Application and explicit solution of recurrence relations with respect to space-time dimension

O.V. Tarasov

aFakultät für Physik, Universität Bielefeld, D-33615, Bielefeld, Germany

A short review of the method for the tensor reduction of Feynman integrals based on recurrence relations w.r.t. space-time dimension $d$ is given. A solution of the difference equation w.r.t. $d$ for the $n$-point one-loop integrals with arbitrary momenta and masses is presented. The result is written as multiple hypergeometric series depending on ratios of Gram determinants. For the 3-point function a new expression in terms of the Appell hypergeometric function $F_1$ is presented.

1. Introduction

The increasing precision of current experiments and the expected precision of future experiments reveal a need to evaluate the two-loop and higher order radiative corrections \[1\]. An evaluation of mass dependent two-loop corrections for many physical parameters such as the $\rho$ parameter $\[2\]$ and $Z \to b\bar{b}$ decay width $\[3\]$ is greatly desirable.

Of special interest is the problem of calculating one-loop corrections to the process $e^- e^+ \to 4$ fermions. To solve this problem efficient algorithms for evaluating multi leg one-loop tensor integrals are needed. Some progress was recently achieved in this direction in \[4\]– \[7\].

In the present paper we consider the algorithms based on recurrence relations w.r.t. space-time dimension or $d$-shift recurrence relations which we hope will be useful for evaluating multi loop and multi leg integrals. In the first part we describe the method for the reduction of tensor integrals to a set of basic integrals proposed in \[8\]. In the second part we present a solution of recurrence relations w.r.t. $d$ for one-loop $n$-point scalar integrals with arbitrary momenta and masses. This is a short report on work done in collaboration with J.Fleischer and F.Jegerlehner \[9\].

2. Reduction of tensor integrals

The reduction of one-loop tensor integrals to scalar integrals with shifted dimensions was given in \[9\]. The reduction of multi loop tensor integrals was considered in \[8\]. In the approach of \[8\] any tensor $L$ loop integral is represented as

$$\prod_{i=1}^{L} \int \frac{d^d k_i}{\pi^{d/2}} \prod_{j=1}^{N} P_{\nu_j, m_j}^{\nu_j} \prod_{r=1}^{n_1 \ldots n_L} k_{1 \mu_r} \ldots k_{L \lambda_s},$$

where $k_j$ is the momentum of the $j$-th line, $p_i$ are external momenta, $T_{\mu_1 \ldots \lambda_n L}$ is a tensor operator and

$$\partial_j \equiv \frac{\partial}{\partial m_j}, \quad d^+ G^{(d)} = G^{(d+2)};$$

$$P_{k, m} = \frac{1}{k^2 - m^2 + i\epsilon}.$$ 

It is assumed that the scalar integral on the r.h.s. of \[8\] has arbitrary masses and only after differentiation w.r.t. $m_i^2$ are these set to their concrete values.

To derive the explicit formula for the operator $T$, $L$ independent auxiliary vectors $a_i$ were introduced to represent the tensor in the integrand as

$$k_{1 \mu_1} \ldots k_{L \lambda_n L} = \frac{1}{i^{n_1 + \ldots + n_L}} \frac{\partial}{\partial a_1 \mu_1} \ldots \frac{\partial}{\partial a_L \lambda_n L} \times \exp \left[i(a_1 k_1 + \ldots + a_L k_L)\right]|_{a_i = 0},$$
and after that the integral was transformed into an α-parametric representation. From the parametric representation it is easy to deduce that

\[
T_{\mu_1...\lambda_nL}(\{p_i\}, \{\partial_j\}, \text{d}^+ \mu) = \frac{e^{-iQ((p), \alpha, (a))\rho}}{i^{n_1+...+n_N}} \times \frac{\partial}{\partial a_{\mu_1}} \cdots \frac{\partial}{\partial a_{L\lambda_nL}} \frac{e^{iQ((p), \alpha, (a))\rho}}{\partial a_{n_1+...+n_N}},
\]

where \(Q\) is a polynomial in \(\alpha, a\) and \(p_i p_j\). Applying \(T\) gives a sum made of external momenta, \(g_{\mu\nu}\) multiplied by scalar integrals with shifted \(d\). Advantages of the proposed tensor reduction are:

- no contractions with external momenta and metric tensor and no solution of a linear system of equations are needed
- it is easy to select a particular tensor structure
- representation in terms of integrals with shifted \(d\) is compact

