Effective Chiral Theory for Radiative Decays of Mesons

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An extended $U(3)_L \otimes U(3)_R$ chiral theory which includes pseudoscalar and vector meson nonets as dynamic degrees of freedom is presented. We combine a hidden symmetry approach with a general procedure of including the $\eta'$ meson into chiral theory. The $U(3)_L \otimes U(3)_R$ and the $SU(3)$ symmetries are broken by the mechanism based on quark mass matrix. Meson radiative decay widths are parameterized in terms of a single and real $\eta - \eta'$ mixing angle $\theta_P$, a $U(3)_V$ symmetry breaking scale parameter $c_W$, and the radiative decay constants $F_\pi$, $F_K$, $F_\eta$, $F_{\eta'}$, for the $\pi, K, \eta, \eta'$ mesons, respectively. Taking $F_\pi = 93 MeV$, a global fit to decay width data yields

$$F_K/F_\pi = 1.16 \pm 0.11, \quad F_\eta/F_\pi = 1.14 \pm 0.04, \quad F_{\eta'}/F_\pi = 1.09 \pm 0.04$$

$$c_W = -0.19 \pm 0.03, \quad \theta_P = -(15.4 \pm 1.8)^o.$$

Key Words : Meson radiative decays, Meson decays, Chiral Perturbation Theory,

I. INTRODUCTION

Meson decays of pseudoscalar and vector mesons have been discussed by several groups, using phenomenological approaches based on effective field theory. In particular, the value of the $\eta - \eta'$ mixing angle, $\theta_P$, was deduced from the analysis of electromagnetic decays of pseudoscalar and vector mesons, $J/\psi$ decays into a vector and a pseudoscalar meson, and some other transitions. Gilman and Kaufman assumed $SU(3)$ symmetry and often the stronger condition of nonet symmetry in order to relate the $SU(3)$-octet wave function to that of the $SU(3)$ singlet, and obtained a value of $\theta_P \approx -20^o$. Less negative a value was extracted by Bramon and Scadron from a rather similar analysis which takes into account small departure from the $\omega - \phi$ ideal mixing. Somewhat different approach was taken by Ball, Frère and Tytgat by relating vector meson decays to the Quantum Electrodynamics (QED) triangle anomaly. More recently, Bramon, Escribano and Scadron have extracted a value $\theta_P = 15.5^o \pm 1.3^o$ from a rather exhaustive analysis of data including strong decays of tensor and higher-spin mesons.

Spontaneously broken chiral symmetry plays a major role in low energy hadron physics. The Quantum Chromodynamics (QCD) Lagrangian exhibits an $SU(3)_L \otimes SU(3)_R$ chiral symmetry which breaks down spontaneously to $SU(3)_V$, giving rise to a light Goldstone boson octet of pseudoscalar mesons. The corresponding effective field theory (EFT) exhibits the symmetry properties of QCD and involves both of the pseudoscalar meson octet and vector meson nonet as dynamical field variables (see, for example, Ref. [1]). The axial $U(1)$ symmetry of the QCD Lagrangian is broken by the anomaly. Though considerably heavier than the octet states, it is rather well accepted that the $\eta'$ meson is the most natural candidate for the corresponding pseudoscalar singlet. In this context, we shall introduce the $\eta'$ meson also as a dynamical field variable. We combine the "hidden symmetry approach" of Bando et al. with a general procedure of including the $\eta'$ meson into chiral theory to construct a most general chiral effective Lagrangian with broken $U(3)_L \otimes U(3)_R$ local symmetry. To this aim we introduce in section II the whole nonets of pseudoscalar and vector mesons interacting with external electroweak fields. In section III we apply this approach to study radiative decays for anomalous processes, like $V^0 \rightarrow P^0 \gamma$, $P^0 \rightarrow V^0 \gamma$ and $P^0 \rightarrow \gamma \gamma$, with $P^0 = \pi, \eta, \eta'$ and $V^0 = \rho, \omega, \phi$. The numerical values of the radiative decay constants, $F_i$ ($i = \pi, \eta, K, \eta'$) and the $\eta - \eta'$ mixing angle, $\theta_P$ are fixed by fitting to experimental rates of these processes. We shall summarize and conclude in section IV.

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II. THE EFFECTIVE LAGRANGIAN

In order to include the η' meson into chiral effective Lagrangian one has to extend the SU(3)_L \( \otimes \) SU(3)_R local symmetry of the QCD Lagrangian into U(3)_L \( \otimes \) U(3)_R local symmetry. This can be achieved by adding to the QCD Lagrangian (herein denoted \( L_{QCD} \)) a term proportional to the topological charge operator, i.e.,

\[
L = L_{QCD} - \Theta(x) \frac{g^2}{16\pi^2} Tr_c \left( G_{\mu\nu} \tilde{G}^{\mu\nu} \right),
\]

where \( \Theta(x) \) represents an auxiliary external field, the so-called QCD vacuum angle. Here, \( \tilde{G}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} G_{\alpha\beta} \); \( G_{\mu\nu} = \partial_{\mu} G_\nu - \partial_{\nu} G_\mu + i[G_\mu, G_\nu] \), \( G_\mu \) being the gluon field, and \( Tr_c \) stands for the trace over color indices. Obviously, the Lagrangian \( L \) of Eqn. (3) has SU(3)_L \( \otimes \) SU(3)_R local symmetry. It can be shown \([8–12]\), that \( L \) would also have U(3)_L \( \otimes \) U(3)_R local symmetry provided \( \Theta(x) \) transforms under axial U(1) rotations as,

\[
\Theta(x) \rightarrow \Theta'(x) = \Theta(x) - 2N_f a \theta,
\]

where \( N_f \) represents the number of flavors and \( \alpha \) the axial U(1) transformation parameter. Indeed, the term generated by the anomaly in the fermion determinant is compensated by the shift in \( \Theta(x) \), so that the overall change in the Lagrangian amounts to a total derivative, giving rise to the well-known anomaly Wess-Zumino term. An effective field theory Lagrangian which involves this integrated form of this anomaly term would also have this same feature. For more details see Ref. [8].

