ON THE HOMOTOPY THEORY FOR LIE $\infty$-GROUPOIDS, WITH AN APPLICATION TO INTEGRATING $L_\infty$-ALGEBRAS

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Abstract. Lie $\infty$-groupoids are simplicial Banach manifolds that satisfy an analog of the Kan condition for simplicial sets. An explicit construction of Henriques produces certain Lie $\infty$-groupoids called “Lie $\infty$-groups” by integrating $L_\infty$-algebras. In order to study the compatibility between this integration procedure and the homotopy theory of $L_\infty$-algebras, we present a homotopy theory for Lie $\infty$-groupoids. Unlike Kan simplicial sets and the higher geometric groupoids of Behrend and Getzler, Lie $\infty$-groupoids do not form a category of fibrant objects (CFO), since the category of manifolds lacks pullbacks. Instead, we show that Lie $\infty$-groupoids form an “incomplete category of fibrant objects” in which the weak equivalences correspond to “stalkwise” weak equivalences of simplicial sheaves. This homotopical structure enjoys many of the same properties as a CFO, such as having, in the presence of functorial path objects, a convenient realization of its simplicial localization. We further prove that the acyclic fibrations are precisely the hypercovers, which implies that many of Behrend and Getzler’s results also hold in this more general context. As an application, we show that homotopy equivalent $L_\infty$-algebras integrate to “Morita equivalent” Lie $\infty$-groups.

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1. INTRODUCTION

Lie $\infty$-groupoids, introduced by Henriques [16], are simplicial Banach manifolds that satisfy a certain diffeo-geometric analog of the “horn filling” condition for Kan simplicial sets. A Lie $\infty$-groupoid for which all horns of dimension $> n$ are filled uniquely is called a “Lie $n$-groupoid”. A Lie 0-groupoid is just a Banach manifold, while a Lie 1-groupoid is the nerve of a Lie groupoid. In general, Lie $n$-groupoids serve as models for differentiable $n$-stacks.

Important examples of Lie $n$-groupoids are “Lie $n$-groups”. These are Lie $n$-groupoids that have a single 0-simplex (i.e. reduced Lie $n$-groupoids). Lie $n$-groups have been used to construct diffeo-geometric models for the higher stages of the Whitehead tower of the orthogonal group. The most famous of these is the “String Lie 2-group”: its geometric realization is a topological group whose homotopy type is the 3-connected cover of the orthogonal group. Initial interest in the String 2-group stemmed from its appearance in string theory and possible applications to geometric models of elliptic cohomology. (See Sec. 1.2 of [16] for a summary and also Sec. 7 of [37].)

With these applications in mind, Henriques developed a smooth analog of Sullivan’s realization functor from rational homotopy theory which produces Lie $n$-groups by “integrating” Lie $n$-algebras. Lie $n$-algebras are non-negatively graded chain complexes concentrated in the first $n – 1$ degrees, equipped with a collection of multi-linear brackets which satisfy a coherent homotopy analog of the Jacobi identity for differential graded Lie algebras. (These are also known as $n$-term $L_\infty$-algebras, or $n$-term homotopy Lie algebras [7, 22].) A Lie 1-algebra is just a Lie algebra.
$L_\infty$-algebras have a good notion of a homotopy theory which can be modeled in a variety of ways, e.g. via enriched category theory [9], or as the bifibrant objects in a model category [34]. In these contexts, morphisms between Lie $n$-algebras are significantly “weaker” than just linear maps which preserve the brackets. But every morphism between Lie $n$-algebras induces a chain map between the underlying complexes. A morphism between Lie $n$-algebras is a weak equivalence when the induced chain map gives an isomorphism on the corresponding homology groups. (Such morphisms are also known as “$L_\infty$-quasi-isomorphisms”.)

What is still missing from this story is a full understanding of the relationship between the homotopy theory of Lie $n$-algebras and the homotopy theory of Lie $n$-groups. The situation is understood in some special cases, for example, for strict Lie 2-algebras and Lie 2-groups [29]. But in general, one would hope that Henriques’ integration functor would send a weak equivalence between Lie $n$-algebras to a weak equivalence between Lie $n$-groups.

It turns out that presenting a user-friendly homotopy theory for Lie $n$-groupoids is a bit subtle. This is mostly due to the fact that the category of Banach manifolds lacks several desirable properties, such as the existence of pullbacks. Recently, Behrend and Getzler [4] showed that higher groupoids internal to certain geometric contexts called “descent categories” form a category of fibrant objects (CFO) for a homotopy theory, in the sense of Brown [6]. Roughly, a descent category is a category of “spaces” which has all finite limits equipped with a distinguished class of morphisms called “covers” satisfying some axioms. Examples include the category of schemes with surjective étale morphisms as covers, as well as the category of Banach analytic spaces, with surjective submersions as covers. The fibrations in the Behrend–Getzler CFO structure for $n$-groupoids are natural generalizations of Kan fibrations, while the acyclic fibrations are precisely the so-called “hypercovers”. As Behrend and Getzler show, this data completely determines the weak equivalences between geometric $n$-groupoids via a very nice combinatorial characterization [4, Thm. 5.1].

Unfortunately, the category of Banach manifolds—regardless of the choice of covers—does not form a descent category, since it lacks finite limits. Hence, Behrend and Getzler’s results do not apply directly in this context. (Nor, unfortunately, does the related work of Pridham [30].) It is worth emphasizing that finite-dimensional Lie $n$-groupoids form a full subcategory of the category of $n$-groupoids internal to the category of $C^\infty$-schemes (in the sense of Dubuc [10]). The latter category, as Behrend and Getzler note, is a descent category. Kan fibrations between Lie $n$-groupoids first appeared in the work of Henriques [16], while hypercovers for Lie $n$-groupoids are featured prominently in the work of the second author [38] and Wolfson [37]. In particular, the second author defines two Lie $n$-groupoids to be “Morita equivalent” if they are connected by a span of hypercovers. Wolfson’s work on $n$-bundles is also quite relevant here: he generalizes aspects of Behrend and Getzler’s machinery to the context Lie $n$-groupoids, but he did not require an explicit presentation of their homotopy theory.

1.1. Summary of main results. In Theorem 7.1, we show that Lie $n$-groupoids form what we call—for lack of better terminology—an “incomplete category of fibrant objects” (iCFO) for a homotopy theory. Just like in a CFO, there are two distinguished classes of morphisms in an iCFO: weak equivalences and fibrations, and as usual, the morphisms which lie in the intersection of the two are called acyclic
fibrations. The axioms of an iCFO (Def. 2.1) are identical to those of a CFO, except we do not assume the existence of pullbacks—even along fibrations. We define the weak equivalences of Lie $n$-groupoids to be those smooth simplicial morphisms which correspond, via the Yoneda embedding, to “stalkwise weak equivalences” between the associated simplicial sheaves. Stalkwise weak equivalences (Def. 5.1) are a natural choice for weak equivalences between simplicial sheaves, appearing in the work of Brown [6], and specifically in diffeo-geometric contexts in the work of Dugger [11], Nikolaus, Schreiber, and Stevenson [27], and Freed and Hopkins [14].

The fibrations in our iCFO structure for Lie $n$-groupoids are the “Kan fibrations” (Def. 3.3) introduced by Henriques. The acyclic fibrations are, by definition, those Kan fibrations which are also stalkwise weak equivalences. However, we show in Prop. 6.7 and Prop. 6.12 that any morphism which is both a Kan fibration and a stalkwise weak equivalence is a hypercover (Def. 6.1). This is because the category of Banach manifolds can be given the structure of what we call a “locally stalkwise pretopology” (Def. 6.5). This is a diffeo-geometric result, and relies on the fact that the inverse function theorem holds for Banach manifolds. And so the acyclic fibrations in our iCFO structure are precisely the hypercovers, just like in the work of Behrend and Getzler. Hence, using their results, we demonstrate in Sec. 7.3 that the weak equivalences between Lie $n$-groupoids can be characterized completely by combinatorial data that only involves maps between Banach manifolds. No reference to simplicial sheaf theory is actually needed.

The main advantage of defining the weak equivalences to be stalkwise weak equivalences is that it allows us to connect to the homotopy theory of Lie $n$-algebras, via Henriques’ integration functor, in a straightforward way. Indeed, we show in Theorem 8.12 that the integration of a weak equivalence ($L\infty$-isomorphism) between Lie $n$-algebras is a stalkwise weak equivalence between the corresponding Lie $n$-groups. In particular, this implies that homotopy equivalent integrable finite-type Lie $n$-algebras integrate to weakly equivalent, and therefore Morita equivalent, Lie $n$-groups (Cor. 8.14). We interpret this result as step one of a larger project in progress whose goal is to prove an analog of “Lie’s Second Theorem” for Lie $n$-groups and Lie $n$-algebras.

1.2. Outline of paper. Throughout, we try to write for a somewhat broader audience, which could include, for example, readers from differential/Poisson geometry with interests in both Lie groupoid theory and $L\infty$-algebras. We attempt a self-contained presentation, within reason, and use only a minimal amount of technical machinery. We recognize that some of the auxiliary results presented here can reside in a more general framework well known to experts in abstract homotopy theory.

We begin in Section 2, where we give the axioms for an incomplete category of fibrant objects (iCFO) for a homotopy theory. We show that many of the nice properties which hold for CFOs also hold for iCFOs. For example, we show that the mapping space between two objects in the Dwyer-Kan simplicial localization of a small iCFO, equipped with functorial path objects and functorial pullbacks of acyclic fibrations, can be described as the nerve of a category of spans.

In Section 3, we recall the definitions of an $n$-groupoid object and a Kan fibration in a large category (such as the category of Banach manifolds) equipped with a “pretopology”. Many of the basic constructions in this section and throughout the paper require taking limits in this category which a priori do not exist. Hence,
limits must be treated as limits of sheaves, and then shown to be representable. Furthermore, we later on define weak equivalences between \( n \)-groupoids in terms of the simplicial Yoneda embedding. We therefore need to deal with sheaves over large categories. To resolve any set-theoretical problems, we employ the standard workaround by passing to a larger Grothendieck universe. In theory, this could introduce a dependence on this “enlargement” (e.g., see [35]). So we show explicitly in Appendix B that, for the case of Lie \( n \)-groupoids, all of our results are independent of choice of Grothendieck universe.

In Section 4, we recall the notion of “points” for categories of sheaves, which generalize the notion of stalks. In particular, we consider a collection of points for sheaves over the category of Banach manifolds, which generalizes those found in the literature (e.g., [11, 27]) for finite-dimensional manifolds. We also collect in this section some useful results regarding matching objects and various notions of epimorphisms and surjections for pretopologies equipped with a collection of points. We use these notions in Section 5 to define stalkwise weak equivalences between \( n \)-groupoids. We also show in this section that any morphism between \( n \)-groups (reduced \( n \)-groupoids) which induces an isomorphism of the corresponding simplicial homotopy group sheaves (in the sense of Joyal [20] and Henriques [16]) is a stalkwise weak equivalence. We use this result in Section 8 where we consider integrating weak equivalences of \( L_\infty \)-algebras.

The definition of hypercover is recalled in Section 6, and we introduce the notion of a category equipped with a “locally stalkwise pretopology”. We show that the category of Banach manifolds equipped with the surjective submersion pretopology is an example. In Section 7, we prove \( \infty \)-groupoids in a category equipped with a locally stalkwise pretopology form an iCFO. We note that this proof can be easily refined to show that \( n \)-groupoids for finite \( n \) also have an iCFO structure. We also demonstrate in this section that the weak equivalences for this iCFO structure can be described without the need of simplicial sheaves, in analogy with a result of Behrend and Getzler.

Finally, in Section 8, we recall some facts concerning Lie \( n \)-algebras and Henriques’ integration functor. We then show that weak equivalences between integrable Lie \( n \)-algebras integrate to weak equivalences between their corresponding Lie \( n \)-groups.

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2. Incomplete categories of fibrant objects

In this section, we introduce a slight generalization of Brown’s definition for a category of fibrant objects (CFO) for a homotopy theory [6, Sec. 1]. In particular, we do not assume the existence of certain limits in the underlying category, hence the term “incomplete”. This is reasonable given our applications to \( \infty \)-groupoid
objects in diffeo-geometric categories. We note that other variations on weakening Brown’s axioms have already appeared in the literature, for example Horel’s “partial Brown categories” [17].

**Definition 2.1.** Let $C$ be a category with finite products and terminal object $* \in C$ equipped with two distinguished classes of morphisms called **weak equivalences** and **fibrations**. A morphism which is both a weak equivalence and a fibration is called an **acyclic fibration**. We say $C$ is an **incomplete category of fibrant objects (iCFO)** iff:

1. Every isomorphism in $C$ is an acyclic fibration.
2. The class of weak equivalences satisfies “2 out of 3”. That is, if $f$ and $g$ are composable morphisms in $C$ and any two of $f, g, g \circ f$ are weak equivalences, then so is the third.
3. The composition of two fibrations is a fibration.
4. If the pullback of a fibration exists, then it is a fibration. That is, if $\begin{array}{ccc} Y & \rightarrow & Z \\ \downarrow \quad & \quad \downarrow f \\ X & \rightarrow & Y \end{array}$ is a diagram in $C$ with $f$ a fibration, and if $X \times_Z Y$ exists, then the induced projection $X \times_Z Y \rightarrow Y$ is a fibration.
5. The pullback of an acyclic fibration exists, and is an acyclic fibration. That is, if $\begin{array}{ccc} Y & \rightarrow & Z \\ \downarrow \quad & \quad \downarrow Z \quad \downarrow f \\ X & \rightarrow & Y \end{array}$ is a diagram in $C$ with $f$ an acyclic fibration, then the pullback $X \times_Z Y$ exists, and the induced projection $X \times_Z Y \rightarrow Y$ is an acyclic fibration.
6. For any object $X \in C$ there exists a (not necessarily functorial) **path object**, that is, an object $X^I$ equipped with morphisms $X \xrightarrow{s} X^I \xrightarrow{(d_0, d_1)} X \times X,$ such that $s$ is a weak equivalence, $(d_0, d_1)$ is a fibration, and their composite is the diagonal map.
7. All objects of $C$ are **fibrant**. That is, for any $X \in C$ the unique map $X \rightarrow *$ is a fibration.

**Remark 2.2.** The only difference between the above and the original definition of Brown is axiom (4). Brown requires that the pullback of a fibration always exists.

2.1. **Factorization.** An important feature of a category of fibrant objects is Brown’s factorization lemma in [6, Sec. 1]. The factorization lemma also holds for any iCFO. Let $Y^I$ be a path object for an object $Y$ in an iCFO $C$. Since $Y \rightarrow *$ is a fibration, Def. 2.1 implies that the projections $\pi_i: Y \times Y \rightarrow Y$ are fibrations. Hence, the morphisms $d_i: Y^I \rightarrow Y$ are also fibrations. Moreover, since $d_i s = \text{id}_Y$, and $s$ is a weak equivalence, the maps $d_i$ are acyclic fibrations. Thus we have proven

**Lemma 2.3.** The projection $d_i: Y^I \rightarrow Y$ is an acyclic fibration for $i = 0, 1$.

The proof of the next lemma is identical to the analogous lemma for categories of fibrant objects. Indeed, the proof does not require the existence of pullbacks along arbitrary fibrations.

**Lemma 2.4.** If $C$ is an iCFO and $f: X \rightarrow Y$ is a morphism in $C$, then $f$ can be factored as $f = p \circ i$, where $p$ is a fibration, and $i$ is a weak equivalence which is a section (right inverse) of an acyclic fibration.
Proof. Let $Y^I$ be a path object for $Y$. Lemma 2.3 implies that the pullback $X \times_Y Y^I$ of the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{d_0} & & \downarrow^{d_0} \\
X & \xrightarrow{d_0} & Y
\end{array}
$$

exists, and hence the projection $pr_1: X \times_Y Y^I \to X$ is an acyclic fibration. Combining this fact with the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{sf} & Y^I \\
\downarrow^{id} & & \downarrow^{d_0} \\
X & \xrightarrow{d_0} & Y
\end{array}
$$

implies that $pr_1$ has a right inverse

$$
i: X \to X \times_Y Y^I
$$

which is necessarily a weak equivalence. Moreover, if $p: X \times_Y Y^I \to Y$ is the composition

$$
X \times_Y Y^I \xrightarrow{pr_2} Y^I \xrightarrow{d_1} Y,
$$

then

$$
f = p \circ i.
$$

To show that $p$ is a fibration, we observe that $X \times_Y Y^I$ is also the pullback of the diagram

$$
\begin{array}{ccc}
X \times_Y Y^I & \xrightarrow{pr_2} & Y^I \\
\downarrow^{(pr_1, p)} & & \downarrow^{(d_0, d_1)} \\
X \times Y & \xrightarrow{(f, id_Y)} & Y \times Y
\end{array}
$$

Since $(d_0, d_1)$ is a fibration, $(pr_1, p)$ is a fibration. Since the projection $X \times Y \to Y$ is a fibration (indeed $Y$ is fibrant), it follows that $p$ is also a fibration.

One simple fact which follows from the factorization lemma is that every weak equivalence in an iCFO yields a span of acyclic fibrations. Also, just like in the case with a CFO [4, Lemma 1.3], the weak equivalences in an iCFO are determined by the acyclic fibrations.

**Proposition 2.5.**

1. If $f: X \to Y$ is a weak equivalence in an iCFO then there exists a span of acyclic fibrations $X \leftarrow Z \to Y$.
2. A morphism $f: X \to Y$ in an iCFO is a weak equivalence if and only if it factors as $f = p \circ i$, where $p$ is an acyclic fibration, and $i$ is a section of an acyclic fibration.

Proof. If $f$ is a weak equivalence, then factor $f = p \circ i$ as in Lemma 2.4. Since $f$ and $i$ are weak equivalences, $p$ is an acyclic fibration. Hence, the legs of the span

$$
\begin{array}{ccc}
X \times_Y Y^I & \xrightarrow{pr_1} & X \\
& \xrightarrow{p} & Y
\end{array}
$$

are acyclic fibrations.

□
**Remark 2.6.** Spans of acyclic fibrations are analogous to the notion of Morita equivalence between Lie groupoids. Indeed, in [38, Def. 2.12], two Lie n-groupoids are considered “Morita equivalent” iff they are connected by a span of maps called hypercovers, which we consider in Sec. 6. We will see in Sec. 7 that hypercovers are the acyclic fibrations in the iCFO structure for Lie n-groupoids.

2.2. **Simplicial localization.** In this section, we show that in certain cases the simplicial localization (or underlying ∞-category) of a small incomplete category of fibrant objects has a simple description in terms of the nerve of a category of spans. An example is a small category of ∞-groupoids equipped with the iCFO structure described in Sec. 7. In particular, the simplicial localization of the category of Lie n-groupoids is discussed in Sec. 7.2.1, and this will be useful for our future work concerning the higher category theory of Lie ∞-groupoids.

Recall that for any small category C and a wide subcategory W of weak equivalences, one may associate to it a simplicial category $L_W C$ (i.e., a category enriched in simplicial sets) via the Dwyer-Kan simplicial localization, or “hammock localization” [13]. The simplicial category $L_W C$ has the same objects as C and is universal with respect to the property that weak equivalences in W are homotopy equivalences in $L_W C$. In particular, $\pi_0 L_W C$, the category whose objects are those of C, and whose hom-sets are

$$\pi_0(\text{Map}_{L_W C}(X,Y)), \quad \forall X, Y \in C,$$

is equivalent to the usual localization $C[W^{-1}]$, the category obtained by formally inverting the morphisms in W.

The mapping space $\text{Map}_{L_W C}(X,Y)$ is the direct limit of nerves of categories [13, Prop. 5.5]. Dwyer and Kan showed that when C is a model category with functorial factorizations, $\text{Map}_{L_W C}(X,Y)$ can be described more simply, up to weak homotopy equivalence, as the nerve of a single category of spans (e.g., Prop. 8.2 in [13] and subsequent corollaries). Weiss [36] later showed that Dwyer and Kan’s proof extends to the case when C is a Waldhausen category (i.e. a category with cofibrations and weak equivalences) equipped with both a cylinder functor, and canonical pushouts of cofibrations. Roughly speaking, the “opposite” of Weiss’s argument is spelled out in detail for CFOs in the work of Nikolaus, Schreiber, and Stevenson [27, Thm. 3.61]. As we show below, an analogous result holds for iCFOs.

2.2.1. **Functorial path objects and pullbacks.** In this section, weak equivalences and fibrations will be denoted by $X \sim Y$ and $X \to Y$, respectively. Furthermore, in this section, C will denote a small incomplete category of fibrant objects equipped with:

- **functorial path objects**: An assignment of a path object $X^I$ to each object $X \in C$ and to each $f : X \to Y$, a morphism $f^I : X^I \to Y^I$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{s} & X^I \\
\downarrow f & & \downarrow (f^I, d_0, d_1) \\
Y & \xrightarrow{s} & Y^I
\end{array}
\begin{array}{ccc}
& & \downarrow (f, f) \\
& & \downarrow \\
& & \text{Y} \times Y
\end{array}
$$

- **functorial pullbacks of acyclic fibrations**: An assignment to each diagram of the form $X \xrightarrow{f} Y \xleftarrow{g} Z$, in which g is an acyclic fibration, a
universal cone $X \xleftarrow{\tilde{g}} X \times_Y Z \xrightarrow{f} Z$ (which exists via the iCFO axioms, and in which $\tilde{g}$ is necessarily an acyclic fibration). The universal property implies that to each commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\alpha} & \downarrow{\sim} & \downarrow{\sim} \\
X' & \xrightarrow{f'} & Z
\end{array}
\begin{array}{ccc}
Y & \xleftarrow{\sim} & Z \\
\downarrow{\beta} & \downarrow{\gamma} & \\
Y' & \xleftarrow{\sim} & Z'
\end{array}
$$

in which the right horizontal morphisms are acyclic fibrations, we obtain a unique commutative diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\sim} & X \times_Y Z \\
\downarrow{\alpha} & \downarrow{\sim} & \downarrow{\sim} \\
X' & \xleftarrow{\sim} & X' \times_{Y'} Z' \\
\end{array}
\begin{array}{ccc}
Y & \xrightarrow{f} & Z \\
\downarrow{\gamma} & \\
Y' & \xrightarrow{f'} & Z'
\end{array}
$$

in which the left horizontal morphisms are acyclic fibrations.

Functorial path objects and pullbacks as above provide an iCFO with functorial factorizations in the sense of [31, Def. 12.1.1].

**Proposition 2.7.** Let $C$ be an iCFO with functorial path objects and functorial pullbacks of acyclic fibrations. Then each morphism $f : X \to Y$ in $C$ can be canonically factored as

$$X \xrightarrow{i_f} X \times_Y Y^I \xrightarrow{p_f} Y$$

where $p_f$ is a fibration and $i_f$ is a right inverse of an acyclic fibration. Moreover, this factorization is natural: Given a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\alpha} & \downarrow{\beta} & \\
X' & \xrightarrow{f'} & Y'
\end{array}
$$

there exists a unique morphism $\gamma : X \times_Y Y^I \to X' \times_{Y'} Y'^I$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i_f} & X \times_Y Y^I \\
\downarrow{\sim} & & \downarrow{\sim} \\
X' & \xrightarrow{i_f'} & X' \times_{Y'} Y'^I \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
Y & \xrightarrow{p_f} & Y \\
\downarrow{\beta} & & \downarrow{\sim} \\
Y' & \xrightarrow{p_{f'}} & Y'
\end{array}
$$

commutes.

**Proof.** One simply repeats the proof of the factorization lemma for an iCFO (Lemma 2.4) using the functorial path objects and pullbacks. □

2.2.2. *Categories of spans in $C$.* Denote by $W_f \subseteq W \subseteq C$ the subcategories of $C$ consisting of acyclic fibrations and weak equivalences, respectively. For each pair of objects $X, Y \in C$ we denote by $CW_f^{-1}(X, Y)$ (respectively, $CW^{-1}(X, Y)$) the category whose objects are spans in $C$ of the form

$$X \xleftarrow{C} \to Y$$
in which the left arrow is an acyclic fibration (respectively, weak equivalence), and whose morphisms are commutative diagrams of the form

\[
\begin{array}{ccc}
X & \xleftarrow{f} & C \\
& \mathllap{\sim} & \mathrlap{\Downarrow \ell} & \mathrlap{\Rightarrow} & Y \\
C' & \xrightarrow{g} & \\
& \mathllap{\Downarrow g'} & \\
\end{array}
\]

in which the vertical arrow is a weak equivalence.

