A special class of solutions
of
the truncated Hill’s equation

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Abstract

This work investigates the existence and properties of a certain class of solutions of the Hill’s equation truncated in the interval \([\tau, \tau + L]\) - where \(L = Na\), \(a\) is the period of the coefficients in Hill’s equation, \(N\) is a positive integer and \(\tau\) is a real number. It is found that the truncated Hill’s equation has two different types of solutions which vanish at the truncation boundaries \(\tau\) and \(\tau + L\): There are always \(N - 1\) solutions in each stability interval of Hill’s equation, whose eigen value is dependent on the truncation length \(L\) but not on the truncation boundary \(\tau\); There is always one and only one solution in each finite conditional instability interval of Hill’s equation, whose eigen value might be dependent on the boundary location \(\tau\) but not on the truncation length \(L\).

The results obtained are applied to the physics problem on the electronic states in one dimensional crystals of finite length. It significantly improves many known results and also provides more new understandings on the physics in low dimensional systems.
1 Introduction

Hill’s equation is a class of second-order ordinary differential equations with coefficients of period $a$. The properties of solutions of Hill’s equation have been investigated in detail by many mathematicians\cite{1, 2}. It is well known that the eigenvalues and eigenfunctions of an ordinary second order differential equation depend on both the region in which the equation is given and the boundary conditions. If the equation is given in a finite interval and we are interested in a special class of the solutions - they vanish at the boundaries - we expect that both the eigenvalues and the eigenfunctions are dependent on the boundary locations and the properties of the finite interval. For an ordinary differential equation with periodic coefficients, in general it is usually expected that both the eigenvalues and the eigenfunctions are dependent on both the boundary locations and the interval length.

By using the general theory of Hill’s equation, this work investigates the existence and properties of this class of solutions of Hill’s equation truncated in the interval $[\tau, \tau + L]$ - where $L = Na$, $a$ is the period of the coefficients in Hill’s equation and $N$ is a positive integer and $\tau$ is a real number. It is found that the truncated Hill’s equation have two different types of solutions which vanish at the truncation boundaries $\tau$ and $\tau + L$: There are always $N - 1$ solutions in each stability interval of Hill’s equation, whose eigen value is dependent on the truncation length $L$ but not on the truncation boundary $\tau$; There is always one and only one solution in each finite conditional instability interval, whose eigen value might be dependent on the boundary location $\tau$ but not on the truncation length $L$.

The Schrödinger differential equation on the electronic states in one dimensional crystals\cite{3} can be considered as a special case of Hill’s equation. Due to the fact that any real crystal always has a finite size, the electronic states in a real crystal of finite size can be better described as solutions of a special case of Hill’s equation truncated in a finite length which vanish at the boundaries. The general results obtained here are applied to the physics problem on the electronic states in one dimensional crystals of finite length. It significantly improves many known results and also provides many new understandings on the physics in low dimensional systems.
2 Hill’s equation

We consider the Hill’s equation having the following form:
\[ \{p(x)y'(x)\}' + \{\lambda s(x) - q(x)\}y(x) = 0, \quad -\infty < x < +\infty \] (1)

where \(\lambda\) is a real parameter, \(p(x)\), \(q(x)\), and \(s(x)\) are real-valued and have the same period \(a\). It is assumed that \(p(x)\) is continuous and nowhere zero and that \(p'(x)\), \(s(x)\) and \(q(x)\) are piecewise continuous with period \(a\) and there is a constant \(s > 0\) such that \(s(x) \geq s[2]\).

A periodic eigenvalue problem and a semi-periodic eigenvalue problem play a significant role in the theory of Hill’s equation (1)[2]:

(i). The periodic eigenvalue problem comprises (1), considered to hold in \([0, a]\) and the periodic conditions:
\[ y(a) = y(0), \quad y'(a) = y'(0). \]
The eigenfunctions are denoted by \(\zeta_n(x)\) and eigenvalues by \(\nu_n\) (n = 0, 1,...).

