Continuity of the path delay operator for dynamic network loading with spillback

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Abstract
This paper establishes the continuity of the path delay operators for dynamic network loading (DNL) problems based on the Lighthill-Whitham-Richards model, which explicitly capture vehicle spillback. The DNL describes and predicts the spatial-temporal evolution of traffic flow and congestion on a network that is consistent with established route and departure time choices of travelers. The LWR-based DNL model is first formulated as a system of partial differential algebraic equations (PDAEs). We then investigate the continuous dependence of merge and diverge junction models with respect to their initial/boundary conditions, which leads to the continuity of the path delay operator through the wave-front tracking methodology and the generalized tangent vector technique. As part of our analysis leading up to the main continuity result, we also provide an estimation of the minimum network supply without resort to any numerical computation. In particular, it is shown that gridlock can never occur in a finite time horizon in the DNL model.

Keywords: path delay operator, continuity, dynamic network loading, LWR model, spillback, gridlock

1. Introduction
Dynamic traffic assignment (DTA) is the descriptive modeling of time-varying flows on traffic networks consistent with traffic flow theory and travel choice principles. DTA models describe and predict departure rates, departure times and route choices of travelers over a given time horizon. It seeks to describe the dynamic evolution of traffic in networks in a fashion consistent with the fundamental notions of traffic flow and travel demand; see Peeta and Ziliaskopoulos (2001) for some review on DTA models and recent developments. Dynamic user equilibrium (DUE) of the open-loop type, which is one type of DTA, remains a major modern perspective on traffic modeling that enjoys wide scholarly support. It captures two aspects of driving behavior quite well: departure time choice and route choice (Friesz et al., 1993). Within the DUE model, travel cost for the same trip purpose is identical for all utilized route-and-departure-time choices. The relevant notion of travel cost is effective travel delay, which is a weighted sum of actual travel time and arrival penalties.

In the last two decades there have been many efforts to develop a theoretically sound formulation of dynamic network user equilibrium that is also a canonical form acceptable to scholars and practitioners alike. There are two essential components within the DUE models: (1) the mathematical expression of Nash-like equilibrium conditions;
and (2) a network performance model, which is, in effect, an embedded dynamic network loading (DNL) problem. The DNL model captures the relationships among link entry flow, link exit flow, link delay and path delay for any given set of path departure rates. The DNL gives rise to the notion of path delay operator, which is viewed as a mapping from the set of feasible path departure rates to the set of path travel times or, more generally, path travel costs.

Properties of the path delay operator are of fundamental importance to DUE models. In particular, continuity of the delay operators plays a key role in the existence and computation of DUE models. The existence of DUEs is typically established by converting the problem to an equivalent mathematical form and applying some version of Brouwer’s fixed-point existence theorem; examples include Han et al. (2013c), Smith and Wisten (1995), Wie et al. (2002) and Zhu and Marcotte (2000). All of these existence theories rely on the continuity of the path delay operator. On the computational side of analytical DUE models, every established algorithm requires the continuity of the delay operator to ensure convergence; an incomplete list of such algorithms include the fixed-point algorithm (Friesz et al., 1993), the route-swapping algorithm (Huang and Lam, 2002), the descent method (Han and Lo, 2003), the projection method (Han and Lo, 2002) Ukkusuri et al. (2012), and the proximal point method (Han et al., 2013a).

It has been difficult historically to show continuity of the delay operator for general network topologies and traffic flow models. Over the past decade, only a few continuity results were established for some specific traffic flow models. Zhu and Marcotte (2000) use the link delay model (Friesz et al., 1993) to show the continuity of the path delay operator. Their result relies on the a priori boundedness of the path departure rates, and is later improved by a continuity result that is free of such an assumption (Han et al., 2012). In Bressan and Han (2013), continuity of the delay operator is shown for networks whose dynamics are described by the LWR-Lax model (Bressan and Han, 2011; Friesz et al., 2013), which is a variation of the LWR model that does not capture vehicle spillback. Their result also relies on the a priori boundedness of path departure rates. Han et al. (2013c) consider Vickrey’s point queue model (Vickrey, 1969) and show the continuity of the corresponding path delay operator for general networks without invoking the boundedness on the path departure rates.

All of these aforementioned results are established for network performance models that do not capture vehicle spillback. To the best of our knowledge, there has not been any rigorous proof of the continuity result for DNL models that allow queue spillback to be explicitly captured. On the contrary, some existing studies even show that the path travel times may depend discontinuously on the path departure rates, when physical queue models are used. For example, Szeto (2003) uses the cell transmission model and signal control to show that the path travel time may depend on the path departure rates in a discontinuous way. Such a finding suggests that the continuity of the delay operator could very well fail when spillback is present. This has been the major hurdle in showing the continuity or identifying relevant conditions under which the continuity is guaranteed. This paper bridges this gap by articulating these conditions and providing accompanying proof of continuity.

This paper presents, for the first time, a rigorous continuity result for the path delay operator based on the LWR network model, which explicitly captures physical queues and vehicle spillback. In showing the desired continuity, we propose a systematic approach for analyzing the well-posedness of two specific junction models: a merge and a diverge model, both originally presented by Daganzo (1995). The underpinning analytical framework employs the wave-front tracking methodology (Dafermos, 1972; Holden and Risebro, 2002) and the technique of generalized tangent vectors (Bressan, 1993; Bressan et al., 2000). A major portion of our proof involves the analysis of the interactions between kinematic waves and the junctions, which is frequently invoked for the study of well-posedness of junction models; see Garavello and Piccoli (2006) for more details. Such analysis is further complicated by the fact that vehicle turning ratios at a diverge junction are determined endogenously by drivers’ route choices within the DNL procedure. As a result, special tools are developed in this paper to handle this unique situation.

As we shall later see, a crucial step of the process above is to estimate and bound from below the minimum network supply, which is defined in terms of local vehicle densities in the same way as in Lebacque and Khoshyaran (1999). In fact, if the supply somewhere tends to zero (that is, when traffic approaches the jam density), the well-posedness of the diverge junction may fail, as we demonstrate in Section 4.2.1. This has also been confirmed by the earlier study of Szeto (2003), where a wave of jam density is triggered by a signal red light and causes spillback at the upstream junction, leading to a jump in the path travel times. Remarkably, in this paper we are able to show that (1) if the supply

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1Well-posedness of a model refers to the property that the behavior of the solution hardly changes when there is a slight change in the initial/boundary conditions.
is bounded away from zero (that is, traffic is bounded away from the jam density), then the diverge junction model is well posed; and (2) the desired boundedness of the supply is a natural consequence of the dynamic network loading procedure that involves only the merge and diverge junction models we study here. This is a highly non-trivial result because it not only plays a role in the continuity proof, but also implies that gridlock can never occur in the network loading procedure in any finite time horizon.

The final continuity result is presented in Section 5.4 following a number of preliminary results set out in previous sections. Although our continuity result is established only for networks consisting of simple merge and diverge nodes, it can be extended to networks with more complex topologies using the procedure of decomposing any junction into several simple merge and diverge nodes [Daganzo, 1995]. Moreover, the analytical framework employed by this paper can be invoked to treat other and more general junction topologies and/or merging and diverging rules, and the techniques employed to analyze wave interactions will remain valid.

The main contributions made in this paper include:

• formulation of the LWR-based dynamic network loading (DNL) model with spillback as a system of partial differential algebraic equations (PDAEs);
• a continuity result for the path delay operator based on the aforementioned DNL model;
• a novel method for estimating the network supply, which shows that gridlock can never occur within a finite time horizon.

The rest of this paper is organized as follows. Section 2 recaps some essential knowledge and notions regarding the LWR network model and the DNL procedure. Section 3 articulates the mathematical contents of the DNL model by formulating it as a PDAE system. Section 4 introduces the merge and diverge models and establishes their well-posedness. Section 5 provides a final proof of continuity and some discussions. Finally, we offer some concluding remarks in Section 6.

2. LWR-based dynamic network loading

2.1. Delay operator and dynamic network loading

Throughout this paper, the time interval of analysis is a single commuting period expressed as [0, T] for some T > 0. We let P be the set of all paths employed by travelers. For each path p ∈ P we define the path departure rate which is a function of departure time t ∈ [0, T]:

\[ h_p(\cdot) : [0, T] \rightarrow \mathbb{R}_+ \]

where \( \mathbb{R}_+ \) denotes the set of non-negative real numbers. Each path departure rate \( h_p(t) \) is interpreted as a time-varying path flow measured at the entrance of the first arc of the relevant path, and the unit for \( h_p(t) \) is vehicles per unit time.

We next define \( h(\cdot) = \{ h_p(\cdot) : p \in P \} \) to be a vector of departure rates. Therefore, \( h = h(\cdot) \) can be viewed as a vector-valued function of t, the departure time.

The single most crucial ingredient is the path delay operator, which maps a given vector of departure rates \( h \) to a vector of path travel times. More specifically, we let

\[ D_p(t, h) \quad \forall t \in [t_0, t_f], \quad \forall p \in P \]

be the path travel time of a driver departing at time t and following path p, given the departure rates \( h \) associated with all the paths in the network. We then define the path delay operator \( D(\cdot) \) by letting \( D(h) = \{ D_p(\cdot, h) : p \in P \} \), which is a vector consisting of time-dependent path travel times \( D_p(t, h) \).

\[ \text{For notation convenience and without causing any confusion, we will sometimes use } h \text{ instead of } h(\cdot) \text{ to denote path flow vectors.} \]
2.2. The Lighthill-Whitham-Richards model on networks

We recap the network extension of the LWR model (Lighthill and Whitham [1955], Richards [1956]), which captures the formation, propagation, and dissipation of spatial queues and vehicle spillback. Discussion provided below relies on general assumptions on the fundamental diagram and the junction model, and involves no ad hoc treatment of flow propagation, flow conservation, link delay, or other critical model features.

We consider a road link expressed as a spatial interval \([a, b] \subset \mathbb{R}\). The partial differential equation (PDE) representation of the LWR model is the following scalar conservation law

\[
\frac{\partial}{\partial t}\rho(t, x) + \frac{\partial}{\partial x} f(\rho(t, x)) = 0 \quad (t, x) \in [0, T] \times [a, b] \tag{2.1}
\]

with appropriate initial and boundary conditions, which will be discussed in detail later. Here, \(\rho(t, x)\) denotes vehicle density at a given point in the space-time domain. The fundamental diagram \(f(\cdot) : [0, \rho_{jam}] \to [0, C]\) expresses vehicle flow at \((t, x)\) as a function of \(\rho(t, x)\), where \(\rho_{jam}\) denotes the jam density, and \(C\) denotes the flow capacity. Throughout this paper, we impose the following mild assumption on \(f(\cdot)\):

\(\text{(F)}\) The fundamental diagram \(f(\cdot)\) is continuous, concave and vanishes at \(\rho = 0\) and \(\rho = \rho_{jam}\).

An essential component of the network extension of the LWR model is the specification of boundary conditions at a road junction. Derivation of the boundary conditions should not only obey physical realism, such as that enforced by entropy conditions (Garavello and Piccoli, 2006; Holden and Risebro, 1995), but also reflect certain behavioral and operational considerations, such as vehicle turning preferences (Daganzo, 1995), driving priorities (Coclite et al., 1999), which represent the link’s sending and receiving capacities. Models following this approach include Daganzo (1995), Jin and Zhang (2003) and Jin (2010). The solution of a Riemann Problem is given by the Riemann Solver (RS), to be defined below.

2.2.1. The Riemann Solver

We consider a general road junction \(J\) with \(m\) incoming roads and \(n\) outgoing roads, as shown in Figure 1.

![Figure 1: A road junction with \(m\) incoming links and \(n\) outgoing links.](image)

We denote by \(I_1, \ldots, I_m\) the incoming links and by \(I_{m+1}, \ldots, I_{m+n}\) the outgoing links. In addition, for every \(i \in \{1, \ldots, m + n\}\), the dynamic on \(I_i\) is governed by the LWR model

\[
\frac{\partial}{\partial t}\rho_i(t, x) + \frac{\partial}{\partial x} f(\rho_i(t, x)) = 0 \quad (t, x) \in [0, T] \times [a_i, b_i] \tag{2.2}
\]

where link \(I_i\) is expressed as the spatial interval \([a_i, b_i]\), and we always use the subscript ‘\(i\)’ to indicate dependence on the link \(I_i\). The initial condition for this conservation law is

\[
\rho_i(0, x) = \hat{\rho}_i(x) \quad x \in [a_i, b_i] \tag{2.3}
\]
Notice that the above \((m+n)\) initial value problems are coupled together via the boundary conditions to be specified at the junction. Such a system of coupling equations is commonly analyzed using the Riemann Problem.

