SOME ALGORITHMS FOR SEMI-INVARIANTS OF QUIVERS.

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Abstract. We present some theorems and algorithms for calculating perpendicular categories and locally semi-simple decompositions. We implemented a computer program TETIVA based on these algorithms and we offer this program for everybody’s use.

1. Introduction

Let $Q$ be a finite quiver with the set $Q_0$ of vertices and $Q_1$ of arrows; for an arrow $\varphi \in Q_1$ denote by $t\varphi$ and $h\varphi$ its tail and its head, respectively. For $\alpha \in \mathbb{Z}^{|Q_0|}$ denote by $R(Q, \alpha)$ the set of the representations of $Q$ of dimension $\alpha$ over an algebraically closed field $k$ of characteristic 0, i.e., $R(Q, \alpha) = \bigoplus_{\varphi \in Q_1} \text{Hom}(k^{\alpha t\varphi}, k^{\alpha h\varphi})$. For $V \in R(Q, \alpha), a \in Q_0, \varphi \in Q_1$ denote by $V(a)$ the vector space $k^{\alpha a}$ sitting at $a$ and by $V(\varphi) : V(t\varphi) \rightarrow V(h\varphi)$ the component of $V$. For representations $U, V$ of $Q$ a homomorphism $f : U \rightarrow V$ is a collection $f = (f_a | a \in Q_0, f_a \in \text{Hom}(U(a), V(a)))$ such that for each $\varphi \in Q_1$ holds $V(\varphi)f(t\varphi) = f(h\varphi)U(\varphi)$. By $\text{Hom}(U, V)$ denote the vector space of all homomorphisms.

Clearly, $R(Q, \alpha)$ is a $k$-vector space and there is a natural linear representation $\rho$ of the group $GL(\alpha) = \prod_{i \in Q_0} GL(\alpha_i)$ in $R(Q, \alpha)$:

$$\rho(g)(V)(\varphi) = g(h\varphi) V(\varphi)(g(t\varphi))^{-1},$$

such that the orbits of $GL(\alpha)$ are the isomorphism classes of representations.

Recall that $\beta \in \mathbb{Z}^{|Q_0|}$ is called a root if $R(Q, \beta)$ contains an indecomposable representation and a Schur root if generic element $U \in R(Q, \beta)$ is indecomposable, and in this case $\text{Aut}(U) = k^*$ ([Kac]). By the Krull-Schmidt theorem for any representation $V \in R(Q, \alpha)$ there is a decomposition $V = R_1 \oplus R_2 \oplus \cdots \oplus R_t$ into a sum of indecomposable summands and it is unique up to permutations and isomorphisms of summands. In particular, $V$ yields a decomposition $\alpha = \dim R_1 + \cdots + \dim R_t$ into the sum of roots and this decomposition does not change over the isomorphism class of $V$. This decomposition is an important invariant of $V$ and in some cases it allows to recover the group $\text{Aut}(V)$ or even the isomorphism class of $V$. The main subject of this paper is to study several special classes of representations and to calculate the corresponding decompositions.

It is shown in [Kac] that among all the decompositions of $\alpha$ into the sum of roots there is a generic element such that there is an open dense subset in $R(Q, \alpha)$ consisting of representations with such a decomposition; V.Kac called this canonical decomposition. We however follow another tradition and prefer the name generic for the same object. By [Kac] each root in the generic decomposition is a Schur root, and moreover a decomposition $\alpha = \beta_1 + \cdots + \beta_k$ is generic if and only if for generic
$B_i \in R(Q, \beta_i)$ holds: $\text{Aut}(B_i) = k^*$ and $\text{Ext}(B_i, B_j) = 0$ for $i, j = 1, \cdots, t, i \neq j$.

Recall that C.M.Ringel introduced in [4] the Euler bilinear form
\begin{equation}
\langle \alpha, \beta \rangle = \sum_{a \in Q_0} \alpha_a \beta_a - \sum_{\varphi \in Q_1} \alpha_{t\varphi} \beta_{h\varphi},
\end{equation}
such that the Tits form can be written as $q_Q(\alpha) = \langle \alpha, \alpha \rangle$, and proved the formula:
\begin{equation}
\dim \text{Ext}(U, V) = \dim \text{Hom}(U, V) - \langle \dim U, \dim V \rangle.
\end{equation}
Applying the above formula to the generic decomposition $\alpha = \beta_1 + \cdots + \beta_t$, we have
\begin{equation}
\langle \beta_i, \beta_j \rangle = \dim \text{Hom}(B_i, B_j) \geq 0, \text{ and in particular, } \beta_i = \beta_j = \beta \text{ implies } q_Q(\beta) \geq 0,
\end{equation}
so imaginary roots $\beta$ with $q_Q(\beta) < 0$ may occur in the generic decomposition only with multiplicity 1. V.Kac found the above properties of the generic decomposition and addressed the problem to find an algorithm for calculation of the generic decomposition in terms of the Euler form. This is done in [2].

Denote by $SL(\alpha)$ the commutator subgroup of $GL(\alpha)$, $SL(\alpha) = \prod_{i \in Q_0} SL(\alpha_i) \subseteq GL(\alpha)$. A natural task of the Invariant Theory in the quiver setup is to study the $GL(\alpha)$-semi-invariant functions on $R(Q, \alpha)$, which are also $SL(\alpha)$-invariant. To be precise, the character group of $GL(\alpha)$ is isomorphic to $Z^{Q_0}$ such that $\chi \in Z^{Q_0}$ gives rise to the character $\chi = \prod a \in Q_0 \chi_a \det_a^a$ and we have:
\begin{equation}
k[R(Q, \alpha)]^{SL(\alpha)}_{\chi} = \bigoplus_{\chi \in Z^{Q_0}} k[R(Q, \alpha)]^{(GL(\alpha))}_{\chi},
\end{equation}
where $k[R(Q, \alpha)]^{(GL(\alpha))}_{\chi} = \{ f \in k[R(Q, \alpha)] \mid \forall \chi f(\alpha) f = \chi(\alpha) f, \forall \alpha \in GL(\alpha) \}$. Note also that $k[R(Q, \alpha)]^{(GL(\alpha))}_{\chi} \neq \{0\}$ implies $\chi(\alpha) = \sum_{a \in Q_0} \chi_a \alpha_a = 0$.

A.Schofield introduced in [Sch91] a correspondence between representations and semi-invariants. Namely, for any representation $W$ there is a determinantal semi-invariant $c_W$ such that $c_W(V) \neq 0$ if and only if $\langle \dim V, \dim W \rangle = 0$ and $\text{Hom}(V, W) = 0$, hence also $\text{Ext}(V, W) = 0$ by Ringel formula (3). Moreover, the weight of $c_W$ is equal $-\langle ., \dim W \rangle$. Besides, the representations $V$ such that $c_W(V) \neq 0$ constitute an abelian subcategory closed under homomorphisms, extensions, direct sum and summands and this subcategory is denoted by $W$. Similarly, the subcategory $W$ consists of those $V$ such that $\text{Hom}(W, V) = 0$ and $\text{Ext}(W, V) = 0$.

Assume that $Q$ has no oriented cycles. Then for any $\alpha$ and $\chi$ the vector space $k[R(Q, \alpha)]^{(GL(\alpha))}_{\chi}$ is finite dimensional and it is proved in [3] that this vector space is generated by semi-invariants $c_W$, where $W$ is a representation such that $-\langle ., \dim W \rangle = \chi$.

In [2] we introduced a class of representations, which help to study the semi-invariants of quiver from the geometrical point of view:

**Theorem 1.1.** (Sh) Let $V = m_1 S_1 + \cdots + m_t S_t \in R(Q, \alpha)$ be a decomposition into indecomposable summands. The following properties of $V$ are equivalent:
(i) the $SL(\alpha)$-orbit of $V$ is closed in $R(Q, \alpha)$
(ii) the $GL(\alpha)$-orbit of $V$ is closed in $R(Q, \alpha)$
(iii) $S_1, \cdots, S_t$ are simple objects in $W$ for a representation $W$.

