OPERATOR INEQUALITIES IMPLYING SIMILARITY TO A CONTRACTION

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Abstract. Let $T$ be a bounded linear operator on a Hilbert space $H$ such that
\[ \alpha[T^*, T] := \sum_{n=0}^{\infty} \alpha_n T^n T^* \geq 0 \]
where $\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n$ is a suitable analytic function in the unit disc $\mathbb{D}$ with real coefficients. We prove that if $\alpha(t) = (1 - t)\tilde{\alpha}(t)$, where $\tilde{\alpha}$ has no roots in $[0, 1]$, then $T$ is similar to a contraction.

Operators of this type have been investigated by Agler, Müller, Olofsson, Pott and others, however, we treat cases where their techniques do not apply.

We write down an explicit Nagy-Foias type model of an operator in this class and discuss its usual consequences (completeness of eigenfunctions, similarity to a normal operator, etc.). We also show that the limits of $\|T^n h\|$ as $n \to \infty$, $h \in H$, do not exist in general, but do exist if an additional assumption on $\alpha$ is imposed.

Our approach is based on a factorization lemma for certain weighted $\ell^1$ Banach algebras.

1. Introduction

Let $H$ be a separable complex Hilbert space and denote by $L(H)$ the set of bounded linear operators on $H$. Let $T \in L(H)$ and let
\[ \alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n \]
be an analytic function on the open unit disc $\mathbb{D} = \{ z : |z| < 1 \}$ such that $\alpha_n$ are real and $\sum_n |\alpha_n| \|T^n\|^2 < \infty$. Then we define the so-called hereditary calculus
\[ \alpha[T^*, T] := \sum_{n=0}^{\infty} \alpha_n T^n T^* \]
This work is devoted to the study of operators $T \in L(H)$ that satisfy an operator inequality
\[ \alpha[T^*, T] \geq 0. \]
Notice that for $\alpha(t) = 1 - t$, this is just the class of all contractions on $H$.

The study of operator inequalities of this type was originated in the work by Agler [1], where he studied more general inequalities of the form $\sum_{j,k} \alpha_{jk} T^j T^k \geq 0$. Suppose that $k(w, z) = 1/(\sum_{j,k} \alpha_{jk} w^j z^k)$ is analytic in $\mathbb{D} \times \mathbb{D}$ and $k(\bar{w}, z)$ is a reproducing kernel that defines a functional Hilbert space $H_k(\mathbb{D})$ of functions on $\mathbb{D}$. Let $M$ be the operator $Mu(z) = zu(z)$, acting on $H_k(\mathbb{D})$, and assume that it is bounded. Agler’s main result in [1] asserts that an operator $T$ whose spectrum $\sigma(T)$ is contained in $\mathbb{D}$ satisfies the above inequality if and only if it is unitarily equivalent to the operator $\bigoplus_{j=1}^{\infty} M^*$ restricted to an invariant subspace. This is what Agler calls a coanalytic model of $T$.

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The condition \( \sigma(T) \subset \mathbb{D} \) is too restrictive; it has been shown in subsequent papers that in many cases it can be replaced by \( \sigma(T) \subset \overline{\mathbb{D}} \). Then, in general, instead of the operator \( \bigoplus_{j=1}^{\infty} M^* \), the above coanalytic model involves a direct sum \( \bigoplus_{j=1}^{\infty} M^* \oplus U \), where \( U \) is a unitary. For instance, in [2], Agler proved that an operator \( T \) is a hypercontraction of order \( n \) (that is, it satisfies (1) for \( \alpha(t) = (1 - t)^k \), \( k = 1, \ldots, n \)) if and only if it can be extended to an operator \( \bigoplus_{j=1}^{\infty} M^* \oplus U \), where \( M \) acts on the Bergman space, whose reproducing kernel is \( 1/(1 - \bar{w}z)^n \). If \( T \) is of class \( C_0 \), (that is, \( T^n \to 0 \) strongly), the above unitary summand is absent.

The case where \( \alpha \) is a polynomial, say \( \alpha = p \), was studied further by Müller in [16]. He considers the class \( C(p) \) of operators \( T \in L(H) \) such that (1) is satisfied and proves that \( T \) has a coanalytic model whenever \( p(1) = 0 \), \( 1/p(t) \) is analytic in \( \mathbb{D} \) and \( 1/p(\bar{w}z) \) is a reproducing kernel. Notice that the last condition is equivalent to the fact that all Taylor coefficients of \( 1/p(t) \) at the origin are positive.

In [21], Olofsson deals with a more general setting, when \( \alpha \) is not a polynomial. Suppose an analytic function \( \alpha(t) \) on \( \mathbb{D} \) satisfies \( \alpha \neq 0 \) in \( \mathbb{D} \) and \( 1/\alpha \) has positive Taylor coefficients at the origin. Olofsson studies contractions \( T \) on \( H \) that satisfy \( \alpha[rT^*, rT] \geq 0 \) for all \( r, 0 \leq r < 1 \) (he imposes some more assumptions on \( \alpha \)). He obtains the coanalytic model for this class of operators.

Certain types of operator inequalities like (1) have also been studied for commuting tuples of operators in [3], [22], [8] and other papers. Pott in [22] considered positive regular polynomials. These are polynomials of several complex variables with non-negative coefficients such that the constant term is 0 and the coefficients of the linear terms are positive. Given such polynomial \( p \), Pott constructed a dilation model for commuting tuples of operators satisfying the positivity conditions \( (1 - p)^k[T^*, T] \geq 0 \) for \( 1 \leq k \leq m \) (see Theorem 3.8 in [22]). In [8], Bhattacharyya and Sarkar define the characteristic function \( \theta_T \) for this class of tuples and construct a functional model in the pure case (see Theorem 4.2 in [8]).

A general framework of Agler’s theory in case of operator inequalities for commuting tuples of operators has been given in [3] and further generalized in [4]: the latter work treats general analytic models, which involve multi-dimensional analogues of operators \( \bigoplus_{j=1}^{\infty} M^* \oplus U \) attached to a domain in \( \mathbb{C}^n \).

Here we restrict ourselves to a single operator, but for this case, we can deal with a large class of operator inequalities, for which the original Agler’s approach does not seem to apply.

In what follows, we will say that an analytic function \( \alpha(t) \) is admissible if it has the form \( \alpha(t) = (1 - t)\tilde{\alpha}(t) \), where \( \sum_{n=0}^{\infty} |\tilde{\alpha}_n| < \infty \) and \( \tilde{\alpha} \) is positive on \( [0, 1] \) (in particular, \( \alpha_0 > 0 \)).

One of our main results is as follows.

**Theorem 1.1.** Let \( T \in L(H) \) be an operator whose spectrum is contained in the closed unit disc \( \overline{\mathbb{D}} \). Let \( \alpha(t) = (1 - t)\tilde{\alpha}(t) \) be an admissible function such that \( \sum_{n=0}^{\infty} |\tilde{\alpha}_n|(1 + \|T^n\|^2) < \infty \). If \( \alpha[T^*, T] \geq 0 \), then \( T \) is similar to a contraction.

Notice that \( \sum_{n=0}^{\infty} |\tilde{\alpha}_n|(1 + \|T^n\|^2) < \infty \) implies that \( \sum_{n=0}^{\infty} |\alpha_n|(1 + \|T^n\|^2) < \infty \), so that the operator \( \alpha[T^*, T] \) is well-defined.

If \( \alpha(t) \) is an admissible function and an operator \( T \) on \( H \) is related to \( \alpha \) as in the above theorem, then we will say that \( T \) belongs to the class \( C_\alpha \) (see the definition at the beginning of Section 3).

An important particular case is when \( \alpha \) is analytic on a disc \( |t| < R \) of radius \( R > 1 \), in particular, if \( \alpha \) is a polynomial or is rational. In this case, \( \alpha \) is admissible whenever \( \alpha(t) > 0 \) on \( [0, 1] \) and \( \alpha(t) \) has a simple root at \( t = 1 \). We get that given a function \( \alpha \) of this type and a Hilbert space operator \( T \), whose spectral radius is less than or equal to 1, \( T \) is similar to a contraction whenever \( \alpha[T^*, T] \geq 0 \).

We remark that the condition that \( \tilde{\alpha} \) has no roots in \( [0, 1] \) has a clear spectral meaning. Indeed, the eigenvalues and, more generally, the approximate point spectrum of \( T \) are contained in \( \{ z \in \mathbb{D} : \alpha(|z|^2) \geq 0 \} \). As it is seen from the example of normal operators, under the above
condition, the approximate point spectrum of $T$ can be whatever closed subset of the closed unit disc.

The main difference with the approach originated by Agler is that here we do not need to assume that $1/\alpha(\bar{w}z)$ is a reproducing kernel. That is, we admit that some of the Taylor coefficients of $1/\alpha(t)$ at $t = 0$ can be negative and we also allow $\alpha$ to have zeros in $\bar{D}$ (excluding the interval $[0, 1]$). Notice that we only get similarity to a contraction; in fact, there are many operators with $\|T\| > 1$, to which our results apply. Notice that the majority of papers based on Agler’s approach (for the case of a single operator) deal only with contractions (see the Remark 4.3 below).

Our main tool for proving Theorem 1.1 is related with Banach algebras. We say that a operator $T$ is

\[ \omega \]

a good weight if

\[ \omega_n \geq 1 \quad \text{for every } n, \]

\[ \omega_n \omega_m \geq \omega_{n+m} \] (submultiplicative property) for every $n, m$,

\[ \omega_n^{1/n} \to 1. \]

Given a good weight $\omega$, we define the corresponding weighted Wiener algebra $A_\omega$ as the following set of analytic functions:

\[ A_\omega := \{ \alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n : \sum_{n=0}^{\infty} |\alpha_n| \omega_n < \infty \}. \]

It is immediate to check that $A_\omega$ is a commutative, unital Banach algebra of analytic functions in $\bar{D}$.

For analytic functions $f(t) = \sum_{n=0}^{\infty} f_n t^n$ and $g(t) = \sum_{n=0}^{\infty} g_n t^n$, we will use the notation $f \succ g$ when $f_n \geq g_n$ for every $n \geq 0$ and the notation $f \succcurlyeq g$ when $f \succ g$ and $f_0 \geq g_0$. To prove Theorem 1.1, the following lemma on factorization in the algebra $A_\omega$ will be used.