**Definition 2.1. (Riemann Problem)** The Riemann Problem at the junction \(J\) is defined to be an initial value problem on a network consisting of the single junction \(J\) with \(m\) incoming links and \(n\) outgoing links, all extending to infinity, such that the initial densities are constants on each link:

\[
\begin{align*}
\rho_i(0, x) &\equiv \hat{\rho}_i, & x \in (-\infty, b_i], & i \in \{1, \ldots, m\} \\
\rho_j(0, x) &\equiv \hat{\rho}_j, & x \in [a_j, +\infty), & j \in \{m+1, \ldots, m+n\}
\end{align*}
\]

where \(\hat{\rho}_k \in [0, \rho_{jam}^k]\) are constants, \(k = 1, \ldots, m+n\).

A Riemann Solver for the junction \(J\) is a mapping that, given any \((m+n)\)-tuple of Riemann initial conditions \((\hat{\rho}_1, \ldots, \hat{\rho}_{m+n})\), provides a unique \((m+n)\)-tuple of boundary conditions \((\rho_1, \ldots, \rho_{m+n})\) such that one can solve the initial-boundary value problem for each link, and the resulting solutions constitute a weak entropy solution of the Riemann Problem at the junction. A precise definition of the Riemann Solver is given as follows.

**Definition 2.2. (Riemann Solver)** A Riemann Solver for the junction \(J\) with \(m\) incoming links and \(n\) outgoing links is a mapping

\[
RS : \prod_{k=1}^{m+n} [0, \rho_{jam}^k] \rightarrow \prod_{k=1}^{m+n} [0, \rho_{jam}^k]
\]

\[
(\hat{\rho}_1, \ldots, \hat{\rho}_{m+n}) \mapsto (\rho_1, \ldots, \rho_{m+n})
\]

which relates Riemann initial data \(\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_{m+n})\) to boundary conditions \(\rho = (\rho_1, \ldots, \rho_{m+n})\), such that the following hold.

(i) The solution of the Riemann Problem restricted on each link \(I_k\) is given by the solution of the initial-boundary value problem with initial condition \(\hat{\rho}_k\) and boundary condition \(\rho_k\), \(k = 1, \ldots, m+n\).

(ii) The Rankine-Hugoniot condition (flow conservation) holds:

\[
\sum_{i=1}^{m} f_i(\rho_i) = \sum_{j=m+1}^{m+n} f_j(\rho_j)
\]

(iii) The consistency condition holds:

\[
RS[RS[\hat{\rho}]] = RS[\hat{\rho}]
\]

Three conditions must be satisfied by the Riemann Solver (RS). Item (i) above requires that the boundary condition on each link must be properly given so that the initial value problems not only have well-defined solutions, but these solutions must also be compatible and form a sensible solution at the junction. (2.4) simply stipulates the conservation of flow across the junction. (2.5) is a desirable property and is sometimes referred to as the invariance property [Jin, 2010].

**Remark 2.3.** For the same Riemann Problem, there exist many Riemann Solvers that satisfy conditions (i)-(iii) above. Despite their varying forms, most existing Riemann Solvers rely on a flow maximization problem at the relevant junction subject to constraints related to turning ratio, right-of-way, or signal controls; see [Coclite et al., 2005; Han et al., 2014; Holden and Risebro, 1999; Jin and Zhang, 2003] and [Jin, 2010].
2.2.2. The link demand and supply

For each link $I_i$, we let $\rho^c_i$ be the critical density at which the flow is maximized. The demand $D_i(t)$ for incoming links $I_i$ and the supply $S_j(t)$ for outgoing links $I_j$ are defined in terms of the density near the exit and entrance of the link, respectively (Lebacque and Khoshyaran, 1999):

$$D_i(t) = D_i(\rho_i(t, b_i-)) = \begin{cases} C_i & \text{if } \rho_i(t, b_i-) \geq \rho^c_i \\ f_i(\rho_i(t, b_i-)) & \text{if } \rho_i(t, b_i-) < \rho^c_i \end{cases}$$ (2.6)

$$S_j(t) = S_j(\rho_j(t, a_j+)) = \begin{cases} C_j & \text{if } \rho_j(t, a_j+) \leq \rho^c_j \\ f_j(\rho_j(t, a_j+)) & \text{if } \rho_j(t, a_j+) > \rho^c_j \end{cases}$$ (2.7)

In prose, the demand represents the maximum flow at which cars can be discharged from the incoming link; and the supply represents the maximum flow at which cars can enter the outgoing link. Notice that the demand and supply are both expressed as functions of density, and they are always greater than or equal to the fundamental diagram $f_i(\cdot)$ or $f_j(\cdot)$; see Figure 2 for an illustration. In our subsequent presentation, without causing confusion we will use notations $D_i(t)$ and $D_i(\rho)$ interchangeably where the former indicates the demand as a time-varying quantity, and the latter emphasizes demand as a function of density. The same applies to the supply.

![Figure 2: Demand and supply as functions of density.](image)

3. Dynamic network loading problem formulated as a PDAE system

The aim of this section is to formulate the LWR-based dynamic network loading (DNL) problem as a system of partial differential algebraic equations (PDAEs). The proposed PDAE system uses vehicle density and queues as the primary unknown variables, and computes link dynamics, flow propagation, and path delay for any given vector of departure rates. The PDAE system captures vehicle spillback explicitly, and accommodates a wide range of junction types and Riemann Solvers.

We consider a network $G(\mathcal{A}, \mathcal{V})$ expressed as a directed graph with $\mathcal{A}$ being the set of links and $\mathcal{V}$ being the set of nodes. Let $\mathcal{P}$ be the set of paths employed by travelers, and $\mathcal{W}$ be the set of origin-destination pairs. Each path $p \in \mathcal{P}$ is expressed as an ordered set of links it traverses:

$$p = \{I_1, I_2, \ldots, I_{m(p)}\}$$

where $m(p)$ is the number of links in this path. There are several crucial components of a complete network loading model, each of which is elaborated in a subsection below. Throughout the rest of this paper, for each node $v \in \mathcal{V}$, we denote by $I^v$ the set of incoming links and $O^v$ the set of outgoing links.
3.1. Within-link dynamics

For each \( I_i \in \mathcal{A} \), the density dynamic is governed by the scalar conservation law

\[
\partial_t \rho_i(t, x) + \partial_x \left[ \rho_i(t, x) \cdot v_i \left( \rho_i(t, x) \right) \right] = 0 \quad (t, x) \in [0, T] \times [a_i, b_i]
\]

subject to initial condition and boundary conditions to be determined in Section 3.2. The fundamental diagram \( f_i(\rho_i) = \rho_i \cdot v_i(\rho_i) \) satisfies condition (F) stated at the beginning of Section 2.2. In order to explicitly incorporate drivers’ route choices, for every \( p \in \mathcal{P} \) such that \( I_i \in p \) we introduce the function \( \mu_{ij}^p(t, x), (t, x) \in [0, T] \times [a_i, b_i] \), which represents, in every unit of flow \( f_i(\rho_i(t, x)) \), the fraction associated with path \( p \). We call these variables path disaggregation variables (PDV). For each car moving along the link \( I_i \), its surrounding traffic can be distinguished by path (e.g. 20% following path \( p_1 \), 30% following path \( p_2 \), 50% following path \( p_3 \)). As this car moves, such a composition will not change since its surrounding traffic all move at the same speed under the first-in-first-out (FIFO) principle (i.e. no overtaking is allowed). In mathematical terms, this means that the path disaggregation variables, \( \mu_{ij}^p(t, \cdot) \), are constants along the trajectories of cars \((t, x(t))\) in the space-time diagram, where \( x(\cdot) \) is the trajectory of a moving car on link \( I_i \). That is,

\[
\frac{d}{dt} \mu_{ij}^p(t, x(t)) = 0 \quad \forall p \text{ such that } I_i \in p,
\]

which, according to the chain rule, becomes

\[
\partial_t \mu_{ij}^p(t, x(t)) + \partial_x \mu_{ij}^p(t, x(t)) = 0,
\]

which further leads to another set of partial differential equations on link \( I_i \):

\[
\partial_t \mu_{ij}^p(t, x(t)) + v_i(\rho_i(t, x(t))) \cdot \partial_x \mu_{ij}^p(t, x(t)) = 0 \quad \forall p \text{ such that } I_i \in p
\]

Here, \( \rho_i(t, x) \) is the solution of (3.8). The following obvious identity holds

\[
\sum_{p \in I_i} \mu_{ij}^p(t, x(t)) = 1 \quad \text{whenever } \rho_i(t, x(t)) > 0
\]

where \( p \supset I_i \) means “path \( p \) contains (or traverses) link \( I_i \)”, and the summation appearing in (3.10) is with respect to all such \( p \). By convention, if \( \rho_i(t, x) = 0 \), then \( \mu_{ij}^p(t, x) = 0 \) for all \( p \supset I_i \).

3.2. Boundary conditions at an ordinary node

For reason that will become clear later, we introduce the concept of an ordinary node. An ordinary node is neither the origin nor the destination of any trip. We use the notation \( \mathcal{V}^o \) to represent the set of ordinary nodes in the network.

As mentioned earlier, the partial differential equations on links incident to \( J \) are all coupled together through a given junction model, i.e., a Riemann Solver. A common prerequisite for applying the Riemann Solver is the determination of the flow distribution (turning ratio) matrix \( \mathcal{C} {\text{[Cocite et al. 2005]}} \), which relies on knowledge of the PDVs \( \mu_{ij}^p(t, b_i) \) for all \( I_i \in J \). We define the time-dependent flow distribution matrix associated with \( J \):

\[
A^J(t) = \{ \alpha_{ij}^J(t) \} \in [0, 1]^{\lvert J \rvert \times \lvert \mathcal{O}^J \rvert}
\]

where by convention, we use subscript \( i \) to indicate incoming links, and \( j \) to indicate outgoing links. Each element \( \alpha_{ij}^J(t) \) represents the turning ratios of cars discharged from \( I_i \) that enter downstream link \( I_j \). Then, for all \( p \) that traverses \( J \), the following holds.

\[
\alpha_{ij}^J(t) = \sum_{p \in I_i, I_j} \mu_{ij}^p(t, b_i) \quad \forall I_i \in J^I, I_j \in \mathcal{O}^J
\]

It can be easily verified that \( \alpha_{ij}^J(t) \in [0, 1] \) and \( \sum_j \alpha_{ij}^J(t) \equiv 1 \) according to (3.10).

We are now ready to express the boundary conditions for the ordinary junction \( J \in \mathcal{V}^o \). Let

\[
\text{RS}^J: \prod_{k=1}^{\lvert J \rvert \times \lvert \mathcal{O}^J \rvert} [0, \rho_{k}^{jam}] \rightarrow \prod_{k=1}^{\lvert J \rvert \times \lvert \mathcal{O}^J \rvert} [0, \rho_{k}^{jam}]
\]
be a given Riemann Solver. Notice that the dependence of the Riemann Solver on \( A^I \) has been indicated with a superscript. The boundary conditions for PDEs (3.9) read

\[
\rho_k(t, b_\xi) = RS_k^{A^I} \left[ (\rho_i(t, b_\xi^{-})), (\rho_j(t, a_\eta^{+})) \right] , \quad \forall I_k \in I^I
\]

\[
\rho_i(t, a_\eta) = RS_i^{A^I} \left[ (\rho_i(t, b_\xi^{-})), (\rho_j(t, a_\eta^{+})) \right] , \quad \forall I_i \in O^I
\]

where \( RS_k^{A^I}[\cdot] \) denotes the \( k \)-th component of the mapping, \( k = 1, \ldots, |I^I| + |O^I| \).

**Remark 3.1.** Intuitively, (3.13)–(3.14) mean that, given the current traffic states \((\rho_i(t, b_\xi^{-})), (\rho_j(t, a_\eta^{+}))\)\( \forall I_k \in I^I \) adjacent to the junction \( J \), the Riemann Solver \( RS^{A^I} \) specifies, for each incident link \( I_k \) or \( I_i \), the corresponding boundary conditions \( \rho_i(t, b_\xi) \) or \( \rho_j(t, a_\eta) \). In prose, at each time instance the Riemann Solver inspects the traffic conditions near the junction, and proposes the discharging (receiving) flows of its incoming (outgoing) links. Such a process is based on the flow distribution matrix \( A^I \) and often reflects traffic control measures at junctions. Furthermore, the Riemann Solver operates with knowledge of every link incident to the junction, thus the boundary condition of any relevant link is determined jointly by all the links connected to the same junction. Therefore, the LWR equations on all the links are coupled together through this mechanism. For this reason the LWR-based DNL models are highly challenging, both theoretically and computationally.

