UNIFORM INTERPOLATION IN PROVABILITY LOGICS

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ABSTRACT. We prove the uniform interpolation theorem in modal provability logics GL and Grz by a proof-theoretical method, using analytical and terminating sequent calculi for the logics. The calculus for Gödel-Löb's logic GL is a variant of the standard sequent calculus of [20], in the case of Grzegorczyk's logic Grz, the calculus implements an explicit loop-preventing mechanism inspired by work of Heuerding [13, 12].

1. Introduction

1.0.1. Uniform interpolation. Uniform Interpolation Property for a logic L is a strong interpolation property, stating that, for any formula α and any propositional variable p, there is a post-interpolant ∃p(α) not containing p, such that ⊢L α → ∃p(α), and ⊢L α → β implies ⊢L ∃p(α) → β for each β not containing p. Similarly, for any β and p there is a pre-interpolant ∀p(β) not containing p, such that ⊢L ∀p(β) → β and ⊢L α → β implies ⊢L α → ∀p(β) for each α not containing p. Uniform interpolation property entails Craig interpolation property, and uniform interpolants are unique up to the provable equivalence, they are the minimal and the maximal interpolants of a given implication w.r.t. the provability ordering.

While for classical propositional logic, and also for other locally tabular logics like modal logic S5, uniform interpolation property is easily obtained, in other logics it is not the case. Interest in the topic arose with a seminal work by Pitts [16], who proved uniform interpolation for intuitionistic propositional logic using a terminating sequent calculus. For modal logic K uniform interpolation was first proved by Visser [27] and Ghilardi [4], for provability logic GL by Shavrukov [22]. The failure of uniform interpolation in modal logic S4, which applies to K4 as well, was proved by Ghilardi and Zawadowski [7]. More recent is a proof for monotone modal logic by Venema and Santocanale [21], using a coalgebraic perspective: uniform interpolants are constructed via erasing variable in a disjunctive normal form. This relates to the way the problem of computing uniform interpolants is understood in Artificial intelligence, which is variable forgetting. Similar motivation, but different approach based on resolution calculi and conjunctive normal forms, is applied to modal logic K by Herzig et al. [11].

As the notation suggests, uniform interpolants relate to a certain type of propositional quantifiers: if propositional quantifiers satisfying at least the usual quantifier axioms and rules are expressible in the language, they are the uniform interpolants.
On the other hand, if we construct uniform interpolants so that the construction commutes with substitutions, we can use them to interpret the propositional quantifiers, precisely as was done by Pitts’ in [16]. Visser [26] proved uniform interpolation for various modal logics, provability logics GL and Grz among them, via a semantical argument which yields a semantic characterization of the resulting quantifiers as *bisimulation* quantifiers: from the semantic point of view, quantifying over formula \( p \), they quantify over possible valuations of \( p \) in models bisimilar to the current one up to \( p \). A complexity bound of uniform interpolants in terms of \( \Box \)-depth is obtained in the proof, however, the proof does not provide us with a direct construction of the interpolants.

Using a method similar to Pitts’ and using sequent calculi for modal logics, the author proved effective uniform interpolation for modal logics K and T in [3, 2]. The thesis [2] also contains proofs of uniform interpolation for provability logics GL and Grz which are reconsidered in this paper. The reason we came back to the topic is a recent interest in uniform interpolation in modal and modal intuitionistic logic by Iemhoff [14].

1.0.2. *Provability modal logics.* In this paper, we concentrate solely on the Gödel-Löb’s provability logic GL, and Grzegorczyk's logic Grz, also known as S4Grz. The main reference for provability modal logics and their properties is Boolos’ book [4], for a history of provability logic see also [19].

The logic GL is a normal modal logic, extending the basic modal logic K with the Löb’s axiom

\[ L : \Box(\Box p \rightarrow p) \rightarrow \Box p. \]

It is known to be complete with respect to transitive and conversely well-founded Kripke frames. The logic Grz is a normal modal logic, extending the basic modal logic K with the axiom T: \( \Box p \rightarrow p \), and the Grzegorczyk’s axiom

\[ Grz : \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow \Box p. \]

It is known to be complete with respect to transitive, reflexive and conversely well-founded Kripke frames. Both logics have the finite model property as well and are therefore decidable, as was shown e.g. in [4].

The \( \Box \) modality of Gödel-Löb’s logic GL can be interpreted as formalized provability in an arithmetical recursively axiomatizable theory \( T \): assume an axiomatization of \( T \) is expressed by a sentence \( \tau \) and consider a standard proof predicate \( \Pr_\tau(\varphi) \) for \( T \). An arithmetical interpretation of modal formulas is a function from propositional variables to arithmetical sentences such that it commutes with logical connectives, and \( e(\bot) = (0 = S(0)) \), and \( e(\Box \alpha) = \Pr_\tau(e(\alpha)) \). Arithmetical completeness is established as the following statement:

\[ \Downarrow_{HGL} \alpha \text{ iff } \forall e(T \Downarrow e(\alpha)). \]

Gödel-Löb’s logic GL was proved to be arithmetically complete for Peano arithmetic by Solovay [24]. Later it was shown that it is the logic of provability of a large family of reasonable formal theories.

Using the above interpretation of GL, we obtain the following arithmetical interpretation of Grzegorczyk’s logic: an arithmetical interpretation of modal formulas is as before, only now \( e(\Box \alpha) = \Pr_\tau(\overline{e(\alpha)}) \land \overline{e(\alpha)} \).
2. Calculi

To prove the uniform interpolation theorem we use sequent calculi with good structural properties. The particular form of sequent calculi has been chosen for proof-search related manipulations. In particular, we use finite multisets of formulas to formulate a sequent, a notation which does not hide contractions (contraction rules are not part of the definition and are to be proved admissible rules), we use a definition without the cut rule (which is to be proved admissible), and structural rules of contraction and weakening are built in logical rules and axioms. Since the proof of the uniform interpolation theorem contained in the next section is closely related to termination of a proof-search in the calculi, we will devote some space in this section to explain proof-search in provability logics and its termination. Namely, we employ simple implicit loop-preventing mechanisms provided naturally by diagonal formulas, and in the case of Grzegorczyk’s logics also an explicit syntactic loop-preventing mechanism to avoid reflexive loops due to the presence of the $T$ axiom.

We assume the reader is familiar with basics on sequent calculi as contained e.g. in the Schwichtenberg’s and Troelstra’s book [25]. For sequent calculi of modal logics having arithmetical interpretation we refer to Sambin and Valentini’s paper [20], or Avron’s paper [1].

2.1. Preliminaries. Formulas are given by the following grammar of the basic modal language, where atoms $p$ are taken from a fixed countable set of propositional variables:

$$\alpha ::= \bot \mid p \mid \neg \alpha \mid \alpha \land \alpha \mid \alpha \lor \alpha \mid \Box \alpha, \Box \alpha,$$

the notion of subformulas is standard, and with the term atomic formula we refer to atoms as well as the constant $\bot$. We moreover define $\top = \neg \bot$ and $\alpha \rightarrow \beta = \neg \alpha \lor \beta$, $\Diamond \alpha = \neg \Box \neg \alpha$, and we use $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$. The weight $w(\alpha)$ of a formula $\alpha$ is the number of symbols it contains, and the box-depth $d(\alpha)$ of a formula $\alpha$ is defined as the maximum number of boxes along a branch in the corresponding formula tree. By capital Greeks we denote finite multisets of formulas. Formally, $\Gamma$ is a function from the set of formulas to natural numbers with finite support (finitely many non-zero values), but we mostly use a relaxed notation and treat multisets as sets with multiple occurrences, in particular by $\alpha \in \Gamma$ we mean that $\Gamma(\alpha) > 0$. For a multiset $\Gamma$, we denote the underlying set by $\Gamma^0$. $\Box \Gamma$ denotes the multiset resulting from prefixing elements of $\Gamma$ with box while keeping the multiplicities intact. A sequent is a syntactic object of the form $\Gamma \Rightarrow \Delta$, or, in the case of Grzegorczyk’s logic, of the form $\Box \Sigma \Gamma \Rightarrow \Delta$. The weight $w(\Gamma)$ of a multiset $\Gamma$ is the sum of the weights of elements in $\Gamma$, the weight $w(\Gamma, \Delta)$ of a sequent $\Gamma \Rightarrow \Delta$ is the sum $w(\Gamma) + w(\Delta)$.

A rule consists of a finite set of sequents called premises and a single sequent called the conclusion, rules with zero premises are called axioms. A calculus is given by a set of rule-schemes, a proof in the calculus is then a finite rooted tree labeled with sequents in such a way that leaves are labeled with axioms and labels of parent-children nodes respect correct instances of the rules of the calculus. The height of a proof is the height of the tree. A sequent is provable if there is a proof whose root is labeled with the sequent.

We call a rule invertible if whenever the conclusion is provable, then all its premises are provable as well, we call a rule height-preserving invertible if moreover
the premises have proofs of at most the height of the proof of the conclusion. We call a rule \textit{admissible} if whenever its premises are provable, so is the conclusion, and height-preserving admissible if moreover the conclusion has a proof of at most the maximum of the heights of the proofs of the premises.

By a \textit{proof-search} in a calculus we mean a procedure based on applying rules of the calculus backwards to a sequent in such a way that, for a provable sequent, the resulting tree contains a proof of the sequent. We call a proof-search \textit{terminating} if it results in a finite tree. Particular instances of proof search will be defined later.

2.2. \textbf{Sequent calculus for GL.} The following calculus is a variant of the sequent calculus introduced in [20], and reconsidered in [9, 10] using multisets in place of sets.

\textbf{Definition 2.1.} Sequent calculus $G_{GL}$:

- $\Gamma, p \Rightarrow p, \Delta$
- $\Gamma, \perp \Rightarrow \Delta$
- $\Gamma, \alpha, \beta \Rightarrow \Delta$ \quad $\land$-l
- $\Gamma, \alpha \Rightarrow \Delta, \beta \Rightarrow \Delta$ \quad $\land$-r
- $\Gamma, \alpha \Rightarrow \Delta$ \quad $\neg$-l
- $\Gamma \Rightarrow \neg \alpha, \Delta$ \quad $\neg$-r
- $\Gamma \Rightarrow \alpha, \Delta$ \quad $\lor$-r
- $\Gamma, \alpha \Rightarrow \Delta$ \quad $\land$-l
- $\Gamma, \alpha \Rightarrow \Delta$ \quad $\lor$-l
- $\Box \Gamma, \Box \alpha \Rightarrow \Delta$ \quad $\Box_{GL}$

In the $\Box_{GL}$ rule, $\Pi$ contains only propositional variables and $\Delta$ contains only propositional variables and boxed formulas, and we call formulas $\Box \alpha$ as well as $\Box \Gamma$ principal formulas (formula occurrences). In the case of axioms, $p$ (resp. $\perp$) are principal formulas, and in the remaining rules, the principal formula is the one to which a connective is introduced.

The propositional (non-modal) part of the calculus is a slight variant of the propositional part of the calculus $G_{3c}$ from [25]. The propositional rules of the calculus are height-preserving invertible, for a proof of this fact we refer to [25]. It is also not hard to prove that sequents of the form $\Gamma, \alpha \Rightarrow \alpha, \Delta$ are provable for arbitrary $\alpha$.

\textbf{Lemma 2.2.} \textit{Weakening and contraction rules are height-preserving admissible in $G_{GL}$}.

\textbf{Proof of Lemma 2.2.} Weakening and contraction rules are:

- $\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}$ \quad $w$-l
- $\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}$ \quad $w$-r
- $\frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}$ \quad $c$-l
- $\frac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}$ \quad $c$-r

Proof is by induction on the weight of the principal formula $\alpha$ of the weakening (resp. contraction) inference, and for each weight on the height of the proof of the premise of the rule. We prove the admissibility of both the left and right weakening rules simultaneously, and the same applies to the two contraction rules.
**Weakening:** For an atomic weakening formula the proof is obvious - note that atomic weakening is built in axioms as well as in the $\square_{GL}$-rule. (The case of weakening-r by $\square\alpha$ when the last inference is a $\square_{GL}$-inference is then also obvious since it is built-in the rule as well.) For a non atomic and not boxed formula we consider its main connective and use height-preserving invertibility of the corresponding propositional rule, then weakening by subformula(s) of lower weight admissible by the induction hypothesis, and finally apply the propositional rule.

Let us therefore only spell out the step for a formula of the form $\alpha = \square\beta$. If it is not principal in the last step of the proof of the premise of the weakening, we simply permute the weakening upwards and use the induction hypothesis. So consider $\square\beta$ being the principle formula of a $\square_{GL}$ inference. Notice that weakening-right by boxed formulas is built in the $\square_{GL}$ rule, therefore it is enough to consider weakening-left rule.

- Weakening-l by $\square\beta$, the last step is a $\square_{GL}$ inference — we permute the proof as follows:

  \[
  \frac{\square\Gamma, \Gamma, \square\gamma \Rightarrow \gamma}{\square\beta, \square\Gamma, \Pi \Rightarrow \square\gamma, \Lambda} \quad \text{w-l} \quad \Rightarrow \quad \frac{\square\beta, \square\beta, \square\Gamma, \square\gamma \Rightarrow \gamma}{\square\Gamma, \Gamma, \Pi \Rightarrow \square\gamma, \Lambda} \quad \text{w-l, i.h.}
  \]

**Contraction:** For $\alpha$ atomic, if the premise is an axiom, the conclusion is an axiom as well. If not, $\alpha$ is not principal and we use the induction hypothesis and permute contraction one step above or, in the case of $\square_{GL}$ rule, we apply the rule so that the conclusion is weakened by only one occurrence of $\alpha$. For $\alpha$ not atomic and not boxed we consider its main connective and use the height preserving invertibility of the appropriate rule and by i.h. we apply contraction on formula(s) of lower weight and then the rule again.