We now turn to construct a general chiral effective Lagrangian with U(3)_L \( \otimes \) U(3)_R local symmetry for pseudoscalar and vector meson nonets interacting with external electroweak fields. As a non-linear representation of a Goldstone nonet we take \([8,9]\),

\[
U(P, \eta_0 + F_0 \Theta) = \xi^2(P, \eta_0 + F_0 \Theta) = \exp \left\{ \frac{i}{F_8} \sqrt{2} P + i \frac{1}{3} \sqrt{2} F_0 (\eta_0 + F_0 \Theta) \right\},
\]

where \( P \) is the Goldstone pseudoscalar octet,

\[
P = \begin{pmatrix}
\pi^0 + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\
\pi^- & -\frac{\pi^0}{\sqrt{2}} - \frac{\eta_8}{\sqrt{6}} & K^0 \\
K^- & \frac{2\eta_8}{\sqrt{6}} & -\frac{\eta_8}{\sqrt{6}}
\end{pmatrix},
\]

and \( \eta_0 \) the pseudoscalar singlet. The unimodular part of the field \( U \) contains the octet degrees of freedom while the phase \( \det U = \exp i X = \exp \left\{ i \sqrt{2}(\eta_0 + F_0 \Theta) \right\} \) describes that of the singlet. The auxiliary field \( \Theta(x) \) ascertains that \( \det U \) is invariant under U(3)_L \( \otimes \) U(3)_R transformations \([8,9]\). The U(3)_L \( \otimes \) U(3)_R group does not have a dimension-nine irreducible representation; the quantity in the curly bracket of Eqn. (3) does not exhibit a nonet symmetry so that the octet (\( F_8 \)) and singlet (\( F_0 \)) decay constants are not necessarily identical. As in Refs. [13–15] we define vector type \( \Gamma_\mu \) and axial-vector type \( \Delta_\mu \) covariants

\[
\Gamma_\mu = \frac{i}{2} \left[ \xi D_\mu \xi - \xi D_\mu \xi^\dagger \right] = \frac{i}{2} \left[ \xi^\dagger, D_\mu \xi \right] + \frac{1}{2} \left( \xi^\dagger r_\mu \xi + \xi l_\mu \xi^\dagger \right),
\]

\[
\Delta_\mu = \frac{i}{2} \left( \xi D_\mu \xi + \xi D_\mu \xi^\dagger \right) = \frac{i}{2} \left[ \xi^\dagger, D_\mu \xi \right] + \frac{1}{2} \left( \xi^\dagger r_\mu \xi - \xi l_\mu \xi^\dagger \right),
\]

with,

\[
D_\mu \xi = \partial_\mu \xi + i r_\mu \xi - i l_\mu \xi.
\]

Here \( r_\mu \) and \( l_\mu \) are the relevant external gauge fields of the standard model; \( l_\mu = v_\mu + a_\mu \) and \( l_\mu = v_\mu - a_\mu \), with \( v_\mu \) and \( a_\mu \) being the vector and axial vector external electroweak fields, respectively. Electroweak interactions are contained in the covariant derivative \( D_\mu \xi \). For pure electromagnetic interactions these fields are related to the quark charge operator \( Q = diag(2/3, -1/3, -1/3) \) and the photon field \( A_\mu \); \( l_\mu = r_\mu = -eQA_\mu \).

Under U(3)_L \( \otimes \) U(3)_R the field, Eqn. (3) transforms as,

\[
U' = RUL^\dagger.
\]
with $R \in U(3)_R$, $L \in U(3)_L$. The vector $(\Gamma_\mu)$ and axial-vector $(\Delta_\mu)$ like quantities transform, respectively, as a gauge and matter fields, i.e.,

$$\Gamma_\mu' = K\Gamma_\mu K^{\dagger} + iK\partial_\mu K^{\dagger}, \quad (9)$$

$$\Delta_\mu' = K\Delta_\mu K^{\dagger}, \quad (10)$$

where $(K, U, R, L)$ is a compensatory field representing an element of conserved vector subgroup $U(3)_V$.

The dynamical gauge bosons are defined as a $3 \times 3$ matrix vector field $V_\mu$ which transforms as,

$$V_\mu' = KV_\mu K^{\dagger} + \frac{i}{g}K\partial_\mu K^{\dagger}. \quad (11)$$

Clearly, the vectors $\Gamma_\mu - gV_\mu$ and $\Delta_\mu$ transform homogeneously and at lowest order the Lagrangian can be constructed from the traces $Tr\Delta_\mu$, $Tr(\Gamma_\mu - gV_\mu)^2$, $Tr\Delta_\mu$, $Tr(\Gamma_\mu - gV_\mu)$ and arbitrary functions of the variable $X = \sqrt{2N_f}\eta_0/F_0 + \Theta(x)$, all being invariant under $U(3)_L \otimes U(3)_R$ transformations. We may thus conclude that a most general lowest order (i.e. with the smallest number of derivatives) effective chiral Lagrangian can be written in the form $[3]$,.

$$L = L_A + aL_V - \frac{1}{2}Tr(V_\mu V^{\mu\nu}) , \quad (12)$$

where,

$$L_A = W_1(X)Tr(\Delta_\mu \Delta^\mu) + W_4(X)Tr(\Delta_\mu)Tr(\Delta^\mu) + W_5(X)Tr(\Delta_\mu)D_\mu \Theta + W_6(X)D_\mu \Theta \Theta \Theta + W_7(X)D_\mu \Theta \Theta \Theta \Theta \Theta + W_8(X)Tr(\Gamma_\mu - gV_\mu)Tr(\Gamma^\mu - gV^\mu), \quad (13)$$

and,

$$\dot{D_\mu \Theta} = \partial_\mu \Theta + Tr(v_\mu - t_\mu), \quad (15)$$

$$V_\mu^\nu = \partial_\mu V_\nu - \partial_\nu V_\mu - ig[V_\mu, V_\nu]. \quad (16)$$

All three terms of the lagrangian $L$ in Eqn. 12 are invariant under $U(3)_L \otimes U(3)_R$ transformations. Though in form this Lagrangian is similar to that of Bando et al. $[4]$, the expressions for $L_A$ and $L_V$ are different. Namely, the inclusion of the $\eta'$ meson as a dynamical variable involves additional terms with $Tr(\Delta_\mu)$, $D_\mu \Theta$ and $Tr(\Gamma_\mu - gV_\mu)$ and coefficient functions $W_i(X)$ which are absent in the $SU(3)$ limit. We note though that as in Bando et al. $[4]$, the Lagrangian $L_A + aL_V$ contains, amongst other contributions, a vector meson mass term $\sim V_\mu V^{\mu}$, a vector-photon conversion factor $\sim V_\mu A^\mu$ and coupling of both vectors and photons to pseudoscalar pairs. The latter can be eliminated by choosing $a = 2$, a choice which allows incorporating the conventional vector-dominance in electromagnetic form-factors of pseudoscalar mesons $[4]$.