The upstream boundary conditions associated with PDEs (3.9) are:

\[
\mu_j^p(t, a_\eta) = \frac{f_j(\rho_j(t, a_\eta)) - \rho_j(t, a_\eta)}{f_j(\rho_j(t, a_\eta))} \quad \forall p \text{ such that } \{I_j, I_j\} \subset p, \quad \forall I_j \in O^I
\]

where the numerator \( f_j(\rho_j(t, a_\eta)) - \rho_j(t, a_\eta) \) expresses the exit flow on link \( I_j \) associated with path \( p \), which, by flow conservation, is equal to the entering flow of link \( I_j \) associated with the same path \( p \); the denominator represents the total entering flow of link \( I_j \).

**Remark 3.2.** Unlike the density-based PDE, the PDV-based PDE does not have any downstream boundary condition due to the fact that the traveling speeds of the PDVs are the same as the car speeds (they can be interpreted as Lagrangian labels that travel with the cars); thus information regarding the PDVs does not propagate backwards or spills over to upstream links.

### 3.3. Flow distribution at origin or destination nodes

We consider a node \( v \in \mathcal{V} \) that is either the origin or the destination of some path \( p \). One immediate observation is that the flow conservation constraint (2.4) no longer holds at such a node since vehicles either are ‘generated’ (if \( v \) is an origin) or ‘vanish’ (if \( v \) is a destination). A simple and effective way to circumvent this issue is to introduce a virtual link. A virtual link is an imaginary road with certain length and fundamental diagram, and serves as a buffer between an ordinary node and an origin/destination; see Figure 3 for an illustration. By introducing virtual links to the original network, we obtain an augmented network \( G(\tilde{A}, \tilde{\mathcal{V}}) \) in which all road junctions are ordinary, and hence fall within the scope of the previous section.

Figure 3: Illustration of the virtual links. Left: a virtual link connecting an origin (s) to an ordinary junction \( J \). Right: a virtual link connecting a destination (t) to an ordinary junction \( J \).

Let us denote by \( S \) the set of origins in the augmented network \( G(\tilde{A}, \tilde{\mathcal{V}}) \). For any \( s \in S \), we denote by \( \mathcal{P}^s \subset \mathcal{P} \) the set of paths that originate from \( s \), and by \( I_s \) the virtual link incident to this origin. For each \( p \in \mathcal{P}^s \) we denote by \( h_p(t) \)
the departure rate (path flow) along \( p \). It is expected that a buffer (point) queue may form at \( s \) in case the receiving capacity of the downstream \( I_s \) is insufficient to accommodate all the departure rates \( \sum_{p \in \mathcal{P}} h_p(t) \). For this buffer queue, denoted \( q_s(t) \), we employ a Vickrey-type dynamic \(^4\) \cite{vickrey1969}, that is,

\[
\frac{d}{dt} q_s(t) = \sum_{p \in \mathcal{P}} h_p(t) - \begin{cases} S_s(t) & \text{if } q_s(t) > 0 \\ \min \left\{ \sum_{p \in \mathcal{P}} h_p(t), S_s(t) \right\} & \text{if } q_s(t) = 0 \end{cases}
\]  

(3.16)

where \( S_s(t) \) denotes the supply of the virtual link \( I_s \). The only difference between (3.16) and Vickery’s model is the time-varying downstream receiving capacity provided by the virtual link.

It remains to determine the dynamics for the path disaggregation variables (PDV). More precisely, we need to determine \( \mu^p_i(t, a_i) \) for the virtual link \( I_p \) where \( p \in \mathcal{P}^i \), and \( x = a_i \) is the upstream boundary of \( I_p \). This will be achieved using the Vickrey-type dynamic (3.16) and the FIFO principle. Specifically, we define the queue exit time function \( \lambda_p(t) \) where \( t \) denotes the time at which drivers depart and join the point queue, if any; \( \lambda_p(t) \) expresses the time at which the same group of drivers exit the point queue. Clearly, FIFO dictates that

\[
\int_0^\tau \sum_{p \in \mathcal{P}} h_p(\tau) \, d\tau = \int_0^{\lambda_p(t)} f_p(\rho(\tau, a_i)) \, d\tau
\]  

(3.17)

where the two integrands on the left and right hand sides of the equation are flow entering the queue and flow leaving the queue, respectively. We may determine the path disaggregation variables as:

\[
\mu^p_i(\lambda_p(t), a_i) = \frac{h_p(t)}{\sum_{q \in \mathcal{P}} h_q(t)} \quad \forall p \in \mathcal{P}^i
\]  

(3.18)

Notice that, if \( \sum_{q \in \mathcal{P}} h_q(t) = 0 \), then the flow leaving the point queue at time \( \lambda_p(t) \) is also zero; thus there is no need to determine the path disaggregation variables. Therefore, the identity (3.18) is well defined and meaningful.

### 3.4. Calculation of path travel times

With all preceding discussions, we may finally express the path travel times, which are the outputs of a complete DNL model. The path travel time consists of link travel times plus possible queuing time at the origin. Mathematically, the link exit time function \( \lambda_i(t) \) for any \( I_i \) is defined, in a way similar to (3.17), as

\[
\int_0^\tau f_l(\rho(\tau, a_i)) \, d\tau = \int_0^{\lambda_i(t)} f_l(\rho(\tau, b_i)) \, d\tau
\]  

(3.19)

For a path expressed as \( p = \{I_1, I_2, \ldots, I_{m(p)}\} \), the time to traverse it is calculated as

\[
\lambda_{I_1} \circ \lambda_{I_2} \circ \cdots \circ \lambda_{I_{m(p)}}(t)
\]  

(3.20)

where \( f \circ g(t) = g(f(t)) \) means the composition of two functions. This is due to the assumption that cars leaving the previous link (or queue) immediately enter the next link without any delay.

### 3.5. The PDAE system

We are now ready to present a generic PDAE system for the dynamic network loading procedure. Let us begin by summarizing some key notations.

\(^4\)The right hand side of the ordinary differential equation (3.16) is discontinuous. An analytical treatment of this irregular equation is provided by Han et al. \cite{han2013a} using the variational formulation.
Given any vector of path departure rates $h = (h_p(\cdot) : p \in \mathcal{P})$, the proposed PDAE system for calculating path travel times is summarized as follows.

$$\frac{d q_s(t)}{dt} = \sum_{p \in \mathcal{P}} h_p(t) - \left\{ \begin{array}{ll} S_s(t) \min \left\{ \sum_{p \in \mathcal{P}} h_p(t), S_s(t) \right\} & q_s(t) > 0 \\ q_s(t) = 0 & q_s(t) = 0 \end{array} \right. \forall s \in S \tag{3.21}$$

$$\int_0^t \sum_{p \in \mathcal{P}} h_p(\tau) d\tau = \int_0^t f_s(\rho_s(\tau, a_s)) d\tau \quad \forall s \in S \tag{3.22}$$

$$\int_0^t f_l(\rho_l(\tau, a_l)) d\tau = \int_0^t f_l(\rho_l(\tau, b_l)) d\tau \quad \forall I_l \in \mathcal{A} \tag{3.23}$$

$$\partial_t \rho_l(t, x) + \partial_x \left[ \rho_l(t, x) \cdot v_l(\rho_l(t, x)) \right] = 0 \quad (t, x) \in [0, T] \times [a_l, b_l] \tag{3.24}$$

$$\partial_t \mu^l(t, x) + v_l(\rho_l(t, x)) \cdot \partial_x \mu^l(t, x) = 0 \quad (t, x) \in [0, T] \times [a_l, b_l] \tag{3.25}$$

$$\mu^l_i(t, a_i) = \frac{h_p(t)}{\sum_{p \in \mathcal{P}} h_p(t)} \quad \forall s \in S, p \in \mathcal{P}_s \tag{3.26}$$

$$\lambda^{l_j}(t) = \left[ a^{l_j}(t) \right], \quad a^{l_j}(t) = \sum_{p \in \mathcal{P}_l} \mu^l_i(t, b_i) \quad \forall I_l \in I^l, I_j \in O^j \tag{3.27}$$

$$\rho_l(t, b_l) = RS_{l_j}^{l_i} \left[ (\rho_l(t, b_l))_{l \in \mathcal{T}^j}, (\rho_l(t, a_l+))_{l \in \mathcal{O}^j} \right] \quad \forall I_l \in I^j \tag{3.29}$$

$$\rho_l(t, a_l) = RS_{l_j}^{l_i} \left[ (\rho_l(t, a_l))_{l \in \mathcal{T}^j}, (\rho_l(t, a_l+))_{l \in \mathcal{O}^j} \right] \quad \forall I_l \in O^j \tag{3.30}$$

$$D_p(t, h) = \lambda_p \circ \lambda_1 \circ \lambda_2 \cdots \circ \lambda_{\mathcal{P}}(t) \quad \forall p \in \mathcal{P}, \forall t \in [0, T] \tag{3.31}$$
Eqn (3.21) describes the (potential) queuing process at each origin. Eqns (3.22) and (3.23) express the queue exit time function for a point queue, and the link exit time function for a link, respectively. Eqns (3.24)-(3.25) express the link dynamics in terms of car density and PDV; Eqns (3.26)-(3.27) specifies the upstream boundary conditions for the PDV as these variables can only propagate forward in space. Eqns (3.28)-(3.30) determine the boundary conditions at junctions. Finally, Eqn (3.31) determines the path travel times.

The above PDAE system involves partial differential operators \( \partial_t \) and \( \partial_x \). Solving such a system requires solution techniques from the theory of numerical partial differential equations (PDE) such as finite difference methods (Godunov, 1959; LeVeque, 1992) and finite element methods (Larsson and Thomée, 2005).

4. Well-posedness of two junction models

In mathematical modeling, the term well-posedness refers to the property of having a unique solution, and the behavior of that solution hardly changes when there is a slight change in the initial/boundary conditions. Examples of well-posed problems include the initial value problem for scalar conservation laws (Bressan, 2000), and the initial value problem for the Hamilton-Jacobi equations (Daganzo, 2006). In the context of traffic network modeling, well-posedness is a desirable property of network performance models capable of supporting analyses and computations of DTA models. It is also closely related to the continuity of the path delay operator, which is the main focus of this paper.

This section investigates the well-posedness (i.e. continuous dependence on the initial/boundary conditions) of two specific junction models. These two junctions are depicted in Figure 4, and the corresponding merge and diverge rules are proposed initially by Daganzo (1995) in a discrete-time setting with fixed vehicle turning ratios and driving priority parameters.

![Figure 4: The diverging (left) and merge (right) junctions.](image)

4.1. The two junction models

4.1.1. The diverge model

We first consider the diverge junction shown on the left part of Figure 4 with one incoming link \( I_1 \) and two outgoing links \( I_2 \) and \( I_3 \). The demand \( D_1(t) \) and the supplies, \( S_2(t) \) and \( S_3(t) \), are defined by (2.6) and (2.7) respectively. The Riemann Solver for this junction relies on the following two conditions.

(A1) Cars leaving \( I_1 \) advance to \( I_2 \) and \( I_3 \) according to some turning ratio which is determined by the PDV \( \mu_1(t, b_1) \) in the DNL model.

(A2) Subject to (A1), the flow through the junction is maximized.

In the original diverge model (Daganzo, 1995), the vehicle turning ratios, denoted \( \alpha_{1,2} \) and \( \alpha_{1,3} \) with obvious meaning of notations, are constants known a priori. This is not the case in a dynamic network loading model since they are determined endogenously by drivers’ route choices, as expressed mathematically by Eqn (3.28). The diverge junction model, described by (A1) and (A2), can be more explicitly written as:

\[
\begin{align*}
  f_{1i}^{	ext{in}}(t) &= \min \left\{ D_1(t), \frac{S_2(t)}{\alpha_{1,2}(t)}, \frac{S_3(t)}{\alpha_{1,3}(t)} \right\} \\
  f_{2i}^{	ext{in}}(t) &= \alpha_{1,2}(t) \cdot f_{1i}^{	ext{in}}(t) \\
  f_{3i}^{	ext{in}}(t) &= \alpha_{1,3}(t) \cdot f_{1i}^{	ext{in}}(t)
\end{align*}
\]
where $f_{1}^{\text{out}}(t)$ denotes the exit flow of link $I_1$, and $f_{j}^{\text{in}}(t)$ denotes the entering flow on link $I_j$, $j = 2, 3$.