(ii). The semi-periodic eigenvalue problem comprises (1), considered to hold in \([0, a]\) and the semi-periodic conditions:
\[ y(a) = -y(0), \quad y'(a) = -y'(0). \]
The eigenfunctions are denoted by \(\xi_n(x)\) and eigenvalues by \(\mu_n\) (n = 0, 1,...).

The numbers \(\nu_n\) and \(\mu_n\) occur in the order[2]
\[ \nu_0 < \mu_0 \leq \mu_1 < \nu_1 \leq \nu_2 < \mu_2 \leq \mu_3 \leq \nu_3 \leq \nu_4 < \ldots \]

Let \(\phi_1(x, \lambda)\) and \(\phi_2(x, \lambda)\) be the linearly independent solutions of (1), such that \(\phi_1(0, \lambda) = 1, \ \phi_1'(0, \lambda) = 0; \ \phi_2(0, \lambda) = 0, \ \phi_2'(0, \lambda) = 1\), the real number \(D(\lambda)\) defined by \(D(\lambda) = \phi_1(a, \lambda) + \phi_2'(a, \lambda)\) is called the discriminant of (1) and it plays a significant role in determining the properties of the solutions of (1).

\(D(\lambda)\) is an analytical function of \(\lambda[2]\). The values of \(\lambda\) for which \(|D(\lambda)| < 2\) form an open set on the real \(\lambda\)-axis. This set can be expressed as the union of countable collection of disjoint open intervals \((\nu_{2m}, \mu_{2m})\) and \((\mu_{2m+1}, \nu_{2m+1})\). Equation (1) is stable when \(\lambda\) lies in these intervals thus the intervals are called the stability intervals of (1). The set
on the real $\lambda$-axis with the stability intervals excluded can be expressed as the union of countable collection of disjoint intervals in which $|D(\lambda)| \geq 2$: $(-\infty, \nu_0]$, $[\mu_{2m}, \mu_{2m+1}]$ and $[\nu_{2m+1}, \nu_{2m+2}]$. Those intervals are called conditional instability intervals. Thus as $\lambda$ increases from $-\infty$ to $+\infty$, the conditional instability intervals and the stability intervals occur alternatively.

### 3 The truncated Hill’s equation

A special class of solutions of the truncated Hill’s equation are the main interest in this paper. We are trying to find out the eigenvalues $\Lambda$ and eigenfunctions $y(x)$, which are solutions of

$$
\left\{ p(x) y'(x) \right\}' + \{\Lambda s(x) - q(x)\} y(x) = 0, \quad \tau \leq x \leq \tau + L
$$

(2)

under the following boundary condition:

$$
y(\tau, \Lambda) = y(\tau + L, \Lambda) = 0,
$$

(3)

where $\tau$ is a real number, $L = Na$ and $N$ is a positive integer.

Suppose $y_1(x, \lambda)$ and $y_2(x, \lambda)$ are two linearly independent solutions of (1), in general, the solution of (2) which satisfies (3) if it exists, can be expressed as

$$
y(x, \Lambda) = c_1 y_1(x, \Lambda) + c_2 y_2(x, \Lambda), \quad \tau \leq x \leq \tau + L
$$

(4)

In the following we try to find out the eigenvalues of (2) under the boundary condition (3) by assuming all non-trivial solutions of (1) are known. After each eigenvalue is found, the corresponding eigenfunction can be obtained easily.

In principle, we should consider solutions of Equations (2) and (3) for $\lambda$ in $(-\infty, +\infty)$. However, according to the Theorem 3.2.2 of [2], there is not a nontrivial solution of (2) satisfying (3) for $\Lambda$ in $(-\infty, \nu_0]$. Thus we need only to consider $\lambda$ in $(\nu_0, +\infty)$.

