TAUBERIAN KOREVAAR

N. H. BINGHAM

Abstract We focus on the Tauberian work for which Jaap Korevaar is best known, together with its connections with probability theory. We begin (§1) with a brief sketch of the field up to Beurling’s work. We follow with three sections on Beurling aspects: Beurling slow variation (§2); the Beurling Tauberian theorem for which it was developed (§3); Riesz means and Beurling moving averages (§4). We then give three applications from probability theory: extremes (§5), laws of large numbers (§6), and large deviations (§7). We turn briefly to other areas of Korevaar’s work in §8. We close with a personal postscript (whence our title).

Keywords Tauberian theorem, Beurling slow variation, Beurling Tauberian theorem, Beurling moving average, Riesz mean, large deviations

MSC Classification 01A70, 40E05

1. Tauberian theorems: from Tauber to Beurling

The origins of Tauberian theory can be traced to Abel’s continuity theorem of 1826, which we teach to students. In 1828, Abel described divergent series as the invention of the devil, and denounced any use of them. But, it was found that some divergent series could be made convergent by suitable smoothing or averaging, leading to what we now call summability theory (see e.g. [ZelB1970]). Then in 1897 Tauber [Tau1897] gave the first of what Hardy and Littlewood later named a Tauberian theorem: a ‘corrected converse’ leading from convergence of a smoothed sequence to that of the original sequence, under a suitable condition, a Tauberian condition. In his Preface to Hardy’s Divergent series [Har1949], Littlewood comments of the resulting theory that ‘in the early years of the century the subject, while in no way mystical or unrigorous, was regarded as sensational, and about the present title, now colourless, there hung an aroma of paradox and audacity’.

At first, attention focused on special methods of summation, such as the Abel, Cesàro, Euler and Borel [Har1949, I-XI]. One culminating result was the Hardy-Littlewood-Karamata (Tauberian) theorem of 1930 for Laplace transforms [Kor2004, I]. The whole area changed with the introduction of Wiener’s general Tauberian theory [Wie1932]; [Har1949, XII], [Kor2004, II]. One of Korevaar’s early contributions here was his strikingly short proof
of Wiener’s Tauberian theorem by Schwartz distributions (generalised functions) [Kor1965].

The Borel (exponential) method $B$ of summability is defined by writing

$$s_n \to s \quad (n \to \infty) \quad (B)$$

(handling series $\sum a_n$ and sequences $s_n$ together by $s_n := \sum_0^n a_k$) for

$$e^{-x} \sum_0^\infty s_n x^n/n! \to s \quad (x \to \infty).$$

This method is useful for analytic continuation of power series [Har1949, VIII, IX], [Kor2004, VI] and in number theory [Ten1980]. It is also useful in probability theory, since the weights $e^{-x}x^n/n!$ are those of a Poisson law $P(x)$ with parameter (mean, variance) $x$. It is intimately linked with the Valiron method $V_1/2$ [BinT1986] (or just $V$ here for convenience, defined by writing

$$s_n \to s \quad (n \to \infty) \quad (V)$$

for

$$\frac{1}{\sqrt{2\pi x}} \sum_0^\infty s_n \exp\{-\frac{1}{2}(x-n)^2/x\} \to s \quad (x \to \infty) \quad [HarL1916].$$

Such a link is probabilistically indicated by the central limit theorem, as for large $x$ the Poisson law approximates a normal, $P(x) \sim N(x, x)$.

2. Beurling slow variation

For reasons we turn to below regarding his Tauberian theorem, Beurling (in unpublished lectures of 1957, published by others in 1972 [Kor2004, §IV.11]) introduced what is now called Beurling slow variation. This is a variant on Karamata slow variation (for which see e.g. [BinGT1987], ‘BGT’), and as there can be formulated for measurable functions (as was done by Korevaar et al. [KorvA1949]), or topologically for functions with the Baire property [Mat1964] (in brief, Baire functions), and indeed more generally.

A positive Baire or measurable (Baire/measurable below) function $\phi$ on $\mathbb{R}_+$ is called Beurling slowly varying if $\phi(x) = o(x)$ as $x \to \infty$ and

$$\phi(x + t\phi(x))/\phi(x) \to 1 \quad (x \to \infty) \quad \forall t. \quad (BSV)$$

Writing

$$x \circ_\phi t := x + t\phi(x),$$

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this is
\[ \phi(x \circ_t f(x) \to 1 \quad \forall t. \]

In recent papers (see e.g. [BinO14], [BinO16], [BinO20a]) the author and Ostaszewski take the view that it is (Baire) category rather than measure which is the principal case, the reverse of the chronological order, whence our ordering above.

