ASYMPTOTIC HILBERT POLYNOMIAL AND A BOUND FOR WALDSCHMIDT CONSTANTS

MARcin DumnickI, Łucja Farnik, and Halszka Tutaj-Gasińska

(Communicated by Igor Dolgachev)

Abstract. In the paper we give a method to compute an upper bound for the Waldschmidt constants of a wide class of ideals. This generalizes the result obtained by Dumnicki, Harbourne, Szemberg and Tutaj-Gasińska, Adv. Math. 2014, [5]. Our bound is given by a root of a suitable derivative of a certain polynomial associated with the asymptotic Hilbert polynomial.

1. Introduction

Recently the asymptotic invariants of ideals have aroused great interest and have been studied by many researchers, see, for example, [8], [9], [6], [7], [19], [18], [20], and others. One of these asymptotic invariants is the so-called Waldschmidt constant of an ideal $I \subset K[\mathbb{P}^n]$ denoted by $\hat{\alpha}(I)$, see, for example, [3], [4], [2]. The constant is the limit of a sequence of quotients of the initial degrees of the $m$-th symbolic power of the ideal by $m$ (see Definition 4). Computing this constant is a hard task in general, as it is difficult to compute the initial degree of a symbolic power of an ideal. For example, if $I$ is the ideal of $s$ points in $\mathbb{P}^2$ in generic position, finding $\hat{\alpha}(I)$ means finding the Seshadri constant of these points. The Seshadri constant is defined as the infimum of the quotients $\deg C \over m_1 + \ldots + m_s$, where $C$ is a curve passing through $P_1, \ldots, P_s$ with multiplicities $m_1, \ldots, m_s$. By the famous Nagata conjecture (see, e.g., [5] or [1] and the references therein) we expect in this situation the equality $\hat{\alpha}(I) = \sqrt{s}$, for $s \geq 10$. Here we know that $\hat{\alpha}(I) \leq \sqrt{s}$. In the Preliminaries we briefly recall the already known lower and upper bounds for the Waldschmidt constants.

In [5] the authors give an upper bound for $\hat{\alpha}(I)$ when $I$ is the ideal of a sum of disjoint linear subspaces of $\mathbb{P}^n$, see Theorem 12. In the present paper we generalize this result and give a method to compute an upper bound of $\hat{\alpha}(I)$ for a wide class of ideals (namely, radical ideals with linearly bounded regularity of symbolic powers, see Preliminaries for the definitions). To find this bound we use $aHP_I(t)$, the so-called asymptotic Hilbert polynomial of $I$, defined in [7]. The bound is given by a

Received by the editors December 7, 2015 and, in revised form, April 14, 2016.

2010 Mathematics Subject Classification. 13P10, 14N20.

Key words and phrases. Symbolic powers, asymptotic invariants.

The authors are grateful to the referee for asking many important questions and thus helping us to improve the paper.

The first and third author’s research was partially supported by the National Science Centre, Poland, grant 2014/15/B/ST1/02197.

©2016 American Institute of Mathematical Sciences
root of a suitable derivative of the polynomial \( \Lambda_I(t) := \frac{t^n}{n!} - aHP_I(t) \). The main result of the present paper is the following theorem:

**Main Theorem.** Let \( I \) be a radical homogeneous ideal in \( \mathbb{k}[P^n] \) with linearly bounded regularity of symbolic powers. Assume that in the sequence \( \{\text{codepth } I^m\} \) there exists a constant subsequence of value \( n - c \). Then

\[
\Lambda^{(c)}_I(\hat{\alpha}(I)) \leq 0,
\]

where \( \Lambda^{(c)}_I \) denotes the \( c \)-th derivative of \( \Lambda_I \).

In particular \( \hat{\alpha}(I) \leq \gamma \Lambda^{(c)}_I \), where \( \gamma \Lambda^{(c)}_I \) is the largest real root of the polynomial \( \Lambda^{(c)}_I(t) \).

The notions of \( aHP_I, \Lambda_I, \hat{\alpha}(I), \) and codepth \( I \) are defined in the Preliminaries.

The paper is organized as follows. In the second section we recall the necessary notions. In the third we prove the main result. The fourth section contains some interesting examples. In particular, in Example 10 we show that it is necessary to take the root of a derivative of the polynomial \( \Lambda_I \), not of the polynomial itself, as \( \Lambda_I(\hat{\alpha}(I)) > 0 \). Example 8 shows that we may get worse bounds on \( \hat{\alpha}(I) \) by computer-aided computations than by application of the Main Theorem.

