EXAMPLES OF NON-TRIVIAL RANK IN LOCALLY CONFORMAL KÄHLER GEOMETRY

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Abstract. We consider locally conformal Kähler geometry as an equivariant, homothetic Kähler geometry \((K, \Gamma)\). We show that the de Rham class of the Lee form can be naturally identified with the homomorphism projecting \(\Gamma\) to its dilation factors, thus completing the description of locally conformal Kähler geometry in this equivariant setting. The rank \(r_M\) of a locally conformal Kähler manifold is the rank of the image of this homomorphism. Using algebraic number theory, we show that \(r_M\) is non-trivial, providing explicit examples of locally conformal Kähler manifolds with \(1 \leq r_M \leq b_1\). As far as we know, these are the first examples of this kind. Moreover, we prove that locally conformal Kähler Oeljeklaus-Toma manifolds have either \(r_M = b_1\) or \(r_M = b_1/2\).

1. Introduction

For many reasons, Kähler manifolds are considered the most interesting objects of complex geometry. However, strong topological properties -like formality- even Betti numbers of odd index and others, obstruct the existence of Kähler metrics on many compact manifolds, some of them very simple ones, like the Hopf or Kodaira surfaces. From the Riemannian viewpoint, the natural place to look for metrics with a given property is a conformal class. When this is not possible, then local metrics with the said property can be searched for, subject to some condition on the overlaps.

This is exactly the way Izu Vaisman arrived to the notion of locally conformal Kähler (briefly, LCK) metric [Vai76]. The original definition puts the accent on a fixed metric which is locally conformal with local Kähler ones. Equivalently, it requires the existence of a closed one-form (the Lee form) which, together with the fundamental two-form, generates a differential ideal. On the other hand, any metric globally conformal with a LCK metric is again LCK. This allows talking about a LCK structure, in which no metric is fixed and only the cohomology class of the Lee form is given. This understanding of LCK geometry is consistent with the fact that any Kähler cover of a LCK manifold bears a Kähler metric with respect to which the covering group acts by holomorphic homotheties. LCK geometry can thus be seen as the pair \((K, \Gamma)\) of a Kähler manifold and a group of holomorphic homotheties. This viewpoint has been suggested in [GOP05], and then developed in [GOPP06], where two key notions were introduced: the presentation (in this paper called LCK-presentation), which is the pair described above, and the rank
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of the subgroup of $\mathbb{R}^+$ given by dilation factors of $\Gamma$, which measures the “true” homothety part of the group.

In the present paper we go a bit further, showing that the Lee form can also be read in these terms. This completes the description of LCK geometry in terms of presentations. Moreover, we show that the examples of LCK manifolds constructed in \cite{OT05} have highly non-trivial rank: their rank is either equal to the first Betti number or to half of it. In particular, this provides a first example of LCK manifold of rank $\neq 1$ and strictly less than $b_1$.

The structure of the paper is as follows. In Section 2, we recall the basic definitions and properties of LCK geometry, presentations and rank. In Section 3 we show how the Lee form can be reconstructed from a presentation. Section 4 is devoted to a detailed description of the complex manifolds defined by Oeljeklaus-Toma in \cite{OT05}, and to the computation of their first Betti number. In Section 5, we recall how Oeljeklaus-Toma manifolds can be LCK-presented, in terms of a global potential, and we compute the dilation factors of $\Gamma$. Then we prove the following Theorem:

**Theorem.** Let $M$ be an LCK Oeljeklaus-Toma manifold. Then its rank is either $b_1(M)$ or $b_1(M)/2$.

Using this Theorem we then compute explicit examples of LCK manifolds with non-trivial rank.

Since Section 4 and 5 makes strong use of tools from Algebraic Number Theory, in Section 6 we make a short summary of these tools.

2. LCK-PRESENTATIONS FOR COMPLEX MANIFOLDS

For convenience of the reader, we here briefly review notation established in \cite{GOPP06}.

Let $M$ be a complex manifold. A *locally conformal Kähler* metric is a conformal class $[g]$ of Hermitian metrics on $M$ such that $[g]$ is given locally by Kähler metrics. The conformal class $[g]$ corresponds to a unique de Rham cohomology class $[\omega_g] \in H^1(M)$, whose representative $\omega_g$ is defined as the unique closed 1-form satisfying $d\Omega_g = \omega_g \wedge \Omega_g$, where $\Omega_g$ denotes the fundamental form of $g$. The 1-form $\omega_g$ is called the *Lee form* of $g$.

Taking into account that a locally conformal Kähler metric on a manifold of Kähler type must be globally conformal Kähler \cite{Val79}, it is a trivial task to show that a complex manifold $M$ (of complex dimension at least 2) admits a locally conformal Kähler metric if and only if there is a complex covering space $K$ of Kähler type such that $\pi_1(M)$ acts on $K$ by holomorphic homotheties with respect to the Kähler metric.

More explicitly, if $g$ is a locally conformal Kähler metric on $M$, and $\omega_g$ its Lee form, then the pull-back of $\omega_g$ to any Kähler covering $\tilde{K}$ of $M$ is exact, say $\omega_g = df$. Denoting by $\tilde{g}$ a lift of $g$ to $\tilde{K}$, then $e^{-f}\tilde{g}$ turns out to be a Kähler metric on $K$. 
such that $\pi_1(M)$ acts on it by holomorphic homotheties. According to the fact that the Kähler metric $e^{-f}\hat{g}$ is defined up to homotheties (because $f$ is defined up to a constant), usually in locally conformal Kähler geometry one is interested in the homothety class of a Kähler manifold.

The above discussion motivates the following definitions, first given in [GOPP06]. For the notion of minimal cover in the more general setting of conformal geometry, see also [BM09].

