SCHATTEN CLASS HANKEL AND $\overline{\partial}$-NEUMANN OPERATORS ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

NİHAT GÖKHAN GÖĞÜŞ AND SÖNMEZ ŞAHUTOĞLU

ABSTRACT. Let $\Omega$ be a $C^2$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$ and let $\varphi$ be a holomorphic function on $\Omega$ that is $C^2$-smooth on the closure of $\Omega$. We prove that if $H_{\varphi}$ is in Schatten $p$-class for $p \leq 2n$ then $\varphi$ is a constant function. As a corollary, we show that the $\overline{\partial}$-Neumann operator on $\Omega$ is not Hilbert-Schmidt.

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $A^2(\Omega)$ denote the Bergman space, the set of square integrable holomorphic functions on $\Omega$. We define the Hankel operator $H_{\varphi} : A^2(\Omega) \to L^2(\Omega)$ with symbol $\varphi \in L^\infty(\Omega)$ as follows: $H_{\varphi}f = (I - P)(\varphi f)$ for $f \in A^2(\Omega)$, where $I$ is the identity map and $P : L^2(\Omega) \to A^2(\Omega)$ is the Bergman projection.

In this paper we study Schatten $p$-class membership of Hankel operators. The Hankel operator $H_{\varphi}$ is said to be in the Schatten $p$-class, $S_p$, if the operator $(H_{\varphi}^*H_{\varphi})^{p/2}$ is in the trace class, $S_1$. We recall that a self-adjoint compact operator on a separable Hilbert space is in $S_1$ if its eigenvalues are absolutely summable. We note that $S_2$ is the class of Hilbert-Schmidt operators and we refer the reader to [Zhu07] for more information about these notions.

On the unit disc, $\mathbb{D} \subset \mathbb{C}$, Arazy-Fisher-Peetre [AFP88] (see also [Zhu07, Theorem 8.29]) showed that for $\varphi \in A^2(\mathbb{D})$ the Hankel operator $H_{\varphi}$ is in the Schatten $p$-class if and only if $\varphi$ is in the Besov space $B_p$ consisting of holomorphic functions $\varphi$ on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |\varphi'(z)|^p (1 - |z|^2)^{p-2}dV(z) < \infty$$

where $dV$ is the Lebesgue measure.

In higher dimensions, that is $\Omega \subset \mathbb{C}^n$ for $n \geq 2$, the first result is due to Kehe Zhu. He [Zhu90] showed that in case $\Omega$ is the unit ball and $\varphi$ is holomorphic, $H_{\varphi} \in S_p$ for $p \leq 2n$ if and only if $\varphi$ is constant. Since then Schatten $p$-class membership of Hankel operators has been studied by many authors. For example, to list a few, it has been studied on the unit ball [Zhu91, Xia02, Pau16], strongly pseudoconvex domains [Li93], finite type pseudoconvex domains in $\mathbb{C}^2$ [KLR97], Reinhardt domains [Le14, CZ13, CZ17], and the Fock spaces [Sch04].

Date: September 13, 2017.

2010 Mathematics Subject Classification. Primary 47B35; Secondary 32W05.

Key words and phrases. Hankel operators, $\overline{\partial}$-Neumann problem, Hilbert-Schmidt, Schatten $p$-class, pseudoconvex domains.
In this paper, we study it on $C^2$-smooth bounded pseudoconvex domains in $\mathbb{C}^n$ for $n \geq 2$. Throughout the paper $O(\Omega)$ denotes the space of holomorphic functions on $\Omega$.

Our main result is the following theorem.

**Theorem 1.** Let $\Omega$ be a $C^2$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$ and $\varphi \in O(\Omega) \cap C^2(\overline{\Omega})$. Then $H\varphi$ is in $S_p$ for $p \leq 2n$ if and only if $\varphi$ is a constant function.

The following is a trivial corollary of Theorem 1.

