Generalized Chain Products of MTL-chains

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Abstract

In this paper we present a different approach to ordinal sums of MTL-chains, as extensions of finite chains in the category of semihoops. In addition we prove in a very simple way that every finite locally unital MTL-chain can be decomposed as an ordinal sum of archimedean MTL-chains. Furthermore, we introduce Generalized Chain Products of MTL-chains and we show that ordinal sums of locally unital MTL-chains are particular cases of those.

Introduction

In [5] Esteva and Godo proposed a new Hilbert-style calculus called Monoidal T-norm based Logic (MTL, for short) in order to find the fuzzy logic corresponding to the bigger class of all left-continuous t-norms. In [6] Jenei and Montagna proved that MTL was in fact, the weakest logic which is complete with respect to a semantics given by a class of t-norms and their residua. Since in MTL the contraction rule in general does not hold, such logic can be regarded not only as a fuzzy logic and as a many-valued logic, but also as a substructural logic. These results motivated to introducing a new class of algebras with an equivalent algebraic semantics for MTL, the variety of MTL-algebras. MTL-algebras are essentially integral commutative residuated lattices with bottom satisfying the prelinearity equation:

\[(x \to y) \lor (y \to x) \approx 1\]

In [4] Castiglioni and Zuluaga characterized the class of finite MTL-chains which can be decomposed as an ordinal sum of archimedean MTL-chains. Such a class of finite MTL-algebras was called \textit{locally unital}. Nevertheless, the general problem of decompositions by ordinal sums of (arbitrary) MTL-chains still remains open. It is worth mentioning that the general problem was already solved for the subvariety of BL-chains (c.f [1],[2]) by proving that (arbitrary) BL-chains can be decomposed as an ordinal sum of Wajsberg hoops.

The aim of this paper is to exhibit a self-contained and very intuitive proof of the decomposition of locally unital MTL-chains as an ordinal sum of archimedean MTL-chains in terms of poset products given in [4], by employing extensions of finite chains in the category of semihoops. Additionally, we propose a construction called Generalized Chain Product and we prove that such construction turns out to be a suitable generalization of
ordinal sums for locally unital MTL-chains.

The first section is devoted to introduce all the concepts required to read this work. The second section is intended to give a characterization of those extensions of finite totally ordered semihoops which are isomorphic to an ordinal sum. In the third section we show that every finite locally unital MTL-chain can be decomposed as the ordinal sum of archimedean MTL-chains. Finally, in the last section, we introduce the Generalized Chain Products and we prove that the ordinal sum construction for finite totally ordered semihoops is indeed a particular case of these.

1 Preliminaries

A semihoop \( \mathbf{A} \) is an algebra \( \mathbf{A} = (A, \cdot, \rightarrow, \land, \lor, 1) \) of type \((2, 2, 2, 2, 0)\) such that \((A, \land, \lor)\) is lattice with 1 as greatest element, \((A, \cdot, 1)\) is a commutative monoid and for every \(x, y, z \in A\) the following conditions hold:

\[
\begin{align*}
(\text{residuation}) & \quad xy \leq z \text{ if and only if } x \leq y \rightarrow z \\
(\text{prelinearity}) & \quad (x \rightarrow y) \lor (y \rightarrow x) = 1
\end{align*}
\]

Equivalently, a semihoop is an integral, commutative and prelinear residuated lattice. We write \( \mathcal{SH} \) for the algebraic category of semihoops. A semihoop \( A \) is bounded if \((A, \land, \lor, 1)\) has a least element 0. An MTL-algebra is a bounded semihoop, hence, MTL-algebras are prelinear integral bounded commutative residuated lattices, as usually defined \([5, 8]\) and semihoops are basically “MTL-algebras without zero”. An MTL-algebra (or semihoop) \( A \) is an MTL-chain (SH-chain) if its semihoop reduct is totally ordered. It is a well known fact, that the class of MTL-algebras is a variety. We write \( \mathcal{MTL} \) for the category of MTL-algebras and MTL-homomorphisms. A totally ordered MTL-algebra is archimedean if for every \( x \leq y < 1 \), there exists \( n \in \mathbb{N} \) such that \( y^n \leq x \). A submultiplicative monoid \( F \) of \( M \) is called a filter if is an up-set with respect to the order of \( M \). For every \( x \in F \), we write \( \langle x \rangle \) for the filter generated by \( x \); i.e.,

\[
\langle x \rangle = \{a \in M \mid x^n \leq a \text{ for some } n \in \mathbb{N}\}.
\]

