On Calderón’s inverse inclusion problem with smooth shapes by a single partial boundary measurement

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Received 18 June 2020, revised 28 January 2021
Accepted for publication 18 March 2021
Published 15 April 2021

Abstract
We consider Calderón’s inverse inclusion problem of recovering the shape of an anomalous inhomogeneity embedded in a homogeneous conductivity by the associated electric boundary measurements. It is a longstanding problem whether one can establish the unique recovery result by a single boundary measurement. In several existing works, it is shown that corner singularities can help to resolve the problem within polygonal or polyhedral shapes. In this paper, under a generic technical condition, we show that the corner singularity can be relaxed to be a certain high-curvature condition and derive novel unique recovery results within smooth shapes.

Keywords: Calderon’s inverse problem, electric impedance tomography, conductive inclusion, smooth shape, high curvature, single partial boundary measurement

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Background and motivation

Initially focusing on the mathematics, but not the physics, we present the Calderón inverse inclusion problem associated with the conductivity equation. Let $\Omega$ and $D$ be bounded Lips-
chitz domains in \( \mathbb{R}^n, n = 2, 3 \), such that \( D \subseteq \Omega \) and \( \Omega \setminus \overline{D} \) is connected. Let \( \eta \in L^\infty(D) \) be such that \( \eta(x) > \eta_0 \in \mathbb{R}_+ \), \( x = (x_j)_{j=1}^n \in D \), and \( |\eta(x) - 1| > 0, x \in \text{neigh}(\partial D) \), where \( \text{neigh}(\partial D) \) denotes an open neighbourhood of \( \partial \Omega \). Set
\[
\gamma(x) = 1 + (\eta - 1)\chi_D(x), \quad x \in \Omega. \tag{1.1}
\]
Introduce
\[
H_0^{-1/2}(\partial \Omega) := \left\{ f \in H^{-1/2}(\partial \Omega); \int_{\partial \Omega} f \, ds = 0 \right\},
\]
and consider the following elliptic PDE problem for \( u \in H^1(\Omega) \),
\[
\begin{aligned}
\text{div}(\gamma \nabla u) &= 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= g \in H_0^{-1/2}(\partial \Omega) & \text{on } \partial \Omega,
\end{aligned} \tag{1.2}
\]
where and also in what follows \( \nu \in \mathbb{S}^{n-1} \) denotes the exterior unit normal vector to the domain concerned. Associated with (1.2), we introduce the following Neumann-to-Dirichlet map, \( \Lambda_\gamma : H_0^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega), \)
\[
\Lambda_\gamma(g) = u|_{\partial \Omega}, \tag{1.3}
\]
where \( u \in H^1(\Omega) \) is the solution to (1.2). For a given \( g \in H_0^{-1/2}(\partial \Omega) \), we are concerned with the inverse problem of determining \( D \) by a pair of boundary data \( (g, \Lambda_\gamma(g)) \), independent of \( \eta \). That is,
\[
(g, \Lambda_\gamma(g)) \to \partial D, \quad \text{independent of } \eta. \tag{1.4}
\]
In the physical setting, \( \gamma \) signifies the conductivity of a medium body \( \Omega \) and \( u \) signifies an electric potential field that fulfills the conductivity equation (1.2). \( g \) and \( \Lambda_\gamma(g) \), respectively, measure the electric current and voltage on the boundary of \( \Omega \). The inverse problem (1.4) corresponds to the electric probing of the interior of the conductive body \( \Omega \) by non-destructive boundary observations. The problem forms the basis for many important real applications including the electrical impedance tomography (EIT) in medical imaging and the geophysical exploration; see [22, 45] and the references cited therein. The problem (1.4) is also referred to as the Calderón inverse inclusion problem in the literature and it is named after Calderón who proposed and studied the inverse conductivity problem in a broader context [25]. There are many theoretical and numerical developments on Calderón’s inverse problem and we refer to [22, 45] for surveys and reviews.

In (1.4), if the boundary data \( (g, \Lambda_\gamma(g)) \) are given associated with a fixed \( g \in H_0^{-1/2}(\partial \Omega) \), it is referred to as a single measurement, otherwise it is referred to as multiple measurements. If multiple measurements are used, there are rich unique recovery results in the literature for the inverse problem (1.4), and we refer to [22, 45] for surveys on the existing results as well as [3, 9, 10, 12, 21] for some recent studies. If only a single measurement is used, one can see that the inverse problem (1.4) is formally determined. In fact, in such a case, it is noted that both the cardinalities of the unknown \( \partial D \) and the known \( (g, \Lambda_\gamma(g))|_{\partial \Omega} \) associated with a fixed \( g \) are \( n - 1 \). Here, by cardinality, we mean the number of independent variables associated with a given quantity. Since both \( \partial D \) and \( \partial \Omega \) and \( (n - 1) \)-dimensional manifolds, their cardinalities are \( n - 1 \) in \( \mathbb{R}^n \). By a single measurement, it remains to be a longstanding problem whether one can establish the uniqueness result in recovering a general-shape inclusion. The existing
results for the single-measurement case are mainly concerned with specific shapes including discs/balls and polygons/polyhedrons \cite{11, 12, 30, 40, 41, 44} as well as the other general shapes but with a priori conditions \cite{2, 4, 5, 23, 32, 34, 39}. In a recent paper \cite{41}, the first two authors of the present article proved a logarithmic type stability in determining polygonal inclusions. In those studies mentioned above by a single measurement, it is a technical requirement that the content of the inclusion has to be uniform; that is, the conductivity $\eta$ of the inclusion in (1.4) is a positive constant. Moreover, in all of the aforementioned literature except \cite{41}, full boundary measurements are required. Here, by full boundary measurements, we mean that the measurement dataset $(g, \Lambda, (g))$ is given over the whole boundary $\partial \Omega$. For comparison, the following partial boundary measurement was used in \cite{41}: $(\psi, \gamma \partial_\nu u|_{\Gamma_0})$ with supp$(\psi) \subset \Gamma_0$, where $\psi = u|_{\partial \Omega}$ and $\Gamma_0 \Subset \partial \Omega$ is a proper subset. We would like to mention that the partial-data Calderón problem constitutes another challenging topic in the field of inverse problems (cf \cite{27, 38}). Nevertheless, it is pointed out that a mild condition was imposed for the study in \cite{41} which depends on the a priori knowledge of the underlying inclusion as well as the corresponding boundary input.

