Results on the Erdős-Falconer distance problem in $\mathbb{Z}^d_q$ for odd $q$

David J. Covert
University of Missouri - St. Louis
covertdj@umsl.edu

Abstract

The Erdős-Falconer distance problem in $\mathbb{Z}_q^d$ asks one to show that if $E \subset \mathbb{Z}_q^d$ is of sufficiently large cardinality, then $\Delta(E) := \{(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2 : x, y \in E\}$ satisfies $\Delta(E) = \mathbb{Z}_q$. Here, $\mathbb{Z}_q$ is the set of integers modulo $q$, and $\mathbb{Z}_q^d = \mathbb{Z}_q \times \cdots \times \mathbb{Z}_q$ is the free module of rank $d$ over $\mathbb{Z}_q$. We extend known results in two directions. Previous results were known only in the setting $q = p^\ell$, where $p$ is an odd prime, and as such only showed that all units were obtained in the distance set. We remove the constriction that $q$ is a power of a prime, and despite this, shows that the distance set of $E$ contains all of $\mathbb{Z}_q$ whenever $E$ is of sufficiently large cardinality.

1 Background and Results

A large portion of geometric combinatorics asks one to show that if a set is sufficiently large, then it exhibits some specific type of geometric structure. One of the best known such results is the Erdős-distance problem. Let $f(n)$ be the minimum number of Euclidean distances determined by any set of $n$ points in $\mathbb{R}^d$. The classical Erdős-distance problem asks one to show that there exists a constant $c$ such that

$$f(n) \geq \begin{cases} cn^{1-\frac{2}{d}} & d = 2 \\ cn^{1-\frac{1}{d}} & d \geq 3 \end{cases}$$

As usual, we write $X = O(Y)$ if $X/Y$ tends to zero as some common parameter of $X$ and $Y$ tends to infinity. The case $d = 2$ was recently resolved by Guth-Katz ([9]) with an ingenious application of the polynomial method. The conjecture remains open for $d \geq 3$. See [6, 8, 13, 16, 17] and the references therein for more background and a thorough treatise of the problem.

A continuous analog of the Erdős-distance problem is due to Falconer ([15]). For a compact set $E \subset [0,1]^d$, let $\Delta(E) = \{|x-y| : x, y \in E\}$ be the set of pairwise Euclidean distances in $E$. Falconer showed ([7]) that if $\text{dim}_H(E) > \frac{d+1}{2}$, then $\mathcal{L}^1(\Delta(E)) > 0$, where $\text{dim}_H(\cdot)$ denotes Hausdorff dimension and $\mathcal{L}^1(\cdot)$ denotes 1-dimensional Lebesgue measure. Further, he constructed a compact set $E \subset [0,1]^d$ with $\text{dim}_H(E) = \frac{d}{2}$ such
that $\Delta(E)$ did not have positive Lebesgue measure. This led him to conjecture that if $E \subset [0,1]^d$ is compact and $\dim_H(E) > \frac{d}{2}$, then $\mathcal{L}^1(\Delta(E)) > 0$. The best know results in this direction are due to Wolff (in the plane) and Erdős (for $d \geq 3$) who showed that $\mathcal{L}^1(\Delta(E)) > 0$ whenever $E \subset [0,1]^d$ is compact with $\dim_H(E) > \frac{d}{2} + \frac{1}{4}$ ([5, 18]).

A finite field analogue of the distance problem was first considered by Bourgain-Katz-Tao ([2]). Let $\mathbb{F}_q^d$ denote the $d$-dimensional vector space over the finite field with $q$ elements. For $E \subset \mathbb{F}_q^d$, define

$$\Delta(E) = \{|x-y| : x, y \in E\} \subset \mathbb{F}_q$$

where

$$||x-y|| = (x-y)(x-y)^t = (x_1-y_1)^2 + \ldots + (x_d-y_d)^2.$$

Clearly, $|| \cdot ||$ is not a norm, though this notion of distance is preserved under orthogonal transformations. More precisely, let $O_d(\mathbb{F}_q)$ denote the set of $d \times d$ orthogonal matrices with entries in $\mathbb{F}_q$. One can readily check that for $O \in O_d(\mathbb{F}_q)$, we have $||x|| = ||Ox||$. Once a suitable notion of distance has been established in $\mathbb{F}_q^d$, the problem proceeds just as before.

