A DOLBEAULT-GROTHENDIECK RESOLUTION FOR SINGULAR SPACES

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Abstract. We construct a generalization of the Dolbeault-Grothendieck resolution on a singular complex space. The same construction yields, for each morphism of analytic spaces, a pullback mapping between the respective Dolbeault-Grothendieck resolutions. As in the smooth case, the terms of the resolution are soft sheaves with stalks which are flat with respect to the sheaf of holomorphic sections. If, moreover, the complex space \((X, \mathcal{O}_X)\) is countable at infinity then the global section spaces of the terms of the resolution are endowed with natural Fréchet-Schwarz topologies which induce the natural topology on the cohomology groups \(H^\bullet(X, \mathcal{O}_X)\). The construction is an exercise in globalization using semi-simplicial techniques. Using the above construction one can produce, for instance, a soft resolution with \(\mathcal{O}_X\)-flat stalks for the de Rham complex on the analytic space \(X\).

1. Introduction

Let \(X\) be an \(n\)-dimensional complex manifold and let

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_X^{0,0} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{E}_X^{0,n} \rightarrow 0
\]

be the Dolbeault-Grothendieck resolution on \(X\). Here, as usual, \(\mathcal{O}_X\) is the sheaf of holomorphic functions and \(\mathcal{E}_X^{p,q}\) the sheaf of \((p, q)\) smooth differential forms on \(X\). The problem is that in the singular case the complex (1) is no longer a resolution for \(\mathcal{O}_X\). The purpose of this paper is to construct an analogue for the Dolbeault-Grothendieck resolution on a complex space with singularities.

There exist two recent constructions of analogues of Dolbeault-Grothendieck resolutions under supplementary hypothesis on the singular space. Ancona and Gaveau [A-G] considered analytic spaces with smooth singular locus; their solution is based on Hironaka desingularization. Andersson and Samuelsson [A-S] considered the case of a reduced analytic space; the resolution is obtained as a subcomplex of the complex of smooth currents on the space; their construction uses Koppelman representation formulas.

Our construction is based on working in a category larger than that of analytic spaces—the category of semi-simplicial analytic spaces. Recall that a s.s.analytic space (throughout this paper s.s.is short for semi-simplicial) is a contravariant functor from a simplicial complex (seen as a category) to the category of analytic spaces, or, equivalently, a family of analytic spaces indexed by the simplexes of a simplicial complex, together with a family of compatible connecting morphisms (see Section 3 for the definitions). S.s.analytic spaces and the corresponding analytic modules proved a very flexible tool. They appeared implicitly or explicitly, for instance, in Forster, Knorr [F-K] for the proof of Grauert’s direct image theorem, in Verdier [V], Baran [B1] for the introduction of natural topologies on the global (hyper)cohomological invariants of analytic sheaves, in Ramis, Ruget [R-R] for the proof of relative analytic duality, in Flenner [E] and Bănică, Putinar, Schumacher [BPS] for computations linked to deformation theory.

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Our construction solves the problem for any analytic space. In fact for each pair \((X, A)\), where \(A\) is an embedding atlas of the analytic space \(X\) (i.e. a family of local closed embeddings of \(X\) in complex manifolds - see paragraph 1.2) we produce a resolution for \(\mathcal{O}_X\), denoted by \(\text{Dolb}(A; \mathcal{O}_X)\). The pair \((X, A)\) will be called a \textit{locally embedded analytic space}. In particular, if \(X\) is a complex manifold and \(A\) the obvious atlas with one chart, then one gets the usual resolution on \(X\). For each morphism of locally embedded analytic spaces:

\[ f : (X, A) \to (Y, B) \]

one constructs a pullback morphism which extends the pullback morphism from the smooth case:

\[ f^* : \text{Dolb}(B; \mathcal{O}_Y) \to f_* \text{Dolb}(A; \mathcal{O}_X) \quad (2) \]

Moreover, the same construction produces a resolution for each \(\mathcal{O}_X\)-module \(F\), denoted \(\text{Dolb}(A; F)\).

The resolution \(\text{Dolb}(A; F)\) depends on the embedding atlas \(A\). However the resolution is unique up to unique isomorphism in the derived category of \(\mathcal{O}_X\)-modules, \(D(\mathcal{O}_X)\). More precisely, if \((X, A), (Y, B)\) are locally embedded analytic spaces and \(f : X \to Y\) is a morphism of analytic spaces (but not necessarily of locally embedded analytic spaces) then there exists in \(D(\mathcal{O}_Y)\) a unique pullback morphism similar to \((2)\) (see Theorem 6.1.6). In particular, if \(A\) and \(B\) are two embedding atlases on the same analytic space \(X\) then there is a unique isomorphism between \(\text{Dolb}(A; F)\) and \(\text{Dolb}(B; F)\) in \(D(\mathcal{O}_X)\).

The main result of the paper is:

**Theorem 1.1.1.**  
(1) Let \((X, A)\) be a locally embedded analytic space and \(F \in \text{Mod}(\mathcal{O}_X)\). Then there is a functor

\[ \text{Dolb}(A; \bullet) : \text{Mod}(\mathcal{O}_X) \to C^+(X) \quad (3) \]

such that:

(a) \(\text{Dolb}(A; \bullet)\) is an exact functor

(b) There is a functorial morphism \(F \to \text{Dolb}(A; F)\) and \(\text{Dolb}(A; F)\) is a resolution of \(F\).

(c) \(\text{Dolb}(A; F)\) has soft components.

(d) \(\text{Dolb}(A; \mathcal{O}_X)\) has \(\mathcal{O}_X\)-flat components

(e) One has a natural quasi-isomorphism:

\[ \text{Dolb}(A; \mathcal{O}_X) \otimes_{\mathcal{O}_X} F \to \text{Dolb}(A; F) \quad (4) \]

Moreover, if \(F \in \text{Coh}(\mathcal{O}_X)\) then the above morphism is an isomorphism.

(f) If \(X\) is a complex manifold and \(A\) consists of only one chart, namely \((X, \text{id}, X)\), then \(\text{Dolb}(A; \bullet)\) coincides with the usual Dolbeault-Grothendieck resolution on \(X\).

(2) Let \(f : (X, A) \to (Y, B)\) be a morphism of locally embedded analytic spaces, \(F \in \text{Mod}(\mathcal{O}_X), G \in \text{Mod}(\mathcal{O}_Y)\) and \(u : G \to f_* F\) a morphism of \(\mathcal{O}_Y\)-modules. Then there exists a natural pullback morphism:

\[ f^*(u) : \text{Dolb}(B; G) \to f_* \text{Dolb}(A; F) \quad (5) \]

such that the following diagram commutes:

\[ \begin{array}{ccc}
\text{Dolb}(B; G) & \xrightarrow{f^*(u)} & f_* \text{Dolb}(A; F) \\
\uparrow^{\nu} \quad & & \uparrow^{f_* b} \\
G & \xrightarrow{u} & f_* F
\end{array} \quad (6) \]

In particular there is a natural morphism:

\[ f^* : \text{Dolb}(B; \mathcal{O}_Y) \to f_* \text{Dolb}(A; \mathcal{O}_X) \quad (7) \]

over the mapping \(f^* : \mathcal{O}_Y \to f_* \mathcal{O}_X\).
(3) Let \((X, A) \rightarrow (Y, B) \rightarrow (Z, C)\) be morphisms of locally embedded analytic spaces and \(h = g \circ f\). Let moreover \(\mathcal{F} \in \text{Mod}(O_X), \mathcal{G} \in \text{Mod}(O_Y), \mathcal{H} \in \text{Mod}(O_Z)\) and morphisms \(u : \mathcal{G} \rightarrow f_* \mathcal{F} O_Y\)-linear, \(v : \mathcal{H} \rightarrow g_* \mathcal{G}\), \(w : \mathcal{H} \rightarrow h_* \mathcal{F} O_Z\)-linear, such that \(g_*(u) \circ v = w\) then one has the commutative diagram:

\[
\begin{array}{ccc}
\text{Dolb}(A; \mathcal{F}) & \xleftarrow{\quad g_* (f^*(u)) \quad} & g_* \text{Dolb}(B; \mathcal{G}) \\
\downarrow{\quad h^*(w) \quad} & & \downarrow{\quad g^*(v) \quad} \\
\text{Dolb}(C; \mathcal{H}) & & \\
\end{array}
\]

(8)

The proof is based on two simple remarks:

(1) Let \(X = ((X_a)_{a \in S})\) be a s.s.complex manifold relative to the simplicial complex \((I, S)\). The compatibility of the pullback of differential forms with the composition of mappings ensures that the Dolbeault-Grothendieck resolutions on the manifolds \(X_a\) form a complex of \(X\)-modules. We denote it \(\text{Dolb}(X; O_X)\). Moreover, if \(F : X \rightarrow Y\) is a morphism of s.s.complex manifolds (see Definition \(\S 3.1.3\)) then one defines a pullback morphism:

\[F^\sharp : \text{Dolb}(Y; O_Y) \rightarrow F_* \text{Dolb}(X; O_X)\]

(9)

where \(F^\sharp\) is a variant of the direct image functor which associates to each \(X\)-module a complex of \(Y\)-modules.

(2) Let \(X \hookrightarrow D\) be an analytic subspace of the complex manifold \(D\), given by the coherent ideal \(\mathcal{I} \subset O_D\) (we say that \((X, i, D)\) is an embedding triple - see paragraph \(\S 1.1\)). The complex obtained by tensoring the Dolbeault-Grothendieck resolution on \(D\) with \(O_D/\mathcal{I}\) (which comes to restricting the coefficients of the differential forms on \(D\) to \(X\)) is a resolution of \(O_X\), since the \(O_D\)-modules \(E_{D}^{p,q}\) are \(O_D\)-flat (see Malgrange \([Ma]\)). We consider this complex as the analogue for the Dolbeault-Grothendieck resolution on \(X\). Note that the complex described here appears in the proof of the duality theorems of Serre-Malgrange (see Malgrange \([Ma]\) or Bănică, Stănășilă \([B-S]\) Ch 7 \(\S 4.2\)). If \((X, k, D)\) is a s.s.embedding triple (see Remark \(\S 4.1.3\)) then the Dolbeault-Grothendieck resolutions on each component form a complex of \(X\)-modules that we denote by \(\text{Dolb}(k; O_X|\mathcal{U})\).

Let \((X, A)\) be a locally embedded analytic space. By Lemma \(\S 2.2.2\) and Example \(\S 3.1.6\) one associates to \((X, A)\) a s.s.embedding triple \((\mathcal{U}, k, \mathcal{D})\) and a natural morphism of s.s.analytic spaces \(b : \mathcal{U} \rightarrow X\). According to \(\S 2.1\), there is a \(\text{Dolb}\)-resolution on \(\mathcal{U}\). We need to define a \(\text{Dolb}\)-resolution on \(X, \text{Dolb}(A; O_X)\), such that a pullback mapping similar to \((11)\) exist for \(b\), i.e. a mapping:

\[b^\sharp : \text{Dolb}(A; O_X) \rightarrow b_* \text{Dolb}(k; O_X|\mathcal{U})\]

For this we simply set:

\[\text{Dolb}(A; O_X) = b_* \text{Dolb}(k; O_X|\mathcal{U})\]

and check that all the properties are verified.

Here are some applications of Theorem \(\S 1.1.1\).

Since the terms of the Dolbeault-Grothendieck resolution are soft sheaves one can use it to define representatives for derived functors and morphisms. In particular the complex \(\Gamma(X, \text{Dolb}(A; \mathcal{F}))\) computes the cohomology of \(X\) with coefficients in \(\mathcal{F}\); furthermore, if \(\mathcal{F}\) is a coherent sheaf then the terms of \(\Gamma(X, \text{Dolb}(A; \mathcal{F}))\) are endowed with Fréchet-Schwarz topologies which induce the natural topologies on the cohomology groups of \(\mathcal{F}\)(see Corollary \(\S 6.1.4\)). Note that since for each open covering of the analytic space one produces a resolution and the construction has good functorial properties, it follows that the resolution is suitable to produce good representatives for derived functors and morphisms.
If \( X \) is a reduced analytic space then, by using a direct limit argument, one can construct on \( X \) an analogue of the Dolbeault-Grothendieck resolution which coincides with the classical one on \( \text{Reg}(X) \), the regular locus of \( X \). However, in this case the topologies on the global sections of the resolution are more complicated.

One can link the complex \( \mathcal{Dolb}(A; \mathcal{O}_X) \) to the complex of smooth differential forms on \( X \), namely there is a natural surjective morphism between \( \mathcal{Dolb}(A; \mathcal{O}_X) \) and a suitable Čech complex of the complex of smooth differential forms on \( X \) (see theorem 6.4.1).

As in the smooth case, by using the \( \mathcal{Dolb}(A; \bullet) \) -functor one can construct a resolution with soft sheaves for the de Rham complex on an analytic space \( X \) (see Theorem 6.3.2).

The main result of this note was announced in [B2]. In the same paper the functor \( F^\# \) (denoted there by \( F^* \)) is defined.

In a future paper, using roughly the same technique as here, but replacing the functor \( F^\# \) with the direct image with proper supports we shall give a construction of the dualizing complex of an analytic space.

2. Preliminaries

**Review and notations.** Throughout this paper analytic space will mean complex analytic space.

Let \((X, \mathcal{O}_X)\) be an analytic space. We use the following notations:

- \( \text{Mod}(\mathcal{O}_X) \) - the abelian category of \( \mathcal{O}_X \)-modules; \( \text{Coh}(\mathcal{O}_X) \) - the subcategory of coherent \( \mathcal{O}_X \)-modules
- \( C(X) \) the abelian category of complexes of \( \mathcal{O}_X \)
- As usual, \( C^*(X) \), respectively \( D^*(X) \), where \( * = +, -, b \), denote the subcategories of complexes bounded below, bounded above, respectively bounded

Let \( X \) be an \( n \)-dimensional complex manifold. We denote by \( \mathcal{E}^{p,q}_X \) the sheaf of \((p,q)\)-differential forms with \( C^\infty \) coefficients on \( X \). It is a soft sheaf and, according to [Ma], it is an \( \mathcal{O}_X \)-flat module. The complex of \( \mathcal{O}_X \)-modules:

\[
0 \rightarrow \mathcal{E}^{0,0}_X \rightarrow \cdots \rightarrow \mathcal{E}^{0,n}_X \rightarrow 0
\]

is the Dolbeault-Grothendieck resolution of \( \mathcal{O}_X \).

For \( f : X \rightarrow Y \) a holomorphic mapping between two complex manifolds we denote \( f^* : \mathcal{E}^{p,q}_Y \rightarrow \mathcal{E}^{p,q}_X \) the \( \mathcal{O}_Y \)-linear morphism given by the pullback of forms.