Let us therefore only spell out the step for a formula of the form $\alpha = \square\beta$. If neither of its two occurrences is principal in the last step of the proof of the premise of the contraction, we simply permute the weakening upwards and use the induction hypothesis. So consider one occurrence of $\square\beta$ is the principle formula of a $\square_{GL}$ inference.

- Contraction-right on $\square\beta$, with one of the occurrences of $\square\beta$ principle of a $\square_{GL}$ inference: we use the $\square_{GL}$ rule so that we do not weaken by the other occurrence of $\square\beta$ in the conclusion.
- Contraction-left on $\square\beta$, with the occurrences of $\square\beta$ principle of a $\square_{GL}$ inference: we permute the proof as follows and use a contraction on a simpler formula, h.p. admissible by the induction hypothesis:

  \[
  \frac{\beta, \beta, \Gamma, \square\gamma \Rightarrow \gamma}{\square\beta, \square\beta, \square\Gamma, \Pi \Rightarrow \square\gamma, \Sigma} \quad \text{c-l} \quad \Rightarrow \quad \frac{\beta, \Gamma, \square\gamma \Rightarrow \gamma}{\beta, \Pi \Rightarrow \square\gamma, \Sigma} \quad \text{c-l, i.h.}
  \]

All the above permutations are easily seen, using the induction hypothesis, to be height-preserving.

**2.3. Terminating proof-search in $G_{GL}$**. The proof-search strategy we adopt is based on applying the rules of $G_{GL}$ backwards to a given sequent, so that we always first apply the invertible rules and then, when it is no longer possible and if we haven’t reached an axiom or a sequent with no boxed formulas on the right, we
perform a modal jump — we apply the $\Box_{GL}$ rule backwards. We prefer to pack all the invertible steps into a single step, therefore it is useful to define the following notions of a critical sequent and a closure of a sequent first:

**Definition 2.3.** A sequent is called *critical*, if no invertible rule can be applied to it backwards. For a sequent $(\Gamma \Rightarrow \Delta)$, consider the smallest set of sequents containing $(\Gamma \Rightarrow \Delta)$ and closed under backward applications of the invertible rules of $G_{GL}$. The *closure* of a sequent $(\Gamma \Rightarrow \Delta)$, denoted $\text{Cl}(\Gamma; \Delta)$, is then the subset of all critical sequents contained in the set.

Note that the closure of any sequent is finite, and that a critical sequent is of the form $(\Pi, \Box \Gamma \Rightarrow \Box \Delta, \Lambda)$, with $\Pi, \Lambda$ multisets of atomic formulas, and its closure is the singleton of the sequent itself.

For a sequent $S = (\Gamma \Rightarrow \Delta)$ and finite multisets $\Theta, \Omega$, let $S(\Theta; \Omega)$ denote the sequent $\Gamma, \Theta \Rightarrow \Delta, \Omega$. The closure satisfies the following lemma, proof of which is immediate from the definition of the closure:

**Lemma 2.4.** Let $S$ be a sequent, $\text{Cl}(S) = \{S_1, \ldots, S_n\}$, and $\Theta, \Omega$ arbitrary finite multisets of formulas. Then:

- $S_1, \ldots, S_n \vdash_{GL} S$
  if $\vdash_{GL} S$ then $\vdash_{GL} S_i$ for each $i$.
- $S_1(\Theta; \Omega), \ldots, S_n(\Theta; \Omega) \vdash_{GL} S(\Theta; \Omega)$
  if $\vdash_{GL} S(\Theta; \Omega)$ then $\vdash_{GL} S_i(\Theta; \Omega)$ for each $i$.

The proof-search procedure can now be described by creating a proof-search tree as follows: we start with creating a root and labeling it with the given sequent. For every node we have created, we proceed as follows: if it is labeled with a non-critical sequent, we compute its closure, and create a child-node for each sequent in the closure and label it with the sequent (thus creating a finite conjunctive branching). If a node is labeled with a critical sequent of the form $(\Pi, \Box \Gamma \Rightarrow \Box \Delta, \Lambda)$, we distinguish the following cases: if $\Pi \cap \Lambda \neq \emptyset$ or $\bot \in \Pi$ or $\Gamma \cap \Delta \neq \emptyset$ we mark the node a provable leaf, if it is not the case and $\Delta = \emptyset$ we mark the node an unprovable leaf, and in the remaining case we apply the $\Box_{GL}$ rule backwards: we create $|\Delta|$ children nodes and label them, for each $\Box \alpha \in \Delta$, with the premise of the $\Box_{GL}$ inference with $\Box \alpha$ principal (thus creating a finite disjunctive branching).

Checking whether $\Gamma \cap \Delta \neq \emptyset$ before applying the modal rule backwards works as a simple loop preventing mechanism — we do not apply the $\Box_{GL}$ rule backwards with $\Box \alpha$ principal if the diagonal formula $\Box \alpha$ is already in the antecedent (in which case the sequent in question is clearly provable). This is crucial since it enables us to bound the number of $\Box_{GL}$ inferences along each branch, and consequently also the weight of sequents occurring in a proof search for a fixed sequent.

**Lemma 2.5.** Proof search in the calculus $G_{GL}$ always terminates.

*Proof of Lemma 2.5.* Consider a proof search for a sequent $(\Phi \Rightarrow \Psi)$. Let $n$ be the number of boxed subformulas contained in multisets $\Phi, \Psi$. This is, by the subformula property, by the nature of the $\Box_{GL}$ rule, and by the loop-preventing mechanism described above, an upper bound of the number of $\Box_{GL}$ inferences along each branch, and consequently also the weight of sequents occurring in a proof search for a fixed sequent.
the number of such steps along a single branch enables us to give an upper bound on
the weight of sequents occurring in the fixed proof search: namely, \( c = 2^n w(\Phi, \Psi) \)
is an upper bound of the weight of a sequent occurring in the proof search for
a sequent \( (\Phi \Rightarrow \Psi) \).

For any multiset \( \Gamma \) occurring in the proof-search tree, let now \( b(\Gamma) \) denote
the number of boxed formulas in \( \Gamma \) counted as a set. For a sequent \( (\Gamma \Rightarrow \Delta) \)
occurring during the proof-search, consider an ordered pair \( (c - b(\Gamma), w(\Gamma, \Delta)) \).
This measure strictly decreases in every backward application of a rule in terms of
the lexicographical ordering: \( c \) is certainly greater or equal to the maximal number
of boxed formulas in the antecedent which can occur during the proof search, so
the first number does not decrease below zero. When an invertible rule is applied
backwards, the weight of a sequent strictly decreases, therefore for a non-critical
sequent, all sequents from its closure are of strictly smaller weight. When the \( \square_{GL} \)
rule is applied backwards, \( b \) increases, and so \( c - b \) decreases, therefore the measure
decreases.

2.3.1. Extracting a proof. From a proof-search tree for a provable sequent we are
expected to be able to extract an actual proof of the sequent. The tree is finite,
and all the leaves are marked either provable, or not provable. We can extend
the marking in an obvious way to all the nodes: if a node is a parent node of a
conjunctive branching, we mark it provable if and only if all its children are marked
provable, if a node is a parent node of a disjunctive branching, we mark it provable
if and only if at least one of its children is marked provable. If the root is marked
provable, the sequent we started with has a proof presented by a tree of nodes
marked provable, generated by the root. It is a routine induction to see it is indeed
a proof.

2.4. Sequent calculi for Grz. We present a calculus for Grzegorczyk’s logic with
a loop-preventing mechanism built into the syntax of sequents. Namely we include
a third multiset of boxed formulas in a sequent, thus sequents are now of the form
\( \square \Sigma \mid \Gamma \Rightarrow \Delta \). The third multiset is used to store the boxed formulas of the \( \square_{Grz}^+ \)
inferences and the diagonal formulas of the \( \square_{\text{Grz}}^+ \) inferences when the rules are
applied backwards to prevent unnecessary looping. This strategy was inspired by
work of Heuerding [13, 12].

To improve readability, we denote the diagonal formula \( \Box (\alpha \rightarrow \Box \alpha) \) of the
Grzegorczyk’s axiom by \( D(\alpha) \) in the following text.

Definition 2.6. Sequent calculus \( G_{Grz}^+ \):
\[
\begin{align*}
\square \Sigma \mid \Gamma, p \Rightarrow p, \Delta & \quad \square \Sigma \mid \Gamma, \bot \Rightarrow \Delta \\
\square \Sigma \mid \Gamma, \alpha, \beta \Rightarrow \Delta & \quad \square \Sigma \mid \Gamma, \alpha \land \beta \Rightarrow \Delta \\
\square \Sigma \mid \Gamma, \alpha \lor \beta \Rightarrow \Delta & \quad \square \Sigma \mid \Gamma \Rightarrow \alpha, \beta, \Delta \\
\square \Sigma \mid \Gamma \Rightarrow \alpha \lor \beta, \Delta & \quad \square \Sigma \mid \Gamma \Rightarrow \alpha, \beta, \Delta
\end{align*}
\]

Another way (closer to the approach of [13] or [12]) how to formulate a measure is the following:
for a sequent \( (\Gamma; \Delta) \) consider the function \( f(\Gamma; \Delta) = c^2 - cb(\Gamma) + w(\Gamma, \Delta) \).
The function (values of which are non-negative integers) decreases in every backward application of a rule in a proof search
for \( (\Phi \Rightarrow \Psi) \). (\( c^2 \) is included to ensure that \( f \) doesn’t decrease below zero, and \( cb(\Gamma) \) balances
the possible increase of \( w(\Gamma \Rightarrow \Delta) \) in the case of a backward application of the \( \square_{GL} \)-rule.)
\[
\begin{align*}
\Box \Sigma | \Gamma, \alpha & \Rightarrow \Delta \quad \Rightarrow \neg_r \\
\Box \Sigma | \Gamma & \Rightarrow \neg \alpha, \neg \Delta
\end{align*}
\]
\[
\begin{align*}
\Box \Sigma | \Gamma & \Rightarrow \alpha, \neg \Delta \\
\Box \Sigma | \Gamma & \Rightarrow \neg \alpha, \Delta
\end{align*}
\]
\[
\begin{align*}
\Box \Sigma | \Gamma & \Rightarrow \alpha, \Delta \\
\Box \Sigma | \Gamma & \Rightarrow \neg \alpha, \neg \Delta
\end{align*}
\]
\[
\begin{align*}
\Box \Sigma | \Gamma & \Rightarrow \alpha \wedge \beta, \Delta \\
\Box \Sigma | \Gamma & \Rightarrow \alpha \vee \beta, \Delta
\end{align*}
\]
\[
\begin{align*}
\Box \alpha, \Box \Sigma | \Gamma & \Rightarrow \Delta \\
\Box \Sigma | \Gamma, \Box \alpha & \Rightarrow \Delta
\end{align*}
\]

In the \( \Box^+ \) rules, \( \Pi \) contains only propositional variables and \( \Delta \) contains only propositional variables and boxed formulas. The notion of a principal formula is similar to the previous case. All the propositional rules are easily seen to be height-preserving invertible, there is one additional invertible rule here:

**Lemma 2.7.** \( \Box^+ \) is height-preserving invertible.

**Proof of Lemma 2.7.** Proof is a routine induction on the height of the proof of the premise and we leave it to the reader. \( \text{QED} \)

**Lemma 2.8.** Weakening rules are admissible in \( G_{Grz}^+ \).

**Proof of Lemma 2.8.** The weakening rules we consider in this paper are:

\[
\begin{align*}
\Box \Sigma | \Gamma & \Rightarrow \Delta \\
\Box \Sigma | \Gamma, \alpha & \Rightarrow \Delta \\
\Box \Sigma | \Gamma & \Rightarrow \Delta, \alpha \\
\Box \Sigma, \Box \alpha | \Gamma & \Rightarrow \Delta
\end{align*}
\]

The proof is by induction on the weight of the principal weakening formula \( \alpha \) and, for each weight, on the height of the proof of the premise. We prove admissibility of the three weakening rules simultaneously.

For an atomic weakening formula the proof is obvious - note that atomic weakening is built in axioms as well as in the \( \Box^+ \) rules. (The case of weakening-r by \( \Box \alpha \) when the last inference is one of the \( \Box^+ \) rules is then also obvious since it is built-in the rules as well.) For non atomic and not boxed formula we consider its main connective and use height-preserving invertibility of the corresponding propositional rule, weaken by formula(s) of lower weight (admissible by the induction hypothesis), and then apply the rule. We next consider the weakening formula being of the form \( \Box \beta \).

**Weakening-right:** consider the last step of the proof of the premise of the weakening inference. If it is a \( \Box^+ \) rule, we can use the rule so that the weakening by \( \Box \beta \) is built-in its conclusion. If it is an invertible rule, we permute the weakening upwards.

