The provably total recursive functions and the MRDP theorem in Basic Arithmetic and its extensions

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Abstract

We study Basic Arithmetic, \(\text{BA}\) introduced by W. Ruitenburg. \(\text{BA}\) is an arithmetical theory based on basic logic which is weaker than intuitionistic logic. We show that the class of the provably total recursive functions of \(\text{BA}\) is a proper sub-class of the primitive recursive functions. Three extensions of \(\text{BA}\), called \(\text{BA} + U\), \(\text{BA}_c\), and \(\text{EBA}\), are investigated with relation to their provably total recursive functions. It is shown that the provably total recursive functions of these three extensions of \(\text{BA}\) are exactly the primitive recursive functions. Moreover, among other things, it is shown that the well-known MRDP theorem does not hold in \(\text{BA}\), \(\text{BA} + U\), \(\text{BA}_c\), but holds in \(\text{EBA}\).

1 Introduction

Basic Arithmetic, \(\text{BA}\) is an arithmetical theory introduced by W. Ruitenburg in [12], based on Basic Predicate Calculus, \(\text{BQC}\), as Heyting Arithmetic, \(\text{HA}\) is based on Intuitionistic Predicate Calculus, \(\text{IQC}\) and Peano Arithmetic, \(\text{PA}\) based on Classical Predicate Calculus, \(\text{CQC}\). \(\text{BQC}\) is a weaker logic than \(\text{IQC}\), in which the rule of Modus Ponens is weakened. It was motivated by a revision of the Brouwer-Heyting-Kolmogorov interpretation (see [11] and [13]). Formally, basic logic is an extension of the well-known geometric logic \(^1\) (see [17]) by adding implication and universal quantification to the language, in a way that they reflect the meaning of “derivability” at the object language level. We will use this close relation between basic logic and geometric logic to obtain some of our results about \(\text{BA}\) and its extensions. We will also briefly discuss some properties of theories formalized over basic logic that are absent in the context of geometric logic, including those of the subtheory \(\text{BA}_w\) of \(\text{BA}\).

Although the arithmetical axioms of \(\text{BA}\) are essentially the same as Peano axioms, \(\text{BA}\) is weaker than \(\text{HA}\). For instance, \(\text{BA}\) does not prove the cancellation law, i.e., \(x + y = x + z \Rightarrow y = z\) (See Lemma 3.12). More interestingly, every provably total function of \(\text{BA}\) is primitive recursive. This result is not new; it has already been proved using primitive recursive realizability introduced in [14]. We use a different analysis based on functionality of the relations given by the defining formulas, instead of their totality. Moreover, we show that every provably total function of \(\text{BA}\) is definable in \(\text{BA}\) by a geometric formula. However, there are primitive recursive functions that are not provably total in \(\text{BA}\). One such primitive recursive function is the cut-off subtraction. That means that the set of all the provably total recursive functions of \(\text{BA}\), indicated by \(\text{PTRF}(\text{BA})\) is a proper subset of all the primitive functions, \(\text{PR}\).

\(^1\)In the literature, the term “geometric logic” is usually used for a logic with infinitary disjunction, and the formulas with only finitary disjunction are called coherent geometric formulas. In this paper, we only work with finitary languages, and only nullary and binary disjunction (\(\bot\) and \(\lor\), respectively) are included in the language, which are sufficient for making up all the finitary disjunctions. Also, derivability in geometric logic is often considered in a context of a finite sequence of variables, which for example allows for an empty domain of discourse when considering semantics. We will use the term “geometric” only for reference to the finitary part of the language and logic, and consider derivability without variable contexts.
We consider three extensions of BA. One is a logical extension and the other two are arithmetical extensions. The logical extension we study is EBA, an extension of BA with the logical axiom \( \top \rightarrow \bot \rightarrow \bot \). This extension is introduced in [1] and it is shown there that its behavior is very close to HA, but still weaker than that. We show that PTRF(EBA) = \( \mathcal{PR} \). The two arithmetical extensions we will consider are the following. The first one is an extension of BA by the arithmetical axiom \( U : x + y = x + z \Rightarrow y = z \) (the cancellation law). It turns out that PTRF(\( BA + U \)) = \( \mathcal{PR} \). The other one is an extension of BA by adding a symbol for the cut-off subtraction in the language, and adding its properties as extra axioms to BA. Again, it turns out that the provably total recursive functions of this extension, denoted by BA\(_{e} \), are exactly the class of primitive recursive functions.

It is shown that the well-known Matiyasevich-Robinson-Davis-Putnam, MRDP theorem does not hold in BA and the two arithmetical extensions of BA mentioned above. However, EBA proves the MRDP theorem. More properties of EBA are considered in the last section of this paper. For instance, it is shown that EBA proves every \( \Pi_{2} \) theorem of the classical arithmetical theory \( \Sigma_{1} \). Moreover, it is shown that PA is \( \Pi_{1} \)-conservative over EBA, and in particular, EBA \( \vdash \text{Con}_{\Sigma_{1}} \), for any \( n \).

In the course of our investigation of the provably total functions and the MRDP theorem for BA and its extensions, we use geometric logic and arithmetic as a bridge, to connect our theories of interest to some classical theories. Most notably, we consider two classical arithmetical theories \( \Sigma_{1}^{+} \) and \( \Sigma_{1}^{+} + U \). It turns out that the class of provably total recursive functions of both of these theories are exactly the class of primitive recursive functions, and while the MRDP theorem holds for the latter theory, it does not hold for the former. We did not find any indication of these results in the literature, and they are interesting on their own.

Finally, based on our analysis in this paper, we suggest that BA + U is a better candidate for the title “basic arithmetic” than BA. The same holds for the theory which we have denoted in this paper by GA and named “geometric arithmetic”: GA + U would be more suitable for the title.

2 Preliminaries to Basic Arithmetic

In this section, we introduce the logical and arithmetical theories we will work with, including BQC and BA. The semantics of Kripke model theory for BQC, BA and their extensions is introduced and its basic properties that are used in this paper are stated. We assume that the reader is familiar with the semantics of first-order classical and intuitionistic theories (See Chapters 3 and 6 of [3], for example). For motivations and basic properties of BQC and BA, see [12] and [1].

2.1 Axioms and Rules of Basic Predicate Calculus

The logical vocabulary of Basic Predicate Calculus, BQC is the same as that of Intuitionistic Predicate Calculus, IQC. It was originally axiomatized in sequent notation, i.e., using pairs of formulas called sequents, denoted by \( A \rightarrow B \) where \( A \) and \( B \) are formulas in the logical language \( \{ \forall, \wedge, \rightarrow, \bot, \top, \forall \} \). Since Modus Ponens is not a rule in BQC, a universally quantified formula like \( \forall x A \) is different from \( \forall x y A \). The first one is read \( \forall x (\top \rightarrow \forall y (\top \rightarrow A)) \) and the second one is read \( \forall x y (\top \rightarrow A) \). In BQC, when we write \( \forall x (A \rightarrow B) \), we mean \( x \) to be a finite sequence of variables once quantified. For existential quantification no such problems occur over BQC. So, as usual over IQC, we may occasionally write \( \exists x A \) as short for \( \exists x_{1} \ldots x_{n} A \). Beside a set of predicate and function symbols of possibly different finite arity, we also include the binary predicate “=” for equality. Terms and atomic formulas are defined as usual. A prime formula is either an atomic formula, \( \top \) or \( \bot \). Formulas are defined as usual except for implication and universal quantification: if \( A \) and \( B \) are formulas, and \( x \) is a (possibly empty) finite sequence of variables, then \( \forall x (A \rightarrow B) \) is a formula. In case \( A \) is not of the form \( B \rightarrow C \), \( \forall x A \) will be used as an abbreviation for \( \forall x (\top \rightarrow A) \). An implication \( A \rightarrow B \) is a universal quantification \( \forall x (A \rightarrow B) \) where \( x \) is the empty sequence. \( \neg A \) means \( A \rightarrow \bot \), and \( \forall x (A \leftrightarrow B) \) means \( \forall x (A \rightarrow B) \wedge \forall x (B \rightarrow A) \). The concepts of free and bound variables are defined as usual. A fresh variable is a variable that does not occur in any of the terms and formulas in the context being discussed. A sentence is a formula with no free variables. Given a sequence of variables \( x \) without repetitions and a sequence of terms \( t \) with the same length as \( x \), substitutability of \( t \) for \( x \) in a formula is defined as usual. \( s[x/t] \) and \( A[x/t] \) stand for, respectively, the term and formula that results from substituting the terms \( t \) for all free occurrences of the variables of \( x \) in the term \( s \) and the formula \( A \). For details, see [12] and [1]. We often simply write \( A \) as an abbreviation for the sequent \( \top \Rightarrow A \), and \( A \leftrightarrow B \) for \( A \Rightarrow B \) and \( B \Rightarrow A \) together.

A rule \( R \) assigns a sequent \( \alpha \) to finitely many sequents \( \alpha_{1}, \ldots, \alpha_{n} \), and is denoted by \( \frac{\alpha_{1} \ldots \alpha_{n}}{\alpha} \). A
is called the lower sequent of $R$, and $\alpha_1, \ldots$ and $\alpha_n$ are called the upper sequents of $R$. To any given first-order language, one can add several new propositional symbols, and define substitution of formulas of the original language in place of the added propositional symbols appearing in a given formula in the extended language, as usual. This definition of substitution can be extended to the sequents and rules, in the obvious way. A sequent schema is a collection of substitution instances of a certain sequent. A rule schema is defined similarly. A theory $T$ is given by a set of sequents called the axioms, and a set of rules. An axiom schema is a sequent schema, all of the instances of which are axioms. We may simply refer to axiom schemas and rule schemas as axioms and rules, respectively.

In the following subsections, we introduce several theories that are of interest in this paper, by listing their axioms and rules. Unless specified otherwise, the formulas, variables, sequences of variables and sequences of terms appearing in the listed axioms and rules are arbitrary. A double horizontal line denotes rules. A double horizontal line denotes reversible rules; i.e. it denotes several rules together, where each converse rule has one of the upper sequents of the original rule as the lower sequent, and the lower sequent of the original rule as the only upper sequent.

A derivation $D$ is a finite rooted tree with a sequent assigned to each node. We often identify the nodes of a derivation with the sequent assigned to them. Considering the partial order of the tree as an above-below relation (so that the root is the lower-most node), the upper-most nodes are called the initial nodes of the derivation, and the sequents assigned to them are called the initial sequents. For a theory $T$, a set of sequents $\Gamma$ and a sequent $\alpha$, A derivation of $\alpha$ from $\Gamma$ in $T$ is a derivation with $\alpha$ as the root, initial sequents either belonging to $\Gamma$ or the set of axioms of $T$, and each non-initial sequent the lower sequent of a rule of $T$ such that the sequents immediately above it are the upper sequents of that rule. We say that $\alpha$ is derivable or provable in $T$ from $\Gamma$, or alternatively, $T$ proves $\alpha$ from $\Gamma$, whenever such a derivation exists. We denote this relation by $T \vdash \alpha$. In case $\Gamma$ is empty, we drop the “from $\Gamma$”, and simply write $T \vdash \alpha$. $T$ proves a rule $R = \frac{\alpha_1 \ldots \alpha_n}{\alpha}$, or $R$ is derivable in $T$, whenever $T + \{\alpha_1, \ldots, \alpha_n\} \vdash \alpha$. Derivability of a rule in a theory from a set of sequents is defined similarly. For theories $T$ and $T'$, $T + T'$ denotes the theory whose sets of axioms and rules are the union of axioms of $T$ and $T'$ and the union of the rules of $T$ and $T'$, respectively (note that this notation is consistent with the previous notation for derivability from a set of sequents). $T$ is an extension of $T'$, denoted by $T \supset T'$, whenever the language of $T$ extends that of $T'$, and we have $T \vdash \alpha$ and $T' \vdash R$ for all axioms $\alpha$ and all rules $R$ of $T'$. $T \supset T'$ means $T \vdash T'$ and $T' \vdash T$. A theorem of $T$ is a sentence $\alpha$ such that $T \vdash \alpha$. $\Th(T)$ denotes the set of all theorems of $T$. For a class $C$ of formulas, $\Th_C(T)$ denotes the set of all $C$-theorems of $T$, $\Th(T) \cap C$. $T$ is a conservative extension of $T'$ whenever $T \vdash T'$ and $\Th_C(T) = \Th(T')$, with $C$ being the class of all formulas in the language of $T'$. $T$ is said to be $C$-conservative over $T'$ whenever $T \vdash T'$ and $\Th_C(T) = \Th_C(T')$.

Axioms and rules of logical theories
We first introduce Geometric Predicate Calculus, GQC, which is the weakest logical theory we need. GQC is formalized in a language without implication and universal quantification. Formulas in such a language are called geometric formulas. A sequent $A \Rightarrow B$ is called geometric whenever both $A$ and $B$ are geometric formulas. A rule is called geometric whenever its lower sequent and all its upper sequents are geometric. GQC is a theory with geometric axioms and rules. In the following list of axioms and rules of GQC, all the formulas are supposed to be geometric.

1. $A \Rightarrow A$;
2. $A \Rightarrow \top$;
3. $\bot \Rightarrow A$;
4. $A \land (B \lor C) \Rightarrow (A \land B) \lor (A \land C)$;
5. $A \land \exists x B \Rightarrow \exists x (A \land B)$; where $x$ is not free in $A$;
6. $x = x$;
7. $x = y \land A \Rightarrow A[x/y]$, where $A$ is atomic;
8. $\frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$.

Note: This notation “Th” is different from that used in [12], which is for sequents and not just sentences.
9. \[
\frac{A \Rightarrow B}{A} \quad \frac{A}{A \Rightarrow C}
\];
10. \[
\frac{B \Rightarrow A}{B \lor C \Rightarrow A}
\];
11. \[
\frac{A \Rightarrow B}{A[x/t] \Rightarrow B[x/t]}
\]; where \( t \) is substitutable for \( x \) in both \( A \) and \( B \);
12. \[
\frac{B \Rightarrow A}{\exists x \, B \Rightarrow A}
\], where \( x \) is not free in \( A \).

**Basic Predicate Calculus.** BQC extends GQC first by including all the instances of the above schemas in the language containing implication and universal quantification, and second by adding the following axioms and rules:

13. \( \forall x \, (A \rightarrow B) \land \forall x \, (B \rightarrow C) \Rightarrow \forall x \, (A \rightarrow C) \);
14. \( \forall x \, (A \rightarrow B) \land \forall x \, (A \rightarrow C) \Rightarrow \forall x \, (A \rightarrow B \land C) \);
15. \( \forall x \, (B \rightarrow A) \land \forall x \, (C \rightarrow A) \Rightarrow \forall x \, (B \lor C \rightarrow A) \);
16. \( \forall x \, (A \rightarrow B) \Rightarrow \forall x \, (A[x/t] \rightarrow B[x/t]) \), where \( t \) is substitutable for \( x \) in both \( A \) and \( B \);
17. \( \forall x \, (A \rightarrow B) \Rightarrow \forall y \, (A \rightarrow B) \), where no variable in \( y \) is free on the left hand side;
18. \( \forall y \, (B \rightarrow A) \Rightarrow \forall y \, (\exists x \, B \rightarrow A) \), where \( x \) is not free in \( A \);
19. \( A \land B \Rightarrow C \Rightarrow \forall x \, (B \rightarrow C) \), where no variable in \( x \) is free in \( A \).

There are three other logical theories that we are interested in. **Extended Basic Predicate Calculus.** EBQC is axiomatized by adding the axiom \( \top \rightarrow \bot \Rightarrow \bot \) to BQC. **Intuitionistic Predicate Calculus.** IQC is axiomatized by adding the schema \( \top \rightarrow A \Rightarrow A \) to BQC, and **Classical Predicate Calculus.** CQC is axiomatized by adding the schema \( A \lor \neg A \) to IQC.

1-19 are collectively called the logical axioms and rules; 1-4 are the propositional axioms; 5 and 13-18 are the predicate axioms; 6 and 7 are the equality axioms; 8-10 are the propositional rules; 11, 12 and 19 are the predicate rules.

### 2.2 Axioms and Rules of Basic Arithmetic

The arithmetical theories we consider are all formalized in a first-order language containing non-logical symbols from the set \( \{0, S, +, \cdot\} \), where \( 0 \) is a constant symbol for zero, \( S \) is a unary function symbol for successor, and \(+\) and \(\cdot\) are binary function symbols for addition and multiplication, respectively.

The ordering relation can be formalized in arithmetical theories by defining \( s < t \) for any terms \( s \) and \( t \) to mean \( \exists x \, (s + Sx = t) \), with \( x \) being a fresh variable. \( s \leq t \) can be defined as \( s < t \lor s = t \), and \( s > t \) and \( s \geq t \) can be defined in the obvious ways. Alternatively, one can include \( "<" \) as a binary predicate symbol in the language, and \( s < t \Leftrightarrow \exists x \, (s + Sx = t) \) as axioms in the theory. For the properties of the arithmetical theories that we are interested in, these alternatives are essentially equivalent. For convenience, we sometimes the latter alternative, and the arithmetical theories will be considered in the language \( \mathcal{L} = \{0, S, +, \cdot, <\} \) (in the case of BA\(_c\), the language \( \mathcal{L} \cup \{\top\} \) and with the mentioned axioms, without further mentioning of it. While theories like Q, G\(_A\), B\(_A\), E\(_B\), H\(_A\), P\(_A\) and I\(_C\) (see the following subsection for definitions) were originally formalized in the language without the primitive symbol \( < \), we denote the theories in the language \( \mathcal{L} \) by the same names, since each of them is a conservative extension of the corresponding theory. This can be seen by the fact that if one substitutes each appearance of a formula of the form \( s < t \) in the sequents of a derivation in the extended theory by a formula of the form \( \exists x \, (s + Sx = t) \), the result will be a derivation in the original theory.
Axioms and rules of arithmetical theories

The weakest set of arithmetical axioms that we consider consists of the ones in the following list.

20. \( Sx = 0 \Rightarrow \bot; \)
21. \( Sx = Sy \Rightarrow x = y; \)
22. \( x + 0 = x; \)
23. \( x + Sy = S(x + y); \)
24. \( x \cdot 0 = 0; \)
25. \( x \cdot Sy = x + y. \)

These axioms, together with \( x = 0 \lor \exists y (x = Sy) \), are collectively known as Robinson’s axioms. The theory axiomatized by Robinson’s axioms over CQC is known as Robinson Arithmetic, \( Q. \) The exclusion of the last of the Robinson’s axioms from the above list is due to the fact that all the arithmetical theories considered in this paper prove it. To see why, note that it is derivable by geometric logic and the instance of the rule of induction with the induction formula \( x = 0 \lor \exists y (y < x \land x = Sy) \) (see below for the definitions). All the arithmetical theories we consider extend GQC and have the mentioned instance of the rule of induction.

Basic Arithmetic, \( BA \) is formalized over BQC by the axioms in the above list together with the following axiom and rule, respectively called the induction axiom and the rule of induction.

26. \( \forall y x (A \Rightarrow A[x/Sx]) \Rightarrow \forall y x (A[x/0] \Rightarrow A), \)
27. \( A \Rightarrow A[x/Sx], \quad A[x/0] \Rightarrow A \).

Weakened Basic Arithmetic, \( BA^w \) is the theory obtained by dropping the rule of induction from \( BA. \) Any extension \( T \) of \( BA^w \) is in fact closed under the rule \( \dfrac{A \Rightarrow A[x/Sx]}{A[x/0] \Rightarrow \forall x (T \Rightarrow A)}. \) To see why, assume that \( T \vdash A \Rightarrow A[x/Sx]. \) This implies \( T \vdash \forall x (A \Rightarrow A[x/Sx]), \) which combining with the induction axiom gives \( T \vdash \forall x (A[x/0] \Rightarrow A). \) As \( T \vdash A[x/0] \Rightarrow \forall x (T \Rightarrow A[x/0]), \) we get \( T \vdash A[x/0] \Rightarrow \forall x (T \Rightarrow A). \)

Extended Basic Arithmetic, \( EBA \) is formalized by the same arithmetical axioms and rules as \( BA \) over EBC. Heyting Arithmetic, \( HA \) and Peano Arithmetic, \( PA \) are respectively formalized over IQC and CQC by the same arithmetical axioms, but without the rule of induction. Note that by the above discussion about \( BA^w, \) any theory extending \( HA \) is automatically closed under the rule of induction, since \( HA \vdash \forall x (T \Rightarrow A[x/0]) \Rightarrow A[x/0]. \)

The formula \( A \) appearing in the induction axiom and the rule of induction is called the induction formula, and the variable \( x \) in it is called the eigenvariable. The induction formula can be restricted to belong to a certain class \( C \) of formulas, resulting in the \( C \)-induction axiom and the rule of \( C \)-induction. We will consider several theories with such restricted inductions. The first one is Geometric Arithmetic, \( GA \) which is axiomatized (in the language restricted to geometric formulas) over GQC by the axioms 20–25 and the rule of geometric induction; i.e. the rule of induction restricted to geometric induction formulas. Before introducing the other theories, we first need to define several classes of formulas.

Definition 2.1.

- \( \Delta_0 \) is the smallest set of formulas such that:
  - If \( A(x) \) is prime, then \( A(x) \in \Delta_0, \)
  - If \( A(x), B(x) \in \Delta_0, \) then \( A(x) \circ B(x) \in \Delta_0, \) where \( \circ \) is \( \land, \lor \text{ or } \to, \)
  - If \( A(x, y) \in \Delta_0 \) and \( s \) is a term in which \( x \) does not occur, then \( \exists x (x < s \land A(x, y)) \in \Delta_0, \)
  - If \( A(x, y) \in \Delta_0 \) and \( s \) is a term in which \( x \) does not occur, then \( \forall x (x < s \to A(x, y)) \in \Delta_0. \)
- For any natural number \( n, \) \( \Sigma_n \) and \( \Pi_n \) are defined as follows:
  - \( \Sigma_0 = \Pi_0 = \Delta_0, \)
\[ \Sigma_{n+1} \] is the smallest set of formulas such that if \( A(x, y) \in \Pi_n \) then \( \exists x A(x, y) \in \Sigma_{n+1} \).

\[ \Pi_{n+1} \] is the smallest set of formulas such that if \( A(x, y) \in \Sigma_n \) then \( \forall x A(x, y) \in \Pi_{n+1} \).

- **Open** is the set of formulas with no existential quantification and no universal quantification over nonempty sequence of variables (i.e., it may contain implications). A formula in **Open** may be called quantifier-free or open.

- **∃\_1** is the set of formulas of the form \( \exists x A(x, y) \), where \( A(x, y) \) is a quantifier-free formula. A formula in **∃\_1** is sometimes called existential or Diophantine.

- **∃\_1^+** is the set of formulas of the form \( \exists x A(x, y) \), where \( A(x, y) \) is a geometric quantifier-free formula; meaning that \( A(x, y) \) does not contain implications. A formula in **∃\_1^+** is sometimes called positive existential, hence the notations **∃\_1^+** and **∃^+** are used for the class of formulas.

**EB\Delta_0** is defined to be the theory axiomatized over EBQC by the axioms 20-25, the \( \Delta_0 \)-induction axiom and the rule of \( \Delta_0 \)-induction.