We called the representations meeting the equivalent properties of the above Theorem *locally semi-simple*. In particular, these representations meet the formula
\begin{equation}
\dim \text{Hom}(S_i, S_j) = \delta_{ij}.
\end{equation}
This property yields an equality $\text{Aut}(V) \cong GL(m_1) \times \cdots \times GL(m_t)$ so the decomposition corresponding to $V$ completely defines the embedding $\text{Aut}(V) \subseteq GL(\alpha)$. 

\begin{thebibliography}{9}
\bibitem{Sh} D.A. Shmelkin, "..."..."...
\end{thebibliography}
Recall that D.Luna introduced in [Lu] a stratification of the quotient $L//G$ of a finite dimensional module $L$ over a reductive group $G$, $L//G = \sqcup (L//G)_{(G)}$, where

$$(L//G)_{(G)} = \bigcup \big( (L//G)_{(G)} \big)$$

is the subset of those $\xi$ such that the unique closed orbit over $\xi$ is $G$-isomorphic to $G/M$. We introduced in [Sh] a sort of specification of the Luna stratification of $L//G$ by the strata $(L//G)_{(G)}$ such that the $G$-orbit of the points on the unique closed orbit in the fibers are $G$-isomorphic to $G/M$. Each usual Luna stratum of $L//G$ is therefore decomposed into finitely many locally closed substrata. By Theorem 1.1, in the case of $G = GL(\alpha), L = R(Q,\alpha)$ this $GL(\alpha)$-stratification of $R(Q,\alpha)//SL(\alpha)$ is equivalent to the description of all locally semi-simple decompositions of $\alpha$. Of particular interest are the open stratum and the corresponding decomposition that we called generic locally semi-simple.

The results of the paper are as follows. Firstly, using the results of [DW06] we get in Theorem 2.6 a sufficient condition for a decomposition to be locally semi-simple. Next, we consider an important particular case of a prehomogeneous dimension vector $\beta$, i.e., such that $R(Q,\beta)$ contains a dense orbit $GL(\beta)W$. Actually, we revisit an important theorem in [Sch91] saying that the category $W^\perp$ is isomorphic to the category of representations of a quiver $\Sigma$ without oriented cycles. We analyze the proof of that theorem and find that it yields Algorithm 3.6 for calculating the dimensions of the simple objects in $W^\perp$. This Algorithm can be viewed, firstly, as a tool for calculating the algebraically independent generators of $k[R(Q,\beta)]^GL(\beta)$ (see Theorem 3.7). Secondly, with the help of the Algorithm we get in Theorem 3.9 a complete description of the Luna $GL(\beta)$-stratification, and in particular, the generic locally semi-simple decomposition of $\beta$ in Corollary 3.10. Finally, we provide an Algorithm 4.8 for calculating the generic locally semi-simple decomposition for arbitrary dimension vector $\alpha$. This Algorithm is based on one hand, on the idea of that for the generic decomposition from [DW00], and on the other hand, on Corollary 3.10.

Besides proving theorems and algorithms we implemented a computer program for doing all these types of calculations, namely, allowing to calculate generic and generic locally semi-simple decompositions for arbitrary dimension vectors and perpendicular categories for a prehomogeneous vector. This program is called TETIVA and is available at [Te].

2. Schur sequences and locally semi-simple representations.

In this section we relate locally semi-simple decompositions of dimension vector to various other decompositions and start with the $\sigma$-stable ones.

Let $\alpha$ be a dimension vector and $\sigma \in \mathbb{Z}^{Q\alpha}$ be a weight such that $\sigma(\alpha) = 0$. Recall that King introduced in [K1] the notion of (semi-)stability of representations of dimension $\alpha$. Assume that generic representation of dimension $\alpha$ is $\sigma$-semi-stable, or, equivalently, $k[R(Q,\alpha)]^{GL(\alpha)} \neq \{0\}$. Then each $\sigma$-semi-stable $V \in R(Q,\alpha)$ has a filtration in the subcategory of $\sigma$-semi-stable representations with the $\sigma$-stable factors, that is, Jordan-Hölder factors. So $V$ yields a decomposition of $\alpha$ into the linear combination of the dimensions of $\sigma$-stable representations, and for $V$ generic we get the so-called $\sigma$-stable decomposition of $\alpha$:

$$\alpha = m_1\alpha_1 + \cdots + m_s\alpha_s.$$  (6)
Note that for $Q$ being a tame quiver and $\sigma$ the defect the $\sigma$-stable representations are the regular ones and the $\sigma$-stable decomposition is Ringel’s canonical one, see [Ri]. By [Sh] Proposition 10, Theorem 11, we get:

**Proposition 2.2.** The $\sigma$-stable decomposition is locally semi-simple.

**Definition 2.2.** For dimension vectors $\alpha, \beta$ denote by $\text{hom}(\alpha, \beta)$ and $\text{ext}(\alpha, \beta)$ the dimensions of $\text{Hom}(A, B)$ and $\text{Ext}(A, B)$ for generic $A \in R(Q, \alpha), B \in R(Q, \beta)$, respectively. Further, write $\alpha \perp \beta$ if $\text{hom}(\alpha, \beta) = 0 = \text{ext}(\alpha, \beta)$.

**Definition 2.3.** A sequence $\alpha_1, \cdots, \alpha_s$ of dimension vectors is called (right) perpendicular if it consists of Schur roots and for $1 \leq i < j \leq s$ holds $\alpha_i \perp \alpha_j$.

Now we introduce an important notion from [DW06]:

**Definition 2.4.** A perpendicular sequence $\alpha_1, \cdots, \alpha_s$ is called a Schur sequence if
\begin{equation}
\alpha_i \circ \alpha_j = \dim k[R(Q, \alpha_i)]^{\langle GL(\alpha_i) \rangle} = 1, 1 \leq i < j \leq s.
\end{equation}
This sequence is called quiver Schur if additionally $\langle \alpha_j, \alpha_i \rangle \leq 0$ for $i < j$.

The notion of quiver Schur sequence can be interpreted in terms of local quiver. The idea of the latter goes back to [LBP], where it was applied for semi-simple representations and we used it in [Sh] for locally semi-simple ones, too. Moreover, the definition works for any representation $V = m_1S_1 + \cdots + m_tS_t$ that meets condition (5) and we define $\Sigma_V$ to be the quiver with $t$ vertices corresponding to the summands $S_1, \cdots, S_t$ and $\delta_{ij} - (\dim S_i, \dim S_j)$ arrows from $i$ to $j$. Note that, thanks to the condition (5) and the Ringel formula (3), $\delta_{ij} - (\dim S_i, \dim S_j) = \dim \text{Ext}(S_i, S_j) \geq 0$. On the other hand, the definition of $\Sigma_V$ only depends on the sequence $\underline{\alpha} = (\dim S_1, \cdots, \dim S_t)$, not on the multiplicities, and even not on the summands themselves, provided the homomorphism spaces are trivial. So it is possible and even more correct to denote the quiver $\Sigma_{\underline{\alpha}}$. This local quiver plays a crucial role in [LBP] and [Sh] because for locally semi-simple $V$ the slice representation at $V$ is (by formula (9) in [Sh]):
\begin{equation}
\langle \text{Aut}(V), \text{Ext}(V, V) \rangle \cong (GL(\gamma), R(\Sigma_{\underline{\alpha}} \gamma)).
\end{equation}
where $\gamma = (m_1, \cdots, m_t)$ is a dimension vector for $\Sigma_{\underline{\alpha}}$. Therefore Luna’s étale slice Theorem relates the local equivariant structure of a neighborhood of $V$ in $R(Q, \alpha)$ with that of $0$ in $R(\Sigma_{\underline{\alpha}} \gamma)$.

**Proposition 2.5.** Let $\underline{\alpha} = (\alpha_1, \cdots, \alpha_t)$ be a sequence of Schur roots.

1. If $\underline{\alpha}$ is a quiver Schur sequence then $\text{hom}(\alpha_i, \alpha_j) = 0$ for any $i \neq j$ and the quiver $\Sigma_{\underline{\alpha}}$ has no oriented cycles except loops.

