Unlimited Budget Analysis of Randomised Search Heuristics∗

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Abstract
Performance analysis of all kinds of randomised search heuristics is a rapidly growing and developing field. Run time and solution quality are two popular measures of the performance of these algorithms. The focus of this paper is on the solution quality an optimisation heuristic achieves, not on the time it takes to reach this goal, setting it far apart from runtime analysis. We contribute to its further development by introducing a novel analytical framework, called unlimited budget analysis, to derive the expected fitness value after arbitrary computational steps. It has its roots in the very recently introduced approximation error analysis and bears some similarity to fixed budget analysis. We present the framework, apply it to simple mutation-based algorithms, covering both, local and global search. We provide analytical results for a number of pseudo-Boolean functions for unlimited budget analysis and compare them to results derived within the fixed budget framework for the same algorithms and functions. There are also results of experiments to compare bounds obtained in the two different frameworks with the actual observed performance. The study show that unlimited budget analysis may lead to the same or more general estimation beyond fixed budget.

Keywords: Randomised search heuristics, performance measures, solution quality, algorithm analysis, working principles of evolutionary computing

1. Introduction
Randomised search heuristics (RSHs), such as simulated annealing and evolutionary algorithms (EAs), are general purpose methods that implement some idea of how search should be conducted to find some desirable solution to a problem. Often they follow a natural paradigm and are inspired by some aspects of this paradigm. For example, considering natural evolution one can arrive at evolutionary algorithms, studying the behaviour of flocks of animals and arrive at particle swarm optimisation, imitating the foraging behaviour of ants and arrive at ant colony optimisation, mimicing the immune system of vertebrates and use artificial immune systems. One possible application domain for these general heuristics is optimisation where one looks for solutions to a maximisation or minimisation problem in a domain that is usually called search space in this context.

While the application of randomised search heuristics is driven by practical needs to find good enough solutions in reasonable time in situations where no good problem-specific algorithm is known there is a growing body of theoretical work that provides insights into how and why these heuristics work or fail. In particular in the area of evolutionary algorithms such analyses have often been concentrated on the aspect of run time, analysing how long an evolutionary algorithm needs to find an optimal solution or a solution with a defined approximation ratio [1].

An alternative perspective is to consider the progress an algorithm makes in a pre-defined number of steps. This is particularly common for algorithms operating on continuous optimisation problems where

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exact optimisation is normally not feasible, for example evolution strategies \cite{2}. One can concentrate on the function value of a solution that can be achieved or on the distance to an optimal solution. The former is done in the fixed budget setting \cite{3}, the latter when analysing the approximation error \cite{4}.

The idea of fixed budget analysis is to consider the performance of an RSH and to derive results about the expected function value that can be achieved with this computational budget. In this paper we present a technique following a similar idea: analysing the distance of the achieved function value to the optimal value depending on the number of computational steps that the algorithm was allowed to take. Since the approach does not consider a computational budget that is fixed in advance and explicitly works for any number of computational steps we call it unlimited budget analysis.

We consider the origin of the method and its relationship to fixed budget analysis in the next section. In Section 3 we introduce the method and present general analytical tools that allow to perform unlimited budget analysis for a range of RSHs on different kinds of optimisation problems. We show how these tools can be applied by using them to derive results for random local search and the \( (1+1) \) EA, in Section 4. We present results for a number of well-known pseudo-Boolean functions. Where available we compare the unlimited budget results to results derived within the fixed budget framework. We also include results of experiments to empirically compare the analytical upper and lower bounds with observed behaviour of the algorithms. We conclude and show directions of possible future research in Section 6.

2. Related work

The framework we introduce has its roots in the recently introduced approximation error analysis Based on the approximation error estimation \cite{5}. Considering a class of elitist evolutionary algorithms He, Chen, and Zhou analyse the underlying Markov chain to prove that the approximation error can always be bounded by a function that is exponential in the number of steps. We follow this idea and prove for algorithms where the underlying random optimisation process corresponds to an homogeneous Markov chain a general upper and lower bound on the expected distance in function value from an optimal solution after an arbitrary number of steps.

The approach bears obvious similarity to fixed budget analysis where the focus of the analysis is also on the function value that can be achieved after a specific number of steps. There are, however, a number of significant differences. The most significant difference is that the starting point of fixed budget analysis is fixing a computational budget \( b \). Only computations of up to \( b \) steps are considered. Fixed budget results hold for any number of steps \( t \leq b \). In principle, \( b \) can be set to arbitrary values but it is recommended to concentrate on budgets that are bounded above by the expected function value \cite{3}. Fixed budget analysis is often carried out in a way that takes into account not only the specific algorithm and objective function but also the computational budget \( b \) itself (see the results for the \((1+1)\) EA on LeadingOnes in \cite{3} for an example). Results using more general methods like drift analysis have also been obtained \cite{6} and there is also a method to transfer results from optimisation time analysis to fixed budget results \cite{7}. However, no general result to derive fixed budget results for a wide class of algorithms and functions is known. This is what is presented in this paper within the framework of unlimited budget analysis.

One key aspect of one of our results is considering the expected relative progress in a single step of the algorithm. Since we consider the quotient between the current and the next expected function value this analysis is similar in spirit to multiplicative drift analysis \cite{8}. Note that multiplicative drift has also been applied to derive results in the fixed budget setting \cite{8}.

So far, existing work on the solution quality of RSHs is still limited and most focuses on the fixed budget performance. Lengler and Spooner \cite{6} analysed the fixed budget performance of the \((1+1)\) EA on linear functions. They adopted two methods, (i) drift analysis and (ii) differential equation plus Chebyshev’s inequality, to derive general results for linear functions and tight fixed budget results for the OneMax function. Nallaperuma, Neumann and Sudholt \cite{9} applied the fixed budget analysis to the well-known Traveling Salesperson problem. They bounded the expected fitness gain of random local search, the \((1+1)\) EA and \((1+\lambda)\) EA within a specified number of generations. Recently, Vinokurov et al. \cite{10} analysed the \((1+1)\)EA with resampling on the OneMax and BinVal problems and obtained some improved fixed budget results on
them. They claimed that for linear functions, the traditional approach via drift analysis cannot easily be extended to the fixed budget performance. As a consequence, it is necessary to investigate other approaches in this field. In the current paper, we consider a completely different approach based on the approximation error estimation [5].

3. Unlimited budget analysis through approximation error estimation

3.1. Randomised search heuristics and mathematical models

A maximisation problem is considered in this paper, which is formalised as follows:

$$\max f(x), \quad \text{subject to } x \in S,$$

where $f : S \rightarrow \mathbb{R}$ is called a fitness function and $S$ is its definition domain. $S$ is a finite state set or a closed set in $\mathbb{R}^n$. Denote $f^* := \max \{f(x); x \in S\}$ and $X^* := \{x \mid f(x) = f^*\}$.

RSHs, described in Algorithm 1, are often applied to the above optimisation problem. An individual $x \in S$ is a single solution and a population $X \subset S$ is a collection of individuals.

Algorithm 1 A randomised search Heuristic

1: population $X^{[0]} \leftarrow$ initialise a population of solutions subject to an initial probability distribution $\Pr(X^{[0]})$ on $S$;
2: for $t = 0, 1, \cdots$ do
3: population $X^{[t+1]} \leftarrow$ generate a new population of solutions subject to a conditional transition probability $\Pr(X^{[t+1]} \mid X^{[0]}, \cdots, X^{[t]});$
4: end for

Definition 1. The fitness of population $X$ is $f(X) := \max \{f(x); x \in X\}$. The fitness of $X^{[t]}$ is denoted by $f(X^{[t]})$ and its expected value by $f^{[t]} := \mathbb{E}[f(X^{[t]})]$.

Besides the fitness value, the approximation error is an alternative measure of solution quality which is defined as follows [1].

Definition 2. The approximation error of $x$ is $e(x) := |f(x) - f^*|$. The approximation error of $X^{[t]}$ is denoted by $e(X^{[t]}) := |f(X^{[t]}) - f^*|$ and its expected value by $e^{[t]} := \mathbb{E}[e(X^{[t]})]$.

From the approximation error $e^{[t]} = |f^{[t]} - f^*|$, we have $f^{[t]} = f^* - e^{[t]}$ for maximisation problem or $f^{[t]} = e^{[t]} - f^*$ for minimisation problem. Both $f^{[t]}$ and $e^{[t]}$ are functions of $t$. They depend on $X^{[0]}$ although we do not explicitly express this dependency in the notation.

