ON IHS FOURFOLDS WITH $b_2 = 23$

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(with an Appendix by Michal Kapustka)

Abstract. The present work is concerned with the study of 4-dimensional irreducible holomorphic symplectic manifolds with the second Betti number 23. We present some ideas concerning their classification and describe their relations with EPW sextics. In particular we study the related O’Grady conjecture.

1. Introduction

By an irreducible holomorphic symplectic (IHS) 4-fold we mean (see [B1]) a four dimensional simply connected Kähler manifold with trivial canonical bundle that admits a unique (up to constant) closed non-degenerated holomorphic 2-form and is not a product of two manifolds. These manifolds are building blocks of Kähler manifolds with trivial first Chern class [B1, Thm. 2]. In the case of four dimensional examples their second Betti number $b_2$, is bounded and $3 \leq b_2 \leq 8$ or $b_2 = 23$ (see [Gu]). There are however only two known families of IHS in this dimension, one with $b_2 = 7$ and the other with $b_2 = 23$ [B1]. The first is the deformation of the Hilbert scheme of two points on a K3 surface and the second is the deformation of the Hilbert scheme of three points that sum to 0 on an abelian surface. In this paper we address the problem of classification of IHS 4-folds $X$ with $b_2 = 23$. It is known by [V] and [Gu] that in this case the cup product induces an isomorphism

$$\text{Sym}^2 H^2(X, \mathbb{Q}) \simeq H^4(X, \mathbb{Q})$$

and that $H^3(X, \mathbb{Q}) = 0$. By [F] the Hodge diamond admits additional symmetries, by [S] it has the following shape

\[
\begin{array}{ccccccc}
1 \\
0 & 0 \\
1 & 21 & 1 \\
0 & 0 & 0 & 0 \\
1 & 21 & 232 & 21 & 1 \\
0 & 0 & 0 & 0 \\
1 & 21 & 1 \\
1
\end{array}
\]

Recall that for an IHS 4-fold $X$ we can find a (Fujiki) constant $c$, such that for $\alpha \in H^2(X, \mathbb{Z})$, we have $cq(\alpha)^2 = \int \alpha^4$ where $q$ is a primitive integral quadric form called the Beauville-Bogomolov form defining a lattice structure on $H^2(X, \mathbb{Z})$ called the Beauville-Bogomolov (for short B-B) lattice.

In order to classify IHS 4-folds with $b_2 = 23$ we have to find the possible lattices and the possible Fujiki invariants for the given lattice. Next for fixed Fujiki invariant and B-B lattice find all deformation families of IHS manifolds with the given numerical data. Note that the lattices for the known examples are even but not unimodular.

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The plan of the paper is the following. We show that each ample divisor on an IHS fourfold $X$ with $b_2(X) = 23$ has self-intersection being an integer number of the form $12k^2$ for some $k \in \mathbb{N}$. Next we study the case when $X$ admits a divisor $H$ with $H^4 = 12$, i.e. the minimal possible self-intersection. In this case we have $h^0(O_X(H)) = 6$ and the first possibility to consider is when $H$ defines a birational morphism $\varphi_H : X \to \mathbb{P}^5$ into a hypersurface of degree 12. Recall that the ideal of the conductor of $\varphi_H$ defines then a scheme structure $C$ on the singular locus of the image $\varphi_H(X)$. It is known that $C \subset \mathbb{P}^5$ is Cohen–Macaulay of pure dimension 3. We obtain the following:

**Theorem 1.1.** Suppose that an IHS 4-fold $X$ with $b_2 = 23$ admits an ample divisor with $H^4 = 12$ such that $H$ defines a birational morphism $\varphi_H : X \to \mathbb{P}^5$. Then the unique sextic containing the above defined scheme $C \subset \mathbb{P}^5$ is an EPW sextic that we denote by $S_A$ (we call it the adjoint hypersurface to the image $\varphi_H(X)$).

Our approach to the study of $C \subset X$ is to use methods of homological algebra described in [EFS], [EPW]. In section 5 we obtain that the unique EPW sextic $S_A$ obtained in Theorem 1.1 has to be special.

**Proposition 1.2.** Under the assumptions of Theorem 1.1 let $S_A$ be the sextic adjoint to the image $\varphi_H(X)$. If we now denote by $\Theta_A$ the set defined by equation (2.2), then we have $\Theta_A \neq \emptyset$.

The Proposition above suggests in fact that the morphism $\varphi_H$ is never birational. Indeed, the Proposition implies that for a fixed sextic $S_A$, being adjoint to the image of an IHS manifold, there is an at least one dimensional family of IHS fourfolds $X$ such that $S_A$ is the adjoint hypersurface to $\varphi_H(X)$.

The idea of the proof of the Proposition is the following: suppose that $S_A$ with $\Theta_A = \emptyset$ is the adjoint hypersurface to $\varphi_H(X)$. Then we show that $S_A$ is normal and construct a natural desingularisation $\pi : V \to S_A$ described in Section 4.1. We obtain a contradiction by considering the pull-back $\pi^*(X' \cap S_A)$ on $V$ using the knowledge of the Picard group of $V$ and the natural duality of the picture.

This work is motivated by the O'Grady conjecture [O]:

**Conjecture 1.3.** Show that if a IHS 4-folds $X$ is numerically equivalent to $K3^{[2]}$ (i.e. such that $c = 3$ and $(H^2(X, \mathbb{Z}), q)$ is isometric to $U^3 \oplus E_8^2 \oplus \langle -2 \rangle$ with the standard notation) then it is deformation equivalent to it.

Under the assumptions of the above O'Grady conjecture we have $b_2(X) = 23$ and there exists an ample divisor $H$ on $X$ with $H^4 = 12$. O'Grady proved that either $(X, H)$ is deformation equivalent to $K3^{[2]}$ or to a manifold $(X_0, H_0)$ (satisfying the conditions from [O6; Claim 4.4]) such that the image of $\varphi_H(X)$ is a hypersurface of degree $6 \leq d \leq 12$. He conjectured that the latter case cannot happen. In [K] we showed that $d \geq 9$ and that $|H|$ has at most three isolated base points. The case were $\varphi_H$ is birational is the case where the method of [K] cannot work and also the most difficult for the point of view of O’Grady (see [O6; Claim 4.9]). In this work we show that the adjoint EPW sextic $S_A$ to the image of such a birational morphism would have to be very special as described in Proposition 2.2 (in particular $2 \geq \dim \Theta_A \geq 1$). The main missing problem in the proof of the O’Grady conjecture, is to find whether the double determinantal cubic can be an adjoint EPW sextic as above. See section 2.2 for a general discussion about the conjecture.

In the appendix we present technical results used in the proofs concerning the geometry of the orbits of the natural $PGL(6)$ action on $\mathbb{P}(\wedge^3 \mathbb{C}^6)$.

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2. Preliminary

It was shown in [Hu] that there is a finite number of deformation types of hyperkähler manifolds with fixed form $H^2(X, Z) \ni \alpha \to \int \alpha^2 c_2 \in \mathbb{Z}$. We obtain in a similar way the following:

**Proposition 2.1.** Let $X$ be an IHS 4-fold with $b_2 = 23$. The Fujiiki constant of $X$ is an integer number of the form $3n^2$ where $n \in \mathbb{N}$. In particular the minimal degree of the self-intersection $H^4$ of an ample divisor $H \subset X$ is 12 and in this case we have $h^0(O_X(H)) = 6$.

**Proof.** First by the H-R-R theorem for IHS 4-folds we infer.

\[ h^0(O_X(H)) = \chi(O_X(H)) = \frac{1}{24} H^4 + \frac{1}{24} c_2(X) H^2 + \chi(O_X). \]

Next, by the formula of Hitchin and Sawon we deduce that

\[ (c_2(X), \alpha)^2 = 192 \int \sqrt{A(X)} \cdot \int \alpha^4, \]

for any class $\alpha \in H^2(X, \mathbb{R})$ where the $\hat{A}$-genus in our case is just the Todd genus of $X$.

We claim that $\int \sqrt{A(X)}$ is independent of $X$ with $b_2(X) = 23$. Indeed by the R-R formula as in [HS] we have

\[ \sqrt{A(X)} = \frac{1}{2} \hat{A}_2(X) - \frac{1}{8} \hat{A}_1^2(X), \]

where $\hat{A}_1(X) = \frac{1}{12} c_2$ and $\hat{A}_2(X) = \frac{1}{720} (3c_2^2 - c_4)$. It remains to show that $c_2^2(X) = 828$. But this follows from the fact that $c_4 = 324$ and $\hat{A}_2 = 3$ and proves the claim.

We deduce also that

\[ \frac{(H^4, c_2(X))^2}{H^4} = 300 \]

so $\sqrt{300H^4} \in \mathbb{N}$. It follows that $H^4 = 3k^2$. On the other hand from equation (2.1) we deduce that $\frac{k^2}{8} + \frac{10k}{8} \in \mathbb{N}$ thus $k$ is even; this finish the proof.

For an IHS manifold $X$ with $b_2(X) = 23$ to admit an ample divisor with $H^4 = 12$ there are two possibilities:

- the Fujiiki invariant is 3 and the B-B lattice is even such that there exist a $h \in H^2(X, Z)$ with $(h, h) = 2$.
- the Fujiiki invariant is 12 and there exist a $h \in H^2(X, Z)$ with $(h, h) = 1$.

It is a natural problem to decide whether the latter case can occur.

2.1. IHS fourfolds with $b_2 = 23$ satisfying condition O. Let us present an approach that aims to classify polarized IHS fourfold $(X, H)$ with $b_2 = 23$ such that $H^4 = 12$.

If for all $D_1, D_2, D_3 \in |H|$ that are independent the intersection $D_1 \cap D_2 \cap D_3$ is a curve then we say that $(X, H)$ satisfy conditions O.

Note that this is one of the conditions from [O6] Claim 4.4]. Moreover, each hyperkähler manifold numerically equivalent to $(K3)^2$ can be deformed to one that satisfies condition O. Motivated by this we can the following:

**Problem 2.2.** Is each IHS fourfolds with $b_2 = 23$ deformation equivalent with a polarized IHS fourfold $(X_0, H_0)$ satisfying condition O such that $H_0^4 = 12$.

If we find such deformation we can repeat the arguments from [O], in order to show that $\varphi_{|H_0|}$ is either the double cover of an EPW sextic (thus a deformation of $K3^{[2]}$) or $X_0$ is birational to a hypersurface of degree $12 \geq d \geq 7$, or a 4 to 1 morphism to a cubic hypersurface with isolated singularities, or a 3 to 1 morphism to a normal quartic hypersurface, or $\dim \varphi_{|H_0|}(X_0) \leq 3$.

It is an natural geometric problem to decide which one of those cases can occur.
2.2. The O’Grady conjecture. The motivation to study IHS fourfolds with divisors of small self-intersection comes from the following construction of O’Grady that aims to prove his conjecture\cite{OG}. Let $X$ be an IHS manifold numerically equivalent to $K3^{[2]}$. Consider $M'_{\chi}$, a connected component of the moduli space of marked IHS fourfolds deformation equivalent to $X$ and the surjective period map

$$P: M'_{\chi} \to \Omega_L.$$ 

Then choose an appropriate $\rho \in \Omega_L$ such that $P^{-1}(\rho)$ is an IHS manifold $X$ deformation equivalent to $X_0$ and $\text{Pic}(X_0) = \mathbb{Z}H$ where $H$ is an ample divisor with $H^4 = 12$. The special choice of $\rho$ requires $X_0$ to satisfy more properties that are described in \cite{OG} Claim 4.4. Then O’Grady proved that the linear system $|H|$ gives a map $\varphi_{|H|}$ of degree $\leq 2$ that is either birational onto its image or a special double cover of an EPW sextic. Since this double cover is deformation equivalent with $K3^{[2]}$, his conjecture follows if we prove that $\deg \varphi \neq 1$. So suppose that $\deg \varphi = 1$ then O’Grady remarked that the image of $\varphi_{|H|}$ is a hypersurface of degree $6 \leq d \leq 12$. In \cite{K} we showed that $d \geq 9$ and that $|H|$ has only isolated base points. Note that in the case when $d$ has one isolated points the scheme defined by the ideal of the conductor is contained in a unique quintic. With the method of this paper we can find a Beilinson monad giving this quintic. There is a lot of geometry appearing as discussed in \cite{G}.

