INTERIORS OF CONTINUOUS IMAGES OF THE MIDDLE-THIRD CANTOR SET

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ABSTRACT. Let \( C \) be the middle-third Cantor set, and \( f \) a continuous function defined on an open set \( U \subset \mathbb{R}^2 \). Denote the image

\[
f_U(C, C) = \{ f(x, y) : (x, y) \in (C \times C) \cap U \}.
\]

If \( \partial_x f, \partial_y f \) are continuous on \( U \), and there is a point \( (x_0, y_0) \in (C \times C) \cap U \) such that

\[
1 < \left| \frac{\partial_x f(x_0, y_0)}{\partial_y f(x_0, y_0)} \right| < 3 \quad \text{or} \quad 1 < \left| \frac{\partial_y f(x_0, y_0)}{\partial_x f(x_0, y_0)} \right| < 3,
\]

then \( f_U(C, C) \) has a non-empty interior. As a consequence, if

\[
f(x, y) = x^\alpha y^\beta (\alpha \beta \neq 0), \ x^\alpha \pm y^\alpha (\alpha \neq 0) \text{ or } \sin(x) \cos(y),
\]

then \( f_U(C, C) \) contains a non-empty interior.

1. Introduction

The middle-third Cantor set, denoted by \( C \), is an elegant set in set theory. Usually, it is used to construct some counterexamples in analysis. However, there are still many open problems for this set. For instance, the multiplication on the Cantor sets [2, 17], the sections of the products of the Cantor sets [6], and so forth.

One of the main motivations of this paper is due to a result of Hochman and Shmerkin [10]: Let \( K_1 \) and \( K_2 \) be two self-similar sets with IFS’s \( \{ f_i(x) = r_i x + a_i \}_{i=1}^n \) and \( \{ g_j(x) = r_j' x + b_j \}_{j=1}^m \) respectively, if there are some \( r_i, r_j' \) such that

\[
\log |r_i|/\log |r_j'| \notin \mathbb{Q},
\]

then

\[
\dim_H(K_1 + K_2) = \min\{\dim_H(K_1) + \dim_H(K_2), 1\},
\]

where \( K_1 + K_2 = \{ x + y : x \in K_1, y \in K_2 \} \). The condition in the above result is called the irrational condition. In [17], Shmerkin stated

\[
\dim_H(K_1 \cdot K_2) = \min\{\dim_H(K_1) + \dim_H(K_2), 1\},
\]

where \( K_1 \cdot K_2 = \{ xy : x \in K_1, y \in K_2 \} \). It is natural to consider that without the irrational condition, how large \( K_1 \cdot K_2 \) is in the sense of Hausdorff dimension or Lebesgue measure.

Let \( f \) be a continuous function defined on an open set \( U \subset \mathbb{R}^2 \). Denote the image

\[
f_U(C, C) = \{ f(x, y) : (x, y) \in (C \times C) \cap U \}.
\]

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For convenience, we write $f(C, C) = f_{g_2}(C, C)$. Steinhaus [18] proved that if $f(x, y) = x - y$, then $f(C, C) = [-1, 1]$. As a result, $C + C = [0, 2]$ since $C$ is symmetric at $1/2$, i.e. $C = 1 - C$. In [2], Athreya, Reznick and Tyson considered the multiplication on $C$, and proved that $17/21 \leq \mathcal{L}(C \cdot C) \leq 8/9$, where $\mathcal{L}$ denotes the Lebesgue measure and $C \cdot C = \{xy : x, y \in C\}$. One can find more results in [7, 15, 19, 13, 12] for the arithmetic representations of real numbers.

Motivated by Athreya, Reznick and Tyson’s result, it is natural to ask whether $C \cdot C$ contains a non-empty interior. To the best of our knowledge, we, up to now, cannot find an answer to this kind of question for two general self-similar sets. In fact, whether a fractal set contains a non-empty interior is a crucial problem in fractal geometry and dynamical systems. This is the second reason why we analyze the existence of the interior. Schief [16], Bandt and Graf [3] showed the relation among the open set condition, positive Hausdorff measure and non-empty interior. Dajani et al. [4], Hare and Sidorov [9, 8] found that the existence of the non-empty interior of a class of self-affine sets is equivalent to the existence of the simultaneous expansions. In dynamical system there is a celebrated conjecture posed by Palis [14], i.e. whether it is true (at least generically) that the arithmetic sum of dynamically defined Cantor sets either has measure zero or contains a non-empty interior. This conjecture was solved in [5]. However, for the general self-similar sets this conjecture is still open.