We will also need generalizations of the above categories of spans. In what follows, we use Dwyer and Kan’s notation for “hammock graphs” [13, Sec. 5.1]. Let \( w \) be a word of length \( n \geq 1 \) consisting of letters \( \{ C, W, W^{-1}, W^{-f}\} \). For each pair of objects \( X, Y \in C \) we denote by \( w(X,Y) \) the category whose objects are diagrams in \( C \) of the form

\[
X \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \cdots \xrightarrow{f_{n-2}} C_{n-1} \xrightarrow{f_{n-1}} Y
\]

in which the morphism \( f_i \) goes to the right and is in \( C \) (resp. \( W \)) iff the \( (n-i) \)th letter in \( w \) is \( C \) (resp. \( W \)). Otherwise, the morphism \( f_i \) goes to the left and is in \( W \) (resp. \( W^{-1} \)) iff the \( (n-i) \)th letter in \( w \) is \( W^{-1} \) (resp. \( W^{-f} \)). Morphisms in \( w(X,Y) \) are commuting diagrams in which all vertical arrows are weak equivalences.

**Proposition 2.8.** Let \( X, Y \) be objects of \( C \). The inclusion functors

\[
\begin{align*}
\text{CW}_f^{-1}(X,Y) & \xhookrightarrow{\iota} \text{CW}^{-1}(X,Y) \\
\text{WW}_f^{-1}(X,Y) & \xhookrightarrow{\iota} \text{WW}^{-1}(X,Y)
\end{align*}
\]

induce simplicial homotopy equivalences between the corresponding nerves

\[
N\text{CW}_f^{-1}(X,Y) \xrightarrow{\sim} N\text{CW}^{-1}(X,Y), \quad N\text{WW}_f^{-1}(X,Y) \xrightarrow{\sim} N\text{WW}^{-1}(X,Y)
\]

**Proof.** We use the fact that a natural transformation between functors induces a homotopy between the corresponding simplicial maps between nerves. Denote by \( F : \text{CW}^{-1}(X,Y) \to \text{CW}_f^{-1}(X,Y) \) the functor which assigns to a span of the form

\[
X \xleftarrow{f \sim} C \xrightarrow{g} Y
\]

an object in \( \text{CW}_f^{-1}(X,Y) \) via the following. First, apply the functorial factorization (Prop. 2.7) to the morphism \( (f, g) : C \to X \times Y \) to obtain

\[
C \xrightarrow{i \sim} C' \xrightarrow{p} X \times Y
\]

Then composing the fibration \( p \) with the projections gives a span of fibrations:

\[
(1) \quad X \xleftarrow{f'} C' \xrightarrow{g'} Y
\]
along with a commutative diagram

![Diagram](image)

which, combined with the "2 out of 3" axiom implies that $f'$ is an acyclic fibration. Hence, the diagram (1) is an object of $\text{CW}^{-1}_t(X,Y)$. It is easy to see that this assignment is indeed functorial, due to the use of functorial factorizations. Moreover, the weak equivalence $i$ in the diagram (2) gives natural transformations

$$\text{id}_{\text{CW}^{-1}_t(X,Y)} \to \iota \circ F, \quad \text{id}_{\text{CW}^{-1}_t(X,Y)} \to F \circ \iota$$

Hence, $N\iota: N\text{CW}^{-1}_t(X,Y) \to N\text{CW}^{-1}_t(X,Y)$ is a homotopy equivalence. To show $N\WW^{-1}_t(X,Y) \simeq N\WW^{-1}_t(X,Y)$ is a homotopy equivalence, we observe that the restriction of the functor $F$ to the subcategory $\WW^{-1}_t(X,Y)$ has as its target $\WW^{-1}_t(X,Y)$, thanks to the commutative diagram (2) and the "2 out of 3" axiom.

2.2.3. $\mathcal{C}$ admits a homotopy calculus of right fractions. In the terminology of [13, Sec. 5.1], the $k$-simplices in $N\text{CW}^{-1}_t(X,Y)$ are hammocks between $X$ and $Y$ of width $k$ and type $\text{CW}^{-1}$. There is simplicial map (i.e., the reduction map)

$$r: N\text{CW}^{-1}_t(X,Y) \to \text{Map}_{L,W}(X,Y)$$

sending such a hammock to a reduced hammock, in the sense of [13, Sec. 2.1]. The main theorem of this section is:

**Theorem 2.9.** Let $\mathcal{C}$ be a small incomplete category of fibrant objects with functorial path objects and functorial pullbacks of acyclic fibrations. Then for all objects $X,Y$ of $\mathcal{C}$, the maps

$$N\text{CW}^{-1}_t(X,Y) \xrightarrow{\iota} N\text{CW}^{-1}_t(X,Y) \xrightarrow{\iota} \text{Map}_{L,W}(X,Y)$$

are weak homotopy equivalences of simplicial sets.

Prop. 2.8 implies, of course, that the first map in (4) is a weak homotopy equivalence. To prove Thm. 2.9, we just need to show that the reduction map is a weak equivalence. We do this by showing that $\mathcal{C}$ admits a homotopy calculus of right fractions. Then our result will follow from a result of Dwyer and Kan [13, Prop. 6.2].

Let $i,j \geq 0$ be integers. Given objects $X,Y \in \mathcal{C}$ there is functor

$$j: C^{i,j}W^{-1}(X,Y) \to C^iW^{-1}C^jW^{-1}(X,Y)$$

which sends a diagram of the form

$$X \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots C_j \xrightarrow{f_j} C_{j+1} \xrightarrow{f_{j+1}} \cdots C_{i+j} \xrightarrow{f_{i+j}} Y$$

to the diagram

$$X \xleftarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots C_j \xrightarrow{f_j} C_{j+1} \xrightarrow{id} C_{j+1} \xrightarrow{f_{j+1}} \cdots C_{i+j} \xrightarrow{f_{i+j}} Y$$
We abuse notation and denote by $j : W^{i+j}W^{-1}(X,Y) \to W^iW^{-1}W^{j}W^{-1}(X,Y)$ the restriction of (5) to the subcategory $W^{i+j}W^{-1}(X,Y)$. We recall [13, Sec. 6.1] that the pair $(C,W)$ admits a homotopy calculus of right fractions iff the induced maps on nerves

\[ N\mathcal{C}^{i+j}W^{-1}(X,Y) \xrightarrow{N\mathcal{f}} N\mathcal{C}W^{-1}C^{i}W^{-1}(X,Y), \]

\[ NW^{i+j}W^{-1}(X,Y) \xrightarrow{N\mathcal{f}} NW^iW^{-1}C^{j}W^{-1}(X,Y) \]

are weak homotopy equivalences for all $i,j \geq 0$ and objects $X,Y \in C$.

**Proof of Thm. 2.9.** We show $(C,W)$ admits a homotopy calculus of right fractions by adopting the strategy used by Nikolaus, Schreiber, and Stevenson to prove their Thm. 3.61 in [27]. First, we observe that the proof of Prop. 2.8 can be easily generalized to show that the inclusions of subcategories

\[ C^iW^{-1}_f \hookrightarrow C^iW^{-1}, \quad W^iW^{-1}_f \hookrightarrow W^iW^{-1} \]

and

\[ C^iW^{-1}_f C^jW^{-1}_f \hookrightarrow C^jW^{-1}C^iW^{-1}_f, \quad W^iW^{-1}_f W^jW^{-1}_f \hookrightarrow W^jW^{-1}W^iW^{-1}_f \]

induce homotopy equivalences on the corresponding nerves. Next, we consider the restriction of the functor (5) $j : C^{i+j}W^{-1}(X,Y) \to C^iW^{-1}C^jW^{-1}(X,Y)$ to the following subcategories:

\[ j : C^{i+j}W^{-1}_f(X,Y) \xrightarrow{j} C^iW^{-1}_fC^jW^{-1}_f(X,Y), \]

and

\[ W^{i+j}W^{-1}_f(X,Y) \xrightarrow{j} W^iW^{-1}_fW^jW^{-1}_f(X,Y). \]

Let $F : C^iW^{-1}_f C^jW^{-1}_f(X,Y) \to C^{i+j}W^{-1}_f(X,Y)$ be the functor that assigns to the diagram

\[ X \xleftarrow{j_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots C_j \xrightarrow{f_j} C_{j+1} \xrightarrow{f_{j+1}} C_{j+2} \xrightarrow{f_{j+2}} \cdots C_{i+j+1} \xrightarrow{f_{i+j+1}} Y \]

a diagram

\[ X \xleftarrow{g_0} D_1 \xrightarrow{g_1} D_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{i+j}} Y \]

obtained by taking the iterated pullback of the acyclic fibration $f_{j+1}$:

\[ X \xleftarrow{f_{j+1}} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots \xrightarrow{f_j} C_{j-2} \xrightarrow{f_{j-2}} \cdots \]

Hence, $D_k := C'_{k+1}$ if $k \leq j$, otherwise $D_k := C_{k+1}$. And $g_0 := f_0 \circ \tilde{f}_1$, $g_k := f_{k+1}$ if $1 \leq k \leq j$, otherwise $g_k := f_{k+1}$. Note $F$ is indeed a functor, since all the pullbacks in (10) are functorial. Furthermore, if all the $f_k$ are morphisms in $W$, then so are the $g_k$ by the 2 out of 3 axiom. Hence $F$ restricts to a functor

\[ W^iW^{-1}_f W^jW^{-1}_f(X,Y) \to W^{i+j}W^{-1}_f(X,Y) \]
There is a natural transformation $F \circ j \to \text{id}_{C^{\ast+1}W^{-1}(X,Y)}$, where $j$ is the functor (8), whose components are all identity morphisms. Indeed, if $f_{j+1} = \text{id}_{C_{j+1}}$ in the diagram (10), then $f_{k} = \text{id}_{C_{k-1}}$ for all $k \geq 1$. There is also a natural transformation $p \circ F \to \text{id}_{C^{\ast+1}W^{-1}(X,Y)}$ whose components are the vertical maps in the following diagram:

$$
\begin{array}{cccccccc}
C_{2} & \xrightarrow{f_{j}} & C_{3} & \xrightarrow{f_{j}} & \cdots & C_{j+1} & \xrightarrow{f_{j+1}} & C_{j+2} & \sim \xrightarrow{id} & C_{j+2} & \sim \xrightarrow{id} & C_{j+3} & \sim & \cdots
\end{array}
$$

$$
X \sim \xrightarrow{f_{0}} C_{1} \xrightarrow{f_{1}} C_{2} \xrightarrow{f_{2}} \cdots \xrightarrow{f_{j}} C_{j} \sim \xrightarrow{id} C_{j+1} \sim \xrightarrow{id} C_{j+2} \sim \cdots
$$

The existence of these natural transformations implies that the functors (8) and (9) induce homotopy equivalences on the corresponding nerves. Combining these with functors (6), (7) and then taking the nerve, gives us commutative diagrams of simplicial sets.

$$
\begin{array}{ccc}
NC^{i+j}W^{-1} & \xrightarrow{N_{j}} & NC^{i+j}W_{f}^{-1}C/W_{r}^{-1} \\
\sim \xrightarrow{i} & \sim \xrightarrow{i} & \sim \xrightarrow{i} \\
NC^{i+j}W^{-1} & \xrightarrow{N_{j}} & NC^{i+j}W^{-1}C/W_{r}^{-1}
\end{array}
$$

$$
\begin{array}{ccc}
NW^{i+j}W_{f}^{-1} & \xrightarrow{N_{j}} & NW^{i+j}W_{f}^{-1}W/W_{r}^{-1} \\
\sim \xrightarrow{i} & \sim \xrightarrow{i} & \sim \xrightarrow{i} \\
NW^{i+j}W^{-1} & \xrightarrow{N_{j}} & NW^{i+j}W^{-1}W/W_{r}^{-1}
\end{array}
$$

By 2 out of 3, the bottom horizontal morphisms in these diagrams are weak equivalences of simplicial sets. Therefore, $(C,W)$ admits a homotopy calculus of right fractions.

\[\square\]

3. Higher groupoids and Kan fibrations

In this section, we recall Henriques’ definition [16] of a higher groupoid object in a (large) category $M$ equipped with a pretopology $\mathcal{T}$.

3.1. Preliminaries and notation. If $M$ is a small category, we denote by $\text{PSh}(M)$ the category of presheaves on $M$. The functor $y : M \to \text{PSh}(M)$, $X \mapsto yX = \text{hom}_{M}(\_, X)$ denotes the Yoneda embedding, which identifies objects in $M$ with the representable presheaves.

We denote by $sM$ the category of simplicial objects in $M$, i.e. the category of contravariant functors $\Delta \to M$, where $\Delta$ is the category of finite ordinals

$$
[0] = \{0\}, \quad [1] = \{0, 1\}, \quad \ldots, \quad [n] = \{0, 1, \ldots, n\}, \quad \ldots,
$$

with order-preserving maps. In particular, $s\text{Set}$ is the category of simplicial sets. For $m \geq 0$, the simplicial sets $\Delta^{m}$ and $\partial \Delta^{m}$ are the simplicial $m$-simplex and its boundary, respectively:

$$
(\Delta^{m})_{n} = \{f : (0, 1, \ldots, n) \to (0, 1, \ldots, m) \mid f(i) \leq f(j) \text{ for all } i \leq j\},
$$

$$
(\partial \Delta^{m})_{n} = \{f \in (\Delta^{m})_{n} \mid \{0, \ldots, m\} \nsubseteq \{f(0), \ldots, f(n)\}\}.
$$

For $m > 0$ and $0 \leq j \leq m$, the horn $\Lambda_{j}^{m}$ is the simplicial set obtained from the $m$-simplex $\Delta^{m}$ by taking away its interior and its $j$th face:

$$
(\Lambda_{j}^{m})_{n} = \{f \in (\Delta^{m})_{n} \mid \{0, \ldots, j - 1, j + 1, \ldots, m\} \nsubseteq \{f(0), \ldots, f(n)\}\}.
$$
3.2. Pretopologies. An $n$-groupoid in $M$ is a special kind of simplicial object in $M$. The precise definition requires us to equip $M$ with extra structure, which also allow us to define sheaves on $M$.

**Definition 3.1.** Let $M$ be a category with coproducts and a terminal object $\ast$. A *pretopology* on $M$ is a collection $T$ of arrows, called *covers*, with the following properties:

1. isomorphisms are covers;
2. the composite of two covers is a cover;
3. pullbacks of covers are covers; more precisely, for a cover $U \to X$ and an arrow $Y \to X$, the pull-back $Y \times_X U$ exists in $M$ and the canonical map $Y \times_X U \to Y$ is a cover;
4. for any object $X \in M$, the map $X \to \ast$ is a cover.

What we call a pretopology is called a “singleton Grothendieck pretopology” in [38], and was first defined in [16, Def. 2.1]. Every pretopology in our sense gives a Grothendieck pretopology in the classical sense.

Let $(M, T)$ be a category equipped with a pretopology. A presheaf $F \in \text{PSh}(M)$ is a sheaf if and only if for every cover $U \to X$, $F(X)$ is the equalizer of the diagram

$$F(U) \rightrightarrows F(U \times_X U).$$

We denote by $\text{Sh}(M) \subseteq \text{PSh}(M)$ the full subcategory of sheaves on $(M, T)$. A pretopology is *subcanonical* iff every representable presheaf is a sheaf. In this paper, all pretopologies are assumed to be subcanonical.

Also, we will never assume $M$ has limits. Therefore, we take limits of diagrams in $M$ by first showing that the limit of the corresponding diagram in $\text{Sh}(M)$ of representable presheaves is representable, and then using the fact that the functor $y$ preserves limits.

3.3. Pretopologies for Banach manifolds. We denote by $\text{Mfd}$ the category whose objects are Banach manifolds, in the sense of [23, Ch 2.1], and whose morphisms are smooth maps. (We could also consider $C^r$ maps as well.) A morphism $f: X \to Y$ between manifolds is a *submersion* iff for all $x \in X$, there exists an open neighborhood $U_x$ of $x$, an open neighborhood $V_{f(x)}$ of $f(x)$, and a local section $\sigma: V_{f(x)} \to U_x$ of $f$ at $x$. That is, $\sigma$ is a morphism in $\text{Mfd}$ such that $f \circ \sigma = \text{id}$ and $\sigma(f(x)) = x$. Note that we may always take $U_x$ to be the connected component of $f^{-1}(V_{f(x)})$ containing $x$.

It is a result of Henriques ([16, Cor. 4.4]), that the collection $T_{\text{subm}}$ of surjective submersions is a subcanonical pretopology for the category $\text{Mfd}$.

**Remark 3.2.** Another example of a subcanonical pretopology on $\text{Mfd}$ is the pretopology of open covers $T_{\text{open}}$. Since every cover in the surjective submersion pretopology can be refined by a cover in the pretopology of open covers (see Example B.6), every sheaf on $(\text{Mfd}, T_{\text{subm}})$ is also a sheaf on $(\text{Mfd}, T_{\text{open}})$ and vice versa. See also [28, Prop. 2.17].

Examples of subcanonical pretopologies for other categories of geometric interest can be found in Table 1 of [38].

3.3.1. Technicalities involving large categories. Strictly speaking, the categories $\text{Sh}(M)$ and $\text{PSh}(M)$ are not well-defined if $M$ is not small. In our main example of interest, $M$ will be the large category of Banach manifolds, so we will need a good
theory of sheaves over a large category. The set-theoretic technicalities involved
with sheaves over large categories can be subtle. For example, the sheafification
functor may not be well-defined for presheaves over a large site since a priori it
requires taking colimits over proper classes. The standard workaround is to use
Grothendieck universes, and in particular, to appeal to the Universe Axiom, that
allows one to take colimits in an ambient larger universe in which classes are sets.
However, this larger universe is by no means canonical, and the resulting colimit
may very well depend on the choice of larger universe. See [35] for such an example
involving sheaves for the fpqc topology in algebraic geometry.

We show in Appendix B that all results in this paper are independent of choice
of universe, provided our pretopology \((M, \mathcal{T})\) on the large category
\(M\) admits what we
call a “small refinement” (Def. B.5). Indeed, an example of such a pretopology is
the surjective submersion pretopology on the category of Banac
h manifolds. (See
Example B.6.)

From here on, we always assume that the pretopology being considered admits a
small refinement. This allows the reader to ignore all set-theoretic issues, and treat
\((M, \mathcal{T})\) as if it was a pretopology on a small category.

3.4. Kan fibrations and higher groupoids in \((M, \mathcal{T})\). Here we recall Henriques’
definition of Kan fibration and \(n\)-groupoid, which is based on the work of Duskin
[12] and Glenn [15]. Let \((M, \mathcal{T})\) be a category equipped with a pretopology. In
what follows, if \(K\) is a simplicial set and \(X\) is a simplicial object in \(M\), then we
denote by \(\text{Hom}(K, X)\) the sheaf

\[
\text{Hom}(K, X)(U) := \text{hom}_{\mathcal{M}}(K \otimes U, X)
\]

where \(K \otimes U\) is the simplicial object \((K \otimes U)_n := \bigsqcup_{K_n} U\). (See Prop. 4.5 for other
characterizations of this sheaf.) Note that \(\text{Hom}(K, X)\) may not be representable.

More generally, suppose \(i: A \to B\) is a map between simplicial sets and \(f: X \to Y\) is a morphism of simplicial objects in \(M\). Denote by

\[
\text{Hom}(A \xrightarrow{i} B, X \xrightarrow{f} Y) := \text{Hom}(A, X) \times_{\text{Hom}(A, Y)} \text{Hom}(B, Y);
\]

the sheaf which assigns to an object \(U \in M\) the set of commuting squares in \(sM\) of the form

\[
\begin{array}{ccc}
A \otimes U & \xrightarrow{i} & X \\
\downarrow & & \downarrow f \\
B \otimes U & \xrightarrow{\text{id}} & Y
\end{array}
\]

There is a canonical map

\[
\text{Hom}(B, X) \xrightarrow{(i^*, f_*)} \text{Hom}(A \to B, X \to Y)
\]

induced by pre and post composition with \(i: A \to B\) and \(f: X \to Y\), respectively.

**Definition 3.3** (Def. 2.3 [16]). A morphism \(f: X \to Y\) of simplicial objects in a cat-

gory equipped with a pretopology \((M, \mathcal{T})\) satisfies the **Kan condition** \(\text{Kan}(m, j)\)
if the sheaf \(\text{Hom}(\Delta^m \to \Delta^m, X \to Y)\) is representable and the canonical map (i.e.,
the horn projection)

\[
X_m = \text{Hom}(\Delta^m, X) \xrightarrow{(i^*_m, f_*)} \text{Hom}(\Delta^m \to \Delta^m, X \to Y)
\]
is a cover. The morphism $f: X \to Y$ satisfies the unique Kan condition
Kan!(m,j) iff the canonical map in (14) is an isomorphism. We say $f: X \to Y$
is a Kan fibration iff it satisfies Kan(m,j) for all $m \geq 1$, $0 \leq j \leq m$.

**Definition 3.4** (Def. 2.3 [16]). A simplicial object $X \in sM$ is a higher groupoid
in $(M, T)$, or more precisely, an $n$-groupoid object in $(M, T)$ for $n \in \mathbb{N} \cup \{\infty\}$ iff
the unique morphism $X \to \ast$
satisfies the Kan condition Kan(m,j) for $1 \leq m \leq n$, $0 \leq j \leq m$, and the unique
Kan condition Kan!(m,j) for all $m > n$, $0 \leq j \leq m$. An $n$-group object in $(M, T)$
is an $n$-groupoid object $X$, such that $X_0 = \ast$, where $\ast$ is the terminal object in $M$.

In other words, $X$ is an $n$-groupoid if the sheaf Hom($\Lambda^m_n$, $X$) is representable
and the restriction map

$\text{Hom}(\Delta^m, X) \to \text{Hom}(\Lambda^m_n, X)$

is a cover for all $1 \leq m \leq n$, $0 \leq j \leq m$ and an isomorphism for all $m > n$,
$0 \leq j \leq m$.

**3.5. Representability results.** Now we record some useful tools for proving rep-
resentability. Similar results can be found in the work of Behrend and Getzler
[4], Wolfson [37], and Zhu [38]. What follows is reminiscent of the use of anodyne
extensions in simplicial sets.

**Definition 3.5.** The inclusion $\iota: S \hookrightarrow T$ of a simplicial subset $S$ into a finitely
generated simplicial set $T$ is a collapsible extension iff it is the composition of
inclusions of simplicial subsets

$S = S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_l = T$

where each $S_i$ is obtained from $S_{i-1}$ by filling a horn. That is, for each $i = 1, \ldots, l$,
there is a horn $\Lambda^m_i$ and a map $\Lambda^m_i \to S_{i-1}$ such that $S_i = S_{i-1} \cup \Lambda^m_i \Delta^m$. If
the inclusion of a point into a finitely-generated simplicial set $T$ is a collapsible
extension, then we say $T$ is collapsible. Similarly, we say $\iota: S \hookrightarrow T$ is a boundary
extension iff it is the composition of inclusions of simplicial subsets

$S = S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_l = T$

where each $S_i$ is obtained from $S_{i-1}$ by filling a boundary. That is, for each $i = 1, \ldots, l$,
there is an $m \geq 0$ and a map $\partial \Delta^m \to S_{i-1}$ such that $S_i = S_{i-1} \cup \partial \Delta^m \Delta^m$.

Collapsible extensions are called “expansions” in [37]. We follow the terminology
that appears in [24, Sec. 2.6]. (A more detailed study of these morphisms can be
found there.) Other examples of collapsible extensions are the “$m$-expansions” in
[4, Def. 3.7].

Clearly a collapsible extension is a boundary extension since the natural inclusion
$\Lambda^m_j \to \Delta^n$ can be decomposed into two boundary extensions $\Lambda^m_j \to \partial \Delta^n \to \Delta^n$.
Also, if $S \hookrightarrow T$ and $T \hookrightarrow U$ are both collapsible extensions (or boundary extensions),
then so is their composition $S \hookrightarrow U$.

We have the following fact:

**Lemma 3.6** (Lemma 2.44, [24]). The inclusion of any face $\Delta^k \to \Delta^n$ is a collapsible
extension for $0 \leq k < n$. 

We will use the following two lemmas to solve most of the representability issues in this paper. They are similar to Lemma 2.4 in [16].

**Lemma 3.7.** Let $S \hookrightarrow T$ be a collapsible extension and let $X$ be a higher groupoid in $(\mathcal{M}, \mathcal{T})$. If $\text{Hom}(S, X)$ is representable, then $\text{Hom}(T, X)$ is representable as well and the restriction map $\text{Hom}(T, X) \to \text{Hom}(S, X)$ is a cover.

