ON $L_1$-WEAK ERGODICITY OF NONHOMOGENEOUS DISCRETE MARKOV PROCESSES AND ITS APPLICATIONS

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Abstract. In the present paper we investigate the $L_1$-weak ergodicity of nonhomogeneous discrete Markov processes with general state spaces. Note that the $L_1$-weak ergodicity is weaker than well-known weak ergodicity. We provide a necessary and sufficient condition for such processes to satisfy the $L_1$-weak ergodicity. Moreover, we apply the obtained results to establish $L_1$-weak ergodicity of discrete time quadratic stochastic processes. As an application of the main result, certain concrete examples are also provided.

Keywords: weak ergodicity; nonhomogeneous discrete Markov process; the Dobrushin’s Condition; quadratic stochastic process.

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1. Introduction

Markov processes with general state space have become a subject of interest due to their applications in many branches of mathematics and natural sciences. One of the important notions in these studies is ergodicity of Markov processes, i.e. the tendency for a chain to forget the distant past. In many cases, a huge number of investigations were devoted to such processes with countable state space (see for example, [1]-[7],[8],[18]). For nonhomogeneous Markov processes with countable state space, investigation of the general conditions of weak ergodicity leads to the definition of a special subclass of regular matrices. In many papers (see for example, [6, 11, 15, 18]) the weak ergodicity of nonhomogeneous Markov process are given in terms of Dobrushin’s ergodicity coefficient [1]. In general case, one may consider several kinds of convergence [10]. In [19] some sufficient conditions for weak and strong ergodicity of nonhomogeneous Markov processes are given and estimates of the rate of convergence are proved. Lots of papers were devoted to the investigation of ergodicity of nonhomogeneous Markov chains (see, for example [1]-[7],[18],[20]).

In the present paper we are going to investigate the $L_1$-weak ergodicity of nonhomogeneous discrete Markov processes, in general state spaces, without using Dobrushin’s ergodicity coefficient. Note that the $L_1$-weak ergodicity is weaker than usual weak ergodicity (see next section). We shall provide necessary and sufficient conditions for such processes to satisfy the $L_1$-weak ergodicity. As application of the main result, certain concrete examples are provided. Note that in [1] similar conditions were found for nonhomogeneous Markov processes to satisfy weak ergodicity. It is worth to mention that in [17] a necessary and sufficient condition was found for
homogeneous Markov processes to satisfy $L_1$-ergodicity. Our condition recovers the mentioned condition when the processes is homogeneous. Moreover, we will provide some applications of the main result to $L_1$-weak ergodicity of discrete quadratic stochastic processes which improves the result of [16]. Note that such processes relate to quadratic operators [9] as Markov processes relate to linear operators. For the recent review on quadratic operator we refer to [5].

2. $L_1$-Weak ergodicity

Let $(X, \mathcal{F}, \mu)$ be a probability space. In what follows, we consider the standard $L^1(X, \mathcal{F}, \mu)$ and $L^\infty(X, \mathcal{F}, \mu)$ spaces. Note that $L^1(X, \mathcal{F}, \mu)$ can be identified with the space of finite signed measures on $X$ absolutely continuous with respect to $\mu$. By $\mathfrak{M}$ we denote the set of all probability measures on $X$ which are absolutely continuous w.r.t. $\mu$. Recall that transition probabilities $P^{[k,m]}(x,A), x \in X, A \in \mathcal{F}$, form a nonhomogeneous discrete Markov process (NHDMP) iff the following conditions are satisfied:

1. for each $k, n$ the function of two variables $P^{[k,n]}(x,A)$ is a Markov kernel, and it is $\mu$-measurable, i.e. $\mu(A) = 0$ implies $P^{[k,n]}(x,A) = 0$ a.e. on $X$.
2. one has Kolmogorov-Chapman equation: for every $k \leq m \leq n$

\begin{equation}
P^{[k,n]}(x,A) = \int P^{[k,m]}(x,dy)P^{[m,n]}(y,A).
\end{equation}

In the sequel, we will deal with $\mu$-measurable NHDMP. In this case, for each $k, n$ such one can define a positive linear contraction operator on $L^1$ (resp. $L^\infty$) denoted by $P_{\nu}^{[k,n]}$ (resp. $P^{[k,n]}$). Namely,

\begin{equation}
(P_{\nu}^{[k,n]}\nu)(A) = \int P^{[k,n]}(x,A)d\nu(x), \quad \nu \in L^1
\end{equation}

\begin{equation}
(P^{[k,n]}f)(x) = \int P^{[k,n]}(x,dy)f(y), \quad f \in L^\infty.
\end{equation}

It is clear that $\|P_{\nu}^{[k,n]}\nu\|_1 = \|\nu\|_1$ for every positive measure $\nu \in L^1$.

From (2.2) it follows that (2.1) can be rewritten as follows

\[ P_{\nu}^{[k,n]} = P_{\nu}^{[m,n]}P_{\nu}^{[k,m]} \]

where $k \leq m \leq n$.

Recall that if for a NHDMP $P^{[k,n]}(x,A)$ one has $P_{\nu}^{[k,n]} = (P_{\nu}^{[0,1]})^{n-k}$, then such process becomes homogeneous, and therefore, it is denoted by $P^n(x,A)$.