The mathematical argument in \cite{41} is of a localized feature, which is based on carefully studying the singular behaviors (in the phase space) of the solution to the conductivity problem (1.2) around a corner point on the polygonal inclusion. In this paper, we show that the corner singularity in \cite{41} can be relaxed to be a certain high-curvature condition. Indeed, the corner singularity can be regarded as having an extrinsic curvature being infinity. Our argument in tackling the singular behaviors of the solution to (1.2) around an admissible high-curvature point on the boundary of the conductive inclusion is mainly motivated by a recent article \cite{17}. However, the study in \cite{17} mainly deals with high-curvatures occurring on the support of a parameter $q$, which is the coefficient for the lower-order term of an elliptic partial differential operator, namely $-\Delta + q$. In the current study, the high-curvatures enter the coefficient of the leading-order term, namely $\gamma$ associated with $\nabla \cdot (\gamma \nabla u)$. It is pointed out that in \cite{41}, quantitative stability estimates are established in determining polygonal inclusions in two dimensions, whereas in this paper, we are mainly concerned with the qualitative unique identifiability issue in both two and three dimensions. Finally, we would like to mention in passing some recent related works \cite{8, 13–16, 18–20, 24, 26, 28, 42} on characterizing the geometric singularities in the coefficients of certain partial differential operators and their implications to the related inverse inclusion problems.

### 1.2. Summary of the main result and discussion

The statement of our main result is technically involved. Nevertheless, in order to give the readers a global picture of our study, we briefly summarize the major finding in the following theorem.

**Theorem 1.1.** Let $(D, \eta)$ and $(\tilde{D}, \tilde{\eta})$ be two conductive inclusions in $\Omega$ as described in (1.1), where both $\eta$ and $\tilde{\eta}$ are assumed to be constants. Let $u$ and $\tilde{u}$ be the solutions to (1.2) associated respectively to $(D, \eta)$ and $(\tilde{D}, \tilde{\eta})$ with a given $g \in H_0^{1/2}(\partial \Omega)$. Suppose that both $D$ and $\tilde{D}$ are convex and of class $C^{3,1}$. Suppose further that $u$ and $\tilde{u}$ fulfill two generic technical conditions in assumptions A and B in what follows that depend on the a priori knowledge of the underlying conductive inclusions as well as the input $g$. Let $\Gamma_0 \subset \partial \Omega$ be an open subset. If $u = \tilde{u}$ on $\Gamma_0$, then $D \Delta \tilde{D} := (D \setminus \tilde{D}) \cup (\tilde{D} \setminus D)$ cannot possess an admissible $K$-curvature point as defined in definition 2.1 with a sufficiently large $K$; see figure 1 for a schematic illustration where $p$ signifies an admissible $K$-curvature point.

The full technical details of the unique recovery result shall be given in theorem 4.2 in what follows.
Remark 1.2. Theorem 1.1 presents a local unique recovery result by a single partial boundary measurement. However, we would like to point out that in certain specific scenarios, one can also obtain the global recovery results. To illustrate this point, let us consider a special case in $\mathbb{R}^2$. Suppose that the conductive inclusion is of an ellipse shape which can be parameterized as follows:

$$x = (x_1, x_2) \in \partial D; \quad \frac{(x_1 - x_0^1)^2}{a^2} + \frac{(x_2 - x_0^2)^2}{b^2} = 1,$$  \hspace{1cm} (1.5)

where $x_0 = (x_0^1, x_0^2)$ signifies the centre of the ellipse and $a, b \in \mathbb{R}_+$ are the corresponding semi-axes. It is assumed that either $a/b = K$ or $b/a = K$ with $K \gg 1$, so that the vertices of the ellipse on the longer semi-axis are two admissible $K$-curvature points (cf definition 2.1). That means, the ellipse is very ‘slender’ and its eccentricity is close either to 1 or to 0. Then if assumptions A and B in theorem 1.1 are fulfilled (they are indeed fulfilled according to our discussion in what follows), such an elliptical inclusion can be uniquely determined by a single boundary measurement. In fact, assuming that $D$ and $\tilde{D}$ are two ‘slender’ elliptical inclusions as described above, if $D \neq \tilde{D}$, then it can be easily seen that $D \Delta \tilde{D}$ possesses a $K$-curvature point, which readily yields a contradiction according to theorem 1.1.

As discussed earlier, the uniqueness in determining a convex polygonal inclusion has been established in the literature by a single measurement. The corner singularity is critical for resolving the inverse problem. In fact, the geometrical singularity induces a certain singularity (in terms of smoothness) in the electric field $u$, which enables one to tackle the inverse problem; see [41] for more related discussion. However, if the corner is mollified (becoming a $K$-curvature point), no such singularities occur and instead one needs to track the curvature effect in the electric field $u$, which becomes highly challenging and intriguing. Hence, though still specific, it is unobjectionable to claim that our study brings new insights to the longstanding Calderon’s inverse inclusion problem.

Remark 1.3. Theorem 1.1 can be generalized in several aspects. It is a local uniqueness result and indeed its proof is also localized in a neighbourhood of the high curvature point $p$. That is,
it is sufficient for us to require in theorem 1.1 that $\eta$ is constant in an open neighbourhood of the point $p$, and $\partial D$ is $C^{3,1}$-continuous in that neighbourhood as well. The same remark holds for $(\tilde{D}, \tilde{\eta})$. Moreover, the convexity of $D$ and $\tilde{D}$ can also be relaxed as long as the local uniqueness is concerned. However, in order to have a focusing study and a concise presentation, we stick to the relatively simple setup as described in theorem 1.1. The corresponding extensions as mentioned above can be easily seen from our localized arguments in what follows.

It is also remarked that we mainly consider our study in two and three dimensions. In principle, all the results established in this paper can be extended to any dimension bigger than 3, including the notion of the $K$-curvature point and the local uniqueness result summarized in theorem 1.1. However, some of our arguments are dimension dependent, say proposition 3.1 where the Sobolev embedding is used, and hence necessary modifications will be needed. This will bring extra treatment and discussion. Since only the two- and three-dimensional cases are physically meaningful and moreover in order to have a focusing study, we stick to the two and three dimensions.

