On Integer Programming With Bounded Determinants
(extended version)

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Abstract Let $A$ be an $(m \times n)$ integral matrix, and let $P = \{ x : Ax \leq b \}$ be an $n$-dimensional polytope. The width of $P$ is defined as $w(P) = \min \{ x \in \mathbb{Z}^n \setminus \{ 0 \} : \max_{x \in P} x^\top u - \min_{x \in P} x^\top v \}$. Let $\Delta(A)$ and $\delta(A)$ denote the greatest and the smallest absolute values of a determinant among all $r(A) \times r(A)$ sub-matrices of $A$, where $r(A)$ is the rank of a matrix $A$.

We prove that if every $r(A) \times r(A)$ sub-matrix of $A$ has a determinant equal to $\pm \Delta(A)$ or 0 and $w(P) \geq (\Delta(A) - 1)(n + 1)$, then $P$ contains $n$ affine independent integer points. Also we have similar results for the case of $k$-modular matrices. The matrix $A$ is called totally $k$-modular if every square sub-matrix of $A$ has a determinant in the set $\{ 0, \pm k^r : r \in \mathbb{N} \}$.

When $P$ is a simplex and $w(P) \geq \delta(A) - 1$, we describe a polynomial time algorithm for finding an integer point in $P$.

Finally we show that if $A$ is almost unimodular, then integer program $\max \{ c^\top x : x \in P \cap \mathbb{Z}^n \}$ can be solved in polynomial time. The matrix $A$ is called almost unimodular if $\Delta(A) \leq 2$ and any $(r(A) - 1) \times (r(A) - 1)$ sub-matrix has a determinant from the set $\{ 0, \pm 1 \}$.

Keywords Empty Lattice · Simplex · Polytope · Integer Programming · Feasibility Problem · Flatness Theorem · Group Minimization

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1 Introduction

Let $A$ be an $m \times n$ integral matrix. Its $ij$-th element is denoted by $A_{ij}$, $A_{i*}$ is the $i$-th row of $A$, and $A_{*j}$ is the $j$-th column of $A$. For a vector $b \in \mathbb{Z}^n$, by $P(A, b)$ (or by $P$) we denote the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$. The set of all vertices of a polyhedron $P$ is denoted by $\text{vert}(P)$.

Let $r(A)$ be the rank of a matrix $A$. Let $\Delta(A)$ and $\delta(A)$ denote the greatest and the smallest absolute values of the determinant among all $r(A) \times r(A)$ sub-matrices of $A$. Let $\Delta_{\text{cm}}(A)$ and $\Delta_{\text{gcd}}(A)$ be the least common multiple and greatest common divisor (resp.) of absolute values of a determinant among all $r(A) \times r(A)$ sub-matrices of $A$.

Now we refer to the notion of the $k$-modular matrices that has been introduced in [21]. The matrix $A$ is called totally $k$-modular if every square sub-matrix of $A$ has a determinant in the set $\{0, \pm k^r : r \in \mathbb{N}\}$. The matrix $A$ is called $k$-modular if every $r(A) \times r(A)$ sub-matrix of $A$ has a determinant in the set $\{0, \pm k^r : r \in \mathbb{N}\}$.

Also we refer to the notion of almost unimodular matrices that was introduced in [6] for square case. The matrix $A$ is called almost unimodular if $\Delta(A) \leq 2$ and any $(r(A) - 1) \times (r(A) - 1)$ sub-matrix has a determinant from the set $\{0, \pm 1\}$.

The following theorem was proved in [35].

**Theorem 1** If every $n \times n$ determinant of $A$ belongs to $\{-2,-1,0,1,2\}$ and $P(A, b)$ is full-dimensional, then
1. $P(A, b) \cap \mathbb{Z}^n \neq \emptyset$.
2. One can check the emptiness of the set $P(A, b) \cap \mathbb{Z}^n$ in polynomial time.
3. For every row $a$ of $A$, the problem $\max\{a^T x : Ax \leq b, x \in \mathbb{Z}^n\}$ can be solved in polynomial time.
4. For every $v \in \text{vert}(P(A, b) \cap \mathbb{Z}^n)$, there exists $u \in \text{vert}(P)$, such that $v$ lies on some edge which contain $u$ ($v$ lies on an edge of $N(u)$).
5. For every $v \in \text{vert}(P)$, $\text{max}\{c^T x : Ax \leq b, x \in \mathbb{Z}^n\}$ can be solved in polynomial time.

The remarkable result was obtained by V.E. Alekseev and D.V. Zakharova in [1] for $\{0,1\}$-matrices.