The aim of this work is to consider the case $d = 12$; this is the case where the method of \cite{K} does not work and also the most difficult for the point of view of O’Grady, see \cite{OG} Claim 4.9. Then the image of $\varphi_{|H|}$ is a non-normal degree 12 hypersurface $X' \subset \mathbb{P}^5 = \mathbb{P}(W)$. Our idea is to show that the adjoint hypersurface to $X' \subset \mathbb{P}(W)$ is an EPW sextic and then exclude case by case such degenerated sextics (worked out in \cite{O2}, \cite{O3}, \cite{O4} and \cite{IM}). Recall that an EPW sextic $S_A$ is a special sextic hypersurface defined as the determinant of the morphism

$$A \otimes O_{\mathbb{P}5} \to \Omega^3_{\mathbb{P}5}(3) \subset \mathbb{P}(W) \times \bigwedge^3 W$$

corresponding to the choice of a Lagrangian $A \subset \bigwedge^3 W$ (as in \cite{EPW} Ex. 9.3]). Furthermore, following O’Grady we denote by

$$(2.2) \quad \Theta_A = \{ V \in G(3, 6) : V \in G(3, 6) \cap \mathbb{P}(A) \subset \mathbb{P}(\bigwedge^3 W) \}.$$ 

The set $\Theta_A$ is empty for a generic choice of $A$ and measures how singular the EPW sextic is. For special $A$ all the values $0 \leq \dim \Theta_A \leq 6$ can be obtained.

The adjoint EPW sextic constructed in the context of the O’Grady conjecture must be very special. Recall that each hyperkähler manifold numerically equivalent to $(K3)^{[2]}$ can be deformed to one that satisfies condition O. We obtain the following:

**Proposition 2.3.** Suppose that a hypersurface $X' \subset \mathbb{P}^5$ of degree 12 is the birational image of a polarized IHS manifold $(X, H)$ with $b_2 = 23$ satisfying O through a morphism given by the complete linear system $|H|$ such that $H^4 = 12$. Let $S_A \subset \mathbb{P}^5$ be the adjoint EPW sextic to the image $X'$. Then for $S_A$ we have either $\dim \Theta_A = 1$ or $S_A$ is the double determinantal cubic or $S_A$ has a non-reduced linear component.

The idea of the proof is as follows: When $\dim \Theta_A \geq 2$ the EPW sextic must have a non-reduced component. Then for each point $U \in \Theta_A$ the plane $\mathbb{P}(U) \subset \mathbb{P}^5$ is contained in $S_A$ such that $S_A$ is singular along it. We consider in this case after O’Grady (see \cite{O4}) the sets $C_{U, A} \subset \mathbb{P}(U)$ defined in the beginning of section\cite{OG}. Each $C_{U, A} \subset \mathbb{P}(U)$ is either the whole plane or is the support of some sextic curve $C_{U, A}$. We show that $C_{U, A}$ has to be contained in $X'$ thus cannot be a plane (by the condition O). We show also that the support of $C_{U, A}$ must have degree $\leq 3$ (see Lemma \cite{L2}). The important remark is that the intersection $S_A \cap X'$ supports the scheme defined by the conductor of the normalization of $X'$. Checking case by case we exclude all the possibilities with $\dim \Theta_A \geq 2$ except when either $S_A$ is the double determinantal cubic and $X'$ has generically tacnodes along the intersection $S_A \cap X'$ or $S_A$ is reducible and equal to $2H_0 + Q$ where $H_0$ is a hyperplane and $Q$ a quartic such that $H_0 \cap Q$ is supporting the scheme $C$ defined by the conductor. In particular $X'$ has triple points along $C$.
that are not ordinary triple points (see the end of Section 7 for a precise description). A new idea is needed to conclude in those cases.

We believe that by the methods of this paper we can also exclude the case \( \dim \Theta_A = 1 \) however the problem becomes more technical (see Section 8). Note that such \( A \) should satisfy for each \( U \in \Theta_A \) the property \( C_U,A \neq \mathbb{P}(U) \) and \( \deg C_U,A \leq 3 \).

3. The Proof of Theorem 1.1

We assume that \( \varphi: X \to X' \subset \mathbb{P}^5(W) \) is a birational morphism and a finite map onto a hypersurface of degree 12. Let us consider the Beilinson monad \( \mathcal{M} \) applied to \( \varphi_*(\mathcal{O}_X(2)) \). This is the following complex

\[
\cdots \to \bigoplus_{j=0}^{5} H^j(\varphi_*(\mathcal{O}_X(2 + e - j)) \otimes \Omega_{P_5}^{-j}(j - e)) \to \cdots
\]

see [EFS] and [DE]. We have \( H^j(\varphi_*(\mathcal{O}_X(2 - k))) = H^j(\mathcal{O}_X(2 - k)) \) since \( \varphi \) is finite. Let us write the monad \( \mathcal{M} \) in the following form

\[
\begin{array}{ccc}
H^4(\mathcal{O}_X(-3)) & H^4(\mathcal{O}_X(-2)) & H^4(\mathcal{O}_X(-1)) \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{C} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{C} & H^0(\mathcal{O}_X(1)) \\
0 & 0 & 0 & 0 & 0 & H^0(\mathcal{O}_X(2))
\end{array}
\]

From [EFS] Cor. 6.2] the maps in the last row corresponds to the natural multiplication map \( W \otimes H^0(\mathcal{O}(k)) \to H^0(\mathcal{O}(k + 1)) \). Since by a result of Guan [Gn] we have \( \text{Sym}^2 H^0(\mathcal{O}_X(1)) = H^0(\mathcal{O}_X(2)) \) the maps in the last row corresponds to the maps in the Beilinson monad of \( \mathcal{O}_{P_5}(2) \). Moreover, we denote by \( A \) the vector space such that \( A \oplus \text{Sym}^3 H^0(\mathcal{O}_X(1)) = H^0(\mathcal{O}_X(3)) \). Then by analogy the natural complex

\[
0 \to \Omega^3(3) \to \Omega^2(2) \otimes W \to \Omega^1(1) \otimes \text{Sym}^2 W \to \mathcal{O} \otimes \text{Sym}^3 W
\]

is exact and is a free resolution of \( \mathcal{O}_{P_5}(3) \). Its Serre dual can be seen as a part of the first row of the monad. We claim that our Beilinson monad is cohomologous to the following (cf. [CS]):

\[
\begin{array}{cccccc}
\Omega^5(5) \otimes (A \oplus \text{Sym}^3 H^0(\mathcal{O}_X(1)))^\ast & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Omega^2(2) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathcal{O}_{P_5}(2)
\end{array}
\]

Indeed, it is enough to consider the bottom row of \( \mathcal{M} \). First observe that it defines a subcomplex of \( \mathcal{M} \). The quotient complex is denoted by \( \mathcal{M}' \). The complex \( \mathcal{N}' \) obtained by replacing the bottom row by \( \mathcal{O}_{P_5}(2) \) also maps surjectively to \( \mathcal{M}' \); we obtain in such a way two exact sequences of complexes. The claim follows from the long homology exact sequences associated with the exact sequences the complexes.

So from [EFS] Thm. 6.1] we obtain an exact sequence

\[
(3.1) \quad 0 \to \mathcal{O}_{P_5}(-6) \oplus A \otimes \mathcal{O}_{P_5}(-3) \xrightarrow{\xi} \Omega^2_{P_5} \oplus \mathcal{O}_{P_5} \to \varphi_*(\mathcal{O}_X) \to 0.
\]

where \( A \) is the 10-dimensional vector space, dual of the quotient of \( H^0(\mathcal{O}_X(3)) \) by the cubics of \( \mathbb{P}^5 \).

The conductor of the finite map \( \varphi: X \to X' \) is the annihilator of the \( \mathcal{O}_{X'} \)-module \( \varphi_*(\mathcal{O}_X) / \mathcal{O}_{X'} \).

Since the conductor is isomorphic to the sheaf \( \mathcal{H}om(\varphi_*(\mathcal{O}_X), \mathcal{O}_{X'}) \) we can deduce the following resolution

\[
(3.2) \quad 0 \to 10\mathcal{O}_{P_5}(-9) \xrightarrow{b} \Omega^2_{P_5} \oplus \mathcal{O}_{P_5}(-6) \to \mathcal{I}_C \to 0,
\]

where \( C \subset \mathbb{P}^5 \) is the subscheme defined by the conductor. Recall that \( C \) is supported on the singular locus of \( X' \), moreover it is locally Cohen–Macaulay and has pure dimension 3 and degree 36 (see [K]).
Consider the part of $F$ given by
\[
A \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \xrightarrow{f} \Omega_{\mathbb{P}^5}^2
\]
the determinant of this map gives the unique (see [K]) sextic $S_A \subset \mathbb{P}^5$ containing $C$.

**Claim 1.** The sextic $S_A$ is an (maybe degenerated) EPW sextic.

**Proof.** We need the following Lemma.

**Lemma 3.1.** There exists an isomorphism $a : \varphi_*(\mathcal{O}_X(3)) \to \text{Ext}^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(3)), \mathcal{O}_{\mathbb{P}^5})$.

**Proof.** From the relative duality $\text{Hom}_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \mathcal{O}_X) = \varphi_*(\mathcal{O}_X(3))$. Now applying the $\text{Hom}_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), .)$ functor to the exact sequence
\[
0 \to \mathcal{O}_{\mathbb{P}^5}(-12) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_{X'} \to 0
\]
we obtain
\[
\text{Hom}_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \mathcal{O}_{X'}) \to \text{Ext}^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \mathcal{O}_{\mathbb{P}^5}(-6)) \xrightarrow{k} \text{Ext}^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \mathcal{O}_{\mathbb{P}^5}(6))
\]
Where $k$ is locally given by the multiplication by the equation of $Y \subset \mathbb{P}^5$ thus it is 0. From the projection formula we obtain an isomorphism
\[
\text{Hom}_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \mathcal{O}_X) \to \text{Ext}^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(3)), \mathcal{O}_{\mathbb{P}^5})
\]

Since $S^2(\mathcal{O}_{\mathbb{P}^5}(-3) \oplus A \otimes \mathcal{O}_{\mathbb{P}^5})$ is a sum of line bundles we deduce that
\[
\text{Ext}^1_{\mathcal{O}_{\mathbb{P}^5}}(S^2(\mathcal{O}_{\mathbb{P}^5}(-3) \oplus A \otimes \mathcal{O}_{\mathbb{P}^5}), \mathcal{O}_{\mathbb{P}^5}) = 0.
\]
Thus we deduce as in the proof of [EPW] thm. 9.2 that there is no obstruction for $a$ to be a chain map, we find a map $\psi$ that close the following diagram
\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_{\mathbb{P}^5}(-3) \oplus \Omega^3_{\mathbb{P}^5}(3) \\
\downarrow \psi^* & & \downarrow \psi \\
\mathcal{O}_{\mathbb{P}^5}(3) & \to & \mathcal{O}_{\mathbb{P}^5}(-6) \oplus \mathcal{O}_{\mathbb{P}^5} \to \text{Ext}^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(3)), \mathcal{O}_{\mathbb{P}^5}) \to 0.
\end{array}
\]
(3.3)

such that $\psi F^*$ is a symmetric map. Since there is no nonzero map $\mathcal{O}_{\mathbb{P}^5}(3) \to \Omega^3_{\mathbb{P}^5}(3)$, the restriction of the above diagram gives
\[
\begin{array}{ccc}
\Omega^3_{\mathbb{P}^5}(3) & \xrightarrow{F} & A' \otimes \mathcal{O}_{\mathbb{P}^5} \\
\downarrow \rho^* & & \downarrow \rho \\
A \otimes \mathcal{O}_{\mathbb{P}^5} & \xrightarrow{f} & \Omega^2_{\mathbb{P}^5}(3),
\end{array}
\]
where $\rho f^*$ is a symmetric map being the restriction of $\psi F^*$ to $\Omega^3_{\mathbb{P}^5}(3)$. We conclude as in the proof of [EPW] thm. 4.1 that $det(f)$ is an EPW sextic.

