Any four orthogonal ququad–ququad maximally entangled states are locally markable

Li-Yi Hsu
Department of Physics, Chung Yuan Christian University, Chungli 320314, Taiwan
E-mail: lyhsu@cycu.edu.tw

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Abstract
In quantum state discrimination, the observers are given a quantum system and aim to verify its state from the two or more possible target states. In the local quantum state marking as an extension of quantum state discrimination, there are composite quantum systems and possible orthogonal target quantum states. Distant Alice and Bob are asked to correctly mark the states of the given quantum systems via local operations and classical communication. Here we investigate the local state marking with systems, \( N = 4, 5, 6, \) and \( 7 \). Therein, Alice and Bob allow for three local operations: measuring the local observable either \( \sigma_z \) or \( \sigma_x \) simultaneously, and entanglement swapping. It shows that, given arbitrary four \( 4 \otimes 4 \) systems, Alice and Bob can perform the perfect local quantum state marking. In the \( N = 5, 6 \) cases, they can perform perfect local state marking with specific target states. We conjecture the impossibility of the local quantum state marking given any seven target states since Alice and Bob cannot fulfill the task in the simplest case.

1. Introduction

In quantum communication, an information carrier is a quantum system with information encoded in its quantum state, which should be faithfully read out at the end of the quantum information processing. If classical information is encoded in non-orthogonal quantum states, the no-cloning theorem prevents the readability without any ambiguity. Quantum state discrimination (QSD) refers to the task of distinguishing the state of a given quantum system with a set of possible target non-orthogonal states \([1–4]\). There are two well-known strategies for optimal QSD. One is to minimize the unavoidable errors of the state distinction \([5–8]\), and the other is unambiguous discrimination where conclusive result with no error can be achieved, but an inconclusive answer may occur \([9–14]\). In the scenario of local state discrimination (LSD), a multipartite quantum system is distributed among the distant parties that are tasked with QSD. To distinguish the given quantum state, these spatially separated parties each are limited to performing the local operations and classical communication (LOCC). Even when all the possible states of a composite system are mutually orthogonal, error-free LSD is not guaranteed. For example, regarding all target states as a special complete set of orthogonal product states in a two-qutrit system, there is nonlocality without entanglement that two distant parties cannot implement perfect distinction by any sequence of LOCC \([15]\). Further, a set of multipartite orthogonal quantum states is strongly nonlocal it is locally irreducible for each bipartition of the subsystems \([16]\).

During the past two decades, there is considerable effort devoted to a variant of LSD problems \([17–21]\). An important result in \([17]\) states that arbitrary two orthogonal states, entangled or otherwise, can be locally distinguished with certainty by LOCC. Regarding quantum entanglement as resource in quantum information processing, local discrimination of maximally entangled states is studied in great detail. Ghosh et al proved three 2-qubit Bell states cannot be distinguished with certainty by LOCC \([18]\). In general, \( d + 1 \) or more \( d \otimes d \) maximally entangled states are locally indistinguishable by LOCC \([21, 22]\). However, \( d + 1 \) is not the tight bound for the number of local indistinguishability of \( d \otimes d \) maximally entangled states. It is demonstrated that a specific set of four \( 4 \otimes 4 \) maximally entangled states are locally indistinguishable by...
LOCC [23]. On the other hand, if an extra Bell state is supplied as entanglement discrimination catalyst, perfect LSD on these four orthogonal ququad–ququad maximally entangled states can be achieved. In [23], a specific set of four $4 \otimes 4$ maximally entangled states is investigated. Later a set of six ququad–ququad states is explored, where any four $4 \otimes 4$ maximally entangled states in the set are locally indistinguishable [24]. Very recently, an alternative QSD scheme termed local quantum state marking (LQSM) is proposed [25]. Unlike LSD with only a quantum system, in LQSM, there are $N$ states of the given quantum systems and $N$ possible target states are given. The task of LQSM is to correctly certify the one-to-one correspondence between the given quantum systems and the target states with certainty. In the $N = 4$ case [25], a specific set of four $4 \otimes 4$ maximally entangled states that are locally indistinguishable can be correctly marked by a sequence of LOCC.

In this paper, we generalize the result in [25] as follows. Any four ququad–ququad maximally entangled states can be locally marked with certainty. In the following, four Bell states each can be stabilized by the operators $\pm \sigma_+ \otimes \sigma_-$ and $\pm \sigma_- \otimes \sigma_+$. An unknown Bell state can be distinguished once its eigenvalues of two joint observables $\sigma_+ \otimes \sigma_-$ and $\sigma_- \otimes \sigma_+$ as the two-bit information can be learned simultaneously. A ququad–ququad maximally entangled state is a product state of two Bell states. It is important to learn the knowledge of the Bell states only by LOCC, while the quantum measurements result in the consumption of the measured Bell states. In the strategy of successful LQSM, there are two major ways of distinguishing two-qubit Bell states. Alice and Bob each usually implement one-qubit measurement to learn the eigenvalue of the joint observable either $\sigma_+ \otimes \sigma_-$ or $\sigma_- \otimes \sigma_+$. In this case, even though the partial information of an unknown Bell state can be learned, it can be exploited for the coarse-grained Bell-state classification. The other way is entanglement swapping. After the coarse-grained classification, Alice and Bob may infer and know the Bell states of some composite systems without measurements. Such known and undisturbed Bell states can be exploited as resource of Bell state discrimination. For example, given a ququad–ququad state with its half being known, Alice and Bob can definitely verify the Bell state of the other half by entanglement swapping. In this case, Alice and Bob each perform the Bell state measurement and hence gain full information of the other unknown Bell state. Given any four ququad–ququad maximally entangled states, there are always sufficient known and undisturbed Bell states consumed for further fine-grained classification, which is impossible in LSD. Consequently, Alice and Bob can perform perfect LQSM in $N = 4$ case. It will be shown that only some specific 5, 6 ququad–ququad states are locally markable.

2. Preliminaries and notations

Denote four 2-qubit Bell states by $|\chi_x\rangle$, $x, z \in \{0, 1\}$. Explicitly, $|\chi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$, $|\chi_1\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)$, and $|\chi_T\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$. Note that $|\chi_x\rangle$ is a stabilizer state stabilized by the product operators $(-1)^x \sigma_x \otimes \sigma_z$ and $(-1)^x \sigma_z \otimes \sigma_x$, where $\sigma_x$ and $\sigma_z$ are Pauli matrices. The sets of four ququad–ququad states are defined as $S_{xz} = \{|\chi_x\rangle \otimes |\chi_z\rangle | x_1 + x_2 = X; z_1 + z_2 = Z\}$, where $X, Z \in \{0, 1\}$ and hereafter the addition is modulo 2.