**Theorem 2** Let $A \in \{0, 1\}^{m \times n}$, $b \in \{0, 1\}^m$, $c \in \{0, 1\}^n$. Let all rows of $A$ have at most 2 ones. Then, the problem $\max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n\}$ can be solved in polynomial time when $\Delta\left(\begin{smallmatrix} c \\ A \end{smallmatrix}\right)$ is fixed.

Width of $P$ is defined as $w(P) = \min\{x \in \mathbb{Z}^n \setminus \{0\} : \max_{x \in P} x^T u - \min_{x \in P} x^T v\}$. 
Now we refer to the classical flatness theorem due to Khinchine [20]. Let $P$ be a convex body. Khinchine shows that if $P \cap \mathbb{Z}^n = \emptyset$, then $w(P) \leq f(n)$, where $f(n)$ is a value that depends only on a dimension. There are many estimates on $f(n)$ in the works [4, 5, 7, 17, 20, 31]. There is a conjecture claiming that $f(n) = O(\sqrt{n})$ [7, 17]. The best known upper bound on $f(n)$ is $O(n^{4/3} \log^c(n))$ due to Rudelson [31], where $c$ is some constant that does not depend on $n$.

The paper [11] contains an estimate of the width for a special class of polytopes.

**Theorem 3** Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $P(A, b)$ is a polytope and every $r(A) \times r(A)$ sub-determinant of matrix $A$ is equal to $\pm \Delta(A)$ or 0. If $w(P(A, b)) > (\Delta(A) - 1)(n + 1)$, then $|P(A, b) \cap \mathbb{Z}^n| \geq n + 1$. Moreover we can find an integer point in $P(A, b) \cap \mathbb{Z}^n$ using a polynomial time algorithm.

We give here another proof of this result.

An interesting problem is estimating of $f(n)$ for empty lattice simplices [1, 4, 13, 18, 32]. A simplex $S$ is called empty lattice if $\text{vert}(S) \subseteq \mathbb{Z}^n$ and $S \cap \mathbb{Z}^n \setminus \text{vert}(S) = \emptyset$. Best known estimate of $f(n)$ for the empty lattice simplices is $O(n \log(n))$ due to [4].

In this paper we will prove that the width of a simplex (not necessary with integer vertices) without lattice points is at most $\delta(A) - 1$, where $A$ is the restriction matrix of the simplex. Moreover, if its width is at least $\delta(A) - 1$, then we can find an integer point in the simplex by a polynomial-time algorithm presented in this paper.

The authors consider this paper as a part of general problem for finding out critical values of parameters, when a given problem changes complexity. For example, the integer programming problem is polynomial time solvable on polyhedrons with integer vertices, due to [19]. On the other hand, it is NP-complete in the class of polyhedrons with denominators of extreme points equal 1 or 2, due [30]. The famous $k$-satisfiability problem is polynomial for $k \leq 2$ but is NP-complete for all $k > 2$. In the papers [25, 26] some graph parameters (the density and packing number) were considered and it was described how its growth in terms of the number of vertices affects on the complexity of the independent set problem. A theory, when an NP-complete graph problem becomes easier, is investignted applying to the family of hereditary classes in the papers [2, 3, 22, 23, 24, 25, 26]. Our main interest is to determine a dependence of the integer programming problem complexity on spectrum of sub-determinants of the restriction matrix.

2 The polytopes with bounded determinants

The main result of this section is
Lemma 2 If \( w(P) > (\Delta(A) - 1) \frac{\Delta(A)}{\delta(A)}(n + 1) \), then \(|P(A, b) \cap \mathbb{Z}^n| \geq n + 1\). We can find an integer point in \( P(A, b) \cap \mathbb{Z}^n \) using a polynomial time algorithm.

This theorem has two trivial corollaries:

**Corollary 1** Let \( A \in \mathbb{Z}^{m \times n} \) be a \( k \)-modular matrix, \( b \in \mathbb{Z}^m \) and \( P = P(A, b) \) is a polytope.

If \( w(P) > (\Delta(A) - 1) \frac{\Delta(A)}{\delta(A)}(n + 1) \), then \(|P(A, b) \cap \mathbb{Z}^n| \geq n + 1\). We can find an integer point in \( P(A, b) \cap \mathbb{Z}^n \) using a polynomial time algorithm.

**Corollary 2** Let \( A \in \mathbb{Z}^{m \times n} \), \( b \in \mathbb{Z}^m \), \( P = P(A, b) \) is a polytope and each \( r(A) \times r(A) \) sub-determinant of \( A \) is equal to \( \pm \Delta(A) \) or 0.

If \( w(P) > (\Delta(A) - 1)(n + 1) \), then \(|P(A, b) \cap \mathbb{Z}^n| \geq n + 1\). We can find an integer point in \( P(A, b) \cap \mathbb{Z}^n \) using a polynomial time algorithm.

