GLOBAL PROPERTIES OF FAMILIES OF PLANE CURVES

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Abstract. We describe degenerations of projective plane curves to curves containing a fixed line \( l \) as a component, and show that \( H^1(V_{n,d,m}, \mathcal{O}(r)) = 0 \), \( r \in \mathbb{Z} \), where \( V_{n,d,m} \subset \mathbb{P}^N \) \((N = n(n + 3)/2)\) is the subscheme consisting of irreducible plane curves having smooth contact of order at least \( m \) with \( l \) at a fixed point \( p \in l \) and \( d \) nodes and no other singularities.

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0. Introduction

An arbitrary smooth projective curve is birationally equivalent to a plane nodal curve. We denote by \( \mathbb{P}^N \) the projective space parametrizing all projective plane curves of degree \( n \) \((N = n(n + 3)/2)\). Let \( l \subset \mathbb{P}^2 \) be a fixed line, and \( p \in l \) a fixed point. Let \( n, d, m \) be three integers with \( 0 \leq m \leq n \) and \( 1 \leq d \leq (n-1)(n-2)/2 \). Let \( V_{n,d,m} \subset \mathbb{P}^N \) be the (locally closed) subscheme consisting of irreducible curves having smooth contact of order at least \( m \) with \( l \) at \( p \) and \( d \) nodes and no other singularities. These schemes are smooth and \( \dim V_{n,d,m} = N - d - m \) [H, Sect. 2].

We denote by \( U_m(n, g) \) the closure in \( \mathbb{P}^N \) of the locus of reduced plane curves of degree \( n \) and geometric genus \( g \), not containing \( l \) and having contact of order \( m \) with \( l \) at \( p \).

In this paper we study global properties of \( V_{n,d,m} \subset \mathbb{P}^N \) and related schemes. In Theorem 2.2, we describe degenerations of a general member of a component of \( U_m(n, g) \) to a curve containing the line \( l \). In Proposition 3.9, we restrict ourselves to \( V_{n,d,m} \) and obtain a more precise result. Fortunately, even the case when \( d = 1 \) is not trivial (Lemma 3.8). We prove both theorems by induction on the number of nodes. Finally, in Theorem 5.2, we prove the vanishing of \( H^1(V, \mathcal{O}(r)) \), \( r \in \mathbb{Z} \), where \( V \) is an arbitrary \( V_{n,d,m} \)-admissible scheme (see Definition 4.5). In particular \( H^1(V_{n,d,m}, \mathcal{O}(r)) = 0 \), \( r \in \mathbb{Z} \). The vanishing theorem is proved by induction on the size of admissible schemes by taking suitable hyperplane sections.

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This paper suggests that instead of dealing with a single $\nabla_{n,d,0}$ one should consider all $\nabla_{n,d,m}$-admissible schemes for $0 \leq m \leq n$. We observe that $\nabla_{n,d,m}$-admissible schemes are often generically non-reduced, and this plays a key role in the proof of the vanishing theorem.

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1. Preliminaries

1.1. Notation. Let

$$\Sigma_{n,d,m} \subset \mathbb{P}^N \times \text{Sym}^d(\mathbb{P}^2)$$

be the closure of the locus of pairs $(E, \sum_{i=1}^d R_i)$, where $E$ is an irreducible nodal curve having smooth contact of order at least $m$ with $l$ at $p$, and $R_1, \ldots, R_d$ are its nodes. We denote by $\pi_N$ and $\pi_{n,d}$ the projections of the product to $\mathbb{P}^N$ and $\text{Sym}^d(\mathbb{P}^2)$, respectively, and identify $V_{n,d,m}$ with $\pi_{N}^{-1}(V_{n,d,m})$. For $1 \leq d \leq (n-2)(n-3)/2$, we set

$$V_{n,d,n+1} = \{ C + l \in \mathbb{P}^N \mid C \in V_{n-1,d,0} \setminus (C_p \cup M_2) \subset \mathbb{P}^{N_1} \},$$

where $N_1 = (n-1)(n+2)/2$, $C_p$ is the divisor in $\mathbb{P}^{N_1}$ of curves containing $p$, and $M_2$ is the divisor in $\mathbb{P}^{N_1}$ of curves having multiple points of intersection with $l$ ($M_2$ is described in [DH1, I, Sect. 3]). We get a chain of closed subschemes in $\mathbb{P}^N$ (see Lemma 3.3 below):

$$\nabla_{n,d,n+1} \subset \nabla_{n,d,n} \subset \cdots \subset \nabla_{n,d,0}.$$

Let $X, Y, Z$ be a coordinate system in $\mathbb{P}^2$ such that $l = \{ X = 0 \}$ and $p = [0 : 1 : 0]$. Let

$$f_C(X, Y, Z) = \sum a_{jk} X^j Y^k Z^{n-j-k} = X(\cdots) + \sum a_{0k} Y^k Z^{n-k} = 0$$

be an equation of a curve $C$. For $m \geq 1$, the condition $a_{on} = \cdots = a_{on+1-m} = 0$ means that $C$ has contact of order at least $m$ with $l$ at $p$ (we do not exclude the case $l \subset C$).

The standard action of $\text{PGL}(2)$ on $\mathbb{P}^2$ induces an action of $\text{PGL}(2)$ on the parameter space $\mathbb{P}^N$, and we can identify $\text{PGL}(2)$ with the corresponding subgroup of $\text{PGL}(N)$. We denote by $\varphi_{s_1,s_2}$ ($\phi_{t_1,t_2}$) the transformation of $\mathbb{P}^N$ induced by the transformation

$$[X : Y : Z] \rightarrow [X : Y : s_1 Z + s_2 X] \quad ([X : Y : Z] \rightarrow [X : t_1 Y + t_2 X : Z]).$$

Let $G \subset \text{PGL}(N)$ denote the subgroup generated by all the elements $\varphi_{s_1,s_2}$ and $\phi_{t_1,t_2}$ ($s_1, s_2, t_1, t_2 \in \mathbb{C}^*$).

We denote by $g(E)$ the geometric genus of a reduced irreducible curve $E$. The geometric genus of a reduced curve $E' + E''$ is defined inductively: $g(E' + E'') = g(E') + g(E'') - 1$. Clearly $g(E) \geq -\deg(E) + 1$ with equality if and only if $E$ is a union of lines. For a point $Q$ of an arbitrary reduced curve $E$, we set $\delta_Q(E) = \dim_{\mathbb{C}} \tilde{O}_Q / O_Q$, where $O_Q$ is the local ring of $E$ at $Q$ and $\tilde{O}_Q$ its normalization.
By a point of a scheme we mean a closed point unless stated otherwise, and by a branch (of a scheme) through a point we mean a local branch. By a component of a scheme we mean an isolated component. We denote by $T_v(W)$ the Zariski tangent space of $W$ at a point $v$. Finally, to a family $W$ of curves of degree $n$ in $\mathbb{P}^2$ one can canonically associate a closed subscheme $\overline{W} \subset \mathbb{P}^N$; we assume that $\overline{W}$ derived from a family is always reduced.

We will frequently use the following basic facts.

1.2. Proposition ([H, Lemma 2.4]). Let $W$ be a component of $U_m(n, g)$. For $m \geq 1$, we have:
   a) $\dim W = 3n + g - 1 - m$; and
   b) if $E$ is a general member of $W$, then in a neighborhood of $p$, $E$ is a union of smooth arcs having contact of orders $a_1, \ldots, a_r$ with $l$, $\sum a_i = m$, and having minimal order of contact among themselves; the remaining singularities of $E$ are all nodes, and $(E \cdot l)_Q \leq 1$ if $Q \neq p$; moreover $\delta_p(E) = \sum b_j(b_j - 1)/2$ and $\sum b_j = m$, where $b_1 = b_1(W)$ is the multiplicity of $E$ at $p$ and $b_2 = b_2(W), \ldots, b_m = b_m(W)$ are the multiplicities of $E$ at its infinitely near points lying over $p$ in the direction of $l$.

1.3. Proposition. i) $\Sigma_{n,d,0}$ is unibranch everywhere [Tr].
   ii) For $d \leq n$, $\Sigma_{n,d,m}$ is unibranch everywhere.
   iii) $V_{n,d,0}$ is irreducible [H].

Proof. ii) We get $n(n+3)/2 - 3d - m \geq 0$ with equality if and only if $n = d = m = 5$. Hence the map $\pi_{n,d} : \Sigma_{n,d,m} \to \text{Sym}^d(\mathbb{P}^2)$ is surjective and one can repeat an elementary argument from [Tr, Theorem (Case: $n >> d$)].

2. Splitting off a line: a theorem

2.1. Assigned and virtually non-existent nodes. Let $F$ be a curve of degree $n$ with $d$ nodes. If $F$ is regarded as the limit of a variable curve of degree $n$ with $d' < d$ nodes, then it is said that the $d'$ nodes of $F$ that are very near the $d'$ nodes of the variable curve are assigned nodes of $F$, while the remaining $d - d'$ nodes of $F$ are considered as virtually non-existent.

One may also consider assigned cycles on arbitrary curves. An assigned $d$-cycle $b$ on a curve $E$ is said to be connected (in the sense of Severi) if there is an irreducible nodal curve $E'$ whose $d$ nodes approach $b$ as $E'$ degenerates to $E$. Further, a branch of $V_{n,d,m}$ through an arbitrary reduced $E \in V_{n,d,m}$ determines a $d$-cycle of assigned singularities on $E$.

Let $W$ be a component of $U_m(n, g)$ with $m \geq 1$. One can construct a maximal irreducible subfamily $W_1 \subset U_m(n, g+1)$ such that $W \subset W_1$. Take a general member $F$ of $U_0(n, g)$ that is very near a general member of $W$. To obtain a family of curves of genus $g + 1$, we regard one node of $F$ as virtually non-existent. We then cut that family by the $m$ hyperplanes in $\mathbb{P}^N$, $a_{0n} = \cdots = a_{0n+1-m} = 0$, and take a suitable irreducible subfamily.