By Bloom's theorem [Blo1976], for \( \phi \) Beurling slowly varying (\( \phi \in \text{BSV} \)) and continuous, the convergence in (BSV) is locally uniform (uniform on compact \( t \)-sets in \( \mathbb{R}_+ \)). Continuity is weakened to the Darboux (or intermediate-value) property in [BinO2014]. There, a number of properties are shown to be sufficient, including monotonicity, but whether continuity may simply be dropped here remains open.

When the convergence in (BSV) is is locally uniform, \( \phi \) is called self-neglecting, \( \phi \in \text{SN} \).

For \( \phi \in \text{SN} \), a Baire/measurable \( f \) is Beurling regularly varying [BinO2014] with limit \( g \) if
\[ f(x + t\phi(x))/f(x) \to g(t) \quad (x \to \infty) \quad \forall t. \quad (BRV) \]

The condition \( \phi(x) = o(x) \) may be usefully weakened to \( \phi(x) = O(x) \) [Ost2015]. For \( \phi(x) = O(x) \), call \( \phi > 0 \) self-equivarying with limit \( \lambda \) if
\[ \phi(x + t\phi(x))/\phi(x) \to \lambda(t) \quad \text{locally uniformly.} \quad (SE) \]
The limit \( \lambda \) then satisfies the Golab-Schinzel functional equation
\[ \lambda(x)\lambda(y) = \lambda(x + y\lambda(x)) \quad \forall x, y. \quad (GS) \]
It is continuous, and of the form
\[ \lambda(t) = 1 + at \]
for some \( a \geq 0 \). To within re-scaling, there are only two cases: the ‘small-order limit’ \( \lambda(t) \equiv 1 \), and ‘large-order limit’ \( \lambda(t) = 1 + t \) [Ost2015, Th. 0].

We note that as \( \phi(x) = O(x) \), \( \int_1^x \frac{dt}{\phi(t)} \) diverges (logarithmically) for large \( x \). Its unboundedness will be important in §4 in connection with Riesz means.

3. Beurling’s Tauberian theorem

The Borel method [Bor1899], while very useful and interesting, is notably
more difficult to handle than most of the other classical summability methods in common use (Abel, Cesàro, Euler etc.), witness its extended treatment in [Har1949] and [Kor2004]. Beurling’s Tauberian theorem, to which we now turn, was developed to extend the Wiener Tauberian theorem conveniently to cover the Borel method.

Recall that $K \in L_1(\mathbb{R})$ satisfies the Wiener condition, or is a Wiener kernel, if its Fourier transform $\hat{K}(t)$ has no real zeros $t$. For $H$ bounded and $F \in L_1(\mathbb{R})$, define the Beurling convolution (w.r.t $\phi$ as in (BSV)) by

$$F *_\phi H(x) := \int_{\mathbb{R}} F\left(\frac{x-u}{\phi(x)}\right) H(u) \frac{du}{\phi(x)} = \int F(-y) H(x \circ_\phi t) dt.$$  

**Theorem B (Beurling’s Tauberian theorem, 1957).** For $K$ a Wiener kernel, $\phi$ Beurling slowly varying: if $H$ is bounded and

$$K *_\phi H(x) \to c \int K(y) dy \quad (x \to \infty),$$

then for all $F \in L_1(\mathbb{R}),$

$$F *_\phi H(x) \to c \int F(y) dy \quad (x \to \infty).$$

The case $\phi \equiv 1$ gives the Wiener Tauberian theorem, Theorem W say ([Wie1932]; BGT §4.8, [Kor2004,II]):

**Wiener’s Tauberian theorem, 1932.** Theorem B holds with $\phi$-convolution $*_\phi$ replaced by ordinary convolution $*$.

Recall that Theorem W is proved from the Wiener approximation theorem (or closure theorem): $K$ is a Wiener kernel if and only if linear combinations of its translates $K(a + \cdot)$ are dense in $L_1(\mathbb{R})$ (see e.g. [Kor2004, V.3]). Korevaar [Kor2004, IV.11] gives a very short proof of Theorem B by reducing it to the Wiener approximation theorem.

