Abstract

In the first half of this study, we introduce a notion of simplicial systems that generalize the Rauzy graphs of interval exchange maps. We then show an effective criterion on them which imply many dynamical properties of Rauzy–Veech induction in this broader setting.

In the second half, we show, on a large set of examples, that this formalism contains many classical multidimensional continued fraction algorithms. As a consequence, we obtain ergodicity as well as existence and uniqueness of a measure of maximal entropy on a canonical suspension of Brun, Selmer algorithms in all dimensions and of Arnoux–Rauzy–Poincaré in dimension three.

In parallel, we consider fractal sets described in this formalism and show an explicit upper bound on their Hausdorff dimension as well as a construction of a measure of maximal entropy equivalent to its Hausdorff measure. This implies in particular that the Rauzy gasket in all dimensions has Hausdorff dimension strictly smaller than its ambient space.
1 Introduction

To compute the best rational approximations of a real number $0 < x < 1$, one classically uses the continued fraction algorithm, also known as the Gauss algorithm. Let the Gauss map

$$G : x \mapsto \left\{ \frac{1}{x} \right\}$$

be the map that associates to any positive number the integer part of its inverse. The Gauss algorithm consists in associating to $x$ the sequence of positive integers $a_n := \lfloor 1/G^{n-1}(x) \rfloor$ for $n \geq 1$. The corresponding sequence of rational numbers

$$\frac{p_n}{q_n} := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n + \frac{1}{a_n}}}}}$$
converges to $x$ as $n \to \infty$ and produces the best approximations of $x$ in the sense that for all integer $a, b > 0$, if $|bx - a| \leq |q_n x - p_n|$ then $b \geq q_n$.

The attempt to generalize this property to simultaneous approximation of vectors by rational numbers — together with other algebraic motivations on characterization of elements in finite extensions of $\mathbb{Q}$ — has been the starting point of the theory of Multidimensional Continued Fraction algorithms (MCF). Jacobi and Poincaré in the 19th century have suggested two different generalizations and a large variety of algorithms have been introduced ever since. Surprisingly enough, even the question of convergence on each coordinate of a vector for these algorithms does not have a straightforward answer. This fact is greatly illustrated by Nogueira [Nog95] who has showed that the algorithm introduced by Poincaré does not converge for almost every vector.

For more than 30 years, a large community of mathematicians have been working on proving dynamical properties of MCF, such as convergence [Fis72], [Nog95], [BL13], as well as further dynamical properties like ergodicity [Sch90], [MNS09], [BFK15], construction of invariant measures [ALT85], [AST74] and estimates on the speed of convergence through Lyapunov exponents [Lag93], [BAG01], [FS19].

The Gauss algorithm is an accelerated version of the projectivization of the Euclidean algorithm on $(x, 1 - x)$, by the map,

$$F : (x, y) \in \mathbb{R}^2_+ \to \begin{cases} (x - y, y) & \text{if } x > y \\ (x, y - x) & \text{if } x < y \end{cases}.$$ 

All of the MCF algorithms (e.g. all the cases mentioned in [Sch00]) can be described as an acceleration of such a map where we subtract some coordinates to others, depending on the ordering of the coordinates.

Whereas MCF are usually defined (c.f. Section 1 in [Sch90]) as iterates of a single map on the positive cone of some $\mathbb{R}^n$, another natural generalization of the Gauss algorithm is given by Rauzy–Veech induction on interval exchanges which acts on several copies of the positive cone placed at each vertex of a combinatorial graph called a Rauzy graph. This induction is fundamental in the field of Teichmüller dynamics and is a key tool for most of the dynamical results on translation surfaces and Teichmüller flow. Let us mention some of the results in the field obtained by studying the dynamics of Rauzy–Veech induction: ergodicity of the Teichmüller flow [Vee82] (also proved by [Mas82] with different techniques), introduction of Lyapunov exponents on translation surfaces [Zor96], existence and uniqueness of a measure of maximal entropy for the Teichmüller flow [BG11] and exponential mixing [AGY06]. See [FM14] for a nice survey about these results.

The Gauss algorithm was the only example so far of MCF which had a known — even though very simple — Rauzy graph as represented on Figure 3. The goal of the following work is to introduce a notion of Rauzy graph for a general MCF, which will be a directed graph labeled
by the index of the coordinates of the manipulated vectors. A path in
this graph will then encode the combinatorial settings of each steps of
the MCF. A label on an edge corresponds in the case of Rauzy–Veech
induction to the losing letter; which is slightly different from the classical
representation where edges are labeled by top and bottom. Consistently
with this latter case, we say a given letter loses if a path goes through
an edge of the corresponding label and wins if it appears as the label of
another outgoing edge of the vertex it leaves. While following a path in
the graph, we act on a positive vector whose coordinates are labeled by
the same set of letters as the edges, and when a letter wins, one subtracts
on the corresponding coordinate its value by the value of the losing letter.
One chose the winner and loser by comparing their value beforehand in
the vector.

This definition is closely related to the idea of simplicial systems intro-
duced in [Ker85] to study unique ergodicity of interval exchange maps,
we will give the same name to the corresponding graphs. The induction
on the vectors, corresponding to Rauzy–Veech, will be called a win-lose
induction.

These graphs keep track of comparisons and subtractions on pairs of
coordinates that are performed at each step of a given algorithm. Intu-
itively, the appearance of several vertices in the graph and thus of several
copies of the initial simplex are a consequence of the fact that the domains
of definition of a MCF often depend on the relative order of more than
two coordinates. For instance the graph of Brun algorithm in dimension
3 is represented on Figure 2. A computation of this graph can be found
in Section 4.2.1.

After giving a general definition of a simplicial system in Section 2 we
develop a criterion on the graph to show ergodicity of the corresponding
algorithm. This criterion will be called unstability of the graph and con-
sists in showing that trajectories will go out of degenerate subgraphs in a
small time with high probability. The degenerate subgraphs correspond
to cases when a subset $L$ of the labels can be though as infinitesimally
small compared to the others. In that case, any time there is a compari-
son between labels in $L$ and its complementary, the trajectory will almost
surely take the edge labeled in $L$. Thus the degenerate subgraph associ-
ated to $L$ consists in removing these latter edges not labeled in $L$.

Our main theorem generalizes the results of [Ker85], [BG11] and [ACY06]
to all unstable simplicial systems.
Theorem 1. Every unstable simplicial system has a unique ergodic measure equivalent to Lebesgue measure and it induces the unique invariant measure of maximal entropy on its canonical suspension.

In the case of Brun algorithm unstability is easy to check. We only need to show unstability for strongly connected components of the degenerate graphs. But such strongly connected components for Brun are always composed of a single loop on a vertex (see Figure 3). In other words, unstability reduces to showing that one letter cannot be the only one losing. This cannot happen because the coordinates of a given vector are finite.

The proof of unique ergodicity of a generic interval exchange transformation of [Ker85] can be interpreted as a proof of a weak form of unstability for Rauzy–Veech induction on interval exchanges. This property has been proved in a different formalism and a stronger sense in [AGY06] (see especially Appendix A) which was the main inspiration for this work.

Two key properties on classical Rauzy graphs, noticed in [Ker85] and later in [CN13], are first that every letter has to lose infinitely often and second that in the degenerate case with labels in $\mathcal{L}$, a letter in $\mathcal{L}$ will
alway lose eventually to a letter in the complementary set. This latter property can be checked directly by considering the labeling of edges in the strongly connected components of all degenerate subgraphs. We say a simplicial system satisfying these two properties is of Rauzy type and show the other main theorem of this work:

**Theorem 2.** Simplicial systems of Rauzy type are unstable.

The Rauzy type property is rather general. In Section 4, we explain a general strategy to associate a simplicial system to a MCF and show that for a large class of algorithms that these graphs are of Rauzy type.

**Proposition 1.** Brun and Selmer algorithms in all dimension and Arnoux–Rauzy–Poincaré algorithm in dimension 3 are of Rauzy type.

Which provides a unified proof of ergodicity of these algorithms as well as new results on existence and uniqueness of a measure of maximal entropy.

**Corollary 1.** Brun, Selmer and Arnoux–Rauzy–Poincaré algorithms have a unique ergodic measure equivalent to Lebesgue measure and it induces
the unique invariant measure of maximal entropy on its canonical suspension.

This point of view may also bring a new perspective on Poincaré algorithm in all dimensions, which are the only examples of MCF which are not of Rauzy types and for which it is not clear that they have stable degenerate subgraphs (except for the case of dimension 3). Studying ergodicity of Poincaré algorithm reduces in this formalism to compute fine estimates of the time a path in the graph stays in the degenerate subgraph. Moreover, this formalism gives a lot of freedom to introduce new examples of ergodic MCF and find algorithms closer to optimality.

Another application of these simplicial systems is given by a generalization of [AHS16]. We obtain a result on the Hausdorff dimension of sets of points with restrictive combinatorics.

**Theorem 3.** The Hausdorff dimension of length parameters at a vertex of a simplicial system for which the path of the win-lose induction stays in a given unstable strict subgraph with the same number of labels is strictly smaller than the dimension of its ambient space.

As a consequence we are able to generalize the result on the Hausdorff dimension of the Rauzy gasket in [AHS16] to Rauzy gaskets of arbitrary dimensions, as introduced in [AS13].

**Corollary 2.** The Rauzy gasket in all dimension has Hausdorff dimension strictly smaller than its ambient space.

## 2 Definitions

### 2.1 Simplicial systems

Let $G = (V, E)$ be a graph labeled on an alphabet $A$ by a function $l : E \to A$ such that for all $\pi \in V$ the restriction of $l$ to edges starting at $\pi$ is injective. We write $e : \pi \to \pi'$ if an edge $e$ goes from vertices $\pi$ to $\pi'$.

Let $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$ and let us consider the norm on $\mathbb{R}_+^A$ defined by $\|\lambda\| = \sum_{\alpha \in A} \lambda_\alpha$. Let $\Delta := \{\lambda \in \mathbb{R}_+^A \mid \|\lambda\| = 1\}$ be a simplex of dimension $|A| - 1$. We associate to graph $G$ as above a piecewise projective map $T : \Delta^G \to \Delta^G$, on the parameter space $\Delta^G := V \times \Delta$.

Let $\pi_{\text{out}}$ be the set of all edges going out of $\pi$ and for all $e \in \pi_{\text{out}}$ let

$$\Delta^e := \{(\lambda_\alpha)_{\alpha \in A} \in \Delta \mid \lambda_{l(e)} < \min_{\alpha \in l(\pi_{\text{out}} \setminus e)} \lambda_\alpha\},$$

the Rauzy matrix associated to this edge is

$$R_e := \text{Id} + \sum_{\alpha \in l(\pi_{\text{out}} \setminus e)} E_{e, \alpha} l(e).$$
Where $E_{a,b}$ is the elementary matrix with coefficient 1 at row $a$ and column $b$.

This implies a partition of $\Delta$. Let $T : \Delta^G \mapsto \Delta^G$, such that for all $\lambda \in \Delta^e$ with $e : \pi \mapsto \pi'$,

$$T(\pi, \lambda) = (\pi', T_e(\lambda)),$$

where

$$T_e : \begin{cases} \Delta^e & \mapsto \Delta^e \\ \lambda & \mapsto \frac{R_e^{-1}\lambda}{\|R_e^{-1}\lambda\|} \end{cases}.$$

![Figure 4: Action of $T_a$ on $\Delta_a$ when $\pi$ has three outgoing edges.](image)

We call the graph $G$ a simplicial system and the map $T$ its associated win-lose induction.

If there is a point in the graph that has no outgoing vertices, the map is not defined and the induction stops.

Notice that these maps are not well defined on the boundaries of $\Delta^e$. We will always study the restriction of this map to points for which the induction is defined at all times or stops. This is the complementary set of countably many codimension one subsets and thus a full Lebesgue measure set.

**Remark 1.** The maps $T_e$ are a projectivized version of linear maps on the cones,

$$\tilde{T}_e : \begin{cases} \mathbb{R}_+ \cdot \Delta^e & \mapsto \mathbb{R}_+^A \\ \lambda & \mapsto \frac{R_e^{-1}\lambda}{\|R_e^{-1}\lambda\|} \end{cases}.$$

Similarly, we have a map $\tilde{T}$ on $V \times \mathbb{R}_+^A$, which will be useful when we will consider suspensions of $T$.

Some dynamical properties of this linear map were studied for a more restrictive generalization of Rauzy–Veech induction in [CN13] which applies to Selmer and Jacobi–Perron algorithms. Although it does not admit a finite invariant measure, they show ergodicity of the linear map with respect to Lebesgue measure.

