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On a DGL-map between derivations of Sullivan minimal models

Abstract For a map \( f : X \to Y \), there is the relative model \( M(Y) = (\Lambda V, d) \to (\Lambda V \otimes \Lambda W, D) \simeq M(X) \) by Sullivan model theory (Félix et al., Rational homotopy theory, graduate texts in mathematics, Springer, Berlin, 2007). Let \( \text{Baut}_1 X \) be the Dold–Lashof classifying space of orientable fibrations with fiber \( X \) (Dold and Lashof, Ill J Math 3:285–305, 1959). Its DGL (differential graded Lie algebra)-model is given by the derivations \( \text{Der} M(X) \) of the Sullivan minimal model \( M(X) \) of \( X \). Then we consider the condition that the restriction \( b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \to \text{Der}(\Lambda V, d) \) is a DGL-map and the related topics.

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1 Introduction

Let \( X \) (and also \( Y \)) be a connected and simply connected finite CW complex with \( \dim \pi_*(X)_\mathbb{Q} < \infty \) \( (G_\mathbb{Q} = G \otimes \mathbb{Q}) \) and \( \text{Baut}_1 X \) be the Dold–Lashof classifying space of orientable fibrations \([5]\). Here \( \text{aut}_1 X = \text{map}(X, X; i d_X) \) is the identity component of the space \( \text{aut} X \) of self-equivalences of \( X \). Then any orientable fibration \( \xi \) with fibre \( X \) over a base space \( B \) is the pull-back of a universal fibration \( X \to E\text{aut}_1 X \to \text{Baut}_1 X \) by a map \( h : B \to \text{Baut}_1 X \) and equivalence classes of \( \xi \) are classified by their homotopy classes \([2,5,23]\). The Sullivan minimal model \( M(X) \) \([24]\) determines the rational homotopy type of \( X \), the homotopy type of the rationalization \( X_0 \) \([14]\) of \( X \). Notice that \( (\text{Baut}_1 X)_0 \simeq \text{Baut}_1 (X_0) \) \([17]\). The differential graded Lie algebra (DGL) \( \text{Der} M(X) \), the negative derivations of \( M(X) \) (see \( \S 2 \)), gives rise to the DGL model for \( \text{Baut}_1 X \) due to Sullivan \([24]\) \( (\text{cf.\,[10,25]} \), i.e., the spatial realization \( ||\text{Der} M(X)|| \) is \( (\text{Baut}_1 X)_0 \). Therefore, we obtain a map \( (\text{Baut}_1 X)_0 \to (\text{Baut}_1 Y)_0 \) if there is a DGL-map \( \text{Der} M(X) \to \text{Der} M(Y) \). However, it does not exist in general.

Let \( f : X \to Y \) be a map whose homotopy fibration \( \xi_f : F_f \to X \to Y \) is given by the relative model (Koszul–Sullivan extension)

\[
M(Y) = (\Lambda V, d) \hookrightarrow (\Lambda V \otimes \Lambda W, D) \simeq M(X)
\]

for a certain differential \( D \mid_{\Lambda V} = d \), where \( M(F_f) \cong (\Lambda W, D) \) for the homotopy fiber \( F_f \) of \( f \) \([7]\). In this paper, we propose

**Question 1.1** When is the restriction map given by \( b_f(\sigma) = \text{proj}_V \circ \sigma \circ i \)

\[
b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \to \text{Der}(\Lambda V, d)
\]

a DGL-map?
Here \( \text{proj}_V : \Lambda V \otimes \Lambda W \to \Lambda V \) is the algebra map with \( \text{proj}_V(w) = 0 \) for \( w \in W \) and \( \text{proj}_V |_{\Lambda V} = id_{\Lambda V} \).

**Definition 1.2** We say that a \( \mathbb{Q} \)-w.t. map \( f : X \to Y \) strictly induces the map
\[
a_f : (\text{Baut}_1 X)_0 \to (\text{Baut}_1 Y)_0
\]
if its DGL model is given by the DGL-map \( b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \to \text{Der}(\Lambda V, d) \) with \( ||b_f|| = a_f \).

Let \( \min \pi_*(S)_\mathbb{Q} := \min\{i > 0 \mid \pi_i(S)_\mathbb{Q} \neq 0\} \) and \( \max \pi_*(S)_\mathbb{Q} := \max\{i \geq 0 \mid \pi_i(S)_\mathbb{Q} \neq 0\} \) for a space \( S \). In particular, \( \min \pi_*(S)_\mathbb{Q} = \infty \) when \( S \) is the one point space.

**Definition 1.3** A fibration \( \xi : F_f \to X \to Y \) or a map \( f : X \to Y \) with homotopy fiber \( F_f \) is said to be \( \pi_0 \)-separable if \( \min \pi_0(F_f)_\mathbb{Q} \geq \max \pi_0(Y)_\mathbb{Q} \).

We say a map is rationally weakly trivial (abbr., \( \mathbb{Q} \)-w.t.) if \( \xi \) is rationally weakly trivial, i.e., \( \pi_*(X)_\mathbb{Q} = \pi_*(F_f)_\mathbb{Q} \oplus \pi_*(Y)_\mathbb{Q} \). Then \( (\Lambda V \otimes \Lambda W, D) \) is just the minimal model \( M(X) \) of \( X \). If a map \( f : X \to Y \) is \( \pi_0 \)-separable, it is \( \mathbb{Q} \)-w.t. The condition to be \( \pi_0 \)-separable is equivalent to the condition that \( \min W = \min\{i > 0 \mid W^i \neq 0\} \leq \max V = \max\{i > 0 \mid V^i \neq 0\} \) in the relative minimal model \( M(Y) = (\Lambda V \otimes \Lambda W, D) \) of \( \xi \).

**Proposition 1.4** For a \( \mathbb{Q} \)-w.t. map \( f : X \to Y \) with relative model \( M(Y) = (\Lambda V, d) \to (\Lambda V \otimes \Lambda W, D) \), the restriction \( b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \to \text{Der}(\Lambda V, d) \) is a DGL-map if and only if \( f \) is \( \pi_0 \)-separable.

That means

**Theorem 1.5** A \( \mathbb{Q} \)-w.t. map \( f : X \to Y \) strictly induces \( a_f : (\text{Baut}_1 X)_0 \to (\text{Baut}_1 Y)_0 \) if and only if \( f \) is \( \pi_0 \)-separable.

In §2, we give the proofs under some preparations of models of [7] and [25]. In this paper, we consider only \( \mathbb{Q} \)-w.t. maps. For example, we do not consider the inclusion map \( i_X : X \to X \times Y \), which is not \( \mathbb{Q} \)-w.t. However \( i_X \) induces the monoid map \( \psi : \text{aut}_1 X \to \text{aut}_1(X \times Y) \) by \( \psi(g) = g \times 1_Y \) and, therefore, there exists the induced map \( B\psi : \text{Baut}_1 X \to \text{Baut}_1(X \times Y) \) without rationalization. The DGL model is given by the natural inclusion \( \text{Der}(M(X)) \to \text{Der}(M(X) \otimes (\Lambda V)) \), which is a DGL-map.

In §3, we give some conditions that the strictly induced map \( a_f : (\text{Baut}_1 X)_0 \to (\text{Baut}_1 Y)_0 \) admits a section. Some results are obtained by Proposition 3.1 induced from [25, VI.1.1(3) Proposition] that the DGL-model of the homotopy fibration \( \chi_f : F_{a_f} \to (\text{Baut}_1 X)_0 \xrightarrow{i} (\text{Baut}_1 Y)_0 \) is given by
\[
\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \to \text{Der}(\Lambda V \otimes \Lambda W, D) \xrightarrow{b_f} \text{Der}(\Lambda V, d).
\]

Let \( \text{aut}_1 f \) be the identity component of the space of all fibre-homotopy self-equivalences of \( f \), i.e., \( \{ g : X \to X \mid f \circ g = f \} \) and \( \text{Baut}_1 f \) be the classifying space of this topological monoid. It is known that \( \text{Baut}_1 f \simeq \text{map}(Y, \text{Baut}(F_f); h) \), where \( h : Y \to \text{Baut}(F_f) \) is the classifying map of the fibration \( F_f \to X \to Y \) and map denotes the universal cover of the function space [3]. Notice that
\[
\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) = \text{Der}_\Lambda(\text{Der}(\Lambda V \otimes \Lambda W)),
\]
where \( \text{Der}_\Lambda(\text{Der}(\Lambda V \otimes \Lambda W)) \) is the sub DGL of \( \text{Der}(\Lambda V \otimes \Lambda W) \) sending the elements of \( \Lambda V \) to zero and it is a DGL model of \( \text{Baut}_1 f \) when \( Y \) and \( F_f \) are finite [4, Theorem 1], [8]. Thus, we note

**Theorem 1.6** If the homotopy fiber \( F_f \) is finite for a \( \pi_0 \)-separable map \( f \), the fiber of \( \chi_f \) has the rational homotopy type of \( \text{Baut}_1 f \).