2.2. Theorem. Let $W$ be a component of $U_m(n, g)$. Let $W'$ be a component of $W \cap \{a_{0n-m} = 0\}$ whose general member is of the form $C + l$. Then $C$ is a general member of a component of $U_{m'}(n - 1, g')$, where

$$m' = m - (g - g') - 2.$$
Further, let $F(E, C+l) \subset W$ be an arbitrary local irreducible 1-dimensional family through $C + l$ such that all members of $F(E, C+l) \setminus C+l$, denoted by $E$, are general points of $W$. Then

$$\sum_{Q \to p} \delta_Q(E) = \delta_p(C+l), \quad \sum_{Q \neq l} \delta_Q(E) + \sum_{Q \to l \setminus p} \delta_Q(E) = \sum_{R \notin l} \delta_R(C+l) + n - m.$$ 

Proof. We will prove the theorem by induction on $n$ and $\Delta_W = \sum_Q \delta_Q(E)$, where $E$ is a general member of $W$. The theorem is trivial if $n \leq 2$ or $\Delta_W = 0$. We assume that $\Delta_W > 0$. The induction is based on the construction described in (2.1). Let $W_1 \subset U_m(n, g + 1)$ be any maximal irreducible subfamily such that $W \subset W_1$. Let $E_1 \in W_1$ be a general member that is very near $E$.

A subvariety of $W$ consisting of nonreduced curves has codimension strictly greater than 1 in $W$. In case general members of the subvariety do not contain $l$, this follows from the semi-stable reduction theorem for families of curves and standard dimension counts ([DH1, Sect. 1(a)], [H], [N]). Otherwise, we take $W_1 \supset W$, as above, and use the induction on $n$ and $\Delta_W$. In particular $C + l$ is a reduced curve.

Since $g(C + l) = g(C) - 1$ and $\sum_Q \delta_Q(E) \leq \sum_Q \delta_Q(C+l)$, we have $g(C) = g(E) - k$ with $k \geq -1$ [Hi].

Claim 1: $m(C) : = (C \cdot l)_p \leq m - k - 2$. Indeed, since $W \notin \{a_{0n-m} = 0\}$, $C$ is moving in a family of dimension $3n + g(E) - m - 2$ by Proposition 1.2; moreover

$$(1) \quad 3n + g(E) - m - 2 \leq 3(n - 1) + g(C) - 1 - m(C) = 3n + g(E) - k - 4 - m(C),$$

and the claim follows.

Let $\gamma$ be the maximal number of nodes of $E$ approaching $l \setminus p$ as $E$ degenerates to $C + l$ along various $F(E, C+l)$’s. Set

$$\Gamma = \sum_Q \delta_Q(C+l), \quad (Q \in l \cap C \setminus p).$$

Claim 2: For $\gamma = 0$, we have $m = n$, $\Gamma = k + 1$, and the theorem follows. Indeed, by the genus formula [Hi],

$$(2) \quad g(E) - k - 1 = g(C+l) = (n - 1)(n - 2)/2 - \Gamma - \delta_p(C+l) - \sum_{R \notin l} \delta_R(C+l)$$

$$g(E) = (n - 1)(n - 2)/2 - \sum_{Q \to l \setminus p} \delta_Q(E) - \sum_{Q \to p} \delta_Q(E) - \sum_{Q \neq l} \delta_Q(E)\Gamma - k - 1 + \sum_{R \notin l} \delta_R(C+l) + \delta_p(C+l) = \sum_{Q \to l \setminus p} \delta_Q(E) + \sum_{Q \neq l} \delta_Q(E) - \sum_{Q \to p} \delta_Q(E).$$

We have $\sum_{Q \to p} \delta_Q(E) \leq \delta_p(C+l)$ and $\sum_{Q \neq l} \delta_Q(E) \leq \sum_{R \notin l} \delta_R(C+l)$. Hence

$$\sum_{Q \to l \setminus p} \delta_Q(E) \geq \Gamma - k - 1 \geq n - 1 - m(C) - k - 1 \geq n - m$$

(the last inequality follows from Claim 1). Therefore $\Gamma \leq k + 1$ and $\sum_{Q \neq p} (C \cdot l)_Q \leq k + 1$. Since $n \geq m$, we get $m(C) = n - k - 2 = m - k - 2$, and Claim 2 follows.
We proceed by induction on $\Delta_W$. Let $\mathcal{F}(E, C+l) \subset W$ be a 1-dimensional family as in the statement of the theorem. Since $C+l$ is a general member of $W$, one can find a branch $\mathcal{W}$ of $W$ such that $\mathcal{F}(E, C+l)$, $\mathcal{W}' \subset \mathcal{W}$, where $\mathcal{W}' \subset W'$ is a branch through $C+l$. We can now choose a family $W_1 \subset U_m(n, g+1)$, as above, and a branch $\mathcal{W}_1$ of $W_1$ such that $\mathcal{W}_1 \supset \mathcal{W}$. There are two cases to consider.

Case 1: A general member $E'_1$ of some branch $\mathcal{W'}$ of $\mathcal{W}_1 \cap \{a_{on-m} = 0\}$ containing $\mathcal{W}'$ does not contain $l$. Then $\gamma \neq 0$, and $g(E'_1) = g(E_1)$ by Proposition 1.2(a). By a trivial dimension count, some general member of $\mathcal{W} \cap \{a_{on-m-1} = 0\}$ is of the form $C + l$. By induction hypothesis, the only singularities of $C \setminus l$ are nodes and $m(C) = m + k_1 - 1$, where $k_1 = g(E_1) - g(C) = k + 1$. Therefore $m(C) = m - k - 2$ and (1) is an equality.

By induction hypothesis and (2), $\Gamma - k_1 - 1 = n - m - 1$ so $\Gamma - k - 1 = n - m$. Since

$$\delta_p(C+l) = \sum_{Q \to p} \delta_Q(E'_1) \leq \sum_{Q \to p} \delta_Q(E) \leq \delta_p(C+l),$$

we get

$$\sum_{Q \to p} \delta_Q(E) = \delta_p(C+l), \quad \sum_{Q \neq l} \delta_Q(E) + \sum_{Q \to l \setminus p} \delta_Q(E) = \sum_{R \notin l} \delta_R(C+l) + n - m.$$

Case 2: The general members of all branches of $\mathcal{W}_1 \cap \{a_{on-m} = 0\}$ that contain $\mathcal{W}'$ are of the form $C_1 + l$. We assume $\gamma \neq 0$. There are two possibilities.

1) $g(C) < g(C_1)$. Set $k_1 = g(E_1) - g(C_1)$ and $m(C_1) = (C_1 \cdot l)_p$. Then $k_1 \leq k$ and $m(C) \geq m(C_1) = m - k_1 - 2 \geq m - k - 2$. Hence $m(C) = m - k - 2$ by Claim 1. Moreover, $C$ is a general member of $U_{m(C)(n-1, g(C))}$, and

$$m(C) = m(C_1), \quad g(C) = g(C_1) - 1, \quad k = k_1, \quad \Gamma = n - 1 - m(C), \quad \Gamma - k - 1 = n - m.$$

We will need the following claim.

Claim 3: $m(C) = \delta_p(C+l) - \delta_p(C)$, and one can assume that at list one of the following conditions holds: $m(C) \neq m(C_1)$, $\sum_{Q \to p} \delta_Q(E) = \delta_p(C+l)$, or $\sum_{Q \to p} \delta_Q(E) = 0$. The first assertion follows at once from the genus formula [Hi]. Further, if $\sum_{Q \to p} \delta_Q(E) \neq 0$, we can assume that for the general member $E_1 \in \mathcal{W}_1$,

$$\sum_{Q \to p} \delta_Q(E_1) < \sum_{Q \to p} \delta_Q(E) \leq \delta_p(C+l).$$

By induction hypothesis, we have $\sum_{Q \to p} \delta_Q(E_1) = \delta_p(C_1+l)$ hence

$$\delta_p(C_1+l) < \sum_{Q \to p} \delta_Q(E).$$

If $m(C) = m(C_1)$, then $\delta_p(C_1+l) + 1 \geq \delta_p(C+l)$, hence $\sum_{Q \to p} \delta_Q(E) = \delta_p(C+l)$. This proves Claim 3.

If $\sum_{Q \to p} \delta_Q(E) = 0$ then $\sum_{Q \to p} \delta_Q(E_1) = \delta_p(C_1+l) = 0$, hence $m(C) = m(C_1) = 0$ and $\delta_p(C+l) = 0$. Therefore $\sum_{Q \to p} \delta_Q(E) = \delta_p(C+l)$. Now, by (2),

$$\sum_{Q \neq l} \delta_Q(E) + \sum_{Q \to l \setminus p} \delta_Q(E) = \sum_{R \notin l} \delta_R(C+l) + n - m.$$
ii) \(g(C) = g(C_1)\). Then \(k_1 = k + 1\). Assume \(m(C) \geq m(C_1) + 1\). Then \(m(C) > m - k_1 - 2 = m - k - 3\), hence \(m(C) = m - k - 2\) by Claim 1, and \(m(C) = m(C_1) + 1\). So \(C\) is moving in a family of dimension 

\[3(n - 1) + g(C_1) - 1 - m(C_1) - 1 = 3(n - 1) + g(C) - 1 - m(C)\]

Moreover \(\Gamma = n - 1 - m(C)\) and \(\Gamma - k - 1 = n - m\). If \(\sum \delta_Q(E) \neq 0\), then we can assume that \(\sum \delta_Q(E_1) = \delta_p(C_1 + l) < \sum \delta_Q(E)\). It follows from (2) that \(\sum \delta_Q(E) = \delta_p(C + l)\), and we are done. But if \(\sum \delta_Q(E) = 0\) then \(\sum \delta(E_1) = \delta_p(C_1 + l) = 0\), hence \(m(C_1) = \delta_p(C + l) = 1\) and exactly one node of \((C_1 + l)\backslash p\) tends to \(p\) as \(C + l\) tends to \(C + l\) (Proposition 1.2(b)). We will derive a contradiction. By (2), \(\sum \delta_R(C + l) = \sum \delta_R(C_1 + l)\).

First, we assume that \(C\) and \(C_1\) are smooth. We proceed by induction on \(n - m\). If \(n = m\), we degenerate \(C\) into a sufficiently general nodal curve which is a sum \(\sum l_i\) of \(n - 1\) lines (since \(p \in C\), one of the lines contains \(p\)). Clearly \(\sum l_i\) has \((n - 1)(n - 2)/2\) nodes. Each node of \(\sum l_i\) determines a branch of \(V_{n,1,0} \subset \mathbb{P}^N\) through \(l_i + l\). We get a degeneration of a reducible nodal curve \(F\) into \(\sum l_i + l\), where \(F\) has degree \(n\) and contact of order \(n\) with \(l\) at \(p\); moreover, \(F\) contains a line through \(p\), which is absurd. If \(n = m + 1\), we can split a line off \(C\), and apply the preceding argument, etc.