The mass degeneracy is removed via the additional of pseudoscalar mass term. The most general expression of a (local) $U(3)_L \otimes U(3)_R$ symmetry violating term reads $[3][3][3][3]$.  

$$L_m = -W_0(X) + W_2(X)Tr\chi_+ + iW_3(X)Tr\chi_-, \quad (17)$$

with,

$$\chi_\pm = 2B_0(\xi\mathcal{M}\xi \pm \xi^\dagger \mathcal{M}\xi^\dagger), \quad B_0 = m^2/2(m_u + m_d), \quad (18)$$

and $\mathcal{M} = diag(m_u, m_d, m_s)$ is the quark mass matrix. Parity conservation implies that all $W_i$ and $\tilde{W}_i$ are even functions of the variable $X$ except $W_3$ which is odd. The correct normalization of the $U(3)_L \otimes U(3)_R$ invariant kinetic term requires that $W_1(0) = F_0^2$, $W_4(0) = (F_0^2 - F_8^2)/3$ and $W_2(0) = F_8^2$ to ensure the standard $\chi$PT pion mass term.

One possible way to incorporate $SU(3)$ symmetry breaking is to introduce a universal matrix $B$ proportional to $\chi_+$, i.e.,

$$B = \frac{1}{4B_0(2m + m_s)}\chi_+. \quad (19)$$
For simplicity we take the exact isospin symmetry limit \( m_u = m_d = m \). Symmetry breaking terms to be added to \( L_A \) and \( L_V \) can be constructed either as conserving, or alternatively, as non-conserving the quadratic form of the Golstone meson kinetic terms. In what follows we develop the former alternative by including terms which break the octet symmetry only. The latter procedure is worked out in the Appendix. Let,

\[
U_8 = \xi_8^2 = \exp(i\frac{\sqrt{2}}{F_8}P)
\]  

be the pure octet matrix and let,

\[
\bar{\Delta}_\mu = \frac{i}{2} \left\{ \xi_8^\dagger, \partial_\mu \xi_8 \right\} + \frac{1}{2} \left( \xi_8^\dagger r_\mu \xi_8 - \xi_8 l_\mu \xi_8 \right),
\]  

be the octet covariant. Then general \( SU(3) \) symmetry breaking Lagrangians \( \bar{L}_A \) and \( \bar{L}_V \) would be,

\[
\bar{L}_A = \begin{align*}
W_1(X)(c_A Tr((B, \bar{\Delta}_\mu)\bar{\Delta}^\mu) + d_A Tr(B\bar{\Delta}_\mu B\bar{\Delta}^\mu) + \left. W_4(X) d_A Tr(B\bar{\Delta}_\mu) Tr(B\bar{\Delta}^\mu) \right),
\end{align*}
\]

\[
\bar{L}_V = \begin{align*}
W_1(X)(c_V Tr(B[\Gamma_\mu - g V_\mu][\Gamma^\mu - g V^\mu]) + \left. d_V Tr(B[\Gamma_\mu - g V_\mu] B[\Gamma^\mu - g V^\mu]) \right) + \left. W_4(X)(c_V Tr(\Gamma_\mu - g V_\mu) Tr(B[\Gamma^\mu - g V^\mu]) + \left. d_V Tr(\Gamma_\mu - g V_\mu) Tr(B[\Gamma^\mu - g V^\mu]) \right).\end{align*}
\]

where \( c_A, c_V, d_A, d_V \) are arbitrary constants. We stress that \( \bar{L}_A \) and \( \bar{L}_V \) differ from those of Bramon et al. [6] and Bando et al. [6]. First, like our symmetric \( L_A \) and \( L_V \) the asymmetric \( \bar{L}_A \) and \( \bar{L}_V \) parts involve additional terms which are absent in the \( SU(3) \) limit. Secondly, the terms proportional to \( d_A \) and \( d_V \) were included by Bando et al. [6] but with \( d_i = c_i^2 \). Thirdly, our symmetry breaking matrix \( B \) is not constant as in ref. [7] though similar (but not identical) to that of Bramon et al. [6].

The full Lagrangian may now be written in the form,

\[
L = L_A + \bar{L}_A + a(L_V + \bar{L}_V) + L_m + L_{ZW}\frac{1}{2} Tr(V_{\mu\nu}V^{\mu\nu}),
\]

where we included the well known Wess-Zumino-Witten term \( L_{ZW} \). This corresponds to the action defined as [21,22]

\[
S_{ZW} = -\frac{i}{80\pi^2} \int d^4 x e^{ijklm} Tr(\Sigma^L_{\mu}\Sigma^L_{\nu}\Sigma^L_{k}\Sigma^L_{l}\Sigma^L_{m})
\]

\[
-\frac{i}{16\pi^2} \int d^4 x \epsilon^{\mu\nu\alpha\beta}[W(U,l,r)_{\mu\nu\alpha\beta} - W(1,l,r)_{\mu\nu\alpha\beta}],
\]

with,

\[
W(U,l,r)_{\mu\nu\alpha\beta} = Tr(Ul_\mu l_\nu l_\alpha l_\beta + \frac{1}{4} U l_\mu U^\dagger r_\nu U l_\alpha U^\dagger r_\beta
\]

\[
iU\partial_\mu l_\nu r_\alpha U^\dagger r_\beta + iU\partial_\nu l_\mu r_\alpha U^\dagger r_\beta - i\Sigma^L_{\mu} l_\nu U^\dagger r_\alpha U l_\beta
\]

\[
+ \Sigma^L_{\nu} U^\dagger \partial_\alpha r_\mu U l_\beta - \Sigma^L_{\alpha} U^\dagger \partial_\mu r_\nu U l_\beta + \Sigma^L_{\mu} l_\alpha \partial_\nu r_\beta + \Sigma^L_{\nu} l_\alpha \partial_\mu r_\beta
\]

\[
- \Sigma^L_{\alpha} \partial_\nu l_\mu l_\alpha l_\beta + \frac{1}{2} \Sigma^L_{\mu} \Sigma^L_{\nu} l_\alpha l_\beta - i\Sigma^L_{\mu} l_\nu \Sigma^L_{\alpha} l_\beta - (L \leftrightarrow R),
\]

where \( \Sigma^L_{\mu} = U^\dagger \partial_\mu U, \Sigma^R_{\mu} = U \partial_\mu U^\dagger \) and, \( (L \leftrightarrow R) \) stands for a similar expression with \( U, l \) and \( \Sigma \) interchanged according to,

\[
(U \leftrightarrow U^\dagger), \quad (l \leftrightarrow r), \quad (\Sigma^L_{\mu} \leftrightarrow \Sigma^R_{\mu}).
\]

