A LERAY MODEL FOR THE ORLIK–SOLOMON ALGEBRA

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Abstract. We construct a combinatorial generalization of the Leray models for hyperplane arrangement complements. Given a matroid and some combinatorial blowup data, we give a presentation for a bigraded (commutative) differential graded algebra. If the matroid is realizable over \( \mathbb{C} \), this is the familiar Morgan model for a hyperplane arrangement complement, embedded in a blowup of projective space. In general, we obtain a \( \text{CDGA} \) that interpolates between the Chow ring of a matroid and the Orlik–Solomon algebra. Our construction can also be expressed in terms of sheaves on combinatorial blowups of geometric lattices. As a key technical device, we construct a monomial basis via a Gröbner basis for the ideal of relations. Combining these ingredients, we show that our algebra is quasi-isomorphic to the classical Orlik–Solomon algebra of the matroid.

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1. Introduction

For a given arrangement of complex linear hyperplanes, a number of complex algebraic varieties have been defined and extensively studied over the years. Notably, there is the projective complement of $A$, $U(A) := \mathbb{P}^d \setminus \bigcup_{H \in A} H$, and the wonderful compactification $Y(A, \mathcal{G})$ obtained by blowing up $\mathbb{P}^d$ along proper transforms of suitably chosen subspaces $\mathcal{G}$. The cohomology algebras of both have been studied and have been described in a purely combinatorial manner in terms of the matroid associated with the arrangement. These are the projective Orlik–Solomon algebra, $\overline{OS}(A) \cong H^*(U(A))$ [Bri73, OS80, Kaw04], and the De Concini–Procesi algebra, $DP(A, \mathcal{G}) \cong H^*(Y(A, \mathcal{G}))$ [DCP95], respectively.

Though they seem to be rather disparate algebraic invariants of arrangements, we place them here in one and the same scene. Moreover, in the spirit of recent work of Adiprasito, Huh, and Katz [AHK18] as well as [ADH20, BHM+20], we lift objects of geometric origin to the purely combinatorial context of matroid theory, where phenomena that are based on the geometry of spaces miraculously persist. Combinatorial blowups, Gröbner bases, and sheaves on posets thereby replace heavy geometric machinery.

In the geometric setting, the wonderful compactification $Y(A, \mathcal{G})$ is constructed through a sequence of blowups dictated by a special set $\mathcal{G}$, called a building set. The projective complement $U(A)$ is a dense open subset of $Y(A, \mathcal{G})$, in which it is realized as the complement of a normal crossings divisor. The cohomology algebras of $Y(A, \mathcal{G})$ and $U(A)$ are linked through the Leray spectral sequence

$$E_2^{pq} = H^p(Y(A, \mathcal{G}), R^q j_! \mathbb{Q}) \Rightarrow H^{p+q}(U(A), \mathbb{Q})$$

given by the inclusion $j : U(A) \hookrightarrow Y(A, \mathcal{G})$: the spectral sequence degenerates at $E_3$, so the cohomology of $(E_2, d_2)$ agrees with that of $U(A)$. Computing with differential forms, this is the Morgan cdga model for $U(A)$ and coincides with the model constructed by De Concini and Procesi [DCP95, §5.3]. Arrangement complements are well-known to be rationally formal [Bri73], and the mixed Hodge structure on their cohomology is pure [Sha93]. Work of Dupont [Dup15] explains how these observations together are equivalent to the fact that the edge map $H^*(U(A), \mathbb{Q}) \to (E_2, d_2)$ is a quasi-isomorphism.

In the more general combinatorial setting, we introduce the notion of a Leray or Morgan model of a matroid. We start with a matroid $M$, its lattice of flats $L$, and a combinatorial building set $\mathcal{G} \subset L$. We study the effect of the blowup of one element of $\mathcal{G}$ at a time, and we build objects indexed by partial building sets $\mathcal{H} \subseteq \mathcal{G}$. We obtain semilattices $L(L, \mathcal{H})$ that interpolate between the geometric lattice $L$ and a simplicial poset $L(L, \mathcal{G})$. Geometrically, this interpolates between the combinatorics of the original arrangement and that of the normal crossings divisor obtained through the sequence of blowups. We combine elements of the Orlik–Solomon algebras and the De Concini–Procesi algebras to define differential graded algebras $B(L, \mathcal{H})$, which play the role of $(E_2, d_2)$ above. Indeed, our main result (Theorem 5.5.1) states that each $B(L, \mathcal{H})$, and in particular $B(L, \mathcal{G})$, is quasi-isomorphic to $\overline{OS}(L)$. 

References

1. 6.1. A complex of sheaves
2. 6.2. Blowups induce quasi-isomorphisms
3. 5.5. Blowups and the cdga
4. 6. The combinatorial Leray model and formality
The bookkeeping that goes along with the sequence of combinatorial blowups is notationally intensive. Some of the arguments in the paper are, we believe, unavoidably quite technical. In order to help the reader navigate the paper without getting lost in the details we provide a roadmap.

**Outline of the paper.** In §2 we provide some combinatorial basics tailored to our use. We summarize the background on sheaves on posets and the poset of intervals. We elaborate on combinatorial blowups and the notions of building sets and nested sets, which form the combinatorial core of De Concini–Procesi arrangement models.

In §3, we construct an Orlik–Solomon algebra $O(S(L))$ from what we call a locally geometric semilattice $L$. Such semilattices include the ones that appear by iteratively blowing up a geometric lattice. In the realizable case, these algebras model the left edge of the Leray spectral sequence: that is, they are the global sections of the cohomology sheaf obtained by restricting the constant sheaf from a partial blowup to the hyperplane arrangement complement. The algebra $O(S(L))$ has a well-known monomial basis called the nbc basis.

In §4, we construct a De Concini–Procesi algebra $D(P(L, H))$ from a geometric lattice $L$ and partial building set $H$. In the realizable case, it is isomorphic to the cohomology of the wonderful De Concini–Procesi model of an arrangement complement obtained by blowing up $P'$ along the subspaces $H$. The algebra is also isomorphic to the bottom edge of the Leray spectral sequence. Regardless of realizability, it is also the Chow ring of a smooth toric variety associated with a subfan of the Bergman fan [FY04].

Our main object of study is introduced in §5, the commutative differential graded algebra $B(L, H)$ associated with a geometric lattice $L$ and a partial building set $H$. We define it by means of a presentation that combines the relations from the Orlik–Solomon and De Concini–Procesi algebras. Using Gröbner basis theory, we show that $B(L, H)$ has a monomial basis that specializes in one direction to the nbc basis for the Orlik–Solomon algebra, and in the other to the basis for the De Concini–Procesi algebra of [FY04]. This basis is essential for obtaining injective maps between the algebras $B(L, H)$ (Theorem 5.5.6).

We show (Proposition 5.1.4) that $B(L, H)$ has a bigraded direct-sum decomposition, indexed over the semilattice $L(L, H)$, where the summands are tensor products of “local” Orlik–Solomon algebras with “local” De Concini–Procesi algebras. In the geometric setting, this is a familiar picture: the compactification is stratified by intersections of hypersurfaces. Each stratum contributes to the Leray spectral sequence a tensor product of the cohomology of the hypersurface near the stratum (Orlik–Solomon), and the cohomology of the stratum itself (De Concini–Procesi): see [Loo93, Dup15, Bib16]. In general, though, there is no Leray spectral sequence, and no geometric reason why there should be such a direct sum decomposition. Instead, we make use of our Gröbner basis to show that expected decomposition exists in all cases.

Similarly, we show that the “local” De Concini–Procesi algebras $D_{pi}(L, H)$, for elements $y \in L(L, H)$, can be decomposed as tensor products (Theorem 4.1.7). This reflects the fact that the strata in De Concini and Procesi’s compactification are themselves products of De Concini–Procesi compactifications of arrangements of lower dimension [DCP95, §4.3].

In an effort to arrive at a combinatorial explanation of phenomena like this, we use the classical notion of sheaves on posets, inspired by work of Yuzvinsky [Yuz95]. We topologize a finite poset $P$ with the order topology, in which basic open sets are principal order ideals. We consider sheaves of Orlik–Solomon algebras, which model the cohomology sheaf $j_* Q$ in the realizable case. For technical reasons, it turns out to be more convenient to work with the homology version of the Orlik–Solomon algebra, the flag complex introduced by Schechtman.
and Varchenko [SV91]. Flag complexes assemble into a graded sheaf $\mathcal{F}(\mathcal{L})$ on the poset $\mathcal{L} := \mathcal{L}(L, \mathcal{H})$. A standard differential makes $\mathcal{F}(\mathcal{L})$ a cochain complex which is, in fact, a flasque resolution of a skyscraper sheaf. Similarly, we define a De Concini–Procesi sheaf $\mathcal{D}(\mathcal{P})$ of algebras on $\mathcal{L}^{\text{op}}$, by letting $\mathcal{D}(\mathcal{L}_{\geq y}) = \mathcal{D}_y(L, \mathcal{H})$. In the realizable case, this is the sheaf of cohomology algebras of strata.

For any poset $P$, the *poset of intervals* $I(P)$ is the poset on pairs $(x, y) \in P \times P^{\text{op}}$ with the order relation given by containment. The poset $I(\mathcal{L})$ turns out to be a key organizational device for understanding our combinatorial Leray model. We define a graded sheaf $\mathcal{C}(L, \mathcal{H})$ on $I(\mathcal{L})$ as the tensor product of the pullbacks of $\mathcal{F}$ and $\mathcal{D}$ along the projections to $\mathcal{L}$ and $\mathcal{L}^{\text{op}}$, respectively. This is, in fact, a cochain complex of coherent sheaves of $\mathcal{D}(\mathcal{P})$-modules. We show (Theorem 6.1.3) that the vector space of global sections of $\mathcal{C}(L, \mathcal{H})$ is the $\mathbb{Q}$-dual of our cdga $B(L, \mathcal{H})$. Moreover, this complex turns out to be a $\Gamma$-acyclic resolution of a sheaf on $I(\mathcal{L})$ obtained from $\mathcal{D}(\mathcal{P})$ (Theorem 6.1.5).

Section §5.5 is devoted to showing that our algebras $B(L, \mathcal{H})$ are quasi-isomorphic (Theorem 5.5.1) to $\overline{\mathcal{OS}}(L)$, regarded as a cdga with zero differential. In the sense of rational homotopy theory, then, these are rational models. This quasi-isomorphism is established by induction, by showing that each blowup gives an injective map of cdgas (Theorem 5.5.6). Arguing with sheaves on the poset of intervals shows that the map at each step is a quasi-isomorphism (Theorem 6.2.1).

## 2. Combinatorial foundations

In this section, we review the basic combinatorial background, as well as build new tools to be used in our setting. Most of this section is technical. Before we turn our attention to the main objects of study (a sequence of blowups of semilattices) in §2.3, we quickly discuss some generalities on posets (sheaves on posets and the poset of intervals) which will not be used until §6.1. For more general background references, we refer to the book of Oxley [Ox11] on matroid theory and the book of Orlik and Terao [OT92] on hyperplane arrangements.

### 2.1. Sheaves on posets

The notion of sheaves on posets and their cohomology turns out to be convenient for analyzing the algebras that we study in this paper. Our approach is inspired by that of Yuzvinsky [Yuz95], and we generalize his work on the (classical) Orlik–Solomon algebra here.

Let $P$ be a finite, partially ordered set. For $x \in P$, we will let $P_{\geq x} = \{ y \in P : y \geq x \}$, and define the obvious variations analogously. We will denote closed intervals by $[x, y]$ for $x, y \in P$. If $P$ is a ranked poset, we let $P_q$ denote the subset of elements of rank $q$, for $q \in \mathbb{Z}$. We give $P$ the topology in which downward-closed sets (order ideals) are open. (We note that this is opposite to Yuzvinsky’s convention.) A basis for the topology is given by the principal order ideals, $\{ P_{\geq x} : x \in P \}$. We will also let $|P|$ denote the *order complex* on $P$, the abstract simplicial complex whose simplices are the totally ordered subsets of $P$.

Then a sheaf $\mathcal{F}$ on $P$ in an abelian category $\mathcal{C}$ is just a diagram of objects of $\mathcal{C}$ over $P$: more precisely, regarding $P$ as a category with morphisms $y \rightarrow x$ whenever $y \geq x$, a functor $\mathcal{F} : P \rightarrow \mathcal{C}$ is equivalent to a (topological) sheaf on $P$ for which the sections over $P_{\leq x}$ equal $\mathcal{F}(x)$. The restriction maps on principal open sets are the morphisms $\mathcal{F}(x \leq y)$. Since each $x \in P$ is contained in a unique minimal open set ($P_{\leq x}$), the stalk of $\mathcal{F}$ at $x$ also equals $\mathcal{F}(x)$.

We will take [Deh62, Bac75] as fundamental references, but prove here some basic facts about sheaves and maps between posets. First, suppose $f : P \rightarrow Q$ is a map of posets with the property that, for all $x, y \in Q$ with $x \leq y$, there exist $u, v \in P$ with $u \leq v$ satisfying $f(u) = x$ and $f(v) = y$. Then $f$ is called a *quotient*. 


If \( f : P \to Q \) is a map of posets for which the order complex \( |f^{-1}(y)| \) is nonempty and connected, for each \( y \in Q \), we will say \( f \) has connected fibres for short.

**Example 2.1.1.** A quotient \( f : P \to Q \) is a surjective map of underlying sets, but not conversely. However, if \( f \) admits a splitting, it is easily seen to be a quotient. An example of a split surjection is a poset \( P \) with a closure operation \( x \mapsto P \). In this case, if \( x = P = y \) for some \( x, y \in P \), then \( y \leq P = x \), so the fibres of the quotient each have a cone vertex; in particular, they are connected and contractible.

**Lemma 2.1.2.** Let \( f : P \to Q \) be a map of posets for which the preimage of each principal order ideal is again a principal order ideal. Then \( f_* : \text{Sheaves}(P) \to \text{Sheaves}(Q) \) is an exact functor.

**Proof.** The direct image is left-exact, so this amounts to checking \( f_* \) is also right-exact. Let us suppose \( \phi : \mathcal{F} \to \mathcal{G} \) is a surjective map of sheaves on \( P \). For any \( y \in Q \), consider the stalk at \( y \) of the map \( f_* \phi : f_* \mathcal{F} \to f_* \mathcal{G} \). By hypothesis, \( f^{-1}(Q_{\leq y}) \) has a unique maximum element \( x \in P \), so \( (f_* \mathcal{F})(y) \to (f_* \mathcal{G})(y) \) equals \( \phi(x) : \mathcal{F}(x) \to \mathcal{G}(x) \). This shows \( f_* \phi \) is surjective on stalks, so \( f_* \phi \) is surjective.

**Lemma 2.1.3.** Suppose \( f : P \to Q \) is a quotient map with connected fibres. Then, for any sheaf \( \mathcal{F} \) on \( Q \), the adjunction \( \eta : \mathcal{F} \to f_* f^* \mathcal{F} \) is an isomorphism.

**Proof.** For any \( y \in Q \) and sheaf \( \mathcal{F} \) on \( Q \), we compute

\[
\begin{align*}
(f_* f^* \mathcal{F})(Q_{\leq y}) &= (f^* \mathcal{F})(f^{-1}(Q_{\leq y})) \\
&= \lim_{x \in P; f(x) \leq y} (f^* \mathcal{F})(P_{\leq x}) \\
&= \lim_{x \in P; f(x) \leq y} \mathcal{F}(Q_{\leq f(x)}).
\end{align*}
\]

(1)

The adjunction map \( \eta_y : \mathcal{F}(Q_{\leq y}) \to f_* f^* \mathcal{F}(Q_{\leq y}) \) is induced by the restriction maps from \( \mathcal{F}(y) = \mathcal{F}(Q_{\leq y}) \) to terms in the limit (1). For any \( x \in P \) with \( f(x) \leq y \), because \( f \) is a quotient, there exist elements \( x', x'' \in P \) for which

\[
x' \leq x'', \quad f(x') = f(x), \quad \text{and} \quad f(x'') = y.
\]

Since the fibres of \( f \) are connected, the diagram (1) contains maps which compose to an isomorphism \( \mathcal{F}(f(x')) \to \mathcal{F}(f(x)) \). Pre-composing with the restriction \( \mathcal{F}(y) = \mathcal{F}(f(x'')) \to \mathcal{F}(f(x')) \), we obtain a map \( \mathcal{F}(y) \to \mathcal{F}(f(x)) \). Then \( \mathcal{F}(y) \) is initial in the diagram, so \( \eta_y \) is an isomorphism to the diagram’s limit. □

**Proposition 2.1.4.** Let \( f : P \to Q \) be a poset quotient. Suppose that, for each \( y \in Q \), the order complex \( |f^{-1}(Q_{\geq y})| \) is contractible, and the fibre \( f^{-1}(y) \) is connected. Then, for any sheaf \( \mathcal{F} \) on \( Q \), the cohomology pullback

\[
f^* : H^*(Q; \mathcal{F}) \to H^*(P, f^* \mathcal{F})
\]

is an isomorphism.

**Proof.** The cohomology pullback is induced by applying (derived) global sections to the composition \( \mathcal{F} \to f_* f^* \mathcal{F} \to Rf_* f^* \mathcal{F} \), identifying \( R\Gamma(Q, Rf_* f^* \mathcal{F}) \cong R\Gamma(P, f^* \mathcal{F}) \) by means of the Leray spectral sequence.

Let

\[
0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E}^1 \longrightarrow \cdots
\]
be the Godement resolution of $\mathcal{F}$. Explicitly, for $y \in Q$, let

$$\mathcal{F}_y(x) = \begin{cases} \mathcal{F}(y) & \text{for } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

so that each $\mathcal{C}^p$ is a product of copies of the skyscraper sheaves $\mathcal{F}_y$, indexed by chains in the poset of length $p + 1$.

We claim that

$$(2) \quad 0 \longrightarrow f^*\mathcal{F} \longrightarrow f^*\mathcal{C}^0 \longrightarrow f^*\mathcal{C}^1 \longrightarrow \cdots$$

is a $\Gamma$-acyclic resolution of $f^*\mathcal{F}$. Exactness is a property of inverse image; for acyclicity, it is enough to check that each factor $H^p(P, f^*\mathcal{F}_y)$ is zero for $p > 0$, for $y \in Q$.

For this, we use the Godement resolution again. Note $f^*\mathcal{F}_y(x) = \mathcal{F}(y)$ if $f(x) \geq y$, and zero otherwise. Global sections of the Godement resolution may be identified with the simplicial cochain complex on the order complex of $f^{-1}(Q_{\geq y})$, with constant coefficients, so

$$H^p(P, f^*\mathcal{F}_y) \cong H^p(|f^{-1}(Q_{\geq y})|, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{F}(y),$$

which is zero for $p > 0$ by hypothesis.

We complete the proof as follows. Applying $H(\eta)$ to $\mathcal{C}$,

$$H^*(Q, \mathcal{F}) \cong H^*\Gamma \mathcal{C} \xrightarrow{\cong} H^*\Gamma f_*f^*\mathcal{C} \quad \text{by Lemma 2.1.3; }$$

$$= H^*\Gamma f_*f^*\mathcal{C} \cong H^*(P, f^*\mathcal{F}),$$

using the resolution (2).

Everitt and Turner [ET19, Cor. 2] establish a similar result with slightly different conventions: the novelty here is the criterion that makes $\eta$ an isomorphism.

### 2.2. The poset of intervals.

Here is a construction which will be of central importance for us in §6. We recall its definition and refer to [Wal88] and [Koz08, Ch. 10.4] for more details.

**Definition 2.2.1 (The poset of intervals).** For any poset $P$, let

$$I(P) = \{(x, y) : x, y \in P, x \leq y\},$$

ordered by $(x_0, y_0) \leq (x_1, y_1)$ if and only if $x_0 \leq x_1 \leq y_1 \leq y_0$. Since the maximal elements of $I(P)$ are of the form $(x, x)$ for $x \in P$, $I(P)$ has a (unique) minimal open cover consisting of the principal order ideals \{I(P)_{\leq (x,x)} : x \in P\}.

The construction is easily seen to be functorial: that is, if $f : P \to Q$ is a map of posets, so is the induced map

$$\text{I}(f) : I(P) \to I(Q)$$

defined by letting $\text{I}(f)(x, y) = (f(x), f(y))$. Translating definitions, we see that $f : P \to Q$ is a quotient if and only if the map $\text{I}(f) : I(P) \to I(Q)$ is surjective.

**Definition 2.2.2.** Say a poset quotient $f : P \to Q$ is initial if, whenever $f(x) = u \leq v$ for $x \in P$ and $u, v \in Q$, there exists some $y \geq x$ with $f(y) = v$.

**Proposition 2.2.3.** If $f : P \to Q$ is an initial quotient, then $\text{I}(f) : I(P) \to I(Q)$ is a quotient.

**Proof.** Suppose $f$ is an initial quotient. If $(u, v) \leq (u', v')$ in $I(Q)$, we have $u \leq u' \leq v' \leq v$. Since $f$ is surjective, there exists some $x \in P$ with $f(x) = u$. Because $f$ is initial, we may find $x'$, $y'$, and $y \in P$, successively, with $x \leq x' \leq y' \leq y$ mapping to $u'$, $v'$, and $v$, respectively. That is,

$$(x, y) \leq (x', y') \quad \text{and} \quad I(f)(x, y) = (u, v), \quad I(f)(x', y') = (u', v').$$
Now let $\text{pr}_1 : I(P) \to P$ and $\text{pr}_2 : I(P) \to P^{\text{op}}$ denote the two coordinate projections. Suppose further that $P$ has a minimum element $\hat{0}$. Then there is also a natural inclusion $\iota : P^{\text{op}} \to I(P)$, given by letting $\iota(y) = (\hat{0}, y)$ for $y \in P^{\text{op}}$. Then $\text{pr}_2 \circ \iota = 1_{P^{\text{op}}}$.

The maps $\text{pr}_i$ and $\iota$ are, in fact, natural transformations. In the case of $\iota$, this means, explicitly, that:

**Proposition 2.2.4.** If $f : P \to Q$ is an order-preserving map of posets with minimum elements and $f(\hat{0}) = \hat{0}$, then the following diagram commutes:

$$
\begin{array}{ccc}
P^{\text{op}} & \xrightarrow{f} & Q^{\text{op}} \\
\downarrow \iota & & \downarrow \iota \\
I(P) & \xrightarrow{I(f)} & I(Q)
\end{array}
$$

Using a result from the previous section, we obtain cohomology isomorphisms:

**Lemma 2.2.5.** For any sheaf $\mathcal{F}$ on $P$, the cohomology pullback $H^*(P, \mathcal{F}) \to H^*(I(P), \text{pr}_1^* \mathcal{F})$ is an isomorphism.

**Proof.** For each $x \in P$,

$$
\text{pr}_1^{-1}(P_{\geq x}) = \{(x', y') : x \leq x' \leq y'\} = I(P_{\geq x}).
$$

By [Wal88, Thm. 4.1], the order complex of $I(P_{\geq x})$ is homeomorphic to that of $P_{\geq x}$. Since the latter has a cone vertex, both order complexes are contractible.

Likewise, the map $(x, y) \mapsto (x, x)$ is easily seen to be a closure operation on $I(P)$, so if we identify its image with $P$, we see $\text{pr}_1$ is a quotient with contractible fibres (Example 2.1.1). The result then follows by Proposition 2.1.4.

\[\square\]

### 2.3. Combinatorial blowups

The notion of a combinatorial blowup was introduced in [FK04]. An expository account appears in [Fei05], and we recall here some definitions.

**Definition 2.3.1 (Locally geometric semilattice).** A finite poset $\mathcal{L}$ is a *meet-semilattice* if every pair of elements has a unique maximal lower bound. Let $\hat{0}$ denote the (unique) least element, and let $\mathcal{L}_+ = \mathcal{L}_{>0}$. We will say a meet-semilattice $\mathcal{L}$ is *locally geometric* if, for every $x \in \mathcal{L}$, the order interval $\mathcal{L}_{\leq x} = [\hat{0}, x]$ is a geometric lattice. This implies that for every $x \leq y$ in $\mathcal{L}$, the interval $[x, y]$ is a geometric lattice, and we let $d(x, y) := \text{rank}([x, y])$.

In what follows, we will often just write “semilattice” for a finite meet-semilattice. These posets are ranked by $\mathbb{Z}_{\geq 0}$. We also note that, in a finite meet-semilattice, if a set of elements $S$ of $\mathcal{L}$ has an upper bound, that upper bound is unique, in which case we will denote it by $\bigvee S$.

**Definition 2.3.2 (The atomic complex).** For a semilattice $\mathcal{L}$, let $\text{at}(\mathcal{L})$ denote its set of atoms (elements which cover $\hat{0}$). Let $\text{at}(\mathcal{L})$ be the abstract simplicial complex on the set $\text{at}(\mathcal{L})$ consisting of subsets $S \subseteq \text{at}(\mathcal{L})$ for which the elements of $S$ have a common upper bound in $\mathcal{L}$.
The join of a simplex defines a natural map \( \vee : \text{at}(\mathcal{L}) \to \mathcal{L} \). Moreover, there is a natural map \( \text{supp} : \mathcal{L} \to \text{at}(\mathcal{L}) \) given by
\[
\text{supp}(y) = \{ a \in \text{a}(\mathcal{L}) : a \leq y \}.
\]

Observe that for \( S \in \text{at}(\mathcal{L}) \), \( \text{supp}(\vee S) \supseteq S \) but equality does not in general hold. However, for \( x \in \mathcal{L} \), we have \( \vee(\text{supp}(x)) = x \).

**Example 2.3.3.** In a geometric lattice \( \mathcal{L} \), every set of atoms has a common upper bound. This implies that the atomic complex \( \text{at}(\mathcal{L}) \) of a geometric lattice \( \mathcal{L} \) is just a simplex of dimension \( |\text{a}(\mathcal{L})| - 1 \). \( \Diamond \)

Any geometric lattice \( \mathcal{L} \) is the lattice of flats of a matroid \( \text{M}(\mathcal{L}) \) defined on its set of atoms. By construction, \( \text{M}(\mathcal{L}) \) has no loops or parallel edges. We note that \( \text{M}(\mathcal{L}_{\leq x}) = \text{M}(\mathcal{L})|_{x} \), the restriction matroid. When \( \mathcal{L} \) is a locally geometric semilattice, there is not necessarily a matroid associated to \( \mathcal{L} \), even though we have a matroid \( \text{M}(\mathcal{L}_{\leq x}) \) associated to each \( x \in \mathcal{L} \).

