Weak limit of iterates of some random-valued functions and its application

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Dedicated to Professor János Aczél on his 95th birthday.

Abstract. Given a probability space \((\Omega, A, P)\), a complete and separable metric space \(X\) with the \(\sigma\)-algebra \(B\) of all its Borel subsets, a \(B \otimes A\)-measurable and contractive in mean \(f : X \times \Omega \to X\), and a Lipschitz \(F\) mapping \(X\) into a separable Banach space \(Y\) we characterize the solvability of the equation

\[
\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega) + F(x)
\]

in the class of Lipschitz functions \(\varphi : X \to Y\) with the aid of the weak limit \(\pi^f\) of the sequence of iterates \((f_n(x, \cdot))_{n \in \mathbb{N}}\) of \(f\), defined on \(X \times \Omega^N\) by \(f^0(x, \omega) = x\) and \(f^n(x, \omega) = f(f^{n-1}(x, \omega), \omega_n)\) for \(n \in \mathbb{N}\), and propose a characterization of \(\pi^f\) for some special rv-functions in Hilbert spaces.

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1. Introduction

Fix a probability space \((\Omega, A, P)\) and a complete and separable metric space \((X, \rho)\).

Let \(\mathcal{B}\) denote the \(\sigma\)-algebra of all Borel subsets of \(X\). We say that \(f : X \times \Omega \to X\) is a random-valued function (shortly: an rv-function) if it is measurable with respect to the product \(\sigma\)-algebra \(\mathcal{B} \otimes A\). The iterates of such an rv-function are given by

\[
f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n)
\]
for $n \in \mathbb{N}, x \in X$ and $(\omega_1, \omega_2, \ldots) \text{ from } \Omega^\infty \text{ defined as } \Omega^\mathbb{N}$. Note that $f^n : X \times \Omega^\infty \to X$ is an rv-function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More exactly, for $n \in \mathbb{N}$ the $n$th iterate $f^n$ is $\mathcal{B} \otimes \mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the $\sigma$-algebra of all sets of the form

$$\{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A\}$$

with $A$ from the product $\sigma$-algebra $\mathcal{A}$.

Let $f : X \times \Omega \to X$ be an rv-function. A result on the a.s. convergence of $(f^n(x, \cdot))_{n \in \mathbb{N}}$ for $X$ being the unit interval may be found in [8, Sec. 1.4]. The paper [6] by Rafał Kapica brings theorems on the convergence a.s. and in $L^1$ of those sequences of iterates in the case where $X$ is a closed subset of a Banach lattice. A simple criterion for the convergence in law of $(f^n(x, \cdot))_{n \in \mathbb{N}}$ to a random variable independent of $x \in X$ was proved in [1] and applied to the equation

$$\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega) + F(x)$$

with $\varphi$ as the unknown function. This criterion reads as follows.

(H) There exists a $\lambda \in (0, 1)$ such that

$$\int_{\Omega} \varrho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \lambda \varrho(x, z) \quad \text{for } x, z \in X$$

and

$$\int_{\Omega} \varrho(f(x, \omega), x) P(d\omega) < \infty \quad \text{for } x \in X.$$  \hspace{1cm} (3)

Thus, denoting by $\pi^f_n(x, \cdot)$ the distribution of $f^n(x, \cdot)$, i.e.,

$$\pi^f_n(x, B) = P^\infty(f^n(x, \cdot) \in B) \quad \text{for } n \in \mathbb{N} \cup \{0\}, \ x \in X \text{ and } B \in \mathcal{B},$$

hypothesis (H) guarantees the existence of a probability Borel measure $\pi^f$ on $X$ such that

$$\lim_{n \to \infty} \int_X u(z) \pi^f_n(x, dz) = \int_X u(z) \pi^f(dz)$$

holds for $x \in X$ and for any continuous and bounded $u : X \to \mathbb{R}$; more exactly, cf. also [2, Theorem 3.1],

$$\int_X \varrho(x, z) \pi^f(dz) < \infty \quad \text{for } x \in X$$

and

$$\left| \int_X u(z) \pi^f_n(x, dz) - \int_X u(z) \pi^f(dz) \right| \leq \frac{\lambda^n}{1 - \lambda} \int_X \varrho(f(x, \omega), x) P(d\omega) \quad \text{for } x \in X, \ n \in \mathbb{N} \text{ and a non-expansive } u \text{ mapping } X \text{ into } [-1, 1].$$

