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Cohomology of Quaternionic Foliations and Orbifolds

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Abstract
Starting with a concise review of quaternionic geometry and quaternion Kähler manifolds, we define a transversely quaternion Kähler foliation. Then we formulate and prove the foliated versions of the now classical results of V.Y. Kraines and A Fujiki on the cohomology of quaternion Kähler manifolds. Finally, as any orbifold can be realized as the leaf space of a suitably defined Riemannian foliation we reformulate our results for quaternion orbifolds.

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MSC Classification: 53C12, 53C28, 57R30

1 Introduction

In recent decades some geometrical theories have been particularly successfully developed and implemented in modern physics making the relationship between these sciences even stronger. The string theory is one of them. It is closely related to the study of a particular class of Riemannian 4n-manifolds whose holonomy group is a subgroup of $Sp(n)\cdot Sp(1)$. They are called quaternion Kähler and these manifolds arise naturally in the study of supergravity, for more details please see [2] and references therein. V.Y. Kraines in [14] gave
an analogue of the Hodge decomposition theorem for a quaternionic manifold. Moreover, using some results of Chern of [7], she demonstrated inequalities on Betti numbers. Later A. Fujiki [10] formulated analogues of the Hodge and Lefschetz decompositions theorems for the cohomology of some special manifolds, in particular quaternion Kähler manifolds.

In this paper we reformulate these results for the case of a foliated quaternion Kähler structure on a Riemannian manifold. In Section 2 we recall some of the definitions regarding quaternions and in particular a 4-form Ω defined by Kraines which is of importance as it is used to define two operators \( L \) and \( Λ \) on the space of all forms. In Section 3, we discuss quaternion Kähler manifolds and quaternion Kähler analogues of the Hodge star operator \( * \) as well as the operators \( L \) and \( Λ \). We give the definition of a transversely quaternion Kähler foliation in Section 4. Next in Sections 5 and 6 we discuss basic forms and basic Hodge theory on foliated Riemannian manifolds. Moreover, the analogues of the operators \( L \) and \( Λ \) are introduced, using which we define basic effective forms. Twistor spaces over a manifold have proven to be important in the study of the properties of the base manifold. In particular in the case of quaternion Kähler manifolds the twistor space was studied by S. Salamon [21] and independently by L. Bérard Bergery [3, 4]. Salamon shows how using cohomology groups of the base manifold some of the characteristic classes of the twistor space can be computed; hence, the study of these spaces can be fruitful in both directions. Having this in mind, in Section 7 we define the transversal quaternion twistor space \( ZF \) on a foliated manifold \( (M^{p+4q}, F) \) of codimension \( 4q \). The basic cohomology for transversely quaternion Kähler foliations and the counterparts of the results of Kraines for this case are the subject of Section 8. The characteristic foliation of a 3-Sasakian is a nice example transversely quaternion Kähler foliations. Since in this case the transversal twistor space is a trivial foliated bundle one can strengthen the general results, which is the main topic of Section 9. The important result of A. Haefliger et al. [12] ensures that for any orbifold, one can find a Riemannian foliated manifold with compact leaves such that its leaf space is the original orbifold. In Section 10 using this fact we reformulate our results for orbifolds.

# 2 Preliminaries

In this section, for the convenience of the reader, we recall some basic definitions and constructions from [14].

Let \( H^n \) be the \( n \)-dimensional right module over the field of quaternions \( H \). The canonical bilinear form on \( H^n \) is defined as

\[
\langle u, v \rangle = \frac{1}{2} \sum_{i=1}^{n} (u_i \bar{v}_i + v_i \bar{u}_i)
\]

where \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in H^n \). The 2-form \( \langle \ldots, \ldots \rangle \) is a scalar product on the \( 4n \)-dimensional real vector space \( H^n \). The group \( Sp(n) \) can be defined as the linear group preserving the “symplectic product”.
\((u, v) = \sum_{i=1}^{n} u_i \bar{v}_i\) on \(\mathbb{H}^n\). Then \(\langle u, v \rangle = \frac{1}{2}\{(u, v) + (v, u)\}\) thus immediately the natural action of \(Sp(n)\) preserves the scalar product \(\langle \cdot, \cdot \rangle\). The group \(Sp(1)\) is identified with the quaternions of length 1, so its right action preserves the product \(\langle \cdot, \cdot \rangle\).

As \(\mathbb{H}\) can be identified with \(\mathbb{R}^4\), it makes it possible to write any quaternion \(x \in \mathbb{H}\) as \(x = x^0 1 + x^1 \mathbf{i} + x^2 \mathbf{j} + x^3 \mathbf{k}\) where \(1, \mathbf{i}, \mathbf{j}, \mathbf{k}\) form the standard base of \(\mathbb{H}\) as a real vector space. The right multiplication by \(\mathbf{i}, \mathbf{j}\) and \(\mathbf{k}\) define three complex structures on \(\mathbb{H}^n\) denoted by the same letters, respectively. In turn, they permit us to define three skew-symmetric 2-forms

\[
\Omega_I (u, v) = \langle u \mathbf{i}, v \rangle,
\Omega_J (u, v) = \langle u \mathbf{j}, v \rangle,
\Omega_K (u, v) = \langle u \mathbf{k}, v \rangle.
\]

In [14] V.Y. Kraines demonstrated that the 4-form \(\Omega\) on \(\mathbb{H}^n\) defined by

\[
\Omega = \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K.
\]

is invariant under the natural action of \(Sp(n).Sp(1)\).

Let \((\mathbb{H}^n)'\) be the dual space of \(\mathbb{H}^n\) as the quaternionic vector space. Let \(z_1, \ldots, z_n\) be a basis of \((\mathbb{H}^n)'\). Each \(z_\alpha\) can be represented as

\[
z_\alpha = a_\alpha 1 + b_\alpha \mathbf{i} + c_\alpha \mathbf{j} + d_\alpha \mathbf{k},
\]

thus the 1-forms \(a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots, d_n\) form a basis of \((\mathbb{H}^n)'\) as a real vector space. Then

\[
\Omega = \sum_{\alpha, \beta, \gamma, \delta} a_\alpha \wedge b_\beta \wedge c_\gamma \wedge d_\delta,
\]

hence

\[
\Omega^n \neq 0.
\]

On \(\bigwedge^* (\mathbb{H}^n)'\) we can define 3 operators \(*, L\) and \(\Lambda\) as follows, cf. [14]: If \(\omega\) is a simple \(p\)-form, then \(*\omega\) is the simple \((4n - p)\)-form that

\[
\omega \wedge * \omega = a_1 \wedge b_1 \wedge c_1 \wedge d_1 \wedge \ldots \wedge a_n \wedge b_n \wedge c_n \wedge d_n.
\]

Then \(*\) is extended by \(\mathbb{R}\)-linearity to \(\bigwedge (\mathbb{H}^n)'\). Moreover, \(* * \omega = \omega\). For an arbitrary form \(\omega\) we define \(L\) and \(\Lambda\) operators as

\[
L \omega = \Omega \wedge \omega\text{ and }\Lambda \omega = *(\Omega \wedge * \omega).
\]

Thus \(L\) and \(\Lambda\) are linear transformations

\[
L: \bigwedge^p (\mathbb{H}^n)' \to \bigwedge^{p+4} (\mathbb{H}^n)',
\Lambda: \bigwedge^p (\mathbb{H}^n)' \to \bigwedge^{p-4} (\mathbb{H}^n)',
\]

On \(\bigwedge^p (\mathbb{H}^n)'\) we define a bilinear form by

\[
(w, w') = *(w \wedge * \omega').
\]
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where $\omega, \omega' \in \bigwedge^p (H^n)'$. Then

$$(Lw, w') = (w, \Lambda \omega').$$

for any $\omega \in \bigwedge^p (H^n)'$ and $\omega' \in \bigwedge^{p+4} (H^n)'$.