**Definition 2.1.** Let $K$ be a homothetic Kähler manifold and $\Gamma$ a discrete Lie group of biholomorphic homotheties acting freely and properly discontinuously on $K$.

- The pair $(K, \Gamma)$ is called a LCK-presentation.
- If $M$ is a complex manifold and $M = K/\Gamma$ as complex manifolds, $(K, \Gamma)$ is called a LCK-presentation for $M$.
- If $\Gamma$ does not contain isometries other than the identity, then $(K, \Gamma)$ is called minimal, and if $K$ is simply connected then $(K, \Gamma)$ is called maximal.

**Remark 2.2.** Given a complex manifold $M$, the statement “$(K, \Gamma)$ is a LCK-presentation for $M$” is just a shortcut for “$K$ is a complex covering space of $M$, and $\Gamma$ are its covering transformations, and there is a Kähler metric on $K$ which is conformally equivalent to a $\Gamma$-invariant metric”. Due to the 1-1 correspondence existing between locally conformal Kähler manifolds and minimal presentations, we will often abuse of this language by saying “the locally conformal Kähler manifold $(K, \Gamma)$”.

In a homothetic Kähler manifold $K$ we denote by $\text{Hmt}(K)$ the group of its biholomorphic homotheties, and by

$$\rho_K: \text{Hmt}(K) \to \mathbb{R}^+$$

the group homomorphism associating to a homothety its dilation factor. For any locally conformal Kähler manifold $M$, LCK-presented as $(K, \Gamma)$, the rank of the free abelian group $\rho_K(\Gamma)$ depends only on $M$ [GOPP06, Proposition 2.10].

**Definition 2.3.** The rank of $\rho_K(\Gamma)$ is called the rank of $M$, and is denoted by $r_M$.

**Remark 2.4.** The rank $r_M$ measures “how much” the locally conformal Kähler manifold is far from the Kähler geometry.

3. The Lee form

Let $M$ be a locally conformal Kähler manifold LCK-presented as $(K, \Gamma)$. The question if the de Rham class of any Lee form of $M$ can be completely described in terms of LCK-presentations has been left open in [GOPP06]. In this Section we fill this gap.

Consider the following exact sequence:

$$1 \to \pi_1(K) \to \pi_1(M) \to \Gamma \to 1$$
Recalling that the abelianization doesn’t preserve exactness on the left, we get:

\[(3.1)\]  
\[H_1(K, \mathbb{Z}) \to H_1(M, \mathbb{Z}) \to \frac{\Gamma}{[\Gamma, \Gamma]} \to 0\]

Using the universal coefficient theorem for cohomology and the de Rham theorem, we can translate the above sequence in de Rham cohomology language:

\[(3.2)\]  
\[0 \to \text{Hom}(\frac{\Gamma}{[\Gamma, \Gamma]}, \mathbb{R}) \xrightarrow{i} H^1_{dR}(M) \to H^1_{dR}(K)\]

Now, observe that \(\Gamma \subset \text{Hmt}(K)\) (by definition), and \([\Gamma, \Gamma] \subset \ker \rho_K\) (because \(\mathbb{R}\) is abelian). We thus obtain a homomorphism from \(\frac{\Gamma}{[\Gamma, \Gamma]}\) to \(\mathbb{R}\) by

\[(3.3)\]  
\[\frac{\Gamma}{[\Gamma, \Gamma]} \xrightarrow{\rho_K} \mathbb{R} + \text{log} \to \mathbb{R}\]

For the sake of simplicity, we still denote by \(\rho_K\) this element of \(\text{Hom}(\frac{\Gamma}{[\Gamma, \Gamma]}, \mathbb{R})\).

We are now ready to state the following Theorem.

**Theorem 3.1.** Let \(M\) be a locally conformal Kähler manifold \(\text{LCK}-\)presented as \((K, \Gamma)\), and let \([\omega] \in H^1_{dR}(M)\) be its Lee form. Let \(i\) be the map given by \((3.2)\), and \(\rho_K\) the element of \(\text{Hom}(\frac{\Gamma}{[\Gamma, \Gamma]}, \mathbb{R})\) given by \((3.3)\). Then

\([\omega] = i(\rho_K)\)

**Proof:** Denote by \(p\) the projection from \(K\) to \(M\), and by \(g\) the Riemannian metric on \(M\) associated to \(\omega\). Thus, \(p^*g\) is a \(\Gamma\)-invariant metric on \(K\), \(p^*\omega = df\) is an exact 1-form on \(K\), and the metric \(g_K = e^{-f}p^*g\) on \(K\) is Kähler.

For any \(\gamma \in \Gamma\), denote by \([\gamma]\) the corresponding element of \(\frac{\Gamma}{[\Gamma, \Gamma]}\). We then have:

\[\gamma^*g_K = \gamma^*e^{-f}p^*g = e^{-f\circ\gamma}\gamma^*p^*g = e^{-f\circ\gamma + f - f\circ\gamma}p^*g = e^{-f\circ\gamma + f}g_K\]

and thus (remember that \(\rho_K\) is defined as in \((3.3)\)):

\[(3.4)\]  
\[\rho_K([\gamma]) = -f \circ \gamma + f\]

Remark in particular that since \(\gamma \in \text{Hmt}(K)\), then \(-f \circ \gamma + f\) is constant.