Remark 1.4. In theorem 1.1, assumptions A and B are technically involved, which shall be given in full details in sections 3 and 4, respectively. Nevertheless, in order to provide a full picture of the main result summarized in the theorem, we briefly discuss these two technical requirements. Both assumptions A and B are related to the quantitative properties of the direct problem (1.2). Assumption A roughly states that the gradient of the electric field $u$, namely $\nabla u$, should be bounded in terms of its curvature around the boundary of a conductive inclusion, which is the subject of several existing studies [29, 31]. However, we require a sharper estimate of such a dependence, and in order to appeal for a more general study, we make it as an assumption. Assumption B basically states that the gradient field $\nabla u$ is non-vanishing everywhere, which is also the subject of several existing studies [1, 6] in two dimensions. We make it as a technical assumption in order to suit our study in the three-dimensional case. More relevant remarks about these two assumptions shall be made in what follows, and as can be seen, they are generically fulfilled.

Finally, we would like to emphasize that the current paper is a piece of theoretical work and its main value is the establishment of novel unique recovery results for a challenging inverse problem. This also brings some new insights on the mathematical study of this topic. We would like to mention in passing some related studies on the reconstruction methods for the Calderón inverse inclusion problem from practical viewpoints, including the monotonicity-based method [36], the factorization method [35], the enclosure method [37], and the method of using the generalized polarization tensors deduced from the layer potential techniques [7]. We expect that our theoretical study in this paper may be used to show the super-resolution imaging of the high-curvature parts of a conductive inclusion; see [8] for a similar study in the context of the inverse acoustic scattering problem.

The rest of the paper is organized as follows. In section 2, we present the geometric and mathematical setups of our study including the definition of the $K$-curvature point and a specific transmission problem associated with the conductivity equation that arises in the proof of the local unique recovery result in theorem 1.1. Section 3 presents the full details of the unique recovery result as well as its proof.

2. Geometric and mathematics setups

In this section, we present the geometric and mathematical setups of our study including the definition of the $K$-curvature point and a particular transmission problem associated with the conductivity equation that arises in the proof of the local unique recovery result in theorem 1.1.
Following the treatment in [17], we first introduce the geometric notion of the $K$-curvature point.

**Definition 2.1.** Let $K, L, M, \delta$ be positive constants and $D$ be a bounded domain in $\mathbb{R}^n, n \geqslant 2$. A point $p \in \partial D$ is said to be an admissible $K$-curvature point with parameters $L, M, \delta$ if the following conditions are fulfilled.

(a) Up to a rigid motion, the point $p$ is the origin $x = 0$ and $e_n := (0, \ldots, 0, 1)$ is the interior unit normal vector to $\partial D$ at $0$.

(b) Set $b = \sqrt{M}/K$ and $h = 1/K$. There is a $C^{2,1}$ function $w : B(0, b) \rightarrow \mathbb{R}_+ \cup \{0\}$ with $B(0, b) \subset \mathbb{R}^{n-1}$ such that if

$$D_{b,h} = B(0, b) \times (-h, h) \cap D,$$

then

$$D_{b,h} = \{x \in \mathbb{R}^n; |x'| < b, -h < x_n < h, w(x') < x_n < h\},$$

where $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. Here and also in what follows, $B(x, b)$ signifies a ball centered at $x$ and of radius $b$.

(c) The function $w$ in (b) satisfies

$$w(x') = K|x'|^2 + O(|x'|^3), \quad x' \in B(0', b).$$

(d) $M \geqslant 1$ and there are $0 < K_- \leqslant K \leqslant K_+ < \infty$ such that

$$K_-|x'|^2 \leqslant w(x') \leqslant K_+|x'|^2, \quad |x'| < b,$$

$$M^{-1} \leqslant \frac{K_+}{K} \leqslant M, \quad K_+ - K_- \leqslant LK^{1-\delta}.$$ (2.4)

(e) The intersection $V = D_{b,h} \cap \mathbb{R}^{n-1} \times \{h\}$ is a Lipschitz domain.

In what follows, we suppose that $\partial D$ possesses an admissible $K$-curvature point $p$. It is easily seen that if $K$ is large, then both the mean and Gaussian curvatures of $\partial D$ at the point $p$ are high. We shall work within an Euclidean system of coordinates, which is transformed from the cardinal one after a rigid motion such that $p$ is the origin. In the simplest scenario, one might think of $w(x') = K|x'|^2$ in (2.3). In such a case, $\partial D$ locally around $p$ is a parabola in the two dimensions and a paraboloid in the three dimensions. In what follows, we denote by $U_{b,h} := B(0', b) \times (-h, h)$ the cylinder centered at $p'$, and by $D_{b,h} := U_{b,h} \cap D$ the neighbourhood of $p$, and by $\partial S := \partial D_{b,h} \setminus \partial D$ the complementary part of its surface.

In order to establish the local unique recovery result in theorem 1.1, we shall make use of an absurdity argument. Let $(D, \eta)$ and $(\tilde{D}, \tilde{\eta})$ be introduced in the theorem, and $u$ and $\tilde{u}$ be the corresponding electric fields. Without loss of generality, we assume that $D \setminus \tilde{D}$ possesses an admissible $K$-curvature point, namely $p \in \partial D \setminus \partial \tilde{D}$. It is clear that

$$d_0 := \text{dist}(p, \partial \tilde{D}) > 0.$$ (2.5)

Since

$$u = \tilde{u} \quad \text{and} \quad \partial_p u = \partial_p \tilde{u} \quad \text{on} \ \Gamma_0,$$ (2.6)

we have from the classical unique continuation principle that

$$u = \tilde{u} \quad \text{in} \ G := \Omega \cap D \cup \tilde{D},$$ (2.7)
which in particular implies that
\[ u = \bar{u}, \quad \partial_{\nu} u = \partial_{\nu} \bar{u} \quad \text{on } \partial G. \tag{2.8} \]

We assume that \( b, h \in \mathbb{R}_+ \) are sufficiently small such that \( U_{b,h} \subset B(p, d_0/2) \). Clearly, \( U_{b,h} \Subset \Omega \setminus \overline{D} \) and \( D_{b,h} \subset D \).

