General realization of $N = 4$ supersymmetric quantum mechanics and its applications

Dong Ruan

Department of Physics, Tsinghua University, Beijing 100084, P. R. China, Center of Theoretical Nuclear Physics, National Laboratory of Heavy Ion Accelerator, Lanzhou, 730000, P. R. China The Key Laboratory of Quantum Information and Measurements of Ministry of Education, Tsinghua University, Beijing 100084, P. R. China, and

Weicheng Huang

Institute of Applied Chemistry, Xingjiang University, Urumqi 830046, P. R. China

*Email: dongruan@tsinghua.edu.cn
Abstract

Based upon the general supercharges which involve not only generators $C_j$ of the Clifford algebra $C(4,0)$ with positive metric, but also operators of third order, $C_jC_kC_l$, the general form of $N = 4$ supersymmetric quantum mechanics (SSQM), which brings out the richer structures, is realized. Then from them, an one-dimensional physical realization and a new multi-dimensional physical realization of $N = 4$ SSQM are respectively obtained by solving the constraint conditions. As applications, $N = 4$ dynamical superconformal symmetries, which possess both the $N = 4$ supersymmetries and the usual dynamical conformal symmetries, are studied in detail by considering two simple superpotentials $k/x$ and $\omega x$, and their corresponding superalgebraic structures, which are spanned by eight fermionic generators and six bosonic generators, are established as well.

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I INTRODUCTION

Since the idea of supersymmetry was applied to quantum mechanical systems \([1, 2]\) to discover dynamical supersymmetry in ordinary quantum mechanics in order to explain the degeneracies of energy spectra extensively studies have been undertaken over the last twenty years in many aspects such as atomic physics, \([3, 4, 5]\) nuclear physics, \([6]\) many-body systems, \([7, 8]\) and so on. According to Witten, \([2]\) a supersymmetric quantum mechanical system is characterized by the existence of \(N\) Hermitian supercharges \(Q^\alpha\) which, together with the supersymmetric Hamiltonian \(H\) of this system, satisfy the following superalgebraic structure

\[
\{Q^\alpha, Q^\beta\} = 2\delta_{\alpha \beta} H, \quad \alpha, \beta = 1, 2, \ldots, N, \\
(Q^\alpha)^\dagger = Q^\alpha, \quad [H, Q^\alpha] = 0,
\]

where \(\{ , \}\) and \([ , \]\) denote an anticommutator and a commutator respectively. We call Eq. (1) a supersymmetric quantum mechanical algebra, denoted by SS(\(N\)). When \(N = 2\), the simplest non-trivial realization of SS(2) was first given by Witten \([2]\]

\[
Q^1 = \frac{1}{\sqrt{2}}\{p\sigma_1 + U(x)\sigma_2\}, \\
Q^2 = \frac{1}{\sqrt{2}}\{p\sigma_2 - U(x)\sigma_1\}, \\
H^w = \frac{1}{2}\{p^2 + [U(x)]^2 + U'(x)\sigma_3\},
\]

where \(p = -i\frac{\partial}{\partial x}\), \(U'(x) \equiv \frac{d}{dx}U(x)\), \(U(x)\) is generally called a superpotential, and \(\sigma_i (i = 1, 2, 3)\) are the usual Pauli matrices. \([9]\) This supersymmetric Hamiltonian \(H^w\) describes a quantum mechanical system of a spin-\(\frac{1}{2}\) particle moving on a line (x-axis).

While \(N = 2\) supersymmetric quantum mechanics (SSQM) has drawn much attention \([10, 11]\) due to its simpler mathematical structure, however, there were only a few attempts at studying \(N = 4\) SSQM, \([12, 13, 14, 15, 16]\) which possesses the higher degeneracies than the \(N = 2\) SSQM. For example, in one dimension, four-fold degeneracies of energy spectrum may typically occur in the \(N = 4\) SSQM, whereas double degeneracies in the \(N = 2\) SSQM. All the supercharges considered in Refs. \([12, 13, 14, 15, 16]\) have the following form

\[
Q^\alpha = \frac{1}{\sqrt{2}}\sum_{j=1}^{r} A^\alpha_j C_j, \quad \alpha = 1, 2, \ldots, N = 4,
\]

where \(A^\alpha_j\) are first-order differential operators with respect to, ‘bosonic’ degrees of freedoms, the Cartesian coordinates \(\{x_n | n = 1, 2, \ldots, d\}\) and the corresponding momentum operators \(\{p_n = -i\frac{\partial}{\partial x_n} \equiv -i\partial_n | n = 1, 2, \ldots, d\}\) in \(d\)-dimensional space, and \(C_j\), ‘fermionic’ degrees of freedoms, are generators of the Clifford algebra \(C(r, 0)\) with positive metric in \(r\)-dimensional flat carrier space. \([17, 18]\) They satisfy

\[
[x_n, p_m] = i\delta_{nm}, \quad n, m = 1, 2, \ldots, d; \\
\{C_j, C_l\} = 2\delta_{jl}, \quad C_j^\dagger = C_j, \quad j, l = 1, 2, \ldots, r; \\
[x_n, C_l] = [p_n, C_j] = 0.
\]

Obviously, for arbitrary \(N\), the supercharges \([3]\), being linear combinations of the fermionic operators \(C_j\) multiplied by the bosonic operators \(A^\alpha_j\), are natural generalizations of the supercharges in Witten’s realization \([2]\) of \(N = 2\) SSQM.
In fact, when $r \geq 4$, the Clifford algebra $C(r,0)$, after a graded structure introduced, may yield a superalgebra.\cite{19, 20} The generators $C_j$ of $C(r,0)$, together with operators of odd orders in $C_j$, span the ‘odd’ space of this superalgebra, in which anticommuting operations among all these ‘odd’ elements are allowed. For example, the odd space of the superalgebra associated with $C(4,0)$ is spanned by the odd elements $C_j$ ($j = 1, 2, 3, 4$) and $C_jC_kC_l$ ($1 \leq j < k < l \leq 4$). The purpose of this paper is to realize the general form of $N = 4$ SSQM in arbitrary dimension starting from the general supercharges in which the fermionic degrees of freedoms include all the odd elements of the superalgebra associated with $C(4,0)$. As we shall see below, this realization brings out the richer structures.

This paper is arranged as follows. In Sec. II, the general form of $N = 4$ SSQM is studied in detail by means of the Clifford algebras $C(4,0)$ and $C(0,3)$. In Sec. III, an one-dimensional physical realization and a new multi-dimensional physical realization for the $N = 4$ SSQM are respectively obtained by solving the constraint conditions. In Sec. IV, as applications, $N = 4$ superconformal quantum mechanics in one dimension, which is expanded from the one-dimensional realization of $N = 4$ SSQM obtained in Sec. III, is discussed in detail by considering two simple superpotentials $k/x$ and $\omega x$, and their corresponding superalgebraic structures are established. A simple summary is given in the final section.

Throughout this paper we shall adopt units wherein $\hbar = m = 1$, the symbol $[x]$ means taking an integer part of the real number $x$, and $T$ in the expression $A^T$ is referred to as transpose of the matrix $A$.

II GENERAL FORM OF $N = 4$ SSQM

For $N = 4$, the four supercharges take the following general form

$$Q^\alpha = \frac{1}{\sqrt{2}} \left( \sum_j A_j^\alpha C_j + \frac{i}{3!} \sum_{jklm} \epsilon_{jklm} D_j^\alpha C_kC_lC_m \right),$$ \hspace{1cm} (5)

where $\epsilon_{jklm}$ is a four-dimensional Levi-Civita symbol, $C_j$ are the generators of the Clifford algebra $C(4,0)$, $A_j^\alpha$ and $D_j^\alpha$ are the Hermitian first-order differential operators of the $d$-dimensional coordinates $\{x_n\}$ and momentum operators $\{p_n\}$. Clearly, the supercharges (3) are the special cases of those given by Eq. (5) with setting $D_j^\alpha = 0$.