**Problem 1.1 (Erdős-Falconer Distance Problem).** Find the minimal exponent $\alpha$ such that if $q$ is odd then there exists a constant $C$ (independent of $q$) so that for any set $E \subset \mathbb{F}_q^d$ of cardinality $|E| \geq Cq^\alpha$, we have $\Delta(E) = \mathbb{F}_q$.

Note that when $q = p^2$, then $\mathbb{F}_{p^2}$ contains a subfield isomorphic to $\mathbb{F}_p$, and hence, there exists a set $E \subset \mathbb{F}_p^d$ such that $|E| = q^{d/2}$ and $|\Delta(E)| = \sqrt{q}$. This shows that the exponent $\alpha$ from Problem 1.1 cannot be less than $\frac{d}{2}$ in general.

**Remark 1.2.** The methods used to attack the Erdős-Falconer distance problem are of a much different flavor when $q$ is even. For example, in $\mathbb{F}_2^d$, we can construct a large set that contains only 0 in its distance set as follows. Let $E \subset \mathbb{F}_2^d$ be the set of vectors which have an even number of nonzero components. Then, $||x-y|| = 0$ in $\mathbb{F}_2$ since

$$\|x-y\| = (x_1-y_1)^2 + \ldots + (x_d-y_d)^2 = x_1 + y_1 + \ldots + x_d + y_d.$$

Note also that $E$ has cardinality

$$|E| = \sum_{k=0}^{\lfloor d/2 \rfloor} C_{d/2}^k = 2^{d-1}.$$

Thus, we have explicitly constructed a set of size $|E| = 2^{d-1}$ such that $\Delta(E) = \{0\}$. Then, taking any set of size $|E| > 2^{d-1}$, gives $\Delta(E) = \mathbb{F}_2$ by the pigeonhole principle. This gives the sharp exponent for the Erdős-Falconer distance problem in $\mathbb{F}_2^d$.

We shall henceforth assume $q$ is odd. Iosevich-Rudnev ([11]) gave the first explicit exponent for the Erdős-Falconer distance problem in $\mathbb{F}_q^d$:

**Theorem 1.3.** There exists a constant $C$ so that if $E \subset \mathbb{F}_q^d$ has cardinality $|E| \geq Cq^{\frac{d+1}{2}}$ then $\Delta(E) = \mathbb{F}_q$. 

2
It would be reasonable to expect that whenever $E \subset \mathbb{F}_q^d$ with $|E| \geq Cq^{\frac{d}{2}}$ for a sufficiently large constant $C$, then $\Delta(E) = \mathbb{F}_q$, in line with the Falconer distance problem. However, it was shown in [10] that Theorem 1.3 is sharp in odd dimensions in the sense that the exponent $\frac{d+1}{d}$ cannot be replaced by any smaller value. It may still be the case that $\frac{d}{2}$ is the proper exponent in even dimensions. The only known improvement occurs in the case $d = 2$, where it has been shown ([1, 3]) that if $E \subset \mathbb{F}_q^2$, then there exists a constant $C$ such that whenever $|E| \geq Cq^{4/3}$, then $|\Delta(E)| \geq cq$ for some $0 < c \leq 1$. Note that the exponent $\alpha = 4/3$ is in line with Wolff’s exponent for the Falconer distance problem. See [8, 14] and the references contained therein for more on the Erdős-Falconer problem and related results.