Set
\[ u_e := \bar{u}|_{U_{b,h}} \quad \text{and} \quad u_i := u|_{D_{b,h}}. \tag{2.9} \]

We know from (1.2) that \( u_e \) and \( u_i \) are both harmonic in \( U_{b,h} \) and \( D_{b,h} \), respectively. Moreover, by making use of the transmission conditions of \( u \) across \( \partial D \) as well as (2.8), one can deduce that
\[ u_i = u_e, \quad \eta \partial_{\nu} u_i = \eta \partial_{\nu} u_e \quad \text{on } U_{b,h} \cap \partial D. \tag{2.10} \]

Hence, we have the following transmission problem:
\[ \begin{cases} \Delta u_e = 0 & \text{in } U_{b,h}, \\ \Delta u_i = 0 & \text{in } D_{b,h}, \\ u_i = u_e & \text{on } U_{b,h} \cap \partial D, \\ \eta \partial_{\nu} u_i = \eta \partial_{\nu} u_e & \text{on } U_{b,h} \cap \partial D. \end{cases} \tag{2.11} \]

Moreover, we note that \( u = u_e \chi_{D_{b,h}} + u_i \chi_{U_{b,h} \setminus D_{b,h}} \in H^1(U_{b,h}) \) satisfies
\[ \text{div} \cdot [(1 + (\eta - 1) \chi_{D_{b,h}}) \nabla u] = 0 \quad \text{in } U_{b,h}. \tag{2.12} \]

The rest of the paper shall be mainly devoted to analyzing the quantitative behaviours of the solutions \( u_i \) and \( u_e \) to (2.11) and (2.12), especially their dependence on the curvature \( K \). In principle, we shall show that generically \( |\nabla u_e(p)| \) is nearly vanishing if the curvature \( K \) is sufficiently large. This will enable us to establish a contradiction in proving the local unique recovery result in theorem 1.1.

3. Unique determination results for Calderón’s inverse inclusion problem

3.1. Local behaviours near the high curvature point \( p \)

As remarked earlier, we shall study the quantitative behaviours of the solutions to the transmission problem (2.11) and (2.12) and in particular their asymptotic dependence on the curvature \( K \) as \( K \to +\infty \). First, by the standard regularity estimates, we have the following proposition.

**Proposition 3.1.** Consider the transmission problem (2.11). One has that \( u_e \) is real analytic in \( U_{b,h} \) and \( u_i \in C^{1,\alpha}(D_{b,h}) \) for some \( 0 < \alpha < 1 \) in two and three dimensions.

**Proof.** The real analyticity of \( u_e \) is obvious since \( u_e = \bar{u} \) in \( \Omega \setminus \overline{D} \) and \( \bar{u} \) is real analytic outside \( D \). On the other hand, from (1.2) and (2.11), one sees that \( u_i \in H^1(D_{b,h}) \) satisfies \( \Delta u_i = 0 \) in \( D_{b,h} \), and
\[ u_i = u_e \quad \text{and} \quad \partial_{\nu} u_i = \eta^{-1} \partial_{\nu} u_e \quad \text{on } \partial D_{b,h} \cap \partial D. \]

According to definition 2.1, we know that \( \partial D_{b,h} \cap \partial D \) is \( C^{2,1} \). Hence, by the regularity estimate up the boundary for elliptic PDEs (cf [43, theorem 4.18]), we have that \( u_i \in H^2(D_{b,h}) \). Finally,
by the Sobolev embedding, we readily see that \( u_t \in C^{1,\alpha}(\overline{D_{p,b}}) \) for some \( 0 < \alpha \leq 1 \) in the two and three dimensions.

By proposition 3.1, we know that \( u_t \in C^{1,\alpha}(\overline{D_{p,b}}) \) and \( u_e \in C^{1,\alpha}(\overline{U_{p,b} \setminus D_{p,b}}) \) for some \( 0 < \alpha \leq 1 \). However, the Hölder norms of the solution \( u_t \) on each side intricately depend on the geometric shape, and particularly depend on the local geometric parameter \( K \) of the admissible \( K \)-curvature point \( p \). The following lemmas give an estimate of these Hölder norms in terms of the interface’s local curvature [29].

**Lemma 3.2.** Let \( Q_R \) be a cube in \( \mathbb{R}^n \) of side length \( R \), centred at \( p \in \partial D \). We denote the two sub-domains of \( Q_R \) lying on the two sides of \( \partial D \) by \( Q_R^+ := Q_R \cap \{ x_n > w(x') \} \) and \( Q_R^- := Q_R \cap \{ x_n < w(x') \} \), respectively. We consider the conductivity equation,

\[
\text{div}[(1 + (\eta - 1)\chi_n) \nabla u] = 0 \quad \text{in } Q_R,
\]

where \( \chi_n(x) = 1 \) in \( Q_R^+ \) and \( \chi_n(x) = 0 \) in \( Q_R^- \). Then there exist positive constants \( \alpha \in (0, 1) \), \( \mu, C \) independent of \( u_t \) and \( w \) such that

\[
\| \nabla u_t \|_{C^\alpha(Q_R^\pm)} \leq C(1 + \| \nabla w \|_{C^1(Q_R)}^\mu) \| \nabla u_t \|_{L^2(Q_R)}.
\]

**Remark 3.3.** In the context of our study, we let \( R_0 > 0 \) be such that \( U_{b,h} \subseteq Q_{R_0} \subseteq \Omega \). Then lemma 3.2 implies that there exist \( C_{n,b,R_0} \) and \( \mu \) which depend only on the \( a \text{ priori} \) data such that

\[
\| \nabla u_t \|_{C^\alpha(Q_{R_0}^\pm)} \leq C_{n,b,R_0} K^\mu \| g \|_{H^{-1/2}(\partial \Omega)}.
\]

It is noted that the Hölder index \( \alpha \) in lemma 3.2 is the same as the one in proposition 3.1.