**Lemma 1.2.** Let $\omega$ be a good weight. If $f \in A_\omega$ is a positive function on $[0, 1]$, then there exists a function $g \in A_\omega$ such that $g > 0$ and $fg \succ 0$.

In Section 3 we collect some elementary properties of classes $C_\alpha$; Proposition 3.1 and Lemma 3.2 give some examples. In particular, we show that any diagonalizable matrix with spectrum on the unit circle belongs to $C_\alpha$ for some admissible function $\alpha$.

Given an operator $T$ of class $C_\alpha$, in Section 4 we will write down its concrete conanalytic model, in other words, an explicit Nagy-Foias-like functional model of $T$ up to similarity.

To construct a functional model of a contraction, first one has to single out its unitary part (recall that the Nagy-Foias construction “forgets” this part). Section 5 is devoted to defining the unitary part of an operator $T \in C_\alpha$, which is a necessary first step to passing to the Nagy-Foias transcription.

The standard Nagy-Foias model of a contraction $S \in L(H)$ makes use of its defect operator, which is defined as a nonnegative square root $D_S = (I - S^*S)^{1/2}$. This model is related with the following well-known identity

\[ \|h\|^2 = \sum_{n=0}^{\infty} \|D_S S^n h\|^2 + \lim_{n \to \infty} \|S^n h\|^2, \quad h \in H, \]

valid for any contraction $S$ (see [18, Section 1.10]). This motivates the next definition, which will be useful for us.

**Definition 1.3.** Let $T \in L(H)$ be a power bounded operator (that is, $\sup_{n \geq 0} \|T^n\| < \infty$), and let $D : H \to F$, where $H, F$ are Hilbert spaces. We will say that $D$ is an abstract defect operator for $T$ if there are some positive constants $c, C$ such that for any $h \in H$,

\[ c\|h\|^2 \leq \sum_{n=0}^{\infty} \|DT^n h\|^2 + \limsup_{n \to \infty} \|T^n h\|^2 \leq C\|h\|^2. \]
By applying the Banach limit, we will show that a power bounded operator is similar to a contraction if and only if it has an abstract defect operator. More precisely, we will prove the following.

**Lemma 1.4.** Let $T \in L(H)$ be a power bounded operator. Then an operator $D \in L(H)$ is an abstract defect operator for $T$ if and only if there exists an invertible operator $W \in L(H)$ such that $\tilde{T} := WTW^{-1}$ is a contraction and $\|Dh\| = \|D_{\tilde{T}}Wh\|$ for any $h \in H$.

The Nagy-Foias-like model we give is in some aspects close to [24]. However, here we deal with a general case and not only with a $C_0$ case, as in [24]. Since the model is only up to similarity, it is not unique, but we suggest a reasonable choice. It will be proven that for an operator $T \in C_\alpha$, $(\alpha|T^*,T|)^{1/2}$ can be taken as an abstract defect operator of $T$. This will permit us to write down explicitly an analogue of the character function of $T^*$.

Our Nagy-Foias-like transcription implies the major part of usual consequences of the Nagy-Foias theory (such as criteria for completeness of eigenvectors of $T$ in terms of the determinant of $\Theta_\alpha$, criteria for these to form a Riesz basis, similarity to a normal operator, etc). These criteria are formulated in terms of the determinant of $\Theta_\alpha(z)$. In Section 7 we show that, roughly speaking, $\Theta_\alpha$ has a determinant whenever $\alpha|T^*,T|$ is of trace class and discuss briefly the above-mentioned consequences.

In Section 8 we will give necessary and sufficient conditions for the inclusion of operator classes $C_\alpha \subset C_\tau$. It will follow, in particular, that there are many functions $\tau$ such that the class $C_\tau$ strictly contains $C_{1-1}$, the class of all contractions on $H$.

As compared with Agler’s case, the construction of the Nagy-Foias model in our case has some extra difficulties. They are related with the fact that for operators of class $C_\alpha$ in general, the limit $\lim_{n} \|T^n h\|^2$, $h \in H$, does not exist (and therefore we need in general Banach limits). In Section 9, we prove, that these limits do exist under the additional assumption that $\alpha$ has no roots on the unit circle (except for the root at $t = 1$).

2. **Proof of Lemma 1.2** on factorization in the Banach algebra $A_\omega$

If $\omega_n = 1$ for all $n$, then the algebra $A_\omega$ is just the usual Wiener algebra (which we denote $A_W$) of analytic functions in $\mathbb{D}$ with absolutely summable Taylor coefficients. In fact, if we denote by $H(\overline{\mathbb{D}})$ the set of functions analytic on (a neighborhood of) $\overline{\mathbb{D}}$, then, obviously,

$$H(\overline{\mathbb{D}}) \subset A_\omega \subset A_W$$

for every good weight $\omega$. The first inclusion is due to the exponential decay of Taylor coefficients of functions in $H(\overline{\mathbb{D}})$.

**Lemma 2.1.** If $q(t) = (t - \lambda)(t - \overline{\lambda})$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then there exists a polynomial $p$ such that $p > 0$ and $pq > 0$.

**Proof.** Let $m$ be the smallest nonnegative integer such that $\text{Re}(\lambda^{2m}) \leq 0$. We define

$$p(t) := \prod_{j=0}^{m-1} (t^{2j} + \lambda^{2j})(t^{2j} + \overline{\lambda}^{2j})$$

(so that $p(t) = 1$ if $m = 0$). Note that by the minimality of $m$, for each factor we have

$$(t^{2j} + \lambda^{2j})(t^{2j} + \overline{\lambda}^{2j}) = t^{2j+1} + 2\text{Re}(\lambda^{2j})t^{2j} + |\lambda^{2j}| > 0.$$  

Therefore $p > 0$. Moreover

$$(pq)(t) = (t^{2m} - \lambda^{2m})(t^{2m} - \overline{\lambda}^{2m}) = t^{2m+1} - 2\text{Re}(\lambda^{2m})t^{2m} + |\lambda^{2m}| > 0. \qed$$

**Corollary 2.2.** If $q$ is a real polynomial without real roots and $q(0) > 0$, then there exists a polynomial $p$ such that $p > 0$ and $pq > 0$.  

Proof. Note that \( q = Cq_1 \cdots q_k \), where \( C \) is a positive constant and each factor \( q_i \) has the form \( q_j(t) = (t - \lambda_j)(t - \lambda_j^*) \) for some \( \lambda_j \in \mathbb{C} \setminus \mathbb{R} \). Then for each factor \( q_j \) we can construct the polynomial \( p_j \) as in the previous lemma and we just take \( p = p_1 \cdots p_k \). \( \square \)

**Corollary 2.3.** If \( q \) is a real polynomial such that \( q(t) > 0 \) for every \( t \in [0, 1] \), then there exists a function \( u \in \mathcal{H}(\mathbb{D}) \) such that \( u > 0 \) and \( uq > 0 \).

Proof. Decompose \( q \) as the product of polynomials \( q_{nr}, q_+ \) and \( q_- \), where the roots of \( q_{nr} \) are nonreal, the roots of \( q_+ \) are positive and the roots of \( q_- \) are negative. Without loss of generality, \( q_+(0) = 1 \), \( q_-(0) = 1 \) and \( q_{nr}(0) = q(0) > 0 \). Therefore \( q_- > 0 \) and by Corollary 2.2 there exists a polynomial \( p \) such that \( p > 0 \) and \( pq_{nr} > 0 \). Notice that \( 1/q_+ \in \mathcal{H}(\mathbb{D}) \) and \( 1/q_+ > 0 \). Hence, we can take \( u := p/q_+ \), and the statement follows. \( \square \)

**Proof of Lemma 1.2.** Let \( f(t) > \varepsilon > 0 \), for \( t \in [0, 1] \). Take \( N \in \mathbb{N} \) such that \( \sum_{n=N+1}^{\infty} |f_n| \omega_n < \varepsilon/2 \). Hence it is obvious that \( \sum_{n=N+1}^{\infty} |f_n| < \varepsilon/2 \). Put

\[
f_N(t) = \sum_{n=0}^{N} f_n t^n - \frac{\varepsilon}{2}, \quad h(t) = \frac{\varepsilon}{2} + \sum_{n=N+1; f_n < 0} \ f_n t^n.
\]

Then \( f_N \) is a polynomial and \( h \in A_\omega \). Since \( |f_N(t)| > \varepsilon - \varepsilon/2 - \varepsilon/2 = 0 \) for \( t \in [0, 1] \), we can apply Corollary 2.3 to obtain a function \( u \in \mathcal{H}(\mathbb{D}) \) such that \( u > 0 \) and \( uf_N > 0 \). Hence \( u \in A_\omega \).

On the other hand, for \( t \in \overline{\mathbb{D}} \) we have

\[
\left| \sum_{n \geq N+1; f_n < 0} f_n t^n \right| \leq \sum_{n=N+1}^{\infty} |f_n| < \varepsilon/2.
\]

Hence \( h(t) \neq 0 \) for \( t \in \overline{\mathbb{D}} \). Notice that the properties of the weight imply that the characters of \( A_\omega \) are exactly the evaluation functionals at the points of \( \overline{\mathbb{D}} \). It follows that \( v := 1/h \in A_\omega \). Note that \( v > 0 \), because it has the form \( c/(1-a) \), where \( a \in A_\omega \), \( a > 0 \) and \( c = \varepsilon/2 \). Put \( g := uv \in A_\omega \). Then \( g > 0 \) and since \( f \succ f_N + h \), we have

\[
gf \succ g(f_N + h) = vu f_N + u > 0. \quad \square
\]

3. THE CLASSES \( C_\alpha \) AND PROOF OF THEOREM 1.1

In what follows, \( \alpha(t) = (1-t)\tilde{\alpha}(t) \) will be an admissible function. We associate to it the class of operators

\[
C_\alpha := \{ T \in L(H) : \sigma(T) \subset \overline{\mathbb{D}}, \sum_{n=0}^{\infty} |\tilde{\alpha}_n|(1 + \|T^n\|^2) < \infty, \ \alpha[T^*, T] \geq 0 \}.
\]

For example, \( C_{1-t} \) is just the set of all contractions in \( L(H) \). Notice that any admissible function \( \alpha \in \mathcal{H}(\overline{\mathbb{D}}) \) has a simple root at \( t = 1 \), and the corresponding \( \tilde{\alpha} \) is also in \( \mathcal{H}(\overline{\mathbb{D}}) \). In this case, any \( T \in L(H) \) with \( \sigma(T) \subset \overline{\mathbb{D}} \) that satisfies \( \alpha[T^*, T] \geq 0 \) is in \( C_\alpha \).