Definition 3. A RSH is called convergent in mean if for any initial population $X^{[0]}$,

$$\lim_{t \rightarrow \infty} e^{[t]} = 0, \quad \text{i.e.,} \quad \lim_{t \rightarrow \infty} f^{[t]} = f^*.$$  \hspace{1cm} (2)

Definition 4. A RSH is called elitist if $e(X^{[t+1]}) \leq e(X^{[t]})$ for any $t$, or strictly elitist if $e(X^{[t+1]}) < e(X^{[t]})$ for any $t$.

Two mathematical models are often used in the study of RSHs, which provide necessary mathematical tools for analysing RSHs.

1. Supermartingales. For many RSHs, the expected fitness of $X^{[t+1]}$ is not less than that of $X^{[t]}$. Such a RSH can be modelled by a supermartingale, that is, for any $t$, $\mathbb{E}[e(X^{[t+1]})] \leq e(X^{[t]})$. Any elitist RSH is a supermartingale because $e(X^{[t+1]}) \leq e(X^{[t]})$. A non-elitist EA could be a supermartingale too because the condition $\mathbb{E}[e(X^{[t+1]})] \leq e(X^{[t]})$ does not require $e(X^{[t+1]}) \leq e(X^{[t]})$.

2. Markov chains. For many RSHs, the state of $X^{[t+1]}$ is determined only by $X^{[t]}$. Such a RSH can be modelled by a Markov chain, that is, for any $t$, the conditional probability $\Pr(X^{[t+1]} \mid X^{[0]}, \cdots, X^{[t]} = X^{[t]}$.
3.2. Unlimited budget analysis

Given a sequence \(\{X[t]; t = 0, 1, \cdots\}\), unlimited budget analysis aims to find a bound (lower or upper) on the fitness value \(f[\cdot]\), which is a function of \(t\) satisfying two conditions:

1. the bound holds for any \(t \in [0, +\infty)\);
2. the bound converges to \(f^*\) as \(t \to +\infty\) if \(\lim_{t \to +\infty} f[t] = f^*\).

Because \(f[t] = f^* - e[t]\) for maximisation problem or \(f[t] = e[t] - f^*\) for minimisation problem, unlimited budget analysis is equivalent to the task of finding a bound (lower or upper) on the approximation error \(e[t]\), which is a function of \(t\) satisfying two conditions:

1. the bound holds for any \(t \in [0, +\infty)\);
2. the bound converges to 0 as \(t \to +\infty\) if \(\lim_{t \to +\infty} e[t] = 0\).

Fixed budget analysis \([3]\) does not have the above requirements, so it is different from unlimited budget analysis. This paper focuses on drawing a bound on \(e[\cdot]\) first, then a bound on \(f[\cdot]\). This is different from fixed budget analysis, which aims to estimate a bound on \(f[\cdot]\) directly without considering \(e[\cdot]\).

The emphasis of the current paper is to seek a bound represented by an exponential function such that \(e[t] \leq e[0]X^t\) for an upper bound or \(e[t] \geq e[0]X^t\) for a lower bound. In this case, unlimited budget analysis is a straightforward application of the convergence rate \([11, 5]\) of the error sequence \(\{e[t]; t = 0, 1, \cdots\}\).

**Definition 5.** Given a sequence \(\{e[t]; t = 0, 1, \cdots\}\), its convergence rate at the \(t\)-th generation is

\[
R[t] = \begin{cases} 
\frac{e[t+1]}{e[t]} , & \text{if } e[t] \neq 0, \\
0, & \text{otherwise.} 
\end{cases} 
\]

Its average (geometric) convergence rate for \(t\) generations is

\[
R[t] = \begin{cases} 
1 - \left(\frac{e[0]}{e[t]}\right)^{1/t} , & \text{if } e[0] \neq 0, \\
1, & \text{otherwise.} 
\end{cases} 
\]

In the above definition, in order to understand the convergence rate as the speed in the usual sense, \(R[t]\) normalises the geometric average \(\left(e[t]/e[0]\right)^{1/t}\) to the range \((-\infty, 1]\). The larger \(R[t]\) is, the faster \(e[t]\) converges to 0. A negative value of \(R[t]\) means that \(X[t]\) moves away from the optimum \(X^*\).

It must be indicated that the definition of the convergence rate does not require that an algorithm is convergent or elitist. However, non-convergent algorithms are beyond our interest, therefore, we always assume that algorithms are convergent in the current paper.

If \(e(x[t])\) is a supermartingale, then it is easy to bound \(e[t]\) and \(f[t]\) from the convergence rate of \(e[t]\). The theorem below originates from \([12]\) Theorem 2.

**Theorem 1.** Given an error sequence \(\{e[t]; t = 0, 1, \cdots\}\),

1. if there exists some \(\delta \in [0, 1]\), \(e[t+1]/e[t] \leq 1 - \delta\) for any \(t\), then \(e[t] \leq e[0](1 - \delta)^t\), i.e., \(f[t] \geq f^* - e[0](1 - \delta)^t\);
2. if there exists some \(\delta \in [0, 1]\), \(e[t+1]/e[t] \geq 1 - \delta\) for any \(t\), then \(e[t] \geq e[0](1 - \delta)^t\), i.e., \(f[t] \leq f^* - e[0](1 - \delta)^t\).

**Proof.** We only prove the first claim because the second one can be proven in a similar way. From the condition \(e[t+1]/e[t] \leq (1 - \delta)\), we get \(e[t+1] \leq e[t](1 - \delta)\) and then \(e[t] \leq e[0](1 - \delta)^t\). Equivalently \(f[t] \geq f^* - e[0](1 - \delta)^t\). \(\square\)

If a RSH is modelled by a Markov chain, we can further estimate \(e[t+1]/e[t]\) from one-step transition. For the sake of analysis, the domain \(S\) is assumed to be a finite state set.
Proof. We only prove the lower bound because the upper bound can be proven in a similar way. From the assumption, we get an error sequence \( \{e[t]\} \).

**Corollary 1.** Given an error sequence \( \{e[t]\} \), we can modify it for evaluating the fixed budget performance of RSHs.

**Theorem 2.** Assume that the sequence \( \{X[t]\}; t = 0, 1, \cdots \) is a Markov chain on a finite state set \( S \). Let

\[
\Delta e(X[t]) := \mathbb{E}[e(X[t]) - e(X[t+1]) | X[t] = X].
\]

The average of error change at the \( t \)th generation is defined as

\[
\Delta e[t] := \mathbb{E}[\mathbb{E}[e(X[t]) - e(X[t+1]) | X[t]]].
\]

Then we get an upper bound on the approximation error as

\[
\delta_{\text{min}} = \inf_{t=1,2,\ldots} \min_{X \cap X^{*} = \emptyset} \frac{\Delta e(X[t])}{e(X[t])} = \frac{1}{\delta_{\text{min}}}.
\]

**Theorem 2** provides a method of bounding \( f[t] \) from the convergence rate of the error sequence \( \{e[t]; t = 0, 1, \cdots\} \).

\[
\delta_{\text{max}} = \sup_{t=1,2,\ldots} \max_{X \cap X^{*} = \emptyset} \frac{\Delta e(X[t])}{e(X[t])}.
\]

Then

1. \( e[t] \leq e[0] (1 - \delta_{\text{min}})^{t} \), i.e., \( f[t] \geq f^{*} - e[0] (1 - \delta_{\text{min}})^{t} \);
2. \( e[t] \geq e[0] (1 - \delta_{\text{max}})^{t} \), i.e., \( f[t] \leq f^{*} - e[0] (1 - \delta_{\text{max}})^{t} \).

**Proof.** We only prove the lower bound because the upper bound can be proven in a similar way. From the definition of \( \delta_{\text{min}} \), we have

\[
\frac{e[t+1]}{e[t]} = 1 - \frac{\Delta e[t]}{e[t]} \leq 1 - \min_{X \cap X^{*} = \emptyset} \frac{\Delta e(X[t])}{e(X[t])} = 1 - \delta_{\text{min}}.
\]

Then we get an upper bound on the approximation error as

\[
e[t] \leq e[0] (1 - \delta_{\text{min}})^{t},
\]

and a lower bound on the fitness value as

\[
f[t] \geq f^{*} - e[0] (1 - \delta_{\text{max}})^{t}.
\]

The last inequality uses the equality \( e[t] = f^{*} - f[t] \).

