Positive-Unlabeled Demand-Aware Recommendation

Jinfeng Yi¹, Cho-Jui Hsieh², Kush Varshney¹, Lijun Zhang³, and Yao Li²

¹IBM Thomas J. Watson Research Center
²Department of Computer Science, University of California, Davis
³Department of Computer Science, Nanjing University

Abstract

Recommendation for e-commerce with a mix of durable and nondurable goods has characteristics that distinguish it from the well-studied media recommendation problem. The demand for items is a combined effect of form utility and time utility, i.e., a product must both be intrinsically appealing to a consumer and the time must be right for purchase. In particular for durable goods, time utility is a function of inter-purchase duration within product category because consumers are unlikely to purchase two items in the same category in close temporal succession. Moreover, purchase data, in contrast to ratings data, is implicit with non-purchases not necessarily indicating dislike. Together, these issues give rise to the positive-unlabeled demand-aware recommendation problem that we pose via joint low-rank tensor completion and product category inter-purchase duration vector estimation. We further relax this problem and propose a highly scalable alternating minimization approach with which we can solve problems with millions of users and items. We also show superior prediction accuracies on multiple real-world data sets.

1 Introduction

E-commerce recommender systems aim to present items with high utility to the consumers [20]. Utility may be decomposed into form utility: the item is desired as it is manifested, and time utility: the item is desired at the given point in time [30]; recommender systems should take both types of utility into account. Economists define items to be either durable goods or nondurable goods based on how long they are intended to last before being replaced [29]. A key characteristic of durable goods is the long duration of time between successive purchases within item categories whereas this duration for nondurable goods is much shorter, or even negligible. Thus, durable and nondurable goods have differing time utility characteristics which lead to differing demand characteristics.

Although we have witnessed great success of collaborative filtering in media recommendation, we should be careful when expanding its application to general e-commerce recommendation involving both durable and nondurable goods due to the following reasons:

1. Since media such as movies and music are nondurable goods, most users are quite receptive to buying or renting them in rapid succession. However, users only purchase durable goods when the time is right. For instance, most users will not buy televisions the day after they have already bought one. Therefore, recommending an item for which a user has no immediate demand can hurt user experience and waste an opportunity to drive sales.
2. A key assumption made by matrix factorization- and approximation-based collaborative filtering algorithms is that the underlying rating matrix is of low-rank since only a few factors typically contribute to an individual’s form utility \[5, 34\]. However, a user’s demand is not only driven by form utility, but is the combined effect of both form utility and time utility. Hence, even if the underlying form utility matrix is of low-rank, the overall purchase intention matrix is usually of high-rank, and thus cannot be directly recovered by existing approaches. To see this, we construct a toy example with 50 users and 100 durable goods. Following the low-rank assumption, we assume the form utility matrix \(Z \in \mathbb{R}^{50 \times 100}\) is generated by \(UV^\top\), where \(U \in \mathbb{R}^{50 \times 10}\) and \(V \in \mathbb{R}^{100 \times 10}\) are both Gaussian random matrices with mean 1 and standard deviation 0.5. As discussed above, user \(i\)’s purchase intention of item \(j\) is mediated by a time utility factor \(h_{ij}\). We assume this factor is sampled from a rectified Gaussian distribution with mean 1 and standard deviation 0.5\[5\]. Then the purchase intention matrix \(B \in \mathbb{R}^{50 \times 100}\) is given by \(B = Z - H\), where \(H \in \mathbb{R}^{50 \times 100}\) is the time utility factor matrix. Figure 1 shows the distributions of singular values for matrices \(Z\) and \(B\). We observe that although the form utility matrix \(Z\) is of low-rank, the purchase intention matrix \(B\) is a full-rank matrix since all its singular values are greater than 0. This simple example illustrates that considering users’ demands can make the underlying matrix no longer of low-rank, thus violating the key assumption made by many collaborative filtering algorithms.

An additional challenge faced by many real-world recommender systems is the one-sided sampling of implicit feedback \[17, 25\]. Unlike the Netflix-like setting that provides both positive and negative feedback (high and low ratings), in many e-commerce systems no negative feedback is available. For example, a user might not purchase an item because she does not derive utility from it, or just because she was simply unaware of it or plans to buy it in the future. In this sense, the labeled training data only draws from the positive class, and the unlabeled data is a mixture of positive and negative samples, a problem usually referred to as positive-unlabeled (PU) learning \[11, 22, 15\].

\[1\] We use a rectified Gaussian distribution to ensure non-negative time utility factors. As introduced in Section 3, \(h_{ij}\) is a function of item \(j\)’s inter-purchase duration \(d\) and the time gap \(t\) of user \(i\)’s most recent purchase within the item \(j\)’s category. If \(d\) and \(t\) are Gaussian random variables, then \(h_{ij} = \max(0, d - t)\) follows a rectified Gaussian distribution.
To address these limitations, we study the problem of demand-aware recommendation. Given purchase triplets (user, item, time) and item categories, the objective is to make recommendations based on users’ overall predicted combination of form utility and time utility.

