THE SEMI-LINEAR TORSIONAL RIGIDITY ON A COMPLETE RIEHMANNIAN TWO-MANIFOLD

JIE XIAO

ABSTRACT. This note is concerned with some essential properties (optimal isoperimetry, first variation, and monotonicity formula) of the so-called $[0, 1] \ni \gamma$-torsional rigidity $T_{\gamma, g}$ on a complete Riemannian two-manifold $(M^2, g)$. Even in the special case of $\mathbb{R}^2$, major results are new.

1. INTRODUCTION

Throughout this note, on $(M^2, g)$ — a two-dimensional manifold $M^2$ with a complete Riemannian metric $g$, we denote by
d_g(\cdot, \cdot); \langle \cdot, \cdot \rangle_g; \ | \cdot |_g; \ K_g(\cdot, \cdot); \ dA_g(\cdot); \ dL_g(\cdot); \ \Delta_g(\cdot); \ \nabla_g(\cdot),
the distance function; the inner product between two vectors in the tangent bundle; the norm of a vector; the Gauss curvature; the area element; the length element; the Laplace-Beltrami operator; the gradient, respectively. Moreover, $B_g(o, r) = \{ z \in M^2 : d_g(z, o) < r \}$ denotes the geodesic disk centered at $o$ with radius $r$, and the isoperimetric constant of $(M^2, g)$ is determined by

$$\tau_g = \inf_{O \in \mathcal{F}(M^2)} \frac{(L_g(\partial O))^2}{A_g(O)}.$$ 

When $M^2$ is the flat Euclidean plane $\mathbb{R}^2$, we naturally equip it with the standard Euclidean metric $e$ and therefore the previous notations will be changed correspondingly, i.e., $g$ is replaced by $e$. In particular, $\tau_e = 4\pi$.

For a parameter $\gamma \in [0, 1)$ and a relatively compact domain $O \subseteq M^2$ with $C^\infty$ smooth boundary $\partial O$, denoted by $O \in \mathcal{F}(M^2)$, let $u$ be the solution of the following semi-linear boundary value problem (see [18], [6], [8], [7], [4], and their related references for the Euclidean case $\mathbb{R}^2$):

$$\begin{cases}
\Delta_g u = -u^\gamma \quad \& \quad u > 0 \quad \text{in} \quad O; \\
u = 0 \quad \text{on} \quad \partial O,
\end{cases}$$

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where the second identity follows from Green’s theorem. Then the semi-linear (or $\gamma$-) torsional rigidity of $O$ as the cross section of the cylindrical beam $O \times \mathbb{R}$ is defined as

$$T_{\gamma, g}(O) = \int_O \left| \nabla_g u \right|^2_g dA_g = \int_O u^{1+\gamma} dA_g.$$  

Note that if $\gamma = 0$ then (1) is just the classical torsion problem and the resulting $0$-torsional rigidity is standard. As well-known, under $\gamma = 1$ the problem (1) has more than one non-trivial solutions, and thus the following eigenvalue problem is instead considered:

$$\begin{cases}
\Delta_g u = -\lambda u & \text{in } O; \\
u = 0 & \text{on } \partial O,
\end{cases}$$  

whose principal (or first) eigenvalue is determined through

$$\Lambda_g(O) := \inf_{v \in W^{1,2}_0(O)} \left\{ \int_O \left| \nabla_g v \right|^2_g dA_g : \int_O v^2 dA_g = 1 \right\},$$  

where $W^{1,2}_0(O)$ stands for the Sobolev space of all compactly-supported $C^\infty$ functions $v$ on $O$ with $v^2$ and $|\nabla_g v|^2$ being $dA_g$-integrable on $O$.

