Inductive quantum learning: Why you are doing it almost right

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In supervised learning, an inductive learning algorithm extracts general rules from observed training instances, then the rules are applied to test instances. We show that this splitting of training and application arises naturally, in the classical setting, from a simple independence requirement with a physical interpretation of being non-signalling. Thus, two seemingly different definitions of inductive learning happen to coincide. This follows from very specific properties of classical information, which break down in the quantum setup. We prove a quantum de Finetti theorem for quantum channels, which shows that in the quantum case, the equivalence holds in the asymptotic setting (for large number of test instances). This reveals a natural analogy between classical learning protocols and their quantum counterparts, thus allowing to naturally enquire about standard elements in computational learning theory, such as structural risk minimization, model and sample complexity.

I. INTRODUCTION

In supervised learning, we are given a sample of a distribution called training instances. These instances have further fine structure, and we often think of them as pairs consisting of an input object and a matching output value or label [16, 20]. Induction infers general rules—a function from a hypothesis space, that is, a parametric function family—from this sample. Once the function is identified, we can apply it to an arbitrary (possibly infinite) number of unlabelled instances that we call test instances.

Machine learning, being a pragmatic discipline, dedicates a tremendous effort to finding the optimal function—in regards to what kind of objectives we consider, or whether we are looking at empirical or structural risk—and to defining a good hypothesis space to optimize over. Quantum machine learning, an emergent line of research that introduces quantum resources in the learning algorithms [1, 36], is similarly pragmatic, with a strong emphasis on speedup [2, 8, 22, 27]. Before devising further quantum protocols for learning, however, we should notice that quantum physics imposes very different limits than what we are used to in statistical learning theory. To begin with, we cannot assume a sampling of random variables that are independent and identically distributed, and this case is seldom studied even in the classical setting. Symmetrization [33] or assuming the exchangeability of random variables and using de Finetti-type theorems [15, 24, 26, 32, 35] helps establish results.

We investigate induction in quantum protocols with the aim of understanding their limits and differences compared to the classical case. Our results show that in the asymptotic limit, that is, when the number of test instances goes to infinity, there is a natural correspondence between classical and quantum inductive learning algorithms.

The paper is organized as follows. In Section II we introduce a formal definition of classical inductive learning, and establish our framework and notation. We identify the relevant feature that needs to be revised in its quantum generalization, namely the non-signalling character of inductive learning algorithms. We show that such property is equivalent to the standard assumption of considering separate training and test stages. In Section III we define inductive quantum learning protocols and investigate the implications of a non-signalling restriction in terms of their performance. Building on the quantum de Finetti theorem for quantum states [13], we prove a de Finetti-type theorem for non-signalling quantum channels and show that the difference in performance between unrestricted and restricted quantum protocols vanishes in the limit of many test instances. Our result shows that one can safely implement any inductive quantum learning protocol in two separate stages and have a performance that approaches that of the most general protocol, despite the potential advantage that coherent collective quantum operations may present. Finally, we end the paper with a discussion of the implications of our result in Section IV. Unless stated otherwise, all proofs can be found in the appendix.
II. INDUCTIVE CLASSICAL LEARNING

Consider a learning problem characterized by an unknown joint probability distribution $P_{XY}$, where $X$ and $Y$ are random variables that model the test data and the label associated to it, respectively. We denote its respective marginals by $P_X$ and $P_Y$. We are given a finite set of i.i.d. test instances $\{x_i\}_{i=1}^n$ and a training set, that we denote by the random variable $A$, and we are set to solve the task of assigning a label $y_i$ to each test instance $x_i$, based on the information contained in the training set $A$. Potentially, $A = \{(X_1,Y_1),\ldots,(X_m,Y_m)\}$, but our result is independent of this. We define a learning protocol that implements this task by a stochastic map $P(y_1,\ldots,y_n|A,x_1,\ldots,x_n)$.

The natural figure of merit for assessing the performance of a learning protocol is the expected risk, given by

**Definition 1.** The conditional expected risk, $E[P|A]$, of a learning protocol $P$ given a training set $A$ is given by

$$E[P|A] = \sum_{y_{1:n},y'_{1:n}} \frac{1}{n} (s_{y_1,y'_1} + \cdots + s_{y_n,y'_n}) P(y_{1:n}|A,x_{1:n}) \times P_{X|Y}(x_1,y'_1) \cdots P_{X|Y}(x_n,y'_n),$$

where $y'_i$ are the true labels, accessible to a “referee” for evaluation purposes, $s_{ab} = 1 - \delta_{ab}$, and we have introduced the short-hand notation $x_{1:n} = \{x_1,\ldots,x_n\}$ (likewise for $y$ and $y'$). The expected risk is then defined as the average conditional expected risk over realizations $a$ of the training set, i.e. $E[P] = \sum_a p_A(a)E[P|a]$.

It is convenient to define the marginal maps of $P$: $P_i(y_i|A,x_{1:n}) \equiv \sum_{y_{1:i-1},y_{i+1:n}} P(y_{1:i-1},y_{i+1:n}|A,x_{1:n})$.

**Definition 2.** We call a learning protocol inductive if it satisfies the non-signalling condition

$$P_i(y_i|A,x_{1:i-1},x_i,x_{i+1:n}) = P_i(y_i|A,x'_{1:i-1},x_i,x'_{i+1:n}), \quad \forall x_{1:i-1},x_{i+1:n}, \forall i,$$

namely, that $P_i(y_i|A,x_{1:n})$ is actually independent of all the $X$ random variables but $X_i$, for all $i$.

Note that this definition establishes an independence among the $X$ random variables as far as the learning protocol is concerned, but not from the training set $A$. In other words, we keep $A$ as an explicit dependence as it still affects each marginal map $P_i$. Definition 2 encompasses the standard assumption of inductive learning, where a classifier $f$ is extracted from the training set, and only $f$ determines the label to be assigned to each test instance.

In contraposition, consider a transductive learning scenario, where the topology of all of the unlabelled instances can have an impact on the assignment of any of the labels. The independence condition in Eq. (3) is thus violated, hence a transductive protocol is potentially signalling. Instead, it is a de facto assumption that all non-signalling protocols are inductive.

The following lemma pinpoints the feature of classical inductive learning that is relevant for our goal. In the next section we explore its extension to quantum settings.

**Lemma 1.** For every inductive learning protocol $P$ that learns a random variable $Y$ from $X$, there exists a set $F$ of functions $f : X \rightarrow Y$ and stochastic maps $T(f|A)$, $Q(y|x,f) = \delta_{y,f(x)}$ such that the inductive protocol $\tilde{P}$

$$\tilde{P}(y_{1:n}|A,x_{1:n}) = \sum_f \prod_{i=1}^n Q(y_i|x_i,f) T(f|A)$$

has expected risk

$$E[\tilde{P}|A] = E[P|A], \quad \forall A.$$  (5)

We refer the reader to the appendix for a proof. The key insight behind this lemma is that the error rate is a symmetric function of the joint inputs and outputs, thus randomly permuting the inputs—and its corresponding outputs—does not affect the expected risk. Under this randomization, the resulting protocol $\tilde{P}$ remains non-signaling, and thus it is meaningful to consider the marginal protocol $\tilde{P}_|a(y|x) = \tilde{P}_i(y|a,x)$, in terms of which the expected risk can be neatly expressed. Therefore,
applying the map $\bar{P}_{\{\mu\}}$ on each of the test instances will yield the same expected risk as the original protocol $\bar{P}$. It is enlightening for our purpose to realize that all maps $\bar{P}_{\{\mu\}}$ on each of the test instances are applied on the same sample of the training set $a$. Thus, each realization of the training set $a$ selects a probabilistic function which is then applied to each test instance independently.

From a fundamental point of view, this shows that every conceivable inductive protocol can be decomposed into a training phase and a test phase with no effect on its performance. On the converse, any learning problem which splits into a training phase and a test phase satisfies the non-signaling condition, so one can arguably think of the non-signaling condition as a definitive trait of inductive learning. The advantage of this approach is many-fold. On one hand, it allows one to focus on a much simpler set of features which fully characterize the performance of the protocol. In addition, the training phase can be extended to provide further information relevant to assess, in advance of the test stage, the expected performance of the protocol. This is the case in, e.g., structural risk minimization, where not only a function is chosen but also an estimator of the expected risk itself is provided [34], and confidence intervals are obtained.

### III. QUANTUM LEARNING

The situation in quantum learning protocols is potentially very different. In classical probability theory, the set of point mass indicator functions constitutes the finest grained measurement that one can perform on a random variable, from which all other observables can be obtained, and from which all observables can be simultaneously derived. This is not the case in quantum theory. There exist incompatible measurements which cannot be implemented simultaneously. The notion of a quantum state then captures all the relevant information of a quantum system, and is a generalization of the classical probability distribution, not reducible to it. Each quantum system $A$ has an associated Hilbert space, denoted $\mathcal{H}_A$, and the associated space of Hermitian operators will be denoted in calligraphic letters $\mathcal{A} = \mathcal{L}(\mathcal{H}_A)$.