To prove the claim, we thus need to show that \(\omega([\alpha]) = \rho_K(\gamma_\alpha^{-1})\) for every loop \(\alpha\) in \(M\). As for \(\omega([\alpha])\), we have:

\[\omega([\alpha]) = \int_\alpha \omega = \int_{\tilde{\alpha}_{y_0}} p^*\omega = \int_{\tilde{\alpha}_{y_0}} df = f(\tilde{\alpha}_{y_0}(1)) - f(\tilde{\alpha}_{y_0}(0))\]

As for \(\rho_K(\gamma_\alpha^{-1})\), we use \((3.4)\):

\[\rho_K(\gamma_\alpha^{-1}) = -f \circ \gamma_\alpha^{-1} + f = -f \circ \gamma_\alpha^{-1}(\tilde{\alpha}_{y_0}(1)) + f(\tilde{\alpha}_{y_0}(1)) = -f(\tilde{\alpha}_{y_0}(0)) + f(\tilde{\alpha}_{y_0}(1))\]

and this proves the claim. \(\blacksquare\)
Remark 3.2. In the proof of Theorem 3.1 we have shown that the rank can be defined in terms of the Lee form $\omega$, as the rank of the image of the natural map

$$H_1(M, \mathbb{Z}) \longrightarrow \mathbb{R}$$

$$\alpha \longmapsto \int \omega$$

Remark 3.3. The rank $r_M$ satisfies $0 \leq r_M \leq b_1(M)$, and $r_M = 0$ if and only if $M$ is globally conformal Kähler.

4. Oeljeklaus-Toma manifolds

In their beautiful paper [OT05], the authors construct locally conformal Kähler manifolds using tools from Algebraic Number Theory, which are summarized in Section 6. In this section, we assume these tools are known.

We will denote by $F$ an algebraic number field, by $O_F$ the ring of algebraic integers of $F$ and by $O_F^*$ the multiplicative group of units of $O_F$. If $[F: \mathbb{Q}] = n = s + 2t$ is the degree of $F$ over $\mathbb{Q}$, we denote by $\{\sigma_i: F \to \mathbb{C}\}_{i=1, \ldots, n}$ the complex embeddings of $F$, where the first $s$ embeddings are real, and the last $2t$ satisfy $\sigma_s+i = \overline{\sigma_s+i}+t$. The units which are positive in all real embeddings of $F$ are denoted by $O_F^*$.+

We are now ready to describe Oeljeklaus-Toma construction. For details, see [OT05].

All together, the embeddings $\sigma_i$ give the natural map

$$F \longrightarrow \mathbb{C}^{s+t}$$

$$\sigma(x) \overset{\text{def}}{=} (\sigma_1(x), \ldots, \sigma_{s+t}(x))$$

The image $\sigma(O_F)$ is a lattice of rank $n$ in $\mathbb{C}^{s+t}$ [Mil09, Proposition 4.26], and in this way we get a properly discontinuous action of $O_F$ on $\mathbb{C}^{s+t}$ given by translations. We denote this action by $T$: if $a \in O_F$ and $(z_1, \ldots, z_{s+t}) \in \mathbb{C}^{s+t}$, then

$$(4.1) \quad T_a(z_1, \ldots, z_{s+t}) \overset{\text{def}}{=} (z_1 + \sigma_1(a), \ldots, z_{s+t} + \sigma_{s+t}(a))$$

We also have a multiplicative action of $O_F^*$ on $\mathbb{C}^{s+t}$, denoted by $R$ (as in “rotation”): if $u \in O_F^*$, then

$$(4.2) \quad R_u(z_1, \ldots, z_{s+t}) \overset{\text{def}}{=} (\sigma_1(u)z_1, \ldots, \sigma_{s+t}(u)z_{s+t})$$

Pairs $(a, u)$ act then on $\mathbb{C}^{s+t}$ by $T_a \circ R_u$, and from $T_a \circ R_u \circ T_b \circ R_v = R_{uv} \circ T_{a+uv}$ one gets $(a, u)(b, v) = (a + ub, uv)$. In other words, the inclusion $O_F^* O_F \subset O_F$ defines a semidirect product $O_F \rtimes O_F^*$ acting on $\mathbb{C}^{s+t}$.

Since for any $(a, u) \in O_F \rtimes O_F^*$ the equation $T_a(R_u(z_1, \ldots, z_{s+t})) = (z_1, \ldots, z_{s+t})$ has one solution $\sigma(\frac{a}{1-u})$, the action is not free, with fixed point set contained in
\[ \sigma(F) \subset \mathbb{R}^s \times \mathbb{C}^t. \] Thus, consider the upper complex half-plane \( \mathbb{H} \) of complex numbers with strictly positive imaginary part, and observe that \( \mathbb{H}^s \) is \( \mathcal{O}_F^{s,+} \)-invariant. Hence, \( \mathcal{O}_F \times \mathcal{O}_F^{s,+} \) acts freely on \( \mathbb{H}^s \times \mathbb{C}^t \).

The above action is free, but not properly discontinuous in general (naively, one could say that “there are too many generators” for the group to act so: indeed \( \mathcal{O}_F \) has rank \( s + 2t \), \( \mathcal{O}_F^{s,+} \) has rank \( s + t - 1 \) so there are \( 2s + 3t - 1 \) generators, while we expect this number to be \( 2s + 2t \)). Still, Oeljeklaus and Toma show that one can always find a suitable subgroup \( U \subset \mathcal{O}_F^{s,+} \) such that the action of \( \mathcal{O}_F \times U \) is properly discontinuous and moreover, the quotient is compact. Subgroups of these kind are called admissible, and it is furthermore shown that when \( t = 1 \) every subgroup of finite index of \( \mathcal{O}_F^{s,+} \) is admissible.

Both \( \mathcal{O}_F \) and \( \mathcal{O}_F^{s,+} \) act holomorphically on \( \mathbb{H}^s \times \mathbb{C}^t \), so the quotient inherits a complex structure.