For a class \( C \) of formulas, by \( I^C \) we mean the classical arithmetical theory \( Q \) together with induction restricted to \( C \) formulas. I.e., \( I^C \) is axiomatized over CQC by the axioms 20-25 and the \( C \)-induction axiom. Note that we can argue similar to what we did for \( BA^\omega \) to prove that in fact \( I^C \) is closed under the rule of \( C \)-induction.

Some authors prefer to define \( I^C \) based on another theory \( PA^- \) instead of \( Q \). \( PA^- \) is the classical theory of nonnegative parts of discretely ordered rings, which is formulated in the language of ordered rings, i.e., \( \{0, 1, +, \cdot, <\} \).

By letting \( 1 \equiv S0 \), \( PA^- \) becomes an arithmetical theory. Conversely, one can consider arithmetical theories in this language by letting \( St \equiv t+1 \). \( PA^- \) can be axiomatized over CQC by the following list of axioms, taken from Section 2.1 of [10].

\[
\begin{align*}
28. \quad (x + y) + z &= x + (y + z); \\
29. \quad x + y &= y + x; \\
30. \quad (x \cdot y) \cdot z &= x \cdot (y \cdot z); \\
31. \quad x \cdot y &= y \cdot x; \\
32. \quad x \cdot (y + z) &= x \cdot y + x \cdot z; \\
33. \quad x + 0 &= x; \\
34. \quad x \cdot 0 &= 0; \\
35. \quad x \cdot 1 &= x; \\
36. \quad x < y \land y < z \Rightarrow x < z; \\
37. \quad x < x \Rightarrow \bot; \\
38. \quad x < y \lor x = y \lor y < x; \\
39. \quad x < y \Rightarrow x + z < y + z; \\
40. \quad 0 < z \land x < y \Rightarrow x \cdot z = y \cdot z; \\
41. \quad x < y \Rightarrow \exists z (x + z = y); \\
42. \quad 0 < 1; \\
43. \quad x > 0 \Rightarrow x \geq 1; \\
44. \quad x \geq 0.
\end{align*}
\]

\(^3\)Every \( \exists^+_1 \) formula is geometric, and every geometric formula is equivalent to a \( \exists^+_1 \) formula over geometric logic, which is the weakest logic we are considering. We could simply skip this definition, and work only with geometric formulas instead. Our purpose of the definition is only to use the notation that already has been used in the literature.
There are other axiomatizations of \( \text{PA}^- \), for example the list appearing in Theorem 1.10 of \([8]\) or in the beginning of §2 of \([9]\). For a classical theory, these axiomatizations are equivalent. It is rather straightforward to see that \( \text{PA}^- \vdash \text{Q} \). Since we have \( \text{IOpen} \vdash \text{PA}^- \) (see Theorem 1.10 in \([8]\)), there is no difference in defining \( \text{C} \) over either of \( \text{Q} \) and \( \text{PA}^- \), when \( \text{C} \supseteq \text{Open} \). Our choice of \( \text{Q} \) over \( \text{PA}^- \) is due to relations between \( \text{BA} \) and classical arithmetical theories with induction restricted to geometric formulas (see Corollary \([8,5]\)). Note that by Lemmas 2.5 and 2.8 and Proposition 2.6 in \([11]\), \( \text{BA} \) proves all the axioms 28-44 except 37.

We consider adding another axiom \( U \) to theories that were discussed in the previous paragraph, to get around the mentioned problems. \( U \) is defined to be

\[
45. \ x + y = x + z \Rightarrow y = z,
\]

which just expresses the cancellation law for addition. This is in fact equivalent to axiom 37 and some other candidates (see Lemma 3.39). The choice of \( U \) over axiom 37 is due to the fact that it is formulated in the language without a symbol for the ordering relation. We chose \( U \) over the remaining candidates because of its relation to provable totality of the cut-off subtraction, which is a recurring subject in this paper. We chose the name “\( U \)” because of the relation between the cancellation law for addition and the uniqueness sequent for cut-off subtraction, later denoted by \( \text{“U}(A_\epsilon)\)”.

Finally, we consider the theory \( \text{BA}_\epsilon \) in the language \( \mathcal{L}_\epsilon = \mathcal{L} \cup \{\epsilon\} \), where \( \epsilon \) is a binary function symbol for \textit{truncated subtraction} or \textit{cut-off subtraction}, which is the function with the following definition:

\[
x \div y = \begin{cases} 
0 & x < y; \\
 x - y & x \geq y.
\end{cases}
\]

\( \text{BA}_\epsilon \) is the theory in the language \( \mathcal{L}_\epsilon \) axiomatized over \( \text{BQC} \) by the axioms 20-25, the induction axiom and the rule of induction, together with the following two axioms:

\[
46. \ x \leq y \Rightarrow x \div y = 0;
\]

\[
47. \ y \leq x \Rightarrow Sx \div y = S(x \div y).
\]

2.3 Kripke semantics of Basic Logic and Arithmetic

We introduce Kripke semantics for logical and arithmetical theories. For simplicity of notation, we may not distinguish between a closed term and the object denoted by it in the model. In particular, we may identify a natural number \( n \) with the \textit{numeral} \( S \ldots S0 \) consisting of \( n \) consecutive appearances of \( S \). Similarly, we may identify a constant symbol denoting an individual in the domain of discourse with the individual itself, when working in a language extended with parameters from a given model.

A \textit{Kripke model} for Basic predicate logic is a quadruple \( K = (K, \prec, D, \vdash) \), like the one for intuitionistic predicate logic, except that the relation \( \prec \) is not needed to be reflexive. I.e. \( \prec \) is a transitive relation on \( K \), \( D(k) \) is nonempty for each \( k \in K \), \( D(k) \) is embedded in \( D(k') \) when \( k \prec k' \), and if \( k \prec k' \) then \( k \vdash A \) implies \( k' \vdash A \) for an atomic sentence \( A \) in the language extended by parameters from \( D(k) \). See \([14]\) and \([12]\) for more details. We write \( \preceq \) for reflexive closure of \( \prec \), and \( \succeq \) and \( \triangleright \) for the converse relations of \( \prec \) and \( \preceq \), respectively. The \textit{forcing relation} \( \vdash \) between a \textit{node} \( k \) in \( K \) and a sentence of the form \( \forall x (A \rightarrow B) \) (in the language extended by parameters from \( D(k) \)) is defined as

\[
k \vdash \forall x (A \rightarrow B) \text{ if and only if for all } k' \succ k \text{ and all } a \in D(k'), 
\]

\[
k' \vdash A[x/a] \text{ implies } k' \vdash B[x/a].
\]

This for implication reduces to

\[
k \vdash A \rightarrow B \text{ if and only if for all } k' \succ k, k' \vdash A \text{ implies } k' \vdash B.
\]

For a formula \( A \) with free variables \( x \), we define

\[
k \vdash A \text{ if and only if for all } k' \succeq k \text{ and all } a \in D(k'), k' \vdash A[x/a].
\]

We extend \( \vdash \) to all sequents and rules. For a sequent \( A \Rightarrow B \), it is defined by

\[
k \vdash A \Rightarrow B \text{ if and only if for all } k' \succeq k \text{ and all } a \in D(k'), k' \vdash A[x/a] \text{ implies } k' \vdash B[x/a].
\]
For a rule $R = \frac{\alpha_1 \ldots \alpha_n}{\alpha}$, it is defined by

$$k \models R \text{ if and only if for all } k' \geq k, \text{ if } k' \models \alpha_i \text{ for all } i \text{ with } 1 \leq i \leq n, \text{ then } k' \models \alpha.$$ 

A formula $A$ is valid in $K$, denoted by $K \models A$, if $A$ is forced in all nodes of $K$, i.e. $k \models A$ for all $k \in K$. $K \models \alpha$ and $K \models R$ are defined similarly for a sequent $\alpha$ and a rule $R$. For a set $\Gamma$ of sequents, $K \models \Gamma$ means that every sequent in $\Gamma$ is valid in $K$. For a theory $T$, $K \models T$ means that all the axioms and rules of $T$ are valid in $K$. By $T + \Gamma \models \alpha$ we mean that for all Kripke models $K$, if $K \models T$ and $K \models \Gamma$ then $K \models \alpha$; $T + \Gamma$ is said to force $\alpha$.

**Proposition 2.2** (Soundness). Let $T$ be a theory, $\Gamma$ a set of sequents, $\alpha$ a sequent and $R$ a rule. If $T + \Gamma \models \alpha$ then $T + \Gamma + R \models \alpha$. If $T + \Gamma \models R$ then $T + \Gamma + R$.

*Proof.* Proposition 5.3 in [12].

A theory $T$ is *functional* whenever for all sequences of formulas $A_0, B_0, A_1, B_1, \ldots, A_n, B_n$ and sentences $A$, if

$$T + \{A_i \Rightarrow B_i\}_{i=1}^n \models A_0 \Rightarrow B_0,$$

then

$$T + \{A \land A_i \Rightarrow B_i\}_{i=1}^n \models A \land A_0 \Rightarrow B_0.$$

$T$ is *well-formed* whenever for all sequences of sentences $\forall x (A_0 \Rightarrow B_0), \ldots, \forall x (A_n \Rightarrow B_n)$ and all formulas $A$ where no free variable of $A$ occurs in $x$, if

$$T + \{A_i \Rightarrow B_i\}_{i=1}^n \models A_0 \Rightarrow B_0,$$

then

$$T \models \bigwedge_{i=1}^n \forall x (A \land A_i \Rightarrow B_i) \Rightarrow \forall x (A \land A_0 \Rightarrow B_0).$$

**Proposition 2.3** (Completeness). Let $T$ be a functional well-formed theory extending BQC, $\Gamma$ a set of sequents, $\alpha$ a sequent and $R$ a rule. If $T + \Gamma \models \alpha$ then $T + \Gamma + R \models \alpha$. If $T + \Gamma \models R$ then $T + \Gamma + R$.

*Proof.* Theorem 5.8 in [12].

**Proposition 2.4.**

1. BQC, EBQC, IQC and CQC are functional and well-formed.

2. BA*, BA, BA+U, EBA, HA and PA are functional and well-formed. For any class $C$ of formulas, $\forall C$ and $\forall C + U$ are functional and well-formed.

3. $BA_c$ and $EB\Delta_0$ are functional and well-formed.

*Proof.*

1. Corollaries 4.10 and 4.14 in [12].

2. The cases of BA*, HA, PA, $\forall C$ and $\forall C + U$ are consequences of Corollaries 4.10 and 4.14 in [12]. The other cases follow from Proposition 6.1 in [12].

3. The proof is similar to that of the previous item. By Propositions 4.9 and 4.13 in [12], it suffices to prove that each theory has a functional and well-formed axiomatization. The only arithmetical rules of each theory are instances of the induction rule. In case of $BA_c$,

$$A \land B(x) \Rightarrow B(Sx) \text{ and } \forall y(A \land B(x) \Rightarrow B(Sx)) \Rightarrow \forall y(A \land B(0) \Rightarrow B(x))$$

are derivable by applying the induction rule and axiom, respectively, with $A \land B(x)$ as the induction formula. The case of $EB\Delta_0$ will be covered by Lemma 4.8.

A Kripke model $K = (K, \prec, D, \models)$ is called *rooted* whenever there is a node $k \in K$, called the *root*, such that $k \leq k'$, for all $k' \in K$. $K$ is called *reflexive* (respectively, *irreflexive*) whenever $\prec$ is reflexive (respectively, irreflexive). A node $k \in K$ is called *reflexive* (respectively, *irreflexive*) whenever $k \prec k$ (respectively, $k \not\prec k$). $k$ is called *terminal* whenever for all $k' \in K$, if $k \prec k'$ then $k = k'$.

A theory $T$ is *sound* with respect to a class of Kripke models whenever all of the axioms and rules derivable in it are valid in all models in the class. $T$ is *complete* with respect to a class of Kripke models whenever any sequent or rule that is valid in all of the models in the class, is provable in $T$. 


Proposition 2.5.

1. BQC is sound and complete with respect to the class of all Kripke models.
2. EBQC is sound and complete with respect to the class of all Kripke models in which every terminal node is reflexive.
3. IQC is sound and complete with respect to the class of all reflexive Kripke models.
4. CQC is sound and complete with respect to the class of all single-node reflexive Kripke models.

Proof.

1. Consider Proposition 2.3 for $T = BQC$.
2. Theorem 3.7 in [1].
3. This is the soundness and completeness theorem of the usual Kripke semantics for intuitionistic logic.
4. The forcing relation in a single-node reflexive Kripke model is equivalent to the satisfaction relation in the corresponding classical model. Thus, this is equivalent to soundness and completeness theorem of the usual classical semantics.

Proposition 2.6.

1. $BA^w$, $BA$, $BA + U$ and $BA_c$ are sound and complete with respect to the class of all Kripke models which force their corresponding arithmetical axioms and rules.
2. $EBA$ and $EB\Delta_0$ are sound and complete with respect to the class of all Kripke models with reflexive terminal nodes which force their corresponding arithmetical axioms and rules.
3. HA is sound and complete with respect to the class of all reflexive Kripke models which force its corresponding arithmetical axioms.
4. PA, $IC$ and $IC + U$ (for any class $C$ of formulas) are sound and complete with respect to the class of all single-node reflexive Kripke models which force their corresponding arithmetical axioms.

Proof. Combine Propositions 2.2, 2.3, 2.4 and 2.5.

Proposition 2.7.

1. Over a language containing at least one constant symbol, BQC and EBQC are fully rooted, and IQC is reflexively rooted.
2. BA, BA + U, BA_c, EBA and $EB\Delta_0$ are fully rooted. HA is reflexively rooted.

Proof.

Consider a theory $T$ in any first-order language including at least one constant symbol, so that the set of closed terms is nonempty. Let $D_T$ be the quotient set of the set of closed terms by the equivalence relation of provable equality in $T$; i.e. the relation defined with $T \vdash s = t$ for any closed terms $s$ and $t$. For each set of models $\{K_i\}_{i \in I}$ of $T$ we can construct two new models, denoted by $K_\bullet$ and $K_\circ$, as follows. Both models are formed by taking the disjoint union of the models $\{K_i\}_{i \in I}$ and then adding a new root. In $K_\bullet$, the new root $\bullet$ is reflexive with $D(\bullet) = D_T$, and in $K_\circ$ the new root $\circ$ is irreflexive with $D(\circ) = D_T$. In case we are working in the languages $L$ and $L_c$ of arithmetic and $T$ is any of the theories defined above, any closed term is provably equal in $T$ to a numeral, and instead of $D_T$ we can simply take $D(\bullet)$ and $D(\circ)$ to be equal to $\mathbb{N}$, the set of standard natural numbers. We call $T$ a reflexively rooted theory if for each set $\{K_i\}_{i \in I}$ of rooted Kripke models of $T$, the model $K_\bullet$ is also a model of $T$. $T$ is called irreflexively rooted if $K_c$ is a model of $T$, and fully rooted if both $K_\bullet$ and $K_\circ$ are models of $T$. 

Proposition 2.7.

1. Over a language containing at least one constant symbol, BQC and EBQC are fully rooted, and IQC is reflexively rooted.
2. BA, BA + U, BA_c, EBA and $EB\Delta_0$ are fully rooted. HA is reflexively rooted.

Proof.
1. The reflexive rootedness of IQC is well known. The full rootedness of BQC follows from item 1 of [23]. The full rootedness of EBQC follows from the fact that $\exists T \models \bot \Rightarrow \bot$ is valid in a Kripke model iff all terminal nodes in the model are reflexive. Consider $K*$ and $K*_{0}$ built from the set $\{K_{i}\}_{i \in I}$ of models of EQC. Since the new root in $K*$ and $K*_{0}$ is not terminal, all terminal nodes in $K*$ and $K*_{0}$ are in some $K_{i}$, and therefore reflexive. By full rootedness of BQC, it follows that both $K*$ and $K*_{0}$ are models of EBQC.

2. For BA and HA, see Proposition 6.2 in [12]. This will immediately imply the statements for BA + U and EBA, using the previous item. The proofs for BA* and EBA0 are identical to those of BA and EBA, only focusing on their corresponding induction formulas in their corresponding languages. □

A theory $T$ is faithful whenever for all sequences of sentences $\forall x (A_{0} \rightarrow B_{0}), \ldots, \forall x (A_{n} \rightarrow B_{n})$, if

$$T \vdash \bigwedge_{i=1}^{n} \forall x (A_{i} \rightarrow B_{i}) \Rightarrow \forall x (A_{0} \rightarrow B_{0}),$$

then

$$T + \{A_{i} \Rightarrow B_{i}\}_{i=1}^{n} \vdash A_{0} \Rightarrow B_{0}.$$  

$T$ has the disjunction property, if $T \vdash A \lor B$ implies $T \vdash A$ or $T \vdash B$, for all sentences $A$ and $B$. $T$ has the existence property, if $T \vdash \exists x A$ implies $T \vdash A[x/t]$ for some closed term $t$ in $L$, for all sentences $\exists x A$.

**Proposition 2.8.** A functional well-formed theory which is either reflexively rooted or irreflexively rooted is faithful, and has the disjunction and existence properties.

**Proof.** Proposition 5.14 in [12]. □

**Corollary 2.9.** BQC, EBQC, IQC, BA, BA + U, BA*, EBA, and EBA0 are faithful, and have the disjunction and existence properties.

**Proof.** Combine Propositions 2.4, 2.7 and 2.8. For the case of logical theories over languages without constant symbols, extra care is needed. See Propositions 5.14 and 4.12 in [12]. □

A useful way of looking at a Kripke model $K = (K, \prec, D, \models)$ is by considering it as a frame such that at any node $k \in K$, a classical model $M_{k}$ is attached. The domain of discourse of $M_{k}$ is $D(k)$, and the positive diagram of $M_{k}$ consists of the atomic formulas that are forced at $k$. If $k \prec k'$, then $M_{k}$ is a substructure of $M_{k'}$.

For a theory $T$, a Kripke model $K$ is said to be $T$-normal if for every node $k$ of $K$, $M_{k} \models T$.

**Corollary 2.10.** Let $T$ be given by all the geometric sentences true in the standard model $\mathbb{N}$. Then BA is $T$-normal. Consequently, for any geometric sentence $A$, if $\mathbb{N} \models A$ then BA $\vdash A$. Hence BA, EBA, HA, and PA prove the same geometric sentences.

**Proof.** Let $A$ be a geometric sentence and $\mathbb{N} \models A$. Let $K$ be an arbitrary model of BA. The model $K_{0}$ formed by $\{K\}$ is also a model of BA, by Propositions 2.7 and 2.11. $M_{0} \models A$, as $M_{0} = \mathbb{N}$, and $M_{k} \models A$ for any node $k$ of $K$, as $A$ is geometric and $M_{0}$ is a substructure of $M_{k}$. So $k \models A$ for any node $k$ of $K$, because $A$ is geometric, and for geometric $A$ we have $k \models A$ if $M_{k} \models A$. Therefore $K \models A$, and as $K$ was an arbitrary model of BA, BA $\models A$. Hence BA $\vdash A$, by Proposition 2.6. □

### 3 The Provably Total Recursive Functions of BA and its extensions

In this section, we investigate the classification of the provably total recursive functions of BA and its three extensions. In the first subsection, we show that the set of all the provably total recursive functions of BA, indicated by PTRF(BA), is a proper subset of all the primitive recursive functions, $\mathcal{PR}$. Half of this result, i.e. $\text{PTRF}(\text{BA}) \subseteq \mathcal{PR}$, has already been proved in [14], and we give an alternative proof in this paper. The other half of our result, i.e. $\text{PTRF}(\text{BA}) \neq \mathcal{PR}$, is a consequence of our Theorem 3.15 (Remark 3.16).

In the second subsection, we consider three extensions of BA. As is explained in Section 1, one of these extensions is a logical extension and the other two are arithmetical extensions. It turns out that the provably total recursive functions of all these three extensions of BA are exactly the primitive recursive functions.
3.1 About the Provably Total Recursive Functions of BA

Definition 3.1. For a formula $A$, the geometric part of $A$ is denoted by $A^3$ and is defined recursively as follows:

- $A^3 \equiv A$ if $A$ is prime,
- $A^3 \equiv B^3 \circ C^3$ if $A$ is of the form $B \circ C$ and $\circ$ is $\vee$ or $\wedge$,
- $A^3 \equiv \exists u \, B^3$ if $A$ is of the form $\exists u \, B$,
- $A^3 \equiv \top$ if $A$ is of the form $\forall x \, (B \rightarrow C)$.

$(A \Rightarrow B)^3$ is defined as $A^3 \Rightarrow B^3$. For a set of sequents $\Gamma$, $\Gamma^3$ is defined as the set $\{\alpha^3 \mid \alpha \in \Gamma\}$. For a rule $R$, $R^3$ is defined as the rule obtained by replacing any of the upper and lower sequents with its geometric part. For a theory $T$, $T^3$ is the theory formalized by all geometric sequents and rules derivable in $T$; i.e., axioms of $T^3$ are the geometric sequents $\alpha$ such that $T \vdash \alpha$, and rules of $T^3$ are the geometric rules such that $T \vdash R$.

Proposition 3.2.

1. For any formula $A$, $A^3$ is geometric.
2. If $A$ is geometric, $A^3 = A$.
3. For any formula $A$, $BQC \vdash A \Rightarrow A^3$.

Proof. Straightforward by induction on the complexity of $A$. \hfill \square

Proposition 3.3.

1. If $BQC + \Gamma \vdash \alpha$ then $GQC + \Gamma^3 \vdash \alpha^3$. Consequently, $BQC^3 \vdash GQC$, and $BQC$ is conservative over $GQC$.
2. If $BA + \Gamma \vdash \alpha$ then $GA + \Gamma^3 \vdash \alpha^3$. Consequently, $BA^3 \vdash GA$, and $BA$ is conservative over $GA$.

Proof.

1. This is a generalization of Proposition 4.2 in [12], and the proof goes along the similar lines. For any axiom $\alpha$ and any rule $R$ of $BQC$ in the items 1-12 of the list of the axioms and rules, $\alpha^3$ and $R^3$ are geometric instances of the same axiom and rule, respectively, and therefore they are axioms and rules of $GA$. For an axiom $\alpha$ in the items 13-18 of the list, $\alpha^3 = \top \Rightarrow \top$, and therefore an axiom of $GQC$. For rule 19, the lower sequent of its geometric part is of the form $A^3 \Rightarrow \top$, and therefore an axiom of $GQC$. Hence, for any derivation $\mathcal{D}$ of $\alpha$ from $\Gamma$ in $BQC$, one can replace any sequent in the derivation with its geometric part, and the sub-derivations ending in an application of rule 19 by a single geometric axiom, and the result will be a derivation of $\alpha^3$ from $\Gamma^3$ in $GQC$.

