Finite element model based on an efficient layerwise theory for dynamics and active vibration control of smart functionally graded beams

M Yaqoob Yasin, Bhanu Prakash and Arshad Hussain Khan
Smart Structures Lab, Department of Mechanical Engineering, Z. H. College of Engineering and Technology, Aligarh Muslim University, Aligarh, UP 202002, India
E-mail: yaqoob.yasin@gmail.com

Abstract
In this work, we present a two-noded efficient finite element (FE) model incorporating the layer-wise mechanics for the dynamics and active vibration control of smart functionally graded (FG) beams. The material properties in the FG beam are assumed to vary smoothly in the thickness direction according to power law variation. The effective properties are computed using Mori-Tanaka homogenization scheme. Electric potential profile in the electroded piezoelectric layers/patches is assumed quadratic across its thickness. The equations of motion are derived using extended Hamilton’s principle. Due to the complex algebraic expressions involved in the effective properties of FG system, the inertia and stiffness parameters are computed numerically using six point Gauss integration method. To fulfill the convergence requirements for weak integration of various energy terms in the variational formulation, the transverse displacement is interpolated with Hermite interpolation function possessing $C^1$-continuity while the inplane displacement, shear rotation and quadratic component of the electric potential are interpolated with linear Lagrangian functions of $C^0$-continuity. The equipotential condition of the electroded piezoelectric sensors and actuators is conveniently modelled using electric node concept. The control system is designed for constant gain velocity feedback (CGVF), and optimal LQR and LQG control strategies for a reduced order model using state space approach. The control performance is studied for single-input-single-output (SISO) and multi-input-multi-output (MIMO) configurations under various excitations. The effect of material inhomogeneity on stability/instability of the closed-loop response in the CGVF control has been discussed.

1. Introduction

The research on active vibration control of lightweight structures has gained impetus in recent years due to the requirements of efficient/ultra-reliable design of structures with wide ranging applications particularly in aerospace and automotive sectors. The vibration sensing and control makes use of thin layers of piezo electric materials (lead zirconate titanate (PZT) or polyvinylidene fluoride (PVDF)) as sensors and/or actuators which may be embedded or mounted on the surface. One of the earliest studies on the use of piezoelectric materials for actuation and sensing is presented by (Bailey and Hubbard 1985). After then, large number of studies have been conduction to achieve sensing and actuation mechanics in flexible structures (Rao and Sunar 1994, Chopra 2002). The modelling, analysis and design of such structural components integrated with sensors/actuators require efficient coupled electro-mechanical modelling together with an optimal control strategy. The various modelling techniques related to laminate mechanics for smart structures are discussed in (Kapuria et al 2010). From these reviews, the research on the active control of smart functionally graded material (FGM) structures are limited compared to the isotropic/laminated composite counterparts. In recent few decades, FGM materials receives significant research interest due to decreased jumps in inplane stresses unlike the...
laminated composites and better thermal response. Studies related to synthesis, modelling and analysis of FGM structures can be seen in state of the art reviews (Birman and Byrd 2007, Gupta and Talha 2015). We present review studies related to modelling and analysis of smart FGM structures related to active shape and vibration control.

Finite element model based on classical plate theory (CLPT) has been presented for active shape and vibration control of smart FGM plates using constant velocity feedback (CGVF) control algorithm by (He et al. 2001). This study is applicable for thin plates due to neglect of shear deformation effects. Finite element analysis of smart FGM plates and shells subjected to temperature gradient based on first order shear deformation theory (FSDT) has been presented for active vibration control employing constant displacement-cum-velocity feedback control by (Liew et al. 2003, 2004). The mesh free method based on element-free Galerkin method has been employed for active shape and vibration control of FGM plates using displacement field of FSĐT and displacement-cum-velocity feedback (Dai et al. 2005). Recently, active vibration control of Functionally graded beam with surface mounted layers of piezoelectric actuator and sensor is investigated using Reddy’s third order shear deformation theory (TOT) and velocity feedback control algorithm (Bendine et al. 2016).

The large amplitude vibration control (Fakhari and Ohadi 2011) and forced vibration response (Fakhari et al. 2011) of functionally graded material (FGM) plates with integrated piezoelectric sensor/actuator layers based on Reddy’s TOT under thermal gradient and transverse mechanical loads has been investigated; Yiqi and Yiming (2010) analysed the nonlinear dynamic response and active vibration control of piezoelectric functionally graded plate using a higher order shear deformation theory. The exact free vibration solution for FGM plate with top and bottom piezoelectric layers with Levy type boundary based on FSĐT has been obtained by (Farsangi and Saidi 2012). Jadhav and Bajoria (2014) reviewed the stability and vibration control aspects of piezo-laminated composite and FGM plates. Finite element model has been developed for the open-loop and closed-loop nonlinear dynamics of FG laminated composite plates (Panda and Ray 2009) and beams (Panda et al. 2016) integrated with a patches/layers of piezoelectric fiber reinforced composite (PFRC) material. Navier solution for free vibration analysis of isotropic, transversely isotropic and FG plates integrated with piezoelectric layers has been obtained by Rouzegar and Abad (2015) based on four-variable refined plate theory. Selim et al. (2016) employed the element-free IMLS-Ritz method based on Reddy’s HSĐT for the free vibration analysis and active vibration control of FGM plates with piezoelectric layers. Cell-based smoothed discrete shear gap method (CS-DSG) based on the FSĐT is employed for active vibration control of FGM plates integrated with piezoelectric sensors and actuators based on the displacement and velocity feedbacks (Nguyen-Quang et al. 2017). Moita et al. (2018) carried out the free vibration analysis of active-passive damped multilayer sandwich plates/shells with a viscoelastic core. Both the FGM and the piezoelectric layers are modelled using the CLPT and the core is modelled using Reddy’s third-order shear deformation theory. Dogan (2015) carried out active vibration control of FGM plates with collocated piezoelectric sensors and actuators subjected to random excitation. The inertia and stiffness of the piezo patches is neglected and the analysis is based on classical plate theory. Gharib et al. (2008) obtained an analytical solution for the dynamic response of FGM beam with piezoelectric sensor/actuator layers based on FSĐT. Kapuria and Yasin (2010) employed efficient layerwise theory for the active vibration control of laminated composite beam. The finite element modeling and analysis of isotropic/orthotropic composite beams bonded with piezoelectric actuators has been carried out by Elshafei and Alraies (2013). Nguyen et al. (2013) obtained the Navier solution for static and free vibration of axially loaded functionally graded beams based on the first-order shear deformation theory. Bruant and Proslier (2015) carried out active vibration control of FGM beam based on state space formulation using trigonometric shear deformation theory. Ebrahimi and Hashemi (2017) carried out vibration analysis of non-uniform porous FGM beam under temperature gradient. El Harti et al. (2019) explored the dynamics of a FGM sandwich beam based on Timoshenko’s assumptions and the finite element method using LQR control. From the detailed literature review, it is observed that most studies uses the simple rule of mixture to homogenize the FGM materials which give inaccurate effective properties when compared with the experimental studies (Kapuria et al. 2008). Also, these studies are carried out using equivalent single layer theories which do not account for the slope discontinuity of inplane displacement due to widely different material properties. Kapuria et al. (2006) presented a layer-wise theory for smart multilayered FGM beam integrated with piezoelectric sensors and actuators. To the best of authors knowledge, there is no study available in the literature for the dynamic analysis and active vibration control of smart functionally graded beams using a layer-wise theory.