In analogy with the standard Rauzy induction on interval exchange transformations (see [Yoc10] for an introduction to the subject), we define a loser and winners labels for each edge in the graph.
**Definition.** At a given state \( \pi \) we say a letter \( \alpha \in A \) loses along an edge \( e \) going out of \( \pi \) if \( l(e) = \alpha \). On the contrary, we say a letter \( \beta \) wins along an edge \( e \) based at \( \pi \) if there exists another edge \( e' \) going out of \( \pi \) such that \( l(e) \neq l(e') = \beta \).

One can describe the linear Rauzy map as the map which compares the coordinates of all the edges going out of a given vertex \( \pi \) and subtract the smallest to the others, in other terms subtract the losing coordinate to the winning ones.

To any orbit of the win-lose induction we can thus associate an infinite path in the graph. For any \( \gamma \) in the set \( \Pi(\pi) \) of all finite paths in \( G \) starting at \( \pi \) we denote the product of matrices
\[
R_\gamma := R_{e_1} \ldots R_{e_n},
\]
if \( \gamma \) ends at \( \pi' \) we define the subsimplex of \( \Delta \),
\[
\Delta^\gamma := R_{\pi} \Delta.
\]
This corresponds to points which associated path starts with \( \gamma \). In the following we will often use the transpose of these Rauzy matrices, hence we introduce the notation
\[
B_e := R_e^T
\]
for every edge \( e \in E \) and similarly for finite paths.

### 2.2 Projective measures

**Definition.** Let \( q \in \mathbb{R}^A_+ \), let \( \nu_q \) be the Borel measure on the projective space \( P\mathbb{R}^A_+ \), such that for any subset \( A \subset P\mathbb{R}^A_+ \),
\[
\nu_q(A) := \text{Leb}(\mathbb{R}^+_A \cap \Lambda_q)
\]
where \( \Lambda_q = \{ v \in \mathbb{R}^A_+ \mid \langle v, q \rangle < 1 \} \).

In the following we will make the abuse of writing \( \nu_q(\Delta) \) for some \( \Delta \subset \mathbb{R}^A_+ \) while meaning \( \nu_q(\mathbb{R}^+_\Delta) \).

A fundamental equality is given by
\[
\nu_q(R_\gamma \cdot \Delta) = \text{Leb}(R_\gamma \Delta \cap \Lambda_q) = \text{Leb}(\Delta \cap \Lambda_{B_\gamma q}) = \nu_{B_\gamma q}(\Delta).
\]
The vector \( q \) keeps track of the way the measure is changed along the induction, we call it the *distortion vector*.

An other fundamental equation comes from a computation that can be found in [Vee78] Formula (5.5). **Proposition 2** (Veech). For \( \pi \in V \), \( \gamma \in \Pi(\pi) \) and \( q \in \mathbb{R}^A_{\leq 0} \),
\[
\nu_q(\Delta^\gamma) = \frac{1}{n!} \frac{1}{(B_\gamma q)_1 \ldots (B_\gamma q)_n}.
\]
If \( \pi \) is a vertex and \( \gamma \) is a path starting at \( \pi \) we define the probability measure,

\[
P_q^\pi(\gamma) = \frac{\nu_q(\Delta \gamma)}{\nu_q(\Delta)}.
\]

According to Formula (2),

\[
P_q^\pi(\gamma) = \frac{N(q)}{N(B, q)}
\]

where \( N(q) = \prod_{a \in A} q_a \).

**Proposition 3.** Let \( e \in E \) such that the label \( l(e) = \alpha \), then

\[
P_q^\pi(e) = \frac{q_\alpha}{(B, q)_{\alpha}}.
\]

**Proof.** Just notice that for all \( \beta \neq l(e) \), \( (B, q)_{\beta} = q_{\beta} \).

If \( \gamma = \gamma_s \gamma_e \) and \( \gamma_s \) ends at \( \pi' \), we can define conditional probabilities, using Formula (1),

\[
P_q^\pi(\gamma | \gamma_s) = \frac{\nu_q(\Delta \gamma)}{\nu_q(\Delta \gamma_s)} = \frac{\nu_q(R_{\gamma_s} \Delta \gamma_e)}{\nu_q(R_{\gamma_s} \Delta)} = P_{\beta, \gamma_s q}^\gamma(\gamma_e). \tag{3}
\]

For a set of paths \( \Gamma \subset \Pi(\pi) \) we define

\[
P_q^\pi(\Gamma) = \frac{\nu_q(\bigcup_{\gamma \in \Gamma} \Delta \gamma)}{\nu_q(\Delta)}.
\]

In words, it is the measure of the set of points whose Rauzy path has a prefix in the set \( \Gamma \). We say that \( \Gamma \) is *disjoint* if there does not exist two elements \( \gamma, \gamma' \in \Gamma \) such that \( \gamma' \) is a prefix of \( \gamma \). In this case the simplex are disjoint and

\[
P_q^\pi(\Gamma) = \sum_{\gamma \in \Gamma} \frac{\nu_q(\Delta \gamma)}{\nu_q(\Delta)}.
\]

If \( \Gamma_s \) is a disjoint set we can decompose the probability,

\[
P_q^\pi(\Gamma) = \sum_{\gamma_s \in \Gamma_s} P_q^\pi(\Gamma \cap \Pi(\gamma_s) | \gamma_s) \cdot P_q^\pi(\gamma_s). \tag{4}
\]

### 2.3 Stopping times

If a path \( \gamma_1 \) is a strict prefix of \( \gamma_2 \) we define the order \( \gamma_1 < \gamma_2 \). Let \( \mathcal{P} \) be a property on finite paths, we denote by \( \Gamma_s(\mathcal{P}) \) the set of minimal paths for this order, starting at \( \pi \) and satisfying \( \mathcal{P} \). These sets are disjoint and will be useful for computations. In the following we will often make the abuse of writing the property instead of \( \Gamma_s \) of the property.

Using Formula (3) remark that, for \( \gamma_s \) a path from \( \pi \) to \( \pi' \),

\[
P_q^\pi(\mathcal{P} | \gamma_s) = P_{\beta, \gamma_s q}^{\gamma} (\gamma_s^{-1} \mathcal{P}) \tag{5}
\]

where we define \( \gamma_e \in \gamma_s^{-1} \mathcal{P} \) iff \( \gamma_s \cdot \gamma_e \in \mathcal{P} \).
If $Q$ is another property on paths, we denote by
$$\Gamma_\pi(\mathcal{P} < Q) \quad (\text{resp. } \Gamma_\pi(\mathcal{P} \leq Q))$$
the set of minimal paths $\gamma = \gamma_1\gamma_2\ldots\gamma_n$ starting at $\pi$ satisfying $\mathcal{P}$ such that no intermediate path $\gamma_1\ldots\gamma_k$ for $0 \leq k \leq n$ (resp. $0 \leq k < n$) satisfy $Q$. In terms of stopping time, those are the path that are stopped by property $\mathcal{P}$ before property $Q$.

**Remark 2.** These definition are in fact stopping times for the $\sigma$-algebra filtration $(F_n)_{n \in \mathbb{N}}$ generated by all $\Delta_\gamma$ for $\gamma$ of length less than a given $n$. In particular this justifies the notation for the comparison of two such stopping times.

This remark is also a motivation for denoting by $\mathcal{P} = \infty$ the set of infinite paths such that none of their (finite) prefix satisfies $\mathcal{P}$.

With this notations, for all $q \in \mathbb{R}_A^+$,
$$P_q(\mathcal{P} < Q) + P_q(\mathcal{Q} \leq \mathcal{P}) + P_q(\mathcal{P} = \infty \text{ and } \mathcal{Q} = \infty) = 1. \quad (6)$$

### 2.4 Suspension semi-flow

Given a measurable function $f : \Delta^G \mapsto \mathbb{R}_+$, one can define a suspension of the parameter space,
$$\tilde{\Delta}^G := (\Delta^G \times \mathbb{R}) / \sim$$
where we use the equivalence relation $(x,t) \sim (Tx,t + f(x))$. On $\tilde{\Delta}^G$ we define a suspension semi-flow
$$\phi_t : (x,s) \mapsto (x,s + t).$$
Notice that these semi-flows are defined such that the first return map to the section $\Delta^G \times \{0\}$ is $T$ and its return time is $f$.

In the case of simplicial systems, there is a canonical suspension coming from the fact that the space
$$\left( V \times \mathbb{R}^A \right) / \sim$$
where we identify $(x,s) \sim \tilde{T}(x,s)$, with $\tilde{T}$ the homogeneous win-lose induction defined in Remark [1]. This suspension has a natural semi-flow given for $t \geq 0$ by
$$\psi_t : (x,s) \mapsto (x,e^t \cdot s).$$

The first return map to the section $\Delta^G \times \{0\}$ for this semi-flow is also equal to $T$ and its first return time is given by a function $r : \Delta^G \rightarrow \mathbb{R}_+$ defined for $e : \pi \rightarrow \pi'$ by
$$r_e : (\lambda, \pi) \mapsto -\log \|R_e^{-1}\lambda\|. $$

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Thus this semi-flow is the suspension semi-flow on $\hat{\Delta}_G^r$ that we will call the canonical suspension semi-flow associated to a simplicial system. And the function $r$ will be called the roof function for the simplicial system. A lot of dynamical properties will be induced by the study of this function using thermodynamic formalism in Section 3.3.2.

3 Dynamical properties

3.1 Kerckhoff lemma

We start by introducing some useful properties for which we will compare the stopping times. Let $L \subset A$, $\tau > 0$, $K > 0$ and $\gamma^*$ be a finite path in $G$.

Let $J^\tau$ be the property of a finite path $\gamma$ along which the distortion vector has jumped by a factor $\tau$ i.e. it satisfies

$$\max B_\gamma(q) \geq \tau \max q.$$  

**Proposition 4.** Assume that $G$ is strongly connected and has not all vertices with only one outgoing edge. For all $\tau > 0$ and all $q \in \mathbb{R}_+^A$,

$$P_\gamma(J^\tau = \infty) = 0.$$  

**Proof.** For a given path $\gamma$ in the graph, let $n$ be the number of times it passes through a vertex of degree strictly larger than 1. Then

$$\max B_\gamma(q) \geq n \min q$$  

and $n$ goes to infinity as the length of the path grows. \hfill \square

Let $M_L$ be the property of a finite path $\gamma$ for which

$$\min_L B_\gamma(q) \geq \max_A q.$$  

We denote by $M$ the case $M_A$.

Let $S_L$ be the property of a finite path $\gamma$ for which

$$\max_{A \setminus L} B_\gamma(q) \geq \min_L B_\gamma(q).$$  

Let $E_{\gamma^*}$ be the property of a finite path which admits $\gamma^*$ as a suffix.

An important property to consider on the distortion vector is the balance between its coordinates given by the following definition.

**Definition.** For $L \subset A$ and $K > 1$, we say a distortion vector $q \in \mathbb{R}_+^L$ is $(L, K)$-balanced if and only if

$$\max_A q < K \min_L q.$$  

We say it is $K$-balanced in the case $L = A$. 

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This property will be very useful due to the fact that, when a distortion
vector satisfies it, the probability that an edge labeled in $\mathcal{L}$ is taken is
bounded from below by a constant depending only on $K$.

**Remark 3.** If we start with a $(\mathcal{L}, K)$-balanced distortion vector, the set
$\mathcal{L}$ form the largest $|\mathcal{L}|$ coordinates. In particular if there is a letter in $\mathcal{L}$
which wins against a letter in its complementary set, it implies the event
$S_\mathcal{L}$.

**Definition** (Unstability property). We say a simplicial system is unstable
if its graph is strongly connected and it satisfies the following condition.
For all non-empty subset $\mathcal{L} \subseteq \mathcal{A}$ and all $K > 1$ there exist $\tau > 1$ and
$\delta > 0$ such that for all $(\mathcal{L}, K)$-balanced distortion vector $q$,

$$P_q(S_\mathcal{L} < J^\tau) > \delta.$$  

**Lemma 1.** If a simplicial system is unstable, there exists $\tau > 1$ and $\delta > 0$
such that for all distortion vector $q \in \mathbb{R}_+^A$,

$$P_q(M < J^\tau) > \delta.$$  

**Proof.** We show by recurrence on $n$ that there exists $\tau_n > 1$ and $\delta_n > 0$
such that, for all distortion vector $q$, there exists a subset $\mathcal{L} \subset \mathcal{A}$ of
cardinal $n$ which satisfies

$$P_q(M_\mathcal{L} < J^{\tau_n}) > \delta_n.$$  

**Initialization.** For $n = 1$, we just have to take $\mathcal{L}$ to be the singleton of
the largest coordinate of $q$.