2. The proof is similar to the previous item. One just needs to additionally note that the axioms 20-25 are already geometric, the geometric part of the induction axiom is $\top \Rightarrow \top$, and the geometric part of the rule of induction is a geometric instance of the same rule, which is a rule of $GA$. \hfill \square

As a technical tool for a part of the proof of the next lemma, we temporarily adopt the standard language of first-order classical theories, in which for any formulas $A$ and $B$ and any variable $x$, $\neg A$, $A \rightarrow B$ and $\forall x \, A$ are formulas in their own right (not the abbreviations we have considered so far). The proof system $LK$ is a sequent calculus formalizing first-order classical logic over the standard language. Unlike the sequents we have considered so far, the sequents in $LK$ are of the form $\Delta \Rightarrow \Delta'$, where $\Delta$ and $\Delta'$ are finite lists of formulas (possibly empty). Classical arithmetical theories can be formalized over $LK$ by adding the axioms of equality, arithmetical axioms and the induction rule (with restricted induction formulas, if necessary). In particular, for any class $C$ of formulas, $\mathcal{I}C$ can be formalized by considering the $C$-induction rule. See the Appendix for definitions.

Lemma 3.4. Let $\Gamma$ be a set of geometric sequents, and $\alpha$ a geometric sequent.

1. $CQC + \Gamma \vdash \alpha$ iff $GQC + \Gamma \vdash \alpha$. Consequently, if $T$ is any of $CQC$, $IQC$ and $EBQC$, then $T^3 \vdash GQC$ and $T$ is conservative over $GQC$.

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2. \( \exists_1^+ + \Gamma \vdash \alpha \iff GA + \Gamma \vdash \alpha \). Consequently, \((\exists_1^+)^3 \vdash GA\) and \(\exists_1^+\) is conservative over \(GA\).

Proof.

1. As \(GQC\) is a sub-system of \(CQC\), one direction is trivial. For the other direction, consider a derivation for \(\alpha\) from \(\Gamma\) in the system \(LK\). By free-cut elimination of \(LK + \Gamma\) (Theorem 2.4.5 in [4]), there is a derivation of \(\alpha\) from \(\Gamma\) in \(LK\), every formula appearing in which is a geometric formula (since every formula in \(\Gamma \cup \{\alpha\}\) is geometric, and the set of geometric formulas is closed under subformulas and term substitution). As a consequence, none of the rules \((\neg \Rightarrow), (\Rightarrow \neg), (\Rightarrow \Rightarrow), (\neg \Rightarrow \Rightarrow)\) and \((\Rightarrow \forall)\) is used in \(\mathcal{D}\). So, \(GQC + \Gamma\) proves \(\alpha\), since the other rules and axioms of \(LK\) are valid in \(GQC\), in the sense that one can substitute \(\Delta \Rightarrow \forall \Delta'\) for \(\Delta \Rightarrow \Delta'\), and the rules and axioms remain valid. Here, \(\Delta \Rightarrow \forall \varnothing\) are defined, respectively as \(\top\) and \(\bot\).

2. The proof is similar to the previous item. For one direction, note that every geometric formula is provably equivalent over \(GQC\) to a formula in \(\exists_1^+\), which is nothing but its prefix normal form. Therefore, restricting the rule of induction in \(GA\) to \(\exists_1^+\)-formulas results in an equivalent theory. The rule of \(\exists_1^+\)-induction is derivable in \(\exists_1^+\), and every other axiom/rule of that theory is an axiom/rule of \(\exists_1^+\). For the other direction, we can use Corollary 2.4.7 in [4], which implies that since the set of geometric formulas is closed under subformulas and substitution, and all the formulas in \(\Gamma \cup \{\alpha\}\) are geometric, if \(\exists_1^+ + \Gamma \vdash \alpha\) then there is a derivation for it in which all the formulas are geometric. Again, this means that the derivation is valid in \(GA\), in the same sense as the previous item.

Corollary 3.5. For a geometric \(\alpha\) and a set \(\Gamma\) of geometric sequents, \(BA + \Gamma \vdash \alpha \iff \exists_1^+ + \Gamma \vdash \alpha\).

Proof. Combine Proposition 3.3 and Lemma 3.4.

Definition 3.6.

- For a function \(f: \mathbb{N}^n \rightarrow \mathbb{N}\) and a formula \(A(x, y)\) in the language of \(BA\), we say \(A\) defines \(f\) if for every \(\mathbf{a} \in \mathbb{N}^n\) and every \(b \in \mathbb{N}\), \(\mathbb{N} \models A(\mathbf{a}, b) \iff f(\mathbf{a}) = b\).

- For a formula \(A(x, y)\) and a variable \(y\), the existence sequent related to \(A(x, y)\) and \(y\), or simply the existence sequent of \(A\), is denoted by \(\mathcal{E}(A(x, y), y)\), or simply by \(\mathcal{E}(A)\), and is defined to be \(\top \Rightarrow \exists y A(x, y)\). If \(A(x, y)\) defines a function \(f\), we also call \(\mathcal{E}(A)\) the existence sequent of \(f\). The existence sequent is the statement that the relation defined by \(A(x, y)\) over a given model is total.

- For a formula \(A(x, y)\) and a variable \(y\) and fresh variables \(u\) and \(v\), the uniqueness sequent related to \(A(x, y), y, u\) and \(v\), or simply the uniqueness sequent of \(A\), is denoted by \(\mathcal{U}(A(x, y), y, u, v)\), or simply by \(\mathcal{U}(A)\), and is defined to be \(A(x, u) \land A(x, v) \Rightarrow u = v\). If \(A(x, y)\) defines a function \(f\), we also call \(\mathcal{U}(A)\) the uniqueness sequent of \(f\). The uniqueness sequent is the statement that the relation defined by \(A(x, y)\) over a given model is functional; i.e. it coincides with the graph of a partial function.

- For a formula \(A(x, y)\) and a function \(f: \mathbb{N}^n \rightarrow \mathbb{N}\), we say \(A(x, y)\) defines \(f\) in a theory \(T\), if \(f\) is defined in \(T\) by some formula. \(f\) is defined in \(T\) by a \(\Sigma_1\) formula. \(PTF(T)\) and \(PTRF(T)\) respectively denote the set of all the provably total functions and the set of all the provably total recursive functions of \(T\).

Theorem 3.7. Let \(\Gamma\) be a set of geometric sequents such that \(\mathbb{N} \models \Gamma\). If a function \(f: \mathbb{N}^n \rightarrow \mathbb{N}\) is defined in \(BA + \Gamma\) by a formula \(A(x, y)\), then it is also defined in \(BA + \Gamma\) by the geometric formula \(A^3(x, y)\). Consequently \(PTF(BA + \Gamma) = PTRF(BA + \Gamma) = PTRF(GA + \Gamma) = PTF(GA + \Gamma)\).

Proof. Assume that a function \(f: \mathbb{N}^n \rightarrow \mathbb{N}\) is defined by a formula \(A(x, y)\) in the language of \(BA\). If \(BA + \Gamma\) proves existence and uniqueness sequents related to \(A(x, y)\), then by Proposition 3.3 it also proves the existence and uniqueness sequents related to \(A^3(x, y)\). Since \(\mathbb{N}\) is a model of \(BA + \Gamma\), \(A^3(x, y)\) defines a function \(g: \mathbb{N}^n \rightarrow \mathbb{N}\). But \(g\) is just equal to \(f\), since by Proposition 3.3, \(BQC + A(x, y) \vdash A^3(x, y)\) and that means if \(\mathbb{N} \models A(\mathbf{a}, f(\mathbf{a}))\) for \(\mathbf{a} \in \mathbb{N}^n\), then \(\mathbb{N} \models A^3(\mathbf{a}, f(\mathbf{a}))\). So \(f\) is defined in \(BA + \Gamma\) by \(A^3(x, y)\), which is a \(\exists_1^+\) formula.

Every provably total recursive function in \(BA + \Gamma\) is clearly provably total, and thus \(PTRF(BA + \Gamma) \subseteq PTF(BA + \Gamma)\). Conversely, as is shown above, every provably total function in \(BA + \Gamma\) is definable in \(BA + \Gamma\) by an \(\exists_1^+\) formula.
Since $\exists^+_1 \subseteq \Sigma_1$, every such function is provably total recursive in $\text{BA} + \Gamma$, and thus $\text{PTF}(\text{BA} + \Gamma) \subseteq \text{PTRF}(\text{BA} + \Gamma)$. Also, the above application of Proposition 3.3 guarantees that the derivations of existence and uniqueness sequents can be done in the language of geometric logic, which completes the proof. 

**Theorem 3.8.** The provably total recursive functions of $\Sigma_1$ are exactly the primitive recursive functions, i.e. $\text{PTF}(\Sigma_1) = \text{PR}$. 

*Proof.* [8], Chapter 4, Corollary 3.7. 

**Theorem 3.9.** The provably total functions of $\text{BA}$ are primitive recursive, i.e. $\text{PTF}(\text{BA}) \subseteq \text{PR}$. Furthermore these functions are definable in $\text{BA}$ by $\exists^+_1$ formulas. 

*Proof.* Let $f$ be a provably total function in $\text{BA}$. Then by Theorem 3.7 it is definable in $\text{BA}$ by a geometric formula. So its existence and uniqueness sequents consist of geometric formulas, and by Corollary 3.5 $\exists^+_1$ proves these sequents as well. Then $f$ is defined in $\exists^+_1$ by a geometric formula. As $\exists^+_1 \subseteq \Sigma_1$, $f$ is provably total recursive in $\Sigma_1$. Hence by Theorem 3.8 $f$ is primitive recursive. 

We now show that not all primitive recursive functions are provably total in $\text{BA}$. For this purpose, we need to take a look at $\mathbb{N}^*$, a specific structure, that is a model of $\exists^+_1$, but not a model of $\text{IOpen}$. 

**Definition 3.10.** 

- $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ is a classical structure, where $\infty$ is a nonstandard element such that $S\infty = \infty + \infty = \infty \cdot \infty = \infty$, $0 \cdot \infty = \infty \cdot 0 = 0$, $0 < \infty$, $n < \infty$, $\infty \not= n$, and $n + \infty = \infty + n = (n + 1) \cdot \infty = \infty \cdot (n + 1) = \infty$, for every $n \in \mathbb{N}$. 
- $\mathbb{K}^*$ is the Kripke model with just one irreducible node with the structure $\mathbb{N}^*$. 

$\mathbb{N}^*$ is not a model of $\text{IOpen}$, because using induction on the open formula $\neg Sx = x$, one can get $\text{IOpen} \vdash Sx = x \Rightarrow \bot$, which is not satisfied in $\mathbb{N}^*$. To show that $\mathbb{N}^*$ is a model of $\exists^+_1$, we need the observation that terms in the language of arithmetic correspond to polynomials with non-negative integer coefficients. Univariate polynomials of this kind are either constant or strictly increasing over the set of natural numbers. Also, if two such polynomials agree on infinitely many points, they are identical. When extending the domain and codomain to $\mathbb{N}^*$, we see that in the constant case, the polynomial takes the same value at $\infty$ as at other points, and in the strictly increasing case, it takes the value $\infty$ at $\infty$. We can generalize these observations by considering terms in the language of $\mathbb{N}^*$ (the language of arithmetic extended with parameters from $\mathbb{N}^*$), in the following way. Such terms correspond to multivariate polynomials with coefficients from $\mathbb{N}^*$. In the univariate case, two such polynomials agreeing on infinitely many points are equal, possibly with the exception at 0. Even in the multivariate case, either the polynomial is constantly equal to a natural number, or it takes the value $\infty$ at $\infty$, where $\infty = (\infty, \ldots, \infty)$ with the length equal to the number of variables of the polynomial. We use these observations to prove that $\mathbb{N}^*$ has the overspill property for geometric formulas. 

**Lemma 3.11.** For every $\exists^+_1$ formula $A(x)$ in the language of $\mathbb{N}^*$ with $x$ as the only free variable, if the set $\{a \in \mathbb{N}^* \models A(a)\}$ is infinite, then $\mathbb{N}^* \models A(\infty)$. 

*Proof.* Since $A(x)$ is geometric, we can assume that it is of the form $\bigvee_i \exists y \bigwedge_j s_{ij}(x, y) = t_{ij}(x, y)$. Because $\{a \in \mathbb{N}^* \models A(a)\}$ is infinite, $\{a \in \mathbb{N}^* \models \exists y \bigwedge_j s_{ij}(a, y) = t_{ij}(a, y)\}$ is infinite for some index $k$. For the sake of simplicity, we shall omit the index $k$ and write the formula $\exists y \bigwedge_j s_{ij}(x, y) = t_{ij}(x, y)$ as $B(x) \equiv \exists y \bigwedge_j s_{ij}(x, y) = t_{ij}(x, y)$. Let $S = \{a \in \mathbb{N}^* \models B(a)\}$. We show $\mathbb{N}^* \models B(\infty)$ by induction on $|y|$ (number of variables appearing in $y$), which implies $\mathbb{N}^* \models A(\infty)$. For the base case, $|y| = 0$ and hence $B(x)$ is of the form $\bigwedge_j s_{ij}(x) = t_{ij}(x)$. Because $S$ is infinite, for every $j$, the polynomials corresponding to $t_{ij}(x)$ and $s_{ij}(x)$ agree on infinitely many points, and thus give equal values for nonzero inputs. Hence $t_{ij}(\infty) = s_{ij}(\infty)$ for every $j$, which means $\mathbb{N}^* \models B(\infty)$. 

For the induction step, assume the statement is true when $|y| = n$, and suppose $|y| = n + 1$ in $B(x)$. Let $\infty = (\infty, \ldots, \infty)$, with $|\infty| = |y|$. If $\mathbb{N}^* \models \bigwedge_j s_{ij}(\infty, \infty) = t_{ij}(\infty, \infty)$, we are done. Otherwise, there exists an index $u$ such that $\mathbb{N}^* \models s_{iu}(\infty, \infty) \neq t_{iu}(\infty, \infty)$. Note that in this case, one of $s_{iu}(x, y)$ and $t_{iu}(x, y)$ must be constantly equal to some $c \in \mathbb{N}$ and the other one must take the value $\infty$ at $(\infty, \infty)$ (they cannot both take the value $\infty$ at $(\infty, \infty)$, and if they are both constant, then the constants must be different and $S$ will be empty, contradicting the assumption).
Without loss of generality, assume \( t_u(x, y) = c \). We can represent \( s_u(x, y) \) as \( \sum_{i=0}^{m} x^{i} p_i(y) \), where \( m \in \mathbb{N} \) and each \( p_i(y) \) is a term with variables only from \( y \). We claim that there exists \( v \) such that \( p_v(y) \) is not a constant polynomial. Suppose this is not the case, which means \( s_u(x, y) = \sum_{i=0}^{m} a_i x^i \), for some \( a_0, \ldots, a_m \in \mathbb{N}^+ \). Because \( S \) is infinite, there exists \( d \in S \) such that \( d > c \). We know \( \mathbb{N}^+ \models B(d) \), so in particular, \( s_u(d, y) = \sum_{i=0}^{m} a_i d^i \). This implies that \( a_0 = c \) and for all \( i > 0 \), \( a_i = 0 \). Thus \( s_u(x, y) = c \), which leads to a contradiction as \( s_u(\infty, \infty) = \infty \). Hence our assumption was false and there exists \( v \) such that \( p_v(y) \) is not a constant polynomial. Let \( f : S \to (\mathbb{N}^+)^{n+1} \) be a function such that for every \( a \in S \), \( \mathbb{N}^+ \models \bigwedge_{j} sj(a, f(a)) = t_j(a, f(a)) \). The following two cases can happen:

1. \( \text{range}(f) \) is finite:

   In this case there exists \( b \in \text{range}(f) \) such that \( \{ a \in \mathbb{N}^+ | \mathbb{N}^+ \models \bigwedge_{j} sj(a, b) = t_j(a, b) \} \) is infinite, hence we can use the base step and the proof is complete.

2. \( \text{range}(f) \) is infinite:

   For every natural numbers \( 1 \leq l \leq n + 1 \) and \( 0 \leq l' \leq c \), define \( W_{l,l'} = \{ b \in \text{range}(f) | b = l' \} \), and let \( W = \bigcup_{l,l'} W_{l,l'} \). We claim that at least one of \( W_{l,l'} \)'s must be infinite. Suppose this is not the case. By the assumption, \( \text{range}(f) \setminus W \) is infinite. This implies that there exists a nonzero \( a \in S \) such that \( f(a) \notin W \). But this leads to a contradiction, because \( a > 0 \) and \( (f(a))_{l'} > c \) for every \( 1 \leq l' \leq n + 1 \), hence \( a^s p_v(f(a)) > c \), which implies \( s_u(a, f(a)) > c \). Therefore for some \( h, h', W_{h,h'} \) is infinite. Let \( B'(x) \) be the formula obtained by removing the existential quantifier over \( y_h \) in \( B(x) \) and substituting every free occurrence of \( y_h \) by \( h' \). By the fact that \( W_{h,h'} \) is infinite we get \( \{ a \in \mathbb{N}^+ | B'(a) \} \) is also infinite. Note that \( B'(x) \) has less \( y \) variables, so by the induction hypothesis \( \mathbb{N}^+ \models B'(\infty) \), which implies \( \mathbb{N}^+ \models B(\infty) \).

**Lemma 3.12.**

1. \( \mathbb{N}^+ \) is a (classical) model of \( \mathbb{I}^+ \).

2. \( \mathbf{K}^+ \) is a model of \( \mathbf{BA} \).

**Proof.**

1. It is easy to check that \( \mathbb{N}^+ \) satisfies Robinson’s axioms and thus we only need to show that it satisfies the induction axiom. Suppose \( A(x, y) \) is a \( \mathbb{I}^+ \) formula and \( \mathbb{N}^+ \models \forall xy (A(x, y) \rightarrow A(Sx, y)) \). If \( \mathbb{N}^+ \models A(0, b) \) for some \( b \in \mathbb{N}^+ \), then \( \{ a \in \mathbb{N}^+ | \mathbb{N}^+ \models A(a, b) \} = \mathbb{N} \) and by Lemma 3.11 \( \mathbb{N}^+ \models A(\infty, b) \). Hence \( \mathbb{N}^+ \models \forall xy (A(x, y) \rightarrow A(Sx, y)) \Rightarrow \forall xy (A(0, y) \rightarrow A(x, y)) \).

2. By Theorem 2.18 in [1], a Kripke model with a single irreflexive node is a model of \( \mathbf{BA} \) if it classically satisfies \( \mathbb{I}^+ \). Use this together with the previous item.

**Lemma 3.13.** Let \( A \) be a formula in the language of arithmetic.

1. If \( A \) is geometric and \( \mathbb{N} \models A \) then \( \mathbb{N}^+ \models A \).

2. If \( \mathbb{N} \models \exists y (y > x \land A) \), then \( \mathbf{K}^+ \models A[y/\infty] \).

**Proof.**

1. We use induction on the number of free variables of \( A \). If \( A \) is a sentence, then since it is an existential sentence, and \( \mathbb{N} \) is a classical substructure of \( \mathbb{N}^+ \), we get \( \mathbb{N}^+ \models A \). If \( xy \) is the sequence of variables free in \( A \), then by induction hypothesis, for every \( b \in \mathbb{N} \) we have \( \mathbb{N}^+ \models A[y/b] \). This shows that for every \( a \in \mathbb{N}^+ \), \( \{ b \in \mathbb{N} | \mathbb{N}^+ \models A[x/a][y/b] \} = \mathbb{N} \). Thus by Lemma 3.11 we have \( \mathbb{N}^+ \models A \).

2. By Proposition 3.2 we have \( \mathbb{N} \models \exists y (y > x \land A^3) \). By item 1 we get \( \mathbb{N}^+ \models \exists y (y > x \land A^3) \), which shows that \( \mathbb{N}^+ \models A^3[y/\infty] \), and thus \( \mathbf{K}^+ \models A^3[y/\infty] \). Since \( \mathbf{K}^+ \) consists of just one irreflexive node, we have \( \mathbf{K}^+ \models A^3 \leftrightarrow A \), which yields what is desired.

**Definition 3.14.** A set \( A \subseteq \mathbb{N} \) is provably decidable in a theory \( T \) if \( \chi_A \), i.e., its characteristic function, is provably total recursive in \( T \).

**Theorem 3.15.** The provably decidable sets of \( \mathbf{BA} \) are exactly the sets \( A \subseteq \mathbb{N} \) that are either finite or co-finite (i.e. \( A^c = \mathbb{N} \setminus A \) is finite).
Proof. Let \( A \) be a provably decidable set in \( BA \), so \( \chi_A \) is a provably total function in \( BA \). Then there exists a \( \exists_1^+ \) formula \( A(x, y) \) such that

- \( BA \vdash \mathcal{E}(A) \),
- \( BA \vdash \mathcal{U}(A) \),
- \( \mathbb{N} \models A(a, \chi_A(a)) \), for every \( a \in \mathbb{N} \).

If both \( A \) and \( A^c \) are infinite, we have

- \( \mathbb{N} \models \exists y (y > x \land A(x, 0)) \),
- \( \mathbb{N} \models \exists y (y > x \land A(x, 1)) \).

Then by Lemma 3.13, \( K^* \models A(\infty, 0) \) and \( K^* \models A(\infty, 1) \), which show that \( BA \not\models U(A) \) and this leads to a contradiction. Hence either \( A \) or \( A^c \) is finite.

Now suppose \( A \subseteq \mathbb{N} \) is finite. Then \( A \) has a maximum element, say \( M \). Define the formula \( A(x, y) \equiv (x > M \land y = 0) \lor \bigvee_{i=0}^M (x = i \land y = \chi_A(i)) \). It is easy to see that \( A(x, y) \) is \( \Sigma_1 \) and defines \( \chi_A \) in \( BA \). A similar argument works when \( A^c \) is finite.

Remark 3.16. Consider the primitive recursive function \( \text{Even}(n) = \begin{cases} 1 & n = 2k \\ 0 & n = 2k + 1 \end{cases} \). If it is provably total recursive in \( BA \), then \( A = \{a \in \mathbb{N} \mid \text{Even}(a) = 1\} \) is a provably decidable set in \( BA \). Then by Theorem 3.15, \( A \) is either finite or co-finite, which leads to a contradiction in either case. Hence \( \text{Even}(n) \) is not provably total in \( BA \), which shows that \( \text{PTRF}(BA) \neq \text{PR} \).

One of the most important consequences of Theorem 3.15 is that given any (definable) pairing function, the corresponding projection function is not provably total recursive in \( BA \). In the following Corollary, think of \( C(x, y, z) \) and \( D(x, y) \) as formulas defining the graphs of a pairing function and projection on the first entry, respectively.

Corollary 3.17. There are no formulas \( C(x, y, z) \) and \( D(x, y) \) with presented free variables such that:

1. \( \mathbb{N} \models \exists z C(x, y, z) \),
2. \( \mathbb{N} \models C(x, u, z) \land C(x, v, z) \Rightarrow u = v \),
3. \( \mathbb{N} \models C(x, y, z) \Rightarrow D(z, x) \),
4. \( BA \vdash \mathcal{E}(D) \),
5. \( BA \vdash \mathcal{U}(D) \).

Proof. Let \( A_n = \{a \in \mathbb{N} \mid \mathbb{N} \models D(a, n)\} \). By (1) and (3) \( A_0 \) and \( A_1 \) are not empty and because of (2) they are infinite. Define \( B(x, y) \equiv \exists z ((D(x, z) \land ((z = 0 \land y = 1) \lor (z > 0 \land y = 0))) \lor \bigvee_{i=0}^M (x = i \land y = \chi_A(i))) \). Then by (4) and (5) we have:

- \( BA \vdash \mathcal{E}(B) \),
- \( BA \vdash \mathcal{U}(B) \),
- \( \mathbb{N} \models B(a, \chi_{A_0}(a)) \), for every \( a \in \mathbb{N} \).