**Proof.** Let $S = S_0 \subset S_1 \subset \cdots \subset S_l = T$ be a filtration as in Definition 3.5. Since composites of covers are again covers, we may assume without loss of generality that $l = 1$, i.e., $T = S \cup_{\Lambda^j_n} \Delta^n$ for some $n, j$. Note that the functor $\text{Hom}(\cdot, X)$ (12) sends colimits of simplicial sets to limits of sheaves. Hence, we have

$$\text{Hom}(T, X) = \text{Hom}(S, X) \times_{\text{Hom}(\Lambda^j_n, X)} \text{Hom}(\Delta^n, X).$$

Since $X$ is a higher groupoid, $\text{Hom}(\Delta^n, X) \to \text{Hom}(\Lambda^j_n, X)$ is a cover between representable sheaves, and hence, the axioms of a pretopology imply that $\text{Hom}(T, X)$ is representable and that $\text{Hom}(T, X) \to \text{Hom}(S, X)$ is a cover. \hfill $\square$

**Remark 3.8.** In particular, if $X$ is a $k$-groupoid for $k < \infty$, and $T = S \cup_{\Lambda^j_n} \Delta^n$ with $n > k$, and $\text{Hom}(S, X)$ is representable, then the proof of Lemma 3.7 implies that $\text{Hom}(T, X) \to \text{Hom}(S, X)$ is not just a cover, but an isomorphism.

The next lemma concerns the representability of the sheaf (13).

**Lemma 3.9.** Let $S$ be a collapsible simplicial subset of $\Delta^k$, $X$ a simplicial object in $\mathcal{M}$, and $Y$ a higher groupoid in $\mathcal{M}$. If $f: X \to Y$ is a morphism which satisfies $\text{Kan}(m, j)$ for all $m < k$ and $0 \leq j \leq m$, then the sheaf $\text{Hom}(S \hookrightarrow \Delta^k, X \xrightarrow{f} Y)$ is representable.

**Proof.** First note that the statement is identical to that of [16, Lemma 2.4] except that we do not require $X_0 = Y_0 = \ast$. So, we can proceed exactly as in the proof of [16, Lemma 2.4], but with a different verification of the base case for the induction.

For this, we consider $S_0 = \ast \hookrightarrow \Delta^k$ and observe that the sheaf $\text{Hom}(\ast \hookrightarrow \Delta^k, X \xrightarrow{f} Y)$ is represented by the pullback $X_0 \times_{Y_0} Y_k$ in $\mathcal{M}$, which exists in $\mathcal{M}$ since Lemmas 3.6 and 3.7. imply that $Y_k \to Y_0$ is a cover. \hfill $\square$

**Remark 3.10.** The horn $\Lambda^j_n \subset \Delta^n$ is collapsible. Hence, if $Y$ is a higher groupoid and $f: X \to Y$ is a simplicial morphism satisfying the Kan condition $\text{Kan}(m, j)$ for $1 \leq m < n$, then Lemma 3.9 implies that $\text{Hom}(\Lambda^j_n \hookrightarrow \Delta^n, X \xrightarrow{f} Y)$ is automatically representable. Similarly, if $X$ is a simplicial object and if $X \to \ast$ satisfies the Kan condition $\text{Kan}(m, j)$ for $1 \leq m < n$, then Lemma 3.9 implies that $\text{Hom}(\Lambda^j_n, X)$ is automatically representable.

### 4. Points for categories with pretopologies

We begin this section by considering certain functors called “points” for a category equipped with a pretopology. This will allow us to make comparisons between the homotopy theory of higher groupoid objects and that of simplicial sets. A point can be thought of as a generalization of the functor which sends a sheaf on a space to its stalk at a particular point. The notion originates in topos theory. See, for example, [21, C.2.2, p. 555] and [26, VII.5].

We will not need all of the topos theory formalism here, so our presentation is quite abbreviated and self-contained. Our motivation stems from the use of points
in the homotopy theory of simplicial sheaves over the category of finite-dimensional manifolds (e.g. [11, 27]).

**Definition 4.1.** Let \((M, \mathcal{T})\) be a category equipped with a pretopology.

1. A **point** of \((M, \mathcal{T})\) is a functor
   \[ p : \text{Sh}(M) \to \text{Set} \]
   which preserves finite limits and small colimits.

2. A collection of points \(\mathcal{P}\) of \((M, \mathcal{T})\) is **jointly conservative** iff a morphism
   \[ \phi : F \to G \text{ in Sh}(M) \]
   is an isomorphism if and only if for all \(p \in \mathcal{P}\)
   \[ p(\phi) : p(F) \to p(G) \]
   is an isomorphism of sets.

If \(X\) is a simplicial object in \((M, \mathcal{T})\) and \(p : \text{Sh}(M) \to \text{Set}\) is a point, we denote by \(pX\) the simplicial set
\[ pX_n := p(yX_n) \]

4.1. **Points for Banach manifolds.** Our main example, the category of Banach manifolds equipped with the pretopology of surjective submersions, admits a jointly conservative collection of points. If we were only considering finite-dimensional manifolds, then these points would be the same as those used in [11, Def. 3.4.6].

Let \(V \in \text{Ban}\) be a Banach space, and denote by \(B_V(r)\) the open ball of radius \(r\) about the origin in \(V\). Let \((\text{Mfd}, \mathcal{T}_\text{subm})\) denote the category of Banach manifolds equipped with the surjective submersion pretopology, and denote by \(p_V : \text{Sh}(\text{Mfd}) \to \text{Set}\) the functor
\[ p_V(F) = \colim_{r \to 0} F(B_V(r)). \]

**Proposition 4.2.**

1. For every Banach space \(V\), the functor \(p_V : \text{Sh}(\text{Mfd}) \to \text{Set}\) preserves finite limits and small colimits.

2. The collection of points \(\mathcal{P}_\text{Ban} := \{p_V \mid V \in \text{Ban}\}\) is jointly conservative.

*Proof.* Since \(p_V\) is a filtered colimit, it commutes with finite limits and small colimits. Now let \(\phi : F \to G\) be a morphism of sheaves such that for all \(V \in \text{Ban}\)
\[ p_V(\phi) : p_VF \to p_VG \]

is bijection. In particular, injectivity implies that if \(x \in F(B_V(r_x))\) and \(y \in F(B_V(r_y))\) such that \(p_V(\phi)(\bar{x}) = p_V(\phi)(\bar{y})\), then there exists \(r \leq r_x\) and \(r \leq r_y\) such that \(i_x^* x = i_y^* y\), where \(i_x : B_V(r) \to B_V(r_x)\) and \(i_y : B_V(r) \to B_V(r_y)\) are the inclusions.

We show \(\phi : F \to G\) is injective. Let \(M \in \text{Mfd}\) and \(x, y \in F(M)\) such that \(\phi_M(x) = \phi_M(y)\). By the usual arguments, for each \(m \in M\) there exists an open neighborhood \(U_m\) of \(m\), a Banach space \(V_m\), a radius \(r_m > 0\) and a diffeomorphism \(\psi_m : B_{V_m}(r_m) \xrightarrow{\cong} U\) such that \(\psi_m(0) = m\). Hence, we have a cover
\[ \coprod_{m \in M} B_{V_m}(r_m) \xrightarrow{(i_m)} M, \]

where \(i_m : B_{V_m}(r_m) \to M\) is the composition of \(\psi_m\) with the inclusion. So for each \(m \in M\)
\[ \phi_{U_m}(i_m^* x) = \phi_{U_m}(i_m^* y) \]
which implies
\[ p_{V_m}^*(\phi_{U_m})(i_m x) = p_{V_m}^*(\phi_{U_m})(\overline{i_m y}). \]
Therefore, there exists a smaller ball \( B_{V_m}(r'_m) \subseteq B_{V_m}(r_m) \) such that the restrictions of \( x \) and \( y \) to \( B_{V_m}(r'_m) \) are equal. Since
\[ \prod_{m \in M} B_{V_m}(r_m)(i_m) \rightarrow M \]
is a cover, and \( F \) is a sheaf, we conclude \( x = y \).

Now we show \( \phi : F \rightarrow G \) is surjective. Let \( M \in \text{Mfd} \) and \( y \in G(M) \). We use the cover (17) as before, so that \( \overline{i_m^* y} \in G(B_{V_m}(r_m)) \). Since \( p_{V_m}^* \phi : p_{V_m}^* F \rightarrow p_{V_m}^* G \) is onto, there exists \( r'_m \leq r_m \) and \( x_m \in F(B_{V_m}(r_m)) \) such that \( \phi(x_m) = j_m^* y \) where \( j_m : B_{V_m}(r'_m) \rightarrow M \) is the composition of \( i_m \) with the inclusion \( B_{V_m}(r'_m) \hookrightarrow B_{V_m}(r_m) \).

Consider the pullback \( W \times_M W \rightrightarrows W \), where \( W \) is the cover \( \coprod_{m \in M} B_{V_m}(r'_m) \).

We claim \( p_1^*(\{x_m\}) = p_2^*(\{x_m\}) \). Indeed, since \( y \) is a global section over \( M \), we have the equalities
\[ \phi(p_1^*(\{x_m\})) = p_1^*\phi(\{x_m\}) = p_1^*(\{j_m^*y\}) = p_2^*(\{j_m^*y\}) = \phi(p_2^*(\{x_m\})). \]

Therefore, since \( \phi \) is one to one, we conclude \( p_1^*(\{x_m\}) = p_2^*(\{x_m\}) \). And since \( F \) is a sheaf, there exists a section \( x \in F(M) \) such that \( j_m^* \phi(x) = j_m^* y \), and hence \( \phi(x) = y \).

Remark 4.3. It follows from Remark 3.2 that Prop. 4.2 also implies that the collection of functors (16) is jointly conservative for \( (\text{Mfd}, \mathcal{T}_{\text{open}}) \).

4.2. Matching objects and stalkwise Kan fibrations. We next recall how points of \( (M, \mathcal{T}) \) take “matching objects” for simplicial sheaves to matching objects for simplicial sets. We also describe the relationship between the sheaf \( \text{Hom}(K, X) \), defined in (12), for a simplicial object \( X \in sM \) and the corresponding matching object for the representable simplicial sheaf \( yX \).

Definition 4.4. Given a simplicial set \( K \in s\text{Set} \) and a simplicial sheaf \( F \in s\text{Sh}(M) \), their matching object \( M_K(F) \) is the sheaf
\[ M_K(F)(U) := \text{hom}_{s\text{Set}}(K, F(U)). \]

If \( F \in s\text{Sh}(M) \) is a simplicial sheaf and \( K \in s\text{Set} \) is a simplicial set, then we denote by \( F^K_n \) the sheaf
\[ U \mapsto F^K_n(U) := \prod_{K_n} F_m(U). \]

Hence, for each \( m, n \), and element \( x \in K_n \) we have the canonical projection \( \pi^m_n(x) : F^K_{m+n} \rightarrow F_m \). For any such \( F \) and \( K \), there are two maps of sheaves:
\[ \alpha_F, \alpha_K : \prod_{m \geq 0} F^K_m \rightarrow \prod_{\theta_{mn}} F^K_{m+n}, \]
where the latter product is taken overall morphisms \( \theta_{mn} : [m] \rightarrow [n] \) in the category \( \Delta \). The maps \( \alpha_F \) and \( \alpha_K \) are defined in the following way: If \( U \in M \), then \( F^K_m(U) = \text{hom}_{s\text{Set}}(K_m, F_m(U)) \). So let \( f = (f_m) \in \prod_m F^K_m(U) \). Then, the projections of \( \alpha_F(f) \) and \( \alpha_K(f) \) to the factor \( F^K_{m+n}(U) \) labeled by a morphism \( \theta_{mn} : [m] \rightarrow [n] \) are \( F(\theta_{mn})(f_n) \) and \( f_m \circ K(\theta_{mn}) \), respectively.
We now record some basic facts about the matching object $M_K(F)$.

**Proposition 4.5.**

1. For any simplicial sheaf $F$ and morphism of simplicial sets $\gamma: K \to L$, there is a natural morphism of sheaves $M_\gamma: M_L(F) \to M_K(F)$.
2. For any simplicial sheaf $F$ and simplicial set $K$, $M_K(F)$ is the equalizer of the diagram

$$\prod_{m \geq 0} F^K_m \xrightarrow{\alpha_F} \prod_{\theta: [m] \to [n]} F^K_n.$$

Moreover, if $K$ is a finitely generated simplicial set of dimension $M$, then $M_K(F)$ is the equalizer of the diagram

$$\prod_{0 \leq m \leq M} F^K_m \xrightarrow{\alpha_F} \prod_{\theta: [m] \to [n]} F^K_n.$$

3. If $X$ is a simplicial object in $M$, and $yX$ is the representable simplicial sheaf $yX(U)_n = \text{hom}_M(U, X_n)$, then there is a unique natural isomorphism of sheaves $M_K(yX) \cong \text{Hom}(K, X)$, where $\text{Hom}(K, X)$ is the sheaf

$$\text{Hom}(K, X)(U) := \text{hom}_{\text{Set}}(K \otimes U, X)$$

previously introduced in (12), and $K \otimes U$ is the simplicial object $(K \otimes U)_n := \prod_{[n]} U$.

**Proof.** Statement (1) is obvious, as is the proof that $M_K(F)$ is the equalizer of (18). If $K$ is finitely generated, then one shows $M_K(F)$ is the equalizer of (19) by using the fact that every simplex in $K$ can be uniquely written as a non-degenerate simplex composed with a degeneracy map (the “Eilenberg–Zilber Lemma”). Finally, (3) follows by showing that $\text{Hom}(K, X)$ is the equalizer of (18) when $F = yX$.

**Corollary 4.6.** Let $(M, T, P)$ be a category equipped with a pretopology and a collection of jointly conservative points. Let $X$ be a higher groupoid object in $(M, T)$ and $K$ a finitely generated simplicial set. Then for each $p \in P$, there is an unique natural isomorphism of sets

$$p \text{Hom}(K, X) \cong \text{hom}_{\text{Set}}(K, pX)$$

**Proof.** For any simplicial set $K$, Prop. 4.5 implies that

$$p \text{Hom}(K, X) \cong pM_K(yX).$$

If $K$ is finitely generated, then for any $n, m \geq 0$, $yX^K_m$ is a finite product of sheaves, and so Prop. 4.5 implies that $M_K(yX)$ is an equalizer of finite limits of sheaves. By definition, the functor $p$ preserves finite limits. Hence,

$$pM_K(yX) \cong \text{eq} \left( \prod_{0 \leq m \leq M} (pX_m)^{K_m} \xrightarrow{p\alpha_{pX}} \prod_{\theta: [m] \to [n]} \prod_{0 \leq m, n \leq M} (pX_m)^{K_n} \right).$$
A direct computation shows that the equalizer on the right-hand side of (21) is simply \( \hom_{\mathcal{Set}}(K, pX) \). □

A jointly conservative collection of points allows us to compare Kan fibrations of higher groupoids in \((\mathcal{M}, \mathcal{T})\) to the usual Kan fibrations of simplicial sets. We first recall a few different notions of “surjection” for sheaves.

**Definition 4.7.** Let \((\mathcal{M}, \mathcal{T})\) be a category equipped with a pretopology.

1. A morphism of sheaves \( \phi: F \to G \) is a **local surjection** iff for every object \( C \in \mathcal{M} \) and every element \( y \in G(C) \), there exists a cover \( U \rightarrowtail C \) such that the element \( f^*y \in G(U) \) is in the image of \( \phi_U: F(U) \to G(U) \).

2. If \( \mathcal{P} \) is a collection of jointly conservative points for \((\mathcal{M}, \mathcal{T})\), then a morphism of sheaves \( \phi: F \to G \) is a **stalkwise surjection** with respect to \( \mathcal{P} \) iff for all \( p \in \mathcal{P} \) the function \( p(\phi): pF \to pG \) is a surjection.

**Lemma 4.8.** Let \((\mathcal{M}, \mathcal{T})\) be a category equipped with a pretopology.

1. If \( \phi: F \to G \) is a local surjection of sheaves, then \( \phi \) is an epimorphism of sheaves.

2. If \( \phi: F \to G \) is an epimorphism of sheaves, then \( \phi \) is a stalkwise surjection with respect to any collection of jointly conservative points \( \mathcal{P} \) of \((\mathcal{M}, \mathcal{T})\).

3. Let \( f: U \to C \) be a cover in \((\mathcal{M}, \mathcal{T})\). The induced morphism of sheaves \( f_*: y(U) \to y(C) \) is a stalkwise surjection with respect to any collection of jointly conservative points.

**Proof.** (1) Suppose \( \alpha, \beta: G \to H \) are morphisms of sheaves such that \( \alpha \circ \phi = \beta \circ \phi \). We wish to show \( \alpha = \beta \). Let \( C \in \mathcal{M} \) and \( y \in G(C) \). Let \( U \rightarrow C \) be a cover such that there exists \( x \in F(U) \) such that \( \phi_U(x) = f^*(y) \in G(U) \). Since \( \alpha_U \circ \phi_U = \beta_U \circ \phi_U \), we have

\[ \alpha_U(f^*y) = \beta_U(f^*y) \in H(U). \]

Hence, \( f^*\alpha_C(y) = f^*\beta_C(y) \). Since \( H \) is a sheaf, and \( U \) is a cover, we conclude \( \alpha_C(y) = \beta_C(y) \).

(2) A morphism \( \phi: F \to G \) is an epimorphism if and only if the diagram

\[
\begin{array}{ccc}
  F & \xrightarrow{\phi} & G \\
  \downarrow \phi & & \downarrow \text{id} \\
  G & \xrightarrow{\text{id}} & G
\end{array}
\]

is a pushout. By definition, a point \( p: \text{Sh}(\mathcal{M}) \to \mathcal{Set} \) preserves small colimits. Hence, the proof follows.

(3) We will show \( f_* \) is a local surjection of sheaves. Then (1) and (2) will imply the result. Let \( A \in \mathcal{M} \) and \( g \in y(C)(A) = \hom(A, C) \). By axioms of a pretopology,
the pullback
\[
\begin{array}{ccc}
A \times_C U & \xrightarrow{pr_2} & U \\
\downarrow pr_1 & & \downarrow f \\
A & \xrightarrow{g} & C
\end{array}
\]
is a cover of \(A\). Since \(pr_2 \in y(U)(A \times_C U)\), we can apply \(f_*\):
\[
f_* (pr_2) = f \circ pr_2 = g \circ pr_1 = pr_1^*(g) \in y(C)(A \times_C U).
\]
Hence, \(f_*\) is a local surjection. \(\square\)

Now we will start connecting the homotopy theory of higher groupoids in \((M, T)\) with that of Kan simplicial sets. The next proposition says that a Kan fibration of higher groupoids is a “stalkwise Kan fibration” with respect to any collection of jointly conservative points.

**Proposition 4.9.** If \(f: X \to Y\) is a Kan fibration of higher groupoids in \((M, T)\) and \(P\) is a collection of jointly conservative points of \((M, T)\), then for all \(p \in P\) the map \(pf: pX \to pY\) is a Kan fibration of simplicial sets.

**Proof.** Since \(f: X \to Y\) is a Kan fibration, the horn projections
\[
\text{Hom}(\Delta^n, X) \xrightarrow{(\iota^n_j, f_*)} \text{Hom}(\Lambda^n_j \to \Delta^n, X \to Y)
\]
are covers for all \(n \geq 1\) and \(0 \leq j \leq n\). Hence, Prop. 4.8 implies that for each point \(p \in P\)
\[
(22) \quad p\text{Hom}(\Delta^n, X) \xrightarrow{p(\iota^n_j, f_*)} p\text{Hom}(\Lambda^n_j \to \Delta^n, X \to Y)
\]
is a surjection. Corollary 4.6 implies that \(p(\text{Hom}(\Delta^n, X)) \cong \text{hom}_{\text{Set}}(\Delta^n, pX)\), and since each \(p\) preserves finite limits, we have
\[
p(\text{Hom}(\Lambda^n_j \to \Delta^n, X \to Y)) \cong \text{hom}_{\text{Set}}(\Lambda^n_j, pX) \times_{\text{hom}_{\text{Set}}(\Delta^n, pY)} \text{hom}_{\text{Set}}(\Delta^n, pY).
\]
Combining these natural isomorphisms with (22) implies that \(pX \xrightarrow{pf} pY\) is a Kan fibration of simplicial sets. \(\square\)

**Corollary 4.10.** If \(X\) is a higher groupoid in \((M, T)\) and \(P\) is a collection of jointly conservative points for \((M, T)\), then the simplicial set \(pX\) is a Kan complex for each \(p \in P\).

5. Stalkwise weak equivalences

In this section, we introduce the morphisms between higher groupoids which will eventually become the weak equivalences in an incomplete category of fibrant objects. These stalkwise weak equivalences are a natural choice for weak equivalences between simplicial sheaves in diffeo-geometric contexts, e.g. [11, Def. 3.4.6].

**Definition 5.1.** Let \((M, T)\) be a category equipped with a pretopology and a collection of jointly conservative points \(P\). A morphism \(f: X \to Y\) of higher groupoids in \((M, T)\) is a **stalkwise weak equivalence** iff \(pf: pX \to pY\) is a weak homotopy equivalence of simplicial sets for all \(p \in P\).
5.1. Simplicial homotopy groups for higher groups. Following Joyal [20], Henriques gave a definition of simplicial homotopy group sheaves for \( n \)-groups in a category equipped with a pretopology. (See also related work of Jardine [19].) We show here that a morphism \( f : X \rightarrow Y \) of \( n \)-groups which induces an isomorphism between the simplicial homotopy groups is a stalkwise weak equivalence. We will need this result for integrating quasi-isomorphisms of Lie \( n \)-algebras in Section 8.

Let \( S^n := \Delta[n]/\partial \Delta[n] \) be the simplicial \( n \)-sphere, and let \( \text{cyl}(S^n) \) denote the simplicial set

\[
\text{cyl}(S^n) := \Delta[n + 1]/\left( \bigcup_{i=2}^{n+1} F^i \cup (F^0 \cap F^1) \right),
\]

where \( F^i \) is the simplicial set generated by the \( i \)th face of \( \Delta[n + 1] \). There are two inclusions \( i_0, i_1 : S^n \rightarrow \text{cyl}(S^n) \) induced by the maps \( \Delta[n] \rightarrow F^0 \) and \( \Delta[n] \rightarrow F^1 \), which are homotopy equivalences as well as homotopic to one another. Let \( X \) be a reduced Kan simplicial set, i.e. \( X_0 = * \). Since we have a unique basepoints, for \( n \geq 1 \) we can define the \( n \)th-homotopy group \( \pi_n(X) \) as the coequalizer

\[
\pi_n(X) = \text{coeq} \left( \text{hom}_\text{Set}(\text{cyl}(S^n), X) \rightrightarrows \text{hom}_\text{Set}(S^n, X) \right).
\]

In [16], Eq. (24) is used to provide an analogous definition of simplicial homotopy groups for higher groups in \( (M, \mathcal{T}) \). The suitable analog of \( \text{hom}_\text{Set}(K, -) \) in (24) is the matching object described in Sec. 4.2.

**Definition 5.2** (Def. 3.1 [16]). Let \( X \) be an \( k \)-group in \( (M, \mathcal{T}) \). Let \( n \geq 1 \) and let \( S^n \) be the simplicial \( n \)-sphere, and \( \text{cyl}(S^n) \) the simplicial set (23). The **simplicial homotopy groups** \( \pi_n^{\text{spl}}(X) \) are the sheaves

\[
\pi_n^{\text{spl}}(X) := \text{coeq} \left( \text{Hom}(\text{cyl}(S^n), X) \rightrightarrows \text{Hom}(S^n, X) \right),
\]

where \( \text{Hom}(-, X) \) is the sheaf (20).