At the end of this section, it is worth mentioning that if \( t \) is restricted to a range \( [0, b] \), then Theorem 2 can be modified for evaluating the fixed budget performance of RSHs.

**Corollary 1.** Given an error sequence \( \{e[t]; t = 0, 1, \cdots\} \),

1. if there exists some fixed budget \( b \) and \( \delta \in [0, 1] \), \( e[t+1]/e[t] \leq 1 - \delta \) for any \( t \leq b \), then \( e[t] \leq e[0] (1 - \delta)^{t} \), i.e., \( f[t] \geq f^{*} - e[0] (1 - \delta)^{t} \) for any \( t \leq b \);
2. if there exists some fixed budget \( b \) and \( \delta \in [0, 1] \), \( e[t+1]/e[t] \geq 1 - \delta \) for any \( t \leq b \), then \( e[t] \geq e[0] (1 - \delta)^{t} \), i.e., \( f[t] \leq f^{*} - e[0] (1 - \delta)^{t} \) for any \( t \leq b \).

4. **Case studies**

The applicability of unlimited budget analysis is demonstrated through several examples of maximising pseudo-Boolean functions.
Algorithm 2 Random local search (RLS)

1: \(x[0] \leftarrow \text{initialise a solution;}
2: \text{for } t = 0, 1, \cdots \text{ do}
3: \quad \text{Onebit Mutation: } y[t+1] \leftarrow \text{choose one bit of } x[t] \text{ at random and flip it;}
4: \quad \text{Elitist Selection: } x[t+1] \leftarrow \text{select the best from } y[t] \text{ and } x[t].
5: \text{end for}

4.1. Randomised search heuristics for pseudo-Boolean functions

Consider the problem of maximising a function,
\[
\max f(x), \quad x \in \{0, 1\}^n.
\]

We introduce two RSHs for the above optimisation problem, which were used in fixed budget perfor-
mance [3]. The first one is a (1+1) EA with 1-bit mutation and elitist selection (Algorithm 2). The algorithm
is often called random local search or RLS in short.

Secondly, we consider a (1+1) EA with bitwise mutation and elitist selection (Algorithm 3). The algo-
rithm is often called the (1+1) EA in short.

Algorithm 3 The (1+1) evolutionary algorithm

1: \(x[0] \leftarrow \text{initialise a solution;}
2: \text{for } t = 0, 1, \cdots \text{ do}
3: \quad \text{Bitwise Mutation: flip each bit of } x[t] \text{ with probability } \frac{1}{n} \text{ and generate } y[t];
4: \quad \text{Elitist Selection: } x[t+1] \leftarrow \text{select the best from } y[t] \text{ and } x[t].
5: \text{end for}

4.2. RLS on OneMax function

In this example, we analyze RLS for maximising the OneMax function,
\[
\max f(x) = |x|, \quad x \in \{0, 1\}^n,
\]
where \(|x| := x_1 + \cdots + x_n\). The optimal solution \(x^* = (1 \cdots 1)\) and the optimal fitness value \(f^* = n\). RLS on
the OneMax function has been studied in [3] by a different method. We show that using unlimited budget
analysis, we can draw the same result as that in [3].

We assume that \(x[t] = x\) such that \(|x[t]| = i\) where integer \(i < n\). The fitness value \(f(x[t]) = i\) and
approximation error \(e(x[t]) = n - i\). The event of \(f(x[t+1]) > f(x[t])\) happens if one of \(n - i\) zero-valued bit
in \(x[t]\) is flipped. The probability of this event is \((n-i)/n\).

Thus, the average of error change is
\[
\Delta e(x[t]) = \frac{n - i}{n},
\]
and the ratio of error change is
\[
\frac{\Delta e(x[t])}{e(x[t])} = \frac{n - i}{n(n - 1)} = \frac{1}{n}.
\]

Then we get the approximation error as
\[
e[t] = \left(1 - \frac{1}{n}\right)^t e[0].
\]
\[6\]
and the fitness value as
\[ f[t] = n - \left(1 - \frac{1}{n}\right)^t e[0]. \] (17)

The above result is the same as that in [3]. It is an exact expression of \( f[t] \) and holds for all \( x[0] \). We compare the derived bound on \( e[t] \) with the experimental result and present it in Figure 1.

![Figure 1: Observed mean \( e[t] \) over 100 independent runs of RLS with random initialisation on OneMax with \( n = 200 \) and the bound (16).](image)

4.3. RLS on linear functions

Linear functions are a family of functions widely used in the theoretical study of RSHs [13]. An instance of linear functions is described as
\[ \max f(x) = \sum_{i=1}^{n} c_i x_i, \quad \text{where } c_i > 0. \] (18)

The optimal solution \( x^* = (1 \ldots 1) \) and the optimal fitness value \( f^* = \sum_{i=1}^{n} c_i \). The (1+1) EA on linear functions has been analysed in [6] using fixed budget analysis. For linear functions, the range of their coefficients can be chosen arbitrarily large or small, so it is difficult to draw a bound on \( f[t] \) directly. However, it is quite simple to draw a bound on \( e[t] \) using unlimited budget analysis. We show it in this section.

We consider RLS on linear functions. We assume that \( x[t] \) is a non-optimal solution such that
\[ x_i[t] = \begin{cases} 1, & \text{if } i \in I, \\ 0, & \text{otherwise}, \end{cases} \] (19)

where \( I \) is a subset of \( \{1, \ldots, n\} \) with \( |I| < n \). The approximation error \( e[t] = \sum_{i \notin I} c_i \). The event of \( f(x[t+1]) > f(x[t]) \) happens if one bit \( x_i[t] \notin I \) is flipped. The probability of this event is \( 1/n \).

The average of error change (over all bits \( j \notin I \)) equals to
\[ \Delta e(x[t]) = \sum_{j \notin I} c_j \frac{1}{n}. \] (20)

The ratio of error change equals to
\[ \frac{\Delta e(x[t])}{e(x[t])} = \frac{\sum_{j \notin I} c_j \frac{1}{n}}{\sum_{j \notin I} c_j} = \frac{1}{n}. \] (21)
Then we get the approximation error as

\[ e[t] = e[0] \left( 1 - \frac{1}{n} \right)^t, \quad (22) \]

and the fitness value as

\[ f[t] = f^* - e[0] \left( 1 - \frac{1}{n} \right)^t. \quad (23) \]

Notice that bound (23) is the same as (17). Since OneMax is a special case of linear functions, the above results generalise (16) and (17) in section 4.2. Surprisingly variant coefficients do not affect the formula.

\[ (22) \] is an exact expression on \( e[t] \) for any \( x[0] \). We compare the derived upper bound (22) on \( e[t] \) with the experimental result on linear function \( f(x) = \sum_{i=1}^{n} i x_i \) and present it in Figure 2.

![Figure 2: Observed mean \( e[t] \) over 100 independent runs of RLS with random initialisation on linear function \( f(x) = \sum_{i=1}^{n} i x_i \) with \( n = 100 \) and upper bound (22).](image)

4.4. \((1+1)\) EA on linear functions (1): upper bound on approximation error

The analysis of the \((1+1)\) EA on linear function is similar to that of RLS. We assume that \( x[t] \) is a non-optimal solution such that

\[ x_i[t] = \begin{cases} 1, & \text{if } i \in I, \\ 0, & \text{otherwise}, \end{cases} \quad (24) \]

where \( I \) is a subset of \( \{1, \cdots, n\} \) with \( |I| < n \). For the \((1+1)\) EA, the event of \( f(x[t+1]) > f(x[t]) \) happens if one bit \( x_j[t] \notin I \) is flipped and other bits are unchanged. The probability of this event is \( \frac{1}{n}(1 - \frac{1}{n})^{n-1} \).

The average of error change (over all bits \( j \notin I \)) satisfies

\[ \Delta e(x[t]) \geq \sum_{j \notin I} c_j \frac{1}{n}(1 - \frac{1}{n})^{n-1}. \quad (25) \]

The ratio of error change satisfies

\[ \frac{\Delta e(x[t])}{e(x[t])} = \frac{\sum_{j \notin I} c_j \frac{1}{n}(1 - \frac{1}{n})^{n-1}}{\sum_{j \notin I} c_j} = \frac{1}{n}(1 - \frac{1}{n})^{n-1}. \quad (26) \]

Then we get the approximation error as

\[ e[t] \leq e[0] \left( 1 - \frac{1}{n}(1 - \frac{1}{n})^{n-1} \right)^t. \quad (27) \]

and the fitness value as

\[ f[t] \geq f^* - e[0] \left( 1 - \frac{1}{n}(1 - \frac{1}{n})^{n-1} \right)^t. \quad (28) \]
Compared with existing fixed budget analysis of linear functions \[6\], unlimited budget analysis is simple and straightforward.

