ENSEMBLE KALMAN SAMPLING: MEAN-FIELD LIMIT AND CONVERGENCE ANALYSIS

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Abstract. Ensemble Kalman sampling (EKS) is a method introduced in [17] to find i.i.d. samples from a target distribution. As of today, why the algorithm works and how it converges is mostly unknown. In this paper, we analyze the continuous version of EKS, a coupled SDE system, and justify its mean-filed limit is a Fokker-Planck equation, whose equilibrium state is the target distribution. This proves that in long time, the samples generated by EKS indeed are approximately i.i.d. samples from the target distribution. We further show the ensemble distribution of EKS converges, in Wasserstein-2 sense, to the target distribution with a near-optimal rate ($J^{-1/2}$). We emphasize that even when the forward map is linear, due to the ensemble nature of the method, the SDE system and the corresponding Fokker-Planck equation are still nonlinear.

1. Introduction

How to generate i.i.d. samples from a target distribution has been studied extensively over many years. A lot of algorithms have been proposed. Traditional methods such as Markov chain Monte Carlo and sequential Monte Carlo (SMC) have garnered a large amount of investigations [10, 30, 36], and new methods such as Ensemble Kalman Inversion [15, 22] (derived from Ensemble Kalman filter) and Stein Variational Gradient Descent (SVGD) [24] have been attracting attention the moment they became available.

In this paper, we investigate a new method, termed Ensemble Kalman sampling (EKS), proposed in [17], and we prove, using mean-field limit argument, the convergence rate is almost optimal in the linear setup.

1.1. Problem setup. EKS is an algorithm to find i.i.d. samples from a target distribution. Suppose $u \in X$ is the to-be-reconstructed parameter and $\mathcal{G} : X \to Y$ is the parameter-to-observable map, namely:

$$y = \mathcal{G}(u) + \eta,$$

where $y \in Y$ collects the observed data with $\eta$ denoting the noise in the measurement-taking. The inverse problem amounts to reconstructing $u$ from $y$. Without loss of generality, we assume $X = \mathbb{R}^L$, and $Y = \mathbb{R}^K$ and $\eta \sim \mathcal{N}(0, \Gamma)$ is a Gaussian noise independent of $u$. Denoting the loss functional $\Phi(\cdot; y) : \mathbb{R}^K \to \mathbb{R}$ by

$$\Phi(u; y) = \frac{1}{2} \| y - \mathcal{G}(u) \|^2_\Gamma,$$

then the Bayes’ theorem, derived simply from the equivalence of the joint probability, states that the posterior distribution is the (normalized) product of the prior distribution and the likelihood function:

$$\rho_{\text{pos}}(u) du = \frac{1}{Z} \exp (-\Phi(u; y)) \rho_0(u) du,$$

where $Z := \int_X \exp (-\Phi(u; y)) \rho_0(u) du$.

Here $Z$ serves as the normalization factor, $\exp (-\Phi(u; y))$ is the likelihood function and $\rho_0$ is the prior distribution that collects people’s prior knowledge about the distribution of $u$. This so-called posterior distribution represents the probability measure of the to-be-reconstructed parameter $u$, blending the prior knowledge and the collected data $y$, taking $\eta$, the measurement error into account. More details on Bayesian inversion can be found in [11, 24].

It is very typical to assume $\rho_{\text{prior}}$ is a Gaussian distribution with mean $u_0$ and covariance $\Gamma_0$:

$$\rho_{\text{prior}}(u) = \frac{1}{Z(0)} \exp \left( -\frac{1}{2} (u - u_0)^\top \Gamma_0^{-1} (u - u_0) \right). \quad (1)$$

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Denoting
\[ \Phi_R(u; y) = \Phi(u; y) + \frac{1}{2} |u - u_0|^2, \]
the posterior distribution \( \rho_{\text{pos}} \) becomes:
\[ \rho_{\text{pos}}(u) du = \frac{1}{Z} \exp (-\Phi_R(u; y)) du, \]
where \( Z \) is the normalizer. For the conciseness of the notation, we abbreviate \( \Phi_R(u; y) \) to \( \Phi_R(u) \).

This formula can be made more explicit if the forward map \( G \) is linear. Assuming \( G \) is linear, meaning: there exists a matrix \( A \) so that
\[ G(\cdot) = A \cdot, \quad \text{with} \quad A \in \mathcal{L}(\mathbb{R}^L, \mathbb{R}^K). \]
For later use, we denote the “closest” solution \( u^\dagger \) with noise \( r \) such that:
\[ y = \Gamma^{-1/2} A u^\dagger + r, \quad \text{with} \quad r \perp \text{range} \{ \Gamma^{-1/2} A \}. \]
In some sense, \( u^\dagger \) is regarded as the optimal solution without regularization imposed from the prior distribution.

Define:
\[ B = A^\top \Gamma^{-1} A + \Gamma_0^{-1}, \quad u^* = B^{-1} \left( A^\top \Gamma^{-1} A u^\dagger + \Gamma_0^{-1} u_0 \right), \]
then
\[ \rho_{\text{pos}}(u) du = \frac{1}{Z} \exp \left( -\frac{1}{2} |u - u^*|^2_{B^{-1}} \right) du, \]
and the expectation and covariance become
\[ \mathbb{E}_{\rho_{\text{pos}}} = u^*, \quad \text{Cov}_{\rho_{\text{pos}}} = B^{-1}. \]

1.2. Algorithm description. The goal of EKS is to generate samples that are approximately i.i.d. sampled from the target distribution \( \mu_{\text{pos}} \).

Sampling from a target distribution is considered a challenging problem. Traditional methods such as Markov chain Monte Carlo (MCMC) converges in the sense that the target distribution is the invariant measure of the MCMC transition kernel, but the method suffers from curse of dimensionality, meaning the number of samples grows exponentially with respect to the dimension of \( u \) and that samples are not i.i.d making redundant samples wasting computational resources [1]. A lot of newer methods are introduced in recent years, including Ensemble Kalman inversion [32, 33, 4, 5, 13] and SVGD [24, 25]. These works are largely inspired by the earlier works in data assimilation on the celebrated Kalman filter, its continuous counterpart (Kalman-Bucy filter), and its ensemble version: Ensemble Kalman filter [14, 18, 15]. For Ensemble Kalman Filter specifically, the continuous-in-time derivation was formulated and investigated in [3, 2, 12], also see a very insightful review and the connection with a set of SDE in [29].

The newer methods seem to work well in practice. One immediate advantage is that: due to the ensemble nature of these methods, the large covariance matrix is no longer stored and evolved, but rather computed from the ensembles, saving computational cost. However, replacing the analytical covariance by the ensemble version does not come for free, and to today, the justification of such “replacement” and the proof for the smallness of the introduced error is still mostly up in the air, except a few exceptions [13, 23, 19]. Such limitation necessarily bounds our understanding and evaluation of the efficiency and convergence rate.

Most of these methods require the computation of the derivatives \( \nabla_u G \) at each step. In practice such derivatives usually call for the computation of both forward and adjoint solvers and can be hard to find. In [17], the authors proposed a new method termed Ensemble Kalman sampling, aiming at avoiding computing derivatives. We investigate the convergence rate of the method in this article. In particular we are interested in quantitively understand in what sense the ensemble covariance can replace the true covariance, and with what trend the ensemble distribution converges to the invariant measure (the target distribution).

The method can be summarized in Algorithm

There are a few parameters in the algorithm:

1. \( T = Nh \) is the stopping time, with \( h \) being the time step, and \( N \) being the number of iterations performed in the algorithm. The hope is to show the convergence to the target distribution is exponentially fast in \( T \).
2. \( J \) is the number of particles fixed ahead of time. The hope is to show that in the large \( J \) limit, the convergence of the sampling to the target distribution is at the order of \( 1/\sqrt{J} \) for any finite \( T \).
Algorithm 1 Ensemble Kalman sampling

Preparation:
1. Input: \( J \gg 1; h \) (stepsize); \( N \) (stopping index); \( \Gamma; \Gamma_0; \) and \( y \) (data).
2. Initial: \( \{u_0^j\} \) sampled from a initial distribution \( \rho_0 \).

Run: Set time step \( n = 0 \);
While \( n < N \):
1. Define empirical means and covariance:
\[
\overline{u}_n = \frac{1}{J} \sum_{j=1}^{J} u_n^j, \quad \text{and} \quad \overline{\mathcal{G}}_n = \frac{1}{J} \sum_{j=1}^{J} \mathcal{G}(u_n^j),
\]
\[
\text{Cov}_{u_n,u_n} = \frac{1}{J} \sum_{j=1}^{J} (u_n^j - \overline{u}_n) \otimes (u_n^j - \overline{u}_n), \quad \text{and} \quad \text{Cov}_{u_n,\mathcal{G}_n} = \frac{1}{J} \sum_{j=1}^{J} (u_n^j - \overline{u}_n) \otimes (\mathcal{G}(u_n^j) - \overline{\mathcal{G}}_n).
\]
2. Update ensemble particles (\( \forall 1 \leq j \leq J \))
\[
\begin{align*}
    u_{n+1}^j &= u_n^j - h \text{Cov}_{u_n,u_n} \Gamma^{-1} \left( \mathcal{G}(u_n^j) - y \right) - h \text{Cov}_{u_n,u_n} \Gamma_0^{-1} \left( u_{n+1}^j - u_0 \right), \\
    u_{n+1}^j &= u_n^j + \sqrt{2h \text{Cov}_{u_n,u_n}} \xi_n^j, \quad \text{with} \quad \xi_n^j \sim \mathcal{N}(0,1).
\end{align*}
\]
3. Set \( n \to n + 1 \).

end

Output: Ensemble particles \( \{u_N^j\} \).

3. \( \rho_0 \) is the initial distribution. It is not necessarily required that \( \rho_0 \) being equivalent to \( \rho_{\text{prior}} \). As will be shown in the later sections, the mean-field limit argument holds true as long as \( \rho_0 \) is smooth and have higher moments.

There are several important factors about analysis we need to emphasize:

- For each iteration, computing the ensemble covariance \( \text{Cov}_{u,u} \) and \( \text{Cov}_{u,\mathcal{G}} \) are the most expensive step. The perturbation \( \Xi \) is added on \( u \) directly. The EKS can be seen as a variation of Ensemble Kalman inversion (EKI) \[24\], which is a single step within Ensemble Kalman Filter \[29\]. EKI is another method for generating i.i.d. samples from the target distribution \( \rho_{\text{pos}} \). In comparison, EKI stops at finite time, and perturb observational data \( y \), but EKS looks for solution at infinite time and perturb particles directly. In \[13\], we proved the convergence of EKI.
- It is not our goal to compare different methods, but rather to provide the fundamental theoretical justification for one particular method: EKS. Without quantitative understanding of numerical error, we believe it is impossible to compare different sampling methods.
- Our result does assume linearity of \( \mathcal{G} \). However, due to the “ensemble” nature of the method, linear forward map still induces a nonlinear coupled SDE, and showing the mean-field limit to the corresponding Fokker-Planck equation still presents big mathematical challenges.
- In the near future we do plan to extend the results obtained in this paper to treat weak nonlinear forward map \( \mathcal{G} \) (we believe the method breaks down in the case of strong nonlinear forward map), and the results from this work lays the foundation.
- Compared to existing sampling strategies \[25\] \[31\], the Brownian term here is self-imposed, meaning the coefficient in front of \( \xi \) in \[4\] depends on the ensemble covariance. When carrying the mean-field limit, such coefficient introduces error as well, and thus the traditional way of bounding \( L^\infty \) is no longer valid, forcing us to use \( L^2 \) type quantization.