In the LQSM scenario with four ququad–ququad maximally entangled states, let the ququad–ququad state of the $i$th 4-qubit composite system be

$$|\chi^{(i)}\rangle = x_1^{(i)} z_1^{(i)} A_1^{(i)} B_1^{(i)} \otimes x_2^{(i)} z_2^{(i)} A_2^{(i)} B_2^{(i)},$$

where $1 \leq i \leq 4$, the qubits $A_1^{(i)}$ and $A_2^{(i)}$ are with Alice and qubits $B_1^{(i)}$ and $B_2^{(i)}$ with Bob. In the following, the qubit indices $A_1^{(i)}$, $A_2^{(i)}$, $B_1^{(i)}$, and $B_2^{(i)}$ are omitted without causing confusion, and the first and second halves of $|\chi^{(i)}\rangle$ are denoted by $|\chi_1^{(i)}\rangle = x_1^{(i)} z_1^{(i)}$ and $|\chi_2^{(i)}\rangle = x_2^{(i)} z_2^{(i)}$, respectively. Denote the target state set $S_T = \{ |\chi_1\rangle, |\chi_2\rangle \}$ and the system state set $S = \{ |\chi_1^{(i)}\rangle, |\chi_2^{(i)}\rangle | i = 1, 2, 3, 4 \}$. Alice and Bob are tasked to mark the quantum system with the correct target state with certainty. That is, they are to find the bijection relation between $S_T$ and $S$. Restricted to LOCC, Alice and Bob allow for three different local quantum operations as follows.

2.1. Entanglement swapping

Given a ququad–ququad state $|\chi_x\rangle \otimes |\chi_z\rangle$ with $|\chi_x\rangle = |\chi^{(i)}\rangle$ and $|\chi_z\rangle = |\chi^{(j)}\rangle$, Alice (Bob) performs Bell-state measurement on qubits $A_1$ and $A_2$ ($B_1$ and $B_2$). Note that the state $|\chi_x\rangle \otimes |\chi_z\rangle$ is a result of combining the $j$ and $j'$ halves from the $i$- and $i'$ composite systems. Let the post-selected states be $|\chi_{x_1^{(i)}}\rangle_{A_1 A_2}$ and $|\chi_{z_2^{(j)}}\rangle_{B_1 B_2}$, respectively. We have

$$x_1 + x_2 = x_1' + x_2', z_1 + z_2 = z_1' + z_2'.$$
For instance, it is easy to verify that $\left\langle 00 \right|_{A_i B_i} \otimes \left| 00 \right|_{A_j B_j} = \frac{1}{4} \left( \sum_{i' j'=0}^{1} \left| i' j' \right|_{A_i A_j} \otimes \left| i' j' \right|_{B_i B_j} \right)$. As a result, Alice knows the bits $x'_1$ and $z'_1$ and Bob knows the bits $x'_2$ and $z'_2$. If the state $\left| x'_1 z'_1 \right>$ is known, Alice and Bob can deduce $x_2$, $z_2$, and hence $\left| x'_1 z'_1 \right>$ and $\left| x'_2 z'_2 \right>_{B_i B_j}$, they can conclude definitely that the pre-selected state belongs to the set $S(x'_1 + x'_2)(z'_1 + z'_2)$ and hence distinct the state with certainty.

2.2. Local Pauli measurements (LPM)

To resolve the eigenvalues of Pauli product observables, two kinds of local Pauli measurements are exploited. (a) Local Pauli-\(X\) measurements (LPxM). Alice and Bob each measure $\sigma_z$ on the unknown $\left| x'_1 z'_1 \right>$ with the local outcomes $\sigma_x^{(A)}$ and $\sigma_x^{(B)} \in \{1, -1\}$, respectively. Using two-way communication, Alice and Bob both learn that $x_i = \frac{1 - \sigma_x^{(A)} \sigma_x^{(B)}}{2}$, and then derive the bit values $x'_1 = x_x + x_z$ and $x'_2 = x_x - x_z$. On the other hand, even though Alice and Bob may or may not assure $|\chi\rangle$ or $|\chi'\rangle$ after local operations on $|\chi\rangle$, it is still possible to deduce the untouched state $|\chi'\rangle$ with certainty. In this case, $|\chi'\rangle$ can be exploited as resource of Bell state discrimination. Alice and Bob perform the entanglement swapping on $|\chi'\rangle$ and an unknown Bell state $|\chi''\rangle$ to gain full two-bit information of $|\chi''\rangle$. It will be shown that, if they cannot certify $|\chi\rangle$ at first, they can identify $|\chi\rangle$ by later deduction. It is also possible to deduce an intact unknown state based on the previous measurement results. In this case, its undisturbed halves can be exploited as resource of Bell state discrimination.

As the end, denote the Bell-state set by $H_j = \{ | i 0 0 \rangle \mid i = 1, 2, 3, 4 \}$, $j = 1, 2$, with the cardinality denoted by $| H_j |$. Hereafter the $(| H_1 |, | H_2 |)$ case indicates that there are $| H_1 |$ and $| H_2 |$ elements in the sets $H_1$ and $H_2$. Without loss of generality, let $2 \leq | H_2 | \leq | H_1 | \leq 4$. For brevity, let the Bell-states $|a\rangle$, $|b\rangle$, $|c\rangle$, $|d\rangle$ be $H_1$, and $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$, $|\delta\rangle$ be $H_2$, where the states $|a\rangle$, $|b\rangle$, $|c\rangle$, $|d\rangle$ are pairwise different, and similarly for the states $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$, $|\delta\rangle$.

3. Strategies for perfect marking on four ququad–ququad states

(2, 2) case. These four states read

$$
\left| \chi_p \right> = |a\rangle |\alpha\rangle,
\left| \chi_q \right> = |a\rangle |\beta\rangle,
\left| \chi_r \right> = |b\rangle |\alpha\rangle,
\left| \chi_s \right> = |b\rangle |\beta\rangle.
$$

In this simple case, LPM for distinguishing the states $|b\rangle$ and $|a\rangle$ $(|\alpha\rangle$ and $|\beta\rangle)$ are performed on the first (second) half. The non-adaptive two-LPM strategy can be employed. For example, Alice and Bob can assure that the state of the first half is $|a\rangle$ and $|\alpha\rangle$ according to the outcomes of the LPMs on the first and second halves of the state $\left| \chi^{(i)} \right>$, Alice and Bob mark the state of ith composite system as $\left| \chi_p \right>$. That is, $\left| \chi^{(i)} \right> = \left| \chi_p \right>$. (4, 1) case. As addressed before, since these four ququad–ququad states must belong to four different sets $S_{00}$, $S_{01}$, $S_{10}$, and $S_{11}$, Alice and Bob can mark these unknown states by performing entanglement swapping. Here the second halves are exploited as resource of Bell state discrimination.
The four unmarked states
\[ |\chi_P\rangle = |a\rangle |\alpha\rangle, \]
\[ |\chi_q\rangle = |b\rangle |\beta\rangle, \]
\[ |\chi_r\rangle = |c\rangle |\gamma\rangle, \]
\[ |\chi_i\rangle = |d\rangle |\delta\rangle. \] (4)

