GRADED AR SEQUENCES AND THE HUNEKE-WIEGAND CONJECTURE

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ABSTRACT. We let $R = \bigoplus_{i \geq 0} R_i$ be a one-dimensional graded complete intersection, finitely-generated over $R_0$ an infinite field, satisfying certain degree conditions which are satisfied whenever $R$ is a numerical semigroup ring of embedding dimension at least three. We show that a broad class of graded maximal Cohen-Macaulay $R$-modules $M$ satisfies the Huneke-Wiegand Conjecture, namely when there exists an Auslander-Reiten sequence ending in $M$ whose middle term has at least two nonfree direct summands.

1. INTRODUCTION

The following is called the Huneke-Wiegand Conjecture in, e.g., [6] and [7].

Conjecture 1.1. ([10]) Let $D$ be a Gorenstein local domain of dimension one and $M$ a nonzero finitely-generated torsionfree $D$-module, that is not free. Then $M \otimes_D M^*$ has a nonzero torsion submodule.

As shown in [9, Theorem 5.9], the above condition on $M \otimes_D M^*$ may be replaced by the condition that $\text{Ext}^1_D(M, M) \neq 0$.

Notation 1.2. If $R$ is a Gorenstein ring of dimension one, and $M$ is a nonfree finitely-generated maximal Cohen-Macaulay module, we will say that $M$ “satisfies the Huneke-Wiegand Conjecture” if $\text{Ext}^1_R(M, M) \neq 0$.

This paper is structured as follows. In Section 2 we define some general notions, and state lemmas in the setting of a graded one-dimensional Gorenstein ring, whose use in Section 3 will usually be explicitly mentioned, so these lemmas need not be read beforehand. In Section 3, we let $R$ be a one-dimensional graded complete intersection satisfying a certain degree condition (Definition 3.1), and prove (Theorem 3.8) that $M$ satisfies the Huneke-Wiegand Conjecture if $M$ is any module in a broad class (see Remark 3.5), namely when there exists an Auslander-Reiten sequence $s(M)$ ending in $M$ such that the middle term of $s(M)$ has at least two nonfree direct summands. In Section 4 we show that the degree condition is satisfied by all numerical semigroup rings of embedding dimension at least three.

2. BACKGROUND: AUSLANDER-REITEN THEORY

Throughout, $R$ will be a commutative, one-dimensional, Gorenstein, graded ring $R = \bigoplus_{i \geq 0} R_i$ graded over the nonnegative integers, where $R_0 = k$ is a field and $R$ is a finitely-generated algebra over $k$. Denote the graded maximal ideal $\bigoplus_{i \geq 1} R_i$ by $m$. For any graded $R$-module $M$, $\Omega M$ will denote the graded syzygy of $M$, i.e., the kernel of a minimal graded surjection onto $M$ by a free $R$-module. We use $M^*$ to denote $\text{Hom}_R(M, R)$.

We let $\text{CM}(R)$ denote the category of graded, finitely-generated, maximal Cohen-Macaulay modules (and all $R$-linear homomorphisms between such), and let $L_p(R)$ denote the full
subcategory \( M \in \text{CM}(R) \mid M_p \) is \( R_p \)-free for all nonmaximal prime ideals \( p < R \). Let \( \text{CM}(R)_0 \) denote the category whose objects are those of \( \text{CM}(R) \) and whose morphisms are the degree zero maps. We say that a module \( M \in \text{CM}(R) \) without free direct summands is \emph{periodic} if \( M \cong \Omega^2 M \), up to a shift in the grading.

**Notation 2.1.** For \( R \)-modules \( X \) and \( Y \), we let \( \text{Hom}_R(X,Y) \) denote the quotient of \( \text{Hom}_R(X,Y) \) by those maps which factor through free \( R \)-modules.

**Lemma 2.2.** Let \( X, Y \in \text{CM}(R) \). Then \( \text{Ext}^1_R(X,Y) \cong \text{Hom}_R(\Omega X,Y) \).

**Proof.** Start with the short exact sequence

\[
0 \to \Omega X \overset{i}{\to} F \overset{j}{\to} X \to 0,
\]

where \( F \) is a free module. Since \((\cdot)^*\) is a duality on \( \text{CM}(R) \), \( i^* \) is epi. Therefore any map from a free module to \((\Omega X)^*\) factors through \( i^* \). Then the duality implies that any map from \( \Omega X \) to a free module factors through \( i \). Therefore we have a right exact sequence

\[
\text{Hom}_R(F,Y) \to \text{Hom}_R(\Omega X,Y) \to \text{Hom}_R(\Omega X,Y) \to 0.
\]

On the other hand, applying \( \text{Hom}_R(\_, Y) \) to \((2.1)\) yields

\[
\text{Hom}_R(F,Y) \to \text{Hom}_R(\Omega X,Y) \to \text{Ext}^1_R(X,Y) \to \text{Ext}^1_R(F,Y) = 0,
\]

and the lemma follows. \( \square \)

**Notation 2.3.** The symbol \( a(R) \) stands for the integer called the \emph{a-invariant} of \( R \), a formula for which is given in \((8.1)\).

**Lemma 2.4.** There exists a short exact sequence in \( \text{CM}(R)_0 \) of the form

\[
0 \to R(a) \to m^*(a) \to k \to 0,
\]

where \( a = a(R) \).

**Proof.** As \( R \) is Gorenstein of dimension one, we have \( \text{Ext}^1_R(k,R(a)) = k \). We can obtain the desired sequence by applying \( \text{Hom}_R(\_, R(a)) \) to the short exact sequence \( 0 \to m \to R \to k \to 0 \).