**Corollary 1.** Let $\Omega$ be a $C^2$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$ and $\varphi \in O(\Omega) \cap C^2(\overline{\Omega})$. Then $H\varphi$ is Hilbert-Schmidt on the Bergman space $A^2(\Omega)$ if and only if $\varphi$ is a constant function.

Hankel operators, through the Kohn’s formula, are connected to the $\overline{\partial}$-Neumann operator, an important tool in several complex variables. Now we explain this connection.

Let $\overline{\partial}^\ast + \partial^\ast \overline{\partial}$ be the complex Laplacian on $L^2_{(0,1)}(\Omega)$, the square integrable $(0,1)$-forms on $\Omega$. This is an unbounded, self-adjoint, closed operator. Hörmander [Hör65] showed that (see also [CS01, Theorem 4.4.1]), if $\Omega$ is a bounded pseudoconvex domain in $\mathbb{C}^n$, then the complex Laplacian has a bounded solution operator $N_1$, called the $\overline{\partial}$-Neumann operator. Furthermore, Kohn [Koh63] (see also [CS01, Theorem 4.4.5]) proved that the Bergman projection and $N_1$ are connected through the following formula

$$P = I - \overline{\partial}^\ast N_1 \overline{\partial}.$$  

Therefore, one can show that if $\Omega$ is a bounded pseudoconvex domain and $\varphi \in C^1(\overline{\Omega})$ then $H\varphi f = \overline{\partial}^\ast N_1(f \overline{\partial} \varphi)$ for $f \in A^2(\Omega)$. So it is reasonable to expect $H\varphi$ to be closely connected to $N_1$. Indeed this is true in terms of compactness of the operators. We refer the reader to [Str10, Proposition 4.1] and [ČS09, CS14, SZ17] for some recent results in this direction, and to books [CS01, Str10, Has14] for more information about the $\overline{\partial}$-Neumann problem.

In terms of Schatten $p$-class membership of $N_1 : L^2_{(0,1)}(\Omega) \to L^2_{(0,1)}(\Omega)$ we have the following corollary, which will be proven at the end of the paper. We note that the result in Corollary 2 below also holds for the restriction of $N_1$ onto $A^2_{(0,1)}(\Omega)$, the space of $(0,1)$-forms with square integrable holomorphic coefficients on $\Omega$. Furthermore, while $\overline{\partial}^\ast N_1$ (canonical solution operator to $\overline{\partial}$) is Hilbert-Schmidt for $\Omega = \mathbb{D} \subset \mathbb{C}$, it fails to be Hilbert-Schmidt when $\Omega$ is the unit ball in $\mathbb{C}^n$ for $n \geq 2$. We refer the reader to [Has14, Chapter 2] and the references therein for results about Schatten $p$-class membership of $\overline{\partial}^\ast N_1$.

**Corollary 2.** Let $\Omega$ be a $C^2$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$ and $N_1$ denote the $\overline{\partial}$-Neumann operator. Then $\overline{\partial}^\ast N_1$ is not in $S_4$ and $N_1$ is not Hilbert-Schmidt.
The rest of the paper is organized as follows. In the next section we will present some necessary basic results that are well known. We include them here for the convenience of the reader. In the last section we give the proofs of Theorem 1 and Corollary 2.

**Preparatory Results**

In this section we will include some preparatory results that will be useful in the proof of Theorem 1. We include them here for the convenience of the reader but we don’t claim any originality about these results.

Let $\Omega$ be a bounded domain and $\varphi \in L^\infty(\Omega)$. Then the Berezin transform of $\varphi$ is defined as

$$\tilde{\varphi}(z) = \int_{\Omega} |k_z(\xi)|^2 \varphi(\xi) dV(\xi)$$

where $k_z(\xi) = \frac{K(\xi, z)}{\sqrt{K(z, z)}}$. Furthermore, we define

$$MO(\varphi, z) = |\varphi|^2(z) - |\tilde{\varphi}(z)|^2.$$  

We denote $H^\infty(\Omega) = O(\Omega) \cap L^\infty(\Omega)$. In case $\varphi \in H^\infty(\Omega)$ we have

$$MO(\varphi, z) = |\varphi|^2(z) - |\varphi(z)|^2$$

as $\tilde{\varphi} = \varphi$.