Here the two-LPM adaptive strategies are usually employed, where Alice and Bob take their local operations based on previous outcomes. Without loss of generality, let \(|a\rangle = |00\rangle, |b\rangle = |01\rangle, |c\rangle = |10\rangle, \text{ and } |d\rangle = |11\rangle\). Given the \(|\chi^{(i)}\rangle\), Alice and Bob perform LPM on its first half for the coarse-grained classification. Then they perform conditional LPxM based on the collective local outcomes. If they learn that \(x^{(i)}_1 = 0 \text{ or } 1\), they can conclude that \(|\chi^{(i)}\rangle \in \{|\chi_P\rangle, |\chi_q\rangle\} \text{ or } |\chi^{(i)}\rangle \in \{|\chi_r\rangle, |\chi_i\rangle\}\). As an example of fine-grained classification, let \(|\chi^{(i)}\rangle \in \{|\chi_P\rangle, |\chi_q\rangle\}\) and they can perform the LPM to distinguish between \(|\alpha\rangle = |\Xi\rangle\) and \(|\beta\rangle = |\Xi^2\rangle\) as follows. If \(x \neq x'(z \neq z')\), they can perform LPxM (LPzM) to learn \(x^{(i)}_2 (z^{(i)}_2)\). If the measurement outcomes indicates that \(x^{(i)}_2 = x (z^{(i)}_2 = z)\), they can verify that \(|\chi^{(i)}\rangle = |\chi_P\rangle\). Otherwise, \(|\chi^{(i)}\rangle = |\chi_q\rangle\).

As addressed before, if these four ququad–ququad states belong to four different sets \(S_00, S_{01}, S_{10}, \text{ and } S_{11}\), Alice and Bob can perform entanglement swapping to mark any unknown ququad–ququad state with certainty.

(4, 2) case. There are two subcases. For the simpler case, let the four unmarked states
\[ |\chi_P\rangle = |a\rangle |\alpha\rangle, \]
\[ |\chi_q\rangle = |b\rangle |\alpha\rangle, \]
\[ |\chi_r\rangle = |c\rangle |\beta\rangle, \]
\[ |\chi_i\rangle = |d\rangle |\beta\rangle. \] (5)

Alice and Bob distinguish between \(|\alpha\rangle\) and \(|\beta\rangle\). If the state of the second half is assured as \(|\alpha\rangle\), next they can perform the LPM that distinguishes between \(|a\rangle\) and \(|b\rangle\). In this way, they can correctly mark \(|\chi_P\rangle\) and \(|\chi_q\rangle\) with certainty. On the other hand, if they assure that the second half is \(|\beta\rangle\), next they perform the LPM that distinguishes between \(|c\rangle\) and \(|d\rangle\) with certainty. In this way, they can correctly mark \(|\chi_r\rangle\) and \(|\chi_i\rangle\) with certainty.

In the less trivial (4, 2) case, the four unmarked states are
\[ |\chi_P\rangle = |a\rangle |\alpha\rangle, \]
\[ |\chi_q\rangle = |b\rangle |\alpha\rangle, \]
\[ |\chi_r\rangle = |c\rangle |\alpha\rangle, \]
\[ |\chi_i\rangle = |d\rangle |\beta\rangle. \] (6)

In the \(k\)th operation on \(|\chi^{(i)}\rangle\), Alice and Bob perform LPM on its second half to distinguish between \(|\alpha\rangle\) and \(|\beta\rangle\) with the first half intact. Let Alice and Bob assure \(|\beta\rangle\) and hence they ascertain \(|\chi^{(k)}\rangle = |\chi_i\rangle\) in \(k\)th operation. Next, \(|d\rangle\) is exploited as discrimination in the following steps.

(a) The \(k = 1\) case. The second intact halves of the other three states \(|\chi^{(2)}\rangle, |\chi^{(3)}\rangle, \text{ and } |\chi^{(4)}\rangle\) are known as \(|\alpha\rangle\), which can be exploited as resource of Bell state discrimination. In addition, Alice and Bob then perform entanglement swapping on the remaining three ququad–ququad states since these three ququad–ququad states each must belong to three different sets among \(S_{00}, S_{01}, S_{10}, \text{ and } S_{11}\).

(b) The \(k \neq 1\) case. There are at most two intact second halves can be exploited as resource of quantum state discrimination. As a result, an alternative strategy is adopted as follows. Alice and Bob perform entanglement swapping on \(|d\rangle\) and \(|\chi^{(k-1)}_1\rangle\). As a result, Alice and Bob can distinguish \(|\chi^{(k-1)}\rangle\). Finally, Alice and Bob can perform some LPM on the first halves of the other two states for further distinction.

(3, 3) case. There are two classes. One includes the following four target states
\[ |\chi_P\rangle = |a\rangle |\beta\rangle, \]
\[ |\chi_q\rangle = |a\rangle |\gamma\rangle, \]
\[ |\chi_r\rangle = |b\rangle |\alpha\rangle, \]
\[ |\chi_i\rangle = |c\rangle |\alpha\rangle. \] (7)
In the first class, without loss of generality, let the states $|a\rangle$ and $|b\rangle$ have some common eigenvalue of either $\sigma_x \otimes \sigma_x$ or $\sigma_y \otimes \sigma_y$, different from that of $|c\rangle$. Let $|\chi^{(k)}\rangle = |c\rangle |\alpha\rangle = |\chi_i\rangle$. The state-marking strategy can be stated as follows.

Step 1: In the $i$th operation, $1 \leq i \leq k$, Alice and Bob perform LPM to distinguish $|c\rangle$ from $|a\rangle$ and $|b\rangle$ in the first half. For example, let $|a\rangle = |00\rangle$, $|b\rangle = |01\rangle$, $|c\rangle = |10\rangle$. We have $\sigma_x \otimes \sigma_x |a\rangle = |a\rangle$, $\sigma_x \otimes \sigma_x |b\rangle = |b\rangle$, and $\sigma_y \otimes \sigma_y |c\rangle = -|c\rangle$. In the $i$th operation, they perform the LPM on $|\chi_i^{(k)}\rangle$. Alice and Bob can perfectly mark the state $|\chi^{(k)}\rangle = |\chi_i\rangle$ with its intact second half, $|\alpha\rangle$ in the $k$-operation. Note that so far the second halves of the other three unmarked states are all different from each other.

