The simplicial interpretation of bigroupoid 2-torsors

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Abstract

Actions of bicategories arise as categorification of actions of categories. They appear in a variety of different contexts in mathematics, from Moerdijk’s classification of regular Lie groupoids in foliation theory [34] to Waldmann’s work on deformation quantization [38]. For any such action we introduce an action bicategory, together with a canonical projection (strict) 2-functor to the bicategory which acts. When the bicategory is a bigroupoid, we can impose the additional condition that action is principal in bicategorical sense, giving rise to a bigroupoid 2-torsor. In that case, the Duskin nerve of the canonical projection is precisely the Duskin-Glenn simplicial 2-torsor, introduced in [25].
1 Introduction

There are several different ways to characterize those simplicial sets which arise as nerves of categories, and the most of this (equivalent) ways rely on the Quillen closed model structure on the category $\mathcal{S}Set$ of simplicial sets. Simplicial sets which are fibrant objects for the closed model structure on $\mathcal{S}Set$ are called Kan complexes, and they are characterized by certain horn filling conditions describing their exactness properties. This conditions for a simplicial set $X_\bullet$ explicitly use a simplicial kernel $K_n(X_\bullet)$ in dimension $n$

$$K_n(X_\bullet) = \{(x_0, x_1, \ldots, x_i, \ldots, x_n-1, x_n)|d_i(x_j) = d_{j-1}(x_i), i < j\} \subseteq X_{n+1}^n$$

which is interpreted as the set of all possible sequences of (n-1)-simplices which could possibly be the boundary of any n-simplex. There exists a natural boundary map

$$\partial_n: X_n \to K_n(X_\bullet) \quad (1.1)$$

which takes any n-simplex $x \in X_n$ to the sequence $\partial_n(x) = (d_0(x), d_1(x), \ldots, d_{n-1}(x), d_n(x))$ of its (n-1)-faces. The set $\bigwedge_n^k(X_\bullet)$ of k-horns in dimension $n$

$$\bigwedge_n^k(X_\bullet) = \{(x_0, x_1, \ldots, x_k-1, x_{k+1}, \ldots, x_{n-1}, x_n)|d_i(x_j) = d_{j-1}(x_i), i < j, i, j \neq k\} \subseteq X_{n-1}^n$$

is the set of all possible sequences of (n-1)-simplices which could possibly be the boundary of any n-simplex, except that we $k^{th}$ face is missing. The k-horn map in dimension $n$

$$p_n^k(x): X_n \to \bigwedge_n^k(X_\bullet) \quad (1.2)$$

is defined by the composition of the boundary map (1.1), with the natural projection $q_n^k(x): K_n(X_\bullet) \to \bigwedge_n^k(X_\bullet)$, which just omits the $k^{th}$ (n-1)-simplex from the sequence. Then we say that for $X_\bullet$ the $k^{th}$ Kan condition in dimension $n$ is satisfied (exactly) if the k-horn map (1.2) is surjection (bijection). If Kan conditions are satisfied for all $0 < k < n$ and for all $n$, then we say that $X_\bullet$ is a weak Kan complex, and if Kan conditions are satisfied for extremal horns as well $0 \leq k \leq n$ and for all $n$, then we say that $X_\bullet$ is a Kan complex.

One of the above mentioned characterizations of nerves of categories, first observed by Street, is that the simplicial set $X_\bullet$ is the nerve of a category if and only if it is a weak Kan complex in which the weak Kan conditions are satisfied exactly. Weak Kan complexes were introduced by Boardman and Vogt [12] in their work on homotopy invariant algebraic structures. These objects are fundamental in the recent work of Joyal [29], which is so far the most advanced form of the interplay between the category theory and the simplicial theory. He even used the name quasicategory, instead of the weak Kan complex, in order to emphasize that "most concepts and results of category theory can be extended to quasicategories".
Similar characterization of nerves of groupoids leads to the fundamental simplicial objects introduced by Duskin in [21]. An \( n \)-dimensional Kan hypergroupoid, is a Kan complex \( X_\bullet \) in which Kan conditions (1.2) are satisfied exactly for all \( m > n \) and \( 0 \leq k \leq m \). Glenn used the name \( n \)-dimensional hypergroupoid in [25] for any simplicial set in which Kan conditions are satisfied exactly above dimension \( n \), while Beke called them in [9] exact \( n \)-types, in order to emphasize their homotopical meaning. These simplicial sets morally play the role of nerves of weak \( n \)-groupoids, which is known to be valid for small \( n \). Consequently, a simplicial set \( X_\bullet \) is the nerve of a groupoid if and only if it is a 1-dimensional Kan hypergroupoid, and similar characterization holds for nerves of bigroupoids.

Bigroupoids and bicategories, introduced by Bénabou [10] in 1967, are weakest possible generalization of ordinary groupoids and categories, respectively, to the immediate next level. In a bicategory (bigroupoid), Hom-sets become categories (groupoids) and the composition becomes functorial instead of functional. This changes properties of associativity and identities which only hold up to coherent natural isomorphisms. The coherence laws which this natural isomorphisms satisfy, are the deep consequence of the process called categorification, invented by Crane [18], [19], in which we find category theoretic analogs of set theoretic concepts by replacing sets with categories, equations between elements of the sets by isomorphisms between objects of the category, functions by functors and equations between functions by natural isomorphisms between functors.

The categorification become an essential tool in many areas of modern mathematics. By generalizing algebraic concepts from the classical set theory to the context of higher category theory, Baez developed a program [3] of higher dimensional algebra in an attempt to unify quantum field theory with traditional algebraic topology. Later, Baez and Schreiber developed a higher gauge theory [4], [5] which describes the parallel transport of strings using 2-connections on principal 2-bundles, as the categorification of the usual gauge theory which describes the parallel transport of point particles using connections on principal bundles. Vector 2-spaces arose as a categorification of vector spaces in the work of Kapranov and Voevodsky [32], and they were used by Baas, Dundas and Rognes [2], who defined vector 2-bundles in a search for a geometrically defined elliptic cohomology. Later, Baas, Bökstedt and Kro used topological bicategories and vector 2-bundles [1] in order to develop 2-categorical K-theory as the categorification of the usual K-theory.

Another essential tool which we used is an internalization. This is a process of generalizing concepts from the category \( \text{Set} \) of sets, which are described in terms of sets, functions and commutative diagrams, to concepts in another category \( \mathcal{E} \) by describing them in terms of objects, morphisms, and commutative diagrams in \( \mathcal{E} \). The internalization of the particular algebraic or geometric structure in the category \( \mathcal{E} \) rely on exactness properties of \( \mathcal{E} \) needed to describe corresponding commutative diagrams. Therefore, the choice of the category \( \mathcal{E} \) will depend on the algebraic or geometric structure one wants to describe.

The most natural choice for an internalization and a categorification of algebraic and geometric structures is a topos, which is according to Grothendieck, the ultimate generalization of the concept of space.
Let us now describe the content and the main results of the paper.

In Chapter 2 we recall some basic simplicial methods which we will extensively use in the thesis. Most of this material is standard and can be found in a classical book [33] by May, or in a modern treatment in [26]. However, we also recall some more exotic endofunctors on a category $SSet$ of simplicial sets, such as the $n$-Coskeleton $\text{Cosk}^n$ and the shift functor or décalage $\text{Dec}$ which can be find in [20]. Actions and n-torsors over n-dimensional Kan hypergroupoids are defined by Glenn in [25] using simplicial maps which we call exact fibrations. A simplicial map $\lambda_\bullet: E_\bullet \rightarrow B_\bullet$ is an exact fibration in dimension $n$, if for all $0 \leq k \leq n$, the diagrams

\[
\begin{array}{ccc}
E_n & \xrightarrow{\lambda_n} & B_n \\
\downarrow p_k & & \downarrow p_k \\
\text{\wedge}^k_n(E_\bullet) & \rightarrow & \text{\wedge}^k_n(B_\bullet)
\end{array}
\]

are pullbacks. It is called an exact fibration if it is an exact fibration in all dimensions. At the end of this chapter, we describe two crucial concepts from [25] which we will use later in the thesis. An action of the n-dimensional hypergroupoid $B_\bullet$ is given in Definition 2.13 as a simplicial map $\lambda_\bullet: P_\bullet \rightarrow B_\bullet$ which is an exact fibration for all $m \geq n$, and an n-dimensional hypergroupoid n-torsor over $X$ in $E$ is given in Definition 2.14 as a simplicial map $\lambda_\bullet: P_\bullet \rightarrow B_\bullet$ such that $P_\bullet$ is augmented over $X$, aspherical and $n-1$-coskeletal.

In Chapter 3, definitions of a bicategory, their homomorphisms, pseudonatural transformations and modifications are given as they were defined by Bénabou in his classical paper [10]. Then Chapter 4 describes the Duskin nerve for bicategories as a geometric nerve defined by the singular functor of the fully faithful embedding

\[
i: \Delta \rightarrow \text{Bicat}
\]

of the skeletal simplicial category $\Delta$ into the category $\text{Bicat}$ of bicategories and strictly unital homomorphism of bicategories, constructed by Bénabou in [10]. This embedding regards any ordinal $[n]$ as the locally discrete 2-category, in the sense that Hom-categories are discrete, so there exist only trivial 2-cells. We show that the Duskin nerve functor

\[
N_2: \text{Bicat} \rightarrow SSet
\]

is fully faithful in Theorem 4.1 based on the result that the geometric nerve provides a fully faithful functor on the category $2-\text{Cat}_{lax}$ of 2-categories and normal lax 2-functors given in [11]. The sets of n-simplices of the nerve $N_2B$ of a bicategory $B$ are defined by $\text{Hom}_{\text{Bicat}}(i[n],B)$, which were explicitly described by Duskin [24] in a geometric form.
In Chapter 5, we introduce the second new concept of this paper, action of a bicategory, in Definition 5.1 as a categorification of an action of a category. For an internal bicategory \( \mathcal{B} \) given by a bigraph in a finitely complete category \( \mathcal{E} \), and an internal category \( \mathcal{P} \),

\[
\begin{array}{c}
P_1 \downarrow t \downarrow \Lambda_0 \\
\downarrow s \downarrow t_0 \\
P_1 \downarrow s_0 \\
B_0 \\
\end{array}
\]

(1.5)

together with the momentum functor \( \Lambda : \mathcal{P} \to \mathcal{B}_0 \) to a discrete category \( \mathcal{B}_0 \) of objects of the bicategory \( \mathcal{B} \), an action functor

\[
A : \mathcal{P} \times \mathcal{B}_0 \mathcal{B}_1 \to \mathcal{P}
\]

is a categorification of an action of the category. We introduce coherence laws for this action, which express the fact that categories with an action of the bicategory \( \mathcal{B} \) are pseudoalgebras over a pseudomonad \( [28, 30, 31] \) naturally defined by \( \mathcal{B} \). We give a description of an Eilenberg-Moore 2-category of actions of the bicategory \( \mathcal{B} \), without details of the construction for corresponding pseudoalgebras over a pseudomonad. In Chapter 6, for each action (1.5) of a bicategory \( \mathcal{B} \) on a category \( \mathcal{P} \), we define the third new concept, an action bicategory \( \mathcal{P} \ltimes \mathcal{B} \) whose construction is given in Theorem 6.1. Then we see in Proposition 6.1 that an action bicategory \( \mathcal{P} \ltimes \mathcal{B} \) comes with a canonical projection

\[
\Lambda : \mathcal{P} \ltimes \mathcal{B} \to \mathcal{B}
\]

(1.7)
to the bicategory \( \mathcal{B} \), which is a strict homomorphism of bicategories.