**Lemma 1.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $\varphi \in H^\infty(\Omega)$. Then $P\varphi k_z = \varphi(z)k_z$ for $z \in \Omega$.

**Proof.** Let $z, w \in \Omega$. Then

$$P\varphi k_z(w) = \int_{\Omega} K(w, \xi)\overline{\varphi(\xi)}k_z(\xi)dV(\xi)$$

$$= \int_{\Omega} K(w, \xi)\frac{K(\xi, z)}{\sqrt{K(z, z)}}\overline{\varphi(\xi)}dV(\xi)$$

$$= \frac{1}{\sqrt{K(z, z)}}\int_{\Omega} K(z, \xi)K(\xi, w)\varphi(\xi)dV(\xi)$$

$$= \frac{1}{\sqrt{K(z, z)}}K(z, w)\varphi(z)$$

$$= \varphi(z)k_z(w).$$

Hence the proof of the lemma is complete. 

**Corollary 3.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $\varphi \in H^\infty(\Omega)$. Then

$$H\varphi k_z(w) = (\varphi(w) - \varphi(z))k_z(w)$$

for $z, w \in \Omega$. 

Lemma 2. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $\varphi \in H^\infty(\Omega)$. Then $\| H_\varphi k_z \|^2 = \text{MO}(\varphi, z)$.

Proof. Let $z \in \Omega$. Lemma 1 implies that $P\varphi k_z = \overline{\varphi(z)} k_z$. Then
\[
\| H_\varphi k_z \|^2 = \langle H_\varphi k_z, H_\varphi k_z \rangle \\
= \langle \overline{\varphi(z)} k_z, \overline{\varphi(z)} k_z \rangle - \langle P\varphi k_z, \overline{\varphi(z)} k_z \rangle \\
= \langle |\varphi(z)|^2 k_z, k_z \rangle - |\varphi(z)|^2 \\
= \text{MO}(\varphi, z).
\]
Hence the proof of the lemma is complete. $\square$

We note that even though Lemmas 1 and 2 in [Zhu91] (used in the proof below) are stated for the ball, they are actually true on any domain. The following corollary can also be deduced from [Li93, Theorem 3.1]. We present a proof here for the convenience of the reader.

Corollary 4. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, $p \geq 2$, and $\varphi \in H^\infty(\Omega)$. Then $H_\varphi \in S_p$ implies that $\int_{\Omega} (\text{MO}(\varphi, z))^{p/2} K(z, z) dV(z) < \infty$.

Proof. Let us assume that $H_\varphi \in S_p$ for $p \geq 2$. Then $(H_\varphi^* H_\varphi)^{p/2}$ is in trace class on $A^2(\Omega)$ (see [Zhu07, Theorem 1.26]). Then [Zhu91, Lemma 1] (see also proof of [Zhu07, Theorem 6.4]) implies that
\[
\int_{\Omega} \langle (H_\varphi^* H_\varphi)^{p/2} k_z, k_z \rangle K(z, z) dV(z) < \infty.
\]
Next we use Lemma 2 and [Zhu91, Lemma 2] (see also [Zhu07, Proposition 1.31]) to conclude that
\[
\int_{\Omega} (\text{MO}(\varphi, z))^{p/2} K(z, z) dV(z) = \int_{\Omega} \| H_\varphi k_z \|^p K(z, z) dV(z) \\
= \int_{\Omega} \langle H_\varphi^* H_\varphi k_z, k_z \rangle^{p/2} K(z, z) dV(z) \\
\leq \int_{\Omega} \langle (H_\varphi^* H_\varphi)^{p/2} k_z, k_z \rangle K(z, z) dV(z) \\
< \infty.
\]
Therefore, the proof of the corollary is complete. $\square$

