The Geometry of (Super) Conformal Quantum Mechanics

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Abstract

$N$-particle quantum mechanics described by a sigma model with an $N$-dimensional target space with torsion is considered. It is shown that an $SL(2,\mathbb{R})$ conformal symmetry exists if and only if the geometry admits a homothetic Killing vector $D^a\partial_a$ whose associated one-form $D_a dX^a$ is closed. Further, the $SL(2,\mathbb{R})$ can always be extended to $Osp(1|2)$ superconformal symmetry, with a suitable choice of torsion, by the addition of $N$ real fermions. Extension to $SU(1,1|1)$ requires a complex structure $I$ and a holomorphic $U(1)$ isometry $D^a I_a \partial_b$. Conditions for extension to the superconformal group $D(2,1;\alpha)$, which involve a triplet of complex structures and $SU(2) \times SU(2)$ isometries, are derived. Examples are given.

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1. Introduction

Conformal and superconformal field theories in various dimensions have played a central role in our understanding of modern field theory and string theory. Oddly, the subject of this paper—one dimension—is one of the least well understood cases. The simplest example of conformally invariant single-particle quantum mechanics was pioneered in [1], following the general analysis of [2–4]. Supersymmetric generalizations were discussed in [5–11]. The quantum mechanics case has taken on renewed interest because superconformal quantum mechanics may provide a dual description of string theory on $\text{AdS}_2$ [12].

Most of the discussions so far have concerned relatively simple systems either with small numbers of particles or exact integrability. In this paper we consider a more general class of models with $N$ particles.

We begin in section 2 with a bosonic sigma model with an $N$-dimensional target space. It is shown that the model has a nonlinearly-realized conformal symmetry if and only if the target space metric has a vector field $D^a \partial_a$ whose Lie derivative obeys

$$\mathcal{L}_{D^a} g_{ab} = 2g_{ab}, \quad (1.1)$$

and whose associated one-form is closed

$$d(D_a dX^a) = 0. \quad (1.2)$$

Given (1.1) and (1.2) it is shown that, in a Hamiltonian formalism, the dilations are (roughly) generated by $D^a P_a$ while the special conformal transformations are generated by

$$K = \frac{1}{2} D_a D^a. \quad (1.3)$$

The conformal symmetry persists in the presence of a potential $V$ obeying $\mathcal{L}_D V = -2V$. A general class of examples is given.

In section 3 we turn to the supersymmetric case. The geometry of Poincaré-supersymmetric quantum mechanics with a variety of supermultiplets was discussed by Coles and Papadopoulos [13]. We restrict our attention to the case for which the multiplet structure with respect to the Poincaré super-subgroup consists of $N$ bosons $X^a$ with $N$ real superpartners $\lambda^a$. Such multiplets arise in the reduction of two-dimensional chiral $(0, N)$ multiplets, where $N$ is the number of supersymmetries, to one dimension, and give rise to what is sometimes referred to as “type $B$” models (most of the literature concerns “type
A" \((\mathcal{N}/2,\mathcal{N}/2)\) multiplets). In section 3.1 we show that every bosonic conformal model can be extended to an \(\mathcal{N} = 1B\) theory with \(Osp(1|2)\) superconformal symmetry provided the torsion obeys certain constraints. In section 3.2 we consider \(\mathcal{N} = 2B\) and find that the extension to \(SU(1,1|1)\) requires a complex structure \(I\) with respect to which \(D\) must be holomorphic. \(D^a I_a^b \partial_b\) is found to generate a \(U(1)\) isometry. In section 3.3 we first derive a simplified version of the conditions for \(\mathcal{N} = 4B\) Poincaré supersymmetry with an \(SU(2)\) \(R\)-symmetry as first-order differential relations between the triplet of complex structures \(I^r\). We further show that an \(\mathcal{N} = 4B\) model has a \(D(2,1;\alpha)\) superconformal symmetry if the vector fields \(D^a I_a^b \partial_b\) generate an \(SU(2)\) isometry group and obey generalizations of the identities required for \(SU(1,1|1)\). The parameter \(\alpha\) is determined by the constant in the \(SU(2)\) Lie bracket algebra. In section 3.4 we construct a large class of \(\mathcal{N} = 4B\) theories in terms of an unconstrained potential \(L\). \(D(2,1;\alpha)\) superconformal symmetry then follows if \(L\) is a homogeneous and \(SU(2)\) rotationally-invariant function of the coordinates. Related results in four dimensions were recently discussed in [14].

Throughout the paper we use a Hamiltonian formalism. In appendix A we give a Lagrangian derivation of the supercharges used in the text. We use real coordinates throughout the body of the text, but appendix B gives various useful formulae for the geometry and supercharges in complex coordinates. In appendix C we discuss the conditions under which an \(\mathcal{N} = 4B\) geometry can be written in terms of a potential \(L\).

A primary motivation for this work is the expectation that quantum mechanics on the five-dimensional multi-black hole moduli space is an \(\mathcal{N} = 4B\) theory with a \(D(2,1;\alpha)\) superconformal symmetry at low energies [15].

2. \(\mathcal{N} = 0\) Conformally Invariant \(N\)-Particle Quantum Mechanics

In this section we find the conditions under which a general \(N\)-particle quantum mechanics admits an \(SL(2,\mathbb{R})\) symmetry. We will adopt a Hamiltonian formalism, and derive the conditions for the existence of appropriate operators generating the symmetries. The general Hamiltonian is

\[
H = \frac{1}{2} P_a^\dagger g^{ab} P_b + V(X). \tag{2.1}
\]

\(^2\) The canonical momentum \(P_a = g_{ab} \dot{X}^b = -i \partial_a\) obeys \([P_a, X^b] = -i \delta_a^b\) and \(P_a^\dagger = \frac{1}{\sqrt{g}} P_a \sqrt{g}\) (for the norm \(\langle f_1, f_2 \rangle = \int d^N X \sqrt{g} f_1^* f_2\)). In this and all subsequent expressions, the operator ordering is as indicated.

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where \( a, b = 1, \ldots, N \). We now determine the conditions under which the theory, defined by equation (2.1), admits an \( SL(2, \mathbb{R}) \) symmetry.