‘Wiener’s second Tauberian theorem’, involving Lebesgue-Stieltjes integrals [Kor2004, II.13], also has a Beurling counterpart [BinO16, §4].

4. Riesz means and Beurling moving averages

First, recall the Riesz (typical) means $R(\lambda, 1)$ of order 1, or simply $R(\lambda)$
here as we shall only need order 1, where $\lambda := (\lambda_n)$ is a real sequence $\lambda_n \uparrow \infty$: we write

$$s_n \to s \quad R(\lambda)$$

for

$$\frac{1}{x} \int_0^x \left( \sum_{\lambda_n \leq y} a_n \right) dy \to s \quad (x \to \infty). \quad (R(\lambda))$$

The first consistency theorem for Riesz means [ChaM1952, I] relates Riesz means of different orders, so need not detain us as we use only order 1 here. The second consistency theorem for Riesz means (Riesz in 1909 and 1916, Hardy in 1910 and 1915; see [ChaM1952, II]), important for us, tells us that two Riesz means $R(\lambda)$, $R(\mu)$ whose logarithms are comparable,

$$0 < \liminf \log \lambda_n/\log \mu_n \leq \limsup \log \lambda_n/\log \mu_n < \infty, \quad (\log \lambda)$$

are equivalent. Thus the order of magnitude of the logarithm is all that matters.

Since a $\phi \in SN$ is Beurling slowly varying, in

$$\frac{1}{t\phi(x)} \sum_{x < n \leq x + t\phi(x)} s_n \to s \quad (x \to \infty) \quad \forall t \quad (BMA)$$

the left-hand side is called a Beurling moving average. Then ([BinGo1988]; cf. [BinT1986]), writing

$$\lambda(x) := \exp \{ \int_1^x dt/\phi(t) \},$$

$(R(\lambda))$ and $(BMA)$ are equivalent.

As noted in §2, all $\int_1^x dt/\phi(t) \uparrow \infty$ as $x \to \infty$. In view of the second consistency theorem, any choice of sequence $\lambda := (\lambda_n), \lambda_n \uparrow \infty$ agreeing with the function $\lambda(x)$ above at all points $x = \lambda_n$ give equivalent Riesz means, and so we are free to choose any and then abbreviate $\lambda(.)$ to $\lambda$ without ambiguity. More is true: agreement may be weakened to approximation within reasonable limits, subject only to $(\log \lambda)$.

The title of [Bin2019] coincides with that of this section, and we can refer to it (or its ArXiv version) for more detail than we give here. It has been known since [Bin1981] that Beurling moving averages are special cases of Riesz means. In particular, [Bin2019 Th. 8.3] gives a transparent proof of
the general case of results in [BinT1986 Th. 3] and [BinGo1983, Th. 5]: that convergence as in (BMA) is equivalent to (ordinary) convergence of an $o(1)$-perturbation with rate $1/\phi(x)$. As noted in [Bin2019], this result, though transparent when seen as a representation theorem for a form (Beurling) of regular variation, has distinguished antecedents, going back to Hardy in 1904 [Har1949, Th. 149]; see [Bin2019, §2] for its full history.

5. Extremes

The mathematics (probability and statistics) of extremes is a vast and topical area, recently surveyed in [BinO2021], so we can be brief here, referring there for detail and references. The subject essentially dates from Fisher and Tippett in 1928. There, they obtained the three classical limit laws of maxima $M_n := \max\{X_1, \cdots, X_n\}$ of independent and identically distributed (iid) random variables, which are to within type (location and scale) the Fréchet (heavy-tailed, $\Phi_\alpha$, $\alpha > 0$), Gumbel (light-tailed, $\Lambda$) and Weibull (bounded tail, $\Psi_\alpha$, $\alpha > 0$). The corresponding domains of attraction (laws $F$ sampling from which can give such a limit) $D(\Phi_\alpha)$, $D(\Lambda)$, $D(\Psi_\alpha)$ are simple to describe in the Fréchet case (in terms of Karamata regular variation, ‘at infinity’) and the Weibull case (regular variation ‘at the finite end-point’). The Gumbel domain of attraction is more complicated, and best described nowadays in the language of Beurling regular variation (below). We note that Fisher and Tippett (who gave no references and merely sketches of proofs) noticed and studied in detail that for $F = \Phi$ the (standard) normal, the convergence is extremely slow – so slow that for numerical purposes it is better to avoid the ‘ultimate’ approximation (to the Gumbel) and use instead the ‘penultimate’ approximation (to the Fréchet). This was both astute theoretically and impressive numerically in the pre-computer days of desk machines (the penultimate approximation has been developed more recently by R. L. Smith and J. P. Cohen).