We need the following lemmas to prove the theorem.

**Lemma 1** Let \( B \in \mathbb{Z}^{n \times s} \), \( M = \text{conv}(B) \cap \mathbb{Z}^n \setminus \{0\} \), and \( c^\top x^* = \min_{x \in MC^\top} x \), then \( x^* \in \text{conv}(0B) \).

Proof Suppose, that \( c^\top B_{*1} = \min_{i} c^\top B_{i1} \) and \( x^* = Bt \notin \text{conv}(0, b_1, ..., b_s) \), then \( t_1 + \cdots + t_s > 1 \). Therefore \( c^\top x^* = c^\top Bt \geq (t_1 + \cdots + t_s)c^\top B_{*1} > c^\top B_{*1} \), this is a contradiction.

**Lemma 2** Let \( z' = b^\top x' = \min\{b^\top x : x \in M\} \), \( z'' = b'^\top x'' = \min\{b'^\top x : x \in M\} \), then \( |z' - z''| \leq ||b' - b''||_\infty ||x''||_1 \).

Proof Let \( z' \geq z'' \), then \( z' - z'' \leq b'^\top x'' - b'^\top x'' \leq ||b' - b''||_\infty ||x''||_1 \).

**Lemma 3** Let \( P' = P(A, b') \) and \( P'' = P(A, b'') \) are nonempty polytopes, then \(|w(P') - w(P'')| \leq \frac{\Delta(A)}{\Delta_{gcd}(A)}(n + 1)||b' - b''||_\infty \).

Proof \( w(P') = \min\{\max_{P'} c^\top x - \min_{P'} c^\top x : c \in \mathbb{Z}^n \setminus \{0\}\} \). According to the duality theorem of the linear programming

\[
\max_{P'} c^\top x = \min\{b^\top y : A^\top y = c, \ y \geq 0\}
\]

\[
\min_{P'} c^\top x = -\min\{b'^\top z : -A^\top z = c, \ z \geq 0\}.
\]

Therefore,

\[
w(P') = \min\{b^\top y + b'^\top z : A^\top y + A^\top z = 0, \ y \geq 0, \ z \geq 0, \ A^\top y \in \mathbb{Z}^n \setminus \{0\}\}.
\]

Let's consider the cone \( C = \{y \in R^{2n} : A^\top y + A^\top z = 0, \ (y, z) \geq 0\} \). From the equation \( A^\top y + A^\top z = 0 \) and the Kramer’s rule it follows that \( C \) is generated by the vectors with components in the set of all \( r(A) \times r(A) \) sub-determinants.
of \( A \). Therefore, the maximal absolute value of coordinates of these vectors is at most \( \Delta(A) \). This value can be decreased by a dividing each of components by \( \Delta_{gcd}(A) \). According to Carateodory’s theorem, every vector has at most \( n + 1 \) non-zero coordinates. So, by the lemma [1] there are \( y', z' \) such that \( w(P') = b'^\top y' + b'^\top z' \) and \( \sum_{i=1}^{m} (y'_i + z'_i) \leq \frac{\Delta(A)}{\Delta_{gcd}(A)} (n + 1) \). Similarly we can show that \( w(P'' \mid P) = b''^\top y'' + b''^\top z'' \) and \( \sum_{i=1}^{m} (y''_i + z''_i) \leq \frac{\Delta(A)}{\Delta_{gcd}(A)} (n + 1) \) for some \( y'', z'' \).

Suppose, for clearness, that \( w(P') \geq w(P'') \). We have
\[
|w(P') - w(P'')| = (b'^\top y' + b'^\top z') - (b''^\top y'' + b''^\top z'') \leq (b'^\top y'' + b'^\top z'') - (b''^\top y'' + b''^\top z'') = (b' - b'')^\top y'' + (b' - b'')^\top z'' \leq ||b' - b''||_\infty \| (y'', z'') \|_1 \leq \frac{\Delta(A)}{\Delta_{gcd}(A)} (n + 1) ||b' - b''||_\infty.
\]

Now, we are ready to prove the main result (theorem [3]) of this section.

**Proof** Let \( b' \in \mathbb{Z}^m \), such that \( b'_i = b_i - (b_i \mod \Delta_{gcd}(A)) \) and \( P' = P(A, b') \).

If \( w(P) > (\Delta_{gcd}(A) - 1) \frac{\Delta(A)}{\Delta_{gcd}(A)} (n + 1) \), then by the lemma [3] \( w(P') > 0 \).

Thus, \( P' \) is full-dimensional and each component of \( b' \) is divided by \( \Delta_{gcd}(A) \).