Kraines shows that $L: \bigwedge^p (H^n)' \rightarrow \bigwedge^{p+4} (H^n)'$ is an isomorphism into for $p + 4 \leq n + 1$. Next she defines effective forms: a $p$-form $\omega$ is called effective if $\Lambda \omega = 0$. The space of all effective $p$-forms is denoted by $\Lambda^p_e$. With all these notions in place she formulates and proves the following decomposition, cf. [14], Theorem 2.6:

**Theorem 1** There is the following direct sum decomposition of $\bigwedge^p (H^n)'$ for $p \leq n + 1$, $r = [p/4]$,

$$\bigwedge^p (H^n)' = \Lambda^p_e + L\Lambda^{p-4}_e + \ldots + L^r \Lambda^{p-4r}_e.$$

3 Quaternion Manifolds

In the Preliminaries of [19] the authors recall several basic definitions related to manifolds of dimension $4n$, namely, the (almost) hypercomplex structure and (almost) quaternionic structure, which we will recall next. Let $I_1, I_2, I_3$ be three almost complex structures on a $4n$-dimensional manifold $M$, such that they satisfy $I_1 \circ I_2 = I_3$ and its cyclic permutations, then the ordered triple $H = (I_1, I_2, I_3)$ on $M$ is called an almost hypercomplex structure. An almost quaternionic structure on the manifold $M$ is a rank 3 vector subbundle $Q$ of the endomorphism bundle $\text{End}(TM)$ which locally is spanned by an almost hypercomplex structure $H = (I_1, I_2, I_3)$ which are transformed by $SO(3)$ on the their respective domains of existence. A quaternionic structure on the manifold $M$ is an almost quaternionic structure $Q$ such that there exists a torsionless connection $\nabla$ whose extension to $\text{End}(TM)$ preserves the subbundle $Q$, i.e. $\nabla Q \subset Q$. On an almost quaternionic manifold $(M, Q)$ the metric $g$ is quaternion Hermitian if it is Hermitian with respect to the local basis $(I_1, I_2, I_3)$ of $Q$. It is quaternion Kähler if it is quaternion Hermitian and $Q$ is $\nabla$-parallel for the Levi-Civita connection of $g$.

Quaternion Kähler manifolds are Riemannian manifolds $(M, g)$ of real dimension $4n$ whose holonomy group can be reduced to $Sp(n)Sp(1)$. In dimension $4(n = 1)$ this condition means only that the manifold is Riemannian as $Sp(1)Sp(1) = SO(4)$. Therefore this condition is meaningful for $n \geq 2$. Quaternion Kähler manifolds can be characterized in terms of local endomorphisms of the tangent bundle, cf., e.g., [4] Proposition 14.36:

**Proposition 2** A Riemannian manifold $(M, g)$ is quaternion Kähler if and only if there exist a covering of $M$ by open sets $U_i$ and, for each $i$, two almost complex structures $I$ and $J$ on $U_i$ such that

1. $g$ is Hermitian for $I$ and $J$ on $U_i$,
2. \( IJ = -JI \),
3. the Levi-Civita derivatives of \( I \) and \( J \) are linear combinations of \( I \), \( J \) and \( K = IJ \),
4. for any \( x \in U_i \cap U_j \) the vector space of endomorphisms of \( T_xM \) generated by \( I \), \( J \) and \( K \) is the same for \( i \) and \( j \).

In fact, the condition (iv) states that these local endomorphisms \( I \), \( J \) and \( K \) defined on each open subset \( U_i \) generate a global subbundle of \( \text{End}(TM) \) which is parallel with respect to the induced action of the Levi-Civita connection.

The proposition itself is a consequence of the fact that the subgroup \( Sp(n).Sp(1) \) of \( SO(4n) \) can be characterized as the group of orientation preserving linear isometries which leave invariant the 3-dimensional subspace of endomorphisms of \( \mathbb{R}^{4n} \) generated by right multiplication by \( i \), \( j \) and \( k \) when \( \mathbb{R}^{4n} \) is identified with \( \mathbb{H}^n \). \( Sp(n) \) acts on the left by \( (n,n) \)-quaternion matrices and \( Sp(1) \) acts on the right by multiplication by quaternions of norm 1. Unfortunately, the endomorphisms \( I \), \( J \) and \( K \) cannot be globally defined.

At any point \( x \in M \) the tangent space \( T_xM \) can be identified with \( \mathbb{H}^n \), and using this identification we can define a global closed 4-form \( \Omega \) of maximal rank by pulling back the form \( \Omega \in (\mathbb{H}^n)' \) defined in the previous section.

As in the case of \( (\mathbb{H}^n)' \) we can define operators \( * \), \( L \) and \( \Lambda \) on the space \( A^*(M) \) of differential forms on the manifold \( M \):

\[
* : A^k(M) \to A^{4n-k}(M) \\
L : A^k(M) \to A^{k+4}(M); \quad L(\alpha) = \Omega \wedge \alpha \\
\Lambda : A^k(M) \to A^{k-4}(M); \quad \Lambda(\alpha) = *(\Omega \wedge *\alpha)
\]

A differential form \( \alpha \) is called **effective** if \( \Lambda \alpha = 0 \).

The decomposition theorem of the previous section, applied point by point, permits to formulate and prove the following decomposition theorem for differentiable form on \( M \), cf. [14], Theorem 3.5.

**Theorem 3** Let \( M \) be a 4n-dimensional quaternionic Kähler manifold and \( \omega \) a differential form on \( M \) of degree \( p \leq n + 1 \). Then

\[
w = \sum_{i=0}^{[p/4]} L^i \omega^p_{4i}
\]

where \( \omega^k_{4i} \) is an effective \( k \)-form.

### 4 Foliations

Let \( \mathcal{F} \) be a foliation on Riemannian \( m \)-manifold \( (M,g) \) of codimension \( q \) and of leaves of dimension \( p \). Then \( \mathcal{F} \) is defined by a cocycle \( \mathcal{U} = \{U_i, f_i, g_{ij}\}_{i,j \in I} \) modeled on a \( q \)-manifold \( N_0 \) such that
1. \( \{ U_i \}_{i \in I} \) is an open covering of \( M \),
2. \( f_i : U_i \to N_0 \) are submersions with connected fibers,
3. \( g_{ij} : N_0 \to N_0 \) are local diffeomorphisms of \( N_0 \) with \( f_i = g_{ij} f_j \) on \( U_i \cap U_j \).

The connected components of the trace of any leaf of \( F \) on \( U_i \) consist of the fibers of \( f_i \). The open subsets \( N_i = f_i(U_i) \subset N_0 \) form a q-manifold \( N_U = \bigsqcup N_i \), which can be considered as a transverse manifold of the foliation \( F \). The pseudogroup \( \mathcal{H}_U \) of local diffeomorphisms of \( N_U \) generated by \( g_{ij} \) is called the holonomy pseudogroup of the foliated manifold \((M,F)\) defined by the cocycle \( U \). Two different cocycles can define the same foliation, then we have two different transverse manifolds and two holonomy pseudogroups. In fact, these two holonomy pseudogroups are equivalent in the sense of Haefliger, cf. [13].

According to Haefliger, cf. [13], a transverse property of a foliated manifold is a property of foliations which is shared by any two foliations with equivalent holonomy pseudogroups. For example, being Riemannian, transversely symplectic, transversely almost-complex, transversely Kähler, etc., is a transverse property. A Riemannian foliation, i.e., admitting a bundle-like metric, is defined by a cocycle \( U \) modeled on a Riemannian manifold whose local submersions are Riemannian submersions. Then the associated transverse manifold \( N_U \) is Riemannian and the associated holonomy pseudogroup \( \mathcal{H}_U \) is a pseudogroup of local isometries. Any foliation defined by a cocycle \( V \) whose holonomy pseudogroup \( \mathcal{H}_V \) is equivalent to \( \mathcal{H}_U \) is also Riemannian, as an equivalence of pseudogroups transports the Riemannian metric from \( N_U \) to \( N_V \) and ensures that the pseudogroup \( \mathcal{H}_V \) is a pseudogroup of local isometries of the transported metric. This metric can be lifted to a bundle-like metric (not unique) on the other foliated manifold making the second foliation Riemannian. The same procedure can be applied to any geometrical structure, for the discussion of this general procedure see [25].