2. If $\text{hom}(\alpha_i, \alpha_j) = 0$ for any $i \neq j$ and $\Sigma_{\underline{\alpha}}$ has no oriented cycles except loops, then after a reordering we have $\alpha_i \perp \alpha_j$ for $i < j$. If moreover $\alpha_i$ is imaginary for at most one $i$, then after a reordering $\underline{\alpha}$ becomes a quiver Schur sequence.

**Proof.** 1. For $i < j$, $\alpha_i \perp \alpha_j$ implies $\text{hom}(\alpha_i, \alpha_j) = 0$ and $\text{ext}(\alpha_i, \alpha_j) = 0$. The latter together with [Sch92] Theorem 4.1] implies that either $\text{hom}(\alpha_j, \alpha_i) = 0$ or $\text{ext}(\alpha_j, \alpha_i) = 0$. By Ringel formula, $\text{hom}(\alpha_j, \alpha_i) - \text{ext}(\alpha_j, \alpha_i) = \langle \alpha_j, \alpha_i \rangle \leq 0$, hence $\text{hom}(\alpha_j, \alpha_i) = 0$ as well. Since all non-loop arrows of $\Sigma_{\underline{\alpha}}$ go from $j$ to $i$ with $j > i$, this quiver does not contain non-loop oriented cycles.

2. Each oriented graph having no oriented cycles admits an order such that all arrows go from bigger to smaller indices; forget the loops of $\Sigma_{\underline{\alpha}}$ and fix such an
order. Since \( \text{hom}(\alpha_i, \alpha_j) = 0 \) for any \( 1 \leq i \neq j \leq t \) and \( \text{ext}(\alpha_i, \alpha_j) = 0 \) for any \( i < j \) in our order, we have \( \alpha_i \perp \alpha_j \). If moreover, at least one from \( \alpha_i, \alpha_j \) is real then \( \alpha_i \circ \alpha_j = 1 \) by [DW06, Lemma 4.2].

Therefore the quiver Schur sequences are very close to the sequences with trivial mutual homorphisms and without oriented cycles. Of course, not each locally semi-simple representation meets the latter condition: for instance take a tame quiver as \( Q \) and the sum of the simple non-homogeneous regular representations over an orbit of Coxeter functor; then this representation is locally semi-simple by [Sh, Proposition 20] but the local quiver is a oriented cycle by [Sh, Proposition 21].

**Theorem 2.6.** For a quiver Schur sequence \( \alpha = (\alpha_1, \ldots, \alpha_t) \) and a tuple \((m_1, \ldots, m_t)\) the decomposition \( \beta = m_1\alpha_1 + \cdots + m_t\alpha_t \) is locally semi-simple.

**Proof.** By Theorem [1.1] the fact that the decomposition is locally semi-simple does not depend on the multiplicities so we may assume \( m_1 = m_2 = \cdots = m_t = 1 \). Then we apply Theorem 5.1 from [DW06]. Denote by \( \Sigma(Q, \beta) \) the set of weights of the semi-invariants in \( k[R(Q, \alpha)]^{\text{SL}(\alpha)} \). Theorem 5.1 asserts that the cone \( R_+ \Sigma(Q, \beta) \) has a face \( F = R_+ \Sigma(Q, \beta) \cap \{ \sigma \in R^{Q_0} | \sigma(\alpha_1) = \cdots = \sigma(\alpha_t) = 0 \} \). Moreover, the Theorem guarantees that for \( \sigma \) from the relative interior of \( F \), \( \beta = \alpha_1 + \cdots + \alpha_t \) is the \( \sigma \)-stable decomposition of \( \beta \). Now the assertion follows from Proposition 2.1. \( \square \)

### 3. Prehomogeneous dimension vectors.

Recall that a dimension vector \( \beta \) is called prehomogeneous if \( R(Q, \beta) \) contains a dense \( GL(\beta) \)-orbit. By [Kac] this is equivalent to the generic decomposition of \( \beta \) containing only real Schur roots. For this particular case we are able to calculate the Luna stratification completely.

**Proposition 3.1.** If \( \beta \) is prehomogeneous and \( \alpha \) is a sequence of Schur roots such that \( \beta = m_1\alpha_1 + \cdots + m_t\alpha_t \) then this decomposition is locally semi-simple if and only if \( \alpha \) is a quiver Schur sequence up to order.

**Proof.** By Theorem [2.6] and Proposition [2.5] we only need to prove that if the decomposition is locally semi-simple then \( \Sigma_{\Sigma_2} \) does not contain oriented cycles (in particular, the absence of loops means that all summands are real roots). Indeed, take \( W = m_1S_1 + \cdots + m_tS_t \) the locally semi-simple representation corresponding to the decomposition. Then the orbit \( GL(\beta)W \) is closed in an open affine neighborhood \( R_0 \subseteq R(Q, \beta), R_0 \ni W \) and by formula (8) the slice representation at \( W \) is isomorphic to \( (GL(\gamma), R(\Sigma_{\Sigma_2}, \gamma)) \). The étale slice Theorem yields an étale morphism of \( R(\Sigma_{\Sigma_2}, \gamma)/GL(\gamma) \) to \( R_0/\text{GL}(\beta) \). Since \( GL(\beta) \) has an open orbit in \( R(Q, \beta) \), it is contained in \( R_0 \) and so \( R_0/\text{GL}(\beta) \) is a point. Consequently, \( R(\Sigma_{\Sigma_2}, \gamma)/GL(\gamma) \) is a point, hence, \( \Sigma_{\Sigma_2} \) does not have oriented cycles. \( \square \)

The way we compute the locally semi-simple decompositions of prehomogeneous dimension vectors is based on Schofield’s Theorem:

**Theorem 3.2.** ([Sch91, Theorem 2.5]) If \( \beta \) is prehomogeneous and \( W = m_1S_1 + \cdots + m_tS_t \) is the decomposition into indecomposable summands of a representation from the dense orbit, then \( W^\perp \) is isomorphic to the category of representations of a quiver with \( n - t \) vertices and without oriented cycles, where \( n = \vert Q_0 \vert \). The same is true for \( ^\perp W \).
We want to generalize this Theorem and to give an algorithm for calculation of the perpendicular category based on the original proof. The above Theorem can be reformulated as follows: there are \( n - t \) representations \( R_1, \ldots, R_{n-t} \), which are all non-isomorphic simple objects in \( W^\perp \). By Theorem 1 \( R = R_1 + \cdots + R_{n-t} \) is locally semi-simple and the above Theorem additionally asserts that the local quiver \( \Sigma \) of \( R \) does not have oriented cycles. A particular case of the Theorem is when \( \beta \) is a real Schur root, and actually the proof in [Sch91] deduces the general case from this particular one, where the proof is based on the notion of projective and injective representations that we recall following [Sch91].

For \( i, j \in Q_0 \) denote by \([i, j]\) the \( k\)-vector space on the basis of oriented paths from \( i \) to \( j \) in \( Q \). For \( a \in Q_0 \) consider the representation \( P_a \) such that \( P_a(i) = [a, i], i \in Q_0 \) and for any arrow \( \varphi \in Q_1 \), the map \( P_a(\varphi) \) takes a path \( T \) from \( a \) to \( t\varphi \) to the concatenation \( T\varphi \), which is a path \( a \) to \( h\varphi \). Dually, consider the representation \( I_a \) such that \( I_a(i) = [i, a]^*, i \in Q_0 \) and \( I_a(\varphi) \) is the dual map to the natural one from \([h\varphi, a]\) to \([t\varphi, a]\). These representations have nice properties with respect to the homomorphisms and extensions: for any representation \( V \) of \( Q \) hold

\[
\text{(9) } \text{Hom}(P_a, V) \cong V(a), \text{Hom}(V, I_a) \cong (V(a))^*, \text{Ext}(P_a, V) = 0, \text{Ext}(V, I_a) = 0.
\]

**Theorem 3.3.** Let \( \beta \) be a real Schur root for a quiver \( Q \) with \( n \) vertices and without oriented cycles and let \( W \) be a generic representation of dimension \( \beta \). Then:

1. If \( \beta = \dim P_a \) for some \( a \in Q_0 \), then the simple objects of the category \( W^\perp \) are all the simple representations of \( Q \) but \( S_a \). Otherwise, the dimensions of \( n - 1 \) projective objects of \( W^\perp \) are the indecomposable summands of the generic decomposition for dimension vectors \( \dim P_a - \langle \beta, \dim P_a \rangle \beta \), where a runs over \( Q_0 \).