A morphism \( f : X \to Y \) in \( \text{CM}(R)_0 \) is called \emph{irreducible} if (1) \( f \) is neither a split monomorphism nor a split epimorphism, and (2) given any pair of morphisms \( g \) and \( h \) in \( \text{CM}(R)_0 \) satisfying \( f = gh \), either \( g \) is a split epimorphism or \( h \) is a split monomorphism.

Let \( M \) be a nonfree indecomposable in \( L_p(R) \). Then [1, Theorem 3] the category \( \text{CM}(R)_0 \) admits an Auslander-Reiten (AR) sequence ending in \( M \). That is, there exists a short exact sequence

\[
0 \to N \overset{f}{\to} X \overset{g}{\to} M \to 0
\]

in \( \text{CM}(R)_0 \) such that \( N \) is indecomposable and the following property is satisfied: Any map \( L \to M \) in \( \text{CM}(R)_0 \) which is not a split epimorphism factors through \( g \) (equivalently, any map \( N \to L \) in \( \text{CM}(R)_0 \) which is not a split monomorphism factors through \( f \)). In this paper \( R \) is Gorenstein, and therefore (1) there is also an AR sequence \emph{beginning} at \( M \), and (2) the modules in \((2.2)\) all lie in \( L_p(R) \) (to see this, use Lemma \([2.11]\)).
Definition 2.5. Given an AR sequence (2.2), N is called the Auslander-Reiten translate of M, written \( \tau M \). One may equivalently write \( \tau^{-1}N = M \).

Lemma 2.6. Let \( 0 \rightarrow \tau M \xrightarrow{f} X \xrightarrow{g} M \rightarrow 0 \) be an AR sequence in \( \text{CM}(R)_0 \). Then, given any \( Y \in \text{CM}(R) \), a degree zero map \( h : \tau M \rightarrow Y \) is irreducible if and only if there exists a split epimorphism \( p \in \text{Hom}_R(X,Y)_0 \) such that \( h = pf \). Dually, a degree zero map \( h' : Y' \rightarrow M \) is irreducible if and only if there exists a split monomorphism \( i \in \text{Hom}_R(Y,X)_0 \) such that \( h' = gi \).

Proof. Cf. [14, Lemma 2.13], [2, Ch. V, Theorem 5.3]. \( \square \)

Lemma 2.7. ([13] Lemma 4.1.8) If \( f : M \rightarrow N \) is an irreducible map in \( \text{CM}(R)_0 \), then \( f \) must be either a monomorphism or an epimorphism.

Notation 2.8. Several short exact sequences in this paper will include a written direct-sum-decomposition (not necessarily into indecomposables) of the middle term, for example

\[
0 \rightarrow X \xrightarrow{f} Y_1 \oplus Y_2 \xrightarrow{g} Z \rightarrow 0.
\]

Given such a diagram, if we write \( f = [f_1, f_2]^T \) and \( g = [g_1, g_2] \), this means that \( g \) restricted to \( Y_i \) is \( g_i \), and similarly \( f \) induces maps \( f_1 : X \rightarrow Y_1 \) and \( f_2 : X \rightarrow Y_2 \) with respect to the direct sum decomposition \( Y_1 \oplus Y_2 \).

Lemma 2.9. If \( 0 \rightarrow X \xrightarrow{[f_1, f_2]^T} Y_1 \oplus Y_2 \xrightarrow{[g_1, g_2]} Z \rightarrow 0 \) is any short exact sequence of abelian groups, then \( f_1 \) is an epimorphism if and only if \( g_2 \) is an epimorphism. If it is an AR sequence in \( \text{CM}(R)_0 \), then each \( f_i \) and \( g_i \) is irreducible, and: \( f_1 \) is a monomorphism if and only if \( g_2 \) is a monomorphism.

Proof. The first sentence is straightforward, and the second follows from Lemmas 2.6 and 2.7. \( \square \)

Notation 2.10. If \( M \) is a graded \( R \)-module, and \( n \in \mathbb{Z} \), then the graded shift \( M(n) \) is the graded \( R \)-module given by \( M(n)_i = M_{n+i}, \forall i \in \mathbb{Z} \).

Lemma 2.11. For any indecomposable nonfree \( M \in L_p(R) \), we have \( \tau M = \Omega M(a) \), where \( a = a(R) \) is the \( a \)-invariant of \( R \).

Proof. Cf. [14, Proposition 3.11] or the proof of [11, Theorem 3]. \( \square \)

Notation 2.12. Let \( f : M \rightarrow N \) be a morphism in \( \text{CM}(R)_0 \). By extending \( f \) to a map between the minimal free resolutions of \( M \) and \( N \), we get induced morphisms \( \Omega^n f : \Omega^n M \rightarrow \Omega^n N \in \text{CM}(R)_0 \), for each integer \( n \). (Of course, these are not quite uniquely determined.) Likewise, we have morphisms \( \tau^n f : \Omega^n M(a(R)) \rightarrow \Omega^n N(a(R)) \).

Lemma 2.13. ([11, Theorem 3.1], [13, Lemma 4.1.7]) Let \( f : M \rightarrow N \) be a morphism in \( \text{CM}(R)_0 \), and assume \( M \) and \( N \) contain no free direct summands. If \( f \) is irreducible, then so is any choice of \( \Omega^n f \), for all \( n \in \mathbb{Z} \).