**Proposition 5.3.** Let \( (M, \mathcal{T}, \mathcal{P}) \) be a category equipped with a pretopology and a collection of jointly conservative points. Let \( X \) be an \( n \)-group in \( (M, \mathcal{T}) \). Then for all \( p \in \mathcal{P} \), there is a unique natural isomorphism of groups

\[
\phi_X : p\pi_k^{\text{spl}}(X) \rightarrow \pi_k(pX)
\]

**Proof.** Since \( p \) preserves colimits, we have a natural isomorphism

\[
p\pi_k^{\text{spl}}(X) \cong \text{coeq} \left( p\text{Hom}(\text{cyl}(S^k), X) \rightrightarrows p\text{Hom}(S^k, X) \right).
\]

Since \( S^n \) and \( \text{cyl}(S^n) \) are finitely generated, Cor. 4.6 implies that there are natural isomorphisms

\[
p\text{Hom}(S^n, X) \cong \text{hom}_\text{Set}(S^n, pX),
p\text{Hom}(\text{cyl}(S^n), X) \cong \text{hom}_\text{Set}(\text{cyl}(S^n), pX).
\]

Combining these natural isomorphisms and using the fact that \( \text{Hom}(-, X) \) is functorial (Prop. 4.5), we obtain a natural isomorphism

\[
\phi_X : p\pi_k^{\text{spl}}(X) \rightarrow \text{coeq} \left( \text{hom}_\text{Set}(\text{cyl}(S^k), pX) \rightrightarrows \text{hom}_\text{Set}(S^k, pX) \right) = \pi_k(pX).
\]

\( \square \)
5.2. Stalkwise acyclic fibrations. Recall that an acyclic fibration of simplicial sets is a morphism of simplicial sets which is both a weak equivalence and a fibration. Equivalently, \( X \to Y \) is an acyclic fibration of simplicial sets iff the boundary projections

\[
\text{hom}_{sSet}(\Delta^n, X) \to \text{hom}_{sSet}(\partial\Delta^n, X) \times_{\text{hom}_{sSet}(\partial\Delta^n, Y)} \text{hom}_{sSet}(\Delta^n, Y)
\]

are surjective for all \( n \geq 0 \). In Prop. 4.9 we showed that a Kan fibration \( X \to Y \) between higher groupoids in \((M, T)\) equipped with a jointly conservative collection of points \( P \) is a stalkwise Kan fibration, i.e. induces a Kan fibration \( pX \to pY \) for all \( p \in P \). So clearly a Kan fibration which is also a stalkwise weak equivalence is obviously a “stalkwise acyclic fibration”. In fact, more is true, as the following proposition shows:

**Proposition 5.4.** Let \((M, T, P)\) be a category equipped with a pretopology and a collection of jointly conservative points. A morphism \( f : X \to Y \) of higher groupoids in \((M, T)\) is both a stalkwise Kan fibration and a stalkwise weak equivalence (i.e., a stalkwise acyclic fibration) if and only if the boundary projections

\[
\text{Hom}(\Delta^n, X) \xrightarrow{(j_n^*, f_*)} \text{Hom}(\partial\Delta^n, X) \to \Delta^n, X \xrightarrow{f} Y
\]

are stalkwise surjections for \( n \geq 0 \).

**Proof.** Recall that the sheaf \( \text{Hom}(\partial\Delta^n \to \Delta^n, X \to Y) \) (13) is the pullback

\[
\text{Hom}(\partial\Delta^n, X) \times_{\text{Hom}(\partial\Delta^n, Y)} \text{Hom}(\Delta^n, Y).
\]

Corollary 4.6 implies that \( p(\text{Hom}(\Delta^n, X)) \cong \text{hom}_{sSet}(\Delta^n, pX) \) for all \( p \in P \), and since each \( p \) preserves finite limits, we have

\[
p(\text{Hom}(\partial\Delta^n, X) \times_{\text{Hom}(\partial\Delta^n, Y)} \text{Hom}(\Delta^n, Y)) \\
\cong \text{hom}_{sSet}(\partial\Delta^n, pX) \times_{\text{hom}_{sSet}(\partial\Delta^n, pY)} \text{hom}_{sSet}(\Delta^n, pY).
\]

Hence, by comparison with (28), we see that \( pf : pX \to pY \) is an acyclic fibration of simplicial sets if and only if \( p(j_n^*, f_*) \) is surjective, i.e. \((j_n^*, f_*))\) is a stalkwise surjection. \( \square \)

6. Hypercovers

Hypercovers were introduced in [2] and subsequently have been used throughout the homotopy theory of simplicial sheaves, e.g. [6]. Hypercovers for Lie \( n \)-groupoids play an important role in the work of Zhu [38] and Wolfson [37]. Also, the acyclic fibrations in the Behrend–Getzler CFO structure for \( n \)-groupoids objects in a descent category are hypercovers.

**Definition 6.1.** Given a category and pretopology \((M, T)\), a morphism \( f : X \to Y \) of simplicial objects in \( M \) is a hypercover iff it satisfies the condition \( \text{Acyc}(m) \) for all \( 0 \leq m \), which means the sheaf \( \text{Hom}(\partial\Delta^m \to \Delta^m, X \to Y) \) is representable and the canonical boundary projection:

\[
\text{Hom}(\Delta^m, X) \xrightarrow{j_m^*, f_*} \text{Hom}(\partial\Delta^m, X) \to \Delta^m, X \xrightarrow{f} Y,
\]

is a cover in \( T \). For \( m = 0 \), \( \text{Hom}(\partial\Delta^0 \to \Delta^0, X \to Y) := Y_0 \) by definition.

**Remark 6.2.**
(1) As shown in [38, Lemma 2.4], the representability of the fiber product
\[ \text{Hom}(\partial \Delta^m \to \Delta^m, X \to Y) \]
is automatic as long as \( Y \) is a higher groupoid in \((M, T)\).

(2) A hypercover is a Kan fibration automatically, since maps in \( \{ \Lambda^m_j \to \Delta^m \mid m > 0, 0 \leq j \leq m \} \)
can be reconstructed as pushouts of ones in \( \{ \partial \Delta^m \to \Delta^m \mid m \geq 0 \} \).

The \( \text{Acyc}(m) \) condition is obviously analogous to the previously discussed characterization (28) of acyclic fibrations of simplicial sets. Indeed, every hypercover of higher groupoids is a stalkwise acyclic fibration in the sense of Prop. 5.4:

**Corollary 6.3.** If a morphism of higher groupoids \( f : X \to Y \) in \((M, T, P)\) is a hypercover, then it is a Kan fibration and a stalkwise weak equivalence.

**Proof.** Remark 6.2 implies that \( f : X \to Y \) is a Kan fibration. Since the boundary projections (29) are covers, Prop. 4.8 implies that they are stalkwise surjections with respect to the points \( P \). Hence, Prop. 5.4 implies that \( f : X \to Y \) is a stalkwise acyclic fibration, so in particular, it is a stalkwise weak equivalence. \( \Box \)

The natural question to ask is whether the converse of Cor. 6.3 is true. That is, is every morphism of higher groupoids which is both a Kan fibration and a stalkwise weak equivalence necessarily a hypercover? The answer will be yes, if our pretopology satisfies some additional conditions. Our main example, the category \( \text{Mfd} \) of Banach manifolds equipped with the surjective submersion pretopology \( T_{\text{subm}} \) and the collection of points \( P_{\text{Ban}} \), satisfies these additional conditions.

### 6.1. Locally stalkwise pretopologies

**Definition 6.4.** Let \((M, T, P)\) be a category equipped with a pretopology and a jointly conservative collection of points. Let \( X \in M \). A morphism \( F \xrightarrow{g} yX \) of sheaves is a **local stalkwise cover** iff there exists an object \( Y \in M \) and a stalkwise surjection (Def. 4.7) \( yY \xrightarrow{f} F \) such that the composition \( g \circ f : yY \to yX \) is a cover in \((M, T)\).

**Definition 6.5.** A pretopology \( T \) on a category \( M \) is a **locally stalkwise pretopology** iff there exists a jointly conservative collection of points \( P \) of \((M, T)\) such that:

1. (2-out-of-3) If \( U \xrightarrow{f} V \xrightarrow{g} W \) are morphisms in \( M \), and \( g \circ f \) is a cover and \( y(f) \) is a stalkwise surjection in \( \text{Sh}(M) \) with respect to \( P \), then \( g \) is also a cover,

2. (locality of covers) If \( W \xrightarrow{q} V \) and \( U \xrightarrow{p} V \) are morphisms in \( M \), \( y(q) \) is a stalkwise surjection with respect to \( P \), and the base change \( U \times_V W \xrightarrow{p} W \) is a local stalkwise cover, then \( p \) is a cover, and therefore \( U \times_V W \) is representable.

We have several remarks about Def. 6.5:

**Remark 6.6.**

1. It is not really precise to call the first condition in Def. 6.5: “2-out-of-3”, because \( g \circ f \) being a cover and \( g \) being a cover will not imply anything (just like surjective maps for sets). We note that our “2-out-of-3” property holds automatically for stalkwise surjections in \( \text{Sh}(M) \).
(2) If \((M, T, P)\) is a category equipped with a locally stalkwise pretopology then a local stalkwise cover \(F \xrightarrow{\varphi} yX\) of \(X\) is a cover in \(T\) whenever \(F\) is representable.

We now can give a converse to Cor. 6.3. In fact, we have:

**Proposition 6.7.** Let \((M, T)\) be a category equipped with a locally stalkwise pretopology with respect to a jointly conservative collection of points \(P\), and let \(f : X \to Y\) be a morphism of higher groupoids in \((M, T)\). The following are equivalent:

1. \(f : X \to Y\) is a Kan fibration and a stalkwise weak equivalence with respect to the collection of points \(P\).
2. \(f : X \to Y\) is a Kan fibration and a stalkwise weak equivalence with respect to any jointly conservative collection of points of \((M, T)\).
3. \(f : X \to Y\) is a hypercover.

**Proof.** (3)\(\Rightarrow\)(2) is the content of Cor. 6.3, and (2)\(\Rightarrow\)(1) is obvious. So we focus on (1)\(\Rightarrow\)(3).

First, let \(k \geq 1\) and consider the following pullback diagram in \(\text{Sh}(M)\)

\[
\begin{array}{ccc}
\text{Hom}(\Delta^k, X) & \xrightarrow{\eta_k} & \text{Hom}(\partial\Delta^k \to \Delta^k, X \to Y) \\
\downarrow & & \downarrow \\
\text{Hom}(\Delta^{k-1}, X) & \xrightarrow{\eta_{k-1}} & \text{Hom}(\partial\Delta^{k-1} \to \Delta^{k-1}, X \to Y)
\end{array}
\]

The projection \(\text{Hom}(\partial\Delta^k \to \Delta^k, X \to Y) \to \text{Hom}(\Delta^{k-1}, X) = X_{k-1}\) is induced by \(d_0^X\). The composition \(\eta_{k-1} \circ \eta_k : \text{Hom}(\Delta^k, X) \to \text{Hom}(\Delta^0 \to \Delta^k, X \to Y)\) is the horn projection, and therefore a cover, since \(f : X \to Y\) is a Kan fibration. Since \(f\) is also a stalkwise weak equivalence, Prop. 5.4 implies that \(q_k\) is a stalkwise surjection. Hence, \(q_{k-1}\) is a local stalkwise cover (Def. 6.4).

Next, we observe that the composition \(pr_0 \circ q_k : \text{Hom}(\Delta^k, X) \to \text{Hom}(\Delta^0 \to \Delta^k, X \to Y)\) is simply the face map \(d_0 : X_k \to X_{k-1}\), which is a cover (Lemmas 3.6 and 3.7), hence a stalkwise surjection (Lemma 4.8). Combining this with the fact that \(q_k\) is a stalkwise surjection implies that \(pr_0 \circ q_k\) is a stalkwise cover, and hence also a stalkwise surjection. Thus \(d_0^Y \times (d_k^X)^k \circ q_{k-1} = q_{k-1} \circ pr_0\) is stalkwise surjective, hence \(d_0^Y \times (d_k^X)^k\) is stalkwise surjective.

Now, we show via induction that the boundary maps \(\text{Hom}(\Delta^k, X) \xrightarrow{\partial_k} \text{Hom}(\partial\Delta^k \to \Delta^k, X \to Y)\) are covers for all \(k \geq 0\). For the base case, let \(k = 1\). Then Def. 6.1 implies that \(\text{Hom}(\partial\Delta^{k-1} \to \Delta^{k-1}, X \to Y) = \Delta^1 \times X_0\) is representable, and since \(f\) is a Kan fibration, \(\text{Hom}(\Delta^1 \to \Delta^1, X \to Y)\) is also representable. The results of the previous paragraphs imply that \(q_0\) is a local stalkwise cover and \(d_0^Y \times (d_k^X)^k\) is an stalkwise surjection. Since \(T\) is a locally stalkwise pretopology, Def. 6.5 (2) implies that \(q_0\) is a cover and \(\text{Hom}(\partial\Delta^1 \to \Delta^1, X \to Y)\) is representable.

Finally, suppose \(k \geq 2\), \(\text{Hom}(\Delta^{k-2}, X) \xrightarrow{\partial_{k-2}} \text{Hom}(\partial\Delta^{k-2} \to \Delta^{k-2}, X \to Y)\) is a cover, and \(\text{Hom}(\partial\Delta^{k-1} \to \Delta^{k-1}, X \to Y)\) is representable. As previously shown, the pullback of \(q_{k-1}\) in diagram (30) along the stalkwise surjection \(d_0^Y \times (d_k^X)^k\) is a local stalkwise cover. Hence, Def. 6.5(2) implies that \(q_{k-1} \circ \text{Hom}(\partial\Delta^{k-2}, X) \to \text{Hom}(\partial\Delta^{k-1} \to \Delta^{k-1}, X \to Y)\) is a cover, and that \(\text{Hom}(\partial\Delta^k \to \Delta^k, X \to Y)\) is representable. This completes the proof. \(\square\)
6.2. Locally stalkwise pretopologies for Banach manifolds. Here we show that the pretopology \( \mathcal{T}_{\text{subm}} \) of surjective submersions equipped with the collection of jointly conservative points \( \mathcal{T}_{\text{Ban}} \) (16) is a locally stalkwise pretopology for the category of Banach manifolds. We first give an explicit description of stalkwise surjective maps in \( (\text{Mfd}, \mathcal{T}_{\text{subm}}, \mathcal{P}_{\text{Ban}}) \).

**Lemma 6.8.** Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be composable morphisms in \( \text{Mfd} \). If \( f \) is surjective and \( g \circ f \) is a surjective submersion, then \( g \) is also a surjective submersion.

**Proof.** Clearly, \( g \) is surjective. Let \( y \in Y \), \( z = g(y) \), and \( x \in f^{-1}(y) \). Since \( g \circ f \) is a surjective submersion, there exists open neighborhoods \( U \) and \( V \) of \( z \) and \( x \), respectively, and a section \( \sigma_{XZ} : U \to V \) such that \( \sigma_{XZ}(z) = x \), and \( g \circ f \circ \sigma_{XZ} = \text{id}_U \). Let \( W = g^{-1}(U) \). Then \( \sigma := f \circ \sigma_{XZ} : U \to W \) is a section of \( g \) such that \( \sigma(z) = y \). Hence \( g \) is a surjective submersion. \( \square \)

**Lemma 6.9.** Let \( \phi : X \to Y \) be a morphism in \( (\text{Mfd}, \mathcal{T}_{\text{subm}}, \mathcal{P}_{\text{Ban}}) \). Then the following are equivalent:

1. The morphism of sheaves \( y(\phi) : yX \to yY \) is stalkwise surjective.
2. For any morphism \( f : U \to Y \) in \( \text{Mfd} \), there exists an open cover \( \{U_i\}_{i \in I} \) of \( U \) and morphisms \( f_i : U_i \to X \) such that \( \phi \circ f_i = f | U_i \) for each \( i \in I \).
3. For each \( y \in Y \), there exists: a preimage \( x \in \phi^{-1}(y) \), an open neighborhood of \( V \) of \( y \), an open neighborhood of \( W \) of \( x \), and a morphism \( \sigma : V \to W \) such that \( \sigma(y) = x \) and \( \phi \circ \sigma = \text{id}_V \).

**Proof.**

(1 \( \Rightarrow \) 2): Let \( f : U \to Y \) be a morphism of Banach manifolds. Since \( \phi \) is stalkwise surjective, for each \( z \in U \), we can find a Banach space \( V_z \), an \( \epsilon_z > 0 \) and a local diffeomorphism \( i_z : B_{V_z}(r_z) \to U \) with \( i_z(0) = z \) such that there exists a map \( f_z : B_{V_z}(r_z) \to X \) with the property that \( \phi \circ f_z = f \circ i_z \). We then use collection of open balls \( \{B_{V_z}(r_z)\}_{z \in U} \) to obtain the desired open cover of \( U \), and the collection \( \{f_z\}_{z \in U} \) as our desired collection of maps.

(2 \( \Rightarrow \) 1): Let \( V \) be a Banach space and \( \bar{g} \in p_V yY \). This class is represented by a map \( g : B_{V'}(r') \to Y \). By hypothesis, there exists an open cover \( \{U_i\} \) of \( B_{V'}(r') \) and maps \( h_i : U_i \to X \) such that \( \phi \circ h_i = g | U_i \). Let \( U_{i_0} \) be an open subset containing \( 0 \), and \( r' > 0 \) such that \( B_{V'}(r') \subseteq U_{i_0} \). Then \( g | B_{V'}(r') \) is the composition of \( \phi \) with \( f = h_{i_0} | B_{V'}(r') \), and hence \( p_V \phi(f) = \bar{g} \).

(2 \( \Rightarrow \) 3): Let \( y \in Y \) and consider the identity map \( \text{id}_Y : Y \to Y \). There exists an open cover \( \{U_i\} \) of \( Y \) and maps \( f_i : U_i \to Y \) such that \( \phi \circ f_i = \text{id}_U | U_i \). Let \( U_{i'} \) be an element of the cover containing \( y \), and let \( x = f_{i'}(y) \). Then \( \sigma := f_{i'} : U_{i'} \to X \) is a desired local section of \( \phi \) which maps \( y \) to \( x \).

(3 \( \Rightarrow \) 2): Let \( f : U \to Y \) be a map. For each \( y \in Y \), there exists an \( x \in \phi^{-1}(y) \), an open subset \( V_y \) containing \( x \), and an open subset \( W_y \) mapping \( y \) to \( x \). The collection \( \{U_y := f^{-1}(V_y)\}_{y \in Y} \) is an open cover of \( U \). For each \( y \in Y \), let \( f_y : U_y \to X \) be the composition \( f_y = \sigma_y \circ f | U_y \). Then, by construction, \( \phi \circ f_y = f | U_y \). \( \square \)

**Remark 6.10.** Note that Lemma 6.9 implies that stalkwise surjective maps are weaker than surjective submersions. Also note that Remark 4.3 implies that Lemma 6.9 also holds if we replace \( \mathcal{T}_{ss} \) with \( \mathcal{T}_{open} \).
Lemma 6.11. Let $W \xrightarrow{q} V \xrightarrow{p} U$ be morphisms in $\text{Mfd}$ such that $y(q)$ is stalkwise surjective, and consider the pullback diagram in the category of topological spaces

\[
\begin{array}{ccc}
U \times_V W & \xrightarrow{\tilde{p}} & W \\
\downarrow q & & \downarrow q \\
U & \xrightarrow{p} & V
\end{array}
\]

Suppose that there exists a manifold $X \in \text{Mfd}$ and a continuous surjective map $X \xrightarrow{\pi} U \times_V W$ such that $q \circ \pi: X \to U$ is smooth and the composition $\tilde{p} \circ \pi: X \to W$ is a smooth surjective submersion. Then $p: U \to V$ is a surjective submersion and therefore $U \times_V W$ is representable in $\text{Mfd}$.

Proof. The surjectivity of $p$ follows from an easy set-theoretical argument, so we will show it is a surjective submersion. Let $u \in U$ and $v = p(u)$. Since $y(q)$ is stalkwise surjective, Lemma 6.9 implies there exists an open neighborhood $O_v$ of $v$, an element $w \in q^{-1}(v) \subseteq W$, and a smooth section

\[\sigma_{WV}: O_v \to O_w := q^{-1}(O_v),\]

such that $q \circ \sigma_{WV}(w) = w$. Hence the restriction $q|_{O_w}$ is a submersion at $w$ (i.e., $T_wq$ is surjective and its kernel splits). The inverse function theorem \cite[Cor. 1.5.2]{23} then implies that, by taking $O_v$ to be a small enough neighborhood, we can express $q|_{O_w}$ as a projection. Therefore, the following pullback diagram exists in the category $\text{Mfd}$:

\[
\begin{array}{ccc}
O_u \times_{O_v} O_w & \xrightarrow{\tilde{p}} & O_w \\
\downarrow \tilde{q} & & \downarrow q \\
O_u & \xrightarrow{p} & O_v
\end{array}
\]

where $O_u := p^{-1}(O_v)$, and we suppress restrictions of the morphisms.

Now, by hypothesis, we have a surjective continuous map $\pi: X \to U \times_V W$ such that $\pi \circ \tilde{q}$ is smooth and $\pi \circ \tilde{p}$ is a surjective submersion. Let $x \in X$ such that $\pi(x) = (u, w)$. We may shrink $O_v$ further if necessary, so that we have a smooth section

\[\sigma_{XW}: O_w \to O_x := (\tilde{p} \circ \pi)^{-1}(O_w)\]

such that $(\tilde{p} \circ \pi) \circ \sigma_{XW} = \text{id}_{O_w}$ and $\sigma_{XW}(w) = x$. It is not difficult to see that we have the following equalities of open sets:

\[\tilde{p}^{-1}(O_w) = O_u \times_{O_v} O_w, \quad \pi^{-1}(O_u \times_{O_v} O_w) = O_x.\]

Since $\tilde{p} \circ \pi \circ \sigma_{XW} = \text{id}_{O_w}$, we see that

\[\pi \circ \sigma_{XW}: O_w \to O_u \times_{O_v} O_w\]

is a continuous section of $\tilde{p}: O_u \times_{O_v} O_w \to O_w$. In fact, $\pi \circ \sigma_{XW}$ is smooth since the factors $\tilde{q} \circ \pi \circ \sigma_{XW}$ and $\tilde{p} \circ \pi \circ \sigma_{XW}$ are compositions of smooth maps.

The commutativity of the pushout diagram gives us $p \circ \tilde{q} \circ \pi \circ \sigma_{XW} = q$. Hence, $\tilde{q} \circ \pi \circ \sigma_{XW} \circ \sigma_{WV}: O_v \to O_u$ is our desired smooth section of $p$. Therefore $p$ is a submersion. \qed
Proposition 6.12. The category of Banach manifolds $\text{Mfd}$ equipped with the surjective submersion pretopology $\mathcal{T}_{\text{subm}}$, and the collection of jointly conservative points $\mathcal{P}_{\text{Ban}}$ is a locally stalkwise pretopology.

Proof. First, we show the “2-out-of-3” property of Def. 6.5 is satisfied. Indeed, this follows immediately from Lemma 6.8, since if $y(f): y(X) \to y(Y)$ is stalkwise surjective with respect to $\mathcal{P}_{\text{Ban}}$ then $f: X \to Y$ is surjective.

Next, we show the locality property of Def. 6.5 is satisfied. Let $W \xrightarrow{q} V \xleftarrow{p} U$ be morphisms in $\text{Mfd}$ and suppose we have the diagram of sheaves

$$
\begin{array}{ccc}
\mathbf{y}X & \xrightarrow{f} & \mathbf{y}U \times_{\mathbf{y}V} \mathbf{y}W \\
\downarrow \mathbf{yq} & & \downarrow \mathbf{yp} \\
\mathbf{y}U & \xrightarrow{\mathbf{y}p} & \mathbf{y}V
\end{array}
$$

in which $\mathbf{y}q$ and $f$ are stalkwise surjections, and $\mathbf{y}p$ is represented by a surjective submersion. The Yoneda lemma implies that there exists smooth maps $g: X \to U$ and $h: X \to W$ such that $\mathbf{y}g = \mathbf{y}p \circ f$ and $\mathbf{yh} = \mathbf{y}q \circ f$. Moreover, since $f$ is stalkwise surjective, $g$ and $h$ are surjective maps. This gives us a surjective map of topological spaces $\pi := (g, h): X \to U \times V W$. We now have satisfied all the hypotheses of Lemma 6.11, and therefore we can conclude that $p: U \to V$ is a cover. □

7. Higher groupoids as an incomplete category of fibrant objects

We now fix a category $(\text{M}, \mathcal{T}, \mathcal{P})$ equipped with a locally stalkwise pretopology and prove that the corresponding category of $\infty$-groupoids admits an iCFO structure. The results in Sec. 6.2 then imply that we obtain an iCFO structure for the category of Lie $\infty$-groupoids as a special case.

After discussing some aspects of the simplicial localization of category of Lie $\infty$-groupoids in Sec. 7.2.1, we analyze in Sec. 7.3 the weak equivalences for this iCFO structure in more detail. In particular, we recall that the weak equivalences are completely characterized by the acyclic fibrations, which in this case are, respectively, stalkwise weak equivalences and hypercovers. Thanks to a result of Behrend and Getzler [4], we obtain a sheaf-theoretic independent characterization of weak equivalences, without any mention of the collection of points $\mathcal{P}$.