**Weakening-left:** consider the last step of the proof of the premise of the weakening inference. If it is an invertible rule, we permute the weakening upwards. Let us consider last step is a \( \Box^+ \) rule, then we permute as follows:
Proof of the premise of the weakening is a propositional inference or a

\[ D(\gamma), \Box \Sigma \emptyset \Rightarrow \gamma \quad \Box^+_{\text{Grz1}} \]
\[ D(\gamma), \Box \Sigma \Pi \Rightarrow \Box \gamma, \Delta \quad \text{w-1} \]
\[ D(\gamma), \Box \Sigma_{\Box \beta}, \Pi \Rightarrow \Box \gamma, \Delta \quad \text{w-1+} \]

\[ D(\gamma), \Box \Sigma_{\Box \beta} \Pi \Rightarrow \Box \gamma, \Delta \quad \Box^+_{\text{Grz1}} \]
\[ D(\gamma), \Box \Sigma_{\Box \beta} \Pi \Rightarrow \Box \gamma, \Delta \quad \Box^+_{\text{T}} \]
\[ D(\gamma), \Box \Sigma \emptyset \Rightarrow \gamma \quad \text{w-1+} \]

\[ D(\gamma), \Box \Sigma_{\Box \beta} \Pi \Rightarrow \Box \gamma, \Delta \quad \Box^+_{\text{Grz2}} \]

\[ \Box \Sigma_{\Box \alpha} \Pi \Rightarrow \Box \beta, \Delta \quad \text{w-1} \]

Remark: The two transformations above are clearly not height-preserving, therefore
weakens are in general not height-preserving admissible. However, one can show,
that weakening rules with \( \alpha \) principal are admissible and the height only increases
by the box depth \( d(\alpha) \).

Weakening-\( l+ \): Notice that w-1+ is built in the axioms. If the last inference
of the proof of the premise of the weakening is a propositional inference or a \( \Box^+_{\text{T}} \) inference,
we just use the i.h., a weakening one step above, and use the appropriate rule again.

Let the last inference of the proof of the premise of the weakening be a \( \Box^+_{\text{Grz1}} \)
inference, w-1+ permutes over the inference as follows:

\[ D(\gamma), \Box \Sigma \emptyset \Rightarrow \gamma \quad \Box^+_{\text{Grz1}} \]
\[ D(\gamma), \Box \Sigma \Pi \Rightarrow \Box \gamma, \Delta \quad \text{w-1} \]
\[ D(\gamma), \Box \Sigma_{\Box \beta} \Pi \Rightarrow \Box \gamma, \Delta \quad \text{w-1+} \]
\[ D(\gamma), \Box \Sigma_{\Box \beta} \Pi \Rightarrow \Box \gamma, \Delta \quad \Box^+_{\text{Grz1}} \]
\[ D(\gamma), \Box \Sigma_{\Box \beta} \Pi \Rightarrow \Box \gamma, \Delta \quad \Box^+_{\text{T}} \]

Let the last inference be a \( \Box^+_{\text{Grz2}} \) inference, w-1+ permutes over the inference as follows:

\[ D(\gamma), \Box \Sigma \Pi \Rightarrow \Box \gamma, \Delta \quad \Box^+_{\text{Grz2}} \]
\[ D(\gamma), \Box \Sigma_{\Box \beta} \Pi \Rightarrow \Box \gamma, \Delta \quad \text{w-1} \]
\[ D(\gamma), \Box \Sigma_{\Box \beta} \Pi \Rightarrow \Box \gamma, \Delta \quad \text{w-1+} \]
\[ D(\gamma), \Box \Sigma_{\Box \beta} \Pi \Rightarrow \Box \gamma, \Delta \quad \Box^+_{\text{Grz2}} \]

The two permutations above are in fact height-preserving.

QED

Lemma 2.9. Contraction rules are height-preserving admissible in \( G^+_\text{Grz} \).

Proof of Lemma 2.9. The contraction rules are:

\[ \Sigma | \Gamma, \alpha \Rightarrow \Delta \quad \text{c-1} \]
\[ \Sigma | \Gamma, \alpha \Rightarrow \Delta \quad \text{c-r} \]
\[ \Sigma, \Box \alpha | \Gamma \Rightarrow \Delta \quad \text{c-1+} \]

The proof is by induction on the weight of the contraction formula and, for each
weight, on the height of the proof of the premise. The induction runs simultaneously
for all the contraction rules. We use the height preserving invertibility of rules. Note
that in the contr-\( l+ \) rule the contraction formula is always of the form \( \Box \alpha \).

For \( \alpha \) atomic, if the premise is an axiom, the conclusion is an axiom as well. If
not, \( \alpha \) is not principal and we use the induction hypothesis and apply contraction
one step above or, in the case of $\Box_{Grz}^+$ rules, we apply the rule so that the conclusion is weakened by only one occurrence of $\Box$. For $\alpha$ not atomic and not boxed we consider its main connective and use the height preserving invertibility of the corresponding propositional rule and by the induction hypothesis we apply contraction on subformula(s) of lower weight and then the rule again. The third multiset does not make any difference here and it works precisely as in the classical logic. All the steps described so far are height preserving.

Now suppose the contraction formula to be of the form $\Box\beta$. We distinguish the following cases:

(i) Both occurrences of the contraction formula are principal of a $\Box_{Grz}^{+1}$ inference in the antecedent, in this case the only possibility is $c-l^{+1}$. Then we permute the proof as follows using the induction hypothesis:

\[
\begin{align*}
\Box\Sigma, \Box\beta, \Box\beta, \gamma \Rightarrow \Box_{Grz}^{+1} \\
\Box\Sigma, \Box\beta, \Box\beta, \Pi \Rightarrow \Box\gamma, \Delta \\
\Box\Sigma, \Box\beta, \Pi \Rightarrow \Box\gamma, \Delta
\end{align*}
\]

In this case, $D(\gamma) \in \Box\Sigma$, or $D(\gamma) = \Box\beta$. The permutation is obviously height preserving.

(ii) Both occurrences of the contraction formula are principal of a $\Box_{Grz}^{+2}$ inference in the antecedent, again $c-l^{+1}$ is the only possibility. Then we permute the proof as follows:

\[
\begin{align*}
D(\gamma), \Box\Sigma, \Box\beta, \Box\beta, \Sigma, \beta, \gamma \Rightarrow \Box_{Grz}^{+2} \\
\Box\Sigma, \Box\beta, \Pi \Rightarrow \Box\gamma, \Sigma
\end{align*}
\]

Here, $D(\gamma) \notin \Box\Sigma$ and $D(\gamma) \notin \Box\beta$. The permutation is obviously height preserving.

(iii) One occurrence of the contraction formula is the principal formula of a $\Box_{Grz}^+$ inference in the antecedent. Then we permute the proof as follows using the i.h. and the height preserving invertibility of the $\Box_{Grz}^+$ rule:

\[
\begin{align*}
\Sigma, \Box\beta, \Box\beta, \beta, \Gamma \Rightarrow \Delta \\
\Sigma, \Box\beta, \beta, \Gamma \Rightarrow \Delta \\
\Sigma, \Box\beta, \Gamma \Rightarrow \Delta
\end{align*}
\]

The permutation is height preserving since the steps $c-l$, $c-l^{+}$, and invert. do not increase the height of the proof.

(iv) One occurrence of the contraction formula is the principal formula in the succedent and the last inference is a $\Box_{Grz}^+$ inference. Then we use the $\Box_{Grz}^+$ rule so that the conclusion is not weakened by the other occurrence of $\Box\beta$. This step is obviously height preserving. If the contraction formula is not the principal formula and the last step is a $\Box_{Grz}^+$ inference, $\Box\beta$ is in $\Delta$. Then we use the $\Box_{Grz}^+$ rule so that the conclusion is weakened by only one occurrence of the contraction formula. If the last step is another inference, we use contraction one step above on the proof of
lower height. If it is an axiom, the conclusion of the desired contraction is an axiom as well. Again, all the steps are height preserving.

Next we want to relate the calculus $G_{Grz}^+$ to the standard sequent calculus $G_{Grz}$ of [1]. For this, we consider a multiset variant of the latter. The calculus is known to be complete, and the rules of weakening are easily proved to be admissible by a similar argument that used in Lemma 2.2.

**Definition 2.10.** Sequent calculus $G_{Grz}$ results from the non-modal part of the calculus $G_{GL}$ adding the following two modal rules:

$$
\frac{\Gamma, \Box \alpha, \alpha \Rightarrow \Delta}{\Gamma, \Box \alpha \Rightarrow \Delta} \quad \Box_T
$$

$$
\frac{\Box \Gamma, D(\alpha) \Rightarrow \alpha}{\Box \Gamma, \Pi \Rightarrow \Box \alpha, \Delta} \quad \Box_{Grz}
$$

**Lemma 2.11.** The calculi $G_{Grz}$ and $G_{Grz}^+$ are equivalent:

$$\vdash_{G_{Grz}} \Gamma \Rightarrow \Delta \quad \text{iff} \quad \vdash_{G_{Grz}^+} \emptyset | \Gamma \Rightarrow \Delta.$$

**Proof of Lemma 2.11.**

The right-left implication: deleting the "|" symbol from a $G_{Grz}^+$ proof of $(\emptyset | \Gamma \Rightarrow \Delta)$ yields correct instances of rules of $G_{Grz}$, except the $\Box_{Grz1}^+$ rule. It has to be simulated as follows:

$$
\frac{\Box \Gamma, D(\alpha) \Rightarrow \alpha}{\Box \Gamma, \Pi \Rightarrow \Box \alpha, \Delta} \quad \text{admiss. w-l}
$$

We end up with a $G_{Grz}$ proof of $\Gamma \Rightarrow \Delta$. (Lemma 2.14 below states that the calculus $G_{Grz}$ is complete, and soundness of weakening entails that weakening is indeed admissible in $G_{Grz}$.)

The left-right implication: the idea is to add a third, empty multiset to all the sequents in a proof. This yields correct instances of the axioms as well as the propositional rules. The $\Box_T$ rule has to be simulated as follows, using invertibility of $\Box_T^+$ rule and admissibility of contraction:

$$
\frac{\emptyset | \Box \alpha, \alpha | \Gamma \Rightarrow \Delta}{\Box \alpha | \alpha, \alpha, \Gamma \Rightarrow \Delta} \quad \text{inv. of } \Box_T^+
$$

$$
\frac{\Box \alpha, \alpha, \Gamma \Rightarrow \Delta}{\Box \alpha, \Gamma \Rightarrow \Delta} \quad \Box_T
$$

The $\Box_{Grz}$ rule is simulated as follows ($D(\alpha) \notin \Box_T$):

$$
\frac{\emptyset | \Box \Gamma, \Box (\neg \alpha \lor \Box \alpha) \Rightarrow \alpha}{\Box \Gamma, \Box (\neg \alpha \lor \Box \alpha) | \Gamma \Rightarrow \alpha} \quad \text{inv. of } \Box_T^+
$$

$$
\frac{\Box \Gamma, \Box (\neg \alpha \lor \Box \alpha) | \Gamma \Rightarrow \alpha, \alpha}{\Box \Gamma, \Box (\neg \alpha \lor \Box \alpha) | \Gamma \Rightarrow \alpha} \quad \text{admiss. c-r}
$$

$$
\frac{\Box \Gamma | \Pi \Rightarrow \Box \alpha, \Delta}{\Box \Gamma | \Pi \Rightarrow \Box \alpha, \Delta} \quad \text{admiss. w-l inferences}
$$

If $D(\alpha) \in \Gamma$, we use some admissible c-l+ inferences before the $\Box_{Grz1}^+$ inference is used.

QED
2.5. Terminating proof-search in $G_{Grz}^+$. We will restrict ourselves to proof-search for sequents of the form $(\emptyset \Phi \Rightarrow \Psi)$ with the third multiset empty and $\Phi$ and $\Psi$ arbitrary finite multisets of formulas (Lemma 2.11 justifies this restriction).

The notion of critical sequent and the closure of a sequent for $G_{Grz}^+$ is the same as given in Definition 2.3, only with a third multiset added. Recall that also the $\Box^+_T$ rule is invertible, so critical sequents are of the form: $\Box \Gamma | \Pi \Rightarrow \Box \Delta, \Lambda$ with $\Pi, \Lambda$ atomic.

For a sequent $S = (\emptyset | \Phi \Rightarrow \Psi)$ and finite multisets $\Lambda, \Theta, \Omega$, let $S(\Lambda | \Theta; \Omega)$ denote the sequent $(\emptyset | \Phi, \Theta \Rightarrow \Delta, \Omega)$. The closure satisfies the following lemma, essentially the same as Lemma 2.4:

Lemma 2.12. Let $S$ be a sequent, $\text{Cl}(S) = \{S_1, \ldots, S_n\}$, and $\Lambda | \Theta, \Omega$ arbitrary finite multisets of formulas. Then:

- $S_1, \ldots, S_n \vdash_{GL} S$
  if $\vdash_{GL} S$ then $\vdash_{GL} S_i$ for each $i$.

- $S_1(\Lambda | \Theta; \Omega), \ldots, S_n(\Lambda | \Theta; \Omega) \vdash_{GL} S(\Lambda | \Theta; \Omega)$
  if $\vdash_{GL} S(\Lambda | \Theta; \Omega)$ then $\vdash_{GL} S_i(\Lambda | \Theta; \Omega)$ for each $i$.

Before we continue to describe proof-search and prove its termination, we briefly discuss forms of looping we prevent by using the specific form of the calculus.