2. If \( \beta = \dim I_a \) for some \( a \in Q_0 \), then the simple objects of the category \( W^\perp \) are all the simple representations of \( Q \) but \( S_a \). Otherwise, the dimensions of \( n - 1 \) injective objects of \( W^\perp \) are the indecomposable summands of the generic decomposition for dimension vectors \( \dim I_a - \langle \dim I_a, \beta \rangle \beta \), where a runs over \( Q_0 \).

**Proof.** We prove assertion 1, the proof for 2 being similar. First of all, for \( \beta = \dim P_a \), the assertion follows from formulae (9). In the opposite case, we apply the proof of Theorem 2.3 and Theorem 3.1 from [Sch91]. From the proof of Theorem 2.3 we learn that all indecomposable projective objects in \( W^\perp \) can be obtained as the indecomposable summands of a generic extension of \( sW \) by \( \Lambda \), where \( \Lambda = \sum_{a \in Q_0} P_a, s = \dim \text{Ext}(W, \Lambda) \). For each individual \( P = P_a \) Theorem 3.1 considers a generic exact sequence \( 0 \to P \to P^\sim \to sW \to 0 \), where \( s = \dim \text{Ext}(W, P) \) and states that \( P^\sim \) is projective in \( W^\perp \). Then mutual extensions of the indecomposable summands of \( P^\sim \) are trivial in \( W^\perp \) by (9), hence, trivial in \( \text{Mod}(Q) \), because \( W^\perp \) is closed under extensions. Therefore, \( P^\sim \) is generic in its dimension and the dimensions of the indecomposable summands of \( P^\sim \) are the summands of the generic decomposition of \( \dim P^\sim = \dim P + \dim \text{Ext}(W, P) \beta = \dim P - \langle \beta, \dim P \rangle \beta \), the latter equality following from \( \text{Hom}(W, P) = 0 \).

The above Theorem yields a quick algorithm as follows:

**Algorithm 3.4.** **Right perpendicular category of a Schur root**

**INPUT:** a quiver \( Q \) with \( n \) vertices and without oriented cycles, a real Schur root \( \gamma \)

**OUTPUT:** \( n - 1 \) dimensions of the simple objects in \( W^\perp \) such that \( GL(\gamma)W \) is dense in \( R(Q, \gamma) \).
1. Calculate the dimensions $\rho_1, \cdots, \rho_n$ of indecomposable projectives. 
   If $\gamma = \rho_j$, then return $\varepsilon_1, \cdots, \varepsilon_j, \cdots, \varepsilon_n$.
2. Loop on $i = 1, \cdots, n$
   Calculate the generic decomposition of $\rho_i - \langle \gamma, \rho_i \rangle$.
   Add each summand to the array provided it is not yet there.
   After step 2 we must have distinct summands $\beta_1, \cdots, \beta_{n-1}$ in the array.
3. Color the entries $1, \cdots, n-1$ white, the number of black entries $b = 0$.
   Loop while $b < n-1$
   Loop on $j = 1, \cdots, n-1$
   If $j$-th entry is white and for each other white entry $k$, $\langle \beta_k, \beta_j \rangle = 0$, then remember this entry $j$, which is going to be black. Break the loop.
   Set next simple dimension $\alpha_j = \beta_j$.
   Loop on $k = 1, \cdots, n-1$
   If $k$-th entry is black then $\alpha_j = \alpha_j - \langle \beta_k, \beta_j \rangle \alpha_k$.
   $b = b + 1$.
4. Return $\alpha_1, \cdots, \alpha_{n-1}$.

Proof. After Theorem 3.3 only the step 3 of the algorithm needs to be explained. In that step we do the inverse to step 1, i.e., we recover the dimensions of the simple objects from those of indecomposable projectives. The idea of the step is that, though we do not know the quiver $\Sigma$ of the simple objects, we know that the Euler form on $\Sigma$ is the same that inherited from $Q$. But by formulae $\langle \beta_j, \beta_j \rangle = 0$, hence again by those formulæ, $\langle \beta_i, \beta_j \rangle$ with respect to the Euler form of $\Sigma$ is equal to the number of paths from $j$ to $i$ in $\Sigma$. So the first vertex becoming black is a sink of $\Sigma$, hence, the corresponding projective is simple. Furthermore, each next vertex becoming black is the sink of $\Sigma$ with removed black vertices, hence the corresponding projective is the simple plus the sum over black vertices of the already obtained simple dimensions multiplied by the number of paths to there. This completes the proof.

Remark 3.1. Dualizing Algorithm 3.4 as in Theorem 3.3 we get one for the left perpendicular category.

Now we generalize Theorem 3.5 as follows:

**Theorem 3.5.** If $W = m_1 S_1 + \cdots + m_t S_t$, where $S_1, \cdots, S_t$ are Schur indecomposable summands and $(\dim S_1, \cdots, \dim S_t)$ is a perpendicular sequence of real Schur roots, then there is a sequence $\alpha = (\alpha_1, \cdots, \alpha_{n-t})$, $n = |Q_0|$ of real Schur roots such that the corresponding indecomposable representations are all simple objects in $W^\perp$ and $\Sigma_\alpha$ has no oriented cycles. The same is true for $W^\perp$.

Proof. We just present an algorithm for calculation of $\alpha$ based on Algorithm 3.4.

**Algorithm 3.6.**

Input: a quiver $Q$ with $n$ vertices and without oriented cycles,
   a perpendicular sequence $(\dim S_1, \cdots, \dim S_t)$ of real Schur roots
Output: dimensions $\alpha_1, \cdots, \alpha_{n-t}$ of the simple objects in $W^\perp$.
Loop on $i = 1, \cdots, t$
   We have dimensions $\alpha_1, \cdots, \alpha_{n+1-i}$ of the simple objects of the current category. For $i = 1$ the category is the whole of $\text{Mod}(Q)$.
   Calculate the quiver $\Sigma_i$ of the current category by means of Euler form.
   Expand $\dim S_i$ as the linear combination of $\alpha_1, \cdots, \alpha_{n+1-i}$,
Coefficients yield a dimension vector $\gamma_i$ for $\Sigma_i$.
Find the dimensions $\alpha_1, \cdots, \alpha_{n-i}$ of the simple objects in the right perpendicular category to $\gamma_i$ for $\Sigma_i$.
calculate new $\alpha_1, \cdots, \alpha_{n-i}$ as the linear combinations of old $\alpha_1, \cdots, \alpha_{n+1-i}$ with coefficients from $\alpha_1, \cdots, \alpha_{n-i}$.
RETURN $\alpha_1, \cdots, \alpha_{n-i}$.

The key idea of this clear algorithm is that we obtain the perpendicular category to $W$ as a sequence of subcategories, where the next is obtained as the perpendicular category to a Schur root. The algorithm for the left perpendicular category is similar, we only go from $S_i$ to $S_1$.

The above Theorem makes possible to introduce an operation $\perp$ mapping a real (i.e., consisting of real roots) perpendicular sequence $\alpha$ of $t$ roots to real perpendicular sequences $\alpha \perp$ and $\perp \alpha$ of $n - t$ roots being the sequence of dimensions of simple objects in the right and left perpendicular categories, respectively. The first natural application of Algorithm 3.6 is given by the following interpretation of the result from [Sch91]:

**Theorem 3.7.** Let $\beta$ be a prehomogeneous dimension vector and $\beta = (\beta_1, \cdots, \beta_t)$ be a perpendicular sequence of the summands for the generic decomposition of $\beta$. For each member $\gamma_i \in \beta \perp$ pick a generic $T_i \in R(Q, \gamma_i)$. Then the determinantal semi-invariants $c_{\gamma_i} = c_{T_i}, i = 1, \cdots, n - t$ constitute an algebraically independent system of generators for $k[R(Q, \beta)]^{SL(\beta)}$.