At this point, we note that the locally geometric semilattices considered here include the geometric semilattices studied by Wachs and Walker [WW86], and we briefly compare the two notions. We will say a simplex \( S \in \text{at}(\mathcal{L}) \) is an independent set if \( |S| = d(0, \vee S) \). That is, \( S \) is independent when it is a basis in the matroid \( \text{M}(\mathcal{L}_{\leq \vee S}) \) of the geometric lattice \( \mathcal{L}_{\leq \vee S} \).

A geometric semilattice \( \mathcal{L} \) is characterized by properties “(G3)” and “(G4)” in [WW86]: the first is that its intervals are geometric, and the second is that the independent sets in \( \text{at}(\mathcal{L}) \), taken all together, are the independent sets of a matroid on all of \( \text{a}(\mathcal{L}) \). We are led to relax this second condition, because the poset of boundary strata in the wonderful compactification is, in general, locally geometric but not geometric. However, we will show that some of the combinatorics of geometric semilattices (e.g., \( \text{nbc} \)-bases) can be generalized without difficulty.

**Definition 2.3.4 (A combinatorial blowup).** For a semilattice \( \mathcal{L} \) and an element \( p \in \mathcal{L}_{+} \), we define a poset \( \text{Bl}_{p}(\mathcal{L}) \) together with subposets \( \mathcal{L}_{(p)} \) and \( \mathcal{L}'_{(p)} \) as follows. Let

\[
\mathcal{L}_{(p)} = \{ x \in \mathcal{L} : x \not\geq p, \text{ and } p \vee x \text{ exists in } \mathcal{L} \},
\]

\[
\mathcal{L}'_{(p)} = \mathcal{L}_{(p)} \cup \{ (p, x) : x \in \mathcal{L}_{(p)} \},
\]

and

\[
\text{Bl}_{p}(\mathcal{L}) = \{ x \in \mathcal{L} : x \not\geq p \} \cup \{ (p, x) : x \in \mathcal{L}_{(p)} \},
\]

with order relations

(i) \( y > x \in \text{Bl}_{p}(\mathcal{L}) \) if \( y > x \in \mathcal{L} \),
(ii) \( (p, y) > (p, x) \in \text{Bl}_{p}(\mathcal{L}) \) if \( y > x \in \mathcal{L} \),
(iii) \( (p, y) > x \in \text{Bl}_{p}(\mathcal{L}) \) if \( y \geq x \in \mathcal{L} \),

where in all three cases \( y, x \not\geq p \).

The poset \( \text{Bl}_{p}(\mathcal{L}) \) is called the combinatorial blowup of \( \mathcal{L} \) at \( p \). Note that the atoms in \( \text{Bl}_{p}(\mathcal{L}) \) are the atoms in \( \mathcal{L} \) together with the new element \( (p, 0) \) that can be thought of as the result of blowing up \( p \).

The construction comes with an order-preserving map \( \pi : \text{Bl}_{p}(\mathcal{L}) \to \mathcal{L} \) given by \( \pi(x) = x \) and \( \pi(p, x) = p \vee x \) for all elements \( x \) and \( (p, x) \) in \( \text{Bl}_{p}(\mathcal{L}) \).

Here we record some basic properties.

**Lemma 2.3.5.** Let \( \mathcal{L} \) be a semilattice and \( p \in \mathcal{L}_{+} \). Let \( \alpha \) denote the poset map
\[
\alpha : \mathcal{L}_{(p)} \times \{ 0 < 1 \} \cong \mathcal{L}'_{(p)} \to \text{Bl}_{p}(\mathcal{L}).
\]
Then $\alpha$ is an open embedding, and

$$\text{Bl}_p(\mathcal{L}) - \text{im}(\alpha) = \{ x \in \mathcal{L} : p \lor x \text{ does not exist} \} .$$

**Proof.** The second statement follows from the definition. For the first, we check that the complement of the image is a closed set, which is to say that if $y \geq x$ for some $x \notin \text{im}(\alpha)$, then $y \notin \text{im}(\alpha)$. For this we just observe that, if $p \lor x$ does not exist and $y \geq x$, then $p \lor y$ does not exist either. \hfill \Box

**Proposition 2.3.6.** If $\mathcal{L}$ is a locally geometric semilattice, so is $\text{Bl}_p(\mathcal{L})$.

**Proof.** The fact that $\text{Bl}_p(\mathcal{L})$ is a semilattice appears as [FK04, Lem. 3.2]. To see that it is locally geometric, we make the following two observations. When $x \in \mathcal{L}_{\geq p}$, the intervals $[0, x]$ in $\mathcal{L}$ and $[0, x]$ in $\mathcal{L}'$ are isomorphic. When $x \in \mathcal{L}_{(p)}$, the interval $[0, (p, x)]$ in $\mathcal{L}'$ is isomorphic to $[0, x] \times \{ 0 < 1 \}$. \hfill \Box

**Proposition 2.3.7.** If $\mathcal{L}$ is a locally geometric semilattice, the map $\pi : \text{Bl}_p(\mathcal{L}) \to \mathcal{L}$ is surjective.

**Proof.** In fact, we only require $\mathcal{L}$ to be atomic. If $x \not\geq p$, then $x = \pi(x)$. If $x \geq p$, it follows easily from the atomic property that $x = p \lor y$ for some $y \not\geq p$, in which case $x = \pi(y)$. \hfill \Box

**Example 2.3.8.** Let $\mathcal{L}$ be a geometric lattice with maximum element $\hat{1}$. Then

$$\text{Bl}_1(\mathcal{L}) = \text{im}(\alpha) = (L_{<1}) \cup \{ (\hat{1}, p) : p \in L_{<1} \} ,$$

$\text{Bl}_1(\mathcal{L})$ is a geometric semilattice, and $\text{at}(\text{Bl}_1(\mathcal{L})) \cong \{ \hat{1} \} \star \text{at}(L_{<1})$. The simplices of $\text{at}(L_{<1})$ are the non-spanning sets of the matroid $M(\mathcal{L})$.

### 2.4. Building sets.

Here we recall what we need from [FK04] about combinatorial building sets and nested sets in the special case of locally geometric semilattices.

Recall that an element $x$ in a semilattice $\mathcal{L}$ is said to be irreducible if $[0, x]$ is not isomorphic to the direct product of two (nontrivial) posets. The reduced Euler characteristic of the order complex of the open interval $(\hat{0}, x)$ is known as the $\beta$-invariant of the geometric lattice $L_{\leq x}$, written $\beta(L_{\leq x})$. It is a classical fact that $\beta(L_{\leq x}) \neq 0$ if and only if $x$ is irreducible [Cra67].

Let $\mathcal{L}_{\text{irr}}$ denote the set of irreducible elements of $\mathcal{L}_{+}$. We note $a(\mathcal{L}) \subseteq \mathcal{L}_{\text{irr}}$ for any $\mathcal{L}$. For any $x \in \mathcal{L}_{+}$, the set of elementary divisors of $x$ is defined to be

$$\text{div}(x) := \text{max}(L_{\text{irr}} \cap [\hat{0}, x]) :$$

obviously, $|\text{div}(x)| = 1$ if and only if $x$ is irreducible. In fact, for any $x \in \mathcal{L}_{+}$, the join map

$$\nabla : \prod_{y \in \text{div}(x)} [\hat{0}, y] \to [\hat{0}, x]$$

is an isomorphism [FK04, Prop. 2.1]. In the language of matroids, the set $\text{div}(x)$ indexes the connected components of $M(L_{\leq x})$. In particular, $M(L_{\leq x})$ is connected when $x$ is irreducible.

The isomorphism (5) inspires the definition of *combinatorial building sets* as stated in [FK04, Def. 2.2]. The geometric version of this definition goes back to work of De Concini and Procesi [DCP95], where a building set indexes the irreducible components of the normal crossings divisor in a wonderful compactification $Y(\mathcal{A}, \mathcal{G})$ of an arrangement complement $U(\mathcal{A})$. 
Definition 2.4.1 (Combinatorial building sets). Let $\mathcal{L}$ be a semilattice. A subset $\mathcal{G}$ in $\mathcal{L}_+$ is called a (combinatorial) building set if for any $x \in \mathcal{L}_+$ and $\max \mathcal{G}_{\leq x} = \{x_1, \ldots, x_k\}$ there is an isomorphism of posets

$$\varphi_x : \prod_{j=1}^k [\hat{0}, x_j] \rightarrow [\hat{0}, x]$$

with $\varphi_x(\hat{0}, \ldots, x_j, \ldots, \hat{0}) = x_j$ for $j = 1, \ldots, k$. We let $F(\mathcal{L}, \mathcal{G}; x) = \max \mathcal{G}_{\leq x}$, the set of factors of $x$.

It is always the case that, if $\mathcal{G}$ is a building set in $\mathcal{L}$, we have $\mathcal{L}_\text{irr} \subseteq \mathcal{G}$, and in particular $a(\mathcal{L}) \subseteq \mathcal{G}$.

For expository simplicity, we will always assume that, if $\mathcal{L}$ has a maximum element $\hat{1}$, then $\hat{1} \in \mathcal{G}$. Having a maximum element means $\mathcal{L}$ is the lattice of flats of a matroid, and $\hat{1}$ is irreducible exactly when the matroid is connected, in which case the condition is satisfied automatically.

We highlight a useful property of building sets, which states that the set of factors of $x \in \mathcal{L}$ induce a partition of $\mathcal{G}_{\leq x}$:

Proposition 2.4.2 ([FK04, Prop. 2.5(1)]). Suppose that $x \in \mathcal{L}$ and $p \in \mathcal{G}$ satisfies $p \leq x$. Then there is a unique $g \in F(\mathcal{L}, \mathcal{G}; x)$ for which $p \leq g$.

For instance, this provides a tool to compare join decompositions of comparable elements.

Proposition 2.4.3. Let $y \in \mathcal{L}_+$ and $h \in a(\mathcal{L})$ for which $h \not\leq y$. Write the factors of $y \lor h \in \mathcal{L}$ as $F(\mathcal{L}, \mathcal{G}; y \lor h) = \{x_1, \ldots, x_t\}$ where $h \leq x_t$. Then there is a unique $y' \in [\hat{0}, x_t]$ such that $F(\mathcal{L}, \mathcal{G}; y) = \{x_1, \ldots, x_{t-1}\} \cup F(\mathcal{L}, \mathcal{G}; y')$.

Proof. By the join decomposition (6) in Definition 2.4.1, there exist unique $z_i \in [\hat{0}, x_i]$ such that $y = \hat{1} \lor \cdots \lor z_t$. Since $h \not\leq y$ and $h \leq x_t$, we must have $z_i \leq x_t$ and $z_i = x_i$ for $i < t$. Thus, $y = x_1 \lor \cdots \lor x_{t-1} \lor y'$ with $y' = z_t$. Write $F(\mathcal{L}, \mathcal{G}; y') = \{y_1, \ldots, y_k\}$, so that our claim is thus $F(\mathcal{L}, \mathcal{G}; y) = \{x_1, \ldots, x_{t-1}, y_1, \ldots, y_k\}$. This fact was proven in [FK04, p. 44].

2.5. Sequences of combinatorial blowups. De Concini and Procesi’s wonderful compactification can be constructed by iterated blowups of $\mathbb{P}^d$, along (proper transforms of) linear subspaces indexed by a building set. The combinatorial version of this applies combinatorial blowups (as in §2.3) sequentially.

From now on, our initial semilattice will be, in fact, a geometric lattice, which we will denote by $\mathcal{L}$. We will also restrict ourselves to blowups of elements of $\mathcal{L}$. The intermediate semilattices in this process are indexed by partial building sets, which we define below.

Definition 2.5.1 (Partial building sets). Let $\mathcal{L}$ be a geometric lattice, and let $\mathcal{G} \subseteq \mathcal{L}_+$ be a building set. A partial building set is a subset $\mathcal{H} \subseteq \mathcal{G}$ for which $a(\mathcal{L}) \subseteq \mathcal{H}$ and $\mathcal{H}^\circ := \mathcal{H} \setminus a(\mathcal{L})$ is an order filter (upward-closed subset) of $\mathcal{G}$.

Fix a (reverse) linear extension $\prec$ of $\mathcal{H}$ so that $p \geq q$ implies $p \prec q$. Write $\mathcal{H}^\circ = \{p_1, \ldots, p_m\}$ where $p_1 \prec \cdots \prec p_m$. The partial blowup of $\mathcal{L}$ is defined as the semilattice

$$\mathcal{L}(\mathcal{L}, \mathcal{H}) := \text{Bl}_{p_m} \circ \cdots \circ \text{Bl}_{p_2} \circ \text{Bl}_{p_1}(\mathcal{L}).$$

The semilattice $\mathcal{L}(\mathcal{L}, \mathcal{H})$ does not depend on the choice of linear extension $\prec$. We point out that even though $a(\mathcal{L}) \subseteq \mathcal{H}$, we are only blowing up the elements of $\mathcal{H}^\circ = \mathcal{H} \setminus a(\mathcal{L})$ in our sequence. It is, however, straightforward to check that $\text{Bl}_p(\mathcal{L}(\mathcal{L}, \mathcal{G})) \cong \mathcal{L}(\mathcal{L}, \mathcal{G})$ when $p \in a(\mathcal{L})$. 

By construction, \( a(\mathcal{L}(\mathcal{L}, \mathcal{H})) = \mathcal{H} \) and if \( \mathcal{H}' = \mathcal{H} \cup \{p\} \) is also a partial building set, \( \mathcal{L}(\mathcal{L}, \mathcal{H}) = \text{Bl}_p(\mathcal{L}(\mathcal{L}, \mathcal{H})) \). It will be convenient simply to write \( p \) for the atom \((p, 0)\) in \( \mathcal{L}(\mathcal{L}, \mathcal{H}) \).

In order to avoid confusion, we will only do so when it is clear that \( p \) refers to an atom, rather than an element of higher rank.

Recall that for a semilattice \( \mathcal{L} \), we have a natural map \( \pi : \text{Bl}_p(\mathcal{L}) \to \mathcal{L} \). If \( \mathcal{H} \subseteq \mathcal{H}' \) are partial building sets, then \( \mathcal{L}(\mathcal{L}, \mathcal{H}') \) is obtained from \( \mathcal{L}(\mathcal{L}, \mathcal{H}) \) through a sequence of combinatorial blowups, and we let

\[
\pi_{\mathcal{H}'}^\mathcal{H} : \mathcal{L}(\mathcal{L}, \mathcal{H}') \to \mathcal{L}(\mathcal{L}, \mathcal{H})
\]

denote the composite of these natural maps. If \( \mathcal{H} = a(\mathcal{L}) \), then \( \mathcal{L}(\mathcal{L}, \mathcal{H}) = \mathcal{L} \), and we will simply write \( \pi_{\mathcal{H}'} : \mathcal{L}(\mathcal{L}, \mathcal{H}') \to \mathcal{L} \).

Remark 2.5.2. Let \( \mathcal{H} \subseteq \mathcal{G} \) be a partial building set of a geometric lattice \( \mathcal{L} \). Since \( \mathcal{H} \) is an order filter of \( \mathcal{G} \), the elements of \( \mathcal{G} \setminus \mathcal{H} \) are elements of \( \mathcal{L} \) which survive the sequence of blowups prescribed by \( \mathcal{H} \). Identifying \( \mathcal{H} \) as the set of atoms in \( \mathcal{L}(\mathcal{L}, \mathcal{H}) \), we may thus view \( \mathcal{G} \) as a subset of \( \mathcal{L}(\mathcal{L}, \mathcal{H}) \). In fact, by [FK04, Prop. 3.3], the set \( \mathcal{G} \) is a building set for the semilattice \( \mathcal{L}(\mathcal{L}, \mathcal{H}) \). \( \top \)

In contrast to arbitrary sequences of combinatorial blowups, our constructions have several useful properties. The first follows from [FK04, Prop. 2.3], while the second is a straightforward generalization of [DCP95, Thm. 2.3(3b')], and the third follows from the fact that a partial building set is an order filter.

Proposition 2.5.3. Let \( \mathcal{H} \) be a partial building set in a building set \( \mathcal{G} \) in \( \mathcal{L} \).

(a) Suppose \( \mathcal{H}' = \mathcal{H} \cup \{p\} \) is also a partial building set. Then the (open) upper order interval \( \mathcal{L}(\mathcal{L}, \mathcal{H})_{\geq p} \) is disjoint from \( \mathcal{G} \). In particular, it contains no irreducible elements.

(b) Suppose \( x, q \in \mathcal{G} \) for which \( x \nleq q \) and \( x \land q \neq 0 \). If \( q \in \mathcal{H} \), then \( x \lor q \in \mathcal{H} \) too.

(c) If \( p \in \mathcal{G} \setminus \mathcal{H} \) and \( S \subseteq \mathcal{H} \), then \( S_{<p} \subseteq a(\mathcal{L}) \).

We can keep track of how the factors of an element \( y \in \mathcal{L}(\mathcal{L}, \mathcal{H}') \) behave under a single blow-down, viewing \( \mathcal{G} \) as a building set in both \( \mathcal{L}(\mathcal{L}, \mathcal{H}) \) and \( \mathcal{L}(\mathcal{L}, \mathcal{H}') \) (via Remark 2.5.2).

Proposition 2.5.4. In the notation above, let \( y \in \mathcal{L}(\mathcal{L}, \mathcal{H}') \). Then

\[
F(\mathcal{L}(\mathcal{L}, \mathcal{H}), \mathcal{G}; \pi_{\mathcal{H}'}(y)) = \begin{cases}
F(\mathcal{L}(\mathcal{L}, \mathcal{H}'), \mathcal{G}; y) & \text{if } \pi(y) \geq p;
F(\mathcal{L}(\mathcal{L}, \mathcal{H}'), \mathcal{G}; y) & \text{otherwise}.
\end{cases}
\]

Proof. We abbreviate \( \pi = \pi_{\mathcal{H}'}^\mathcal{H} \), \( \mathcal{L} = \mathcal{L}(\mathcal{L}, \mathcal{H}) \) and \( \mathcal{L}' = \mathcal{L}(\mathcal{L}, \mathcal{H}') = \text{Bl}_p(\mathcal{L}) \). First, if \( \pi(y) \nleq p \), then the restriction of \( \pi : [0, y] \to [0, \pi(y)] \) is an isomorphism, and the conclusion follows.

Otherwise, \( y = (p, x) \) for some \( x \in \mathcal{L}(p) \) and \( p \in F(\mathcal{L}', \mathcal{G}; y) \). The set \( F(\mathcal{L}', \mathcal{G}; y)_{<p} = F(\mathcal{L}', \mathcal{G}; x) \cup \{p\} \) contains pairwise incomparable elements in \( \mathcal{L} \) whose join is \( \pi(y) = p \lor x \), so by [FK04, Prop. 2.8], it equals \( F(\mathcal{L}, \mathcal{G}; \pi(y)) \).

If the element \( 1 \) is contained in a partial building set \( \mathcal{H} \), clearly it must come first in any linear order. This leads to the following observation:

Proposition 2.5.5. If \( \hat{1} \in \mathcal{H} \), then \( \hat{1} \) is a cone vertex in the atomic complex: that is,

\[
\text{at}(\mathcal{L}(\mathcal{L}, \mathcal{H})) \cong \{\hat{1}\} \ast \text{lk}_1(\text{at}(\mathcal{L}(\mathcal{L}, \mathcal{H}))).
\]

Proof. If \( \mathcal{H} = \{\hat{1}\} \cup a(\mathcal{L}) \), this is Example 2.3.8. In general, it follows by induction on \( |\mathcal{H}| \). \( \square \)

We conclude this section with a simple example.
Example 2.5.6. Let $L$ be the lattice of the rank-3 matroid on $[5]$ with two 3-element flats, $\{1, 2, 4\}$ and $\{1, 3, 5\}$. This is the intersection lattice of the hyperplane arrangement defined, in order, by the linear forms $(x, y, z, x - y, x - z)$. Let $G = G_{\text{min}}$ be the minimal building set, ordered

$\hat{1} \prec 124 \prec 135 \prec 1 \prec \cdots \prec 5$.

$G$ contains three (nontrivial) partial building sets. Their atomic complexes are cones over the complexes drawn in Figure 1.

2.6. Nested sets. An important concept related to building sets are nested sets, which form an abstract simplicial complex. In the geometric setting, the faces of the nested set complex index the non-trivial intersections of divisor components in the wonderful compactification. Here, we extend the notion of nested sets to partial building sets. Once again they form an abstract simplicial complex, and in Theorem 2.6.7 we will show that this is none other than the atomic complex for the semilattice $L(L, H) = \text{Bl}_p \circ \cdots \circ \text{Bl}_p \circ \text{Bl}_p(L)$. In the classical case, nested sets are independent sets in the semilattice of the blowup; in our generalization, nested sets are no longer necessarily independent, and the nested set complex is no longer pure in general (see Figure 1 for examples).

Throughout this section, let $H$ be a partial building set in a geometric lattice $L$.

Definition 2.6.1 (Nested set complex). A subset $S$ of $H$ is called $H$-nested if, for any set of incomparable elements $x_1, \ldots, x_t$ in $S$ of cardinality at least two, the join $x_1 \lor \cdots \lor x_t$ is not contained in $H$. The $H$-nested sets form an abstract simplicial complex $n(L, H)$, called the nested set complex with respect to $H$.

A set of pairwise incomparable elements will be called an antichain, and an antichain with at least two elements will be called a nontrivial antichain.

In the definition of a nested set, we emphasize that we are comparing elements and using the join in the original lattice $L$, rather than in $L(L, H)$. Observe that if $H$ is chosen to be the maximal building set $L_+$, then the nested set complex coincides with the order complex of $L$.

First, we recall a useful relationship between nested sets and the join decomposition of Definition 2.4.1(6).

Proposition 2.6.2 ([FK04, Prop. 2.8]). Let $G$ be a (full) building set, and let $S \in n(L, G)$. Then $\max S = F(L, G; y)$ where $y = \lor S \in L$. That is, the maximal elements of $S$ in $L$ are the factors of $y$ in the join decomposition.

Our immediate goal is to show that $n(L, H) = \text{at}(L(L, H))$, which we do in Theorem 2.6.7. The two extremes for choices of $H$ will be fundamental, and we state these here:
Proposition 2.6.3. If either \( H = G \) is a building set or \( H = a(L) \), then \( \text{at}(L(H)) = n(L, H) \).

Proof. When \( H = G \), [FK04, Thm. 3.4] states that \( L(G) \) is the face poset of the simplicial complex \( n(L, G) \), which means that \( \text{at}(L(G)) = n(L, G) \).

On the other hand, when \( H = a(L) \), both \( \text{at}(L(H)) \) and \( n(L, H) \) consist of all subsets of \( H \), hence they are both full simplices of dimension \( |a(L)| - 1 \). \( \square \)

In order to deduce Theorem 2.6.7 from these extremes, we will use the following map to “blow-down” subsets of a partial building set. Suppose that \( H \subseteq H' \) are partial building sets obtained from a building set \( G \) in a geometric lattice \( L \). Recall that we may identify the atoms \( a(L, H') \) with \( H' \) (and similarly for \( H \)): through this identification, define for \( S \subseteq H' \) a set

\[
\Pi(S) = \Pi_H^H(S) = \left\{ h \in H : h \leq \pi_H^H(g) \text{ for some } g \in S \right\}.
\]

That is, one first blows down the elements of \( S \) from \( L(H') \) to \( L(H) \) and then collects the atoms of \( L(H, H') \) lying underneath. We can view \( \Pi \) as a function on the atomic complexes via

\[
\Pi : \text{at}(L(H')) \rightarrow L(H') \xrightarrow{\pi_H^H} L(L(H)) \xrightarrow{\text{supp}} \text{at}(L(L,H)),
\]

where supp was defined in Definition 2.3.2(3). Clearly, in the case where \( H' = H \cup \{ p \} \), we have

\[
(8) \quad \Pi(S) = \begin{cases} S & \text{if } p \notin S; \\ (S - \{ p \}) \cup \text{supp}(p) & \text{otherwise}. \end{cases}
\]

In Lemma 2.6.5 below, we will see that \( \Pi \) can also be viewed as a function on the nested set complexes. Note, however, that \( \Pi \) is not a simplicial map; that is, an \( i \)-simplex is not necessarily sent to an \( i \)-simplex.

Example 2.6.4. Recall the lattice \( L \) from Example 2.5.6. The nested set complex (equivalently, atomic complex) for each partial building set \( H \subseteq G = \{1, 124, 135\} \) is a cone over the corresponding complex depicted in Figure 1.

The function \( \Pi \) sends the vertex 135 in \( n(L, G) \) to the 2-simplex \( \{1, 3, 5\} \) in \( n(L, H) \), where \( H \) is either \( \{1\} \) or \( \{1, 124\} \). Observe that the set \( \{1, 3, 5\} \) is \( H \)-nested but not \( G \)-nested.

As another example, the 1-simplex \( S = \{2, 124\} \in n(L, G) \) is preserved when blowing down to \( n(L, \{1, 124\}) \); that is, \( \Pi(S) = S \). But then applying \( \Pi \) again, one obtains the 2-simplex \( \{1, 2, 4\} \) in \( n(L, \{1\}) \). \( \diamond \)

Lemma 2.6.5. Let \( H \subseteq H' \) be partial building sets in a geometric lattice \( L \). If \( S \) is \( H' \)-nested, then \( \Pi_H^H(S) \) is \( H \)-nested. That is, \( \Pi_H^H \) defines a function \( n(L, H') \rightarrow n(L, H) \).

Proof. Let \( \Pi = \Pi_H^H \) for short. It suffices to check the case that \( H' = H \cup \{ p \} \). Suppose we have a set \( S \subseteq H' \) for which \( \Pi(S) \) is not \( H \)-nested: that is, there is a nontrivial antichain \( T \) contained in \( \Pi(S) \) whose join \( \bigvee T \) is an element of \( H \). We will show that this implies \( S \) is not \( H' \)-nested. We have two cases, depending on whether \( T \subseteq S \).

First, suppose that \( T \subseteq S \). Then \( S \) contains a nontrivial antichain \( T \) whose join is an element of \( H \), hence also \( H' \) since \( H \subseteq H' \). Thus, \( S \) is not \( H' \)-nested.