For $\lambda$ in $(\nu_0, +\infty)$, $D(\lambda)$ may have five different cases depending on the value of $\lambda$. Correspondingly, the two linearly independent solutions of (1) are also different:

Case A. $|D(\lambda)| < 2$: 
This is in the stability intervals of (1). Two linearly independent solutions of (1) can be expressed as[2]
\[ y_1(x, \lambda) = e^{i\alpha(\lambda)x} p_1(x, \lambda), \quad y_2(x, \lambda) = e^{-i\alpha(\lambda)x} p_2(x, \lambda), \]
where \( \alpha(\lambda) \) is a real number depending on \( \lambda \) and
\[ 0 < \alpha(\lambda)a < \pi \]
and \( p_1(x, \lambda) \) and \( p_2(x, \lambda) \) have period \( a \). All \( \alpha(\lambda) \) and \( p_1(x, \lambda) \) and \( p_2(x, \lambda) \) are functions of \( \lambda \).

From (3) and (4) we have
\[ c_1 e^{i\alpha \tau} p_1(\tau, \lambda) + c_2 e^{-i\alpha \tau} p_2(\tau, \lambda) = 0, \]
and
\[ c_1 e^{i\alpha(\tau+L)} p_1(\tau + L, \lambda) + c_2 e^{-i\alpha(\tau+L)} p_2(\tau + L, \lambda) = 0. \]
Due to that
\[ p_1(\tau, \lambda) = p_1(\tau + L, \lambda), \quad p_2(\tau, \lambda) = p_2(\tau + L, \lambda), \]
we obtain either
\[ e^{i\alpha(\lambda)L} - e^{-i\alpha(\lambda)L} = 0. \] (A.1)
or
\[ c_1 p_1(\tau, \lambda) = 0 \quad \text{and} \quad c_2 p_2(\tau, \lambda) = 0. \] (A.2)
We can easily prove that neither \( p_1(\tau, \lambda) \) nor \( p_2(\tau, \lambda) \) can be zero. Suppose \( p_1(\tau, \lambda) = 0 \), we have \( p_1(\tau + a, \lambda) = 0 \) and thus \( y_1(\tau, \lambda) = y_1(\tau + a, \lambda) = 0 \). Then according to the theorem 3.1.3 of [2], \( \lambda \) must be in the range of \( [\mu_{2m}, \mu_{2m+1}] \) or \( [\nu_{2m+1}, \nu_{2m+2}] \), in which \( |D(\lambda)| \geq 2 \). This is in contradictory to the condition \( |D(\lambda)| < 2 \) here. Similarly \( p_2(\tau, \lambda) \) can not be zero. Thus we must have \( c_1 = c_2 = 0 \) and the non-trivial solutions obtained from (A.2) do not exist.

Thus the existence of non-trivial solution of (2) and (3) requires
\[ \alpha(\lambda)L = j\pi, \quad j = 1, 2, 3, \ldots, N - 1. \]
Therefore for each stability interval, there are $N - 1$ values of $\Lambda_j$, where $j = 1, 2, ..., N - 1$, for which

$$\alpha(\Lambda_j) = \frac{j \pi}{L}. \quad (5)$$

Each eigen value for this case, is a function of $L$, the truncation length. But they all do not depend on the location of the truncation boundary $\tau$.

Case B. $D(\lambda) = 2$.

There could be two possibilities[2]:

B1. $\phi_2(a, \lambda)$ and $\phi'_1(a, \lambda)$ are both zero:

$$\phi_2(a, \lambda) = \phi'_1(a, \lambda) = 0.$$  

This corresponds to $\nu_{2m+1} = \nu_{2m+2}$. Two linearly independent solutions of (1) can be expressed as[2]

$$y_1(x, \lambda) = p_1(x, \lambda), \quad y_2(x, \lambda) = p_2(x, \lambda),$$

and $p_1(x, \lambda)$ and $p_2(x, \lambda)$ have period $a$. It is always possible to choose $c_1$ and $c_2$, which are not both zero, to make $y(\tau, \lambda) = c_1 y_1(\tau, \lambda) + c_2 y_2(\tau, \lambda) = 0$ and naturally we have $y(\tau + L, \lambda) = 0$. Thus we have a solution $\Lambda = \lambda$ of (2) and (3) in this case and the corresponding eigenfunction is a periodic function.