**Definition 1** A foliation \( F \) is transversely quaternion Kähler if it is defined by a cocycle \( U = \{ U_i, f_i, g_{ij} \}_{i,j \in I} \) modeled on a quaternion Kähler manifold \((N_0, g_0, Q_0)\) and the local diffeomorphisms \( g_{ij} \) are local automorphisms of the quaternion Kähler structure of \((N_0, g_0, Q_0)\), i.e., the \( g_{ij} \) are local isometries and the induced mappings \( \tilde{g}_{ij} \) of \( \text{End}(T N_0) \) preserve the subbundle \( Q_0 \) of rank 3.

In the language of foliated structures this condition can be formulated as follows, cf. [19]. Let \( N(M,F) = TM/T F \) be the normal bundle of the foliation \( F \). The vector bundle \( \text{End}(N(M,F)) \) admits the natural foliation \( F_{\text{End}} \) of dimension \( p \) which is defined by a cocycle \( F_{\text{End}} = \{ V_i, \tilde{f}_i, \tilde{g}_{ij} \}_{i,j \in I} \) modeled on \( \text{End}(T N_0) \) where \( \tilde{f}(A) = df \circ A \circ (df|_{N(M,F)})^{-1} \). With this in mind we can define a foliated quaternion Kähler structure.

**Definition 2** A foliated quaternion Kähler structure on a foliated Riemannian manifold \((M,F)\) is given by the following data:
1. $g$ is a foliated Riemannian metric in $N(M,\mathcal{F})$;
2. a 3-dimensional foliated subbundle $Q$ of $\text{End}(N(M,\mathcal{F}))$ which is locally spanned by 3 almost complex foliated structures;
3. the metric $g$ is Hermitian with respect to these local almost complex structures;
4. the subbundle $Q$ is parallel with respect to the foliated Levi-Civita connection of $g$.

Therefore a foliated quaternion Kähler structure on a foliated Riemannian manifold $(M,\mathcal{F})$ will be denoted by $(M,\mathcal{F},g,Q)$. Let $g$ be a foliated Riemannian metric on $N(M,\mathcal{F})$ and $\bar{g}$ the corresponding holonomy invariant metric on the transverse manifold $N$. At each point $x \in U_i$ there exist 3 foliated almost complex structures $I_x, J_x,$ and $K_x$ on an open neighbourhood $U_x$ which project to 3 almost complex structures $\bar{I}_x, \bar{J}_x,$ and $\bar{K}_x$ on a neighborhood of $f_i(x) \in N_i$. Then on $U_x$ we define the 2-forms

$$\Omega_I(u,v) = g(Iu,v), \Omega_J(u,v) = g(Ju,v), \text{ and } \Omega_K(u,v) = g(Ku,v),$$

where $u,v \in N(M,\mathcal{F})$. Using the same argument as in [14] one can show that the 4-form $\Omega$

$$\Omega = \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K$$

is well-defined, i.e., it is independent of the choice of the structures $I, J,$ and $K$. As these structures were foliated the form $\Omega$ is basic. Moreover, $\Omega$ is closed and of maximal rank and parallel with respect to the foliated Levi-Civita connection. In exactly the same way using the transverse metric $\bar{g}$ we define the 4-form $\Omega$ of the transverse manifold $N$. The 4-form $\Omega$ closed and maximal rank. It is also holonomy invariant as the Riemannian metric $\bar{g}$ is. The forms $\Omega$ and $\bar{\Omega}$ correspond to each other under the correspondence between foliated and transverse objects, cf. [25].

## 5 Hodge theory for basic forms

In this section, we gather some of the definitions and properties of Hodge theory for basic forms. On a foliated Riemannian manifold $(M,g,\mathcal{F})$ the set of all basic $k$-forms is

$$A^k(M,\mathcal{F}) = \{ \alpha \in A^k(M) : i_X\alpha = i_Xd\alpha = 0 \text{ for all vectors } X \in T\mathcal{F} \}$$

which is a subcomplex of $A^k(M)$ and we denote its cohomology by $H^k(M,\mathcal{F})$. The restriction of the bundle-like metric to the normal bundle of the foliation of the Riemannian foliated manifold $(M,g,\mathcal{F})$ defines $\check{*}$ operator, cf. [24],

$$\check{*}: A^k(M,\mathcal{F}) \rightarrow A^{4n-k}(M,\mathcal{F}).$$
The corresponding transverse metric on the transverse manifold \((N, \bar{g})\) defines the star operator denoted by the same symbol \(\bar{*}\). As elements of the holonomy pseudogroup are local isometries they commute with \(\bar{*}\), thus \(\bar{*}\) sends holonomy invariant forms into holonomy invariant forms:

\[
\bar{*} : A^k_u(N) \to A^{4n-k}_u(N).
\]

These two star operators correspond to each other under the isomorphism of the differential forms complexes.

On the Riemannian manifold \((M, g)\) we have the \(*\)-operator acting on the complex of smooth forms:

\[
*: A^k(M) \to A^{m-k}(M)
\]

On the subcomplex \(A^k(M, F)\) of basic forms these two operators are related by the formula

\[
\bar{*}\alpha = (-1)^{p(q-k)} * (\alpha \wedge \chi_F)
\]

for any \(\alpha \in A^k(M, F)\), where \(\chi_F\) is the volume form of leaves.

In \(A^k(M, F)\) we have the standard scalar product

\[
\langle \alpha, \beta \rangle_b = \int_M \alpha \wedge \bar{*} \beta \wedge \chi_F
\]

which is the restriction of the standard scalar product on \(A^k(M)\). A Riemannian foliation on a compact manifold is said to be taut if there exists a Riemannian metric that makes all its leaves minimal submanifolds. Tautness is characterized by the nonvanishing of the top dimensional basic cohomology, i.e., \(H^q(M, F) \neq 0\). In this case we say that the foliation is cohomologically taut. In fact, this Riemannian metric can be chosen to be bundle-like. Moreover, one can make the modification only along leaves, cf. [24], Chapter 7.

The formal adjoint \(\delta_b\) of \(d\) in the complex \(A^k(M, F)\) with the scalar product \(\langle \cdot, \cdot \rangle_b\) is the operator

\[
\delta_b = (d - \kappa \wedge)^* : A^k(M, F) \to A^{k-1}(M, F),
\]

where \(\kappa\) is the mean curvature form of the leaves, and

\[
(d - \kappa \wedge)^* (\beta) = (-1)^{q(k+1)+1} \bar{*}(d - \kappa) \bar{*}\beta,
\]

for any \(\beta \in A^k(M, F)\). If the leaves of \(F\) are minimal submanifolds for the bundle-like metric \(g\), then \(\kappa = 0\) and \(\delta_b = d\bar{*}\). We define the basic Laplacian as

\[
\Delta_b = \delta_b d + d\delta_b.
\]
A basic form $\alpha$ is called harmonic iff $\Delta_b \alpha = 0$. The basic Hodge theorem for compact Riemannian foliated manifolds asserts that $\alpha$ is harmonic iff $d\alpha = 0 = \delta_b \alpha$.