Next, if \(C\backslash l\) has only unassigned nodes as singularities, we can argue as above (note that \(C_1\) may have at most \(n - 2\) such nodes). But if \(C\backslash l\) contains assigned nodes, we can assume \(\sum \delta_R(C + l) \neq \sum \delta_R(C_1 + l)\), a contradiction. This concludes the argument in the case \(m(C) \geq m(C_1) + 1\).

Now, we suppose that \(m(C) = m(C_1)\) for any choice of \(\mathcal{W}_i\) and derive a contradiction. If \(\sum \delta_Q(E) = \delta_p(C + l) \neq 0\), we can assume \(\delta_p(C + l) \neq \delta_p(C)\) (see Claim 3). Hence \(g(C) \neq g(C_1)\), a contradiction.

If \(\sum \delta_Q(E) = 0\), then \(\sum \delta_Q(E_1) = \delta_p(C_1 + l) = 0\) and \(m(C) = m(C_1) = 0\) so \(\delta_p(C + l) = 0\). If \(C\backslash l\) contains assigned points, we can assume \(\sum \delta_R(C + l) \neq \sum \delta_R(C_1 + l)\) and derive a contradiction with a help of (2) and Claim 1.

Throughout the rest of the proof, we will exploit that \(\sum \delta_Q(E)\) is too large.

To begin with, we assume, in addition, that \(C\backslash l\) has no singularities; the case when \(C\backslash l\) has only unassigned nodes as singularities is similar. By a trivial dimension count, there are two possibilities for \(C + l\) provided \(\text{deg}(C) \geq 2\): \(C + l\) has either an ordinary triple point or a tacnode.

First, we consider the case when a general curve \(C_1\) with one node tends to a curve \(C\) with a node along \(l\). Then \(d = n - m + 2\) by (2) and Claim 1. If \(d = 2\), we degenerate \(C\) into a sum \(\sum l_i\) of \(n - 1\) general lines with \(l_1 \cap l_2 \cap l \neq \emptyset\). Then \(\sum l_i\) has \((n - 1)(n - 2)/2 - 1\) nodes outside \(l\). Such a node determines a branch of \(V_{n,1,0} \subset \mathbb{P}^N\) through \(l_i + l\). We get a degeneration of \(F\) into \(\sum l_i + l\), where \(F\) is a curve of degree \(n\) having contact of order \(n\) with \(l\) at \(p\) and \((n - 1)(n - 2)/2 - 1 + d\) nodes and no other singularities. Such an \(F\) must contain a line through \(p\), which is absurd.

If \(d = n - m + 2 > 2\), we degenerate \(C\) into a curve \(C' + l'\), where \(l'\) is a general line. We then split \(l'\) off the general curve of \(V_{n,d,m}\) and apply the preceding argument to curves of \(V_{n-1,d,m}\) if \(n - 1 = m\), etc.

Next, we consider the case when a smooth curve \(C_1\) tends to a smooth curve \(C\) tangent to \(l\) at one point. Then \(d = n - m + 1\) by (2) and Claim 1. If \(d = 1\),
we take a general line \( l' \subset \mathbb{P}^2 \) and a point \( Q' = l' \cap l \). The projection \( \pi_{n,1} \) gives a natural fibering \( f : \Sigma \to l' \), where \( \Sigma \subset \Sigma_{n,1,n} \) is an irreducible subvariety: \( f^{-1}(Q) \) is a projective space for every \( Q \in l' \setminus Q' \). By imposing appropriate conditions on plane curves, we get a similar fibering \( g : \Sigma' \to l' \) whose fibers have dimension 1.

Thus we can assume that \( n = 3 \), and \( g \) is a 1-dimensional fibering over \( l' \), as above. Then \( g^{-1}(Q') \) consists of two (genuine) projective lines intersecting in one point plus another projective line of plane curves of degree 2 having contact of order \( 2 \) with \( l \) at \( Q' \) and passing through 2 general points. The latter line will intersect each of the former lines. One intersection point corresponds to a curve \( l_1 + l_2 + l \) with \( Q' = l_1 \cap l_2 \cap l \), and the other one to a curve \( l_3 + 2l \) with \( Q' \not\in l_3 \). However, there are no such fiberings over a projective line, a contradiction.

If \( d = n - m + 1 > 1 \), we split off a line, as in the case \( d = n - m + 2 > 2 \) above, and apply the preceding argument provided \( n - 1 = m \), etc.

It remains to consider the case when \( C \) is a general curve with unassigned singularities only, one of which is a tacnode or an ordinary triple point and the rest are nodes (clearly \( \sum Q \delta_Q(C) \leq n - 2 \)). We will treat the case when \( C \) has a tacnode and no other singularities; the remaining cases are similar. We proceed by induction on \( n - m \). If \( n = m \), we degenerate \( C \) into a sufficiently general curve \( C' \) which is a sum of \( n - 3 \) lines and a quadric tangent to one of the lines. As before, we get a degeneration of \( F \) into \( C' + l \), where \( F \) is a nodal curve of degree \( n \) having contact of order \( n \) with \( l \) at \( p \); moreover, \( F \) contains a line through \( p \), which is absurd. If \( n - 1 = m \), we can split a line off \( C \), and apply the preceding argument, etc. This proves the theorem.

2.3. Remark. \( m' = \delta_p(C + l) - \delta_p(C) \) by the genus formula [Hi].

2.4. Remark. \( m' \leq m - 1 \) with equality if and only if \( E \) contains a line through \( p \) that tends to \( l \) as \( E \) tends to \( C + l \).

3. Splitting off a line: lemmas and a proposition

3.1. Lemma. We keep the notation of Theorem 2.2 and assume, in addition, that \( C \) is irreducible and \( C \setminus p \) is smooth. Then \( W'_\text{red} \simeq \mathbb{P}^r \subset \mathbb{P}^{N_1} \) where \( N_1 = (n - 1)(n + 2)/2 \) and \( r = 3(n - 1) + g(C) - 1 - m(C) \). Let \( \tilde{U} \subset \mathbb{P}^r \) \((U \subset \mathbb{P}^r)\) be the open subset of curves \( B \) such that \( g(B) = g(C) \) and the corresponding multiplicities are equal: \( b_j(B) = b_j(C) = b_j(W') \) (in addition, \( B \) intersects \( l \setminus p \) transversely). We set \( \Gamma = C \cap (l \setminus p) \) and assume length(\( \Gamma \)) = deg(\( C \)) - \( m(C) \) \( \geq 2 \). Consider the incidence correspondence \( I \subset (l \setminus p) \times U \) and the projection \( I \to U \). Then the image of the monodromy map

\[ \mu : \pi_1(U, C) \to \text{Aut}(\Gamma) \]

is the full symmetric group.

Proof. Since the multiplicities \( b_j(W') \)’s determine \( g(C) \), \( W'_\text{red} \) is a linear system by Proposition 1.2 and its proof. We set

\[ I(2) = \{ (p_1, p_2, D) \mid p_1, p_2 \in (l \setminus p) \cap D, p_1 \neq p_2 \} \subset l \times l \times \mathbb{P}^r \]

where \( g(D) = g(C) \) and \( b_j(D) = b_j(C) \) for all \( j \). The image of \( \mu \) is the full symmetric group, provided it is twice transitive and contains a simple transposition. As in [ACGH, pp. 111 – 112], it will suffice to verify that the fibers of the projection \( I(2) \to (l \times l) \setminus \Delta \) are projective spaces and \( \tilde{U} \) contains curves simply tangent to \( l \setminus p \) at one point. Both properties follow at once from Proposition 1.2 and its proof.
3.2. Lemma. Let $W'$ be a component of $\nabla_{n,d,m} \cap \{a_{0n-m} = 0\}$ with a general member $D = C + l$ such that $\delta_p(C) = 0$. Then $\nabla_{n,d,m}$ contains a branch through $(D, b)$ with a cycle $b$ containing all nodes of $C$.

Proof. First, one can find a connected cycle on $D$ of the form

$$b = \delta_p(C) \cdot p + b_1 + b_2,$$

where $b_1$ is a sum of $n - m$ nodes of $D$ along $l \setminus p$, and $b_2$ is a sum of the nodes of $C$. Indeed, consider any branch of $\nabla_{n,d,m}$. We get a cycle of assigned singularities on $D : \delta_p(C) \cdot p + c_1 + c_2$, where $c_1$ is a sum of nodes of $D$ along $l \setminus p$. By Theorem 2.2, $C$ has unassigned nodes if and only if $c_1 > n - m$.

If $c_1 > n - m$, we proceed as follows. Let $C'$ be a component of $C$ with an unassigned point $Q_1$ on $l$. If all nodes of $D$ along $C' \setminus l$ are assigned, we consider $D - C'$ in place of $D$. Otherwise, let $Q_2$ be an unassigned node of $D$ along $C' \setminus l$, and $Q_2$ also belongs to a component $C''$. If $C''$ has an assigned point $Q_3$ along $l \setminus p$, we can interchange $Q_2$ with $Q_3$. If $C''$ has an unassigned point along $l \setminus p$, we can interchange $Q_2$ with any assigned node of $D \setminus p$. Now assume $C''$ is a line through $p$. If $C'' \setminus Q_2$ has no unassigned points, we consider $D - C''$ in place of $D$. Finally, suppose $C'' \setminus Q_2$ has an unassigned node $Q_4$, and $Q_4$ also belongs to a component $F$. Then we interchange $Q_4$ either with $F \cap l$ (if $F \cap l$ is assigned) or with any assigned node of $D$.

Now, let $W$ be a branch of $\nabla_{n,d,0}$ through $(D, b)$. We claim that $W \cap \{a_0m = \cdots = a_{0n-m+1} = 0\}$ contains the required branch of $\nabla_{n,d,m}$. Indeed, if a general member of $W \cap \{a_0m = \cdots = a_{0n-m+r} = 0\}$, $r \geq 2$, has the form $F + l$, then $F$ and $C$ have the same number of nodes, a contradiction.

3.3. Lemma. For $0 \leq g \leq (n-1)(n-2)/2$ and $0 \leq m \leq n$, there exists a unique component of $\mathcal{U}_m(n, g)$, denoted by $\nabla_{n,d,m}$ ($d = (n-1)(n-2)/2 - g$), whose general members are smooth at $p$ and irreducible.

Proof. The existence is a classical fact and the uniqueness is proved, for instance, in [R, Irreducibility Theorem (bis)]. We will present a proof of the existence of $\nabla_{n,d,m}$.

