COMPLEXITY OF THE CLASSICAL KERNEL
FUNCTIONS OF POTENTIAL THEORY

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ABSTRACT. We show that the Bergman, Szegő, and Poisson kernels associated to
a finitely connected domain in the plane are all composed of finitely many easily
computed functions of one variable. The new formulas give rise to new methods
for computing the Bergman and Szegő kernels in which all integrals used in the
computations are line integrals; at no point is an integral with respect to area measure
required. The results mentioned so far can be interpreted as saying that the kernel
functions are simpler than one might expect. However, we also prove that the kernels
cannot be too simple by showing that the only finitely connected domains in the plane
whose Bergman or Szegő kernels are rational functions are the obvious ones. This
leads to a proof that the classical Green’s function associated to a finitely connected
domain in the plane is the logarithm of a rational function if and only if the domain
is simply connected and rationally equivalent to the unit disc.

1. Introduction. The Bergman and Szegő kernels associated to a bounded do-
main in the plane with smooth boundary carry encoded within them an astonishing
amount of information about the domain. Conformal mappings onto canonical do-
mains, classical domain functions, and other important objects of potential theory
can be expressed simply in terms of the Bergman and Szegő kernels. It is therefore
tempting to believe that these kernels are extremely complex and difficult to com-
pute. The purpose of this paper is to show that the kernel functions are not nearly
as complex as one might suspect.

Suppose that Ω is a bounded finitely connected domain in the plane with $C^\infty$
smooth boundary, i.e., that the boundary $\partial \Omega$ of Ω is given by finitely many non-
intersecting $C^\infty$ simple closed curves. The Bergman kernel $K(z, w)$ and the Szegő
kernel $S(z, w)$ associated to such a domain are both known to extend to be in the
space $C^\infty((\overline{\Omega} \times \overline{\Omega}) - D)$ where $D$ denotes the boundary diagonal $\{(z, z) : z \in \partial \Omega\}$. Our problem is to determine a method to compute $K(z, w)$ and $S(z, w)$ at any given
ordered pair of points $(z, w)$ in $(\overline{\Omega} \times \overline{\Omega}) - D$. We shall see that, once the boundary
values of finitely many basic functions of one variable have been determined, the
kernels become known at all points $(z, w)$. Furthermore, the basic functions which
comprise the kernel functions are all solutions to explicit Kerzman-Stein integral
equations, and as such, are easy to compute. All elements of the kernel functions
may be computed by means of simple linear algebra and one dimensional integrals
and one dimensional integral equations. At no point is a double integral needed.

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Although the words, “numerical method,” appear in this paper, this is not a paper on numerical analysis; no examples of numerical computations are given. However, the results of this paper should be interesting to numerical analysts.

Our results are particularly surprising for the Bergman kernel function. A traditional way to attempt to compute the Bergman kernel has been to orthonormalize a set of rational functions that span a dense subset of the Bergman space. This is a numerical nightmare compared to the methods we establish below.

In §6, we give conditions on a domain for its Bergman or Szegő kernel function to be a rational function. We prove, for example, that the Bergman kernel associated to a finitely connected domain is rational exactly when the domain is simply connected and is biholomorphic to the unit disc via a rational mapping. Thus, the domains whose Bergman kernels are rational are precisely those domains whose kernels can be seen to be rational by means of an elementary application of the transformation formula for the kernel functions under biholomorphic maps and the fact that the kernel for the unit disc is rational. This result about the Bergman kernel has as a corollary that the Green’s function associated to a finitely connected domain is the logarithm of a rational function if and only if the domain is simply connected and is rationally equivalent to the unit disc.

In §7, we show how the Poisson kernel can be expressed in terms of the Szegő kernel, and thereby shed light on the degree of complexity of the Poisson kernel.

Our results are most interesting in case the domain under study is multiply connected. However, to illustrate the flavor of our results, we take a moment here to state analogues of our theorems for a bounded simply connected domain \(\Omega\) with \(C^\infty\) smooth boundary. How difficult is it to compute the Szegő kernel \(S(z, w)\) at any given pair of points? Is it so difficult that we would need to follow a separate numerical procedure to compute \(h(z) = S(z, w_0)\) for each individual point \(w_0 \in \Omega\)? The following formula shows that it is not nearly so difficult. It is only as difficult as computing the boundary values of a single function. Let \(a\) be a fixed point in \(\Omega\) and let \(f_a(z)\) denote the Riemann mapping function mapping \(\Omega\) one-to-one onto the unit disc \(D_1(0)\) with \(f_a(a) = 0\) and \(f'_a(a) > 0\). This Riemann map can easily be written down in terms of the boundary values of the function \(S(z, a)\) (see [2]), and \(S(z, a)\) is the solution to a simple Fredholm integral equation of the second kind with \(C^\infty\) kernel and inhomogeneous term (see [2,3,7,10,11,14]). The kernel \(S(z, w)\) may be expressed as

\[
S(z, w) = \frac{c S(z, a) S(w, a)}{1 - f_a(z) f_a(w)},
\]

where \(c = 1/S(a, a)\). This shows that, once the boundary values of the single function of one variable \(S(z, a)\) are known, the Szegő kernel can be evaluated at an arbitrary pair of points. A similar identity holds for the Bergman kernel,

\[
K(z, w) = \frac{4\pi S(z, a) S(w, a)}{(1 - f_a(z) f_a(w))^2}.
\]

This shows that the Bergman kernel is composed of the same basic functions that make up the Szegő kernel. Finally, the Poisson kernel \(p(z, w)\) is given by

\[
p(z, w) = \frac{S(z, w) S(w, a) S(z, a)}{S(z, a) f_a(w)} + \frac{S(z, w) S(w, a) f_a(z)}{S(z, a) f_a(w)},
\]
where $z$ is a point in $\Omega$ and $w$ is a point in the boundary (see [2, page 37]). Thus, the Poisson kernel is also composed of the same basic functions. None of these formulas for the kernel functions in a simply connected domain could be considered very new. However, we shall prove analogous results for $n$-connected domains that are new. The new results show that there are $n + 1$ basic functions that comprise all the kernels. An interesting feature of all the results in this paper is the central role played by the zeroes of the Szegő kernel.

2. The Ahlfors map and zeroes of the Szegő kernel. Before we start stating and proving our main theorems, we must review some basic facts about the kernel functions. Most of these facts are proved in Bergman’s book [5]; all of them are proved in [2].

Suppose that $\Omega$ is a bounded $n$-connected domain in the plane with $C^\infty$ smooth boundary. Let $\gamma_j$, $j = 1, \ldots, n$, denote the $n$ non-intersecting $C^\infty$ simple closed curves which define the boundary of $\Omega$, and suppose that $\gamma_j$ is parameterized in the standard sense by $z_j(t)$, $0 \leq t \leq 1$. We shall use the convention that $\gamma_n$ denotes the outer boundary curve of $\Omega$. Let $T(z)$ be the $C^\infty$ function defined on $\partial \Omega$ such that $T(z)$ is the complex number representing the unit tangent vector at $z \in \partial \Omega$ pointing in the direction of the standard orientation. This complex unit tangent vector function is characterized by the equation $T(z_j(t)) = z_j'(t)/|z_j'(t)|$.

We shall let $A^\infty(\Omega)$ denote the space of holomorphic functions on $\Omega$ that are in $C^\infty(\overline{\Omega})$. The space of complex valued functions on $\Omega$ that are square integrable with respect to Lebesgue area measure $dA$ will be written $L^2(\Omega)$, and the space of complex valued functions on $\partial \Omega$ that are square integrable with respect to arc length measure $ds$ will be denoted by $L^2(\partial \Omega)$. The Bergman space of holomorphic functions on $\Omega$ that are in $L^2(\Omega)$ shall be written $H^2(\Omega)$ and the Hardy space of functions in $L^2(\partial \Omega)$ that are the $L^2$ boundary values of holomorphic functions on $\Omega$ shall be written $H^2(\partial \Omega)$. The inner products associated to $L^2(\Omega)$ and $L^2(\partial \Omega)$ shall be written

$$\langle u, v \rangle_{\Omega} = \int \int_{\Omega} u \bar{v} \, dA \quad \text{and} \quad \langle u, v \rangle_{\partial \Omega} = \int_{\partial \Omega} u \bar{v} \, ds,$$

respectively.

For each fixed point $a \in \Omega$, the Szegő kernel $S(z, a)$, as a function of $z$, extends to the boundary to be a function in $A^\infty(\Omega)$. (An even stronger smoothness property is mentioned in the introduction.) Furthermore, $S(z, a)$ has exactly $(n - 1)$ zeroes in $\Omega$ (counting multiplicities) and does not vanish at any points $z$ in the boundary of $\Omega$. The Garabedian kernel $L(z, a)$ is a kernel related to the Szegő kernel via the identity

$$\frac{1}{\pi} L(z, a) T(z) = S(a, z) \quad \text{for } z \in \partial \Omega \text{ and } a \in \Omega. \quad (2.1)$$

For fixed $a \in \Omega$, the kernel $L(z, a)$ is a holomorphic function of $z$ on $\Omega - \{a\}$ with a simple pole at $a$ with residue $1/(2\pi)$. Furthermore, as a function of $z$, $L(z, a)$ extends to the boundary and is in the space $C^\infty(\overline{\Omega} - \{a\})$. In fact, $L(z, a)$ extends to be in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z, z) : z \in \overline{\Omega}\})$. Also, $L(z, a)$ is non-zero for all $(z, a)$ in $\overline{\Omega} \times \Omega$ with $z \neq a$.