A space \( X \) is said to be elliptic if the dimensions of the rational cohomology algebra and homotopy group are both finite [7]. An elliptic space \( X \) is said to be pure if \( dM(X)^{even} = 0 \) and \( dM(X)^{odd} \subset M(X)^{even} \). A pure space is said to be an \( F_0 \)-space (or positively elliptic) if \( \dim \pi_0(X) \otimes \mathbb{Q} = \dim \pi_0(\mathbb{Q}) \otimes \mathbb{Q} \) and \( H^{odd}(X; \mathbb{Q}) = 0 \). Then it is equivalent to \( H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \), in which \( x_i \), the degree of \( x_i \), is even and \( f_1, \ldots, f_n \) forms a regular sequence in the \( \mathbb{Q} \)-polynomial algebra \( \mathbb{Q}[x_1, \ldots, x_n] \), where \( M(X) = (\mathbb{Q}[x_1, \ldots, x_n] \otimes \Lambda(y_1, \ldots, y_n), d) \) with \( dx_i = 0 \) and \( dy_i = f_i \). In 1976, S. Halperin [12] conjectured that the Serre spectral sequences of all fibrations \( X \to E \to B \) of simply connected CW complexes collapse at the \( E_2 \)-term for any \( F_0 \)-space \( X \) [7]. For compact connected Lie groups \( G \) and \( H \) where \( H \) is a subgroup of \( G \), when rank \( G = \text{rank} \, H \), the homogeneous space \( G/H \) satisfies the Halperin conjecture [21]. Also the Halperin conjecture is true when \( n \leq 3 \) [16]. Finally, we note some relations with the Halperin conjecture [7, §39] due to Meier [18] as
Theorem 1.7 Let $Y$ be an $F_0$-space. Then the fibration $\chi_f$ is fibre-trivial for any $\pi_Q$-separable map $f : X \to Y$ if and only if $Y$ satisfies the Halperin's conjecture.

In §4, we observe the cellular obstruction for the lifting $\tilde{h}$ for a map $h : B \to (\text{Baut}_1Y)_0$ for a simply connected CW complex $B$ of finite type:

$$
\begin{array}{c}
B \\
\downarrow h \\
\text{(Baut}_1X)_0
\end{array}
\quad
\begin{array}{c}
\text{(Baut}_1Y)_0 \\
\downarrow af \\
\text{h}
\end{array}
\quad
\begin{array}{c}
\text{(Baut}_1X)_0 \\
\downarrow B \cup_a e^N \\
\text{(Baut}_1Y)_0
\end{array}
$$

for a $\pi_Q$-separable map $f : X \to Y$. Of course, it is sufficient to define as $\tilde{h} = s \circ h$ if $af$ admits a section $s$. Specifically, for a $\pi_Q$-separable map $f : X \to Y$, let

$$
\begin{array}{c}
B \\
\downarrow h_x \\
\text{(Baut}_1X)_0
\end{array}
\quad
\begin{array}{c}
\text{(Baut}_1Y)_0 \\
\downarrow af \\
\text{h_y}
\end{array}
\quad
\begin{array}{c}
B \cup_a e^N \\
\downarrow B \cup_a e^N \\
\text{(Baut}_1Y)_0
\end{array}
$$

be a commutative diagram. Then, from Proposition 1.6, we define an obstruction class by derivations in $\pi$. Recall the problem of lifting (up to homotopy) of Gottlieb [11]:

Theorem 1.8 Let $f : X \to Y$ be a $\pi_Q$-separable map with $Y$ and $F_f$ finite. There is a lift $h$ such that

$$
\begin{array}{c}
B \\
\downarrow h_x \\
\text{(Baut}_1X)_0
\end{array}
\quad
\begin{array}{c}
\text{(Baut}_1Y)_0 \\
\downarrow af \\
\text{h_y}
\end{array}
\quad
\begin{array}{c}
B \cup_a e^N \\
\downarrow B \cup_a e^N \\
\text{(Baut}_1Y)_0
\end{array}
$$

is commutative if and only if $O_a(h_X, h_Y) = 0$ in $\pi_{N-1}(\text{Baut}_1f)_Q$.

In §5, we consider an application to lifting actions. Let $G$ be a topological group and acts on a CW complex $X$. Recall the problem of (up to homotopy) of Gottlieb [11]:

Problem 1.9 When is a fibration $F_f : X \to Y$ a $G$-fibration? i.e., when is there a fibration $f' : X' \to Y$ such that $f'$ is fibre homotopy equivalent to $f$ and there is a $G$-action on $X'$ such that $f'$ is equivariant?

Suppose that $G$ is a compact connected Lie group. Since $H^*(BG; \mathbb{Q})$ is evenly graded, the obstruction classes of Theorem 1.8 are contained in $\pi_{odd}(\text{Baut}_1f)_Q$ when $B = BG$. If $\pi_{odd}(\text{Baut}_1f)_Q = 0$, they vanish and there exists a lift $h : BG \to (\text{Baut}_1X)_0$. Then from Theorem 2.6 in the case that $B = B' = BG$ and $g = (id_{BG})_0$, we obtain using Theorem 5.1 of D. H. Gottlieb.

Theorem 1.10 Let $f : X \to Y$ be a $\pi_Q$-separable map with $Y$ and $F_f$ finite. Suppose that a compact Lie group $G$ acts on $Y$. If $\pi_{odd}(\text{Baut}_1f)_Q = 0$, the action on $Y$ is rationally lifted to $X$, i.e., $f$ is rationally fibre homotopy equivalent to a $G$-equivariant map $f' : X' \to Y$ for a $G$-space $X'$.

Due to Theorem 1.7 and the result of Shiga and Tezuka [21], we have

Corollary 1.11 Let $f : X \to Y$ be a $\pi_Q$-separable map such that $Y$ is a homogeneous space $G/H$ with rank $G = \text{rank } H$. Then any group action on $Y$ is rationally lifted to $X$. In particular, the natural $G$-action on $Y$ is rationally lifted to $X$.

Furthermore, we apply the obstruction argument to a rational homotopical invariant: let $r_0(X)$ be the rational toral rank of a simply connected complex $X$ of dim $H^*(X; \mathbb{Q}) < \infty$, i.e., the largest integer $r$ such that an $r$-torus $T^r = S^1 \times \cdots \times S^1(r$-factors) can act continuously on a CW-complex $X'$ in the rational homotopy type of $X$ with all its isotropy subgroups finite (almost free action) [1,9,13]. It is very difficult to calculate $r_0(\ )$ in general. From the definition, we have the inequality $r_0(X \times Y) \geq r_0(X) + r_0(Y)$. Notice that it may sometimes be a strict inequality since there is an example that $r_0(X \times S^{12}) > 0$ even though $r_0(X) = r_0(S^{12}) = 0$ [15, Example 3.3]. For a map $f : X \to Y$, we see $r_0(Y) \leq r_0(X)$ when $X = F \times Y$ for any space $F$ and $f$ is the projection $F \times Y \to Y$. In general, when does a map $f : X \to Y$ induce $r_0(Y) \leq r_0(X)$?
Corollary 1.12 Let \( f : X \to Y \) be a \( \pi_\mathbb{Q} \)-separable map with \( Y \) and \( F_f \) finite. If \( \pi_{\text{odd}}(\text{Baut}_1 f)_\mathbb{Q} = 0 \), we have \( r_0(Y) \leq r_0(X) \).

2 Sullivan models and derivations

Let \( M(X) = (\Lambda V, d) \) be the Sullivan minimal model of simply connected CW complex \( X \) of finite type [24]. It is a free \( \mathbb{Q} \)-commutative differential graded algebra (DGA) with a \( \mathbb{Q} \)-graded vector space \( V = \bigoplus_{i \geq 1} V^i \) where \( \dim V^i < \infty \) and a decomposable differential, i.e., \( d(V^i) \subset (\Lambda^+ V \times \Lambda^+ V)^{i+1} \) and \( d \circ d = 0 \). Here \( \Lambda^+ V \) is the ideal of \( \Lambda V \) generated by elements of positive degree. The degree of a homogeneous element \( x \) of a graded algebra is denoted as \( |x| \). Then \( xy = (-1)^{|x||y|}yx \) and \( d(xy) = d(x)y + (-1)^{|x|}xd(y) \). Note that \( M(X) \) determines the rational homotopy type of \( X \), namely the spatial realization is given as \( ||M(X)|| \approx X_0 \). In particular,

\[
V^n \cong \text{Hom}(\pi_n(X), \mathbb{Q}) \quad \text{and} \quad H^*(\Lambda V, d) \cong H^*(X, \mathbb{Q}).
\]

Here the second is an isomorphism as graded algebras. Refer to [7] for detail.