Note that this expression involves Lagrangian terms up to fifth chiral order, only. Other terms which accounts for the regularization of the one loop contributions are listed in Refs. [8,18,19].
By substituting these expressions into Eqn. 24, the kinetic terms of the pseudoscalar mesons is,

\[ W_0 = \text{const} + F_8^2 w_0 \frac{\eta_0^2}{F_0^2} + \ldots , \]

(28)

\[ W_1 = F_8^2 (1 + w_1 \frac{\eta_0^2}{F_0^2} + \ldots ) , \]

(29)

\[ W_2 = \frac{F_8^2}{4} (1 + w_2 \frac{\eta_0^2}{F_0^2} + \ldots ) , \]

(30)

\[ W_3 = \frac{F_8^2}{2} (w_3 \frac{\eta_0}{F_0} + \ldots ) , \]

(31)

\[ W_4 = \frac{F_8^2}{3} (1 + w_4 \frac{\eta_0^2}{F_0^2} + \ldots ) , \]

(32)

\[ \tilde{W}_1 = \frac{F_8^2}{2} (1 + \tilde{w}_1 \frac{\eta_0^2}{F_0^2} + \ldots ) , \]

(33)

\[ \tilde{W}_4 = \frac{F_8^2}{2} (\tilde{w}_4 + \tilde{w}_4 \frac{\eta_0^2}{F_0^2} + \ldots ) . \]

(34)

(35)

By substituting these expressions into Eqn. 24, the kinetic terms of the pseudoscalar mesons is,

\[ L_{\text{kin}} = \frac{1}{2} \left( 1 + c_A \frac{2m}{2m + m_s} + d_A \frac{m^2}{(2m + m_s)^2} \right) (\partial_\mu \vec{\pi})^2 + \right. \]

\[ \frac{1}{2} \left( 1 + c_A \frac{m + m_s}{2m + m_s} + d_A \frac{m m_s}{(2m + m_s)^2} \right) \sum_i (\partial_\mu K_i)^2 + \right. \]

\[ \frac{1}{2} \left( 1 + c_A \frac{1}{3} \frac{m + 2m_s}{2m + m_s} + d_A \frac{1}{3} \frac{m^2 + 2m_s^2}{(2m + m_s)^2} \right) (\partial_\mu \eta_8)^2 + \frac{1}{2} (\partial_\mu \eta_8)^2 . \]

(36)

To restore the standard normalization of the kinetic term we rescale the pseudoscalar fields according to,

\[ \pi \Rightarrow z_\pi \pi, \quad K \Rightarrow z_s K, \quad \eta_8 \Rightarrow z_8 \eta_8 , \]

(37)

with,

\[ z_\pi = \frac{F_8}{F_\pi} = \frac{1}{\sqrt{1 + c_A \frac{2m}{2m + m_s} + d_A \frac{m^2}{(2m + m_s)^2}}} , \]

(38)

\[ z_s = \frac{F_8}{F_K} = \frac{1}{\sqrt{1 + c_A \frac{m + m_s}{2m + m_s} + d_A \frac{m m_s}{(2m + m_s)^2}}} , \]

(39)

\[ z_8 = \frac{F_8}{F_\eta} = \frac{1}{\sqrt{1 + c_A \frac{1}{3} \frac{m + 2m_s}{2m + m_s} + d_A \frac{1}{3} \frac{m^2 + 2m_s^2}{(2m + m_s)^2}}} . \]

(40)

After some algebraic manipulations the octet field matrix can be written in the form,

\[ P = \begin{pmatrix} z_0^+ & \bar{z}_8^+ m_8 \sqrt{6} & \bar{z}_s^+ K^+ \\ z_\pi^- & \sqrt{2} & 0 \\ \bar{z}_s K^- & 0 & \sqrt{2} \end{pmatrix} , \]

(41)

where,

\[ z_8 = z_8 / z_\pi, \quad \bar{z}_s = z_s / z_\pi . \]

(42)

In addition to the usual quadratic terms \( \eta_8^2 \) and \( \eta_0^2 \), the quantity \( W_2(X)Tr\chi_+ + iW_3(X)Tr\chi_- \) in the mass term, Eqn. 17 gives rise to a mixing term \( \sim \eta_8\eta_0 \) which violates the orthogonality of the \( \eta_8 \) and \( \eta_0 \) states. The mass matrix is diagonalized via the usual unitary transformation 24.
\[ \eta_8 = \eta \cos \theta_P + \eta' \sin \theta_P, \]  
\[ \eta_0 = -\eta \sin \theta_P + \eta' \cos \theta_P, \]  
where \( \theta_P \) is the \( \eta - \eta' \) mixing angle.

In terms of the physical fields \( \eta \) and \( \eta' \), the nonlinear representation of the pseudoscalar particles can now be written as,

\[ U = \exp i \sqrt{2} \frac{F_\pi}{F_\eta} \mathcal{P}, \]  
where \( \mathcal{P} \) stands for the pseudoscalar nonet matrix,

\[
\mathcal{P} = \begin{pmatrix}
\frac{s^0_\pi}{\sqrt{2}} + \frac{1}{\sqrt{6}}(X_\eta \eta + X_\eta' \eta') & \frac{\pi^+}{\sqrt{6}} & \bar{z}_s K^+ \\
-\frac{s^0_\eta}{\sqrt{2}} + \frac{1}{\sqrt{6}}(X_\eta \eta + X_\eta' \eta') & \frac{\bar{z}_s K^0}{\sqrt{6}} & \frac{1}{\sqrt{6}}(Y_\eta \eta + Y_\eta' \eta') \\
\bar{z}_s K^- & \bar{z}_s K^0 & \phi
\end{pmatrix},
\]

with,

\[
X_\eta = \cos \theta_P (z_8 - \sqrt{2} \bar{r} \tan \theta_P), \quad X_\eta' = \cos \theta_P (z_8 \tan \theta_P + \sqrt{2} \bar{r}),
\]

\[
Y_\eta = \cos \theta_P (-2 \bar{r} z_8 - \sqrt{2} \bar{r} \tan \theta_P), \quad Y_\eta' = \cos \theta_P (-2 \bar{r} z_8 \tan \theta_P + \sqrt{2} \bar{r}),
\]

and \( \bar{r} = F_\pi / F_\eta \).