Thus the problem is reduced to the evaluation of scalar integrals with different powers of scalar propagators and shifts in \(d\). They can be evaluated by applying generalized recurrence relations \([8]\). To obtain these relations we start with the identity:

\[
\prod_{i=1}^{L} \int d^4 k_i \frac{\partial}{\partial k_{\mu j}} \left\{ R_{\mu j} (\{k\}, \{p\}) \prod_{j=1}^{N} P_{k_j, m_j}^{\nu_j} \right\} = 0,
\]

where \(R\) is an arbitrary tensor polynomial. Such kind of identities for finding relations between Feynman integrals was used in \([11, 12]\). A systematic method was proposed in \([13]\). Upon differentiating two representations for scalar products can be used:

a) **integration by parts method** \([13]\):

\[
k_i p_i = \frac{1}{2} (k_i^2 + p_i^2) - (k_i - p_i)^2,
\]

b) **generalized recurrence relations** \([8]\):

\[
p_{\mu j} T_{\mu}(\{p\}, \{\partial\}, \text{d}^+) \int_{r=1}^{L} d^4 k_r \prod_{j=1}^{N} P_{k_j, m_j}^{\nu_j} = 
\]

Combining these methods gives many recurrence relations connecting integrals with different indices \(\nu_j\) and shifts of \(d\).

### 2.1. One-loop \(n\)-point integrals

The method for the reduction of one-loop tensor integrals

\[
I_{n}^{(d)} = \int \frac{d^d q}{i \pi^{d/2}} \frac{1}{\prod_{j=1}^{n} c_j},
\]

with \(c_j = (q - p_j)^2 - m_j^2 + i\epsilon\), was presented in \([8]\). The integral \(I_{n}^{(d)}\) can be written as a combination of scalar integrals by using the relation

\[
I_{n}^{(d)} = T_{\mu_1...\mu_n}(\{p_i\}, \{\partial_j\}, \text{d}^+) \ I_{n}^{(d)},
\]

where

\[
I_{n}^{(d)} = \int \frac{d^d q}{i \pi^{d/2}} \prod_{j=1}^{n} \frac{1}{c_j},
\]

and

\[
T_{\mu_1...\mu_n}(\{p_i\}, \{\partial_j\}, \text{d}^+) = \frac{1}{i} \prod_{j=1}^{n} \frac{\partial}{\partial a_{\mu_j}} \exp \left[ i \left( \sum_{k=1}^{n} (a p_k) \alpha_k - \frac{a^2}{4} \right) \right] \frac{1}{a_{\mu_j}},
\]

Two recurrence relations are needed to reduce any scalar integral \(I_{n}^{(d)}\) to basic ones:

\[
2\Delta_n \nu_j j^+ I_{n}^{(d)} = \sum_{k=1}^{n} (1 + \delta_{jk}) \\
\times \left( \frac{\partial \Delta_n}{\partial Y_{jk}} \left[ d - \sum_{i=1}^{n} \nu_i (k^{-1} + 1) \right] \right) I_{n}^{(d)},
\]

\[
(d - \sum_{i=1}^{n} \nu_i + 1) I_{n}^{(d+2)} = \\
\left[ 2\Delta_n G_{n-1} + \sum_{k=1}^{n} \frac{(\partial_k \Delta_n)}{G_{n-1}} k \right] I_{n}^{(d)},
\]

where \(j^+\) etc. shift the indices \(\nu_j \rightarrow \nu_j \pm 1\) and
therefore it is worthwhile to introduce the short-
tions will be ratios of Gram determinants and
characteristic variables occurring in our deriva-
through the

\[ \Delta_n = \Delta_n(\{p_1, m_1\}, \ldots, \{p_n, m_n\}) = \begin{vmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{vmatrix}, \]

(5)

\[ G_{n-1} \equiv G_{n-1}(p_1, \ldots, p_n) = -2^n \begin{vmatrix} p_{1n}p_{1n} & \cdots & p_{1n}p_{n-1} \\ p_{1n}p_{2n} & \cdots & p_{2n}p_{n-1} \\ \vdots & \vdots & \vdots \\ p_{1n}p_{n-1} & \cdots & p_{n-1}n p_{n-1} \end{vmatrix}, \]

(6)

\[ Y_{ij} = -p_i^2 + m_i^2 + m_j^2, \quad p_{ij} = p_i - p_j. \]

Here \( p_i \) is the external momentum flowing through the \( i \)-th line and \( m_j \) is the mass of the propagator corresponding to the \( j \)-th line.

Relations similar to (4) for the integrals with zero Gram determinants were given in [6], [7].