**Induction.** Assume now that the property is true for some $n \geq 1$. With
probability larger than $\delta_n$, the distortion vector will satisfy $\min_\mathcal{L}(B_\gamma q) \geq
\max_\mathcal{L} q$ and $\max_\mathcal{L}(B_\gamma q) \leq \tau_n \cdot \max_\mathcal{L} q$, in particular it is $(\mathcal{L}, \tau_n)$-balanced.

Using the chain rule in Formula (4), we only need to show the induction
property with such a distortion vector. The unstability hypothesis tells
us that there exists $\tau > 1$ and $\delta > 0$ such that, with probability larger
than $\delta$, there is a letter $\alpha$ in $\mathcal{L}$ and a letter $\beta$ outside of this set such that
$(B_\gamma q)_\beta \geq (B_\gamma q)_\alpha$ before $q$ jumps by $\tau$. But then $(B_\gamma q)_\beta \geq (B_\gamma q)_\alpha \geq
\max_\mathcal{L} q$ and obviously $(B_\gamma q)_\beta \geq q_\beta$ thus $M_{\mathcal{L} \cup \{\beta\}}$ is satisfied.

**Corollary 3.** If a simplicial system is unstable, for all finite path $\gamma^*$,
there exists $K > 1$ and $\delta > 0$ such that for all $q \in \mathbb{R}_+^A$

$$P_q(E_{\gamma^*} < J^K) > \delta.$$  

**Proof.** According to the previous lemma, with probability bounded from
below by some constant independent of the distortion vector $q$, we have

$$\min B_\gamma q \geq \max q > \frac{1}{K} \cdot \max B_\gamma q.$$  

Thus the distortion vector is $K$-balanced. Now remark that in this case,
the probability that the word $\gamma$ appears in a minimal number of steps —
If $q$ diverges for any choice of path in the induction, this corollary implies in particular that all finite path appears almost surely in the induction. In particular one can define a first return map for the win-lose induction to the subsimplex of paths starting with $\gamma^*$. 

**Corollary 4.** If a simplicial system is unstable, for all path $\gamma^*$, there exists $C > 1, \eta > 0$ such that for all $\tau > 1$ and all $q \in \mathbb{R}^A_+$

$$P_q(J^\tau \leq \mathcal{E}_{\gamma^*}) < C \cdot \tau^{-\eta}.$$ 

**Proof.** Consider $K$ and $\delta$ as in the previous corollary. For all $\tau > 1$, if $\tau > K^n$ for some integer $n$, we have, using the chain rule and Proposition 4,

$$P_q(J^\tau \leq \mathcal{E}_{\gamma^*}) \leq P_q(J^K_n \leq \mathcal{E}_{\gamma^*}) < (1 - \delta)^n.$$ 

Thus, taking $n = \lceil \log \tau / \log K \rceil$,

$$(1 - \delta)^n \leq (1 - \delta)^{\frac{\log \tau}{\log K}} = \frac{1}{1 - \delta} \cdot \exp \left( \log(1 - \delta) \cdot \frac{\log \tau}{\log K} \right).$$ 

Hence, for $C = \frac{1}{1 - \delta}$ and $\eta = -\frac{\log(1 - \delta)}{\log K}$, we have

$$P_q(J^\tau \leq \mathcal{E}_{\gamma^*}) < C \cdot \tau^{-\eta}.$$ 



### 3.2 Unstability criterion

We introduce a property on the graph of a simplicial system that implies unstability. This property is true for a very large class of known examples such as the classical Rauzy induction and most of multidimensional continued fractions algorithms.

The idea of the criterion we develop here is to consider degenerations of the induction where some subset of labels $\mathcal{L} \subset \mathcal{A}$ the distortion vector at these corresponding coordinates is infinitely larger than for other labels. In particular, when we are on a vertex that has an outgoing edge labeled in $\mathcal{L}$, any edges with a label outside of $\mathcal{L}$ will almost surely not be chosen.

Let us denote by $G_\mathcal{L}$ the subgraph of $G$, with the same set of vertices $V$ and a set of edges defined as follows. For any $\pi \in V$,

- if there is at least one edge in the outgoing edges $\pi_{\text{out}}$ labeled in $\mathcal{L}$, the set of outgoing edges in $G_\mathcal{L}$ is

$$\pi_{\text{out}}^\mathcal{L} = \{ e \in \pi_{\text{out}} \mid l(e) \in \mathcal{L} \},$$

- otherwise,

$$\pi_{\text{out}}^\mathcal{L} = \pi_{\text{out}}.$$
**Definition 1.** We say a simplicial system is of Rauzy type if:

- it is strongly connected,
- every letter wins or loses infinitely many times almost surely with respect to Lebesgue measure,
- for all $\mathcal{L} \subseteq A$, in any strongly connected component of $G_{\mathcal{L}}$ there is no vertex out of which two outgoing edges have their label in $\mathcal{L}$.

The last property can be reformulated as: no letter in $\mathcal{L}$ can win against another letter in $\mathcal{L}$ in any strongly connected component of $G_{\mathcal{L}}$.

The second condition is a dynamical property, which makes it more difficult to check. We give at the end of the section an equivalent definition that is purely graph theoretic.

**Proposition 5.** Rauzy–Veech induction on an irreducible interval exchange is of Rauzy type.

*Proof.* The strong connection comes from the fact that for each vertex there are exactly two edges going in and two going out.

The second property comes from the observation that after a finite number of steps the subset of letters that never lose or win must form an invariant subinterval and would contradict the irreducibility (see [Yoc10] for details).

The last one is due to the fact that if we have a subset of labels always losing, eventually they lose to an interval labeled in the complementary set and this label will keep winning afterwards.

As an illustration, the reader can check directly these properties on the Rauzy graph for 3-interval exchange transformations represented on Figure 5.

![Figure 5: Rauzy graph for 3-IET.](image)

**Remark 5.** The fully subtractive algorithm in dimension 3 or larger (see Section 4) provides a simple example of a simplicial system that is neither of Rauzy type nor unstable nor ergodic. The Poincaré algorithm in dimension 4 is a case that is not of Rauzy type but which is conjecturally ergodic.
Let $\alpha \in A$, $q \in \mathbb{R}^A_+$ and $\tau > 1$. We introduce some other useful properties on path for the following.

Let $W_\alpha$ be the property of a path for which the letter $\alpha$ wins at its last step. The set $\Gamma_\pi(P_\alpha)$ can be thought of as the set of all paths stopping whenever $\alpha$ wins.

Similarly $W_\alpha^k$ corresponds to whether the letter $\alpha$ wins at the last step of $\alpha$ or the length of the path is larger or equal to $k$. It stops whenever $\alpha$ has won or the length of the path is $k$.

Let $J_{\tau,q}$ be the property of a finite path $\gamma$ along which the distortion vector $q$ has jumped by a factor $\tau$ i.e. it satisfies

$$(B_\gamma(q))_\alpha \geq \tau \cdot q_\alpha.$$ 

We write $J_\tau^*$ when $q$ corresponds to the distortion vector defining the measure.

**Lemma 2.** In a simplicial system, for all $\alpha \in A$, all $\pi \in V$, $\tau > 1$ and $q \in \mathbb{R}^A_+$, the probability that the vector $q$ jumps by $\tau$ on coordinate $\alpha$ before the letter $\alpha$ wins is bounded:

$$P_q(J_{\tau,q} < W_\alpha) < \frac{1}{\tau}.$$ 

**Proof.** We prove the result by induction on $k$, for all $k \in \mathbb{N}$,

$$P_q(J_{\tau,q} < W_\alpha^k) < \frac{1}{\tau}.$$ 

For $k = 1$,

$$P_q(J_{\tau,q} < W_\alpha^1) = 0,$$

since $J_{\tau,q}^*$ is never satisfied by the empty path which keeps the vector $q$ unchanged.

Assume now that the inequality is true for some $k$,

$$P_q(J_{\tau,q} < W_\alpha^{k+1}) = \sum_{e \in \pi_{out}} P_q(J_{\tau,q} < W_\alpha^{k+1} | e) \cdot P_q(e) = \sum_{e \notin W_\alpha} P_q(J_{\tau,q} < W_\alpha^{k+1} | e) \cdot P_q(e)$$

If $e = (\pi, \pi') \notin W_\alpha$ and $q' := B_{e,q}$, observe that

$$e^{-1} J_{\tau,q} = J_{\tau,q'}$$

$$e^{-1} W_\alpha^{k+1} = W_\alpha^k$$

where $\tau' = \tau \cdot \frac{q'}{q_\alpha}$. 

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Thus according to (5) and the recurrence hypothesis applied to the constant $\tau'$,

$$P_q \left( J_{\alpha,q}^\tau < W_{\alpha}^{k+1} \mid e \right) = P_q' \left( J_{\alpha,q}^{\tau'} < W_{\alpha}^k \right) \leq \frac{1}{\tau} \cdot \frac{q'_{\alpha}}{q_{\alpha}}.$$  

First assume that the label $\alpha$ appears in the vertices going out of $\pi$, then there is only one edge that does not satisfies $W_{\alpha}$: it is the unique edge such that $l(e_{\alpha}) = \alpha$. Thus

$$P_q \left( J_{\alpha}^\tau < W_{\alpha}^{k+1} \right) < \frac{1}{\tau} \cdot \frac{q'_{\alpha}}{q_{\alpha}} \cdot P_q(e_{\alpha}) = \frac{1}{\tau}. \quad \square$$

On the contrary, if the label $\alpha$ does not appear in the vertices leaving $\pi$, we always have $q'_{\alpha} = q_{\alpha}$ and

$$P_q \left( J_{\alpha}^\tau < W_{\alpha}^{k+1} \right) < \sum_{e \notin W_{\alpha}} \frac{1}{\tau} \cdot P_q(e) \leq \frac{1}{\tau}. \quad \square$$

Using Formula (6) we obtain,

**Corollary 5.** If for almost all point $\alpha$ wins infinitely many times,

$$P_q (W_{\alpha} \leq J_{\alpha}^\tau) \geq \frac{\tau - 1}{\tau}$$

**Theorem 4.** A Rauzy type simplicial system is unstable.

**Proof.** Let $\mathcal{L} \subset \mathcal{A}$, $K > 1$ and $q$ a $(\mathcal{L}, K)$-balanced distortion vector. We consider the largest $\tau$ induced by Corollary 5 for letters in $\mathcal{L}$. There exists a constant $N$ such that if we start from anywhere, there is a path with labels in $\mathcal{L}$ in less than $N$ steps that ends up in a strongly connected component. The balance hypothesis on $q$ implies that there is a lower bound on the probability of these paths.

In this strongly connected component, Rauzy type last property and Corollary 5 imply that a letter in $\mathcal{L}$ wins against a letter in its complementary set before the distortion vector jumps on a coordinate in $\mathcal{L}$ with probability bounded from below by a positive number depending only on $K$ and $\tau$.

If we condition the probability to the fact that the distortion vector does not jump on a coordinate outside of $\mathcal{L}$ before it jumps on $\mathcal{L}$, this implies the lower bound.

Let us condition the probability by the complementary event. Let $\gamma$ be the minimal path such that the jump property $J^\tau$ is satisfied. By assumption, there exists $\beta$ not in $\mathcal{L}$ such that $(B_\gamma q)_\beta \geq \tau \max \mathcal{L} q$ and for every other letter $\alpha$ in $\mathcal{A}$, $(B_\gamma q)_\alpha < \tau \max \mathcal{L} q$. This implies that $(B_\gamma q)_\beta \leq 2 \tau \max \mathcal{L} q$, thus the conditional probability of the event $S_{\mathcal{L}} < J^{2\tau}$ is bounded from below.

The chain rule in Formula (4) implies the theorem. \quad \square
The theorem enables us to relax the condition on winning and losing letters in Definition 1 to a purely graph theoretic one.

**Definition 1'.** We say a simplicial system is of Rauzy type if:
- it is strongly connected,
- every letter in $\mathcal{A}$ appears as a label,
- for all $\mathcal{L} \subseteq \mathcal{A}$, in any strongly connected component of $G_{\mathcal{L}}$ there is no vertex out of which two outgoing edges have their label in $\mathcal{L}$.

**Proposition 6.** Both definitions are equivalent.

*Proof.* Assume there is a subset $\mathcal{A}'$ of letters in the graph such that for almost every path these labels win and lose a finite number of times. Then there exists a number $N$ of steps such that for a set of positive measure, the letters in $\mathcal{A}'$ never win or lose of $N$ steps of the induction. Up to applying $N$ times the induction, one can assume that $N = 0$.