2. Preliminaries

In the paper [7] the authors define the so-called asymptotic Hilbert function and asymptotic Hilbert polynomial. Namely, let \( \mathbb{k} \) be an algebraically closed field of characteristic zero. We denote by \( \mathbb{k}[P^n] = \mathbb{k}[x_0, \ldots, x_n] \) the homogeneous coordinate ring of the projective space \( P^n \). Let \( I \) be a homogeneous radical ideal in \( \mathbb{k}[P^n] \), and let \( I^m \) denote its \( m \)-th symbolic power, defined as

\[
I^m = \mathbb{k}[P^n] \cap \bigcap_{Q \in \text{Ass}(I)} (I^m)_Q,
\]

where localizations are embedded in a field of fractions of \( \mathbb{k}[P^n] \) ([11]). By the Zariski-Nagata theorem, for a radical homogeneous ideal \( I \) in a polynomial ring over an algebraically closed field, the \( m \)-th symbolic power \( I^m \) is equal to

\[
I^m = \bigcap_{p \in V(I)} m^m_p,
\]

where \( m_p \) denotes the maximal ideal of a point \( p \) and \( V(I) \) denotes the set of zeroes of \( I \). In characteristic zero, the symbolic power (of a radical ideal) can also be described as the set of polynomials which vanish to order \( m \) along \( V(I) \); this (compare [21]) can be written as

\[
I^m = \left\{ f : \frac{\partial^{\mid \alpha \mid} f}{\partial x^\alpha} \in I \text{ for } \mid \alpha \mid \leq m - 1 \right\}.
\]

To define the asymptotic Hilbert polynomial, recall that the Hilbert function \( HF_I \) of a homogeneous ideal \( I \) is defined as

\[
HF_I(t) = \dim_{\mathbb{k}}(\mathbb{k}[P^n]_t/I_t).
\]

For \( t \) big enough the above function behaves as a polynomial, the Hilbert polynomial \( HP_I \) of \( I \).

Let us define ideals with linearly bounded symbolic regularity (i.e., satisfying the LBSR condition):
Definition 1. Let $I$ be a homogeneous ideal in $\mathbb{K}[\mathbb{P}^n]$. We say that $I$ satisfies linearly bounded symbolic regularity, or is LBSR for short, if there exists a constant $a > 0$ such that 

$$\text{reg}(I^{(m)}) \leq am.$$ 

It is worth observing that we do not know, so far, any example of a homogeneous ideal that is not LBSR. The list of ideals which are proved to be LBSR may be found, e.g., in [7].

Now we recall the definitions of the asymptotic Hilbert function and the asymptotic Hilbert polynomial of an ideal $I$.

Definition 2. The asymptotic Hilbert function of $I$ is 

$$aHF_I(t) := \lim_{m \to \infty} \frac{HF_I(m^m)}{m^n}$$ 

if the limit exists.

In [7] it is shown that if $I$ is a radical ideal then the limit exists.

Definition 3. The asymptotic Hilbert polynomial of $I$ is 

$$aHP_I(t) := \lim_{m \to \infty} \frac{HP_I(m^m)}{m^n}$$ 

if the limit exists.

In [7] it is shown that if $I$ is a radical LBSR ideal then the limit exists and for all $t$ big enough $aHP_I(t) = aHF_I(t)$.

Recall the definition of the Waldschmidt constant of an ideal $I$:

Definition 4.

$$\hat{\alpha}(I) := \lim_{m \to \infty} \frac{\alpha(I^{(m)^m})}{m} = \inf_m \frac{\alpha(I^{(m)^m})}{m},$$

where $\alpha(J)$ is the least degree of a nonzero polynomial appearing in $J$ (called the initial degree of $J$).

Obviously, for each $m \geq 1$ the number $\frac{\alpha(I^{(m)^m})}{m}$ is an upper bound for the Waldschmidt constant of $I$; however, computing $\alpha(I^{(m)^m})$ is usually very hard. For a homogeneous ideal in $\mathbb{K}[\mathbb{P}^n]$ there is a lower bound implied by the containment results of Ein-Lazarsfeld-Smith [10] and Hochster-Huneke [17], namely $\hat{\alpha}(I) \geq \frac{\alpha(I^{(m)^m})}{m+n-1}$ for each $m$, with the Waldschmidt-Skoda bound $\hat{\alpha}(I) \geq \frac{\alpha(I)}{n}$ as a consequence. The result of Esnault and Viehweg from [12] gives a lower bound of the Waldschmidt constant of an ideal of distinct points in $\mathbb{P}^n$.

Recall that the generic initial ideal $\text{gin}(I)$ of $I$ is the initial ideal of $I$, with respect to the degree reverse lexicographical order, of a generic coordinate change of $I$. Galligo [13] assures that for a homogeneous ideal $I$ and a generic choice of coordinates, the initial ideal of $I$ is fixed and hence $\text{gin}(I)$ is well-defined.

Consider the sequence of monomial ideals $\text{gin}(I^{(m)^m})$. Note that a radical ideal in $\mathbb{K}[\mathbb{P}^n]$ not equal to the irrelevant ideal is saturated, moreover its $m$-th symbolic power is also saturated due to the formula (1). Hence by Green [15, Theorem 2.21] no minimal generator of $\text{gin}(I^{(m)^m})$ contains as a factor the last variable $x_n$. Therefore these monomial ideals can be naturally regarded as ideals in $\mathbb{K}[x_0, \ldots, x_{n-1}]$. 
Definition 5. The codepth of an ideal \( I \) is the number of variables which appear as factors in the minimal set of generators of \( \text{gin}(I) \).