Lemma 2.14. Let \( M \) and \( N \) be finitely-generated, graded \( R \)-modules, and assume \( M \) has no free direct summand. If \( f : M \rightarrow N \) factors through a free \( R \)-module, then \( f(M) \subseteq mN \).
Proof. We may assume $N$ is free. As $f$ is necessarily a sum of graded maps (cf. [4] Exercise 1.5.19 (f)), we may assume $f$ is homogeneous. Now if $m \in M$ is homogeneous, and $f(m) \notin mN$, then $f(m)$ can be extended to a basis of $N$, and we may obtain a surjection $M \longrightarrow Rf(m)$, contradicting our assumption that $M$ has no free direct summand. □

Lemma 2.15. Let $f : M \longrightarrow N$ be a morphism in $\text{CM}(R)_0$, and let $n$ be an integer. If $g$ and $h \in \text{Hom}_R(\Omega^n M, \Omega^n N)$ are two choices for $\Omega^n f$, and $g$ is epi, then so is $h$.

Proof. For $n \geq 0$, it is well known that $g - h$ must factor through a free module, and it is easy to see that this still holds for $n < 0$ by Gorenstein duality. Therefore $(g - h)(\Omega^n M) \subseteq m \Omega^n N$ by Lemma 2.14. The assertion now follows from Nakayama’s Lemma. □

Lemma 2.16. ([13], Lemma 4.1.13) Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a short exact sequence in $\text{CM}(R)_0$. Then $\Omega g$ is an epimorphism if and only if $m X = X \cap m Y$.

Lemma 2.17. (cf. [8], Lemma 2.1) Let $0 \longrightarrow X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z \longrightarrow 0$ be a short exact sequence in $\text{CM}(R)_0$ with $g$ irreducible, and suppose that $\Omega g$ is a monomorphism. Then there exist graded maps $i : X \longrightarrow m^*$ and $j : m^* \longrightarrow Y$ in $\text{CM}(R)$ such that $f = ji$; in particular, $X$ is isomorphic to a graded ideal of $R$, up to a graded shift.

Proof. It is part of the general (Auslander-Reiten) folklore that if $f' : X \longrightarrow Y'$ is any map in $\text{CM}(R)_0$, then either $f$ factors through $f'$ or $f'$ factors through $f$ (in $\text{CM}(R)_0$). To see this, assume that $f'$ does not factor through $f$. This says that the pushout of $0 \longrightarrow X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z \longrightarrow 0$ by $f'$ does not split. Therefore, the irreducibility of $g$ implies that the middle map in the diagram

$$
\begin{array}{ccc}
0 & \overset{f}{\longrightarrow} & X \\
\longrightarrow & & \downarrow \text{id} \\
0 & \overset{f'}{\longrightarrow} & Y' \\
\end{array}
\begin{array}{ccc}
Y' & \overset{g}{\longrightarrow} & Z \\
\downarrow \text{id} & & \downarrow \text{id} \\
W & \longrightarrow & Z \\
\end{array}
\begin{array}{ccc}
& & 0 \\
\end{array}
$$

is a split monomorphism, and it follows that $f$ factors through $f'$.

From the short exact sequence $0 \longrightarrow R(a) \longrightarrow m^*(a) \longrightarrow k \longrightarrow 0$ (Lemma 2.4), we obtain a commutative square

\[
\begin{array}{ccc}
\text{Hom}_R(Y, m^*(a)) & \longrightarrow & \text{Hom}_R(Y, k) \\
\downarrow & & \downarrow \\
\text{Hom}_R(X, m^*(a)) & \longrightarrow & \text{Hom}_R(X, k)
\end{array}
\]

where the horizontal maps are surjective since $\text{Ext}_R^1(Y, R) = \text{Ext}_R^1(X, R) = 0$ (as $R$ is one-dimensional Gorenstein). From Lemma 2.16 it follows that $\dim_k(Y \otimes_R k) < \dim_k(X \otimes_R k) + \dim_k(Hom_R(Y, k))$, i.e., $\dim_k(Hom_R(Y, k)) < \dim_k(Hom_R(X, k)) + \dim_k(Hom_R(Z, k))$, and therefore the map $\text{Hom}_R(f, k) : \text{Hom}_R(Y, k) \longrightarrow \text{Hom}_R(X, k)$ is not an epimorphism. Then, in turn, the map $\text{Hom}_R(f, m^*(a)) : \text{Hom}_R(Y, m^*(a)) \longrightarrow \text{Hom}_R(X, m^*(a))$ is not epi (because the horizontal maps in (2.3) are epi). Therefore, we may pick $f' \in \text{Hom}_R(X, m^*)$ which does not factor through $f$. We may also assume $f$ is homogeneous of some degree, cf. [4], Exercise 1.5.19 (f). It then follows from our first sentence in this proof that $f = jf'$ for some homogeneous $j : m^* \longrightarrow Y$. We know that $f' : X \longrightarrow m^*$ is mono, because $f$ is mono. Lastly, note that there
exists a monomorphism $X \hookrightarrow R$, because there exists a monomorphism $m^* \hookrightarrow R$ (since $m^*$ is a finitely-generated submodule of $R[\text{nonzerodivisors}]^{-1}$).

**Definition 2.18.** If $X$ and $Y$ are modules in $\text{CM}(R)$ having no free direct summands, we say that an irreducible map $g: X \twoheadrightarrow Y$ is **eventually $\Omega$-perfect** if either $\Omega^n g$ is epi for all large $n$, or $\Omega^n g$ is mono for all large $n$.

**Lemma 2.19.** (cf. [8] Proposition 2.4) Assume that $R$ is a complete intersection, and let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a short exact sequence in $\text{CM}(R)_0$, such that $g$ is irreducible and not eventually $\Omega$-perfect. Then, for all $n \geq 0$ such that $\Omega^n g$ is epi and $\Omega^{n+1} g$ is mono, we have that $\ker(\Omega^n g)$ is isomorphic to a periodic ideal.

