Soft S-Matrices, Defects and the Double Copy on the Celestial Sphere

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We recast the soft S-matrices on the celestial sphere as correlation functions of certain 2-dimensional models of topological defects. In pointing out the double copy structure between the soft photon and soft graviton cases, we arrive at a putative classical double copy between the corresponding topological models and a rederivation of gauge invariance and the equivalence principle as Ward identities of the 2-dimensional theories.

I. INTRODUCTION

The study of scattering amplitudes as analytical objects in their own right has been a rather fertile one in recent years. Of particular interest has been the soft sector of the S-matrix for theories with massless particles, which as it turns out reflects underlying symmetries which are manifest at null infinity (see [1–22] and references therein).

Emerging from this analysis is a picture of flat space holography, in which (massless) particles on the celestial sphere control most of the analytic properties of the entire S-matrix. The representation of the S-matrix in terms of operators on the celestial sphere is realized by a change of basis brought about by the Mellin transform [23, 24]. In this basis, the infrared sector of the S-matrix can be understood with relative ease in terms of operator product expansions (OPEs) of insertions on \(\mathbb{C}P^1\) [25–28]. This has led to a conjectural 4d/2d duality, which relates the S-matrix to correlation functions of operators of an alleged CFT on the celestial sphere.

In this short note, in focusing our attention on the soft S-matrix, we will realize an amusing web of dualities. The soft S-matrix for photons is recast as a correlation function of a Coulomb gas, which is a manifestly two-dimensional system. A double copy construction relating the soft photon S-matrix to the one in gravity is manifested by squaring the kinetic operator of the dual 2d model. The resulting model in two dimensions is identified with a dynamical system of crystal disclinations. These observations lead to a rather satisfying web of dualities which to our knowledge has been stated in this part of the infrared finite. Indeed, it has been argued recently that this part of the S-matrix can be defined unambiguously [30, 31].

In the eikonal limit, this portion of the S-matrix is especially simple, and can be written in dimensional regularization\(^1\) (\(d = 4 + \epsilon\)) as [32]

\[
\ln \left( A_{\text{soft}}^{n,s=1} |_{\text{vir}} \right) = -\frac{1}{8\pi^2\epsilon} \sum_{i \neq j} e_i e_j \ln |z_i - z_j|^2 .
\]

Here, \(z_i\) are points on \(\mathbb{C}P^1\), determined by directions of the external momenta according to the prescription

\[
p_k = \omega_k (1 + z_k \mathcal{E}_k, z_k + \mathcal{E}_k, -i(z_k - \mathcal{E}_k), 1 - z_k \mathcal{E}_k) .
\]

We point out that the divergence manifested here cancels in suitably inclusive processes [32–35]. It has been

\(^1\) The dimensional regularization scheme used here makes application of the fact that IR divergences can be controlled beyond 4 dimensions.
established that this object can be represented as correlation functions of Wilson lines defined using asymptotic values of the gauge field and metric for the spin-1 [25] and spin-2 cases [28] respectively.

Moving now to the first leg of the dualities, let the following action be noted

\[ S_1 = 8\pi \varepsilon \int d^2 z \varphi(z, \overline{z}) \partial \overline{\partial} \varphi(z, \overline{z}). \] (4)

This is easily identified as the Coulomb gas CFT in stereographic coordinates. It exhibits the well known BKT transition due to vortex binding below a critical temperature\(^2\).

The two point function is simply the Green’s function on \( \mathbb{CP}^1 \)

\[ \langle \varphi(z_1, \overline{z}_1), \varphi(z_2, \overline{z}_2) \rangle = \frac{1}{8\pi^2 \varepsilon} \ln |z_1 - z_2|^2. \] (5)

We find by direct computation that the vertex operators defined as

\[ V_j(z_j, \overline{z}_j) =: e^{i e_j \varphi(z_j, \overline{z}_j)} ; \] (6)

lead to the equivalence

\[ \langle V_1(z_1, \overline{z}_1) \cdots V_n(z_n, \overline{z}_n) \rangle = A^{soft}_n|_{vir, s=1}. \] (7)

We should like to emphasize that this is not a correlator of Wilson lines - it computes the correlation function of vertex operators in the Coulomb gas theory. In other words, the soft S-matrix can be derived equivalently as a correlator of vertex operators belonging to a dual two-dimensional model of interacting electrons.