### 4.1.2. The merge model

We now turn to the merge junction in Figure 4 with two incoming links $I_4$ and $I_5$ and one outgoing link $I_6$. In view of this merge junction, assumption (A1) becomes irrelevant as there is only one outgoing link; and assumption (A2) cannot determine a unique solution.

To address this issue, we consider a right-of-way parameter $p \in (0, 1)$ and the following priority rule:

(R1) The actual link exit flows satisfy $(1 - p) \cdot f_{4}^{\text{out}}(t) = p \cdot f_{5}^{\text{out}}(t)$.

![Figure 5: Illustration of the merge model. The shaded areas represent the feasible domain of the maximization problem (4.33), and the thick line segments (Ω) are the optimal solution sets of (4.33). Left: Rule (R1) is compatible with (A2); and there exists a unique point $Q$ satisfying both (A2) and (R1). Right: (R1) is incompatible with (A2); in this case, the model selects point $Q$ within the set $\Omega$ that is closest to the ray ($R$) from the origin with slope $\frac{1 - p}{p}$. Notice that (R1) may be incompatible with assumption (A2); and we refer the reader to Figure 5 for an illustration.](image)

Mathematically, we let $\Omega$ be the set of points $(f_{4}^{\text{out}}, f_{5}^{\text{out}})$ that solves the following maximization problem:

$$
\left\{ \begin{array}{l}
\max \ f_{4}^{\text{out}} + f_{5}^{\text{out}} \\
\text{such that } 0 \leq f_{4}^{\text{out}} \leq D_4(t), \ 0 \leq f_{5}^{\text{out}} \leq D_5(t), \ f_{4}^{\text{out}} + f_{5}^{\text{out}} \leq S_6(t)
\end{array} \right. 
\quad (4.33)
$$

Moreover, we define the ray $R \equiv \{(f_{4}^{\text{out}}, f_{5}^{\text{out}}) \in \mathbb{R}^2 : (1 - p) \cdot f_{4}^{\text{out}} = p \cdot f_{5}^{\text{out}}\}$. Then the solution of the merge model is defined to be the projection of $R$ onto $\Omega$, that is,

$$
(f_{4}^{\text{out},*}, f_{5}^{\text{out},*}) = \arg \min_{(f_{4}^{\text{out}}, f_{5}^{\text{out}}) \in \Omega} d\left[(f_{4}^{\text{out}}, f_{5}^{\text{out}}), R\right] 
\quad (4.34)
$$

where $d\left[(f_{4}^{\text{out}}, f_{5}^{\text{out}}), R\right]$ denotes the Euclidean distance between the point $(f_{4}^{\text{out}}, f_{5}^{\text{out}})$ and the ray $R$:

$$
d\left[(f_{4}^{\text{out}}, f_{5}^{\text{out}}), R\right] = \min_{(x, y) \in R} \left\| (f_{4}^{\text{out}}, f_{5}^{\text{out}}) - (x, y) \right\|_2
$$

---

5 More generally, as pointed out by Coclite et al. (2005), when the number of incoming links exceeds the number of outgoing links, (A1) and (A2) combined are not sufficient to ensure a unique solution.
4.2. Well-posedness of the diverge junction model

In this section, we investigate the well-posedness of the diverge junction model. Unlike previous studies (Garavello and Piccoli [2006], [Han et al. 2015b]), a major challenge in this case is to incorporate drivers’ route choices, expressed by the path disaggregation variable \( \mu \), into the model and the analysis. In effect, we need to establish the continuous dependence of the model on the initial/boundary conditions in terms of both \( \rho \) and \( \mu \). As we shall see in Section 4.2.1 below, such a continuity does not hold in general. Following this, Section 4.2.2 provides sufficient conditions for the continuity to hold. These sufficient conditions are crucial for the desired continuity of the delay operator.

4.2.1. An example of ill-posedness

It has been shown by Han et al. [2015b] that the diverge model (Section 4.1.1) with constant turning ratios, \( \alpha_{1,2} \) and \( \alpha_{1,3} \), is well-posed. However, the assumption of fixed and exogenous turning ratios does not hold in DNL models; see (3.28). As a result the well-posedness is no longer true, which is demonstrated by the following counterexample.

We consider the diverge junction (Figure 4) and assume the same fundamental diagram \( f(\cdot) \) for all the links for simplicity. We consider a series of constant initial data parameterized by \( \varepsilon \) on the three links \( I_1, I_2, \) and \( I_3 \):

\[
\rho_1(0, x) \equiv \hat{\rho}_1 \in (\rho^c, \rho^{	ext{jam}}), \quad \rho_2(0, x) \equiv \hat{\rho}_2^\varepsilon \in (\rho^c, \rho^{	ext{jam}}), \quad \rho_3(0, x) \equiv \hat{\rho}_3^\varepsilon \in (0, \rho^c)
\]

where \( \rho^c \) and \( \rho^{	ext{jam}} \) are the critical density and the jam density, respectively. \( \hat{\rho}_1, \hat{\rho}_2^\varepsilon, \) and \( \hat{\rho}_3^\varepsilon \) satisfy:

\[
f(\hat{\rho}_1^\varepsilon) = \varepsilon f(\hat{\rho}_1), \quad f(\hat{\rho}_2^\varepsilon) = (1-\varepsilon) f(\hat{\rho}_1)
\]

where \( \varepsilon \geq 0 \) is a parameter. This configuration of initial data implies that link \( I_1 \) and link \( I_2 \) are both in the congested phase, while link \( I_3 \) is in the uncongested (free-flow) phase, see Figure 6 for an illustration.

Two paths exist in this example: \( p_1 = \{I_1, I_2\} \) and \( p_2 = \{I_1, I_3\} \). The initial conditions for \( \mu_1^\rho \) and \( \mu_1^\varepsilon \), which arise from travelers’ route choices, are as follows.

\[
\mu_1^\rho(0, x) \equiv \varepsilon, \quad \mu_1^\varepsilon(0, x) \equiv 1-\varepsilon
\]

Notice that these initial conditions are part of the initial value problem at the junction. The solution of this initial value problem depends on the value of \( \varepsilon \); in particular, we have the following two cases.

- When \( \varepsilon > 0 \), we claim that the initial conditions \( \hat{\rho}_1, \hat{\rho}_2^\varepsilon \) and \( \hat{\rho}_3^\varepsilon \) satisfying (4.35)-(4.36) constitute a constant solution at the junction. To see this, we follow the junction model (4.32) and the definitions of demand and supply (2.6)-(2.7) to get

\[
\begin{align*}
&f_1^\text{out}(t) = \min \left\{ D_1(t), \frac{S_2(t)}{\alpha_{1,2}}, \frac{S_1(t)}{\alpha_{1,3}} \right\} = \min \left\{ C, \frac{f(\hat{\rho}_2^\varepsilon)}{\varepsilon}, \frac{C}{1-\varepsilon} \right\} = \min \left\{ C, f(\hat{\rho}_1), \frac{C}{1-\varepsilon} \right\} = f(\hat{\rho}_1) \quad (4.37) \\
&f_2^\text{in}(t) = \alpha_{1,2} \cdot f_1^\text{out}(t) = \varepsilon f(\hat{\rho}_1) = f(\hat{\rho}_2^\varepsilon) \quad (4.38) \\
&f_3^\text{in}(t) = \alpha_{1,3} \cdot f_1^\text{out}(t) = (1-\varepsilon) f(\hat{\rho}_1) = f(\hat{\rho}_3^\varepsilon) \quad (4.39)
\end{align*}
\]
where $C$ denotes the flow capacity. Thus $\hat{\rho}_{1}, \hat{\rho}_{2}^{\epsilon}$ and $\hat{\rho}_{3}^{\epsilon}$ form a constant solution at the junction.

- When $\epsilon = 0$, the turning ratios satisfy $a_{1,2}(t) \equiv 0$, $a_{1,3}(t) \equiv 1$. Effectively, link $I_{1}$ is only connected to link $I_{3}$. We easily deduce that

\[
\begin{align*}
    f_{1}^{\text{out}}(t) & \equiv C \\
    f_{2}^{\text{out}}(t) & \equiv 0 \\
    f_{3}^{\text{out}}(t) & \equiv C 
\end{align*}
\]  

(4.40)  
(4.41)  
(4.42)

As a result, the solution on link $I_{1}$ is given by a backward-propagating rarefaction wave (or expansion wave, fan wave) with $\hat{\rho}_{1}$ and $\rho^{e}$ on the two sides. On link $I_{3}$, a forward-propagating rarefaction wave with $\rho^{e}$ and $\hat{\rho}_{3}^{0}$ (that is, the limit of $\hat{\rho}_{3}^{\epsilon}$ as $\epsilon \to 0$) on the two sides is created. Link $I_{2}$ remains in a completely jam state with full density $\rho^{\text{jam}}$. See Figure 7 for an illustration of the solutions.

![Figure 7](http://www.sciencedirect.com/science/article/pii/S0191261515002039)

Figure 7: Comparison of solutions on $I_{1}$, $I_{2}$ and $I_{3}$ for the two cases: $\epsilon > 0$ and $\epsilon = 0$. The dashed lines represent the kinematic waves (characteristics), and darker color indicates higher density. Notice that jumps in the boundary conditions across these two cases exist on $I_{1}$ and $I_{3}$.

The two sets of solutions corresponding to $\epsilon > 0$ and $\epsilon = 0$ are shown in Figure 7. We first notice that in these two cases, the boundary flows are very different, despite the infinitesimal difference in the PDV, namely from $\epsilon > 0$ to $\epsilon = 0$. In particular, both $f_{1}^{\text{out}}(t)$ and $f_{3}^{\text{out}}(t)$ jump from $f(\hat{\rho}_{1})$ and $f(\hat{\rho}_{3})$ (when $\epsilon > 0$) to $C$ (when $\epsilon = 0$). This is a clear indication of the discontinuous dependence of the diverge junction model on its initial conditions.

Let us zoom in on the mechanism that triggers such a discontinuity. According to (4.37), the expression for $f_{1}^{\text{out}}(t)$ when $\epsilon > 0$ is

\[
    f_{1}^{\text{out}}(t) = \min \left\{ C, \frac{f(\hat{\rho}_{1})}{\epsilon} \right\} = \min \left\{ C, \frac{\epsilon f(\hat{\rho}_{1})}{\epsilon} \right\}
\]

As long as $\epsilon$ is positive, the fraction $\frac{f(\hat{\rho}_{1})}{\epsilon} = f(\hat{\rho}_{1}) < C$. However, when $\epsilon = 0$ we have an expression of $\frac{\epsilon f(\hat{\rho}_{1})}{\epsilon}$, which should be equal to $\infty$ since link $I_{2}$ has effectively no influence on the junction, and we have $\min(C, 0) = C$. The above argument amounts to the following statement:

- If $\epsilon > 0$, then $C > \frac{\epsilon f(\hat{\rho}_{1})}{\epsilon}$.
- If $\epsilon = 0$, then $C < \frac{\epsilon f(\hat{\rho}_{1})}{\epsilon}$.

which explains the jump in the solutions when $\epsilon$ tends to zero.
Remark 4.1. The fact that $\hat{\rho}_2^\varepsilon$ tends to $\rho_{jam}$ as $\varepsilon \to 0$ plays a key role in this example. As we shall see later in Theorem 4.2 bounding $\hat{\rho}_2^\varepsilon$ away from the jam density is essential for the well-posedness.

4.2.2. Sufficient conditions for the well-posedness of the diverge model

As we have previously demonstrated, the diverge model with time-varying vehicle turning ratios may not depend continuously on its initial conditions, which are defined in terms of the two-tuple $(\rho, \mu)$. In this section, we propose additional conditions that guarantee the continuous dependence with respect to the initial data at the diverge junction. Our analysis relies on the method of wave-front tracking (Bressan, 2000; Garavello and Piccoli, 2006) and the technique of generalized tangent vectors (Bressan, 1993; Bressan et al., 2000). In order to be self-contained while keeping our presentation concise, we move some general background on these subjects and essential mathematical details to Appendix A.

Theorem 4.2. (Well-posedness of the diverge model) Consider the diverge junction (Figure 4) and assume that

1. there exists some $\delta > 0$ such that the supplies $S_2(t) \geq \delta, S_3(t) \geq \delta$ for all $t$;
2. the path disaggregation variable $\mu(t, x)$ has bounded total variation in $t$.

Then the solution of the diverge junction depends continuously on the initial and boundary conditions in terms of $\rho$ and $\mu$.

