Criticality of models interpolating between the sine– and the sinh–Gordon Lagrangians

N. Defenu, V. Bacsó, I. G. Máríán, I. Nándori, and A. Trombettoni

1 Institut für Theoretische Physik, Universität Heidelberg, D-69120 Heidelberg, Germany
2 University of Debrecen, P.O.Box 105, H-4010 Debrecen, Hungary
3 MTA-DE Particle Physics Research Group, P.O.Box 51, H-4001 Debrecen, Hungary
4 MTA Atomki, P.O. Box 51, H-4001 Debrecen, Hungary
5 CNR-ION DEMOCRITOS Simulation Center, Via Bonomea 265, I-34136 Trieste, Italy
6 SISSA and INFN, Sezione di Trieste, via Bonomea 265, I-34136 Trieste, Italy

PACS numbers: 11.10.Hi, 05.70.Fh, 64.60.-i, 05.10.Cc

I. INTRODUCTION

Symmetries and dimensionality play a crucial role in the determination of critical properties and phase diagrams. As an example, in quantum field theory one of the most studied model is the Ising one with interaction terms $\phi^4$ which is known to have two phases in $d=1+1$ dimensions in one of which the $Z_2$ symmetry has been broken spontaneously \[1\]. Another paradigmatic and well studied instance of phase transition in $d=2$ dimensions is provided by the sine-Gordon (SG) scalar theory where the interaction Lagrangian contains a periodic self-interaction $\cos(\beta \phi)$. The SG model has been widely studied for the properties of its soliton solutions \[2, 3\] and it is known to exhibit a Berezinskii-Kosterlitz-Thouless (BKT) phase transition \[4, 5\]. Replacing the real valued frequency $\beta$ of the SG model by an imaginary one, $\beta \rightarrow i \beta$, one arrives at the sinh-Gordon (ShG) model with a self-interaction term $\cos(i \beta \phi) = \cosh(\beta \phi)$ which is in turn a well studied scalar field theory \[6\].

For the ShG model the periodicity is lost and there no BKT type transition is expected. One could argue that, due to its non-periodic nature, the interaction potential can be expanded in Taylor series which generates $\phi^{2N}$ terms, so that one could naively expect an Ising type phase structure. However, this is not the case. The ShG model is known to possess a single phase, and the explanation of this fact is related to the preservation of the functional form of its potential \[6\], which is connected with the special properties of the exponentials entering the hyperbolic sine.

Another way to relate the Ising, SG and ShG models is based on their conformal properties. It is known that systems at criticality, where they are scale-invariant, may give rise to invariance under the larger group of conformal transformations \[7\] locally acting as scale transformations \[8\]. The conformal group in $d = 2$ dimensions is infinitely dimensional \[8\] and the occurrence and consequences of conformal invariance for 2-dimensional field theories have been deeply investigated and exploited to obtain a variety of exact results \[1, 8\]. As a consequence of conformal invariance the central charge $c$ is well defined at any fixed point in the phase structure of the model and its difference $\Delta c$ between the one at the Gaussian and the non-trivial fixed point characterizes the theory. For example, it is known that $\Delta c = 1/2$, 1, for the Ising, SG and ShG models respectively. It is clear that the peculiarities of the SG and ShG models based on the preservation of the functional form of its potential along renormalization group (RG) flows are at the basis of the fact that in both cases $\Delta c = 1$, with the result for the ShG model differing from that of the Ising although it is not periodic (and it can be considered as a $\phi^{2N}$ theory).

The goal of the present work is to introduce and discuss a class of models interpolating between the SG and the ShG models. The proposed models can be studied by functional renormalization group (FRG), which allows well to clarify from the point of view of the interpolation the characteristics of the ShG model discussed above. The considered interpolation, that we term SnG model, is based on Jacobi functions \[9\]. The SG periodic potential $V_{SG}(\phi) = u \cos(\beta \phi)$ is modified as

$$V_{SnG}(\phi) = u \cosh(\beta \phi, m) \sinh(\beta \phi, m). \quad (1)$$

The interpolation relies on the fact that, with $m$ between 0 and 1, the Jacobi function $sn(\beta \phi, m)$ reduces to $\sin(\beta \phi)$ for $m = 0$ and to $\tanh(\beta \phi)$ for $m = 1$. Consequently, the SnG potential \[4\] for $m = 0$ reads $u \cos(\beta \phi)$, while for
$m = 1$ it is $u \cosh(\beta \phi)$ reducing to the ShG potential $V_{\text{ShG}}$. We observe that the interpolating potential is periodic (except for $m = 1$).