**4. The adjoint EPW sextics**

We shall see how to translate geometrical properties of the map $\varphi : X \to X' \subset \mathbb{P}^5$ into geometrical properties of the adjoint EPW sextic. Let us also consider the subschemes $N_r \subset X'$ defined by the $\text{Fitt}^{2r}_4(\varphi_*(\mathcal{O}_X))$ (we have for example $N_1 = C$). Recall that from the results of [MP] &4] the scheme $N_2$ has a symmetric presentation matrix and is of codimension $\leq 3$ if it is non-empty. Moreover $N_2$ is supported on points where $C$ is not a locally complete intersection (see [MP] p.131). Denote by $M_r$ the degeneracy loci of rank $\leq 10 - r$ of the map
\[
A \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \xrightarrow{f} \Omega^2_{\mathbb{P}^5}.
\]

**Proposition 4.1.** The subschemes $N_2$ and $M_2$ of $\mathbb{P}^5$ are equal. Moreover, the radical of the scheme $N_r$ and $M_r$ are equal for $r \geq 2$. 
Proof. This is an analogous statement to the rank condition see [CS] rem. 2.8. We claim that locally the map $F$ can be seen as a symmetric map. Indeed in the diagram \[3.3\] using alternating homotopies as in [EPW] p. 447 we have a freedom of choice of the map $\Psi$. In particular restricting to an affine neighborhood we can assume that the matrix $A := F\Psi$ is symmetric and that $\Psi$ is an isomorphism. Remark that the matrix $B$ consisting of the last 9 columns of $A$ and the matrix $B'$ being the last 9 rows of $B$ have maximal degeneracy loci defining locally the scheme $C$ and the sextic $S_A$ respectively (see the sequence \[3.2\]). Since we know that $X'$ has a non-singular normalization, we can conclude with [KU] Prop. 3.6(3). For the second part we use [KU] Lem. 2.8. \[□\]

4.1. Desingularisation of EPW sextic. We shall need the following construction of desingularisation of the EPW sextic $S_A$: From [EPW] Thm. 9.2 we can see that \((f^V)\) define an embedding of $\Omega^2_{\mathbb{P}^2}(3)$ as a symplectic subbundle of \((A \oplus A^\vee) \otimes \mathcal{O}_{\mathbb{P}^5} = \wedge^3W \otimes \mathcal{O}_{\mathbb{P}^5} \). On the other hand from [O1] \& 5.2 we deduce that we can look at $\wedge^3W \otimes \mathcal{O}_{\mathbb{P}^5}$ as a symplectic vector bundle with the symplectic form induced from the wedge product $\wedge^3W \oplus \wedge^3W \rightarrow \wedge^6W = \mathbb{C}$ such that the fiber of the subbundle $\Omega^2_{\mathbb{P}^2}(3)$ over $v \in \mathbb{P}^5$ corresponds to the 10-dimensional linear space $F_v = \{v \wedge \gamma \in \wedge^3W: \gamma \in \wedge^2W\} \subset \wedge^3W$.

Then, $f^V$ is given by the above embedding composed with the quotient map $\wedge^3W \otimes \mathcal{O}_{\mathbb{P}(W)} \rightarrow (\wedge^3W/A) \otimes \mathcal{O}_{\mathbb{P}(W)}$, where $A$ is a Lagrangian subspace of $\wedge^3W$ (there is a canonical isomorphism $\wedge^3W/A = A^\vee$). More precisely we have a diagram

$$\begin{array}{ccc}
\mathbb{P}^5 & \xrightarrow{\pi} & \mathbb{P}(\Omega^2_{\mathbb{P}^2}(3)) \\
\downarrow{\pi} & & \searrow{\alpha} \\
W \subset \mathbb{P}(\wedge^3W)
\end{array}$$

Such that the image of $\alpha$ is the variety $W_2$ and $\pi_1$ is the rational map (regular outside $G(3,6)$) with fiber $\mathbb{P}(F_v)$ considered in the Appendix. Next, $\alpha$ is given by the complete linear system of the line bundle $T := \mathcal{O}_{\mathbb{P}(\Omega^2_{\mathbb{P}^2}(3))}(1)$. If $\mathbb{P}(A) \cap W_2$ is nonsingular in codimension 1 (c.f. in the case dim $\Theta \leq 0$, denote $V := \alpha^{-1}(\mathbb{P}(A) \cap W_2)$.

Note that $V$ map by $\pi$ onto $S_A$. Moreover, by Proposition \[3.3\] and the description of the tangent space of the EPW sextic given in [O2] Cor. 1.5 we have that $V$ is smooth over points where $S_A$ smooth. We shall need the following:

Lemma 4.2. Suppose that $P \in M_k - M_{k+1}$ then the dimension of the intersection $\mathbb{P}(F_P) \cap \mathbb{P}(A)$ is $k - 1$ for $k \geq 1$.

Proof. It follows from the discussion below that the dimension of the fiber $V \cap \pi^{-1}(P)$ is equal to $k - 1$. We conclude by observing that the map $\alpha$ does not contract curves on $\pi^{-1}(P)$. \[□\]

Recall that the sequence \[3.2\] defines a codimension 1 subscheme $C \subset S_A$. Let us apply Kempf’s idea and pull-back $b^\vee$ (where $b$ is defined by equation \[3.2\]) on $\mathbb{P}(10\mathcal{O}_{\mathbb{P}^5})$. Then as in [L] Appendix B we obtain a diagram

$$
\begin{array}{ccc}
\pi^*\Omega^2_{\mathbb{P}^2}(3) & \oplus \pi^*\mathcal{O}_{\mathbb{P}^5}(-3) & \xrightarrow{\pi^*b^\vee} & \pi^*(10\mathcal{O}_{\mathbb{P}^5}) \\
\downarrow{\pi^*} & & \downarrow{\pi^*} \\
\mathcal{O}(10\mathcal{O}_{\mathbb{P}^5})(1) \oplus \pi^*\mathcal{O}_{\mathbb{P}^5}(3) & \oplus (\mathcal{O}(10\mathcal{O}_{\mathbb{P}^5})(1) \oplus \pi^*\Omega^2(3)).
\end{array}
$$

we see that the degeneracy locus of $b^\vee$ can be seen on $\mathbb{P}(10\mathcal{O}_{\mathbb{P}^5})$ as the degeneracy of $v$, thus as a zero section of

$$
(\mathcal{O}(10\mathcal{O}_{\mathbb{P}^5})(1) \oplus \pi^*\mathcal{O}_{\mathbb{P}^5}(3)) \oplus (\mathcal{O}(10\mathcal{O}_{\mathbb{P}^5})(1) \oplus \pi^*\Omega^2(3)).$$
Finally, remark that the zero scheme of the bundle $\mathcal{O}_{\mathcal{O}_{P^3}}(1) \otimes \pi^*(\mathcal{O}_{P^3}(3))$ defines set theoretically $V$ and the following restrictions $\mathcal{O}_{\mathcal{O}_{P^3}}(1)|_V = \mathcal{O}_{(\mathcal{O}_{P^3}(3))(1)|_V}$ are equal. We proved the following:

**Proposition 4.3.** Suppose that $V$ is normal. Then there exists a divisor $D \subset V$ given by the vanishing of a section of the vector bundle $10\mathcal{O}_{P(\mathcal{O}_{P^3}(3))}(1) \oplus (\mathcal{O}_P(\mathcal{O}_{P^3}(3)) \otimes \mathcal{O}_{P^3}(3))$ on $\mathbb{P}(\mathcal{O}_{P^3}(3))$ (it defines a subscheme of $V$) that project through $\pi$ to $C$ such that the fiber over $p \in C$ has dimension equal to the corank of the map $b$.

In particular it follows from Lemma 4.2 that over points $P \in N_k - N_{k+1}$ the fiber of $\pi|_D$ have dimension $k - 1$ for $k \geq 2$.

**Remark 1.** Denote by $E$ the exceptional divisor of $\alpha$. It maps to $G(3, 6) \subset W_2$ such that the fiber over a point $U \in G(3, 6)$ is a projective plane that maps through $\pi$ to $\mathbb{P}(U) \subset \mathbb{P}(W)$. Moreover, $E$ is isomorphic to the projectivisation of the tautological bundle on $G(3, 6)$. By Lemma 5.1 we deduce that the pull-back $(\alpha \circ \pi_2)^*(H_2)$ is a Cartier divisor in the linear system $|2T - H|$ on $\mathbb{P}(\mathcal{O}_{P^3}(3))$, where $H := \pi^*(H_1)$. Moreover, $E$ is the base locus of $|2T - H|$ such that after blowing-up $E \subset \mathbb{P}(\mathcal{O}_{P^3}(3))$ this linear system become base-point-free and factorize through $\pi_2$.

4.2. **The duality.** Since we have a second fibration $\pi_2$ (see the Appendix) of the variety $W_2$ it is natural to consider the following picture.

\[
\begin{array}{ccc}
\mathbb{P}(W) & \overset{\pi_1}{\leftarrow} & W_2 \subset \mathbb{P}(\wedge^3 W) \\
\mathcal{O}_{\mathbb{P}(W)}(3) & \overset{\pi_2}{\leftarrow} & \mathbb{P}(W^\vee) \\
\end{array}
\]

Denote by $F'_w$ the fiber of $\pi_2$ and by $S'_A \subset \mathbb{P}(W^\vee)$ the corresponding EPW sextic constructed from $A$. It follows from the definition of $\pi_1$ and $\pi_2$ that $\pi_2(F'_w)$ is a hyperplane in $\mathbb{P}(W^\vee)$ that is dual to $v \in \mathbb{P}(W)$. Next it follows from the description from [O2 Cor. 1.5(2)] of the tangent space $T$ to $S_A$ at a smooth point that there is a point $w \in S'_A$ such that $\pi_1(F'_w) = T$. We proved the following:

**Lemma 4.4.** Assume that the sextic $S_A \subset \mathbb{P}(W)$ is irreducible. Then $S'_A \subset \mathbb{P}(W^\vee)$ is irreducible and dual to $S_A$.

**Remark 2.** As remarked by O’Grady in [O2 & 1.3] the map $\pi_2|_{F_v}$ is given by the linear system of Plücker quadrics defining the intersection $F_v \cap G(3, W) = G(2, 5) \subset \mathbb{P}^9$. Thus the fibers of $\pi_2|_{F_v}$ are 5-dimensional linear spaces spanned by $G(2, 4) \subset G(2, 5) \subset \mathbb{P}^9$.

5. **The general case**

The aim of this section is to prove Proposition 1 that an EPW sextic constructed by choosing $\mathbb{P}(A)$ disjoint from $G(3, 6)$ (i.e. such that $\Theta_A = \emptyset$), cannot be the adjoint hypersurface of a birational image of an IHS manifold with $b_2 = 23$. The proof is by contradiction. Suppose that such a sextic $S_A$ can be the adjoint hypersurface as above. Then for the corresponding Lagrangian space $A$ we have that $W_2 \cap \mathbb{P}(A)$ is isomorphic to $V$ (we can thus identify $\pi$ and $\pi_1$). Next from [O3 claim 3.7] we obtain that there are only a finite number of planes contracted to points by $\pi$ and there are no higher dimensional contracted linear spaces. The following Lemma follows from Proposition 4.4.

**Lemma 5.1.** If $L$ is a 9-dimensional Lagrangian subspace of $\mathbb{P}^{19} = \mathbb{P}(\wedge^3 W)$ such that $L \cap G(3, 6) = \emptyset$ then $L \cap W_2$ is smooth and of dimension 4.

Thus $V$ is smooth. Denote by $E_1$ an $E_2$ the exceptional locus of $\pi_1|_V$ and $\pi_2|_V$ respectively, by $T$ the restriction of the hyperplane in $\mathbb{P}(\wedge^3 W)$, and by $H_1$ and $H_2$ the pull-back by $\pi_1$ and $\pi_2$ respectively of the hyperplane sections. By [O2 Prop. 1.9] it follows that the singular locus of $S_A$ is a surface $G$ of degree 40 that is smooth outside of the image of the contracted planes. Moreover, $S_A$ has ODP singularities along the smooth locus of $G$, thus $E_1$ and $E_2$ are reduced.
Using Proposition 9.2 and the Lefschetz theorem [RS Thm. 1], that works when $V$ is smooth and omits the singular locus of $W_2$, we deduce that the Picard group of $V$ has rank 2 and is generated by the restrictions of $H_1$ and $H_2$. We denote the restrictions by the same symbols.

Lemma 5.2. The following equalities holds $H_2 = 5H_1 - E_1$ and $H_1 + H_2 = 2T$.

Proof. The first follows from [Dol] & 1.2.2] and the second from Lemma 9.3. \hfill \Box

Now, we find using Proposition 4.3 a divisor $D \subset V$ in the linear system $|3H_1 + T|$ such that $p(D) = C$. It follows from Proposition 4.1 that $D - E_1$ is an effective divisor $D_1$. Let $l \subset V$ be a line contracted by $\pi_2$ (such lines cover $E_2$). Since $l.T = 1$ thus from Proposition 5.2 we obtain $l.H_1 = 2$. It follows that $l.(D - E_1) = l.(3H_1 + T - E_1) = l.(T + H_2 - 2H_1) = -3$. Since $D_1$ is effective we infer that $l \subset D_1$ thus $E_2 \subset D_1$ and $D_1 - E_2$ is effective ($E_2$ is reduced). This is a contradiction since $D_1 - E_2 = T - 4H_2 - H_1 = -3H_1 - T$ cannot be effective.