Similarly, the vector nonet with ideal mixing has the form [20],

\[
V = \begin{pmatrix}
\frac{s^0_\rho}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\
\rho^- & \frac{K^{*0}}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^* \\
K^{*-} & \frac{1}{\sqrt{2}} K^{*0} & \phi
\end{pmatrix}.
\]

In order to account for a strange (non strange) admixture in \( \omega(\phi) \) we substitute

\[
\omega \rightarrow \omega + \epsilon' \phi, \quad \phi \rightarrow \phi + \epsilon \omega.
\]

### III. RADIATIVE DECAY WIDTHS

We now turn to calculate radiative decay widths for \( P^0 \rightarrow \gamma \gamma, V^0 \rightarrow P^0 \gamma \) and \( P^0 \rightarrow V^0 \gamma \), with \( P^0 = \pi, \eta, \eta', K \) and \( V^0 = \rho, \omega, \phi, K^* \) using the formalism outlined above. We generalize the treatment of Ref. [3] by incorporating "indirect" symmetry breaking effects via pseudoscalar and vector nonet matrices Eqs. [47] and [48] and "direct" symmetry breaking terms (such as \( \bar{L}A \) and \( \bar{L}V \)). The Lagrangian is factorized in the form,

\[
L_{P \gamma \gamma} = L^{(s)}_{P \gamma \gamma} + c_W L^{(b)}_{P \gamma \gamma},
\]

\[
L_{V P \gamma} = L^{(s)}_{V P \gamma} + c_W L^{(b)}_{V P \gamma},
\]

where \( L^{(s)} \) and \( L^{(b)} \) are generic for indirect and direct symmetry breaking contributions, and \( c_W \) is a symmetry breaking parameter. In order to write these terms explicitly, we consider first the indirect anomalous Lagrangian [3],

\[
L^{(s)}_{\text{anomalous}} = L^{(0)}_{V V P} + L_{WZ}(P \gamma \gamma).
\]

From the four covariants \( \Delta_\mu, \Gamma_\mu - g V_\mu, V_\mu \) and \( \Gamma_\mu = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu - i[\Gamma_\mu, \Gamma_\nu] \), the Lagrangian \( L^{(0)}_{V V P} \) can have at most six terms,
With the quark mass ratios advocated by Weinberg [21]

where \( g_i \), \( i = 1, \ldots, 6 \) are arbitrary functions of the variable \( X \). We recall that \( \Gamma_\mu \) involves a term proportional to the photon field \( A_\mu \). By rearranging contributions to \( V P_\gamma \) and \( P_\gamma \gamma \) interaction terms we obtain,

\[
L^{(s)}_{V P_\gamma} = g_V \frac{e}{F_\pi} \epsilon_{\mu \nu \alpha \beta} \partial_\mu A_\nu \text{Tr}(Q \{ \partial_\alpha V_\beta, P \}) ,
\]

\[
L^{(s)}_{P_\gamma \gamma} = g_P \frac{e^2}{2 F_\pi} \epsilon_{\mu \nu \alpha \beta} \partial_\mu A_\nu \partial_\alpha A_\beta \text{Tr}(\{ Q^2, P \}) .
\]

For convenience we have introduced coupling constants \( g_V \) and \( g_P \) which incorporate all relevant contributions to \( L^{(s)}_{V P_\gamma} \) and \( L^{(s)}_{P_\gamma \gamma} \). It is now rather easy to obtain the direct symmetry breaking terms by introducing the quantity \( B \), Eqn. 43 as described in the previous section, i.e.,

\[
L^{(b)}_{V P_\gamma} = g_V \frac{e}{F_\pi} \epsilon_{\mu \nu \alpha \beta} \partial_\mu A_\nu \text{Tr}(Q \{ B, \{ \partial_\alpha V_\beta, P \} \}) ,
\]

\[
L^{(b)}_{P_\gamma \gamma} = g_P \frac{e^2}{2 F_\pi} \epsilon_{\mu \nu \alpha \beta} \partial_\mu A_\nu \partial_\alpha A_\beta \text{Tr}(\{ Q^2, \{ B, P \} \}) .
\]

### A. The \( V \to P \gamma \) and \( P \to V \gamma \) Processes

The relevant vertices are,

\[
V(V P_\gamma) = -i g_V \frac{e}{F_\pi} w(V P) \epsilon^{\mu \nu \alpha \beta} k_\mu e_\nu^{(V)} p_\alpha e_\beta^{(V)} ,
\]

where \( e_\nu^{(V)} (p) \) and \( e_\nu^{(s)} (k) \) are the polarization (four-momentum) of the vector meson and final photon, respectively. With the quark mass ratios advocated by Weinberg [21] \( m_u : m_d : m_s = 0.55 : 1.0 : 20.3 \), the ratio \( m : m_s = (m_u + m_d) : 2m_s = 0.038 \) is rather small and terms proportional to \( c_W m / (2m + m_s) \) can be neglected [7]. Then for the \( P \)- matrix of Eqns. 46 [7] one obtains,

\[
w(\rho \pi) = \frac{1}{3} , \quad w(\rho \eta) = \frac{1}{\sqrt{3}} X_\eta , \quad w(\rho \eta') = \frac{1}{\sqrt{3}} X_{\eta'} ,
\]

\[
w(\omega \pi) = 1 , \quad w(\omega \eta) = -\frac{1}{3 \sqrt{3}} X_\eta , \quad w(\omega \eta') = \frac{1}{3 \sqrt{3}} X_{\eta'} ,
\]

\[
w(\phi \pi) = e' , \quad w(\phi \eta) = -\frac{\sqrt{2}}{3 \sqrt{3}} Y_\eta (1 + c_W \frac{m_s}{2m + m_s}) , \quad w(\phi \eta') = -\frac{\sqrt{2}}{3 \sqrt{3}} Y_{\eta'} (1 + c_W \frac{m_s}{2m + m_s}) ,
\]

\[
w(K^0 \bar{K}^0) = w(K^0 \bar{K}^0) = -\frac{2}{3} \bar{z}_s (1 + \frac{1}{2} c_W \frac{m_s}{2m + m_s}) ,
\]

\[
w(K^+ \bar{K}^+) = w(K^+ \bar{K}^+) = \frac{1}{3} \bar{z}_s (1 - c_W \frac{m_s}{2m + m_s}) .
\]

\[1\] The Weinberg’s ratios give apparently the lowest limit for \( m_s / m \). The current algebra prediction is \( m_s / m = (2m_K^2 - m^2_s) / m^2_s = 25.6 \) while recent estimations [10] give the value \( m_s / m \approx 26.6 \).
In terms of these vertices the decay widths of $V \rightarrow P\gamma$ and $P \rightarrow V\gamma$ are,