We first look for a dilational symmetry of the general form

\[
\delta_D X^a = \epsilon D^a(X), \\
\delta_D t = 2\epsilon t.
\]  

(2.2)

This is generated by an operator

\[
D = \frac{1}{2} D^a P_a + \text{h.c.}
\]  

(2.3)

which should obey

\[
[D, H] = 2iH.
\]  

(2.4)

From the definitions (2.3) and (2.1) one finds

\[
[D, H] = -\frac{i}{2} P_a^b (\mathcal{L}_D g^{ab}) P_b - i\mathcal{L}_D V - i\frac{1}{4} \nabla^2 \nabla^a D^a,
\]  

(2.5)

where \( \mathcal{L}_D \) is the usual Lie derivative obeying:

\[
\mathcal{L}_D g^{ab} = D^c g_{ab,c} + D^c a g_{cb} + D^c b g_{ac}.
\]  

(2.6)

Therefore, given a metric \( g \) and potential \( V \) a dilational symmetry exists if and only if there exists a conformal killing vector \( D \) obeying

\[
\mathcal{L}_D g_{ab} = 2g_{ab}
\]  

(2.7)

and

\[
\mathcal{L}_D V = -2V.
\]  

(2.8)

Note that (2.7) implies the vanishing of the last term of equation (2.5). A vector field \( D \) obeying (2.7) is known as a \textit{homothetic} vector field, and the action of \( D \) is known as a \textit{homothety}.

Next we look for a special conformal symmetry generated by an operator \( K(X) \) obeying

\[
[D, K] = -2iK,
\]  

(2.9)

and

\[
[H, K] = -iD.
\]  

(2.10)
Equations (2.9) and (2.10) together with (2.3) is an \(SL(2, \mathbb{R})\) algebra. Equation (2.9) is equivalent to
\[ \mathcal{L}_D K = 2K, \] (2.11)
while (2.10) can be written
\[ D_a dX^a = dK. \] (2.12)
Hence the one-form \(D\) is exact. One can solve for \(K\) as
\[ K = \frac{1}{2} g_{ab} D^a D^b. \] (2.13)

We shall adopt the phrase “closed homothety” to refer to a homothety whose associated one-form is closed and exact.

An alternate basis of \(SL(2, \mathbb{R})\) generators is
\[ L_0 = \frac{1}{2} (H + K), \]
\[ L_{\pm 1} = \frac{1}{2} (H - K \mp iD). \] (2.14)
In this basis the generators obey the standard commutation relations
\[ [L_1, L_{-1}] = 2L_0, \]
\[ [L_0, L_{\pm 1}] = \mp L_{\pm 1}. \] (2.15)

The nature of these geometries can be illuminated by choosing coordinates such that \((X^0)^2 = 2K\) and \(g_{i0} = 0\) for \(i = 1, \ldots, N-1\). This is always locally possible away from the zeros of \(D\). One then finds
\[ ds^2 = (dX^0)^2 + (X^0)^2 g_{ij}(X^k)dX^i dX^j, \]
\[ D^a \partial_a = X^0 \frac{\partial}{\partial X^0}. \] (2.16)
Hence, given any metric \(g_{ij}\) in \(N-1\) dimensions, one can construct a geometry with a closed homothety in \(N\) dimensions by dressing it with an extra radial dimension. Similar comments pertain to the potential \(V\).

An alternate useful choice is dilational coordinates, in which
\[ D^a = \frac{2}{\hbar} X^a, \] (2.17)
where $h$ is an arbitrary constant. These are related to the coordinates in (2.16) by $X^0 = (X^0)^{2/h}$, $X^i = (X^0)^{2/h}X^i$. In such dilational coordinates one finds

$$X^a \partial_a g_{bc} = \frac{h}{2} \mathcal{L}_D g_{bc} - X^a, c g_{ba} - X^a, b g_{ac} = (h - 2)g_{bc}. \quad (2.18)$$

Hence in dilational gauge the metric components are homogeneous functions of degree $h - 2$. (It is not, however, the case that every homogeneous metric admits an $SL(2, \mathbb{R})$ symmetry.) At this point $h$ can be changed by transformations which take the coordinates to powers of themselves, and so has no coordinate independent meaning. However, it turns out that for $\mathcal{N} = 4B$ supersymmetry, a preferred value of $h$ is obtained in quaternionic coordinates, when such coordinates exist and coincide with dilational gauge, as in the class of examples considered in section 3.4.

In conclusion, the Hamiltonian (2.1) describes an $SL(2, \mathbb{R})$ invariant quantum mechanics if and only if the metric admits a closed homothety

$$\mathcal{L}_D g_{ab} = 2g_{ab},$$

$$d(D_a dX^a) = 0, \quad (2.19)$$

under which the potential transforms according to (2.8).

3. The Supersymmetric Case

In the following we supersymmetrize the bosonic sigma model by extending the boson $X^a$ to the supermultiplet $(X^a, \lambda^a)$ with $\lambda^a = \lambda^a$. A number of other multiplets exist \[13\] which will not be considered in the following. Furthermore we will set the potential $V = 0$. An operator approach to a similar system can be found in \[10\].

3.1. $\mathcal{N} = 1B$ Poincaré supersymmetry and $Osp(1|2)$ superconformal symmetry

Let us supersymmetrize the bosonic sigma-model (2.1) for $V = 0$ with $N$ fermions $\lambda^\alpha$ where $\alpha = 1, \ldots, N$ is a tangent space index. These obey the standard anticommutation relations

$$\{\lambda^\alpha, \lambda^\beta\} = \delta^\alpha\beta, \quad (3.1)$$

and of course commute with $P_a$ and $X^b$. It is convenient to make the field redefinitions

$$\lambda^a \equiv e^{a, \alpha} \lambda^\alpha,$$

$$\Pi_a \equiv P_a - \frac{i}{2} \omega_{ab} \lambda^b \lambda^c + \frac{i}{2} c_{abc} \lambda^b \lambda^c, \quad (3.2)$$
where $\omega$ is related to the usual spin connection by 
\[ \omega_{abc} = \omega_a^{\beta\gamma} e_{\beta} e_{\gamma}^c. \]

A supercharge can then be constructed as
\[ Q = \lambda^a \Pi_a - \frac{i}{3} c_{abc} \lambda^a \lambda^b \lambda^c, \quad (3.3) \]
where $c$ is a 3-form, which at this point is arbitrary. A derivation of the supercharge from
a supersymmetric Lagrangian is given in appendix A. The supercharge obeys
\[ \{Q, Q\} = 2H, \quad (3.4) \]
where the bosonic part of $H$ agrees with (2.1) for $V = 0$.