On the basis of Section 5— a $\gamma$-torsional rigidity Schwarz’s lemma for the conformal mappings on $\mathbb{R}^2$, we shall present some fundamental properties of $T_{\gamma, g}$ in: Section 2— the optimal isoperimetric inequality in terms of $\tau_g$; Section 3— the first variational formula arising from a domain deformation; Section 4— the monotonicity for the $\gamma$-torsional rigidity of a geodesic disk.

2. ISOPERIMETRY

Whenever $\mathbb{M}^2 = \mathbb{R}^2$, a famous problem posed by St. Venant in 1956 and settled by G. Pólya in 1948 (cf. [16, p. 121]) was to prove that among all simply connected domains of given area, a disk of the area has the largest $0$-torsional rigidity. Such an isoperimetric result can be naturally extended to the $\gamma$-torsional rigidity.

**Proposition 1.** Given $\gamma \in [0, 1)$. Let $(\mathbb{M}^2, g)$ be a complete Riemannian two-manifold with $\tau_g > 0$. If $u$ is the solution of (1) with $O \in \mathcal{F}(\mathbb{M}^2)$ being simply-connected, then

$$\int_O u^{1+\gamma} dA_g \leq \left( \frac{1 + \gamma}{2\tau_g} \right) \left( \int_0 u^\gamma dA_g \right)^2,$$

equivalently,

$$\int_O \left| \nabla_g u \right|^2_g dA_g \leq \left( \frac{1 + \gamma}{2\tau_g} \right) \left( \int_{\partial O} \left| \nabla_g u \right|^2_g dL_g \right)^2.$$

Moreover, if $\mathbb{M}^2 = \mathbb{R}^2$ and $O = B_g(o, r)$, then equality of (3) or (4) is valid.
Proof. Partially inspired by R. Sperb’s exposition in [18, pp. 190-196], we make the following argument.

Given a simply-connected domain \( O \in \mathcal{F}(\mathbb{M}^2) \). For \( 0 \leq t \leq S := \sup_{z \in O} u(z) \) let

\[
O_t = \{ z \in O : u(z) > t \}; \quad \partial O_t = \{ z \in O : u(z) = t \}; \quad a(t) = A_g(O_t).
\]

Without loss of generality, we may assume that the set of the critical points of \( u \) is finite. An application of the well-known co-area formula gives

\[
\frac{da(t)}{dt} = - \int_{\partial O_t} |\nabla_g u|^{-1}_g dL_g.
\]

Using (5), Cauchy-Schwarz’s inequality and \( \tau_g > 0 \), we find

\[
\tau_g a(t) \leq \left( L_g(\partial O_t) \right)^2 \leq \left( - \frac{da(t)}{dt} \right) \int_{\partial O_t} |\nabla_g u|^{-1}_g dL_g.
\]

For convenience, set

\[
I_\gamma(t) = \int_{O_t} u^\gamma dA_g \quad \& \quad I_{1+\gamma}(t) = \int_{O_t} u^{1+\gamma} dA_g.
\]

Then, using the layer-cake formula, the integration-by-part and (5), we get

\[
I_\gamma(t) = \int_t^S \left( \int_{\partial O_s} |\nabla_g u|^{-1}_g dL_g \right) s^\gamma ds,
\]

whence finding

\[
\frac{dI_\gamma(t)}{dt} = -t^\gamma \int_{\partial O_t} |\nabla_g u|^{-1}_g dL_g = t^\gamma \left( \frac{da(t)}{dt} \right)
\]

and so

\[
\frac{dI_\gamma(t)}{da(t)} = t^\gamma.
\]

On the other hand, an application of (6), Green’s formula, (1), and \( \tau_g > 0 \) implies

\[
I_\gamma(t) = - \int_{O_t} \Delta_g u dA_g = \int_{\partial O_t} |\nabla_g u|^{-1}_g dL_g \geq \tau_g a(t) \left( - \frac{dt}{da(t)} \right).
\]

By (7)-(8) we obtain

\[
I_\gamma(t) \left( \frac{dI_\gamma(t)}{da(t)} \right) + \tau_g t^\gamma a(t) \left( \frac{dt}{da(t)} \right) \geq 0.
\]