7.1. Path object. We first construct a candidate for a path object. This will require several steps. Later in Sec. 7.2, we verify that this gives a path object as part of the iCFO structure for higher groupoids. Our construction, in particular the proof of Prop. 7.4, is essentially identical to that of Behrend and Getzler [4, Thm. 3.21]. However, as previously mentioned, we do not assume the existence of finite limits in our category $\text{M}$. Moreover, our definition of weak equivalences, necessary for our application in Sec. 8 is different than the one given in [4]. Hence, it is necessary to present a verification that the Behrend–Getzler construction works in our context.

Given a simplicial sheaf $F \in s\text{Sh}(\text{M})$ and a simplicial set $K \in s\text{Set}$, denote by $F^K$ the simplicial sheaf

$$
F^K(U)_n := \text{hom}_{\text{Set}}(\Delta^n \times K, F(U)) = M_{\Delta^n \times K}(F),
$$

(31)
where $M_{\Delta^m \times \Delta^1}(F)$ is the aforementioned matching object (Def. 4.4). A natural path object for $F$ is the simplicial sheaf $F^{\Delta^1}$. The inclusion of simplicial sets

$$\partial \Delta^1 \cong \Delta^0 \cup \Delta^0$$

induces a morphism of simplicial sheaves

$$F^{\Delta^1} \xrightarrow{(d^0\ast, d^1\ast)} F \times F \cong F^{\Delta^0 \cup \Delta^0}.$$  

Also, the constant map $s^0: \Delta^1 \to \Delta^0$ gives us a map of simplicial sheaves

$$F \cong F^{\Delta^0} \xrightarrow{s^0\ast} F^{\Delta^1}.$$  

We now make the following observation which justifies our consideration of $F^{\Delta^1}$:

**Proposition 7.1.** Let $X$ be a higher groupoid in $(M, \mathcal{T}, \mathcal{P})$. In the diagram

$$yX \xrightarrow{s^0\ast} (yX)^{\Delta^1} \xrightarrow{(d^0\ast, d^1\ast)} yX \times yX.$$  

the map $s^0\ast$ is a stalkwise weak equivalence and the map $(d^0\ast, d^1\ast)$ is a stalkwise Kan fibration.

**Proof.** Recall that Corollary 4.10 implies that $pX := pyX$ is a Kan complex for any point $p \in \mathcal{P}$. Equation 31 and Prop. 4.5 imply that the image of the above diagram under $p$ is naturally isomorphic to the diagram of simplicial sets

$$pX \xrightarrow{s^0\ast} (pX)^{\Delta^1} \xrightarrow{(d^0\ast, d^1\ast)} pX \times pX.$$  

This is the usual diagram which exhibits the Kan complex $(pX)^{\Delta^1} = \text{Map}_{\text{Set}}(\Delta^1, pX)$ as the path object of $pX$. Hence, $s^0\ast$ is a weak equivalence and $(p(d^0\ast, p(d^1\ast))$ is a fibration of simplicial sets for any point $p$. \qed

The above proposition suggests that, to construct a path object for an $n$-groupoid $X$ in $M$, we should show that the simplicial sheaf $(yX)^{\Delta^1}$ is representable by a higher groupoid $X^{\Delta^1}$, and that the map of higher groupoids $X^{\Delta^1} \to X \times X$ is not just a stalkwise Kan fibration, but a Kan fibration in the sense of Def. 3.3.

7.1.1. **Total décalage and its adjoints.** We next need to recall a minimal number of facts concerning augmented simplicial objects and biaugmented bisimplicial objects. We follow the expositions given in the introductions of [32, 33].

Let $\Delta_+ = \Delta \cup \{-1\}$ denote the augmented simplicial category where $\{-1\}$ is the empty ordinal. Let $D$ be a complete and cocomplete category. (In what follows, we will only be concerned with the case $D = \text{Set}$ or $D = \text{Sh}(M)$.) The category $sD^+$ of **augmented simplicial objects** in $D$ is the category of functors $\Delta_+^{op} \to D$. The inclusion $i: \Delta \hookrightarrow \Delta_+$ induces an adjoint pair

$$i^*: sD^+ \rightleftarrows D: i_*.$$  

Above $i^*$ forgets the augmentation and $i_*(X)_{-1} = \ast$, where $\ast$ is the terminal object of $D$.

Analogously, the category $ssD^+$ of **biaugmented bisimplicial objects** in $D$ is the category of functors $\Delta_+^{op} \times \Delta_+^{op} \to D$, and there is an adjoint pair

$$i^* \times i^*: ssD^+ \rightleftarrows sD^*: i_* \times i_*.$$
Above $i^* \times i^*$ produces a bisimplicial object from a biaugmented one by forgetting both the negative vertical and horizontal dimensions, and $(i_* \times i_*)(X)_{p,q} := *$ if $p = -1$ or $q = -1$.

Next, denote by $\sigma: \Delta_+ \times \Delta_+ \to \Delta_+$ the **ordinal sum** bifunctor which is defined on objects as $\sigma([n],[m]) := [n + m + 1]$ and on morphisms as
\[
\sigma(\alpha, \beta): \alpha([p],[q]) \to \sigma([p'],[q'])
\]
\[
\sigma(\alpha, \beta)(k) = \begin{cases} 
\alpha(k), & \text{if } 0 \leq k \leq p \\
\beta(k - p - 1) + p' + 1, & \text{if } p + 1 \leq k.
\end{cases}
\]

The ordinal sum induces the **(total) augmented décalage functor**
\[
\text{Dec}^+: sD^+ \to ssD^+
\]
\[
\text{Dec}^+(X)_{p,q} := (X \circ \sigma^{op})_{p,q} = X_{p+q+1}, \quad p, q \geq -1,
\]
which via composition with the adjunctions (34) and (35) gives the **(total) décalage functor**
\[
\text{Dec}: sD \to ssD
\]
\[
\text{Dec} := (i^* \times i^*) \circ \text{Dec}^+ \circ i_*
\]
\[
\text{Dec}(X)_{p,q} = X_{p+q+1} \quad p, q \geq 0.
\]

Since $D$ is complete and cocomplete, both $\text{Dec}$ and $\text{Dec}^+$ have left and right adjoints given by left and right Kan extension along $\sigma^{op}$. We will, in particular, be interested in the left adjoint of $\text{Dec}^+$:
\[
T: ssD^+ \rightleftarrows sD^+: \text{Dec}^+
\]
and the right adjoint of $\text{Dec}$:
\[
\text{Dec}: sD \rightleftarrows ssD: \nabla.
\]

The functor $\nabla$ is known as the **Artin-Mazur codiagonal** [3]. Both of these functors have explicit descriptions obtained from the usual end/coend formulations for Kan extensions.

The reason why we introduce this machinery is to prove the following:

**Proposition 7.2.** If $F \in s\text{Sh}(M)$ is a simplicial sheaf, then
\[
F^{\Delta^1} \cong (\nabla \circ \text{Dec})(F).
\]

We will need a few lemmas before proving Prop. 7.2. We first consider the left adjoint $T: ssD^+ \to D^+$ in more detail. Given an biaugmented bisimplicial object $X \in ssD$ we have
\[
T(X)_m = \prod_{p+q+1 = m} X_{p,q}.
\]

Let $\theta: [n] \to [m]$ be a morphism in $\Delta_+$. Then
\[
T(X)(\theta): T(X)_m \to T(X)_n
\]
is defined in the following way: If $p + q + 1 = m$, then we have $\theta: [n] \to \sigma([p],[q])$. It is not hard to show that there exists unique ordinals $[p'], [q']$ and morphisms $\alpha: [p'] \to [p], \beta: [q'] \to [q]$ such that $\sigma([p'],[q']) = [n]$ and $\sigma(\alpha, \beta) = \theta$. Hence, we have a composition of morphisms in $D$
\[
X_{p,q} \xrightarrow{X_{\alpha,\beta}} X_{p',q'} \xrightarrow{T(X)_\theta} T(X)_n,
\]
which via the universal property induces the map $T(X)(\theta)$. 
Lemma 7.3. For all \( n \geq 0 \) there is an equality of augmented simplicial sets
\[
i_*\Delta^n \times i_*\Delta^1 \cong T(\text{Dec}^+(i_*\Delta^n)).
\]

Proof. If \( k = -1 \), then clearly we have \((i_*\Delta^n \times i_*\Delta^1)_{-1} = * = T(\text{Dec}^+(i_*\Delta^n))_{-1}\). Now let \( k \geq 0 \). We have the following equalities of sets
\[
(i_*\Delta^n \times i_*\Delta^1)_k = \Delta^n_k \times \Delta^1_k = \coprod_{1 \leq i \leq k+2} \Delta^n_i = \coprod_{i+j+1 = k} \Delta^n_{i+j+1} = T(\text{Dec}^+(i_*\Delta^n))_k.
\]
Moreover, if \( \theta \) is a morphism in \( \Delta^+ \), then using the description (38) for \( T(\text{Dec}^+(X))(\theta) \), it is not hard to see that the above equalities extend to an equality of augmented simplicial sets.

Now we give the proof of Prop. 7.2. In what follows, we use the same notation for the décalage functors, their adjoints, and related functors for \( \mathcal{D} = \text{Set} \) or \( \mathcal{D} = \text{Sh}(\mathcal{M}) \). It will be clear from the context which functors we mean.

Proof of Prop. 7.2. Let \( F \in \text{sSh}(\mathcal{M}) \). If \( U \in \mathcal{M} \) then, for any \( n \geq 0 \), we have a string of equalities and natural isomorphisms:
\[
F^\Delta_n(U) = \text{hom}_{\text{Set}}(\Delta^n \times \Delta^1, F(U)) \cong \text{hom}_{\text{Set}}(\Gamma^* \circ i_*(\Delta^n \times \Delta^1), F(U)) \\
\cong \text{hom}_{\text{Set}^+}(i_*(\Delta^n \times \Delta^1), i_*F(U)).
\]
Using Lemma 7.3 and the fact that \( T \) is left adjoint, we obtain:
\[
F^\Delta_n(U) \cong \text{hom}_{\text{Set}^+}(T(\text{Dec}^+(i_*\Delta^n)), i_*F(U)) \\
\cong \text{hom}_{\text{Set}^+}(\text{Dec}^+(i_*\Delta^n), \text{Dec}^+(i_*F(U))).
\]
Now recall that if \( X \) and \( Y \) are simplicial sets, any collection of functions \( \{f_k: X_k \to Y_k\}_{k \geq 1} \) which are compatible with the simplicial structure can be extended to a simplicial morphism \( f: X \to Y \) by defining \( f_0(x) := d_0 f_1(s_0 x) \). The same argument gives a natural isomorphism
\[
\text{hom}_{\text{Set}^+}(\text{Dec}^+(i_*\Delta^n), \text{Dec}^+(i_*F(U))) \cong \text{hom}_{\text{Set}^+}(\text{Dec}(\Delta^n), \text{Dec}(F(U))).
\]
We then use the right adjoint \( \nabla \) of \( \text{Dec} \) to obtain
\[
F^\Delta_n(U) \cong \text{hom}_{\text{Set}}(\Delta^n, \nabla \circ \text{Dec}(F(U))) \cong \left(\nabla \circ \text{Dec}(F(U))\right)_n.
\]
Finally, since all of the above isomorphisms are natural, we obtain the desired isomorphism of simplicial sheaves
\[
F^\Delta \cong \left(\nabla \circ \text{Dec}(F)\right).
\]

7.1.2. Representability. Our next goal is to show that if \( X \) is an \( \infty \)-groupoid in \( (\mathcal{M}, \mathcal{T}) \) then our candidate for the path object of \( X \) is also an \( \infty \)-groupoid. Below in Remark 7.8, we also note that if \( X \) is an \( n \)-groupoid for finite \( n \), then \( X^\Delta^1 \) is as well.

Proposition 7.4. If \( X \) is an \( \infty \)-groupoid in \( (\mathcal{M}, \mathcal{T}) \), then there is a canonical \( \infty \)-groupoid \( X^\Delta^1 \) in \( (\mathcal{M}, \mathcal{T}) \) representing the simplicial sheaf \( (yX)^\Delta^1 \).
Before we prove this, let us first recall that if $F \in ssSh(M)$ is a bisimplicial sheaf, then $\nabla F$ can be expressed explicitly in terms of iterated pullbacks. Denote by $d^h_i : F_{p,q} \to F_{p-1,q}$ and $d^v_i : F_{p,q} \to F_{p,q-1}$ the horizontal and vertical face maps of $F$. Then we have

$$\left(\nabla F\right)_n = F_{0,n} \; d^v_0 \times d^h_0 \; F_{1,n-1} \; d^v_0 \times d^h_0 \cdots d^v_n \times d^h_n \; F_{1,n-i} \; d^v_0 \times d^h_0 \cdots d^v_n \times d^h_n \; F_{n,0}$$

(40)

The face and degeneracy maps $d_i : \left(\nabla F\right)(U) \to \left(\nabla F\right)(U)_{n-1}$, $s_i : \left(\nabla F\right)(U)_n \to \left(\nabla F\right)(U)_{n+1}$ can be written as [32]

$$d_i = (d_i^v p_0, d_i^v p_1, \ldots, d_i^v p_{n-1}, d_i^h p_{i+1}, \ldots, d_i^h p_n),$$

$$s_i = (s_i^v p_0, s_i^v p_1, \ldots, s_i^v p_{i-1}, s_i^h p_i, s_i^h p_{i+1}, \ldots, s_i^h p_n)$$

(41)

where $p_i : \nabla F_n \to F_{i,n-i}$ is the canonical projection.

To show that $X^{\Delta^1}$ is a higher groupoid, will need the following Lemma whose proof we will postpone for the moment.

**Lemma 7.5.** The inclusion

$$\Lambda^n_j \times \Delta^1 \to \Delta^n \times \Delta^1,$$

is a collapsible extension (Def. 3.5).

**Proof of Prop. 7.4.** Let $X$ be an $\infty$-groupoid in $M$. The bisimplicial sheaf $\text{Dec}(yX)$ is represented by the bisimplicial object $\text{Dec}(X)_{p,q} = X_{p+q+1}$ whose horizontal face maps $d^h_i : \text{Dec}(X)_{p,q} \to \text{Dec}(X)_{p-1,q}$ are the face maps $d_i : X_{p+q+1} \to X_{p+q}$, and whose vertical face maps $d^v_i : \text{Dec}(X)_{p,q} \to \text{Dec}(X)_{p,q-1}$ are the face maps $d_i : X_{p+q+1} \to X_{p+q}$.

Since $X$ is a higher groupoid, Lemmas 3.6 and 3.7 imply that the face maps of $X$ are covers. Hence, for each $n \geq 0$, the iterated pullback in $M$

$$X_n^{\Delta^1} := \text{Dec}(X)_{0,n} \; d^v_0 \times d^h_0 \; \text{Dec}(X)_{1,n-1} \; d^v_0 \times d^h_0 \cdots d^v_n \times d^h_n \; \text{Dec}(X)_{1,n-i} \; d^v_0 \times d^h_0 \cdots d^v_n \times d^h_n \; \text{Dec}(X)_{n,0}$$

exists in $M$. Hence, $X^{\Delta^1}$ is a simplicial object in $M$ whose face and degeneracy maps are given by formulas analogous to (41). Proposition 7.2 combined with Eq. (40) imply that $X^{\Delta^1}$ represents the simplicial sheaf $(yX)^{\Delta^1}$.

Next we prove that $X^{\Delta^1}$ is an $\infty$-groupoid in $M$. By definition, we need to show the condition $\text{Kan}(n,j)$ holds, i.e., $\text{Hom}(\Lambda^n_j, X^{\Delta^1})$ is representable and the map induced by the inclusion

$$\text{Hom}(\Delta^n, X^{\Delta^1}) \to \text{Hom}(\Lambda^n_j, X^{\Delta^1})$$

is a cover, for all $n \geq 1$ and $0 \leq j \leq n$. Observe that Prop. 7.2 and Prop. 4.5 imply that we have natural isomorphisms of sheaves:

(42) $\text{Hom}(\Delta^n, X^{\Delta^1}) \cong \text{Hom}(\Delta^n \times \Delta^1, X)$, \hspace{0.5cm} $\text{Hom}(\Lambda^n_j, X^{\Delta^1}) \cong \text{Hom}(\Lambda^n_j \times \Delta^1, X)$.

We proceed by induction. For the $n = 1$ case, we have

$$\text{Hom}(\Delta^1 \times \Delta^1, X) \to \text{Hom}(\Lambda^1_j \times \Delta^1, X) \cong \text{Hom}(\Delta^1, X) \cong X_1.$$

Since $X$ is a higher groupoid, Lemma 7.5 combined with Lemma 3.7 imply that $\text{Kan}(1,j)$ is satisfied.
Now assume \( n > 1 \) and that Kan\((m,j)\) holds for all \( m < n \) and \( 0 \leq j \leq m \). This plus the fact that \( \Lambda^n_j \) is a collapsible subset of \( \Delta^n \), allows us to apply Lemma 3.9 to conclude that \( \text{Hom}(\Lambda^n_j, X^{\Delta^1}) \) is representable. Hence, Lemma 7.5 again combined with Lemma 3.7 imply that Kan\((n,j)\) is satisfied. This completes the proof. \( \square \)

It remains to prove Lemma 7.5. We start with the following auxiliary Lemma:

**Lemma 7.6.** Let \( f: \Lambda^1_i \to \Delta^1 \), for either \( i = 0 \) or \( i = 1 \) be the usual inclusion. For every boundary extension \( \iota: S \to T \) the induced map

\[
(S \times \Delta^1) \sqcup_{S \times \Lambda^1_i} (T \times \Lambda^1_i) \to T \times \Delta^1,
\]

is a collapsible extension.

**Proof.** The case when \( \iota: S \to T \) is the standard inclusion \( \partial \Delta^n \to \Delta^n \) is proven in [18, Lemma 3.3.3]. We next observe the following following fact: If \( F: s\text{Set} \times s\text{Set} \to s\text{Set} \) is a co-continuous functor and

\[
\begin{array}{c}
A \\
\downarrow \\
S
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow \\
T
\end{array}
\]

is a pushout square of simplicial sets, then the diagram

\[
\begin{array}{c}
F(A, \Delta^1) \sqcup_{F(A, \Lambda^1_i)} F(B, \Lambda^1_i) \\
\downarrow \\
F(S, \Delta^1) \sqcup_{F(S, \Lambda^1_i)} F(T, \Lambda^1_i)
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow \\
F(B, \Delta^1) \\
\longrightarrow \\
F(T, \Delta^1)
\end{array}
\]

is again a pushout square. (See [24, Lemma 2.42], where \( F \) is taken to be the join functor, and observe that only the co-continuity of the join is used in the proof.)

Hence, to prove the statement, we take \( F \) above to be the product functor and proceed by induction. \( \square \)

Since any collapsible extension is a boundary extension, we have the following corollary:

**Corollary 7.7.** Let \( f: \Lambda^1_i \to \Delta^1 \), for either \( i = 0 \) or \( i = 1 \) be the usual inclusion. For every collapsible extension \( \iota: S \to T \) the induced map

\[
(S \times \Delta^1) \sqcup_{S \times \Lambda^1_i} (T \times \Lambda^1_i) \to T \times \Delta^1,
\]

is a collapsible extension.

**Proof of Lemma 7.5.** We observe that the morphism \( \Lambda^n_j \times \Delta^1 \to \Delta^n \times \Delta^1 \) is a composition of the following maps

\[
\Lambda^n_j \times \Delta^1 \to (\Lambda^n_j \times \Delta^1) \sqcup_{\Lambda^n_j \times \Lambda^1_i} (\Delta^n \times \Lambda^1_i) \to \Delta^n \times \Delta^1.
\]

The first map is clearly a collapsible extension (note that \( \Lambda^1_i = \Delta^0 \)). The second map is also a collapsible extension by Cor. 7.7. \( \square \)
Remark 7.8. If $X$ in Prop. 7.4 is actually a $k$-groupoid for $k < \infty$, then the above proof can be refined to show that $X^\Delta^1$ is also an $k$-groupoid. We just need to verify that the cover

$$\text{Hom}(\Delta^n \times \Delta^1, X) \to \text{Hom}(\Lambda^n_\partial \times \Delta^1, X)$$

is an isomorphism for $n > k$. By Remark 3.8, this will be true provided we can show that the collapsible extension

$$\Lambda^n_\partial \times \Delta^1 \to \Delta^n \times \Delta^1$$

is the pushout of maps of the form $\Lambda^m_\partial \to \Delta^m$ with $m > k$. Indeed, this is the case. The collapsible extension $\Lambda^n_\partial \to \Delta^n$ is the composition of two boundary extensions $\Lambda^n_\partial \to \partial \Delta^n \to \Delta^n$. We obtain $\partial \Delta^n$ from $\Lambda^n_\partial$ by attaching $\Delta^{n-1}$ along $\partial \Delta^{n-1} \to \Lambda^n_\partial$.

Therefore, the collapsible extension obtained from $S = \Lambda^n_\partial \to \Delta^n = T$ from the auxiliary Lemma 7.6 above involves pushouts of maps of the form $\Lambda^m_\partial \to \Delta^m$ and $\Lambda^{n+1}_\partial \to \Delta^{n+1}$. (See the proof of Lemma 3.3.3 in [18].) From the proof of Lemma 7.5, we can then conclude that the collapsible extension (43) only involves pushouts along inclusions of horns which have dimension $> k$.

7.2. The iCFO structure. We arrive at our first main result:

**Theorem 7.1.** Let $(\mathcal{M}, \mathcal{T}, \mathcal{P})$ be a category equipped with a locally stalkwise pretopology with respect to a jointly conservative collection of points. The category $\text{Gpd}_\infty(\mathcal{M}, \mathcal{T})$, whose objects are $\infty$-groupoids in $(\mathcal{M}, \mathcal{T})$, and whose morphisms are simplicial maps is an incomplete category of fibrant objects in which:

- the weak equivalences are the stalkwise weak equivalences (Def. 5.1),
- the fibrations are the Kan fibrations (Def. 3.3),
- the acyclic fibrations are hypercovers (Def. 6.1).

In particular, the category of Lie $\infty$-groupoids

$$\text{Lie}_\infty \text{Gpd} := \text{Gpd}_\infty(\text{Mfd}, \mathcal{T}_{\text{subm}}, \mathcal{P}_{\text{Ban}})$$

is an incomplete category of fibrant objects in this way.

Let us make a number of important remarks before we proceed to the proof.

**Remark 7.9.**

1. The acyclic fibrations are obviously determined by the weak equivalences and fibrations, i.e., those Kan fibrations that are also stalkwise weak equivalences. Since we are working with a locally stalkwise pretopology, it follows from Prop. 6.7 that acyclic fibrations are precisely the hypercovers.

2. In particular, if $\mathcal{M}$ is small and has all finite limits, then $(\mathcal{M}, \mathcal{T})$ is a “descent category” in the sense of Behrend–Getzler. (See proof of Cor. 7.18 for more details.) In this case, the iCFO structure on $\text{Gpd}_\infty(\mathcal{M}, \mathcal{T})$ agrees with the CFO structure of Thm. 3.6 in [4].

3. In the proof of Thm 7.1, we use the assumption that $\mathcal{T}$ is a locally stalkwise pretopology only in the proof of Prop. 7.14, where we show that pullbacks of acyclic fibrations always exist.

4. The fact that the category of Lie $\infty$-groupoids is an example of such an iCFO follows immediately from Prop. 6.12.

5. The proof below of Thm. 7.1 can be enhanced to show that the category $\text{Gpd}_n(\mathcal{M}, \mathcal{T})$ of $n$-groupoids in $(\mathcal{M}, \mathcal{T})$ for $n < \infty$ also forms an iCFO. It
follows from Remark 7.8 that the path object $X^{\Delta^1}$ associated to an $n$-groupoid $X$ will also be an $n$-groupoid. The only other modifications needed are in the proof of Prop. 7.12 below. (See Remark 7.13.)

The following collection of propositions proves Thm. 7.1 by directly verifying the axioms of Def. 2.1. We begin with the easiest axioms to verify:

**Proposition 7.10** (Axioms 1, 2, 7).

1. Every isomorphism in $\text{Gpd}_\infty[M, T]$ is a stalkwise weak equivalence and a Kan fibration.
2. If $f$ and $g$ are composable morphisms in $\text{Gpd}_\infty[M, T]$, and any two of $f$, $g$, or $g \circ f$ are stalkwise weak equivalences, then so is the third.
3. If $X$ is an $\infty$-groupoid in $(M, T)$, then $X \to \ast$ is a Kan fibration.