The kernel $S(z, w)$ is holomorphic in $z$ and antiholomorphic in $w$ on $\Omega \times \Omega$, and $L(z, w)$ is holomorphic in both variables for $z, w \in \Omega$, $z \neq w$. We shall need to know
that \(S(z, z)\) is real and positive for each \(z \in \Omega\), and we shall need to use the basic identities \(S(z, w) = \overline{S(w, z)}\) and \(L(z, w) = -L(w, z)\). The Szegő kernel reproduces holomorphic functions in the sense that

\[
h(a) = \langle h, S(\cdot, a) \rangle_{b\Omega}
\]

for all \(h \in H^2(b\Omega)\) and \(a \in \Omega\).

Given a point \(a \in \Omega\), the Ahlfors map \(f_a\) associated to the pair \((\Omega, a)\) is a proper holomorphic mapping of \(\Omega\) onto the unit disc. It is an \(n\)-to-one mapping (counting multiplicities), it extends to be in \(A^\infty(\Omega)\), and it maps each boundary curve \(\gamma_j(\Omega)\) one-to-one onto the unit circle. Furthermore, \(f_a(a) = 0\), and \(f_a\) is the unique function mapping \(\Omega\) into the unit disc maximizing the quantity \(|f'_a(a)|\) with \(f'_a(a) > 0\). The Ahlfors map is related to the Szegő kernel and Garabedian kernel via

\[
f_a(z) = \frac{S(z, a)}{L(z, a)}.
\]

Also, \(f'_a(a) = 2\pi S(a, a) \neq 0\). Because \(f_a\) has \(n\) zeroes, and because the simple pole of \(L(z, a)\) at \(a\) accounts for the simple zero of \(f_a\) at \(a\), it follows that \(S(z, a)\) has \((n - 1)\) zeroes in \(\Omega - \{a\}\). Let \(a_1, a_2, \ldots, a_{n-1}\) denote these \(n - 1\) zeroes (counted with multiplicity). I proved in [4] (see also [2, page 105]) that, if \(a\) is close to one of the boundary curves, then the zeroes \(a_1, \ldots, a_{n-1}\) become distinct simple zeroes. It follows from this result that, for all but at most finitely many points \(a \in \Omega\), \(S(z, a)\) has \(n - 1\) distinct simple zeroes in \(\Omega\) as a function of \(z\).

**3. A special orthonormal basis for the Hardy space.** The zeroes of the Szegő kernel give rise to a particularly nice basis for the Hardy space of an \(n\)-connected domain with \(C^\infty\) smooth boundary. We shall use the notation that we set up in the preceding section. We assume that \(a \in \Omega\) is a fixed point in \(\Omega\) that has been chosen so that the \(n - 1\) zeroes, \(a_1, \ldots, a_{n-1}\, of \(S(z, a)\) are distinct and simple. We shall let \(a_0\) denote \(a\) and we shall use the shorthand notation \(f(z)\) for the Ahlfors map \(f_a(z)\).

We shall now prove that the set of functions \(\{h_{ik}(z) : 0 \leq i \leq n - 1, \text{ and } k \geq 0\}\) where \(h_{ik}\) is defined via

\[
h_{ik}(z) = S(z, a_i)f(z)^k
\]

forms a basis for the Hardy space \(H^2(b\Omega)\). Furthermore,

\[
\langle h_{ik}, h_{jm} \rangle_{b\Omega} = \begin{cases} 0, & \text{if } k \neq m \\ S(a_j, a_i), & \text{if } k = m. \end{cases}
\]

The proof of these assertions consists of two parts. First, we must prove that these functions span a dense subset of \(H^2(b\Omega)\), and second, we must prove the identity (3.1) to see that the set forms a basis. To prove the density of the span, suppose that \(g \in H^2(b\Omega)\) is orthogonal to the span. Notice that

\[
\langle g, S(\cdot, a_j) \rangle_{b\Omega} = g(a_j),
\]

and therefore \(g\) vanishes at \(a_0, a_1, \ldots, a_{n-1}\). Suppose we have shown that \(g\) vanishes to order \(m\) at each \(a_j, j = 0, 1, \ldots, n - 1\). It follows that \(g/f^m\) is in \(H^2(b\Omega)\) and the
value of $g/f^m$ at $a_j$ is $g^{(m)}(a_j)/f'(a_j)^m$. Since $|f(z)| = 1$ when $z \in b\Omega$, it follows that $1/f(z) = \overline{f(z)}$ when $z \in b\Omega$, and we may write

$$\langle g, S(\cdot, a_j)f^m \rangle_{b\Omega} = \langle g/f^m, S(\cdot, a_j) \rangle_{b\Omega} = g^{(m)}(a_j)/f'(a_j)^m.$$  

We conclude that $g$ vanishes to order $m+1$ at each $a_j$. By induction, $g$ vanishes to infinite order at each $a_j$ and hence, $g \equiv 0$. This proves the density. To prove (3.1), let us suppose first that $k > m$. The fact that $\overline{f(z)}$ on $b\Omega$ and the reproducing property of the Szegő kernel now yield that

$$\langle h_{ik}, h_{jm} \rangle_{b\Omega} = \int_{z \in b\Omega} S(z, a_i)f(z)^{k-m} \overline{S(z, a_j)} \, ds = \int_{z \in b\Omega} S(a_j, z) S(z, a_i)f(z)^{k-m} \, ds = S(a_j, a_i)f(a_j)^{k-m}.$$  

The identity now follows because $f(a_j) = 0$ for all $j$. If $k = m$, then

$$\langle h_{ik}, h_{jm} \rangle_{b\Omega} = \int_{z \in b\Omega} S(a_j, z) S(z, a_i) \, ds = S(a_j, a_i),$$

and identity (3.1) is proved. We remark here that it easy to see that the functions $h_{ik}$ are linearly independent. Indeed, identity (3.1) reveals that we need only check that, for fixed $k$, the $n$ functions $h_{ik}$, $i = 0, 1, \ldots, n - 1$, are independent, and this is easy because a relation of the form

$$\sum_{i=0}^{n-1} C_i S(z, a_i) \equiv 0$$

implies, via the reproducing property of the Szegő kernel, that every function $g$ in the Hardy space satisfies

$$\sum_{i=0}^{n-1} \overline{C_i} g(a_i) = 0,$$

and it is easy to construct polynomials $g$ that violate such a condition.

We next orthonormalize the sequence $\{h_{ik}\}$ via the Gram-Schmidt procedure. Identity (3.1) shows that most of the functions in the sequence are already orthogonal, and so our task is quite easy. We need only fix $k$ and orthonormalize the $n$ functions $h_{ik}$, $i = 0, 1, \ldots, n - 1$. We obtain an orthonormal set $\{H_{ik}\}$ given by

$$H_{0k}(z) = b_{00}S(z, a)f(z)^k \quad \text{and,}$$

$$H_{ik}(z) = \sum_{j=1}^{i} b_{ij}S(z, a_j)f(z)^k, \quad i = 1, \ldots, n - 1$$

where $b_{ii} \neq 0$ for each $i = 0, 1, \ldots, n - 1$. Because $|f| = 1$ on $b\Omega$, it follows that the coefficients $b_{ij}$ do not depend on $k$. Notice that $H_{ij}$ does not contain a term involving $S(z, a)$ if $i > 0$ because of (3.1) and the fact that $S(a_i, a) = 0$. 


The Szegő kernel can be written in terms of our orthonormal basis via

\[ S(z, w) = \sum_{i=0}^{n-1} \sum_{k=0}^{\infty} H_{ik}(z) \overline{H_{ik}(w)}. \]

The sum

\[ \sum_{k=0}^{\infty} f(z)^k \overline{f(w)^k} = \frac{1}{1 - f(z)\overline{f(w)}} \]

can be factored from the expression for \( S(z, w) \) to yield the formula in the following theorem.

**Theorem 3.1.** The Szegő kernel can be evaluated at an arbitrary pair of points \((z, w)\) in \(\Omega\) via the formula

\begin{equation}
S(z, w) = \frac{1}{1 - f(z)\overline{f(w)}} \left( c_0 S(z, a) S(w, a) + \sum_{i,j=1}^{n-1} c_{ij} S(z, a_i) S(w, a_j) \right)
\end{equation}

where \( f(z) \) denotes the Ahlfors map \( f_a(z) \), \( c_0 = 1/S(a, a) \), and the coefficients \( c_{ij} \) are given as the coefficients of the inverse matrix to the matrix \([S(a_j, a_k)]\).

The only part of Theorem 3.1 that remains unproved is the statement about the coefficients in the formula. We have shown that these coefficients exist and that they are given as certain combinations of the Gram-Schmidt coefficients used above. That \( c_0 = 1/S(a, a) \) can be seen by setting \( z = a \) and \( w = a \) in (3.2). To complete the proof of Theorem 3.1, we shall now describe how to determine the coefficients \( c_{ij} \). Suppose \( 1 \leq k \leq n-1 \). Set \( w = a_k \) in (3.2) and note that \( f(a_k) = 0 \) and \( S(a, a_k) = 0 \) to obtain

\[ S(z, a_k) = \sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} c_{ij} S(a_j, a_k) \right) S(z, a_i). \]

We saw an identity like this when we showed above that the functions \( h_{jk} \) are linearly independent. The same reasoning we used there yields that such a relation can only be true if

\[ \sum_{j=1}^{n-1} c_{ij} S(a_j, a_k) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k. \end{cases} \]

This shows that the \((n-1) \times (n-1)\) matrix \([S(a_j, a_k)]\) is invertible and \([c_{ij}]\) is its inverse.