**Definition 2.1.** A NHDMP $P^{[k,n]}(x,A)$ is said to satisfy

(i) the weak ergodicity if for any $k \in \mathbb{Z}_+$ one has

\[ \lim_{n \to \infty} \sup_{x,y \in X} \|P^{[k,n]}(x,\cdot) - P^{[k,n]}(y,\cdot)\|_1 = 0; \]
(ii) the $L_1$-weak ergodicity if for any probability measures $\lambda, \nu \in \mathcal{M}$ and $k \in \mathbb{Z}_+$ one has
\[
\lim_{n \to \infty} \|P_{k,n}^* \lambda - P_{k,n}^* \nu\|_1 = 0;
\]
(iii) the strong ergodicity if there exists a probability measure $\mu_1$ such that for every $k \in \mathbb{Z}_+$ one has
\[
\lim_{n \to \infty} \sup_{x \in X} \|P_{k,n}(x, \cdot) - \mu_1\|_1 \to 0;
\]
(iv) the $L_1$-strong ergodicity if there exists a probability measure $\mu_1$ such that for every $k \in \mathbb{Z}_+$ and $\lambda \in \mathcal{M}$ one has
\[
\lim_{n \to \infty} \|P_{k,n}^* \lambda - \mu_1\|_1 \to 0.
\]
One can see that the weak (resp. strong) ergodicity implies the $L_1$-weak (resp. $L_1$-strong) ergodicity. Indeed, let us consider the following example.

**Example.** Let $X = \{1, 2, 3, 4\}$ and $\mu = (1/2, 1/2, 0, 0)$. In this case, the set $\mathcal{M}$ coincides with $\{(\alpha, 1-\alpha, 0, 0) : \alpha \in [0,1]\}$. Consider stochastic matrix
\[
\mathbb{P} = \begin{pmatrix} p & q & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad p \in (0,1), \ p + q = 1,
\]
which is clearly $\mu$-measurable. One can check that for any $\lambda \in \mathcal{M}$ (i.e. $\lambda = (\alpha, 1-\alpha, 0, 0), \alpha \in [0,1]$) we have
\[
\mathbb{P}_n^* \lambda \to (1/2, 1/2, 0, 0) \quad \text{as} \quad n \to \infty,
\]
this means $\mathbb{P}$ satisfies the $L_1$-strong ergodicity. On the other hand, the matrix $\mathbb{P}$ has another two invariant measures, i.e.
\[
\mu_1 = (0, 0, 1, 0), \quad \mu_2 = (0, 0, 0, 1)
\]
which implies that $\mathbb{P}$ is not strong ergodic.

Therefore, it is natural to find certain necessary and sufficient conditions for the satisfaction $L_1$-weak ergodicity of NHDMP. So, in the paper we will deal with $L_1$-weak ergodicity. Note that historically, one of the most significant conditions for the weak ergodicity is the Doeblin’s Condition (for homogeneous Markov processes), which is formulated as follows: there exist a probability measure $\nu$, an integer $n_0 \in \mathbb{N}$ and constants $0 < \varepsilon < 1, \delta > 0$ such that for every $A \in \mathcal{F}$ if $\nu(A) > \varepsilon$ then
\[
\inf_{x \in X} P_{n_0}^n(x, A) \geq \delta.
\]
Such a condition does not imply either the aperiodicity or the ergodicity of the process. In [13] the aperiodicity is studied by minorization type conditions, i.e. there exist a non-trivial positive measure $\lambda$ and $n_0 \in \mathbb{N}$ such that
\[
P_{n_0}^n(x, A) \geq \lambda(A), \quad \forall x \in X, \forall A \in \mathcal{F}.
\]
But this condition is not sufficient for the strong ergodicity. In [17] it was introduced a variation of the above condition, i.e. Condition (C₀): there exists a non-trivial positive measure \( \mu_0 \in L^1 \), \( \|\mu_0\|_1 \neq 0 \), and for every \( \lambda \in \mathcal{M} \) one can find a sequence \( \{X_n\} \subset \mathcal{F} \) with \( \mu(X \setminus X_n) \to 0 \), as \( n \to \infty \), and \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) one has\(^1\)

\[
P^n \lambda \geq \mu_0 1_{X_n},
\]

where \( 1_Y \) stands for the indicator function of a set \( Y \). It has been proved that such a condition is necessary and sufficient for the \( L^1 \)-strong ergodicity of homogeneous processes. In the present paper we shall introduce a simple variation of the above condition (C₀) for NHDMP, and prove that the introduced condition is a necessary and sufficient for the \( L^1 \)-weak ergodicity. Note that another direction of variation of the Doeblin’s Condition has been studied in [2].

3. Main results

In this section we are going to introduce a simple variation of condition (C₀).

Definition 3.1. We say that a NHDMP \( P^{[k,n]}(x, A) \) given on \( (X, \mathcal{F}, \mu) \) satisfies condition (C₁) if for each \( k \in \mathbb{Z}_+ \) there exist a positive measure \( \mu_k \in L^1 \), \( \|\mu_k\|_1 \neq 0 \), and for every \( \delta > 0 \) and \( \lambda, \nu \in \mathcal{M} \) one can find sets \( X_k, Y_k \in \mathcal{F} \) with \( \mu(X \setminus X_k) < \delta \), \( \mu(X \setminus Y_k) < \delta \) and an integer \( n_k \in \mathbb{N} \) such that

\[
P^{[k,k+n_k]} \lambda \geq \mu_k 1_{X_k}, \quad P^{[k,k+n_k]} \nu \geq \mu_k 1_{Y_k},
\]

here as before \( 1_Y \) stands for the indicator function of a set \( Y \).