A key concept in quantum information theory is that of a pure state, namely a state which cannot be non-trivially expressed as a convex combination of other states. These are the analogs of the point mass functions in classical probability. Non-trivial convex combinations of pure states are called mixed states. A generic state of a quantum system $A$ is a positive semidefinite Hermitian operator $\rho_{A}$ in $\mathcal{A}$ with trace equal to 1, $\text{tr}[\rho] = 1$ and is usually written with the Greek letter $\rho$, or $\rho_A$. Pure states are rank-one projectors $\rho = |\psi\rangle \langle \psi|$. A measurement on system $A$ is described by a positive operator-valued measure (POVM) $\mathcal{M}(dg)$, over a set of outcomes $G$, such that $\int_G M(dg) = 1$ and the probability density of a given outcome is given by $p(dg) = \text{tr}[\rho_{A}M(dg)]$. It will be convenient to define for any operator $X \in \mathcal{A}$, its Hilbert-Schmidt dual $X : \mathcal{A} \rightarrow \mathbb{C}$, defined by $X(Y) = \text{tr}[XY]$. An observable $X$ on $A$ is a Hermitian operator in $\mathcal{A}$, which can be implemented by its spectral decomposition, $X = \sum_i x_i P_i$, where $P_i$ are orthogonal projectors and $\{P_i\}$ is a POVM, i.e. $\sum_i P_i = 1$. The expectation value of $X$ is given by $\text{tr}[\rho_{A}X] = \sum_i x_i \text{tr}[\rho_{A}P_i]$. In the spirit of learning protocols, independent inputs and outputs of a quantum protocol can be regarded to be independent quantum systems, which are commonly called parties ($A$ for Alice, $B$ for Bob, ...). The multipartite system is described by the tensor product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, and multipartite quantum states are positive semidefinite operators $\rho_{AB}$ in $\mathcal{A} \otimes \mathcal{B} \simeq \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Different quantum systems can show correlations, in the sense that the probability distribution that describes the multipartite system is not the product of single-party distributions. However, unlike classical distributions, pure quantum states need not be separable. Reduced states are the quantum analogs of marginal distributions, and an extension of a state $\rho_A$ is another state $\rho_{AB}$ which yields $\rho_A = \text{id}_A \otimes tr_B[\rho_{AB}]$ upon reduction. In general, we will denote this partial trace operation by omitting the identity map, and specifying the systems on which the trace operates, $tr_B \equiv \text{id}_A \otimes tr_B$.

The fact that not all pure states are separable has remarkable consequences, the most important being the phenomenon of quantum entanglement: a pure state $\rho_{AB} = |\Psi_{AB}\rangle \langle \Psi_{AB}|$ is said to be entangled if there does not exist a decomposition of the form $|\Psi_{AB}\rangle = \sum_j c_j |\psi_A\rangle_j \otimes |\psi_B\rangle_j$, where $c_j \geq 0$. Unlike pure classical distributions, the marginal of a pure quantum state need not be pure. This results in the so-called monogamy of entanglement. As it turns out, for finite dimensional systems, the amount of quantum entanglement existing between two parties is always upper-bounded. In addition, the operational character of entanglement implies that this maximum value must be attained by certain non-separable pure bipartite states. In fact, consider a system of three parties Alice, Bob and Charlie. If Alice is maximally entangled with Bob, tracing out Alice will leave Bob and Charlie in a mixed state, thus preventing them from sharing maximal entanglement among each
other. This idea is rigorously known as entanglement monogamy \cite{14,25} and prevents Alice from sharing a given amount of entanglement with too many parties.

For various identical parties, we will differentiate them either by primes \( \mathcal{Y}, \mathcal{Y}', \) or by subscript indices \( B_1, \ldots, B_n. \) A many-fold tensor product of such parties (say from \( k \) till \( n \)) will be denoted by the notation \( k : n, B_{k:n} = B_k \otimes \cdots \otimes B_n. \) A permutation of \( n \) subsystems is implemented by the unitary transformation \( \Pi^B_\sigma(X) = U_\sigma X U^\dagger_\sigma, \) where \( \sigma \in S_n \) and \( U_\sigma \) is the \( n \)-fold tensor product representation of \( S_n \) over \( B_{1:n}. \) A generic operator \( X \in B \) can be symmetrized as \( X = \frac{1}{n!} \sum_{\sigma \in S_n} \Pi^B_\sigma(X) = \frac{1}{n} (X \otimes 1_{B_{2:n}} + \cdots + 1_{B_{1:n-1}} \otimes X). \) A quantum state \( \rho_{B_{1:n}} \) is symmetric under permutations iff \( \rho_{B_{1:n}} = \Pi^B_\sigma(\rho_{B_{1:n}}) \forall \sigma \in S_n. \) The monogamy of entanglement applied to symmetric states leads to the celebrated quantum de Finetti theorems \cite{7,3,10,13,21}, which asserts that if a symmetric quantum multipartite state \( \rho_{B_{1:k}} \) admits a symmetric extension \( \rho_{B_{1:n}} \) \( (k < n), \) then \( \rho_{B_{1:k}} \) must be close to separable, i.e. only so much entanglement can exist between any two \( B_i,B_j, \) \( (i,j \leq k), \) compatible with the existence of that much entanglement with all other parties \( B_{k+1}, \ldots, B_n. \)

**Theorem 1** (Quantum de Finetti theorem). Let \( A \) and \( B \) be quantum systems and let \( \omega_{AB_{1:k}} \) be a symmetric quantum state under exchange of the \( B \) systems. If \( \omega_{AB_{1:k}} \) admits a symmetric extension \( \omega_{AB_{1:n}} \) then there is a set \( G, \) a POVM \( \mathcal{M}(dg) \) over \( A, \) and a map \( \phi : G \to B \) such that

\[
\left\| \omega_{AB_{1:k}} - \int \frac{\mathcal{M}(dg)}{d_A} \otimes \phi_g \right\|_1 \leq \frac{4d^2k}{n} \tag{6}
\]

where \( d = \dim(B), d_A = \dim(A), \phi_g \geq 0, \forall g \in G, \mathcal{M}(dg) \) only depends on the \( n \)-extension \( \omega_{AB_{1:n}} \) and, in particular, is independent of \( k. \) \( \|X\|_1 = \text{tr}|X| \) denotes the trace-norm of operator \( X. \) In general, one can take \( G = SU(d^2). \) \( G \) and the accuracy of the approximation is independent of the dimension of \( A. \)

This theorem, which can be found in Ref. \cite{13}, is the fundamental ingredient in our result. We refer the reader to the original paper, or to the appendix for an adapted proof.

Quantum information cannot be cloned. Indeed, it is this very feature (not bug) of quantum theory that makes the argument supporting Lemma \cite{3} break down. However, it is possible to approximately clone a quantum state, and the quality of the clones will depend on how many copies must be produced, reaching an asymptotic limit in which each copy contains no more information than that which can be obtained by a single quantum measurement on the original system. This idea is reflected in the seminal paper \cite{25} which asserts that asymptotic cloning is equivalent to state estimation succeeded by state preparation. Therefore, one possible approach when trying to use a training set \( A \) to address a large number of test instances is to perform a measurement \( \mathcal{M} \) on \( A, \) distribute the measurement outcome across all test instances \( B_{1:n}, \) and then use it to handle each test instance independently. This approach has the property of being non-signalling by construction. We will show that any symmetric non-signalling protocol can be well approximated by this strategy when the number of test instances is large.

A quantum channel from \( X \) to \( Y \) is a completely-positive (CP) trace-preserving (TP) map \( \Phi : X \to Y. \) The trace-preserving condition is equivalent to \( \Phi^*(\mathbb{1}_Y) = \mathbb{1}_X, \) where \( \Phi^* \) is the adjoint map w.r.t. the Hilbert-Schmidt product. Quantum information theory has taught us that collective approaches are potentially much more powerful than local ones \cite{25,26}. Multipartite channels that can be implemented by local operations and classical communication are called LOCC, and constitute a proper subset of all physical channels.

In analogy to a classical learning problem, where an unknown probability distribution \( p_{X \otimes Y} \) must be mimicked by attaching appropriate labels to given random variables \( X, \) one may consider the most general quantum learning problem as the task of generating bipartite quantum states \( p_{X \otimes Y} \) by the action of a quantum channel on the marginal \( p_X. \) The learning protocol can be thought of as a collective quantum channel \( Q \) which takes a training register \( A \) and the set of test instance registers \( X^{\otimes n} \) as inputs, and yields a corresponding set of output registers \( Y^{\otimes n} \) (see Fig. \cite{14} for an illustrative description of the setup).