So far the relationship discussed is purely formal and not especially surprising; the form of the integrated soft contributions lends itself quite easily to this dual picture. However, we would now like to understand how this observation can help inform a new perspective into this dual picture. The following section is dedicated to laying out the argument that realizes this hope. To do so, we turn our attention to the soft S-matrix for gravitationally interacting particles.

### III. A DOUBLE COPY ON THE CELESTIAL SPHERE

The soft part of the S-matrix can be defined for any theory with massless quanta. In this section, we are concerned with the case of interacting spin-2 quanta. Accordingly, the virtual particles circulating in soft loops are gravitons. The soft S-matrix for such theories takes the form

\[ \ln \left( A^{soft}_{n,s=2}|_{vir} \right) = -\frac{1}{8\pi^2} \sum_{i \neq j} \kappa_i \kappa_j |\omega_i - \omega_j|^2 \ln |z_i - z_j|^2 \] (8)

where \( \kappa_i \) is the coupling constant characterizing the strength of interaction between the corresponding particle and the graviton. We know however that \( \kappa_i \) must be the same for all \( i \) by the equivalence principle. It will be seen that this need not be assumed; it arises as an implication of Ward identities of the two-dimensional model to be described.

In the appendix, we relate this expansion of the soft factor in gravity to those in the case of photons and scalars to derive a soft version of the KLT relation, which holds here at the level of integrated soft factors rather than the integrand. However, it is worthwhile to ask if there is a version of the double copy on the celestial soft factors that can be realized without appealing to a nonlinear relation like the KLT relations. The purpose of this section is to provide an affirmative answer to this question.

Take the kernel

\[ K(z_{ij}) = |z_{ij}|^2 \ln |z_{ij}|^2 \] (9)

where \( z_{ij} = z_i - z_j \). Now, by direct calculation we have the result,

\[ \partial \overline{\partial} K(z_{ij}) \sim \ln |z_{ij}|^2 \] (10)

where the \( \sim \) is used to indicate equality up to an additive constant. From this we have

\[ (\partial \overline{\partial})^2 K(z) = \delta^2(z, \overline{z}) \] (11)

which indicates that the kernel \( K \) is the Green’s function of the square of the Laplacian \( \Delta = (\partial \overline{\partial})^2 \). In light with this, suppose we have a nonlinear sigma model defined by the action,

\[ S_2 = 8\pi \varepsilon \int d^2 z e(z, \overline{z}) (\partial \overline{\partial})^2 e(z, \overline{z}). \] (12)

With the vertex operators

\(^2\) Wilson lines can also be used to extract the soft theorems as well, and were applied in proofs of soft-collinear factorization in the past - see [36–40] and references therein.

\(^3\) The BKT transition, named after Brezenskii, Kosterlitz and Thouless is a topological phase transition. Unlike conventional Landau transitions, it is precipitated by the formation of topologically nontrivial excitations, which condense below a critical temperature. See [41] for an overview of numerical estimates of the BKT transition temperature.

\(^4\) See [42] for a Mellin space perspective at low multiplicity.
\[ U_j(z_j, z_j) = e^{i\kappa_j \omega_j (z_j, z_j)} \]  

it is the case that

\[ A_{n,s=2}^{\text{soft}}|_{\text{vir}} = \langle U_1(z_1, z_1) \cdots U_n(z_n, z_n) \rangle. \]  

It is generally expected that there is some double copy structure relating the S-matrices of gauge theory and gravity. Here we see a natural manifestation of this in the form of the soft factors. Unlike standard double copy relations however, this one is not kinematic in any sense. Instead, the double copy is manifested by an abstract replacement of the kinetic operator in an auxiliary theory by its square. In summary, the double copy between the spin-1 and spin-2 soft S-matrices is obtained by the replacements,

\[ \partial \overrightarrow{\partial} \rightarrow (\partial \overrightarrow{\partial})^2 \]  

\[ e_i \rightarrow \kappa_i. \]  

