Commutation Structures

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Abstract For fixed object $X$ in a monoidal category, an $X$-commutation structure on an object $A$ is just a map $X \otimes A \to A \otimes X$. We study aspects of such structures in case $A$ has a dual object.

We consider a monoidal category $\mathcal{V}, \otimes, I$; for simplicity we let it be strict (the application we have in mind is anyway a category of endofuncors on a category, with composition as $\otimes$).

Let $X$ be an object in $\mathcal{V}$, fixed throughout. An $X$-commutation structure on an object $A \in \mathcal{V}$ is an arrow

$$\alpha : X \otimes A \to A \otimes X.$$ 

A morphism of $X$-commutation structures $(A, \alpha) \to (B, \beta)$ is an arrow $A \to B$ such that the obvious square

$$\begin{array}{ccc}
X \otimes A & \xrightarrow{\alpha} & A \otimes X \\
\downarrow X \otimes f & & \downarrow f \otimes X \\
X \otimes B & \xrightarrow{\beta} & B \otimes X
\end{array}$$

commutes.

In this way, we get a category of $X$-commutation structures; denote it $X\mathcal{V}$. There is a faithful forgetful functor $X\mathcal{V} \to \mathcal{V}$: to $(A, \alpha)$, associate $A$.

There is also a monoidal structure on $X\mathcal{V}$, preserved strictly by the forgetful functor $X\mathcal{V} \to \mathcal{V}$: If $(A, \alpha)$ and $(B, \beta)$ are objects in $X\mathcal{V}$, we get a commutation structure $\gamma$ on $A \otimes B$ in an obvious way: $\gamma$ is taken to be the composite

$$X \otimes A \otimes B \xrightarrow{\alpha \otimes B} A \otimes X \otimes B \xrightarrow{A \otimes \beta} A \otimes B \otimes X.$$
Also \( I \) carries a canonical \( X \)-commutation structure, namely the canonical \( X \otimes I \to I \otimes X \) (an identity, in fact, since we assumed \( \mathcal{V} \) strict).

Recall that a right dual for an object \( A \) in a monoidal category is an object \( B \) together with arrows (“unit and counit”)

\[
\eta : I \to B \otimes A \quad \text{and} \quad \epsilon : A \otimes B \to I
\]
satisfying the usual two triangle equations.

We are interested in the monoidal category \( X \cdot \mathcal{V} \):

**Theorem 1** If an object \((A, \alpha) \in X \cdot \mathcal{V}\) admits a right dual, then \( \alpha \) is an invertible arrow in \( \mathcal{V} \).

**Proof.** The assumption on \((A, \alpha)\) can be expressed: \( A \) admits a right dual \( B \) in the category \( \mathcal{V} \), and there is an \( X \)-commutation structure \( \beta \) on \( B \) in such a way that the unit and counits are morphisms of commutation structures.

The proof is now purely equational: we exhibit a two sided inverse for \( \alpha : X \otimes A \to A \otimes X \). We shall prove that the following composite \( \gamma \) will serve. (To save space, we write \( \otimes \) just as a dot, \( A \cdot B \) for \( A \otimes B \), etc.)

\[
A \cdot X \xrightarrow{A \cdot X \cdot \eta} A \cdot X \cdot B \cdot A \xrightarrow{A \cdot \beta \cdot A} A \cdot B \cdot X \cdot A \xrightarrow{\epsilon \cdot X \cdot A} X \cdot A.
\]

To prove that \( \alpha \circ \gamma \) is the identity means to prove that the clockwise composite in the following diagram is the identity, and this is proved by considering the rest of the diagram, as we shall argue:

\[
\begin{array}{ccc}
A \cdot X & \xrightarrow{A \cdot X \cdot \eta} & A \cdot X \cdot B \cdot A \\
& \downarrow{A \cdot \eta \cdot X} & \downarrow{A \cdot B \cdot \alpha}
\end{array}
\]

Here the square commutes, because \( \otimes \) is a functor in two variables. For the left hand cell, consider the square (with top map the identity)

\[
\begin{array}{ccc}
X \cdot I & \xrightarrow{X \cdot \eta} & I \cdot X \\
\downarrow{X \cdot \eta} & & \downarrow{\eta \cdot X}
\end{array}
\]

(1)
It commutes by the assumption that $\eta : I \to B \cdot A$ is a morphism of $X$-commutation structures. If we apply the functor $A \cdot -$ we get (by a geometric reflection) the desired commutativity of left hand cell. Finally, the lower (counter-clockwise) composite is an identity arrow: it is just the functor $\cdot X$ applied to one of the triangle equations for $\eta, \epsilon$. Thus $\alpha \circ \gamma$ is the identity arrow of $A \cdot X$.