If no letter in $\mathcal{A}'$ ever loses or wins the corresponding paths remain in a subgraph whose vertices are all labeled in $\mathcal{A} \setminus \mathcal{A}'$. Up to applying a finite number of inductions, one can assume that the paths remain in a strongly connected subgraph with positive probability.

If we restrict the simplicial system conditionally to staying in this subgraph, we obtain generic paths in a Rauzy type simplicial system on alphabet $\mathcal{A}'$. Thus almost every path will be $K$-balanced infinitely many times for some $K > 1$. This implies that for almost every of these points, the path goes out of the subgraph. Which is a contradiction.

### 3.3 Ergodic measures for a win-lose induction

#### 3.3.1 A uniformly expanding acceleration

The win-lose induction associated to a simplicial system is not uniformly hyperbolic. As for finite Markov shifts, a useful idea is to wait until a given path $\gamma^*$ appears in the coding. Remark 4 states that such an acceleration of the win-lose induction can be defined for unstable simplicial systems. For a good choice of path, this acceleration will be uniformly hyperbolic.

If we assume that the chosen path starts and ends at the same vertex, this is exactly considering the first return map on the given vertex to the subsimplex $\Delta_{\gamma^*} := R_{\gamma^*} \Delta$, which we denote by

$$T_{\gamma^*} : \Delta_{\gamma^*} \to \Delta_{\gamma^*}.$$

**Proposition 7.** If $R_{\gamma^*}$ is a positive matrix, $T_{\gamma^*}$ is uniformly expanding. We say in this case that $\gamma^*$ is a positive path.

Let $S$ be the set of finite words in $\mathcal{A}$, including the empty word, for which $\gamma^*$ is not a factor i.e. such that it cannot be written as $w_1 \cdot \gamma^* \cdot w_2$. Every point in $\Delta_{\gamma^*}$ follows a path $\gamma^* \cdot \gamma_0$ with $\gamma_0 \in S$ before returning to $\Delta_{\gamma^*}$. This simplex is thus partitioned into subsimplices indexed by $S$ which are sent bijectively to $\Delta_{\gamma^*}$ by $T_{\gamma^*}$. 

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Corollary 6. The accelerated win-lose induction $T_{\gamma^*}$ is weak-Bernoulli and thus conjugated to a Bernoulli shift on a countable alphabet $S$.

Before giving a proof of this proposition, we need to introduce some definitions. For any two vectors $v, w \in \mathbb{R}^+_A$, let
\[
\alpha(v, w) := \max_{a \in A} \frac{v_a}{w_a}, \quad \beta(v, w) := \min_{a \in A} \frac{v_a}{w_a}
\]

and
\[
d(v, w) := \log \frac{\alpha(v, w)}{\beta(v, w)}.
\]

One can check that $d$ is a complete metric on the projectivization of $\mathbb{R}^+_A$ called the Hilbert metric. This metric has the very useful feature that any linear map induced by a positive matrix is contracting with respect to it.

Lemma 3. For any non-negative matrix $M$, we have
\[
d(Mv, Mw) \leq d(v, w),
\]
moreover if $M$ is positive, there exists $\theta < 1$ such that,
\[
d(Mv, Mw) \leq \theta d(v, w).
\]

Proof. This is a well known property of Hilbert metrics, the proof can be found e.g. in [Via97].

Proposition 7. Let $\gamma^* \cdot \gamma_e$ be the Rauzy path for a given point in $\Delta_{\gamma^*}$ until its first return. The inverse of the Rauzy map is a projectivization of the linear map $R_{\gamma^*} R_{\gamma_e}$, which is, according to Lemma 3 the composition of a weakly contracting map and a contraction a contracting map with coefficient $\theta < 1$ for the Hilbert metric on $\Delta$. Hence the inverse of the Rauzy map is contracting for the coefficient $\theta$ depending only on $\gamma^*$. Moreover, by positivity, $\Delta_{\gamma^*}$ is precompact in $\Delta$, thus the Hilbert metric is equivalent to all finite metric on this space.

We will use the notations $\Delta^*$ for $\Delta_{\gamma^*}$, $T_*$ for $T_{\gamma^*}$ and $\Delta^*_w$ for the domain corresponding to $w \in S$ in $\Delta^*$. Similarly $\Delta^*_w$ stands for the sub-simplex for which the coding of $T_*$ starts by $w$.

In the case of unstable simplicial system one can always find such an acceleration.

Lemma 4. If $G$ is an unstable simplicial system is admits a positive path.

Proof. Notice that the unstability property implies in particular that $G_{\mathcal{L}}$ is a strict subgraph of $G$. Thus for any subset of letter $\mathcal{L}$ there is a vertex such there is an outgoing edge labeled in $\mathcal{L}$ and an other labeled in the complementary set. We can then construct a positive path by recurrence.

We will assume in the following that we are given a positive path $\gamma^*$ that starts and ends at the same vertex of the graph.
3.3.2 Thermodynamic formalism

In the following, we show that there exists a unique invariant measure of maximal entropy for the canonical suspension of the win-lose induction. The general strategy consists in showing existence of Gibbs measures for a family of potentials parametrized in $\mathbb{R}$. To these potentials is associated a pressure for which, when zero, the associated Gibbs measure induces a measure of maximal entropy on the suspension.

**Properties on the norm** We list some easy but nonetheless useful properties for the $L^1$ norm on the simplex.

**Proposition 8.** Let $\gamma$ a path in $G$, $\lambda \in \Delta$ and $\lambda' = R_\gamma^{-1} \lambda$.

If $\|\lambda\| = 1$ then

$$\lambda = \frac{R_\gamma \lambda'}{\|R_\gamma \lambda'\|}.$$ 

If $\|\lambda'\| = 1$ then

$$\|R_\gamma^{-1} \lambda\| = \frac{1}{\|R_\gamma \lambda'\|}.$$ 

Moreover if we can decompose $\gamma = \gamma_1 \cdot \gamma_2$ and if $\lambda_1 = \frac{R_\gamma^{-1} \lambda}{\|R_\gamma^{-1} \lambda\|}$,

$$\|R_\gamma^{-1} \lambda\| = \|R_{\gamma_2}^{-1} \lambda_1\| \cdot \|R_{\gamma_1}^{-1} \lambda\|.$$ 

**Proposition 9.** Let $v, w \in \mathbb{R}^A^+$,

$$\frac{\|v\|}{\|w\|} \leq \max_{\alpha \in A} \frac{v_\alpha}{w_\alpha}.$$ 

**Roof function** We now use thermodynamic formalism to define an invariant measure for the map $T^*$. As noticed in Section 2.4, there is a natural suspension flow associated to $T$ whose first return time is the roof function.

The canonical suspension flow on $\Delta^G$ can be defined on the base $\Delta^*$ with an accelerated roof function defined, for $x \in \Delta^*$, as

$$r_*(x) = r(x) + r(Tx) + \cdots + r(T^{n-1}x) = -\log \|R_{\gamma}^{-1}x\|$$

where $n \geq 1$ is the smallest integer such that $T^n x \in \Delta^*$ and $\gamma$ is the finite path in the graph associated to $x$ until it returns to $\Delta^*$. The second equality uses Proposition 8.

**Remark 6.** This potential is similar to the geometric potential in the context of thermodynamic formalism.

Let $0 < \theta < 0$ be the constant associated to the matrix $R_\gamma^*$ by Lemma 3. We show that the accelerated roof function is H"{o}lder of order $\beta := \log(1/\theta)$.

**Lemma 5.** For all $x, y \in \Delta^*$ in the same $n$-cylinder $\Delta^*_w$, where $n \geq 1$,

$$|r_*(x) - r_*(y)| \leq \theta^n \cdot \text{diam}(\Delta^*).$$
Proof. Let \( x, y \in \Delta^* \), in the same cylinder \( \Delta^*_w \) which corresponds to the path
\[
\gamma = \gamma^* \cdot w_1 \cdot \gamma^* \cdot w_2 \cdots \gamma^* \cdot w_n,
\]
then according to Proposition 8,
\[
|r_*(x) - r_*(y)| = \left| \log \frac{\| R_{\gamma^{-1}} y \|}{\| R_{\gamma^{-1}} x \|} \right| = \left| \log \frac{\| R_{\gamma} x \|}{\| R_{\gamma} y \|} \right|.
\]
By Proposition 9 up to switching \( x \) and \( y \), we can bound the distance by the Hilbert metric,
\[
|r_*(x) - r_*(y)| \leq d(R_{\gamma} x, R_{\gamma} y).
\] \( (8) \)
Let \( x', y' \in \Delta^* \), such that \( x = R_{\gamma} x' \) and \( y = R_{\gamma} y' \). Using Lemma 3,
\[
d(R_{\gamma} x', R_{\gamma} y') = d(R_{\gamma} \cdot R_{w_1} R_{\gamma} x', R_{\gamma} \cdot R_{w_1} R_{\gamma} y')
\]
\[
\leq \theta \cdot d(R_{w_1} R_{\gamma} x', R_{w_1} R_{\gamma} y')
\]
\[
\leq \theta \cdot d(R_{\gamma} x', R_{\gamma} y'),
\]
where \( \gamma = \gamma^* \cdot w_1 \cdot \gamma' \). By induction on \( n \), we obtain
\[
d(x, y) \leq \theta^n \cdot \text{diam}(\Delta^*).
\]
We now prove a key lemma for existence to apply thermodynamic formalism theorems.

**Lemma 6.** The accelerated roof function \( r_* \) has exponential tail, i.e. there exists \( 0 < \sigma \) such that, for all \( q \in \mathbb{R}^A_+ \),
\[
\int_{\Delta^*} e^{\sigma r_*} d\nu_q < \infty.
\]

**Proof.** Notice that there exists \( C, \eta > 0 \) such that, for any \( q \in \mathbb{R}^A_+ \) and all \( \tau > 1 \),
\[
\nu_q \{ x \in \Delta^* \mid r_*(x) \geq \log \tau \} \leq C \tau^{-\eta}.
\] \( (9) \)
It follows from Corollary 4 since the above set is included in the subset of \( \Delta^* \) that satisfy the property \( \mathcal{J} \leq E_{\gamma^*} \).

Now Formula \( (9) \) implies that, cutting into pieces where \( \log \tau^n < r_*(x) \leq \log \tau^{n+1} \), for all \( \sigma < \eta \),
\[
\int_{\Delta^*} e^{\sigma r_*} d\nu_q \leq \sum_{n=0}^{\infty} (\tau^{n+1})^\sigma \cdot C \cdot (\tau^n)^{-\eta}
\]
\[
= C \cdot \tau \cdot \sum_{n=0}^{\infty} (\tau^{\sigma - \eta})^n = C \cdot \tau \cdot \frac{1}{1 - \tau^{\sigma - \eta}}
\]
\[
\square
\]
In the following we denote by \( \sigma_0 \) the supremum of such \( \sigma \). As \( e^{\sigma r_*} \) is positive and increasing in \( \sigma \), the integral for \( \sigma = \sigma_0 \) is infinite.
Estimates on the Jacobian  We give some useful properties on the Jacobian of the win-lose induction, which follow from a computation that can be found e.g. in [Vee78].

**Proposition 10.** For all $x \in \Delta^*$, the Jacobian of the win-lose induction satisfies

$$|DT_*(x)| = e^{\text{A}|r_*(x)|}.$$

**Corollary 7.** There exists $Q > 0$, such that for all 1-cylinder $\Delta^*_w$ and all $x \in \Delta^*_w$,

$$\frac{1}{Q} \cdot |DT_*(x)|^{-1} \leq \nu(\Delta^*_w) \leq Q \cdot |DT_*(x)|^{-1}. $$

**Proof.** If $x, y$ are in the same 1-cylinder $\Delta^*_w$, using (8),

$$|r_*(x) - r_*(y)| \leq \text{diam}(\Delta^*) < \infty.$$

Thus, using Proposition 10, there exists $Q' > 0$ such that,

$$\frac{1}{Q'} \cdot |DT_*(x)| \leq |DT_*(y)| \leq Q' \cdot |DT_*(x)|.$$  \hfill (10)

The restriction $T_* w := T_*|\Delta^*_w$ is invertible. Thus,

$$\frac{1}{Q'} |DT_*(x)|^{-1} \cdot \nu(\Delta^*) \leq \int_\Delta |DT_*(T_* w y)|^{-1} \cdot dv(y) \leq Q' \cdot |DT_*(x)|^{-1} \cdot \nu(\Delta^*).$$

And

$$\int_\Delta |DT_*(T_* w y)|^{-1} \cdot dv(y) = \int_\Delta |DT_*(x)|^{-1} \cdot dv = \nu(\Delta^*_w).$$

**Corollary 8.** There exists an ergodic $T_*$-invariant measure $\mu$ equivalent to Lebesgue measure $\nu$ such that $\log \frac{d\mu}{d\nu}$ is bounded by a constant at almost every point.