Finally, in Chapter 7 we define the fourth new concept, and our main geometric object - a bigroupoid 2-torsor. In Definition 7.2 we define a bigroupoid 2-torsor as a bundle of groupoids \( \pi : \mathcal{P} \to X \) over an object \( X \) in the category \( \mathcal{E} \), for which the induced functor

\[
(Pr_1, A) : \mathcal{P} \times \mathcal{B}_0 \mathcal{B}_1 \to \mathcal{P} \times X \mathcal{P}
\]

(1.8)

for an action (1.5) is a strong equivalence of groupoids. The first main result of the paper is Theorem 7.1 in Chapter 7 which proves that for an action (1.5) of an internal bigroupoid \( \mathcal{B} \) on groupoid \( \mathcal{P} \), the simplicial map \( \Lambda_* = N_2(\Lambda) : \mathcal{Q}_* \to \mathcal{B}_* \) which arise as an application of a Duskin nerve for bicategories (1.6) on a canonical homomorphism of bicategories (1.7) is a (simplicial) action of the bigroupoid \( \mathcal{B} \) on the groupoid \( \mathcal{P} \), i.e. it is an exact fibration for all \( n \geq 2 \). The second main result of the paper is Theorem 7.2 which proves that for any \( \mathcal{B} \)-2-torsor \( \mathcal{P} \) over \( X \), the simplicial map \( \Lambda_* = N_2(\Lambda) : \mathcal{Q}_* \to \mathcal{B}_* \) is a Glenn’s 2-torsor, which is an internal simplicial map \( \Lambda_* : \mathcal{P}_* \to \mathcal{B}_* \) in \( \mathcal{S}(\mathcal{E}) \), which is an exact fibration for all \( n \geq 2 \), and where \( \mathcal{P}_* \) is augmented over \( X \), aspherical and 1-coskeletal (\( \mathcal{P}_* \simeq \text{Cosk}^1(\mathcal{P}_*) \)).
2 Simplicial objects

In this section we will review some standard notions from the theory of simplicial sets. Most of the statements and proofs may be found in standard textbooks [26] or [33].

Definition 2.1. Skeletal simplicial category $\Delta$ consists of the following data:

- objects are finite nonempty ordinals $[n] = \{0 < 1 < ... < n\}$,
- morphisms are monotone maps $f: [n] \to [m]$, which for all $i, j \in [n]$ such that $i \leq j$, satisfy $f(i) \leq f(j)$.

We also call $\Delta$ the topologist’s simplicial category, and this is a full subcategory of the algebraist’s simplicial category $\overline{\Delta}$, which has an additional object $[-1] = \emptyset$, given by a zero ordinal, that is an empty set.

Skeletal simplicial category $\Delta$ may be also given by means of generators given by the diagram

$$
\begin{array}{cccc}
0 & \xrightarrow{\partial_1} & 1 & \xrightarrow{\partial_2} \quad 2 & \xrightarrow{\partial_3} \quad 3 \\
\xrightarrow{\partial_0} & & \xrightarrow{\partial_0} & & \xrightarrow{\partial_0} \\
\end{array}
$$

and relations given by the maps $\partial_i: [n - 1] \to [n]$ for $0 \leq i \leq n - 1$, called coface maps, which are injective maps that omit $i$ in the image, and the maps $\sigma_i: [n] \to [n - 1]$ for $0 \leq i \leq n - 1$, called codegeneracy maps, which are surjective maps which repeat $i$ in the image. These maps satisfy following cosimplicial identities:

$$
\begin{align*}
\partial_j \partial_i &= \partial_i \partial_{j-1} & (i < j) \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1} & (i \leq j) \\
\sigma_j \partial_i &= \partial_i \sigma_{j-1} & (i < j) \\
\sigma_j \partial_i &= \partial_i \sigma_{j+1} & (i > j + 1) \\
\end{align*}
$$

We will use the following factorization of monotone maps by means of cofaces and codegeneracies.

Lemma 2.1. Any monotone map $f: [m] \to [n]$ has a unique factorization given by

$$
f = \partial_{i_1}^{m} \sigma_{i_2}^{m-1} ... \partial_{i_s}^{m-s+1} \sigma_{i_1}^{m-t} ... \partial_{i_t}^{m-t} \sigma_{j_1}^{m-1}
$$

where $0 \leq i_s < i_{s-1} < ... < i_1 \leq n$, $0 \leq j_t < j_{t-1} < ... < j_1 \leq m$ and $n = m - t + s$.

Proof. The proof follows directly from the injective-surjective factorization in Set and simplicial identities. \qed
Definition 2.2. Simplicial object \( X \) in a category \( C \) is a functor \( X : \Delta^{op} \to C \). This is an object of the category \( S(C) \) whose morphisms are natural transformations, which we call internal simplicial morphisms. In the case when the category \( C = \text{Set} \) is the category of sets (in a fixed Grothendieck universe), then we call \( X \) a simplicial set, and we denote the corresponding category of simplicial sets by \( S\text{Set} \).

Thus we can view a simplicial object \( X \) in \( C \) as a diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{d_1} & X_1 \\
\downarrow{d_0} & & \downarrow{d_0} \\
X_2 & \xrightarrow{d_3} & X_3 \\
\end{array}
\]

in \( C \), where we denoted just extremal face operators, and left the signature for inner face operators, and degeneracies.

Then the following simplicial identities hold:

\[
\begin{align*}
d_i d_j &= d_{j-1} d_i & (i < j) \\
s_i s_j &= s_{j+1} s_i & (i \leq j) \\
d_i s_j &= s_{j-1} d_i & (i < j) \\
d_i s_j &= \text{id} & (i = j, i = j + 1) \\
d_i s_j &= s_{j+1} d_i & (i > j + 1)
\end{align*}
\]

where \( d_i := X(\partial_i) \) and \( s_i := X(\sigma_i) \).

Definition 2.3. An augmented simplicial object \( X \) in a category \( C \) is a functor \( X : \bar{\Delta}^{op} \to C \). This is an object of the category \( S_a(C) \) whose morphisms are natural transformations, which we call simplicial maps of augmented simplicial objects.

In order to define basic endofunctors on the category \( S(C) \), which we will use in the thesis, we first need to describe the process of a truncation of internal simplicial objects. For any natural number \( n \), we have the full subcategory \( \Delta_n \) of the simplicial category \( \Delta \), whose objects are the first \( n + 1 \) ordinals. Then we have the following definition.

Definition 2.4. Let \( X \) be a simplicial object in \( C \). An \( n \)-truncated simplicial object \( \text{tr}^n(X) \) in a category \( C \) is a functor \( X_i : \Delta_n^{op} \to C \) given by the precomposition with an embedding \( i_n : \Delta_n \to \Delta \). This is an object of the category \( S^n(C) \), and we have an \( n \)-truncation functor

\[
\text{tr}^n : S(C) \to S^n(C)
\]

from the category \( S(C) \) of simplicial objects in \( C \), to the category \( S^n(C) \) of \( n \)-truncated simplicial objects in \( C \).

If \( C \) is a finitely complete category, an \( n \)-truncation functor \( \text{tr}^n : S(C) \to S^n(C) \) has a right adjoint \( \text{cosk}^n : S^n(C) \to S(C) \), and if \( C \) is a finitely cocomplete category, it has a left adjoint \( \text{sk}^n : S^n(C) \to S(C) \).
The corresponding comonad $Sk^n = sk^n tr^n: SSet \to SSet$ for $\mathcal{C} = \text{Set}$ is easy to describe. For any simplicial set $X_\bullet$, its skeleton $Sk^n(X_\bullet)$ is a simplicial subset of $X_\bullet$, which is identical to $X_\bullet$ in all dimensions $k \leq n$, and has only degenerate simplices in all higher dimensions.

The monad $Cosk^n = cosk^n tr^n: S(\mathcal{C}) \to S(\mathcal{C})$ is described by the simplicial kernel.

**Definition 2.5.** The $n^{th}$ simplicial kernel of the simplicial object $X_\bullet$ is an object $K_n(X_\bullet)$ in $\mathcal{C}$, together with morphisms $pr_j: K_n(X_\bullet) \to X_{n-1}$ for $j = 0, \ldots, n$, which is universal with respect to relations $d_i pr_j = pr_{j-1} d_i$, for all $0 \leq i < j \leq n$.

Now, let we describe in more detail the monad $Cosk^n = cosk^n tr^n: SSet \to SSet$ in the case $\mathcal{C} = \text{Set}$, that is when we deal with simplicial sets.

The simplicial kernel of the simplicial set $X_\bullet$ in dimension $n$ is a set $K_n(X_\bullet)$ defined by

$$K_n(X_\bullet) = \{(x_0, x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{n-1}, x_n) | d_i(x_j) = d_{j-1}(x_i), \ i < j \} \subseteq X_{n-1}^n$$

so that we can interpret it as the set of all possible sequences of $(n-1)$-simplices which could possibly be the boundary of any $n$-simplex. If $x \in X_n$ is an $n$-simplex in a simplicial set $X_\bullet$, its boundary $\partial_n(x)$ is a sequence of its $(n-1)$-faces

$$\partial_n(x) = (d_0(x), d_1(x), \ldots, d_{n-1}(x), d_n(x)).$$

Then, for the simplicial set $X_\bullet$, the simplicial set $Cosk^n(X_\bullet)$ is identical to $X_\bullet$ in all dimensions $k \leq n$, and the set of $(n+1)$-simplices of $Cosk^n(X_\bullet)$ is defined by

$$Cosk^n(X_\bullet)_{n+1} = K_{n+1}(X_\bullet)$$

while the face operators are given by the projections $d_i = pr_i: K_{n+1}(X_\bullet) \to X_n$ for all $0 \leq i \leq n + 1$. All of the higher dimensional set of simplices of $Cosk^n(X_\bullet)$ are obtained just by inductively iterating the simplicial kernels

$$Cosk^n(X_\bullet)_{n+2} = K_{n+2}(tr^{n+1}Cosk^n(X_\bullet))$$

and so on.

From the universal property of the $n^{th}$ simplicial kernel $K_n(X_\bullet)$, we have a canonical morphism $\delta_n = (d_0, d_1, \ldots, d_{n-1}, d_n): X_n \to K_n(X_\bullet)$, called the boundary of the object of $n$-simplices, or briefly the $n^{th}$ boundary morphism.

The first nontrivial component of the unit $\eta: Id_{S\mathcal{C}} \to Cosk^n$ of the adjunction is given by $(n + 1)^{th}$ boundary morphism

$$\delta_{n+1} = (d_0, d_1, \ldots, d_n, d_{n+1}): X_{n+1} \to Cosk^n(X_\bullet)_{n+1} = K_{n+1}(X_\bullet)$$

and we have following definitions.
Definition 2.6. We say that the simplicial object $X_\bullet$ in $\mathcal{C}$ is coskeletal in dimension $n$, or $n$-coskeletal, if the unit $\eta: \text{Id}_{\mathcal{S}_{\text{Set}}} \to \text{Cosk}^n$ of the adjunction is a natural isomorphism. Similarly, we say that the simplicial object $X_\bullet$ in $\mathcal{C}$ is skeletal in dimension $n$, or $n$-skeletal, if the counit $\epsilon: \text{Sk}^n \to \text{Id}_{\mathcal{S}_{\text{Set}}}$ of the adjunction is a natural isomorphism.

Definition 2.7. We say that the simplicial object $X_\bullet$ in $\mathcal{C}$ is aspherical in dimension $n$ if the $n^{\text{th}}$ boundary morphism $\delta_n: X_n \to K_n(X_\bullet)$ is an epimorphism. If $X_\bullet$ is aspherical in all dimensions, then we say that it is aspherical.

In order to define Kan complexes later, we use another universal construction which formally describe ‘hollow’ simplices, or simplices in which the $k^{\text{th}}$ face is missing.

Definition 2.8. The $k$-horn in dimension $n$ of the simplicial object $X_\bullet$ is an object $\bigwedge_n^k(X_\bullet)$ in $\mathcal{C}$, together with morphisms $p_i: \bigwedge_n^k(X_\bullet) \to X_{n-1}$ for $i = 0, \ldots, n$ and $i \neq k$, which is universal with respect to relations $d_i p_j = p_{j-1} d_i$, for all $0 \leq i < j \leq n$ and $i,j \neq k$.

The set $\bigwedge_n^k(X_\bullet)$ of k-horns in dimension $n$

$$\bigwedge_n^k(X_\bullet) = \{(x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1}, x_n)|d_i(x_j) = d_{j-1}(x_i), i < j, i,j \neq k\} \subseteq X_{n-1}$$

is the set of all possible sequences of (n-1)-simplices which could possibly be the boundary of any n-simplex, except that we $k^{\text{th}}$ face is missing. Then for the simplicial set $X_\bullet$, the $k$-horn map in dimension $n$

$$p_n^k(x): X_n \to \bigwedge_n^k(X_\bullet)$$

is defined by the composition of the boundary map $\partial_n: X_n \to K_n(X_\bullet)$, with the projection $q_n^k(x): K_n(X_\bullet) \to \bigwedge_n^k(X_\bullet)$, and it just omits the $k^{\text{th}}$ (n-1)-simplex from the sequence.