Remark 3.2. In (3.18), (3.1) without loss of generality we may assume that \( \|\mu_k\|_1 < 1/2 \), otherwise we will replace \( \mu_k \) with \( \mu'_k = \mu_k/2 \).

Proposition 3.3. Let a NHDMP \( P^{[k,n]}(x, A) \) given on \( (X, \mathcal{F}, \mu) \). Then for the following assertions

(i) \( P^{[k,n]}(x, A) \) satisfies condition (C₁);

(ii) for any \( \lambda, \nu \in \mathcal{M} \) and \( k \in \mathbb{Z}_+ \) there is a sequence \( \{n_i\} \) such that for all \( n \geq K_\ell := \sum_{i=1}^\ell n_i \) \( (K_0 = k) \) one has\(^2\)

\[
\|P^{[k,n]} \lambda - P^{[k,n]} \nu\|_1 = \left( \prod_{i=1}^\ell \gamma_i \right) \|P^{[K_\ell,n]} \lambda - P^{[K_\ell,n]} \nu\|_1, \quad \gamma_i \in \mathcal{M}, \quad \ell \in \mathbb{N},
\]

where \( \lambda_\ell, \nu_\ell \in \mathcal{M} \), and

\[
\frac{1}{2} \leq \gamma_i \leq 1 - \frac{\|\mu_{K_i-1}\|_1}{2}, \quad i = 1, \ldots, \ell.
\]

the implication hold true: (i)⇒(ii).

\(^1\)Here and in what follows, a given \( B \in \mathcal{F} \) the measure \( \mu 1_B \) is defined by \( \mu 1_B(Y) = \mu(Y \cap B) \) for any \( Y \in \mathcal{F} \).
Proof. Take any \( \lambda, \nu \in M \) and fix \( k \in \mathbb{Z}_+ \). Let us prove (3.2) by induction. Due to condition \((C_1)\) there is a measure \( \mu_k \). Then according to absolute continuity of Lebesgue integral, there is \( \delta_1 > 0 \) such that for any \( Z \in \mathcal{F} \) with \( \mu(Z) < 2\delta_1 \) one has

(3.4) \[
\int \mu_k 1_Z d\mu < \frac{\|\mu_k\|_1}{2}.
\]

Now again due to condition \((C_1)\) there are \( X_1, Y_1 \subset \mathcal{F} \) and \( n_1 \in \mathbb{N} \) such that one has \( \max\{\mu(X \setminus X_1), \mu(X \setminus Y_1)\} < \delta \) and

(3.5) \[
P_*^{[k,k+n_1]} \lambda \geq \mu_k 1_{X_1}, \quad P_*^{[k,k+n_1]} \nu \geq \mu_k 1_{Y_1}.
\]

Denoting \( Z_1 = X_1 \cap Y_1 \), one has \( \mu(X \setminus Z_1) < 2\delta \), and from (3.5) we find

(3.6) \[
P_*^{[k,k+n_1]} \lambda \geq \mu_k 1_{Z_1}, \quad P_*^{[k,k+n_1]} \nu \geq \mu_k 1_{Z_1}.
\]

It follows from (3.6) that

\[
\|P_*^{[k,k+n_1]} \lambda - \mu_k 1_{Z_1}\|_1 = \int (P_*^{[k,k+n_1]} \lambda - \mu_k 1_{Z_1}) d\mu
\]

\[
= \int P_*^{[k,k+n_1]} \lambda d\mu - \int \mu_0 1_{Z_1} d\mu
\]

\[
= 1 - \int \mu_0 1_{Z_1} d\mu
\]

\[
= \int P_*^{[k,k+n_1]} \nu d\mu - \int \mu_0 1_{Z_1} d\mu
\]

\[
= \|P_*^{[k,k+n_1]} \nu - \mu_0 1_{Z_1}\|_1.
\]

(3.7)

Therefore, let us denote

\[
\gamma_1 = \|P_*^{[k,k+n_1]} \lambda - \mu_k 1_{Z_1}\|_1.
\]

One can see that

(3.8) \[
1 - \int \mu_k 1_{Z_1} d\mu \geq 1 - \int \mu_k d\mu \geq \frac{1}{2}.
\]

Due to \( \mu(X \setminus Z_1) < 2\delta_1 \) from (3.11) we have

\[
\frac{1}{2} \int \mu_k d\mu \geq \int \mu_k 1_{X \setminus Z_1} d\mu = \int \mu_k d\mu - \int \mu_k 1_{Z_1} d\mu
\]

which yields

\[
\int \mu_k 1_{Z_1} d\mu \geq \frac{\|\mu_k\|_1}{2}.
\]

Therefore, one finds

(3.9) \[
1 - \int \mu_k 1_{Z_1} d\mu \leq 1 - \frac{\|\mu_k\|_1}{2}.
\]

Hence, from (3.8), (3.9) we infer

\[
\frac{1}{2} \leq \gamma_1 \leq 1 - \frac{\|\mu_k\|_1}{2}
\]
Thus, at \( n \geq k + n_1 \) we obtain
\[
\|P^{[k,n]} - P^{[k,n_1]}\|_1 = \|P^{[k,n+1,n]}(P^{[k,k+n]} - \mu_k 1_{Z_k}) - P^{[k,n]}\|_1,
\]
where
\[
\lambda_1 = \frac{1}{\gamma_1} (P^{[k,k+n]} - \mu_k 1_{Z_k})
\]
and
\[
\nu_1 = \frac{1}{\gamma_1} (P^{[k,k+n]} - \mu_k 1_{Z_k}).
\]
It is clear that \( \lambda_1, \nu_1 \in \mathcal{M} \), so we have proved (3.2) for \( \ell = 1 \).