For any real perpendicular sequence $\alpha$ the sequence $\alpha \perp$ is a real quiver Schur sequence, so if $\alpha$ is not a quiver Schur sequence, then $\perp (\alpha \perp)$ is different from $\alpha$ (cf. Corollary 3.10). Otherwise we have:

**Proposition 3.8.** If $\alpha$ is a real quiver Schur sequence, then $\alpha = \perp (\alpha \perp)$.

**Proof.** Denote $\alpha \perp$ by $\beta = (\beta_1, \cdots, \beta_{n-t})$ and $\perp \alpha$ by $\gamma = (\gamma_1, \cdots, \gamma_t)$. Each $\alpha_i \in \alpha$ has the property $\alpha_i \perp \beta_j$ for each $\beta_j \in \beta$, hence, $\alpha_i$ decomposes as $\alpha_i = \rho_1^j \gamma_1 + \cdots + \rho_t^j \gamma_t$ with non-negative integers $\rho_1^j, \cdots, \rho_t^j$. By [Sch, Proposition 13] the sequence $\rho = (\rho_1, \cdots, \rho_t)$ is a real quiver Schur sequence for the quiver $\Sigma_{\beta}$. Since this quiver has $t$ vertices, we have $\perp \rho$ is empty and therefore, there are no semi-invariants non-vanishing on the representation $O$ corresponding to the decomposition $\rho_1 + \cdots + \rho_t$. However, by Theorem 2.6 $O$ is locally semi-simple, hence $O$ is the zero point in $R(\Sigma_{\beta}, \dim O)$. In other words, $\rho_1, \cdots, \rho_t$ are the dimensions of simple representations, so $\alpha = \gamma$ up to order.

Recall that the set of all locally semi-simple decompositions of a dimension vector $\beta$ is the same as the Luna $GL(\beta)$-stratification of $R(Q, \beta)//SL(\beta)$. For $\beta$ prehomogeneous we are able to describe this stratification completely:

**Theorem 3.9.** Let $\beta$ be a prehomogeneous dimension vector and $\beta = (\beta_1, \cdots, \beta_t)$ be a perpendicular sequence of the summands for the generic decomposition of $\beta$.

1. There is a bijection of the Luna $GL(\beta)$-stratification of $R(Q, \beta)//SL(\beta)$ with the set of subsequences in $\beta \perp = (\gamma_1, \cdots, \gamma_{n-t})$

$$\gamma \subseteq \beta \perp \rightarrow \beta = m_1 \rho_1 + \cdots + m_s \rho_s, (\rho_1, \cdots, \rho_s) = \perp \gamma_1, \cdots, m_s \in \mathbb{Z}_+.$$ 

2. The stratum corresponding to $\gamma$ is $\{ \xi \in R(Q, \beta)//SL(\beta) | c_{\gamma}(\xi) \neq 0 \iff \gamma_i \in \gamma \}$.
3. This bijection preserves the order on the sets: if \( \gamma_1 \subseteq \gamma_2 \), then the stratum corresponding to \( \gamma_1 \) is contained in the closure of that for \( \gamma_2 \).

Before proving the Theorem we state an important

**Corollary 3.10.** The generic locally semi-simple decomposition of \( \beta \) is \( \beta = m_1 \beta'_1 + \cdots + m_t \beta'_t \), where \( \beta' = \perp \beta \).

**Proof.** 1. First of all we show that the map is well-defined, i.e., there is a unique decomposition of \( \beta \) in \( \perp \gamma \) and this decomposition is locally semi-simple. By definition of \( \beta' \) we have for each \( \beta_i \in \beta \) and each \( \gamma_j \in \gamma, \beta_i \perp \gamma_j \). Then by definition of \( \perp \gamma \), \( \beta_i \) is a \( \mathbb{Z}_+ \)-linear combination of elements of \( \perp \gamma \), hence, \( \beta \) is. This decomposition is locally semi-simple by Theorem 3.1 and is unique, because \( \perp \gamma \) is clearly linear independent. To prove that our correspondence is a bijection we only need to check that each locally semi-simple decomposition \( \beta = p_1 \rho_1 + \cdots + p_s \rho_s \) can be obtained this way. First of all by Proposition 3.8 we may assume that \( \rho = (\rho_1, \cdots, \rho_s) \) is a real quiver Schur sequence. In particular, there is a unique representation \( V \subseteq R(Q, \beta) \) corresponding to this decomposition, up to isomorphism. Then the sequence \( \rho' \) is a subsequence in \( \beta' \). Indeed, each semi-invariant that does not vanish on \( V \) does not vanish generically on \( R(Q, \beta) \) so each element of \( \rho' \) is presented as a \( \mathbb{Z}_+ \)-linear combination over \( \beta' \). Then \( \rho' \) can be identified with the subsequence \( \{i \in \{1, \cdots, n-t\}| c_{\gamma_i}(V) \neq 0\} \). So we may set \( \gamma = \rho' \) and recover \( \rho \) as \( \perp \gamma \) by Proposition 3.8.

2. The locally semi-simple representations over the stratum corresponding to \( \gamma \) constitute one orbit \( GL(\beta) V \), where \( V \) corresponds to the decomposition \( \beta = m_1 \rho_1 + \cdots + m_s \rho_s \), \( \rho = \perp \gamma \). By definition of \( \perp \gamma \), \( c_{\gamma_j}(V) \neq 0 \) for \( \gamma_j \in \gamma \). For any \( j = 1, \cdots, n-t \) it follows from the properties of determinantal semi-invariants that \( c_{\gamma_j}(V) \neq 0 \) implies \( \langle \rho_i, \gamma_j \rangle = 0 \) for each \( i = 1, \cdots, s \). Since \( \gamma_1, \cdots, \gamma_{n-t} \) are linear independent, the number of semi-invariants \( c_{\gamma_j} \) non-vanishing on \( V \) is less than or equal to the codimension of the common kernel of the corresponding forms \( \langle \cdot, \gamma_j \rangle \) on \( \mathbb{Q} Q^e \). Since \( \rho_1, \cdots, \rho_s \) are linear independent and belong to that kernel, its codimension cannot be more than \( n-s \), so \( c_{\gamma_j}(V) = 0 \) for \( \gamma_j \notin \gamma \). Assertion 3 clearly follows from 2. \( \square \)

**Remark 3.2.** This assertion is closely related with [DW06, Theorem 5.1] because quiver Schur sequences are in bijection with the Luna \( GL(\beta) \)-strata by Proposition 3.1 and the subsequences in \( \beta' \) are in bijection with the faces of the cone \( R_+ \Sigma(Q, \beta) \), which is simplicial in this case.

4. **Generic locally semi-simple decomposition**

We start with an obvious observation:

**Proposition 4.1.** \( R(Q, a)^{GL(a)} \) is generated by the scalar endomorphisms of \( V(a), a \in Q_0 \) corresponding to the loops \( \varphi \in Q_1, t\varphi = h\varphi = a \).

Recall that the multiplicities of the summands in generic decompositions and in locally semi-simple ones are of slightly different meaning. For the latter the summand \( m_\beta \) means that the corresponding representation has a direct summand \( m S, \text{dim} S = \beta \); on the other hand, there can be summands as \( m_1 \beta + m_2 \beta \) and this means that the two indecomposable summands of the representation of dimension
\( \beta \) are non-isomorphic (hence, \( \beta \) is imaginary). In the generic decomposition the different summands are assumed to be distinct, the multiplicity of \( \beta \) with \( q_Q(\beta) < 0 \) is 1 by [Kac], and for \( q_Q(\beta) = 0 \) the direct summand \( m\beta \) stands for the sum of \( m \) pairwise non-isomorphic Schur representations of dimension \( \beta \). We now restrict the set of locally semi-simple decomposition we consider, as follows:

**Definition 4.2.** We call a locally semi-simple decomposition almost loopless if the multiplicity of each imaginary Schur root \( \beta \) is 1 and if \( q_Q(\beta) < 0 \), then this root occurs one time.