If $x \in X_n$ is an n-simplex, its k-horn $p_n^k(x)$ is defined by the image of the projection of its boundary to the sequence of faces in which the $k^{\text{th}}$ face is omitted

$$p_n^k(x) = (d_0(x), d_1(x), \ldots, d_{k-1}(x), d_{k+1}(x), \ldots, d_{n-1}(x), d_n(x))$$

Let $(x_0, x_1, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_{n-1}, x_n) \in \bigwedge_n^k(X_\bullet)$ be a k-horn in dimension $n$. If there exists an n-simplex $x \in X_n$ such that

$$p_n^k(x) = (x_0, x_1, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_{n-1}, x_n)$$

then we say that n-simplex $x$ is a filler of the horn.

Definition 2.9. Let $X_\bullet$ be an simplicial object in the category $\mathcal{C}$. We say that the $k^{\text{th}}$ Kan condition in dimension $n$ is satisfied for $X_\bullet$ if the k-horn morphism

$$p_n^k(x): X_n \to \bigwedge_n^k(X_\bullet)$$
is an epimorphism. The condition is satisfied exactly if the above morphism is an isomorphism. If Kan conditions are satisfied for all $0 < k < n$ and for all $n$, then we say that $X_\bullet$ is a weak Kan complex. Finally, if Kan conditions are satisfied for extremal horns as well $0 \leq k \leq n$ and for all $n$, then we say that $X_\bullet$ is a Kan complex.

This condition can be stated entirely in the topos theoretic context by using the sieves

$$\bigwedge^k [n] \hookrightarrow \Delta[n] \hookrightarrow \Delta[n]$$

in $\mathcal{S}Set$, where $\Delta[n]$ is the standard $n$-simplex, which is just the simplicial set represented by the ordinal $[n]$. The simplicial set $\Delta[n]$ is the boundary of the standard $n$-simplex which is identical to standard $n$-simplex in all dimensions below $n$, and has only degenerate simplices in higher dimensions. It is defined by the $(n-1)$-skeleton $\Delta[n] = S^{n-1}(\Delta[n])$ of the standard $n$-simplex. The simplicial set $\bigwedge^k [n]$ is the $k$-horn of the standard $n$-simplex, which is identical to $\Delta[n]$ except that it is not generated by the simplex $\delta_k : [n-1] \to [n]$.

Using the Yoneda lemma

$$\text{Hom}_{\mathcal{S}Set}(\Delta[n], X_\bullet) \simeq X_n$$

the $n^{th}$ Kan condition says that for any simplicial map $\bar{x} : \bigwedge^k [n] \to X_\bullet$, there exist a simplicial map $x : \Delta[n] \to X_\bullet$ such that the diagram

$$\begin{array}{ccc}
\bigwedge^k [n] & \xrightarrow{\bar{x}} & X_\bullet \\
\downarrow & & \downarrow x \\
\Delta[n] & \xrightarrow{x} & X_\bullet
\end{array}$$

commutes.

**Remark 2.1.** The $n^{th}$ Kan condition is equivalent to the injectivity of the simplicial set $X_\bullet$ with respect to monomorphisms $\bigwedge^k [n] \hookrightarrow \Delta[n]$ for all $0 \leq k \leq n$. In this terms, Kan complex $X_\bullet$ is a simplicial set which is injective with respect to all monomorphisms $\bigwedge^k [n] \hookrightarrow \Delta[n]$ for all $0 \leq k \leq n$, and all $n \geq 0$.

**Proposition 2.1.** Every aspherical simplicial object $X_\bullet$ is a Kan simplicial object.

*Proof.* We will use the Barr embedding theorem and prove it in Set. Consider the diagram
and a k-horn \((x_0, x_1, \ldots, x_{k-1}, -x_{k+1}, \ldots, x_n, x_{n+1}) \in \bigwedge^k_{n+1}(X_*)\). If there exists a filler \(x \in X_{n+1}\) for which \(p^k_{n+1}(x) = (x_0, x_1, \ldots, x_{k-1}, -x_{k+1}, \ldots, x_n, x_{n+1})\) then its k-face \(d_k(x) = x_k\) has a boundary uniquely determined by the simplices \(x_i\) for \(i \neq k\) since

\[
d_i(x_k) = \begin{cases} 
d_{k-1}(x_i) & 0 \leq i < k \leq n + 1 \\
d_k(x_{i+1}) & 0 \leq k \leq i \leq n + 1
\end{cases}
\]

and therefore \((d_0(x_k), d_1(x_k), \ldots, d_{n-1}(x_k), d_n(x_k)) \in K_n(X_*)\). Since we supposed that \(\delta_n : X_n \to K_n(X_*)\) is an epimorphism, then such a simplex \(x_k \in X_n\) really exists, and we conclude that the morphism \(q^k_{n+1} : K_{n+1}(X_*) \to \bigwedge^k_{n+1}(X_*)\) is also an epimorphism. But this is true for all \(n\), and it follows that \(p^k_{n+1} : X_{n+1} \to \bigwedge^k_{n+1}(X_*)\) is an epimorphism as a composition of epimorphisms, and therefore \(X_*\) is a Kan simplicial set.

\[\square\]

**Remark 2.2.** For any simplicial set \(X_*\) the simplicial kernel \(K_1(X_*)\) in dimension 1 is equal to the product \(K_1(X_*) = X_0 \times X_0\). For the augmented simplicial set \(X_0 \to X_{-1}\), when we have \(K_1(X_*) = X_0 \times X_{-1} X_0\). The set of k-horns is given by \(\bigwedge^k_{-1}(X_*) = X_0\) for \(k = 0, 1\), and in each case maps \(p^k_1 : X_1 \to \bigwedge^k_{-1}(X_*)\) and \(q^k_1 : K_1(X_*) \to \bigwedge^k_{-1}(X_*)\) are always epimorphisms.

**Definition 2.10.** A simplicial object \(X_*\) in \(\mathcal{C}\) is said to be split if there exist a family of morphisms \(s_{n+1} : X_n \to X_{n+1}\) for all \(n \geq 0\), called the contraction for \(X_*\), which satisfy all the simplicial identities involving degeneracies. When a simplicial object is augmented \(p : X_0 \to X_{-1}\) then the contraction includes also a morphism \(s_0 : X_{-1} \to X_0\) such that \(p s_0 = \text{id}_{X_{-1}}\).

**Remark 2.3.** Any augmented split simplicial set \(X_* \to X_{-1}\) may be seen as the simplicial set \(X_*\) together with the homotopy equivalence \(d_* : X_* \to K(X_{-1}, 0)\) to the constant simplicial set \(K(X_{-1}, 0)\) which has \(X_{-1}\) at each dimension and the identity maps for faces and degeneracies. This means that there exists a simplicial map \(s_* : K(X_{-1}, 0) \to X_*\) such that the compositions \(s_0 d_* \simeq \text{id}_{X_*}\) and \(d_* s_* \simeq \text{id}_{K(X_{-1}, 0)}\) are homotopic to respective identity simplicial maps.

**Proposition 2.2.** Every augmented aspherical simplicial set \(X_* \to X_{-1}\) is split.

**Proof.** The proof follows by induction. Let’s take any section \(s_0 : X_{-1} \to X_0\) and we assume that we have the \(n^\text{th}\) contraction \(s_n : X_{n-1} \to X_n\). Let \(q_i(x) : X_n \to K_{n+1}(X_*)\) be the \(i^\text{th}\) degeneracy for the \(n^\text{th}\) simplicial kernel of \(X_*\), and we define \(q_{n+1}(x) : X_n \to K_{n+1}(X_*)\) by

\[
q_{n+1}(x) = (s_n d_0(x), s_n d_1(x), \ldots, s_n d_{n-1}(x), s_n d_n(x)).
\]

Now let’s choose the splitting \(s : K_{n+1}(X_*) \to X_{n+1}\) of the \((n + 1)^{\text{th}}\) boundary map \(\delta_{n+1}(x) : X_{n+1} \to K_{n+1}(X_*)\), which is a surjection by assumption, such that \(s_i = sq_i\) for all \(0 \leq i \leq n\). Then the contraction \(s_{n+1} : X_n \to X_{n+1}\) defined by \(s_{n+1} = sq_{n+1}\) satisfies all the identities involving degeneracies since \(q_{n+1} = \delta_{n+1} s q_{n+1} = \delta_{n+1} s_{n+1}\). \[\square\]
An $n$-truncation functor has the extension to the augmented $n$-truncation functor

$$tr^n_a: S_a(C) \rightarrow S^n_a(C)$$

from the category $S_a(C)$ of augmented simplicial objects in $C$ to the category $S^n_a(C)$ of $n$-truncated augmented simplicial objects in $C$. Since $C$ is finitely complete, it has a right adjoint $cosk^n_a: S^n_a(C) \rightarrow S_a(C)$, called the augmented $n$-coskeleton functor. If we regard any augmented simplicial object $X_\bullet \rightarrow X_{-1}$ in $C$ as the ordinary simplicial object in the slice category $(C, X_{-1})$, then the augmented $n$-coskeleton functor becomes ordinary $n$-coskeleton functor in the slice category $(C, X_{-1})$.

**Example 2.1.** The category $C$ may be identified with the category $S^{-1}_a(C)$ of $-1$-truncated augmented simplicial objects in $C$, and the augmented $-1$-truncation functor $tr^{-1}_a: S_a(C) \rightarrow S^{-1}_a(C)$ assigns to any augmented simplicial object $X_\bullet \rightarrow X_{-1}$ the object $X_{-1}$ of $C$. Its right adjoint is augmented $-1$-coskeleton functor $cosk^{-1}_a: S^{-1}_a(C) \rightarrow S_a(C)$ which assigns to any object $X$ in $C$ the constant augmented simplicial object

$$X \leftarrow id X \cong id X_1 \cong id X \cong id X \ldots$$

denoted by $K(X, 0) \rightarrow X$.

**Example 2.2.** The category of morphisms $C^I$ of $C$ may be identified with the category $S^0_a(C)$ of $0$-truncated augmented simplicial objects in $C$, and the augmented $0$-truncation functor $tr^0_a: S_a(C) \rightarrow S^0_a(C)$ assigns to any augmented simplicial object $X_\bullet \rightarrow X_{-1}$ the morphism $d: X_0 \rightarrow X_{-1}$ of $C$. Its right adjoint is augmented $0$-coskeleton functor $cosk^0_a: S^0_a(C) \rightarrow S_a(C)$ which assigns to any morphism $d: X_0 \rightarrow X_{-1}$ in $C$ the simplicial kernel of the morphism

$$X_{-1} \leftarrow d X_0 \cong pr_1 X_0 \times X_{-1} X_0 \cong pr_2 X_0 \times X_{-1} X_0 \times X_{-1} X_0$$

denoted by $cosk^0_a(X_0 \rightarrow X_{-1})$.

The corresponding monad and the comonad on the category $S_a(C)$ of augmented simplicial objects in $C$ are denoted by $Cosk_a: S_a(C) \rightarrow S_a(C)$ and $Sk_a: S_a(C) \rightarrow S_a(C)$ respectively, in accordance with the case of nonaugmented simplicial objects in $C$.