**Proof.** (1) Obvious. (2) Follows from the fact that weak equivalences of simplicial sets satisfy the analogous 2 out of 3 axiom. (3) Follows immediately from Def. 3.4. □

This next proposition implies that the composition of two Kan fibrations is again a Kan fibration. (The same result for Lie $n$-groupoids appears as [16, Lemma 2.7]).

**Proposition 7.11** (Axiom 3). Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of simplicial objects in $M$. If $f$ and $g$ satisfy $\text{Kan}(n, j)$ and the sheaf $\text{Hom}(\Lambda^n_j \to \Delta^n, X \xrightarrow{g \circ f} Z)$ is representable, then $g \circ f$ also satisfies $\text{Kan}(n, j)$. Similarly, if $f$ and $g$ satisfy $\text{Kan}!(n, j)$, then $g \circ f$ also satisfies $\text{Kan}!(n, j)$.

**Proof.** We have the following diagram containing three pullback squares:

\[
\begin{array}{ccc}
\text{Hom}(\Delta^n, X) & \xrightarrow{(\iota^*, f_*)} & \text{Hom}(\Lambda^n_j \to \Delta^n, X \xrightarrow{f} Y) \\
\downarrow^{\text{pr}} & & \downarrow^{f_*} \\
\text{Hom}(\Delta^n, Y) & \xrightarrow{(\iota^*, g_*)} & \text{Hom}(\Lambda^n_j \to \Delta^n, Y \xrightarrow{g} Z) \\
\downarrow^{\text{pr}} & & \downarrow^{g_*} \\
\text{Hom}(\Delta^n, Z) & \xrightarrow{\iota^*} & \text{Hom}(\Lambda^n_j, Z)
\end{array}
\]

If $g$ satisfies $\text{Kan}(n, j)$, then $\text{Hom}(\Delta^n, Y) \xrightarrow{(\iota^*, g_*)} \text{Hom}(\Lambda^n_j \to \Delta^n, Y \xrightarrow{g} Z)$ is a cover. Hence, $\text{Hom}(\Lambda^n_j \to \Delta^n, X \xrightarrow{f} Y) \xrightarrow{f_*} \text{Hom}(\Lambda^n_j \to \Delta^n, X \xrightarrow{g \circ f} Z)$ is a cover. If $f$ satisfies $\text{Kan}(n, j)$, then $\text{Hom}(\Delta^n, X) \xrightarrow{(\iota^*, f_*)} \text{Hom}(\Lambda^n_j \to \Delta^n, X \xrightarrow{f} Y)$ is a cover. Hence, the composition

$g_* \circ (\iota^*, f_*) = (\iota^*, (g \circ f)_*) : \text{Hom}(\Delta^n, X) \to \text{Hom}(\Lambda^n_j \to \Delta^n, X \xrightarrow{g \circ f} Z)$

is a cover, and so $g \circ f$ satisfies $\text{Kan}(n, j)$. The same argument *mutatis mutandis* shows that if $f$ and $g$ satisfy $\text{Kan}!(n, j)$, then so does $g \circ f$. □

We next show that the pullback of a Kan fibration is a Kan fibration, provided the pullback exists in $\text{Gpd}_\infty[M, T]$. As previously mentioned, this is the only difference between the axioms for an iCFO and Brown’s axioms for a CFO: we do not require
the pullback along a fibration to exist in general. It turns out, for \( \infty \)-groupoids, this generalization is in fact quite mild. The pullback of along a Kan fibration will always exist provided the corresponding pullback of the 0-simplices exists in \( M \).

(The same result appears as Thm. 2.17d in [37], where Kan fibrations are called "\( n \)-stacks").

**Proposition 7.12** (Axiom 4). Let \( f : X \to Y \) be a Kan fibration in \( \text{Gpd}_\infty[M, T] \) and \( g : Z \to Y \) a morphism in \( \text{Gpd}_\infty[M, T] \). If the pullback \( Z_0 \times_{Y_0} X_0 \) exists in \( M \), then:

1. the pullback \( Z_n \times_{Y_n} X_n \) exists in \( M \) for all \( n \geq 0 \),
2. the morphism \( Z \times_Y X \xrightarrow{p_f} Z \) induced by pulling back \( f \) along \( g \) is a Kan fibration between simplicial objects in \( M \),
3. the pullback \( Z \times_Y X \) is an object of \( \text{Gpd}_\infty[M, T] \).

**Proof.** For convenience, we use the following notation below: If \( K \) is a simplicial set and \( W \) is a simplicial object in \( M \), then \( K(W) \) is the sheaf Hom(\( K, W \)). Also, we denote by Hom(\( \iota, pf \)) the sheaf Hom(\( \Lambda^n_j \to \Delta^n, Z \times_Y X \xrightarrow{pf} Z \)). Finally, we do not distinguish between a simplicial object \( W \) in \( M \), and the representable sheaf \( yW \).

We shall prove statements (1) and (2) simultaneously: For all \( n \geq 0 \), and \( 0 \leq j \leq n \), we wish to show that morphism of sheaves

\[
Z_n \times_{Y_n} X_n \xrightarrow{(\iota^*, pf^*)} \text{Hom}(\iota, pf)
\]

is represented by a cover in \( M \). It follows from the definition Def. 12, that for a fixed simplicial set \( K \), the functor Hom(\( K, - \)) : sM \to Sh(M) preserves limits. Therefore, we have an isomorphism of sheaves

\[
\text{Hom}(\iota, pf) := \Lambda^n_j (Z \times_Y X) \times_{\Lambda^n_j(Z)} Z_n \\
\cong (\Lambda^n_j(Z) \times_{\Lambda^n_j(Y)} \Lambda^n_j(X)) \times_{\Lambda^n_j(Z)} Z_n,
\]

and a commuting diagram of pullback squares

\[
\begin{array}{ccc}
\text{Hom}(\iota, pf) & \xrightarrow{pr_1} & Z_n \\
\downarrow^{pr_2} & & \downarrow^{\iota^*} \\
\Lambda^n_j(Z) \times_{\Lambda^n_j(Y)} \Lambda^n_j(X) & \xrightarrow{pf^*} & \Lambda^n_j(Z) \\
\downarrow^{pr_3} & & \downarrow^{g^*} \\
\Lambda^n_j(X) & \xrightarrow{f^*} & \Lambda^n_j(Y)
\end{array}
\]

Hence, the pasting law for pullbacks gives an isomorphism of sheaves

\[
\text{Hom}(\iota, pf) \cong \Lambda^n_j(X) \times_{\Lambda^n_j(Y)} Z_n.
\]
The above isomorphism gives another commuting diagram of pullback squares, which via the universal property, contains the morphism (44):

$$(48)$$

![Diagram](attachment:diagram.png)

Note that since $f: X \to Y$ is a Kan fibration, the morphism $X_n \xrightarrow{(\iota^*, f_*)} Y_n \times_{\Lambda^n_j(Y)} \Lambda^n_j(X)$ is represented by a cover. Hence, to show that the morphism (44) is represented by a cover, the above diagram implies that it is sufficient to show that the sheaf

$$(49)$$

is representable for all $n \geq 1$ and $0 \leq j \leq n$.

First consider the $n = 1$ case. Diagram (46) and the isomorphism (47) imply that we have the pullback diagram

$$(50)$$

The sheaf $Z_0 \times_{Y_0} X_0$ is representable by hypothesis, and since $Z$ is an $\infty$-groupoid, $Z_1 \xrightarrow{d_j} Z_0$ is a cover. Therefore, for $n = 1$ and $j = 0, 1$, the sheaf (49) is representable, $Z_1 \times_{Y_1} X_1$ is representable, and $p_f: Z \times_Y X \to Z$ satisfies Kan($1, j$).

Now suppose $p_f$ satisfies Kan($m, j$) for all $1 \leq m < n$ and $0 \leq j \leq m$. This plus the fact that $\Lambda^n_j$ is a collapsible subset of $\Delta^n$, allows us to apply Lemma 3.9 to conclude that $\text{Hom}(\iota_{n,j}, p_f) \cong \Lambda^n_j(X) \times_{\Lambda^n_j(Y)} Z_n$. is representable. Hence, $p_f: Z \times_Y X \to Z$ satisfies Kan($n, j$), and so it is a Kan fibration.

Statement (3) immediately follows. Indeed, since $Z$ is an $\infty$-groupoid, $Z \to \ast$ is a Kan fibration. Therefore, Prop. 7.11 implies that $Z \times_Y X \to \ast$ is also a Kan fibration.

**Remark 7.13.** If $X$, $Y$, and $Z$ in the statement of Prop. 7.12 are $k$-groupoids, $k < \infty$, then one can show that the pullback $Z \times_Y X$, if it exists, is also a $k$-groupoid. As previously mentioned in Remark 7.9, this fact helps show that the category $\text{Gpd}_k[M, T]$ forms an iCFO. We just need to verify that the morphism

$$(51)$$

is an isomorphism for $n > k$. Since $Z$ is a $k$-groupoid, the pullback diagram (46) implies that

$$(52)$$
is an isomorphism. Since \( f: X \to Y \) is a Kan fibration between \( k \)-groupoids, it is not hard to show that the morphism

\[
X_n \xrightarrow{\left( \iota^* f_\ast, \right)} Y_n \times \Lambda^n_\partial Y \Lambda^n_\partial X
\]

is an isomorphism for all \( n > k \). Hence, the pullback diagram (48) implies that

\[
Z_n \times Y_n \xrightarrow{\left( \iota^*_n, f_\ast \right)} X_n \times \Lambda^n_\partial X
\]

is an isomorphism. Composing this with the isomorphism (52), we conclude that (51) is an isomorphism.

Proposition 7.12 also makes it easy to show that the pullbacks of acyclic fibrations always exist in \( \text{Gpd}_\infty[M, T] \), and are always acyclic fibrations. (Since acyclic fibrations turn out to be equivalent to hypercovers, this result is equivalent to [38, Lemma 2.8].)

**Proposition 7.14** (Axiom 5). Let \( f: X \to Y \) be an acyclic fibration in \( \text{Gpd}_\infty[M, T] \) and \( g: Z \to Y \) a morphism in \( \text{Gpd}_\infty[M, T] \). Then the morphism \( Z \times_Y X \xrightarrow{q_f} Z \) induced by pulling back \( f \) along \( g \) is an acyclic fibration.

**Proof.** Since \( f: X \to Y \) is an acyclic fibration, Prop. 6.7 implies that \( f \) is a hypercover. Then, by definition, \( f_0: X_0 \to Y_0 \) is a cover, and hence the pullback \( Z_0 \times Y_0 X_0 \) exists in \( M \). Proposition 7.12 therefore implies that the morphism \( Z \times_Y X \xrightarrow{q_f} Z \) is a Kan fibration in \( \text{Gpd}_\infty[M, T] \).

Let \( p \) be a point. Then \( p f: p X \to p Y \) is an acyclic fibration of simplicial sets. By definition, points preserve finite limits. Hence, \( p(Z \times_Y X) \xrightarrow{p q_f} p Z \) is the pullback of \( p f \), and is therefore a weak equivalence of simplicial sets. So we conclude \( q_f \) is a stalkwise weak equivalence, hence an acyclic fibration. \( \square \)

Finally, we show that if \( X \in \text{Gpd}_\infty[M, T] \), then the \( \infty \)-groupoid \( X^{\Delta^1} \) constructed in Prop. 7.4 is a path object for \( X \). Let

\[
X^{\Delta^1} \xrightarrow{(d^{0*}, d^{1*})} X \times X \cong X^{\Delta^0 \cup \Delta^0},
\]

and

\[
s^{0*}: X \to X^{\Delta^1},
\]

denote the morphisms (32) and (33), respectively, induced by the inclusions \( \partial \Delta^1 \cong \Delta^0 \cup \Delta^0 \xrightarrow{d^{0*}} \Delta^1 \) and the constant map \( s^0: \Delta^1 \to \Delta^0 \), respectively.

**Proposition 7.15** (Axiom 6). Let \( X \in \text{Gpd}_\infty[M, T] \). Then

\[
X \xrightarrow{s^{0*}} X^{\Delta^1} \xrightarrow{(d^{0*}, d^{1*})} X \times X.
\]

is a factorization of the diagonal map \( X \times X \) into a stalkwise weak equivalence \( s_0^*: X \to X^{\Delta^1} \) followed by a Kan fibration \( X^{\Delta^1} \xrightarrow{(d^{0*}, d^{1*})} X \times X \).

To prove the above, we'll need the following lemma, whose proof we postpone to Appendix A.

**Lemma 7.16.** The inclusion of simplicial sets

\[
\Lambda^n_\partial \times \Delta^1 \sqcup \Lambda^n_\partial \times \Delta^1 \rightarrow \Delta^n \times \Delta^1
\]

is a collapsible extension.
Proof of Prop. 7.14. First, observe that \( d^* \circ s^* = \text{id}_X \). This follows from fact that the assignment \( K \mapsto (yX)^K \), where \((yX)^K\) is the simplicial sheaf (31), is a contravariant functor from simplicial sets to simplicial sheaves. Hence, (53) is a factorization of the diagonal map.

Next, since we have an isomorphism of sheaves \( y(X^1) \cong (yX)^1 \), Prop. 7.1 implies that \( s_0^* : X \to X^1 \) is a stalkwise weak equivalence.

Now we show \( f := (d^*, d^1*) : X^1 \to X \times X \) satisfies the condition Kan\((n, j)\) for all \( n \geq 1 \) and \( 0 \leq j \leq n \), i.e. the morphism of sheaves

\[
\begin{align*}
X^1_n & \xrightarrow{(\tau_n, j, f_\ast)} \text{Hom}(\tau_n, f),
\end{align*}
\]

where \( \text{Hom}(\tau_n, f) := \text{Hom}(\Lambda^n_\ast \to \Delta^n, X^1 \to X \times X) \) is represented by a cover. It follows from Prop. 4.5 that for any finitely generated simplicial sets \( K \) and \( L \), we have an isomorphism of sheaves

\[
\text{Hom}(L, X^K_\ast) \cong \text{Hom}(L \times K, X).
\]

Therefore, we have the following isomorphisms of sheaves

\[
\begin{align*}
\text{Hom}(\tau_n, f) & \cong \text{Hom}(\Lambda^n_\ast \times \partial \Delta^1, X) \times \text{Hom}(\Lambda^n \times \partial \Delta^1, X) \text{Hom}(\Lambda^n_\ast \times \Delta^1, X) \\
& \cong \text{Hom}(\Lambda^n_\ast \times \Delta^1 \cup \Lambda^n_\ast \times \partial \Delta^1, \Delta^n \times \partial \Delta^1, X)
\end{align*}
\]

Hence, showing that \( f \) satisfies Kan\((n, j)\) is equivalent to showing that the morphism of sheaves

\[
\text{Hom}(\Delta^n \times \Delta^1, X) \xrightarrow{(\tau_n, j, f_\ast)} \text{Hom}(\Lambda^n_\ast \times \Delta^1 \cup \Lambda^n_\ast \times \partial \Delta^1, \Delta^n \times \partial \Delta^1, X)
\]

is a cover. Lemma 7.16 implies that the inclusion \( \Lambda^n_\ast \times \Delta^1 \cup \Lambda^n_\ast \times \partial \Delta^1 \hookrightarrow \Delta^n \times \Delta^1 \) is a collapsible extension. Hence, Lemma 3.7 implies that in order to show (55) is a cover, it suffices to show that

\[
\text{Hom}(\tau_n, f) \cong \text{Hom}(\Lambda^n_\ast \times \Delta^1 \cup \Lambda^n_\ast \times \partial \Delta^1, \Delta^n \times \partial \Delta^1, X)
\]

is representable for all \( n \) and \( j \).

Consider the \( n = 1 \) case. Then we have the pullback square

\[
\begin{array}{ccc}
\text{Hom}(\tau_1, f) & \to & X_1 \times X_1 \\
\downarrow & & \downarrow (d_0, d_1) \\
X_1 & \to & X_0 \times X_0
\end{array}
\]

Since \( X \times X \) is an \( \infty \)-groupoid, the map \( X_1 \times X_1 \xrightarrow{(d_0, d_1)} X_0 \times X_0 \) is a cover. Hence, the pullback \( \text{Hom}(\tau_1, f) \) exists and so \( f \) satisfies Kan\((1, j)\). Now if \( n > 1 \) and \( f \) satisfies Kan\((m, j)\) for all \( m < n \) and \( 1 \leq j \leq m \), then Lemma 3.9 implies that \( \text{Hom}(\tau_n, f) \) is representable. \( \square \)

7.2.1. Simplicial localization for \( \text{Gpd}_\infty [M, T] \). We note that the path object \( X^1 \) used in the proof of Thm. 7.1 is functorial, in the sense of Sec. 2.2. This can be easily deduced from the fact that \( X^1 \) represents the sheaf \((yX)^1\) (Prop. 7.4). The iCFO structure on \( \text{Lie}_\infty \text{Gpd} \) in particular is equipped with both functorial path objects, as well as functorial pullbacks of acyclic fibrations. Indeed, the pullbacks in this case are characterized in each simplicial dimension by the unique Banach manifold structure on the set-theoretic fiber product. (See, for example, Prop.
2.5 and Prop. 2.6 of [23].) Hence, for a small full subcategory of Lie \( n \)-groupoids closed under the iCFO structure, Thm. 2.9 would provide a convenient description of its simplicial localization. A potentially useful example of this sort, which will be studied in future work, is the category of \( n \)-groupoids internal to the category of separable Banach manifolds\(^1\).

### 7.3. Alternative characterization of weak equivalences

The incorporation of stalkwise weak equivalences into our iCFO structure for \( \text{Gpd}_\infty[M, T] \) turns out to be quite convenient for some applications. For example, as we will see in Section 8, the integration of an \( L_\infty \)-quasi-isomorphism is a stalkwise weak equivalence of Lie \( \infty \)-groups.

However, in general, verifying directly that a morphism is a stalkwise weak equivalence could be cumbersome. Furthermore, we have the aesthetically inelegant fact that the stalkwise weak equivalences are the only piece of the iCFO structure on \( \text{Gpd}_\infty[M, T] \) which requires us to leave the realm of simplicial objects in \( M \) for the larger world of simplicial sheaves on \( M \).

Fortunately, as was mentioned in Sec. 2, the weak equivalences in an iCFO are completely determined by the acyclic fibrations. This very useful fact is emphasized in the work of Behrend and Getzler [4] on CFOs for higher geometric groupoids in descent categories. What this implies in particular for the iCFO structure on \( \text{Gpd}_\infty[M, T] \), is the following: If \( f : X \to Y \) is a morphism in \( \text{Gpd}_\infty[M, T] \), we consider the pullback diagram

\[
\begin{array}{ccc}
X \times_Y Y & \to & Y^\Delta^1 \\
\downarrow & & \downarrow d_0 \\
X & \to & Y
\end{array}
\]

Then it follows from Lemma 2.4 and Prop. 2.5 (and Thm. 7.1) that \( f : X \to Y \) is a stalkwise weak equivalence if and only if the composition

\[
pf : X \times_Y Y^\Delta^1 \to Y^\Delta^1 \to Y
\]

is a hypercover. Moreover, the path object construction can be avoided altogether, and weak equivalences can be characterized directly in terms of \( f \) and covers between representable sheaves.

To give just a simple example, denote by \( \iota_a : \Delta^n \to \Delta^1 \times \Delta^n \) for \( a = 0, 1 \) the inclusions \( m \mapsto (a, m) \). Similarly, there are the inclusions \( \partial \iota_a : \partial \Delta^n \to \Delta^1 \times \partial \Delta^n \). There is the pushout diagram

\[
\begin{array}{ccc}
\Lambda^1_1 \times \partial \Delta^n & \to & \Delta^1 \times \partial \Delta^n \\
\downarrow & & \downarrow j_1 \\
\Lambda^1_1 \times \Delta^n & \to & \Delta^1 \times \partial \Delta^n \cup \Lambda^1_1 \times \Delta^n
\end{array}
\]

The following is parallel to the first step in the proof of Thm. 5.1 in [4]:

\(^1\)We thank E. Getzler for this observation.
Proposition 7.17. A morphism \( f: X \to Y \) in \( \text{Gpd}_\infty (\mathcal{M}, \mathcal{T}) \) is a stalkwise weak equivalence if and only if, for all \( n \geq 0 \) the morphism in \( \mathcal{M} \)

\[
\text{Hom}(\Delta^n \overset{j_1}{\to} \Delta^1 \times \Delta^n, X \overset{f}{\to} Y) \to \text{Hom}(\partial \Delta^n \overset{j_2 \circ \partial_1}{\to} \Delta^1 \times \partial \Delta^n \cup \Lambda^1_1 \times \Delta^n, X \overset{f}{\to} Y)
\]

is a cover.

Proof. First, it follows from Lemma 2.4 in [38] that the sheaf

\[
\text{Hom}(j_1, p_f) := \text{Hom}(\partial \Delta^n \overset{j_1}{\to} \Delta^n, X \times Y \Delta^1 \overset{p_f}{\to} Y)
\]

is representable. From the discussion preceding the proposition, we know \( f: X \to Y \) is a weak equivalence if and only if for all \( n \geq 0 \), the morphism in \( \mathcal{M} \)

\[
X_n \times_{Y_n} \Delta^1 \to \text{Hom}(j_1, p_f)
\]

is a cover. Since \( Y^\Delta \cong \text{Hom}(\Delta^1 \times \Delta^n, Y) \), there is the pullback square

\[
\begin{array}{ccc}
X_n \times_{Y_n} \Delta^1 & \to & \text{Hom}(\Delta^1 \times \Delta^n, Y) \\
\downarrow & & \downarrow \\
\text{Hom}(\Delta^n, X) & \overset{f_\ast}{\to} & \text{Hom}(\Delta^n, Y)
\end{array}
\]

Hence,

\[
X_n \times_{Y_n} \Delta^1 \cong \text{Hom}(\Delta^n \overset{j_1}{\to} \Delta^1 \times \Delta^n, X \overset{f}{\to} Y)
\]

To complete the proof, we just need to show

\[
\text{Hom}(j_1, p_f) \cong \text{Hom}(\partial \Delta^n \overset{j_1 \circ \partial_1}{\to} \Delta^1 \times \partial \Delta^n \cup \Lambda^1_1 \times \Delta^n, X \overset{f}{\to} Y)
\]

This follows from pasting together the following pullback squares:

\[
\begin{array}{ccc}
\text{Hom}(j_1, p_f) & \to & \text{Hom}(\Delta^1 \times \partial \Delta^n \cup \Lambda^1_1 \times \Delta^n, Y) \\
\downarrow & & \downarrow \\
\text{Hom}(\partial \Delta^n, X) \times_{Y \Delta^1} \text{Hom}(\Delta^1 \times \partial \Delta^n, Y) & \overset{pr_2}{\to} & \text{Hom}(\Lambda^1_1 \times \partial \Delta^n, Y) \\
\downarrow & & \downarrow \\
\text{Hom}(\partial \Delta^n, X) & \overset{f_\ast}{\to} & \text{Hom}(\partial \Delta^n, Y)
\end{array}
\]

Theorem 5.1 in [4] further shows that if the category of \( n \)-groupoids in \( (\mathcal{M}, \mathcal{T}) \) form a category of fibrant objects, then \( f: X \to Y \) is a weak equivalence if and only if the morphism

\[
\text{Hom}(\Delta^n \to \Delta^{n+1}, X \to Y) \to \text{Hom}(\partial \Delta^n \to \Lambda^{n+1}_n + 1, X \to Y)
\]

is a cover for \( n \geq 0 \). This turns out to be true in our iCFO case as well.

Corollary 7.18. A morphism \( f: X \to Y \) in \( \text{Lie}_\infty \text{Gpd} \) is a stalkwise weak equivalence if and only if the natural morphism (61) is a cover for \( n \geq 0 \).
Proof. By Proposition 7.17, we see that \( f : X \to Y \) in \( \text{Lie}_\infty \text{Gpd} \) is a stalkwise weak equivalence if and only if the morphism (56) is a cover for all \( n \geq 0 \). The morphisms (56) and (61) are exactly the morphisms (5.2) and (5.1), respectively, in [4]. The sources and targets for the morphisms in [4] are \( n \)-groupoids in a descent category of spaces: a small category with finite limits, equipped with a subcategory of covers closed under pullback, which satisfy a “2 of 3” property. (Axioms D1, D2, and D3, respectively in [4].) A category equipped with a locally stalkwise pretopology satisfies all of these axioms, except D1. Indeed, D2 is included in the definition of a pretopology, and D3 follows from Def. 6.5. Even though D1 is not satisfied in this context, the proof of Thm. 5.1 in [4] still applies. A direct verification shows that all limits appearing in the proof exist in \((M, T)\). And clearly, the proof works for \( n = \infty \). □

Remark 7.19. We also mention that a characterization of weak equivalences similar to (61) between Lie 2-groupoids can be deduced using properties of the join construction of simplicial sets and the theory of Morita bibundles developed in Li’s Ph.D. thesis [24]. This fact is generalized to all Lie \( n \)-groupoids in [5], which provides another interpretation of the combinatoric formula (61).