Step 2. If $k \neq 2$, in the following operation, Alice and Bob perform entanglement swapping on the states $|\chi^{(k)}_2\rangle$ and $|\chi^{(k+1)}_2\rangle$. As a result, Alice and Bob can mark $|\chi^{(k+1)}\rangle$ with certainty. If $k = 4$, Alice and Bob perform entanglement swapping on $|\chi^{(4)}_2\rangle$ and $|\chi^{(3)}_2\rangle$. As a result, Alice and Bob can correctly mark $|\chi^{(3)}\rangle$ with certainty.

Step 3. Regarding the other two unmarked states, Alice and Bob can correctly mark them with certainty by performing the LPM on their second halves.

As for the other class of $(3, 3)$ case, it includes the following four unmarked states.

\[
\begin{align*}
|\chi_p\rangle &= |a\rangle |\alpha\rangle, \\
|\chi_q\rangle &= |a\rangle |\beta\rangle, \\
|\chi_r\rangle &= |b\rangle |\alpha\rangle, \\
|\chi_s\rangle &= |c\rangle |\gamma\rangle.
\end{align*}
\]

Denote the set $D = \{|\chi_p\rangle, |\chi_q\rangle, |\chi_r\rangle, |\chi_s\rangle\}$. If $\{|\beta\rangle, |\gamma\rangle\} \not\in D$, Alice and Bob can adopt the adaptive strategy similar to the former case to mark these states with certainty. That is, since $|a\rangle$ and $|c\rangle$ must have some common eigenvalue of either $\sigma_x \otimes \sigma_x$ or $\sigma_y \otimes \sigma_y$, different from that of $|b\rangle$ and again let $|\chi^{(k)}\rangle = |\chi_i\rangle$. As a result, Alice and Bob go through Step 1 to 3 in the former case to mark these states with certainty.

On the other hand, let the condition either $\{|\beta\rangle, |\gamma\rangle\} \not\in D$. Without loss of generality, let $\{|\beta\rangle, |\gamma\rangle\} \not\in D$. In this case, the states $|b\rangle$ and $|c\rangle$ must have some common eigenvalue of either $\sigma_x \otimes \sigma_x$ or $\sigma_y \otimes \sigma_y$ that is different from that of $|a\rangle$. Hence Alice and Bob can adopt the following strategy. In the $j$-operation, Alice and Bob perform LPM on the first half of $|\chi^{(j)}\rangle$ to distinguish the set $|a\rangle$ from states $|b\rangle$ and $|c\rangle$. If its first half is $|a\rangle$ and hence $|\chi^{(j)}\rangle \in \{|\chi_p\rangle, |\chi_q\rangle\}$, Alice and Bob perform some LPM on the second half to distinguish between $|\alpha\rangle$ and $|\beta\rangle$; if the first half is either $|b\rangle$ and $|c\rangle$ and hence $|\chi^{(j)}\rangle \in \{|\chi_r\rangle, |\chi_s\rangle\}$, Alice and Bob perform the some LPM to distinguish between $|\alpha\rangle$ and $|\gamma\rangle$. As a result, they can correctly mark these four states with certainty. Finally, we conclude that any four orthogonal ququad–ququad maximally entangled states are locally markable.

### 4. Strategies for perfect marking on five ququad–ququad states

Unlike the four ququad–ququad states case, it is not always locally markable given any five ququad–ququad state. Here we investigate the markability of all possible five-state cases. In the following discussion, the notation $(|H_{i_1}|, |H_{i_2}|, \ldots, |H_{i_l}|)$ indicates that the first state (e.g. $|\alpha\rangle$) appears $i_1$ times, the second state (e.g. $|\beta\rangle$) appears $i_2$ times, …, and the $|H_{i_l}|$-th state appears $i_{l_{|H_{i_l}|}}$ times. Consequently, $i_1 + \cdots + i_{|H_{i_l}|} = 5$.

(4, 2, 1, 4) case. These five unknown states are

\[
\begin{align*}
|\chi_p\rangle &= |a\rangle |\alpha\rangle, \\
|\chi_q\rangle &= |a\rangle |\beta\rangle, \\
|\chi_r\rangle &= |b\rangle |\beta\rangle, \\
|\chi_i\rangle &= |c\rangle |\beta\rangle, \\
|\chi_s\rangle &= |d\rangle |\beta\rangle.
\end{align*}
\]

The strategy is very similar to that taken for the $(3, 3)$ case in (7). In detail, Alice and Bob perform LPM on to distinguish between $|\alpha\rangle$ and $|\beta\rangle$ in the second halves. Let $|\chi^{(k)}\rangle = |a\rangle |\alpha\rangle$, and note the other four states, $|a\rangle |\beta\rangle, |b\rangle |\beta\rangle, |c\rangle |\beta\rangle,$ and $|d\rangle |\beta\rangle$, belong to four different sets $S_0, S_{01}, S_{01}, S_{11}$. If $k \neq 5$, Alice and Bob take the following strategy. Therein, $|\chi^{(4)}_1\rangle = |\alpha\rangle$ is exploited as resource of Bell state discrimination.

Step 1: In the $j$-operation on the state $|\chi^{(j)}_1\rangle$, $j = 1, \ldots, k \leq 4$, Alice and Bob perform LPMs on its second half that distinguishes between $|\alpha\rangle$ and $|\beta\rangle$. In the $k$-operation, Alice can Bob can correctly mark $|\chi^{(k)}\rangle$ as $|\chi_p\rangle$ since they can ascertain that its second half is $|\beta\rangle$. 


Step 2: In the j-operation on the state $|\chi^{(j)}\rangle$ with $j = (k + 1), \ldots, 5$, then Alice and Bob correctly mark the $|\chi^{(j)}\rangle$ by performing entanglement swapping on its first and second halves.

Step 3: If $k \geq 2$, Alice and Bob perform entanglement swapping on $|\chi^{1}_{1}\rangle$ and $|\chi^{(k-1)}_{1}\rangle$. They can infer $|\chi^{(k-1)}_{1}\rangle$ and then learn $|\chi^{(k-1)}\rangle$ with certainty. As a result, Alice and Bob can correctly mark the state $|\chi^{(k-1)}\rangle$.

Step 4: Now Alice and Bob can correctly mark the states $|\chi^{(k-1)}\rangle$, $|\chi^{(l)}\rangle$, and $|\chi^{(l+1)}\rangle$ at least, and then there are at most $l (l = 1, 2)$ intact second halves. Specifically, if $k = 3$ and hence $l = 1$, Alice and Bob can correctly deduce the state $|\chi^{(1)}\rangle$ since $|\chi^{(2)}\rangle$, $|\chi^{(3)}\rangle$, $|\chi^{(4)}\rangle$ and $|\chi^{(5)}\rangle$ can be correctly marked after Step 3; if $k = 4$ and hence $l = 2$, there are some LPM to distinguish between $|\chi^{(1)}_{1}\rangle$ and $|\chi^{(2)}_{1}\rangle$ with certainty, and then $|\chi^{(1)}\rangle$ and $|\chi^{(2)}\rangle$ can be correctly marked.