An important comment is that, while the SG and the ShG models are integrable both at classical and quantum level, models interpolating between them are in general not integrable (we refer to [10] for a discussion of $1 + 1$ classical integrable models). The approach followed here, with the interpolation inserted via the potential (1) in the Lagrangian, is therefore different from the models in which the interpolation is done directly in the $S$-matrix, as the staircase model in which an analytic continuation of the ShG $S$-matrix is performed to describe interpolating flows between minimal models in $2D$ [11]. These interpolating models, studied in relation to the so-called “roaming”, are integrable by construction. In the staircase model a real parameter $\theta_0$ encodes the distance of the continued $S$-matrix from the ShG self-dual point: in the limit of large $\theta_0$, the ground-state energy found by thermodynamic Bethe ansatz exhibits a sequence of scaling behaviours approximating those of the minimal conformal field theories. Several aspects of staircase and related models were studied [12–15], including a study of the form factors of the ShG field [19] when the real parameter $\theta_0$ is sent to infinity [14] (see more references in [15]). In these models one typically does not work with the Lagrangian (and to reconstruct the Lagrangian corresponding to their $S$-matrices is not straightforward) – at variance the model with the SnG potential (1) defines a bare Lagrangian, but anyway one can ask the fate of RG flow in the interpolation between SG and ShG models.

At the end of the paper we will briefly comment on the relation between interpolating Lagrangians and roaming phenomena.

We also discuss the comparison of the SnG model with another interpolation done at Lagrangian level, builded by considering the coupling constant $\beta = \beta_1 + i\beta_2$ as a complex quantity, giving

$$V_{\text{ShineG}}(\phi) = u \cos(\beta_1 \phi) \cosh(\beta_2 \phi), \quad (2)$$

where $\beta_1$ and $\beta_2$ are real value frequencies. We note that the resulting class of theories can be treated for each non-zero $\beta_2$ as a scalar polynomial field theory. For the sake of clarity, we refer to the Lagrangian with potential (2) as the Shine-Gordon (ShineG) model. A disclaimer here is certainly due: as mentioned in [3], the convention of denoting the generalization of the Klein-Gordon model to sinusoidal potential as “sine-Gordon” generated a certain amount of controversy. If from this point of view the proliferation of similar abbreviations should be avoided, from the other the use of sine-Gordon and sinh-Gordon models has become so widespread that in this paper devoted to Lagrangian interpolations between these two limits we decided for the purpose of compactness to refer to models [1] and [2] as “sn-Gordon” and “shine-Gordon” respectively.

The paper is organized as follows. Section II is devoted to introduce the FRG formalism for the study of the SnG model, and the linearized RG equations at Local Potential Approximation (LPA) level are discussed. A discussion of the FRG equations for the considered SG models is presented in Section III, including a discussion of the specific properties of the ShG model and of the qualitative comparison with the ShnG model. Section IV is devoted to our conclusions, while some FRG results for the ShnG model are contained in the Appendix.

II. LINEARIZED RG EQUATIONS FOR THE SN-GORDON MODEL

In this section we briefly summarize the FRG approach for scalar models, and its application to the ShG and the SnG models.

The FRG equation has the following form [20–23]

$$k\partial_k \Gamma_k[\varphi] = \frac{1}{2} \text{Tr} \left[ \frac{k\partial_k R_k}{\Gamma_k^{(2)}[\varphi] + R_k} \right] \quad (3)$$

for the effective action $\Gamma_k[\varphi]$. $\Gamma_k^{(2)}[\varphi]$ denotes the second functional derivative of the effective action and the trace $\text{Tr}$ stands for the integration over all momenta. The RG equation (3) is a functional equation, that should be handled by truncations. Truncated RG flows depend on the choice of the regulator function $R_k$, i.e. on the renormalization scheme. Regulator functions have already been discussed in the literature by introducing its dimensionless form

$$R_k(p) = p^2 r(y), \quad y = p^2/k^2,$$

where $r(y)$ is dimensionless. Various types of regulator functions can be chosen, but a general choice is the so called CSS regulator [24, 25] which recovers all major types of regulators in appropriate limits: the Litim [26], the power-law [27] and the exponential [20] ones. The mass cutoff is the power-law regulator with $b = 1$.