In this work, an electromechanically coupled FE modeling for dynamics and active vibration control of smart functionally graded beams has been developed by extending the previously developed FE model for piezolaminated laminated composite beams (Kapuria and Yasin 2010). The properties are assumed to vary smoothly in transverse direction in terms of ceramic volume fraction according to power law variation. The effective properties of the FGM substrate at any point is obtained using Mori–Tanaka approach. The inertia and stiffness parameters are obtained numerically using Gauss quadrature integration rule taking six integration points. Quadratic variation of electric potential is assumed in the thickness direction of the PZT patches/layers.
Transverse deformation of the piezoelectric sensors and actuators due to larger values of the piezoelectric coefficients $d_{33}$ has been incorporated. The electric potential conditions on the piezoelectric sensors/actuators has been achieved using the concept of electric node. The element has two nodes with three mechanical degrees of freedom for inplane, transverse and shear rotation components and one electric degree of freedom for quadratic component of the electric potential which enables to model the induced electric field due to direct piezoelectric effects. The control system has been designed based on reduced-order model using state space approach. Control performance for cantilever beam with single and multiple piezoelectric sensors and actuators using classical CGVF and optimal LQR and LQG control approaches is presented. The FE model has been validated with the experimental results for elastic FGM beam. The efficacy of various control strategies in the active vibration control of smart FGM beam has been analyzed for the first time based on an efficient layerwise theory. The effect of electromechanical coupling on the natural frequencies of smart FGM beam with different boundary conditions has been studied. The effect of material inhomogeneity on the instability/stability of the closed loop response obtained using CGVF control with ‘conventionally’ and ‘truly’ collocated actuator-sensor pairs has been discussed. The influence of excitations on the open-loop and closed-loop tip-deflection and actuation potential for CGVF, LQR and LQG control algorithms are studied.

2. Micro-mechanical modeling

Consider a FGM beam having PZT patches/layers bonded on the top and bottom surfaces (figure 1). The length, width and thickness are denoted by $l$, $b$ and $h_{fg}$ respectively. The PZT layers/patches are poled along the thickness and are acting as actuators and sensors. The FGM substrate is formed by mixing ceramic and a metal in such a way that the concentration is smoothly varying from one phase at the bottom surface to the other phase at the top surface. In this study, otherwise stated, we employ the Mori-Tanaka model (MTM) to homogenize the FGM material substrate.

The Mori-Tanaka homogenization method (Mori and Tanaka 1973) is based on the Eshelby’s fundamental work. This scheme accounts for the interactions among the adjacent inclusions where the effective bulk modulus $K$ and shear modulus $G$ are expressed as

$$K(z) = \frac{V_c(z)(K_c - K_m)}{1 + (1 - V_c(z)) \frac{K_c - K_m}{K_m + \frac{4}{3} G_m}} + K_m,$$

$$G(z) = \frac{V_c(z)(G_c - G_m)}{1 + (1 - V_c(z)) \frac{G_c - G_m}{G_m + f_1}} + G_m$$

where $K_c$, $G_c$, $K_m$, $G_m$ are the bulk modulus and shear modulus for ceramic and metal phases respectively, $f_1 = G_m(9K_m + 8G_m)/6(K_m + 2G_m)$ and $V_c(z)$ is the volume fraction profile of the ceramic phase which is defined according to power law variation as:

$$V_c(z) = (0.5 + z/h)^p$$

The effective Young’s modulus $Y$ and Poisson’s ratio $\nu$ of the FGM are obtained from effective $K$ and $G$ using the following equations

$$Y(z) = \frac{9K(z)G(z)}{3K(z) + G(z)}, \quad \nu(z) = \frac{3K(z) - 2G(z)}{2(3K(z) + G(z))},$$

Figure 1. Geometry of smart functionally graded piezoelectric beam.
The Voigt’s rule of mixture ROM can also be used easily to obtain the effective properties of the FGM. However, this method is based on the neglect of the interactions among the neighboring inclusions and therefore unable to predict accurate effective properties. For this method, the effective value of any property \( P \) is obtained using

\[
P = P_m V_m + P_c V_c
\]

where \( P_m \) and \( P_c \) denote metal and ceramic phases and \( V_m \) and \( V_c \) are the respective volume fractions with \( V_c + V_m = 1 \). The effective mass density \( \rho \) is always determined using the ROM.

3. Efficient layerwise theory for smart FGM beams

One dimensional beam theory and its FE model for smart FGM beam is presented in this section. The width of the beam is very small so that the plane stress condition can be applied, i.e., \( \sigma_y = \tau_{yz} = \tau_{xy} = 0 \). Transverse pressure load has been applied. The transverse normal stress \( \sigma_z \) is neglected. Using these assumptions, the general 3D constitutive relations for smart FGM beam can be written as

\[
\sigma_z = Q_{33}(z) \varepsilon_z, \quad \tau_{xx} = \hat{Q}_{13}(z) \varepsilon_x, \quad D_x = \hat{Q}_{15}(z) \varepsilon_x + \hat{Q}_{13}(z) \varepsilon_z
\]

where \( \sigma_z, \varepsilon_z, \) are the normal stress and normal strain, \( \tau_{xx}, \gamma_{xx} \) are the transverse shear stress and transverse shear strain. \( \hat{Q}_{13}(z), \hat{Q}_{15}(z), \) are the reduced stiffness coefficients, piezoelectric strain constants and electric permittivities for the piezoelectric material. These are functions of Young’s moduli \( Y \), shear moduli \( G \), piezoelectric strain constants \( d \), and electric permittivities \( \varepsilon \). \( Q_{ij} \) are the reduced stiffness coefficients for the FGM substate which are smooth functions of \( z \) whose effective values at any point along thickness coordinate are determined using \( V(z) \) and \( \nu(z) \) following a micromechanics model described in section 2. The mid surface of the host FGM is considered as the reference plane \( (z = 0) \).

In earlier studies based on the exact three dimensional elasticity theory (Dube et al 1996), it is shown that the electric potential \( \phi \) is varying quadratically across the thickness of the PZT patches/layers. In this study we assume \( \phi \) as piecewise quadratic between \( n_j \) points at \( z_j^l, j = 1, 2, \ldots, n_\phi \), across the thickness as

\[
\phi(x, z, t) = \sum_{j=1}^{n_\phi} \psi^j l(z) \phi^j_l(x, t) + \sum_{l=1}^{n_l} \psi^l(z) \phi^l(x, t)
\]

where \( \phi^j_l(x, t) \) is the surface electric potentials at the piezoelectric layer surfaces/interfaces, and \( \phi^l(x, t) \) is its quadratic component at \( z = (z_j^l + z_j^{l+1})/2 \) with \( j = 1, 2, \ldots, n_j \), \( l \in [1, 2, \ldots, n_l] \), and \( n_j^l = n_\phi - 1 \). \( \psi^j l(z) \) and \( \psi^l(z) \) are piecewise linear and quadratic interpolation functions respectively.

The surfaces of the piezoelectric sensors and the actuators are always electrode with the metallic coating which makes the surfaces equipotential. Linear component of the electric potential \( \phi \) contributed by surface potentials is taken constant over the surfaces because of equipotential conditions of the electroded surfaces. The quadratic component \( \phi^j_l \) which is induced because of direct piezoelectric effect is a function of \( x \).

The efficient layerwise theory (Kapuria and Yasin 2010) employed here for the kinematics of smart FGM beam. In this theory, the transverse deflection \( w \) is obtained by incorporating the effect of the deformations due to electric field component \( E_z \) because of larger values of \( d_{33} \) and is given by

\[
w(x, z, t) = w_0(x, t) - \sum_{j=1}^{n_\phi} \psi^j l(z) \phi^j_l(x, t) - \sum_{l=1}^{n_l} \psi^l(z) \phi^l(x, t)
\]

where

\[
\psi^j l(z) = \int_0^z d_{33} \psi^j l(\zeta) d\zeta, \quad \psi^l(z) = \int_0^z d_{33} \psi^l(\zeta) d\zeta
\]

The inplane displacement \( u \) follow a linear layerwise variation of \( z \) superimposed with a third order polynomial of \( z \) having \( 2L + 2 \) variable where \( L \) is the number of layers. The final expression of \( u \) is obtained using the \( 2(L - 1) \) continuity conditions for \( \tau_{yy} \) and \( u \) at the layer interfaces, and the two shear stress free conditions at the top and bottom surfaces. Thus, the displacement field for the smart FGM is expresses as

\[
u(x, z, t) = u_0(x, t) - z w_0(x, t) + R^k(z) \psi_0(x, t)
\]
where
\[ R^k(z) = \ddot{R}^k_t + z\ddot{R}^k_e + z^2\ddot{R}_3 + z^3\ddot{R}_4 \]  
(11)

where the coefficients \( \ddot{R}^k_t, \ddot{R}^k_e, \ddot{R}_3 \) and \( \ddot{R}_4 \) are dependent on the lay-up and material properties of the layers. A subscript comma denotes differentiation.

