Displacement-noise-free gravitational-wave detection with a single Fabry-Perot cavity: a toy model

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We propose a detuned Fabry-Perot cavity, pumped through both the mirrors, as a toy model of the gravitational-wave (GW) detector partially free from displacement noise of the test masses. It is demonstrated that the noise of cavity mirrors can be eliminated, but the one of lasers and detectors cannot. The isolation of the GW signal from displacement noise of the mirrors is achieved in a proper linear combination of the cavity output signals. The construction of such a linear combination is possible due to the difference between the reflected and transmitted output signals of detuned cavity. We demonstrate that in low-frequency region the obtained displacement-noise-free response signal is much stronger than the $f_{\text{sc}}$-limited sensitivity of displacement-noise-free interferometers recently proposed by S. Kawamura and Y. Chen. However, the loss of the resonant gain in the noise cancelation procedure results is the sensitivity limitation of our toy model by displacement noise of lasers and detectors.

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I. INTRODUCTION

Currently the search for gravitational radiation from astrophysical sources is conducted with the first-generation Earth-based laser interferometers [1,2] (LIGO in USA [3,4,5], VIRGO in Italy [6,7], GEO-600 in Germany [8,9], TAMA-300 in Japan [10,11] and ACIGA in Australia [12,13]). The development of the second-generation GW detectors (Advanced LIGO in USA [14,15], LCGT in Japan [16]) is underway.

The sensitivity of the first-generation detectors is limited by a great amount of noises of various nature: seismic and gravity-gradient noise at low frequencies (below $\sim 50$ Hz), thermal noise in suspensions, bulks and coatings of the mirrors ($\sim 50 \div 500$ Hz), photon shot noise (above $\sim 500$ Hz), etc. It is expected that the sensitivity of the second-generation detectors will be limited by the noise of quantum nature arising due to Heisenberg’s uncertainty principle: the more precise is the measurement of the test mass coordinate, the more disturbed becomes its momentum which in turn evolves into the disturbance of the coordinate, thus ultimately limiting the sensitivity [17].

The optimum between measurement noise (photon shot noise) and back-action noise (radiation pressure noise) is called the Standard Quantum Limit (SQL) [18,19,20].

Though the start of operation of the second-generation detectors is planned for the next decade, theoretical investigations of the third-generation prototypes have already begun [21,22,23,24,25,26]. It is expected that the barrier of SQL will be overcome and the sensitivity of the third-stage detectors will be at least an order of magnitude better than the SQL of a free mass.

Recently in a series of papers [21,28,29] S. Kawamura and Y. Chen proposed several topologies of the GW detectors, both ground- and space-based, which are free from displacement noise of the test masses — the noise produced by external fluctuative forces. For the ground-based optical interferometers this implies the insusceptibility to seismic noise, thermal noise in suspensions of the mirrors, etc. However, the most intriguing feature of displacement-noise-free interferometry (DFI) is the straightforward overcoming of SQL (since radiation pressure noise is canceled) without the need of implementation of very complicated and vulnerable schemes for Quantum-Non-Demolition (QND) measurements [21,31,31,32]. One only needs to increase laser power to suppress quantum shot noise and achieve the arbitrarily high sensitivity.

The isolation of the GW signal from fluctuating displacements of the test masses in the DFI schemes proposed by S. Kawamura et al. is possible due to the fact that the interaction of GWS with a laser interferometer is distributed, as viewed from both the transverse-traceless (TT) gauge [33,34,35] and the local Lorentz (LL) gauge [34,36,37].

In the TT gauge test masses are immovable, i.e. have fixed spacial coordinates and thus do not sense the gravitational wave. However, GW couples to the light wave in this gauge producing a non-vanishing phase shift. This can be thought of as an apparent change of the coordinate velocity of light. Even if the test masses are not ideally inertial and follow non-geodesic motion then the interferometer will respond differently to the test masses motions and the gravitational wave. This difference allows the cancelation of displacement noise in a proper linear combination of the interferometer response signals.

From the viewpoint of local observer (the LL gauge) interaction of the GW with a laser interferometer adds up to two effects. The first one is the motion of the test masses in the GW tidal force-field. In this aspect GWs are indistinguishable from any non-GW forces since both are sensed by the light wave only at the moments of re-
flection from the test masses. If the linear scale $L$ of a GW detector is much smaller than the gravitational wavelength $\lambda_{gw}$ (the so-called long-wave approximation) then the effect of the GW force-field is of the order of $h(L/\lambda_{gw})^\alpha$, where $h$ is the absolute value of the GW amplitude. Relative motion of the test masses, separated by a distance $L$, in any force field cannot be sensed by one of them faster than $L/c$, thus resulting in the rise of terms of the order of $O[h(L/\lambda_{gw})^3]$ describing time delays which take the light wave to travel between the masses. Second, GW directly couples to the light wave effectively changing the coordinate velocity of light (but in a different manner as compared to the TT gauge). In long-wave approximation this effect has the order of $O[h(L/\lambda_{gw})^2]$. Therefore, from the viewpoint of local observer displacement-noise-free interferometry necessarily implies the cancelation of non-GW forces along with the GW force-field leaving a non-vanishing information about the direct coupling of the GW to light.

It was pointed in Refs. [28, 29] that in order the GW detector to be a truly displacement-noise-free interferometer it should be also free from optical laser noise which is indistinguishable from laser displacement noise. Cancelation of the optical noise in interferometric experiments is usually achieved by implementing the differential (balanced) schemes of measurements: in conventional interferometers (such as LIGO) it is the Michelson topology and in DFIs proposed in Ref. [29] it is the Mach-Zehnder (MZ) topology.

The analysis performed by S. Kawamura et al. in Ref. [28] showed, however, that though it is possible to eliminate all the information about displacement and laser noises from the data, the sensitivity to GWs at low frequencies turns out to be limited by the $(\omega_{gw}L/c)^2$-factor for 3D (space-based) configurations and $(\omega_{gw}L/c)^3$-factor for 2D (ground-based) configurations. In the latter case this means the cancelation of all the terms of the order of $O[h(L/\lambda_{gw})^\alpha]$, $\alpha = 0, 1, 2$. For the signals around $\omega_{gw}/2\pi \approx 100 \text{ Hz}$ and $L \approx 4 \text{ km}$, the DFI sensitivity of the ground-based detector is $\sim 10^6$ times worse than the one of the conventional Michelson interferometer (i.e. a single round-trip detector). The proposed MZ-based configurations could be modified with power- and signal-recycling mirrors, artificial time-delay devices [38], but nevertheless, the potentially achievable sensitivity remains incomparable with conventional non-DFI detectors. However, it is worth noting that the basic features of DFI concept has been recently demonstrated experimentally [40].

In this paper we continue investigation of the noise cancelation issue in large-scale interferometric experiments and present the result of intensive discussions inside the GW community [41, 42]. Namely, we propose a simple toy model of the GW detector partially free from displacement noise of the test masses with strong enough GW response. It should be stressed that our model is now widely discussed with regard to its implementation in the 3rd generation GW detectors and may soon be realized in a prototype lab experiment [43]. The basic element of our model is a single detuned Fabry-Perot (FP) cavity pumped through both of its movable, partially transparent mirrors; lasers and detectors are assumed to be located on auxiliary (also movable) platforms. Pump waves in different input ports are assumed to be orthogonally polarized in order the corresponding output waves to be separately detectable and to exclude nonlinear coupling of the corresponding intracavity waves. By properly combining the signals of all four output ports of the cavity (a pair of reflection and transmission ports for each of the pumps) an experimentalist can remove the information about the fluctuations of the mirrors coordinates from the data. Below we call the proposed scheme a double-pumped Fabry-Perot (DPFP) cavity. In this paper we do not consider the problem of optical laser noise cancelation in full detail and thus “displacement noise” refers only to mechanical motions of the test masses further. However, after detailed analysis of a single DPFP cavity we propose and consider qualitatively one of the possible DPFP-based balanced optical setups, namely a LIGO-type topology of Michelson interferometer with two DPFP cavities in its arms.

The isolation of the GW signal from displacement noise in a DPFP cavity is achieved in a different manner as compared to MZ-based interferometers. The basic idea is that when a detuned FP cavity is pumped through one of the mirrors (mirror $a$ for definitness), the reflected and transmitted waves respond differently to the motion of mirrors $a$ and $b$. The physical reason for this is that the reflected wave, in contrast to the transmitted one, includes the component due to the prompt reflection from mirror $a$. This component measures only the position of mirror $a$ but not the position of mirror $b$. By properly combining both the response signals one can eliminate the information about the fluctuating coordinate of mirror $a$ completely, leaving only the part of the signal containing the displacement noise of mirror $b$ plus its displacement due to GW (assuming we work in the local Lorentz frame of mirror $a$). By pumping the cavity through mirror $b$ and performing the similar operations, one can eliminate the information about displacement noise of mirror $b$. Ultimately, the proper linear combination of all four output signals cancels displacement noise of both the mirrors leaving a non-vanishing GW signal.