Let \( \text{Der}_i M(X) \) be the set of \( \mathbb{Q} \)-derivations of \( M(X) \) decreasing the degree by \( i \) with \( \sigma(xy) = \sigma(x)y + (-1)^{|x|}x \sigma(y) \) for \( x, y \in M(X) \). The boundary operator \( \partial : \text{Der}_i M(X) \to \text{Der}_{i-1} M(X) \) is defined by

\[
\partial(\sigma) = d \circ \sigma - (-1)^{|\sigma|} \sigma \circ d
\]

for \( \sigma \in \text{Der}_i M(X) \). We denote \( \bigoplus_{i \geq 0} \text{Der}_i M(X) \) by \( \text{Der} M(X) \) in which \( \text{Der}_i M(X) \) is \( \partial \)-cycles. Then \( \text{Der} M(X) \) is a (non-free) DGL by the Lie bracket

\[
[\sigma, \tau] = \sigma \circ \tau - (-1)^{|\sigma||\tau|} \tau \circ \sigma.
\]

Note that \( H_* (\text{Der} M) = H_* (\text{Der} N) \) when free DGAs \( M \) and \( N \) are quasi-isomorphic [20]. Furthermore, recall the definition of Tanrè [25, p. 25]: let \( (L, \partial) \) be a DGL of finite type with positive degree. Then \( C^*(L, \partial) = (\Lambda s^{-1} \mathbb{Z} L, D = d_1 + d_2) \) with

\[
(d_1 s^{-1} z; sx) = -\langle z; \partial x \rangle \quad \text{and} \quad (d_2 s^{-1} z; sx_1, sx_2) = (-1)^{|x_1|} \langle z; [x_1, x_2] \rangle,
\]

where \( s^{-1} z; sx \) is \( (-1)^{|z|} \langle z; x \rangle \) and \( 2L \) is the dual space of \( L \).

Theorem 2.1 [24, §11],[10, 25] For a Sullivan model \( M(X) \) of \( X \), \( \text{Der} M(X) \) is a DGL-model of \( \text{Baut}_1 X \). In particular, there is an isomorphism of graded Lie algebras \( H_* (\text{Der} M(X)) \cong \pi_* (\Omega \text{Baut}_1 X)_{\mathbb{Q}} \) where the right hand has the Samelson bracket. Furthermore, \( C^* (\text{Der} M(X)) \) is a DGA-model of \( \text{Baut}_1 X \).

Two fibrations \( \xi_{f_1} \) and \( \xi_{f_2} \) are fibre homotopy equivalent if there is a diagram:

\[
\begin{array}{ccc}
F_{f_1} & \xrightarrow{i_1} & X_1 & \xrightarrow{f_1} & Y \\
\sim & \psi & \sim & \psi & \\
F_{f_2} & \xrightarrow{i_2} & X_2 & \xrightarrow{f_2} & Y,
\end{array}
\]

where \( \psi \circ i_2 \simeq i_1 \circ \psi \) and \( f_1 \circ \psi = f_2 \). Then its Sullivan model is given as

\[
\begin{array}{cccc}
\Lambda V, d & \rightarrow & \Lambda V \otimes \Lambda W, D_1 & \rightarrow & \Lambda W, D_1 \\
\phi \downarrow & \cong \psi & \uparrow \psi & \cong & \uparrow \psi \\
\Lambda V, d & \rightarrow & \Lambda V \otimes \Lambda W, D_2 & \rightarrow & \Lambda W, D_2,
\end{array}
\]

where the left square is DGA-commutative and the right square is DGA-homotopy commutative.

Lemma 2.2 Suppose that two maps \( f_1 \) and \( f_2 \) strictly induce \( a_{f_1} \) and \( a_{f_2} \), respectively. If \( \xi_{f_1} \) and \( \xi_{f_2} \) are fibre homotopy equivalent, there is a DGL-isomorphism \( \phi : \text{Der}(\Lambda V \otimes \Lambda W, D_1) \cong \text{Der}(\Lambda V \otimes \Lambda W, D_2) \) such that \( \phi(\sigma) = \psi \circ \sigma \circ \psi^{-1} \) and \( b_{f_2} \circ \phi = b_{f_1} \). Thus, there is a homotopy equivalence map \( \phi : (\text{Baut}_1 X_1)_0 \sim (\text{Baut}_1 X_2)_0 \) such that \( a_{f_2} \circ \phi = a_{f_1} \), i.e., \( a_{f_1} \) and \( a_{f_2} \) are fibre homotopy equivalent as fibrations.
Proof Recall the DGA-diagram of §1. Then $D_2 = \psi \circ D_1 \circ \psi^{-1}$. Therefore, there is a DGL-isomorphism $\phi$ given by $\phi(\sigma) = \psi \circ \sigma \circ \psi^{-1}$ and

$$
\begin{array}{c}
\text{Der}(\Lambda V \otimes \Lambda W, D_1), \partial_1 b_{f_1} \rightarrow \text{Der}(\Lambda V) \\
\phi \downarrow \\
\text{Der}(\Lambda V \otimes \Lambda W, D_2), \partial_2 b_{f_2} \rightarrow \text{Der}(\Lambda V)
\end{array}
$$

is DGL-commutative since $\psi|_{\Lambda V} = i d_{\Lambda V}$. In particular, we can check $\partial_2 \circ \phi = \phi \circ \partial_1$ by

$$
\partial_2 \phi(\sigma) = D_2 \psi \sigma \psi^{-1} - (-1)^i (\psi \sigma \psi^{-1}) D_2 = \psi \psi_1 \sigma \psi^{-1} - (-1)^i \psi \sigma D_1 \psi^{-1}
$$

$$
= \phi(D_1 \sigma - (-1)^i \sigma D_1) = \phi(\partial_1(\sigma))
$$

for $\sigma \in \text{Der}(\Lambda V \otimes \Lambda W, D_1)$. Similarly, we have $\phi([\sigma, \tau]) = [\phi(\sigma), \phi(\tau)]$.

\textbf{Convention.} For a DGA-model $(\Lambda V, d)$ the symbol $(v, f)$ means the \textit{elementary derivation} that takes a generator $v$ of $V$ to an element $f$ of $\Lambda V$ and the other generators to 0. Note that $|(v, f)| = |v| - |f|$. 

\textit{Proof of Proposition 1.4} Let $M(Y) = (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda W, D)$ be the model of $f$. Note that $b_f(\sigma)$ is a derivation on $\Lambda V$ for $\sigma \in \text{Der}(\Lambda V \otimes \Lambda W)$ since

$$
b_f(\sigma)(a \cdot b) = (\text{proj}_V \circ \sigma \circ i)(a \cdot b) = \text{proj}_V(\sigma(a) \cdot b + (-1)^{[\sigma][a]} a \cdot \sigma(b))
$$

$$
= (\text{proj}_V \circ \sigma \circ i)(a) \cdot b + (-1)^{[\sigma][a]} a \cdot (\text{proj}_V \circ \sigma \circ i)(b) = b_f(\sigma)(a) \cdot b + (-1)^{[\sigma][a]} a \cdot b_f(\sigma)(b)
$$

for $a, b \in \Lambda V$. Thus, $b_f$ is well defined.

(i) When $\min W \geq \max V$, there is a decomposition of vector spaces

$$
\text{Der}(\Lambda V \otimes \Lambda W) = \text{Der}(\Lambda V) \oplus \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)
$$

from degree arguments. Then there is a DGL-map $b_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \rightarrow \text{Der}(\Lambda V, d)$ by $b_f(\sigma_1) = \sigma_1$ and $b_f(\sigma_2) = 0$ for $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 \in \text{Der}(\Lambda V)$ and $\sigma_2 \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$. Then $b_f$ is a Lie algebra map since

$$
b_f(\{\sigma_1, \tau_1\}) = b_f(\{\sigma_1 + \sigma_2, \tau_1 + \tau_2\}) = b_f(\{\sigma_1, \tau_1\})
$$

$$
= \text{proj}_V \circ \sigma_1 \circ \tau_1 \circ i + (-1)^{[\sigma_1][\tau_1]} \text{proj}_V \circ \sigma_1 \circ i
$$

$$
= (\text{proj}_V \circ \sigma_1 \circ i) \circ (\text{proj}_V \circ \tau_1 \circ i) + (-1)^{[\sigma_1][\tau_1]} (\text{proj}_V \circ \sigma_1 \circ i) \circ (\text{proj}_V \circ \sigma_1 \circ \tau_1 \circ \tau_1)
$$

$$
= [b_f(\sigma_1), b_f(\tau_1)] = [b_f(\sigma), b_f(\tau)]
$$

for $\sigma, \tau \in \text{Der}(\Lambda V \otimes \Lambda W)$. Furthermore, it preserves the differential since

$$
(b_f \circ \partial_X)(\sigma) = b_f(\partial_X(\sigma_1 + \sigma_2)) = b_f(\partial_X(\sigma_1)) + b_f(\partial_X(\sigma_1))
$$

$$
= b_f(\partial_X(\sigma_1)) = \partial_Y(\sigma_1) = \partial_Y(b_f(\sigma_1)) = (\partial_Y \circ b_f)(\sigma)
$$

for $\sigma \in \text{Der}(\Lambda V \otimes \Lambda W)$ with some $\tau \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)$.