Now assume that (3.2) holds for \( i = \ell \), i.e. there are numbers \( \{n_i\} \) such that for any \( n \geq K_\ell := \sum_{i=1}^{\ell} n_i \) one has
\[
\|P^{[k,n]} - P^{[k,n]}\|_1 = \|\prod_{i=1}^{\ell} \gamma_i \|_1,
\]
where \( \lambda_\ell, \nu_\ell \in \mathcal{M} \), and
\[
\frac{1}{2} \leq \gamma_i \leq 1 - \frac{\|\mu_{K_{i-1}}\|_1}{2}, \quad i = 1, \ldots, \ell.
\]

Let us prove (3.2) at \( i = \ell + 1 \). According to condition \((C_1)\) there is a positive measure \( \mu_{K_\ell} \). One can find \( \delta_{\ell+1} > 0 \) such that for any \( Z \in \mathcal{F} \) with \( \mu(Z) < 2\delta_{\ell+1} \) one has
\[
\int \mu_{K_\ell} 1_Z d\mu < \frac{\|\mu_{K_\ell}\|_1}{2}.
\]

For \( \lambda_\ell \) and \( \nu_\ell \) from condition \((C_1)\) one finds \( X_\ell + 1, Y_\ell + 1 \subset \mathcal{F} \) and \( n_{\ell+1} \in \mathbb{N} \) such that one has
\[
\max \{ \mu(X \setminus X_\ell + 1), \mu(X \setminus Y_\ell + 1) \} < \delta_{\ell+1}
\]
and
\[
P^{[K_\ell,K_\ell + n_{\ell+1}]} \lambda_\ell \geq \mu_{K_\ell} 1_{X_{\ell+1}}, \quad P^{[K_\ell,K_\ell + n_{\ell+1}]} \nu_\ell \geq \mu_{K_\ell} 1_{Y_{\ell+1}}.
\]
Denote \( Z_{\ell+1} = X_{\ell+1} \cap Y_{\ell+1} \), then one can see that \( \mu(X \setminus Z_{\ell+1}) < 2\delta_{\ell+1} \) and
\[
P^{[K_\ell,K_\ell + n_{\ell+1}]} \lambda_\ell \geq \mu_{K_\ell} 1_{Z_{\ell+1}}, \quad P^{[K_\ell,K_\ell + n_{\ell+1}]} \nu_\ell \geq \mu_{K_\ell} 1_{Z_{\ell+1}}.
\]
Denoting \( K_{\ell+1} = K_\ell + n_{\ell+1} \), and similarly to (3.7) we get
\[
\|P^{[K_\ell,K_\ell + K_{\ell+1}]} - \mu_{K_\ell} 1_{Z_{\ell+1}}\|_1 = \|P^{[K_\ell,K_\ell + K_{\ell+1}]} \nu_\ell - \mu_{K_\ell} 1_{Z_{\ell+1}}\|_1
\]
\[
= 1 - \int \mu_{K_\ell} 1_{Z_{\ell+1}} d\mu
\]
Denote
\[
\gamma_{\ell+1} = 1 - \int \mu_{K_\ell} 1_{Z_{\ell+1}} d\mu,
\]
hence using $\mu(X \setminus Z_{\ell+1}) < 2\delta_{\ell+1}$ and the same argument as \(3.8\), \(3.9\) one finds
\[
\frac{1}{2} \leq \gamma_{\ell+1} \leq 1 - \frac{\|\mu_{K_{\ell}}\|_1}{2}.
\]

Now at $n \geq K_{\ell+1}$ we get
\[
\|P_{k,n}^{[K_{\ell+1},n]} \lambda_{\ell+1} - P_{k,n}^{[K_{\ell+1},n]} \nu_{\ell+1}\|_1 = \|P_{k,n}^{[K_{\ell+1},n]} (P_{k,n}^{[K_{\ell+1},n]} \lambda_{\ell+1} - \mu_{K_{\ell+1}} 1_{Z_{\ell+1}}) - P_{k,n}^{[K_{\ell+1},n]} (P_{k,n}^{[K_{\ell+1},n]} \nu_{\ell+1} - \mu_{K_{\ell+1}} 1_{Z_{\ell+1}})\|_1
\]
\[
= \gamma_{\ell+1} \|P_{k,n}^{[K_{\ell+1},n]} \lambda_{\ell+1} - P_{k,n}^{[K_{\ell+1},n]} \nu_{\ell+1}\|_1,
\]
where
\[
\lambda_{\ell+1} = \frac{1}{\gamma_{\ell+1}} (P_{k,n}^{[K_{\ell+1},n]} \lambda_{\ell+1} - \mu_{K_{\ell+1}} 1_{Z_{\ell+1}})
\]
\[
\nu_{\ell+1} = \frac{1}{\gamma_{\ell+1}} (P_{k,n}^{[K_{\ell+1},n]} \nu_{\ell+1} - \mu_{K_{\ell+1}} 1_{Z_{\ell+1}}).
\]

It is clear that $\lambda_{\ell+1}, \nu_{\ell+1} \in \mathcal{M}$. Thus, taking into account \(3.10\) we derive the desired equality. \(\Box\)

Next theorem shows that condition $(C_1)$ is equivalent to the satisfaction of the $L_1$-weak ergodicity of NHDDMP.