Substituting Eq. (5) into the first equation of Eq. (3), we may obtain the corresponding supersymmetric Hamiltonian

$$H = \frac{1}{2} (U + VC_1C_2C_3C_4) + \frac{1}{2} \sum_{l=1}^q \sum_{j<k} \phi_{jlk} B_l \Gamma_{jk},$$ \hspace{1cm} (6)
where

1. \( U = \sum_j [(A_j^\alpha)^2 + (D_j^\alpha)^2] \), for any \( \alpha \);
2. \( V = i \sum_j [A_j^\alpha, D_j^\alpha] \), for any \( \alpha \);
3. \( i[A_j^\alpha, A_k^\beta] + i[D_j^\alpha, D_k^\beta] + \frac{1}{2} \sum_{mn} \epsilon_{jkmn} \{A_m^\alpha, D_n^\alpha\} \)
   \(-\{A_n^\alpha, D_m^\alpha\} = -q \sum_{l=1}^q \phi_{jk}^l B_l \), for any \( \alpha \);
4. \( \sum_j \{\{A_j^\alpha, A_j^\beta\} + \{D_j^\alpha, D_j^\beta\}\} = 0, \alpha \neq \beta \);
5. \( [A_j^\alpha, A_k^\beta] - [A_k^\alpha, A_j^\beta] + [D_j^\alpha, D_k^\beta] - [D_k^\alpha, D_j^\beta] \)
   \(= -i \sum_{mn} \epsilon_{jkmn} \{A_m^\alpha, D_n^\beta\} + \{A_m^\beta, D_n^\alpha\}, \alpha \neq \beta \);
6. \( \sum_j ([A_j^\alpha, D_j^\beta] + [A_j^\beta, D_j^\alpha]) = 0, \alpha \neq \beta \);
7. \( \Gamma_{jk} \equiv \frac{i}{4}[C_j, C_k] \).

In Eq. (5), \( q \) antisymmetric matrices \( \phi^l \) \( (\phi_{jk}^l = -\phi_{kj}^l) \) and Hermitian operators \( B_l \) have to be determined by the third equation in Eq. (7).

Now introduce the following linear transformations

\[
A_i^\bar{\alpha} = \Xi_{ij}^\bar{\alpha} A_j^4, \quad A_j^4 \equiv A_j, \quad \bar{\alpha} = 1, 2, 3; \\
D_i^\bar{\alpha} = \Xi_{ij}^\bar{\alpha} D_j^4, \quad D_j^4 \equiv D_j, 
\]

In order to satisfy the first, second, fourth and sixth equations in Eq. (7), \( \Xi_{ij}^\bar{\alpha} \) should be real antisymmetric,

\( (\Xi^\bar{\alpha})^T = (\Xi^\bar{\alpha})^{-1} = -\Xi^\bar{\alpha}, \)

and satisfy

\( \{\Xi^\bar{\alpha}, \Xi^\bar{\beta}\} = -2\delta_{\bar{\alpha}\bar{\beta}}, \)

that is, \( \Xi^\bar{\alpha} \) constitute the Clifford algebra C(0,3) with negative metric. The third and fifth equations in Eq. (7) require further that \( \Xi^\bar{\alpha} \) and \( \phi^l \) satisfy

\[
\Xi_{jk}^\bar{\alpha} \epsilon_{jkmn} = -2\Xi_{mn}^\bar{\alpha}, \\
[\Xi^\bar{\alpha}, \phi^l] = 0. \]

Eq. (12) shows that the number of antisymmetric matrices \( \phi^l \) is three, i.e., \( l = 1, 2, 3 \).

It follows that matrix representations of \( \Xi^\bar{\alpha} \) and \( \phi^l \) that satisfy Eqs. (5), (10) and (12) may be taken as \( [12, 18] \)

\[
\Xi^1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad \Xi^2 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \quad \Xi^3 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}; \\
\phi^1 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \phi^2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \phi^3 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \]
Correspondingly, the supersymmetric Hamiltonian (6) becomes
\[ \mathcal{H}(4,0) \text{ as} \]
A constraint conditions that

By making use of Eqs. (8), (13) and (14), we can obtain from the third equation in Eq. (7)

Thus, the four supercharges (15) and the supersymmetric Hamiltonian (17) can be written

With the help of Eqs. (8) and (13), the four supercharges (5), in which the differential

\[ Q^1 = \frac{1}{\sqrt{2}}[A_2C_1 - A_1C_2 + A_4C_3 - A_3C_4 + i(D_2C_2C_3C_4 + D_1C_3C_4C_1 + D_4C_1C_2C_4 + D_3C_1C_2C_3)], \]
\[ Q^2 = \frac{1}{\sqrt{2}}[A_3C_1 - A_4C_2 - A_1C_3 + A_2C_4 + i(D_3C_1C_2C_3 - D_1C_1C_2C_4 - D_2C_1C_2C_3)], \]
\[ Q^3 = \frac{1}{\sqrt{2}}[A_1C_1 + A_2C_2 + A_3C_3 + A_4C_4 + i(D_4C_3C_4C_1 + D_2C_1C_2C_4 + D_1C_1C_2C_3)], \]
\[ Q^4 = \frac{1}{\sqrt{2}}[A_1C_1 + A_2C_2 + A_3C_3 + A_4C_4 + i(D_1C_2C_3C_4 - D_2C_3C_4C_1 + D_3C_1C_2C_4 - D_4C_1C_2C_3)]. \]

By making use of Eqs. (8), (13) and (14), we can obtain from the third equation in Eq. (7) constraint conditions that \( A_j \) and \( D_j \) need satisfying

\[ B_1 = -i[A_1, A_4] - i[D_1, D_4] - \{A_2, D_3\} + \{A_3, D_2\}, \]
\[ B_2 = -i[A_2, A_4] + i[D_2, D_4] + \{A_1, D_3\} - \{A_3, D_1\}, \]
\[ B_3 = -i[A_3, A_4] - i[D_3, D_4] - \{A_1, D_2\} + \{A_2, D_1\}, \]
\[ B_4 = -i[A_4, A_4] + i[D_4, D_4] + \{A_1, D_2\} - \{A_2, D_1\}. \]

Correspondingly, the supersymmetric Hamiltonian (3) becomes

\[ H = \frac{1}{2} \sum_{j=1}^{4} [A_j^2 + D_j^2] + [B_1(\Gamma_{14} + \Gamma_{32}) + B_2(\Gamma_{24} + \Gamma_{13}) + B_3(\Gamma_{34} + \Gamma_{21})] + \frac{1}{2} \sum_{j=1}^{4} [A_j, D_j]C_1C_2C_3C_4. \]

Similar to \( N = 2 \) SSQM, (10) we may further rewrite the four supercharges (15) in the raising/lowering form, which are closely related to the factorization method, (21)

\[ Q^\pm_1 = \frac{1}{\sqrt{2}}(Q_4 \mp iQ_1), \quad Q^\pm_2 = \frac{1}{\sqrt{2}}(Q_2 \mp iQ_3), \]

and satisfy \( Q^\pm_\mu = (Q_\mu^\pm)^\dagger \) (\( \mu = 1, 2 \), after redefining the four generators \( C_j \) (\( j = 1, 2, 3, 4 \)) of \( C(4,0) \) as

\[ C^\pm_1 = \frac{1}{\sqrt{2}}(C_1 \pm iC_2), \quad C^\pm_2 = \frac{1}{\sqrt{2}}(C_3 \pm iC_4). \]

Thus, the four supercharges (15) and the supersymmetric Hamiltonian (17) can be written
respectively in the forms

\[ Q_1^+ = (A_1 - iA_2)C_1^+ + (A_3 - iA_4)C_2^+ - i(D_1 - iD_2)[C_1^+, C_2^+]C_1^+ + i(D_3 - iD_4)[C_1^+, C_2^-]C_2^+ , \]

\[ Q_1^- = (A_1 + iA_2)C_1^- + (A_3 + iA_4)C_2^- + i(D_1 + iD_2)[C_1^-, C_2^-]C_1^- - i(D_3 + iD_4)[C_1^- , C_2^-]C_2^- , \]

\[ Q_2^+ = (A_3 - iA_4)C_1^+ - (A_1 - iA_2)C_2^+ - i(D_3 - iD_4)[C_1^+, C_2^+]C_1^- + i(D_1 - iD_2)[C_1^+, C_2^-]C_2^- , \]

\[ Q_2^- = (A_3 + iA_4)C_1^- - (A_1 + iA_2)C_2^- + i(D_3 + iD_4)[C_1^- , C_2^-]C_1^+ - i(D_1 + iD_2)[C_1^- , C_2^-]C_2^+ , \]

and

\[ H = \frac{1}{2} \sum_{j=1}^{4} [A_j^2 + D_j^2] + \left\{ -B_1(C_1^+ C_2^- - C_1^- C_2^+) + iB_2(C_1^+ C_2^- + C_1^- C_2^+) \right\} + \frac{i}{2} \sum_{j=1}^{4} [A_j, D_j][C_1^+, C_1^-][C_2^-, C_2^-] . \]

It is easy to check that Eqs. (20) and (21) satisfy SS(4) [see Eq. (1)], which now becomes

\[ \{Q_\mu^+, Q_\nu^-\} = 2\delta_{\mu\nu}H, \quad \mu, \nu = 1, 2, \]

\[ \{Q_\mu^\pm, Q_\nu^\pm\} = 0, \quad [H, Q_\mu^\pm] = 0. \]  

III PHYSICAL REALIZATIONS OF N = 4 SSQM

By virtue of the results obtained in Sec. I, we shall discuss in this section two physical realizations of N = 4 SSQM through choosing the concrete forms for A_j and D_j in Eqs. (20) and (21) to ensure the kinetic energy and potential energy terms appear in the corresponding N = 4 supersymmetric Hamiltonian.