1.3. Continuum limit of Ensemble Kalman sampling. In this paper, our results are based on the continuum limit of \[4\]. Setting \( h \to 0 \), \( 4 \) becomes
\[
du_t^j = -\text{Cov}_{u_t,\mathcal{G}_t} \Gamma^{-1} (\mathcal{G}(u_t) - y) - \text{Cov}_{u_t,u_t} \Gamma_0^{-1} (u_t - u_0) + \sqrt{2\text{Cov}_{u_t,u_t}} dW_t^j, \quad (\forall j = 1, \cdots, J),
\]
where $\text{Cov}_{u_t, \mathcal{G}_t}$, $\text{Cov}_{u_t, u_t}$ are empirical variances:

$$\overline{u}_t = \frac{1}{J} \sum_{j=1}^{J} u^j_t, \quad \text{and} \quad \overline{\mathcal{G}}_t = \frac{1}{J} \sum_{j=1}^{J} \mathcal{G}(u^j_t),$$

$$\text{Cov}_{u_t, u_t} = \frac{1}{J} \sum_{j=1}^{J} \left( u^j_t - \overline{u}_t \right) \otimes \left( u^j_t - \overline{u}_t \right), \quad \text{and} \quad \text{Cov}_{u_t, \mathcal{G}_t} = \frac{1}{J} \sum_{j=1}^{J} \left( u^j_t - \overline{u}_t \right) \otimes \left( \mathcal{G}(u^j_t) - \overline{\mathcal{G}}_t \right).$$

In the linear setting, assuming (1) and (3), taking gradient of (2):

$$\nabla_u \Phi_R(u) = \nabla_u \mathcal{G}(u) \Gamma^{-1} (\mathcal{G}(u) - y) + \Gamma^{-1}_u (u - u_0) = B(u - u^*),$$

we reduce (5) to, for all $1 \leq j \leq J$

$$du_j^t = -\text{Cov}_{u_t, u_t} \nabla_u \Phi_R(u) \div 2\text{Cov}_{u_t, u_t} dW^j_t = -\text{Cov}_{u_t, u_t} B(u_j^t - u^*) \div 2\text{Cov}_{u_t, u_t} dW^j_t,$$ (7)

**Remark 1.1.** Since (5) involves no computation of derivatives $\nabla_u \mathcal{G}$, the authors in [17] argued this sampling method is superior than other methods. We do think moving from (4) to (5) is a big approximation. Since $\text{Cov}_{u, \mathcal{G}} \neq \text{Cov}_{u, \nabla_u \mathcal{G}}$, unless $\mathcal{G}$ has some kind of linearity. In this paper, however, we do not argue about the linear assumption but start with (4) directly, and show that this continuous version of the algorithm does provide samples approximately i.i.d. drawn from the target distribution $\mu_{\text{pos}}$.

We further define $M_u(u, t)$ to be the ensemble distribution:

$$M_{u_t}(u) = \frac{1}{J} \sum_{j=1}^{J} \delta_{u^j_t}(u).$$ (8)

The goal of this paper is to give a quantitative estimate of how this empirical distribution, in the Wasserstein distance, converges to the target distribution in both time $t$ and as the number of particles $J$ increases to infinity.

**Remark 1.2.** It has been a tradition to design sampling method that converges as $J \to \infty$, namely as $J \to \infty$ in long time the ensemble distribution becomes the invariant measure (the target distribution). In a five-page small note [27], the authors provide a very insightful adjustment to the "flux" term so that the invariant measure can be achieved by any finite number of samples in long time as well.

In the later sections we denote

$$\text{Cov}_{m, n} = \frac{1}{J} \sum_{j=1}^{J} \left( m^j_t - \overline{m}_t \right) \otimes \left( n^j_t - \overline{n}_t \right),$$

the covariance of any vectors $\{m^j_t\}_{j=1}^{J}$ and $\{n^j_t\}_{j=1}^{J}$, and abbreviate $\text{Cov}_{m} = \text{Cov}_{m, m}$. Set $\Omega$ be the sample space and $\mathcal{F}_0 = \sigma \{w^j(t = 0), 1 \leq j \leq J\}$, then the filtration introduced by (7) is:

$$\mathcal{F}_t = \sigma \{w^j(t), W^j_s, 1 \leq j \leq J, s \leq t\}.$$

### 1.4. Main results and strategy of our proof

We now present our main results and the roadmap of proof.

Throughout the paper we use the Wasserstein distance to quantify the error. It is a standard measurement that evaluates the distance between two measures.

**Definition 1.** Let $\nu_1, \nu_2$ be two probability measures in $(\mathbb{R}^L, \mathcal{B}_{\mathbb{R}^L})$, then the $W_2$-Wasserstein distance between $\nu_1, \nu_2$ is defined as

$$W_2(\nu_1, \nu_2) := \left( \inf_{\gamma \in \Gamma(\nu_1, \nu_2)} \int_{\mathbb{R}^L \times \mathbb{R}^L} |x - y|^2 d\gamma(x, y) \right)^{1/2},$$

where $\Gamma(\nu_1, \nu_2)$ denotes the collection of all measures on $\mathbb{R}^L \times \mathbb{R}^L$ with marginals $\nu_1$ and $\nu_2$.

Three major steps are to be taken.

**Step 1** In this step we justify the wellposedness of the SDE system (7). We mainly follow stochastic Lyapunov theory to show:
Theorem 1.1. If \( \{ u_j^0 : \Omega \to \mathcal{X} \}_{j=1}^J \) is independent almost surely, then for all \( t \geq 0 \), there exists a unique strong solution \( \{ u_j^t \}_{j=1}^J \) (up to \( \mathbb{P} \)-indistinguishability) of the set of coupled SDEs (7).

The wellposedness further gives upper bounds for high moments:

Theorem 1.2. For the solution \( \{ u_j^t \}_{j=1}^J \) of (7), if initial condition has finite higher moments, meaning there exists \( M > 0 \) independent of \( J \) such that

\[
\left( \mathbb{E} |u_0^j|_p^p \right)^{1/p} < M, \quad \forall 1 \leq j \leq J
\]

for \( p \geq 2 \), then the boundedness still holds true for any \( t \geq 0 \) and \( 1 \leq j \leq J \), namely:

1. \( \left( \mathbb{E} |u_j^t - \bar{u}^t|_p^p \right)^{1/p} \leq C e^{C t}, \quad \text{and} \quad \left( \mathbb{E} \left\| \text{Cov}_{u_t} u_t \right\|_2^p \right)^{1/p} \leq C e^{C t}, \) (9)

2. \( \left( \mathbb{E}|u_j^t|^p \right)^{1/p} \leq C e^{C e^{C t}}, \) (10)

with \( C > 0 \) depending only on \( p \) and \( M \).

Remark 1.3. As can be clearly seen in the statement of Theorem 1.2, the bounds of the high moments increase in time rather quickly. This bound may not be sharp. However, finite time bound is sufficient for showing the convergence in the later sections, and thus we do not pursue tighter bound.

Step 2 In this step we investigate the limiting Fokker-Planck system. Define the following Fokker-Planck equation:

\[
\begin{cases}
\partial_t \rho = \nabla \cdot \left( \rho \text{Cov}_{\rho(t)} \nabla \Phi_R(u) \right) + \text{Tr} \left( \text{Cov}_{\rho(t)} D^2 \rho \right), \\
\rho(u,0) = \rho_0
\end{cases}
\]

then one can show that the target distribution \( \rho_{\text{true}} \) is the equilibrium of the PDE, and the convergence rate is exponentially fast.

Theorem 1.3. Under assumptions (1) and (3), for arbitrary initial distribution \( \rho_0 \) that is smooth and has finite high moments,

– KL divergence converges to zero exponentially fast.

– If \( \rho_0 \) is a Gaussian distribution, \( W_2(\rho_t, \rho_{\text{true}}) \) converge to zero exponentially fast.

Naturally, one can define particle system that follows the flow of (11). Let

\[
dv_j^t = -\text{Cov}_{\rho(t)} \nabla \Phi_R(v_j^t) dt + \sqrt{2 \text{Cov}_{\rho(t)}} dW^t, \quad 1 \leq j \leq J,
\]

with \( v_0^j = u_0^j \) drawn from \( \rho_0 \). Define the ensemble distribution to be:

\[
M_{v_t}(u) = \frac{1}{J} \sum_{j=1}^J \delta_{v_j^t}(u).
\]

We can show:

Theorem 1.4. \( \{ v_j \} \) from (12) can be seen as i.i.d. samples from \( \rho(t,u) \) that solves (11). In particular, under assumption (1), (3), let \( \{ v_j^\epsilon \} \) drawn i.i.d. from \( \rho_0 \) that is smooth and has finite higher moments, then for any \( t > 0 \) and \( \epsilon \) such that, for all \( J \geq 1 \):

\[
\mathbb{E} \left( W_2(M_{v_t}(u)du, \rho(t,u)du) \right) \leq C \begin{cases}
J^{-1/2+\epsilon}, & L \leq 4 \\
J^{-2/L}, & L > 4
\end{cases}
\]

Here \( \rho \) is the solution to the Fokker-Planck equation, and \( M_{v_t} \) is the ensemble distribution (11).

Step 3 In this step we show the mean-field limit, namely we show that the two particle flows are asymptotically equivalent.
Theorem 1.5. Under assumptions \([11, 3]\), and assume the solution \(\rho(t, u)\) to \((11)\) satisfies
\[
\lambda_{\text{min}}(B)\lambda_{\text{min}}(\text{Cov}_{\rho(t)}) \geq 1, \quad \forall 0 \leq t \leq T.
\]
Suppose \(u^1_0 = v^0_0\) are i.i.d. drawn from \(\rho_0\) (smooth and has finite high moments), then \(\{u^1_0\}\) that solves \((7)\) and \(\{v^0\}\) that solves \((12)\) are close in the sense that for any \(0 < \epsilon \ll 1\), there exists a constant \(C\) depending only on \(L, T\) and \(\epsilon\) such that
\[
\mathbb{E}(W_2(M_{u^1_0}du, M_{v^0}du)) \leq \left( \frac{1}{J} \sum_{j=1}^J \mathbb{E}|u^1_T - v^0|_2^2 \right)^{1/2} \leq CJ^{-1/2+\epsilon}.
\]
\[
\text{(Main result)}
\]
(15)

Remark 1.4. We should note that due to technical reasons, the constant in front of the decay rate blows up with \(\epsilon \to 0\), and thus \((15)\) can’t be improved to \(J^{-1/2}\).