**Theorem 3.4.** Let a NHDDMP $P^{[k,n]}(x, A)$ be given on $(X, \mathcal{F}, \mu)$. Then the following assertions are equivalent

(i) $P^{[k,n]}(x, A)$ satisfies condition $(C_1)$ with

\[
\sum_{n=1}^{\infty} \|\mu_{k,n}\|_1 = \infty
\]

for any increasing subsequence $\{k_n\}$ of $\mathbb{N}$.

(ii) $P^{[k,n]}(x, A)$ satisfies the $L_1$-weak ergodicity.

**Proof.** $(i) \Rightarrow (ii)$. Then due to Proposition 3.3 there is a subsequence $\{K_{\ell}\}$ such that

\[
\|P_{k,n}^{[k,n]} \lambda - P_{k,n}^{[k,n]} \nu\|_1 = \left(\prod_{i=1}^{\ell} \gamma_i\right) \|P_{k,n}^{[k,n]} \lambda - P_{k,n}^{[k,n]} \nu\|_1,
\]

where $\lambda_{\ell}, \nu_{\ell} \in \mathcal{M}$. Now from \(3.3\) one gets

\[
\|P_{k,n}^{[k,n]} \lambda - P_{k,n}^{[k,n]} \nu\|_1 \leq 2 \prod_{i=1}^{\ell} \left(1 - \frac{\|\mu_{K_{i-1}}\|_1}{2}\right)
\]

According to \(3.13\) we get the desired assertion.

Now consider the implication $(ii) \Rightarrow (i)$. Fix $1 > \varepsilon > 0$. Then given $k \in \mathbb{N}$ and $\lambda, \mu_0 \in \mathcal{M}$, (here $\mu_0$ is fixed) one has

\[
\|P_{k,n}^{[k,n]} \lambda - P_{k,n}^{[k,n]} \mu_0\|_1 \to 0 \text{ as } n \to \infty.
\]
Then there is a sequence \( \{Y_n\} \subset \mathcal{F} \) such that \( \mu(X \setminus Y_n) \to 0 \), as \( n \to \infty \), and 
\[
\| (P_*^{[k,n]} \lambda - P_*^{[k,n]} \mu_0) 1_{Y_n} \|_\infty \to 0 \quad \text{as} \quad n \to \infty.
\]
Therefore, there exists an \( n_k \in \mathbb{N} \) such that \( \mu(X \setminus Y_{n_k}) < \varepsilon \) and
\[
(3.15) \quad \| (P_*^{[k,k+n_k]} \lambda - P_*^{[k,k+n_k]} \mu_0) 1_{Y_{n_k}} \|_\infty < \frac{\varepsilon}{2}
\]
Now denote \( \nu_k = P_*^{[k,k+n_k]} \mu_0 \). Hence, from (3.15) we get
\[
P_*^{[k,k+n_k]} \lambda \geq P_*^{[k,k+n_k]} \mu_0 1_{Y_{n_k}} \geq \nu_k 1_{Y_{n_k}} - \frac{\varepsilon}{2} 1_{Y_{n_k}} \geq \mu_k 1_{Y_{n_k}},
\]
where
\[
\mu_k = \frac{1}{2} \nu_k 1_{A_k}, \quad A_k = \left\{ x \in X : \nu_k(x) \geq \frac{\varepsilon}{2} \right\}.
\]
Since \( \nu_k \) is a probability measure, therefore, we have \( 0 < \| \mu_k \|_1 \leq 1/2 \), so
\[
1 - \frac{\| \mu_k \|_1}{2} \geq \frac{3}{4}.
\]
Hence, this completes the proof. \( \square \)

If one takes \( n_k = k + 1 \) in condition \( C_1 \), then we get the following

**Corollary 3.5.** Let \( P^{[k,n]}(x,A) \) be a NHDP on \( (X, \mathcal{F}, \mu) \). If for each \( k \in \mathbb{Z}^+ \) there exist a positive measure \( \mu_k \in L^1, \| \mu_k \|_1 \neq 0 \), and for every \( \delta > 0 \) and \( \lambda \in \mathcal{M} \) one can find a set \( X_k \in \mathcal{F} \) with \( \mu(X \setminus X_k) < \delta \) such that
\[
(3.16) \quad P_*^{[k,k+1]} \lambda \geq \mu_k 1_{X_k},
\]
with
\[
(3.17) \quad \sum_{n=1}^{\infty} \| \mu_n \|_1 = \infty
\]
then the NHDP satisfies the \( L_1 \)-weak ergodicity.