III.1 ONE-DIMENSIONAL REALIZATION

Take

\[ A_4 = p = -i \frac{d}{dx}; \quad D_\bar{n} = 0, \quad \bar{n} = 1, 2, 3, \]  

and the other components A_\bar{n} and D_4 are real functions of x. Substituting them into the constraint conditions (14) gives rise to

\[ D_4 = A_4' = A_2' \]  

\[ 2A_1 = A_3' \] \[ 2A_3' \]  

where the symbol “′” means derivation with respect to x. In order to satisfy Eq. (24), we may choose for simplicity

\[ A_\bar{n} = k_\bar{n}W, \]  

where k_\bar{n} are constants, and W, here referred to as a superpotential, is an ‘arbitrary’ real function of x. Accordingly, we have

\[ D_4 = \frac{W'}{2W}. \]
It follows by inserting Eqs. (25) and (26) into Eqs. (24) and (21) that the four supercharges and $N = 4$ supersymmetric Hamiltonian in one dimension have respectively the following forms

\begin{align*}
Q_1^+ &= (-ip + k_3 W)C_2^+ + k^- WC_1^+ + \frac{W'}{2W'}[C_1^+, C_1^-]C_2^+, \\
Q_1^- &= (+ip + k_3 W)C_2^- + k^+ WC_1^- + \frac{W'}{2W'}[C_1^+, C_1^-]C_2^-, \\
Q_2^+ &= (-ip + k_3 W)C_1^+ - k^- WC_2^+ - \frac{W'}{2W'}[C_2^+, C_2^-]C_1^+, \\
Q_2^- &= (+ip + k_3 W)C_1^- - k^+ WC_2^- - \frac{W'}{2W'}[C_2^+, C_2^-]C_1^-,
\end{align*}

where $k^\pm = k_1 \pm ik_2$, and

\begin{equation}
H = \frac{1}{2}p^2 + \frac{1}{2}k^2W^2 + \frac{1}{2} \left( \frac{W'}{2W'} \right)^2 + \left\{ -k_1W'(C_1^+C_2^- - C_1^-C_2^+) \\
+ik_2W'(C_1^+C_2^- + C_1^-C_2^+ + \frac{1}{2}k_3W'[C_1^+, C_1^-] - [C_2^+, C_2^-]) \right\} + \frac{1}{2}[C_1^+, C_1^-][C_2^+, C_2^-] \left( \frac{W'}{2W'} \right)',
\end{equation}

with $k^2 = k_1^2 + k_2^2 + k_3^2$. Obviously, the Hamiltonian (28) possesses a usual kinetic energy term $\frac{1}{2}p^2$ and a potential function $V(x;C_j)$, so this resulting realization, Eqs. (27) and (28), may be applied to the real quantum mechanical systems provided that $W$ and $C_j$ are appropriately taken.

For the sake of convenient applications, let us further discuss the explicit matrix form for the one-dimensional $N = 4$ SSQM given by Eqs. (27) and (28). In fact the Clifford algebra $C(4,0)$ is isomorphic to the well-known Dirac algebra in the relativistic quantum mechanics, here we may take the following matrix representation for $C_j$

\begin{equation}
C_\bar{m} = \begin{pmatrix}
0 & i\sigma_\bar{m} \\
-i\sigma_\bar{m} & 0
\end{pmatrix}, \quad \bar{m} = 1, 2, 3, \quad C_4 = \begin{pmatrix}
0 & I_{2\times2} \\
I_{2\times2} & 0
\end{pmatrix},
\end{equation}

where $I_{2\times2}$ is a $2 \times 2$ unit matrix, then the four supercharges (27) and the supersymmetric Hamiltonian (28) read respectively

\begin{align*}
Q_1^+ &= \begin{pmatrix}
0 & 0 & \eta^+ & \varepsilon^- \\
0 & 0 & 0 & 0 \\
0 & -\varepsilon^- & 0 & 0 \\
0 & \zeta^+ & 0 & 0
\end{pmatrix}, \\
Q_2^+ &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \eta^+ & \varepsilon^- \\
\varepsilon^- & 0 & 0 & 0 \\
-\zeta^+ & 0 & 0 & 0
\end{pmatrix},
\end{align*}

where $\eta^\pm \equiv p \pm i(k_3 W + \frac{W'}{2W'})$, $\zeta^\pm \equiv p \pm i(k_3 W - \frac{W'}{2W'})$, $\varepsilon^\pm \equiv \mp ik^\pm W$, and

\begin{equation}
H = \frac{1}{2}p^2 + \frac{1}{2}k^2W^2 + \frac{1}{2} \left( \frac{W'}{2W'} \right)^2 \\
+ \frac{1}{2} \begin{pmatrix}
-\left( \frac{W'}{2W'} \right)' & \left( \frac{W'}{2W'} \right)' + 2k_3W' & 2k^-W' & \frac{W'}{2W'} - 2k_3W'
\end{pmatrix},
\end{equation}
A comparison between Eq. (31) and the third equation in Eq. (2) shows that in one dimension the matrix form of $N = 4$ supersymmetric Hamiltonian is quasi-diagonal, whereas the one of $N = 2$ supersymmetric Hamiltonian is completely diagonal. [The quasi-diagonal form of $N = 2$ supersymmetric Hamiltonian appear in multi-dimension only, for example, see Ref. [23].]

Especially, when $k^\pm = 0$, it follows from Eq. (31) that we may obtain a completely diagonal $N = 4$ supersymmetric Hamiltonian

$$\ddot{H} = \frac{1}{2} \dot{p}^2 + \frac{1}{2} k_3^2 \dot{W}^2 + \frac{1}{2} \left( \frac{W'}{2W} \right)^2 \begin{pmatrix} -\left( \frac{W'}{2W} \right)' & \left( \frac{W'}{2W} \right)' + 2k_3 W' \\ \left( \frac{W'}{2W} \right)' - 2k_3 W' \end{pmatrix} + \frac{1}{2} \sum_{\mu} [C_{\mu}^+, C_{\mu}^-] \
\equiv \text{diag} [\ddot{H}_1, \ddot{H}_2, \ddot{H}_3, \ddot{H}_4].$$

Note that in Eq. (32) the first and second diagonal component Hamiltonians $\ddot{H}_1$ and $\ddot{H}_2$ are identical, but neither of them can be abandoned because of requirement of SS(4). The corresponding supercharges $\ddot{Q}_\mu^\pm (\mu = 1, 2)$ may be directly obtained by setting $k^\pm = 0$ in Eq. (30). Since $\ddot{H}$ commutes with $\ddot{Q}_\mu^\pm$, the energy spectra of $\ddot{H}_i (i = 1, 2, 3, 4)$ are identical except for the ground state, which is also an elementary property of SSQM. Consequently, $\ddot{H}_i$ are called superpartner Hamiltonians.