We wish to extend this $\mathcal{N} = 1$ $B$ Poincaré-superalgebra to the $Osp(1|2)$ superconformal
algebra whose non-vanishing commutation relations are
\[
\begin{align*}
[H, K] &= -iD, \quad [H, D] = -2iH, \quad [K, D] = 2iK, \\
\{Q, Q\} &= 2H, \quad [Q, D] = -iQ, \quad [Q, K] = -iS, \\
\{S, S\} &= 2K, \quad [S, D] = iS, \quad [S, H] = iQ, \\
\{S, Q\} &= D.
\end{align*} \quad (3.5)
\]
As before, the bosonic subalgebra requires a closed homothety. The new supercharge can
then be constructed as
\[ S = i[Q, K] = \lambda^a D_a. \quad (3.6) \]
with $K$ given by (2.13). The $\{S, Q\}$ anticommutator is then used to find
\[ \{S, Q\} = D = \frac{1}{2} (D^a \Pi_a + \text{h.c.}). \quad (3.7) \]
Then, $[S, D] = iS$ is satisfied, but the $[Q, D]$ commutator is
\[ [Q, D] = -iQ - i c_{abc} D^a \lambda^b \lambda^c + O(\lambda^3). \quad (3.8) \]

---

3 We note that $[\Pi_a, \Pi_b] = -\frac{1}{2} R_{abcd}^a \lambda^c \lambda^d$ where $R_{abcd}^a$ is constructed from the connection $\Gamma_{ac}^b$ + $c_{ac}^b$; $[P_a, \lambda^b] = -i(\omega_{ac}^b - \Gamma_{ac}^b)\lambda^c$ and $[\Pi_a, \lambda^b] = i(\Gamma_{ac}^b + c_{ac}^b)\lambda^c$, where $\Gamma$ is the Christoffel
connection. The Hilbert space can be viewed as a spinor (as is seen by identifying equation (3.1)
with the $\gamma$-matrix algebra) and $\Pi_a$ as the covariant derivative (with torsion $c$) on Hilbert space
states.

4 Despite the non-hermiticity of $\Pi_a$, this expression is hermitian with the indicated operator
ordering.
Agreement with (3.5) then requires $c$ to be orthogonal to $D$:

$$D^a c_{abc} = 0. \quad (3.9)$$

Given (3.9), the full commutator becomes

$$[Q, D] = -iQ - \frac{1}{6} \chi^a \chi^b \chi^c (\mathcal{L}_D - 2) c_{abc}. \quad (3.10)$$

We therefore demand that $c$ transform under dilations as

$$\mathcal{L}_D c_{abc} = 2c_{abc}. \quad (3.11)$$

The remaining commutators in (3.5) then follow from the Jacobi identities, with no further constraints on the geometry.

In summary any $\mathcal{N} = 0$ conformal quantum mechanics can be promoted to $Osp(1|2)$, but the torsion $c$ appearing in the supercharges must obey

$$D^a c_{abc} = 0,$$
$$\mathcal{L}_D c_{abc} = 2c_{abc}. \quad (3.12)$$

3.2. $\mathcal{N} = 2B$ Poincaré supersymmetry and $SU(1,1|1)$ superconformal symmetry

$\mathcal{N} = 2B$ supersymmetry requires a complex structure $I$ and a hermitian metric on the target space [3]. The relevant formulae are simplest in complex coordinates. However complex coordinates are less useful in the extension to the $4B$ case (which has an $SU(2)$ triplet of complex structures) considered in the next subsection. Accordingly we continue with real coordinates, but give the complex version in appendix B.

The second supercharge is given by

$$\bar{Q} = \lambda^a I_a b \Pi_b - \frac{i}{2} \lambda^a I_a b c_{bcd} \lambda^c \lambda^d - \frac{i}{6} \lambda^a \lambda^b \lambda^c I_a d I_b e I_c f c_{def} - \frac{i}{2} \lambda^a c_{abc} I^{bc}. \quad (3.13)$$

A derivation is given in appendix B. Whereas $c$ is unconstrained for $\mathcal{N} = 1B$, for $\mathcal{N} = 2B$ the vanishing of $\{\bar{Q}, Q\}$ requires [3]

$$\nabla^+ (b_I c)^a = 0, \quad (3.14)$$

where the torsion connection $\nabla^+$ involves the Christoffel connection plus the torsion $c$ as $\Gamma^b_{ac} + c^b_{ac}$. In complex coordinates (3.14) can be solved for the $(1,2)$ part of $c$ as

$$c_{(1,2)} = -\frac{i}{2} \partial J, \quad (3.15)$$
with
\[ J = \frac{1}{2} I_a^c g_{bc} dX^a \wedge dX^b. \] (3.16)
The (0, 3) part of \( c \) must be closed under \( \bar{\partial} \) but is otherwise unconstrained, and the (2, 1) and (3, 0) parts are obtained by complex conjugation.

We wish to promote the \( \mathcal{N} = 2B \) algebra to \( SU(1, 1|1) \). This involves an additional bosonic generator \( R \) which is the generator of the \( R \) symmetry group of the \( \mathcal{N} = 2B \) subalgebra. The non-vanishing commutation relations are given by (3.5), an identical set of relations with both \( Q \) and \( S \) replaced by \( \tilde{Q} \) and \( \tilde{S} \), together with
\[
\begin{align*}
\{ \tilde{Q}, S \} &= R, & \{ \tilde{S}, Q \} &= -R, \\
[R, Q] &= -i \tilde{Q}, & [R, \tilde{Q}] &= i Q, \\
[R, S] &= -i \tilde{S}, & [R, \tilde{S}] &= i S.
\end{align*}
\] (3.17)

As before closure of the algebra requires that the geometry must admit a closed homothety, as well as the constraints (3.12) on \( c \). Commutation of the supercharges with \( K \) leads to the superconformal charges
\[
\begin{align*}
S &= \lambda^a D_a, \\
\tilde{S} &= \lambda^a I_a^b D_b.
\end{align*}
\] (3.18)