2.5.1. Reflexive looping. This simple looping occurs when one searches for proofs in the calculus $G_{Grz}$ and applies the $\Box_T$ rule backwards repeatedly with the same principal formula. Such looping is prevented by the presence of the third storage multiset in sequents and by the particular form of $\Box^+_T$ rule we use — when this rule is applied backwards, it remembers that the principle formula has already been treated.

2.5.2. Transitive looping. Another looping phenomenon arises when one tries to search for a proof of the sequent $\Box \neg \Box p \Rightarrow \Box p$ in the calculus $G_{Grz}$ — it loops on the sequent $\Box \neg \Box p, D(p) \Rightarrow p, \Box p$.

Such looping can be avoided and the diagonal formula plays a crucial role here as a natural loop-preventing mechanism again. We have made this mechanism explicit by splitting the $\Box_{Grz}$ rule into two cases distinguishing if the diagonal formula is present in the antecedent or not. Consider the $\Box^+_{Grz1}$ rule bottom up. When the diagonal formula is already in the third multiset, we apply the rule so that we neither add the diagonal formula to the third multiset, nor we add $\Gamma$ to the antecedent.

The proof-search procedure for Grzegorczyk logic is fully analogous to that for logic GL: we create a proof-search tree, using the strategy of alternating the closure step for non-critical sequents and a modal jump step for critical sequents. Also the labeling and extraction of an actual proof is carried out similarly.

Lemma 2.13. Proof search in $Gm^+_{Grz}$ for sequents of the form $(\emptyset | \Phi \Rightarrow \Psi)$ always terminates.

Proof of Lemma 2.13. Consider a proof search for a sequent $(\emptyset | \Phi \Rightarrow \Psi)$. Let $n$ be the number of boxed subformulas occurring in the sequent $(\emptyset | \Phi \Rightarrow \Psi)$. This number, as in the case of GL, is an upper bound on the number of the $\Box^+_{Grz}$
rules applied backwards along one branch of the proof search tree. Each backward application of the $\Box^+_2$ rule adds a new boxed formula in the storage multiset, but also a $\Box^+_1$ rule does so during the closure steps. Therefore $n^2$ is an upper bound of the number of formulas stored in $\Sigma$ if we do not duplicate them and count them as a set. (If we allowed duplicate formulas in $\Sigma$, we would need an exponential function of $n$.)

With each sequent $(\Box \Sigma | \Gamma \Rightarrow \Delta)$ occurring during the proof search, we associate an ordered pair $\langle n^2 - |\Sigma^o|, w(\Gamma, \Delta) \rangle$. Therefore the first number does not decrease below zero. The measure obviously decreases in every backward application of a rule of the calculus. For the $\Box^+_2$ rule, $|\Sigma^o|$ increases and so $n^2 - |\Sigma^o|$ decreases, while for other rules the weight $w(\Gamma, \Delta)$ decreases.

QED

2.6. Cut admissibility via completeness. We do not give a constructive proof of completeness of the two calculi without the cut rule in this paper. Such a proof can be established using a proof-search method described in the previous subsections. One can argue that, for any given sequent, the proof-search tree either yields a proof, or can be used to construct a (finite) counterexample. Instead, we state the completeness without the cut rule, and refer for a proof to Avron [1] who proved that the calculi $G_{GL}$ and $G_{Grz}$ are complete without the cut rule w.r.t. their respective Kripke semantics. Then an easy semantical argument of soundness of the cut rule entails its admissibility. Lemma 2.11 then yields admissibility in $G^+_{Grz}$ of the cut rule we will use later in the proof of Theorem 3.5.

Proofs of completeness can be found in [1] for $GL$ and Grzegorczyk’s logic, and [28] or [20] for $GL$, where redundancy of the cut rule is established through a decision procedure which either creates a cut-free proof, or a Kripke counterexample to a given sequent. Although both authors use a formulation via sets of formulas, observe, that a cut-free proof with sets can be equivalently formulated using multisets and contraction rules, which are, as we have proved, admissible in our cut-free calculi. Equivalently, if a sequent does not have a cut-free proof in the system based on multisets, its set-based counterpart sequent does not have a cut-free proof in the system based on sets.

Lemma 2.14. (Avron [1]:) There are a canonical Kripke model $(W, <)$ and a canonical valuation $V$ such that:

- $<$ is irreflexive and transitive
- for every $w \in W$, the set $\{v | v < w\}$ is finite
- if $(\Gamma \Rightarrow \Delta)$ has no cut-free proof in $G_{GL}$, then there is a $w \in W$ such that $w \Vdash_V \alpha$ for every $\alpha \in \Gamma$ and $w \Vdash_V \beta$ for every $\beta \in \Delta$.

There are a canonical Kripke model $(W, \leq)$ and a canonical valuation $V$ such that:

- $\leq$ partially orders $W$
- for every $w \in W$, the set $\{v | v \leq w\}$ is finite
- if $(\Gamma \Rightarrow \Delta)$ has no cut-free proof in $G_{Grz}$, then there is a $w \in W$ such that $w \Vdash_V \alpha$ for every $\alpha \in \Gamma$ and $w \Vdash_V \beta$ for every $\beta \in \Delta$.

Proof of Lemma 2.14. See [1]. The canonical model is built from all saturated sequents (sequents closed under subformulas) that have no cut-free proof in appropriate calculi. The lemma entails completeness of $G_{GL}$ w.r.t. transitive, conversely well-founded Kripke models; and completeness of $G_{Grz}$ w.r.t. transitive, reflexive and conversely well-founded Kripke models.
Corollary 2.15. The cut rule

\[ \frac{\Gamma \Rightarrow \Delta, \gamma \quad \gamma, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \]

is admissible in \( G_{GL} \) and \( G_{Grz} \).

Proof of Cor. 2.15. It is easy to give a semantic argument of soundness of the cut rule. Given a counterexample of the conclusion \((\Gamma, \Pi \Rightarrow \Delta, \Lambda)\) of a cut inference, there is a counterexample to one of its premises: consider the counterexample \((W, R)\) and a world \( w \in W \) in it such that \( w \Vdash V \alpha \) for every \( \alpha \in \Gamma \cup \Pi \) and \( w \not\Vdash V \beta \) for some \( \beta \in \Delta \cup \Lambda \). For any formula \( \gamma \) it is either the case that \( w \Vdash V \gamma \), and then \( w \) refutes \((\gamma, \Pi \Rightarrow \Lambda)\), or \( w \not\Vdash V \beta \), and then \( w \) refutes \((\Gamma \Rightarrow \Delta, \gamma)\).

Now Lemma 2.14 (completeness of \( G_{GL} \) and \( G_{Grz} \)) entails admissibility of the cut rule in the calculi.

Corollary 2.16. The cut rule

\[ \frac{\emptyset \Vdash \Gamma, \gamma \quad \emptyset, \gamma, \Pi \Rightarrow \Lambda}{\emptyset, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \]

is admissible in \( G_{Grz}^+ \).

Proof of Cor. 2.16. Follows from Corollary 2.15 and Lemma 2.11.

Remark 2.17. The above cut rule cannot be replaced by the expected form of cut:

\[ \Box \Sigma \Gamma \Rightarrow \Delta, \alpha \quad \Box \Theta \alpha, \Pi \Rightarrow \Lambda \]

\[ \Box \Sigma \Theta \Gamma \Rightarrow \Delta, \Lambda \]

since it is not admissible. The counterexample is the following instance of cut':

\[ \begin{array}{l}
\Box \phi, \Box \phi | \emptyset \Rightarrow \Box \phi \\
\Box \phi | \emptyset \Rightarrow \Box \phi \\
\Box \phi | \emptyset \Rightarrow \Box \phi \\
\Box \phi | \emptyset \Rightarrow \Box \phi \\
\end{array} \]

which results in the sequent \((\Box \phi | \emptyset \Rightarrow \Box \phi)\) unprovable in \( G_{Grz}^+ \).

A fixed-point trick: Before we proceed to the proof of uniform interpolation, we state a lemma which later will play a crucial role in a termination argument for the definition of the interpolants. The lemma is a simple example of a particular fixed point existence in modal logic \( K4 \). Namely, a recursive equivalence \( \Diamond((\alpha \land x) \land \beta) \equiv x \) has a solution \( x = \Diamond(\alpha \land \beta) \). We will however need a bit more complicated form of the statement, which is the following lemma:

Lemma 2.18. Let \( \delta = \bigwedge_i \alpha_i \land \beta, \ i = 1 \ldots k \). Then the following sequents are provable:

\[ \vdash_{G_{GL}} \Diamond(\bigwedge_i (\alpha_i \lor \Diamond \delta) \land \beta) \iff \Diamond(\bigwedge_i (\alpha_i \land \beta)) \]

\[ \vdash_{G_{Grz}^+} \emptyset | \Diamond(\bigwedge_i (\alpha_i \lor \Diamond \delta) \land \beta) \iff \Diamond(\bigwedge_i (\alpha_i \land \beta)). \]

Proof of Lemma 2.18. The corresponding two implications (same in both cases) are easily seen to hold on transitive models. The claim now follows from the completeness results of Lemma 2.14. It is however also not hard to write down proofs in \( G_{GL} \) and \( G_{Grz}^+ \), which we omit for space reasons.

QED
3. Uniform interpolation

3.1. Uniform interpolation in GL. We will prove the uniform interpolation by constructing, for each formula $\alpha$, the pre-interpolant $\forall p(\alpha)$. The post-interpolant can be defined by $\exists (\alpha) = \neg \forall p(\neg \alpha)$. The construction is based on a proof-search for the sequent $\emptyset \Rightarrow \alpha$. To make it work we need to define interpolants for sequents instead of formulas. The uniform interpolation is then obtained via $\forall p(\alpha) = \forall p(\emptyset; \alpha)$.

**Theorem 3.1.** Let $\Gamma, \Delta$ be finite multisets of formulas. For every propositional variable $p$ there exists a formula $\forall p(\Gamma; \Delta)$ such that:

(i) $\text{Var}(\forall p(\Gamma; \Delta)) \subseteq \text{Var}(\Gamma, \Delta) \setminus \{p\}$

(ii) $\vdash_{GL} \Gamma, \forall p(\Gamma; \Delta) \Rightarrow \Delta$

(iii) moreover let $\Phi, \Psi$ be multisets of formulas not containing $p$ and $\vdash_{GL} \Phi, \Gamma \Rightarrow \Delta, \Psi$.

Then $\vdash_{GL} \Phi \Rightarrow \forall p(\Gamma; \Delta), \Psi$.

**Proof of Theorem 3.1.** In the following construction of the interpolant, it is instructive to imagine that with the formula $\forall p(\Gamma; \Delta)$ we are describing (a relevant part of) the proof-search tree for the sequent $(\Phi, \Gamma \Rightarrow \Delta, \Psi)$ for any context $\Phi, \Psi$ not containing $p$, namely the part only depending on $\Gamma, \Delta$. This description has to be finite. The interpolant is defined recursively, closely following the proof-search strategy: for a non-critical sequent we simply use its closure, while for a critical sequent we apply a matching argument, similar to the strategy used by Pitts [16]. We start with a definition of the formula $\forall p(\Gamma; \Delta)$, then we prove that the definition terminates, and proceed with proving it satisfies items (i)-(iii) of the Theorem 3.1.

**Definition of the interpolant.** We describe the construction of the interpolant recursively. The formula $\forall p(\Gamma; \Delta)$ is for a noncritical $(\Gamma; \Delta)$ defined by

$$(3.1) \quad \forall p(\Gamma; \Delta) = \bigwedge_{(\Gamma_i; \Rightarrow \Delta_i) \in C(\Gamma; \Delta)} \forall p(\Gamma_i; \Delta_i)$$

The recursive steps for $\Gamma \Rightarrow \Delta$ being a critical sequent of the form $(\Box \Gamma', \Pi; \Box \Delta', \Lambda)$, with $\Pi, \Lambda$ atomic, are given below in Table 1. The first line of Table 1 corresponds to some of the cases when the critical sequent is provable - it is either an axiom or the diagonal formula is already in the antecedent (here we are using the loop preventing mechanism from the termination argument in Lemma 2.5). The line 2 of Table 1 corresponds to a critical step in a proof-search, the corresponding disjunction covering:

- propositional variables from multisets $\Pi, \Lambda$, where all $q, r \neq p$,
- all the possibilities of a $\Box_{GL}$ inference with the principal formula from $\Box \Delta'$,
- and, by the diamond formula $\Diamond \bigwedge N(\Box \Gamma', \Gamma'; \emptyset)$, which we will define below, also the possibility of a $\Box_{GL}$ inference with the principal formula not from $\Box \Delta'$ (i.e. from a context not containing $p$). Morally, we should include $\Diamond \forall p(\Box \Gamma', \Gamma'; \emptyset)$ instead, but then the definition would not terminate. This is the trick we describe below in Remark 3.2.
For a sequent of the form \((\Box \Gamma'; \Gamma'; \emptyset)\), a set of formulas \(N(\Box \Gamma', \Gamma'; \emptyset)\) is defined as the smallest set given by Table 2.

| \(\Box \Gamma', \Pi; \Box \Delta', \Lambda\) matches | \(\forall p(\Box \Gamma', \Pi; \Box \Delta', \Lambda)\) equals |
| --- | --- |
| 1 | if \(p \in \Pi \cap \Lambda\) or \(\bot \in \Pi\) or \(\Gamma' \cap \Delta' \neq \emptyset\) | \(\top\) |
| 2 | otherwise (here all \(q, r \neq p\)) | \(\bigvee q \in \Lambda \bigvee r \in \Pi \neg r \bigvee \Box \forall p(\Box \Gamma', \Gamma' ; \emptyset)\) \(\bigvee \beta \in \Delta' \bigvee \Box \forall p(\Box \Sigma, \Sigma' ; \emptyset)\) |

Table 1.