Another important construction on simplicial objects is given by the so called shift functor. For any simplicial object $X_\bullet$ in $C$, we restrict the corresponding functor $X: \Delta^{op} \rightarrow C$ to the subcategory of $\Delta^{op}$ with the same objects, and with the same generators except for the injections $\partial_n: [n-1] \rightarrow [n]$. If we renumber the objects in $\Delta^{op}$, so that the ordinal $[n-1]$ becomes $[n]$, we obtain a simplicial object in $C$, denoted by $Dec(X_\bullet)$, which is augmented.
to the object $X_0$ (or to the constant simplicial object $S_k^0(X_\bullet)$ in $C$) and is contractible with respect to the simplicial map obtained from the family $(s_n)_{n \geq 0}$ of extremal degeneracies, as is shown in the diagram

\[ \begin{array}{cccccc}
X_0 & \rightarrow & X_0 & \rightarrow & X_0 & \rightarrow & \cdots & S_k^0(X_\bullet) \\
\downarrow^{d_0} & & \downarrow^{d_0} & & \downarrow^{d_0} & & \downarrow^{d_0} \\
X_1 & \rightarrow & X_2 & \rightarrow & X_3 & \rightarrow & \cdots & Dec(X_\bullet) \\
\downarrow^{s_0} & & \downarrow^{s_1} & & \downarrow^{s_2} & & \downarrow^{s_3} \\
X_0 & \rightarrow & X_1 & \rightarrow & X_2 & \rightarrow & \cdots & X_\bullet
\end{array} \]

where the simplicial map $S_0: S_k^0(X_\bullet) \rightarrow Dec(X_\bullet)$ on the right side of the diagram is defined by $(S_0)_n = (s_0)^n = s_0s_0\cdots s_0$, and the simplicial map $D_0: Dec(X_\bullet) \rightarrow S_k^0(X_\bullet)$ is defined by $(D_0)_n = (d_0)^n = d_0d_0\cdots d_0$, for each level $n$. The other two simplicial maps $S_1: X_\bullet \rightarrow Dec(X_\bullet)$ and $D_1: Dec(X_\bullet) \rightarrow X_\bullet$ are defined by $(S_1)_n = s_n$ and $(D_1)_n = d_n$ respectively.

The above construction extends to a functor

\[ Dec: S(C) \rightarrow S_{as}(C) \]

from the category of simplicial objects in $C$, to the category $S_{as}(C)$ of augmented split simplicial objects in $C$. This functor has a left adjoint, given by the forgetful functor

\[ U: S_{as}(C) \rightarrow S(C) \]

which forgets the augmentation and a splitting. Thus, for any split augmented simplicial object $A_\bullet \rightarrow A_{-1}$ in $S_{as}(C)$, and any simplicial object $X_\bullet$ in $S(C)$, we have a natural bijection

\[ \theta_{A_\bullet, X_\bullet}: Hom_{S(C)}(U(A_\bullet), X_\bullet) \cong Hom_{S_{as}(C)}(A_\bullet, Dec(X_\bullet)) \]

which takes any simplicial map $f_\bullet: U(A_\bullet) \rightarrow X_\bullet$ to its composite with the splitting

\[ \begin{array}{cccccc}
A_{-1} & \rightarrow & A_0 & \rightarrow & A_1 & \rightarrow & \cdots \\
\downarrow^{s_0} & & \downarrow^{d_0} & & \downarrow^{d_0} & & \downarrow^{d_0} \\
X_0 & \rightarrow & X_1 & \rightarrow & X_2 & \rightarrow & \cdots \\
\downarrow^{d_1} & & \downarrow^{d_1} & & \downarrow^{d_1} & & \downarrow^{d_1} \\
A_0 & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & \cdots \end{array} \]

as in the above diagram.
In order to compare later our 2-torsors with Glenn’s simplicial 2-torsors we will recall some basic definitions from [25].

**Definition 2.11.** A simplicial map $\Lambda_\bullet : E_\bullet \to B_\bullet$ is said to be an exact fibration in dimension $n$, if for all $0 \leq k \leq n$, the diagrams

\[
\begin{array}{ccc}
E_n & \xrightarrow{\lambda_n} & B_n \\
\downarrow{p_k} & & \downarrow{p_k} \\
\Lambda_n^k(E_\bullet) & \xrightarrow{\Lambda_n^k(B_\bullet)} & \Lambda_n^k(B_\bullet)
\end{array}
\]

are pullbacks. It is called an exact fibration if it is an exact fibration in all dimensions $n$.

Using the language of simplicial fibration algebra, Glenn defined actions and n-torsors over n-dimensional hypergroupoids. This objects morally play the role of the n-nerve of weak n-groupoids, and we give their formal definition.

**Definition 2.12.** An n-dimensional Kan hypergroupoid is a Kan simplicial object $G_\bullet$ in $E$ such that the canonical map $G_m \to \Lambda_m^k(G_\bullet)$ is an isomorphism for all $m > n$ and $0 \leq k \leq m$.

**Remark 2.4.** The term n-dimensional hypergroupoid was introduced by Duskin [21], for any simplicial object satisfying the above condition without being Kan simplicial object. One of his motivational examples was the standard simplicial model for an Eilenberg-MacLane space $K(A,n)$, for any abelian group object $A$ in $E$. In [9], Beke used the term an exact n-type to emphasize the meaning of these objects as algebraic models for homotopy n-types.

**Definition 2.13.** An action of the n-dimensional hypergroupoid is an internal simplicial map $\Lambda_\bullet : P_\bullet \to B_\bullet$ in $E$ which is an exact fibration for all $m \geq n$.

**Definition 2.14.** An action $\Lambda_\bullet : P_\bullet \to B_\bullet$ is the n-dimensional hypergroupoid n-torsor over $X$ in $E$ if $P_\bullet$ is augmented over $X$, aspherical and $n$-1-coskeletal ($P_\bullet \simeq \text{Cosk}^{n-1}(P_\bullet)$).

### 3 Bicategories

Bicategories were defined by Benabou [10], and from the modern perspective, we could call them weak 2-categories. Instead of stating their original definition we will use Batanin’s approach to weak n-categories given in [7]. In this approach a bicategory $\mathcal{B}$, given by the reflexive 2-graph
\[
B = (B_2 \xrightarrow{d_1} B_1 \xrightarrow{d_0} B_0)
\]

is a 1-skeletal monoidal globular category, given by the diagram of categories and functors

\[
\begin{array}{ccc}
B_1 & \xrightarrow{D_1} & B_0 \\
\downarrow{D_0} & & \\
B_0 & \xrightarrow{D_1} & B_1
\end{array}
\]

where the category \(B_1\) is the category of morphisms of the bicategory \(B\) and the category \(B_0\) is the image \(\mathcal{D}(B_0)\) of the discrete functor \(\mathcal{D}: \text{Set} \to \text{Cat}\) which just turns an object of \(\mathcal{E}\) into a discrete internal category in \(\mathcal{E}\). Source functor \(D_1\) is defined by \(D_1 := d_1^0: B_1 \to B_0\) and \(D_1 := d_0^0d_1^0 = d_0^0d_1^0: B_2 \to B_0\), and a target functor \(D_0\) is defined by \(D_0 := d_1^0: B_1 \to B_0\) and \(D_0 := d_0^0d_1^0 = d_0^0d_1^0: B_2 \to B_0\), where we used the same notation for objects and morphisms parts of the functor. Also, the unit functor \(I: B_0 \to B_1\) is defined by \(I := s_0: B_0 \to B_1\) on the level of objects, and \(I := s_1: B_1 \to B_2\) on the level of morphisms, where \(s_0: B_0 \to B_1\) and \(s_1: B_1 \to B_2\) are section morphisms in the above 2-graph from left to right, which we didn’t label to avoid too much indices.

In the lower definition of a bicategory we will denote the vertex \(B_1 \times_{B_0} B_1\) of the following pullback of functors

\[
\begin{array}{ccc}
B_1 \times_{B_0} B_1 & \xrightarrow{\text{Pr}_2} & B_1 \\
\downarrow{\text{Pr}_1} & & \\
B_1 & \xrightarrow{D_1} & B_0
\end{array}
\]

by \(B_2 := B_1 \times_{B_0} B_1\) and likewise \(B_3 := B_1 \times_{B_0} B_1 \times_{B_0} B_1\), and so on. Thus we will adopt the following convention: for any functor \(P: \mathcal{E} \to B_0\), the first of the symbols

\[
\mathcal{E} \times_{B_0} B_1 \text{ and } B_1 \times_{B_0} \mathcal{E}
\]

will denote the pullback of \(P\) and \(D_0\), and the second one that of \(D_1\) and \(P\).

**Definition 3.1.** A bicategory \(\mathcal{B}\) consists of the following data:

- two categories, a discrete category \(\mathcal{B}_0\) of objects, and a category \(\mathcal{B}_1\) of morphisms of the weak 2-category \(\mathcal{B}\),
• functors $D_0, D_1 : B_1 \to B_0$, called target and source functors, respectively, a functor $I : B_0 \to B_1$, called unit functor, and a functor $H : B_2 \to B_1$, called the horizontal composition functor,

• natural isomorphism

\[
\begin{array}{ccc}
B_3 & \xrightarrow{H \times Id_{B_1}} & B_2 \\
\downarrow Id_{B_1} \times H & & \downarrow H \\
B_2 & \xrightarrow{H} & B_1 \\
\end{array}
\]

• natural isomorphisms

\[
\begin{array}{ccc}
B_2 & \xrightarrow{S_1} & B_1 \\
\downarrow H & & \downarrow S_0 \\
B_1 & \xrightarrow{\rho} & B_1 \\
\end{array}
\]

where the functor $S_0 : B_1 \to B_2$ is defined by the composition

\[
B_1 \xrightarrow{(D_0, Id_{B_1})} B_1 \times_{B_0} B_0 \xrightarrow{I \times Id_{B_1}} B_1 \times_{B_0} B_1,
\]

and the functor $S_1 : B_1 \to B_2$ is defined by the composition

\[
B_1 \xrightarrow{(Id_{B_1}, D_1)} B_0 \times_{B_0} B_1 \xrightarrow{Id_{B_0} \times I} B_1 \times_{B_0} B_1,
\]

or more explicitly for any 1-morphism $f : x \to y$ in $B$ (i.e. object in $B_1$) we have $S_0(f) = (f, i_x)$ and $S_1(f) = (i_y, f),$

such that following axioms are satisfied:
which for any object \((k, h, g, f)\) in \(\mathcal{B}_4\) becomes the commutative pentagon

\[
\begin{array}{c}
(k \circ (h \circ g)) \circ f \\
\alpha_{k,h,g}\circ f \\
(k \circ (h \circ g)) \circ f \\
\alpha_{k,h,g}\circ f \\
k \circ ((h \circ g) \circ f)
\end{array}
\quad
\begin{array}{c}
(k \circ h) \circ (g \circ f) \\
\alpha_{k,h,g} \circ f \\
(k \circ h) \circ (g \circ f) \\
\alpha_{k,h,g} \circ f \\
k \circ (h \circ (g \circ f))
\end{array}
\]

of components of natural transformations
• the commutative pyramid

which for any object \((g, f)\) in \(B_2\) becomes the triangle diagram

\[
\begin{align*}
(g \circ i_y) \circ f & \xrightarrow{\alpha_{g,i_y,f}} g \circ (i_y \circ f) \\
\rho_g \circ f & \leq g \circ \lambda_f
\end{align*}
\]

**Remark 3.1.** Note that in the above definition of the horizontal composition functor \(H : B_2 \to B_1\), for any diagram of 2-arrows (i.e. a morphism in a category \(B_2 \times_{B_1} B_2\))

\[
\begin{align*}
\psi_2 \circ \psi_1 (\phi_2 \circ \phi_1) = (\psi_2 \psi_1) \circ (\phi_2 \phi_1).
\end{align*}
\]
Example 3.1. (Strict 2-categories) A weak 2-category in which associativity and left and right identity natural isomorphisms are identities is called (strict) 2-category.

Example 3.2. (Monoidal categories) Monoidal category is a bicategory $B$ in which $B_0 = 1$ is terminal discrete category (or one point set). Strict monoidal category is a one object strict 2-category.