So, it is easy to see, that \( P' \) is a full dimensional polytope and all components of any vertex of \( P' \) are integer. Now the theorem follows from the fact that \( P' \subseteq P \).

We can use any polynomial algorithm of linear programming (Khachiyan’s algorithm [15,19]) to find some vertex of \( P' \) as an integer point of \( P \).

### 3 The simplices with bounded determinants

A part of this section describes results of the R. E. Gomory [8,10,9,14]. We will repeat some of the Gomory’s arguments, slightly modifying them for our purposes.

Let \( A \in \mathbb{Z}^{n \times n} \), \( B \in \mathbb{Z}^{n \times s} \), \( b \in \mathbb{Z}^s \), and \( |\det(A)| = \Delta > 0 \). Consider the system

\[
\begin{align*}
Ax + By &= b \\
x &\in \mathbb{Z}^n, \ y \in \mathbb{Z}^s^+.
\end{align*}
\]

Let \( D \) be the Smith normal form [34] of the matrix \( A \), then \( A = P^{-1} D Q^{-1} \), where \( P^{-1} \), \( Q^{-1} \) are integer unimodular matrices. So system (1) becomes

\[
\begin{align*}
DQ^{-1}x + PB y &= Pb \\
x &\in \mathbb{Z}^n, \ y \in \mathbb{Z}^s^+.
\end{align*}
\]
After the unimodular map $Q^{-1}x \rightarrow x$ and removing $x$’s variables the system becomes

\[
\begin{align*}
PB y & \equiv Pb \pmod{D} \\
y & \in \mathbb{Z}_+^s.
\end{align*}
\] (2)

There is a bijection between variables $x$ and $y$ giving by the formula $x = A^{-1}(b - By)$.

Let $M(A, B, b)$ be a polyhedron induced by the system (2).

**Definition 1 (Gomory)** Let $y \in M(A, B, b)$. Hence $PB y \equiv Pb \pmod{D}$.

We say that $y$ is an irreducible point of $M(A, B, b)$ if for any $u \neq v$ such that $u \leq y$ and $v \leq y$ we have $PB u \neq PB v \pmod{D}$.

**Theorem 5 (Gomory)** Columns of the matrix $PB$ induce an additive group, the group operation is an addition by a modulo $D$. The power of this group is at most $\Delta$. If $M(A, B, b) \neq \emptyset$ and $y$ is an irreducible point of $M(A, B, b)$, then $\prod_{k=1}^s (1 + y_k) \leq \Delta$.

**Proof** It is easy to see that columns of $PB$ induce an additive group. Let $g$ be element of this group, so $g = PB t \mod D$ for some $t \in \mathbb{Z}_+^s$. Hence $0 \leq g_k \leq D_k - 1$, where $1 \leq k \leq n$. So a total number of group elements is at most $\prod_{k=1}^n D_k = \Delta$.

Let $y$ be an irreducible point of $M(A, B, b)$, and $t \leq y$. From the definition of the point $y$ it follows that all group elements $g = PB t \mod \Delta$ are different for different $t$, so different combinations of $t$ induce different elements of a group. Since the total number of $t$ combinations is $\prod_{k=1}^s (1 + y_k)$ and the number of distinct group elements is at most $\Delta$ then we have $\prod_{k=1}^s (1 + y_k) \leq \Delta$.

**Theorem 6 (Gomory)** Let $y \in \text{vert}(M(A, B, b))$, then $y$ is an irreducible point of $M(A, B, b)$.

**Corollary 3 (Gomory)** Let $y \in \text{vert}(M(A, B, b))$, then $\prod_{k=1}^s (1 + y_k) \leq \Delta$.

**Lemma 4** Let $I$ be an identity matrix. There is polynomial time algorithm to find the point $y \in M(A, I, b)$ with properties $\sum_{k=1}^s y_k \leq \Delta - 1$ and $\prod_{y_k > 0} y_k \leq \Delta/2^{s-1} - 1$.

**Proof** The system for $M(A, I, b)$ has very simple structure:

\[
\begin{align*}
P y & \equiv Pb \pmod{D} \\
y & \in \mathbb{Z}_+^s.
\end{align*}
\]
So \( y = b + P^{-1}Dt \), for any \( t \in \mathbb{Z}^s \). We can assume that the matrix \( P^{-1} \in \mathbb{Z}^{n \times s} \) has the following form:

\[
P^{-1} = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
p_{21} & 1 & 0 & \ldots & 0 \\
p_{31} & p_{32} & 1 & \ldots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
p_{s1} & p_{s2} & p_{s3} & \ldots & 1
\end{pmatrix}.
\]

Hence \( y_1 = b_1 + D_{11}t_1 \) and \( y_k = b_k + (p_{k1}D_{11}t_1 + p_{k2}D_{22}t_2 + \cdots + D_{kk}t_k) = b_k + D_{kk}t_k + \sum_{j=1}^{k-1} p_{kj}D_{jj}t_j \) for \( 1 < k \leq s \). Then we can put \( y_1 = b_1 \mod D_{11} \) and \( y_k = (b_k + \sum_{j=1}^{k-1} p_{kj}D_{jj}t_j) \mod D_{kk} \) for \( 1 < k \leq s \).