The proof that $\gamma \circ \alpha$ is the identity is much similar. The map $\gamma \circ \alpha$ appears as the counterclockwise composite in the diagram

$$
\begin{array}{ccc}
X \cdot A & \xrightarrow{X \cdot A \cdot \eta} & X \cdot A \cdot B \cdot A \\
\alpha \downarrow & & \alpha \cdot B \cdot A \\
A \cdot X & \xrightarrow{A \cdot X \cdot \eta} & A \cdot X \cdot B \cdot A \\
\end{array}
$$

The square on the left commutes because $\otimes$ is a functor in two variables, and the cell on the right commutes because $\epsilon$ was assumed to be a morphism of $X$-commutation structures (we omit the diagram, which is analogous to (\text{II})). Finally, the upper (clockwise) composite is the identity: it is just the functor $\cdot X$ applied to one of the triangle equations for $\eta, \epsilon$. Thus $\gamma \circ \alpha$ is the identity arrow of $X \cdot A$. This proves the Theorem.

**Example.** Let $\mathcal{C}$ be a category with coproducts. Let $\mathcal{V}$ be the monoidal category of endofunctors on $\mathcal{C}$. Let $J$ be a fixed set, and let $X$ be the endofunctor on $\mathcal{C}$ given by $C \mapsto \coprod_J C$ (coproduct of $J$ copies of $C$). Let $A : \mathcal{C} \to \mathcal{C}$ be any endofunctor. There is for each $C \in \mathcal{C}$ a map

$$
\prod_j A(C) \to A(\prod_j C)
$$

which on the $j$'th summand of $\prod_J A(C)$ returns $A(\text{incl}_j) : A(C) \to A(\prod_j C)$. This is natural in $C$, and thus is a natural transformation

$$
\alpha : X \circ A \to A \circ X,
$$

in other words, an $X$-commutation structure on $A$ in $\mathcal{V}$. It is clear that any natural transformation $A_1 \to A_2$ between endofunctors on $\mathcal{C}$ is a morphism of $X$-commutations. In particular, if $A$ has a right adjoint ($\cdot$ = a right dual
in the monoidal category $\mathcal{V}$ of endofunctors), this adjointness (duality) lifts to an adjointness/duality in $X\mathcal{V}$.

From the Theorem then follows that $\alpha$ is actually an isomorphism. This can be restated: “If $A$ has a right adjoint, then $A$ commutes with copowers”; which is of course no big surprise.

More generally, if $\mathcal{C}$ is a category enriched over a category $\mathcal{S}$, it makes sense to say that it is tensored over $\mathcal{S}$. To say that an endofunctor $A$ on $\mathcal{C}$ is enriched can expressed in terms of existence of a tensorial strength $\alpha$, cf. [1], which is a map, natural in $(J$ and) $C$,

$$J \otimes A(C) \to A(J \otimes C)$$

for $J \in \mathcal{S}$ and $C \in \mathcal{C}$. Now an adjointness $A \dashv B$ is no longer automatically enriched/strong, but if it is, the Theorem implies that $A$ commutes with tensors $J \otimes -$ up to isomorphism.

Consider a Cartesian Closed Category $\mathcal{S}$. Being closed, it is enriched over itself, and the tensors $J \otimes C$ are just $J \times C$. An enrichment/strength of an endofunctor $A : \mathcal{S} \to \mathcal{S}$ then can be encoded as a “tensorial strength”, i.e. as a natural family of maps

$$J \times A(C) \to A(J \times C),$$

equivalently as a commutation (for each $J$)

$$\alpha : X \circ A \to A \circ X$$

where $X$ denotes the endofunctor $J \times -$.

Consider in particular an endofunctor of the form $(\cdot)^D : \mathcal{S} \to \mathcal{S}$. If this endofunctor has a right adjoint, $D$ is called an atom, cf. e.g. [2], [3], or [5] (who calls such objects $D$ tiny). A Corollary of the Theorem is now a result (due to Yetter):