**Remark 4.1.** Each locally semi-simple decomposition yields a loopless one with the automorphism group of a smaller dimension. Namely, the piece \( m_1\beta_1 + \cdots + m_t\beta_t \) of the decomposition with \( \beta_1 = \cdots = \beta_t = \beta \) such that \( \beta \) is imaginary can be replaced either by the imaginary Schur root \( (m_1 + \cdots + m_t)\beta \), if \( q_Q(\beta) < 0 \), or by the sum of \( m_1 + \cdots + m_t \) times \( \beta \), if \( \beta \) is isotropic.

**Remark 4.2.** If a locally semi-simple decomposition is almost loopless, then we write it down in the style of generic one, with different summands being distinct.

**Proposition 4.3.** Assume that \( Q \) has no oriented cycles. Let \( \alpha = m_1\gamma_1 + \cdots + m_t\gamma_t \) be the generic decomposition. Then \( k | R(\alpha) |^{SL(\alpha)} = k \) if and only if \( |Q_0| = t \).

**Proof.** Assume that \( k | R(\alpha) |^{SL(\alpha)} \neq k \) so that there is a non-trivial determinantal semi-invariant \( c_W, \dim W = \delta \). Then by [DW00] \( c_W \) does not vanish on the summands of generic representation, in particular, \( (\gamma_i, \delta) = 0, i = 1, \ldots, t \). On the other hand, by Remark 4.6 and Corollary 4.12 from [DW06], \( \gamma_1, \ldots, \gamma_t \) are linear independent; since these are contained in a proper \( \mathbb{Q} \)-vector subspace of \( Q^{Q_0} \), we conclude \( t < |Q_0| \).

Conversely, assume that \( k | R(\alpha) |^{SL(\alpha)} = k \). This is equivalent to \( SL(\alpha) \) acting with a dense orbit on \( R(\alpha) \) and, in particular \( \alpha \) is a prehomogeneous dimension vector. But \( (\gamma_1, \ldots, \gamma_t)^t \) must be empty because there are no non-trivial semi-invariants. Then by Theorem [Kac], \( t = |Q_0| \).

**Corollary 4.4.** A locally semi-simple decomposition \( \alpha = p_1\delta_1 + \cdots + p_s\delta_s \) is generic if and only if it is almost loopless, the local quiver \( \Sigma_\alpha \) has no oriented cycles except for the loops, and \( s = t \).

**Proof.** By Remark 4.1, the generic locally semi-simple decomposition is almost loopless. By formula [Sh] the slice representation corresponding to the decomposition is \( (GL(\rho), R(\Sigma_\delta, \rho)) \). By [Sh] Corollary 5] the decomposition is generic locally semi-simple if and only if \( k | R(\Sigma_\delta, \rho) |^{SL(\rho)} \) is generated by the \( GL(\rho) \)-invariant submodule in \( R(\Sigma_\delta, \rho) \). The condition that \( \delta \) is almost loopless is equivalent to the loops of \( \Sigma_\delta \) existing only at vertices with dimension 1. Removing the loops of \( \Sigma_\delta \) we define a quiver \( \Sigma \) such that the generic and the generic locally semi-simple decompositions of \( \rho \) with respect to \( \Sigma \) are the same as for \( \Sigma_\delta \). By Proposition 4.1 the above condition on \( \Sigma_\delta \) and \( \rho \) is equivalent to \( k | R(\Sigma, \rho) |^{SL(\rho)} = k \). If \( \Sigma \) has loops, then the latter is false, in the opposite case we apply the Proposition.

In what follows we will intensively use the following well-known fact

**Proposition 4.5.** An exact sequence of homomorphisms \( 0 \to U \to V \to W \to 0 \) for representations of \( Q \) yields for any representation \( X \) exact sequences:

\[
0 \to \text{Hom}(W, X) \to \text{Hom}(V, X) \to \text{Hom}(U, X) \to 0
\]
Proof. Assertions decomposition of \((a, q)\) one real. However, it is shown in [DW02] for this type of quiver that either a where \(\delta\) We prove that decomposition has \(r\), generic decomposition for \((\Sigma + \alpha\), \(\beta\)). By [Sh, Proposition 14] the generic decomposition for \((\Sigma, \alpha)\) is similar to that for generic decomposition from [DW02]. That algorithm works with perpendicular sequences and glue and permute two items each time there is a non-trivial extension between them. We proceed as follows: starting from the generic decomposition, we transform it into the generic locally semi-simple one by slightly splitting summands by each other on the fact of non-trivial homomorphism. More precisely, we do it only when at least one of the items is imaginary, for the homomorphisms between real summands we apply Corollary 3.10. The most simple step of this procedure is: having Schur roots \(\alpha, \beta\) with \(\alpha \perp \beta, \text{ext}(\beta, \alpha) = 0\) but \(\text{hom}(\beta, \alpha) \neq 0\), we "factorize" the imaginary root by the real one to get an imaginary Schur root with trivial homomorphism spaces with the real root:

**Proposition 4.6.** Let \(\alpha\) and \(\beta\) be Schur roots such that \(\text{ext}(\alpha, \beta) = 0 = \text{ext}(\beta, \alpha)\).

1. If both \(\alpha\) and \(\beta\) are imaginary, then \(\text{hom}(\alpha, \beta) = 0 = \text{hom}(\beta, \alpha)\).
2. In any case either \(\text{hom}(\alpha, \beta) = 0\) or \(\text{hom}(\beta, \alpha) = 0\).
3. Assume that \(\text{hom}(\alpha, \beta) = 0, \dim \text{hom}(\beta, \alpha) = p > 0\).

   **A:** If \(\alpha\) is imaginary, then for generic \(A \in R(Q, \alpha), B \in R(Q, \beta)\) there is an exact sequence of homomorphisms: \(0 \to pB \to A \to C \to 0, C\) is Schurian, \(q_Q(\dim C) = q_Q(\alpha), \text{Hom}(C, B) = 0, \text{Hom}(B, C) = 0, \text{Ext}(B, C) = 0\).

   **B:** If \(\beta\) is imaginary, then for generic \(A \in R(Q, \alpha), B \in R(Q, \beta)\) there is an exact sequence of homomorphisms: \(0 \to C \to B \to pA \to 0, C\) is Schurian, \(q_Q(\dim C) = q_Q(\beta), \text{Hom}(A, C) = 0, \text{Hom}(C, A) = 0, \text{Ext}(C, A) = 0\).

Moreover, in both cases \(A, B\), if \(\gamma \perp \alpha\) and \(\gamma \perp \beta\), then \(\gamma \perp \dim C\); if \(\alpha \perp \gamma\) and \(\beta \perp \gamma\), then \(\dim C \perp \gamma\).

**Proof.** Assertions 1 and 2 follow from Theorems 4.1 and 2.4 of [Sch12], respectively. We prove 3A the proof for 3B being similar. Consider the decomposition \(\rho = \alpha + \beta\); the conditions \(\text{ext}(\alpha, \beta) = 0 = \text{ext}(\beta, \alpha)\) imply that this is a generic one. Then by Corollary 4.3 the generic locally semi-simple decomposition of \(\rho\) has two summands, \(\rho = m_1\gamma + m_2\delta\). If both summands \(\gamma, \delta\) would be imaginary, then by 1 that decomposition would be generic and different from \(\rho = \alpha + \beta\). We claim that at least one of \(\gamma, \delta\) is imaginary. Assume not, then the local quiver \(\Sigma\) of that decomposition has no loops, so it only has \(r\) arrows of the same orientation.

By [Sh] Proposition 14 the generic decomposition for \((\Sigma, (m_1, m_2))\) would give \(\rho = \alpha + \beta\) when dimension vectors for \(\Sigma\) are converted in those for \(Q\). So the generic decomposition for \((\Sigma, (m_1, m_2))\) has two summands, one imaginary and one real. However, it is shown in [DW02] for this type of quivers that either a dimension vector is an imaginary Schur root, or it has two real summands in the generic decomposition. This contradiction proves our claim.