(only if) Suppose that $\min W < \max V$. There are elements $w \in W$ and $v \in V$ with $|w| < |v|$. Then $b_f$ is not a DGL-map since $b_f(\{(w, 1), (v, v)\}) = b_f(v, 1) \neq 0$ but $[b_f(w, 1), b_f(v, v)] = [0, 0] = 0$ from the definition of $b_f$. □

\textbf{Example 2.3} Consider the case that $f$ is not $\pi_\mathbb{Q}$-separable (not \mathbb{Q}-w.t.). Let $f : S^7 \rightarrow S^4$ be the Hopf map. Then the model is given by

$$
M(S^4) = (\Lambda(x, y), \nabla) \rightarrow (\Lambda(x, y, z), D) \cong M(S^7)
$$

with $|x| = 4$, $|y| = 7$, $|z| = 3$, $dx = Dx = 0$, $Dy = dy = x^2$, $Dz = x$, $Dy = x^2$. Then the bases of derivations are given as,
where $H_n(Der(\Lambda(x, y)) = \mathbb{Q}[(y, 1)]$. By degree reason, any DGL-map

$$\psi : (Der(\Lambda(x, y, z), D) \to (Der(\Lambda(x, y), d))$$

is given by $\psi(y, 1) = a_1(y, 1), \psi(x, 1) = a_2(x, 1), \psi(y, z) = a_3(x, 1), \psi(y, x) = a_4(y, x), \psi(z, 1) = a_5(y, x)$ and $\psi_f(x, z) = 0$ for some $a_i \in \mathbb{Q}$.

From $(x, 1) = [(z, 1), (x, z)]$ and $(y, z) = [(x, z), (y, x)]$ we have $a_2 = 0$ and $a_3 = 0$, respectively. Then from $2(y, 1) = [(z, 1), (y, z)] + [(x, 1), (y, x)]$, we obtain $a_1 = 0$. Thus, $||\psi||$ is homotopic to the constant map.

Let $f : X \to Y$ be a map with a section $s$, i.e., there is a map $s : Y \to X$ with $f \circ s \sim id_Y$. Then there is a map $\psi_f : \text{aut}_1 X \to \text{aut}_1 Y$ with $\psi_f(g) := f \circ g \circ s$ for $g \in \text{aut}_1 X$. In general, this does not preserve the monoid structures.

**Theorem 2.4** If a $\pi_Q$-separable map $f$ admits a section, there is a commutative diagram:

$$
\begin{array}{ccc}
\pi_n(\text{aut}_1 X)_Q & \xrightarrow{\cong} & \pi_n(\text{aut}_1 Y)_Q \\
\downarrow \cong & & \downarrow \cong \\
H_n(Der(\Lambda V \otimes \Lambda W, D)) & \xrightarrow{H_n(b_f)} & H_n(Der(\Lambda V, d)) \\
\downarrow \cong & & \downarrow \cong \\
\pi_n(\Omega \text{Baut}_1 X)_Q & \xrightarrow{\pi_n(\Omega a_f)_Q} & \pi_n(\Omega \text{Baut}_1 Y)_Q.
\end{array}
$$

**Proof** The map $\pi_n(\psi_f) : \pi_n(\text{aut}_1 X) \to \pi_n(\text{aut}_1 Y)$ is given by $\pi_n(\psi_f)([\sigma]) = [\tau] := [f \circ \sigma \circ (s \times 1_{\mathbb{S}^n})]$ in the following homotopy commutative diagram:

![Diagram](image)

from adjointness. That is the pointed homotopy classes of maps $S^n \to \text{aut}_1 X = \text{map}(X, X; id_X)$ are in bijection with the homotopy classes of those maps $X \times S^n \to X$ that composed with the inclusion $i_X : X \to X \times S^n$ yield the identity [20, p. 43–44]. Let $M(Y) = (\Lambda V, d) \xrightarrow{f} (\Lambda V \otimes \Lambda W, D) \simeq M(X)$ be the model of $f$. We identify the rationalized map $\sigma_0 : (X \times S^n)_0 \to X_0$ to an element of $\text{Der}_n(\Lambda V \otimes \Lambda W)$ [24, p. 313] (cf. [20, Proposition 11]). Then there is a chain map

$$c_f : \text{Der}(\Lambda V \otimes \Lambda W, D) \to \text{Der}(\Lambda V, d)$$
given by $c_f(\sigma_0) = \text{proj}_V \circ \sigma_0 \circ i$. It is well defined, i.e., $\partial_Y \circ c_f = c_f \circ \partial_X$, from $DW \subset \Lambda V \otimes \Lambda^+ W$ [26] since it admits a section. (However, $c_f$ is not a DGL-map in general.) Notice that

$$\pi_n(\psi_f)([\sigma_0]) \equiv H_n(c_f)([\sigma_0]).$$

If $f$ is a $\pi_\mathbb{Q}$-separable map, $c_f$ is the DGL-map $b_f$. \hfill \Box

The following is obvious from the definition of $b_f$ and useful:

Claim 2.5 For any $\pi_\mathbb{Q}$-separable map $f : X \to Y$, we have $b_f(C) = 0$ and $b_f |_{\text{Der}(\Lambda V)} = id_{\text{Der}(\Lambda V)}$ for $\text{Der}(\Lambda V \otimes \Lambda W) = C \oplus \text{Der}(\Lambda V)$.

Theorem 2.6 For a $\pi_\mathbb{Q}$-separable map $f : X \to Y$, let

$$
\begin{array}{c}
B_0 \xrightarrow{h} (\text{Baut}_1X)_0 \\
\downarrow g \\
B'_0 \xrightarrow{h'} (\text{Baut}_1Y)_0
\end{array}
$$

be a commutative diagram. Then there exists a map between total spaces $k : E \to E'$ in the diagram:

$$
\begin{array}{c}
X_0 \xrightarrow{i} E \xrightarrow{p} B_0 \\
\downarrow f_0 \\
Y_0 \xrightarrow{i'} E' \xrightarrow{p'} B'_0
\end{array}
$$

where $k \circ i \simeq i' \circ f_0$ and $g \circ p = p' \circ k$. Here $p : E \to B_0$ and $p' : E' \to B'_0$ are induced by the rationalized classifying maps $h$ and $h'$, respectively.

Proof Let $X \to E^X_\infty \xrightarrow{p^X_\infty} \text{Baut}_1X$ and $Y \to E^Y_\infty \xrightarrow{p^Y_\infty} \text{Baut}_1Y$ be the universal fibrations of $X$ and $Y$, respectively. Let $C^*(\text{Der}(\Lambda V)) \otimes \Lambda V, D_Y$ be the DGA-model of $E^Y_\infty$ and $C^*(\text{Der}(\Lambda V \otimes \Lambda W)) \otimes \Lambda V \otimes \Lambda W, D_X$ be the DGA-model of $E^X_\infty$. For a $\pi_\mathbb{Q}$-separable map $f : X \to Y$, there exists a DGA-inclusion map $\psi$ such that

$$
\begin{array}{c}
C^*(\text{Der}(\Lambda V)) \xrightarrow{C^*(b_f)} C^*(\text{Der}(\Lambda V)) \otimes \Lambda V, D_Y \\
\downarrow \psi \\
C^*(\text{Der}(\Lambda V \otimes \Lambda W)) \xrightarrow{C^*(b_f)} C^*(\text{Der}(\Lambda V \otimes \Lambda W)) \otimes \Lambda V \otimes \Lambda W, D_X
\end{array}
$$

is commutative from the universality. Indeed, $C^*(\text{Der}(\Lambda V)) \otimes \Lambda V, D_Y$ is a sub-DGA of $C^*(\text{Der}(\Lambda V \otimes \Lambda W)) \otimes \Lambda V \otimes \Lambda W, D_X$ from Claim 2.5 and $\text{max } V \leq \text{min } W$. Thus, there is a map $\tilde{\phi}_f := |\psi| : (E^X_\infty)_0 \to (E^Y_\infty)_0$ such that $(p^X_\infty)_0 \circ \tilde{\phi}_f = a_f \circ (p^X_\infty)_0$. Since $p'$ is the pull-back of $(p^X_\infty)_0$ by $h'$, there exists a map $k : E \to E'$ such that

$$
\begin{array}{c}
E \xrightarrow{h} (E^X_\infty)_0 \\
\downarrow k \\
E' \xrightarrow{h'} (E^Y_\infty)_0
\end{array}
$$

is commutative from the universality since $h' \circ g \circ p = a_f \circ h \circ p = a_f \circ (p^X_\infty)_0 \circ \tilde{h} = (p^Y_\infty)_0 \circ \tilde{\phi}_f \circ \tilde{h}. \hfill \Box
Example 2.7 Let \( X = K(\mathbb{Q}, n) \times K(\mathbb{Q}, 2n) \) and \( Y = K(\mathbb{Q}, n) \) for some even integer \( n \). Then \( M(X) = \Lambda(x, y), 0 \) and \( M(Y) = \Lambda(z), 0 \) with \( |x| = |z| = n \) and \( |y| = 2n \). Let a map \( f : X \to Y \) be given by \( M(f) : \Lambda(z) \to \Lambda(x, y) \) with \( M(f)(z) = x \). The homotopy fibration of any \( \pi_0 \)-separable map is given by \( \Lambda(z), 0 \to \Lambda(z, y), 0 \cong \Lambda(x, y), 0 \) from the degree reason. Therefore, the DGL-map \( \psi : \text{Der}\Lambda(x, y) \to \text{Der}\Lambda(z) \) such that \( \psi((x, y)) = \psi((x, 1)) = (z, 1) \) is not DGL-homotopic to \( b_f \) from Claim 2.5.