Let four-component spinor eigenfunction of $\ddot{H}$ be $\ddot{\psi} = [\ddot{\psi}_1, \ddot{\psi}_2, \ddot{\psi}_3, \ddot{\psi}_4]^T$, in which $\ddot{\psi}_i (i = 1, 2, 3, 4)$ are respectively eigenfunctions of $\ddot{H}_i$ belonging to energy eigenvalues $\ddot{E}_i$. In terms of the third equation in Eq. (22), the four eigenfunctions $\ddot{\psi}_1, \ddot{\psi}_2, \ddot{\psi}_3, \ddot{\psi}_4$ may be related by the four supercharges $\ddot{Q}_\mu^\pm$, or, more concretely, by the four first-order differential operators $\eta^\pm$ and $\zeta^\pm$ given in Eq. (30)

$$\ddot{\psi}_3 \eta^+ \ddot{\psi}_1 \rightarrow \ddot{\psi}_2, \ddot{\psi}_4 \zeta^+ \ddot{\psi}_1 \rightarrow \ddot{\psi}_2 \zeta^- \ddot{\psi}_4.$$  (33)

Similar to $N = 2$ case, [10] the $N = 4$ supersymmetry of some $N = 4$ supersymmetric quantum mechanical system is broken if this system has no zero-energy ground state, and is unbroken if this system has a zero-energy ground state. A typical structure of the four-fold degenerate energy spectrum of $\ddot{H}$ is illustratively depicted in Fig. 1, which $N = 4$ supersymmetry is broken.

Furthermore, introduce

$$X_3 = \frac{1}{2} \sum_{\mu} [C_{\mu}^+, C_{\mu}^-], \quad X^\pm = \mp C_1^\pm C_2^\pm,$$

which satisfy

$$[X_3, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = X_3.$$  (35)

that is, $X_3$ and $X^\pm$ span an internal SO(3) algebra. Due to the fact $[\ddot{H}, X_3] = 0$, we may use the values of $X_3$ to label the energy spectra of $\ddot{H}_i (i = 1, 2, 3, 4)$. With the help of Eq.
The matrix representation of $X_3$ is

$$X_3 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$  \hfill (36)

Denoting the values of $X_3$ by $\beta = 1, -1, 0$ and 0, then the energy spectrum of $\hat{H}_1$ belongs to the $\beta = 1$ sector, that of $\hat{H}_2$ to the $\beta = -1$ sector and so on. Hence, though the first and second superpartner Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ are identical, their energy spectra belong to the different sectors respectively; though the third and forth superpartner Hamiltonians $\hat{H}_3$ and $\hat{H}_4$ are different, their energy spectra belong to the same $\beta = 0$ sector.

### III.2 NEW MULTI-DIMENSIONAL REALIZATION

The $N = 4$ SSQM obtained in Sec. II itself is valid for the arbitrary dimensions. In this subsection we shall put forward a new multi-dimensional physical realization of $N = 4$ SSQM by taking the following matrix forms for the Hermitian operators $A_4$ and $D_4$ in Eq. (20)

$$A_4 = \sum_{j=1}^{d}(p_j + L_j)\tau_j, \quad D_4 = \sum_{j=1}^{d}F_j\tau_j,$$  \hfill (37)

and the other components, $D_\bar{n} = 0$ ($\bar{n} = 1, 2, 3$), $A_\bar{n}$, together with $L_j$ and $F_j$ in Eq. (37), are the real functions of the coordinates $\{x_n\}$ in $d$-dimensional space. Here, $\tau_j$ are a set of Hermitian matrices which we assume to commute with $p_j$, $L_j$, $F_j$ and $C_j$. The fact that $\tau_j$ commute with $C_j$ implies that they should be respectively considered as

$$\tau_j \sim I_{4 \times 4} \otimes \tau_j, \quad C_j \sim C_j \otimes I_{t \times t},$$  \hfill (38)

where the subscript 4 is the dimension of the matrix representation of $C(4,0)$, and the subscript $t$ stands for the order of the matrices $\tau_j$. Note that in the present realization the number of matrices $\tau_j$ is equal to the dimensions of space. In order to produce the usual kinetic energy term, we may, after substituting Eq. (37) into Eq. (21), take $\tau_j$ ($j = 1, 2, \ldots, d$) so that

$$\{\tau_j, \tau_i\} = 2\delta_{jl},$$  \hfill (39)

that is, $\tau_j$ constitute the Clifford algebra $C(d,0)$ as well.

The constraint conditions (16) require that

$$2A_{\bar{n}}F_j = \partial_j A_{\bar{n}}, \quad \bar{n} = 1, 2, 3, \quad j = 1, 2, \ldots, d.$$  \hfill (40)

Similar to the one-dimensional case [see Eq. (24)], a simple choice for Eq. (40) is

$$A_{\bar{n}} = k_{\bar{n}}W,$$  \hfill (41)
where $k_n$ are constants, the superpotential $W$ is a real function of $\{x_n\}$. In consequence, we have

$$F_j = \frac{\partial_j W}{2W}.$$  \hfill (42)

It follows by substituting Eqs. (37), (41) and (42) into Eq. (21) that we have the $N = 4$ supersymmetric Hamiltonian in $d$-dimensional space

$$H = \frac{1}{2} \sum_j (-i\partial_j + L_j)^2 + \frac{1}{2} k^2 W^2 + \frac{1}{2} \sum_j \left( \frac{\partial_j W}{2W} \right)^2 - \frac{1}{2} \sum_{k<l} F_{kl} \tau_{kl}$$

$$+ \sum_j \tau_j \left( \frac{\partial_j W}{2W} \right) \left\{ -k_1 (C_1^+ C_2^- - C_1^- C_2^+) + i k_2 (C_1^+ C_2^- + C_1^- C_2^+) \right.$$

$$+ \frac{1}{2} k_3 [(C_1^+, C_1^-) - (C_2^+, C_2^-)] \left. - \frac{1}{2} [C_1^+, C_1^-][C_2^+, C_2^-] \right\} \sum_{j<k} \left( \frac{\partial_j \partial_k W}{2W} \right)$$

$$+ \sum_{k<l} \left\{ -i \partial_k, \frac{\partial_l W}{2W} + 2L_k \frac{\partial_l W}{2W} \right\} \tau_{kl} \right\},$$

where $\tau_{kl} \equiv \frac{i}{2}[\tau_k, \tau_l]$, and $F_{kl} \equiv \partial_k L_l - \partial_l L_k$. The vector potential $L_i$ naturally generates a gauge field interaction structure in $d$-dimensional space so that $F_{kl}$ may be seen as the strength of vector field. The terms $F_{kl} \tau_{kl}$ and $\{-i\partial_k, \frac{\partial_l W}{2W}\} \tau_{kl}$ generalize the Pauli coupling and the orbit-spin coupling interactions respectively. For the simple three-dimensional case, these interpretations are more distinct. We take conveniently $\tau_j$ as the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  \hfill (44)

then the supersymmetric Hamiltonian (43) becomes

$$H = \frac{1}{2} (\vec{p} + \vec{L})^2 + \frac{1}{2} k^2 W^2 + \frac{1}{2} \left( \frac{\nabla W}{2W} \right)^2 + \frac{1}{2} \vec{\nabla} \cdot \left( \frac{\nabla W}{2W} \right) \left( \begin{array}{cc} -I_{4\times 4} & 0 \\ 0 & I_{4\times 4} \end{array} \right)$$

$$+ \frac{1}{2} \vec{B} \cdot \vec{r} - \frac{1}{2} \left( \frac{\nabla W}{2W} \right) \times (-i\nabla + \vec{L}) \cdot \vec{r},$$  \hfill (45)

where $\nabla$ is a three-dimensional gradient operator, $\vec{B} \cdot \vec{r} \equiv \nabla \times \vec{L} \cdot \vec{r}$ and $\nabla \frac{\nabla W}{2W} \times (-i\nabla) \cdot \vec{r}$ are the usual Pauli coupling term and the orbit-spin coupling interaction respectively. It can be seen that in three dimension the new realized $N = 4$ supersymmetric Hamiltonian (45) is an eight-by-eight matrix, whereas the original one (17) or (21), in which $A_j$ and $D_j$ are taken as some appropriate first-order differential operator functions of the three-dimensional coordinates and momentum operators, is a four-by-four matrix.