In the resonant regime both the response signals (corresponding to one of the pumps) carry identical information about the mirrors coordinates and thus cannot be combined to cancel their fluctuations. This happens because the prompt reflection does not occur for the resonant pump.

Note that the LL-effect of GW direct coupling to light plays no role in this noise-cancelation scheme: the notion of the GW in our analysis can be approximated with the corresponding tidal force-field. This means that the leading order of the DFI signal we obtain will be $h(L/\lambda_{gw})^0$.

The “payment” for isolation of the GW signal from displacement noise in our case is the loss of the optical
resonant gain of the order of $c/(\gamma L)$, where $\gamma$ is the cavity half-bandwidth. In conventional interferometers this resonant factor describes the accumulation of the low-frequency GW signal by the light wave circulating in a FP cavity. The DFI response signal of a DPFP cavity becomes limited with the factor of the order of unity as compared to the limiting factor $(\omega_{gw}L/c)^3 \approx 6 \times 10^{-7}$ of the double Mach-Zehnder configuration for $L \approx 4$ km and $\omega_{gw}/2\pi \approx 100$ Hz. This difference between the MZ-based topologies and the DPFP topology arises due to the different mechanisms of noise cancelation: the former utilizes the LL-effect of direct interaction between the GW and light, while the latter utilizes the asymmetry between the output signals of detuned cavity.

However, the most dramatic consequence which the loss of the resonant gain results in is that the displacement noise of the auxiliary platforms (where lasers and detectors are mounted) becomes comparable to the DFI response. The reason for this is the relative principle itself: only relative measurements of the test masses positions and velocities are allowed; in our case we are able to measure the positions of cavity mirrors only with respect to the mentioned auxiliary platforms. It is natural then that the precision of the coordinate measurements is limited with the noises of reference test masses (see Sec. II below). Remind also that in conventional non-DFI (LIGO) topology these noises are negligible since they are suppressed finesse times as compared to the GW signal (and displacement noise of the mirrors). The incomplete cancelation of displacement noise is the major (fundamental) limitation of our model. To increase its SNR in practice one will need to install lasers and detectors on heavy platforms (to suppress displacement noise due to external forces) cooled down to cryogenic temperatures (to suppress internal thermal noise).

Note that the non-resonant regime implies the rise of the electromagnetic ponderomotive force (and corresponding optical rigidity) acting on the mirrors of a FP cavity $\gamma_L$ $\approx$ $3\times10^{-7}$, $\approx$ $4\times10^{-7}$. However, in this paper we do not take into account the effects of radiation pressure. In particular, optical rigidity vanishes if pump waves in different input ports have detunings with equal absolute values but opposite signs.

This paper is split into two logical parts for convenience. First, in Sec. II using a simple mathematical model of the cavity we illustrate the basic physics underlying the proposed method of noise cancelation. Next, in Secs. III—V we perform strict calculations. Finally, in Sec. VI we discuss some issues associated with displacement noise cancelation, consider Michelson/DPFP balanced optical setup for laser optical noise cancelation and briefly outline further prospects.

II. BASIC PHYSICAL MECHANISM OF NOISE CANCELATION

Before analyzing our scheme in full detail we consider the simplest (Newtonian) model of a FP cavity to demonstrate the basic physics underlying the mechanism of noise cancelation.

Let us derive the response signals of a FP cavity from the intuitive reasonings assuming that the cavity is short enough (we neglect time delays) and the GW can be treated as a classical force acting on the test masses. Consider a system illustrated in Fig. 1: FP cavity assembled of two movable, partially transparent, mirrors $a$ and $b$ is pumped by laser $L_1$ through mirror $a$. Detectors $D_1$ and $D_2$ measure the phases of reflected and transmitted waves correspondingly. $b$. The same cavity is pumped by laser $L_2$ through mirror $b$. Detectors $D_3$ and $D_4$ measure the phases of reflected and transmitted waves correspondingly.

FIG. 1: A simple noise-cancelation setup. a. Fabry-Perot cavity assembled of two movable, partially transparent, mirrors $a$ and $b$ is pumped by laser $L_1$ through mirror $a$. Detectors $D_1$ and $D_2$ measure the phases of reflected and transmitted waves correspondingly. b. The same cavity is pumped by laser $L_2$ through mirror $b$. Detectors $D_3$ and $D_4$ measure the phases of reflected and transmitted waves correspondingly.
to the prompt reflection of the pump wave from the input mirror. For instance, if the cavity is pumped through mirror \( a \) then this component is proportional to \( \xi_a - \xi_{P_1} \). The reflected signal is then
\[
a^r_{\text{out}} = p(\xi_a - \xi_{P_1}) + q_2(\xi_{\text{gw}} + \xi_b - \xi_a). \tag{1b}
\]
Here \( q_2 \) also describes the resonant gain (multiple reflections inside the cavity), while \( p \) is the quantity of the order of unity since it describes a single reflection from the input mirror. Equations (1a) and (1b) tell us that we are unable to measure absolute values of \( \xi_a \) and \( \xi_b \), only relative measurements, e.g. with respect to platform \( P_1 \), are allowed.

Now consider the situation illustrated in Fig. 1: the same cavity is pumped by laser \( L_2 \) through mirror \( b \) with the wave polarized normally to the wave emitted by laser \( L_1 \). Detectors \( D_3 \) and \( D_4 \) measure the phases of reflected and transmitted waves correspondingly. Again we assume that laser \( L_2 \) and detector \( D_1 \) are rigidly mounted on platform \( P_2 \) and detector \( D_3 \) is rigidly mounted on platform \( P_1 \). The second pair of response signals can be derived in full similarity. Let us consider the simplest case of equal pumps (equal amplitudes and detunings). Then due to the symmetry of the system and plane GW wavefront the second pair of responses can be written as:
\[
\begin{align*}
b^t_{\text{out}} &= q_1(\xi_{\text{gw}} + \xi_b - \xi_a), \tag{2a} \\
b^r_{\text{out}} &= p(\xi_{P_2} - \xi_b) + q_2(\xi_{\text{gw}} + \xi_b - \xi_a). \tag{2b}
\end{align*}
\]
Here displacements of the mirrors are measured with respect to platform \( P_2 \).

Now constructing the following linear combination of the responses
\[
s = a^r_{\text{out}} + \frac{p - q_2}{q_1} a^t_{\text{out}} + b^r_{\text{out}} - \frac{q_2}{q_1} b^t_{\text{out}},
\]
we are able to cancel displacement noise of both mirrors:
\[
s = p(\xi_{\text{gw}} + \xi_{P_2} - \xi_{P_1}). \tag{3}
\]

Note that displacement noise of the platforms cannot be eliminated. This is the direct consequence of the relativity principle which states that no absolute coordinate or velocity measurements are allowed: one can measure the coordinates of the mirrors only with respect to the positions of reference test masses, platforms \( P_1 \) and \( P_2 \) in our case. Therefore, it is natural that displacement noise of the reference masses imposes the sensitivity limit of the coordinate measurements.

According to formula (3), noise cancelation in a DPFP cavity is possible due to the effect of prompt reflection from the input mirror which is described by the \( p \)-multiplier. The obtained DFI response is similar to the response of a simple single-pass GW detector: an observer sends the light wave to the reflective mirror and receives it back measuring the phase shift. The noise-cancelation algorithm that we perform for a DPFP cavity in some sense can be interpreted as removal of the cavity “by hands”. Evidently, this results in the loss of the optical resonant gain: signal \( s \) in formula (3) includes neither \( q_1 \) nor \( q_2 \).

Two special cases when noise cancelation is impossible can be immediately “predicted” from Eqs. (1a — 2a): (i) \( p = 0 \), meaning that the prompt reflection does not occur (this takes place for the resonant pump, see below) and (ii) \( \xi_a = \xi_{P_1} \) and simultaneously \( \xi_b = \xi_{P_2} \), meaning that the mirrors are rigidly attached to the platforms.

It is evident now that the relativity principle and the notion of the reference frame play significant roles in our analysis. The simplest model of the cavity presented above has been considered in the laboratory (globally inertial) reference frame. However, such a consideration is not free from certain drawbacks. In particular, formula (1a) for the transmitted signal cannot be derived from the simple assumption that the phase of transmitted wave (after emission at \( P_1 \)) is measured at platform \( P_2 \) with respect to the local clocks (i.e. clocks located at \( P_2 \)). To justify formula (1a) certain manipulations with the reference wave, produced by laser \( L_1 \), need to be performed. However, if one wishes to calculate in the laboratory frame the phase of transmitted signal at \( P_2 \) with respect to the local clocks (i.e. the reference oscillation produced at \( P_2 \)), he will inevitably run into a “forward-trip paradox” described and resolved in Ref. 53: if platforms \( P_1 \) and \( P_2 \) move as a single body (i.e. \( \xi_{P_1} = \xi_{P_2} \)) and cavity mirrors are either absent or attached to the platforms, then the phase shift carried by the transmitted wave will contain information about the velocity of the whole system with respect to some “absolute space”, that is forbidden by the relativity principle.