6. $\dim \Theta_A = 0$

Let us study the geometry of EPW sextics that are adjoint to the birational image of a IHS fourfolds that satisfy condition $O$. Our aim in this section is to prove the lower bound of $\dim \Theta_A$ in Proposition 2.2. For $U \in G(3, W)$ we see that $\pi(\alpha^{-1}(U)) = \mathbb{P}(U) \subset \mathbb{P}(W)$ is the corresponding plane contained in $S_A$. Let us consider after O’Grady the set

$$C_{U,A} := \{[v] \in \mathbb{P}(U) : \dim(F_v \cap \mathbb{P}(A)) \geq 1\}.$$ 

There is a natural scheme structure on $C_{U,A}$ described in [O2 & 3.1] such that $C_{U,A}$ is either a septic curve or the whole plane $\mathbb{P}(U)$.

Proposition 6.1. The set $C_{U,A}$ is contained in $X' \subset \mathbb{P}(W)$. In particular $C_{U,A}$ is never equal to $\mathbb{P}(U)$ if $X$ satisfy condition $O$.

Proof. First over the points from the set $C_{U,A}$ the map

$$A \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \xrightarrow{f} \Omega^2_{\mathbb{P}^5},$$

has corank $\geq 2$; so they are points from $M_2$. But from Proposition 4.1 we have $N_2 = M_2$ thus $C_{U,A} \subset M_2 = N_2 \subset X'$. Finally it follows from the condition $O$ that $X' \subset \mathbb{P}(W)$ cannot contain any plain. \hfill \Box

Property 1. If $U \in \Theta_A$ then from [O2 Cor. 1.5] the sextic $S_A \subset \mathbb{P}(W)$ is singular along $\mathbb{P}(U)$. Moreover, it follows that $S_A$ has singularities with constant multiplicity along $\mathbb{P}(U) - C_{U,A}$.

We can describe the picture more precisely in our case.

Lemma 6.2. If $\dim \Theta_A = 0$ then $S_A$ is normal. Moreover, $V$ and $\mathbb{P}(A) \cap W_2$ are irreducible.

Proof. Since $S_A$ is locally a complete intersection the normality of $S_A$ follows from the Serre criterium if $S_A$ is nonsingular in codimension 1. It follows from [O2 & 1.3] that $S_A$ is only singular along the sum of planes $\mathbb{P}(U)$ for $U \in \Theta_A$ and along the set $\mathcal{G}$ such that for $v \in \mathcal{G}$ we have

$$\mathbb{P}(F_v) \cap \mathbb{P}(A) \cap G(3, 6) = \emptyset \quad \text{and} \quad \dim(F_v \cap \mathbb{P}(A)) \geq 1.$$ 

From [O2 Prop. 1.9] we infer that $\mathcal{G}$ is a surface.

Since the intersection of $\mathbb{P}(A)$ with the tangent to $W_2$ at $P$ is 5-dimensional isotropic, we deduce from Proposition 4.1 that the intersection $\mathbb{P}(A) \cap W_2$ is smooth at $P \in ((F_v \cap \mathbb{P}(A)) - G(3, W))$ when $F_v \cap \mathbb{P}(A) \cap G(3, W) = \emptyset$. Thus we have to show that the dimension of the exceptional set of $\pi : V \to S_A$ that maps to $\bigcup_{U \in \Theta_A} \mathbb{P}(U)$ is smaller then 4. Since $\Theta_A$ is a finite set it is enough to consider the exceptional set below $C_{U_0,A} \subset \mathbb{P}(U_0)$ for a fixed $U_0 \in \Theta_A$. Since $\Theta_A$ is finite, the fiber $\alpha(\pi^{-1}(v)) \subset F_v$ for a given $v \in C_{U_0,A}$ cut $G(3, W) \cap F_v$ at a finite number of points. Since the dimension of $G(3, 5) \subset \mathbb{P}^9$ is 6, we infer $\dim \pi^{-1}(v) \leq 3$ for all $v \in C_{U_0,A}$ and that $\dim \pi^{-1}(v) \leq 2$ for a generic $v \in C_{U_0,A}$. It follows that $\mathbb{P}(A) \cap W_2$ and $V$ are irreducible. \hfill \Box
The map $V \xrightarrow{\alpha} (\mathbb{P}(A) \cap W_2)$ is an isomorphism outside $\alpha^{-1}(G(3, W))$. Thus from the proof below we deduce that $V$ can only be singular at points that map to a curve $C_{U,A}$ for some $U \in \Theta_A$.

**Proposition 6.3.** The variety $V$ and $\mathbb{P}(A) \cap W_2$ are nonsingular in codimension 1. Moreover, $V$ is normal.

**Proof.** Note that $V$ is locally a complete intersection thus it is enough to show the first part. Our aim is to show that the singular points of $\mathbb{P}(A) \cap W_2$ are contained in the sum of tangent spaces to $G(3, 6)$ at points from $\Theta_A$. Next we show that the intersection of $\mathbb{P}(A) \cap W_2$ with those tangent spaces is of codimension 2.

We need to consider points in the pre-image $B := \pi^{-1}(C_{U,A})$.

Suppose that for a given $U_0$ this set is 3-dimensional, then either there is an one parameter family of planes parameterized by $C_{U_0,A}$ or there is a $\mathbb{P}^3$ mapping to a point on $C_{U_0,A}$. Let us consider the first case such that the other is treated similarly.

Suppose that $V$ is singular along $B$. Then it follows that $\mathbb{P}(A)$ does not cut transversally the tangent plane to $W_2$ at each $p \in \alpha(B) - G(3, W)$. By Proposition 6.1 the intersection $\mathbb{P}(A) \cap F_v \cap F_{w_v}$, where $v = \pi_1(p)$ and $w = \pi_2(p)$, contains the line $[p, U'] \subset F_v$ where $U'$ is one of the finite number of points in the intersection $\mathbb{P}(A) \cap G(3, W)$.

We claim that for a generic choice of $p \in B$ this line $[p, U]$ is contained in the tangent space to $G(2, 5) \subset F_v$ at $U$. Since $\Theta_A$ is finite we infer that for a generic choice of $p \in B$ the lines $[p, U']$, for $U' \in \mathbb{P}(A) \cap G(3, W)$, cuts $G(3, W)$ in one point. From Remark 2 the line is contained in a five dimensional linear space $L_p = F_v \cap F_{w_v}$ such that $L_p \cap G(2, 5)$ is a quadric. Since this line cuts $G(3, W)$ in one point it have to be tangent to $G(2, 5)$. The claim follows.

Let $T_U$ be the projective tangent space to $G(3, W) \ni U$ and $R_U := \mathbb{P}(A) \cap T_U$. The following is a nice exercise.

**Lemma 6.4.** The intersection $K_U := T_U \cap G(3, W)$ can be seen as a set of plane in $\mathbb{P}(W)$ that intersects the plane $\mathbb{P}(U)$ along a line. In particular the intersection $K_U$ has dimension 5 and is a cone over the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. Moreover, the sum of linear spaces $F_v \cap T_U$ for $v \in \mathbb{P}(U)$ is a cone over the determinantal cubic $E_U$ that contains $K_U$ as the singular set.

First we have $\dim R_U \leq 4$ since otherwise we infer
\[
\dim \Theta_A \geq \dim(R_U \cap K_U) \geq 1
\]
a contradiction with $\dim \Theta_A = 0$. We have three possibilities $\dim R_U = 2, 3, \text{or } 4$. Note that the intersection $F_v \cap T_U$ is the tangent space to $G(2, 5)$ at $U$ thus has dimension 6. If $\dim R_U = 4$ we have that each linear spaces $F_v \cap T_U$ for $v \in \mathbb{P}(U)$ intersects $\mathbb{P}(A)$ along a linear space of dimension at least 1 since this intersection contain $F_v \cap R_U$. Thus $C_{U,A} = \mathbb{P}(U)$ and we obtain a contradiction by Proposition 6.1.

Assume that $\dim R_U \leq 3$. We saw above that the generic fiber of $B \rightarrow C_{U,A}$ is a plane contained in $T_U$. Since these fibers are contained in $R_U$ and disjoint outside $U$ we obtain a contradiction.

The closure in $V$ of the exceptional set of the morphism $V \rightarrow S_A - (\bigcup_{U \in \Theta_A} \mathbb{P}(U))$ is a reduced Weil divisor $E_G$ that maps to the surface $\text{supp } N_2$. We have also exceptional sets over points from $\bigcup_{U \in \Theta_A} \mathbb{P}(U)$. Since the intersection $W_2 \cap \mathbb{P}(A)$ is irreducible we deduce that there are two kinds of irreducible components of the exceptional set of $\pi$:

- either one parameter families of planes such that the image through $\pi$ is a curve contained in $C_{U,A}$.
- or 3-dimensional linear spaces $E_i$ for $i = 1, \ldots, s$ mapping to points in $C_{U,A} \subset S_A$ for some $U \in G(3, 6)$.

We believe that such exceptional sets cannot exist. However we only prove that the first type exceptional set cannot occur. For this we need to understand better the duality between $S_A$ and $S_A'$.
**Proposition 6.5.** Suppose that the set of points $v \in \mathbb{P}(W)$ with $\dim(F_v \cap \mathbb{P}(A)) \geq 2$ is a curve $C \subset \mathbb{P}(W)$ then the tangent space $T_{v_0}$ to $C$ at $v_0$ is perpendicular to the linear space spanned by the image $\pi_2(\mathbb{P}(A) \cap F_v)$.

**Proof.** Denote after O’Grady $\tilde{\Delta}(0) := \{(A, v) \in LG(10, \wedge^3 W) : \dim(F_v \cap \mathbb{P}(A)) = 2\}$. It was observed by O’Grady that $\tilde{\Delta}(0)$ is smooth and is an open subset of $\tilde{\Delta}$ where we have $\dim(F_v \cap \mathbb{P}(A)) \geq 2$. We know from [O3, Prop. 2.3] the description of the tangent space to $\tilde{\Delta}$. In particular we deduce that $T_{v_0} = \text{Ker} \tau_{K_v}$, where $K := A \cap F_{v_0}$, with the notation of [O3, eq. 2.1.11]. It remains to show that the linear space spanned by $\pi_2(K)$ is perpendicular to $\text{Ker} \tau_{K_v}$. To see this it is enough to remark that $\pi_2|_{\Pi(K)}$ is given by the system of Plücker quadrics $\delta_v$ and use the definition [O3, eq. 2.1.11]. □

It would be nice to find a simpler proof of the following:

**Proposition 6.6.** There are no exceptional set of the first kind.

**Proof.** Suppose that such exceptional set exists and denote it by $G' \subset V$. Denote by $G \subset \mathbb{P}(A) \cap W_2$ the image of $G'$ through $\alpha$ such that each fiber $G \supset G_v = G \cap F_v$ is a plane and $G$ map to a curve $C_0 \subset \mathbb{P}(U_0)$.

We claim that $G_v$ cuts $U_{v_0}$ along a line contained in $E_{v_0}$. Indeed, from the proof of Proposition 6.3 it follows that the generic fiber $G_v$ cannot be contained in $U_{v_0}$. Next from [O4, Prop. 3.2.6 (3)] we infer that if $G_v$ cuts the tangent space to $T_{U_0} \cap F_v$ only at $U_0$ and is disjoint from $\Theta_A$ then $C_0$ has a node at $v$. The claim follows since the nodes on $C_0$ are at isolated points. We deduce also that $C_0$ is a triple component of $C_{U,A}$ thus it is either a conic or a line.

We infer that the intersection $G \cap T_{U_0}$ has dimension $\geq 2$. Since we know from the proof of Proposition 6.3 that $\dim(T_{U_0} \cap \mathbb{P}(A)) \leq 3$. We deduce that $G \cap T_{U_0}$ is either a plane or $\dim(T_{U_0} \cap \mathbb{P}(A)) = 3$.