Of course, in Eq. (37), we may also take $A_4 = \sum_j (ip_j + \tilde{L}_j)\tilde{r}_j$, $D_4 = \sum_j \tilde{F}_j \tilde{r}_j$, and $A_n$, $\tilde{L}_j$ and $\tilde{F}_j$ are the functions of $\{x_n\}$ as well. Thus, the Hermiticities of the supercharges $Q_a^\pm$ require that $\tilde{L}_j$ and $\tilde{F}_j$ should be pure imaginary, and $\tilde{r}_j$ should be anti-Hermitian. The convenient choices $\tilde{L}_i = iL_j$, $\tilde{F}_j = iF_j$ and $\tilde{r}_j = i\tau_j$ [i.e., $\tau_j$ constitute the Clifford algebra $\text{C}(0,d)$ with negative metric] will lead to the same results as Eq. (43).
In this section, using the diagonal matrix realization (32) of \( N = 4 \) SSQM in one dimension, we shall study in detail \( N = 4 \) SCQM, which, discussed first by Fubini \textit{et al.}, \cite{24} is a generalization of \( N = 4 \) SSQM, by considering two simple superpotentials \( k/x \) and \( \omega x \). Here the main task is to find a set of special exactly solvable potentials which can be brought into the framework of \( N = 4 \) SCQM and the corresponding superalgebraic structures.

1. The first example is a one-dimensional superpotential

\[
W(x) = \frac{k}{x},
\]

where \( k \) is a real constant, and \( x \in (-\infty, \infty) \). Substituting Eq. (46) into Eq. (30) combined with \( k^\pm = 0 \) and \( k_3 = 1 \) and Eq. (32), we may obtain respectively the supercharges

\[
\tilde{Q}_1^+ = \begin{pmatrix}
0 & 0 & \tilde{\eta}^+ & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \tilde{\zeta}^+ & 0 & 0
\end{pmatrix}, \quad \tilde{Q}_2^+ = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \tilde{\eta}^+ & 0 \\
0 & 0 & 0 & 0 \\
-\tilde{\zeta}^+ & 0 & 0 & 0
\end{pmatrix},
\]

(47)

where \( \tilde{\eta}^\pm = p \pm i(k - \frac{1}{4})\frac{1}{x} \), \( \tilde{\zeta}^\pm = p \pm i(k + \frac{1}{4})\frac{1}{x} \), and the supersymmetric Hamiltonian

\[
H^{sc} = \frac{1}{2}p^2 + \frac{1}{2x^2} \begin{pmatrix}
k^2 - \frac{1}{4} & k^2 - \frac{1}{4} & k^2 - 2k + \frac{3}{4} & k^2 + 2k + \frac{3}{4} \\
0 & k^2 - \frac{1}{4} & k^2 - 2k + \frac{3}{4} & k^2 + 2k + \frac{3}{4} \\
0 & 0 & k^2 - \frac{1}{4} & k^2 - 2k + \frac{3}{4} \\
0 & 0 & 0 & k^2 - \frac{1}{4}
\end{pmatrix}
\equiv \text{diag}[H_1^c, H_2^c, H_3^c, H_4^c].
\]

(48)

It may be further verified that \( H^{sc} \), together with the dilatation generator \( D \) and the conformal generator \( K \), which are given explicitly by

\[
D = -\frac{1}{4}\{p, x\}, \quad K = \frac{1}{2}x^2
\]

fulfills the same commutation relations of the conformal algebra \( \text{SO}(2,1) \) \cite{23, 26, 27}

\[
[D, H^{sc}] = -iH^{sc}, \quad [D, K] = iK, \quad [H^{sc}, K] = 2iD,
\]

(50)

as its four superpartner Hamiltonians \( H_i^c \) \((i = 1, 2, 3, 4)\). Hence, \( H^{sc} \) given by Eq. (48) is the so-called superconformal Hamiltonian, \cite{24} which possesses not only the \( N = 4 \) supersymmetry but also the dynamical conformal symmetry. Different from the results of Fubini \textit{et al.}, \cite{24} here we realize successfully a \( N = 4 \) superconformal quantum mechanics (SCQM) in one dimension. However, the realization of \( N = 4 \) SCQM obtained in Ref. \cite{24}, holding uniquely in two dimension, can not be reduced to the one-dimensional or extended to more than two-dimensional cases. Furthermore, in the quartet structure of \( H^{sc} \), three superpartner Hamiltonians \( H_i^c (= H_5^c) \), \( H_2^c \), \( H_4^c \) are different, whereas in the quartet structure in Ref. \cite{24} only two different superpartner Hamiltonians, \( H_1^c (= H_2^c) \) and \( H_3^c (= H_4^c) \), appear.
Let the four-component spinor eigenfunction of $H^{sc}$ be $\psi^{sc} = [\psi^c_1, \psi^c_2, \psi^c_3, \psi^c_4]^T$, where $\psi^c_i$ ($i = 1, 2, 3, 4$) are respectively the eigenfunctions of $H^c_i$ belonging to the energy eigenvalues $E^c_i$. According to the transformation property (33), $\psi^c_i$ are related by $\bar{\eta}^\pm$ and $\bar{\zeta}^\pm$ given in Eq. (47). In order to look for the eigenfunctions and energy eigenvalues of $H^{sc}$, we check first whether or not a zero-energy ground state exists by solving the following four first-order differential equations

$$\bar{Q}^\pm_\mu \psi^{sc}_0 = 0, \quad \mu = 1, 2,$$

where $\psi^{sc}_0 \equiv [\psi^{sc}_{0,1}, \psi^{sc}_{0,2}, \psi^{sc}_{0,3}, \psi^{sc}_{0,4}]^T$ stands for a zero-energy eigenfunction. It is clear that neither of four solutions to Eq. (51)

$$\psi^c_{0,1} \sim x^{-(2k-1)/2},$$
$$\psi^c_{0,2} \sim x^{+(2k+1)/2},$$
$$\psi^c_{0,3} \sim x^{+(2k-1)/2},$$
$$\psi^c_{0,4} \sim x^{-(2k+1)/2}$$

is normalizable on $(-\infty, \infty)$ so that neither of $H^c_i$ has zero-energy level. Hence, the supersymmetry of the $N = 4$ superconformal quantum mechanical system described by $H^{sc}$ is broken. From the SSQM point of view, we know that either of $H^c_i$ ($i = 1, 2, 3, 4$) has the same energy spectrum as the other three superpartner Hamiltonians, with $E^c_i$ being larger than zero. Consider the special case of $k = 1/2$, it follows immediately from Eq. (48) that the conformal Hamiltonian $\hat{H}^c_4 = \frac{1}{2}p^2 + \frac{1}{12}$ is the superpartner of the Hamiltonian $\hat{H}_1 = \frac{1}{2}p^2$ ($= \hat{H}_2 = \hat{H}_3$) of a free particle. Therefore, the normalizable eigenfunctions of $H^c_i$ in Eq. (48) corresponding to some positive definite energy $E^c_i > 0$ are the normalizable wave plane eigenfunctions, i.e., the Bessel functions, $\psi^c_i = \sqrt{\frac{1}{\pi}}J_{\lambda_i}(x\sqrt{2E^c_i})$ ($i = 1, 2, 3, 4$), with $\lambda_i = k + (-)^i \left[\frac{1}{2}\right]$.

Now let us establish a superalgebra that governs the above $N = 4$ superconformal quantum mechanical system described by $H^{sc}$, in which both SS(4) and SO(2,1) should be contained. Direct calculations show that the five generators of SS(4) and three generators of SO(2,1) are not closed under the anticommutation and commutation relations, for example, commuting the generators $\bar{Q}^\pm_\mu$ of SS(4) and the generator $K$ of SO(2,1) yields new operators

$$S^\pm_\mu = xC^\pm_\mu, \quad \mu = 1, 2.$$
Thus, we obtain the following closed superalgebraic structure