**Proof.** This is a direct application of Theorem 3.1 in [ADU93] (see alternatively Lemma 4.4.1 in [Aar97]) and Formula (10).

Potential functions and Gibbs measures.

**Definition 3.** Let $\mu$ be a $\sigma$-invariant (shift) Borel probability measure on a countable Markov chain $\Sigma$. For any continuous function $\phi : \Sigma \to \mathbb{R}$, $\mu$ will be called a Gibbs measure for the potential $\phi$ if there exists $Q > 0$ and $P$ such that for every path $\gamma = \gamma_0 \cdot \gamma_n$ and every $x$ in the cylinder $[x_1, \ldots, x_n]$,

$$\frac{1}{Q} \leq \frac{\mu([x_1, \ldots, x_n])}{\exp\left(\sum_{k=0}^{n-1} \phi(\sigma^k(x)) - Pm\right)} \leq Q.$$  \hfill (11)

$P$ is called the topological pressure of $\phi$.

In the following, we will consider the function $\phi_\kappa = -\kappa r$ for some $\kappa > 0$. For convenience, when there is no ambiguity we will denote it by $\phi$. We will show that it satisfies good properties to induce existence and uniqueness of Gibbs measures of potential $\phi$ which will be conjugate to the win-lose induction.
Gurevic–Sarig pressure. To show the existence of a Gibbs measure for the considered potential, we need two more definitions.

As we deal with a coding on a countable alphabet, we need to check a technical property on this coding namely that it has "Big Image and big Preimage".

**Definition 4.** The BIP property is the existence of $w_1, \ldots, w_m \in S$ tiles of the Markov partition, such that for all $v \in S$, there exists $1 \leq k, l \leq m$ such that $T_* w_k \cap v$ and $T_* v \cap w_l$ are not empty.

This property is obviously true in our case, since each Markov tile is sent to the whole domain by $T_*$.

Consider now the Ruelle operator, acting on continuous functions, associated to a potential function $\phi$, for $f$ a function on $\Delta^*$ and $x \in \Delta^*$,

$$(L_\phi f)(x) = \sum_{T_* (y) = x} e^{\phi(y)} f(y).$$

As explained in [Sar99]: "the analysis of thermodynamic limits reduces to the study of the asymptotic behavior of $L^n\phi f$ as $n \to \infty$ for sufficiently many functions $f$ ". One of the key to understand this behavior is to first understand the limit of $\frac{1}{n} \log L^n\phi f$. In particular, it can be compared to the following quantities. For $w \in S$, let

$$Z_n(\phi, w) = \sum_{T^n_*(x) = x, x_0 = w} e^{\phi_n(x)},$$

with $\phi_n = \phi + \phi \circ T_1 + \cdots + \phi \circ T^{n-1}_*$ and $x_0$ is the tile in $S$ to which $x$ belongs.

According to Theorem 4.3 in [Sar99] the limit

$$P_G(\phi) := \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi, w)$$

exists for all $w \in S$ and is independent of $w$. Moreover, if $\|L\phi 1\| < \infty$, then $P_G(\phi) < \infty$.

**Definition 5.** $P_G(\phi)$ is called the Gurevic–Sarig pressure of $\phi$.

This is a relevant quantity to consider according to Theorem 4.4 of [Sar99], since when $P_G(\phi)$ is finite, it is equal to the limit of $\frac{1}{n} \log L^n\phi f$, for a large class of functions. As a consequence, for all $x \in \Delta^*$,

$$\frac{1}{n} \log (L^n_\phi 1)(x) \to P_G(\phi).$$

Let $r^{(n)} := r_* + r_* \circ T_* + \cdots + r_* \circ T_*^{n-1}$, then these iterates of the transfer operator on 1 have the following form,

$$(L^n_\phi 1)(x) = \sum_{T^n_*(y) = x} e^{-r^{(n)}(y)}.$$

Let us introduce the $n$-th variation of a potential function,

$$\text{var}_n(\phi) = \sup \{|\phi(x) - \phi(y)| : x_i = y_i, i = 1, \ldots, n\}.$$
Definition 6. The potential function $\phi$ has bounded variations if and only if $\sum_{n=2}^{\infty} \text{var}_n(\phi) < \infty$.

The Hölder property proved in Lemma 5 implies that, for all $\kappa$, $\phi$ has bounded variations and $\text{var}_1(\phi) < \infty$.

These two definitions enable us to state the key theorem in this section. It gives a criterion for uniqueness of a Gibbs measure for a given potential function.

Theorem (Sarig [Sar03]). Assume that the potential $\phi$ has summable variations. Then $\phi$ admits a unique $T^*$-invariant Gibbs measure $\mu_\phi$ if and only if

- $X$ satisfies the BIP property;
- the Gurevic–Sarig pressure $P_G(\phi) < \infty$ and $\text{var}_1(\phi) < \infty$.

In this case, the topological pressure and Gurevic–Sarig pressure coincide.

Uniqueness will be true whenever the pressure is finite. In particular it true for large $\kappa$ according to the following lemma which implies finiteness of the pressure.

Lemma 7. The pressure $P_G(\phi)$ is finite if and only if $\kappa > |A| - \sigma_0$.

Proof. Let us first prove the necessary condition for the lemma. As noticed above, we only need to show that $L_\phi 1$ is finite for all $\kappa > |A| - \sigma_0$. By definition,

$$L_\phi 1 = \sum_{T^* (y) = x} e^{\phi(y)} \leq \left( Q' \right)^{\kappa/|A|} \cdot \sum_{w \in S} e^{-\kappa r_* (w)}.$$ 

Where the last inequality is a consequence of Formula (10) and $S$ is a choice of representative of every 1-cylinders.

Let $Y(N)$ be the set of representative $w \in S$ for which $N \leq r_* (w) < N + 1$, then,

$$\sum_{w \in S} e^{-\kappa r_* (w)} = \sum_{N=0}^{\infty} \sum_{w \in Y(N)} e^{-\kappa r_* (w)} \leq \sum_{N=0}^{\infty} |Y(N)| e^{-\kappa N}. \quad (15)$$

Using Lemma 6, let $0 < \sigma < \sigma_0$,

$$\sum_{N=0}^{\infty} \sum_{w \in Y(N)} \int_{\Delta_w} e^{\sigma r_* (w)} d\nu < \infty,$$

where $\Delta_w$ is the 1-cylinder corresponding to $w$.

Thus

$$\sum_{w \in Y(N)} \int_{\Delta_w} e^{\sigma r_* (w)} d\nu < I_\sigma := \int_\Delta e^{\sigma r_*} d\nu,$$

and

$$\sum_{w \in Y(N)} \int_{\Delta_w} e^{\sigma r_* (w)} d\nu \geq e^{\sigma N} \sum_{w \in Y(N)} \nu(\Delta_w).$$

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Moreover, according to Corollary 7 for all \( w \in Y(N) \),
\[
\nu(\Delta_w) \geq Q^{-1} \cdot e^{-|A|(N+1)}.
\]
Hence
\[
I_{\sigma} > Q^{-1} \cdot |Y(N)| \cdot e^{\sigma N - |A|(N+1)},
\]
and
\[
|Y(N)| < Q \cdot I_{\sigma} \cdot e^{(|A| - \sigma)N}.
\]
Thus the geometric sum in (14) is bounded for \( \kappa > |A| - \sigma_0 \) by
\[
Q \cdot I_{\sigma} \cdot e^{(|A| - \sigma_0)} \cdot e^{-\kappa N} \sum_{N=0}^{\infty} e^{-\kappa N} = Q \cdot I_{\sigma} \cdot \frac{e^{(|A| - \sigma_0)}}{1 - e^{(|A| - \sigma_{0})N}}. \tag{16}
\]
Passing to the limit \( \sigma \to \sigma_0 \) we get that for all \( \kappa > |A| - \sigma_0 \) the pressure is finite.

For the sufficient condition, we will show that if the tail integral is infinite, for some \( x \in \Delta^* \),
\[
\frac{1}{n} \log(L^n_{\phi}(x)) \to \infty.
\]
By (14) and (10),
\[
\sum_{T^*_n(y)=x} e^{-\kappa r_*(n)} \geq (Q')^{n \kappa / |A|} \sum_{w_1, \ldots, w_n \in S} e^{-\kappa r_*(w_1)} \cdots e^{-\kappa r_*(w_n)}.
\]
Hence, we only need to show that, for \( \kappa \leq |A| - \sigma_0 \),
\[
\sum_{w \in S} e^{-\kappa r_*(w)} = \infty.
\]
As previously we split the sum,
\[
\sum_{w \in S} e^{-\kappa r_*(w)} = \sum_{N=0}^{\infty} \sum_{w \in Y(N)} e^{-\kappa r_*(w)} \geq \sum_{N=0}^{\infty} |Y(N)| e^{-\kappa (N+1)}.
\]
Now,
\[
I_{\sigma} < Q \cdot \sum_{N=0}^{\infty} |Y(N)| \cdot e^{\sigma (N+1) - |A| N}
\]
\[
= Q \cdot e^\sigma \cdot \sum_{N=0}^{\infty} |Y(N)| \cdot e^{-|A| - \sigma N}.
\]
In particular,
\[
\sum_{N=0}^{\infty} |Y(N)| \cdot e^{-|A| - \sigma_0 N} = \infty
\]

**Proposition 11.** For all \( \kappa > |A| - \sigma_0 \) there exists a unique Gibbs measure \( \mu_\phi \) of potential \( \phi = -\kappa \cdot r_* \).

**Proof.** The existence of the Gibbs measure is a direct consequence of the Hölder property in Lemma 5 and Theorem 1.25 in [Bow08]. Uniqueness is a consequence of Sarig’s theorem and Lemma 7. 

\[ \square \]
3.3.3 Suspension flow

There is an easy way to construct a natural extension for a full shift on a countable alphabet by extending it to bi-infinite words. The canonical suspension then extends to a flow on the suspension of the natural extension.

Any Borel probability measure \( \tilde{\mu} \) invariant for this suspension flow can be written as a product of a Borel probability measure on \( \Delta^* \), invariant by \( T \), denoted by \( \mu \) and Lebesgue measure on the fiber. We denote by \( \mathcal{M}_T \) this latter set of Borel invariant probability measures. The Kolmogorov–Sinai entropy of the flow for this measure is written \( h(\Phi, \tilde{\mu}) \) and satisfies Abramov’s formula

\[
h(\Phi, \tilde{\mu}) = h(T, \mu)(r) \frac{\mathcal{L}(\Phi, \tilde{\mu})}{\mu(r)},
\]

where \( \mu(r) = \int_{\Delta^G} r d\mu \) and \( h(T, \mu) \) is the Kolmogorov–Sinai entropy, for \( T \). In this setting the topological entropy can be defined as

\[
h_{\text{top}}(\Phi) = \sup_{\mathcal{M}_T} h(\Phi, \tilde{\mu}).
\]

A measure \( \mu \in \mathcal{M}_T \) at which this supremum is achieved is referred to as a measure of maximal entropy.

As noticed before, the suspension flow on \( \Delta^G \) for the roof function and the one on \( \Delta^* \) for the accelerated roof function are conjugate. Moreover, the exponential tail integral are equal for these two suspension flows, with the same measure on the base restricted to \( \Delta^* \).

In the following we will use the representation of the suspension on the base \( \Delta^* \) with the map \( T^* \) for its nice dynamical properties.

**Proposition 12.** There exists \( \kappa_0 > |A| - \sigma_0 \) such that \( P_G(\phi_{\kappa_0}) = 0 \). The Gibbs measure \( \mu_0 \), as in the previous proposition, is the unique measure of maximal entropy for the suspension of win-lose induction.