Let us show that the second case cannot happen. Suppose first that $G \cap T_{U_0}$ is a cone over a cubic $A$ (being a section of $E_{U_0}$). Denote by $N$ a generic hyperplane section of $G$. Note that $N$ is smooth because it maps by $\pi_2$ to a smooth curve with linear spaces as fibers. It follows that $N$ is the projection of a rational normal scroll. General properties of such surfaces are described in [H &V 2]. In particular $A$ cannot be irreducible because a generic hyperplane section containing the plane spanned by $A$ should be a rational normal curve. Thus $A$ is reducible; we need the following:

**Claim 2.** The curve $C_0$ cannot be a line.

**Proof.** Suppose the contrary and fix a $v \in C_0$. Since the morphism $\pi_2|_{G_v}$ is given by a linear sub-system of conic with base point $G_v \cap G(3, 6)$ is birational and contract the line $G_v \cap T_{v_0}$ to a point, we deduce that $\pi_2(G_v)$ is a surface being an irreducible quadric cone $Q_v \subset \mathbb{P}^2$ tangent to $\mathbb{P}(U_0)$ along a line with vertex at the image of the contracted line (because the image of a line passing through $G_v$ is a line passing through the image of $G_v \cap T_{v_0}$). Consider the rational scroll $N$ and denote by $f$ his fiber of $\pi_1|N$ and by $c_0$ the section $T_{U_0} \cap N$. We saw that $c_0$ is either a line or a conic. We have $H_1|N = f$ and $H_2|N = a.f + b.c_0$ for some $a, b \in \mathbb{Z}$.

Two possibilities are possible either $\pi_2(G)$ is a quadric surface or a threefold. Let us treat the first case: Suppose that $Q_{v_1}$ and $Q_{v_2}$ are equal for $v_1 \neq v_2$. Since $C_0$ is a line we deduce that $H_1|c_0$ have degree 1. Next, from $2T = H_1 + H_2$ and the fact that, $\pi_2(c_0) \subset \mathbb{P}(U_0') \cap Q_{v_0}$ we infer that $H_2|c_0$ have degree 2 and $c_0$ is a line. It follows that $N \subset \mathbb{P}(A)$ is embedded by $c_0 + (e + 1)f$ where $f^2 = -e$ on $N$. Observe that $\pi_2|N$ have connected linear fibers being linear sections of the spaces $F_v$. On the other hand we have $\pi_2(N) = \pi_2(G) = Q_v$ so $2 = (H_2|N)^2 = \pi_2(G)^2$ because $\pi_2|N$ is birational. So using $2T = H_1 + H_2$ we infer $H_2 = 2c_0 + (2e + 1)f$ thus a contradiction with $4(2e + 1) = (H_2|N)^2$.

It follows that the dimension of $\pi_2(G)$ is 3 and $\pi_2|N$ is birational. It is good to have in mind that $\pi_2|G$ is an isomorphism outside the double point locus $G = G' \cup \bigcup_{U \in \Theta_A} \mathbb{P}(U')$ of $S_A'$. From Proposition 6.3 the tangent line $T_{C_0}$ to $C_0$ at $r \in C$ is projectively dual to the space $\mathbb{P}^3$ spanned by $Q_r$. Since we assumed that $C_0$ is a line the image of $\pi_2(G)$ is a projective space that we denote by $\mathbb{P}$. Since the double point locus of $S_A'$ is of codimension 2 we have that $\pi_2|G$
is birational. Consider the locus $G'$ of points $p \in \mathbb{P}$ such that there are two different $v_1, v_2 \in C_0$ with $p \in Q_{v_1} \cap Q_{v_2}$ such that $G' \subset G$. We obtain the contradiction by proving that $G' = \mathbb{P}$. Fix a generic $v_0 \in C_0$; it is enough to prove that $Q_{v_0} \subset G'$. Recall that moving $v \in C_0$ the center of the cone $Q_v$ moves along a curve in $\mathbb{P}(U''_0) \subset \mathbb{P}$ such that $Q_v$ is tangent to $\mathbb{P}(U''_0)$. We conclude by observing that it is impossible that such quadrics is in the same pencil determined by a common quartic curve.

We deduce that $C_0$ is a triple conic. Since $\dim(T_{U_0} \cap \mathbb{P}(A)) \neq 3$ we infer that $T_{U_0} \cap \mathbb{P}(A)$ is a plane. Let us consider again the ruled surface $\pi$ common quartic curve. containing $P$ embedded by $H$ a degree 4 curve, we deduce that $\pi$ is covered by projective spaces $\mathbb{P}^3$ dual to the tangent lines to $C_0$. It follows that $\pi_2|G$ is an isomorphism outside $G \cap T_{U_0}$. Consider the pull-back by $\pi_2|N$ of a generic hyperplane containing $\mathbb{P}(U''_0)$. Since the intersection of the hyperplane with $Q$ are two projective spaces we infer that the class of the pull-back $H_2|N$ is $a.c_0 + 2f$. Using $2T = H_1 + H_2$ we compute that $a = 2$ and $e = 1$ thus $N$ is the blow-up of $\mathbb{P}^2$ in one point with $c_0$ as exceptional line. Moreover, $\pi_2|N$ contract $c_0$ and maps $N$ to a projective plane. We infer that $\pi_2(N)$ intersects $\mathbb{P}(U''_0)$ at only one point being the image of $c_0$. It follows also that $\pi_2(N)$ is either the second Veronese embedding of $\mathbb{P}^2$ or a smooth central projection of this second Veronese (because $\pi_2(N)$ can be singular only at one point). Consider the curve $D_0$ being the generic fiber of the projection of $\pi_2(N)$ with center $\mathbb{P}(U''_0)$ to the curve $W$. The curve $D_0$ can be seen as the intersection $\pi_2(N) \cap \mathbb{P}^3$ for some generic $v \in C_0$. Since there are no lines nor degree three curve contained in the projection of the double Veronese and a hyperplane section cuts $\pi_2(N)$ along a degree 4 curve, we deduce that $D_0$ is an irreducible plane conic. We obtain a contradiction since $D_0 \subset Q_v \subset \mathbb{P}^3$ cannot pass through the center of the cone $Q_v$.

We can now return to the proof of Proposition 6.6. To obtain a contradiction we can now proceed as in the general case. By [10, 1.2.2] the rational map between the sextic $S_A$ and his dual $S_A'$ is given by the partial derivatives of the sextic $s_A$ defining $S_A$. The composition

$$V \to S_A' \to S_A' \subset \mathbb{P}(W')$$

is given by the linear system induced by the pull-back of quintics being the partial derivatives of $s_A$ on $V$. On the other hand, by Remark 1.1 each such generic quintic $q'$ corresponds to an irreducible Cartier divisor $Q' \in |2T - H|$ on $V$. The divisor $Q'$ coincide with the proper transform of the zero locus $\{q' = 0\} \cap S_A$ on $V$ (they are equal on an open subset of of $Q'$). Recall that $S_A$ has ordinary double points along a generic point of supp $N_2$. It follows from Proposition 6.6 that the Cartier divisor $\pi^*(Q') = E_G + \sum a_i E_i + B$ where $a_i \geq 0, B \in |2T - H|$ is effective Cartier divisor on the normal variety $V$, $E_i$ are exceptional divisors mapping to points on $C$, and $E_G$ is the exceptional divisor over supp $N_2$. We infer $E_G + \sum a_i E_i \in |6H - 2T|$.

By Proposition 4.3 we find, similarly as in the general case, a divisor $D \subset V$ in the linear system $|3H - T|$ that maps to $C \subset S_A$. From Proposition 4.1 we deduce that $D$ is decomposable such that $D - E_G$ is an effective Weil divisor. We saw that

$$D' = D - (E_G + \sum a_i E_i)$$

is an Cartier divisor in the linear system $|3T - 3H|$. Since the Weil divisors $E_i$ cuts $a^{-1}(U)$ in isolated points we infer that $D'$ restricts to an effective curve on the plane $a^{-1}(U)$, where $U \in \Theta_A$ is fixed. On the other we have $O_V(T)|_{a^{-1}(U)} = O_{a^{-1}(U)}$ and $O_V(H)|_{a^{-1}(U)} = O_{a^{-1}(U)}(1)$. Thus the restriction of a divisor from $|3T - 3H|$ cannot be an effective curve on $\mathbb{P}(U)$ (see [KM prop. 1.35(1)])). It follows that $D$ contains $a^{-1}(U)$ so $X'$ contains $\mathbb{P}(U)$, this is a contradiction by Proposition 6.4.

7. $\dim \Theta_A \geq 2$

We show first that $\dim \Theta_A \geq 3$ cannot happen. Choose an irreducible component $\Theta_A'$. Denote by $G$ the reduced sum of planes $\mathbb{P}(U)$ for $U \in \Theta_A'$.
Lemma 7.1. If \( \Theta_A' \) has dimension \( k \) and \( \mathcal{G} \) has dimension \( \leq k + 1 \) then there is a point \( U \in \Theta_A' \) such that \( \mathcal{C}_{U,A} \) is a plane.

Proof. First, \( (\alpha^{-1}(\Theta_A'))_{\text{red}} \) is irreducible of dimension \( k + 2 \), so the image

\[ \mathcal{G} = (\pi(\alpha^{-1}(\Theta_A')))_{\text{red}} \]

is irreducible. Suppose it has dimension \( \leq k + 1 \) and that all the \( \mathcal{C}_{U,A} \) are curves. Then there exists an open \( U \subset (\alpha^{-1}(\Theta_A'))_{\text{red}} \) set such that \( \pi_U \) is 1 : 1 onto a proper subset of \( \mathcal{G} \). This is a contradiction since \( (\alpha^{-1}(\Theta_A'))_{\text{red}} \) is irreducible.

If \( \dim \Theta_A \geq 3 \) (thus \( \dim \mathcal{G} \leq 4 \)) then \( X' \) have to contain a plane and we obtain a contradiction with the condition \( O \).

7.1. \( \dim \Theta_A = 2 \). The strategy in this case is to show that in many cases the support \( \mathcal{C}_{U,A} \)
has degree \( \geq 4 \). We then obtain a contradiction showing that \( \mathbb{P}(U) = \mathcal{C}_{U,A} \subset X' \) using the following:

Lemma 7.2. If the intersection \( \mathbb{P}(U) \cap X' \) have dimension 1 then it supports a cubic curve.

Proof. If \( \dim \Theta_A \leq 1 \) then this is a consequence of Proposition 4.3. If \( \dim \Theta_A = 2 \) similar arguments apply: For a fixed \( U \in \Theta_A \) the plane \( \alpha^{-1}(U) \subset \mathbb{P}(O_{P^5}^2(3)) \) is a plane that maps through \( \pi \) to \( \mathbb{P}(U) \). On the other hand we see that \( \alpha^{-1} \) is contained in \( \mathbb{P}(10O_{P^5}) \) such that \( \pi'^{-1}(O_{P^5}(1)) \) is equal to the pull back of \( O_{P^5}(1) \) on \( \mathbb{P}(10O_{P^5}) \) are equal and \( \mathcal{O}_{10O_{P^5}}(1|_{\alpha^{-1}(U)}) = \mathcal{O}_{10O_{P^5}}(1)_{\alpha^{-1}(U)} \). Thus we can conclude as in Proposition 4.3.

Recall that O’Grady defined for \( A \in LG(10, \wedge^3 W) \) and \( U \in \Theta_A \) the set \( B(U, A) \subset \mathbb{P}(U) \), of \( v \) such that either:
- There exists \( U' \in (\Theta_A - \{U\}) \) such that \( v \in \mathbb{P}(W') \) or
- \( \dim(\mathbb{P}(A) \cap F_v \cap T_U) \geq 1 \),
where \( T_U \) is the projective tangent space to \( G(3, 6) \) at \( U \).

Property 2. By [O4, Cor. 3.2.7] we know that \( \mathcal{C}_{U,A} \) can have only isolated singularities outside \( B(U, A) \). Next, if \( \mathbb{P}(U) \neq \mathcal{C}_{U,A} \) then we have \( B(U, A) \subset \text{sing} \mathcal{C}_{U,A} \). Moreover, if \( U_1, U_2 \in \Theta_A \) then \( \mathbb{P}(U_1) \) and \( \mathbb{P}(U_2) \) intersect as plane in \( \mathbb{P}^5 \) in a point from \( \mathcal{C}_{U_1,A} \cap \mathcal{C}_{U_2,A} \).

O’Grady observed also that we can apply the Morin theorem [M]. Indeed, if \( \Theta_A' \) is a irreducible component of \( \Theta_A \) of dimension \( \geq 1 \) then it parameterize mutually intersecting planes. By the Morin theorem it follows then that \( \Theta_A' \) is a linear section of one of the following sets:

1. \( \mathbb{P}^2 \) embedded in \( G(3, 6) \subset \mathbb{P}^9 \) by the double Veronese embedding
2. \( G(2, 5) \subset G(3, W) \) embedded as fibers of \( \pi_1 \)
3. \( G(2, 5) \subset G(3, W) \) embedded as fibers of \( \pi_2 \)
4. \( T_P \cap G(3, W) \) where \( T_P \) is the projective tangent space at \( P \) to \( G(3, W) \subset \mathbb{P}(\wedge^3 W) \)
5. \( \mathbb{P}^2 \) embedded in \( G(3, W) \subset \mathbb{P}(\wedge^3 W) \) by the triple Veronese embedding.