One of the commonly used systematic approximation is the truncated derivative expansion where the action is expanded in powers of the derivative of the field,

$$\Gamma_k[\varphi] = \int_x \left[ V_k(\varphi) + Z_k(\varphi) \frac{1}{2} (\partial_\mu \varphi)^2 + \ldots \right].$$

In LPA, higher derivative terms are neglected and the wave-function renormalization is set equal to constant, i.e. $Z_k \equiv 1$. In this case (3) reduces to the partial differential equation for the dimensionless blocked potential ($\tilde{V}_k = k^{-2} V_k$) which has the following form for $d = 2$ dimensions

$$(2 + k\partial_k)\tilde{V}_k(\varphi) = -\frac{1}{4\pi} \int_0^\infty dy \frac{y^2 \frac{d^2 \varphi}{dy^2}}{(1 + r)y + \tilde{V}_k''(\varphi)}, \quad (4)$$

where $\tilde{V}_k''(\varphi)$ is the second derivative of the potential with respect to the field.
Before going into the details of the solution of the exact FRG equation, in this section we take the linearized form (around the Gaussian fixed point) of the equation \( (4) \) obtained in the LPA level which reads as

\[
(2 + k \partial_k) \bar{V}_k(\varphi) = -\frac{1}{4\pi} \bar{V}_k''(\varphi) + \mathcal{O}(\bar{V}_k'^2),
\]  

(5)

independently of the choice of the regulator functions \( r(y) \) and apply it to the Ising, SG, ShG and to the two interpolating SnG models.

A. The Ising model

Although it is not the goal of the present work to consider the FRG study of the Ising model, since it is useful in the following let us first apply (5) for the Ising model by substituting

\[
\bar{V}_{\text{Ising}}(\phi) = \sum_{n=1}^{N_{\text{CUT}}} \frac{\tilde{g}_{2n}(k)}{(2n)!} \phi^{2n},
\]

(6)

into Eq. (5). One can then read the RG flow equations for the scale dependent dimensionless couplings \( \tilde{g}_{2n}(k) \). For any finite \( N_{\text{CUT}} \), the linearized FRG equation does not preserve the functional form of the bare theory (6), i.e., the l.h.s of (5) contains polynomial terms \( \phi^{2n} \) of order \( n = N_{\text{CUT}} \). The r.h.s of (5) has terms of order \( n < N_{\text{CUT}} \). Let us note that the same holds for the case where the linearization of the FRG equation (4) is performed in terms of the field-dependent part of \( \bar{V}_k(\phi) \) which results in a regulator-dependent linearized FRG equation.

B. The SG model

The situation is different for the SG model where the bare potential is defined by (for the sake of simplicity keeping only the fundamental Fourier mode)

\[
\bar{V}_{\text{SG}}(\phi) = \tilde{u}_k \cos(\beta \phi),
\]

(7)

where the dimensionless Fourier amplitude carries the scale-dependence since in LPA the frequency \( \beta \) does not depend on the running momentum cutoff \( k \). It is clear that the linearized FRG equation (5) preserve the functional form of the bare potential (no higher harmonics are generated):

\[
(2 + k \partial_k) \tilde{u}_k \cos(\beta \phi) = \frac{1}{4\pi} \beta^2 \tilde{u}_k \cos(\beta \phi).
\]

(8)

The RG flow equation for the Fourier amplitude reads

\[
k \partial_k \tilde{u}_k = \tilde{u}_k \left( -2 - \frac{1}{4\pi} \beta^2 \right),
\]

(9)

with a solution

\[
\tilde{u}_k = \tilde{u}_0 \left( \frac{k}{\Lambda} \right)^{-2+\frac{\beta^2}{4\pi}},
\]

(10)

which determines the critical frequency \( \beta_c^2 = 8\pi \), where the model undergoes a BKT-type phase transition [28]. It is important to note that even if the bare theory of the SG model contains higher harmonics, the linearized FRG equation (5) reduces to decoupled flow equations for the Fourier amplitudes of various modes.