It is worth noting that the phrase double copy has only been used schematically. No relation is to be construed between this prescription and traditional double copies like BCJ or KLT. All we have indicated is that one moves from the spin-1 case to the spin-2 case by 'double copying', or squaring, a dynamical quantity, namely the kinetic operator. Instead, the double copy is manifested by an abstract replacement of the kinetic operator in an auxiliary theory by its square. In summary, the double copy between the spin-1 and spin-2 soft S-matrices is obtained by the replacements,

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\[ e_i \rightarrow \kappa_i. \]  

The action (12) determines a genuine two-dimensional model. Indeed, singularities of the sort exhibited by (9) arise in the theory of crystal dislocations. Consider the tensor,

\[ \sigma_{ij} = \varepsilon_{ik} \varepsilon_{jl} \partial_k \partial_l \chi, \]  

where \( \chi := \chi(x, y) \) depends on a source \( \eta(x, y) \) through

\[ \nabla^4 \chi(x, y) = \eta(x, y) \]  

and the tensor \( \varepsilon_{ij} \) is the Levi-Civita tensor in two dimensions. \( \sigma_{ij} \) is the two-dimensional analogue of the stress tensor; it measures the stress and shear suffered by the system supporting dislocated lattice points. Given a suitably singular \( \eta \) of the form

\[ \eta(x, y) = \sum_a b_a(x, y) \]  

\[ \frac{1}{2} \int \chi(x, y) \eta(x, y) d^2x \]  

characterizes that of a solid with a line defect normal to the plane charted by the \((x, y)\) coordinates. The vectors \( b_a(x, y) \) are known as Burgers vectors, which characterize the direction of dislocation at each defect site. We note however that in our model the source function \( \eta \) never quite becomes that singular, since we only need to consider sources of the form

\[ \eta(x, y) = \sum_a \kappa_a \delta^2(x, y). \]  

This tells us that the theory we have on our hands is not of dislocations, which are line defects, but of objects known as disclinations, which experience a strongly confining interaction controlled by a biharmonic field equation and are point defects. We do not consider the theory of defects in larger detail here, reserving potential expansions of this point of view for future research. For interested readers however additional technical details on defects can be found in [44] and references therein.

With the results of this section and the last, we have determined that there exists a double copy construction that relates the soft S-matrix for photons to that of gravitons, albeit in a manner that is markedly different from the kinematical identities normally employed. Such a double copy structure is significant, since it appears to exist at the level of soft factors post phase space integration. It has been confirmed by recasting the S-matrices in terms of conformal correlators of auxiliary theories with topological degrees of freedom, thereby positing putative relationships between the respective duals as well. Diagrammatically we have

\[ A_{n,s=1}^{\text{soft}}|_{\text{vir}} \xrightarrow{\text{Double Copy}} A_{n,s=2}^{\text{soft}}|_{\text{vir}} \]

\[ (V_1 \cdots V_n) \xrightarrow{\Delta \rightarrow \Delta^2} \langle U_1 \cdots U_n \rangle. \]

IV. THE SOFT THEOREMS

We have one more piece of the story to discuss, namely that of the soft theorems. So far, we have analyzed dualities between topological models that have been shown

\[ \text{Thanks to Radu Roiban for comments that encouraged me to clarify this point}. \]

\[ \text{Note also that we have switched to two-dimensional Cartesian coordinates}. \]
to reproduce the virtual soft $S$-matrices. The goal of this section is to establish that Ward identities in the two-dimensional model are responsible for the soft photon and graviton theorems, arising due to the emission of real soft particles.