If $k = 5$, the above strategy is adapted as follows. In Step 1, Alice and Bob perform the j-operation on $|\chi^{(j)}_{2}\rangle$, $j = 1, \ldots, 4$. After these four operations, Alice and Bob can infer that the only intact state is $|\chi^{(5)}\rangle = |\chi_{p}\rangle = |a\rangle |\alpha\rangle$. They exploit the first and second halves of $|\chi_{p}\rangle$, $|a\rangle$ and $|\alpha\rangle$ respectively, as resource of Bell state discrimination. For example, Alice and Bob perform entanglement swapping on the states $|a\rangle$ ($|\alpha\rangle$) and $|\chi^{(4)}_{1}\rangle$ ($|\chi^{(1)}_{1}\rangle$). In this way, they can mark the states $|\chi^{(3)}\rangle$ and $|\chi^{(4)}\rangle$ with certainty. Finally, they perform LPM that distinguishes the first halves of the remaining unmarked states $|\chi^{(1)}\rangle$ and $|\chi^{(2)}\rangle$.

Before processing further, we introduce the following strategy $\mathcal{E}$ for perfect state-marking.

Step 1: In the j-operation on the state $|\chi^{(j)}\rangle$, $j = 1, \ldots, 5$, Alice and Bob perform LPM either first or second halves of all these five states. Based on the measurement outcome and the reordering, these unmarked states are coarse-grained into two sets: $\mathcal{E}_{1} = \{|\chi^{(1)}\rangle, |\chi^{(2)}\rangle, |\chi^{(3)}\rangle\}$ and $\mathcal{E}_{2} = \{|\chi^{(4)}\rangle, |\chi^{(5)}\rangle\}$.

Step 2: Alice and Bob perform the LPM on the other intact half of $|\chi^{(j)}\rangle$ that distinguishes between $|\chi^{(4)}\rangle$ and $|\chi^{(5)}\rangle$. Consequently, they can correctly mark $|\chi^{(4)}\rangle$ with certainty and then $|\chi^{(5)}\rangle$ by deduction. In addition, the intact half of $|\chi^{(5)}\rangle$ is exploited as discrimination resource in Step 3.

Step 3: Alice and Bob perform entanglement swapping on the intact halves of $|\chi^{(5)}\rangle$ and $|\chi^{(3)}\rangle$. Consequently, $|\chi^{(3)}\rangle$ can be correctly marked.

Step 4: Alice and Bob LPM on the intact half of $|\chi^{(2)}\rangle$ that distinguish between the intact halves of the states $|\chi^{(1)}\rangle$ and $|\chi^{(2)}\rangle$. Consequently, they can correctly mark $|\chi^{(2)}\rangle$ and then $|\chi^{(1)}\rangle$ by deduction. It should be emphasized that, after Step 1, the intact halves in $\mathcal{E}_{2}$ and $\mathcal{E}_{3}$ must be pairwisely different

(4, 2, 2, 2) case. Here the five possible target states are

$$
|\chi_{p}\rangle = |a\rangle |\alpha\rangle, \\
|\chi_{q}\rangle = |a\rangle |\beta\rangle, \\
|\chi_{r}\rangle = |b\rangle |\alpha\rangle, \\
|\chi_{s}\rangle = |a\rangle |\beta\rangle, \\
|\chi_{t}\rangle = |d\rangle |\beta\rangle.
$$

(10)

Here Alice and Bob employ strategy $\mathcal{E}$. In Step 1, Alice and Bob perform LPMs on all second halves that distinguish between $|\alpha\rangle$ and $|\beta\rangle$. After the reordering, the states with the same second half are coarse-grained into the same state sets. Without loss of generality, after Step 1, these two state sets are $\mathcal{E}_{3} = \{|\chi^{(1)}\rangle, |\chi^{(2)}\rangle, |\chi^{(3)}\rangle\}$ and $\mathcal{E}_{2} = \{|\chi^{(4)}\rangle, |\chi^{(5)}\rangle\}$, respectively.

Next, in Step 2, Alice and Bob Alice and Bob perform LPMs that distinguish between $|a\rangle$ and $|b\rangle$ on the first half of $|\chi^{(4)}\rangle$. If Alice and Bob learn $|\chi^{(4)}_{1}\rangle = |a\rangle |\beta\rangle$, they can ensure that $|\chi^{(5)}\rangle = |\chi_{p}\rangle (|\chi_{q}\rangle)$ and then deduce that $|\chi^{(5)}\rangle = |\chi_{p}\rangle (|\chi_{q}\rangle)$ with its first half $|\chi^{(5)}_{1}\rangle = |b\rangle (|a\rangle)$ intact. In Step 3, Alice and Bob perform entanglement swapping on $|\chi^{(5)}_{1}\rangle$ and the unknown state $|\chi^{(5)}\rangle$. In this way, $|\chi^{(3)}_{1}\rangle$ and hence $|\chi^{(3)}\rangle$ can be ensured. Finally, Alice and Bob perform LPM that distinguishes between $|\chi^{(1)}_{1}\rangle$ and $|\chi^{(2)}_{1}\rangle$.

Finally, Alice and Bob can correctly mark $|\chi^{(1)}\rangle$ and $|\chi^{(2)}\rangle$ with certainty.

(4, 4, 2, 2, 2) case. Here the five possible target states are

$$
|\chi_{p}\rangle = |a\rangle |\alpha\rangle, \\
|\chi_{q}\rangle = |a\rangle |\gamma\rangle, \\
|\chi_{r}\rangle = |d\rangle |\beta\rangle, \\
|\chi_{s}\rangle = |a\rangle |X\rangle, \\
|\chi_{t}\rangle = |a\rangle |X\rangle.
$$

(11)
where \( X \in \{\beta, \gamma, \delta\} \). Without loss of generality, let \(|a\rangle = |00\rangle, |b\rangle = |11\rangle, |c\rangle = |10\rangle, \) and \(|d\rangle = |11\rangle\), where \(|\{a\rangle, |b\rangle\} \notin D \) and \(|\{c\rangle, |d\rangle\} \notin D\). If \( X \neq \beta \) and \( X \in \{\gamma, \delta\}\), according to Step 1 of strategy \( \mathcal{E} \), Alice and Bob perform \( \sigma_x \otimes \sigma_y \) on the first halves to certify the unknown Bell states belonging to \(|\{a\rangle, |b\rangle\} \) or \(|\{c\rangle, |d\rangle\} \).