Obviously for this case, we always have

$$y(\tau, \lambda) = y(\tau + a, \lambda) = 0.$$  

B2. $\phi_2(a, \lambda)$ and $\phi'_1(a, \lambda)$ are not both zero.

Two linearly independent solutions of (1) can be expressed as[2]

$$y_1(x, \lambda) = p_1(x, \lambda), \quad y_2(x, \lambda) = xp_1(x, \lambda) + p_2(x, \lambda),$$

and $p_1(x, \lambda)$ and $p_2(x, \lambda)$ have period $a$.

Equations (3) and (4) become

$$c_1 p_1(\tau, \lambda) + c_2(\tau p_1(\tau, \lambda) + p_2(\tau, \lambda)) = 0,$$

and

$$c_1 p_1(\tau + L, \lambda) + c_2((\tau + L)p_1(\tau + L, \lambda) + p_2(\tau + L, \lambda)) = 0.$$
Thus, we must have,
\[ c_2 L p_1(\tau, \lambda) = 0, \]
and
\[ c_1 p_1(\tau, \lambda) + c_2 (\tau p_1(\tau, \lambda) + p_2(\tau, \lambda)) = 0. \]

These two lead to
\[ p_1(\tau, \lambda) = 0, \quad \text{and} \quad c_2 = 0. \] (B2.1)

Thus if we have a solution for this case, we must have (B2.1).

Case C. \( D(\lambda) > 2 \).

Two linearly independent solutions of (1) can be expressed as \[ 2 \]
\[ y_1(x, \lambda) = e^{\beta(\lambda)x} p_1(x, \lambda), \quad y_2(x, \lambda) = e^{-\beta(\lambda)x} p_2(x, \lambda), \]
where \( \beta(\lambda) \) is a real non-zero number depending on \( \lambda \) and \( p_1(x, \lambda) \) and \( p_2(x, \lambda) \) have period \( a \).

Equations (3) and (4) lead to
\[ c_1 e^{\beta \tau} p_1(\tau, \lambda) + c_2 e^{-\beta \tau} p_2(\tau, \lambda) = 0, \]
and
\[ c_1 e^{\beta (\tau + L)} p_1(\tau + L, \lambda) + c_2 e^{-\beta (\tau + L)} p_2(\tau + L, \lambda) = 0. \]

Due to
\[ p_1(\tau, \lambda) = p_1(\tau + L, \lambda), \quad p_2(\tau, \lambda) = p_2(\tau + L, \lambda), \]
we must have
\[ c_1 p_1(\tau, \lambda) = 0, \quad \text{and} \quad c_2 p_2(\tau, \lambda) = 0. \]

By the Sturm Separation Theorem[4], the zeros of \( p_1(x, \Lambda) \) are separated from the zeros of \( p_2(x, \Lambda) \). Thus \( p_1(\tau, \lambda) \) and \( p_2(\tau, \lambda) \) can not be zero simultaneously, either \( c_1 \) or \( c_2 \) must be zero. Whether we have a solution or not in this case is dependent on whether either one of
\[ p_1(\tau, \lambda) = 0 \quad \text{and} \quad c_2 = 0 \] (C.1)
and
\[ p_2(\tau, \lambda) = 0 \quad \text{and} \quad c_1 = 0 \] (C.2)
is true. From the discussion on (B1), (B2), (C), we can see that in the conditional
instability intervals in which $D(\lambda) \geq 2$, if we have a solution of (2) and (3), we must
have either $\nu_{2m+1} = \nu_{2m+2}$ or one of (B2.1), (C.1) and (C.2) when $\nu_{2m+1} < \nu_{2m+2}$. Thus
we always have $y(\tau + a, \Lambda) = 0$ if $y(\tau, \Lambda) = 0$. Therefore for conditional instability
intervals $D(\lambda) \geq 2$, the following equation is a necessary condition for having a solution
of Equations (2) and (3):

$$y(\tau + a, \Lambda) = y(\tau, \Lambda) = 0. \quad (6)$$

Case D. $D(\lambda) = -2$.
There could be two possibilities[2]:

D1. $\phi_2(a, \lambda)$ and $\phi'_1(a, \lambda)$ are both zero:

$$\phi_2(a, \lambda) = \phi'_1(a, \lambda) = 0.$$  

This corresponds to $\mu_{2m} = \mu_{2m+1}$. Two linearly independent solutions of (1) can be
expressed as[2]

$$y_1(x, \lambda) = s_1(x, \lambda), \quad y_2(x, \lambda) = s_2(x, \lambda),$$

and $s_1(x, \lambda)$ and $s_2(x, \lambda)$ have semi-period $a$: $s_i(x + a) = -s_i(x)$. It is always possible
to choose $c_1$ and $c_2$, they are not both zero, to make $y(\tau, \lambda) = c_1y_1(\tau, \lambda) + c_2y_2(\tau, \lambda) = 0$
and we naturally have $y(\tau, \lambda) = y(\tau + L, \lambda) = 0$. Thus we always have a solution of (2)
and (3) in this case. The corresponding eigenfunction is a semiperiodic function.

Obviously for this case, we have

$$y(\tau, \lambda) = y(\tau + a, \lambda) = 0.$$  

D2. $\phi_2(a, \lambda)$ and $\phi'_1(a, \lambda)$ are not both zero.

Two linearly independent solutions of (1) can be expressed as[2]

$$y_1(x, \lambda) = s_1(x, \lambda), \quad y_2(x, \lambda) = x \cdot s_1(x, \lambda) + s_2(x, \lambda),$$

and $s_1(x, \lambda)$ and $s_2(x, \lambda)$ have semi-period $a$.

Equations (3) and (4) become

$$c_1 s_1(\tau, \lambda) + c_2 (\tau s_1(\tau, \lambda) + s_2(\tau, \lambda)) = 0,$$
and 
\[ c_1 s_1(\tau + L, \lambda) + c_2 ((\tau + L) s_1(\tau + L, \lambda) + s_2(\tau + L, \lambda)) = 0. \]
Thus, we must have, 
\[ c_2 L s_1(\tau, \lambda) = 0, \]
and 
\[ c_1 s_1(\tau, \lambda) + c_2 (\tau s_1(\tau, \lambda) + s_2(\tau, \lambda)) = 0. \]
These two lead to 
\[ s_1(\tau, \lambda) = 0 \quad \text{and} \quad c_2 = 0. \quad (D2.1) \]

Case E. \( D(\lambda) < -2 \):
Two linearly independent solutions of (1) can be expressed as[2]
\[ y_1(x, \lambda) = e^{\beta(\lambda)x} s_1(x, \lambda), \quad y_2(x, \lambda) = e^{-\beta(\lambda)x} s_2(x, \lambda), \]
where \( \beta(\lambda) \) is a non-zero real number depending on \( \lambda \) and \( s_1(x, \lambda) \) and \( s_2(x, \lambda) \) are semiperiodic functions.

Equations (3) and (4) lead to 
\[ c_1 e^{\beta(\lambda)\tau} s_1(\tau, \lambda) + c_2 e^{-\beta(\lambda)\tau} s_2(\tau, \lambda) = 0, \]
and 
\[ c_1 e^{\beta(\lambda)(\tau + L)} s_1(\tau + L, \lambda) + c_2 e^{-\beta(\lambda)(\tau + L)} s_2(\tau + L, \lambda) = 0. \]
Due to 
\[ s_1(\tau, \lambda) = (-1)^N s_1(\tau + L, \lambda), \quad s_2(\tau, \lambda) = (-1)^N s_2(\tau + L, \lambda), \]
we must have 
\[ c_1 s_1(\tau, \lambda) = 0, \quad \text{and} \quad c_2 s_2(\tau, \lambda) = 0. \]