That means \( A_0 \) is a provably decidable set in \( BA \), and then by Theorem 3.15 \( A_0 \) is either finite or co-finite. But \( A_0 \) and \( A_1 \) are infinite which leads to a contradiction.

Corollary 3.18. The cut-off subtraction is not provably total in \( BA \).

Proof. Suppose the cut-off subtraction is defined in \( BA \) by a formula \( A(x, y, z) \), which by Theorem 3.7 can be assumed to be geometric. By definition of the cut-off subtraction we have:

- \( \mathbb{N} \models \exists y (y > x \land A(y, y, 0)) \),
• \(\mathbb{N} \models \exists y(y > x \land A(Sy, y, 1))\).

Hence by Lemma 3.13, \(K^* \models A(\infty, \infty, 0)\) and also \(K^* \models A(S\infty, \infty, 1)\). By the fact that \(K^* \models S\infty = \infty\), we have \(K^* \not\models U(A)\) which shows \(BA \not\models U(A)\) and this leads to a contradiction. Hence the cut-off subtraction is not provably total in \(BA\).

**Corollary 3.19.** Suppose \(P(x)\) is a formula that defines prime numbers (or even an infinite subset of prime numbers). Then \(BA \not\models P(x) \land y|x \Rightarrow y = 1 \lor y = x\), where \(s|t \equiv \exists z(s \cdot z = t)\), with \(z\) being a fresh variable.

**Proof.** Because \(\mathbb{N} \models \exists y(y > x \land P(y))\), by Lemma 3.13, we have \(K^* \models P(\infty)\). Moreover, \(K^* \models 2|\infty\), and hence \(K^* \not\models P(\infty) \land 2|\infty \Rightarrow 1 \lor 2 = \infty\). Thus \(BA \not\models P(x) \land y|x \Rightarrow y = 1 \lor y = x\). \(\blacksquare\)

We close the discussion about the provably total functions of \(BA\) by showing that the predecessor function, defined with \(pd(0) = 0\) and \(pd(S(x)) = x\), is provably total recursive in \(BA\). This will give an explicit instance of primitive recursion applied to a provably total recursive function of \(BA\) resulting in a function not provably total in \(BA\). In other words, it shows that the class of provably total recursive functions of \(BA\) is not closed under primitive recursion. Note that the cut-off subtraction can be defined by iterating the predecessor function: \(x \div 0 = x\) and \(x \div S(y) = pd(x \div y)\). In the next proposition, note that since \(\mathbb{N} \models BA\), the formula considered is actually defining the predecessor function.

**Proposition 3.20.** There is a geometric open formula \(A_{pd}(x, y)\) such that:

- \(BA \vdash A_{pd}(0, 0)\)
- \(BA \vdash A_{pd}(Sx, x)\)
- \(BA \vdash E(A_{pd})\)
- \(BA \vdash U(A_{pd})\)

**Proof.** Let \(A_{pd}(x, y) \equiv (x = 0 \land y = 0) \lor x = Sy\). The first two items follow immediately. The third item follows from \(BA \vdash x = 0 \lor \exists y(x = Sy)\). The last item is a consequence of \(BA \vdash Sy = 0 \Rightarrow \bot\) and \(BA \vdash Sy = Sz \Rightarrow y = z\). \(\blacksquare\)

In the remaining part of this subsection, we analyze BQC and BA on a level beyond what can be done inside GQC and GA. The main method we have used until now ignores the additional expressive power of BA in comparison to GA. To emphasize on this aspect of BA, let’s consider semi-geometric formulas, in which every subformula of the form \(\forall x(A \rightarrow B)\) is such that \(A\) and \(B\) are geometric. In other words, semi-geometric formulas are those in which there are no nested implications/universal quantifications. A sequent \(A \Rightarrow B\) is called semi-geometric whenever both \(A\) and \(B\) are semi-geometric formulas. A rule is called semi-geometric whenever its lower sequent and all its upper sequents are semi-geometric. We show that studying semi-geometric formulas can give us some results on BA which are similar to, yet different than, what we have already proven about BA using geometric formulas. To give concrete examples, we consider certain modifications of the notion of provably total functions. These modifications are equivalent to what we have previously considered, if one is working over HA or PA, but not over BA. We first show that semi-geometric formulas give us ways to recover some properties of provably total functions of BA in the modified senses, which are analogous to our previous results, yet cannot be derived by only looking at geometric formulas. We then consider the weaker theory \(BA^w\), and apply both methods (involving geometric and semi-geometric formulas) to study its provably total functions in different senses. Unlike what was obtained for BA, the results on \(BA^w\) will not be all similar to one another, which gives another example of the contrast between basic arithmetic and geometric arithmetic.

**Definition 3.21.** For a formula \(A\), the semi-geometric part of \(A\) is denoted by \(A^\diamond\) and is defined recursively as follows:

- \(A^\diamond = A\) if \(A\) is prime,
- \(A^\diamond = B^\diamond \circ C^\diamond\) if \(A\) is of the form \(B \circ C\) and \(\circ\) is \(\lor\) or \(\land\),
- \(A^\diamond = \exists u B^\diamond\) if \(A\) is of the form \(\exists u B\),
- \(A^\diamond = \forall x(B^\exists \rightarrow C^\exists)\) if \(A\) is of the form \(\forall x(B \rightarrow C)\).
\( (A \Rightarrow B)^v \) is defined as \( A^v \Rightarrow B^v \). For a rule \( R \), \( R^v \) is defined as the rule obtained by replacing any of the upper and lower sequents with its semi-geometric part.

**Proposition 3.22.**

1. For any formula \( A \), \( A^v \) is semi-geometric.
2. For any formula \( A \), \( (A^v)^3 = A^3 \).
3. If \( A \) is semi-geometric, \( A^v = A \).
4. If \( A \) is semi-geometric, \( \text{BQC} \vdash A \Rightarrow A^3 \). Moreover, it can be so that only semi-geometric formulas appear in the derivation of \( A \Rightarrow A^3 \) in \( \text{BQC} \).

**Proof.** Straightforward by induction on the complexity of \( A \).

**Proposition 3.23.** Let \( \Gamma \) be a set of geometric sequents.

1. If \( \text{BQC} + \Gamma \vdash \alpha \) then \( \text{BQC} + \Gamma \vdash \alpha^v \). Moreover, it can be so that only semi-geometric formulas appear in the derivation of \( \alpha^v \) from \( \Gamma \) in \( \text{BQC} \).
2. If \( \text{BA} + \Gamma \vdash \alpha \) then \( \text{BA} + \Gamma \vdash \alpha^v \). Moreover, it can be so that only semi-geometric formulas appear in the derivation of \( \alpha^v \) from \( \Gamma \) in \( \text{BA} \).

**Proof.**

1. We prove the claim by a structural induction on the derivation of \( \alpha \) from \( \Gamma \) in \( \text{BQC} \). If \( \alpha \) is an axiom, then \( \alpha^v \) is a semi-geometric instance of the same axiom. If \( \alpha \in \Gamma \), then \( \alpha \) is geometric and thus equal to its semi-geometric parts, which means \( \alpha^v \in \Gamma \). Thus the base case is proved. Now suppose that \( \alpha \) is proved by the rule 19. Then \( \alpha \) is of the form \( A \Rightarrow \forall x(B \Rightarrow C) \), and we have \( \text{BQC} + \Gamma \vdash A \land B \Rightarrow C \) from the immediate sub-derivation. So by Proposition 3.22, \( \text{GQC} + \Gamma \vdash A^3 \land B^3 \Rightarrow C^3 \) and by applying rule 19, \( \text{BQC} + \Gamma \vdash A^3 \Rightarrow \forall x(B^3 \Rightarrow C^3) \). Then by item 4 of Proposition 3.22, \( \text{BQC} + \Gamma \vdash \alpha^v \), with all the formulas appearing in the derivation being semi-geometric. If \( R \) is any other rule of \( \text{BQC} \), \( R^v \) is a semi-geometric instance of the same rule, and the proof of the induction step is complete.

2. The proof is similar to the previous item. One just needs to additionally note that the axioms 20-25 are all geometric, and the semi-geometric part of the induction axiom and the rule of induction are semi-geometric instances of the same axiom and rule, respectively.

The uniqueness sequent related to a formula \( A \) can be defined in different ways which are very close in meaning from the intuitionistic and/or classical point of view, yet not so in the context of basic logic. We list some examples of alternative definitions for \( \mathcal{U}(A) \) in the following. Taking any of these definitions results in a different notion of provably total functions of a theory.

- \( \mathcal{U}_0(A) \equiv \forall x\forall u(A[y/u] \land A[y/v] \Rightarrow u = v) \), where \( xy \) consists of all the free variables of \( A \);
- \( \mathcal{U}_1(A) \equiv A[y/u] \land A[y/v] \Rightarrow u = v \);
- \( \mathcal{U}_2(A) \equiv \neg u = v \Rightarrow \neg A[y/u] \lor \neg A[y/v] \);
- \( \mathcal{U}_3(A) \equiv A[y/u] \Rightarrow A[y/v] \Rightarrow u = v \);
- \( \mathcal{U}_4(A) \equiv A[y/u] \land A[y/v] \Rightarrow \top \Rightarrow (\top \Rightarrow u = v) \).

For \( i = 0, \ldots, 4 \) and a theory \( T \), one can consider \( \text{PTF}_i(T) \) and \( \text{PTRF}_i(T) \) by replacing \( \mathcal{U} \) with \( \mathcal{U}_i \) in the definition of \( \text{PTF}(T) \) and \( \text{PTRF}(T) \), respectively. By functionality and faithfulness of \( \text{BA} \) (Proposition 2.4 and Corollary 2.9), provability of \( \mathcal{U}_0(A) \) is equivalent to provability of \( \mathcal{U}(A) \) in \( \text{BA} \). Consequently we get the same results regarding \( \text{PTF}_0(\text{BA}) \) and \( \text{PTRF}_0(\text{BA}) \). We use the above Proposition to obtain some results about \( \text{PTF}_i(\text{BA}) \) and \( \text{PTRF}_i(\text{BA}) \) for \( i = 1, 2, 3 \). However, the case of \( \mathcal{U}_4 \) may need a different treatment, as \( (\top \Rightarrow (\top \Rightarrow u = v))^v = \top \Rightarrow \top \), and the technique fails to deliver what is required.

In the course of the proof of the next Theorem, we use the following Lemma about \( \mathbf{I}_\Sigma_1 \). The proof of the Lemma is based on well-known methods applicable in \( \mathbf{I}_\Sigma_1 \); introducing a new variable as a bound for existentially quantified variables in a \( \Sigma_1 \) formula to get a \( \Delta_0 \) variant, and using a pairing function to encode two variables into one.
Lemma 3.24. Let $T$ be a theory extending $\Sigma_1$ and $A(x, y)$ be a $\Sigma_1$ formula such that $T \vdash E(A)$. Then there exists a $\Sigma_1$ formula $B(x, y)$ such that

- $T \vdash E(B)$;
- $T \vdash U(B)$;
- $T \vdash B \Rightarrow A$.

Proof. Assume that $A(x, y)$ is of the form $\exists z D(x, y, z)$ where $D$ is a $\Delta_0$ formula and $z = (z_1, \ldots, z_k)$. Let $C(x, y, z)$ be a $\Delta_0$ formula which defines the Cantor pairing function $\langle x, y \rangle = \frac{1}{2}(x + y)(x + y + 1) + y$ in $T$ and moreover

- $T \vdash \exists xy C(x, y, z)$;
- $T \vdash C(x, y, z) \land C(u, v, z) \Rightarrow x = u \land y = v$.

Define

$$E(x, w) \equiv \exists uvz (u \leq w \land v \leq w \land \bigwedge_{i=1}^{k} z_i < v \land C(u, v, w) \land D(x, u, z))$$

and let $F(x, w) \equiv E(x, w) \land \forall z (z < w \Rightarrow \neg E(x, z))$. Note that $F(x, w)$ is provably equivalent in $T$ to a $\Delta_0$ formula. As $T \vdash A(x, y) \Rightarrow \exists w F(x, w)$ and $T \vdash E(A)$, we have $T \vdash E(F)$ by the fact that $I\Sigma_1$ proves the least number principle for $\Delta_0$ formulas. Also, by the way $F$ has been defined, $T \vdash U(F)$. Finally, define $B(x, y, z) \equiv \exists w (C(y, v, w) \land F(x, w))$. It is easy to see that $B$ is provably equivalent in $T$ to a $\Sigma_1$ formula and $T \vdash B(x, y) \Rightarrow A(x, y)$. The provability of the existence and uniqueness sequents of $B$ follow from those of $F$. \hfill $\square$

Theorem 3.25. Let $\Gamma$ be a set of geometric sequents such that $\mathbb{N} \models \Gamma$.

1. For $i = 1, 2$, if $BA + \Gamma \vdash U_i(A)$ then $BA + \Gamma \vdash U_i(A^3)$.

2. If $BA + \Gamma \vdash U_3(A)$ then $BA + \Gamma \vdash U_3(A^3)$. If additionally $BA + \Gamma \vdash E(A)$ then $BA + \Gamma \vdash E(A^3)$, $BA + \Gamma \vdash U_1(A^3)$ and $BA + \Gamma \vdash U_3(A^3)$.

3. For $i = 1, 2, 3$, $PTRF_i(BA + \Gamma) = PTRF_i(BA + \Gamma) \subseteq PTRF(I\Sigma_1 + \Gamma)$.

Proof.

1. Apply Proposition 3.23 and note that $(U_i(A))^\forall \equiv U_i(A^3)$.

2. If $BA + \Gamma \vdash U_3(A)$ then by Proposition 3.23 we have $BA + \Gamma \vdash (U_3(A))^\forall \equiv A^3[y/u] \Rightarrow A^3[y/v] \Rightarrow u = v$, which together with item 4 of Proposition 3.22 implies $BA + \Gamma \vdash U_3(A^3)$. Assume that we additionally have $BA + \Gamma \vdash E(A)$. First, note that by Proposition 3.23 we must have $BA + \Gamma \vdash E(A^3)$. This together with $BA + \Gamma \vdash (U_3(A))^\forall$ shows that $BA + \Gamma \vdash \exists y (A^3[y/u] \land A^3[y/v] \Rightarrow y = u \land y = v)$, which implies $BA + \Gamma \vdash U_3(A^3)$. Also, since $BA + \Gamma \vdash A^3[y/u] \Rightarrow A^3[y/v] \Rightarrow A^3[y/u] \land A^3[y/v]$, we can conclude $BA + \Gamma \vdash U_3(A^3) \equiv A^3[y/u] \Rightarrow A^3[y/v] \Rightarrow u = v$.

3. Suppose $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is in $PTRF_i(BA + \Gamma)$; i.e. there exists a formula $A(x, y)$ such that:

- $BA + \Gamma \vdash E(A)$;
- $BA + \Gamma \vdash U_i(A)$;
- $\mathbb{N} \models A(a, f(a))$, for every $a \in \mathbb{N}^k$.

Then, by the previous items and item 3 of Proposition 3.22 we have

- $BA + \Gamma \vdash E(A^3)$;
- $BA + \Gamma \vdash U_i(A^3)$.

\footnote{We can take $C(x, y, z) \equiv 2z = (x + y) \cdot S(x + y) + 2y$, and then formalize the usual arguments for the desired properties of the Cantor pairing function (totality and uniqueness properties, as well as being bijective) inside $I\Sigma_1$.}
As $\mathbb{N} \models \Gamma$, this means that $A^3$ also defines a total function, which by item 3 of Proposition 3.22 must be the same as $f$. Thus, $f$ is in PTF$_i(\mathsf{BA} + \Gamma)$, which shows PTF$_i(\mathsf{BA} + \Gamma) = \text{PTRF}_i(\mathsf{BA} + \Gamma)$.

Now, note that $\mathsf{BA} + \Gamma \vdash \mathcal{E}(A^3)$ implies $\exists i^* + \Gamma \vdash \mathcal{E}(A^3)$, using Lemma 3.4. Thus $\Sigma_1 + \Gamma \vdash \mathcal{E}(A^3)$, and by Lemma 3.21 we get a $\Sigma_1$ formula $B(x, y)$ such that

- $\Sigma_1 + \Gamma \vdash \mathcal{E}(B)$;
- $\Sigma_1 + \Gamma \vdash \mathcal{U}(B)$;
- $\Sigma_1 + \Gamma \vdash B \Rightarrow A^3$.

Thus, $B(x, y)$ defines a total function, that must be the same as $f$. Hence $f$ is in PTF($\Sigma_1 + \Gamma$), which completes the proof.

\[\Box\]

**Remark 3.26.** The case $\Gamma = \emptyset$ of Theorem 3.26 and the previous observations show that PTF$_i(\mathsf{BA}) \subseteq \mathcal{P}\mathcal{R}$ for $i = 0, 1, 2, 3$, by Theorem 1.8. One can get the results PTF(\mathsf{BA}) \subseteq \mathcal{P}\mathcal{R}$ and PTF$_i(\mathsf{BA}) \subseteq \mathcal{P}\mathcal{R}$ for $i = 0, \ldots, 4$, by an analysis of provability of the existence sequent, and completely disregard the uniqueness sequent, as is done in [14]. See Theorem 3.53 where we use that method for the even stronger theory EBA.

The importance of taking the provably total functions of BA in the sense of $\mathcal{U}_1$ comes from the fact that the cut-off subtraction becomes provably total. For the next proposition, note that since $\mathbb{N} \models \mathsf{BA}$, the formula considered is actually defining cut-off subtraction.

**Proposition 3.27.** There is a geometric open formula $A_c(x, y, z)$ such that:

- $\mathsf{BA} \vdash x < y \Rightarrow A_c(x, y, 0)$
- $\mathsf{BA} \vdash x = y + z \Rightarrow A_c(x, y, z)$
- $\mathsf{BA} \vdash \mathcal{E}(A_c)$
- $\mathsf{BA} \vdash \mathcal{U}_1(A_c)$

**Proof.** Let $A_c(x, y, z) \equiv (x < y \land z = 0) \lor x = y + z$. The first two items follow immediately. The third item is a consequence of $\mathsf{BA} \vdash x < y \lor y \leq x$. To prove the last item, we first use the rule of induction to show that $\mathsf{BA} \vdash y + u = y + v \rightarrow u = v$, noting that $\mathsf{BA} \vdash 0 + u = 0 + v = u = v$ and $\mathsf{BA} \vdash y + u = y + v \rightarrow u = v = Sy + u = Sy + v \rightarrow u = v$.

Next, we show how the analysis can be done for the weakened basic arithmetic $\mathsf{BA}^w$, in which the rule of induction is dropped from $\mathsf{BA}$. For this, consider the Weakened Geometric Arithmetic, $\mathsf{GA}^w$, in which the rule of geometric induction is dropped from $\mathsf{GA}$. Note that $\mathsf{GA}^w$ is a theory even weaker than Robinson arithmetic $\mathsf{Q}$; not only its language is restricted to geometric formulas, but also it lacks the axiom $x = 0 \lor \exists y x = Sy$. An important property of $\mathsf{GA}^w$ is that its arithmetical axioms contain only (geometric) open formulas. As geometric logic is a subtheory of classical logic, this lets us use well-known classical methods for analyzing $\mathsf{GA}^w$, most notably the famous Herbrand’s theorem. In the next Proposition, we use these facts to characterize the provably total functions of $\mathsf{GA}^w$, which will become handy in our investigation of the provably total functions of $\mathsf{BA}^w$.

**Proposition 3.28.** Let $\Gamma$ be a set of sequents of (geometric) open formulas.

1. If $\mathsf{GA}^w + \Gamma \vdash \exists y \exists z A(x, y, z)$ for some (geometric) open formula $A(x, y, z)$, then there exists a (geometric) open formula $B(x, y)$ and a finite list $s_1(x), \ldots, s_r(x)$ of terms such that $\mathsf{GA}^w + \Gamma \vdash B(x, y) \Rightarrow \exists z A(x, y, z)$ and $\mathsf{GA}^w + \Gamma \vdash \forall x \exists z \exists y \exists z B(x, y, s_i(x))$.

2. If $\mathbb{N} \models \Gamma$, then provably total functions of $\mathsf{GA}^w + \Gamma$ are definable in $\mathsf{GA}^w + \Gamma$ by (geometric) open formulas.

**Proof.**
1. As already discussed in the proof of Lemma 3.24, $GA^w$ can be formalized by the fragment of $LK$ without the rules for implication and universal quantification, together with the axioms of equality and arithmetic (see the Appendix), in all of which only quantifier-free formulas appear. By Herbrand's theorem (Theorem 2.5.1 of [1]), this implies that if $GA^w + \Gamma \vdash \exists y \exists z A(x, y, z)$, then there are finite lists $s_1(x), \ldots, s_r(x)$ of terms and $t_1(x), \ldots, t_s(x)$ of sequences of terms with the same length as $y$, such that $GA^w + \Gamma \vdash \bigvee_{i=1}^r A(x, s_i(x), t_i(x))$. Letting $B(x, y) \equiv \bigvee_{i=1}^r A(x, y, t_i(x))$, it is straightforward to verify that $B(x, y)$ has the desired properties.

2. Let $f : \mathbb{N}^n \to \mathbb{N}$ be defined in $GA^w + \Gamma$ by a (geometric) formula $A(x, y)$. Without loss of generality, $A(x, y)$ can be considered to be $\exists y$. Choosing $B(x, y)$ as in the previous item, we can see that $GA^w + \Gamma$ proves the existence and uniqueness sequents of $B(x, y)$, since it does so for $A(x, y)$. Since $\mathbb{N} \models GA^w + \Gamma$, this means that $B(x, y)$ defines a function $g : \mathbb{N}^n \to \mathbb{N}$. By the provability of the uniqueness sequent of $B(x, y)$ and the fact that $GA^w + \Gamma \vdash B(x, y) \Rightarrow A(x, y)$, it turns out that $g$ is the same function as $f$, and thus $f$ is definable in $GA^w + \Gamma$ by the (geometric) open formula $B(x, y)$.

\[ \square \]

**Proposition 3.29.** Let $\Gamma$ be a set of sequents.

1. If $BA^w + \Gamma \vdash \alpha$ then $GA^w + \Gamma \vdash \alpha$. Consequently, $(BA^w)^3 \vdash GA^w$, and $BA^w$ is conservative over $GA^w$.

2. Assuming that every sequent in $\Gamma$ is geometric, if $BA^w + \Gamma \vdash \alpha$ then $BA^w + \Gamma \vdash \alpha^G$. Moreover, it can be so that only semi-geometric formulas appear in the derivation of $\alpha^G$ from $\Gamma$ in $BA^w$.

**Proof.**

1. Same as the proof of item 2 of Proposition 3.23 without considering the rule of induction.

2. Same as the proof of item 2 of Proposition 3.23 without considering the rule of induction. \[ \square \]

**Proposition 3.30.** Let $\Gamma$ be a set of sequents of geometric open formulas.

1. If $BA^w + \Gamma \vdash \exists y \exists z A(x, y, z)$ for some geometric open formula $A(x, y, z)$, then there exists a geometric open formula $B(x, y)$ and a finite list $s_1(x), \ldots, s_r(x)$ of terms such that $BA^w + \Gamma \vdash B(x, y) \Rightarrow \exists z A(x, y, z)$ and $BA^w + \Gamma \vdash \bigvee_{i=1}^r A(x, s_i(x))$.