It is well known that if \( X \xrightarrow{f} Y \xrightarrow{g} Z \) are holomorphic mappings between complex manifolds and \( h = g \circ f \), then one has the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{E}^{p,q}_X & \xrightarrow{h^*} & \mathcal{E}^{p,q}_Z \\
\downarrow{\phi^*} & & \downarrow{\psi^*} \\
\mathcal{E}^{p,q}_Y & \xrightarrow{g^*} & \mathcal{E}^{p,q}_Z
\end{array}
\]

3. Semi-simplicial Objects

**3.1 Semi-simplicial analytic spaces.** Let \((I, \mathcal{S})\) be a simplicial complex, i.e. \( I \) is a set and \( \mathcal{S} \) is a family of non-empty finite parts of \( I \), called simplexes, such that:

1. \( \{i\} \in \mathcal{S} \) for all \( i \in I \)
2. if \( \alpha' \subset \alpha \in \mathcal{S} \) then \( \alpha' \in \mathcal{S} \)

If \( \alpha \in \mathcal{S} \) we denote by \( |\alpha| = \text{Card}(\alpha) - 1 \) the length of the simplex \( \alpha \). Recall that \( \dim((I, \mathcal{S})) = \sup\{|\alpha| \mid \alpha \in \mathcal{S}\} \).

A morphism of simplicial complexes \( f : (I, \mathcal{S}) \rightarrow (J, \mathcal{T}) \) is simply a mapping \( f : I \rightarrow J \) such that \( f(\alpha) \in \mathcal{T} \) whenever \( \alpha \in \mathcal{S} \). If \( K(pt) \) is the simplicial complex over the set with one element \( \{pt\} \), then we denote by \( a_S : (I, \mathcal{S}) \rightarrow K(pt) \) the morphism induced by the unique mapping \( I \rightarrow \{pt\} \).
Definition 3.1.1.  
(1) Let $\mathcal{C}$ be a category. A semi-simplicial (s.s.) system of objects in $\mathcal{C}$ indexed by the simplicial complex $(I, S)$ consists of:
- a family $(X_\alpha)_{\alpha \in S}$ of objects in $\mathcal{C}$
- a family $(\rho_{\alpha \beta})_{\alpha \subset \beta}$ of connecting morphisms, $\rho_{\alpha \beta}: X_\beta \to X_\alpha$, such that $\rho_{\alpha \alpha} = \text{id}$ for $\alpha \in S$, and $\rho_{\alpha \beta} \circ \rho_{\beta \gamma} = \rho_{\alpha \gamma}$ whenever $\alpha \subset \beta \subset \gamma$.

(2) Let $\mathcal{X} = ((X_\alpha)_{\alpha \in S}, (\rho_{\alpha \beta})_{\alpha \subset \beta})$, $\mathcal{Y} = ((Y_\alpha)_{\alpha \in T}, (\rho'_{\alpha \beta})_{\alpha \subset \beta})$ be s.s.systems of objects in $\mathcal{C}$ indexed by $(I, S)$. A morphism $F: \mathcal{X} \to \mathcal{Y}$ consists of a family of morphisms in $\mathcal{C}$, $(F_\alpha)_{\alpha \in S}$, $F_\alpha: X_\alpha \to Y_\alpha$, such that $F_\alpha \circ \rho_{\alpha \beta} = \rho'_{\alpha \beta} \circ F_\beta$.

If the simplicial complex is clear from the context we shall omit mentioning it.

If $\mathcal{C}$ is the category of analytic spaces then we say for short s.s.analytic space instead of s.s.system of analytic spaces. Let $U = ((\alpha_i)_{\alpha \in S}, \rho_{\alpha \beta})$ be a semi-simplicial analytic space. Here $X_\alpha$ is short for $(X_\alpha, O_\alpha)$, where $O_\alpha$ denotes the sheaf of holomorphic sections of $X_\alpha$, and $\rho_{\alpha \beta}$ is short for $(\rho_{\alpha \beta}, \rho'_{\alpha \beta})$ where $\rho_{\alpha \beta}: X_\beta \to X_\alpha$ is the topological part and $\rho'_{\alpha \beta}: O_\alpha \to \rho_{\alpha \beta}(O_\beta)$ is the sheaf level part. If $X_\alpha$ is a complex manifold for all $\alpha \in S$ then $X$ will be called a s.s.complex manifold.

Remark 3.1.2. An analytic space can be regarded as a s.s.analytic space indexed by $K(pt)$, the simplicial complex constructed over the index set with one element.

Example 3.1.3. Let $X$ be an analytic space and $U = (U_i)_{i \in I}$ an open covering of $X$. One associates to $U$
- the simplicial complex $(I, N(U))$, where $N(U)$ denotes the nerve of $U$
- the s.s.analytic space indexed by $(I, N(U))$,
$$\mathcal{U} = ((U_\alpha)_{\alpha \in N(U)}, (i_{\alpha \beta})_{\alpha \subset \beta})$$
where $U_\alpha$ denotes, as usual, the intersection $\bigcap_{i \in \alpha} U_i$, and $i_{\alpha \beta}: U_\beta \to U_\alpha$ is the natural inclusion.

Example 3.1.4. Let $(I, S)$ be a simplicial complex and $(X_i)_{i \in I}$ a family of analytic spaces. For $\alpha \in S$ let $X_\alpha = \prod_{i \in \alpha} X_i$. Then $\mathcal{X} = ((X_\alpha)_{\alpha \in S}, (\rho_{\alpha \beta})_{\alpha \subset \beta})$ is a semi-simplicial analytic space, where $\rho_{\alpha \beta}: X_\beta \to X_\alpha$ is the natural projection.

Definition 3.1.5. Let $\mathcal{C}$ be a category, $f: (I, S) \to (J, T)$ a morphism of simplicial complexes, $\mathcal{X} = ((X_\alpha)_{\alpha \in S}, (\rho_{\alpha \beta})_{\alpha \subset \beta}), \mathcal{Y} = ((Y_\alpha)_{\alpha \in T}, (\rho'_{\alpha \beta})_{\alpha \subset \beta})$ s.s.systems of objects in $\mathcal{C}$ indexed by $(I, S)$, respectively $(J, T)$. A morphism $F: \mathcal{X} \to \mathcal{Y}$ of s.s.systems of objects in $\mathcal{C}$ over $f$ consists of a family of morphisms in $\mathcal{C}$, $(F_\alpha)_{\alpha \in S}$, $F_\alpha: X_\alpha \to Y_{f(\alpha)}$, such that $F_\alpha \circ \rho_{\alpha \beta} = \rho'_{f(\alpha) \beta} \circ F_\beta$.

Example 3.1.6. Let $X$ be an analytic space, $U = (U_i)_{i \in I}$ an open covering of $X$, and $\mathcal{U} = ((U_\alpha)_{\alpha \in N(U)}, (i_{\alpha \beta})_{\alpha \subset \beta})$ the semi-simplicial analytic space associated to $U$ (see Example 3.1.3). Then the inclusion mappings $i_\alpha: U_\alpha \to X$ determine a morphism of semi-simplicial spaces $i: \mathcal{U} \to X$ over $N(U): N(U) \to K(pt)$.

3.2 Modules over s.s.analytic spaces. Unless otherwise stated, in this section $\mathcal{X} = ((X_\alpha, O_\alpha)_{\alpha \in S}, (\rho_{\alpha \beta}, \rho'_{\alpha \beta})_{\alpha \subset \beta})$ will denote a semi-simplicial analytic space indexed by the simplicial complex $(I, S)$.

Definition 3.2.1. (1) An $\mathcal{X}$-module consists of
- a family $(F_\alpha)_{\alpha \in S}$ where $F_\alpha$ is an $O_\alpha$-module for each $\alpha \in S$
- a family of connecting morphisms $(\varphi_{\beta \alpha})_{\alpha \subset \beta}$, where
$$\varphi_{\beta \alpha}: F_\alpha \to \rho_{\alpha \beta \ast}(F_\beta)$$
is a morphism of $O_\alpha$-modules such that $\varphi_{\alpha \alpha} = \text{id}$ for all $\alpha \in S$, and $\rho_{\beta \gamma} \ast (\varphi_{\beta \delta}) \circ \varphi_{\alpha \beta} = \varphi_{\alpha \gamma}$ whenever $\alpha \subset \beta \subset \gamma$. 

(2) If $\mathcal{F} = (\{F_{\alpha}\}_{\alpha \in S}, \{\varphi_{\beta\alpha}\}_{\alpha \subseteq \beta})$, $\mathcal{G} = (\{G_{\alpha}\}_{\alpha \in S}, \{\psi_{\beta\alpha}\}_{\alpha \subseteq \beta})$ are $X$-modules, then a morphism of $X$-modules $u : \mathcal{F} \to \mathcal{G}$ consists of a family $(u_{\alpha})_{\alpha \in S}$, where $u_{\alpha} : F_{\alpha} \to G_{\alpha}$ is a morphism of $\mathcal{O}_{\alpha}$-modules, such that for $\alpha \subseteq \beta$ $\rho_{\alpha\beta} \circ \varphi_{\beta\alpha} = \psi_{\beta\alpha} \circ u_{\alpha}$.

We denote by $\text{Mod}(X)$ the abelian category of $X$-modules and by $C(X)$ the category of complexes with terms in $\text{Mod}(X)$.

**Example 3.2.2.** $(\{\mathcal{O}_{\alpha}\}_{\alpha \in S}, (\rho_{\alpha\beta})_{\alpha \subseteq \beta})$ is obviously an $X$-module that we denote by $\mathcal{O}_X$.

**Example 3.2.3.** In the context of Example 3.1.3 let $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$. Then $(\mathcal{F}U_{\alpha})_{\alpha \in S}$ with the obvious connecting morphisms is an $\mathcal{O}_X$-module that we denote by $\mathcal{F}U$.

The tensor product induces a bifunctor:

$$\otimes : \text{Mod}(X) \times \text{Mod}(X) \to \text{Mod}(X)$$

namely if $\mathcal{F} = (\{F_{\alpha}\}_{\alpha \in S}, \{\varphi_{\beta\alpha}\}_{\alpha \subseteq \beta})$, $\mathcal{G} = (\{G_{\alpha}\}_{\alpha \in S}, \{\psi_{\beta\alpha}\}_{\alpha \subseteq \beta}) \in \text{Mod}(X)$ then $((\mathcal{F} \otimes \mathcal{G})_{\alpha \in S}, (\varphi_{\beta\alpha} \otimes \psi_{\beta\alpha})_{\alpha \subseteq \beta})$ is an $X$-module.

### 3.3 Alternate $X$-modules.

In order to define the $F_i$ functor (see paragraph 3.5) we need to construct an alternate version for the notion of $X$-module. For this, let $(I, S)$ be a simplicial complex and fix a total order on $I$. We use the following notations:

- if $\alpha \in S$ and $j \in [0, |\alpha|]$, then
  $$v(\alpha; j) = \text{the } j\text{-th vertex of } \alpha \text{ with respect to the order on } I,$$
  the counting starting from 0
  $$\sigma(\alpha; j) = \alpha \setminus \{v(\alpha; j)\}$$
- if $\alpha \in S$, $|\alpha| \geq 1$ and $j, k \in [0, |\alpha|], j \neq k$, then
  $$\sigma(\alpha; j, k) = \alpha \setminus \{v(\alpha; j), v(\alpha; k)\}$$

Thus, if for instance $j < k$ then $\sigma(\alpha; j, k) = \sigma(\sigma(\alpha; j); j) = \sigma(\sigma(\alpha; j); k - 1)$.

If $\alpha = \{i_0, ..., i_n\}$ and $i_0 < i_1 < ... < i_n$, one checks immediately that

$$|\alpha| = n$$

$$v(\alpha; j) = v_j$$

$$\sigma(\alpha; j) = \{i_0, ..., i_j, ..., i_n\}$$

$$\sigma(\alpha; j, k) = \{i_0, ..., i_j, ..., i_k, ..., i_n\}$$

Let $\mathcal{X} = (\{X_{\alpha}\}_{\alpha \in S}, (\rho_{\alpha\beta})_{\alpha \subseteq \beta})$ be a s.s.analytic space. We use the subscript $(\alpha; j)$ to refer to the mappings along the edge $[\alpha, \sigma(\alpha; j)]$ of the simplicial complex $(I, S)$. Thus we shall write $\rho(\alpha; j)$ instead of $\rho_{\sigma(\alpha; j)} : X_{\alpha} \to X_{\sigma(\alpha; j)}$. Similarly, if $\mathcal{F} = (\{F_{\alpha}\}_{\alpha \in S}, (\varphi_{\beta\alpha})_{\alpha \subseteq \beta})$ is an $X$-module we write $\varphi(\alpha; j)$ instead of $\varphi_{\sigma(\alpha; j)} : F_{\alpha} \to F_{\sigma(\alpha; j)}$.

**Remark 3.3.1.** The family of commuting morphisms $(\rho_{\alpha\beta})_{\alpha \subseteq \beta}$ can be "reconstructed" (by finite compositions) from the subfamily $(\rho_{\sigma(\alpha; j)})_{(\alpha; j)}$. More precisely, If $\alpha \in S$, $|\alpha| \geq 1$, $j, k \in [0, |\alpha|]$, and, for instance, $j < k$, then the following rectangular diagram commutes:

$$\begin{array}{ccc}
X_{\alpha} & \xrightarrow{\rho(\alpha; j)} & X_{\sigma(\alpha; j)} \\
\downarrow \rho(\alpha; k) & & \downarrow \rho(\sigma(\alpha; j), k-1) \\
X_{\sigma(\alpha; k)} & \xrightarrow{\rho(\sigma(\alpha; j), k)} & X_{\sigma(\alpha; j, k)}
\end{array}$$

Conversely, any family of morphisms $(\rho_{\sigma(\alpha; j)})_{(\alpha; j)}$ such that the diagrams $D(\alpha; j, k)$ commute, generates a family of connecting morphisms for the family of analytic spaces $(X_{\alpha})_{\alpha \in S}$. 
Remark 3.3.3. Indeed, if \( \varphi(\alpha;j) \) do not depend on the total order on \( \alt \) denote by \( \varphi(\alpha;j) \) there is a (non-unique) functorial isomorphism between the two corresponding \( \alt \) \( \alt \)

\[
\rho(\alpha;j,k) \ast \mathcal{F}_\alpha \leftarrow \rho(\sigma(\alpha,k),j) \ast \mathcal{F}(\alpha;j) \leftarrow \rho(\sigma(\alpha;k),j) \ast \mathcal{F}(\sigma(\alpha;j),k)
\]

Definition 3.3.2. (1) An alternate \( \mathcal{X} \)-module consists of a family \( (\mathcal{F}_\alpha)_{\alpha \in S} \), where each \( \mathcal{F}_\alpha \) is an \( \mathcal{O}_\alpha \)-module, together with the family of connecting morphisms \( (\varphi(\alpha;j))_{(\alpha;j)} \), \( \varphi(\alpha;j) : \mathcal{F}(\alpha;j) \rightarrow \rho(\alpha;j) \ast (\mathcal{F}_\alpha) \) such that the diagrams \( D(\mathcal{F};\alpha;j,k) \) anti-commute. (2) Let \( \mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha \in S}, (\varphi(\alpha;j))_{(\alpha;j)}) \), \( \mathcal{G} = ((\mathcal{G}_\alpha)_{\alpha \in S}, (\psi(\alpha;j))_{(\alpha;j)}) \) be alternate \( \mathcal{X} \)-modules. A morphism of alternate \( \mathcal{X} \)-modules \( u : \mathcal{F} \rightarrow \mathcal{G} \) consists of a family \( (u_\alpha)_{\alpha \in S} \), \( u_\alpha : \mathcal{F}_\alpha \rightarrow \mathcal{G}_\alpha \) morphism of \( \mathcal{O}_\alpha \)-modules, such that for each pair \( (\alpha;j) \) the diagram commutes:

\[
\begin{array}{c}
\mathcal{F}(\alpha;j) \xrightarrow{u(\alpha;j)} \mathcal{G}(\alpha;j) \\
\varphi(\alpha;j) \downarrow \quad \psi(\alpha;j) \\
\rho(\alpha;j) \ast \mathcal{F}_\alpha \xrightarrow{u_\alpha} \rho(\alpha;j) \ast \mathcal{G}_\alpha
\end{array}
\]

(3) With the notations at point 2 an anti-morphism of alternate \( \mathcal{X} \)-modules is a family of morphisms \( u = (u_\alpha)_{\alpha \in S} \) such that the diagrams \( D(\mathcal{F},\mathcal{G};\alpha;j) \) anti-commute. A complex of alternate \( \mathcal{X} \)-modules with anti-morphism differentials will be called an alternate complex of alternate \( \mathcal{X} \)-modules. One denotes by \( aC(\mathcal{X}) \) the category of alternate complexes of alternate \( \mathcal{X} \)-modules.