**Definition 4.1.** Given a finite field extension \( F \) of \( \mathbb{Q} \) and an admissible subgroup \( U \subset \mathcal{O}_F^{s,+} \), the compact complex manifold

\[ M(F, U) \overset{\text{def}}{=} \frac{\mathbb{H}^s \times \mathbb{C}^t}{\mathcal{O}_F \times U} \]

is called an Oeljeklaus-Toma manifold.

The first Betti number of \( M(F, U) \) is computed in [OT05]. Since we also need this fact, we include here a different proof than the original one, which makes no use of group cohomology, spectral sequences and Hurewicz’s Theorem.

**Theorem 4.2.** [OT05, Proposition 2.3] Let \( M = M(F, U) \) be an Oeljeklaus-Toma manifold. Then \( b_1(M) = s \).

**Proof:** We identify \( \pi_1(M) \) with the deck transformation group \( \mathcal{O}_F \times U \), which is generated by \( \{ T_a, R_u \}_{a \in \mathcal{O}_F, u \in U} \) see (4.1), (4.2). Since \( \pi_1(M)/\mathcal{O}_F \simeq U \) is abelian, and \( H_1(M, \mathbb{Z}) \) is the maximal abelian quotient of \( \pi_1(M) \), we have a commutative diagram:

\[
\begin{array}{cccc}
1 & \to & [\pi_1(M), \pi_1(M)] & \to & \pi_1(M) & \to & H_1(M, \mathbb{Z}) & \to & 1 \\
& & i\downarrow & & \| & & & & \downarrow p \\
1 & \to & \mathcal{O}_F & \to & \pi_1(M) & \to & U & \to & 1
\end{array}
\]

From \( p \) we get that \( \text{Rank}(H_1(M, \mathbb{Z})) = \text{Rank}(U) - \text{Rank}(\ker p) \). But the Snake Lemma gives \( \ker p \simeq \text{coker } i \), thus it is enough to show that \( \text{coker } i = \mathcal{O}_F/[[\pi_1(M), \pi_1(M)] \) is finite.

By direct computation we see \( [T_a, R_u] = T_{(1-u)a} \), for any \( a \in \mathcal{O}_F \) and any \( u \in U \). In particular, this shows that for any \( u \in U \), the principal ideal \( (1 - u)\mathcal{O}_F \) is a
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subgroup of \([\pi_1(M), \pi_1(M)]\). But if \(u \neq 1\), then \((1 - u)O_F\) has finite index, as \(O_F\) is a Dedekind ring.

Since for the rest of this paper we will be concerned only with the case \(t = 1\) and \(U = O_F^{s,+}\), we skip the details about \(U\).

5. The rank of Oeljeklaus-Toma manifolds

The following result was the starting point for this paper.

**Theorem 5.1.** [OT05, Page 7] Consider an Oeljeklaus-Toma manifold \(M(F, O_F^{s,+})\), with \(t = 1\) and \(s > 0\).

1. The real function

   \[
   \phi(z) = \prod_{j=1}^{s} \frac{i}{z_j - \bar{z}_j} + |z_{s+1}|^2
   \]

   is a global Kähler potential on \(\mathbb{H}^s \times \mathbb{C}\).

2. When \(\mathbb{H}^s \times \mathbb{C}\) is equipped with the Kähler metric \(i\partial \bar{\partial} \phi\) given above, the pair \((\mathbb{H}^s \times \mathbb{C}, O_F \rtimes O_F^{s,+})\) is a lck-presentation for \(M(F, O_F^{s,+})\).

**Proof:** To prove (1), one has to show that \(\phi\) is strictly plurisubharmonic. By direct calculation, we see that

\[
(\partial_{z_l} \partial_{\bar{z}_k} \phi) = \begin{pmatrix} \partial_{z_l} \partial_{\bar{z}_k} \phi_1 & 0 \\ 0 & 2 \end{pmatrix}
\]

where

\[
\phi_1 = \prod_{j=1}^{s} \frac{i}{z_j - \bar{z}_j}
\]

thus it suffices to look only at \((\partial_{z_l} \partial_{\bar{z}_k} \phi_1)\).

One has

\[
\partial_{z_l} \partial_{\bar{z}_k} \phi_1 = \frac{-1}{(z_k - \bar{z}_k)(z_l - \bar{z}_l)} \phi_1 \quad l \neq k = 1, \ldots, s
\]

\[
\partial_{z_k} \partial_{\bar{z}_k} \phi_1 = \frac{-2}{(z_k - \bar{z}_k)^2} \phi_1 \quad k = 1, \ldots, s
\]

thus \((\partial_{z_l} \partial_{\bar{z}_k} \phi_1)\) is proportional to the matrix

\[
A = \begin{pmatrix}
\frac{2}{4y_1^2} & \frac{1}{4y_1y_2} & \cdots & \frac{1}{4y_1y_k} \\
\frac{1}{4y_2y_1} & \frac{2}{4y_2^2} & \cdots & \frac{1}{4y_2y_k} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{4y_ky_1} & \frac{1}{4y_ky_2} & \cdots & \frac{2}{4y_k^2}
\end{pmatrix}
\]

But \(A\) is the sum between a diagonal, positive definite matrix, and a positive semidefinite one, thus \(A\) is positive definite.
Alternatively, one can directly notice that \( \frac{1}{z_{\bar{s}}} \) defines a Kähler potential in \( \mathbb{H} \) and then use [Dem09, Theorem 5.6].