C. The ShG model

By using the replacement \( \beta \to i\beta \) in Eq. (7), one finds the bare potential for the ShG model

\[
\bar{V}_{\text{ShG}}(\phi) = \tilde{u}_k \cos(i\beta \phi) = \tilde{u}_k \cos(\beta \phi)
\]

(11)

which is inserted into (5) preserving again the functional form of the bare potential:

\[
(2 + k \partial_k) \tilde{u}_k \cos(\beta \phi) = -\frac{1}{4\pi} \beta^2 \tilde{u}_k \cos(\beta \phi).
\]

(12)

The RG flow equation for the Fourier amplitude reads

\[
k \partial_k \tilde{u}_k = \tilde{u}_k \left( -2 - \frac{1}{4\pi} \beta^2 \right),
\]

(13)

with a solution

\[
\tilde{u}_k = \tilde{u}_0 \left( \frac{k}{\Lambda} \right)^{-2+\frac{\beta^2}{4\pi}},
\]

(14)

showing that in case of \( \beta^2 = 8\pi \) the exponent does not change sign, hence, the ShG model has no BKT-type phase transition. In other words, the linearized FRG of the ShG model can be derived from the the SG model by using the replacement \( \beta \to i\beta \) which results in a sign change of \( \beta^2 \) and no BKT-type phase transition.

D. The SnG model

In the SnG model, the dimensionless bare potential reads

\[
\bar{V}_{\text{SnG}}(\phi) = \tilde{A}_k \text{cn}(\beta \phi, m) \text{nd}(\beta \phi, m),
\]

(15)

where the amplitude \( \tilde{A}_k \) is scale-dependent. By using the properties of the Jacobi functions \( \text{cd}(u, m) = \text{cn}(u, m)/\text{dn}(u, m) \) and \( \text{nd}(u, m) = 1/\text{dn}(u, m) \) it can also be written as

\[
\bar{V}_{\text{SnG}}(\phi) = \tilde{A}_k \text{cn}(\beta \phi, m) [\text{nd}(\beta \phi, m)]^2.
\]

(16)

Inserting Eq. (15) or Eq. (16) into the linearized FRG equation (5) one observes that the functional form is not preserved since the second derivatives of the potential has the following form

\[
\bar{V}_{\text{SnG}}''(\phi) = \beta^2 \tilde{A}_k \frac{\text{cn}(\beta \phi, m)}{\text{dn}(\beta \phi, m)^3} \left( 6(m - 1) + (5 - 4m) \text{dn}(\beta \phi, m)^2 \right).
\]
However, it is important to note that the Jacobi function is a periodic function, so, it can be expanded in Fourier series. One has

$$cn(u, m) = \frac{2\pi}{K\sqrt{1-m}} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \cos \left( (2n + 1) \frac{\pi u}{2K} \right),$$

$$nd(u, m) = \frac{\pi}{2K\sqrt{1-m}} + \frac{2\pi}{K\sqrt{1-m}} \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1 + q^{2n}} \cos \left( 2n \frac{\pi u}{2K} \right),$$

where \( q = \exp[-\pi K(1-m)/K(m)] \) and \( K(m) \) is the quarter period which can be expressed by the hypergeometric function

$$K = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1-m \sin^2(\theta)}} = \frac{\pi}{2} 2F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, m \right).$$

It follows then

$$\tilde{V}_{\text{SnG}}(\phi) = \sum_{n=1}^{\infty} \tilde{u}_n(k) \cos(n b \phi), \quad b = \frac{\beta}{2F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, m \right)}.$$

Inserting (17) into the linearized FRG equation (5), one can derive a set of uncoupled differential equations for the Fourier modes

$$k \partial_k \tilde{u}_n(k) = \tilde{u}_n(k) \left( -2 + \frac{1}{4\pi} n^2 b^2 \right).$$

Similarly to the SG model the critical frequency corresponds to the fundamental mode, i.e., for \( n = 1 \) where one finds \( b^2_c = 8\pi \) and the higher harmonics do not modify it [29, 30]. Thus, one can read the \( m \)-dependence of the original frequency

$$\beta^2_c(m) = 8\pi \left[ 2F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, m \right) \right]^2,$$

which clearly signals the existence of a BKT-type phase transition if \( m \neq 1 \). In the limit \( m \to 0 \) one gets back \( \beta^2_c = 8\pi \), while for \( m \to 1 \) the original frequency blows up and thus the system is always in the so called massive (ionized) phase, see Fig. 1 where the fundamental Fourier amplitude is increasing in the IR limit. Thus, the \( m = 1 \) case the SnG model undergoes no BKT phase transition. In the next section we discuss whether if it undergoes or not an Ising-type transition in the FRG formalism.