6 Basic form complex for transversely quaternion Kähler foliations

Using the 4-forms $\Omega$ and $\bar{\Omega}$ we define $L$ and $\Lambda$ operators on the complexes $A^*(M,F)$ and $A^*_U(N)$, respectively:

$L : A^k(M,F) \to A^{k+4}(M,F); \quad L(\alpha) = \Omega \wedge \alpha$

$\Lambda : A^k(M,F) \to A^{k-4}(M,F); \quad \Lambda(\alpha) = \bar{\ast}(\Omega \wedge \bar{\ast} \alpha)$

and

$\bar{L} : A^*_U(N) \to A^{k+4}_U(N); \quad \bar{L}(\alpha) = \bar{\Omega} \wedge \alpha$

$\bar{\Lambda} : A^*_U(N) \to A^{k-4}_U(N); \quad \bar{\Lambda}(\alpha) = \bar{\ast}(\Omega \wedge \bar{\ast} \alpha)$

The space of $H_U$-invariant $k$-forms on the manifold $N_U$ can be identified with the space of foliated sections of the bundle $\bigwedge^k Q^*$ which in turn is isomorphic to the space of basic $k$-forms. Under this identification the operators $\ast$, $L$, and $\Lambda$ correspond to $\bar{\ast}$, $\bar{L}$, and $\bar{\Lambda}$, respectively. Basic or transverse forms which are annihilated by $\Lambda$ and $\bar{\Lambda}$, respectively, are called effective.

On a compact manifold with a taut foliation one can define scalar products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_b$ on $A^k(M)$ and $A^k(M,F)$, respectively, as

1. $\langle \omega, \omega' \rangle = \int_M \ast(\omega \wedge \ast \omega') = \int_M \omega \wedge \ast \omega'$
2. $\langle \omega, \omega' \rangle_b = \int_M \bar{\ast}(\omega \wedge \bar{\ast} \omega') = \int_M \omega \wedge \bar{\ast} \omega' \wedge \chi_F$

Using this scalar product we have for any $\omega \in A^k(M,F)$ and $\omega' \in A^{k+4}(M,F)$

$\langle L\omega, \omega' \rangle_b = \langle \omega, \Lambda \omega' \rangle_b$

As the proof of the decomposition theorem, cf. [14], Theorem 3.5, was based on the pointwise application of the decomposition theorem for forms on the quaternions, the same argument is valid in the case of basic forms.

**Theorem 4** Let $(M,g,F)$ be a $(4n + p)$-dimensional Riemannian foliated manifold whose $p$-dimensional foliation $F$ is transversely quaternion Kähler. Let $\omega$ be a basic differential form on $(M,F)$ of degree $p \leq n + 1$. Then

$$\omega = \sum_{i=0}^{\lfloor p/4 \rfloor} L^i \omega_e^{p-4i}$$

where $\omega_e^k$ is an effective basic $k$-form.
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Proof The operator $L$ is an isomorphism into and $\Lambda$ as the adjoint of $L$ is onto. The proof can be done using induction on $p$. It is easy to see that by definition of $\Lambda$ we have $\bigwedge^p = \bigwedge^p_c$ for $p = 0, 1, 2, 3$. Now assume that for $i < p$ the theorem holds we shall prove it for $i = p$. We want to show that the space $\bigwedge^p_c$ is the orthogonal complement of the subspace $L \bigwedge^{p-4}(\mathbb{H}^n)'$ in $\bigwedge^p(\mathbb{H}^n)'$. Let $w \in \bigwedge^p_c$ and $Lw' \in L \bigwedge^{p-4}(\mathbb{H}^n)'$ then since $\Lambda w = 0$ we have

$$(w, Lw') = \Lambda(w, w') = 0,$$

therefore orthogonality is proved. Now let $w \in \bigwedge^p(\mathbb{H}^n)'$ and $(w, Lw') = 0, \forall w' \in \bigwedge^{p-4}(\mathbb{H}^n)'$ then

$$(\Lambda w, w') = 0,$$

hence we have $\Lambda w = 0$ and by induction the proof is complete. □

7 Twistor space

Let $(M^{4n}, g)$ be a quaternion Kähler manifold and following notation of Salamon [21], let $H$ be the quaternionic line bundle associated to the representation of $Sp(1)$ on $C^2$. Salamon showed that on such a quaternion Kähler manifold $(M, g)$ there exists a twistor fibration $q : Z \rightarrow M$. $Z$ is a complex manifold of dimension $2n + 1$ which can be viewed as an $S^2$-bundle generated by anticommuting almost complex structures $I, J, K$ or as $Z = \mathbb{P}(H)$, i.e. the projectivization of the bundle $H$ and $Z$ can be referred to as the twistor space of the quaternion Kähler manifold $M$. It is worth noting that $H$ is not always globally well-defined but $Z = \mathbb{P}(H)$ is. Twistor space of quaternion Kähler manifolds and in general quaternionic manifolds can be used to study some of the properties of the base manifold. In particular Salamon showed how some of the characteristic classes of the twistor space $Z$ can be computed using cohomology groups of the manifold $M$.

Let $F$ be a foliation of codimension $4q$ on the manifold $M^{p+4q}$ and on $(M, F)$ we have a foliated quaternion Kähler structure $Q$ with local bases $(I_1, I_2, I_3)$ on the normal bundle $N(M, F) = TM/TF$. We define the transversal quaternion twistor space of $F$ by

$$ZF = \{ J \in Q, J = \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3, \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \}$$

i.e. $ZF$ is the sphere bundle associated with the foliated vector bundle $Q$, and $Q$ carries a Riemannian structure such that it makes $\{I, J, K\}$ an orthonormal basis.

In our earlier work [18], we discussed the twistor spaces of foliated manifolds and the corresponding transverse manifold and how these two are related. Now as we discussed in Section 4 the quaternion Kähler foliation is defined by a cocycle modeled on a quaternion Kähler manifold $N_0$ and one can expect that since the twistor space $Z(N_0)$ of the manifold $N_0$ is a useful tool in studying its differential geometric properties, one should by using the relation between $Z(N_0)$ and $ZF$ be able to study properties of the foliated manifold $(M, F)$. 
8 Basic cohomology of transversely quaternion Kähler foliations

At the beginning we formulate and provide the proofs of some important technical results for foliated manifolds which are foliated counterparts of well-known theorems for Riemannian manifolds. Let \((M,\mathcal{F})\) be a compact Riemannian foliated manifold. Assume that

1. its foliated normal bundle \((N(M,\mathcal{F}),\mathcal{F}_N)\) admits a reduction to a connected subgroup \(G\) of \(O(q)\),
2. the corresponding foliated \(G\)-reduction \(B((M,\mathcal{F}),G,\mathcal{F}_G)\) of the foliated frame bundle \(L((M,\mathcal{F}),\mathcal{F}_L)\) admits a foliated connection without torsion. This condition is equivalent to the vanishing of the structure tensor of \(B((M,\mathcal{F}),\mathcal{F}_G)\), cf. \([26]\).

These two foliated conditions correspond to the following transverse ones:

1. the tangent bundle of the transverse manifold \(N_U\) admits a reduction to a connected subgroup \(G\) of \(O(q)\),
2. the corresponding \(G\)-reduction \(B(N_U,G)\) of the frame bundle \(L(N_U)\) is holonomy invariant, i.e., \(\mathcal{H}_U\)-invariant, and admits a connection without torsion. This condition is equivalent to the vanishing of the structure tensor of the \(G\)-structure. Since a \(G\)-connection without torsion is unique, it is \(\mathcal{H}_U\)-invariant, cf. \([11, 26]\).

Let \(\pi : L \to M\) denote the projection from the foliated frame bundle \(L((M,\mathcal{F}),\mathcal{F}_L)\) to the foliated manifold \((M,\mathcal{F})\) and \(\pi_N : TM \to N(M,\mathcal{F})\) be the natural projection from the tangent bundle of the foliated manifold to the normal bundle of the foliation. We define an \(\mathbb{R}^q\)-valued canonical 1-form on the total space \(L((M,\mathcal{F}),\mathcal{F}_L)\) or \(B((M,\mathcal{F}),G,\mathcal{F}_G)\) as follows

\[
\theta_p = p^{-1} \pi_N d_p \pi, \quad x = \pi(p)
\]

where \(p^{-1} : N_x(M,\mathcal{F}) \to \mathbb{R}^q\) is the inverse of the natural isomorphism defined by the linear normal frame \(p\).