In the next sections we will also use an index notation for \( \Delta_n \) and \( G_n \):

\[ \lambda_{i_1 i_2 \ldots i_n} = \Delta_n(\{p_i, m_i\}, \ldots, \{p_n, m_n\}), \]

\[ g_{i_1 \ldots i_n} = G_{n-1}(p_i, \ldots, p_n). \]

(7)

We shall use the index notation for integrals obtained from \( I_n^{(d)} \) by contracting some lines. The characteristic variables occurring in our derivations will be ratios of Gram determinants and therefore it is worthwhile to introduce the shorthand notation

\[ r_{i_1 \ldots i_n} = -\frac{\lambda_{i_1 \ldots i_n}}{g_{i_1 \ldots i_n}}. \]

(8)

### 2.2. Two-loop propagator integrals

The reduction of the two-loop propagator type tensor integrals to basic scalar integrals can be done by using the relation [4]

\[ \int \int \frac{d^dk_1 d^dk_2}{c_1^{\nu_1} c_2^{\nu_2} \nu_3 c_4^{\nu_4}} k_{1\mu_1} \ldots k_{1\mu_2} k_{2\lambda_1} \ldots k_{2\lambda_2}, \]

\[ T_{\mu_1 \ldots \lambda_n}(q, \{\partial_j\}, d^+) \int \int \frac{d^dk_1 d^dk_2}{c_1^{\nu_1} c_2^{\nu_2} \nu_3 c_4^{\nu_4}} k_{1\mu_1} \ldots k_{1\mu_2} k_{2\lambda_1} \ldots k_{2\lambda_2}, \]

(9)

\[ c_1 = k_1^2 - m_1^2 + i\epsilon, \quad c_2 = (k_1 - q)^2 - m_2^2 + i\epsilon, \]

\[ c_3 = (k_2 - q)^2 - m_3^2 + i\epsilon, \quad c_4 = (k_1 - k_2)^2 - m_4^2 + i\epsilon, \]

An explicit form for the operator \( T \) and all necessary recurrence relations are given in [13]. The recursive procedure allows one to transform any diagram into a sum over 30 basic integrals:

\[ I_n^{(d)}(q^2) = \sum_{j=1}^{30} R_j(q^2, \{m_i^2\}, \{d\}) I_j^{(d)}(q^2). \]

This algorithm was implemented in [13] as a Mathematica package.

The method of tensor reduction [8] was used for the evaluation of radiative corrections to several important physical quantities. For example, for the evaluation of the 2-loop correction to the static potential in QCD [20] and the evaluation of the 2-loop correction in the gauge Higgs system in 3-dimensions [17]. Another important application may be the evaluation of 2-loop correction to the Bhabha scattering [18], [19].

### 3. Explicit solution of \( d \)-shift recurrence relations

The solution of the \( d \)-shift relations for one-loop propagator type integral was first discussed in [8]. Here we present the solution of the relation [4] for the integral \( I_n^{(d)} \) with arbitrary momenta and masses and first powers of scalar propagators. In this case

\[ (d-n+1)I_n^{(d+2)} = \frac{2\Delta_n}{G_{n-1}} + \sum_{k=1}^{n} \frac{\partial_k \Delta_n}{G_{n-1}} k^- I_n^{(d)}, \]

(10)

where the operator \( k^- \) removes the \( k \)-th line from \( I_n^{(d)} \). If we assume that \( n-1 \) point functions are already known then relation (10) is an inhomogeneous first order difference equation w.r.t. \( d \). Methods of solution of this kind of equations are well described in the mathematical literature [20]. The redefinition

\[ I_n^{(d)} = \frac{1}{\Gamma\left(\frac{d-n+1}{2}\right)} \frac{\Delta_n}{G_{n-1}} \frac{2}{\Delta_n} T_n^{(d)} \]

(11)

leads to a simpler equation

\[ T_n^{(d+2)} = T_n^{(d)} + \Gamma\left(\frac{d-n+1}{2}\right) \frac{2}{

\[ \Delta_n} \sum_{k=1}^{n} (\partial_k \Delta_n) k^- T_n^{(d)}. \]

(12)
Without loss of generality $d$ can be parametrized as

$$d = 2l - 2\varepsilon,$$

with $l$ being an integer number and $\varepsilon$ an arbitrary small parameter. The solution of equation (12) then reads

$$I_n^{(2l-2\varepsilon)} = b_n(\varepsilon) + \sum_{r=0}^{l} \frac{\Gamma(r-1-\varepsilon - \frac{n-1}{2})}{2\Delta_n}$$

$$\times \left( \frac{G_{n-1}}{\Delta_n} \right)^{r-1-\varepsilon} \sum_{k=1}^{n} (\partial_k \Delta_n) k^{-I_n^{(2r-2-2\varepsilon)}},$$

where $b_n(\varepsilon)$ is an $l$ independent constant. By changing $b_n(\varepsilon)$ the solution (13) can be rewritten in another form

$$I_n^{(2l-2\varepsilon)} = b_n(\varepsilon) - \sum_{r=0}^{\infty} \frac{\Gamma(r + d - \frac{n+1}{2})}{2\Delta_n}$$

$$\times \left( \frac{G_{n-1}}{\Delta_n} \right)^{r+\frac{d}{2}} \sum_{k=1}^{n} (\partial_k \Delta_n) k^{-I_n^{(d+2r)}},$$

The result for $I_n^{(d)}$ then reads

$$I_n^{(d)} = \frac{1}{\Gamma\left(\frac{d-n}{2}\right)} \frac{\Delta_n}{(G_{n-1})^{\frac{d}{2}}} b_n(\varepsilon) - \sum_{k=1}^{n} \frac{\partial_k \Delta_n}{2\Delta_n}$$

$$\times \sum_{r=0}^{\infty} \frac{(d-n+1)}{2} \left( \frac{G_{n-1}}{\Delta_n} \right)^{r} k^{-I_n^{(d+2r)}},$$

where $\bar{b}_n$ can be determined from the asymptotic value of $I_n^{(d)}$ in the limit $d \to \infty$.