To avoid the paradox one should perform the calculations in the proper reference frame of detector which is non-inertial in general due to the action of external fluctuating forces. The accelerated frame necessarily implies the use of general relativity (GR). Therefore, to obtain strict and consistent description of the cavity in general case we need to complete fully general relativistic calculations.

Another reason to implement GR is the notion of the GW itself which is a purely GR effect. Even though in this paper one may reduce the action of the GW to the effective (Newtonian) tidal force-field, further development of the DPFP idea suggests that displacement noise of the auxiliary platforms may be eliminated but at the cost of GW response reduction, leading to the \( (L/\lambda_{gw})^n \) limiting factors which have a purely GR nature.

Furthermore, it is widely known that the fundamental limit of the sensitivity of optical interferometers is imposed by the vacuum photon shot noise: it will be the only limiting factor left when other noises are canceled or suppressed. Therefore, in order to analyze the ultimate sensitivity of our GW detector we need to quantize the electromagnetic wave circulating inside the cavity since vacuum noise cannot be obtained in the framework of classical electrodynamics.

Summing up, one may conclude that the most gen-
eral and strict problem definition would be the boundary problem for quantized electromagnetic wave in the space-time of accelerated observer in the field of the GW. Therefore, in Sec. III A we define the mentioned space-time, then in Sec. III B we remind the formalism of the optical wave quantization and finally in Sec. IV we set and solve the corresponding boundary problem for a FP cavity.

III. SPACE-TIME OF ACCELERATED OBSERVER IN THE GRAVITATIONAL WAVE FIELD

A. Motion of the test masses

In the Earth-bound GW observatories all the test masses including lasers and detectors undergo fluctuative motions. Since it is the detector that produces an experimentally observable quantity one should consider the operation of an interferometer in its proper reference frame, which is non-inertial in general. For this purpose we first introduce the space-time associated with an observer having non-geodesic 3-acceleration $\dot{\xi}_i(t) = \{\dot{\xi}_x(t), \dot{\xi}_y(t), \dot{\xi}_z(t)\}$ and falling in the GW field $h = h(t-z/c)$. We assume the latter to be weak, plane, $^{+1}$-polarized and propagating along the z-axis. The case of generic GW polarization and direction of propagation does not introduce any significant changes (in the context of this work) to our further analysis. Therefore, in the proper reference frame of such an observer space-time metric takes the following form [23, 36, 53, 54, 55, 56]:

$$ds^2 = -(c \, dt)^2 \left[1 + \frac{2}{c^2} \dot{\xi}_i(t)x^i\right] + dx^2 + dy^2 + dz^2 + \frac{1}{2} \frac{x^2 - y^2}{c^2} \dot{h}(t-z/c)(c \, dt - dz)^2.$$  

Latin indices run over 1, 2, 3. In this paper we consider only one-dimensional motion of the test masses, thus without the loss of generality we may assume $y = z = 0$ and denote $\xi_x(t) = \xi(t)$. In practice fluctuative forces acting on the test masses are very weak, as the GW itself, thus it is natural to require that for all reasonable $x$ and $t$ conditions $|2\dot{\xi}_x/c^2| \ll 1$ and $|h| \ll 1$ are fulfilled so we can use the methods of linearized theory. Metric (4) has two special cases.

1. $\dot{\xi}(t) = 0$ and the proper reference frame coincides with the local Lorentz frame (also called the LL gauge in literature) of the observer freely falling in the GW field. It is worth noting that the LL gauge is free from the requirement of the distance $L$ between the test masses to be much smaller than the gravitational wavelength $\lambda_{gw} [23, 36, 57]$. Corresponding approximation $L \ll \lambda_{gw}$ will be called below the long-wave approximation.

2. $h(t) = 0$ and the proper reference frame is simply a non-inertial frame in Newtonian sense. Note that the curvature of space-time with metric (4) equals to zero under this condition, since it can be made globally flat with the coordinate transformation that brings us from the non-inertial frame to the inertial one.

Remind, that due to the relativity principle an observer is unable to measure his non-geodesic displacement $\xi(t)$ absolutely, in contrast to the corresponding acceleration $\ddot{\xi}(t)$. To avoid the ambiguity associated with the choice of initial conditions $\xi(0)$ and $\dot{\xi}(0)$ we assume below that $\xi(t)$ is measured in such a globally inertial (laboratory) reference frame in the absence of the GW, that $\ddot{\xi}(t) = 0$ results in $\ddot{\xi}(t) = 0$.

The solution to geodesic equation corresponding to metric (4) can be found in Ref. 52. If the $j$th test mass and an observer are separated by a distance $L$ on the average (the 0th order solution) then the test mass displacement relative to an observer (the 1st order solution) equals to $X_j(t) = \frac{1}{2}Lh(t) - \xi(t)$. If, in addition, the test mass is subjected to some non-GW forces and undergoes corresponding fluctuative displacement $\xi_j(t)$ (measured in the globally inertial reference frame) then

$$X_j(t) = \frac{1}{2}Lh(t) + \xi_j(t) - \xi(t).$$  

Below we assume that for any test mass both its displacements $X_j(t)$ and $\xi_j(t)$ obey the relation $|X_j|, |\xi_j| \ll L$. We will also widely use the spectral domain where

$$\left[X_j(\Omega)\right] = \int_{-\infty}^{+\infty} \left[\xi_j(\Omega)\right] e^{-i\Omega t} \frac{d\Omega}{2\pi}.$$  

The introduced proper reference frame is the best suited for analysis of the GW detectors with the test masses undergoing non-geodesic motion, in contrast to the transverse-traceless (TTT) gauge, where such an analysis should be additionally validated. In addition, proper reference frame is the natural frame used by Newtonian experimentalists performing measurements in the laboratory and recording the obtained data from detectors.

B. Quantized electromagnetic wave interacting with the weak gravitational wave in a non-inertial frame

In the interferometric experiments an observer studies the motion of the test masses by sending and receiving the reflected light waves. Thus it is necessary to take into account the effects imposed by the GW and acceleration fields on the optical field for a complete description of an interferometer. Here we briefly remind the formalism used to describe the quantized electromagnetic wave (EMW) propagating in the space-time with metric (4).

First, we start from the simplest case of Minkowski space-time. It is convenient to represent the electric field operator of the EMW as a sum of (i) the “strong” (classical) plane monochromatic wave (which approximates...
the light beam with cross-section $S$) with amplitude $A_0$ and frequency $\omega_0$ and (ii) the “weak” wave describing quantum fluctuations of the electromagnetic field (see Appendix A):

\[
A(x, t) = \sqrt{\frac{2\pi \hbar \omega_0}{Sc}} \left[ A_0 + a(x, t) \right] e^{-i(\omega_0 t + k_0 x)} + \text{h.c.},
\]

\[
a(x, t) = \int_{-\infty}^{\infty} a(\omega_0 + \Omega)e^{-i(\xi x + \omega_0 t + \Omega t)} \frac{d\Omega}{2\pi},
\]

with amplitude $a(\omega_0 + \Omega)$ (Heisenberg operator to be strict) obeying the commutation relations:

\[
[a(\omega_0 + \Omega), a(\omega_0 + \Omega')] = 0,
\]

\[
[a(\omega_0 + \Omega), a^\dagger(\omega_0 + \Omega')] = 2\pi\delta(\Omega - \Omega').
\]

This notation for quantum fluctuations $a(x, t)$ will be the most suitable for us since it coincides exactly with the Fourier-representation of the classical fields. For briefness throughout the paper we omit the $\sqrt{2\pi\hbar\omega_0/Sc}$ multiplier and notation “h.c.” We call $A(x, t)$ the vacuum-state wave if $A_0 = 0$.

Electromagnetic wave propagating in space-time with metric (4) directly couples to the GW and acceleration fields. We will study only the 1st order (in metric (4)) directly couples to the GW and acceleration of $\Omega$ posed on the EMW by the non-inertiality of the reference frame. Both terms are accurate up to the order of $(\Omega/\omega_0)^0$, vanish at $x = 0$ and in long-wave (or low-frequency) approximation have the $O((\Omega x/c)^2)$ asymptotics. It is also straightforward to verify that both $g_\pm(x, t)$ and $w_\pm(w, t)$ are the pure imaginary values; sometimes it will be convenient to use the following approximate formulas:

\[
1 + g_\pm(x, t) = 1 + i\mathcal{I}[g_\pm(x, t)] \approx e^{i\frac{3}{2}[g_\pm(x, t)]}, \quad (8a)
\]

\[
1 + w_\pm(w, t) = 1 + i\mathcal{I}[w_\pm(x, t)] \approx e^{i\frac{3}{2}[w_\pm(x, t)]}. \quad (8b)
\]

### IV. RESPONSE OF A FABRY-PEROT CAVITY TO A PLANE GRAVITATIONAL WAVE

**A. Input, circulating and output waves**

Let us consider the operation of the optical scheme, illustrated in Fig. 2 which consists of platforms $P_{1,2}$ and a FP cavity assembled of two movable mirrors $a$ and $b$, both lossless and having the amplitude transmission coefficient $T, |T| \ll 1$. We put distance between the mirrors in the absence of the gravitational wave and optical radiation to be equal to $L$. Without the loss of generality we assume the cavity to be lying in the plane $z = 0$ along one of the GW principal axes, coinciding with the $x$-axis.