In this paper, we prove the following result.

**Theorem 1.** If $\partial_x f, \partial_y f$ are continuous on $U$, and there is a point $(x_0, y_0) \in (C \times C) \cap U$ such that

$$1 < \frac{\partial_x f(x_0, y_0)}{\partial_y f(x_0, y_0)} < 3 \quad \text{or} \quad 1 < \frac{\partial_y f(x_0, y_0)}{\partial_x f(x_0, y_0)} < 3,$$

then $f_U(C, C)$ contains a non-empty interior.

**Corollary 1.** Let $C$ be the middle-third Cantor set. If $f(x, y)$ is one of the following functions,

$\alpha, \beta \neq 0, x^\alpha \pm y^\beta (\alpha \neq 0), x \pm y^2, \sin(x) \cos(y),$

then $f_U(C, C)$ contains a non-empty interior.

This paper is arranged as follows. In Section 2, we give a proof of Theorem 1. In Section 3, we give some remarks.

## 2. Proof of Theorem 1

The middle-third Cantor set can be generated by an iterated function system, i.e. $C$ is the unique non-empty compact set satisfying the equation:

$$C = f_1(C) \cup f_2(C),$$

where $f_1(x) = \frac{x}{3}, f_2(x) = \frac{x + 2}{3}$, see [11]. Given $J = [a, b]$, let

$$\bar{J} = [a, a + \frac{b - a}{3}] \cup [b - \frac{b - a}{3}, b].$$
Let $H = [0, 1]$. For any $(i_1, \cdots, i_n) \in \{1, 2\}^n$, we call $f_{i_1,\cdots,i_n} = f_{i_1} \circ \cdots \circ f_{i_n}(H)$ a basic interval of rank $n$, which has length $3^{-n}$. We say that $I \times J$ is a basic square of $C \times C$, if $I$ and $J$ are basic intervals of the same rank. Denote by $H_n$ the collection of all these basic intervals of rank $n$. Let $J \in H_n$, then $\tilde{J} = \bigcup_{i=1}^n I_{n+1,i}$, where $I_{n+1,i} \in H_{n+1}$ and $I_{n+1,i} \subset J$ for $i = 1, 2$. Let $[A, B] \subset [0, 1]$, where $A$ and $B$ are the left and right endpoints of some basic intervals in $H_k$ for some $k \geq 1$, respectively. $A$ and $B$ may not be in the same basic interval. Let $F_k$ be the collection of all the basic intervals in $[A, B]$ with length $3^{-k}, k \geq k_0$ for some $k_0 \in \mathbb{N}^+$, i.e. the union of all the elements of $F_k$ is denoted by $G_k = \bigcup_{i=1}^t I_{k,i}$, where $t_k \in \mathbb{N}^+$, $I_{k,i} \in H_k$ and $I_{k,i} \subset [A, B]$. Clearly, by the definition of $G_n$, it follows that $G_{n+1} \subset G_n$ for any $n \geq k_0$. Similarly, suppose that $M$ and $N$ are the left and right endpoints of some basic intervals in $H_k$. Denote by $G_k'$ the union of the all the basic intervals with length $3^{-k}$ in the interval $[M, N]$, i.e. $G_k' = \bigcup_{j=1}^{t_k'} J_{k,j}$, where $t_k' \in \mathbb{N}^+$, $J_{k,j} \in H_k$ and $J_{k,j} \subset [M, N]$.

The following Lemma comes from [2] and [19], here we give its proof just for the self-containedness of the paper.

**Lemma 1.** Let $F : U \to \mathbb{R}$ be a continuous function. Suppose $A$ and $B$ (and $N$) are the left and right endpoints of some basic intervals in $H_{k_0}$ for some $k_0 \geq 1$ respectively such that $[A, B] \times [M, N] \subset U$. Then $C \cap [A, B] = \cap_{n=k_0}^\infty G_n$, and $C \cap [M, N] = \cap_{n=k_0}^\infty G_n'$. Moreover, if for any $n \geq k_0$ and any two basic intervals $I \subset G_n$, $J \subset G_n'$ such that

$$F(I, J) = F(\tilde{I}, \tilde{J}),$$

then $F(C \cap [A, B], C \cap [M, N]) = F(G_{k_0}, G_{k_0}')$.