The strains and electric field are obtained from linear strain displacement and electric field potential relations as

\[
\varepsilon_x = u_x = u_{0,x} - zw_{0,xx} + R^k(z)\psi_{0,x} \\
\gamma_{zz} = u_{x,x} + w_{0,xx} = R^k(z)\psi_{0,x} + \sum_{l=1}^{n_l} \hat{\Psi}_l(z)\phi_{l,x}^i \\
E_x = -\ddot{\phi}_x = -\sum_{l=1}^{n_l} \hat{\Psi}_l(z)\phi_{l,x}^i \\
E_\phi = -\ddot{\phi}_z = -\sum_{l=1}^{n_l} \hat{\Psi}_l(z)\phi_{l,z}^i \\
\]  
(12)

4. Variational principle

The equations governing the dynamics of smart functionally graded beams are obtained using the extended Hamilton’s principle (Tiersten 1969), which can be expressed as

\[
\int_{-0.5h}^{0.5h} \left[ (\rho^k(z)(\dddot{u} + \dddot{w}) + \sigma_x \dddot{\varepsilon}_x + \tau_{zz} \dddot{\gamma}_{zz} - D_x \ddot{E}_x - D_z \ddot{E}_z) dz - b(p^e)\ddot{w}(x, z_0, t) + bD_x(x, z_0, t)\ddot{\phi}_x - bD_z(x, z_0, t)\ddot{\phi}_z \right] dx = -\int_{-0.5h}^{0.5h} \left[ (\sigma_x \dddot{\varepsilon}_x + \tau_{zz} \dddot{\gamma}_{zz}) dz \right] dx = 0 \]  
(13)

\( \forall \delta u_0, \delta w_0, \delta \psi_0, \delta \phi_x, \delta \phi_z, \) and \( \rho^k(z) \) is the material mass density of the \( k \)th layer. Over dot represents the derivative of the respective variable with time. \( p^e \) is the applied pressure load intensity on top surface. Substituting displacements, strains and electric filed from equations (8), (10), (12), and neglecting the contribution of electric potential terms in \( w \) in inertia matrices, the extended Hamilton’s principle can be expressed as

\[
\int_{-0.5h}^{0.5h} \left[ \delta \bar{u}_0^T \hat{\bar{u}}_0 + \delta \bar{w}_0^T \hat{\bar{w}}_0 + \delta \bar{\psi}_0^T \hat{\bar{\psi}}_0 + \delta \bar{\phi}_x^T \hat{\bar{\phi}}_x + \delta \bar{\phi}_z^T \hat{\bar{\phi}}_z + \hat{\bar{G}}^T \delta \phi_x + \hat{\bar{G}}^T \delta \phi_z + \hat{\bar{H}}^T \delta \phi_x + \hat{\bar{H}}^T \delta \phi_z \right] dx = -\int_{-0.5h}^{0.5h} \left[ \delta \bar{u}_0^T \bar{F}_1 + \delta \bar{w}_0^T \bar{F}_2 + \delta \bar{\psi}_0^T \bar{F}_3 + \delta \bar{\phi}_x^T \bar{F}_4 + \delta \bar{\phi}_z^T \bar{F}_5 \right] dx + \left[ \delta \bar{u}_0^T \bar{I}_{11} \right] = 0
\]  
(14)

where an over-bar means the values are known at the beam ends. In this equation, \( I (3 \times 3 \) matrix) and \( \bar{I} \) are the inertia properties, \( N_x, M_x, P_x; Q_x, Q_x^i, V_x, V_x^i, V_x^{ij} \) are the stress resultants of \( \sigma_x \) and \( \tau_{zz} \); \( H_x, H_x^i, G_x, G_x^i \) are the electric displacement resultants of \( D_x \) and \( D_z \), respectively which have been defined as

\[
I = \int \begin{bmatrix} 1 & z & R^k(z) \\ z & z^2 & zR^k(z) \\ R^k(z) & zR^k(z) & (R^k(z))^2 \end{bmatrix} \rho^k(z) dz, \quad \bar{I} = \int \rho^k(z) dz = \bar{I}_{11}
\]  
(15)

and

\[
\begin{bmatrix} N_x \\ M_x \\ P_x \end{bmatrix} = \int \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \sigma_x dz = F_1, \quad \begin{bmatrix} Q_x \\ Q_x^i \end{bmatrix} = \int \begin{bmatrix} R^k(z) \\ -\hat{\Psi}_l(z) \end{bmatrix} \tau_{zz} dz = F_2, \\
\begin{bmatrix} G_x^i \\ G_x^i \\ G_x^i \end{bmatrix} = \int \begin{bmatrix} \hat{\Psi}_l(z) \\ \hat{\Psi}_l(z) \end{bmatrix} D_x dz = F_4, \quad \begin{bmatrix} V_x^i \\ V_x^i \end{bmatrix} = \int \begin{bmatrix} \hat{\Psi}_l(z) \\ \hat{\Psi}_l(z) \end{bmatrix} \tau_{zz} dz = F_5, \\
V_x = \int \tau_{zz} dz, \quad H_x^i = \int \hat{\Psi}_l(z) D_x dz = F_3, \quad H_x^i = \int \hat{\Psi}_l(z) D_x dz.
\]  
(16)

and \( P_2 \) and \( P_\phi^i \) are mechanical and electromechanical load terms defined by

\[
P_2 = b p^e, \quad P_\phi^i = b + p^e \hat{\Psi}_l(z) + D_x \delta \phi,- D_z \delta \phi]
\]  
(17)

where \( \delta \phi \) is Kronecker’s delta, \( D_{z_0} = D_x(x, z_0, t) \) and \( D_{z} = D_x(x, z, t) \).
Using constitutive relations in equation (16) and integrating over z from \(-h/2\) to \(h/2\) yields

\[
F_1 = A\varepsilon_1 + e_1^T e_4 \\
F_2 = \tilde{A}\varepsilon_2 + \varepsilon_2^T \varepsilon_3 \\
F_3 = e_3\varepsilon_1 - \eta_1\varepsilon_3 \\
F_4 = e_2\varepsilon_1 - \eta_2\varepsilon_4
\]

where

\[
A = \int \begin{bmatrix}
1 & z & R^k(z) \\
z & z^2 & zR^k(z) \\
R^k(z) & zR^k(z) & (R^k(z))^2
\end{bmatrix} Q_{1,1} dz \\
\tilde{A} = \int \begin{bmatrix}
R^k(z)R^k(z) & -R^k(z) & \Psi_i^j \\
R^k(z) & R^k(z) & \Psi_i^j \\
-\Psi_i^j & \Psi_i^j & \Psi_i^j
\end{bmatrix} Q_{5,5} dz \\
e_2 = \int \begin{bmatrix}
\Psi_{c,(x)}(z) & z\Psi_{c,(x)}(z) & R^k(z)\Psi_{c,(x)}(z) \\
\Psi_{o,(x)}(z) & z\Psi_{o,(x)}(z) & R^k(z)\Psi_{o,(x)}(z)
\end{bmatrix} \tilde{e}_{1,3} dz \\
e_3 = \int \begin{bmatrix}
\Psi_{c,(x)}^j(\tilde{z}) & -\Psi_{c,(x)}^j(\tilde{z}) & \Psi_{o,(x)}^j(\tilde{z})
\end{bmatrix} \tilde{e}_{1,3} dz \\
\eta_{1,3} = \int \begin{bmatrix}
\Psi_{c,(x)}^j(\tilde{z}) & \Psi_{c,(x)}^j(\tilde{z}) & \Psi_{o,(x)}^j(\tilde{z})
\Psi_{o,(x)}^j(\tilde{z}) & \Psi_{o,(x)}^j(\tilde{z}) & \Psi_{o,(x)}^j(\tilde{z})
\end{bmatrix} \eta_{1,3} dz
\]

and

\[
\varepsilon_1 = \begin{bmatrix}
u_{0,x} \\
-w_{0,xx} \\
\psi_{0,x}
\end{bmatrix} \\
\varepsilon_2 = \begin{bmatrix}
\psi_{0} \\
-\phi_{0,x}^j
\end{bmatrix} \\
\varepsilon_3 = \phi_{c,(x)}^j \\
\varepsilon_4 = \phi_{o,(x)}^j
\]

It is noted that in the case of FGM beams, the mass density and reduced stiffness coefficients are functions of \(z\). Therefore for obtaining the inertia matrix \(I\) and beam stiffness coefficients \(A\), \(\tilde{A}\) in equation (19), through-the-thickness integration is required. As observed from micromechanics approach, Mori-Tanaka method yields the effective values of \(Q_{1,1}\) and \(Q_{5,5}\) in terms of complex function of \(z\) which cannot be explicitly expressed in terms of simple polynomials. Therefore, the closed form integration over the thickness for such function cannot be performed. In this study, we adopt numerical integration using Gauss integration rule taking six integration point.