For $m = 0$ or 1, the existence is well known and follows from the deformation theory. We assume $m \geq 2$. Let $D + l$ be a curve, where $D$ is a sum of $n - 1$ general lines. We choose $d - n + m$ nodes of $D$ and $n - m$ nodes of $D + l$ along $l$ such that $D + l$ remains connected after blowing up these nodes. Regarding these $d$ nodes as assigned, we get a family of irreducible curves with $d$ nodes and no other singularities in a neighborhood of $D + l$. Intersecting this family with $m$ hyperplanes, $a_0n = \cdots = a_{0n+1-m} = 0$, we derive the existence of a required component by Theorem 2.2.

3.4. Lemma. Let $W$ be a component of $\mathcal{U}_m(n, g)$, and $E$ a general member of $W$. Let $W'$ be a component of $W \cap \{a_{0n-m} = 0\}$ and $D$ a general member of $W'$. If $l \notin D$, then $g(D) = g$, and $\sum_{Q \to p} \delta_Q(E) = \delta_p(D)$ as $E$ tends to $D$. If, in addition, $\delta_p(D) = 0$ then $W$ is smooth at $D$.

Proof. The first assertion follows from Proposition 1.2 and the genus formula [Hi]. Further, if $E$ is smooth, then $W' = W'_{\text{red}}$ and $W$ is smooth at $D$. But if $E$ has nodes, we get $\mathcal{O}_{D,W'} = (\mathcal{O}_{D,W'})_{\text{red}}$ by a simple induction on the number of nodes of $E$, hence $W$ is smooth at $D$ by Proposition 1.2.
3.5. Lemma. With the notation of Lemma 3.2,
\[ \dim T_D(\nabla_{n,d,m}) \geq \dim \nabla_{n,d,m} + n - m. \]

Proof. For simplicity, we assume \( C \) is irreducible and \( n \neq m \). Each branch \( W \) of \( \nabla_{n,d,m} \) through \( D \) determines the corresponding cycle of assigned singularities of \( D \), denoted by \( c(W) \). By Lemma 3.2, \( \nabla_{n,d,m} \) contains a branch, denoted by \( W_1 \), through \( (D,b) \) with \( c(W_1) \) containing all nodes of \( C \). Then \( c(W_1) \) contains exactly \( n - m \) nodes of \( D \) along \( l \).

Now, we exhibit \( n - m + 1 \) branches \( W_1, \ldots, W_i \) such that the span of the tangent spaces to \( W_1, \ldots, W_{i-1} \) does not contain the tangent space to \( W_i, i = 2, \ldots, n - m + 1 \). We take \( n - m + 1 \) nodes of \( D \setminus p \), containing \( n - m \) assigned (with respect to \( W_1 \)) nodes. Such nodes exist by the Principle of Degenerations because \( \sum_{l \rightarrow p} \delta_{l}(E) = \delta_{p}(D) \). By standard deformation theory, we obtain \( n - m + 1 \) branches \( \{W_i\} \) of \( \nabla_{n,d,m} \) with

\[ \sup c(W_1) \cap \cdots \cap \sup c(W_{i-1}) \not\subset \sup c(W_i) \quad (i = 2, \ldots, n - m + 1) \]

and the desired tangent spaces. (Note: if \( C \) is reducible, then each component of \( C \) contains unassigned nodes of \( D \) along \( l \).)

3.6. Remark. With the notation of Lemma 3.2, we assume that \( \delta_{p}(D) = 0 \). Let \( W \) be a branch of \( \nabla_{n,d,m} \) through \( D \) with \( n - m \) assigned nodes along \( l \) and \( d - n + m \) assigned nodes away from \( l \). Then \( T_D(W) \subset P^N \) consists of all curves of degree \( n \) having contact of order at least \( m \) with \( l \) at \( p \) and passing through all the assigned nodes of \( D \) (cf. [H, Sect. 2], [Z]); moreover, \( \{a_{0n-m} = 0\} \) intersects \( W \) transversely.

3.7. By abuse of notation, we will denote by \( \pi_N \) and \( \pi_{n,d} \) the restrictions of the corresponding projections to \( \Sigma_{n,d,m} \) and its tangent spaces.

The following lemma is a special case of Proposition 3.9 below.

3.8. Lemma. For \( 2 \leq m \leq n \) and \( n - m + 1 \leq d \leq (n - 1)(n - 2)/2 \), the scheme \( \nabla_{n,d,m} \cap \{a_{0n-m} = 0\} \) contains a component \( W' \) whose general members have the form \( D = C + l \), where \( m(C) = 1 \) and \( C \) has \( d - 1 - n + m \) nodes. Moreover, there exists \( (D,b) \subset \Sigma_{n,d,m} \) such that \( b \) contains all the nodes of \( C \) and

\[ \dim T_{(D,b)}(\Sigma_{n,d,m}) = \dim W' + 2. \]

Proof. Case: \( d = n - m + 1 \). First, we assume that \( n - m = h \) and \( d = 1 \). By a trivial dimension count, a general member of a component of \( \nabla_{n,1,n} \cap \{a_{00} = 0\} \) is a curve of the form \( F + l \), where \( F \) may be either a smooth curve through \( p \), or a curve with one node. By Theorem 2.2, \( \nabla_{n,1,n} \cap \{a_{0n-m} = 0\} \) contains the component \( K \) whose general members have the form \( F + l \) with \( F \in U_0(n - 1, (n - 2)(n - 3)/2 - 1) \). Indeed, \( \nabla_{n,1,n} \cap \{a_{0n-m} = 0\} \) contains a member that is a sum \( \sum l_i + l \) of \( n \) distinct lines meeting in a point of \( l \setminus p \) [Z, Lemma 2]. In fact, for \( 0 \leq m \leq n \), \( 1 \leq d \leq (n - 1)(n - 2)/2 \), and an arbitrary point of \( l \), \( \nabla_{n,d,m} \) contains all curves of the form \( \sum l_i + l \) that are sums of \( n \) distinct lines meeting in that point (see (3.3)).

Now, we consider \( \Sigma_{n,1,n-1} \) and its point \( \alpha = (\sum l_i + l, p) \), where \( l_i \) are general lines meeting in \( p \) \((1 \leq i \leq n - 1)\). Set \( H_0 = \pi_N^{-1}(\{a_{01} = 0\}) \). In a small neighborhood of \( \alpha \) in \( \Sigma_{n,1,n-1}, \Sigma_{n,1,n-1} \cap H_0 \) is an analytic subset, denoted by \( W \), connected
in codimension 1. This follows at once from Proposition 1.3 and Grothendieck’s connectedness theorem [G, Exp XII, Theorem 2.1] (compare [Tr, Lemma 1]). Let $U$ be the closure in $\mathbb{P}^N$ of $U_n(n, (n - 1)(n - 2)/2 - 1) \setminus \mathcal{V}_{n,1,n}$. Clearly $U$ is a linear system. We can describe the branches of $\mathcal{W}_{\text{red}}$:

a) a (unique) branch $\mathcal{A}$ of $\Sigma_{n,1,n}$ (see Lemma 3.3 and Proposition 1.3);

b) the branches of $\pi_{n,1}^{-1}(U)$ inside $\mathcal{W}_{\text{red}}$; and

c) a (unique) branch whose general point is $(E + l, c)$, where $E$ is a general curve of degree $n - 1$ through $c$ ($c \neq p$).

Since $U$ is a hyperplane in $\mathcal{V}_{n,0,0}$, $H = U \cap \mathcal{V}_{n,1,n}$ is a hypersurface in the projective space $U$. We will show that $H_{\text{red}}$ is the required $W'_{\text{red}}$. By a trivial dimension count, it will suffice to show that $H_{\text{red}}$ contains a component that lies in $\{a_{00} = 0\}$. Suppose to the contrary that all general points of $H \cap \{a_{00} = 0\}$ are curves of the form $F' + l$, where $F'$ is a general curve of degree $n - 1$ through $p$ with one node. Let $K'$ denote the corresponding component of $H \cap \{a_{00} = 0\}$. The key point is that $K'_{\text{red}}$ is not a linear system. We get

$$\dim H = \dim K' + 1 = n(n + 3)/2 - n - 2 = n(n - 1)/2 + n - 2.$$  

First, we will show that each component of $H_{\text{red}}$ is a hyperplane in $U$. Let $E' = l'_1 + \cdots + l'_{n-1} + l \in K'$ be a nodal curve, where $l'_1, \ldots, l'_{n-1}$ are general lines and $p \in l'$. We choose one general point on each $l'_i$ ($1 \leq i \leq n - 1$), and consider the set $M$ consisting of these points and the nodes of $E' \setminus p$. Thus $M$ has dim $H$ points. To each point, there corresponds a hyperplane in $\mathbb{P}^N$ of curves passing through that point. Clearly those hyperplanes will intersect $K'$ in a reduced scheme consisting of one point, namely $E'$. On the other hand, if a component $H' \subset H_{\text{red}}$ is not a hyperplane, then $H' \cap \{a_{00} = 0\}$ intersects those dim $H$ hyperplanes in a subscheme containing more than one point, because $H' \cap \{a_{00} = 0\}$ belongs to the hyperplane corresponding to the point $l'_{n-1} \cap l$.

Now take any component of $H_{\text{red}}$, say $H' \subset U$. Clearly $\dim(H' \cap \{a_{00} = 0\}) \geq \dim(H' \cap K')$ and we derive a contradiction. We have also proved that

$$H_{\text{red}} = U \cap \mathcal{V}_{n,1,n} \cap \{a_{00} = 0\} = U \cap \{a_{00} = 0\} = W'_{\text{red}}.$$  

We set $F = \sum l_i$, where $l_i$ are general lines meeting in $p$ ($1 \leq i \leq n - 1$), and take an additional general line $l'$ through $p$. To obtain two lines in $\Sigma_{n,1,n}$ through $\alpha$, we move $F$ along the two lines, namely $l$ and $l'$. Thus

$$\pi_{n,1} : T_\alpha(\Sigma_{n,1,n}) \to T_p(\mathbb{P}^2)$$

is surjective hence $\dim T_\alpha(\Sigma_{n,1,n}) \geq \dim W' + 2$. Since

$$\mathcal{V}_{n,1,n} \cap \{a_{00} = a_{1n-1} = 0\} = W'_{\text{red}},$$  

$\{a_{00} = 0\}$ is transversal to $\mathcal{V}_{n,1,n}$ at $F + l$, and $\{a_{1n-1} = 0\}$ is transversal to $\mathcal{V}_{n,1,n} \cap \{a_{00} = 0\}$ at $F + l$. Therefore $a_{00}$ and $a_{1n-1}$ produce linearly independent elements in $T_{(D,p)}(\mathcal{V}_{n,d,m})$ and $\dim T_{(D,p)}(\Sigma_{n,1,n}) = \dim W' + 2$.