Let \( h : S_0^{n+1} \to (\text{Baut}_1 X)_0 \) and \( h' : S_0^{n+1} \to (\text{Baut}_1 Y)_0 \) be given by \( L(h) : \mathbb{L}(u) \to \text{Der}(\Lambda(x, y)) \) with \( |u| = n \), \( L(h)(u) = (y, x) \) and \( L(h') : \mathbb{L}(u) \to \text{Der}(\Lambda(z)) \) with \( L(h')(u) = (z, 1) \), respectively. Then the commutative diagram

\[
\begin{array}{ccc}
S_0^{n+1} & \xrightarrow{h} & (\text{Baut}_1 X)_0 \\
\downarrow & & \downarrow \|\psi\| \\
S_0^{n+1} & \xrightarrow{h'} & (\text{Baut}_1 Y)_0
\end{array}
\]

does not induce a map between total spaces \( f' : E \to E' \) such that

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & E \xrightarrow{p} S_0^{n+1} \\
\downarrow & & \downarrow p' \\
Y_0 & \xrightarrow{f'} & E' \xrightarrow{p'} S_0^{n+1}
\end{array}
\]

is homotopy commutative. Indeed, there does not exist a DGA-map \( h : \Lambda(v, z), D' \to \Lambda(v, x, y), D \) with \( D'z = v, D_y = vx \) and \( Dx = 0 \) such that

\[
\begin{array}{ccc}
\Lambda v, 0 & \xrightarrow{h} & \Lambda(v, z), D' \\
\downarrow & & \downarrow p \\
\Lambda v, 0 & \xrightarrow{h} & \Lambda(v, x, y), D \xrightarrow{p'} \Lambda(x, y), 0,
\end{array}
\]

where \( |v| = n + 1 \) is homotopy commutative since \( h \) cannot be a DGA-map from \( Dh(z) = 0 \) but \( hD'(z) = v \).

3 When does \( a_f \) admit a section?

Let \( f : X \to Y \) be a \( \pi_\mathbb{Q} \)-separable map with homotopy fiber \( F_f \) and \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \) the sub-DGL of \( \text{Der}(\Lambda V \otimes \Lambda W) \) restricted to derivations out of \( \Lambda W \).

Proposition 3.1 Let \( F_{a_f} \) be the homotopy fiber of \( a_f \). Then the DGL-model of the fibration \( \chi_f : F_{a_f} \xrightarrow{j} (\text{Baut}_1 X)_0 \xrightarrow{a_f} (\text{Baut}_1 Y)_0 \) is given by

\[
\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \xrightarrow{\text{incl.}} \text{Der}(\Lambda V \otimes \Lambda W) \xrightarrow{b_f} \text{Der}(\Lambda V).
\]

Proof Since \( b_f \) is surjective and \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) = \text{Ker} b_f \), it follows from [25, VI.1.3.(3) Proposition].

Let \( L(F) = \oplus_{i>0} L(F)_i \) be the degree decomposition of a DGL-model of a space \( F \).

Theorem 3.2 \( L(F_{a_f})_n \cong \oplus_{i-j=n} \text{Der}_i(\Lambda W) \otimes H^j(\Lambda V) \).

Proof A chain-map \( \rho : \text{Der}_i(\Lambda W) \otimes H^j(\Lambda V) \to \text{Der}_i(\Lambda W, \Lambda W \otimes (\Lambda V)^j) \) is given by \( \rho(\sigma \otimes [f])((u) := (-1)^{|u||f|}\sigma(u) : f \) induced by an inclusion \( H^j(\Lambda V) \to (\Lambda V)^j \). It is quasi-isomorphic, i.e., there is a decomposition \( \text{Der}(\Lambda W, \Lambda W \otimes \Lambda V) = (\text{Der}(\Lambda W) \otimes H^*(\Lambda V)) \oplus C \) for a complex \( C \) of derivations with \( H_n(C) = 0 \).
The rational homotopy exact sequence of the strictly induced fibration \( \chi_f \):
\[
\cdots \to \pi_{n+2}(\text{Baut}_1 X)_{\mathbb{Q}} \xrightarrow{a_f} \pi_{n+2}(\text{Baut}_1 Y)_{\mathbb{Q}} \xrightarrow{\delta_f} \pi_{n+1}(F_{a_f})_{\mathbb{Q}} \xrightarrow{b_f} \pi_{n+1}(\text{Baut}_1 X)_{\mathbb{Q}} \xrightarrow{a_f} \pi_{n+1}(\text{Baut}_1 Y)_{\mathbb{Q}} \xrightarrow{\delta_f} \cdots
\]
is equivalent to the homology exact sequence:
\[
\cdots \to H_{n+1}(\text{Der}(\Lambda V \otimes \Lambda W)) \to H_n(\text{Der}(\Lambda V \otimes \Lambda W)) \to H_n(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)) \to H_n(\text{Der}(\Lambda V)) \to \cdots
\]

Then we have the following from an ordinary chain complex property:

**Claim 3.3** The connecting map \( \delta_f \) is given by \( \delta_f(\sigma) = [\tau] \) when \( \partial_Y \sigma = \tau \) for a \( \partial_Y \)-cycle \( \sigma \) of \( \text{Der}(\Lambda V) \) and a \( \partial_X \)-cycle \( \tau \) of \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \).

Recall that the following implications hold for a general fibration \( \chi : F \to E \xrightarrow{p} B \) of simply connected spaces:

\( \chi \) is fibre-trivial \( \Rightarrow \) \( p \) admits a section \( \Rightarrow \) \( \chi \) is weakly trivial \( \Leftrightarrow \delta = 0 \). \((*)\)

Here \( \delta : \pi_{\ast}(B) \to \pi_{\ast-1}(F) \) is the connecting map of the homotopy exact sequence for \( \chi \). The following may be a characteristic phenomenon in our fibration \( \chi_f \).

**Proposition 3.4** \( a_f \) admits a section if and only if \( \delta_f = 0 \).

**Proof** \( (i) \) Let the DGA-model of the fibration \( \chi_f : F_{a_f} \xrightarrow{j} (\text{Baut}_1 X)_0 \xrightarrow{a_f} (\text{Baut}_1 Y)_0 \) be given as the commutative diagram:

\[
\begin{array}{cccccc}
\Lambda U, d & \xrightarrow{\rho_2} & \Lambda U \otimes \Lambda Z, D_2 & \xrightarrow{\rho_2} & \Lambda Z, D_2 \\
\rho_2|_{\Lambda U} \cong & & & & \\
C^*(\text{Der}\Lambda V) & \xrightarrow{\rho_1} & C^*(\text{Der}\Lambda V \otimes \Lambda Z, D_1) & \xrightarrow{\rho_1} & C^*(\Lambda Z, D_1) \\
\end{array}
\]

where \( M(\text{Baut}_1 Y) \cong (\Lambda U, d) \) with \( U^{n+1} = H_0(\text{Der}(\Lambda V)) \) and \( M(F_{a_f}) \cong (\Lambda Z, D_2) \) with \( Z^{n+1} = H_0(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)) \). Here

\[
(\Lambda U, d) \to (\Lambda U \otimes \Lambda Z, D_2) \to (\Lambda Z, D_2)
\]

is the model of \( \chi_f \). From the assumption \( \chi_f \) is weakly equivalent, i.e., \( M(\text{Baut}_1 X) \cong (\Lambda U \otimes \Lambda Z, D_2) \) as a minimal model. Let \( D = d_1 + d_2 \) as in \( \S 2 \). Notice that a linear component of any bracket representation of \( \sigma \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \) is not contained in \( \{\text{Der}(\Lambda V), \text{Der}(\Lambda W)\} \), where \( [\cdot, \cdot] \) is the Lie bracket. That means

\[
d_2([s^{-1}\sigma^*]) \in I(C^*(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W))) \quad (**).
\]

Here \( I(S) \) is the ideal in \( C^*(\text{Der}(\Lambda V \otimes \Lambda W)) \) generated by a basis of a vector space \( S \). Let \( \sigma \) be a non-exact \( \partial_X \)-cycle of \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \). Then

\[
H^*(C^*(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W))), d_1^* / H^* \cdot H^* \cong Z \ni [s^{-1}\sigma^*]
\]

for \( D' = d_1^2 + d_2^2 \) as in \( \S 2 \). Since \( \rho_1(D_1 ([s^{-1}\sigma^*])) \) and \( d_2([s^{-1}\sigma^*]) \) are \( D \)-cohomologous in \( C^*(\text{Der}(\Lambda V \otimes \Lambda W)) \), we can put

\[
D_1([s^{-1}\sigma^*]) \in I(Z)
\]

from (**), i.e., \( D_1(Z) \subset C^*(\text{Der}\Lambda V) \otimes \Lambda^+ Z \). By \( \rho_2, D_2(Z) \subset \Lambda U \otimes \Lambda^+ Z \). Then we have done from [26]. (only if) It holds from the above implications (**). \( \square \)
Theorem 3.5  If a \( \pi_Q \)-separable map \( f : X \to Y \) is rationally fibre-trivial (i.e., \( X_0 \sim (F_f)_0 \times Y_0 \), \( a_f \) admits a section.