Rafał Kapica strengthened this estimation showing, see [7, Corollary 5.6 and Lemma 3.1], that (5) holds for $x \in X, \ n \in \mathbb{N}$ and a non-expansive $u : X \to \mathbb{R}$. Since it is explicitly stated there only for a non-expansive and bounded
$u : X \to \mathbb{R}$ we prove it for a non-expansive $u$ mapping $X$ into a separable Banach space making use of (5) for non-expansive and bounded $u : X \to \mathbb{R}$ only. Having done that we characterize the solvability of (1) in the class of Lipschitz functions with the aid of this limit distribution $\pi_f$. Moreover, we propose a characterization of $\pi_f$ for some special rv-functions in Hilbert spaces.

2. Solvability of the equation

Following [3] given an rv-function $f : X \times \Omega \to X$ such that (H) holds and a Lipschitz $F$ mapping $X$ into a separable Banach space $Y$ define

$$F_0(x) = F(x), \quad F_n(x) = \int_\Omega F_{n-1}(f(x, \omega)) P(d\omega)$$

for $x \in X$ and $n \in \mathbb{N}$, and note that according to [3, Theorem 2.1] there exists a $y_0 \in Y$ such that for every $x \in X$ the sequence $(F_n(x))_{n \in \mathbb{N}}$ converges to $y_0$, and any Lipschitz solution $\varphi : X \to Y$ of (1) has the form

$$\varphi(x) = c + \sum_{n=0}^{\infty} F_n(x) \quad \text{for} \ x \in X,$$

where $c$ is a constant from $Y$. With the aid of the limit distribution $\pi_f$ we characterize this pointwise limit of $(F_n)_{n \in \mathbb{N}}$ and, making use of [3, Theorem 2.1(ii)], the solvability of (1) as follows (cf. [1, Corollary 4.1]).

**Theorem 2.1.** Assume (H). If $F$ is a Lipschitz mapping of $X$ into a separable Banach space $Y$, then

$$\lim_{n \to \infty} F_n(x) = \int_X F(z) \pi_f(dz) \quad \text{for} \ x \in X \tag{6}$$

and Eq. (1) has a Lipschitz solution $\varphi : X \to Y$ if and only if

$$\int_X F(z) \pi_f(dz) = 0. \tag{7}$$

As announced above, we start with the following lemma.

**Lemma 2.2.** If $f : X \times \Omega \to X$ is an rv-function such that (2) holds with a $\lambda \in (0,1)$ and (3) is satisfied, then

$$\left\| \int_X u(z) \pi_n(dz) - \int_X u(z) \pi_f(dz) \right\| \leq \frac{\lambda^n}{1-\lambda} \int_X \varphi(f(x, \omega), x) P(d\omega) \tag{8}$$