Now assume $n \neq m$. We proceed by induction on $d$. Consider $\mathcal{V}_{n,d-1,m} \cap \{a_{0n-m} = 0\}$. By Lemma 3.3, it contains $\mathcal{V}_{n,d-1,m+1}$. By induction hypothesis, $\mathcal{V}_{n,d-1,m+1} \cap \{a_{0n-m} = 0\}$ contains a component $W'$ that is also the required
component of $\nabla_{n,d,m} \cap \{a_{0n-m} = 0\}$ (Theorem 2.2). Indeed, intersect an appropriate branch of $\nabla_{n,d-1,m}$ with a branch of a hypersurface $\nabla_{n,1,0} \subset \mathbb{P}^N$ corresponding to an unassigned node along $l$ of a general member of $W'$. The remaining assertions of the lemma also follow at once (see Lemma 3.2).

General case. We consider $\nabla_{n,d',m}$ with $d' = n-m+1$. One can find a component of $\nabla_{n,d',m} \cap \{a_{0n-m} = 0\}$ with a general member $D' = C' + l$ that satisfy the lemma. We degenerate $C'$ into a nodal curve $C$ with $d-d'$ nodes and $m(C') = m(C)$ (Lemma 3.3), and obtain a curve $D = C + l$ that is a general member of the required component of $\nabla_{n,d,m} \cap \{a_{0n-m} = 0\}$. The remaining assertions of the lemma also follow at once (see Lemma 3.2).

3.9. Proposition. For $m \leq n$, let $W'$ be a component of $\nabla_{n,d,m} \cap \{a_{0n-m} = 0\}$ with a general member $D = C + l = C_1 + \cdots + C_q + l$, where each $C_r$ is irreducible $(1 \leq r \leq q)$. Let $a = ep + b \in \text{Sym}^d(\mathbb{P}^2)$, $p \notin \text{sup}(b)$, be a cycle of assigned singularities of $D$ (with respect to a branch of $\nabla_{n,d,m}$) such that $b$ contains all nodes of $D \setminus l$. Then

\begin{enumerate}
  \item $\dim T_{(D,a)}(\Sigma_{n,d,m}) = \dim W' + e + 1$;
  \item $\dim T_D(\nabla_{n,d,m}) \geq \dim \nabla_{n,d,m} + e + n - m$; and
  \item $\delta_p(C) = 0$, that is, $C$ is smooth at $p$.
\end{enumerate}

Moreover, for every integer $e$, $0 \leq e \leq \min\{d - n + m, m - 2\}$, there exist components $W'$ and points $(D,a) \in \Sigma_{n,d,m}$ as above.

Proof. For $e = 0$, the proposition follows at once from Theorem 2.2 and Lemma 3.3. In fact, there is then only one (smooth) branch of $\Sigma_{n,d,m}$ through $(D,a)$. We assume that $e \geq 1$ and proceed by induction on $e$. We have two natural inclusions:

$$\mathbb{P}^e = \text{Sym}^e(l) \subset \text{Sym}^e(\mathbb{P}^2), \quad T_{ep}(\text{Sym}^e(\mathbb{P}^2)) \subset T_a(\text{Sym}^d(\mathbb{P}^2)),$$

and a natural map $T_{(D,a)}(\Sigma_{n,d,m}) \to T_{ep}(\text{Sym}^e(\mathbb{P}^2))$.

Case: sup$(b) \subset l$. We observe that $T_{(D,a)}(\Sigma_{n,d,m}) \subset T_{(D,a)}(\Sigma_{n,d,0})$, $\Sigma_{n,d,0}$ is smooth at $(D,a)$, and $\pi_N$ induces a one-to-one map of a small neighborhood of $(D,a)$ in $\Sigma_{n,d,0}$ onto its image in $\nabla_{n,d,0}$ ([AC, Sect. 4], [DH2, Sect. 4], [Ta]). So $\pi_N$ induces an embedding $T_{(D,a)}(\Sigma_{n,d,m}) \subset T_D(\nabla_{n,d,m})$. By Lemma 3.5, $\nabla_{n,d,m}$ may have several branches through $D$ (at least if $\delta_p(C) = 0$ and $n > m$), and we will see, among other things, that

$$\dim T_D(\nabla_{n,d,m}) \geq \dim T_{(D,a)}(\Sigma_{n,d,m}) + n - m.$$  

By Theorem 2.2 and Remark 2.3,

$$m(C) = m - (g - g(C)) - 2 \leq e = \delta_p(C+l)$$

with equality only if $\delta_p(C) = 0$. By induction hypothesis, we get $d - 1 \leq n - 2$. We will also show that

$$W' \cap \{a_{0n-m-1} = \cdots = a_{00} = a_{1n-1} = \cdots = a_{1n-e} = 0\} = W'_\text{red}.$$  

Consider a general point of $\Sigma_{n,d,m}$ that is very near $(D,a)$. Regarding one of its nodes approaching $p$ as virtually non-existent, we get a component $W''$ of
\[ \nabla_{n,d-1,m} \cap \{ a_{0n-m} = 0 \} \text{ such that } W' \subset W'' \text{ and a general member of } W'' \text{ contains } l \text{ (Theorem 2.2)}. \] By induction hypothesis

\[ \dim W'' \cap \{ a_{0n-m-1} = \cdots = a_{00} = a_{1n-1} = \cdots = a_{1n-e+1} = 0 \} = \dim W'' (= \dim W' + 1). \]

Since

\[ W' \cap \{ a_{0n-m-1} = \cdots = a_{1n-e} = 0 \} \subset W'' \cap \{ a_{0n-m-1} = \cdots = a_{1n-e} = 0 \} = W_{red} \cap \{ a_{1n-e} = 0 \}, \]

we get

\[ \dim W' \cap \{ a_{0n-m-1} = \cdots = a_{1n-e} = 0 \} \leq \dim W'. \]

This inequality is an equality only if \( W' \cap \{ a_{0n-m-1} = \cdots = a_{1n-e} = 0 \} = W_{red} \).

Now we assume, in addition, that \( e \geq 2 \) and \( n = m \), and generalize Lemma 3.8 where the case \( e = 1 \) was discussed. Consider \( \Sigma_{n,e,n} \) and its point \( \alpha_0 = (nl, ep) \). Set \( H_0 = \pi_N^{-1}(\{a_{01} = 0\}) \). Let \( U \) be the closure in \( \mathbb{P}^N \) of

\[ U_n(n, (n-1)(n-2)/2 - e) \setminus \nabla_{n,e,n}. \]

In a small neighborhood of \( \alpha_0 \) in \( \Sigma_{n,e,n} \cap H_0 \) is an analytic subset, denoted by \( \mathcal{W} \), connected in codimension 1. We can describe the branches of \( \mathcal{W}_{red} \):

a) a (unique) branch \( \mathcal{A} \) of \( \Sigma_{n,e,n} \) (see Lemma 3.3 and Proposition 1.3);

b) the branches of \( \pi_N^{-1}(U) \) inside \( \mathcal{W}_{red} \); and

c) for each \( t, 0 \leq t \leq e - 1 \), the branches, denoted by \( \mathcal{A}_t, \mathcal{A}'_t, \ldots \), whose general points have the form \((E + l, tp + c)\), where \( p \notin \text{sup}(c) \), \( \delta_p(E) = 0 \), and \( E \) is a general member of a component of \( U_t(n - 1, (n - 2)(n - 3)/2 - e + 1 + t) \).

In (c), we used the induction hypothesis; note that \((E + l, tp + c)\) have at least one assigned node along \(l, p \) (Theorem 2.2).

To prove (iii), we consider the branch \( \mathcal{W}'' \) of \( W'' \) whose general member has the form \( D'' = C'' + l \) with \( m(C'') = e - 1 \). Let \( \mathcal{V} \) be a branch of \( \pi_N(\mathcal{A}) \cap \{ a_{00} = 0 \} \) whose general member is the curve \( D = C + l \). We may assume that \((D', (e - 1)p)\) tends to \((D, (e - 1)p)\) (note that \( \delta_p(D) = e \) while \( \delta_p(D') = e - 1 \)).

We claim that \( \mathcal{V}_{red} \subset \pi_N(\mathcal{A}_{e-1}) \) for a suitable \( \mathcal{A}_{e-1} \), and \((E + l, (e - 1)p + c)\) tends to \((D, a)\). Then \( c \) will approach \( p \), and we get \( m(C) = e \) hence \( C \) is a smooth curve. To prove the claim, we observe that

\[ \mathcal{V} \subset \pi_N(\Sigma_{n,e,n} - \Sigma_{n,e-1,n-1}) \cap \pi_N(H_0). \]

As before, we can describe the branches of \( \pi_N(\Sigma_{n,e-1,n-1}) \cap \pi_N(H_0) \). By induction hypothesis, these branches have general members of the form \( E' + l \) with \( m(E') \leq e - 2 \), provided those members contain \( l \). By a trivial dimension count, only the branch with \( m(E') = e - 2 \) can contain \( \mathcal{W}''_{red} \) and \( \pi_N(\mathcal{A}_{e-1})_{red} \). Therefore \( \mathcal{V}_{red} = \pi_N(\mathcal{A}_{e-1})_{red} \). So \( \mathcal{V}_{red} \subset \pi_N(\mathcal{A}_{e-1}) \). Note that \( \mathcal{W'}_{red} \) is a linear system.

Next, we will prove (i) and (ii) in case \( n = m \) (that is, \( a = ep \)). Let

\[ \alpha = (l_1 + \cdots + l_{n-2} + 2l, ep) \in \Sigma_{n,e,n}, \]
where $l_1, \ldots, l_{n-2}$ are general lines meeting in $p$. Moving $l_1$ along $l$, we get a point

$$(l_1' + l_2 + \cdots + l_{n-2} + 2l, \Sigma p_i) \in \Sigma_{n,e,n}, \quad p_1 = l_1' \cap l, \quad p_i = l_1' \cap l_i \ (2 \leq i \leq e).$$

We then take arbitrary $e$ points on $l$: $q_1, \ldots, q_e$. Consider $e$ general lines $l_i'$ with $q_i = l_i' \cap l$ ($1 \leq i \leq e$). Then

$$(l_1' + \cdots + l_e' + l_{e+1} + \cdots + l_{n-2} + 2l, \Sigma q_i) \in \Sigma_{n,e,n};$$

see the proof of Lemma 3.3 (existence). This shows that the image of the natural map

$$T_{\alpha}(\Sigma_{n,e,n}) \to T_p(\text{Sym}^e(\mathbb{P}^2))$$

contains $T_p(\text{Sym}^e(l))$ and has dimension at least $e + 1$. It follows that

$$\dim T_{\alpha}(\Sigma_{n,e,n}) \geq \dim W' + e + 1.$$ 

Since

$$\nabla_{n,d,m} \cap \{a_{00} = a_{1n-1} = \cdots = a_{1n-e} = 0\} = W'_{\text{red}},$$

we get $\dim T_{(D,\text{ep})}(\Sigma_{n,e,n}) \geq \dim W' + e + 1$, as in the proof of Lemma 3.8. This inequality, however, must be an equality.

