Ordinary differential equations (ODE): metric entropy and nonasymptotic theory for noisy function fitting

Ying Zhu† and Mozhgan Mirzaei‡

Abstract

This paper establishes novel results on the metric entropy of ODE solution classes. Most of our theory is built upon or expands the smoothness structure in the Picard-Lindelöf theorem but we also provide a couple of results that do not require such a structure. In addition, we establish a nonasymptotic theory concerning noisy function fitting for nonparametric least squares and least squares based on Picard iterations. We also discuss the estimation of initial values with local polynomials. Our results on the metric entropy provide answers to “how do the degree of smoothness and the “size” of a class of ODEs affect the “size” of the associated class of solutions?” We establish a general upper bound on the covering number of solution classes associated with the higher order Picard type ODEs, \( y^{(m)}(x) = f(x, y(x), y'(x), ..., y^{(m-1)}(x)) \).

This result implies, the covering number of the underlying solution class is (basically) bounded from above by the covering number of the class \( F \) that \( f \) ranges over. This general bound (basically) yields a sharp scaling when \( f \) is parameterized by a \( K \)-dimensional vector of coefficients belonging to a ball and the noisy recovery is essentially no more difficult than estimating a \( K \)-dimensional element in the ball. For \( m = 1 \), when \( F \) is an infinitely dimensional smooth class, the solution class ends up with derivatives whose magnitude grows factorially fast – “a curse of smoothness”. We introduce a new notion called the “critical smoothness parameter” to derive an upper bound on the covering number of the solution class. When the sample size is large relative to the degree of smoothness, the rate of convergence associated with the noisy recovery problem obtained by applying this “critical smoothness parameter” based approach improves the rate obtained by applying the general upper bound on the covering number (and vice versa when the sample size is small relative to the degree of smoothness).

1 Introduction

Differential equations enjoy a long standing history in mathematics and have numerous applications in science and engineering. Differential equations also play a crucial role in social science and business, for example, the famous Solow growth model (which is a first order ODE) in economics (see, Barro and Sala-i-Martain, 2004), and the famous Bass product diffusion model (also a first order ODE) in marketing (see, Bass 1969, 2004). Since the COVID-19 pandemic, lots of attention has been given to the compartmental models in epidemiology for predicting the spread of infectious diseases (see, Vynnycky and White, 2010).

Statisticians have proposed useful methods for inference of structural parameters in differential equations and addressed important computational challenges (see, e.g., Ramsay, et. al, 2007;
Campbell, et. al, 2012). In contrast to the existing statistical literature on differential equations, this paper focuses on the metric entropy of ODE solution classes and a nonasymptotic theory for noisy function fitting problems, both of which, to the best of our knowledge, have not been examined in the literature. Results on metric entropy of ODE solution classes are crucial for building an empirical process theory for differential equations. We choose to focus on ODEs in this paper as this is a natural and necessary first step before one delves into more complicated differential equations (such as partial differential equations). Even in ODEs, there are plenty of theoretical challenges and we are able to develop many new perspectives. Moreover, because of the deep connections between ODEs and contraction mapping, understanding the “size” of ODE solution classes could pave the way for understanding the “size” of systems that involve fixed points (such as Markov decision processes and game theoretic models in economics).

When talking about the theory of ODEs, one must mention the Picard-Lindelöf theorem, a central result in ODEs. This theorem hinges on a Lipschitz continuity condition, which ensures the existence of a solution to an initial value problem and the uniqueness of this solution. In this paper, we consider ODEs in the form

\[ y^{(m)}(x) = f\left(x, y(x), y'(x), \ldots, y^{(m-1)}(x)\right). \tag{1} \]

A natural question to ask is, how do the degree of smoothness and the “size” of a class of \( f \) affect the “size” of the associated class of \( y \) in (1)? The answers to this question turn out quite complicated and the analyses require substantial efforts as well as deep thinking, as our paper suggests. By letting \( f \) in (1) range over various function classes, we establish novel results on the metric entropy of the associated solution classes \( \mathcal{Y} \). Most of our theory is built upon or expands the smoothness structure in the Picard-Lindelöf theorem although we also provide a couple of results that do not require such a structure.

We also establish a nonasymptotic theory concerning the following noisy function fitting/recovery problem

\[ Y_i = y^*(x_i) + \varepsilon_i, \quad i = 1, \ldots, n, \tag{2} \]

where \( \{Y_i\}_{i=1}^n \) are response variables and \( \{x_i\}_{i=1}^n \) is a collection of fixed design points (both observed), the unobserved noise terms \( \{\varepsilon_i\}_{i=1}^n \) are i.i.d. draws from \( \mathcal{N}(0, \sigma^2) \), and \( y^*(\cdot) \in \mathcal{Y} \). We consider the so-called nonparametric least squares estimator

\[ \hat{y} \in \arg\min_{\tilde{y} \in \mathcal{Y}} \frac{1}{2n} \sum_{i=1}^n (Y_i - \tilde{y}(x_i))^2 \]

and establish upper bounds on the prediction error \( \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n (\hat{y}(x_i) - y^*(x_i))^2 \right] \).

Besides its theoretical importance, we point out that a Picard-type Lipschitz condition can be quite useful for practical implementations, as it paves the way for computing an approximate solution to an initial value problem in an iterative manner. If one can solve for \( y^*(x) \) in an explicit form, nonlinear least squares and maximum likelihood procedures can obviously be applied. However, in many ODEs (even when the ODE is parameterized by a finite vector of coefficients), the solutions may be in implicit forms. In the situation where a quality estimator of the initial value is available, one can exploit the Picard iteration to circumvent this issue. We propose estimation strategies based on Picard iterations for noisy recovery of \( y^*(\cdot) \in \mathcal{Y} \) associated with first order

\(^1\)The metric entropy here refers to the concept (proposed by Kolmogorov, Tikhomirov, and others; see, e.g., Kolmogorov and Tikhomirov, 1959) that measures the “size” of a class with possibly infinitely many members.
ODEs ranging over both parametric and nonparametric classes, and establish upper bounds for the estimation errors of these strategies. We also discuss the estimation of initial values with local polynomials.

Below we highlight the novelty in some of our results. We establish a general upper bound on the covering number of solution classes associated with the higher order Picard type ODEs. This result implies, the covering number of the underlying solution class $\mathcal{Y}$ is bounded from above by the covering number of the class $\mathcal{F}$ that $f$ ranges over, as long as $\mathcal{F}$ satisfies a Lipschitz condition with respect to the $y, ..., y^{(m-1)}$ coordinates (which we will refer to as the Picard type Lipschitz condition). If the initial values are fixed, this general bound yields a sharp scaling when $f$ is parameterized by a $K$–dimensional vector of coefficients belonging to a ball and further satisfies a Lipschitz condition with respect to the coefficients. As a result, the noisy recovery is essentially no more difficult than estimating a $K$–dimensional element in the ball.

When $\mathcal{F}$ is an infinitely dimensional smooth class, the rate of convergence associated with the noisy recovery problem obtained by applying the general bound on the covering number can be improved when the sample size $n$ is large relative to the degree of smoothness. In the paper we focus on the first order ODE

$$y'(x) = f(x, y(x)) \quad (3)$$

to showcase the improvement. In particular, when $\mathcal{F}$ is the standard smooth class of degree $\beta + 1$ (where the absolute values of all partial derivatives of $f \in \mathcal{F}$ are bounded from above by 1 and the $\beta$th derivative of $f$ is 1–Lipschitz$^3$), we show that the solution class $\mathcal{Y}$ is a subset of a variant smooth class $\mathcal{S}^\dagger_{\beta+2}$, where the absolute value of the $k$th derivative of $y \in \mathcal{Y}$ is bounded from above by $2^{k-1} (k-1)!$ for $k = 1, ..., \beta + 1$ and the $(\beta + 1)$th derivative of $y$ is $\rho_{\beta}$–Lipschitz with $\rho_{\beta} = 2^{\beta+1} (\beta + 1)!$.

Given the $\rho_{\beta}$–Lipschitz continuity of the $(\beta + 1)$th derivative of $y \in \mathcal{Y}$, existing results (e.g., Wainwright, 2019) would suggest

$$\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \hat{y}(x_i) - y^*(x_i) \right)^2 \right] \lesssim \rho_{\beta} \frac{2^{2(\beta+2) + 1}}{n} \frac{\sigma^2}{\rho_{\beta}^{2(\beta+2) + 1}}. \quad (4)$$

The minimax lower bounds in Tsybakov (2009) also imply the same scaling $\rho_{\beta}^{2(\beta+2) + 1} \frac{\sigma^2}{n} \rho_{\beta}^{2(\beta+2) + 1}$. When $\beta$ is small relative to $\frac{n}{\sigma^2}$, the bound $\rho_{\beta}^{2(\beta+2) + 1} \frac{\sigma^2}{n} \rho_{\beta}^{2(\beta+2) + 1}$ is clearly useful. However, if $\beta$ is large enough such that $\rho_{\beta}^{2(\beta+2) + 1} \frac{\sigma^2}{n} \rho_{\beta}^{2(\beta+2) + 1} \rightarrow \infty$, the lower bound result suggests that there exist no consistent estimators in the worst case.

Classical results like the above do not deliver a useful bound in this case because they concern a bigger class than $\mathcal{S}^\dagger_{\beta+2}$ and fail to take into account the special structure of $\mathcal{S}^\dagger_{\beta+2}$. Indeed, for estimations of functions in $\mathcal{S}^\dagger_{\beta+2}$, one can easily obtain a rate much better than $\infty$ even if $\rho_{\beta}^{2(\beta+2) + 1} \frac{\sigma^2}{n} \rho_{\beta}^{2(\beta+2) + 1} \rightarrow \infty$. To see this, note that the first derivative of $h \in \mathcal{S}^\dagger_{\beta+2}$ is 2–Lipschitz and the optimal rate concerning estimations of twice smooth functions scales as $n^{-\frac{5}{2}}$, for which consistent estimators apparently exist in the worst case. What makes the class $\mathcal{S}^\dagger_{\beta+2}$ interesting

\footnote{Our results on the covering numbers include an additional term related to the “size” of the sets where initial values belong when they are not fixed. When they are fixed, this term is zero.}

\footnote{We say that a function $g$ on $[0, 1]^d$ is $L$–lipschitz if $|g(w) - g(w')| \leq L \cdot |w - w'|_\infty$ for all $w, w' \in [0, 1]^d$, where $|\cdot|$ is the $l_\infty$–norm.}

\[ \text{3} \]
is that it exhibits a “curse of smoothness” as \( \beta \) increases because of the factorial growth in the derivatives. Motivated by this observation, we introduce a new notion called the “critical smoothness parameter” to derive an upper bound on the covering number of \( \mathcal{S}_{\beta+2}^\dagger \). The resulting upper bound along with the aforementioned general upper bound for the covering number of \( \mathcal{Y} \) allows us to obtain the following rate:

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \right] \lesssim \min \left\{ \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^* + 2)}{2(\gamma^* + 2) + 1}}, \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 2}} \right\}
\]  

(5)

where the non-negative integer \( \gamma^* \in [0, \beta] \) is the critical smoothness parameter (whose derivation is detailed in Section 2). In theory, our analysis suggests that \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^* + 2)}{2(\gamma^* + 2) + 1}} \) in (3) may be improved even further but an analytical form of this sharper bound is difficult to obtain.

Nevertheless, the rate \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^* + 2)}{2(\gamma^* + 2) + 1}} \) can be smaller than the rate \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 2}} \). Obviously, if \( \frac{\sigma^2}{n} \leq 1 \), \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 2}} \) is only useful when \( \beta > 3 \). We also show that \( \gamma^* = \beta \) when \( \frac{n}{\sigma^2} \gtrsim (\beta \log \beta)^{4(\beta + 2) / 2} \). In this case, (5) becomes

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \right] \lesssim \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 2)}{2(\beta + 1) + 1}}.
\]

On the other hand, when \( \beta \) is large enough relative to \( \frac{n}{\sigma^2} \), the rate \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 2}} \) can be smaller than the rate \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^* + 2)}{2(\gamma^* + 2) + 1}} \). Obviously, as we mentioned before, if \( \beta \) is much smaller than \( \frac{n}{\sigma^2} \), the classical bound \( \rho_{\beta} \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 2)}{2(\beta + 1) + 1}} \) in (4) can be also useful.

We also consider a special case of (3) where its right-hand-side has a “single index” structure

\[
y'(x) = f(y(x)).
\]

(6)

In the ODE literature, this structure is referred to as an autonomous system while (3) is referred to as a nonautonomous system. In particular, when \( \mathcal{F} \) is a standard smooth class of degree \( \beta + 1 \), we show that the solution class \( \mathcal{Y} \subseteq \mathcal{AS}_{\beta+2}^\dagger \), where the absolute value of the \( k \)th derivative of \( y \in \mathcal{Y} \) is bounded from above by \( (k - 1)! \) for \( k = 1, \ldots, \beta + 1 \) and the \( (\beta + 1) \)th derivative of \( y \) is \( (\beta + 1)! \)-Lipschitz. These factorial bounds are tight in the case of (6). Like in the nonautonomous system, we derive an upper bound on the covering number of \( \mathcal{AS}_{\beta+2}^\dagger \) using the approach of “critical smoothness parameter”. This result along with the general upper bound for the covering number of \( \mathcal{Y} \) allows us to obtain the following rate:

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \right] \lesssim \min \left\{ \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^* + 2)}{2(\gamma^* + 2) + 1}}, \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 2}} \right\}.
\]

(7)

Despite the tight factorial bounds and the “single index” structure in the autonomous system, we find that the critical smoothness parameters \( \gamma^* \) associated with (7) and (5) can coincide if \( \frac{n}{\sigma^2} \) is large relative to \( \beta \). Consequently, when the rate \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^* + 2)}{2(\gamma^* + 2) + 1}} \) is smaller than \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 2}} \), bounds
(5) and (7) exhibit the same scaling. On the other hand, in situations where \( \left( \frac{a^2}{n} \right)^{2(\beta+1)} \) improves upon \( \left( \frac{a^2}{n} \right)^{2(\gamma+2)} \) in (7) and \( \left( \frac{a^2}{n} \right)^{\frac{\beta+1}{\beta+2}} \) improves upon \( \left( \frac{a^2}{n} \right)^{\frac{2(\gamma+2)}{\beta+2}} \) in (5) (as a result of small enough \( \frac{n}{\gamma} \) and large enough \( \beta \)), we can clearly see the difference in the upper bounds for the prediction errors between the autonomous system and the nonautonomous system.

While most of our theoretical results are built upon a Picard-type Lipschitz condition, we provide an oracle inequality for the prediction errors without requiring (the true) \( f^* \) in the ODE \( y''(x) = f^*(x, y^*(x)) \) to be Lipschitz continuous with respect to the \( y^* \) coordinate. In addition, we provide results on the covering number of separable first order ODEs with monotonic \( f \), as well as the VC dimension of ODEs with polynomial potential functions.

2 Main results

General notation. The \( l_q \)-norm of a \( K \)-dimensional vector \( \theta \) is denoted by \( |\theta|_q, 1 \leq q \leq \infty \) where \( |\theta|_q := \left( \sum_{j=1}^{K} |\theta_j|^q \right)^{1/q} \) when \( 1 \leq q < \infty \) and \( |\theta|_q := \max_{j=1,...,K} |\theta_j| \) when \( q = \infty \). Let \( \mathbb{B}_q(1) := \left\{ \theta \in \mathbb{R}^K : |\theta|_q \leq 1 \right\} \) with \( q \geq 1 \). Define \( \mathbb{P}_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \) that places a weight \( \frac{1}{n} \) on each observation \( x_i \) for \( i = 1, ..., n \), and the associated \( \mathcal{L}^2(\mathbb{P}_n) \)-norm of the vector \( v := \{v(x_i)\}_{i=1}^{n} \), denoted by \( |v|_n \), is given by \( \frac{1}{n} \sum_{i=1}^{n} (v(x_i))^2 \). For two functions \( f \) and \( g \) on \([a, b]^d \subseteq \mathbb{R}^d\), we denote the supremum metric by \( |f - g|_\infty := \sup_{x \in [a, b]^d} |f(x) - g(x)| \). For functions \( f(n) \) and \( g(n) \), \( f(n) \gtrsim g(n) \) means that \( f(n) \geq c g(n) \) for a universal constant \( c \in (0, \infty) \); similarly, \( f(n) \lesssim g(n) \) means that \( f(n) \leq c' g(n) \) for a universal constant \( c' \in (0, \infty) \); and \( f(n) \succ g(n) \) means that \( f(n) \gtrsim g(n) \) and \( f(n) \lesssim g(n) \) hold simultaneously. As a general rule for this paper, the various \( c \) and \( C \) constants (all \( \gtrsim 1 \)) denote positive universal constants that are independent of the sample size \( n \) and the smoothness parameter \( \beta \), and may vary from place to place.

General definitions. Given a set \( \mathbb{T} \), a set \( \{t^1, t^2, ..., t^N\} \subset \mathbb{T} \) is called a \( \delta \)-cover of \( \mathbb{T} \) with respect to a metric \( \rho \) if for each \( t \in \mathbb{T} \), there exists some \( i \in \{1, ..., N\} \) such that \( \rho(t, t^i) \leq \delta \). The cardinality of the smallest \( \delta \)-cover is denoted by \( N_\rho(\delta; \mathbb{T}) \), namely, the \( \delta \)-covering number of \( \mathbb{T} \). For example, \( N_\infty(\delta, \mathcal{F}) \) denotes the \( \delta \)-covering number of a function class \( \mathcal{F} \) with respect to the supremum metric \( |.|_\infty \). Let \( \mathcal{V} \) be a class of binary-valued functions. The class \( \mathcal{V} \) shatters the collection of points \( \{w_1, ..., w_d\} := w_1^d \) if the cardinality of \( \mathcal{V}(w_1^d) \) is \( 2^d \). The VC dimension is defined as the largest \( d \) such that there exists some collection \( w_1^d \) which can be shattered by \( \mathcal{V} \).

We begin by refreshing on a number of classical existence and/or uniqueness results about first ODEs

\[ y'(x) = f(x, y(x)) \quad \text{with} \quad y(a_0) = y_0. \]  

(8)

These results can certainly be extended to higher order ODEs but require us to introduce extra notations. Therefore, we refer the readers to Coddington and Levinson (1955) for a more thorough treatment. To distinguish this paper’s contributions from existing results, we label the cited results with capital Roman numerals (I, II, etc.) and our contributed results by section specific numbers (2.1, 2.2, etc., A.1, B.1, etc).