The fiber bundle \(\bigwedge^k N_x(M,\mathcal{F})^*\) can be understood as the associated bundle of \(L((M,\mathcal{F}),\mathcal{F}_L)\) with the standard fiber \(\bigwedge^k (\mathbb{R}^q)^*\). The space of sections of this bundle we denote by \(A^k(N)\). Since the normal frame bundle \(L(M,\mathcal{F})\) is foliated, the foliation \(\mathcal{F}_L\) induces a foliation \(\mathcal{F}_L^k\) of the fiber bundle \(\bigwedge^k N_x(M,\mathcal{F})^*\). The space of \(k\)-basic forms \(A^k(M,\mathcal{F})\) is a subspace of \(A^k(N)\). If the normal frame bundle \(L(M,\mathcal{F})\) admits a foliated \(G\)-reduction \(B((M,\mathcal{F}),G,\mathcal{F}_G)\), the bundle \(\bigwedge^k N_x(M,\mathcal{F})^*\) can be understood as the associated fiber bundle of \(B((M,\mathcal{F}),G,\mathcal{F}_G)\) with the standard fiber \(\bigwedge^k (\mathbb{R}^q)^*\). The natural induced foliations coincide.

Let \(W \subset \bigwedge^k (\mathbb{R}^q)^*\) be an invariant subspace of \(\bigwedge^k (\mathbb{R}^q)^*\) under the standard action of \(G\). There is the standard scalar product on \(\bigwedge^k (\mathbb{R}^q)^*\) for
which the induced action of $G$ is isometric. The associated fiber bundle $W$ of $B((M, F), G, F_G)$ with the standard fiber $W$ can be understood as a foliated vector subbundle of the foliated vector bundle $(\bigwedge^k N_x(M, F)^*, F^k_L)$. Therefore a $k$-differential form $\alpha$ which which corresponds to a section of $W$ is said to be of type $W$. The space of these $W$-valued sections we denote also by $W$. The projection $P_W: A^k(N) \to W$ sends basic forms into basic forms as the operation is done point by point. Next we show that the result of S.S. Chern in [7] can be reformulated for the basic Laplacian $\Delta_b$.

**Proposition 5** (cf. [7]) Let $W \subset \bigwedge^k (R^q)^*$ be an invariant subspace of $\bigwedge^k (R^q)^*$ under the standard action of $G$, $P_W$ be the projection $P_W: A^k(M, F) \to W$ and $\Delta_b$ be the basic Laplacian, then

$$P_W \Delta_b = \Delta_b P_W.$$ 

Moreover, let $W_1, \ldots, W_s$ be irreducible invariant subspaces of $\bigwedge^k (R^q)^*$ for the action of the group $G$. Then if $\alpha$ is a harmonic basic $k$-form, the $k$-forms $P_{W_1} \alpha, \ldots, P_{W_s} \alpha$ are basic and harmonic. Moreover, if $\alpha$ is a basic $k$-form of type $W$ so is the form $\Delta_b \alpha$.

**Proof** Our proof follows Chern’s original proof, cf. [7]. The unique torsionless connection in the foliated $G$-structure is foliated, which in particular means that the tangent bundle to the foliation $F_G$ is a subbundle of the horizontal bundle, cf. [26]. One of key points of Chern’s proof is the identification of forms on the base manifold with horizontal forms on the total space $B$ of the $G$-bundle. The previous remark about foliations ensures that basic forms on $(M, F)$ correspond to horizontal basic forms on $B((M, F), F_G)$. Having that in mind Chern’s proof can be easily adapted to the foliated case. For reader’s convenience we recall the main steps of the proof.

Since the canonical torsionless connection of $B((M, F), F_G)$ is foliated the structure equation for the canonical 1-form $\theta$ can be written as

$$d\theta^i = -\sum_{\rho, a} a_{\rho a}^i \theta^a \wedge \pi^\rho$$

(2)

where $\theta = (\theta^1, \ldots, \theta^q)$ is a local representation of the canonical 1-form of the foliated bundle $B((M, F), F_G)$ and $\pi^i$ is the set of linearly independent left-invariant 1-forms of $G$ lifted to the total space of the $G$-bundle. The forms $\theta^i$ can be chosen to be orthogonal.

The foliated curvature form of the foliated connection can be written as

$$\Omega^i = \frac{1}{2} \sum_{j,a} R_{ija}^j \theta^j \wedge \theta^a, \quad R_{ija}^j = -R_{aij}^j.$$ 

Let us next introduce new tensors which shall be of use later on

$$S_{jal}^i = \sum_{\rho} a_{\rho j}^i R_{jal}^\rho.$$ 

(3)

Since $G$ is a subgroup of $O(q)$, there exists a metric and using it we can lower or raise the indices, so we shall from now on use only subscripts and we have the following relations

$$S_{ijal} = -S_{jial} = -S_{ijla},$$

(4)
Let $\alpha$ be a basic $k$-form which can be written as

$$
\alpha = \frac{1}{k!} \sum_{i_1, \ldots, i_k} \alpha_{i_1 \ldots i_k} \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}.
$$

(6)

where $(\theta^1, \ldots, \theta^q)$ is the local basis and $\alpha_{i_1 \ldots i_k}$ is anti-symmetric in any two of its indices. In order to compute the basic Laplacian $\Delta_b \alpha$, initially we need to compute $d\alpha$ and to do this, first using (2) and differentiating the terms in (6) we get the relation

$$
d\alpha_{i_1 \ldots i_k} + \sum_{l=1}^{k} \sum_{\rho} \alpha_{i_1 \ldots i_{l-1} j i_{l+1} \ldots i_k} a^j_{\rho l} \pi^\rho = \sum_m \alpha_{i_1 \ldots i_k} |m \theta^m}.
$$

(7)

the notation used for the right-hand side is originally used by Chern and almost resembles the notation usually used for the derivative of the coefficients in an ordinary exterior derivative. Now we can write the exterior derivative as

$$
d\alpha = \frac{1}{k!} \sum_{i_1, \ldots, i_k, j} \alpha_{i_1 \ldots i_k} |j \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}.
$$

Next to have anti-symmetric coefficients we write it as

$$
d\alpha = \frac{(-1)^k}{(k+1)!} \sum_{i_1, \ldots, i_k+1} (\alpha_{i_1 \ldots i_k} |i_k+1 \alpha_{i_k+1 \ldots i_k} \theta^{i_1} \wedge \ldots \wedge \theta^{i_k+1}.
$$

We also have

$$
\star \alpha = \frac{1}{k!(q-k)!} \sum_{i_1, \ldots, i_q} \epsilon_{i_1 \ldots i_k \ldots i_q} \alpha_{i_1 \ldots i_k} \theta^{i_k+1} \wedge \ldots \wedge \theta^{i_q}.
$$

where $\epsilon_{i_1 \ldots i_q}$ is equal to +1(-1) if $i_1, \ldots, i_q$ form an even(odd) permutation of $1, \ldots, q$ and is otherwise equal to zero.