Obtaining the correct commutator \([D, \tilde{Q}] = i \tilde{Q}\) requires that the action of \( D \) preserves the complex structure:
\[
\mathcal{L}_D I_a^b = 0.
\] (3.19)
This is equivalent to the statement that \( D \) acts holomorphically. Alternate forms of (3.19) are
\[
D^I_f I^a f c_{abc} I_d^b I_e^c = D^I_f I^a c_{abc};
\]
\[
\partial_i D^j = 0.
\] (3.20)

It follows from (3.20), together with (3.9) and (3.15) that \( \tilde{D}^a = D^b I_b^a \) generates a holomorphic isometry
\[
\mathcal{L}_D I_a^b = 0,
\]
\[
\mathcal{L}_{\tilde{D}} g_{ab} = 0,
\] (3.21)
as expected from \([R, H] = 0\). Moreover the (2, 1) part of the torsion \( c \) is annihilated by \( \mathcal{L}_{\tilde{D}} \) while the (3, 0) part has weight \(-2i\).
\( R \) is determined from the commutator of \( Q \) and \( \tilde{S} \) as

\[
R = \tilde{D}^a \Pi_a - i I_{ab} \lambda^a \lambda^b - i \tilde{D}^a c_{abc} \lambda^b \lambda^c
\]

where we used equation (3.20). One finds

\[
\begin{align*}
[R, \lambda^a] &= -i (I^a b + \tilde{D}^a, b) \lambda^b, \\
[R, D^a] &= -i \tilde{D}^b D^a, b. 
\end{align*}
\]

(3.23)

In complex coordinates and dilational gauge \( hD^a = 2X^a \), when such coordinates exist, this reduces to

\[
\begin{align*}
[R, \lambda^a] &= -i (1 + \frac{2}{h}) \lambda^b I^a b, \\
[R, X^a] &= -\frac{2i}{h} X^b I^a b. 
\end{align*}
\]

(3.24)

Notice that \( R \) commutes with \( \lambda \) in complex coordinates with \( h = -2 \).

All the remaining commutators (3.17) and (3.5) are satisfied without any additional constraints.

In summary, there is an \( SU(1, 1|1) \) symmetry if and only if, in addition to the \( Osp(1|2) \) constraints (2.19) and (3.12), and the \( \mathcal{N} = 2B \) constraints, \( D \) preserves the complex structure:

\[
\mathcal{L}_D I^a b = 0.
\]

(3.25)

It further follows that \( \tilde{D}^a = D^b I^a b \) generates a holomorphic isometry.

3.3. \( \mathcal{N} = 4B \) Poincaré supersymmetry and \( D(2, 1; \alpha) \) superconformal symmetry

Remarks on \( \mathcal{N} = 4B \) Poincaré supersymmetry

Extending the algebra to include 4 supersymmetries requires 3 complex structures \( I^r \), \( r = 1, 2, 3 \). With each \( I^r \) one can associate a generalized exterior derivative

\[
d^r = dX^a I^r a b \nabla^r b, 
\]

(3.26)

where the connection \( \Omega^r \) appearing in \( \nabla^r \) is\(^5\)

\[
\Omega^r a b c = -I^r a d \partial_c I^r b d.
\]

(3.27)

\(^5\) \( \Omega^r \) defined in this way gives a connection acting on forms as described but not on general tensors.
One of the conditions for $\mathcal{N} = 4$ supersymmetry found in \cite{13,17} can be expressed

$$\{d^r, d^s\} = 0. \quad (3.28)$$

These are the vanishings of the Nijenhuis tensors and concomitants. In complex coordinates adapted to $I^r$, $\Omega^r$ vanishes and $d^r = i(\partial - \bar{\partial})$. Equation (3.28) further implies

$$\{d^r, d\} = 0. \quad (3.29)$$

Additional requirements for supersymmetry discussed in \cite{13,17} are

$$g_{ab} = I_a^r c_i I^r_d g_{cd} \quad (\forall r), \quad (3.30)$$

$$\{I^r, I^s\} = -2\delta^{rs}, \quad (3.31)$$

$$\partial_{[a} (I^r_{b c]} e_{c d]} - 2I^r_{[a} \partial_{e c} g_{b d]} = 0, \quad (3.32)$$

$$\nabla^+_{(b} I^r_{c)} = 0. \quad (3.33)$$

In this last equation, we used the covariant derivative with torsion $\nabla^+$ defined just below equation (3.14).

The commutators of $I^r$ are related to the $R$-symmetry group. We shall consider the $SU(2)$ case\footnote{So, Theorem 3.9 of \cite{18} implies that the vanishing of any two of these equations yields the vanishing of all six.}

$$[I^r, I^s] = 2\epsilon^{rst}I^t. \quad (3.34)$$

This case is sometimes referred to as $\mathcal{N} = 4B$ supersymmetry, and arises in the reduction of $(0, 4)$ supersymmetry from two dimensions.

We now show, defining the two-forms

$$J^r = \frac{1}{2} I^r_{a c} g_{bd} dX^a \wedge dX^b, \quad (3.35)$$

that the necessary and sufficient conditions for $\mathcal{N} = 4B$ supersymmetry can be recast in the simpler form

$$\{d^r, d^s\} = 0, \quad (3.36)$$

\footnote{We have employed an obvious summation convention in this equation. We hope that it will be clear from the context when repeated indices should or should not be summed over.}
\[ g_{ab} = I^r_c I^d_b g_{cd} \quad (\forall r), \]  

\[ I^r I^s = -\delta^r_s + \epsilon^{rst} I^t, \]  

\[ d^3 J^1 = d^2 J^2 = d^3 J^3. \]  

Note that the last two conditions (3.32) and (3.33) which involve the torsion \( c \) have been replaced by the condition (3.39) which is independent of \( c \). Let us write the torsion appearing in (3.33) as

\[ c = \frac{1}{2} d^3 J^3 + e \]  

for some three-form \( e \). It can be checked that the torsion connection with \( e \) set to zero is the unique such connection annihilating \( I^3 \), and therefore has holonomy contained in \( U(N/2) \). It follows that, in complex coordinates adapted to \( I^3 \), the condition (3.33) for \( r = 3 \) reduces to

\[ e_{\bar{i}jk} = e_{ijk} = 0. \]  

(This is the argument that led to equation (3.15).) On the other hand, adding the \( r = 1 \) plus or minus \( i \) times the \( r = 2 \) component of (3.33) yields

\[ e_{ijk} = e_{\bar{i}jk} = 0. \]  

We conclude that \( e = 0 \) and \( c = \frac{1}{2} d^3 J^3 \). By symmetry we must also have \( c = \frac{1}{2} d^1 J^1 \) and \( c = \frac{1}{2} d^2 J^2 \), from which (3.39) follows. Conversely given (3.39), adding the torsion \( c = \frac{1}{2} d^3 J^3 \) to the Christoffel connection implies (3.33). It can be further checked that this choice of \( c \) satisfies (3.32).