Clearly, (3.3) gives the estimates of \( \| \nabla u_t \|_{C^\alpha(U_{b,h})} \) and \( \| \nabla u_e \|_{C^\alpha(U_{b,h} \setminus D_{b,h})} \). We next further derive the estimates of \( \| u_t \|_{C^\alpha(U_{b,h})} \) and \( \| u_e \|_{C^\alpha(U_{b,h} \setminus D_{b,h})} \) in terms of the local curvature of the admissible \( K \)-curvature point, which then yield the desired estimates of \( \| u_t \|_{C^\alpha(U_{b,h})} \) and \( \| u_e \|_{C^\alpha(U_{b,h} \setminus D_{b,h})} \).

**Lemma 3.4.** Let \( u = u_t \chi_D + u_e \chi_{\Omega \setminus D} \in H^1(\Omega) \) be the solution to (2.12), or equivalently (1.2). Then one has that

\[
\| u_t \|_{C^\alpha(U_{b,h} \setminus D_{b,h})} \leq CK^\mu \| g \|_{H^{-1/2}(\partial \Omega)},
\]

and

\[
\| u_e \|_{C^\alpha(U_{b,h} \setminus D_{b,h})} \leq CK^\mu \| g \|_{H^{-1/2}(\partial \Omega)},
\]

where the positive constant \( C \) depends on the generic constant in the estimate (3.3) as well as \( \Omega \) and \( D \), but independent of \( K \).

**Proof.** We first prove (3.5), and by virtue of (3.3), we see that

\[
\| \nabla u_t \|_{C^\alpha(U_{b,h} \setminus D_{b,h})} \leq CK^\mu \| g \|_{H^{-1/2}(\partial \Omega)},
\]

and hence it suffices for us to show that

\[
\| u_e \|_{C^\alpha(U_{b,h} \setminus D_{b,h})} \leq CK^\mu \| g \|_{H^{-1/2}(\partial \Omega)}.
\]
Recall that $Q_{R_0}$ is introduced in remark 3.3. We fix an $\tilde{x} \in \partial Q_{R_0}$ such that $\text{dist}(\tilde{x}, \partial D) \sim 1$. By the standard elliptic PDE estimate, we know

$$|u_e(\tilde{x})| \leq C_1 \|g\|_{H^{-1/2}(\partial D)},$$

(3.8)

where $C_1$ is a positive constant depending only on $\eta, \Omega$ and $D$, but independent of $K$. For any $x \in \overline{U_{b,h}} \setminus D_{b,h}$, we let $l(x, \tilde{x})$ be the line segment connecting $x$ and $\tilde{x}$ such that $l(x, \tilde{x}) \subseteq \Omega$. Then by the intermediate value theorem, we have

$$u_e(x) - u_e(\tilde{x}) = \nabla u_e(\xi) \cdot (x - \tilde{x}),$$

(3.9)

where $\xi \in l(x, \tilde{x})$. By combining (3.6) and (3.8), we readily have from (3.9) that

$$|u_e(x)| \leq C_2 (1 + K^\mu) \|g\|_{H^{-1/2}(\partial D)} \leq 2C_2 K^\mu \|g\|_{H^{-1/2}(\partial D)},$$

(3.10)

where we have made use of the facts that $|x - \tilde{x}| \leq \text{diam}(\Omega)$, and without loss of generality that $K^\mu \geq 1$. (3.10) clearly implies (3.7), which in combination with (3.6) immediately yields (3.5).

(3.4) can be proved by following a completely similar argument. Indeed, for any $x \in D_{b,h}$, one can take $\hat{x} \in \partial D \cap U_{b,h}$, and then it holds that

$$u_e(x) - u_e(\hat{x}) = u_e(x) - u_e(\hat{x}) = \nabla u_e(\xi) \cdot (x - \hat{x}),$$

(3.11)

where $\xi \in l(x, \hat{x}) \subseteq \overline{D_{b,h}}$. Finally, by combining remark 3.3, (3.10) and (3.11), one can show (3.4).

The proof is complete.

As mentioned earlier, $u_e$ is harmonic in $U_{b,h}$ and hence is real analytic in $U_{b,h}$. Therefore for $b, h \in \mathbb{R}_+$ sufficiently small, we can have from (3.5) that

$$\|u_e\|_{C^1(\overline{U_{b,h}})} \leq CK^\mu \|g\|_{H^{-1/2}(\partial D)},$$

(3.12)

where $C$ depends on the same a priori data as those in (3.5). Next, we introduce a technical condition in our study.

**Assumption A.** Throughout the rest of our study, we assume that the exponent $\mu$ in (3.4) and (3.5) (or equivalently in (3.3)) satisfies

$$\mu < \frac{\min(1, \delta)}{2},$$

(3.13)

where $\delta$ is the a priori parameter associated to $p$ (cf definition 2.1).

It is emphasized that the a priori parameter $\delta$ in (2.4) is not a restrictive requirement which can actually approach 1 in certain generic scenarios. Hence, it is essential for our subsequent analysis that (3.13) holds with $\mu < 1/2$. This requires a much more accurate estimate of the curvature effect on the electric field $u$ in (1.2). However, a rigorous proof of this much subtle estimate is fraught with significant difficulties. Nevertheless, we have conducted extensive numerical experiments to verify that such a technical condition indeed holds in generic scenarios. We shall present two typical examples in the appendix. Moreover, in a recent paper [31] by the first named author of the present article, an accurate dependence of the conductivity equation on the curvature of a conducive rod is derived in a different context, which corroborates our assumption A.
Based on assumption A, we proceed to further study the quantitative behaviours of the solution $u$ to (2.11) and (2.12) at $p$.

**Proposition 3.5.** Let $u_0 \in H^1_{loc}(U_{b,h})$ be harmonic in $U_{b,h}$. Then,

$$
(\eta - 1) \int_{D_{b,h}} \nabla u \cdot \nabla u_0 \, dx = \int_{\partial S} (\eta \partial_n u_i - \partial_i u \partial_n u_0 - (u_i - u_0) \partial_j u_0) \, ds,
$$

where $u = u_0 \chi_D + u_\varepsilon \chi_{\Omega \setminus \overline{S}} \in H^1(U_{b,h})$ is the solution to (2.11).

**Proof.** The integral identity (3.14) can be directly verified by using Green’s formula and the transmission conditions across $\partial D$ of $u$.