2. If $\mathbb{N} \models \Gamma$, then provably total functions of $BA^w + \Gamma$ (in the sense of the original formulation for the uniqueness sequents) are definable in $BA^w + \Gamma$ by geometric open formulas.

**Proof.** Combine Proposition 3.29 with item 1 of Proposition 3.29. \[ \square \]

**Theorem 3.31.** Let $\Gamma$ be a set of geometric sequents such that $\mathbb{N} \models \Gamma$.

1. If $BA^w \vdash \mathcal{U}(A)$ then $GA^w \vdash \mathcal{U}(A^3)$; $PTF(BA^w + \Gamma) = PTF(GA^w + \Gamma) = PTF(GA^w + \Gamma)$.

2. For $i = 0, 1, 2$, if $BA^w + \Gamma \vdash \mathcal{U}_i(A)$ then $BA^w + \Gamma \vdash \mathcal{U}_i(A^3)$; $PTF_i(BA^w + \Gamma) = PTF_i(GA^w + \Gamma) \subseteq PTF_i(GA^w + \Gamma)$.

3. If $BA^w \vdash \mathcal{U}_3(A)$ then $BA^w \vdash \mathcal{U}_3(A^3)$; $PTF_3(BA^w) \subseteq PTF_3(GA^w)$.

**Proof.** The first item can be proven by an argument similar to the proof of Theorem 3.7, only using item 1 of Proposition 3.29 instead of Proposition 3.28. The other items can be proven by an argument similar to the proof of Theorem 3.25, only using item 2 of Proposition 3.29 instead of Proposition 3.28. Just note that this time the induction formulas are only appearing in the induction axioms, which means that they are geometric, as the induction axiom appearing in the derivation is supposed to be semi-geometric. \[ \square \]

Note that we have also included $i = 0$ in the second item of the above Theorem. In fact, the provabilities of $\mathcal{U}_0(A)$ and $\mathcal{U}(A)$ are not equivalent in $BA^w$, unlike $BA$. That is because $BA^w$ is not faithful, which we now set out to prove.

**Lemma 3.32.** Let $\mathcal{T}$ and $\mathcal{T}'$ be theories in a first-order language such that
• T’ extends BQC;

• for any n (including 0), any sequence ∃x(A₀ → B₀), ∃x(A₁ → B₁), . . . , ∃x(Aₙ → Bₙ) of sentences such that
  \( A₁ \rightarrow B₁ \) . . . \( Aₙ \rightarrow Bₙ \) \( A₀ \Rightarrow B₀ \) is a rule of T (or in case n = 0, \( A₀ \Rightarrow B₀ \) is an axiom of T), and any
  formula A no free variable of which appears in x., \( T' \vdash \bigwedge_{i=1}^{n} \exists x(A \land A_i \rightarrow B_i) \Rightarrow \exists x(A \land A₀ \rightarrow B₀) \).

Then for any sequence ∃x(A₀ → B₀), ∃x(A₁ → B₁), . . . , ∃x(Aₙ → Bₙ) of sentences such that \( T + \{ A_i \Rightarrow B_i \}_{i=1}^{n} \vdash A₀ \Rightarrow B₀ \), and any formula A no free variable of which appears in x., \( T' \vdash \bigwedge_{i=1}^{n} \exists x(A \land A_i \rightarrow B_i) \Rightarrow \exists x(A \land A₀ \rightarrow B₀) \).

Proof. The proof is essentially the same as the one showing that a theory with a well-formed axiomatization is well-formed (Proposition 4.13 of [12]). Note that the assumptions on T’ made explicit in the statement of this Lemma are exactly what is needed for the proof to work.

Corollary 3.33. Let A and B be formulas in the language of arithmetic. The following are equivalent:

1. BA ⊢ A ⇒ B;
2. BA⁺ ⊢ ∃x(A → B) for some sequence of variables x containing all the free variables of A and B;
3. BA⁺ ⊢ ∃x(A → B) for all sequences of variables x.

Proof. To prove (2) assuming (1), apply Lemma 3.32 with T = BA and T’ = BA⁺. Note that other than the rule of induction, all the axioms and rules of BA are already axioms and rules of BA⁺, and the presence of the induction axiom in BA⁺ covers all that is needed for applying the Lemma. (3) follows from (2) by using axiom 17, and (1) follows from (3) by the faithfulness of BA (Corollary 2.9) and the fact that BA extends BA⁺.

Corollary 3.34.

1. BA⁺ ⊢ ∃x∃yA(x, y) iff BA ⊢ E(A(x, y)).
2. BA⁺ ⊢ U₀(A) iff BA ⊢ U(A).

Proof. Straightforward by Corollary 3.32.

The fact that BA⁺ is not faithful comes from the fact that the provability of ∃x∃yA(x, y) in the first item of Corollary 3.34 cannot be strengthened to the provability of E(A(x, y)). To see this, we need to consider a new model for BA⁺, as in the following definition.

Definition 3.35.

• \( \mathbb{R}_0^+ \) is the classical structure of the set of non-negative real numbers. The symbols 0, +, - and < of the language of arithmetic are interpreted standardly. The symbol S is interpreted as Sa = a + 1 for every a ∈ \( \mathbb{R}_0^+ \).

• \( K_0^+ \) is the Kripke model with just one irreflexive node with the structure \( \mathbb{R}_0^+ \).

Lemma 3.36.

1. \( \mathbb{R}_0^+ \) is a (classical) model of GA⁺. \( K_0^+ \) is a model of BA⁺.
2. \( \mathbb{R}_0^+ \not\models x = 0 \lor \exists y x = Sy \). \( K_0^+ \not\models x = 0 \lor \exists y x = Sy \).
3. \( GA⁺ \not\models x = 0 \lor \exists y x = Sy \). \( BA⁺ \not\models x = 0 \lor \exists y x = Sy \).

Proof.

1. It is straightforward to verify that \( \mathbb{R}_0^+ \) satisfies the arithmetical axioms of GA⁺, and hence \( \mathbb{R}_0^+ \models GA⁺ \). As \( K_0^+ \) is a single-node irreflexive Kripke model, we have \( K_0^+ \models A \iff A^2 \) for all formulas A, which implies \( K_0^+ \models BA⁺ \).
2. \( \frac{1}{2} \) is neither equal to 0 nor a successor of any element of \( \mathbb{R}_0^+ \).
3. By combining the previous items.
Corollary 3.37.

1. $BA^w$ is not faithful.
2. $BA^w$ is neither reflexively rooted nor irreflexively rooted.

Proof.

1. By Proposition 3.20 item 1 of Corollary 3.34, $BA^w \vdash \forall x \exists y A_{pd}(x, y)$. If $BA^w$ is faithful, we must have $BA^w \vdash E(A_{pd})$, which contradicts item 3 of Lemma 3.36.

2. By the previous item and Proposition 2.8.

Looking at the proof of item 1 of Corollary 3.37, we can see that $PTF_0(BA^w) \subseteq PTRF(BA)$. By item 2 of Corollary 3.34, this is due to unprovability of some existence sequent $s$, and not that of uniqueness sequents. That shows the important role of the rule of induction of $BA$ in the provability of totality of functions.

As a closing remark, we note that while $BA^w$ lacks rootedness, which was the main tool for proving disjunction and existence properties for theories like $BA$ and $HA$ (see Proposition 2.8), it still satisfies some versions of these properties. The question of $BA^w$ having disjunction and existence properties is still open to us.

Proposition 3.38.

1. For all sentences $A$ and $B$, $BA^w \vdash T \rightarrow A \lor B$ implies $BA^w \vdash T \rightarrow A$ or $BA^w \vdash T \rightarrow B$.
2. For all sentences $\exists x A$, $BA^w \vdash T \rightarrow \exists x A$ implies $BA^w \vdash T \rightarrow A[x/t]$ for some closed term $t$.

Proof.

1. If $BA^w \vdash T \rightarrow A \lor B$, then by Corollary 3.34, $BA \vdash A \lor B$. By the disjunction property of $BA$, we have either $BA \vdash A$ or $BA \vdash B$, which using Corollary 3.34 again, implies $BA^w \vdash T \rightarrow A$ or $BA^w \vdash T \rightarrow B$.

2. If $BA^w \vdash T \rightarrow \exists x A$, then by Corollary 3.34, $BA \vdash \exists x A$. By the existence property of $BA$, we have $BA \vdash A[x/t]$ for some closed term $t$, which using Corollary 3.34 again, implies $BA^w \vdash T \rightarrow A[x/t]$.

3.2 The Provably Total Recursive Functions of Extensions of $BA$

In the last section, we showed that the provably total recursive functions of $BA$ are primitive recursive, however there are some primitive recursive functions that are not provably total in $BA$. In this section, we consider three extensions of $BA$ that their provably total recursive functions are exactly the primitive recursive functions. The first extension is by adding the cancellation law (the axiom $U: x + y = x + z \Rightarrow y = z$) to $BA$, the second one is obtained by adding a symbol for the cut-off subtraction to the language of $BA$, and the third one is the theory $EBA$, introduced in [1]. As is known, the cut-off subtraction is a primitive recursive function, and it turns out that adding a symbol for it to the language of $BA$, and its properties as additional axioms, will result then that the provably total recursive functions of this extension captures all the primitive recursive functions. We show that the first two extensions of $BA$ coincide in some sense, see Theorem 3.55. The theory $EBA$ that is an extension of $BA$ by adding the sequent axiom $T \rightarrow \bot \Rightarrow \bot$, is a stronger theory than the previous two extensions of $BA$, and is very close to $HA$ in some respects, however still weaker than that. For more details on motivations and some properties of $EBA$, that we will use in this paper, see [1].

We start with adding the axiom $U$ to our theories. The name “$U$” comes from the equivalence of the cancellation law of addition and the uniqueness sequent of the cut-off subtraction, $U(A_c)$. We take a look at this relation in the next Lemma. We say that two sequents $\alpha$ and $\beta$ are equivalent over a theory $T$, whenever $T + \alpha \vdash \beta$ and $T + \beta \vdash \alpha$. It turns out that $U$ is equivalent to the only axiom of $PA^-$ that is not provable in $BA$.

Lemma 3.39. The following sequents are equivalent over $BA$.

1. $A_c(x, y, z) \land A_c(x, y, w) \Rightarrow z = w$
Lemma 3.41. A

It is routine to check the other cases for

BA

\( \exists z \) (N

Remark 3.42. Definition 3.40. For a quantifier-free formula A, the geometric equivalent and the geometric negation of A are respectively denoted by \( A^+ \) and \( A^- \), and are defined recursively as follows:

\begin{itemize}
  \item \( T^+ \equiv T \) and \( T^- \equiv \perp \);
  \item \( \perp^+ \equiv \perp \) and \( \perp^- \equiv T \);
  \item \( (s = t)^+ \equiv s = t \) and \( (s = t)^- \equiv s \neq t \lor t \neq s \), if s and t are terms;
  \item \( (s < t)^+ \equiv s < t \) and \( (s < t)^- \equiv s = t \lor t = s \), if s and t are terms;
  \item \( A^+ \equiv B^+ \land C^+ \) and \( A^- \equiv B^- \lor C^- \), if A is of the form \( B \land C \);
  \item \( A^+ \equiv B^+ \lor C^+ \) and \( A^- \equiv B^- \land C^- \), if A is of the form \( B \lor C \);
  \item \( A^+ \equiv B^- \land C^- \) and \( A^- \equiv B^+ \lor C^+ \), if A is of the form \( B \rightarrow C \).
\end{itemize}

Lemma 3.41. For every quantifier free formula A, \( A^+ \) and \( A^- \) are geometric open formulas such that \( \exists_1^+ + U \vdash A \iff A^+ \) and \( \exists_1^+ + U \vdash \neg A \iff A^- \).

Proof. The first part of the lemma is obvious. For the second part, we first note that \( \exists_1^+ \vdash x < y \lor x = y \lor y < x \). Now we prove the lemma by induction on complexity of A.

\begin{itemize}
  \item If A is of the form \( s = t \), for some terms s and t, then \( A^- \) is \( s < t \lor t < s \). Hence \( \exists_1^+ + U \vdash \neg(s = t) \Rightarrow s < t \lor t < s \).
    \begin{itemize}
      \item On the other hand, \( \exists_1^+ + U \vdash x < x \Rightarrow \perp \), so \( \exists_1^+ + U \vdash s < t \lor t < s \Rightarrow \neg(s = t) \).
    \end{itemize}
  \item If A is of the form \( s < t \), for some terms s and t, then \( A^- \) is \( s = t \lor t = s \). Hence \( \exists_1^+ + U \vdash \neg(s < t) \Rightarrow t = s \).
    \begin{itemize}
      \item On the other hand, since \( \exists_1^+ + U \vdash x < x \Rightarrow \perp \) and \( \exists_1^+ + U \vdash x < y \land y < x \Rightarrow \perp \), we get \( \exists_1^+ + U \vdash s = t \lor t = s \Rightarrow \neg(s < t) \).
    \end{itemize}
  \item If A is of the form \( B \rightarrow C \), then \( \exists_1^+ + U \vdash B \rightarrow C \iff \neg B \lor C \) and \( \exists_1^+ + U \vdash \neg(B \rightarrow C) \iff B \land \neg C \). By the induction hypothesis, \( \exists_1^+ + U \vdash B \rightarrow C \iff B^- \lor C^- \) and \( \exists_1^+ + U \vdash \neg(B \rightarrow C) \iff B^+ \land C^+ \).
\end{itemize}

It is routine to check the other cases for A.

Remark 3.42. Note that \( \exists_1^+ \not\models \neg(x = y) \iff (x = y)^- \) and \( \exists_1^+ \not\models \neg(x = y) \iff (\neg(x = y))^+ \), for the following reason. By Lemma 3.12, \( \mathbb{N}^* \) is a model of \( \exists_1^+ \) such that \( \mathbb{N}^* \models (\infty = \infty)^- \) and \( \mathbb{N}^* \models (\neg(\infty = \infty))^+ \). So the axiom U is necessary for the proof of Lemma 3.41.
Lemma 3.43. For every $\exists_1$ formula $A$, there exists a $\exists_1^+$ formula $B$ with the same free variables as $A$, such that $\exists_1^+ + U \vdash A \iff B$.

Proof. Since $A$ is $\exists_1$, we have $\exists_1^+ + U \vdash A \iff \exists y C$ for some quantifier free formula $C$. If we define $B \equiv \exists y C^+$, then by Lemma 3.44 $\exists_1^+ + U \vdash A \iff B$. \hfill $\square$

We now turn to the well-known MRDP theorem. This theorem is about the existence of Diophantine equivalents for $\Sigma_1$ formulas in a strong enough arithmetical theory. We will study our theories of interest in this relation, and show which are strong enough in this sense and which are not. Our main results on this matter are postponed until the next Section, but we need to consider some instances here, for the purpose of characterizing the provably total recursive functions of our theories. A well-known classical instance of the MRDP theorem appears in the proof of the following Theorem, which is an instance of the MRDP theorem itself.

Theorem 3.44. For every $\Sigma_1$ formula $A$, there exists a $\exists_1^+$ formula $B$ with the same free variables as $A$, such that $\exists_1^+ + U \vdash A \iff B$.

Proof. Given a $\Sigma_1$ formula $A$, by Corollary 4.11 in [9], there exists a $\exists_1$ formula $A'$ with the same free variables as $A$, such that $\exists_1 \vdash A \iff A'$. By Lemma 3.43 for every $\exists_1$ formula $C$ there exists an $\exists_1^+$ formula $C'$ such that $\exists_1^+ + U \vdash C \iff C'$. Also $\exists_1^+ \vdash \forall x y (C' \to C'(x/Sx)) \Rightarrow \forall x y (C' \to C')$, so $\exists_1^+ + U \vdash \forall x y (C \to C(\exists y C')) \Rightarrow \forall x y (C(\exists y C) \to C)$. Thus $\exists_1^+ + U \vdash \exists_1$, and hence $\exists_1^+ + U \vdash A \iff A'$. By Lemma 3.43 there exists an $\exists_1^+$ formula $B$ with the same free variables as $A'$ such that $\exists_1^+ + U \vdash A' \iff B$. Hence $\exists_1^+ + U \vdash A \iff B$. \hfill $\square$

Corollary 3.45. $\exists_1^+ + U \vdash \exists_1$

Proof. By Theorem 3.44 $\exists_1^+ + U \vdash \exists_1$. Also $\exists_1 \vdash U$ and $\exists_1 \subseteq \Sigma_1$, so $\exists_1 \vdash \exists_1^+ + U$. \hfill $\square$

Corollary 3.46. The provably total recursive functions of $\exists_1^+ + U$ are exactly the primitive recursive functions, i.e. $\text{PTF}(\exists_1^+ + U) = \mathcal{PR}$. Furthermore these functions are definable in $\exists_1^+ + U$ by $\exists_1^+$ formulas.

Proof. Combine Corollary 3.44 and Theorems 3.8 and 3.44. \hfill $\square$

Theorem 3.47. The provably total functions of $\mathcal{B} + U$ are exactly the primitive recursive functions, and they are definable in $\mathcal{B} + U$ by $\exists_1^+$ formulas. Consequently, $\text{PTF}(\mathcal{B} + U) = \mathcal{PR}$.

Proof. If a function $f : \mathbb{N}^n \to \mathbb{N}$ is provably total in $\mathcal{B} + U$, then by Theorem 3.7 there exists a $\exists_1^+$ formula $A(x, y)$ such that:

- $\mathcal{B} + U \vdash E(A)$,
- $\mathcal{B} + U \vdash U(A)$,
- $\mathbb{N} \models A(a, f(a))$, for every $a \in \mathbb{N}^n$.

Then by Corollary 3.5

- $\exists_1^+ + U \vdash E(A)$,
- $\exists_1^+ + U \vdash U(A)$.

So $f$ is provably total recursive in $\exists_1^+ + U$ and by Corollary 3.46 $f$ is primitive recursive. Therefore $\text{PTF}(\mathcal{B} + U) \subseteq \mathcal{PR}$. Now suppose $g : \mathbb{N}^m \to \mathbb{N}$ is a primitive recursive function. Then by Corollary 3.46 there exists a $\exists_1^+$ formula $B(x, y)$ such that:

- $\exists_1^+ + U \vdash E(B)$,
- $\exists_1^+ + U \vdash U(B)$,
- $\mathbb{N} \models B(b, g(b))$, for every $b \in \mathbb{N}^m$.

Thus by Corollary 3.5

- $\mathcal{B} + U \vdash E(B)$,
• \( BA + U \vdash U(B) \),

and hence \( \mathcal{PR} \subseteq \mathsf{PTRF}(BA + U) \).

In the next theorem, we show that the provably total recursive functions of \( \Sigma_1^+ \) are also exactly the primitive recursive functions. Our method of characterizing the provably total functions of \( \Sigma_1^+ + U \) did rely on the axiom \( U \), and the result was that the provably total recursive functions of \( \Sigma_1^+ + U \) are definable in \( \Sigma_1^+ + U \) by geometric formulas. This fails to hold in \( \Sigma_1^+ \), and the MRDP theorem also fails. To characterize the provably total recursive functions of \( \Sigma_1^+ \), we consider \( \Sigma_1 \) defining formulas that are not geometric. The idea of the proof is essentially due to \( \mathcal{PR} \), and goes by the following lines. Given an \( \exists \Sigma_1 \) formula \( \exists \Sigma \) \( A \) defining a primitive recursive function in \( \Sigma_1^+ + U \), we first modify \( A \) by adding to it bounded versions of its uniqueness sequent and the axiom \( U \), to get \( B \). Then we take the new defining formula for the function to be \( \exists \Sigma (B \lor C) \), where \( C \) roughly states that both \( U \) and \( B \) are false. While \( \Sigma_1^+ \) is not strong enough to refute \( C \), it can prove that \( B \) and \( C \) cannot happen together, which paves the way for a proof of the uniqueness sequent. The proof of the existence sequent relies on derivability of the principle of excluded middle in \( \Sigma_1^+ \). The fact that \( \exists \Sigma (B \lor C) \) defines the same function as before comes from \( B \) being true and \( C \) being false in the standard model \( \mathbb{N} \).

**Theorem 3.48.** The provably total recursive functions of \( \Sigma_1^+ \) are exactly the primitive recursive functions, i.e. \( \mathsf{PTRF}(\Sigma_1^+) = \mathcal{PR} \).

**Proof.** Let \( f : \mathbb{N}^n \to \mathbb{N} \) be a primitive recursive function. By 3.46 there is a geometric open formula \( A(x, y, z) \) such that \( f \) is definable in \( \Sigma_1 + U \) by the formula \( \exists z A(x, y, z) \), where \( z = (z_1, \ldots, z_m) \). Without loss of generality, we can assume that \( z \) is non-empty, because we can consider \( A(x, y, z) \lor z = z \) instead of \( A(x, y, z) \). Suppose \( x, y', y'', z_1', \ldots, z_m', z_1'', \ldots, z_m'' \), \( u, v, w \) are pairwise distinct fresh variables, \( z' = (z_1', \ldots, z_m') \), \( z'' = (z_1'', \ldots, z_m'') \), \( t = y + z_1 + \cdots + z_m \), \( t' = y + z_1' + \cdots + z_m' \), \( t'' = y' + z_1'' + \cdots + z_m'' \), and let:

\[
U(x) \equiv \forall u \forall v (u + v + w \leq x \land u + w = v + w \rightarrow u = v);
\]

\[
B(x, y, z) \equiv U(t) \land A(x, y, z) \land \forall y' z'(t' \leq t \land A(x, y', z') \rightarrow y = y');
\]

\[
C(x, y, z) \equiv \neg U(t) \land y = 0 \land \forall y' z'(t' \leq t \rightarrow \neg B(x, y', z'));
\]

\[
D(x, y) \equiv \exists z (B(x, y, z) \lor C(x, y, z)).
\]

It’s easy to see that \( U(x), B(x, y, z) \) and \( C(x, y, z) \) are provably equivalent in \( \Sigma_1^+ \) to \( \Delta_0 \) formulas, and thus \( D(x, y) \) is provably equivalent in \( \Sigma_1^+ \) to a \( \Sigma_1 \) formula.

Since \( \Sigma_1^+ + U \vdash U(\exists z A(x, y, z)) \), we have \( \Sigma_1^+ + U \vdash A(x, y, z) \Rightarrow \forall y' z'(A(x, y', z') \rightarrow y = y') \), and because \( \Sigma_1^+ + U \vdash E(\exists z A(x, y, z)) \), we get \( \Sigma_1^+ \vdash \forall u \forall v (u + v + w = v + w \rightarrow u = v) \Rightarrow \exists y \exists z B(x, y, z) \). Hence \( \Sigma_1^+ \vdash \forall y \forall z B(x, y, z) \Rightarrow \exists x \forall u (U(x)) \). Now we note that

\[
\Sigma_1^+ \vdash \neg U(x) \land y = 0 \land z_1 = 0 \land \cdots \land z_{m-1} = 0 \land z_m = x \rightarrow \neg U(t) \land y = 0,
\]

which together with the previous result, yields \( \Sigma_1^+ \vdash \forall y \exists z B(x, y, z) \Rightarrow \exists y \exists z C(x, y, z) \). As \( \Sigma_1^+ \vdash \forall y \forall z \neg B(x, y, z) \lor \exists y \exists z B(x, y, z) \), we get \( \Sigma_1^+ \vdash E(D) \).