Now let us consider a nonhomogeneous version of condition \( (C_0) \). Namely, a NHDP \( P^{[k,n]}(x,A) \) given on \( (X, \mathcal{F}, \mu) \) is said to satisfy condition \( (C_2) \) if for each \( k \in \mathbb{Z}^+ \) there exists a positive measure \( \mu_k \in L^1, \| \mu_k \|_1 \neq 0 \), and for every \( \lambda \in \mathcal{M} \) one can find a sequence \( \{X_n^{(k)}\} \subset \mathcal{F} \) with \( \mu(X \setminus X_n^{(k)}) \to 0 \), as \( n \to \infty \), and \( n_0(\lambda,k) \in \mathbb{N} \) such that for all \( n \geq n_0(\lambda,k) \) one has
\[
(3.18) \quad P_*^{[k,n]} \lambda \geq \mu_k 1_{X_n^{(k)}};
\]
From Proposition (3.3) and Theorem 3.4 we immediately see that condition \( (C_2) \) with (3.13) is sufficient for the \( L_1 \)-weak ergodicity. One the other hand, if NHDP becomes homogeneous then condition \( (C_2) \) reduces to \( C_0 \), but in [17] it has been proved that the last condition (i.e. (2.4)) is equivalent to the \( L_1 \)-strong ergodicity.
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of the homogeneous process. Therefore, one can formulate the following:

**Problem.** Is Condition ($C_2$) with (3.13) necessary for the $L_1$-weak ergodicity?

4. Applications

In this section we provide some application of the main result for concrete cases.

4.1. **Discrete case.** Let us consider a countable state space NHDM. Namely, let $X = \mathbb{N}$ and $\mu$ be the Poisson measure. Then NHDM can be given in a form of stochastic matrices $\{p_{i,j}^{[k,n]}\}_{i,j \in \mathbb{N}}$.

**Theorem 4.1.** Let $\{p_{i,j}^{[k,n]}\}_{i,j \in \mathbb{N}}$ be a NHDM. If there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$, $0 \leq \lambda_n \leq 1$ satisfying

\[(4.1)\]

$$\sum_{n=1}^{\infty} (1 - \lambda_n) = \infty$$

and such that for some sequence of states $\{n_k\}$

\[(4.2)\]

$$p_{i,n_k}^{[k-1,k]} \geq \lambda_k \quad \text{for all } i, k \in \mathbb{N},$$

then the NHDM satisfies the $L_1$-weak ergodicity.

**Proof.** Now we show that the process satisfies the condition ($C_1$). Indeed, for each $k \in \mathbb{Z}$ we first define a measure $\mu_i^{(k)}$ on $X$ as follows:

$$\mu_i^{(k)} = \begin{cases} \lambda_k, & i = n_k \\ 0, & i \neq n_k \end{cases}$$

It is clear that $\|\mu_i^{(k)}\|_1 \neq 0$. From (4.2) it follows that

\[(4.3)\]

$$P_{i,j}^{[k-1,k]} \geq \mu_j^{(k)}, \quad \text{for all } i, j \in \mathbb{N}.$$  

Now take any $\nu \in \mathcal{M}$ and each $k \in \mathbb{Z}_+$ we put $X_k = X$, then from (4.3) one finds

$$P_{i,j}^{[k-1,k]} \nu \geq \mu_j^{(k)} \quad \text{for all } k \in \mathbb{N}.$$  

Hence, the condition ($C_1$) is satisfied. So, taking into account (4.1), from Corollary 3.3 we get the desired assertion.  

We note that the proved theorem extends a result of [4, 15].

**Example.** Let us consider more concrete examples. Assume that the transition probability $p_{i,j}^{[k,k+1]}$ is defined by

\[(4.4)\]

$$p_{i,j}^{[k,k+1]} = q_{i,j}^{(k)} \lambda_{k,j} + r_{k,i} \delta_{ij}, \quad i, j \in \mathbb{N}, \quad k \in \mathbb{N},$$

Here $\lambda_{k,j}, q_{i,j}^{(k)}, r_{k,i}$ are positive numbers with the following constrains

\[(4.5)\]

$$\sum_{j=1}^{\infty} (q_{i,j}^{(k)} \lambda_{k,j} + r_{k,i} \delta_{ij}) = 1, \quad \text{for all } i \in \mathbb{N}.$$
It is clear that \( p_{ik}^{[k,k+1]} \geq \lambda_{k,k}q_{ik}^{(k)} \). Now assume that
\[
\inf\{q_{ik}^{(k)} : i \in \mathbb{N}\} := \gamma_k > 0
\]
and
\[
\sum_{k=1}^{\infty} (1 - \lambda_{k,k}\gamma_k) = \infty.
\]

Then one can see that \( p_{ik}^{[k,k+1]} \geq \lambda_{k,k}\gamma_k \), this means that conditions of Theorem 4.1 are satisfied with \( n_k = k \), \( \lambda_k = \lambda_{k,k}\gamma_k \). Hence, the defined NHDMP is \( L_1 \)-weak ergodic.

Now consider more exact values of \( \lambda_{k,j}, q_{ij}^{(k)}, r_{k,i} \).

Define
\[
(4.6) \quad r_{k,i} = \frac{1}{k+i}, \quad \lambda_{k,j} = \begin{cases} 0, & 1 \leq j \leq k - 2 \text{ or } j \geq k + 1 \\ \alpha_k, & j = k - 1 \\ \beta_k, & j = k \end{cases}
\]

Note that \( \alpha_k, \beta_k \) will be chosen later on.

Let \( q_{ik}^{(k)} = \beta_k \) for all \( i \in \mathbb{N} \), and \( q_{ij}^{(k)} = 0 \) for every \( 1 \leq j \leq k - 2 \) and \( j \geq k + 1 \).