**Theorem 3.6.** Let $u_0 \in H^1_{loc}(U_{b,h})$, known as the complex geometric optics solution, be constructed in the form

$$
u_0 = \exp(\xi \cdot x),
$$

with the parameter

$$
\xi = i\tau \hat{\nu} - \tau \varepsilon_n \in \mathbb{C}^n, \quad \tau \in \mathbb{R}_+,
$$

where

$$
\hat{\nu} := \begin{cases} 
\frac{\nabla \nu_i(p) - (\nabla \nu_i(p) \cdot \varepsilon_n) \varepsilon_n}{|\nabla \nu_i(p) - (\nabla \nu_i(p) \cdot \varepsilon_n) \varepsilon_n|} & \text{if } \nabla \nu_i(p) \parallel \varepsilon_n, \\
\varepsilon_1 := (1, 0, \ldots, 0) & \text{if } \nabla \nu_i(p) \parallel \varepsilon_1.
\end{cases}
$$

Then $u_0$ is a harmonic function. Moreover, it holds that

$$
C_{\eta, \alpha, \varepsilon} |\nabla \nu_i(p)| \leq ||u_i||_{C^1, \alpha, \Omega_{b,h}}(1 + (\tau h)^{(n-1)/2} e^\left(\frac{\tau}{4} - \frac{\tau}{2}\eta\right))
$$

$$
+ \|u_i\|_{C^1, \alpha, \Omega_{b,h}} \left(\frac{K}{K_+} - \left(\frac{K}{K_-}\right)^{\frac{n-1}{2}}\right) e^{\frac{\tau}{2}}
$$

$$
+ \|u_i\|_{C^1, \alpha, \Omega_{b,h}} (h + K^{-1})^{(n+1)(1+\alpha)/2} (K/K_-)^{(n-1)/2} \frac{\tau}{\tau} e^{\frac{\tau}{2}}
$$

$$
+ \|u_i\|_{C^1, \alpha, \Omega_{b,h}} + \|u_\varepsilon\|_{C^1, \alpha, \Omega_{b,h}} h^{n+1/2} (K/K_-)^{(n-1)/2}
$$

$$
\times (1 + \tau h)^{(n-1)/2} e^{\left(\frac{\tau}{4} - \frac{\tau}{2}\eta\right)},
$$

where $C_{\eta, \alpha, \varepsilon}$ is a positive constant depending on $n, \eta$, and $\alpha$.

**Proof.** It is directly verified that $u_0$ constructed in (3.15)–(3.17) is a harmonic function. Next, we apply the constructed $u_0$ to the integral identity (3.14). By straightforward calculations, we split the integral at the left-hand side of (3.14) into the following identity,

$$
\nabla \nu_i(p) \cdot \xi \int_{x_0 \geq K|\xi|^2} e^{\xi \cdot x} dx = \nabla \nu_i(p) \cdot \xi \int_{x_0 \geq \max(K, h)|\xi|^2} e^{\xi \cdot x} d\sigma_x
$$

$$
+ \nabla \nu_i(p) \cdot \xi \left(\int_{K|\xi|^2 \leq x_0 h} e^{\xi \cdot x} d\sigma_x - \int_{D_{b,h}} e^{\xi \cdot x} d\sigma_x\right)
$$
+ \int_{D_{b_h}} \xi \cdot (\nabla u_i(x) - \nabla u_i(p))e^{\xi \cdot x} d\sigma_x \\
+ \frac{1}{\eta - 1} \int_{\partial S} (\eta \partial_\nu u_i - \partial_\nu u_e)e^{\xi \cdot x} - (u_i - u_e)\partial_\nu e^{\xi \cdot x} ds. \quad (3.19)

For notational convenience, we rewrite the integral identity (3.19) in the following form,

\[ \nabla u_i(p) \cdot \xi I_0 = \nabla u_i(p) \cdot \xi I_1 + \nabla u_i(p) \cdot \xi I_2 + I_3 + \frac{1}{\eta - 1} I_4. \quad (3.20) \]

where \( I_j, j = 1, \ldots, 4 \), are respectively defined as

\[ I_0 = \int_{\kappa > K|x|^2} e^{\xi \cdot x} d\sigma_x, \]
\[ I_1 = \int_{\kappa > \max(h,K|x|^2)} e^{\xi \cdot x} d\sigma_x, \]
\[ I_2 = \int_{K|x|^2 < \kappa < h} e^{\xi \cdot x} d\sigma_x - \int_{D_{b_h}} e^{\xi \cdot x} d\sigma_x, \quad (3.21) \]
\[ I_3 = \int_{D_{b_h}} \xi \cdot (\nabla u_i(x) - \nabla u_i(p))e^{\xi \cdot x} d\sigma_x, \]
\[ I_4 = \int_{\partial S} (\eta \partial_\nu u_i - \partial_\nu u_e)e^{\xi \cdot x} - (u_i - u_e)\partial_\nu e^{\xi \cdot x} ds. \]

Using lemmas 2.17–2.20 in [17] we have the following estimates of the integrals \( I_0, I_1, I_2, I_3, I_4 \):

\[ I_0 = \frac{1}{-\xi} \left( \frac{\pi}{-\xi K} \right)^{(n-1)/2} \exp \left( -\frac{\xi' \cdot \xi'}{4\xi K} \right), \]
\[ |I_1| \leq C_n \frac{1 + (\tau h)^{n+1}}{\tau^{n+2} K} e^{-\tau h}, \quad (3.22) \]
\[ |I_2| \leq C_n \left( K^{\frac{n+1}{2}} - e^{\frac{n+1}{2}} \right) \tau^{\frac{n+1}{2}}, \]
\[ |I_3| \leq C_n \|u_i\|_{C^{1,\alpha}(D_{b_h})} \|h + K^{-1} H^{(n+\alpha+1)/2} K^{-\alpha/2}, \]
where \( \xi = (\xi_j)_{j=1}^n \) and \( \xi' = (\xi_j')_{j=1}^n \). By following a similar argument to the proof of proposition 2.21 in [17], the last integral \( I_4 \) can be estimated as follows,

\[ |I_4| \leq C_n \frac{h^{\alpha+1/2} K^{-\alpha/2} (1 + \tau h)}{\|u_i\|_{C^{1,\alpha}(D_{b_h})} + \|u_e\|_{C^{1,\alpha}(D_{b_h})}}. \quad (3.23) \]

Finally, by applying the estimates in (3.20)–(3.23), together with grouping similar terms, one can arrive at the estimate (3.18).