Now, choosing \( a = a(t) \) as an independent variable, we get \( A = a(0) \) and \( 0 = a(S) \). Then, integrating (9) over the interval \((0, A)\), taking an
integration-by-part, and using (5) once again, as well as the layer-cake formula, we achieve

\[ 0 \leq \int_0^A \left( \frac{dI}{da} \right) I_\gamma \, da + \tau_g \int_0^A t^{1+\gamma} \, da \]

\[ = 2^{-1} \int_0^A dI_\gamma^2 - \left( \frac{\tau_g}{1 + \gamma} \right) \int_0^A t^{1+\gamma} \, da \]

\[ = 2^{-1} (I_{\gamma}(0))^2 - \left( \frac{\tau_g}{1 + \gamma} \right) \int_0^S t^{1+\gamma} \left( \int_{\partial O_t} |\nabla_g u|^{-1}_g \, dL_g \right) \, dt \]

\[ = 2^{-1} (I_{\gamma}(0))^2 - \left( \frac{\tau_g}{1 + \gamma} \right) I_{1+\gamma}(0), \]

thereby finding (3) right away.

Clearly, (4) follows from (3) and

\[ \int_O u^\gamma \, dA_g = - \int_O \Delta_g u \, dA_g = - \int_{\partial O} \frac{\partial u}{\partial \nu} \, dL_g = \int_{\partial O} |\nabla_g u| \, dL_g \]

in which the Green formula has been used and \( \partial/\partial \nu \) represents the partial derivative along the unit outward normal to the boundary \( \partial O \).

The equality case of (3) or (4) under \( M^2 = \mathbb{R}^2 \) and \( O = B_e(o, r) \) (the origin-centered disk of radius \( r \)) can be verified via a direct calculation with the radial solution \( u \) (cf. [7]) to

\[ \begin{cases} 
\Delta_e u = -\kappa_\gamma u^\gamma \, \& \, u > 0 \, \text{in} \, B_e(o, r); \\
u|_{\partial B_e(o, r)} = 0 \, \text{and} \int_{B_e(o, r)} u^{1+\gamma} \, dA_e = 1, 
\end{cases} \]

where

\[ \kappa_\gamma := \inf_{v \in W^{1,2}_0(B_e(o, r))} \left\{ \int_{B_e(o, r)} |\nabla_e v|^2_e \, dA_e : \int_{B_e(o, r)} |v|^{1+\gamma}_e \, dA_e = 1 \right\}. \]

Remark 2. Under the same hypothesis on \( (M^2, g) \) as Proposition [7] we can discover two interesting facts:

(i) If \( \gamma = 0 \), \( K_g \geq 0 \), and

\[ \inf_{(o, r) \in M^2 \times (0, \infty)} \frac{2\tau_g T_{0,g}(B_g(o, r))}{(\pi r^2)^2} \geq 1 \]

which, plus the special case \( \gamma = 0 \) of (3), implies

\[ \inf_{(o, r) \in M^2 \times (0, \infty)} \frac{A_g(B_g(o, r))}{\pi r^2} \geq 1, \]

then \( M^2 \) is isometric to \( \mathbb{R}^2 \) due to E. Hebey’s explanation on [12, p. 244].
(ii) When $\gamma = 1$, the corresponding formulation of (3) (cf. [18, p. 195, (11.24)]) for $\mathbb{M}^2 = \mathbb{R}^2$ is: if $u$ denotes the Laplace-Beltrami eigenfunction associated to $\Lambda_g(O)$, then

\begin{equation}
\int_O u^2 \, dA_g \leq \frac{\Lambda_g(O)}{\tau_g} \left( \int_O u \, dA_g \right)^2,
\end{equation}

amounting to,

\begin{equation}
\int_O |\nabla u|^2 \, dA_g \leq \frac{1}{\tau_g} \left( \int_{\partial O} |\nabla_g u|_g \, dL_g \right)^2.
\end{equation}

Moreover, equality in (10) or (11) holds for $\mathbb{M}^2 = \mathbb{R}^2$ and $O = B_e(o, r)$.