Our aim of is to check case by case the possible two dimensional irreducible component \( \Theta_A' \) of \( \Theta_A \) and find that either:

- \( \Theta_A' \) is the third Veronese embedding of \( \mathbb{P}^2 \) in \( G(3, 6) \) or
- \( \Theta_A' \) is a irreducible component supported on a hyperplane.

The last case happen for example when \( \Theta_A' \) is a plane. Note that by Lemma 7.1 we can assume that \( \mathcal{G} \) is a hypersurface of degree \( \leq 3 \) (because \( \mathcal{G} \) is a non-reduced component of \( \mathcal{S}_A \)).

Case (1) From Lemma 7.1 we deduce that \( \Theta_A' \) is a hyperplane section of the double Veronese embedding of \( \mathbb{P}^3 \) (this is the only possibility because there are no plane contained in this double Veronese). It follows from [O1, Claim 1.14] that \( \mathcal{G} \) is a smooth quadric moreover we have the following:

- from [O5 Prop. 2.1] it follows that \( \mathcal{G} \) has multiplicity 2 in the EPW sextic \( \mathcal{S}_A \) (thus \( \mathcal{S}_A \) can be written in the following form \( 2 \mathcal{G} + R \) where \( R \) is a quadric),
- the intersection \( R \cap \mathcal{G} \) is contained in the sum of \( \mathcal{C}_{U,A} \) for \( U \in \Theta_A' \),
the restriction of $\pi$: $(\alpha^{-1}(\Theta_A'))_{\text{red}} \rightarrow \mathcal{G}$ is the blow-up of a plane $F$ contained in $\mathcal{G}$.

Since the curves $C_{U,A}$ cover $F$ we have $F \subset X'$ (each curve $C_{U,A}$ is contained in $X'$). This is a contradiction with the condition $\mathcal{O}$.

**Case (2)** Denote by $v \in \mathbb{P}(W)$ the image $\pi_1(\Theta_A')$. The planes parameterized by $\Theta_A'$ contain the point $v$ and are defined by a line $l_p \subset (2, V/\langle v \rangle)$. Using [O2 Prop. 2.31] we deduce that $\Theta_A'$ is either

a) a plane or $\Theta_A' \subset G(2, T) \subset G(2, 5)$ where $T \in G(4, 5)$ or
b) $\Theta_A'$ is a linear section of $G(2, 5)$ being a del Pezzo surface or
c) there is a line $l_0 \subset \mathbb{P}(V/\langle v \rangle)$ that intersects all the lines $\mathbb{P}(V/\langle v \rangle)$ parameterized by $\Theta_A'$.

We shall treat each case separately.

- If we assume a) then the planes parameterized by $\Theta_A'$ cover a hyperplane. From Property 1 this hyperplane have to be a multiple component of $S_A$.

- Assume b) such that $\Theta_A'$ is a linear section of $G(2, 5) \subset F_v$. Then $\Theta_A'$ is a possibly singular del Pezzo surface $D_5$ of degree 5 (observe that $D_5$ cannot be reduced if it has one component because of the degree). Then the sum of planes parameterized by $\Theta_A'$ is a cone over a cubic hypersurface; denote it by $Q$. More precisely these planes are spanned by the lines corresponding to points on $D_5 \subset G(2, 5)$ (the sum of this lines is a cubic threefold denote it by $Q' \subset \mathbb{P}(V/\langle v \rangle)$). It follows that the corresponding EPW sextic is the double cubic. Since, $\dim(\mathbb{P}(A) \cap F_v) = 5$ from [O4 Prop. 3.1.2] and [O4 Claim 3.2.2] it follows that $v$ is a point of multiplicity 6 on $C_{U,A}$ for $U \in D_5$. Thus $C_{U,A}$ is a sum of multiple lines passing through $v$ (if it is the whole plane we obtain a contradiction).

Let us now identify the sets $B(U,A)$ in order to prove that $C_{U,A}$ have to be reduced for a generic $U \in D_5$. Let us fix such a generic point $U$ of $D_5$ then the intersection $\mathbb{P}(A) \cap T_{U,G(3,W)}$, where $T_{U,G(3,W)}$ is the projective tangent space to $G(3,W)$ at $U$, has dimension 2. Moreover, $\dim(F_v \cap \mathbb{P}(A) \cap T_{U,G(3,W)}) = 2$ because the space contains the tangent space to the del Pezzo surface $D_5 \subset F_v$ and is contained in the previous intersection. It follows also that the set of $w \in \mathbb{P}(U)$ such that $\dim(\mathbb{P}(A) \cap F_w \cap T_{U,G(3,W)}) \geq 1$ is equal to the singleton $\{v\}$. Now observe that $U$ does not belong to any line on $D_5 \subset \mathbb{P}^5$ since such lines cannot cover the whole $D_5$ when $D_5$ is irreducible of dimension 2. Thus it follows that for $U' \in D_5 - \{U\}$ we have $\mathbb{P}(U') \cap \mathbb{P}(U) = \{v\}$.

So the set $B(U,A)$ is equal to the sum of intersections $\mathbb{P}(U) \cap \mathbb{P}(V_0)$ where $V_0 \in \Theta_A - D_5$ and $\{v\}$. Let us show that for each such $V_0$ we have $C_{V_0,A} = \mathbb{P}(V_0)$; thus we obtain a contradiction with condition $\mathcal{O}$. For a fixed $V_0$ we have that $\mathbb{P}(V_0)$ intersects $\mathbb{P}(U)$ outside $v$ (because $F_v \cap G(3,W) = G(2,5)$) and from Property 2 at one point (since $C_{U,A}$ is a sum of lines passing through $v$). Since the plane $\mathbb{P}(V_0)$ have to be contained in our cubic hypersurface $S$, the set $C_{V_0,A}$ must be the whole $\mathbb{P}(V_0)$.

It follows that $C_{U,A}$ is a reduced sum of six lines for a generic choice of $U \in D_5$. We obtain a contradiction by Lemma 7.2

- Assume c) then $\Theta_A'$ is linear section of the cone with vertex $U_0$ over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2$. The planes parameterized by points in $\Theta_A'$ are spanned by the point $v$ and a line in $\mathbb{P}^4(V/\langle v \rangle)$. Let us describe this lines geometrically. The first factor of $\mathbb{P}^1 \times \mathbb{P}^2$ corresponds to a choice of a point on the line $l_0$ and the second factor corresponds to a choice of plane containing $l_0 \subset \mathbb{P}^4$ (and the directrix of our cone gives a choice of a line on this plane passing through our point).

We will obtain a contradiction by showing that $\mathbb{P}(U_0)$ must be contained in $X'$. Thus it is enough to show that the sum of curves $C_{U,A}$ for $U \in \Theta_A'$ covers the line $l_0$. By Property 2 it is enough to prove that for each point of $l_0$ there are at least two lines parameterized by $\Theta_A$ that contain this point. If $\Theta_A'$ contains $U_0$ then it is a cone and we obtain a contradiction unless $\Theta_A'$ is a plane spanned by $U_0$ and a line contained in the second factor of $\mathbb{P}^1 \times \mathbb{P}^2$. Indeed, the planes in $\mathbb{P}(W)$ parameterized by the point from $\Theta_A'$ intersects in this case along a line spanned by $v$ and the fixed point from $l_0$ and cover a hyperplane. If $\Theta_A'$ does not contain the vertex $U_0$ we obtain similarly a contradiction unless the image of the projection $\Theta_A' \rightarrow \mathbb{P}^1$ is a point $Q_0$. 

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In the remaining case \( \Theta'_A \) is a plane and the planes parameterized by \( \Theta'_A \) pass through a line \( l \) (determined by \( v \) and \( Q_0 \)) and cover a hyperplane \( H_0 \) being a non-reduced component of \( S_A \).

**Case (3)** Suppose that \( G(2, 5) \) is embedded in \( G(3, W) \) as a fiber of \( \pi_2 \). This embedding is given by choosing a point \( L \in G(5, W) \) that gives a natural embedding \( G(3, L) \subset G(3, W) \). In this case the sum of planes corresponding to points in \( \Theta'_A \) is contained in the hyperplane \( \mathbb{P}(L) \subset \mathbb{P}(W) \). By Lemma 6.1 we can assume that this sum covers \( \mathbb{P}(L) \). It follows from Property 1 that \( S_A \) has a non-reduced linear component.

**Case (4)** Then from Lemma 6.4 the component \( \Theta'_A \) is two-dimensional linear section of the cone over \( \mathbb{P}^2 \times \mathbb{P}^2 \) in \( \mathbb{P}^3 \) with vertex \( U_0 \). It is useful to have in mind the following:

**Lemma 7.3.** Geometrically the first factor of \( \mathbb{P}^2 \times \mathbb{P}^2 \) corresponds to a choice of line in \( \mathbb{P}(U_0) \) and the second factor the choice of a \( \mathbb{P}^3 \) containing \( \mathbb{P}(U_0) \). The directrix of the cone corresponds to planes containing the fixed line in a fixed \( \mathbb{P}^3 \).

Suppose first that \( \Theta'_A \) contain the vertex of the cone \( U_0 \in G(3, 6) \). Then the plane \( \mathbb{P}(U_0) \) is covered by the intersection with other planes corresponding to points from \( \Theta'_A \) unless \( \Theta'_A \) maps to a point by the projection \( \Theta_A \rightarrow \mathbb{P}^2 \). Thus, in the first case, we obtain a contradiction from Proposition 6.1. But in the second case we see that \( \Theta'_A \) is a plane; then we are also in Case (2) that was described before.

We can assume that \( \Theta'_A \) does not contain the vertex of the cone so we can use [O2 Prop. 2.33]. We want to obtain a contradiction by showing that \( \mathbb{P}(U_0) \subset X' \). For this it is enough to see that the sums of the curves \( C_{U, A} \) for \( U \in \Theta'_A \) contain \( \mathbb{P}(U_0) \). Consider the projections to the factors \( \mathbb{P}^2 \leftarrow \Theta'_A \rightarrow \mathbb{P}^2 \). Since by Property 2 the intersection of two planes \( \mathbb{P}(U) \) and \( \mathbb{P}(V) \) is contained in the curve \( C_{U, A} \) and \( C_{V, A} \) we obtain a contradiction when the dimensions of the images of both projections have dimension \( \geq 1 \). The remaining case is when \( \Theta'_A = v \times \mathbb{P}^2 \), where \( v \) corresponds to a fixed line in \( \mathbb{P}(U_0) \). But then we are in Case (2).

**Case (5)** We assume that \( \Theta'_A \) is the triple Veronese embedding of \( \mathbb{P}^2 \). Then from [O1 Claim 1.16] we have that \( \mathcal{G} \) is the secant cubic of the Veronese surface in \( \mathbb{P}^5 \). It follows from [O2 4.4] that for all \( U \in \Theta'_A \) we have that \( C_{U, A} \) is a triple smooth conic. Consider the restriction \( \mathcal{E}_0 \rightarrow \Theta'_A \) of the tautological bundle on \( G(3, 6) \). In this case we obtain \( \mathcal{E}_0 = S^2 \Omega^1_{\mathbb{P}^2}(1) \) and the following diagram:

\[
\begin{array}{ccc}
\mathbb{P}(\Omega^2_{\mathbb{P}^2}(3)) & \supset & \mathbb{P}(S^2 \Omega^1_{\mathbb{P}^2}(1)) \\
\downarrow \pi & & \downarrow \Theta'_A \\
\mathbb{P}^5 & \supset & \mathcal{G}
\end{array}
\]

The system of quadrics containing the Veronese surface give the Cremona transformation

\[
\begin{array}{ccc}
\mathbb{P}^5 & \leftarrow & \mathbb{P}^5 \\
_{c_1} & & \quad_{c_2}
\end{array}
\]

where \( c_1 \) and \( c_2 \) are the blow up of the Veronese surface \( V_i \subset \mathbb{P}^5 \) for \( i = 1, 2 \) respectively. Then the exceptional divisor \( E \) of \( c_1 \) maps through \( c_2 \) to the determinantal cubic singular along \( V_2 \). Moreover, the exceptional divisor \( F \) of the induced map \( E \rightarrow \mathcal{G} \) is naturally isomorphic to the projective bundle \( \mathbb{P}(\Omega^2_{\mathbb{P}^2}(1)) \). We see also that \( \pi|_{\mathbb{P}(S^2 \Omega^1_{\mathbb{P}^2}(1))} \) can be seen as the blow-up of \( \mathcal{G} \) along his singular locus, thus we can identifies it with \( c_2|_E \).