In summary, one can conclude that only the SG and ShG model has a special structure such that their functional forms are preserved by the linearized FRG equation. A BKT-type phase transition is observed for the SG and the SnG models, for the latter with a condition \( m \neq 1 \).

### III. FRG EQUATIONS

Here we consider the study of the models introduced in the previous section. The FRG equations are taken in LPA for the Ising model with \( N_{\text{CUT}} = 2 \) and beyond LPA for the other models (keeping only the fundamental mode).

#### A. Ising model

Here we repeat briefly the FRG study of the Ising model where apart from the trivial mass term, a \( \phi^4 \) self-interaction is taken into account (\( N_{\text{CUT}} = 2 \)). The FRG equations are taken in the LPA level, reading in \( d = 2 \) dimensions as

$$\partial_t g_2 = -2g_2 - \frac{1}{4\pi} \frac{g_4}{(1 + g_2)}$$

$$\partial_t g_4 = -2g_4 + \frac{3}{4\pi} \frac{g_4^2}{(1 + g_2)^2}$$

for the mass cutoff and

$$\partial_t g_2 = -2g_2 - \frac{1}{4\pi} \frac{g_4}{(1 + g_2)^2}$$

$$\partial_t g_4 = -2g_4 + \frac{6}{4\pi} \frac{g_4^2}{(1 + g_2)^3}$$

for the Litim cutoff. The above equations have a trivial Gaussian and a non-trivial (cutoff-dependent) Wilson-Fisher (WF) fixed point, where the latter indicates the existence of two phases. The \( c \)-function along the trajectory starting at the Gaussian and terminating at the WF fixed points is known to decrease by \( \Delta c = 1/2 \). However, if one consider the massive deformation of the Gaussian fixed point \( \Delta c = 1 \).
B. SG model

If the SG model (7) is studied beyond LPA, the RG equation has to be solved over the functional subspace spanned by the following ansatz

$$\Gamma_k = \int d^2x \left[ \frac{1}{2} \frac{\partial_k^2 \varphi^2}{u_k} + V_k(\varphi) \right],$$

(24)

where the local potential contains a single Fourier mode

$$V_k(\varphi) = -u_k \cos(\varphi),$$

(25)

and the following notation is introduced

$$z \equiv 1/\beta^2$$

(26)

via the rescaling of the field $\varphi \rightarrow \varphi/\beta$ in (7), with $z_k$ the field-independent wave-function renormalization. Then Eq. (3) leads to the evolution equations for the coupling constants

$$k \partial_k u_k = \frac{1}{2} \int_p \frac{p(k \partial_k R_k)}{u_k} P \left( \frac{P}{\sqrt{P_k^2 - u_k^2}} - 1 \right),$$

(27)

$$k \partial_k z_k = \frac{1}{2} \int_p \frac{p(k \partial_k R_k)}{u_k} \left( \frac{u_k^2 P_k^2 (\partial_p^2 P_k)^2 (4P_k^2 + u_k^2)}{4(P_k^2 - u_k^2)^{1/2}} - 4u_k^2 P_k (\partial_p^2 P_k + p^2 \partial_p^2 P_k) \right)$$

(28)

with $P_k = z_k p^2 + R_k$. In general, the momentum integrals have to be performed numerically, however, in some cases analytical results are available. Indeed, by using the mass cutoff, i.e. power-law type regulator with $b = 1$, the momentum integrals can be performed and the RG equations reads as,

$$(2 + k \partial_k) \tilde{u}_k = \frac{1}{2\pi z_k u_k} \left[ 1 - \sqrt{1 - \tilde{u}_k^2} \right]$$

$$k \partial_k z_k = -\frac{1}{24\pi} \frac{\tilde{u}_k^2}{[1 - \tilde{u}_k^2]^2}$$

(29)

with the dimensionless coupling $\tilde{u} = k^{-2} u$. The phase structure of the SG model based on Eqs. (29) is plotted on Fig. 2 which indicates two phases with a critical value for the frequency $\beta_c^2 = 8\pi$. Thus, the ShG model can be considered as an Ising-type model but with restricted initial values for the coupling. The key point is that with ShG-type initial values the RG flow always starts from the symmetric phase, see Fig. 3. Therefore, the ShG model has a single phase, so, it does not go through a BKT or other type of phase transitions.