Combining Theorem 1.4 and Theorem 1.5 we quickly have:

Theorem 1.6. Under the conditions as in Theorem 1.5, for any \(0 < \epsilon \ll 1\), \(T > 0\) there exists \(J_\epsilon > 0\), such that for any \(J > J_\epsilon\)
\[
\mathbb{E}(W_2(\rho(T, u)du, M_{u^1_T}(u)du)) \leq \epsilon,
\]
where \(\rho(u, T)\) is the solution to \((11)\) at \(T\) using \(\rho_0\) as initial data, and \(M_{u^1_T}(u)\) is the ensemble distribution of \(\{u^1_T\}\) defined in \([8]\), with \(u^1_T\) solving \((7)\) and initially drawn from \(\rho_0\).

Proof. Considering \((10)\) and \((13)\), by triangle inequality, one has:
\[
\mathbb{E}(W_2(M_{u^1_T}du, \rho(T, u)du)) \leq \mathbb{E}(W_2(M_{u^1_T}du, M_{c_T}(du))) + \mathbb{E}(W_2(M_{c_T}du, \rho(T, u)du))
\]
\[
\leq C \begin{cases} 
J^{-1/4}, & L \leq 4 \\
J^{-2/4}, & L > 4 
\end{cases}.
\]
with \(C\) independent of \(J\). Setting this less than \(\epsilon\) gives \(J_\epsilon\) which concludes.
\[\square\]

Furthermore, combining this with Theorem 1.3 we have our main result:

Theorem 1.7 (Main result). Under the condition as in Theorem 1.5, for any \(0 < \epsilon \ll 1\), there exists \(T_\epsilon > 0\) and \(J_{T_\epsilon, \epsilon} > 0\) so that
\[
\mathbb{E}(W_2(M_{u^1_{T_\epsilon}}(u)du, \mu_{\text{pos}}du)) \leq \epsilon.
\]

Proof. For all \(0 < \epsilon \ll 1\), according to Theorem 1.3 there exists a time \(T_\epsilon > 0\) so that:
\[
W_2(\rho(T_\epsilon, u), \mu_{\text{pos}}du) \leq \epsilon/2.
\]
For this fixed \(T_\epsilon\), then according to Theorem 1.6 there is a \(J_{T_\epsilon, \epsilon} > 0\), such that for any \(J > J_{T_\epsilon, \epsilon}\)
\[
\mathbb{E}(W_2(\rho(T, u)du, M_{u^1_{T_\epsilon}}(u)du)) \leq \epsilon/2.
\]
The statement of the theorem is immediate with the triangle inequality.
\[\square\]

Remark 1.5. Several comments are in line:

- Theorem 1.7 gives a qualitative convergence result. The convergence depends on the stopping time \(T_\epsilon\) and number of ensembles \(J_\epsilon\). In contrast, Theorem 1.5 gives a quantitative result that states the convergence rate \(O(J^{-1/2+\epsilon})\) of obtaining i.i.d. samples. Further note the result of Theorem 1.6 doesn’t depend on \(L\) the dimension of \(u\), while that of Theorem 1.7 does.
- In fact, the initial condition for \((11)\) can be much more relaxed. The solution still converges to \(\rho_{\text{pos}}\) even if \(\rho(0, u)\) is not \(\rho_0\). That complicates notations and we avoid the discussion in this paper.

The later three sections are designated to the three steps described above. In particular, we show the wellposedness of the SDE system in Section 2 and prove Theorems 1.1 and 1.2. In Section 3 we investigate the convergence of the Fokker-Planck solution and its particle flow \(\{u^j\}\) with proof of Theorem 1.3 and Theorem 1.4. Finally in Section 4 we prove the equivalence between the two particle flows and finalize the proof of Theorem 1.5, Theorem 1.6 and 1.7 naturally follow.
2. Wellposedness of Noisy Ensemble Kalman Flow

In this section, we study the wellposedness of the SDE system \( \text{(7)} \). Considering each \( u^j \) is a vector of \( L \)-length, we stack them up to have a coupled SDE:

\[
dU_t = F(U_t)dt + G(U_t)dW_t,
\]

where \( U_t = \left( u^j_t \right)_{j=1}^J \in \mathbb{R}^{LJ \times 1} \), \( W_t = \left( W^j_t \right)_{j=1}^J \in \mathbb{R}^{LJ \times 1} \) and

\[
F(U_t) = \left( -\text{Cov}_u B \left( u^j_t - u^* \right) \right)_{j=1}^J \in \mathbb{R}^{LJ \times 1}, \quad G(U_t) = \text{diag} \left( \sqrt{2\text{Cov}_u} \right)_{j=1}^J,
\]

where \( \text{Cov}_u \) is the empirical covariance and \( \text{diag}(D_j)_{j=1}^J \) is a diagonal block matrix with matrices \( (D_j)_{j=1}^J \) on the diagonal.

We first show the wellposedness of the SDE system \( \text{(7)} \) by following the standard Lyapunov theory.

**Proof of Theorem 2**. According to the stochastic Lyapunov theory (See for example Theorem 4.1 [23]), strong solution exists if one finds local Lipschitz property of the drift \( F \) and the diffusion \( G \), namely we need to find a function \( V \in C^2(\mathbb{R}^{LJ};\mathbb{R}_+) \) so that:

- there is a \( c > 0 \) so that:
  \[
  LV(U) := \nabla V(U) \cdot F(U) + \frac{1}{2} \text{Tr} \left( G^T(U) \text{Hess}[V](U) G(U) \right) \leq c V(U), \tag{17}
  \]

- the function blows up at infinity:
  \[
  \inf_{|U| > R} V(U) \to \infty \text{ as } R \to \infty. \tag{18}
  \]

For that we define the Lyapunov function:

\[
V(U) = V_1(U) + V_2(U) = \frac{1}{J} \sum_{j=1}^J |u^j - \bar{u}|^2 + |\bar{u} - u^*|^2 = V_1 + V_2.
\]

To justify (17), we first notice that

\[
\nabla V_1(U) \cdot F(U) = -\frac{2}{J} \sum_{j=1}^J \langle u^j - \bar{u}, \text{Cov}_u B(u^j - u^*) \rangle = -\frac{2}{J} \sum_{j=1}^J \langle u^j - \bar{u}, \text{Cov}_u B(u^j - \bar{u}) \rangle \leq 0,
\]

\[
\nabla V_2(U) \cdot F(U) = -\frac{2}{J} \sum_{j=1}^J \langle B(\bar{u} - u^*), \text{Cov}_u B(u^j - u^*) \rangle = -2 \langle B(\bar{u} - u^*), \text{Cov}_u B(\bar{u} - u^*) \rangle \leq 0,
\]

where we used the facts that \( \text{Cov}_u \) and \( B \) are positive definite, and

\[
\frac{1}{2} \text{Tr} \left( G^T(U) \text{Hess}[V_1](U) G(U) \right) = \sum_{j=1}^J \frac{2}{J} \left( 1 - \frac{1}{J} \right) \langle u^j - \bar{u}, \rangle \leq 2 V_1(U),
\]

\[
\frac{1}{2} \text{Tr} \left( G^T(U) \text{Hess}[V_2](U) G(U) \right) = \sum_{j=1}^J \frac{2}{J} \langle u^j - \bar{u}, B(\bar{u} - u) \rangle \leq 2 \|B\|_2 V_1(U),
\]

Therefore we have

\[
LV(U) = \nabla V(U) \cdot F(U) + \frac{1}{2} \text{Tr} \left( G^T(U) \text{Hess}[V](U) G(U) \right) \leq 2(1 + \|B\|_2) V_1(U) \leq 2(1 + \|B\|_2) V(U).
\]

showing (17). To show (18), we run the argument of contradiction. Assume there exists \( M > 0 \) and a sequence \( \{U_n\}_{n=1}^\infty \) such that

\[
\lim_{n \to \infty} |U_n| = \infty, \quad V_1(U_n) + V_2(U_n) < M, \tag{19}
\]

then

\[
|u^j_n - \bar{u}_n| < \sqrt{MJ}, \quad |u^* - \bar{u}_n| < \sqrt{M},
\]
meaning:
\[ |U_n| = \left( \sum_{j=1}^{J} |u_n^j|^2 \right)^{1/2} < \left( \sum_{j=1}^{J} \left( |u^*| + \sqrt{M} (\sqrt{J} + 1) \right)^2 \right)^{1/2}, \]

contradicting (7).

We now move to show the boundedness of high moments of the SDE system. We firstly define
\[ e_t^j = u_t^j - \bar{u}_t. \]

Then it is obvious that
\[ \text{Cov}_{u_t} = \text{Cov}_{e_t}. \]

The following lemma shows the boundedness of any high moments of \( \{e_t^j\}_{j=1}^{J} \) at any finite time \( T \).

**Lemma 2.1.** If initial condition is finite, meaning there is a constant \( M > 0 \) independent of \( J \) so that
\[ \left( \mathbb{E} \left| u_0^j \right|^2 \right)^{1/2} < M, \quad \forall 1 \leq j \leq J, \]
then the boundedness holds true for all finite time for the SDE system (7), namely there is \( C > 0 \) so that:
\[ \left( \mathbb{E} |e_t^j|^2 \right)^{1/2} < M e^{Ct}. \]

**Proof.** By (20), we can write SDEs satisfied by \( \{e_t^j\}_{j=1}^{J} \) as
\[ de_t^j = -\text{Cov}_{e_t} B e_t^j dt + \sqrt{2\text{Cov}_{e_t}} d(W_t^j - \bar{W}_t), \quad \text{with} \quad \bar{W}_t = \frac{1}{J} \sum_{j=1}^{J} W_t^j. \]

With Itô’s formula we have:
\[ d|e_t^j|^2 = 2 \left\langle e_t^j, de_t^j \right\rangle + \left\langle de_t^j, de_t^j \right\rangle = -2 \left\langle e_t^j, \text{Cov}_{e_t} B e_t^j \right\rangle dt + 2 \left\langle e_t^j, \sqrt{2\text{Cov}_{e_t}} d(W_t^j - \bar{W}_t) \right\rangle 
+ 2 \left\langle \sqrt{\text{Cov}_{e_t}} d(W_t^j - \bar{W}_t), \sqrt{\text{Cov}_{e_t}} d(W_t^j - \bar{W}_t) \right\rangle, \]

leading to:
\[ d \left( \frac{1}{J} \sum_{j=1}^{J} |e_t^j|^2 \right) = \left( -\frac{2}{J} \sum_{j=1}^{J} \left\langle e_t^j, \text{Cov}_{e_t} B e_t^j \right\rangle + \frac{2(J-1)}{J} \text{Tr}(\text{Cov}_{e_t}) \right) dt + \frac{2}{J} \sum_{j=1}^{J} \left\langle e_t^j, \sqrt{2\text{Cov}_{e_t}} d(W_t^j - \bar{W}_t) \right\rangle \]

Noticing that
\[ \frac{2}{J} \sum_{j=1}^{J} \left\langle e_t^j, \text{Cov}_{e_t} B e_t^j \right\rangle = 2 \text{Tr} \left( \left[ \frac{1}{J} \sum_{j=1}^{J} e_t^j \otimes e_t^j \right] \left[ \frac{1}{J} \sum_{j=1}^{J} e_t^j \otimes e_t^j \right] B \right) \geq 0, \]

and
\[ \frac{2(J-1)}{J} \text{Tr}(\text{Cov}_{e_t}) = \frac{2}{J^2} \sum_{j=1}^{J} \left\langle e_t^j, e_t^j \right\rangle, \]

we take expectation of (21), and denote \( h_2(t) := \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^{J} |e_t^j|^2 \right), \) to have
\[ h_2(t) \leq h_2(0) + \int_{0}^{t} \frac{2(J-1)}{J} h_2(s) ds. \]