**Definition 3.** A quantum learning protocol for a training set \( A \) and \( n \) quantum states \( p_{X \otimes Y} \in X \otimes Y \) is a multipartite quantum channel \( Q : A \otimes X^{\otimes n} \to Y^{\otimes n}. \) A non-signalling quantum protocol is a quantum channel \( Q : A \otimes X^{\otimes n} \to Y^{\otimes n} \) such that

\[
\text{try}_Y[Q(p_{AX^{\otimes n}})] = \text{tr}_X[Q(p_{AX^{\otimes n}})], \forall i. \tag{7}
\]
Fig. 1: Diagramatic representation of (a) a generic quantum learning protocol \( Q \), as per Definition 4, and (b) the approximate protocol \( \tilde{Q} \), given in Eq. (12). Both setups take training and test instances \( \rho_A \) and \( \rho_{XY} \) as inputs. We distinguish two agents in the diagrams: the performer of the learning protocol or “learner”, placed above the dashed horizontal line, and the referee, placed below. The learner obtains as output of the learning channel the classical labels \( Y_{1:n} \), and sends them to the referee, who contrasts them with the true labels \( Y'_{1:n} \) and evaluates the performance of the channel by its average risk \( \tilde{S} \) (see Definition 4). While the most general approach (a collective quantum channel \( Q \)) in principle acts globally on all its inputs (Fig. 1a), its approximation \( \tilde{Q} \) comprises two separate stages: first the training stage consists on a measurement \( M \), performed on the training set \( \rho_A \), and second, in the test stage the classical information \( g \) obtained from the measurement is distributed among all test instances, and corresponding quantum channels \( \Phi_g \) are applied locally to each test instance.

This approach serves as a good starting point for generalizing several quantum learning problems, both classificative and generative. In particular, a quantum state classification problem may be expressed as \( \rho_{XY} = \sum_y p_y \rho_X^{(y)} \otimes |y\rangle \langle y| \), where register \( X \) contains the quantum state, and \( Y \) holds a classical label corresponding to the state in \( X \). This naturally encompasses the programmable quantum discriminator [29, 30], but admits a much wider class of setups. In these situations \( X \) is a genuinely quantum state, but the \( Y \) register only holds classical labels. Another relevant approach is that of quantum state tomography, i.e., where a classical label \( x \) is taken as a predictor for certain quantum states \( \rho_X^{(x)} \), thus \( \rho_{XY} = \sum_x p_x |x\rangle \langle x| \otimes \rho_X^{(x)} \). The task of the protocol is to learn from the training set each one of the quantum states and then produce a similar copy for each \( X \) instance. More generally, one could consider the task of generating genuine bipartite quantum states \( \rho_{XY} \) starting from their reduced states \( \rho_X^{(x)} \).

Definition 4. Given a risk observable \( S \in Y \otimes Y' \), the expected risk of protocol \( Q \) is the expectation value of the symmetrized risk observable \( \tilde{S} \in (Y \otimes Y')_{1:n} \) on the output of the channel \( Q \),

\[ E[Q] = \text{tr}[Q \otimes \text{id}_{Y_{1:n}} (\rho_A \otimes (\rho_{XY})^{(1:n)}) \tilde{S}]. \]  

(8)

For every quantum protocol \( Q \) we can define the symmetrized protocol \( \tilde{Q} \),

\[ \tilde{Q} = \frac{1}{n!} \sum_{\sigma \in S_n} \Pi_{g}^{(Y')} \circ Q \circ (\text{id}_A \otimes \Pi_{g}^{(X)}). \]  

(9)

\( \tilde{Q} \) is non-signalling if \( Q \) is. In analogy with the classical case, a non-signalling quantum channel \( \tilde{Q} : A \otimes X^{\otimes n} \rightarrow Y^{\otimes n} \) naturally admits a notion of marginalization, thereby inducing channels for a reduced number of registers, \( k \leq n \), \( Q_k : A \otimes X^{\otimes k} \rightarrow Y^{\otimes k} \) (Lemma 2 in the appendix). Then, the expected risk can be expressed in terms of \( Q_1 \),

\[ E[Q] = \text{tr}[Q_1 \otimes \text{id}_{Y} (\rho_A \otimes \rho_{XY'}) \tilde{S}_{Y'}], \]  

(10)

or the conditional channel \( \tilde{Q}_{1|\rho_A} : \rho_X \rightarrow \tilde{Q}_1(\rho_A \otimes \rho_X) \).

It is clear that the line of reasoning so far is a simple reformulation of the ideas involved in the classical arguments. If one could implement the protocol \( Q_1 \) on each of the test instances then one could perform with an average performance \( E[Q] \). At this point, however, we encounter the fundamental roadblock that motivates this work. The map

\[ \rho_A \otimes \rho_{X_1} \otimes \cdots \otimes \rho_{X_n} \mapsto \tilde{Q}_{1|\rho_A}(\rho_{X_1}) \otimes \cdots \otimes \tilde{Q}_{1|\rho_A}(\rho_{X_n}). \]  

(11)
is non-linear in \( a \), so it does not reflect a physically realizable transformation. Physically, this impossibility reflects the non-clonable nature of quantum information \([17,37]\). It is the impossibility of cloning the training set which prevents the simultaneous application of the map \( \tilde{Q} \) on the \( n \) test instances. Therefore, a generic quantum channel \( \tilde{Q} : A \otimes X_{1:n} \rightarrow Y_{1:n} \) that distributes symmetrically the system \( A \) across \( n \) identical parties \( X \), can, at best, perform some sort of approximate cloning, which then is acted upon independently and symmetrically. Since asymptotically, this cloning operation becomes a measure and prepare process, we can reduce it to a measurement on the training set, and consider the preparation of the clones as part of the task to be performed on each test instance. This argument can be regarded as a form of de Finetti theorem for quantum channels. We note that a de Finetti theorem for fully symmetric quantum channels can be found in the literature \([18]\). To conclude this section, we make our statement rigorous.

**Theorem 2** (Main result). Let \( Q : A \otimes X_{1:n} \rightarrow Y_{1:n} \) be a non-signalling quantum channel, and let \( S \in Y \otimes Y' \) be a local operator. Then, there exists a POVM \( \mathcal{M}(dg) \) on \( A \) and a set of quantum channels \( \Phi_g : X \rightarrow Y \) such that the quantum channel \( \tilde{Q} \),

\[
\tilde{Q} = \int \mathcal{M}(dg) \otimes \Phi_g^{\otimes n},
\]

satisfies

\[
\left| E[Q] - E[\tilde{Q}] \right| \leq O\left(\frac{1}{n^{1/6}}\right).
\] (13)

In its strict sense, the theorem shows that, for any local operator \( S \), its symmetrized expectation under the action of a non-signalling quantum channel \( Q \) can be approximated by a 1-way LOCC quantum channel \( \tilde{Q} \). This channel amounts to performing a measurement \( \mathcal{M}(dg) \) yielding outcome \( g \) over the training set, and applying simultaneously \( \Phi_g \) on each of the test instances (see Fig. 1). The resulting performance of both protocols, as measured by their expected risk, approach as \( n \) tends to infinity.

**IV. DISCUSSION**

The main result reported in this paper, Theorem 2, is a natural consequence of the symmetry implicit in the problem. Given the fact that the performance on a multiple-instance inductive learning task is symmetric under simultaneous exchange of the test/answer pairs, a randomized permutation of the test instances will yield the same average performance. Therefore, each protocol performs equally well as its randomized permutation protocol. We have used this symmetry and the fact that the quantum information contained in the training set cannot be perfectly distributed over an arbitrarily large number of parties, to show that any such protocol must, effectively, be well approximated by the training set which prevents the simultaneous application of the map \( \tilde{Q} \) on the \( n \) test instances. Therefore, a generic quantum channel \( \tilde{Q} : A \otimes X_{1:n} \rightarrow Y_{1:n} \) that distributes symmetrically the system \( A \) across \( n \) identical parties \( X \), can, at best, perform some sort of approximate cloning, which then is acted upon independently and symmetrically. Since asymptotically, this cloning operation becomes a measure and prepare process, we can reduce it to a measurement on the training set, and consider the preparation of the clones as part of the task to be performed on each test instance. This argument can be regarded as a form of de Finetti theorem for quantum channels. We note that a de Finetti theorem for fully symmetric quantum channels can be found in the literature \([18]\). To conclude this section, we make our statement rigorous.
variety of relevant quantum learning tasks. Also, our quantitative approximation bounds allow for single-copy algorithms to be used as benchmarks for coherent multi-instance ones.