(27) is an upper bound on \(e[t]\) for any \(x[0]\) and is reached at \(|x[0]| = n - 1\). Again, we compare the derived upper bound (27) on \(e[t]\) with the experimental result on a linear function (called BinVal) \(f(x) = \sum_{i=1}^{n} 2^i x_i\) and present it in Figure 3.

![Figure 3](image)

Figure 3: Observed mean \(e[t]\) over 100 independent runs of the (1+1) EA with random initialisation on the BinVal function with \(n = 100\) and the upper bound (27).

4.5. (1+1) EA on linear functions (2): lower bound on approximation error

Following a procedure similar to the analysis of the upper bound on \(e[t]\), we derive a lower bound on \(e[t]\). We assume that \(x[t]\) is a non-optimal solution such that

\[
x_i[t] = \begin{cases} 1, & \text{if } i \in I, \\ 0, & \text{otherwise,} \end{cases}
\]

where \(I\) is a subset of \(\{1, \ldots, n\}\) with \(|I| < n\). For the (1+1) EA, the event of \(f(x[t+1]) > f(x[t])\) can be split into several mutually exclusive sub-events:

1. one bit \(x_j[t] \notin I\) is flipped and other bits are unchanged. The probability of this event is \(\frac{1}{n}(1 - \frac{1}{n})^{n-1}\);
2. two mutually different bits \(x_j[t], x_j[t] \notin I\) are flipped and other bits are unchanged. The probability of this event is \(\frac{1}{n^2}(1 - \frac{1}{n})^{n-2}\);
3. three mutually different bits \(x_j[t], x_j[t], x_j[t] \notin I\) are flipped and other bits are unchanged. The probability of this event is \(\frac{1}{n^3}(1 - \frac{1}{n})^{n-3}\);
4. and so on.

The average of error change (over all bits \(\notin I\)) satisfies

\[
\Delta e(x[t]) = \sum_{j \notin I} c_j \left(1 - \frac{1}{n}\right)^{n-1} + \sum_{j_1, j_2 \notin I; j_1 \neq j_2} (c_{j_1} + c_{j_2}) \frac{1}{n^2} \left(1 - \frac{1}{n}\right)^{n-2} + \cdots
\]

The ratio of error change satisfies

\[
\frac{\Delta e(x[t])}{e(x[t])} = \frac{\sum_{j \notin I} c_j \left(1 - \frac{1}{n}\right)^{n-1} + \sum_{j_1, j_2 \notin I; j_1 \neq j_2} (c_{j_1} + c_{j_2}) \frac{1}{n^2} \left(1 - \frac{1}{n}\right)^{n-2} + \cdots}{\sum_{j \notin I} c_j} \leq \frac{1}{n}.
\]

Then we get the approximation error as

\[
e[t] \geq e[0] \left(1 - \frac{1}{n}\right)^t,
\]

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and the fitness value as
\[ f[t] \leq f^{*} - e^{[0]} \left(1 - \frac{1}{n}\right)^t. \] (33)

Combining the upper bound (27) and lower bound (32) together, we obtain tight bounds for all linear functions as
\[ e^{[0]} \left(1 - \frac{1}{n}\right)^t \leq e^{[t]} \leq e^{[0]} \left(1 - \frac{1}{n}\right)^{n-1} \left(1 - \frac{1}{n}\right)^t. \] (34)

We compare the derived upper bound (32) on \( e^{[t]} \) with the experimental result on BinVal function \( f(x) = \sum_{i=1}^{n} 2^{x_i+1} \) and present it in Figure 4.

\[
\begin{align*}
\text{Figure 4: Observed mean } & e^{[t]} \text{ over 100 independent runs of the (1+1) EA with random initialisation on BinVal function with } n = 100 \text{ and the lower bound (32).}
\end{align*}
\]

4.6. RLS on Square function: scaling OneMax

It is obvious that scaling a function changes the fitness value and then will affect the bounding on the fitness value. This is different from the running time analysis. Scaling a function usually doesn’t change the running time of an EA. Scaling might cause a trouble if we bound \( f[t] \) directly. However, if we turn to estimate the convergence rate \( e^{[t+1]} / e^{[t]} \), scaling doesn’t bring a big trouble. We show this by an example.

Consider the problem of maximising the square function which scales the OneMax function,
\[ \text{max } f(x) = |x|^2, \quad x \in \{0,1\}^n. \] (35)

The optimal solution \( x^* = (1 \cdots 1) \) and the optimal fitness value \( f^* = n^2 \).

We set \( x^{[0]} \) to be a non-optimal solution such that \( |x^{[0]}| < n \). The approximation error \( e(x^{[0]}) = n^2 - |x^{[0]}|^2 \). Because of elitist selection, \( |x^{[t]}| \geq |x^{[0]}| \) is true for any \( t \geq 0 \).

We assume that \( x^{[t]} \) is a non-optimal solution such that \( |x^{[t]}| = i \) where integer \( i \) satisfies: \( |x^{[0]}| \leq i < n \). The event of \( f(x^{[t+1]}) > f(x^{[t]}) \) happens if one of \( n-i \) zero-valued bits in \( x^{[t]} \) is flipped. The probability of this event is \( (n-i)/n \).

The average of error change satisfies
\[ \Delta e(x^{[t]}) \geq \frac{n-i}{n} \cdot [(i+1)^2 - i^2]. \] (36)

The ratio of error change satisfies
\[ \frac{\Delta e(x^{[t]})}{e(x^{[t]})} \geq \frac{n-i}{n} \cdot \frac{(i+1)^2 - i^2}{n^2 - i^2} = \frac{2i + 1}{n(n+i)}, \] (37)
Because $i = |x^{[t]}| \geq |x^{[0]}|$, the ratio of error change is lower-bounded by

$$\frac{\Delta e(x^{[t]})}{e(x^{[t]})} \geq \frac{2|x^{[0]}| + 1}{n(n + |x^{[0]}|)}$$

Then we get an upper bound on the approximation error as

$$e^{[t]} \leq e^{[0]} \left( 1 - \frac{2|x^{[0]}| + 1}{n(n + |x^{[0]}|)} \right)^t$$

and a lower bound on the fitness value as

$$f^{[t]} \geq f^* - e^{[0]} \left( 1 - \frac{2|x^{[0]}| + 1}{n(n + |x^{[0]}|)} \right)^t.$$  \hspace{1cm} (38)

The bound (39) on the square function is different from the bound (17) on the OneMax function. This reveals that scaling affects the bound on the fitness value.

(16) is a upper bound on $e^{[t]}$ which holds for any $x^{[t]}$. Again, we compare the derived bound (16) on $e^{[t]}$ with the experimental result and present it in Figure 5.

![Figure 5: Observed mean $e^{[t]}$ over 100 independent runs of RLS with random initialisation on the square function with $n = 100$ and upper bound (38).](image)

### 4.7. (1+1) EA on LeadingOnes function (1): upper bound on approximation error

In this example, we show an advantage of unlimited budget analysis over fixed budget analysis, that is, a bound holds on all $t$. Consider the (1+1) EA for maximising the LeadingOnes function.

$$\max f(x) = \sum_{i=1}^{n} \prod_{j=1}^{i} x_j, \quad x \in \{0, 1\}^n.$$  \hspace{1cm} (40)

The optimal solution $x^* = (1 \cdots 1)$ and the optimal fitness value $f^* = n$. This problem has been analysed in [3] using fixed budget analysis. According to [3], a bound on $f^{[t]}$ is given as follows: if $x^{[0]}$ is chosen uniformly at random, $t = (1 - \beta)n^2/\alpha(n)$ for any $\beta$ with $(1/2) + \beta' < \beta < 1$ where $\beta'$ is a positive constant and $\alpha(n) = \omega(1)$, $\alpha(n) \geq 1$, then

$$f^{[t]} = 1 + \frac{2t}{n} - o\left( \frac{t}{n} \right),$$

$$e^{[t]} = n - 1 - \frac{2t}{n} + o\left( \frac{t}{n} \right).$$  \hspace{1cm} (41)

(41) is true for a small $t$. But the formula is obviously incorrect for a large $t$ because $\lim_{t \to +\infty} 1 + \frac{2t}{n} - o\left( \frac{t}{n} \right) = +\infty$ contradicting with $f(x) \leq n$. 