for $x \in X$, $n \in \mathbb{N}$ and for any non-expansive $u$ mapping $X$ into a separable Banach space.
Proof. First of all let us observe that for every $x \in X$, $n \in \mathbb{N}$ and $(\omega_1, \omega_2, \ldots) \in \Omega^\infty$ we have
\[
\rho \left( f^n(x, \omega_1, \omega_2, \ldots), x \right) = \rho \left( f^{n-1}(f(x, \omega_1), \omega_2, \omega_3, \ldots), x \right)
\leq \sum_{k=1}^{n} \rho \left( f^{n-k}(f(x, \omega_k), \omega_{k+1}, \omega_{k+2}, \ldots), f^{n-k}(x, \omega_{k+1}, \omega_{k+2}, \ldots) \right),
\]
where the value $f^n(x, \omega_1, \omega_2, \ldots)$ depends only on $x$ and $(\omega_1, \ldots, \omega_n)$, and by (2) we have
\[
\int_{\Omega^\infty} \varrho \left( f^n(x, \omega), f^n(z, \omega) \right) P^\infty(d\omega) \leq \lambda^n \varrho(x, z) \text{ for } x, z \in X \text{ and } n \in \mathbb{N}.
\]
Hence, applying the Fubini theorem, for $x \in X$ and $n \in \mathbb{N}$ we get
\[
\int_X \rho(x, z) \pi^f_n(x, dz) = \int_{\Omega^\infty} \rho \left( f^n(x, \omega), x \right) P^\infty(d\omega)
\leq \sum_{k=1}^{n} \int_{\Omega^\infty} \rho \left( f^{n-k}(f(x, \omega_1), \omega_2, \omega_3, \ldots), f^{n-k}(x, \omega_2, \omega_3, \ldots) \right) P^\infty(d\omega)
\leq \sum_{k=1}^{n} \lambda^{n-k} \int_{\Omega^\infty} \rho \left( f(x, \omega), x \right) P(d\omega) \leq \frac{1}{1 - \lambda} \int_{\Omega^\infty} \varrho(x, \omega) P(d\omega).
\]
Consequently, for any non-expansive $u$ mapping $X$ into a separable Banach space and for every $x \in X$, $n \in \mathbb{N}$ we obtain
\[
\int_X \|u(z)\| \pi^f_n(x, dz) \leq \frac{1}{1 - \lambda} \int_{\Omega^\infty} \varrho \left( f(x, \omega), x \right) P(d\omega) + \|u(x)\|; \quad (9)
\]
moreover,
\[
\int_X \|u(z)\| \pi^f_n(dz) \leq \int_X \varrho(x, z) \pi^f_n(dz) + \|u(x)\|. \quad (10)
\]
Let $u$ be a non-expansive mapping of $X$ into a separable Banach space $Y$. To show that (8) holds for $x \in X$ and $n \in \mathbb{N}$ we may assume that $Y$ is a real space.

Fix $x \in X$, $n \in \mathbb{N}$ and then a $y^* \in Y^*$ such that $\|y^*\| \leq 1$ and
\[
\left\| \int_X u(z) \pi^f_n(x, dz) - \int_X u d\pi^f \right\| = y^* \left( \int_X u(z) \pi^f_n(x, dz) - \int_X u d\pi^f \right). \quad (11)
\]
For every $k \in \mathbb{N}$ the function $\tau_k : \mathbb{R} \rightarrow \mathbb{R}$ given by $\tau_k(t) = -k$ for $t \in (-\infty, -k)$, $\tau_k(t) = t$ for $t \in [-k, k]$, $\tau_k(t) = k$ for $t \in (k, \infty)$ is non-expansive and $|\tau_k(t)| \leq |t|$ for $t \in \mathbb{R}$. Consequently, since (5) holds for every non-expansive and bounded $u : X \rightarrow \mathbb{R}$, for every $k \in \mathbb{N}$ we have
\[
\left| \int_X \tau_k(y^* u(z)) \pi^f_n(x, dz) - \int_X \tau_k(y^* u(z)) \pi^f(dz) \right| \leq \frac{\lambda^n}{1 - \lambda} \int_X \varrho \left( f(x, \omega), x \right) P(d\omega)
\]
(12)
and, by (9) and (10),

\[
\int_X |\tau_k (y^* u(z))| \pi_n^f (x, dz) \leq \frac{1}{1 - \lambda} \int_\Omega \varrho (f(x, \omega), x) P(d\omega) + \|u(x)\|,
\]

\[
\int_X |\tau_k (y^* u(z))| \pi^f (dz) \leq \int_X \varrho(x, z) \pi^f (dz) + \|u(x)\|.
\]

Hence, taking (3) and (4) into account, applying the Lebesgue dominated convergence theorem and passing with \( k \) to the limit in (12) we get

\[
\left| \int_X y^* u(z) \pi_n^f (x, dz) - \int_X y^* u(z) \pi^f (dz) \right| \leq \lambda^n \int_X \varrho (f(x, \omega), x) P(d\omega)
\]

and (8) follows now from (11).

\[\square\]

**Proof of the theorem.** An easy induction shows that

\[F_n(x) = \int_{\Omega^\infty} F(f^n(x, \omega)) P^\infty(d\omega),\]

i.e.,

\[F_n(x) = \int_X F(z) \pi_n^f (x, dz) \quad \text{for } x \in X, \ n \in \mathbb{N}.
\]

Let \( L \) be a Lipschitz constant for \( F \). Putting \( u = \frac{1}{L} F \) we have (8), whence

\[\|F_n(x) - \int_X F(z) \pi^f (dz)\| \leq \frac{L\lambda^n}{1 - \lambda} \int_X \varrho (f(x, \omega), x) P(d\omega)
\]

for \( x \in X \) and \( n \in \mathbb{N} \). It proves (6) and according to [3, Theorem 2.1(ii)] Eq. (1) has a Lipschitz solution \( \varphi : X \rightarrow Y \) if and only if (7) holds. \[\square\]