With the Grönwall inequality, we have:
\[ h_2(t) = \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^{J} |e_t^j|^2 \right) \leq e^{\frac{2(J-1)}{J}} \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^{J} |e_0^j|^2 \right) \leq h_2(0) e^{2t}, \]

concluding the lemma. \( \square \)
The result holds true for higher moments as well, namely we have:

**Lemma 2.2.** If initial condition is finite, there is a constant \( M > 0 \) independent of \( J \) so that

\[
\left( \mathbb{E}\left| u^j_0 \right|^p \right)^{1/p} < M, \quad \forall 1 \leq j \leq J,
\]

for some \( p > 2 \), then the boundedness holds true for all finite time for the SDE system (7), namely there is \( C > 0 \) depending on \( p \) only so that:

\[
\left( \mathbb{E}|e^j_t|^p \right)^{1/p} < Me^{Ct}.
\]

**Proof.** We first define

\[
e^j_t = \sqrt{B}e^j_t, \quad V_p(e) = \frac{1}{J} \sum_{j=1}^{J} \langle e^j_t, e^j_t \rangle^p,
\]

and

\[
h_p(t) = \mathbb{E}\left( \frac{1}{J} \sum_{j=1}^{J} \langle e^j_t, e^j_t \rangle^p \right) = \mathbb{E}V_p.
\]

Because \( \lambda_{\min}(B) > 0 \), it suffices to prove \( h_p(t) \) is bounded.

According to Itô’s formula, it holds that

\[
dV_p(e_t) = \sum_{j=1}^{J} \frac{\partial V_p(e_t)}{\partial e^j_t} de^j_t + \frac{1}{2} \sum_{i,j=1}^{J} \left( de^i_t \right)^\top \frac{\partial^2 V_p(e_t)}{\partial e^i_t \partial e^j_t} de^j_t,
\]

which implies

\[
dV_p(e_t) = -\frac{2p}{J} \sum_{j=1}^{J} \left( \langle e^j_t, e^j_t \rangle \right)^{p-1} \langle e^j_t, \text{Cov}_{e_t} e^j_t \rangle dt + \frac{2p}{J} \sum_{j=1}^{J} \left( \langle e^j_t, e^j_t \rangle \right)^{p-1} \langle e^j_t, \sqrt{B} \text{Cov}_{e_t} d \left( W^j_t - W_t \right) \rangle
\]

\[
+ \frac{8(J-1)p(p-1)}{J^2} \sum_{j=1}^{J} \left( \langle e^j_t, e^j_t \rangle \right)^{p-2} \text{Tr} \left\{ \left( e^j_t \otimes e^j_t \right) \text{Cov}_{e_t} \right\} dt + \frac{2(J-1)p}{J^2} \sum_{j=1}^{J} \left( \langle e^j_t, e^j_t \rangle \right)^{p-1} \text{Tr} \left\{ \text{Cov}_{e_t} \right\} dt.
\]

Then taking the expectation and eliminate the nonpositive first term:

\[
h_p(t) - h_p(0) \leq \frac{8(J-1)p(p-1)}{J^3} \int_0^t \sum_{j,k=1}^{J} \mathbb{E}\left( \langle e^j_s, e^j_s \rangle \right)^{p-2} \langle e^j_s, e^k_s \rangle^2 ds + \frac{2(J-1)p}{J^3} \int_0^t \sum_{j,k=1}^{J} \mathbb{E}\left( \langle e^j_s, e^j_s \rangle \right)^{p-1} \langle e^k_s, e^k_s \rangle^2 ds
\]

\[
+ \frac{8(J-1)p(p-1)}{J^3} \int_0^t \sum_{j,k=1}^{J} \mathbb{E}\left( \langle e^j_s, e^j_s \rangle \right)^{p-2} \langle e^j_s, e^k_s \rangle^2 ds + \frac{2(J-1)p}{J^3} \int_0^t \sum_{j,k=1}^{J} \mathbb{E}\left( \langle e^j_s, e^j_s \rangle \right)^{p-1} \langle e^k_s, e^k_s \rangle ds
\]

\[
\leq \frac{4(J-1)p(p-1)}{J} \int_0^t h_p(s) ds + \frac{4(J-1)p(p-1)}{J^3} \int_0^t \sum_{j,k=1}^{J} \mathbb{E}\left( \langle e^j_s, e^j_s \rangle \right)^{p-2} \langle e^k_s, e^k_s \rangle^2 ds
\]

\[
+ \frac{2(J-1)p}{J^3} \int_0^t \sum_{j,k=1}^{J} \mathbb{E}\left( \langle e^j_s, e^j_s \rangle \right)^{p-1} \langle e^k_s, e^k_s \rangle ds.
\]
Using the Hölder’s inequality we can control the second and third term, namely:

\[
\sum_{j,k=1}^{J} \mathbb{E} \langle e_j^k, e_j^k \rangle^{p-2} \langle e_j^k, e_j^k \rangle^2 = \mathbb{E} \left[ \sum_{j=1}^{J} \langle e_j^j, e_j^j \rangle^{p-2} \right] \leq J \mathbb{E} \left[ \sum_{j=1}^{J} \langle e_j^j, e_j^j \rangle^p \right] = J \mathbb{E} \left[ \sum_{j=1}^{J} \langle e_j^j, e_j^j \rangle^2 \right] = J^2 h_p(t),
\]

and

\[
\sum_{j,k=1}^{J} \mathbb{E} \langle e_j^k, e_j^k \rangle^{p-1} \langle e_j^k, e_j^k \rangle = \mathbb{E} \left[ \sum_{j=1}^{J} \langle e_j^j, e_j^j \rangle^{p-1} \right] \leq J \mathbb{E} \left[ \sum_{j=1}^{J} \langle e_j^j, e_j^j \rangle^p \right] = J^2 h_p(t).
\]

Plug (23)–(24) into (22), we finally have \( h_p(t) - h_p(p) \leq \frac{(J-1)2^{p-6}}{(J-1)p} \int_0^t h_p(s)ds \), which leads to the conclusion using the Grönwall inequality:

\[
h_p(t) \leq h_p(0) e^{\frac{2(J-1)p(4p-3)\bar{t}}{J}},
\]

to conclude.

Now, we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2**. The first inequality of equation (9) is already shown in Lemma 2.2. The second inequality is a direct consequence:

\[
\left( \mathbb{E} \| \text{Cov}_u \|_2 \right)^{1/p} \leq \left( \frac{1}{J} \sum_{j=1}^{J} \mathbb{E} \| (u_j^t - \overline{u}_t) \otimes (u_j^t - \overline{u}_t) \|_2^p \right)^{1/p} \leq \left( \frac{1}{J} \sum_{j=1}^{J} \mathbb{E} \| u_j^t - \overline{u}_t \|_2^{2p} \right)^{1/p} \leq C \sqrt{Ct/2}.
\]

To show (10), define:

\[
u_j^t = \sqrt{B} u_j^t, \quad \nu^* = \sqrt{B} u^*, \quad K_p(u) = \frac{1}{J} \sum_{j=1}^{J} \langle u_j^j, u_j^j \rangle^p,
\]

and

\[
g(t) = \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^{J} \langle u_j^j, u_j^j \rangle^p \right) = \mathbb{E} (K_p(u_t)).
\]

Then it’s suffices to control the growth of \( g(t) \) because \( \lambda_{\text{min}}(B) > 0 \). We first multiply \( \sqrt{B} \) onto both sides of (13) to obtain

\[
du_j^t = -\text{Cov}_u (u_j^t - u^*) + \sqrt{B} \sqrt{2\text{Cov}_u} dW_j^t.
\]

Using Itô’s lemma to have:

\[
dK_p(u) = \sum_{j=1}^{J} \frac{\partial K_p(u)}{\partial u_j} du_j^t + \frac{1}{2} \sum_{i,j=1}^{J} du_i^t \frac{\partial^2 K_p(u)}{\partial u_i^t \partial u_j^t} du_j^t,
\]

which implies

\[
dK_p(u) = -\frac{2p}{J} \sum_{j=1}^{J} \langle u_j^t, u_j^t \rangle^{p-1} \langle u_j^t, \text{Cov}_u (u_j^t - u^*) \rangle dt + \frac{2p}{J} \sum_{j=1}^{J} \langle u_j^t, u_j^t \rangle^{p-1} \langle u_j^t, \sqrt{B} \sqrt{2\text{Cov}_u} dW_j^t \rangle
\]

\[+ \frac{8p(p-1)}{J} \sum_{j=1}^{J} \langle u_j^t, u_j^t \rangle^{p-2} \text{Tr} \left\{ (u_j^t \otimes u_j^t) \text{Cov}_u \right\} dt + \frac{2p}{J} \sum_{j=1}^{J} \langle u_j^t, u_j^t \rangle^{p-1} \text{Tr} \left\{ \text{Cov}_u \right\} dt.
\]
The expectation of the second term vanishes, and to control the first term, we note:

\[-\frac{2p}{J} \sum_{j=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^{p-1} \left( u_i^j, \text{Cov}_{u_i} \left( u_i^j - u^* \right) \right) \]

\[= -\frac{2p}{J} \sum_{j=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^{p-1} \left( u_i^j, \text{Cov}_{u_i} u_i^j \right) + \frac{2p}{J} \sum_{j=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^{p-1} \left( u_i^j, \text{Cov}_{u_i} u^* \right) \]

\[\leq \frac{2p}{J^2} \sum_{j,k=1}^{J} \mathbb{E} \left\{ \left( u_i^j, u_i^k \right)^{p-1} \left( u_i^j, e_i^k \right) \langle e_i^k, u_i^* \rangle \right\} \]

\[\leq \frac{2p}{J} \sum_{j,k=1}^{J} \mathbb{E} \left\{ \left( u_i^j, u_i^j \right)^{p-1} \left| u_i^j \right| \left| e_i^k \right| \left| e_i^k \right| \left| u_i^* \right| \right\} \]

\[\leq 2p \left| u^* \right| \mathbb{E} \left\{ \left( \frac{1}{J} \sum_{j=1}^{J} \left( u_i^j, u_i^j \right)^{p-1} \right) \left( \frac{1}{J} \sum_{k=1}^{J} \left( e_i^k, e_i^k \right) \right) \right\} \]

\[\leq 2p \left| u^* \right| \left( \frac{1}{J} \sum_{j=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^p \right)^{(p-1)/p} \left( \frac{1}{J} \sum_{k=1}^{J} \mathbb{E} \left( e_i^k, e_i^k \right)^{2p} \right)^{1/(2p)} \leq Ce^{Ct} \left( \frac{1}{J} \sum_{j=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^p \right)^{(p-1)/p} = Ce^{Ct} g^{(p-1)/p}(t), \]

where the second last inequality comes from Hölder’s inequality, and we used the estimate from Lemma 2.2.