### 3.1. Asymptotic form of $I_n^{(d)}$ at $d \to \infty$

The asymptotic form of $I_n^{(d)}$ as $d \to \infty$ can be derived from its parametric integral representation. $I_n^{(d)}$ can be written in a parametric form by employing, for example, the following formula

$$\frac{1}{c_1c_2\ldots c_n} = \Gamma(n) \int_{0}^{1} \ldots \int_{0}^{1} dx_1 \ldots dx_{n-1}$$

$$\times \frac{x_1^{2n-3}x_2^{n-3} \ldots x_{n-2}}{D^n},$$

with

$$D = [c_1x_1 \ldots x_{n-1}$$

$$+ c_2x_1 \ldots x_{n-2}(1-x_{n-1}) + \ldots + c_n(1-x_1)](17)$$

then shifting integration variable $q$ and applying the formula

$$\int \frac{d^q q}{(\pi^d/2) (q^2 - m_n^2)^n} = (-1)^{\alpha} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)(m_n^2)^{\alpha - \frac{d}{2}}}.$$ (18)

Any $n$-point function will be represented by a multiple parametric integral of the form:

$$I_n^{(d)} = \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(\alpha)} \int_{0}^{1} \ldots \int_{0}^{1} dx_1 \ldots dx_{n-1}$$

$$\times f \left( \{x\} \right) (h_n(p_j, m_s, \{x\}))^{\frac{d}{2} - n},$$ (19)

where $\{x\} = \{x_1, \ldots, x_{n-1}\}$ and $h_n$ is a polynomial. In our opinion this particular parametric integral representation is very convenient for performing asymptotic expansions. The behavior of this integral, as $d \to \infty$, can be established by using asymptotic methods [21]. The main contribution to the integral will come from the maximum of $h_n$. The maximum can be either on the boundary or in the interior of the integration region. In case the solution $\{x\} = \{\pi\}$ of the system of equations

$$\frac{\partial h_n}{\partial x_i} = 0, \quad (1 < i < n - 1)$$

lie in the interior of integration region then it is possible to show that

$$\pi_i = \frac{\sum_{k=1}^{n-1} \partial_k \Delta_n}{\sum_{j=1}^{n-1} \partial_j \Delta_n},$$ (21)

and at this point

$$h_n(p_j, m_s, \{\pi\}) = r_1 \ldots r_n.$$ (22)

In the domain of analyticity $h_n(p_j, m_s, \{x\}) \geq 0$ and $I_n^{(d)}$ is an integral of Laplace type. In the domain of non-analyticity there are subdomains of the integration region where $h_n(p_j, m_s, \{x\}) < 0$ and so the integral $I_n^{(d)}$ has an imaginary part. For this kinematic configuration $I_n^{(d)}$ may be represented as

$$I_n^{(d)} = \Gamma\left(n - \frac{d}{2}\right) \left\{ \int \frac{dx_1 \ldots dx_{n-1}}{h_n(p_j, m_s, \{x\})^{\alpha - \frac{d}{2}}} \right\}$$

$$+ \left( \cos \frac{\pi}{2} (d - 2n) + i \sin \frac{\pi}{2} (d - 2n) \right)$$

$$\times \sum_{j} \int_{0}^{1} \frac{dx_j f \left( \{x\} \right)}{|h_n(p_j, m_s, \{x\})^{\alpha - \frac{d}{2}}}.$$ (23)
Here \( \Omega_\lambda \) are subdomains of the integration region where \( h_n < 0 \). The boundaries of these subdomains can be found from the solutions of the equation

\[
h_n((p_j, m_s), \{ \mathcal{F} \}) = 0. \tag{24}
\]

In (23) we took the main branch of \( h_n \) and assumed that the momenta and masses are real parameters. A concrete example of representation (23) will be given in the next section.

The imaginary part of \( I_n^{(d)} \)

\[
\text{Im} I_n^{(d)} = \frac{-\pi}{\Gamma\left(\frac{d}{2} - n + 1\right)} \times \sum \int_{\Omega_\lambda} \frac{\{dx\} f(\{x\})}{|h_n((p_j, m_s), \{x\})|^{n-\frac{d}{2}}} \tag{25}
\]

is a sum of integrals of Laplace type and therefore its asymptotic form at \( d \to \infty \) can be established.