Another benefit of this reduction is the ability to access, without disturbance, the state of the learner in between the training and test stages. This information is essential for several machine learning tasks. For structural risk minimization, one uses an estimate of the expected risk, produced by evaluating the performance of a given classifier on the training set. In the quantum setup, this approach is not directly applicable. As the training set can only be accessed once, one can either extract information to determine the best classifier, or to assess the performance of one given classifier. However, both tasks will generally be incompatible. Therefore, a "quantum black box" – e.g. a fully quantum processor that takes all the inputs (training and tests) – will, despite being the most general approach, provide only the required answers. It is unclear how one can adapt a generic quantum black box to provide an assessment of its own performance. Our result, nevertheless, opens the door to assessing the performance of any classifier by suitably processing the intermediate measurement outcome $g$.

\section{Conclusions}

We proved a natural analogy between classical learning protocols and their quantum counterparts in the asymptotic setting. We expect the result reported here will shed light on the potential and limitations of learning from quantum sources, and ultimately serve as a starting ground for developing a fully quantum theory of risk bounds in statistical learning.

A few comments on the degree of generality of our result are in order. The convergence rate of our approximation is potentially not tight, and we expect better bounds to be achievable. For simplicity, the approach presented here uses the operator form of Chebyshev inequality (Lemma 5 in the appendix), which ultimately hinders us from obtaining a better bound. We expect a more detailed study will yield better approximations.

More importantly, our result can be extended in various ways. A potentially very relevant practical problem is to learn quantum operations rather than states. This, however, can be easily addressed within the Choi matrix formalism. A related result for learning quantum unitary operations already shows the same splitting reported here \cite{Bae2006}. Indeed, the formalism of quantum combs \cite{Bengtsson2006} provides the theoretical framework for this extension, but essentially, the most general such process will also be described by a suitable multipartite quantum system $\omega_{AB_1,\ldots}$, where $A$ will now consist of input and output ports, and the maps $\Phi_\varphi$ will be potential implementations of the learned operations. This leads us to another question, which regards the degree of generality of considering linear risk functions. In general, several figures of merit of interest cannot be expressed as a linear function of the density operator. However, any polynomial function of the density operator $\rho$ can be implemented linearly if one has access to multiple copies of $\rho$. It is not difficult to extend our approximation results for higher-order figures of merit.

\section*{Acknowledgements}

The authors wish to thank useful discussions with Andreas Winter and Giulio Chiribella. A.M. is supported by the ERC (Advanced Grant IRQUAT, project number ERC-267386). G.S. is supported by the Spanish MINECO (project FIS2015-67161-P) and the ERC (Starting Grant 258647/GEDENTQOPT). P.W. acknowledges financial support from the ERC (Consolidator Grant QITBOX), MINECO (Severo Ochoa grant SEV-2015-0522 and FOQUS), Generalitat de Catalunya (SGR 875), and Fundació Privada Cellex.

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Appendix A: Proof of our main result

Our main result, Theorem 2, consists in showing that for every non-signalling CPTP map $Q : \mathcal{A} \otimes \mathcal{X}_{1:n} \rightarrow \mathcal{Y}_{1:n}$ there is a symmetric one-way LOCC map $Q : \mathcal{A} \rightarrow \mathcal{Y}_{1:n}$ that approximately reproduces all local expectation values, and is non-signalling by construction. The backbone of our result is the quantum de Finetti theorem, specifically in its form as it appears in [13], Theorem 4. In order to apply the de Finetti theorem to our problem, we also use the Choi-Jamiolkowski identification between quantum states and quantum channels [3].

**Theorem 3 (Choi).** Every CP map $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ can be represented by a positive semidefinite operator $\phi \in \mathcal{X} \otimes \mathcal{Y}$, such that

$$\phi = \text{id}_{\mathcal{X}} \otimes \Phi(\Omega),$$

where $|\Omega\rangle = \sum_{i,j} |i, j\rangle \in \mathcal{X} \otimes \mathcal{X}$, and $\Omega = |\Omega\rangle \langle \Omega|$. In addition, for any $X \in \mathcal{X}$ we have

$$\Phi(X) = d_X \text{tr}_X[\phi^\top_X X \otimes \mathbb{I}_Y].$$

The adjoint map $\Phi^* : \mathcal{Y} \rightarrow \mathcal{X}$ is given by (we use the customary identification between $\mathcal{X}^*$ and $\mathcal{X}$ induced by the Hilbert-Schmidt product)

$$\Phi^*(Y) = d_X \text{tr}_Y[\phi^\top_X X \otimes \mathbb{I}_Y].$$

In addition, if $\Phi$ is trace-preserving, then $\Phi^*(\mathbb{I}_Y) = d_X \text{tr}_Y[\phi^\top_Y Y] = \mathbb{I}_X$.

This allows us to characterize properties of channels by referring to properties of their respective Choi matrices. The non-signalling property of a quantum channel has a direct relation with the reduced states of its Choi matrix.

**Lemma 2.** Let $Q : \mathcal{A} \otimes \mathcal{X}_{1:n} \rightarrow \mathcal{Y}_{1:n}$ be a non-signalling quantum channel, and let $\omega_{\mathcal{A}(\mathcal{X}Y)_{1:n}}$ be its Choi matrix. Then

$$\text{tr}_{\mathcal{Y}_{n-k:n}}[\omega_{\mathcal{A}(\mathcal{X}Y)_{1:n}}] = \omega_{\mathcal{A}(\mathcal{X}Y)_{1:k}} \otimes \mathbb{I}_{\mathcal{Y}_{n-k:n}} \Big/ d^{n-k}$$

and $\omega_{\mathcal{A}(\mathcal{X}Y)_{1:k}} = \text{tr}_{\mathcal{X}_{n-k:n}}[\omega_{\mathcal{A}(\mathcal{X}Y)_{1:n}}]$ is the Choi matrix of the induced channel $Q_k : \mathcal{A} \otimes \mathcal{X}_{1:k} \rightarrow \mathcal{Y}_{1:k}$.

Lemma 2 is proved by straightforward evaluation.

Applying the quantum de Finetti theorem to the Choi matrix of the CPTP map $Q$, $\omega_{\mathcal{A}(\mathcal{X}Y)_{1:n}}$, we get an approximation to $Q$ as described by the Choi matrix $\eta_{\mathcal{A}(\mathcal{X}Y)_{1:k}}$,

$$\eta_{\mathcal{A}(\mathcal{X}Y)_{1:k}} = \int_G M(dg) \otimes \Phi^k_g.$$

For $k = 0$ the approximation is exact, so $\int_G M(dg) = \text{tr}_{\mathcal{X}_{1:n}}[\omega_{\mathcal{A}(\mathcal{X}Y)_{1:n}}] = \mathbb{I}_A/d_A$, therefore $M(dg) = d_A M(dg)$ is a POVM. The positive semidefinite quantum states $\phi_g$ describe a family of completely positive maps $\Phi_g : \mathcal{X} \rightarrow \mathcal{Y}$.

The state $\eta_{\mathcal{A}(\mathcal{X}Y)_{1:k}}$ does not, however, represent a quantum operation which is deterministically realizable, in the first place because $\text{tr}_{\mathcal{X}_{1:k}}[\eta_{\mathcal{A}(\mathcal{X}Y)_{1:k}}]$ may not be $\mathbb{I}_{\mathcal{A} \mathcal{X}_{1:k}}/d_A d_{\mathcal{X}_{1:k}}$, as is required for a trace-preserving channel. Furthermore, a quantum channel can be implemented by 1-way LOCC iff its Choi matrix is of the form

$$\tilde{\eta}_{\mathcal{A}(\mathcal{X}Y)_{1:k}} = \int_G M(dg) \otimes \Phi^k_g,$$

where $\text{tr}_Y[\tilde{\phi}_g] = 1/d$, for all $g \in G$. This would ensure that all corresponding CP maps $\tilde{\Phi}_g$ are trace-preserving, and thus the channel described by $\tilde{\eta}_{\mathcal{A}(\mathcal{X}Y)_{1:k}}$ can be implemented by first performing measurement $M$ on $A$ and then applying $\tilde{\Phi}_g : \mathcal{X} \rightarrow \mathcal{Y}$ on each of the systems $X$.