Example 3.3. (Bicategory of spans) Let $C$ be a cartesian category (that is a category with pullbacks). First we make a choice of the pullback:

\[
\begin{array}{ccc}
  u \times_y v & \overset{q}{\longrightarrow} & v \\
  ^p \downarrow & & \downarrow ^h \\
  u & \underset{g}{\longrightarrow} & z \\
\end{array}
\]

for any such diagram $x \overset{f}{\longrightarrow} z \overset{g}{\leftarrow} y$ in a category $C$. We construct the weak 2-category $\text{Span}(C)$ of spans in the category $C$. The objects of $\text{Span}(C)$ are the same as objects of $C$. For any two objects $x, y$ in $\text{Span}(C)$, a 1-morphism $u: x \rightarrow y$ is a span:

\[
\begin{array}{ccc}
  f & \overset{g}{\longrightarrow} & y \\
  x \downarrow & & \downarrow ^y \\
  u & \underset{y}{\longrightarrow} & y \\
\end{array}
\]

and a 2-morphism $a: z \Rightarrow w$ is given by the commutative diagram:

\[
\begin{array}{ccc}
  u & \overset{g}{\longrightarrow} & y \\
  f \downarrow & & \downarrow ^m \\
  x & \underset{a}{\longrightarrow} & y \\
\end{array}
\]

from which we easily see that vertical composition of 2-morphisms is given by the composition in $C$. Horizontal composition of composable 1-morphisms:

\[
\begin{array}{ccc}
  f & \overset{u}{\longrightarrow} & v \\
  x \downarrow & & \downarrow ^z \\
  x & \underset{y}{\longrightarrow} & z \\
\end{array}
\]
is given by the pullback

\[
\begin{array}{c}
\times \v
\end{array}
\]

and from here we have obvious horizontal identity \(i_x: x \to x\)

Example 3.4. (Bimodules) Let \(\text{Bim}\) denote the bicategory whose objects are rings with identity. For any two rings \(A\) and \(B\), \(\text{Bim}(A,B)\) will be a category of \(A-B\) bimodules and their homomorphisms. Horizontal composition is given by the tensor product, and associativity and identity constraints are the usual ones for the tensor product.

4 Nerves of bicategories

In this section, we describe the nerve construction for bicategories, first given by Duskin in [24]. This construction is a natural outcome of various attempts to describe nerves of higher dimensional categories and groupoids, whose origin is a conjecture on a characterization of the nerve of strict \(n\)-category, in an unpublished work of Roberts. This conjecture was published by Street in [36], and it was finally proved by Verity [37], who characterized nerves of strict \(n\)-categories by means of special simplicial sets, which he called complicial sets.

We will derive the construction of the Duskin nerve for bicategories from the standard description of the geometric nerve. First we have a fully faithful functor

\[
\begin{array}{c}
i: \Delta \to \text{Bicat}
\end{array}
\]

where \(\text{Bicat}\) is a category of bicategories and their homomorphisms, as it is given in [10], so we consider each ordinal as a locally discrete 2-category. Thus the nerve of the bicategory \(\mathcal{B}\) is a simplicial set \(N_2\mathcal{B}\), which is defined via the embedding (4.1) by

\[
N_2\mathcal{B}_n := Hom_{\text{Bicat}}(i[n], \mathcal{B}).
\]
The 0-simplices of \( N_2(\mathcal{B}) \) are the objects of \( \mathcal{B} \) and 1-simplices are directed line segments

\[
x_0 \xrightarrow{f_{01}} x_1
\]

which may be seen as homomorphisms \( f: [1] \to \mathcal{B} \) from the locally discrete bicategory \([1]\) to \( \mathcal{B} \). Face maps are defined by \( d_0(f_{01}) = x_1 \) and \( d_1(f_{01}) = x_0 \). If \( x_0 \) is a 0-cell of \( \mathcal{B} \) then we define the corresponding degenerate 1-simplex \( s_0(x_0) \) by

\[
x_0 \xrightarrow{id_{x_0}} x_0.
\]

A typical 2-simplex is given by the triangle filled with a 2-morphism \( \beta_{012}: f_{12} \circ f_{01} \Rightarrow f_{02} \)

\[
\begin{array}{c}
x_0 \\
\downarrow f_{01} \\
x_1 \\
\downarrow f_{12} \\
x_2
\end{array}
\]

where \( f_{ij}: [1] \to \mathcal{B} \) is a homomorphism for which \( f_{ij}(0) = x_i \) and \( f_{ij}(1) = x_j \). The face operators are defined as usual by

\[
d_i(f_{12}, f_{02}, f_{01}, \beta_{012}) = \begin{cases} f_{12} & i = 0 \\ f_{02} & i = 1 \\ f_{01} & i = 2 \end{cases}
\]

while for a 1-cell \( x_0 \xrightarrow{f_{01}} x_1 \) the degeneracy operators are defined by

\[
s_0(f_{01}) = \rho_{f_{01}} \\
s_1(f_{01}) = \lambda_{f_{01}}
\]

which are the two 2-simplices

\[
\begin{array}{c}
x_0 \\
\downarrow f_{01} \\
x_1 \\
\downarrow f_{01} \\
x_0
\end{array}
\begin{array}{c}
x_0 \\
\downarrow f_{01} \\
x_1 \\
\downarrow f_{01} \\
x_1
\end{array}
\]
respectively, where the 1-morphisms \( \rho_{f_{01}} : f_{01} \circ \text{id}_{x_0} \to f_{01} \) and \( \lambda_{f_{01}} : \text{id}_{x_1} \circ f_{01} \to f_{01} \) are the components of the right and left identity natural isomorphisms in \( B \). The general 3-simplex is of the form

\[
\begin{array}{c}
\downarrow \beta_{023} \\
\downarrow f_{02} \\
\downarrow f_{01} \\
\downarrow f_{13} \\
\downarrow \beta_{013} \\
\downarrow f_{23} \\
\downarrow f_{12} \\
\downarrow \beta_{123} \\
x_0 \\
x_1 \\
x_2 \\
x_3
\end{array}
\]

such that we have an identity

\[
\beta_{023}(\beta_{012} \circ f_{23})\alpha_{0123} = \beta_{013}(\beta_{123} \circ f_{01})
\]

where \( \alpha_{0123} : (f_{23} \circ f_{12}) \circ f_{01} \Rightarrow f_{23} \circ (f_{12} \circ f_{01}) \), and this condition follows directly from the coherence for the composition. Since this construction is given by the geometric nerve \([1,2]\), it follows immediately that the Duskin nerve is functorial with respect to homomorphisms of bicategories, which leads us to the following result.

**Theorem 4.1.** The Duskin nerve functor \( N_2 : \text{Bicat} \to \text{SSet} \) is fully faithful.

**Proof.** An analogous proof that the geometric nerve provides a fully faithful functor on the category \( 2-\text{Cat}_{\text{lax}} \) of 2-categories and normal lax 2-functors is given in \([11]\). Then the statement of the theorem follows immediately for a category \( \text{Bicat} \) of bicategories and normal homomorphisms. \( \square \)

## 5 Actions of bicategories

Now, we will introduce actions of bicategories. It will be clear from the definition that such actions are categorification of actions of categories.

**Definition 5.1.** A right action of a bicategory \( B \) is quintuple \((C, \Lambda, A, \kappa, \iota)\) given by:

- a category \( C \) and a functor \( \Lambda : C \to B_0 \) to the discrete category of objects \( B_0 \) of the bicategory \( B \), called the momentum functor,
- a functor \( A : C \times_{B_0} B_1 \to C \), called the action functor, and we usually write \( A(p, f) := p \triangle f \) for any object \((p, f) \) in \( C \times_{B_0} B_1 \), and \( A(a, \phi) := a \triangle \phi \) for any morphism \((a, \phi) : (p, f) \to (q, g) \) in \( C \times_{B_0} B_1 \),
• a natural isomorphism

\[
\begin{array}{c}
\downarrow
\end{array}
\]

whose components are denoted by \( \kappa_{p,f,g} : (p \triangleleft f) \triangleright g \to p \triangleleft (f \circ g) \) for any object \((p, f, g)\) in \( C \times B_0 B_1 \times B_0 B_1 \)

• a natural isomorphism

\[
\begin{array}{c}
\downarrow
\end{array}
\]

whose components are denoted by \( i_p : p \triangleleft i_{\Lambda(p)} \to p \) for each object \( p \) in \( C \)

such that following axioms are satisfied:

• equivariance of the action

\[
\begin{array}{c}
\downarrow
\end{array}
\]

which means that for any object \((p, f)\) in \( C \times B_0 B_1 \), we have \( \Lambda(p \triangleleft f) = D_1(f) \), and for any morphism \((a, \phi) : (p, f) \to (q, g)\) in \( C \times B_0 B_1 \), we have \( \Lambda(a \triangleleft \phi) = D_1(\phi) \),
• for any object \((p, f, g, h)\) in \(\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1\) the following diagram

\[
\begin{array}{c}
(p \triangleleft (f \circ g)) \triangleleft h \\
(p \triangleleft (f \circ h)) \triangleleft g \\
p \triangleleft ((f \circ g) \circ h) \\
p \triangleleft ((f \circ h) \circ g)
\end{array}
\]

commutes,

• for any object \((p, f)\) in \(\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1\) following diagrams

\[
\begin{array}{c}
(p \triangleleft i_{\Lambda_0(p)}) \triangleleft f \triangleleft p \triangleleft (i_{\Lambda_0(p)} \circ f) \\
p \triangleleft (i_{\Lambda_0(f)} \circ f) \\
p \triangleleft (f \circ i_{\Lambda_0(f)})
\end{array}
\]

commute.

**Remark 5.1.** Note the fact that \(A : \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \rightarrow \mathcal{C}\) is a functor, immediately implies an interchange law

\[(b \triangleleft \psi)(a \triangleleft \phi) = (ba) \triangleleft (\psi \phi)\]

**Definition 5.2.** Let \(\pi : \mathcal{C} \rightarrow M\) be a bundle of categories over an object \(M\) in \(\mathcal{E}\). A (fiberwise) right action of a bicategory \(\mathcal{B}\) on a bundle of categories \(\pi : \mathcal{C} \rightarrow M\) is given by
the action of the bicategory $\mathcal{B}$ on a category $\mathcal{C}$ for which the diagram

\[
\begin{array}{ccc}
\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow{A} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{B}_1 & \xrightarrow{\pi} & \mathcal{M}
\end{array}
\]

commute. We call a bundle $\pi: \mathcal{C} \to M$, a $\mathcal{B}$-2-bundle over $M$.

**Definition 5.3.** Let $(\mathcal{C}, \Lambda, A, \kappa, \iota)$ and $(\mathcal{D}, A', \Omega, \kappa', \iota')$ be two $\mathcal{B}$-categories. A $\mathcal{B}$-equivariant functor is a pair $(F, \theta): (\mathcal{C}, \Lambda, A, \kappa, \iota) \to (\mathcal{D}, A', \Omega, \kappa', \iota')$ consisting of

- a functor $F: \mathcal{C} \to \mathcal{D}$
- a natural transformations $\theta: A' \circ (F \times Id_{\mathcal{B}_1}) \Rightarrow F \circ A$

\[
\begin{array}{ccc}
\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow{F \times Id_{\mathcal{B}_1}} & \mathcal{D} \times_{\mathcal{B}_0} \mathcal{B}_1 \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
\]

such that following conditions are satisfied:

- $\Omega \circ F = \Lambda$

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow & & \downarrow \\
\mathcal{B}_0 & \xrightarrow{\Lambda} & \mathcal{M}
\end{array}
\]

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commutes, which means that we have an identity of natural transformations

$$(F \circ \kappa)[\theta \circ (A \times Id_{B_1})][A' \circ (\theta \times Id_{B_1})] = [\theta \circ (Id_C \times H)][\kappa' \circ (F \times Id_{B_1} \times Id_{B_1})]$$

when evaluated at object $(p, f, g)$ in $C \times_{B_0} B_1 \times_{B_0} B_1$, becomes a commutative diagram

in the category $D$. 

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• the diagram

\[ \begin{array}{ccc}
\mathcal{C} \times_{B_0} B_1 & \xrightarrow{F \times \text{Id}_{B_1}} & \mathcal{D} \times_{B_0} B_1 \\
\downarrow (\text{Id}_C, \Lambda) & & \downarrow (\text{Id}_D, \Omega) \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow F & & \downarrow \psi \\
\mathcal{C} & \xrightarrow{\iota} & \mathcal{D} \\
\end{array} \]

commutes, which means that we have identity of natural transformations

\[ (\iota' \circ \text{Id}_F) \text{Id}_F = (F \circ \iota) [\theta \circ (\text{Id}_C, \Lambda)] \text{Id}_{(F, \Lambda)} \]

when evaluated at object \( p \) in \( \mathcal{C} \), becomes a commutative diagram

\[ \begin{array}{ccc}
F(p) \circ i_{\Lambda(p)} & \xrightarrow{\psi_{p,i_{\Lambda(p)}}} & F(p \circ i_{\Lambda(p)}) \\
\downarrow \iota_{F(p)} & & \downarrow F(i_{p}) \\
F(p) & & \\
\end{array} \]

in the category \( \mathcal{D} \).