C. ShG model

It is important to note that (11) has a $Z_2$ symmetry, and that the ShG model is not periodic. Therefore, in order to study the RG flow of the ShG model and to map out its phase structure one can use the Taylor-expanded form of Eq. (11)

$$\tilde{V}_k(\varphi) = \tilde{u}_k \left[ 1 + \frac{1}{2} \beta_2 \varphi^2 + \frac{1}{4!} \beta_4 \varphi^4 + ... \right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} g_{2n} \varphi^{2n}, \quad g_{2n} = \tilde{u}_k \beta_c^{2n}.$$ 

(30)

The ShG model has a special structure that no $2 \rightarrow 2n$ particle production is allowed, i.e. the production amplitudes of any $2$ particles decay into $2n$ ones are zero.
at tree-level (and also at 1-loop level) \cite{16}. This special structure of the bare Lagrangian of the ShG model results in a single phase.

The phase structure of the ShG model can also be mapped out by using analytic continuation. The simplest way of doing that if one try the replacement of the frequency by an imaginary one directly. For example, the RG flow equations for the ShG model can be constructed from \cite{29}

\begin{equation}
(2 + k\partial_k)\tilde{u}_k = -\frac{\beta^2}{2\pi}\left[1 - \sqrt{1 - \tilde{u}_k^2}\right], \quad (31)
\end{equation}

\begin{equation}
k\partial_k\beta_k^2 = -\frac{1}{24\pi}\frac{\beta_k^2\tilde{u}_k^2}{[1 - \tilde{u}_k^2]^2}. \quad (32)
\end{equation}

The RG flow of the ShG model based on \cite{31} and \cite{32} is obtained numerically and shown in Fig. 4, which also indicates a single phase for the ShG model. We observe that due to the poor convergence properties of the regulator ($b = 1$ power-law), similarly to the SG case, the RG trajectories does not converge properly, specially in the limit of vanishing $\beta^2$.

Following the method discussed in \cite{34}, the $c$-function of the ShG model can be determined in the framework of FRG based on the flow equations \cite{34} and \cite{35}, which is identical to that of the SG model in the limit of $\beta \to 0$. Thus, the flows for the $c$-function of the ShG and the SG models are identical in the limit of vanishing frequency, consequently they give us the same result which recovers the known value $\Delta c = 1$ ($\Delta c = c_{UV} - c_{IR}$) \cite{34}.

D. SnG model

The FRG study of the SnG model is based on its Fourier decomposition \cite{17}, where the frequency $b^2$ of the fundamental mode play a crucial role in the determination of the phase structure. Thus, beyond LPA, the SnG model can be treated the way as the SG model, so the RG equation has to be solved over the functional subspace spanned by the following ansatz

\begin{equation}
\Gamma_k = \int d^2x \left[\frac{1}{2} z_k (\partial_\mu \varphi_x)^2 + V_k(\varphi_x)\right], \quad (36)
\end{equation}

where the local potential contains infinitely many Fourier modes

\begin{equation}
V_k(\varphi) = -\sum_{n=1}^{\infty} u_n(k) \cos(n \phi), \quad (37)
\end{equation}

and the following notations are introduced

\begin{equation}
z \equiv \frac{1}{b^2} = \frac{(2F_1(1/2, 1/2, 1, m))^2}{\beta^2} \quad (38)
\end{equation}

via the rescaling of the field $\varphi \to \varphi/b$ in \cite{17} and $z_k$ again standing for the field-independent wave-function renormalization. It is important to note that $m$ remains a non-scaling parameter even beyond LPA.