We remark here that formula (3.2) has an interesting application. It is quite easy to prove that \( S(z, a) \) is in \( C^\infty(\overline{\Omega}) \) as a function of \( z \) for each fixed \( a \in \Omega \) (see [2, page 22]). The more difficult result that \( S(z, w) \) is in \( C^\infty(\overline{\Omega} \times \overline{\Omega} - \{(z, z) : z \in \partial \Omega\}) \) follows directly from the the smoothness of \( S(z, a) \) for fixed \( a \) and formula (3.2).

**4. Complexity of the Szegő kernel.** Formula (3.2) reveals that the Szegő kernel associated to an \( n \)-connected domain is composed of the \( n + 1 \) functions, \( S(z, a), S(z, a_1), S(z, a_2), \ldots, S(z, a_{n-1}), \) and \( L(z, a) \) (because \( f(z) = S(z, a)/L(z, a) \)). If
one knows the boundary values of these \( n + 1 \) functions, then the Szegő kernel may be evaluated at any pair of points \((z, w)\) in \( \Omega \times \Omega \) by applying the Cauchy integral formula twice, once to evaluate the functions on the right hand side of (3.2) at \( z \), and once to evaluate the functions at \( w \). In this section, we show how much effort is required to numerically compute the boundary values of the \( n + 1 \) functions comprising \( S(z, w) \).

Kerzman and Stein [10] discovered an effective method for computing the Szegő kernel (see also [2,3,7,11,14]). They proved that the function \( S_{a}(z) = S(z, a) \) is the solution to a Fredholm integral equation of the second kind given by

\[
S_{a}(z) - \int_{w \in b\Omega} A(z, w)S_{a}(w)\, ds = C_{a}(z),
\]

where \( A(z, w) \) is the Kerzman-Stein kernel and \( C_{a}(z) \) is the Cauchy kernel. To be precise,

\[
A(z, w) = \frac{1}{2\pi i} \left( \frac{T(w)}{w - z} - \frac{T(z)}{w - \bar{z}} \right)
\]

if \( z, w \in b\Omega, z \neq w, \) and \( A(z, w) = 0 \) if \( z = w, \) and

\[
C_{a}(z) = \frac{1}{2\pi i} \frac{T(z)}{\bar{a} - \bar{z}}.
\]

The Kerzman-Stein kernel is skew-hermitian and, in spite of the apparent singularity at \( z = w \) in the formula above, it is in \( C^{\infty}(b\Omega \times b\Omega) \). (Kerzman and Stein discovered that the apparent singularities in the formula for \( A(z, w) \) exactly cancel.) The Cauchy kernel is in \( C^{\infty}(b\Omega) \). It follows from standard theory that this integral equation has a unique \( C^{\infty} \) smooth solution. (See Kerzman and Tummer [14] and [3,7] for descriptions of convenient ways to write and to solve this integral equation.)

The Kerzman-Stein equation produces the boundary values of \( S(z, a) \). The boundary values of the Garabedian kernel \( L(z, a) \) can be computed via identity (2.1), and the boundary values of the Ahlfors map \( f_{a}(z) \) can now be gotten from (2.2). The remaining functions in expansion (3.2) can be computed via the Kerzman-Stein integral equation once the zeroes \( a_{1}, \ldots, a_{n-1} \) have been located. Since \( S(z, a) \) does not vanish on \( b\Omega \), we may use the residue theorem to compute the symmetric sums

\[
\sum_{j=1}^{n-1} a_{j}^{k} = \int_{z \in b\Omega} \frac{z^{k}(\partial / \partial z)S(z, a)}{S(z, a)}\, dz
\]

for \( k = 1, \ldots, n-1 \). Newton’s identities can now be used to compute the elementary symmetric functions of \( a_{1}, \ldots, a_{n-1} \), and hence, the coefficients of the polynomial \( \prod_{j=1}^{n-1}(\zeta - a_{j}) \) are determined. We have therefore shown that the problem of locating the zeroes of \( S(z, a) \) is equivalent to computing \( n - 1 \) line integrals and finding the roots of a polynomial of degree \( n - 1 \).

5. The Bergman kernel. In this section, we shall prove that the Bergman kernel of an \( n \)-connected domain in the plane with \( C^{\infty} \) smooth boundary is composed of the same basic functions that comprise the Szegő kernel. We shall also prove that the Bergman kernel can be computed at every pair of points by solving \( n \) one
dimensional $C^\infty$ Fredholm integral equations of the second kind, and by solving a linear system. At no point is it necessary to evaluate a double integral.

The Bergman kernel $K(z, w)$ is related to the Szegö kernel via the identity

$$K(z, w) = 4\pi S(z, w)^2 + \sum_{i, j=1}^{n-1} \lambda_{ij} F_i'(z) F_j'(w),$$

where the functions $F_i'(z)$ are classical functions of potential theory described as follows. The harmonic function $\omega_j$ which solves the Dirichlet problem on $\Omega$ with boundary values one on the boundary curve $\gamma_j$ and zero on $\gamma_k$ if $k \neq j$ has a multivalued harmonic conjugate. The function $F_j'(z)$ is a globally defined single valued holomorphic function on $\Omega$ which is locally defined as the derivative of $\omega_j + iv$ where $v$ is a local harmonic conjugate for $\omega_j$. The Cauchy-Riemann equations reveal that $F_j'(z) = 2(\partial \omega_j/\partial z)$.

Let $F'$ denote the vector space of functions given by the complex linear span of the set of functions $\{F_j'(z) : j = 1, \ldots, n-1\}$. It is a classical fact that $F'$ is $n-1$ dimensional. Notice that $S(z, a_i)L(z, a)$ is in $A^\infty(\Omega)$ because the pole of $L(z, a)$ at $z = a$ is cancelled by the zero of $S(z, a_i)$ at $z = a$. A theorem due to Schiffer (see [12,2,4]) states that the $n-1$ functions $S(z, a_i)L(z, a)$, $i = 1, \ldots, n-1$ form a basis for $F'$. We may now write

$$K(z, w) = 4\pi S(z, w)^2 + \sum_{i, j=1}^{n-1} \lambda_{ij} S(z, a_i)L(z, a) S(w, a_j)L(w, a), \tag{5.1}$$

which, together with (3.2) allows us to write down a formula which sheds light on the degree of complexity of the Bergman kernel.

**Theorem 5.1.** The Bergman kernel is composed of the same basic functions that make up the Szegö kernel, as evidenced by the following formula.

$$K(z, w) = \frac{1}{(1 - f(z)f(w))^2} \left( \sum_{0 \leq i \leq j \leq n-1 \atop 0 \leq k \leq m \leq n-1} C_{ijkm} S(z, a_i)S(z, a_j) S(w, a_k)S(w, a_m) \right) + \sum_{i, j=1}^{n-1} \lambda_{ij} S(z, a_i)L(z, a) S(w, a_j)L(w, a).$$

We shall now discuss the amount of computational effort required to compute the Bergman kernel. To streamline the argument, it will be convenient to use the fact that the linear span of $\{S(z, a_i)L(z, a) : i = 1, \ldots, n-1\}$ is the same as the linear span of $\{L(z, a_i)S(z, a) : i = 1, \ldots, n-1\}$ (see [2, page 80]). Hence, formula (5.1) can also be written in the form

$$K(z, w) = 4\pi S(z, w)^2 + \sum_{i, j=1}^{n-1} \lambda_{ij} L(z, a_i)S(z, a)L(w, a_j)S(w, a), \tag{5.2}$$

(where, here, the coefficients $\lambda_{ij}$ represent different constants than they do in (5.1)). The difficulty of computing the functions appearing in (5.2) has been discussed. We
now describe a method for computing the coefficients \( \lambda_{ij} \). We shall write \( K_w(z) \) in place of \( K(z, w) \) and \( S_w(z) \) in place of \( S(z, w) \) to emphasize that we are thinking of \( w \) as being fixed and we are viewing these kernels as functions of \( z \). Let us also write \( \mathcal{L}_i(z) = L(z, a_i)S(z, a) \). Thus, formula (5.2) may be rewritten as

\[
(5.3) \quad K_w - 4\pi S_w^2 = \sum_{i,j=1}^{n-1} \lambda_{ij} \mathcal{L}_j(w) \mathcal{L}_i.
\]

The complement of \( \Omega \) in \( \mathbb{C} \) is the union of domains \( D_j, j = 1, \ldots, n \), where the boundary of \( D_j \) is described by the boundary curve \( \gamma_j \). Recall that \( \gamma_n \) denotes the outer boundary curve of \( \Omega \). For \( j = 1, \ldots, n-1 \), pick a point \( b_j \) in \( D_j \). We now consider the effect of integrating (5.3) against the function \( 1/(z-b_k) \). Notice that

\[
\langle (z-b_k)^{-1}, K_w \rangle_{\Omega} = \frac{1}{w-b_k}
\]

because the Bergman kernel reproduces holomorphic functions. Since \( 1/(z-b_k) = (\partial/\partial z) \ln |z-b_k|^2 \), we may use the complex Green’s identity to compute

\[
\langle (z-b_k)^{-1}, S_w \rangle_{\Omega} = \int_{z \in \Omega} (\partial/\partial z) \ln |z-b_k|^2 \overline{S_w(z)}^2 (i/2) dz \wedge d\bar{z} = i \int_{z \in b\Omega} \ln |z-b_k| \overline{S_w(z)}^2 d\bar{z}.
\]