This single choice of torsion connection annihilates all three complex structures

\[ \nabla^+_b I^r_c = 0. \]  

In fact the condition (3.43) is equivalent to (3.39). It differs from (3.33) by the absence of symmetrization but is nevertheless equivalent for \( \mathcal{N} = 4B \). Equation (3.43) is referred to in [17] as the weak HKT (hyperkähler with torsion) condition. We have shown that \( \mathcal{N} = 4B \) (which includes the condition (3.38)) implies weak HKT.

**Extension to \( D(2,1;\alpha) \) superconformal symmetry**

We now turn to superconformal symmetry. It turns out that the relevant supergroup is \( D(2,1;\alpha) \), where the parameter \( \alpha \neq -1 \) will be determined by the geometry. In order to write down the commutators, it is convenient to define the four-component supercharges
\( Q^m = (Q^r, Q) \) and \( S^m = (S^r, S) \) for \( m = 1, 2, 3, 4 \); these transform in the \((2, 2)\) of the \( SU(2) \times SU(2) \) \( R \)-symmetry group of \( \mathcal{N} = 4B \). Operators \( Q^m, S^m, H, D, K \) and \( R^r_\pm \) (to be described) then comprise the \( D(2, 1; \alpha) \) algebra. The non-vanishing commutators are

\[
\begin{align*}
[H, K] &= -iD, \\
[H, D] &= -2iH, \\
[K, D] &= 2iK, \\
\{Q^m, Q^n\} &= 2H\delta^{mn}, \\
\{Q^m, D\} &= -iQ^m, \\
\{Q^m, K\} &= -iS^m, \\
\{S^m, S^n\} &= 2K\delta^{mn}, \\
\{S^m, D\} &= iS^m, \\
\{S^m, H\} &= iQ^m, \\
[R^r_\pm, Q^m] &= it^{\pm r}_m Q^n, \\
[R^r_\pm, S^m] &= it^{\pm r}_m S^n, \\
[R^r_\pm, R^s_\pm] &= i\epsilon^{rst} R^t_\pm.
\end{align*}
\]

The \( t^\pm \) matrices defined by

\[
t^{\pm r}_m \equiv \mp\delta^r_m \delta^4_n + \frac{1}{2}\epsilon_{rmn}
\]

obey

\[
\{t^{+ r}, t^{- s}\} = 0, \quad \{t^{+ r}, t^{\pm s}\} = -\epsilon^{rst} t^{\pm t}, \quad \{t^{\pm r}, t^{\pm s}\} = -\frac{1}{2}\delta^{rs}.
\]

Notice that when \( \alpha = 0 \) or \( \alpha = \infty \), one of the two \( SU(2) \)s can be decoupled, and there is an \( SU(1, 1|2) \) subalgebra.

Since \( D(2, 1|\alpha) \) has three \( SU(1, 1|1) \) and one \( \mathcal{N} = 4B \) subalgebra, the (previously discussed) conditions on the geometry for the existence of those subgroups can all be assumed. In particular, \( D \) must now be holomorphic with respect to all three complex structures

\[
\mathcal{L}_D I^r_a b = 0.
\]

Expressions for \( Q^r \) and \( S^r \) are then of the \( SU(1, 1|1) \) forms (3.13) and (3.18) with \( I \) replaced by \( I^r \). Somewhat lengthy expressions for \( R^r_\pm \) as a function of \( \alpha \) then follow from linear combinations of \( \{Q^m, S^n\} \) anticommutators as determined by (3.44). Obtaining properly normalized \( SU(2) \) algebras for the operators \( R^r_\pm \) so determined requires

\[
[L^r_D, L^s_D] = \frac{4}{h}\epsilon^{rst} L^{r'}_D,
\]

with \( D^r = D^a I^r_a b \partial_b \) and

\[
h = -2\alpha - 2.
\]

\[8\] In principle, we should treat \( \alpha = 0 \) or \( \alpha = \infty \) as special cases. In fact, \( \alpha = \infty \) cannot be realized with the supermultiplet we are considering. For \( \alpha = 0 \), the logic is slightly different but the results are the same.
Equation (3.48) can be taken as the definition of the constant $h$. Since the normalization of $D^r$ is fixed in terms of $D$, $h$ is a coordinate-invariant parameter associated to the geometry.

Reproducing the proper $[R, Q]$ commutators leads to the stronger requirement

$$\mathcal{L}_{D^r} I^{sb}_a = \frac{4}{h} \epsilon^{rst} I^{tb}_a. \quad (3.50)$$

In fact (3.50) (including $r = s$) implies both (3.48) and (3.47). Using (3.50) one then finds $R^r_\pm$ are given by

$$R^r_- = -\frac{h}{4} D^r a \Pi_a + i \frac{h-2}{8} \lambda^a I^r b \lambda_b + i \frac{h}{4} \lambda^a \lambda^b D^{rc} c_{cab} \quad (3.51)$$

$$R^r_+ = i \frac{4}{h} \lambda^a I^r b \lambda_b. \quad (3.52)$$

The torsion $c$ can be eliminated from (3.51) using the identity $D^{rc} c_{cab} = \frac{1}{2} (d^r dK)_{ab} - J^r_{ab}$.

Using the Jacobi identity, the remaining commutators follow with no further constraints on the geometry.