To prove (2), one has to show that \( (\mathcal{O}_F \rtimes \mathcal{O}_F^*)^+ \) act by homoteties on \( \mathbb{H}_s \times \mathbb{C} \). Let \( a \in \mathcal{O}_F, u \in \mathcal{O}_F^* \), and consider the generators \( T_a, R_u \) of \( (\mathcal{O}_F \rtimes \mathcal{O}_F^*)^+ \) given by (1.1), (1.2). Then, using (5.2) above and the fact that the embeddings \( \{\sigma_j\}_{j=1,\ldots,s} \) are real, one obtains \( T_a^*(i\partial \bar{\partial} \phi) = i\partial \bar{\partial} \phi \), whereas using (6.3) one obtains

\[
R_u^*(\phi) = |\sigma_{s+1}(u)|^2 \phi
\]

that is, \( R_u \) acts by homotheties on the potential itself. \( \blacksquare \)

**Remark 5.2.** The Kähler potential \( \phi \) given by (5.1) corrects a minor typo present in the original paper. We acknowledge a useful email exchange with Mătei Toma.

**Remark 5.3.** In the proof of Theorem 5.1, we have shown that the rank of \( M(F, \mathcal{O}_F^*)^+ \) is the rank of the multiplicative subgroup of \( \mathbb{R}_+^* \) given by

\[
\{|\sigma_{s+1}(u)|^2 \text{ such that } u \in \mathcal{O}_F^*\}
\]

(see also the proof of [OT05, Proposition 2.9]).

The following result describes the rank of Oeljeklaus-Toma manifolds.

**Theorem 5.4.** Let \( F \) be a number field with \( s > 0 \) real embeddings \( \sigma_1, \ldots, \sigma_s : F \to \mathbb{R} \) and exactly two non-real embeddings \( \sigma_{s+1}, \bar{\sigma}_{s+1} : F \to \mathbb{C} \). Let \( M = M(F, \mathcal{O}_F^*)^+ \) be an Oeljeklaus-Toma manifold with the locally conformal Kähler structure given by Theorem 5.1. Let \( n = [F : \mathbb{Q}] \), so that \( n = s + 2 \), \( \dim_{\mathbb{C}} M = n - 1 \) and \( b_1(M) = n - 2 \).

1. If \( n \) is odd, then \( M \) has maximal rank, that is to say, \( r_M = b_1(M) = n - 2 \).
2. If \( n \) is even, then either \( M \) has again maximal rank or \( r_M = \frac{b_1(M)}{2} \); the last situation occurs if and only if \( F \) is a quadratic extension of a totally real number field.

**Proof:** (1) By Remark 5.3, it is enough to show that the map

\[
(5.3) \quad u \mapsto |\sigma_{s+1}(u)|^2
\]

is injective. Let \( u \in \mathcal{O}_F^* \) be a unit with \( |\sigma_{s+1}(u)|^2 = 1 \) (that is to say, \( |\sigma_{s+1}(u)| = 1 \)), and let \( P_u \) be its minimal polynomial over \( \mathbb{Q} \). Since \( p \overset{\text{def}}{=} \deg P_u = [\mathbb{Q}(u) : \mathbb{Q}] \) divides \( n = [F : \mathbb{Q}] \) and \( n \) is odd, we see that also \( p \) is odd. So \( P_u \) is given by

\[
P_u(X) = X^p + a_{p-1}X^{p-1} + \cdots + a_1X + a_0
\]

Moreover, using (6.1), (6.2) and (6.3) we obtain

\[
a_0^{n/p} = -Nm(u) = -1
\]

and since \( a_0 \in \mathbb{Z} \), we get \( a_0 = -1 \). Now observe that \( |\sigma_{s+1}(u)| = 1 \) implies \( \bar{\sigma}_{s+1}(u) = \frac{1}{\sigma_{s+1}(u)} \). But \( \bar{\sigma}_{s+1}(u) \) is a root of \( P_u \), hence \( P_u \left( \frac{1}{\sigma_{s+1}(u)} \right) = 0 \). This means
that $\sigma_{s+1}(u)$ satisfies the equation of degree $p$

$$1 + a_{p-1}X + \cdots + a_1X^{p-1} - X^p = 0$$

Thus, the uniqueness of the minimal polynomial forces $a_k = -a_{p-k}$ for all $k$: but then $P_u(1) = 0$, hence $u = 1$.

(2) We consider the case when $M$ is not of maximal rank. This means that the map $(5.3)$ is not injective, so that there exists a unit $u \in O_F^{\times}+ u \neq 1$, such that $|\sigma_{s+1}(u)| = 1$. We claim that deg $P_u$ is even. In fact, if deg $P_u$ was odd, by (6.1) we would get

$$a_0^{n/p} = Nm(u) = 1$$

that is, $a_0 = \pm 1$. If $a_0 = -1$, arguing the same as in point (1) above, we would have $u = 1$, whereas if $a_0 = 1$, we would have $P_u(-1) = 0$, that is, $u = -1 \notin O_F^{\times}+$, which is a contradiction in both cases. Thus, deg $P_u$ is even.

Lemma 5.5. If $F$ admits exactly 2 complex non-real embeddings, then any proper intermediate field extension $F \supseteq E \supseteq Q$ is totally real.

Proof: Assume $E$ is not totally real, and let $\varsigma$ a complex non-real embedding. Let $d \overset{\text{def}}{=} [F : E]$. Then $F$ admits $d$ embeddings fixing $E$ pointwise, and their composition with $\varsigma$ gives $2d$ complex non-real embeddings of $F$. Thus $d = 1$, and $F = E$. \(\blacksquare\)

In what follows, we need to assume that $u$ is non-real: since we can always replace $u$ with $\sigma_{s+1}(u)$, this assumption is not restrictive. Then, using Lemma 5.5 we get $F = Q(u)$.

Consider now the intermediate field extension $E \overset{\text{def}}{=} Q(u + \frac{1}{u})$. Clearly, $F \supseteq E \supseteq Q$. Using again Lemma 5.5 we get that $E$ is totally real, whereas from $(u + \frac{1}{u}) u = u^2 + 1$ we get $[F = Q(u) : E] = 2$.