Define numbers \( A_{ik} = \langle (z-b_k)^{-1}, \mathcal{L}_i \rangle_{\Omega} \). We may use the complex Green’s identity again to obtain

\[
A_{ik} = i \int_{z \in b\Omega} \ln |z-b_k| \overline{\mathcal{L}_i(z)} d\bar{z}.
\]

We now collect the integrals above as dictated by (5.3), and we set \( w = a_m, m = 1, \ldots, n-1 \) to obtain the system,

\[
\frac{1}{a_m-b_k} - 4\pi i \int_{z \in b\Omega} \ln |z-b_k| S(a_m, z)^2 d\bar{z} = \sum_{i,j=1}^{n-1} \lambda_{ij} A_{ik} \overline{\mathcal{L}_j(a_m)}.
\]

To show that this system determines the numbers \( \lambda_{ij} \), we need only check that the matrices given by \( A = [A_{ik}] \) and \( \mathcal{L} = [\mathcal{L}_j(a_m)] \) are invertible. That \( \mathcal{L} \) is invertible is obvious because

\[
L(w, a_j)S(w, a) = \begin{cases} 0, & \text{if } w = a_m, m \neq j \\ \frac{1}{2\pi i} \frac{\partial}{\partial z}, & \text{if } w = a_j \end{cases}
\]

and \( (\partial/\partial z)S(a_j, a) \neq 0 \) because \( a \) has been chosen so that the zeroes of \( S(z, a) \) are simple zeroes. To show that \( A \) is invertible, we shall need to use an argument from [4]. If \( G = \sum_{k=1}^{n-1} c_k F_k \), then \( G = 2(\partial/\partial z)\omega \) where \( \omega = \left( \sum_{k=1}^{n-1} c_k \omega_k \right) \). It is proved in [4, page 12] that the constants \( c_k \) are given by the integral

\[
c_k = -\frac{1}{2\pi i} \int_{z \in b\Omega} \ln |z-b_k| G(z) dz,
\]
where $b_k$ is the fixed point chosen from $D_k$. Notice that $c_k$ is the value of $\omega$ on $\gamma_k$. Suppose $A$ is not invertible. Then there would exist constants $\sigma_i$, not all zero, such that
\[ \sum_{i=1}^{n-1} A_{ik} \sigma_i = 0 \]
for each $k$, and the complex conjugate of this equality yields that
\begin{equation}
(5.4) \quad \int_{z \in b\Omega} \ln |z - b_k| \left( \sum_{i=1}^{n-1} \sigma_i \mathcal{L}_i(z) \right) \, dz = 0.
\end{equation}
Let $G = \sum_{i=1}^{n-1} \sigma_i \mathcal{L}_i$. Since $G$ is in the linear span of $\{F_j'\}_{j=1}^{n-1}$, condition (5.4) and the fact from [4] imply that $G = 2(\partial / \partial z)\omega$ where $\omega$ is a harmonic function on $\Omega$ that vanishes on each boundary curve of $\Omega$, i.e., that $G \equiv 0$. Now each $\sigma_k$ must be zero because the functions $\mathcal{L}_i$ are linearly independent. This contradiction yields that the matrix $A$ must be non-singular and the proof is finished.

6. **Characterization of domains with rational kernel functions.** In the previous sections, we have shown that the kernel functions are not as complex as one might expect them to be. In this section, we shall prove theorems that say roughly that the only domain whose Bergman or Szegő kernels are so simple as to be rational functions is the disc.

A function $R(z, w)$ of two complex variables is called rational if there are relatively prime polynomials $P(z, w)$ and $Q(z, w)$ such that $R(z, w) = P(z, w)/Q(z, w)$. It is not hard to prove that a function $H(z, w)$, which is holomorphic in $z$ and $w$ on a product domain $\Omega_1 \times \Omega_2$ is rational if and only if, for each fixed $b \in \Omega_2$, the function $H(z, b)$ is rational in $z$, and for each fixed $a \in \Omega_1$, the function $H(a, w)$ is rational in $w$ (see Bochner and Martin [6, page 201]). We shall say that the Bergman kernel function $K(z, w)$ associated to a domain $\Omega$ is rational if it can be written as $R(z, w)$ where $R$ is a holomorphic rational function of two variables. Because the Bergman kernel is hermitian, the facts above imply that $K(z, w)$ is rational if and only if, for each point $a \in \Omega$, the function $K(z, a)$ is a rational function of $z$. In fact, $K(z, w)$ is rational if and only if there exists a small disc $D_\varepsilon(w_0) \subset \Omega$ such that $K(z, a)$ is a rational function of $z$ for each $a \in D_\varepsilon(w_0)$. Similar statements hold for the other kernel functions.

**Theorem 6.1.** Suppose $\Omega$ is a bounded $n$-connected domain, $n > 1$, with $C^\infty$ smooth boundary. Neither the Bergman kernel nor the Szegő kernel associated to $\Omega$ can be rational functions.

The assumption in Theorem 6.1 that the boundary of $\Omega$ is $C^\infty$ smooth can be relaxed. For example, the conclusion about the Szegő kernel holds if the boundary is only assumed to be $C^2$ smooth. The conclusion about the Bergman kernel holds if the domain is only assumed to be finitely connected and such that no boundary component is a point. We shall explain how to relax the smoothness assumptions later in this section.

Before we proceed to prove Theorem 6.1, let us consider the case of a one-connected domain $\Omega \neq \mathbb{C}$. If $f_a$ is a Riemann mapping $f_a : \Omega \to D_1(0)$ such that $f_a(a) = 0$ and $f_a'(a) > 0$, the Bergman kernel for $\Omega$ can be expressed via
\[ K(z, w) = \frac{f_a'(z) f_a'(w)}{\pi (1 - f_a(z) f_a(w))^2}. \]
If we set \( w = a \) in this formula, we obtain the identity

\[
K(z, a) = C f'_a(z),
\]

where \( C = f'_a(a)/\pi \) is a positive constant. If we differentiate the formula with respect to \( \bar{w} \) and then set \( w = a \), we obtain

\[
\frac{\partial}{\partial \bar{w}} K(z, a) = f'_a(z)(C_1 + C_2 f_a(z)),
\]

where \( C_1 \) and \( C_2 \) are constants, and \( C_2 \neq 0 \). (In fact, \( C_2 = 2f'_a(a)^2/\pi \).) It can easily be deduced from these formulas that the Bergman kernel is rational if and only if the Riemann map is rational.

To study the Szegő kernel, assume that \( \Omega \) is a bounded simply connected domain with \( C^2 \) smooth boundary and let \( f_a \) denote a Riemann map as above. The Szegő kernel is given by

\[
S(z, w) = \frac{\sqrt{f'_a(z)} \sqrt{f'_a(w)}}{2\pi(1 - f_a(z)f_a(w))}.
\]

Set \( w = a \) in this formula to obtain

\[
S(z, a) = c \sqrt{f'_a(z)},
\]

where \( c = \sqrt{f'_a(a)/(2\pi)} \) is a positive constant. Now differentiate the formula with respect to \( \bar{w} \) and then set \( w = a \) to obtain

\[
\frac{\partial}{\partial \bar{w}} S(z, a) = \sqrt{f'_a(z)(c_1 + c_2 f_a(z))},
\]

where \( c_1 \) and \( c_2 \) are constants, and \( c_2 \neq 0 \). (In fact, \( c_2 = f'_a(a)^3/(2\pi) \).) These formulas reveal that Szegő kernel is rational if and only if the Riemann map and the square root of its derivative are rational. Let us summarize these results in the following theorem

**Theorem 6.2.** Suppose \( \Omega \neq \mathbb{C} \) is a simply connected domain. The Bergman kernel associated to \( \Omega \) is rational if and only if there is a rational biholomorphic mapping \( f(z) \) mapping \( \Omega \) one-to-one onto the unit disc. If \( \Omega \) is further assumed to be bounded and have \( C^2 \) smooth boundary, then the Szegő kernel associated to \( \Omega \) is rational if and only if there is a rational biholomorphic mapping \( f(z) \) mapping \( \Omega \) one-to-one onto the unit disc such that \( f'(z) \) is the square of a rational function.

**Proof of Theorem 6.1.** We shall use the notations that we set up previously to describe our \( n \)-connected domain \( \Omega \). Hence, \( \gamma_n \) denotes the outer boundary of \( \Omega \). Since we are assuming that \( n > 1 \), we may let \( \gamma_1 \) denote one of the inner boundary curves of \( \Omega \), and we let \( D_1 \) denote the bounded region enclosed by \( \gamma_1 \).