8. Integration of Lie \( n \)-algebras

In this section, we exhibit a relationship between the iCFO structure on \( \text{Lie}_\infty \text{Gpd} \) and the homotopy theory of Lie \( n \)-algebras by showing that Henriques’ integration functor sends \( L_\infty \)-quasi-isomorphisms to weak equivalences of Lie \( n \)-groups.

8.1. Lie \( n \)-algebras. A non-negatively graded \( L_\infty \)-algebra [22, Def. 2.1] is a graded real vector space \( L = L_0 \oplus L_1 \oplus \cdots \) equipped with a collection of graded skew-symmetric \( k \)-ary brackets of degree \( k - 2 \)

\[
l_k : \Lambda^k L \to L \quad 1 \leq k < \infty,
\]

satisfying an infinite sequence of Jacobi-like identities of the form:

\[
\sum_{\sigma \in \text{Sh}(i, m - i)} (-1)^\epsilon(\sigma)(-1)^{i(j-1)}l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(m)}) = 0.
\]

for all \( m \geq 1 \). Above, the permutation \( \sigma \) ranges over all \((i, m - i)\) unshuffles, and \( \epsilon(\sigma) \) denotes the Koszul sign. (See Sec. 2 in [22].) In particular, Eq. 63 implies that \( l_1 \) is a degree \(-1\) differential on \( L \).

If \( L \) is concentrated in the first \( n - 1 \) degrees (\( L_i = 0 \) for \( i \geq n \)), then \( L \) is called a Lie \( n \)-algebra [7, Def. 4.3.2]. If each \( L_i \) is finite-dimensional, we say \( L \) is finite type. In this case, one can associate to \( L \) its Chevalley–Eilenberg algebra \( C^*(L) \), a commutative differential graded algebra (cdga), in the following way. Let \( L^\vee \) denote the non-negatively graded vector space \( L^\vee : = \text{hom}_R(L_i, \mathbb{R}) \), and define \( L^\vee[1] := L^\vee[-1] \). As a commutative graded algebra, \( C^*(L) := S(L^\vee[1]) \) is the free commutative graded algebra generated by the graded vector space \( L^\vee[1] \). The brackets (62) induce a degree +1 derivation \( \delta : C^*(L) \to C^{*+1}(L) \), and the higher Jacobi-like identities (63) imply that \( \delta^2 = 0 \).
8.2. Morphisms of Lie $n$-algebras. A morphism of $L_{\infty}$-algebras

\[ \phi = \{ \phi_k \geq 1 \}: (L, l_k) \to (L', l'_k) \]

is a collection of graded skew-symmetric $k$-linear maps of degree $k - 1$

\[ \phi_k: \Lambda^k L \to L' \quad 1 \leq k < \infty \]

satisfying an infinite sequence of equations of the form:

\[
\sum_{j+k=m+1} \sum_{\sigma} \pm \phi_j(l_k(x_{\sigma}(1), \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots x_{\sigma(m)}) \\
+ \sum_{i_1 + \cdots + i_t = m} \sum_{\tau} \pm l'_1(\phi_{i_1}(x_{\tau(1)}, \ldots, x_{\tau(i_1)}), \phi_{i_2}(x_{\tau(i_1+1)}, \ldots, x_{\tau(i_1+i_2)}), \\
\ldots, \phi_{i_t}(x_{\tau(i_1+\cdots+i_t+1)}, \ldots, x_{\tau(m)})) = 0.
\]

Above $\sigma$ ranges over all $(k, m-k)$ unshuffles, and $\tau$ ranges through certain $(i_1, \ldots, i_t)$ unshuffles. (See [1, Def. 2.3] for further details and the precise signs.) If $\phi_k = 0$ for all $k \geq 2$, then we say $\phi = \phi_1$ is a strict $L_{\infty}$-morphism.

For any $L_{\infty}$-morphism, the equations (66) imply that the degree 0 map $\phi_1$ is a chain map:

\[ \phi_1: (L, l_1) \to (L', l'_1). \]

Furthermore, if $x, y \in L$, then the equations (66) also imply that

\[ \phi_1(l_2(x, y)) - l'_2(\phi_1(x), \phi_1(y)) = l'_1(\phi_2(x, y)) \]

Remark 8.1. The bilinear bracket $l_2$ on a $L_{\infty}$-algebra $L$ induces a Lie algebra structure on $H_0(L)$. It follows from Eq. (68) that if $\phi: L \to L'$ is a $L_{\infty}$-morphism, then the induced map $H_0(\phi_1): H_0(L) \to H_0(L')$ is a morphism of Lie algebras.

Definition 8.2. A morphism of $L_{\infty}$-algebras $\phi: (L, l_k) \to (L', l'_k)$ is a weak equivalence or $L_{\infty}$-quasi-isomorphism iff the chain map $\phi_1: (L, l_1) \to (L', l'_1)$ is a quasi-isomorphism, i.e. the induced map on homology

\[ H_*(\phi_1): H_*(L) \to H_*(L') \]

is an isomorphism.

Remark 8.3. There is a notion of homotopy between two $L_{\infty}$-morphisms which can be expressed (equivalently) either in terms of model categories [34, Def. 3.4] or simplicial categories [9, Def. 4.7]. Hence, we have a notion of homotopy equivalent Lie $n$-algebra. It is a fact that $L_{\infty}$-quasi-isomorphisms are invertible up to homotopy, and hence if $L \xrightarrow{\sim} L'$ is a weak equivalence, then $L$ and $L'$ are homotopy equivalent Lie $n$-algebras.

Technically, a morphism between two $L_{\infty}$-algebras $L$ and $L'$ is defined as a dg coalgebra morphism between their corresponding Chevalley–Eilenberg coalgebras. (See Remark 5.3 in [22].) This is the origin of the equations (66). Hence, if $L$ and $L'$ are finite-type $L_{\infty}$-algebras, then a morphism between them is equivalently a morphism of edgas

\[ \phi^\vee: C^*(L') \to C^*(L). \]

between the corresponding Chevalley–Eilenberg algebras. The components (65) are recovered by first dualizing $\phi^\vee$ and then desuspending the composition of the restriction $\phi|_{S^*(L[1])}$ with the projection $S(L'[1]) \to L'[1]$. 
For a fixed \( n \in \mathbb{N} \), we denote by \( \text{Lie}_n \text{Alg}^{\text{fin}} \) the category whose objects are finite-type Lie \( n \)-algebras, and whose morphisms are \( L_\infty \)-morphisms (64), or equivalently cdga morphisms (70) between the associated Chevalley–Eilenberg algebras. We note that a morphism which induces a quasi-isomorphism between CE algebras is not necessarily a weak equivalence of Lie \( n \)-algebras.

8.3. **Postnikov tower for Lie \( n \)-algebras.** Let \( (L, l_k) \) be a Lie \( n \)-algebra. Following [16, Def. 5.6], we consider two different truncations of the underlying chain complex \( (L, d = l_1) \). For any \( m \geq 0 \), denote by \( \tau_{\leq m} L \) and \( \tau_{< m} L \) the following \((m + 1)\)-term complexes:

\[
(\tau_{\leq m} L)_i = \begin{cases} L_i & \text{if } i < m, \\
\text{coker}(d_{m+1}) & \text{if } i = m, \\
0 & \text{if } i > m,
\end{cases} \quad (\tau_{< m} L)_i = \begin{cases} L_i & \text{if } i < m, \\
\text{im}(d_m) & \text{if } i = m, \\
0 & \text{if } i > m.
\end{cases}
\]

In degree \( m \), the differentials for \( \tau_{\leq m} L \) and \( \tau_{< m} L \) are \( d_m : L_m / \text{im}(d_{m+1}) \to L_{m-1} \), and the inclusion \( \text{im}(d_m) \hookrightarrow L_{m-1} \), respectively. The homology complexes of \( \tau_{\leq m} L \) and \( \tau_{< m} L \) are

\[
H_i(\tau_{\leq m} L) = \begin{cases} H_i(L) & \text{if } i \leq m, \\
0 & \text{if } i > m,
\end{cases} \quad H_i(\tau_{< m} L) = \begin{cases} H_i(L) & \text{if } i < m, \\
0 & \text{if } i \geq m.
\end{cases}
\]

We have the following obvious surjective chain maps

\[
(71) \quad p_{\leq m} : L \to \tau_{\leq m} L \quad p_{< m} : L \to \tau_{< m} L
\]

where in degree \( m \), \( p_{\leq m} : L_m \to \text{coker}(d_{m+1}) \) is the usual surjection, and \( p_{< m} = d_m : L_m \to \text{im}(d_m) \) is the differential. There are also the similarly defined surjective chain maps

\[
(72) \quad \tau_{\leq m} L \to \tau_{< m} L, \quad \tau_{< m+1} L \to \tau_{\leq m} L.
\]

The latter of these is a quasi-isomorphism of complexes.

**Proposition 8.4.** Let \( (L, l_k) \) be a Lie \( n \)-algebra.

1. The Lie \( n \)-algebra structure on \( (L, l_k) \) induces Lie \((m + 1)\)-structures on the complexes \( \tau_{\leq m} L \) and \( \tau_{< m} L \) whose brackets are given by

\[
\tau_{\leq m} l_k(\bar{x}_1, \ldots, \bar{x}_k) := p_{\leq m} l_k(x_1, \ldots, x_k), \quad \tau_{< m} l_k(\bar{y}_1, \ldots, \bar{y}_k) := p_{< m} l_k(y_1, \ldots, y_k),
\]

where \( \bar{x}_i = p_{\leq m}(x_i) \) and \( \bar{y}_i = p_{< m}(y_i) \).

2. The assignments \( (L, l_k) \mapsto (\tau_{\leq m} L, \tau_{< m} l_k) \) and \( (L, l_k) \mapsto (\tau_{< m} L, \tau_{< m} l_k) \) are functorial.

3. An \( L_\infty \)-morphism \( \phi : (L, l_k) \to (L', l'_k) \) induces a morphism of towers of Lie \( n \)-algebras

\[
(73) \quad L = \tau_{\leq n-1} L \rightarrow \tau_{< n-1} L \rightarrow \tau_{\leq n-2} L \rightarrow \cdots \rightarrow \tau_{\leq 1} L \rightarrow \tau_{< 1} L \rightarrow \tau_{\leq 0} L
\]

\[
\phi \bigg| \begin{array}{c|c}
\tau_{< n-1} \phi & \tau_{\leq n-2} \phi \\
\tau_{\leq 1} \phi & \tau_{< 1} \phi \\
\tau_{\leq 0} \phi
\end{array}
\]

\[
L' = \tau_{\leq n-1} L' \rightarrow \tau_{< n-1} L' \rightarrow \tau_{\leq n-2} L' \rightarrow \cdots \rightarrow \tau_{\leq 1} L' \rightarrow \tau_{< 1} L' \rightarrow \tau_{\leq 0} L'
\]

in which the horizontal arrows are the (strict) \( L_\infty \)-morphisms induced by the surjective chain maps (72).
Proof: We prove (1) and (2) for $\tau_{\leq m}L$. The same arguments apply for $\tau_{<m}L$. First, we verify that the brackets $\tau_{\leq m}l_k$ are well-defined. For degree reasons, the only non-trivial case to check is $\tau_{\leq m}l_2(\bar{x}_1, \bar{x}_2)$ when $\bar{x}_1$ is in degree $m$ and $\bar{x}_2$ is in degree $0$. Suppose $x_1 = x'_1 + d_{m+1}z$, where $d_{m+1}$ is the differential $l_1$ in degree $m + 1$. The Jacobi-like identities (63) for the $L_\infty$-structure imply that the degree 0 bracket $l_2$ satisfies $l_1l_2(x_1, x_2) = l_2(l_1x_1, x_2) + l_2(l_1x_1, x_2) = l_2(x_1, l_1x_2)$. Hence, $l_2(x_1, x_2) = l_2(x'_1, x_2) + d_{m+1}l_2(z, x_2)$, and so $\tau_{\leq m}l_2$ is well-defined. The fact that the brackets $l_k$ satisfy the identities (63) immediately implies that the brackets $\tau_{\leq m}l_k$ satisfy them as well.

Next, let $\phi = \{\phi_k\}: (L, l_k) \to (L', l'_k)$ be a morphism of Lie $n$-algebras. Define maps $\tau_{\leq m}\phi_k: \Lambda\tau_{\leq m}L \to \tau_{\leq m}L'$ by

$$\tau_{\leq m}\phi_k(\bar{x}_1, \ldots, \bar{x}_k) := p_{\leq m}'\phi_k(x_1, \ldots, x_k),$$

where $p_{\leq m}': L' \to \tau_{\leq m}L'$ is the projection (71). We verify that these are well-defined. Again, for degree reasons, the only non-trivial case to check is $\tau_{\leq m}\phi_1(\bar{x})$ with $\bar{x}$ in degree $m$. The defining equations (66) for an $L_\infty$-morphism imply that $\phi_1$ is a chain map. Hence, if $x_1 = x'_1 + d_{m+1}z$, then $\tau_{\leq m}\phi_1(\bar{x}) = \tau_{\leq m}\phi_1(x')$. The fact that the maps $\phi_k$ satisfy the defining equations (66) immediately implies that the maps $\tau_{\leq m}\phi_k$ form an $L_\infty$-morphism $\tau_{\leq m}\phi: \tau_{\leq m}L \to \tau_{\leq m}L'$.

For (3), since the $L_\infty$ brackets for $\tau_{\leq m}L$ and $\tau_{\leq m}L'$ are defined using the projection maps (71), it is easy to see that the horizontal projections in the diagram (73) are strict $L_\infty$-morphisms. Since the vertical morphisms $\tau_{\leq m}\phi$ and $\tau_{\leq m}\phi$ are also defined using the projection maps (71), the diagram indeed commutes. □

Remark 8.5. The Lie $n$-algebra $\tau_{\leq 0}L$ is just the Lie algebra $H_0(L)$ concentrated in degree zero. Given a morphism of Lie $n$-algebras $\phi: L \to L'$, the induced morphism $\tau_{\leq 0}\phi: H_0(L) \to H_0(L')$ of Lie algebras is the morphism $H_0(\phi_1)$ from Remark 8.1.

8.4. Lie $n$-groups. A Lie $n$-group is an $n$-group object (Def. 3.4) in $(\mathcal{Mfd}, \mathcal{T}_{\text{subm}})$. We denote by $\mathbf{Lie}_n\mathbf{Grp}$ the category of Lie $n$-groups. Let us recall Henriques’ construction of Lie $n$-groups from Lie $n$-algebras of finite-type. First, fix an integer $r \geq 1$, and denote by $\Omega^r(\Delta^n)$ the differential graded Banach algebra of $r$-times continuously differentiable forms on the geometric $n$-simplex. (See Sec. 5.1 of [16]).

Proposition-Definition 8.6 (Def. 5.2, Thm. 5.10, Thm. 6.4 [16]). Let $L \in \mathbf{Lie}_n\mathbf{Alg}^{\text{fin}}$ be a finite-type Lie $n$-algebra and $C^*(L)$ its associated Chevalley–Eilenberg cdga.

1. The assignment

$$L \mapsto \left( \int L \right)_m := \text{hom}_{\text{cdga}}\left( C^*(L), \Omega^*\left(\Delta^m\right) \right)$$

induces a functor

$$\int : \mathbf{Lie}_n\mathbf{Alg}^{\text{fin}} \to \mathbf{Lie}_\infty\mathbf{Grp}$$

(74)

from the category of finite-type Lie $n$-algebras to the category of Lie $\infty$-groups.

2. Let $G$ be the simply connected Lie group integrating the Lie algebra $H_0(L)$. Then $\pi_1^m\left( \int L \right) \cong G$, and there is a long exact sequence of (sheaves of) groups
\[ \cdots \to \pi_{n+1}^{spl}(\int L) \to \pi_{n+1}(G,e) \to H_{n-1}(L) \to \pi_n^{spl}(\int L) \to \pi_n(G,e) \to \cdots \]

\[ \cdots \to \pi_3^{spl}(\int L) \to \pi_3(G,e) \to H_1(L) \to \pi_2^{spl}(\int L) \to \pi_2(G,e). \]

Note that the integration functor (76) above a priori assigns to a Lie \( n \)-algebra not a Lie \( n \)-group but a Lie \( \infty \)-group. To resolve this, a truncation functor is introduced.

**Definition 8.7** (Def. 3.5 [16]). Let \( X \) be a Lie \( \infty \)-group. For \( n \geq 0 \), the truncation \( \tau_{\leq n}X \) is the simplicial sheaf given by

\[ (\tau_{\leq n})_m := \text{coeq} \left( \text{Hom}(P,X) \rightrightarrows \text{Hom}(\Delta[m],X) \right), \]

where \( P \) is the pushout of the following diagram of simplicial sets

\[
\begin{array}{ccc}
\Delta[1] \times \text{sk}_{n-1} \Delta[m] & \longrightarrow & \text{sk}_{n-1} \Delta[m] \\
\downarrow & & \downarrow \\
\Delta[1] \times \Delta[m] & \longrightarrow & P
\end{array}
\]

and \( \text{Hom}(\cdot,X) \) is the sheaf (20).

Henriques observed that the simplicial sheaf \( \tau_{\leq n}X \) may not be representable for every Lie \( \infty \)-group \( X \). So, we will focus on the subcategory of “integrable” Lie \( n \)-algebras. An important example of such a Lie \( n \)-algebra is the “string Lie 2-algebra”.

**Definition 8.8.** Let \( L \) be a finite-type Lie \( n \)-algebra. We say \( L \) is integrable iff the simplicial sheaf \( \tau_{\leq n}(\int L) \) is representable. We denote by

\[ \text{Lie}_n\text{Alg}_{\text{fin}}^{\text{int}} \subseteq \text{Lie}_n\text{Alg}_{\text{fin}} \]

the full subcategory of integrable finite-type Lie \( n \)-algebras.

**Proposition 8.9** (Lemma 3.6 [16]). Let \( L \) be a finite-type integrable Lie \( n \)-algebra. The assignment

\[ L \mapsto \int_{\leq n} L := \tau_{\leq n}(\int L) \]

induces a functor

\[ \int_{\leq n} : \text{Lie}_n\text{Alg}_{\text{fin}}^{\text{int}} \to \text{Lie}_n\text{Grp} \]

from the category of finite-type Lie \( n \)-algebras to the category of Lie \( n \)-groups \( \text{Lie}_n\text{Grp} \).
8.5. Integration of $L_\infty$-quasi-isomorphisms. Our proof of the main result in this section, Thm. 8.12, crucially relies on the fact that Henriques’ long exact sequence (75) is functorial with respect to morphisms of Lie $n$-algebras. Although this is not shown in [16], it does follow from the arguments made there, as we now explain.

The sequence (75) can be constructed by applying the integration functor of Prop/Def. 8.6 to the Postnikov tower (73) of the Lie $n$-algebra $L$. This gives a tower of Kan fibrations between Lie $\infty$-groups [16, Thm. 5.10]. As discussed in the remarks preceding Cor. 6.5 in [16], the spectral sequence associated to this tower is

\begin{equation}
E^1_{m,k} = \pi^{spl}_m \left( \int H_{k-1}(L)[k-1] \right) \Rightarrow \pi^{spl}_{m+k} \left( \int L \right)
\end{equation}

where $H_{k-1}(L)[k-1]$ is the Lie $n$-algebra with only $H_{k-1}(L)$ in degree $k-1$. Prop. 8.4 implies that the spectral sequence is functorial. The spectral sequence is also sparse. As a result, it reduces to a long exact sequence, which becomes (75) after identifications are made between certain simplicial homotopy groups, and the Lie group $G$ and its homotopy groups. Therefore, to verify the functoriality of (75), it remains to check the functoriality of these identifications.

Following [16, Example 5.5], we consider the Lie $\infty$-group $\int g$, where $g$ is the Lie algebra $\tau_{\leq 0} L = H_0(L)$. For each $n \geq 0$, there is a natural 1-1 correspondence

\begin{equation}
\left( \int g \right)_m := \text{hom}_{cdga}( C^*(g), \Omega^*(\Delta^n)) \cong \left( \Omega^1(\Delta^n) \otimes g \right)^b
\end{equation}

between cdga morphisms $C^*(g) \to \Omega^*(\Delta^n)$ and flat connections on the trivial bundle $G \times \Delta^n \to \Delta^n$. There is also an identification

\begin{equation}
\rho^*_g : \text{Map}(\Delta^n, G)/G \cong \left( \Omega^1(\Delta^n) \otimes g \right)^b
\end{equation}

\begin{equation}
f : \Delta^n \to G \mapsto \alpha_f = f^{-1} df
\end{equation}

between the set $\text{Map}(\Delta^n, G)/G$ of $G$-valued $C^{r+1}$ maps, modulo constants, and the set of flat connections.

**Lemma 8.10.** Let $\phi : L \to L'$ be a morphism of Lie $n$-algebras. Let $\Phi : G \to G'$ denote the unique homomorphism of simply connected Lie groups that integrates the Lie algebra morphism $\tau_{\leq 0}(\phi) = H_0(\phi_1) : g \to g'$, where $g = H_0(L)$ and $g' = H_0(L')$. Then the following diagram commutes

\begin{equation}
\begin{array}{ccc}
\text{Map}(\Delta^n, G)/G & \xrightarrow{f \mapsto \Phi \circ f} & \text{Map}(\Delta^n, G')/G' \\
\rho^*_g \downarrow & & \downarrow \rho^*_{g'} \\
\left( \Omega^1(\Delta^n) \otimes g \right)^b & \xrightarrow{\text{id} \otimes \tau_{\leq 0}(\phi)} & \left( \Omega^1(\Delta^n) \otimes g' \right)^b
\end{array}
\end{equation}

**Proof.** The commutativity of the diagram is verified by using the following elementary facts: (1) $\Phi$, being a homomorphism, intertwines the left multiplication on $G$ with that of $G'$, and (2) the differential of $\Phi$ at the identity is $H_0(\phi_1)$. \qed

**Remark 8.11.** Lemma 8.10 also implies that the identifications made in Example 6.2 of [16]:

\begin{equation}
\pi^{spl}_1 \left( \int H_0(L) \right) \cong G, \quad \pi^{spl}_{k \geq 2} \left( \int g \right) \cong \pi_{k \geq 2}(G, e).
\end{equation}
are also natural with respect to the zeroth truncation $\tau_{\leq 0}\phi$ of a Lie $n$-algebra morphism $\phi: L \to L'$.

Now that the functoriality of the long exact sequence (75) has been clarified, we can prove the second main result of the paper:

**Theorem 8.12.** If $\phi: L \sim \to L'$ is a weak equivalence between integrable finite-type Lie $n$-algebras, then the morphism

$$
\int_{\leq n} \phi: \int_{\leq n} L \to \int_{\leq n} L'
$$

is a weak equivalence of Lie $n$-groups.

**Proof.** It is sufficient to show that $\int_{\leq n} \phi$ induces an isomorphism of simplicial homotopy groups. Indeed, if that is the case, then Prop. 5.3 implies that $\int_{\leq n} \phi$ is a stalkwise weak equivalence. For $i \leq n$, the $i$th simplicial homotopy groups of $\int_{\leq n} L$ and $\int_{\leq n} L'$ are equal to those of $\int L$ and $\int L'$, respectively. (For $i > n$, the groups are trivial.) So let $\psi: X \to X'$ denote the pre-truncated morphism of Lie $\infty$-groups

$$\int \phi: \int L \to \int L'.$$

To complete the proof, we will show that the morphisms

$$\tilde{\psi}_n := \pi^{\text{spl}}_n(\psi): \pi^{\text{spl}}_n(X) \to \pi^{\text{spl}}_n(X') \quad n > 0$$

induced by $\psi$ are isomorphisms. Let $G$ and $G'$ denote the simply connected Lie groups integrating the Lie algebras $H_0(L)$ and $H_0(L')$, respectively. From Remark 8.1, it follows that

$$H_0(\phi_1): H_0(L) \xrightarrow{\sim} H_0(L')$$

is an isomorphism of Lie algebras. Let $\Phi: G \xrightarrow{\sim} G'$ denote the corresponding isomorphism of Lie groups induced by $H_0(\phi)$. It follows from Lemma 8.10 and Remark 8.11 that we have a commuting diagram

$$
\begin{array}{ccc}
\pi^{\text{spl}}_1(X) & \xrightarrow{\tilde{\psi}_1} & \pi^{\text{spl}}_1(X') \\
\downarrow & & \downarrow \\
G & \xrightarrow{\Phi} & G'
\end{array}
$$

Hence, $\tilde{\psi}_n$ is an isomorphism for $n = 1$.