![FIG. 2: Emission-detection scheme. Pump wave is radiated by laser L and reflected wave is detected with the homodyne detector HD1. Transmitted wave is redirected towards platform $P_1$ and is detected with the homodyne detector HD2. Laser L and both the homodyne detectors are assumed to be rigidly mounted on platform $P_1$. Mirrors which redirect the transmitted wave (and the reference wave) towards detector HD2 are assumed to be rigidly mounted on platform $P_2$.](image_url)
optical scheme elements. However, in practice these motions will result in some additional displacement noise.

In this section we will work in the proper reference frame of (the center of mass of) platform $P_1$ at which the origin of the coordinate system is set: $x_{P_1}(t) = 0$. Then the coordinates (their operators to be strict) of the mirrors are $x_a(t) = l_1 + X_a(t) \approx X_a(t)$ and $x_b(t) = L + l_2 + X_b(t) \approx L + X_b(t)$, where $l_1 \ll L$ is the negligible distance between the center of mass of platform $P_1$ and mirror $a$. The coordinate of (the center of mass of) platform $P_2$ is $X_{a, b, P_2}(t) \approx L - L + l_2 \approx L$, where $l_2 \ll L$ is the negligible distance between the centers of mass of platform $P_2$ and mirror $b$. Remind that $X_{a, b, P_2}(t)$ are the displacements with respect to non-inertial reference frame of platform $P_1$ and obey the relation $|X_{a, b, P_2}| \ll L$.

Let the cavity be pumped by laser $L$ through mirror $a$ with the input wave $A_{in}(x, t)$ and through mirror $b$ with the vacuum-state wave $A_{vac}(x, t)$. Optical field inside the cavity is represented as a sum of the wave $A_+(x, t)$, running in the positive direction of the $x$-axis, and the wave $A_-(x, t)$, running in the opposite direction. The reflection-output signal is $A'_{out}(x, t)$ and transmission-output signal is $A_{out}(x, t)$.

\[ A_{out}(x, t) = A_{out0} \left[ 1 + g-(x, t) + w-(x, t) \right] e^{-i(\omega_1 t + k_1 x)} + a_{out}(x, t) e^{-i(\omega_1 t + k_1 x)}, \]  

(12)

Quadrature components (see Appendix A) of this wave are assumed to be measured with the homodyne detector HD$_1$ (see Fig. 2). The reference oscillation is produced by laser $L$.

Output wave transmitted through the cavity

\[ A'_{out}(x, t) = A_{out0} \left[ 1 + g+(x, t) + w+(x, t) \right] e^{-i(\omega_1 t - k_1 (x - L))} + a'_{out}(x, t) e^{-i(\omega_1 t - k_1 (x - L))}, \]

(13)

is redirected towards platform $P_1$ by the small auxiliary mirrors mounted on platform $P_2$. Quadratures of the transmitted wave are measured with the homodyne detector HD$_2$ (see Fig. 2). The reference oscillation is produced by laser $L$ which commits a single round trip along the $P_1 - P_2 - P_1$ path (see below).

It should be mentioned that since both the reflected and transmitted waves commit round-trips and are detected at location of the source, one may perform all the calculations in the TT gauge (see Ref. [52]). However, for the sake of generality we work in the proper reference frame of detector.

Note that the complex amplitudes $a^\pm_{out}(x, t)$ are the unknown function of their arguments and are obtained as the solutions of the corresponding boundary problem for a FP cavity (see below). Obviously, they should vanish in the limit $R \to 0$, i.e. in the absence of the cavity, if $a_{in} = a_{vac} \equiv 0$. Therefore, below we call functions $a^\pm_{out}(x, t)$ or $a^\pm_{out}(\omega_1 + \Omega)$ the cavity response (or output) signals, meaning that they describe the influence of a FP cavity on the light propagation. The summand proportional to $A'_{out0}$ in formula (12) and the one proportional to $A_{out0}$ in (13) thus correspond to the “no-cavity” case and are unimportant for us. In order to make our analysis more transparent we construct our detection scheme in such a way that these terms become unmeasurable.

In the case of reflected wave both $g_-(x, t)$ and $w_-(x, t)$ vanish at $x = 0$ and the only measurable quantities left are the quadratures of $a_{out}(x, t)$.

The case of transmitted wave is more complex. Note that the $A'_{out0}$-summand in formula (13) at point $x = x_{P_2}(t)$ describes a single forward trip of light along the cavity:

\[ \left[ 1 + g_+(x_{P_2}, t) + w_+(x_{P_2}, t) \right] e^{ik_1 x_{P_2}(t)} \approx \exp \left\{ i k_1 x_{P_2}(t) + i \Omega \left[ g_+(L, t) + w+(L, t) \right] \right\}. \]

Here we used formulas (83) and (93). Remind also, that the transmitted wave is redirected towards platform $P_1$.

FIG. 3: Fabry-Perot cavity assembled of two movable mirrors $a$ and $b$. Cavity is pumped through mirror $a$ with the input wave $A_{in}(x, t)$ and through mirror $b$ with the vacuum-state wave $A_{vac}(x, t)$.
for detection and thus commits a backward trip. Clearly, the whole round trip will result in phase shift
\[ 2k_1X_{P_2}(t) + \mathcal{J}\left[g_+(L, t) - g_-(L, t) + w_+(L, t) - w_-(L, t)\right]. \]

In order to make this phase shift unmeasurable we make the reference wave, produced by laser L, to travel the same round trip before returning to the homodyne detector HD. Ultimately, both the additional phases of the transmitted wave and of the reference oscillation are completely subtracted in the homodyne measurement. Therefore, the only measurable quantities left in the transmitted wave are the quadratures of \( a_{\mathrm{out}}^+ (x, t) \).

It is worth noting that such a detection scheme (illustrated in Fig. [2]) only serves the purpose of making the theoretical (rather general) analysis of our toy model more transparent. Experimentalists may want to change it in the way to simplify this or that specific experimental setup; small changes in formulas need to be introduced then, depending on it.

### B. Response signals of a Fabry-Perot cavity

To obtain the response functions of a Fabry-Perot cavity we substitute fields [9, 13] into the set of bound-

\[
\begin{align*}
a_{\mathrm{out}}^+ &= \frac{R - R_0 e^{2i(\delta_1 + \Omega)\tau}}{1 - R_0 e^{2i(\delta_1 + \Omega)\tau}} a_{\mathrm{in}} + \frac{T^2 e^{i(\delta_1 + \Omega)\tau}}{1 - R_0 e^{2i(\delta_1 + \Omega)\tau}} a_{\mathrm{vac}} - \frac{R T^2 A_{\mathrm{in}0} e^{2i\delta_1 \tau}}{1 - R_0 e^{2i(\delta_1 + \Omega)\tau}} e^{i\Omega t}, \\
a_{\mathrm{out}}^- &= \frac{T^2 e^{i(\delta_1 + \Omega)\tau}}{1 - R_0 e^{2i(\delta_1 + \Omega)\tau}} a_{\mathrm{in}} + \frac{R - R_0 e^{2i(\delta_1 + \Omega)\tau}}{1 - R_0 e^{2i(\delta_1 + \Omega)\tau}} a_{\mathrm{vac}} + \frac{RT^2 A_{\mathrm{in}0} e^{2i\delta_1 \tau}}{1 - R_0 e^{2i(\delta_1 + \Omega)\tau}} e^{i\Omega t}. 
\end{align*}
\]

Here phase shift \( \delta \Psi_{\mathrm{emw}} = \delta \Psi_{\mathrm{gw+emw}} \) calculated in the approximation \( \Omega/\omega_1 \ll 1 \), describes the direct coupling of the optical wave to the GW and acceleration fields:

\[
\begin{align*}
\delta \Psi_{\mathrm{gw+emw}}(\Omega) &= -k_1 L h(\Omega) \left(1 - \frac{\sin \Omega \tau}{\Omega \tau}\right) e^{i\Omega t}, \\
\delta \Psi_{\mathrm{acc+emw}}(\Omega) &= -k_1 \xi_{\mathrm{P_1}}(\Omega) \left(1 - 2e^{i\Omega t} + e^{2i\Omega t}\right). 
\end{align*}
\]

Remind also, that the transmitted wave is redirected towards platform \( P_1 \) for detection. Therefore, the truly measured quantity is \( a_{\mathrm{out}}^+ e^{i(\delta_1 + \Omega)\tau} \). However, keeping this in mind, below we deal only with \( a_{\mathrm{out}}^- \). The additional phase can be taken into account straightforwardly.