For simplicity, we work to within type (centre and scale, to have mean 0 and variance 1). We may then combine the three cases above into a single one-parameter family, the extreme-value distributions (EVD) $G_\alpha$, $\alpha \in \mathbb{R}$, where $\alpha > 0$ for Fréchet, $\alpha = 0$ for Gumbel, $\alpha < 0$ for Weibull, using the ‘L’Hospital convention’: as $(1 + x/n)^n \to e^x$ as $n \to \infty$, we interpret the $\alpha = 0$ case of $(1 + \alpha x)^{1/\alpha}$ as $e^x$. This gives

$$G_\alpha(x) := \exp(-g_\alpha(x)), \quad g_\alpha(x) := [1 + \alpha x]^{1/\alpha}. \quad (EVD)$$

Here the parameter $\alpha \in \mathbb{R}$ is called the extreme-value index (EVI) or extremal
The upper end-point \( x_+ \) of \( F \) is \( \infty \) for \( \alpha \geq 0 \) (with a power tail for \( \alpha > 0 \) and an exponential tail for \( \alpha = 0 \)); for \( \alpha < 0 \) \( x_+ = -1/\alpha \), with a power tail to the left of \( x_+ \).

The Gumbel domain of attraction, due to de Haan in 1970-71 (see e.g. BGT, Th. 8.13.4; cf. BGT, Ch. 3, De Haan theory) is given by \( F \in D(\Lambda) \) iff

\[
\frac{F(t + xa(t))}{F(t)} \to g_0(x) := e^{-x} \quad (t \to \infty),
\]

for some auxiliary function \( a > 0, a \in SN \), which may be taken [EmbK, (3.34)] as

\[
a(t) := \int_t^{x_+} \frac{F(u)}{F(t)} du / F(t) \quad (t < x_+), \quad (aux)
\]

and satisfies (in the usual case, \( x_+ = \infty \))

\[
a(t + xa(t))/a(t) \to 1 \quad (t \to \infty), \quad \text{locally uniformly.} \quad (Beu)
\]

The three domain-of-attraction conditions may be unified (again using the L’Hospital convention): \( F \in D(G_\alpha) \) iff

\[
\frac{F(t + xa(t))}{F(t)} \to g_\alpha(x) := (1 + \alpha x)^{-1/\alpha} \quad (t \to \infty) \quad (**)
\]

for some auxiliary function \( a \), and then

\[
a(t + xa(t))/a(t) \to 1 + \alpha x \quad (t \to \infty), \quad (\alpha Beu)
\]

extending the \( \alpha = 0 \) case (Beu) above (see e.g. [BieG, §2.6]).

As explained in [BinO2021], there is no essential loss in assuming (following von Mises) that the density \( f \) of the law \( F \) exists, in which case the inverse hazard function \( i = 1/h \) of the hazard function \( h \),

\[
i(t) := \int_t^{\infty} f(u) du / f(x),
\]

may be used as auxiliary function in place of \( a \) if preferred. So may the mean excess function

\[
e(t) := \mathbb{E}[X - t | X > t].
\]

For an application of Theorem B to records, see El Arrouchi [ElA2017].

6. Laws of large numbers

For \( X, X_1, \ldots, X_n, \ldots \) iid, we recall Kolmogorov’s classic strong law of
large numbers (his *Grundbegriffe*, 1933):

\[ \mathbb{E}[ |X| ] < \infty \; \& \; \mathbb{E}[X] = \mu \iff \frac{1}{n} \sum_{k=1}^{n} X_k \to \mu \; (n \to \infty) \; a.s. \]

The summability method used here is the Cesàro \( C_1 \). This is embedded in a one-parameter family \( \{ C_\alpha : \alpha > 0 \} \) (see e.g. [Har1949, V, VI]; \( \alpha > -1 \) is possible), ordered by inclusion: increasing \( \alpha \) sums more series but gives a weaker conclusion. Here

\[ s_n \to s \quad (C_\alpha) \]

means

\[ \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k \to s \]