**Proof.** Notice that the accelerated roof function is bounded away from zero, since \( R_{\gamma_*} \) is a positive integer matrix,

\[
r_+(x) = -\log \| R_{\gamma_*}^{-1} x \| = \log \| R_{\gamma_*} x \| \geq \log |A|.
\]

Thus \( r_+^{(n)} \geq n \cdot \log |A| \) and for \( \kappa > |A| - \sigma_0 \) and \( \epsilon > 0 \), by (13),

\[
(L_{\phi_{\kappa_0}}^n 1)(x) = \sum_{T^n_{\gamma}(y) = x} e^{-(\kappa+\epsilon) r_+^{(n)}} \leq |A|^{-\kappa \epsilon} \cdot \sum_{T^n_{\gamma}(y) = x} e^{-\kappa r_+^{(n)}}.
\]

By Lemma 7 the pressure is bounded for \( \kappa \) and

\[
\lim_{n \to \infty} \frac{1}{n} \log (L_{\phi_{\kappa_0}+\epsilon}^n 1)(x) \leq P_G(\phi_{\kappa_0}) - \epsilon \cdot \log |A|.
\]

Thus for \( \epsilon \to \infty \), using (13), \( P_G(\phi_{\kappa_0}+\epsilon) \to -\infty \).

Moreover, \( P_G(\phi_0) = \infty \) and \( P_G \) is a decreasing continuous function of \( \kappa \) (see Theorem 4.6 of [Sar99]). Thus there exists \( \kappa_0 > 0 \) such that
$P_G(\phi_{\kappa_0}) = 0$.

By the variational principal for the topological pressure and Theorem 1.1 in [BS03], the associated $\mu_0$ measure is the unique measure that maximizes the quantity

$$h(T^*,\mu_0) - \int_{\Delta G} \kappa_0 r d\mu_0 = 0.$$  

Thus $h(\Phi, \tilde{\mu}_0) = \frac{h(T^*,\mu_0)}{\mu_0(r)} = \kappa_0$ is maximal. \qed

**Theorem 5.** The measure of maximal entropy for the suspension of an unstable win-lose induction is the suspension of the unique ergodic $T^*$-invariant Borel measure equivalent to Lebesgue measure such that $\log \frac{d\mu}{d\nu}$ is bounded by a constant at almost every point. Moreover, its entropy is equal to $|A|$.

**Proof.** Let $\mu$ be a measure as in Corollary 8, we show that it the unique Gibbs measure for potential $\phi_{\kappa_0}$. Indeed, according to Corollary 7 we have a constant $Q > 0$ such that

$$\frac{1}{Q} \leq \frac{\nu([x_1, \ldots, x_n])}{\exp\left(\sum_{k=0}^{n-1} |A| r(\sigma^k(x))\right)} \leq Q.$$  

If $\mu$ is such that $\log \frac{d\mu}{d\nu}$ is bounded at almost every point, then it also satisfies the same property for another constant $Q$. Thus it is by definition a Gibbs function for the potential $-|A| \cdot r$ and it has zero topological pressure. Propositions 11 and 12 imply the result. \qed

### 3.3.4 Subgraph parameter space

Let $F$ be a subgraph of $G$. Similarly to $\Delta^G$, we denote by $\Delta^G(F) \subset \Delta^G$ or simply $\Delta(F)$ the subset of points whose path belongs to $F$. There is a natural bijection

$$\iota : \Delta^F \to \Delta^G(F).$$  

Let us assume that $F$ is strongly connected and all letter in $A$ is a label of the subgraph. There exists a positive path $\gamma^* \in F$ which we will use to acceleration win-lose induction. Notice that

$$r_{\gamma^*} \circ \iota = r_F + \delta$$

where $r_*$ is the accelerated roof function on $F$ and $\delta$ is a non-negative function.

We show in the remaining of the section that the Hausdorff dimension of the space $\Delta(F)$ can be expressed in terms of a zero of an equation similar to what is obtained by Bowen in [Bow79].

We can reproduce the argument in the proof of Proposition 12 to show the existence of a unique $\kappa_F$ such that $P_G(-\kappa_F \cdot (r_F + \delta)) = 0$.  

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Theorem 6. The Hausdorff dimension of $\Delta(F)$ satisfies

$$\dim_H \Delta(F) \leq |A| - 1 + \frac{\kappa_F}{|A|}.$$ 

Proof. Let $\tilde{\mu}$ be the Gibbs measure associated to the potential $-\kappa_F \cdot (r_F + \delta)$. We consider the pushed forward measure $\mu := \iota_* \tilde{\mu}$ which is also a Gibbs measure for the potential equal to $-\kappa_F \cdot r_F$ on $\Delta(F)$ and 0 elsewhere.

According to Formula (11) in the definition of Gibbs measures, there exists $Q > 0$ such that for all $x$ in the intersection of the cylinder $w = [w_1, \ldots, w_n]$ and $\Delta(F)$,

$$\frac{1}{Q} \exp \left( - \sum_{k=0}^{m-1} \kappa_F r_F(T^k_F(x)) \right) \leq \mu(\Delta^*_w) \leq Q \cdot \exp \left( - \sum_{k=0}^{m-1} \kappa_F r_F(T^k_F(x)) \right).$$

Corollary 7 implies that,

$$\exp \left( |A| \sum_{k=0}^{m-1} r_F(T^k_F(x)) \right) = |DT^m_F(x)| \simeq \frac{1}{\nu(\Delta^*_w)}.$$ 

Thus, for all cylinder intersecting $\Delta(F)$,

$$\mu(\Delta^*_w) \simeq \nu(\Delta^*_w)^{\kappa_F/|A|}. \quad (17)$$

Let us introduce the notation $\alpha := \kappa_F / |A|$.

Let $\mathcal{F}_n$ be a family of cylinders for $T_F$ that intersect $\Delta(F)$ and such that $\nu(\Delta^*_w) < e^{-n}$. Such a family exists thanks to Lemma 3. For any $C > 1$, let $\mathcal{F}_n^C$ be the subset of cylinders $w \in \mathcal{F}_n$ such that $\nu(\Delta^*_w) \leq e^{-nC}$ and $\mathcal{F}_n^{>C}$ the complementary set $\mathcal{F}_n \setminus \mathcal{F}_n^C$.

For $\mathcal{F}$ a family of cylinders of $T_F$ we denote

$$\Delta^*_\mathcal{F} := \bigcup_{w \in \mathcal{F}} \Delta^*_w.$$ 

Lemma 8. The following limit exists and satisfies

$$\lim_{n \to \infty} - \frac{1}{n} \log \nu(\Delta^*_\mathcal{F}_n) = 1 - \alpha.$$ 

Proof. Assume for simplicity that $\nu$ is normalized to be a probability measure, as $\mu$ is one by construction, by Formula (17),

$$\sum_{w \in \mathcal{F}_n} \nu(\Delta^*_w)^\alpha \simeq 1.$$ 

Thus

$$\nu(\Delta^*_\mathcal{F}_n) = \sum_{w \in \mathcal{F}_n} \nu(\Delta^*_w)^\alpha \cdot \nu(\Delta^*_w)^{1-\alpha} \lesssim e^{-n(1-\alpha)}.$$ 

If $\kappa_F = |A|$, the inequality is obvious.
If $\kappa F < |A|$, there exists some $\sigma < \sigma_0$ such that $\kappa F > |A| - \sigma$. Let $\epsilon > 0$ such that $\kappa F - \epsilon \cdot |A| > |A| - \sigma$. Then, using Formula (16) in Lemma 7 there exists $K > 0$ such that
\[
\sum_{w \in F_n} \nu(\Delta_w^\epsilon)^{\alpha - \epsilon} \leq K.
\]
Thus
\[
\sum_{w \in F_n^{<C}} \nu(\Delta_w^\epsilon)^{\alpha} \leq K \cdot e^{-nC\epsilon},
\]
and
\[
\sum_{w \in F_n^{\geq C}} \nu(\Delta_w^\epsilon)^{\alpha} \geq 1 - K \cdot e^{-nC\epsilon}.
\]
Notice that for any $C > 1$, one can take $n$ large enough such that the previous bound is larger than $1/2$. We now have,
\[
\nu(\Delta_{F_n}^\epsilon) \geq e^{-nC(1-\alpha)} \cdot \sum_{w \in F_n^{>C}} \nu(\Delta_w^\epsilon)^{\alpha} \geq e^{-nC(1-\alpha)}.
\]
Letting $C$ go to 1 induces the result.

As noticed in [AD16], simplices satisfy a useful property to bound Hausdorff dimensions.

**Proposition 13.** There exists $K > 0$ such that for all simplex $\Delta$ of dimension $d$, measure $m$ and diameter less than 1, the minimal number of ball of radius $0 < \rho \leq m$ required to cover $\Delta$ satisfies
\[
N_{\rho} \leq K \cdot \frac{m}{\rho^d}.
\]

Let $N_{\rho}(\Delta_{F_n}^>)$ be the minimal number of ball of radius $\rho = e^{-C'n}$ one needs to cover $\Delta_{F_n}^>$ where $C' > C$. By Proposition 13
\[
N_{\rho}(\Delta_{F_n}^>) \leq K \cdot e^{dC'n} \cdot \sum_{w \in F_n^{>C}} \nu(\Delta_w^\epsilon) \leq K \cdot e^{dC'n} \cdot \nu(\Delta_{F_n}^>)
\]
Thus
\[
\frac{\log N_{\rho}(\Delta_{F_n}^>)}{C'n} = d - \frac{\log \nu(\Delta_{F_n}^>)}{C'n} + o(1).
\]
As $n$ goes to infinity, one obtains,
\[
\lim_{n \to \infty} \frac{\log N_{\rho}(\Delta_{F_n}^>)}{C'n} \leq d - \frac{1 - \alpha}{C'} := \delta.
\]
Using Proposition 13 with $m = \rho$, one can cover each $\Delta_w^\epsilon$ with $w \in \Delta_{F_n}^{>C}$ by at most $K \cdot \nu(\Delta_w^\epsilon)^{1-d}$ balls of radius $\nu(\Delta_w^\epsilon)$. Let $\{B_i\}$ be the resulting cover. Thence
\[
\sum_{i} \text{diam } B_i^\delta \leq K \cdot \sum_{w \in F_n} \nu(\Delta_w^\epsilon)^{1-d+\delta} \leq K \cdot \sum_{w \in F_n} \nu(\Delta_w^\epsilon)^{\alpha'},
\]
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where \( \alpha' = \frac{C' + 1}{C} \geq \alpha - \epsilon \) for \( C' \) close to 1.

Letting \( C \) and \( C' \) go to infinity, we bound the Hausdorff dimension from above by \( d - 1 + \alpha \). \( \square \)

**Corollary 9.** For any \( F \) strict subgraph of \( G \), if \( F \) is unstable, the Hausdorff dimension of the parameter subset \( \Delta(F) \) is strictly smaller than the dimension of \( \Delta^G \).

**Proof.** The pulled back of \( r_{\Delta(F)} \) has still bounded variations and the pressure by assumption is zero at \( \kappa_F \). Thus by Buzzi–Sarig theorem, there exists two unique measures \( \tilde{\mu}_0 \) and \( \tilde{\mu}_1 \) such that

\[
P_G (- |A| \cdot (r_{\ast F} + \delta)) = h(T_{\ast F}^{\tilde{\mu}_0}) - \int_{\Delta_F} \kappa_F \cdot (r_{\ast F} + \delta) \, d\tilde{\mu}_0
\]

and

\[
P_G (- |A| \cdot r_{\ast F}) = h(T_{\ast F}^{\tilde{\mu}_1}) - \int_{\Delta_F} \kappa_F \cdot r_{\ast F} \, d\tilde{\mu}_1.
\]

If \( \tilde{\mu}_0 \neq \tilde{\mu}_1 \),

\[
h(T_{\ast F}^{\tilde{\mu}_0}) - \int_{\Delta_F} \kappa_F \cdot r_{\ast F} \, d\tilde{\mu}_0 < h(T_{\ast F}^{\tilde{\mu}_1}) - \int_{\Delta_F} \kappa_F \cdot r_{\ast F} \, d\tilde{\mu}_1,
\]

and

\[
P_G (- |A| \cdot (r_{\ast F} + \delta)) < P_G (- |A| \cdot r_{\ast F}).
\]

The Birkhoff sums of the map \( \delta \) diverge for almost every orbits since, by unstability, they pass through a vertex to which we have removed an edge infinitely often.

If \( \tilde{\mu}_0 = \tilde{\mu}_1 \), the measure is equivalent to Lebesgue measure according to Theorem 5. And

\[
\int_{\Delta_F} \delta \, d\tilde{\mu}_0 > 0.
\]

Again we have,

\[
P_G (- |A| \cdot (r_{\ast F} + \delta)) < P_G (- |A| \cdot r_{\ast F}).
\]

In either case \( P_G (- |A| \cdot (r_{\ast F} + \delta)) < 0 \) hence \( \kappa_F < |A| \). \( \square \)
4 Classical MCF algorithms

In this section, we describe how to associate to a large set of examples of linear simplex-splitting MCF algorithm (in the sense of Lagarias [Lag93]) a conjugate simplicial system. This hopefully will make the general algorithm clear. We are able to check unstability using the criterion introduced in Section 3.2 for all known ergodic algorithms we are considering. The only limit case in which our criterion does not apply is given by the Poincaré algorithm in dimensions larger or equal to 4.