This set of equations is accurate up to the 0th order of \( \Omega/\omega_1 \) since it does not take into account the relativistic terms proportional to \( X_{a,b}^0 \). The solution of this set is obtained in Appendix [13] using the method of successive approximations. Since we do not consider the effect of parametric excitation of the additional optical modes under the influence of the GW [37], it will be convenient to introduce the detuning \( \delta_1 = \omega_1 - n_0 \omega_0/\tau \), where \( n_0 \) is integer, even (for simplicity) and fixed; \( \tau = L/c \). Then the solution of the 1st order takes the following form (all spectral arguments are omitted):

\[
\begin{align*}
A_+(x, t) &= T \Lambda_{\mathrm{in}}(x, t) - R \Lambda_{-}(x, t), \\
A_{-}^0(x, t) &= R \Lambda_{\mathrm{in}}(x, t) + T \Lambda_{-}(x, t), \\
A_{-}(x, b, t) &= T \Lambda_{\mathrm{vac}}(x, t) - R \Lambda_{+}(x, t), \\
A_{+}^0(x, b, t) &= R \Lambda_{\mathrm{vac}}(x, t) + T \Lambda_{+}(x, b, t). 
\end{align*}
\]

This set of equations is accurate up to the 0th order of \( \Omega/\omega_1 \) since it does not take into account the relativistic terms proportional to \( X_{a,b}^0 \). The solution of this set is obtained in Appendix [13] using the method of successive approximations. Since we do not consider the effect of parametric excitation of the additional optical modes under the influence of the GW [37], it will be convenient to introduce the detuning \( \delta_1 = \omega_1 - n_0 \omega_0/\tau \), where \( n_0 \) is integer, even (for simplicity) and fixed; \( \tau = L/c \). Then the solution of the 1st order takes the following form (all spectral arguments are omitted):

\[
\begin{align*}
A_+(x, t) &= T \Lambda_{\mathrm{in}}(x, t) - R \Lambda_{-}(x, t), \\
A_{-}^0(x, t) &= R \Lambda_{\mathrm{in}}(x, t) + T \Lambda_{-}(x, t), \\
A_{-}(x, b, t) &= T \Lambda_{\mathrm{vac}}(x, t) - R \Lambda_{+}(x, t), \\
A_{+}^0(x, b, t) &= R \Lambda_{\mathrm{vac}}(x, t) + T \Lambda_{+}(x, b, t). 
\end{align*}
\]

Remind also, that the transmitted wave is redirected towards platform \( P_1 \) for detection. Therefore, the truly measured quantity is \( a_{\mathrm{out}}^+ e^{i(\delta_1 + \Omega)\tau} \). However, keeping this in mind, below we deal only with \( a_{\mathrm{out}}^- \). The additional phase can be taken into account straightforwardly.

We should now express the obtained result in terms of (i) the fluctuative displacements measured in the laboratory frame and (ii) the GW displacement measured in the local Lorentz frame of platform \( P_1 \). According to formula [13] the transformation law is:

\[
\begin{align*}
X_a(t) &= \xi_a(t) - \xi_{\mathrm{P_1}}(t), \\
X_{b}(t) &= \frac{1}{2} L h(t) + \xi_b(t) - \xi_{\mathrm{P_1}}(t). 
\end{align*}
\]

Here we denoted the fluctuative motions of mirrors \( a \) and \( b \) as \( \xi_{a,b} \). These formulas are strict for any separation between the mirrors. Substituting \( X_a \) and \( X_b \) into the response signals [15a] and [15b] we rewrite them in terms of the GW signal

\[
\xi_{\mathrm{gw}}(\Omega) = \frac{1}{2} L h(\Omega) \frac{\sin \Omega \tau}{\Omega \tau},
\]
and fluctuating displacements $\xi_{a,b,p_1}$:

$$a_{\text{out}}^r = R_1 a_{\text{in}} + T_1 a_{\text{vac}}$$

$$- \frac{RT^2 A_{\text{in}} e^{2i\delta_1} \tau}{T^2_{\delta_1 + \Omega}} 2ik_1 \left[ \xi_b e^{i\Omega \tau} - \sigma_1 \xi_a + \xi_{gw} e^{i\Omega \tau} \right]$$

$$- \frac{RT^2 A_{\text{in}} e^{2i\delta_1} \tau}{T^2_{\delta_1 + \Omega}} ik_1 \xi_{p_1} (2\sigma_1 - 1 - e^{2i\Omega \tau}), \quad (18a)$$

$$a_{\text{out}}^r = R_1 a_{\text{in}} + T_1 a_{\text{vac}}$$

$$+ \frac{RT^2 T^2 A_{\text{in}} e^{2i\delta_1} \tau}{T^2_{\delta_1 + \Omega}} 2ik_1 \left[ \xi_b e^{i\Omega \tau} - \xi_a + \xi_{gw} e^{i\Omega \tau} \right] e^{i\Omega \tau}$$

$$+ \frac{RT^2 T^2 A_{\text{in}} e^{2i\delta_1} \tau}{T^2_{\delta_1 + \Omega}} ik_1 \xi_{p_1} (1 - e^{-2i\Omega \tau}) e^{i\Omega \tau}. \quad (18b)$$

The following notations have been introduced above:

$$T_{\delta_1}^2 = 1 - R^2 e^{2i\delta_1 \tau}, \quad T_{\delta_1 + \Omega}^2 = 1 - R^2 e^{2i(\delta_1 + \Omega) \tau},$$

$$R_1 = \frac{R - R e^{2i(\delta_1 + \Omega) \tau}}{1 - R^2 e^{2i(\delta_1 + \Omega) \tau}}, \quad T_1 = \frac{T e^{i(\delta_1 + \Omega) \tau}}{1 - R^2 e^{2i(\delta_1 + \Omega) \tau}},$$

having the following physical meaning: $1/T_{\delta_1}^2$ describes the resonant amplification of the input amplitude $A_{\text{in}}$ inside the cavity, $1/T_{\delta_1 + \Omega}^2$ describes the frequency-dependent resonant amplification of the variation of the circulating light wave, $R_1$ and $T_1$ are the generalized coefficients of reflection (from a FP cavity) and transmission (through a FP cavity).

It is convenient to analyze the physical meaning of the obtained formulas. First we consider the reflected wave rewriting it in the following form:

$$a_{\text{out}}^r = R_1 (a_{\text{in}} - A_{\text{in}} tk_{1} \xi_{p_1}) + T_1 a_{\text{vac}}$$

$$+ TA_{\text{out}} 2ik_1 \left[ (\xi_b + \xi_{gw}) e^{i\Omega \tau} - \xi_a \right] / T_{\delta_1 + \Omega}^2$$

$$+ A_{\text{out}}^r 2ik_1 \xi_a - A_{\text{out}}^r 2ik_1 \xi_{p_1}. \quad (18c)$$

The 1st term states that the optical laser noise $a_{\text{in}}$ is indistinguishable from laser displacement noise $\xi_{p_1}$, so they always come together. The 2nd summand describes the propagation of the vacuum noise through a FP cavity. The 3rd term is the light wave flowing out of the cavity containing the accumulated phase shift. The 4th summand, which is responsible for $\Delta \sigma_1$, describes the prompt reflection from the input mirror $a$. The last term describes the phase shift acquired by the light wave due to displacement noise of detector on platform $P_1$.

In a similar way one can consider the transmitted wave. The only difference which should be taken into account is the following: the term proportional to $\xi_{p_1}$ in formula $(18c)$ cannot be reduced to $-T_1 A_{\text{in}} tk_{1} \xi_{p_1}$ due to the detection scheme we use for the transmitted wave. If one adds the $A_{\text{out}}^r$-summand in formula $(13)$ to $a_{\text{out}}$ then $-T_1 A_{\text{in}} tk_{1} \xi_{p_1}$ is recovered.

V. DOUBLE-PUMPED FABRY-PEROT CAVITY

A. Response signals of a double-pumped Fabry-Perot cavity

Let a single Fabry-Perot cavity be pumped through both of its mirrors (see Fig. 4). We assume the pump wave through mirror $a$ to have amplitude $A$, detuning $\delta_1$ (carrier frequency $\omega_1$), polarization in the plane of incidence and denote it with $A_{\text{in}}$; the pump wave through mirror $b$ is assumed to have amplitude $B$, detuning $\delta_2$ (carrier frequency $\omega_2$), polarization orthogonal to the plane of incidence and is denoted with $B_{\text{in}}$. Corresponding vacuum pumps through mirrors $b$ and $a$ are denoted with $A_{\text{vac}}$ and $B_{\text{vac}}$.