Proof. The proof is long and technical, and thus will be presented in Appendix B.1 following necessary mathematical preliminaries in Appendix A.

4.3. Well-posedness of the merge model

For the merge junction depicted in Figure 4, the path disaggregation variables $\mu$ become irrelevant since there is only one downstream link. According to Theorem 5.3 of Han et al. (2015b), the solution at the merge junction, in terms of density $\rho$, depends continuously on the initial and boundary conditions. Moreover, according to (3.9), the propagation speed of $\mu$ is the same as the vehicle speed $v(\rho)$, we thus conclude that $\mu$ also depends continuously on the initial and boundary conditions. This shows the well-posedness of the merge junction.

5. Continuity of the delay operator

The proof of the continuity of the path delay operator is outlined as follows. We first show that the first hypothesis of Theorem 4.2 holds for networks that consists of only the merge and diverge junctions discussed in Section 4.1; this will be done in Section 5.1. Then, we propose some mild conditions on the fundamental diagram and the departure rates in Section 5.2 to ensure that the second hypothesis of Theorem 4.2 holds. It then follows that both the merge and diverge models are well posed. Finally, in Section 5.3 we show the well-posedness of the model in terms of $\rho$ and $\mu$ at each origin, where a point queue may be present. Put altogether, these results lead to the desired continuity for the delay operator.

5.1. An estimation of minimum network supply

This section provides a lower bound on the supply, which is a function of density, on any link in the network during the entire time horizon $[0, T]$. Our finding is that the jam density can never be reached within any finite time horizon $T$, and the supply anywhere in the network is bounded away from zero. As a consequence, gridlock will never occur in the dynamic network.

We denote by $D$ the set of destinations in the augmented network with virtual links (see Section 3.3); that is, every destination $d \in D$ is incident to a virtual link that connects $d$ to the rest of the network; see Figure 8. For each $d \in D$, we introduce its supply, denoted $S^d$, to be the maximum rate at which cars can be discharged from the virtual link connected to $d$. Effectively, there exists a bottleneck between the virtual link and the destination; and the supply of the destination is equal to the flow capacity of this bottleneck; see Figure 8. Notice that in some literature such a bottleneck is completely ignored and the destination is simply treated as a sink with infinite receiving capacity. This is of course a special case of ours once we set the supply $S^d$ to be infinity. However, such a supply may be finite and

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even quite limited under some circumstances due to, for example, ramp metering, limited parking spaces, or the fact that the destination is an aggregated subnetwork that is congested.

We introduce below a few more concepts and notations.

\[ L = \min_{l \in \mathcal{A}} L_l : \text{the minimum link length in the network, including virtual links.} \]

\[ C_{\min} = \min_{l \in \mathcal{A}} C_l : \text{the minimum link flow capacity in the network, including virtual links.} \]

\[ \lambda = \max_{l \in \mathcal{A}} \left| f'(\rho_l^{\text{jam}}) \right| : \text{the maximum backward wave speed in the network.} \]

\[ \bar{p} = \min_{J \in \mathcal{V}_M} \left\{ p_J, 1 - p_J \right\} > 0, \text{ where } p_J \text{ is the priority parameter for the merge junction } J \text{ (see Section 4.1.2).} \]

\[ \bar{p}_C_{\min} : \text{the minimum supply among all the destination nodes.} \]

\[ \delta_k = \min_{l \in \mathcal{A}} \inf_{x \in [a_i, b_i]} S_l(\rho_l(t, x)) : \text{the minimum supply at any location in the network during time interval} \]

\[ \left[ k\frac{L}{\lambda}, (k + 1)\frac{L}{\lambda} \right], \text{ where } k = 0, 1, \ldots \]

**Theorem 5.1.** Consider a network consisting of only merge and diverge junctions as shown in Figure 4. Given any vector of path departure rates, the dynamic network loading procedure described by the PDAE system yields the following property in the solution:

\[ \delta_k \geq \bar{p} \min \left\{ \delta^D, \bar{p}C_{\min} \right\} \quad \forall k = 0, 1, \ldots \] (5.43)

**Proof.** We postpone the proof until Appendix B.2 for a more compact presentation.

Theorem 5.1 guarantees that the supply values anywhere in the network during the period \( [k\frac{L}{\lambda}, (k + 1)\frac{L}{\lambda}] \) are uniformly bounded below by some constant that depends only on \( k \), although such a constant may decay exponentially as \( k \) increases. Given that any dynamic network loading problem is conceived in a finite time horizon, we immediately obtain a lower bound on the supplies, as shown in the corollary below.

**Corollary 5.2.** Under the same setup as Theorem 5.1, we have

\[ \min_{l \in \mathcal{A}} \inf_{x \in [a_i, b_i]} S_l(\rho_l(t, x)) \geq \bar{p} \inf_{t \in [0, T]} \left\{ \delta^D, \bar{p}C_{\min} \right\} \quad \forall t \in [0, T] \] (5.44)

and

\[ \min_{l \in \mathcal{A}} \inf_{x \in [a_i, b_i]} \inf_{t \in [0, T]} S_l(\rho_l(t, x)) \geq \bar{p} \min \left\{ \delta^D, \bar{p}C_{\min} \right\} \] (5.45)

where the operator \( \lfloor \cdot \rfloor \) rounds its argument to the nearest integer from below.
Proof. Both inequalities are immediately consequences of \(5.43\).

Remark 5.3. Corollary 5.2 shows that, for any network consisting of the merge and diverge junctions, the supply values are uniformly bounded from below at any point in the spatial-temporal domain. In particular, a jam density can never occur. Moreover, such a network is free of complete gridlock.\footnote{A complete gridlock refers to the situations where a non-zero static solution of the PDAE system exists. Intuitively, it means that a complete jam \(\rho = \rho^{\text{jam}}\) is formed somewhere in the network and the static queues do not dissipate in finite time.}

5.2. Estimation regarding the path disaggregation variables

In this section we establish some properties of the path disaggregation variable \(\mu\), which serve to validate the second hypothesis of Theorem 4.2.

Lemma 5.4. Assume that there exists \(M > 0\) and \(\epsilon > 0\) so that the following hold:

1. For all links \(I_i\), the fundamental diagrams \(f_i(\cdot)\) are uniformly linear near the zero density; more precisely, \(f_i'(\rho)\) is constant for \(\rho \in [0, \epsilon]\) for all \(I_i \in \mathbb{A}\).
2. Each \(f_i(\cdot)\) has non-vanishing derivative, i.e. \(|f_i'(\rho)| \geq \epsilon\), \(\forall \rho \in [0, \rho_i^{\text{jam}}]\).
3. The departure rates \(\{h_p(\cdot), p \in \mathcal{P}\}\) are uniformly bounded, have bounded total variation (TV), and bounded away from zero when they are non-zero; i.e. \(TV(h_p) < M\) and \(h_p(t) \in [0] \cup [\epsilon, M]\) for all \(p \in \mathcal{P}\) and almost every \(t \in [0, T]\).

Then the path disaggregation variables \(\mu(t, x)\) are either zero, or uniformly bounded away from zero. Moreover, they have bounded variation.

Proof. The proof is moved to Appendix B.3

Remark 5.5. Notice that assumptions 1 and 2 of Lemma 5.4 are satisfied by the Newell-Daganzo (triangular) fundamental diagrams, where both the free-flow and congested branches of the FD are linear. Moreover, given an arbitrary fundamental diagram, one can always make minimum modifications at \(\rho = 0\) and \(\rho = \rho^c\) to comply with conditions 1 and 2. Assumption 3 of Lemma 5.4 is satisfied by any departure rate that is a result of a finite number of cars entering the network. And, again, any departure rate can be adjusted to satisfy this condition with minor modifications.

5.3. Well-posedness of the queuing model at the origin with respect to departure rates

In this section we discuss the continuous dependence of the queues \(q_i(t)\) and the solutions \(\rho_i\) and \(\mu_i^p\) with respect to the path departure rates \(h_p(t), p \in \mathcal{P}\). The analysis below relies on knowledge of the wave-front tracking algorithm and generalized tangent vector, for which an introduction is provided in Appendix A.

Following Herty et al. (2007), we introduce a generalized tangent vector \((\eta, \xi, \xi^p)\) for the triplet \((q_i, \rho_i, \mu_i^p)\) where \(\eta \in \mathbb{R}\) is a scalar shift of the queue \(q_i(\cdot)\), i.e. the shifted queue is \(q_i(\cdot) + \eta\); while the tangent vectors of \(\rho_i\) and \(\mu_i^p\) are defined in the same way as in Appendix A.2. The tangent vector norm of \(\eta\) is simply its absolute value \(|\eta|\).

Lemma 5.6. Assume that the departure rates \(\{h_p(\cdot), p \in \mathcal{P}\}\) are piecewise constant, and let \(\xi_p\) be a tangent vector defined via shifting the jumps in \(h_p(\cdot)\). Then the tangent vector \((\eta, \xi, \xi^p)\) is well defined, and its norm is equal to that of \(\xi_p\) and bounded for all times.

Proof. The proof is postponed until Appendix B.4
5.4. Final proof of continuity for the delay operator

At the end of this paper, we are able to prove the continuity result for the delay operator based on a series of preliminary results presented so far.

Theorem 5.7. (Continuity of the delay operator) Consider a network consisting of merge and diverge junctions described in Section 4.1 under the same assumptions stated in Lemma 5.4, the path delay operator, as a result of the dynamic network loading model (3.21)-(3.31), is continuous.

Proof. We have shown that at each node (origin, diverge node, or merge node), the solution depends continuously on the initial and boundary values. In addition, between any two distinct nodes, the propagation speeds of either $ρ$-waves or $µ$-waves are uniformly bounded. Thus such well-posedness continues to hold on the network level. Consequently, the vehicle travel speed $v_i(ρ_i)$ for any $I_i$ depends also continuously on the departure rates. We thus conclude that the path travel times depend continuously on the departure rates.

The assumption that the network consists of only merge and diverge nodes is not restrictive since junctions with general topology can be decomposed into a set of elementary junctions of the merge and diverge type (Daganzo, 1995). In addition, junctions that are also origins/destinations can be treated in a similar way by introducing virtual links.

Here, we would like to comment on the assumptions made in the proof of the continuity. In a recent paper (Bressan and Yu, 2015), a counterexample of uniqueness and continuous dependence of solutions is provided under certain conditions. More precisely, the authors construct the counterexample by assuming that the path disaggregation variables $µ^p_i$ have infinite total variation. This shows the necessity of the second assumption of Theorem 4.3. Moreover, again in Bressan and Yu (2015), the authors prove that for density oscillating near zero the solution may not be unique (even in the case of bounded total variation), which shows the necessity of the third assumption in Lemma 5.4.

Remark 5.8. Szeto (2003) provides an example of discontinuous dependence of the path travel times on the path departure rates using the cell transmission model representation of a signal-controlled network. In particular, the author showed that when a queue generated by the red signal spills back into the upstream junction, the experienced path travel time jumps from one value to another. This, however, does not contradict our result presented here for the following reason: the jam density caused by the red signal in Szeto (2003) does not exist in our network, which has only merge and diverge junctions (without any signal controls). Indeed, as shown in Corollary 5.2, the supply functions at any location in the network are uniformly bounded below by a positive constant, and thus the jam density never occurs in a finite time horizon.

The reader is reminded of the example presented in Section 4.2.1, where the ill-posedness of the diverge model is caused precisely by the presence of a jam density. The counterexample from Szeto (2003) is constructed essentially in the same way as our example, by using signal controls that create the jam density.

We offer some further insights here on the extension of the continuity result to second-order traffic flow models. To the best of our knowledge, second-order models on traffic networks are mainly based on the Aw-Rascle-Zhang model (Garavello and Piccoli, 2006; Haut and Bastin, 2007; Herty and Rascle, 2006) and the phase-transition model (Colombo et al., 2010). Solutions of the Aw-Rascle-Zhang system may present the vacuum state, which, in general, may prevent continuous dependence (Godvik and Hanche-Olsen, 2008). On the other hand, the phase transition model on networks behaves in a way similar to the LWR scalar conservation law model. However, continuous dependence may be violated for density close to maximal; see Colombo et al. (2010). Therefore, extensions of the continuity result to second-order models appear possible only for the phase-transition type models with appropriate assumptions on the maximal density achievable on the network, but this is beyond the scope of this paper.