Hence we have vector \( y \in M(A, I, b) \) with the property \( \prod_{k=1}^{s} y_k \leq \prod_{k=1}^{s} (D_{kk} - 1) \). We know that \( \prod_{k=1}^{s} D_{kk} = \Delta \) and a maximum of the sum \( \sum_{k=1}^{s} D_{kk} \), that equals to \( \Delta + s - 1 \) is reached when \( D_{kk} = 1 \) for \( k \in [1, s - 1] \) and \( D_{ss} = \Delta \). Hence the maximum of the sum \( \sum_{k=1}^{s} (D_{kk} - 1) \) is \( \Delta - 1 \).

The maximum of the product \( \prod_{k=1}^{s} (D_{kk} - 1) \) is reached when \( D_{kk} = 2 \) for \( k \in [1, s - 1] \) and \( D_{ss} = \Delta/2^{s-1} \), so we have \( \prod_{k=1}^{s} (D_{kk} - 1) \leq \Delta/2^{s-1} - 1 \). The additional constraints on \( D_{kk} \) to be diagonal coefficients of the Smith normal form can only decrease an estimate. Next we trivially have \( \prod_{y_k > 0} y_k \leq \prod_{k=1}^{s} (D_{kk} - 1) \).

Moreover finding the Smith normal form \( D \) is a polynomial problem \([34]\).

Now we can prove the main result of this section.

**Theorem 7** Let \( A \in \mathbb{Z}^{(n+1) \times n} \), \( b \in \mathbb{Z}^{n+1} \), \( P = P(A, b) \) be a simplex. If \( w(P(A, b)) \geq \delta(A) - 1 \), then \( P(A, b) \cap \mathbb{Z}^n \neq \emptyset \). There is polynomial time algorithm to find some integer point in \( P \).

**Proof** Suppose that \( \delta(A) > 1 \) (the case of \( \delta(A) = 1 \) is trivial). Let \( \bar{A}x \leq \bar{b} \) be the subsystem of the system \( Ax \leq b \), where \( \bar{A} \in \mathbb{Z}^{n \times n} \) and \( \bar{b} \in \mathbb{Z}^n \) such that \( |\det(\bar{A})| = \delta(A) \). Let \( C = \{ x \in \mathbb{R}^n : \bar{A}x \leq \bar{b} \} \). Then \( P \subset C \). Let \( v \in \mathbb{Q}^n \) be the vertex of \( P \) such that \( \bar{A}v = \bar{b} \), so \( v = \bar{A}^{-1} \bar{b} \).

Let \( B = -\delta(A) - 1 \bar{A}^{-1} \), then from elementary theory of polyhedrons it follows that \( C = v + \text{cone}(B) \). So columns of the matrix \( B \) defines edges of the cone \( C \). Let \( S = v + \text{cone}(0 B) \). So \( S \subset C \).

Also from elementary theory of polyhedrons it follows that \( n + 1 \) lines of the system \( \bar{A}x \leq \bar{b} \) correspond to \( n + 1 \) facets of \( P \) by the following way: if \( F \) is a facet of \( P \) then \( F = P \cap \{ x \in \mathbb{R}^n : A_kx = b_k \} \) for some line \( (A_k, b_k) \) of the system \( \bar{A}x \leq \bar{b} \).

Let \( F_1, F_2, \ldots, F_s \) be the faces of \( P \) corresponding to the system \( \bar{A}x \leq \bar{b} \) and let \( F_{n+1} \) be the last facet of \( P \) corresponding to the line of the system \( \bar{A}x \leq \bar{b} \) that is not included to the system \( \bar{A}x \leq \bar{b} \).

First we need to prove that \( S \cap \mathbb{Z}^n \neq \emptyset \). Set \( C \cap \mathbb{Z}^n \) is induced by the following equivalent systems:
\[
\begin{aligned}
\begin{cases}
\hat{A}x \leq \hat{b} \\
x \in \mathbb{Z}^n
\end{cases}
&\sim
\begin{cases}
\hat{A}x + y = \hat{b} \\
x \in \mathbb{Z}^n \\
y \in \mathbb{Z}^n_+
\end{cases}
\end{aligned}
\]

By the lemma there is a polynomial time algorithm to find a solution of this system \(y^* \in \mathbb{Z}^n_+\) with the property that \(\sum_{k=1}^{n} y_k \leq \delta(A) - 1\). Hence \(\exists x^* \in \mathbb{Z}^n\) that \(\hat{A}x^* + y^* = \hat{b}\) and \(x^* = \hat{A}^{-1}\hat{b} - \hat{A}^{-1}y^*\). Finally \(x^* = v + B \frac{1}{\delta(A) - 1} y^*\), it is equivalent to the statement that \(x^* \in S\).