To the edge \( (\alpha;j) \) of the simplicial complex \( (I,S) \) we associate the alternating coefficient \( \varepsilon(\alpha;j) = (-1)^j \). Note that if \( \mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha \in S}, (\varphi(\alpha;j))_{(\alpha;j)}) \) is an \( \mathcal{X} \)-module then \( \alt(\mathcal{F}) = ((\mathcal{F}_\alpha)_{\alpha \in S}, (\varepsilon(\alpha;j)\varphi(\alpha;j))_{(\alpha;j)}) \) is an alternate \( \mathcal{X} \)-module. One checks easily that \( \alt : \text{Mod}(\mathcal{X}) \rightarrow a\text{Mod}(\mathcal{X}) \) is an isomorphism of categories with an obvious inverse that we denote by \( \alt^{-1} \). The functor \( \alt \) extends to an isomorphism of categories \( C(\mathcal{X}) \rightarrow aC(\mathcal{X}) \).

Indeed, if \( \mathcal{F}^* \in C(\mathcal{X}) \), \( \mathcal{F}^* = ((\mathcal{F}_\alpha^*)_{\alpha \in S}, (\varphi(\alpha;j))_{(\alpha;j)}) \) then the terms of \( \alt(\mathcal{F}^*) \) are obtained from the terms of \( \mathcal{F}^* \) via the functor \( \alt \), while the differentials of each complex \( \mathcal{F}_\alpha^* \) are multiplied by \( (-1)^{|\alpha|} \).

Remark 3.3.3. The notions of alternate \( \mathcal{X} \)-module and alternate complex of \( \mathcal{X} \)-modules do not depend on the total order on \( I \). The \( \alt \) functors do. However for two total orders on \( I \) there is a (non-unique) functorial isomorphism between the two corresponding \( \alt \) functors.

3.4 Inverse images. Consider the following setting:

- \( f : (I,S) \rightarrow (J,T) \) a morphism of simplicial complexes
- fixed total orders on \( I \) and \( J \) such that \( f : I \rightarrow J \) is increasing
- \( X = ((X_\alpha)_{\alpha \in S}, (\rho_{\alpha \beta})_{\alpha \leq \beta}), \ Y = ((Y_\gamma)_{\gamma \in T}, (\rho'_{\gamma \delta})_{\gamma \leq \delta}) \) s.s.analytic spaces indexed by \( (I,S) \), respectively \( (J,T) \)
- \( F : X \rightarrow Y \) a morphism of s.s.analytic spaces over \( f \) (see Definition 3.3.1), that is \( F = (F_\alpha,F_\alpha^*)_{\alpha \in S} \) with \( F_\alpha : X_\alpha \rightarrow Y_{f(\alpha)} \) morphism of analytic spaces such that for \( \alpha \leq \beta \) the following diagram commutes:

\[
\begin{array}{c}
X_\beta \xrightarrow{F_\beta} Y_{f(\beta)} \\
\downarrow \rho_{\alpha \beta} \quad \downarrow \rho'_{f(\alpha)f(\beta)} \\
X_\alpha \xrightarrow{F_\alpha} Y_{f(\alpha)}
\end{array}
\]
Note that $S$ is the disjoint union of the sets $(S_{\gamma})_{\gamma \in \mathcal{T}}$, with
\[ S_{\gamma} = \{ \alpha \in S | f(\alpha) = \gamma \} \quad (14) \]
Moreover, each $S_{\gamma}$ is the union of the sets $(I(\gamma, i))_{i \geq 0}$ where
\[ I(\gamma, i) = \{ \alpha \in S | f(\alpha) = \gamma, |\alpha| = |\gamma| + i \} \quad (15) \]
We also set:
\[ I(i) = \bigcup_{\gamma \in \mathcal{T}} I(\gamma, i) \quad (16) \]
In particular $I(\gamma, 0)$ consists of all the simplexes of $S$ which are in a one-to-one correspondence with $\gamma$ via $f$.

Let $G \in \text{Mod}(\gamma)$ with $G = ((G_{\gamma})_{\gamma \in \mathcal{T}}, (\psi_{\gamma})_{\gamma \subset \delta})$. The inverse image $F^*(G)$ of $G$ is, by definition, the $X$-module with the components:
\[ F^*(G)_{\alpha} = F^*_\alpha(G_{f(\alpha)}) \quad \text{for} \ \alpha \in S \quad (17) \]
and connecting morphisms for all $\alpha \subset \beta$
\[ \mu_{\beta \alpha} = F^*_\beta(\tilde{\psi}_{f(\beta)f(\alpha)}) \quad (18) \]
where
\[ \tilde{\psi}_{f(\beta)f(\alpha)} : F_{f(\beta)f(\alpha)}(G_{f(\alpha)}) \rightarrow G_{f(\beta)} \quad (19) \]
is the morphism which corresponds via the usual adjunction isomorphism to the connecting morphism
\[ \psi_{f(\beta)f(\alpha)} : G_{f(\alpha)} \rightarrow \rho'_{f(\beta)f(\alpha)}(G_{f(\beta)}) \quad (20) \]
One checks easily that the family of morphisms $(\mu_{\beta \alpha})_{\alpha \subset \beta}$ satisfies the required conditions.

**Remark 3.4.1.** Let $X$ be an analytic space, $U = (U_i)_{i \in I}$ an open covering of $X$, $\mathcal{U}$ the s.s. analytic space determined by $U$ (see Example 3.1.3) and $i : \mathcal{U} \rightarrow X$ the morphism of s.s. analytic spaces given by the natural inclusions (see Example 3.1.6). If $F \in \text{Mod}(\mathcal{O}_X)$ then the $\mathcal{U}$-module $F|\mathcal{U}$ (see Example 3.2.3) coincides with $i^*(F)$.

Let $G \in \text{Mod}(\gamma)$ as above, $F \in \text{Mod}(X)$ with $(F_\alpha)_{\alpha \in S}$, $(\varphi_{\alpha \beta})_{\alpha \subset \beta}$, and $v : F^*(G) \rightarrow F$ a morphism of $X$-modules. One remarks that $v$ is completely determined by the family of morphisms $(v_\alpha)_\alpha$
\[ v_\alpha : F_\alpha^*(G_{f(\alpha)}) \rightarrow F_\alpha \]
where $\alpha \in S$ is such that $f|\alpha$ is injective. More precisely one checks directly the following lemma:

**Lemma 3.4.2.** Let $F \in \text{Mod}(X)$, $G \in \text{Mod}(\gamma)$ as above. Then the morphisms $(v_\alpha)_{\alpha \in I(0)}$ determine a morphism of $X$-modules $v : F^*(G) \rightarrow F$ iff they verify the following conditions:

(1) For $\gamma \in \mathcal{T}$, $\beta \in I(\gamma, 1)$ let $\alpha_1$, $\alpha_2 \in I(\gamma, 0)$ be the only two simplexes s.t. $\alpha_1$, $\alpha_2 \subset \beta$. Then the following diagram commutes:
\[ F^*_\beta(G_{\gamma}) \xrightarrow{\rho^*_\alpha\beta(v_{\alpha_1})} F^*_\alpha(F_{\alpha_1}) \xrightarrow{\tilde{\varphi}_{\beta\alpha_1}} F^*_\beta(F_{\alpha_1}) \quad (21) \]
\[ \xrightarrow{\tilde{\varphi}_{\beta\alpha_2}} F^*_\beta(F_{\alpha_2}) \]
\[ (2) \quad \text{For} \ \gamma, \delta \in \mathcal{T}, \ \gamma \subset \delta, \ \text{and} \ \alpha \in I(\gamma, 0), \ \beta \in I(\delta, 0) \ \text{with} \ \alpha \subset \beta \ \text{the following diagram commutes:} \]
\[ \rho^*_{\alpha\beta}F^*_\alpha(G_{\gamma}) \xrightarrow{\rho^*_{\alpha\beta}(v_\alpha)} F^*_\alpha(F_{\alpha}) \xrightarrow{\tilde{\varphi}_{\beta\alpha}} F^*_\beta(F_{\alpha}) \quad (22) \]
Remark 3.4.3. In the same way as above (i.e. componentwise) one can construct an inverse image functor $F^{-1}$ for s.s.sheaves of abelian groups.

3.5 The $F^\sharp$ functor. We use the setting described at the beginning of paragraph 3.4. Let $\mathcal{F} \in a\text{Mod}(X)$, $\mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha \in S}, (\varphi_{(\alpha;j)})_{(\alpha;j)})$. For $\gamma \in \mathcal{T}$

$$(F^{\alpha*}(\mathcal{F}_\alpha))_{\alpha}, (\mathcal{F}_{\alpha(\alpha;j)})_{(\alpha;j)})_{f(\alpha)=f(\sigma(\alpha;j))=\gamma}$$

is a multicomplex of $Y_{\gamma}$-modules (recall that multicomplex means anti-commuting rectangles) and consider the following simple complex associated to this multicomplex:

$$... \to \prod_{\alpha \in I(\gamma,i)} F^{\alpha*}(\mathcal{F}_\alpha) \to \prod_{\alpha \in I(\gamma,i+1)} F^{\alpha*}(\mathcal{F}_\alpha) \to ...$$

with the product indexed by $I(\gamma,0)$ in degree 0. The connecting morphisms of $\mathcal{F}$ induce anti-morphisms $C^*(\sigma;\gamma;i)) \to \rho(\gamma;j)* (C^*(\gamma))$ and one checks that $(C^*(\gamma))_{\gamma \in \mathcal{T}}$ is an alternated complex of alternated $\mathcal{Y}$-modules.

If we start with an alternated complex of alternated $\mathcal{X}$-modules $\mathcal{F}$ instead of an alternated $\mathcal{X}$-module, then $C^*(\gamma)$ is a double complex where the product for $I(\gamma,0)$ is considered in bidegree $(0,0)$.

Definition 3.5.1. (1) If $\mathcal{F} \in a\text{Mod}(X)$ then $F^\sharp_{\mathcal{F}}(\mathcal{F})$ is the alternated complex of alternated $\mathcal{Y}$-modules with

$$F^\sharp_{\mathcal{F}}(\mathcal{F})_\gamma = C^*(\gamma)$$

and with connecting morphisms induced by those of $\mathcal{F}$

(2) If $\mathcal{F} \in aC(X)$ then $F^\sharp_{\mathcal{F}}(\mathcal{F})$ is the alternated complex of alternated $\mathcal{Y}$-modules where $F^\sharp_{\mathcal{F}}(\mathcal{F})_\gamma$ is the simple complex associated to the double complex $C^*(\gamma)$ and the connecting morphisms are induced by those of $\mathcal{F}$

(3) If $\mathcal{F} \in \text{Mod}(X)$ (respectively $\mathcal{F} \in C(X)$) then $F^\sharp_{\mathcal{F}}(\mathcal{F}) = \text{alt}^{-1}(F^\sharp_{\mathcal{F}}(\text{alt}(\mathcal{F})))$

One checks easily that the definition of $F^\sharp_{\mathcal{F}}$ is compatible with the natural inclusion functors $a\text{Mod}(X) \to aC(X)$ and $\text{Mod}(X) \to C(X)$.

Example 3.5.2. The components $(F^\sharp_{\mathcal{F}})_\alpha$ of the morphism $F : X \to Y$ determine a morphism of $\mathcal{Y}$-modules $F^\sharp : \mathcal{O}_Y \to F^\sharp_{\mathcal{O}_X}$

Example 3.5.3. If $f : (I,S) \to (J,T)$ is bijective (in particular if $f$ is the identity of $(I,S)$) then $F^\sharp_{\mathcal{F}}(\gamma) = F^{\alpha*}(\mathcal{F}_\alpha)$ where $f(\alpha) = \gamma$. Remark that if $F : X \to Y$ is a morphism of analytic spaces and $\mathcal{F} \in \text{Mod}(O_X)$ then the usual direct image $F_*(\mathcal{F})$ coincides with $F^\sharp_{\mathcal{F}}$ as module over $X$ seen as a s.s.analytic space indexed by $K(\text{pt})$ (see Example 3.1.2).

Example 3.5.4. Let $\mathcal{X}$ be an analytic space, $\mathcal{U} = (U_i)_{i \in I}$ an open covering of $X$, and $\mathcal{F} \in \text{Mod}(O_X)$. If $i : \mathcal{U} \to X$ is the morphism of s.s.analytic spaces over $a_{N(\mathcal{U})} : N(\mathcal{U}) \to K(\text{pt})$ given by the inclusions (see Example 3.1.2) then $i_*(\mathcal{F}|\mathcal{U})$ is the Čech complex of $\mathcal{F}$ with respect to the covering $\mathcal{U}$. Note that the natural morphism $\mathcal{F} \to i^\sharp_{\mathcal{F}}(\mathcal{F}|\mathcal{U})$ is a quasi-isomorphism (see e.g. [S-FAC], chap 1, §4, Lemme 1).

Since the Cartesian product is associative, the form of the terms of the complex $(C^*(\gamma))$ implies that the functor $F^\sharp$ of s.s.modules commutes with the composition of morphisms of s.s.analytic spaces. Thus the following lemma holds:

Lemma 3.5.5. Let $f : (I_1, S_1) \to (I_2, S_2)$, $g : (I_2, S_2) \to (I_3, S_3)$ be morphisms of simplicial complexes and assume that we have fixed total orders on $I_1$, $I_2$, $I_3$. Let $F : X \to Y$, $G : Y \to Z$ be morphisms of s.s.analytic spaces over $f$, respectively $g$, where $X$, respectively $Y$, respectively $Z$ is a s.s.analytic space relative to $(I_1, S_1)$, Respectively to $(I_2, S_2)$, and $(I_3, S_3)$ Then if $\mathcal{F} \in a\text{Mod}(X)$ or $\mathcal{F} \in aC(X)$ or $\mathcal{F} \in \text{Mod}(X)$ or $\mathcal{F} \in C(X)$ or $\mathcal{F} \in aC(X)$ one has

$$(G \circ F)^\sharp_{\mathcal{F}} = G^\sharp_{F^\sharp_{\mathcal{F}}}$$

□
For $\mathcal{F} \in \text{Mod}(\mathcal{X})$ set

$$F_*(\mathcal{F}) = Z^0(F_1(\mathcal{F}))$$

where $Z^0$ denotes the sheaf of 0 degree cocycles. One checks that the definition of $F_*(\mathcal{F})$ agrees with the one given in [F] §2.A.