To prove (i) - (iii) in case $n \neq m$, we proceed by induction on the number of assigned nodes along $l\setminus p$, as in the proof of Lemma 3.8 (case: $n \neq m$). Note that if $\sum a_{jk}x^jy^kz^{n-j-k} = 0$ is an equation of a general member of $\nabla_{n,d,m}$, then $a_{0n-m}, \ldots, a_{00}, a_{1n-1}, \ldots, a_{1n-e}$ produce $n - m + e + 1$ linearly independent elements in $T_D(\nabla_{n,d,m})$.

To establish the existence of $W'$, we generalize the corresponding argument from Lemma 3.8. We assume $n = m$, because the case $n \neq m$ will follow by induction on $d$, as in Lemma 3.8. Consider the branches of $\Sigma_{n,e,n} \cap \pi_N^{-1}(\{a_{00} = 0\})$ through the point $\alpha_0$. It is known what kind of branches one may expect (Theorem 2.2). By induction hypothesis, for each $t, 0 \leq t \leq e - 1$, we get branches whose general points have the form $(E + l, \text{ep} + c)$, where $p \notin \text{sup}(c), \delta_{\text{ep}}(E) = 0$, and $E$ is a general member of a component of $U_t(n - 1, (n - 2)(n - 3)/2 - e + t)$. We claim that for $t = e$, we get similar branches.

Let $\gamma(e, t), 0 \leq t \leq e$, denote the number of conditions imposed on curves of sufficiently large degree to have contact of order at least $t$ with $l$ at $p$ and a singularity at $p$ with $\delta_p \geq e - t$ [S, Lemma 6]. Here the general members of the ambient variety are curves, say of degree $r$, having smooth contact of order $t$ with $l$ at $p$ and $e - t$ nodes and no other singularities; we also consider the corresponding general points $(F, \text{ep} + c) \in \mathbb{P}^R \times \text{Sym}^e(\mathbb{P}^2)$, where $R = r(r + 3)/2$ and $p \notin \text{sup}(c)$.

First, we assume that $n - 1$ is sufficiently large. Thus the dimension of the subfamily of the corresponding ambient variety, whose general members have degree $n - 1$, contact of order at least $t$ with $l$ at $p$, and a singularity at $p$ with $\delta_p \geq e - t$, equals

$$(n - 1)(n + 2)/2 - e - \gamma(e, t) = n(n + 3)/2 - e - n - \gamma(e, t) - 1 \quad (0 \leq t \leq e).$$

For $0 \leq t \leq e - 1$, let $\Sigma_t \subset \mathbb{P}^N \times \text{Sym}^e(\mathbb{P}^2)$ be the closed subvariety of the corresponding ambient variety, whose general members have the form $(F, \text{ep})$, where $F$ has degree $n$, contact of order at least $t$ with $l$ at $p$, and a singularity at $p$ with
\[ \delta_p \geq e - t. \] In particular, \( F \) has a singularity at \( p \). Utilizing the proof of the existence in Lemma 3.8, we get

\[
\dim(\Sigma_{n,e,n} \cap \pi_N^{-1}(\{a_{00} = 0\}) \cap \Sigma_t) = \dim(\Sigma_{n,e,n} \cap \Sigma_t) \geq n(n+3)/2 - e - n - \gamma(e, t),
\]

where \( \dim(\cdot) \) means, as usual, the dimension of the components of maximal dimension. Comparing this estimate with the previous one for \( t \leq e - 1 \), we obtain a required \( W' \).

Finally, if \( n - 1 \) is not sufficiently large, we take a sufficiently large integer \( r \) and apply the preceding argument to \( \Sigma_{r+1,e,r+1} \) in place of \( \Sigma_{n,e,n} \). We then split \( r - n + 1 \) general lines off.

**General case.** As we have already seen before, the nodes of \( D \setminus l \) do not play an essential role. The existence is established by induction with the help of Lemma 3.3 (uniqueness); see the corresponding argument in Lemma 3.8. To prove (iii), we do not need a generalization of Proposition 1.3. Assume \( D \setminus l \) has \( v \) nodes. Regarding those nodes as virtually non-existent, we apply the above discussion to \( \nabla_{n,d,m} (d' = d - v) \) and the original curve \( D \). We now consider \( \Sigma_{n,d',n-1} \cap H_0 \cap H_1 \cap \cdots \cap H_v \) in place of \( \Sigma_{n,e,n-1} \cap H_0 \), where \( H_1, \ldots, H_v \) are the branches of a hypersurface in \( \mathbb{P}^N \) corresponding to the virtually non-existent nodes. In fact, the singular curves form a hypersurface \( \nabla_{n,1,0} \subset \mathbb{P}^N \), and each node of a curve of degree \( n \) determines a unique branch of \( \nabla_{n,1,0} \) through that curve. By Proposition 1.2, the multiplicities \( b_j \)'s remain unchanged when we intersect the ambient families with \( H_1, \ldots, H_v \).

The same argument establishes (i) and (ii) as well.

**3.10 Remark.** For \( 2 \leq m \leq n \), \( \nabla_{n,d,m} \cap \{a_{0n-m} = 0\} \) contains components with general members of the form \( C + l = C_1 + \cdots + C_q + l, \ l \notin C, \) and each \( C_r \) is irreducible, where either \( q = 1 \) or \( \deg(C_r) = 1 \) for \( 2 \leq r \leq q \). Indeed, if \( q \geq 2 \) and \( m(C_1) \neq 0 \), then \( m(C_2) = 0 \) by Proposition 3.9(iii). We suppose \( \deg(C_2) \geq 2 \). The following surgeries establish the existence of the required components.

We will decrease the degree of \( C_2 \) and increase the degree of \( C_1 \). First, we degenerate \( C_2 \) into a nodal curve \( C'_2 + L \), where \( L \) is a general line and \( C'_2 \) a sufficiently general curve [H]. By the Principle of Degenerations, \( C'_2 + L \) has acquired several additional nodes. We then consider \( C_1 + C'_2 + L \) and smooth several nodes of \( C_1 + C'_2 \) and \( C_1 + L \).

Applying similar surgeries, one can easily obtain a list of all components of \( (\nabla_{n,d,m} \cap \{a_{0n-m} = 0\})_{red} \) whose general members contain \( l \). We omit details.

4. **Admissible schemes: notation, definitions, and lemmas**

**4.1. The hyperplanes \( \{H_\sigma\} \).** We consider the following sequence of \( n + 1 \) hyperplanes in \( \mathbb{P}^N \): \( a_{0n} = 0, a_{0n-1} = 0, \ldots, a_{00} = 0 \). If \( G \in \mathbb{P}^N \) belongs to the intersection of these hyperplanes, then

\[
f_G(X, Y, Z) = X\left(\sum a_{jk}X^{j-1}Y^kZ^{n-j-k}\right) = X\{\cdots\} + \sum a_{1k}Y^kZ^{n-1-k}\}.
\]

Next, we consider the following \( n \) hyperplanes in \( \mathbb{P}^N \): \( a_{1n-1} = 0, a_{1n-2} = 0, \ldots, a_{10} = 0, \) etc. We obtain \( \kappa \) hyperplanes in \( \mathbb{P}^N \), where \( \kappa = n + 1 + n + \cdots + 4 \). If \( G \in \mathbb{P}^N \) belongs to the intersection of these \( \kappa \) hyperplanes, then \( G = (n - 2)l + D \). We shall employ a sequence \( \{H_\sigma\} \) of \( \kappa + 3 \) hyperplanes in \( \mathbb{P}^N \), where

\[
H_1 = \{a_{0n} = 0\}, H_2 = \{a_{0n-1} = 0\}, \ldots, H_{\kappa+3} = \{a_{n-20} = 0\}.
\]
4.2. Standard exact sequences. Let $V \subset \mathbb{P}^N$ be an arbitrary projective scheme and $H \subset \mathbb{P}^N$ a hyperplane. In the sequel, we denote by $h$ a form defining $H$. For a positive integer $k$, let $A^k(h) \subset \mathcal{O}_V$ denote the annihilator sheaf of the ideal sheaf $(h^k) \subset \mathcal{O}_V$. We have a standard exact sequence of sheaves on $V$:

\[(3_k)\quad 0 \rightarrow \mathcal{O}_{V^k}(r - k) \otimes_{h^k} \mathcal{O}_V(r) \rightarrow \mathcal{O}_{V^k}(r) \rightarrow 0,\]

where $\mathcal{O}_{V^k} = \mathcal{O}_V/(h^k)$ and $\mathcal{O}_{V^k} = \mathcal{O}_V/A^k(h)$. This sequence yields a cohomology sequence

\[(4_k)\quad H^i(\mathcal{O}_{V^k}(r - k)) \rightarrow H^i(\mathcal{O}_V(r)) \rightarrow H^i(\mathcal{O}_{V^k}(r)).\]

Therefore $H^i(\mathcal{O}_V(r)) = 0$ if $H^i(\mathcal{O}_{V^k}(r - k)) = H^i(\mathcal{O}_{V^k}(r)) = 0$.

Given a scheme $V$, we denote by $\min \sup(V)$ the set of minimal associated points of $V$.

4.3. Lemma. With the above notation, let $s = s(V, H)$ be the smallest integer such that $A^s(h) = A^{s+1}(h) = \ldots$. Then $V \setminus V_s = V^s \setminus V_s$ and

$$\min \sup(V^s) = \min \sup(V) \setminus \sup(V_s) \subset \min \sup(V^k), \quad k \in \mathbb{Z}_+.\$$

Proof. The problem is local and we may restrict everything to $\mathcal{O}_{v,V}$; we denote the restrictions by $[\cdot]$. Clearly $\min \sup(V^s) \setminus \sup(V_s) = \min \sup(V) \setminus \sup(V_s)$. It remains to show that $\min \sup(V^s) \subset \sup(V) \setminus \sup(V_s)$. Let $[A^s(h)] = \cap_i J_i$ be a minimal primary decomposition. Suppose $\text{rad}(J_1) \in [\min \sup(V^s)]$ and $[h] \in \text{rad}(J_1)$. Take $a \in \cap_{i \geq 2} J_i \setminus J_1$. Then $a[h]^e \in [A^s(h)]$ for $e >> 0$. Hence $a \in [A^s(h)]$, a contradiction.