After Step 1, these unmarked states are coarse-grained as two sets \( \mathcal{E}_1 = \{|\chi^{(1)}\rangle, |\chi^{(2)}\rangle, |\chi^{(3)}\rangle\} = \{|\chi_p\rangle, |\chi_q\rangle, |\chi_i\rangle\} \) and \( \mathcal{E}_2 = \{|\chi^{(4)}\rangle, |\chi^{(5)}\rangle\} = \{|\chi_i\rangle, |\chi_i\rangle\} \); if \( X = \beta \), Alice and Bob perform \( \sigma_x \otimes \sigma_z \) on the second halves to certify such unknown Bell state belonging to the state set \(|\{a\rangle, |c\rangle\} \) or \(|\{b\rangle, |d\rangle\} \). After Step 1, these unmarked states are coarse-grained as two sets \( \mathcal{E}_1 = \{|\chi^{(1)}\rangle, |\chi^{(2)}\rangle, |\chi^{(3)}\rangle\} = \{|\chi_p\rangle, |\chi_i\rangle, |\chi_i\rangle\} \) and \( \mathcal{E}_2 = \{|\chi^{(4)}\rangle, |\chi^{(5)}\rangle\} = \{|\chi_i\rangle, |\chi_i\rangle\} \). In Step 2, Alice and Bob perform the LPM that distinguishes between the untouched halves of \(|\chi^{(1)}\rangle \) and \(|\chi^{(2)}\rangle \) on the untouched half of \(|\chi^{(4)}\rangle \), and hence Alice and Bob can definitely mark \(|\chi^{(4)}\rangle \) and \(|\chi^{(5)}\rangle \). In Step 3, Alice and Bob can correctly mark the state \(|\chi^{(3)}\rangle\) by performing entanglement swapping on the intact halves of \(|\chi^{(1)}\rangle \) and \(|\chi^{(3)}\rangle \). Finally, Alice and Bob can correctly mark the states in \( \mathcal{E}_2 \) by performing the LPM that distinguishes between the intact halves of the states \(|\chi^{(1)}\rangle \) and \(|\chi^{(2)}\rangle \) on that half of \(|\chi^{(2)}\rangle \).

In the following \((k, 3)\) cases with \( k = 3, 4 \), there are some unmarkable five-state sets and conditionally markable ones.

\((4, 3_{1+1+3})\) case. There are two types of these five target states, where one reads

\[
|\chi_p\rangle = |a\rangle |\alpha\rangle, \\
|\chi_q\rangle = |a\rangle |\beta\rangle, \\
|\chi_\gamma\rangle = |b\rangle |\gamma\rangle, \\
|\chi_\delta\rangle = |c\rangle |\gamma\rangle, \\
|\chi_i\rangle = |d\rangle |\gamma\rangle,
\]

and the other reads

\[
|\chi_p\rangle = |a\rangle |\alpha\rangle, \\
|\chi_q\rangle = |a\rangle |\gamma\rangle, \\
|\chi_\delta\rangle = |b\rangle |\beta\rangle, \\
|\chi_i\rangle = |c\rangle |\alpha\rangle, \\
|\chi_i\rangle = |d\rangle |\beta\rangle.
\]

Without loss of generality, let \(|\{a\rangle, |b\rangle\} \notin D \) and hence \(|\{c\rangle, |d\rangle\} \notin D\). Similarly, Alice and Bob can LPMs on the first halves to classify these states in either (12) or (13) into some three-state set \( \mathcal{E}_1 \) and the two-state set \( \mathcal{E}_2 = \{|\chi^{(1)}\rangle, |\chi^{(3)}\rangle\} = \{|\chi_i\rangle, |\chi_i\rangle\} \). However, one cannot correctly mark the states \(|\chi^{(3)}\rangle\) and \(|\chi^{(5)}\rangle\) since their intact second halves are the same as \(|\gamma\rangle\). As a result, these five ququad-ququad states in the \((4, 3_{1+1+3})\) case are unmarkable.

\((4, 3_{1+2+2})\) case. Similarly, there are two types of these five target states, where one reads

\[
|\chi_p\rangle = |a\rangle |\alpha\rangle, \\
|\chi_q\rangle = |a\rangle |\gamma\rangle, \\
|\chi_\gamma\rangle = |b\rangle |\beta\rangle, \\
|\chi_i\rangle = |c\rangle |\alpha\rangle, \\
|\chi_i\rangle = |d\rangle |\beta\rangle,
\]

and the other reads

\[
|\chi_p\rangle = |a\rangle |\alpha\rangle, \\
|\chi_q\rangle = |a\rangle |\beta\rangle, \\
|\chi_\delta\rangle = |b\rangle |\gamma\rangle, \\
|\chi_i\rangle = |c\rangle |\alpha\rangle, \\
|\chi_i\rangle = |d\rangle |\beta\rangle.
\]

Notably, at least one of these two conditions \(|\{a\rangle, |b\rangle\} \notin D \) and \(|\{a\rangle, |d\rangle\} \notin D \) must hold. Let \(|\{a\rangle, |b\rangle\} \notin D \) \((|\{a\rangle, |d\rangle\} \notin D \) and hence \(|\{c\rangle, |d\rangle\} \notin D \) \((|\{b\rangle, |c\rangle\} \notin D \)). In Step 1 of strategy \( \mathcal{E} \), Alice and Bob can perform LPM on the first halves to certify the unknown Bell states belonging to the state set \(|\{a\rangle, |b\rangle\} \) or \(|\{c\rangle, |d\rangle\} \) \(|\{a\rangle, |d\rangle\} \) or \(|\{b\rangle, |c\rangle\} \). As a result, five states in (14) can be coarse-grained into two sets

\[
\mathcal{E}_3 = \{|\chi^{(1)}\rangle, |\chi^{(2)}\rangle, |\chi^{(3)}\rangle\} = \{|\chi_p\rangle, |\chi_q\rangle, |\chi_i\rangle\} \{(|\chi_p\rangle, |\chi_q\rangle, |\chi_i\rangle\} and \mathcal{E}_2 = \{|\chi^{(4)}\rangle, |\chi^{(5)}\rangle\} = \{|\chi_i\rangle, |\chi_i\rangle\}\].
In Step 2, Alice and Bob can correctly mark the states $|x^4\rangle$ and $|x^5\rangle$ by performing the LPM distinguishing between the states $|\alpha\rangle$ and $|\beta\rangle$ on the second half of $|x^4\rangle$. In Step 3, they perform entanglement swapping on $|x^{(3)}\rangle$ and $|x^{(5)}\rangle$, which they can deduce that $|x^{(3)}\rangle$ with certainty. Finally, Alice and Bob can correctly mark the states $|x^{(1)}\rangle$ and $|x^{(2)}\rangle$ after the LPM that distinguishes the states on $|x^{(1)}\rangle$ and on $|x^{(2)}\rangle$ is performed on $|x^{(2)}\rangle$. As a result, these five ququad–ququad states in (14) are markable.