There is an obvious natural inclusion morphism:

$$F_*(\mathcal{F}) \hookrightarrow F'_*(\mathcal{F})$$

Remark 3.5.6. One checks that the morphism $(27)$ induces an isomorphism between the respective derived functors.

Remark 3.5.7. Lemma 3.5.5 immediately implies that $(G \circ F)_*(\mathcal{F}) = G_* F_*(\mathcal{F})$

Let $\mathcal{F} \in \text{Mod}(\mathcal{X})$ be as above, $\mathcal{G} \in \text{Mod}(\mathcal{Y})$ with $\mathcal{G} = ((G_\gamma)_{\gamma \in \mathcal{T}}, (\psi_\delta)_{\gamma \subset \delta})$ and let $u : \mathcal{G} \to F_1(\mathcal{F})$ be a morphism of $\mathcal{Y}$-modules. Note that $u$ factors through $F_*(\mathcal{F})$. Hence $u$ is completely determined by the family of morphisms $(u_{\gamma\alpha})_{\gamma\alpha}$ where $\gamma \in \mathcal{T}$, $\alpha \in I(\gamma, 0)$, with

$$u_{\gamma\alpha} : G_\gamma \to F_{\alpha*}(\mathcal{F}_\alpha)$$

Conversely, one checks directly the following lemma that describes the families $(u_{\gamma\alpha})_{\gamma\alpha}$ as above that give a morphism of $\mathcal{Y}$-modules $u : \mathcal{G} \to F'_*(\mathcal{F})$:

Lemma 3.5.8. The morphisms $(u_{\gamma\alpha})_{\gamma\alpha}$ are the components of a morphism of $\mathcal{Y}$-modules $u : \mathcal{G} \to F'_*(\mathcal{F})$ iff they verify the following conditions:

1. For $\gamma \in \mathcal{T}, \beta \in I(\gamma, 1)$ let $\alpha_1, \alpha_2 \in I(\gamma, 0)$ be the only two simplexes s.t. $\alpha_1, \alpha_2 \subset \beta$. Then the following diagram commutes:

$$\begin{array}{ccc}
G_\gamma & \xrightarrow{u_{\gamma\alpha_1}} & F_{\alpha_1*}(\mathcal{F}_{\alpha_1}) \\
\downarrow u_{\gamma\alpha_2} & & \downarrow F_{\alpha_1*}(\varphi_{\beta\alpha_1}) \\
F_{\alpha_2*}(\mathcal{F}_{\alpha_2}) & \xrightarrow{F_{\alpha_2*}(\varphi_{\beta\alpha_2})} & F_{\beta*}(\mathcal{F}_{\beta})
\end{array}$$

2. For $\gamma, \delta \in \mathcal{T}, \gamma \subset \delta$, and $\alpha \in I(\gamma, 0), \beta \in I(\delta, 0)$ with $\alpha \subset \beta$ the following diagram commutes:

$$\begin{array}{ccc}
G_\gamma & \xrightarrow{u_{\gamma\alpha}} & F_{\alpha*}(\mathcal{F}_{\alpha}) \\
\downarrow \psi_\gamma & & \downarrow F_{\alpha*}(\varphi_{\beta\alpha}) \\
\rho'_\gamma(\mathcal{G}_\delta) & \xrightarrow{\rho'_{\gamma*}(\alpha_{\beta\delta})} & F_{\alpha*}(\mathcal{G}_\delta)
\end{array}$$

Remark 3.5.9. Conditions in Lemma 3.5.5 and those in Lemma 3.4.2 can be obtained from one another by adjunction. Using the two lemmas one checks that the functor $F_*$ on $\text{Mod}(\mathcal{X})$ is indeed a right adjoint for $F^*$.

4. Embedding Atlases

4.1 Embedding triples. Let $i : X \hookrightarrow D$ be a closed embedding of the analytic space $X$ in the complex manifold $D$. $(X, i, D)$ will be called an embedding triple. A morphism of embedding triples $(f, \tilde{f}) : (X_1, i_1, D_1) \to (X_2, i_2, D_2)$ is a pair of morphisms of analytic spaces such that the following diagram commutes:

$$\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow i_1 & & \downarrow i_2 \\
D_1 & \xrightarrow{\tilde{f}} & D_2
\end{array}$$

If $\tilde{f}$ is clear from the context we sometimes write $f$ instead of $(f, \tilde{f})$. 
A complex manifold $D$ will be identified with the embedding triple $(D, id, D)$.

If $(X, i, D)$, $(Y, i', D')$ are embedding triples and $f : X \to Y$ is a morphism of analytic spaces, then, in general, there does not exist $\tilde{f} : D \to D'$ such that $(f, \tilde{f})$ is a morphism of embedding triples. However the following result can be checked easily:

**Lemma 4.1.1.** 1. Let $(X, i, D)$, $(Y, i', D')$ be embedding triples and $f : X \to Y$ a morphism of analytic spaces. Then $j = (i, i' \circ f) : D \to X \times D'$ is a closed embedding and we have natural morphisms:

$$
(X, i, D) \xleftarrow{(id, p_1)} (X, j, D \times D') \xrightarrow{(f, p_2)} (Y, i', D')
$$

where $p_1, p_2$ are the projections. Moreover, assume we have another embedding triple $(X, i_1, D_1)$ and morphisms

$$(X, i, D) \xleftarrow{(id, q_1)} (X, i_1, D_1) \xrightarrow{f, q_2} (Y, i', D')$$

Then $\alpha = (id, (q_1, q_2)) : (X, i_1, D_1) \to (X, j, D \times D')$ is the unique morphism s.t. the following diagram commutes:

$$
\begin{array}{ccc}
(X, i, D) & \xleftarrow{(id, p_1)} & (X, j, D \times D') & \xrightarrow{(f, p_2)} & (Y, i', D') \\
& \alpha \uparrow & & \downarrow \alpha & \\
&(X, i_1, D_1) & & (X, j, D \times D') & \xrightarrow{(f, q_2)} & (Y, i', D')
\end{array}
$$

2. For $s = 1, 2$ let $(Y_s, i'_s, D'_s)$ be embedding triples and $f_s : X \to Y_s$ morphisms of analytic spaces. Then $j_{12} = (i, i'_1 \circ f_1, i'_2 \circ f_2) : X \to D \times D'_1 \times D'_2$ is a closed embedding and there exists unique morphisms $\alpha_s$ s.t. the following diagrams commute:

$$
\begin{array}{ccc}
(X, i, D) & \xleftarrow{(id, p_1)} & (X, j, D \times D'_1 \times D'_2) & \xrightarrow{(f_s, p_{2s})} & (Y, i'_s, D'_s) \\
& \alpha_s \downarrow & & \downarrow \alpha_s & \\
&(X, j_s, D \times D'_s) & & (X, j, D \times D') & \xrightarrow{(f, q_2)} & (Y, i', D')
\end{array}
$$

where $j_s = (i, i'_s \circ f_s)$ and $p_1, p_2, p_{2s}$ are the obvious projections.

**Remark 4.1.2.** Under the hypothesis of Lemma 4.1.1, if $D_1 \subset \mathbb{C}^n$, $D_2 \subset \mathbb{C}^m$ are Stein open sets then one checks that there exists a Stein open subset $D'_1 \subset D_1$ and $\tilde{f} : D'_1 \to D_2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
D_1 & \xleftarrow{i} & D'_1 & \xrightarrow{f} & D_2 \\
\uparrow i_1 & & \uparrow \tilde{i}_1 & & \uparrow i_2 \\
X & \xleftarrow{id} & X & \xrightarrow{f} & Y
\end{array}
$$

i.e we have the diagram of embedding triples:

$$
(X, i_1, D_1) \xleftarrow{(id, i)} (X, i_1, D'_1) \xrightarrow{(f, \tilde{f})} (Y, i_2, D_2)
$$

For brevity we shall say s.s.embedding triple instead of s.s.system of embedding triples.

**Remark 4.1.3.** A s.s.embedding triple $(X_\alpha, k_\alpha, D_\alpha)_{\alpha \in S}$ can be seen as a triple $(X, k, D)$ where $X = (X_\alpha)_{\alpha \in S}$ is s.s.analytic space, $D = (D_\alpha)_{\alpha \in S}$ is a s.s.complex manifold, and $k : X \to D$ is a morphism of s.s.analytic spaces such that each $k_\alpha : X_\alpha \to D_\alpha$ is a closed embedding.
4.2 Embedding atlases.

Definition 4.2.1. (1) Let $X$ be an analytic space. An embedding atlas of $X$ consists of a family of embedding triples $A = (U_i, k_i, D_i)_{i \in I}$ such that the family $\text{cov}(A) = (U_i)_{i \in I}$ is an open covering of $X$. An embedding triple $(U_i, k_i, D_i)$ of $A$ will be called a chart. The pair $(X, A)$ will be called a locally embedded analytic space or, sometimes, a local embedding of $X$.

(2) Let $A = (U_i, k_i, D_i)_{i \in I}$ and $B = (V_j, k'_j, D'_j)_{j \in J}$ be embedding atlases of the analytic space $X$, respectively $Y$. A morphism of locally embedded analytic spaces $F : (X, A) \to (Y, B)$ consists of a triple $(f, \tau, (\tilde{f}_i)_{i \in I})$ where:
- $f : X \to Y$ is a morphism of analytic spaces
- $\tau : I \to J$ is a refinement mapping such that $f(U_i) \subset V_{\tau(i)}$ for all $i \in I$
- $\tilde{f}_i : D_i \to D'_{\tau(i)}$ is a morphism of complex manifolds such that $(f|_{U_i}, \tilde{f}_i) : (U_i, k_i, D_i) \to (V_{\tau(i)}, k'_{\tau(i)}, D'_{\tau(i)})$ is a morphism of embedding triples.

We say that $A$ is locally finite if the open covering $\text{cov}(A)$ is locally finite.

In particular, an embedding triple $(X, i, D)$ can be seen as an embedding atlas of $X$ with one chart.

Lemma 4.2.2. 1. Let $(X, A)$ be a locally embedded analytic space with $A = (U_i, k_i, D_i)_{i \in I}$. There exists a s.s.embedding triple $(\Omega, k, D) = (U_\alpha, k_\alpha, D_\alpha)_{\alpha \in N(\Omega)}$ indexed by the simplicial complex $(I, N(\text{cov}(A)))$ such that the embedding triples corresponding to 0-length simplexes coincide with the embedding triples of the atlas $A$.

2. If $f : (X, A) \to (Y, B)$ is a morphism of embedded analytic spaces then $f$ induces a morphism $F : (\Omega, k, D) \to (\Omega', k', D')$ between the respective associated s.s.embedding triples. Moreover the following diagram commutes:

$$
\begin{array}{ccc}
\Omega & \xrightarrow{F} & \Omega' \\
\downarrow b & & \downarrow b' \\
X & \xrightarrow{f} & Y
\end{array}
$$

Proof. Take $\Omega = \{(U_\alpha)_{\alpha \in N(\text{cov}(A))}, (\iota_{\alpha\beta})_{\alpha \subset \beta}\}$ to be the s.s.analytic space corresponding to the open covering $(U_i)_{i \in I}$ (see Example 3.1.3), $D = \{(D_\alpha)_{\alpha \in N(\text{cov}(A))}, (\rho_{\alpha\beta})_{\alpha \subset \beta}\}$ the s.s.complex manifold associated to the family $(D_i)_{i \in I}$ (see Example 3.1.4) and $k : \Omega \to D$ the morphism deduced from the closed embeddings $k_i : U_i \to D_i$(one checks that each $k_\alpha : U_\alpha \to D_\alpha$ is also a closed embedding).

The s.s.embedding triple from Lemma 4.2.2 will be called the s.s.embedding triple associated to $(X, A)$.

Remark 4.2.3. One checks easily that the correspondence in Lemma 4.2.2 gives an equivalence between the category of locally embedded analytic spaces and a subcategory of the category of s.s.embedding triples.

Lemma 4.2.4. (1) Let $f : (X, A) \to (Y, B)$ be a morphism of locally embedded analytic spaces, $F \in \text{Mod}(\mathcal{O}_X)$, $G \in \text{Mod}(\mathcal{O}_Y)$ and $u : G \to f_\ast F$ a morphism of $\mathcal{O}_Y$-modules. If $F : (\Omega, k, D) \to (\Omega', k', D')$ is the morphism induced between the s.s.systems of embedding triples associated to $(X, A)$, $(Y, B)$ then $u$ induces a natural morphism:

$$
F^\ast(u) : G|_\Omega \to F^\ast(F|_\Omega) \quad (38)
$$

(2) Let $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ be morphisms of locally embedded analytic spaces and $h = g \circ f$. Let moreover $F \in \text{Mod}(\mathcal{O}_X)$, $G \in \text{Mod}(\mathcal{O}_Y)$, $H \in \text{Mod}(\mathcal{O}_Z)$ and
mappings \( u : \mathcal{G} \to f_* \mathcal{F} \mathcal{O}_Y \)-linear, \( v : \mathcal{H} \to g_* \mathcal{G}, \) \( w : \mathcal{H} \to h_* \mathcal{F} \mathcal{O}_Z \)-linear, such that \( g_*(u) \circ v = w \) then one has the commutative diagram:

\[
\begin{array}{ccc}
H_* (\mathcal{F} | \mathcal{U}) & \xleftarrow{G_* (F^*(u))} & G_* (\mathcal{G} | \mathcal{U}) \\
\xrightarrow{H^*(w)} & & \xrightarrow{G^*(v)} \\
\mathcal{H} | \mathcal{U} & \xrightarrow{G^*(v)} & \mathcal{G} | \mathcal{U}
\end{array}
\] (39)

Proof. 1. Let \( \text{cov}(\mathcal{A}) = (U_i)_{i \in I}, \) \( \text{cov}(\mathcal{B}) = (V_j)_{j \in J} \). For \( \alpha \in \mathcal{N}(\text{cov}(\mathcal{A})), \gamma \in \mathcal{N}(\text{cov}(\mathcal{B})) \) such that \( \tau(\alpha) \subset \gamma \) let

\[
f_{\gamma \alpha} : U_\alpha \to V_\gamma
\]

be the restriction of \( f \) and

\[
u_{\gamma \alpha} : \mathcal{G} | V_\gamma \to f_{\gamma \alpha} (\mathcal{F} | U_\alpha)
\]

the restriction of \( u \). One verifies that the family of morphisms \( (u_{\gamma \alpha})_{\gamma \alpha} \), where \( \gamma \in \mathcal{N}(\text{cov}(\mathcal{B})), \alpha \in I(\gamma, 0) \), satisfies the hypothesis of Lemma 3.5.8 and consequently they determine a morphism \( F^*(u) \).