It remains to check that the rank in this case is $\frac{b_1(M)}{2}$. For any unit $u \in O_F^{\times}$ we have $|\sigma_{s+1}(u)|^2 = Nm_{F/E}(u)$. This means that the rank of the group

$$\{|\sigma_{s+1}(u)|^2 \text{ such that } u \in O_F^{\times}\}$$

is the rank of the image of the norm map $Nm_{F/E} : O_F^{\times} \rightarrow O_E^{\times}$. But $(O_E^{\times})^2 \subset \text{Im}(Nm_{F/E})$, thus $Nm_{F/E}$ has the same rank as $O_E^{\times}$, which is $n/2 + 0 - 1 = n/2 - 1$ by Dirichlet Unit Theorem. \(\blacksquare\)

Remark 5.6. Theorem 5.4 holds for the general Oeljeklaus-Toma manifold $M(F, U)$, whenever $U \subset O_F^{\times}$ has finite index.

Remark 5.7. Theorem 5.4 shows that [GOPP06, Example 2.13] holds only for some Oeljeklaus-Toma manifolds.

The following two examples describe the case $[F : Q]$ even.

Example 5.8. Pick monic polynomials $f_1$, $f_2$ and $f_3$ in $\mathbb{Z}[X]$ of degree $2n$ such that:
\begin{itemize}
\item $f_1$ is irreducible modulo 2;
\item $f_2$ splits as a product of a linear factor and an irreducible polynomial modulo 3;
\item $f_3$ is a product of an irreducible polynomial of degree 2 and of two irreducible polynomials of odd degree modulo 5.
\end{itemize}

Then for every polynomial $g \in \mathbb{Z}[X]$ of degree $2n - 1$ the polynomial
\[
f = -15f_1 + 10f_2 + 6f_3 + 30g
\]
is monic, is irreducible (since its reduction modulo 2 is irreducible), and has maximal Galois group $S_{2n}$ (proceed as in [Moh08, Example 4.31], noting that $30 \equiv 0$ modulo 2, 3 and 5). For suitable choices of $g$ we will have that $f$ has exactly 2 non-real roots (proceed as in [OT05, Remark 1.1], observing that the set of polynomials \{\(-15f_1 + 10f_2 + 6f_3 + 30g, \deg g = 2n - 1\) is a lattice in $\mathbb{Q}^{2n}$). Let $F_f$ be the splitting field of $f$ and fix an isomorphism between $\text{Gal}(F_f/\mathbb{Q})$ and $S_{2n}$; let $H \subset \text{Gal}(F_f/\mathbb{Q})$ be the subgroup corresponding to $S_{2n-1}$ viewed as the set of all permutations fixing 1. Then $F_f^H$ has no proper subfields as $S_{2n-1} \subset S_{2n}$ is a maximal subgroup, and by Theorem 5.4, point 2 the corresponding Oeljeklaus-Toma manifold $M(F_f^H, \mathcal{O}_{F_f^H}^*)$ has maximal rank $r_M = b_1(M) = 2n - 2$.

**Example 5.9.** Pick an arbitrary totally real number field $E$ of degree $n$. Let $\alpha$ be a primitive element of $E$ over $\mathbb{Q}$ and let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ be the conjugates of $\alpha$: we can assume $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. Let $\sigma_i$ be the embedding corresponding to $\alpha_i$, and let $q \in \mathbb{Q}$ such that $\alpha_{n-1} > q > \alpha_n$. Take $F \overset{\text{def}}{=} E(\sqrt[2]{\alpha - q})$. Then $[F : E] = 2$ (otherwise, $\alpha - q = e^2$ for some $e \in E$; but then $\sigma_n(\alpha) - q = \sigma_n(e^2)$ so $\alpha_n - q > 0$ since $\sigma_n(e) \in \mathbb{R}$ as $E$ is totally real), and $F$ admits exactly 2 complex non-real embeddings (the $[F : E]$ extensions of $\sigma_n$ to $F$). Then by Theorem 5.4, point 2 the corresponding Oeljeklaus-Toma manifold $M(F, \mathcal{O}_F^*)$ has rank $r_M = \frac{b_1(M)}{2} = n - 1$.

**Remark 5.10.** Example 5.9 relies on the existence of a totally real number field of degree $n$, for an arbitrary $n$. This can be shown this way. First, recall that if $p$ is an arbitrary prime number, and $\zeta$ be a primitive root of unity of order $p$, then $\mathbb{Q} \subset \mathbb{Q}((\zeta))$ is a Galois extension of degree $p - 1$, with Galois group cyclic of order $p - 1$. Moreover, $\mathbb{Q} \subset \mathbb{Q}((\zeta + \frac{1}{\zeta}))$ is a totally real Galois extension of $\mathbb{Q}$, with Galois group cyclic of order $\frac{p - 1}{2}$. Now, choose a prime $p$ such that $n$ divides $\frac{p - 1}{2}$ (Dirichlet’s Theorem on prime numbers in arithmetic progressions), and choose a subgroup $H$ of $\text{Gal}(\mathbb{Q}((\zeta + \frac{1}{\zeta}))/\mathbb{Q})$ of index $n$. Then $\mathbb{Q}((\zeta + \frac{1}{\zeta})^H$ is a subfield of $\mathbb{Q}((\zeta + \frac{1}{\zeta}))$, hence totally real, and $\mathbb{Q} \subset \mathbb{Q}((\zeta + \frac{1}{\zeta})^H$ is Galois. Thus $[\mathbb{Q}((\zeta + \frac{1}{\zeta})^H : \mathbb{Q}]$ is the cardinality of its Galois group, that is exactly $n$.