Now, we show that \( \Sigma_1^+ \vdash \mathit{U}(D) \). For this purpose, we argue as follows.

1. We first show \( \Sigma_1^+ \vdash B(x, y, z) \land C(x, y', z'') \Rightarrow \bot \). This can be proven using \( \Sigma_1^+ \vdash t \leq t'' \lor t'' \leq t, \Sigma_1^+ \vdash t \leq t'' \land C(x, y', z'') \Rightarrow \neg B(x, y, z) \) (by the definition of \( C(x, y', z'') \)) and \( \Sigma_1^+ \vdash t'' \leq t \land B(x, y, z) \Rightarrow U(t'') \) (by definitions of \( B(x, y, z) \) and \( U(t'') \)).

2. Then, we show that \( \Sigma_1^+ \vdash B(x, y, z) \lor B(x, y', z'') \Rightarrow y = y'' \). This is true because \( \Sigma_1^+ \vdash t \leq t'' \lor t'' \leq t, \Sigma_1^+ \vdash t \leq t'' \land B(x, y', z'') \land A(x, y, z) \Rightarrow y = y'' \) (by definition of \( B(x, y', z'') \)) and \( \Sigma_1^+ \vdash t'' \leq t \land B(x, y, z) \land A(x, y', z'') \Rightarrow y = y'' \) (for a similar reason).

3. At last, it is easy to see that \( \Sigma_1^+ \vdash C(x, y, z) \land C(x, y', z'') \Rightarrow y = y'', \Sigma_1^+ \vdash C(x, y, z) \Rightarrow y = 0 \) and \( \Sigma_1^+ \vdash C(x, y', z'') \Rightarrow y'' = 0 \).

Combining these three facts, we get what was claimed.

Finally, as \( \Sigma_1^+ + U \vdash A(x, y, z) \iff B(x, y, z) \) and \( \Sigma_1^+ + U \vdash C(x, y, z) \Rightarrow \bot \), we get \( \Sigma_1^+ + U \vdash \exists z A(x, y, z) \iff D(x, y) \). As \( \mathbb{N} \models \Sigma_1^+ + U \), we see that \( D(x, y) \) defines \( f \). Thus \( f \) is definable in \( \Sigma_1^+ \) by a \( \Sigma_1 \) formula. Consequently \( \mathcal{PR} \subseteq \mathsf{PTRF}(\Sigma_1^+) \).

For the other way around, i.e. \( \mathsf{PTRF}(\Sigma_1^+) \subseteq \mathcal{PR} \), it is enough to note that \( \Sigma_1 \vdash \Sigma_1^+ \) and use Theorem 3.8.
Corollary 3.49. The MRDP theorem does not hold in $\Sigma_1^+$, i.e. there is a $\Sigma_1$ formula $A$ with no geometric formula $B$ such that $\Sigma_1^+ \vdash A \leftrightarrow B$.

Proof. The cut-off subtraction is primitive recursive, and thus by Theorem 3.48 it is provably total in $\Sigma_1^+$. Let $A$ be a $\Sigma_1$ formula that defines the cut-off subtraction in $\Sigma_1^+$. If there is a geometric formula $B$ such that $\Sigma_1^+ \vdash A \leftrightarrow B$, then $\Sigma_1^+$ proves existence and uniqueness sequents of $B$. By Corollary 3.49 BA proves these sequents as well. This contradicts Corollary 3.18.

Now, we consider another extension of BA. In this new extension $BA_c$, we augment the language $L$ with a new symbol "∸" for the cut-off subtraction, to get the language $L_c$, and extend the theory with the axioms $x \leq y \Rightarrow x \div y = 0$ and $y \leq x \Rightarrow Sx \div y = S(x \div y)$. It turns out that this extension of BA coincides with $BA + U$ over $L$.

Note that by Corollary 3.49 the cut-off subtraction is not provably total in BA, hence $BA_c$ might be stronger than $BA$ (over the language $L$). The next Theorem states that it is indeed the case, by showing that $BA_c$ and $BA + U$ prove the same $L$-sequents. We need the following definitions and lemmas concerning elimination of the cut-off subtraction function symbol to prove the Theorem.

Definition 3.50.

- For an $L_c$-term $t$ and a variable $x$ not occurring in $t$, $t^{=x}$ is defined recursively as follows:
  - $t^{=x} \equiv t = x$, if $t$ is a variable or 0;
  - $t^{=x} \equiv \exists y(r^{=y} \land sy = x)$ for a fresh variable $y$, if $t$ is of the form $Sr$;
  - $t^{=x} \equiv \exists yz(r^{=y} \land s^{=z} \land y + z = x)$ for fresh variables $y$ and $z$, if $t$ is of the form $r + s$;
  - $t^{=x} \equiv \exists yz(r^{=y} \land s^{=z} \land y \cdot z = x)$ for fresh variables $y$ and $z$, if $t$ is of the form $r \cdot s$;
  - $t^{=x} \equiv \exists yz(r^{=y} \land s^{=z} \land A_c(y, z, x))$ for fresh variables $y$ and $z$, if $t$ is of the form $r \div s$.

- For an $L_c$-formula $A$, $A^*$ is defined recursively as follows:
  - $A^* \equiv \exists xy(s^{=x} \land t^{=y} \land x = y)$ for fresh variables $x$ and $y$, if $A$ is of the form $s = t$;
  - $A^* \equiv \exists xy(s^{=x} \land t^{=y} \land x < y)$ for fresh variables $x$ and $y$, if $A$ is of the form $s < t$;
  - $A^* \equiv A$, if $A$ is $\top$ or $\bot$;
  - $A^* \equiv B^* \circ C^*$, if $A$ is of the form $B \circ C$ and $\circ$ is $\lor$ or $\land$;
  - $A^* \equiv \exists u B^*$, if $A$ is of the form $\exists u B$;
  - $A^* \equiv \forall x (B^* \rightarrow C^*)$, if $A$ is of the form $\forall x (B \rightarrow C)$.

$(A \Rightarrow B)^*$ is defined as $A^* \Rightarrow B^*$. For a set $\Gamma$ of $L_c$-sequents, $\Gamma^*$ is defined as $\{ \alpha^* | \alpha \in \Gamma \}$. For a rule $R$, $R^*$ is defined as the rule obtained by replacing any upper or lower sequent $\alpha$ of $R$ with $\alpha^*$.

Lemma 3.51.

1. For every $L_c$-term $t$ and every variable $x$ not occurring in $t$, $t^{=x}$ is an $L$-formula. Moreover, in case $t$ is an $L$-term, $BQC \vdash t^{=x} \leftrightarrow t = x$.

2. For every $L_c$-formula $A$, $A^*$ is an $L$-formula. Moreover, in case $A$ is an $L$-formula, $BQC \vdash A^* \leftrightarrow A$.

Proof.

1. Straightforward by induction on $t$.

2. Straightforward by induction on $A$. □

Lemma 3.52.

1. $BA_c \vdash U$.

2. $BA_c \vdash A_c(x, y, x \div y)$.  

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Proof.

1. We show that $BA_c \vdash (x + y) \div x = y$ using the rule of induction on the formula $(x + y) \div x = y$ with $y$ as the eigenvariable. It is easy to see that $BA_c \vdash (x + 0) \div x = 0$. Note that $BA \vdash x \leq x + y$, hence $BA_c \vdash (S(x + y)) \div x = S((x + y) \div y)$, which implies $BA_c \vdash (x + y) \div x = y \Rightarrow (x + Sy) \div x = Sy$. By the rule of induction, $BA_c \vdash (x + 0) \div x = 0 \Rightarrow (x + y) \div x = y$, which yields what was claimed. As $BA_c \vdash x + y = x + z \Rightarrow (x + y) \div x = (x + z) \div x$, the proof is complete.

2. Since $BA_c \vdash x < y \Rightarrow x + y = 0$, we have $BA_c \vdash x < y \Rightarrow A_c(x, y, x = y)$. As $BA_c \vdash x < y \land y < x$, it is now sufficient to prove $BA_c \vdash y \leq x \Rightarrow x = y$, which implies $BA_c \vdash x < y \Rightarrow A_c(x, y, x = y)$. To show this, note that $BA_c \vdash y < x \Rightarrow \exists z (z = y + z)$, and thus proving $BA_c \vdash x = y + z \Rightarrow z = x - y$ does the job. This last statement follows from the proof of the previous item: $BA_c \vdash (y + z) \div y = z$.

Lemma 3.53.

1. For every $L_c$-term $t$ and any variable $x$ not occurring in it, $BA_c \vdash t^{=x} \iff t = x$.

2. For every $L_c$-formula $A$, $BA_c \vdash A^* \iff A$

Proof.

1. The proof is by induction on the term $t$. The only nontrivial case is when $t$ is of the form $r \div s$. By the induction hypothesis, we have $BA_c \vdash t^{=x} \iff \exists y (r = y \land s = z \land A_c(y, z, x))$. On the one hand, by item 2 of Lemma 3.52, we have $BA_c \vdash A_c(r, s, t)$, and therefore $BA_c \vdash t = x \Rightarrow \exists y (r = s \land s = z \land A_c(r, s, t))$. On the other hand, by Lemma 3.39 and item 1 of Lemma 3.52, $BA_c \vdash U(A_c)$, and therefore $BA_c \vdash A_c(r, s, t) \Rightarrow t = x$. This yields $BA_c \vdash \exists y (r = s \land s = z \land A_c(y, z, x)) \Rightarrow t = x$, which together with the previous results completes the proof.

2. The proof is by induction on the formula $A$. The basis is covered by the previous item, and the induction steps are straightforward.

Lemma 3.54.

1. Let $t$ be an $L_c$-term, and $x$ be a variable not occurring in $t$. Then $BA \vdash \exists x(t^{=x})$.

2. Let $t$ be an $L_c$-term, and $x$ and $y$ be variables not occurring in $t$. Then $BA + U \vdash t^{=x} \land t^{=y} \Rightarrow x = y$.

3. Let $s$ and $t$ be $L_c$-terms, $z$ be a variable occurring neither in $s$ nor in $s[y/t]$, and $x$ be a variable not occurring in $t$ and substitutable for $y$ in $s^{=z}$. Then $BA + U \vdash (s[y/t])^{=z} \iff \exists x(t^{=x} \land (s^{=z})[y/x])$.

4. Let $A$ be an $L_c$-formula, $t$ be an $L_c$-term substitutable for $y$ in $A$, and $x$ be a variable substitutable for $y$ in $A^*$. Then $BA + U \vdash (A[y/t])^* \iff \exists x(t^{=x} \land A^*[y/x])$.

Proof.

1. Straightforward by induction on $t$, using Proposition 3.27 in the case where $t$ is of the form $r \div s$.

2. The proof is by induction on $t$. The only nontrivial case is when $t$ is of the form $r \div s$. By induction hypothesis, we have $BA + U \vdash r^{=z} \land s^{=w} \land A_c(z, w, x) \land r^{=u} \land s^{=v} \land A_c(u, v, y) \Rightarrow A_c(u, v, x) \land A_c(u, v, y)$, for suitable variables $z$, $w$, $u$ and $v$. Since $BA + U \vdash U(A_c)$, we get $BA + U \vdash r^{=z} \land s^{=w} \land A_c(z, w, x) \land r^{=u} \land s^{=v} \land A_c(u, v, y) \Rightarrow x = y$, which in turn proves $BA + U \vdash t^{=x} \land t^{=y} \Rightarrow x = y$.

3. The proof is by induction on $s$. The only nontrivial case is when $s$ is of the form $s_1 \div s_2$. $s^{=z}$ is of the form $\exists u_1 u_2 (s_1^{=u_1} \land s_2^{=u_2} \land A_c(u_1, u_2, z))$ for some fresh variables $u_1$ and $u_2$, and $(s[x/t])^{=z}$ is of the form $\exists v_1 v_2 ((s_1[x/t])^{=v_1} \land (s_2[x/t])^{=v_2} \land A_c(v_1, v_2, z))$ for some fresh variables $v_1$ and $v_2$, which respectively can
be assumed to be the same as $u_1$ and $u_2$, without loss of generality. By induction hypothesis, we have

$$BA + U \vdash (s_i[y/t])^{=z} \iff \exists x(t^{=x} \land (s_1^{=u_1})[y/x])$$

for $i = 1, 2$. Therefore

$$BA + U \vdash (s[x/t])^{=z} \iff \exists u_1, u_2 \exists x(t^{=x} \land (s_1^{=u_1})[y/x]) \land A_c(u_1, u_2, z),$$

which by the previous item gives

$$BA + U \vdash (s[x/t])^{=z} \iff \exists u_1, u_2 \exists x(t^{=x} \land (s_1^{=u_1})[y/x]) \land (s_2^{=u_2})[y/x] \land A_c(u_1, u_2, z)).$$

Using logical rules, this statement is equivalent to

$$BA + U \vdash (s[x/t])^{=z} \iff \exists x(t^{=x} \land (\exists u_1, u_2)((s_1^{=u_1})[y/x] \land (s_2^{=u_2})[y/x] \land A_c(u_1, u_2, z))).$$

which completes the proof.

4. The proof is by induction on $A$. The basis is covered by the previous item. The induction steps are straightforward using item 2. In the case where $A$ is of the form $\forall z (B \rightarrow C)$, also use item 1.

Theorem 3.55. For every $\mathcal{L}_c$-sequent $\alpha$ and every set $\Gamma$ of $\mathcal{L}_c$-sentences, $BA_c + \Gamma \vdash \alpha$ iff $BA + U + \Gamma^* \vdash \alpha^*$.

Proof. Since $BA_c$ is an extension of $BA$ and $BA_c \vdash \Gamma$, if $BA + U + \Gamma^* \vdash \alpha^*$ then $BA_c + \Gamma^* \vdash \alpha^*$, which by item 2 of Lemma 3.53 implies $BA_c + \Gamma \vdash \alpha$. For the other direction, note that for any $\beta$ in $\Gamma$, $\beta^*$ is in $\Gamma^*$. Therefore, if we show $BA + U \vdash \beta^*$ for any axiom $\beta$ of $BA_c$ and $BA + U \vdash \beta^*$ for any rule $R$ of $BA_c$, the intended statement follows by induction on the derivations of $\alpha$ from $\Gamma$ in $BA_c$. Let $\beta$ be one of the arithmetical axioms 20-25 or the equality axiom 6. Then, by item 2 of Lemma 3.51 we have $BA + U \vdash \beta^*$. For the equality axiom 7, Lemma 3.53 implies $BA + U \vdash (A[x/y])^* \iff \exists z(y = z \land A^*[x/z])$, and consequently $BA + U \vdash (A[x/y])^* \iff A^*[x/y]$. As we also have $BA + U \vdash (x = y)^* \iff x = y$, we get $BA + U \vdash (x = y \land A \Rightarrow A[x/y])^*$. For the logical axiom 16, consider $x = (x_1, \ldots, x_n)$ and $t = (t_1, \ldots, t_n)$. Without loss of generality, assume that $x_i$ does not occur in $t_i$ for any $i$, and let $C \equiv \bigwedge_{i=1}^n t_i^{=x_i}$. Note that $BA + U \vdash \forall x(A \rightarrow B^*) \Rightarrow \forall x(A \land C \land A^* \Rightarrow B^*)$, which by item 3 of Lemma 3.53 shows $BA + U \vdash \forall x(A \rightarrow B) \Rightarrow \forall x(A[x/t] \rightarrow B[x/t])^*$. A similar argument proves $BA + U \vdash \Gamma^*$, where $R$ is the logical rule 11. If $\beta$ is any of the other logical axioms or the induction axiom, then $\beta^*$ is an instance of the same axiom, and if $R$ is any of the other logical rules or the rule of induction, then $R^*$ is an instance of the same rule. Therefore, it only remains to prove $BA + U \vdash (x \leq y \Rightarrow x + y = 0)^*$ and $BA + U \vdash (y \leq x \Rightarrow Sx + y = S(x + y))^*$. Note that $BQA \vdash (x + y = 0)^* \iff x < y \lor x = y + 0$ and $BQA \vdash (Sx + y = S(x + y))^* \iff \exists z(A_c(Sx, y, Sz) \land A_c(x, y, z))$, by directly checking Definition 3.5. Thus, by Lemma 3.51 it is sufficient to prove $BA + U \vdash x \leq y \Rightarrow x < y \lor x = y + 0$ and $BA + U \vdash y \leq x \Rightarrow z(A_c(Sx, y, Sz) \land A_c(x, y, z))$. The first one is trivial. For the second one, note that $BA + U \vdash y \leq x \Rightarrow \exists z(y + z = x)$ and $BA + U \vdash y + z = x \Rightarrow y + Sz = Sz$.

Corollary 3.56. The provably total functions of $BA_c$ are exactly the primitive recursive functions, and they are definable in $BA_c$ by $\exists^* z_i$ formulas in the language $\mathcal{L}$. Consequently $PTRF(BA_c) = PTF(BA_c) = \mathcal{P}R$.

Proof. Assume that a function $f : \mathbb{N}^n \rightarrow \mathbb{N}$ is definable in $BA_c$ by an $\mathcal{L}_c$-formula $A$. By item 2 of Lemma 3.53, $f$ is also definable in $BA_c$ by the $\mathcal{L}$-formula $A^*$. By Theorem 3.53, $f$ is definable in $BA + U$ by the same formula, and by Theorem 3.7, it is definable in $BA + U$ by the geometric formula $(A^*)^3$. Since $BA_c$ is an extension of $BA + U$, the result follows by Theorem 3.47.

Now we consider a third extension of $BA$, i.e., $EBA = BA + \top \rightarrow \bot \Rightarrow \bot$, introduced in [11]. To find out the provably total recursive functions of $EBA$, we use a result form [11], in which the author introduced the notion of primitive recursive realizability for the language of $BA$, and showed that its provably total functions are primitive recursive. The following definition is from [13].

Definition 3.57. (Primitive Recursive Realizability) Let $\varphi_n$ be the unary partial recursive function with Gödel code $n$, $\pi_1$ and $\pi_2$ be the primitive recursive projections of a fixed primitive recursive pairing function $\langle\cdot,\cdot\rangle$, and $PR(x)$ be the formula expressing that “in the program $x$ there is no use of minimization”. For a sequence $x = (x_1, \ldots, x_m)$, $\varphi_n(x)$ is understood as $\varphi_n((x_1, (x_2, \ldots, (x_{m-1}, x_m)))$. For a formula $A$, $x \mathbb{Q}^{PR} A$ is defined by induction on complexity of $A$.

---

\footnote{The program of a partial recursive function shows how it is defined in terms of zero, successor and projection functions by repeatedly applying composition, primitive recursion and minimization. Note that by choosing a suitable coding, we can assume that every natural number is the code of a program.}

\footnote{It is true that $\mathbb{N} \models PR(n)$ implies $\varphi_n \in \mathcal{P}R$, but not vice versa. However, for every primitive recursive function $f$ there is a natural number $n$ such that $\varphi_n = f$ and $\mathbb{N} \models PR(n)$.}
Theorem 3.58. $A$ is the sequence of all free variables in $B$ of Theorem 4.4 in [14], which shows the soundness of

Proof. Suppose $A \Rightarrow B$, for a sequent $A \Rightarrow B$.

Proposition 3.61. $(EBA$ functions are defined in $PA$.

Proof. By Corollary 3.59, the provably total functions of $EBA$ are primitive recursive, i.e. $PTF(EBA) \subseteq PR$. On the other hand, $PA \vdash U$, we have $EBA \vdash U$ by Proposition 3.61. So by Theorem 3.47 all primitive recursive functions are defined in $EBA$ by $\exists^+_1$ formulas, and thus $PR \subseteq PTF(EBA)$.

Definition 3.60. For a formula $A$, the Gödel negative translation of $A$ is denoted by $A^\neg$ and defined recursively as follows:

- $A^\neg = A$, if $A$ is prime;
- $A^\neg \equiv B^\neg \land C^\neg$, if $A$ is of the form $B \land C$;
- $A^\neg \equiv \neg(\neg B^\neg \land \neg C^\neg)$, if $A$ is of the form $B \lor C$;
- $A^\neg \equiv \forall u \neg B^\neg$, if $A$ is of the form $\exists u B$;
- $A^\neg \equiv \forall x (B^\neg \rightarrow C^\neg)$, if $A$ is of the form $\forall x (B \rightarrow C)$.

$(A \rightarrow B)^\neg$ is defined as $A^\neg \Rightarrow B^\neg$.

Proposition 3.61. If $PA \vdash a$, then $EBA \vdash a^\neg$.

Proof. Theorem 3.14 of [1].

Theorem 3.62. The provably total recursive functions of $EBA$ are exactly the provably total functions of $EBA$, and coincide with the primitive recursive functions, i.e. $PTF(EBA) = PTF(EBA) = PR$. Furthermore these functions are definable in $EBA$ by $\exists^+_1$ formulas.

Proof. By Corollary 3.59 the provably total functions of $EBA$ are primitive recursive, i.e. $PTF(EBA) \subseteq PR$. On the other hand, Since $PA \vdash U$, we have $EBA \vdash U$ by Proposition 3.61. So by Theorem 3.47 all primitive recursive functions are defined in $EBA$ by $\exists^+_1$ formulas, and thus $PR \subseteq PTF(EBA)$.

Note the slight difference between the current definition and the usual negative translation, in which $A^\neg = \neg \neg A$ for atomic $A$. 

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4 The MRDP theorem in BA and some of its extensions

In this section, we consider the well-known MRDP theorem in BA and its extensions defined in the last section. We show that in BA and in two of its extensions, i.e., BA augmented by the cancellation law and also BA augmented by the cut-off subtraction, the MRDP theorem does not hold. However, EBA is strong enough to have the MRDP theorem.

In the last part of this section, we will have a closer look at EBA, and obtain some of its nice properties with relation to some classical fragments of PA.

Notation 4.1. Let \( T \) be a theory. Then:

- \( T \vdash \text{MRDP} \) means that for every \( \Sigma_1 \) formula \( A \) there exists a \( \exists^+_1 \) formula \( B \) with the same free variables as \( A \) such that \( T \vdash A \iff B \);
- \( T \vdash \text{MRDP}^\text{w} \) means that for every \( \Sigma_1 \) formula \( A \) there exists a \( \exists^+_1 \) formula \( B \) with the same free variables as \( A \) such that \( T \vdash A \leftrightarrow B \).

Theorem 4.2. Let \( \Gamma \) be a set of geometric sequents. If \( \text{HA} + \Gamma \) is consistent, then \( \text{BA} + \Gamma \not\vdash \text{MRDP}^\text{w} \).

Proof. Let \( A \) be the \( \Sigma_1 \) formula defined as \( A \equiv \top \to \bot \). Suppose that \( \text{BA} + \Gamma \vdash \text{MRDP}^\text{w} \). Then there exists a \( \exists^+_1 \) formula \( B \) such that \( \text{BA} + \Gamma \vdash A \iff B \). In particular, \( \text{BA} + \Gamma \vdash (\top \to \bot) \to B \). Hence by Proposition 3.23, \( \text{BA} + \Gamma \vdash \top \to B \). So \( \text{HA} + \Gamma \vdash A \). Then \( \text{HA} + \Gamma \vdash \bot \), which leads to a contradiction.