Remark 1. We will use [BBCZ90, Theorem F] in the proof of Theorem 1. So we take this opportunity to comment that even though [BBCZ90, Theorem F] is stated for bounded symmetric domains, observation of the proof (see [BBCZ90, Remark on pg 321] reveals that it is actually true on all bounded domains in $\mathbb{C}^n$. Indeed, let $\psi : \mathbb{C} \to [0, \infty)$ be a rotation-invariant $C^\infty$-smooth function with $\text{supp}(\psi) \subset D$ and $\int_D \psi(\xi) dV(\xi) = 1$. Then for $z \in \Omega$ and sufficiently
small $\varepsilon > 0$ we define
\[ \chi_z(w) = \frac{1}{\varepsilon^{2n}} \psi \left( \frac{w_1 - z_1}{\varepsilon} \right) \cdots \psi \left( \frac{w_n - z_n}{\varepsilon} \right) \in C_0^\infty(\Omega) \]
where $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$. Then we have $K(w,z) = P\chi_z(w)$ (see, for instance, [JP13, Remark 12.1.5]). To prove that $\frac{\partial}{\partial x_j} K(.,z) \in A^2(\Omega)$, it is enough to show that
\[ \frac{\partial}{\partial x_j} P\chi_z = P \left( \frac{\partial}{\partial x_j} \chi_z \right) \quad \text{and} \quad \frac{\partial}{\partial y_j} P\chi_z = P \left( \frac{\partial}{\partial y_j} \chi_z \right) \]
where $z_j = x_j + iy_j$. We will show only the first equality as the second one is similar. Let $h_j = (0, \ldots, 0, h, 0, \ldots, 0)$ where $h$ is a real number at the $j$th spot. Since we are dealing with holomorphic functions, it is enough to prove that $\| P\chi_{z+h_j} - P\chi_z - hP\partial_{x_j}\chi_z \| = o(h)$ where $\partial_{x_j} = \frac{\partial}{\partial x_j}$. Since $P$ is a bounded linear operator with norm equal to 1 and $\chi_z \in C_0^\infty(\Omega)$ we have
\[ \left\| \frac{P\chi_{z+h_j} - P\chi_z - hP\partial_{x_j}\chi_z}{h} \right\| \to 0 \]
as $h \to 0$. Therefore, $\frac{\partial}{\partial x_j} P\chi_z = P \left( \frac{\partial}{\partial x_j} \chi_z \right)$. Furthermore, using induction we conclude that $\frac{\partial^a}{\partial z^a} K(.,z) \in A^2(\Omega)$ for any multi-index $a$.

The following is a version of [BBCZ90, Theorem F] for bounded domains in $\mathbb{C}^n$.

**Theorem 2** ([BBCZ90]). Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $\gamma : [0, 1] \to \Omega$ be a $C^1$-smooth curve. Assume that $s(t)$ denote the arc-length of $\gamma$ with respect to the Bergman metric of $\Omega$ and $\varphi \in L^\infty(\Omega)$. Then
\[ \left| \frac{d}{dt} \tilde{\varphi}(\gamma(t)) \right| \leq 2\sqrt{2} \left( \frac{ds}{dt} \right) \sup_{0 \leq t \leq 1} (MO(\varphi, \gamma(t)))^{1/2}. \]

Then we have the following useful corollary.

**Corollary 5.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, $\varphi \in H^\infty(\Omega)$, and $X = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Then
\[ \left| \sum_{j=1}^n a_j \frac{\partial \varphi(z)}{\partial z_j} \right| \leq 2\sqrt{2} (MO(\varphi, z))^{1/2} B(X,z) \]
where $B(X,z)$ denotes the Bergman metric applied to the vector $X$ at $z$.