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In this paper, we present a new lower bound on $f[t]$ which is true for any $t$ and identical to (41) if $t = o(n^2)$. Similar to [3] Theorem 13, we assume that the initial solution $x[0]$ is chosen uniformly at random. Then we can draw a simple lemma as follows.

**Lemma 1.** Assume the initial solution $x[0]$ is chosen uniformly at random, that is, $Pr(x[0] = 1) = 1/2$. Then for any $t \geq 0$ and any $x[t] = (1 \cdots 1,0 \cdots *)$ where $i \in \{0, \cdots, n\}$ and $* \in \{0,1\}$ is a random variable, it holds $Pr(*) = 1/2$.

**Proof.** We prove it by induction. Because $x[0]$ is chosen uniformly at random, we know $Pr(x[0] = 1) = 1/2$ for any $i \in \{1, \cdots, n\}$. We assume that the lemma is true at some $t \geq 0$, that is, $x[i] = (1 \cdots 1,0 \cdots *)$ where $i \in \{0, \cdots, n\}$ and $Pr(x[i] = 1) = 1/2$ for any $j \geq i + 1$. Now we let $x[t+1] = (1 \cdots 1,0 \cdots *)$. Thanks to elitist selection, it holds where $k \geq i$. For each bit $x[j]$ such that $j \geq k + 1$, the change (0 $\rightarrow$ 1 or 1 $\rightarrow$ 0) by bitwise mutation makes no contribution to the value of $f(x[t+1])$. Thus

$$Pr(x[j] = 1) = Pr(x[j] = 1 | x[j] = 1) + Pr(x[j] = 1 | x[j] = 0) = 1/2.$$ 

This means the lemma is true at $t + 1$. By induction, the lemma is proven. □

We assume that $x[i]$ is a non-optimal solution such that $x[i] = (1 \cdots 1,0_{i+1} \cdots *)$ for some $i \in \{0, \cdots, n - 1\}$. According to the above lemma, $Pr(*) = 1/2$.

- **Case 1:** $i < n - 1$. We have

$$Pr(x[t+1] = (1 \cdots 1, i+2 \cdots *) | x[t]) = (1 - \frac{1}{n})^i \frac{1}{n} \times \frac{1}{2^i}, \quad (43)$$

The formula is explained as follows. The first $i$ one-valued bits are unchanged with probability $(1 - 1/n)^i$. The $(i + 1)$th zero-valued bit is flipped to one-valued with probability $1/n$. The flipping of bits labelled by $*$ doesn’t affect the fitness value. From bitwise mutation, the $(i + 1)$th bit satisfies $Pr(x_{t+1,i+2} = 0 | x[i]_{i+2} = 0) = 1 - 1/n$ and $Pr(x_{t+1,i+2} = 0 | x[i]_{i+2} = 1) = 1/n$. According to Lemma 1, $Pr(x[i]_{i+2} = *) = 1/2$, then we get $Pr(x_{t+1,i+2} = 0 | x[i]_{i+2} = *) = 1/2$.

Following a similar argument, we draw that

$$Pr(x[t+1] = (1 \cdots 1, i+2 \cdots *) | x[t]) = (1 - \frac{1}{n})^i \frac{1}{n} \times \frac{1}{2^i}, \quad (44)$$

The average of error change is

$$\Delta e(x[t]) = \sum_{j=1}^{n-i} j \cdot Pr(x[t+1] = (1 \cdots 1, j+1 \cdots *) | x[t]) = \frac{1}{n} (1 - \frac{1}{n})^i \left( \frac{1}{2^i} + \frac{3}{2^i} + \cdots + \frac{n-i-1}{2^i} \right) = \Theta \left( \frac{1}{n} \right). \quad (45)$$

Then the ratio of error change is

$$\frac{\Delta e(x[t])}{e(x[t])} = \frac{1}{n} \left( 1 - \frac{1}{n} \right)^i \left( \frac{1}{2^i} + \frac{3}{2^i} + \cdots + \frac{n-i-1}{2^i} \right) \frac{1}{n-i}. \quad (46)$$

Since for a large $n$ (say $n \geq 100$),

$$\left( 1 - \frac{1}{n} \right)^i \left( \frac{1}{2} + \frac{3}{2^2} + \cdots + \frac{n-i-1}{2^{i-1}} \right) \frac{1}{n-i} \geq \left( \frac{1}{2} + \frac{3}{2^2} + \cdots + \frac{n-1}{2^{n-1}} \right) \frac{1}{n}, \quad (47)$$

the ratio of error change is lower-bounded by

$$\frac{\Delta e(x[t])}{e(x[t])} \geq \left( \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^2} + \cdots + \frac{n-1}{2^{n-1}} \right) \frac{1}{n^2} = \left( 2 - o \left( \frac{1}{n} \right) \right) \frac{1}{n^2}. \quad (48)$$

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• **Case 2:** $i = n - 1$. In this case, $x^{[t]} = (1 \cdots 10)$.

$$\Pr(x^{[t+1]} = (1 \cdots 1) \mid x^{[t]}) = \left(1 - \frac{1}{n}\right)^{n-1} \frac{1}{n}.$$  \hspace{1cm} (49)

The average of error change is lower-bounded by

$$\Delta e(x^{[t]}) = \left(1 - \frac{1}{n}\right)^{n-1} \frac{1}{n}. $$  \hspace{1cm} (50)

Then the ratio of error change is

$$\frac{\Delta e(x^{[t]})}{e(x^{[t]})} = \left(1 - \frac{1}{n}\right)^{n-1} \frac{1}{n}. $$  \hspace{1cm} (51)

Summarising the two cases, we get

$$\frac{\Delta e(x^{[t]})}{e(x^{[t]})} \geq \left(2 - o\left(\frac{1}{n}\right)\right) \frac{1}{n^2}. $$  \hspace{1cm} (52)

Then we get an upper bound on the approximation error as

$$e^{[t]} \leq e^{[0]} \left(1 - \frac{2}{n^2} + o\left(\frac{1}{n^3}\right)\right)^t,$$  \hspace{1cm} (53)

and a lower bound on the fitness value as

$$f^{[t]} \geq f^* - e^{[0]} \left(1 - \frac{2}{n^2} + o\left(\frac{1}{n^3}\right)\right)^t.$$  \hspace{1cm} (54)

Now we estimate $e^{[0]}$. Because each bit in $x^{[0]}$ is set to 0 or 1 uniformly at random, we have for $i = 0, \cdots, n$,

$$\Pr(x^{[0]} = (1 \cdots 1, 0 \cdots *)) = \left(\frac{1}{2}\right)^{i+1}.$$  \hspace{1cm} This

Thus the initial fitness value

$$f^{[0]} = \sum_{i=1}^{n} i \Pr(x^{[0]} = (1 \cdots 1, 0 \cdots *)) = \sum_{i=1}^{n} i \left(\frac{1}{2}\right)^{i+1} = 1 - o\left(\frac{1}{n}\right),$$

and the initial error

$$e^{[0]} = n - 1 + o\left(\frac{1}{n}\right). $$  \hspace{1cm} (55)

Then we get a lower bound on the fitness value as

$$f^{[t]} \geq n - \left(1 - \frac{2}{n^2} + o\left(\frac{1}{n^3}\right)\right)^t \left(n - 1 + o\left(\frac{1}{n}\right)\right).$$  \hspace{1cm} (56)

Thus we get a lower bound on $f^{[t]}$ which holds for any $t$.

Next we draw a linear approximation of (56) under the condition $t = o(n^2)$. This condition implies $\frac{\epsilon^2}{n^2} = \frac{\epsilon^2}{n^2} o(1)$. From the binomial theorem, we get

$$\left(1 - \frac{2}{n^2} + o\left(\frac{1}{n^3}\right)\right)^t = 1 - \frac{2t}{n^2} + \frac{2t}{n^2} o(1),$$  \hspace{1cm} (57)
Then a linear approximation of (56) is given as

\[ f(t) \geq 1 + \frac{2t}{n} - o\left(\frac{t}{n} \right) - o\left(\frac{1}{n} \right). \]  

(58)

This lower bound is the same as (41). But it must be mentioned that it holds under the condition \( t = o(n^2) \). Thus, the bound (53) derived from unlimited budget is an extension of the bound (41) derived from fixed budget.