To finish we need to prove that \(S \subseteq P\). Suppose that \(S \not\subseteq P\). Let \(g^{(1)}, g^{(2)}, \ldots, g^{(n)}\) be the vertices of \(S\) that are adjacent to the vertex \(v\). If \(S \not\subseteq P\) then the facet \(F_{n+1}\) intersects some edge \([v, g^{(k)}]\) in \(S\) such that \([v, g^{(k)}] \cap F_{n+1} = \{u\}\), \(u \in \text{vert}(P)\) and \(u \neq g^{(k)}\). Let \(F_1\), where \(1 \leq j \leq n\), be an opposite facet to the vertex \(u\) and \((A_j, b_j)\) be the line of an initial system correspondent to \(F_1\). Then \(\max\{A_j x : x \in P\} - \min\{A_j x : x \in P\} = |b_j - A_j u| < |b_j - A_j z^{(k)}| = \delta(A) - 1\). The last statement contradicts to an assumption of Theorem that \(w(P) \geq \delta(A) - 1\).

**Corollary 4** Let \(A \in \mathbb{Z}^{(n+1) \times n}, b \in \mathbb{Z}^{n+1}, c \in \mathbb{Z} = P(A, b)\) be a simplex. Let the vertex \(v\) be an optimal solution of the linear problem \(\max \{c^\top x : x \in P\}\). Let \(J = \{j : A_j v = b_j\}\) and \(\Delta = |\det(A_J)|\). If \(w(P(A, b)) \geq \Delta - 1\), then \(P(A, b) \cap \mathbb{Z}^n \neq \emptyset\) and there is an algorithm with the computational complexity \(O(n \Delta)\) that solves an integer problem \(\max \{c^\top x : x \in P \cap \mathbb{Z}^n\}\).

**Proof** We have already proved that \(S \subseteq P\), where \(S = v + \text{conv}(0B)\) and \(B = -(\Delta - 1)\hat{A}^{-1}\) (see the previous Theorem). It is easy to see that \(\max \{c^\top x : x \in P \cap \mathbb{Z}^n\} = \max \{c^\top x : x \in S \cap \mathbb{Z}^n\}\). Also from Theorem it follows that all vertices of the system are in the set \(S\). But we already have a remarkable algorithm proposed by Gomory, Hu [8,14,16] for integer optimization in systems of the type \((\mathbf{3})\) or \((\mathbf{3})\) with a computational complexity \(O(n \Delta)\).

**Corollary 5** Let \(A \in \mathbb{Z}^{(n+1) \times n}, b \in \mathbb{Z}^{n+1}, c \in \mathbb{Z} = P(A, b)\) be a simplex. If \(w(P(A, b)) \geq \Delta(A) - 1\), then \(P(A, b) \cap \mathbb{Z}^n \neq \emptyset\) and there is an algorithm with the computational complexity \(O(n \Delta)\) that solves an integer problem \(\max \{c^\top x : x \in P \cap \mathbb{Z}^n\}\).

### 4 Integer Programs with Almost Unimodular Matrices

We will use the theorem to prove the following result.

**Theorem 8** Let \(A \in \mathbb{Z}^{n \times n}\) be almost unimodular, \(b \in \mathbb{Z}^n, c \in \mathbb{Z}^n\). Then, there is a polynomial-time algorithm to solve \(\max \{c^\top x : x \in P(A, b) \cap \mathbb{Z}^n\}\).
Proof Let \( P = P(A, b) \). We can check an emptiness of \( P \cap \mathbb{Z}^n \) in polynomial time by the theorem. Then, if \( P \cap \mathbb{Z}^n \neq \emptyset \), we need to find a vertex \( v \) of \( P \), that is an optimal solution of the relaxed problem \( \max \{ c^\top x : Ax \leq b, \ x \in \mathbb{Q}^n \} \).

To this end, one can use the Khachiyan’s polynomial algorithm for linear programming [19]. If \( v \) is integer, then it is an optimal solution of the integer problem. If not, we consider the shifted cone \( N(v) = \{ x \in \mathbb{R}^n : A_j x \leq b \} \), where \( J = \{ k : A_k v = b \} \).

