Half-quantum vortex and $d$-soliton in Sr$_2$RuO$_4$

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Assuming that the superconductivity in Sr$_2$RuO$_4$ is described by a planar $p$-wave order parameter, we consider possible topological defects in Sr$_2$RuO$_4$. In particular, it is shown that both of the $d$-soliton and half-quantum vortex can be created in the presence of the magnetic field parallel to the $a$-$b$ plane. We discuss how one can detect the $d$-soliton and half-quantum vortex experimentally.

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It has been suggested that the unconventional superconducting state of Sr$_2$RuO$_4$ is described by the planar spin-triplet $p$-wave order parameter with broken time reversal symmetry in analogy to the $^3$He A-phase. As known from the example of superfluid $^3$He, one of the hallmarks of the triplet superconductivity is the presence of a manifold of topological defects. Thus, we expect that the creation and detection of topological defects in Sr$_2$RuO$_4$ (or spin-triplet superconductor) will provide further insights about the nature of the unconventional superconducting state of Sr$_2$RuO$_4$. In this sense, the study of the topological defects in the planar $p$-wave superconducting state with broken time reversal symmetry is of great interest.

More specifically, it has been proposed that the superconducting order parameter in this system is described by the planar $p$-wave form written as,

$$\Delta_{\alpha\beta}(\hat{k}) = \vec{d}(\hat{k}) \cdot (\vec{\sigma}_{i\alpha\beta})$$

with

$$\vec{d}(\hat{k}) = \Delta \hat{d}(\hat{k}_1 \pm i\hat{k}_2),$$

where $\sigma_{\mu}(\mu = 1, 2, 3)$ are Pauli matrices and $\alpha, \beta$ represent spin $\uparrow$ or $\downarrow$. Here $\hat{k}_j (j = 1, 2)$ represent the projection of the unit wave vector $\hat{k}$ along two perpendicular directions $\hat{e}_1$ and $\hat{e}_2$ in two dimensional space. This order parameter describes the Cooper pair state with the zero spin projection on $\hat{d}$ and with the unique projection of the pair orbital angular momentum given by $\hat{l} = \hat{e}_1 \times \hat{e}_2$. In Sr$_2$RuO$_4$, due to the spin-orbit coupling, $\hat{d}$ is forced to be parallel to $\pm \hat{c}$ and $\hat{k}$ is the quasi-particle momentum in the $a$-$b$ plane.

Indeed the spontaneous magnetization seen by muon spin relaxation experiment and flat Knight shift seen by NMR seem to be consistent with this picture. On the other hand, the origin of the spontaneous magnetization seen by muon spin relaxation experiment is somewhat mysterious since we do not expect such a magnetization in a homogeneous system.

It is important to notice that the superconducting ground state described by the order parameter of Eq.(2) is doubly degenerate. We can designate these two ground states, which we shall call $\hat{l}$-soliton in analogy to the case of superfluid $^3$He-A. It is likely that such a soliton is magnetically active, so it may be an origin of the spontaneous magnetization seen in muon spin relaxation experiment. In particular, in a magnetic field $H \parallel \hat{e}$, only one of these degenerate states is favored. Therefore, it is possible to control $\hat{l}$-solitons by a magnetic field parallel to the $e$-axis. They also proposed that these $\hat{l}$-solitons would provide very efficient barriers for the vortex motion and this effect is possibly related to the pinning of vortices observed in Sr$_2$RuO$_4$ below $T = 30mK$. However, in this experiment the magnetic field is applied in a direction perpendicular to the $c$-axis. As we will see later, so-called $d$-solitons appear to be more appropriate than $\hat{l}$-solitons in this configuration. Also, for the $\hat{l}$-solitons, it is rather difficult to estimate the soliton energy and to make a further quantitative prediction.

The purpose of this paper is to propose an alternative model for the appearance of the spontaneous magnetization and the mechanism of the pinning of vortices; $d$-soliton and half-quantum vortex. The $d$-soliton is a domain wall between $\hat{d} \parallel \hat{c}$ and $\hat{d} \parallel -\hat{c}$ as in superfluid $^3$He-A. We believe that $\hat{d}$ is parallel and antiparallel to $\hat{c}$, because they are forced to be parallel to the angular momentum $\hat{l}$ (or $-\hat{l}$) due to the spin-orbit coupling characterized by an energy scale $\Omega_d$. Therefore, if we use $\Omega_d$ as a parameter, we can calculate the energy and shape of the $d$-soliton provided that $\Omega_d \ll \Delta(T)$, where $\Delta(T)$ is the superconducting gap. Unfortunately we do not know the precise value of $\Omega_d$. 