The finite element modeling of smart FGM beam has been performed using a two noded beam element. As observed from variation equation (14), the transverse displacement \(w_0\) has second order derivative, therefore, as per continuity requirement, it is interpolated using \(C^1\) continuous cubic Hermite interpolation functions. The other variables \(u_0\), \(\psi_{0}\), \(\phi_{0}^j\) possess first order derivative, therefore, these are interpolated using linear Lagrangian interpolation functions having \(C^0\) continuity. The surfaces of the PZT sensors and actuators are electroded with metallic coating to make them equipotential surfaces. The equipotential condition exists on an equipotential point. This equipotential condition spanning over several elements of piezoelectric patches in a beam section is modeled using electric nodes. The electric node can be connected to multiple elements belonging to the same electroded surface and does not have any \(x\)-coordinate unlike the conventional nodes, and therefore have \(n_o\) degrees of freedom. This concept models the equipotential condition conveniently with significant reduction in the over all number of DOFs. Let \(U\) is the generalized displacement vector of the system, the equations of motion can be expresses as

\[
MU + Ku = P
\]

in which \(M\) and \(K\) are global mass and stiffness matrices and \(P\) is the global load vector. Partitioning the mechanical displacements, unknown output voltages of the sensory surfaces \(\Phi_i\) known actuation potentials \(\Phi_o\), the equation (21) can be rearranged as

\[
\begin{bmatrix}
M^{uu} & 0 & 0 & 0 & \Phi^u \\
0 & M^{uu} & K^{uu} & K^{uu} & 0 \\
0 & K^{uu} & K^{uu} & K^{uu} & 0 \\
0 & K^{uu} & K^{uu} & K^{uu} & 0
\end{bmatrix}
\begin{bmatrix}
\dot{U} \\
\Phi^u \\
\Phi^u \\
\Phi^u
\end{bmatrix}
= 
\begin{bmatrix}
P \\
0 \\
0 \\
0
\end{bmatrix}
\]

In the present analysis, the sensors and actuators does not share any common interface, the coupling matrix \(K^{uu}\) is a null matrix. Also at the sensor surfaces, the applied electric charge is zero, \(Q_s = 0\). The sensory output from equation (22) is evaluated as
\[ \Phi = -[K^T]^{-1}K^wU \]  

Adding Rayleigh viscous damping of the form \( C^w = \alpha M + \beta K \), the equations of motion can be expressed as

\[ M^w\ddot{U} + C^w\dot{U} + [K^w - K^w(\cdot^{(1)})^{-1}K^u]U = P - K^w\Phi. \]  

### 5. Control system design

The size of finite element system equations is huge due to large number of degrees of freedom (DOF) required for an accurate determination of dynamic response. Therefore, the solution of the dynamical system directly in nodal coordinates require high computational cost. A reduced order model can be constructed to reduce computational time with sufficient level of accuracy. The effect of unmodelled dynamics (residual modes) such as controller and observer spillover can be nullified by giving a small amount of damping to the system (Meirovitch 1989). Also, the response of dynamic system can be assumed to be dependent on first few modes because of their lower associated energy. Let \( \Psi \) represent the truncated modal matrix containing first few modes. The response vector is obtained as \( \hat{U} = \sum_{i=1}^{n_r} \Psi_i \hat{\eta}_i \) where \( n_r \) is the number of modes considered in the reduced order model, \( \Psi = [\Psi_1 \ \Psi_2 \ ... \ \Psi_{n_r}] \) and \( \eta = [\hat{\eta}_1 \ \hat{\eta}_2 \ ... \ \hat{\eta}_{n_r}] \) is the modal coordinates vector. Employing the transformation, the equations of motion (24) are expressed in reduced order modal space as

\[ \ddot{\eta}(t) + \Lambda \dot{\eta}(t) + \Omega \eta(t) = \Psi^T \bar{P}(t) - \Psi^T \bar{K}^w \Phi, \]  

where \( \Lambda = \Psi^T C^w \Psi \) and \( \Omega = \Psi^T [K^w - \bar{K}^w(\cdot^{(1)})^{-1}K^u] \Psi \) are the diagonal matrices which are defined as

\[
\Lambda = \begin{bmatrix}
2\xi_1 \omega_1 & 0 & \cdots & 0 \\
0 & 2\xi_2 \omega_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 2\xi_{n_r} \omega_{n_r}
\end{bmatrix}, \quad \Omega = \begin{bmatrix}
\omega_1^2 & 0 & \cdots & 0 \\
0 & \omega_2^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega_{n_r}^2
\end{bmatrix}.
\]  

with \( \xi_i \) and \( \omega_i \) represent the modal damping ratio and undamped natural frequency corresponding to \( i \)th mode. The control system is designed in space state format where the second order system equation (25) is expresses in terms of first order derivative (Burl 2000) as

\[
\begin{bmatrix}
\dot{\eta}(t) \\
\ddot{\eta}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\Lambda & -\Omega
\end{bmatrix}
\begin{bmatrix}
\eta(t) \\
\dot{\eta}(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
\Psi^T \bar{P}(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
\Psi^T \bar{K}^w
\end{bmatrix} \Phi, \tag{27}
\]

and sensor output

\[
\begin{bmatrix}
\Phi_3 \\
\delta \Phi_3
\end{bmatrix} =
\begin{bmatrix}
-\bar{K}^w \Phi & 0 \\
0 & -\delta \bar{K}^w \Phi
\end{bmatrix}
\begin{bmatrix}
\eta(t) \\
\dot{\eta}(t)
\end{bmatrix}, \tag{28}
\]

where \( \delta = 1 \) for CGVF control and \( \delta = 0 \) for LQR and LQG control systems where the derivatives of the sensory potentials are not required. Defining the state vector \( x(t) = \text{col}(\eta, \dot{\eta}) \), equations (27) and (28) can be expressed in compact form as

\[
x(t) = Ax(t) + Bu(t) + Bu(t), \quad y(t) = Cx(t), \tag{29}
\]

In equation (29), \( A \) is the state system matrix, \( B_u \) and \( B_v \) are the mechanical and control input matrices, \( C \) is the output matrix, \( u_m(t) \) and \( u_e(t) \) are mechanical disturbances and electrical control input vectors respectively.

We employ output based direct and optimal feedback control algorithms to achieve active damping for vibration suppression and study their relative performance. In constant gain velocity feedback (CGVF) control system, the control input can be easily obtained by multiplying the system output with the constant values of gains as

\[
u_v(t) = -G y(t) \tag{30}
\]

with \( G = [0 \ \ G_v] \), where \( G_v \) is the velocity gain matrix. For a set of actuator-sensor pairs, it is a diagonal matrix.

The closed loop feedback system is obtained by substituting the control input from equation (30) into equation (29). This feedback modifies the state system matrix \( A \) by enhancing the damping \( \Lambda_A = \Psi^T \bar{K}^w G_v \bar{C}_w \bar{K}^w \Psi \). The stability of the system is dependent on the nature of the active damping matrix. For the dynamic system possessing the extension modes in the reduced order model, the active damping matrix \( \Lambda_A \) is negative due to stretching bending coupling in the conventionally collocated actuator sensor pairs. However, for the truly collocated actuator sensor pairs, \( \Lambda_A \) is always symmetric positive definite and yield an asymptotic stability of the closed loop deflection and control voltages.