We denote purchases by the sparse binary tensor \( P \). To model implicit feedback, we assume that \( P \) is obtained by thresholding an underlying real-valued utility tensor to a binary tensor \( Y \) and then revealing a subset of \( Y \)'s positive entries. The key to demand-aware recommendation is defining an appropriate utility measure for all (user, item, time) triplets. To this end, we quantify purchase intention as a combined effect of form utility and time utility. Specifically, we model a user's time utility for an item by comparing the time \( t \) since her most recent purchase within the item’s category and the item category’s underlying inter-purchase duration \( d \); the smaller the value of \( t - d \), the less likely she needs this item. In contrast, \( t \geq d \) may indicate that the item needs to be replaced, and she may be open to related recommendations. Therefore, the hinge loss \( h = \max(0, d - t) \) can be employed to measure the time utility factor for a (user, item) pair. Then the purchase intention for a (user, item, time) triplet is given by \( x - h \), where \( x \) denotes the user’s form utility.

This observation allows us to cast demand-aware recommendation as the problem of learning users’ form utility tensor \( X \) and items’ inter-purchase durations vector \( d \) given the binary tensor \( P \).

Although the learning problem can be naturally formulated as a tensor nuclear norm minimization problem, the high computational cost significantly limits its application to large-scale recommendation problems. To address this limitation, we first relax the problem to a matrix optimization problem with a class-dependent loss. We note that the problem after relaxation is still non-trivial to solve since it is a highly non-smooth problem with nested hinge losses. More severely, the optimization problem involves \( mnl \) entries, where \( m \), \( n \), and \( l \) are the number of users, items, and time slots, respectively. Thus a naive optimization algorithm will take at least \( O(mnl) \) time, and is intractable for large-scale recommendation problems. To overcome this limitation, we develop an efficient alternating minimization algorithm and show that its time complexity is only approximately proportional to the number of nonzero elements in the purchase records tensor \( P \). Since \( P \) is usually very sparse, our algorithm is extremely efficient and can scale up to problems with millions of user and items. For instance, our synthetic study shows that the proposed algorithm is able to make demand-aware recommendations for a trillion (user, item) pairs in less than 2 hours.

Compared to existing recommender systems, our work has the following contributions and advantages: (i) to the best of our knowledge, this is the first work that makes demand-aware recommendation by considering inter-purchase durations for durable and nondurable goods; (ii) the proposed algorithm is able to simultaneously infer items’ inter-purchase durations and users’ real-time purchase intentions, which can help e-retailers make more informed decisions on inventory planning and marketing strategy; (iii) by effectively exploiting sparsity, the proposed algorithm is extremely efficient and able to handle problems with a huge number of users and items.

2 Related Work

Our contributions herein relate to three different areas of prior work: consumer modeling from a microeconomics and marketing perspective [6], time-aware recommender systems [4, 31, 9, 21], and PU learning [22, 10, 15, 16, 25, 2]. The extensive consumer modeling literature is concerned with descriptive and analytical models of choice rather than prediction or recommendation, but nonetheless forms the basis for our modeling approach. A variety of time-aware recommender systems have been proposed to exploit time information, but none of them explicitly consider the notion of time utility derived from inter-purchase durations in item categories. Much of the PU learning literature is focused on the binary classification problem, e.g. [22, 10], whereas we are in the collaborative filtering setting. For the papers that do examine collabo-
rative filtering with PU learning or learning with implicit feedback [16,25,2], they mainly focus on media recommendation and overlook users’ demands, thus are not suitable for durable goods recommendation.

Temporal aspects of the recommendation problem have been examined in a few ways: as part of the cold-start problem [3], to capture dynamics in interests or ratings over time [19,8,31], and as part of the context in context-aware recommenders [1]. However, the problem we address in this paper is different from all of those aspects, and in fact could be combined with the other aspects in future solutions. To the best of our knowledge, there is no existing work that tries to take inter-purchase durations into account to better time recommendations as we do herein.

3 Positive-Unlabeled Demand-Aware Recommendation

Throughout the paper, we use boldface Euler script letters, boldface capital letters, and boldface lower-case letters to denote tensors (e.g., \(\mathcal{A}\)), matrices (e.g., \(\mathbf{A}\)) and vectors (e.g., \(\mathbf{a}\)), respectively. Scalars such as entries of tensors, matrices, and vectors are denoted by lowercase letters, e.g., \(a\). In particular, the \((i,j,k)\) entry of a third-order tensor \(\mathcal{A}\) is denoted by \(a_{ijk}\).