Theorem 1.1 asserts that if \( T \in C_\alpha \) for some admissible function \( \alpha \) then \( T \) is similar to a contraction.

Here are some elementary properties of the classes \( C_\alpha \).

**Proposition 3.1.**

(a) If \( N \in L(H) \) is a normal operator with \( \|N\| \leq 1 \), then \( N \in C_\alpha \) for every admissible function \( \alpha \). In particular, all unitary operators are in \( C_\alpha \).

(b) If \( T_1, T_2 \in C_\alpha \), then the orthogonal sum \( T_1 \oplus T_2 \) also is in \( C_\alpha \).

(c) If \( T \in C_\alpha \), then \( CT \in C_\alpha \) for every \( C \) on the unit circle \( \mathbb{T} = \{ z : |z| = 1 \} \).

(d) If \( T \in C_\alpha \), then \( T[L \subset C_\alpha \) for every \( T \)-invariant subspace \( \mathbb{D} \subset H \).

(e) If \( T \) is Hilbert space operator, whose spectral radius is less than one, then \( T \in C_{1-t} \) for any sufficiently large \( n > 0 \).
and let $\lambda$.

Notice that it only has roots on the unit circle; it follows that $\sigma$.

Suppose $\sigma(T) \subset T$, there exists an admissible polynomial $p(t)$ such that $p[T^*, T] = 0$.

**Proof.** Suppose $T$ is of size $n \times n$. Let $\{v_j\} (1 \leq j \leq n)$ be a basis of eigenvectors of $T$ in $\mathbb{C}^n$ and let $\lambda_j \in \mathbb{C}$ be the corresponding eigenvalues. Consider the polynomial

$$p(t) = (1 - t) \prod_{1 \leq k < \ell \leq n} (1 - 2 \text{Re}(\lambda_k \bar{\lambda}_\ell) t + t^2).$$

Notice that it only has roots on the unit circle; it follows that $p$ is admissible. We will prove that $p[T^*, T] = 0$. Each $h \in \mathbb{C}^n$ can be written as $h = \sum h_j v_j$. We get

$$\langle p[T^*, T]h, h \rangle = \sum_j p_j \langle T^j \sum_k h_k v_k, T^j \sum_\ell v_\ell \rangle = \sum_j p_j \langle \sum_k \lambda_j^k h_k v_k, \sum_\ell \lambda_j^\ell h_\ell v_\ell \rangle$$

$$= \sum_{k, \ell} \sum_j p_j \lambda_j^k \lambda_j^\ell h_k h_\ell \langle v_k, v_\ell \rangle = \sum_{k, \ell} p(\lambda_k \bar{\lambda}_\ell) h_k h_\ell (v_k, v_\ell) = 0,$$

because $p(\lambda_k \bar{\lambda}_\ell) = 0$ for all $k, \ell$. \qed

For the proof of Theorem \[\text{Theorem 1.1}\] we need some technical results. Let $\omega$ be a good weight and let $f \in A_\omega$. If $T, B \in L(H)$ and $T$ satisfies the condition $\|T^n\|^2 \lesssim \omega_n$ (i.e., $\|T^n\|^2 \lesssim C \omega_n$ for some positive constant $C$), then the operator

$$f[T^*, T](B) := \sum_{n=0}^\infty f_n T^n B T^n$$

is well defined. Indeed, $\|f[T^*, T](B)\| \lesssim \|B\| \|f\|_{A_\omega}$. Note that, in particular, $f[T^*, T](I) = f[T^*, T]$.

**Lemma 3.3.** Let $\omega$ be a good weight, $f, g, h \in A_\omega$ such that $fg = h$ and let $T, B \in L(H)$. If $\|T^n\|^2 \lesssim \omega_n$ then one has

(i) $h[T^*, T](B) = g[T^*, T](f[T^*, T](B)).$

(ii) $h[T^*, T] = g[T^*, T](f[T^*, T]).$

**Proof.** Let us define

$$g_{[N]}(t) := \sum_{n=0}^N g_n t^n \quad \text{and} \quad h_N := f g_{[N]} \quad (N \geq 0).$$

Then, $\|g - g_{[N]}\|_{A_\omega} \xrightarrow{N \to \infty} 0$ and $\|h - f g_{[N]}\|_{A_\omega} = \|f g - f g_{[N]}\|_{A_\omega} \xrightarrow{N \to \infty} 0$. It easily implies that

$$\|(g_{[N]} f)[T^*, T](B) - h[T^*, T](B)\|_{L(H)} \xrightarrow{N \to \infty} 0.$$

Note that $(z^n f)[T^*, T](B) = T^n f[T^*, T](B) T^n$ for every $n \geq 0$. Hence

$$(g_{[N]} f)[T^*, T](B) = \sum_{n=0}^N g_n T^n f[T^*, T](B) T^n,$$

and therefore,

$$\|(g_{[N]} f)[T^*, T](B) - g[T^*, T](f[T^*, T](B))\|_{L(H)} \xrightarrow{N \to \infty} 0.$$

Formulas (4) and (5) give (i). To get (ii), one just has to put $B = I$. \qed
If $A, T \in L(H)$ and $A$ is positive, it is said that $T$ is an $A$-contraction if $T^*AT \leq A$.

**Remark 3.4.** $T$ is similar to a contraction if and only if $T$ is an $A$-contraction for some $A \geq \varepsilon I$, where $\varepsilon > 0$.

**Proof of Theorem 1.3**. Put $\omega_n = 1 + \|T^n\|^2$. Since $\|T^{m+n}\| \leq \|T^m\|\|T^n\|$, the weight $\omega$ satisfies (GW 1) − (GW 3). By Lemma 1.2, there exists a function $\tilde{\beta} \in AW$ such that $\tilde{\beta} > 0$ and $f := \tilde{\beta}(t)\alpha(t)$ and by Lemma 3.3 (ii) we get

$$f[T^*, T] - T^*f[T^*, T] = \sum_{n=0}^{\infty} \tilde{\beta}_n T^{*n} \alpha[T^*, T]{\|T^n\|} \geq 0.$$  \hfill(6)

Hence $T$ is a $f[T^*, T]$-contraction and the theorem follows from Remark 3.4. \hfill $\square$

### 4. The Abstract Defect Operator of $T$

Let us begin by recalling the notion of a Banach limit. If we denote by $c$ the set of all convergent complex sequences then we can define the linear functional $L : c \to \mathbb{C}$ given by $L(x) = \lim x_n$ for every $x = \{x_n\}_{n=1}^\infty \in c$. It is immediate that $\|L\| = 1$, $L(x') = L(x)$ if $x' = \{x_n\}_{n=2}^\infty$, and also $L(x) \geq 0$ if $x \geq 0$ (i.e., $x_n \geq 0$ for every $x$). Using the Hahn-Banach Theorem, these properties of the limit functional can be extended to $\ell^\infty$.

**Theorem A.** There is a linear functional $L : \ell^\infty \to \mathbb{C}$ such that

(a) $\|L\| = 1$;
(b) $L(x) = \lim x_n$ for every $x \in c$;
(c) $L(x') \geq 0$ for every $x \in \ell^\infty$ such that $x \geq 0$;
(d) $L(x') = L(x)$ if $x \in \ell^\infty$ and $x' = \{x_n\}_{n=2}^\infty$;
(e) $\liminf x_n \leq L(x) \leq \limsup x_n$ if $x \in \ell^\infty$ is a real sequence.

**Proof.** Statements (a)-(d) are contained in [9, Theorem III.7.1] and assertion (e) is their easy consequence (and it is also standard). \hfill $\square$

A functional $L$ with the above properties is called a Banach limit.

**Proof of Lemma 1.4.** We remark first that by a lemma by Gamal [10, Lemma 2.1], for any power bounded operator $T$, one has

$$\liminf_{n \to \infty} \|T^n h\|^2 = \limsup_{n \to \infty} \|T^n h\|^2, \quad h \in H$$  \hfill(7)

(we say that two quantities $A, B$, depending on $h$ or some other parameter, are comparable and write $A \asymp B$ if there are two positive constants $c, C$ such that $cA \leq B \leq CA$).

Suppose first that there exists a linear isomorphism $W \in L(H)$ with the properties stated in Lemma. Since $T = W T W^{-1} \in L(H)$ is a contraction, by (2) we have

$$\|h\|^2 = \sum_{n=0}^{\infty} \|D_T T^n h\|^2 + \lim_{n \to \infty} \|T^n h\|^2.$$  \hfill(8)

Note that $T^n = W T^n W^{-1}$, and thus

$$\|h\|^2 = \sum_{n=0}^{\infty} \|D_T W T^n W^{-1} h\|^2 + \lim_{n \to \infty} \|W T^n W^{-1} h\|^2.$$  \hfill

Since $W$ is invertible and $\|D h\| = \|D_T W h\|$, we get

$$\|h\|^2 \asymp \|W h\|^2 = \sum_{n=0}^{\infty} \|D T^n h\|^2 + \lim_{n \to \infty} \|T^n h\|^2.$$  \hfill

By (7), $\lim_{n \to \infty} \|W T^n h\|^2 \asymp \limsup_{n \to \infty} \|T^n h\|^2$. We deduce that $D$ is an abstract defect operator for $T$. 


Conversely, suppose now that \( D \) is an abstract defect operator for \( T \). Fix a Banach limit \( L \) and put

\[
\|h\|^2 := \sum_{n=0}^{\infty} \|DT^nh\|^2 + L\left( \{\|T^nh\|^2\} \right).
\]

(9)

Notice that (14) and Theorem A (e) give that

\[
l \lim_{n \to \infty} \|T^nh\|^2 = \|\widehat{T}^n\|, \quad h \in H.
\]

The relation (15) implies that \( \|h\| \cong \|h\|, \) \( h \in H \). It follows that \( \| \cdot \| \) is an equivalent Banach space norm on \( H \). By applying the Cauchy-Schwarz inequality, it is easy to see that

\[
[x, y] := \sum_{n=0}^{\infty} \langle DT^nx, DT^ny \rangle + L\left( \{\langle T^nx, T^ny \rangle\} \right)
\]

absolutely converges for any \( x, y \in H \). It is a semi-inner product on \( H \), which induces the norm \( \| \cdot \| \). So, in fact, \( \| \cdot \| \) is a Hilbert space norm equivalent to \( \| \cdot \| \). (see [12] and [17] for a similar argument).