**Proof.** For each $n \geq 0$ we can apply the Horseshoe Lemma to obtain a short exact sequence

$$0 \longrightarrow \Omega^n X \stackrel{[f_1, f_2]^T}{\longrightarrow} F^n \oplus \Omega^n Y \stackrel{[\xi, \Omega^n g]}{\longrightarrow} \Omega^n Z \longrightarrow 0$$

for some free module $F^n$, and $\xi \in \text{Hom}_R(F^n, \Omega^n Z)_0$. If $\Omega^n g$ is surjective, then so is $f_1$ (Lemma 2.9) and this implies $F^n = 0$, since $\Omega^n X$ cannot have a free direct summand. So if $\Omega^{n+1} g$ is mono while $\Omega^n g$ is surjective, then

$$\Omega^n X \cong \ker(\Omega^n g)$$

is isomorphic to an ideal, by Lemma 2.17. Since $\Omega^n X$ is thus isomorphic to an ideal for infinitely many $n \geq 0$, it follows that the free modules in a minimal resolution of $X$ have bounded rank (cf. [13, Lemma 4.1.17]). Therefore $X$ is eventually periodic, by [5, Theorem 4.1]. As $X \in \text{CM}(R)$ and $R$ is Gorenstein, $X$ is in fact already periodic. \hfill \Box

### 3. Huneke-Wiegand Conjecture

Let $S = k[x_1, \ldots, x_e] = \bigoplus_{i \geq 0} S_i$ be a graded polynomial ring over an infinite field $k$, where each $x_i$ is homogeneous (of any positive degree), $k = S_0$, and $e \geq 2$. Let $f_1, \ldots, f_{e-1} \in S$ be a regular sequence of homogeneous polynomials, and let $R = S/(f_1, \ldots, f_{e-1})$. We assume that $e = \text{edim} R$, the embedding dimension of $R$ (i.e., each $f_i \in (x_1, \ldots, x_e^2)$). Let $a_i = \deg x_i$, and $d_i = \deg f_i$. Then the $a$-invariant of $R$ is ([4, 3.6.14-15])

$$a(R) = \sum_{i=1}^{e-1} d_i - \sum_{i=1}^{e} a_i.$$  

**Definition 3.1.** Let $d_{\text{max}} = \max(d_1, \ldots, d_{e-1})$. We will say that $R$ “satisfies Condition (*)” if $a(R) > d_{\text{max}}/2$.

In Section 4 we show that Condition (*) is satisfied by all complete intersection numerical semigroup rings of embedding dimension at least three.

**Definition 3.2.** We will call an indecomposable module $M \in L_p(R)$ **elevated** if

$$\min\{i | (\tau M)_i \neq 0\} < \min\{i | M_i \neq 0\}.$$  

(Recall that $\tau$ denotes the AR translate; see Definition 2.5 and Lemma 2.11)

Notice that if $M$ is elevated, then there exists no monomorphism $\tau M \hookrightarrow M$ in $\text{CM}(R)_0$.

**Lemma 3.3.** Assume $R$ satisfies Condition (*), and let $M$ be an indecomposable nonfree module in $L_p(R)$. Then there exists $n_0$, depending on $M$, such that for all $n \geq n_0$, either $\Omega^n M$ is elevated or $\Omega^{n+1} M$ is elevated (or both).
Proof. Let \((F, \delta) : \cdots \xrightarrow{\delta} F_1 \xrightarrow{\delta} F_0\) be a graded \(R\)-free resolution of \(M\). Following Eisenbud’s construction of cohomology operators \([5]\), take a graded lifting \((\hat{F}, \hat{\delta})\) of \((F, \delta)\) to free \(S\)-modules, that is, \((\hat{F}, \hat{\delta})\) is a graded sequence of free \(S\)-modules such that \(\hat{\delta} \otimes_S R = \delta\). Then for \(j \in \{1, \ldots, e - 1\}\), and all \(n\), we can choose graded maps \(t_j : \hat{F}_{n+2} \rightarrow \hat{F}_n\) such that

\[
\hat{\delta}^2 = \sum_j f_j t_j, \quad \text{and} \quad \deg t_j = -d_j \quad \text{for each} \quad j \in \{1, \ldots, e - 1\}.
\]

Now let \(\hat{R} = \prod_{i \geq 0} R_i\) (the completion of \(R\) with respect to \(m\)) and consider the resolution of free \(\hat{R}\)-modules \((\hat{F}, \hat{\delta})\) induced by \(\delta\), as well as the maps \(\hat{t}_j : \hat{F}_{n+2} \rightarrow \hat{F}_n\) induced by the \(t_j\)’s. From the proof of \([5]\), Theorem 3.1, there exist \(g_1, \ldots, g_e \in \hat{R}\) such that \(\hat{f}_1 + \sum_{j=2}^{e-1} g_j \hat{t}_j : \hat{F}_{n+2} \rightarrow \hat{F}_n\) is an epimorphism for all large \(n\).

It follows that, for large \(n\), \(\min(i | (F_{n+2})_i \neq 0) - \min(i | (F_n)_i \neq 0) \leq d_{\text{max}}\). In other words, \(\min(i | (\Omega^{n+2}M)_i \neq 0) - \min(i | (\Omega^n M)_i \neq 0) \leq d_{\text{max}}\), which implies that either \(\min(i | (\Omega^{n+2}M)_i \neq 0) - \min(i | (\Omega^{n+1}M)_i \neq 0) \leq d_{\text{max}}/2\) or \(\min(i | (\Omega^n M)_i \neq 0) \leq d_{\text{max}}/2\). The result now follows from Condition (*), since \(\tau^n M = \Omega^n M(\cdot | a(R))\), by Lemma 2.11.