So the generic locally semi-simple decomposition of \(\rho\) has the form \(\rho = a\gamma + \delta\), where \(\delta\) is an imaginary Schur root and \(\gamma\) is real. Again, the local quiver \(\Sigma\) of that decomposition has \(r\) arrows of the same orientation between two vertices and besides \(1 - q_Q(\delta)\) loops at the vertex corresponding to \(\delta\). Consider the generic decomposition of \((a, 1)\) on \(\Sigma\), it must have a form \((a, 1) = (a - q)(1, 0) + (q, 1), q > 0\).
The first necessary condition on \( q \) is that \((q, 1)\) should be a root, hence \( q \leq r \). Further, by \([\text{Kac}]\) the Euler form is non-negative on the pairs of summands of the generic decomposition; \((q, 1), (1, 0)\) and \((1, 0), (q, 1)\) are equal \( q \) and \( q - r \) up to transposition, so \( q = r \). On the other hand, by Propositions 14, 15 from \([\text{SB}]\) the pair of values of the Euler form for these summands must be same as for \( \alpha \) and \( \beta \) on \( Q \), so \( r = p \). We therefore proved that the generic locally semi-simple decomposition of \( \rho \) is \( \rho = (p+1)\beta + \alpha - p\beta \). In particular, \( \alpha - p\beta \) is a Schur root and \( \beta \perp \alpha - p\beta \). Hence, by \([\text{Sch} 92 \text{ Theorem 3.3}]\) generic representation of dimension \( \alpha \) has a subrepresentation of dimension \( p\beta \), so isomorphic to \( pB \) because generic. Thus we have the claimed exact sequence. Write: \( qQ(\alpha - p\beta) = (\alpha - p\beta, \alpha - p\beta) = qQ(\alpha) + p^2 - p(\langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle) = qQ(\alpha) \).

The rest of the assertion will be deduced from formulae \((10,11)\) for \( U = pB, V = A, W = C \). Since \( \text{Ext}(V, U) = 0 \) and \( \text{Hom}(V, U) = 0 \), \((11)\) with \( X = V \) yields \( \text{Hom}(V, W) \cong \text{Hom}(V, V) = k \). Then \((10)\) with \( X = W \) yields \( \text{Hom}(W, W) = k \), so \( C \) is Schurian. Next, \( \text{Hom}(V, U) = 0 \) and \((10)\) with \( X = U \) yield \( \text{Hom}(W, U) = 0 \), hence, \( \text{Hom}(C, B) = 0 \). Finally, apply \((11)\) with \( X = U \) and note that \( \text{Ext}(U, V) = 0 \) by assumption, \( \text{Ext}(U, U) = 0 \) because \( \beta \) is real, \( \text{Hom}(U, U) \cong \text{Hom}(V, V) \cong k^e \); then \( \text{Ext}(U, W) \) and \( \text{Hom}(U, W) \) vanish. That \( \text{dim} C \) is perpendicular to any dimension vector \( \gamma \), which is perpendicular to \( \alpha \) and \( \beta \) follows directly from formulae \((10,11)\) with \( X \) being generic representation of dimension \( \gamma \); with such a choice of \( X \) four members in the exact sequence vanish, hence all six vanish. \( \square \)

**Definition 4.7.** Assume that \( \alpha, \beta \) are the subsequent members of a decomposition. If \( \alpha, \beta \) meet the conditions of \([4.6,3A](\text{resp. } 4.6,3B)\), we call the replacement of \( \alpha, \beta \) by \( \beta, \alpha - p\beta \) (resp. \( \beta - p\alpha, \alpha \)) **pushing \( \alpha \) right** (resp. **pushing \( \beta \) left**). This operation also includes the obvious recalculation of multiplicities on the decomposition. We also may apply both terms to the transposition of the members \( \alpha, \beta \) (even when both are real) such that \( \alpha \perp \beta \) and \( \beta \perp \alpha \).

Now we present our algorithm for the generic locally semi-simple decomposition of a dimension vector \( \alpha \) with a given generic decomposition \( \alpha = m_1\gamma_1 + \cdots + m_t\gamma_t \) such that \( \gamma \) is a Schur sequence. Recall again that the decomposition of this sort is the result of the algorithm from \([\text{DW} 02]\).

**Algorithm 4.8.**

**INPUT:** a quiver \( Q \) with \( n \) vertices and without oriented cycles, a quiver sequence \((\gamma_1, \cdots, \gamma_t)\) of the summands for the generic decomposition, multiplicities \((m_1, \cdots, m_t)\)

**OUTPUT:** a quiver sequence \((\alpha_1, \cdots, \alpha_t)\) of the summands for the generic locally semi-simple decomposition, multiplicities \((p_1, \cdots, p_t)\)

\[ \text{ shifts in a circle } \]

\( \square \) **First stage:**

\text{ FOR } i = 2, \cdots, t \\
\text{ Set } j = i \\
\text{ WHILE } \gamma_j \text{ is imaginary and } \gamma_{j-1} \text{ is real} \\
\text{ push } \gamma_j \text{ left and } j = j - 1 \\
\text{ Result of the first stage: first } s \leq t \text{ members of } \gamma \text{ are imaginary, } \\
\text{ last } t - s \text{ are real. } \\
\text{ Second stage: } \\
\text{ Replace the subsequence } \gamma' = (\gamma_{s+1}, \cdots, \gamma_t) \text{ by } \perp \gamma' \]
Result of the second stage: homorphisms for new \( (\gamma_{s+1}, \cdots, \gamma_t) \) are trivial.

Result of the second stage: homorphisms for new \( (\gamma_{s+1}, \cdots, \gamma_t) \) are trivial.

Third stage:

- While there is \( 1 \leq i < j \leq t \) with \( \langle \gamma_j, \gamma_i \rangle > 0 \)
  - Guarantee that the segment \([i, j]\) is minimal with such a property.
  - Transfer \( \gamma_j \) to position \( i + 1 \)
  - Push \( \gamma_i \) right.

Result of the third stage: the sequence \( \gamma \) is now a quiver Schur sequence.

Now we are going to prove the algorithm.

**Proposition 4.9.** While the first stage of the algorithm the sequence \( \gamma \) remains perpendicular and each time when \( \gamma_j \) is imaginary and \( \gamma_{j-1} \) is real \( \text{Ext}(\gamma_j, \gamma_{j-1}) \neq 0 \).

**Proof.** By Proposition [4.6] the sequence \( \gamma \) remains orthogonal after pushing \( \gamma_j \) left provided \( \gamma \) was perpendicular before it. We need to prove additionally that \( \gamma \) in the exact sequence from Proposition [4.6]B has the property \( \text{Ext}(R, D) = 0 \) for \( D \) being generic representation of dimension \( \gamma_q, q < j - 1 \). This follows from [10] and the fact that \( \text{Ext}(B, D) = 0 \) by induction. \( \square \)

**Proposition 4.10.** After the first stage \( \gamma_i \perp \gamma_j \) for \( 1 \leq i \neq j \leq s \).

**Proof.** Assume that \( i < j \). Then \( \gamma_i \perp \gamma_j \) follows from the previous Proposition and \( \text{Hom}(\gamma_j, \gamma_i) = 0 \) follows from [Sch92, Theorem 4.1], because both \( \gamma_i \) and \( \gamma_j \) are imaginary. Assume that \( \text{Ext}(\gamma_j, \gamma_i) > 0 \). Then we can apply the Algorithm for generic decomposition from [DW02] to the perpendicular sequence \( \gamma \). From that Algorithm follows that in such a situation we can glue together \( \gamma_i \) and \( \gamma_j \) and get a perpendicular sequence with \( t - 1 \) members and, continuing the Algorithm, we get the generic decomposition with less than \( t \) members. Contradiction. \( \square \)

**Proposition 4.11.** After the second stage of the algorithm the sequence \( \gamma \) remains orthogonal and \( \text{Hom}(\gamma_i, \gamma_j) > 0 \) implies \( i > s, j \leq s \).