Therefore there exists a linear isomorphism \( W : H \to H \) such that \( \|Wh\| = \|h\| \). Observe that

\[
\|Th\|^2 = \|h\|^2 - \| Dh\|^2 \leq \|h\|^2.
\]

Let \( \widehat{T} := WTW^{-1} \in L(H) \) (similar to \( T \)). Take \( x \in H \) and put \( h := W^{-1}x \). We get

\[
\|WT<h\| \leq \|Wh\|
\]

so \( \widehat{T} \) is a contraction. Since \( \|Dh\|^2 = \|h\|^2 - \|Th\|^2 \) and

\[
\|D_{\widehat{T}}Wh\|^2 = \|Wh\|^2 - \|\widehat{T}Wh\|^2 = \|h\|^2 - \|Th\|^2,
\]

we get \( \|Dh\| = \|D_{\widehat{T}}Wh\| \) for every \( h \in H \). \( \square \)

Let \( \alpha \) be an admissible function and let \( T \in C_\alpha \). We know already that \( T \) is similar to a contraction. Since \( \alpha \in A_W \), by Lemma [12] there exists a function \( \tilde{\beta} \in A_W \) such that \( \tilde{\beta} > 0 \) and \( f := \tilde{\beta}\alpha > 0 \). Hence \( (1-t)f(t) = \tilde{\beta}(t)\alpha(t) \). Set

\[
B := (f[T^*, T])^{1/2},
\]

where the positive square root has been taken. Then \( B > \varepsilon I \) for some \( \varepsilon > 0 \). We will assume, without loss of generality, that \( \sum f_k = \|f\|_{A_W} = 1 \). We put

\[
D := (\alpha[T^*, T])^{1/2}.
\]

Theorem 4.1. If \( T \in C_\alpha \) for some admissible function \( \alpha \in A_W \), then \( D \) is an abstract defect operator for \( T \). More specifically, if \( \tilde{\beta}, f \) and \( B \) are as above, then the expression

\[
\|h\|^2 := \sum_{n=0}^{\infty} \|DT^nh\|^2 + \lim_{n \to \infty} \|BT^nh\|^2
\]

(11)

defines an equivalent Hilbert space norm in \( H \) and \( T \) is a contraction with respect to this norm. In particular, the limit in (11) exists for every \( h \in H \). Moreover,

\[
\|h\|^2 - \|Th\|^2 = \|Dh\|^2 \quad (\forall h \in H).
\]

Proof. Since \( (1-t)f(t) = \tilde{\beta}(t)\alpha(t) \), we have

\[
B^2 - T^*B^2T = \sum_{n=0}^{\infty} \tilde{\beta}_nT^*nD^2T^n.
\]
Therefore, for every $h \in H$ we have
\[ \|Bh\|^2 - \|BT^j h\|^2 = \sum_{n=0}^{\infty} \hat{\beta}_n \|DT^n h\|^2. \]
Changing $h$ by $T^j h$ we obtain
\[ \|BT^j h\|^2 - \|BT^{j+1} h\|^2 = \sum_{n=0}^{\infty} \hat{\beta}_n \|DT^n h\|^2, \]
for every $j \geq 0$. Summing these equations for $j = 0, 1, \ldots, N - 1$ we obtain
\[ (13) \quad \|Bh\|^2 - \|BT^N h\|^2 = \sum_{n=0}^{N-1} \beta_n \|DT^n h\|^2 + \sum_{n=N}^{\infty} \left( \sum_{j=n-N+1}^{\infty} \beta_j \right) \|DT^n h\|^2, \]
where $\beta_n = \sum_{j=0}^{n} \hat{\beta}_j$. In particular, $0 < \tilde{\beta}_0 \leq \beta_n$ and therefore by (13) we get
\[ \|Bh\|^2 \geq \sum_{n=0}^{N-1} \beta_n \|DT^n h\|^2 \geq \tilde{\beta}_0 \sum_{n=0}^{N-1} \|DT^n h\|^2. \]
Hence the series $\sum_{n=0}^{\infty} \|DT^n h\|^2$ converges. On the other hand, since
\[ \sum_{j=n-N+1}^{n} \beta_j \leq \sum_{j=0}^{\infty} \beta_j = \|\tilde{\beta}\|_{AW} < \infty, \]
we obtain that
\[ \sum_{n=N}^{\infty} \left( \sum_{j=n-N+1}^{n} \beta_j \right) \|DT^n h\|^2 \leq \|\tilde{\beta}\|_{AW} \sum_{n=N}^{\infty} \|DT^n h\|^2 \to 0 \]
when $N \to \infty$. Therefore, taking limit in (13) when $N$ goes to infinity we obtain that
\[ (14) \quad \|Bh\|^2 = \sum_{n=0}^{\infty} \beta_n \|DT^n h\|^2 + \lim_{N \to \infty} \|BT^N h\|^2 \]
(and the limit in the right hand side exists for any $h$). Since $\tilde{\beta}_0 \leq \beta_n \leq \|\tilde{\beta}\|_{AW} < \infty$, it follows that $\|h\|$ defines an equivalent Hilbert space norm on $H$. Formula (12) is immediate from (11). \hfill \Box

Remark 4.2. Notice that Theorems 4.1 and 4.4 give two methods of finding an equivalent norm such that $T$ is a contraction in this norm. With the method of Theorem 4.1 we obtained in addition that $D$ is an abstract defect operator.

Remark 4.3. Assume that $\alpha(t) \neq 0$ for $t \in \overline{B}$, $t \neq 1$, and $1/\hat{\alpha}$ has positive Taylor coefficients at the origin. Then in the above calculations, we could set $f(t) \equiv 1$, so that $B = I$ and $\tilde{\beta} = 1/\hat{\alpha} > 0$. In this case, formula (13) yields a unitarily equivalent (coanalytic) model of $T$, which represent it as a part of an operator $\bigoplus_{j=1}^{\infty} M_j \ominus U$, where $M$ is a weighted shift and $U$ is unitary. In this situation, there is no need to assume that $\alpha$ is admissible and one can deal with more general functions. For the case when $\alpha$ is a polynomial, this a result by Müller [16, Theorem 3.10]. Most general result of this kind was given recently by Olofsson, see Theorem 6.6 in [21]. On the other hand, in this case $\tilde{\beta} > 0$, so that $\{\beta_n\}$ is an increasing sequence and therefore the backward shift $M^*$ has to be a contraction.

In fact, in this setting, one has not to assume that $\alpha$ is admissible and can deal with more general functions. However, in both results cited above, $T$ has to be a contraction. Theorem 4.4 requires $\alpha$ to be admissible, but applies to operators $T$ that are not contractions.
Remark 4.4. Using the notation above, note that
\[ \|BT^n h\|^2 = \sum_{k=0}^{\infty} f_k \langle T^{n+k}h, T^{n+k}h \rangle. \]
So if we put
\[ (15) \lim^* x_n = \lim_{n \to \infty} \sum_{k=0}^{\infty} f_k x_{n+k}, \]
then (11) can be written as
\[ (16) \|h\|^2 = \sum_{n=0}^{\infty} \|DT^n h\|^2 + \lim_{n \to \infty}^* \|T^n h\|^2. \]
So, on the contrary to the general formula (9), the use of a Banach limit is unnecessary in the context of an operator \( T \) in the class \( C_\alpha \), and one can use instead the “regularized” limit \( \lim^* \).

We observe that \( \lim^* \) coincides with the usual limit when the sequence is convergent (here we use the above normalization assumption that \( \sum f_k = 1 \)).

On the other hand, the example of matrices \( T \), meeting the requirements of Lemma 3.2, shows that in general, the limit \( \lim \|T^n h\|^2 \) does not exist for \( T \in C_\alpha \). In Section 9 we will show that this limit does exist if \( \alpha \) satisfies an extra requirement.

In what follows, for \( T \in C_\alpha \), the operator \( D \), given by (10), will be called the \textit{defect operator of} \( T \).

5. On definition of \textit{unitary part of a} \( C_\alpha \) \textit{operator}

Definition 5.1. Let \( K \) be a Hilbert space. We say that \( T \in L(K) \) is a \textit{completely nonunitary operator} if there is no nonzero reducing subspace \( L \) for \( T \) such that \( T|L \) is unitary.

It is well-known that every contraction \( S \) can be decomposed into an orthogonal sum of a unitary operator and a completely nonunitary operator (called the \textit{unitary part} and the \textit{completely nonunitary part} of \( S \), respectively). We recall that the standard construction of the Nagy-Foias model applies only to completely nonunitary contractions.

If one takes an operator \( T \) in the class \( C_\alpha \) and applies to it Theorem 4.1, then one gets a \textit{direct sum} decomposition
\[ (17) H = H_0 + H_1 \]
such that \( T|H_0 \) is similar to a unitary operator and \( T|H_1 \) is similar to a completely non-unitary operator.

For a general admissible function \( \alpha \), we cannot say much more. However, some extra properties hold if \( \alpha \) is in following subclass.

Definition 5.2. A function \( \alpha \in A_W \) will be called \textit{strongly admissible} if it has the form \( \alpha(t) = (1-t)\tilde{\alpha}(t) \) for some function \( \tilde{\alpha} \in A_W \) with real Taylor coefficients, which has no roots on the unit circle \( \mathbb{T} \) and satisfies \( \tilde{\alpha}(0) = \alpha(0) > 0 \).

Notice that any strongly admissible function is admissible.

For the sequel, let us recall the following characterization of the unitary part of a contraction.

Theorem B (See [18], Theorem I.3.2). To every contraction \( S \) on the space \( H \) there corresponds a decomposition of \( H \) into an orthogonal sum of two subspaces reducing \( S \), say \( H = H_0 \oplus H_1 \), such that the part of \( S \) on \( H_0 \) is unitary, and the part of \( S \) on \( H_1 \) is completely nonunitary; \( H_0 \) or \( H_1 \) may equal the trivial subspace \( \{0\} \). This decomposition is uniquely determined. Indeed, \( H_0 \) consists of those elements \( h \) of \( H \) for which
\[ \|S^n h\| = \|h\| = \|S^n h\| \quad (n = 1, 2, \ldots). \]

\( S_0 = S|H_0 \) and \( S_1 = S|H_1 \) are called the unitary part and the completely nonunitary part of \( S \), respectively, and \( S = S_0 \oplus S_1 \) is called the canonical decomposition of \( S \).
The next theorem is an analogue of this decomposition for our operators in the case of a strongly admissible function $\alpha$.