1. \( \{ \tilde{Q}_\nu^+, \tilde{Q}_\nu^- \} = 2\delta_{\mu\nu}H^{2\mu\nu}, \quad \{ \tilde{Q}_\mu^+, \tilde{Q}_\mu^- \} = 0, \quad \mu, \nu = 1, 2; \)
2. \( \{ S^+_{\mu}, S^-_{\nu} \} = 2\delta_{\mu\nu}K, \quad \{ S^+_{\mu}, S^+_{\nu} \} = 0; \)
3. \( \{ Q^+_{\mu}, S^-_{\nu} \} = \delta_{\mu\nu}(k + (-)^\mu X_3 \pm 2iD) + 2(\delta_{\mu\nu} \mp 1)(\delta_{\mu1} X^\pm + \delta_{\mu2} X^\mp), \quad \{ Q^+_{\mu}, S^+_{\nu} \} = 0; \)
4. \( [D, H^{2\mu\nu}] = -iH^{2\mu\nu}, \quad [D, K] = iK, \quad [H^{2\mu\nu}, K] = 2iD; \)
5. \( [X_3, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = X_3; \)
6. \( \begin{align*}
[H^{2\mu\nu}, Q^\pm_\mu] &= 0, \quad [D, Q^\pm_\mu] = -\frac{1}{2}iQ^\pm_\mu, \quad [K, Q^\pm_\mu] = \pm 2S^\pm_\mu; \\
H^{2\mu\nu}, S^\pm_\mu &= \pm 2Q^\pm_\mu, \quad [D, S^\pm_\mu] = \frac{1}{2}iS^\pm_\mu, \quad [K, S^\pm_\mu] = 0;
\end{align*} \)
7. \( [H^{2\mu\nu}, X_3] = 0, \quad [D, X_3] = 0, \quad [K, X_3] = 0; \)
8. \( \begin{align*}
[X^\pm, Q^\pm_\mu] &= \mp (1 - \delta_{\mu2}) Q^\pm_\mu, \quad [X^\pm, Q^\pm_\mu] = \mp (1 - \delta_{\mu1}) Q^\pm_\mu; \\
[X^\pm, S^\pm_\mu] &= \mp (1 - \delta_{\mu2}) S^\pm_\mu, \quad [X^\pm, S^\pm_\mu] = \mp (1 - \delta_{\mu1}) S^\pm_\mu.
\end{align*} \)

We denote the above superalgebra by SC(4), which is spanned by eight fermionic generators \( S^\pm_\mu, Q^\pm_\mu \) \( (\mu = 1, 2) \) and six bosonic generators \( H^{2\mu\nu}, D, K, X_3 \) and \( X^\pm \). The first equation in the second set of equations in Eq. (54)] indicates that the fermionic generators \( S^\pm_\mu \) may be seen as square roots of the conformal generator \( K \), which is similar as the supercharges are the square roots of the supersymmetric Hamiltonian. Similar to SO(2,1), the superconformal symmetry described by SC(4) is dynamical since \( H^{2\mu\nu} \) does not commute with \( S^\pm_\mu \) and \( D \) and \( K \). It is obvious that besides SS(4), SO(2,1), and SO(3) [see the fifth set of equations in Eq. (54)], SC(4) contains a Lie superalgebra OSp(2,1) as its subalgebra, which, spanned by either \( \{ S^\pm_\mu, Q^\pm_\mu, H^{2\mu\nu}, D, K, X_3 \} \) or \( \{ S^\pm_\mu, Q^\pm_\mu, H^{2\mu\nu}, D, K, X_3 \} \), has been used to study \( N = 2 \) SQM. Since the generators of SO(2,1) commute with those of SO(3) [see the seventh set of equations in Eq. (54)], SC(4) contains a maximum Lie subalgebra \( SO(3) \times SO(2,1) \). Consequently, we have the following two canonical group chains

\[
\text{SC}(4)_1 \supset \left\{ \begin{array}{c}
\text{OSp}(2,1) \supset \text{SO}(2) \times \text{SO}(2,1) \\
\text{SO}(3) \times \text{SO}(2,1)
\end{array} \right\} \supset \text{SO}(2) \times \text{SO}(2). \quad (55)
\]

2. The second example is a linear superpotential on the half line,

\[
W(x) = \omega x,
\]

where \( \omega \) is a real constant, and \( x \in (0, \infty) \). Substitution into Eq. (56) combined with \( k^\pm = 0 \) and \( k_3 = 1 \) and Eq. (54) gives respectively the supercharges

\[
\tilde{Q}_1^+ = \begin{pmatrix}
0 & 0 & \tilde{\eta}^+ & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \tilde{\zeta}^+ & 0 & 0
\end{pmatrix}, \quad \tilde{Q}_2^+ = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \tilde{\eta}^+ & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\tilde{\zeta}^+ & 0 & 0 & 0
\end{pmatrix}, \quad (57)
\]
where $\tilde{\eta}^\pm \equiv p \pm i(\omega x + \frac{1}{2\sigma})$, $\tilde{\zeta}^\pm \equiv p \pm i(\omega x - \frac{1}{2\sigma})$, and the corresponding supersymmetric Hamiltonian

$$\tilde{H} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 + \left(\begin{array}{cc}
\frac{3}{8\sigma^2} & \frac{3}{8\sigma^2} \\
\frac{1}{8\sigma^2} & -\omega
\end{array}\right)$$

$$\equiv \text{diag}[\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4].$$

Note that the set of potential functions $\tilde{V}_i$ $(i = 1, 2, 3, 4)$ corresponding to the four superpartner Hamiltonians $\tilde{H}_i$ is different from the well-known radial harmonic oscillator potential $V_{\text{ho}}(l) = \frac{1}{2}\omega^2 x^2 l(l+1)/2\sigma^2$, in which the angular momentum quantum number $l$ must be a positive integer, though, from the point of view of mathematicians, $\tilde{V}_i$ in Eq. (58) are the special cases of $V_{\text{ho}}(l)$ for $l$ taking some special values: $\tilde{V}_1 = \tilde{V}_2 = V_{\text{ho}}(l = \frac{1}{2})$, $\tilde{V}_3 = V_{\text{ho}}(l = -\frac{1}{2}) + \omega$, and $\tilde{V}_4 = V_{\text{ho}}(l = -\frac{1}{2}) - \omega$. It may be easily inferred by using Theorem X.7 in Ref. [29] that all $\tilde{H}_i$ are hermitian on the half line $(0, \infty)$ since either of $\tilde{V}_i$ is in the limit point case in both zero and infinity. Let the four-component spinor eigenfunction of $\tilde{H}$ be $\tilde{\psi} = [\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4]^T$, where $\tilde{\psi}_i$ $(i = 1, 2, 3, 4)$ are respectively the eigenfunctions of $\tilde{H}_i$ belonging to the energy eigenvalues $\tilde{E}_i$, and are related by the four first-order differential operators $\tilde{\eta}^\pm$ and $\tilde{\zeta}^\pm$ given in Eq. (57)

$$\tilde{\psi}_3 \xrightarrow{\tilde{\eta}^+} \tilde{\psi}_1 \xleftarrow{\tilde{\eta}^-} \tilde{\psi}_2 \xrightarrow{\tilde{\zeta}^+} \tilde{\psi}_4 \quad \text{and} \quad \tilde{\psi}_3 \xleftarrow{\tilde{\zeta}^-} \tilde{\psi}_1 \xrightarrow{\tilde{\eta}^-} \tilde{\psi}_2 \xleftarrow{\tilde{\zeta}^-} \tilde{\psi}_4.$$

Note that the quantum mechanical system described by $\tilde{H}$ has no additional symmetry that can be, together with the $N = 4$ supersymmetry, embedded in a larger superconformal symmetry (for example the superconformal symmetry). If we rewrite $\tilde{H}$ as

$$\tilde{H} = \tilde{H}_0^{\text{sc}} + \omega^2 K + \omega Y_3,$$

where $\tilde{H}_0^{\text{sc}}$ is the supersymmetric Hamiltonian given by Eq. (51) combined with $k_3 = k^\pm = 0$ and $W(x) = \omega x$, and $Y_3$ is a constant matrix,

$$\tilde{H}_0^{\text{sc}} = \frac{1}{2}p^2 + \left(\begin{array}{cc}
\frac{3}{8\sigma^2} & \frac{3}{8\sigma^2} \\
\frac{1}{8\sigma^2} & -\omega
\end{array}\right), \quad Y_3 = \left(\begin{array}{cc}
0 & 0 \\
0 & 1
\end{array}\right),$$

then using the same analysis employed in the last example, it is easy to find that $\tilde{H}_0^{\text{sc}}$ is a superconformal Hamiltonian since it satisfies not only SO(2,1), with the dilatation generator $D = -\frac{1}{4}\{p, x\}$ and the conformal generator $K = \frac{1}{2}x^2$, but also SS(4), with the four supercharges $Q^\pm_\mu$ $(\mu = 1, 2)$

$$Q^+_1 = \left(\begin{array}{cccc}
0 & 0 & p + \frac{i}{2\sigma} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & p - \frac{i}{2\sigma} & 0 & 0
\end{array}\right), \quad Q^+_2 = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & p + \frac{i}{2\sigma} & 0 \\
0 & 0 & 0 & 0 \\
-p + \frac{i}{2\sigma} & 0 & 0 & 0
\end{array}\right).$$
Introducing extra three operators

\[ Y_3 = \frac{1}{2} \sum_\mu (-1)^{\mu+1}[\dot{C}_\mu^+, C_\mu^-], \quad Y^\pm = \pm C_2^\pm C_1^\pm, \]  