2. follows also from Lemma 3.5.8 since one verifies that travelling both ways along the edges of diagram (39) the two morphisms are determined by the same family of morphisms. \( \Box \)

Definition 4.2.5. Let \( f : X \to Y \) be a morphism of analytic spaces and let \( \mathcal{A} = (U_i, k_i, D_i)_{i \in I} \) and \( \mathcal{B} = (V_j, k'_j, D'_j)_{j \in J} \) be embedding atlases of \( X \), respectively \( Y \). \( \mathcal{A} \) and \( \mathcal{B} \) are said to be \( f \)-compliant if \( \text{cov}(\mathcal{A}) \prec f^{-1}(\text{cov}(\mathcal{B})) \), i.e. there exists a refinement mapping \( \tau : I \to J \) such that \( f(U_i) \subset V_{\tau(i)} \) for all \( i \in I \). Obviously the refinement mapping \( \tau \) need not be unique.

Note that if \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are embedding atlases of \( X \), to say that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are \( \text{id}_X \)-compliant simply means that \( \text{cov}(\mathcal{A}_1) \prec \text{cov}(\mathcal{A}_2) \).

If \( \mathcal{A} \) and \( \mathcal{B} \) are \( f \)-compliant then, in general, there does not exist a morphism of locally embedded analytic spaces \( F : (X, \mathcal{A}) \to (Y, \mathcal{B}) \) over \( f \). However Lemma 4.1.1 immediately implies:

Lemma 4.2.6. (1) In the context of Definition 4.2.5 let

\[
\mathcal{A} \times_{\tau} \mathcal{B} = (U_i, k_{i\tau(i)}, D_i \times D'_{\tau(i)})_{i \in I}
\]

Then \( \mathcal{A} \times_{\tau} \mathcal{B} \) is an embedding atlas on \( X \), and the family of morphisms of embedding triples

\[
(U_i, k_i, D_i) \xrightarrow{(id,p_1)} (U_i, k_{i\tau(i)}, D_i \times D'_{\tau(i)}) \xrightarrow{(f,p_2)} (V_{\tau(i)}, k'_{\tau(i)}, D'_{\tau(i)})
\]

where \( p_1, p_2 \) are the projections, give the diagram of locally embedded analytic spaces:

\[
(X, \mathcal{A}) \xrightarrow{(id, id, p_1)} (X, \mathcal{A} \times_{\tau} \mathcal{B}) \xrightarrow{(f_{\tau}, p_2)} (Y, \mathcal{B})
\]

Moreover, assume \( \mathcal{A}' = (U_k, k'_k, D'_k)_{k \in K} \) is another embedding atlas of \( X \), and there exist morphisms

\[
(X, \mathcal{A}) \xrightarrow{(id, v, q_1)} (X, \mathcal{A}') \xrightarrow{(f_{\tau}, q_2)} (Y, \mathcal{B})
\]

Then there is a unique morphism \( \alpha = (X, \mathcal{A}') \to (X, \mathcal{A} \times_{\tau} \mathcal{B}) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
(X, \mathcal{A}) & \xleftarrow{(id, id, p_1)} & (X, \mathcal{A} \times_{\tau} \mathcal{B}) \\
\xrightarrow{(id, \tau, q_1)} & & \xrightarrow{(f_{\tau}, p_2)} \\
(X, \mathcal{A}') & \xrightarrow{\alpha} & (Y, \mathcal{B})
\end{array}
\] (40)
(2) Let $\tau_1, \tau_2 : I \to J$ be refinement mappings. Then

$$A \times_{\tau_1 \tau_2} B = (U_i, k_i, D_i)_{i \in I}, D_i \times D'_i \times D''_i$$

is an embedding atlas on $X$ and, for $s = 1, 2$, one has natural morphisms:

$$\alpha : (X, A, A \times_{\tau_1 \tau_2} B) \to (Y, B)$$

such that the following diagram commutes:

$$\begin{array}{c}
(X, A) \\ (id, id, p_1) \\
(id, id, p_1) \\
\end{array} \quad \begin{array}{c}
(X, A \times_{\tau_1 \tau_2} B) \\
\alpha_s \\
(f, \tau_s, p_2) \\
\end{array} \quad \begin{array}{c}
(Y, B) \\
(id, id, p_1) \\
(id, id, p_1) \\
\end{array}$$

where $\alpha_s = (id, id, p_{1s})$

**Corollary 4.2.7.** Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of analytic spaces, $h = g \circ f$, and let $A = (U_i, k_i, D_i)_{i \in I}$, $B = (V_j, k'_j, D'_j)_{j \in J}$, $C = (W_k, k''_k, D''_k)_{k \in K}$ be embedding atlases of $X$, respectively $Y$, and $Z$. Assume that $A, B$ are $f$-compliant, $B, C$ are $g$-compliant and let $\tau : I \to J$, $\upsilon : J \to K$ be refinement mappings. Set

$$A \times_{\tau} B \times_{\upsilon} C = (U_i, k_i, D_i) \times D'_i \times D''_i$$

Then $A \times_{\tau} B \times_{\upsilon} C$ is an embedding atlas on $X$, and the following diagram commutes:

$$\begin{array}{c}
(X, A) \\
(id, id, p_{12}) \\
(id, id, p_{12}) \\
\end{array} \quad \begin{array}{c}
(X, A \times_{\tau} B \times_{\upsilon} C) \\
(f, \tau, p_{23}) \\
(id, id, p_{23}) \\
\end{array} \quad \begin{array}{c}
(Y, B \times_{\upsilon} C) \\
(id, id, p_{23}) \\
(id, id, p_{23}) \\
\end{array} \quad \begin{array}{c}
(Z, C) \\
(id, id, p_3) \\
(id, id, p_3) \\
\end{array}$$

where $p_{12}, p_{23}, p_1, p_2, p_3$ are the obvious projections.

**Remark 4.2.8.** By Lemma 4.2.6 there is a unique morphism

$$\alpha : (X, A \times_{\tau} B \times_{\upsilon} C) \to (X, A \times_{\upsilon} C)$$

such that the following diagram commutes:

$$\begin{array}{c}
(X, A) \\
(id, id, p_{1}) \\
(id, id, p_{1}) \\
\end{array} \quad \begin{array}{c}
(X, A \times_{\upsilon} C) \\
(gof, \upsilon, p_2) \\
(gof, \upsilon, p_2) \\
\end{array} \quad \begin{array}{c}
(Z, C) \\
(id, id, p_{3}) \\
(id, id, p_{3}) \\
\end{array}$$

**Remark 4.2.9.** Let $(X, A)$, $(Y, B)$ be locally embedded analytic spaces, $A = (U_i, k_i, D_i)_{i \in I}$ and $B = (V_j, k'_j, D'_j)_{j \in J}$, and $f : X \to Y$ a morphism of analytic spaces. If $A_1$ is an embedding atlas of $X$ over the open covering $(U_i \cap f^{-1}(V_j))_{i,j}$ then $A_1$, $A$ are $id_X$-compliant and $A_1, B$ are $f$-compliant. Moreover, if $A_2$ is another embedding atlas of $X$ s.t. $A_2$, $A$ are $id_X$-compliant and $A_2, B$ are $f$-compliant then $A_2, A_1$ are $id_X$-compliant.

5. **Construction of the Dolbeault resolution**

We shall extend successively the definition of the Dolbeault-Grothendieck resolution from the classical case of complex manifolds to that of embedding triples, then to s.s. of embedding triples and, finally, to the case of a general analytic space with a fixed embedding atlas. It is essential that at each extension the definition be compatible with $\sharp$-direct images (i.e. there exists a commutative diagram similar to diagram [S]). For reference purposes
property 1.a, for instance, will be called 1.a-mfld in the smooth case, 1.a-emb in the embedded case, and 1.a-ss in the semi-simplicial case.

5.1 The smooth case. Let $X$ be an $n$-dimensional complex manifold. We regard $X$ as a locally embedded analytic space with one chart given by the identity map. For $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$ denote by $\text{Dolb}(X;\mathcal{F})$ the complex:

$$0 \to \mathcal{E}_X^{0,0} \otimes \mathcal{O}_X \mathcal{F} \to \ldots \to \mathcal{E}_X^{0,n} \otimes \mathcal{O}_X \mathcal{F} \to 0$$

(46)

obtained by applying the functor $\bullet \otimes \mathcal{O}_X$ to the resolution $\mathcal{E}_X$ with the term containing $\mathcal{E}_X^{0,0}$ considered in degree 0. This complex appeared first in the proof of the duality theorems of Serre-Malgrange.

Note that $\text{Dolb}(X;\bullet)$ is a functor $\text{Mod}(\mathcal{O}_X) \to C^b(X)$. We check that this functor satisfies the properties of the Theorem.

1.a-mfld is proved in Malgrange [Ma]. 1.a-mfld and 1.b-mfld follow immediately from 1.d-mfld, and 1.c-mfld is well known. In 1.e-mfld the morphism

$$\text{Dolb}(X;\mathcal{O}_X) \otimes \mathcal{O}_X \mathcal{F} \to \text{Dolb}(X;\mathcal{F})$$

(47)

is obviously an isomorphism for any $\mathcal{F}$.

For 2-mfld let $f : X \to Y$ be a morphism of complex manifolds. Let

$$f^* : \text{Dolb}(Y;\mathcal{O}_Y) \to f_*\text{Dolb}(X;\mathcal{O}_X)$$

(48)

be the morphism given by the pullback of differential forms - it is a morphism over the mapping $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$. For $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$, combining (47) and (48), one gets a functorial morphism

$$\text{Dolb}(Y;f_*\mathcal{F}) \to f_*\text{Dolb}(X;\mathcal{F})$$

(49)

and moreover the following diagram commutes:

$$\text{Dolb}(Y;f_*\mathcal{F}) \to f_*\text{Dolb}(X;\mathcal{F})$$

(50)

Let moreover $\mathcal{G} \in \text{Mod}(\mathcal{O}_Y)$ and $u : \mathcal{G} \to f_*\mathcal{F}$ a morphism of $\mathcal{O}_Y$-modules. Then the morphism (49) together with $u$ induces the morphism of complexes:

$$f^*(u) : \text{Dolb}(Y;\mathcal{G}) \to f_*\text{Dolb}(X;\mathcal{F})$$

(51)

and combining (50) with 1.b-mfld one checks that the diagram (50) commutes in the smooth case.

3-mfld One starts from the obvious commutative diagram:

$$h_*\text{Dolb}(X;\mathcal{O}_X) \leftarrow g_*(f^*) \text{Dolb}(Y;\mathcal{O}_Y) \rightarrow g_*\text{Dolb}(Y;\mathcal{O}_Y)$$

(52)

Using isomorphism (47) and the above diagram one checks the case $\mathcal{G} = f_*\mathcal{F}$, $\mathcal{H} = h_*\mathcal{F}$, and each of $u, v, w$ equals the respective identity. For the general case one uses the functorial morphism (49).

Finally we consider the case of open and closed embeddings of manifolds.

Remark 5.1.1. Let $i : X \hookrightarrow Y$ be an open embedding of complex manifolds, $\mathcal{F} \in \text{Mod}(\mathcal{O}_Y)$, and denote by $u : \mathcal{F} \to i_*i^{-1}(\mathcal{F})$ the canonical adjunction morphism. Then obviously $\text{Dolb}(X;i_*i^{-1}\mathcal{F}) \simeq i^{-1}\text{Dolb}(Y;\mathcal{F})$ and, hence, $i^*(u)$ coincides with the adjunction morphism $\text{Dolb}(Y;\mathcal{F}) \to i_*i^{-1}\text{Dolb}(Y;\mathcal{F})$. 
Proposition 5.1.2. Let $i : X \hookrightarrow Y$ be a closed embedding of manifolds and $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$ Then the natural morphism

$$i^*(id) : \text{Dolb}(Y; i_*\mathcal{F}) \to i_*\text{Dolb}(X; \mathcal{F})$$

(53)

is a quasiisomorphism.

Proof. $i^*(id)$ is a morphism between two soft resolutions of $i_*\mathcal{F}$. □

5.2 The embedded case. Let $(X, i, D)$ be an embedding triple. We regard $(X, i, D)$ as a locally embedded analytic space with one chart. Let $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$. We use the notation $\text{Dolb}(i; \mathcal{F})$ for the restriction to $X$ of the complex $\text{Dolb}(D; i_*\mathcal{F})$:

$$0 \to \mathcal{E}^{0,0}_D \otimes_{\mathcal{O}_D} i_*\mathcal{F} \to ... \to \mathcal{E}^{0,n}_D \otimes_{\mathcal{O}_D} i_*\mathcal{F} \to 0$$

(54)

Remark that each term of $\text{Dolb}(D; i_*\mathcal{F})$, is null outside $X$ and, furthermore, has a structure of $i_*\mathcal{O}_X$-module deduced via the natural morphism $\mathcal{O}_D \to i_*\mathcal{O}_X$. If $\mathcal{I}_X \subset \mathcal{O}_D$ is the ideal that defines $X$ as an analytic subspace of $D$ (i.e. $i_*\mathcal{O}_X \approx \mathcal{O}_D/\mathcal{I}_X$) then it is immediate to see that $\text{Dolb}(i; \mathcal{O}_X)$ is isomorphic with the restriction to $X$ of the complex

$$0 \to \mathcal{E}^{0,0}_D/\mathcal{I}_X \mathcal{E}^{0,0}_D \to ... \to \mathcal{E}^{0,n}_D/\mathcal{I}_X \mathcal{E}^{0,n}_D \to 0$$

(55)

and, moreover, $\text{Dolb}(i; \mathcal{F})$ is isomorphic with the restriction to $X$ of the complex:

$$0 \to \mathcal{E}^{0,0}_D/\mathcal{I}_X \mathcal{E}^{0,0}_D \otimes_{\mathcal{O}_D} i_*\mathcal{F} \to ... \to \mathcal{E}^{0,n}_D/\mathcal{I}_X \mathcal{E}^{0,n}_D \otimes_{\mathcal{O}_D} i_*\mathcal{F} \to 0$$

(56)

Thus $\text{Dolb}(i; \mathcal{F})$ is a functor $\text{Mod}(\mathcal{O}_X) \to \text{C}^b(X)$ and

$$\text{Dolb}(i; \mathcal{F}) = i^{-1}\text{Dolb}(D; i_*\mathcal{F})$$

(57)

Moreover, applying $i_*$ one obtains an isomorphism on $D$:

$$\text{Dolb}(D; i_*\mathcal{F}) \approx i_*\text{Dolb}(i; \mathcal{F})$$

(58)

The functor $\text{Dolb}(i; \mathcal{F})$ coincides with $\text{Dolb}(X; \mathcal{F})$ if $X$ is a complex manifold (recall that we identify $X$ with the embedding triple $(X, id, X)$).