$$\Gamma(VP\gamma) = G \frac{(m_V^2 - m_P^2)^3}{m_V^2 F_8^2} |w(VP)|^2,$$

$$\Gamma(PV\gamma) = 3G \frac{(m_P^2 - m_V^2)^3}{m_P^2 F_8^2} |w(VP)|^2,$$

with,

$$G = \frac{e^2 g_V^2}{4\pi 24}.$$

**B. The $P \rightarrow \gamma\gamma$ decays**

The relevant vertices are,

$$V(P\gamma\gamma) = -2ig_P \frac{e^2}{F_8} \bar{w}(P) \epsilon^{\mu\nu\alpha\beta} k_{1\mu} \epsilon_\nu^{(\gamma)} k_{2\alpha} \epsilon_\beta^{(\gamma)},$$

where $\epsilon^{(\gamma)}$ and $\epsilon^{(\gamma)}$ are the polarizations of the final photons , $k_1$ and $k_2$ are their corresponding four-momenta. For the $\pi, \eta$ and $\eta'$ one has,

$$\bar{w}(\pi) = \frac{3}{\sqrt{2}},$$

$$\bar{w}(\eta) = \frac{3 \cos \theta_P}{\sqrt{6}} \left[ z_8 \left(1 - \frac{4}{3} c_w \frac{m_s}{2m + m_s}\right) - 2\sqrt{2} \left(1 + \frac{1}{3} c_w \frac{m_s}{2m + m_s}\right) \bar{r} \tan \theta_P \right],$$

$$\bar{w}(\eta') = \frac{3 \cos \theta_P}{\sqrt{6}} \left[ z_8 \left(1 - \frac{4}{3} c_w \frac{m_s}{2m + m_s}\right) \tan \theta_P + 2\sqrt{2} \left(1 + \frac{1}{3} c_w \frac{m_s}{2m + m_s}\right) \bar{r} \right].$$

With these vertices the decay rate is given by,

$$\Gamma(P\gamma\gamma) = \bar{G} \frac{m_P^3}{F_8^2} |\bar{w}(P)|^2,$$

with,

$$\bar{G} = \frac{\pi}{2} \left(\frac{e^2}{4\pi}\right)^2 \left(\frac{g_P}{9}\right)^2.$$

**C. Numerical Analysis and Results**

The decay width of the anomalous processes mentioned above are described in terms of coupling constants ($g_V, g_P$), pseudoscalar singlet-octet mixing angle ($\theta_P$), relative radiative decay constants ($\bar{r}, \bar{z}, \bar{z}$) and direct $U(3)_V$ symmetry breaking scale ($c_w$). The numerical values of these parameters can be fixed from experimental decay rates. The value of $g_V$ is determined from the $\omega \rightarrow \pi \gamma$ decay, $\Gamma(\omega\pi\gamma) = G \frac{(m_\omega^2 - m_\pi^2)^3}{m_\omega^2 F_8^2} = (716 \pm 43) KeV$, to have,

$$G = (1.44 \pm 0.04) \cdot 10^{-5}, \quad g_V = 0.22 \pm 0.006.$$

From the $\rho \rightarrow \pi \gamma$ decay, $\Gamma(\rho\pi\gamma) = 76 \pm 10 KeV$ one obtains practically identical value for $g_V$. Similarly, the decay $\pi \rightarrow \gamma\gamma$ can serve to fix $g_P$. From the experimental decay width, $\Gamma(\pi^0\gamma\gamma) = 9\bar{G} m_\pi^3 / 2 F_8^2 = (7.8 \pm 0.55)eV$, one obtains,
The value of the symmetry breaking scale $c_W$ can be fixed from the ratio,

$$\frac{\Gamma(K^{*0}K^0)}{\Gamma(K^{*+}K^{+})} = 4 \left[ \frac{1 + \frac{1}{2}c_W}{1 - c_W} \right]^2 = \frac{(117 \pm 10) KeV}{(50 \pm 5) KeV} = 2.34 \pm 0.43 ,$$

which yields $c_W = -0.19 \pm 0.04$. From equating $\Gamma(K^{*0}K^0)$ to its experimental value one finds $\tilde{z}_s = 0.86 \pm 0.08$ a value corresponding to $F_K = (1.16 \pm 0.11) F_\pi$. In fact, from the ratio,

$$\frac{\Gamma(\phi\pi^0\gamma)}{\Gamma(\omega\pi^0\gamma)} = e^2 \left[ \frac{(m_\phi^2 - m_\pi^2)m_\omega}{(m_\phi^2 - m_\pi^2)m_\phi} \right]^3 = \frac{(5.8 \pm 0.6) KeV}{(716 \pm 43) KeV} = 0.008 \pm 0.001$$

one obtains, $|w(\phi\pi^0)| = 0.059 \pm 0.005$, a value identical to that quoted previously by Bramon et al. \([4]\) from using the vector meson dominance.

The remaining parameters $\tilde{z}$, $\tilde{r}$ and $\phi_P$ can now be calculated using Eqs. \([55, 60, 22]\) and the experimental decay widths $\Gamma(\eta'\gamma\gamma)$, $\Gamma(\phi\eta\gamma)$, $\Gamma(\phi\gamma\gamma)$ and $\Gamma(\eta'\gamma\gamma)$, one obtains,

$$\tilde{z} = 0.92 \pm 0.06 \ , \ \tilde{r} = 0.97 \pm 0.06 \ , \ \theta_P = -(15 \pm 2.4)^{\circ} .$$

These values (hereafter we refer to as solution I) are listed in Table \[I\]. Rather similar values (solution II of Table \[I\]) are obtained from a global fit of the data listed in Table \[I\]. Predictions of decay widths as obtained with the solutions I and II of Table \[I\] are summarized in Table \[I\].

From Eqs. \([40, 42]\) and Eqs. \([23, 27]\), one can see that for fixed $c_W$ the width ratios $\Gamma(\eta'\gamma\gamma)/\Gamma(\eta\gamma\gamma)$, $\Gamma(\phi\eta\gamma)/\Gamma(\phi\gamma\gamma)$, $\Gamma(\phi\gamma\gamma)/\Gamma(\pi^\prime\gamma\gamma)$ and $\Gamma(\eta'\gamma\gamma)/\Gamma(\eta\gamma\gamma)$ depend on $\tan \theta_P$ and $r/z$. Precision measurements of these ratios would be very useful to obtain more accurate values for $\theta_P$ and $r/z$. Perhaps even more attractive quantities are the ratios $\Gamma(\eta'\gamma\gamma)/\Gamma(\eta\gamma\gamma)$, $\Gamma(\phi\eta\gamma)/\Gamma(\phi\gamma\gamma)$ which can now be determined at DAΦNE with high precision.