**Remark 5.11.** We summarize in the following table what we presently know about relations between $r_M$ and properties of $M$. If $M = (K, \Gamma)$, by “Potential” in the Table we mean there exists a global Kähler potential on $K$, and by “Automorphic
potential" we mean there exists a global Kähler potential on $K$ such that $\Gamma$ acts on it by homotheties (see [OV09] [OV10]).

| Statement                          | True/False | Proof or Refutation                                      |
|------------------------------------|------------|----------------------------------------------------------|
| $b_1 = 1 \Rightarrow r_M = 1$      | True       | $1 \leq r_M \leq b_1$                                   |
| $r_M = 1 \Rightarrow b_1 = 1$      | False      | Induced Hopf bundles over curves of large genus in $\mathbb{C}^2$: $r_M = 1$, arbitrarily large $b_1$ |
| Vaisman $\Rightarrow r_M = 1$      | True       | [GOPP06, Corollary 4.7]                                  |
| Automorphic potential $\Rightarrow M$ can be deformed to $M'$, with $r_M' = 1$ | True       | [OV10] Proposition 2.5 and [OV09] Proofs of 5.2 and 5.3 |
| $r_M = 1 \Rightarrow$ Automorphic potential | False      | Inoue surfaces                                           |
| Potential $\Rightarrow r_M = 1$    | False      | Oeljeklaus-Toma manifolds as in Theorem 5.3 with $s > 2$: $\phi$ is a potential, and $r_M = b_1$ or $b_1/2$ |
| $r_M = 1 \Rightarrow$ Potential    | False      | Diagonal Hopf surface blown up in one point: $b_1 = 1$, so $r_M = 1$, and no potential because the universal covering contains compact complex submanifolds |
| $\exists M$ with $r_M = b_1 > 1$    | True       | Take $n = 2$, $f_1 = X^4 + X + 1 = f_2$, $f_3 = (X^2 + 2)(X-1)(X+1)$ and $g = 0$ in Example 5.8 |
| $\exists M$ with $b_1 > 1$ and $r_M = b_1/2$ | True | Take $n = 2$, $L = \mathbb{Q}(\sqrt{2})$, $\alpha = \sqrt{2}$ and $q = 0$ in Example 5.9 |

6. Algebraic Number Theory background

Let $F$ be a number field, that is, a finite extension of $\mathbb{Q}$. Such an extension is algebraic [Mil08 Proposition 1.30], that is, elements $x$ in $F$ satisfy $P(x) = 0$, where $P$ is a polynomial in $\mathbb{Q}[X]$. Whenever $P$ can be chosen monic and with coefficients in $\mathbb{Z}$, $x$ is said to be an algebraic integer, and all algebraic integers in
\( F \) form a ring usually denoted by \( \mathcal{O}_F \):
\[
\mathcal{O}_F \overset{\text{def}}{=} \{ x \in F \text{ such that } x^l + a_{l-1}x^{l-1} + \cdots + a_1x + a_0 = 0, \ a_i \in \mathbb{Z} \}
\]

Algebraic integers \( \mathcal{O}_F \) are a Dedekind domain \([\text{Mil09} \text{ Theorem 3.29}]\). Moreover, it is well-known that \( \mathcal{O}_F/I \) is a finite ring whenever \( I \) is a proper ideal in \( \mathcal{O}_F \).

For any \( x \in F \), there is one and only monic, irreducible \( P_x \in \mathbb{Q}[X] \) such that \( P_x(x) = 0 \). Such a \( P_x \) is called the minimal polynomial of \( x \) over \( \mathbb{Q} \), and algebraic integers are characterized by having minimal polynomial in \( \mathbb{Z}[X] \) \([\text{Mil09} \text{ Proposition 2.11}]\):
\[
\mathcal{O}_F = \{ x \in F \text{ such that } P_x \in \mathbb{Z}[X] \}
\]
The quotient ring \( \mathbb{Q}(x) = \mathbb{Q}[X]/(P_x) \) is a field because \( P_x \) is irreducible, and it is the smallest field containing \( \mathbb{Q} \) and \( x \). By \([\text{Mil08} \text{ Primitive Element Theorem, 5.1}]\), every number field is obtained this way, so it is not restrictive to think of \( F \) as \( \mathbb{Q}(\alpha) \), for a fixed \( \alpha \in \mathbb{C} \). The degree \( \deg P_\alpha = n = [F : \mathbb{Q}] \) of \( P_\alpha \) is then the dimension of \( F \) as a vector space over \( \mathbb{Q} \), a basis for \( F \) being \( \{1, \alpha, \ldots, \alpha^{n-1}\} \). Any intermediate field between \( F \) and \( \mathbb{Q} \) is of the form \( \mathbb{Q}(x) \), for some \( x \in F \), and \( \deg P_x \) is a divisor of \( \deg P_\alpha \) \([\text{Mil09} \text{ Proposition 1.20}]\). The ring of algebraic integers \( \mathcal{O}_F \) is a free \( \mathbb{Z} \)-module of rank \( n \) \([\text{Mil09} \text{ Page 29}]\).

Any root \( \alpha_i \) of \( P_\alpha \) induces a field embedding \( \sigma_i : F \to \mathbb{C} \), by
\[
x_0 + x_1\alpha + \cdots + x_{n-1}\alpha^{n-1} \mapsto x_0 + x_1\alpha_i + \cdots + x_{n-1}\alpha_i^{n-1}
\]
Clearly, \( \sigma_i(x) = x \) for every \( x \in \mathbb{Q} \), and the \( \sigma_i \) are the only embeddings of \( F \) into \( \mathbb{C} \) with this property \([\text{Mil08} \text{ Proposition 2.1b}]\). Moreover, \( \sigma_i(F) \subset \mathbb{R} \) if and only if \( \alpha_i \in \mathbb{R} \), and \( F \) is called totally real if \( \sigma_i(F) \subset \mathbb{R} \) for all \( i \).