6. Conclusion

This paper presents, for the first time, a rigorous continuity result for the path delay operator based on the LWR network model with spillback explicitly captured. This continuity result is crucial to many dynamic traffic assignment models in terms of solution existence and computation. Similar continuity results have been established in a number of studies, all of which are concerned with non-physical queue models. As we show in Section 4.2.1 the well-posedness...
of a diverge model may not hold when spillback occurs. This observation, along with others made in previous studies (Szeto 2003), have been the major source of difficulty in showing continuity of the delay operator. In this paper, we bridge this gap through rigorous mathematical analysis involving the wave front tracking method and the generalized tangent vectors. In particular, by virtue of the finite propagation speeds of $\rho$-waves and $\mu$-waves, the continuity of the delay operator boils down to the well-posedness of nodal models, including models for the origins, diverge nodes, and merge nodes. Minor assumptions are made on the fundamental diagram and the path departure rates in order to provide an upper bound on the total variations of the density $\rho$ and the path disaggregation variables $\mu$, which subsequently leads to the desired well-posedness of the nodal models and eventually the continuity of the operator.

A crucial step of the above process is to estimate and bound from below the minimum network supply, which is defined in terms of local vehicle densities. In fact, if the supply of some link tends to zero, the well-posedness of the diverge junction may fail as we demonstrate in Section 4.2.1. This has also been confirmed by an earlier study (Szeto 2003), where a wave of jam density is triggered by a signal red light and causes spillback at the upstream junction, leading to a jump in the path travel times. Remarkably, in this paper we are able to show that (1) if the supply is bounded away from zero, then the diverge junction model is well posed; and (2) the desired boundedness of the supply is a natural consequence of the dynamic network loading procedure that involves only the simple merge and diverge junction models. This is a highly non trivial result because it not only plays a role in the continuity proof, but also implies that gridlock can never occur in the network loading procedure in a finite time horizon. However, we note that in numerical computations gridlock may very well occur due to finite approximations and numerical errors, while our no-gridlock result is conceived in a continuous-time and analytical (i.e., non-numerical) framework.

It is true that, despite the correctness of our result regarding the lower bound on the supply, zero supply (or gridlock) can indeed occur in real-life traffic networks as a result of signal controls. In fact, this is how [Szeto 2003] constructs the counter example of discontinuity. Nevertheless, signal control is an undesirable feature of the dynamic network loading model for far more obvious reasons: the on-and-off signal control creates a lot of jump discontinuities in the travel time functions, making the existence of dynamic user equilibria (DUE) almost impossible. Given that the main purpose of this paper is to address fundamental modeling problems pertinent to DUEs, it is quite reasonable for us to avoid junction models that explicitly involve the on-and-off signal controls. Nevertheless, there exist a number of ways in which the on-and-off signal control can be approximated using continuum (homogenization) approaches. Some examples are given in [Aboudolas et al. (2009); Gazis and Potts (1963); Han et al. (2014) and Han and Gayah (2015)].

Regarding the exponential decay of the minimum supply, we point out that, as long as the jam density (gridlock) does not occur, our result is still applicable. From a practical point of view most networks reach congestion during day time but are almost empty during night time. In other words, the asymptotic gridlock condition is never reached in reality. Therefore, we think that our result is a reasonable approximation of reality for DUE problems.

We would also like to comment on the continuity result for other types of junction models. Garavello and Piccoli (2006) prove that Lipschitz continuous dependence on the initial conditions may fail in the presence of junctions with at least two incoming and two outgoing roads. Notably, the counterexample that they use to disprove the Lipschitz continuity is valid even when the turning ratio matrix $A'$ is constant and not dependent on the path disaggregation variables $\mu^p$. It is interesting to note that, in the same book, continuity results for general networks are provided, but only for Riemann Solvers at junctions that allow re-directing traveling entities – a feature not allowed in the dynamic network loading model but instead is intended for the modeling of data networks.

The findings made in this paper has the following important impacts on DTA modeling and computation. (1) The analytical framework proposed for proving or disproving the well-posedness of junction models and the continuity property can be applied to other junction types and Riemann Solvers, leading to a set of continuity/discontinuity results of the delay operators for a variety of networks. (2) The established continuity result not only guarantees the existence and computation of continuous-time DUEs based on the LWR model, but also sheds light on discrete-time DUE models and provides useful insights on the numerical computations of DUEs. In particular, the various findings made in this paper provide valuable guidance on numerical procedures for the DNL problem based on the cell transmission model (Daganzo 1994, 1995), the link transmission model (Yperman et al. 2005), or the kinematic wave model (Han et al. 2015b, Lu et al. 2013), such that the resulting delay operator is continuous on finite-dimensional spaces. As a result, the existence and computation of discrete-time DUEs will also benefit from this research.
Appendix A. Essential mathematical background

Appendix A.1. Wave-front tracking method

The wave-front tracking (WFT) method was originally proposed by Dafermos (1972) as an approximation scheme for the following initial value problem

\[
\begin{align*}
\partial_t \rho(t, x) + \partial_x f(\rho(t, x)) &= 0 \\
\rho(0, x) &= \hat{\rho}(x)
\end{align*}
\]  
(A.1)

where the initial condition \( \hat{\rho}(\cdot) \) is assumed to have bounded variation (BV) (Bressan, 2000). The WFT approximates the initial condition \( \hat{\rho}(\cdot) \) using piecewise constant (PWC) functions, and approximates \( f(\cdot) \) using piecewise affine (PWA) functions. It is an event-based algorithm that resolves a series of wave interactions, each expressed as a Riemann Problem (RP). The WFT method is primarily used for showing existence of weak solutions of conservation laws by successive refinement of the initial data and the fundamental diagram; see Bressan (2000) and Holden and Risebro (1995). Garavello and Piccoli (2006) extend the WFT to treat the network case and show the existence of the weak solution on a network.

To show that the procedure described above indeed produces a well-defined approximate solution on the network, one needs to ensure that the following three quantities are bounded: (1) the total number of waves; (2) the total number of junction interactions; and (3) the total variation (TV) of the piecewise constant solution at any point in time. These quantities are known to be bounded in the single conservation law case; in fact, they all decrease in time (Bressan, 2000). However, for the network case, one needs to proceed carefully in estimating these quantities as they may increase as a result of a wave interacting with a junction, which may produce new waves in all other links incident to the same junction.

To show that the total variation of \( \rho^\varepsilon \) is uniformly bounded, then as \( \varepsilon \to 0 \) a weak entropy solution on the network is obtained.

Appendix A.2. Generalized tangent vector

The generalized tangent vector is a technique proposed by Bressan (1993) to show the wellposedness of conservation laws, and is used later by Garavello and Piccoli (2006) to show the well-posedness of junction models in connection with the LWR model. Its mathematical contents are briefly recapitulated here.

Given a piecewise constant function \( F(x): [a, b] \to \mathbb{R} \), a tangent vector is defined in terms of the shifts of the discontinuities of \( F(\cdot) \). More precisely, let us indicate by \( \{x_k\}_{k=1}^N \) the discontinuities of \( F \) where \( a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b \), and by \( \{F_k\}_{k=1}^N \) the values of \( F \) on \( (x_{k-1}, x_k) \). A tangent vector of \( F(\cdot) \) is a vector \( \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \) such that for each \( \varepsilon > 0 \), one may define the corresponding perturbation of \( F(\cdot) \), denoted by \( F^\varepsilon(\cdot) \) and given by

\[
F^\varepsilon(x) = F_k \quad x \in [x_{k-1} + \varepsilon \xi_{k-1}, x_k + \varepsilon \xi_k)
\]

for \( k = 1, \ldots, N+1 \), where we set \( \xi_0 = \varepsilon x_{N+1} = 0 \). The norm of the tangent vector \( \xi \) is defined as

\[
||\xi|| = \sum_{k=1}^N |\xi_k| \cdot |F(x_k) - F(x_{k-1})| \quad \text{A.2}
\]
In other words, the norm of the tangent vector is the sum of each $|\xi_i|$ multiplied by the magnitude of the shifted jumps.

The procedure of showing the well-posedness of a junction model using the tangent vectors is as follows. Given piecewise constant initial/boundary conditions on each link incident to the junction, one considers their tangent vectors. By showing that the norms of their tangent vectors are uniformly bounded in time among all approximate wave-front tracking solutions, it is guaranteed that the $L^1$ distance of any two solutions is bounded, up to a multiplicative constant, by the $L^1$ distance of their respective initial/boundary conditions. More precisely, we have the following theorem

**Theorem Appendix A.1.** If the norm of a tangent vector at any time is bounded by the product of its initial norm and a positive constant, among all approximate wave-front tracking solutions, then the junction model is well posed and has a unique solution.

**Proof.** See Garavello and Piccoli [2006].

Throughout this paper, we employ the notation $(\rho_i, \rho_j^-)$ to represent a wave interacting with the junction from road $I_i$, where $\rho_i$ is the density value in front of the wave (in the same direction as the traveling wave) and $\rho_j$ is the density behind the wave. After the interaction, a new wave may be created on some road $I_j$ (it is possible that $I_j = I_i$), and we use $\rho_j^+$ and $\rho_j^-$ to denote the density at $J$ on road $I_j$ before and after the interaction, respectively.

**Lemma Appendix A.2.** Consider the diverge model in Section 4.1.1. If a wave $(\rho_i, \rho_j^-)$ on $I_i$ interacts with $J$ then the shift $\xi_j$ produced on any $I_j$, as a result of the shift $\xi_i$ on $I_i$, satisfies

$$\xi_j(\rho_j^+ - \rho_j^-) = \frac{\Delta q_i}{\Delta q_j} \xi_i(\rho_i^+ - \rho_i^-)$$  \hspace{1cm} (A.3)

where

$$\Delta q_i \doteq f(\rho_i^+) - f(\rho_i^-), \quad \Delta q_j \doteq f(\rho_j^+) - f(\rho_j^-)$$

are the jumps in flow across these two waves.

**Proof.** To fix the ideas, we assume that $I_j$ is the incoming road $I_1$ (the other cases can be treated in a similar way). Let $(\rho_1, \rho_j^-)$ be the interacting wave, then it must be that $\rho_1 \leq \sigma$.

If no wave is reflected on road $I_1$ after the interaction (i.e. no new backward wave $(\rho_1, \rho_j^-)$ is created on $I_1$ as a result of the interaction), then we can conclude the lemma by following the same estimate as shown in Garavello and Piccoli [2006]. If, on the other hand, a wave $(\rho_1, \rho_j^-)$ is reflected then $\rho_j^- < \rho_1$, $(\rho_1, \rho_j^-)$ is a rarefaction wave, and $f(\rho_j^-) \leq f(\rho_j^+) < \rho_j$. In other words, part of the change in the flow caused by the interaction passes through the junction, and part of it is reflected back onto $I_1$. We may then apply the same argument as in Garavello and Piccoli [2006] to the part that passes through the junction $(f(\rho_j^+) - f(\rho_j^-))$, and conclude the lemma.

**Definition Appendix A.3.** If a $\rho$-wave from $I_i$ interacts with the junction, producing a $\rho$-wave on link $I_j$, we call $I_i$ and $I_j$ the source and recipient of this interaction, respectively. Such an event is denoted $I_i \rightarrow I_j$.

We observe that $|\xi_i(\rho_i^- - \rho_j)|$ is precisely the $L^1$-distance of two initial conditions on $I_i$ with and without the shift $\xi_i$ at the discontinuity $(\rho_i, \rho_j^-)$. Similarly, $|\xi_j(\rho_j^+ - \rho_j^-)|$ is the $L^1$-distance of the two solutions on $I_j$ as a result of having or not having the initial shift $\xi_i$ on $I_i$, respectively. Furthermore, if $\xi_i$ is the only shift in the initial condition on $I_i$, then $|\xi_i(\rho_i^- - \rho_j)|$ is the norm of the tangent vector for $I_i$ (see definition (A.2) before the interaction, and $|\xi_j(\rho_j^+ - \rho_j^-)|$ is the norm of the tangent vector for $I_j$ after the interaction. From (A.3), we have

$$|\xi_j(\rho_j^+ - \rho_j^-)| = \frac{|\Delta q_j|}{|\Delta q_i|} |\xi_i(\rho_i^- - \rho_j)|$$  \hspace{1cm} (A.4)

Therefore, to bound the norm of the tangent vectors one has to check that the multiplication factors $|\Delta q_j|/|\Delta q_i|$ remain uniformly bounded, regardless of the number of interactions that may occur at this junction.