Optimal control (LQR & LQG) laws does not show the instability of the closed loop responses if the sensor-actuator pairs are not truly collocated. In the LQR, the gain need for a full state feedback is obtained by minimizing a quadratic performance measure or cost function as
where $Q_Y$ and $R$ are the output and control input weighing matrices, which should be real symmetric positive definite. Assuming the control input for feedback as $u_{f}(t) = -RX(t)$, the optimal gain is obtained as

$$G = -R^{-1}B_{x}P X(t)$$

(32)

where $P$ is obtained by solving the following steady state matrix Ricatti equation

$$A^{T}P + PA - PB_{y}R^{-1}B_{y}^{T}P + C^{T}Q_{f}C = 0.$$

(33)

Equations (32) and (33) are the Lyapunov equations which are solved using MATLAB® function to estimate the optimal gain matrix $G$. In comparison to direct CGVF control, the LQR control scheme provides a
well-defined rule for the selection of the feedback gain matrix. It can be noted from equation (32) that LQR design require full state vector for feedback. In practical, it is not possible to derive full state vector from few sensor outputs. Also, the presence of sensor/measurement noise and system/process noise may affect the accuracy in the estimation of state vector. Thus, the full state vector is estimated from the available sensor outputs using an optimal observer or kalman-Bucy filter. The LQG control algorithm combines the LQR with an optimal observer, where the control input for the feedback is obtained using the estimated states \( \hat{X} \)

\[
\mathbf{u}_f(t) = -G\hat{X}(t)
\]

The state space format of the system equation (29) in the presence of process noise \( \mathbf{u}_w \) and sensor noise \( \mathbf{u}_v \), can be written as

\[
\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_p\mathbf{u}_f(t) + \mathbf{B}_v\mathbf{u}_v(t) + \mathbf{B}_w\mathbf{u}_w(t)
\]

\[
\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{u}_v(t)
\]

where \( \mathbf{B}_w \) is the plant noise input matrix. Substituting the control input in equation (35), the system equation for the closed-loop control can be expressed as

\[
\begin{bmatrix}
\dot{\mathbf{x}} \\
\dot{\mathbf{e}}
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{A} - \mathbf{B}_p G & \mathbf{B}_v G \\
0 & \mathbf{A} - \mathbf{L}^* \mathbf{C}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x} \\
\mathbf{e}
\end{bmatrix}
+ 
\begin{bmatrix}
\mathbf{B}_p \\
0
\end{bmatrix}
\mathbf{u}_f
+ 
\begin{bmatrix}
\mathbf{B}_w \\
\mathbf{B}_w - \mathbf{L}^* \mathbf{C}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_w \\
\mathbf{u}_v
\end{bmatrix}
\]

In equation (36) \( \mathbf{L}^* \) optimal gain matrix of the observer which minimizes \( E[\mathbf{e}^T \mathbf{e}] \) of the steady state estimation error, \( \mathbf{e} = \mathbf{X} - \hat{\mathbf{X}} \), and is obtained as

\[
\mathbf{L}^* = -\mathbf{M}^T \mathbf{V}^{-1}
\]

where \( \mathbf{M} \) is the correlation matrix of the steady state estimation error \( \mathbf{e} \) and is determined by solving the following Riccati equation:

\[
\mathbf{A}^T \mathbf{M} + \mathbf{M} \mathbf{A}^T + \mathbf{B}_p \mathbf{W} \mathbf{B}_p^T + \mathbf{M} \mathbf{C}^T \mathbf{V}^{-1} \mathbf{C} \mathbf{M} = 0
\]

with \( \mathbf{W}(t) = E[\mathbf{u}_w(t)\mathbf{u}_w^T(t)] \), \( \mathbf{V}(t) = E[\mathbf{u}_v(t)\mathbf{u}_v^T(t)] \). In this study, the control system is implemented using MATLAB®.

### Table 1. Natural frequencies (Hz) for smart simply supported FGM beam.

| Inhomogeneity index | Closed circuit condition, 1/5 PZT segment | Open circuit condition, 1 PZT segment | Open circuit condition, 5 PZT segment |
|---------------------|----------------------------------------|--------------------------------------|---------------------------------------|
| P                   | \( f_1 \) | \( f_2 \) | \( f_3 \) | \( f_4 \) | \( f_5 \) | \( f_6 \) | \( f_7 \) | \( f_8 \) |
| 0                   | 247.44  | 988.16  | 2 217.42 | 3 927.63 | 6 108.72 | 7 993.88  |
| 0.25                | 213.55  | 850.39  | 1 909.06 | 3 380.71 | 5 258.27 | 6 916.50  |
| 0.5                 | 198.84  | 794.13  | 1 782.17 | 3 156.86 | 4 908.32 | 6 382.99  |
| 1                   | 186.48  | 744.66  | 1 670.81 | 2 958.70 | 4 597.05 | 5 814.95  |
| 2                   | 178.33  | 711.94  | 1 596.69 | 2 825.64 | 4 385.43 | 5 293.18  |
| 4                   | 172.69  | 689.21  | 1 545.00 | 2 732.51 | 4 237.61 | 4 876.78  |
| \( \infty \)       | 152.80  | 609.84  | 1 367.19 | 2 418.50 | 3 753.42 | 4 235.10  |

In equation (36) \( \mathbf{L}^* \) optimal gain matrix of the observer which minimizes \( E[\mathbf{e}^T \mathbf{e}] \) of the steady state estimation error, \( \mathbf{e} = \mathbf{X} - \hat{\mathbf{X}} \), and is obtained as

\[
\mathbf{L}^* = -\mathbf{M}^T \mathbf{V}^{-1}
\]

where \( \mathbf{M} \) is the correlation matrix of the steady state estimation error \( \mathbf{e} \) and is determined by solving the following Riccati equation:

\[
\mathbf{A}^T \mathbf{M} + \mathbf{M} \mathbf{A}^T + \mathbf{B}_p \mathbf{W} \mathbf{B}_p^T + \mathbf{M} \mathbf{C}^T \mathbf{V}^{-1} \mathbf{C} \mathbf{M} = 0
\]

with \( \mathbf{W}(t) = E[\mathbf{u}_w(t)\mathbf{u}_w^T(t)] \), \( \mathbf{V}(t) = E[\mathbf{u}_v(t)\mathbf{u}_v^T(t)] \). In this study, the control system is implemented using MATLAB®.
Table 2. Natural frequencies (Hz) for smart cantilevered FGM beam.

| Inhomogeneity index | Closed circuit condition, 1/5 PZT segment |
|---------------------|------------------------------------------|
|                     | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ |
| 0                   | 88.18 | 551.77 | 1 541.30 | 3 010.05 | 4 954.34 | 7 362.71 |
| 0.25                | 76.04 | 475.17 | 1 326.93 | 2 591.62 | 4 266.50 | 6 342.14 |
| 0.5                 | 70.86 | 443.45 | 1 238.95 | 2 420.21 | 3 984.81 | 5 924.00 |
| 1                   | 66.45 | 415.85 | 1 161.71 | 2 268.93 | 3 734.92 | 5 550.76 |
| 2                   | 63.55 | 397.61 | 1 110.33 | 2 167.39 | 3 565.29 | 5 287.19 |
| 4                   | 61.55 | 384.91 | 1 074.44 | 2 095.97 | 3 444.97 | 4 881.84 |
| $\infty$            | 54.45 | 340.58 | 950.58 | 1 854.23 | 3 047.38 | 4 253.10 |

Table 3. Natural frequencies (Hz) for smart clamped-simply supported FGM beam.

| Inhomogeneity index | Closed circuit condition, 1/5 PZT segment |
|---------------------|------------------------------------------|
|                     | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ |
| 0                   | 88.53 | 552.46 | 1 541.96 | 3 010.72 | 4 955.00 | 7 363.37 |
| 0.25                | 76.30 | 475.67 | 1 327.41 | 2 592.11 | 4 266.99 | 6 342.68 |
| 0.5                 | 71.33 | 444.35 | 1 239.82 | 2 421.08 | 3 985.68 | 5 924.90 |
| 1                   | 66.96 | 416.84 | 1 162.66 | 2 269.88 | 3 735.88 | 5 551.81 |
| 2                   | 64.10 | 398.66 | 1 111.34 | 2 168.40 | 3 566.30 | 5 291.91 |
| 4                   | 62.10 | 386.02 | 1 074.44 | 2 097.01 | 3 446.01 | 4 904.50 |
| $\infty$            | 55.08 | 341.79 | 950.58 | 1 855.39 | 3 048.53 | 4 279.49 |

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Table 4. Open-loop natural frequencies (Hz) of smart FGM beams with conventionally collocated actuator/sensor pairs.