**Proof.** Before the second stage \( \gamma \) consists of two segments, \( (\gamma_1, \cdots, \gamma_s) \) and \( \gamma' = (\gamma_{s+1}, \cdots, \gamma_t) \). Both segments are orthogonal sequences with trivial \( \text{Ext} \) spaces, the first segment by the previous Proposition and \( \gamma' \) because it is a subsequence in the original \( \gamma \). Consider two dimension vectors \( \rho_1, \rho_2 \) being the linear combinations of the two segments with the multiplicities. The fact that \( \gamma \) is orthogonal is equivalent to \( \rho_1 \perp \rho_2 \), that is, generic representation \( R_1 \) of dimension \( \rho_1 \) is perpendicular to \( \rho_2 \) and \( R_2 \). The second stage consists in replacing the generic decomposition for \( \rho_2 \) by the generic locally semi-simple one. Therefore \( \gamma \) remains orthogonal because \( R_1 \) is perpendicular to generic locally semi-simple representation of dimension \( \rho_2 \) otherwise the determinantal semi-invariant defined by \( R_1 \) vanishes on \( R(Q, \rho_2) \). The fact about Hom-spaces follows from the previous Proposition and the feature of generic locally semi-simple decompositions. \( \square \)

The third stage of the algorithm is based on the following

**Lemma 4.12.** If \( \alpha, \beta, \gamma \) is a perpendicular sequence of Schur roots such that \( \alpha \) is imaginary and \( \text{Hom}(\gamma, \alpha) > 0 \), then \( \text{Ext}(\gamma, \beta) = 0 \).

**Proof.** Assume \( \text{Ext}(\gamma, \beta) > 0 \). Pick generic representations \( U \in R(Q, \alpha), V \in R(Q, \beta), W \in R(Q, \gamma) \) and consider a non-split exact sequence \( 0 \rightarrow V \rightarrow X \overset{p}{\rightarrow} \).
$W \to 0$. Then $X$ is Schurian (see the proof of [DW06 Corollary 12]) and $U \perp V, U \perp W$ together with (11) imply $U \perp X$. Pick a non-trivial homomorphism $h \in \text{Hom}(W, U)$. By [Sch92 Lemma 2.3] $h$ is either injective or surjective. Since $\alpha$ is imaginary, by Proposition 4.6.3A $h$ must be injective. Then the composition $h \circ h \in \text{Hom}(X, U)$ is not surjective. But the kernel of $h \circ h$ contains the image of $V$ so $h \circ h$ is neither injective nor surjective in contradiction with [Sch92, Lemma 2.3]. □

**Proposition 4.13.** While the third stage of the algorithm assume $\langle \gamma_j, \gamma_i \rangle > 0$

1. if $[i, j]$ is minimal with the property and $\gamma_i$ is imaginary, then we have $\langle \gamma_j, \gamma_k \rangle = 0$ for $i < k < j$.

2. $\gamma_j$ is real and $\gamma_i$ is imaginary.

3. The third stage of the algorithm finishes after finitely many steps.

**Proof.** First of all, for any $i < j$ holds $\text{ext}(\gamma_i, \gamma_j) = 0$ and by [Sch92 Theorem 4.1] either $\text{ext}(\gamma_j, \gamma_i) = 0$ or $\text{hom}(\gamma_j, \gamma_i) = 0$, and the latter is the case if both $\gamma_i$ and $\gamma_j$ are imaginary. So either $\langle \gamma_j, \gamma_i \rangle < 0$ and in this case $\text{ext}(\gamma_j, \gamma_i) > 0$ and $\text{hom}(\gamma_j, \gamma_i) = 0$ or $\langle \gamma_j, \gamma_i \rangle > 0$ and in this case $\text{ext}(\gamma_j, \gamma_i) = 0$ or $\text{hom}(\gamma_j, \gamma_i) = 0$.

1. Applying Lemma 4.12 to the sequence $\gamma_i, \gamma_k, \gamma_j$, $i < k < j$, we get $\text{ext}(\gamma_j, \gamma_k) = 0$. So we have $\langle \gamma_j, \gamma_k \rangle \geq 0$ and $\langle \gamma_j, \gamma_k \rangle > 0$ contradicts to the minimality of $[i, j]$.

2. In the third stage we do not change the real roots, only permute them, hence, exactly one of $\gamma_i$ and $\gamma_j$ is imaginary. We prove that $\gamma_j$ is real and $\gamma_i$ is imaginary applying induction. At the beginning each imaginary is to the left of each real.

Then we apply the step of the third type to a minimal segment $[i, j]$ and have $\gamma_{i'}$ imaginary by induction. Then by 1 $\gamma_{i'}$ is perpendicular from both sides to $\gamma_k, i' < k < j'$ so the property remains true for all pairs containing $\gamma_{j'}$. As for the imaginary root being the result of pushing $\gamma_{i'}$ right, it is given by Proposition 4.6.3A for $\alpha = \gamma_{i'}, \beta = \gamma_{j'}$ and we have the exact sequence $0 \to pB \to A \to C \to 0$ from 4.6.3A. Taking $X$ to be a generic representation of dimension $\gamma_i, l < i'$, we have $\text{Hom}(A, X) = 0$, because $\langle \gamma_{i'}, \gamma_i \rangle \leq 0$ by induction. Then by (10), $\text{Hom}(C, X) = 0$, so $\text{hom}(\gamma_{i'}, p\gamma_{j'}, \gamma_i) = 0$, hence, $\langle \gamma_{i'} - p\gamma_{j'}, \gamma_i \rangle \leq 0$ and the property remains true.

3. Throughout the stage we decrease the imaginary members of $\gamma$ so we can not do it infinitely many times. □

**Theorem 4.14.** Algorithm 4.8 yields the generic locally semi-simple decomposition.

**Proof.** The above Propositions convinced us that after finitely many steps we obtain a decomposition with $\gamma$ being a perpendicular sequence of Schur roots such that $\text{hom}(\gamma_i, \gamma_j) = 0$ if $i \neq j$. We also claim that this is a Schur sequence, that is, $\gamma_i \circ \gamma_j = 1$ for $i < j$. Indeed this was true for the starting sequence and by [DW06 Lemma 4.2] this needs to be checked only for $\gamma_i$ and $\gamma_j$ being imaginary.

We now show that the property is preserved by any pushing of the imaginary root. So assume that $\gamma_i$ and $\gamma_j$ are imaginary $\gamma_k$ is real, $i < k$ and we replace $\gamma_j$ by $\gamma_j - p\gamma_k$. Then the vector space of the semi-invariants on $R(Q, \gamma_j)$ with the weight $-\langle \cdot, \gamma_j - p\gamma_k \rangle$ is embedded to that of the weight $-\langle \cdot, \gamma_j \rangle$ by multiplying with a semi-invariant of the weight $-\langle \cdot, p\gamma_k \rangle$. So $\gamma_i \circ \gamma_j - p\gamma_k > 1$ would imply $\gamma_i \circ \gamma_j > 1$. A similar argument works for pushing $\gamma_i$ so we proved that the output of the algorithm is a quiver Schur sequence. Hence, by Theorem 2.6 the resulting decomposition is locally semi-simple and by Corollary 4.4 this is the generic locally semi-simple decomposition. □
References

[DW00] H. Derksen and J. Weyman, Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients, J. AMS, 13 (2000), 3, 467-479.
[DW02] H. Derksen and J. Weyman, On the canonical decomposition of quiver representations, Composition Math. 133 (2002), 245-265.
[DW06] H. Derksen and J. Weyman, The combinatorics of quiver representations, preprint arXiv:math.RT/0606288.
[Kac] V. Kac, Infinite root systems, representations of graphs, and Invariant theory, II, J. Algebra 78 (1982), 141-162.
[Ki] A. D. King, Moduli of representations of finite dimensional algebras, Quart. J. Math. Oxford (2), 45 (1994), 515-530.
[LBP] L. Le Bruyn and C. Procesi, Semisimple representations of quivers, Transactions of AMS, 317 (1990), 2, 585-598.
[Lu] D. Luna, Slices étalés, Bull. Soc. Math. France 33 (1973), 81-105.
[Ri] C. M. Ringel, Rational invariants of the tame quivers, Inv.math. 58 (1980), 217-239.
[Sch91] A. Schofield, Semi-invariants of quivers, J. London Math. Soc. 43 (1991), 383-395.
[Sch92] A. Schofield, General representations of quivers, Proc. London Math. Soc. (3) 65 (1992), 46-64.
[Sh] D. A. Shmelkin, Locally semi-simple representations of quivers, Transf. Groups 12 (2007), 153-183.
[Te] D. A. Shmelkin, TETIVA a computer program available at http://www.mccme.ru/~mitia

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