![FIG. 4: Fabry-Perot cavity pumped through both of its mirrors (a DPFP cavity). Lasers $L_1$ and $L_2$ are rigidly mounted on platforms $P_1$ and $P_2$ respectively. The pump wave through mirror $a$ is denoted with $A_{\text{in}}$ and is assumed to be polarized in the plane of incidence. The pump wave through mirror $b$ is denoted with $B_{\text{in}}$ and is assumed to be polarized normally to the plane of incidence. Corresponding vacuum pumps are $A_{\text{vac}}$ and $B_{\text{vac}}$.](image)

The response functions corresponding to the pump through mirror $b$ are straightforwardly obtained from functions $(18a)$, $(18b)$ replacing $\delta_1 \rightarrow \delta_2$, $\xi_a \rightarrow -\xi_b$, $\xi_b \rightarrow -\xi_a$, $\xi_{p_1} \rightarrow -\xi_{p_2}$ and keeping the GW term unchanged due to the symmetry of the system and plane GW wavefront. For convenience we gather signals in all the four output ports of the DPFP cavity omitting spectral arguments and taking into account the relation $k_1 \approx k_2 \equiv k_0$ valid for the corresponding carrier frequencies $\omega_1$ and $\omega_2$ lying within the same resonance curve:

$$a_{\text{out}}^r = R_1 a_{\text{in}} + T_1 a_{\text{vac}}$$

$$- \frac{RT^2 A_2 e^{2i\delta_1 \tau}}{T^2_{\delta_1 + \Omega}} 2ik_0 \left[ \xi_b + \xi_{gw} \right] e^{i\Omega \tau} - \sigma_1 \xi_a \right]$$

$$- \frac{RT^2 A_2 e^{2i\delta_1 \tau}}{T^2_{\delta_1 + \Omega}} ik_0 \xi_{p_1} (2\sigma_1 - 1 - e^{2i\Omega \tau}), \quad (19a)$$

$$a_{\text{out}}^t = T_1 a_{\text{in}} + R_1 a_{\text{vac}}$$

$$+ \frac{RT^2 T^2 A_2 e^{2i\delta_1 \tau}}{T^2_{\delta_1 + \Omega}} 2ik_0 \left[ \xi_b + \xi_{gw} \right] e^{2i\Omega \tau} - \xi_a e^{i\Omega \tau}]$$

$$+ \frac{RT^2 T^2 A_2 e^{2i\delta_1 \tau}}{T^2_{\delta_1 + \Omega}} 2ik_0 \left[ \xi_b + \xi_{gw} \right] e^{2i\Omega \tau} - \xi_a e^{i\Omega \tau}].$$
\[ \frac{R^2T^2Ae^{3i\delta_1\tau}}{T_{s_1}^2 T_{s_1}^2 + \Omega} - \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} 2ik_0 \left( (-\xi_a + \xi_{gw})e^{2i\Omega\tau} + \sigma_2 \right) = s_1^{\text{fl}} + \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} 2ik_0 \left( \xi_b + \xi_{gw} \right) e^{2i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{R^2T^2Ae^{3i\delta_1\tau}}{T_{s_1}^2 T_{s_1}^2 + \Omega} - \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} 2ik_0 \Delta \sigma_1 \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau} \]

\[ s_1^{\text{fl}} + \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} 2ik_0 \left( \xi_b + \xi_{gw} \right) e^{2i\Omega\tau}, \]

\[ = s_1^{\text{fl}} + \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} 2ik_0 \left( \xi_b + \xi_{gw} \right) e^{2i\Omega\tau}, \]

\[ = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} 2ik_0 \left( \xi_b + \xi_{gw} \right) e^{2i\Omega\tau}, \]

\[ + \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ + \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ + \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]

\[ s_1^{\text{fl}} = \frac{RT^2Be^{2i\delta_2\tau}}{T_{s_2}^2 T_{s_2}^2 + \Omega} \frac{A}{T_{s_1}^2} \xi_{p_1} \left( 1 + e^{2i\Omega\tau} \right)e^{i\Omega\tau}, \]
In the simplest case of equal pumps we have $A = B$ and $\delta_1 = \delta_2$. Then in the narrow-band approximation ($T^2 = 2\gamma\tau \ll 1$, $\delta_1,2\tau \ll 1$, where $\gamma$ is the cavity half-bandwidth):

$$s|_{\delta_2 = \delta_1} \approx a_{in} + b_{in} + a_{vac} + b_{vac} - \frac{i\delta_1}{\gamma - i\delta_1} A 2i\kappa_0 \left( \frac{1}{2} Lh + \xi P_2 - \xi P_1 \right). \quad (23)$$

Remind [43, 44, 45, 46, 47, 48], that due to the significant amplification of the input laser power inside a FP cavity test masses are subjected to the force of radiation pressure. It is known that the sign of the induced ponderomotive rigidity depends on the sign of detuning. Therefore, in order to cancel the effects of radiation pressure we should consider the pumps with opposite detunings $\delta_2 = -\delta_1$. In this case both the pumps create ponderomotive rigidities with the opposite signs and total rigidity vanishes. The DFI signal in this case is:

$$s|_{\delta_2 = -\delta_1} \approx a_{in} - b_{in} + a_{vac} - b_{vac} - \frac{i\delta_1}{\gamma - i\delta_1} A 2i\kappa_0 \left( \frac{1}{2} Lh + \xi P_2 - \xi P_1 \right). \quad (24)$$

Obviously, in the previous case of equal detunings total ponderomotive rigidity does not vanish and, strictly speaking, the effects of radiation pressure in the DPFP cavity require separate detailed analysis.

From formulas (23) and (24) we conclude that the signal-to-noise ratio of the DPFP cavity operating as the displacement-noise-free detector is of the same order as for the configuration with two test masses and only one round trip of light between them (i.e. without the resonant gain).

VI. DISCUSSION

Let us now discuss several issues concerning the noise cancelation in the proposed model.

A. Special cases

First, it is useful to consider two special cases when noise cancelation is impossible.

1. Resonant pump. One can derive from formula (18a) that the coefficient $p$ in formula (11) is proportional to the amplitude of reflected wave $A_{out}$. In Appendix [11] it is found that $A_{out} = R A_{in0} \left( 1 - e^{2i\delta_1 \tau} \right) / T_s^2$. Thus in the resonant regime ($\delta_1 = 0$) reflected wave has no “strong” component meaning that the prompt reflection from the input mirror does not occur and $p = 0$. As a result, both the reflected and transmitted signals become indistinguishable, i.e. they carry equal amount of information about the coordinates of the mirrors (see equations (1a) and (1b)). In general case (formulas (18a) and (18b)) the resonant regime corresponds to $\Delta \sigma_1 = 0$, resulting in the relation $a_{out} = -R a_{out} e^{i2\tau}$, neglecting the optical noise.

2. Mirrors mounted on the platforms. One may think of mounting the mirrors on the platforms to reduce the additional fluctuative degrees of freedom associated with the platforms. For instance, if the mirror $a$ is mounted on platform $P_1$ then $\xi_a = \xi_{P_1}$ and from equation (11) it is evident that both the responses become equivalent. In general case (see formulas (15a) and (15b) it is evident that for $X_a = \xi_a - \xi_{P_1} = 0$ again $a_{out} = -R a_{out} e^{i2\tau}$.

B. Optical power requirements

The loss of the resonant gain in a DPFP cavity also results in increase of the optical power needed to reach the SQL level of sensitivity. In conventional (LIGO) topology both the mean amplitude and the signal are resonantly amplified resulting in less power needed to reach SQL as compared to any single-round-trip detector. For instance, in Advanced LIGO detectors (utilizing also the power recycling mirrors) SQL will be reached with $\approx 1$ MW of circulating optical power corresponding to $\approx 100$ W laser. In contrast, in a DPFP cavity the same level of sensitivity will be reached at $\approx 1$ GW of laser power. This number might not seem so dramatic if one reminds that the squeezed light allows to decrease the power needed. To achieve the high factors of squeezing one must provide the mirrors with the coefficient of optical losses as small as possible; according to J.M. Makowsky there is a strong evidence that the loss coefficient $\approx 10^{-9}$ will be reached in the near future.