The definition above is stated in accordance with [11, p. 351] where depth \( K[x_0, \ldots, x_n]/I \) is defined to be the maximal \( t \) such that \( x_{n-t+1}, \ldots, x_n \) do not appear as factors in the minimal set of generators of the generic initial ideal of \( I \). For a general definition of depth we refer to [11, p. 451].

Let \( \text{gin}(I) = (g_1, \ldots, g_s) \) where \( \{g_1, \ldots, g_s\} \) is a minimal set of generators. Then \( \text{codepth} I = c + 1 \) is equivalent to the fact that \( g_i \in K[x_0, \ldots, x_c] \) for all \( i \), and there exists \( j \) such that \( g_j \not\in K[x_0, \ldots, x_{c-1}] \).

In [5] the authors defined a polynomial \( \Lambda_{n,r,s}(t) \). Namely, let \( L \) be a sum of \( s \) disjoint linear subspaces of dimension \( r \) in \( \mathbb{P}^n \) (called a flat; by a fat flat we denote such subspaces with multiplicities), as in [5]. Let \( I \) be the ideal of \( L \). Define

\[
P_{n,r,s,m}(t) := \left( \frac{t + n}{n} \right) - \text{HP}_{I(m)}(t).
\]

Substitute \( t \) by \( mt \) into \( P_{n,r,m,s}(t) \) and regard it as a polynomial in \( m \) (this is indeed a polynomial, see [5]). The leading term of this polynomial is denoted by \( \Lambda_{n,r,s}(t) \).

In [7] it is shown that

\[
\text{aHP}_I(t) = \frac{t^n}{n!} - \Lambda_{n,r,s}(t),
\]

where \( I \) is the ideal of the fat flat.

In this paper we define \( \Lambda_I(t) \) for any radical LBSR ideal as

\[
\Lambda_I(t) := \frac{t^n}{n!} - \text{aHP}_I(t).
\]

3. MAIN RESULT

The main result of our paper is Theorem 7 giving a method to compute an upper bound for \( \hat{\alpha}(I) \), where \( I \) is a radical homogeneous ideal satisfying the LBSR condition. Thus, our theorem generalizes [5, Theorem 2.5], where the bound is proved for ideals of linear subspaces only. More comments on this generalization are in Remark 11.

Before we prove the main theorem we make the following claim:

Claim 6. Let \( Q_\bullet = \{Q_m\}_{m \in \mathbb{N}} \) be a graded sequence (i.e., such that \( Q_i \cdot Q_j \subset Q_{i+j} \)) of homogeneous, monomial ideals in \( K[x_0, \ldots, x_n] \) satisfying

(i) for all \( t \) big enough and for all \( m \),

\[
\text{HF}_{Q_m}(mt) = \text{HP}_{Q_m}(mt);
\]

(ii) for each \( m \) there exists a set of generators of \( Q_m \) belonging to \( K[x_0, \ldots, x_{n-1}] \) with all variables \( x_0, \ldots, x_{n-1} \) appearing as factors of some of these generators.

Then \( \text{aHP}_{Q_\bullet}(t) = \lim_{m \to \infty} \frac{\text{HP}_{Q_m}(mt)}{m^n \text{ exists}, \hat{\alpha}(Q_\bullet) = \lim_{k \to \infty} \frac{\alpha(Q_k)}{k} \text{ exists and } \Lambda_{Q_\bullet}(\hat{\alpha}(Q_\bullet)) \leq 0, \text{ where } \Lambda_{Q_\bullet}(\hat{\alpha}(Q_\bullet)) = \frac{t^n}{n!} - \text{aHP}_{Q_\bullet}(t).}

Proof. We use the argumentation from [7]. Note that by condition (ii) there is a set of generators of each \( Q_m \) not involving the last variable \( x_n \), as in the assertion of [7, Lemma 4].
Let $L_m = \{ \alpha + \mathbb{R}_+^n : x^n \in Q_m \}$. Notice that if $\alpha \in L_m$ then $\alpha + \mathbb{R}_+^n \subset L_m$, and $L_i + L_j \subset L_{i+j}$. Observe that in the proofs of Lemma 6 and Theorem 5 from [7] only these properties of $L_m$ are used. Therefore from the proof of [7, Theorem 7] applied for $Q_m$ we get that $\text{aHF}_{Q_m} = \lim_{m \to \infty} \frac{\text{HF}_{Q_m}(mt)}{m^n}$ exists.

By [7, Theorem 11] $\text{aHF}_{Q_m}(t)$ is equal to the volume of the restricted closure of the complement of the limiting shape $\Delta_{Q_m}$. Moreover by [7, Theorem 13] for $t$ big enough $\text{aHF}_{Q_m}(t)$ is equal to this volume. Note that we need the property (i) here as an analogue of the LBSR condition.

The limit $\hat{\alpha}(Q_m) = \lim_{k \to \infty} \frac{\alpha(Q_m)}{k}$ is well-defined by Fekete’s subadditivity lemma.