Given a set of \(m\) users, \(n\) items, and \(l\) time slots, we construct a third-order binary tensor \(\mathcal{P} \in \{0,1\}^{m \times n \times l}\) to represent the purchase history. Specifically, entry \(p_{ijk} = 1\) indicates that user \(i\) has purchased item \(j\) in time slot \(k\). We denote \(\|\mathcal{P}\|_0\) as the number of nonzero entries in tensor \(\mathcal{P}\). Since \(\mathcal{P}\) is usually very sparse, we have \(\|\mathcal{P}\|_0 \ll mnl\). Also, we assume that the \(n\) items belong to \(r\) item categories, with items in each category sharing similar inter-purchase durations. We use an \(n\)-dimensional vector \(\mathbf{c} \in \{1,2,\ldots,r\}^n\) to represent the category membership of each item. Given \(\mathcal{P}\) and \(\mathbf{c}\), we further generate a tensor \(\mathcal{T} \in \mathbb{R}^{m \times r \times l}\) where \(t_{icjk}\) denotes the number of time slots between user \(i\)’s most recent purchase within item category \(c_j\) until time \(k\). If user \(i\) has not purchased within item category \(c_j\) until time \(k\), \(t_{icjk}\) is set to \(+\infty\).

3.1 Inferring Purchase Intentions from Users’ Purchase Histories

In this work, we formulate users’ utility as a combined effect of form utility and time utility. To this end, we use an underlying third-order tensor \(\mathcal{X} \in \mathbb{R}^{m \times n \times l}\) to quantify form utility. In addition, we employ a non-negative vector \(\mathbf{d} \in \mathbb{R}^+_l\) to measure the underlying inter-purchase duration times of the \(r\) item categories. It is understood that the inter-purchase durations for durable good categories are large, while for nondurable good categories are small, or even zero. In this study, we focus on items’ inherent properties and assume that the inter-purchase durations are user-independent. The problem of learning personalized durations will be studied in our future work.

As discussed above, the demand is mediated by the time elapsed since the last purchase of an item in the same category. Let \(d_{cj}\) be the inter-purchase duration time of item \(j\)’s category \(c_j\), and let \(t_{icjk}\) be the time gap of user \(i\)’s most recent purchase within item category \(c_j\) until time \(k\). Then if \(d_{cj} > t_{icjk}\), a previously purchased item in category \(c_j\) continues to be useful, and thus user \(i\)’s utility from item \(j\) is weak. Intuitively, the greater the \(d_{cj} - t_{icjk}\), the weaker the utility. On the other hand, \(d_{cj} < t_{icjk}\) indicates the item is nearing the end of its lifetime and the user may be open to recommendations in category \(c_j\). We use a hinge loss \(h(i,c_j,k) = \max(0,d_{cj} - t_{icjk})\) to model such time utility. The overall utility can be obtained by comparing form utility and time utility. In more detail, we model a binary utility indicator tensor \(\mathcal{Y} \in \{0,1\}^{m \times n \times l}\) as

\[\mathcal{Y}_{ijk} = \begin{cases} 1 & \text{if } h(i,c_j,k) > 0 \\ 0 & \text{otherwise} \end{cases}\]

\[h(i,c_j,k) = \max(0,d_{cj} - t_{icjk})\]
being generated by the following thresholding process:

\[ y_{ijk} = \mathbf{1}[x_{ijk} - h(i, c_j, k) > \tau], \]  

where \( \mathbf{1}(\cdot) : \mathbb{R} \to \{0, 1\} \) is the indicator function, and \( \tau > 0 \) is a predefined threshold.

Note that the positive entries of \( \mathcal{Y} \) denote high purchase intentions, while the positive entries of \( \mathcal{P} \) denote actual purchases. Generally speaking, a purchase only happens when the utility is high, but a high utility does not necessarily lead to a purchase. This observation allows us to link the binary tensors \( \mathcal{P} \) and \( \mathcal{Y} \): \( \mathcal{P} \) is generated by a one-sided sampling process that only reveals a subset of \( \mathcal{Y} \)'s positive entries. Following the setting of PU learning [11, 22, 15], we assume that the positive entries of \( \mathcal{P} \) are uniformly sampled from positive entries of \( \mathcal{Y} \).

It is impossible to directly recover tensor \( \mathcal{X} \) from \( \mathcal{P} \) but fortunately, we need not recover \( \mathcal{X} \) to make recommendations. Instead, we only need to infer the binary tensor \( \mathcal{Y} \) whose entries \( y_{ijk} \) can be obtained by comparing \( x_{ijk} - h(i, c_j, k) \) to \( \tau \). In this sense, our goal becomes learning a tensor \( \mathcal{Z} \in \mathbb{R}^{m \times n \times l} \) such that the binary tensors obtained by thresholding \( \mathcal{Z} \) and \( \mathcal{X} \) to be the same. We note that tensor \( \mathcal{Z} \) should be of low-rank to capture temporal dynamics of users’ interests, which are generally believed to be dictated by a small number of latent factors [24].

In addition, since \( \mathcal{P} \) is generated by \( \mathcal{Y} \) via a one-sided sampling process, we follow [15] and include an asymmetric margin loss with a parameter \( \eta \) trading the relative cost of positive and unlabeled samples [28]:

\[
\mathcal{L}(\mathcal{Z}, \mathcal{P}) = \eta \sum_{ijk: \rho_{ijk}=1} \max[1 - (z_{ijk} - \max(0, d_{c_j} - t_{ic_j,k})), 0]^2 + (1 - \eta) \sum_{ijk: \rho_{ijk}=0} l(z_{ijk}, 0),
\]

where \( l(x, c) = (x - c)^2 \) denotes the squared loss.