In the first line of Table 2, we use the fact that, given \(\Sigma^o \supset \Gamma^o\), the sequent \((\Box \Sigma, \emptyset; \Box \Omega, \Theta)\) is strictly simpler than \((\Box \Gamma', \Gamma' ; \emptyset)\) in terms of the measure we will use below to prove termination of the definition, and therefore it is safe to recursively call the procedure. In the remaining cases the sequent is provable and the value in the second column therefore equals \(\top\). The second line of Table 2 covers the case when \(\Sigma^o = \Gamma^o\), and resembles the line 2 of Table 1, only to ensure termination, we have omitted the diamond-part of the disjunction.

**Termination.** Let us see that the definition terminates. The argument is similar to that we have used to prove termination of the calculus \(G_{GL}\) in 2.5. Consider a run of the procedure for \(\forall p(\Phi; \Psi)\) and let \(n\) be the number of boxed subformulas occurring in \((\Phi; \Psi)\), which bounds the maximal number of critical steps occurring along a branch in the tree corresponding to the run of the procedure. This is crucial since it enables us to consider an upper bound of the weight of an argument of \(\forall p\) occurring during this run of the procedure. Put \(c = 4^n w(\Phi; \Psi)\), i.e. an upper bound of the weight of an argument of \(\forall p\) occurring during the run of the procedure for \((\Phi; \Psi)\). Here, in contrast to the termination argument for the calculus \(G_{GL}\), we need \(4^n\) since the weight of a recursively called argument of \(\forall p\) can increase more. This is because in the construction of \(N(\Box \Gamma', \Gamma'; \emptyset)\) we look one level deeper. Let,
for any multiset $\Gamma$, $b(\Gamma)$ be the number of boxed formulas in $\Gamma$ counted as a set. For a $\forall p$ argument $(\Gamma; \Delta)$ occurring during the construction, consider an ordered pair $\langle c - b(\Gamma), w(\Gamma, \Delta) \rangle$. This measure decreases in every recursive step of the procedure in terms of the lexicographical ordering:

- It is obvious that, for each noncritical sequent $(\Gamma' \Rightarrow \Delta') \in Cl(\Gamma; \Delta)$, $w(\Gamma', \Delta') < w(\Gamma, \Delta)$ and that $b$ does not decrease.
- for a critical argument $(\Box \Gamma', \Pi; \Box \Delta', \Lambda)$, i.e., line 2 of Table 1 and whole Table 2 constructing $N(\Box \Gamma', \Gamma'; \emptyset)$. For all the three recursively called arguments $b$ increases, thus $c - b$, and therefore the whole measure, decreases.

**Remark 3.2.** [termination trick] To retain termination of the definition, we cannot replace

$$\Diamond \bigwedge S N(\Box \Gamma', \Gamma'; \emptyset)$$

in the line 2 of Table 1 with $\Diamond \forall p(\Box \Gamma', \Gamma'; \emptyset)$, which in fact seems to be needed to prove the part (iii) of the theorem. The reason is that its recursively called argument need not be simpler than the sequent in question. However, in the prove of (ii) $\Diamond \forall p(\Box \Gamma', \Gamma'; \emptyset)$ and will be used, because, as we show next, it is the case that

$$(3.2) \quad \vdash_{GL} \Diamond \bigwedge S N(\Box \Gamma', \Gamma'; \emptyset) \iff \Diamond \forall p(\Box \Gamma', \Gamma'; \emptyset).$$

To see this, we use the fixed point observation we made earlier in Lemma 2.18. Consider sequents $(\Box \Sigma, \Theta; \Box \Omega, \Theta)$ in the closure of $(\Box \Gamma', \Gamma'; \emptyset)$, and refer by $S$ to sequents with $\Sigma^c = \Gamma^c$, and by $S'$ to sequents in the closure with $\Sigma^c \supset \Gamma^c$, i.e. strictly simpler then $(\Box \Gamma', \Gamma'; \emptyset)$. Since

$$\forall p(\Box \Gamma', \Gamma'; \emptyset) = \bigwedge S \forall p(S) \land \bigwedge S' \forall p(S'),$$

and for each $S = (\Box \Sigma, \Theta; \Box \Omega, \Theta)$ with $\Sigma^c = \Gamma^c$ we obtain by the line 2 of Table 1

$$\forall p(S) = \bigvee q \vee \bigvee r \neg r \lor \bigvee \Box \forall p(\Box \Sigma, \Sigma; \Box \beta; \beta) \lor \Diamond \bigwedge S N(\Box \Sigma, \Sigma; \emptyset),$$

and using the definition of $N(\Box \Gamma', \Gamma'; \emptyset)$ in the Table 2, the above equivalence (3.2) becomes the following:

$$\vdash_{GL} \Diamond \bigwedge S \forall p(S) \lor \bigwedge S' \forall p(S') \iff \Diamond \bigwedge S \forall p(\Box \Sigma, \Sigma; \Box \beta; \beta) \lor \bigwedge S N(\Box \Sigma, \Sigma; \emptyset) \land \bigwedge S' \forall p(S').$$

Observe, that $N(\Box \Sigma, \Sigma; \emptyset)$ is equivalent to $N(\Box \Gamma', \Gamma'; \emptyset)$ by $\Sigma^c = \Gamma^c$. Therefore the result now follows by Lemma 2.18 instantiated with

$$\alpha_S = \bigvee q \lor \bigvee r \neg r \lor \bigvee \Box \forall p(\Box \Sigma, \Sigma; \Box \beta; \beta)$$

$$\beta = \bigwedge S' \forall p(S')$$

$$\delta = \bigwedge S N(\Box \Gamma', \Gamma'; \emptyset)$$

We have thus established the termination of the definition of the uniform interpolants. Now we proceed in proving the three items of Theorem 3.1.
The item (i) follows easily by induction on \( \Gamma, \Delta \), because we never use \( p \) during the definition of the formula \( \forall p(\Gamma; \Delta) \).

**ii.** We proceed by induction on the complexity of \( \Gamma, \Delta \) given by the measure function defined above, and prove that \( \vdash_{GL} \Gamma, \forall p(\Gamma; \Delta) \Rightarrow \Delta \).

First, let \((\Gamma \Rightarrow \Delta)\) be a noncritical sequent. Then sequents \((\Gamma_i \Rightarrow \Delta_i) \in Cl(\Gamma; \Delta)\) are of lower complexity and by the induction hypotheses \( \vdash_{GL} \Gamma_i, \forall p(\Gamma_i; \Delta_i) \Rightarrow \Delta_i \) for each \( i \). Then by admissibility of weakening and by Lemma 2.4

\[
\vdash_{GL} \Gamma, \forall p(\Gamma_1; \Delta_1), \ldots, \forall p(\Gamma_k; \Delta_k) \Rightarrow \Delta,
\]

therefore by a \( \land \)-inference

\[
\vdash_{GL} \Gamma, \bigwedge_{(\Gamma_i \Rightarrow \Delta_i) \in Cl(\Gamma; \Delta)} \forall p(\Gamma_i; \Delta_i) \Rightarrow \Delta,
\]

which is by (5.1)

\[
\vdash_{GL} \Gamma, \forall p(\Gamma; \Delta) \Rightarrow \Delta.
\]

Second, let \((\Gamma \Rightarrow \Delta)\) be a critical sequent. If \((\Gamma \Rightarrow \Delta)\) is a critical sequent matching the line 1 of Table 1 then (ii) is an axiom or a provable sequent. Let \((\Gamma \Rightarrow \Delta)\) be a critical sequent matching the line 2 of Table 1. We prove \( \vdash_{GL} \Pi, \delta, \Box \Gamma' \Rightarrow \Box \Delta', \Lambda \) for each disjunct \( \delta \) used in the line 2 of Table 1 to define the interpolant:

- For each \( r \in \Pi \) obviously \( \vdash_{GL} \Pi, \neg r, \Box \Gamma' \Rightarrow \Box \Delta', \Lambda \), therefore
  \[ \vdash_{GL} \Pi, \bigvee_{r \in \Pi} \neg r, \Box \Gamma' \Rightarrow \Box \Delta', \Lambda \].
- For each \( q \in \Lambda \) obviously \( \vdash_{GL} \Pi, q, \Box \Gamma' \Rightarrow \Box \Delta', \Lambda \), therefore
  \[ \vdash_{GL} \Pi, \bigvee_{q \in \Lambda} q, \Box \Gamma' \Rightarrow \Box \Delta', \Lambda \].
- For each \( \beta \in \Delta' \), \( \vdash_{GL} \Box \Gamma', \Pi', \forall p(\Box \Gamma', \Pi', \Box \beta) \Rightarrow \beta \) by the induction hypothesis, which gives \( \vdash_{GL} \Box \Gamma', \Pi, \forall p(\Box \Gamma', \Pi, \Box \beta) \Rightarrow \Box \Delta', \Lambda \) by a \( \Box_L \)-inference.

It remains to be proved that

\[
\vdash_{GL} \Box \Gamma', \Pi, \Diamond \bigwedge N(\Box \Gamma', \Pi'; \emptyset) \Rightarrow \Box \Delta', \Lambda.
\]

For each \((\Box \Sigma, \Upsilon \Rightarrow \Box \Omega, \Theta) \in Cl(\Box \Gamma', \Pi'; \emptyset)\) from the first line of Table 2 we know by the induction hypotheses that

\[
\vdash_{GL} \Box \Sigma, \forall p(\Box \Sigma, \Upsilon; \Box \Omega, \Theta) \Rightarrow \Box \Omega, \Theta
\]

For each \((\Box \Sigma, \Upsilon \Rightarrow \Box \Omega, \Theta) \in Cl(\Box \Gamma', \Pi'; \emptyset)\) from the second line of Table 2 we have the following:

- \( \vdash_{GL} \Box \Sigma, \forall q \Rightarrow \Box \Omega, \Theta \).
- \( \vdash_{GL} \Box \Sigma, \forall q \Rightarrow \Box \Omega, \Theta \).
- For each \( \beta \in \Omega \) by the induction hypotheses
  \[ \vdash_{GL} \Box \Sigma, \forall p(\Box \Sigma, \Box \beta) \Rightarrow \beta \]
  and by weakening and a \( \Box_{GL} \)-inference
  \[ \vdash_{GL} \Box \Sigma, \forall p(\Box \Sigma, \Box \beta) \Rightarrow \Box \beta. \]
Together this yields, using $\lor$-l inferences,
\[ \vdash_{GL} □ Σ, Y, \bigvee_{q \in Θ} \lor \bigvee_{r \in Π} \lor □ \forall p(□ Σ, □ β; □ β) \Rightarrow □ Ω, Θ. \]

Therefore, for each $(□ Σ, Y \Rightarrow □ Ω, Θ) \in Cl(□ Γ', Γ'; \emptyset)$, we obtain, using weakening and $\land$-r inferences,
\[ \vdash_{GL} □ Σ, Y, \bigwedge N(□ Γ', Γ'; \emptyset) \Rightarrow □ Ω, Θ. \]

Now, by Lemma 2.4
\[ \vdash_{GL} □ Γ', Γ', \bigwedge N(□ Γ', Γ'; \emptyset) \Rightarrow \emptyset. \]

By negation and weakening inferences
\[ \vdash_{GL} □ Γ', Γ', □ \neg \bigwedge N(□ Γ', Γ'; \emptyset) \Rightarrow \neg \bigwedge N(□ Γ', Γ'; \emptyset) \]
and by a $\square_{GL}$ inference
\[ \vdash_{GL} □ Γ', Π \Rightarrow □ \neg \bigwedge N(□ Γ', Γ'; \emptyset), □ Δ', Λ. \]

Now, using a negation inference again, we obtain
\[ \vdash_{GL} □ Γ', Π, □ ◊ \bigwedge N(□ Γ', Γ'; \emptyset) \Rightarrow □ Δ', Λ. \]

Putting finally all the above disjuncts together then yields, using $\lor$-l inferences,
\[ \vdash_{GL} Π, □ Γ', \bigvee_{q \in Λ} q \lor □ \forall p(□ Γ', Γ', □ β; □ β) \lor □ \bigwedge N(□ Γ', Γ'; \emptyset) \Rightarrow □ Δ', Λ, \]
that is, by the line 2 of Table 1
\[ \vdash_{GL} Π, □ Γ', □ P(Π, □ Γ'; □ Δ', Λ) \Rightarrow □ Δ', Λ. \]

(iii). We proceed by induction on the height of a proof of $(Φ, Γ ⇒ △, Ψ)$, and by sub-induction on the measure of the sequent $(Γ; △)$ used to show termination of the definition. We show that $\vdash_{GL} Φ ⇒ □ p(Γ; △), Ψ$.