(using the full range \( \alpha > -1 \) here), where

\[ A_n^\alpha := (\alpha + 1) \cdots (\alpha + n)/n! \sim n^\alpha / \Gamma(1 + \alpha) \; (n \to \infty). \]

The key role of Kolmogorov’s strong law is reflected in a striking discontinuity of the behaviour of \( C_\alpha \) here across \( \alpha = 1 \) [Bin1989, Th. 1]:

**Theorem K.** (i) For \( 0 < \alpha \leq 1 \),

\[ X_n \to \mu \; a.s. \; (C_\alpha) \iff \mathbb{E}[ |X|^{1/\alpha} ] < \infty \; \& \; \mathbb{E}[X] = \mu; \]

(ii) For \( \alpha \geq 1 \),

\[ X_n \to \mu \; a.s. \; (C_\alpha) \iff \mathbb{E}[ |X| ] < \infty \; \& \; \mathbb{E}[X] = \mu. \]

**Proof.** (i). The case \( \alpha = 1 \) is Kolmogorov’s result (so, Theorem K in his honour).

For \( \alpha \in (\frac{1}{2}, 1) \) \( (p := 1/\alpha \in (1, 2)) \), this is Theorem 3 of Lorentz [Lor1955].

For \( \alpha \in (0, \frac{1}{2}) \) \( (p := 1/\alpha > 2) \), this is the special case \( c_n = A_n^{\alpha-1} \) of Theorem 1 of Chow and Lai [ChoL1973]: in the zero-mean case, \( X \in L_{1/\alpha} \) if and only if

\[ n^{-\alpha} \sum_{k=0}^{n} c_{n-k} X_k \to 0 \; a.s. \]

for some (all) \( c = (c_n) \in \ell_2 \) (as \( \alpha - 1 < -\frac{1}{2} \) here).

For \( \alpha = \frac{1}{2} \), the result is due to Déniel and Derriennic [DenD].
(ii) This is contained in Lai’s Theorem L below [Lai1974], that in this iid setting for $\alpha \geq 1$ the Cesàro methods $C_\alpha$ are all equivalent, to each other and to the Abel method $A$. □

**Theorem L (Lai, 1974).** The following are equivalent:

$$\mathbb{E}[|X|] < \infty, \quad \mathbb{E}[X] = \mu,$$

$$X_n \to \mu \quad a.s. \quad (C_\alpha) \quad \text{for some (all) } \alpha \geq 1;$$

$$X_n \to \mu \quad a.s. \quad (A).$$

For the Euler methods $E_p$, $p \in (0,1)$ and the Borel method $B$ [Har1949, VIII, IX], one has:

**Theorem C (Chow, 1973).** The following are equivalent:

$$\text{var } X < \infty, \quad \mathbb{E}[X] = \mu,$$

$$X_n \to \mu \quad a.s. \quad (E_p) \quad \text{for some (all) } p \in (0,1);$$

$$X_n \to \mu \quad a.s. \quad (B);$$

$$X_n \to \mu \quad a.s. \quad (R(e^{\sqrt{n}}));$$

$$X_n \to \mu \quad a.s. \quad (V_{1/2}).$$

**Proof.** The first four statements are in [Cho1973], with the fourth in Beurling moving average (‘delayed sum’) language, equivalent as above to the Riesz language used here; the fifth is in [Bin1984a, Th. 3]. □

Thus the Cesàro-Abel family of summability methods corresponds in this setting to means and $L_1$, the Euler-Borel family of methods to variances and $L_2$. Similarly for the Riesz and Valiron methods (above), the circle methods (Kreisverfahren: [Har1949, IX], Meyer-König [MeyK1949]; [Bin1984b]), and the random-walk methods [Bin1984c]. These are matrix methods $A = (a_{nk})$, whose elements give the distribution of integer-valued random walks $S_n = \sum_1^n X_k$:

$$a_{nk} = P(S_n = k).$$

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When the step-length law is Poisson, this gives the discrete Borel method. Interestingly, whereas the Borel method has a pure gap (high-indices) theorem (Gaier, [Gai1953]; Turán [Tur1984]), the discrete Borel does not (Meyer-König and Zeller [MeyKZ1960]).