We note that equations (3.50) and (3.43) imply

$$2(h+2)J^r = h(d^r dK - \frac{1}{2} \epsilon^{rst} d^s d^t K),$$

$$4(h+2)c = -h d^1 d^2 d^3 K. \quad (3.53)$$

Properties of $d^r$ and $d$ then imply equation (3.39), which thus needs not be taken as a further condition.

We also find, in quaternionic coordinates and dilational gauge, when such coordinates exist, that

$$[R^-_r, \lambda^a] = 0, \quad [R^r_-, X^a] = \frac{i}{2} X^b I^r b a, \quad (3.54)$$

$$[R^+_r, \lambda^a] = \frac{i}{2} \lambda^b I^r b a, \quad [R^r_+, X^a] = 0.$$ 

In summary, a quantum mechanical theory has $\mathcal{N} = 4B$ supersymmetry if and only if the complex structure and metric obey equations (3.36)–(3.39). The torsion $c$ is then uniquely determined as

$$c = \frac{1}{2} d^3 J^3. \quad (3.55)$$

Note that the two excluded values $\alpha = -1$ and $\alpha = \infty$, correspond respectively to $h = 0$ and $h = \infty$, for which the algebra (3.48) is clearly singular.
A $D(2, 1; \alpha)$ symmetry arises if and only if in addition there is a vector field $D$ obeying

$$\mathcal{L}_D g_{ab} = 2g_{ab},$$

$$d(D_a dX^a) = 0,$$

$$\mathcal{L}_{D^r} I^b_a = \frac{4}{h} \epsilon^{rst} I^r_a,$$

$$\mathcal{L}_{D^r} g_{ab} = 0,$$

where $D^r = D^a I^r_a$ and $h$ is a constant characterizing the geometry. The parameter $\alpha$ in the superconformal algebra is related to the constant $h$ in (3.56) by

$$\alpha = -\frac{h + 2}{2}.$$

### 3.4. Examples of $D(2, 1; \alpha)$ Quantum Mechanics

In this subsection, we show that a large class of examples of quantum mechanical systems with $D(2, 1; \alpha)$ symmetry (and an integrable quaternionic structure) can be constructed from a potential $L$. In an $\mathcal{N} = 2$ superspace formalism (not described here, but similar to the ones in [19,20]) $L$ turns out to be the superspace integrand.

$\mathbb{R}^4$ has an obvious $SU(2)$ triplet of complex structures associated to self-dual two-forms obeying (3.38). Let $I^r$ be the generalizations to $\mathbb{R}^{4\mathcal{N}}$. We may then define a triplet of fundamental two-forms by

$$J^r = \frac{1}{8} (2d^r dL - \epsilon^{rst} d^s d^t L).$$

(3.58)

It follows immediately from this definition and $\{d^r, d^s\} = 0$ that the $J^r$ obey (3.39). Moreover the associated metric $g_{ab} = I^r_b I^r_a (\forall r)$ can be written (in a coordinate system in which the $I^r$ are constant)

$$g_{ab} = \frac{1}{4} (\delta^c_a \delta^d_b + I^r_c I^r_d) \partial_c \partial_d L.$$

(3.59)

This expression is manifestly hermitian. In other words for any $L$ we can construct an $\mathcal{N} = 4B$ quantum mechanics.\footnote{Although one may wish in addition to impose positivity of the metric $g$, which further constrains $L$.} It is natural to ask whether or not every weak HKT geometry is described by some potential $L$. This is related to the integrability of the quaternionic structure, as discussed in appendix C.
The full $D(2,1;\alpha)$ symmetry follows by imposing
\[ X^a \partial_a L = hL, \quad (3.60) \]
where $h$ is an arbitrary constant and
\[ X^a I^b_a \partial_b L = 0. \quad (3.61) \]
The first condition implies that $L$ is a homogeneous function of degree $h$ on $\mathbb{R}^{4N}$, while the second states that it is invariant under $SU(2)$ $R$-symmetry rotations. These conditions manifestly ensure the existence of the required homothety
\[ D^a \partial_a = \frac{2}{h} X^a \partial_a \quad (3.62) \]
as well as the $SU(2)$ isometries. Remarkably, it follows from (3.60) and (3.61) with a little algebra that $D$ is automatically a closed homothety,
\[ D_a dX^a = \frac{(h + 2)}{2h} (\partial_a L) dX^a. \quad (3.63) \]
As discussed in section 2 this implies the existence of special conformal transformations generated by
\[ K = \frac{1}{2} g_{ab} D^a D^b = \frac{(h + 2)}{2h} L. \quad (3.64) \]
In fact, all the requirements of (3.56) are automatically satisfied with these conditions, and so indeed the full $D(2,1;\alpha = -\frac{h+2}{2})$ algebra is obtained.

The conditions (3.60) and (3.61) are sufficient but not necessary to insure $D(2,1;\alpha)$ invariance. More generally one could add to the right hand side anything which is in the kernel of the second-order differential operator in (3.58). This is especially relevant for the interesting case $h = -2$, for which equations (3.63) and (3.64) show that the metric is otherwise degenerate. An example of this will appear in [15].

The simplest case is
\[ L = \frac{1}{2} \delta_{ab} X^a X^b, \quad (3.65) \]
where $a, b = 1, \ldots, 4N$. This has $h = 2$. The metric is then simply the flat metric on $\mathbb{R}^{4N}$
\[ ds^2 = \delta_{ab} dX^a dX^b, \quad (3.66) \]
while the torsion \( c \) vanishes. The generators of \( D(2, 1; -2) \sim Osp(4|2) \) are then
\[
\begin{align*}
H &= \frac{1}{2} P^a P_a, & K &= \frac{1}{2} X^a X_a, & D &= X^a P_a, \\
Q &= \lambda^a P_a, & Q^r &= \lambda^a I^b_a P_b, \\
S &= \lambda^a X_a, & S^r &= \lambda^a I^b_a X_b, \\
R^c_- &= -\frac{1}{2} X^a I^b_a P_b, & R^c_+ &= \frac{i}{4} \lambda^a I^b_a \lambda_b.
\end{align*}
\tag{3.67}
\]

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Appendix A. Lagrangian Derivation of the Supercharges