3. VARIATION

Following the first variation formula of the principal eigenvalue (i.e., $\gamma = 1$) discovered in P. R. Garabedian and M. Schiffer [10] when $\mathbb{M}^2 = \mathbb{R}^2$ and in A. El Soufi and S. Ilias [9] for the general setting which was reformulated by F. Pacard and P. Sicbaldi in [14, Proposition 2.1], we can establish an extension from $\Lambda_g$ to $T_{\gamma,g}$ with $\gamma \in [0, 1)$.

**Proposition 3.** Let $\gamma \in [0, 1)$ and $(\mathbb{M}^2, g)$ be a complete Riemannian two-manifold. For a given time interval $|t| < t_0$ suppose that $O_t = \xi(t, O_0)$ is the flow on a domain $O_0 \in \mathcal{F}(\mathbb{M}^2)$ associated to the vector field $\Xi(t, z)$, i.e,

\begin{equation}
\begin{cases}
\partial_t(t, z) = \Xi(\xi(t, z)) \\
\xi(0, z) = z \in O_0.
\end{cases}
\end{equation}

If $u_t$ is the solution of (7) with $O$ replaced by $O_t$ and $\nu_t$ is the unit outward normal vector field to $\partial O_t$, then

\begin{equation}
\left. \frac{d}{dt} T_{\gamma,g}(O_t) \right|_{t=0} = \left( \frac{1 + \gamma}{1 - \gamma} \right) \int_{\partial O_0} \langle \nabla_g u_0, \nu_0 \rangle_g^2 \langle \nabla_g \Xi, \nu_0 \rangle_g \, dL_g.
\end{equation}

**Proof.** Note that $u_t(\xi(t, z)) = 0$ holds for any $z \in \partial O_0$. So, a differentiation with respect to $t = 0$ gives $\partial_t u_0|_{t=0} = -\langle \nabla_g u_0, \Xi \rangle_g$ on $\partial O_0$. Because $u_0$ vanishes on $\partial O_0$, only the normal component of $\Xi$ plays a role in the last formula. As a result, one gets

\begin{equation}
\partial_t u_0|_{t=0} = -\langle \nabla_g u_0, \nu_0 \rangle_g = \langle \Xi, \nu_0 \rangle_g \quad \text{on} \quad \partial O_0.
\end{equation}

Next, since $-\Delta_g u_t = u_t^\gamma$ holds in $O_t$, taking the partial derivative of this last equation at $t = 0$ yields

\begin{equation}
0 = \Delta_g \partial_t u_0|_{t=0} + \gamma u_0^{\gamma - 1} \partial_t u_0|_{t=0} \quad \text{in} \quad O_0.
\end{equation}
Now, an application of the definition of $\mathcal{T}_{\gamma,g}(O_t)$, (14), (15), (1) with $O_0$, and Green’s formula derives

\[ \frac{d}{dt} \mathcal{T}_{\gamma,g}(O_t) \bigg|_{t=0} = \left( \gamma + 1 \right) \int_{O_0} u^\gamma \partial_t u_0 \bigg|_{t=0} dA_g \]

\[ = \left( \frac{\gamma + 1}{\gamma - 1} \right) \int_{O_0} \left( \partial_t u_0 \bigg|_{t=0} \Delta_g u_0 - u_0 \Delta_g \partial_t u_0 \bigg|_{t=0} \right) dA_g \]

\[ = \left( \frac{1 + \gamma}{1 - \gamma} \right) \int_{\partial O_0} \langle \nabla_g u_0, \nu_0 \rangle \langle \nabla_g \Xi, \nu_0 \rangle_g dL_g. \]