Although one does not expect that each $\tilde{\phi}_g$ in Eq. (A5) satisfies

$$\text{tr}_Y[\tilde{\phi}_g] \overset{\triangle}{=} \mathbb{I}_X/d_X,$$

on average they approximately do. More importantly, we now show that the outcomes $g$ are concentrated with high probability on those $\tilde{\phi}_g$ which almost satisfy the condition. Let $\| \cdot \|_1$ be the trace-norm and $\| \cdot \|_\infty$ be the operator norm.
Lemma 3. Let $Q$ be a non-signalling CPTP map $Q : A \otimes X^\otimes n \to Y^\otimes n$ with Choi matrix $\omega_{A(XY)}$, and let $M(dg)$ and $\{\phi_g \in A \otimes Y\}_G$ be such that

$$\eta_{A(XY)} = \int_G M(dg) \otimes \phi_g^\otimes k$$

is a separable approximation of $\omega_{A(XY)}$ such that

$$\|\eta_{A(XY)} - \omega_{A(XY)}\|_1 \leq k\delta.$$  \hspace{1cm} (A8)

Define for all $0 \leq k \leq n$ and for any subset $R \subseteq G$,

$$E_k[R] = \int_R \text{tr}[M(dg)]\text{tr}G[\phi_g]^{\otimes k}.$$  \hspace{1cm} (A9)

Then, the following holds

1. $\|E_k[G] - \mathbb{I}_{X^1/k}/d_X^{k}\|_1 \leq k\delta$.
2. For any $\epsilon > 0$, let $R_\epsilon = \{g \in G\|\text{tr}G[\phi_g] - E_1[G]\|_\infty < \epsilon\}$, $R_\epsilon \equiv R \setminus G$. Then

$$E_0[R_\epsilon] \leq \frac{d_X^2}{\epsilon^2} \left(2\delta \left(1 + \frac{1}{d_X}\right) + \delta^2\right).$$  \hspace{1cm} (A10)

Consider the measurement $M(dg) = d_A M(dg)$ is performed on the state $\mathbb{I}/d_A$ yielding outcome $g$, and $\Phi_g : X \to Y$ is to be applied on each of the test instances. Of course, for this to be deterministically implementable, one needs that $\Phi^* (\mathbb{I}/d_A) = I_X$, which amounts to $\tau_g = \text{tr}G[\phi_g^{\otimes X}] = \mathbb{I}/d_X$. If this condition is met approximately, one can implement a suitably modified map $\tilde{\Phi}_g$ at the expense of actually implementing a slightly worse approximation to $Q$. However, if the condition is not met even approximately, the implementation cannot be expected to approximate $Q$. Lemma 3 shows that this case is unlikely to occur, since

$$\Pr[\|\tau - \text{E}[\tau]\|_\infty \geq \epsilon] \leq \text{E}[\tilde{R}_\epsilon] \leq \frac{\epsilon^2}{d_X^2} \left(2\delta \left(1 + \frac{1}{d_X}\right) + \delta^2\right)$$

$$= \frac{\epsilon^2}{d_X^2} \left(2d_A(d_X + d_X^2) + \left(d_A d_X^2 \frac{n}{n}\right)^2\right).$$  \hspace{1cm} (A11)

Hence, one can slightly modify the operators $\phi_g$ into $\tilde{\phi}_g$ in order to satisfy Eq. (A7) and ensure that in all cases, either $\phi_g$ and $\tilde{\phi}_g$ are close enough, or $g$ is unlikely enough so that the approximation still converges in $n$ to the actual channel given by $\omega_{A(XY)}$. We call this a 1-way LOCC approximation.

Lemma 4 (1-way LOCC approximation). Let $Q$ be a symmetric, non-signalling CPTP map $Q : A \otimes X^\otimes n \to Y^\otimes n$ with Choi matrix $\omega_{A(XY)}$. Then there is a POVM $d_A M(dg)$ and there are states $\tilde{\phi}_g$ such that $\text{tr}G[\tilde{\phi}_g] = \mathbb{I}/d_X$ and the quantum state

$$\eta_{AXY} = \int_G M(dg) \otimes \tilde{\phi}_g$$  \hspace{1cm} (A12)

is a separable approximation to $\omega_{AXY}$,

$$\|\omega_{AXY} - \eta_{AXY}\|_1 \leq c n^{-1/6} + O(n^{-1/3}).$$  \hspace{1cm} (A13)

where $c$ is a constant depending on $\text{dim}(X)$ and $\text{dim}(Y)$.

Proof of Lemma 4. Let $M(dg)$ and $\{\phi_g\}$ be the factors in the de Finetti approximation to $\omega_{A(XY)}$, which admits a symmetric $n$-extension by assumption. Then they satisfy Eq. (A8) with $\delta = 4d_X^2 d_Y^2/n$. From statement 1 in Lemma 3 we have

$$\|E_1[G] - \mathbb{I}_X/d_X\|_\infty \leq \|E_1[G] - \mathbb{I}_X/d_X\|_1 \leq \delta,$$  \hspace{1cm} (A14)

so that

$$E_1[G] \geq \left(\frac{1}{d_X} - \delta\right) \mathbb{I}_X.$$  \hspace{1cm} (A15)
Therefore, for $\epsilon < \frac{1}{d^2} - \delta$ we have
\[ g \in R_\epsilon \Rightarrow \tau_g \equiv \text{tr}_Y[\phi_g] \geq E_1[G] - \epsilon \mathbb{1} > 0. \] (A16)

Thus, we can ensure that all $g \in R_\epsilon$ satisfy $\tau_g > 0$. We can define
\[ \tilde{\phi}_g = \begin{cases} \frac{1}{\epsilon}(\tau_g^{-1/2} \otimes \mathbb{1}_Y)\phi_g(\tau_g^{-1/2} \otimes \mathbb{1}_Y) & \text{if } g \in R_\epsilon, \\ \phi_g & \text{if } g \notin R_\epsilon, \end{cases} \] (A17)

where $\varphi$ is the Choi matrix of any CPTP map $X \rightarrow Y$. By definition every $g \in R_\epsilon$ has $\tau_g \geq E_1[G] - \epsilon \mathbb{1}$, and using $E_1[G] \geq (\frac{1}{d} - \delta) \mathbb{1}$ we can write
\[ \text{tr}[\tau_g^{1/2}]/2 \geq \left( \text{tr}\left[ E_1[G] - \epsilon \mathbb{1}_X \right] \right)^2 \]
\[ \geq \left( \text{tr}\left[ (\frac{1}{d^2} - \delta - \epsilon)^{1/2} \mathbb{1}_X \right] \right)^2 \]
\[ = d^2X - d^2_1(\delta + \epsilon), \] (A18)

Thus, Lemma 6 shows that for all $g \in R_\epsilon$,
\[ \|\phi_g - \tilde{\phi}_g\|_1 \leq \sqrt{d^2X} \sqrt{\epsilon + \delta}, \] (A19)

and the subadditivity of the trace distance ($\|\rho^{\otimes k} - \sigma^{\otimes k}\|_1 \leq k\|\rho - \sigma\|_1$) leads to
\[ \|\phi^{\otimes k}_g - \tilde{\phi}^{\otimes k}_g\|_1 \leq \begin{cases} k\sqrt{d^2X} \sqrt{\epsilon + \delta} & \text{if } g \in R_\epsilon, \\ \|\phi^{\otimes k}_g - \tilde{\phi}^{\otimes k}_g\|_1 & \text{if } g \notin R_\epsilon. \end{cases} \] (A20)

Combining this with $\|A \otimes (B - B')\|_1 = \text{tr}[A]\|B - B'\|_1$ for all $A \geq 0$,
\[ \|M(dg) \otimes (\phi^{\otimes k}_g - \tilde{\phi}^{\otimes k}_g)\|_1 = \text{tr}[M(dg)]\|\phi^{\otimes k}_g - \tilde{\phi}^{\otimes k}_g\|_1 \] (A21)

and the triangle inequality we get
\[ \left\| \int_G M(dg) \otimes (\phi^{\otimes k}_g - \tilde{\phi}^{\otimes k}_g) \right\|_1 \leq \int_G \text{tr}[M(dg)]\|\phi^{\otimes k}_g - \tilde{\phi}^{\otimes k}_g\|_1 \]
\[ \leq k\sqrt{d^2X} \sqrt{\epsilon + \delta} \int_{R_\epsilon} \text{tr}[M(dg)] + 2 \int_{R_\epsilon} \text{tr}[M(dg)] \]
\[ \leq k\sqrt{d^2X} \sqrt{\epsilon + \delta + 2E_0[R_\epsilon]} \]
\[ \leq k\sqrt{d^2X} \sqrt{\epsilon + \delta + 2d^2X \left( 2\delta \left( 1 + \frac{1}{d^2X} \right) + \delta^2 \right)}, \] (A22)

Taking $k = 1$ and using the triangle inequality we get
\[ \|\omega_{AXY} - \eta_{AXY}\|_1 \leq \left\| \omega_{AXY} - \int_G M(dg) \otimes \phi_g \right\|_1 + \left\| \int_G M(dg) \otimes (\phi_g - \tilde{\phi}_g) \right\|_1 \]
\[ \leq \delta + \sqrt{d^2X} \sqrt{\epsilon + \delta + 2d^2X \left( 2\delta \left( 1 + \frac{1}{d^2X} \right) + \delta^2 \right)}, \] (A23)

Chosing $\epsilon = \delta^{1/3}$ and expanding around $\delta = 0$ up to leading order we get
\[ \left\| \omega_{AXY} - \int_G M(dg) \otimes \tilde{\phi}_g \right\|_1 \leq \sqrt{d^2X} \delta^{1/6} + O(\delta^{1/3}), \] (A24)

which using $\delta = 4(d^2dy)^2/n$ leads to
\[ \left\| \omega_{AXY} - \int_G M(dg) \otimes \tilde{\phi}_g \right\|_1 \leq 4^{1/6} d^{5/6} dV_{Y}^{1/3} \frac{1}{n^{1/6}} + O(n^{-1/3}) \] (A25)

the desired result.