### 3. A characterization of the limit distribution

Obviously the problem of characterization of the limit distribution \( \pi^f \) arises. The following theorem provides a characterization via functional equations for some special rv-functions in Hilbert spaces. More exactly, we characterize \( \pi^f \) via a functional equation for its characteristic function \( \varphi^f : X \rightarrow \mathbb{C} \),

\[\varphi^f(u) = \int_X e^{i(u \cdot z)} \pi^f (dz),\]

cf. [10] by O. K. Zakusilo. Note that any two probability Borel measures on \( X \) with the same characteristic function are equal, see [9, Ch. VI, Th. 2.1(2)].
Theorem 3.1. Assume $X$ is a real separable Hilbert space, $\Lambda : X \to X$ is linear and continuous, $\xi : \Omega \to X$ is $\mathcal{A}$-measurable, and

$$f(x, \omega) = \Lambda x + \xi(\omega) \text{ for } (x, \omega) \in X \times \Omega.$$ 

If

$$\|\Lambda\| < 1 \text{ and } \int_{\Omega} \|\xi(\omega)\| P(d\omega) < \infty,$$

then the characteristic function of $\pi f$ is the only solution $\varphi : X \to \mathbb{C}$ of the equation

$$\varphi(u) = \gamma(u) \varphi(\Lambda^* u) \quad (13)$$

which is continuous at zero and fulfils $\varphi(0) = 1$, where $\gamma$ stands for the characteristic function of $\xi$.

Proof. For $n \in \mathbb{N}$ define $\xi_n : \Omega^\infty \to X$ by $\xi_n(\omega_1, \omega_2, \ldots) = \xi(\omega_n)$ and note that $\xi_n$, $n \in \mathbb{N}$, are identically distributed: Denoting by $\rho$ the distribution of $\xi$ we have

$$P^\infty(\xi_n \in B) = P(\xi \in B) = \rho(B)$$

for $n \in \mathbb{N}$ and $B \in \mathcal{B}$. Since

$$f^n(x, \omega) = \Lambda f^{n-1}(x, \omega) + \xi_n(\omega) \text{ for } \omega \in \Omega^\infty,$$

and the random variables $\Lambda \circ f^{n-1}(x, \cdot)$, $\xi_n$ are independent, we see that

$$\pi_n^f(x, \cdot) = \left(\pi_{n-1}^f(x, \cdot) \circ \Lambda^{-1}\right) \ast \rho \text{ for } n \in \mathbb{N}, x \in X.$$ 

Hence, passing to the limit (cf. [9, Ch. III, Th. 1.1]),

$$\pi^f = (\pi^f \circ \Lambda^{-1}) \ast \rho.$$

Consequently, see also [9, p. 58], for $u \in X$,

$$\varphi^f(u) = \int_X e^{i(u|x)} \left((\pi^f \circ \Lambda^{-1}) \ast \rho\right)(dz)$$

$$= \int_{X \times X} e^{i(u|x+y)} \left((\pi^f \circ \Lambda^{-1}) \times \rho\right)(d(x, y))$$

$$= \int_X \left(\int_X e^{i(u|x)} \cdot e^{i(u|y)}(\pi^f \circ \Lambda^{-1})(dx)\right) \rho(dy)$$

$$= \left(\int_X e^{i(u|x)}(\pi^f \circ \Lambda^{-1})(dx)\right) \left(\int_X e^{i(u|y)} \rho(dy)\right)$$

$$= \left(\int_X e^{i(u|\Lambda x)} \pi^f(dx)\right) \gamma(u) = \varphi^f(\Lambda^* u) \gamma(u).$$
To prove the uniqueness consider a continuous at zero solution $\varphi : X \rightarrow \mathbb{C}$ of (13) such that $\varphi(0) = 1$. Then

$$\varphi(u) = \varphi((\Lambda^*)^n u) \prod_{k=0}^{n-1} \gamma((\Lambda^*)^k u) \quad \text{for } n \in \mathbb{N}, \ u \in X,$$

and $\lim_{n \to \infty} (\Lambda^*)^n u = 0$ for $u \in X$, whence

$$\varphi(u) = \prod_{n=0}^{\infty} \gamma((\Lambda^*)^n u) \quad \text{for } u \in X.$$

\[\square\]