| Inhomogeneity $p$ | 1 segment of sensor/actuator pair | 5 segment of sensor/actuator pair |
|-------------------|----------------------------------|----------------------------------|
|                   | $f_1$  | $f_2$  | $f_3$  | $f_4$  | $f_5$  | $f_6$  | $\xi_1$  | $\xi_2$  | $\xi_3$  | $\xi_4$  | $\xi_5$  | $\xi_6$  |
| 0                 | 88.36  | 552.11 | 1 541.63 | 3 010.38 | 4 954.67 | 7 363.04 | 88.46  | 553.40  | 1 545.15 | 3 015.62 | 4 957.90 | 7 376.55 |
| 0.25              | 76.04  | 475.17 | 1 326.93 | 2 591.62 | 4 266.50 | 6 342.14 | 76.15  | 476.37  | 1 330.21 | 2 596.50 | 4 269.51 | 6 354.79 |
| 0.5               | 71.02  | 443.76 | 1 239.25 | 2 420.50 | 3 985.10 | 5 924.22 | 71.12  | 444.94  | 1 242.45 | 2 425.27 | 3 988.04 | 5 936.60 |
| 1                 | 66.62  | 416.16 | 1 162.00 | 2 269.22 | 3 735.21 | 5 550.92 | 66.71  | 417.33  | 1 165.18 | 2 273.95 | 3 738.12 | 5 563.26 |
| 2                 | 63.74  | 397.94 | 1 110.64 | 2 167.69 | 3 565.38 | 5 288.16 | 63.83  | 399.16  | 1 113.97 | 2 172.64 | 3 568.62 | 5 299.93 |
| 4                 | 61.74  | 385.31 | 1 074.80 | 2 096.32 | 3 445.30 | 4 892.95 | 61.85  | 386.69  | 1 078.55 | 2 101.89 | 3 448.73 | 4 895.71 |
| $\infty$         | 54.77  | 341.18 | 951.16  | 1 854.81 | 3 047.95 | 4 266.29 | 54.96  | 343.43  | 957.27  | 1 863.83 | 3 053.48 | 4 269.31 |
Table 5. Closed-loop natural frequencies (Hz) and damping ratios of smart FGM beams with conventionally collocated actuator/sensor pairs using CGVF control.

| Inhomogeneity | Frequencies $f_i$ (Hz) | Damping ratios $\xi_i$ |
|---------------|------------------------|------------------------|
| $p$           | $f_1$      | $f_2$  | $f_3$  | $f_4$  | $f_5$  | $f_6$  |
| One segment of sensor/actuator pair |
| 0            | 88.37     | 552.46 | 1542.70 | 3011.80 | 4953.72 | 7350.20 |
| 0.25         | 76.06     | 475.58 | 1328.18 | 2593.24 | 4265.23 | 6327.49 |
| 0.5          | 71.04     | 444.20 | 1240.57 | 2422.18 | 3983.55 | 5909.09 |
| 1            | 66.63     | 416.62 | 1163.37 | 2270.83 | 3732.89 | 5536.44 |
| 2            | 63.74     | 398.18 | 1110.11 | 2160.87 | 3533.97 | 5350.54 |
| 4            | 61.74     | 384.40 | 1066.02 | 2066.51 | 3404.37 | 5076.63 |
| $\infty$     | 54.77     | 340.06 | 940.72  | 1822.49 | 3099.86 | 4460.37 |
| Five segment of sensor/actuator pair |
| 0            | 88.46     | 553.68 | 1557.53 | 3283.79 | 4801.73 | 6935.44 |
| 0.25         | 76.15     | 476.69 | 1344.70 | 2909.70 | 3479.90 | 4055.49 |
| 0.5          | 71.12     | 445.28 | 1257.92 | 2756.02 | 2816.97 | 3751.18 |
| 1            | 66.71     | 417.69 | 1181.70 | 2353.13 | 2620.40 | 3481.34 |
| 2            | 63.82     | 399.41 | 1129.16 | 2508.21 | 3430.36 | 4709.15 |
| 4            | 61.82     | 386.38 | 1091.97 | 2210.70 | 3233.95 | 4909.38 |
| $\infty$     | 54.92     | 343.07 | 974.60  | 1993.33 | 2798.40 | 3928.08 |
6. Results and discussion

6.1. Validation
To illustrate the accuracy of the present FE model, we consider a cantilever layered FGM beam for which experimental values of the natural frequencies are available in the literature. The beam has five layers of equal thickness. Each layer of the beam has different concentrations of metal and ceramic phases and are assumed to be isotropic. The volume fraction of the ceramic in the individual layers vary linearly from 0% to 40%. The bottom layer has 100% aluminum concentration while the upper layer has 40% SiC and 60% aluminum concentrations. The thickness, width and effective length of the individual layers are 2 mm, 15 mm and 110 mm respectively. The material properties of aluminum and silicon carbide (SiC) are taken from (Kapuria et al. 2008):

\[ Y_m = 67 \text{ GPa}, \nu_m = 0.33 \text{ and } \rho = 2700 \text{ kg m}^{-3} \text{ and } Y_c = 302 \text{ GPa}, \nu_c = 0.17 \text{ and } \rho = 3200 \text{ kg m}^{-3} \] where symbols have their usual meaning. The effective values of elastic properties of individual layers are obtained from both MTM and ROM. Natural frequencies obtained using the present FE formulation based on MTM and ROM for first five modes are plotted in figure 2 and compared with the experimental values presented by (Kapuria et al. 2008). It is observed that the frequencies predicted by the present FE formulation employing MTM for homogenization are closely matching with the experimental predictions while the frequencies based on ROM scheme are deviating from the experimental values. Therefore, the results presented in the next section are based on MTM to maintain the accuracy.

6.2. Free vibration response of smart FGM beam
A FGM beam made of Al/SiC FG system and equipped with identical PZT G1195 layers bonded to bottom and top surfaces is considered in figure 3(a). The interfaces between the host FGM beam and PZT layers are
The properties in the FGM substrate are smoothly varying along the thickness direction according to the power law variation of the volume fraction in such a way that the top surface of the FGM substrate is purely metallic (Al) and bottom surface is purely ceramic (SiC). The properties of the PZT and the constituent of the FG system are selected as PZT-G1195N (Jiang and Batra 2005):

\[
[Y_1, Y_2, Y_3, G_{12}, G_{23}, G_{13}] = [63, 63, 63, 24.2, 24.2, 24.2] \text{ GPa},
\]

\[
[\nu_{12}, \nu_{23}, \nu_{13}] = [0.3, 0.3, 0.3]
\]

\[
[d_{31}, d_{32}, d_{33}, d_{15}, d_{24}] = [254, 254, 374, 584, 584] \times 10^{-12} \text{ mV}^{-1},
\]

\[
[\eta_{11}, \eta_{22}, \eta_{33}] = [1 728.8, 1 728.8, 1 694.9],
\]

\[
\rho = 7500 \text{ Kg m}^{-3}, \quad \epsilon_0 = 8.85 \times 10^{-12} \text{ Fm}^{-1};
\]

The length and width of the smart FGM beam are 200 mm and 10 mm respectively. The thickness of the FGM substrate is 4 mm and thickness of each PZT layers/patches is 0.5 mm. The beam has been discretized using 20 equal elements. The free surfaces of the PZT layers are divided into one as well as five equal segments. The effect of segmentation of the PZT layers has been studied for the free vibration response. First, the free surfaces of the PZT layers are short circuited making the electro-mechanical coupling ineffective. For this case, the over all effect of the electromechanical terms in the stiffness of the beam are nullified and the segmentation of the PZT layers does not produce any change in the frequencies of the free vibration. Secondly, the free surfaces of the PZT layers are in open circuit conditions. In this way, both the PZT layers are working in sensory mode and

Figure 5. First six modeshapes of smart cantilever beam for \( p = 4 \).
therefore enhances the stiffness of the smart FGM beam due to electromechanical coupling. The effect of closed-
and open-circuit conditions on the natural frequencies are shown in tables 1, 2 and 3 for simply supported,
clamped-free (cantilevered) and clamped-simply supported boundary conditions with 1 and 5 segments of PZT
sensors. It is observed that the frequencies corresponding to open circuit conditions are larger than those of the
closed circuit conditions. This happens because the stiffness of the smart beam increases during open circuit
conditions due to electromechanical coupling. Also observed from these tables, the segmentation also lead to
increase in the frequencies of open circuit conditions.