In this section we derive the supercharges used in the body of the text, from the component action \[ 13, 17 \]
\[
S = \int dt \left\{ \frac{1}{2} g_{ab} \dot{X}^a \dot{X}^b + i \lambda^a \left( g_{ab} \frac{D\lambda^b}{dt} - \dot{X}^c c_{abc} \lambda^b \right) - \frac{1}{6} \partial_d c_{abc} \lambda^d \lambda^a \lambda^b \lambda^c \right\} \tag{A.1}
\]
where the covariant derivative is
\[
\frac{D\lambda^b}{dt} \equiv \dot{\lambda}^b + \dot{X}^c \Gamma^b_{cd} \lambda^d, \tag{A.2}
\]
with \( \Gamma \) the Christoffel connection, and we use dots to denote time derivatives. Although we have, for ease of manipulation, written the fermions with spacetime indices, in deriving commutators it is better to use \( \lambda^\alpha \), where \( \alpha \) is a tangent index, because, unlike \( \lambda^a \), it will commute with the momentum conjugate to \( X \). In terms of \( \lambda^\alpha \), the kinetic term for the fermions is
\[
\frac{i}{2} g_{ab} \lambda^a \frac{D\lambda^b}{dt} = \frac{i}{2} (\delta_{\alpha\beta} \lambda^\alpha \dot{\lambda}^\beta + \dot{X}^c \omega_{\alpha\beta} \lambda^\alpha \lambda^\beta), \tag{A.3}
\]
and the momentum conjugate to \( X \) is
\[
P_a = g_{ab} \dot{X}^b + \frac{i}{2} (\omega_{abc} - c_{abc}) \lambda^b \lambda^c, \tag{A.4}
\]
or, using the definition in \[ 3.2 \],
\[
\Pi_a = g_{ab} \dot{X}^b. \tag{A.5}
\]
The action (A.2) is invariant under the supersymmetry transformation

\[ \delta_\epsilon X^a = -i\epsilon \lambda^a \quad \delta_\epsilon \lambda^a = \epsilon \dot{X}^a, \quad (A.6) \]

where \( \epsilon \) is a real anticommuting parameter. Note that

\[ [\delta_\epsilon, \delta_\eta] = -2i\eta \epsilon \frac{d}{dt}, \quad (A.7) \]

as required of a supersymmetry transformation. It is straightforward to compute the Noether charge corresponding to this symmetry; we find

\[ Q = \lambda^a \Pi_a - \frac{i}{3} c_{abc} \lambda^a \lambda^b \lambda^c, \quad (A.8) \]

which is the origin of equation (3.3). Actually, the Noether procedure determines the charge only up to operator ordering. We have fixed this ambiguity by demanding hermiticity and target space covariance.

Appendix B. \( \mathcal{N} = 2B \) Supersymmetry in Complex Coordinates

In this appendix we revisit the \( \mathcal{N} = 2B \) supersymmetry of section 3.2 in complex coordinates, which simplifies the formulae and calculations. Equation (3.15) for the (1,2) part of the torsion is

\[ c_{ij\bar{k}} = g_{i[j,\bar{k}]} \quad (B.1) \]

The (3,0) part of the torsion is constrained by the relation

\[ c_{[l,ijk]} = 0 \quad (B.2) \]

Identities required of \( D^a \) are

\[ D^k g_{ij,k} + D^k i g_{k\bar{i}} = g_{ij}, \]
\[ D^i c_{ijk} = 0, \]
\[ D^\bar{j} c_{ijk} = 0, \]
\[ D^\bar{j} c_{\bar{i}jk} = -D^i c_{ij\bar{k}}. \quad (B.3) \]

It is convenient to define a complex supercharge

\[ Q = \frac{1}{2}(Q - i\bar{Q}) \quad \bar{Q} = \frac{1}{2}(Q + i\bar{Q}) \quad (B.4) \]

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\( \mathcal{Q} \) can be determined by the requirement \( \mathcal{Q} = \mathcal{Q} + \bar{\mathcal{Q}} \) together with

\[
\{ \mathcal{Q}, \lambda^k \} = 0. \tag{B.5}
\]

Equation (B.5) is a manifestation of the separation of the holomorphic and antiholomorphic parts of the theory. One finds

\[
\mathcal{Q} = \lambda^i \Pi_i - ic_{ijk} \lambda^i \lambda^j \lambda^k - \frac{i}{3} c_{ijk} \lambda^i \lambda^j \lambda^k - ic^i_{jk} \lambda^k.
\tag{B.6}
\]

Using

\[
(\lambda^k \Pi_k)^\dagger = \lambda_k^\dagger \Pi_k,
\tag{B.7}
\]

one finds the hermitian conjugate is

\[
\bar{\mathcal{Q}} = \lambda^i \Pi^i - ic_{ijk} \lambda^i \lambda^j \lambda^k - \frac{i}{3} c_{ijk} \lambda^i \lambda^j \lambda^k - ic^i_{jk} \lambda^k.
\tag{B.8}
\]

After reordering the operators these expressions agree with that for \( \tilde{\mathcal{Q}} \) in the text.

It is straightforward, though quite tedious, to obtain this expression for \( \tilde{\mathcal{Q}} \) as the (hermitian) Noether charge for the second supersymmetry

\[
\delta_{\tilde{\epsilon}} X^a = -i \epsilon I_b^a \lambda^b \quad \delta_{\tilde{\epsilon}} \lambda^a = -\epsilon [I_b^a \dot{X}^b - i \lambda^c (\partial_c I_b^a) \lambda^b].
\tag{B.9}
\]

We note that the extra term in the transformation of \( \lambda \) not only appears naturally from the \( \mathcal{N} = 1 \) superspace formulation of [13], but also is necessary to obtain the algebra

\[
[\tilde{\delta}_{\tilde{\epsilon}}, \delta_{\eta}] = 0 \quad [\tilde{\delta}_{\tilde{\epsilon}}, \tilde{\delta}_{\eta}] = -2i \epsilon \eta \frac{d}{dt},
\tag{B.10}
\]

for \( I \) a complex structure with vanishing Nijenhuis tensor. Note that equation (B.9) implies equation (B.5).

**Appendix C. More on the Geometry of Weak HKT Manifolds**

In this appendix we will show that, given a weak HKT manifold with integrable complex structures, we can find a potential \( L \). We will prove this shortly but first we should elaborate on the assumption that the quaternionic structure is integrable.