**Definition 5.4.** A \( \mathcal{B} \)-equivariant natural transformation \( \tau: (F, \theta) \Rightarrow (G, \zeta) \) between \( \mathcal{B} \)-covariant functors \( (F, \theta), (G, \zeta): (\mathcal{C}, \Lambda, \Phi, \alpha, \iota) \to (\mathcal{D}, \Psi, \Omega, \beta, \kappa) \) is a natural transformation \( \tau: F \Rightarrow G \) such that diagram

\[ \begin{array}{ccc}
\mathcal{C} \times_{B_0} B_1 & \xrightarrow{F \times \text{Id}_{B_1}} & \mathcal{D} \times_{B_0} B_1 \\
\downarrow \tau \times \text{Id}_{B_1} & & \downarrow G \times \text{Id}_{B_1} \\
\mathcal{C} & \xrightarrow{\psi} & \mathcal{D} \\
\downarrow F & & \downarrow G \\
\mathcal{C} & \xrightarrow{\iota} & \mathcal{D} \\
\end{array} \]

(5.1)
commutes, which means that we have a following identity

$$\zeta[A' \circ (\tau \times Id_{B_1})] = (\tau \circ A)\theta$$

that becomes a commutative diagram

\[
\begin{array}{ccc}
F(p) \triangleleft f & \xrightarrow{\theta_{p,f}} & F(p \triangleleft f) \\
\tau \circ f & & \tau \circ f \\
G(p) \triangleleft f & \xrightarrow{\zeta_{p,f}} & G(p \triangleleft f)
\end{array}
\]

in the category $\mathcal{D}$, when evaluated at object $p$ in $\mathcal{C}$.

The above construction gives rise to the 2-category in an obvious way, so we have a following theorem.

**Theorem 5.1.** The class of $\mathcal{B}$-categories, $\mathcal{B}$ equivariant functors and their natural transformations form a 2-category.

**Proof.** The vertical and horizontal composition in a 2-category is induced from the composition in $\text{Cat}$. \qed

Let $\mathcal{B}$ be a bicategory and $\mathcal{P}$ a category together with a momentum functor $\Lambda : \mathcal{P} \to B_0$

\[
\begin{array}{ccc}
P_1 & \xrightarrow{\tau_1} & B_2 \\
\downarrow{s_1} & & \downarrow{s_1} \\
P_0 & \xrightarrow{\Lambda_0} & B_0
\end{array}
\]

and let $\mathcal{B}$ acts on $\mathcal{P}$ via an action functor

$$A : \mathcal{P} \times_{B_0} B_1 \to \mathcal{P}$$

which satisfies coherence axioms from Definition 13.1. Such actions allows us to introduce a fundamental objects which we will use later.
Example 5.1. (Regular Lie groupoids) Recall that a regular Lie groupoid $\mathcal{G}$ is Lie groupoid

$$
\begin{array}{ccc}
G & \xrightarrow{s} & M \\
& \searrow^t & \downarrow \\
& & M
\end{array}
$$

such that for each $x \in M$, the target map $t: G \to M$ restricts to a map $t: s^{-1}(x) \to M$ of locally constant rank. Regular Lie groupoids cover many important classes of Lie groupoids like transitive Lie groupoids, étale Lie groupoids and bundles of Lie groups. Groupoids which arise in foliation theory are always regular, and Moerdijk gave a classification of regular Lie groupoids in [34]. Any regular Lie groupoid (5.4) fits into extension of Lie groupoids

$$
\begin{array}{ccc}
K & \xrightarrow{j} & G & \xrightarrow{\pi} & E \\
& \searrow^t & & \downarrow & \\
M & \xleftarrow{k} & & \xleftarrow{s} & \xleftarrow{s}
\end{array}
$$

where $E$ is the manifold of morphisms of a foliation groupoid $\mathcal{E}$ over $M$, which means that the fibers of the map $(s,t): E \to M \times M$ are discrete. Therefore, the category of regular Lie groupoids $\mathcal{RLie}(M)$ over $M$ is equipped with a canonical projection functor

$$
\begin{array}{ccc}
\mathcal{RLie}(M) & \xrightarrow{\Pi} & \mathcal{Fol}(M) \\
& \searrow & \\
& & \mathcal{Fol}(M)
\end{array}
$$

to the category $\mathcal{Fol}(M)$ of foliation groupoids over $M$. There is a bigroupoid $\mathcal{B}(M)$ whose discrete groupoid of objects $\text{BunGr}(M)$ consists of bundles of Lie groups over $M$, and whose groupoid of morphisms $\text{Bitors}(M)$ has bitorsors as objects and their isomorphisms as morphisms. The bigroupoid $\mathcal{B}(M)$ acts on the projection (5.6) with respect to the diagram

$$
\begin{array}{ccc}
\text{Bitors}(M) & \xrightarrow{T} & \mathcal{RLie}(M) \\
& \searrow & \downarrow \Lambda & \searrow \Pi \\
\text{BunGr}(M) & \xleftarrow{S} & \mathcal{Fol}(M)
\end{array}
$$

in which a momentum functor $\Lambda: \mathcal{RLie}(M) \to \text{BunGr}(M)$ is defined for a regular Lie groupoid $\mathcal{G}$ by $\Lambda(\mathcal{G}) = K$, where $k: K \to M$ is bundle of Lie groups from an extension (5.5). The (left) action of the bigroupoid $\mathcal{B}(M)$ on the category $\mathcal{RLie}(M)$ is defined by a functor

$$
\mathcal{A}: \text{Bitors}(M) \times_{\text{BunGr}(M)} \mathcal{RLie}(M) \to \mathcal{RLie}(M)
$$

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on pairs \((L_P K, G)\) where \(L_P K\) is just the notation for \(L - K\) bitorsor over \(M\), and \(G\) fits into an exact sequence of groupoids \((5.3)\), and produces a new regular Lie groupoid \(P \otimes_K G \otimes_K P^{-1}\) which fits into an exact sequence of groupoids

\[
\begin{array}{cccccc}
L & \xrightarrow{j} & P \otimes_K G \otimes_K P^{-1} & \xrightarrow{\pi_P} & E \\
\downarrow{t} & & \downarrow{t} & & \downarrow{s} & \downarrow{s} \\
M & & \xrightarrow{p} & & \end{array}
\] (5.9)

Here, \(P \otimes_K G \otimes_K P^{-1}\) denotes the usual contracted product of bitorsors (which also defines a horizontal composition in the bigroupoid \(\mathcal{B}(M)\)) whose elements are equivalence classes \((q \otimes g \otimes p^{-1})\) by the equivalence relation which identifies points \((qk, g, p^{-1})\) and \((q, kg, p^{-1})\) (as well as points \((q, gk, p^{-1})\) and \((q, g, kp^{-1})\)) in a fiber product \(P \times_K G \times_K P^{-1}\). We use the notation \(\mathcal{K}_P L^{-1}\) for a \(K - L\) bitorsor opposite to \(L_P K\), which has the same underlying space \(P\), but we denote its points by \(p^{-1} \in P^{-1}\) just to distinguish \(P^{-1}\) from \(P\). The left action of \(K\) on \(P^{-1}\) is defined by means of the right action of \(K\) on \(P\) by

\[kp^{-1} = (pk^{-1})^{-1}.\]

Then the map \(p_P : P \times_K G \times_K P^{-1} \to E\) in the exact sequence \((5.9)\) is defined by

\[\pi_P(q \otimes g \otimes p^{-1}) = p(g)\]

and its kernel is an adjoint bundle \(P \otimes_K P^{-1}\) of Lie groups, whose group law is given by

\[(s \otimes r^{-1})(q \otimes p^{-1}) = s(r^{-1}q) \otimes p^{-1} = s \otimes (r^{-1}q)p^{-1}\]

where \((r^{-1}q) \in L\) denotes a value of a division map \(\delta : P \times_M P^{-1} \to L\) for the right action of \(K\) on \(P\), which factors through the tensor product \(P \otimes_K P^{-1}\) by an \(\text{(unique)}\) isomorphism

\[\tilde{\delta} : P \otimes_K P^{-1} \to L\]

which shows that the sequence \((5.9)\) is exact and its kernel is canonically isomorphic to \(L\). The map \(j_P : L \to P \otimes_K G \otimes_K P^{-1}\) is defined for any point \(l \in L_x\) in the fiber over \(x \in M\) by

\[j_P(l) = (lq \otimes 1_x \otimes q^{-1})\]

for any point \(q \in P_x\), and this map is independent of the choice of \(q \in P_x\). Now, if we denote the above action of \(P\) on \(G\) by \(P \triangleright G\) then for any \(D - L\) bitorsor \(Q\) and any \(L - K\) bitorsor \(P\) we have a canonical isomorphism

\[\kappa_{Q,P,G} : (Q \otimes L P) \triangleright G \to Q \triangleright (P \triangleright G)\]

given by the coherence associativity for the tensor product of bitorsors.
6 Action bicategory

Theorem 6.1. For any action \([5,2]\) of the bicategory \(B\) on the category \(P\), there exists an action bicategory \(P \triangleleft B\) consisting of the following data:

- Objects of \(P \triangleleft B\) are given by objects \(P_0\) of the category \(P\)
- a 1-morphism is a pair \((\psi, h): q \rightarrow p\) which we draw as an arrow

\[ q \xrightarrow{(\psi, h)} p \]

where \(h: \Lambda_0(q) \rightarrow \Lambda_0(p)\) is a 1-morphism in the bicategory \(B\), and \(\psi: q \rightarrow p \triangleleft h\) is a morphism in the category \(P\), thus it is an element of \(P_1\).

- a 2-morphism \(\gamma: (\psi, h) \Rightarrow (\xi, l)\)

\[ q \xrightarrow{\gamma} p \]

is a 2-morphism \(\gamma: h \Rightarrow l\) in \(B_2\), such that the diagram of morphisms in \(P\)

\[ q \xrightarrow{\psi} p \triangleleft h \]

commutes.

Proof. We define the composition for any two composable 1-morphisms

\[ r \xrightarrow{(\phi, g)} q \xrightarrow{(\psi, h)} p \]

by \((\psi, h) \circ (\phi, g) = (\psi \circ \phi, h \circ g): r \rightarrow p\), where \(\psi \circ \phi: r \rightarrow p \triangleleft (h \circ g)\) is a morphism in \(P\), defined by the composition

\[ r \xrightarrow{\phi} q \triangleleft g \xrightarrow{\psi \circ g} (p \triangleleft h) \triangleleft g \xrightarrow{\kappa_{p,h,g}} p \triangleleft (h \circ g) \]
and we will show that this composition is a coherently associative. For any three composable 1-morphisms
\[
\begin{array}{c}
 s \stackrel{(\varphi,f)}{\rightarrow} \stackrel{(\phi,g)}{\rightarrow} q \stackrel{(\psi,h)}{\rightarrow} p
\end{array}
\]
first we have a morphism \((\psi \circ \phi) \circ \varphi, (h \circ g) \circ f\), where the first term is a composite of
\[
\begin{array}{c}
 s \varphi \rightarrow r \circ f \stackrel{(\psi \circ \phi) \circ f}{\rightarrow} (p \circ (h \circ g)) \circ f \stackrel{\kappa_{p,h,g,f}}{\rightarrow} p \circ ((h \circ g) \circ f)
\end{array}
\]
Also we have the composition \((\psi \circ (\phi \circ \varphi), h \circ (g \circ f))\), and the first term is given by a composite
\[
\begin{array}{c}
 s \stackrel{\phi \circ \varphi}{\rightarrow} q \circ (g \circ f) \stackrel{\psi \circ (g \circ f)}{\rightarrow} (p \circ h) \circ (g \circ f) \stackrel{\kappa_{p,h,g,f}}{\rightarrow} p \circ ((h \circ g) \circ f)
\end{array}
\]
and the component of the associativity \(\alpha_{h,g,f} : (h \circ g) \circ f \rightarrow h \circ (g \circ f)\), defines a 2-morphism
\[
\begin{array}{c}
s \downarrow \alpha_{h,g,f} \downarrow p
\end{array}
\]
which we see from the commutativity of the diagram
\[
\begin{array}{c}
s \stackrel{\varphi}{\rightarrow} r \circ f \stackrel{(\psi \circ \phi) \circ f}{\rightarrow} (p \circ (h \circ g)) \circ f \stackrel{\kappa_{p,h,g,f}}{\rightarrow} p \circ ((h \circ g) \circ f)
\end{array}
\]
that follows from the definition of the horizontal composition, the naturality and the coherence for quasiasociativity of the action. The horizontal composition of 2-morphisms
\[
\begin{array}{c}
(\phi,g) \downarrow \pi \downarrow (\delta,k)
\end{array}
\]
\[
\begin{array}{c}
(\psi,h) \downarrow \rho \downarrow (\xi,l)
\end{array}
\]
is given by the horizontal composition in \( B_2 \)