4.4. Definition. A closed irreducible family of curves of degree $n$ is maximal if its general members are of the form $D + (n - w)l$, where $D$ is either a general member of a component $W \subset U_m(w, g)$ or $D = 0$.

4.5. Definition. We consider $\overline{V}_{n,d,m}$ with $m \leq n$. A closed subscheme $W \subset \mathbb{P}^N$ of dimension $\geq 2$ is $\overline{V}_{n,d,m}$-admissible (or simply admissible) if either $W = \overline{V}_{n,d,m}$, or $W = V^k$ or $V_k$ in (3_k), where $V$ is admissible, $V_\text{red} \not\s H = H_\sigma$ for the smallest possible $\sigma = \sigma(V)$, and $k$ is the smallest integer such that $\min \sup(V^k) = \min \sup(V) \setminus \sup(V_k)$.

4.6. Remark. Let $V$ be an admissible scheme and $V_\text{red} \not\s H = H_\sigma$ for the smallest possible $\sigma$. For every positive integer $s$ in (3_s), each component of $(V_s)_\text{red}$ is a maximal family by Proposition 1.2(a) and Theorem 2.2. So, by Lemma 4.3, the integer $k \leq s(V, H)$ and each component of $(V^k)_\text{red}$ is a maximal family.

Admissible schemes can be generically non-reduced as the following example illustrates; see also Section 3.

4.7. Example. The family $\overline{V}_{4,2,2}$ lies in $\mathbb{P}^{12}$ (see $\mathbb{P}^{14}$). The scheme $\overline{V}_{4,2,2} \cap H_3$ contains an irreducible subscheme $W$ whose general member is a curve of the form $D = C + l$, where $C$ is a general cubic. Exactly three transversal branches of $\overline{V}_{4,2,2}$ are passing through $D$, one for each pair of the nodes of $D$. By a simple dimension count $\dim T_D(\overline{V}_{4,2,2}) = 12$. Hence $\overline{V}_{4,2,2} \cap H_3$ is non-reduced at $D$, and $\dim T_D(W) = 11$. 

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4.8. Definition. Set $H_0 = \nabla_{n,0,0} = \mathbb{P}^N$. Let $V$ be a $\nabla_{n,d,m}$-admissible subscheme, and $K$ a component of $V$. We define a subset $I(K) = I(K, V) \subset \mathbb{Z}$ as follows:

$$i \in I(K) \iff K_{\text{red}} \subset H_i, \ V \cap H_0 \cap \cdots \cap H_{i-1} \not\subset H_i.$$

4.9. Lemma. $G(\nabla_{n,d,m}) = \nabla_{n,d,m}$ and $G(H_\sigma) = H_\sigma$ for $\sigma = 1, \ldots, \kappa + 3$; moreover, $G(W) = W$ for every admissible $W$.

Proof. Since $G \subset \text{PGL}(2) \subset \text{PGL}(N)$, we get $G(\nabla_{n,d,0}) = \nabla_{n,d,0}$. By looking at equations of curves (see Section 1), we conclude that $G(H_\sigma) = H_\sigma$, hence $G(\nabla_{n,d,p}) = \nabla_{n,d,p}$ if $p \leq n$. Consider (3) with an admissible $V$ such that $G(V) = V$ and $H = H_\sigma$. Since $G(H) = H$, we get $G(A^k(h)) = A^k(h)$ for every $k$. Thus $G(W) = W$ for every admissible $W$.

4.10. Notation. Let $V$ be an admissible scheme, and $K$ an irreducible subvariety of $V$ with a general member

$$C + \mu l = C_1 + \cdots + C_q + \mu l, \ l \not\subset C,$$

where each $C_r$ is irreducible ($1 \leq r \leq q$). We say that $K$ is a subvariety of level $\mu$, contact $m = m(C)$, and genus $g = g(C)$. Consider (3) as in Definition 4.5. A component $K'$ of $V_k$ is said to be new if $K'_\text{red}$ is not a component of $V_{\text{red}}$.

4.11. Definition. a) With the above notation, a component $K$ of $V$ is said to be $\delta_p$-nice if $\delta_p(C_r) = 0$ ($1 \leq r \leq q$).

b) A $\delta_p$-nice component $R$ of a $\nabla_{n,d,m}$-admissible scheme $W$ is said to be nice if either $R = W = \nabla_{n,d,m}$, or $W = V_k$ or $V^k$ in (3) of Definition 4.5 and $R$ is a component of $W$ coming from a nice component $M$ of $V$, that is, $R_{\text{red}}$ is a component of $(M \cap H_\sigma(V))_{\text{red}}$ or $R_{\text{red}} = M_{\text{red}}$.

c) If $K$ is a nice ($\delta_p$-nice) component of $V$, then $K_{\text{red}}$ is said to be a nice ($\delta_p$-nice) component of $V_{\text{red}}$.

4.12 Lemma. With the above notation, we have:

a) If an admissible scheme $V$ contains a component of dimension $v$, level $\mu$, contact $m$, and genus $g$ whose general members contain $u$ general lines through $p$ and $v$ general lines, then it contains an irreducible subvariety with the same data of a nice component of $V$.

b) One and only one of the following conditions holds:

i) all nice components of $V_k$ are new and $K_{\text{red}} \not\subset H_\sigma(V)$ for every nice component $K$ of $V$;

ii) a nice component $K'$ of $V_k$ is new if and only if $K'_{\text{red}} \not\subset H_\sigma(V_k)$; in addition, $V$ and $V_k$ have nice components of different levels, and all nice components of $V_k$ of smaller level are new; or

iii) $V_k$ has no new nice components and all its nice components have the same level.

c) If $V$ has two nice components of level $\mu$ and contacts $m$ and $m_2$, respectively, then it has a nice component of level $\mu$ and contact $e$ for every $e$ ($m_1 \leq e \leq m_2$). Moreover $m_1 - g_1 = m_2 - g_2$, where $g_i$ is the genus of the component of contact $m_i$ ($i = 1,2$).

Proof. The lemma is trivial if $V = \nabla_{n,d,m}$. We proceed by induction on the size of an admissible scheme. Assume the lemma for all admissible $W \supset V$. Then $V_k$ satisfies the lemma.
Now we turn to \( V_k \). To begin with, we make the following two remarks.

First, consider \( \overline{V}_{n,d,m} \) and \( \overline{V}_{n,d,m+1} \) with \( 2 \leq m \leq n - 1 \). Let \( U' \) denote a union of the components of \( (\overline{V}_{n,d,m} \cap \{ a_{0n-m} = 0 \})_{\text{red}} \) that lie in \( \{ a_{0n-m-1} = 0 \} \).

By Proposition 1.2(a), the general members of \( U' \) contain \( l \). Furthermore, every component of \( (\overline{V}_{n,d,m+1} \cap \{ a_{0n-m-1} = 0 \})_{\text{red}} \), whose general members contain \( l \), lies in \( U' \) by Proposition 3.9.

Now, consider a component \( W' \) of \( \overline{V}_{n,d,m} \cap \{ a_{0n-m} = 0 \} \), \( m \leq n \), whose general members contain \( l \). We assume \( W' \) has contact \( m' < \min \{ d - n + m, m - 2 \} \); so \( W'_{\text{red}} \subseteq \{ a_{1n-m'} = 0 \} \). Let \( U'' \) denote a union of the components of \( (\overline{V}_{n,d,m} \cap \{ a_{0n-m} = 0 \})_{\text{red}} \) that lie in \( \{ a_{1n-m'-1} = 0 \} \). Then \( (W' \cap \{ a_{1n-m'-1} = 0 \})_{\text{red}} \subseteq U'' \). In view of Proposition 3.9, this follows at once by induction on \( d \), the case \( d = n - m + m' + 1 \) being trivial.

**Completion of the proof.** First, we assume that \( K_{\text{red}} \not\subset H_{\sigma(V)} \) for every nice component \( K \) of \( V \). Then all nice components of \( V_k \) are new, and \( V_k \) satisfies (a) by induction hypothesis and Proposition 3.9. Note that if an irreducible curve \( E \) degenerates into \( C + l \) as in Theorem 2.2, the curve \( C \) cannot contain lines through \( p \) by Theorem 2.2 and the Principle of Degenerations, although it may contain general lines provided \( E \) has sufficiently small genus (compare Remark 3.10). Furthermore \( m(C) \leq m(E) - 2 \) by Remark 2.4. The assertion (c) also follows from Proposition 3.9 and the formula \( m' - g' = m - g - 2 \) of Theorem 2.2.

Now, we assume that \( V \) contains nice components of different levels. Then, by (i) or (ii) for \( V \) (in place of \( V_k \)), all nice components of \( V \) of smaller level are new and \( K_{\text{red}} \not\subset H_{\sigma(V)} \) for every nice component \( K \) of \( V \) of smaller level. By the remarks, \( V_k \) satisfies (ii) or (iii); \( V_k \) satisfies (iii) if and only if all nice components of \( V_k \) have the same level. Furthermore, \( V_k \) satisfies (a) and (c) as in the preceding case.

Finally, we assume that all nice components of \( V \) have the same level and \( V \) has nice components of different contacts. By the remarks, \( V_k \) has no new nice components so it satisfies (iii). This proves the lemma.

Now, we come to the following key lemma which will enable us to prove Theorem 5.2 (Vanishing Theorem).

**4.13. Lemma.** Let \( W \) be an arbitrary \( \overline{V}_{n,d,m} \)-admissible scheme, and \( g \in W \) a general point. Let \( W(g) \subset W \) be a nice component whose general point is \( g \). Then

\[ \dim T_g(W(g)) \geq \dim W(g) + \# I(W(g)). \]

Furthermore, a basis of \( T_g(W(g))_{\text{red}} \) together with \( \# I(W(g)) \) elements of \( T_g(W(g)) \) corresponding to the hyperplanes \( H_i \) \( (i \in I(W(g))) \) form a linearly independent subset of \( T_g(W(g)) \).