Now suppose that \( T \nsubseteq S \). Then \( p \in S \) and \( T_{<p} \neq \emptyset \) (by (8)), so \( \emptyset \nsubseteq \bigvee T_{<p} \leq (\bigvee T) \wedge p \). Since \( \bigvee T \in H \subseteq H' \) and \( p \in H' \), this implies that \( (\bigvee T) \vee p \in H' \) by Proposition 2.5.3(b). Now, since \( T \subseteq H \) and \( p \in H' \setminus H \), we must have \( T_{<p} \neq \emptyset \). It follows that the set \( T_{<p} \cup \{ p \} \) is a nontrivial antichain contained in \( S \). But \( (\bigvee T_{<p}) \vee p = (\bigvee T) \vee p \in H' \), which means that \( S \) cannot be \( H' \)-nested. \( \square \)
Lemma 2.6.6. Let $\mathcal{H} \subseteq \mathcal{H}'$ be partial building sets in a geometric lattice $L$. For any simplex $S \in \mathfrak{at}(L(H))$, there is some simplex $T \in \mathfrak{at}(L(H'))$ such that $S \subseteq \Pi_{H}^{H'}(T)$.

**Proof.** Let us abbreviate $\Pi = \Pi_{H}^{H'}$. Again, it suffices to check the case that $\mathcal{H}' = \mathcal{H} \cup \{p\}$, where $\mathcal{L}(L, \mathcal{H'}) = \text{Bl}_{p}(\mathcal{L}(L, \mathcal{H}))$. Let $S \in \mathfrak{at}(L(L, \mathcal{H}))$, and write $S = S_{\not \in \mathcal{H}} \cup S_{\in \mathcal{H}}$. We have two cases: either $\bigvee S_{\in \mathcal{H}} = p$ or $\bigvee S_{\in \mathcal{H}} < p$.

If $\bigvee S_{\in \mathcal{H}} = p$, then we claim $T = S_{\notin \mathcal{H}} \cup \{(p, \emptyset)\}$ is a simplex in $\mathfrak{at}(L(L, \mathcal{H}'))$ with $S \subseteq \Pi(T)$. It is clear that $S \subseteq \Pi(T)$. To show that $T$ is indeed a simplex, we first note that $\bigvee S_{\in \mathcal{H}}$ exists in $\mathcal{L}(L, \mathcal{H}')$, since this join exists and is not above $p$ in $\mathcal{L}(L, \mathcal{H})$. Moreover, since $\bigvee S_{\in \mathcal{H}} = p$ the join $\bigvee S_{\in \mathcal{H}} \vee p = \bigvee S$ exists in $\mathcal{L}(L, \mathcal{H})$, so $\bigvee T = (p, \bigvee S_{\in \mathcal{H}}) \in \mathcal{L}(L, \mathcal{H}')$.

Otherwise, $\bigvee S_{\in \mathcal{H}} < p$, and we claim $T = S$ is in $\mathfrak{at}(L(L, \mathcal{H}'))$. This is true because $\bigvee S$ exists and is not above $p$ in $\mathcal{L}(L, \mathcal{H})$, hence the join exists in $\mathcal{L}(L, \mathcal{H}')$. □

We are now ready to prove that the nested set complex and atomic complex of $\mathcal{L}(L, \mathcal{H})$ agree, which we noted in Proposition 2.6.3 is a fundamental fact in the case where $\mathcal{H}$ is a full building set.

**Theorem 2.6.7.** Let $L$ be a geometric lattice and $\mathcal{H} \subseteq L_{+}$ a partial building set. Then a set $S \subseteq \mathcal{H}$ is $\mathcal{H}$-nested if and only if $S \in \mathfrak{at}(L(L, \mathcal{H}))$.

**Proof.** Let $\mathcal{G}$ be a building set that contains $\mathcal{H}$.

First, assume that $S \in \mathfrak{at}(L(L, \mathcal{H}))$. By Lemma 2.6.6, there is a simplex $T \in \mathfrak{at}(L(L, \mathcal{G}))$ for which $\Pi_{H}^{G}(T) \supseteq S$. Since $\mathcal{G}$ is a (full) building set, $T$ is $\mathcal{G}$-nested (see Proposition 2.6.3), which implies that $\Pi_{H}^{G}(T)$ is $\mathcal{H}$-nested by Lemma 2.6.5. The property of being $\mathcal{H}$-nested is inherited by subsets, so $S$ must also be $\mathcal{H}$-nested.

For the converse, we use induction on $|H|$. The base case, when $\mathcal{H} = a(L)$, was mentioned in Proposition 2.6.3. Now assume that every $\mathcal{H}$-nested set is a simplex of $\mathfrak{at}(L(L, \mathcal{H}))$, and we will show that the same holds for the next partial building set $\mathcal{H}' = \mathcal{H} \cup \{p\} \subseteq \mathcal{G}$. Assume that $S$ is $\mathcal{H}'$-nested, and let $\Pi = \Pi_{H}^{H'}$. By Lemma 2.6.5, $\Pi(S)$ is $\mathcal{H}$-nested hence a simplex in $\mathfrak{at}(L(L, \mathcal{H}))$. This means that $\bigvee \Pi(S)$ exists in $\mathcal{L}(L, \mathcal{H})$. We have two cases, depending on whether $p \in S$.

If $p \in S$, then $\Pi(S) = S_{\not \in \mathcal{H}} \cup \{g \in a(L): g < p\}$. Consider $\bigvee S_{\not \in \mathcal{H}}$ in $\mathcal{L}(L, \mathcal{H})$, which exists because $\bigvee \Pi(S)$ exists. Since $S$ is $\mathcal{H}'$-nested, we must have $\bigvee S_{\in \mathcal{H}} \neq p$ and hence $\bigvee S_{\not \in \mathcal{H}} \neq p$. Thus, $\bigvee S_{\not \in \mathcal{H}}$ exists in $\mathcal{L}(L, \mathcal{H})$. Moreover, since $\bigvee S_{\not \in \mathcal{H}} \vee p = \bigvee \Pi(S)$ exists in $\mathcal{L}(L, \mathcal{H})$, we have $\bigvee S = (p, \bigvee S_{\not \in \mathcal{H}}) \in \mathcal{L}(L, \mathcal{H}')$. Therefore, $S \in \mathfrak{at}(L(L, \mathcal{H}'))$.

Otherwise, $p \notin S$ and $\Pi(S) = S$. The order of blowups implies that $S_{\in \mathcal{H}} \subseteq a(L)$ (Proposition 2.5.3(c)); in particular, $S_{\in \mathcal{H}}$ is an antichain. We claim that $\bigvee S_{\in \mathcal{H}} \neq p$. This is immediate if it has less than two elements, and otherwise $S_{\in \mathcal{H}}$ is a nontrivial antichain and the claim follows from the assumption that $S$ is $\mathcal{H}'$-nested. Now since $\bigvee S_{\in \mathcal{H}} \neq p$, we have $\bigvee S_{\in \mathcal{H}} < p$; in particular, $\bigvee S_{\in \mathcal{H}}$ exists and is not above $p$ in $\mathcal{L}(L, \mathcal{H})$. By the construction of a blowup, then $\bigvee S$ exists in $\mathcal{L}(L, \mathcal{H}')$, which is to say $S \in \mathfrak{at}(L(L, \mathcal{H}'))$. □

In the next section, we will study how blowups affect nested sets. Before doing so, though, we include one more easy property for later reference.

**Proposition 2.6.8.** Let $\mathcal{H}$ be a partial building set in a geometric lattice $L$, and let $S \subseteq \mathcal{H}$ be a nontrivial antichain. If $S$ is $\mathcal{H}$-nested, then $\land S = 0$.

**Proof.** For a contradiction, assume $S$ is $\mathcal{H}$-nested and $\land S \neq 0$. Let $g, g' \in S$. Since $S$ is an antichain with $\land S \neq 0$, and $g$ and $g'$ are incomparable with $g \land g' \neq 0$. By Proposition 2.5.3(b), $g \lor g' \in \mathcal{H}$, which would contradict nestedness of $\{g, g'\} \subseteq S$. □
2.7. Blowing up nested sets. There is another map on nested sets which we will use, for example, to deal with the join decomposition in Definition 2.4.1(6) throughout our sequence of blowups. This map will be of particular importance when working with local building sets in §2.8. To define it, suppose that \( \mathcal{H} \) and \( \mathcal{H}' = \mathcal{H} \cup \{p\} \) are partial building sets in \( \mathcal{L} \). For a subset \( S \subseteq \mathcal{H} \), define a subset \( \eta(S) \subseteq \mathcal{H}' \) by

\[
\eta(S) = \eta^\mathcal{H}_\mathcal{H}'(S) = \begin{cases} S & \text{if } \bigvee S_{<p} \neq p \\ (S_{<p}) \cup \{p\} & \text{if } \bigvee S_{<p} = p \end{cases}
\]

If \( \mathcal{H} \) and \( \mathcal{H}' = \mathcal{H} \cup \{p_{m+1}, \ldots, p_n\} \) are partial building sets, we can define \( \eta_{\mathcal{H}}^\mathcal{H}' \) as a composition of these maps, and it is particularly useful to consider \( \mathcal{H}' = \mathcal{G} \) (a full building set). Note that just as with \( \Pi \), while \( \eta \) is a function on simplicial complexes (via Lemma 2.7.4 below), it is not a simplicial map.

Example 2.7.1. Recall the lattice \( \mathcal{L} \) from Example 2.5.6 and Figure 1. For the 2-simplex \( S = \{1, 2, 4\} \) and the 1-simplex \( T = \{2, 4\} \) in \( \mathcal{n}(\mathcal{L}, a(\mathcal{L})) \), we have \( \eta(S) = \eta(T) = \{124\} \) in both \( \mathcal{n}(\mathcal{L}, \mathcal{H}) \) (with \( \mathcal{H}' = \{1, 124\} \)) and \( \mathcal{n}(\mathcal{L}, \mathcal{G}) \).

The 1-simplex \( \{2, 3\} \) is preserved by \( \eta \).

Proposition 2.7.2. For \( S \subseteq \mathcal{H} \), we have

\[
\eta_{\mathcal{H}}^\mathcal{H}'(S) = \eta_{\mathcal{H}}^\mathcal{G}(S \cap a(\mathcal{L})) \cup (S \cap \mathcal{H}^\circ).
\]

In particular, if \( S \subseteq \mathcal{H}^\circ \), then \( \eta_{\mathcal{H}}^\mathcal{H}'(S) = S \).

Proof. The fact that \( S \cap \mathcal{H}^\circ \) does not interact with the sequence of blowups follows from the observation that \( S_{<p} \subseteq a(\mathcal{L}) \) whenever \( p \in \mathcal{G} - \mathcal{H} \) (Proposition 2.5.3(c)).

Remark 2.7.3. Recall from Remark 2.5.2, we may view \( \mathcal{G} \) as a building set for the semilattice \( \mathcal{L}(\mathcal{L}, \mathcal{H}) \). For an \( \mathcal{H} \)-nested set \( S \), the factors of \( z = \bigvee S \in \mathcal{L}(\mathcal{L}, \mathcal{H}) \) with respect to the building set \( \mathcal{G} \subseteq \mathcal{L}(\mathcal{L}, \mathcal{H}) \) are given by \( \eta_{\mathcal{H}}^\mathcal{G}(S) \). More explicitly,

\[
F \left( \mathcal{L}(\mathcal{L}, \mathcal{H}), \mathcal{G}; \bigvee S \right) = \eta_{\mathcal{H}}^\mathcal{G}(S).
\]

Indeed, by Proposition 2.4.2, the factors of \( z \) index a partition of the atoms in \( \mathcal{L}(\mathcal{L}, \mathcal{H}) \) that lie below \( z \). As we apply \( \eta \) is the order of blowups, we replace the atoms below each not-yet-blown-up factor \( g \in \mathcal{G} - \mathcal{H} \) with \( g \) itself, leaving other atoms untouched.

We will show in the next two lemmas that \( \eta \) preserves nestedness as well as incomparability, leading to a useful tool in Proposition 2.7.6.

Lemma 2.7.4. Let \( \mathcal{H} \subseteq \mathcal{H}' \) be partial building sets in a geometric lattice \( \mathcal{L} \). If \( S \) is also \( \mathcal{H} \)-nested, then \( \eta_{\mathcal{H}}^\mathcal{H}'(S) \) is \( \mathcal{H}' \)-nested. That is, \( \eta_{\mathcal{H}}^\mathcal{H}' \) defines a function \( \mathcal{n}(\mathcal{L}, \mathcal{H}) \to \mathcal{n}(\mathcal{L}, \mathcal{H}') \).

Proof. It suffices to check the case that \( \mathcal{H}' = \mathcal{H} \cup \{p\} \), which we do by contrapositive. Suppose that \( S \subseteq \mathcal{H} \) with \( \eta(S) \notin \mathcal{n}(\mathcal{L}, \mathcal{H}') \); we will show that then \( S \) is not \( \mathcal{H} \)-nested. Since \( \eta(S) \) is not \( \mathcal{H}' \)-nested, there is a nontrivial antichain \( T \subseteq \eta(S) \) for which \( \bigvee T \in \mathcal{H}' \). We have two cases, depending on whether \( p \in T \).

If \( p \notin T \), then by definition of \( \eta \), \( T \subseteq S \) and \( \bigvee T \neq p \). Thus, \( \bigvee T \in \mathcal{H}' \setminus \{p\} = \mathcal{H} \) and hence \( S \) is not \( \mathcal{H} \)-nested.

Otherwise, we have \( p \in T \). Since \( T \) is a nontrivial antichain, it contains at least one element that is incomparable with \( p \). This implies that \( \bigvee T > p \) and hence \( \bigvee T \in \mathcal{H}' \setminus \{p\} = \mathcal{H} \). Consider the set

\[
U = \{ h \in S_{<p} : h \leq y \text{ for any } y \in T_{\neq p} \}
\]
Then \( U \cup T_{\not\in p} \subseteq S \) is an antichain with \( \bigvee (U \cup T_{\not\in p}) = \bigvee T \) in \( \mathcal{H} \). It remains to show that \( U \cup T_{\not\in p} \) is a nontrivial antichain, proving that \( S \) is not \( \mathcal{H} \)-nested. Since \( T_{\not\in p} \neq \emptyset \), this is immediate if \( U \neq \emptyset \). If \( U = \emptyset \), then we have \( \bigvee T_{\not\in p} = \bigvee T > p \) which implies that \( T_{\not\in p} \) has more than one element. \( \square \)

**Lemma 2.7.5.** Let \( \mathcal{H} \subseteq \mathcal{H}' \) be partial building sets in a geometric lattice \( L \). If \( S \subseteq \mathcal{H} \) is an antichain, then so is \( \eta_{\mathcal{H}'}(S) \).

Proof. Again, it suffices to check the case that \( \mathcal{H}' = \mathcal{H} \cup \{ p \} \). The statement is trivial if \( \bigvee S_{\not\in p} \neq 0 \). So we write \( S = S_{\not\in p} \cup S_{\in p} \), where \( \bigvee S_{\in p} = p \), and hence \( \eta(S) = S_{\not\in p} \cup \{ p \} \). The set \( S_{\not\in p} \) is an antichain by assumption; to show that \( \eta(S) \) is an antichain we need only show that any \( y \in S_{\not\in p} \) is incomparable with \( p \). It is clear that \( y \not\leq p \), and we also have \( y \not\geq p \) since \( y \) is incomparable with every element of \( S_{\in p} \). \( \square \)

**Proposition 2.7.6.** Let \( \mathcal{H} \) be a partial building set in a geometric lattice \( L \). Let \( S \subseteq \mathcal{H} \) and \( g \in \mathcal{H} \setminus S \). If \( S \) is a nontrivial antichain with \( \bigvee S \geq g \), and \( g > h \) for all but at most one element \( h \in S \), then \( S \) is not \( \mathcal{H} \)-nested.

Proof. For a contradiction, assume that \( S \) is an \( \mathcal{H} \)-nested nontrivial antichain with \( \bigvee S \geq g \) and \( g > h \) for all but at most one element \( h \in S \). We will consider \( \eta = \eta_{\mathcal{H}}^{\mathcal{H}} \), where \( G \supseteq \mathcal{H} \) is a (full) building set. By Lemma 2.7.4, \( \eta(S) \) is \( \mathcal{G} \)-nested. Note that by the assumption \( g > h \) for all but at most one element \( h \in S \), we must have \( g \in \mathcal{H}^{E} \). We consider two cases, depending on the size of \( \eta(S) \).

If \( \eta(S) = \{ p \} \) for some \( p \in G \), then (since \( S \) has more than one element) we must have \( p \in G \setminus \mathcal{H} \) and hence \( g \not< p \). But by assumption, \( p = \bigvee S \geq g \) and \( g \in \mathcal{H}^{E} \), contradicting the compatible order property.

So assume that \( \eta(S) \) has more than one element. By Lemma 2.7.5, the elements of \( \eta(S) \) are pairwise incomparable. Then by Proposition 2.6.2, the elements of \( \eta(S) \) must be the factors of their join \( \bigvee \eta(S) \). But since \( g > h \) for all but maybe one \( h \in S \) and \( |\eta(S)| > 1 \), there is some \( y \in \eta(S) \) such that \( g > y \). This contradicts the factors being maximal under \( \bigvee \eta(S) \), since \( \bigvee \eta(S) = \bigvee S \geq g \). \( \square \)

We conclude by remarking that while \( \eta \) and \( \Pi \) are extremely useful tools for going between the simplicial complexes along a sequence of blowups, they are not inverses of each other.

**Example 2.7.7.** Recall the lattice \( L \) from Examples 2.5.6, 2.6.4, and 2.7.1, and Figure 1. Let \( \mathcal{H}^{E} = \{ 1, 124 \} \), and let \( S = \{ 2, 124 \} \), a 2-simplex in \( n(L, \mathcal{H}) \). Then \( \Pi(S) = \{ 1, 2, 4 \} \in n(L, a(L)) \) but \( \eta(\Pi(S)) = \{ 124 \} \neq S \).

On the other hand, consider the 1-simplex \( T = \{ 2, 4 \} \) in \( n(L, a(L)) \). Then \( \eta(T) = \{ 124 \} \) in \( n(L, G) \) and \( \Pi(\eta(T)) = \{ 1, 2, 4 \} \neq T \) in \( n(L, a(L)) \). \( \square \)

### 2.8. Building set decompositions

In the geometric case, a sequence of blowups produces an arrangement of hypersurfaces whose intersections are themselves products of blowups of minors of the original arrangement. In the case where \( \mathcal{H} = G \) is a building set, this is well-known [DCP95, p. 482]. In our purely combinatorial and partially blown up setting, we will want to show that the generalized cohomology algebras we construct have analogous tensor product decompositions. Here, we establish the notation and basic results needed for such decompositions that appear in §4.1.

**Definition 2.8.1** (Local intervals). Let \( L(L, \mathcal{H}) \) be a partial blowup of a geometric lattice \( L \). For each \( y \in L(L, \mathcal{H}) \), let

\[
F^+(L(L, \mathcal{H}), G; y) := F(L(L, \mathcal{H}), G; y) \cup \{ 1 \} \subseteq G,
\]
where we recall the notation for the factors of $y$ from Definition 2.4.1, regarding $\mathcal{G}$ as a building set in the semilattice $\mathcal{L}(L, H)$ via Remark 2.5.2. Since in our discussion the building set $\mathcal{G}$ is now regarded as a fixed choice, we will often abbreviate $F^+(y) = F^+(\mathcal{L}(L, H), \mathcal{G}; y)$.

The set $F^+(y)$ is $\mathcal{G}$-nested and may alternatively be written as

$$F^+(y) = \eta^\mathcal{G}_H(\text{supp}_{\mathcal{G}}(y)) \cup \{1\},$$

using the point of view of the previous section, where $\text{supp} : \mathcal{L}(L, H) \to \mathcal{G}(L, H)$ is the map from Definition 2.3.2(3).

For each $g \in F^+(y)$, we define an interval in $L$ by

$$L_{y,g} := [z_y(g), g], \quad \text{where } z_y(g) := \bigvee_{f \in F^+(y) \atop f \leq g} f.$$

As usual, we will write $z(g)$ in place of $z_y(g)$ when $y$ is understood.

Each closed interval is $L_{y,g}$ a geometric lattice, and clearly the half-open intervals $(L_{y,g})_+ = (z_y(g), g]$ for $g \in F^+(y)$ are disjoint. In this section, we will describe an induced partial building set $H_{y,g}$ on $L_{y,g}$.

For a simplex $S$ of a simplicial complex $K$, let $\text{st}_K(S) := \{T \in K : S \subseteq T\}$ denote the star of $S$ in $K$, and $\text{st}_K(S)$ the smallest subcomplex of $K$ containing it. Let $K_0$ denote the vertices of a simplicial complex $K$. In our setting with $K = \mathcal{G}(L, H)$, we give the explicit description

$$\text{st}_{\mathcal{G}}(\mathcal{G}(L, H))(S)_0 = \{p \in H : S \cup \{p\} \in \mathcal{G}(L, H)\}.$$

**Proposition 2.8.2.** Suppose that $y \in \mathcal{L}(L, H)$ and let $S = \text{supp}(y) \in \mathcal{G}(L, H)$. For each $p \in H$, the set $\{g \in F^+(y) : p \leq g\}$ has a unique minimum element, which we denote by $\hat{p}$. Then $z_y(\hat{p}) < p \lor z_y(\hat{p})$, and the assignment $p \mapsto p \lor z_y(\hat{p}) \in (L_{y,g})_+$ defines a map

$$\zeta = \zeta_{y,H} : H \to \bigsqcup_{g \in F^+(y)} (L_{y,g})_+.$$

Furthermore, for every $q \in \text{im}(\zeta)$, there exists $f \in \text{st}_{\mathcal{G}}(\mathcal{G}(L, H))(S)_0 \cup a(L)$ such that $\zeta(f) = q$. If, in addition, $f \in \mathcal{H}^c$ and $p \neq f$ such that $\zeta(p) = q$, then $p < f$ and $p \notin \text{st}_{\mathcal{G}}(\mathcal{G}(L, H))(S)_0$.

**Proof.** The set $T := \{g \in F^+(y) : p \leq g\}$ is nonempty because $\hat{1} \in F^+(y)$. Since $\mathcal{H}_T$ is a $\mathcal{G}$-nested antichain with $\wedge(\mathcal{H}_T) \geq p > \hat{0}$, Proposition 2.6.8 implies that $T$ has a unique minimum. The assertion that $z(\hat{p}) < p \lor z(\hat{p})$ amounts to showing $p \notin z(\hat{p})$. If instead $p \leq z(\hat{p}) = \bigvee(F^+(y))_{<\hat{p}}$, then $p \leq f$ for some $f \in (F^+(y))_{<\hat{p}}$ by Proposition 2.4.2, contradicting the minimality of $\hat{p}$. Thus, the map $\zeta$ is well-defined.

Now let $q \in \text{im}(\zeta)$. If $\zeta^{-1}(q) \subseteq a(L)$, there is nothing to show, so suppose $q = p \lor z_y(\hat{p})$ for some $p \in L^c$. Let $f \in F(L, \mathcal{G}; q)$ such that $p \leq f$ (guaranteed by Proposition 2.4.2). It is clear that $f \in \mathcal{H}^c$ and $\zeta(f) = q$, and we will show that $\{f\} \cup S$ is $\mathcal{H}$-nested. Suppose that $A \subseteq S$ such that $\{f\} \cap A$ is an antichain with $f \lor \bigvee A = h \in H$. Since $S$ is $\mathcal{H}$-nested, we have $\bigvee A < h$. Note that for $a \in A \subseteq S$, $a \leq \hat{a}$ in $\mathcal{L}(L, H)$, so $\hat{a} \in H$ implies $a \leq \hat{a}$. It follows that, for $a \in A$, if $\hat{a} = \hat{h}$ then $a = \hat{a} = \hat{h} \geq h$, contradicting $a < h$. So we must have $\hat{a} < \hat{h}$ for each $a \in A$, thus $\bigvee A \leq z_y(\hat{h})$. Then $z_y(\hat{h}) < h \lor z_y(\hat{h}) = f \lor z_y(\hat{h})$, implying that $f = \hat{h}$ and $q = \zeta(h)$. By our choice of $f$, this means that $h \leq f$, and thus $A = \emptyset$.

For our final claim, we still assume $\zeta^{-1}(q) \subseteq a(L)$ and let $f \in \mathcal{H}^c$ be as in the last paragraph. Suppose that $p \in \zeta^{-1}(q)$ such that $p \neq f$. Abbreviate $z = z_y(f) = z_y(\hat{p})$. Then since $f$ is the only element of $F(L, \mathcal{G}; q)$ not below $z$, we have $p < f$ and $f = p \lor (f \land z)$. But then
\( U = \{ s \in S : \hat{s} < f \} \) has \( \bigvee U = f \wedge z \), which implies \( \{ p \} \cup U \subseteq \{ p \} \cup S \) is not nested, as desired. \( \square \)

In fact, the above proposition implies that \( \mathcal{H} \) has a partition with blocks indexed by \( F^+(y) \), and we obtain a partial building set for \( L_{y,g} \) by letting \( \mathcal{H}_{y,g} \) be the image of a block under the map \( \zeta \):

**Definition 2.8.3** (Local partial building sets). For \( y \in \mathcal{L}(L, \mathcal{H}) \) and \( g \in F^+(y) \), let

\[
\mathcal{H}_{y,g} = \zeta(y, \mathcal{H}) \cap (L_{y,g})^+
\]

The fact that \( \mathcal{H}_{y,g} \) is indeed a partial building set will be proved in Proposition 2.8.6 below; first we provide an example.

**Example 2.8.4.** Let \( L \) be the lattice of set partitions of \( [7] = \{1, 2, \ldots, 7\} \), and consider the building set \( \mathcal{G} = L_{irr} \) of irreducibles (partitions with one nonsingleton block). Let us denote \( \hat{1} = 1234567, p = 1234567, q = 1234567 \in L \), and consider the partial building set \( \mathcal{H} = a(L) \cup \{ \hat{1}, p \} \).

Let \( y = (p, q) \in \mathcal{L}(L, \mathcal{H}) \), for which \( S = \sup \mathcal{H} = \{12, 13, 23, p\} \). Then \( \mathcal{H}_{y} = \{1, p\} \cup \{ij : 1 \leq i < j < 7\} \) and \( F^+(y) = \{q, p, \hat{1}\} \), and we obtain three intervals:

- \( L_{y,q} = [\hat{0}, q] \), with \( \mathcal{H}_{y,q} = \{12, 13, 23\} \) and \( \mathcal{G}_{y,q} = \{12, 13, 23, q\} \).
- \( L_{y,p} = [q, p] \), with \( \mathcal{H}_{y,p} = \{p\} \cup a(L_{y,p}) \). Note that the atoms include elements such as \( 1234567 = \{1\} \cap \{q\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\} \cup \{7\} \) and \( 1234567 \cup \{q\} \cup \{7\} = q \wedge 45 \) (which is not itself in \( \mathcal{G} \)).
- \( L_{y,1} = [p, \hat{1}] \) with \( \mathcal{H}_{y,1} = \mathcal{G}_{y,1} = \{1\} \).