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but it may be about $\frac{1}{\rho_s} \Delta(T)$. In this picture, moving $d$-soliton generates the local magnetization which can result in the spontaneous magnetization seen in muon spin relaxation experiment. One can also generate a large number of $d$-solitons by applying a burst of high frequency microwave with frequency $\sim \Omega_d$ sent parallel to the $a$-$b$ plane. \cite{8}

As in superfluid $^3$He-A, each $d$-soliton is terminated by a pair of half-quantum vortices. \cite{13} We find that these pairs of half-quantum vortices are more stable than the usual single quantum vortex in the superconducting state in the presence of the magnetic field parallel to the $a$-$b$ plane. This means that the usual single quantum vortex would split into a pair of half-quantum vortices connected by the $d$-soliton. \cite{8,11} In this case, these objects would provide an extremely efficient pinning mechanism of vortices in Sr$_2$RuO$_4$. Also the half-quantum vortices should exhibit a clear electron spin resonance (ESR) signature. Further we believe that these objects are visible by the scanning tunneling microscopy (STM) imaging and by micromagnetometry developed by Kirtley et al. \cite{15} used in high $T_c$ cuprate compounds.

**Free energy of the conventional single vortex when the magnetic field is parallel to the $a$-$b$ plane**

Let us assume that the magnetic field is parallel to the $a$-axis. Then the free energy of the conventional vortex with the flux quantum $\phi_0 = hc/2e$ is obtained within the London approximations as

$$f_v = (\frac{\phi_0}{4\pi \lambda})^2 \ln(\frac{\lambda}{\xi}),$$

where $\xi$ is the coherence length and $\lambda$ is the magnetic penetration depth. The magnetic penetration depth $\lambda$ is related to the superfluid density $\rho_s(T)$ by $\lambda^{-2} = \frac{4\pi e^2}{mc^2} \rho_s(T)$. When the magnetic field is parallel to the $a$-axis, in the anisotropic system like Sr$_2$RuO$_4$, $\lambda$ and $\xi$ should be reinterpreted as $\lambda = \sqrt{\lambda_y \lambda_z}$ and $\xi = \sqrt{\xi_b \xi_c}$. Here $\lambda_b, c$ and $\xi_b, c$ are the magnetic penetration depth and coherence length in the $b$ and $c$ directions respectively.

**$d$-soliton and a pair of half-quantum vortices**

There exists huge anisotropy in the in-plane and out-of-plane transport properties in Sr$_2$RuO$_4$. Thus Sr$_2$RuO$_4$ may be regarded as an effectively two dimensional system. The large anisotropy or the effective two-dimensionality of the system forces the angular momentum of the Cooper pair to be parallel or antiparallel to the $c$-axis. In the $p$-wave superconducting state described by the order parameter described by Eq.\(\ref{eq:1}\) and Eq.\(\ref{eq:2}\), the $d$ vector is oriented along $\pm \hat{c}$ in the presence of the spin-orbit coupling. Here we consider the case that the angular momentum $\hat{l}$ is uniform in the entire system. We can assume, without loss of generality, that $\hat{l} \parallel \hat{c}$. We are interested in the deformation of the $d$ configuration from the uniform case; for example, $\hat{d} \parallel \hat{c}$. Any deviation from the uniform state would cost the energy associated with the spin-orbit coupling characterized by an energy scale $\Omega_d$ \cite{12}. However, we will show that the so-called $d$-soliton (a particular form of the $d$ configuration) with a pair of half-quantum vortices can have lower energy than the conventional single vortex. Thus it is easier to excite a pair of half-quantum vortices with a $d$-soliton compared to single conventional vortex. In particular, a magnetic field parallel to the $a$-$b$ plane generates very likely pairs of half-quantum vortices rather than usual vortices when the formers are stable.