(63)

which constitute SO(3) as well [see the fifth set of equations in Eq. (64)], it follows that \( \dot{Q}_\mu^\pm, S_\mu^\pm, \tilde{H}^{\text{sc}}_0, D, K, Y_3 \) and \( Y^\pm \) satisfy the following closed superalgebraic structure

1. \( \{ \dot{Q}_\mu^+, \dot{Q}_\nu^- \} = 2\delta_{\mu\nu} \hat{H}^{\text{sc}}_0, \quad \{ \dot{Q}_\mu^+, \dot{Q}_\nu^- \} = 0, \quad \mu, \nu = 1, 2; \)
2. \( \{ S_\mu^+, S_\nu^- \} = 2\delta_{\mu\nu} K, \quad \{ S_\mu^+, S_\nu^- \} = 0; \)
3. \( \{ \dot{Q}_\mu^\pm, S_\nu^\pm \} = \delta_{\mu\nu}(\pm 2iD + Y_3), \quad \{ \dot{Q}_\mu^\pm, S_\nu^\pm \} = (-)^{\mu}(1 - \delta_{\mu\nu})2Y^\mp; \)
4. \( [D, \tilde{H}^{\text{sc}}_0] = -i\tilde{H}^{\text{sc}}_0, \quad [D, K] = iK, \quad [\tilde{H}^{\text{sc}}_0, K] = 2iD; \)
5. \( Y_3, Y^\pm = \pm 2Y^\pm, \quad [Y^+, Y^-] = Y_3; \)
6. \( [\tilde{H}^{\text{sc}}_0, \dot{Q}_\nu^\pm] = 0; \quad [D, \dot{Q}_\nu^\pm] = -\frac{1}{2}i\dot{Q}_\nu^\pm, \quad [K, \dot{Q}_\nu^\pm] = \pm S_\nu^\pm; \)  
(64)
\( [\tilde{H}^{\text{sc}}_0, S_\nu^\pm] = \pm \dot{Q}_\nu^\pm, \quad [D, S_\nu^\pm] = \frac{1}{2}iS_\nu^\pm, \quad [K, S_\nu^\pm] = 0; \)
7. \( [\tilde{H}^{\text{sc}}_0, Y_3] = 0, \quad [D, Y_3] = 0, \quad [K, Y_3] = 0; \)
8. \( [\tilde{H}^{\text{sc}}_0, Y^\pm] = 0, \quad [D, Y^\pm] = 0, \quad [K, Y^\pm] = 0; \)

where \( S_\mu^\pm \) has been given by Eq. (53). We denote the above superalgebra by \( \text{SC}(4)_2 \), which, different from \( \text{SC}(4)_1 \) defined by Eq. (54), has the same subgroup structure as \( \text{SC}(4)_1 \), i.e.,

\[
\text{SC}(4)_2 \supset \left\{ \begin{array}{c}
\text{OSp}(2,1) \supset \text{SO}(2) \times \text{SO}(2,1) \\
\text{SO}(3) \times \text{SO}(2,1)
\end{array} \right\} \supset \text{SO}(2) \times \tilde{H}_0^{\text{sc}}. \]  
(65)

To determine the eigenfunctions and energy eigenvalues of \( \tilde{H} \) by algebraic method, it is convenient to regroup the previous operators \( \tilde{H}_0^{\text{sc}}, D, K, \dot{Q}_\mu^\pm \) and \( S_\mu^\pm \) as

\[
T_3 = \frac{1}{\omega} \dot{H}_0^{\text{sc}} + \frac{\omega}{2} K, \quad T^\pm = \frac{1}{2\omega} \dot{H}_0^{\text{sc}} - \frac{\omega}{2} K \mp iD, \\
L^\pm_\mu = \frac{1}{2\sqrt{\omega}} \dot{Q}_\mu^\pm - \frac{\sqrt{\omega}}{2} S_\mu^\pm, \quad R^\pm_\mu = \frac{1}{2\sqrt{\omega}} \dot{Q}_\mu^\pm + \frac{\sqrt{\omega}}{2} S_\mu^\pm. \]  
(66)

Note that \( R^\pm_\mu (\mu = 1, 2) \) are up to normalization constants the supercharges \( \dot{Q}_\mu^\pm \) associated with \( \tilde{H} \), i.e., \( R^\pm_\mu = \frac{1}{2\sqrt{\omega}} \dot{Q}_\mu^\pm \). Owing to the fact \( \tilde{H} = 2\omega(T_3 + \frac{1}{2} Y_3) \), there exists a simple relation between the energy eigenvalues \( \tilde{E}_i \) \((i = 1, 2, 3, 4)\) of \( \tilde{H} \) and the energy eigenvalues \( e_i \) \((i = 1, 2, 3, 4)\) of \( T_{i,3} \), the ith diagonal component of \( T_3 \),

\[
\tilde{E}_i = e_i + \Delta_i, \quad \Delta_i = (-)^i \frac{1}{2} \left[ \frac{i}{3} \right], \]  
(67)

and their corresponding eigenfunctions are identical. Therefore, the eigenfunctions and energy eigenvalues of \( \tilde{H} \) may be directly obtained provided that those of \( T_3 \) are known. With
the help of Eq. (64), the closed anticommutation and commutation relations satisfied by the six bosonic operators $T_3, T^\pm, Y_3, Y^\pm$ and eight fermionic operators $L^\pm_\mu, R^\pm_\mu$, read

1. $[T_3, T^\pm] = \pm T^\pm, \quad [T^+, T^-] = -2T_3;$
2. $[Y_3, Y^\pm] = \pm 2Y^\pm, \quad [Y^+, Y^-] = Y_3;$
3. $\{L^\pm_\mu, L^\pm_\nu\} = T_3 - \frac{1}{2}Y_3, \quad \{L^\pm_\mu, L^\mp_\nu\} = \{L^\pm_\mu, L^\pm_\nu\} = 0;$
4. $\{R^\pm_\mu, R^-_\mu\} = T_3 + \frac{1}{2}Y_3, \quad \{R^\pm_\mu, R^\mp_\nu\} = \{R^\pm_\mu, R^\pm_\nu\} = 0;$
5. $\{L^\pm_\mu, R^\pm_\nu\} = \delta_{\mu\nu}T^\pm, \quad \{L^\pm_\mu, R^\mp_\nu\} = (-1)^{\mu}(1 - \delta_{\mu\nu})Y^\mp;$
6. $[T^\pm, Y_3] = 0, \quad [T^\pm, Y^\pm] = 0, \quad [T^\pm, Y_3] = 0,$

$$[T^\pm, Y^\pm] = 0;$$
7. $[T_3, L^\pm_\mu] = \pm \frac{1}{2}L^\pm_\mu, \quad [T^\pm, L^\pm_\mu] = 0, \quad [T^\pm, L^\pm_\mu] = \pm L^\pm_\mu;$
8. $[T_3, R^\pm_\mu] = \pm \frac{1}{2}R^\pm_\mu, \quad [T^\pm, R^\pm_\mu] = 0, \quad [T^\pm, R^\pm_\mu] = \pm R^\pm_\mu;$

where $\mu, \nu = 1, 2$. We observe from Eq. (68) that (1) the first set of equations indicates that $T_3, T^\pm$ constitute $SO(2,1)$ also, where $T_3$ is a compact operator with a discrete spectrum, and $T^+$ ($T^-$) raises (lowers) the energy eigenvalues of $T_3$ by 1 unit. (2) Similar to SC(4)$_1$, the supealgebra determined by Eq. (68) contains $SO(2,1) \times SO(3)$ as its maximum Lie subalgebra as well, moreover, the values of $Y_3$ [see Eq. (51)] may be used to label the energy eigenvalues $\epsilon$ of $T_3$ or $\tilde{H}$. (3) $T_3$ and $R^\pm_\mu$ ($R_\mu^\pm$) do not form SS(4), therefore, the energy spectrum of $T_3$ is not four-fold degenerate. (4) The first column of equations in the seventh and eighth sets of equations show that $R^+_\mu$ and $L^\pm_\mu$ raise the energy eigenvalues of $T_3$ by $\frac{1}{2}$ unit meanwhile lower the values of $Y_3$ by 1 unit, whereas $R^-_\mu$ and $L^-_\mu$ lower the energy eigenvalues of $T_3$ by $\frac{1}{2}$ unit meanwhile raise the values of $Y_3$ by 1 unit.