For general $p > 1$, one has [BinT1986, Th.5]:

**Theorem BT.** For $p > 1$, the following are equivalent:

\[ \mathbb{E}[|X|^p] < \infty, \quad \mathbb{E}[X] = \mu; \]

\[ X_n \to \mu \quad \text{a.s.} \quad (R(\exp(n^{1/(p)})); \]

\[ X_n \to \mu \quad \text{a.s.} \quad (V_{1/p}). \]

The results above assume at least as much integrability as existence of the mean. But one can also assume less [BinGa2015]; here one can no longer centre at means as means do not exist, but must use alternative centring. Recall the Lambert $W$-function, the solution to the functional equation

\[ z = W(z) \exp\{W(z)\}, \quad (W) \]

and the logarithmic summability method ($\ell$), defined by writing

\[ s_n \to s \quad (\ell) \]

for

\[ \frac{1}{\log n} \sum_{0}^{n} s_k/(k + 1) \to s \quad (n \to \infty) \]

(the method ($\ell$) is equivalent to the Riesz mean $R(\log(n + 1))$ [Har1949, Th. 37]. Note that the limit in the Riesz mean is continuous, while that in the logarithmic method is discrete. This contrast is explored in detail in [BinGa2015] (where other logarithmic methods are also treated), and later in [BinO2020b]. Discontinuous Riesz means, together with Voronoi means and non-regular (not necessarily convergence-preserving) summability methods, are considered in [BinGa2017].

**Theorem BG.** For $m_k := \mathbb{E}[X_k I(|X_k| \leq (k + 1) \log(k + 1)]$, the following are equivalent:

(i) $\mathbb{E}[\exp\{W(|X|)\}] < \infty$, i.e. $\mathbb{E}\left[\frac{|X|}{\log(|X|)}\right] < \infty$, i.e. $\mathbb{E}\left[\frac{|X|}{1 + \log_2(|X|)}\right] < \infty$;
There are numerous other equivalences; applications include the Almost-Sure Central Limit Theorem and the Prime Number Theorem.

Far-reaching generalizations of results of this type are given in [BinGa2017].

7. Large deviations

We turn now to more detailed results, in which the law $F$, equivalently its characteristic function (Fourier-Stieltjes transform) $\phi(t) := \mathbb{E}[e^{itX}] = \int e^{itx}dF(x)$, has progressively better behaviour. More moments $m_n := \mathbb{E}[X^k]$ may exist; all moments may exist; the moment sequence may uniquely determine the law $F$; $\phi$ may be analytic (in $t$, now complex, in a neighbourhood of the origin, or a strip in the complex $t$-plane – Cramér’s condition); $\phi$ may be entire. We can say things about events further from typical behaviour the more we progress up this hierarchy, hence the name large deviations for the area.

We mention first the Berry-Esseen theorem (see e.g. Petrov [Pet1975, V.2], giving a uniform (in $x$) bound between the law of the sum $S_n := \sum_1^n X_k$ when centred and scaled and the standard normal (Gaussian) $\Phi = N(0, 1)$, in terms of the third moment of $F$. Non-uniform versions are valuable for large $|x|$ [IbrL1971, §3.6], [Pet1975, V.4].

Expansions related to the central limit theorem when moments beyond the second exist are called Edgeworth expansions (from his work of 1907); see Hall [Hal1992] (who traces them back to Chebyshev in 1890 – very appropriately, as the area has been dominated by the Russian school; cf. [Bin2021]). For detailed accounts, see e.g. Feller [Fel1971, XVI], [Pet1975, VI].

Returning to the Borel method $B$ of §1: Hardy [Har1949, (9.1.8)] gives a detailed approximation of the ratio of Borel weights to Valiron. In probability language, for $X \sim P(\lambda)$, he estimates this in the range $\{|X - \mathbb{E}[X]| = O(\lambda^\zeta)\}$, for

$$\frac{1}{2} < \zeta < \frac{2}{3}$$

(Korevaar [Kor2004, 292]) – the ‘signature of large deviations’ (cf. [IbrL1971, Ch. 12]).

The Tauberian theorems of exponential type in BGT, §4.12 are of large-deviations type; cf. [Kor2004, 208, 292], [Bin2008].