**Remark 1.** The lemma is also valid when we replace $C \cap [A, B]$ (or $C \cap [M, N]$) by $\cap_{n=k_0}^\infty G_n$ (or $\cap_{n=k_0}^\infty G_n'$) due to the symmetry of $C$.

**Proof.** By the construction of $G_n (G_n')$, i.e. $G_{n+1} \subset G_n (G_{n+1}' \subset G_n')$ for any $n \geq k_0$, it follows that

$$C \cap [A, B] = \cap_{n=k_0}^\infty G_n \text{ and } C \cap [M, N] = \cap_{n=k_0}^\infty G_n'.$$

The continuity of $F$ yields that $F(C \cap [A, B], C \cap [M, N]) = \cap_{n=k_0}^\infty F(G_n, G_n')$.

In terms of the relation $G_{n+1} = \tilde{G}_n$, $G_{n+1}' = \tilde{G}_n'$ and the condition in the lemma, it follows that

$$F(G_n, G_n') = \bigcup_{1 \leq i \leq t_n} \bigcup_{1 \leq j \leq t_n'} F(I_{n,i}, J_{n,j})$$

$$= \bigcup_{1 \leq i \leq t_n} \bigcup_{1 \leq j \leq t_n'} F(I_{n,i}, J_{n,j})$$

$$= F(\bigcup_{1 \leq i \leq t_n} I_{n,i}, \bigcup_{1 \leq j \leq t_n'} J_{n,j})$$

$$= F(G_{n+1}, G_{n+1}')$$

Therefore, $F(C \cap [A, B], C \cap [M, N]) = F(G_{k_0}, G_{k_0}')$. \hfill \Box

**Proposition 1.** Assume that $f : U \to \mathbb{R}$ is a function such that $\partial_x f$, $\partial_y f$ are continuous on $U$. If there is a point $(x_0, y_0) \in (C \times C) \cap U$ such that

$$\begin{cases}
\partial_x f|_{(x_0, y_0)} > 0, \\
\partial_y f|_{(x_0, y_0)} > 0
\end{cases}$$

then $f_U(C, C)$ contains a non-empty interior.
Proof. Let $E$ be a basic square of $C \times C$ containing $(x_0, y_0)$ such that the diameter of $E$ is small enough. By Lemma 1 it suffices to show that for any basic square $I \times J = [a_1, a_1+3t] \times [a_1, a_1+3t] \subset E$, we have

$$f(\bar{I}, \bar{J}) = \bigcup_{i,j = 1}^{2} f(I_i, J_j) = f(I, J),$$

where $\bar{I} = I_1 \cup I_2 = [a_1, a_1+t] \cup [a_2, a_2+t]$ and $\bar{J} = J_1 \cup J_2 = [b_1, b_1+t] \cup [b_2, b_2+t]$.

![Figure 1](image1.png)

Note that $f$ is differentiable since $\partial_x f$, $\partial_y f$ are continuous. Since $t$ is small enough, when $(x, y) \to (x_0, y_0)$, by the above conditions we obtain that

- $H_1(x, y) = f(x + 3t, y + 3t) - f(x, y) = 3(\partial_x f + \partial_y f)t + o(t) > 0$,
- $H_2(x, y) = f(x + t, y + t) - f(x, y) = (\partial_x f + \partial_y f)t + o(t) > 0$,
- $H_3(x, y) = f(x + y + 3t) - f(x + t, y) = (3\partial_y f - \partial_x f)t + o(t) > 0$,
- $H_4(x, y) = f(x + t, y + t) - f(x, y + t) = (\partial_x f - \partial_y f)t + o(t) > 0$,

where $o(t)/t \to 0$ uniformly as $t \to 0$, i.e. $o(t)$ is independent of the choice of $(x, y)$ as $\partial_x f$ and $\partial_y f$ are continuous. Using $\partial_x f, \partial_y f > 0$, $H_1(x, y) > 0$ and $H_2(x, y) > 0$ we have

$$f(I, J) = [f(a_1, b_1), f(a_1 + 3t, b_1 + 3t)],$$

$$f(I_i, J_j) = [f(a_i, b_j), f(a_i + t, b_j + t)]$$

for all $i, j$, and

$$f(I_1, J_1) \cap f(I_1, J_2) \neq \emptyset$$ since $f(a_1 + t, b_1 + t) - f(a_1, b_2) = H_4(a_1, b_1 + t) > 0$,

$$f(I_1, J_2) \cap f(I_2, J_1) \neq \emptyset$$ since $f(a_1 + t, b_2 + t) - f(a_2, b_1) = H_3(a_1 + t, b_1) > 0$,

$$f(I_2, J_1) \cap f(I_2, J_2) \neq \emptyset$$ since $f(a_2 + t, b_1 + t) - f(a_2, b_2) = H_4(a_2, b_1 + t) > 0$.