Proof From the assumption and Claim 3.3, we have \( \delta_f = 0 \). Then it holds from Proposition 3.4. \( \square \)

Refer [19, page 292] for related topics. Conversely, when \( Y \) is an odd-sphere,

Theorem 3.6  If a \( \pi_Q \)-separable map \( f : X \to Y = S^{2n+1} \) is not rationally fibre-trivial, \( a_f \) does not admit a section. Furthermore, \( a_f \sim \#(\text{the constant map}) \).

Proof Let \( M(S^{2n+1}) = (\Lambda v, 0) \). Since there exists an element \( w \in W \) such that \( Dw \in \Lambda v \otimes \Lambda^+W \) from the assumption, \( \partial_X(v, 1) = \pm(w, \partial Dw/\partial v + \cdots) \neq 0 \) in \( \text{Der}(\Lambda W) \). From Claim 3.3 \( \delta_f \) is injective since \( \delta_f([(v, 1)]) = [\pm(w, \partial Dw/\partial v + \cdots)] \neq 0 \). Then the former holds from Proposition 3.4. Furthermore, from the homotopy exact sequence, we have \( a_{f,z} = 0 \). Thus, the latter holds. \( \square \)

Example 3.7  (1) Let \( S^5 \times S^7 \to X \to Y = S^3 \) be a non-( fibre-)trivial \( \pi_Q \)-separable fibration given by the model

\[
(\Lambda(v_1), 0) \to (\Lambda(v_1, w_1, w_2), D) \to (\Lambda(w_1, w_2), 0)
\]

with \(|v_1| = 3, |w_1| = 5, |w_2| = 7, Dw_1 = 0 \) and \( Dw_2 = v_1 w_1 \). Then \( a_f \) does not admit a section from Theorem 3.6. Indeed \( \delta_f : H_3(\text{Der}(\Lambda^v)) \to H_2(\text{Der}(\Lambda(w_1, w_2), \Lambda(v_1, w_1, w_2))) \) is non-trivial from \( \delta_f([(v_1, 1)]) \neq 0 \).

(2) Let \( S^5 \times S^7 \to X' \to Y' \) be a non-( fibre-)trivial \( \pi_Q \)-separable fibration given by the model

\[
(\Lambda(v_1, v_2, v_3), Dv) \to (\Lambda(v_1, v_2, w_1, w_2), D') \to (\Lambda(w_1, w_2), 0)
\]

with \(|v_1| = |v_2| = 3, |v_3| = 5, |w_1| = 7, |w_2| = 9, Dw_1 = dv_1, Dw_2 = v_1 w_1 \). Then \( a_f \) admits a section from Proposition 3.4 since \( \delta_f([v_3, 1]) = 0 \) for \( H_3(\text{Der}(\Lambda(v_1, v_2, v_3))) = \mathbb{Q}[[v_3, 1]] \). However, \( \chi_f \) is not trivial from \( [v_3, 1], (w_1, w_2v_3)] = (w_2, v_2) \).

Indeed, then

\[
D(s^{-1}(w_2, v_2)^*) = d_2(s^{-1}(w_2, v_2)^*) = s^{-1}(v_3, 1)^* \cdot s^{-1}(w_2, v_2v_3)^*
\]

for \( (C^* (\text{Der}(\Lambda(v_1, v_2, w_1, w_2)), D)) \) with \( D = d_1 + d_2 \). Refer the proof of Proposition 3.4.

Proof of Theorem 1.7  Let \( M(Y) = (\Lambda V, d) = (Q[x_1, \ldots, x_n] \otimes \Lambda(y_1, \ldots, y_m), d) \) with \( dx_i = 0 \) and \( dy_i = f_i \) for \( i = 1, \ldots, n \).

(if) Let \( M(Y) = (\Lambda V, d) \to (\Lambda V \otimes \Lambda W, D) \) be the model of \( f \). From the regularity of \( f_1, \ldots, f_n \), \( \text{Im} D \subset Q[x_1, \ldots, x_n] \otimes \Lambda W \). Thus,

\[
\partial_X(x_i, h_i) = \sum_{j=1}^n (y_j, (df_j/\partial x_i) \cdot h_i) + \theta \quad \text{and} \quad \partial_X(y_i, h_i) = 0 \quad (i = 1, \ldots, n)
\]

for any \( h_i \in \Lambda V^{even} = Q[x_1, \ldots, x_n] \) with suitable degree and some \( \theta \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \). Then we have \( \delta_f = 0 \) from Claim 3.3 since \( H_{even}(\text{Der}(M(Y))) = 0 \) [18] from the assumption. Then \( a_f \) admits a section from Proposition 3.4. Furthermore, from Theorem 3.2, we can suppose that the Lie bracket decomposition of an element of \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \) does not have an element of \( \text{Der}(\Lambda V) \) as a factor since \( \text{Der} H^*(Y; \mathbb{Q}) = 0 \) [18] again. Thus, we have

\[
D_2 = d \otimes 1 \pm 1 \otimes \overline{D}_2
\]

for the Sullivan minimal model \( (\Lambda U, d) \to (\Lambda U \otimes \Lambda Z, D_2) \to (\Lambda Z, \overline{D}_2) \) of \( \chi_f \) (in the proof of Proposition 3.4).

(only if) Suppose that \( Y \) does not satisfies the Halperin conjecture, i.e., there is a non-zero element \( [\sum_i(x_i, h_i) + \sum_j(y_j, g_j)] \in H_{2m}(\text{Der}(M(Y))) \) for \( h_i \in Q[x_1, \ldots, x_n], g_j \in \Lambda V \) and some \( m \) [18]. Let \( (S^a \times S^b \cong 0) F \to X \to Y \) be a fibration of the model:

\[
(\Lambda V, d) \to (\Lambda V \otimes \Lambda(w_1, w_2, w_3), D) \to (\Lambda(w_1, w_2, w_3), d_F),
\]
where \(|w_1| = a\) is even and \(|w_2| = b\) is odd with \(b - a = |x_k| - 1\) for some \(k\), \(d_Fw_1 = d_Fw_2 = 0\), \(d_Fw_3 = w_1^2\), \(Dw_1 = 0\) and \(Dw_2 = x_kw_1\). When \(h_k\) is not \(d_Y\)-exact, the element \(h_kw_1\) is not \(D\)-exact. Then
\[
\delta_f \left( \sum_i (x_i, h_i) + \sum_j (y_j, g_j) \right) = [(w_2, h_kw_1)] \neq 0
\]
for \(\delta_f : H_{2m}(\text{Der} \Lambda V) \to H_{2m-1}(\text{Der}(\Lambda(w_1, w_2, w_3), \Lambda V \otimes \Lambda(w_1, w_2, w_3))\) from Claim 3.3. In particular, \(\chi_f\) is not fibre-trivial. □

**Example 3.8** Let \(Y\) be the homogeneous space \(SU(6)/SU(3) \times SU(3)\). Then \(Y\) is a pure space but not an \(F_0\)-space since \(\text{rank } SU(6) = 5 > 4 = \text{rank } SU(3) \times SU(3)\). Let \(\xi : (S^{11} \times S^{23} \simeq_0 F) \to X \xrightarrow{f} Y\) be a fibration whose relative model is given as
\[
(\Lambda(x_1, x_2, y_1, y_2, y_3, d_Y) \to (\Lambda(x_1, x_2, y_1, y_2, y_3) \otimes \Lambda(w_1, w_2), D) \to (\Lambda(w_1, w_2), 0),
\]
where \(|x_1| = 4, |x_2| = 6, |y_1| = 7, |y_2| = 9, |y_3| = 11, |w_1| = 11, |w_2| = 23, d_Y y_1 = x_1^2, d_Y y_2 = x_1x_2, d_Y y_3 = x_2^2, Dw_1 = 0\) and \(Dw_2 = (x_1y_2 - x_2y_1)w_1\). Then \(\delta_X((y_1, 1)) = (w_2, x_2w_1)\), i.e., \(\delta_f((y_1, 1)) = [(w_2, x_2w_1)] \neq 0\) from Claim 3.3. In particular, \(\chi_f\) is not trivial. Refer [19, Example 1.14(2)] for the Sullivan minimal model of \(\text{Baut}_1 Y\).