To control the third and fourth term, we have:

\[\frac{8p(p-1)}{J} \sum_{j=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^{p-2} \text{Tr} \left\{ \left( u_i^j \otimes u_i^j \right) \text{Cov}_{u_i} \right\} = \frac{8p(p-1)}{J} \sum_{j=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^{p-2} \left( \frac{1}{J} \sum_{k=1}^{J} \left( u_i^j, e_i^k \right)^2 \right) \]

\[\leq \frac{8p(p-1)}{J^2} \sum_{j,k=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^{p-1} \left( e_i^k, e_i^k \right) \]

\[\leq 8p(p-1) \left( \frac{1}{J} \sum_{k=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^p \right)^{(p-1)/p} \left( \frac{1}{J} \sum_{k=1}^{J} \mathbb{E} \left( e_i^k, e_i^k \right)^p \right)^{1/p} \leq Ce^{Ct} \left( \frac{1}{J} \sum_{k=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^p \right)^{(p-1)/p} = Ce^{Ct} g^{(p-1)/p}(t), \]

and

\[\frac{2p}{J} \sum_{j=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^{p-1} \text{Tr} \left\{ \text{Cov}_{u_i} \right\} \leq \frac{2p}{J} \sum_{j,k=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^{p-1} \left( e_i^k, e_i^k \right) \]

\[\leq 2p \left( \frac{1}{J} \sum_{k=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^p \right)^{(p-1)/p} \left( \frac{1}{J} \sum_{k=1}^{J} \mathbb{E} \left( e_i^k, e_i^k \right)^p \right)^{1/p} \leq Ce^{Ct} \left( \frac{1}{J} \sum_{k=1}^{J} \mathbb{E} \left( u_i^j, u_i^j \right)^p \right)^{(p-1)/p} = Ce^{Ct} g^{(p-1)/p}(t). \]

In conclusion, we obtain

\[\frac{dg}{dt} \leq Ce^{Ct} \left[ g^{(p-1)/p}(t) + g^{(p-1)/p}(t) \right] \Rightarrow g(t) \leq g(t = 0)Ce^{Ct}. \]
3. Fokker-Planck equation and corresponding particle system

In this section, we study the wellposedness of the limiting PDE (11), its convergence rate to the equilibrium, and its i.i.d. samples, the \( \{v_i^j\} \) system.

We first adopt a result from [17].

**Lemma 3.1** (Lemma 3, Proposition 4 from [17]). Under the assumption of (11) and (3), let \( \rho(t, u) \) solve (11) with initial distribution \( \rho_0 \) that is smooth has finite second moments, then the mean \( m \) and the covariance \( C \) of the solution to (11) is governed by

\[
\frac{d}{dt} m(t) = -C(t)(Bm(t) - r), \quad \frac{d}{dt} C(t) = -2C(t)BC(t) + 2C(t).
\]

Furthermore, \( m(t) \to E(\rho_{pos}) \) and \( C(t) \to \text{Cov}(\rho_{pos}) \) exponentially as \( t \to \infty \).

Now we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** To show the convergence in KL-divergence, we directly cite the result from Proposition 2 of [17]. If \( \rho_0 \) is a Gaussian distribution, then \( \rho_t \) is always a Gaussian:

\[
\rho(t, u) = \frac{1}{(2\pi)^{L/2} (\text{det} C(t))^{1/2}} \exp \left( -\frac{1}{2} \| u - m_t \|_C^2 \right),
\]

and that

\[
W_2(\rho_t, \rho_{pos}) \leq \| E(\rho_t) - E(\rho_{pos}) \|^2 + B^2 \left( \text{Cov}(\rho_t), \text{Cov}(\rho_{pos}) \right),
\]

where

\[
B^2 \left( \text{Cov}(\rho_t), \text{Cov}(\rho_{pos}) \right) = \text{Tr} \left( \text{Cov}(\rho_t) \right) + \text{Tr} \left( \text{Cov}(\rho_{pos}) \right) - 2\text{Tr} \left[ \left( \text{Cov}^{1/2}(\rho_t) \text{Cov}^{1/2}(\rho_{pos}) \right)^2 \right].
\]

According to Lemma 3.1, the result of Theorem 1.3 follows. \( \square \)

Finally, since the solution to the PDE is a Gaussian function for all time, with known dynamics of the moments, we can easily obtain upper bounds for higher moments:

**Lemma 3.2.** If \( \rho_0 \) is smooth and has finite high moments, then for any \( p \geq 2, t > 0 \), there exists a constant \( C \) depending on \( p \) and \( t \) such that

\[
\int |u|^p \rho(t, u) du \leq C(p, t) < \infty, \quad \text{and} \quad \| \text{Cov}(\rho_t) \|_2^p \leq C(p, t) < \infty.
\]

**Proof.** According to Lemma 3.1, the covariance of \( \rho(t, u) \) is uniformly bound, namely:

\[
\| \text{Cov}(\rho_t) \|_F \leq M, \quad \forall t > 0,
\]

for a constant \( M \) independent of \( t \). This means the transport and Hessian coefficient of (11) are uniformly bounded, considering the formula in (11):

\[
\| \text{Cov}(\rho_t) \nabla_u \Phi_R(u) \|_2 = \| \text{Cov}(\rho_t) \nabla_u [B(u - u^*)] \|_2 = \| \text{Cov}(\rho_t) B \|_2 \leq \| B \|_2 M, \quad \text{for all} \ t > 0,
\]

and this implies, using (11), that high moments of \( \rho(t) \) are also finite for any time \( t < \infty \). \( \square \)

The properties shown above describe the dynamics and the boundedness of \( \rho(t) \), the solution to the Fokker-Planck equation (11). With these, it is relatively easy to obtain properties of samples i.i.d. drawn from \( \rho(t, u) \). In particular, the SDE system (12) generates i.i.d. samples from \( \rho(t, u) \).

**Proposition 3.1.** For the SDE system (12), under assumption (11), (3), if \( \rho_0 \) is smooth and has finite high moments, then for any \( J \), the bound holds true for all finite time \( t \), namely there is \( C > 0 \) depending on \( p, M, t \) so that for all \( 1 \leq j \leq J \):

\[
\left( \mathbb{E} |v_j^j|^p \right)^{1/p} \leq C, \quad \left( \mathbb{E} \| \text{Cov}(v_j^j(t)) \|_2^p \right)^{1/p} \leq C, \quad \left( \mathbb{E} |v_j^j - \bar{v}_t|^p \right)^{1/p} \leq C.
\]

Furthermore we have central limit theorem,

\[
\left( \mathbb{E} \| \text{Cov}(v_j^j - \text{Cov}(\rho(t))) \|_2^p \right)^{1/p} \leq C J^{-1/2}.
\]
Proof. The bounds in (20) are immediate considering Lemma 3.2. We only show (27) here. Without loss of generality, assume $E(v_i^1) = 0$, then we write $\text{Cov}_{v_t}$ as

$$\text{Cov}_{v_t} = \frac{J - 1}{J^2} \left( \sum_{j=1}^{J} v_i^j \otimes v_i^j \right) - \frac{1}{J^2} \sum_{j \neq k}^{J} v_i^j \otimes v_i^k.$$

Now we divide (27) into three parts

$$\left( E \| \text{Cov}_{v_t} - \text{Cov}_{\rho(t)} \|_2 \right)^{1/p} \leq \left( E \left\| \frac{J - 1}{J^2} \left( \sum_{j=1}^{J} v_i^j \otimes v_i^j \right) - \frac{J - 1}{J^2} \left( \sum_{j=1}^{J} \text{Cov}_{\rho(t)} \right) \right\|_2 \right)^{1/p}$$

$$+ \left( E \left\| \frac{1}{J^2} \sum_{j \neq k}^{J} v_i^j \otimes v_i^k \right\|_2 \right)^{1/p}$$

$$+ \left( E \| \text{Cov}_{\rho(t)} \|_2 \right)^{1/p}. \quad (28)$$

The third term is of $O(\lesssim J^{-2})$ since $\| \text{Cov}_{\rho(t)} \|_2$ is bounded according to Lemma 3.2. To analyze the first term, we have

$$\left( E \left\| \frac{J - 1}{J^2} \left( \sum_{j=1}^{J} v_i^j \otimes v_i^j \right) - \frac{J - 1}{J^2} \left( \sum_{j=1}^{J} \text{Cov}_{\rho(t)} \right) \right\|_2 \right)^{1/p} \leq 2 \left( E \left\| \frac{1}{J} \left( \sum_{j=1}^{J} v_i^j \otimes v_i^j \right) - \text{Cov}_{\rho(t)} \right\|_F \right)^{1/p}$$

$$\leq C_{p,L} \sum_{m,n=1}^{L} \left( E \left\| \frac{1}{J} \sum_{j=1}^{J} v_i^j \otimes v_i^j - \text{Cov}_{\rho(t)} \right\|_{m,n} \right)^{1/p}$$

$$= \frac{C_{p,L}}{J^{1/2}} \sum_{m,n=1}^{L} \left\{ E \left| \frac{\sum_{j=1}^{J} (v_i^j \otimes v_i^j - \text{Cov}_{\rho(t)})_{m,n}}{\sqrt{J}} \right|^p \right\}^{1/p}, \quad (29)$$

where $\left( \frac{1}{J} \sum_{j=1}^{J} v_i^j \otimes v_i^j - \text{Cov}_{\rho(t)} \right)_{m,n}$ means the $(m,n)^{th}$ entry of the matrix, and the constant $C_{p,L}$ has $p$ and $L$ dependence. With the central limit theorem, we have

$$\frac{\sum_{j=1}^{J} (v_i^j \otimes v_i^j - \text{Cov}_{\rho(t)})_{m,n}}{\sqrt{J}} \xrightarrow{d} N(0, V_{m,n}),$$

where $V_{m,n}$ can be calculated using the first four moments of $\rho$, and is also an $O(1)$ term. This implies

$$E \left( \frac{\sum_{j=1}^{J} (v_i^j \otimes v_i^j - \text{Cov}_{\rho(t)})_{m,n}}{\sqrt{J}} \right)^p \sim O(1). \quad (30)$$
For second term of (28), we first use Hölder’s inequality to obtain
\[
\left( \mathbb{E} \left( \frac{1}{J^2} \sum_{j \neq k} v_j^q \otimes v_k^q \right)^{\frac{1}{p}} \right)^p \leq \left( \mathbb{E} \left( \frac{1}{J^{2p}} \left( \sum_j \left\| \sum_{j \neq k} v_j^q \otimes v_k^q \right\|_2^p \right)^{\frac{1}{p}} \right) \right) \leq \left( \mathbb{E} \left( \frac{1}{J} \sum_j \sum_{j \neq k} v_j^q \otimes v_k^q \right)^{\frac{1}{p}} \right) \]
\[
= \left( \mathbb{E} \left\| \sum_{k=2}^J v_j^q \otimes v_k^q \right\|_2 \right)^{\frac{1}{p}},
\]
where the equality comes from symmetry of particles. Similar to (29), we further bound it by
\[
\left( \mathbb{E} \left\| \sum_{j \neq k} v_j^q \otimes v_k^q \right\|_2 \right)^{\frac{1}{p}} \leq C_{p,L} \sum_{m,n=1}^L \left( \mathbb{E} \left( \left| \frac{1}{J} \sum_{k=2}^J (v_j^q)_{m} \right|^p \mathbb{E} \left| \frac{1}{J} \sum_{k=2}^J (v_k^q)_{n} \right|^p \right) \right)^{\frac{1}{p}} \]
\[
\leq C_{p,L} \sum_{m,n=1}^L \left( \mathbb{E} \left| \sum_{k=2}^J (v_k^q)_{m} \right|^p \mathbb{E} \left| \sum_{k=2}^J (v_k^q)_{n} \right|^p \right)^{\frac{1}{p}} \approx O(J^{-1/2}),
\]
where we use uniform boundedness of high moments to bound \( \mathbb{E} \left( (v_j^q)_{m} \right|^p \) and central limit theorem again in the last inequality. Plug (30)-(31) into (28), we finally obtain (27). \( \square \)