**Definition 3.4.** Given an AR sequence \(0 \rightarrow \tau M \rightarrow X \rightarrow M \rightarrow 0\), and writing \(X = \bigoplus_i X^i\) as a direct sum of indecomposable modules \(X^i\), define \(a(M)\) to be the number of (not necessarily nonisomorphic) summands \(X^i\) which are nonfree.

**Remark 3.5.** Let \(X\) and \(Y\) be nonfree indecomposables in \(L_p(R)\), such that \(a(X)\) and \(a(Y)\) are both equal to 1. For the sake of simplicity, assume also that \(X\) is not a direct summand of \(m\) (so that there exists no irreducible map \(X \rightarrow R\)), and \(Y\) is not a direct summand of \(m^*\) (so that there exists no irreducible map \(R \rightarrow Y\)). Then there exists no irreducible morphism \(X \rightarrow Y\) (i.e., \(X\) and \(Y\) are not adjacent in the stable AR quiver). Indeed, any irreducible map \(X \rightarrow X'\) is mono and any irreducible map \(Y' \rightarrow Y\) is epi, by Lemma 2.6. Thus the nonfree indecomposables \(M \in L_p(R)\) having \(a(M) \geq 2\) form a broad class.

**Lemma 3.6.** Assume \(R\) satisfies Condition (*), and let \(M\) be a nonfree indecomposable in \(L_p(R)\), with \(a(M) \geq 2\). Then there exists a nonfree indecomposable \(X \in \text{CM}(R)\) and an irreducible morphism \(p : \tau M \rightarrow X\) in \(\text{CM}(R)_0\), such that the set

\[
N := \{n \geq 0 | \Omega^n p \text{ is epi and } \Omega^n M \text{ is elevated}\}
\]

is infinite.

Proof. Let \(0 \rightarrow \tau^l M \xrightarrow{\bigoplus_i X^i} \bigoplus_i X^i \oplus F^l \rightarrow 0\) be the AR sequence in \(\text{CM}(R)_0\) ending in \(M\), where \(X^1, X^2, \ldots, X^l\) are nonfree indecomposables in \(L_p(R)\), and \(F\) is a (possibly zero) free module. Then it follows from Lemmas 2.6 and 2.13 that the AR sequence ending in \(\tau^n M\) has the form

\[
0 \rightarrow \tau^{n+1} M \xrightarrow{f^l} \bigoplus_i X^i \oplus F^l \xrightarrow{\bigoplus_i g_i} \tau^n M \rightarrow 0
\]

where \(F^l\) is free and \(f^l = [\tau^n f_1, \ldots, \tau^n f_l, \xi]^T\), for some \(\xi : \tau^{n+1} M \rightarrow F^l\).

Certainly \(\tau^n M\) is elevated for infinitely many \(n \geq 0\), by Lemma 3.3. We claim that for each such \(n\), \(\tau^n f_i\) is epi for some \(i \in \{1, \ldots, l\}\). Note that, given the AR sequence (3.2), the map \(\xi : \tau^{n+1} M \rightarrow F^l\) must be mono, since otherwise it would be epi (Lemma 2.7) and therefore split, contradicting the fact that it is irreducible. So if we write \(g^l = [g_1', \ldots, g_l']\), and if \(\tau^n f_i\) is mono for each \(i \in \{1, \ldots, l\}\), then it follows from Lemma 2.9 that each of \(g_1', \ldots, g_l'\) is mono.
Then \( g'_1 \circ (\tau^n f_1) \), for example, is a degree zero monomorphism, which contradicts that \( \tau^n M \) is elevated. Thus the claim is thus proved, and the lemma follows (by applying a version of the “pigeonhole principle”). □

**Lemma 3.7.** If \( \text{edim} R \geq 3 \), then \( m^* \) is not periodic.

**Proof.** See [3, Theorem 8.1.2]. □

**Theorem 3.8.** Assume that either \( R \) is isomorphic to a numerical semigroup ring \( k[t^{a_1}, \ldots, t^{a_r}] \), or that \( \text{edim} R \geq 3 \) and \( R \) satisfies Condition (*) . If \( M \in L_R(R) \) is a nonfree indecomposable with \( a(M) \geq 2 \), then \( \text{Hom}_R(\tau M, M)_0 \neq 0 \), and \( M \) satisfies the Huneke-Wiegand Conjecture.

**Proof.** If \( \text{edim} R < 3 \), and \( R \) is a domain (for example, a numerical semigroup ring), then \( M \) satisfies the Huneke-Wiegand Conjecture by [10, Theorem 3.7]. If \( R \) is a numerical semigroup ring with \( \text{edim} R \geq 3 \), then we will see in Proposition 4.10 that \( R \) satisfies Condition (*).

First we set about proving the following claim: \( \tau^{n+1} p \) is epi for all sufficiently large \( n \). If \( p \) is eventually \( \Omega \)-perfect then the claim holds (by choice of \( p \)), so assume otherwise. Let \( n \in \mathcal{N} \), i.e., \( \tau^n p: \tau^{n+1} M \to \tau^n X \) is epi and \( \tau^n M \) is elevated. Let us first observe that:

\[
\tau^n X \to \tau^n M \text{ in CM}(R)_0.
\]

Indeed, there exists an irreducible map \( \tau^n X \to \tau^n M \) by Lemma 2.6 and if it were epi then the composition with the epimorphism \( \tau^n p \) would (by Lemma 2.14) give a nonzero element of \( \text{Hom}_R(\tau^{n+1} M, \tau^n M)_0 \cong \text{Hom}_R(\tau M, M)_0 \). (Indeed, \( \text{Hom}_R(\Omega^{n+1} M, \Omega^n M) \cong \text{Hom}_R(\Omega M, M) \) for all \( n \in \mathbb{Z} \) since \( R \) is Gorenstein.)