\( \Diamond \)

**Lemma 2.8.5.** Let \( \mathcal{G} \) be a building set for a geometric lattice \( L \), and let \( y \in \mathcal{L}(L, \mathcal{G}) \). For each \( g \in F^+(y) \), the set \( \mathcal{G}_{y,g} \) is a building set for \( L_{y,g} \).

**Proof.** Fix \( g \in F^+(y) \), and let \( x \in (z(g), g) \). Write \( F(L, \mathcal{G}; x) = \{h_1, \ldots, h_k\} \), so that

\[
(10) \quad \emptyset, x \cong \bigvee_{i=1}^{k} [0, h_i].
\]

Write \( z(g) = \bigvee_{i=1}^{k} f_i \) for some unique \( f_i \in [0, h_i] \), via the join decomposition in (10). Let \( T = \{i : f_i \neq h_i\} \), so that \( h_i \vee z(g) = \zeta(h_i) \in \mathcal{G}_{y,g} \) for each \( i \in T \). We claim that \( \{h_i \vee z(g) : i \in T\} \) is the set of maximal elements of \( \mathcal{G}_{y,g} \) which lie below \( x \). Indeed, if \( p \vee z(g) \in \mathcal{G}_{y,g} \) with \( p \in \mathcal{G} \) and \( z(g) \leq p \vee z(g) \leq x = \bigvee_{i=1}^{k} h_i \), then \( p \leq h_i \) for some unique \( i \in T \) by Proposition 2.4.2, and hence \( p \vee z(g) \leq h_i \vee z(g) \). It remains to see that these index factors in a join decomposition of \( [z(g), x] \), as follows:

\[
[z(g), x] \cong \bigvee_{i=1}^{k} [f_i, h_i] \cong \bigvee_{i=1}^{k} [z(g), z(g) \vee h_i] \cong \bigvee_{i \in T} [z(g), z(g) \vee h_i].
\]

Therefore, \( \mathcal{G}_{y,g} \) is a building set for \( L_{y,g} = [z(g), g] \). \( \square \)

**Proposition 2.8.6.** Let \( \mathcal{H} \subseteq \mathcal{G} \) be a partial building set for a geometric lattice \( L \), and let \( y \in \mathcal{L}(L, \mathcal{H}) \). For each \( g \in F^+(y) \), the set \( \mathcal{H}_{y,g} \) is a partial building set for \( L_{y,g} \).

**Proof.** Let us abbreviate \( F(y) := F(L, \mathcal{L}(L, \mathcal{H}), \mathcal{G}; y) \). Viewing this set of factors \( F(y) \subseteq \mathcal{G} \) as a set of atoms in \( \mathcal{L}(L, \mathcal{G}) \), let \( y' = \vee F(y) \in \mathcal{L}(L, \mathcal{G}) \). Then \( F^+(y) = F^+(\mathcal{L}(L, \mathcal{G}), \mathcal{G}; y') \), and for every \( g \in F^+(y) \), \( \mathcal{G}_{y',g} \) is a building set in \( L_{y',g} = L_{y,g} \) by Lemma 2.8.5. Our claim is that, for every \( g \in F^+(y) \), \( \mathcal{H}_{y,g}^{\circ} \) is an order filter in \( \mathcal{G}_{y',g}^{\circ} \).
For this, let \( g \in F^+(y) \) and \( p, q \in \zeta_{y',g}^{-1}(L_{y',g}) \cap G^0 \). Assume that \( \zeta_{y',g}(p) \leq \zeta_{y',g}(q) \). That is, we assume \( \hat{p} = g = q \) and \( p \lor z(g) \leq q \lor z(g) \). Then \( p \leq q \lor z(g) \) and, by Proposition 2.4.2, this implies that either \( p \leq q \) or \( p \leq f \) for some \( f \in (F^+(y))_{<g} \). By minimality of \( \hat{p} \) in Proposition 2.8.2 and \( g = \hat{p} \), the latter case cannot happen, thus \( p \leq q \). Since \( H^0 \) is an order filter of \( G^0 \), this means that \( p \in H^0 \) then \( q \in H^0 \). In particular, this means that if \( \zeta_{y',g}(p) = \zeta_{y',g}(q) \in H_{y,g}^0 \), then \( \zeta_{y',g}(q) = \zeta_{y',g}(q) \in H_{y,g}^0 \).

Since \( H_{y,g} \) is a partial building set for \( L_{y,g} \), we may also consider the complex of \( H_{y,g} \)-nested sets, \( n(L_{y,g}, H_{y,g}) \). Just as before, this is isomorphic to the atomic complex at \( (L(L_{y,g}, H_{y,g})) \). The next statement relates these local building sets to \( H \)-nested sets.

**Proposition 2.8.7.** Let \( y \in L(L, \mathcal{H}) \) and \( S = \text{supp}_n(L, \mathcal{H})(y) \), and recall \( \zeta = \zeta_{\mathcal{H},T} \) from Proposition 2.8.2. Given \( T \subseteq \pi_n(L, \mathcal{H})(S) \), the set \( T \cup S \) is \( H \)-nested if and only if, for every \( g \in F^+(y) \), the set \( \zeta(T) \cap H_{y,g} \) is \( H_{y,g} \)-nested.

**Proof.** Suppose that \( g \in F^+(y) \) and \( \zeta(T) \subseteq H_{y,g} \) is a nontrivial antichain with \( \bigvee \zeta(T) \in H_{y,g} \), and we argue that \( T \cup S \) is \( H \)-nested. Let \( h \in H^0 \) such that \( \zeta(h) = \bigvee \zeta(T) \). Then \( (\bigvee T) \lor z(y) = \bigvee \Rightarrow p \lor z(y) = h \lor z'(y) \) where \( z'(y) = \bigvee f \in (F^+(y))_{<g} : f \neq h \). Let \( Z = \{ s \in S : s \leq z(g) \} \), noting that \( \bigvee Z = z(g) \), and let \( A = \max(T \cup Z) \subseteq T \cup S \). Since \( z \not\leq z(g) \) for any \( p \in T \), we have \( T \subseteq A \) and hence \( A \) is a nontrivial antichain. We further note that \( (A \lor z'(y)) \lor z'(y) = \bigvee (T \cup Z) \lor z'(y) = h \lor z'(y) \), contradicting \( A \lor z'(y) = \bigvee (T \cup Z) \lor z'(y) = h \lor z'(y) \). Since \( h \land z'(y) = 0 \), it follows that \( \bigvee A = h \in H \), thus \( T \cup S \) is not \( H \)-nested.

Conversely, suppose that \( U \subseteq S \) such that \( T \cup U \) is a nontrivial antichain with \( \bigvee (T \cup U) = h \in H \), and we argue that for \( g = h \), the set \( \zeta(T) \cap H_{y,g} \) is not \( H_{y,g} \)-nested. Since \( \{ p \} \subseteq U \subseteq S \) for every \( p \in T \), the set \( T \) is necessarily a nontrivial antichain. For \( p \subseteq T \cup U \), we have \( p < h \) and hence either \( \hat{p} = \hat{h} \) or \( p \leq z(g) \). In the case \( p \subseteq U \), then \( p \leq z(g) \) (as in the proof of Proposition 2.8.2, since \( U \subseteq S \)). Now let \( A = \{ p \in T : \hat{p} = \hat{h} \} \). If \( A = \emptyset \), then \( p \leq z(g) \) for all \( p \in T \cup U \), leading to a contradiction between \( h = \bigvee (T \cup U) \leq z(g) \) and \( z(g) < h \lor z'(y) \). If \( A \neq \emptyset \), then \( h \lor z'(y) = p \lor z(y) \) while \( p < h \), contradicting Proposition 2.8.2 when \( p \in \text{st}(S) \). It follows that \( \zeta(A) \subseteq \zeta(T) \cap H_{y,g} \) is a nontrivial antichain with \( \bigvee \zeta(A) = \zeta(h) = \zeta \in H_{y,g} \), completing the proof.

Finally, we examine how blowups affect the local building sets.

**Proposition 2.8.8.** Let \( \mathcal{H} \) and \( \mathcal{H}' = \mathcal{H} \cup \{ p \} \) be partial building sets in a geometric lattice \( L \), let \( y \in L(L, \mathcal{H}') \), and let \( \hat{p} = \min \{ f \in F^+(y) : p \leq f \} \). Then for any \( g \in F^+(\pi_{H,y}(y)) \),

\[
H_{\pi_{H,y}(y),g} = \begin{cases} \left(\mathcal{H}_{y,g}'\right)^\circ & \text{if } g \neq \hat{p} \\ \left(\mathcal{H}_{y,p}\right)^\circ \setminus \{ p \lor z(\hat{p}) \} & \text{if } g = \hat{p}. \end{cases}
\]

Note that \( \hat{p} \) exists by Proposition 2.8.2, and \( H_{\pi_{H,y}(y),g} \) is well-defined by Proposition 2.5.4.

**Proof.** We abbreviate \( \pi = \pi_{H,y} \). If \( \pi(y) \not\leq p \), then \( F^+(y) = F^+(\pi(y)) \) (Proposition 2.5.4) and the statement follows from the simple fact that \( \mathcal{H}' = \mathcal{H} \cup \{ p \} \). So assume \( \pi(y) \geq p \), that is, \( y = (p, x) \) for some \( x \in L(p) \). In this case, \( F^+(\pi(y)) = F^+(x)_{<p} \cup \{ p \} \) (Proposition 2.5.4) and \( \hat{p} = p \). Then \( H_{\pi(y),p} = 0 \) and \( H_{\pi_{H,y}(y),0} = \{ p \} \), while \( H_{\pi_{H,y}(y),g} = \left(\mathcal{H}_{y,g}'\right)^\circ \) for \( g \in F^+(x)_{<p} \).

**Example 2.8.9.** Let \( L \) be the partition lattice for \( \{ 7 \} \), \( \mathcal{H} = \text{a}(L) \cup \{ \hat{1} \} \), \( p = 1234567 \), \( q = 1234560 \), and \( y = \{ p, q \} \in L(L, \mathcal{H} \cup \{ p \}) \). Recall from Example 2.8.4 that \( F^+(y) = \{ q, p, \hat{1} \} \). Now note that \( \pi(y) = p \) and \( F^+(\pi(y)) = \{ p, 1 \} \), so that \( L_{\pi_{H,y}(y),p} = [0, p] \) with \( H_{\pi_{H,y}(y),p} = 0 \) while \( L_{y,p} = \{ q, p \} \) with \( \hat{p} \in H_{y,p} \).
For a different flavor, consider the same $L$, $H$, $p$, and $q$, but now $y = q \vee 456|7 = 123|456|7$ with $\pi(y) = y$. Then $F^+(y) = F^+(\pi(y)) = \{123, 456, 1\}$. The difference in local building sets in passing from $H$ to $H' = H \cup \{p\}$ is in the interval $L_{y,1}$, where $H_{\pi(y),1} = \emptyset$ while $H'_{y,1} \ni p$. \hfill \Box

3. A combinatorial model for open neighborhoods

Let $\mathcal{A}$ be a collection of affine hyperplanes in $\mathbb{C}^d$, and let $M(\mathcal{A}) := \mathbb{C}^{d+1} - \bigcup_{H \in \mathcal{A}} H$, its complement. The Orlik–Solomon algebra of $\mathcal{A}$ is a combinatorial presentation of the (integral) cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ that appears in the literature with various levels of generality. It depends only on the intersection poset $\mathcal{L}(\mathcal{A})$, so we will denote it by $\text{OS}(\mathcal{L}(\mathcal{A}))$. If $\mathcal{A}$ is a central hyperplane arrangement, then $\mathcal{L}(\mathcal{A})$ is a geometric lattice, and $\text{OS}(\mathcal{L}(\mathcal{A}))$ is defined by generators and relations from the underlying matroid. If $\mathcal{A}$ is an affine-linear arrangement, $\mathcal{L}(\mathcal{A})$ is a geometric semilattice, and the presentation of $\text{OS}(\mathcal{L}(\mathcal{A}))$ acquires monomial relations from subsets of atoms with no upper bound: we refer to Yuzvinsky [Yuz01] and Kawahara [Kaw04] for details.

Here, then, we define a yet more general Orlik–Solomon algebra, for any locally geometric semilattice $\mathcal{L}$, which we denote $\text{OS}(\mathcal{L})$.

3.1. The Orlik–Solomon algebra of a semilattice. For a semilattice $\mathcal{L}$, let $E(\mathcal{L})$ denote the exterior algebra over $\mathbb{Q}$ on the generators $\{e_J : g \in a(\mathcal{L})\}$. Define a derivation $\partial$ on $E(\mathcal{L})$ of degree $-1$ by letting $\partial(e_i) = 1$ for each $i$, then extending via the Leibniz rule.

If $\mathcal{L}$ is a geometric lattice, define an ideal of $E(\mathcal{L})$ by

$$I(\mathcal{L}) = (\partial(e_C) : C \text{ is a circuit in } M(\mathcal{L})), \tag{11}$$

where $e_J := e_{j_1} \cdots e_{j_k}$ for any subset $J = \{j_1, \ldots, j_k\}$ with entries in increasing order. More generally, though, suppose that $\mathcal{L}$ is a locally geometric semilattice. For each $x \in \mathcal{L}$, the interval $[0, x]$ is a geometric lattice, and we let

$$I_1(\mathcal{L}) = \sum_{x \in \mathcal{L}} I([0, x]) \tag{12}$$

$$= (\partial(e_C) : C \text{ is a circuit in } M(\mathcal{L}_{\leq x}) \text{ for some } x \in \mathcal{L}).$$

Now let

$$I_2(\mathcal{L}) = (e_J : J \subseteq a(\mathcal{L}) \text{ but } J \notin a(\mathcal{L})). \tag{13}$$

**Definition 3.1.1** (The algebra OS). The *Orlik–Solomon algebra* of a locally geometric semilattice $\mathcal{L}$ is, by definition,

$$\text{OS}(\mathcal{L}) := E(\mathcal{L})/(I_1(\mathcal{L}) + I_2(\mathcal{L})).$$

**Definition 3.1.2** (The projective Orlik–Solomon algebra). If $\mathcal{L}$ is a geometric lattice, we let $\overline{\text{OS}}(\mathcal{L})$ denote the subalgebra of $\text{OS}(\mathcal{L})$ generated in degree $1$ by $\ker \partial = (e_i - e_j : 1 \leq i < j \leq n)$. It is known (see [Kaw04]) that $\text{OS}(\mathcal{L}) \cong \overline{\text{OS}}(\mathcal{L}) \otimes \mathbb{Q}[e_0]$, where the latter is a $1$-dimensional exterior algebra.

**Remark 3.1.3.** The motivation comes from the realizable case. If $\mathcal{A}$ is a complex hyperplane arrangement with intersection lattice $\mathcal{L}$, the algebras $\text{OS}(\mathcal{L})$ and $\overline{\text{OS}}(\mathcal{L})$ are, respectively, the cohomology algebras of the affine and projective arrangement complements, $M(\mathcal{A}) = \mathbb{C}^{d+1} \setminus \bigcup_{H \in \mathcal{A}} H$ and $U(\mathcal{A}) = \mathbb{P}^d \setminus \bigcup_{H \in \mathcal{A}} \mathbb{P}H$. The isomorphism $\text{OS}(\mathcal{L}) \to H^*(M(\mathcal{A}), \mathbb{Q})$ is realized on the level of forms by $e_j \mapsto \frac{1}{2\pi i} f_j^\wedge$, where $f_j$ is a linear form for which $H_j = f_j^{-1}(0)$: see [OT92].
for details. The differential $\partial : \text{OS}(L) \to \text{OS}^{+1}(L)$ is realized by contraction along the Euler vector field, and $H (U(A), \mathbb{Q})$ is the kernel.

**Example 3.1.4.** If $L$ is obtained from a geometric lattice $L$ by blowing up a building set in a compatible order, then $\text{at}(L) = n(L, G)$ (Proposition 2.6.3). Since the order intervals in the face poset of a simplicial complex are Boolean, their matroids have no circuits, and $I_1(L) = 0$. This is the combinatorial abstraction of the normal crossings property of the boundary divisor in the wonderful compactification. We find that $\text{OS}(L) = E(L)/I_2(L)$, the exterior face ring of $n(L, G)$.

Now we show that the following well-known properties of Orlik–Solomon algebras extend to our slightly more general context. First, an additive basis for $\text{OS}(L)$ when $L$ is geometric goes back to a result of Björner [Bjö82]. Recall that an independent set $J$ in a matroid $M$ with a totally ordered ground set $E$ is a broken circuit if there exists some $g < \min J$ for which $\{g\} \cup J$ is a circuit. The collection of all subsets of $E$ which do not contain a broken circuit, denoted $\text{nbc}(M)$, is a pure simplicial complex on $E$ of dimension one less than the rank of $M$.

**Definition 3.1.5 (The no-broken-circuit complex).** Let $L$ be a locally geometric semilattice and $\prec$ a total order on $a(L)$. We say that a simplex $S \in \text{at}(L)$ is a broken circuit if there exists some $g \prec \min S$ for which $\{g\} \cup S$ is a circuit in $M(L_{\leq x})$, for some $x \geq g \lor \lor S$.

Let $\text{nbc}_{\prec}(L)$ denote the set of simplices in $\text{at}(L)$ which do not contain a broken circuit. Clearly $\text{nbc}_{\prec}(L)$ is a subcomplex of $\text{at}(L)$. We will simply write $\text{nbc}(L)$ if the order on which it depends is understood.

An order of the atoms $a(L)$ gives an order of the variables of $E(L)$. We extend it lexicographically to a term order on $E(L)$. In particular, if $H$ is a partial building set for a geometric lattice $L$, a compatible order on $H$ gives a lexicographic term order for $E(L(H))$.

**Example 3.1.6 (Example 2.5.6, continued).** The $\text{nbc}$ complexes for the three semilattices in Example 2.5.6 are shown in Figure 1: they are the cones at the vertex $\hat{1}$ of the respective 1-complexes shown in bold.

This leads to a monomial basis for the Orlik–Solomon algebra, given in the next theorem. This theorem and the corollaries we state here are well-known for geometric semilattices, but the arguments are “local” in nature and hence hold without change for locally geometric semilattices.

**Theorem 3.1.7 (Basis of $\text{OS}(L)$, [Yuz01, Thm. 2.8]).** Let $L$ be a locally geometric semilattice with a fixed linear order on $a(L)$. The generators of the ideal $I_1(L) + I_2(L)$ form a Gröbner basis with respect to the graded lexicographic order. The corresponding additive basis for $\text{OS}^i(L)$ consists of monomials

$$\{ e_J : J \in \text{nbc}(L) \text{ and } |J| = i \}.$$  

In the case of hyperplane arrangements, the following is due to Brieskorn [Bri73].

**Corollary 3.1.8 (The Brieskorn decomposition).** Let $L$ be a locally geometric semilattice, and let $i \geq 0$. Then

$$\text{OS}^i(L) = \bigoplus_{x \in \mathcal{L}_i} \text{OS}^i(L_{\leq x}),$$

where $\mathcal{L}_i$ denotes the set of elements in $L$ of rank $i$.

**Proof.** Broken circuits have a local definition, so $\text{nbc}(L_{\leq x})$ is the full subcomplex of $\text{nbc}(L)$ on the vertices $a(L_{\leq x})$, for each $x \in L$. The decomposition follows by comparing the additive bases provided by Theorem 3.1.7. \qed
Corollary 3.1.9. If \( x, y \in \mathcal{L} \) and \( x \leq y \), the inclusion \( a(\mathcal{L}_{\leq x}) \subseteq a(\mathcal{L}_{\leq y}) \) induces an injective algebra homomorphism \( \text{OS}(\mathcal{L}_{\leq x}) \rightarrow \text{OS}(\mathcal{L}_{\leq y}) \).

Remark 3.1.10. The derivations \( \partial \) on \( \text{OS}(\mathcal{L}_{\leq x}) \), for each \( x \), are compatible with inclusions \( \text{OS}(\mathcal{L}_{\leq x}) \hookrightarrow \text{OS}(\mathcal{L}_{\leq y}) \) for \( x \leq y \), but not with their left-inverses \( \text{OS}(\mathcal{L}_{\leq y}) \rightarrow \text{OS}(\mathcal{L}_{\leq x}) \). \( \diamond \)

Given its local nature, it is natural to consider a sheaf of Orlik–Solomon algebras.

Definition 3.1.11 (The sheaf \( \text{OS} \)). Let \( \mathcal{L} \) be a locally geometric semilattice, and define a graded sheaf of vector spaces \( \text{OS}(\mathcal{L}) \) on \( \mathcal{L}^{op} \) as follows. For each \( i \geq 0 \), define a sheaf of algebras \( \text{OS}^i(\mathcal{L}) \) for \( x \in \mathcal{L} \) by

\[
\text{OS}^i(\mathcal{L})(x) = \text{OS}^i(\mathcal{L}_{\leq x}),
\]

with restriction maps for \( x \leq y \) given by \( \text{OS}^i(\mathcal{L})(x) \rightarrow \text{OS}^i(\mathcal{L})(y) \).

By the remark above, the map \( \partial \) extends to sheaves to make a chain complex

\[
\text{OS}^0 \leftarrow \text{OS}^1 \leftarrow \cdots \leftarrow \text{OS}^i \leftarrow \cdots \leftarrow \text{OS}^r.
\]

We note that the surjections \( \text{OS}(\mathcal{L}_{\leq y}) \rightarrow \text{OS}(\mathcal{L}_{\leq x}) \) also allow us to define a graded sheaf on \( \mathcal{L} \), a special case of which plays a key role in \([\text{Yuz95}]\). With this structure, though, \( \partial \) is not a map of sheaves (Remark 3.1.10).

Proposition 3.1.12. The complex (14) is exact except in degree 0.

Proof. We check the claim on stalks. The stalk at \( x \) is the complex \( (\text{OS}(\mathcal{L}_{\leq x}), \partial) \) geometric lattice of rank \( \geq 1 \). For \( x \neq 0 \), this is exact by \([\text{OT92}, \text{Lem. 3.13}]\). For \( x = 0 \), the complex is concentrated in degree 0. \( \square \)

3.2. The flag complex. Classically, the graded \( \mathbb{Q} \)-dual of the Orlik–Solomon algebra is identified with a vector space spanned by flags in the intersection lattice. The flags index explicit homology cycles in a hyperplane arrangement complement. As usual, the combinatorics of the flag complex extends beyond its topological origins, as we show here. The following construction appears first for hyperplane arrangements in \([\text{SV91}, \S 2]\).

Definition 3.2.1 (The flag complex). Let \( \mathcal{L} \) be a locally geometric semilattice. For each \( i \geq 0 \), let \( \hat{\text{Fl}}^i(\mathcal{L}) \) denote the \( \mathbb{Q} \)-span of all chains \( Y := (y_0 < y_1 < \cdots < y_i) \), where \( y_j \in \mathcal{L}_j \) for \( 0 \leq j \leq i \). Let \( \hat{\text{Fl}}^i(\mathcal{L})^\vee \) be the \( \mathbb{Q} \)-dual, and denote the dual basis vectors by \( Y^\vee \).

If \( Y \in \hat{\text{Fl}}^i(\mathcal{L}) \) is a chain as above, \( 0 < j < i \), and \( y \in \mathcal{L}_j \) satisfies \( y_{j-1} < y < y_{j+1} \), let \( Y(\hat{y}_j; y) := (y_0 < y_1 < \cdots < y_{j-1} < y < y_{j+1} < \cdots < y_i) \). Define \( \text{Fl}^i(\mathcal{L}) \) to be the quotient of \( \hat{\text{Fl}}^i(\mathcal{L}) \) by the sums \( \sum_y Y(\hat{y}_j; y) \), for each chain \( Y \) of length \( i+1 \) and each \( 0 < j < i \).

An ordered, independent set of atoms \((h_1, \ldots, h_i)\) of \( \mathcal{L} \) defines a flag \( Y(h_1, \ldots, h_i) \) by letting \( y_j = h_1 \vee \cdots \vee h_j \), for \( 0 \leq j \leq i \), provided the join exists. This extends to a map \( \hat{\Phi} : E(\mathcal{L}) \rightarrow \hat{\text{Fl}}(\mathcal{L})^\vee \) by setting

\[
\hat{\Phi}(e_S) = \sum_{\sigma \in \Sigma_i} (-1)^{|\sigma|} Y(g_{\sigma(1)}, \ldots, g_{\sigma(i)})^\vee,
\]

where \( e_S \) is the monomial indexed by an independent set \( S = \{g_1, \ldots, g_i\} \), provided \( S \in \text{at}(\mathcal{L}) \), and zero otherwise. Here, \( \Sigma_i \) denotes the symmetric group on \( \{1, \ldots, i\} \).

We will make use of the dual version of Brieskorn’s decomposition. This appeared in the arrangement case as \([\text{SV91}, (2.1.2)]\) and follows directly from Corollary 3.1.8.
Lemma 3.2.2. If $\mathcal{L}$ is a locally geometric semilattice, for each $i \geq 0$,
\begin{equation}
\text{Fl}^i(\mathcal{L}) \cong \bigoplus_{x \in \mathcal{L}_i} \text{Fl}^i(\mathcal{L}_{\leq x}).
\end{equation}

Lemma 3.2.3. The map $\tilde{\Phi}$ induces a map $\Phi: \text{OS}(\mathcal{L}) \to \text{Fl}(\mathcal{L})^\vee$ whose corestriction to $\text{Fl}(\mathcal{L})^\vee$ is an isomorphism.

Proof. This was noted for hyperplane arrangements in [SV91, §2]; however, the argument there applies without change for any geometric lattice. That is, $\Phi: \text{OS}(\mathcal{L}_{\leq x}) \to \text{Fl}(\mathcal{L}_{\leq x})^\vee$ is an isomorphism for each $x \in \mathcal{L}$. The global result then follows using the decompositions of Corollary 3.1.8 and Lemma 3.2.2. \qed

The graded vector space $\text{Fl}(\mathcal{L})$ is a cochain complex with respect to a differential $\delta$ defined for chains $Y = (y_0 < \cdots < y_i)$ by the formula
\begin{equation}
\delta(Y) = (-1)^i \sum_{y_{i+1} \in \mathcal{L}_{i+1}} (y_0 < y_1 < \cdots < y_{i+1}).
\end{equation}

The next result was stated for arrangements as [SV91, Thm. 2.4(b)]; again, the proof is combinatorial and applies any geometric lattice.