Finally, (13) follows. \qed

**Remark 4.** Two comments are in order:
(i) Under $\mathbb{M}^2 = \mathbb{R}^2$ and $\gamma = 0$, an early form of (13) was established by J. Hadamard [11] (cf. [13]), but also a convex-body-based variant of (13) was stated in A. Colesanti [6, Proposition 18].
(ii) Clearly, (13) does not allow $\gamma = 1$ whose corresponding formula for the principal eigenvalue is the following: (cf. [14, Proposition 2.1]):

\[ \frac{d}{dt} \Lambda_g(O_t) \bigg|_{t=0} = -\int_{\partial O_0} \langle \nabla_g u_0, \nu_0 \rangle \langle \nabla_g \Xi, \nu_0 \rangle_g dL_g. \]

Of course, $O_t$ in (16) is generated by the solution $u_t$ of (2) with $\lambda$ replaced by $\Lambda_g(O_t)$.

4. MONOTONICITY

According to [6, p. 132], we have that if $\mathbb{M}^2 = \mathbb{R}^2$, $g = e$, and $O$ is a convex domain containing the origin in its interior, then $v_r(z) = r^{\frac{2}{1-\gamma}} u(r^{-1}z)$ solves (11) with $O$ replaced by its $r$-dilation $rO$ and hence

\[ \mathcal{T}_{\gamma,e}(rO) = \int_{rO} |\nabla_e v|^2 dA_e = r^{\frac{4}{1-\gamma}} \int_{O} |\nabla_e u|^2 dA_e = r^{\frac{4}{1-\gamma}} \mathcal{T}_{\gamma,e}(O). \]

This observation leads to the following monotonicity formula for the $\gamma$-torsional rigidity of a geodesic disk.

**Proposition 5.** Given $\gamma \in [0, 1)$. Let $(\mathbb{M}^2, g)$ be a complete Riemannian two-manifold with $K_g \geq 0$ and $\tau_g > 0$. If $o \in \mathbb{M}^2$ is fixed, then

\[ r \mapsto Q_{\gamma,g}(o, r) := \frac{\mathcal{T}_{\gamma,g}(B_{g}(o, r))}{r^{\frac{4}{1-\gamma}}} \]

is monotone increasing in $(0, \infty)$. Consequently,

\[ \lim_{r \downarrow 0} Q_{\gamma,g}(o, r) \leq Q_{\gamma,g}(o, r) \leq \lim_{r \uparrow \infty} Q_{\gamma,g}(o, r) \quad \forall \quad r \in (0, \infty) \]

holds with equalities for $\mathbb{M}^2 = \mathbb{R}^2$. 
Proof. Suppose that \( u \) is the solution of (1) with \( O = B_g(o, r) \). Since \( K_g \geq 0 \), a generalized version of the well-known Bishop-Gromov comparison theorem (cf. [15, p. 41, Theorem 2.14]) yields

\[
\frac{d}{dr} \left( r^{-1} L_g(\partial B_g(o, r)) \right) \leq 0 \quad \& \quad L_g(\partial B_g(o, r)) \leq 2\pi r.
\]

Applying \( \tau_g > 0 \), (4), Green’s formula, Cauchy-Schwarz’s inequality, and (18), we get

\[
T_{\gamma, g}(B_g(o, r)) \leq \left( \frac{1 + \gamma}{2\tau_g} \right) \left( \int_{\partial B_g(o, r)} |\nabla_g u|_g dL_g \right)^2
\]

(19)

On the other hand, consider a vector field induced by a normal shift \( \delta \nu \), counted positively in the direction of the outward normal to \( \partial B_g(o, r) \). More precisely, for \( t > -r \) and \( z \in \partial B_g(o, r) \) let \( \xi = \xi(t, z) \) be the point on the geodesic radius starting at \( o \) of \( B_g(o, r) \) with \( d_g(o, \xi) = (1 + tr^{-1})d_g(o, z) \). Consequently, if \( B_g(o, r) \) is chosen as the initial domain \( O_0 \) in Proposition 3, then

\[
\xi(0, B_g(o, r)) = O_0 \quad \& \quad \xi(t, B_g(o, r)) = O_t = B_g(o, r + t).
\]