If \( E \) is any intermediate field \( F \supset E \supset \mathbb{Q} \), we can consider the finite group of all automorphisms of \( F \) fixing \( E \) pointwise, denoted by \( \text{Aut}(F/E) \). Then, for any subgroup \( H \) of \( \text{Aut}(F/E) \), we have the subfield of \( F \) given by \( F^H = \{ x \in F \text{ such that } Hx = x \} \). The key point of Galois Theory is that this “subfield-subgroup-subfield” correspondence is \( 1 \to 1 \), for a certain class of extensions \( F \), called Galois extensions \([\text{Mil08} \text{ Theorem 3.16}]\).

Galois extensions are characterized by the following equivalent conditions \([\text{Mil08} \text{ Theorem 3.10}]\):

- \( F = E(\alpha) \) contains all roots of \( P_\alpha \), where \( P_\alpha \) is the minimal polynomial of \( \alpha \) over \( E \).
- \( F \) contains all roots of an irreducible polynomial \( P \in E[X] \) (we say that \( F \) is the splitting field of \( P \)).
- \( \text{Aut}(F/E) \) contains \( n \) elements, where \( n = \deg P_\alpha = [F : E] \).
- \( E = F^{\text{Aut}(F/E)} \).

Whenever \( F \) is a Galois extension of \( E \), the group \( \text{Aut}(F/E) \) is called Galois group of \( F \) over \( E \), and it is denoted by \( \text{Gal}(F/E) \). If \( x \in F \) and \( g \in \text{Gal}(F/E) \), then \( g(x) \) is called a Galois conjugate of \( x \) (briefly, a conjugate of \( x \)). One of the many nice properties of Galois extensions is that for any \( x \in F \) one has
If $F = \mathbb{Q}(\alpha)$ and $\sigma_i : F \to \mathbb{C}$ are defined as above, we see that $\mathbb{Q} \subset F$ is Galois if and only if $\alpha_i \in F$, and in this case $\sigma_i \in \text{Gal}(F/\mathbb{Q})$. This implies that $\mathbb{Q} \subset F$ in Section 5 is never a Galois extension, since there are both real and complex non-real embeddings.

An example of Galois extension is $E \subset F$ in proof of Theorem 5.4, point 2, since $[F : E] = 2$. The non-trivial element of $\text{Gal}(F/E)$ is the complex conjugation.

Multiplication by any $x \in F$ can be viewed as a $\mathbb{Q}$-linear map on $F$: the norm of $x \in F$, denoted by $\text{Nm}(x)$, is the determinant of this linear map. Since the characteristic polynomial $c_x$ of $x$ as a linear map is related to the minimal polynomial $P_x$ by \cite[Proposition 5.40]{Mil08}

\begin{equation}
 c_x = P_x^{[F: \mathbb{Q}(x)]}
\end{equation}

one can prove that, for any $x \in F$, one has \cite[Remark 5.43]{Mil08}

\[ \text{Nm}(x) = \prod_{i=1, \ldots, n} \sigma_i(x) \]

The norm can be defined the same way as above for a field extension $E \subset F$. One obtains this way a map $\text{Nm}_{F/E} : F \to E$.

The norm can be used to distinguish elements of the multiplicative group $\mathcal{O}_F^*$ of units of $\mathcal{O}_F$, using the following result: $x \in \mathcal{O}_F$ is a unit if and only if $\text{Nm}(x) = \pm 1$ \cite[Lemma 5.2]{Mil09}.

A positive unit is a unit which is positive in all real embeddings of $F$: if $n = s + 2t$, with $s$ the number of real roots, and $t$ the number of conjugate pairs of complex roots of $P_\alpha$ (with $\alpha_{s+i}$ the complex conjugate of $\alpha_{s+i+t}$), one defines

\begin{equation}
 \mathcal{O}_F^{*,+} \overset{\text{def}}{=} \{ u \in \mathcal{O}_F^* \text{ such that } \sigma_i(u) > 0 \text{ for } i = 1, \ldots, s \}
\end{equation}

Thus, for any positive unit $u$, one has:

\begin{equation}
 \prod_{i=1, \ldots, n} \sigma_i(x) = 1
\end{equation}

It is a classical fact that the norm takes units to units, and positive units to positive units:

\[ \text{Nm}_{F/E}|_{\mathcal{O}_F^*} : \mathcal{O}_F^* \to \mathcal{O}_E^*, \quad \text{Nm}_{F/E}|_{\mathcal{O}_F^{*,+}} : \mathcal{O}_F^{*,+} \to \mathcal{O}_E^{*,+} \]

The units group $\mathcal{O}_F^*$ is a finitely generated abelian group with rank $s + t - 1$ \cite[Dirichlet Unit Theorem, 5.1]{Mil09}. Its torsion is the set of roots of 1 contained in $\mathcal{O}_F$, so whenever $s > 0$ (that is, $P_\alpha$ has at least one real root), it must be $\{ \pm 1 \}$. Clearly, $\mathcal{O}_F^{*,+}$ doesn’t contain $-1$, so it is free. Moreover, it contains the subgroup $\{ u^2 \text{ such that } u \in \mathcal{O}_F^* \}$, which has finite index in $\mathcal{O}_F^*$. Thus, $\mathcal{O}_F^{*,+}$ has rank $s + t - 1$. 

\[ e \overset{\text{def}}{=} \prod_{g \in \text{Gal}(F/E)} g(x) \in E: \text{ this appears evident from the fact that } e \text{ is fixed by } \text{Gal}(F/E) \text{ and the fact that } E = F^{\text{Gal}(F/E)}. \]
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