Taking into account that for a taut foliation $\mathcal{F}$ leaves are minimal submanifolds and hence $\kappa = 0$, we can write the adjoint $\delta_b$ of $d$

$$
\delta \alpha = \frac{(-1)^k}{(k-1)!} \sum_{i_1, \ldots, i_{k-1}, j} \alpha_{i_1 \ldots i_{k-1} j} \theta^{i_1} \wedge \ldots \wedge \theta^{i_{k-1}}.
$$

Further differentiating (7) with some extra steps taken gives us

$$
\alpha_{i_1 \ldots i_k} \theta^{i_1} \wedge \ldots \wedge \theta^{i_k} = \sum_l \sum_j \alpha_{i_1 \ldots i_{l-1} j i_{l+1} \ldots i_k} S_{i_l j i_{l+1} \ldots i_k} S_{i_l j i_{l+1} \ldots i_k}
$$

and this formula helps us to shorten the expression for the basic Laplacian given as follows

$$
\Delta_b \alpha = \frac{1}{k!} \sum_{i_1, \ldots, i_k, j} \alpha_{i_1 \ldots i_k} |j \theta^{i_1} \wedge \ldots \wedge \theta^{i_k} - \frac{1}{(k-1)!} \sum_{i_1, \ldots, i_k, r, j} \alpha_{i_1 \ldots i_{k-1} r} S_{i_k j i_{k+1} \ldots i_k} \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}
$$
\[ + \frac{1}{(k-2)!} \sum_{i_1, \ldots, i_k, r, j} \alpha_{i_1 \ldots i_{k-2} r j} S_{r i_{k-1} j i_k} \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}. \]

Now we are facing an issue which is the coefficients in this formula are not anti-symmetric. In order to solve this issue we need to introduce the following quantities

\[
S(i_1 \ldots i_k, j_1 \ldots j_k; r_1 \ldots r_k, m_1 \ldots m_k) = \sum \epsilon(i_1 \ldots i_k; n_1 \ldots n_{k-1} g) \times \epsilon(j_1 \ldots j_k; n_1 \ldots n_{k-1} h) \epsilon(r_1 \ldots r_k; l_1 \ldots l_{k-1} u) \epsilon(m_1 \ldots m_k; l_1 \ldots l_{k-1} v) S_{ghuv}
\]

where \( \epsilon(i_1 \ldots i_k; j_1 \ldots j_k) \) is +1 or −1 if \( j_1 \ldots j_k \) is an even or odd permutation of \( i_1 \ldots i_k \) respectively and is zero otherwise, moreover all the indices run from 1 to \( q \) and the summation is over all the repeated ones. For the sake of brevity we will use the notation \( S((i)(j); (r)(m)) \) instead of (8), using which we define yet another quantity as follows

\[
S((i)(r)) = S(i_1 \ldots i_k, r_1 \ldots r_k) = \frac{1}{k!} \sum_{j_1 \ldots j_k} S(i_1 \ldots i_k, j_1 \ldots j_k; r_1 \ldots r_k, j_1 \ldots j_k),
\]

which is anti-symmetric in the indices belonging to each one of the sets \( i_1, \ldots, i_k \) and \( r_1, \ldots, r_k \) and \( S((i)(r)) = S((r)(i)) \). Using (9) we are able to rewrite the expression for the basic Laplacian

\[
\Delta_b \alpha = -\frac{1}{k!} \sum_{i_1, \ldots, i_k, j} \alpha_{i_1 \ldots i_k j} \theta_{i_1} \wedge \ldots \wedge \theta_{i_k}
- \frac{1}{(k!(q-k)!)^2} \sum_{i_1, \ldots, i_k, r_1, \ldots, r_k} \alpha_{i_1 \ldots i_k} S(i_1, \ldots, i_k, r_1, \ldots, r_k) \theta^{r_1} \wedge \ldots \wedge \theta^{r_k}
\]

Let \( \theta^{i_1} \wedge \ldots \wedge \theta^{i_k} \) be the basis of \( \bigwedge^k (R^{q*}) \) where \( 1 \leq i_1 < \ldots < i_k \leq q \). Suppose \( W_1 \) is an invariant subspace of \( \bigwedge^k (R^{q*}) \) under the standard action of \( G \) and \( W_2 \) be its orthogonal space, then \( W_2 \) is also invariant under this action. There exist base vectors \( \gamma_1, \ldots, \gamma_K \subset \bigwedge^k (R^{q*}) \), \( K = \binom{q}{k} \) which are related to \( \theta^{i_1} \wedge \ldots \wedge \theta^{i_k} \) via an orthogonal transformation

\[
\theta^{i_1} \wedge \ldots \wedge \theta^{i_k} = \sum_{\mu=1}^{K} g_{i_1 \ldots i_k, \mu} \gamma_{\mu},
\]

such that \( \gamma_1 \ldots \gamma_{\nu} \) and \( \gamma_{\nu+1} \ldots \gamma_K \) span the subspaces \( W_1 \) and \( W_2 \) respectively. From (11) one can obtain

\[
\gamma_{\mu} = \frac{1}{k!} \sum_{i_1, \ldots, i_k} g_{i_1 \ldots i_k, \mu} \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}
\]

Using (2) we compute the exterior derivative

\[
d \gamma_{\mu} = \frac{1}{(k-1)!} \sum_{i_1, \ldots, i_k, m, \rho} g_{i_1 \ldots i_{k-1}, m, \rho} a_{m \rho i_k} \pi^\rho \wedge \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}
= \frac{1}{(k-1)!} \sum_{i_1, \ldots, i_{k-1}, m, A, \lambda, \rho} g_{i_1 \ldots i_{k-1}, m, \mu} g_{i_1 \ldots i_{k-1}, A, \lambda} a_{m \rho A} \pi^\rho \wedge \gamma^\lambda,
\]
here $1 \leq \mu \leq \nu$.

The condition of $W_1$ being invariant under the action of $G$ can be obtained as

$$\sum_{i_1,\ldots,i_{k-1},m,A} g_{i_1\ldots i_{k-1},m,A}g_{i_1\ldots i_{k-1},A,\zeta}S_{m,\lambda_{j_{\nu}}} \nu + 1 \leq \zeta \leq K. \quad (12)$$

Now suppose that $\alpha$ is a basic $k$-form of type $W_1$ and write it as

$$\alpha = \sum_{\mu} \alpha_{\mu} \gamma_{\mu} = \frac{1}{k!} \sum_{i_1,\ldots,i_{k-1},\mu} \alpha_{\mu} g_{i_1\ldots i_{k-1},\mu} \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}.$$

and the basic Laplacian is the following

$$\Delta_b \alpha = -\sum_{\mu \mu j} \alpha_{\mu} \gamma_{\mu} - \frac{1}{(k!(k-1)!)^2} \sum_{\mu} \alpha_{\mu} g_{i_1\ldots i_{k-1},\mu} g_{r_1\ldots r_k,\lambda} S((i)(r)) \gamma_{\lambda}. \quad (12)$$

Therefore in order for $\Delta_b \alpha$ to be of type $W_1$ the second term needs to vanish for $\nu + 1 \leq \lambda \leq K$, which follows from the properties of $S((i)(r))$ and (12). \qed

Let $\{e_i\}_{i=1,\ldots,q}$ be a local orthonormal frame of the normal bundle $Q$ and $\{e^i\}_{i=1,\ldots,q}$ be its dual coframe. Let $\alpha \in A^k(M,F)$ be a basic $k$-form, the exterior derivative preserves the basic forms and the restriction to basic forms denoted by $d_b = d|_{A^k(M,F)}$ is well-defined. It is possible to write $d_b$ and its adjoint $\delta_b$ for $\kappa = 0$ as follows

$$d_b = \sum_i e^i \wedge \nabla e_i, \quad \delta_b = -\sum_j i_{e_j} \nabla e_j, \quad i, j = 1,\ldots,q. \quad (13)$$

For a basic $k$-form $\alpha$ let us define a normal $k$-tensor field $\rho$ on $(M,F)$ given at each point $x \in Q$ by

$$\rho_\alpha(X_1,\ldots,X_k) = \sum_{i=1}^q \sum_{j=1}^k (R(e_i,X_j)\alpha)(X_1,\ldots,X_{j-1},e_i,X_{j+1},\ldots,X_k), \quad (14)$$

where $X_j \in (\Gamma Q)_x$ and $R$ is the normal Riemann curvature operator. Using (13) one can get the Weitzenböck formula for the Laplacian of a basic $k$-form $\alpha$

$$\Delta \alpha = -\nabla \nabla \alpha + \rho_\alpha, \quad (15)$$

for more details see [20].