**Theorem I** (Cauchy-Peano Existence). Assume \( f(x, y) \) is continuous on

\[ [a_0, a_0 + a] \times [y_0 - b, y_0 + b] =: \Lambda \]
for some \(a, b > 0\). Then there exists a solution to (8) in \(C[a_0, a_0 + \alpha]\) where \(\alpha \leq \min \left\{a, \frac{b}{M} \right\}\) and

\[
M = \max_{(x,y) \in \Lambda} |f(x, y)|.
\]

**Theorem II** (Picard local existence). Assume \(f(x, y)\) is continuous on \(\Lambda\) and is uniformly Lipschitz continuous with respect to \(y\) on \(\Lambda\). Then there exists a solution to (8) in \(C[a_0, a_0 + \alpha]\) where \(\alpha \leq \min \left\{a, \frac{b}{M} \right\}\).

**Theorem III** (Local existence and uniqueness). Assume \(f(x, y)\) is continuous on \(\Lambda\) and satisfies

\[
|f(x, y) - f(x, \tilde{y})| \leq L |y - \tilde{y}| \quad \forall (x, y), (x, \tilde{y}) \in \Lambda.
\]

Then for some \(\alpha \in (0, a]\), there is a unique solution to (8) in \(C[a_0, a_0 + \alpha]\). Moreover, \(\alpha\) can be chosen to be any positive number such that \(\alpha \leq L^{-1}\) and \(\alpha \leq \min \left\{a, \frac{b}{M} \right\}\).

To proceed, we expand the structures in Theorems I-III and establish upper bounds on the covering numbers (or VC dimensions) of the classes of solutions to the ODEs under various assumptions: (1) when \(f\) is parameterized by a (finite dimensional) vector of coefficients (subsection 2.1), (2) when \(f\) ranges over (infinitely dimensional) smooth classes (subsections 2.2 and 2.3), (3) separable first order ODEs with monotonic \(f\) that need not be uniformly Lipschitz continuous (the end of subsection 2.3), as well as (4) ODEs with polynomial potential functions (subsection 2.4). We end this section with a general upper bound on the covering number of higher order Picard type ODEs (subsection 2.5). Besides the results on metric entropy, we also establish a nonasymptotic theory for noisy function recovery.

### 2.1 Parametric first order ODEs

#### 2.1.1 Covering numbers

**Proposition 2.1.** Consider the ODE

\[
y'(x) = f(x, y(x); \theta), \quad y(0) = y_0 \tag{9}
\]

where \(|y_0| \leq C_0\), \((x, y(x)) \in [0, 1] \times [-C_0 - b, C_0 + b]\) \((C_0, b > 0)\), and \(f\) is parameterized by a \(K\)-dimensional vector of coefficients \(\theta \in \mathbb{B}_q(1)\) with \(q \geq 1\). Suppose \(f\) is continuous on \([0, 1] \times [-C_0 - b, C_0 + b]\), \(f(x, y; \theta)\) is parameterized by \(a, b > 0\) and \(\alpha \leq \min \left\{a, \frac{b}{M} \right\}\).

where \(\alpha \leq \min \left\{a, \frac{b}{M} \right\}\). Moreover,

\[
|f(x, y; \theta) - f(x, \tilde{y}; \theta)| \leq |y - \tilde{y}| \tag{10}
\]

for all \((x, y)\) and \((x, \tilde{y})\) in \([0, 1] \times [-C_0 - b, C_0 + b]\), and \(\theta \in \mathbb{B}_q(1)\); moreover,

\[
|f(x, y; \theta) - f(x, y; \theta')| \leq L_{K,q} |\theta - \theta'|_q, \tag{11}
\]

for all \((x, y) \in [0, 1] \times [-C_0 - b, C_0 + b]\) and \(\theta, \theta' \in \mathbb{B}_q(1)\). Then we have

\[
\log N_\infty(\delta, \mathcal{Y}) \leq K \log \left(1 + \frac{2C_{\text{max}} L_{K,q}}{\delta} \right) + \log \left(\frac{C_0 C_{\text{max}}}{\delta} + 1\right) \tag{12}
\]

where \(C_{\text{max}} = \sup_{x \in [0, a]} \{\exp(x) [1 + \int_0^x \exp(-s) ds]\}\) with \(\alpha = \min \{1, b\}\) and \(\mathcal{Y}\) is the class of solutions (to (8) with \(\theta \in \mathbb{B}_q(1)\)) on \([0, a]\).

\[\text{In (12), } L_{K,q} \text{ can depend on } K \text{ and } q.\]
Remark. Note that \([y_0 - b, y_0 + b] \subseteq [-C_0 - b, C_0 + b]\) for all \(y_0\) such that \(|y_0| \leq C_0\). By the Cauchy-Peano Existence Theorem (Theorem I) or the Picard Existence Theorem (Theorem II), there exists a solution to every ODE in the form of (12) on \([0, \alpha]\) such that \(\alpha = \min\{1, b\}\). We will use this definition of \(\alpha\) throughout the results in this subsection and subsections 2.2-2.3. In all these results, the existence of a solution to their underlying ODE is guaranteed by Theorem I or Theorem II.

In (12), the part “\(K \log \left(1 + \frac{2C_{\max}L_K}{\theta}\right)\)” is related to the “size” of \(B_0\) (1) that \(\theta\) ranges over, and the part “\(\log \left(\frac{C_{\max}L_K}{\theta}\right) + 1\)” is related to the “size” of \([-C_0, C_0]\) that the initial value \(y_0\) ranges over. The part “\(\log \left(\frac{C_{\max}L_K}{\theta}\right) + 1\)” reveals an interesting feature of differential equations: even if the equation is fixed and known (for example, \(y' (x) = y (x)\) with solutions \(y (x) = c^e x^y\)), the solution class is still not a singleton (and has infinitely many solutions) unless a fixed initial value is given. As a consequence, in the simple example \(y' = y\), the noisy recovery of a solution to this ODE still requires the estimation of \(c^e\).

The result in Proposition 2.1 implies that if the class \(f\) ranges over a “parametric” class, then the associated solution class \(\mathcal{Y}\) also behaves like a “parametric” class. Suppose the class of ODEs \(\mathcal{G}\) has a fixed initial value. Then the term “\(\log \left(\frac{C_{\max}L_K}{\theta}\right) + 1\)” can be dropped while the scaling of “\(K \log \left(1 + \frac{2C_{\max}L_K}{\theta}\right)\)” can be attained as the following example suggests. Consider the simple ODE \(y' (x) = -\theta y (x)\) with \(\theta \in [0, 1]\), \(x \in [0, 1]\) and initial value \(y (0) = 1\), which has solutions in the form \(y (x) = e^{-\theta x}\). It can be easily verified that \(\log N_{\infty} (\delta, \mathcal{Y}) \asymp \log \left(\frac{c}{\delta} + c'\right)\) for some positive universal constants \(c\) and \(c'\).

### 2.1.2 Least squares based on explicit solutions

Our next result concerns the estimation of the following ODE

\[
y^* (x; \theta^*) = f (x, y^* (x; \theta^*); \theta^*), \quad y^* (x_0; \theta^*) = y_0^*
\]

where \(\theta^* \in \mathbb{B}_0 (1)\) and \(|y_0^*| \leq C_0\). We first consider the setup where one can solve for \(y^* (x; \theta^*, y_0)\) in an explicit form from (14) and then obtain an estimator \(\hat{y}^* (x; \hat{\theta}, \hat{y}_0)\) with

\[
(\hat{\theta}, \hat{y}_0) \in \arg \min_{\theta \in \mathbb{B}_0 (1), |y_0| \leq C_0} \frac{1}{2n} \sum_{i=1}^{n} (Y_i - y^* (x_i; \theta, y_0))^2.
\]

In particular, we assume that all the \(n\) design points are sampled from \([0, \alpha]\). Under (11), the Picard local existence and uniqueness theorem (Theorem III) implies that there is a unique solution to (9) on \([0, \alpha]\) and therefore, given \(\theta\) and \(y_0\), \(y^* (x_i; \theta, y_0)\) in program (15) has a unique form for all \(x_i\) \((i = 1, ..., n)\) on \([0, \alpha]\).

\(5\) In terms of the homogeneous higher order linear ODEs

\[
a_0 (x) y + a_1 (x) y^{(1)} + a_2 (x) y^{(2)} + \cdots + a_m (x) y^{(m)} = 0,
\]

it is well understood that the solutions of (13) (where \(a_j (\cdot)\)s are fixed continuous functions) form a vector space of dimension \(m\). A classical result on the VC dimension of a vector space of functions (Steele, 1978 and Dudley, 1978) implies that the solution class associated with (13) has VC dimension at most \(m\), when \(a_j (\cdot)\)s \((j = 0, ..., m)\) are fixed continuous functions.
Proposition 2.2. Suppose the conditions in Proposition 2.1 are satisfied with \( q = \infty \), and all the \( n \) design points are sampled from the interval \([0, \alpha]\). Let us consider (2) where \( y^*(\cdot) \) is the (unique) solution to (14) on \([0, \alpha]\). Letting \( B_{L_K, \infty} = (L_K, \infty \lor 1) \), if

\[
K \log \left( 1 + \frac{2C_{\text{max}} L_K q}{\delta} \right) \gtrsim \log \left( \frac{C_0 C_{\text{max}}}{\delta} + 1 \right) , \forall \delta \gtrsim B_{L_K, \infty} \sigma \sqrt{\frac{K}{n}} \tag{16}
\]

\[
\max \{ \alpha, C_0 \} \geq c_0 B_{L_K, \infty} \sigma \sqrt{\frac{K}{n}} \tag{17}
\]

for a sufficiently large positive universal constant \( c_0 \), then we have

\[
\frac{1}{n} \sum_{i=1}^{n} \left( y^*(x_i; \hat{\theta}, \hat{y}_0) - y^*(x_i; \theta^*, y_0^*) \right)^2 \gtrsim B_{L_K, \infty}^2 \frac{\sigma^2 K}{n} \tag{18}
\]

with probability at least \( 1 - c_1 \exp \left( -c_2 B_{L_K, \infty}^2 K \right) \), where \((\hat{\theta}, \hat{y}_0)\) is a solution to (15).

Remark. Condition (16) simply restricts \( C_0 \) from being too large, and as a consequence, (12) implies that \( \log N(\delta, \mathcal{Y}) \gtrsim K \log \left( 1 + \frac{2C_{\text{max}} L_K q}{\delta} \right) \). This upper bound implies that \( \mathcal{Y} \) is no “larger” than the class of \( f \)s parameterized by \( \theta \in \mathcal{B}_\infty (1) \). As long as \( B_{L_K, \infty} \gtrsim 1 \), Proposition 2.2 suggests that the noisy recovery of (14) is essentially no more difficult than estimating a \( K \)-dimensional element in \( \mathcal{B}_\infty (1) \), where the optimal rate is \( \frac{\sigma^2 K}{n} \).

Remark. Condition (17) in Proposition 2.2 simply excludes the case where \( b \) and \( C_0 \) are “too small”. Without such a condition, as long as \( f \) is bounded from above, we would simply replace (18) with

\[
\left[ \frac{1}{n} \sum_{i=1}^{n} \left( y^*(x_i; \hat{\theta}, \hat{y}_0) - y^*(x_i; \theta^*, y_0^*) \right)^2 \right]^{\frac{1}{2}} \leq c_1 \min \left\{ B_{L_K, \infty} \sigma \sqrt{\frac{K}{n}}, \max \{ \alpha, C_0 \} \right\}
\]

Later conditions (22) in Proposition 2.3, (57) in Theorem 2.2, (12) in Proposition 2.4, (17) in Proposition 2.5, (59) in Theorem 2.4, and (63) in Proposition 2.6 serve similar purposes.

2.1.3 Least squares based on Picard iterations

In many ODEs, the solutions may be in implicit forms so it is not possible to write down (15). In the situation where we have a quality estimator \( \hat{y}_0 \) of \( y_0^* \), we can exploit the Picard iteration below to circumvent this issue:

\[
y_{r+1}(x; \theta) = y_0 + \int_0^x f(s, y_r(s; \theta); \theta) \, ds, \quad \text{integer } r \geq 0, y(0; \theta) = y_0. \tag{19}
\]

An estimator based on (19) performs the following steps: first, we compute

\[
\hat{y}_1(x; \theta) = \hat{y}_0 + \int_0^x f(s, \hat{y}_0; \theta) \, ds,
\]

\[
\hat{y}_2(x; \theta) = \hat{y}_0 + \int_0^x f(s, \hat{y}_1(s; \theta); \theta) \, ds,
\]

\[
\vdots
\]

\[
\hat{y}_{R+1}(x; \theta) = \hat{y}_0 + \int_0^x f(s, \hat{y}_R(s; \theta); \theta) \, ds;
\]

\[
8
\]
second, we solve the following program
\[
\hat{\theta} \in \arg \min_{\theta \in \mathbb{B}_{\infty}(1)} \frac{1}{2n} \sum_{i=1}^{n} (Y_i - \hat{y}_{R+1}(x_i; \theta))^2.
\] (21)

**Proposition 2.3.** Suppose the conditions in Proposition 2.1 hold with \( q = \infty \), and all the \( n \) design points are sampled from \([0, \tilde{\alpha}]\) where \( \tilde{\alpha} < \alpha \). Let us consider (2) where \( y^*(\cdot) \) is the (unique) solution to (14) on \([0, \tilde{\alpha}]\). In terms of \( \hat{y}_{R+1}(x_i; \tilde{\theta}) \) where \( \hat{\theta} \) is obtained from solving (21), if
\[
\max \{ \tilde{\alpha}, C_0 \} \geq c_0 \bar{b} \sqrt{\frac{K}{n}}
\] (22)

for a sufficiently large positive universal constant \( c_0 \), then we have
\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{y}_{R+1}(x_i; \hat{\theta}) - y^*(x_i; \theta^*) \right] \right\} \frac{1}{2} \lesssim \sigma \left( \bar{b} \sqrt{\frac{K}{n}} + \frac{1}{1-\tilde{\alpha}} |\hat{y}_0 - y_0^*| + \frac{\tilde{\alpha}^{R+1}}{1-\tilde{\alpha}} \max \{ C_0, \tilde{\alpha} \} \right)
\] (23)
with probability at least \( 1 - c_1 \exp(-c_2n) - c_3 \exp(-c_4\tilde{b}^2) \), where \( \bar{b} = \left( \frac{\tilde{\alpha}L}{1-\alpha} \lor 1 \right) \).

**Remark.** If the integral in (19) is hard to compute analytically, numerical integration can be used in (20). This would introduce an additional approximation error \( \sigma (R + 1) \cdot \text{Err} \), where \( \text{Err} \) is an upper bound on the error incurred in each iteration of (20), depending on the smoothness of \( f \) and which numerical method is used. For example, if \( f \) is twice differentiable with bounded first and second derivatives, and the integral is approximated with the midpoint rule with \( T \) slices, then \( \text{Err} \lesssim T^{-2} \).

**Remark.** When our results concern a Picard-iteration-based estimation (Proposition 2.3 above and later Propositions 2.5-2.6), we assume the sampling of all the \( n \) design points from \([0, \tilde{\alpha}]\) with \( \tilde{\alpha} < \alpha \). This assumption ensures \( \frac{1}{1-\tilde{\alpha}} |\hat{y}_0 - y_0^*| + \frac{\tilde{\alpha}^{R+1}}{1-\tilde{\alpha}} \max \{ C_0, \tilde{\alpha} \} \) in (23) to be well defined in the case where \( \alpha = 1 \).

The bound (23) reflects three sources of errors: \( \bar{b} \sqrt{\frac{K}{n}} \) is due to the estimation error in \( \hat{\theta} \), \( \frac{1}{1-\alpha} |\hat{y}_0 - y_0^*| \) is due to the estimation error in \( \hat{y}_0 \), and \( \frac{\tilde{\alpha}^{R+1}}{1-\tilde{\alpha}} \max \{ C_0, \tilde{\alpha} \} \) is due to the error from the finite \((R+1)\) Picard iterations.

### 2.2 Nonparametric smooth first order ODEs

#### 2.2.1 Covering numbers

To facilitate presentations of the upcoming results, we introduce additional definitions below.

**Definitions.** Given \( C_0 > 0, b > 0, \) and \( \alpha = \min \{ 1, b \} \), let
\[
\mathcal{C} := C_0 + b, \\
\Xi := [0, 1] \times [-\mathcal{C}, \mathcal{C}], \\
C_{\text{max}} := \sup_{x \in [0, \alpha]} \left\{ \exp(x) \left[ 1 + \int_0^x \exp(-s) \, ds \right] \right\}.
\]
Let $p = (p_1, ..., p_d)$ and $[p] = \sum_{k=1}^{d} p_k$ where $p_k$s are non-negative integers. We write

$$D^p h (z_1, ..., z_d) := \partial^{[p]} h / \partial z_1^{p_1} ... \partial z_d^{p_d}.$$

Given a non-negative integer $\gamma$, we let $S_{\gamma+1, d} (\rho, [\underline{a}, \overline{a}]^d)$ denote the class of functions such that any function $h \in S_{\gamma+1, d} (\rho, [\underline{a}, \overline{a}]^d)$ satisfies:

1. $h$ is continuous on $[\underline{a}, \overline{a}]^d$, and all partial derivatives of $h$ exist for all $p$ with $[p] \leq \gamma$;
2. $|D^p h (X)| \leq \rho$ for all $X \in [\underline{a}, \overline{a}]^d$ and all $p$ with $[p] \leq \gamma$, where $D^0 h (X) = h (X)$;
3. $|D^p (h (X)) - D^p (h (X'))| \leq \rho |X - X'|_{\infty}$ for all $X, X' \in [\underline{a}, \overline{a}]^d$ and all $p$ with $[p] = \gamma$.