Therefore we obtain $f(\bar{I}, \bar{J}) = f(I, J)$. The proposition follows from Lemma 1. \qed

Proof of Theorem 1.

1 Case 1: Suppose $\partial_x f|_{(x_0, y_0)} > 0$ and $\partial_y f|_{(x_0, y_0)} > 0$. If

$$\partial_x f|_{(x_0, y_0)} < \partial_y f|_{(x_0, y_0)} < 3\partial_x f|_{(x_0, y_0)},$$

we replace $f(x, y)$ by $g_1(x, y) = f(y, x)$, then Theorem 1 follows from Proposition 1 in this case.
(2) Case 2: Suppose \( \partial_x f \mid (x_0, y_0) < 0 \) and \( \partial_y f \mid (x_0, y_0) < 0 \). In this case, we can replace \( f(x, y) \) by \( g_2(x, y) = -f(x, y) \) or \( g_3(x, y) = -f(y, x) \).

(3) Case 3: Suppose \( \partial_x f \mid (x_0, y_0) < 0 \) and \( \partial_y f \mid (x_0, y_0) > 0 \). In this case, we can replace \( f(x, y) \) by \( g_4(x, y) = f(-x, y) \), we obtain \( \partial_x g_4 \mid (x_0, y_0) > 0 \) and \( \partial_y g_4 \mid (x_0, y_0) > 0 \). By the symmetry of \( C \) and \((−C)\), applying Lemma \( \ref{lem:interior} \) to \((−C) \times C\), Theorem \( \ref{thm:interior} \) follows from Proposition \( \ref{prop:interior} \).

(4) Case 4: Suppose \( \partial_x f \mid (x_0, y_0) > 0 \) and \( \partial_y f \mid (x_0, y_0) < 0 \). We can replace \( f(x, y) \) by \( g_5(x, y) = f(x, -y) \) in this case. \( \square \)

Proof of Corollary \( \ref{cor:interior} \) It suffices to check the conditions in Theorem \( \ref{thm:interior} \).

(1) If \( f(x, y) = x^\alpha y^\beta \) with \( \alpha \beta \neq 0 \), using \( (x^\alpha y^\beta) = (xy^{\beta/\alpha})^\alpha \), we only need to deal with \( f(x, y) = xy \). Using the symmetry, we may assume that \(|\gamma| \geq 1\). Now we have

\[
\partial_x f = y^\gamma \quad \text{and} \quad \partial_y f = \gamma xy^{\gamma - 1}
\]

with \( \left| \frac{\partial f}{\partial y} \right| = \frac{1}{|\gamma|} |x| \). If \(|\gamma| = 3^k\) for some integer \( k \geq 0 \), we take \( y = 1 \) and \( x = (2/3) \cdot 3^{-k} \), hence \( \left| \frac{\partial f}{\partial y} \right| = 3/2 \in (1, 3) \) in this case. Otherwise, if \( 3^k < |\gamma| < 3^{k+1} \) for some integer \( k \geq 0 \), then we take \( y = 1 \) and \( x = 3^{-(k+1)} \), then \( \left| \frac{\partial f}{\partial y} \right| \in (1, 3) \). Now, \( f_U(C, C) \) contains a non-empty interior for \( f(x, y) = x^\alpha y^\beta \) with \( \alpha \beta \neq 0 \).

(2) If \( f(x, y) = x^\alpha \pm y^\beta \) with \( \alpha \neq 0 \), then

\[
|\partial_x f| = |\alpha| x^{\alpha - 1} \quad \text{and} \quad |\partial_y f| = |\alpha| y^{\beta - 1}
\]

with \( \left| \frac{\partial f}{\partial y} \right| = \left| \frac{x^{\alpha - 1}}{y^{\beta - 1}} \right| \). When \( \alpha \neq 1 \), take \( x, y \in C \) such that \( y/x \) is close to 1 enough, then \( 1 < \left| \frac{\partial f}{\partial y} \right| < 3 \) or \( 1 < \left| \frac{\partial f}{\partial y} \right| < 3 \). When \( \alpha = 1 \), the classical result \( C + C = [0, 2] \) implies there is a non-empty interior in \( f(C, C) = C + C \).