It remains to prove that $\Lambda_{Q_m}(\hat{\alpha}(Q_m)) \leq 0$, i.e., that

$$\Lambda_{Q_m}(t) = \lim_{m \to \infty} \frac{(\frac{n+mt}{m}) - \text{HP}_{Q_m}(mt)}{m^n}$$

is less than or equal to zero for $t = \hat{\alpha}(Q_m)$. Take

$$t_m := \frac{\alpha(Q_m) - 1}{m}.$$ 

If we prove that

$$\lim_{m \to \infty} \frac{(\frac{n+mtm}{m}) - \text{HP}_{Q_m}(mtm)}{m^n} \leq 0,$$

then we are done as $\Lambda_{Q_m}$ is a limit of uniformly convergent polynomials with bounded degrees.

Observe that

$$\left(\frac{n+mtm}{m}\right) - \text{HP}_{Q_m}(mtm) = \left(\frac{n+\alpha(Q_m) - 1}{m}\right) - \text{HP}_{Q_m}(\alpha(Q_m) - 1) = 0$$

from the definition of $t_m$. So, it is enough to show that in our case

$$\text{HF}_{Q_m}(t) \leq \text{HP}_{Q_m}(t)$$

for all nonnegative integers $t \geq \alpha(Q_m) - 1$.

Fix $m$. Let $K = Q_m$.

Let $M(n)$ denote the set of monomials in variables $x_0, \ldots, x_n$. Observe that

$$\#\{\mu \in M(n) : \mu \in K, \deg \mu = t\} = \#\{\mu \in M(n-1) : \mu \in K, \deg \mu = t\}.$$ 

In other words,

$$\text{HF}_{K}(t) = \#\{\mu \in M(n-1) : \deg \mu \leq t, \mu \notin K\}.$$ 

Let $K = (\mu_1, \ldots, \mu_k)$ where $\mu_1, \ldots, \mu_k$ are monomial generators of $K$. Then let $\hat{\mu} := \gcd(\mu_1, \ldots, \mu_k)$, let $J = (\hat{\mu})$ be the ideal generated by $\hat{\mu}$, and let

$$\Delta := \{\mu \in M(n-1) : \mu \in J \setminus K\}.$$ 

We will show by induction on $n$ that $\Delta$ is finite. For $n = 1$ the claim is obvious (since $K = J$ in this case). So let $n$ be arbitrary. By $\deg_j(\mu)$ we denote the degree of $\mu$ with respect to $x_j$. For each $j = 0, \ldots, n-1$ let

$$d_j := \max_{i} \{\deg_j(\mu_i)\}, \quad \Delta_j := \Delta \cap \{\mu : \deg_j(\mu) \geq d_j\}.$$ 

Observe that each $d_j$ is well-defined, as each variable appears as a factor in some generator. Let $K_j$ be the dehomogenization of $K$ with respect to $x_j$, and let $\hat{\mu}_j$ be the dehomogenization of $\hat{\mu}$. Observe that the dehomogenization of each $\mu \in \Delta_j$
belongs to the set $\Delta$ defined for $K_j$ and $\tilde{\mu}_j$ (the assumption that $\deg_j(\mu) \geq d_j$ plays a crucial role here). Hence, by inductive assumption (and since the degree of $\mu$ is bounded) each set $\Delta_j$ is finite. Observe also that
\[
\Delta \subset \bigcup_{j=0}^{n-1} \Delta_j \cup (\Delta \cap \{\mu : \deg_j(\mu) \leq d_j \text{ for } j = 0, \ldots, n-1\}).
\]
Each of these sets is finite, and thus $\Delta$ is finite, as claimed.

From finiteness of $\Delta$ it follows that, for $t$ big enough (it is enough to take $t$ bigger than the maximum degree of a monomial in $\Delta$),
\[
HF_K(t) = HF_J(t) + \#\Delta.
\]
Since $HF_K(t) = HP_K(t)$ for all $t$ big enough (and the same for $J$), for all $t$ we have
\[
HP_K(t) = HP_J(t) + \#\Delta.
\]
Observe that
\[
HF_J(t) = \begin{cases} \binom{n+t}{n} - \binom{n+t-deg(\tilde{\mu})}{n} & \text{for } t \geq deg(\tilde{\mu}) = \alpha(J), \\ \binom{n+t}{n} & \text{for } t < deg(\tilde{\mu}). \end{cases}
\] (3)
Therefore, for $t \geq \alpha(J)$, and as a consequence for $t \geq \alpha(K) - 1$,
\[
HP_K(t) = HP_J(t) + \#\Delta \quad \text{by (3)} \quad HF_J(t) + \#\Delta \geq HF_K(t).
\]
The last inequality follows from the fact that
\[
\{\mu \in M(n-1) : \mu \notin K\} = \{\mu \in M(n-1) : \mu \notin J\} \cup \Delta. \quad \square
\]
Now we are ready to prove the main theorem.

**Theorem 7 (Main Theorem).** Let $I$ be a radical homogeneous LBSR ideal. Assume that in the sequence $\{\text{codepth } I^{(m)}\}$ there exists a constant subsequence of value $n-c$. Then
\[
\Lambda_I^{(c)}(\hat{\alpha}(I)) \leq 0,
\]
where $\Lambda_I^{(c)}$ denotes the $c$-th derivative of $\Lambda_I$.