Despite the Erdős-distance problem having been resolved in $\mathbb{R}^2$, the Falconer distance problem is open in all dimensions, and the finite field analogue is open in all even dimensions. To try and obtain a better understanding of why this is the case, the author along with Iosevich and Pakianathan extended ([4]) the Erdős-Falconer distance problem to $\mathbb{Z}_q$, the integers modulo $q$. Here we let $\mathbb{Z}_q^d = \mathbb{Z}_q \times \cdots \times \mathbb{Z}_q$ denote the free module of rank $d$ over $\mathbb{Z}_q$. For $E \subset \mathbb{Z}_q^d$, define $\Delta(E) = \{ \|x - y\| : x, y \in E\} \subset \mathbb{Z}_q$, where as before $\|x - y\| = (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2$. We obtain the following results in this setting.

**Theorem 1.4.** Suppose that $E \subset \mathbb{Z}_q^d$, where $q = p^\ell$ is a power of an odd prime. Then there exists a constant $C$ such that $\Delta(E) \supset \mathbb{Z}_q^d \cup \{0\}$ whenever $|E| \geq C\ell(\ell + 1)q^{\frac{(2\ell - 1)d + 3}{d}}$.

This result is a nice extension of Theorem 1.3 in the sense that when $\ell = 1$, $\mathbb{Z}_q$ is a field, and the exponents match those of Theorem 1.3 exactly. Since Theorem 1.3 is sharp in odd dimensions, then Theorem 1.4 is sharp in odd dimensions as well, at least in the case $\ell = 1$. In [4] it was shown that Theorem 1.4 is close to optimal in the sense that there exists a value $b = b(p)$ such that $|E| = bq^{(\frac{2\ell - 1}{d})d}$, and yet $\Delta(E) \cap \mathbb{Z}_q^d = \emptyset$. This shows that for these constructed sets $E$, we have $|\Delta(E)| \leq p^{\ell - 1} = q(q^\ell)$.

It is of interest to extend Theorem 1.4 to non-units in $\mathbb{Z}_q$ and to the case $q \neq p^\ell$. This is the purpose of the article, and our main result is the following.

**Theorem 1.5.** Suppose that $q$ has the prime decomposition $q = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $2 < p_1 < \cdots < p_k$ and $\alpha_i > 0$ for each $i = 1, \ldots, k$. Suppose that $E \subset \mathbb{Z}_q^d$ for some $d > 2$. Let $\tau(q) = \sum_{d|q} 1$ be the number of positive divisors of $q$. Then, there exists a constant $C$ such that $\Delta(E) = \mathbb{Z}_q$ whenever

$$|E| \geq C\tau(q)q^d p_1^{\frac{d-2}{2}}.$$  

**Remark 1.6.** As $\tau(q) = q(q^\ell)$ for all $\ell > 0$, our result is always nontrivial for $d \geq 3$.

Theorem 1.5 immediately implies the following. 

**Corollary 1.7.** Suppose that $E \subset \mathbb{Z}_q^d$, where $q = p^\ell$ is odd and $d > 2$. Then there exists a constant $C$ such that $\Delta(E) = \mathbb{Z}_q$ whenever $|E| \geq C(\ell + 1)q^{\frac{(2\ell - 1)d + 2}{d}}$.
1.0.1 Fourier Analysis in \( \mathbb{Z}_q^d \)

For \( f : \mathbb{Z}_q^d \to \mathbb{C}, \) we define the (normalized) Fourier transform of \( f \) as

\[
\hat{f}(m) = q^{-d} \sum_{x \in \mathbb{Z}_q^d} f(x) \chi(-x \cdot m)
\]

where \( \chi(x) = \exp(2\pi i x / q). \) Since \( \chi \) is a character on the additive group \( \mathbb{Z}_q, \) we have the following orthogonality property.

Lemma 1.8. We have

\[
q^{-d} \sum_{x \in \mathbb{Z}_q^d} \chi(x \cdot m) = \begin{cases} 1 & m = (0, \ldots, 0) \\ 0 & \text{otherwise} \end{cases}
\]

In turn Lemma 1.8 gives Plancherel and inversion-like identities.