Proposition 3.2.4. For any $x \in \mathcal{L}$, the map $\Phi: (\text{OS}(\mathcal{L}_{\leq x}), \partial) \to (\text{Fl}(\mathcal{L}_{\leq x}), \delta)^\vee$ is an isomorphism of chain complexes.

Just as with the Orlik–Solomon algebra, the flag complex is a local construction. Thus, we consider the following sheaf on $\mathcal{L}$:

Definition 3.2.5 (Sheaf $\mathcal{F}^\ell$). Let $\mathcal{L}$ be a locally geometric semilattice, and define a graded sheaf $\mathcal{F}^\ell$ on $\mathcal{L}$ by $\mathcal{F}^\ell(x) = \text{Fl}(\mathcal{L}_{\leq x})$ for $x \in \mathcal{L}$, using restriction maps dual to those of $\text{OS}$, $\rho_{x_1,x_0}: \text{Fl}(\mathcal{L}_{\leq x_1}) \to \text{Fl}(\mathcal{L}_{\leq x_0})$ for all $x_0 \leq x_1$ and $j \geq 0$. We note that the maps $\rho_{x_1,x_0}$ are coordinate projections, with respect to the direct sum decomposition (16).

Proposition 3.2.6. The differential $\delta$ from (17) makes $\mathcal{F}^\ell$ a cochain complex of sheaves. The augmented complex
\begin{equation}
0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F}^\ell^0 \longrightarrow \mathcal{F}^\ell^1 \longrightarrow \cdots \longrightarrow \mathcal{F}^\ell^i \longrightarrow \cdots \longrightarrow \mathcal{F}^\ell^r \longrightarrow 0
\end{equation}
is flasque and exact, where $\mathcal{K}(x) := \mathbb{Q}$ for $x = \hat{0}$ and $\mathcal{K}(x) := 0$ otherwise.

Proof. The fact that $\delta$ is compatible with restrictions is dual to Remark 3.1.10. Likewise, the exactness of (18) is obtained from Proposition 3.1.12 by taking $\mathbb{Q}$-duals.

We check each $\mathcal{F}^\ell^i$ is flasque directly. Suppose $a \in \mathcal{F}^\ell^i(U) \subseteq \bigoplus_{y \in U} \text{Fl}^i(\mathcal{L}_{\leq y})$ is a section over a downward-closed set $U \subseteq \mathcal{L}$, and suppose $x$ is a minimal element of $\mathcal{L} - U$. Then $x$ covers some elements $x_1, \ldots, x_k \in U$. We show we can extend $a$ to $x$ by considering two cases. If $i \geq \text{rank}(x)$, then $i > \text{rank}(x_j)$ for each $1 \leq j \leq k$, so $\mathcal{F}^\ell^i(x_j) = 0$ for each $j$, and we let $a_x = 0$. Otherwise, we write each
\[ \mathcal{F}^\ell^i(x_j) = \bigoplus_{y \leq x_j, y \in \mathcal{L}_i} \text{Fl}^i(\mathcal{L}_{\leq y}) \]
using Lemma 3.2.2(16). Since it is a section, $a \in \mathcal{F}^\ell^i(U)$ has the property that $(a_{x_j})_y = (a_{x'_j})_y$ whenever $y \leq x_j$ and $y \leq x'_j$. So for every $y \in \mathcal{L}_i$ with $y \leq x$, let $(a_x)_y = (a_{x_j})_y$ if $y \leq x_j \leq x$, and 0 otherwise, which determines an element $a_x \in \text{Fl}^i(\mathcal{L}_{\leq x})$ that restricts to each $a_{x_j}$. \qed
Corollary 3.2.7. As graded vector spaces, \( \text{Fl}(L) \cong \Gamma(\mathcal{F}(L)) \).

Proof. We fix \( i \geq 0 \) and use Brieskorn’s Lemma 3.2.2 to construct a map \( \theta: \text{Fl}(L) \to \Gamma(\mathcal{F}(L)) \) using maps \( \theta_x: \text{Fl}(L) \to \mathcal{F}(L)(x) \) for each \( x \in L \); we let

\[
\theta_x: \text{Fl}^i(L) \cong \bigoplus_{y \in L_i} \text{Fl}^i(L_{\leq y}) \to \text{Fl}^i(L_{\leq x}) \cong \bigoplus_{y \in L_i \cap L_{\leq x}} \text{Fl}^i(L_{\leq y})
\]

be the obvious coordinate projection. Clearly for all \( x, z \geq 0 \), we have \( \rho_{x,z} \circ \theta_x = \theta_z \), so this induces a map \( \theta: \text{Fl}(L) \to \Gamma(\mathcal{F}(L)) \). The inverse of \( \theta \) is given for each \( x \in L \) by a corresponding coordinate inclusion.

3.3. Blowups and the Orlik–Solomon algebra. In this section, we examine the effect of a single blowup on Orlik–Solomon algebras of semilattices.

Theorem 3.3.1. Suppose \( \mathcal{H} \) and \( \mathcal{H}' = \mathcal{H} \cup \{p\} \) are partial building sets associated to a building set \( \mathcal{G} \) in a geometric lattice \( L \), and consider the blown-up semilattices \( L = L(L, \mathcal{H}) \) and \( L' = B_p(L) = L(L, \mathcal{H}') \).

The homomorphism of exterior algebras \( \phi_E: E(L) \to E(L') \), defined by letting

\[
\phi_E(e_g) = \begin{cases} 
 e_g & \text{if } g \not< p \\
 e_g + e_p & \text{if } g \leq p
\end{cases}
\]

for each \( g \in a(L) \), induces an injective homomorphism \( \phi_{OS}: \text{OS}(L) \to \text{OS}(L') \).

We break up the proof of Theorem 3.3.1 into two pieces: Lemma 3.3.3 (which shows that \( \phi_{OS} \) is well-defined) and Lemma 3.3.5 (which shows that \( \phi_{OS} \) is injective). We will often write \( \phi \) in place of both \( \phi_E \) and \( \phi_{OS} \). We start with an elementary observation.

Lemma 3.3.2. If \( C = \{g_1, \ldots, g_k\} \) and \( g_1 \prec \cdots \prec g_k \), then

\[
\partial(e_C) = (e_{g_2} - e_{g_1}) \cdots (e_{g_{i-1}} - e_{g_i}) \cdots (e_{g_k} - e_{g_{k-1}}).
\]

This leads to the first main lemma:

Lemma 3.3.3. The map \( \phi_E \) induces a well-defined homomorphism \( \phi_{OS}: \text{OS}(L) \to \text{OS}(L') \).

Proof. We show that \( \phi \) sends relations to relations by considering the two cases.

If \( \partial(e_C) \in I_1(L) \) for a circuit \( C \), let \( x = \sqrt{C} \in L \). The closure of a circuit is irreducible, so \( x \not< p \) by Proposition 2.5.3(a). If \( x < p \), then \( g < p \) for each \( g \in C \). Using the expression in Lemma 3.3.2(20), we find \( \phi(\partial(e_C)) = \partial(e_C) \in I_1(L') \), since \( C \) remains a circuit in \( L' \). If \( x = p \), again \( \phi(\partial(e_C)) = \partial(e_C) \), which is a signed sum of monomials \( e_{C-g_i} \) for \( 1 \leq i \leq k \). Since, for each \( i \), \( \sqrt{(C - \{g_i\})} = p \) in \( L \), the set \( C - \{g_i\} \) has no upper bound in \( L' \). It follows that each term \( e_{C-g_i} \in I_2(L') \).

Last, if \( x \) and \( p \) are incomparable, reorder \( C \) to assume \( g_i \leq p \) for \( 1 \leq i \leq r \), and \( g_i \not\leq p \) for \( r + 1 \leq i \leq k \). We may assume \( r \geq 1 \); if not, again \( \phi(\partial(e_C)) = \partial(e_C) \in I_1(L') \). Then by Lemma 3.3.2,

\[
\partial(e_C) = (e_{g_2} - e_{g_1}) \cdots (e_{g_{r+1}} - e_{g_r}) \cdots (e_{g_{k+1}} - e_{g_{k+1}}),
\]

so

\[
\phi(\partial(e_C)) = \partial(e_C) + (e_{g_{r+1}} - e_{g_{r+1}}) \cdots (e_{g_{k+1}} - e_{g_{k+1}}).
\]

Since \( x \in L' \), \( C \) remains a circuit and \( \partial(e_C) \in I_1(L') \). Each monomial in the right summand is indexed by a set \( \{p\} \cup (C - \{g, h\}) \), where \( g \leq p \) and \( h \not\leq p \) in \( L \). Since \( C \) is a circuit, in \( L \)
we have
\[ p \lor \bigvee (C - \{g, h\}) = p \lor \bigvee (C - \{h\}) = p \lor x, \]
if these upper bounds exist. But \( p \land x \neq 0 \), so by Proposition 2.5.3(b), \( p \lor x \notin \mathcal{L} \). So each \( \{p\} \cup (C - \{g, h\}) \notin \mathfrak{at}(\mathcal{L}’) \), and the remaining monomials are in \( I_2(\mathcal{L}’) \).

Thus, for circuits \( C \) of \( \mathcal{L} \) we have \( \partial(e_C) \in I_1(\mathcal{L}’) + I_2(\mathcal{L}’) \).

If \( e_J \in I_2(\mathcal{L}) \), monomials in \( \phi(e_J) \) are indexed by the sets \( J \) and \( J_g := J \cup \{p\} - \{g\} \) for each \( g \in J \) with \( g \leq p \). If \( J_g \in \mathfrak{at}(\mathcal{L}’) \), then \( \Pi(J_g) \in \mathfrak{at}(\mathcal{L}) \), by Lemma 2.6.5. But \( J \subseteq \Pi(J_g) \), so \( J \in \mathfrak{at}(\mathcal{L}) \), a contradiction.

We use the deg-lex monomial order on \( \text{OS}(\mathcal{L}) \), with the order on generators induced by the order on \( G \). For an element \( f \) in an Orlik–Solomon algebra, let \( \text{In}(f) \) denote its initial monomial when written in terms of the monomial basis from Theorem 3.1.7.

**Lemma 3.3.4.** For a monomial \( e_J \) where \( J \in \text{nbc}(\mathcal{L}) \), we have
\[
\text{In}(\phi(e_J)) = \begin{cases} 
  e_J & \text{if } \bigvee J_{<p} \neq p \\
  e_{J - \{g\}} e_p & \text{if } \bigvee J_{<p} = p
\end{cases}
\]
where \( g = \min_{\prec} J_{<p} \) and, as usual, \( J_{<p} := \{ h \in J : h < p \} \).

**Proof.** If \( J \in \text{nbc}(\mathcal{L}) \), then
\[
\phi(e_J) = e_J + e_p \sum_{h \in J_{<p}} \pm e_{J - \{h\}}.
\]
Since \( e_p \prec e_h \) for each \( h \leq p \), the leading term in \( \phi(e_J) \) is \( e_J \), provided that \( J \in \mathfrak{at}(\mathcal{L}’) \). Since \( J \in \mathfrak{at}(\mathcal{L}’) \) if and only if \( \bigvee J_{<p} \neq p \), we are done with the first case.

Suppose \( J \notin \mathfrak{at}(\mathcal{L}’) \), then, which equivalently means \( \bigvee J_{<p} = p \). Let \( J_h := J \cup \{p\} - \{h\} \) for each \( h \in J_{<p} \). Since \( J \) is independent, \( \bigvee (J - \{h\})_{<p} \leq \bigvee J_{<p} = p \) and hence \( J_h \in \mathfrak{at}(\mathcal{L}’) \) for each \( h \in J_{<p} \). This implies that the lead term of \( \phi(e_J) \) is \( \pm e_{J_h} \), where \( g = \min_{\prec} J_{<p} \).

**Lemma 3.3.5.** The homomorphism \( \phi_{\text{OS}} : \text{OS}(\mathcal{L}) \to \text{OS}(\mathcal{L}’) \) is injective.

**Proof.** Finally, if \( J \in \text{nbc}(\mathcal{L}) \) with \( \bigvee J_{<p} = p \), and \( g = \min_{\prec} J_{<p} \), then the \text{nbc} property implies \( g = \min_{\prec} a(\mathcal{L}_{<p}) \) as well. It follows that for no two \text{nbc} sets \( J \) are the lead terms of \( \phi(e_J) \) the same. We conclude that \( \phi \) is injective.

4. A COMBINATORIAL MODEL FOR THE CLOSED STRATA

At this point we recall a second algebra which is also given by a geometric lattice and partial building set. We will show that, like the Orlik–Solomon algebras in §3, its local versions form a sheaf, and we study how the algebra behaves under a blowup.

In the case where the geometric lattice is the intersection lattice of a complex hyperplane arrangement, this is the cohomology ring of the De Concini–Procesi compactification from [DCP95]. The form of the presentation here follows [FY04] in the case where \( \mathcal{H} = \mathcal{G} \), a full building set. The cohomology of a partial blowup is understood in the same way by the work of Dupont [Dup15]. Beyond the realizable setting, it is also the Chow ring of a smooth toric variety associated with a subfan of the Bergman fan, an observation that has its origins with Feichtner and Yuzvinsky [FY04]. It is not clear that the Chow ring interpretation is important to us here, but we mention it because of the central role it plays for \( \mathcal{H} \subset \mathcal{G} \) and \( \mathcal{G} = L_+ \) in the recent paper of Adiprasito, Huh and Katz [AHK18].
4.1. The De Concini–Procesi algebra for a partial building set. We begin with an algebra presentation. Suppose that $\mathcal{H}$ is a partial building set for a geometric lattice $\mathcal{L}$. For each $g \in \mathcal{H}$, we define an element
\begin{equation}
\tilde{c}_g^\mathcal{H} = \sum_{\substack{h \in \mathcal{H}; \\ g \leq h}} x_h \in \mathbb{Q}[x_g; \ g \in \mathcal{H}],
\end{equation}
which we will abbreviate by $c_g$ when the choice of partial building set is clear.

**Definition 4.1.1** (The algebra $\text{DP}$). For a partial building set $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{L}_+$, we define
$\text{DP}(\mathcal{L}, \mathcal{H}) := \mathbb{Q}[x_g; \ g \in \mathcal{H}] / J(\mathcal{H}),$
where the ideal $J(\mathcal{H})$ is generated by:
(i) $x_T$ whenever $T \not\in \mathcal{N}(\mathcal{L}, \mathcal{H})$;
(ii) $c_g^\mathcal{H}$ for each $g \in a(\mathcal{L})$.

We will write $\text{DP}(\mathcal{H})$ in place of $\text{DP}(\mathcal{L}, \mathcal{H})$ when the geometric lattice $\mathcal{L}$ is understood. The algebra $\text{DP}(\mathcal{H})$ is graded by assigning degree 2 to each variable $x_g$.

For every element of the semilattice $\mathcal{L}(\mathcal{L}, \mathcal{H})$, we define a quotient of $\text{DP}(\mathcal{L}, \mathcal{H})$ as follows.

**Definition 4.1.2.** Let $y \in \mathcal{L}(\mathcal{L}, \mathcal{H})$, and suppose $y = \bigvee S$ for a nested set $S \subseteq \mathcal{N}(\mathcal{L}, \mathcal{H})$. Define
$\text{DP}_y(\mathcal{L}, \mathcal{H}) := \mathbb{Q}[x_g; \ g \in \mathcal{H}] / J_y(\mathcal{H}),$
where the ideal $J_y(\mathcal{H})$ is generated by:
(i) $x_T$ whenever $S \cup T \not\in \mathcal{N}(\mathcal{L}, \mathcal{H})$;
(ii) $c_g^\mathcal{H}$ for each $g \in a(\mathcal{L})$.

We remark that the definition is independent of the choice of $S$, since if $\bigvee S = \bigvee S'$, then $S \cup T \in \mathcal{N}(\mathcal{L}, \mathcal{H})$ if and only if $S' \cup T \in \mathcal{N}(\mathcal{L}, \mathcal{H})$ (in view of Theorem 2.6.7). We also note that the monomials $x_T$ in (i) are just the Stanley-Reisner relations for $\text{DP}(\mathcal{L}, \mathcal{H})$.

Just as for Orlik–Solomon algebras (in Definition 3.1.11), we obtain a sheaf, but this time on $\mathcal{L}(\mathcal{L}, \mathcal{H})^{\text{op}}$. We note that, if $y \leq z$ in $\mathcal{L}(\mathcal{L}, \mathcal{H})$, then $J_y(\mathcal{H}) \subseteq J_z(\mathcal{H})$, so there is an obvious surjection $\text{DP}_y(\mathcal{L}, \mathcal{H}) \to \text{DP}_z(\mathcal{L}, \mathcal{H})$.

**Definition 4.1.3** (The sheaf $\mathcal{D}\mathcal{P}$). Let $\mathcal{H}$ be a partial building set in a geometric lattice $\mathcal{L}$, and let $\mathcal{L} = \mathcal{L}(\mathcal{L}, \mathcal{H})$. Define a sheaf of graded rings $\mathcal{D}\mathcal{P}(\mathcal{L}, \mathcal{H})$ on $\mathcal{L}^{\text{op}}$ by setting, for each $y \in \mathcal{L}^{\text{op}}$, $\mathcal{D}\mathcal{P}(\mathcal{L}, \mathcal{H})(y) = \text{DP}_y(\mathcal{L}, \mathcal{H})$, with restrictions given by the obvious maps.

**Remark 4.1.4.** In Definition 4.1.1, we may use the relations (ii) to eliminate the variables $x_h$ for $h \in a(\mathcal{L})$. Then $\text{DP}(\mathcal{L}, \mathcal{H}) \cong \mathbb{Q}[x_g; \ g \in \mathcal{H}^\circ] / J(\mathcal{H}^\circ)$, where the ideal $J(\mathcal{H}^\circ)$ is generated by (non-monomial) elements of type (i).

**Remark 4.1.5.** Later, in Corollary 5.3.3, we will see that the set of monomials $x_T^b$ for which $T \in \mathcal{N}(\mathcal{L}, \mathcal{H})$, and $0 < b(g) < d \left( \bigvee_{f \in T_<g} f, g \right)$ for $g \in T$, is a monomial basis for $\text{DP}(\mathcal{L}, \mathcal{H})$. This is an easy generalization of the monomial basis for full building sets given in [FY04]. In fact, we have everything we need to prove this now, but we defer the proof until we are in a more general setting in §5. We will, however, use this monomial basis in the proof of Theorem 4.1.7 next.

The algebras $\text{DP}_y(\mathcal{L}, \mathcal{H})$ can be further decomposed as a tensor product of the algebras derived from intervals in the geometric lattice $\mathcal{L}$. In the geometric setting, this corresponds to the fact that the closed strata in the wonderful compactification are themselves products of...
wonderful compactifications. We will be interested in $y \in \mathcal{L}$ whose blow-down $\pi(y) \neq \hat{1}$, and we refer back to the definitions and notation introduced in §2.8, namely Definitions 2.8.1 and 2.8.3. We need one more lemma before proceeding to the tensor decomposition in Theorem 4.1.7.

**Lemma 4.1.6.** Let $y \in \mathcal{L}(\mathcal{L}, \mathcal{H})$, and recall $\zeta = \zeta_y, \mathcal{H}$ from Proposition 2.8.2. Suppose that $a, b \in \mathcal{H}$ such that $\zeta(a) = \zeta(b)$.

1. If $a \in a(L)$, then $c_a^H = c_b^H$ in $DP_y(L, \mathcal{H})$.
2. If $a, b \in a(L)$, then $x_a = x_b$ in $DP_y(L, \mathcal{H})$.

**Proof.** We first claim that for $p \in s\mathfrak{m}_{n,L}(\mathcal{H})(y)_a$, if $p > a$ then $p \geq b$. By our hypotheses, $b \leq a \vee z_g(b) = a \vee z_g(b) \leq p \vee z_g(b)$. Similar to the proof of Proposition 2.8.2, we have $F(L, G; p \vee z_g(b)) = \{p\} \cup F(L, G; z_g(b)) \not\subseteq \mathcal{P}$, and so minimality of $\hat{b}$ implies $b \leq p$.

It follows that $c_p^H = c_b^H$ in $DP_y(L, \mathcal{H})$. Moreover, if $a, b \in a(L)$, then $c_a^H - x_a = c_b^H - x_b$ and hence $x_a = x_b$ in $DP_y(L, \mathcal{H})$. \hfill $\Box$

**Theorem 4.1.7 (Local tensor decompositions).** Let $\mathcal{H}$ be a partial building set for a geometric lattice $L$, and let $y \in \mathcal{L}(\mathcal{L}, \mathcal{H})$ such that $\pi(y) \neq \hat{1}$ in $L$. As graded algebras,

$$DP_y(L, \mathcal{H}) \cong \bigotimes_{g \in F^+(y)} DP(L_{y,g}, \mathcal{H}_{y,g}) \cong \bigotimes_{g \in F^+(y) \cap H^0} DP(L_{y,g}, \mathcal{H}_{y,g}).$$

**Proof.** First, note that the two tensor decompositions are equivalent, since for $g \in F^+(y) \cap H^0$ we have $\mathcal{H}_{y,g} = \emptyset$ and hence $DP_y(L_{y,g}, \mathcal{H}_{y,g}) \cong \mathbb{Q}$.

We will define a map $\psi_y : DP(L_{y,g}, \mathcal{H}_{y,g}) \to DP_y(L, \mathcal{H})$ for each $g \in F^+(y)$. Then we will prove that $\psi := \bigotimes \psi_y$ is an isomorphism.

Write $S = \text{supp}(y)$ and recall $\zeta = \zeta_y, \mathcal{H}$ from Proposition 2.8.2. Fix a section

$$\sigma : \bigsqcup_{g \in F^+(y)} \mathcal{H}_{y,g} \to \mathcal{H}$$

of $\zeta$ such that $\sigma(q) \in \text{max} \zeta^{-1}(q)$. Then for $g \in F^+(y)$ and $q \in \mathcal{H}_{y,g}$, define

$$\psi_y(x_q) = \begin{cases} x_{\sigma(q)} & \text{if } q \neq g \\ \sum_{h \geq g} x_h & \text{if } q = g. \end{cases}$$

To see that $\psi_y$ does not depend on the choice of $\sigma$, consider $q \in \mathcal{H}_{y,g}$. By Proposition 2.8.2, the choice of $\sigma(q)$ is unique unless $\zeta^{-1}(q) \subseteq a(L)$. In this case, the fact that $\psi_y(x_q)$ does not depend on choice follows from Lemma 4.1.6(2). Proposition 2.8.7 implies that $\psi_y$ is well-defined on the monomial relations. For the linear relations, let $q \in a(L_{y,g})$ and pick $a \in a(L)$ such that $\zeta(a) = q$. Then by Lemma 4.1.6(1), $\psi_y(c_{a}^{\mathcal{H}_{y,g}}) = c_{\sigma(q)}^{\mathcal{H}} = c_{a}^{\mathcal{H}}$ in $DP_y(L, \mathcal{H})$. Therefore, $\psi_y$ is well-defined.

Now, to show that $\psi = \bigotimes \psi_y$ is an isomorphism, we first check surjectivity. Here, it suffices to show that $x_h \in \text{im}(\psi)$ for all $h \in \mathcal{H} \cap \text{s}\mathfrak{m}_{n,L}(\mathcal{H})(S)$. If $h \in \mathcal{H} \cap F^+(y)$ then $\psi(x_{\zeta(h)}) = x_h$. Note that $\psi(x_1) = x_1$, and for the remaining $g \in \mathcal{H} \cap F^+(y)$, we proceed inductively. If $x_h \in \text{im}(\psi)$ for each $h > g$, then $x_g = \psi(x_g) - \sum_{h > g} x_h \in \text{im}(\psi)$.

It remains to show that $\psi$ is injective, which we do next by analyzing basic monomials. Suppose that, for each $g \in F^+(y)$, $x_g^b$ is a basic monomial in $DP(L_{y,g}, \mathcal{H}_{y,g})$. This means, by Corollary 5.3.3 (see also Remark 4.1.5), that $T_g \subseteq \mathcal{H}_{y,g}$ and $0 < b_g(p) < d(\sqrt{T_g})_{<p}, p)$ for
each $p \in T_d$. Let $T = \cup g \sigma(T_g) \subseteq \mathcal{H}^o$ and $b(h) = b_g(\zeta(h))$ for $h \in T$; we prove that $x_T^b$ is a basic monomial in $DP_y(L, \mathcal{H})$. For this, we claim that for each $g \in F^+(y)$ and $h \in \sigma(T_g)$,

$$(22) \quad z \lor \bigvee_{f \in \mathcal{S} \cup T: f < h} f = z \lor \bigvee_{f \in \sigma(T_g): f < h} f$$

where $S = \text{supp}_H(L, \mathcal{H})(y)$ and $z = z_y(g)$. This then implies

$$d(\bigvee_{f \in \mathcal{S} \cup T: f < h} f, h) \geq d(\bigvee_{f \in \mathcal{S} \cup T: f < h} f \lor z, h \lor z) = d(\bigvee_{f \in \sigma(T_g): f < h} f \lor z, h \lor z)$$

$$= d(\bigvee_{f \in \sigma(T_g): f < h} \zeta(f), \zeta(h)) = d(\bigvee_{f \in \sigma(T_g): \sigma(f) < h} f, \zeta(h)) = d(\bigvee_{f \in \sigma(T_g): f < h} f, \zeta(h)),$$

so that for every $h \in T$, $0 < b(h) < d(\bigvee(S \cup T)_{< h}, h)$. Since $S \cup T$ is $\mathcal{H}$-nested by Proposition 2.8.7, the monomial $x_T^b$ is basic in $DP_y(L, \mathcal{H})$. Since $\sigma$ is injective, the assignment $\otimes x_T^b \mapsto x_T^b$ is an injection on basic monomials.

Now to show $(22)$, we first observe that each join factor on the right is clearly also a factor on the left. In the other direction, there are two cases to consider. If $f \in S$, then $f < h \leq g$, so $f \leq z$. Otherwise, $\zeta(f) \in T_g$ for some $g' \in S$. This means, in particular, that $g' = f$, the minimum element of $F^+(y)$ above $f$ (Proposition 2.8.2). Since $f < h \leq g$, this implies $g' \leq g$. If $g' = g$, then $f$ is also a factor on the right; otherwise, $g' < g$ implies that $f \leq g' \leq z$. \hfill \Box

4.2. Blowups and the De Concini–Procesi algebra. The effect of a blowup on the algebras $DP(L, \mathcal{H})$ is analogous to what happens for Orlik–Solomon algebras.