We first assume that the Szegő kernel associated to \( \Omega \) is rational. Formula (3.2) shows that it then follows that the Ahlfors mapping \( f_a(z) \) is a rational function of \( z \) for each point \( a \) in \( \Omega \) minus the finite set where the zeroes of \( S(z, a) \) might not all be simple zeroes. (It was proved earlier by M. Jeong [8,9], using other techniques, that the Ahlfors maps are all rational.) We may now use formula (2.2) to deduce that the Garabedian kernel \( L(z, a) \) is a rational function of \( z \) for each \( a \) in an open
subset of $\Omega$, and hence that $L(z,a)$ is a rational function of $(z,a)$. It is clear that the Ahlfors maps $f_\alpha$ can have no poles on $b\Omega$. Since, the boundary of $\Omega$ is assumed to be smooth, the Hopf lemma implies that $f'_\alpha(z) \neq 0$ for $z \in b\Omega$. Since the boundary curves of $\Omega$ are described by the equation $|f_\alpha(z)|=1$, it follows that the boundary curves of $\Omega$ are all real analytic curves. From this it follows that $S(z,w)$ extends holomorphically in $z$ and antiholomorphically in $w$ to an open set in $\mathbb{C} \times \mathbb{C}$ containing $\overline{\Omega} \times \overline{\Omega} - \{(z,z) : z \in b\Omega\}$, and that $L(z,w)$ extends holomorphically in $z$ and $w$ to an open set in $\mathbb{C} \times \mathbb{C}$ containing $\overline{\Omega} \times \overline{\Omega} - \{(z,z) : z \in \Omega\}$. This may seem like a silly thing to say in light of the fact that $S(z,w)$ and $L(z,w)$ are rational, however it implies that singularities must stay away from $b\Omega \times b\Omega - \{(z,z) : z \in b\Omega\}$.

We shall be concerned with the number of zeroes and poles of $S(z,a)$ and $L(z,a)$ as functions of $z$ which lie in $D_1$, and we shall consider how these numbers vary as $a$ moves from a point on the outer boundary of $\Omega$ to a point on $\gamma_1$. First, however, we shall need to review some properties of the zeroes of the Szegő kernel proved in [4] (see also [2]). We mentioned earlier that if $a \in \Omega$, then $S(z,a) \neq 0$ and $L(z,a) \neq 0$ for all $z \in b\Omega$. We also mentioned that neither $S(z,a)$ nor $L(z,a)$ can have poles on $b\Omega$. We shall use these facts to see that zeroes and poles of $S(z,a)$ and $L(z,a)$ which lie in $D_1$ cannot exit $D_1$ through $\gamma_1$ as $a$ varies in $\Omega$. We also mentioned earlier that $S(z,a)$ has $n-1$ zeroes in $\Omega$ as a function of $z$, and $L(z,a) \neq 0$ for all $z \in \overline{\Omega} - \{a\}$. It is proved in [4] that, if $a \in \Omega$ is allowed to tend to a point $A_k$ in a boundary curve $\gamma_k$, then the $n-1$ zeroes of $S(z,a)$ separate into simple zeroes which migrate to distinct points on the boundary in such a way that there is a point on each boundary curve $\gamma_j$, $j \neq k$ to which exactly one of the zeroes tends. To be precise, there exist points $\{A_j : 1 \leq j \leq n, j \neq k\}$ with $A_j \in \gamma_j$ such that the $n-1$ zeroes of $S(z,a)$ can be listed as $a_j, 1 \leq j \leq n, j \neq k$, where $a_j$ tends to $A_j$ for each $j \neq k$ as $a$ tends to $A_k$.

Since $S(z,w)$ is rational, there exist relatively prime polynomials $P(z,w)$ and $Q(z,w)$ such that $S(z,w) = P(z,w)/Q(z,w)$. There are at most finitely many points $w_0 \in \mathbb{C}$ for which the equations $P(z,w_0) = 0$ and $Q(z,w_0) = 0$ have a common root (see Ahlfors [1, page 300]). Let $B_S$ denote the (possibly empty) set of such points $w_0$. Similarly, there are relatively prime polynomials $p(z,w)$ and $q(z,w)$ such that $L(z,w) = p(z,w)/q(z,w)$, and there is a finite set $B_L$ of points $w_0$ where the equations $p(z,w_0) = 0$ and $q(z,w_0) = 0$ have a common root. Let $B = B_S \cup B_L$.

Let $S_a(z) = S(z,a)$. It is a simple exercise using the argument principle that the zeroes and poles of $S_a$ are continuous functions of $a$ when $a \notin B$ in the following sense. Suppose $z_0$ is a zero of multiplicity $m$ of $S(z,a_0)$ where $a_0 \notin B$. Given $\epsilon > 0$ such that $z_0$ is the only zero of $S(z,a_0)$ in $D_\epsilon(z_0)$, there is a $\delta > 0$ such that $S(z,a)$ has precisely $m$ zeroes in $D_\epsilon(z_0)$ as a function of $z$ (counting multiplicities) when $a \in D_\delta(a_0)$. A similar statement holds for poles of $S(z,a)$, and for zeroes and poles of $L(z,a)$.

We have stated all the necessary facts to be able to assert that there exist nonnegative integers $Z_S$, $Z_L$, $P_S$, and $P_L$ such that, for any point $a \in \Omega - B$, $Z_S$ is equal to the number of zeroes of $S(z,a)$ in $\overline{D_1}$, $Z_L$ is equal to the number of zeroes of $L(z,a)$ in $\overline{D_1}$, $P_S$ is equal to the number of poles of $S(z,a)$ in $\overline{D_1}$, and $P_L$ is equal to the number of poles of $L(z,a)$ in $\overline{D_1}$.

Let $\sigma$ denote a curve in $\overline{\Omega} - B$ which starts at a point $A_n$ on the outer boundary $\gamma_n$ of $\Omega$, travels through $\Omega$, and terminates at a point $A_1$ in $\gamma_1$. We shall be
able to deduce relationships between the four integers, \( Z_S, Z_L, P_S, \) and \( P_L \), by letting \( a \) tend to the two endpoints of \( \sigma \). The relationships shall turn out to be contradictory. To find relationships between these numbers, we shall need to use an argument from [4]. Since the boundary curves of \( \Omega \) are real analytic curves, there exists an antiholomorphic reflection function \( R(z) \) with the properties that \( R(z) \) is defined and is antiholomorphic on a neighborhood \( \mathcal{O} \) of \( \partial \Omega \), \( R(z_0) = z_0 \) when \( z_0 \in \partial \Omega \). \( R'(z) \) is non-vanishing on \( \mathcal{O} \), and \( R(z) \) maps \( \mathcal{O} \cap \Omega \) one-to-one onto \( \mathcal{O} - \overline{\Omega} \).

Let \( w_k \) be a sequence of points in \( \Omega \) that tend to \( A_n \) along \( \sigma \), and let \( a \) be a fixed point in \( \Omega - B \). By (2.1), we have \(-i L(z, a)T(z) = S(a, z) \) and \(-i L(z, w_k)T(z) = S(w_k, z) \) for \( z \in \partial \Omega \). Divide the second of these identities by the first and use the fact that \( R(z) = z \) on \( \partial \Omega \) to obtain

(6.1) \[
\frac{S(w_k, z)}{S(a, z)} = \frac{L(R(z), w_k)}{L(R(z), a)} \quad \text{for } z \in \partial \Omega.
\]

The function on the left hand side of (6.1) is antiholomorphic in \( z \) on a neighborhood of \( \partial \Omega \); so is the function on the right hand side. Since these functions agree on \( \partial \Omega \), they must be equal on a neighborhood of \( \partial \Omega \). In fact, because \( S \) and \( L \) are rational, these two functions are equal as meromorphic functions on the neighborhood \( \mathcal{O} \) of \( \partial \Omega \) on which \( R(z) \) is defined. We may assume that \( \mathcal{O} \) is small enough that \( S(z, a) \) and \( L(z, a) \) have no poles or zeroes in \( \mathcal{O} \). Formula (6.1) now allows us to read off the following facts (keep in mind that \( w_k \) is close to \( A_n \in \gamma_n \)). If \( S(z, w_k) \) has a zero \( z_0 \in \Omega \) near \( \gamma_1 \), then \( L(R(z_0), w_k) = 0 \), i.e., \( L(z, w_k) \) has a zero at the reflected point \( R(z_0) \in D_1 \) near \( \partial \Omega \). Neither \( S(z, w_k) \) nor \( L(z, w_k) \) can have a pole \( z_0 \in D_1 \) near \( \gamma_1 \), because neither \( L(z, w_k) \) nor \( S(z, w_k) \) has a pole at the reflected point \( R(z_0) \in \Omega \).

Finally, notice that (2.1) yields that

\[-i L(A_n, z)T(A_n) = S(z, A_n) \quad \text{for } z \in \Omega,\]

and consequently \(-i L(A_n, z)T(A_n) = S(z, A_n) \) for \( z \in \overline{D_j} \). Hence, the functions \( L(A_n, z) \) and \( S(z, A_n) \) have the same number of zeroes and poles in \( \overline{D_j} \). It is proved in [4] that \( S(z, A_n) \) has a single simple zero on \( \gamma_1 \) and this zero is approached by single simple zeroes of \( S(z, w_k) \) as \( k \to \infty \). No other zeroes of \( S(z, w_k) \) can migrate near \( \gamma_1 \). Our remarks above yield that \( L(z, w_k) \) has a simple zero at the reflection of the zero of \( S(z, w_k) \) near \( \gamma_1 \). By letting \( k \to \infty \), we obtain the relations

\[
Z_L = Z_S + 1
\]

\[
P_L = P_S.
\]

We now take a sequence of points \( w_k \) in \( \Omega \) that tend to \( A_1 \) along \( \sigma \). Formula (6.1) remains valid and, if we reason as above, we deduce that, since \( S(z, w_k) \) has no zeroes \( z_0 \) with \( z_0 \) near \( \gamma_1 \), \( L(z, w_k) \) has no zeroes in \( D_1 \) near \( \gamma_1 \). However, since \( L(z, w_k) \) has a simple pole at \( z = w_k \), it follows that \( S(z, w_k) \) has a simple pole at the reflected point \( R(w_k) \) in \( D_1 \). We now let \( k \to \infty \) and use the facts that \( S(z, A_1) \) and \( L(z, A_1) \) have the same zeroes and poles in \( \overline{D_1} \) and that one of those poles is a simple pole at \( A_1 \) to obtain

\[
Z_L = Z_S
\]

\[
P_L + 1 = P_S.
\]
These relationships contradict the ones we obtained by letting $w_k$ tend to $A_n$, and we conclude that $\Omega$ cannot be multiply connected.