The properties of $\text{Dolb}(i; \mathcal{F})$ are obtained from those of $\text{Dolb}(D; \mathcal{F})$ via $i^{-1}$ and $i_*$ (see 1.d-emb and 1.a-emb). Indeed, since $i^{-1}$ is an exact functor the statements 1.a-emb, through 1.d-emb for $\text{Dolb}(i; \mathcal{F})$ follow immediately from the respective statements in the smooth case. Moreover, for any $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$ the isomorphism [47] implies that one has an isomorphism

$$\text{Dolb}(i; \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \to \text{Dolb}(i; \mathcal{F})$$

(59)

2-emb Let $f = (f, \tilde{f}) : (X, i_1, D_1) \to (Y, i_2, D_2)$ be a morphism of embedding triples. On $D_2$ one has the sequence of morphisms:

$$\text{Dolb}(D_2; i_2;\mathcal{O}_Y) \to \text{Dolb}(D_2; \tilde{f}_*i_1;\mathcal{O}_X) \to \tilde{f}_*\text{Dolb}(D_1; i_1;\mathcal{O}_X)$$

(60)

By definition $f^*$ is the morphism obtained by restricting to $X$ the composition of morphisms above (which comes down to applying $i_{2}^{-1}$):

$$f^* : \text{Dolb}(i_2;\mathcal{O}_Y) \to \tilde{f}_*\text{Dolb}(i_1;\mathcal{O}_X)$$

(61)

Combining (59) and (60), one gets for $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$ a functorial morphism

$$\text{Dolb}(i_2; f_*\mathcal{F}) \to \tilde{f}_*\text{Dolb}(i_1; \mathcal{F})$$

(62)

Let moreover $\mathcal{G} \in \text{Mod}(\mathcal{O}_Y)$ and $u : \mathcal{G} \to f_*\mathcal{F}$ a morphism of $\mathcal{O}_Y$-modules Then $f^*(u)$ is by definition the composition of the above morphism with the morphism induced by $u$:

$$f^*(u) : \text{Dolb}(i_2; \mathcal{G}) \to \tilde{f}_*\text{Dolb}(i_1; \mathcal{F})$$

(63)

Finally, diagram [4] on $Y$ commutes iff its extension by $0$ to $D_2$ commutes, and this is true by 2-mfld.
Example 5.2.1. The embedding morphism $i : X \to D$ can be seen as the morphism of embedding triples $(i, i_D^*)$ and $i^*$ is the natural quotient morphism

$$\text{Dolb}(D; \mathcal{O}_D) \to i_! \text{Dolb}(i : \mathcal{O}_X)$$

(64)

For two embeddings of the same analytic space one checks easily the following result:

Remark 5.2.2. Let $f = (i_D, \tilde{f}) : (X, i_1, D_1) \to (Y, i_2, D_2)$ be a morphism of embedding triples over the same analytic space. If $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$ then one checks easily that the natural morphism $f^*(\mathcal{F}) : \text{Dolb}(i_2; \mathcal{F}) \to \text{Dolb}(i_1; \mathcal{F})$ is a quasi-isomorphism.

Remark 5.2.3. In the above setting assume we only have a morphism of analytic spaces $f : X \to Y$ instead of a morphism of embedding triples $(f, \tilde{f}) : (X, i_1, D_1) \to (Y, i_2, D_2)$. Then there may not exist a "direct" morphism $f^*(\mathcal{F}) : \text{Dolb}(i_2; \mathcal{F}) \to \text{Dolb}(i_1; \mathcal{F})$. However, by Lemma [4.1.2] we have the sequence of mappings:

$$(X, i_1, D_1) \xrightarrow{(id, p_1)} (X, i_2, D_1 \times D_2) \xrightarrow{(f, p_2)} (Y, i_2, D_2)$$

and consequently the sequence of morphisms on $Y$:

$$f_* \text{Dolb}(i_1; \mathcal{F}) \xrightarrow{f^*(id)} f_* \text{Dolb}(i_2; \mathcal{F}) \xrightarrow{f^*(u)} \text{Dolb}(i_2; \mathcal{G})$$

(65)

Since the components of $\text{Dolb}(\bullet; \mathcal{F})$ are soft sheaves, Remark 5.2.2 implies that the first morphism above is a quasi-isomorphism and hence one has a morphism $\text{Dolb}(i_2; \mathcal{G}) \to f_* \text{Dolb}(i_1; \mathcal{F})$ in the derived category $D^b(\mathcal{O}_Y)$, that we also denote by $f^*(u)$. Moreover, if $(X, i_3, D_3)$ is another embedding triple with mappings:

$$(X, i_1, D_1) \xleftarrow{(id, q_1)} (X, i_3, D_3) \xrightarrow{(f, q_2)} (Y, i_2, D_2)$$

(66)

then Lemma 4.1.2 also implies that the morphism in the derived category $D^b(\mathcal{O}_Y)$:

$$f_* \text{Dolb}(i_1; \mathcal{F}) \xleftarrow{f^*(id)} f_* \text{Dolb}(i_3; \mathcal{F}) \xrightarrow{f^*(u)} \text{Dolb}(i_2; \mathcal{G})$$

(67)

(68)

coincides with the morphism (66).

To prove 3-emb in the embedded case extend the sheaves $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ with 0 to $D_1$, $D_2$, respectively $D_3$. It is easy to see that $i_* \mathcal{F}$, $i_* \mathcal{G}$, $i_* \mathcal{H}$ satisfy the hypothesis of 3-mfld. Thus we get the commutative diagram on $D_3$:

$$\tilde{g}_* \text{Dolb}(D_1; i_* \mathcal{F}) \xleftarrow{\text{Dolb}(D_3; i_* \mathcal{H})} \tilde{g}_* \text{Dolb}(D_2; i_* \mathcal{G})$$

(69)

Restricting the above diagram to $Z$ (i.e. applying $i_3^{-1}$) one gets the result.

5.3 The semi-simplicial case. We extend the functor $\text{Dolb}$ to the semi-simplicial context for technical reasons. While a s.s.analytic space is not a particular case of analytic space with an embedding atlas, most of the properties in Theorem 1.1.1 apply.

Let $\mathcal{X} = (X_\alpha)_{\alpha \in S}, (\rho_{\alpha \beta})_{\alpha \subseteq \beta}$ be a s.s.complex manifold indexed by the simplicial complex $(I, S)$, and let $\mathcal{F} \in \text{Mod}(\mathcal{X})$, $\mathcal{F} = ((F_\alpha)_{\alpha \in S}, (\varphi_{\beta \alpha})_{\alpha \subseteq \beta})$. We set:

$$\text{Dolb}(\mathcal{X}; \mathcal{F}) = (\text{Dolb}(X_\alpha; F_\alpha)_{\alpha \in S}, (\rho_{\alpha \beta}^* \varphi_{\beta \alpha})_{\alpha \subseteq \beta})$$

(70)

By property 3-mfld, $\text{Dolb}(\mathcal{X}; \mathcal{F})$ is a complex of $\mathcal{X}$-modules; furthermore, it is null in degrees $< 0$.

Note that if $\mathcal{X}$ is a s.s.complex manifold relative to the simplicial complex $K(pt)$ (i.e. a complex manifold - see Example 3.1.2) the functor $\text{Dolb}(\mathcal{X}; \bullet)$ coincides with the functor $\text{Dolb}$ from the smooth case.

Let now $(X, k, \mathcal{D})$ be a s.s.embedding triple indexed by the simplicial complex $(I, S)$ (see Remark 4.1.3), where $\mathcal{X} = (X_\alpha)_{\alpha \in S}$ is a s.s.analytic space, $\mathcal{D} = (D_\alpha)_{\alpha \in S}$ is a s.s.complex.
manifold, and each component $k_\alpha : X_\alpha \to D_\alpha$ of the morphism $k : X \to D$ is a closed embedding.

If $\mathcal{F} \in Mod(X)$, $\mathcal{F} = (\mathcal{F}_\alpha)_{\alpha \in S}$ we set:

$$\text{Dolb}(k; \mathcal{F}) = k^*\text{Dolb}(D; k_1\mathcal{F})$$

(71)

Note that in this case $k_2 = k_1$ and for each $\alpha$, $(k_2\mathcal{F})_\alpha = k_{\alpha*}\mathcal{F}_\alpha$. Hence, for $\alpha \in S$

$$\text{Dolb}(k; \mathcal{F})_\alpha = \text{Dolb}(k_\alpha; \mathcal{F}_\alpha)$$

(72)

Thus $\text{Dolb}(k; \bullet)$ is a functor $\text{Mod}(X) \to C^+(X)$. Furthermore, $\text{Dolb}(k; \mathcal{F})$ is null in degrees $< 0$.

In what follows we shall check the properties of the functor $\text{Dolb}(k; \bullet)$. The treatment for $\text{Dolb}(X; \bullet)$, where $X$ is a s.s.complex manifold, is similar.

Properties 1.a-ss, 1.b-ss, and 1.e-ss follow from the corresponding properties in the embedded case applied for each component $\alpha \in S$. Properties 1.c and 1.d have no sense in this context. However, they can be replaced by the following statements that follow immediately from the embedded case:

1.c'-ss The terms of $\text{Dolb}(k; \mathcal{F})_\alpha = \text{Dolb}(X_\alpha; \mathcal{F}_\alpha)$ are soft sheaves for all $\alpha \in S$.

1.d'-ss The terms of $\text{Dolb}(k; \mathcal{O}_X)_\alpha = \mathcal{O}_{X_\alpha}$ are flat for all $\alpha \in S$.

2-ss Consider the following data:

- $\tau : (I, S) \to (J, T)$ a mapping of simplicial complexes
- $F : (X, k, D) \to (Y, k', D')$ a morphism of s.s.embedding triples over $\tau$
- $F \in \text{Mod}(X)$, $\mathcal{F} = (\mathcal{F}_\alpha)_{\alpha \in S}$, $G \in \text{Mod}(Y)$, $G = (G_\gamma)_{\gamma \in T}$, and $u : G \to F_\tau\mathcal{F}$ a morphism of $\gamma$-modules

For $\gamma \in T$ and $\alpha \in I(\gamma, 0)$ consider the natural morphism:

$$u_{\tau\alpha} : \text{Dolb}(k_\gamma'; G_\gamma \to F_{\alpha*}\text{Dolb}(k_\alpha; \mathcal{F}_\alpha)$$

(73)

According to property 3-emb one sees that the family of morphisms $(u_{\tau\alpha})_{\gamma\alpha}$ satisfies the hypothesis of Lemma 3.5.3. Hence it induces a morphism

$$F^2(u) : \text{Dolb}(k'; G) \to F_\tau F_\tau \text{Dolb}(k; \mathcal{F})$$

(74)

3-ss One checks directly that on each component $\gamma \in T$ the diagram commutes, which comes down to property 3-emb.

5.4 The case of a locally embedded analytic space. Let $(X, A)$ be a locally embedded analytic space, where $A = (U_i, k_i, D_i)_{i \in I}$. We fix the following notations (see paragraph 4.2):

- $\mathcal{U} = (U_i)_{i \in I}$ be the open covering of $X$ corresponding to the atlas $A$
- $(U, k, D)$ be the s.s.embedding triple associated to $(X, A)$ and $b : \mathcal{U} \to X$ the natural morphism given by the inclusions (see Lemma 4.2.2)

For $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$ set:

$$\text{Dolb}(A; \mathcal{F}) = b^*\text{Dolb}(k; \mathcal{F}|_\mathcal{U})$$

(75)

Thus $\text{Dolb}(A; \mathcal{F})$ is the simple complex associated to the double complex:

$$0 \to \prod_{|\alpha| = 0} b_{\alpha*}\text{Dolb}(k_\alpha; \mathcal{F}|_{U_\alpha}) \to \prod_{|\alpha| = 1} b_{\alpha*}\text{Dolb}(k_\alpha; \mathcal{F}|_{U_\alpha}) \to \ldots$$

(76)

where $\alpha \in N(\mathcal{U})$.

1.f If the atlas $A$ consists of only one chart $(X, i, D)$ (i.e. $X$ is embedded in the complex manifold $D$) then the functor $\text{Dolb}(A; \mathcal{F})$ coincides with the functor $\text{Dolb}(i; \mathcal{F})$ defined at paragraph 5.2 (the embedded case). In particular, if $X$ is a complex manifold and $A$ consists of the chart $(X, id, X)$, then $\text{Dolb}(A; \mathcal{O}_X)$ is the usual Dolbeault-Grothendieck resolution on $X$.

1.a Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of $O_X$-modules. By properties 1.a-ss (exactness of $\text{Dolb}(k; \bullet)$) and 1.c*-ss it follows that
\[ 0 \to \text{Dolb}(k; F_1|U) \to \text{Dolb}(k; F_2|U) \to \text{Dolb}(k; F_3|U) \to 0 \]
is an exact sequence of complexes of $\mathcal{U}$-modules such that the terms of every $\text{Dolb}(k; F_i)_\alpha$ are soft sheaves. Hence the image by $i_\alpha^*$ of the exact sequence on $U_\alpha$ is an exact sequence on $X$, and the exactness of $\text{Dolb}(\mathcal{A}; \bullet)$ follows by using (76).

1.c $b_{\alpha*}(\text{Dolb}(k; F)_\alpha) = b_{\alpha*}(\text{Dolb}(k; F|U_\alpha))$ has soft terms. Thus the terms of $\text{Dolb}(\mathcal{A}; F)$ are cartesian products of soft sheaves and consequently soft.

1.b The morphism is given by the composition:
\[ F \xrightarrow{j_\alpha} b_2(F|U) \xrightarrow{j'_\alpha} b_2Dolb(k; F|U) = \text{Dolb}(\mathcal{A}; F) \] (77)
As remarked in Example 3.5.4, $b_2(F|U)$ coincides with the Čech complex of $F$ with respect to the open covering $\mathcal{U}$ and $u$ is a quasi-isomorphism.

To see that $v$ is also a quasi-isomorphism we restrict ourselves to an open set $U_j$ of the covering $\mathcal{U}$. $\text{Dolb}(\mathcal{A}; F)|U_j$ is the simple complex associated to the double complex (see formula (77)):
\[ K^{p,q} = \prod_{|\alpha| = p} b_{j\alpha*}\text{Dolb}^g(k_\alpha, F|U_\alpha)|U_\alpha \cap U_j \] (78)
where $b_{j\alpha} : U_\alpha \cap U_j \rightarrow U_j$ is the natural inclusion.

The terms of the second drawer of the spectral sequence associated to $K^-$ are:
\[ E_2^{p,q} = \begin{cases} \mathcal{F}|U_j & \text{if } p = q = 0 \\ 0 & \text{otherwise} \end{cases} \] (79)
Indeed, taking the cohomology of $K^-$ in the $q$-direction, one obtains for $q \geq 0$, the following Čech-type complex relative to the covering $\mathcal{U} \cap U_j$:
\[ \cdots \to \prod_{|\alpha| = p} R^q b_{j\alpha*}(\mathcal{F}|U_\alpha \cap U_j) \to \prod_{|\alpha| = p+1} R^q b_{j\alpha*}(\mathcal{F}|U_\alpha \cap U_j) \to \cdots \] (C(q))
Note that $U_j$ is among the open sets of the covering $\mathcal{U} \cap U_j$ and that $b_{jj} = \text{id}_{U_j}$. Using a homotopy argument similar to that in [S-FAC], chap 1, §3 Proposition 3 and §4, Lemma 1 one checks that the cohomology of $C(q)$:
\[ H^p(C(q)) = \begin{cases} R^p b_{j\alpha*}(\mathcal{F}|U_j) = R^p \text{id}_*(\mathcal{F}|U_j) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases} \]
which proves the claim. Moreover, one also checks that $v$ induces an isomorphism between the second drawers of the spectral sequences associated to $\mathcal{C}^\bullet(\mathcal{U}, F)|U_j$ and $K^-$ and, consequently, is a quasi-isomorphism.