Once setting \( \Xi(\xi(t, z)) \) be the point on the geodesic (radial) direction from \( o \) to \( \xi(t, z) \) with \( (r + t)^{-1}d_g(o, \xi) \) as its distance from \( o \), we see that (12) holds. Obviously, the unit outward normal vector to the boundary \( \partial O_0 \) at \( \xi \in \partial O_0 \) is the unit vector formed by \( \xi \) and so equal to \( \Xi(\xi) \). Suppose now that \( u \) is the solution of (1) with \( O = B_g(o, r) \). Then, an application of (13) gives

\[
\frac{d}{dr} \mathcal{T}_{\gamma, g}(B_g(o, r)) = \left( \frac{1 + \gamma}{1 - \gamma} \right) \int_{\partial B_g(o, r)} |\nabla_g u|_g dL_g.
\]

(20)

Next, we employ (19) and (20) to achieve

\[
\frac{d}{dr} Q_{\gamma, g}(r) = r \frac{d}{dr} \mathcal{T}_{\gamma, g}(B_g(o, r)) - \left( \frac{\tau_g}{\pi(1 - \gamma)} \right) \mathcal{T}_{\gamma, g}(B_g(o, r)) \geq 0,
\]

thereby reaching the desired monotonicity. Of course, the consequence part is immediate. \( \square \)
Remark 6. When $\gamma = 1$, by (16) and the foregoing proof we can establish that under the same hypothesis on $(\mathbb{M}^2, g)$ as in Proposition 5,

$$r \mapsto Q_g(o, r) := \frac{\Lambda_g(B_g(o, r))}{r^{-\frac{2\gamma}{2\gamma+2}}}$$

is monotone decreasing in $(0, \infty)$. Consequently,

$$\lim_{r \uparrow \infty} Q_g(o, r) \leq Q_g(o, r) \leq \lim_{r \downarrow 0} Q_g(o, r) \quad \forall \quad r \in (0, \infty)$$

holds with equalities for $\mathbb{M}^2 = \mathbb{R}^2$—this follows immediately from the well-known fact (see e.g. [6, p. 110]) that $\Lambda_e$ is homogeneous of order $-2$.

5. Appendix

In their 2008 paper [3], R. Burckel, D. Marshall, D. Minda, P. Poggi-Corradini and T. Ransford discovered the area-capacity-diameter versions of the following Schwarz’s lemma variant: For a holomorphic map $f$ from the origin-centered unit disk $B_e(o, 1)$ into $\mathbb{R}^2$,

$$r \mapsto \sup_{z \in B_e(o, r)} \frac{|f(z) - f(o)|_e}{r}$$

is strictly increasing in $(0, 1)$ unless $f$ is linear. Soon after, their results were extended differently by A. Y. Solynin [17], D. Betsakos [1]-[2], and J. Xiao and K. Zhu [19]. While, as a new complement to [3], T. Carroll and J. Ratzkin’s 2010 article [4] on the Schwarz type lemma for $\Lambda_e$ has partially stimulated us to carry out our current project. In contrast to the monotone-decreasing-principle (i.e., the backward Schwarz type lemma) in [4] saying that

$$r \mapsto \frac{\Lambda_e(f(B_e(o, r)))}{\Lambda_e(B_e(o, r))}$$

is strictly decreasing in $(0, 1)$ unless $f$ is a linear map, we have the forward Schwarz type lemma for the $\gamma$-torsional rigidity:

Lemma 7. Given $\gamma \in [0, 1)$. If $f$ is a conformal mapping from $B_e(o, 1)$ into $\mathbb{R}^2$, then

$$r \mapsto \Phi_{\gamma,e}(f; r) := \frac{\mathcal{T}_{\gamma,e}(f(B_e(o, r)))}{\mathcal{T}_{\gamma,e}(B_e(o, r))}$$

is strictly increasing in $(0, 1)$ unless $f$ is linear. Consequently,

$$\lim_{r \downarrow 0} \Phi_{\gamma,e}(f; r) \leq \Phi_{\gamma,e}(f; r) \leq \lim_{r \uparrow 1} \Phi_{\gamma,e}(f; r) \quad \forall \quad r \in (0, 1)$$

holds with equalities when $f$ is linear.
Proof. The argument for the monotonicity of $Q_{\gamma,e}(f; r)$ in $(0, 1)$ is similar to that proving [4, Theorem 1]. The key point is to construct a proper vector field via the given conformal map $f$. More precisely, if $g$ stands for the inverse map of $f$, then

$$\xi = \xi(t, w) = f((1 + tr^{-1})g(w)) \quad \forall \quad w \in f(B_e(o, r))$$

and

$$\Xi(\xi) = \frac{g(\xi)f'(g(\xi))}{r + t}$$

are selected for (12). Note that the unit outward normal vector to the boundary $\partial f(B_e(o, r))$ at $\xi$ is

$$\nu(\xi) = \left(\frac{g(\xi)}{r}\right) \left(\frac{f'(g(\xi))}{|f'(g(\xi))|_e}\right)$$

and so that

$$\langle \Xi, \nu \rangle_e = |f'(g(\xi))|_e \quad \forall \quad \xi \in \partial f(B_e(o, r)).$$

Next, suppose that $u_r$ is the solution of (1) with $O = f(B_e(o, r))$. Then the chain rule yields

$$|\nabla_e u_r(\xi)|_e = |\nabla_e u_r(f(z))|_e |f'(z)|_e \quad \forall \quad \xi = f(z) \in f(B_e(o, r)),$nabla_e u_r(f(z))|_e |f'(z)|_e \quad \forall \quad \xi = f(z) \in f(B_e(o, r)),$

whence giving (by Proposition 3):

$$\frac{d}{dr} T_{\gamma,e}(f(B_e(o, r))) = \left(\frac{1 + \gamma}{1 - \gamma}\right) \int_{\partial B_e(o,r)} |\nabla_e u_r|^2_e dL_e. \quad (21)$$

Meanwhile, Proposition 1 plus Cauchy-Schwarz’s inequality derives

$$T_{\gamma,e}(f(B_e(o, r))) \leq \left(\frac{1 + \gamma}{4r^{r-1}}\right) \int_{\partial B_e(o,r)} |
abla_e u_r|^2_e dL_e. \quad (22)$$

Finally, putting (17), (21) and (22) together, we get that $\frac{d}{dr} Q_{\gamma,e}(f; r) \geq 0$ holds with the strict inequality unless $f$ is linear, whence reaching the desired result. Since the consequence part is straightforward, our proof is complete. \[\square\]

Remark 8. Lemma 7 can be extended to a slightly general form: For a holomorphic map $f$ from $B_e(o,1)$ into $\mathbb{R}^2$, let $f(B_e(o,r))$ be its Riemann surface with constant Gauss curvature $-1$. Then

$$r \mapsto \frac{T_{\gamma,e}(f(B_e(o,r)))}{T_{\gamma,e}(B_e(o,r))}$$
is strictly increasing in $(0, 1)$ unless $f$ is linear. This is in contrast to [4, Corollary 2] which reads as:

$$r \mapsto \frac{\Lambda_e(f(B_e(o, r)))}{\Lambda_e(B_e(o, r))}$$

is strictly decreasing in $(0, 1)$ unless $f$ is linear.

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Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John’s, NL A1C 5S7, Canada

E-mail address: jxiao@mun.ca