We now turn to the study of the Bergman kernel. Assume that $K(z, w)$ is rational. We shall use an argument similar to the one above for the Szegő kernel, however, many of the underlying facts are different. Before we can begin, we must review some facts about the Bergman kernel (see [2] for proofs of these facts).

We first must prove that if the Bergman kernel associated to a bounded domain is rational, then any proper holomorphic mapping of the domain onto the unit disc must be rational. Suppose $f : \Omega \to D_1(0)$ is a proper holomorphic map. Such a map must be in $A^\infty(\Omega)$ and there is a positive integer $m$ such that $f$ is an $m$-to-one mapping of $\Omega$ onto $D_1(0)$ (see [2, page 62–70]). The branch locus $B = \{z \in \Omega : f'(z) = 0\}$ is a finite set, and for each point $w_0$ in $D_1(0) - f(B)$, there are exactly $m$ distinct points in $f^{-1}(w_0)$. Near such a point $w_0$, there is an $\epsilon > 0$ such that it is possible to define $m$ holomorphic maps $F_1(w), \ldots, F_m(w)$ on $D_\epsilon(w_0)$ which map into $\Omega - B$ such that $f(F_k(w)) = w$. These local inverses appear in the following transformation formula for the Bergman kernels under a proper holomorphic mapping. Let $K_1(z, w) = \pi^{-1}(1 - z\bar{w})^{-2}$ denote the Bergman kernel of the unit disc (and recall that $K(z, w)$ denotes the Bergman kernel for $\Omega$). It is proved in [2, page 68] that the kernels transform via

$$f'(z)K_1(f(z), w) = \sum_{k=1}^{m} K(z, F_k(w))\overline{F'_k(w)}.$$

Although the functions $F_k$ are only locally defined on $D_1(0) - f(B)$, the function on the right hand side of the transformation formula, being symmetric in the $F_k$, is globally well defined. In fact, the function is holomorphic in $z$ and antiholomorphic in $w$ for $(z, w) \in \Omega \times (D_1(0) - f(B))$. (The set $f(B)$ can be seen to be a removable singularity set, but we shall not need to know this.) If the origin is in $D_1(0) - f(B)$, we replace $f$ by its composition with a Möbius transformation so that $0 \not\in D_1(0) - f(B)$. We now set $w = 0$ in the transformation formula for the Bergman kernels to obtain

$$f'(z) = \pi \sum_{k=1}^{m} K(z, F_k(0))\overline{F'_k(0)}.$$

This shows that $f'(z)$ is a rational function. Now differentiate the transformation formula with respect to $\bar{w}$ and then set $w = 0$ to obtain

$$2f'(z)f(z) = \pi \sum_{k=1}^{m} \frac{\partial}{\partial \bar{w}}K(z, F_k(0))\overline{F'_k(0)}^2 + \pi \sum_{k=1}^{m} K(z, F_k(0))\overline{F''_k(0)}.$$

We may now deduce that $f'(z)f(z)$ is rational, and so it follows that $f(z)$ is rational.

Since the Ahlfors mappings $f_a(z)$ are proper mappings of $\Omega$ onto the unit disc, they are rational functions of $z$. As above, this implies that the boundary curves of $\Omega$ are all real analytic curves, and from this it follows that $K(z, w)$ extends holomorphically in $z$ and antiholomorphically in $w$ to an open set in $\mathbb{C} \times \mathbb{C}$ containing $\overline{\Omega} \times \overline{\Omega} - \{(z, z) : z \in b\Omega\}$.

The Bergman kernel is related to the classical Green’s function via

$$K(z, w) = -\frac{2}{\pi} \frac{\partial^2 G(z, w)}{\partial z \partial \bar{w}}.$$
Define another function $\Lambda(z, w)$ on $\Omega$ via

$$
\Lambda(z, w) = -\frac{2}{\pi} \frac{\partial^2 G(z, w)}{\partial z \partial w}.
$$

(This function is sometimes written $L(z, w)$ in the literature; we have chosen the symbol $\Lambda$ here to avoid confusion with our notation for the Garabedian kernel above.) It follows from known properties of the Green’s function that $\Lambda(z, w)$ extends holomorphically in $z$ and $w$ to an open set in $\mathbb{C} \times \mathbb{C}$ containing $\overline{\Omega} \times \overline{\Omega} - \{(z, z) : z \in \overline{\Omega}\}$, and that, if $a \in \overline{\Omega}$, then $\Lambda(z, a)$ has a double pole at $z = a$ as a function of $z$.

We shall need to use the following real variable theorem. Suppose that $R(x, y)$ is a real analytic function of $(x, y)$ on a product domain $U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $R(x, y_0)$ is a rational function of $x$ on $U_1$ for each $y_0 \in U_2$, and $R(x_0, y)$ is a rational function of $y$ on $U_2$ for each $x_0 \in U_1$. It then follows that $R(x, y)$ is a rational function of $(x, y)$ (the proof in Bochner and Martin [6, page 201] works in the real case, too).

We now consider the function $|f_a(z)|^2 = |S(z, a)|^2/|L(z, a)|^2$. We know that for each fixed $a \in \Omega$, the function $f_a(z)$ is a rational function of $z$, and hence $|f_a(z)|^2$ is a rational function of $(x, y)$ where $z = x + iy$. Since $|f_a(z)| = |f_z(a)|$, the real variable theorem mentioned above implies that $|f_a(z)|^2$ is a rational function of the variables $\text{Re } z$, $\text{Im } z$, $\text{Re } a$, and $\text{Im } a$.

Assume that $a \in \Omega$ is such that the zeroes of $S(z, a)$ are all simple zeroes. Because Ahlfors maps are proper holomorphic maps, it is easy to verify that

$$
\frac{1}{2} \ln |f_a(z)|^2 = G(z, a) + \sum_{i=1}^{n-1} G(z, a_i)
$$

(6.2)

where the points $a_i$, $i = 1, \ldots, n-1$ are the zeroes of $S(z, a)$ (which, together with $a$, are the zeroes of $f_a$). We now consider the way in which the zeroes $a_i$ depend on $a$, and we write $a_i(a)$ in order to regard $a_i$ as a function of $a$. Let $A_0$ be a fixed point in $\Omega$ such that the zeroes of $S(z, A_0)$ are simple. Since the points $a_i(A_0)$ are distinct, we may choose an $\epsilon > 0$ such that $D_\epsilon(a_i(A_0)) \subset \Omega$ for each $i$ and $D_\epsilon(a_i(A_0)) \cap D_\epsilon(a_j(A_0)) = \emptyset$ if $i \neq j$. Thus, $a_i(A_0)$ is the only zero of $S(z, A_0)$ in $D_\epsilon(a_i(A_0))$. The dependence of the zeroes of $S(z, a)$ on $a$ can be described by the formula,

$$
a_i(a) = \frac{1}{2\pi i} \int_{|z-a_i(A_0)|=\epsilon} z \frac{\partial}{\partial z} \frac{S(z, a)}{S(z, a_i)} \, dz,
$$

which is valid when $a$ is close to $A_0$. Because $S(z, a)$ is antiholomorphic in $a$, this formula shows that $a_i(a)$ is an antiholomorphic function of $a$ near $A_0$. We now differentiate (6.2) with respect to $z$ to obtain

$$
\frac{f_a'(z)}{2f_a(z)} = \frac{\partial}{\partial z} G(z, a) + \sum_{i=1}^{n-1} \frac{\partial}{\partial z} G(z, a_i).
$$

Next, we differentiate with respect to $a$ and use the complex chain rule to obtain

$$
\frac{\partial}{\partial a} \left( \frac{f_a'(z)}{2f_a(z)} \right) = \frac{\partial^2 G(z, a)}{\partial z \partial a} + \sum_{i=1}^{n-1} \frac{\partial^2 G(z, a_i)}{\partial z \partial a_i} \frac{\partial a_i}{\partial a}.
$$

(6.3)
We now claim that the function on the left hand side of (6.3) is a rational function $R(z, a)$ of $z$ and $a$. Indeed, because $|f_a(z)|^2$ is rational in the real and imaginary parts of $z$ and $a$, it follows that $R(z, a)$ is rational in the real and imaginary parts of $z$ and $a$. It is clear that $R(z, a)$ is holomorphic in $z$. Since $|f_a(z)| = |f_z(a)|$, it follows that $R(z, a) = R(a, z)$, and so $R(z, a)$ is holomorphic in $a$, too. Consequently, $R(z, a)$ is a rational function of $z$ and $a$. The function on the right hand side of (6.3) can be rewritten to yield

$$R(z, a) = -\frac{\pi}{2} \Lambda(z, a) - \frac{\pi}{2} \sum_{i=1}^{n-1} K(z, a_i) \frac{\partial \bar{a}_i}{\partial a}.$$ 

This last formula shows that, for each fixed $a$ in an open subset of $\Omega$, the function $\Lambda(z, a)$ is a rational function of $z$. Since $\Lambda(z, a) = \Lambda(a, z)$, we conclude that $\Lambda(z, a)$ is a rational function of $(z, a)$.