We consider the $d$-soliton that is a topological planar defect in the $d$ configuration. The orientation of $d$ changes by $\pi$ across the planar defect while $d$ vectors at far distances are still along the $c$-axis. Typical configurations of $d$-soliton in the $y$-$z$ plane can be found in Fig.1 and Fig.2 which we will explain later. We take $y$ and $z$ as the coordinates along $b$-axis and $c$-axis respectively. Now let us attach a pair of half-quantum vortices to the end points of the $d$-soliton of length $R$ in $y$-$z$ plane. In the case of an isolated half-quantum vortex, we have $e^{i\pi} = -1$ factor in the order parameter due to phase winding around the half-quantum vortex. Therefore, an isolated half-quantum vortex cannot occur. On the other hand, if the half-quantum vortex is attached to the end points of the $d$-soliton, the disyrgation in $d$ at the same point compensates the phase $\pi$ so that there is no net change in the overall phase of the order parameter.

In order to show that a pair of half-quantum vortices with the $d$-soliton is a lower energy excitation compared to single conventional vortex, we have to compare the free energies of two cases. The free energy required to create the $d$-soliton is obtained from

$$f_d = \frac{1}{2} \chi_N C^2 \int d^3 r \left[ \sum_{ij} |\partial_i \hat{d}_j|^2 + \xi_d^{-2} (1 - d_z^2) \right],$$

where $\chi_N$ is the spin susceptibility, $\xi_d(T) = C(T)/\Omega_d(T)$ where $C(T)$ is the spin wave velocity, and $\Omega_d(T)$ is the longitudinal spin resonance frequency. \cite{12}

On the other hand, $d$ vector of the $d$-soliton can be parametrized by the following expression.

$$\hat{d} = \cos \psi \hat{z} + \sin \psi \hat{y},$$

(5)
\[ \psi(y, z) = \frac{1}{2} \left( \arctan \frac{z + R/2}{y} - \arctan \frac{z - R/2}{y} \right), \]

where we put two half-quantum vortices at \((y, z) = (0, R/2)\) and \((0, -R/2)\).

In the past, similar form of \(\psi\) was also discussed in a different context in regard to \(^3\text{He}\). As one can see, there is a discontinuity in \(\psi\) across the line defined by \(-R/2 < z < R/2\) and \(y = 0\). The spatial configuration of the corresponding \(\hat{d}\) around a pair of half-quantum vortices is shown in Fig.1. As one can see from the figure, the planar defect is parallel to the \(\hat{z}\)-direction or \(c\)-axis. One can also consider the planar defect lying along the \(\hat{y}\)-axis given by

\[ \psi = \frac{1}{2} \left( \arctan \frac{y + R/2}{z} - \arctan \frac{y - R/2}{z} \right), \]

where two half-quantum vortices are located at \((y, z) = (R/2, 0)\) and \((-R/2, 0)\). The configuration of the \(\hat{d}\) vector using the above \(\psi\) is shown in Fig.2. One can easily see that the free energies, \(f_d\), associated with two possible \(\hat{d}\) configurations are the same.

The total free energy of the \(\hat{d}\)-soliton and a pair of half-quantum vortices is given by

\[ f_{\text{pair}} = \frac{1}{2} \chi_N C^2 \int dydz \left[ K(\nabla \Phi)^2 + \sum_{ij} |\partial_i \hat{d}_j|^2 + \xi_d^2 \sin^2 \psi \right] \]
\[ = \frac{1}{2} \chi_N C^2 (\pi K \ln \frac{\lambda}{R} + I_1 + I_2), \]

where \(\Phi\) represents the phase of the order parameter which couples to the external electromagnetic field. The parameter \(K\) is defined by

\[ K = \frac{\rho_s}{\rho_{sp}} = \frac{1 + 1/3F_1(1 + 1/3F_1^0(1 - \rho_0^s))}{1 + 1/3F_1^0(1 - \rho_0^s)}, \]

where \(\rho_s\) and \(\rho_{sp}\) are the superfluid density and the spin superfluid density respectively. \(F_1\) and \(F_1^0\) are the Landau Parameters and \(\rho_0^s\) \((\equiv 1 - Y(T)\) and \(Y(T)\) is the Yosida function) is the superfluid density without the Fermi liquid correction. Notice that \(K(T_c) = 1\) at \(T = T_c\) and \(K(0) = \frac{1 + 1/3F_1}{1 + 1/3F_1^0}\) at \(T = 0\). The temperature dependence of the parameter \(K\) is shown in Fig.3 assuming that \(F_1 = 9\) and \(F_1^0 = 0\). This choice of the parameters will be explained later.