Proposition 1.9. Let \( f \) and \( g \) be complex-valued functions defined on \( \mathbb{Z}_q^d. \) Then,

\[
\begin{align*}
\sum_{m \in \mathbb{Z}_q^d} \chi(x \cdot m) \hat{f}(m) &= f(x) \quad (1.1) \\
q^{-d} \sum_{x \in \mathbb{Z}_q^d} f(x)g(x) &= \sum_{m \in \mathbb{Z}_q^d} \hat{f}(m)\hat{g}(m) \quad (1.2)
\end{align*}
\]

2 Proof of Theorem 1.5

The proof of Theorem 1.5 follows a similar approach as that in [4]. We write

\[
\nu(t) = |\{(x, y) \in E \times E : \|x - y\| = t\}|
\]

and we will demonstrate that \( \nu(t) > 0 \) for each \( t \in \mathbb{Z}_q. \) To this end write

\[
\begin{align*}
\nu(t) &= \sum_{x, y} E(x)E(y)S_t(x - y) \\
&= \sum_{x, y, m} E(x)E(y)\hat{S}_t(m)\chi(m \cdot (x - y)) \\
&= q^{2d} \sum_m \left| \hat{E}(m) \right|^2 \hat{S}_t(m) \\
&= q^{-d} |E|^2 |S_t| + q^{2d} \sum_{m \neq 0} \left| \hat{E}(m) \right|^2 \hat{S}_t(m) \\
&= M + R_t.
\end{align*}
\]

We will utilize the following Lemmas.

Lemma 2.1. For \( d > 2 \) and \( t \in \mathbb{Z}_q \) for odd \( q, \) we have

\[
|S_t| = q^{d-1}(1 + o(1)).
\]
Lemma 2.2. Let $d > 2$, and $q = p_1^{a_1} \cdots p_k^{a_k}$, where $q$ is odd. Then for $m \neq 0$, we have

$$|\hat{S}_t(m)| \leq q^{-1} \tau(q) p_1^{-\frac{d-2}{2}}.$$ 

Applying Lemma 2.1 it is immediate that

$$M = q^{-1}|E|^2 (1 + o(1)).$$

In order to deal with the error term $R_t$, we note that

$$|R_t| \leq q^d \max_{m \neq 0} |\hat{S}_t(m)| \sum_{m \neq 0} |\hat{E}(m)|^2 \leq q^d |E| \cdot \max_{m \neq 0} |\hat{S}_t(m)|$$

where the last inequality follows from adding back the zero element and applying Proposition 1.9. Applying Lemma 2.2 and putting the estimates for $M$ and $R_t$ together, we see that

$$\nu(t) = q^{-1}|E|^2 (1 + o(1)) + R_t,$$

where

$$|R_t| \leq |E| \cdot \tau(q) q^{d-1} p_1^{-\frac{d-2}{2}}$$

and this shows that $\nu(t) > 0$ whenever

$$|E| \geq C \tau(q) q^{d} p_1^{-\frac{d-2}{2}}$$

for a sufficiently large constant $C$. It remains to prove Lemmas 2.1 and 2.2.

2.1 Gauss Sums

Before we prove Lemmas 2.1 and 2.2, we will need the following well known result which we provide for completeness.

Definition 2.3 (Quadratic Gauss sums). For positive integers $a, b, n$, we denote by $G(a, b, n)$ the following sum

$$G(a, b, n) := \sum_{x \in \mathbb{Z}_n} \chi(ax^2 + bx).$$

where $\chi(x) = e^{2\pi i x/n}$. For convenience, we denote the sum $G(a, 0, n)$ by $G(a, n)$.

Proposition 2.4 ([12]). Let $\chi(x) = e^{2\pi i x/n}$. For $a \in \mathbb{Z}_n$ with $(a, n) = 1$, we have

$$G(a, n) = \begin{cases} 
\varepsilon_n \left( \frac{a}{n} \right) \sqrt{n} & n \equiv 1 \pmod{2} \\
0 & n \equiv 2 \pmod{4} \\
(1 + i)\varepsilon_n^{-1} \left( \frac{a}{n} \right) \sqrt{n} & n \equiv 0 \pmod{4}, a \equiv 1 \pmod{2}
\end{cases}$$

where $(\cdot)$ denotes the Jacobi symbol and

$$\varepsilon_n = \begin{cases} 
1 & n \equiv 1 \pmod{4} \\
i & n \equiv 3 \pmod{4}
\end{cases}$$