Our set of parameters agree with the results of Bramon et al. \([4]\) and Venugopal et al. \([24]\) except for the mixing angle which differs significantly from Bramon et al. \([4]\) and Venugopal et al. \([22]\) but agrees with more recent analysis of Bramon et al. \([4]\) and Esciriano et al. \([25]\).

How significant are the departures of these parameters from their values in the limit exact $U(3)_L \otimes U(3)_R$ symmetry? Clearly, the exact $SU(3)$ limit, i.e., $c_W = 0$, $F_\pi = F_K = F_\phi$ is inconsistent with the ratio $\Gamma(K^{*0}K^0)/\Gamma(K^{*+}K^{+})$; a value of $c_W = 0$ predicts a ratio equals 4 as opposed to the experimental value of 2.34 ± 0.43. Furthermore, with $c_W = -0.19$ and with $F_K = F_\pi$ one obtains $\Gamma(K^{*0}K^0)$ about 35% higher than experimental value and far beyond the measurement accuracy. We may thus conclude that data requires $SU(3)$ symmetry to be broken directly ($c_W \neq 0$) and indirectly ($F_K \neq F_\pi$). If either direct or indirect symmetry breaking is not included, the quality of the fit deteriorates significantly. The value of the mixing angle $\theta_P$ is rather sensitive to direct symmetry breaking. At the limit $F_0 = F_\phi$ the mixing angle varies from $\theta_P = -23^\circ$ at $c_W = 0$ to $\theta_P = -16^\circ$ at $c_W = -0.19$. The mixing angle is less sensitive to indirect symmetry breaking. Indeed, a global fit which neglects indirect symmetry breaking, i.e., with $F_K = F_\pi = F_\phi = F_0$ gives $c_W = -0.22$ and $\theta_P = -16.2^\circ$. Also, a global fit which assumes broken $SU(3)$ symmetry but with nonet symmetry, i.e., with $F_0 = F_\phi = F_K = F$ but $F \neq F_\pi$ yields $F = 1.1F_\pi$, $c_W = -0.20$ and $\theta_P = -14.6^\circ$, rather close to the values of solution II. Upon concluding we stress that confidence criterion favors our solution II, i.e., with direct and indirect symmetry breaking.

### IV. SUMMARY AND DISCUSSION

In this paper, using the hidden symmetry approach of Bando et al. \([4]\) combined with general procedure of including the $\eta'$ meson into $\chi$PT \([11]\) we have constructed an effective Lagrangian which incorporates pseudoscalar and vector meson nonets as dynamical degrees of freedom interacting with external electroweak fields. At lowest order the Lagrangian $L$ is a linear combination of three parts $L_A$, $L_V$ and a vector nonet "kinetic" term $\frac{1}{2} Tr (V_{\mu\nu} V^{\mu\nu})$, all of which possessing a $U(3)_L \otimes U(3)_R$ and a local (hidden) $U(3)_V$ symmetry. The $L_A$ and $L_V$ parts involve the pseudoscalar and vector meson fields and their interactions with external electroweak fields, respectively. Though in form this division of the Lagrangian is identical to that of Bando et al. \([4]\), the expressions for $L_A$ and $L_V$ are different, as they include the $\eta'$ meson as a dynamical variable also. The symmetry breaking effects are included via
the pseudoscalar meson mass term as well as direct symmetry breaking terms $L_A$ and $L_V$ in a fashion similar to that proposed by Bramon et al. [6]. These terms are constructed by introducing a universal matrix $B$ which is proportional to pseudoscalar meson mass matrix into our general expressions for $L_A$ and $L_V$. The symmetry breaking leads to the mass splitting for the pseudoscalar and vector meson nonets, $\eta - \eta'$ and $\omega - \phi$ mixing effects, $F_\eta \neq F_K \neq F_\pi \neq F_{\eta'}$ etc. We may thus conclude that the Lagrangian of Eqn (3) provides a basis for an effective perturbative chiral theory capable to describe interacting pseudoscalar and vector mesons. To demonstrate that, we have considered anomalous radiative decay processes within our approach. Namely, the decay widths of anomalous processes are calculated by taking into account indirect as well as direct symmetry breaking effects. The widths were parameterized in terms of five parameters, including a symmetry breaking scale $c_W$, pseudoscalar meson weak decay constants $F_K$, $F_\eta$, $F_{\eta'}$ and the $\eta - \eta'$ meson mixing angle $\theta_P$. Our analysis show that the value of the mixing angle $\theta_P$ is rather sensitive to the presence of a direct symmetry breaking. The best solution was obtained with $c_W = -0.19$ suggesting a value $\theta_P \approx -(15.4 \pm 1.8)^\circ$. This agrees with the value extracted by Bramon et al. [5] from rather exhaustive analysis of data. Our analysis provides evidence for a broken $U(3)$ symmetry with $F_0 \neq F_8$ and $F_K \neq F_\eta \neq F_\pi$.

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| Decays | $\Gamma_{\text{exp}}$(KeV) | $\Gamma_{\text{calc}}$(KeV) | $\Gamma_{\text{calc}}$(KeV) |
|--------|----------------|----------------|----------------|
| $\rho \to \pi\gamma$ | 76 ± 10 | 76 ± 12 | 76 ± 12 |
| $\omega \to \pi\gamma$ | 716 ± 43 | *716 ± 43 | *716 ± 43 |
| $\rho \to \eta\gamma$ | 58 ± 11 | 47 ± 11 | 42 ± 10 |
| $\omega \to \eta\gamma$ | 7.0 ± 1.8 | 6 ± 1.6 | 5.5 ± 1.5 |
| $\phi \to \eta\gamma$ | 56.7 ± 2.8 | 61.4 ± 3 | 56.2 ± 2.8 |
| $\phi \to \eta'\gamma$ | $0.54 \pm 0.29^{\dagger}$ | 0.5 ± 0.14 | 0.44 ± 0.14 |
| $\eta' \to \rho\gamma$ | 60.7 ± 7.4 | 74 ± 12 | 62 ± 9.1 |
| $\eta' \to \omega\gamma$ | 6.07 ± 0.74 | 6.8 ± 0.8 | 6.0 ± 0.8 |
| $\pi^0 \to \gamma\gamma$ | 7.8 ± 0.55 | *7.8 ± 0.55 | *7.8 ± 0.55 |
| $\eta \to \gamma\gamma$ | 460 ± 40 | 550 ± 70 | 490 ± 65 |
| $\eta' \to \gamma\gamma$ | 4290 ± 190 | 4430 ± 280 | 4050 ± 260 |
| $K^{\ast0} \to K^{\ast}\gamma$ | 117 ± 10 | 117 ± 10 | 117 ± 10 |
| $K^{\ast\pm} \to K^{\ast}\gamma$ | 50 ± 5 | 50 ± 5 | 50 ± 5 |