**Proofs of Theorem 1 and Corollary 2**

Before we start the proof of Theorem 1 we present two results in several complex variables. We note that $B_{z_0}(r)$ denotes the open ball centered at $z_0$ with radius $r$. We will use the notion of CR functions in the following proposition. We refer the reader to [CS01, Chapter 3] for the definition and properties of CR functions.
Proposition 1. Let Ω be a domain in C^n for n ≥ 2, z₀ ∈ bΩ, and φ ∈ O(Ω) ∩ C^2(Ω). Assume that there exists r > 0 such that bΩ is C^2-smooth in the ball B_{z₀}(r), the Levi form of bΩ has at least one positive eigenvalue at z₀, and ‖φ‖ is CR function on bΩ ∩ B_{z₀}(r). Then φ is constant.

Proof. Using a holomorphic change of coordinates we may assume that z₀ is the origin, y_n-axis is the real normal direction and X₁ = (0, . . . , 0, 1, 0) is complex tangential (corresponding to a positive eigenvalue of the Levi form, and the two dimensional slice H₀ at z₀, and

\[ H₀ = \{(ξ₁, ξ₂) ∈ C^2 : (0, . . . , 0, ξ₁, ξ₂) ∈ Ω\} \]

is strictly convex at the origin. Furthermore, since small C^2 perturbations of strictly convex surfaces are strictly convex, the slices \( \{(ξ₁, ξ₂) ∈ C^2 : (z₁, . . . , z_{n-2}, ξ₁, ξ₂) ∈ Ω\} \) are strictly convex for sufficiently small |z₁| + . . . + |z_{n-2}|. Then we conclude that there exists 0 < c < 1 such that Ω ∩ B_{z₀}(cr) is union of discs parallel to z₁-axis whose boundaries lie in bΩ ∩ B_{z₀}(r).

Since ‖φ‖_{bΩ ∩ B_{z₀}(r)} is a CR function, [CS01] Theorem 3.3.2] implies that it has a holomorphic extension φ_{z₀,r} onto Ω ∩ B_{z₀}(cr) for some c > 0 (here we shrink c if necessary). Then the fact that φ_{z₀,r} and ‖φ‖ are harmonic and they match on bΩ ∩ B_{z₀}(r) imply that φ_{z₀,r} = ‖φ‖ on Ω ∩ B_{z₀}(cr). Hence, φ and ‖φ‖ are holomorphic on Ω ∩ B_{z₀}(cr). Therefore, φ is constant. □

In the following theorem (see also [Ohs02] Theorem 6.8] for a statement) π(z) denotes the point in bΩ closest to z and d_{bΩ}(z) denotes the distance from z to bΩ. We note that the function π is well defined near C^2-smooth portion of the boundary.

Theorem 3 (Diederich [Die70]). Let Ω be a pseudoconvex domain in C^n and z₀ ∈ bΩ. Assume that there exists an open neighborhood U of z₀ such that bΩ is C^2-smooth in U and bΩ ∩ U is composed of strongly pseudoconvex points. Then there exists a neighborhood V ⊆ U of z₀ and C > 0 such that

\[ B(X, z) ≤ C \left( \frac{|X_τ|}{(d_{bΩ}(z))^{1/2}} + \frac{|X_v|}{d_{bΩ}(z)} \right) \]

for z ∈ V ∩ Ω where X_τ and X_v denote that complex tangential and complex normal component of X at π(z), respectively.

Now we are ready to present the proof of Theorem 1. We will use the fact that every bounded C^2-smooth pseudoconvex domain has some strongly pseudoconvex boundary points (see, for instance, [Bas77]). Then we will follow the ideas in [Li93] and localize the estimate near a strongly pseudoconvex point in the boundary to get a contradiction in case H_φ ∈ S_p for p ≤ 2n.