On the other hand, the states in (15) are conditionally markable. Specifically, if $\{a, b\} \notin D$, Alice and Bob can perform state-marking using the strategy $S$ as in the previous (4, $S_{1+1+3}$) case. It is because Alice and Bob can coarse-grain these five ququad–ququad states into the sets $E_1 = \{\{x_P\}, \{x_q\}, \{x_i\}\}$ and $E_2 = \{\{x_s\}, \{x_t\}\}$ at the end of Step 1. In this case, the states in (15) are unmarkable. However, if $\{a, b\} \in D$, such coarse-graining is impossible. It is because we have $E_3 = \{\{x_P\}, \{x_q\}, \{x_i\}\}$ or $E_3 = \{\{x_P\}, \{x_q\}, \{x_i\}\}$ after Step 1. However, the second halves of the states in $E_3$ are not pairwise different. As a result, the states in (15) are unmarkable.

(3, 3) case. There are three different types for this case: ($S_{1+1+3}$, $S_{1+1+3}$), ($S_{1+1+3}$, $S_{1+2+2}$), and ($S_{1+2+2}$, $S_{1+2+2}$).

(3, 1+1+3, 1+1+3) case. Five states are

$$
|\chi_P\rangle = |a\rangle |\alpha\rangle,
|\chi_q\rangle = |a\rangle |\beta\rangle,
|\chi_s\rangle = |a\rangle |\gamma\rangle,
|\chi_t\rangle = |b\rangle |\alpha\rangle,
|\chi_i\rangle = |c\rangle |\beta\rangle.
$$

It is impossible to achieve the perfect marking using strategy $S$. Specifically, it can be easy to verify that either $E_2 = \{\{x_P\}, \{x_q\}\}$ or $E_2 = \{\{x_s\}, \{x_t\}\}$, where the intact halves are the same.

As for the subcase ($S_{1+1+3}$, $S_{1+2+2}$), these five states are

$$
|\chi_P\rangle = |a\rangle |\alpha\rangle,
|\chi_q\rangle = |a\rangle |\beta\rangle,
|\chi_s\rangle = |a\rangle |\gamma\rangle,
|\chi_t\rangle = |b\rangle |\alpha\rangle,
|\chi_i\rangle = |c\rangle |\beta\rangle.
$$

These states are conditionally markable. In details, if $\{b, c\} \notin D$, Alice and Bob can perform the LPM that distinguishes the state $|a\rangle$ from $|b\rangle$ and $|c\rangle$. As a result, the states in (17) after Step 1 of the strategy $S$ can be classified into two sets $E_4 = \{\{x_P\}, \{x_q\}, \{x_i\}\}$ and $E_5 = \{\{x_s\}, \{x_t\}\}$, which Alice and Bob can achieve perfect state-marking through Steps 2 to Step 4. On the other hand, if $\{b, c\} \in D$, Alice and Bob cannot coarse-grain these states in Step 1 and hence perfect state marking is impossible.

($S_{1+2+2}$, $S_{1+2+2}$) case. There are two types. The five states in one type are

$$
|\chi_P\rangle = |a\rangle |\alpha\rangle,
|\chi_q\rangle = |a\rangle |\beta\rangle,
|\chi_s\rangle = |b\rangle |\alpha\rangle,
|\chi_t\rangle = |b\rangle |\beta\rangle,
|\chi_i\rangle = |c\rangle |\gamma\rangle,
$$

where the condition either $\{a\}, \{b\} \notin D$ or $\{b\}, \{c\} \notin D$ must hold. Without loss of generality, let $\{b\}, \{c\} \notin D$, Alice and Bob perform the LPM distinguishing the state $|a\rangle$ from $|b\rangle$ and $|c\rangle$ on the first halves of these ququad–ququad states that distinguish the state that in Step 1 of strategy $S$. Consequently, the states in (18) can be coarse-grained as two sets $E_5 = \{\{x_s\}, \{x_t\}\}$ and $E_5 = \{\{x_P\}, \{x_q\}\}$. As a result, Alice and Bob can achieve perfect state-marking through Steps 2 to Step 4 since the intact halves in the same set are pairwise different.

On the other hand, these five states in the other type are
second halves of these five states into the sets
of the only unmeasured state is impossible.

Although they can reach the coarse-graining state sets $\mathcal{C}_3 = \{ |\chi_1\rangle, |\chi_2\rangle, |\chi_3\rangle \}$ and $\mathcal{C}_2 = \{ |\chi_4\rangle, |\chi_5\rangle \}$. However, if $\{ |b\rangle, |c\rangle \} \notin D$, Alice and Bob can perform the LPM distinguishing the state $b$ from the states $|a\rangle$ and $|c\rangle$ in Step 1 of strategy $\mathcal{S}$. Although they can reach the coarse-graining state sets $\mathcal{C}_3 = \{ |\chi_1\rangle, |\chi_2\rangle, |\chi_3\rangle \}$ and $\mathcal{C}_2 = \{ |\chi_4\rangle, |\chi_5\rangle \}$. However, two of the second halves of the states in $\mathcal{C}_3$ are in the same state, $|\beta\rangle$, and the perfect state-marking is impossible.

5. Discussion

We have thoroughly investigated local state-marking of $N$ ququad–ququad states with $N = 4, 5$. The state-marking of six and more ququad–ququad states is more complicated and beyond our scope. Here we explore the simplest case with $|H_2| = 2$ for $N = 6$ and 7. In the case of unmarkable five ququad–ququad states, the remaining intact Bell states in the set either $\mathcal{C}_3$ or $\mathcal{C}_2$ are the same and fail the fine-graining from Step 2 to Step 4. It will be shown that even in the simplest case of seven ququad–ququad states, perfect state-marking is impossible. It is because there are three unknown Bell states in the set either $\mathcal{C}_3$ or $\mathcal{C}_4$, which is impossible to distinguish them perfectly. On this basis, we conjecture that perfect state-marking of any seven ququad–ququad states is impossible.