It is well known that the two-dimensional Coulomb gas model exhibits a so-called shift symmetry (see [45] for details). Specifically, the invariance of the action (4) under global shifts of the form $\varphi \rightarrow \varphi + a$ implies the existence of a conserved holomorphic current, which may be readily verified. Indeed, this is precisely the soft photon theorem expressed in stereographic coordinates. It remains now only to verify the implication of the soft theorem due to Weinberg [29].

For thoroughness, we provide one more proof of charge conservation and the equivalence principles as implications of invariance under shift symmetry. The line of reasoning follows that of [45].

Let us consider a shift $\varphi \rightarrow \varphi + a$. Since this is a global symmetry, we demand that under it the correlator (7) be invariant. However, performing this operation on (7), we encounter a phase,

$$e^{ia(e_1 + \cdots + e_n)}.$$ 

For this to equal unity for arbitrary $a$, we are led to require

$$e_1 + \cdots + e_n = 0$$

which is the conservation of charge.

Applying this argument to the spin-2 case means that we study the effects of a shift $c \rightarrow c + a$ on the correlator (14). Indeed, we notice a familiar phase factor of

$$e^{ia(\omega_1\kappa_1 + \cdots + \omega_n\kappa_n)}.$$ 

Insisting that this equal unity again forces us to set

$$\omega_1\kappa_1 + \cdots + \omega_n\kappa_n = 0,$$

which is only commensurate with generic kinematics when the $\kappa_i$ are identical, again yielding the equivalence principle.

V. DISCUSSION

In this short note we have studied a dual model of soft divergences in quantum field theory. We related the soft $S$-matrix for theories with massless spin-1 particles to a dual model of topological defects on the worldsheet,
known variously as the Coulomb gas model or XY model. This construction made it possible to identify a double copy prescription that automatically yields the soft $S$-matrix for theories with gravitons. The corresponding dual model when recognized as a model of crystal dislocations completes a satisfying web of dualities, summarized in (21).

We suggest some possible avenues for future work. There has been much recent interest in worldsheet [46–55] and geometric models [56–63] of scattering amplitudes in quantum field theory. In particular, certain two dimensional models at null infinity have been shown to reproduce as correlation functions scattering amplitudes in gauge theory [64] and gravity [65]. A synthesis of this work with the ideas in this paper may reveal a consistent picture of massless amplitudes at null infinity.

It is important to keep in mind that the results in this paper hold in the so-called eikonal limit. While gauge theory soft amplitudes are corrected at higher loop order and the gravitational analogues are not, the double copy has been shown to hold at the level of Feynman diagrams [66]. Having only dealt with abelian interactions in this note, it remains to be seen with future work how the picture developed here plays out beyond the eikonal limit.

Vertex operators in the Coulomb gas picture are conformal primaries. While we have defined vertex operators by analogy for the gas of disclinations, a simple interpretation of the state generated by the vertex operator is unclear. One may regard such operators as point sources by analogy for the gas of disclinations, a simple interpretation of the state generated by the vertex operator is unclear. One may regard such operators as point sources of disclinations, as the correlation function (14) generates the free energy of a gas of dislocations. Finding a more satisfying interpretation can be a topic of further study.

Finally, one natural and concrete step forward would be understanding how higher-order soft theorems can be derived in this framework.

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APPENDIX

Here we provide some contextual details for those unfamiliar with the holographic representation of the soft $S$-matrix. We will first explain some of the salient features of the expression of soft factors in terms of Wilson lines alluded to in the main text. For purposes of clarity, we will present the relevant information separately for the spin-1 and spin-2 cases.

Let us start with the case of massless spin-1 particles being communicated as soft virtual particles in some scattering process. For purposes of simplicity, we stick to massless scattering, although the generalization to massive particles is straightforward. Recall that the celestial sphere is defined implicitly, by expanding the momentum $p_k$ of a massless particle in the form,

$$p_k = \omega_k \left( 1 + z_k \bar{z}_k, z_k + \bar{z}_k, -i(z_k - \bar{z}_k), 1 - z_k \bar{z}_k \right).$$

Furthermore we have,

$$p_k \cdot p_l = \omega_k \omega_l |z_k - z_l|^2.$$

Lorentz invariant quantities can then be expressed up to energy factors entirely in terms of such distances on the celestial sphere. Indeed, it can be shown that similar expansions can be used for polarization vectors as well. For extensive details on such computations, see [18].