Having established a 1-way LOCC approximation bound for any symmetric non-signalling channel, we can now proceed to prove Theorem 2.
Theorem 2 (Main result). Let $Q: \mathcal{A} \otimes \mathcal{X}_{1:n} \rightarrow \mathcal{Y}_{1:n}$ be a non-signalling quantum channel, and let $S \in \mathcal{Y} \otimes \mathcal{Y'}$ be a local operator. Then, there exists a POVM $M(dg)$ on $\mathcal{A}$ and a set of quantum channels $\Phi_g: \mathcal{X} \rightarrow \mathcal{Y}$ such that the quantum channel $\hat{Q}$,

$$\hat{Q} = \int \hat{M}(dg) \otimes \Phi_g^{\otimes n},$$

satisfies

$$|E[Q] - E[\hat{Q}]| \leq O \left( \frac{1}{n^{1/6}} \right).$$

Proof of Theorem 2. We want to obtain approximation bounds for

$$E[Q] = \text{tr}[\hat{S}(Q \otimes \text{id}_{Y_{1:n}})(\rho_{\mathcal{A}(XY')_{1:n}})].$$

The specific form of $\rho_{\mathcal{A}(XY')_{1:n}}$ is irrelevant for our purposes, besides symmetry among the $XY'$ parties. Expressing $E[Q]$ in terms of the symmetrized local channel $\hat{Q}$, and in turn, in terms of its Choi matrix, we have

$$\begin{align*}
E[Q] &= \text{tr}[\hat{S}(Q \otimes \text{id}_{Y_{1:n}})(\rho_{\mathcal{A}(XY')_{1:n}})] \\
&= d_{AXY'} \text{tr}_Y[S \text{tr}_AX[(\omega_{AXY} \otimes \mathbb{I}_{Y'})^T AX (\rho_{AXY} \otimes \mathbb{I}_{Y'})^T AX]] \\
&= d_{AXY'} \text{tr}_Y[S \text{tr}_AX[(\omega_{AXY} \otimes \mathbb{I}_{Y'})^T AX (\rho_{AXY} \otimes \mathbb{I}_{Y'})^T AX]] \\
&= d_{AXY'} \text{tr}_AX \text{tr}[\hat{S}(Q \otimes \text{id}_{Y_{1:n}})(\rho_{AXY'} \otimes \mathbb{I}_{Y'})^T AX] \\
&= d_{AXY'} \text{tr}_AX \text{tr}[\hat{S}(Q \otimes \text{id}_{Y_{1:n}})(\rho_{AXY} \otimes \mathbb{I}_{Y})^T AX] \\
&= d_{AXY'} \text{tr}_AX \text{tr}[\hat{S}(Q \otimes \text{id}_{Y_{1:n}})(\rho_{AXY} \otimes \mathbb{I}_{Y})^T AX].
\end{align*}$$

(A27)

To ease the notation, it is convenient to define $R = \text{tr}_Y[(\rho_{AXY'} \otimes \mathbb{I}_{Y'})^T AX (\mathbb{I}_{AX} \otimes S)] \in \mathcal{A} \otimes \mathcal{X} \otimes \mathcal{Y'}$, so that Eq. (A27) reads

$$E[Q] = d_{AXY} \text{tr}[\omega_{AXY} R].$$

(A28)

Using Lemma 4 we can replace $\omega_{AXY}^{1:n}$ by its 1-way LOCC approximation $\eta_{AXY}$,

$$E[Q] = d_{AXY} \text{tr}[\eta_{AXY} R].$$

(A29)

which satisfies

$$|E[Q] - E[\hat{Q}]| \leq d_{AXY} \text{tr}[(\omega_{AXY} - \eta_{AXY}) R]$$

(A30)

$$\leq d_{AXY} \left\| \omega_{AXY} - \int M(dg) \otimes \tilde{\phi}_g \right\|_1 R \| \|_\infty$$

$$\leq \left(4^{1/6} d_{AXY} d_{AX}^{1/6} d_{AX}^{1/3} \frac{1}{n^{1/6}} + O(1/n^{1/3}) \right) R \| \|_\infty.$$ 

(A31)

Finally, we can absorb the constant $\| R \|_\infty$ into the factors preceeding $1/n^{1/6}$. □

Appendix B: Proofs of Lemmas and Theorem

Proof of Lemma 4. Consider the expected risk $E[P]$ of protocol $P$. Let $\sigma \in S_n$ be any permutation of $n$ elements, and let the $P^{(\sigma)}$ be the accordingly permuted protocol

$$P^{(\sigma)}(y_{1:n} | A, x_{1:n}) = P(y_{\sigma(1):\sigma(n)} | A, x_{\sigma(1):\sigma(n)}).$$

(B1)

Furthermore, let $P$ be the symmetrized protocol,

$$P(y_{1:n} | A, x_{1:n}) = \frac{1}{n!} \sum_{\sigma \in S_n} P^{(\sigma)}(y_{1:n} | A, x_{1:n}).$$

(B2)
It follows trivially that
\[ E[\mathbb{P}|A] = E[\mathbb{P}^{(\sigma)}|A] = E[\mathbb{P}|A], \quad \forall \sigma \in S_n, A. \]  
(B3)

Since \( \mathbb{P} \) is non-signalling, so is \( \mathbb{P}^{(\sigma)} \), and one can meaningfully define the marginal maps \( \mathbb{P}_1 \), which are all equal, so we refer to them as \( \tilde{P}_1 \),
\[ \tilde{P}_1(y|A, x) = \sum_{y_{2:n}} \tilde{P}(y, y_{2:n}|A, x, \cdots). \]  
(B4)

The conditional expected risk can be expressed in terms of \( \tilde{P}_1 \),
\[ E[\mathbb{P}|A] = \sum_{x, y, y'} \delta_{y, y'} \tilde{P}_1(y|A, x) P_{XY}(x, y'). \]  
(B5)

Considering \( A \) fixed, \( \tilde{P}_1(y|A, x) \) is a stochastic map from \( X \) to \( Y \), and thus it is a convex combination of deterministic maps \( Q_f(y|x) = \delta_{y, f(x)} \) for some set of functions \( f \in F \), i.e.
\[ \tilde{P}_1(y|A, x) = \sum_f \mu_A(f) Q_f(y|x), \]  
(B6)

where \( \mu_A \) is a probability measure that depends on \( A \). Then
\[ E[\mathbb{P}|A] = \sum_f \mu_A(f) E[Q_f|A]. \]  
(B7)

Thus, the stochastic maps \( Q(y|x, f) = Q_f(y|x) \) and \( T(f|A) = \mu_A(f) \) can be combined into the protocol
\[ \tilde{P}(y_{1:n}|A, x_{1:n}) = \sum_f \left[ \prod_{i=1}^n Q(y_i|x_i, f) \right] T(f|A), \]  
(B8)

which achieves
\[
E[\mathbb{P}|A] = E \left[ \sum_f \prod_{i=1}^n Q(y_i|x_i, f) T(f|A) \right| A \right]
= \sum_{x_{1:n}, y_{1:n}, y'_{1:n}} \frac{1}{n} (s_{y_1, y'_1} + \cdots + s_{y_n, y'_n}) \prod_{i=1}^n P_{XY}(x_1, y'_1) \cdots P_{XY}(x_n, y'_n)
= \sum_{x_{1:n}, y_{1:n}, y'_{1:n}} \delta_{y_1, y'_1} \tilde{P}_1(y_1|A, X) P_{XY}(x_1, y'_1)
= \sum_{x_{1:n}, y_{1:n}, y'_{1:n}} \delta_{y_1, y'_1} \tilde{P}_1(y_1|A, x) P_{XY}(x_1, y'_1)
= E[\mathbb{P}|A]. \]  
(B9)

The following proof of Theorem \[ \boxed{\text{(4)}} \] reproduces that of the original paper \[ \boxed{\text{(13)}} \], where, as suggested, a probability measure is replaced by an operator-valued measure.

**Proof of Theorem \[ \boxed{\text{(4)}} \]** Let us start by assuming \( \omega_{AB_{1:k}} \) admits a pure state extension \( \Psi_{AB_{1:n}} = |\Psi_{AB_{1:n}}\rangle \langle \Psi_{AB_{1:n}}| \). Then
\[ |\Psi_{AB_{1:n}}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{\text{sym}(n)}, \]  
(B10)

where \( \mathcal{H}_{\text{sym}(n)} \) is the symmetric subspace of \( \mathcal{H}_{B_{1:n}} \). Let also \( d \equiv \dim(\mathcal{H}_B) \).