4. Examples

1. Fix an integer $n \geq 2$ and a Lipschitz mapping $F$ of $\mathbb{R}$ into a separable Banach space $Y$ and consider the equation

$$\varphi(x) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi\left(\frac{x+k}{n}\right) + F(x). \quad (14)$$

In this case

$$f(x, \omega) = \frac{1}{n} x + \frac{\omega}{n} \quad \text{for } x \in \mathbb{R}, \ \omega \in \{0, 1, \ldots n-1\},$$

$\xi(\omega) = \frac{\omega}{n}$ and $P(\{\omega\}) = \frac{1}{n}$ for $\omega \in \{0, 1, \ldots n-1\}$. Hence the characteristic function $\gamma$ of $\xi$ is given by

$$\gamma(u) = \frac{1}{n} \sum_{k=0}^{n-1} \exp\left(iu\frac{k}{n}\right) \quad \text{for } u \in \mathbb{R}.$$

A simple calculation shows that the characteristic function of the uniform distribution $U(0, 1)$, i.e. the function $\phi : \mathbb{R} \to \mathbb{C}$ given by

$$\phi(u) = \frac{e^{iu} - 1}{iu} \quad \text{for } u \neq 0, \ \phi(0) = 1,$$

satisfies

$$\phi(u) = \gamma(u)\phi\left(\frac{1}{n} u\right) \quad \text{for } u \in \mathbb{R}.$$

Hence and from Theorem 3.1 we infer that $\pi^f = U(0, 1)$, i.e., $\pi^f(B) = \lambda_1(B \cap [0, 1])$ for Borel $B \subset \mathbb{R}$, where $\lambda_1$ denotes the one-dimensional Lebesgue measure. Applying now Theorem 2.1 we see that Eq. (14) has a Lipschitz solution $\varphi : \mathbb{R} \to Y$ if and only if
\[
\int_{[0,1]} F(z)dz = 0.
\]

Cf. [3, Example 2.2].

2. Fix an \( \alpha \in (-1,1) \) and a Lipschitz mapping \( F \) of \( \mathbb{R} \) into a separable Banach space \( Y \) and consider the equation

\[
\varphi(x) = \int_{\Omega} \varphi(\alpha x + \xi(\omega)) P(d\omega) + F(x)
\]

assuming that \( \xi : \Omega \to \mathbb{R} \) is a random variable with the Gaussian law \( N(m, \sigma^2) \), i.e. the equation

\[
\varphi(x) = \frac{1}{\sigma \sqrt{2\pi}} \int_{\mathbb{R}} \varphi(\alpha x + y) e^{-\frac{(y-m)^2}{2\sigma^2}} dy + F(x),
\]

where \( m \) is a real number and \( \sigma \) is a positive real number. In this case

\[
f(x, \omega) = \alpha x + \xi(\omega) \quad \text{for} \ (x, \omega) \in \mathbb{R} \times \Omega,
\]

\[
\gamma(u) = \exp\left(imu - \frac{1}{2} \sigma^2 u^2\right) \quad \text{for} \ u \in \mathbb{R},
\]

and, as a simple calculation shows, the function \( \phi : \mathbb{R} \to \mathbb{C} \) given by

\[
\phi(u) = \exp\left(i \frac{m}{1-\alpha} u - \frac{1}{2} \frac{\sigma^2}{1-\alpha^2} u^2\right)
\]

satisfies

\[
\phi(u) = \gamma(u) \phi(\alpha u) \quad \text{for} \ u \in \mathbb{R}.
\]

Hence and from Theorem 3.1 we infer that

\[
\pi_f = N\left(\frac{m}{1-\alpha}, \frac{\sigma^2}{1-\alpha^2}\right).
\]

Applying now Theorem 2.1 we see that Eq. (15) has a Lipschitz solution \( \varphi : \mathbb{R} \to Y \) if and only if

\[
\int_{\mathbb{R}} F(z) \exp\left(-\frac{(z - \frac{m}{1-\alpha})^2}{2 \frac{\sigma^2}{1-\alpha^2}}\right) dz = 0.
\]