**Theorem 5.3.** Let $T \in \mathcal{C}_\alpha$, where $\alpha$ is strongly admissible. Denote by $H_0$ the elements $h \in H$ for which there exists a two-sided sequence $\{h_n\}_{n \in \mathbb{Z}}$ such that $h_0 = h, Th_0 = h_{n+1}$ and $\|h_n\| = \|h\|$ for every $n \in \mathbb{Z}$. Let $\tilde{T}$ be the operator $T$ acting on $\tilde{H} := (H, \cdot, \cdot)$ (the new norm, which was given in (11)). Then $H_0$ is a closed subspace of $H$ and there exists a direct sum decomposition $H = H_0 + H_1$ with the following properties:

(i) $T|H_0$ is unitary;

(ii) $H_0$ and $H_1$ are invariant subspaces of $T$;

(iii) $H_0$ and $H_1$ are orthogonal in $\tilde{H}$;

(iv) $\tilde{T}|H_0$ and $\tilde{T}|H_1$ are the unitary part and the completely nonunitary part of the contraction $\tilde{T}$, respectively.

**Remark 5.4.** In Lemma 5.2 we saw that if $T$ is a finite matrix without Jordan blocks and $\sigma(T) \subset \mathbb{T}$, then $T \in \mathcal{C}_p$ for some admissible polynomial $p$. In this case, $H_0 = H$ and $H_1 = H$ in the decomposition (17), but $T|H_0 = T$ is non-unitary, as a rule. As a consequence we get:

(i) Theorem 5.3 is not valid if we do require $\alpha$ to be strongly admissible;

(ii) A non-unitary finite matrix $T$ with $\sigma(T) \subset \mathbb{T}$ cannot belong to $\mathcal{C}_\alpha$ if $\alpha$ is strongly admissible.

Observe that a completely nonunitary contraction can be similar to a unitary operator. For a general operator $T$, one cannot single out the largest direct summand which is similar to a unitary.

**Proof of Theorem 5.3.** Since $\tilde{T}$ is a contraction on $\tilde{H}$, Theorem B gives us that for the decomposition $H = H_0 \oplus H_1$, where $H_0$ consists of those elements $h$ of $H$ for which

$$\|\tilde{T}^n h\| = \|h\| = \|\tilde{T}^* h\| \quad (n = 1, 2, \ldots),$$

we have that $\tilde{T}_0 := \tilde{T}|H_0$ is unitary and $\tilde{T}_1 := \tilde{T}|H_1$ is completely nonunitary.

The goal of the following two claims is to prove that $H_0 = \tilde{H}_0$.

**Claim 1.** Let $h \in H$. Then $h \in H_0$ if and only in there exists a sequence $\{h_n\}_{n \in \mathbb{Z}}$ such that $h_0 = h, \tilde{T} h_0 = h_{n+1}$ and $\|h_n\| = \|h\|$ for every $n \in \mathbb{Z}$.

(In fact, this claim is a variation of Theorem B and is valid for any contraction $\tilde{T}$ on a Hilbert space.)

Indeed, suppose that $h \in \tilde{H}_0$. Define the sequence $\{h_n\}_{n \in \mathbb{Z}}$ by $h_0 := h, h_n := \tilde{T}^n h$ and $h_{n-1} := \tilde{T}^* h$, for $n \geq 1$. Since $\tilde{T}|\tilde{H}_0$ is unitary we obtain that $\tilde{T} h_{n-1} = \tilde{T}^* h = \tilde{T}^* h_{n-1}$, for $n \geq 1$. It follows that the sequence $\{h_n\}_{n \in \mathbb{Z}}$ satisfies the conditions of the statement.

Reciprocally, suppose that a sequence $\{h_n\}_{n \in \mathbb{Z}}$ satisfies the conditions of the statement. Fix $n \geq 1$. Since $\tilde{T}^n h = h$, we have that $\|\tilde{T}^n h - n\| = \|h\|^2 = \|h_0\|^2$. Hence $\langle (I - \tilde{T}^n) h, h \rangle = 0$. But using that $\tilde{T}^n$ is a contraction on $\tilde{H}$ it follows that $(I - \tilde{T}^n) h = h$. Therefore, using that $\tilde{T}^n h = h$, we obtain that $\tilde{T}^n h = h$. Then $h \in \tilde{H}_0$. This finishes the proof of Claim 1.

**Claim 2.** Let $h \in H$. Then $h \in \tilde{H}_0$ if and only in there exists a sequence $\{h_n\}_{n \in \mathbb{Z}}$ such that $h_0 = h, Th_n = h_{n+1}$ and $\|h_n\| = \|h\|$ for every $n \in \mathbb{Z}$.

(Note that this claim is stated in terms of the original norm in $H$.)

Indeed, we apply Claim 1. Let $\{h_n\}_{n \in \mathbb{Z}}$ be a sequence such that $h_0 = h$ and $Th_n = h_{n+1}$. We have to show that the sequence $\{\|h_n\|^2\}_{n \in \mathbb{Z}}$ is constant if and only if the sequence $\{\|h_n\|^2\}_{n \in \mathbb{Z}}$ is constant. Since the two norms are equivalent, if either of these two sequences is constant, the other is in $\ell^\infty(\mathbb{Z})$.

By (12) we have that

$$\|h_{n+1}\|^2 = \|Th_n\|^2 = \|h_n\|^2 - \|Dh_n\|^2.$$
Therefore \( \|h_n\| = \|h\| \) for every \( n \in \mathbb{Z} \) if and only if \( \| Dh_n \|^2 = 0 \) for every \( n \in \mathbb{Z} \). We will use the backward shift operator \( \nabla \), acting on \( \ell^\infty(\mathbb{Z}) \) by \( [\nabla a]_n = a_{n+1}, a \in \ell^\infty(\mathbb{Z}) \). For any \( f \in A_W, f(\nabla) \) is well-defined by \( (f(\nabla) a)_n := \sum_{j=0}^{\infty} f_j a_{n+j} \).

Denote by \( h \) the sequence \( \{\|h_n\|^2\}_{n \in \mathbb{Z}} \). Then it is easy to obtain that \( \|Dh_n\|^2 = [\alpha(\nabla)h]_n \). Hence \( \|h_n\| = \|h\| \) for every \( n \in \mathbb{Z} \) if and only if \( \alpha(\nabla)h = (\ldots, 0, 0, 0, \ldots) = 0 \).

So we have to show that \( h \) is constant if and only if \( \alpha(\nabla)h = 0 \). The direct implication is obvious. For the converse, fix a factorization \( \alpha(t) = (1-t)q(t)\gamma(t) \), where \( q \) is a polynomial with zeros in \( D \) and \( \gamma \in A_W(\mathbb{D}) \) without zeros in \( \mathbb{D} \) (recall that \( \alpha \) is assumed to be strongly admissible). Then \( 1/\gamma \in A_W(\mathbb{D}) \). Hence \( \gamma(\nabla)[q(\nabla)](1-\nabla)h] = 0 \) implies that \( q(\nabla)(1-\nabla)h] = 0 \) (just multiply by \( 1/\gamma \)). Now let \( g = (1-\nabla)h \). We want to proof that \( g = 0 \). When \( q \) has just a single root, say \( q(t) = 1 - at \) for some \( a \) with \( |a| > 1 \), the result is immediate. For a general \( q \) we just need to apply induction on the number of roots of \( q \). This finishes the proof of Claim 2.

Therefore we have that \( H_0 = \tilde{H}_0 \). Put \( H_1 := \tilde{H}_1 \). Now (i) is obvious, since \( T|H_0 \) is a surjective isometry. The rest of items of the theorem follow immediately using Theorem B. □

**Remark 5.5.** In Lemma 3.2 we saw that if \( T \) is a finite matrix without Jordan blocks and \( \sigma(T) \subset T \), then \( T \in C_\alpha \) for some admissible polynomial \( p \). In this case, \( H_0 = H \) and \( H_1 = H \) in the decomposition \( 17 \), but \( T|H_0 = T \) is non-unitary, as a rule. As a consequence we get:

(i) Theorem 5.3 is not valid if we do require \( \alpha \) to be strongly admissible;

(ii) A non-unitary finite matrix \( T \) with \( \sigma(T) \subset \mathbb{T} \) cannot belong to \( C_\alpha \) if \( \alpha \) is strongly admissible.

### 6. The Nagy-Foiaş Model of \( T \)

Let \( T \in C_\alpha \) for some \( \alpha \in A_W \). In this section, for simplicity, we will assume that \( \alpha \) is strongly admissible and \( T \) is a completely nonunitary operator.

Then, using the notation of the previous section, we have that \( \tilde{T} \) is a completely nonunitary contraction on \( \tilde{H} \). Let \( D_{\tilde{T}} \) and \( D_{\tilde{T}_*} \) be the defect operators of \( T \) and let \( D_{\tilde{T}}^* \) and \( D_{\tilde{T}_*}^* \) be its defect spaces. We recall that the defect operator of \( T \) has been defined by \( 19 \). We define the defect space of \( T \) as \( D_T = \text{clos} \, DH \), where the closure is taken with respect to \( \| \cdot \| \).