**Proof.** Observe that the minimal possible value of $c$ is 0 since there exists a set of generators of $I^{(m)}$ with no generator containing the last variable $x_n$ as a factor.

We divide the proof into three cases:

**Case c = 0.** Let $Q_m = \text{gin}(I^{(m)})$. For any ideal $J$ in $K[\mathbb{P}^n]$, we have that $HF_J = HF_{\text{gin}(J)}$, $HP_J = HP_{\text{gin}(J)}$, see, e.g., [15, Proposition 1.11]. Therefore we have that $\Lambda_I(t) = \Lambda_Q(t)$ and $\hat{\alpha}(I) = \hat{\alpha}(Q)$. Since $I$ is LBSR, the condition (i) of Claim 6 is fulfilled. By the codepth assumption, the condition (ii) is also fulfilled. We apply the claim.

**Case c = 1.** Let $I^{(m)}$ be a sequence of ideals in $K[\mathbb{P}^n]$ with a subsequence of codepth $n-1$. By the assumption that $c = 1$, each ideal $P_m = \text{gin}(I^{(m)})$ has a set of generators belonging in $K[\mathbb{P}^{n-2}]$. Therefore we may consider each $P_m$ as an ideal in $K[\mathbb{P}^n]$ or as an ideal generated by the same set of generators in $K[\mathbb{P}^{n-1}]$ denoted by $Q_m$. Notice that codepth $Q_m = n-1$ in $K[\mathbb{P}^{n-1}]$.

Observe that for all $m$ and for $t$ big enough
\[
HP_{Q_m}(mt) = HP_{P_m}(mt) - HP_{P_m}(mt - 1).
\]
Since $aHP_J(t)$ is a polynomial by [7, Theorem 13], the derivative $aHP_J'(t)$ exists and is equal to the limit of the difference quotient. Moreover for any $T \in \mathbb{R}$ and
for \( t \in [0,T] \), \( \text{aHP}_I \) is a limit of uniformly convergent polynomials with bounded degrees. Hence

\[
\text{aHP}_I'(t) = \lim_{h \to 0} \frac{\text{aHP}_I(t) - \text{aHP}_I(t - h)}{h} = \lim_{h \to 0} \lim_{m \to \infty} \frac{1}{m} \left( \frac{\text{HP}_{P_m}(mt)}{m} - \frac{\text{HP}_{P_m}(m(t - 1))}{m} \right) = \lim_{m \to \infty} \left( \frac{\text{HP}_{Q_m}(mt)}{m} - \frac{\text{HP}_{P_m}(m(t - 1))}{m} \right) = \frac{\text{aHP}_{Q_\bullet}(t)}{m}. 
\]

The sequence \( Q_\bullet = \{ Q_m \}_{m \in \mathbb{N}} \) of ideals in \( \mathbb{K}[\mathbb{P}^{n-1}] \) satisfies the assumptions of Claim 6, thus we use the claim to get the assertion.

**Case \( c > 1 \).** Let \( Q_m = \text{gin}(I^{(m)}) \) where for an ideal \( J \subset \mathbb{K}[\mathbb{P}^n] \) generated by elements belonging to \( \mathbb{K}[\mathbb{P}^{n-1}] \) the ideal \( J^c \) denotes the ideal generated by the same set of generators in \( \mathbb{K}[\mathbb{P}^{n-c}] \).

By induction we get that \( \text{aHP}_I^{(c)}(t) = \text{aHP}_{Q_\bullet}(t) \). The sequence \( Q_\bullet \) satisfies the assumptions of Claim 6 hence we are done.

\[ \square \]

### 4. Examples

In this section we present some interesting examples. There are two types of examples.

The first type of examples concerns “crosses.” By a “cross” we mean two intersecting lines. We show in Example 8 that our theorem gives a better bound than those we could obtain with the help of a computer.

The second type of examples are examples of some star configurations in \( \mathbb{P}^4 \). In particular, in Example 10 we show that it is necessary to take the root of a derivative of the polynomial \( \Lambda_I \), not of the polynomial itself. We also formulate a problem that may be viewed as a generalization of Nagata and Nagata-type conjectures, see [5].

In the sequel we will need the notion of a limiting shape and some results from [6] and [7]. Let us start with recalling the notion of limiting shapes.

The Newton polytope of a monomial ideal is defined as the convex hull of the set of exponents:

\[ P(J) := \text{conv}(\{ \alpha \in \mathbb{R}^n : x^\alpha \in J \}). \]

The limiting shape of an ideal \( I \) as above is defined as

\[ \Delta(I) = \bigcup_{m=1}^\infty \frac{P(\text{gin}(I^{(m)}))}{m} \]

(see Mayes [18]).