When $\rho = 1$, $d = 1$, $\underline{a} = 0$ and $\overline{a} = 1$, we use the shortform $S_{\gamma+1} : = S_{\gamma+1, 1} (1, [0, 1])$. In addition, when $d = 1$, $\underline{a} = 0$ and $\overline{a} = 1$, we denote the class of functions satisfying parts 1 and 3 above by $S'_{\gamma+1} (\rho)$. Lastly, we let $S^\dagger_{\beta+2}$ denote the class of functions such that any function $h \in S^\dagger_{\beta+2}$ satisfies the following properties:

1. $h$ is continuous on $[0, 1]$ and differentiable $\beta + 1$ times;
2. $|h(x)| \leq C$, and $|h^{(k)}(x)| \leq 2^{k-1} (k - 1)!$ for all $k = 1, ..., \beta + 1$ and $x \in [0, 1]$;
3. $|h^{(\beta+1)}(x) - h^{(\beta+1)}(x')| \leq 2^{\beta+1} (\beta + 1)! |x - x'|$ for all $x, x' \in [0, 1]$.

The results in this section concern the ODE in the form

$$y' (x) = f (x, y (x)), \quad y (0) = y_0$$

(24)

which is subject to the following assumption.

**Assumption F.** In (24), $|y_0| \leq C_0$ and $(x, y (x)) \in [0, 1] \times [\overline{\Theta}, \overline{\Theta}] = \Xi$ (where $\overline{\Theta} > 1$); $f$ ranges over $S_{\beta+1, 2} (1, \Xi)$, the class of $2$–dimensional smooth functions of degree $\beta + 1$. That is, $f$ is continuous on $\Xi$, and all partial derivatives $D^p$ of $f$ exist for all $p$ with $[p] = p_1 + p_2 \leq \beta$; $|D^p f (x, y)| \leq 1$ for all $p$ with $[p] \leq \beta$ and $(x, y) \in \Xi$, where $D^0 f (x, y) = f (x, y)$; and

$$|D^p f (x, y) - D^p f (x', \tilde{y})| \leq \max \left\{ |x - x'|, |y - \tilde{y}| \right\}$$

for all $p$ with $[p] = \beta$ and $(x, y), (x', \tilde{y}) \in \Xi$.

**Theorem 2.1.** Let us consider the ODE (24). Suppose Assumption F holds.

(i) We have

$$\log N_{\infty} (\delta, \mathcal{Y}) \leq c \left[ \left( \frac{\delta}{C_{\max}} \right)^{\frac{\beta+1}{2}} + (\beta + 1) \log \left( \frac{C_{\max}}{\delta} \right) + \log \left( \frac{C_0 C_{\max}}{\delta} + 1 \right) \right]$$

(25)

where $\mathcal{Y}$ is the class of solutions (to (24) with $f \in S_{\beta+1, 2} (1, \Xi)$) on $[0, \alpha]$.

10
(ii) For a given $\frac{\delta}{5} \in (0, 1)$, let $\beta^* (\delta) = \gamma (\leq \beta)$ be the largest non-negative integer such that

$$\log \left( \prod_{i=0}^{\gamma} i! \right) + \frac{\gamma^2 + \gamma}{2} \log 2 - \frac{\gamma + 3}{2} \log \frac{\delta}{5} \leq \left( \frac{\delta}{5} \right)^{\frac{1}{\gamma^2}} \log 21 + \max \left\{ 0, \log \left( 4C \right) \right\}.$$  (26)

We have

$$\log N_\infty (\delta, \mathcal{Y}) \leq C_1 \left( \frac{\delta}{5} \right)^{\frac{1}{\gamma + 1}} + C_2 \quad \text{if } \beta > \beta^* (\delta),$$  (27)

$$\log N_\infty (\delta, \mathcal{Y}) \leq C_1 \left( \frac{\delta}{5} \right)^{\frac{1}{\gamma}} + C_2 \quad \text{if } \beta = \beta^* (\delta).$$  (28)

Consequently, $\log N_\infty (\delta, \mathcal{Y})$ is bounded from above by the minimum of (25) and (27)-(28).

**Remark.** We can take $C_1 = 2 \log 21$ and $C_2 = 2 \max \left\{ 0, \log \left( 4C \right) \right\}$ in Theorem 2.1.

**Remark.** Note that the LHS of (26) is a strictly increasing function of $\gamma$ (since $\frac{\delta}{5} \in (0, 1)$) and the RHS is a strictly decreasing function of $\gamma$, and the LHS is no greater than the RHS for any $\frac{\delta}{5} \in (0, 1)$ when $\gamma = 0$ (to see this, note that $LHS = 3 \log \left( \left( \frac{\delta}{5} \right)^{-\frac{1}{\gamma}} \right)$ and $RHS \geq \left( \frac{\delta}{5} \right)^{-\frac{1}{\gamma}} \log 21 \geq 3 \left( \frac{\delta}{5} \right)^{-\frac{1}{\gamma}}$, where “log” is the natural logarithm.). Therefore, the largest non-negative solution $\beta^* (\delta) = \gamma (\leq \beta)$ to (26) exists (i.e., $\beta^* (\delta)$ is well defined).

Essentially, bound (25) implies that $\mathcal{Y}$ is no “larger” than $S_{\beta+1,2} (1, \Xi)$, the class where $f$ in (24) ranges over. There are situations where (27)-(28) is more useful for deriving bounds on estimation errors than (25), as we will see in the subsequent subsection (2.2.2). In proving (27)-(28), we first establish an intermediate lemma (Lemma A.1(ii) in Section A.6.2 of the supplementary materials) which shows that $\mathcal{Y} \subseteq S_{\beta+2}^\dagger$.

One can clearly see that $S_{\beta+2}^\dagger \subseteq S_{\beta+2}^* (\rho_\beta)$ where $\rho_\beta := 2^{\beta+1} (\beta + 1)!$. For this bigger class $S_{\beta+2}^* (\rho_\beta)$, the minimax lower bounds in Tsybakov (2009) would imply that

$$\lim_{n \to \infty} \inf_{g_n} \inf_{x_0 \in [0,1]} \sup_{\rho_\beta} \mathbb{E} \left\{ \frac{\beta}{2} \rho_\beta^2 n^{2(\beta+2)/2+1} e^{2(\beta+1)\rho_\beta^2} \right\} \leq c$$  (29)

where $\rho_\beta^2 n^{2(\beta+2)/2+1} e^{2(\beta+1)\rho_\beta^2} \geq \left( \frac{2^{(\beta+1)}}{e} \right)^{2(\beta+1)/2+1}$, by the elementary inequality $(\beta + 1)! \geq \frac{(\beta + 1)^{\beta + 1}}{e^{\beta + 1}} \geq \frac{(\beta + 1)^{\beta + 1}}{e^{\beta + 1}}$. If $\beta$ tends to infinity at a faster rate than $n$, then the lower bound above suggests that there exist no consistent estimators in the worst case. However, this conclusion cannot hold true for estimations of functions in $S_{\beta+2}^\dagger$ (which is a proper subset of $S_{\beta+2}^* (\rho_\beta)$). To see this, note that an easy calculation shows that $S_{\beta+2}^\dagger \subseteq S_{2,1} (\mathcal{C} \vee 2, [0,1])$ and the optimal rate concerning estimations of functions in $S_{2,1} (\mathcal{C} \vee 2, [0,1])$ scales as $n^{-\frac{1}{5}}$, for which consistent estimators apparently exist in the worst case.

Motivated by this observation, we introduce a new notion called the $\delta$–dependent “critical smoothness parameter”, $\beta^* (\delta) = \gamma (\leq \beta)$, which is the largest non-negative integer solution to (26). The naming “critical smoothness parameter” echoes the so called “critical radius” in the existing literature, although these two definitions are fundamentally different. A “critical radius”
typically determines the optimal estimation rate; see, e.g., Wainwright (2019) and Zhu (2017) for a number of applications involving critical radiiuse. Finding a “critical radius” often boils down to finding a sharp enough \( \delta \) resolution in the covering numbers. This sharp \( \delta \) resolution generally becomes smaller as the sample size \( n \) gets larger in statistical estimation problems. Meanwhile, as \( \delta \) becomes smaller, \( \gamma \) in (20) may be expected to increase such that the inequality still holds, and hence the “critical smoothness parameter” increases. Intuitions on why (27)-(28) can lead to sharper bounds on estimation errors than (25) can be gained from a simple example: If \( \delta \) is small enough (possibly as a result of large enough \( n \)) so that \( \beta^* (\delta) = \beta - 1 \) (or even \( \beta^* (\delta) = \beta \)), then (27)-(28) yields scaling \( \delta^{-\frac{1}{\beta+1}} \) (or even \( \delta^{-\frac{1}{\beta+2}} \)), while (25) yields \( (\delta)^{-\frac{1}{\beta+1}} \).

In the next two subsections (2.2.2 and 2.2.3), we establish estimation theory with the assistance of the bounds in Theorem 2.1. Our results concern the estimation of the following ODE

\[
y' (x) = f^* (x, y^* (x)), \quad y^* (0) = y_0^*
\]

with \( |y_0^*| \leq C_0 \) and \( (x, y^* (x)) \in \Xi \).

### 2.2.2 A theory for nonparametric least squares

In this subsection, we establish upper bounds for the the constrained least squares estimator

\[
\hat{y} \in \arg \min_{\hat{y} \in \mathcal{Y}} \frac{1}{2n} \sum_{i=1}^{n} (Y_i - \hat{y} (x_i))^2 .
\]

As we have mentioned after Theorem 2.1, Lemma A.1(ii) in Section A.6.2 shows that \( \mathcal{Y} \subseteq \mathcal{S}_{\beta+2}^1 \). Our next result (Theorem 2.2) bounds \( \text{E} \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y} (x_i) - y^* (x_i))^2 \right] \) using the idea of critical smoothness parameter we introduced in Theorem 2.1.

To motivate Theorem 2.2, we look at a classical result in the literature applied to our context. Let \( | \cdot |_F := \sqrt{\int_0^1 [h^{(\beta+2)} (t)]^2 \, dt} \). Given \( \mathcal{S}_{\beta+2}^1 \subseteq \mathcal{S}_{\beta+2}^\dagger (\rho_{\beta}) \) (recalling \( \rho_{\beta} := 2^{\beta+1} (\beta + 1)! \)), we may consider the alternative estimator

\[
\tilde{y} \in \arg \min_{\tilde{y} \in \mathcal{F}_{\beta+2}} \frac{1}{2n} \sum_{i=1}^{n} (Y_i - \tilde{y} (x_i))^2 + \lambda_n |\tilde{y}|_F^2
\]

where \( \mathcal{F}_{\beta+2} \) is the \( (\beta + 2) \)th order Sobolev class. If we solve (32) with \( \lambda_n \propto \left[ \frac{\sigma^2}{n \rho_{\beta}} \right]^{2(\beta+2)+1} \), Theorem 13.17 and Corollary 13.18 in Wainwright (2019) imply that

\[
\text{E} \left[ \frac{1}{n} \sum_{i=1}^{n} (\tilde{y} (x_i) - y^* (x_i))^2 \right] \lesssim \rho_{\beta}^{2(\beta+2)+1} \left[ \frac{\sigma^2}{n} \right]^{\frac{2(\beta+2)}{2(\beta+2)+1}} + c' \exp \left\{ -c'' n \sigma^2 \rho_{\beta}^{2(\beta+2)+1} \left[ \frac{\sigma^2}{n} \right]^{\frac{2(\beta+2)}{2(\beta+2)+1}} \right\}
\]

which essentially matches the lower bound (29). Note that our earlier calculation in Section 2.2.1 suggested that \( \rho_{\beta}^{2(\beta+2)+1} \left[ \frac{\sigma^2}{n} \right]^{\frac{2(\beta+2)}{2(\beta+2)+1}} \) is clearly useful. However, if \( \beta \) is large enough, the bound \( \rho_{\beta}^{2(\beta+2)+1} \left[ \frac{\sigma^2}{n} \right]^{\frac{2(\beta+2)}{2(\beta+2)+1}} \) can tend to infinity even as \( n \to \infty \). Classical results like the above do not deliver a useful bound in this case because they concern a bigger class than \( \mathcal{S}_{\beta+2}^1 \) and fail to take into account the special structure of \( \mathcal{S}_{\beta+2}^1 \). The prediction error of (31) can have a much better rate than \( \infty \), as we show
**Assumption G.** For every \( \gamma \in \{0, \ldots, \beta\} \) and a positive universal constant \( c \), let \( \delta (\gamma) = c \left( \frac{\sigma^2}{n} \right)^{\frac{\gamma+2}{2(\gamma+2)+1}} \) and \( \gamma^* (\leq \beta) \) be the largest non-negative integer such that

\[
\log \left( \prod_{i=0}^{\gamma^*} i! \right) + \frac{\gamma^*+2+3\gamma^*}{2} \log 2 - \frac{\gamma^*+3}{2} \log \left( \frac{\delta (\gamma^*)}{7} \right) \leq \left( \frac{\delta (\gamma^*)}{7} \right)^{\frac{\gamma^*+2}{2(\gamma^*+2)+1}} \log 21 + \max \left\{ 0, \log \left( 36C \right) \right\}.
\]

(34)

The sample size \( n \) is large enough such that

\[
\left( \frac{\delta (\gamma^*)}{7} \right)^{\frac{\gamma^*+2}{2(\gamma^*+2)+1}} \log 21 \geq -\frac{\gamma^*+3}{2} \log \left( \frac{\delta (\gamma^*)}{7} \right) \quad \forall \delta \leq \delta (\gamma^*)
\]

(35)

and

\[
\left( \frac{\delta (\gamma^*)}{C_{\max}} \right)^{\frac{\gamma^*+2}{2(\gamma^*+2)+1}} \geq \max \left\{ \log \left( \frac{C_0 C_{\max}}{\delta} + 1 \right), (\beta + 1) \log \left( \frac{c' C_{\max}}{\delta} \right) \right\} \quad \forall \delta \leq c'' \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{\gamma^*+2}{2(\gamma^*+2)+1}}
\]

(36)

for some positive universal constants \( c' \) and \( c'' \).

**Remark.** Assumption G is very common in the literature. For \( S_{\gamma+1, d} (1, [0, 1]^d) \), the standard smooth class of degree \( \gamma + 1 \),

\[
\log N_\infty \left( \delta, S_{\gamma+1, d} \left( 1, [0, 1]^d \right) \right) \leq \delta^{-\frac{d}{\gamma+1}} + (\gamma + 1) \log \left( \frac{c_0}{\delta} \right).
\]

Almost all results (as far as we are aware) in the existing literature implicitly assume that \( \delta^{-\frac{d}{\gamma+1}} \) dominates \( (\gamma + 1) \log \left( \frac{c_0}{\delta} \right) \), and hence one typically sees the following expression

\[
\log N_\infty \left( \delta, S_{\gamma+1, d} \left( 1, [0, 1]^d \right) \right) \leq \delta^{-\frac{d}{\gamma+1}}.
\]

The upper bound for \( \log N_\infty \left( \delta, S_{\gamma+1, d} \left( 1, [0, 1]^d \right) \right) \) differs from the upper bound for \( \log N_\infty \left( \delta, S_{\beta+2} \right) \) mainly in \( \log \left( \prod_{i=0}^{\beta} i! \right) + \frac{\beta^2+3\beta}{2} \log 2 \), which comes from the factorial growth in the derivatives of functions in \( S_{\beta+2} \) and gives arise to the “curse of smoothness” evidenced by \( \rho_{\beta}^{2(\gamma+2)+1/2} \left( \frac{\sigma^2}{n} \right)^{2(\beta+2)+1} \) in the classical bound (33).

**Theorem 2.2.** Let Assumption F hold for \( y^* \) in (30), \( |y_0| \leq C_0 \) and \( (x, y^*(x)) \in \Xi \), and Assumption G hold. Suppose all the \( n \) design points are sampled from \([0, \alpha]\). Let us consider (2) where \( y^*(\cdot) \) is a solution to (30) on \([0, \alpha]\). In terms of (31), if \( \frac{\sigma^2}{n} \leq 1 \), \( \left( \frac{\delta(\gamma)}{7} \right)^{\frac{\gamma+2}{2(\gamma+2)+1}} \geq \max \left\{ 0, \log \left( 36C \right) \right\} \), and

\[
\max \{ \alpha, C_0 \} \geq c_0 \min \left\{ \left( \frac{\sigma^2}{n} \right)^{\frac{\gamma^*+2}{2(\gamma^*+2)+1}}, \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{\gamma^*+2}{2(\gamma^*+2)+1}} \right\},
\]

(37)
for a sufficiently large positive universal constant $c_0$, then we have
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \right] \lesssim \min \{ \mathcal{T}_1, \mathcal{T}_2 \}
\] (38)

where
\[
\mathcal{T}_1 = \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\gamma^*+2)+1}} + \exp \left\{ -c_1 n \sigma^{-2} [\delta (\gamma^*)]^2 \right\},
\]
\[
\mathcal{T}_2 = \left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\gamma+2}} + \exp \left\{ -c_1 n \sigma^{-2} \left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\gamma+2}} \right\}.
\]

**Remark.** Unlike Propositions 2.2 and 2.3, here in Theorem 2.2 and Proposition 2.4 below, we allow the $n$ design points to be sampled from the closed interval $[0, \alpha]$ in (31). By the Picard local existence theorem (Theorem II), there exists a solution to (30) on $[0, \alpha]$.

**Remark.** In theory, an inspection of our proof for Theorem 2.2 (Section A.7 of the supplementary materials) suggests that $\mathcal{T}_1$ may be tightened even further by finding $\bar{r}_n \lesssim \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\gamma^*+2)+1}}$ that solves the following inequality
\[
\frac{1}{\sqrt{n}} \int_{\delta_n}^{\bar{r}_n} \sqrt{\log N_{\infty} (\delta, \mathcal{F})} \, d\delta \lesssim \frac{1}{\sqrt{n}} \int_{\delta_n}^{\bar{r}_n} \left( \frac{\delta}{\gamma+2} \right) \frac{1}{n \sigma^2} \, d\delta \lesssim \frac{\sigma^2}{n},
\]
where $\beta^* (\delta)$ is defined in Theorem 2.1 and $\mathcal{F} := \{ g = g_1 - g_2 : g_1, g_2 \in \mathcal{V} \}$. However, computing the second integral in the above analytically is difficult and therefore we use the bound $\bar{r}_n \lesssim \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\gamma^*+2)+1}}$.