By combining asymmetric sampling and the low-rank property together, we recover the tensor \( \mathcal{Z} \) and the inter-purchase duration vector \( \mathbf{d} \) by solving the following tensor nuclear norm minimization (TNNM) problem:

\[
\min_{\mathcal{Z} \in \mathbb{R}^{m \times n \times l}, \mathbf{d} \in \mathbb{R}^n} \eta \sum_{ijk: \rho_{ijk}=1} \max[1 - (z_{ijk} - \max(0, d_{c_j} - t_{ic_j,k})), 0]^2 + (1 - \eta) \sum_{ijk: \rho_{ijk}=0} z_{ijk}^2 + \lambda \|\mathcal{Z}\|_*,
\]

where \( \|\mathcal{Z}\|_* \) denotes the tensor nuclear norm, a convex combination of nuclear norms of \( \mathcal{Z} \)'s unfolded matrices [23]. Given the learned \( \mathcal{Z} \) and \( \mathbf{d} \), the underlying binary tensor \( \mathcal{Y} \) can be recovered by

\[ y_{ijk} = \mathbf{1}[\hat{z}_{ijk} - \max(0, \hat{d}_{c_j} - t_{ic_j,k}) > \tau]. \]

We note that although the TNNM problem (2) can be solved by optimization techniques such as block coordinate descent [23] or ADMM [12], they suffer from high computational cost since they need to be solved iteratively with multiple SVDs at each iteration. An alternative way to solve the problem is tensor factorization [18]. However, this also involves iterative singular vector estimation, and is thus not scalable enough. As a typical example, recovering a rank 10 tensor of size \( 500 \times 500 \times 500 \) takes the state-of-the-art tensor factorization algorithm TenALS [4] more than 20,000 seconds on an Intel Xeon 2.40 GHz processor with 32 GB main memory.

\footnote{To see this, consider a tensor \( \mathcal{X} \) with all entries equaling \( \rho \), which leads to an all one tensor \( \mathcal{Y} \) if \( \rho > \tau + \phi \), where \( \phi \) is the upper bound of \( h(i, c_j, k) \) As is apparent, \( \rho \) cannot be uniquely identified by just observing \( \mathcal{Y} \).}

\footnote{http://web.engr.illinois.edu/~swoh/software/optspace/code.html}
3.2 A Scalable Relaxation

In this subsection, we discuss how to significantly improve the scalability of the proposed demand-aware recommendation model. To this end, we assume that an individual’s form utility does not change over time, an assumption widely-used in many collaborative filtering methods [27]. Under this assumption, the tensor $Z$ is a repeated copy of its frontal slice $Z_{:,1}$, i.e.,

$$Z = Z_{:,1} \odot e,$$

where $e$ is an $l$-dimensional all-one vector and the symbol $\odot$ represents the outer product operation. In this way, we can relax the problem of learning a third-order tensor $Z$ to the problem of learning its frontal slice, which is a second-order tensor (matrix). For notational simplicity, we use a matrix $Z$ to denote the frontal slice $Z_{:,1}$, and use $z_{ij}$ to denote the entry $(i, j)$ of the matrix $Z$.

Since $Z$ is a low-rank tensor, its frontal slice $Z$ should be of low-rank as well. With the assumption, the minimization problem (2) simplifies to:

$$\min_{Z \in \mathbb{R}^{m \times n}} \eta \sum_{ijk: p_{ijk}=1} \max[1 - (z_{ij} - \max(0, d_{c j} - t_{icj,k})), 0]^2$$

$$+ (1 - \eta) \sum_{ijk: p_{ijk}=0} z_{ij}^2 + \lambda \|Z\|_* := f(Z, d),$$

where $\|Z\|_*$ stands for the matrix nuclear norm, the convex surrogate of the matrix rank function. By relaxing the optimization problem (2) to the problem (4), we recover a matrix instead of a tensor to infer users’ purchase intentions. In Section 4, we discuss how to efficiently optimize the problem (4) to make demand-aware recommendations for a huge number of users and items.

4 Optimization

Although the learning problem has been relaxed, optimizing (4) is still very challenging for two reasons: (i) the objective is highly non-smooth with nested hinge losses, and (ii) it contains $mnl$ terms: a naive optimization algorithm will take at least $O(mnl)$ time.