Define \( \Gamma_I \) as the closure of the complement of \( \Delta(I) \) in \( \mathbb{R}^n_{\geq 0} \). [7, Corollary 14] says that for a radical LBSR ideal \( I \) and for \( T_t = \{(x_1, \ldots, x_n) : x_1 + \ldots + x_n \leq t \} \)
we have
\[ aHP_I(t) = \text{vol}(\Gamma_I \cap T), \quad t \gg 0. \] (4)

Now we present the first type of examples. First, take a “cross,” i.e., two intersecting lines. A cross is a complete intersection of type (1, 2) in \( \mathbb{P}^3 \). From the results of Mayes [19, Theorem 1.1 and Proposition 3.3] we have that for the ideal \( I \) of a cross in \( \mathbb{P}^3 \),
\[ \Gamma_I = T \times \mathbb{R}, \]
where \( T \) is the convex hull of \((0, 0), (1, 0), \) and \((0, 2)\) in \( \mathbb{R}^2 \). We compute the volume of \( \Gamma_I \) cut by the plane \( x + y + z = t \), i.e.,
\[ \int_T \int t - (x + y) dxdy, \]
and thus by equation (4) we obtain that the asymptotic Hilbert polynomial of a cross in \( \mathbb{P}^3 \) is
\[ aHP_I(t) = t - 1. \] (5)

In the examples below we will also need the following, rather obvious fact: if \( Z_I \) and \( Z_J \) are two disjoint sets given as zero sets of radical LBSR ideals \( I \) and \( J \), respectively, then
\[ aHP_{I \cap J} = aHP_I + aHP_J. \] (6)

The formula follows from the definitions of the Hilbert polynomial and the asymptotic Hilbert polynomial, from the exact sequence
\[ 0 \rightarrow J/I \cap J \rightarrow R/I \cap J \rightarrow (R/I \cap J)/(J/I \cap J) \simeq R/J \rightarrow 0 \]
and from the fact that for the ideals of disjoint sets \( R/I = (I + J)/I \simeq J/I \cap J \).

Consider \( s \) generic crosses in \( \mathbb{P}^3 \) with the ideal \( I_s \). Take the polynomial \( \Lambda_s \), see (5) and (6):
\[ \Lambda_s(t) = \frac{t^3}{6} - st(t - 1). \]

Denote the largest real root of \( \Lambda_s \) by \( \gamma_s \). For \( s = 2, 3, 4 \), we have that \( \gamma_2 = 2.76873... \), \( \gamma_3 = 3.60687... \), and \( \gamma_4 = 4.29021... \). Since \( \alpha(I_s) = s \) for \( s = 2, 3, 4 \), we have that \( \hat{\alpha}(I_s) \leq s \), which is less than the root of \( \Lambda_s \).

In case \( s = 5 \) the situation is different:

Example 8. Consider 5 generic crosses in \( \mathbb{P}^3 \) with the ideal \( I_5 \). It is an LBRs ideal since \( \text{dim}(R/I_5) \leq 2 \) (see [7], remarks following Definition 12).

We have that \( \Lambda_5(t) = \frac{t^3}{6} - 5(t - 1) \) and it is easy to observe that \( \alpha(I_5) = 5 \).

Assuming that there exists a subsequence of a suitable codepth, we compute that \( \gamma_{\Lambda_5^{(0)}} = 4.88447... \), \( \gamma_{\Lambda_5^{(1)}} = \sqrt{10} \), \( \gamma_{\Lambda_5^{(2)}} = 0 \). The latest root gives an absurd upper bound on \( \hat{\alpha}(I_5) \). Since \( \sqrt{10} < 4.88447... \) we obtain that \( \hat{\alpha}(I_5) \leq \gamma_{\Lambda_5^{(0)}} = 4.88447... \) without the help of a computer.

We were curious if we can get a better result using computer approach. We have checked that for \( m = 2, \ldots, 10 \) we still have \( \alpha(I_5^{(m)}) = 5m \) (and then the time of computations grows rapidly). However, combining the fact that \( \alpha(I_5^{(4)}) = 20 \) with Ein-Lazarsfeld-Smith result, gives a lower bound \( \hat{\alpha}(I_5) \geq \frac{40}{3} \) which is greater than \( \sqrt{10} \). Hence we eliminated \( \sqrt{10} \) as an upper bound, and, moreover, we have computed that for \( m \) big enough codepth \( (I_5^{(m)}) \) is equal to 3.
Remark 9. We may want to compute (with help of a computer) the expected initial degree \( e_{\alpha_m} \) of \( I^{(m)} \) (in the similar way as it was done for fat flats in [5]). This could be the way to bound \( \hat{\alpha} \) by finding the lowest possible term (or infimum) of \( e_{\alpha_m} \). There are two problems with this method. The first is that the expected degree \( e_{\alpha_m} \) may go down very slowly. For five crosses \( e_{\alpha_m} = 5 \) up to \( m = 12 \) and \( e_{\alpha_{13}} = \frac{64}{13} = 4.92307... > 4.88447... \).

The second problem is more important. We do not know if the expected initial degree is properly computed. The formula for a dimension of a system of forms of degree \( d \) vanishing along a given set with multiplicity \( m \) may be correct only for \( d \) big enough. We have an unpublished result that for a cross in \( \mathbb{P}^3 \) the formula is correct for \( d \geq 2m - 2 \).