First, consider $(Φ, Γ ⇒ △, Ψ)$ is an axiom. The following cases apply:

- $⊥$ is principal and $⊥ \in Φ$, then (iii) is an axiom.
- $⊥$ is principal and $⊥ \in Γ$, then $∀ p(Γ; △) = T$ and $Φ ⇒ T, Ψ$ is provable.
- $p$ is principal, i.e. $p \in Γ \cap △$ and $∀ p(Γ; △) = T$ and $Φ ⇒ T, Ψ$ is provable.
- $q \neq p$ is principal, and $q \in Φ \cap Ψ$. Then $Φ ⇒ □ p(Γ; △), Ψ$ is an axiom.
- $q \neq p$ is principal, and $q \in Φ \cap △$. Then $\vdash_{GL} q ⇒ □ p(Γ; △)$ by the line 1 of Table 1 and we obtain the result by weakening.
- $q \neq p$ is principal, and $q \in Γ \cap Ψ$. Then $\vdash_{GL} \neg q ⇒ □ p(Γ; △)$ by the line 1 of Table 1 and $\vdash_{GL} Φ ⇒ □ p(Γ; △), q$ by $\neg$-l invertibility, and we obtain the result by weakening.
- $q \neq p$ is principal, and $q \in Γ \cap △$. Then $\vdash_{GL} q \lor \neg q ⇒ □ p(Γ; △)$ by the line 1 of the table, and therefore $\vdash_{GL} Φ ⇒ □ p(Γ; △)$, and we obtain the result by weakening.
Consider then \((\Phi, \Gamma \Rightarrow \Delta, \Psi)\) is not an axiom. We distinguish two main cases: Consider first \((\Gamma; \Delta)\) is a noncritical sequent. Then all \((\Gamma' \Rightarrow \Delta') \in Cl(\Gamma; \Delta)\) are strictly simpler in terms of the measure, and for all of them we have \(\vdash_{GL} \Phi, \Gamma' \Rightarrow \Delta', \Psi\) by Lemma 2.4. Then, using the induction hypothesis and (3.1), the following are equivalent:

\[ \vdash_{GL} \Phi \Rightarrow \forall p(\Gamma', \Delta'), \Psi \quad \text{for all} \quad (\Gamma' \Rightarrow \Delta') \in Cl(\Gamma; \Delta) \]

\[ \vdash_{GL} \Phi \Rightarrow \bigwedge_{(\Gamma' \Rightarrow \Delta') \in Cl(\Gamma; \Delta)} \forall p(\Gamma'; \Delta'), \Psi \]

Consider \((\Gamma; \Delta)\) is a critical sequent and the last inference is an instance of an invertible rule. Then the principal formula of the inference is in \(\Phi, \Psi\). We apply the induction hypothesis to the premise of the last inference, and then the invertible rule in question again.

Finally assume that \((\Gamma; \Delta)\) is a critical sequent and the last inference is a \(\Box_L\) inference:

Consider first that the principal formula \(\Box \alpha \in \Psi\), in particular, \(\alpha\) doesn’t contain \(p\). Then the proof ends with the step:

\[ \Box \Phi', \Box \Gamma', \Phi', \Gamma', \Box \alpha \Rightarrow \alpha \]

\[ \Box \Pi', \Box \Gamma', \Phi', \Gamma' \Rightarrow \Box \alpha, \Psi' \]

where \(\Box \Phi', \Phi''\) is \(\Phi\); \(\Box \Gamma', \Gamma''\) is \(\Gamma\); and \(\Box \alpha, \Psi'\) is \(\Psi\). Consider \(\Box \Gamma' \cap \Delta = \emptyset\) (otherwise the line 1 of Table 1 applies and \(\forall p(\Gamma; \Delta) = \top\), and therefore (iii) holds). So we can use the line 2 of Table 1. Then the induction hypothesis gives

\[ \vdash_{GL} \Phi' \Rightarrow \forall p(\Box \Gamma', \Gamma'; \emptyset), \alpha \]

and by a \(\neg\)l inference we obtain

\[ \vdash_{GL} \Phi' \Rightarrow \forall p(\Box \Gamma', \Gamma'; \emptyset), \alpha \]

Now, by a \(\Box_L\) and a negation inferences, we obtain

\[ \vdash_{GL} \Box \Phi', \Phi'' \Rightarrow \forall p(\Box \Gamma', \Gamma'; \emptyset), \Box \alpha, \Psi' \]

By the line 2 of Table 1 invertibility of the \(\lor\)l rule, and by (3.2) we have

\[ \vdash_{GL} \forall p(\Box \Gamma', \Gamma'; \emptyset) \Rightarrow \forall p(\Box \Gamma', \Gamma''; \Delta) \]

The two sequents above yield (iii) by cut admissibility.

Consider the principal formula \(\Box \alpha \in \Delta\). Again, consider \(\Box \Gamma' \cap \Delta = \emptyset\) so we can use the line 2 of Table 1. Then the proof ends with:

\[ \Box \Phi', \Box \Gamma', \Phi', \Gamma', \Box \alpha \Rightarrow \alpha \]

\[ \Box \Phi', \Box \Gamma', \Phi', \Gamma' \Rightarrow \Box \alpha, \Box \Gamma', \Psi \]

where \(\Box \Phi', \Phi''\) is \(\Phi\); \(\Box \Gamma', \Gamma''\) is \(\Gamma\); and \(\Box \alpha, \Box \Gamma'\) is \(\Delta\). Now the induction hypothesis gives

\[ \vdash_{GL} \Box \Phi', \Phi' \Rightarrow \forall p(\Box \Gamma', \Gamma'; \Box \alpha; \alpha) \]

and by weakening and a \(\Box_L\) inference we obtain

\[ \vdash_{GL} \Box \Phi', \Phi' \Rightarrow \Box \forall p(\Box \Gamma', \Gamma'; \Box \alpha; \alpha), \Psi. \]
The line 2 of Table 1 and invertibility of the ∨-l rule yields
$$\vdash_{GL} \Box \forall p(\Box \Gamma', \Gamma'', \Box \alpha; \alpha) \Rightarrow \forall \exists \Gamma''; \Box \alpha, \Delta').$$
Finally, we obtain (iii) by cut admissibility. QED

Remark 3.3 (Constructivity of the proof). We have given a construction of uniform interpolant which is effective and implementable. However, since we argued semantically to claim cut-free completeness of the calculus, reader might object that our proof of the uniform interpolation theorem is not fully constructive. To this point we say the following: one can look at the cut-elimination proof in [9, 10] and prove constructively that the two calculi are equivalent. Another way is to use the proof-search procedure described in subsection 2.3 and prove completeness via decidability. The point is that an unsuccessful proof-search tree can be used to construct a counterexample to a given sequent, in spirit of the proof contained in [28]. We have not included such an argument here mainly for space reasons and because it is not essential to understand the proof of uniform interpolation.

3.1.1. Fixed points. Uniform interpolation theorem for GL entails Sambin’s and de Jongh’s fixed point theorem. Our proof then presents an alternative constructive proof of the fixed point theorem:

Theorem 3.4. Fixed point theorem: Suppose p is modalized in β (i.e., any occurrence of p is in the scope of a □). Then we can find a formula γ in the variables of β without p such that
$$\vdash_{GL} \gamma \leftrightarrow \beta(\gamma).$$

Already Craig interpolation entails fixed point theorem: a fixed point of a formula β is an interpolant of a sequent expressing the uniqueness of the fixed point
$$\Box(p \leftrightarrow \beta(p)) \land \Box(q \leftrightarrow \beta(q)) \Rightarrow p \leftrightarrow q,$$
which is provable in GL - proofs of this fact in [20] and [4] are easily adaptable to our variant of the calculus. However, to construct the fixed point using this method requires to have an actual proof of the sequent expressing the uniqueness.

Direct proofs of fixed point theorem were given by Sambin [18], Sambin and Valentini in [20] (a construction of explicit fixed points which is effective and implementable), Smoryński [23] from Beth’s definability property, Reidhar-Olson [17], Gleit and Goldfarb [8]. A proof from Beth’s property can be found also in Kracht’s book [15], for three different proofs see Boolos’ book [4]. A different and effective constructive proof of fixed point theorem is the one by Sambin and Valentini in [20]. We present a proof of fixed point theorem based on uniform interpolation, it is an effective proof alternative to those above. We learnt this simple argument from Albert Visser, and we found it an interesting application of the uniform interpolation theorem.

Proof of Theorem 3.4 Let us consider a formula β(p, q) with p modalized in β. The fixed point of β then would be the simulation of
$$\exists p(\Box(p \leftrightarrow \beta(p)) \land \beta(p))$$
or, equivalently, of
$$\forall r(\Box(r \leftrightarrow \beta(r)) \Rightarrow \beta(r)).$$
Let us denote them γ₁ and γ₂ and observe they are both interpolants of the sequent
$$(\Box(p \leftrightarrow \beta(p)) \land \beta(p) \Rightarrow \Box(r \leftrightarrow \beta(r)) \Rightarrow \beta(r))$$
and that neither of them contains $p, r$. We show that any of them is the fixed point of $\beta(p)$ and that they are indeed equivalent. To keep readability we just sketch the proofs in $G_{GL}$ below.

First we show that $\Box(p \leftrightarrow \beta(p)) \land \beta(p) \Rightarrow \Box(r \leftrightarrow \beta(r)) \rightarrow \beta(r))$ is provable from the uniqueness statement:

$$
\Box(p \leftrightarrow \beta(p)) \land \Box(r \leftrightarrow \beta(r)) \Rightarrow p \leftrightarrow r \quad p \leftrightarrow r, p \leftrightarrow \beta(p), r \leftrightarrow \beta(r), \beta(p) \Rightarrow \beta(r) \quad \text{cut}
$$

Now let us see that any of $\gamma_i$ is a fixed point and thus, by the uniqueness, $\gamma_1 \iff \gamma_2$. First observe, that whenever $(\Gamma(p) \Rightarrow \Delta(p))$ is provable, $(\Gamma[p/\alpha] \Rightarrow \Delta[p/\alpha])$ where we substitute $\alpha$ for $p$ is provable as well (we substitute everywhere in the proof, to treat $\Box_{GL}$ inferences can require some admissible weakenings, and we add proofs of sequents $(\Gamma, \alpha \Rightarrow \Delta, \alpha)$ in place of axioms with $p$ principal). The label ”subst.” in the following proof-tree refers to such a substitution, the label ”inv.” refers to invertibility of a rule:

$$
\Box(p \leftrightarrow \beta(p)) \land \beta(p) \Rightarrow \gamma_i \quad \Box(\gamma_i \leftrightarrow \beta(\gamma_i)) \land \beta(\gamma_i) \Rightarrow \gamma_i \quad \text{inv. subst.}
$$

Now by a cut

$$
\Box(\gamma_i \leftrightarrow \beta(\gamma_i)) \Rightarrow \gamma_i \leftrightarrow \beta(\gamma_i) \quad \text{cut}
$$

From this proof one can see that already ordinary interpolation does the job. The point of using uniform interpolation here is that we do not need to have an actual proof of $(\Box(p \leftrightarrow \beta(p)) \land \beta(p) \Rightarrow \Box(r \leftrightarrow \beta(r)) \rightarrow \beta(r))$ to construct a fixed point - we just need to know that the sequent is provable to show that we have indeed constructed a fixed point.

QED

3.2. Uniform interpolation in $\text{Grz}$. The proof of uniform interpolation in $\text{Grz}$ follows the same ideas and is very similar to the previous one, only syntactically a bit more more complicated.

**Theorem 3.5.** Let $\Gamma, \Delta, \Sigma$ be finite multisets of formulas. For every propositional variable $p$ there exists a formula $\forall p(\Box\Sigma|\Gamma; \Delta)$ such that:

(i) $\text{Var}(\forall p(\Box\Sigma|\Gamma; \Delta)) \subseteq \text{Var}(\Sigma, \Gamma, \Delta) \setminus \{p\}$
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(ii) \[ \vdash_{G_{Grz}^+} \Box \Sigma | \forall p (\Box \Sigma | \Gamma ; \Delta) \Rightarrow \Delta \]

(iii) moreover let \( \Phi, \Psi, \Theta \) be multisets of formulas not containing \( p \) and

\[ \vdash_{G_{Grz}^+} \Box \Theta, \Box \Sigma | \Phi, \Gamma \Rightarrow \Psi, \Delta. \]

Then

\[ \vdash_{G_{Grz}^+} \emptyset \Box \Theta \Rightarrow \forall p (\Box \Sigma | \Gamma ; \Delta), \Psi. \]

Proof of Theorem 3.5. We start with a definition of the formula \( \forall p (\Box \Sigma | \Gamma ; \Delta) \), then we prove that the definition terminates, and proceed with proving it satisfies items (i)-(iii) of the Theorem. We remark that the item (iii) is formulated with the third multiset empty because we only have a particular form of cut admissible, see Remark 2.17.