To address these challenges, we adopt an alternating minimization scheme that iteratively fixes one of $d$ and $Z$ and minimizes with respect to the other. Specifically, we apply an alternating minimization scheme to iteratively solve the following subproblems:

$$d \leftarrow \arg \min_d f(Z, d).$$

$$Z \leftarrow \arg \min_Z f(Z, d)$$

We note that both subproblems are non-trivial to solve because subproblem (5) is a nuclear norm minimization problem, and both subproblems involve nested hinge losses. In the following we discuss how to efficiently optimize subproblems (5) and (6):

4.1 Update $d$

Eq (5) can be written as

$$\min_d \sum_{ijk: p_{ijk}=1} \left\{ \max \left(1 - (z_{ij} - \max(0, d_{c j} - t_{icj,k})), 0\right)^2 \right\} := g(d) := \sum_{ijk: p_{ijk}=1} g_{ijk}(d_{c j}).$$
We then analyze the value of each $g_{ijk}$ by comparing $d_{c_j}$ and $t_{ic_jk}$:

1. If $d_{c_j} \leq t_{ic_jk}$, we have
   \[ g_{ijk}(d_{c_j}) = \max(1 - z_{ij}, 0)^2 \]

2. If $d_{c_j} > t_{ic_jk}$, we have
   \[ g_{ijk}(d_{c_j}) = \max(1 - (z_{ij} - d_{c_j} + t_{ic_jk}), 0)^2, \]
   which can be further separated into two cases:
   \[ g_{ijk}(d_{c_j}) = \begin{cases} 
   1 - (z_{ij} - d_{c_j} + t_{ic_jk})^2, & \text{if } d_{c_j} > z_{ij} + t_{ic_jk} - 1 \\
   0, & \text{if } d_{c_j} \leq z_{ij} + t_{ic_jk} - 1 
   \end{cases} \]

Therefore, we have the following observations:

1. If $z_{ij} \leq 1$, we have
   \[ g_{ijk}(d_{c_j}) = \begin{cases} 
   \max(1 - z_{i,j}, 0)^2, & \text{if } d_{c_j} \leq t_{ic_jk} \\
   (1 - (z_{ij} - d_{c_j} + t_{ic_jk}))^2, & \text{if } d_{c_j} > t_{ic_jk} 
   \end{cases} \]

2. If $z_{ij} > 1$, we have
   \[ g_{ijk}(d_{c_j}) = \begin{cases} 
   (1 - (z_{ij} - d_{c_j} + t_{ic_jk}))^2, & \text{if } d_{c_j} > t_{ic_jk} + z_{ij} - 1 \\
   0, & \text{if } d_{c_j} \leq t_{ic_jk} + z_{ij} - 1 
   \end{cases} \]

This further implies
\[ g_{ijk}(d_{c_j}) = \begin{cases} 
   \max(1 - z_{ij}, 0)^2, & \text{if } d_{c_j} \leq t_{ic_jk} + \max(z_{ij} - 1, 0) \\
   (1 - (z_{ij} - d_{c_j} + t_{ic_jk}))^2, & \text{if } d_{c_j} > t_{ic_jk} + \max(z_{ij} - 1, 0) 
   \end{cases} \]

For notational simplicity, we let $s_{ijk} = t_{ic_jk} + \max(z_{ij} - 1, 0)$ for all triplets $(i, j, k)$ satisfying $p_{ijk} = 1$.

**Algorithm.** For each category $\kappa$, we collect the set $Q = \{(i, j, k) \mid p_{ijk} = 1 \text{ and } c_j = \kappa\}$ and calculate the corresponding $s_{ijk}s$. We then sort $s_{ijk}s$ such that $(s_{i_1,j_1,k_1}) \leq \cdots \leq s_{(i_{|Q|},j_{|Q|},k_{|Q|})}$. For each interval $[s_{i_q,j_q,k_q}, s_{i_{q+1},j_{q+1},k_{q+1}}]$, the function is
\[ g_\kappa(d) = \sum_{t=q+1}^{\left|Q\right|} \max(1 - z_{i_t j_t}, 0)^2 + \sum_{t=1}^{q} (d + 1 - z_{i_t j_t} - t_{i_t c_t k_t})^2 \]

By letting
\[ R_q = \sum_{t=q+1}^{\left|Q\right|} \max(1 - z_{i_t j_t}, 0)^2, \]
\[ F_q = \sum_{t=1}^{q} (1 - z_{i_t j_t} - t_{i_t c_t k_t}), \]
\[ W_q = \sum_{t=1}^{q} (1 - z_{i_t j_t} - t_{i_t c_t k_t})^2, \]
we have

\[ g_\kappa(d) = qd^2 + 2F_qd + W_q + R_q \]

\[ = q\left(d + \frac{F_q}{q}\right)^2 - \frac{F_q^2}{q} + W_q + R_q. \]

Thus the optimal solution in the interval \([s(i_qj_qk_q), s(i_q+1j_q+1k_q+1)]\) is given by

\[ d^* = \max\left(s(i_qj_qk_q), \min\left(s(i_q+1j_q+1k_q+1), -\frac{F_q}{q}\right)\right), \]

and the optimal function value is \(g_\kappa(d^*)\). By going through all the intervals from small to large, we can obtain the optimal solution for the whole function. We note that each time when \(q \Rightarrow q + 1\), the constants \(R_q, F_q, W_q\) only change by one element. Thus the time complexity for going from \(q \Rightarrow q + 1\) is \(O(1)\), and the whole procedure has time complexity \(O(|Q|)\).