Let \( g \) be a generic \( SU(d) \) element, \( |0\rangle \) a reference state in \( \mathcal{H}_B \), and \( dg \) the Haar measure on \( SU(d) \).

Let \( |\phi_g\rangle = U_g |0\rangle \) and use \( \phi_g = |\phi_g\rangle \langle \phi_g| \).

For any \( 1 \leq k \leq n \), let \( E^g_k = \dim \mathcal{H}_{\text{sym}(k)} \phi^g_k \) be a POVM in \( \mathcal{H}_{\text{sym}(k)} \), such that
\[
\int dg \ E^g_k = \mathbb{1}_{\mathcal{H}_{\text{sym}(k)}}, \]  
(B11)
This allows to write
\[
\omega_{AB_1:k} = \int w_g \omega^0_{A \otimes B_1:k} dg, \tag{B12}
\]
where \(w_g \omega^0_{AB_1:k}\) is the residual state on \(AB_1:k\) when measuring \(E_{n-k}\)
\[
w_g \omega^0_{AB_1:k} = \text{tr}_{B_{k+1:n}}[\mathbb{1}_A \otimes \mathbb{1}_{B_1:k} \otimes E^g_{n-k} \Psi_{AB_1:n}]. \tag{B13}
\]
Then \(\omega_{AB_1:k}\) is close to a convex combination of separable and \(B\)-iid states \(\int M(dg) \otimes \phi^g_k\), with a distribution \(M(dg)\) independent of \(k\), namely
\[
\Delta_k = \omega_{AB_1:k} - \int M(dg) \otimes \phi^g_k \tag{B14}
\]
is close to zero in trace-norm. The operator-valued measure \(M(dg)\) is given by
\[
M(dg) = \text{tr}_{B_{1:n}}[\mathbb{1}_A \otimes E^g_n \Psi_{AB_1:n}]dg. \tag{B15}
\]
We now bound \(\|\Delta_k\|_1 = \|S - \delta\|_1\), where
\[
S = \int w_g \omega^0_{AB_1:k} dg - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}} \int M(dg) \otimes \phi^g_k, \tag{B16}
\]
\[
\delta = \left(1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}}\right) \int M(dg) \otimes \phi^g_k. \tag{B17}
\]
One can readily check that
\[
\|\delta\|_1 = \left(1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}}\right) \left\|\int M(dg) \otimes \phi^g_k\right\|_1
= \left(1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}}\right) \int \text{tr}[M(dg) \otimes \phi^g_k]
= \left(1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}}\right) \int \text{tr}[M(dg)]
= \left(1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}}\right) \int \text{tr}[E^g_n \Psi_{AB_1:n}]dg
= \left(1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}}\right) \text{tr}[\Psi_{AB_1:n}]
= 1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}} \tag{B18}
\]
On the other hand,
\[
\frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}} \text{tr}_{B_{1:n}}[\mathbb{1}_A \otimes E^g_n \Psi_{AB_1:n}] = \dim \mathcal{H}_{\text{sym}(n-k)} \text{tr}_{B_{1:n}}[\mathbb{1}_A \otimes \phi^g_n \Psi_{AB_1:n}]
= \dim \mathcal{H}_{\text{sym}(n-k)} \text{tr}_{B_{1:n}}[\mathbb{1}_A \otimes \phi^g_k \otimes \phi^g_{(n-k)} \Psi_{AB_1:n}]
= \langle \phi^g_k | \text{tr}_{B_{k+1:n}}[\mathbb{1}_A \otimes \mathbb{1}_{B_1:k} \otimes E^g_{(n-k)} \Psi_{AB_1:n}] | \phi^g_k \rangle
= w_g \langle \phi^g_k | \omega^0_{AB_1:k} | \phi^g_k \rangle. \tag{B19}
\]
Notice that this is an operator in \(A\). With this we have
\[
S = \int w_g \omega^0_{AB_1:k} dg - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}} \int \text{tr}_{B_{1:n}}[\mathbb{1}_A \otimes E^g_n \Psi_{AB_1:n}] \otimes \phi^g_k dg
= \int w_g \omega^0_{AB_1:k} dg - \int \langle \phi^g_k | w_g \omega^0_{AB_1:k} | \phi^g_k \rangle \otimes \phi^g_k dg
= \int w_g \left(\omega^0_{AB_1:k} - [\mathbb{1}_A \otimes \phi^g_k] \omega^0_{AB_1:k} [\mathbb{1}_A \otimes \phi^g_k]\right) dg. \tag{B20}
\]
We now use $A - BAB = (A - BA) + (A - AB) - (1 - B)A(1 - B)$, so that we are interested in expressions of the form
\[
\alpha = \int w_g \left[ \mathbb{1}_{AB_{1:k}} - \mathbb{1}_A \otimes \phi_g^{\otimes k} \right] \omega_{AB_{1:k}}^g dg, \quad \text{(B21)}
\]
\[
\gamma = \int w_g \left[ \mathbb{1}_{AB_{1:k}} - \mathbb{1}_A \otimes \phi_g^{\otimes k} \right] \omega_{AB_{1:k}}^g \left[ \mathbb{1}_{AB_{1:k}} - \mathbb{1}_A \otimes \phi_g^{\otimes k} \right] dg, \quad \text{(B22)}
\]
so that $S = \alpha + \alpha^\dagger + \gamma$. Using
\[
w_g \left[ \mathbb{1}_A \otimes \phi_g^{\otimes k} \right] \omega_{AB_{1:k}}^g = \text{tr}_{B_{k+1:n}} \left[ \mathbb{1}_A \otimes \phi_g^{\otimes k} \otimes E_{n-k}^g \Psi_{AB_{1:n}} \right]
\]
\[
= \dim \mathcal{H}_{\text{sym}(n-k)} \text{tr}_{B_{k+1:n}} \left[ \mathbb{1}_A \otimes E_{n}^g \Psi_{AB_{1:n}} \right] \quad \text{(B23)}
\]
we have
\[
\alpha = \left( 1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}} \right) \text{tr}_{B_{k+1:n}} \left[ \Psi_{AB_{1:n}} \right], \quad \text{(B25)}
\]
thus
\[
||\alpha||_1 = ||\alpha^\dagger||_1 = 1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}}. \quad \text{(B26)}
\]
For $\gamma$ we use the relation $\text{tr}[PX] = \|PX\|_1$. This together with convexity of the trace yields
\[
||\gamma||_1 \leq \int \left\| \left[ \mathbb{1}_A \otimes \phi_g^{\otimes k} \right] \omega_{AB_{1:k}}^g \left[ \mathbb{1}_A \otimes \phi_g^{\otimes k} \right] \right\|_1 dg
\]
\[
= \text{tr} \left( \int w_g \left[ \mathbb{1}_A \otimes \phi_g^{\otimes k} \right] \omega_{AB_{1:k}}^g \right)
\]
\[
= ||\alpha||_1. \quad \text{(B27)}
\]
In summary, we can write
\[
||\Delta_k||_1 \leq ||S||_1 + ||\delta||_1 \leq ||\alpha||_1 + ||\alpha^\dagger||_1 + ||\gamma||_1 + ||\delta||_1,
\]
where each of the norms is upper-bounded by
\[
\frac{1}{2} \left( 1 - \frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}} \right). \quad \text{(B29)}
\]
Using
\[
\frac{\dim \mathcal{H}_{\text{sym}(n-k)}}{\dim \mathcal{H}_{\text{sym}(n)}} \geq 1 - \frac{dk}{n}, \quad \text{(B30)}
\]
we finally get
\[
||\Delta_k||_1 \leq \frac{4dk}{n}. \quad \text{(B31)}
\]
This proves the statement in case the extension $\omega_{AB_{1:n}} = |\Psi_{AB_{1:n}}\rangle \langle \Psi_{AB_{1:n}}|$ is a pure state. In case no such pure state exists, let $\omega_{AB_{1:n}}$ be a symmetric purification, generally mixed, and let
\[
|\Psi_{AB_{1:n}A'B_{1:n}}\rangle = \sqrt{\omega_{AB_{1:n}}} \otimes \mathbb{1}_{A'B_{1:n}} |\Phi\rangle \quad \text{(B32)}
\]
be its purification, with $|\Phi\rangle \in (\mathcal{H}_A \otimes \mathcal{H}_{B_{1:n}}^\otimes) \otimes (\mathcal{H}_A \otimes \mathcal{H}_{B_{1:n}}^\otimes)$ being the maximally entangled state among $AB_{1:n}$ and $A'B_{1:n}$. The state $\Psi_{AB_{1:n}A'B_{1:n}}$ is symmetric under exchange of $BB'$ systems, and has reductions
\[
\omega_{AB_{1:k}A'B_{1:k}} = \text{tr}_{B_{k+1:n}} \left[ \Psi_{AB_{1:n}A'B_{1:n}} \right], \quad \text{(B33)}
\]
and there is a measure $\tilde{M}(dg)$ over $\text{SU}(d^2)$ in $\mathcal{A}A'$ such that
\[
\left\| \omega_{AB_{1:k}A'B_{1:k}} - \int \tilde{M}(dg) \otimes \phi_g^{\otimes k} \right\|_1 \leq \frac{4dk}{n}, \quad \text{(B34)}
\]
where $|\phi_g\rangle = U_g \{0\} \in \mathcal{H}_B \otimes \mathcal{H}_{B'}$. $|\{0\}\rangle$ is a reference pure state in $\mathcal{H}_B \otimes \mathcal{H}_{B'}$, and $U_g$ is a generic $\text{SU}(d^2)$ element. Since the partial trace does not increase the trace-norm, the statement of the theorem is recovered by tracing out the primed systems.
**Lemma 5** (Operator Chebyshev inequality). Let $X \in \mathcal{B}$ be an self-adjoint operator-valued random variable with expectation $\mathbb{E}[X] = \mu$. Then,