C. Limitations due to the relativity principle

Remind that in formula (11) $\xi_{gw} \approx Lh/2$ (see also formulas (23) and (24)), thus direct coupling of the GW to the light wave plays no role in our noise-cancelation scheme. From the obtained results (see also reasonings in Sec. II) it seems that it is hardly possible (without contradicting the relativity principle) to completely eliminate the displacement noise, keeping simultaneously the $h(L/\lambda_{gw})^0$ or $h(L/\lambda_{gw})^1$ order of the DFI signal, since these orders correspond to coordinate and velocity measurements. Relativity principle forbids absolute coordinate and velocity measurements; only acceleration, in principle, can be measured absolutely, corresponding to complete DFI of the $h(L/\lambda_{gw})^n$, $n \geq 2$ order proposed by Kawamura et al. Thus we are left to choose either sacrifice with the GW sensitivity but completely eliminate displacement noise, or keep good GW sensitivity at the expense of incomplete noise cancelation. To suppress the fluctuations associated with the platforms (where lasers
and detectors are mounted) one will need to increase their masses and cool them down to cryogenic temperatures. The only limiting factors will be left then are the classical (laser) and vacuum optical noises.

D. Further prospects: cancelation of laser noise and detection schemes

From formula (23) or (24) one may conclude that the fundamental limitations of the proposed scheme are (i) the vacuum shot noise (\(a_{\text{vac}}\) and \(b_{\text{vac}}\) terms) due to the uncertainty principle and (ii) the residual displacement noise (\(\xi_{P_1}\) and \(\xi_{P_2}\) terms) due to the relativity principle as discussed above. It is also known that laser noise can be eliminated in differential (balanced) optical setup, for instance Mach-Zehnder or Michelson interferometer. Since laser noise dominates over vacuum shot noise in practice, one needs to implement the proposed DPFP cavity into some balanced scheme to increase the overall SNR. Here we propose one of the obvious modifications of LIGO topology, namely a Michelson interferometer with two DPFP cavities in its arms, which utilizes a “round-trip ideology” widely used in this paper (see Fig. 5).

First, we describe the operation of the scheme as a whole and then consider noise cancelation issue. Let laser \(L_1\) emit the optical wave polarized in the plane of incidence. Upon arrival to beamsplitter BS optical wave is splitted into two beams: the one traveling in the horizontal arm towards FP cavity assembled of mirrors \(a\) and \(b\) and the other traveling in the vertical arm towards FP cavity \(cd\). Both reflected waves then reunite at beamsplitter and the resulting optical field is detected by homodyne detector HD\(_1\). The wave transmitted through \(ab\) cavity is redirected towards beamsplitter by auxiliary mirrors \(M_1\) and \(M_2\). Similarly, the wave transmitted through \(cd\) cavity is redirected towards beamsplitter by mirrors \(M_3\) and \(M_4\). Ultimately, both transmitted waves interfere at beamsplitter and are detected by homodyne detector HD\(_2\).

Let the second pump be produced by laser \(L_2\) emitting the radiation polarized normally to the plane of incidence. Input wave inside the horizontal arm produces the reflected wave via \(BS - M_2 - M_1 - ab\) cavity \(- M_1 - M_2 - BS\) optical path and the transmitted wave via \(BS - M_2 - M_1 - ab\) cavity \(- BS\) path. Similarly, reflected and transmitted waves are produced in the vertical arm. Interfering reflected waves are detected then by homodyne detector HD\(_2\) and transmitted waves are detected by HD\(_1\) detector.

Following the consideration of a single DPFP cavity, we may assume that several optical elements are rigidly attached to each other. For instance, let us assume that both lasers, beamsplitter and both detectors are rigidly mounted on platform \(P_{\text{BS}}\); mirror \(b\) and small auxiliary mirrors \(M_1\) and \(M_2\) are mounted on platform \(P_b\); mirror \(d\) and mirrors \(M_3\) and \(M_4\) are mounted on platform \(P_d\). Then there are left only six essential degrees of freedom: displacement of \(P_{\text{BS}}\) along \(x\)- and \(y\)-axes, displacements of \(a\) and \(P_b\) along the \(x\)-axis, and displacements of \(c\) and \(P_d\) along the \(y\)-axis. Let us denote the coordinate fluctuations of the \(j\)th test mass corresponding to the motion along the \(x\)- and \(y\)-axes as \(\xi_j\) and \(\eta_j\). Each of four interferometer responses \(a_i\) contains displacement noise in the combinations of the following type:

\[
a_i \sim a_{\text{HD}_1} + a_{\text{HD}_2} + k_0(\xi_{P_{\text{BS}}} - \eta_{P_{\text{BS}}}) + k_0(\xi_{P_b} - \xi_a + \frac{1}{2} L h) - k_0(\eta_{P_d} - \eta_c - \frac{1}{2} L h),
\]

where \(a_{\text{HD}_i}\) is the vacuum shot noise in the dark port of detector HD\(_i\). Note that the terms describing optical laser noise are absent since it vanishes due to the interference of the waves at beamsplitter. In fact it is not necessary to demand that lasers are rigidly attached to beamsplitter: since optical and displacement noise of a laser are indistinguishable (there sum is usually called laser phase noise) both noises are canceled simultaneously. Here we do not calculate explicitly the coefficients before each noise term in \(a_i\) since they depend on specific

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**FIG. 5:** A Michelson/DPFP optical setup. DPFP cavities \(ab\) and \(cd\) are inserted into the horizontal and vertical arms of Michelson interferometer correspondingly. Lasers \(L_1\) and \(L_2\), beamsplitter BS and homodyne detectors HD\(_1\) and HD\(_2\) are rigidly mounted on platform \(P_{\text{BS}}\). Cavity mirror \(b\) and auxiliary mirrors \(M_1\) and \(M_2\) are rigidly mounted on platform \(P_b\); cavity mirror \(d\) and auxiliary mirrors \(M_3\) and \(M_4\) are rigidly mounted on platform \(P_d\). Detector HD\(_1\) measures the quadratures of reflected wave corresponding to laser \(L_1\) and the ones of transmitted wave corresponding to laser \(L_2\). Detector HD\(_2\) measures the quadratures of reflected wave corresponding to laser \(L_1\) and the ones of transmitted wave corresponding to laser \(L_2\).
details of the optical setup.

Excluding $\xi_p - \eta_p$, and $\xi_a - \eta_a$ from the linear combination of responses one obtains signal with only fundamental noises (for $g(\Omega) - \text{DFI}$) left:

$$s_{\text{DFI}} \sim a_{\text{HD}} + a_{\text{HD}} + k_0(\xi_{\text{p}} - \eta_{\text{p}} + Lh).$$

Obviously, the major drawback of the proposed scheme is the significant amount of additional optical elements such as beamsplitter and mirrors used to split and redirect laser beams. Our assumption that several elements could be rigidly installed on the platforms (i.e. to be noiseless) should be validated in practice.

The related problem is the construction of the most practical measurement schemes. In particular, when analyzing the transmitted wave in a single DPFP cavity above, we dealt only with the round-trip measurement schemes, i.e. redirected the transmitted radiation for detection towards the location (approximately) of emitting device. To clarify our analysis we also made the corresponding reference oscillation to perform a round trip. This may seem inconvenient (but certainly not impossible) to the experimentalists, thus other possibilities could be explored. For instance, one may think of forward-trip measurement schemes [53], i.e. the situation when transmitted wave is detected straightforwardly (without any redirection). Corresponding balanced schemes could be proposed then.

VII. CONCLUSION

In this paper we have analyzed the operation of a Fabry-Perot cavity pumped through both the mirrors (a DPFP cavity) performing the mirrors-displacement-noise-free gravitational-wave detection. We have demonstrated that due to the asymmetry between the reflection and transmission output ports of detuned cavity it is possible to construct a linear combination of four response signals which cancels displacement fluctuations of the mirrors. At low frequencies the GW response of the DPFP cavity turns out to be far better than that of the Mach-Zehnder-based DFIs proposed by S. Kawamura et al. due to the different mechanisms of noise-cancellation. However, the effective loss of the resonant gain results in the sensitivity limitation of the DPFP cavity by displacement noise of lasers and detectors.

The performed analysis suggests that though addressed as a toy model in this paper, DPFP cavity can be considered a promising candidate for constituent part of the future generation GW detectors, provided the noises of lasers and detectors are suppressed: it allows the significant extension of the frequency band of the ground-based detectors and by elimination of back-action noise straightforwardly avoids the standard quantum limitation.

The problems of (i) DPFP-based laser-noise-cancellation schemes, (ii) practical measurement schemes and (iii) radiation pressure effects in a DPFP cavity require future investigation. We hope that presented analysis will stimulate the search for new configurations of FP-based displacement-noise-free GW detectors.

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APPENDIX A: QUANTIZED ELECTROMAGNETIC WAVE

In this Appendix we introduce the notations for the quantized field of electromagnetic wave which will be used throughout the paper.