It should be noted that representations like (23) are valid for multi loop integrals including vacuum ones and also they can be used as an alternative to the method for evaluating imaginary parts (23) does not influence our considerations corresponding to integrals one can obtain from \( I_n^{(d)} \) by contracting two lines and so on. Splitting the integral for \( h_n < 0 \) into real and imaginary parts (23) does not influence our considerations because splitting adds a boundary where \( h_n \) vanishes and therefore there will be no contribution to the asymptotic value of the integral.

According to the previous discussion and from Eq. (15) it follows that in order to find \( \mathcal{T}_n \) the contribution of \( I_n^{(d)} \) at \( d \to \infty \) is needed only if it is proportional to \( r_{i_1...i_n}^{d/2} \). Such an asymptotic value will occur for kinematic configurations when all \( \mathcal{F} \leq 1 \) and \( h_n \) has either maximum or negative minimum. In the latter case \( I_n^{(d)} \) must be split into real and imaginary parts according to (23) and the asymptotic value of the term giving a contribution proportional to \( r_{i_1...i_n}^{d/2} \) are to be determined. It should be noted that the contribution of order \( r_{i_1...i_n}^{d/2} \) may come from the analytic continuation of the sums in (15). As will be seen in the next sections these sums represent hypergeometric functions.
From the formula for the leading asymptotic of the multiple integral \cite{23} we obtain:

\[ \mathcal{R}_n = -(2\pi)^{\frac{d}{2}} \frac{\Gamma \left(1 - \frac{d}{2}\right) \Gamma \left(\frac{d}{2}\right) i^d}{\sqrt{\pi |G_{n-1}|}} \frac{i^d}{r_n^{d-n}}, \]  

(30)

and therefore the general solution of Eq. (11) for the kinematic configuration satisfying the conditions $0 \leq \varpi_i \leq 1$ will be

\[ I_n^{(d)} = -(2\pi)^{\frac{d}{2}} \frac{\Gamma \left(1 - \frac{d}{2}\right) \Gamma \left(\frac{d}{2}\right)}{\Gamma \left(\frac{d-n+1}{2}\right)} \frac{i^d}{r_n^{d-n-1}} \]

\[ \times \sum_{k=1}^{n} \frac{\partial_k \Delta_n}{2\Delta_n} \sum_{r=0}^{\infty} \frac{d-n+1}{\frac{d}{2} - r} \left( \frac{2}{\Delta_n} \right)^r x_n^{-d+2r}, \]  

(31)

where $(a)_r \equiv \Gamma(r + a)/\Gamma(a)$ is the Pochhammer symbol. A representation of one-loop integrals in terms of Lauricella functions was given in \cite{23}.

4. 2-point function

Expression \cite{13} for $I_2^{(d)}$ includes two one-fold infinite sums over tadpole integrals $I_1^{(d)}$ (see formula \cite{13}):

\[ I_1^{(d)} = -\Gamma \left(1 - \frac{d}{2}\right) (m_1^2)^{\frac{d}{2} - 1}. \]  

(32)

It is convenient to label the lines of $I_2^{(d)}$ by $i, j$ because the integrals $I_2^{(d)}$ will be encountered in calculating $I_3^{(d)}, \ldots$ as a result of contraction of different lines.

At $n = 2$, substituting \cite{32} into \cite{13} gives

\[ \frac{2\lambda_{ij} I_2^{(d)}}{\Gamma \left(1 - \frac{d}{2}\right)} = b_2 + \frac{\partial_i \lambda_{ij}}{(m_1^2)^{\frac{d}{2} - \frac{d}{2}} \Gamma \left(\frac{d-d}{2}\right) r_{ij}} \sum_{r=0}^{\infty} \frac{d-1}{\left(\frac{d}{2} - 1\right)_r} \left( \frac{m_2^2}{r_{ij}} \right)^r \]

\[ + \frac{\partial_j \lambda_{ij}}{(m_2^2)^{\frac{d}{2} - \frac{d}{2}} \Gamma \left(\frac{d-d}{2}\right) r_{ij}} \sum_{r=0}^{\infty} \frac{d-1}{\left(\frac{d}{2} - 1\right)_r} \left( \frac{m_1^2}{r_{ij}} \right)^r. \]  

(33)

One can easily recognize that each of the two infinite sums can be written in terms of Gauss’ hypergeometric function $_2F_1$. The parametric integral for $I_2^{(2)}$ is

\[ I_2^{(2)} = \Gamma \left(2 - \frac{d}{2}\right) \int_0^1 dx_1 b_2^{\frac{d}{2} - 2}, \]  

(34)

with

\[ h_2 = p_{ij}^2 x_1^2 - x_1 (p_{ij}^2 - m_1^2 + m_2^2) + m_2^2. \]

An extremum of $h_2$ is at $\varpi_1 = (p_{ij}^2 - m_1^2 + m_2^2)/(2p_{ij}^2)$. The maximum of $h_2$ exists if

\[ \frac{\partial^2 h_2}{\partial x_1^2} = 2p_{ij}^2 < 0, \]  

(35)

i.e. only for Euclidean momentum.