* TABLE II. Calculated decay widths with the parameter set solution I and solution II of Table I. Widths marked with asterisk were used to fix $g_P$ and $g_V$. Data are taken from [24]. A value from the KLOE collaboration [24], $\Gamma_{\text{exp}}(\phi \to \eta'\gamma) = (0.36 \pm 0.12)$ KeV yield practically the same results.
V. APPENDIX

The symmetry breaking terms $\bar{L}_A$ and $\bar{L}_V$ can be constructed by using the nonet (rather than the octet) covariant $\Delta_\mu$. We demonstrate that for $L_A$. We write,

$$
\bar{L}_A =
W_1(X) (c_A \text{Tr}((B, \Delta_\mu)\Delta^\mu) + d_A \text{Tr}(B\Delta_\mu B\Delta^\mu)) +
W_2(X) (c_A \text{Tr}(B\Delta_\mu)\text{Tr}(\Delta^\mu) + d_A \text{Tr}(B\Delta_\mu)\text{Tr}(B\Delta^\mu))
$$

(79)

From this expression the contributions of the $\eta_0$ and $\eta_8$ to the kinetic term is,

$$
L_\text{kin}^{\eta_8} = \kappa_8 (\partial_\mu \eta_8)^2 + \kappa_0 (\partial_\mu \eta_0)^2 + \kappa_8 \partial_\mu \eta_8 \partial^\mu \eta_0 +
m_{\eta_8}^2 \eta_8^2 + m_{\eta_0}^2 \eta_0^2 + 2m_{\eta_0}^2 \eta_8 \eta_0,
$$

(80)

where the matrix $\kappa$ depends on the parameters $c_A$ and $d_A$. This expression gives rise to twofold $\eta - \eta'$ mixing, one from the kinetic term and one from the nondiagonal mass matrix. We first diagonalize the matrix $\kappa$ using the unitary transformation

$$
\begin{pmatrix}
\eta_8 \\
\eta_0
\end{pmatrix}
= \begin{pmatrix}
\cos \lambda & \sin \lambda \\
-\sin \lambda & \cos \lambda
\end{pmatrix}
\begin{pmatrix}
\bar{\eta}_8 \\
\bar{\eta}_0
\end{pmatrix} = \Upsilon
\begin{pmatrix}
\bar{\eta}_8 \\
\bar{\eta}_0
\end{pmatrix}
$$

(81)

This leads to,

$$
L_\text{kin}^{\eta_8} = \kappa_8 (\partial_\mu \bar{\eta}_8)^2 + \kappa_0 (\partial_\mu \bar{\eta}_0)^2 + (\bar{\eta}_8, \bar{\eta}_0) \Upsilon^{-1} \bar{M}^2 \Upsilon
\begin{pmatrix}
\bar{\eta}_8 \\
\bar{\eta}_0
\end{pmatrix}
$$

(82)

where $\kappa_i$ are the eigenvalues of the matrix $\kappa$. Now to restore the standard normalization of the kinetic term we rescale the pseudoscalar fields

$$
\pi \Rightarrow z_\pi \pi, \quad K \Rightarrow z_K K, \quad \bar{\eta}_8 \Rightarrow z_\bar{\eta}_8, \quad \bar{\eta}_0 \Rightarrow f \bar{\eta}_0
$$

(83)

where $z = 1/\sqrt{\kappa_8}$, $f = 1/\sqrt{\kappa_0}$. In other words the fields $\bar{\eta}_8$ and $\bar{\eta}_0$ are related by nonunitary transformation (matrix $R = \text{diag}(z, f)$). Therefore, the $\bar{\eta}$ mass matrix has the (nondiagonal) form

$$
\bar{M}^2 = R \Upsilon^{-1} \bar{M}^2 \Upsilon R
$$

(84)

and the Goldstone field kinetic term now reads $(1/2)[(\partial_\mu \pi)^2 + (\partial_\mu K)^2 + (\partial_\mu \bar{\eta}_8)^2 + (\partial_\mu \bar{\eta}_0)^2]$. As a last step we relate the $\bar{\eta}$ to the physical fields $\eta$ and $\eta'$ which are eigenvectors of the mass matrix $\bar{M}^2$,

$$
\begin{pmatrix}
\bar{\eta}_8 \\
\bar{\eta}_0
\end{pmatrix} = \Omega
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix}
$$

(85)

where,

$$
\Omega = \begin{pmatrix}
\cos \chi & \sin \chi \\
-\sin \chi & \cos \chi
\end{pmatrix}
$$

(86)

The relations between the $\eta_8$ and $\eta_0$ and physical fields $\eta$, $\eta'$ is then given by,

$$
\begin{pmatrix}
\eta_8 \\
\eta_0
\end{pmatrix} = \Theta
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix}
$$

(87)

where,

$$
\Theta = \begin{pmatrix}
z \cos \lambda \cos \chi - f \sin \lambda \sin \chi & z \cos \lambda \sin \chi + f \sin \lambda \cos \chi \\
-z \sin \lambda \cos \chi - f \cos \lambda \sin \chi & -z \sin \lambda \sin \chi + f \cos \lambda \cos \chi
\end{pmatrix}
$$

(88)

Clearly, the transformation $\Theta$ is nonunitary and can not be written in the so called two angle form of Refs. 3,23 since $\Theta_{11}^2 + \Theta_{22}^2 \neq 1$.

The pseudoscalar meson matrix has the form of Eqn.40 but with $X$ and $Y$ defined as,

$$
X_\eta = z \cos \lambda \cos \chi - f \sin \lambda \sin \chi + \sqrt{2} r(-z \sin \lambda \cos \chi - f \cos \lambda \sin \chi),
$$

$$
X_{\eta'} = z \cos \lambda \sin \chi + f \sin \lambda \cos \chi + \sqrt{2} r(-z \sin \lambda \sin \chi + f \cos \lambda \cos \chi),
$$

$$
Y_\eta = -2(z \cos \lambda \cos \chi - f \sin \lambda \sin \chi) + \sqrt{2} r(-z \sin \lambda \cos \chi - f \cos \lambda \sin \chi),
$$

$$
Y_{\eta'} = -2(z \cos \lambda \sin \chi + f \sin \lambda \cos \chi) + \sqrt{2} r(-z \sin \lambda \sin \chi + f \cos \lambda \cos \chi).
$$

(89)

Clearly, for $\lambda = 0$ and $f = 1$ the expressions reduce to the ones in Eqns. 17.
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