For any filter \( F \) of \( M \), we can define the relation \( \sim_F \), on \( M \) by \( a \sim_F b \) if and only if \( a \rightarrow b \in F \) and \( b \rightarrow a \in F \). It follows that \( \sim_F \) is indeed a congruence on \( M \). On MTL-algebras there exists a well known correspondence between filters and congruences (c.f. \([8]\)) so we write the quotient \( M/\sim_F \) by \( M/F \). For every \( a \in M \), we write \([a]_F\) for the equivalence class of \( a \) in \( M/F \). If there is no ambiguity, we simply write \([a]\). A filter \( F \) of \( M \) is prime if \( 0 \notin F \) and \( x \lor y \in F \) entails \( x \in F \) or \( y \in F \), for every \( x, y \in M \).

Let \( I = (I, \leq) \) be a totally ordered set and \( \mathcal{F} = \{A_i\}_{i \in I} \) a family of semihoops. Let us assume that the members of \( \mathcal{F} \) share (up to isomorphism) the same neutral element; i.e., for every \( i \neq j \), \( A_i \cap A_j = \{1\} \). The ordinal sum of the family \( \mathcal{F} \), is the structure \( \bigoplus_{i \in I} A_i \).

\footnote{Also called basic semihoop in \([9]\).}
whose universe is \( \bigcup_{i \in I} A_i \) and whose operations are defined as:

\[
x \cdot y = \begin{cases} 
  x \cdot_i y, & \text{if } x, y \in A_i \\
  y, & \text{if } x \in A_i, \text{ and } y \in A_j - \{1\}, \text{ with } i > j \\
  x, & \text{if } x \in A_i - \{1\}, \text{ and } y \in A_j, \text{ with } i < j.
\end{cases}
\]

\[
x \rightarrow y = \begin{cases} 
  x \rightarrow_i y, & \text{if } x, y \in A_i \\
  y, & \text{if } x \in A_i, \text{ and } y \in A_j, \text{ with } i > j \\
  1, & \text{if } x \in A_i - \{1\}, \text{ and } y \in A_j, \text{ with } i < j.
\end{cases}
\]

where the subindex \( i \) denotes the application of operations in \( A_i \).

Furthermore, if \( I \) has a minimum \( \bot \), \( A_i \) is a totally ordered semihoop for every \( i \in I \) and \( A_\bot \) is bounded then \( \bigoplus_{i \in I} A_i \) becomes an MTL-chain. In order to clarify notation, we will use the symbol \( \oplus \) to denote the usual linear sum of lattices (as defined in Section 1.24 of \[3\]).

Let \( M \) be an MTL-algebra. Write 0, 1 for the trivial idempotents, and \( \mathcal{I}(M) \) for the set of all idempotent elements of \( M \) and \( \mathcal{I}^*(M) \) for \( \mathcal{I}(M) - \{0\} \). We say that \( e \in \mathcal{I}^*(M) \) is a **local unit** if \( xe = x \), for all \( x \leq e \). Clearly 1 is a local unit. If \( M \) is archimedean, 1 is in fact the only local unit of \( M \). Notice that there may be idempotents that are not local units. As an example, one may consider the MTL-algebra \( A \) whose underlying set is the totally ordered set \( A = \{0, x, e, 1\} \) and the product is determined by the following table:

|     | 0 | 1 | e | x |
|-----|---|---|---|---|
| 0   | 0 | 0 | 0 | 0 |
| 1   | 1 | 1 | e | x |
| e   | e | e | e | 0 |
| x   | x | 0 | 0 | 0 |
| 0   | 0 | 0 | 0 | 0 |

Table 1: MTL-algebra without local units.

Let us to consider the following quasi-identity in the language of MTL-algebras (or semihoops):

\((LU)\) If \( e^2 = e \) and \( e \lor x = e \) then \( ex = x \).

In \[4\] the corresponding quasivariety resulting from adding \((LU)\) to the theory of MTL-algebras was called **locally unital** MTL-algebras (luMTL, for short). In this this paper, locally unital MTL-algebras will be denoted by \( \text{luMTL} \). Observe that every archimedean chain is a member of \( \text{luMTL} \).

## 2 Another perspective on finite luMTL-chains

We start this section with the following observation: Let \( M \) be a semihoop and \( F \) be a filter of \( M \). A straightforward verification shows that \( F \) is a subalgebra of \( M \), which basically sais that in \( SH \), filters always belong to the same category of the original algebra. Such situation does not hold for MTL-algebras, since the only filter of \( M \) which
is a subalgebra is $M$ itself. Hence, taking advantage of the fact that MTL-algebras are “semihoops with zero”, along this section we will place our selves in the context of semihoops, rather than MTL-algebras.