For the prediction error, Theorem 2.2 suggests that the noisy recovery of (30) is no more difficult than estimating a function in $\mathcal{S}_{\beta+1, 2} \left( 1, \alpha \right)^2$, whose optimal rate is $\left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\gamma+2}}$. The rate $\left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\gamma^*+2)+1}}$ can be sharper than the rate $\left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\gamma+2}}$. Obviously, if $\frac{\sigma^2}{n} \leq 1$, $\left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\gamma+2}}$ is only useful when $\beta > 3$.

Note that (34) always holds for $\gamma^* = 0$ and any $\frac{1}{\gamma}$-resolution within $(0, 1)$, so we easily have $\left( \frac{\sigma^2}{n} \right)^{\frac{1}{\gamma}} \leq \left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\gamma+2}}$ as long as $\frac{\sigma^2}{n} \leq 1$ and $\beta \leq 3$. As another example, it is easy to show that when $\frac{\sigma^2}{n} \lesssim (\beta \sqrt{\log \beta})^{4(\beta+2)+2}$, (34) holds for $\gamma^* = \beta$. In this case, (38) becomes
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \right] \lesssim \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\gamma^*+2)+1}} + \exp \left\{ -cn \sigma^{-2} [\delta (\beta)]^2 \right\}.
\]

On the other hand, when $\beta$ is large enough relative to $\frac{n}{\sigma^2}$, it is possible that the rate $\left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\gamma+2}}$ would be dominated by $\left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\gamma^*+2)+1}} = \left( \frac{\sigma^2}{n} \right)^{\frac{1}{\gamma}}$.

\textsuperscript{7}In fact, $\left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\gamma+2}}$ is derived only for $\beta > 0$ in the proof for technical reasons. On the other hand, even for the case $\beta = 0$, this expression does not affect the result as $\left( \frac{\sigma^2}{n} \right)^{\frac{1}{\gamma}}$ would be dominated by $\left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\gamma^*+2)+1}} = \left( \frac{\sigma^2}{n} \right)^{\frac{1}{\gamma}}$. 

14
is smaller than the rate \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\beta+2)+1}} \). Obviously, as we mentioned before, if \( \beta \) is much smaller than \( \frac{n}{\sigma^2} \), the classical bound \( \rho_0^{\frac{2(\beta+2)}{2(\beta+2)+1}} \left[ \frac{\sigma^2}{n} \right]^{\frac{2(\gamma^*+2)}{2(\beta+2)+1}} \) in (33) can be also useful.

So far, our results have focused on the ODE (24) where \( f \) at least satisfies a Picard-type Lipschitz condition

\[
|f(x, y) - f(x, \tilde{y})| \leq |y - \tilde{y}| \quad \forall \,(x, y), (x, \tilde{y}) \in \Xi. \tag{39}
\]

It is possible to bound \( \frac{1}{n} \sum_{i=1}^n (\hat{y}(x_i) - y^*(x_i))^2 \) without requiring (39). In the following, we provide an oracle inequality that achieves this purpose.

**Proposition 2.4.** Suppose all the \( n \) design points are sampled from \([0, \alpha]\). Let us consider (2) where \( y^*(\cdot) \) is a solution to

\[
y^*(x) = g(x, y^*(x)), \quad y^*(0) = y_0^* \tag{40}
\]

on \([0, \alpha]\). In (40), \( |y_0^*| \leq C_0 \) and \((x, y^*(x)) \in \Xi\). Suppose \( g \) is continuous on \( \Xi \). Let \( \mathcal{Y} \) denote the class of the solutions (on \([0, \alpha]\)) to (24) where \((x, y(x)) \in \Xi, |y_0| \leq C_0, \) and \( f \in \mathcal{S}_{1,2}(1, \Xi) \) satisfying Assumption F with \( \beta = 0 \). Let (35) in Assumption G hold with \( \gamma^* = 0 \). Assume that there is a continuous function \( \varphi : [0, 1] \mapsto [0, \infty) \) such that

\[
|f(x, y(x)) - g(x, y(x))| \leq \varphi(x). \tag{41}
\]

In terms of (34), if \( \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{1}{2}} \gtrsim \max \left\{ 0, \log \left( \frac{36C}{\sigma^2} \right) \right\} \) and

\[
\max \{\alpha, C_0\} \geq c_0 \left( \frac{\sigma^2}{n} \right)^{\frac{1}{4}}, \tag{42}
\]

then we have

\[
\frac{1}{n} \sum_{i=1}^n (\hat{y}(x_i) - y^*(x_i))^2 \leq \inf_{\kappa \in (0,1)} \left\{ \frac{1+\kappa}{1-\kappa} |y - y^*|^2 + \frac{c}{\kappa(1-\kappa)} \left( \frac{\sigma^2}{n} \right)^{\frac{3}{4}} \right\}, \quad \forall y \in \mathcal{Y}. \tag{43}
\]

with probability at least \( 1 - c_1 \exp \left\{ \frac{-c_2 \sigma^2}{\sigma^2} \left( \frac{\sigma^2}{n} \right)^{\frac{1}{4}} \right\} \), where

\[
|y - y^*|^2 \leq 2 \frac{1}{n} \sum_{i=1}^n [E_1 (x_i) + E_2 (x_i)],
\]

\[
E_1 (x_i) := (y_0 - y_0^*)^2 \exp (2x_i),
\]

\[
E_2 (x_i) := \left[ \int_{x_i}^{x} \exp (-s) \varphi (s) ds \right]^2 \exp (2x_i).
\]

### 2.2.3 Least squares based on Picard iterations

If an estimator \( \hat{y}_0 \) of \( y_0^* \) is available, then we can exploit the Picard iteration in an analogous way as in (21):

\[
y_{r+1} (x; f) = y_0 + \int_0^x f \left( s, y_r (s; f) \right) ds, \quad \text{integer } r \geq 0, y(0) = y_0. \tag{44}
\]
Suppose all the $n$ design points are sampled from $[0, \tilde{\alpha}]$ where $\tilde{\alpha} < \alpha$. An estimator based on (44) performs the following steps: first, we compute
\begin{align*}
g_1(x; f) &= \hat{y}_0 + \int_0^x f(s, \hat{y}_0) \, ds, \\
g_2(x; f) &= \hat{y}_0 + \int_0^x f(s, \hat{y}_1(s; f)) \, ds,
\end{align*}
and
\begin{align*}
g_{R+1}(x; f) &= \hat{y}_0 + \int_0^x f(s, \hat{y}_{R}(s; f)) \, ds; \\
\end{align*}
second, we solve the following program
\begin{equation}
\hat{f} \in \arg \min_{f \in S_{\beta+1,2}(1, \Xi)} \frac{1}{2n} \sum_{i=1}^n (Y_i - \hat{y}_{R+1}(x_i; f))^2. \tag{46}
\end{equation}

**Proposition 2.5.** Let Assumption F hold for $f^*$ in (30), and $|y_0^*| \leq C_0$ and $(x, y^*(x)) \in \Xi$. Suppose all the $n$ design points are sampled from $[0, \tilde{\alpha}]$ where $\tilde{\alpha} < \alpha$. Let us consider (2) where $y^*(\cdot)$ is the (unique) solution to (30) on $[0, \tilde{\alpha}]$. In terms of $\hat{y}_{R+1}(x_i; \hat{f})$ where $\hat{f}$ is obtained from solving (46), if $\beta > 0$ and
\begin{equation}
\left( \frac{\delta}{b} \right)^{\frac{\alpha}{\beta+1}} \geq (\beta + 1) \log \left( \frac{c_0 b}{\sigma} \right), \quad \forall \delta \leq c_1 \frac{1}{b^{\frac{\alpha}{\beta+2}}} \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{\alpha+1}{\beta+2}},
\end{equation}
then we have
\begin{equation}
\left\{ \frac{1}{n} \sum_{i=1}^n \left[ \hat{y}_{R+1}(x_i; \hat{f}) - y^*(x_i; f^*) \right]^2 \right\}^{\frac{1}{2}} \overset{\mathbb{P}}{\lesssim} b^{\frac{1}{\beta+2}} \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{\alpha+1}{\beta+2}} + \frac{\sigma}{1 - \tilde{\alpha}} |y_0 - y_0^*| + \frac{\sigma \tilde{\alpha} R^2}{1 - \tilde{\alpha}} \max \{ C_0, \tilde{\alpha} \}
\end{equation}
with probability at least $1 - c_1 \exp(-c_2 n) - c_3 \exp \left( -c_4 n \sigma^{-2} b^{\frac{\alpha}{\beta+2}} \left( \frac{\sigma^2}{n} \right)^{\frac{\alpha+1}{\beta+2}} \right)$, where $b = \frac{\tilde{\alpha}}{1 - \alpha}$.

**Remark.** For technical reasons, Proposition 2.5 excludes the trivial case $\beta = 0$. For this case, the results in Section 2.2.2 suggest that a direct estimator
\begin{equation}
\hat{y} \in \arg \min_{\hat{y} \in S_{2,1}(C \sqrt{2}, [0, 1])} \frac{1}{2n} \sum_{i=1}^n (Y_i - \hat{y}(x_i))^2
\end{equation}
immediately yields
\begin{equation}
\frac{1}{n} \sum_{i=1}^n (\hat{y}(x_i) - y^*(x_i))^2 \lesssim \left( \frac{\sigma^2}{n} \right)^{\frac{1}{2}}
\end{equation}
with probability at least $1 - c' \exp \left\{ -c'' n \sigma^{-2} \left( \frac{\sigma^2}{n} \right)^{\frac{1}{2}} \right\}$.

If $|y_0 - y_0^*|$ is small relative to $b^{\frac{1}{\beta+2}} \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{\alpha+1}{\beta+2}}$, then Proposition 2.5 suggests recovering (30) in (2) based on Picard iterations is essentially no more difficult than estimating a function in $S_{\beta+1,2}(1, [0, 1]^2)$, whose optimal rate is $\left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{\alpha+1}{\beta+2}}$. \hfill 16
Estimation of the initial value

Without knowing $y_0^*$, the fact that $Y \subseteq S_{\beta+2}^*$ allows us to apply existing estimators such as the local polynomials to estimate $y_0^*$:

$$\hat{y}_{0, \omega}^* = U_\omega^T (0) \hat{\theta}_\omega (0)$$

where

$$\hat{\theta}_\omega (0) = \arg \min_{\theta \in \mathbb{R}^{k+2}} \sum_{i=1}^{n} \left[ Y_i - \theta^T U_\omega \left( \frac{x_i}{h} \right) \right]^2 K \left( \frac{x_i}{h} \right)$$

$$U_\omega (t) = \left( 1, t, \frac{t^2}{2}, \ldots, \frac{t^{\omega+1}}{(\omega + 1)!} \right)$$

$K (\cdot)$ is a kernel, $h$ is a bandwidth parameter, and $\omega$ is a non-negative integer.

Since $S_{\beta+2}^* \subseteq S_{\beta+2}^* (\rho_\beta)$ (recalling $\rho_\beta = 2^{\beta+1} (\beta + 1)!$), it is natural to consider the estimator $\hat{y}_{0, \beta}^*$ constructed as above with $\omega = \beta$. Proposition 1.13 and Theorem 1.6 in Tsybakov (2009) imply that

$$\limsup_{n \to \infty} \sup_{g \in S_{\beta+2}^* (\rho_\beta)} \sup_{x_0 \in [0, 1]} \mathbb{E} \left\{ \lambda_{\min, \beta}^2 A^{-(\beta+1) n^2 (\beta+2)+1} \left[ \tilde{g}_n (x_0) - g (x_0) \right]^2 \right\} \leq c'$$

(48)

where $\lambda_{\min, \beta}$ is the smallest eigenvalue of

$$B_{x_0, \beta} := \frac{1}{nh} \sum_{i=1}^{n} U_\beta \left( \frac{x_i - x_0}{h} \right) U_\beta^T \left( \frac{x_i - x_0}{h} \right) K \left( \frac{x_i - x_0}{h} \right).$$

Suppose the conditions in Lemma 1.5 of Tsybakov (2009) hold; that is, $x_i = \frac{i}{n}$ for $i = 1, \ldots, n$; for some constants $C_\min > 0$ and $\Delta > 0$, $K (t) \geq C_\min \|t\|_2 \Delta$ for all $t \in \mathbb{R}$. Then one has

$$\inf_{v \in \mathbb{R}^2} \|v\|_2 \geq \frac{C_\min}{2} \min \left\{ \inf_{|v|_2 = 1} \int_{|v|_2 = 1}^{\Delta} (v^T U_\beta (t))^2 \, dt, \inf_{|v|_2 = 1} \int_{-\Delta}^{0} (v^T U_\beta (t))^2 \, dt \right\}$$

for large enough $n$. Note that if $\beta$ tends to $\infty$ at a much faster rate than $n$, $\lambda_{\min, \beta}$ can become quite small and the resulting upper bound for $\mathbb{E} \left\{ \tilde{g}_n (x_0) - g (x_0) \right\}^2$ based on (48) may become rather large. On the other hand, since $S_{\beta+2}^* \subseteq S_{2,1} (\overline{C} \setminus 2, [0, 1])$, $\mathbb{E} \left\{ \tilde{g}_n (x_0) - g (x_0) \right\}^2$ can always be bounded from above by roughly $\left( \frac{1}{n} \right)^{\frac{4}{3}}$. In theory, the rate $\left( \frac{1}{n} \right)^{\frac{4}{3}}$ may be improved by finding a smoothness parameter $\omega^*$ such that

$$\lambda_{\min, \omega^*}^{-2} A^{\omega^*+1} n^{\frac{-2(\omega^*+2)}{\omega^*+2}+1} = \min_{\omega \leq \beta} \left\{ \lambda_{\min, \omega}^{-2} A^{\omega+1} n^{\frac{-2(\omega+2)}{\omega+2}+1} \right\}$$

where $\lambda_{\min, \omega}$ is the smallest eigenvalue of $B_{x_0, \omega}$.

2.3 Nonparametric autonomous smooth first order ODEs

In this subsection, we consider the ODE in the form

$$y' (x) = f (y (x)), \quad y (0) = y_0.$$

(49)

The right-hand-side of the above has a “single index” structure. In the ODE literature, this structure is referred to as an autonomous system. The main results concerning (49) are presented in a
similar fashion as those in Section 2.2. As we show in these results, the autonomous system \([49]\) and the nonautonomous system \([24]\) in noisy recovery problems sometimes exhibit the same behavior while other times, a smaller upper bound can be obtained for the estimation error associated with \([49]\) than with \([24]\).

At the end of this subsection, we present a special case of \([49]\), where \(f\) need not be Lipschitz continuous, yet the associated solution class is essentially no “larger” than \(S_2\).

### 2.3.1 Covering numbers

**Assumption AF.** In \([49]\), \(|y_0| \leq C_0\) and \((x, y(x)) \in \Xi\); \(f\) ranges over \(S_{\beta+1,1}(1, \left[-\overline{C}, \overline{C}\right] )\) (where \(\overline{C} \geq 1\)); that is, \(f\) is continuous on \([-\overline{C}, \overline{C}]\), and differentiable \(\beta\) times; \(f^{(k)}(y) := \left| \frac{\partial^k f(y)}{\partial y^k} \right| \leq 1\) for all \(k = 0, \ldots, \beta\) and \(y \in [-\overline{C}, \overline{C}]\), where \(f^{(0)}(y) = f(y)\); and

\[
\left| f^{(\beta)}(y) - f^{(\beta)}(\tilde{y}) \right| \leq |y - \tilde{y}|
\]

for all \(y, \tilde{y} \in [-\overline{C}, \overline{C}]\).

**Theorem 2.3.** Let us consider the ODE \([49]\). Suppose Assumption AF holds.

(i) We have

\[
\log N_\infty (\delta, \mathcal{Y}) \leq c \left[ \left( \frac{\delta}{C_{\max}} \right)^{\frac{1}{\beta+1}} + (\beta + 1) \log \left( \frac{c' C_{\max}}{\delta} \right) + \log \left( \frac{C_0 C_{\max}}{\delta} + 1 \right) \right]
\]  

(50)

where \(\mathcal{Y}\) is the class of solutions (to \([24]\) with \(f \in S_{\beta+1,1}(1, \left[-\overline{C}, \overline{C}\right] )\) on \([0, \alpha]\).

(ii) For a given \(\frac{\delta}{5} \in (0, 1)\), let \(\beta^*(\delta) = \gamma(\leq \beta)\) be the largest non-negative integer such that

\[
\log \left( \prod_{i=0}^{\gamma} \delta ! \right) - \frac{\gamma + 3}{2} \log \frac{\delta}{5} \leq \left( \frac{\delta}{5} \right)^{\frac{1}{\gamma+2}} \log 21 + \max \{0, \log \left( 4\overline{C} \right) \}.
\]  

(51)

We have

\[
\begin{align*}
\log N_\infty (\delta, \mathcal{Y}) &\leq C_1 \delta^{\frac{1}{\beta^*(\delta)+2}} + C_2 & \text{if } \beta > \beta^*(\delta), \\
\log N_\infty (\delta, \mathcal{Y}) &\leq C_1 \delta^{\frac{1}{\beta+2}} + C_2 & \text{if } \beta = \beta^*(\delta).
\end{align*}
\]  

(52) \hspace{1cm} (53)

Consequently, \(\log N_\infty (\delta, \mathcal{Y})\) is bounded from above by the minimum of \((50)\) and \((52)-(53)\).

**Remark.** We can take \(C_1 = 2 \log 21\) and \(C_2 = 2 \max \{0, \log \left( 4\overline{C} \right) \}\) in Theorem 2.3.

**Remark.** In proving \((52)-(53)\) of Theorem 2.3, we first show that \(\mathcal{Y} \subseteq AS_{\beta+2}^I\) (see Lemma A.1(i) in Section A.6.2), where any function \(h \in AS_{\beta+2}^I\) satisfies the following properties:

- \(h\) is continuous on \([0, 1]\) and differentiable \(\beta + 1\) times;
- \(|h(x)| \leq \overline{C}\), and \(|h^{(k)}(x)| \leq (k - 1)!\) for all \(k = 1, \ldots, \beta + 1\) and \(x \in [0, 1]\);
- \(|h^{(\beta+1)}(x) - h^{(\beta+1)}(x')| \leq (\beta + 1)! |x - x'|\) for all \(x, x' \in [0, 1]\).
In the next two subsections (2.3.2 and 2.3.3), we establish estimation theory with the assistance of the bounds in Theorem 2.3. Our results concern the estimation of the following ODE
\[ y''(x) = f^*(y^*(x)), \quad y^*(0) = y_0^* \] (54)
with \( |y_0^*| \leq C_0 \) and \((x, y^*(x)) \in \Xi\).