Furthermore, for general values of $a \in \mathbb{Z}_n$, we have

$$G(a, b, n) = \begin{cases} 
(a, n)G \left( \frac{a}{(a,n)}, \frac{b}{(a,n)}, \frac{n}{(a,n)} \right) & (a, n) \mid b \\
0 & \text{otherwise}
\end{cases}$$
2.2 Proof of Lemma 2.1

We first note that by the Chinese Remainder Theorem, it is enough to prove Lemma 2.1 in the case that \( q = p^t \) is a power of a prime. Write

\[
|S_t| = \sum_{x \in \mathbb{Z}_q^d} S_t(x) = q^{-1} \sum_{x \in \mathbb{Z}_q^d} \sum_{s \in \mathbb{Z}} \chi(s(x_1^2 + \cdots + x_d^2 - t))
\]

\[
= q^{d-1} + q^{-1} \sum_{s \neq 0} \chi(-st) (G(s, p^t))^d
\]

\[
= q^{d-1} + I_{tt}.
\]

Put \( \text{val}_p(s) = k \), if \( p^k \) is the largest power of \( p \) dividing \( s \), in which case we let \( s = p^k u \), where \( u \in \mathbb{Z}_{p^t-k}^\times \) is uniquely determined. Then,

\[
I_{tt} = q^{-1} \sum_{k=0}^{t-1} \sum_{\text{val}_p(s) = k} \chi(-st) (G(s, p^k))^d
\]

\[
= q^{-1} \sum_{k=0}^{t-1} \sum_{u \in \mathbb{Z}_{p^t-k}^\times} \chi(-p^k ut) (G(p^k u, p^k))^d
\]

\[
= q^{-1} \sum_{k=0}^{t-1} p^{kd} \sum_{u \in \mathbb{Z}_{p^t-k}^\times} \chi(-p^k ut) (G(u, p^k))^d
\]

\[
= q^{-1} \sum_{k=0}^{t-1} p^{kd} \left( p^{\frac{t-k}{2}} \right)^d \sum_{u \in \mathbb{Z}_{p^t-k}^\times} \chi(-p^k ut) \left( \frac{u}{p} \right)^{d(t-k)}.
\]

Applying the trivial bound

\[
\left| \sum_{u \in \mathbb{Z}_{p^t-k}^\times} \chi(-p^k ut) \left( \frac{u}{p} \right)^{d(t-k)} \right| \leq p^{\ell-k},
\]
we have

\[
|II_1| \leq q^{-1} \sum_{k=0}^{\ell-1} p^{k\delta} p^d p_{\delta} \delta^{-k} d^{-k} = p^{\delta} \sum_{k=0}^{\ell-1} p^{k(d-2)} d^{-k} \leq \ell p^{\delta} p^{\ell(d-2)} d^{-1} = \ell q^{\ell(d-2)} q^{1-1} = \ell q^{d-1} q^{1-1} \]

which shows that \( II_1 = o(q^{d-1}) \) whenever \( d > 2 \).

### 2.3 Proof of Lemma 2.2

Writing \( m = (m_1, \ldots, m_d) \) and unraveling the definition, we see

\[
\hat{S}_t(m) = q^{-d-1} \sum_{s \in \mathbb{Z}_q} \sum_{x \in \mathbb{Z}_q^d} \chi(s(x_1^2 + \cdots + x_d^2)) \chi(-m \cdot x) = q^{-d-1} \sum_{s \in \mathbb{Z}_q \setminus \{0\}} \sum_{x \in \mathbb{Z}_q^d} \chi(-st) \chi(sx_1 - m_1x_1) \cdots \chi(sx_d - m_dx_d)
\]

\[
= q^{-d-1} \sum_{s \in \mathbb{Z}_q \setminus \{0\}} \chi(-st) \prod_{i=1}^d G(s, -m_i, q),
\]