\[ r \xrightarrow{(\psi \circ \phi, h \circ g)} p \]

since we have a commutative diagram

\[ r \xrightarrow{\phi} q \xleftarrow{\theta} k \]

\[ q \xrightarrow{\psi \circ g} (p \circ h) \]

\[ \xrightarrow{(p \circ g) \circ \pi} p \circ (h \circ g) \]

\[ q \xleftarrow{\xi \circ k} (p \circ l) \]

\[ \xleftarrow{p \circ (p \circ l) \circ \pi} p \circ (l \circ k) \]

which follows from the interchange law and the naturality of the coherence for the quasi-associativity of the action. The vertical composition of 2-morphisms in \( \mathcal{P} \circ \mathcal{B} \) is similarly induced from the one in \( \mathcal{B} \). The coherence of the horizontal composition in \( \mathcal{P} \circ \mathcal{B} \) is immediately given by the coherence of the horizontal composition in \( \mathcal{B} \).

**Proposition 6.1.** There exists a canonical projection

\[ \Lambda: \mathcal{P} \circ \mathcal{B} \to \mathcal{B} \]

which is a strict homomorphism of bicategories.

**Proof.** A homomorphism \( \Lambda: \mathcal{P} \circ \mathcal{B} \to \mathcal{B} \) is defined by (the component of) the momentum functor \( \Lambda_0(p) = \lambda_0(p) \), for any object \( p \) in \( \mathcal{P} \circ \mathcal{B} \). For any 1-morphism \( (\psi, h) \) it is defined by \( \Lambda_1(\psi, h) = h \), and for any 2-morphism \( \gamma: (\psi, h) \Rightarrow (\xi, l) \) in \( \mathcal{P} \circ \mathcal{B} \), it is given simply by \( \Lambda_2(\gamma) = \gamma \). Then we have a following identity

\[ \Lambda((\psi, h) \circ (\phi, g)) = \Lambda(\psi \circ \phi, h \circ g) = h \circ g = \Lambda(\psi, h) \circ \Lambda(\phi, g) \]

which means that this homomorphism is strict (it preserves a composition strictly). \( \square \)
Example 6.1. The right action of a bicategory $\mathcal{B}$ on itself is given by a diagram

\[
\begin{array}{ccc}
B_2 & \to & B_2 \\
\downarrow^{s_1} & & \downarrow^{s_1} \\
B_1 & \to & B_1 \\
\downarrow^{s_0} & & \downarrow^{t_0} \\
B_0 & & B_0
\end{array}
\]

where a momentum functor is given by the source $S: \mathcal{B}_1 \to \mathcal{B}_0$ and an action functor is given by a horizontal composition $H: \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{B}_1$. Any object of an action bicategory $\mathcal{B}_1 \triangleleft \mathcal{B}$ is an element of $\mathcal{B}_1$, which which is a 1-morphism

\[ x \xrightarrow{f} y. \]

A 1-morphism from an object $f$ to an object $f'$ is a pair $($\(\phi, g\): $f \to f'$$)$ as in the diagram

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow^{g} & & \downarrow^{\phi} \\
z & \xleftarrow{f'} & y
\end{array}
\]

where $\phi: f \Rightarrow f' \circ g$ is a 2-morphism in $\mathcal{B}$. A 2-morphism $\gamma: ($\(\phi, g\)$) \Rightarrow ($\psi, h$)$ is a diagram

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow^{g} & \Rightarrow \downarrow^{\psi} & \downarrow^{h} \\
z & \xleftarrow{f'} & y
\end{array}
\]

where $\gamma: g \Rightarrow h$ is a 2-morphism in $\mathcal{B}$ such that identity $\psi = (f' \circ \gamma) \phi$ holds. We will denote an action bicategory $\mathcal{B}_1 \triangleleft \mathcal{B}$ by $TB$, and we call it a tangent bicategory because the 2-bundle

\[ T: TB \to \mathcal{B}_0 \] (6.2)

(which associates to all above diagrams an object $y$) is a generalization of a tangent 2-bundle introduced by Roberts and Schreiber in [35] in the case of strict 2-categories. This example of an action bicategory plays a crucial role in understanding of universal 2-bundles.
7 Bigroupoid 2-torsors

Definition 7.1. A right action of a bigroupoid \( B \) on a groupoid \( P \) is given by the action of the underlying bicategory \( B \) on a category \( P \) given as previously by \((P, B, \Lambda, A, \alpha, \iota)\).

Definition 7.2. Let \( B \) be an internal bigroupoid in \( \mathcal{E} \), and \( \pi: P \to X \) a right \( B \)-2-bundle of groupoids over \( X \) in \( \mathcal{E} \). We say that \((P, \pi, \Lambda, A, X)\) is a right \( B \)-principal-2-bundle (or a right \( B \)-torsor) over \( X \) if the following conditions are satisfied:

- the projection morphism \( \pi_0: P_0 \to X \) is an epimorphism,
- the action morphism \( \lambda_0: P_0 \to B_0 \) is an epimorphism,
- the induced internal functor
  \[(Pr_1, A): P \times_{B_0} B_1 \to P \times X \]
  (7.1)
is a (strong) equivalence of internal groupoids over \( P \) (where both groupoids are seen as objects over \( P \) by the first projection functor).

Example 7.1. (The trivial 2-torsor) The trivial 2-torsor is given by the triple \((B_1, T, S, H, B_0)\) where the momentum is given by the source functor \( S: B_1 \to B_0 \), and the action is given by the horizontal composition \( H: B_1 \times B_0 B_1 \to B_1 \).

Example 7.2. For any \( B \)-2-torsor \((P, \pi, \Lambda, A, X)\) over \( X \), and any morphism \( f: M \to B_0 \), we have a pullback \( B \)-2-torsor over \( M \), defined by \((f^*(P), Pr_1 \circ Pr_2, f^*(A), X)\).

Let us describe the simplicial set \( P_* \) arising by the application of the Duskin nerve functor
\[N_2: \text{Bicat} \to \mathcal{SSet}\]
to the action bicategory \( P \cdot B \). The set of 0-simplices is \( P_0 \) and any 1-simplex is an arrow
\[p_j \xrightarrow{(\pi_{ij}, f_{ij})} p_i\]
and face operators are defined by \( d^0_0(\pi_{ij}, f_{ij}) = p_i \) and \( d^1_0(\pi_{ij}, f_{ij}) = p_j \), while the degeneracy is defined by \( s^0_0(p_i) = (\iota_{p_i}, \iota_{p_i}) \) and it is given by the arrow
\[p_i \xrightarrow{(\iota_{p_i}, \iota_{p_i})} p_i\]
where the morphism \( \iota_{p_i}: p_i \to p_i \cdot \Lambda_0(p_i) \) is an identity coherence of the action. A 2-simplex in \( P_* \) is of the form
\[p_k \xrightarrow{(\pi_{ik}, f_{ik})} p_j \]
\[\xrightarrow{L_{ij}^k} \]
\[p_i \xrightarrow{(\pi_{ij}, f_{ij})} p_j \]
\[\xrightarrow{(\pi_{ik}, f_{ik})} p_i \]
where the diagram

\[
\begin{array}{ccc}
  p_k & \xrightarrow{\pi_{ij} \circ \pi_{jk}} & p_i \\
  & \downarrow {\pi_{ik}} & \downarrow {p \beta_{ijk}} \\
  & p_i \triangleleft f_{ik} & \\
\end{array}
\]

of morphisms in \( \mathcal{P} \) commutes, and the morphism \( \pi_{ij} \circ \pi_{jk} : p_k \to p_i \triangleleft (f_{ij} \circ f_{jk}) \) is the composite of

\[
\begin{array}{ccc}
p_k & \xrightarrow{\pi_{jk}} & p_j \triangleleft f_{jk} \\
\pi_{ij} \circ f_{jk} & \xrightarrow{(p_i \triangleleft f_{ij}) \triangleleft f_{jk}} & (p_i \triangleleft f_{ij}) \triangleleft (f_{ij} \circ f_{jk}) \\
\end{array}
\]

of morphisms in \( \mathcal{P} \). Face operators are defined by

\[
\begin{align*}
d_0^2(\beta_{ijk}) &= (\pi_{jk}, f_{jk}) \\
d_1^2(\beta_{ijk}) &= (\pi_{ik}, f_{ik}) \\
d_2^2(\beta_{ijk}) &= (\pi_{ij}, f_{ij})
\end{align*}
\]

and the degeneracy operators are given by

\[
\begin{align*}
s_0^2(\pi_{ij}, f_{ij}) &= \rho_{f_{ij}} \\
s_1^2(\pi_{ij}, f_{ij}) &= \lambda_{f_{ij}}
\end{align*}
\]

which are the two 2-simplices

\[
\begin{array}{ccc}
p_j & \xrightarrow{(i_{p_j}, i_{p_j})} & p_j \\
& \downarrow {\rho_{f_{ij}}} & \downarrow \circ \rho_{f_{ij}} \\
(\pi_{ij}, f_{ij}) & \xrightarrow{p_i} & \triangleleft (\pi_{ij}, f_{ij}) \\
& \downarrow {\lambda_{f_{ij}}} & \downarrow \circ \lambda_{f_{ij}} \\
& p_i & \xrightarrow{p_i} \triangleleft (p_{i}, i_{p_i}) \\
\end{array}
\]

respectively, where the 1-morphisms \( \rho_{f_{ij}} : f_{ij} \circ i_{p_j} \to f_{ij} \) and \( \lambda_{f_{ij}} : i_{p_i} \circ f_{ij} \to f_{ij} \) are the components of the right and left identity natural isomorphisms in \( \mathcal{B} \).
A general 3-simplex is of the form

\[ \beta_{ijkl}(\beta_{ijk} \circ f_{kl}) = \alpha_{ijkl} \beta_{ijl}(\beta_{jkl} \circ f_{ij}) \]

which is just a nonabelian 2-cocycle condition.

**Example 7.3.** Let \( B_* \) be Duskin nerve for a bicategory \( \mathcal{B} \). The tangent bicategory \( \mathcal{T}_B \) from Example 6.1. is action bicategory for the right action of \( \mathcal{B} \) on itself and a décalage construction \((2.1)\) from Chapter 2 becomes the diagram of simplicial sets

\[
\begin{array}{ccccccccccc}
B_0 & \overset{d_0}{\longrightarrow} & B_0 & \overset{d_1}{\longrightarrow} & B_0 & \overset{d_2}{\longrightarrow} & B_0 & \overset{d_3}{\longrightarrow} & B_0 & \overset{d_4}{\longrightarrow} & \cdots & \overset{Sk^0(B_*)}{\longrightarrow} \\
\downarrow{s_0} & & \downarrow{s_1} & & \downarrow{s_2} & & \downarrow{s_3} & & \downarrow{s_4} & & & \\
B_1 & \overset{d_0}{\longrightarrow} & B_2 & \overset{d_1}{\longrightarrow} & B_3 & \overset{d_0}{\longrightarrow} & \cdots & \overset{D_0}{\longrightarrow} & B_0 \\
\downarrow{s_0} & & \downarrow{s_1} & & \downarrow{s_2} & & \downarrow{s_3} & & \downarrow{s_4} & & & \\
B_0 & \overset{d_0}{\longrightarrow} & B_1 & \overset{d_1}{\longrightarrow} & B_2 & \overset{d_0}{\longrightarrow} & B_3 & \overset{d_0}{\longrightarrow} & B_4 & \overset{d_0}{\longrightarrow} & \cdots & B_0 \\
\end{array}
\]

in which \( D_1: \text{Dec}(B_*) \to B_* \) is a simplicial map which is the Duskin nerve of the canonical projection \( \Lambda: \mathcal{T}_B \to \mathcal{B} \) and \( D_0: \text{Dec}(B_*) \to B_* \) is a simplicial map which is the Duskin nerve of the tangent 2-bundle \( T\mathcal{B} \to B_0 \).