### 2.3.2 A theory for nonparametric least squares

In this subsection, we establish upper bounds on the the constrained least squares estimator
\[ \hat{y} \in \arg \min_{\hat{y} \in \mathcal{Y}} \frac{1}{2n} \sum_{i=1}^{n} (Y_i - \hat{y}(x_i))^2. \] (55)

**Assumption AG.** For every \( \gamma \in \{0, ..., \beta\} \) and a positive universal constant \( c \), let \( \delta(\gamma) = c \left( \frac{\sigma^2}{n} \right)^{\frac{1}{\gamma+2}} \) and \( \gamma^* (\leq \beta) \) be the largest non-negative integer such that

\[ \log \left( \prod_{i=0}^{\gamma^*} i! \right) + \gamma^* \log 2 - \frac{\gamma^* + 3}{2} \log \frac{\delta(\gamma^*)}{2} \leq \left[ \frac{\delta(\gamma^*)}{7} \right] \frac{\gamma^* + 1}{\gamma + 2} \log 21 + \max \{0, \log \left( 36C_n \right) \}. \] (56)

The sample size \( n \) is large enough such that

\[ \left( \frac{\delta(\gamma^*)}{7} \right)^{\frac{1}{\gamma + 2}} \log 21 \geq -\frac{\gamma^* + 3}{2} \log \frac{\delta(\gamma^*)}{7} \quad \forall \delta \leq \delta(\gamma^*) \] (57)

and

\[ \left( \frac{\delta}{C_{\max}} \right)^{\frac{1}{\gamma + 1}} \simeq \max \left\{ \log \left( \frac{C_0 C_{\max}}{\delta} + 1 \right), (\beta + 1) \log \left( \frac{c' C_{\max}}{\delta} \right) \right\} \quad \forall \delta \leq c'' \left( \frac{\sigma^2}{n} \right)^{\frac{1}{\gamma + 1}} \] (58)

for some positive universal constants \( c' \) and \( c'' \).

**Theorem 2.4.** Let Assumption AF hold for \( f^* \) in (54), \( |y_0^*| \leq C_0 \) and \((x, y^*(x)) \in \Xi\), and Assumption AG hold. Suppose all the \( n \) design points are sampled from \([0, \alpha]\). Let us consider (2) where \( y^*(\cdot) \) is a solution to (54) on \([0, \alpha]\). In terms of (55), if \( \frac{\sigma^2}{n} \leq 1 \), \( \left[ \frac{\delta(\gamma^*)}{7} \right]^{\frac{1}{\gamma + 2}} \simeq \max \{0, \log \left( 36C_n \right) \} \) and

\[ \max \{ \alpha, C_0 \} \geq c_0 \min \left\{ \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma + 2)}{2(\gamma + 2) + 1}}, \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 2)}{2(\beta + 2) + 1}} \right\}, \] (59)

for a sufficiently large positive universal constant \( c_0 \), then we have

\[ \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \right] \lesssim \min \{ \mathcal{T}_1, \mathcal{T}_2 \} \] (60)

where

\[ \mathcal{T}_1 = \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma + 2)}{2(\gamma + 2) + 1}} + \exp \left\{ -cn\sigma^{-2} [\delta(\gamma^*)]^2 \right\}, \]

\[ \mathcal{T}_2 = \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 2)}{2(\beta + 2) + 1}} + \exp \left\{ -cn\sigma^{-2} \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 2)}{2(\beta + 2) + 1}} \right\}. \]
Given the “single index” structure of (49), bound (50) in Theorem 2.3 implies that \( \gamma \) is basically no “larger” than \( S_{\beta+1,1} (1, [-\overline{C}, \overline{C}]) \), the class where \( f \) in (49) ranges over. Like how we motivated (27)–(28) in Theorem 2.1, there are situations where (52)–(53) can be more useful for deriving bounds on the estimation errors than (50). For example, when \( n \sigma^2 \gtrsim (\beta \sqrt{\log \beta})^{4(\beta+2)+}\) such that (50) holds for \( \gamma^* = \beta \), (60) becomes

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \right] \lesssim \left( \frac{\sigma^2}{n} \right) \frac{2^{(\beta+2)}}{n^{\beta+1}} + \exp \left\{ -cn\sigma^{-2} \delta(\beta)^2 \right\}.
\]

### Comparing the autonomous system with the nonautonomous system

Recalling the definition of \( \mathcal{A} \mathcal{S}^T_{\beta+2} \), the bound \((k-1)!\) on the absolute value of the \( k \)th derivative of \( y(x) \) (for all \( k = 1, \ldots, \beta+1 \)) and the bound \((\beta+1)!\) on \( |y^{(\beta+1)}(x) - y^{(\beta+1)}(x')| \) tight in the autonomous system (49). Despite these tight bounds and the “single index” structure in the autonomous system, the left-hand-sides of (51) and (56) have the same scaling as those of (26) and (31), as the terms \( \frac{2^{\gamma^*+\gamma}}{\gamma^*+\gamma} \log 2 \) in (26) and \( \frac{\gamma^*+3\gamma}{2} \log 2 \) in (31) are dominated by \( c \cdot \log (\prod_{i=0}^{\gamma^*+1} i!) \) and \( c' \cdot \log (\prod_{i=0}^{\gamma^*+1} i!) \), respectively (for some positive universal constants \( c \) and \( c' \)). As a consequence, \( T_l \) in bounds (38) and (60) can coincide if \( \frac{n}{\sigma^2} \) is large enough relative to \( \beta \).

On the other hand, in situations where (51) and the rate \( \left( \frac{\sigma^2}{n} \right) \frac{2^{(\gamma^*+\gamma)+1}}{n^{\gamma^*+\gamma+1}} \) improve upon (52)–(53) and the rate \( \left( \frac{\sigma^2}{n} \right) \frac{2^{(\gamma^*+\gamma)+1}}{n^{\gamma^*+\gamma+1}} \) (as a result of small enough \( \frac{n}{\sigma^2} \) and large enough \( \beta \)), we can clearly see the difference in the upper bounds for the prediction errors between the autonomous system (49) and the nonautonomous system (24); that is, \( \frac{1}{\sigma^{\gamma^*+1}} \) in Theorem 2.3 versus \( \frac{\delta^2}{\sigma^{\gamma^*+1}} \) in Theorem 2.1, and \( \left( \frac{\sigma^2}{n} \right) \frac{2^{(\gamma^*+\gamma)+1}}{n^{\gamma^*+\gamma+1}} \) in Theorem 2.4 versus \( \left( \frac{\sigma^2}{n} \right) \frac{2^{(\gamma^*+\gamma)+1}}{n^{\gamma^*+\gamma+1}} \) in Theorem 2.2.

#### 2.3.3 Least squares based on Picard iterations

If an estimator \( \hat{y}_0 \) of \( y_0 \) is available, then we can exploit the Picard iteration in an analogous way as in (44):

\[
y_{r+1}(x; f) = y_0 + \int_0^x f (y_r(s; f)) \, ds, \quad \text{integer } r \geq 0, \quad y(0) = y_0.
\]

Suppose all the \( n \) design points are sampled from \([0, \alpha]\) where \( \alpha < \alpha \). An estimator based on (61) performs the following steps: first, we compute

\[
\hat{y}_1(x; f) = \hat{y}_0 + \int_0^x f (\hat{y}_0) \, ds,
\]

\[
\hat{y}_2(x; f) = \hat{y}_0 + \int_0^x f (\hat{y}_1(s; f)) \, ds,
\]

\[
\vdots
\]

\[
\hat{y}_{R+1}(x; f) = \hat{y}_0 + \int_0^x f (\hat{y}_R(s; f)) \, ds;
\]

\[
20
\]
second, we solve the following program

$$\hat{f} \in \arg \min_{f \in \mathcal{S}_{\beta + 1, 1} \left(1, \left[-\hat{\nu}, \hat{\nu}\right]\right)} \frac{1}{2n} \sum_{i=1}^{n} (Y_i - \hat{y}_{R+1}(x_i; f))^2. \tag{62}$$

**Proposition 2.6.** Let Assumption AF hold for \( f^* \) in (57), and \( |y_0^*| \leq C_0 \) and \((x, y^*(x)) \in \Xi\). Suppose all the \( n \) design points are sampled from \([0, \tilde{\alpha}]\) where \( \tilde{\alpha} < \alpha \). Let us consider (54) where \( y^*(\cdot) \) is the (unique) solution to (57) on \([0, \tilde{\alpha}]\). In terms of \( \hat{y}_{R+1}(x_i; \hat{f}) \) where \( \hat{f} \) is obtained from solving (62), if

$$\left( \frac{\delta}{b} \right)^{1 + \frac{1}{\beta + 1}} \gtrsim (\beta + 1) \log \left( \frac{c_1 b}{\delta} \right), \quad \forall \delta \leq c_1 b^{2(\beta + 1)} \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 1}},$$

$$\max \{\tilde{\alpha}, C_0\} \geq c_0 b^{2(\beta + 1)} \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 1}}, \tag{63}$$

then we have

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{y}_{R+1}(x_i; \hat{f}) - y^*(x_i; f^*) \right]^2 \right\}^{\frac{1}{2}} \lesssim b^{2(\beta + 1) + 1} \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 1}} + \frac{\sigma |\hat{y}_0 - y_0^*|}{1 - \tilde{\alpha}} + \frac{\sigma \tilde{\alpha} R + 1}{1 - \tilde{\alpha}} \max \{C_0, \tilde{\alpha}\}$$

with probability at least \( 1 - c_1 \exp(-c_2 n) - c_3 \exp(-c_4 n\sigma^{-2} b^{2(\beta + 1)} \left( \frac{\sigma^2}{n} \right)^{\frac{2(\beta + 1)}{2(\beta + 1) + 1}}) \), where \( b = \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \).

**Remark.** In contrast to Proposition 2.5, Proposition 2.6 is able to accommodate the case \( \beta = 0 \).

### 2.3.4 A special case allowing continuous (but not uniformly Lipschitz continuous) \( f \)

So far all our results for the nonparametric smooth ODEs have required \( f \) to be at least uniformly Lipschitz continuous on the relevant domain, in which case we have (at least)

$$\log N_\infty(\delta, \mathcal{Y}) \leq C_1 \delta^{-\frac{1}{2}} + C_2. \tag{64}$$

The autonomous system \( y'(x) = f(x) \) with \( x \in [0, 1] \) and the initial value \( y(0) = y_0^* \) (known as the separable ODE) indicates that uniform Lipschitz continuity of \( f \) is clearly not necessary for (54) to hold. Suppose \( f \) is continuous and non-decreasing (or non-increasing) on \([0, 1] \). Then the solution \( y(x) = y_0^* + \int_{0}^{1} f(x) \, dx \) is convex (respectively, concave) on \([0, 1] \) (see our Proposition B.1 in Section B of the supplementary materials for the proof). As a result, the existing results (e.g., Guntiboyina and Sen, 2013) imply that \( \log N_\infty(\delta, \mathcal{Y}) \lesssim \delta^{-\frac{1}{2}} \). Essentially, \( \mathcal{Y} \) is no “larger” than \( \mathcal{S}_2 \).

### 2.4 ODEs with polynomial potential functions

In this subsection, we study ODEs with polynomial potential functions. These ODEs have natural geometric interpretations, and the concept of VC dimensions turns out to be most useful for understanding the “size” of such ODEs. If a class \( \mathcal{F} \) has a finite VC dimension \( \nu \) and \( \sup_{f,g \in \mathcal{F}} |f - g|_n \leq 2\tilde{r} \), then existing results in the literature (e.g., van der Vaart and Wellner, 1996; Wainwright, 2019) imply that

$$N_n(\delta, \mathcal{F}) \lesssim \nu \left( 16 e \nu \right) \left( \frac{\tilde{r}}{\delta} \right)^{2\nu} \tag{65}$$

---

For example, \( f(x) = \sqrt{x} \) defined on \([0, 1]\) is continuous and non-decreasing, but not Lipschitz continuous.
where \( N_n (\delta, \mathcal{F}) \) is the \( \delta \)-covering number of \( \mathcal{F} \) with respect to the norm \( | \cdot |_p \).

We begin by briefly reviewing the exact ODEs and inexact ODEs that can be transformed into exact ODEs. The system
\[
F_1 (x, y) y' + F_2 (x, y) = 0
\]
with the functions \( F_1 \) and \( F_2 \) satisfying
\[
\partial_x F_1 (x, y) = \partial_y F_2 (x, y)
\]
is an exact ODE. For example, a separable ODE is exact. If \( \partial_x F_1 (x, y) \neq \partial_y F_2 (x, y) \) in (66), but
\[
\left( e^{F_0} F_1 \right) y' + \left( e^{F_0} F_2 \right) = 0
\]
where \( F_0 = \int f_0 (s) \, ds \) and \( f_0 = \frac{\partial_y F_2 - \partial_x F_1}{F_1} (F_1 \neq 0) \) does not depend on \( y \), then (67) is exact. Given (67), we can write
\[
\frac{d\phi}{dx} (x, y (x)) = 0
\]
where \( \phi \) has the name “potential function” satisfying \( \partial_y \phi = e^{F_0} F_1 \) and \( \partial_x \phi = e^{F_0} F_2 \). Consequently, the solutions of (66) have the following form:
\[
\phi (x, y (x)) = c^*
\]
where \( c^* \in \mathbb{R} \) depends on the initial value.

While it is difficult to derive a meaningful general theory for noisy recovery problems associated with the implicit form (68), we can develop results for special (but useful) cases. Often the potential function \( \phi \) is a polynomial. As a simple example, the (exact) ODE \( yy' + x = 0 \) has a potential function \( \phi = x^2 + y^2 \) and solutions in the form of 2-dimensional spheres \( x^2 + y^2 = c^* \).

In what follows, we restrict our attention to the class of ODEs that satisfy the following:

1. the solutions of the ODEs have the form (68), where the potential function \( \phi (x, y) \) associated with each member of this class is a polynomial (in 2 variables) of degree at most \( D \);

2. writing (67) as \( y' (x) = \frac{- e^{F_0 (x)} F_2 (x, y)}{e^{F_0 (x)} F_1 (x, y)} =: f (x, y) \), the associated initial value \( |y_0| \leq C_0 \) and \( f \) is continuous on \([0, 1] \times [-C_0 - b, C_0 + b]\) and satisfies
\[
|f (x, y) - f (x, \tilde{y})| \leq |y - \tilde{y}| \quad \forall (x, y), (x, \tilde{y}) \in [0, 1] \times [-C_0 - b, C_0 + b].
\]

Let us refer to this class of ODEs satisfying the two properties above as \( \mathcal{E} \). We would like to recover \( y^* \) in (2) where \( y^* \) is a solution to an ODE in \( \mathcal{E} \). In this setup, a solution often has an implicit form and therefore, the estimation strategy based on Picard iterations is the most natural method if an estimator \( \hat{y}_{R+1} \) of the initial value \( y_0^* \) is available. We can adapt the idea in Section 2.2.3 to obtain an estimator \( \hat{y}_{R+1} \) for \( y^* \). If we can bound the VC dimension of the solution class \( \mathcal{Y} \) underlying \( \mathcal{E} \), then arguments similar to those in the proof for Proposition 2.5 along with (65) can be adapted for deriving an upper bound on \( \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{y}_{R+1} (x_i) - y^* (x_i) \right] \right\}^{\frac{1}{2}} \).

The following result (from Matoušek, 2002) provides an upper bound on the VC dimension of the set of polynomials: let
\[
\mathcal{P}_{d, D} := \left\{ \{ w \in \mathbb{R}^d : h (w) \geq 0 \} : h \in \mathbb{R} [w_1, w_2, ..., w_d]_{\leq D} \right\}
\]
where \( R[w_1, w_2, ..., w_d] \leq D \) is the set of all real polynomials of degree at most \( D \) in \( d \) variables. Then the VC-dimension of \( P_{d,D} \) is no greater than \( \left( \frac{d+D}{d} \right) \). Therefore, if \( \phi(x, y) \) is a polynomial (in 2 variables) of degree \( D \), then the VC-dimension of the solution class in the form of (68) is no greater than \( \left( \frac{2+D}{2} \right) \). Note that this result by itself does not require (69), which, however, would be quite useful for estimating \( y^* \) as discussed above.

### 2.5 A general upper bound for higher order ODEs

We end this section with a general upper bound on the covering number of higher order Picard type ODEs. Previously in (12), (25) and (50), we have seen that the underlying solution class \( \mathcal{Y} \) is “essentially” no “larger” than the class \( f \) ranges over. This phenomenon also holds for higher order ODEs as long as they satisfy a Picard type condition. We would like to point out that this general bound overlooks additional structures in the solution class. For establishing an estimation theory, entropy bounds that take into account special structures (such as those we have seen in Sections 2.2-2.4) may also be useful.

We let

\[
Y(x) := \begin{bmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(m-1)}(x) \end{bmatrix} \quad \text{and} \quad Y_0 := \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(m-1) \end{bmatrix}.
\]

Let us consider the ODE

\[
y^{(m)}(x) = f \left( x, y(x), y'(x), ..., y^{(m-1)}(x) \right),
\]

\[
y(0) = y(0), \quad y'(0) = y(1), \quad ..., \quad y^{(m-1)}(0) = y(m-1),
\]

with \(|Y_0|_2 \leq C_0\) and \((x, Y(x)) \in \bar{\Gamma}\) where

\[
\bar{\Gamma} := \{(x, Y) : x \in [0, 1], |Y|_2 \leq b + C_0\}.
\]

**Proposition 2.7.** In terms of (70), suppose \( f \) is continuous on \( \bar{\Gamma} \), and

\[
\left| f(x, Y) - f(x, \bar{Y}) \right| \leq L_m |Y - \bar{Y}|_2
\]

for all \((x, Y) := (x, y, ..., y^{(m-1)}) \) and \((x, \bar{Y}) := (x, \bar{y}, ..., \bar{y}^{(m-1)}) \) in \( \bar{\Gamma} \). If \( f \) ranges over a function class \( \mathcal{F} \) with \( N_\infty(\delta, \mathcal{F}) \) on \( \bar{\Gamma} \), and \( \max_{(x, Y) \in \bar{\Gamma}} |f(x, Y)| \leq M \) for all \( f \in \mathcal{F} \), then we have

\[
\log N_\infty(\delta, \mathcal{Y}) \leq \log N_\infty \left( \frac{\delta}{L_{\max}}, \mathcal{F} \right) + m \log \left( \frac{2C_0L_{\max}}{\delta} + 1 \right)
\]

where \( L_{\max} = \sup_{x \in [0, \alpha]} \{ \exp(x\sqrt{L_{m+1}}) \left[ 1 + \int_0^x \exp(-s\sqrt{L_{m+1}}) ds \right] \} \) with \( \alpha = \min \{ 1, \frac{b}{M} \} \), and \( \mathcal{Y} \) is the class consisting of solutions (to (70) with \( f \) ranging over \( \mathcal{F} \)) on \([0, \alpha]\).