The main result of A. Lichnerowicz in [15] has its foliated counterpart, however, first we need to show that (15) is equal to the Laplacian used in [15]. A careful look at how the Riemann curvature operator acts on forms, one can from (14) obtain

$$\rho_\alpha(X_1,\ldots,X_k) = -\sum_{i,j} \sum_{l \neq j} \alpha(X_1,\ldots,R(e_i,X_j)X_l,\ldots,e_i,\ldots,X_k) \quad (16)$$
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\[- \sum_{i,j} \alpha(X_1, \ldots, R(e_i, X_j)e_i, \ldots, X_k).\]

Now if we apply the Ricci identity we get

\[
\rho_\alpha(X_1, \ldots, X_k) = - \sum_{i,j,l} R(e_i, X_j, e_l, X_l) \alpha(X_1, \ldots, e_l, \ldots, X_k).
\]

(17)

where in the second line we have used the fact that

\[
R(e_i, X_j, e_l, e_l) = -R(e_i, X_j, e_l, e_l) = -Ric(X_j, e_l) = -Ric(e_l, X_j).
\]

Let \( \nabla \) be a torsionfree foliated connection in \((N(M, F), F_N)\), if one replaces \( \rho_\alpha \) in (15) with the one in (17), it is clear to be the same as the Laplacian used by Lichnerowicz in [15], hence the basic Laplacian of a normal \( p \)-tensor field can be written locally using the Ricci tensor and Riemann curvature tensor of the foliated manifold

\[
\Delta_b U_{\alpha_1 \ldots \alpha_p} = - \nabla^i \nabla_i U_{\alpha_1 \ldots \alpha_p} + \sum_j R_{\alpha_j A} U_{\alpha_1 \ldots \alpha^A \ldots \alpha_p} - \sum_{j \neq k} R_{\alpha_j A, \alpha_k B} U_{\alpha_1 \ldots \alpha^A \ldots \alpha^B \ldots \alpha_p}.
\]

Theorem 6 Let \( T \) be a normal tensor field. If the covariant derivative of the normal tensor field \( T \) vanishes, then for any normal tensor field \( U \) the following holds

\[
\Delta_b(T \otimes U) = T \otimes \Delta_b U.
\]

Proof Let \( T \) and \( U \) be two normal tensor fields with the ranks \( p \) and \( q \) respectively. Applying the basic Laplacian on the tensor product \( T \otimes U \) and using \( \nabla T = 0 \), we obtain

\[
\Delta_b(T \otimes U) = T \otimes \Delta_b U + V \otimes U + W.
\]

where the tensors \( V \) and \( W \) are

\[
V_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} = \sum_j \sum_A R_{\mu_j A} T_{\mu_1 \ldots A \ldots \mu_p \nu_1 \ldots \nu_q} - \sum_{j \neq k} \sum_{A,B} R_{\mu_j A, \mu_k B} T_{\mu_1 \ldots A \ldots B \ldots \mu_p \nu_1 \ldots \nu_q},
\]

\[
W_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} = - \sum_{j,k} \sum_{A,B} R_{\mu_j A, \mu_k B} T_{\mu_1 \ldots \nu_1 \ldots \nu_q}.\]
So the aim is now to show that $W$ and $V \otimes U$ vanish. Applying the Ricci identity on the normal tensor $T$ we get

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)T_{\mu_1...\mu_p} = \sum_j \sum_A R_{\mu_j A,\alpha\beta} T_{\mu_1...A...\mu_p} = 0$$

which implies that $W$ vanishes. The basic Laplacian commutes with the contraction and if in (18) we contract $\mu_j$ with $\beta$ and change $\alpha$ to $\mu_k$ we obtain

$$\sum_A R_{\mu_k A} T_{\mu_1...A...\mu_p} = \sum_j \sum_{A,B} R_{\mu_k A,\mu_k B} T_{\mu_1...A...B...\mu_p} = 0$$

which results in $V = 0$ and this completes the proof. □

As a consequence we obtain the following corollary.

**Corollary 1** If $T$ is a linear mapping of the module of normal $r$-tensor fields into normal $s$-tensor fields defined by a tensor field $T$ with $\Delta_b T = 0$, then $T$ commutes with $\Delta_b$.

Kraines noticed that the Chern decomposition theorem, see [7], can be applied in the context she studied. Thus a harmonic form $\omega$ can be represented as

$$\omega = \sum_{i=0}^{[p/4]} L^i \omega_e^{p-4i},$$

and then the forms $L^i \omega_e^{p-4i}$ must be harmonic.

The original Chern’s proof of the decomposition theorem is just a very subtle linear algebra plus the Hodge decomposition theorem for differential forms. The theory of harmonic basic forms for compact Riemannian foliated manifolds permits to extend the result to basic forms. Therefore we have the basic counterpart of Kraines’ Theorem 3.6.

**Theorem 7** Let $(M,F)$ be a compact Riemannian foliated manifold of codimension $4q$. If the foliation $F$ is cohomologically taut and transversely quaternionic Kähler then the basic Betti numbers $B^i_F$ of $(M,F)$ satisfy the inequalities:

$$B^i_F \leq B^{i+4}_F \leq \ldots \leq B^{i+4r}_F$$

for $i + 4r \leq q + 1$, $i = 0, 1, 2$ or 3.

**Proof.** Since the basic 4-form $\Omega$ is invariant under $G$, so are the subspaces $L^i \bigwedge_e^{p-4i}$ of $\bigwedge^p$. Therefore they can be written as sum of the subspaces...
$W_1, \ldots, W_s$ and the projection of a harmonic form into these subspaces is again harmonic.

The previous results of the section combined with the proof provided in \cite{10} allow us to formulate the following foliated version of Theorem 3.22 of \cite{10}.

\textbf{Theorem 8} Let $(M, g, Q, \mathcal{F})$ be a cohomologically taut quaternion Kähler foliated manifold of codimension $4q$. Then

1. for any $k < q$ the linear map $L: H^k(M, \mathcal{F}) \to H^{k+4}(M, \mathcal{F})$ is injective,
2. and there is the direct sum decomposition

$$H^k(M, \mathcal{F}) = \sum_{0 \leq s \leq \lfloor k/4 \rfloor} L^s H^{k-4s}(M, \mathcal{F}), \ k \leq q + 3.$$ 

\section{Example - 3-Sasakian manifolds}

In the paper \cite{10} A. Fujiki presents a generalization of Kraines’s Theorem, cf. Theorem 3.22. The proof uses two important facts, namely that the total space of the twistor bundle is a compact Kähler manifold and that the pull-back mapping induces an injection of the cohomology of the base into the cohomology of the total space. The fact is due to the Leray-Hirsch theorem. It has been very tempting to formulate and prove a foliated version of Fujiki’s theorem. A careful reading of Salomon’s theorem, cf. Theorem 6.1, \cite{21}, confirms that the canonical foliation of transversely Kähler. One should also mentions that a foliated version of the Calabi conjecture has been demonstrated by Aziz El Kacimi, cf. \cite{8}. However, there is not any version of the Leray-Hirsch theorem for foliated bundles and their basic cohomology.

On the other hand, it is possible for one interesting class of foliated manifolds, namely 3-Sasakian manifolds. Indeed, their canonical foliation is transversely quaternion. The main reference for the theory of 3-Sasakian manifolds is Ch. Boyer’s and K. Galicki’s book \cite{5} or their paper \cite{6}. Let us recall some useful fact about 3-Sasakian manifolds.

i) The characteristic foliation of a 3-Sasakian manifold is transversely quaternion Kähler, cf. \cite{6} Theorem 2.8.

ii) The twistor space of a 3-Sasakian manifold is a trivial bundle as a foliated bundle due to the fact on the normal bundle of the characteristic foliation we have 3 linearly independent foliated almost-complex structures, cf. \cite{6}, pp.190 ff.

iii) The characteristic foliation of a 3-Sasaki manifold is defined by an isometric action of the group $S^3$. Therefore the foliation is totally geodesic, and in particular taut. Consequently the Poincaré duality holds for the basic cohomology of this foliation, cf. \cite{9} and \cite{16}.

