally notice that an obvious and useful transformation of the function $I$ in eq. (1) can be obtained with the change integration variable $y = x^{r}$ and $r > 0$

$$I(\lambda, \mu, \nu; z) = \frac{1}{r} I \left( \frac{\lambda - r + 1}{r}; \frac{\mu}{r}, \nu; z^{r} \right);$$

in particular, by setting $r = \mu$ it is always possible to reduce to the case of integration of eq. (1) with $E_{\nu}$, depending linearly on the integration variable, and this case is evaluated by Mathematica.
Figure 1 The region in the $(\lambda, \mu)$ parameter space for existence of the function $I(\lambda, \mu, \nu; z)$, as determined by eq. (10). For values of $\nu > 1$ all points above the the horizontal dotted line are acceptable. At decreasing $\nu$ the existence region reduces to the points above the dashed line, here represented for $\nu = 0$ and $\nu = -1/2$. Notice that for all points above the $\nu = 0$ line, also the expression in eq. (15) can be used, and that the value $\nu = -1/2$ is the minimum value required for existence of the function $H$ in eqs. (16)-(17) when $\lambda = \mu = 2$ (solid dot).

4. A proof of identity (1)

We are now in position to prove the indentity in eq. (1) by elementary methods. First, as $\mu > 0$, and considering that from eq. (10) certainly $\lambda > -1$, we can integrate by parts with $x^\lambda$ as differential factor, obtaining a recursion identity

$$I(\lambda, \mu, \nu; z) = \frac{z^{\nu+1}E_\nu(z^\mu) + \mu I(\lambda + \mu, \mu, \nu - 1; z)}{\lambda + 1}. \quad (12)$$

The first term follows from eq. (9) and the limitations in eq. (10), while the second term from the second identity in eq. (6). Then, from the first identity in eq. (6), and restricting (for the moment) to $\nu \neq 1$ we have

$$x^\mu E_{\nu-1}(x^\mu) = e^{-x^\mu} - (\nu - 1)E_\nu(x^\mu). \quad (13)$$

We now multiply the identity above for $x^\lambda$ and integrate over $x$, so that

$$\mu I(\lambda + \mu, \mu, \nu - 1; z) = \gamma \left( \frac{1 + \lambda}{\mu}, z^\mu \right) - \mu(\nu - 1)I(\lambda, \mu, \nu, z) : \quad (14)$$

notice that the identity also holds for $\nu = 1$, so we can relax the restriction $\nu \neq 1$. Therefore we have a second identity that can be used with eq. (12) to obtain the function $I(\lambda, \mu, \nu; z)$ and finally prove eq. (1), QED.
Notice that the procedure is the same used (for example) in standard exercises to integrate products of trigonometric functions and exponentials. The correctness of eq. (1) can be verified with some work from the second of eq. 1.2.1.1 of Prudnikov et al. (1990, Volume 2). In particular, first express the Exponential Integral in terms of the incomplete Gamma function from eq. (2), then change the parameters in Prudnikov’s equation as \( \lambda \rightarrow \lambda + \mu (\nu - 1) \), \( \alpha \rightarrow 1 - \nu \), \( a \rightarrow 1 \), and \( \nu \rightarrow \mu \), and finally combine two Gamma functions in the incomplete \( \gamma \) function from identity (5). Reassuringly, notice how the limitations on the parameters given in Prudnikov, once expressed in terms of our parameters, coincide with those given in eq. (10).

Note that for \((1 + \lambda)/\mu > 1\), i.e. \( \lambda > -1 + \mu \), it is possible to apply the first recursion formula in eq. (4) to the the incomplete \( \gamma \) function appearing in eq. (1), and successively reduce the resulting formula from the second identity in eq. (7), obtaining

\[
I(\lambda, \mu, \nu; z) = \frac{1+\lambda-\mu \gamma \left(1+\frac{\lambda-\mu}{\mu}, z\mu\right) - \nu z^{1+\lambda-\mu}E_{\nu+1}(z\mu)}{1 + \lambda + \mu (\nu - 1)}.
\]  

(15)

Of course, if \( \lambda > -1 + 2\mu \), the argument can be applied again to eq. (15), and so on, but the resulting formulae become increasingly complicated (even if of trivial construction), and not reported here.

With the aid of eq. (15) we can easily prove eqs. (30)-(31) in Ciotti (2010), that were derived by using “ad hoc” integration based on the properties of the Error function. Starting from eqs. (16)-(21)-(29) in Ciotti (2010), the two integrals to be solved can be written as

\[
H(y) := ac^a \frac{4}{\sqrt{\pi}} \int_c^\infty r^{-\nu} dr \int_0^y t^2 e^{-rt^2} dt, \quad a > 1, \quad c = 1 - \frac{1}{a};
\]  

(16)

and the two functions \( H_1 \) and \( H_2 \) of interest in Stellar Dynamics correspond to \( \nu = a - 3/2 \) and \( \nu = a - 5/2 \), respectively. In the original work the integration was performed considering first the inner integral, and then integrating by parts over \( r \) a term involving the Error function. Here instead we invert order of integration in eq. (16) so that

\[
H(y) = ac^{a-\nu-1/2} \frac{4}{\sqrt{\pi}} \int_0^{\sqrt{cy}} x^2 E_{\nu}(x^2) dx = ac^{a-\nu-1/2} \frac{4}{\sqrt{\pi}} I(2, 2, \nu; \sqrt{cy}),
\]  

(17)

where in the integral we changed variable as \( x = \sqrt{cy} \). As \( \lambda = 2 > -1 + \mu = 1 \), it is then possible to use eq. (15), and finally from the identity

\[
\frac{1}{\sqrt{\pi}} \gamma \left(1, \frac{1}{2}, z\right) = \text{Erf}(z),
\]  

(18)

eqs. (30)-(31) in Ciotti (2010) are recovered. Figure 1 immediately shows (solid dot) that \( a > 1 \) is required for existence of \( H_1 \), and \( a > 2 \) for existence of \( H_2 \).
5. Conclusions

Prompted by a problem of Stellar Dynamics, an elementary derivation is presented for the closed-form expression of a family of indefinite integrals involving powers and Exponential Integrals. Well known computer algebra systems seem unable to obtain the primitive in closed form in the general case, and also for numerical values of (some) of the parameters the resulting formulae can be quite complicated and not easily simplified to the compact expressions in eqs. (1)-(15), though the numerical values are in perfect agreement. However, from the two last identities and eqs. (4), (7) and (11), it is expected that general and uniform simplification procedures for the integrals $I(\lambda, \mu, \nu; z)$ could be easily implemented in computer algebra systems.

6. References

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