In summary, we can solve the subproblem (5) by the following steps:

1. generate the set \(U_\kappa = \{(i, j, k) \mid p_{ijk} = 1 \text{ and } c_j = \kappa\}\) for each category \(r\),
2. sort each list (costing \(O(|Q_\kappa| \log |Q_\kappa|)\) time),
3. compute \(R_0, F_0, W_0\) (costing \(O(|Q_\kappa|)\) time), and then
4. search for the optimal solution for each \(q = 1, 2, \ldots, |Q_\kappa|\) (costing \(O(|Q_\kappa|)\) time).

The above steps lead to an overall time complexity \(O(|P|_0 \log(|P|_0))\), where \(|P|_0\) is the number of nonzero elements in tensor \(P\). Therefore, we can efficiently update \(d\) since \(P\) is a very sparse tensor with only a small number of nonzero elements.

### 4.2 Update Z

By defining

\[ a_{ijk} = \begin{cases} 1 + \max(0, d_{c_j} - t_{ic_jk}), & \text{if } p_{ijk} = 1 \\ 0, & \text{otherwise} \end{cases} \]

the subproblem (6) can be written as

\[
\min_{Z \in \mathbb{R}^{m \times n}} h(Z) + \lambda \|Z\|_* \quad \text{where } h(Z) := \eta \sum_{i,j,k: \ p_{ijk} = 1} \max(a_{ijk} - z_{ij}, 0)^2 + (1 - \eta) \sum_{i,j,k: \ p_{ijk} = 0} z_{ij}^2.
\]

Since there are \(O(mnl)\) terms in the objective function, a naive implementation will take \(O(mnl)\) time, which is computationally inefficient when the data is large. To address this issue, We use proximal gradient descent to solve the problem. At each iteration, \(Z\) is updated by

\[ Z \leftarrow S_\lambda(Z - \alpha \nabla h(Z)), \quad (7) \]

where \(S_\lambda(\cdot)\) is the soft-thresholding operator for singular values.
Algorithm 1: Proximal Gradient Descent for Updating $Z$

**Input**: $\mathcal{P}$, $Z^0$ (initialization), step size $\gamma$

**Output**: A sequence of $Z^t$ converges to the optimal solution

1. for $t = 1, \ldots, \text{maxiter}$ do
2. \[ [U, \Sigma, V] = \text{rand_svd}(Z - \gamma \nabla h(Z^t)) \]
3. \[ \bar{\Sigma} = \max(\Sigma - \gamma \lambda, 0) \]
4. $k$: number of nonzeros in $\Sigma$
5. $Z^{t+1} = U(:, 1:k)\bar{\Sigma}(1:k, 1:k) V(:, 1:k)^T$

In order to efficiently compute the top singular vectors of $Z - \alpha \nabla h(Z)$, we rewrite it as

$$Z - \alpha \nabla h(Z) = [1 - 2(1 - \eta)l] Z + \left(2(1 - \eta) \sum_{ijk: p_{ijk}=1} z_{ij} - 2\eta \sum_{ijk: p_{ijk}=1} \max(a_{ijk} - z_{ij}, 0) \right). \tag{8}$$

Since $Z$ is a low-rank matrix, $[1 - 2(1 - \eta)l] Z$ is also of low-rank. Besides, since $\mathcal{P}$ is very sparse, the term

$$\left(2(1 - \eta) \sum_{ijk: p_{ijk}=1} z_{ij} - 2\eta \sum_{ijk: p_{ijk}=1} \max(a_{ijk} - z_{ij}, 0) \right)$$

is also sparse because it only involves the nonzero elements of $\mathcal{P}$. In this case, when we multiply $(Z - \alpha \nabla h(Z))$ with a skinny $m$ by $k$ matrix, it can be computed in $O(nk^2 + mk^2 + \|\mathcal{P}\|_0 k)$ time.

As shown in [14], each iteration of proximal gradient descent for nuclear norm minimization only requires a fixed number of iterations before convergence, thus the time complexity to update $Z$ is $O(nk^2T + mk^2T + \|\mathcal{P}\|_0 kT)$, where $T$ is the number of iterations.

### 4.3 Overall Algorithm

Combining the two subproblems together, the time complexity of each iteration of the proposed algorithm is:

$$O(\|\mathcal{P}\|_0 \log(\|\mathcal{P}\|_0) + nk^2T + mk^2T + \|\mathcal{P}\|_0 kT).$$

**Remark**: Since each user should make at least one purchase and each item should be purchased at least once to be included in $\mathcal{P}$, $n$ and $m$ are smaller than $\|\mathcal{P}\|_0$. Also, since $k$ and $T$ are usually very small, the time complexity to solve problem (4) is dominated by the term $\|\mathcal{P}\|_0$, which is a significant improvement over the naive approach with $O(mnl)$ complexity.

Since our problem has only two blocks $d$, $Z$ and each subproblem is convex, our optimization algorithm is guaranteed to converge to a stationary point [13]. Indeed, it converges very fast in practice. As a concrete example, it takes only 10 iterations to optimize a problem with 1 million users, 1 million items, and more than 166 million purchase records.

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5If $X$ has the singular value decomposition $X = UV^T$, then $S_\lambda(X) = U(\Sigma - \lambda I)_+ V^T$ where $a_+ = \max(0, a)$. 