**Proof.** The lemma is trivial if \( W = \overline{V}_{n,d,m} \). We proceed by induction on the size of \( W \). Consider (3) as in Definition 4.5 (with \( s = k \)). Assume the lemma for all admissible \( W \supseteq V \). Then \( V^s \) satisfies the lemma.

Now we turn to \( V_s \). Let \( V_s(g_s) \) be a nice component of \( V_s \) with a general point \( g_s \). Then \( V_s(g_s) \) is coming from a nice component of \( V \) denoted by \( V(g_s) \).

First, we consider the case when \( V_s(g_s)_{\text{red}} \) is a component of \( V_{\text{red}} \), that is, \( V_s(g_s)_{\text{red}} \subset H_{\sigma} \). If \( s = 1 \) then \( \# I(V_s(g_s)) \leq \# I(V(g_s)) - 1 \), and the lemma follows by induction. But if \( s \geq 2 \), then \( \dim T_{g_s}(V_s(g_s)) = \dim T_{g_s}(V(g_s)) \) and the lemma follows.
Next, we assume that \( g_s \) is not a general point of \( V \). Let \( D_s \) denote the curve corresponding to \( g_s \). Let \( D \) be a general point of \( V(g_s) \); \( D \) degenerates into \( D_s \). We have
\[
D = C + \mu l, \quad D_s = C_s + \mu_sl, \quad (l \notin C, \, C_s).
\]
By (5) for \( V(g_s) \), we get
\[
\dim \mathbf{T}_{D_s}(V(g_s)) \geq \dim V(g_s) + \#I(V(g_s)).
\]
If \( \mu = \mu_s \), then we consider two cases, \( s = 1 \) and \( s \geq 2 \), and deduce the lemma by induction. For instance, if \( s = 1 \) then \( \#I(V_s(g_s)) \leq \#I(V(g_s)) \), since \( V(g_s)_{\text{red}} \subset H_i \) for \( i \leq \sigma - 1 \) (see Definition 4.5).

Throughout the rest of the proof, we assume that \( \mu \neq \mu_s \), that is, \( \mu_s = \mu + 1 \). Let
\[
C = E + F, \quad C_s = E_s + F,
\]
where \( E \) is an irreducible curve that degenerates into \( E_s + l \). We claim:
\[
(6) \quad \dim \mathbf{T}_{D_s}(V(g_s)) \geq \dim \mathbf{T}_D(V(g_s)) + e + \deg(E) - m(E),
\]
where \( e \) is the number of nodes of \( E \) approaching \( p \) as \( E \) tends to \( E_s + l \). In the proof of the claim, we can assume, without loss of generality, that \( V(g_s) = V(g_s)_{\text{red}} \).

Indeed, \( \dim \mathbf{T}_v(V(g_s)) \) is an upper semicontinuous function on the set of points of \( V(g_s) \) and nilpotents can only "improve" the inequality. Assuming \( C \) is reducible, we consider the product
\[
\Pi = \mathbb{P}^{N_E} \times \mathbb{P}^{N_F}, \quad N_E = \deg(E)(\deg(E) + 3)/2, \quad N_F = \deg(F)(\deg(F) + 3)/2,
\]
and a natural morphism \( \eta : \Pi \to \mathbb{P}^M \), where \( M = \deg(C)(\deg(C) + 3)/2 \). (If \( C = E \) we set \( \Pi = \mathbb{P}^{N_E} \).) Then \( E + F = \eta(E \times F) \), and we can apply Proposition 3.9 to \( \nabla_{n,d,m} \subset \mathbb{P}^N \), where \( n = \deg(E) \), \( m = m(E) \), and \( d \) is the number of nodes of \( E \). This proves the claim.

Now we will verify the following inequality:
\[
(7) \quad \#I(V(g_s)) + e + \deg(E) - m(E) + 1 \geq \begin{cases} \#I(V_s(g_s)) + 1, & s = 1 \\ \#I(V_s(g_s)), & s \geq 2. \end{cases}
\]
Let \( i \) be the minimal integer in \( I(V_s(g_s)) \). Then either \( i \in I(V(g_s)) \) or we get the vanishing of the corresponding coefficient in the equation of \( E \) (see the proof of Proposition 3.9). We then take the next integer in \( I(V_s(g_s)) \) and repeat the argument, etc.

Finally, combining (6) with (5) for \( V(g_s) \) and (7), we get (5) for \( V_s(g_s) \). The above discussion also proves the last assertion of the lemma in case \( \mu_s = \mu + 1 \).

4.14. Remark. Let \( W \) be a component of \( U_m(n, g) \). One can now describe all components of \( W \cap \{a_{0n} = 0\} \). Further, utilizing standard exact sequences it is not difficult to calculate the Hilbert polynomial of \( \nabla_{n,d,m} \subset \mathbb{P}^{N-m} \) as well as some other interesting schemes of curves.
5. A vanishing theorem for admissible schemes

5.1. Lemma. We consider (3_1) such that V is admissible, V^1 \simeq V, H^0(V_1, \mathcal{O}(-r)) = 0 for r >> 0, and H^1(V_1, \mathcal{O}(r)) = 0 for all r \in \mathbb{Z}. Then H^0(V, \mathcal{O}(-r)) = 0 for r >> 0, and H^1(V, \mathcal{O}(r)) = 0 for all r \in \mathbb{Z}.

Proof. Recall first the following classical lemma of Enriques - Severi - Zariski [G, Exp XII, Corollary 1.4]: for an arbitrary projective scheme V \subseteq \mathbb{P}^N, \text{depth}(\mathcal{O}_v, V) \geq 2 for every point v \in V if and only if H^i(V, \mathcal{O}(-r)) = 0 for r >> 0 and i = 0, 1.

It follows that depth(\mathcal{O}_{C,V_1}) \geq 2 for every point C \subseteq V_1, hence depth(\mathcal{O}_{C,V}) \geq 2 for every point C \subseteq V_1 \subseteq V.

Let C \subseteq V be an arbitrary point. We will show that [1 : 0 : 0] \notin \gamma(C) for a suitable \gamma \in G \subseteq \text{PGL}(N). Assuming [1 : 0 : 0] \in C, take a point [1 : a : b] \notin C with a, b \in \mathbb{Z}_+. It is easy to find \gamma \in G \subseteq \text{PGL}(2) such that \gamma([1 : 0 : 0]) = [1 : a : b]. Then [1 : 0 : 0] \notin \gamma^{-1}(C).

Now we assume that [1 : 0 : 0] \notin C. Consider the points (\varphi_t \cdot \phi_t)(C) for t \in \mathbb{C}^*.
As t goes to 0, (\varphi_t \cdot \phi_t)(C) tends to a point nl of the subscheme V_1 \subseteq V. Since depth(\mathcal{O}_{C,V}) is an upper semicontinuous function of C, depth(\mathcal{O}_{C,V}) \geq 2 for every point C \subseteq V. Thus, Lemma 5.1 follows from the Enriques - Severi - Zariski lemma and (4_k).

5.2. Theorem (Vanishing Theorem). Let V \subseteq \mathbb{P}^N be a \n,d,m-admissible subscheme. Then H^1(V, \mathcal{O}(r)) = 0 for all r \in \mathbb{Z}, and H^0(V, \mathcal{O}(-r)) = 0 for r >> 0.

Proof. Case: dim V = 2. We consider six closed irreducible families in \mathbb{P}^5 (\subseteq \mathbb{P}^N), whose general members are of the form D + (n - 2)l with l \notin D, by specifying D:
\begin{align*}
F_1: & \text{ D is a general quadric;} \\
F_2: & \text{ D is a general quadric with } (D \cdot l)_p = 1; \\
F_3: & \text{ D is a general quadric with } (D \cdot l)_p = 2; \\
F_4: & \text{ D is a union of two general lines;} \\
F_5: & \text{ D is a union of two general lines with } (D \cdot l)_p = 1; \\
F_6: & \text{ D is a union of two general lines with } (D \cdot l)_p = 2.
\end{align*}

We also consider the closed irreducible family F_7 \subseteq \mathbb{P}^5 whose general member is of the form D + (n - 1)l, where D is a general line. There are only four maximal families of dimension at most 2, namely: F_6, F_7, \{(n - 1)l + l' \mid p \in l'\}, and \{nl\}. The latter two families are contained in any admissible scheme. We get
\[ V_{\text{red}} \subseteq F_6 \cup F_7 \subseteq \bigcap_{1 \leq \sigma \leq \kappa + 2} H_\sigma = \mathbb{P}^3 \subset \mathbb{P}^5 \subset \mathbb{P}^N, \quad F_7 = \mathbb{P}^3 \cap H_{\kappa + 3}. \]
Moreover H^1(V_{\text{red}}, \mathcal{O}(r)) = 0 for all r. Thus we get the vanishing of the cohomology groups provided V \subseteq \mathbb{P}^3. If V_{\text{red}} = F_6 \cup F_7, we take H = H_{\kappa + 3} and consider (3_k) for an appropriate integer k. Then (V_k)_{\text{red}} = F_7 and (V^k)_{\text{red}} = F_6.

Now, we assume that V is irreducible, that is, V_{\text{red}} = F_6 or F_7. First we take the hyperplane H = H_1. Let h = 0 be an equation of H. Consider the following exact sequence similar to (3_1):
\[ 0 \to K \xrightarrow{\otimes h} \mathcal{O}_V(r) \to \mathcal{O}_{V_1}(r) \to 0, \]
where K = 0 if h |_V = 0, and K = \mathcal{O}_{V_1}(r - 1) otherwise. In the latter case, dim V^1 = 2 by Lemma 4.13. Indeed, the restriction of h to the local ring of V at
its general point is a nontrivial element. Hence the stalk of $A^1(h)$ at this point is a nontrivial ideal in the corresponding local ring. This ideal is nilpotent, so $\dim V^1 = 2$. Next, we consider $V_1$ (or $V^1$) in place of $V$ and take the hyperplane $H = H_2$ (or $H = H_1$), etc. After a finite number of steps, we obtain subschemes in $\mathbb{P}^3$.

**General case.** Assume the theorem for all smaller admissible schemes. If $\dim V \geq 3$, we take the hyperplane $H_\sigma$ with the smallest possible $\sigma$ such that $V_{\text{red}} \not\subset H_\sigma$. For an appropriate $k$, we consider $(3_k)$ with $H = H_\sigma$ and get two admissible schemes, $V_k$ and $V^k$. Since $V_k \neq V$, we can apply Lemma 5.1 if $V^k \simeq V$. This proves the theorem.

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