Define \( V : D_{\tilde{T}} \to D_T \) by \( V(D_{\tilde{T}} h) = Dh, h \in H \). Then, using equation \( 12 \), we obtain that \( V \) is an isometry from \( D_{\tilde{T}} \) to \( D_T \). Let us define the functions \( \Theta_\ast \in H^\infty(D_{\tilde{T}_*} \to D_T) \) and \( \Delta_\ast : T \to D_{\tilde{T}_*} \) given by

\[
\Theta_\ast(z) := V(-\tilde{T}^* + zD_{\tilde{T}}(I-zT)^{-1}D_{\tilde{T}_*})h, \quad h \in D_{\tilde{T}_*}, \; z \in \mathbb{D},
\]

\[
\Delta_\ast(\zeta) := (I - \Theta_\ast(\zeta)^*\Theta_\ast(\zeta))^{1/2}, \quad \zeta \in \mathbb{T}.
\]

Now put

\[
K_{\Theta_\ast} := \left( \frac{H^2(D_T)}{\text{clos} \, \Delta_\ast L^2(D_{\tilde{T}_*})} \right) \ominus \left( \Theta_\ast \Delta_\ast \right)^H(\Delta_\ast).
\]

Finally, let us define \( \Phi_1 : H \to H^2(D_T) \) by \( \Phi_1 h(z) = D(I-zT)^{-1}h \) and \( M_\ast : K_{\Theta_\ast} \to K_{\Theta_\ast} \) by

\[
M_\ast \left( \begin{array}{c} u \\ v \end{array} \right) := \left( \begin{array}{c} u(z) - u(0) \\ z^{-1}v(z) \end{array} \right).
\]

**Theorem 6.1.** Let \( T \in C_\alpha \), where \( \alpha \) is strongly admissible. With the notation used above, there exists a linear map \( \Phi_2 : H \to \text{clos} \, \Delta_\ast L^2(D_{\tilde{T}_*}) \) such that \( \|\Phi_2 h(z)\|^2 = \lim_{n \to \infty} \|T^nh\|^2 \) for every \( h \in H \), and

(i) \( \Phi := (\Phi_1) : H \to K_{\Theta_\ast} \) is an isometric isomorphism;

(ii) \( \Phi T = M_\ast \Phi \).
It is a common point that this kind of result implies a variant of von Neumann inequality: if \( T \) satisfies the conditions of the above theorem, then for any \( f \in \mathcal{H}(\mathbb{D}) \),
\[
\|f(T)\| \leq C \max_{\mathbb{D}} |f|,
\]
where \( C \) is a constant depending only on \( T \) (in fact, \( C = \|f(\Phi)\| \cdot \|\Phi^{-1}\| \)).

Here we should mention the works by Olofsson (see [20]), where he relates certain transfer functions associated with \( n \)-hypercontractions with Bergman inner functions, which are crucial in the description of invariant subspaces of Bergman spaces. His results were further generalized in the work by Ball and Bolotnikov [5], [6].

7. Operators in \( C_\alpha \) whose characteristic function has a determinant

In what follows, \( \mathcal{S}_p \) \((0 < p \leq \infty)\) will denote the Schatten-von Neumann class of operators.

**Lemma 7.1.** Let \( T \in C_\alpha \) for some admissible function \( \alpha \) and let \( p \in [1, \infty] \).

(i) If \( I - T^*T \in \mathcal{S}_p \), then \( D^2 \in \mathcal{S}_p \).

(ii) If \( D^2 \in \mathcal{S}_p \) and \( \tilde{\alpha} \) has no zeros in \( \mathbb{D} \), then \( I - T^*T \in \mathcal{S}_p \).

**Proof.** (i) It is immediate, since \( D^2 = \alpha[T^*, T] = \tilde{\alpha}[T^*, T](I - T^*T) \).

(ii) Since \( \alpha \) has no zeros in \( \mathbb{D} \), \( 1/\tilde{\alpha} \in \mathcal{A}_W \), and we obtain that
\[
(1/\tilde{\alpha})(T^*, T)(D^2) = (1/\tilde{\alpha})[T^*, T]((\tilde{\alpha}[T^*, T](I - T^*T)) = I - T^*T,
\]
which proves the result. \( \Box \)

**Lemma 7.2.** Let \( T \in C_\alpha \) for some admissible function \( \alpha \) and let \( p \in [1, \infty] \). If \( \sigma(T) \neq \mathbb{D} \), then \( D_T \in \mathcal{S}_p \) if and only if \( D_{\tilde{T}_\alpha} \in \mathcal{S}_p \).

**Proof.** In the case when \( 0 \notin \sigma(T) \), \( D_T \) is unitarily equivalent to \( D_{\tilde{T}_\alpha} \), see [18] the proof of Theorem VIII.1.1] or [13] Lemma 9]; this implies our assertion. The general case follows from this one. Indeed, take any \( \lambda \in \mathbb{D} \setminus \sigma(T) \) and consider the M"obius self-map of \( \mathbb{D} \), given by \( b_\lambda(z) = (z - \lambda)/(1 - \lambda z) \). Let \( \tilde{T}_\lambda \) be the contraction, defined by \( T_\lambda = b_\lambda \circ T \). Then the formula
\[
I - \tilde{T}_\lambda^* \tilde{T}_\lambda = W^*(I - \tilde{T}_\lambda^* \tilde{T}_\lambda)W \quad \text{with} \quad W = (1 - |\lambda|^2)^{1/2}(I - \lambda \bar{T})^{-1}
\]
implies that \( D_{\tilde{T}_\lambda} \in \mathcal{S}_p \) if and only if \( D_{\tilde{T}_\lambda} \in \mathcal{S}_p \). Since \( 0 \notin \sigma(\tilde{T}_\lambda) \), the general case follows. \( \Box \)

We remark that the previous lemma applies to a more general situation when \( T \) is a power bounded operator with \( \sigma(T) \neq \mathbb{D} \) and \( D \) is its abstract defect operator. Then, by Lemma 1.3, one gets a contraction \( \tilde{T} \), similar to \( T \), and so for any \( p \), \( D \in \mathcal{S}_p \) if \( D_T \in \mathcal{S}_p \) if \( D_{\tilde{T}_\alpha} \in \mathcal{S}_p \).

We recall the well-known fact that the characteristic function of a contraction \( S \) has the determinant whenever \( \sigma(S) \neq \mathbb{D} \) and \( I - S^*S \in \mathcal{S}_1 \) (this is the so-called class of weak contractions). It follows that \( \Theta_\alpha \) has a determinant whenever \( \sigma(T) \neq \mathbb{D} \) and \( D \in \mathcal{S}_2 \). Notice that \( \det \Theta_\alpha \) is an \( H^\infty \) function such that \( \|\det \Theta_\alpha\| \leq 1 \). We obtain the following statement.

**Proposition 7.3.** Suppose \( \alpha \) is strongly admissible and \( T \in C_\alpha \) is a completely nonunitary operator. Suppose also that \( \sigma(T) \neq \mathbb{D} \) and \( \alpha[T^*, T] \in \mathcal{S}_1 \). Denote by \( \sigma_p(T) \) the point spectrum of \( T \). Then the following assertions are equivalent.

(i) \( T \) is complete, that is, \( H = \text{span}\{\ker(\lambda I - T)^k : k \geq 1, \lambda \in \sigma_p(T)\} \);

(ii) \( T^* \) is complete;

(iii) \( \det \Theta_\alpha(z) \) is a Blaschke product.

This follows from the above observations and from an analogous fact for completely nonunitary contractions, see [19] p. 134].

In a similar way, one can extend the results by Treil [23], Nikolski - Benamara [7], Kupin [14], [15] and others to the setting of operators in \( C_\alpha \), where \( \alpha \) is strongly admissible.
8. A result on the inclusion of classes $C_\alpha \subset C_\tau$

The goal of this section is to prove the following result on the containment of classes.

Let $\alpha$ and $\tau$ be admissible functions and let $\gamma := \tau/\alpha$. Note that $\gamma$ is analytic on a neighbourhood of the origin.

**Theorem 8.1.** Let $\alpha, \tau$ and $\gamma$ be as above.

(i) If $C_\alpha \subset C_\tau$, then $\gamma > 0$.

(ii) If $\gamma \in AW$, then $C_\alpha \subset C_\tau$ if and only if $\gamma > 0$.

(iii) If $\gamma$ is not bounded in $\mathbb{D}$ (in particular, if $\gamma$ has poles in $\mathbb{D}$), then $C_\alpha \not\subset C_\tau$.

The following lemma is in the spirit of Pringsheim’s Theorem (see [11]).

**Lemma 8.2.** Let $g$ be a meromorphic function in $\mathbb{D}$, analytic in the origin, such that $g \not\preceq 0$. If $g$ is bounded on $[0,1)$, then $g$ is a bounded analytic function on $\mathbb{D}$.

**Proof.** Since $g \not\preceq 0$, we have

$$|g(z)| \leq g(|z|)$$

whenever the series for $g(|z|)$ converges. Let $r \in [0,1]$ be the radius of convergence of $g$. If $r < 1$, then $g$ has poles on the circle $\{|z| = r\}$, and this contradicts the boundedness of $g$ on $[0,r)$. Hence $r = 1$, and therefore by [13], $g$ is bounded on $\mathbb{D}$. $\square$

**Lemma 8.3.** Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis in $H$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers which are eventually 1. Define the sequence $\Lambda = \{\Lambda_n\}_{n=0}^{\infty}$ by $\Lambda_n := (\lambda_1 \cdots \lambda_{n+1})^2$. Let $T$ be the weighted shift operator, given by $T_e = \lambda_{n+1}e_{n+1}$, and let $\alpha \in AW$. Then $T \in C_\alpha$ if and only if $\alpha(\nabla)\Lambda > 0$ (here $\nabla$ is the backward shift on one-sided sequences $\{A_n\}_{n \geq 0}$).

**Proof.** Note that $\Lambda$ is a bounded sequence and that $T$ is a power bounded operator. Hence $\alpha[T^*,T]$ and $\alpha(\nabla)\Lambda$ are well defined, so we need to prove that $\alpha[T^*,T] \geq 0$ if and only if $\alpha(\nabla)\Lambda > 0$.

It is immediate that

$$\alpha[T^*,T] \geq 0 \iff \sum_{n=0}^{\infty} \alpha_n \|T^n h\|^2 \geq 0 \quad (\forall h \in H).$$

Next we observe that

$$\sum_{n=0}^{\infty} \alpha_n \|T^n h\|^2 \geq 0 \quad (\forall h \in H) \iff \sum_{n=0}^{\infty} \alpha_n \|T^n e_j\|^2 \geq 0 \quad (\forall j \geq 0).$$

Indeed, the direct implication is obvious. The converse is seen from the following formula, valid for any vector $h = \sum_{n=0}^{\infty} h_n e_n \in H$, where $\{h_n\}_{n=0}^{\infty} \in \ell^2$:

$$\sum_{n=0}^{\infty} \alpha_n \|T^n h\|^2 = \sum_{n=0}^{\infty} \alpha_n \left(\sum_{j=0}^{\infty} |h_j|^2 \|T^n e_j\|^2\right) = \sum_{j=0}^{\infty} |h_j|^2 \left(\sum_{n=0}^{\infty} \alpha_n \|T^n e_j\|^2\right)$$

(we can change the order of summation because the series converge absolutely).