In particular (cf., e.g., [5, pp. 299–300]):

2.1. If \( \alpha \in (-1,1) \) and \( \xi : \Omega \to \mathbb{R} \) is a random variable with the Gaussian law \( N(m, \sigma^2) \), then the equation

\[
\varphi(x) = \int_{\Omega} \varphi(\alpha x + \xi(\omega)) P(d\omega) + x
\]

has a Lipschitz solution \( \varphi : \mathbb{R} \to \mathbb{R} \) if and only if \( m = 0 \).
2.2. If \( \alpha \in (-1, 1) \), \( \xi : \Omega \to \mathbb{R} \) is a random variable with the standard Gaussian law \( N(0, 1) \), \( n \in \mathbb{N} \) and \( \alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R} \), then the equation

\[
\phi(x) = \int_{\Omega} \phi(\alpha x + \xi(\omega)) P(d\omega) + \sum_{k=0}^{n} \alpha_k x^k
\]

has a Lipschitz solution \( \phi : \mathbb{R} \to \mathbb{R} \) if and only if

\[
\frac{[n/2]}{\sum_{k=0}^{[n/2]} \alpha_{2k} (2k)!}{k!2^k (1 - \alpha^2)^k} = 0.
\]

3. Let \( X \) be a real separable Hilbert space. Following [4] denote by \( L_1^+(X) \) the set of all linear, symmetric and positive self-mappings of \( X \) with finite trace.

Given a linear and symmetric \( \Lambda : X \to X \) with \( \| \Lambda \| < 1 \) and a Lipschitz mapping \( F \) of \( X \) into a separable Banach space \( Y \) consider the equation

\[
\phi(x) = \int_{\Omega} \phi(\Lambda x + \xi(\omega)) P(d\omega) + F(x)
\]

assuming now that \( \xi : \Omega \to X \) is a random variable with the Gaussian law \( N(m, Q) \), where \( m \in X \) and \( Q \in L_1^+(X) \), and

\[
\Lambda Q = Q \Lambda.
\]

In this case

\[
f(x, \omega) = \Lambda x + \xi(\omega) \quad \text{for} \ (x, \omega) \in X \times \Omega
\]

and (see [4, Sec. 1.2])

\[
\gamma(u) = \exp \left( i(m|u) - \frac{1}{2} (Qu|u) \right) \quad \text{for} \ u \in X.
\]

Put

\[
A = (I - \Lambda^2)^{-1} Q.
\]

Since

\[
(I - \Lambda^2)^{-1} = \sum_{n=0}^{\infty} \Lambda^{2n},
\]

the operator \( (I - \Lambda^2)^{-1} \) is symmetric, and by (17) we have

\[
(I - \Lambda^2)^{-1} Q = Q (I - \Lambda^2)^{-1} \quad \text{and} \quad \Lambda A = A \Lambda.
\]

Consequently \( A \) is symmetric and positive. As \( Q \) has finite trace, so has \( A \) and \( A \in L_1^+(X) \). Moreover, the function \( \phi : X \to \mathbb{C} \) given by
\( \phi(u) = \exp\left( i((I - \Lambda)^{-1}m|u) - \frac{1}{2}(Au|u) \right) \)
satisfies
\[ \phi(u) = \gamma(u)\phi(\Lambda u) \quad \text{for} \quad u \in X. \]

Hence and from Theorem 3.1 and \cite[Sec. 1.2]{4} we infer that
\[ \pi f = N((I - \Lambda)^{-1}m, (I - \Lambda^2)^{-1}Q). \]

Applying now Theorem 2.1 we see that Eq. (16) has a Lipschitz solution \( \varphi : X \to Y \) if and only if
\[ \int_X FdN((I - \Lambda)^{-1}m, (I - \Lambda^2)^{-1}Q) = 0. \]

In particular, if \( \Lambda : X \to X \) is linear and symmetric with \( \|\Lambda\| < 1 \) and \( \xi : \Omega \to X \) is a random variable with the Gaussian law \( N(m, Q) \), where \( m \in X, Q \in L_1^+(X) \) and (17) holds, then the equation
\[ \varphi(x) = \int_\Omega \varphi(\Lambda x + \xi(\omega)) P(d\omega) + x \]
has a Lipschitz solution \( \varphi : X \to X \) if and only if \( m = 0 \).

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