The formula \cite{23} gives the imaginary part of $I_2^{(d)}$ on the cut

\[ \text{Im} I_2^{(d)} = -\pi \int p_{ij}^2 dx_1 \frac{((x_1 - x_1^-)(x_1^+ - x_1^-))^{\frac{d}{2} - 2}}{p_{ij}^{d-d} \Gamma \left(\frac{d-d}{2}\right)}, \]  

(36)

where

\[ x_1^+ = \frac{p_{ij}^2 m_1^2 + m_2^2 \pm \sqrt{-\lambda_{ij}}}{2p_{ij}^2} \]

is the solution of the equation

\[ h_2(\{p_{ij}, m_i\}, x_1) = 0. \]  

(37)

The change of the integration variable $x_1 = (x_1^- - x_1^-) + x_1$ makes the integration trivial and gives

\[ \text{Im} I_2^{(d)} = -\pi \frac{\Gamma \left(\frac{d-d}{2}\right)}{p_{ij}^{d-d} \Gamma \left(\frac{d-d}{2}\right)} \left( \frac{\sqrt{-\lambda_{ij}}}{p_{ij}^2} \right)^{d-3}. \]  

(38)

The real part of $I_2^{(d)}$ coming from the region where $h_2 < 0$ is

\[ \text{Re} I_2^{(d)} = \text{Im} I_2^{(d)} \cot \frac{\pi}{2}(d-4). \]

At large $d$ the asymptotic value of $\text{Im} I_2^{(d)}$ and $\text{Re} I_2^{(d)}$ can be easily found. At $p_{ij}^2 > (m_1 + m_2)^2$ both $0 \leq x_1^\pm \leq 1$.

Now several remarks concerning the evaluation of $b_2$ are in order. From \cite{13} we see that at large dimension $b_2 \sim r_{ij}^{d/2}$. Such a contribution may come either from the asymptotic value of $I_2^{(d)}$ (this may happen only if $0 < \varpi_1 < 1$) at large $d$ or from the analytic continuation of infinite sums when their expansion parameter exceeds 1. To proceed further let us assume that $m_1^2 > m_2^2$. At $p_{ij}^2 < m_1^2 - m_2^2$ and $p_{ij}^2 > (m_1 + m_2)^2$ the value of $b_2$
is determined from the asymptotic of \( I^{(d)}_2 \sim r_{ij}^{d/2} \). At \( m_i^2 - m_j^2 < p_{ij}^2 < (m_i - m_j)^2 \) both \( I^{(d)}_2 \) and the infinite sums have no contributions \( \sim r_{ij}^{d/2} \) and therefore in this region \( b_2 = 0 \). In the region \( (m_i - m_j)^2 < p_{ij}^2 < (m_i + m_j)^2 \) the value of \( b_2 \) is determined from the asymptotic value of infinite sums at large \( d \).

Combining all values of \( b_2 \) and writing infinite sums in terms of hypergeometric functions we obtained the following result

\[
\frac{2\lambda_{ij}}{\Gamma(1 - \frac{d}{2})} I^{(d)}_2 = \frac{-\sqrt{\pi} \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \sum_{r_{ij}} \left[ \frac{\frac{d}{2}}{\sqrt{1 - \frac{m_i^2}{r_{ij}}}} + \frac{\frac{d}{2}}{\sqrt{1 - \frac{m_j^2}{r_{ij}}}} \right] + \frac{\partial_1 \lambda_{ij}}{m_i^2} \right] 2F_1 \left[ 1, \frac{d-1}{2}; \frac{m_j^2}{r_{ij}} \right] \]

The analytic continuation of \( \sum_{r_{ij}} \) leads to a compact form for \( I^{(d)}_2 \) (see also [23])

\[
g_{ij} I^{(d)}_2(\Gamma(\frac{2-d}{2}) = \frac{\partial_1 \lambda_{ij}}{m_i^2} \right] 2F_1 \left[ 1, \frac{4-d}{2}; \frac{1}{2} - \frac{r_{ij}}{m_j^2} \right] + \frac{\frac{d}{2}}{\sqrt{1 - \frac{m_i^2}{r_{ij}}}} \right] 2F_1 \left[ 1, \frac{4-d}{2}; \frac{1}{2} - \frac{r_{ij}}{m_i^2} \right].
\]

The expansion of \( I^{(d)}_2 \) to all orders in \( \varepsilon = (4 - d)/2 \) is given in [21].