The first term in the first and second lines of Eq.8 is the contribution from two half-quantum vortices and represents the fact that these half-quantum vortices repel each other. \(I_{1,2}\) are the contributions from the second and third terms in the first line of Eq.8. These contributions come from the disclination of the \(\hat{d}\)-vector.

Using the form of \(\psi(y, z)\) discussed above, Eq.8, \(I_1\) and \(I_2\) can be obtained as follows.

\[ I_1 = \frac{1}{4} \int dydz \frac{R^2}{|y^2 + (z + R/2)^2| |y^2 + (z - R/2)^2|} \]
\[ = \pi \ln \frac{R}{\xi}, \]

\[ I_2 = \frac{1}{2\xi_d^2} \int dydz \left( 1 - \frac{y^2 + z^2 - R^2/4}{\sqrt{(y^2 + z^2 - R^2/4)^2 + y^2 R^2}} \right) \]
\[ = \pi \left( \frac{R}{2\xi_d} \right)^2 \ln \frac{4\xi_d}{R}, \]

where \(\xi_d\) is the length scale associated with the spin-orbit coupling defined by \(\xi_d(T) = C(T)/\Omega_d(T)\).

By minimizing \(f_{\text{pair}}\) with respect to \(R\), we obtain the optimal \(R_0\) for the lowest free energy configuration of a pair of half-quantum vortices and the \(\hat{d}\)-soliton. The optimal \(R_0\) is given by

\[ R_0^2 = \frac{(K - 1)2\xi_d^2}{\ln \frac{4\xi_d}{\sqrt{\Omega_0}}} > 0. \]
Here we have assumed that the $\xi_d > \sqrt{\epsilon R_0}/4$. Notice that the half-quantum vortices with a $\hat{d}$-soliton is possible only when $K > 1$ in order to have $R_0 > 0$. Although we have no information about $F_{\alpha}$, it is most likely that $F_{\alpha} \sim 0$. The ratio between the effective mass and the bare mass, $\frac{m^*}{m}$, is about 4, which means that $F_1 \sim 9$. Therefore, $K > 1$ in the superconducting state, as one can see from Eq.\[14.\] Thus this condition is always satisfied below $T_c$. However, the existence of the solution for $R_0$ depends on the value of $K$. We find that the solution exists only if $1 < K \leq 1.5$. For example, for $K = 1.5$, $\xi_d/R = 0.85$. Since the parameter $K$ depends on temperature as shown in Fig.3, we find that a pair of half-quantum vortices with $\hat{d}$-soliton exist only for $0.78 \leq T/T_c < 1$.

Now the free energy of a pair of half-quantum vortices and the $\hat{d}$-soliton at the optimal $R_0$ can be obtained as

$$f_{\text{pair}} = \frac{1}{2} \pi \chi N C^2 [\ln \frac{\lambda}{\xi} + \frac{(K-1)}{2} \ln \frac{\lambda \xi^2}{2(K-1) \xi_d^2} + \frac{K-1}{2}]$$

(12)

where $\Lambda = \ln \frac{\xi_d}{\sqrt{\epsilon R_0}}$. In order to examine the stability of the half-quantum vortices, we have to compare $f_{\text{pair}}$ and the free energy of single vortex, $f_v$. The difference is given by

$$f_v - f_{\text{pair}} = \frac{1}{2} \pi \chi N C^2 [\ln \frac{\lambda}{\xi} + \frac{(K-1)}{2} \ln \frac{\lambda \xi^2}{\Lambda \lambda^2} - \frac{K-1}{2}]$$

(13)

If $f_v - f_{\text{pair}} > 0$ for some values of $K > 1$, a pair of half-quantum vortices are more stable than the conventional single vortex. This condition can be rewritten as

$$\frac{\lambda}{\xi} \left( \frac{\xi_d}{\lambda} \right)^{K-1} > \frac{e^{(K-1)/2} \Lambda^{(K-1)/2}}{2^{(K-1)/2} (K-1)^{(K-1)/2}}$$

(14)

Recalling that the solution of Eq. \[11\] exists if $1 < K \leq 1.5$, one can investigate the stability condition given by Eq.\[14.\]. One can see from Eq.\[11\] and Eq.\[14\] that, as long as $K > 1$, a pair of half-quantum vortices can be stabilized over single vortex under certain conditions for the ratio between $\xi_d$ and $\lambda$. For example, for $K = 1.1$ and $K = 1.5$, the conditions for the stability of a pair of half-quantum vortices over single vortex are given by

$$\frac{\xi_d}{\lambda} > 10^{-11} \quad \text{and} \quad \xi_d / \lambda > 0.0094$$

(15)

respectively. One can see that these conditions are easily satisfied. Here we use $\lambda/\xi = \sqrt{(\lambda \xi_d)/(\xi_b \xi_c)} = 12.186$ which is appropriate for Sr$_2$RuO$_4$. One can also see that the stability of a pair of half-quantum vortices with the $\hat{d}$-soliton is determined by the value of $K$ which depends on temperatures as shown in Fig.3.