1.e and 1.d The morphism (41) is induced by the natural morphisms:
\[ \prod_{|\alpha| = p} b_{\alpha*}\text{Dolb}(k_\alpha; O_X|U_\alpha) \otimes_{O_X} \mathcal{F} \to \prod_{|\alpha| = p} b_{\alpha*}\text{Dolb}(k_\alpha; F|U_\alpha) \] (80)
If $\mathcal{F}$ is a coherent $O_X$-module then (80) is an isomorphism. Indeed if $\mathcal{F} = O_X$ or $\mathcal{F} = O_X^p$ then the statement is clear. Hence, using local presentations of $\mathcal{F}$
\[ O_X^p \rightarrow O_X^p \rightarrow \mathcal{F} \rightarrow 0 \]
it follows that (80) is a local isomorphism, and consequently an isomorphism. Note that if $\mathcal{F}$ is not coherent then (80) may not be an isomorphism.

To prove the flatness of $\text{Dolb}(\mathcal{A}; O_X)$ it is enough to check that the following sequence:
\[ 0 \to \text{Dolb}(\mathcal{A}; O_X) \otimes_{O_X} I \to \text{Dolb}(\mathcal{A}; O_X) \] (81)
is exact, where \( \mathcal{I} \subset \mathcal{O}_X|U \) is any coherent ideal on some open set \( U \subset X \). This is implied by the following commutative diagram:

\[
\begin{array}{cc}
\text{Dolb}(\mathcal{A};\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{I} & \rightarrow \text{Dolb}(\mathcal{A};\mathcal{O}_X) \\
\downarrow & \\
0 & \rightarrow \text{Dolb}(\mathcal{A};\mathcal{I}) & \rightarrow \text{Dolb}(\mathcal{A};\mathcal{O}_X)
\end{array}
\]

(82)

since the lower row is exact (exactness of \( \text{Dolb}(\mathcal{A};\bullet) \)) and the left hand side vertical arrow is an isomorphism by property 1.e in the coherent case.

Finally, to prove that the morphism (4) is a quasi-isomorphism consider the commutative diagram:

\[
\begin{array}{c}
\text{Dolb}(\mathcal{A};\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \\
\uparrow \\
\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} \\
\uparrow \\
\mathcal{F}
\end{array}
\]

(83)

By the \( \mathcal{O}_X \)-flatness of \( \text{Dolb}(\mathcal{A};\mathcal{O}_X) \) and 1.c, the vertical arrows are quasi-isomorphisms, and the lower horizontal arrow is an isomorphism which yields the result.

2 Let \((\mathcal{U}, k, \mathcal{D}), (\mathcal{Y}, k', \mathcal{D}')\) be the s.s.embedding triples associated to \((X, \mathcal{A}), (Y, \mathcal{B})\) and \(b: \mathcal{U} \rightarrow X, b': \mathcal{U} \rightarrow Y\) the morphisms given by the inclusions. Let

\[
F: (\mathcal{U}, k, \mathcal{D}) \rightarrow (\mathcal{Y}, k', \mathcal{D}')
\]

(84)

be the morphism induced by \(f: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})\) (see Remark 4.2.4) and

\[
F^*(u): \mathcal{G}|\mathcal{U} \rightarrow F_*\mathcal{F}|\mathcal{U}
\]

(85)

be the morphism induced by \(u: \mathcal{G} \rightarrow f_*\mathcal{F}\) (see Lemma 4.2.1). By property 2-ss there exists a natural morphism:

\[
F^!(F^*(u)): \text{Dolb}(k'; \mathcal{G}|\mathcal{U}) \rightarrow F'_2\text{Dolb}(k; \mathcal{F}|\mathcal{U})
\]

(86)

Now apply \(b'_2\) and use Lemma 3.5.5 to obtain the morphism:

\[
f^*(u): \text{Dolb}(\mathcal{B}; \mathcal{G}) \rightarrow f_1\text{Dolb}(\mathcal{A}; \mathcal{F})
\]

(87)

**Remark 5.4.1.** Assume \(Y = X\), \(f = (id, \tau, (\tilde{f}_i)_{i \in I}) : (X, \mathcal{A}) \rightarrow (X, \mathcal{B}), \mathcal{G} = \mathcal{F}\), and \(u = id\). Then the morphism defined above

\[
f^*(id): \text{Dolb}(\mathcal{B}; \mathcal{F}) \rightarrow \text{Dolb}(\mathcal{A}; \mathcal{F})
\]

(88)

is a quasi-isomorphism, since it is a morphism between two resolutions of \(\mathcal{F}\).

3 Let \((\mathcal{U}, k, \mathcal{D}), (\mathcal{W}, k', \mathcal{D}'), (\mathcal{Y}, k'', \mathcal{D}'')\) be the s.s.embedding triples associated to \((X, \mathcal{A}), (Y, \mathcal{B})\), and \((Z, \mathcal{C})\) and \(b: \mathcal{U} \rightarrow X, b': \mathcal{U} \rightarrow Y, b'': \mathcal{W} \rightarrow Z\) the morphisms given by the inclusions. By Lemma 4.2.2 \(\mathcal{F}|\mathcal{U}, \mathcal{G}|\mathcal{W}, \mathcal{H}|\mathcal{W}\) satisfy the hypothesis of property 3-ss and hence the following diagram commutes:

\[
\begin{array}{ccc}
H_2\text{Dolb}(k; \mathcal{F}|\mathcal{U}) & \leftarrow & G_2\text{Dolb}(k'; \mathcal{G}|\mathcal{W}) \\
\downarrow & & \downarrow \\
\text{Dolb}(k''; \mathcal{H}|\mathcal{W})
\end{array}
\]

(89)

To get the result apply \(b'_2\) to the above diagram and use Lemma 3.5.3.

**Remark 5.4.2.** By properties 2 and 3 of Theorem 1.1.1 the functor \(\text{Dolb}\) extends to the category of s.s.locally embedded analytic spaces, with the pullback morphisms defined using \(\sharp\)-direct images (similar to morphism (7)). In particular, with the notations at the beginning of paragraph 5.4, the pullback morphism over the natural mapping \(b: \mathcal{U} \rightarrow X\),

\[
b^\sharp: \text{Dolb}(\mathcal{A}; \mathcal{F}) \rightarrow b_1\text{Dolb}(k; \mathcal{F}|\mathcal{U})
\]

is the identity of \(\text{Dolb}(\mathcal{A}; \mathcal{F})\).
6. Further Results and Applications

6.1 The functor $\text{Dolb}$ in the derived category. To simplify notation in what follows we shall omit to write the localization functors such as

$$Q : K(O_X) \to D(O_X)$$

It should be clear from the context to which category each complex belongs.

**Corollary 6.1.1.** Let $(X, A)$ be a locally embedded analytic space and let $F \in \text{Mod}(O_X)$.

1. The natural functor $F \to \text{Dolb}(A; F)$ gives a functorial isomorphism in the derived category $D^+(O_X)$

2. Let $S(X)$ be the full subcategory of $D^+(O_X)$ consisting of complexes with soft terms and

$$j : S(X) \to D^+(O_X)$$

the inclusion functor. Then the extension of $\text{Dolb}(A; \bullet)$ to $D^+(O_X)$ is a quasi-inverse for $j$.

**Proof.** 1. follows from Theorem 1.1.1 1.b. While 2. follows from 1. and Theorem 1.1.1 1.c. □

**Remark 6.1.2.** Corollary 6.1.1 implies in particular that using $\text{Dolb}(A; \bullet)$ one defines derived functors for any functor $F : \text{Mod}(O_X) \to \text{Mod}(O_X)$ s.t. soft sheaves are $F$-acyclic.

One checks immediately:

**Corollary 6.1.3.** Let $(X, A)$ be a locally embedded analytic space and $F \in \text{Mod}(O_X)$. Then the complex $\Gamma(X, \text{Dolb}(A; F))$ is a representative of $R\Gamma(X, F)$, and hence it computes the cohomology groups $H^\bullet(X, F)$.

**Corollary 6.1.4.** Let $(X, A)$ be a locally embedded analytic space and $F \in \text{Coh}(O_X)$. Assume that $X$ is countable at infinity and that $A$ has at most countably many charts. Then the terms of the complex $\Gamma(X, \text{Dolb}(A; F))$ have natural topologies of type FS and the differentials are continuous. Furthermore, the terms of $\Gamma(X, \text{Dolb}(A; F))$ induce the natural topology on the cohomology groups $H^\bullet(X, F)$.

**Proof.** It is well known that $\Gamma(X, \mathcal{E}_{X}^{p,q} \otimes F)$ has a natural topology of type FS (see e.g. B-S, 78.4.b). Thus the global sections of the terms in (77) are countable products of FS spaces and hence are themselves FS. Note that if $B$ is another embedding atlas of $X$ then, by Remark 5.4.1, $\Gamma(X, \text{Dolb}(A; F))$ and $\Gamma(X, \text{Dolb}(B; F))$ induce the same topology on $H^\bullet(X, F)$ (since $f^\star(id)$ determines a continuous quasi-isomorphism). Thus, to check that this topology coincides with the natural one, we can assume that $cov(A)$ is a Stein covering. The morphism $v$ in diagram (77) determines a continuous quasi-isomorphism on the global sections:

$$C^\bullet(cov(A), F) \to \Gamma(X, \text{Dolb}(A; F))$$

which ends the proof, since the left-hand side complex (the Čech complex with respect to $cov(A)$ ) defines the natural topology on $H^\bullet(X, F)$. □

**Remark 6.1.5.** By [C] and [C-J] if $X$ is a finite dimensional analytic space countable at infinity then it can be covered by finitely many Stein open sets; if moreover $X$ is connected then the Stein open sets can also be chosen connected. Hence if $X$ has also finite embedding dimension then it has an embedding atlas $A$ with finitely many charts, respectively finitely many connected charts, and for any $F \in \text{Mod}(O_X)$ the terms of the complex $\text{Dolb}(A; F)$ consist of products with finitely many factors (see (76)).

**Theorem 6.1.6.** Let $f : X \to Y$ be a morphism of analytic spaces, $F \in \text{Mod}(O_X)$, $G \in \text{Mod}(O_Y)$ and $u : G \to f_*F$ a morphism of $O_Y$-modules. Let moreover $A = (U_i, k_i, D_i)_{i \in I}$ and $B = (V_j, k'_j, D'_j)_{j \in J}$ be embedding atlases of $X$, respectively $Y$. Then
(1) \( f_\ast \text{Dolb}(A; \mathcal{F}) \) is a representative for \( Rf_\ast \mathcal{F} \)
(2) There exists a unique morphism \( f^*(u) \) in \( D^+(\mathcal{O}_Y) \) such that the following diagram commutes (in \( D^+(\mathcal{O}_Y) \)):

\[
\begin{array}{ccc}
\text{Dolb}(\mathcal{B}; \mathcal{G}) & \xrightarrow{f_\ast(u)} & f_\ast \text{Dolb}(A; \mathcal{F}) \\
\uparrow \psi & & \uparrow f_\ast b \\
\mathcal{G} & \xrightarrow{u} & f_\ast \mathcal{F}
\end{array}
\]

(3) The morphism \( f^*(u) \) can be represented as a sequence of pullback morphisms
(4) Let \( g : Y \rightarrow Z \) be another morphism of analytic spaces and \( h = g \circ f \). Let moreover \( \mathcal{H} \in \text{Mod}(\mathcal{O}_Z) \) and \( v : \mathcal{H} \rightarrow g_* \mathcal{G}, w : \mathcal{H} \rightarrow h_* \mathcal{F} \) morphisms of \( \mathcal{O}_Z \)-modules, such that \( g_*(u) \circ v = w \). If \( \mathcal{C} \) is an embedding atlas of \( Z \) then, in the derived category \( D^+(\mathcal{O}_Z) \), one has the commutative diagram:

\[
\begin{array}{ccc}
h_* \text{Dolb}(A; \mathcal{F}) & \xleftarrow{R_h(f^*(u))} & g_* \text{Dolb}(\mathcal{B}; \mathcal{G}) \\
\downarrow h^*(w) & & \downarrow g^*(v) \\
\text{Dolb}(\mathcal{C}; \mathcal{H}) & \xrightarrow{D_h(f^*(u))} & \text{Dolb}(\mathcal{B}; \mathcal{G})
\end{array}
\]

Proof. 1. and 2. are obvious.

3. Assume first that the embedding atlases \( A, \mathcal{B} \) are \( f \)-compliant. If \( \tau \) is a refinement mapping, consider the diagram of locally embedded analytic spaces (see Lemma 4.2.6):

\[
(X, A) \leftrightarrow P_1 \rightarrow (Y, \mathcal{B})
\]

and let

\[
f_\ast \text{Dolb}(A; \mathcal{F}) \xrightarrow{f_\ast P_1(id)} f_\ast \text{Dolb}(A \times_\tau \mathcal{B}; \mathcal{F}) \xrightarrow{P_2(u)} \text{Dolb}(\mathcal{B}; \mathcal{G})
\]

be the corresponding Dolb-diagram (i.e the diagram of pullback morphisms over the arrows in diagram (92)), where

\[
P_1 = (id, id, p_1) \text{ and } P_2 = (f, \tau, p_2)
\]

Note that since the components of the Dolb-resolutions are soft sheaves, Remark 5.4.1 implies that \( f_\ast P_1(id) \) is a quasi-isomorphism, and so diagram (93) gives a morphism in \( D^+(\mathcal{O}_Y) \) which coincides with \( f^*(u) \) (to see this use diagrams similar to (90) for the morphisms in diagram (93)).

Remark 6.1.7. The morphism given by diagram (93) does not depend on the refinement mapping \( \tau \) (use, for instance, the Dolb-diagram over diagram (4) in Lemma 4.2.6). Moreover let \( A' \) be an embedding atlas on \( X \) such that one has the diagram of locally embedded analytic spaces:

\[
(X, A) \xrightarrow{(id, \nu, q_1)} (X, A') \xrightarrow{(f, \tau \circ \nu, q_2)} (Y, \mathcal{B})
\]

Then the corresponding Dolb-diagram is also a representative for \( f^*(u) \) (use, for instance, the Dolb-diagram over diagram (4) in Lemma 4.2.6).

Remark 6.1.8. Assume that \( Y = X, f = id_X, \mathcal{G} = \mathcal{F}, \) and \( u = id \). Remark 5.4.1 implies \( f^*(u) \) is an isomorphism in the derived category.