We deduce from the diagram [7.7] that we have \( (2H - F) = 2B \) on \( \mathbb{P}(S^2 \Omega^1_{\mathbb{P}^2}(1)) \) where \( B \) (resp. \( H \)) is the pull back of the hyperplane from \( \mathbb{P}^2 \) (resp. \( \mathbb{P}^5 \)). The linear system \( |3H + T| \) can be seen on \( E \) as \( |3H + 3B| \). By Proposition 4.1 we infer that \( 3H + 3B - F \) is effective thus it is an element from \( |H + 5B| \).

We can go in the other direction; choose an element from \( |H + 5B| \) map it to \( \mathcal{G} \) and choose a hypersurface of degree 12 singular along the image. Since the conductor is non-reduced the singularities of this hypersurface have to have generically tacnodes (see [Rd & 4.4]) along the intersection with \( S_A \). This can lead to a possible counterexample to the O’Grady conjecture.
Remark 3. Let us describe more precisely the EPW sextic $S_A$ in the missing cases when $\Theta'_A$ is plane. First observe that if $\Theta'_A$ is a plane then it is contained in the tangent space to $G(2, 5) \subset F_3$ at one of his points; we can thus assume that we are in the case c) above. In this case $S_A$ is singular along a hyperplane $H_0$ being a multiple component such that there is a line $l \subset H_0$ contained in all the planes $\mathbb{P}(U)$ for $U \in \Theta'_A$. By Properties 2 the line $l \subset H_0$ is also contained in all the curves $C_{U,A}$ for $U \in \Theta'_A$. Moreover, the divisor $D \in [3H + T]$ from Proposition 1.3 cuts $\alpha^{-1}(\Theta'_A)_{red}$ (this is just the blow-up of $H_0$ along $l$) along a divisor in the system $|4H - 2E| + E$. So there is a quartic on $H_0$ singular along $l$ that define set theoretically the intersection of $H_0$ with the scheme $C$ defined by the conductor. So we can describe the situation (in the generic case) as follows: the EPW sextic is equal to $2H_0 + Q$ such that $Q$ is a quartic intersecting $H_0$ along a quartic that is singular along $l$. Moreover the underlying set of the scheme $C$ is the intersection $H_0 \cap Q$. Since $C$ has multiplicity 3 at a generic point of the image $X'$ have multiplicity 3 along a $C$ and the singularities along $C$ are worst then ordinary triple point (see [Re] & 4.4)).

8. $\dim \Theta_A = 1$

The aim of this section is to show that the adjoint EPW sextic from Theorem 4.4 cannot correspond to a generic $A$ with $\Theta_A$ of dimension 1 i.e. such that $\Theta_A$ is a line with some more conditions. Following [O2 & 2] we denote by $R_{\Theta_A} = \sum_{P \in \Theta_A} \mathbb{P}(P)$ by $E_{\Theta_A} \rightarrow \Theta_A$ the restriction of the tautological bundle from $G(3, 6)$ and by $f_{\Theta_A} : \mathbb{P}(E_{\Theta_A}) \rightarrow R_{\Theta_A}$ the tautological surjective map. Observe that there is a natural embedding of $\mathbb{P}(E_{\Theta_A})$ in $\mathbb{P}(\Omega^2_5(3))$ (in fact into the exceptional set $E$). The divisor $D \in [3H + T]$ (that maps to $C$) cuts $\mathbb{P}(E_{\Theta_A})$ along an effective divisor $D'$ that we shall analyze.

Suppose that $\Theta'_A$ is an irreducible component of $\Theta_A$. O'Grady applied the Morin theorem and showed that $1 \leq \deg(\Theta'_A) \leq 9$. He also presented in [O2] Table 2 the precise description of this curves and of the corresponding three dimensional sets $R_{\Theta_A}$.

If $\deg \Theta_A = 1$ then it is a line that we denote by $t$. Then the variety $R_{\Theta}$ is a 3-dimensional linear space containing a line $l$ such that the exceptional divisor $E$ of $f_{\Theta}$ (in fact $f_{\Theta}$ is the blow-up along $l$) maps to $l$. We compute that on $\mathbb{P}(E_{\Theta})$ we have $T = H - E$ so $D' = 4H - E$. Since the planes $\mathbb{P}(P)$ pass through $l$ and $C_P$ is never a plane, we deduce that the image of $D'$ on $R_{\Theta}$ is an irreducible quartic containing $l$ or a sum of two quadrics (if there is a linear component we obtain a contradiction because it have to be contained in $X'$).

On the other hand let us analyze the reduced sum $Z \subset R_{\Theta}$ of the curves $C_{P,A} \subset \mathbb{P}(P)$ for $P \in \Theta$. As observed before we have $Z \subset \text{supp} D'$. Observe that generically $C_{P,A}$ is a sum of a reduced quartic and a double line $l$, so we obtain a contradiction in this case. The problem are the special choices of $A$. There is a lot of possibilities; we hope to consider them in a future work.

9. Appendix

Let $W$ be a 6-dimensional vector space. The exterior product defines a sympletic form on the 20-dimensional vector space $\Lambda^3 W$. By a result of Segre the natural action of $PGL(6)$ on $\mathbb{P}(\Lambda^3 W)$ has four closed orbits $\mathbb{P}(\Lambda^3 W) \setminus W_1$, $W_1 \setminus W_2$, $W_2 \setminus W_3$ and $W_3$, where $W_1 \supset W_2 \supset W_3$ are subvarieties of dimensions 18, 14, and 9. Moreover, it is known that $W_3 = G(3, 6)$, $W_1$ is a quartic described in [Dol] lem.3.6 and $W_2$ (resp. $W_3$) is the singular locus of $W_1$ (resp. $W_2$). The locus $W_2 \subset \mathbb{P}^9$ can be also seen as the set of points lying on more than one chord of $G(3, 6)$ (see [Dol] lem.3.3) or as the union of spaces spanned by some $G(3, N)$ for $N \subset W$ of dimension 5, which is equal to the union of spaces spanned by some $F(p, 3, N)$ for some $p \in W$. With this interpretation we get a description of $W_2$ as the set of 3-forms $\{[\alpha \wedge \omega] \in \mathbb{P}(\Lambda^3 W) | \alpha \in W, \omega \in \Lambda^2 W\}$. It follows that there are two natural fibrations of $\pi_1, \pi_2 : W_2 \setminus W_3 \rightarrow \mathbb{P}^5$ such that the closure of the fibers are 9-dimensional linear spaces. More precisely $\pi_1$ is defined as the map $W_2 \setminus W_3 \ni \alpha \wedge \omega \mapsto \alpha \in \mathbb{P}(W)$ and $\pi_2$ the map $W_2 \setminus W_3 \ni [\alpha \wedge \omega] \mapsto [\alpha \wedge \omega \wedge \omega] \in \mathbb{P}(W^7)$.

Lemma 9.1. The maps $\pi_1$ and $\pi_2$ are well defined on $W_2 \setminus W_3$. 

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Proof. Assume that $[\alpha_1 \wedge \omega_1] = [\alpha_2 \wedge \omega_2] \in W_2 \setminus W_3$, for some $\alpha_1, \alpha_2 \in V$ and $\omega_1, \omega_2 \in \Lambda^2 W$. To prove the assertion we need to show that $[\alpha_1] = [\alpha_2]$ and $[\alpha_1 \wedge \omega_1 \wedge \omega_1] = [\alpha_2 \wedge \omega_2 \wedge \omega_2]$. Observe that under our assumption we have $\alpha_1 \wedge \alpha_2 \wedge \omega_1 = 0$, but $\alpha_2 \wedge \omega_2$ is not a simple form hence $\alpha_1 \wedge \alpha_2 = 0$ and the first part of the assertion follows. The second part is now clear as $[\alpha_2 \wedge \omega_2 \wedge \omega_2] = [\alpha_1 \wedge \omega_1 \wedge \omega_2] = [\alpha_1 \wedge \omega_2 \wedge \omega_1] = [\alpha_1 \wedge \omega_1 \wedge \omega_1]$. \hfill \qed

Proposition 9.2. The divisor class group of $W_2$ has rank 2 and is generated by the closures of the pull backs of the hyperplane sections by $\pi_1$ and $\pi_2$; denote them by $H_1$ and $H_2$.

Proof. First the Picard group of the projectivised vector bundle

$$P(\Omega^2_W(3)) \subset P(\Lambda^3 W) \times P^5$$

has rank 2 and is generated by $H$ and $T$; the pull back of hyperplanes from $P(W)$ and $P(\Lambda^3 W)$ respectively. So it is enough to consider the map

$$\alpha : P(\Omega^2_W(3)) \to W_2$$

given by the big divisor $O_{P(\Omega^2_W(3))}(1) = H$. By [RS Thm. 1], the divisor class group of $W_2 \subset P^{19}$ is isomorphic to the divisor class group of its generic codimension 4 linear section $W'_2$. Since $W_2$ is smooth the latter is equal to the Picard group of $W'_2$. On the other hand, $\alpha$ restricted to the pre-image $W''_2$ of $W'_2$ is an isomorphism. Since $W''_2$ is the intersection of four generic big divisors from the system $|H|$, we deduce from the generalized Lefschetz theorem [RS Thm. 6] that the Picard group of $W''_2$ is isomorphic to the Picard group of $P(\Omega^2_W(3))$. \hfill \qed

Let us describe the tangent space to $W_2$ in a point $p \in W_2 \setminus W_3$.

Lemma 9.3. Let $p = [\alpha \wedge \omega] \in W_2 \setminus W_3$, where $\alpha \in W$ and $\omega \in \Lambda^3 W$. Then the tangent space $T_p W_2$ is the linear space spanned by the two fibers of $\pi^{-1}_1(\pi_1(p))$ and $\pi^{-1}_2(\pi_2(p))$, passing through $p$, and by the linear space $\Pi = \{[w \wedge \omega] \in P(\Lambda^3 W) | w \in W\}$. \hfill \qed

Proof. It is clear that all three linear spaces are contained in $W_2$ and pass through $p$. It follows that they span a subspace of the tangent $T_p W_2$. Recall that $W_2$ is of dimension 14, and the intersection $\pi^{-1}_1(\pi_1(p)) \cap \pi^{-1}_2(\pi_2(p))$ is a $P^5$. It follows that the two fibers span a hyperplane in $T_p W_2$. It is hence enough to prove that $\Pi$ is not contained in the span of the two fibers.

To do this let us denote by $\Sigma_p$ the hyperplane $\{\beta \in \Lambda^3 W | \beta \wedge \alpha \wedge \omega = 0\}$. We clearly have $\pi^{-1}_1(\pi_1(p)) \cup \pi^{-1}_2(\pi_2(p)) \subset \Sigma_p$ whereas $\Pi \nsubseteq \Sigma$ as $\exists w \in W$ such that $w \wedge \alpha \wedge \omega \wedge \omega \neq 0$. \hfill \qed

Remark 4. Observe that $\Sigma_p \cap T_p W_2$ is the $P^{13}$ spanned by the two fibers.

Proposition 9.4. Let $T_p$ be a tangent space to $W_2$ at a smooth point $p \in W_2$ then there are no 5-dimensional isotropic subspaces $K \subset T_p W_2$ such that $p \in K$ and $K \cap \pi^{-1}_1(\pi_1(p)) \cap \pi^{-1}_2(\pi_2(p)) \cap W_3 = \emptyset$.