Definition of the interpolant. We describe the construction of the interpolant recursively. The formula \( \forall p (\Box \Sigma | \Gamma ; \Delta) \) is defined by

\[ \forall p (\Box \Sigma | \Gamma ; \Delta) = \bigvee_{(\Box \Sigma_i | \Gamma_i \Rightarrow \Delta_i) \in C((\Box \Sigma | \Gamma ; \Delta))} \forall p (\Box \Sigma_i | \Gamma_i ; \Delta_i) \]

The recursive steps for \( (\Box \Sigma | \Gamma \Rightarrow \Delta) \) being a critical sequent of the form \( (\Box \Gamma' | \Pi ; \Box \Delta', \Lambda) \), with \( \Pi, \Lambda \) atomic, are given by the following table:

| \( \Box \Gamma' | \Pi ; \Box \Delta', \Lambda \) matches | \( \forall p (\Box \Gamma' | \Pi ; \Box \Delta', \Lambda) \) equals |
|--------------------------------------------------|--------------------------------------------------|
| 1. if \( p \in \Pi \cap \Lambda \) or \( \perp \in \Pi \) or \( \Gamma' \cap \Delta' \neq \emptyset \) | \( \top \) |
| 2. otherwise (here all \( q, r \neq p \)) | \( \bigvee_{q \in \Lambda} q \bigvee_{r \in \Pi} \neg r \bigvee_{\beta \in \Delta', D(\beta) \notin \Box \Gamma'} (\neg \forall p (\Box \Gamma', D(\beta) | \Gamma' ; \beta)) \bigvee_{\beta \in \Delta', D(\beta) \in \Box \Gamma'} (\forall p (\Box \Gamma' | \emptyset ; \beta)) \bigvee \Box \bigwedge N(\Box \Gamma' | \emptyset) \) |

Table 3.

As in Table 1 before, the first line corresponds to some of the cases when the critical sequent is provable, and the line 2 corresponds to a critical step, the corresponding disjunction covering

- propositional variables from multisets \( \Pi, \Lambda \),
- all the possibilities of \( \Box_{Grz1}^+ \) and \( \Box_{Grz2}^+ \) inferences with the principal formula from \( \Box \Delta' \),
- and, by the diamond formula \( \Box \bigwedge N(\Box \Gamma' | \emptyset) \) defined below in Table 4 also the possibility of a \( \Box_{Grz1}^+ \) or a \( \Box_{Grz2}^+ \) inference with the principal formula not from \( \Box \Delta' \) (i.e. from a context not containing \( p \)). For a sequent of the form \( (\Box \Gamma' | \emptyset) \), a set of formulas \( N(\Box \Gamma' | \emptyset) \) is defined as the smallest set given by the Table 4.
Consider sequents $(\Box|\Psi; \Box|\Theta)$ in the closure of $(\Box)\Gamma'; \emptyset)$, and refer by $S'$ to sequents in the closure with $\Sigma^\circ \supset \Gamma^\circ$, i.e.

\[
(\Box|\Psi; \Box|\Theta) \in Cl(\Box|\Psi; \Box|\Theta) \text{ matches } \bigwedge_{(\Box|\Psi; \Box|\Theta) \in Cl(\Box|\Psi; \Box|\Theta)} N(\Box|\Psi; \Box|\Theta) \text{ contains } \forall p(\Box|\Psi; \Box|\Theta)
\]

or $p \in \emptyset \Rightarrow \emptyset$, or $\emptyset \Rightarrow \emptyset$

or $\emptyset \Rightarrow \emptyset$
strictly simpler then \((\Box \Gamma'; \emptyset)\). Since

\[ \forall p(\Box \Gamma'; \emptyset) \equiv \bigwedge_S \forall p(S) \land \bigwedge_{S'} \forall p(S'), \]

and for each \(S = (\Box \Sigma'; \Box \Omega, \Theta)\) with \(\Sigma' = \Gamma'\) we obtain \(\forall p(S)\) by the line 2 of Table 3 to be the following formula:

\[ \bigvee_{q \in \Theta} q \lor \bigvee_{r \in \Upsilon} \neg r \lor \bigvee_{\beta \in \Omega} \Box \forall p(\Box \Sigma, D(\beta)|\Sigma; \beta) \lor \bigvee_{\delta \in \Theta} \Box \forall p(\Box \Sigma|\emptyset; \beta) \lor \bigvee_{\beta \in \Omega} \Box \forall p(\Box \Sigma|\emptyset; \beta) \lor \bigvee_{\delta \in \Theta} \Box \forall p(\Box \Sigma|\emptyset; \beta) \lor \Box \forall p(\Box \Sigma; \emptyset), \]

which we can shorten as

\[ \alpha_S \lor \bigwedge N(\Box \Sigma, \Sigma; \emptyset). \]

Now using the definition of \(N(\Box \Gamma'; \emptyset)\) in Table 4, the left-hand side of the above sequent becomes the following:

\[ \bigwedge \alpha_S \land \bigwedge_{S'} \forall p(S'). \]

and the right-hand side becomes the following:

\[ \bigwedge (\bigvee \alpha_S \lor \bigwedge N(\Box \Sigma, \Sigma; \emptyset)) \land \bigwedge_{S'} \forall p(S'). \]

Observe, that \(N(\Box \Sigma; \Sigma; \emptyset)\) is equivalent to \(N(\Box \Sigma'; \emptyset)\) by \(\Sigma' = \Gamma'\). The result now follows by Lemma 2.18 putting \(\beta = \bigwedge \forall p(S')\) and \(\delta = \bigwedge N(\Box \Gamma'; \emptyset)\).

(i). The item (i) follows easily by induction on \((\Box \Sigma; \Gamma; \Delta)\) just because we never add \(p\) during the definition of the formula \(\forall p(\Box \Sigma; \Gamma; \Delta)\).

(ii). We proceed by induction on the complexity of \((\Box \Sigma; \Gamma; \Delta)\) given by the measure function used above to prove termination, and prove that

\[ \vdash_{Grz} \Box \Sigma; \forall p(\Box \Sigma; \Gamma; \Delta) \Rightarrow \Delta. \]

First let \((\Box \Sigma; \Gamma \Rightarrow \Delta)\) be a noncritical sequent. Then sequents \((\Box \Sigma; \Gamma_i \Rightarrow \Delta_i) \in \mathcal{Cl}(\Box \Sigma; \Gamma; \Delta)\) are of lower complexity and by the induction hypotheses

\[ \vdash_{Grz} \Box \Sigma_i; \forall p(\Box \Sigma_i; \Gamma_i; \Delta_i) \Rightarrow \Delta_i, \]

for each \(i\). Then by admissibility of weakening and by Lemma 2.12

\[ \vdash_{Grz} \Sigma; \Gamma; \bigwedge \forall p(\Box \Sigma_i; \Gamma_i; \Delta_i), \ldots, \forall p(\Box \Sigma_k; \Gamma_k; \Delta_k) \Rightarrow \Delta, \]

due to each \(\land\)-1 inference

\[ \vdash_{Grz} \Box \Sigma; \Gamma, \bigwedge_{(\Box \Sigma_i; \Gamma_i \Rightarrow \Delta_i) \in \mathcal{Cl}(\Box \Sigma; \Gamma; \Delta)} \forall p(\Box \Sigma_i; \Gamma_i; \Delta_i) \Rightarrow \Delta, \]

which is by \(3.3\)

\[ \vdash_{Grz} \Gamma; \forall p(\Box \Sigma; \Gamma; \Delta) \Rightarrow \Delta. \]

Let \((\Box \Sigma; \Gamma \Rightarrow \Delta)\) be a critical sequent matching the line 1 of Table 3. Then either

(ii) is an axiom in the case that \(p \in \Pi \cap \Lambda\) or \(\bot \in \Pi\), or (ii) is provable in the case that \(\Gamma' \cap \Delta' \neq \emptyset\).

Let \((\Box \Sigma; \Gamma \Rightarrow \Delta)\) be a critical sequent matching the line 2 of Table 3. We prove

\[ \vdash_{Grz} \Box \Gamma'; \Pi, \delta \Rightarrow \Box \Delta', \Lambda \]
for each disjunct \( \delta \) used in the line 2 of Table 3 to define the interpolant.

- For each \( r \in \Pi \) obviously \( \vdash_{G_{Grz}^+} \Box \Gamma' \mid \Pi, \neg r, \Box \Gamma' \Rightarrow \Box \Delta', \Lambda \), therefore
  \( \vdash_{G_{Grz}^+} \Box \Gamma' \mid \Pi, \bigvee_{r \in \Pi} \neg r, \Box \Gamma' \Rightarrow \Box \Delta', \Lambda \).

- For each \( q \in \Lambda \) obviously \( \vdash_{G_{Grz}^+} \Box \Gamma' \mid \Pi, q, \Box \Gamma' \Rightarrow \Box \Delta', \Lambda \), therefore
  \( \vdash_{G_{Grz}^+} \Box \Gamma' \mid \Pi, \bigvee_{q \in \Lambda} q, \Box \Gamma' \Rightarrow \Box \Delta', \Lambda \).

- For each \( \beta \in \Delta' \) with \( D(\beta) \notin \Box \Gamma' \) we have
  \( \vdash_{G_{Grz}^+} \Box \Gamma', D(\beta) \mid \Gamma', \forall p(\Box \Gamma', D(\beta) \mid \Gamma'; \beta) \Rightarrow \beta \)
by the induction hypothesis, which gives
  \( \vdash_{G_{Grz}^+} \Box \Gamma', D(\beta), \neg \forall p(\Box \Gamma', D(\beta) \mid \Gamma'; \beta)) \mid \Gamma', \forall p(\Box \Gamma', D(\beta) \mid \Gamma'; \beta) \Rightarrow \beta \)
by admissible weakening inferences. This yields
  \( \vdash_{G_{Grz}^+} \Box \Gamma', \neg \forall p(\Box \Gamma', D(\beta) \mid \Gamma'; \beta)) \mid \Pi \Rightarrow \Box \Delta', \Lambda \)
by a \( \Box_{Grz}^+ \) inference. Then by weakening and \( \Box \) inferences
  \( \vdash_{G_{Grz}^+} \Box \Gamma' \mid \neg \forall p(\Box \Gamma', D(\beta) \mid \Gamma'; \beta), \Pi \Rightarrow \Box \Delta', \Lambda \).

- For each \( \beta \in \Delta' \) with \( D(\beta) \in \Box \Gamma' \) we have
  \( \vdash_{G_{Grz}^+} \Box \Gamma' \mid \forall p(\Box \Gamma' \mid \emptyset; \beta) \Rightarrow \beta \)
by the induction hypothesis, which gives
  \( \vdash_{G_{Grz}^+} \Box \Gamma', \forall p(\Box \Gamma' \mid \emptyset; \beta)) \mid \forall p(\Box \Gamma' \mid \emptyset; \beta), \Gamma' \Rightarrow \beta \)
by admissible weakening inferences. This yields
  \( \vdash_{G_{Grz}^+} \Box \Gamma', \forall p(\Box \Gamma' \mid \emptyset; \beta)) \mid \Pi \Rightarrow \Box \Delta', \Lambda \)
bay a \( \Box_{Grz}^+ \) inference and a weakening (notice there is an occurrence of
\( D(\beta) \) missing in \( \Gamma' \)). Then by weakening and \( \Box \) inferences
  \( \vdash_{G_{Grz}^+} \Box \Gamma' \mid \forall p(\Box \Gamma' \mid \emptyset; \beta), \Pi \Rightarrow \Box \Delta', \Lambda. \)

It remains to be proved that
  \( \vdash_{G_{Grz}^+} \Box \Gamma' \mid \Pi, \bigvee \land N(\Box \Gamma' \mid \emptyset) \Rightarrow \Box \Delta', \Lambda. \)

For each \( (\Box \Sigma \Rightarrow \Box \Omega, \Theta) \in \text{Cl}(\Box \Gamma' \mid \Gamma'; \emptyset) \) of the first line of Table 3 we know by the induction hypotheses that
  \( \vdash_{G_{Grz}^+} \Box \Sigma \mid \forall p(\Box \Sigma \mid \emptyset; \emptyset, \Theta) \Rightarrow \Box \Omega, \Theta. \)

For each \( (\Box \Sigma \Rightarrow \Box \Omega, \Theta) \in \text{Cl}(\Box \Gamma' \mid \Gamma'; \emptyset) \) of the second line of Table 4 we have the following:

- \( \vdash_{G_{Grz}^+} \Box \Sigma \mid \forall q \Rightarrow \Box \Omega, \Theta. \)

- \( \vdash_{G_{Grz}^+} \Box \Sigma \mid \forall q \Rightarrow \Box \Omega, \Theta. \)

- for each \( \beta \in \Omega \) with \( D(\beta) \notin \Box \Sigma \) by the induction hypotheses
  \( \vdash_{G_{Grz}^+} \Box \Sigma, D(\beta) \mid \Sigma, \forall p(\Box \Sigma, D(\beta)) \mid \Sigma; \beta \Rightarrow \beta \)
and by weakening and a \( \Box_{Grz}^+ \) inference
  \( \vdash_{G_{Grz}^+} \Box \Sigma, \neg \forall p(\Box \Sigma, D(\beta)) \mid \Sigma; \beta \Rightarrow \Box \beta. \)
and by weakening and □?-l
\[ \Gamma^+_G \vdash_D \square \forall p(\square \Sigma, D(\beta) | \Sigma; \beta) \Rightarrow \square \beta. \]

- for each $\beta \in \Omega$ with $D(\beta) \in \square \Sigma$ by the induction hypotheses
\[ \Gamma^+_G \vdash_D \square \Sigma | \emptyset, \forall p(\square \Sigma | \emptyset; \beta) \Rightarrow \beta \]
and by weakening and a $\square G_{\text{refl}}^{+}$ inference
\[ \Gamma^+_G \vdash_D \square \Sigma, \square \forall p(\square \Sigma | \emptyset; \beta) | \emptyset \Rightarrow \square \beta. \]