The above mentioned results combined with the proof provided in \cite{10} allow us to formulate and prove the following version of Theorem 3.23 of \cite{10} in the case of a 3-Sasakian manifold.
Theorem 9 Let $M$ be a compact 3-Sasakian manifold of dimension $3 + 4q$ with positive normal scalar curvature and characteristic foliation $\mathcal{F}$. Then

1. for any $0 \leq k < q$ the induced linear map $L^{q-k}: H^{2k}(M,\mathcal{F}) \to H^{4q-2k}(M,\mathcal{F})$ is an isomorphism.
2. for any $k \geq 0$ we have the direct sum decomposition $H^{2k}(M,\mathcal{F}) = \sum_r L^r H^{2k-4r}(M,\mathcal{F}), (k-q) \leq r \leq [k/2]$.

10 Orbifolds

In 1956, I. Satake [22] introduced a new generalization of the notion of manifolds that he named $V$-manifolds. Currently due to W. Thurston [23], they are known as orbifolds and have applications in both mathematics and physics, especially in the string theory. It is a well-known result that having a Riemannian foliation with compact leaves, its leaf space can be given a structure of an orbifold and any orbifold can be realized as the leaf space of a Riemannian foliation, cf. [12]. This enables us to reformulate some of our results for orbifolds. In this section we follow the notations and borrow some notions from [27], which can be consulted for more discussions on the subject.

Let $X$ be a topological space, $\tilde{U} \subset \mathbb{R}^n$ be a connected open subset, $\Gamma$ be a finite group of smooth diffeomorphisms of $\tilde{U}$, and $\phi: \tilde{U} \to X$ be a map which is $\Gamma$-invariant and induces a homeomorphism of $\tilde{U}/\Gamma$ onto an open subset $U \subset X$. The triple $(\tilde{U}, \Gamma, \phi)$ is called an $n$-dimensional orbifold chart on $X$.

An embedding $\lambda: (\tilde{U}, \Gamma, \phi) \to (\tilde{V}, \Delta, \psi)$ between two orbifold charts is a smooth embedding $\lambda: \tilde{U} \to \tilde{V}$ which satisfies $\psi \circ \lambda = \phi$.

Let $\mathcal{A} = \{(\tilde{U}_i, \Gamma_i, \phi_i)\}_{i \in I}$ be a family of such charts, it is called an orbifold atlas on $X$, if it covers $X$ and any two charts are locally compatible in the following sense: given two charts $\{(\tilde{U}_i, \Gamma_i, \phi_i)\}_{i=1,2}$ and $x \in U_1 \cup U_2$, there exists an open neighborhood $U_3 \subset U_1 \cup U_2$ containing $x$ and a chart $(\tilde{U}_3, \Gamma_3, \phi_3)$, $U_3 = \phi_3(\tilde{U}_3) \subset X$ such that it can be embedded into the other two charts. As in the case of manifolds, one can define a maximal atlas.

Definition 3 A Hausdorff paracompact topological space $X$ together with a maximal orbifold atlas $\mathcal{A}$ is called a smooth $n$-dimensional orbifold.

Let $X$ be the orbifold associated to the foliated Riemannian manifold $(M,\mathcal{F})$ with compact leaves. The foliated geometrical structures on $M$ are in one-to-one correspondence with orbifold geometrical structures on $X$, e.g., any bundle-like Riemannian metric of $M$ induces an orbifold Riemannian metric on $X$ and vice versa, cf. [27].

Definition 4 Let $F$ be a smooth manifold. An orbifold $E$ is called an orbifold frame bundle over the orbifold $X$ with standard fiber $F$ if

i) there exists a smooth orbifold map $p: E \to X$,
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ii) there exists an orbifold atlas \( A = \{ (\tilde{U}_i, \Gamma_i, \phi_i) \} \) of \( X \),

iii) let \( V_i = p^{-1}(U_i) \) and \( \tilde{V}_i = \tilde{U}_i \times F \), then there exist a group \( \Lambda_i \) of fiber preserving diffeomorphisms of \( \tilde{V}_i \) and a homeomorphism \( \psi_i : \tilde{V}_i / \Lambda_i \to V_i \) such that \( \{ (\tilde{V}_i, \Lambda_i, \psi_i) \} \) form an atlas of the orbifold \( E \)

iv) and the following diagram is commutative

\[
\begin{array}{ccc}
\tilde{U}_i \times F & \xrightarrow{\tilde{p} = p \times id} & \tilde{U}_i \\
\downarrow & & \downarrow \\
\tilde{V}_i / \Lambda_i & \xrightarrow{\phi_i} & \tilde{U}_i / \Gamma_i \\
\downarrow & & \downarrow \\
V_i & \xrightarrow{p} & U_i \\
\end{array}
\]

The tangent bundle of an orbifold can be constructed as follows. Let \( \{ (\tilde{U}_i, \Gamma_i, \phi_i) \} \) be the orbifold atlas on \( X \). Take \( \tilde{V}_i = T\tilde{U}_i = \tilde{U}_i \times \mathbb{R}^n \) and for the group \( \Sigma_i \) of local transformations take \( \Sigma_i = \{ d\gamma : \gamma \in \Gamma_i \} \) and the quotient map \( \tilde{V}_i \to \tilde{V}_i / \Sigma_i \) can be taken as \( \psi_i \). It can be shown that \( \{ (\tilde{V}_i, \Sigma_i, \psi_i) \} \) is an orbifold atlas for \( TX \).

Denote by \( L(X) \) the linear frame bundle of the orbifold \( X \). It is constructed similarly as the tangent bundle \( TX \) with fiber \( F \) now being \( GL(n) \) instead of \( \mathbb{R}^n \). It is well-known that \( L(X) \) is in fact a manifold and many of the geometrical structures on \( X \) can be realized as its reduction, e.g., having an orbifold Riemannian metric on \( X \) is equivalent to a choice of an orbifold \( O(n) \) reduction of \( L(X) \). In particular an orbifold quaternionic Kähler structure on \( X \) can be introduced by an orbifold \( Sp(n).Sp(1) \) reduction of \( L(X) \). One can also construct dual vector bundle of any orbifold vector bundle, more details can be found on another article written by the second author [27]. Moreover, the tensor product, skewsymmetric product and exterior product of orbifold vector bundles over a given orbifold can be defined and an orbifold differential \( k \)-form on the orbifold \( X \) can be taken as the section of \( \bigwedge^k T^* X \), i.e., the \( k \)-th exterior power of the cotangent bundle of \( X \).

Let \( \Omega^k(X) \) be the space of all orbifold differential \( k \)-forms on the orbifold \( X \), having the differential \( d : \Omega^k(X) \to \Omega^{k+1}(X) \), one can define the orbifold de Rham cohomology group \( H^*_DR(\Omega^* , d) \) similarly as for manifolds, which for the sake of brevity from now on we shall use \( H^*_DR(X) \), for more details see [1].

**Theorem 10** (Theorem 2.15 [17]) Let \( \mathcal{F} \) be a foliation of codimension \( q \) of a manifold \( M \) such that any leaf of \( \mathcal{F} \) is compact with finite holonomy group. Then the space of leaves \( M / \mathcal{F} \) has a canonical structure of dimension \( q \). The isotropy group of a point in \( M / \mathcal{F} \) is its holonomy group.

An effective orbifold differential \( k \)-form can be defined similarly as for manifolds and we denote its orbifold cohomology group by \( H^*_{e,DR}(X) \). It can be
shown that if $X$ is the orbifold associated to the foliated Riemannian manifold $M$, the de Rham orbifold cohomology group on $X$ is actually isomorphic to the basic cohomology group of $M$, hence, we can reformulate Theorem 8.

**Theorem 11** Let $(M, g, Q, \mathcal{F})$ be a cohomologically taut quaternion Kähler foliated manifold of codimension $4q$ and let $X$ be its associated orbifold with its de Rham cohomology group denoted by $H^*_{DR}(X)$. Then

1. for any $k < q$ the linear map $L: H^k_{DR}(X) \rightarrow H^{k+4}_{DR}(X)$ is injective,
2. and there is the direct sum decomposition $H^k_{DR}(X) = \sum_{0 \leq s \leq [k/4]} L^s H^{k-4s}_{DR}(X), \; k \leq q + 3.$

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