We will start with two simple examples, the fully subtractive and Poincaré algorithms, for which it is easy to derive from their classical description an associated simplicial system. One of the reason that make these examples easier to describe in terms of simplicial systems is the fact that their domains of definition are all sent to the whole simplex by the corresponding map.

We then present a general strategy to compute these simplicial systems and apply it to Brun and Selmer algorithms. We finish by computing a simplicial system which induces the Rauzy gasket in every dimension.

As a consequence we have a unified proof that Brun and Selmer and Arnoux–Rauzy–Poincaré algorithms are ergodic for their unique invariant measure equivalent to Lebesgue. Moreover this measure induces the unique measure of maximal entropy on their canonical suspension.

Ergodicity for Brun and Selmer algorithms in all dimension is due to Schweiger [Sch00], for Arnoux–Rauzy–Poincaré it has been proved in [BL13]. The result on Hausdorff dimension has been proved in dimension 2 in [AHS16].

4.1 Two full-image examples

4.1.1 Fully subtractive algorithms

The fully subtractive algorithm in dimension 3 can be described by the map, defined at almost every point, $F : (x_1, x_2, x_3) \in \mathbb{R}_+^3 \to (x'_1, x'_2, x'_3)$, where if $\{i, j, k\} = \{1, 2, 3\}$ and $x_i > x_j > x_k$,

$$
x'_i = x_i - x_k, \quad x'_j = x_j - x_k, \quad x'_k = x_k.
$$

This map corresponds to a step for the win-lose induction in the graph with one vertex and three edges of distinct labels, represented below.

This first example has stable subgraphs, in which the orbits will eventually be trapped. This corresponds to the behavior proved in [Nog95] for the 3-dimensional Poincaré algorithm where one coordinate remains much bigger than the two others which decrease very fast by applying a continued fraction algorithm to them.
This construction generalizes to fully subtractive algorithms in dimension \( n > 3 \) by taking a single edge with \( n \) loops labeled by \( n \) different letters.

### 4.1.2 Poincaré algorithm

Poincaré algorithm has been introduced by Poincaré as a generalization of the continued fraction algorithm and was later studied and generalized in \[Nog95\]. It can be described by the map

\[ F : (x_1, x_2, x_3) \in \mathbb{R}^3_+ \rightarrow (x'_1, x'_2, x'_3), \]

where if \( \{i, j, k\} = \{1, 2, 3\} \) and \( x_i > x_j > x_k \),

\[ x'_i = x_i - x_j, \quad x'_j = x_j - x_k, \quad x'_k = x_k. \]

This map corresponds to the first return map of the simplicial system represented on Figure 6 to the white node (where all white nodes are identified). The first step is determining which coordinate is the smallest of the three and subtracting it to the other two. The second step is comparing the two initially largest coordinates and subtracting the smallest to the largest. This is precisely describing Poincaré algorithm.

![Figure 6: Poincaré algorithm as a simplicial system.](image)

Notice that the induction associated to the subgraph \( G_{\{1, 2\}} \), where \( G \) is the graph represented on Figure 6, is equivalent to Rauzy induction on two intervals. As for fully subtractive algorithms, this subgraph is stable by a result of Nogueira \[Nog95\]. For a higher dimension \( n \) Poincaré algorithm this tree graph construction generalizes by starting with a vertex of degree
with \( n \) edges labeled by distinct letters and removing the outgoing edge of the ingoing label for each new vertex; when there is only one label left, we identify the vertex to the root.

Dimensions \( n \geq 4 \) are the only classical examples to our knowledge for which the criterion developed in Section 3.2 does not apply and which does not have obvious stable subgraphs.

### 4.2 Other examples

We now deal with examples that do not have full image. Let

\[ I_1, \ldots, I_n \subset \Delta \]

be all the different image sets of the domain on which the given algorithm is a linear map. In the following examples these domains of definition correspond to the different cases depending on the order of the coordinates and will thus be indexed by the corresponding permutations. Moreover, the image sets \( I_k \) will form a finite cover of the set \( \Delta \).

Let \( \pi \) be the finite-to-one projection from the disjoint union of the sets \( I_k \) to \( \Delta \). We will construct a simplicial system for which a first return to a given set of vertices of the win-lose induction map \( T^* \) satisfies \( \pi \circ T^* = F \circ \pi \) and thus has the same dynamical properties as \( F \).

If \( d \) is the dimension of the simplex \( \Delta \), by assumption on simplex-splitting MCF, for all \( k \), there exists a matrix in \( \text{SL}(d + 1, \mathbb{Z}) \) that sends projectively \( I_k \) to \( \Delta \). We make the further assumption that the inverse of these matrices are non-negative. In the examples we consider, the image sets are a union of domain sets up to higher codimension subsets. Consider now the graph whose vertices are all the image sets and draw an edge between \( I_k \) and \( I_l \) if there is a domain contained in \( I_k \) which is sent to \( I_l \) by a matrix in \( \text{SL}(d + 1, \mathbb{Z}) \).

**Remark.** If for some given MCF this condition is not met, one can try to divide the domains of definition in smaller piece.

**Proposition 14.** If two non-negative matrices in \( \text{SL}(d + 1, \mathbb{Z}) \) have the same projective action on the extremal points of \( \Delta \) then they are equal.

**Proof.** Let \( v_1, \ldots, v_{d+1} \) be the vectors defining the extremal points of \( \Delta \). Assume the images of these vectors by the first matrix are \( w_1, \ldots, w_{d+1} \). For the second matrix they must be by assumption \( \alpha_1 w_1, \ldots, \alpha_{d+1} w_{d+1} \). Moreover, as the matrices are both non-negative of determinant 1, we have \( \prod_{k=1}^{d+1} \alpha_k = 1 \), hence \( \alpha_1 = \cdots = \alpha_{d+1} = 1 \).

In particular, it is enough to describe the action of a simplicial system on the extremal points of its linear domains to fully characterize it. As we are reduced to the full image case, it is enough to find a graph that splits each simplex \( I_k \) into the domain subsimplices it contains and to connect the endpoints of this graph with the corresponding image sets. This will define the right simplicial system up to permutation of the extremal points of the simplex. Checking the action on the extremal points of the simplex
will be dealt with in the following by discussing labeling of the length vectors coordinates. This is in general straightforward but will have to be discussed further in the case of Selmer algorithm.

**Remark.** This issue can always been dealt with up to taking a finite number of copies of the image set with different labellings.

### 4.2.1 Brun algorithms

The Brun algorithm, introduced by Brun in 1957, is described in dimension 3 by the map \( F : (x_1, x_2, x_3) \in \mathbb{R}_+^3 \rightarrow (x'_1, x'_2, x'_3) \), where if \( \{i, j, k\} = \{1, 2, 3\} \) and \( x_i > x_j > x_k \),

\[
x'_i = x_i - x_j, \quad x'_j = x_j, \quad x'_k = x_k.
\]

The definition domains of this map are given by the order of the coordinates and the action of the map on these domains is described by Figure 7a. The Figure gives the action on the extremal points up to permutation, to specify it let us remark that each small triangle is sent to the large one which has a common side with the small one and contains the central point of the simplex.

The image sets as introduced above are all the 6 halves of the simplex which we will denote by the relation on two coordinates that define them. They are represented on Figure 7b.

![Graphs](image.png)

(a) action on simplicial domains. (b) image domains.

Each of these halves of the simplex is itself cut into three parts that are sent by Brun algorithm to three different halves. The combinatoric of these domains are represented in Figure 8. Where the dashed arrows and states are identified with the states of same label.

We now can convert the three cuts in the simplex to a sequence of comparison between the three coordinates, as in Figure 9. Where the dashed arrow on left and right are identified with one another.

Now the three actions on the three subsimplices in the image domains can be described by the graph in Figure 9.

Following the arguments developed at the beginning of the section, we obtain the following proposition, which will generalize to higher dimensions.
**Proposition 15.** Let $T_*$ be the first return map of the simplicial system defined on Figure 9 to the white circle vertices, then we have $\pi \circ T_* = F \circ \pi$.

In dimension 3 we only need to check unstability for 2 letters subgraphs. In $G_{1,2}$, the strongly connected components are two loops around $1 \to 2$ and $2 \to 1$ which are clearly unstable. The same is true for any two letters and implies the following proposition.

**Proposition 16.** Brun algorithm in dimension 3 is of Rauzy type.

This construction can be generalized to all dimensions. For any $n \geq 2$, the Brun algorithm is defined by the map,

$$F : (x_1, \ldots, x_n) \in \mathbb{R}_+^n \to (x'_1, \ldots, x'_n),$$

where for $\sigma \in \mathfrak{S}_n$ defined such that $x_{\sigma_1} > \cdots > x_{\sigma_n}$,

$$x'_{\sigma_1} = x_{\sigma_1} - x_{\sigma_2}$$
$$x'_{\sigma_i} = x_{\sigma_i} \text{ for all } i \geq 2.$$

The domains of definition depend again on the order of the coordinates. They can be labeled by permutations in $\mathfrak{S}_n$ and will be denoted by $D_\sigma$ for any
σ ∈ \mathcal{S}_n. For any σ ∈ \mathcal{S}_n the corresponding domain is sent bijectively by F to the subsimplex defined by the equation \( x'_{\sigma_2} > \cdots > x'_{\sigma_n} \), which will be denoted by \( I_{\sigma} \). We change basis to have a simplex corresponding to a whole positive cone and for which the labels are compatible:

\[ y_{\sigma_n} = x'_{\sigma_n}, \quad y_{\sigma_{n-1}} = x'_{\sigma_{n-1}} - x'_{\sigma_n}, \quad \ldots, \quad y_{\sigma_2} = x'_{\sigma_2} - x'_{\sigma_3} \quad \text{and} \quad y_{\sigma_1} = x'_{\sigma_1}. \]

In \( I_{\sigma} \), the coordinate \( x'_{\sigma_1} \) can be in any position, in other words, \( I_{\sigma} = \bigcup_{k=1}^n D_{\{1, \ldots, k\}}^{-1, \sigma} \). Thus the corresponding combinatorial graph has vertices from \( I_{\sigma} \) to all \( I_{\{1, \ldots, k\}}^{-1, \sigma} \) with \( 1 \leq k \leq n \). Now the algorithm can be decomposed into first checking if \( x'_{\sigma_1} \) is smaller than \( x'_{\sigma_n} \), if so, \( F \) sends the domain in \( I_{\{1, \ldots, n\}}^{-1, \sigma} \), otherwise, we check if \( x'_{\sigma_1} \) is smaller than \( x'_{\sigma_{n-1}} \), if so, \( F \) sends the domain to \( I_{\{1, \ldots, (n-1)\}}^{-1, \sigma} \) and so on and so forth. . .

One can check that this corresponds for a simplicial systems on coordinates \( y \), to compare \( y_{\sigma_n} \) and \( y_{\sigma_1} \), then if \( y_{\sigma_1} \) wins, compare \( y_{\sigma_{n-1}} \) and \( y_{\sigma_1} \) (since \( y_{\sigma_1} \) will be equal to \( x'_{\sigma_1} - x'_{\sigma_n} \)), . . .

This description is giving us the corresponding vertices and labels between the image domains, it is represented on Figure 10.

\[ \begin{array}{ccccccc}
I_{\sigma} & \sigma_n & \bullet & \sigma_{n-1} & \cdots & \sigma_4 & \bullet & \sigma_3 & \bullet & \sigma_2 & \bullet & I_{\sigma} \\
\sigma_1 & \sigma_1 & \sigma_1 & \sigma_1 & & & & & & & & & \\
I_{\{1, \ldots, n\}}^{-1, \sigma} & I_{\{1, \ldots, (n-1)\}}^{-1, \sigma} & I_{\{123\}}^{-1, \sigma} & I_{\{12\}}^{-1, \sigma} & & & & & & & & \\
\end{array} \]

Figure 10: Brun algorithm as a simplicial system in dimension \( n \).