Now we move to the second type of examples, concerning star configurations.

We begin with giving an experimentally found formula for the asymptotic Hilbert polynomial of a star configurations given in \( \mathbb{P}^n \) by intersecting every \( c \) out of \( s \) generic hyperplanes. We will denote the ideal of such a configuration by \( I_{c,s,n} \). From [6, Theorem 1.1] we know that \( \Gamma_{I_{c,s,n}} = \Gamma_{c,s,c} \times \mathbb{R}^{n-c} \), where \( \Gamma_{c,n,c} \) is a simplex in \( \mathbb{R}^c \) with vertices \( \frac{1}{c}, \frac{1}{c}, \ldots, \frac{1}{c}, s - (c - 1) \).

Using equation (4) we see that to compute \( \text{aHP}_{I_{c,s,n}}(t) \) it is enough to compute the volume of \( \Gamma_{I_{c,s,n}} \) cut by the plane \( x_1 + \ldots + x_n = t \).

Denote \( a_1 = \frac{s}{c}, a_2 = \frac{s - 1}{c - 1}, \ldots, a_c = s - (c - 1) \).

The volume is the integral

\[
\int_0^t \cdots \int_0^t \int_0^{a_{c+1}-1} \cdots \int_0^{a_2-1} \cdots \int_0^{a_1-1} \cdots (t - x_1 - \ldots - x_n - 1) \, dx_n - 1,
\]

where \( A = -\frac{a_1}{a_1} - \cdots - \frac{a_{c-1}}{a_{c-1}} + a_c \). By computing the integral for small values of \( n \) and \( c \) we found the formula

\[
\text{aHP}_{I_{c,s,n}}(t) = a_1 \cdot a_2 \cdots a_c \cdot \frac{(n-c)!}{n!} \cdot \left( n \choose 0 \right) (-1)^{n-c} \sum_{t=0}^{n-c} \frac{1}{t} + \left( n \choose 1 \right) (-1)^{n-c-1} \sum_{t=2}^{n-c-2} t^2 + \ldots + \left( n \choose n-c \right) (-1)^0 \sum_{t=0}^{n-c} t^{n-c},
\]

where the sum \( \sum t^k \) denotes the sum of all monomials of degree \( k \) in variables \( a_i \). So far we are not able to prove the formula.

The next, important example shows that it is necessary to take an appropriate derivative of the polynomial \( \Lambda_I \).

Example 10. Take star configurations of lines in \( \mathbb{P}^4 \), given by \( s \)-hyperplanes, \( s \geq 4 \), with the ideal \( I_{3,s,4} \). By [14, Theorem 3.1] such an ideal and each of its symbolic powers is arithmetically Cohen-Macaulay and therefore codepth \( I_{3,s,4}^{(m)} = 3 \). From
the considerations above we have that
\[
\Lambda_{I_{3,s,t}}(t) = \frac{t^4}{24} - \text{aHP}_{I_{3,s,t}}(t)
\]
\[
= \frac{t^4}{24} + \frac{1}{864}(-30s + 67s^2 - 48s^3 + 11s^4) + \frac{1}{864}(-48s + 72s^2 - 24s^3)t.
\]
It is easy to see (with computer help) that the polynomial \(\Lambda_{I_{3,s,t}}(t)\) has no real zeros for \(s \geq 4\). Indeed, the value at the zero of the derivative (minimum point) is greater than 0.012 for \(s \geq 10\), and for \(s = 4, \ldots, 9\) we see by a direct check that the value at the minimum point is positive.

The derivative of \(\Lambda_{I_{3,s,t}}(t)\) has a root. This root is approximately equal to \(\frac{s}{\sqrt{6}}\).

Theorem 12. Let \(n, r, s\) be integers with \(n \geq 2r + 1, r \geq 0\) and \(s \geq 1\). Let \(I\) be the ideal of \(s\) disjoint \(r\)-planes in \(\mathbb{P}^n\). Then the polynomial \(\Lambda_I(t)\) has a single real root bigger than or equal to 1. Denote this largest real root by \(\gamma_I\). Then \(\hat{\alpha}(I) \leq \gamma_I\).

In particular the theorem holds for one linear subspace of codimension \(n - r\). Observe that it is in a sense accidental that in this case \(\hat{\alpha}(I)\) is bounded from above by the root of \(\Lambda_I\), as according to Theorem 7 we should take the largest root of the \(r\)-th derivative of the polynomial.

Next, we formulate a problem that may be viewed as a generalization of Nagata-type conjectures (see [5]). Note that one may generalize the conjecture of Nagata asking if there exists a number \(N_0\) such that for \(s \geq N_0\) the Waldschmidt constant of the ideal of \(s\) generic linear subspaces on \(\mathbb{K}[\mathbb{P}^n]\) is maximal possible. The original Nagata conjecture says that \(N_0 = 10\) for \(\mathbb{P}^2\), and the maximal possible value of \(\hat{\alpha}\) is \(\sqrt{s}\), i.e., the largest root of the polynomial \(\Lambda\) for these points.