Now, suppose \( \tau^{n+1} p \) is mono. Letting \( K = \ker(\tau^n p) \), we know \( K \) is isomorphic to a periodic ideal, by Lemma 2.19. Letting \( \kappa \) denote the inclusion map \( \kappa: K \hookrightarrow \tau^{n+1} M \), we have \( \kappa = \kappa'' \kappa' \) for some graded maps \( \kappa': K \to m^* \) and \( \kappa'': m^* \to \tau^{n+1} M \), by Lemma 2.17. Note that \( K \cong K^{**} \cong \text{Hom}_R(K^*, m) \) since \( K \) lies in \( \text{CM}(R) \) and has no free direct summand. This implies that \( K \) is a module over \( \text{End}_R m = m^* \). So if \( 1 \in \kappa'(K) \), then \( \kappa' \) must be a split epi, contradicting that \( K \) is periodic and \( m^* \) is not (Lemma 3.3). So \( 1 \notin \kappa'(K) \), while on the other hand, \( \kappa''(1) \) must be nonzero since otherwise \( \kappa'' \) would be zero.

Let us observe that the latter sentence implies that \( \kappa(K) \cap (\tau^{n+1} M)_{i_0} = 0 \) where \( i_0 = \min\{i| (\tau^{n+1} M)_i \neq 0 \} \). To check this, note that \( a(R) \) is positive by Condition (*), and therefore by the short exact sequence \( 0 \to R \to m^* \to k(-a) \to 0 \) (Lemma 2.4), we have that \( (m^*)_0 = R_0 \) is a one-dimensional \( k \)-vector space, and \( (m^*)_i = R_i = 0 \) for all \( i < 0 \). So indeed we see that since the image of \( K \) does not touch the lowest degree of \( m^* \), and the lowest degree of \( m^* \) does not go to zero in \( \tau^{n+1} M \), therefore \( \kappa(K) \) does not touch the lowest degree of \( \tau^{n+1} M \).

Therefore \( \tau^n p \) restricted to \( (\tau^{n+1} M)_{i_0} \) is injective. But together with (3.3), this implies \( (\tau^n M)_{i_0} \neq 0 \), and this is a contradiction since \( \tau^n M \) is elevated, by definition of \( \mathcal{N} \). So the claim, that \( \tau^{n+1} p \) is epi for all sufficiently large \( n \), is proved.

Therefore, using Lemma 3.3 and the fact that \( \text{Hom}_R(\tau M, M) \cong \text{Hom}_R(\tau^{n+1} M, \tau^n M) \) for all \( n \in \mathbb{Z} \), we may assume that \( p: \tau M \to X \) is itself epi and that \( \tau^{-1} X \) is elevated.

By Lemma 2.6 the AR sequence beginning with \( X \) has the form

\[
\begin{array}{cccc}
0 & \to & X & \xrightarrow{f=[f_1,f_2]^T} & M & \oplus & Z & \xrightarrow{g=[g_1,g_2]} & \tau^{-1} X & \to & 0 ,
\end{array}
\]
where $Z$ may be zero. Moreover, $f_1(X) \subseteq mM$, since otherwise $f_1 p$ gives a nonzero element of $\text{Hom}_R(\tau M, M)_0$, by Lemma 2.14. In particular, $f_1$ is mono by Lemma 2.7. The surjectivity of $p$ also gives $e(\tau M) > e(X)$, where $e()$ denotes multiplicity, $e(N) := \lim_{n \to \infty} \frac{1}{n} \text{length}(N/m^n N)$. (Recall that $e()$ is additive along short exact sequences, and positive on $\text{CM}(R)$.) Therefore, $\tau g_1 : \tau M \twoheadrightarrow X$ is also epi, by Lemma 2.7. This implies that $\Omega g$ is epi.

Now let $0 \neq w \in X_{j_0}$ where $j_0 = \min \{j \in \mathbb{Z} | X_j \neq 0 \}$. Then $f(w) \notin mM \oplus mZ$, by Lemma 2.16, and therefore $f_2(w) \neq 0$, since $f_1(X) \subseteq mM$. Meanwhile $g_2$ is mono by Lemma 2.9, and so $g_2 f_2(w) \neq 0$. This contradicts the assumption that $\tau^{-1}X$ is elevated. So the assertion $\text{Hom}_R(\tau M, M)_0 \neq 0$ is proved. Now by Lemma 2.2, $\text{Ext}^1_R(M, M) \neq 0$, i.e. $M$ satisfies the Huneke-Wiegand Conjecture.

\[\square\]

4. Numerical Semigroups

The purpose of this section is to prove Proposition 4.10.

**Definition 4.1.** Let $\mathbb{N}$ denote the natural numbers $\{0, 1, 2, \ldots \}$. A numerical semigroup $S$ is a subset of $\mathbb{N}$ such that $0 \in S$, $s + s' \in S$ whenever $s$ and $s'$ are in $S$, and $\mathbb{N} \setminus S$ is finite. The numerical semigroup generated by positive integers $a_1, \ldots, a_e$ (such that $\gcd(a_1, \ldots, a_e) = 1$) is the set $\{n_1 a_1 + \cdots + n_e a_e | n_1, \ldots, n_e \in \mathbb{N} \} \cap \mathbb{N}$. If $S = \mathbb{N} a_1 + \cdots + \mathbb{N} a_e$, $k[S]$ will denote the graded $k$-algebra $k[t^{a_1}, \ldots, t^{a_e}] \subseteq k[t]$ where $\deg t = 1$. The Frobenius of a numerical semigroup $S$ is $\max \{n \in \mathbb{Z} | n \notin S \}$, and we denote this integer by $F(S)$. A numerical semigroup $S$ is said to be complete intersection if $k[S]$ is a complete intersection. In this case $F(S) = \alpha(k[S])$ (see, e.g., [13, Proposition 2.3.3]), so in particular we can use formula (3.1) to find $F(S)$.