Our first step is to write \( s = p_1^{\beta_1} \cdots p_k^{\beta_k} u \) where \( u \in \mathbb{Z}_q^\times \) is uniquely determined for \( q' = p_1^{\alpha_1-\beta_1} \cdots p_k^{\alpha_k-\beta_k} \) and \( \beta_i \geq 0 \). Note that \( \beta_i < \alpha_i \) for some \( i \) since \( s \neq 0 \). We will use the notation \( \sum_{\beta} \) to denote the sum over all such \( (\beta_1, \ldots, \beta_k) \). For \( m = (m_1, \ldots, m_d) \) and \( \beta = (\beta_1, \ldots, \beta_k) \), we define \( \lambda_{m, \beta} \) to be 1 if \( p_1^{\beta_1} \cdots p_k^{\beta_k} \mid m_i \) for all \( i \), and zero
otherwise. When \( \lambda_{m, \beta} = 1 \), we put \( \mu_i = \frac{m_i}{p_1^i \cdots p_k^i} \). Applying Proposition 2.4,

\[
\tilde{S}_t(m) = q^{-d-1} \sum_{\beta} \sum_{u \in \mathbb{Z}_{q'}^d} \chi(-p_1^{\beta_1} \cdots p_k^{\beta_k} u t) \prod_{i=1}^{d} \mathcal{G}^{p_i^{\beta_i} \cdots p_k^{\beta_k} u, -m_i, q}
\]

\[
= \lambda_{m, \beta} q^{-d-1} \sum_{\beta} \sum_{u \in \mathbb{Z}_{q'}^d} p_1^{\beta_1} \cdots p_k^{\beta_k} \chi(-p_1^{\beta_1} \cdots p_k^{\beta_k} u t) \prod_{i=1}^{d} \mathcal{G}(u, -\mu_i, q')
\]

\[
= \lambda_{m, \beta} q^{-d-1} \sum_{\beta} \sum_{u \in \mathbb{Z}_{q'}^d} p_1^{\beta_1} \cdots p_k^{\beta_k} \chi(-p_1^{\beta_1} \cdots p_k^{\beta_k} u t) \prod_{i=1}^{d} \mathcal{G}(u, -\mu_i, q')
\]

\[
= \lambda_{m, \beta} q^{-d-1} \sum_{\beta} \sum_{u \in \mathbb{Z}_{q'}^d} p_1^{\beta_1} \cdots p_k^{\beta_k} \chi(-p_1^{\beta_1} \cdots p_k^{\beta_k} u t) \prod_{i=1}^{d} \mathcal{G}(u, -\mu_i, q')
\]

\[
= \lambda_{m, \beta} q^{-d-1} \sum_{\beta} \sum_{u \in \mathbb{Z}_{q'}^d} p_1^{\beta_1} \cdots p_k^{\beta_k} \chi(-p_1^{\beta_1} \cdots p_k^{\beta_k} u t) \prod_{i=1}^{d} \mathcal{G}(u, -\mu_i, q')
\]

\[
= \lambda_{m, \beta} q^{-d-1} \sum_{\beta} \sum_{u \in \mathbb{Z}_{q'}^d} p_1^{\beta_1} \cdots p_k^{\beta_k} \chi(-p_1^{\beta_1} \cdots p_k^{\beta_k} u t) \prod_{i=1}^{d} \mathcal{G}(u, -\mu_i, q')
\]

Applying the trivial bound to the sum in \( u \in \mathbb{Z}_{q'}^d \), we see that

\[
\left| \tilde{S}_t(m) \right| \leq q^{-d-1} \sum_{\beta} p_1^{\left(\frac{\alpha_1 + \beta_1}{2}\right)d} \cdots p_k^{\left(\frac{\alpha_k + \beta_k}{2}\right)d} \prod_{i=1}^{d} p_i^{d + \alpha_i - \beta_i}.
\]

