JULIA OPERATORS AND HALMOS DILATIONS

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Abstract. We offer a simple direct proof of the unitarity of the Julia operator associated to a contraction $A$, from which follow the intertwining identity $(I - AA^*)^{1/2}A = A(I - A^*A)^{1/2}$ and the unitarity of Halmos dilations.

Let $A : \mathbb{K} \to \mathbb{H}$ be a contraction from the complex Hilbert space $\mathbb{K}$ to the complex Hilbert space $\mathbb{H}$. The associated Julia operator is the unitary operator $J_A \in B(\mathbb{H} \oplus \mathbb{K})$ having $2 \times 2$ block form

$$J_A = \begin{bmatrix} (I - AA^*)^{1/2} & A \\ -A^* & (I - A^*A)^{1/2} \end{bmatrix}$$

where $(I - AA^*)^{1/2} \in B(\mathbb{H})$ and $(I - A^*A)^{1/2} \in B(\mathbb{K})$ denote the positive square-roots as usual. This operator is named in recognition of Gaston Julia [3] and features in an operator-theoretic Möbius transformation approach [10] to the theorem of Parrott [4] on contractive completions of partially-filled block operators.

Our primary aim here is extremely modest: to offer a simple direct proof of the fact that $J_A$ is indeed unitary. As immediate consequences, we deduce a simplified proof of the standard intertwining identity

$$(I - AA^*)^{1/2}A = A(I - A^*A)^{1/2}$$

and a simplified proof of the fact that the Halmos [1] dilation associated to a contractive endomorphism of the single Hilbert space $\mathbb{H}$ is unitary. Our simplified proofs make effective use of the $2 \times 2$ block-operator context in which Julia operators and Halmos dilations arise.

Theorem 0. If $A : \mathbb{K} \to \mathbb{H}$ is a contraction, then $J_A \in B(\mathbb{H} \oplus \mathbb{K})$ is unitary.

Proof. Introduce on $\mathbb{H} \oplus \mathbb{K}$ the skew-adjoint operator $C$ with block form

$$\begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}$$

and the positive operator $D$ with block form

$$\begin{bmatrix} (I - AA^*)^{1/2} & 0 \\ 0 & (I - A^*A)^{1/2} \end{bmatrix}$$

so that

$$J_A = D + C.$$ 

The operator $C$ commutes with $D^2$ and hence commutes with its positive square-root $D$. Thus

$$J_A^*J_A = (D - C)(D + C) = D^2 - C^2 = I$$

and $I = J_AJ_A^*$ likewise, so $J_A$ is unitary as claimed. \(\square\)

In some presentations, the Julia operator is defined with columns switched; this produces an operator from $\mathbb{K} \oplus \mathbb{H}$ to $\mathbb{H} \oplus \mathbb{K}$ and thereby obstructs direct application of the square-root argument presented here.
As a first corollary, we deduce at once the standard intertwining identity.

**Theorem 1.** If \( A: K \to \mathbb{H} \) is a contraction, then

\[
(I - AA^*)^{1/2}A = A(I - A^*A)^{1/2}.
\]

**Proof.** Simply compare off-diagonal blocks in the commutative identity \( DC = CD \) of the proof for Theorem [1] \( \square \)

As a second corollary, we deduce at once the unitarity of the Halmos dilation.

**Theorem 2.** If \( A \in B(\mathbb{H}) \) is a contraction, then its Halmos dilation

\[
\begin{pmatrix}
A \\
(I - A^*A)^{1/2}
\end{pmatrix} \in B(\mathbb{H} \oplus \mathbb{H})
\]

is unitary.

**Proof.** Simply note that if \( F \in B(\mathbb{H} \oplus \mathbb{H}) \) is the (unitary) ‘flip’ operator with block form

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\]

then the indicated Halmos dilation is precisely the composite \( J_A F \) \( \square \)

We remark that historically, the traditional approach to these results has been to start from the standard intertwining identity of Theorem 1, whence Theorem 0 and Theorem 2 follow by matrix multiplication. Write \( S = (I - AA^*)^{1/2} \) and \( T = (I - A^*A)^{1/2} \) for convenience. Halmos [1] observes that

\[
S^2 A = (I - AA^*)A = A(I - A^*A) = AT^2
\]

so that (by induction and linearity) \( p(S^2)A = Ap(T^2) \) when \( p \) is any polynomial and therefore (by the Weierstrass approximation theorem) \( f(S^2)A = Af(T^2) \) when \( f \) is any continuous function; the case \( f : [0, 1] \to [0, 1] : t \mapsto t^{1/2} \) yields the standard intertwining identity. This very same approach is taken by Halmos in Problem 222 of *A Hilbert Space Problem Book* [2]. According to Sz.-Nagy [9], each contraction on \( \mathbb{H} \) has a unitary power dilation. The proof of this fact presented in [5] makes use of the intertwining identity, following exactly the traditional justification due to Halmos; see page 467. The simplified construction of a unitary power dilation by Schäffer [8] again rests on this traditional justification. Sarason [6] surveys all of this and more, the traditional justification coming on page 196. The traditional justification also supports the theorem of Parrott [4] on contractive completions of partially-filled \( 2 \times 2 \) block operators; combine (i) at the top of page 313 with the calculation at the top of page 316. Young [10] presents an alternative approach to the Parrott theorem, based on operator-theoretic Möbius transformations; see Theorem 12.20 for the traditional justification there. The intertwining identity is important in the model theory originating with de Branges and Rovnyak: for example, it can be found on page 3 of [7], yet again with the traditional justification. This list of references involving the traditional justification is a mere sampling; it could be lengthened considerably.

The ubiquitous traditional justification of the intertwining identity essentially recapitulates the standard procedure whereby a positive operator \( R \) is shown to have a unique positive square-root \( R^{1/2} \), which commutes with every operator that commutes with \( R \) itself. Our simplified justification forgoes this recapitulation, instead appealing directly to the square-root itself. By first proving Theorem 0 we make entirely natural use of the \( 2 \times 2 \) block-operator setting of the theory. In hindsight and in spirit, our simplified approach is thus kin to the Berberian route from the Fuglede theorem to its Putnam extension, which asserts that an intertwiner of two normal operators likewise intertwines their adjoints.
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