$$
\Pr[\|X - \mu\|_\infty \geq \epsilon] \leq \frac{d_2^2}{\epsilon^2} \mathbb{E}[\|X \otimes X\| - \mu \otimes \mu]_\infty. \tag{B35}
$$

**Proof.** Let us first establish an intermediate result,

$$
\mathbb{E}[\|X - \mu\|_\infty^2] = \int \mu(dX)\|X - \mu\|_\infty^2
\geq \int_{\|X - \mu\|_\infty \geq \epsilon} \mu(dX)\|X - \mu\|_\infty^2
\geq \epsilon^2 \int_{\|X - \mu\|_\infty \geq \epsilon} \mu(dX)
= \epsilon^2 \Pr[\|X - \mu\|_\infty \geq \epsilon], \tag{B36}
$$

Furthermore, we can upper bound $\mathbb{E}[\|X - \mu\|_\infty^2]$ in terms of $\mathbb{E}[X \otimes X]$ and $\mu$,

$$
\|X - \mu\|_\infty^2 = \|(X - \mu)(X - \mu)\|_\infty
\leq \|(X - \mu)(X - \mu)\|_1
= \text{tr}[(X - \mu)(X - \mu)]
= \langle \Phi (X - \mu) \otimes (X - \mu)^\top | \Phi \rangle
= \text{tr}[(X - \mu) \otimes (X - \mu) \Phi^{Tz}]
= \text{tr}[(X \otimes X - \mu \otimes (X - \mu) \otimes (X - \mu)) \Phi^{Tz}]. \tag{B38}
$$

Taking the expectation we can commute it inside the trace,

$$
\mathbb{E}[\|X - \mu\|_\infty^2] \leq \text{tr}[\mathbb{E}[X \otimes X - \mu \otimes (X - \mu) \otimes (X - \mu)] \Phi^{Tz}]
= \text{tr}[\mathbb{E}[X \otimes X] - \mu \otimes \mu] \Phi^{Tz}]
\leq \|\mathbb{E}[X \otimes X] - \mu \otimes \mu\|_\infty \|\Phi^{Tz}\|_1. \tag{B39}
$$

The swap operator $\Phi^{Tz}$ has eigenvalues $\pm 1$, so its trace-norm is $d_2^2$. This concludes the proof.

**Proof of Lemma 3** Statement 1 is proven by straightforward evaluation and exploiting the contractivity of the partial trace,

$$
\left\| \mathbb{E}_k[G] - \frac{\mathbb{E}_k}[X_{1:k}]}{dX} \right\|_1 = \left\| \text{tr}_{SA_{1:k}} \left( \int_G M(dg) \otimes \phi_g^k - \omega_A(XY)_{1:k} \right) \right\|_1
\leq \left\| \text{tr}_{SA_{1:k}} \left[ \eta_A(XY)_{1:k} - \omega_A(XY)_{1:k} \right] \right\|_1
\leq \kappa \delta. \tag{B40}
$$

For Statement 2, notice that $\mathbb{E}_0$ is a probability measure on $G$. It is convenient to define the operator-valued random variable $\tau_p = \text{tr}_1[\phi_g^\otimes_1]$, with $\mathbb{E}_1[G]$ being its mean. We now apply the Chebyshev-type inequality (Lemma 4) to obtain

$$
\mathbb{E}_0[R_t] \leq \frac{d_2^2}{\epsilon^2} \|\mathbb{E}_2[G] - \mathbb{E}_1[G]^\otimes_2\|_\infty. \tag{B41}
$$

Computing the bound,

$$
\mathbb{E}_2[G] - \mathbb{E}_1[G]^\otimes_2 = \mathbb{E}_2[G] - \frac{\mathbb{E}_2}[X]{dX} \otimes \frac{\mathbb{E}_2}[X]{dX} - \left( \mathbb{E}_1[G] - \frac{\mathbb{E}_1}[X]{dX} \right)^\otimes_2
- \left( \mathbb{E}_1[G] - \frac{\mathbb{E}_1}[X]{dX} \right) \otimes \frac{\mathbb{E}_1}[X]{dX} \otimes \left( \mathbb{E}_1[G] - \frac{\mathbb{E}_1}[X]{dX} \right). \tag{B42}
$$
Thus,
\[
\|E_2[G] - E_1[G]\|_\infty^2 \leq \left\| E_2[G] - \frac{d}{dx} \|_\infty + \left\| \left( E_1[G] - \frac{d}{dx} \right) \right\|_\infty^2 
+ 2 \left\| \left( E_1[G] - \frac{d}{dx} \right) \otimes \frac{d}{dx} \right\|_\infty 
\leq 2 \left( 1 + \frac{1}{d^2} \right) \delta + \delta^2,
\]
where the first step just applies the triangle inequality, and the second uses the bound \(\|X\|_\infty \leq \|X\|_1\).

On the other hand we have
\[
E_0[R_e] \leq \frac{d^2}{\epsilon^2} \left[ 2 \left( 1 + \frac{1}{d} \right) \delta + \delta^2 \right].
\]

Lemma 6. Let \(\phi \in X \otimes Y\) be a quantum state with \(\tau = \text{tr}_Y[\phi] > 0\). Then the state
\[
\tilde{\phi} = \frac{1}{d^2} \left( \tau^{-1/2} \otimes \mathbb{1}_Y \right) \phi \left( \tau^{-1/2} \otimes \mathbb{1}_Y \right)
\]
satisfies
\[
\text{tr}_Y[\tilde{\phi}] = \frac{\mathbb{1}_X}{d^2},
\]
\[
\|\phi - \tilde{\phi}\|_1 \leq \sqrt{1 - \frac{1}{d^2} \text{tr}[\tau^{1/2}]}.
\]

Proof. Using \(\|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}\), where \(F(\rho, \sigma) = (\text{tr}\sqrt{\rho^{1/2} \sigma \rho^{1/2}})^2\) and computing the fidelity we have
\[
F(\phi, \tilde{\phi}) = \left( \text{tr}\sqrt{\phi^{1/2} \tilde{\phi} \phi^{1/2}} \right)^2
\]
\[
= \frac{1}{d^2} \left( \text{tr}\sqrt{\phi^{1/2} \left( \tau^{-1/2} \otimes \mathbb{1}_Y \right) \phi \left( \tau^{-1/2} \otimes \mathbb{1}_Y \right) \phi^{1/2}} \right)^2
\]
\[
= \frac{1}{d^2} \left( \text{tr}\sqrt{\phi^{1/2} \left( \tau^{-1/2} \otimes \mathbb{1}_Y \right) \phi^{1/2} \phi^{1/2} \left( \tau^{-1/2} \otimes \mathbb{1}_Y \right) \phi^{1/2}} \right)^2
\]
\[
= \frac{1}{d^2} \left( \text{tr}\left( \phi^{1/2} \left( \tau^{-1/2} \otimes \mathbb{1}_Y \right) \phi^{1/2} \right)^2 \right)
\]
\[
= \frac{1}{d^2} \left( \text{tr}[\phi^{1/2} \left( \tau^{-1/2} \otimes \mathbb{1}_Y \right) \phi^{1/2}]^2 \right)
\]
\[
= \frac{1}{d^2} \text{tr}[\phi \left( \tau^{-1/2} \otimes \mathbb{1}_Y \right)]^2
\]
\[
= \frac{1}{d^2} \text{tr}[\tau^{-1/2} \text{tr}_Y[\phi]]^2
\]
\[
= \frac{1}{d^2} \text{tr}[\tau^{-1/2} \tau]^2
\]
\[
= \frac{1}{d^2} \text{tr}[\tau^{1/2}]^2.
\]
This concludes the proof.