As for $N = 6$, given the $(k, 2k+3)$ case with $k = 3, 4$, the strategy $\mathcal{S}$ can be further exploited for the perfect state-marking. Therein, these six ququad–ququad states read

$$
\begin{align*}
|\chi_P\rangle &= |a\rangle |\alpha\rangle, \\
|\chi_q\rangle &= |a\rangle |\beta\rangle, \\
|\chi_r\rangle &= |b\rangle |\alpha\rangle, \\
|\chi_s\rangle &= |b\rangle |\beta\rangle, \\
|\chi_t\rangle &= |c\rangle |\alpha\rangle, \\
|\chi_u\rangle &= |c\rangle |\beta\rangle.
\end{align*}
$$

(19)

where $|x\rangle = |c\rangle$ if $k = 3$ and $|x\rangle = |d\rangle$ if $k = 4$. For state-marking, Alice and Bob perform the LPM on the second halves that distinguish $|\alpha\rangle$ and $|\beta\rangle$. Without loss of generality, let Alice and Bob perform LPM on the second halves of these five states $|a\rangle |\alpha\rangle, |b\rangle |\alpha\rangle, |c\rangle |\alpha\rangle, |a\rangle |\beta\rangle, |b\rangle |\beta\rangle$. They can deduce that the second half of the only unmeasured ququad–ququad state is $|\beta\rangle$, and then perform entanglement swapping on $|x\rangle |\beta\rangle$ to distinguish the state $|x\rangle$. They can correctly mark the state $|x\rangle |\beta\rangle$. In addition, they can coarse-grain these five states into the sets $\mathcal{C}_3 = \{ |\chi^{(1)}\rangle, |\chi^{(2)}\rangle, |\chi^{(3)}\rangle \} = \{ |\chi_P\rangle, |\chi_q\rangle, |\chi_r\rangle \}$ and $\mathcal{C}_2 = \{ |\chi^{(4)}\rangle, |\chi^{(5)}\rangle \} = \{ |\chi_s\rangle, |\chi_t\rangle \}$. At last, Alice and Bob Alice and Bob can achieve perfect state-marking through Steps 2 to Step 4. It is impossible to perform perfect state-marking in the simplest $(4, 2k+4)$ case of seven ququad–ququad states using the strategy $\mathcal{S}$. In detail, these seven ququad–ququad states are

$$
\begin{align*}
|\chi_P\rangle &= |a\rangle |\alpha\rangle, \\
|\chi_q\rangle &= |b\rangle |\alpha\rangle, \\
|\chi_r\rangle &= |c\rangle |\alpha\rangle, \\
|\chi_s\rangle &= |b\rangle |\beta\rangle, \\
|\chi_t\rangle &= |c\rangle |\beta\rangle, \\
|\chi_u\rangle &= |c\rangle |\beta\rangle, \\
|\chi_v\rangle &= |d\rangle |\beta\rangle.
\end{align*}
$$

(20)

Similar to the strategy in $(k, 2k+3)$ case of six ququad–ququad states, Alice and Bob perform the LPM on the second halves of these six unmarked states that distinguish the states $|\alpha\rangle$ and $|\beta\rangle$. They can deduce the second half of the remaining intact ququad–ququad state and then perform entanglement swapping on this state to correctly mark this state, which can be either $|X\rangle |\alpha\rangle, X \in \{ a, b, c \}$ or $|Y\rangle |\beta\rangle, Y \in \{ a, b, c, d \}$. Without
loss of generality, let the remaining intact ququad–ququad state be $|α⟩$. In this case, they can coarse-grain the other six states into the sets $\mathcal{C}_1 = \{|\chi^{(1)}⟩, |\chi^{(2)}⟩, |\chi^{(3)}⟩, |\chi^{(4)}⟩\}$ and $\mathcal{C}_2 = \{|\chi^{(5)}⟩, |\chi^{(6)}⟩\} = \{|\chi_0⟩, |\chi_1⟩\}$. Alice and Bob can perform the LPM that distinguishes between the states $|b⟩$ and $|c⟩$ on $|\chi^{(5)}⟩$, and then correctly mark the $|\chi^{(5)}⟩$ by deduction in Step 2.

Although $|\chi^{(6)}⟩$ can be exploited as discrimination resource to correctly mark the state $|\chi^{(4)}⟩$ in Step 3, it is impossible to distinguish the remaining three orthogonal Bell states. On the other hand, let the remaining intact ququad–ququad state be $|α⟩|β⟩$, the coarse-grained sets are $\mathcal{C}_3 = \{|\chi_1⟩, |\chi_2⟩, |\chi_3⟩\}$ and $\mathcal{C}_4 = \{|\chi_4⟩, |\chi_5⟩, |\chi_6⟩\}$. Alice and Bob cannot distinguish the remaining three orthogonal Bell states in each of the sets $\mathcal{C}_3$ and $\mathcal{C}_4$.

To reveal the advantage of quantum local state marking over quantum state discrimination in quantum secret communication [25], let three parties—Alice, Bob, and Charlie—be spatially separated. To send a secret message to Alice and Bob, Charlie prepares an ensemble of $N$ orthogonal bipartite states that are not locally indistinguishable. If Alice and Bob are tasked to perform perfect LSM, the average communication bits per ququad are larger than those of being tasked to perform the perfect LSD. Furthermore, since arbitrary four ququad–ququad states can be perfectly marked, one can improve the security in the $N = 4$ case. Charlie can prepare eight Bell states randomly and then the halves of each Bell state are sent to Alice and Bob, respectively. In this way, Alice and Bob have no prior information on these four orthogonal ququad–ququad states. To reveal the secret message before perfect LQSM, Charlie announces which two Bell states are paired in order to constitute a ququad–ququad state, and the target states set $S^T$. Only when Alice and Bob collaborate without any dishonest behavior, they can reveal the secret message. On the other hand, there are Bell-state-based quantum cryptography protocols such as authenticated quantum secret sharing [26], authenticated semi-quantum direct communication protocols [27], arbitrated quantum signature schemes [28], and bidirectional quantum secure communication schemes based on Bell states [29]. It is believed that the LQSM can be also exploited for increasing the efficiency in the security check and message decoding process.

In the end, the LQSM of the ququad–ququad states can be realized using the quantum circuit. Therein, the preparation of any ququad–ququad state requires at least four qubits with pairwise C-NOT operations. Alice and Bob each have to perform the single-qubit conditional unitary operations and measurements, and entanglement swapping on two local qubits. However, the unavoidable noise makes perfect LQSM hardly possible in the current experiments. For example, regarding the qubits as the atoms or photons in the cavity QED [30, 31], it is also interesting to study the LQSM process under the influences of Markovian and non-Markovian effects from the environments, which is however beyond the scope of the paper [32, 33].

In conclusion, we investigate the local quantum state marking of $N$ orthogonal ququad–ququad maximally entangled states with $N$ target states. It is found that any four ququad–ququad entangled states can be perfectly marked, but such task can be only achieved given specific sets of five or six ququad–ququad entangled states. It is shown that in the simplest case of seven ququad–ququad entangled states cannot be correctly marked. There is another unexplored local quantum state marking problem as follows. There are $N$ orthogonal ququad–ququad maximally entangled states with $T$ target states, where $N \neq T$. In this case, some inference without local operations may not be possible, and it is unclear whether there suffices enough resource of Bell state discrimination in entanglement swapping for gaining full information of some unknown Bell states. As the end, let $T = N - 1$. An interesting open question arises as follows. Is it possible for Alice and Bob to verify the unknown ququad–ququad maximally entangled state without the given target state?

**Data availability statement**

No new data were created or analysed in this study.

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**ORCID iD**

Li-Yi Hsu ♦ https://orcid.org/0000-0003-3579-3007
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