**Proposition 17.** Brun algorithm in any dimension \( n \geq 3 \) is of Rauzy type.

**Proof.** The graph is clearly strongly connected and all labels in \([1, \ldots, n]\) appear at least once.

Let us denote by \( G \) the graph in Figure 10 and let \( \mathcal{L} \subset [1, \ldots, n] \) be such that \( 0 < |\mathcal{L}| = k < n \). Notice that in the subgraph \( G_{\mathcal{L}} \), the set of vertices labeled by permutations such that \( \sigma([n-k+1, n]) = \mathcal{L} \) and the vertices between two of them forms a strongly connected component and does not contain vertices labeled by other permutations or any vertex in between. This is clear since the coordinates in \( \mathcal{L} \) are assumed to be infinitesimally small compared to the others and thus will always be the smallest in the permutation describing the order when we subtract the second largest to the largest coordinate. Moreover, the quantity

\[ m_{\mathcal{L}}(\sigma) := \max\{i \geq 0 \mid \sigma([n-i+1, n]) \subset \mathcal{L} \} \]

is non-decreasing. If it is not equal to \( k \), when starting from a white circle vertex, either \( \sigma_1 \) is in \( \mathcal{L} \) and the corresponding vertex points to
a component with a larger $m_L(\sigma')$; or it is not but one can follow the horizontal edges until a label in $L$ and not in $\sigma([n - i + 1, n])$ appears and any path going to a white circle will again point to a component with a larger $m_L(\sigma')$. Hence, the subgraphs with maximal $m_L(\sigma)$ describe all strongly connected components of $G_L$.

Now in a given strongly connected component of $G_L$, for each vertex labeled by a $\sigma$ such that $\sigma([n - k + 1, n]) = L$, there is a sequence of $k$ single edges going to the next vertex and labeled by $L$; at the end of this sequence, there are two vertices labeled in the complementary set $L$ and pointing to a permutation satisfying the same previous property. In particular, there are no vertices with two outgoing edges labeled in $L$.

4.2.2 Selmer algorithms

Introduced by Selmer in 1961 [Sel61], the Selmer algorithm in dimension 3 is defined by

$$F : (x_1, x_2, x_3) \in \mathbb{R}_+^3 \rightarrow (x'_1, x'_2, x'_3),$$

where if $\{i, j, k\} = \{1, 2, 3\}$ and $x_i > x_j > x_k$,

$$x'_i = x_i - x_k, \quad x'_j = x_j, \quad x'_k = x_k.$$

Figure 11 describes the action of Selmer algorithm on its simplicial domains. Notice that unlike Brun algorithm, the image domains are not covering the simplicial domains defining the map. This is related to the fact that the subsimplex $D$ defined by $x_i < x_j + x_k$ for all $\{i, j, k\} = \{1, 2, 3\}$ is an invariant attractive subset of this algorithm.

![Figure 11: Action on simplicial domains.](image)

Restricted to $D$, the algorithm admits a simple description. In Figure 12, we represent the action of the restriction of the Selmer algorithm on $D$, define new labels for a basis of the simplex $D$ and for image domains. On this domain, the vertex 2 is fixed and the central one is sent to 1.

This restriction of the algorithm is described by the graph in Figure 13. This graph is clearly of Rauzy type and corresponds to Cassaigne algorithm given by the map

$$F : (x_1, x_2, x_3) \in \mathbb{R}^3 \rightarrow \begin{cases} 
(x_1 - x_3, x_3, x_2) & \text{if } x_1 > x_3 \\
(x_2, x_1, x_3 - x_1) & \text{if } x_3 > x_1
\end{cases}.$$
Figure 12: Action on the restriction and image domains.

The domains $a, b, c$ correspond to marking the permutation action of the Cassaigne algorithm on the coordinates of the vector.

Figure 13: Cassaigne algorithm as a simplicial system.

As a straightforward consequence we obtain the following proposition.

Proposition 18. Selmer algorithm in dimension 3 restricted to $D$ is of Rauzy type.

Let us consider now the generalization of this algorithm for $n \geq 3$, $F : (x_1, \ldots, x_n) \in \mathbb{R}^n_+ \to (x'_1, \ldots, x'_n)$, where for $\sigma \in \mathfrak{S}_n$ defined such that $x_{\sigma_1} > \cdots > x_{\sigma_n}$,

- $x'_{\sigma_1} = x_{\sigma_1} - x_{\sigma_n}$
- $x'_{\sigma_i} = x_{\sigma_i}$ for all $i \geq 2$.

As for Brun algorithms, the domains of definition are labeled by $\mathfrak{S}_n$ and will be denoted by $D_\sigma$ for any $\sigma \in \mathfrak{S}_n$. Similarly to dimension 3, there is a stable subsimplex $D$ defined by the equations $x_{\sigma_1} < x_{\sigma_{n-1}} + x_{\sigma_n}$. In the following, we consider the map $F|_D$, and the $D_\sigma$ denote their intersection with $D$. 
For any $\sigma \in \mathfrak{S}_n$ the domain $D_\sigma$ in $D$ is sent bijectively by $F|_D$ to the subsimplex defined by the equations $x'_{\sigma_2} > \cdots > x'_{\sigma_n}$ and $x'_{\sigma_1} < x'_{\sigma_{n-1}}$, which will be denoted by $I_\sigma$. In $I_\sigma$ the coordinate $x'_{\sigma_1}$ can either be in position $n - 1$ or $n$, in other words, $I_\sigma = D_{(1\ldots(n-1))}^{-1}\sigma} \cup D_{(1\ldots n)}^{-1}\sigma}$. Thus the corresponding combinatoric graph has vertices pointing from $I_\sigma$ to $I_{(1\ldots(n-1))}^{-1}\sigma}$ and from $I_\sigma$ to $I_{(1\ldots n)}^{-1}\sigma}$.

We first define a labeling for the basis which will help us keep track of the permutation of the extremal points of the simplex. This is a generalization of what we did previously on Selmer algorithm in dimension 3.

The point for which all coordinates but one are equal to 1 and the other is equal to 0 is an extremal point of $D$ and is fixed by the algorithm. We label each of these points by the label corresponding to its zero coordinate:

$$v_\alpha = 11\ldots 10_1\ldots 1.$$  

This is what we did before in Figure 12. Now observe that $I_\sigma$ is the convex hull of $v_{\sigma_n}, v_{\sigma_1}, c$ and $w_k$ for $2 \leq k \leq n - 2$, where $c$ is the point for which all coordinates are equal to 1 and

$$w_k(i) = \begin{cases} 1 & \text{if } i = \sigma_2,\ldots,\sigma_k \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

On each of these subsimplices the algorithm only compare coordinates in $v_{\sigma_1}$ and $v_{\sigma_n}$, these two labels are the only ones that matter. In this labeling, $\sigma_1$ loses when $x_{\sigma_1} > x_{\sigma_n}$ and vice-versa, which may be counter-intuitive. The graph for Selmer algorithm is thus described by Figure 14.

![Figure 14: Selmer algorithm as a simplicial system in dimension n.](image)

**Proposition 19.** Selmer algorithm in any dimension $n \geq 3$, restricted to $D$, is of Rauzy type.

**Proof.** The graph is strongly connected since the permutation group is generated by the two cycles $(1\ldots n)$ and $(1\ldots(n-1))$. Moreover, all labels in $[1,\ldots, n]$ appear at least once.

Let us denote by $G$ the graph in Figure 14 we describe for a given subset $\mathcal{L} \subseteq A$ the strong connected components of $G_{\mathcal{L}}$. The property that $\sigma_n$ is not in $\mathcal{L}$ is invariant in the subgraph $G_{\mathcal{L}}$, and if $\sigma_n$ is in $\mathcal{L}$ one can apply the cycle $(1\ldots n)^{-1}$ until it is not. Hence the vertices of strong
connected components are in a subset of permutations where \( \sigma_n \) is not in \( \mathcal{L} \). This implies that at least one of the two outgoing edges of a vertex of the subgraph is not labeled by a letter in \( \mathcal{L} \).

4.3 Rauzy Gasket and Arnoux-Rauzy-Poincaré

Following [AS13], we define the Rauzy gasket in arbitrary dimension \( n \geq 2 \). Let \( C = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \mid x_j \leq \sum_{i \neq j} x_i, \forall j\} \) and the Arnoux-Rauzy map,

\[
F : (x_1, \ldots, x_n) \in \mathbb{R}_+^n \setminus C \rightarrow (x'_1, \ldots, x'_n),
\]

where for \( \sigma \in S_n \) defined such that \( x_{\sigma 1} > \cdots > x_{\sigma n} \),

\[
x_{\sigma 1}' = x_{\sigma 1} - \sum_{i=2}^{n} x_{\sigma i},
\]

\[
x_{\sigma i}' = x_{\sigma i} \text{ for all } i \geq 2.
\]

Consider the limit set,

\[
R = \bigcap_{n \geq 0} F^{-n}(\mathbb{R}_+^n \setminus C).
\]

The Rauzy gasket is the intersection \( R \cap \Delta \), where

\[
\Delta := \{(x_1, \ldots, x_n) \mid \sum x_i = 1\}.
\]

Observe that in a simplicial system point of view, the map \( F \) first splits the simplex depending on the order of the coordinates then for each ordering \( \sigma \in S_n \) sends the subsimplex defined by \( x_{\sigma 1} > \sum_{i=2}^{n} x_{\sigma i} \) to the whole simplex and is not defined on the other parts. The graph will thus have two main parts: one connecting \( I_{\sigma} \) states to \( \tilde{I}_{\sigma} \) which will be the same as for Brun algorithm and another one which connects states \( \tilde{I}_{\sigma} \) to \( I_{\sigma} \), cutting out the parts on which the algorithm is not defined.

![Graph](image.png)

Figure 15: Part of the graph for Rauzy gasket connecting \( I_{\sigma} \) to \( \tilde{I}_{\sigma} \).

Now consider the compatible basis introduced for Brun algorithm

\[
y_{\sigma n} = x_{\sigma n}, \ y_{\sigma n-1} = x_{\sigma n-1} - x_{\sigma n}, \ldots, \ y_{\sigma 2} = x_{\sigma 2} - x_{\sigma 3} \text{ and } y_{\sigma 1} = x_{\sigma 1}.
\]

In this basis, the condition \( x_{\sigma 1} < \sum_{i=2}^{n} x_{\sigma i} \) is given by

\[
y_{\sigma 1} < y_{\sigma 3} + 2y_{\sigma 4} + \cdots + (n-2)y_{\sigma n}.
\]
This is given by a graph with a sequence of edges from $\tilde{I}_\sigma$ to $I_\sigma$ labeled in the following order: $\sigma_3$, twice $\sigma_4$, \ldots, $n - 2$ times $\sigma_n$. For each vertex in this sequence, starting with $\tilde{I}_\sigma$, there is an edge labeled by $\sigma_1$ and going out.

Figure 16: Part of the graph for Rauzy gasket connecting $\tilde{I}_\sigma$ to $I_\sigma$.

Let $F$ be the subgraph of the graph defined in Figure 15 and Figure 16.

From the construction it is clear that we have the following,

**Proposition 20.** There exists a finite-to-one projection from $\Delta(F)$ to $R$ which is locally the identity map.

Moreover, the subgraph $F$ is dynamically equivalent to the graph defining Brun algorithm (it can be accelerated to the Brun algorithm) since the only added edges are of degree one. Thus we have,

**Theorem 7.** The Rauzy gasket in any dimension $n \geq 3$ has Hausdorff dimension strictly smaller than $n - 1$ and its canonical suspension flow has a unique measure of maximal entropy.

Finally we remark that in dimension 3, the Poincaré algorithm acts on $C$ as described on Figure 17.

Figure 17: Action of Poincaré algorithm on a subdomain of $C$.

This gives us a natural way to describe Arnoux-Rauzy-Poincaré algorithm in dimension 3, consisting in applying the Arnoux-Rauzy map on $R^n_+ \setminus C$ and the restriction of Poincaré map on $C$ (see [BL15]). We only need to make the edges pointing to the hole vertex $\times$ from $I_\sigma$ point to $I_{(123)^-1 \sigma}$ as represented on Figure 18.

As for Brun algorithm in dimension 3, we only need to check unstability for two letter subgraphs, say $G_{1,2}$. Here again the strongly connected components will be two loops around $1 > 2$ and $2 > 1$ formed by 3 edges. Which implies,
Proposition 21. The Arnoux-Rauzy-Poincaré algorithm in dimension 3 is of Rauzy type.

Observe that the generalization of this algorithm to higher dimension will have more complicated combinatorics, since the images induced by the edges going out of the graph of Arnoux-Rauzy will for new sets of images. Perhaps another more natural way to generalize this algorithm in the simplicial system point of view would be to connect all these edges to $I_{(1\ldots n)-1\sigma}$. This will again be an unstable simplicial system.

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