Proof of Theorem 1. We will only prove the non-trivial direction. Since S_α ⊆ S_β for α ≤ β we start the proof by assuming that H_φ ∈ S_{2n}. Then Corollary 4 (see also [Li93, Theorem 3.1])
implies that
\[
\int_{\Omega} (MO(\phi, z))^{n} K(z, z) dV(z) < \infty.
\]
Let \( z_0 \in b\Omega \) be a strongly pseudoconvex point and \( U = B_{z_0}(r) \) so that all points in \( B_{z_0}(2r) \cap b\Omega \) are strongly pseudoconvex. By Corollary 5 we have
\[
\left| \sum_{j=1}^{n} a_j \frac{\partial \phi(z)}{\partial z_j} \right| \leq 2\sqrt{2}(MO(\phi, z))^{1/2} B(X, z)
\]
for \( X = (a_1, \ldots, a_n) \in \mathbb{C}^n \). Furthermore, Theorem 3 implies that there exists \( C > 0 \) such that
\[
B(X, z) \leq \frac{C}{2\sqrt{2}} \left( \frac{|X_T|}{(d_{b\Omega}(z))^{1/2}} + \frac{|X_N|}{d_{b\Omega}(z)} \right)
\]
for \( z \in U \cap \Omega \) where \( X_T \) and \( X_N \) are the tangential and normal components of \( X \), respectively. Combining the previous two estimates, we conclude that for any \( z \in U \cap \Omega \) we have
\[
\left| \sum_{j=1}^{n} a_j \frac{\partial \phi(z)}{\partial z_j} \right| \leq C \cdot (MO(\phi, z))^{1/2} \left( \frac{|X_T|}{(d_{b\Omega}(z))^{1/2}} + \frac{|X_N|}{d_{b\Omega}(z)} \right).
\]
Then
\[
|\partial_b \phi(z)|^2 d_{b\Omega}(z) = |\overline{\partial}_b \overline{\phi}(z)|^2 d_{b\Omega}(z) \leq C^2 \cdot MO(\phi, z).
\]
Combining the previous inequality with (1) we get
\[
\int_{\Omega \cap U} |\partial_b \phi(z)|^{2n} (d_{b\Omega}(z))^n K(z, z) dV(z) < \infty.
\]
We note that \( K(z, z) \) is comparable to \( (d_{b\Omega}(z))^{-n-1} \) near strongly pseudoconvex boundary points (see, for example, [Hör65 Theorem 3.5.1]). Then there exists \( \tilde{C} > 0 \) such that for sufficiently small \( \varepsilon > 0 \) we get
\[
\int_0^\varepsilon \frac{dt}{t} \int_{b\Omega \cap \tilde{U}} |\partial_b \phi(z)|^{2n} d\sigma(z) \leq \tilde{C} \int_{\Omega \cap U} |\partial_b \phi(z)|^{2n} dV(z)
\]
\[
\leq \tilde{C}^2 \int_{\Omega \cap U} |\partial_b \phi(z)|^{2n} (d_{b\Omega}(z))^n K(z, z) dV(z)
\]
\[
< \infty
\]
where \( b\Omega_t = \{ z \in \Omega : d_{b\Omega}(z) = t \} \) and \( \tilde{U} = B_{z_0}(r/2) \). Then \( \int_{b\Omega \cap \tilde{U}} |\partial_b \phi(z)|^{2n} d\sigma(z) = 0 \). Since \( \partial_b \phi \) is continuous on \( b\Omega \cap U \) we conclude that \( \partial_b \phi = 0 \) on \( b\Omega \cap U \). Finally, Proposition 1 implies that \( \phi \) is constant.

Finally we present the proof of Corollary 2.