Proof. Let $K$ be an isotropic subspace of $T_p W_2$ and let $L$ be a Lagrangian (maximal isotropic) subspace of $T_p W_2$ containing $K$. Then, since $p \in K \subset L$, we have $L \subset \Sigma_p$, where $\Sigma_p$ is as in the from the proof of Lemma 9.3. By Remark 4 we get $K \subset L \subset \pi^{-1}_1(\pi_1(p)) + \pi^{-1}_2(\pi_2(p))$. We observe that the projectivised support $S$ of the intersection form on the latter $P^{13}$ has dimension 7 and is disjoint from $\pi^{-1}_1(\pi_1(p)) + \pi^{-1}_2(\pi_2(p))$. It follows that $\dim(L \cap S) = 3$, $\dim(L) = 9$ and $\pi^{-1}_1(\pi_1(p)) \cap \pi^{-1}_2(\pi_2(p)) \subset L$. It is easy to see that $\pi^{-1}_1(\pi_1(p)) \cap \pi^{-1}_2(\pi_2(p)) \cap W_3$ is a quadric hypersurface in $\pi^{-1}_1(\pi_1(p)) \cap \pi^{-1}_2(\pi_2(p))$. It follows that any 5-dimensional subspace of $L$ meets $\pi^{-1}_1(\pi_1(p)) \cap \pi^{-1}_2(\pi_2(p)) \cap W_3$ as it meets $\pi^{-1}_1(\pi_1(p)) \cap \pi^{-1}_2(\pi_2(p))$ in a line. \hfill \qed

Lemma 9.5. Let us keep the notation above. In particular, let $H_1$ and $H_2$ be hyperplanes in $P(W)$ and $P(W^*)$ respectively. Then the linear system $\pi^{-1}_1(H_1) + \pi^{-1}_2(H_2)$ is given by restrictions of quadrics to $W_2$. 

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Proof. Let \( v \in W^\vee \) and \( w \in W = (W^\vee)^\vee \) correspond to hyperplanes \( H_1 \) and \( H_2 \) respectively. Consider the quadric \( Q : \Lambda^3 W : \omega \mapsto \omega(v) \wedge \omega \wedge w \in \Lambda^6 W = \mathbb{C} \). It is enough to prove that \( Q^{-1}(0) \cap W_2 = \pi_1^{-1}(H_1) \cup \pi_2^{-1}(H_2) \) and the equality needs only to be checked outside \( G(3, W) \subset W \).

- We first prove the inclusion \( \supseteq \). Take \( \omega \in \pi_1^{-1}(H_1) \) then there exists \( \alpha \in H_1 \) such that \( \alpha \wedge \omega = 0 \). We then observe that since \( \alpha \in H_1 \) it follows that \( \alpha \wedge \omega(v) = 0 \). The inclusion of the second component follows by duality.

- Let us pass to the inclusion \( \subseteq \). Take \( \omega \in W_2 \setminus (\pi_1^{-1}(H_1) \cup \pi_2^{-1}(H_2) \cup G(n, W)) \). Then \( \omega \) may be written in the form \( \alpha \wedge \beta \) with \( \beta \in \Lambda^2 W \) such that \( \alpha \wedge \beta^2 \wedge w \) is nonzero and \( \omega(\alpha) \) is nonzero. The value of the quadric on \( \omega \) is then the product of these nonzero values.

\[
\square
\]

Proposition 9.6. Let \( L \) be a generic 9-dimensional Lagrangian subspace of \( \mathbb{P}^{19} = \mathbb{P}(\wedge^3 W) \). Then \( X_i = \pi_i(W_2 \cap L) \), for \( i = 1, 2 \), are two projectively dual EPW sextics in \( \mathbb{P}^5 \).

Proof. The varieties \( X_i \) are EPW sextics from [O1]. To prove duality we first observe that by Corollary 5.1 we know that \( W_2 \cap L \) is smooth of dimension 4. It follows that the maps \( \pi_1(W_2 \cap L) \) and \( \pi_2(W_2 \cap L) \) are birational and consider the map \( \varphi := \pi_2 \circ (\pi_1(W_2 \cap L))^{-1} \). It is a birational map between \( X_1 \) and \( X_2 \). We claim that this is the duality map between two dual hypersurfaces i.e. a generic point \( x \in X_2 \) corresponds to the hyperplane \( T_{\varphi^{-1}(x)}(X_1) \) and vice versa. Let \( x \) be a generic point on \( X_2 \) and \( y \in W_2 \) be the unique point in \( \pi_2^{-1}(\{x\}) \cap L \). Then as in the proof of Lemma 9.4 we know that \( T_y(W_2 \cap L) = \mathbb{P}(\Lambda^2 W) \cap \mathbb{P}^9 \) contained in the 13 dimensional projective space \( \pi_2^{-1}(\{x\}) + \pi_1^{-1}(\{y\}) \). Hence it follows that \( T_{\varphi^{-1}(x)}(X_1) = (d_y \pi_1(T_y(W_2 \cap L)) = d_y \pi_1(\pi_2^{-1}(\{x\}) + \pi_1^{-1}(\{y\})) \). The fact that \( T_{\varphi^{-1}(x)}(X_1) \) is the hyperplane perpendicular to \( x \) follows now from the following:

Lemma 9.7. Let \( y \in W_2 \setminus W_3 \). Then \( d_y \pi_1(\pi_2^{-1}(\{y\})) + \pi_1^{-1}(\{y\}) \subset W \) is the hyperplane in \( W \) perpendicular to \( \varphi \) \( \subset \) \( W^\vee \) and \( d_y \pi_2(\pi_2^{-1}(\{y\})) + \pi_1^{-1}(\{y\}) \subset W \) is the hyperplane in \( W^\vee \) perpendicular to \( \varphi \) \( \subset \) \( W^\vee \).

Proof. Denote by \( D_i \) the singular locus of \( X_i \) for \( i = 1, 2 \). It is known (see [EPW]) that \( X_i \) has \( A_1 \) singularities along \( D_i \) and that \( D_i \subset \mathbb{P}^5 \) is a smooth surface of degree 40. It follows that the \( D_i \) is scheme theoretically defined by the six quintics being the partial derivatives of the sextic \( X_i \).

Corollary 9.8. The morphism \( \pi_i : W_2 \cap L \to X_i \) is the blow up of \( D_i \subset X_i \). Moreover the birational map \( \pi_1 \circ \pi_2 : X_1 \to X_2 \) is given by the linear system \( \left| 5H - D_1 \right| \).

The following corollary can be also proved using the methods from [W].

Corollary 9.9. The degree of \( W_2 \subset \mathbb{P}^{19} \) is 42.

Proof. Under the assumptions as above denote by \( E_i \subset C := W_2 \cap L \) the exceptional locus of \( \pi_i \) for \( i = 1, 2 \), and by \( \mathcal{O}_C(H_i) = \pi^*(\mathcal{O}_H(H)) \), where \( H \subset \mathbb{P}^5 \) is the hyperplane section. We have to compute \( \frac{\left(\mathcal{O}_{H_1}(-E_1)\right)^2}{16} \). Thus it is enough to prove that \( H_1^4 = 6, H_1^2E_1 = 0, H_1^2E_2 = -80, H_1E_1^3 = -480, \) and \( E_4 = -1344 \). First from the adjunction formula \( E_4^2H_2 = K_EH_2^2 \),
$E_1^2 H_1 = K_2^2 H$, and $E_1^4 = K_2^3$. Now from [O & 4] we deduce that $p: E_1 = \mathbb{P}(T_{D_1}) \to D_1$. Thus $K_{E_1} = -2\psi$ where $\psi$ is the tautological divisor. Finally we need the following equality

$$\psi^2 - 3\psi \cdot H + c_2(p^*(T_{D_1})) = 0$$

Since $E_1 = 2(3H - T)$ is even in the Picard group of $C$, there exists a double cover of $Y \to C$ ramified along $E_1$ (we can take the double cover ramified along $E_2$). The strict transform of $E_1$ on $Y$ can be blown down such that the image is the irreducible symplectic manifold $Y$ constructed by O’Grady.

References

[B] Beauville, A., Holomorphic symplectic geometry: a problem list. Complex and Differential Geometry, 49-64. Springer Proceedings in Mathematics 8 (2011).

[B1] Beauville, A., Variétés kählériennes dont la première classe de Chern est nulle. J. of Diff. Geometry 18, 755-782 (1983).

[CS] Catanese, F., Schreyer, F.O., Canonical projections of irregular algebraic surfaces Algebraic geometry, 79–116, de Gruyter, Berlin, 2002.

[Ch] Chang, Mei-Chu A filtered Bertini-type theorem. J. Rein e Angew. Math. 397 (1989), 214–219.

[DE] Decker, W., Eisenbud, D., Sheaf algorithms using the exterior algebra. Computations in algebraic geometry with Macaulay 2. Algorithms and Computation in Mathematics. 8 (2002), 215-249.

[Dol] Dolgachev, I., Classical Algebraic Geometry; a modern view preprint to appear in Cambridge University press.

[Don] Donagi, R., On the geometry of grassmanian Duke Math. J. 44 (1977), no. 4, 795-837.

[EFS] Eisenbud, D., Floystad, G., Schreyer, F.O Sheaf cohomology and free resolutions over exterior algebras. Transactions of the AMS. 355 (2003), no.11, 4397-4426.

[EPW] Eisenbud, D., Popescu, S., Walter, Ch., Lagrangian subbundles and codimension 3 subcanonical subschemes Duke Math. J. 107 (2001), no. 3, 427-467.

[EV] Esnault, H., Viehweg, E. Lectures on vanishing theorems. DMV Seminar, vol. 20 Birkhuser, 1992.

[F] Fujiki, A., On the de Rham cohomology group of a compact Kahler symplectic manifold. Algebraic geometry, Sendai, 1985, 105-165, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[Gr] Gruson, L., The case $d = 11$, preprint.

[Gu] Guan, G., On the Betti numbers of irreducible compact hyperkahler manifolds of complex dimension four, Math Research Letters 8, 2001, pp. 663-669.

[H] Hartshorne, R. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. 496 pp.

[HS] Hitchin, N., Sawon, J., Curvature and characteristic numbers of hyper-Kahler manifolds. Duke Math. J. 106 (2001), no. 3, 599-615.

[Hu] Huybrechts, D., Finiteness results for hyperkahler manifolds. J.Reine Angew. Math. 558 (2003), 15-22.

[IM] Iliev, A., Manivel, L., Fano manifolds of degree ten and EPW sextics. Ann. Sci. Ec. Norm. Super. (4) 44 (2011), no. 3, 393-426.

[K] Kapustka, G., On irreducible symplectic 4-folds numerically equivalent to $(K^3)^2$. arXiv:1004.3177v3.

[KM] Kollar, J., Mori, S., Birational geometry of algebraic varieties. Cambridge University Press, Cambridge, 1998. xii+254

[KU] Kleiman, S., Ulrich, B., Gorenstein algebras, symmetric matrices, self-linked ideals, and symbolic powers Trans. Amer. Math. Soc. 349 (1997), no. 12, 4973-5000.

[L] Lazarsfeld, R. Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004. xvii+387

[MP] Mond, D., Pellikaan, R., Fitting ideals and multiple points of analytic maps, Algebraic Geometry and Complex Analysis, Patzcuaro, 1987, Springer Lecture Notes 1414, E. Ramirez de Arellano (ed.), 1989

[M] Morin, U. Sui sistemi di piani a due a due incidenti Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti LXXIX, 1930, pp. 907–926.

[O1] O’Grady, K., Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics, Duke Math. J. 134 (2006), no. 1, 99-137.

[O2] O’Grady, K., EPW-sextics: taxonomy arXiv:1007.3882v2, to appear in Manuscripta.

[O3] O’Grady, K., Double covers of EPW-sextics arXiv:1007.4952v3 to appear in Michigan J.

[O4] O’Grady, K., Moduli of double EPW-sextics arXiv:1111.1395v2

[O5] O’Grady, K., Periods of Double EPW-sextics. arXiv:1203.6495v1 [math.AG]

[O6] O’Grady, K., Higher-dimensional analogues of K3 surfaces, arXiv:1005.3131
[R] Rao, A.P., *Conormal bundles of determinantal curves* Rend. Sem. Mat. Univ. Politec. Torino 50 (1992), no. 3, 277–311 (1993).

[Ra] Ran, Z., *Curvilinear enumerative geometry* Acta Math. 155 (1985), no. 1-2, 81–101.

[Re] Reid, M., *Nonnormal del Pezzo surfaces* Publ. Res. Inst. Math. Sci., 30 (1994) (5). pp. 695-727.

[RS] Ravindra, G.V., Srinivas, V., *The Grothendieck-Lefschetz theorem for normal projective varieties*, J. Algebraic Geom. 15 (2006), 563-590.

[S] Salamon, S. M. *On the cohomology of Kahler and hyper-Kahler manifolds*. Topology 35 (1996), no. 1, 137-155.

[V] Verbitsky, M., *Cohomology of compact hyperkahler manifolds and its applications* Geom. Funct. Anal. 6 (1996), no. 4, 601–611.

[W] Weymann, J., *Cohomology of vector bundles ans syzygies* Cambridge Tracts in Mathematics, 149. Cambridge University Press, Cambridge, 2003. 371 pp.

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