(3) If \( f(x, y) = x \pm y^2 \), then \( \partial_x f = 1, |\partial_y f| = 2y \). Take \( x_0 = 8/9, y_0 = 1/3 \), which implies \( 1 < |1/(2y_0)| < 3 \).

(4) If \( f(x, y) = \sin(x) \cos(y) \), then

\[
|\partial_x f| = |\cos x \cos y|, |\partial_y f| = |\sin x \sin y|
\]

We take \((x_0, y_0) = (2/3, 2/3)\), we obtain that

\[
|\cos(2/3) \cos(2/3)| = 0.6176 \cdots, \quad |\sin(2/3) \sin(2/3)| = 0.3823 \cdots
\]

and thus \( 1 < \left| \frac{\cos(2/3) \cos(2/3)}{\sin(2/3) \sin(2/3)} \right| = 1.615 \cdots < 3 \). \( \square \)

3. Final remarks

Our idea can be implemented for some overlapping self-similar sets. Moreover, in Theorem \( \ref{thm:interior} \) for some functions, we can obtain that \( f_U(C, C) \) contains infinitely many closed intervals.
References

[1] Steve Astels. Cantor sets and numbers with restricted partial quotients. *Trans. Amer. Math. Soc.*, 352(1):133–170, 2000.

[2] Jayadev S. Athreya, Bruce Reznick, and Jeremy T. Tyson. Cantor set arithmetic. *To appear in American Mathematical Monthly*, 2018.

[3] Christoph Bandt and Siegfried Graf. Self-similar sets. VII. A characterization of self-similar fractals with positive Hausdorff measure. *Proc. Amer. Math. Soc.*, 114(4):995–1001, 1992.

[4] Karma Dajani, Kan Jiang, and Tom Kempton. Self-affine sets with positive Lebesgue measure. *Indag. Math. (N.S.)*, 25(4):774–784, 2014.

[5] Carlos Gustavo T. de A. Moreira and Jean-Christophe Yoccoz. Stable intersections of regular Cantor sets with large Hausdorff dimensions. *Ann. of Math. (2)*, 154(1):45–96, 2001.

[6] Kenneth Falconer. *Fractal geometry*. John Wiley & Sons, Ltd., Chichester, 1990. Mathematical foundations and applications.

[7] Marshall Hall, Jr. On the sum and product of continued fractions. *Ann. of Math. (2)*, 48:966–993, 1947.

[8] Kevin G. Hare and Nikita Sidorov. Two-dimensional self-affine sets with interior points, and the set of uniqueness. *Nonlinearity*, 29(1):1–26, 2016.

[9] Kevin G. Hare and Nikita Sidorov. On a family of self-affine sets: topology, uniqueness, simultaneous expansions. *Ergodic Theory Dynam. Systems*, 37(1):193–227, 2017.

[10] Michael Hochman and Pablo Shmerkin. Local entropy averages and projections of fractal measures. *Ann. of Math. (2)*, 175(3):1001–1059, 2012.

[11] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[12] Kan Jiang. Hausdorff dimension of the arithmetic sum of self-similar sets. *Indag. Math. (N.S.)*, 27(3):684–701, 2016.

[13] Kan Jiang and Lifeng Xi. Arithmetic representations of real numbers in terms of self-similar sets. *arXiv:1808.09722*, 2018.

[14] Jacob Palis and Floris Takens. *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, volume 35 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993. Fractal dimensions and infinitely many attractors.

[15] Yuval Peres and Pablo Shmerkin. Resonance between Cantor sets. *Ergodic Theory Dynam. Systems*, 29(1):201–221, 2009.

[16] Andreas Schief. Separation properties for self-similar sets. *Proc. Amer. Math. Soc.*, 122(1):111–115, 1994.

[17] Pablo Shmerkin, https://mathoverflow.net/questions/132445/arithmetic-products-of-cantor-sets. 2013.

[18] Hugo Steinhaus. *Mowa W/suppress lasno/´s´ c Mnogo´sci Cantora*. Wector, 1-3. English translation in: STENIHAUS, H.D. 1985.

[19] Li Tian, Jiaweng Gu, Qianqian Ye, Li-Feng Xi, and Kan Jiang. Multiplication on self-similar sets with overlaps. *arXiv:1807.05368*, 2018.

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