Proof of Corollary 2. Let \( K_{(0,q)}^2(\Omega) \) denote the square integrable \( \overline{\partial} \)-closed \( (0, q) \)-forms on \( \Omega \) and \( N_q \) denote the \( \overline{\partial} \)-Neumann operator on \( L^2_{(0,q)}(\Omega) \). We note that \( K_{(0,1)}^2(\Omega) \) is a closed subspace
(as it is the kernel of $\overline{\partial}$) of $L^2_{(0,1)}(\Omega)$ and $N_1$ maps $K^2_{(0,1)}(\Omega)$ into itself (as $\overline{\partial}N_1 = N_2\overline{\partial}$). Range’s Theorem (see, for instance, [Str10, p.77] and [Ran84]) implies that $N_1 = (\overline{\partial}^*N_1)^*\overline{\partial}^*N_1$ on $\text{Ker}(\overline{\partial})$. Furthermore, $T \in S_p$ if and only if $T^*T \in S_{p/2}$ (see [Zhu07, Theorem 1.26]). If $N_1$ is Hilbert-Schmidt then $\overline{\partial}^*N_1 A^2_{(0,1)}(\Omega) \subset S_4$ where $A^2_{(0,1)}(\Omega)$ is the space of $(0,1)$-forms with square integrable holomorphic coefficients. However, $Hz_1 f = \overline{\partial}^*N_1(fdz_1)$ for $f \in A^2(\Omega)$ and $Hz_1 \not\in S_4$. Therefore, $\overline{\partial}^*N_1 \not\in S_4$ and $N_1$ is not Hilbert-Schmidt. □

ACKNOWLEDGMENT

Part of this work was done while the second author was visiting Sabancî University. He thanks this institution for its hospitality and good working conditions. He also thanks Trieu Le for fruitful discussions. We are thankful to the anonymous referee for constructive comments that improved the presentation of the paper.

REFERENCES

[AFP88] J. Arazy, S. D. Fisher, and J. Peetre, Hankel operators on weighted Bergman spaces, Amer. J. Math. 110 (1988), no. 6, 989–1053.

[Bas77] Richard F. Basener, Peak points, barriers and pseudoconvex boundary points, Proc. Amer. Math. Soc. 65 (1977), no. 1, 89–92.

[BBCZ90] D. Békollé, C. A. Berger, L. A. Coburn, and K. H. Zhu, BMO in the Bergman metric on bounded symmetric domains, J. Funct. Anal. 93 (1990), no. 2, 310–350.

[CS01] So-Chin Chen and Mei-Chi Shaw, Partial differential equations in several complex variables, AMS/IP Studies in Advanced Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2001.

[ČS09] Željko Ćučković and Sönmez Şahutoğlu, Compactness of Hankel operators and analytic discs in the boundary of pseudoconvex domains, J. Funct. Anal. 256 (2009), no. 11, 3730–3742.

[CŞ14] Mehmet Çelik and Sönmez Şahutoğlu, Compactness of the $\overline{\partial}$-Neumann operator and commutators of the Bergman projection with continuous functions, J. Math. Anal. Appl. 409 (2014), no. 1, 393–398.

[CZ13] Mehmet Çelik and Yunus E. Zeytuncu, Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on complex ellipsoids, Integral Equations Operator Theory 76 (2013), no. 4, 589–599.

[CZ17] Mehmet Çelik and Yunus E. Zeytuncu, Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on complete pseudoconvex Reinhardt domains, Czechoslovak Math. J. 67(142) (2017), no. 1, 207–217.

[Die70] Klas Diederich, Das Randverhalten der Bergmanschen Kernfunktion und Metrik in streng pseudo-konvexen Gebieten, Math. Ann. 187 (1970), 9–36.

[Has14] Friedrich Haslinger, The $\overline{\partial}$-Neumann problem and Schrödinger operators, De Gruyter Expositions in Mathematics, vol. 59, De Gruyter, Berlin, 2014.

[Hör65] Lars Hörmander, $L^2$ estimates and existence theorems for the $\overline{\partial}$ operator, Acta Math. 113 (1965), 89–152.

[JP13] Marek Jarnicki and Peter Pflug, Invariant distances and metrics in complex analysis, extended ed., De Gruyter Expositions in Mathematics, vol. 9, Walter de Gruyter GmbH & Co. KG, Berlin, 2013.