It’s a classical result that ensemble distribution of i.i.d. samples approximates the original measure:

**Theorem 3.1** (Theorem 1 in [10]). Let \( \rho(u)du \) be a probability measure on \( \mathbb{R}^d \) and let \( p > 0 \). Assume that
\[
M_q(p) := \int_{\mathbb{R}^d} |x|^q \rho(dx) < \infty
\]
(32)
for some \( q > p \). Consider an i.i.d. sequence \( (X_k)_{k \geq 1} \) of \( \rho \)-distributed random variables and, for \( J \geq 1 \), define the empirical measure
\[
\rho_N := \frac{1}{J} \sum_{k=1}^J \delta_{X_k}.
\]

Then for all \( N \geq 1 \) and \( 0 < \epsilon \ll 1 \), there exists a constant \( C \) depending only on \( p, L, q, \epsilon \) such that
\[
\mathbb{E} (W_p(\rho_N du, \rho du)) \leq C M_{q/p}^p(\rho) \begin{cases} J^{-1/2+\epsilon} + J^{-(q-p)/q}, & \text{if } p \geq L/2 \text{ and } q \neq 2p \\ J^{-p/L} + J^{-(q-p)/q}, & \text{if } p \in (0, L/2), \text{ and } q \neq L/(L-p). \end{cases}
\]

Combine Proposition 3.1 and Theorem 3.1 we can show Theorem 1.4.

**Proof of Theorem 1.4.** Since (32) holds true according to (26), we conclude the proof by choosing \( p = 2 \) and \( q \) large enough in Theorem 3.1. \( \square \)

4. Equivalence of SDE and PDE, mean field limit

In this section we show that the two particle systems are asymptotically equivalent. More specifically, \( \{u^j\} \) system is governed by a coupled SDE, while \( \{v^j\} \) comes from i.i.d. sampling of the Fokker-Planck equation. We will show the \( W_2 \)-Wasserstein distance of the ensemble distribution of \( \{v_j\} \) and \( \{u_j\} \) converge in \( J \) for all \( t > 0 \). This kind of techniques are usually termed the derivation of mean-field limit, and
they have been applied in many applications, including sampling method [6, 7, 13, 25, 26, 35] and particle method for PDE [28, 20, 8, 9].

First, we unify the notations. Let

\[ x_t^j = u_t^j - v_t^j, \quad p_t^j = x_t^j - x_t, \quad q_t^j = v_t^j - x_t, \]

then we have

\[ \text{Cov}_{u_t} = \text{Cov}_{x_t+v_t} = \text{Cov}_{v_t+q_t}, \quad \text{Cov}_{v_t} = \text{Cov}_{q_t}, \quad \text{Cov}_{x_t} = \text{Cov}_{p_t}. \]

As a preparation, we give an priori estimation for the moments of \( x_t^j \):

**Proposition 4.1.** Under assumptions [11, 33] and condition (15), for all \( 2 \leq p < \infty \), we have a constant \( C_p \) independent of \( J \) and \( t \) such that:

\[ \mathbb{E}|x_t^j|^p \leq C_p, \quad \mathbb{E}|p_t^j|^p \leq C_p \quad \forall 1 \leq j \leq J. \]

**Proof.** The first inequality is a direct result from the fact that

\[ \left( \mathbb{E}|x_t^j|^p \right)^{1/p} \leq \left( \mathbb{E}|u_t^j|^p \right)^{1/p} + \left( \mathbb{E}|v_t^j|^p \right)^{1/p} \]

and then applying Theorem [11, 33] and Proposition [33, 26]. Then the second inequality comes from

\[ \left( \mathbb{E}|p_t^j|^p \right)^{1/p} \leq \left( \mathbb{E}|x_t^j|^p \right)^{1/p} + \left( \mathbb{E}|x_t^j|^p \right)^{1/p} + \frac{1}{J} \sum_{j=1}^{J} \left( \mathbb{E}|x_t^j|^p \right)^{1/p} \leq 2 \left( \mathbb{E}|x_t^j|^p \right)^{1/p}. \]

\[ \square \]

The proof of Theorem 4.5 is built upon the following lemma:

**Lemma 4.1.** Under assumptions [11, 33] and condition (15), for any \( 0 \leq \alpha < 1 \) and \( T > 0 \), if:

\[ \mathbb{E}|x_t^j|^2 \lesssim O \left( J^{-\alpha} \right), \quad \forall 1 \leq j \leq J, \quad 0 \leq t \leq T, \]

then we can tighten the decay rate, namely: for any \( \epsilon > 0 \) and \( 1 \leq j \leq J \)

\[ \mathbb{E}\left| p_t^j \right|^2 = \mathbb{E}\left| x_t^j - \frac{1}{J} \sum_{k=1}^{J} x_t^k \right|^2 \lesssim O(J^{-1/2-\alpha/2+\epsilon}), \quad \text{and} \quad \mathbb{E}|x_t^j|^2 \lesssim O \left( J^{-1/2-\alpha/2+\epsilon} \right). \]

**Proof.** First of all, due to the symmetry of the particle system, for all \( 1 \leq j \leq J \) and \( 0 \leq t \leq 1 \):

\[ \mathbb{E}|p_t^j|^2 = \mathbb{E}|p_t^1|^2, \quad \mathbb{E}|x_t^j|^2 = \mathbb{E}|x_t^1|^2. \]

Then condition (35) implies

\[ \left( \mathbb{E}|p_t|^2 \right)^{1/2} \leq \left( \mathbb{E}|x_t^1|^2 \right)^{1/2} + \left( \mathbb{E}|x_t|^2 \right)^{1/2} \leq 2 \left( \mathbb{E}|x|^2 \right)^{1/2} \lesssim (J^{-\alpha/2}). \]

Subtracting the SDEs (7) and (12), we have

\[ dx_t^j = \left( -\text{Cov}_{x_t+v_t} B(x_t^j + v_t^j) + \text{Cov}_{\rho(t)} Bv_t^j \right) dt + \left( \text{Cov}_{x_t+v_t} - \text{Cov}_{\rho(t)} \right) \text{Bu}^t dt \]

\[ + \left( \sqrt{2\text{Cov}_{x_t+v_t}} - \sqrt{2\text{Cov}_{\rho(t)}} \right) dW_t^j. \]

Using Ito’s formula, this becomes

\[ d|x_t|^2 = -2 \left( x_t^j, \text{Cov}_{x_t+v_t} Bx_t^j \right) dt - 2 \left( x_t^j, \left( \text{Cov}_{x_t+v_t} - \text{Cov}_{\rho(t)} \right) Bu_t^j \right) dt \]

\[ + 2 \left( x_t^j, \left( \text{Cov}_{x_t+v_t} - \sqrt{2\text{Cov}_{\rho(t)}} \right) Bu_t^j \right) dt + 2 \text{Tr} \left( \sqrt{\text{Cov}_{x_t+v_t} - \sqrt{\text{Cov}_{\rho(t)}}}^2 \right) dt \]

\[ + 2 \left( x_t^j, \left( \sqrt{2\text{Cov}_{x_t+v_t} - 2\text{Cov}_{\rho(t)}} \right) dW_t^j \right). \]
Replace Cov\(_{\rho(t)}\) with Cov\(_{v_t}\) in second and third terms, we obtain

\[
d|x_t|^2 = -2 \left< x_t^j, \text{Cov}_{x_t+v_t} Bx_t^j \right> dt - 2 \left< x_t^j, (\text{Cov}_{x_t+v_t} - \text{Cov}_{v_t}) Bv_t^j \right> dt \\
+ 2 \left< x_t^j, (\text{Cov}_{x_t+v_t} - \text{Cov}_{v_t}) Bu^* \right> dt + 2 \text{Tr} \left( \sqrt{\text{Cov}_{x_t+v_t}} - \sqrt{\text{Cov}_{v_t}} \right)^2 dt \\
+ 2 \left< x_t^j, \left( \sqrt{2\text{Cov}_{x_t+v_t}} - \sqrt{2\text{Cov}_{v_t}} \right) dW_t^j \right> + R_t^j dt,
\]

where the remainder \(R_t^j\) is introduced to account for the replacement:

\[
R_t^j = 2 \left< x_t^j, (\text{Cov}_{\rho(t)} - \text{Cov}_{v_t}) Bv_t^j \right> dt - 2 \left< x_t^j, (\text{Cov}_{\rho(t)} - \text{Cov}_{v_t}) Bu^* \right> dt.
\]

We then take average of (37) in \(j\) to obtain

\[
d|\mathbf{x}_t|^2 = -2 \left< \mathbf{x}_t, \text{Cov}_{x_t+v_t} B\mathbf{x}_t \right> dt - 2 \left< \mathbf{x}_t, (\text{Cov}_{x_t+v_t} - \text{Cov}_{v_t}) B\mathbf{v}_t \right> dt \\
+ (\sqrt{2\text{Cov}_{x_t+v_t}} - \sqrt{2\text{Cov}_{v_t}}) d\mathbf{W}_t,
\]

so that according to Ito’s formula:

\[
d|\mathbf{x}_t|^2 = -2 \left< \mathbf{x}_t, \text{Cov}_{x_t+v_t} B\mathbf{x}_t \right> dt - 2 \left< \mathbf{x}_t, (\text{Cov}_{x_t+v_t} - \text{Cov}_{v_t}) B\mathbf{v}_t \right> dt \\
+ 2 \left< \mathbf{x}_t, (\text{Cov}_{x_t+v_t} - \text{Cov}_{v_t}) Bu^* \right> dt + \frac{2}{J} \text{Tr} \left( \sqrt{\text{Cov}_{x_t+v_t}} - \sqrt{\text{Cov}_{v_t}} \right)^2 dt \\
+ 2 \left< \mathbf{x}_t, \left( \sqrt{2\text{Cov}_{x_t+v_t}} - \sqrt{2\text{Cov}_{v_t}} \right) d\mathbf{W}_t \right> + \mathbf{R}_t dt,
\]

where the remainder term:

\[
\mathbf{R}_t = 2 \left< \mathbf{x}_t, (\text{Cov}_{\rho(t)} - \text{Cov}_{v_t}) B\mathbf{v}_t \right> dt - 2 \left< \mathbf{x}_t, (\text{Cov}_{\rho(t)} - \text{Cov}_{v_t}) Bu^* \right> dt.
\]