Conjecture 13. Take a radical ideal \(I\) in \(\mathbb{K}[\mathbb{P}^n]\). Take the ideal
\[
J = \bigcap_{j=1}^{s} \phi_j(V(I)),
\]
where \(\phi_j, j = 1, \ldots, s\), is a generic change of coordinates. Then, for \(s\) big enough, codepth \(J^{(m)} = n\) and the Waldschmidt constant of \(J\) is the maximal possible, i.e., equal to the largest root of the polynomial \(\Lambda_J(t) = \frac{t^n}{m!} - s \cdot \text{aHP}_J(t)\).

References

[1] Th. Bauer, S. Di Rocco, B. Harbourne, M. Kapustka, A. L. Knutsen, W. Syzdek and T. Szemberg, A primer on Seshadri constants, in Interactions of Classical and Numerical Algebraic Geometry, Proceedings of a conference in honor of A. J. Sommese, held at Notre Dame, May 22–24 2008 (eds. D. J. Bates, G.-M. Besana, S. Di Rocco and C. W. Wampler), Contemporary Mathematics, 496, American Mathematical Society, Providence, RI, 2009, 362pp. MR 2555945
[2] C. Bocci, S. Cooper and B. Harbourne, Containment results for ideals of various configurations of points in \(\mathbb{P}^N\), J. Pure Appl. Algebra, 218 (2014), 65–75. MR 3120609
[3] C. Bocci and B. Harbourne, Comparing powers and symbolic powers of ideals, J. Algebraic Geometry, 19 (2010), 399–417. MR 2629595
[4] C. Bocci and B. Harbourne, The resurgence of ideals of points and the containment problem, Proc. Amer. Math. Soc., 138 (2010), 1175–1190. MR 2578512
18 MARCIN DUMNICKI, ŁUCJA FARNIK, AND HALSZKA TUTAJ-GASIŃSKA

[5] M. Dumnicki, B. Harbourne, T. Szemberg and H. Tutaj-Gasińska, Linear subspaces, symbolic powers and Nagata type conjectures, Adv. Math., 252 (2014), 471–491. MR 3144238
[6] M. Dumnicki, T. Szemberg, J. Szpond and H. Tutaj-Gasińska, Symbolic generic initial systems of star configurations, J. Pure Appl. Algebra, 219 (2015), 1073-1081. MR 3282126
[7] M. Dumnicki, J. Szpond and H. Tutaj-Gasińska, Asymptotic Hilbert polynomials and limiting shapes, J. Pure Appl. Algebra, 219 (2015), 4446–4457. MR 3346500
[8] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier (Grenoble), 56 (2006), 1701–1734. MR 2282673
[9] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, Restricted volumes and base loci of linear series, Amer. J. Math., 131 (2009), 607–651. MR 2530849
[10] L. Ein, R. Lazarsfeld and K. Smith, Uniform bounds and symbolic powers on smooth varieties, Invent. Math., 144 (2001), 241–252. MR 1826369
[11] D. Eisenbud, Commutative Algebra. With a View Toward Algebraic Geometry, Springer-Verlag, New York, 1995. MR 1322960
[12] H. Esnault and E. Viehweg, Sur une minoration du degré d’hypersurfaces s’annulant en certains points, Math. Ann., 263 (1983), 75–86. MR 697332
[13] A. Galligo, À propos du théorème de préparation de Weierstrass, in Fonctions de Plusieurs Variables Complexes, Lecture Notes in Math., Vol. 409, Springer, Berlin, 1974, 543–579. MR 0402102
[14] A. V. Geramita, B. Harbourne and J. Migliore, Star configurations in $\mathbb{P}^n$, J. Algebra, 376 (2013), 279–299. MR 3003727
[15] M. L. Green, Generic initial ideals, in Six Lectures on Commutative Algebra, Progr. Math., 166, Birkhäuser, Basel, 1998, 119–186. MR 1648665
[16] J. Herzog and H. Srinivasan, Bounds for multiplicities, Trans. Amer. Math. Soc., 350 (1998), 2879–2902. MR 1458304
[17] H. Hochster and C. Huneke, Comparison of symbolic and ordinary powers of ideals, Invent. Math., 147 (2002), 349–369. MR 1881923
[18] S. Mayes, The asymptotic behaviour of symbolic generic initial systems of generic points, J. Pure Appl. Alg., 218 (2014), 381–390. MR 3124204
[19] S. Mayes, The limiting shape of the generic initial system of a complete intersection, Comm. Algebra, 42 (2014), 2299–2310. MR 3169705
[20] M. Mustaţă, On multiplicities of graded sequences of ideals, J. Algebra, 256 (2002), 229–249. MR 1936888
[21] S. Sullivant, Combinatorial symbolic powers, J. Algebra, 319 (2008), 115–142. MR 2378064

E-mail address: Marcin.Dumnicki@uj.edu.pl

E-mail address: Lucja.Farnik@gmail.com

E-mail address: Halszka.Tutaj-Gasinska@uj.edu.pl

Jagiellonian University, Faculty of Mathematics and Computer Science, Łojasiewicza 6, PL-30-348 Kraków, Poland