In quantum electrodynamics the operator of electric field in Heisenberg picture is:

$$A(x, t) = \int_0^\infty \sqrt{\frac{2\pi \hbar \omega}{Sc}} a(\omega) e^{-i\omega(t-x/c)} \frac{d\omega}{2\pi} + \text{h.c.},$$

where $S$ is the effective cross section area of the laser beam and $a(\omega)$ is the annihilation operator obeying the commutation relations

$$[a(\omega), a(\omega')] = 0, \quad [a(\omega), a^\dagger(\omega')] = 2\pi\delta(\omega - \omega').$$

It will be convenient now to introduce the carrier frequency $\omega_0$: $\omega = \omega_0 + \Omega, |\Omega| \ll \omega_0$, and to rewrite the field operator in the following way:

$$A(x, t) = e^{-i(\omega_0 t - k_0 x)} \times \int_{-\omega_0}^{\omega_0} \sqrt{\frac{2\pi \hbar (\omega_0 + \Omega)}{Sc}} a(\omega_0 + \Omega) e^{-i\Omega(t-x/c)} \frac{d\Omega}{2\pi} + \text{h.c.},$$

where $k_0 = \omega_0/c$. Now we split the annihilation operator into two summands:

$$a(\omega_0 + \Omega) = A_0 \delta(0) + a'(\omega_0 + \Omega).$$

For convenience we change notation $a' \rightarrow a$ since we do not need old $a$ any further. Extending now the lower limit of integration to $-\infty$ (since $|\Omega| \ll \omega_0$), we finally obtain the double-sided (from $-\infty$ to $+\infty$) expression for the field operator:

$$A(x, t) = \sqrt{\frac{2\pi \hbar \omega_0}{Sc}} e^{-i(\omega_0 t - k_0 x)}$$
In these notations electric field of the wave is represented as a sum of (i) “strong” (classical) wave with amplitude $A_0$ and (carrier) frequency $\omega_0$ and (ii) “weak” wave describing the quantum fluctuations of the optical field with its amplitude obeying the commutation relations:

$$[a(\omega_0 + \Omega), a(\omega_0 + \Omega')] = 0,$$

$$[a(\omega_0 + \Omega), a^\dagger(\omega_0 + \Omega')] = 2\pi \delta(\Omega - \Omega').$$

The double-sided expression is the one most close to the Fourier representation of the classical fields and will be used throughout the paper. For convenience we omit the $\sqrt{2\pi\hbar\omega_0}/Sc$-multiplier in the main body of the paper since it is the common multiplier in all the equations.

For completeness we also introduce the quadrature components of the wave. Formula (A1) can be rewritten as:

$$A(x, t) = \sqrt{2\pi\hbar\omega_0}/Sc e^{-i(\omega_0 t - k_0 x)}$$

$$\times \left\{ A_0 + \int_0^\infty \left[ a_{\omega_0 + \Omega} e^{-i\Omega(t-x/c)} + a_{\omega_0 - \Omega} e^{i\Omega(t-x/c)} \right] \frac{d\Omega}{2\pi} \right\}$$

$$+ \text{h.c.},$$

(A2)

where $a_{\omega_0 - \Omega}$ obeys the same commutation relation as $a_{\omega_0 + \Omega}$:

$$[a_{\omega_0 + \Omega}, a^\dagger_{\omega_0 + \Omega'}] = [a_{\omega_0 - \Omega}, a^\dagger_{\omega_0 - \Omega'}] = 2\pi \delta(\Omega - \Omega').$$

Next we introduce the so-called correlated two-photon modes with field operators $^{58, 59}$

$$a_c(\Omega) = \frac{a_{\omega_0 + \Omega} + a^\dagger_{\omega_0 - \Omega}}{\sqrt{2}}, \quad a_s(\Omega) = \frac{a_{\omega_0 + \Omega} - a^\dagger_{\omega_0 - \Omega}}{\sqrt{2} i},$$

with the only non-zero commutators

$$[a_c, a_s^\dagger] = [a_c^\dagger, a_s] = 2\pi i \delta(\Omega - \Omega'),$$

where prime denotes the argument with $\Omega'$. In terms of these two-photon modes formula (A2) takes the form:

$$A(x, t) = \sqrt{2\pi\hbar\omega_0}/Sc \left[ \sqrt{2} A_0 \cos(\omega_0 t - k_0 x) + a_c(x, t) \cos(\omega_0 t - k_0 x) + a_s(x, t) \sin(\omega_0 t - k_0 x) \right],$$

where operators

$$a_c(x, t) = \int_0^\infty a_c(\Omega) e^{-i\Omega(t-x/c)} \frac{d\Omega}{2\pi} + \text{h.c.},$$

$$a_s(x, t) = \int_0^\infty a_s(\Omega) e^{-i\Omega(t-x/c)} \frac{d\Omega}{2\pi} + \text{h.c.},$$

in the case $A_0 = 0$ are called the cosine and sine quadratures (or quadrature components) correspondingly.

**APPENDIX B: BOUNDARY CONDITIONS**

In this Appendix we solve the set of equations (14a – 14d).

First we substitute fields (9 – 13) into this set and separate the 0th and the 1st order sets.

The zeroth order set is:

$$A_{x0} = T A_{i0} - RA_{-0},$$

$$A^r_{out0} = RA_{i0} + TA_{-0},$$

$$A_{r-0} = -RA_{i0} e^{2i\omega_1 \tau},$$

$$A^t_{out0} = TA_{i0} e^{2i\omega_1 \tau}.$$

Corresponding solution is:

$$A_{x0} = \frac{T}{1 - R^2 e^{2i\omega_1 \tau}} A_{i0},$$

$$A_{r-0} = -\frac{RT e^{2i\omega_1 \tau}}{1 - R^2 e^{2i\omega_1 \tau}} A_{i0},$$

$$A^t_{out0} = \frac{T^2 e^{2i\omega_1 \tau}}{1 - R^2 e^{2i\omega_1 \tau}} A_{i0},$$

$$A^r_{out0} = \frac{R - Re^{2i\omega_1 \tau}}{1 - R^2 e^{2i\omega_1 \tau}} A_{i0}.$$

Amplitudes $A_{i0}$, $A_{x0}$ and $A^t_{out0}$ are evaluated at point $x = 0$ and amplitude $A^r_{out0}$ at point $x = L$.

The first order solution in spectral domain is:

$$a_+ = Ta_{i-} - Ra_+ + RA_{i0} 2i\omega_1 X_+, a_+ = \pm 2i\omega_1 X_+, a_+ = Ta_{vac} e^{i(\omega_1 + \Omega)\tau} - Ra_+ e^{2i(\omega_1 + \Omega)\tau} - RA_{i0} e^{2i\omega_1 \tau} \left[ 2i\omega_1 X_+ + g_+(L) - g_-(L) \right] e^{i\Omega \tau},$$

$$a^r_{out} = Ra_{vac} + Ta_+ e^{i(\omega_1 + \Omega)\tau}. $$

Here $a_+ = a_i(\omega_1 + \Omega)$, $g_\pm(x) = g_\pm(x, \omega_1 + \Omega)$ and $X_+ = X_i(\Omega)$. Spectral amplitudes $a_{in}$, $a_{\pm}$ and $a^r_{out}$ are evaluated at point $x = 0$ and amplitude $a^r_{out}$ at point $x = L$. The first order solution is:
\[ a_+ = \frac{T}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} a_{\text{in}} + \frac{RT e^{i(\omega_1 + \Omega)\tau}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} a_{\text{vac}} + \frac{R^2 A_{+0} e^{2i\omega_1 \tau}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} A_0 \left[ 2k_1 (X_b e^{i\Omega \tau} - X_a) + \delta \Psi_{\text{emw}} \right], \]
\[ a_- = \frac{RT e^{2i(\omega_1 + \Omega)\tau}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} a_{\text{in}} + \frac{T e^{i(\omega_1 + \Omega)\tau}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} a_{\text{vac}} + \frac{A_{-0}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} \left[ 2k_1 (X_b e^{i\Omega \tau} - \rho_1 X_a) + \delta \Psi_{\text{emw}} \right], \]
\[ a_{\text{out}}^+ = \frac{T^2 e^{i(\omega_1 + \Omega)\tau}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} a_{\text{in}} + \frac{R - R e^{2i(\omega_1 + \Omega)\tau}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} a_{\text{vac}} + \frac{R^2 A_{+0} e^{2i\omega_1 \tau}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} \left[ 2k_1 (X_b e^{i\Omega \tau} - X_a) + \delta \Psi_{\text{emw}} \right] e^{i\Omega \tau}, \]
\[ a_{\text{out}}^- = \frac{R - R e^{2i(\omega_1 + \Omega)\tau}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} a_{\text{in}} + \frac{T^2 e^{i(\omega_1 + \Omega)\tau}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} a_{\text{vac}} + \frac{TA_{-0}}{1 - R^2 e^{2i(\omega_1 + \Omega)\tau}} \left[ 2k_1 (X_b e^{i\Omega \tau} - \sigma_1 X_a) + \delta \Psi_{\text{emw}} \right]. \]

where \( \rho_1 (\Omega) = R^2 e^{2i(\omega_1 + \Omega)\tau} \). Phase shift \( \delta \Psi_{\text{emw}} \) and factor \( \sigma_1 \) are introduced in Sec. IV.
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