Combine (38) and (39), it is a straightforward calculation that:

\[
d \left( \frac{1}{J} \sum_{j=1}^J |x_t^j|^2 - |\mathbf{x}_t|^2 \right) = -\frac{2}{J} \sum_{j=1}^J \left< p_t^j, \text{Cov}_{p_t+q_t} Bp_t^j \right> dt - \frac{2}{J} \sum_{j=1}^J \left< p_t^j, (\text{Cov}_{p_t+q_t} - \text{Cov}_{q_t}) Bq_t^j \right> dt \\
+ 2 \left( 1 - \frac{1}{J} \right) \text{Tr} \left( \sqrt{\text{Cov}_{x_t+v_t}} - \sqrt{\text{Cov}_{v_t}} \right)^2 dt + \left( \frac{1}{J} \sum_{j=1}^J R_t^j - \mathbf{R}_t \right) dt \\
+ \frac{2}{J} \sum_{j=1}^J \left< x_t^j - \mathbf{x}_t, \left( \sqrt{2\text{Cov}_{x_t+v_t}} - \sqrt{2\text{Cov}_{v_t}} \right) d\mathbf{W}_t \right>.
\]

The expectation of the last term is 0 due to the property of the Brownian motion. Since \(R_t^j\) involves the difference between Cov\(_{\rho}\) and Cov\(_{v}\), it is expected that it can be controlled using the central limit theorem. In fact, with Proposition 5.1 we have:

\[
\mathbb{E} \left< \frac{1}{J} \sum_{j=1}^J R_t^j - \mathbf{R}_t \right> \leq 2 \left( \mathbb{E} \|\text{Cov}_{\rho(t)} - \text{Cov}_{v_t} \|^2 \right)^{1/2} \left( \mathbb{E} \|p_t^j\|^2 \|Bq_t^j\|^2 \right)^{1/2} \\
\leq 2 \left( \mathbb{E} \|\text{Cov}_{\rho(t)} - \text{Cov}_{v_t} \|^2 \right)^{1/2} \left( \mathbb{E} \|p_t^j\|^2 - \epsilon \|p_t^j\| \|Bq_t^j\| \right)^{1/2} \\
\leq 2 \left( \mathbb{E} \|\text{Cov}_{\rho(t)} - \text{Cov}_{v_t} \|^2 \right)^{1/2} \left( \mathbb{E} \|p_t^j\|^2 \right)^{(2-\epsilon)/4} \left( \mathbb{E} \|Bq_t^j\|^4 / \epsilon \right)^{\epsilon/4} \\
\leq 2 \left( \mathbb{E} \|\text{Cov}_{\rho(t)} - \text{Cov}_{v_t} \|^2 \right)^{1/2} \left( \mathbb{E} \|p_t^j\|^2 \right)^{(2-\epsilon)/4} \left( \mathbb{E} \|p_t^j\|^4 / \epsilon \right)^{\epsilon/8} \left( \|Bq_t^j\|^8 / \epsilon \right)^{8/8} \\
\leq \frac{C_J}{J^{1/2}} \left( \mathbb{E} \|p_t^j\|^2 \right)^{(2-\epsilon)/4} \leq C_J J^{-1/2 - \alpha/2 + \epsilon \alpha/4},
\]

where the last inequality relies on (27), and the uniform boundedness of high moments, stated in Proposition 5.1, equation (26) and Proposition 4.1, equation (54).
The third term in \((40)\) is expected to contribute a relatively slow-decaying term. For that, we apply Ando-Hemmen inequality (see for instance Theorem 6.2 on page 135 in [21]). Define \(\lambda_0 = \lambda_{\text{min}}(\text{Cov}_{\rho(t)})\), then:

\[
\mathbb{E}\text{Tr} \left( \sqrt{\text{Cov}_{x_t+v_t}} - \sqrt{\text{Cov}_{\rho(t)}} \right)^2 \leq \mathbb{E} \left[ \frac{1}{\lambda_0} \left\| \text{Cov}_{x_t+v_t} - \text{Cov}_{\rho(t)} \right\|_F^2 \right] \\
\leq \frac{1}{\lambda_0} \left\{ \mathbb{E} \left\| \text{Cov}_{x_t+v_t} - \text{Cov}_{v_t} \right\|_F^2 + \mathbb{E} \left\| \text{Cov}_{v_t} - \text{Cov}_{\rho(t)} \right\|_F^2 \right\} \\
+ \frac{2}{\lambda_0} \left( \mathbb{E} \left\| \text{Cov}_{x_t+v_t} - \text{Cov}_{v_t} \right\|_F^2 \right)^{1/2} \left( \mathbb{E} \left\| \text{Cov}_{v_t} - \text{Cov}_{\rho(t)} \right\|_F^2 \right)^{1/2} \\
\leq \frac{1}{\lambda_0} \mathbb{E} \left\| \text{Cov}_{x_t+v_t} - \text{Cov}_{v_t} \right\|_F^2 + \frac{2J^{-1/2}}{\lambda_0} \left( \mathbb{E} \left\| \text{Cov}_{x_t+v_t} - \text{Cov}_{v_t} \right\|_F^2 \right)^{1/2} + \frac{J^{-1}}{\lambda_0},
\]

where we used Proposition 3.1 to control \(\left( \mathbb{E} \left\| \text{Cov}_{v_t} - \text{Cov}_{\rho(t)} \right\|_F^2 \right)^{1/2}\). To estimate \(\left( \mathbb{E} \left\| \text{Cov}_{x_t+v_t} - \text{Cov}_{v_t} \right\|_F^2 \right)^{1/2}\), we notice that

\[
\left( \mathbb{E} \left\| \text{Cov}_{x_t+v_t} - \text{Cov}_{v_t} \right\|_F^2 \right)^{1/2} = \left( \mathbb{E} \left\| \text{Cov}_{p_{t+q_t}-q_{t+q_t}} \right\|_F^2 \right)^{1/2} \\
\leq \left( \mathbb{E} \left\| \text{Cov}_{p_{t+q_t}} \right\|_F^2 \right)^{1/2} + \left( \mathbb{E} \left\| \text{Cov}_{p_{t+q_t}} \right\|_F^2 \right)^{1/2} + \left( \mathbb{E} \left\| \text{Cov}_{q_{t+q_t}} \right\|_F^2 \right)^{1/2}.
\]

Consider the three terms separately:

1. to estimate \(\mathbb{E} \left\| \text{Cov}_{p_{t+q_t}} \right\|_F^2\):

\[
\mathbb{E} \left\| \text{Cov}_{p_{t+q_t}} \right\|_F^2 = \mathbb{E} \left\{ \text{Tr}(\text{Cov}_{p_{t+q_t}}) \right\} \leq \frac{1}{J^2} \mathbb{E} \left\{ \sum_{j,k=1}^{J} |p_j^1|^2 |p_k^1|^2 \right\} \\
= \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^4 \right)^2 \leq \frac{1}{J} \sum_{j=1}^{J} \mathbb{E} \left( |p_j^1|^4 \right) = \mathbb{E} \left( |p_j^1|^4 \right) \]

\[
= \mathbb{E} \left( |p_j^1|^{2-\epsilon} |p_j^1|^{2+\epsilon} \right) \leq \left( \mathbb{E} |p_j^1|^2 \right)^{(2-\epsilon)/2} \left( \mathbb{E} |p_j^1|^{(4+2\epsilon)/\epsilon} \right)^{\epsilon/2} \]

\[
\leq C \cdot J^{-\alpha + \epsilon / \epsilon},
\]

where we used the H"{o}lder’s inequality and the boundedness for high moments (Proposition 3.1 [21]);

2. to estimate \(\mathbb{E} \left\| \text{Cov}_{p_{t+q_t}} \right\|_F^2\), note:

\[
\mathbb{E} \left\| \text{Cov}_{p_{t+q_t}} \right\|_F^2 = \mathbb{E} \left\{ \text{Tr}(\text{Cov}_{p_{t+q_t}}) \right\} \\
= \frac{1}{J^2} \mathbb{E} \left\{ \sum_{i,j=1}^{J} \left( p_i^1, q_i^1 \right) \left( q_j^1, q_j^1 \right) \right\} \leq \frac{1}{J^2} \mathbb{E} \left\{ \sum_{i,j=1}^{J} |p_i^1|^2 |q_i^1|^2 |q_j^1|^2 \right\} \\
= \frac{1}{J^2} \mathbb{E} \left\{ \sum_{i,j=1}^{J} |p_i^1|^2 |q_j^1|^2 \right\} \leq \mathbb{E} \left\{ \left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^2 \right)^2 \left( \frac{1}{J} \sum_{j=1}^{J} |q_j^1|^2 \right)^2 \right\} \\
= \mathbb{E} \left\{ \left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^2 \right) \text{Var}(\rho(t)) \right\} + \mathbb{E} \left\{ \left( \frac{1}{J} \sum_{j=1}^{J} |q_j^1|^2 \right) \left( \frac{1}{J} \sum_{j=1}^{J} |q_j^1|^2 \right) - \text{Var}(\rho(t)) \right\}.
\]
Since

\[
\mathbb{E}\left\{ \left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^2 \right) \left( \frac{1}{J} \sum_{j=1}^{J} |q_j^1|^2 \right) - \text{Var}(\rho(t)) \right\}
\]

\[= \mathbb{E}\left\{ \left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^2 \right)^{1-\epsilon} \left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^2 \right)^{\epsilon} \left( \frac{1}{J} \sum_{j=1}^{J} |q_j^1|^2 \right) - \text{Var}(\rho(t)) \right\} \]

\[\leq \left( \mathbb{E}\left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^2 \right)^{1-\epsilon} \left( \mathbb{E}\left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^2 \right)^{\epsilon} \left( \frac{1}{J} \sum_{j=1}^{J} |q_j^1|^2 \right) - \text{Var}(\rho(t)) \right) \right)^{1/\alpha} \]

\[\leq \left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^2 \right)^{1-\epsilon} \left( \frac{1}{J} \sum_{j=1}^{J} |p_j^1|^2 \right)^{\epsilon/2} \left( \frac{1}{J} \sum_{j=1}^{J} |q_j^1|^2 \right) - \text{Var}(\rho(t)) \right) \]

\[\leq C_\epsilon J^{-1/2-\alpha(1-\epsilon)}, \]

where we use the uniform boundedness of high moments, stated in Proposition 3.1 and Proposition 4.1. Therefore, we have

\[\mathbb{E}[\|\text{Cov}_{p_t, q_t}\|_F^2] \leq \text{Var}(\rho(t)) \mathbb{E}[|p_t|^2] + C_\epsilon J^{-1/2-\alpha+\epsilon} \leq C_\epsilon J^{-\alpha}. \]  

(46)

3. to estimate \(\mathbb{E}[\|\text{Cov}_{q_t, p_t}\|_F^2]\) is the same as (46).