We compare the derived bound (53) on \( e^t \) with the experimental result and present it in Figure 6.

![Figure 6: Observed mean \( e^t \) over 100 independent runs of the (1+1) EA with random initialisation on LeadingOnes with \( n = 100 \) and upper bound (53).](image)

4.8. (1+1) EA on LeadingOnes function (2): general lower bound on approximate error

In this example, we further estimate a lower bound on \( e^t \) for the (1+1) EA on LeadingOnes. We still assume that \( x^0 \) is chosen uniformly at random. Let \( x^t \) be a non-optimal solution such that \( x^t = (1 \cdots 1, 0_{i+1} \cdots 0) \) for some \( i \in \{0, \cdots, n-1\} \). According to Lemma 1, \( \Pr(s = 1) = 1/2 \).

- **Case 1**: \( i < n - 1 \). The ratio of error change is given by (46) as

\[ \frac{\Delta e(x^t)}{e(x^t)} = \frac{1}{n} \left(1 - \frac{1}{n}\right)^i \left(1 + \frac{2}{2^2} + \cdots + \frac{n-i-1}{2^{n-i-1}}\right) \frac{1}{n-i}. \]  

(59)

Since for a large \( n \) (say \( n \geq 100 \)),

\[ \left(1 - \frac{1}{n}\right)^i \left(1 + \frac{2}{2^2} + \cdots + \frac{n-i-1}{2^{n-i-1}}\right) \frac{1}{n-i} \leq \left(1 - \frac{1}{n}\right)^{n-2} \frac{1}{4}. \]  

(60)

the ratio of error change is upper-bounded by

\[ \frac{\Delta e(x^t)}{e(x^t)} \leq \left(1 - \frac{1}{n}\right)^{n-2} \frac{1}{4n}. \]  

(61)

- **Case 2**: \( i = n - 1 \). The ratio of error change is given by (51) as

\[ \frac{\Delta e(x^t)}{e(x^t)} = \left(1 - \frac{1}{n}\right)^{n-1} \frac{1}{n}. \]  

(62)

Summarising the two cases, we get

\[ \frac{\Delta e(x^t)}{e(x^t)} \leq \left(1 - \frac{1}{n}\right)^{n-1} \frac{1}{n} \leq \frac{1}{en}. \]  

(63)
Then from (55), we get a lower bound on \(e^t\) as
\[
e^t \geq e^0 \left(1 - \frac{1}{en}\right)^t = \left(n - 1 + o\left(\frac{1}{n}\right)\right) \left(1 - \frac{1}{en}\right)^t, \tag{64}
\]
and an upper bound on \(f^t\) as
\[
f^t \leq f^* - e^0 \left(1 - \frac{1}{en}\right)^t = n - \left(1 - \frac{1}{en}\right)^t \left(n - 1 + o\left(\frac{1}{n}\right)\right), \tag{65}
\]
The above bounds hold for any \(t \geq 0\). We compare the derived lower bound (64) on \(e^t\) with the experimental result and present it in Figure 7. The bound is tight for \(t \geq 10000\) but not for \(t \leq 10000\). This shows the limitation of Theorem 1 which only gives a bound represented by an exponential function as \(c\lambda^t\). But such an expression is too simple to bound the approximation error for some problems and algorithms. According to the theory of RSHs’ approximation error [5], for a RSH corresponding to a homogeneous Markov chain, its error \(e^t\) is represented by a function as
\[
\sum_i \sum_m c_{im} \left(\frac{t}{l_{i,m}}\right)^{-m+1}, \tag{66}
\]
where some coefficients \(c_{im}\) and binomial coefficients \(\binom{t}{l_{i,m}}\). In many cases, \(e^t\) can be simply represented by the sum of several exponential function as \(\sum_i c_i \lambda_i^t\). This paper will not discuss this topic in depth. This general theory can be referred to He, Chen and Zhou’s paper [5].

![Figure 7: Observed mean \(e^t\) over 100 independent runs of the \((1+1)\) EA with random initialisation on LeadingOnes with \(n = 100\) and lower bound (64).](image)

4.9. \((1+1)\) EA on LeadingOnes function (3): tight lower bound on approximate error within fixed budget

Corollary [4] provides a method to evaluate the fixed budget performance of RSHs. Now we apply it to the \((1+1)\) EA on LeadingOnes. We show we can draw the same tight upper bound on \(f^t\) as that in (41) with a fixed budget \(b = o(n^2)\).

First we estimate the ratio of error change. Define set \(S_{o(n)} := \{x \mid x = (1 \cdots 1, 0 \cdots \ast), i = o(n)\}\). For any \(x = (1 \cdots 1, 0_{1+i+1} \cdots \ast) \in S_{o(n)}\), from [46], we know that the ratio of error change is
\[
\frac{\Delta e(x)}{e(x)} = \frac{1}{n} \left(1 - \frac{1}{n}\right)^i \left(1 - \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n-i-1}{2^{n-i-1}}\right) \frac{1}{n-i}. \tag{67}
\]
For a large \(n\) (say \(n \geq 100\)) and \(|x| = o(n)\),
\[
\frac{\Delta e(x)}{e(x)} = \frac{1}{n} \left(2 - o\left(\frac{1}{n}\right)\right) \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right) = \frac{2}{n^2} - o\left(\frac{1}{n^2}\right). \tag{68}
\]
Next we prove a claim, that is, for the fixed budget \( b = o(n^2) \), \( x^b \) is in the set \( \mathcal{S}_{o(n)} \) with a large probability, i.e., \( \Pr(x^b \in \mathcal{S}_{o(n)}) > 1 - o(1) \).

Let \( g(n) = n^2/b \), then

\[
\lim_{n \to \infty} g(n) = +\infty, \quad \lim_{n \to +\infty} \ln g(n) = +\infty, \quad \lim_{n \to +\infty} \frac{g(n)}{n} = +\infty.
\]

Because the error change at each step is \( \Delta e(x^t) = \Theta(1/n) \), we have

\[
\mathbb{E}[f(x^b)] = b \times \Theta\left(\frac{1}{n}\right).
\]

According to Markov inequality, we get

\[
\Pr(f(x^b) \geq \frac{n}{\ln g(n)}) \leq \frac{\mathbb{E}[f(x^b)]}{n} \leq \frac{b}{n} \Theta(1) \frac{n}{\ln g(n)} = \frac{\ln g(n)}{g(n)} \Theta(1) = o(1). \tag{69}
\]

This means \( \Pr(f(x^b) < \frac{n}{\ln g(n)}) = 1 - o(1) \), Because \( \frac{n}{\ln g(n)} = o(n) \), we have \( \Pr(x^b \in \mathcal{S}_{o(n)}) = 1 - o(1) \).

Thus, \( x^{[0]} , \ldots , x^b \in \mathcal{S}_{o(n)} \) with probability \( 1 - o(1) \).

From (68) and with probability \( 1 - o(1) \), we get a lower bound on \( e^t \) as

\[
e^t = e^{[0]} \left(1 - \frac{2}{n^2} + o\left(\frac{1}{n^2}\right)\right)^t = e^{[0]} \left(1 - \frac{2}{n^2} + o\left(\frac{1}{n^2}\right)\right)^t,
\]

and an upper bound on \( f^t \) as

\[
f^t = n - e^{[0]} \left(1 - \frac{2}{n^2} + o\left(\frac{1}{n^2}\right)\right)^t. \tag{72}
\]

Recall that \( e^{[0]} = (n - 1 + o\left(\frac{1}{n}\right)) \).

Since \( t \leq b = o(n^2) \), we have \( \frac{t^2}{n^2} = \frac{1}{n^2} o(1) \). According to the binomial theorem, we get the following approximation of \( e^t \) and \( f^t \).

\[
e^t = e^{[0]} \left(1 - \frac{2t}{n^2} + o\left(\frac{2t}{n^2}\right)\right), \tag{73}
\]

\[
f^t = 1 + \frac{2t}{n} - o\left(\frac{1}{n}\right) - o\left(\frac{t}{n}\right). \tag{74}
\]

This expression is the same as (41) with fixed budget performance (52). Therefore, Corollary 1 can be used to estimate the fitness value \( e^t \) as fixed budget analysis does. Figure 8 compares the observed mean \( e^t \) and the fixed budget bound (73) on \( e^t \). It is clear that the bound is tight for small \( t \leq 1000 \), but it is useless for large \( t \geq 5000 \) because the bound is negative. The approximation error is always non-negative. This is the limitation of fixed budget performance.