6.3. Active vibration control
Smart FGM beam considered in the previous study has been considered next to study the active control
response. For this, the PZT segments at the top surface of FGM beam are acting as actuators while those at the
bottom surface are acting as sensors. This forms the conventionally collocated actuator sensor arrangement and
has been widely considered in most studies on active vibration control. In this case too, both single segment and
multi-segment sensor and actuator arrangements have been utilized to study the effect of segmentation of PZT
surfaces or control behavior of the smart FGM beam. The control system has been designed using a reduced
order modal model considering first six modes. An initial 0.8% passive damping ratio (He et al 2001) has been
considered for each mode. First six open loop natural frequencies with one and five PZT actuator/sensor
segments are presented in table 4 for different values of the power law exponent. It can be seen that the
segmentation of the PZT layers to form multiple sensor and actuator pairs has small impact on the open loop
natural frequencies of the cantilever beam. As in the previous example of open circuit conditions, the
frequencies with 5 segment case are slightly larger.
Table 6. Open loop and closed loop natural frequencies and modal damping ratios of smart FGM beams with truly collocated sensor/actuator arrangement using CGVF control.

| Mode order | Open loop $G = 0.0$ | Closed loop $G = 1.0$ | Closed loop $G = 3.0$ |
|-------------|------------------|-----------------|------------------|
|             | $f_i$ (Hz) | $f_i$ (Hz) | $\xi_i$ | $f_i$ (Hz) | $f_i$ (Hz) | $\xi_i$ |
| $p = 4$     | $p = \infty$ | $p = 4$ | $p = \infty$ | $p = 4$ | $p = \infty$ | $p = 4$ | $p = \infty$ |
| 1           | 61.79       | 54.73       | 61.87       | 54.82       | 4.21       | 4.71       | 62.09       | 55.54       | 7.63       | 12.53 |
| 2           | 385.41      | 341.10      | 387.47      | 343.52      | 7.42       | 8.40       | 394.15      | 368.20      | 14.10      | 23.72 |
| 3           | 1 074.86    | 951.05      | 1 080.09    | 957.24      | 7.23       | 8.22       | 1 098.98    | 1 057.47    | 14.19      | 26.28 |
| 4           | 2 096.34    | 1 854.64    | 2 097.07    | 1 855.42    | 7.32       | 8.36       | 2 097.06    | 1 766.23    | 15.23      | 36.70 |
| 5           | 3 445.28    | 3 047.71    | 3 405.19    | 3 000.66    | 7.01       | 7.90       | 3 278.29    | 2 627.70    | 12.56      | 14.71 |
| 6           | 4 893.20    | 4 266.33    | 4 893.04    | 4 266.33    | 0.81       | 0.80       | 4 892.69    | 4 266.33    | 0.82       | 0.80 |

Five segments of sensor/actuator pairs

| Mode order | Sensor/actuator pairs |
|-------------|-----------------------|
| 1           | 61.95 54.90           |
| 2           | 387.22 343.12         |
| 3           | 1 079.79 956.54       |
| 4           | 2 103.64 1 862.76     |
| 5           | 3 449.75 3 052.69     |
| 6           | 4 895.75 4 269.31     |
6.4. Stability of closed-loop response in direct feedback control

In this section, we study the effect of material variation on the stability of the closed loop tip deflection and actuation potential for smart FG beam when submitted to direct feedback control such as constant gain velocity feedback.

Figure 7. Tip deflection and actuation potential for smart FGM beam with single segment of truly collocated actuator-sensor pair under step excitation using CGVF control.

Figure 8. Smart FGM beam with five segments of piezoelectric sensor-actuator pairs.
feedback (CGVF) control. The cantilever beam is subjected to a step uniform load of 5 kN/m applied on its top surface in downward direction. The beam has single segment of actuator–sensor pair. Closed-loop natural frequencies and damping ratios corresponding to the first six modes taking $G = 0.01$ are presented in Table 5. It is observed that both closed-loop frequencies and modal damping ratios are strongly effected by changing the value of the inhomogeneity index $p$. It is also observed that, for smaller values of $p$ ($< 2$), the damping ratios corresponding to all the six modes are positive ensuring a stable closed-loop response, but for larger values of $p$ ($p \geq 2$), giving a small values of gain leads to negative active damping ratio corresponding the sixth mode. This negative active damping ratio leads to the positive real part of the corresponding pole and consequently the unstable closed loop–response. The mode shapes corresponding to the first six frequencies are shown in Figures 4 and 5 for the two values of $p$ ($p = 1$ and $p = 4$). It can be seen from these plots that for $p = 1$, the magnitude of the modal displacement corresponding to inplane displacement $u_0$ is very small in comparison to transverse displacement ($w_0$) and shear rotation ($\psi_0$) modes. However, for $p = 4$, modal displacement corresponding to $u_0$ is dominating in the six mode and therefore, the inclusion of the extension mode in the reduced order model will change the over all dynamics of the closed loop response. This extension mode leads an asymmetric modal active damping matrix $\Lambda_a$ and hence an electro–mechanically asymmetric actuator–sensor pairs. The instability in the closed-loop tip deflection and actuation potential with single segment sensor–actuator pair can be seen in figure 6.

The instability observed in the closed-loop tip deflection and actuation potential due to the asymmetry of $\Lambda_a$ can be nullified by employing an electromechanically symmetric pair of sensor and actuator. This electromechanical symmetry is obtained by bonding two similar piezoelectric layers in place of single piezoelectric layer to both top and bottom of the FGM beam. A pair of symmetrically placed piezoelectric layers

Figure 9. Tip deflection and actuation potential for smart FGM beam with five segment of conventionally collocated actuator–sensor pair under step excitation using CGVF control.
together is then considered as the actuator/sensor. The gain is applied on the average of measured potential derivative at the two sensors and the actuation voltage is applied equally on the two actuators. This arrangement ensures a symmetric modal active damping matrix and yields true collocation of actuator-sensor pairs. Table 6 depicts the closed-loop frequencies and active modal damping ratios for the FGM beam with truly collocated actuator-sensor arrangement. It can be seen that this arrangement yields positive active damping ratios corresponding to all the six modes and consequently a stable closed-loop response has been obtained which is shown in figure 7.

The effect of multiple sensor and actuator pairs (figure 8) on the control performance of the CGVF control has been studies next. The closed-loop natural frequencies and active modal damping ratios for the FGM beam corresponding the first six modes of beam having five segments of actuators and sensors in conventionally and truly collocated arrangement are shown in tables 5 and 6. Table 5 shows positive active damping ratio for \( p \leq 2 \), but for larger values of gain \( (G = 0.02) \), the active damping corresponding to six mode for \( p = 2 \) becomes negative. For other larger values of \( p \), the active damping ratios are negative even for small values of gain as observed in case of single pair sensor-actuator arrangement. The truly collocated sensor-actuator arrangement yield positive damping ratios corresponding to each mode. The unstable and stable responses of tip deflection and actuation potential with five segments of conventionally and truly collocated sensor-actuator pairs are shown in figures 9 and 10.

To study the effect of excitation on the performance of the CGVF control, the beam has been excited with an impulse pressure of 10 Ns m\(^{-1}\). The closed-loop tip deflection and actuation potentials using single segment and
Figure 11. Tip deflection and actuation potential for smart FGM beam with single segment of (a) conventionally collocated (b) truly collocated actuator-sensor pair under impulse excitation using CGVF control.