In order to follow the strategy done for the SG model one has to take the single-Fourier mode approximation of the SnG model \cite{37}. The higher harmonics do not change the qualitative picture drawn by the single-Fourier mode approximation (for $m \neq 1$) \cite{29,35}. Indeed, by using the mass cutoff, i.e., the power-law type regulator with $b = 1$, the RG equations for the couplings of the SnG reads as,

\begin{equation}
(2 + k\partial_k)\tilde{u}_k = -\frac{1}{2\pi z_k u_k}\left[1 - \sqrt{1 - \tilde{u}_k^2}\right]
\end{equation}

\begin{equation}
k\partial_k\beta_k^2 = -\frac{1}{24\pi}\frac{\tilde{u}_k^2}{[1 - \tilde{u}_k^2]^2}. \quad (39)
\end{equation}

with the dimensionless coupling $\tilde{u} = k^{-2}u$ which is identical to the flow equations \cite{29} of the SG model but with the different definition for $z$. In order to compare the flow diagrams of the SnG and SG models it is convenient to use the squared frequency $\beta^2$ instead of the wave function renormalization $z$. Then, the flow diagram of the SnG model obtained in the single-Fourier approximation

![Phase structure of the ShG model based on (31) and (32)](image_url)
beyond LPA for the particular value \( m = 0.45 \) is shown in Fig. 5.

We finally comment on the comparison between the SnG model and the ShineG model defined in (2). The latter model has a BKT transition only for \( \beta^2 = 0 \), i.e. for \( \text{Im} \beta = 0 \) (with \( \beta = \beta_1 + i \beta_2 \)). Therefore it is perfectly specular to the SnG model, which has a BKT transition in all points but the ShG model. A very simple analysis in LPA, with \( N_{\text{CUT}} = 2, 3 \), however shows that the model may have a phase diagram rather rich, since as soon as it deviates from the ShG model (with small values of \( \beta_1 \)), the system appears to immediately enter – in the considered approximation – in the Ising phase. As soon as that, at variance, \( \beta_2 \) is small, the system tends to the BKT phase transition. Since the latter transition is obtained only for \( \beta_2 = 0 \), then the system approaches the line \( \beta_2 = 0 \) apparently passing through multi-critical phases, whose order is suggested by the truncated LPA approach to increase (since the sign at infinity changes due to the number of powers in the Taylor truncation). To perform a reliable characterization of the phase diagram one should of course perform a systematic (or different) study of the LPA equations with growing \( N_{\text{CUT}} \), which appears a non-trivial but interesting work to be done in the future.

**IV. SUMMARY**

In the present work the renormalization group (RG) study of a class of models interpolating between the sine-Gordon (SG) and the sinh-Gordon (ShG) theories has been addressed. The study of the functional RG (FRG) equations clearly show that only the SG and ShG model has a special structure such that their functional forms are preserved by the linearized FRG equations. It was discussed that functional RG provides a tool to show that while the SG theory undergoes a phase transition at \( \beta^2 = 8\pi \), this is absent in the ShG model. The construction of the \( c \)-function of the ShG model has been also done. Moreover, we argued that the ShG model has a single phase since it can be considered as an Ising-type model but with restricted initial values for the coupling constants.

We also studied the proposed model, to which we referred as the sn-Gordon model, where the potential is expressed in terms of a product of Jacobi functions. We concluded that the SnG model exhibits a BKT phase transition for all \( m \neq 1 \), and we determined the phase diagram and the critical value of \( \beta \) as a function of the Jacobi parameter \( m \). We also compared the obtained results with the Shine-Gordon (ShineG) model (2), which has a BKT transition only for \( \text{Im} \beta = 0 \). These results clearly shows the peculiarities of the two limiting cases, the ShG and the SG models, and point out to the interest as a future work on the ShineG model, that we argue to have a rich structure, possibly related to roaming phenomena.

Finally we observe that other interpolations between the SG and the ShG models can be considered. It would be interesting to determine the solitonic solutions of the SnG model and discuss the relation with other, possibly integrable, interpolations, as the one based on a Weierstrass potential \([10, 36]\), which include the sine and the sinh potentials.

**Acknowledgement**

This work was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. The authors gratefully thank G. Gori, G. Mussardo, P. Sodano, G. Somogyi and G. Takačs for useful discussions. A.T. is grateful for kind hospitality to the Galileo Galilei Institute (Florence) where the final part of this work has been performed during the Workshop “From Static to Dynamical Gauge Fields with Ultracold Atoms”.

---

[1] G. Mussardo, *Statistical field theory: an introduction to exactly solved models in statistical physics* (Oxford, Oxford University Press, 2010).
[2] P. G. Drazin, *Solitons* (Cambridge, Cambridge University Press, 1983).
[3] R. Rajaraman, *Solitons and instantons: an introduction to solitons and instantons in quantum field theory* (Amsterdam, North-Holland, 1987).