Plug (44) and (46) back into (43), we have the control over the second term in (42):

\[\frac{2J^{1/2}}{\lambda_0} \left( \mathbb{E}[\|\text{Cov}_{x_i, v_i} - \text{Cov}_{v_i, v_i}\|_F^2] \right)^{1/2} \leq C_\epsilon J^{-1/2-\alpha/2+\epsilon} \]

and thus (42) turns into

\[\mathbb{E}(\text{Tr} \left( \sqrt{\text{Cov}_{x_i, v_i}} - \sqrt{\text{Cov}_{v_i, v_i}} \right)^2 \leq \frac{1}{\lambda_0} \mathbb{E}[\|\text{Cov}_{p_t, q_t} - \text{Cov}_{q_t, q_t}\|_F^2] + C_\epsilon J^{-1/2-\alpha/2+\epsilon/4}. \]

(47)

Finally, we deal with first and second term in (46), we first rewrite:

\[-\frac{2}{J} \mathbb{E} \left\{ \sum_{j=1}^{J} \left( p_j^1, \text{Cov}_{p_t, q_t} B p_j^1 \right) + \sum_{j=1}^{J} \left( p_j^1, (\text{Cov}_{p_t, q_t} - \text{Cov}_{q_t}) B q_j^1 \right) \right\} \]

\[= -\mathbb{E} \left\{ 2 \text{Tr} \left[ (\text{Cov}_{p_t, q_t} + \text{Cov}_{q_t, q_t}) B \right] \right\} + \mathbb{E} \left\{ \text{Tr} \left[ (\text{Cov}_{q_t, q_t} - \text{Cov}_{q_t}) B \right] \right\} \]

\[\leq -\mathbb{E} \left\{ \text{Tr} \left[ (\text{Cov}_{p_t, q_t} - \text{Cov}_{q_t}) B \right] \right\} + \mathbb{E} \left\{ \text{Tr} \left[ (\text{Cov}_{p_t, q_t} - \text{Cov}_{q_t}) B \right] \right\}. \]

The first term becomes:

\[-\mathbb{E} \left\{ \text{Tr} \left[ (\text{Cov}_{p_t, q_t} - \text{Cov}_{q_t}) B \right] \right\} \leq -\lambda_{\text{min}}(B) \mathbb{E} \left\{ \text{Tr} \left[ (\text{Cov}_{p_t, q_t} - \text{Cov}_{q_t}) B \right] \right\} \]

\[\leq -\lambda_{\text{min}}(B) \mathbb{E} \left[ \| \text{Cov}_{p_t, q_t} - \text{Cov}_{q_t} \|_F \right]^2 = -\lambda_{\text{min}}(B) \mathbb{E} \left[ \| \text{Cov}_{x_i, v_i} - \text{Cov}_{v_i, v_i} \|_F \right]^2, \]

while the second term can be bounded by applying (46):

\[\mathbb{E} \left\{ \text{Tr} \left[ (\text{Cov}_{q_t, p_t} + \text{Cov}_{p_t, q_t}) B \right] \right\} \leq \|B\|_2 \mathbb{E} \left\{ \text{Tr} \left[ (\text{Cov}_{q_t, p_t} + \text{Cov}_{p_t, q_t}) \right] \right\} \]

\[= \|B\|_2 \mathbb{E} \left[ \| \text{Cov}_{p_t, q_t} \|_F^2 \right] \leq \|B\|_2 \text{Var}(\rho(t)) \mathbb{E}[|p_t|^2] + C_\epsilon J^{-1/2-\alpha(1-\epsilon)}, \]

where the last inequality comes from (46).
And the third term:

$$\left| \mathbb{E} \left[ \text{Tr} \left[ \text{Cov}_{p_t} \text{Cov}_{q_t} B \right] \right] \right| \leq \mathbb{E} \left( \sum_{j=1}^{J} \left\langle p_{t}^{j}, \text{Cov}_{q_t} B p_{t}^{j} \right\rangle \right) = \mathbb{E} \left( \sum_{j=1}^{J} \left\langle p_{t}^{j}, \text{Cov}_{\rho(t)} B p_{t}^{j} \right\rangle + \sum_{j=1}^{J} \left\langle p_{t}^{j}, \left( \text{Cov}_{q_t} - \text{Cov}_{\rho(t)} \right) B p_{t}^{j} \right\rangle \right)$$

$$\leq \| B \|_{2} \text{Var}(\rho(t)) \mathbb{E}|p_{t}|^2 + \mathbb{E} \left( \sum_{j=1}^{J} \| \left( \text{Cov}_{q_t} - \text{Cov}_{\rho(t)} \right) B \|_{2} |p_{t}|^2 \right)$$

$$\leq \| B \|_{2} \text{Var}(\rho(t)) \mathbb{E}|p_{t}|^2 + C \epsilon J^{-1/2 - \alpha(1 - \epsilon)},$$

where we used the same techniques as in (45).

Combine (41), (47)-(48), and note that

$$\mathbb{E} \left( \frac{1}{J} \sum_{j=1}^{J} |x_{t}^{j} - \bar{x}_{t}|^2 \right) = \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^{J} |p_{t}^{j} - \bar{x}_{t}|^2 \right) = \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^{J} |p_{t}^{j}|^2 \right) = \mathbb{E}|p_{t}|^2,$$

we finally have:

$$\frac{d\mathbb{E}|p_{t}|^2}{dt} \leq 2 \| B \|_{2} \text{Var}(\rho(t)) \mathbb{E}|p_{t}|^2 + \left( \lambda_{\text{min}}(B) - \frac{1}{\lambda_{0}} \right) \mathbb{E} \left[ \text{Tr} \left[ \left( \text{Cov}_{p_{t} + q_t} - \text{Cov}_{q_t} \right)^2 \right] \right]$$

$$+ C \epsilon J^{-1/2 - \alpha(1 - \epsilon)} + C \epsilon J^{-1/2 - \alpha/2 + \epsilon/4}.$$

Under the assumption that $\lambda_{\text{min}}(B)\lambda_{0} \geq 1$, and with $\mathbb{E}|p_{t}|^2 = 0$, we apply Grönwall inequality for $\mathbb{E}|p_{t}|^2$ to obtain

$$\mathbb{E}|p_{t}|^2 \leq C \epsilon J^{-1/2 - \alpha/2 + \epsilon/4},$$

for all finite time, finishing the proof for the first inequality of (38).

To prove the second inequality in (38), we first note that the first inequality in (38) with $\epsilon$ small enough, if inserted back in (41) and (46), improves the estimates to:

$$\mathbb{E} \left\| \text{Cov}_{p_{t}} \right\|_{F}^2 \leq \left( \mathbb{E}|p_{t}|^2 \right)^{(2-\epsilon)/2} \left( \mathbb{E}|p_{t}|^2 / (2(2+\epsilon)) \right)^{\epsilon/2} \leq C \epsilon J^{-1/2 - \alpha/2 + (3/4 - \alpha/4 + \epsilon/2)} \leq C \epsilon J^{-1/2 - \alpha/2 + \epsilon},$$

$$\mathbb{E} \left\| \text{Cov}_{p_{t} + q_t} \right\|_{F}^2 = \mathbb{E} \left\| \text{Cov}_{q_t, p_{t}} \right\|_{F}^2 \leq \text{Var}(\rho(t)) \mathbb{E}|p_{t}|^2 + C \epsilon J^{-1/2 - \alpha(1 - \epsilon)} \leq C \epsilon J^{-1/2 - \alpha/2 + \epsilon},$$

which further improves the estimates in (38) to:

$$\left( \mathbb{E} \left\| \text{Cov}_{x_{t} + v_{t}} - \text{Cov}_{v_{t}} \right\|_{2} \right)^{1/2} \leq C \epsilon J^{-1/4 - \alpha/4 + \epsilon/2}.$$

This helps us to control each term in (38):

1. $$\mathbb{E} \left( x_{t}^{j}, (\text{Cov}_{x_{t} + v_{t}} - \text{Cov}_{v_{t}}) B v_{t}^{j} \right) \leq \left( \mathbb{E}|x_{t}^{j}|^2 |B v_{t}^{j}|^2 \right)^{1/2} \left( \mathbb{E}\| \text{Cov}_{x_{t} + v_{t}} - \text{Cov}_{v_{t}} \|_{2} \right)^{1/2} \leq C \epsilon J^{-1/4 - \alpha/4 + \epsilon/2} \left( \mathbb{E}|x_{t}^{j}|^2 \right)^{(2-\epsilon)/4}$$

2. $$\mathbb{E} \left( x_{t}^{j}, (\text{Cov}_{x_{t} + v_{t}} - \text{Cov}_{v_{t}}) B u_{t}^{j} \right) \leq \left( \mathbb{E}|x_{t}^{j}|^2 |B u_{t}^{j}|^2 \right)^{1/2} \left( \mathbb{E}\| \text{Cov}_{x_{t} + v_{t}} - \text{Cov}_{v_{t}} \|_{2} \right)^{1/2} \leq C \epsilon J^{-1/4 - \alpha/4 + \epsilon/2} \left( \mathbb{E}|x_{t}^{j}|^2 \right)^{1/2} \leq C \epsilon J^{-1/4 - \alpha/4 + \epsilon/2} \left( \mathbb{E}|x_{t}^{j}|^2 \right)^{(2-\epsilon)/4}$$

3. $$\mathbb{E} \text{Tr} \left[ (\sqrt{\text{Cov}_{x_{t} + v_{t}}} - \sqrt{\text{Cov}_{v_{t}}})^2 \right] \leq \frac{C \epsilon J^{-1/2 - \alpha/2 + \epsilon}}{\lambda_{0}} \leq C \epsilon J^{-1/2 - \alpha/2 + \epsilon}$$

4. $$\mathbb{E} R_{t}^{2} \leq \left( \mathbb{E}|x_{t}^{j}|^2 |B v_{t}^{j}|^2 \right)^{1/2} \left( \mathbb{E}\| \text{Cov}_{v_{t}} - \text{Cov}_{\rho(t)} \|_{2} \right)^{1/2} \left( \mathbb{E}|x_{t}^{j}|^2 |B u_{t}^{j}|^2 \right)^{1/2} \left( \mathbb{E}\| \text{Cov}_{v_{t}} - \text{Cov}_{\rho(t)} \|_{2} \right)^{1/2} \leq C \epsilon J^{-1/4 - \alpha/4 + \epsilon/2} \left( \mathbb{E}|x_{t}^{j}|^2 \right)^{(2-\epsilon)/4},$$
where we use Hölder’s inequality and uniform boundedness of high moments, stated in Proposition 3.1 [20] and Proposition 4.1 [31] in these estimations. Now, we rewrite:

$$\frac{d\mathbb{E}[x_1^\alpha]}{dt} \leq C_\epsilon J^{-1/4-\alpha/4+\epsilon/2} \mathbb{E}[x_1^1(2-\epsilon)/4] + J^{-1/2-\alpha/2+\epsilon},$$

and with $\mathbb{E}[x_1^1] = 0$, we finally have:

$$\mathbb{E}[x_1^\alpha] \leq C_\epsilon J^{-(1+\alpha-2\epsilon)/(2-\epsilon)},$$

by Grönwall inequality. This finishes the proof. □

Proof of Theorem 1.5. First, by Corollary 4.1, we have the condition (35) holds true for $E_n$ and Proposition 4.1 (34) in these estimations. Now, we rewrite:

$$\text{Lemma 4.1 implies (35) is true for } \alpha \to \infty, (35) \text{ holds true with } \alpha = 1 - 2\epsilon, \text{ (55) holds true with } \alpha = 1 - 2\epsilon \text{ for any } \epsilon > 0, \text{ and this completes the proof.} \square$

Finally, we are ready to prove Theorem 1.5.

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