4 The obstruction class for a lifting

Let \(\text{L}(B) = (\text{L}(B), \partial_B)\) be the Quillen model of a simply connected CW complex \(B\) of finite type. Then \(\text{L}(B \cup_a e^N)\) is given by \(\text{L}(B) \coprod \text{L}(u), \partial_u\) where \(|u| = N - 1, \partial_B|_{\text{L}(B)} = \partial_B\) and \(\partial_u(u) \in \text{L}(B)\) [25, Proposition III.3.6].

**Theorem 4.1** For a \(\pi_G\)-separable map \(f : X \to Y\), let
\[
\begin{CD}
B @> h_X >> (\text{Baut}_1 X)_0 \\
@. \downarrow i \downarrow a_f \\
B \cup_a e^N @> h_Y >> (\text{Baut}_1 Y)_0
\end{CD}
\]
be a commutative diagram. Then there is a lift \(h\) such that
\[
\begin{CD}
B @> h_X >> (\text{Baut}_1 X)_0 \\
@. \downarrow h \downarrow a_f \\
B \cup_a e^N @> h_Y >> (\text{Baut}_1 Y)_0
\end{CD}
\]
is commutative if and only if
\[
\mathcal{O}_u(h_X, h_Y) := [\tau(h_Y(u)) - h_X'(\partial_u(u))] = 0
\]
in \(H_{N-2}(\text{Der}(\Lambda W, \Lambda V \otimes \Lambda W)) = \pi_{N-1}(F_{a_f})_G\) for the DGL-commutative diagram
\[
\begin{CD}
\text{L}(B) @> h_X >> \text{Der}(\Lambda V \otimes \Lambda W), \partial_X \\
@. \downarrow p_f \downarrow a_f \\
\text{L}(B) \coprod \text{L}(u), \partial_u @> h_Y >> \text{Der}(\Lambda V), \partial_Y
\end{CD}
\]
with
Therefore, \( \tau(\partial_{\alpha}(u)) = \partial_{\alpha}(\nu) \) and \( \partial_{\alpha}(\nu) \) is a \( \partial_{\alpha} \)-cycle in \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \). Furthermore, \( \nu = h_{\nu} \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \) for \( b \in L(B) \).

**Proof** Since \( b_f \circ h_{X} = h_{Y} \circ i \) and \( h_{Y} \) is a DGL-map,

\[
h'_{X} \partial_{\alpha}(u) = h_{Y} \partial_{\alpha}(u) = \partial_{\alpha}(h_{Y}(u)) \quad (1)
\]

in \( \text{Der}(\Lambda V) \). Notice that the obstruction element \( \partial_{\alpha}(\nu) = \partial_{\alpha}(u) \) is a \( \partial_{\alpha} \)-cycle in \( \text{Der}(\Lambda V \otimes \Lambda W) \). Therefore, \( \tau(\partial_{\alpha}(u)) \)-cycle in \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \) from (1).

**(if)** Suppose that \( \mathcal{O}_{\alpha}(h_{X}, h_{Y}) = 0 \). Then there is an element \( q \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \) such that

\[
\partial_{\alpha}(q) = \tau(\partial_{\alpha}(u)) - h'_{X} \partial_{\alpha}(u). \quad (2)
\]

Let \( h \mid_{L(B)} := h_{X} \) and \( h(u) := h_{Y}(u) - q \).

Then \( h \) is a DGL-map since

\[
\partial_{\alpha}(h(u)) = \partial_{\alpha}(h_{Y}(u)) - \partial_{\alpha}(q) = (h'_{X} \partial_{\alpha}(u) + \tau h_{Y}(u)) - (\tau h_{Y}(u) - h'_{X} \partial_{\alpha}(u)) = h'_{X} \partial_{\alpha}(u) + h'_{X} \partial_{\alpha}(u) = h_{X}(\partial_{\alpha}(u)) = h(\partial_{\alpha}(u))
\]

from (1) and (2). Furthermore,

\[
\begin{array}{ccc}
L(B) & \xrightarrow{h} & \text{Der}(\Lambda V \otimes \Lambda W), \partial_{\alpha} \\
\downarrow{i} & & \uparrow{b_f} \\
L(B) \bigwedge \mathbb{L}(u), \partial_{\alpha} & \xrightarrow{h_{Y}} & \text{Der}(\Lambda V), \partial_{\alpha}
\end{array}
\]

is commutative since \( b_f(q) = 0 \). Thus, the (if)-part holds from the special realization of (*).

**(only if)** Suppose that there exists a map \( h \) such that (*) is commutative. Since \( h \) is a DGL-map,

\[
\partial_{\alpha}(h(u)) = h(\partial_{\alpha}(u)) \quad (3)
\]

in \( \text{Der}(\Lambda V \otimes \Lambda W) \) and

\[
h'_{X} \partial_{\alpha}(u) = \tau h(u) \quad (4)
\]

from (1) and (3). Furthermore,

\[
\tau h(u) \sim \tau h_{Y}(u) \quad (5)
\]

in \( \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \). Here \( \sim \) means "homologous". Indeed, (5) follows since

\[
h(u) = h_{Y}(u) + x
\]

for some element \( x \in \text{Der}(\Lambda W, \Lambda V \otimes \Lambda W) \) from \( b_f \circ h = h_{Y} \) and then since

\[
\tau h(u) = \tau(h_{Y}(u) + x) = \tau h_{Y}(u) + \partial_{\alpha}(x). \quad \Box
\]

Thus, we obtain that \( \mathcal{O}_{\alpha}(h_{X}, h_{Y}) = [\tau(\partial_{\alpha}(u)) - h'_{X}(\partial_{\alpha}(u))] = 0 \) from (4) and (5).

From Theorem 3.2, we have

**Corollary 4.2** If \( \pi_{N-1}(\text{Baut}_{1}F_{f})_{\mathbb{Q}} = 0 \) for the homotopy fiber \( F_{f} \) of \( f \), there exists a lift \( h \) for the pair \( (h_{X}, h_{Y}) \) of above.
Example 4.3 Let $B = S^2 = \mathbb{C}P^1$. Let $S^3 \times S^5 \to X \overset{f}{\to} Y = S^3$ be the fibration given by the model

$$(\Lambda(v), 0) \to (\Lambda(v, w_1, w_2), D) \to (\Lambda(w_1, w_2), 0)$$

with $|v| = |w_1| = 3, |w_2| = 5$, $Dw_1 = 0$ and $Dw_2 = vw_1$. Let $L(\mathbb{C}P^2) = L(B \cup_a e^4) = (\Lambda(u_1, u_2), \partial)$ with $|u_1| = 1, |u_2| = 3$, $\partial u_1 = 0$ and $\partial u_2 = [u_1, u_1]$ [25]. Let

$$
\begin{array}{ccc}
S^2 & \xrightarrow{h_X} & (\text{Baut}_1X)_0 \\
\downarrow i & & \downarrow a_f \\
S^2 \cup_\alpha e^4 & \xrightarrow{h_Y} & (\text{Baut}_1Y)_0
\end{array}
$$

be a commutative diagram given by the DGL-model

$$
\begin{array}{ccc}
(\Lambda(u_1), 0) & \xrightarrow{h_X} & (\text{Der}(\Lambda(v, w_1, w_2)), \partial_X) \\
\downarrow i & & \downarrow b_f \\
(\Lambda(u_1, u_2), \partial) & \xrightarrow{h_Y} & (\text{Der}(\Lambda(v)), 0)
\end{array}
$$

by $h_X(u_1) = h_Y(u_1) = 0$ and $h_Y(u_2) = (v, 1)$. Then $O(\alpha(h_X, h_Y) \neq 0$ in $H_2(\text{Der}(\Lambda(w_1, w_2), \Lambda(v, w_1, w_2))$ since

$$
\tau h_Y(u_2) = \partial_X(v, 1) = (w_2, w_1) \sim 0 = h_X^\nu([u_1, u_1]) = h_X^\nu(\partial_0(u_2)).
$$

Thus $h_Y : \mathbb{C}P^2 \to (\text{Baut}_1Y)_0$ cannot lift to $h : \mathbb{C}P^2 \to (\text{Baut}_1X)_0$. Note that $h_Y$ is extended to $\mathbb{C}P^\infty \to (\text{Baut}_1Y)_0$. Since $BS^1 = \mathbb{C}P^\infty$, we obtain that any free $S^1$-action on $Y$ cannot lift to $X$.