4.10. (1+1) EA on OneMax function: different mutation rates

The choice of algorithm parameters has a great influence on the performance of a RSH. In this example, we study the mutation rate \( p \) in bitwise mutation, that is to flip each bit with probability \( p \) where \( p \in (0,1) \). For the \((1+1)\) EA on OneMax, we show that \( p = 1/n \) is the best choice with which the upper bound on \( e^t \) is the smallest.

We set \( x^{[0]} \) to be a non-optimal solution such that \( |x^{[0]}| < n \). Assume that \( x^t \) is a non-optimal solution such that \( |x^t| = i \) where \( i < n \). The approximation error \( e(x^t) = n - i \). The event of \( f(x^{[t+1]}) > f(x^t) \) happens if one of \( n - i \) zero-valued bits in \( x^t \) is flipped and other bits are unchanged. The probability of this event is \( (\frac{n-i}{n}) p (1-p)^{n-1} \).
Figure 8: Observed mean $e^{[t]}$ over 100 independent runs of the (1+1) EA with random initialisation on LeadingOnes with $n = 100$ and bound (73) on $e^{[t]}$.

The average of error change satisfies

$$\Delta e(x^{[t]}) \geq \left(\frac{n - i}{1}\right)p(1-p)^{n-1}. \quad (75)$$

The ratio of error change satisfies

$$\frac{\Delta e(x^{[t]})}{e(x^{[t]})} \geq \frac{(n - i)p(1-p)^{n-1}}{n - i} = p(1-p)^{n-1}. \quad (76)$$

The above lower bound is reached when $i = n - 1$. When $i = n - 1$, it means $x^{[t]}$ includes only one zero-valued bit. The event of $f(x^{[t+1]}) > f(x^{[t]})$ happens if and only if this unique zero-valued bit in $x^{[t]}$ is flipped and other bits are unchanged. The probability of this event equals $p(1-p)^{n-1}$. In the ratio of error change equals to

$$\frac{\Delta e(x^{[t]})}{e(x^{[t]})} = \frac{p(1-p)^{n-1}}{1} = p(1-p)^{n-1}. \quad (77)$$

Thus the minimal ratio of error change is

$$\delta_{\min}(p) := \min\{\frac{\Delta e(x)}{e(x)}; |x| < n\} = p(1-p)^{n-1}. \quad (80)$$

Then we get an upper bound on the approximation error as

$$e^{[t]} \leq e^{[0]}(1 - \delta_{\min}(p))^t = e^{[0]}(1 - p(1-p)^{n-1})^t, \quad (78)$$

and a lower bound on the fitness value as

$$f^{[t]} \geq f^* - e^{[0]}(1 - p(1-p)^{n-1})^t. \quad (79)$$

(78) and (79) hold for all $x^{[0]}$ and they are reached when $|x^{[t]}| = n - 1$.

Now we find the value of $p$ such that

$$\max\{\delta_{\min}(p); p \in (0, 1)\}. \quad (80)$$

The derivative of the function $\delta_{\min}(p)$ is

$$\delta'_{\min}(p) = (1-p)^{n-1} - p(1-p)^{n-2}(n-1).$$
Let $\delta'_{\min}(p) = 0$, then we get $p = \frac{1}{n}$. Since $\delta''_{\min}(p) \leq 0$ for $p \in (0, 1)$, we know $\delta'_{\min}(p)$ takes the maximal value at $p = 1/n$. Then the best upper bound on $e^{[t]}$ is achieved at $p = 1/n$ as

$$e^{[t]} \leq e^{[0]} \left( 1 - \frac{1}{n} (1 - \frac{1}{n})^{n-1} \right)^t.$$

Figure 9 shows the observed value of $e^{[t]}$ in computational experiments for mutation rates $p = 1/n, 2/n, 1/(2n)$ respectively. Experimental results confirm that $p = 1/n$ is the best one among the three mutation rates in terms of the approximation error.

![Figure 9](image-url)

**5. Discussions**

5.1. Unlimited budget analysis with multi-step transition probabilities

In unlimited budget analysis, it is possible to improve lower and upper bounds on $e^{[t]}$ using multi-step transition probabilities. For the sake of illustration, only the 2-step transition is considered here. However, the idea is applicable to any multi-step transition.

**Theorem 3.** Given an error sequence $\{e^{[t]}; t = 0, 1, \cdots \}$,

1. if there exists some $\delta \in [0, 1]$, $e^{[t+2]}/e^{[t]} \leq 1 - \delta$ for any $t$, then $e^{[2t]} \leq e^{[0]}(1 - \delta)^t$;
2. if there exists some $\delta \in [0, 1]$, $e^{[t+2]}/e^{[t]} \geq 1 - \delta$ for any $t$, then $e^{[2t]} \geq e^{[0]}(1 - \delta)^t$.

**Proof.** We only prove the first claim. The second one can be proven in a similar way. From the condition $e^{[t+2]}/e^{[t]} \leq 1 - \delta$, we get $e^{[t+2]} \leq e^{[t]}(1 - \delta)$ and then $e^{[2t]} \leq e^{[0]}(1 - \delta)^t$. Then $e^{[2t]} \leq e^{[0]}(1 - \delta)^t$. \hfill $\square$

If the domain $\mathcal{S}$ is a finite state set and $X^{[t]}$ is a Markov chain, then we can estimate the lower bound on $f^{(t)}$ based on the average of error change in two generations.

**Definition 7.** The average of error change in two generations at $X^{[t]} = X$ is defined as

$$\Delta^2 e(X^{[t]}) := \mathbb{E}[e(X^{[t]}) - e(X^{[t+2]}) | X^{[t]} = X].$$

The average of error change in two generations at the $t$th generation is defined as

$$\Delta^2 e^{[t]} := \mathbb{E}[e(X^{[t]}) - e(X^{[t+2]}) | X^{[t]}].$$
Proof. We only prove the first conclusion because the second one can be proven in a similar way. Without Theorem 5.

Assume that the sequence \(e^{[t]}\) satisfies:

\[
e^{[2t]} \leq e^{[0]} \left(1 - \inf_{t \in \{1, 2, \ldots\}} \min_{X : X \cap X^* = \emptyset} \frac{\Delta^2 e(X^{[t]})}{e(X^{[t+1]} = X)}\right) t^2,
\]

\[
e^{[2t]} \geq e^{[0]} \left(1 - \sup_{t \in \{1, 2, \ldots\}} \max_{X : X \cap X^* = \emptyset} \frac{\Delta^2 e(X^{[t]})}{e(X^{[t+1]} = X)}\right) t^2.
\]

The lower bound on the fitness value can be drawn using the equality: \(e^{[0]} = f^* - f^{[0]}\).

Using 2-step transition probabilities is more complex than using 1-step ones, but a potential benefit is a tighter bound. This is proven in the following theorem.

Theorem 5. Assume that the sequence \(\{X^{[t]} : t = 0, 1, \ldots\}\) is a Markov chain on a finite set \(S\). Then

\[
1 - \inf_{t \in \{0, 1, \ldots\}} \min_{X : X \cap X^* = \emptyset} \frac{\Delta^2 e(X^{[t]})}{e(X^{[t+1]} = X)} \leq \left(1 - \inf_{t \in \{0, 1, \ldots\}} \min_{X : X \cap X^* = \emptyset} \frac{\Delta e(X^{[t]})}{e(X^{[t+1]} = X)}\right)^2,
\]

\[
1 - \sup_{t \in \{0, 1, \ldots\}} \max_{X : X \cap X^* = \emptyset} \frac{\Delta^2 e(X^{[t]})}{e(X^{[t+1]} = X)} \geq \left(1 - \sup_{t \in \{0, 1, \ldots\}} \max_{X : X \cap X^* = \emptyset} \frac{\Delta e(X^{[t]})}{e(X^{[t+1]} = X)}\right)^2.
\]

Proof. We only prove the fist conclusion because the second one can be proven in a similar way. Without loss of generality, denote

\[
X^\dagger := \arg \inf_{t \in \{0, 1, \ldots\}} \min_{X : X \cap X^* = \emptyset} \frac{\Delta^2 e(X^{[t]})}{e(X^{[t+1]} = X)}.
\]

We get for \(X^\dagger\)

\[
1 - \frac{\Delta^2 e(X^{[t]})}{e(X^{[t+1]} = X^\dagger)} \leq \frac{\mathbb{E}[e(X^{[t+1]} = X^\dagger)]}{e(X^{[t]} = X^\dagger)} \leq \frac{\mathbb{E}[e(X^{[t+1]} = X^0)]}{e(X^{[t]} = X^0)} \leq \left(1 - \frac{\Delta e(X^{[t]})}{e(X^{[t+1]} = X^\dagger)}\right)^2
\]

\[
\leq \left(1 - \inf_{t \in \{0, 1, \ldots\}} \min_{X : X \cap X^* = \emptyset} \frac{\Delta e(X^{[t]})}{e(X^{[t+1]} = X^\dagger)}\right)^2,
\]

then we come to the first conclusion.