**Definition 4.2.** Consider a numerical semigroup $S$ which is minimally generated by $a_1, \ldots, a_e$. We may take an additive surjection from the free monoid $\text{Free}(Y_1, \ldots, Y_e) = \langle n_1 Y_1 + \cdots + n_e Y_e | n_1, \ldots, n_e \in \mathbb{N} \rangle$, $\varphi: \text{Free}(Y_1, \ldots, Y_e) \twoheadrightarrow S$ sending $Y_i$ to $a_i$. A minimal presentation for $S$ is a set $\rho \subseteq \text{Free}(Y_1, \ldots, Y_e) \times \text{Free}(Y_1, \ldots, Y_e)$ which generates the kernel congruence $\{(u, v) \in \text{Free}(Y_1, \ldots, Y_e) | \varphi(u) = \varphi(v) \}$; see [12, Chapter 7]. For $r = (r_1, r_2) \in \rho$, we will use the notation $|r| = \varphi(r_1) = \varphi(r_2)$. The numbers $\{|r| | r \in \rho \}$ are the same as the $\{d_i \}$ from Section 3, when $R \cong k[S]$.

**Definition 4.3.** For a complete intersection numerical semigroup $S$, we will abuse notation slightly by saying that “$S$ satisfies Condition (*)” if $k[S]$ satisfies Condition (*). Instead of writing $a(R) > d_{\max}/2$, we may alternatively write $F(S) > |r|/2$ for all $r \in \rho$, where $F(S)$ is the Frobenius of $S$ and $\rho$ is a minimal presentation of $S$.

**Definition 4.4.** [12, Ch. 8, §3] Let $S$ and $S'$ be two numerical semigroups minimally generated by $\{a_1, \ldots, a_e \}$ and $\{a'_1, \ldots, a'_{e'} \}$, respectively. Let $\lambda \in S \setminus \{a_1, \ldots, a_e \}$ and $\mu \in S' \setminus \{a'_1, \ldots, a'_{e'} \}$ be such that $\gcd(\lambda, \mu) = 1$. Then $S'' = \mathbb{N} \mu a_1 + \cdots + \mathbb{N} \mu a_e + \mathbb{N} \lambda a'_1 + \cdots + \mathbb{N} \lambda a'_{e'}$ is called a gluing of $S$ and $S'$.

**Lemma 4.5.** In the notation of Definition 4.4, $S''$ is minimally generated by $\{\mu a_1, \ldots, \mu a_e, \lambda a'_1, \ldots, \lambda a'_{e'} \}$, and we have $\lambda \geq 2$ and $\mu \geq 2$.

**Proof.** For the first statement, see [12, Lemma 9.8]. We have for example $\lambda \geq 2$ because $\lambda \in S \setminus \{a_1, \ldots, a_e \}$.

\[\square\]
Lemma 4.6. (See the proof of [12] Proposition 9.9.) Assume we have $S$, $S'$, and $S''$ as in Definition 4.4 and let $\rho$ and $\rho'$ be minimal presentations of $S$ and $S'$, respectively. Assume that $S$ and $S'$ are complete intersections. Then $S''$ is minimally presented by $\rho \cup \rho'$ together with a single additional element $r$, with $|r| \in \lambda \mu \mathbb{N}$.

Theorem 4.7. [12] Theorem 9.10] A numerical semigroup other than $\mathbb{N}$ is a complete intersection if and only if it is a gluing of two complete intersection numerical semigroups.

Lemma 4.8. Let $m$ and $n$ be relatively prime integers $\geq 2$. Then $mn/2 - m - n \geq -2$, and the only cases when $mn/2 - m - n \leq 0$ are when either $2 \in \{m, n\}$ or $\{m, n\} \in \{(3, 4), (3, 5)\}$.

Lemma 4.9. Let $S$ and $S'$ be complete intersection numerical semigroups. Assume that $S$ satisfies Condition (*) and is not equal to $\mathbb{N}$, and either: $S'$ can be generated by 1 or 2 elements, or $S'$ satisfies Condition (*). Then each gluing of $S$ and $S'$ satisfies Condition (*) as well.

Proof. Assume all notation in Definition 4.4. Let $\rho$ and $\rho'$ denote minimal presentations of $S$ and $S'$ respectively, and let $(a_1, \ldots, a_{e-1}) = \{r\}_r \in \rho$ and $(a'_1, \ldots, a'_{e-1}) = \{r\}_r \in \rho'$. Formula (3.1) together with Condition (*) on $S$ say

\[
F(S) = \sum_{i=1}^{e-1} d_i - \sum_{i=1}^{e} a_i > d_j/2
\]

for each $j \in \{1, \ldots, e\}$. Let $\rho''$ denote a minimal presentation for $S''$. By Lemma 4.6, \{r''\}_r \in \rho'' = \{\mu d_1, \ldots, \mu d_{e'}, \lambda d'_1, \ldots, \lambda d'_{e'}, d''\}$, where $d'' \in \lambda \mu \mathbb{N}$, and we have (using (3.1) and Lemma 4.5)

\[
F(S'') = d'' + \sum_{i=1}^{e-1} \mu d_i + \sum_{i=1}^{e'} \lambda d'_i - \sum_{i=1}^{e} \mu a_i - \sum_{i=1}^{e'} \lambda a'_i = d'' + \mu F(S) + \lambda F(S').
\]