By the theorem \( \max \{ c^\top x : Ax \leq b, \ x \in \mathbb{Z}^n \} = \max \{ c^\top x : x \in N(v), \ x \in \mathbb{Z}^n \} \) and an optimal integral point \( v^* \) lies on some edge of \( N(v) \).

Let \( \mathcal{B}_k \) be the set \( \{ x \in \mathbb{R}^n : x_k = \lfloor v_k \rfloor \} \) and \( \mathcal{B}_k \) be the set \( \{ x \in \mathbb{R}^n : x_k = \lceil v_k \rceil \} \).

Now we going to prove that there exists \( k, 1 \leq k \leq n \) such that \( v^* \) is the optimal solution of the linear problem \( \max \{ c^\top x : x \in \mathcal{B}_k \} \) or problem \( \max \{ c^\top x : x \in \mathcal{B}_k \} \).

Let \( L \) be an edge of \( N(v) \). If \( |L \cap \mathcal{B}_k| = 1 \) or \( |L \cap \mathcal{B}_k| = 1 \) for some \( k \) and we know that \( L \cap \mathcal{B}_k = u \) or \( L \cap \mathcal{B}_k = u \) then \( u \in \mathbb{Z}^n \). Indeed, assume that \( L \) is induced by an integer subsystem of the system \( Ax \leq b \) denoted by \( \hat{A} x \leq \hat{b} \). We also can assume that \( \hat{A} \) is \((n - 1) \times n \) matrix. Then the point \( u \) is induced by the system \( \begin{pmatrix} \hat{A} & 1 \end{pmatrix} \begin{pmatrix} x \\sigma \end{pmatrix} = \begin{pmatrix} \hat{b} \end{pmatrix} \) where \( \sigma \) is \( \lfloor v_k \rfloor \) or \( \lceil v_k \rceil \). By assumptions of a theorem the matrix \( \begin{pmatrix} \hat{A} & 1 \end{pmatrix} \) is unimodular, hence \( u \) is integer.

Hence we have only two possibilities for \( \text{vert}(\mathcal{B}_k) \) (the same is true for \( \text{vert}(\mathcal{B}_k) \)):

1. \( \text{vert}(\mathcal{B}_k) = \{ v \} \). This is the case when \( v \in \{ x \in \mathbb{R}^n : x_k = \lfloor v_k \rfloor \} \), so no new vertices were created.

2. \( \text{vert}(\mathcal{B}_k) \subset \mathbb{Z}^n \). Indeed if \( v \notin \{ x \in \mathbb{R}^n : x_k = \lfloor v_k \rfloor \} \), then new vertices must be generated. Each vertex of \( \mathcal{B}_k \) is a result of an intersection of some edge of \( N(v) \) with the hyperplane \( \{ x \in \mathbb{R}^n : x_k = \lfloor v_k \rfloor \} \). By the previous, all these intersections are integer.

Now let \( L^* \) be an edge of \( N(v) \) that contains an optimal point \( v^* \).

There are few possible cases for some fixed \( k \):

1. \( L^* \subset \mathcal{B}_k \). Then it is easy to see that an optimal solution of the linear problem \( \max \{ c^\top x : x \in \mathcal{B}_k \} \) is a point \( v \).

2. \( L^* \cap \mathcal{B}_k = \emptyset \). In this case the linear problem \( \max \{ c^\top x : x \in \mathcal{B}_k \} \) can be inconsistent or it has an integer or a rational solution. By the previous, if a solution is rational then it is the vertex \( v \).

3. \( |L^* \cap \mathcal{B}_k| = 1 \). This is the case of an intersection of a ray or a hyperplane. More precisely we can prove that \( L^* \cap \mathcal{B}_k = \{ v^* \} \). Let \( L^* \cap \mathcal{B}_k = \{ u \} \) and \( u \neq v^* \). By the previous results the point \( u \) must be integral. Hence \( v^* \in [v, u] \) because \( L^* \) is an edge of \( N(v) \) and an objective function only can be increased by the edge \( L^* \). But \([v, u]\) can’t have any other integer points except \( u \) because \( u_k = \lfloor v_k \rfloor \) or \( u_k = \lceil v_k \rceil \). So \( u = v^* \).

4. The same items are true for \( L^* \cap \mathcal{B}_k \).
Since \( L^* \) contains \( v^* \in \mathbb{Z}^n \) then \( \exists k, 1 \leq k \leq n \) such that \( |L^* \cap \overline{B_k}| = 1 \) or \( |L^* \cap B_k| = 1 \). Hence, by the previous, we have \( L^* \cap \overline{B_k} = \{ v^* \} \) or \( L^* \cap B_k = \{ v^* \} \). If this is true, then \( v \notin \{ x \in \mathbb{R}^n : x_k = [v_k] \} \cup \{ x \in \mathbb{R}^n : x_k = [\overline{v}_k] \} \) and all vertices of \( \overline{B_k} \) or \( B_k \) resp. are integer.