The Bergman kernel is related to $\Lambda$ via the identity

$$\Lambda(w, z)T(z) = -K(w, z)\overline{T(z)} \quad \text{for } w \in \Omega \text{ and } z \in b\Omega$$

(see [2, page 135]). Define the set $B$ to be the finite set of points $a$ at which the numerators and denominators of $K(z, a)$ and $\Lambda(z, a)$ have common zeroes as functions of $z$. Let $\sigma$ denote a curve in $\Omega - B$ which starts at a point $A_n$ on the outer boundary $\gamma_n$ of $\Omega$, travels through $\Omega$, and terminates at a point $A_1$ on an inner boundary curve $\gamma_1$. Since $K(z, a)$ and $\Lambda(z, a)$ cannot have poles on the boundary as functions of $z$ when $a \in \Omega$, the number $P_K$ of poles of $K(z, a)$ as a function of $z$ which lie in $D_1$ is constant as $a$ moves along $\sigma$ away from the endpoints of the curve. Also, the number $P_\Lambda$ of poles of $\Lambda(z, a)$ in $D_1$ is constant as $a$ moves along the curve. We shall deduce relationships between $P_K$ and $P_\Lambda$ by letting $a$ tend to the two endpoints of $\sigma$. The relationships shall turn out to be contradictory.

Let $w_k$ be a sequence of points in $\Omega$ that tend to $A_n$ along $\sigma$. Since $K(z, a)$ and $\Lambda(z, a)$ cannot vanish for $z \in b\Omega$ when $a$ is close to the boundary (see [13] or [2, page 132]), we may choose a point $a$ in $\Omega - B$ so that these functions are non-vanishing in $z$ near $b\Omega$. By (6.4), we have $\Lambda(a, z)T(z) = -K(a, z)\overline{T(z)}$ and $\Lambda(w_k, z)T(z) = -K(w_k, z)\overline{T(z)}$ for $z \in b\Omega$. Divide the second of these identities by the first and use the fact that $R(z) = z$ on $b\Omega$ to obtain

$$\frac{\Lambda(w_k, z)}{\Lambda(a, z)} = \frac{K(w_k, R(z))}{K(a, R(z))} \quad \text{for } z \in b\Omega.$$ (6.5)

The function on the left hand side of (6.1) is holomorphic in $z$ on a neighborhood of $b\Omega$; so is the function on the right hand side. Since these functions agree on $b\Omega$, they must be equal on a neighborhood of $b\Omega$. In fact, because $K$ and $\Lambda$ are rational, these two functions are equal as meromorphic functions on the neighborhood $\mathcal{O}$ of $b\Omega$ on which $R(z)$ is defined. We may assume that $\mathcal{O}$ is small enough that $K(z, a)$ and $\Lambda(z, a)$ have no poles or zeroes in $\mathcal{O}$. Formula (6.5) now allows us to read off the following facts. Neither $K(z, w_k)$ nor $\Lambda(z, w_k)$ can have a pole $z_0 \in D_1$ near $\gamma_1$ because neither of these functions has a pole at the reflected point $R(z_0) \in \Omega$.

Notice that (6.4) yields that

$$\Lambda(z, A_n)T(A_n) = -K(z, A_n)\overline{T(A_n)}$$
for \( z \in \Omega \), and hence for \( z \) in \( D_1 \). Hence, \( K(z, A_n) \) and \( \Lambda(z, A_n) \) have the same poles in \( \overline{D_1} \). Because no poles of \( K(z, w_k) \) or \( \Lambda(z, w_k) \) can migrate near the boundary of \( D_1 \) as \( w_k \to A_n \), we deduce that \( P_K = P_\Lambda \).

We now take a sequence of points \( w_k \) in \( \Omega \) that tend to \( A_1 \) along \( \sigma \). Formula (6.5) remains valid and, if we reason as above, we deduce that, since \( \Lambda(z, w_k) \) has a double pole at \( z = w_k \), it follows that \( K(z, w_k) \) has a double pole at the reflected point \( R(w_k) \). We now let \( k \to \infty \) and use the facts that \( K(z, A_1) \) and \( \Lambda(z, A_1) \) have the same poles in \( D_1 \) and that one of those poles is a double pole at \( A_1 \). We deduce that \( P_K = P_\Lambda + 2 \). This relationship contradicts the one we obtained by letting \( w_k \) tend to \( A_n \), and we conclude that \( \Omega \) cannot be multiply connected. The proof is complete.

We shall now explain how to relax the smoothness assumption that the boundary of \( \Omega \) be \( C^\infty \) smooth in Theorem 6.1. The conclusion about the Szegö kernel holds if the boundary is only assumed to be \( C^2 \) smooth because, in this setting, the functions \( S_a(z) \) and \( L_a(z) \) extend continuously to the boundary and \( T(z) \) is continuous on \( \partial \Omega \). All of the arguments carry through as before.

We next show that the conclusion about the Bergman kernel in Theorem 6.1 holds if the domain \( \Omega \) is only assumed to be finitely connected and such that no boundary component is a point. Since the Bergman kernel is related to the Green’s function by \( K(z, w) = \frac{-2}{\pi} \frac{\partial^2}{\partial z \partial \overline{w}} G(z, w) \), it follows that if the Green’s function is the logarithm of a rational function of the real and imaginary parts of \( z \) and \( w \), then the Bergman kernel must be rational, too. Hence, we have the following theorem.

**Theorem 6.3.** Suppose \( \Omega \) is a finitely connected domain such that no boundary component of \( \Omega \) is a point. The Green’s function \( G(z, a) \) associated to \( \Omega \) is the logarithm of a real valued rational function of the four real variables given by the real and imaginary parts of \( z \) and \( a \) if and only if \( \Omega \) is simply connected and equivalent to the disc via a rational biholomorphic mapping. Similarly, the Bergman kernel \( K(z, w) \) associated to \( \Omega \) is rational if and only if \( \Omega \) is simply connected and equivalent to the disc via a rational biholomorphic mapping.

Hence, the only finitely connected domains having Green’s functions as simple as the Green’s function for the disc are the obvious ones. (Of course, the Green’s function itself can never be rational because it has a logarithmic singularity.)

**Proof of Theorem 6.3.** Suppose \( \Omega \) is an \( n \)-connected domain such that no boundary component is a point and assume that the Bergman kernel \( K(z, w) \) associated to \( \Omega \) is rational. It is a standard result in the theory of conformal mapping that \( \Omega \) is biholomorphic to a bounded domain with real analytic boundary. Let \( \tilde{\Omega} \) denote such a bounded \( n \)-connected domain with \( C^\infty \) smooth boundary whose boundary consists of \( n \) non-intersecting simple closed real analytic curves and let \( \Phi : \Omega \to \tilde{\Omega} \) denote the biholomorphic mapping. Let \( \tilde{K}(z, w) \) denote the Bergman kernel associated to \( \tilde{\Omega} \). The transformation formula for the Bergman kernel under biholomorphic mappings gives

\[
K(z, w) = \Phi'(z) \tilde{K}(\Phi(z), \Phi(w)) \overline{\Phi'(w)}.
\]

It will be convenient to operate in the extended complex plane because it is inconvenient if the point at infinity belongs to one of the boundary components of
The transformation formula for the Bergman kernel under biholomorphic maps allows us to replace \( \Omega \) by any domain which is the inverse image of \( \Omega \) under a rational biholomorphic map. By replacing \( \Omega \) by its inverse image under a mapping of the form \( 1/(z - a) \), we may suppose that \( \Omega \) contains the point at infinity in its interior.

Since the transformation formula for the Bergman kernels under proper holomorphic mappings holds in the more general setting of Theorem 6.3, we deduce, as above, that the Ahlfors mappings are rational when the Bergman kernel is rational. Pick a point \( a \in \Omega \) and let \( f_a(z) \) denote the Ahlfors map associated to \( a \). Since \( f_a \) is rational, and since it is clear that \( f_a \) cannot have any poles in \( \Omega \), it follows that the boundary of \( \Omega \) consists of finitely many piecewise real analytic curves. Furthermore, there are at most finitely many points in the boundary where the boundary is not a \( C^\infty \) smooth curve. The non-smooth points in the boundary occur at boundary points where \( f_a' \) vanishes. Suppose \( f_a' \) vanishes to order \( m \) at a boundary point \( z_0 \). The boundary of \( \Omega \) near \( z_0 \) is described by two real analytic curves that cross at \( z_0 \) and make an angle of \( \pi/(m + 1) \). The mapping \( \Phi : \Omega \to \tilde{\Omega} \) described above extends continuously to the boundary of \( \Omega \). Let \( A = \Phi(a) \), and let \( F_A(z) \) denote the Ahlfors map of \( \tilde{\Omega} \) onto the unit disc associated to \( A \).