**Remark 1.** The proof of [7], showing that the category of Heyting semilattices is semi-abelian, also works to show that the variety of integral commutative residuated lattices is semi-abelian. In particular, we can conclude that $\mathcal{SH}$ is semi-abelian.

Let $K, C \in \mathcal{SH}$ with $C$ a chain. By Remark 1, $\mathcal{SH}$ is semi-abelian, so an extension $E$ of $K$ by $C$ is simply a split short exact sequence in $\mathcal{SH}$ of the form

$$
0 \longrightarrow K \xrightarrow{k} E \xrightarrow{s} C \longrightarrow 0
$$

Here $0 = \{1\}$ is the zero object of the category. An interesting problem is to classify all the extensions of this form. Nevertheless, in this paper we give a solution for a more concrete problem. The aim of this section is presenting a characterization of the extensions of $C$ by $K$ which are isomorphic to the ordinal sum of $C$ with $K$.

**Lemma 1.** Let $K, E, C \in \mathcal{SH}$ where $C$ is a chain. Consider the following short exact sequence:

$$
0 \longrightarrow K \xrightarrow{k} E \xrightarrow{p} C \longrightarrow 0
$$

Then, $E \cong (C - \{1\}) \sqcup K$ if and only if the sequence is an extension $E$ of $K$ by $C$, such that:

i) $E = k(K) \cup s(C)$,

ii) $k(K) \cap s(C) = 0$.

**Proof.** Let us assume that the short exact sequence of above is an extension $E$ of $K$ by $C$ satisfying (i) and (ii). Since $s, k$ are morphisms of semihoops, $s(C)$ and $k(K)$ are subobjects of $E$ isomorphic to $K$ and $C$, respectively. So let us simply write $K$ for $k(K)$ and $C$ for $s(C)$. Let $e \in E$. By (i), $e \in K$ or $e \in C$. If $e \in K \cap C$, then, by (ii), $e = 1$. If $e \in K$ and $f \in E$ is such that $e \leq f$, then, from $1 = p(e) \leq p(f)$ we get that $p(f) = 1$. Hence $f \in K$. In consequence, $K$ is a filter in $E$, and every $c \in C - \{1\}$ is below every $k \in K$. Thereby, as a lattice, $E \cong (C - \{1\}) \sqcup K$. Notice that, both $K$ and $C$ are closed by the product and the residuum. So we just have to calculate $ck$, $c \rightarrow k$ and $k \rightarrow c$ for $c \in C$ and $k \in K$. Observe that we can assume $c \neq 1$. From the calculation

$$
p(ck) = p(c)p(k) = p(c) \cdot 1 = p(c)
$$

we can conclude that $ck = c$ for every $c \in C$ and $k \in K$. Taking into account the order of $E$, thus $ck = c \wedge k$. On the other hand, $p(k \rightarrow c) = 1 \rightarrow p(c) = p(c)$ so $k \rightarrow c = c$; $p(c \rightarrow k) = p(c) \rightarrow 1 = 1$, hence $c \rightarrow k \in K$. Finally, since $c \wedge k' = ck' \leq c \leq k$ for every $k' \in K$, then $c \rightarrow k = 1$. 

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We have proved that the binary operations on $E \cong (C - \{1\}) \oplus K$ are given by:

\[
kc = \begin{cases} 
  k \cdot_K c, & k, c \in K \\
  k \cdot_C c, & k, c \in C \\
  k \land c, & \text{otherwise}
\end{cases}
\]

\[
k \rightarrow c = \begin{cases} 
  k \rightarrow_K c, & k, c \in K \\
  k \rightarrow_C c, & k, c \in C \\
  c, & k \in K, c \in C \\
  1, & \text{otherwise}
\end{cases}
\]

That is to say, $E \cong K \oplus C$, the ordinal sum of semihoops. The converse follows directly from the definition of ordinal sum of semihoops.

To conclude this section we would like to remark that in $\mathcal{SH}$ there are extensions that are not of the kind presented in Lemma 1. In order to exhibit one, consider the semihoop $A = \{0, x, e, 1\}$ whose product is defined in Table 1. Let $F = \uparrow e$ and $2$ be the chain of two elements. It is clear that $A/F \cong 2$ and that $s : 2 \to A$, defined by $s(1) = 1_A$ and $s(0) = 0_A$ is a section of the quotient map, so $0 \to F \to A \to 2 \to 0$ is a short split exact sequence. Nevertheless, $x \notin s(2) \cup k(F)$ so $A \not\cong 2 \oplus F$.