When $\phi(t)$ is entire, one can work instead with the moment-generating
function $M(t) = \phi(-it) = \int e^{tx} dF(x)$, also entire, and its logarithm, the cumulant-generating function $K(t) := \log M(t)$, which is convex. So one may take its Fenchel dual (or Legendre transform), $K^*$. By Cramér’s theorem [Cra1938], writing $F_n$ for the $n$-fold convolution of $F$ (law of $S_n = \sum_1^n X_k$), for measurable sets $A$ with interior $A^\circ$ and closure $\bar{A}$,

$$-\inf \{ K^*(x) : x \in A^\circ \} \leq \liminf \frac{1}{n} \log(F_n(A)) \leq \limsup \frac{1}{n} \log(F_n(A)) \leq -\inf \{ K^*(x) : x \in \bar{A} \}.$$ 

One says that the $F_n$ satisfy a large-deviation principle with rate-function $K^*$. An extensive theory of large deviations has developed from this; see e.g. [DeuS1989], [DupE1997], [DemZ1998].

Estimating the rate of decay of exponentially small probabilities, or of the rate of occurrence of extremely rare events, is important in physics – e.g., for the half-life of radioactive elements, which determine how long radioactive contamination persists.

8. Other areas

There is a great deal more to Korevaar the mathematician than being an (or the) expert in Tauberian theorems. In preparing this centenary tribute, I came across parts of his extensive and impressive corpus that I was not (or was no longer) familiar with. I mention a few.

1. Electrostatics and potential theory. Nature likes to arrange herself so to minimise energy (cf. soldiers going from standing at attention to standing at ease, then standing easy). Korevaar studies how discrete charges approximate the continuous situation (Fekete points).

2. Gap (lacunary) series. This has been an ongoing interest of Korevaar’s. For other work, we refer to Levinson’s book [Lev1940], Mary Weiss’s paper [Wei1959], and for probabilistic aspects, Hawkes [Haw1980].

3. Pansions. These are ‘expansions’ in Hermite polynomials arising via the Fourier transform, but which are not strictly expansions in any natural sense. Korevaar [Kor1959] thus calls them pansions; Kahane, in his sympathetic review (MR0104975) in French, correspondingly abbreviates développement to ‘veloppement’. (I heard a course of lectures on pansions at the St. Andrews Colloquium in 1968 by de Bruijn, and had forgotten they stem from Korevaar; it was nice to meet them again.)
4. *Obituaries etc.* His writings on N. G. (Dick) de Bruijn (1918-2012) [Kor2013] and J. G. van der Corput (1890-1975) [Kor2015] stand out, as do his memories of his teacher H. D. Kloosterman (1900-1968) [Kor2013]. (Adam Ostaszewski and I were happy to contribute to the de Bruijn memorial issue of *Indagationes* [BinO2013].)

**Postscript: On Jaap Korevaar**

I close by reminiscing briefly about my dealings with Jaap Korevaar over the years. I fell in love with Tauberian theory in general, and the Wiener and Karamata theories in particular, as a research student in the late 60s, and began to publish on Tauberian theory in the 70s. I attended the 1979 LMS Durham Symposium on Aspects of Contemporary Complex Analysis, where he gave one of the invited talks. Early on, when we were wearing our name-badges, I was approached by a smiling, very well-preserved man (whom I now know to have been 56, to my 34), who put his hand out saying “Tauberian Bingham”. I shook it, saying “Tauberian Korevaar”. We both roared with laughter, and have been firm friends ever since. We followed each other’s work, and met intermittently at conferences. In the early years of this century, I had the great good fortune to be consulted by him while he was preparing his splendid magnum opus, his Tauberian book of 2004. I have his inscribed copy as a treasured possession. I also had the pleasure of speaking at his 80th birthday conference (my talk, 31 January 2003, ‘Tauberian theorems, Korevaar and me’), and of meeting his charming wife Pia (Pfluger, daughter of the complex analyst Albert Pfluger), alas, no longer with us.

Dr Samuel Johnson famously said that every man should strive to be an ornament to his profession. Jaap Korevaar has most certainly done that, and is an example to us all.

I thank the editors for their kind invitation to contribute to this centenary volume in his honour. It gives me great pleasure to do so.

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N. H. Bingham, Mathematics Department, Imperial College, London SW7 2AZ; n.bingham@ic.ac.uk