Let us consider the diverge junction, and assume that a wave interacts with $J$ from road $I_i$ and produces a wave on road $I_j$, where $i, j = 1, 2, 3$. As in the proof of Lemma Appendix A.2 we can restrict to the flow passing through the junction (indeed the reflected part has the same flow variation and smaller shift since the reflected wave is a slow
big shock). According to (4.32), $\Delta q_2 = \alpha_{1,2}\Delta q_1$ and $\Delta q_3 = \alpha_{1,3}\Delta q_1$. Consequently, we have the following matrix of multiplication factors:

$$
\begin{bmatrix}
\frac{|\Delta q_1|}{|\Delta q_1|} = 1,
\frac{|\Delta q_2|}{|\Delta q_1|} = \alpha_{1,2},
\frac{|\Delta q_3|}{|\Delta q_1|} = \alpha_{1,3}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\frac{|\Delta q_1|}{|\Delta q_2|} = \frac{1}{\alpha_{1,2}},
\frac{|\Delta q_2|}{|\Delta q_2|} = 1,
\frac{|\Delta q_3|}{|\Delta q_2|} = \frac{\alpha_{1,3}}{\alpha_{1,2}}
\end{bmatrix}
$$

(A.5)

We denote this matrix by $[Q_{ij}]_{i,j=1,2,3}$. According to Garavello and Piccoli (2006), in order to estimate the tangent vector norm, it suffices to keep track of just one single shift and show the corresponding tangent vector norm is bounded regardless of the number of wave interactions. To this end, we consider the only meaningful sequence of $\beta$-waves. These will be done in Theorem 4.2.

**Appendix B. Technical Proofs**

**Appendix B.1. Proof of Theorem 4.2**

**Proof.** This proof is completed in several steps.

**Step 1.** We invoke the wave-front tracking framework and the generalized tangent vector to show the desired continuous dependence. As discussed in Appendix A.2, it boils down to showing that the increase in the tangent vector norm, as a result of arbitrary number of wave interactions (including both $\rho$-wave and $\mu$-wave), remains bounded.

**Step 2.** First of all, the end of Appendix A.2 shows that, for constant values of $\mu$ (that is, with fixed vehicle turning ratios), the tangent vector norm is uniformly bounded regardless of the interactions between the $\rho$-waves and the junction. Next, one needs to consider the case where $\mu$ changes from one value to another, i.e. when a $\mu$-wave interacts with the junction. In general, we consider two consecutive interaction times of the $\mu$-waves: $t_k$ and $t_{k+1}$, and let $\beta_k$ be the multiplication factor for the tangent vector norm of $\rho$ relative to the times $t_k$ and $t_{k+1}$. More precisely, we have $\nu(t_{k+1}^-) = \beta_k \nu(t_k^+)$, where $\nu(t)$ is the tangent vector norm of $\rho$ at time $t$. Clearly, $\beta_k$ can only take value in the following matrix where $a^{k}_{1,2}$ and $a^{k}_{1,3}$ are given by the constant $\mu$ value during $(t_k, t_{k+1})$; see (A.5).

$$
Q^k = [Q^k_{ij}] = \begin{bmatrix}
1 & a^{k}_{1,2} & a^{k}_{1,3} \\
1/a^{k}_{1,2} & 1 & a^{k}_{1,3}/a^{k}_{1,2} \\
1/a^{k}_{1,3} & a^{k}_{1,3}/a^{k}_{1,2} & 1
\end{bmatrix}, \quad k = 0, 1, 2, \ldots
$$

(B.1)

Similar arguments can be carried out to all other intervals $[t_{k+1}, t_{k+2}], [t_{k+2}, t_{k+3}]$ and so forth. In the rest of the proof, our goal is to estimate the multiplication factor for the tangent norm of $\rho$ under the WFT framework, assuming arbitrary interaction patterns of the $\rho$-waves and $\mu$-waves.

**Step 3.** We define $\gamma \equiv \frac{\delta}{C_1} > 0$ where $\delta$ is stated in the hypothesis of this theorem and $C_1$ is the flow capacity of link $I_1$. Without loss of generality, we let $\gamma < \frac{1}{2}$. Throughout Step 3, we assume that $\alpha_{1,2}(t) \geq \gamma$ and $\alpha_{1,3}(t) \geq \gamma$ (the other cases will be treated in Step 4).
As demonstrated at the end of Appendix A.2, in order to estimate the increase in the tangent vector norms it suffices to consider the following type of sequence in which $\rho$-waves interacts with the junction: $I_1 \rightarrow I_2$ then $I_2 \rightarrow I_3$; in other words, the recipient of the previous $\rho$-wave interaction is the source of the next $\rho$-wave interaction. For each $k \geq 1$, let $I(k) \in \{I_1, I_2, I_3\}$ be the recipient of the last interacting $\rho$-wave during $(t_k, t_{k+1})$. This gives rise to the sequence $\{I(k)\}_{k \geq 1}$; see Figure B.9 for an illustration.

- $\rho$-wave interaction
- Last $\rho$-wave interaction

![Figure B.9](image)

We make the following crucial observation. Consider any three elements in the sequence of the form $I^{(i)} = I_i$, $I^{(k+1)} = I_2$ and $I^{(k+2)} = I_j$ where $i, j \in \{1, 2, 3\}$. By definition, the product of the multiplication factors within $(t_{k+1}, t_{k+3})$ is

$$Q^{k+1}_i \cdot Q^{k+2}_j = A^{k+1}_2 \cdot Q^{k+2}_i$$

where $A^{k+1}_2 \in \left\{ 1, \frac{\alpha^{k+1}_{1,2}}{\alpha^{k+2}_{1,2}}, \frac{\alpha^{k+1}_{1,3}}{\alpha^{k+2}_{1,3}}, \frac{\alpha^{k+1}_{2,3}}{\alpha^{k+2}_{2,3}} \right\}$, $i \in \{k+1, k+2\}$ (B.2)

where we use the superscripts to indicate dependence on a specific time interval. The significance of (B.2) is that the multiplication factor can be decomposed into a term $A_2$ with a very specific structure (to be further elaborated below), and a term that would have been the multiplication factor as if the middle link $I_2 = I^{(k+1)}$ were removed. Clearly, the argument above applies equally to the case where $I^{(k+1)} = I_3$, and we use

$$A^{k+1}_3 \in \left\{ 1, \frac{\alpha^{k+1}_{1,3}}{\alpha^{k+2}_{1,3}}, \frac{\alpha^{k+1}_{2,3}}{\alpha^{k+2}_{2,3}}, \frac{\alpha^{k+1}_{3,3}}{\alpha^{k+2}_{3,3}} \right\}$$

to represent the term factored out if the middle link $I^{(k+1)} = I_3$ is removed. By repeating this procedure, one may eliminate all links $I_2$ and $I_3$ from the sequence $\{I(k)\}$ except when they are the first or possibly the last in this last. As a result, the multiplicative terms $\{A^{k+1}_2\}$ and $\{A^{k+1}_3\}$ are factored out. Therefore, the entire multiplication factor for the tangent norm is bounded from above by

$$\prod_{m=1}^{\infty} A^{k+1}_2, \prod_{n=1}^{\infty} A^{k+1}_3, \frac{1}{\gamma^2}$$

(B.3)

where $\{k_m\}_{m=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ are subsequences of $\{k\}$ such that $I^{(k_m+1)} = I_2$ and $I^{(k_n+1)} = I_3$. The first two terms in (B.3) are results of eliminating $I_2$ and $I_3$ from the sequence $\{I(k)\}$; the third term takes care of the first element and possibly the last element in the sequence, and is derived from the observation that $Q^{k}_{i,j} \leq \frac{1}{\gamma}$ for all $i, j \in \{1, 2, 3\}$ and $k \geq 0$.

It remains to show that the first and second terms of (B.3) are bounded, and hence the entire (B.3) is bounded. We write, without referring explicitly to the multiplication indices, that

$$\prod_{m=1}^{\infty} A^{k+1}_2 = \prod_k \frac{\alpha^{k+1}_{1,2}}{\alpha^{k+2}_{1,2}} \cdot \prod_k \frac{\alpha^{k+1}_{1,3}}{\alpha^{k+2}_{1,3}} \cdot \frac{1}{\gamma^2}$$

23
Case (2). A forward wave from I this link. Moreover, these backward waves can never reach the upstream end of the link within 

\[ TV \rightarrow \text{corresponding multiplication factors are respectively} \]

interaction. So the only interactions that may change the tangent vector norm are

\[ I \rho \text{the same virtual link} \]

Case (3).

\[
\{ \text{Since} \}
\]

Appendix B.2. Proof of Theorem 5.1

\[
\text{where the second multiplicative term deals with the first and the last element in the sequence since} \max \{a_1^{k+1}, \frac{1}{a_1^{k+1}} \} = \frac{1}{a_1^{k+1}} < \frac{1}{1 - \gamma} \text{for all} \ k. \text{We may then proceed in the same way as Step 3 and conclude that this multiplication factor is bounded, provided that} \ TV(\mu) < \infty. \]

Appendix B.2. Proof of Theorem 5.1

Proof. Since \( \delta_2 \) is defined in terms of the supplies, it suffices for us to focus only on those densities such that \( \rho(t, x) > \rho_0 \) for \( I \in \bar{A}, (t, x) \in [0, T] \times [a_i, b_i] \). It is also useful to keep in mind that densities beyond the critical density always propagate backwards in space. The proof is divided into several steps.

[Step 1]. \( k = 0 \). We have \( t \in [0, \frac{1}{\gamma}] \). Since the network is initially empty, all the supplies \( S_i(\rho(t, x)), I \in \bar{A}, (t, x) \in [0] \times [a_i, b_i] \), are maximal and equal to the respective flow capacities at \( t = 0 \). Afterwards, a higher-than-critical density or a lower-than-maximum supply can only emerge from the downstream end of a link and propagate backwards along this link. Moreover, these backward waves can never reach the upstream end of the link within \( [0, \frac{1}{\gamma}] \) since \( \frac{1}{\gamma} \) is the minimum link traversal time for backward waves.

A higher-than-critical density (backward wave) can only arise in one of the following cases.

Case (1). A forward wave from I (see Figure) interacts with the diverge junction and creates a backward wave on I_1.

Case (2). A forward wave from I_4 (or I_5) interacts with the merge junction and creates a backward wave on I_4 or I_5.

Case (3). A forward wave interacts with a destination from the relevant virtual link and creates a backward wave on the same virtual link.

Each individual case will be investigated in detail below.
\textbf{Case (1).} We use the same notation shown in Figure 4. According to the reason provided above, $S_i(t) \equiv C_i$, $i = 2, 3$ for $t \in [0, \frac{1}{4}]$. Let the time of the wave interaction be $\bar{t}$. Recall from (4.32) that

$$f_i^{\text{out}}(\bar{t}) = \min \left\{ D_1(\bar{t}), \frac{C_2}{\alpha_{1,2}(\bar{t})}, \frac{C_3}{\alpha_{1,3}(\bar{t})} \right\}$$ \hspace{1cm} (B.4)

If the minimum is attained at $D_1(\bar{t})$, then the entrance of $I_1$ will remain in the uncongested phase, i.e. $\rho_1(t +, b_1-) \leq \rho'_1$ after the interaction. Hence, no lower-than-maximum supply is generated. On the other hand, if say $\frac{C_2}{\alpha_{1,2}(\bar{t})}$ is the smallest among the three, then

$$S_1(\rho_1(\bar{t} +, b_1-)) \geq f_1(\rho_1(\bar{t} +, b_1-)) = f_1^{\text{out}}(\bar{t}) = \frac{C_2}{\alpha_{1,2}(\bar{t}+)} \geq \rho_2 \geq \bar{C}_i$$ \hspace{1cm} (B.5)

In summary, the supply values corresponding to the backward waves generated at $I_1$, if any, are uniformly above $\bar{C}_i$.

\textbf{Case (2).} We turn to the merge junction shown in Figure 4 and note $S_6(t) \equiv C_6$ for $t \in [0, \frac{1}{2}]$. As usual, we let $\bar{t}$ be the time of interaction. There are two further cases for the merge junction model as shown in Figure 5. We first consider the situation illustrated in Figure 5(a). Clearly, we have that

$$S_4(\rho_4(\bar{t} +, b_4-)) \geq f_4(\rho_4(\bar{t} +, b_4-)) = f_4^{\text{out}}(\bar{t}) = pC_6 \geq \bar{p}C_{\text{min}}$$ \hspace{1cm} (B.6)

$$S_5(\rho_5(\bar{t} +, b_5-)) \geq f_5(\rho_5(\bar{t} +, b_5-)) = f_5^{\text{out}}(\bar{t}) = (1 - p)C_6 \geq \bar{p}C_{\text{min}}$$ \hspace{1cm} (B.7)

That is, the supply values of the backward waves generated at the downstream ends of $I_4$ and $I_5$ are uniformly above $\bar{p}C_{\text{min}}$.