Below we discuss two applications of Proposition 2.7. The first one is an extension of (12) to parametric smooth higher order ODEs and the second one is an extension of (25) to nonparametric
smooth higher order ODEs.

**Example 1.** Consider the ODE

\[ y^{(m)}(x) = f\left(x, y(x), y'(x), \ldots, y^{(m-1)}(x); \theta\right), \quad \text{with initial values } Y_0, \]  

(73)

where \( f \) is parameterized by a \( K \)-dimensional vector of coefficients \( \theta \in \mathbb{B}_2(1) \). Suppose \( f \) is continuous on \( \bar{\Gamma} \), \( \max_{(x,Y) \in \bar{\Gamma}} |f(x, Y; \theta)| \leq M \), and

\[ |f(x, Y; \theta) - f(x, \tilde{Y}; \tilde{\theta})| \leq L_m |Y - \tilde{Y}|_2 \]

for all \((x, Y), (x, \tilde{Y}) \in \bar{\Gamma} \) and \( \theta \in \mathbb{B}_2(1) \); moreover,

\[ |f(x, Y; \theta) - f(x, Y; \theta')| \leq L_{K, 2} |\theta - \theta'|_2, \]  

(74)

for all \((x, Y) \in \bar{\Gamma} \) and \( \theta, \theta' \in \mathbb{B}_2(1) \). Let \( \mathcal{Y} \) be the class consisting of solutions to (73) with \( \theta \in \mathbb{B}_2(1) \) on \([0, \min \{1, \frac{b}{\max \delta}\}]\). Then we have

\[ \log N_\infty(\delta, \mathcal{Y}) \leq K \log \left(1 + \frac{2L_{\max} L_{K, 2}}{\delta}\right) + m \log \left(\frac{2C_0 L_{\max}}{\delta} + 1\right). \]  

**Example 2.** In terms of (70), suppose (71) is satisfied. Assume \( f \) ranges over \( S_{\beta+1, m+1}(1, \bar{\Gamma}) \). That is, \( f \) is continuous on \( \bar{\Gamma} \), and all partial derivatives \( D^p \) of \( f \) exist for all \( p \) with \( [p] = \sum_{k=1}^{m+1} p_k \leq \beta; \quad |D^p f(x, y, \ldots, y^{(m-1)})| \leq 1 \) for all \( p \) with \([p] \leq \beta \) and \((x, y, \ldots, y^{(m-1)}) \in \bar{\Gamma} \), where \( D^0 f(x, y, \ldots, y^{(m-1)}) = f(x, y, \ldots, y^{(m-1)}) \); and

\[ |D^p f(x, Y) - D^p f(x', \tilde{Y})| \leq \max \{|x - x'|, |Y - \tilde{Y}|_\infty\} \]

for all \( p \) with \([p] = \beta \) and \((x, Y), (x', \tilde{Y}) \in \bar{\Gamma} \). Then we have

\[ \log N_\infty(\delta, \mathcal{Y}) \leq c \left[ \left(\frac{2C}{L_{\max}}\right)^m \left(\frac{\delta}{\max L_{\max}}\right)^{(m+1)} + (\beta + 1) \log \left(\frac{\epsilon L_{\max}}{\delta}\right) + m \log \left(\frac{2C_0 L_{\max}}{\delta} + 1\right) \right] \]  

(75)

where \( \mathcal{Y} \) is the class consisting of solutions (to (70) with \( f \) ranging over \( S_{\beta+1, m+1}(1, \bar{\Gamma}) \)) on \([0, \min \{1, b\}]\).

**Remark.** Essentially, bound (75) implies that \( \mathcal{Y} \) is no “larger” than \( S_{\beta+1, m+1}(1, \bar{\Gamma}) \), the class where \( f \) ranges over. It is theoretically possible to derive a bound with a similar flavor to (27)-(28) in Theorem 2.1 but this task is very laborious for smooth higher order ODEs.

# 3 Conclusion and future work

Differential equations have a long standing history in mathematics and are widely applied in science, engineering, economics, business analytics, and public health. While the statistical literature on differential equations has been focusing on methodological and/or computational development, this
paper focuses on the metric entropy of ODE solution classes and a nonasymptotic theory for noisy function fitting problems, both of which, to the best of our knowledge, have not been examined in the literature. Given \( y^{(m)}(x) = f \left( x, y(x), y'(x), \ldots, y^{(m-1)}(x) \right) \) and letting \( f \) range over various function classes, we establish novel results on the metric entropy of the associated solution classes \( Y \). Most of our theory is built upon or expands the smoothness structure in the Picard-Lindelöf theorem although we also provide a number of results that do not require such a structure. In addition, we establish a nonasymptotic theory concerning the noisy function fitting/recovery problem for nonparametric least squares estimators and least squares estimators based on Picard iterations. We also discuss the estimation of initial values with local polynomials.

Here we discuss a number of future directions. First, an obvious extension would be to explore the metric entropy of solution classes associated with partial differential equations. Second, while this paper has delivered upper bounds for covering numbers of solution classes and estimation theory, it would be useful to establish minimax lower bounds to understand the information theoretic limitations of statistical recovery of an ODE solution. Third, this paper focuses on recovery of ODE solutions rather than recovery of structural parameters in an ODE. Nonasymptotic theory for the latter would be also worthwhile pursuing.

References

[1] Barro, R. J. and X. Sala-i-Martin (2004). Economic growth. The MIT Press.

[2] Bartlett, P. and S. Mendelson (2002). “Gaussian and Rademacher complexities: risk bounds and structural results”. Journal of Machine Learning Research, 3, 463–482.

[3] Bass, F. M. (1969). “A new product growth for model consumer durables”. Management Science, 15, 215–227.

[4] Bass, F. M. (2004). “Comments on 'A new product growth for model consumer durables The Bass model'”. Management Science, 50, 1833–1840.

[5] Campbell, D., G. Hooker and K. McAuley (2012). “Parameter estimation in differential equation models with constrained states”, Journal of Chemometrics, 56, 322–332.

[6] Coddington, E. A. and N. Levinson (1955). Theory of ordinary differential equations. McGraw-Hill, New York, NY.

[7] Dudley, R. M. (1978). “Central limit theorems for empirical measures”. Annals of Probability, 6, 899–929.

[8] Gronwall, T. H. (1919). Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. Annals of Mathematics, 20, 292–296.

[9] Guntuboyina, A. and B. Sen (2013). “Covering numbers for convex functions”. IEEE Transactions on Information Theory, 59, 1957–1965.

[10] Howard, R. (1998). “The Gronwall inequality”. Unpublished notes. Department of Mathematics, University of South Carolina.

[11] Kolmogorov, A. and B. Tikhomirov (1959). “\( \varepsilon \)-entropy and \( \varepsilon \)-capacity of sets in functional spaces”. Uspekhi Mat. Nauk., 86, 3-86. Appeared in English as American Mathematical Society Translations, 17, 277–364, 1961.
[12] Koltchinskii, V. 2006. “Local Rademacher complexities and oracle inequalities in risk minimization”. *Annals of Statistics*, 34, 2593–2656.

[13] Matoušek, J. (2002). *Lectures on discrete geometry*. Springer-Verlag, New York.

[14] Ramsay, J., G. Hooker, D. Campbell and J. Cao (2007). “Parameter estimation for differential equations: A generalized smoothing approach”. *Journal of the Royal Statistical Society*, 69, 741–796.

[15] Steele, J. M. (1978). “Empirical discrepancies and sub-additive processes”. *Annals of Probability*, 6, 118–127.

[16] Tsybakov, A. B. (2009). *Introduction to nonparametric estimation*. New York, NY: Springer.

[17] van de Geer, S (2000). *Empirical Processes in M-Estimation*. Cambridge University Press.

[18] van der Vaart, A. W. and J. Wellner (1996). *Weak convergence and empirical processes*. Springer-Verlag, New York, NY.

[19] Vynnycky, E. and R. White (2010). *An introduction to infectious diseases modelling*. Oxford University Press.

[20] Wainwright, M. J. (2019). *High-dimensional statistics: A non-asymptotic viewpoint*. Cambridge University Press.

[21] Zhu, Y. (2017). “Nonasymptotic analysis of semiparametric regression models with high-dimensional parametric coefficients”. *Annals of Statistics*, 45, 2274–2298.
Supplementary materials for “Ordinary differential equations (ODE): metric entropy and nonasymptotic theory for noisy function fitting”

by Ying Zhu\textsuperscript{9} and Mozhgan Mirzaei\textsuperscript{10}

A Proofs for Section 2

A.1 Notation and definitions

The $l_q$-norm of a $K$-dimensional vector $\theta$ is denoted by $|\theta|_q$, $1 \leq q \leq \infty$ where $|\theta|_q := \left(\sum_{j=1}^{K} |\theta_j|^q \right)^{1/q}$ when $1 \leq q < \infty$ and $|\theta|_\infty := \max_{j=1,\ldots,K} |\theta_j|$ when $q = \infty$. Let $\mathcal{B}_q(1) := \{\theta \in \mathbb{R}^K : |\theta|_q \leq 1\}$ with $q \geq 1$. Define $\mathcal{P}_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$ that places a weight $\frac{1}{n}$ on each observation $x_i$ for $i = 1, \ldots, n$, and the associated $L^2(\mathcal{P}_n)$-norm of the vector $v := \{v(x_i)\}_{i=1}^{n}$, denoted by $|v|_n$, is given by $\left(\frac{1}{n} \sum_{i=1}^{n} (v(x_i))^2\right)^{1/2}$. For two functions $f$ and $g$ on $[a, b]^d \subseteq \mathbb{R}^d$, we denote the supremum metric by $|f-g|_\infty := \sup_{x \in [a, b]^d} |f(x) - g(x)|$. For functions $f(n)$ and $g(n)$, $f(n) \gtrsim g(n)$ means that $f(n) \geq cg(n)$ for a universal constant $c \in (0, \infty)$; similarly, $f(n) \lesssim g(n)$ means that $f(n) \leq c'g(n)$ for a universal constant $c' \in (0, \infty)$; and $f(n) \asymp g(n)$ means that $f(n) \gtrsim g(n)$ and $f(n) \lesssim g(n)$ hold simultaneously. As a general rule for this paper, the various $c$ and $C$ constants denote positive universal constants that are independent of the sample size $n$ and the smoothness parameter $\beta$, and may vary from place to place.

Given a set $T$, a set $\{t^1, t^2, \ldots, t^N\} \subseteq T$ is called a $\delta$-cover of $T$ with respect to a metric $\rho$ if for each $t \in T$, there exists some $i \in \{1, \ldots, N\}$ such that $\rho(t, t^i) \leq \delta$. The cardinality of the smallest $\delta$-cover is denoted by $N_\rho(\delta; T)$, namely, the $\delta$-covering number of $T$. For example, $N_\infty(\delta, F)$ denotes the $\delta$-covering number of a function class $F$ with respect to the supremum metric $|.|_\infty$.

Given $C_0 > 0$, $b > 0$, and $\alpha = \min \{1, b\}$, let

\[
\begin{align*}
\overline{C} &= C_0 + b, \\
\Xi &= [0, 1] \times \left[-\overline{C}, \overline{C}\right], \\
C_{\max} &= \sup_{x \in [0, \alpha]} \left\{ \exp(x) \left[ 1 + \int_{0}^{x} \exp(-s) \, ds \right] \right\}.
\end{align*}
\]

Let $p = (p_1, \ldots, p_d)$ and $[p] = \sum_{k=1}^{d} p_k$ where $p_k$ is a non-negative integer. We write

\[
D^p g(z_1, \ldots, z_d) := \partial^{|p|} f / \partial z_1^{p_1} \ldots \partial z_d^{p_d}.
\]

Given a non-negative integer $\gamma$, we let $S_{\gamma+1,d} \left( \rho, [a, \overline{a}]^d \right)$ denote the class of functions such that any function $h \in S_{\gamma+1,d} \left( \rho, [a, \overline{a}]^d \right)$ satisfies:

1. $h$ is continuous on $[a, \overline{a}]^d$, and all partial derivatives of $h$ exist for all $p$ with $[p] \leq \gamma$;
2. $|D^p h(X)| \leq \rho$ for all $X \in [a, \overline{a}]^d$ and all $p$ with $[p] \leq \gamma$, where $D^\theta h(X) = h(X)$;

\textsuperscript{9}Assistant Professor, Department of Economics, UC San Diego. Corresponding author. yiz012@ucsd.edu.
\textsuperscript{10}PhD, former student at Department of Mathematics, UC San Diego. momirzae@ucsd.edu.
3. \( |D^p h(X) - D^p(X')| \leq \rho |X - X'|_\infty \) for all \( X, X' \in [a, \bar{a}]^d \) and all \( p \) with \( [p] = \gamma \).

When \( \rho = 1, d = 1, a = 0 \) and \( \bar{a} = 1 \), we use the shortform \( S_{\gamma + 1} := S_{\gamma + 1, 1} (1, [0, 1]) \).

### A.2 Preliminary

In terms of our least squares estimators, the basic inequality

\[
\frac{1}{2n} \sum_{i=1}^{n} (Y_i - \hat{y}(x_i))^2 \leq \frac{1}{2n} \sum_{i=1}^{n} (Y_i - y^*(x_i))^2
\]

yields

\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \leq \frac{2}{n} \sum_{i=1}^{n} \varepsilon_i (\hat{y}(x_i) - y^*(x_i)).
\] (76)

To bound the right-hand-side of (76), we define the shifted version of the function class:

\[
\bar{\mathcal{F}} := \{ g = g_1 - g_2 : g_1, g_2 \in \mathcal{F} \}.
\] (77)

Given a radius \( \tilde{r}_n > 0 \), define the local complexity

\[
\mathcal{G}_n(\tilde{r}_n; \bar{\mathcal{F}}) := \mathbb{E}_\varepsilon \left[ \sup_{h \in \Omega(\tilde{r}_n; \bar{\mathcal{F}})} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i h(x_i) \right],
\] (78)

where \( \varepsilon_i \sim \mathcal{N}(0, 1), \bar{\mathcal{F}} = \{ h \in \mathcal{F} : |h|_n \leq \tilde{r}_n \} \), and \( |h|_n := \sqrt{\frac{1}{n} \sum_{i=1}^{n} (h(x_i))^2} \).

Our goal is to seek a sharp enough \( \tilde{r}_n > 0 \) that satisfies the critical inequality

\[
\mathcal{G}_n(\tilde{r}_n; \bar{\mathcal{F}}) \lesssim \frac{\tilde{r}_n^2}{\sigma}.
\] (79)

It is known that the complexity \( \mathcal{G}_n(\tilde{r}_n; \bar{\mathcal{F}}) \) in (78) can be bounded above by the Dudley’s entropy integral (see, e.g., van de Geer, 2000; Barlett and Mendelson, 2002; Koltchinski, 2006; Wainwright, 2019). Let \( N_n(\delta, \Omega(\tilde{r}_n; \bar{\mathcal{F}})) \) denote the \( \delta \)-covering number of the set \( \Omega(\tilde{r}_n; \bar{\mathcal{F}}) \) in the \( |h|_n \) norm. Then the critical radius condition (79) holds for any \( \tilde{r}_n \in (0, \sigma] \) such that

\[
\frac{c}{\sqrt{n}} \int_{\tilde{r}_n^2}^{\tilde{r}_n^2} \sqrt{\log N_n(\delta, \Omega(\tilde{r}_n; \bar{\mathcal{F}}))} d\delta \leq \frac{\tilde{r}_n^2}{\sigma}.
\] (80)

### A.3 Proposition 2.1

**Proof.** Let \( N_q(\delta, \mathbb{B}_q(1)) \) denote the covering number of \( \mathbb{B}_q(1) \) with respect to the \( l_q \)-norm and \( N_q(\delta / C_{\max}, [-C_0, C_0]) \) denote the covering number of \([-C_0, C_0] \) with respect to the \( l_q \)-norm. For a given \( \delta > 0 \), let us consider the smallest \( \frac{\delta}{C_{\max} L_{K,q}} \)–covering \( \{ \theta^1, ..., \theta^N \} \) with respect to the \( l_q \)-norm. By (11), note that \( \{ f_{\theta^1}, f_{\theta^2}, ..., f_{\theta^N} \} \) forms a \( \frac{\delta}{C_{\max}} \)–cover of \( \mathcal{F} \) with respect to the \( l_q \)-norm. We also consider the smallest \( \frac{\delta}{C_{\max} L_{K,q}} \)–covering \( \{ y_{0,1}, ..., y_{0,N'} \} \) for the interval \([-C_0, C_0] \) where the initial value lies. By Theorem IV, for a solution \( y \) to the ODE with \( f \) parameterized by any \( \theta \in \mathbb{B}_q(1) \) (and subject to (11)) and \( y_0 \in [-C_0, C_0] \), we can find \( i \in \{ 1, ..., N \} \) and \( i' \in \{ 1, ..., N' \} \) such that

\[
|y(i, i')(x) - y(x)| \leq \frac{\delta}{C_{\max}} \exp(x) \left[ 1 + \int_0^x \exp(-s) ds \right] \leq \delta \quad \forall x \in [0, \alpha]
\]
where \( y(i,i') \) is a solution to the ODE with \( f \) parameterized by \( \theta^i \) and the initial value being \( y_0,i' \). Consequently, we obtain a \( \delta \)-cover of \( \mathcal{Y} \). By the standard volumetric argument which yields

\[
\log N_q (\delta, B_q (1)) \leq K \log \left( 1 + \frac{2C_{\max}L_{K,q}}{\delta} \right),
\]

and

\[
\log N_q \left( \frac{\delta}{C_{\max}}, [-C_0, C_0] \right) \leq \log \left( \frac{C_0C_{\max}}{\delta} + 1 \right),
\]

we conclude that

\[
\log N_\infty (\delta, \mathcal{Y}) \leq K \log \left( 1 + \frac{2C_{\max}L_{K,q}}{\delta} \right) + \log \left( \frac{C_0C_{\max}}{\delta} + 1 \right).
\]