In conclusion, we note that our final algorithm consists one running of a linear programming algorithm to find the vertex \( v \) as an optimal solution of the linear problem \( \max \{ c^\top x : Ax \leq b \} \) and \( 2n \) times running of a linear programming algorithm to solve problems \( \max \{ c^\top x : x \in \overline{B_k} \} \) and \( \max \{ c^\top x : x \in B_k \} \) for each \( 1 \leq k \leq n \). One of this solutions must be \( v^* \) that can be recognized by an integrality and a maximality of the objective function.

Since the linear programming is polynomial due to Khachiyan \[19\] so our algorithm is polynomial too.

Let us to make some generalization of this proof. The matrix \( A \) is called \( k \)-almost unimodular for \( 1 \leq k \leq n−1 \) if \( \Delta(A) \leq 2 \) and any \( (r(A)−k)\times(r(A)−k) \) sub-matrix has a determinant from the set \{0, ±1\}.

**Theorem 9** Let \( A \in \mathbb{Z}^{m \times n} \) be \( k \)-almost unimodular, \( b \in \mathbb{Z}^m \), \( c \in \mathbb{Z}^n \). Then there is a polynomial-time algorithm for fixed \( k \) to solve \( \max \{ c^\top x : x \in P(A, b) \cap \mathbb{Z}^n \} \).

**Proof** To proof this theorem we need to consider sets \( \overline{B_J} = N(v) \cap \{ x \in \mathbb{R}^n : x_J = [v_J] \} \) and \( B_J = N(v) \cap \{ x \in \mathbb{R}^n : x_J = [\overline{v}_J] \} \), for all subsets \( J \subset \{1, n\} \), \( |J| = k \). There are \( \binom{n}{k} \) of these subsets.

Again we have following possibilities for \( \overline{B_J} \):
1. \( \text{vert}(\overline{B_J}) = \{ v \} \).
2. \( v^* \notin \text{vert}(\overline{B_J}) \subset \mathbb{Z}^n \).
3. \( v^* \in \text{vert}(\overline{B_J}) \subset \mathbb{Z}^n \).
4. The same items are true for \( B_J \).

Since \( v^* \in \mathbb{Z}^n \) then \( \exists J \) such that \( v^* \in \text{vert}(\overline{B_J}) \) or \( v^* \in \text{vert}(B_J) \). If this is true, then \( v \notin \{ x \in \mathbb{R}^n : x_J = [v_J] \} \cup \{ x \in \mathbb{R}^n : x_J = [\overline{v}_J] \} \), hence the linear programs \( \max \{ c^\top x : x \in \overline{B_J} \} \) or \( \max \{ c^\top x : x \in B_J \} \) have \( v^* \) as an optimal solution.

Our final algorithm consists one running of a linear programming algorithm to find the vertex \( v \) and \( 2\binom{n}{k} \) times running of a linear programming algorithm to solve problems \( \max \{ c^\top x : x \in \overline{B_J} \} \) and \( \max \{ c^\top x : x \in B_J \} \) for each \( J \subset \{1, n\} \) \( |J| = k \). One of this solutions must be \( v^* \) that can be recognized by an integrality and a maximality of the objective function.

5 Examples of the Corner Polyhedrons with an Exponential Number of Edges

Theorem 11 implies that there is a polynomial-time algorithm for solving an integer program \( \max \{ c^\top x : x \in P(A, b) \cap \mathbb{Z}^n \} \), when \( \Delta(A) \leq 2 \) and the number of edges in shifted cone \( N(v) \) is bounded by a polynomial, where \( v \) is optimal.
solution of relaxed linear problem. But, there is an example of an exponential
number of edges in the shifted cone $P(A, b), \Delta(A) \leq 2$.

Consider the set $D = \text{conv}\{\{0, 1\}^n = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1, \ i \in \overline{1,n}\}$. We use the homogenization and find the matrix $A_D$, such that: $\{(\frac{x}{x_0}) : A_D(\frac{x}{x_0}) \leq 0\}$ = $\{(\frac{x}{x_0}) : x \in D, x_0 \geq 0\}$. Finally, we create the matrix $A_D'$ by multiplying any column of $A_D$ by 2. $A_D$ is unimodular so $A_D'$ is bimodular and we have the following theorem:

**Theorem 10** The cone $\{x \in \mathbb{R}^{n+1} : A_D'x \leq 0\}$ has an exponential number of edges. Any of its edges can be represented by a bimodular submatrix of $A_D'$.

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