### 3 Dealing with finite chains in luMTL

In [2], Busaniche proved that every BL-chain can be decomposed as an ordinal sum of totally ordered Wajsberg hoops. In [4], it was observed that, as a consequence of the divisibility condition for BL-algebras, it follows that every BL-algebra is a locally unital MTL-algebra. Moreover, it can be proved that every totally ordered Wajsberg hoop is in fact an archimedean MTL-algebra. The aim of this section is to generalize the finite case of Busaniche’s result to finite locally unital MTL-chains. That is to say, to give an elementary proof of the fact that every finite locally unital MTL-chain decomposes as an ordinal sum of archimedean MTL-chains (see [4] for another proof).

Let $M$ be a finite MTL-chain (or SH-chain). The following useful characterization was proved in [4].

**Proposition 1.** Let $M$ be a finite MTL-chain. The following are equivalent:

i) $M$ is archimedean,

ii) $M$ is simple,

iii) $\mathcal{I}(M) = \{0, 1\}$.

Write $(X)$ for the filter generated by $X \subseteq M$ and $\mathcal{LU}(M)$ for the set of local units of $M$. Note that in general $\mathcal{LU}(M) \subseteq \mathcal{I}^*(M)$. Furthermore, if $M \in \text{luMTL}$ then $\mathcal{LU}(M) = \mathcal{I}^*(M)$.

**Proposition 2** (Corollary 4, [4]). In any finite MTL-algebra $M$, the following are equivalent:

---

2For every $x, y$ the equation $x(x \to y) = x \land y$ holds.
i) $a \in \mathcal{I}(M)$,

ii) $\langle a \rangle = \uparrow a$.

Note that if $M$ is finite and $F$ is a proper filter of $M$, there exists a unique $e \in \mathcal{I}^*(M)$ such that $F = \uparrow e$. We write $eM$ for the set $eM = \{ex \mid x \in M\}$.

**Remark 2.** Let $M$ be a finite MTL-chain and $e \in \mathcal{LU}(M)$. Then the extension

$$
\begin{array}{c}
0 \\
\longrightarrow \uparrow e \\
\longrightarrow M \\
\longrightarrow M/\uparrow e \\
\longrightarrow 0
\end{array}
$$

is an extension of the type of Lemma 1 and $M/\uparrow e \cong eM$ (which is in fact true for any $e \in \mathcal{LU}(M)$). Hence $M \cong \uparrow e \oplus eM$.

**Lemma 2.** Every finite locally unital MTL-chain is an ordinal sum of archimedean chains.

**Proof.** Let $M$ be a finite locally unital MTL-chain. It follows that $\mathcal{LU}(M)$ is also a finite chain. In order to prove the statement, we proceed by induction over the amount of local units of $M$. If $|\mathcal{LU}(M)| = 1$, then by Proposition $1$ $M$ is archimedean. Let us assume that $|\mathcal{LU}(M)| > 1$ and let us order its elements in an increasing way:

$$
e_1 < ... < e_n = 1
$$

Let us suppose that any locally unital MTL-chain $N$ with $|\mathcal{LU}(N)| = n - 1$, is an ordinal sum of $n - 1$ archimedean chains:

$$
N \cong C_1 \oplus ... \oplus C_{n-1}
$$

Since $e_{n-1} \in \mathcal{LU}(M)$, then, by Remark 2 the short exact sequence

$$
\begin{array}{c}
0 \\
\longrightarrow \uparrow e_{n-1} \\
\longrightarrow M \\
\overset{\uparrow e_{n-1}}{\longrightarrow} e_{n-1}M \\
\longrightarrow 0
\end{array}
$$

is a split extension, and hence, by Lemma $1$ $M \cong \uparrow e_{n-1} \oplus e_{n-1}M$. Furthermore, $\uparrow e_{n-1}$ is archimedean by Proposition $1$ and $\mathcal{LU}(e_{n-1}M) = n - 1$. By inductive hypothesis we can conclude that $e_{n-1}M \cong C_1 \oplus ... \oplus C_{n-1}$. An easy verification shows that

$$
M \cong \uparrow e_{n-1} \oplus e_{n-1}(\uparrow e_{n-2}) \oplus ... \oplus e_2(\uparrow e_1) \oplus e_1M
$$

This concludes the proof. 

\[\square\]

## 4 Generalized Chain Products

In this section we introduce the concept of Generalized Chain Products of MTL-chains and we show its intimate relation with extensions by chains of totally ordered semihoops. Furthermore, we show that generalized chain products are in fact, a generalization of ordinal sums of locally unital MTL-chains.