The functoriality of the long exact sequence (75) gives the commutative diagram (with exact rows)

$$
\begin{array}{ccccccc}
\pi_3(G) & \longrightarrow & H_1(L) & \longrightarrow & \pi_2^{\text{spl}}(X) & \longrightarrow & \pi_2(G) \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
\pi_3(G') & \longrightarrow & H_1(L') & \longrightarrow & \pi_2^{\text{spl}}(X') & \longrightarrow & \pi_2(G')
\end{array}
$$
Since $\pi_2(G) = \pi_2(G') = 0$, a simple diagram chase shows that $\tilde{\psi}_2: \pi_2\text{spl}(X) \to \pi_2\text{spl}(X')$ is an isomorphism. For $n > 2$, we can apply the 5-lemma to
\[ \cdots \longrightarrow \pi_{n+2}\text{spl}(X) \longrightarrow \pi_{n+1}(G) \longrightarrow H_{n-1}(L) \longrightarrow \pi_{n+1}\text{spl}(X) \longrightarrow \pi_n(G) \longrightarrow \cdots \]
\[ \downarrow \tilde{\psi}_{n+1} \quad \text{\rotatebox{90}{\(\longrightarrow\)}} \quad \pi_{n+1}(\Phi) \quad \text{\rotatebox{90}{\(\longrightarrow\)}} \quad H_{n-1}(\phi_1) \quad \text{\rotatebox{90}{\(\longrightarrow\)}} \quad \tilde{\psi}_n \quad \text{\rotatebox{90}{\(\longrightarrow\)}} \quad \pi_n(\Phi) \]
\[ \cdots \longrightarrow \pi_{n+1}\text{spl}(X') \longrightarrow \pi_{n+1}(G') \longrightarrow H_{n-1}(L') \longrightarrow \pi_{n+1}\text{spl}(X') \longrightarrow \pi_n(G') \longrightarrow \cdots \]
and deduce that $\tilde{\psi}_n$ is an isomorphism. This completes the proof. 

Remark 8.13. The same proof holds if $\phi: L \xrightarrow{\sim} L'$ is a weak equivalence between integrable finite-type $L_\infty$-algebra $(n = \infty)$.

We conclude by recalling from [38, Def. 2.12] that two Lie $n$-groupoids $X$ and $Y$ are Morita equivalent iff there exists a Lie $n$-groupoid $Z$ and a span of hypercovers $X \leftarrow Z \rightarrow Y$. Theorem 7.1 implies that the hypercovers between Lie $n$-groupoids are precisely the acyclic fibrations in $\text{Lie}_\infty\text{Gpd}$. Combining this with Prop. 2.5 and Remark 8.3, we have the following corollary to Thm. 8.12:

Corollary 8.14. Homotopy equivalent integrable finite-type Lie $n$-algebras integrate to weakly equivalent, and hence Morita equivalent, Lie $n$-groups.

Appendix A. Proof of Lemma 7.16

We need to prove the following: For all $n \geq 1$ and $0 \leq j \leq n$, the natural inclusion
\[ \Lambda^n_j \times \Delta^1 \cup \Lambda^n_j \times \partial \Delta^1 : \Delta^n \times \partial \Delta^1 \hookrightarrow \Delta^n \times \Delta^1 \]
is a collapsible extension.

Let us first establish some notation. We fix $n$. Via the usual triangulation, we write $\Delta^n \times \Delta^1$ as the union $\bigcup_{0 \leq l \leq n} x_l$ of $n+1$ $(n+1)$-simplices where:
\[ x_l := \Delta^{n+1} \setminus \{(0,0), (1,0), \ldots, (l,0), (l,1), (l+1,1), \ldots, (n,1)\}. \]
Also, for $0 \leq l \leq n$ and $0 \leq j \leq n$ denote by $y^j_l$ the following face of the simplex $x_l$:
\[ y^j_l := \begin{cases} d_{j+1} x_l, & \text{if } 0 \leq l \leq j \\ d_j x_l, & \text{if } j < l. \end{cases} \]

And for $l = -1, \ldots, n$, and $0 \leq j \leq n$, denote by $T_{j,l} \subseteq \Delta^n \times \Delta^1$ the following simplicial subsets: If $l = -1$, then
\[ T_{j,-1} := \Lambda^n_j \times \Delta^1 \cup \Delta^n \times \partial \Delta^1 \]
and for $l \geq 0$
\[ T_{j,l} := T_{j,l-1} \cup x_l. \]

Note that $T_{j,-1}$ is the left side of the inclusion (83), while $T_{j,n} = \Delta^n \times \Delta^1$. Below, we will prove the lemma by showing the inclusions $T_{j,l-1} \subseteq T_{j,l}$ are collapsible extensions.

We will also need the following simple facts:
Figure 1. For \( n = 2 \), the left-hand side depicts the geometric realization of the simplicial subset \( T_{1,-1} \subseteq \Delta^2 \times \Delta^1 \). For \( n = 3 \), the right-hand side depicts the realization of the “face” \( d_2 \Delta^3 \times \Delta^1 \) which is missing from \( T_{2,-1} \).

**Claim A.1.** Let \( 0 \leq j \leq n \) and \( 0 \leq l \leq n \). Every face of the \((n+1)\)-simplex \( x_l \), with the possible exception of \( d_{l+1}x_l \) and \( y_l^j \), is contained in the simplicial subset \( T_{j,l-1} \).

**Proof.** We consider the faces \( d_k x_l \). There are three cases. First, if \( k = l \), then it is easy to verify that:

\[
d_k x_l \subseteq \Delta^n \times \partial \Delta^1 \subseteq T_{j,l-1}, \quad \text{if } l = 0 \text{ or } l = n
\]

\[
d_k x_l = d_{l+1} x_{l-1} \in T_{j,l-1} \quad \text{if } 0 < l < n.
\]

Now, if \( k < l \), then \( d_k x_l \subseteq d_k \Delta^n \times \Delta^1 \). Hence, if \( k \neq j \), then we have \( d_k x_l \subseteq \Lambda_j^n \times \Delta^1 \subseteq T_{j,l-1} \). Otherwise, we have \( d_k x_l = d_j x_l = y_l^j \).

Finally, if \( k \geq l + 2 \), then \( d_k x_l \subseteq d_{k-1} \Delta^n \times \Delta^1 \). So if \( k \neq j + 1 \), then \( d_k x_l \subseteq \Lambda_j^n \times \Delta^1 \subseteq T_{j,l-1} \). Otherwise, we have \( d_k x_l = d_{j+1} x_l = y_l^j \). □

**Claim A.2.** Let \( \Sigma_j := d_j \Delta^n \) for \( 0 \leq j \leq n \). Then for \( 0 \leq i \leq n-1 \), we have the inclusion of simplicial sets

\[
d_i \Sigma_j \times \Delta^1 \subseteq T_{j,l-1}.
\]

**Proof.** This follows directly from the simplicial identities for face maps. □

**Claim A.3.** Let \( 0 \leq j \leq n \), \( 0 \leq l \leq n \), and let \( z_l^j \) denote the \((n-1)\)-simplex

\[
z_l^j := \begin{cases} d_{l+1} y_l^j, & \text{if } l \leq j \\ d_j y_l^j, & \text{if } j < l, \end{cases}
\]

where \( y_l^j \) is the \( n \)-simplex \((85)\). Every face of \( y_l^j \), with the possible exception of \( z_l^j \), is contained in the simplicial subset \( T_{j,l-1} \).

**Proof.** There are a few cases to consider.

**Case** \( k < l \): If \( 0 \leq j \leq l \), then we have \( d_k y_l^j = d_k d_{j+1} x_l = d_j d_k x_l \). Claim A.1 implies that \( d_k x_l \subseteq T_{j,l-1} \), so therefore \( d_k y_l^j \subseteq T_{j,l-1} \). If \( j < l \), then \( d_k y_l^j = d_k d_j x_l \). Hence, we have \( d_k y_l^j \subseteq d_k \Sigma_j \times \Delta^1 \). So Claim A.2 implies that \( d_k y_l^j \subseteq T_{j,l-1} \).

**Case** \( k = l \): If \( j < l \), then \( d_k y_l^j = z_l^j \). If \( l \leq j \), then we have \( d_k y_l^j = d_l d_{j+1} x_l = d_j d_l x_l \). Claim A.1 implies that \( d_l x_l \subseteq T_{j,l-1} \), so therefore \( d_k y_l^j \subseteq T_{j,l-1} \).
Case $k = l + 1$: If $l \leq j$, then we have $d_k y^j_i = z^j_l$. If $j < l$, then we have $d_k y^j_i = d_{i+1} d_j x_l = d_j d_{i+2} x_l$. Claim A.1 implies that $d_l x_l \subseteq T_{j, l-1}$. Hence, $d_k y^j_i \subseteq T_{j, l-1}$.

Case $k \geq l + 2$: If $l \leq j$, then either $d_k y^j_i = d_j d_k x_l$ or $d_k y^j_i = d_{j+1} d_k x_l$, depending on whether $k < j + 1$ or $k \geq j + 1$. In both cases, Claim A.1 implies that $d_k y^j_i \subseteq T_{j, l-1}$. Finally, if $j < l$, then $d_k y^j_i = d_k d_j x_l$, which implies that $d_k y^j_i \subseteq d_k \Sigma_j \times \Delta^1$. Hence, it follows from Claim A.2 that $d_k y^j_i \subseteq T_{j, l-1}$.

We now arrive at:

Proof of Lemma 7.16. Let $0 \leq j \leq n$ and $0 \leq l \leq n$. We will show that the inclusion $T_{j, l-1} \subseteq T_{j, l}$ is a collapsible extension. First, we observe that the boundary of $x_l$ is not contained in $T_{j, l-1}$: either the face $y^j_l$ or the face $d_{l+1} x_l$ is missing. If $y^j_l$ is contained in $T_{j, l-1}$, then Claim A.1 implies that there exists a map

$$
\psi: \Lambda_{l+1}^{n+1} \to T_{j, l-1}
$$

which sends the generators of $\Lambda_{l+1}^{n+1}$ to all the faces of $x_l$ except $d_{l+1} x_l$. Then the pushout of $\Lambda_{l+1}^{n+1} \hookrightarrow \Delta^n$ along $\psi$ is $T_{j, l}$.

On the other hand, if $y^j_l$ is not contained in $T_{j, l-1}$ then let

$$
k = \begin{cases} 
  l + 1 & \text{if } l \leq j, \\
  l & \text{if } j < l.
\end{cases}
$$

We observe that the boundary of $y^j_l$ is not contained in $T_{j, l-1}$. Claim A.3 implies that there exists a map $\phi: \Lambda^n_k \to T_{j, l-1}$ which sends generators of the horn to all faces of $y^j_l$ except $z^j_l$. The pushout of $\Lambda^n_k \hookrightarrow \Delta^n$ along $\phi$ gives a simplicial subset $S_{j, l} := T_{j, l-1} \cup y^j_l$. If $x_l$ is not contained in $S_{j, l}$, then we compose $\psi$ (88) with the inclusion $T_{j, l-1} \subseteq S_{j, l}$. The pushout of $\Lambda_{l+1}^{n+1} \hookrightarrow \Delta^n$ along this composition is $T_{j, l}$.

\section*{Appendix B. Sheaves on large categories}

As mentioned in Section 3.3.1, there can be set-theoretic technicalities when working with sheaves over large categories. For example, in this paper, we take colimits of representable sheaves (e.g., the simplicial homotopy groups in Def. 5.2) and this implicitly requires a sheafification functor. Unfortunately, the usual plus-construction for producing a sheaf from a presheaf is not well-defined for large categories, since it a priori requires taking colimits over proper classes. All of this can be avoided by using Grothendieck universes, and in particular the Universe Axiom, which allows us take colimits in an ambient larger universe in which our classes are sets. However, we would like our formalism to not depend on this ambient larger universe in any way. One could have, for example, the colimit of a diagram of representables be a sheaf that takes values in sets which properly reside in the ambient larger universe.

In this appendix, we verify that colimits of representable sheaves are independent of choice of the ambient universe, for those pretopologies on large categories which admit a “small refinement” (Def. B.5). We conclude by showing that our main example of interest: the surjective submersion pretopology $\mathcal{T}_{subm}$ on the category $\text{Mfd}$ of Banach manifolds is a pretopology which admits a small refinement.
We claim no particular originality for these results: they just provide us with an elementary way to comfortably ignore size issues. Our main reference throughout for the set theory involved is Sec. 1.1 of [8], as well as the preprint [25].

B.1. Grothendieck universes.

**Definition B.1.** A universe is a set \( U \) that satisfies the following axioms:

1. If \( x \in y \) and \( y \in U \), then \( x \in U \).
2. If \( x \) and \( y \) are elements of \( U \), then \( \{x, y\} \in U \).
3. If \( x \in U \) then the power set \( 2^x \) is an element of \( U \).
4. If \( I \in U \) and \( \{x_\alpha\}_{\alpha \in I} \) is a family of elements of \( U \), then the union \( \bigcup_{\alpha \in I} x_\alpha \) is an element of \( U \).
5. The set of all finite von Neumann ordinals is an element of \( U \).

We adopt the following universe axiom:

**Assumption B.2 (Universe Axiom).** For each set \( x \), there exists a universe \( U \) with \( x \in U \).

Any universe \( U \) is a model of ZFC, so all usual set-theoretic constructions apply. A \( U \)-set is a member of \( U \), and a \( U \)-class is a subset of \( U \). A proper \( U \)-class is a \( U \)-class which is not a \( U \)-set.

**Convention B.3.** We fix a universe \( U \) in which we consider as the “usual universe” in which we do our mathematics. We denote by \( \text{Set} \) the category of \( U \)-sets.

B.2. Sheaves on locally \( U \)-small categories. A category \( M \) is \( U \)-small iff \( \text{Ob}(M) \) and \( \text{Mor}(M) \) are \( U \)-sets. We say \( M \) is locally \( U \)-small iff \( \text{hom}_M(x, y) \) is a \( U \)-set for all \( x, y \in \text{Ob}(M) \).

B.2.1. Universe extension. Presheaves are only well-defined over categories that are \( U \)-small. However, by the universe axiom, there exists a universe \( \hat{U} \) such that

\[
U \in \hat{U}.
\]

It follows from Def. B.1 that \( 2^U \) is also a \( \hat{U} \)-set, and hence any \( U \)-class is as well. We denote by \( \hat{\text{Set}} \), the category of \( \hat{U} \)-sets. Therefore, if \( M \) is a locally \( U \)-small category, then it is \( \hat{U} \)-small category. So, for a well-behaved theory of presheaves on \( M \), we consider the category \( \text{PSh}(M) \) of \( \hat{\text{Set}} \)-valued functors

\[
F : M^{\text{op}} \to \hat{\text{Set}}.
\]

The entire theory of presheaves and sheaves can be applied to those on a locally \( U \)-small category without worry of set-theoretic complications, provided we work in \( \hat{U} \). Let \( (M, \mathcal{T}) \) be a locally \( U \)-small category equipped with a pretopology (Def. 3.1). Let \( \mathcal{T}(-) : \text{Ob}(M) \to 2^{\text{Mor}(M)} \) denote the function which assigns to an object \( X \) the \( \hat{U} \)-set of covers \( \mathcal{T}(X) \) of \( X \). (Note that \( \mathcal{T}(X) \) is a priori not a \( U \)-set.)

Let \( \text{Sh}(M) \) denote the category of sheaves on \( M \). We have the adjunction

\[
(89) \quad \ell : \text{PSh}(M) \leftrightarrows \text{Sh}(M) : i
\]

where \( i \) is the inclusion, and \( \ell \) is the sheafification functor. As usual, the functor \( \ell \) preserves finite limits, and the composite \( \ell \circ i \) is naturally isomorphic to the identity functor.
B.2.2. Sheafification. Let us quickly describe the sheafification functor $\ell$ for pretopologies in the sense of Def. 3.1. Let $\alpha: U \to X$ and $\beta: V \to X$ be a covers of an object $X \in M$. We say $\alpha$ refines $\beta$, and write $\alpha \prec \beta$ if there exists a morphism $f: U \to V$ such that $\beta \circ f = \alpha$. The axioms of a pretopology imply that any two covers of an object $X$ have a common refinement, hence the set $T(X)$ is equipped with a directed preorder.

Let $F$ be a presheaf and $\alpha: U \to X$ a cover. A matching family for $\alpha$ is an element $x \in F(U)$ such that the following diagram commutes:

\[ \begin{array}{ccc}
U \times_a U & \longrightarrow & U \\
\downarrow & & \alpha \\
U & \longrightarrow & X \\
\alpha & & \\
& & \downarrow \\
& & F \\
\end{array} \]

Denote by $\text{Match}(\alpha, F)$ the $\tilde{U}$ set of all matching families for the cover $\alpha$.

The plus construction applied to a presheaf $F$ produces a new presheaf $F^+$, which assigns to an object $X \in M$, the $\tilde{U}$-set

$$ F^+(X) := \colim_{\alpha \in T(X)} \text{Match}(\alpha, F). $$

An element of $F(X)^+$ is an equivalence class of matching families $\overline{x_\alpha}$. Matching families $x_\alpha$ and $x_\beta$ for covers $\alpha: U \to X$, $\beta: V \to X$, respectively, are equivalent iff there exists a cover $\gamma: W \to X$ refining $\alpha$ and $\beta$ such that $x_\alpha \circ f = x_\beta \circ g$, where $f: W \to U$, $g: W \to V$ are morphisms such that $\gamma = \alpha \circ f = \beta \circ g$. The sheafification functor $\ell: PSh(M) \to \text{Sh}(M)$ is then defined as

$$ \ell(F) := (F^+)^+. $$

B.3. Independence of choice of ambient universe. Let $F: M^{\text{op}} \to \tilde{\text{Set}}$ be a presheaf. We say $F$ is a $\mathcal{U}$-presheaf iff $F(X)$ is a $\mathcal{U}$-set for all objects $X \in M$. The analogous definition for sheaves is clear: A sheaf $F$ is a $\mathcal{U}$-sheaf iff the presheaf $i(F)$ is a $\mathcal{U}$-presheaf. Since $M$ is locally $\mathcal{U}$-small, the representable sheaves are $\mathcal{U}$-sheaves.

We want $\mathcal{U}$-small colimits and limits involving $\mathcal{U}$- (pre)sheaves to output an object which “stays” in our universe $\mathcal{U}$, and does not depend on the non-canonical choice of the ambient universe $\tilde{\mathcal{U}}$. This is true for $\mathcal{U}$-presheaves, since the inclusion functor

$$ \text{Set} \hookrightarrow \tilde{\text{Set}} $$

reflects colimits and limits for all $\mathcal{U}$-small diagrams [25, Cor. 1.19]. And, indeed, this will also be true for $\mathcal{U}$-sheaves, provided we require our pretopology to satisfy a certain smallness condition.

First, we deal with limits of $\mathcal{U}$-sheaves.

**Proposition B.4.** Let $J$ be a $\mathcal{U}$-small category and $D: J \to \text{Sh}(M)$ a diagram such that $D(j)$ is a $\mathcal{U}$-sheaf for all objects $j \in J$. Then $\lim D$ is a $\mathcal{U}$-sheaf which can be constructed independently from the choice of ambient universe $\tilde{\mathcal{U}}$. 
Proof. Since the inclusion $i: \text{Sh}(C) \to \text{PSh}(C)$ is a right adjoint, it preserves all $U$-small limits. Hence, $\lim i \circ D \cong \lim i \circ \text{D}$. Now $i \circ \text{D}$ is a $U$-small limit of $U$-presheaves. Such a limit is computed point-wise, and since $\text{Set}$ is complete with respect to $U$-small limits, it follows that $\lim i \circ \text{D}$ is a small presheaf. □

B.3.1. A smallness condition for pretopologies.

**Definition B.5.** We say that a pretopology $\mathcal{T}$ on a locally $U$-small category $M$ admits a $U$-small refinement $\mathcal{O}$ iff for every object $X \in M$ there is a $U$-small subset $\mathcal{O}(X) \subseteq \mathcal{T}(X)$ such that every cover in $\mathcal{T}(X)$ is refined by a cover in $\mathcal{O}(X)$.

An example of pretopology which admits a $U$-small refinement is the surjective submersion topology on the category of Banach manifolds.

**Example B.6.** Let $(\text{Mfd}, \mathcal{T}_{\text{subm}})$ denote the category of (locally) Banach manifolds equipped with the surjective submersion pretopology. Let $M \in \text{Mfd}$. If $\{U_i\}_{i \in I}$ is an open cover of $M$, then the morphism $\coprod_{i \in I} U_i \to M$ is a surjective submersion, where $\iota$ is the unique map induced by the inclusions $U_i \subseteq M$. Consider the subset of $\mathcal{T}(M)$

$$\mathcal{O}(M) := \left\{ \coprod_{i \in I} U_i \to M \mid \{U_i\}_{i \in I} \text{ is an open cover of } M \right\}.$$

Since power sets of $U$-sets are $U$-sets, $\mathcal{O}(M)$ is a $U$-set for each $M \in \text{Mfd}$. If $f: N \to M$ is a surjective submersion, then for each $x \in N$ there exists an open neighborhood $U_x \subseteq M$ of $f(x)$ and a map $\sigma_x: U_x \to N$ such that $\sigma_x(f(x)) = x$ and $f \circ \sigma_x = \text{id}_{U_x}$. Therefore via the universal property of the coproduct, we have a commuting diagram in $\text{Mfd}$

$$\begin{array}{ccc}
\coprod_{x \in N} U_x & \xrightarrow{\sigma} & N \\
\downarrow \iota & & \downarrow f \\
M & \xrightarrow{f} & N
\end{array}$$

Hence, every cover in $\mathcal{T}_{\text{subm}}(M)$ is refined by a cover in $\mathcal{O}(M)$.

Note we do not require $\mathcal{O}$ in Def. B.5 to be a pretopology. Nevertheless, taking refinements induces a directed preorder structure on $\mathcal{O}$. If $F$ is a $U$-presheaf on $(M, \mathcal{T})$, and $\mathcal{T}$ admits a $U$-small refinement, then for each $\alpha \in \mathcal{O}(X)$, we have $\text{Match}(\alpha, F) \in \text{Set}$ and moreover, we have

$$\text{colim}_{\alpha \in \mathcal{O}(X)} \text{Match}(\alpha, F) \in \text{Set}$$

since this is $U$-small colimit of $U$-small sets. Then we have the following:

**Theorem B.7.** Let $(M, \mathcal{T})$ be a locally $U$-small category equipped with a pretopology which admits a $U$-small refinement. If $F$ is a $U$-presheaf on $M$, then its sheafification $\ell(F)$ is a $U$-sheaf which can be constructed independently from the choice of ambient universe $\mathcal{U}$.

**Proof.** Let $X \in M$ and define a new $U$-presheaf $F^+_{\mathcal{O}}$ via the colimit

$$F^+_{\mathcal{O}}(X) := \text{colim}_{\alpha \in \mathcal{O}(X)} \text{Match}(\alpha, F).$$
There is the obvious function from $F_\mathcal{O}$ to the plus construction (90) applied to $F$:
\[
F_\mathcal{O}^+(X) \to F^+(X) \quad \text{where } x_\alpha \mapsto \alpha,
\]
where $x_\alpha$ on the right hand side above is the equivalence class in $\text{colim}_{\alpha \in \mathcal{T}(X)} \text{Match}(\alpha, F)$ represented by the matching family $x_\alpha$. The theorem is proved if this assignment is a bijection. But this follows directly from the fact that for every cover $\alpha \in \mathcal{T}(X)$, there exists a cover in $\mathcal{O}(X)$ refining $\alpha$. \hfill \Box

Since the colimit of a diagram of sheaves is constructed by sheafifying the pointwise colimit of the underlying diagram of presheaves, the above theorem gives, as a corollary, the dual of Prop. B.4

**Corollary B.8.** Let $J$ be a $\mathcal{U}$-small category and $D: J \to \text{Sh}(M)$ a diagram such that $D(j)$ is a $\mathcal{U}$-sheaf for all objects $j \in J$. Then $\text{colim} D$ is a $\mathcal{U}$-sheaf which can be constructed independently of the choice of ambient universe $\mathcal{U}$.

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