By the Sturm Separation Theorem[4], the zeros of \( s_1(x, \lambda) \) are separated from the zeros of \( s_2(x, \lambda) \). Because \( s_1(\tau, \lambda) \) and \( s_2(\tau, \lambda) \) can not be zero simultaneously, either \( c_1 \) or \( c_2 \) must be zero. Whether we have a solution or not in this case is dependent on whether either one of 
\[ s_1(\tau, \lambda) = 0 \quad \text{and} \quad c_2 = 0 \quad (E.1) \]
and
\[ s_2(\tau, \lambda) = 0 \quad \text{and} \quad c_1 = 0 \quad (E.2) \]
is true. From the discussion on (D1), (D2), (E), it can be seen that in the conditional instability intervals in which \( D(\lambda) \leq -2 \), if we have a solution of (2) and (3), we must have either \( \mu_{2m} = \mu_{2m+1} \) or one of (D2.1), (E.1), (E.2) when \( \mu_{2m} < \mu_{2m+1} \). From these equations we can see that we also always have \( y(\tau + a, \Lambda) = 0 \) if \( y(\tau, \Lambda) = 0 \). Thus for \( D(\lambda) \leq -2 \), the same Equation (6) is a necessary condition of (4).

It is easy to see that (6) is also a sufficient condition for having a solution of Equations (2) and (3) for \( |D(\lambda)| \geq 2 \): Equation (3) can be obtained by repeating (6) for \( N \) times. Thus the Equation (6) is a necessary and sufficient condition for having a solution of Equations (2) and (3) in a conditional instability interval for which \( |D(\lambda)| \geq 2 \).

Equation (6) does not contain the confinement length \( L \), thus the eigen value \( \Lambda \) in a conditional instability interval \( |D(\lambda)| \geq 2 \) might be dependent on \( \tau \), but not on \( L \).

The Theorem 3.1.3 of [2] indicates that for any real number \( \tau \), there is always one and only one \( \Lambda \) in each conditional instability interval \([\mu_{2m}, \mu_{2m+1}] \) or \([\nu_{2m+1}, \nu_{2m+2}] \), for which we have \( y(\tau, \Lambda) = y(\tau + a, \Lambda) = 0 \). We can label these \( \Lambda \) as \( \Lambda_{\tau,2m} \) and \( \Lambda_{\tau,2m+1} \). According to the theorem 3.1.3 of [2], the range of \( \Lambda_{\tau,2m} \) are in \([\mu_{2m}, \mu_{2m+1}] \) and \( \Lambda_{\tau,2m+1} \) are in \([\nu_{2m+1}, \nu_{2m+2}] \).

It is interesting to see how each of these \( \tau \)-dependent eigenvalues \( \Lambda \) changes as \( \tau \) changes. Here we only discuss \( \Lambda_{\tau,2m} \). \( \Lambda_{\tau,2m+1} \) can be very similarly discussed.

If \( \mu_{2m} = \mu_{2m+1} \), then we have the case D1, the \( \Lambda_{\tau,2m} \) will not change as \( \tau \) changes, we always have \( \Lambda_{\tau,2m} = \mu_{2m} \) and the corresponding eigen function \( y(x, \Lambda_{\tau,2m}) \) will be dependent on \( \tau \), but it will always be a semi-periodic function.

The more interesting case is when \( \mu_{2m} \neq \mu_{2m+1} \). According to the theorem 3.1.2 of [2], \( \xi_{2m}(x) \) and \( \xi_{2m+1}(x) \) have exact \( 2m + 1 \) zeros in \([0,a) \). Then according to the Sturm Comparison Theorem[4], the zeros of \( \xi_{2m}(x) \) and the zeros of \( \xi_{2m+1}(x) \) must be distributed alternatively: There is always one and only one zero of \( \xi_{2m+1}(x) \) between two consecutive zeros of \( \xi_{2m}(x) \), and there is always one and only one zero of \( \xi_{2m}(x) \) between two consecutive zeros of \( \xi_{2m+1}(x) \).