Corollary 4.3. Neither \( \text{MRDP}^\text{w} \) nor \( \text{MRDP} \) can be proved in \( \text{BA} \), \( \text{BA} + U \) or \( \text{BA}_c \).

Proof. For \( \text{BA} \) and \( \text{BA} + U \), the proof is straightforward by Theorem 4.2. For \( \text{BA}_c \), we note that for every geometric formula \( A \) in the language \( \mathcal{L}_c \), \( A^\ast \) is a geometric formula in the language \( \mathcal{L} \), and thus by Theorem 3.55, the claim is reduced to that of \( \text{BA} + U \).

For analyzing the MRDP theorem and other properties for EBA, we first consider the theory \( \text{EB}\Delta_0 \), which is formalized by the axioms and rules of EBA, except that axiom and rule of induction are restricted to \( \Delta_0 \) formulas.

Definition 4.4. For a \( \Delta_0 \) formula \( A \), the bounded negation of \( A \) is denoted by \( A^\neg \) and is defined recursively as follows:

- \( \top^\neg \equiv \bot \),
- \( \bot^\neg \equiv \top \),
- \( (s = t)^\neg \equiv t < s \lor s < t \), if \( s \) and \( t \) are terms,
- \( (s < t)^\neg \equiv t < s \lor s = t \), if \( s \) and \( t \) are terms,
- \( A^\neg \equiv B^\neg \lor C^\neg \) if \( A \) is of the form \( B \land C \),
- \( A^\neg \equiv B^\neg \land C^\neg \) if \( A \) is of the form \( B \lor C \),
- \( A^\neg \equiv B \land C^\neg \) if \( A \) is of the form \( B \rightarrow C \),
- \( A^\neg \equiv \forall x(x < t \rightarrow B(x)^\neg) \) if \( A \) is of the form \( \exists x(x < t \land B(x)) \),
- \( A^\neg \equiv \exists x(x < t \land B(x)^\neg) \) if \( A \) is of the form \( \forall x(x < t \rightarrow B(x)) \).

Lemma 4.5. For every \( \Delta_0 \) formula \( A \), \( \text{EB}\Delta_0 \vdash A \lor A^\neg \) and \( \text{EB}\Delta_0 \vdash A \land A^\neg \Rightarrow \bot \). Consequently \( \text{EB}\Delta_0 \vdash A^\neg \iff \neg A \).

Proof. We prove \( \text{EB}\Delta_0 \vdash A \lor A^\neg \) and \( \text{EB}\Delta_0 \vdash A \land A^\neg \Rightarrow \bot \) simultaneously by induction on the complexity of \( A \). But first, note that \( \text{EB}\Delta_0 \vdash \text{PA}^\neg \), i.e., \( \text{EB}\Delta_0 \) proves all the axioms 28-44. For the those other than axiom 37, the argument is similar to those in Lemmas 2.5 and 2.8 and Proposition 2.6 in [1], which prove the same axioms for \( \text{BA} \), noting that only \( \Delta_0 \)-induction formulas are used. Derivability of axiom 37 follows from applying the \( \Delta_0 \)-induction rule to the formula \( \neg x < x \) and the fact that \( \text{EB}\Delta_0 \vdash \top \to \bot \Rightarrow \bot \).
If \( A \) is of the form \( s = t \) or \( s < t \), then \( A \lor A^\sim \) is provably equivalent to \( s < t \lor s = t \lor t < s \) in BQC, and thus derivable in \( \text{EB}_\Delta \). This also shows \( \text{EB}_\Delta \vdash \neg A \Rightarrow A^\sim \), since \( \text{EB}_\Delta \vdash \top \Rightarrow \bot \Rightarrow \bot \). For the remaining part of the base case, it is sufficient to prove \( \text{EB}_\Delta \vdash s = t \Rightarrow \bot \), which is a consequence of \( \text{EB}_\Delta \vdash x < x \Rightarrow \bot \).

In case \( A \) is of the form \( B \rightarrow C \), we have the induction hypothesis \( \text{EB}_\Delta \vdash B \lor B^\sim \) and \( \text{EB}_\Delta \vdash C \lor C^\sim \), together with \( \text{EB}_\Delta \vdash B \land B^\sim \Rightarrow \bot \) and \( \text{EB}_\Delta \vdash C \land C^\sim \Rightarrow \bot \). Note that we already have the derivability of \( \neg B \lor C \Rightarrow (B \rightarrow C) \) and \( (B \land C) \Rightarrow \bot \lor \bot \) in BQC, which combining with the induction hypothesis, completes this induction step.

Suppose that \( A \) is of the form \( \exists x (s < s \land B(x)) \). Define \( C(y) \equiv \exists x (s < y \land B(x)) \land \forall x (s < y \Rightarrow B(x)) \), where \( y \) does not occur in \( A \). From the induction hypothesis \( \text{EB}_\Delta \vdash B(y) \lor B(y)^\sim \), it can easily be derived that \( \text{EB}_\Delta \vdash C(y) \Rightarrow C(Sy) \). So by the induction rule, \( \text{EB}_\Delta \vdash C(0) \Rightarrow C(y) \). Also from \( \text{EB}_\Delta \vdash x < 0 \Rightarrow B(x)^\sim \), and thus \( \text{EB}_\Delta \vdash C(0) \). Hence \( \text{EB}_\Delta \vdash C(y) \), and \( \text{EB}_\Delta \vdash C(s) \), which means \( \text{EB}_\Delta \vdash A \lor A^\sim \). To prove \( \text{EB}_\Delta \vdash A \land A^\sim \Rightarrow \bot \), note that we have \( \text{BQC} \vdash \exists x (s < s \land B(x)) \land \forall x (s < s \rightarrow B(x)) \Rightarrow \top \Rightarrow (\top \Rightarrow \bot) \).

An argument similar to the one for the previous case works for the case where \( A \) is of the form \( \forall x (s < t \rightarrow B(x)) \). The other cases are easy to verify.

Lemma 4.6. For every \( \Delta \) formulas \( A \) and \( B \), the following hold:

1. \( \text{EB}_\Delta \vdash \top \Rightarrow A \Rightarrow A \);
2. \( \text{EB}_\Delta \vdash A \Leftrightarrow \neg \neg A \);
3. \( \text{EB}_\Delta \vdash A \Rightarrow B \Rightarrow \neg A \lor B \).

Proof. Use Lemma 4.5 and the fact that \( \text{BQC} \vdash (\top \Rightarrow A) \land \neg A \Rightarrow \top \Rightarrow \bot \).

The following Lemma shows that \( \text{EB}_\Delta \) (and consequently any of its extensions, for instance \( \text{EBA} \)) proves the least number principle for \( \Delta \) formulas.

Lemma 4.7. For every \( \Delta \) formula \( A(x) \), \( \text{EB}_\Delta \vdash \exists x A(x) \Rightarrow \exists x (A(x) \land \forall y (y < x \rightarrow \neg A(x))) \).

Proof. Define \( B(x) \equiv \forall y (y < x \rightarrow \neg A(y)) \lor \exists y (y < x \land A(y) \land \forall z (z < y \rightarrow \neg A(z))) \), where \( x \) does not occur in \( A(y) \). We have:

- \( \text{EB}_\Delta \vdash \forall y (y < x \rightarrow \neg A(y)) \land \neg A(x) \Rightarrow \forall y (y < Sx \rightarrow \neg A(y)) \),
- \( \text{EB}_\Delta \vdash \forall y (y < x \rightarrow \neg A(y)) \land A(x) \Rightarrow \exists y (y < Sx \land A(y) \land \forall z (z < y \rightarrow \neg A(z))) \),
- \( \text{EB}_\Delta \vdash \exists y (y < x \land A(y) \land \forall z (z < y \rightarrow \neg A(z))) \Rightarrow \exists y (y < Sx \land A(y) \land \forall z (z < y \rightarrow \neg A(z))) \).

By Lemma 4.5, \( \text{EB}_\Delta \vdash A(x) \lor \neg A(x) \). By the above items we have \( \text{EB}_\Delta \vdash B(x) \Rightarrow B(Sx) \), and by the induction rule, \( \text{EB}_\Delta \vdash B(0) \Rightarrow B(x) \). We also have \( \text{EB}_\Delta \vdash B(0) \), and thus \( \text{EB}_\Delta \vdash B(x) \). Lemma 4.5 implies that \( \text{EB}_\Delta \vdash \forall y (y < x \rightarrow \neg A(y)) \land \exists y (y < x \land A(y)) \Rightarrow \bot \). Therefore \( \text{EB}_\Delta \vdash \exists y (y < x \land A(y)) \Rightarrow \exists y (y < x \land A(y) \land \forall z (z < y \rightarrow \neg A(z))) \), and hence \( \text{EB}_\Delta \vdash \exists y A(y) \Rightarrow \exists y (A(y) \land \forall z (z < y \rightarrow \neg A(z))) \).

The next Lemma shows that \( \text{EB}_\Delta \) and its sequent extensions prove instances of induction (both in the axiom form and the rule form) that are more general than just \( \Delta \)-induction. This is needed to show that these theories are functional and well-formed, and therefore satisfy the soundness and completeness with respect to their corresponding class of Kripke models; see subsection 2.3. Note that the proof of the Lemma and the statements it relies on are all syntactic in nature. In particular, completeness itself hasn’t been used and no cyclic reasoning is happening here.

Lemma 4.8. Let \( A \) be a formula, \( B(x) \) be a \( \Delta \) formula such that \( x \) is not free in \( A \), and \( \Gamma \) be a set of sequents in \( \mathcal{L} \). Then

1. \( \text{EB}_\Delta \vdash \Gamma \vdash A \land B(x) \Rightarrow B(Sx) \vdash A \land B(0) \Rightarrow B(x) \);
2. \( \text{EB}\Delta_0 + \Gamma \vdash \forall y(A \land B(x) \rightarrow B(Sx)) \rightarrow \forall y(A \land B(0) \rightarrow B(x)), \) for any sequence \( y \) of variables.

Proof.

1. Since \( \text{EB}\Delta_0 + x = 0 \lor \exists y x = S\bar{y} \), we have \( \text{EB}\Delta_0 + B(0) \land \neg B(x) \Rightarrow \exists y x = S\bar{y} \). By Lemma 4.7 and item 2 of Lemma 4.6, this implies \( \text{EB}\Delta_0 + B(0) \land \neg B(x) \Rightarrow \exists y B(Sy) \land \forall z (z < Sy \rightarrow B(z)) \), thus \( \text{EB}\Delta_0 + B(0) \land \neg B(x) \Rightarrow \exists y B(Sy) \land B(y) \), and hence \( \text{EB}\Delta_0 + A \land B(0) \land \neg B(x) \Rightarrow \exists y (A \land B(y) \land \neg B(Sy)) \). Therefore \( \text{EB}\Delta_0 + \Gamma + A \land B(x) \Rightarrow B(Sx) \land A \land B(0) \land \neg B(x) \Rightarrow \bot \), implying \( \text{EB}\Delta_0 + \Gamma + A \land B(x) \Rightarrow B(Sx) \land A \land B(0) \Rightarrow \neg B(x) \), which by item 2 of Lemma 4.6 yields \( \text{EB}\Delta_0 + \Gamma + A \land B(x) \Rightarrow B(Sx) \land A \land B(0) \Rightarrow B(x) \).

2. By an argument similar to the beginning parts of what we did for the previous item, we get \( \text{EB}\Delta_0 + \forall y(A \land B(0) \land \neg B(x) \rightarrow \exists z(A \land B(z) \land \neg B(Sz)) \). As \( \text{EB}\Delta_0 + \forall y(A \land B(x) \rightarrow B(Sx)) \rightarrow \forall y \exists z(A \land B(z) \land \neg B(Sz)) \), we have \( \text{EB}\Delta_0 + \Gamma \vdash \forall y(A \land B(x) \rightarrow B(Sx)) \Rightarrow \forall y \neg (A \land B(0) \land \neg B(x)) \). Consequently \( \text{EB}\Delta_0 + \Gamma \vdash \forall y(A \land B(x) \rightarrow B(Sx)) \Rightarrow \forall y(A \land B(0) \rightarrow \neg B(x)) \).

Lemma 4.9. Let \( \mathbf{K} \) be a Kripke model \( \mathbf{K} \models \text{EB}\Delta_0 \), \( k \) be a node of \( \mathbf{K} \) and \( A \) be a \( \Delta_0 \) sentence in \( \mathcal{L}(k) \). Then \( \models_k A \) implies \( \models_k \lnot A \).

Proof. We prove the Lemma by induction on \( A \).

• Assume that \( A \) is prime. By the definition of the forcing relation, \( k \models A \) is equivalent to \( \models_k A \). The result follows by Lemma 4.3.

• Assume that \( A \) is of the form \( B \rightarrow C \). If \( k \models A \), then by item 3 of Lemma 4.6 we have \( k \models \neg B \) or \( k \models C \). The induction hypothesis then gives \( \models_k \neg B \) or \( \models_k C \), or equivalently \( \models_k A \). If \( k \models \neg A \), then by Lemma 4.5 we get \( k \models B \) and \( k \models \neg C \). Applying the induction hypothesis yields \( \models_k B \) and \( \models_k \neg C \), hence \( \models_k \neg A \).

• The case where \( A \) is of the form \( B \circ C \) for \( \circ \in \{\land, \lor\} \) can be done similar to the previous case.

• Assume that \( A \) is of the form \( \forall x(x < t \rightarrow B(x)) \). If \( k \models A \), then for all \( k' \succ k \) and all \( a \in D(k') \), \( k' \models a < t \) implies \( k' \models B(a) \). In particular, this holds for all \( a \in D(k) \), which shows that \( k \models a < t \) implies \( k \models \top \rightarrow B(a) \). Using item 1 of Lemma 4.6 then shows that for all \( a \in D(k) \), \( k \models a < t \) implies \( k \models B(a) \). Since \( a < t \) is atomic, \( k \models a < t \) is equivalent to \( \models_k a < t \). Applying the induction hypothesis will then show that for all \( a \in D(k) \), \( \models_k a < t \) implies \( \models_k B(a) \), hence \( \models_k A \). If \( k \models \neg A \), then by Lemma 4.5 we have \( k \models \exists x(x < t \land \neg B(x)) \). Thus, there is some \( a \in D(k) \) such that \( k \models a < t \) and \( k \models \neg B(a) \). Noting that \( a < t \) is atomic and using the induction hypothesis, we can conclude that there is some \( a \in D(k) \) with \( \models_k a < t \) and \( \models_k \neg B(a) \). This shows \( \models_k \exists x(x < t \land \neg B(x)) \), and consequently \( \models_k \neg A \).

• The case where \( A \) is of the form \( \exists x(x < t \land \neg B(x)) \) can be done similar to the previous case.

Corollary 4.10. For every Kripke model \( \mathbf{K} \) such that \( \mathbf{K} \models \text{EB}\Delta_0 \), every node \( k \) of \( \mathbf{K} \) and

1. every \( \Delta_0 \) sentence \( A \) and every \( \Sigma_1 \) sentence \( B \) in \( \mathcal{L}(k) \), if \( \models_k A \rightarrow B \) then \( k \models A \rightarrow B \);  
2. every \( \Sigma_1 \) formulas \( A(x) \) and \( B(x) \) in \( \mathcal{L}(k) \), if \( k \models A(x) \rightarrow B(x) \) then \( \models_k A(x) \rightarrow B(x) \).

Proof.

1. Suppose that \( B \equiv \exists x C(x) \) and \( \models_k A \rightarrow B \), where \( C(x) \) is a \( \Delta_0 \) formula in \( \mathcal{L}(k) \). This means that \( \models_k A \) implies \( \models_k B \). Assume \( k' \models A \) for any \( k' \geq k \). In case \( k' = k \), we already have \( k \models A \). In case \( k' > k \), note that \( k \models \neg A \) implies \( k' \not\models A \), which cannot happen. Thus, by Lemma 4.3 we must have \( k \models A \). So in any case, by Lemma 4.9 we have \( \models_k A \), and thus \( \models_k B \). Therefore, there is some \( a \in D(k) \) with \( \models_k C(a) \), by Lemma 4.9 \( k \models \neg C(a) \) implies \( \models_k \neg C(a) \), which cannot happen. Hence, by Lemma 4.5 we must have \( k \models C(a) \). Thus \( k \models B \), and consequently \( k' \models A \rightarrow B \).
2. Suppose that \( A(x) \equiv \exists y C(x, y) \), \( B(x) \equiv \exists z D(x, z) \), and \( k \vdash A(x) \Rightarrow B(x) \), where \( C(x, y) \) and \( D(y, z) \) are \( \Delta_0 \) formulas in \( L(k) \). This means that for all \( k' \geq k \) and all \( a \in D(k') \), \( k' \vdash A(a) \) implies \( k' \vdash B(a) \), which in particular holds when \( k' = k \). Assuming \( \mathfrak{M}_k \models A(a) \) for any \( a \in D(k) \), there is some \( b \in D(k) \) such that \( \mathfrak{M}_k \models C(a, b) \). By Lemma 4.7, \( k \vdash \neg C(a, b) \) implies \( \mathfrak{M}_k \models \neg C(a, b) \), which cannot happen. Therefore, by Lemma 4.5, we must have \( k \vdash C(a, b) \), and thus \( k \vdash A(a) \). Hence we get \( k \vdash B(a) \); i.e. there exists \( c \in D(k) \) such that \( k \vdash D(a, c) \). Applying Lemma 4.9 again, we can conclude that \( \mathfrak{M}_k \models D(a, c) \). Hence \( \mathfrak{M}_k \models B(a) \), which completes the proof.

\[ \square \]

**Theorem 4.11.** For every Kripke model \( K \) such that \( K \models \text{EB} \Delta_0 \) and a node \( k \) of \( K \), \( \mathfrak{M}_k \models \upharpoonright \Delta_0 \).

**Proof.** We show that \( \Delta_0 \)-induction holds in \( \mathfrak{M}_k \); the other axioms and rules are easy to check. Let \( A(x) \) be a \( \Delta_0 \) formula. By Lemma 4.7, \( \text{EB} \Delta_0 \vdash \exists x \neg A(x) \Rightarrow \exists x \neg A(x) \land \forall y (y < x \Rightarrow \neg \neg A(x)) \). Then by item 2 of Corollary 4.10, \( \mathfrak{M}_k \models \exists x \neg A(x) \Rightarrow \exists x \neg A(x) \land \forall y (y < x \Rightarrow \neg \neg A(x)) \). If \( \mathfrak{M}_k \models A(0) \land \forall x (A(x) \Rightarrow A(Sx)) \) and \( \mathfrak{M}_k \models \exists x \neg A(x) \), then there exist \( a \in D(k) \) such that \( \mathfrak{M}_k \models \neg A(a) \land \forall y (y < a \Rightarrow \neg A(y)) \), which leads to a contradiction, since we have \( \mathfrak{M}_k \models a = 0 \lor \exists y a = Sy \).

\[ \square \]

**Theorem 4.12.** For every \( \Sigma_1 \) formulas \( A(x) \) and \( B(x) \), if \( \Delta_0 \vdash A(x) \Rightarrow B(x) \), then \( \text{EB} \Delta_0 \vdash A(x) \Rightarrow B(x) \).

**Proof.** Suppose that \( A(x) \equiv \exists y C(x, y) \), where \( C(x, y) \) is a \( \Delta_0 \) formula, and assume that \( \text{EB} \Delta_0 \not\models A(x) \Rightarrow B(x) \). Then there exists a Kripke model of \( \text{EB} \Delta_0 \), a node \( k \) and some \( a \in D(k) \) such that \( k \not\models A(a) \Rightarrow B(a) \). As rule 12 is valid in \( K \), there is a node \( k' \geq k \) and some \( b \in D(k') \) such that \( k' \not\models C(a, b) \Rightarrow B(a) \). So by item 1 of Corollary 4.10, \( \mathfrak{M}_{k'} \not\models C(a, b) \Rightarrow B(a) \). On the other hand, since rule 12 is also valid in \( \upharpoonright \Delta_0 \), we have \( \Delta_0 \models C(x, y) \Rightarrow B(x) \). By Theorem 4.11, \( \mathfrak{M}_k \models \upharpoonright \Delta_0 \), and we must have \( \mathfrak{M}_{k'} \models C(a, b) \Rightarrow B(a) \), which leads to a contradiction.

\[ \square \]

**Theorem 4.13.** There exists a \( \Delta_0 \) formula \( A_{\text{exp}}(x, y, z) \) such that:

- \( \Delta_0 \models A_{\text{exp}}(x, 0, 1) \)
- \( \Delta_0 \models y > 0 \Rightarrow A_{\text{exp}}(0, y, 0) \)
- \( \Delta_0 \models A_{\text{exp}}(x, y, z) \land A_{\text{exp}}(x, y, w) \Rightarrow z = w \)
- \( \Delta_0 \models A_{\text{exp}}(x, y, z) \Rightarrow A_{\text{exp}}(x, Sy, x \cdot z) \)

**Proof.** See appendix in [9].

Let EXP be \( \mathcal{E}(A_{\text{exp}}) \), i.e. \( \exists z A_{\text{exp}}(x, y, z) \).

**Lemma 4.14.** \( \text{EB} \models \text{EXP} \)

**Proof.** By Theorems 4.12 and 4.13 we have \( \text{EB} \models A_{\text{exp}}(x, y, z) \Rightarrow A_{\text{exp}}(x, Sy, x \cdot z) \). Then \( \text{EB} \models \exists z A_{\text{exp}}(x, y, z) \Rightarrow \exists z A_{\text{exp}}(x, Sy, z) \) and so by the induction rule, we have \( \text{EB} \models \exists z A_{\text{exp}}(x, 0, z) \Rightarrow \exists z A_{\text{exp}}(x, y, z) \). Also by Theorems 4.12 and 4.13 we have \( \text{EB} \models A_{\text{exp}}(x, 0, 1) \). Hence \( \text{EB} \models \text{EXP} \).

\[ \square \]

**Theorem 4.15.** For every Kripke model \( K \) such that \( K \models \text{EB} \Delta_0 \) and \( \text{EXP} \) and a node \( k \), \( \mathfrak{M}_k \models \upharpoonright \Delta_0 \) + EXP.

**Proof.** By Theorem 4.11, \( \mathfrak{M}_k \models \upharpoonright \Delta_0 \). Also, since \( k \models \text{EXP} \), for all \( a, b \in D(k) \) there is \( c \in D(k) \) with \( k \vdash A_{\text{exp}}(a, b, c) \), which by Corollary 4.10 implies \( \mathfrak{M}_k \models A_{\text{exp}}(a, b, c) \). Therefore \( \mathfrak{M}_k \models \text{EXP} \), and hence \( \mathfrak{M}_k \models \upharpoonright \Delta_0 \) + EXP.

\[ \square \]

**Theorem 4.16.** For every \( \Sigma_1 \) formulas \( A(x) \) and \( B(x) \), if \( \Delta_0 \models \text{EXP} \vdash A(x) \Rightarrow B(x) \), then \( \text{EB} \Delta_0 \) + EXP \( \vdash A(x) \Rightarrow B(x) \).