Writing \( \beta_i = \alpha_i - \epsilon_i \), we have

\[
\left| \tilde{S}_t(m) \right| \leq q^{-d-1} \sum_{\beta} \prod_{i=1}^{k} p_i^{\left(\frac{2\alpha_i - \epsilon_i}{2}\right)d + \epsilon_i}
\]

\[
= q^{-d-1} \sum_{\beta} \prod_{i=1}^{k} p_i^{\left(\frac{\alpha_i - \epsilon_i}{2}\right)d + \epsilon_i}
\]

\[
= q^{-d-1} \sum_{\beta} q^d \prod_{i=1}^{k} p_i^{-\frac{d\epsilon_i - 2}{2}}
\]

\[
\leq q^{-\tau(q)} p_i^{-\frac{d-2}{2}}
\]

for some \( p_i \) since \( \epsilon_i > 0 \) for at least one value \( i \) as \( s \neq 0 \). The largest value obtained by the quantity occurs when \( \epsilon_1 = 1 \) and \( \epsilon_i = 0 \) for \( i > 1 \). Hence,

\[
\left| \tilde{S}_t(m) \right| \leq q^{-\tau(q)} p_1^{-\frac{d-2}{2}}.
\]
References

[1] M. Bennett, D. Hart, A. Iosevich, J. Pakianathan, M. Rudnev, Group actions and geometric combinatorics in $\mathbb{F}_q^d$ (preprint) ArXiv: 1311.4788. 3
[2] J. Bourgain, N. Katz, T. Tao, A sum-product estimate in finite fields, and applications, Geom. Func. Anal. 14, 27–57, (2004). 2
[3] J. Chapman, M. B. Erdoğan, D. Hart, A. Iosevich and D. Koh, Pinned distance sets, $k$-simplices, Wolff’s exponent in finite fields and sum-product estimates, Mathematische Zeitschrift, Math Z. 271 (2012), 63–93. 3
[4] D. Covert, A. Iosevich, J. Pakianathan, Geometric configurations in the ring of integers modulo $p^\ell$, Indiana Univ. Math. J., 61 (2012), no. 5, 1949–1969. 3, 4
[5] B. Erdoğan, A bilinear Fourier extension theorem and applications to the distance set problem, IMRN (2006). 2
[6] P. Erdős, Integral distances, Bull. Amer. Math. Soc. 51 (1946) 996. 1
[7] K. Falconer, On the Hausdorff dimensions of distance sets Mathematika 32, 206–212, (1986). 1
[8] J. Garibaldi, A. Iosevich, S. Senger, Erdős Distance Problem, AMS Student Library Series, 56, (2011). 1, 3
[9] L. Guth and N. Katz On the Erdős distinct distances problem in the plane. (preprint) arXiv:1011.4105. 1
[10] D. Hart, A. Iosevich, D. Koh and M. Rudnev, Averages over hyperplanes, sum-product theory in vector spaces over finite fields and the Erdős-Falconer distance conjecture, Transactions of the AMS, 363 (2011) 3255–3275. 3
[11] A. Iosevich, M. Rudnev, Erdős distance problem in vector spaces over finite fields. Trans. Amer. Math. Soc. 359, 12, 6127–6142, (2007). 2
[12] H. Iwaniec, and E. Kowalski, Analytic Number Theory, Colloquium Publications 53 (2004). 5
[13] N. Katz and G. Tardos A new entropy inequality for the Erdős distance problem Contemp. Math. 342, Towards a theory of geometric graphs, 119-126, Amer. Math. Soc., Providence, RI (2004). 1
[14] D. Koh, C.Y. Shen, The generalized Erdős-Falconer distance problems in vector spaces over finite fields, to appear, Journal of Number Theory. 3
[15] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, 1995. 1
[16] J. Solymosi and V. Vu, Near optimal bound for the number of distinct distances in high dimensions, Near optimal bound for the distinct distances in high dimensions, Combinatorica 28 (2008), 113-125. 1
[17] T. Tao and V. Vu. Additive Combinatorics. Cambridge University Press, 2006. 1
[18] T. Wolff, Decay of circular means of Fourier transforms of measures. International Mathematics Research Notices 10, 547–567, (1999). 2