Now define \( q_{i,k-1}^{(k)} \) from the equality (4.5) as follows
\[
\alpha_k q_{i,k-1}^{(k)} + \beta_k^2 + r_{k,i} = 1
\]
which implies that
\[
(4.7) \quad q_{i,k-1}^{(k)} = \frac{1}{\alpha_k}(1 - r_{k,i} - \beta_k^2)
\]

Now choose \( \alpha_k \) and \( \beta_k \) as follows
\[
(4.8) \quad \alpha_k = \frac{1}{k}, \quad \beta_k = \sqrt{\frac{k - 1}{k}}, \quad k \in \mathbb{N}.
\]

Then from (4.6)-(4.8) one finds
\[
q_{i,k-1}^{(k)} = \frac{i}{k+i}.
\]

It is clear that \( \gamma_k = \beta_k \), therefore, from (4.6), (4.8) one gets
\[
\sum_{k=1}^{\infty} (1 - \lambda_{k,k}\gamma_k) = \sum_{k=1}^{\infty} (1 - \beta_k^2) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.
\]

Hence, due to Theorem 4.1 the following NHDMP defined by
\[
p_{ij}^{[k,k+1]} = \begin{cases} \delta_{ij}, & 1 \leq j \leq k - 2 \text{ or } j \geq k + 1, \\ \frac{i}{k+1} (\frac{i}{k} + \delta_{i,k-1}), & j = k - 1, \\ \frac{k-1}{k}, & j = k, \end{cases}
\]
satisfies the \( L_1 \)-weak ergodicity.
4.2. Continuous case. Let \((X, \mathcal{F}, \mu)\) be a probability space and \(P^{[k,m]}(x, A)\) be a NHDMP on this space.

**Theorem 4.2.** Let \(P^{[k,m]}(x, A)\) be a NHDMP on \((X, \mathcal{F}, \mu)\). If for every \(k \in \mathbb{Z}_+\) there exists a set \(A_k \in \mathcal{F}\) and a number \(\alpha_k > 0\) such that

\[
P^{[k-1,k]}(x, A_k) \geq \alpha_k \quad \text{for all } x \in X, \ k \in \mathbb{N}
\]

where

\[
\sum_{n=1}^{\infty} \left( 1 - \frac{\alpha_n}{2} \right) = \infty.
\]

Then the NHDMP satisfies the \(L_1\)-weak ergodicity.

**Proof.** To prove the statement it is enough to establish that the process satisfies condition \(C_1\). Indeed, for each \(k \in \mathbb{Z}\) let us define

\[
\nu_k(A) = \bigwedge_{x \in X} P^{[k-1,k]}(x, A \cap A_k), \ A \in \mathcal{F}
\]

Due to Theorem IV.7.5 \[3\] the defined mapping \(\nu_k\) is a measure on \(X\), and moreover, one has \(\nu_k(A_k) \geq \alpha_k\). Now put

\[
\mu_k(A) = \frac{\nu_k(A \cap A_k)}{\nu_k(A_k)}, \ A \in \mathcal{F}.
\]

Then one can see that

\[
P^{[k-1,k]}_* \delta_x \geq \alpha_k \mu_k \quad \text{for all } x \in X, k \in \mathbb{N}.
\]

It is clear that \(\|\mu_k\|_1 \neq 0\).

Denote

\[
\mathcal{M} = \left\{ \nu = \sum_{i=1}^{n} \alpha_i \delta_{x_i} : \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0, \{x_i\}_{i=1}^{n} \subset X, \ n \in \mathbb{N} \right\}
\]

which is convex set. Therefore, from \(4.11\) we immediately find that

\[
P^{[k-1,k]}_* \mu \geq \alpha_k \mu_k \quad \text{for all } \mu \in \mathcal{M}.
\]

Due to the fact (see \[3\]) that the set \(\mathcal{M}\) is a weak dense subset of the set of all probability measures \(\widehat{\mathcal{M}}\) on \((X, \mathcal{F})\), i.e. \(\overline{\mathcal{M}^\prime} = \widehat{\mathcal{M}}\). Hence, from \(4.12\) one gets

\[
P^{[k-1,k]}_* \lambda \geq \alpha_k \mu_k \quad \text{for all } \lambda \in \widehat{\mathcal{M}}.
\]

Now for each each \(k \in \mathbb{Z}_+\) we put \(X_k = X\), then from \(4.13\) it follows condition \(C_1\). So, taking into account \(4.11\), from Corollary \(3.5\) we get the desired assertion. \(\square\)
4.3. **Quadratic stochastic processes.** In this section we apply the obtained results to discrete time quadratic stochastic processes. Note that such kind of processes relate with quadratic operators as well as Markov processes with linear operators (see [5] for review).