The proof is complete. \( \square \)

**Theorem 3.7.** Let \( u \in H^1(\Omega) \) be the solution to (2.11) and \( p \in \partial D \) be an admissible \( K \)-curvature point. Suppose that (3.4) and (3.5) hold. Suppose that assumption A is true. Then it
holds that
\[ |\nabla u_0(p)| \leq \mathcal{E} \|g\|_{H^{-1/2}(\Omega)} (\ln K)^{(n+1)/2} K^{\mu - \min(\alpha, \beta)/2}, \quad (3.24) \]
where \( \mathcal{E} \) depends on the same a priori data as those in (3.4) and (3.5) as well as \( \alpha \) and \( L, M \) in definition 2.1, but independent of \( g \) and \( K \).

Proof. By plugging the estimates (3.4) and (3.12) into (3.18) of theorem 3.6, we can obtain
\[
C|\nabla u_0(p)| \leq K^\nu \|g\|_{H^{-1/2}(\Omega)} (1 + (\tau h)^{(n-1)/2}) e^{\frac{1}{2}(\frac{1}{K} - \frac{1}{K_+})} \\
+ K^\nu \|g\|_{H^{-1/2}(\Omega)} \left( \frac{K}{K_+} \right)^{-\frac{1}{2}} \left( \frac{K}{K_-} \right)^{-\frac{1}{2}} e^{\frac{1}{2}(\frac{1}{K} - \frac{1}{K_+})} \\
+ K^\nu \|g\|_{H^{-1/2}(\Omega)} (h + K_-)^{-\frac{1}{2}} (h + K_-)^{\frac{n+1}{2}} (K/K_-)^{(n-1)/2} e^{\frac{1}{2}(\frac{1}{K} - \frac{1}{K_+})} \\
+ K^\nu \|g\|_{H^{-1/2}(\Omega)} K^{\alpha} (h + K_-)^{(n-1)/2} e^{\frac{1}{2}(\frac{1}{K} - \frac{1}{K_+})}. \quad (3.25)
\]

It is noted that in (3.25), we have absorbed the generic constant involved into the right-hand side term, and it depends on the a priori data as stated in the theorem, which should be clear in the context.

Next, by following a similar argument to the proof of proposition 2.22 in [17], one can show that there exists \( C_{\mu,L,M} > 0 \) such that
\[
\left| \frac{K}{K_-} \right|^{-\frac{1}{2}} \left( \frac{K}{K_+} \right)^{-\frac{1}{2}} e^{\frac{1}{2}(\frac{1}{K} - \frac{1}{K_+})} \leq C_{\mu,L,M} K^{-\delta}. \quad (3.26)
\]
Using \( h = 1/K \) and \( b = \sqrt{M}/K \) in the definition of the \( K \)-curvature point in definition 2.1, the estimate (3.25) further yields
\[
C|\nabla u_0(p)| \leq (1 + (\tau/K)^{(n-1)/2}) K^\nu e^{\frac{3}{2}(\frac{1}{K} - \frac{1}{K_+})} \|g\|_{H^{-1/2}(\Omega)} \\
+ K^\nu \|g\|_{H^{-1/2}(\Omega)} (h + K_-)^{-\frac{1}{2}} (h + K_-)^{\frac{n+1}{2}} (K/K_-)^{(n-1)/2} e^{\frac{1}{2}(\frac{1}{K} - \frac{1}{K_+})} \\
+ K^\nu \|g\|_{H^{-1/2}(\Omega)} K^{\alpha} (h + K_-)^{(n-1)/2} e^{\frac{1}{2}(\frac{1}{K} - \frac{1}{K_+})}. \quad (3.27)
\]

Choosing \( \tau = 4K \ln K^\mu \) for some \( \rho > 0 \) and dividing by \( \|g\|_{H^{-1/2}(\Omega)} \), the left-hand side of (3.27) can be estimated by
\[
(\ln K)^{(n-1)/2} K^{\rho - 3\rho} + K^{\mu - \delta + \rho} + (\ln K)^{3/2} K^{\mu + 1/2 - \alpha + \rho} \\
+ (\ln K)^{(n+1)/2} K^{\mu - \alpha - 3\rho}. \quad (3.28)
\]
By setting \( \rho = \min(\alpha, \delta)/2 \), each of the terms in (3.28) can be estimated by
\[
C(\ln K)^{(n+1)/2} K^{\mu - \min(\alpha, \delta)/2}, \quad (3.29)
\]
and thus the claim of this theorem follows.

The proof is complete. \( \square \)
4. A local uniqueness result for Calderón’s inverse inclusion problem

We are in a position to present the full details of the local unique recovery result in theorem 1.1. To that end, we first introduce another technical condition on the input \( g \in H_0^{-1/2}(\partial \Omega) \).

**Assumption B.** Let \( u \) be the solution to (1.2). The input \( g \) fulfills that for all \( x \in \Omega \),

\[
\lim_{r \to +0} \frac{\int_{B_r(x)} |\nabla u(z)| \, dz}{|B_r(x)|} \geq m_g, \tag{4.1}
\]

where \( m_g > 0 \) is a positive constant independent of the conductivity function \( \gamma \).

**Remark 4.1.** Assumption B means that the electric field generated by the input \( g \) and the conductive inclusion \((D, \eta)\) is positive everywhere within \( \Omega \). This is a mild condition on the input \( g \) which can be fulfilled in practical scenarios. In fact, the condition (4.1) in the two-dimensional case is true if the input data \( g \) has only one local maximum and only one local minimum on \( \partial \Omega \) (see e.g. [1, 6]). Throughout the rest of the paper, we assume that the condition holds true in our study.