5 An application to lifting actions

Let $BG$ and $EG$ be the classifying space and the universal space of a compact connected Lie group $G$ of rank $G = r$, respectively. If $G$ acts on a space $Y$ by $\mu : G \times Y \to Y$, there is the Borel fibration

$$Y \overset{i}{\to} EG \times_G^\mu Y \to BG,$$

where the Borel space $EG \times_G^\mu Y$ (or simply $EG \times_G Y$) is the orbit space of the diagonal action $g(e, y) = (e g^{-1}, g y)$ on the product $EG \times Y$. It is rationally given by the KS extension (model)

$$(\mathbb{Q}[t_1, \ldots, t_r], 0) \to (\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, D_\mu) \to (\Lambda V, d) = M(Y) \quad (*),$$

where $|t_i|$ are even for $i = 1, \ldots, r$, $D_\mu(t_i) = 0$ and $D_\mu(v) \equiv d(v)$ modulo the ideal $(t_1, \ldots, t_r)$ for $v \in V$.

Recall the lifting theorem of D. H. Gottlieb:

**Theorem 5.1** [11, Theorem 1] Let a topological group $G$ acts on a space $Y$. A fibration $Y \overset{i}{\to} EG \times_G^\mu Y$ is fibre homotopy equivalent to a $G$-fibration if and only if it is fibre homotopy equivalent to the pull-back of a fibration over $EG \times_G Y$ induced by the inclusion $i : Y \to EG \times_G Y$.

**Proof of Theorem 1.10.** Let $h_Y : BG \to (\text{Baut}_1Y)_0$ be the rationalization of the classifying map of the Borel fibration $Y \overset{i}{\to} EG \times_G^\mu Y \to BG$ of the action $\mu : G \times Y \to Y$. Let $B^n$ be the $n$-skeleton of a CW complex $B$. From Theorem 1.8, there is a lift $h_X^\nu$ such that

$$
\begin{array}{ccc}
(BG)^n & \xrightarrow{h_X^\nu} & (\text{Baut}_1X)_0 \\
\downarrow h_Y^\nu & & \downarrow a_f \\
(BG)^n \cup_a e_i^{n+1} & \xrightarrow{h_Y^\nu} & (\text{Baut}_1Y)_0
\end{array}
$$
is commutative for all \( n \) and attaching \( \alpha \) since \( O_\alpha(h^n_X, h^n_Y) = 0 \). Indeed, \( \pi_{\text{odd}}(\text{Baut}_1 f) = 0 \) and \( L(BG) \) is oddly graded since \( H^*(BG; \mathbb{Q}) \) is evenly graded. Thus, we have the commutative diagram:

\[
\begin{array}{c}
BG \\ (\text{Baut}_1 X)_0 \\
\downarrow \alpha \\
BG \\ h_f (\text{Baut}_1 Y)_0.
\end{array}
\]

From Theorem 2.6, there is a commutative diagram:

\[
\begin{array}{c}
E \\
\downarrow f \\
(EG \times G Y)_0 \\
\downarrow g \\
BG_0
\end{array}
\]

for some space \( E \). Let \( g': E' \to EG \times G Y \) be the pull-back of \( g \) by the rationalization \( l_0 \) and \( f': X' \to Y \) be the pull-back of \( g' \) by \( i \)

\[
\begin{array}{c}
X' \\
\downarrow f' \\
E' \\
\downarrow g' \\
Y \\
\downarrow i \\
EG \times G Y \\
\downarrow b_0 \\
(EG \times G Y)_0 \\
\downarrow l_0 \\
BG_0
\end{array}
\]

Notice that the model is given by the DGA-commutative diagram:

\[
\begin{array}{c}
R \\
\downarrow M(g') \\
R \otimes \Lambda V \\
\downarrow M(f') \\
R \otimes \Lambda V \otimes \Lambda W \\
\downarrow M(f) \\
R \otimes \Lambda V \otimes \Lambda W \\
\downarrow \Lambda V \otimes \Lambda W
\end{array}
\]

for \( R := H^*(BG; \mathbb{Q}) = \mathbb{Q}[t_1, \ldots, t_r] \). Notice that the third square is given by the push-out [7, Proposition 15.8]. Thus, from Theorem 5.1, we obtain the commutative diagram

\[
\begin{array}{c}
G \times X' \\
\downarrow f' \times id_G \\
G \times Y \\
\downarrow \eta \\
Y
\end{array}
\]

since \( M(X') \cong \Lambda V \otimes \Lambda W = M(X) \).

If the \( r \)-torus \( T' \) acts on a space \( Y \), \( |t_1| = \cdots = |t_r| = 2 \) in (*)

**Proposition 5.2** [13, Proposition 4.2] Suppose that \( Y \) is a simply connected CW-complex with \( \dim H^*(Y; \mathbb{Q}) < \infty \). Put \( M(Y) = (\Lambda V, d) \). Then \( r_0(Y) \geq r \) if and only if there is a KS extension (*) satisfying \( \dim H^*(\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, \omega) < \infty \). Moreover, if \( r_0(Y) \geq r \), then \( T' \) acts freely on a finite complex \( Y' \) that has the same rational homotopy type as \( Y \) and \( M(ET' \times_{T'} Y') \cong (\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, \omega) \).

**Proof of Corollary 1.12.** Let \( r_0(Y) = r \). Notice that \( L(BT') \) is oddly generated since \( H^*(BT'; \mathbb{Q}) = \mathbb{Q}[t_1, \ldots, t_r] \). Since \( \pi_{\text{odd}}(\text{Baut}_1 f) = 0 \), there exists a lift \( (BT')_0 \to (\text{Baut}_1 X)_0 \) from Theorem 1.8. Then we have the homotopy commutative diagram:

\[
\begin{array}{c}
(F_f)_0 \\
\downarrow \cong \\
F_g \\
\downarrow \exists_0 \\
E \\
\downarrow \exists_0 \\
(ET' \times_{T'} Y')_0
\end{array}
\]

\[
\begin{array}{c}
\cong \\
\cong \\
(ET' \times_{T'} Y')_0 \\
\cong \\
(ET' \times_{T'} Y')_0 \\
\cong
\end{array}
\]

\[
\begin{array}{c}
(ET' \times_{T'} Y')_0 \\
\downarrow \\
(ET' \times_{T'} Y')_0 \\
\downarrow \\
(ET' \times_{T'} Y')_0 \\
\downarrow \\
(ET' \times_{T'} Y')_0
\end{array}
\]
from Theorem 2.6. Here is the one point space. We have dim $H^*(\tilde{E}; \mathbb{Q}) < \infty$ since dim $H^*(F_{\tilde{g}}; \mathbb{Q}) < \infty$ and dim $H^*(E; \mathbb{Q}) = \dim H^*(ET' \times T'; \mathbb{Q}) < \infty$ for the fibration $F_{\tilde{g}} \to \tilde{E} \to E$. Thus there is a free $T'$-action on $X'$ with $X'_0 \simeq X_0$ and $\tilde{E} \simeq (ET' \times T'; X')_0$ from Proposition 5.2. Thus, we have $r_0(X) \geq r$. ☐

Example 5.3 Let $S^5 \to X \xrightarrow{f} Y$ be a rationally non-trivial fibration given by the model

$$(\Lambda V, dy) = (\Lambda(v_1, v_2, v_3, v_4, v_5, w), dy) \to (\Lambda(v_1, v_2, v_3, v_4, v_5, w), D)$$

with $|v_1| = |v_2| = 2$, $|v_3| = |v_4| = |v_5| = |w| = 5$, $d_f(v_1) = d_f(v_2) = 0$, $d_f(v_3) = v_1^5$, $d_f(v_4) = v_1^2 v_2$, $d_f(v_5) = v_1^3$ and $D(w) = v_1 v_2^2$. Then

$$\pi_{odd}(\text{Baut}_1 f)_{\mathbb{Q}} \cong H_{even}(\text{Der}_{\Lambda V}(\Lambda V \otimes \Lambda(w))) = 0$$

since there is no element of odd-degree $< 5$ in $\Lambda V$. Therefore, $r_0(Y) \leq r_0(X)$. Indeed, we can directly check that $r_0(Y) = 1$ and $r_0(X) = 2$.

On the other hand, let $S^5 \to X \xrightarrow{f} Y = S^3 \times S^3$ be a rationally non-trivial fibration. Then the model is given by

$$(\Lambda(v_1, v_2), 0) \to (\Lambda(v_1, v_2, w), D)$$

with $|v_1| = |v_2| = 3$, $|w| = 5$, and $D(w) = v_1 v_2$. Then

$$\pi_3(\text{Baut}_1 f)_{\mathbb{Q}} \cong H_2(\text{Der}_{\Lambda V}(\Lambda V \otimes \Lambda(w))) = \mathbb{Q}\{w, v_1\} \oplus \mathbb{Q}\{w, v_2\} \neq 0$$

and $r_0(Y) = 2 > 1 = r_0(X)$.

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