5. 3-point function

The expression for \( I^{(d)}_3 \) includes summation over \( 2F_1 \) coming from \( I^{(d)}_2 \). To derive an explicit formula we first use the relation

\[
(1 - z)^{a} 2F_1 \left[ a, b; \frac{c}{z} \right] = 2F_1 \left[ a, c - b; \frac{z}{z - 1} \right]
\]

in order to remove the factor \( d \) from the second parameter of \( 2F_1 \). Summing over \( I^{(d)}_2 \) gives the Appell hypergeometric functions

\[
F_3(\frac{d - 2}{2}, 1, 1/2, d/2; x, y) =
\sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_{m+n} m! n!} x^m y^n,
\]

which may be reduced to \( F_1 \) by means of \[25\]

\[
F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = (1 - y)^{-\beta'} F_1 \left( \alpha, \beta', \alpha + \alpha'; x, \frac{y}{y - 1} \right)
\]

The Appell hypergeometric function \( F_1 \) defined as

\[
F_1(\alpha, \beta, \gamma; x, y) =
\sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}(\gamma)_{m+n}}{(\gamma)_{m+n} m! n!} x^m y^n.
\]

Similar to \( I^{(d)}_2 \) we will label the lines of \( I^{(d)}_3 \) by \( i, j, k \). The result for \( I^{(d)}_3 \) valid when \( h_3 \) has maximum inside the integration region reads

\[
\left. \frac{\lambda_{ijk}}{\Gamma(2 - \frac{d}{2})} I^{(d)}_3 = 2 \pi \frac{\sqrt{-g_{ijk} r_{ijk}}}{d} \middle| \theta_{ij} \right| \left( \theta_{i} \lambda_{ij} + \theta_{k} \lambda_{ij} + \theta_{j} \lambda_{ij} + \theta_{j} \lambda_{k} + \theta_{i} \lambda_{jk} \right),
\]

where

\[
\lambda_{ij} =
\frac{(m_i^2 - m_j^2 d/2)}{2(d - 2)} \frac{\partial_1 \lambda_{ij}}{m_i^2} \right] 2F_1 \left[ 1, \frac{4-d}{2}; \frac{1}{2} - \frac{r_{ij}}{m_j^2} \right] + \frac{\frac{d}{2}}{\sqrt{1 - \frac{m_i^2}{r_{ij}}}} \right] 2F_1 \left[ 1, \frac{4-d}{2}; \frac{1}{2} - \frac{r_{ij}}{m_i^2} \right],
\]

The function \( F_1 \) has a simple integral representation

\[
F_1 \left( \frac{d - 2}{2}, 1, \frac{1}{2}, \frac{d}{2}; \frac{x}{y} \right) = \frac{d - 2}{2} \times \int_0^1 \frac{a^{d/2}}{(1 - xu)1 - yu} du.
\]
The behavior of the integrand for $I_3^{(d)}$ having the maximum inside the integration region is demonstrated in Fig.1. The polynomial
$$h_3 = -x_1 x_2 (1 - x_1) p_{13}^2 - x_1^2 x_2 (1 - x_2) p_{12}^2$$
$$x_1 x_2 m_1^2 - x_1 (1 - x_1) (1 - x_2) p_{23}^2$$
$$+ x_1 (1 - x_2) m_2^2 + (1 - x_1) m_3^2.$$ \hspace{1cm} (48)
has a maximum at
$$x_1 = \frac{\partial_1 \Delta_3 + \partial_2 \Delta_3}{-G_2}, \quad x_2 = \frac{\partial_1 \Delta_3 + \partial_2 \Delta_3}{-G_2}$$
if
$$\frac{\partial^2 h_3}{\partial x_1 \partial x_2} = 2 x_1^2 p_{12}^2 < 0,$$ \hspace{1cm} (49)
$$\frac{\partial^2 h_3}{\partial x_1 \partial x_1} \frac{\partial^2 h_3}{\partial x_2 \partial x_2} - \left( \frac{\partial^2 h_3}{\partial x_1 \partial x_2} \right)^2$$
$$= -\frac{(\partial_1 \Delta_3 + \partial_2 \Delta_3)^2}{2 G_2} > 0.$$ \hspace{1cm} (50)
From (49), (50) it follows that the maximum will be achieved if
$$p_{12}^2 < 0, \quad \text{and} \quad G_2 < 0.$$ \hspace{1cm} (51)