**Theorem 4.2.** Let \((D, \eta), (\tilde{D}, \tilde{\eta})\) be described in theorem 1.1. Suppose that assumptions A and B hold for \( u \) and \( \tilde{u} \). Let \( d_0 \in \mathbb{R}_+ \) and \( \Gamma_0 \subset \partial \Omega \). If \( u = \tilde{u} \) on \( \Gamma_0 \), then \( D \Delta \tilde{D} = (D \backslash \tilde{D}) \cup (\tilde{D} \backslash D) \) cannot possess an admissible \( K \)-curvature point \( p \) such that

\[
\max \{\dist(p, \partial D), \dist(p, \partial \tilde{D})\} > d_0, \tag{4.2}
\]

and \( K \geq K_0 \), where \( K_0 \in \mathbb{R}_+ \) is sufficiently large and depends on the a priori parameters of \( p \) in definition 2.1 as well as \( \Omega, D, \tilde{D}, d_0, g, \eta, \tilde{\eta} \).

**Proof.** By absurdity, we assume without loss of generality that there exists an admissible \( K \)-curvature point \( p \in \partial D \cup \partial \tilde{D} \), such that \( B(p, d_0) \subset \Omega \backslash \tilde{D} \). Clearly, one can arrive at the transmission problem (2.11) and (2.12). By applying theorem 3.7, we immediately obtain from (3.24) that

\[
|\nabla u(p)| \leq \mathcal{E} \|g\|_{H^{-1/2}(\partial \Omega)} (\ln K)_{n+1/2} K^{n-\min(1, \delta)/2}. \tag{4.3}
\]

Clearly, the right-hand side of the above estimate tends to zero as \( K \to +\infty \). Therefore, we can choose \( K_0 \) such that when \( K > K_0 \), \( |\nabla u(p)| < m_g \), which contradicts to the assumption B in the theorem.

The proof is complete. \( \square \)

**Acknowledgments**

The work of H Liu was supported by a startup grant from City University of Hong Kong, Hong Kong RGC General Research Funds, 12301218, 12302919 and 12301420. The authors would like to express their gratitude to an editorial board member and the anonymous referee for many constructive comments and suggestions, which have led to significant improvements on the presentation as well as the results of our paper.

**Data availability statement**

No new data were created or analysed in this study.
Appendix. Further remark on assumption A

In theorem 4.2, we require that the condition (3.13) holds true, namely assumption A. That is, the exponent $\mu$ in (3.4) and (3.5) (or equivalently in (3.3) in remark 3.3) is required to satisfy (3.13). The theoretical result in [29] only shows that $\mu$ is a positive parameter, whereas we need a more precise upper bound of $\mu$ in order to establish the estimate of $|\nabla u_i(p)|$ in theorem 3.7. It is important to verify if this condition indeed holds in generic scenarios. As remarked earlier, in this appendix, we shall present two typical numerical examples to illustrate this condition indeed holds. As is also noted earlier that the requirement of the positive constant $\delta$ in definition 2.1 is not restrictive, and hence it is sufficient to numerically verify that (3.4) and (3.5) can hold for $\mu < 1/2$.

Our numerical simulations below focus on the local behaviours of $|\nabla u|$ near the $K$-curvature point $p$. For illustration, we only consider the two-dimensional case. We recall the configuration in lemma 3.2 and remark 3.3. Let $p$ be a $K$-curvature point, and the interface in its neighbourhood can be represented by $\{x_2 = w(x_1)\}$ for $x = (x_1, x_2)$. Let $Q \Subset \mathbb{R}^2$ be a domain containing $p$ such that $Q$ is divided into two non-empty sub-domains $Q^\pm$ by the interface. We consider the following conductivity equation,

$$
\begin{aligned}
\text{div}[(1 + (\eta - 1)\chi)\nabla u] &= 0 \quad \text{in } Q, \\
u = f &\in H^{1/2}(\partial Q) \quad \text{on } \partial Q,
\end{aligned}
$$

(A.1)

where $\chi$ is the characteristic function $\chi(x) = 1$ if $x_2 > w(x_1)$ and $\chi(x) = 0$ if $x_2 < w(x_1)$. The Dirichlet boundary condition $f$ is arbitrarily chosen.

Our numerical experiments is to study the relationship between $\max_Q|\nabla u|$ and the curvature $K$. In order to do so, we choose two sets of interfaces, and we test different values of the curvature $K$ in each set of interfaces. The interfaces are precisely described by parametric curves where the point $p$ is the point with the maximum curvature. In each set of interfaces, only the curvature $K$ at $p$ is variable from case to case. To illustrate the relation (3.3), we draw the regression line of $\log(K)$ respect to $\log(\max_Q|\nabla u|)$. The slopes of the regression lines give the estimates of the values $\mu$ in the corresponding scenarios.

The technical settings for numerical simulations are specified as follows:

- We use Freefem++ [33] as the finite element method (FEM) solver of the conductivity equation (A.1).
- $Q$ is the square of side 10 centred at the origin in $\mathbb{R}^2$.
- We choose a Dirichlet boundary condition on $\partial Q$: $f = 2x_1 + 3x_2$.
- We choose two sets of interface functions.
  - Parabolic interface, $w(x_1) = Kx_1^2$.
  - Hyperbolic interface, $w(x_1) = A \sqrt{x_1^2 + (\frac{K}{2})^2}$.
- The values of $K$ are taken as: $K = 1, 1.5, 1.5^2, \ldots, 1.5^9$.
- We choose the conductivity $\eta = 2$ in (A.1).
- For each value of $K$, we solve numerically the equation (A.1) with the Dirichlet condition above.
We trace the regression line of $\log(K)$ respect to $\log(\max_Q |\nabla u|)$ for each set of the interface.

The numerical results are summarized in figures 2 and 3. We can easily observe that the maximum of the gradient $\max_Q |\nabla u|$ indeed increase as $K$ grows. Moreover, we can estimate the value of $\mu$ respectively in the two cases by the regression lines. The regression lines indicate that in those two examples, the value of $\mu$ can be estimated as

(a) $\mu_{\text{parabola}} = 0.027$;
(b) $\mu_{\text{hyperbola}} = 0.1099$.

With those numerical results, we can conclude that the assumption (4) in theorem 4.2 is not void. In the case of parabolic interfaces, we have even observed a relatively weak dependence of $\nabla u$ on $K$. It is thus reasonable to make such an assumption to derive our local uniqueness result.

Figure 2. Parabolic interfaces.
Figure 3. Hyperbolic interfaces.

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