**Theorem 7.1.** Let the bigroupoid \( \mathcal{B} \) acts on a groupoid \( \mathcal{P} \). Then the Duskin nerve of the canonical projection \((6.1)\) is a simplicial map \( \Lambda_* = \mathcal{N}_2(\Lambda): \mathcal{P}_* \to \mathcal{B}_* \) which is a simplicial action of the Duskin nerve \( \mathcal{B}_* \) on the bigroupoid \( \mathcal{B} \), i.e. it is an exact fibration for all \( n \geq 2 \).
Proof. We need to show that for any $n \geq 2$ and for any $k$ such that $0 \leq k \leq n$, the diagram

$$
\begin{array}{ccc}
P_n & \xrightarrow{\lambda_n} & B_n \\
p_k & & p_k \\
\wedge_n^k(P_{\bullet}) & \xrightarrow{\lambda_n^k} & \wedge_n^k(B_{\bullet})
\end{array}
$$

is a pullback. A k-horn $((f_{ij}, \pi_{ij}), \ldots, (f_{j,k-1}, \pi_{j,k-1}), (f_{k,k+1}, \pi_{k,k+1}), \ldots, (f_{n-1,n}, \pi_{n-1,n}))$ in $\wedge_n^k(P_{\bullet})$ is given by the n-tuple of 1-morphisms in $A_B P$, and its image by $\lambda_n^k: \wedge_n^k(P_{\bullet}) \to \wedge_n^k(P_{\bullet})$ is a k-horn in $\wedge_n^k(B_{\bullet})$, given by the n-tuple $(f_{ij}, \ldots, f_{j,k-1}, f_{k,k+1}, \ldots, f_{n-1,n})$ of 1-morphisms in $B$. For example, in the case $n = 2$, any filler of a 1-horn $(f_{ij}, - f_{jk})$ in $\wedge_2^1(B_{\bullet})$, is the 2-simplex

$$
\begin{array}{ccc}
x_k & \xrightarrow{f_{jk}} & x_j \\
\downarrow{\phi_{\beta_{ijk}}} & & \downarrow{f_{ij}} \\
x_i & & f_{ij}
\end{array}
$$

in $B_2$. A 2-simplex in $P_{\bullet}$ is a lifting of the previous 2-simplex if it is of the form

$$
\begin{array}{ccc}
p_k & \xrightarrow{(\pi_{ik}, f_{jk})} & p_j \\
\downarrow{\phi_{\beta_{ijk}}} & & \downarrow{(\pi_{ij}, f_{ij})} \\
(\pi_{ik}, f_{ik}) & & \pi_{ij}
\end{array}
$$

where the diagram

$$
\begin{array}{ccc}
p_k & \xrightarrow{\pi_{ij} \circ \pi_{jk}} & p_i \uplus (f_{ij} \circ f_{jk}) \\
\downarrow{\pi_{ik}} & & \downarrow{p_i \uplus f_{ik}} \\
p_k \circ \beta_{ijk} & & \pi_{ik}
\end{array}
$$

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of morphisms in \( \mathcal{P} \) commutes, and the morphism \( \pi_{ij} \circ \pi_{jk} : p_k \to p_i \circ (f_{ij} \circ f_{jk}) \) is the composite of

\[
p_k \xrightarrow{\pi_{jk}} p_j \xrightarrow{f_{jk}} c_{ijk} \xrightarrow{\kappa_{i,j,k}} p_i \xrightarrow{(f_{ij} \circ f_{jk})}
\]

so we see that a pair \( ((f_{ij}, \pi_{ij}), -, (f_{jk}, \pi_{jk}), \beta_{ijk}) \) in \( \Lambda_2^1(\mathcal{P}_\bullet) \times \Lambda_2^k(\mathcal{B}_\bullet) B_2 \) uniquely determines above 2-simplex in \( \mathcal{P}_2 \). Since \( \mathcal{P} \) is a groupoid, any pair consisting of a k-horn in \( \Lambda_2^k(\mathcal{B}_\bullet) \), for \( k = 0, 2 \), and a 2-simplex in \( \mathcal{B}_2 \) which covers the k-horn, uniquely determines a 2-simplex in \( \mathcal{P}_2 \), and thus provides a canonical isomorphism \( \mathcal{P}_2 \simeq \Lambda_2^k(\mathcal{P}_\bullet) \times \Lambda_2^k(\mathcal{B}_\bullet) B_2 \). Since both simplicial objects are 2-coskeletal, the assertion follows for all \( n \geq 2 \).

**Definition 7.3.** An action of the \( n \)-dimensional Kan complex is an internal simplicial map \( \Lambda_\bullet : \mathcal{P}_\bullet \to \mathcal{B}_\bullet \) in \( \mathcal{E} \) which is a weak exact fibration for all \( m \geq n \).

In the case of the bigroupoid \( \mathcal{B} \), the Duskin nerve functor is a 2-dimensional hypergroupoid \( \mathcal{B}_\bullet = N_2(\mathcal{B}) \) and let \( \mathcal{P}_\bullet = N_2(\mathcal{A}_B \mathcal{P}) \) be the Duskin nerve of an action bigroupoid associated to the action of the bigroupoid \( \mathcal{B} \) on the groupoid \( \mathcal{P} \). Glenn introduced in [25] a simplicial definition of an \( n \)-dimensional hypergroupoid n-torsor in \( \mathcal{E} \).

**Definition 7.4.** An action \( \Lambda_\bullet : \mathcal{P}_\bullet \to \mathcal{B}_\bullet \) is the \( n \)-dimensional hypergroupoid n-torsor over \( X \) in \( \mathcal{E} \) if \( \mathcal{P}_\bullet \) is augmented over \( X \), aspherical and \( n \)-1-coskeletal (\( \mathcal{P}_\bullet \simeq \text{Cosk}^{n-1}(\mathcal{P}_\bullet) \)).

In the case of the bigroupoid \( \mathcal{B} \), the above definition reduces to the following one.

**Definition 7.5.** A bigroupoid \( \mathcal{B}_\bullet \) 2-torsor over an object \( X \) in \( \mathcal{E} \) is an internal simplicial map \( \Lambda_\bullet : \mathcal{P}_\bullet \to \mathcal{B}_\bullet \) in \( S(\mathcal{E}) \), which is an exact fibration for all \( n \geq 2 \), and where \( \mathcal{P}_\bullet \) is augmented over \( X \), aspherical and 1-coskeletal (\( \mathcal{P}_\bullet \simeq \text{Cosk}^1(\mathcal{P}_\bullet) \)).

Thus in the case when an action of \( \mathcal{B} \) on \( \mathcal{P} \) is principal, we have the main result of our paper.

**Theorem 7.2.** Let \( \mathcal{P} \) be a \( \mathcal{B} \)-2-torsor over \( X \). Then simplicial map \( \Lambda_\bullet = N_2(\Lambda) : \mathcal{P}_\bullet \to \mathcal{B}_\bullet \) is a Duskin-Glenn 2-torsor.

**Proof.** The simplicial complex \( \mathcal{P}_\bullet \) is augmented over \( X \) because the action of \( \mathcal{B} \) is fiberwise, since for any 1-simplex \( (f_{ij}, \pi_{ij}) : p_j \to p_i \) in \( \mathcal{P}_0 \), where \( \pi_{ij} : p_j \to p_i \circ f_{ij} \) we have

\[
\pi_0d_0(f_{ij}, \pi_{ij}) = \pi_0(p_i) = \pi_0(p_i \circ f_{ij}) = \pi_1(\pi_{ij}) = \pi_0(p_j) = \pi_0d_1(f_{ij}, \pi_{ij}).
\]

The simplicial complex \( \mathcal{P}_\bullet \) is obviously aspherical and we prove now that it is also 1-coskeletal. A general 2-simplex in \( \text{Cosk}^1(\mathcal{P}_\bullet) \) is a triple \( ((f_{ij}, \pi_{ij}), (f_{ik}, \pi_{ik}), (f_{jk}, \pi_{jk})) \).
which we see as the triangle

\[
\begin{array}{c}
p_k \\
\downarrow \downarrow \downarrow \\
p_i \\
\end{array}
\begin{array}{c}
(p_{ik}, f_{ik}) \\
\downarrow \\
(p_{ij}, f_{ij}) \\
\end{array}
\begin{array}{c}
(p_{ij}, f_{jk}) \\
\downarrow \\
p_j \\
\end{array}
\]

from which we have morphisms \(\pi_{ij} \circ \pi_{jk} : p_k \to p_i \circ (f_{ij} \circ f_{jk})\) and \(\pi_{ik} : p_k \to p_i \circ f_{ik}\) in \(\mathcal{P}\). Now we use the fact that the induced functor

\[
(Pr_1, A) : \mathcal{P} \times B_0 \to \mathcal{P}
\]

is a (strong) equivalence of internal groupoids over \(\mathcal{P}\), and therefore fully faithful. Specially, for the two objects \((p_i, f_{ij} \circ f_{jk})\) and \((p_i, f_{ik})\) of \(\mathcal{P} \times B_0\), this equivalence induces a bijection

\[
\text{Hom}_{\mathcal{P} \times B_0}((p_i, f_{ij} \circ f_{jk}), (p_i, f_{ik})) \simeq \text{Hom}_{\mathcal{P}}((p_i, p_i \circ (f_{ij} \circ f_{jk})), (p_i, p_i \circ f_{ik}))
\]

and therefore for a morphism \(\text{id}_{p_i} \circ (\pi_{ij} \circ \pi_{jk})^{-1} : (p_i, p_i \circ (f_{ij} \circ f_{jk})) \to (p_i, p_i \circ f_{ik}))\)

\[
\begin{array}{c}
p_k \\
\downarrow \downarrow \downarrow \\
p_i \circ f_{ik} \\
\downarrow \\
p_i \circ (f_{ij} \circ f_{jk}) \\
\end{array}
\begin{array}{c}
(p_{ij} \circ \pi_{jk})^{-1} \\
\downarrow \\
p_{ij} \circ f_{jk} \\
\end{array}
\]

there exists a unique 2-morphism \(\beta_{ijk} : f_{ij} \circ f_{jk} \to f_{ik}\) in \(\mathcal{B}\), such that the diagram

\[
\begin{array}{c}
p_k \\
\downarrow \downarrow \downarrow \\
p_i \circ f_{ik} \\
\downarrow \\
p_i \circ (f_{ij} \circ f_{jk}) \\
\end{array}
\begin{array}{c}
\pi_{ik} \\
\downarrow \\
\pi_{ik} \\
\end{array}
\begin{array}{c}
\pi_{ij} \circ \pi_{jk} \\
\downarrow \\
\pi_{ij} \circ f_{jk} \\
\end{array}
\begin{array}{c}
(\pi_{ij} \circ f_{jk})^{-1} \\
\downarrow \\
p_{ij} \circ f_{jk} \\
\end{array}
\begin{array}{c}
\beta_{ijk} \\
\downarrow \\
p_{ij} \circ f_{jk} \\
\end{array}
\]
commutes, and this uniquely determines a 2-simplex

\[
\begin{array}{c}
\pi_{ik}, f_{ik}\\
\phi_{ijk}
\end{array}
\]

\[
\begin{array}{c}
p_k \\
p_i \\
p_j
\end{array}
\]

in \( \mathcal{P}_2 \), which proves that we have a bijection \( \mathcal{P}_2 \cong \text{Cosk}^1(\mathcal{P}_\bullet)_2 \). From here it follows immediately that \( \mathcal{P}_\bullet \cong \text{Cosk}^1(\mathcal{P}_\bullet) \).

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