Figure 12. Tip deflection and actuation potential for smart FGM beam with five segment of (a) conventionally collocated (b) truly collocated actuator-sensor pair under impulse excitation using CGVF control.
Table 7. Closed loop natural frequencies and damping ratios of smart FGM beams using LQG control.

| Inhomogeneity | Frequencies $f_i$ (Hz) | Damping ratios $\xi_i$ |
|---------------|------------------------|------------------------|
|               | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ |
|               | $\xi_1$ | $\xi_2$ | $\xi_3$ | $\xi_4$ | $\xi_5$ | $\xi_6$ |
| One segment of sensor/actuator pair |
| 0 89.02 | 552.57 | 1541.80 | 3010.45 | 4954.68 | 7363.00 | 6.03 | 2.04 | 1.03 | 0.87 | 0.82 | 0.81 |
| 0.25 76.63 | 475.57 | 1327.07 | 2591.68 | 4266.51 | 6342.10 | 6.08 | 2.06 | 1.03 | 0.87 | 0.83 | 0.81 |
| 0.5 71.56 | 443.13 | 1239.38 | 2420.56 | 3985.11 | 5924.19 | 6.07 | 2.03 | 1.03 | 0.87 | 0.82 | 0.81 |
| 1 67.12 | 416.51 | 1162.13 | 2269.27 | 3735.21 | 5580.89 | 6.05 | 2.03 | 1.03 | 0.87 | 0.82 | 0.81 |
| 2 64.16 | 398.17 | 1110.66 | 2167.63 | 3565.42 | 5288.46 | 6.08 | 2.05 | 1.03 | 0.87 | 0.82 | 0.96 |
| 4 61.96 | 385.12 | 1074.41 | 2095.78 | 3444.39 | 4899.32 | 6.18 | 2.07 | 1.03 | 0.87 | 0.82 | 3.40 |
| $\infty$ 54.96 | 341.02 | 950.81 | 1854.31 | 3047.09 | 4272.00 | 6.17 | 2.08 | 1.04 | 0.87 | 0.82 | 3.43 |
| Five segments of sensor/actuator pairs |
| 0 88.80 | 553.19 | 1549.00 | 3021.63 | 4959.71 | 7384.02 | 6.16 | 3.60 | 4.74 | 3.55 | 1.56 | 3.61 |
| 0.25 76.43 | 477.92 | 1333.53 | 2601.70 | 4271.09 | 6361.26 | 6.17 | 3.61 | 4.75 | 3.55 | 1.56 | 3.62 |
| 0.5 71.39 | 446.38 | 1245.35 | 2430.12 | 3989.31 | 5942.64 | 6.17 | 3.61 | 4.75 | 3.55 | 1.56 | 3.62 |
| 1 66.97 | 418.69 | 1168.11 | 2278.52 | 3739.51 | 5568.93 | 6.19 | 3.62 | 4.76 | 3.56 | 1.57 | 3.63 |
| 2 64.03 | 400.44 | 1116.72 | 2176.67 | 3569.67 | 5304.62 | 6.19 | 3.62 | 4.75 | 3.55 | 1.56 | 3.35 |
| 4 61.91 | 387.84 | 1081.12 | 2104.83 | 3449.00 | 4898.83 | 6.20 | 3.61 | 4.74 | 3.54 | 1.56 | 2.64 |
| $\infty$ 55.01 | 344.44 | 939.53 | 1866.41 | 3033.71 | 4272.07 | 6.19 | 3.59 | 4.72 | 3.52 | 1.53 | 2.60 |
five segments of conventionally and truly collocated sensor-actuator pairs for smart FGM beam taking \( p = 4 \) are plotted in figures 11 and 12 respectively. In this case too, the truly collocated sensor-actuator arrangement successfully removes the instability observed in the closed-loop responses using the conventionally collocated actuator/sensor arrangement.

Table 8. Open-loop and closed-loop peak tip deflection and actuation potential for smart FGM beams with single segment of sensor/actuator pair using LQG control.

| Inhomogeneity index | Peak tip deflection (mm) | Peak actuation potential (V) |
|---------------------|--------------------------|-----------------------------|
|                     | Open loop | Closed loop | \( \phi_a \) |
| \( p \)             |           |             |             |
| 0                   | 2.570     | 2.439       | 271.1       |
| 0.25                | 3.525     | 3.343       | 244.7       |
| 0.5                 | 4.086     | 3.874       | 234.4       |
| 1                   | 4.709     | 4.466       | 226.6       |
| 2                   | 5.220     | 4.956       | 229.5       |
| 4                   | 5.625     | 5.371       | 254.7       |
| \( \infty \)        | 7.269     | 6.940       | 294.5       |

Figure 13. Tip deflection and control voltage for LQG control of FGM beam under (a) step excitation (b) impulse excitation (\( p = 4 \)).
6.5. Optimal control

The electromechanical symmetry (true collocation) of the sensor and actuator pairs needed to achieve stable closed loop response in direct feedback control is not required in optimal control laws. In this study, we employ the output feedback based linear quadratic Gaussian control algorithm for the active vibration control of smart FG beam. A step load has been applied on the top surface of FG beam. The PZT layer has been divided into one and five sensor-actuator pairs. The beam is subjected to a step pressure excitation of 5 kN/m applied on the upper surface. Output weighing $Q_y$ is suitable adjusted to achieve a nondimensional settling time of 10 (number of cycles required for reducing deflection to 2% of its peak value) using one and five pairs of sensors and actuators. The control input weighing matrix is considered as $R = I$. For the LQG control, only the sensor voltage has been considered for the state estimation. The sensor noise intensity $V$ and plant noise intensity $W$ are taken as $2.5 \times 10^{-5}$ V$^2$ and 0.03 N$^2$ respectively.

The closed-loop natural frequencies and active damping ratios for smart functionally graded beam with single and five segments of PZT sensor-actuator pairs under LQG control are shown in table 7. It is observed that the active damping ratios are positive corresponding to all the six modes which establishes the stability of the closed-loop response. For achieving the same settling time, the active damping ratio corresponding to first mode

![Image of graphs showing performance analysis of CGVF, LQR and LQG control strategies for active vibration control of smart FGM under impulse excitation ($p = 0.25$).](image-url)
is nearly same for FG beam with single as well as five segments sensor-actuator pairs. However, the multiple segmentation of PZT layers to achieve multi-input multi-output (MIMO) configuration shows better controllability of higher modes as MIMO configuration yields higher active damping ratios corresponding to higher modes. Tables 8 and 9 show the variation of open- and closed loop peak tip-deflection and actuation potential with inhomogeneity index $p$ for smart FG beams with one and five segments of sensor-actuator pairs. It is observed that the peak values of tip deflection increases with increase in the value of $p$ for both single and multi-segment cases but the actuation potential does not follow the same trend. The actuation potential attains a minimum value for $p = 1$ to attain a settling time parameter of 10. Also the peak actuation potential in five segment case appeared on the actuator segment near the fixed end is much larger, but it is much smaller for other actuator segments. It is also shown that the average value of the peak actuator voltages in five segment case to achieve a desired control performance is much lower than the actuation potential required in single segment case establishing the high saving in the over all electrical energy consumed. The displacement and actuation potential time histories for FGM beam with one and five segments of conventionally collocated S/A pairs are shown in figure 13 for step and impulse excitation.

Figure 14 depicts a comparison of direct feedback (CGVF) and optimal (LQR & LQG) control algorithms to achieve a settling time factor of 10 using one and five segments of conventionally collocated S/A pairs under impulse excitation for FGM beam with $p = 0.25$. It is observed, the implementation of CGVF control algorithm is simple than the optimal control laws but require highest control effort to achieve a desired control performance. The corresponding control effort is minimum in LQG control but this method is mathematically more involved.

## 7. Conclusions

A two noded beam finite element based on an efficient layerwise theory is presented for dynamic analysis and active vibration control of functionally graded beam equiped with piezoelectric sensors and actuators. The properties in the FG core were assumed to vary according to power law variation across the thickness direction. The effective properties are obtained according to the Mori-Tanaka homogenization rule. The inertia parameters and beam stiffness coefficients are computed using six point Gauss integration scheme. The equipotential condition on the electric piezoelectric sensor/actuator surfaces is conveniently modeled using the concept of electric node.

1. The natural frequencies obtained using Mori-Tanaka scheme as homogenization rule of the FGM beam show good agreement with the experimental results available in the literature but the predictions based on widely used rule-of-mixtures show significant errors.

2. The segmentation of the sensory surfaces marginally increase the natural frequencies in open circuit condition. The segmentation does not produce any effect on the natural frequencies in closed circuit conditions.

3. The direct feedback control using CGVF control strategy employed for FGM beam having conventionally collocated sensor-actuator pairs shows instability in the closed-loop response if the power law index $p \geq 2$. The instability is conveniently nullified employing the truly collocated sensor-actuator pairs.

4. The MIMO configuration shows better control performance over SISO configuration as it require lesser electrical energy to achieve desired settling time for both CGVF and LQG control.
5. The peak control voltages to achieve a desired control performance are much smaller in LQG then LQR and CGVF control for both SISO and MIMO configurations.

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ORCID iDs

M Yaqoob Yasin  https://orcid.org/0000-0002-7891-0413

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