The celestial sphere in position space tiles the boundaries of the Penrose diagram for flat space. Namely, both past and future null infinity are tilled by copies of the celestial sphere, each copy labelled by the null coordinate on the boundaries, which could be advanced or retarded. Accordingly, the Maxwell field $A_k$ can be expanded at infinity. Consider the component $A_z$, specifically the leading part $A_z^{(0)}|_{\mathcal{I}_z^\pm}^{10}$ when expanded at null infinity in inverse powers of $r$. The linear combination

$$S_z = \frac{i}{4} \left( A_z^{(0)}|_{\mathcal{I}_z^+} + A_z^{(0)}|_{\mathcal{I}_z^+} + A_z^{(0)}|_{\mathcal{I}_z^-} + A_z^{(0)}|_{\mathcal{I}_z^-} \right)$$

defines a current $\phi(z, \bar{z})$ by the implicit relation

$$S_z(z, \bar{z}) = i \partial_z \phi(z, \bar{z}).$$

Given these boundary components, we can now quantize them by developing them in terms of Fourier modes asymptotically (see again [18] for details). When this is done, it can be shown that the operator $\phi(z, \bar{z})$ gives rise to the correlation function

\[10 \mathcal{I}_z^\pm \] denote the past and future boundary of future null infinity. Similarly, $\mathcal{I}_z^\pm$ indicate the past and future boundaries of past null infinity.

9 I thank Radu Roiban for raising this question.
\[ \langle \phi(z, \bar{z}), \phi(w, \bar{w}) \rangle \sim \ln |z - w|^2. \]  

This tells us that the Wilson line defined as,

\[ W_{s=1}(z, \bar{z}) = e^{ic\phi(z, \bar{z})} \]  

reproduces the soft factor upon evaluation of the correlator

\[ (W_{s=1}(z_1, \bar{z}_1) \cdots W_{s=1}(z_n, \bar{z}_n)). \]  

The factorization of soft and hard modes is achieved by decomposing any operator \( \mathcal{O} \) on the celestial sphere into a hard \( \mathcal{O} \), which has IR finite correlation functions and a soft part according to

\[ \mathcal{O} = W_{s=1}\mathcal{O}. \]  

This provides context for the alternative construction we have discussed in this work. In our framework, the hard and soft parts of the theory are putatively described by two different models entirely. The soft part in particular is reproduced by a free field Coulomb gas.

The case of gravity is analogous, but the background required is somewhat more involved, as it involves quantization of the metric for asymptotically flat spacetimes. Briefly, in retarded coordinates \((r, u, z, \bar{z}) := (r, dx^i)\), the metric may be asymptotically developed as a series

\[ ds^2 = -\gamma_{ij}dx^i dx^j + rC_{zz}dz^2 + rC_{\bar{z}z}d\bar{z}^2 + \ldots \]  

where \( \gamma_{ij} \) is the flat metric on null infinity and \( C_{zz} \) and \( C_{\bar{z}z} \) are asymptotic fields that encode gravitational degrees of freedom. For such spaces, Poincaré symmetry is enlarged to a much larger Bondi-Metzner-Sachs (BMS) symmetry. Among other things, it was found in recent analyses ([18] and references therein) that the Ward identities of the so-called supertranslations - angle dependent translations on the celestial sphere contained in the BMS group - are equivalent to the soft photon theorems after asymptotic quantization.