9
Figure 2: Prediction errors $\|\mathbf{d} - \mathbf{d}^*\|_2/\|\mathbf{d}\|_2$ as a function of number of users, items, categories, and noise levels on synthetic data sets

Table 1: CPU time for solving problem (4) with different number of purchase records

| $m$ (# users) | $n$ (# items) | $l$ (# time slots) | $\|\mathbf{P}\|_0$ | $k$ | CPU Time (in seconds) |
|---------------|---------------|--------------------|-----------------|-----|-----------------------|
| 1,000,000     | 1,000,000     | 1,000              | 693,826         | 10  | 250                   |
| 1,000,000     | 1,000,000     | 1,000              | 2,781,040       | 10  | 311                   |
| 1,000,000     | 1,000,000     | 1,000              | 11,112,400      | 10  | 595                   |
| 1,000,000     | 1,000,000     | 1,000              | 43,106,100      | 10  | 1,791                 |
| 1,000,000     | 1,000,000     | 1,000              | 166,478,000     | 10  | 6,496                 |

5 Experiments

5.1 Experiment with Synthesized Data

We first conduct experiments with simulated data to verify that the proposed demand-aware recommendation algorithm is computationally efficient and robust to noise. To this end, we first construct a low-rank matrix $\mathbf{Z} = \mathbf{W}\mathbf{H}^T$, where $\mathbf{W} \in \mathbb{R}^{m \times 10}$ and $\mathbf{H} \in \mathbb{R}^{n \times 10}$ are random Gaussian matrices with entries drawn from $\mathcal{N}(1, 0.5)$, and then normalize $\mathbf{Z}$ to the range of $[0, 1]$. We randomly assign all the $n$ items to $r$ categories, with their inter-purchase durations $\mathbf{d}$ equaling $[10, 20, \ldots, 10r]$. We then construct the high purchase intension set $\Omega = \{(i, j, k) \mid t_{ic, k} \geq d_{c,j} \text{ and } z_{ij} \geq 0.5\}$, and sample a subset of its entries as the observed purchase records. We let $n = m$ and vary them in the range $\{10,000, 20,000, 30,000, 40,000, 100,000\}$. Given the learned durations $\mathbf{d}^*$, we use $\|\mathbf{d} - \mathbf{d}^*\|_2/\|\mathbf{d}\|_2$ to measure the prediction errors.

Accuracy Figure 2(a) and 2(b) clearly show that the proposed algorithm can perfectly recover the underlying inter-purchase durations with varied numbers of users, items, and categories. To further evaluate the robustness of the proposed algorithm, we randomly flip some entries in tensor $\mathbf{P}$ from 0 to 1 to simulate the rare cases of purchasing two items in the same category in close temporal succession. Figure 2(c) shows that when the ratios of noisy entries are not large, the predicted durations $\hat{\mathbf{d}}$ are close enough to the true durations, thus verifying the robustness of the proposed algorithm.

Scalability To verify the scalability of the proposed algorithm, we fix the numbers of users and items to be 1 million, the number of time slots to be 1000, and vary the number of purchase records (i.e., $\|\mathbf{P}\|_0$). Table 1 summarizes the running time of solving problem (4) on a computer with 32 GB main memory using a single thread. We observe that the proposed algorithm is extremely efficient, e.g., even with 1 million users, 1
million items, and more than 166 million purchase records, the running time of the proposed algorithm is less than 2 hours.

5.2 Experiment with Real-World Data

We then evaluate the proposed demand-aware recommendation algorithm on two real-world datasets Tmall and Amazon Review. Due to the high computational costs of some of the baseline algorithms, only subsets of these two datasets are used in our study. For Tmall data set, we pick the top 70 item categories, which contain the purchase records of 438 users and 937 items. For Amazon Review data set, we randomly select 300 users who have provided reviews to at least 5 item categories on Amazon.com. This leads to a total of 5,051 items belonging to 11 categories. Time information for both data sets is provided in days, and we have 184 and 3,420 time slots for Tmall and Amazon Review datasets, respectively.

We compare the proposed recommendation algorithm to the following six state-of-the-art recommendation algorithms: (a) M^3F, maximum-margin matrix factorization [26], (b) PMF, probabilistic matrix factorization [27], (c) WR-MF, weighted regularized matrix factorization [16], (d) CP-APR, Candecomp-Parafac alternating Poisson regression [7], (e) Rubik, knowledge-guided tensor factorization and completion method [32], and (f) BPTF, Bayesian probabilistic tensor factorization [33]. Among them, M^3F and PMF are widely-used static collaborative filtering algorithms. We include these two algorithms as baselines to justify whether traditional collaborative filtering algorithms are suitable for general e-commerce recommendation involving both durable and nondurable goods. Since they require explicit ratings as inputs, we follow [2] to generate numerical ratings based on the frequencies of (user, item) consumption pairs. WR-MF is essentially the positive-unlabeled version of PMF and has shown to be very effective in modeling implicit feedback data. All the other three baselines, i.e., CP-APR, Rubik, and BPTF, are tensor-based methods that can consider time utility when making recommendations. We refer to the proposed recommendation algorithm as Demand-Aware Recommender for One-Sided Sampling, or DAROSS for short.