For the situation depicted in Figure 5(b), we first note that the coordinates of $Q'$ is $(pC_6, (1 - p)C_6)$, and the coordinates of the solution $Q = (f_4^{\text{out}*}, f_5^{\text{out}*})$ either satisfy

$$f_4^{\text{out}*} < pC_6 \quad \text{and} \quad f_5^{\text{out}*} > (1 - p)C_6$$ \hspace{1cm} (B.8)

as shown exactly in Figure 5(b), or

$$f_4^{\text{out}*} > pC_6 \quad \text{and} \quad f_5^{\text{out}*} < (1 - p)C_6$$ \hspace{1cm} (B.9)

Taking (B.8) as an example (the (B.9) case can be treated similarly), we have

$$S_6(\rho_5(\bar{t} +, b_5-)) \geq f_5(\rho_5(\bar{t} +, b_5-)) > (1 - p)C_6 \geq \bar{p}C_{\text{min}};$$ \hspace{1cm} (B.10)

and no increase in density is produced at the downstream end of link $I_4$ since its exit flow is equal to the demand.

To sum up, the supply values corresponding to the backward waves generated at $I_4$ or $I_5$, if any, are uniformly above $\bar{p}C_{\text{min}}$.

\textbf{Case (3).} For each destination $d \in D$, let $S^d$ be its supply and denote by $vl$ the virtual link connected to $d$. If $S^d \geq C_d$ then no backward waves can be generated on this link and the supply value on the link is always equal to $C_d$. If $S^d < C_d$, then only one higher-than-critical density can exist on this virtual link, that is, $\rho$ such that $\rho > \rho''_d$ and $f_0(\rho) = S^d$. Clearly, its supply value is equal to $S^d$.

To sum up, the supply value corresponding to the backward waves generated at any virtual link, if any, is bounded below by $\delta^D$.

Finally, we notice that all the backward waves exhaustively described above originate from the downstream end of a link, and they cannot reach the upstream end of the same link within period $[0, \frac{1}{2})$. Thus, these backward waves cannot bring further reduction in the supply values by means of interacting with the junctions. We conclude that during period $[0, \frac{1}{2})$, the minimum supply in the network, $\delta_0$, is bounded below; that is,

$$\delta_0 \geq \min \left\{ \delta^D, \bar{p}C_{\text{min}} \right\}$$ \hspace{1cm} (B.11)
We move on to Case (4). Without loss of generality, we assume that the backward wave that interacts with the diverge junction is coming from $I_2$, and has the density value $\rho^*_2 \in (\rho^*_2, \rho^{jam}_2)$. Let $\bar{t}$ be the time of the interaction. In view of \[B.4\], if the minimum is attained at $D_1(\bar{t})$, then the interaction does not bring any increase in density at the downstream end of $I_1$, hence no decrease in the supply there. If the minimum is attained at $\frac{S_2(\bar{t})}{\alpha_{1,2}(\bar{t})}$, we deduce in a similar way as \[B.5\] that \[
abla \min \frac{S_1(t, \bar{t}, \bar{b}_{1,-})}{\alpha_{1,2}(\bar{t})} = f_{1}(\rho_1(\bar{t}, \bar{b}_{1,-})) = f_{1}^{out}(\bar{t}+, \bar{b}_{1,-}) = \frac{S_2(\rho^*_2)}{\alpha_{1,2}(\bar{t})} \geq S_2(\rho^*_2) \geq \delta_{k-1}
abla
\]
The last inequality is due to the fact that a backward wave such as $\rho^*_2$ must be created at the downstream end of $I_2$ at a time earlier than $\bar{t}$, thus its supply value $S_2(\rho^*_2)$ must be bounded below by $\delta_{k-1}$. If the minimum is attained at $\frac{S_2(t)}{\alpha_{1,2}(\bar{t})}$, we have \[
abla S_1(\rho_1(\bar{t}, \bar{b}_{1,-})) \geq f_1(\rho_1(\bar{t}, \bar{b}_{1,-})) = f_{1}^{out}(\bar{t}+, \bar{b}_{1,-}) = \frac{S_3(t, \bar{t}, \bar{b}_{2,-})}{\alpha_{1,3}(\bar{t})} \geq S_3(\bar{t}) \geq \delta_{k-1}
abla
\] The last inequality is because any lower-than-maximum supply on $I_1$ must be created in the previous time interval.

To sum up, the supply values corresponding to the backward waves generated at $I_1$, if any, are bounded below by $\delta_{k-1}$.

**Case (5).** For the merge junction, we begin with the case illustrated in Figure 5(a). Assuming the backward wave interacts with the merge junction from $I_2$, with the density value $\rho^*_2 \in (\rho^*_2, \rho^{jam}_2)$. Similar to \[B.6\]-\[B.7\], we have \[
abla S_4(\rho_4(\bar{t}, \bar{b}_{2,-})) = f_4(\rho_4(\bar{t}, \bar{b}_{2,-})) = f_{4}^{out}(\bar{t}+, \bar{b}_{2,-}) = pS_3(\rho^*_2) \geq \bar{p}\delta_{k-1} \tag{B.12}
abla S_4(\rho_4(\bar{t}, \bar{b}_{2,-})) = f_4(\rho_4(\bar{t}, \bar{b}_{2,-})) = f_{4}^{out}(\bar{t}+, \bar{b}_{2,-}) = (1-p)S_3(\rho^*_2) \geq \bar{p}\delta_{k-1} \tag{B.13}
abla
\]
where the last inequalities are due to the same reason provided in **Case (4)**. The case illustrated in Figure 5(b) can be treated in the same way as \[B.8\]-\[B.10\].

To sum up, the supply values corresponding to the backward waves generated at $I_4$ or $I_5$, if any, must be bounded below by $\bar{p}\delta_{k-1}$.

So far, we have shown that for $t \in [k\frac{1}{2}, (k+1)\frac{1}{2})$ where $k \geq 1$, the presence of higher-than-critical densities, as exhaustively illustrated through **Case (1)**-**Case (5)**, brings supply values throughout the entire network that are bounded below by

\[
\begin{align*}
\min \{ \min_{\text{Case (1)-(3)}} \{ \delta^D, \bar{p}C^{\min} \}, \delta_{k-1}, \bar{p}\delta_{k-1} \} &= \min \{ \delta^D, \bar{p}C^{\min}, \bar{p}\delta_{k-1} \} = \min \{ \delta^D, \bar{p}\delta_{k-1} \} \tag{B.14}
\end{align*}
\]

**Step 3.** Recall \[B.11\] and \[B.14\]:

\[
\delta_0 \geq \min \{ \delta^D, \bar{p}C^{\min} \}, \quad \delta_k \geq \min \{ \delta^D, \bar{p}\delta_{k-1} \} \quad \forall k \geq 1 \tag{B.15}
\]
We define $\hat{\delta}_0 = \min \{\delta^D_0, \bar{p}C_{min}^\delta\}$ and $\hat{\delta}_k = \min \{\delta^D_k, \bar{p}\hat{\delta}_{k-1}\}, k \geq 1$. Clearly, $\bar{p}\hat{\delta}_{k-1} \leq \delta^D_k$ for all $k \geq 1$, which implies that $\hat{\delta}_k = \min \{\delta^D_k, \bar{p}\hat{\delta}_{k-1}\}, \forall k \geq 1$. Thus

$$\delta_0 \geq \hat{\delta}_0 = \min \{\delta^D_0, \bar{p}C_{min}^\delta\}, \quad \delta_k \geq \hat{\delta}_k = \bar{p}\hat{\delta}_{k-1} = \bar{p}^k \hat{\delta}_0 = \bar{p}^k \min \{\delta^D_k, \bar{p}C_{min}^\delta\}, \forall k \geq 1$$

\[\square\]

Appendix B.3. Proof of Lemma 5.4

Proof. The proof is divided into a few steps.

[Step 1]. Notice that assumption 3 implies that the departure rates $\{h_p(\cdot), p \in \mathcal{P}\}$ are non-zero on a finite set of intervals. Indeed, let $n$ be the number of intervals where $h_p(\cdot)$ does not vanish then $n \epsilon \leq TV(h_p) < \infty$, which implies that $n$ is finite.

[Step 2]. Using the assumption that each fundamental diagram $f_i(\cdot)$ has nonvanishing derivative, we have, at each origin, that the bounded total variation of the density must imply bounded total variation of the flow, and vice versa. Therefore, the assumption on the bounded variation of $h_p(\cdot)$ implies that the density and path disaggregation variable are of bounded variation on the virtual link incident to that origin.

[Step 3]. Assumption 1 guarantees that, on each link, all waves of the type $(0, \rho)$ or $(\rho, 0)$ are contact discontinuities for $\rho$ sufficiently small and, in particular, travel with a constant speed. Moreover, all $\mu$-waves travel with the same speed for low densities. Therefore, taking into account Step 1, whenever $\mu$ is non-zero, it is uniformly bounded away from zero on the entire network. In particular there exists $\varepsilon' > 0$ so that $\mu^f_r(t, x) \in [0] \cup [\varepsilon', 1]$ for every $t, x, i$ and $p$. Therefore, at diverging junctions, the coefficients $\alpha_{1,2}(t)$ and $\alpha_{1,3}(t)$ satisfy the same properties.

[Step 4]. Let us now turn to the total variation of the path disaggregation variables. We know, from Step 2, that $\mu$ has bounded variation on virtual links incident to the origin. We also know from Step 3 that $\mu$ is bounded away from zero whenever it is non-zero; and the same holds for the turning ratios at divergent junctions. Consider a $\mu$-wave $(\mu_i, \mu_j)$, then its variation $[\mu_j - \mu_i]$ can change only upon interaction with diverge junctions. More precisely, denote by $(\mu^f_r, \mu^c_r)$ and $(\mu^f_r, \mu^c_r)$ the wave respectively before and after the interaction, we have $|\mu^f_r - \mu^c_r| \leq \frac{1}{n} |\mu^c_r - \mu^c_c|$ where $\alpha = \alpha_{1,2}$ or $\alpha = \alpha_{1,3}$ and is bounded away from zero. Since the $\mu$-waves travel only forward on the links with uniformly bounded speed, we must have that the interactions with diverge junctions can occur only finite number of times. Thus we conclude that $\mu$ has bounded total variation.

Appendix B.4. Proof of Lemma 5.6

Proof. Let $\xi^j_i$ be the shift of the $j$-th jump of $h_p(\cdot)$, which occurs at time $t_j$, then the expression of $q_j(\cdot)$ is possibly affected on the interval $[t_j, t_j + \xi^j_i]$ (assuming $\xi^j_i > 0$). More precisely, if $q_j(t) > 0$ then no wave is generated on the virtual link while a shift in the queue is generated with $\eta_k = \Delta h_p \cdot \xi^j_i$, where $\Delta h_p$ is the jump in $h_p(\cdot)$ occurring at time $t_j$. On the other hand, if $q_j(t) = 0$ then a wave $(\rho^-, \rho^+)$ is produced at time $t_j$ on the virtual link with shift $\xi^j_i = \lambda \cdot \xi^j_i$ where $\lambda$ is the speed of the wave $(\rho^-, \rho^+)$. We can then compute:

$$\xi^j_i \cdot (\rho^+ - \rho^-) = \xi^j_i \cdot (\rho^+ - \rho^-) = \xi^j_i \cdot \frac{f(\rho^+ - \rho^-)}{\rho^+ - \rho^-} \cdot (\rho^+ - \rho^-)$$

Moreover, $f(\rho^+ - \rho^-) = \Delta h_p$, thus the norm of the tangent vector generated is the same as before.

To prove that the norm of the tangent vector $(q_j, \xi^j_i, \xi^j_i)$ is bounded for all times, let us first consider a backward-propagating wave $(\rho^-, \rho^+)$ interacting with the queue $q_i$, and with a shift $\xi^j_i$. Then we can write:

$$q^+ - q^- = f(\rho^-) - f(\rho^+)$$
Thus the norm of the tangent vector is bounded.

\[ \eta_i = \frac{\xi^s_i}{\lambda} \left( (f(\rho^-) - f(\rho^+)) \right) = \frac{\xi^s_i}{\lambda} \left( (\rho^- - \rho^+) \cdot (f(\rho^-) - f(\rho^+)) \right) = \xi^s_i \cdot (\rho^- - \rho^+) \]

Thus the norm of the tangent vector is bounded.

The norm of the tangent vector \( (\eta_i, \xi^r_i, \xi^s_i) \) may also change when the queue \( q_i \) becomes empty, but this can be treated in the same way as in [Herty et al. (2007)].

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