Now drop the suplimentary assumption. Let \( A' \) be an embedding atlas of \( X \) s.t. \( A', A \) are \( id_X \)-compliant and \( A', \mathcal{B} \) are \( f \)-compliant (choose for instance an embedding atlas over the open covering \( \text{cov}(A) \cap f^{-1} \text{cov}(\mathcal{B}) \)). Then the Dolb-diagram in \( D^+(\mathcal{O}_Y) \)

\[
f_\ast \text{Dolb}(A; \mathcal{F}) \xrightarrow{f_\ast(id)} f_\ast \text{Dolb}(A'; \mathcal{F}) \xrightarrow{f_\ast(u)} \text{Dolb}(\mathcal{B}; \mathcal{G})
\]

determines a morphism:

\[
\text{Dolb}(\mathcal{B}; \mathcal{G}) \rightarrow f_\ast \text{Dolb}(A; \mathcal{F})
\]
since, by Remark 6.1.8 the left-hand arrow is an isomorphism. Moreover one checks, as in the f-compliant case, that this morphism coincides with $f^*(u)$.

4. follows from the equality $g_c(u) \circ v = w$ by considering the diagrams similar to diagram (90) for each morphism. Alternatively, in the compliant case (i.e. if $\mathcal{A}, \mathcal{B}$ are f-compliant and $\mathcal{B}, \mathcal{C}$ are g-compliant) the claim follows from the Dolb-diagram over diagram (15) in Remark 6.2.5. The general case reduces to the compliant one via isomorphisms. □

6.2 Dolbeault resolutions on reduced analytic spaces. Let $X$ be a reduced analytic space. Using an embedding atlas of a particular form one can construct a Dolbeault-Grothendieck type resolution which coincides with the usual one on the regular part of $X$. For this let $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$, let $(W_j)_j$ be a neighbourhood basis of $\text{Sing}(X)$, and $(U_i,k_i,D_i)_{i \in I}$ a family of embedding triples s.t. $(U_i)_{i \in I}$ are open sets of $X$ which cover $\text{Sing}(X)$. Denote

$$S = \text{Sing}(X) \text{ and } U_0 = \text{Reg}(X)$$

The following Lemma is obvious:

**Lemma 6.2.1.**  
(1) The family of charts $(U_i,k_i,D_i)_{i \in I}$ together with the embedding triple $(U_0,\text{id},U_0)$ give an embedding atlas $\mathcal{A}$ of $X$.

(2) For each $j \in J$, the family of embedding triples $(U_i \cap W_j,k_i|W_j,D'_i)_{i \in I}$, where $D'_i \subset D_i$ is a suitable open subset, together with the embedding triple $(U_0,\text{id},U_0)$ give an embedding atlas $\mathcal{A}^{(j)}$ of $X$.

(3) $\text{Dolb}(\mathcal{A}^{(j)};\mathcal{F})$ does not depend on the choice of the open subsets $D'_i$.

(4) If $W_{j_1} \subset W_{j_2}$ then there is a natural pullback morphism $i^*_{j_1,j_2}(\text{id}) : \text{Dolb}(\mathcal{A}^{(j_2)};\mathcal{F}) \to \text{Dolb}(\mathcal{A}^{(j_1)};\mathcal{F})$ over the identity of $X$, and $(\text{Dolb}(\mathcal{A}^{(j)};\mathcal{F}))_j$ is an inductive system of complexes of $\mathcal{O}_X$-modules.

We set:

$$r\text{Dolb}(\mathcal{A};\mathcal{F}) = \lim_J \text{Dolb}(\mathcal{A}^{(j)};\mathcal{F})$$

It is easy to see that the definition of $r\text{Dolb}(\mathcal{A};\mathcal{F})$ is independent of the neighbourhood basis $(W_j)_j$.

**Corollary 6.2.2.** $r\text{Dolb}(\mathcal{A};\bullet)$ is a functor $\text{Mod}(\mathcal{O}_X) \to C^+(X)$. Moreover properties 1.a - 1.e in Theorem 1.1.1 hold for $\text{Dolb}(\mathcal{A};\bullet)$ replaced by $r\text{Dolb}(\mathcal{A};\bullet)$.

**Proof.** 1.a, 1.b, 1.d, 1.e follow immediately from the respective properties in Theorem 1.1.1 because of the compatibility with inductive limits. For 1.c note that the terms of $r\text{Dolb}(\mathcal{A};\mathcal{F})$ consist of Godement restrictions to the closed set $S$ of soft sheaves, and consequently are also soft. □

**Corollary 6.2.3.** $r\text{Dolb}(\mathcal{A};\mathcal{O}_X)|U_0$ coincides with the Dolbeault-Grothendieck resolution on the manifold $U_0$.

**Remark 6.2.4.** Assume $\mathcal{F} \in \text{Coh}(\mathcal{O}_X)$. Then the topologies on the global sections spaces $\Gamma(X,r\text{Dolb}(\mathcal{A};\mathcal{O}_X))$ are more complicated than the FS topologies of Corollary 6.1.4. However, since the natural quasi-isomorphism

$$\Gamma(X,\text{Dolb}(\mathcal{A};\mathcal{F})) \to \Gamma(X,r\text{Dolb}(\mathcal{A};\mathcal{F}))$$

is continuous, both complexes induce the same topology on the cohomology groups $H^*(X,\mathcal{F})$.

**Remark 6.2.5.** If $f : (X,\mathcal{A}) \to (Y,\mathcal{B})$ is a morphism of locally embedded analytic spaces, $\mathcal{G} \in \text{Mod}(\mathcal{O}_Y)$ and $u : \mathcal{G} \to f^*\mathcal{F}$ a morphism of $\mathcal{O}_Y$-modules then one has a pullback morphism

$$f^*(u) : \text{Dolb}(\mathcal{B};\mathcal{G}) \to f_*r\text{Dolb}(\mathcal{A};\mathcal{F})$$

but, in general, not a morphism

$$r\text{Dolb}(\mathcal{B};\mathcal{G}) \to f_*r\text{Dolb}(\mathcal{A};\mathcal{F})$$

when $Y$ is also a reduced analytic space.
6.3 The functor \( \text{Dolb} \) and the de Rham complex on analytic spaces. Let \( X \) be an analytic space and
\[
0 \rightarrow \Omega^0_X \xrightarrow{\partial^0} \Omega^1_X \xrightarrow{\partial^1} \ldots \tag{98}
\]
be the de Rham complex on \( X \) (see e.g. H.Grauert, H.Kerner [Gr-K] or A.Grothendieck [Gr]). Recall that if \( k : X \hookrightarrow D \) is a closed embedding of \( X \) in the complex manifold \( D \), and \( \mathcal{I}_X \subset \mathcal{O}_D \) is the coherent ideal sheaf which gives \( X \) as a subspace of \( D \), then
\[
\Omega_X^i = \Omega_D^i / \mathcal{N}_X^i \tag{99}
\]
where \( \mathcal{N}_X^i \) is the \( \mathcal{O}_D \)-submodule of \( \Omega_D^i \) generated by \( \mathcal{I}_X \Omega_X^1 \) and \( \partial^j (-) \mathcal{I}_X \Omega_X^{i-1} \), and the differentials
\[
\partial_X^i : \Omega_X^i \rightarrow \Omega_X^{i+1} \tag{100}
\]
are induced by those of \( \Omega^*_D \). To simplify notation in what follows we shall write \( \partial \) instead of \( \partial_X \) if \( i \) and \( X \) are clear from the context.

If \( f : X \rightarrow Y \) is a morphism of analytic spaces, one has a pullback morphism:
\[
f^* : \Omega^*_Y \rightarrow f_* \Omega^*_X \tag{101}
\]
In particular, if \( f : (X, i_1, D_1) \rightarrow (Y, i_2, D_2) \) is a morphism of embedding triples then the morphism \( (101) \) is induced by the usual pullback morphism
\[
\Omega^*_D \rightarrow f_* \Omega^*_D \tag{102}
\]
since one checks that \( f^*(\mathcal{N}_Y^i) \subset \mathcal{N}_X^i \) for all \( i \).

**Remark 6.3.1.** If \( X = (X_a)_\alpha \) is a s.s.analytic space then the functoriality of the pullback morphisms \( (101) \) implies that \( \Omega^*_X = (\Omega^*_X, \partial) \alpha \) is a complex of \( X \)-modules with \( \mathbb{C} \)-linear differentials.

**Theorem 6.3.2.** Let \( (X, \mathcal{A}) \) be a locally embedded analytic space.

1. **The differential** \( \partial_X^i : \Omega_X^i \rightarrow \Omega_X^{i+1} \) **induces a \( \mathbb{C} \)-linear morphism of resolutions:**
\[
\partial^i : \text{Dolb}(\mathcal{A}; \Omega_X^i) \rightarrow \text{Dolb}(\mathcal{A}; \Omega_X^{i+1}) \tag{103}
\]
such that \( \text{Dolb}(\mathcal{A}; \Omega_X^i) \) is a double complex

2. **The simple complex associated to** \( \text{Dolb}(\mathcal{A}; \Omega_X^i) \) **is a resolution of** \( \Omega^*_X \) **with soft, \( \mathcal{O}_X \)-flat sheaves**

**Proof.** 1. The morphism of resolutions is obtained by following the construction of the functor \( \text{Dolb}(\mathcal{A}; \bullet) \) in Section 5

a. **(Smooth case)** Let \( D \) be a complex manifold and consider the morphism of resolutions:
\[
0 \xrightarrow{\partial_D} \Omega_D^1 \xrightarrow{\partial_D^1} \mathcal{E}_D^1,0 \xrightarrow{\partial_D} \mathcal{E}_D^1,1 \xrightarrow{\partial_D} \ldots \tag{104}
\]
Using the natural isomorphisms:
\[
\mathcal{E}_D^{0,j} \otimes_{\mathcal{O}_D} \Omega_D^i \xrightarrow{\sim} \mathcal{E}_D^{i,j} \tag{105}
\]
one gets a morphism of resolutions: \( \text{Dolb}(D; \Omega_D^i) \rightarrow \text{Dolb}(D; \Omega_D^{i+1}) \). Obviously \( \partial^{i+1} \circ \partial^i = 0 \) and \( \text{Dolb}(D; \Omega_D^i) \) is a double complex.

Note that if \( (z_1, \ldots, z_n) \) are local coordinates on \( D \) then the morphism
\[
\partial^i : \mathcal{E}_D^{0,j} \otimes_{\mathcal{O}_D} \Omega_D^i \rightarrow \mathcal{E}_D^{0,j} \otimes_{\mathcal{O}_D} \Omega_D^{i+1} \tag{106}
\]
is given by
\[
\partial^i (\alpha \otimes \omega) = \sum_{k=1}^n \frac{\partial}{\partial z_k}(\alpha) \otimes dz_k \wedge \omega + (-1)^j \alpha \otimes \partial \omega \tag{107}
\]
b. (Embedded case) If \((X, k, D)\) is an embedding triple then one checks that
\[
\partial(\mathcal{E}_D^{0,j} \otimes \mathcal{O}_D \mathcal{N}^i_X) \subset \mathcal{E}_D^{0,j} \otimes \mathcal{O}_D \mathcal{N}^{i+1}_X
\]  
(108)
and consequently one obtains a morphism of resolutions:
\[
\partial : \text{Dolb}(k ; \Omega_X^i) \to \text{Dolb}(k ; \Omega_X^{i+1})
\]  
(109)
and \(\text{Dolb}(k ; \Omega_X^i)\) becomes a double complex. Moreover the differentials \(\partial\) are compatible with the pullback morphisms.

c. (General case) Let \((\mathcal{U}, k, \mathcal{D})\) be the s.s.embedding triple associated to \((X, A)\) and \(b : \mathcal{U} \to X\) the natural morphism given by the inclusions. The morphisms \(\partial\) give a morphism of resolutions
\[
\text{Dolb}(k ; \Omega^i_X | \mathcal{U}) \to \text{Dolb}(k ; \Omega^{i+1}_X | \mathcal{U})
\]  
(110)
and by applying \(b\) the morphism \(\alpha\). It is immediate to check that \(\text{Dolb}(A ; \Omega^i_X)\) is a double complex.

2. is obvious.

6.4 The functor \(\text{Dolb}\) and the complex of smooth differential forms
Let \(X\) be an analytic space and let
\[
0 \to \mathcal{E}_X^0,0 \xrightarrow{\partial^0_X} \mathcal{E}_X^0,1 \xrightarrow{\partial^1_X} \ldots
\]  
(111)
be the complex of smooth differential forms with first degree 0 on \(X\). Recall that if \(k : X \to D\) is a closed embedding of \(X\) in the complex manifold \(D\), and \(\mathcal{I}_X \subset \mathcal{O}_D\) is the coherent ideal sheaf which gives \(X\) as a subspace of \(D\), then
\[
\mathcal{E}_X^{0,i} = \mathcal{E}_D^{0,i} / \mathcal{M}_X^i | \mathcal{X}
\]  
(112)
where \(\mathcal{M}_X^i\) is the \(\mathcal{O}_D\)-submodule of \(\mathcal{E}_D^{0,i}\) generated by \(\mathcal{I}_X \mathcal{E}_D^{0,i}\), \(\mathcal{T}_X \mathcal{E}_D^{0,i}\), and the differentials
\[
\partial^i_X : \mathcal{E}_X^{0,i} \to \mathcal{E}_X^{0,i+1}
\]  
(113)
are induced by those of \(\mathcal{E}_D^{0,i}\). If \(X\) is a reduced analytic space \(X\) then \(\mathcal{M}_X^i\) consists of the forms in \(\mathcal{E}_D^{0,i}\) which have null pullback to \(\text{Reg}(X)\). To simplify notation, in what follows we shall write \(\partial\) instead of \(\partial_X\) if \(i\) and \(X\) are clear from the context.

Note that the natural surjective morphisms
\[
\mathcal{E}_D^{0,i} / \mathcal{I}_X \mathcal{E}_D^{0,i} | \mathcal{X} \to \mathcal{E}_X^{0,i}
\]  
(114)
determine a natural morphism of complexes:
\[
\text{Dolb}(k ; \mathcal{O}_X) \to \mathcal{E}_X^{0,*}
\]  
(115)
In general one proves:

**Theorem 6.4.1.** Let \((X, A)\) be a locally embedded analytic space. Then there is a surjective morphism of complexes of sheaves
\[
\text{Dolb}(A ; \mathcal{O}_X) \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{E}_X^{0,*})
\]  
(116)
where \(\mathcal{U} = \text{cov}(A)\) is the open covering of \(X\) corresponding to the atlas \(A\) and \(\mathcal{C}^\bullet(\mathcal{U}, \bullet)\) denotes the Čech complex on \(\mathcal{U}\).

**Proof.** Let \((\mathcal{U}, k, \mathcal{D})\) be the s.s.embedding triple associated to \((X, A)\) and \(b : \mathcal{U} \to X\) the natural morphism given by the inclusions. For each \(\alpha \in \mathcal{N}(\mathcal{U})\) one has a morphism
\[
\text{Dolb}(k_\alpha ; \mathcal{O}_{U_\alpha}) \to \mathcal{E}_{U_\alpha}^{0,*}
\]  
(117)
and these morphisms give a morphism
\[
\text{Dolb}(k ; \mathcal{O}|\mathcal{N}) \to \mathcal{E}_X^{0,*}|\mathcal{U}
\]  
(118)

The morphism \(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{E}_X^{0,*})\) is obtained by applying \(b\) to \(\mathcal{E}_X^{0,*}\). □
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