### A.4 Proposition 2.2

**Proof.** Let \( \mathcal{F} = \mathcal{Y} \) in (77). We have

\[
\frac{1}{\sqrt{n}} \int_{\mathcal{F}} \log N_n (\delta, \Omega (\bar{r}_n; \bar{\mathcal{F}})) d\delta \leq \frac{1}{\sqrt{n}} \int_{\mathcal{F}} \log N_\infty (\delta, \Omega (\bar{r}_n; \bar{\mathcal{F}})) d\delta
\]

\[
\leq \sqrt{K} \int_{\mathcal{F}} \log \left( 1 + \frac{c\bar{r}_n (L_{K,\infty} \vee 1)}{\delta} \right) d\delta
\]

\[
= \bar{r}_n (L_{K,\infty} \vee 1) \sqrt{K} \int_{\mathcal{F}} \frac{1}{L_{K,\infty} \vee 1} \sqrt{\log \left( 1 + \frac{c}{t} \right)} dt
\]

\[
\leq (L_{K,\infty} \vee 1) \frac{\bar{r}_n}{\sqrt{K}} \left( L_{K,\infty} \vee 1 \right) \sqrt{K} \int_{\mathcal{F}} \frac{1}{L_{K,\infty} \vee 1} \sqrt{\log \left( 1 + \frac{c}{t} \right)} dt
\]

\[
\leq (L_{K,\infty} \vee 1) \frac{\bar{r}_n}{\sqrt{K}} \left( L_{K,\infty} \vee 1 \right) \sqrt{K} \int_{\mathcal{F}} \frac{1}{L_{K,\infty} \vee 1} \sqrt{\log \left( 1 + \frac{c}{t} \right)} dt
\]

where we have applied a change of variable \( t = \frac{\delta}{\bar{r}_n (L_{K,\infty} \vee 1)} \) in the third line. Setting \( B_{L_{K,\infty}} \sqrt{\bar{r}_n} \sqrt{K} \leq \bar{r}_n \) yields \( \bar{r}_n \leq B_{L_{K,\infty}} \sqrt{\bar{r}_n} \sqrt{K} \), where \( B_{L_{K,\infty}} = (L_{K,\infty} \vee 1) \). By Theorem 13.5 in Wainwright (2019), we obtain

\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \leq B_{L_{K,\infty}}^2 \frac{\sigma^2 K}{n},
\]

with probability at least \( 1 - c_1 \exp \left\{ -c_2 B_{L_{K,\infty}}^2 K \right\} \).

### A.5 Proposition 2.3

**Proof.** By (76), we need to bound \( \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \left( \hat{y}_{R+1} (x_i, \hat{\theta}) - y^*(x_i, \theta^*) \right) \). We can write \( \hat{y}_{R+1} (x_i, \hat{\theta}) - y^*(x_i, \theta^*) = \sum_{j=1}^{3} T_j (x_i) \) where

\[
T_1 (x_i) = \hat{y}_{R+1} (x_i; \hat{\theta}) - \hat{y}_{R+1} (x_i; \theta^*), \quad \text{estimation error due to } \hat{\theta}
\]

\[
T_2 (x_i) = \hat{y}_{R+1} (x_i; \theta^*) - y_{R+1}^* (x_i; \theta^*), \quad \text{estimation error due to } \hat{y}_0
\]

\[
T_3 (x_i) = y_{R+1}^* (x_i; \theta^*) - y^* (x_i; \theta^*), \quad \text{estimation error due to the finite iterations}
\]
where
\[ y_0^* = y_0^0, \]
\[ y_1^*(x; \theta) = y_0^0 + \int_0^x f(s, y_0^0; \theta) ds, \]
\[ y_2^*(x; \theta) = y_0^0 + \int_0^x f(s, y_1^*(x; \theta); \theta) ds, \]
\[ \vdots \]
\[ y_{R+1}^*(x; \theta) = y_0^0 + \int_0^x f(s, y_R^*(x; \theta); \theta) ds. \]

As a result, we have
\[
\left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i \left( \hat{y}_{R+1}(x_i, \hat{\theta}) - y^*(x_i, \theta^*) \right) \right| \lesssim \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i T_1(x_i) \right| + \sigma \sqrt{\frac{1}{n} \sum_{i=1}^n [T_2(x_i)]^2} + \sigma \sqrt{\frac{1}{n} \sum_{i=1}^n [T_3(x_i)]^2}
\]  
with probability at least \( 1 - c_1 \exp(-c_2n) \).

We first analyze \( \frac{1}{n} \sum_{i=1}^n \varepsilon_i T_1(x_i) \) in (82). In (81), let
\[
\mathcal{F} = \{ g_{\theta} (x) = \hat{y}_{R+1} (x; \theta) : \theta \in \mathbb{B}_\infty (1), x \in [0, \tilde{\alpha}] \}
\]  
where \( \hat{y}_{R+1} (s; \theta) \) is constructed in the following fashion:
\[ \hat{y}_0 = \hat{y}_0, \]
\[ \hat{y}_1 (x; \theta) = \hat{y}_0 + \int_0^x f(s, \hat{y}_0; \theta) ds, \]
\[ \hat{y}_2 (x; \theta) = \hat{y}_0 + \int_0^x f(s, \hat{y}_1 (s; \theta); \theta) ds, \]
\[ \vdots \]
\[ \hat{y}_{R+1} (x; \theta) = \hat{y}_0 + \int_0^x f(s, \hat{y}_R (s; \theta); \theta) ds. \]

For any \( \theta, \theta' \in \mathbb{B}_\infty (1) \), at the beginning, we have
\[
\left| \hat{y}_1 (x; \theta) - \hat{y}_1 (x; \theta') \right| \leq \bar{\alpha} L_{K, \infty} \left| \theta - \theta' \right|_{\infty} \quad \forall x \in [0, \tilde{\alpha}]
\]  
where the inequality follows from (11). For the second iteration, we have
\[
\left| \hat{y}_2 (x; \theta) - \hat{y}_2 (x; \theta') \right| \leq \left| \int_0^x f(s, \hat{y}_1 (s; \theta); \theta) ds - \int_0^x f(s, \hat{y}_1 (s; \theta'); \theta) ds \right| + \left| \int_0^x f(s, \hat{y}_1 (s; \theta'); \theta) ds - \int_0^x f(s, \hat{y}_1 (s; \theta'); \theta') ds \right| \\
\leq \bar{\alpha} \left( \bar{\alpha} L_{K, \infty} \left| \theta - \theta' \right|_{\infty} \right)_{(i)} + \bar{\alpha} L_{K, \infty} \left| \theta - \theta' \right|_{\infty} \left| \theta - \theta' \right|_{\infty} \quad \forall x \in [0, \tilde{\alpha}]
\]  
\[
\leq \alpha^2 L_{K, \infty} \left| \theta - \theta' \right|_{\infty} + \bar{\alpha} L_{K, \infty} \left| \theta - \theta' \right|_{\infty} \quad \forall x \in [0, \tilde{\alpha}]
\]
where (i) and (ii) in the second inequality follow from (10) with (84) and (11), respectively. Continuing with this pattern until the \((R + 1)\)th iteration, we obtain

\[
\left| \hat{y}_{R+1} (x; \theta) - \hat{y}_{R+1} (x; \theta') \right| \leq \left( L_{K, \infty} \left| \theta - \theta' \right|_\infty \right) \sum_{i=1}^{R+1} \bar{\alpha}^i \\
\leq \frac{\bar{\alpha} L_{K, \infty}}{1 - \bar{\alpha}} \left| \theta - \theta' \right|_\infty \forall x \in [0, \bar{\alpha}] .
\]

(85)

In particular, (85) holds for \(x \in \{x_1, x_2, \ldots, x_n\}\). Consequently,

\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{y}_{R+1} (x_i; \theta) - \hat{y}_{R+1} (x_i; \theta') \right] \right\} \leq \frac{\bar{\alpha} L_{K, \infty}}{1 - \bar{\alpha}} \left| \theta - \theta' \right|_\infty .
\]

Let \(\tilde{b} = \left( \frac{\bar{\alpha} L_{K, \infty}}{1 - \bar{\alpha}} \right) \). For a given \(\delta > 0\), let us consider the smallest \(\frac{\delta}{2\tilde{b}}\)–covering \(\{\theta^1, \ldots, \theta^N\}\) (with respect to the \(\ell_\infty\)–norm), and by (85), for any \(\theta, \theta' \in B_\infty (1)\), we can find some \(\theta^i\) and \(\theta^j\) from the covering set \(\{\theta^1, \ldots, \theta^N\}\) such that

\[
\left| \hat{y}_{R+1} (x; \theta) - \hat{y}_{R+1} (x; \theta') \right| \leq \left( \hat{y}_{R+1} (x; \theta^i) - \hat{y}_{R+1} (x; \theta^j) \right)
\]

\[
\leq \frac{\bar{\alpha} L_{K, \infty}}{1 - \bar{\alpha}} \left| \theta - \theta' \right|_\infty \leq \delta.
\]

Thus, \(\{g_{\theta^1}, g_{\theta^2}, \ldots, g_{\theta^N}\} \times \{g_{\theta^1}, g_{\theta^2}, \ldots, g_{\theta^N}\}\) forms a \(\delta\)–cover of \(\bar{F}\) in terms of \(F\) defined in (83).

Consequently, we have

\[
\frac{1}{\sqrt{n}} \int_{0}^{\tilde{r}_n} \sqrt{\log N_n (\delta; \Omega (\tilde{r}_n; \bar{F}))} d\delta \leq \frac{1}{\sqrt{n}} \int_{0}^{\tilde{r}_n} \sqrt{\log N_\infty (\delta; \Omega (\tilde{r}_n; \bar{F}))} d\delta
\]

\[
\leq \sqrt{\frac{K}{n}} \int_{0}^{\tilde{r}_n} \sqrt{2 \log \left( 1 + \frac{\sqrt{\bar{f}_n}}{\delta} \right)} d\delta
\]

\[
\approx \tilde{b} \sqrt{\frac{K}{n}}.
\]

Setting \(\tilde{b} \sqrt{\frac{K}{n}} \approx \tilde{b}^2 \) yields \(\tilde{r}_n \approx \sigma b \sqrt{\frac{K}{n}}\) and therefore,

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i T_1 (x_i) \right| \approx \sigma^2 b^2 K
\]

(86)

with probability at least \(1 - c_1 \exp \left( -c_2 b^2 K \right)\).

To analyze \(\frac{1}{n} \sum_{i=1}^{n} \left[ T_2 (x_i) \right]^2\) in (82), note that (10) implies

\[
\left| \hat{y}_1 (x_i; \theta^*) - y_1^* (x_i; \theta^*) \right| \leq \left| \hat{y}_0 - y_0^* \right| + \bar{\alpha} \left| \hat{y}_0 - y_0^* \right|,
\]

\[
\left| \hat{y}_2 (x_i; \theta^*) - y_2^* (x_i; \theta^*) \right| \leq \left| \hat{y}_0 - y_0^* \right| + \bar{\alpha} \left| \hat{y}_0 - y_0^* \right| + \bar{\alpha}^2 \left| \hat{y}_0 - y_0^* \right|,
\]

\[
\vdots
\]

\[
\left| \hat{y}_{R+1} (x_i; \theta^*) - y_{R+1}^* (x_i; \theta^*) \right| \leq \left| \hat{y}_0 - y_0^* \right| + \bar{\alpha} \left| \hat{y}_0 - y_0^* \right| + \cdots + \bar{\alpha}^{R+1} \left| \hat{y}_0 - y_0^* \right| \leq \frac{1}{1 - \bar{\alpha}} \left| \hat{y}_0 - y_0^* \right|
\]

31
for all $i = 1, ..., n$. As a result, we have

$$\sqrt{\frac{1}{n} \sum_{i=1}^{n} |T_2 (x_i)|^2} \leq \frac{1}{1 - \alpha} |\tilde{y}_0 - y_0^*|.$$

For $\sqrt{\frac{1}{n} \sum_{i=1}^{n} |T_3 (x_i)|^2}$ in (82), standard argument for the Picard-Lindelöf Theorem implies that

$$\sup_{x \in [0, \alpha]} |y^{*}_r(x; \theta^*) - y^{*}_{r+r'}(x; \theta^*)| \leq \tilde{\alpha}^{r+1} \frac{1 - \tilde{\alpha}^{r'}}{1 - \alpha} \sup_{x \in [0, \alpha]} |y^*_1(x; \theta^*) - y^*_0|$$

for any non-negative integers $r$ and $r'$; as a result, we have

$$\sqrt{\frac{1}{n} \sum_{i=1}^{n} |T_3 (x_i)|^2} \leq \frac{\tilde{\alpha}^{R+1}}{1 - \alpha} \sup_{x \in [0, \alpha]} |y^*_1(x; \theta^*) - y^*_0| \times \frac{\tilde{\alpha}^{R+1}}{1 - \alpha} \max \{C_0, \tilde{\alpha}\}.$$

A.6 Theorem 2.1

A.6.1 Main argument

Proof. Part (i). Let $C_{\max} = \sup_{x \in [0, \alpha]} \{\exp (x) [1 + f_0^x \exp (-s) ds]\}$. For a given $\delta > 0$, let us consider the smallest $\frac{\delta}{C_{\max}}$-covering $\{f_1, ..., f_N\}$ of $S_{\beta+1, 2} (1, \Xi)$ with respect to the sup-norm such that

$$\log N_{\infty} \left( \frac{\delta}{C_{\max}}, S_{\beta+1, 2} (1, \Xi) \right) \leq \left( \frac{\delta}{C_{\max}} \right)^{\frac{2}{\beta+2}} + (\beta + 1) \log \left( \frac{C_{\max}}{\delta} \right).$$

We also consider the smallest $\frac{\delta}{C_{\max}}$-covering $\{y_0, ..., y_{0,N'}\}$ for the interval $[-C_0, C_0]$ where the initial value lies. Note that

$$\log N_{\infty} \left( \frac{\delta}{C_{\max}}, [-C_0, C_0] \right) \leq \log \left( \frac{C_0 C_{\max}}{\delta} + 1 \right).$$

By Theorem IV, for a solution $y$ to the ODE associated with any $f \in S_{\beta+1, 2} (1, \Xi)$ and $y_0 \in [-C_0, C_0]$, we can find $i \in \{1, ..., N\}$ and $i' \in \{1, ..., N'\}$ such that

$$\left| y(x) - y_{(i,i')} (x) \right| \leq \frac{\delta}{C_{\max}} \exp (x) \left[ 1 + \int_0^x \exp (-s) ds \right] \leq \delta \quad \forall x \in [0, \alpha]$$

where $y_{(i,i')}$ is a solution to the ODE associated with $f_i$ and the initial value $y_{0,i'}$ from the covering sets. Consequently, we obtain a $\delta$—cover of $\mathcal{Y}$. We conclude that

$$\log N_{\infty} (\delta, \mathcal{Y}) \leq c \left[ \left( \frac{\delta}{C_{\max}} \right)^{\frac{2}{\beta+2}} + (\beta + 1) \log \left( \frac{C_{\max}}{\delta} \right) + \log \left( \frac{C_0 C_{\max}}{\delta} + 1 \right) \right].$$

Part (ii). The proof for part (ii) consists of three steps.

Step 1: In Lemma A.1(ii) (subsection A.6.2), we establish $\mathcal{Y} \in \mathcal{S}_{\beta+2}^{\dagger}$ where $\mathcal{S}_{\beta+2}^{\dagger}$ denotes the class of functions such that any function $h \in \mathcal{S}_{\beta+2}^{\dagger}$ satisfies the following properties:

- $h$ is continuous on $[0, 1]$ and differentiable $\beta + 1$ times;
\[ |h(x)| \leq C, \text{ and } |h^{(k)}(x)| \leq 2^{k-1} (k-1)! \text{ for all } k = 1, \ldots, \beta + 1 \text{ and } x \in [0, 1]; \]

- \[ |h^{(\beta+1)}(x) - h^{(\beta+1)}(x')| \leq 2^{\beta+1} (\beta + 1)! |x - x'| \text{ for all } x, x' \in [0, 1]. \]

### Step 2: In Lemma A.2, we establish a crude bound as follows:

\[
N_\infty \left( \delta, S_{\beta+2}^\dagger \right) \leq \exp \left[ \log \left( \frac{\beta}{C} \prod_{i=0}^{\beta} i! \right) + \frac{\beta^2 + \beta}{2} \log 2 - \frac{\beta + 3}{2} \log \left( \frac{\delta}{5} \right)^{\frac{1}{\beta+2}} \log 21 + \log 4 \right] \quad (87)
\]

### Step 3: The third step refines the crude bound \((88)\). For a given \(\frac{\delta}{5} \in (0, 1)\), let \(\beta^*(\delta) = \gamma (\leq \beta)\) be the largest non-negative integer such that

\[
\log \left( \prod_{i=0}^{\gamma} i! \right) + \frac{\gamma^2 + \gamma}{2} \log 2 - \frac{\gamma + 3}{2} \log \left( \frac{\delta}{5} \right)^{\frac{1}{\beta+2}} \log 21 + \max \left\{ 0, \log \left( 4C \right) \right\}. \quad (88)
\]

Note that the LHS of \((88)\) is a strictly increasing function of \(\gamma\) (since \(\frac{\delta}{5} \in (0, 1)\)) and the RHS is a strictly decreasing function of \(\gamma\), and the LHS is smaller than the RHS for any \(\frac{\delta}{5} \in (0, 1)\) when \(\gamma = 0\) (to see this, note that \(LHS = 3 \log \left( \frac{\delta}{5} \right)^{\frac{1}{2}} \) and \(RHS \geq \frac{\delta}{5} \log 21 \geq 3 \log \left( \frac{\delta}{5} \right)^{\frac{1}{2}}\), where “log” is the natural logarithm.). Therefore, the largest non-negative solution \(\beta^*(\delta) = \gamma (\leq \beta)\) to \((88)\) exists (i.e., \(\beta^*(\delta)\) is well defined).