**Proposition 4.2.1.** Suppose $\mathcal{H}$ and $\mathcal{H}' = \mathcal{H} \cup \{p\}$ are partial building sets associated to a building set $\mathcal{G}$ in a geometric lattice $L$. Then there is a well-defined ring homomorphism $\phi_{DP}: DP(L, \mathcal{H}) \rightarrow DP(L, \mathcal{H}')$ defined by letting

$$(23) \quad \phi_{DP}(x_g) = \begin{cases} x_g & \text{if } g \not\leq p; \\ x_g + x_p & \text{if } g \leq p, \end{cases}$$

for each $g \in \mathcal{H}$.

**Proof.** The argument for relations of type (i) is the same as the monomial relations for the Orlik–Solomon algebra in the proof of Lemma 3.3.3. Now let $g \in a(L)$ and consider $c^H_g$ in $(21)$. If $h \in \mathcal{H}$ and $h > g$, then $h \in \mathcal{H}^o$ and hence $h \not\leq p$ (since $\mathcal{H}^o$ is an order filter in $\mathcal{G}$). This implies that $c^H_g$ picks up an $x_p$ exactly when $g \leq p$, thus $\phi_{DP}(c^H_g) = c^{H'}_g$. \hfill \Box

**Remark 4.2.2.** By eliminating the generators $x_h$ for $h \in a(L)$, as in Remark 4.1.4, the map $\phi_{DP}$ has the simple formula $\phi_{DP}(x_g) = x_g$ for $g \in \mathcal{H}^o$. This is because $g \in \mathcal{H}^o$ implies that $g \not\leq p$ in $L(L, \mathcal{H})$. \hfill \Diamond

We will now show that this map extends to sheaves, and then fit the sheaves in a short exact sequence. For ease of notation, we will write $DP$ in place of $DP(L, \mathcal{H})$ and $DP'$ for $DP(L, \mathcal{H}')$ as well as $L := L(L, \mathcal{H})$ and $L' := L(L, \mathcal{H}')$. 
Lemma 4.2.3. For each \( y \in L' \), the composite of \( \phi_{DP} \) with the restriction \( DP' \to DP'_y \) factors through the restriction \( DP \to DP_{\pi(y)} \):

\[
\begin{array}{ccc}
DP & \xrightarrow{\phi_{DP}} & DP' \\
\downarrow & & \downarrow \\
DP_{\pi(y)} & \xrightarrow{\phi_{DP,y}} & DP'_y.
\end{array}
\]

We let \( \phi_{DP,y} \) denote the map \( DP_{\pi(y)} \to DP'_y \).

Proof. We lift the generators of \( DP_{\pi(y)} \) to \( DP \) and check that the relations map to zero in \( DP'_y \). We saw in Proposition 4.2.1 that \( \phi_{DP}(c^H_g) = c^H'_g \) for all \( g \in a(L) \), so it remains to check the Stanley-Reisner relations. For this, we express \( y = \bigvee S \) for a minimal nested set \( S \in n(L,H') \). Let \( \Pi = \Pi_{H'} \). Then \( \pi(y) = \bigvee \Pi(S) \) (see, e.g., (8)), so we show simplices of \( st_{n(L,H')}(S) \) map to simplices of \( st_{n(L,H')}(\Pi(S)) \). So suppose \( T \in st_{n(L,H')}(S) \). Then \( S \cup T \in n(L,H') \Rightarrow \Pi(S \cup T) \in n(L,H') \), by Lemma 2.6.5; \( \Leftrightarrow \Pi(S) \cup \Pi(T) \in n(L,H') \), \( \Leftrightarrow \Pi(T) \in st_{n(L,H')}(\Pi(S)) \), as required. \( \square \)

Lemma 4.2.4. The map \( \phi_{DP} \) induces a map of sheaves \( \pi^* DP(L) \to DP(L') \) on \( (L')^{op} \).

Proof. The claim amounts to showing that, for any \( x \leq y \) in \( L' \), the square

\[
\begin{array}{ccc}
DP_{\pi(x)} & \xrightarrow{\phi_{DP,x}} & DP'_x \\
\downarrow & & \downarrow \\
DP_{\pi(y)} & \xrightarrow{\phi_{DP,y}} & DP'_y.
\end{array}
\]

commutes. For \( x = \hat{0} \), this is Lemma 4.2.3. In general, it follows from Lemma 4.2.3 because the map \( DP \to DP_{\pi(x)} \) is surjective. \( \square \)

Recall the notation \( L_{(p)} \) from Definition 2.3.4. We define a sheaf \( Q \) on \( L_{(p)}^{op} \) as follows: for \( y \in L_{(p)}, Q(y) \) is the ideal generated by \( x \) in the ring \( \mathbb{Q}[x]/(x^d) \), where \( d = d(z_y(\hat{p}), z_y(\hat{p}) \vee \hat{p}) \) and \( \hat{p} \) is as in Proposition 2.8.2. We recall from Lemma 2.3.5(4) the embedding \( \alpha : L_{(p)}^{op} \times \{0 < 1\} \to L'^{op} \).

Theorem 4.2.5. If \( L' = Bl_p(L) \), there is a short exact sequence of sheaves on \( (L')^{op} \)

\[
0 \to \pi^* DP(L) \xrightarrow{\phi_{DP}} DP(L') \xrightarrow{\alpha_!} \bigwedge \alpha(Q \otimes \mathbb{Q}) \to 0,
\]

where \( Q \) denotes the constant sheaf on the two-element poset \( \{0 < 1\} \).

Proof. Lemma 4.2.4 says that \( \phi_{DP} \) is a map of sheaves, and so we check that it is injective and has the stated cokernel. Let us examine this on the stalk at \( y \in L' \), \( \phi_{DP,y} : DP_{\pi(y)} \to DP'_y \).
using our tensor decomposition from Theorem 4.1.7:
\[ \text{DP}_{\pi(y)} \cong \bigotimes_{f \in F^+(\pi(y))} \text{DP}(L_{\pi(y),f}, \mathcal{H}_{\pi(y),f}) \to \bigotimes_{f \in F^+(y)} \text{DP}(L_{y,f}, \mathcal{H}_{y,f}) \cong \text{DP}_y. \]

If \( y \notin \mathcal{L}_{(p)} \), then by Proposition 2.8.8, the local building sets are unchanged by the blowup and hence \( \phi_{\text{DP},y} \) is an isomorphism with cokernel \( (\alpha_1(\mathcal{Q} \otimes \mathcal{Q}))_y \).

Now suppose that \( y \in \mathcal{L}_{(p)} \), and let \( \hat{p} = \min\{g \in F^+(y) : g \leq p\} \) as in Proposition 2.8.2. By Proposition 2.8.8, the map \( \phi_{\text{DP},y} \) is an isomorphism on each tensor factor of \( \text{DP}_{\pi(y)} \) except the one indexed by \( \hat{p} \). Furthermore, the tensor factors of \( \text{DP}_y \) indexed by \( f \in F^+(x)_{\leq p} \) are simply \( \text{DP}(L_{y,f}, \mathcal{H}_{y,f}) \cong \mathcal{Q} \) since \( (\mathcal{H}_{y,f})^0 = \emptyset \). So it suffices to understand the map \( \text{DP}(L_{y,p}, \mathcal{H}_{y,p}) \to \text{DP}(L_{y,\hat{p}}, \mathcal{H}_{y,\hat{p}}) \). When \( y = (p,x) \) for some \( x \in \mathcal{L}_{(p)} \), we have \( L_{y,p} = [z_p(p),p] \) with \( \mathcal{H}_{y,p}^0 = \emptyset \) and \( L_{y,p} = [0,p] \) with \( (H_{y,p}^0)^0 = \{p\} \). In this case, the map in question is the inclusion of \( \mathcal{Q} \) into \( \mathcal{Q}[x_p]/(x_p^d) \), where \( d = d(z_p(p), z_p(p) \vee p) \), with cokernel \( (\alpha_1(\mathcal{Q} \otimes \mathcal{Q}))_y \) as desired. The case \( y \in \mathcal{L}_{(p)} \) is similar.

\[ \square \]

5. Combining the Two Models

In this section, we blend together the Orlik–Solomon and De Concini–Procesi algebras into a cdga. The ultimate goal is to show that this is a model for the Orlik–Solomon algebra, agreeing with the Leray model when realizable.

Throughout this section, we continue to let \( L \) be a geometric lattice containing a fixed, building set \( G \). Suppose \( \mathcal{H} \subseteq \mathcal{G} \) is a partial building set, and \( \mathcal{L}(L, \mathcal{H}) \) is the semilattice obtained from \( L \) by blowing up \( \mathcal{H}^0 \).

5.1. Defining the Algebra. Let \( R(\mathcal{H}) = \mathbb{Q}[e_g, x_g : g \in \mathcal{H}] \) be the graded-commutative algebra with generators \( e_g \) in bidegree \((0,1)\) and \( x_g \) in bidegree \((2,0)\). The algebra \( R(\mathcal{H}) \) is equipped with a differential \( d \) of bidegree \((2,-1)\), defined on generators by \( d(e_g) = x_g \) and \( d(x_g) = 0 \), giving it the structure of a cdga. Fixing a (reverse) linear extension of the order on \( \mathcal{H} \) as in Definition 2.5.1 gives an order among the \( e \) variables and among the \( x \) variables; we also require \( x_g \prec e_h \) for each \( g, h \).

\( R(\mathcal{H}) \) has a monomial basis which we denote by
\[ e_S x_T^p := e_{g_1} ... e_{g_s} x_{h_1}^{b_1} ... x_{h_t}^{b_t} \]
where \( S = \{g_1, ..., g_s\} \) with \( g_1 \prec \cdots \prec g_s \) and \( T = \{h_1, ..., h_t\} \) with \( h_1 \prec \cdots \prec h_t \). Recall the derivation \( \partial \) from Definition 3.1.1:
\[ \partial e_S = \sum_{j=1}^k (-1)^{j-1} e_{g_1} ... \hat{e}_{g_j} ... e_{g_k}. \]

Also recall the following defined in (21):
\[ e_g = \sum_{\substack{h \in \mathcal{H} : g \leq h \ \text{in} \mathcal{H} \mathcal{G}}} x_h. \]

**Definition 5.1.1** (The Leray model of a matroid). The algebra \( \tilde{B}(L, \mathcal{H}) \) is defined as the quotient of \( R(\mathcal{H}) \) by the ideal \( I(L, \mathcal{H}) \) generated by:

(i) \( e_S x_T \) whenever \( S \cup T \notin \mathfrak{n}(L, \mathcal{H}) \),
(ii) \( \partial e_S \) whenever \( S \in \mathfrak{n}(L, \mathcal{H}) \) is a circuit,
(iii) \( e_g \) for each \( g \in \mathfrak{a}(L) \).
If \( \hat{1} \in \mathcal{H} \), we also define \( B(L, \mathcal{H}) := \hat{B}(L, \mathcal{H})/(e_1) \). The algebras \( \hat{B}(L, \mathcal{H}) \) and \( B(L, \mathcal{H}) \) inherit the grading and differential \( d \) from \( R(\mathcal{H}) \).

Relation (i) is a Stanley-Reisner relation for our simplicial complex, and relation (ii) is an Orlik–Solomon relation associated to \( \mathcal{L}_{\leq \vee S} \). Note that, if \( L \) is realizable and \( \mathcal{G} = \mathcal{H} \) is a full building set, this agrees with the presentation of the Morgan model given in [DCP95, §5]. It follows from Proposition 2.5.5 that the generator \( e_1 \) does not appear in any of the relations defining \( I(L, \mathcal{H}) \); therefore \( \hat{B}(L, \mathcal{H}) \cong B(L, \mathcal{H}) \otimes \mathbb{Q}[e_1] \). The choice of whether or not to include \( e_1 \) models the difference between blowing up a central arrangement versus a projective arrangement, and we will allow both possibilities: see Remark 3.1.3.

We note that the bigraded algebra \( \hat{B}(L, \mathcal{H}) \) contains the two algebras of §3 and §4. Because of this, much of the work done here is an extension of that in §3, §4, and [FY04]. Explicitly, we have:

**Proposition 5.1.2.** Let \( \mathcal{H} \) be a partial building set in a geometric lattice \( L \).

(a) The subalgebra of \( \hat{B}(L, \mathcal{H}) \) in degree \((0, -)\) is \( \text{OS}(\mathcal{L}(L, \mathcal{H})) \).

(b) The subalgebra of \( \hat{B}(L, \mathcal{H}) \) in degree \((-!, 0)\) is \( \text{DP}(L, \mathcal{H}) \).

(c) \((\hat{B}(L, \mathcal{H}), d)\) is a chain complex of \( \text{DP}(L, \mathcal{H}) \)-modules.

**Proof.** The first statement is a matter of comparing relations in \( B(L, \mathcal{H}) \) with (12), (13). Similarly, the relations in the \( x \) variables agree with Definition 4.1.1, establishing the second statement. The last follows by checking \( d(x_g f) = x_g \cdot d(f) \) by the Leibniz rule for each \( x_g \), since \( d(x_g) = 0 \), so \( d \) is a \( \text{DP}(L, \mathcal{H}) \)-module homomorphism.

It should be clear that the last two statements hold for \( B(L, \mathcal{H}) \) as well. The analogue of the first statement is the following.

**Proposition 5.1.3.** Suppose \( L \) is a geometric lattice. Let \( \mathcal{H} = a(L) \) and \( \mathcal{H}' = \mathcal{H} \cup \{ \hat{1} \} \). Then \( \text{OS}(L) \cong \hat{B}(L, \mathcal{H}) \) via the map \( e_g \mapsto e_g \) for each \( g \in a(L) \), and

\[
\text{OS}(L) \cong \ker d: B^0(\mathcal{L}, \mathcal{H}') \to B^2(\mathcal{L}, \mathcal{H}').
\]

**Proof.** For the first statement, we see from (21) that \( e_g = x_g \), so the defining ideal \( I(L, \mathcal{H}) \) is generated by Orlik–Solomon relations and the variables \( x_g \) for each \( g \in \mathcal{H} \). For the second statement, the kernel of the differential can be seen to be generated by differences \( e_g - e_h \) for atoms \( g, h \), and the result follows from Definition 3.1.2.

We conclude this section by noting that the algebra \( B(L, \mathcal{H}) \) has a fine grading indexed by the semilattice \( \mathcal{L}(L, \mathcal{H}) \) which is induced by Brieskorn’s decomposition of the Orlik–Solomon algebra (Corollary 3.1.8). In the geometric setting, the summands are indexed by strata and obtained as tensor products of the cohomology of the divisor complement near the stratum (an Orlik–Solomon algebra) and the cohomology of the stratum itself (a De Concini–Procesi algebra): see [DCP95, §5.2] as well as [Dup15, Bib16].

**Proposition 5.1.4.** Let \( \mathcal{H} \subseteq L_+ \) be a partial building set, and let \( \mathcal{L} = \mathcal{L}(L, \mathcal{H}) \). We have a decomposition of \( B(L, \mathcal{H}) \) as \( \text{DP}(L, \mathcal{H}) \)-modules: for all \( i, j \geq 0 \),

\[
\hat{B}^{ij}(L, \mathcal{H}) \cong \bigoplus_{y \in \mathcal{L}_j} \text{OS}^i(\mathcal{L}_{\leq y}) \otimes_{\mathbb{Q}} \text{DP}_y^i(L, \mathcal{H}),
\]

where \( \text{DP}_y^i(L, \mathcal{H}) \) is defined in Definition 4.1.2. Similarly,

\[
B^{ij}(L, \mathcal{H}) \cong \bigoplus_{y \in \mathcal{L}_j; \ y \not\subseteq (1, 0)} \text{OS}^i(\mathcal{L}_{\leq y}) \otimes_{\mathbb{Q}} \text{DP}_y^i(L, \mathcal{H}).
\]
Proof. Since the relations defining $\tilde{B}(\mathcal{H})$ are homogeneous in both $e$ and $x$ variables, Brieskorn’s Lemma 3.1.8 gives a decomposition

$$\tilde{B}^{ij}(L, \mathcal{H}) \cong \bigoplus_{y \in \mathcal{L}} \bigoplus_{S \in \mathbf{nbc}(\mathcal{L}) \cap y} e_S \text{DP}_S^{ie},$$

for some DP$(L, \mathcal{H})$-module DP$^i_S$. We note that if $S$ and $S'$ are independent sets with $\bigvee S = \bigvee S'$, the relations of Definition 5.1.1 are unchanged by replacing $S$ by $S'$, so DP$^i_S = \text{DP}^i_{S'}$. Using the homogeneity of the presentation, we compare it with Definition 4.1.2 and find DP$^i_S \cong \text{DP}^i_y(L, \mathcal{H})$, where $y = \bigvee S$. The version (25) is analogous.

5.2. An equivalent presentation. Just as in [FY04, Thm. 1], we must add to relation (iii) from Definition 5.1.1 in order to obtain a Gröbner basis for the ideal $I(L, \mathcal{H})$ and hence a monomial basis for our algebra $B(L, \mathcal{H})$.

**Theorem 5.2.1.** The ideal $I(L, \mathcal{H})$ in $R(\mathcal{H})$, from Definition 5.1.1, is equal to the ideal generated by:

(i) $e_SXT$ whenever $S \cup T \notin \mathbf{n}(L, \mathcal{H})$,

(ii) $\partial e_S$ whenever $S \in \mathbf{n}(L, \mathcal{H})$ is a circuit, and

(iii') $e_{SXTc_y^g}$ whenever $S \cup T \in \mathbf{n}(L, \mathcal{H})$ and $g \in \mathcal{H}$ for which: $S \cap T = \emptyset$, $S \cup T$ is an antichain, $\bigvee(S \cup T) < g$, and $d = d(\bigvee(S \cup T), g)$.

**Proof.** Let $I'$ denote the ideal generated by relations (i), (ii), (iii'). Every relation of type (iii) is also of type (iii'), since for $g \in a(L)$, we can take $S = T = \emptyset$ and have $d(0, g) = 1$. To show $I' = I(L, \mathcal{H})$, then it remains to show every relation of type (iii') is in $I(L, \mathcal{H})$, which we do by induction on $d$.

**Case $d = 1$**. Consider an element $e_{SXTc_y^g}$ of type (iii'), and let $p := \bigvee(S \cup T)$. Then $S \cup T$ is a nested set, $g \in \mathcal{H}$, $p < g$, and $d(p, g) = 1$. Since $d(p, g) = 1$, we can pick $h \in a(L)$ for which $p \lor h = g$. We want to show that if $y \in \mathcal{H}$ has $y \geq h$ but $y \nleq g$, then $\{y\} \cup S \cup T \notin \mathbf{n}(L, \mathcal{H})$, because this would imply that, modulo type (i) relations, we have $e_{SXTc_y^g} \equiv e_{SXTc_h^g} \in I(L, \mathcal{H})$.

Accordingly, assume that $y \geq h$ and $\{y\} \cup S \cup T \notin \mathbf{n}(L, \mathcal{H})$, and we will show that $y \geq g$. By construction, $h \nleq p$, so $y \nleq z$ for any $z \in S \cup T$. Let $Z \subseteq S \cup T$ consist of those elements of $S \cup T$ which are incomparable with $y$. If $Z \neq \emptyset$, then $Z \cup \{y\}$ is a nontrivial antichain that does not contain $g$, and $g > z$ for all $z \in Z$. Since

$$\bigvee(Z \cup \{y\}) = \bigvee(S \cup T \cup \{y\}) = p \lor y \geq p \lor h = g,$$

Proposition 2.7.6, shows that $Z \cup \{y\} \notin \mathbf{n}(L, \mathcal{H})$. However, we assumed $S \cup T \cup \{y\} \in \mathbf{n}(L, \mathcal{H})$ and $\mathbf{n}(L, \mathcal{H})$ is a simplicial complex, which yields a contradiction. Thus, $Z = \emptyset$, and we must have $y \geq g$ for all $z \in S \cup T$. This gives us our desired conclusion that $y \geq p \lor h = g$.

**Case $d > 1$**. Take $e_{SXTc_y^g}^{d-1}$ of type (iii') with $d > 1$, and let $p := \bigvee(S \cup T)$ once again. Pick $h \in a(L)$ so that $p < p \lor h < g$. We want to show that if $y \geq h$ but $y \nleq g$, we have $e_{SXTxy}c_y^{d-1} \in I(L, \mathcal{H})$: then, modulo $I(L, \mathcal{H})$, we have

$$e_{SXTc_y^g} \equiv e_{SXTc_h^g}c_y^{d-1},$$

which is an element of $I(L, \mathcal{H})$ since $c_h$ is of type (iii).

Assume that $y \geq h$ but $y \nleq g$. We may assume that $\{y\} \cup S \cup T \notin \mathbf{n}(L, \mathcal{H})$, since otherwise we’d be done by using type (i) relations. Once again, our choice of $h$ implies $y \nleq z$ for any $z \in S \cup T$. Consider the (possibly empty) subset $Z \subseteq S \cup T$ consisting of elements which are incomparable to $y$. We will first show that $e_{S\cap ZXTy}c_y^{d-1}$ is in the ideal $I(L, \mathcal{H})$ using...
induction, and then show that it is equal to $e_{S \cap T} e_T^d$ modulo $I(L, \mathcal{H})$, which will conclude the proof.

Let $\tilde{y} = \vee (\{y\} \cup Z) = y \vee p$. Note that $\tilde{y} \vee g = y \vee g$. Also, since $0 < h \leq y \wedge g$, we have $y \vee g \in \mathcal{H}$ by Proposition 2.5.3(b). Moreover,$$
abla(d(\tilde{y}, \tilde{y} \wedge g) \leq d(\tilde{y} \wedge g, g) \leq d(p \wedge g, g) < d)
$$where the first inequality follows by [FY04, §3(iv)], and the last two by [FY04, §3(i)], together with the observation that $\tilde{y} \wedge g \geq (y \wedge g) \vee (p \wedge g) \geq h \vee p$. Thus, our first claim follows from the induction hypothesis.

Now we argue that if $q \geq g$ but $q \not\geq y \vee g$, we have $\{q, y\} \notin \mathbf{n}(L, \mathcal{H})$, so that relations of type (i) give us our last claim. So assume that $q \geq g$ and $\{q, y\} \in \mathbf{n}(L, \mathcal{H})$, and we will show that this implies $q \geq y \vee g$. If $q \leq y$, then $q \leq y$, contradicting our choice of $y$. If $q$ and $y$ are incomparable, then by nestedness we have $q \vee y \notin \mathcal{H}$. But since $h \leq g \leq q$ and $h \leq y$, we obtain $0 < h \leq q \wedge y$, from which Proposition 2.5.3(b) yields a contradiction. Thus, $q \geq y$, which implies that $q \geq y \vee g$. □

**Remark 5.2.2.** Observe that the relation $e_{S \cap T} e_T^d$ holds in DP($L, \mathcal{H}$) even without the hypotheses $S \cap T = \emptyset$ and $S \cup T$ an antichain. Indeed, if $S \cup T \in \mathbf{n}(L, \mathcal{H})$ and $g \in \mathcal{H}$ for which $\vee(S \cup T) < q$ and $d = d(\vee(S \cup T), g)$, then $e_{S \cap T} e_T^d$ is divisible by the type (iii) relation $e_{S \cap T} e_T^d$ where $S' = (S \setminus T) \cap \max(S \cup T)$ and $T' = T \cap \max(S \cup T)$. □

5.3. A Gröbner basis. The construction we describe next is modelled after the Gröbner basis [FY04, Thm. 2]. In fact, in the case where $\mathcal{H} = \mathcal{G}$, that Gröbner basis is the subset of ours obtained by restricting to the $x$ variables. On the other hand, by restricting to the $e$ variables, we recover the nbc basis for OS($L(L, \mathcal{H})$) from Theorem 3.1.7.

The corresponding additive basis for $B(L, \mathcal{H})$ will play an essential role in our proof that blowups induce injective quasi-isomorphisms of cDGAs (Theorem 5.5.6 and Theorem 5.5.1.)

**Theorem 5.3.1.** Recall that a fixed linear extension of the order on $\mathcal{H}$ induces an order among the $e$ variables and among the $x$ variables, and we further require $x_g \prec e_h$ for each $g, h$. With this, consider the deg-lx monomial order on $R(\mathcal{H})$.

The relations (i), (ii), and (iii) from Theorem 5.2.1 form a Gröbner basis for the ideal $I(L, \mathcal{H})$ in $R(\mathcal{H})$.

**Proof.** This proof is not very enlightening: we follow the method used by Feichtner and Yuzvinsky and explicitly compute syzygies. We have several cases, depending on the different types of relations.

**Case (i)–(i):** Since type (i) relations are monomial, the syzygy for two of these will be zero.

**Case (ii)–(ii):** The syzygy for two type (ii) relations is zero by Theorem 3.1.7.

**Case (i)–(ii):** If we have $e_{RI} e_S$ of type (i) and $\partial e_T$ of type (ii), we note that $\vee(T - \{g\}) = \vee T$ for each $g \in T$, since $T$ is a circuit. Since $R \cup S$ and hence $R \cup S \cup T$ is not nested, then neither is $R \cup S \cup T - \{g\}$ for any $g \in T$. It follows that each monomial in the syzygy between $e_{RI} e_S$ and $\partial e_T$ is a relation of type (i)

**Case (i)–(iii):** Now consider $e_S e_T$ of type (i) and $e_{AB} e_B^d$ of type (iii). Let $U = S \cup A$ and $V = B \cup T - \{g\}$, so that the syzygy is

$$z = e_U e_V (x_g^d - e_g^d).$$

If $g \not\in T$, then the syzygy $z$ is divisible by the type (i) relation $e_S e_T$. So assume that $g \in T$. Since $S \cup T$ is not nested, then neither is $U \cup V \cup \{g\}$. If $U \cup V$ were not nested, then the syzygy $z$ would be divisible by the type (i) relation $e_U e_V$, so assume that $U \cup V$ is nested.
Modulo $e_Ux_{V∪\{g\}}$, then
\[ z \equiv e_Ux_V\left(\sum_{f>g} x_f\right)^d. \]

Since $U ∪ V \cup \{g\}$ is not nested, it contains a nontrivial antichain $Y$ whose join $y = \bigvee Y$ is in $H$. Moreover, $Y$ must contain $g$ since $U ∪ V$ is nested; let $y' = \bigvee(Y - \{g\})$. Since
\[ d = d(\bigvee_{f∈A∪B} f, g) \geq d(\bigvee_{f<g} f ∨ y', g ∨ y') \geq d(\bigvee_{f<y} f, y), \]
we have that $e_Ux_Vc_g^d$ is divisible by a type (iii') relation. We claim that modulo type (i) relations,
\[ z \equiv e_Ux_Vc_g^d, \]
which, along with Remark 5.2.2, will finish the proof of this case.