[KLR97] Steven G. Krantz, Song-Ying Li, and Richard Rochberg, The effect of boundary geometry on Hankel operators belonging to the trace ideals of Bergman spaces, Integral Equations Operator Theory 28 (1997), no. 2, 196–213.
SCHATTEN CLASS HANKEL AND $\overline{\partial}$-NEUMANN OPERATORS ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

[1] J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds. I*, Ann. of Math. (2) **78** (1963), 112–148.

[2] Trieu Le, *Hilbert-Schmidt Hankel operators over complete Reinhardt domains*, Integral Equations Operator Theory **78** (2014), no. 4, 515–522.

[3] Huiping Li, *Schatten class Hankel operators on the Bergman spaces of strongly pseudoconvex domains*, Proc. Amer. Math. Soc. **119** (1993), no. 4, 1211–1221.

[4] Takeo Ohsawa, *Analysis of several complex variables*, Translations of Mathematical Monographs, vol. 211, American Mathematical Society, Providence, RI, 2002, Translated from the Japanese by Shu Gilbert Nakamura, Iwanami Series in Modern Mathematics.

[5] Jordi Pau, *Characterization of Schatten-class Hankel operators on weighted Bergman spaces*, Duke Math. J. **165** (2016), no. 14, 2771–2791.

[6] R. Michael Range, *The $\overline{\partial}$-Neumann operator on the unit ball in $\mathbb{C}^n$*, Math. Ann. **266** (1984), no. 4, 449–456.

[7] Georg Schneider, *Hankel operators with antiholomorphic symbols on the Fock space*, Proc. Amer. Math. Soc. **132** (2004), no. 8, 2399–2409.

[8] Sönmez Şahutoğlu, *A note on Schatten-class membership of Hankel operators with anti-holomorphic symbols on generalized Fock-spaces*, Math. Nachr. **282** (2009), no. 1, 99–103.

[9] Emil J. Straube, *Lectures on the $L^2$-Sobolev theory of the $\overline{\partial}$-Neumann problem*, ESI Lectures in Mathematics and Physics, vol. 7, European Mathematical Society (EMS), Zürich, 2010.

[10] Kristian Seip and El Hassan Youssfi, *Hankel operators on Fock spaces and related Bergman kernel estimates*, J. Geom. Anal. **23** (2013), no. 1, 170–201.

[11] Sönmez Şahutoğlu and Yunus E. Zeytuncu, *On Compactness of Hankel and the $\overline{\partial}$-Neumann Operators on Hartogs Domains in $\mathbb{C}^2$*, J. Geom. Anal. **27** (2017), no. 2, 1274–1285.

[12] Jingbo Xia, *On the Schatten class membership of Hankel operators on the unit ball*, Illinois J. Math. **46** (2002), no. 3, 913–928.

[13] Ke He Zhu, *Hilbert-Schmidt Hankel operators on the Bergman space*, Proc. Amer. Math. Soc. **109** (1990), no. 3, 721–730.

[14] Ke He Zhu, *Schatten class Hankel operators on the Bergman space of the unit ball*, Amer. J. Math. **113** (1991), no. 1, 147–167.

[15] Ke He Zhu, *Operator theory in function spaces*, second ed., Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, Providence, RI, 2007.

(Nihat Gökhan Göğüş) SABAŃCI UNIVERSITY, TUSLA, 34956, ISTANBUL, TURKEY

E-mail address: nggogus@sabanciuniv.edu

(Sönmez Şahutoğlu) UNIVERSITY OF TOLEDO, DEPARTMENT OF MATHEMATICS & STATISTICS, TOLEDO, OH 43606, USA

Current address: Sabancı University, Tuzla, 34956, Istanbul, Turkey

E-mail address: Sonmez.Sahutoglu@utoledo.edu