Ahlfors maps are solutions to an extremal problem of mapping the domain into the unit disc in such a way so as to maximize the real part of the derivative of the mapping at the associated point, it is easy to see that Ahlfors maps are invariant under biholomorphic mappings modulo unimodular constants to make derivatives real valued at the points of interest. Hence, we may write

\[
f_a = e^{i\theta} F_A \circ \Phi,
\]

where \( \theta \) is a real constant. Since \( \tilde{\Omega} \) has real analytic boundary, the Ahlfors map \( F_A \) extends holomorphically past the boundary and is locally one-to-one near the boundary. Hence, near \( z_0 \), we may write

\[
\Phi = F_A^{-1} \circ (e^{-i\theta} f_a)
\]

to see that \( \Phi \) extends holomorphically past the boundary of \( \Omega \) near \( z_0 \) and \( \Phi' \) vanishes to order \( m \) at \( z_0 \). Hence \( \Phi \) extends holomorphically to a neighborhood of \( \Omega \) and \( \Phi' \) only vanishes at points in \( \Omega \) that are corners in the boundary. Formula (6.6) now yields that \( K(z, w) \) extends holomorphically in \( z \) and antiholomorphically in \( w \) to a neighborhood in \( \mathbb{C} \times \mathbb{C} \) of \( (\overline{\Omega} \times \overline{\Omega}) \) – \( \{(z, z) : z \in b\Omega\} \). The rest of the argument is now a routine transcription of the proof of Theorem 6.1. All of the kernel identities used in the proof of Theorem 6.1 can be deduced by pulling back the identities that are known on \( \tilde{\Omega} \). For example, the fact that \( |f_a(z)| = |f_z(a)| \) can be deduced by using the argument given in the proof of Theorem 6.1 and then pulling back to \( \Omega \) using \( \Phi \). The Green’s function \( G(z, w) \) is related to the Green’s function \( \tilde{G}(z, w) \) on \( \tilde{\Omega} \) via \( \tilde{G}(z, w) = G(\Phi(z), \Phi(w)) \) and the corresponding statement for the \( \Lambda \) kernels is \( \Lambda(x, w) = \Phi'(z)\Lambda(\Phi(z), \Phi(w))\Phi'(w) \). The movement of zeroes and poles of \( K(z, w) \) near the boundary of \( \Omega \) can be read off from (6.6) and the known behavior of the zeroes and poles of \( \tilde{K}(z, w) \) near the boundary of \( \tilde{\Omega} \). The kernel \( K(z, w) \) vanishes identically when \( z = z_0 \) is a corner in the boundary, but this does not interfere with our work because we may choose a curve \( \sigma \) as in the proof of Theorem 6.1 that does not begin or terminate at a corner in the boundary.
of $\Omega$. As $w$ moves along such a curve, the poles of $K(z, w)$ as a function of $z$ that lie in a bounded component $\overline{D_1}$ of the complement of $\Omega$ cannot approach a corner in $b\Omega$. We leave it to the reader to complete the proof.

7. Complexity of the Poisson kernel. I showed in [4] how the Szegő projection can be used to solve the Dirichlet problem. The method gives rise to a formula for the Poisson kernel of a bounded $n$-connected domain $\Omega$ with $C^\infty$ smooth boundary which, in light of results in §4, reveals the level of complexity of that kernel. We shall use the same notation for describing $\Omega$ as we have set up previously, and as before, we also select a point $a \in \Omega$ such that the zeroes $a_1, \ldots, a_{n-1}$ of $S(z, a)$ are all distinct and simple. As before, let $S_a(z) = S(z, a)$ and $L_a(z) = L(z, a)$. The Szegő projection $P$ associated to $\Omega$ is the orthogonal projection of $L^2(b\Omega)$ onto the Hardy space $H^2(b\Omega)$. The Szegő kernel is the kernel for the Szegő projection in the sense that, given a function $u \in L^2(b\Omega)$, the projection $Pu$ is identified with a holomorphic function $h = Pu$ defined on $\Omega$ whose $L^2$ boundary values are equal to $Pu$, and

$$(Pu)(z) = \int_{w \in b\Omega} S(z, w) u(w) \, ds.$$ 

The Szegő projection maps $C^\infty(b\Omega)$ into $C^\infty(\overline{\Omega})$ (see [2] for proofs of these basic facts).

Recall that the set of functions $\{L(z, a_k)S(z, a)\}_{k=1}^{n-1}$ spans the same linear space as the set of functions $\{F'_k\}_{k=1}^{n-1}$. Define an $n-1 \times n-1$ matrix of periods via

$$(7.1) \quad A_{jk} = -i \int_{\gamma_j} L(z, a_k)S(z, a) \, dz,$$

for $j = 1, \ldots, n-1$. Because the matrix of periods of $F'_k$ is non-singular, so is $[A_{jk}]$. The following theorem was proved in [4].

**Theorem 7.1.** Given $\varphi \in C^\infty(b\Omega)$, let $c_j$ solve the linear system

$$\sum_{j=1}^{n-1} A_{jk} c_j = P(S_a \varphi)(a_k), \quad k = 1, \ldots, n-1.$$ 

The harmonic extension $E \varphi$ of $\varphi$ to $\Omega$ is given by

$$E \varphi = h + \overline{H} + \sum_{j=1}^{n-1} c_j \omega_j,$$

where, if we let $\psi = \varphi - \sum_{j=1}^{n-1} c_j \omega_j$, then

$$h = \frac{P(S_a \psi)}{S_a},$$

and

$$H = \frac{P(L_a \overline{\psi})}{L_a}.$$
The functions \( h \) and \( H \) are in \( A^\infty(\Omega) \).

This theorem allows the Poisson kernel to be written down in terms of the Szegő and Garabedian kernels. Let \([B_{jk}]\) denote the inverse of \([A_{jk}]\) so that \( c_j = \sum_{k=1}^{n-1} B_{jk} P(S_a \varphi)(a_k) \), i.e., so that

\[
c_j = \int_{w \in \partial \Omega} \left( \sum_{k=1}^{n-1} B_{jk} S(a_k, w) S(w, a) \right) \varphi(w) \, ds.
\]

The formulas for \( h \) and \( H \) can be written

\[
h(z) = \int_{w \in \partial \Omega} \frac{S(z, w) S(w, a)}{S(z, a)} \psi(w) \, ds,
\]

and

\[
H(z) = \int_{w \in \partial \Omega} \frac{S(z, w) L(w, a)}{L(z, a)} \psi(w) \, ds.
\]

Finally, when all these formulas are collected in one sum, we see that the Poisson extension \( \mathcal{E}u \) of \( u \) to \( \Omega \) is given by an integral

\[
(\mathcal{E}u)(z) = \int_{w \in \partial \Omega} p(z, w) u(w) \, ds,
\]

where \( p(z, w) \) is the Poisson kernel and is given by

\[
p(z, w) = \frac{S(z, w) S(w, a)}{S(z, a)} + \frac{S(z, w) L(w, a)}{L(z, a)} - \sum_{j,k=1}^{n-1} \left( B_{jk} S(a_k, w) S(w, a) \int_{\zeta \in \gamma_j} \frac{S(z, \zeta) S(\zeta, a)}{S(z, a)} \, ds \right)
\]

\[
- \sum_{j,k=1}^{n-1} \left( B_{jk} S(a_k, w) S(w, a) \int_{\zeta \in \gamma_j} \frac{S(z, \zeta) L(\zeta, a)}{L(z, a)} \, ds \right)
\]

\[
+ \sum_{j=1}^{n-1} \omega_j(z) \left( \sum_{k=1}^{n-1} B_{jk} S(a_k, w) S(w, a) \right).
\]

A disappointing feature of this formula for the Poisson kernel is the appearance of the term \( \omega_j(z) \). This function is closely tied to the Szegő kernel, but it is not as easily computed as the other terms in the sum. I gave a method to compute \( F'_j \) in [4, page 12]. The function \( \omega_j \) can be gotten from \( F'_j \) via the identity

\[
\omega_j(z) = \frac{1}{2\pi i} \int \int \frac{F'_j(w)}{w - z} \, dw \wedge d\bar{w}.
\]

This is the one point in this paper where I have not been able to obviate the need to compute an integral with respect to area measure.
8. Formulas for other kernels. The formulas for the Szegő kernel and Bergman kernel are the most interesting results of this paper. Similar formulas may be deduced for the Garabedian kernel $L(z, w)$ and the kernel $\Lambda(z, w)$. We close this paper by writing these formulas down.

Let $z \in \Omega$ and $w \in b\Omega$, and consider formula (3.2). Using identity (2.1) and the fact that $f_a = 1/f_a$ on $b\Omega$, we obtain

$$L(z, w) = \frac{f_a(w)}{f_a(z) - f_a(w)} \left( c_0 S(z, a)L(w, a) + \sum_{i,j=1}^{n-1} c_{ij} S(z, a_i)L(w, a_j) \right).$$

Since both sides of this identity are holomorphic in $z$ and $w$, this identity holds for $z, w \in \Omega, z \neq w$. Note that the constants $c_0$ and $c_{ij}$ are the same as the constants in (3.2).

Similarly, combining (6.4), (5.1), and (2.1) yields the identity

$$\Lambda(z, w) = 4\pi L(z, w)^2 - \sum_{i,j=1}^{n-1} \lambda_{ij} S(z, a_i)L(z, a)L(w, a_j)S(w, a),$$

where the coefficients $\lambda_{ij}$ are the same as those appearing in (5.1), and $z \in \Omega$ and $w \in b\Omega$. Again, since both sides of this identity are holomorphic in $z$ and $w$, the identity holds for $z, w \in \Omega, z \neq w$.

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