To prove this last claim, we will show that if $f \in H$ with $f > g$ and $f \not\geq y$, then $U ∪ V ∪ \{f\}$ is not nested. Suppose that $f > g$ and $f \leq y$. Then $y' ∨ f = y' \vee g = y \in H$, which implies that $U ∪ V ∪ \{f\}$ is not nested. Now suppose that $f > g$ such that $f$ and $g$ are incomparable. Then $f ∨ y' ≥ g ∨ y' = y$, and by Proposition 2.7.6, this means that $U ∪ V ∪ \{f\}$ is not nested.

Case (ii)–(iii') Suppose that we have $\partial e_S$ of type (ii) and $e_Ax_Bc_d^d$ of type (iii'). Let $h = \min_S S$, and $S' = S - \{h\}$. Let $L := e_{S'∪A}x_Bx_g^d$, the lead monomial in the syzygy. Cancelling $L$, we obtain
\[ e_{A-S}x_Bx_g^d(\partial e_S - e_{S'}) - e_{S'∪A}x_B(e_g^d - x_g^d) \equiv = e_{A-S}x_Bx_g^d(\partial e_S - e_{S'}) + e_{A-S}(\partial e_S - e_{S'})x_B(e_g^d - x_g^d) \]
by adding a multiple of $\partial e_S$ with initial term less than $L$;
\[ = e_{A-S}(\partial e_S - e_{S'})x_Bc_g^d, \]
\[ = \sum_{k∈S'} \pm e_{A∪S'-\{k\}}x_Bc_g^d, \]
If $k \notin A$, then $d(\bigvee(A∪B∪S - \{k\}), g) \leq d(\bigvee(A∪B), g)$ and hence this summand is divisible by a relation of type (iii'). Now assume $k ∈ A$, which implies that $k ≤ g$. Since $S$ is a circuit, we have $v := \bigvee S \in G$. Then for any $f ∈ H$ with $f ≥ g$, we have $0 < k ≤ f ∧ v$ and hence $f ∨ v ∈ H$ by Proposition 2.5.3(b). This means that if $f \not≥ v$, the set $S \cup \{f\} \setminus \{k\}$ is not nested, and type (ii) relations then reduce the above expression to
\[ = \sum_{k∈A∩S'} \pm e_{A∪S'-\{k\}}x_Bc_g^d. \]
Since $d(\bigvee(A∪B) \vee g ∨ v) \leq d(\bigvee(A∪B), g)$, this is divisible by a type (iii') relation, which completes the argument.

Case (iii'–(iii')) Let $e_Sx_Tc_g^d$ and $e_Ax_Bc_h^f$ be two (iii') relations. We have different scenarios here:

First, if $g = h$ and $d ≤ f$, then the syzygy is
\[ e_{S∪A}x_Tx_h^c - e_{S∪A}x_Tx_Bx_h^c = e_{S∪A}x_Tx_Bx_h^c (x_f^d - c_f^d), \]
which is divisible by the type (iii') relation $e_{S∪T}c_g^d$.

Second, if $g ≠ h$, $g ≠ B$, and $h ≠ T$, then assume (without loss of generality) that $g > h$. The syzygy is then
\[ z := e_{S∪A}x_Tx_B(x_h^c - x_g^c), \]
Let \( y = e_{S \cup A} x_{T \cup B} c^d_{g}(c^d_h - x^f_l) \), which is divisible by the type (iii') relation \( e_S x_T c^d_g \) and satisfies \( \text{In}(y) \leq \text{In}(z) \). It suffices to check that \( z + y \) reduces to zero, and it does since

\[
z + y = e_{S \cup A} x_{T \cup B} (c^d_g - x^d_g) c^f_h
\]
is divisible by the type (iii') relation \( e_A x_B c^f_h \).

Finally, assume \( g \neq h \) and \( g \in B \), and note that we must also have \( g < h \) (so \( h < g \)) and \( h \notin T \). Let \( U = S \cup A \) and \( V = T \cup B \setminus \{g\} \). The syzygy is then

\[
z := e_U x_V (x_h^f c^d_g - x^d_g c^f_h).
\]

Let \( y = e_U x_V c^d_g (c^f_h - x^f_l) \), which is divisible by the type (iii') relation \( e_S x_T c^d_g \) and has a leading term smaller than or equal to that of \( z \). It suffices to check that

\[
z + y = e_U x_V c^f_h (c^d_g - x^d_g)
\]
reduces to zero. First, through division by the type (iii') relation \( e_A x_B c^f_h \), since \( g \in B \), we obtain

\[
z + y \equiv e_U x_V \left( \sum_{k > g} x_k \right)^d c^f_h.
\]

Then, for \( g < k < h \), since \( d(\sqrt{(U \cup V \cup \{k\})}, h) \leq f \), we may divide by \( e_U x_V x_k c^f_h \) from which we obtain

\[
z + y \equiv e_U x_V \left( \sum_{k > g, k \neq h} x_k \right)^d c^f_h.
\]

Then, since \( d(\sqrt{(U \cup V)}, h) \leq d + f \), we may divide by \( e_U x_V c^d_k c^f_h \). This leaves us with a sum of monomials, each of which is divisible by some \( e_U x_V x_k c^f_h \) where \( k > g \) and \( k \) is incomparable to \( h \). Thus, it remains to show that when \( U \cup V \cup \{k\} \in \mathfrak{n}(L, H) \), \( k > g \), and \( k \) incomparable to \( h \), we get \( e_U x_V x_k c^f_h = 0 \).

For this, we claim that modulo type (i) relations,

\[
e_U x_V x_k c^f_h \equiv e_U x_V x_k c^f_{h \vee k}
\]

and that the right hand side is divisible by a type (iii') relation. The latter claim follows since \( h \wedge k \geq g \) implies \( h \vee k \in H \), and also \( d(\sqrt{(U \cup V \cup \{k\})}, h \vee k) \leq f \). For the first claim, which will finish our proof, we show that if \( p \geq h \) but \( p \npreceq h \vee k \) then \( \{p, k\} \) is not nested so that \( x_k x_p \) is a type (i) relation. Now, if \( p \geq h \) with \( p \npreceq h \vee k \), then \( p \) and \( k \) are incomparable. In this case, we would also have \( p \wedge k \geq g \) implying \( p \vee k \in H \) by Proposition 2.5.3(b). Therefore, \( \{p, k\} \notin \mathfrak{n}(L, H) \). \( \square \)

Since a monomial basis of the quotient \( R(H)/I(L, H) \) is given by the monomials which are not divisible by initial monomials of elements of the Gröbner basis, and since \( \hat{B}(L, H) \cong B(L, H) \otimes \mathbb{Q}[e_1] \), we obtain the following corollary.

**Corollary 5.3.2.** The algebra \( \hat{B}(L, H) \) has an additive basis given by the monomials \( e_S x_T^b \) for which \( S \cup T \in \mathfrak{n}(L, H) \), \( S \in \mathfrak{nc}(L(H), \mathfrak{n}) \), and for each \( g \in T \) we have \( 0 < b(g) < d(\sqrt{(S \cup T)} < g) \). If \( \hat{f} \in H \), then the subset of monomials of this form which do not contain \( e_1 \) is an additive basis for the algebra \( B(L, H) \).
Recalling Remark 4.1.4, we note that, if $c_Sx_T^b$ is a monomial in the basis above, then $T \subseteq \mathcal{H}^\circ$. In the special case where $S = \emptyset$, we obtain a straightforward generalization of the additive basis of [FY04].

**Corollary 5.3.3.** The algebra $\text{DP}(L, \mathcal{H})$ has an additive basis given by monomials $x_T^b$, indexed by $T \in \mathfrak{n}(L, \mathcal{H})$ for which, if $g \in T$, then $0 < b(g) < d(z(g), g)$.

Using this and the tensor decomposition of Theorem 4.1.7, one could also obtain an explicit monomial basis for the local algebras $\text{DP}_y(L, \mathcal{H})$.

5.4. **Poincaré duality.** We conclude this section with a discussion of Poincaré duality in the sheaf of algebras $\text{DP}(L, \mathcal{H})$. First, the existence of Poincaré duality for each of the algebras $\text{DP}_y(L, \mathcal{H})$ is not surprising but, we feel, requires a bit of justification. Using the decomposition of Theorem 4.1.7, it is enough to show that $\text{DP}(L, \mathcal{H})$ itself possesses Poincaré duality. In the case where $\mathcal{G} = L_+$ is the maximal building set, Adiprasito, Huh and Katz accomplished this for any matroid and any partial building set [AHK18, §6]. In [Yuz97, §3], Yuzvinsky explicitly gave an isomorphism $\text{DP}^2(L, \mathcal{G}) \cong \text{DP}^{2(r-p)}(L, \mathcal{G})$, for $0 \leq p \leq r$. Although he assumes that both $\mathcal{G}$ is a (full) building set and that $\mathcal{L}$ is the intersection lattice of a complex arrangement, it is straightforward to extend his approach to any geometric lattice and partial building set $\mathcal{H}$. We will do so here using our Gröbner basis from §5.3.

We will want to use Poincaré duality because the $\mathbb{Q}$-dual of $(B(L, \mathcal{H}), d)$, as a cochain complex, has a technical advantage: its differential is obtained from the differential on the flag complex of §3.2 by extension of scalars.

As usual, let $L$ be a geometric lattice of rank $r + 1$, and $\mathcal{H}$ a partial building set for $L$. For a nested set $T \subseteq \mathcal{H}^\circ$, write $T^+ := T \cup \{\hat{1}\}$ and for $g \in T^+$, write

$$z_T(g) := \bigvee_{f \in T: f < g} f.$$  

Since $T \subseteq \mathcal{H}^\circ$, this is consistent with our notation from §2.8: by considering $y = \bigvee T \in \mathcal{L}(L, \mathcal{H})$ we have $T^+ = F^+(y)$ and $z_T(g) = z_y(g)$. Recall from Corollary 5.3.3 that a monomial basis for $\text{DP}(L, \mathcal{H})$ is given by $x_T^b$, where $T \in \mathfrak{n}(L, \mathcal{H})$ and $0 < b(g) < d(z_T(g), g)$. We define a $\mathbb{Q}$-linear map $\varepsilon$ on this monomial basis by letting

$$\varepsilon(x_T^b) = (-1)^{|T^--\{\hat{1}\}|} \prod_{g \in T^+} x_{\varepsilon(g)-b(g)},$$

where $d_T(g) = d(z_T(g), g)$ if $g \neq \hat{1}$, and $d_T(\hat{1}) = d(z_T(\hat{1}), \hat{1}) - 1$. Up to sign, this is simply Yuzvinsky’s basis involution.

**Lemma 5.4.1.** Let $S \in \mathfrak{n}(L, \mathcal{H})$ be such that $S \subseteq \mathcal{H}^\circ$ and $\hat{1} \in S$. Let $g \in S$ and suppose that $x_{\hat{1}}^b$ is a monomial such that for all $h \in S$ with $h > g$, we have $b(h) = d_S(h)$.

(a) If $b(g) = d_S(g)$, then we have $x_{\hat{1}}^b = (-1)^{|S'-(S')^+|} x_{\hat{1}}^b x_{\varepsilon'}^{d_S(1)}$ in $\text{DP}(L, \mathcal{H})$, where $S' = \{h \in S: h \not\geq g\}$.

(b) If $b(g) > d_S(g)$, then we have $x_{\hat{1}}^b = 0$ in $\text{DP}(L, \mathcal{H})$.

**Proof.** We argue by lexicographic induction, the base case being $x_{\hat{1}}^b$, which is zero if $b > r$. Now assume that $x_{\hat{1}}^b$ and $g \in S$ are such that $b(h) = d_S(h)$ for all $h > g$, and $b(g) \geq d_S(g)$, and assume that the statements are true for all earlier such monomials. First, we note that $x_{\hat{1}}^b x_{g}^{b(g)}$ is a multiple of a type (iii') relation. Thus we can write $x_{\hat{1}}^b = x_{\hat{1}}^b x_{\hat{1}}^{b(g)}$ as a sum
of terms of the form $-x_S^{b}g x^h_T$, where $T \subseteq \{ h \in H : h \geq g \}$ and $T \neq \{ g \}$, the set $(S-\{g\}) \cup T$ is nested, and $\sum_{h \in T} a(h) = b(g)$. We will now examine these monomials more closely.

Write $S_{\geq g} = \{ h \in S : h \geq g \}$ and $S' = S - S_{\geq g}$. Consider one of the above monomials, which can be written in the form $m = -x_S^{b}g x^{d_S + b} h x^{S_1 - S_g}$. Note that since $T$ is nested and $\emptyset < g \leq \sqrt{T}$, it cannot contain a nontrivial antichain (by Proposition 2.6.8). Hence $T$ is a nonempty chain. Similarly, $S_{>g}$ is also a nonempty chain. Let $h$ be the minimum element of $S_{>g}$, and let $\max T$ be the maximum element of $T$ (both with respect to $\leq$). If $\max T \neq h$, then the monomial is zero by the inductive hypothesis (b) with the minimum element of $S_{\geq \max T}$ playing the role of $g$. Thus, the monomial $m$ could only be nonzero if $\max T = h$. Now suppose that $|T| > 1$; i.e. there exists some $f \in T$ for which $f < h$. Taking $f$ to the maximum such, the monomial will be zero by the inductive hypothesis (b) with $h$ playing the role of $g$.

Thus, the only possibly nonzero monomial in our expansion of $x_S^b$ is $m = -x_S^{b}g x^{d_S + b} h x^{S_1 - S_g}$, where $h$ is the minimum element of $S_{>g}$. Since $d_{S-\{g\}}(h) = d_S(g) + d_S(h)$, it follows by induction that if $b(g) > d_S(g)$ then $x_S^b = 0$, and if $b(g) = d_S(g)$ then

$$x_S^{b} = -x_S^{b}g x^{d_S + b} h x^{S_1 - S_g}$$

$$= -x_S^{b}g x^{d_S + b} h x^{S_1 - S_g}$$

$$= -\left(1\right)^{|S-\{g\}|-(S')}+d_S h x^{S_1 - S_g}$$

$$= -\left(1\right)^{|S-\{g\}|-(S')} x_S^{b} h x^{S_1 - S_g}$$

□

**Proposition 5.4.2.** Let $L$ be a geometric lattice of rank $r + 1$ and $H$ a partial building set for $L$. For each basic monomial $x_L^b$ in $DP(L, H)$,

$$x_L^b \varepsilon(x_L^b) = (-1)^r x_1^r.$$  

**Proof.** By definition of $\varepsilon$, we have

$$x_L^b \varepsilon(x_L^b) = (-1)^{|T-\{1\}|+r} \prod_{h \in T^+} x_h^{d_T(h)}.$$  

The statement is clearly true when $T^+ = \{1\}$, and otherwise the result follows by applying Lemma 5.4.1(a) to each minimal element of $T$. □

Accordingly, for homogeneous elements $u \in DP^{2i}(L, H)$ and $v \in DP^{2(r-i)}(L, H)$, we define $(u, v) \in \mathbb{Q}$ to be the coefficient of $\mu := (-1)^r x_1^r$ in the product $uv \in DP^{2r}(L, H) \cong \mathbb{Q}$. Proposition 5.4.2 states that this is a perfect pairing, and the monomials $\varepsilon(x_L^b)$ form a dual basis for $DP^{2i}(L, H)^\vee$.

Using the decomposition from Theorem 4.1.7, we can describe Poincaré duality on the level of monomials in each quotient algebra $DP_y(L, H)$, for each $y \in L(L, H)$ as well: let

$$\mu_y = \prod_{g \in F^+(y) \cap H} (-c_g)^{d(z_y(g), g) - 1} \prod_{g \in F^+(y) \cap H^c} (-c_g)^{d(z_y(g), g) - 1}$$

The two expressions are equivalent since $d(z_y(g), g) = 1$ when $g \in a(L)$. We let $(u,v)_y$ be the coefficient of $\mu_y$ in the product $uv$, for $u,v \in DP_y(L, H)$.

For the rest of the section, we fix $L = L(L, H)$ and $DP = DP(L, H)$. For a graded module $M$, let $M[i]^r = M^{i+r}$ for all integers $i,r$. 


Proposition 5.4.3. For each \( y \in \mathcal{L} \), the pairing \( \langle - , - \rangle_{y} \) gives a graded isomorphism of graded DP-modules \( \text{DP}_{\text{y}}(\mathcal{L}, \mathcal{H})^{\vee} \cong \text{DP}_{\text{y}}(\mathcal{L}, \mathcal{H})[2r - 2\text{rank}(y)] \).

Proof. The Poincaré duality pairing of Proposition 5.4.2 gives an additive isomorphism \( \text{DP}^{\vee} \cong \text{DP}[2r] \). Since \( \langle u, vw \rangle = \langle vw, w \rangle \) for all \( u, v, w \in \text{DP} \), the map is an isomorphism of DP-modules. The corresponding claim for \( \text{DP}_{\text{y}} \) follows from Theorem 4.1.7. \( \square \)

Lemma 5.4.4. Suppose that \( w = y \lor h \) for some \( y, w \in \mathcal{L} \) and atom \( h \in \mathcal{H} \setminus \{1\} \), and let \( \rho : \text{DP}_{\text{y}}(\mathcal{L}, \mathcal{H}) \to \text{DP}_{w}(\mathcal{L}, \mathcal{H}) \) denote the restriction map. There is a DP\(_{\text{y}}(\mathcal{L}, \mathcal{H})\)-module homomorphism

\[
s : \text{DP}_{w}(\mathcal{L}, \mathcal{H}) \to \text{DP}_{\text{y}}(\mathcal{L}, \mathcal{H})
\]

given by letting \( s(u) = x_{h} \tilde{u} \), for any element \( \tilde{u} \) satisfying \( \rho(\tilde{u}) = u \), and an equality

\[
\langle s(u), v \rangle_{y} = \langle u, \rho(v) \rangle_{w}
\]

for all \( u \in \text{DP}_{w}(\mathcal{L}, \mathcal{H}) \) and \( v \in \text{DP}_{\text{y}}(\mathcal{L}, \mathcal{H}) \).

Proof. Let \( K = \ker(\rho) \), and \( S \) a nested set for which \( y = \lor S \). By comparing presentations (Definition 4.1.2), we see that \( K \) is generated by monomials \( x_{T} \) for which \( S \cup T \) is a nested set, and for which \( S \cup \{h\} \cup T \) is not. For such monomials, \( x_{T} : x_{h} \in J_{\text{y}}(\mathcal{H}) \), which is to say that \( x_{h} \in \text{ann}(K) \). It follows that, if \( \rho(\tilde{u}) = \rho(\tilde{v}) \), then \( x_{h} \tilde{u} = x_{h} \tilde{v} \). That is, the map \( s \) is well-defined (and clearly a \( \text{DP}_{\text{y}}(\mathcal{L}, \mathcal{H}) \)-module homomorphism.)

Using the bilinearity of the pairing, it is enough to check (27) for \( u = 1 \) and all \( v \) for which \( \rho(v) = \mu_{w} \), which is to say that \( x_{h} v = \mu_{y} \). If \( \rho(v') = \mu_{w} \) as well, then \( v - v' \in K \), so \( x_{h} (v - v') = 0 \). It suffices, then, to show \( x_{h} v = \mu_{y} \) for a single choice of \( v \).

Furthermore, in view of the tensor decomposition of \( \text{DP}_{\text{y}}(\mathcal{L}, \mathcal{H}) \) (Theorem 4.1.7), it suffices to consider \( y = 0 \) and \( w = h \in \mathcal{H} \). Let \( v = (-c_{h})^{d(0,h) - 1}(-x_{1})^{d(h,1) - 1} \). Using the (iii') relations \( c_{h}^{d(0,h)} = 0 \) and \( x_{p} x_{1}^{d(p,1)} = 0 \) for \( p < 1 \), we have

\[
x_{h} v = x_{h} (-c_{h})^{d(0,h) - 1}(-x_{1})^{d(h,1) - 1}
\]
\[
= \left( - \sum_{p > h} x_{p} \right) (-c_{h})^{d(0,h) - 1}(-x_{1})^{d(h,1) - 1}
\]
\[
= (-x_{1})^{1 + (d(0,h) - 1) + (d(h,1) - 1)}
\]
\[
= (-x_{1})^{d(0,1) - 1}
\]
\[
= \mu.
\]

\( \square \)

In the next section, it will be more convenient to work with the \( \mathbb{Q} \)-dual of \( B \), rather than \( B \). We note that, for any graded DP-module \( M \), \( \text{Hom}_{\text{DP}}(M, \text{DP}) = \text{Hom}_{\mathbb{Q}}(M, \mathbb{Q})[-2r] \). So for each \( j \geq 0 \) we let

\[
C^{j}(\mathcal{L}, \mathcal{H}) = \text{Hom}_{\mathbb{Q}}(B(\mathcal{L}, \mathcal{H}), \mathbb{Q})[2j - 2r]
\]
\[
= \text{Hom}_{\text{DP}}(B(\mathcal{L}, \mathcal{H}), \text{DP})[2j]
\]
as a graded DP-module. Using the direct sum decomposition of Proposition 5.1.4, we may write it as

\[
C^{ij}(\mathcal{L}, \mathcal{H}) = \bigoplus_{y \in \mathcal{L}_j: y \geq (1,0)} \text{OS}^j(\mathcal{L}_{\leq y})^\vee \otimes_{\mathbb{Q}} (\text{DP}_y^j)^{i+2j-2r}
\]

\[
= \bigoplus_{y} \text{Fl}^j(\mathcal{L}_{\leq y}) \otimes_{\mathbb{Q}} \text{DP}_y^i.
\]

(29)

Our pairing from above extends to one for all \(f, f'\) such that \(0 \leq j \leq r\) and \(0 \leq i \leq r - j\): We will also denote it by \(\langle - , - \rangle : B^{2i,j} \otimes C^{2(i+j), j} \rightarrow \mathbb{Q}\). We will let \(d^\vee : C^{ij} \rightarrow C^{i,j+1}\) denote the dual to the differential \(d\). Then \(d^\vee\) makes \((\mathcal{L}, \mathcal{H})\) a complex of DP-modules.

Recall from §3.2 that the flag complex \(\text{Fl}(\mathcal{L})\) has a differential \(\delta\), defined in (17).

**Proposition 5.4.5.** We have \(d^\vee = \delta \otimes_{\mathbb{Q}} \text{DP}(\mathcal{L}, \mathcal{H})\).

**Proof.** We will simply write \(\delta\) in place of \(\delta \otimes_{\mathbb{Q}} \text{DP}(\mathcal{L}, \mathcal{H})\). Suppose \(S\) is an independent set of size \(j \geq 1\) in \(\mathcal{L}(\mathcal{L}, \mathcal{H})\) with \(z = \bigvee S\), and \(Y \in \mathcal{L}^{j-1}(\mathcal{L})\) with top element \(y \prec z\). It suffices to check that, for all \(f, f' \in \text{DP}_y\) in which \(\deg(f) + \deg(f') = 2r - 2\text{rank}(z)\), we have

\[
\langle d(e_S \otimes \rho(f)), Y \otimes f' \rangle_y = \langle e_S \otimes \rho(f), \delta(Y \otimes f') \rangle_z.
\]

Both sides are zero unless \(Y = Y(g_1, g_2, \ldots, g_{j-1})\), where \(S = \{g_1, g_2, \ldots, g_{j-1}, g_j\}\), where \(z = y \vee g_j\), and we abbreviate \(g = g_j\).

Then we have

\[
\langle d(e_S \otimes \rho(f)), Y \otimes f' \rangle_y = \sum_{i=1}^j (-1)^{i-1} e_{S^{-\{g_j\}}} \otimes x_{g_j} Y \otimes f'
\]

\[
= (-1)^{j-1} \langle e_{S^{-\{g_j\}}} \otimes x_{g_j} Y \otimes f' \rangle_y
\]

\[
= (-1)^{j-1} \langle e_{S^{-\{g\}}} \otimes f, Y \otimes x_{g} f' \rangle_y
\]

\[
= (-1)^{j-1} \langle e_{S^{-\{g\}}} \otimes \rho(f), Y \otimes \rho(f') \rangle_z \quad \text{by Lemma 5.4.4;}
\]

\[
= \langle e_S \otimes \rho(f), \delta(Y \otimes f') \rangle_z,
\]

as required. \(\Box\)

### 5.5. Blowups and the cdga

Our last objective is to show that combinatorial blowups induce injective quasi-isomorphisms between the CDGAs we have constructed, hence establishing a model for the Orlik–Solomon algebra. It turns out to be relatively straightforward to verify that the map predicted by topology is indeed well-defined and injective in our more general setting. Our argument that the maps are quasi-isomorphisms requires some additional ideas which we develop in §6.2. We state our main result now, and the rest of this paper is devoted to completing the proof.

**Theorem 5.5.1** (Cohomology of the model). For each partial building set \(\mathcal{H}\) containing \(\hat{1}\), there are isomorphisms \(\text{OS}(L) \simeq H \cdot B(\mathcal{L}, \mathcal{H})\) and \(\overline{\text{OS}}(L) \simeq H \cdot B(\mathcal{L}, \mathcal{H})\) induced by

\[
e_g \mapsto \sum_{h \in \mathcal{H} : s \leq h} e_h.
\]

(30)

**Proof.** In Lemma 5.5.4, we construct for any partial building sets \(\mathcal{H}\) and \(\mathcal{H'} = \mathcal{H} \cup \{p\}\), a CDGA map \(\phi_B : B(\mathcal{L}, \mathcal{H}) \rightarrow \hat{B}(\mathcal{L}, \mathcal{H'})\), motivated by topology and defined as in (30). In Theorem
we prove that \( \phi_B \) is always injective. When \( 1 \in \mathcal{H} \), we have \( \hat{B}(L, \mathcal{H}) \cong B(L, \mathcal{H}) \otimes \mathbb{Q}[, i] \), and hence \( \phi_B \) induces an injective cdga map \( \phi_B : B(L, \mathcal{H}) \to B(L, \mathcal{H}') \).