Let \((X, \mathcal{F}, \mu)\) be a probability space. We recall that a family of functions \(\{Q^{[k,n]}(x, y, A)\}\) defined for \(n > k \ (k, n \in \mathbb{Z}_+\) for all \(x, y \in X, A \in \mathcal{F}\), is called *discrete quadratic stochastic process* (DQSP) if the following conditions are satisfied:

(i) \(Q^{[k,n]}(x, y, A) = Q^{[k,n]}(y, x, A)\) for any \(x, y \in X\) and \(A \in \mathcal{F}\);

(ii) \(Q^{[k,n]}(x, y, \cdot) \in \mathfrak{M}\) for any fixed \(x, y \in X\);

(iii) \(Q^{[k,n]}(x, y, A)\) as a function of \(x\) and \(y\) is measurable on \((X \times X, \mathcal{F} \otimes \mathcal{F})\) for any \(A \in \mathcal{F}\);

(iv) (Analogue of the Chapman-Kolmogorov equation) for the initial measure \(\mu \in \mathfrak{M}\) and arbitrary \(k < m < n, k, m, n \in \mathbb{Z}_+\) we have either

\[
(iv)_A
\]

\[
Q^{[k,n]}(x, y, A) = \int_X \int_X Q^{[k,m]}(x, y, du)Q^{[k,n]}(u, v, A)\mu_m(dv),
\]

where the measure \(\mu_m\) on \((X, \mathcal{F})\) is defined by

\[
\mu_m(B) = \int_X \int_X Q^{[0,m]}(x, y, B)\mu(dx)\mu(dy),
\]

for any \(B \in \mathcal{F}\), or

\[
(iv)_B
\]

\[
Q^{[k,n]}(x, y, A) = \int_X \int_X \int_X \int_X Q^{[k,m]}(x, z, du)Q^{[k,m]}(y, v, dw)Q^{[m,n]}(u, w, A)\mu_k(dz)\mu_k(dw).
\]

If the condition \((iv)_A\) (resp. \((iv)_B\)) holds, then DQSP is called of **type** (A) (resp. (B)).

The process \(Q^{[k,n]}(x, y, A)\) can be interpreted as the probability of the following event: if \(x\) and \(y\) in \(X\) interact at time \(k\), then one of the elements of the set \(A \in \mathcal{F}\) will be realized at time \(n\). All phenomena in physics, chemistry, and biology develop along non-zero finite time intervals. Therefore, we assume that the maximum of these values of time is equal to 1. Hence, \(Q^{[k,n]}(x, y, A)\) is defined for \(n - k \geq 1\) (we refer the reader to [5] for more information).

By \(\mathfrak{M}^2\) we denote the set of all probability measures on \(X \times X\) which are absolutely continuous w.r.t. \(\mu \otimes \mu\), i.e. \(\mathfrak{M}^2\) can be considered as a subset of \(L^1(X \times X, \mathcal{F} \otimes \mathcal{F}, \mu \otimes \mu)\). Given DQSP \(Q^{[k,n]}(x, y, A)\) one can define

\[
(4.14) \quad (Q^{[k,n]}_\ast \tilde{\nu})(A) = \int_X \int_X Q^{[k,n]}(x, y, A)d\tilde{\nu}(x, y), \quad \tilde{\nu} \in L^1(X \times X, \mu \otimes \mu).
\]

We recall that a DQSP \(Q^{[k,n]}(x, y, A)\) is said to satisfy the **L1-weak ergodicity** (or **ergodic principle**) if for any probability measures \(\tilde{\lambda}, \tilde{\nu} \in \mathfrak{M}^2\) and \(k \in \mathbb{Z}_+\) one has

\[
\lim_{n \to \infty} \|Q^{[k,n]}_\ast \tilde{\lambda} - Q^{[k,n]}_\ast \tilde{\nu}\|_1 = 0;
\]
Let $Q^{[k,n]}(x, y, A)$ be a given DQSP. Now define the following transition probability

\begin{equation}
Q^{[k,n]}_Q(x, A) = \int_X Q^{[k,n]}(x, y, A) d\mu_k(y).
\end{equation}

In [12] it has been proved the following

**Theorem 4.3.** Let $Q^{[k,n]}(x, y, A)$ be a given DQSP on $(X, \mathcal{F}, \mu)$. Then the following statements hold true:

(i) the defined $P^{[k,n]}_Q(x, A)$ is a NHDM on $(X, \mathcal{F}, \mu)$;

(ii) the process $P^{[k,n]}_Q(x, A)$ satisfies the $L_1$-weak ergodicity if and only if $Q^{[k,n]}(x, y, A)$ satisfies the $L_1$-weak ergodicity.

This theorem allows us to prove the following result.

**Theorem 4.4.** Let $Q^{[k,n]}(x, y, A)$ be a given DQSP on $(X, \mathcal{F}, \mu)$. If for every $k \in \mathbb{Z}_+$ there exists a set $A_k \in \mathcal{F}$ and a number $\alpha_k > 0$ such that

\begin{equation}
Q^{[k-1,k]}(x, y, A_k) \geq \alpha_k \text{ for all } x, y \in X, k \in \mathbb{N}
\end{equation}

where

\begin{equation}
\sum_{n=1}^{\infty} \left(1 - \frac{\alpha_n}{2}\right) = \infty,
\end{equation}

then the DQSP is $L_1$-weak ergodic.

**Proof.** Consider the process $P^{[k,n]}_Q(x, A)$. Then from (4.15) and (4.16) one finds

\begin{equation}
P^{[k-1,k]}_Q(x, A_k) = \int_X Q^{[k-1,k]}(x, y, A_k) d\mu_k(y) \geq \alpha_k \text{ for all } x \in X, k \in \mathbb{N}.
\end{equation}

Hence, the Markov process $P^{[k,n]}_Q(x, A)$ satisfies the conditions of Theorem 4.2 so it is $L_1$-weak ergodic. Therefore, Theorem 4.3 implies the $L_1$-weak ergodicity of $Q^{[k,n]}(x, y, A)$. \qed

Note that the last theorem improves the result of [16].

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