The field \( C_{zz} \) can be extrapolated to past-future null infinity \( \mathcal{I}^\pm \), where it determines a field \( C \) according to the implicit relation

\[ C_{zz}|_{\mathcal{I}^\pm} = -\partial^2 C. \]  

Additionally, it can be shown that this field is precisely the Goldstone mode arising out of the spontaneous violation of the invariance under the full BMS group. The spin-2 Wilson line is defined according to

\[ W_{s=2}(z, \bar{z}) = e^{i\omega C(z, \bar{z})}. \]

Now it can be inferred directly from the field expansion that the following correlator holds

\[ \langle C(z, \bar{z}), C(w, \bar{w}) \rangle \sim |z - w|^2 \ln |z - w|^2. \]

It is then checked quite easily that the correlator of the form (40) using instead the spin-2 Wilson line operator reproduces the correct soft factor for spin-2 particles traversing loops. Again, the purpose of this work has been to replace this use of Wilson line operators by vertex operators of a dual, auxiliary and free theory, capable of encoding the soft \( S \)-matrix without reference to asymptotic fields.

The form of the soft factors lends itself to the derivation of a cute analogue of the KLT relation, which we will now derive and discuss. Consider the soft factors for the cases of spin 0, spin-1 and spin-2 in order. We have\(^{11}\)

\[ \ln \left( A_{n,s=0|vir}^{soft} \right) = -\frac{1}{8\pi^2} \sum_{i \neq j} \frac{g_{i,j}}{\omega_i \omega_j |z_{ij}|^2} \ln |z_{ij}|^2, \]  

\[ \ln \left( A_{n,s=1|vir}^{soft} \right) = -\frac{1}{8\pi^2} \sum_{i \neq j} c_i c_j \ln |z_{ij}|^2 \]  

and

\[ \ln \left( A_{n,s=2|vir}^{soft} \right) = -\frac{1}{8\pi^2} \sum_{i \neq j} \kappa_i \kappa_j |z_{ij}|^2 |\ln |z_{ij}|^2 |. \]

Here, we have denoted the couplings to the scalars by \( g_i \).

There’s no symmetry principle at play here, so we keep things general. We see here that defining

\[ g_{i,j,s=0} = \frac{1}{|z_{ij}|^2}, \]

\[ g_{i,j,s=1} = \ln |z_{ij}|^2 \]

\[ g_{i,j,s=2} = |z_{ij}|^2 \ln |z_{ij}|^2 \]

we can infer

\[ g_{i,j,s=2} = \frac{g_{i,j,s=1}^2}{g_{i,j,s=0}} \]

which is the analogue of the KLT relation for soft factors. This may strike one as a bit of a triviality, but it is quite

\(^{11}\) The soft factor for scalars can be derived by simply noting that the kinematical dependence of the factors for \( s=1 \) and \( s=2 \) is precisely \( p_i \cdot p_j \) and \( (p_i \cdot p_j)^2 \) prior to phase space integration. The integration removes one factor of \( p_i \cdot p_j \). Since the kinematical dependence of scalars is trivial prior to integration, this operation introduces a denominator.
gratifying that something like this holds post phase space integration. If one follows the derivation of these factors closely, which can be done by simple analogy referring to the spin-2 computation done in [67], the relation isn’t especially surprising. Nevertheless, it is nice to see something like this come up when it does.

We remark parenthetically that it is possible to find the dual model that gives rise to the Green’s function $g_{ij,s=0}$. Unlike the spin-1 and spin-2 cases however, the theory is fairly nonlocal, with rather baroque expressions for the Green’s function (we include the expressions in an attached MATHEMATICA notebook). The nonlocality is more easily seen when the Green’s functions for the various spins are compared. The relevant Green’s functions have been plotted in figure 1.

![Figure 1](image.png)

The upper line represents the Green’s function for spin-2, followed by the spin-1 case and the spin 0 case at the bottom.

The nonlocality of the spin 0 dual model can also be inferred from the behaviour of the Green’s function for large distances. There is no appreciable decay over long ranges, indicating a rather severe nonlocality. It is hard to see that such a system would have any real meaning, and it is interesting that good local behaviour seems to be a property only enjoyed beyond spin-1 (it is a simple case to check that a power $(\partial n)$ supplies the soft factor for the spin n case. We haven’t considered it here - massless spin n particles are ruled out by an analysis of the soft theorem). We defer a more detailed study of these matters to future research.

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