For each user, we randomly sample 90% of her purchase records as the positive class training data, and use the remaining 10% as the test data. In addition, we randomly sample 20% of her non-purchase triplets as the negative class training data and feed them to the baseline algorithms. We select this relatively small ratio since most of the unknown entries in implicit data are caused by the users not being aware of the items as opposed to users disliking the items. For each triplet \((u, i, t)\) in the test set, we evaluate all the
Table 2: Estimated inter-reviewing durations for Amazon Review Data

| Categories               | Instant Video | Apps for Android | Automotive | Baby | Beauty | Digital Music | Grocery | Musical Instruments | Office Products | Patio Suppliers | Pet Supplies |
|--------------------------|---------------|------------------|------------|------|--------|---------------|---------|---------------------|-----------------|----------------|-------------|
| d                        | 0             | 0                | 326        | 0    | 0      | 158           | 0       | 38                  | 94              | 271            | 40          |

algorithms on two tasks: (i) *item prediction*, and (ii) *purchase time prediction*. In the first task, we record the predicted ranking of item $j$ among all items at time $t$; while in the second task, we record the error between the true purchase time $t$ and its nearest predicted purchase time within item $j$’s category. If no purchase is predicted within the item category, we set the error be the total number of time slots $l$. Ideally, good recommendations should have both small item rankings and small time errors. Thus we adopt the top percentages, i.e., $\text{ranking} / n \times 100\%$ and $\text{error} / l \times 100\%$, as the evaluation metrics of item and purchase time prediction tasks, respectively. The algorithms M$^3$F, PMF, and WR-MF are excluded from the purchase time prediction task since they are static models.

Figure 3 displays the predictive performance of the seven recommendation algorithms. Compared to all the baseline algorithms, DAROSS, the proposed demand-aware recommendation algorithm yields the best performance for both datasets and tasks. As expected, we also observe that DAROSS achieves better performance on Tmall than Amazon Review. We conjecture that this is because the time stamps of Amazon Review reflect the review time instead of purchase time, and inter-reviewing durations could be different from inter-purchase durations. Table 2 reports the inter-reviewing durations of Amazon Review estimated by our algorithm DAROSS. Although they may not reflect the true inter-purchase durations, the estimated durations still make sense since they clearly distinguished between durable good categories, e.g., *automotive*, *musical instruments*, and non-durable good categories, e.g., *instant video*, *apps*, and *food*. Apparently, these categories are relatively coarse and may contain multiple sub-categories with different durations. By choosing a more proper category granularity (which is unfortunately not supported by Amazon Review), we expect to achieve more accurate duration estimations and also better recommendation performance. We do not report the estimated durations of Tmall herein since its item categories are anonymized.

We finally test our algorithm on the full set of Amazon Review[8] to evaluate its performance on large-scale real-world data. After removing duplicate items, the data set contains more than 72 million product reviews from 19.8 million users and 7.7 million items that belong to 24 item categories. The collected reviews span a long range of time, i.e., from May 1996 to July 2014, which leads to 6,639 time slots in total. Comparing to the small subset we evaluated earlier, the full set is a much more challenging data set both due to its much larger size and much lower sampling rate, i.e., many reviewers only provided a few reviews, and many items were only reviewed a few times. Indeed, most of our baseline algorithms fail to handle such a large data set and we only obtain the predictive performance of three algorithms DAROSS, WR-MF, and PMF. Their average item ranking percentages are 17.4\%, 21.9\%, and 38.6\%, respectively. In addition, it only takes our algorithm 10 iterations and 37 minutes to find out the optimal solution. All these results clearly validate the scalability and effectiveness of our proposed algorithm.

6 Conclusion

In this paper, we examine the problem of demand-aware recommendation in settings when inter-purchase duration within item categories affects users’ purchase intention in combination with intrinsic properties

[8]http://jmcauley.ucsd.edu/data/amazon/
of the items themselves. Also accounting for the one-sided sampling from implicit feedback present in e-commerce, we formulate a TNNM problem that seeks to learn the form utility tensor and estimate a vector of inter-purchase durations jointly.

The TNNM formulation does not scale to large realistic problem sizes, and thus we develop a tractable matrix optimization relaxation that we solve via an alternating minimization scheme. We show that such an approach converges to a stationary point and, despite challenging subproblems, has tractable time complexity dominated by the number of purchases in the data rather than the cardinalities of users, items, and time slots.

Our empirical studies show that the proposed approach can yield perfect recovery of duration vectors in noiseless settings; it is robust to noise and scalable as analyzed theoretically. On two real-world datasets, Tmall and Amazon Review, we show that our algorithm outperforms five state-of-the-art recommender systems on both metrics of our interest: item prediction and purchase time prediction. Thus overall, the algorithm proposed in this paper ought to be considered for practical deployment in e-commerce recommendation of durable goods both due to its accuracy and its scalability.

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