Now turn to the eigenfunctions and energy eigenvalues of $\tilde{H}$ by means of the similar approach as used in the last example. Solving the following four equations

$$\tilde{Q}^\pm_\mu \tilde{\psi}_0 = 0, \quad \mu = 1, 2,$$

where $\tilde{\psi}_0 \equiv [\tilde{\psi}_{0,1}, \tilde{\psi}_{0,2}, \tilde{\psi}_{0,3}, \tilde{\psi}_{0,4}]^T$ stands for the zero-energy eigenfunction, gives rise to

$$\tilde{\psi}_{0,1} \sim \frac{1}{\sqrt{x}} \exp(-\omega x^2/2), \quad \tilde{\psi}_{0,2} \sim \frac{1}{\sqrt{x}} \exp(+\omega x^2/2), \quad \tilde{\psi}_{0,3} \sim \sqrt{x} \exp(+\omega x^2/2),$$
$$\tilde{\psi}_{0,4} \sim \sqrt{x} \exp(-\omega x^2/2).$$

It is not difficult to find by simple calculations that of the four eigenfunctions $\tilde{\psi}_{0,i}$ ($i = 1, 2, 3, 4$), only $\tilde{\psi}_{0,4}$ is square-integrable on the interval $(0, \infty)$, whose normalized form is

$$\tilde{\psi}_{0,4} = \sqrt{\omega x} \exp(-\omega x^2/2),$$

that is, only $\tilde{H}_4$ has a normalizable zero-energy ground state, whereas the other three superpartner Hamiltonians $\tilde{H}_1, \tilde{H}_2, \tilde{H}_3$ do not have. Of course, $\tilde{\psi}_{0,4}$ is also the normalizable
ground state eigenfunction of the fourth diagonal component \( T_{4,3} \) of \( T_3 \), with its corresponding energy being larger than zero. With the help of the superalgebraic relations (68), the eigenfunctions \( \tilde{\psi}_{n',i} \) \((n' = 1, 2, \ldots; i = 1, 2, 3, 4)\) for arbitrary excited states may be obtained from \( \tilde{\psi}_{0,4} \) through two steps: first applying \( n \) times the raising operator \( T^+ \) to \( \tilde{\psi}_{0,4} \) produces all the excited states \( \tilde{\psi}_{n',4} \) \((n' = 1, 2, \ldots)\) of \( T_{4,3} \), and then acting respectively once and twice on \( \tilde{\psi}_{n',4} \) with \( \tilde{Q}_-^j \) or \( \tilde{R}_-^j \) or \( \tilde{L}_-^j \) gives the other excited states \( \tilde{\psi}_{n,j'} \) \((n = 0, 1, \ldots; j' = 1, 2, 3)\), i.e., all the eigenfunctions of \( \tilde{H}_\ell' \) or \( T_{\ell',3} \). With the help of Rodrigus’ formula for the generalized Laguerre polynomial \( L_n^\alpha(x) \) of positive integer \( n \) and real parameter \( a \) in argument \( x \), we can finally obtain by induction

\[
\begin{align*}
\tilde{\psi}_{n,1} &= \sqrt{2\omega x} \exp(-\omega x^2/2) L_n^0(\omega x^2), \\
\tilde{\psi}_{n,2} &= \sqrt{2\omega x} \exp(-\omega x^2/2) L_n^1(\omega x^2), \\
\tilde{\psi}_{n,3} &= \frac{2x}{n+1} \omega x \exp(-\omega x^2/2) L_n^1(\omega x^2), \\
\tilde{\psi}_{n,4} &= \frac{2x}{n+1} \omega x \exp(-\omega x^2/2) L_n^1(\omega x^2), \\
n &= 0, 1, 2, \ldots
\end{align*}
\]

Thus, the energy eigenvalues, \( \tilde{E}_i \) and \( e_i \) \((i = 1, 2, 3, 4)\) related by Eq. (77), that correspond to the same normalized eigenfunctions (72), of \( \tilde{H}_j \) and \( T_{j,3} \) are respectively

\[
\begin{align*}
\tilde{E}_1 &= 2\omega(e_1 + \Delta_1) = 2\omega(n + 1), \quad e_1 = n + 1, \\
\tilde{E}_2 &= 2\omega(e_2 + \Delta_2) = 2\omega(n + 1), \quad e_2 = n + 1, \\
\tilde{E}_3 &= 2\omega(e_3 + \Delta_3) = 2\omega(n + 1), \quad e_3 = n + 1/2, \\
\tilde{E}_4 &= 2\omega(e_4 + \Delta_4) = 2\omega n, \quad e_4 = n + 1/2, \\
n &= 0, 1, 2, \ldots
\end{align*}
\]

Eq. (73) shows clearly that the \( N = 4 \) supersymmetry of the quantum mechanical system described by \( \tilde{H} \) is unbroken since its ground state energy is zero, and the four-fold degeneracies may be observed above the second level. However, the quartet energy spectrum structure of \( T_3 \), which is not four-fold degenerate, involves two sets of double degenerate spectra, moreover, the corresponding supersymmetry is broken since its ground state energy is \( \frac{1}{2} \). The energy spectrum structures of \( \tilde{H} \) and \( T_3 \) are depicted in Fig. 2 and Fig. 3, respectively. The solutions to the Schrödinger equations with the potentials \( ax^2 + \frac{b}{x^2} \) for different \( a \)'s and \( b \)'s may be obtained by different approaches, see Refs. [24, 25, 30, 31, 32].

V SUMMARY

In this paper, we obtained the general form of \( N = 4 \) SSQM in arbitrary dimension, starting from the general form of four supercharges, in which the fermionic degrees of freedoms include all the odd elements, \( C_j \) and \( C_j C_k C_l \), of the superalgebra associated with the Clifford algebra \( C(4,0) \). Then, from them, we gave the one-dimensional physical realization and the new multi-dimensional physical realization for the \( N = 4 \) SSQM by solving their respective constraint conditions. As applications, we studied in detail, on the base of the one-dimensional realization, two superconformal quantum mechanical systems with their superpotentials being \( k/x \) and \( \omega x \), which possess both the \( N = 4 \) supersymmetries and the
dynamical conformal symmetries, and established their corresponding superalgebraic structures, which are spanned by the eight fermionic generators and six bosonic generators. Our next work is to apply the general realizations of $N = 4$ SSQM obtained in this paper to the other possible (quasi-) exactly solvable potentials, for example, listed in Refs. [10, 33], and to discussing the $N = 4$ supersymmetries in the relativistic quantum mechanical systems such as Dirac equation, Klein-Gordon equation and so on. From the point of view of mathematics, it is also of very interest to investigate the representations of SC(4)$_1$ and SC(4)$_2$ and their relations to the classical Lie superalgebras [13, 34]. This work is now under way.

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Captions

Fig. 1. Typical structure of the four-fold degenerate energy spectrum of $\hat{H}$ given by Eq. (32). The energy spectra of $H_1$, $H_2$, $H_3$, $H_4$ correspond to the $\beta = 1, -1, 0, 0$ sectors, respectively. The eigenstate belonging to some energy level (dot) may be connected with its left or right eigenstate by the supercharges $\hat{Q}_\mu^\pm$ ($\mu = 1, 2$), or, concretely, by the first-order differential operators $\eta^\pm$ and $\zeta^\pm$ along the horizontal line.

Fig. 2. Four-fold degenerate energy spectrum of $\tilde{H}$ given by Eq. (73), which $N = 4$ supersymmetry is unbroken since its ground state energy is zero. Each eigenstate (dot) is connected with its surrounding eigenstates (dots) by the supercharges $\tilde{Q}_\mu^\pm$ ($\mu = 1, 2$) along the horizontal line, and by the raising/lowering operators $T^\pm$ along the vertical line.

Fig. 3. Quartet structure of the energy spectrum of $T_3$ related tightly to $\tilde{H}$ [see Eq. (73)]. Different from that of $\hat{H}$ exhibited in Fig. 2, $T_3$ possesses two sets of double degenerate spectra with the corresponding supersymmetry being broken. Each eigenstate (dot) is connected with its surrounding eigenstates (dots) by the fermionic operators $R_\mu^\pm, L_\mu^\pm$ ($\mu = 1, 2$) along the slanting line, and by the raising/lowering operators $T^\pm$ along the vertical line.
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