- and by weakening and $\square^+_{\text{refl}}$
\[ \Gamma^+_G \vdash_D \square \Sigma | \emptyset, \forall p(\square \Sigma | \emptyset; \beta) \Rightarrow \square \beta. \]

Together this yields, using $\vee$-l inferences,
\[ \Gamma^+_G \vdash_D \square \Sigma | \emptyset, \bigvee_{q \in \emptyset} \bigvee_{\neg r \in \emptyset} \bigvee_{D(\beta) \in \square \Sigma} \forall p(\square \Sigma, D(\beta) | \Sigma; \beta) \vee \bigvee_{D(\beta) \in \square \Sigma} \forall p(\square \Sigma | \emptyset; \beta) \Rightarrow \square \Omega, \Theta. \]

Therefore finally, putting things together for each $(\square \Sigma | \emptyset \Rightarrow \square \Omega, \Theta) \in C(\square \Gamma' | \Gamma'; \emptyset)$, we obtain, using weakening and $\land$-r inferences,
\[ \Gamma^+_G \vdash_D \square \Sigma, \emptyset, \bigwedge N(\square \Gamma' | \Gamma' | \emptyset) \Rightarrow \square \Omega, \Theta. \]

Now, by closure properties in Lemma 2.1
\[ \Gamma^+_G \vdash_D \square \Gamma' | \Gamma', \bigwedge N(\square \Gamma' | \Gamma' | \emptyset) \Rightarrow \emptyset. \]

By negation and weakening inferences
\[ \Gamma^+_G \vdash_D \square \Gamma', D(\neg \bigwedge N(\square \Gamma', \Gamma'; \emptyset) | \Gamma' \Rightarrow \neg \bigwedge N(\square \Gamma', \Gamma'; \emptyset) \]
and by a $\square^+_{\text{refl}}$ inference
\[ \Gamma^+_G \vdash_D \square \Gamma' | \Pi \Rightarrow \square \neg \bigwedge N(\square \Gamma', \Gamma'; \emptyset), \square \Delta', \Lambda. \]

Now, using a negation inference again, we obtain
\[ \Gamma^+_G \vdash_D \square \Gamma' | \Pi, \emptyset, \bigwedge N(\square \Gamma', \Gamma'; \emptyset) \Rightarrow \square \Delta', \Lambda. \]

Putting finally all the above disjuncts together for a critical sequent $(\square \Gamma' | \Pi; \square \Delta', \Lambda)$ yields, using $\vee$-l inferences and the line 2 of Table 3
\[ \Gamma^+_G \vdash_D \square \Gamma' | \Pi, \forall p(\square \Gamma' | \Pi; \square \Delta', \Lambda) \Rightarrow \square \Delta', \Lambda. \]

(iii) We proceed by induction on the height of the proof of the sequent $(\square \Theta, \square \Sigma | \Phi, \Gamma \Rightarrow \Psi, \Delta)$ in $G_{\text{Grz}}^{+}$, and sub-induction on the measure of the sequent $(\square \Sigma | \Theta | \Gamma, \Delta)$. We show that
\[ \Gamma^+_G \vdash_D \square \Theta | \Phi \Rightarrow \forall p(\square \Sigma | \Theta | \Gamma, \Delta), \Psi. \]

Let us first consider the last step of the proof of $(\square \Theta, \square \Sigma | \Phi, \Gamma \Rightarrow \Psi, \Delta)$ is an axiom, or, if it is not an axiom, then $(\square \Sigma | \Gamma, \Delta)$ is a noncritical sequent. In this case we proceed similarly as in Theorem 3.1 (iii), the third multiset makes no difference here.

Let us then consider that the last inference of the proof of $(\square \Theta, \square \Sigma | \Phi, \Gamma \Rightarrow \Psi, \Delta)$ is a $\square^+_{\text{refl}}$ inference. There are two cases to distinguish:
- Consider first the case when the principal formula $\square \alpha \in \Delta$. Then the proof ends with:
Consider next the case when the principal formula \( \square \alpha \), \( \Delta' \) is \( \Delta \). Consider \( \square \Sigma \cap \Delta = \emptyset \) (otherwise \( \forall p(\square \Sigma | \Sigma) \equiv T \) and (iii) holds). Then by the induction hypotheses

\[ \vdash G^+_\text{Grz} \emptyset | \square \Theta, \Theta \Rightarrow \forall p(\square \Sigma, D(\alpha)|\Sigma; \alpha). \]

By invertibility of \( \square^+_T \) inferences, by contraction inferences, and weakening

\[ \vdash G^+_\text{Grz} \square \Theta, D(\forall p(\square \Sigma, D(\alpha)|\Sigma; \alpha)) | \Theta \Rightarrow \forall p(\square \Sigma, D(\alpha)|\Sigma; \alpha), \]

Now, by a \( \square^+_T \) inference, we obtain

\[ \vdash G^+_\text{Grz} \square \Theta | \Phi \Rightarrow \Box \forall p(\square \Sigma, D(\alpha)|\Sigma; \alpha), \Psi. \]

By weakening inferences

\[ \vdash G^+_\text{Grz} \square \Theta | \Phi \Rightarrow \Box \forall p(\square \Sigma, D(\alpha)|\Sigma; \alpha), \Psi. \]

By \( \square^+_T \) inferences we obtain

\[ \vdash G^+_\text{Grz} \emptyset | \square \Theta, \Phi \Rightarrow \Box \forall p(\square \Sigma, D(\alpha)|\Sigma; \alpha), \Psi. \]

By the line 2 of Table 3 and invertibility of the \( \forall \)-rule

\[ \vdash G^+_\text{Grz} \emptyset | \Box \forall p(\square \Sigma, D(\alpha)|\Sigma; \alpha) \Rightarrow \forall p(\square \Sigma, D(\alpha)|\Sigma; \alpha), \]

The two sequents above yield (iii) by admissibility of the cut rule in \( G^+_\text{Grz} \).

- Consider next the case when the principal formula \( \Box \alpha \in \Psi \), i.e., \( \alpha \) doesn’t contain \( p \). Then the proof ends with:

\[ \vdash G^+_\text{Grz} \emptyset | \Box \Theta, \Theta \Rightarrow \forall p(\square \Sigma; \alpha). \]

where \( \Box \alpha, \Psi' \) is \( \Psi \). Then by the induction hypotheses

\[ \vdash G^+_\text{Grz} \emptyset | D(\alpha), \Box \Theta, \Theta \Rightarrow \forall p(\square \Sigma; \emptyset), \alpha. \]

By invertibility of \( \Box^+_T \) inferences and by contraction inferences we obtain

\[ \vdash G^+_\text{Grz} D(\alpha), \Box \Theta | (\alpha \rightarrow \Box \alpha), \Theta \Rightarrow \forall p(\square \Sigma; \emptyset), \alpha. \]

To get rid of \( (\alpha \rightarrow \Box \alpha) \), which is \( (\neg \alpha \lor \Box \alpha) \), we use invertibility of the \( \forall \)-rule and \( \Box \)-rule, and contraction, to obtain

\[ \vdash G^+_\text{Grz} D(\alpha), \Box \Theta | \Theta \Rightarrow \forall p(\square \Sigma; \emptyset), \alpha. \]

By a \( \Box \)-rule and weakening

\[ \vdash G^+_\text{Grz} D(\alpha), \Box \Theta, \Box \neg \forall p(\square \Sigma; \emptyset) | \Theta, \neg \forall p(\square \Sigma; \emptyset) \Rightarrow \alpha. \]

By a \( \Box^+_T \) inference

\[ \vdash G^+_\text{Grz} \Box \Theta, \Box \neg \forall p(\square \Sigma; \emptyset) | \Phi \Rightarrow \Box \alpha, \Psi'. \]

Since weakening is admissible in \( G^+_\text{Grz} \), we obtain

\[ \vdash G^+_\text{Grz} \Box \Theta, \Box \neg \forall p(\square \Sigma; \emptyset) | \Theta, \neg \forall p(\square \Sigma; \emptyset), \Phi \Rightarrow \Box \alpha, \Psi' \]

and now \( \Box^+_T \) inferences and a \( \Box \)-rule yield

\[ \vdash G^+_\text{Grz} \Box \Theta, \Phi \Rightarrow \Box \forall p(\square \Sigma; \emptyset), \Box \alpha, \Psi'. \]
UNIFORM INTERPOLATION IN PROVABILITY LOGICS

By weakening inferences

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \square \Theta \proof \land \Phi \Rightarrow \land p(\square \Sigma \mid \Theta; \emptyset), \square \alpha, \Psi'. \]

By \( \square^+ \) inferences

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \emptyset \proof \square \Theta, \Phi \Rightarrow \land p(\square \Sigma \mid \Theta; \emptyset), \square \alpha, \Psi'. \]

By the line 2 of Table 3, invertibility of the \( \lor \)-l rule, and by (3.4) we have

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \emptyset \proof \land p(\square \Sigma \mid \Theta; \emptyset) \Rightarrow \land p(\square \Sigma \mid \emptyset). \]

The two sequents above yield (iii) by admissibility of the cut rule in \( \mathcal{G}_{\text{Grz}}^+ \).

Let us consider that the last inference of the proof of \((\square \Theta, \square \Sigma \mid \Phi, \Gamma \Rightarrow \Psi, \Delta)\) is a \( \square^+ \) inference. Again, we distinguish two cases:

- Consider first the case when the principal formula \( \square \alpha \in \Delta \). Then the proof ends with:

\[
\begin{align*}
\square \Theta, \square \Sigma \mid \emptyset & \Rightarrow \alpha \\
\square \Theta, \square \Sigma \mid \Gamma ; \Phi & \Rightarrow \square \alpha, \Delta', \Psi
\end{align*}
\]

where \( \square \alpha, \Delta' \) is \( \Delta \). Consider \( \square \Sigma \cap \Delta = \emptyset \) (otherwise \( \land p(\square \Sigma \mid \Gamma; \Delta) \equiv \top \) and (iii) holds). Then by the induction hypotheses

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \emptyset \proof \land p(\square \Sigma \mid \emptyset; \alpha). \]

By invertibility of \( \square^+ \) inferences, by contraction inferences, and weakening

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \square \Theta, D(\land p(\square \Sigma \mid \emptyset; \alpha)) \mid \Theta \Rightarrow \land p(\square \Sigma \mid \emptyset; \alpha), \]

Now, by a \( \square^+ \) inference, we obtain

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \square \Theta \proof \Phi \Rightarrow \land \land p(\square \Sigma \mid \emptyset; \alpha), \Psi. \]

By weakening inferences and \( \square^+ \) inferences we obtain

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \emptyset \proof \land \land p(\square \Sigma \mid \emptyset; \alpha), \Psi. \]

By the line 2 of Table 3 and invertibility of the \( \lor \)-l rule we have

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \emptyset \proof \land \land p(\square \Sigma \mid \emptyset; \alpha) \Rightarrow \land \land p(\square \Sigma \mid \emptyset; \alpha). \]

The two sequents above yield (iii) by admissibility of the cut rule in \( \mathcal{G}_{\text{Grz}}^+ \).

- Consider next the case that the principal formula \( \square \alpha \in \Psi \), i.e., \( \alpha \) doesn’t contain \( p \). Then the proof ends with:

\[
\begin{align*}
\square \Theta, \square \Sigma \mid \emptyset & \Rightarrow \alpha \\
\square \Theta, \square \Sigma \mid \Gamma ; \Phi & \Rightarrow \Delta, \square \alpha, \Psi'
\end{align*}
\]

where \( \square \alpha, \Psi' \) is \( \Psi \). Then by the induction hypotheses

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \emptyset \proof \land \land p(\square \Sigma \mid \emptyset; \emptyset), \alpha. \]

Notice that \((\square \Sigma \mid \emptyset; \emptyset)\) is a critical sequent with all but one multisets empty, and by the table \( \land p(\square \Sigma \mid \emptyset; \emptyset) \equiv \bot \). Thus we have in fact

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \emptyset \proof \land \land p(\square \Sigma \mid \emptyset; \emptyset). \]

By invertibility of \( \square^+ \) inferences we obtain

\[ \vdash_{\mathcal{G}_{\text{Grz}}^+} \square \Theta \proof \theta \Rightarrow \alpha. \]
and by weakening
\[ \vdash_{Grz}^+ \Theta, D(\alpha) | \Theta \Rightarrow \alpha. \]

By a \( \Box_{Grz}^+ \) inference
\[ \vdash_{Grz}^+ \Theta | \Phi \Rightarrow \Box \alpha, \Psi'. \]

By admissibility of weakening we obtain
\[ \vdash_{Grz}^+ \Theta | \Phi \Rightarrow \forall \alpha (\Box \Sigma | \Gamma, \Delta), \Box \alpha, \Psi'. \]

Qed

3.3. Concluding remarks. We have provided an effective construction of uniform interpolants in provability logics. We would like to point out, that even if the proofs as presented are not fully constructive, the only part that is not constructive is the completeness of the two calculi without the cut rule. This can be, in both cases, repaired by completing the proof search argument and make it into a decision procedure.

What we also left open in this paper is to investigate which distribution laws the quantifiers satisfy. For example, it is the case that, in the basic modal logic \( K \), the universal bisimulation quantifier commutes with the diamond modality \( \Box \). In fact, it commutes with (the dual of) the cover modality, which is a principle that, besides the usual axioms and rules for quantification, axiomatizes bisimulation quantifiers over \( K \). Whether a similar insight can be obtained for \( GL \) is not clear at the moment.

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