Let $F$ and $A$ be two finite MTL-chains. Since $A$ is finite, we can order its elements in an increasing way; let us say, $0_A < x_1 < ... < x_{n-1} < 1_A$. Let $C_{(F,A)} = \{C_j \mid j \in A\}$ a family of sets such that:
1. For every $i \in A$, $C_i$ is a finite chain,
2. For $i = 1_A$, $C_{1_A} = F$, and
3. If $i \neq j$, with $i, j \in A$, then $C_i \cap C_j = \emptyset$.

Let $E := \bigcup_{i \in A} C_i$ with the order induced by the ordinal sum of lattices; that is, as a poset, $E = \bigsqcup_{i \in A} C_i$. Hence $E$ is a lattice. Observe that, since each $C_i$ is a finite lattice, then it has a minimum element. Let us write $0_i$ for it.

Now, take a family of functions $M(F,A) = \{ \mu_{ij} : C_i \times C_j \to C_{i \cdot j} \mid i, j \in A \}$ (where $i \cdot j$ denotes the product in $A$), such that:

i) For every $i, j \in A$, $\mu_{ij}$ is monotone in each coordinate,

ii) For every $k_i \in C_i$ and $k_j \in C_j$, $\mu_{ij}(0_i, k_j) = \mu_{ij}(k_i, 0_j) = 0_{ij}$,

iii) $M(F,A)$ is jointly associative; that is, the following diagram

\[
\begin{array}{c}
C_i \times C_j \times C_k \xrightarrow{id_i \times \mu_{jk}} C_i \times C_{j \cdot k} \\
\mu_{ij} \times id_k \downarrow \downarrow \mu_{i \cdot (j \cdot k)} \\
C_{i \cdot j} \times C_k \xrightarrow{\mu_{i \cdot (j \cdot k)}} C_{(i \cdot j) \cdot k}
\end{array}
\]

commutes, for every $i, j, k \in A$.

iv) $M(F,A)$ is jointly commutative; that is, the following diagram

\[
\begin{array}{c}
C_i \times C_j \xrightarrow{\mu_{ij}} C_{i \cdot j} \\
\tau \downarrow \downarrow \\
C_j \times C_i \xrightarrow{\mu_{ji}} C_{j \cdot i}
\end{array}
\]

commutes, for every $i, j, k \in A$.

v) $M(F,A)$ has a global unit; that is, the following diagram

\[
\begin{array}{c}
1 \times C_i \xrightarrow{id_1 \times \mu_{i1}} C_i \\
1_F \times id_1 \downarrow \downarrow \mu_{1_i} \\
C_1 \times C_i
\end{array}
\]

commutes, for every $i \in A$. Here $1$ is the singleton chain and $1$ is the unit of $A$.

The latter presentation lead us to introduce the following concept.

**Definition 1.** Let $F$ and $A$ be two finite MTL-chains. A generalized chain product (induced by $F$ and $A$) is a pair $(C_{(F,A)}, M_{(F,A)})$ such that:

1. The collection $C_{(F,A)} = \{ C_j \mid j \in A \}$ satisfies 1., 2. and 3,
2. The set $M_{(F,A)} = \{ \mu_{ij} : C_i \times C_j \to C_{ij} \mid i, j \in A \}$ is a family of functions satisfying (i) to (v).

Lemma 3. Let $F$ and $A$ be two finite MTL-chains. Then, every generalized chain product $(C_{(F,A)}, M_{(F,A)})$ defines an extension of $A$ by $F$.