Fix $j \geq 0$. For every $n \geq 0$ we have

$$T^n e_j = \lambda_{j+1} \lambda_{j+2} \cdots \lambda_{j+n} e_{j+n} = \frac{\Lambda_{n+j}}{\Lambda_j} e_{j+n},$$

so $\|T^n e_j\|^2 = \Lambda_{n+j}/\Lambda_j$ and therefore

$$\sum_{n=0}^{\infty} \alpha_n \|T^n e_j\|^2 = \sum_{n=0}^{\infty} \alpha_n \frac{\Lambda_{n+j}}{\Lambda_j} = \frac{1}{\Lambda_j} \alpha(\nabla)\Lambda_j.$$
Hence it follows that
\begin{equation}
\sum_{n=0}^{\infty} \alpha_n ||T^n e_j||^2 \geq 0 \quad (\forall j \geq 0) \iff \alpha(\nabla) \Lambda > 0.
\end{equation}

The statement now follows from \([19], \ [20]\) and \([21]\). \qed

Before starting the proof of Theorem 8.1 let us make a final observation.

**Remark 8.4.** If \(T \in L(H)\) is a power bounded operator and \(\alpha \in A_W\) then
\[
\alpha[T^*, T] \geq 0 \iff \sum_{n=0}^{\infty} \alpha_n ||T^n h||^2 \geq 0 \quad (\forall h \in H)
\]
\[
\iff \sum_{n=0}^{\infty} \alpha_n ||T^{n+1} h||^2 \geq 0 \quad (\forall j \geq 0, \forall h \in H),
\]
where in the last equivalence we just change \(h\) by \(T^j h\). Therefore, if we fix \(h \in H\) and define the sequence \(\Lambda = \{\Lambda_n\}_{n=0}^{\infty}\) by \(\Lambda_n := ||T^n h||^2\) then \(\alpha[T^*, T] \geq 0\) implies that \(\alpha(\nabla) \Lambda > 0\).

**Proof of Theorem 8.1** (i) Suppose that \(\gamma \not\succ 0\) and let \(\ell\) be the smallest index such that \(\gamma_\ell < 0\). Note that \(\ell \geq 1\) because \(\gamma_0 > 0\). By Lemma 8.3 we just need to find a sequence of positive numbers \(\Lambda = \{\Lambda_n\}_{n=0}^{\infty}\) that is eventually constant such that \(\alpha(\nabla) \Lambda > 0\) and \(\tau(\Lambda) \not\succ 0\), because in that case if we fix an orthonormal basis \(\{e_n\}_{n=0}^{\infty}\) of \(H\) then the weighted shift operator \(T\) defined by \(T e_n = \sqrt{\Lambda_{n+1}/\Lambda_n} e_{n+1}\) satisfies \(T \in C_\alpha \setminus C_\tau\). Let us construct that sequence.

Consider the sequence \(\Gamma := \{\gamma_\ell, \gamma_{\ell-1}, \ldots, \gamma_0, 0, 0, \ldots\}\) and define the sequence \(\Psi\) by
\[
\Psi := (\tilde{\tau})^{-1}(\nabla) \Gamma.
\]
Note that \(\Psi\) is well defined because \(\Gamma\) has only finitely many nonzero terms and \(\tilde{\tau}\) is invertible in a neighbourhood of the origin.

Since \(\Gamma_n = 0\) for \(n \geq \ell + 1\) we obtain that also \(\Psi_n = 0\) for \(n \geq \ell + 1\). Finally, let \(\Lambda\) be a the sequence that satisfies
\[
\Psi = (1 - \nabla) \Lambda,
\]
with \(\Lambda_0\) large enough so that \(\Lambda_n > 0\) for every \(n\). Note that \(\Lambda_n\) is constant for \(n \geq \ell + 1\) and it also satisfies
\[
\alpha(\nabla) \Lambda = \tilde{\alpha}(\nabla) \Psi = \left(\frac{\tilde{\alpha}}{\tilde{\tau}}\right) (\nabla) \Gamma = \left(\frac{1}{\gamma}\right) (\nabla) \Gamma = (0, \ldots, 0, 1, 0, \ldots) > 0
\]
and
\[
\tau(\nabla) \Lambda = \tilde{\tau}(\nabla) \Psi = \Gamma \not\succ 0,
\]
since \(\Gamma_0 = \gamma_\ell < 0\). Hence (i) is proved.

(ii) It is clear that \(C_\alpha \subset C_\tau\) if \(\gamma > 0\). Indeed, if \(T \in C_\alpha\), then using Lemma 8.2 (ii) we have
\[
\tau[T^*, T] = (\gamma \alpha)[T^*, T] = \sum_{n=0}^{\infty} \gamma_n T^{\alpha n} \alpha[T^*, T] T^n \geq 0,
\]
hence \(T \in C_\tau\). The other implication follows from (i).

(iii) Note that Lemma 8.2 implies that if \(\gamma\) is not bounded on \(D\) then \(\gamma \not\succ 0\) (recall that \(\gamma\) has no poles in \([0, 1]\)), so this statement also follows from (i) and the theorem is proved. \(\square\)

**Remark 8.5.** Note that in general, for a rational admissible function \(r\), it is not possible to find an admissible polynomial \(p\) such that \(C_r \subset C_p\). For example, consider the rational admissible function
\[
r(t) = \frac{1 - t}{1 - t/2}.
\]
If such a \(p\) exists, say \(p(t) = (1 - t)\tilde{p}(t)\), then by Theorem 8.1 (i) we should have \(\tilde{p}(t)(1 - t/2) \succ 0\). But it is immediate to check that this is impossible for any real polynomial \(\tilde{p}\).
Remark 8.6. Let $\alpha$ be an admissible function. Since $\tilde{\alpha} \in A_W$, by Theorem 8.1 (i) we have that $C_{1-t} \subset C_\alpha$ if and only if $\tilde{\alpha} \succ 0$. If moreover $\tilde{\alpha}$ has zeros in $\overline{D}$, then by Theorem 8.1 (ii) we deduce that $C_{1-t} \not\subset C_\alpha$. For example

$$C_{1-t} \not\subset C_{(1-0)(t^2+1/4)}.$$ 

Remark 8.7. Consider the strongly admissible rational function

$$\alpha(t) = \frac{1-t}{\frac{4}{9}(t-\frac{3}{2})^2}.$$ 

Then $C_\alpha \not\subset C_{1-t^n}$ for every $n \geq 1$. Indeed, suppose on the contrary that $C_\alpha \subset C_{1-t^n}$ for some $n \geq 1$. Then, by Theorem 8.1 (i), we should have

$$\frac{4}{9} \left(t - \frac{3}{2}\right)^2 (1 + t + \cdots + t^{n-1}) \succ 0;$$

however, the coefficient of $t$ in the above expression is $-1/3$. This shows that there are classes $C_\alpha$ that are not contained in the union of classes $C_{1-t^n}$ over all $n \geq 1$.

Lemma 8.8. Let $\alpha$ and $\beta$ be admissible functions, and consider the following conditions.

(a) $\alpha, \beta \in \mathcal{H}(D)$;
(b) $\alpha(t)/(1-t)$ or $\beta(t)/(1-t)$ has no zeros on $D$.

If (a) or (b) holds, then there exists an admissible function $\gamma$ such that $C_\alpha \cup C_\beta \subset C_\gamma$.

Proof. (a) Suppose that $\alpha, \beta \in \mathcal{H}(D)$. Then

$$\alpha \beta = ab \varphi$$

for some polynomials $a$ and $b$ without common roots and a function $\varphi \in \mathcal{H}(D)$ which is positive on $[0, 1]$. By Lemma 1.2 there exists a function $\psi \in A_W$ such that $\psi \succ 0$ and $\psi \varphi \succ 0$. Put $a = a_+ a_- a_{nr}$ where $a_+$ contains the positive roots of $a$, $a_-$ contains the negative roots of $a$ and $a_{nr}$ contains the non-real roots of $a$. If for example $a$ does not have any positive root, then we just put $a_- = 1$. In the same way, put $b = b_+ b_- b_{nr}$. Applying Corollary 2.2 twice, notice that there exists a polynomial $p$ without roots in $D$ such that $p \succ 0$, $p a_{nr} \succ 0$ and $p b_{nr} \succ 0$. Let

$$v := \frac{a_- a_{nr} p}{b_+}, \quad w := \frac{b_- b_{nr} p}{a_+}.$$ 

Note that $v \succ 0$, $w \succ 0$ and $a/b = v/w$. Now we simply put $\gamma := aw \varphi = bw \psi \varphi$. Since $w \varphi \succ 0$ and $v \psi \varphi \succ 0$, the result follows from Theorem 8.1 (i).

(b) Suppose that $\beta(t)/(1-t)$ has no zeros on $D$. Then $\alpha/\beta = : \varphi \in A_W$ is positive on $[0, 1]$. By Lemma 1.2 there exists a function $\psi \in A_W$ such that $\psi \succ 0$ and $\psi \varphi \succ 0$. Now we put $\gamma := \alpha \psi = \beta \psi \varphi$. Since $\psi \succ 0$ and $\varphi \succ 0$, the result follows from Theorem 8.1 (ii).

Corollary 8.9. Let $T_1$ be a complex square matrix and $T_2$ be a Hilbert space operator. If $T_1$ has no Jordan blocks and $\sigma(T_1) \subset T$, whereas the spectral radius of $T_2$ is less than 1, then there exists an admissible function $\gamma$ such that $T_1 \oplus T_2 \in C_\gamma$.

Proof. This follows immediately from Lemma 3.2, Proposition 3.1 (b) and (e), and Lemma 8.8.

9. Existence of the limit of $\|T^n h\|^2$

As it was explained at the end of Section 4, in general, the limit of norms $\|T^n h\|$ as $n \to \infty$ (where $h \in H$) does not exist. In this section we prove the following result:

Theorem 9.1. Let $\alpha(t) = (1-t)\tilde{\alpha}(t)$, where $\alpha$ is strongly admissible, and let $T \in C_\alpha$. Then, for every $h \in H$, there exists the limit $\lim_{n \to \infty} \|T^n h\|^2$. 


We will use the backward shift $\nabla^*$ and the shift $\nabla$, acting on one-sided bounded sequences $\{a_n\}_{n=0}^{\infty}$. They are given by $[\nabla a]_n = a_{n+1}$ for every $n \geq 0$ and $[\nabla^* a]_0 = 0$. \[ [\nabla^* a]_n = a_{n-1} \] for every $n \geq 1$.