It is well known that, for almost complex manifolds, the almost complex structure is integrable—that is, there exists a coordinate system in which the components of the almost complex structure are constant—if and only if the Nijenhuis tensor vanishes. The analogous
statement is not true for quaternionic manifolds; rather, the vanishing of the six Nijenhuis concomitants on an almost quaternionic manifold only guarantees the integrability of any one complex structure. To see this (see also [21]), suppose that we work in a complex coordinate system adapted to $I^3$. Then, $I^1$ and $I^2$ have only mixed indices (i.e., as forms they are $(2,0) \oplus (0,2)$ forms). Now consider the connection [22,18]

$$ C^k_{ij} = I^k_i \partial_i I^l_j \quad \text{and} \quad \text{c.c.} \quad (C.1) $$

The vanishing of the Nijenhuis tensor implies that $C^k_{ij}$ is actually a symmetric connection. Furthermore, this connection vanishes in a basis (if one exists) in which $I^1$ and $I^3$ are simultaneously constant, and so its curvature tensor vanishes in such a basis. Thus, a necessary condition for integrability of the quaternionic structure is the vanishing of the curvature associated with the connection $(C.1)$. Obata [22] has shown that this is also a sufficient condition.

If we assume integrability of the quaternionic structure, then we can, without loss of generality, work in a basis in which the complex structures are given by

$$ I^1 = id\bar{w}^A \otimes \frac{\partial}{\partial z^A} - id\bar{z}^A \otimes \frac{\partial}{\partial w^A} - idw^A \otimes \frac{\partial}{\partial \bar{z}^A} + idz^A \otimes \frac{\partial}{\partial \bar{w}^A} $$

$$ I^2 = dw^A \otimes \frac{\partial}{\partial z^A} - dz^A \otimes \frac{\partial}{\partial w^A} + dw^A \otimes \frac{\partial}{\partial \bar{z}^A} - dz^A \otimes \frac{\partial}{\partial \bar{w}^A} \quad (C.2) $$

$$ I^3 = idz^A \otimes \frac{\partial}{\partial z^A} + idw^A \otimes \frac{\partial}{\partial w^A} - idz^A \otimes \frac{\partial}{\partial \bar{z}^A} - idw^A \otimes \frac{\partial}{\partial \bar{w}^A}, $$

where we have split up the complex coordinates into two sets $(z^A, w^A)$, $A = 1, \ldots, \frac{N}{4}$. Hermiticity of the metric with respect to $I^1$ (we do not get any additional information from $I^2$) implies that

$$ g_{z^A \bar{z}^B} = g_{w^A \bar{w}^B}; \quad g_{z^A \bar{w}^B} = -g_{z^B \bar{w}^A} = g_{z^A \bar{w}^B}. \quad (C.3) $$

\[11\] This is not identical to equation (3.27). Equation (3.27) was written in a general coordinate system, whereas the following equation is written in coordinates adapted to the $I^3$. Thus, $C$ is a connection, provided one restricts oneself to holomorphic coordinate transformations, for $C$ depends implicitly on $I^3$, while $\Omega^1$ did not.
The condition that $d^1 J^1 = d^3 J^3$ then becomes:

\begin{align*}
  g_z^{|A, \bar{w}B, wC]} = 0 & \quad g_w^{|A, \bar{z}B, wC]} = 0 \\
  g_w^{|A, \bar{z}B, \bar{z}C]} = 0 & \quad g_z^{|A, \bar{w}B, \bar{z}C]} = 0 \\
  g_z^{|A, \bar{z}|C|, \bar{w}B]} - \frac{1}{2} g_z^{|A, \bar{w}B, \bar{z}C]} = 0 & \quad g_z^{|A, \bar{z}|B, wC]} + \frac{1}{2} g_w^{|B, wC, \bar{z}A]} = 0 \\
  g_z^{|A, \bar{z}|B, \bar{z}C]} - \frac{1}{2} g_w^{|B, wC, \bar{w}A]} = 0 & \quad g_z^{|A, \bar{z}|C|, \bar{z}B]} + \frac{1}{2} g_z^{|A, \bar{w}B, wC]} = 0.
\end{align*}

(C.4)

The first and second lines of equation (C.4), when combined with the antisymmetry in $A$, $B$ of $g_z^{|A, \bar{w}B|}$, allow us to write

\begin{align*}
  g_z^{|A, \bar{w}B|} &= (\partial_A \partial_{\bar{z}B} - \partial_Z \partial_{\bar{w}A}) L; &
  g_w^{|A, \bar{z}B|} &= (\partial_{wA} \partial_{\bar{z}B} - \partial_{wB} \partial_{\bar{z}A}) L
\end{align*}

(C.5)

where $L$ is some real (by hermiticity of the metric—and therefore identical in the two equations (C.5)) function. Inserting equation (C.5) into the third equation of (C.4) gives

\begin{equation}
\partial_{\bar{w}B} (g_z^{|A, \bar{z}C|} - L_{\bar{z}A, \bar{z}C}) - (B \leftrightarrow A) = 0,
\end{equation}

(C.6)

and therefore,

\begin{equation}
  g_z^{|A, \bar{z}B|} = L_{\bar{z}A, \bar{z}B} + \partial_{wA} G_{\bar{z}B}
\end{equation}

(C.7)

for some integration one-form $G_{\bar{z}B}$. Combining this with the fourth equation of (C.4) gives $G_{\bar{z}B} = L_{\bar{w}B}$. Thus we have obtained equation (3.59), which is the desired result.

We have shown that integrability of the quaternionic structure implies the existence of a potential $L$ for the metric. Although equation (3.59) holds only in a coordinate system in which the quaternionic structures are constant, equation (3.58) is coordinate invariant. Equation (3.53) motivates us to ask whether or not the existence of a potential $L$ obeying equation (3.58) is generically implied by the weak HKT conditions, independent of integrability of the quaternionic structure.

---

\footnote{Again, we do not get any additional information from $J^2$, since $d^3 J^3$ is $(1,2) \oplus (2,1)$ and the $(2,1)$ and $(1,2)$ parts of $d^2 J^2$ are trivially equal to those of $d^3 J^1$ and the $(0,3)$ and $(3,0)$ parts of $d^2 J^2$ are just minus those of $d^3 J^1$.}
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