Proof. Take $E(C_{(F,A)}, M_{(F,A)}) := \bigcup C$ with the mentioned order. Endow $E = E(C_{(F,A)}, M_{(F,A)})$ with the following binary operation:

$$\mu : E \times E \to E$$

defined by $\mu(e, f) = \mu_{ij}(e, f)$, if $e \in C_i$ and $f \in C_j$. Observe that conditions (i) to (v) of $M$ guarantee that $(E, \vee, \wedge, \mu, 1_F)$ is an MTL-algebra. Furthermore, $F = C_i$ is a filter of $E$. Let us calculate $E/F$. For $e, f \in E$, $e \sim_F f$ if and only if there is a $g \in F$ such that $ge \leq f$ and $gf \leq e$, so it is clear that $F = [1]$. Now, let us assume $e, f \in C_i$ for some $i \neq 1_A$. From (ii), $\mu_{i1}(0, e) = \mu_{i1}(0, f) = 0$, so $e \sim_F f$. If $e \in C_i$ and $f \in C_j$ for some $i \neq j$, then, since $C_i \cap C_j = \emptyset$, thus for every $g \in F = C_1$ we have that $\mu_{i1}(g, e) \in C_i$ and $\mu_{i1}(g, f) \in C_j$; that is, $e \sim_F f$. Hence $C_i = [e]_F$ for any $e \in C_i$ and $i \in A$. So we have a bijection $E/F \to A$ which clearly is an MTL-morphism. Moreover, the map $s : A \to E$ defined by

$$s(i) = \begin{cases} 1_F, & i = 1_A \\ 0, & \text{otherwise} \end{cases}$$

is a section of $p : E \to A \cong E/F$. Hence, we get an split short exact sequence in $\mathcal{SH}$:

$$0 \to F \xrightarrow{j} E \xrightarrow{s} A \to 0$$

where $j$ is the inclusion. $\square$

The following result is a partial converse of Lemma 3.

Lemma 4. Let

$$0 \to F \xrightarrow{j} E \xrightarrow{p} A \to 0$$

be a split exact sequence in $\mathcal{SH}$ such that:

i) $j(F)$ is a convex subset of $E$,

ii) $E, F$ and $A$ are finite chains.

Then, there exists a generalized chain product $(C_{(F,A)}, M_{(F,A)})$ such that $E \cong E(C_{(F,A)}, M_{(F,A)})$.

Proof. Observe that, since $j(F) \cong F$ is a convex submonoid of a finite partially ordered commutative integral monoid $F$ is a filter of $E$ and hence, there exists a unique idempotent $e_F \in E$ such that $F = \uparrow e_F$. In the future we shall simply write $e$ for $e_F$. Notice that there is an isomorphism between $A$ and $eE$. By definition of extension, we know that $A$ is just $E/F$, so we can define a map $f : E/F \to eE$ by $f([a]) = ea$ which is clearly bijective. From this fact we can conclude that $[a] = \{ x \in E \mid ex = ea \}$. Now, since

\footnote{Also called negative ordered monoid in $\mathcal{SH}$.}
each equivalence class is a convex set of $E$ (because each class is in particular a lattice congruence class), we have a decomposition of $E$ as a lattice as $E \cong \bigoplus_{x \in A} C[x]$, where the $C[x]$’s form a partition of $E$ by convex chains. Clearly, this corresponds to the quotient map. Now take $C = \{ [a] \mid [a] \in A \}$ and define $\mu_{[a][b]} : [a] \times [b] \rightarrow [ab]$ by $\mu_{[a][b]} (\alpha, \beta) = \alpha \beta$; that is, the restriction of the product to the respective $C[x]$’s. It is easy to check that they form a family $M$ satisfying conditions (i) to (v) of the beginning of this section.

To conclude this paper, we show how Lemma 4 justifies the use of the term “generalized chain product” in Definition 1 in the sense that regular sum products (ordinal sums) in locally unital MTL-chains are a very particular sort of generalized chain products.

**Corollary 1.** Ordinal sums of finite locally unital MTL-chains are generalized chain products.

**Proof.** Let $\{M_i\}_{i \in I}$ be a family of finite locally unital MTL-chains, where $I$ is a finite totally ordered set with top $\top$. Take $\{M'_i\}_{i \in I}$ with $M'_\top = M_\top$ and $M'_i = M_i - \{1_{M_i}\}$ for $i \neq \top$. Note that each $M'_i$ is a semigroup. Take $i \cdot j = i \wedge j$, for $i, j \in I$. This makes of $I$ a finite MTL-algebra (in fact, a Heyting algebra). By regarding $C(M'_\top, I) = \{ M'_i \}_{i \in I}$ and $M(M'_\top, I) = \{ \mu_{ij} : M'_i \times M'_j \rightarrow M'_{i \wedge j} \mid i, j \in I \}$, defined as $\mu_{ij}(a, b) = a \cdot_i b$, if $i = j$; $\mu_{ij}(a, b) = a$, if $i < j$; and $\mu_{ij}(a, b) = b$, if $j < i$; from Lemmas 2 and 3 the result follows.

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