We start from a zero \( x_{1,2m} \) of \( \xi_{2m}(x) \). If \( \tau = x_{1,2m} \), we have a solution of (2) and (3)
\( \Lambda_{r,2m} = \mu_{2m}, \) and \( y(x, \Lambda_{r,2m}) = \xi_{2m}(x) \). There is a nearby zero \( x_{1,2m+1} \) of \( \xi_{2m+1}(x) \). If \( \tau = x_{1,2m+1} \), we have a solution of (2) and (3) \( \Lambda_{r,2m} = \mu_{2m+1}, \) and \( y(x, \Lambda_{r,2m}) = \xi_{2m+1}(x) \).

If we treat \( \tau \) as a variable and let \( \tau \) go continuously from a zero \( x_{1,2m} \) of \( \xi_{2m}(x) \) to a nearest neighbor zero \( x_{1,2m+1} \) of \( \xi_{2m+1}(x) \), because \( \Lambda_{r,2m} \) is a continuous function of \( \tau \), the corresponding solution of (2) and (3) \( \Lambda_{r,2m} \) will also go continuously from \( \mu_{2m} \) to \( \mu_{2m+1} \). Similarly, if \( \tau \) goes from the zero \( x_{1,2m+1} \) of \( \xi_{2m+1}(x) \) to the other nearest neighbor zero \( x_{2,2m} \) of \( \xi_{2m}(x) \) continuously, the corresponding \( \Lambda_{r,2m} \) will also go continuously from \( \mu_{2m+1} \) to \( \mu_{2m} \). Since in the interval \([0, a)\), both \( \xi_{2m}(x) \) and \( \xi_{2m+1}(x) \) have exactly \( 2m+1 \) zeros, in general \( \Lambda_{r,2m} \) as function of \( \tau \) will always complete \( 2m+1 \) ups and downs in an interval of length \( a \). Similarly \( \Lambda_{r,2m+1} \) as function of \( \tau \) will always complete \( 2m+2 \) ups and downs in an interval of length \( a \).

Depending on the value of \( \tau \), the solutions of (2) and (3) may have different forms. If \( \tau \) goes in one direction (either in the right or in the left) from a zero \( x_{1,2m} \) of \( \xi_{2m}(x) \) \( (y(x, \Lambda) = \xi_{2m}(x) \) here) to its nearest neighbor zero \( x_{1,2m+1} \) of \( \xi_{2m+1}(x) \) \( (y(x, \Lambda) = \xi_{2m+1}(x) \) now) then again to the next zero \( x_{2,2m} \) of \( \xi_{2m}(x) \) \( (y(x, \Lambda) = \xi_{2m}(x) \) again) continuously, the corresponding \( \Lambda_{r,2m} \) will also goes from \( \mu_{2m} \) to \( \mu_{2m+1} \) then again back to \( \mu_{2m} \) continuously. But for \( \tau \) in the two open intervals \((x_{1,2m}, x_{1,2m+1})\) and \((x_{1,2m+1}, x_{2,2m})\) of this path, the solution of (2) and (3) - the function \( y(x, \Lambda) \) (4) has two different forms: one has the form \( c_1 e^{\beta(\Lambda)} x \) \( s_1(x, \Lambda) \), and the other has the form \( c_2 e^{-\beta(\Lambda)} x \) \( s_2(x, \Lambda) \). In which one section it has which form is dependent on \( p(x), q(x) \) and \( s(x) \) in (1).

A function with the form of \( c_1 e^{\beta(\Lambda)} x \) \( s_1(x, \Lambda) \) or \( c_2 e^{-\beta(\Lambda)} x \) \( s_2(x, \Lambda) \), in which \( \beta(\Lambda) > 0 \), is mainly distributed near either one of the two ends of the truncated region, due to the exponential factor. We can call it as an end solution of (2) and (3). End solutions may also exist in the instability intervals \( D(\Lambda) > 2 (\nu_{2m+1} < \Lambda < \nu_{2m+2}) \) and have the form of either \( c_1 e^{\beta(\Lambda)} x \) \( p_1(x, \Lambda) \) or \( c_2 e^{-\beta(\Lambda)} x \) \( p_2(x, \Lambda) \) in the truncated region, where \( p_1(x, \Lambda) \) and \( p_2(x, \Lambda) \) are period functions. These end solutions are introduced into the instability intervals when the truncated boundary \( \tau \) is not a zero of either one of the two eigen functions \( \xi_n(x) \) of the periodic eigenvalue problem or when \( \tau \) is not a zero of either one of the two eigenfunctions \( \xi_n(x) \) of the semi-periodic eigenvalue problem. That is the case \(|D(\Lambda)| > 2 \).