Since $F(S) \geq 1$ and $F(S') \geq -1$, we obtain $F(S'') - d''/2 \geq d''/2 + \mu - \lambda \geq \lambda \mu/2 + \mu - \lambda$, which is positive since $\mu \geq 2$. Using (4.1) and (4.1), we have $F(S'') = d'' + \mu F(S) + \lambda F(S') > d'' + \mu d_j/2 + \lambda F(S') > \mu d_j/2$ for each $j \in \{1, \ldots, e' - 1\}$; and if $S'$ satisfies Condition (*) then symmetrically $F(S'') > \lambda d_j/2$ for each $j \in \{1, \ldots, e' - 1\}$. It remains to check that $F(S'') > \lambda d_j/2$ whenever $e' = 2$. In this case $d'_1 = a'_1 a'_2$ and the goal amounts to $\lambda (a'_1 a'_2 - a'_1 - a'_2) + d'' + \mu F(S) > 0$. But indeed, $\lambda (a'_1 a'_2 - a'_1 - a'_2) \geq -2 \lambda$ (by Lemma 4.8), $d'' \geq 2 \lambda$, and $F(S) > 0$.

\[\square\]

Proposition 4.10. Let $S$ be a complete intersection numerical semigroup, minimally generated by $\{a_1, \ldots, a_e\}$. Then $S$ satisfies Condition (*), unless $e = 2$ and either $2 \in \{a_1, a_2\}$ or $\{a_1, a_2\} \in \{(3, 4), (3, 5)\}$.

Proof. First suppose $e = 2$. Now $d_1 = a_1 a_2$ and $F(S) = a_1 a_2 - a_1 - a_2$, and we are done by Lemma 4.8.

Now suppose $e = 3$. Then, by [12] Theorem 10.6] (or alternatively Theorem 4.7) there exist relatively prime integers $m_1$ and $m_2$ greater than one, and nonnegative integers $a$, $b$, and $c$ such that $S = \mathbb{N} m_1 + \mathbb{N} m_2 + \mathbb{N} (b m_1 + c m_2)$, and furthermore, $a \geq 2$, $b + c \geq 2$, and $\gcd(a, b m_1 + c m_2) = 1$. In this case, the minimal presentation $\rho$ of $S$ has $\{r\}_r \in \rho = \{a m_1 m_2, a (b m_1 + c m_2)\}$, and

\[
F(S) = a(m_1 m_2 - m_1 - m_2) + (a - 1)(b m_1 + c m_2).
\]

First, notice that $F(S) - a(b m_1 + c m_2)/2 = a(m_1 m_2 - m_1 - m_2) + (a/2 - 1)(b m_1 + c m_2)$ is positive since $m_1 m_2 - m_1 - m_2 > 0$ and $a \geq 2$.
To finish the \( e = 3 \) case, suppose (with the aim of a contradiction) \( F(S) \leq am_1m_2/2 \), i.e.,
\[
(4.4) \quad a(m_1m_2/2 - m_1 - m_2) \leq (1 - a)(bm_1 + cm_2).
\]

However, \( a(m_1m_2/2 - m_1 - m_2) \geq -2a \) by Lemma 4.8, so we get \(-2a \leq (1 - a)(bm_1 + cm_2)\), and thus \( bm_1 + cm_2 \leq 2a/(a - 1) \leq 4 \). Since a strict inequality \( bm_1 + cm_2 < 4 \) is impossible (as \( b + c \geq 2 \)), we get \( bm_1 + cm_2 = 2a/(a - 1) = 4 \). So \( a = 2 \) and \( bm_1 + cm_2 = 4 \), contradicting \( \gcd(a, bm_1 + cm_2) = 1 \).

Having finished cases \( e = 2 \) and \( e = 3 \), we proceed to the induction part, which requires little additional work in light of Theorem 4.7 and Lemma 4.9. In fact the only thing left to check is that any gluing of two-generated numerical semigroups (not counting \( \mathbb{N} \) as two-generated) satisfies Condition (*). So let \( S = \mathbb{N}a_1 + \mathbb{N}a_2, \ S' = \mathbb{N}a'_1 + \mathbb{N}a'_2, \lambda \in S \setminus \{a_1, a_2\}\) and \( \mu \in S' \setminus \{a'_1, a'_2\} \), such that \( \gcd(\lambda, \mu) = 1 \), and let \( S'' = \mathbb{N}a_1 + \mathbb{N}a_2 + \mathbb{N}\lambda a'_1 + \mathbb{N}\lambda a'_2 \). We wish to check that each of \( \{\mu a_1a_2, \lambda a'_1a'_2, d''\} \) is exceeded by \( 2F(S'') \), where \( d'' \geq \mu \lambda \) (recall Lemma 4.10) and \( F(S'') = d'' + \mu(a_1a_2 - a_1 - a_2) + \lambda(a'_1a'_2 - a'_1 - a'_2) = d'' + F(S) + F(S') \). We have \( F(S'') - d''/2 > 0 \) since \( F(S) \) and \( F(S') \) are positive. Meanwhile, by Lemma 4.8, \( F(S'') - \mu a_1a_2/2 = d'' + \mu(a_1a_2 - a_1 - a_2) + F(S') \geq d'' - 2\mu + F(S') \geq \lambda \mu - 2\mu + 1 > 0 \). The proof that \( F(S'') - \lambda a'_1a'_2/2 > 0 \) is of course the same. \( \square \)

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