**Proposition 2** If $(\cdot)^D$ has a strong right adjoint, then $D = 1$.

**Proof.** Let $X$ be an endofunctor of the form $J \times -$ ($J$ an arbitrary object of $\mathcal{S}$). If $(\cdot)^D$ has a strong right adjoint, the Theorem implies that the natural map (the tensorial strength, or commutation structure)

$$J \times Y^D \to (J \times Y)^D$$


is an isomorphism for any \( Y \). This in particular applies to \( Y = 1 \), so that 
\[ J \times 1^D \cong (J \times 1)^D, \]
but since \( 1^D \cong 1 \), we get from this that the natural map 
\[ J \to J^D \]
is an isomorphism, for any \( J \). A suitable enriched Yoneda Lemma now gives the result, but there is an elementary proof:

From the fact that the natural map \( J \to J^D \) is an isomorphism, we conclude that for each \( X \), there is a bijection between \( \text{hom}(X, J) \) and \( \text{hom}(X, J^D) \) (induced by the natural map \( J \to J^D \)); passing to transposes, there is a bijection between \( \text{hom}(X, J) \) and \( \text{hom}(X \times D, J) \) (induced by the projection \( X \times D \to X \)). Now take \( X = 1 \) and \( J = D \) and conclude that the projection \( 1 \times D \to D \) factors across the projection \( 1 \times D \to 1 \). From this follows that \( D \) is a retract of \( 1 \), hence is itself (isomorphic to) \( 1 \).

The notion of \( X \)-co-commutation is obtained by duality, thus a \( X \)-co-commutation on \( A \) is a map \( a : A \cdot X \to X \cdot A \). Similarly for morphisms of co-commutations \( (A, a) \to (B, b) \), and for the monoidal category of co-commutations. In particular, if \( A \dashv B \) by virtue of \( \eta, \epsilon \), as above, and these are compatible with the co-commutations, it follows by Theorem 1 (dualized) that the co-commutation \( b \) on \( B \) is invertible. Thus, the \( b^{-1} \) in the following Theorem makes sense. (I omit the \( \otimes \), formerly abbreviated \( ; \); now they are both denoted just by concatenation.)

**Theorem 3** Let \((A, a) \dashv (B, b)\) in the category of \( X \)-co-commutations. Then \( a \) can be expressed in terms of \( b^{-1} \) as follows: 

\[
\begin{align*}
AX & \xrightarrow{AX \eta} AXBA \xrightarrow{Ab^{-1}A} ABXA \xrightarrow{cXA} XA.
\end{align*}
\]

(2)

**Proof.** Let us denote by \( c \) the composite co-commutation \( BAX \to XBA \) on \( BA \), Thus the middle of the small triangles in the following diagram
commutes by definition:

\[
\begin{array}{cccc}
AX & \xrightarrow{AX\eta} & AXBA & \\
A\eta X & \downarrow & AbA & \xrightarrow{Ab^{-1}A} \\
ABAX & \xrightarrow{ABa} & ABXA & = ABXA \\
\epsilon AX & \downarrow & \epsilon XA & \\
AX & \xrightarrow{a} & XA & 
\end{array}
\]

The upper left little triangle commutes since \( \eta : I \rightarrow BA \) is a morphism of co-commutation structures. The next triangle commutes by definition of \( c \), as observed, and the third triangle evidently commutes. The bottom “square” commutes by bi-functorality of \( \otimes \). Finally, the left hand column is the identity map of \( AX \), by one of the triangle equations for \( \eta, \epsilon \). Thus, the counterclockwise composite in the diagram is \( a \), the clockwise is the arrow in (2). This proves the Theorem.

My motivation for this Theorem was the desire to understand the proof of Proposition XIV.3.1 in [4]; according to this, we have in a braided category that the braiding \( c_{AX} : AX \rightarrowXA \) of a dualizable object \( A \) is determined by the braiding \( c_{BX} \) for its dual \( B \) (\( X \) a fixed object). (For the comparison with Kassel, our \( A \) is his \( V^* \), \( B \) is \( V \), and \( X \) is \( W \).)

More precisely, let \( X \) be a fixed object in a braided monoidal category. Then for any object \( D \), the braiding \( c_{DX} \) defines an \( X \)-co-commutation structure on \( D \), and any map \( D_1 \rightarrow D_2 \) is a homomorphism of co-commutativity structures, just by naturality of \( c_{\_,X} \). Also, the composite co-commutativity structure on \( BA \) is \( c_{BA,X} \), by the “hexagon” axiom for braidings (which here reduces to a triangle), and any duality \( A \perp B \) is automatically compatible with the co-commutations (since any map is).

Applying Theorem 3 to \( a := c_{AX}, b := c_{B,X} \) therefore expresses \( c_{AX} \) in terms of \( c_{B,X} \), as in the statement of the Proposition in [4] (the proof in loc. cit. is given in terms of “graphical calculus”).
References

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