If we identify the sequence $a = \{a_n\}_{n=0}^{\infty}$ with the power series $a(z) = \sum_{n=0}^{\infty} a_n z^n$, then we can identify the operator $\nabla$ and $\nabla^*$ with the operators given by
\[ (\nabla a)(z) = \frac{a(z) - a(0)}{z}, \quad (\nabla^* a)(z) = za(z). \]

It is clear that $\nabla \nabla^* = 1$ and $\nabla^* \nabla a(z) = a(z) - a(0)$.

Given a function $f \in AW$, the operators $f(\nabla)$ and $f(\nabla^*)$ are well-defined. Note that $c = f(\nabla^*)a$ is given by $c_n := \sum_{j=0}^{n} f_j a_{n-j}$. In terms of power series, one just has $c(z) = f(z)a(z)$. We can say, in fact, that in the power series representation, $f(\nabla^*)$ is an analytic Toeplitz operator and $f(\nabla)$ is an anti-analytic Toeplitz operator.

The following formula will be useful:
\[ (22) \quad \nabla^k \nabla^j a(z) = z^{k-j}(a(z) - a_{j-1}z^{j-1} - \cdots - a_1z - a_0). \]

We need some auxiliary lemmas.

**Lemma 9.2.** Let $f, g \in AW$ and let $a \in \ell^\infty$. Then
\[ f(\nabla)g(\nabla)a = (fg)(\nabla)a. \]

The proof is immediate just doing a change of summation indices.

**Lemma 9.3.** Let $f \in AW$ and let $a \in \ell^\infty$ be a convergent sequence, say $a_n \to a_\infty$.

(i) If $b = f(\nabla)a$, then $b_n \to f(1)a_\infty$.

(ii) The same is true for $\nabla^*$ in place of $\nabla$. Namely, if $c = f(\nabla^*)a$, then also $c_n \to f(1)a_\infty$.

**Proof.** Both statements are straightforward, and we will only check (i). Fix $\varepsilon > 0$ and let $|a_n - a_\infty| < \varepsilon/\sum_{j=0}^{\infty} |f_j|$ for every $n \geq N$. Then
\[ |b_n - f(1)a_\infty| \leq \sum_{j=0}^{\infty} |f_j| |a_{n+j} - a_\infty| < \varepsilon \]
for every $n \geq N$. \[ \square \]

We can rephrase part (ii) of last lemma in terms of formal power series as follows.

**Corollary 9.4.** Let $f \in AW$ and let $a(z) = \sum_{n=0}^{\infty} a_n z^n$ be a formal power series where the sequence $\{a_n\}_{n=0}^{\infty}$ converges to some number $a_\infty \in \mathbb{R}$. If $b(z) = f(z)a(z)$, then $b_n \to f(1)a_\infty$.

**Lemma 9.5.** Let $q$ be a real polynomial whose roots are in $\mathbb{D}$. Let $a \in \ell^\infty$ and put $b = q(\nabla)a$. If $b_n \to b_\infty \in \mathbb{R}$, then $a_n \to b_\infty/q(1)$.

**Proof.** Put $q(t) = q_0 t^s + \cdots + q_1 t + q_0$. Then
\[ \nabla^* b = (q_0 \nabla^s + q_1 \nabla^{s-1} + \cdots + q_s \nabla^2 \nabla^s)a, \]
which can be written in formal power series using $(22)$ as
\[ z^s b(z) = q_0 z^s a(z) + q_1 z^{s-1}(a(z) - a_0) + \cdots + q_s (a(z) - a_{s-1}z^{s-1} - \cdots - a_1z - a_0). \]
So if we put $\tilde{q}(t) = q_0 t^s + q_1 t^{s-1} + \cdots + q_s$, then
\[ z^s b(z) = \tilde{q}(z) a(z) - r(z) \]
for some polynomial $r$ of degree at most $s - 1$. Note that $\tilde{q}$ has no roots in $\overline{\mathbb{D}}$, hence $1/\tilde{q} \in AW$ and therefore
\[ a(z) = \frac{z^s}{\tilde{q}(z)} b(z) + \frac{r(z)}{\tilde{q}(z)} \]
Since $r/\tilde{q} \in AW$, its $n$-th Taylor coefficient tends to 0. Now the statement follows using the previous corollary and that $\tilde{q}(1) = q(1)$. \[ \square \]
Lemma 9.6. Let $q$ be a real polynomial whose roots are in $\mathbb{D}$ and put $Q(t) = (1-t)q(t)$. Let $a \in \ell^\infty$. If $b = Q(\nabla)a$ and $b_n \geq 0$, then there exists $\lim a_n$.

Proof. Put $c := q(\nabla)a$. Then $b = q(\nabla)a - \nabla q(\nabla)a$, so $b_n = c_n - c_{n+1} \geq 0$. Hence $\{c_n\}_{n=0}^\infty$ is a decreasing sequence. Since $||c||_\infty \leq (||q_0|| + \cdots + ||q_s||)||a||_\infty$, the sequence $c$ is bounded. Therefore $\{c_n\}_{n=0}^\infty$ converges and by the previous lemma we obtain that $a_n \to c_\infty/q(1)$.

Proof of Theorem 9.4. Let $h \in H$. Since $T \in C_\alpha$, we obtain that $\sum_{n=0}^\infty a_n ||T^n h||^2 \geq 0$. Changing $h$ by $T^jh$ for $j \geq 1$ we get that $\sum_{n=0}^\infty a_n ||T^{n+j} h||^2 \geq 0$. Hence, if we define the sequence $a$ by $a_n = ||T^n h||^2$ then we have that $b := \alpha(\nabla)a$ satisfies $b_n \geq 0$ for every $n \geq 0$.

By Lemma 9.2 we have $b = \alpha(\nabla)a = (1-\nabla)\tilde{\alpha}(\nabla)a$. Since $\alpha$ does not vanish on $T$, we can split it as

$$\tilde{\alpha} = q\tilde{\beta},$$

where $q$ is a polynomial with roots in $\mathbb{D}$ and $\tilde{\beta} \in A_W$ does not vanish on $\mathbb{D}$. Therefore, if we put $Q(t) = (1-t)q(t)$ and $c := \tilde{\beta}(\nabla)a$, then $b = Q(\nabla)c$. By Lemma 9.6 we know that there exists $\lim c_n$, and since $a = (1/\tilde{\beta})(\nabla)c$ and $(1/\tilde{\beta}) \in A_W$, the statement follows by Corollary 9.4.

Corollary 9.7 (of Theorem 9.1). Assume the hypotheses of Theorem 9.1, in particular, that $T \in C_\alpha$ and that $f$, $B$ are defined as in this theorem. If $\alpha$ is strongly admissible, then the norm defined in (1) can be alternatively expressed by

$$||h||^2 = \sum_{n=0}^\infty ||DT^n h||^2 + \lim_{n \to \infty} ||T^n h||^2.$$

This follows from Theorem 9.1 and formula (10).

References

[1] J. Agler. The Arveson extension theorem and coanalytic models. Integral Equations and Operator Theory, 5(1):608–631, 1982.
[2] J. Agler. Hypercontractions and subnormality. J. Operator Theory, 13(2):203–217, 1985.
[3] C. Ambrozie, M. Engliš, and V. Muller. Operator tuples and analytic models over general domains in $\mathbb{C}^n$. Journal of Operator Theory, 47(2):289–304, 2002.
[4] J. Arazy and M. Engliš. Analytic models for commuting operator tuples on bounded symmetric domains. Transactions of the American Mathematical Society, 355(2):837–864, 2003.
[5] J. A. Ball and V. Bolotnikov. Weighted Bergman spaces: shift-invariant subspaces and input/state/output linear systems. Integral Equations and Operator Theory, 76(3):301–356, 2013.
[6] J. A. Ball and V. Bolotnikov. Weighted Hardy spaces: shift invariant and coinvariant subspaces, linear systems and operator model theory. Acta Sci. Math. (Szeged), 79(3-4):623–686, 2013.
[7] N.-E. Benamara and N. Nikolski. Resolvent tests for similarity to a normal operator. Proc. London Math. Soc. (3), 78(3):585–626, 1999.
[8] T. Bhattacharyya and J. Sarkar. Characteristic function for polynomially contractive commuting tuples. J.Math. Anal. Appl., 321:242–259, 2006.
[9] J. B. Conway. A course in functional analysis, volume 96. Springer-Verlag, 2 edition, 1990.
[10] M. F. Gamal. On power bounded operators that are quasifinite transforms of singular unitaries. Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum, 77(3-4):589–606, 2011.
[11] E. Hille. Analytic function theory, Volume I, volume 269. American Mathematical Soc., 2012.
[12] L. Kérchy. Isometric asymptotes of power bounded operators. Indiana University mathematics journal, pages 173–188, 1989.
[13] L. Kérchy. On the functional calculus of contractions with nonvanishing unitary asymptotes. Michigan Math. J., 37:323–338, 1990.
[14] S. Kupin. Linear resolvent growth test for similarity of a weak contraction to a normal operator. Ark. Mat., 39:95–119, 2001.
[15] S. Kupin. Operators similar to contractions and their similarity to a normal operator. Indiana University Mathematics Journal, 52(3):753–768, 2003.
[16] V. Müller. Models of operators using weighted shifts. J. Operator Theory, 20:3–29, 1988.
[17] B. d. S. Nagy. On uniformly bounded linear transformations in hilbert space. Acta Sci. Math.(Szeged), 11(19):7, 1947.
[18] B. S. Nagy, C. Foias, H. Bercovici, and L. Kérchy. *Harmonic analysis of operators on Hilbert space*. Springer Science & Business Media, 2010.

[19] N. K. Nikolski. *Operators, functions, and systems: an easy reading*. Vol. 2, volume 93 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Model operators and systems, Translated from the French by Andreas Hartmann and revised by the author.

[20] A. Olofsson. Operator-valued Bergman inner functions as transfer functions. *St. Petersburg Mathematical Journal*, 19(4):603–623, 2008.

[21] A. Olofsson. Parts of adjoint weighted shifts. *Journal of Operator Theory*, 74(2):249–280, 2015.

[22] S. Pott. Standard models under polynomial positivity conditions. *J. Operator Theory*, 41(2):365–389, 1999.

[23] S. Treil. Unconditional bases of invariant subspaces of a contraction with finite defects. *Indiana University Mathematics Journal*, 46(4):1021–1054, 1997.

[24] D. V. Yakubovich. Linearly similar model of Sz.-Nagy-Foias type in a domain. *Algebra i Analiz*, 15(2):180–227, 2003.

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