In Figure 1 is shown \( \Lambda_{r,0} \) as the function of \( \tau \). In the open interval \((x_{1,0}, x_{1,1})\) and
the open interval \((x_{1,1}, x_{1,0} + a)\) the function \(y(x, \Lambda)\) (4) has different forms: one has the form \(c_1 e^{\beta(x)} x \, s_1(x, \Lambda)\), and the other has the form \(c_2 e^{-\beta(x)} x \, s_2(x, \Lambda)\), shown as a dashed line and a dotted line, indicating that the end solution is mainly distributed near either one of the two different ends of the truncated region.

In summary thus there are two different types of solutions of the truncated Hill’s equation: there are always \(N - 1\) solutions in each stability interval, whose eigen values depend on the truncation length \(L\) but not on the truncation boundary \(\tau\). There is always one and only one solution in each finite conditional instability interval \([\mu_{2m}, \mu_{2m+1}]\) or \([\nu_{2m+1}, \nu_{2m+2}]\), whose eigen value might depend on the boundary \(\tau\) but not on the truncation length \(L\).

4 A special example: Electronic states in one dimensional crystals of finite length

The one-dimensional Schrödinger differential equation[3] with periodic potential is a special case of Hill’s equation (1), in which

\[
p(x) = \frac{\hbar^2}{2m}, \quad s(x) = 1, \quad q(x) = v(x)
\]

is the periodic potential and \(\lambda = E\) is the energy eigen value. Here \(\hbar\) is the Planck’s constant and \(m\) is the electron mass. It has been well known that the corresponding eigenvalues form energy bands and the eigenfunctions are Bloch waves[3]. The more general results obtained here for the truncated Hill’s equation can be directly applied to the electronic states in one dimensional crystals of finite length.

By using a Kronig-Penney model potential, Pedersen and Hemmer found that for a finite crystal of length \(L = Na\), there are \(N - 1\) states corresponding to each energy band \(n > 0\), the energies of these \(N - 1\) states map the energy bands[5] of the infinite crystal exactly and this mapping does not depend on the boundaries of the finite crystal. This is a special case of the more general result obtained in (5).

By using the theorem 3.1.2 of [2], the author has given an analytical solution on the complete quantum confinement of one dimensional Bloch waves[6] in an inversion-symmetric potential. The results obtained in [6] greatly generalized and improved many
results obtained in [5].

The results obtained in this work can be directly applied to the electronic states in one dimensional crystals of finite length. The exact and general results are[7]:

For one dimensional crystals bounded at $\tau$ and $\tau + L$, there are two different types of electronic states: There are $N - 1$ states corresponding to each energy band of the Bloch wave, their eigenvalues $\Lambda$ are given by (5), thus are dependent only on the crystal length $L$ but not on the crystal boundary $\tau$ and map the energy band exactly; There is always one and only one electronic state corresponding to each band gap of the Bloch wave, whose eigenvalue $\Lambda$ might be dependent on the crystal boundary $\tau$ but not on the crystal length $L$. Such a $L$-independent state can be either a constant energy confined band edge state (if $\tau$ is a zero of a band edge wavefunction) or a surface state in the band gap (if $\tau$ is not a zero of either band edge wavefunction). The results obtained in [6] are a special case of the more general results obtained here.

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Figure Captions

Fig. 1. $\Lambda_{\tau,0}$ as a function of $\tau$ when $\mu_0 \neq \mu_1$ in interval $x_{1,0} \leq \tau \leq x_{1,0} + a$. The zeros of $\xi_0(x)$ are shown as solid circles and the zero of $\xi_1(x)$ are shown as an open circle. However, in general, $\Lambda_{\tau,n}$ as a function of $\tau$ does not have to be linear.