**Proof.** The proof is similar to the proof of Theorem 4.12 by using Theorem 4.15.

\[ \square \]

**Theorem 4.17.** \( \Delta_0 \) + EXP \( \vdash \text{MRDP} \).

**Proof.** See section 4 in [6].

\[ \square \]

**Theorem 4.18.** \( \text{EB} \Delta_0 \) + EXP \( \vdash \text{MRDP} \).

\[ \square \]
Proof. Consider a $\Sigma_1$ formula $A$. By Theorem 4.17 there exists a $\exists_1^+ \Sigma_1$ formula $B$ such that $\models \Delta_0 + \text{EXP} \vdash A \iff B$. Since both $A$ and $B$ are $\Sigma_1$, the result follows from Theorem 4.16. \hfill \square

**Corollary 4.19.** EBA $\vdash$ MRDP.

*Proof.* Straightforward by Lemma 4.14 and Theorem 4.18. \hfill \square

**Theorem 4.20.**

1. For every $\Sigma_1$ formulas $A(x)$ and $B(x)$, if $\models \Sigma_1 \vdash A(x) \Rightarrow B(x)$, then EBA $\vdash A(x) \Rightarrow B(x)$.

2. For every $\Sigma_1$ formula $A(x)$ and every $\Delta_0$ formula $B(x)$, if PA $\vdash A(x) \Rightarrow B(x)$, then EBA $\vdash A(x) \Rightarrow B(x)$.

*Proof.*

1. Suppose that $\models \Sigma_1 \vdash A(x) \Rightarrow B(x)$. By Theorem 4.17
   
   - $\models \Delta_0 + \text{EXP} \vdash A(x) \iff C(x)$ for some $\exists_1^+ \Sigma_1$ formula $C(x)$,
   - $\models \Delta_0 + \text{EXP} \vdash B(x) \iff D(x)$ for some $\exists_1^+ \Sigma_1$ formula $D(x)$.

   Since $\models \Sigma_1 \vdash \Delta_0 + \text{EXP}$, we have $\models \Sigma_1 \vdash C(x) \Rightarrow D(x)$. By Corollary 3.45, $\models \Delta_0 + \text{U} \vdash C(x) \Rightarrow D(x)$. So by Corollary 3.3, $\models \Delta_0 \vdash C(x) \Rightarrow D(x)$. Since EBA $\vdash \text{U}$, we get EBA $\vdash C(x) \Rightarrow D(x)$. On the other hand, by Theorem 4.16
   
   - EBA $\vdash A(x) \iff C(x)$,
   - EBA $\vdash B(x) \iff D(x)$.

   Hence EBA $\vdash A(x) \Rightarrow B(x)$.

2. Suppose that $A(x) \equiv \exists y C(x, y)$ for some $\Delta_0$ formula $C(x, y)$, and PA $\vdash A(x) \Rightarrow B(x)$. By rule 12, PA $\vdash C(x, y) \Rightarrow B(x)$, hence by Proposition 3.61 EBA $\vdash C^g(x, y) \Rightarrow B^g(x)$. By Lemma 4.5 every $\Delta_0$ formula is equivalent in EBA to its Gödel translation. Therefore EBA $\vdash C(x, y) \Rightarrow B(x)$, and by rule 12, EBA $\vdash A(x) \Rightarrow B(x)$. \hfill \square

**Corollary 4.21.**

1. EBA proves all $\Pi_2$ theorems of $\Sigma_1$.

2. PA is $\Pi_1$-conservative over EBA.

*Proof.* Straightforward by Theorem 4.20. \hfill \square

It is worth mentioning that every Kripke model of HA is $\Delta_0 + \text{Th}_{\Sigma_2}(\text{PA})$-normal (see Theorem 3.1 in [18]) and also HA is complete with respect to PA-normal Kripke models (see Theorem 6 in [3]). Having these facts, we have the following results.

**Theorem 4.22.**

1. Every Kripke model of EBA is $\Delta_0 + \text{Th}_{\Pi_1}(\Sigma_1) + \text{Th}_{\Pi_1}(\text{PA})$-normal.

2. EBA is not complete with respect to $\text{Th}_{\Pi_2}(\text{PA})$-normal Kripke models. In particular, there exists a Kripke model of EBA which is not $\text{Th}_{\Pi_2}(\text{PA})$-normal.

*Proof.*

1. Straightforward by Theorem 4.14 and Corollary 4.21.

2. It is well-known that the Ackermann function is provably total recursive in PA (see [19]). Therefore by Theorem 4.17 there exists a $\exists_1^+ \Sigma_1$ formula $B(x, y, z)$ such that
   
   - PA $\vdash E(B)$,
   - PA $\vdash U(B)$,

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By lemma 4.23, we have

Proof. 2 of Corollary 4.21, Corollary 4.24.

Lemma 4.23. EBA ⊢ Con_{Σ_n}.

Proof. By Corollary 4.34 of Chapter I in [8], IΣ_{n+1} ⊢ Con_{Σ_n}, and thus PA ⊢ Con_{Σ_n}. Since Con_{Σ_n} is Π_1, by item 2 of Corollary 4.21 EBA ⊢ Con_{Σ_n}.

Corollary 4.24. If IΣ_n is consistent, there is no translation (·)^t from the set of formulas of arithmetic to itself such that:

- for every prime formula A, IΣ_n ⊢ A^t ⇔ A,
- for every formulas A and B, if EBA ⊢ A ⇒ B then IΣ_n ⊢ A^t ⇒ B^t.

Proof. By lemma 4.23 we have EBA ⊢ ¬s_n(z, x, ∨⊥) = t_n(z, x, ∨⊥). This implies that EBA ⊢ s_n(z, x, ∨⊥) = t_n(z, x, ∨⊥) ⇒ ⊥ as EBA ⊢ T ⇒ ⊥ ⇒ ⊥. If such a translation exists, we must have IΣ_n ⊢ ¬s_n(z, x, ∨⊥) = t_n(z, x, ∨⊥) and hence IΣ_n ⊢ Con_{Σ_n}, which contradicts Gödel’s second incompleteness theorem.

Corollary 4.24 shows that a proposition similar to Proposition 3.3 cannot be proved for EBA. Thus to find an upper bound on the class of the provably total functions of EBA, we cannot use techniques similar to those we used for BA and BA + U.

5 Final Remarks

We have proved that the provably total functions of BA are primitive recursive and definable in BA by geometric formulas. Our attempt to characterize the provably total functions of BA was not successful, however we could do it for some extensions of BA. We introduced alternative versions of the uniqueness sequent, for some of which we also proved the geometric definability of the provably total functions. We further proved that provably total functions of BA in those alternative senses are primitive recursive. It is worth mentioning that the primitive recursive realizability technique introduced in [14] (see Definition 3.57), that we also applied to EBA (see Theorem 3.58), is used to analyze the provability of existence sequents. Therefore, regardless of which definition we choose for the uniqueness sequent, the primitive recursive upper bound can be proven to exist for the class of the provably total functions of BA.

One may expect that taking one of the alternative definition for uniqueness sequents mentioned above, characterizing the class of the provably total functions of BA reduces to that of BA + U for the following reason. The formula x + y = x + z ⇒ y = z, which is provable in BA, is actually a weakening of the axiom U. We just note that such a reduction may not be a straightforward procedure, because there are sequents consisting of geometric formulas provable in BA + U, such that the weakened conditional formula is not provable in BA. This may mean that for the mentioned reduction, we can neither use arguments based on derivations containing only geometric formulas, nor arguments based on local analysis of Kripke models. As an example, note that BA + U ⊢ ∃x(x + y = x + z) ⇒ y = z,
but $\mathbf{BA} \not\vdash \exists x(x + y = x + z) \rightarrow y = z$, since the formula is refuted in the Kripke model consisting of two irreflexive nodes, the below one with the structure $\mathbb{N}$ and the above one with the structure $\mathbb{N}^*$.

We cannot reduce the characterization of the provably total functions of $\mathbf{BA}$ to that of $\mathbb{I}^+_1$, as we did between $\mathbf{BA} + \mathbf{U}$ and $\mathbb{I}^+_1 + \mathbf{U}$. The reason is that the MRDP theorem does not hold in $\mathbb{I}^+_1$ (see Corollary 5.49), and thus the defining formulas of its provably total functions may not be equivalent to any geometric formulas, and so we lose the benefits of Corollary 5.5.

Our choice of the additional axioms for $\mathbf{BA}_c$ (axioms 46 and 47) is not canonical. One can fix other axioms for formalizing the properties of cut-off subtraction; but that may change results on the class of provably total functions and the MRDP theorem for $\mathbf{BA}_c$. One alternative is using the axioms $x < y \Rightarrow x + y = 0$ and $y \leq x \Rightarrow y + (x - y) = x$ instead, which may more resemble the definition given for cut-off subtraction at the end of Subsection 2.2. Another alternative would be the three axioms $x \div x = x$, $x \div y = 0 \Rightarrow x \div Sy = 0$ and $y \div x = Sz \Rightarrow x \div Sy = z$, which may more resemble the definition of cut-off subtraction by primitive recursion using the predecessor function, as mentioned before Proposition 5.20. It seems that in both of these alternative cases, $x \div x = 0$ may not be derivable from the resulting theory. An idea for seeing this would be considering the expansion $\mathbb{N}^*_c$ of $\mathbb{N}^*$ in which $n \div n = 0$ and $\infty \div n = \infty$ for all $n \in \mathbb{N}$, and $\infty \div \infty = \infty$. If one could show that $\mathbb{N}^*_c$ is a model of $\mathbb{I}^+_1$, it would result in unprovability of $x \div x = 0$ in the theory. Even adding $x \div x = 0$ to the other axioms might not be satisfactory; consider an expansion $\mathbb{N}^*_0$ of $\mathbb{N}^*$ similar to $\mathbb{N}^*_c$, only this time with $\infty \div \infty = 0$. While this is a model of the additional axiom $x \div x = 0$, it does not satisfy $(\infty + \infty) \div \infty = \infty$. Therefore if $\mathbb{N}^*_0$ is a model of $\mathbb{I}^+_1$, we can show unprovability of $(x + y) \div x = y$ in the corresponding arithmetical theory. Verifying whether $\mathbb{N}^*_c$ and $\mathbb{N}^*_0$ are models of $\mathbb{I}^+_1$ or not is out of the scope of what we intend to do here. The important point we want to make is that our results on $\mathbf{BA}_c$ strongly rely on the fact that $\mathbf{BA}_c$ is capable of proving $\mathbf{U}$, which in turn paves the way for eliminating cut-off subtraction from the language. The alternative axiomatizations might look natural when thinking about cut-off subtraction in the context of classical or intuitionistic arithmetical theories, but not strong enough in the context of basic arithmetic.

And lastly, the realizability technique used in [15] to find a bound on the provably total functions of $\mathbf{BA}$ leads to no satisfactory result. To show this, we define the notion of $D$-bounded recursive realizability, which is a generalization of the notion defined in Definition 3.4 of [15]. It is worth mentioning that the main results of [15] are already disproved in [2], by presenting an explicit sequent which is a counterexample to the soundness of $\mathbf{BA}^w$ with respect to the $D$-bounded recursive realizability, when $D(n, m) = n^m + m$.

**Definition 5.1.** Let $\varphi_n$ be the $n$-th partial recursive function, $\pi_1$ and $\pi_2$ be the projections of the Cantor pairing function $(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y$, and $D : \mathbb{N}^2 \rightarrow \mathbb{N}$ be a primitive recursive function. For a sequence $x = (x_1, \ldots, x_m)$, $\varphi_n(x)$ is understood as $\varphi_n((x_1, (x_2, \ldots, (x_{m-1}, x_m))))$. Let

$$B_D(n) \equiv \varphi_\pi(x)(n) \downarrow \land \forall x(\varphi_\pi(x)(n) \leq D(x, \pi_2(n))),$$

and define $x q^D A$ by induction on complexity of a formula $A$:

- $x q^D A \equiv A$, for prime $A$.
- $x q^D (B \land C) \equiv (\pi_1 x) q^D B \land (\pi_2 x) q^D C$.
- $x q^D (B \lor C) \equiv (\pi_1 x) = 0 \land (\pi_2 x) q^D B \lor (\pi_1 x) \neq 0 \land (\pi_2 x) q^D C$.
- $x q^D \exists y B(y) \equiv \pi_2(x) q^D B(\pi_1(x))$.
- $x q^D \forall z (B(z) \rightarrow C(z)) \equiv B_D(x) \land \forall y z (y q^D B(z) \rightarrow \varphi_{\pi_1(z)}(y, z) q^D C(z)) \land \forall z (B(z) \rightarrow C(z))$.

For a sequent $A \Rightarrow B$, $x q^D (A \Rightarrow B) \equiv B_D(x) \land \forall y z (y q^D A \rightarrow \varphi_{\pi_1(z)}(y, z) q^D B)(A \rightarrow B)$, where $z = (z_1, \ldots, z_n)$ is the sequence of all free variables in $A \Rightarrow B$ in the order of appearance.

The following theorem shows that $\mathbf{BA}^w$ is not sound with respect to the $D$-bounded recursive realizability for any primitive recursive function $D$. Then we may conclude that this type of realizability is not suitable for $\mathbf{BA}$ by Lemma 3.39 and the fact that for every sequent $A \Rightarrow B$, $\mathbb{N} \models \exists n (n q^D (A \Rightarrow B))$ iff $\mathbb{N} \models \exists n (n q^D (\top \Rightarrow \forall x (A \Rightarrow B)))$.

**Theorem 5.2.** For every primitive recursive function $D : \mathbb{N}^2 \rightarrow \mathbb{N}$, there exists a sequent $A \Rightarrow B$ such that $\mathbf{BA} \vdash A \Rightarrow B$, but $\mathbb{N} \not\models \exists n (n q^D (A \Rightarrow B))$. 

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Proof. Let \( h(n, m) \) be the following primitive recursive function:

\[
  h(n, m) := \max_{i\leq n, j\leq m} \{D(i, j)\} + n + m + 1.
\]

It is easy to see that for every \( m, n \in \mathbb{N} \), \( D(n, m) < h(n, m) \). By Theorem 3.47 there is a \( \exists_1^+ \) formula \( A(x, y) \) that defines the function \( g(n) = h((0, n), n) \) and also \( \mathbf{BA} + U \vdash \exists yA(x, y) \), hence by Corollary 2.59

\[
  \mathbf{BA} \vdash \forall xyz(x + z = y + z \rightarrow x = y) \Rightarrow \forall x(\top \rightarrow \exists yA(x, y)).
\]

Let the constant zero function be the \( s \)-th partial recursive function, then we have \( \mathbb{N} \models t q^D \forall xyz(x + z = y + z \rightarrow x = y) \) for \( t = (s, 0) \). Suppose there exists a natural number \( n \) such that \( \mathbb{N} \models n q^D (\forall x, y, z(x + z = y + z \rightarrow x = y) \Rightarrow \forall x(\top \rightarrow \exists yA(x, y))) \). Then, by definition of \( D \)-bounded recursive realizability, we have

1. \( \mathbb{N} \models B_D(n) \land \forall a(q^D \forall xyz(x + z = y + z \rightarrow x = y) \rightarrow \varphi_{\pi_1(n)}(a)) \land q^D \forall x(\top \rightarrow \exists yA(x, y))) \),
2. then \( \mathbb{N} \models u q^D \forall x(\top \rightarrow \exists yA(x, y)) \) for \( u = \varphi_{\pi_1(n)}(t) \),
3. so \( \mathbb{N} \models B_D(u) \land \forall ab(q^D \forall x(\top \rightarrow \exists yA(x, y))) \),
4. hence \( \mathbb{N} \models B_D(u) \land \forall b(\varphi_{\pi_1(u)}(0, b) q^D \exists yA(b, y)) \).

Note that by definition of the realizability, we have \( \mathbb{N} \models \forall b(\varphi_{\pi_1(u)}(0, b) q^D A(b, \pi_1 \varphi_{\pi_1(u)}(0, b))) \), so \( \mathbb{N} \models \forall b A(b, \pi_1 \varphi_{\pi_1(u)}(0, b)) \) and hence \( g(b) = \pi_1 \varphi_{\pi_1(u)}(0, b) \). This implies that \( g(b) \leq \varphi_{\pi_1(u)}(0, b) \). Since \( B_D(u) \) is true, for all \( b \) we have

\[
  g(b) \leq \varphi_{\pi_1(u)}(0, b) \leq D((0, b), \pi_2(u)) < h((0, b), \pi_2(u)).
\]

Let \( b = \pi_2(u) \), then

\[
  g(\pi_2(u)) = h((0, \pi_2(u)), \pi_2(u)) < h((0, \pi_2(u)), \pi_2(u)),
\]

which leads to a contradiction. Hence our assumption is false. \( \square \)

Our final remark in this paper is a proposal for axiomatization of \( \mathbf{BA} \). It is a relatively standard tradition that one axiomatizes an arithmetical theory by the usual Peano axioms. This tradition is applied, for instance, to the intuitionistic arithmetical theory, well-known as Heyting arithmetic, and also to the theory based on basic predicate logic, called basic arithmetic \([12]\). We suggest the axiom \( U \) to be added to the list of axioms and rules of \( \mathbf{BA} \). Our motive is twofold. On one hand, as we have seen in the previous sections, \( \mathbf{BA} + U \) is an arithmetical theory stronger than \( \mathbf{BA} \) with remarkable mathematical properties, and on the other hand, it is still a constructive theory in the sense of \([12]\). By Lemma 3.39, we can also add other equivalent axioms instead of \( U \). One of them, \( S(x + y) = y \Rightarrow \perp \), can in fact replace the axiom \( Sx = 0 \Rightarrow \perp \) of \( \mathbf{BA} \), instead of being added. Just note that \( x + 0 = x \) is already an axiom of \( \mathbf{BA} \), and therefore \( Sx = 0 \Rightarrow \perp \) is equivalent to the instance \( S(x + 0) = 0 \Rightarrow \perp \) of the new axiom. The same remarks can be made in relation with the theories \( \mathbf{GA} \) and \( \mathbf{GA} + U \). While there seems to be no theory widely referred to as “geometric arithmetic” in the literature, we suggest that considering the additional axiom \( U \) is a more suitable choice than just taking the Peano axioms over geometric logic.

6 Appendix

The System LK

The system LK (see 1.2.2 and 2.3.2 of [4]) is a sequent calculus formed by the following axioms and rules: (Here, \( \Delta, \Delta' \) and \( \Delta'' \) are finite lists of formulas)

Structural Axiom

\[
(Ax) \quad \frac{}{A \Rightarrow A}
\]
Structural Rules:

(Ex⇒) \( \Delta, A, B, \Delta' \Rightarrow \Delta'' \)

(⇒Ex) \( \Delta \Rightarrow \Delta', A, B, \Delta'' \)

(W⇒) \( \Delta \Rightarrow \Delta' \)

(⇒) \( \Delta \Rightarrow \Delta', A, \Delta'' \)

(C⇒) \( \Delta, A, A \Rightarrow \Delta' \)

(⇒C) \( \Delta \Rightarrow A, A, \Delta' \)

(Cut) \( \Delta \Rightarrow A, \Delta', \Delta \Rightarrow \Delta' \)

Logical Axioms:

⇒ \( \top \)

⊥ ⇒ \( \bot \)

Logical Rules:

(¬⇒) \( \Delta \Rightarrow A, \Delta' \)

(⇒¬) \( \Delta, A \Rightarrow \Delta' \)

(⇒∧) \( \Delta, A, B \Rightarrow \Delta' \)

⇒ (⇒∧) \( \Delta \Rightarrow A, B, \Delta' \)

⇒ (⇒∨) \( \Delta, A \Rightarrow \Delta', \Delta, B \Rightarrow \Delta' \)

⇒ (⇒) \( \Delta \Rightarrow A, B, \Delta', \Delta \Rightarrow A \Rightarrow B \Rightarrow \Delta' \)

(⇒⇒) \( \Delta \Rightarrow A, \Delta', \Delta, B \Rightarrow \Delta' \)

(⇒⇒) \( \Delta \Rightarrow A \Rightarrow B, \Delta' \)

⇒ (⇒⇒) \( \Delta \Rightarrow A, \Delta', \Delta, B \Rightarrow \Delta' \)

⇒ (⇒⇒) \( \Delta \Rightarrow A \Rightarrow B, \Delta' \)

(⇒⇒) \( \Delta \Rightarrow A \Rightarrow B, \Delta' \)

(∃⇒) \( \Delta, A[x/y] \Rightarrow \Delta', \delta y \) is substitutable for \( x \) in \( A \), and \( y \) is not free in \( \Delta, \Delta' \)

(⇒∃) \( \Delta \Rightarrow A[x/t], \Delta', \delta t \) is substitutable for \( x \) in \( A \)
\((\forall \Rightarrow) \quad \Delta, A[x/t] \Rightarrow \Delta'\), where \(t\) is substitutable for \(x\) in \(A\)

\((\Rightarrow \forall) \quad \Delta \Rightarrow A[x/y], \Delta'\), where \(y\) is substitutable for \(x\) in \(A\), and \(y\) is not free in \(\Delta, \Delta'\)

In each rule, the formulas appearing in \(\Delta, \Delta'\) or \(\Delta''\), are called _context_. Other formulas which appear in the upper sequents are called _active formulas_, and those which appear in the lower sequents are called _principal formulas_. The active formula of the cut rule (Cut) is called the _cut formula_.

**Arithmetical Theories Based on LK**

Arithmetical theories over LK are formalized in the language of arithmetic, with the additional axioms of logic with equality (see 2.3.3 of [4]), and axioms and rules specific to arithmetic (see 2.4.6 of [4]):

**Equality Axioms:**

\((-\text{ref})\)
\[s = s\]

\((-\text{eqv})\)
\[s = t, s' = t', s = s' \Rightarrow t = t'\]

\((S-\text{func})\)
\[s = t \Rightarrow Ss = St\]

\((+\text{-}\text{func})\)
\[s = t, s' = t' \Rightarrow s + t = s' + t'\]

\((-\text{func})\)
\[s = t, s' = t' \Rightarrow s - t = s' - t'\]

\((<\text{-rel})\)
\[s = t, s' = t', s < s' \Rightarrow t < t'\]

**Arithmetical Axioms:**

\((S-\text{pos})\)
\[Ss = 0 \Rightarrow\]

\((S-\text{inj})\)
\[Ss = St \Rightarrow s = t\]

\((+\text{-}0)\)
\[\Rightarrow s + 0 = s\]

\((+\text{-}S)\)
\[\Rightarrow s + St = S(s + t)\]

\((-\text{-}0)\)
\[\Rightarrow s \cdot 0 = 0\]

\((-\text{-}S)\)
\[\Rightarrow s \cdot St = s \cdot t + s\]

**Arithmetical Rule:**

\((\text{ind})\)
\[\Delta, A \Rightarrow A[x/Sx], \Delta'\]
\[\Delta, A[x/0] \Rightarrow A[x/t], \Delta'\]

The formula \(A\) in the induction rule (ind) is called the _induction formula_. If the induction formula is restricted to a class \(C\) of formulas, the rule is called the \(C\)-induction rule.
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