The result for the 4-point function then reads
$$\frac{\lambda_{ijkl}}{\Gamma \left( 2 - \frac{d}{2} \right)} I_4^{(d)} = -4 \pi^{\frac{d}{2}} \sqrt{g_{ijkl}} \ r_{ijkl} \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right)$$
$$\cdot F_1 \left( \frac{x - \beta}{y - \beta}; \frac{x}{y} \right) + \phi_{ijkl} \ b_{ijkl} \phi_{ijkl}$$ \hspace{1cm} (52)
$$+ \phi_{kl} \ b_{ijkl} \phi_{ijkl}.$$ \hspace{1cm} (53)

where
$$\phi_{ijkl} = -\frac{\pi \sqrt{2}}{\sqrt{g_{ijkl}}} \ r_{ijkl} \ 2 F_1 \left( \frac{d-3}{2}; \ r_{ijkl} \right)$$
$$\times \text{terms with } F_1 \text{ and } F_S.$$ \hspace{1cm} (54)

and $F_S$ is the Lauricella-Saran function of three variables. In our case it has a simple integral representation:
$$F_S = \frac{\Gamma \left( \frac{d}{2} \right) (y - z)^{-\frac{d}{2}}}{\Gamma \left( \frac{d-3}{2} \right) \Gamma \left( \frac{d}{2} \right)}$$
$$\times \int_0^1 \arcsin \sqrt{\frac{(y - z)(1 - t)}{1 - ty}} \ dt.$$ \hspace{1cm} (55)

All other details one can find in [3].

6. Conclusions

The recurrence relations with respect to the space-time dimension turn out to be useful for the reduction of tensor integrals. The relations for master integrals in different dimensions are rather
simple even for integrals with several momenta and masses. Preliminary investigation of recurrence relations w.r.t. $d$ for the two-loop propagator type integrals revealed that scalar integral in (9) with all $\nu_j = 1$ can be written as multiple hypergeometric series. Asymptotic expansion at large $d$ may be used as yet another tool for approximate evaluation of Feynman integrals. As is known such an expansion works well in quantum mechanics and in quantum field theory in lattice calculations.

Acknowledgements. I wish to thank organizers of the conference “Loops and Legs 2000” Johannes Blümlein and Tord Riemann for the useful and well organized conference. I am thankful to C. Ford and V. Ravindran for carefully reading the manuscript and useful remarks. I would like also to thank the DFG for financial support.

REFERENCES

1. J. Erler, S. Heinemeyer, W. Hollik, G. Weiglein and P.M. Zerwas, preprint DESY 00-050; hep-ph/0005024.
2. R. Barbieri et al. Nucl. Phys. B409 (1993) 105; J. Fleischer, O.V. Tarasov and F. Jegerlehner, Phys. Lett. B319 (1993) 249.
3. J. Fleischer, O.V. Tarasov, F. Jegerlehner and P. Raczka, Phys. Lett. B293 (1992) 437.
4. Z. Bern, L. Dixon and D.A. Kosower, Nucl. Phys. B412 (1994) 751.
5. J.M. Campbell, E.W.N. Glover and D.J. Miller, Nucl. Phys. B498 (1997) 397.
6. J. Fleischer, F. Jegerlehner and O.V. Tarasov, Nucl. Phys. B566 (2000) 423.
7. T. Binoth, J.P. Guillet and G. Heinrich, Nucl. Phys. B572 (2000) 361.
8. O.V. Tarasov, Phys. Rev. D54 (1996) 6479.
9. J. Fleischer, F. Jegerlehner and O.V. Tarasov, in preparation.
10. A.I. Davydychev, Phys. Lett. B263 (1991) 107.
11. B. Petersson, J. Math. Phys. 6 (1965) 1955.
12. G. ’t Hooft and M. Veltman, Nucl. Phys. B192 (1981) 159.
13. F.V. Tkachov, Phys. Lett. 100B (1981) 65; K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159.
14. O.V. Tarasov, Nucl. Phys. B502 (1997) 455.
15. R. Mertig and R. Scharf, Comput. Phys. Comm. 111 (1998) 265.
16. Y. Schröder, Phys. Lett. B447 (1999) 321.
17. F. Eberlein, Nucl. Phys. B550 (1999) 303.
18. V. Smirnov and O. Veretin, Nucl. Phys. B566 (2000) 469.
19. C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, [hep-ph/0003261].
20. L.M. Milne-Thomson, The calculus of Finite Differences, Macmillan, London, 1960.
21. N. Bleistein and R.A. Handelsman, Asymptotic expansions of integrals Holt, Rinehart and Winston, 1975.
22. A.I. Davydychev, J. Math. Phys. 32 (1991) 1052; 33 (1992) 358.
23. F. Berends, A. Davydychev and V. Smirnov, Nucl. Phys. B478 (1996) 59.
24. A.I. Davydychev, Phys. Rev. D61 (2000) 087-701; A.I. Davydychev and M.Yu. Kalmykov, [hep-th/0005287].
25. Higher Transcendental Functions, Vol. 1, A. Erdélyi, Ed. (McGraw-Hill Book Company, Inc., New York, 1953).
26. E.E. Boos and A.I. Davydychev, Theor. Math. Phys. 89 (1991) 1052.
27. L.G. Cabral-Rosetti and M.A. Sanchis-Lozano, [hep-ph/9809213].
28. C. Anastasiou, E.W.N. Glover and C. Oleari, Nucl. Phys. B572 (2000) 307.
29. S. Saran, Ganita, 5 (1954) 77-91.