Now let us discuss the relation between $\hat{l}$- and $\hat{d}$-solitons. It is difficult to estimate the energy of $\hat{l}$-soliton in terms of the texture free energy given by Eq.\[4.\]. However, it is likely that $\hat{l}$-soliton costs much more energy because, if it exists, the order parameter given by Eq.\[2\] should vanish inside the $\hat{l}$-soliton. Therefore, if there is a natural passage for conversion of $\hat{l}$-solitons to $\hat{d}$-solitons, most of $\hat{l}$-solitons will be converted into $\hat{d}$-solitons.

In summary, assuming that the superconducting state of Sr$_2$RuO$_4$ is characterized by the spin-triplet order parameter with broken time reversal symmetry, we investigated the existence of half-quantum vortices and associated topological defect; $\hat{d}$-soliton. We showed that a pair of half-quantum vortices attached to a $\hat{d}$-soliton can be created in the presence of the magnetic field parallel to the $a$-$b$ plane. It was found that a pair of half-quantum vortices with a $\hat{d}$-soliton is more stable than the conventional single vortex for certain temperatures below $T_c$. As in superfluid $^3$He-A, the presence of $\hat{d}$-soliton may be detected as the deficit in the intensity of electron spin resonance signal at $\omega = \Omega_d$. There should be a clear electron spin resonance signature due to the half-quantum vortices. Detection of the half-quantum vortices by scanning tunneling microscopy (STM) would also provide a convincing evidence for the spin-triplet pairing state with time reversal symmetry breaking.

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[1] D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3* (Tayor & Francis, New York, 1990); A. Leggett, Rev. Mod. Phys. 47, 331 (1975).
[2] G. E. Volovik, *Exotic Properties of Superfluid Helium 3* (World Scientific Pub. Co., Singapore, 1991).
[3] T. M. Rive and M. Sigrist, J. Phys. Cond. Matter, 7, L643 (1995).
[4] M. Sigrist et al, Physica C 317-318, 134 (1999).
[5] G. M. Luke et al, Nature 394, 558 (1998).
[6] K. Ishida et al, Nature 396, 658 (1998).
[7] M. Sigrist and D. F. Agterberg, Prog. Theor. Phys. 102, 965 (1999).
[8] K. Maki, *Solitons* ed. S. E. Trullinger, V. E. Zakharov and V. L. Pokrovskii (North-Holland, Amsterdam, 1986).
[9] D.F.Agterberg, Phys.Rev.Lett 80, 5184 (1998); Phys. Rev. B 58, 14484 (1998)
[10] G.F.Wang and K.Maki, Europhys. Lett 45, 71 (1999)
[11] A. C. Mota, G. Juri, P. Visami, A. Pollini, T. Teruzzi, and K. Aupke, Physica C 185-189, 343 (1999)
[12] H. Y. Kee, Y. B. Kim, and K. Maki, Phys. Rev. B 61, 3584 (2000).
[13] M. M. Salomaa and G. E. Volovik, Phys. Rev. Lett, 55, 1184 (1985).
[14] K. Maki, Phys. Rev. Lett, 56, 1312 (1986).
[15] J. R. Kirtley et al, Phys. Rev. Lett. 81, 2140 (1998); K. A. Moler et al, Science 279, 1193 (1998).
FIG. 1. The spatial configuration of $d$-vector in the $b$-$c$ plane given by Eq. 6. The thick line denotes the domain wall with the length of $R$, which is parallel to the $c$ axis.
FIG. 2. The spatial configuration of $d$-vector in the $b$-$c$ plane given by Eq. 7. The thick line denotes the domain wall with the length of $R$, which is parallel to the $b$ axis.
FIG. 3. The parameter $K$ as a function of the reduced temperature $t = T/T_c$. 