5.2. RLS on square function revisited: using two-step transition probabilities

We revisit the example of RLS on the square function in Section 4.6 and draw a tighter lower bound on \(f^{[t]}\) using two-step transition probabilities.

We set \(x^{[0]}\) to be a non-optimal solution such that \(|x^{[0]}| < n - 1\). The approximation error \(e(x^{[0]}) = n^2 - |x^{[0]}|^2\). Because of elitist selection, \(|x^{[0]}| \geq |x^{[0]}|\) is true for any \(t \geq 0\).

We assume that \(x^{[t]}\) is a non-optimal solution such that \(|x^{[t]}| = i\) where integer \(i\) satisfies: \(|x^{[0]}| \leq i < n\). We split the discussion of \(i\) into two cases: \(i < n - 1\) and \(i = n - 1\).
• **Case 1:** \( i < n - 1 \). After one generation, the event of \(|x^{[i+1]}| = |x^{[i]}| + 1\) happens with probability \((n - i)/n\). The event of \(|x^{[i+1]}| = |x^{[i]}|\) happens with probability \(i/n\). After two generations, the event of \(|x^{[i+2]}| = |x^{[i]}| + 2\) happens with probability \((n - i)^2/n^2\). The event of \(|x^{[i+2]}| = |x^{[i]}| + 1\) happens with probability \(2(n - i)/n\).

The average of error change in two generations satisfies

\[
\Delta^2 e(x^{[i]} \geq \frac{(n - i)}{n} \times \left( [(i + 2)^2 - i^2] + \frac{2i(n - i)}{n^2} \times [(i + 1)^2 - i^2] \right)
\]

\[
= \frac{(n - i)(4in + 4n - 2i)}{n^2}.
\]  

(90)

The ratio of error change in two generations satisfies

\[
\frac{\Delta^2 e(x^{[i]})}{e(x^{[i]})} \geq \frac{(n - i)(4in + 4n - 2i)}{n^2(n - i)^2} = \frac{4in + 4n - 2i}{(n + i)n^2}.
\]

(91)

• **Case 2:** \( i = n - 1 \). After one generation, the event of \(|x^{[i+1]}| = n\) happens with probability \(1/n\). The event of \(|x^{[i+1]}| = n - 1\) happens with probability \((n - 1)/n\). After two generations, the event of \(|x^{[i+2]}| = n - 1\) happens with probability \((n - 1)/n\). The event of \(|x^{[i+2]}| = n\) happens with probability \(\frac{2}{n} - \frac{1}{n^2}\).

The average of error change in two generations satisfies

\[
\Delta^2 e(x^{[i]} \geq \left( \frac{2}{n} - \frac{1}{n^2} \right) \times [n^2 - (n - 1)^2] = \frac{(2n - 1)^2}{n^2}
\]

(92)

The ratio of error change in two generations satisfies

\[
\frac{\Delta^2 e(x^{[i]})}{e(x^{[i]})} \geq \frac{(2n - 1)^2}{n^2(n - 1)^2}.
\]

(93)

Summarising the two cases, we know the ratio of error change in two generations is lower-bounded by

\[
\frac{\Delta^2 e(x^{[i]})}{e(x^{[i]})} \geq \frac{4in + 4n - 2i}{(n + i)n^2} \geq \frac{4|x^{[0]}|n + 4n - 2|x^{[0]}|}{(n + |x^{[0]}|)n^2}.
\]

(94)

Then we get an upper bound on the approximation error as

\[
e^{[2] \leq e^{[0]} \left( 1 - \frac{4|x^{[0]}|n + 4n - 2|x^{[0]}|}{(n + |x^{[0]}|)n^2} \right)^t
\]

(95)

and a lower bound on the fitness value as

\[
f_{2t} \geq f^* - e^{[0]} \left( 1 - \frac{4|x^{[0]}|n + 4n - 2|x^{[0]}|}{(n + |x^{[0]}|)n^2} \right)^t.
\]

(96)

Since

\[
1 - \frac{4|x^{[0]}|n + 4n - 2|x^{[0]}|}{(n + |x^{[0]}|)n^2} \leq \left( 1 - \frac{2|x^{[0]}| + 1}{n(n + |x^{[0]}|)} \right)^2,
\]

(97)

The bound \(96\) is tighter than \(39\), which is obtained from one-step transition probabilities. This example validates the claim that a potentially tighter bound can be obtained using two-step transition probabilities.
5.3. Extension to other measures of solution quality

So far the quality of solutions is measured by the fitness value $f_t$ or the approximation error $e_t$. Nevertheless, other measures also can be used to evaluate the solution quality such as Euclidean distance in continuous optimisation or Hamming distance in pseudo-Boolean optimisation. More generally, consider the distance $d(X) = \| X - X^\ast \|$ where $\| \cdot \|$ is a norm. Let $d_t := E[d(X^t)]$. Then unlimited budget analysis can be extended to the estimation of the sequence $\{d_t; t = 0, 1, \cdots \}$ too.

**Theorem 6.** Given the sequence $\{d_t; t = 0, 1, \cdots \}$,

1. If there exists some $\delta \in [0, 1]$, $d_{t+1}/d_t \leq 1 - \delta$ for any $t$, then $d_t \leq d_0(1 - \delta)^t$.
2. If there exists some $\delta \in [0, 1]$, $d_{t+1}/d_t \geq 1 - \delta$ for any $t$, then $d_t \geq d_0(1 - \delta)^t$.

**Definition 8.** The average of distance change at population $X^t = X$ is defined as

$$\Delta d(X^t) := E[d(X^t) - d(X^{t+1}) \mid X^t = X].$$

The average of distance change at the $t$th generation is defined as

$$\Delta d_t := E[d(X^t) - d(X^{t+1}) \mid X^t].$$

**Theorem 7.** Assume that the sequence $\{X^t; t = 0, 1, \cdots \}$ is a Markov chain on a finite set $S$. Then

$$d_t \leq d_0 \left(1 - \inf_{t=t_0, \cdots, \infty} \min_{X^t \cap X^\ast = \emptyset} d(X^t) - E[X^t = X]\right)^t,$$

$$d_t \geq d_0 \left(1 - \sup_{t=t_0, \cdots, \infty} \max_{X^t \cap X^\ast = \emptyset} d(X^t) - E[X^t = X]\right)^t.$$

6. Conclusions

We have presented unlimited budget analysis, an analytical framework to derive results about the expected performance, measured by means of function values, of RSHs after an arbitrary number of computational steps. We proved that for a RSH where the underlying RSH corresponds to a super-martingale, the expected function value converges to the optimal function value exponentially in the number of steps. We have provided a general result that allows to derive upper and lower bounds for this convergence based on the expected behaviour in one or multiple steps for a RSH where the underlying RSH corresponds to a Markov chain.

We have demonstrated the applicability of our method by considering random local search and the (1+1) EA on a number of pseudo-Boolean functions, namely OneMax; Square; LeadingOnes and linear functions. We have compared our results with those of fixed budget analysis. We observe that for OneMax we obtain the same results as with fixed budget analysis and that for LeadingOnes the results we obtain are an extension of the bounds obtained with fixed budget analysis. Using the new method is easy to obtaining a good result for Square and linear functions, but the same is impossible in the fixed budget framework using the existing methods. Furthermore, for OneMax, we have investigated the relationship between the mutation rate and the approximation error and given the optimal mutation rate.

Considering only a single step is a strength in the sense that it makes the method easy to apply. However, it is also a weakness because the results can be weak if considering a single step does not yield a good indication of the performance overall. We intend to consider multiple steps in future work to see how much bounds can be strengthened and how this compares with increased analytical effort. It also remains subject of future research to apply the method to other problems, in particular combinatorial optimisation problems. Considering more complex algorithms, in particular algorithms that employ a non-trivial population, is subject of future research.
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