We will argue by induction, needing separate base cases for \( B \) and \( \hat{B} \). For the latter, Proposition 5.1.3 establishes an isomorphism between \( OS(L) \) and \( \hat{B}(L, a(L)) \), and the differential is zero. For the former, let \( \mathcal{H} = a(L) \cup \{ \hat{1} \} \), and we calculate directly. Since each \( DP_y(L, \mathcal{H}) \cong \mathbb{Q}[x]/(x^{r+1}-i) \) for \( y \in L \), using (25), we may identify \( B^{0, j}(L, \mathcal{H}) \cong OS(L_{\leq t-i}) \), where \( L_{\leq t-i} \) denotes the truncation of \( L \) to degrees \( j \leq r-i \). Under this identification,

\[
(B(L, \mathcal{H}), d) \cong \bigoplus_{i=0}^{r} (OS(L_{\leq t-i}), \partial),
\]

so \( H^0(B(L, \mathcal{H})) = \ker \partial = \overline{OS}(L) \), and higher cohomology vanishes by [OT92, Lem. 3.13] as in Proposition 3.1.12.

Now we apply Theorems 5.5.6 and 6.2.1 to see that the composition of maps \( \phi_B \) gives an isomorphism \( \overline{OS}(L) \to H^0(B(L, \mathcal{H}), d) \), for any partial building set \( \mathcal{H} \) containing \( 1 \), and \( H^p(B(L, \mathcal{H}), d) = 0 \) for \( p > 0 \). By induction, we observe that this composition agrees with the formula (30). The analogous result for \( \hat{B} \) follows similarly.

Before proceeding, we single out an interesting consequence of our theorem, as well as a question for future work. At the two extremes, we have \( L = \mathbb{L} \) is a geometric lattice and \( \mathcal{L} = \mathcal{L}(L, G) \) is the face poset of the (classical) nested set complex, respectively. When \( G \) is a full building set, the poset \( \mathcal{L}(L, G) \) is simplicial, and \( OS(\mathcal{L}(L, G)) \) is the exterior Stanley-Reisner algebra of the nested set complex (see Example 3.1.4). We immediately obtain the following.

**Corollary 5.5.2.** The map (30) gives an inclusion \( OS(L) \hookrightarrow OS(\mathcal{L}(L, G)) \), where \( OS(\mathcal{L}(L, G)) \) is the exterior face ring of the nested set complex \( n(L, G) \).

**Question 5.5.3.** If the matroid comes from a complex hyperplane arrangement, both \( OS(L) \) and \( OS(\mathcal{L}(L, G)) \) are cohomology algebras of spaces determined by poset combinatorics. The algebra \( OS(\mathcal{L}(L, G)) \) is the cohomology algebra of a complex made up of unions of coordinate subtori in a torus, indexed by \( n(L, G) \), which is in some cases a classifying space for a right-angled Artin group: see [PS09] for details.

For other partial building sets \( \mathcal{H} \subseteq G \), is there a reasonable space for which \( OS(\mathcal{L}(L, \mathcal{H})) \) is its cohomology algebra?

Now we turn back to the efforts of proving Theorem 5.5.1. In §3.3, we found that there was an injective map \( \phi : OS(\mathcal{L}(L, \mathcal{H})) \to OS(\mathcal{L}(L, \mathcal{H}') \). Here, we show that the map extends to our cdga.

**Lemma 5.5.4.** Suppose that \( \mathcal{H} \) and \( \mathcal{H}' = \mathcal{H} \cup \{ p \} \) are partial building sets in a geometric lattice \( L \). There is a cdga map \( \phi : \hat{B}(L, \mathcal{H}) \to \hat{B}(L, \mathcal{H}') \) defined as follows:

\[
\phi_B(e_g) = \begin{cases} 
   e_g & \text{if } g \not\leq p \\
   e_g + e_p & \text{if } g \leq p
\end{cases}
\]

\( \phi_B(x_g) = \begin{cases} 
   x_g & \text{if } g \not\leq p \\
   x_g + x_p & \text{if } g \leq p
\end{cases}
\]

**Proof.** We check that \( \phi \) preserves the relations from Definition 5.1.1. The argument for relations of type (i) and (ii) is the same as the one given for the Orlik-Solomon algebra in Theorem 3.3.1. Now for \( g \in a(L) \), the only term \( x_h \) in the definition of \( e^\mathcal{H}_g \) (21) that could possibly correspond to \( h \leq p \) is \( g \) itself. That implies \( e^\mathcal{H}_g \) picks up an \( x_p \) exactly when \( g \leq p \), hence \( \phi(e^\mathcal{H}_g) = e^\mathcal{H}'_g \).

It follows that \( \phi \) is an algebra homomorphism. To see \( \phi \) is a cdga homomorphism, it is enough to verify that \( \phi \circ d = d \circ \phi \) for the generators \( e_g \) and \( x_g \), where the claim is obvious. \( \square \)
To establish the injectivity of $\phi$, we describe how it behaves on our monomial basis from Corollary 5.3.2. For an expression $f \in \check{B}(L, \mathcal{H})$, we may write its standard representative in the monomial basis and let $\text{In}(f)$ denote the largest monomial which appears.

**Lemma 5.5.5.** For a monomial $e_Sx_T^b$ in the monomial basis of $\check{B}(L, \mathcal{H})$ from Corollary 5.3.2, we have

$$\text{In}(\phi(e_Sx_T^b)) = \begin{cases} e_Sx_T^b & \text{if } \bigvee S < p \neq p; \\ e_S - p e_p x_T^b & \text{if } \bigvee S < p = p, \end{cases}$$

where $g = \min S < p$, and, as usual, $S < p := \{h \in S : h < p\}$.

**Proof.** First recall from Lemma 3.3.4 that $\text{In}(\phi(e_S))$ is either $e_S$ if $S \in n(L, \mathcal{H'})$ (equivalently $\bigvee S < p \neq p$) or $e_S - p e_p$ if $S \notin n(L, \mathcal{H'})$ (equivalently $\bigvee S < p = p$).

Also note that $\phi(x_T^b)$ is a sum of monomials where some of the $x_h$'s with $h < p$ are replaced by $x_p$. Since $h < p$ implies that $x_p < x_h$, we have $\text{In}(\phi(x_T^b)) = x_T^b$.

Next, we take the product of the standard representatives for $\phi(e_S)$ and $\phi(x_T^b)$, and then rewrite its expansion in terms of the monomial basis in order to get the standard representative of $\phi(e_Sx_T^b)$. The largest monomial that could appear is $\text{In}(\phi(e_S))\text{In}(\phi(x_T^b))$, and so it remains to check that this monomial is indeed in the basis. This check is similar to that in the proof of Lemma 3.3.4. □

**Theorem 5.5.6.** Suppose that $\mathcal{H}$ and $\mathcal{H}' = \mathcal{H} \cup \{p\}$ are partial building sets in a geometric lattice $L$. There is an injective cdga map $\phi_B : \check{B}(L, \mathcal{H}) \to \check{B}(L, \mathcal{H}')$. Furthermore, if $\check{I} \in \mathcal{H}$, there is an injective cdga map $\check{\phi}_B : B(L, \mathcal{H}) \to B(L, \mathcal{H}')$.

**Proof.** The initial monomials in Lemma 5.5.5 are distinct, by the same argument in the proof of Lemma 3.3.5. This implies that the map $\phi_B$ is an injective cdga map, and it induces the injective map on $B$. □

6. The combinatorial Leray model and formality

The main objective of this section is to show that combinatorial blowups induce quasi-isomorphisms of cdgas, completing the proof of Theorem 5.5.1. For this, we make use of some sheaf cohomology on the semilattice $L = \mathcal{L}(L, \mathcal{H})$ and, more interestingly, on its associated poset of intervals $I(L)$, whose definition was given in §2.2. In fact, we find that the algebras $\text{DP}_{\nu}(L, \mathcal{H})$ give the poset $L$ the structure of a ringed space, by equipping the poset with the order topology. Then (the dual of) $B(L, \mathcal{H})$ is expressed as the global sections of a complex of locally free coherent sheaves on the poset $I(L)$.

6.1. A complex of sheaves. Inspired by Yuzvinsky’s methods in [Yuz95], we will find that our complex $(C(L, \mathcal{H}), d^\nu)$, defined in (28), is a global version of a simpler, local construction. To begin, we recall our notational conventions from §2.1, and the poset of intervals together with coordinate maps from §2.2:

$$\mathcal{L} \xleftarrow{\text{pr}_1} I(L) \xrightarrow{\text{pr}_2} \mathcal{L}^{\text{op}}.$$

It’s easy to see that $\text{pr}_2^* = \iota_*$; in particular, for $(x, y) \in I(L)$ we have

$$\iota_* \text{DP}(x, y) = \text{DP}^*_\nu(\mathcal{H}).$$

Like $(\mathcal{L}^{\text{op}}, \text{DP})$, we see $(I(L), \iota_* \text{DP})$ is also a ringed space. Because $\mathcal{L}$ has a minimum element, sheaves coming from $\mathcal{L}^{\text{op}}$ are acyclic:
Lemma 6.1.1. For any $\mathcal{G}$ on $\mathcal{L}^\text{op}$, we have $H^q(\mathcal{L}^\text{op}, \mathcal{G}) = H^q(I(\mathcal{L}), \iota_* \mathcal{G}) = 0$ for all $q > 0$.

Proof. Let $\mathcal{G}$ be a sheaf on $\mathcal{L}^\text{op}$, and consider the constant map $p: \mathcal{L}^\text{op} \to \{0\}$. Since $\hat{0}$ is the (unique) minimum element of $\mathcal{L}$, the constant map $p$ satisfies the hypothesis of Lemma 2.1.2, so $p_*$ is exact. But $p_* = \Gamma$, the global sections functor, so $H^i(\mathcal{L}^\text{op}, \mathcal{G}) = 0$ for all $i > 0$.

The pullback $H(\mathcal{L}^\text{op}, \mathcal{G}) \to H(I(\mathcal{L}), \pr^2_2 \mathcal{G})$ is an isomorphism: to see this, we just note $I(\mathcal{L}^\text{op}) \cong I(\mathcal{L})$ under the map $(x, y) \mapsto (y, x)$, and use Lemma 2.2.5. The argument is completed by noting $\pr^2_2 = \iota_*$. \qed

Now we are ready to introduce the local version of our Leray model, using the decomposition (29) as a guide. For a partial building set $\mathcal{H}$, let $\mathcal{L} = \mathcal{L}(\mathcal{L}, \mathcal{H})$, and let $\mathcal{D} \mathcal{P} = \mathcal{D} \mathcal{P}(\mathcal{L}, \mathcal{H})$.

Definition 6.1.2. Let $\mathcal{C}(\mathcal{L}, \mathcal{H}) = \pr^i_1 \mathcal{H}(\mathcal{L}) \otimes_{\mathcal{Q}} \pr^j_2 \mathcal{D} \mathcal{P}$, with differential $\delta_{\mathcal{Q}} \otimes \pr^j_2 \mathcal{D} \mathcal{P}$. That is, for all $(x, y) \in I(\mathcal{L})$ and $i, j \geq 0$, we have

$$C^{ij}(\mathcal{L}, \mathcal{H})(x, y) = \mathcal{F}^i(\mathcal{L}_{\leq x}) \otimes_{\mathcal{Q}} \mathcal{D}^j \mathcal{P}_y.$$  

As usual, we will write $\mathcal{C}$ in place of $\mathcal{C}(\mathcal{L}, \mathcal{H})$ when no ambiguity arises.

By construction, $(\mathcal{C}, \delta)$ is a complex of locally free sheaves of $\iota_* \mathcal{D} \mathcal{P}$-modules on $I(\mathcal{L})$. Our motivation is to study the double complex dual to $B(\mathcal{L}, \mathcal{H})$, which we recover here as global sections:

Theorem 6.1.3. We have an isomorphism of cochain complexes $\Gamma(\mathcal{C}^\cdot, \delta) = (C^\cdot, \delta)$.

Proof. We will omit the cohomological indices for clarity, and let $[\mathcal{L}]$ denote the discrete topological space on $\mathcal{L}$. The diagonal map $\Delta: [\mathcal{L}] \to I(\mathcal{L})$ is continuous, and we can regard the direct sum decomposition (29) as (trivially) making $C^\cdot$ a sheaf on $[\mathcal{L}]$.

Since $\Delta_\mathcal{L}(x, y) = \mathcal{C}(x, y)$ for $x = y$ and zero otherwise, the obvious map $\mathcal{C} \to \Delta_\mathcal{L} \mathcal{C}$ has kernel $\mathcal{Z}$, where $\mathcal{Z}(x, y) = \mathcal{C}(x, y)$ for $x < y$, and $\mathcal{Z}(x, x) = 0$. Then

$$\Gamma(\mathcal{Z}) = \lim_{(x, y) \in I(\mathcal{L})} \mathcal{Z}(x, y) = 0,$$

since the initial objects in the diagram are all zero. Applying global sections to the exact sequence $0 \to \mathcal{Z} \to \mathcal{C} \to \Delta_\mathcal{L} \mathcal{C}$ then gives an injective map $\Gamma(\mathcal{C}) \to \Gamma(\Delta_\mathcal{L} \mathcal{C}) = \mathcal{C}$.

In the other direction, for each $z \in \mathcal{L}$ we give a map $\psi_z: C(z) \to \mathcal{C}(x, y)$. Fix $i, j \geq 0$.

If $j \neq \text{rank}(z)$, we let $\psi_z = 0$. Otherwise, for $x_0 \leq y_0$, then, let $\psi_z(x_0, y_0): C^{ij}(x, y) \to \mathcal{C}^{ij}(x_0, y_0)$ be given by 0 unless $z \leq x_0$, and otherwise the tensor product of the inclusion $\mathcal{F}^i(\mathcal{L}_{\leq x_0}) \hookrightarrow \mathcal{F}^i(\mathcal{L}_{\leq x})$ with the surjection $\mathcal{D}^j \mathcal{P}_z \to \mathcal{D}^j \mathcal{P}_{y_0}$. To check that the maps $\psi_z$ are compatible with the restriction maps, consider elements $x_0 \leq x_1 \leq y_1 \leq y_0$, and consider the diagram

\[
\begin{array}{ccc}
\mathcal{C}^{ij}(x_1, y_1) & \xrightarrow{\psi_z(x_1, y_1)} & \mathcal{C}^{ij}(x_0, y_0) \\
\mathcal{F}^i(\mathcal{L}_{\leq z}) \otimes \mathcal{D}^j \mathcal{P}_z & \xrightarrow{\psi_z(x_0, y_0)} & \mathcal{C}^{ij}(x_0, y_0)
\end{array}
\]

where the horizontal map is the restriction map given by $(x_0, y_0) \leq (x_1, y_1)$ in $I(\mathcal{L})$. Clearly the diagram commutes if $\psi_z(x_0, y_0)$ and $\psi_z(x_1, y_1)$ are both zero. The only remaining possibility is that $z \leq x_1$. But then the composite

$$\mathcal{F}^i(\mathcal{L}_{\leq z}) \hookrightarrow \mathcal{F}^i(\mathcal{L}_{\leq x_1}) \to \mathcal{F}^i(\mathcal{L}_{\leq x_0})$$

is zero when $z \leq x_0$ and inclusion when $z \leq x_0$, using Lemma 3.2.2, so again the diagram is seen to commute.
Since
\[ \Gamma(\mathcal{E}) = \lim_{\to} \mathcal{E}(x, y), \]
this induces a map \( C = \Gamma \Delta_* C \to \Gamma(\mathcal{E}) \). The composite \( C \to \Gamma(\mathcal{E}) \to \Gamma(\Delta_* C) = C \) is easily seen to be an isomorphism. We conclude that \( \Gamma(\mathcal{E}) \to \Gamma(\Delta_* C) = C \) is also surjective, hence an isomorphism. \( \square \)

Lemma 6.1.4. For all \( i, j \), we have \( H^q(I(L), C_{ij}) = 0 \) for all \( q > 0 \).

Proof. In Lemma 6.1.1 we saw \( \Pr^* \mathcal{D}_r = \iota_* \mathcal{D}_r(L) \) is acyclic. In Proposition 3.2.6, we saw \( \mathcal{I}(L) \) is flasque, hence acyclic. By Lemma 2.2.5, this implies \( \Pr^* \mathcal{I}(L) \) is also acyclic, and the result follows by the Künneth formula. \( \square \)

Theorem 6.1.5. The cochain complex
\[
0 \longrightarrow \iota_! \mathcal{D}_r(L, H) \longrightarrow C^0 \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^{i-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^r \longrightarrow 0
\]
is a \( \Gamma \)-acyclic resolution of \( \iota_! \mathcal{D}_r(L, H) \).

Proof. Lemma 6.1.4 showed that the sheaves in the complex are \( \Gamma \)-acyclic. To see the complex is exact, we start with the exact complex \( 0 \to \mathcal{K} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots \) on \( L \) from Proposition 3.2.6. Applying \( \Pr^* \) preserves exactness. The sheaf \( \Pr^* \mathcal{I}(L) \) is free over \( \mathbb{Q} \), hence flat. We conclude \( C = \Pr^* \mathcal{I} \otimes \Pr^* \mathcal{D}_r \) has cohomology concentrated in degree zero, and
\[
\ker(\delta^0) = \mathcal{K} \otimes \Pr^* \mathcal{D}_r = \mathcal{K} \otimes \iota_* \mathcal{D}_r = \iota_! \mathcal{D}_r. \]

\( \square \)

6.2. Blowups induce quasi-isomorphisms. Now we combine the pieces above to prove the following theorem. As usual, let \( L = \mathcal{L}(L, H) \) and \( L' = \mathcal{B}_p(L) \) be locally geometric semilattices as above, and \( \pi: L' \to L \) the blow-down map.

Theorem 6.2.1. For each partial building set \( H \) containing \( \hat{1} \), there is a quasi-isomorphism \( B(\mathcal{L}, H) \to B(\mathcal{L}, H') \).

The proof will make use of some preparatory results. The first two lemmas are of central importance. Initial quotients were defined in Definition 2.2.2.

Lemma 6.2.2. The map \( \pi: L' \to L \) is an initial quotient with connected fibres.

Proof. By the construction of \( L' = \mathcal{B}_p(L) \), we note that \( \pi^{-1}(y) = \{ y \} \) if \( y \not\in \mathcal{L}_{\geq p} \). On the other hand, if \( y \geq p \), we have \( \pi^{-1}(y) = \{(p, x) : p \vee x = y\} \).

We claim that this set has a unique minimal element \( (p, z_y) \). If \( y = p \), clearly \( z_y = \hat{0} \). If \( y > p \), by Proposition 2.5.3(a), \( y \) is not irreducible, and \( y = p \vee g_1 \vee \cdots \vee g_k \) where \( F(L, G; y) = \{p, g_1, \ldots, g_k\} \). Let \( z_y = g_1 \vee \cdots \vee g_k \), so that \( (p, z_y) \in \pi^{-1}(y) \). Now consider any \( (p, x) \in \pi^{-1}(y) \). The join decomposition (5) gives
\[
[\hat{0}, y] \cong [\hat{0}, p] \times \prod_{i=1}^{k} [\hat{0}, g_i].
\]
Since $x \leq p \lor x = y$, the image of $x$ under this isomorphism has the form $(q, q_1, \ldots, q_k)$ for
some $q \in [0, p]$. It follows $x \geq z_y$, and $(p, x) \geq (p, z_y)$, so $(p, z_y)$ is a cone vertex in $\pi^{-1}(y)$,
which proves the fibres are connected.

To show $\pi$ is an initial quotient map, we suppose $\pi(x) \leq y$ and look for $x' \geq x$ for which
$\pi(x') = y$. Again if $y \nmid p$, we may take $x' = y$.

Otherwise $y \nmid p$, and we let $z_y$ be as above. If $x \in \mathcal{L}_{\geq p}$, then take $x' = (p, (x \land p) \lor z_y) \in
\pi^{-1}(y)$. In this case, the fact that $x = \pi(x) \leq y$ along with the join decomposition above imply that
$x = (x \land p) \lor (x \land z_y) \leq (x \land p) \lor z_y$ and hence $x \leq x'$. Similarly, if $x = (p, w)$ for
some $w \in \mathcal{L}_p$, then take $x' = (p, (w \land p) \lor z_y)$.

\begin{lemma}
For any closed interval $[y_0, y_1] \subseteq \mathcal{L}$, the order complexes $|\pi^{-1}(\mathcal{L}_{\geq y_0})|$ and $|\pi^{-1}([y_0, y_1])|$ are contractible.
\end{lemma}

\begin{proof}
We fix a closed interval $[y_0, y_1]$. Let $U = \pi^{-1}(\mathcal{L}_{\leq y_1})$, and consider the set $\pi^{-1}(\mathcal{L}_{\geq y_0})$.

Case 1: If $p \lor y_0$ does not exist, then $\mathcal{L}_{\geq y_0} \cap \mathcal{L}_{\geq p} = \emptyset$, and once again, $\pi^{-1}(\mathcal{L}_{\geq y_0}) \cong \mathcal{L}_{\geq y_0}$. This
order complex has a cone point, $y_0$, and so does its intersection with the open set $U$.

In this case, then, both order complexes are contractible.

Case 2: Next suppose $p \leq y_0$. This time, there is a unique minimal element in $\pi^{-1}(y_0)$, denoted
$(p, z_{y_0})$ in the proof of Lemma 6.2.2. It is a cone point in both order complexes.

Case 3: Finally, if $p \lor y_0$ exists but $p \nmid y_0$, it is easy to check that

\[ \pi^{-1}(\mathcal{L}_{\geq y_0}) = \mathcal{L}'_{\geq y_0} \cup \mathcal{L}'_{\geq (p, z_{y_0} \lor p)}. \]

Since $y_0 \lor (p, z_{y_0} \lor p) = (p, y_0)$, we have $\mathcal{L}'_{\geq y_0} \cap \mathcal{L}'_{\geq (p, z_{y_0} \lor p)} = \mathcal{L}'_{\geq (p, y_0)}$.

Thus $\pi^{-1}(\mathcal{L}_{\geq y_0})$ is a union of contractible simplicial complexes which intersect along
a contractible subcomplex, so again $|\pi^{-1}(\mathcal{L}_{\geq y_0})|$ is contractible. To conclude the proof
we look at the intersection of each of these three conical sets with $U$.

Since $\pi(y_0) = y_0$ and $y_0 \leq y_1$, we see $\mathcal{L}'_{\geq y_0} \cap U$ is nonempty. Now $\pi(p, z_{y_0} \lor p) = y_0 \lor p$.

So if $y_0 \lor p \leq y_1$, then all three cone points are in $U$, and $|\pi^{-1}([y_0, y_1])|$ is contractible
by the same argument. If $y_0 \lor p \nmid y_1$, then $\mathcal{L}'_{(p, z_{y_0} \lor p)} \cap U$ is empty, and the preimage
of $[y_0, y_1]$ equals $\mathcal{L}'_{\geq y_0} \cap U$, a cone.
\end{proof}

\begin{remark}
By Quillen’s fibre lemma (see, e.g., [Koz08, p. 272]), the result above implies
that the combinatorial blowdown $\pi: \mathcal{L}'_+ \to \mathcal{L}_+$ induces a homotopy equivalence of order
complexes. If we compose these equivalences over the whole building set $\mathcal{G}$, we recover
the matroidal case of the main result of [FM05]. When $\mathcal{G} = \mathcal{L}_+$, this was also noted in [AHK18,
Rem. 6.5].
\end{remark}

By Proposition 2.2.4, we have a commuting square:

\[
\begin{array}{ccc}
(L')^{op} & \xrightarrow{\pi} & L^{op} \\
\downarrow{I} & & \downarrow{I} \\
I(L') & \xrightarrow{I(\pi)} & I(L)
\end{array}
\]

\begin{lemma}
For any sheaf $\mathcal{G}$ on $L^{op}$, it is the case that $\iota_1^\ast \pi^\ast \mathcal{G} = I(\pi)^\ast \iota_1^\ast \mathcal{G}$.
\end{lemma}
Lemma 6.2.1.

Let us show the cohomology of \(\mathbb{Q}(\mathbb{Q} \boxtimes \mathbb{Q})\) vanishes identically. First, it is easy to check that, since \(\alpha\) is an open embedding (on \(\mathcal{L}\)), then \(I(\alpha)\) is a closed embedding. So we see \(\nu_{\alpha} = I(\alpha)\nu_1 = I(\alpha)\epsilon_1\). It is also easy to check that the functor \(I(\alpha)\epsilon_1\) commutes with products. Then

\[
H(I(\mathcal{L}'), \nu\alpha_1(\mathcal{Q} \boxtimes \mathbb{Q})) \cong H(I(\mathcal{L}), I(\alpha)\epsilon_1(\mathcal{Q} \boxtimes \mathbb{Q}))
\]

\[
\cong H(I(\mathcal{L}_p)) \times I(\{0 < 1\}), \nu\mathcal{Q} \boxtimes \mathcal{Q}).
\]

A routine calculation shows that \(H(I(\{0 < 1\}), \nu\mathcal{Q}) = 0\) (e.g., as in [Bac75]). It follows by the Künneth formula that the cohomology of \(\nu\alpha_1(\mathcal{Q} \boxtimes \mathcal{Q})\) vanishes identically. Applying the cohomology long exact sequence to (32), we see \(H(\nu\phi_{\mathbb{Q}})\) is an isomorphism.
By Theorem 6.1.5, the cohomology of the cochain complex \((C(L'), \delta)\) is isomorphic to
\[
H^\cdot (I(L'), \iota_! DP') \cong H^\cdot (I(L'), \iota_! \pi^* DP) \text{ by the above;}
\]
\[
= H^\cdot (I(L'), 1(\pi)^* \iota_! DP) \text{ by Lemma 6.2.5;}
\]
\[
\cong H^\cdot (I(L), \iota_! \pi^* DP) \text{ by Proposition 6.2.6,}
\]
\[
= H^\cdot (C(L), \delta) \text{ by Theorem 6.1.5 again.}
\]

\[\Box\]

Remark 6.2.7. We are unable to show directly in the argument above that the quasi-isomorphism above is given by the dual of the cdga map \(\phi_B : B(L) \to B(L')\). Nevertheless, we can conclude this after the fact using Theorem 5.5.1, since we showed there that \(\phi_B\) induces a cohomology isomorphism in degree zero, and (using the result above) that the higher cohomology vanishes.

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**A MODEL FOR THE ORLIK–SOLOMON ALGEBRA**

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