Case (i): For a given \(\delta\), when \(\beta > \beta^*(\delta)\), we can always bound \(\log N_\infty \left( \delta, S_{\beta+2}^\dagger \right)\) by

\[
2 \left( \frac{\delta}{5} \right)^{\frac{1}{\beta^*(\delta)+2}} \log 21 + \max \left\{ 0, \log \left( 4C \right) \right\}
\]

from above, because \(S_{\beta+2}^\dagger \subseteq S_{\beta^*(\delta)+2}^\dagger\) and

\[
\log N_\infty \left( \delta, S_{\beta^*(\delta)+2}^\dagger \right) \leq 2 \left( \frac{\delta}{5} \right)^{\frac{1}{\beta^*(\delta)+2}} \log 21 + \max \left\{ 0, \log \left( 4C \right) \right\}.
\]

Case (ii): For a given \(\delta\), when \(\beta = \beta^*(\delta)\), the LHS of \((88)\) is dominated by the RHS and therefore, we can bound \(\log N_\infty \left( \delta, S_{\beta+2}^\dagger \right)\) by

\[
2 \left( \frac{\delta}{5} \right)^{\frac{1}{\beta^*(\delta)+2}} \log 21 + \max \left\{ 0, \log \left( 4C \right) \right\}
\]

from above.

### A.6.2 Lemma A.1

**Lemma A.1.** (i) Assume \(f\) is \(\beta\)-times differentiable, and for all \(y, \bar{y} \in [-C, C]\),

\[
|f^{(k)}(y)| \leq 1, \quad \forall 0 \leq k \leq \beta, \quad (89)
\]

\[
|f^{\beta}(y) - f^{\beta}(\bar{y})| \leq |y - \bar{y}|, \quad (90)
\]

where \(f^{(0)} = f\). Then for all \(x, x' \in [0, 1]\), we have

\[
|y^{(k)}(x)| \leq (k - 1)!
\]
Lemma A.1(i) concerns the autonomous system (49) and Lemma A.1(ii) concerns the nonautonomous system (50). In Lemma A.1(i), by the mean value theorem, the assumption that $f(0, y) = 0$ implies that

$$
\left| y^{(\beta+1)}(x) - y^{(\beta+1)}(x') \right| \leq (\beta + 1)!|x - x'|.
$$

(ii) Assume that all partial derivatives $D^p f$ of $f$ exist for all $p$ with $[p] = p_1 + p_2 \leq \beta$; $|D^p f(x, y)| \leq 1$ for all $p$ with $[p] \leq \beta$ and $(x, y) \in [0, 1] \times [y_0 - b, y_0 + b]$, where $D^0 f(x, y) = f(x, y)$; and

$$
\left| D^p f(x, y) - D^p f(x', y) \right| \leq \max \left\{ |x - x'|, |y - y'| \right\} \quad \forall (x, y), (x', y) \in [0, 1] \times [y_0 - b, y_0 + b],
$$

for all $p$ with $[p] = \beta$. Then for all $x, x' \in [0, 1]$, we have

$$
\left| y^{(k)}(x) \right| \leq 2^{k-1}(k-1)!
$$

for all $0 \leq k \leq \beta + 1$ and

$$
\left| y^{(\beta+1)}(x) - y^{(\beta+1)}(x') \right| \leq 2^{\beta+1} (\beta + 1)!|x - x'|.
$$

Remark. In Lemma A.1(i), by the mean value theorem, \[59\] with $k = 0$ (i.e., $|f(0)(y(x))| = |f(y(x))| = |y'(x)| \leq 1$) and \[90\] imply that

$$
\left| f^\beta(y(x)) - f^\beta(y(x')) \right| \leq |y(x) - y(x')| \leq |x - x'| \quad \forall x, x' \in [0, 1];
$$

moreover, \[89\] with $1 \leq k \leq \beta$ implies that

$$
\left| f^{k-1}(y(x)) - f^{k-1}(y(x')) \right| \leq |y(x) - y(x')| \leq |x - x'|, \quad \forall 1 \leq k \leq \beta, x, x' \in [0, 1].
$$

In Lemma A.1(ii), by the mean value theorem, the assumption that $|f(x, y(x))| = |y'(x)| \leq 1$ and \[91] imply that

$$
\left| D^p f(x, y(x)) - D^p f(x', y(x')) \right| \leq \max \left\{ |x - x'|, |y(x) - y(x')| \right\} \leq |x - x'|
$$

for all $p$ with $[p] = \beta$ and $(x, y(x)), (x', y(x')) \in [0, 1] \times [y_0 - b, y_0 + b]$; moreover, the assumption that $|D^p f(x, y)| \leq 1$ for all $p$ with $1 \leq [p] \leq \beta$ implies that

$$
\left| D^{p-1} f(x, y(x)) - D^{p-1} f(x', y(x')) \right| \leq |x - x'| + |y(x) - y(x')| \leq 2 |x - x'|,
$$

for all $p$ with $1 \leq [p] \leq \beta$ and $(x, y(x)), (x', y(x')) \in [0, 1] \times [y_0 - b, y_0 + b]$.

Remark. Lemma A.1 can easily handle situations where the bound “1” on the absolute values of the derivatives and the Lipschitz conditions is replaced with general constants.

Lemma A.1(i) concerns the autonomous system \[49\] and Lemma A.1(ii) concerns the nonautonomous system \[24\]. We first prove Lemma A.1(i) and then Lemma A.1(ii). Let us begin with some intuitions for the autonomous system. When $\beta = 0$, $\left| y'(x) \right| = |f(y(x))| \leq 1$ for all $x$ and $y(x)$ on $[0, 1] \times [y_0 - b, y_0 + b]$; moreover, we have
\[ |y'(x) - y'(x')| = |f(y(x)) - f(y(x'))| \leq |x - x'|.\]

When \( \beta = 1 \), note that
\[
|y^{(2)}(x)| = \left| \frac{\partial y^{(1)}(x)}{\partial x} \right| = \left| f^{(1)}(y(x))y^{(1)}(x) \right| \leq 1
\]
and
\[
|y^{(2)}(x) - y^{(2)}(x')| = \left| f^{(1)}(y(x))y^{(1)}(x) - f^{(1)}(y(x'))y^{(1)}(x') \right|
\leq \left| f^{(1)}(y(x))y^{(1)}(x) - f^{(1)}(y(x'))y^{(1)}(x) \right|
+ \left| f^{(1)}(y(x'))y^{(1)}(x) - f^{(1)}(y(x'))y^{(1)}(x') \right|
\leq |y(x) - y(x')| + |y^{(1)}(x) - y^{(1)}(x')|
\leq 2|x - x'|.
\]

When \( \beta = 2 \), we have
\[
|y^{(3)}(x)| = \left| f^{(2)}(y(x)) \left( y^{(1)}(x) \right)^2 + f^{(1)}(y(x)) y^{(2)}(x) \right|
\leq \left| f^{(2)}(y(x)) \right| + \left| f^{(1)}(y(x)) \right| \leq 2
\]
and
\[
|y^{(3)}(x) - y^{(3)}(x')| \leq 6|x - x'|.
\]

After trying \( \beta = 1, \ldots, 4 \), we observe the following pattern:

\[
y^{(1)}(x) = f(y(x))
\]
\[
y^{(2)}(x) = f^{(1)}(y(x))y^{(1)}(x)
\]
\[
y^{(3)}(x) = f^{(2)}(y(x))\left( f(y(x)) \right)^2 + \left( f^{(1)}(y(x)) \right)^2 \cdot f(y(x))
\]
\[
y^{(4)}(x) = f^{(3)}(y(x))\left( f(y(x)) \right)^3 + 2f^{(2)}(y(x))f^{(1)}(y(x))\left( f(y(x)) \right)^2 + 2f^{(2)}(y(x))f^{(1)}(y(x))\left( f(y(x)) \right)^3 \cdot f(y(x))
\]

In what follows, we derive \( b_k \)s (for all \( k = 1, \ldots, \beta + 1 \)) such that
\[
|y^{(k)}(x)| \leq b_k,
\]
and \( L_{\beta+1} \) such that
\[
|y^{(\beta+1)}(x) - y^{(\beta+1)}(x')| \leq L_{\beta+1} |x - x'|.
\]
Proof of Lemma A.1(i). For a $\beta$–times differentiable function $f$, define

$$f^{(a_1, \ldots, a_k)}(x) := f^{(a_1)}(y(x)), f^{(a_2)}(y(x)), \ldots, f^{(a_k)}(y(x)),$$

where $a_1 + a_2 + \cdots + a_k = k - 1$ for all $k = 1, \ldots, \beta + 1$, and $a_i \geq 0$ are all integers. We first show that

$$\frac{d}{dx} f^{(a_1, \ldots, a_k)}(x) = \sum_{j=1}^{k} f^{(a_1, \ldots, a_{j-1}, a_j + 1, 0, a_{j+1}, \ldots, a_k)}(x). \quad (96)$$

The equality (96) follows from the derivations below:

$$\frac{d}{dx}(f^{(a_1)}(y(x)).f^{(a_2)}(y(x)). \ldots. f^{(a_k)}(y(x)))$$

$$= \sum_{j=1}^{k} f^{(a_1)}(y(x)). \ldots. f^{(a_{j-1})}(y(x)). \frac{d}{dx}(f^{(a_j)}(y(x))). f^{(a_{j+1})}(y(x)) \ldots f^{(a_k)}(y(x))$$

by product rule

$$= \sum_{j=1}^{k} f^{(a_1)}(y(x)). \ldots. \left[f^{(a_{j-1})}(y(x)). f^{(a_j)}(y(x)) \right] f^{(a_{j+1})}(y(x)) \ldots f^{(a_k)}(y(x)).$$

by chain rule

Now by induction on $k$, we show that if $f$ is $\beta$–times differentiable, then for each $1 \leq k \leq \beta + 1$, we have

$$y^{(k)}(x) = \sum_{i=1}^{(k-1)!} f^{(a_1, \ldots, a_k)}(x), \quad \text{and for all } i \text{ in } (a_1^1, \ldots, a_k^i), a_1^i + \ldots + a_k^i = k - 1. \quad (97)$$

For the base case, $k = 1 \Rightarrow y'(x) = f(y(x)) = f^{(a_1)}(x) = \sum_{i=1}^{a_1^i} f^{(a_1)}(x)$ where $a_1 = 0$. Now assume $k \leq \beta + 1$ and the induction hypothesis holds for $k - 1$. Then

$$y^{(k)}(x) = \frac{d}{dx} \left(y^{(k-1)}(x)\right)$$

$$= \frac{d}{dx} \left(\sum_{i=1}^{(k-2)!} f^{(a_1^i, \ldots, a_{k-1}^i)}(x)\right) \quad \text{where } \forall i, a_1^i + \ldots + a_{k-1}^i = k - 2,$$

$$= \sum_{i=1}^{(k-2)!} f^{(a_1^i, \ldots, a_{j-1}^i, a_j^i, 0, a_{j+1}^i, \ldots, a_{k-1}^i)}(x) \quad \text{by } (96).$$

Notice that $\left(a_1^i, \ldots, a_{j-1}^i, a_j^i, 0, a_{j+1}^i, \ldots, a_{k-1}^i\right)$ has exactly $k$ terms, and adds up to $a_1^i + \ldots + a_{k-1}^i + 1 + 0 = k - 2 + 1 + 0 = k - 1$, and in total there are $(k - 2)!(k - 1) = (k - 1)!$ terms. This completes the induction. By (89), we have $\left|y^{(k)}(t)\right| \leq (k - 1)!$. 

36
To show the second claim \( |y^{(\beta+1)}(x) - y^{(\beta+1)}(x')| \leq (\beta + 1)!|x - x'| \), we use (97). Note that

\[
|y^{(\beta+1)}(x) - y^{(\beta+1)}(x')| = \left| \sum_{i=1}^{\beta!} f^{(a_1,\ldots,a_{\beta+1})}(x) - \sum_{i=1}^{\beta!} f^{(a_1,\ldots,a_{\beta+1})}(x') \right| \\
\leq \sum_{i=1}^{\beta!} \left| f^{(a_1,\ldots,a_{\beta+1})}(x) - f^{(a_1,\ldots,a_{\beta+1})}(x') \right|,
\]

where

\[
\left| f^{(a_1,\ldots,a_{\beta+1})}(x) - f^{(a_1,\ldots,a_{\beta+1})}(x') \right| = \sum_{j=1}^{\beta+1} \left| f^{(a_1)}(y(x)) \ldots f^{(a_{j-1})}(y(x)). \left( f^{(a_j)}(y(x)) - f^{(a_j)}(y(x')) \right) \ldots f^{(a_{\beta+1})}(y(x')) \right| \\
\leq \sum_{j=1}^{\beta+1} \left| f^{(a_1)}(y(x)) \ldots f^{(a_{j-1})}(y(x)) \right| \left| f^{(a_j)}(y(x)) - f^{(a_j)}(y(x')) \right| \left| f^{(a_{j+1})}(y(x')) \ldots f^{(a_{\beta+1})}(y(x')) \right| \\
\leq \sum_{j=1}^{\beta+1} |x - x'| 
\]

and the third line in the above follows from (92) and (93). Hence,

\[
|y^{(\beta+1)}(x) - y^{(\beta+1)}(x')| = \sum_{i=1}^{\beta!} \left| f^{(a_1,\ldots,a_{\beta+1})}(x) - f^{(a_1,\ldots,a_{\beta+1})}(x') \right| \\
\leq \beta! \cdot (\beta + 1)!|x - x'| = (\beta + 1)!|x - x'|.
\]

Now we show Lemma A.1(ii). As before, let us gain some intuitions first. For \( \beta = 0 \), \( |y'(x)| = |f(x, y(x))| \leq 1 \) for all \( x \) and \( y(x) \) on \([0, 1] \times [y_0 - b, y_0 + b]\); moreover, we have

\[
|y'(x) - y'(x')| = \left| f(x, y(x)) - f(x', y(x')) \right| \\
\leq \left| f(x, y(x)) - f(x', y(x)) \right| + \left| f(x', y(x)) - f(x', y(x')) \right| \\
\leq 2|x - x'|.
\]

For \( \beta = 1 \), note that

\[
|y^{(2)}(x)| = \left| \frac{\partial y^{(1)}(x)}{\partial x} \right| = \left| f_y^{(1)}(x, y(x)) y^{(1)}(x) + f_x^{(1)}(x, y(x)) \right| \\
\leq \left| f_y^{(1)}(x, y(x)) y^{(1)}(x) \right| + \left| f_x^{(1)}(x, y(x)) \right| \leq 2
\]
and
\[
|y^{(2)}(x) - y^{(2)}(x')| \leq \left| f_y^{(1)}(x, y(x))y^{(1)}(x) - f_y^{(1)}(x', y(x'))y^{(1)}(x') \right| \\
+ \left| f_x^{(1)}(x, y(x)) - f_x^{(1)}(x', y(x')) \right| \\
\leq \left| f_y^{(1)}(x, y(x))y^{(1)}(x) - f_y^{(1)}(x', y(x'))y^{(1)}(x) \right| \\
+ \left| f_y^{(1)}(x', y(x'))y^{(1)}(x) - f_y^{(1)}(x', y(x'))y^{(1)}(x') \right| \\
+ \left| f_x^{(1)}(x, y(x)) - f_x^{(1)}(x', y(x')) \right| \\
\leq 6|x - x'|.
\]

In what follows, we derive \(b_k\)s (for all \(k = 1, ..., \beta + 1\)) such that
\[
\left| y^{(k)}(x) \right| \leq b_k
\]
and \(L_{\beta + 1}\) such that
\[
\left| y^{(\beta + 1)}(x) - y^{(\beta + 1)}(x') \right| \leq L_{\beta + 1} |x - x'|.
\]

**Proof of Lemma A.1(ii).** Writing \(\vartheta(x) = (x, y(x))\) and \(\eta_j(t_1, t_2) = \frac{\partial^{a_j+b_j} f}{(\partial t_1)^{a_j}(\partial t_2)^{b_j}}(t_1, t_2)\), we have
\[
\frac{d}{dx} \vartheta(x) = (1, y'(x)) = (1, f(x, y(x))),
\]
\[
\frac{\partial \eta_j}{\partial t_1}(t_1, t_2) = \frac{\partial^{a_j+b_j+1} f}{(\partial t_1)^{a_j+1}(\partial t_2)^{b_j}}(t_1, t_2),
\]
\[
\frac{\partial \eta_j}{\partial t_2}(t_1, t_2) = \frac{\partial^{a_j+b_j+1} f}{(\partial t_1)^{a_j}(\partial t_2)^{b_j+1}}(t_1, t_2).
\]

Hence,
\[
\frac{d}{dx} \left( \frac{\partial^{a_j+b_j} f}{(\partial t_1)^{a_j}(\partial t_2)^{b_j}}(x, y(x)) \right)
\]
\[
= \frac{d}{dx} (\eta_j \circ \vartheta(x))
\]
\[
= \left( \frac{\partial \eta_j}{\partial t_1}(x, y(x)), \frac{\partial \eta_j}{\partial t_2}(x, y(x)) \right) \cdot (1, f(x, y(x)))
\]
\[
= \frac{\partial^{a_j+b_j+1} f}{(\partial t_1)^{a_j+1}(\partial t_2)^{b_j}}(x, y(x)) + \frac{\partial^{a_j+b_j+1} f}{(\partial t_1)^{a_j}(\partial t_2)^{b_j+1}}(x, y(x))f(x, y(x)).
\]  \hspace{1cm} (98)

For a \(\beta\)-times differentiable function \(f\), define
\[
f(a_1, b_1; ..., a_k, b_k)(x) := \prod_{i=1}^{k} \frac{\partial^{a_i+b_i} f}{(\partial t_1)^{a_i}(\partial t_2)^{b_i}}(x, y(x))
\]
where \(\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i = k - 1\) for all \(k = 1, ..., \beta + 1\), and \(a_i, b_i \geq 0\) are all integers. We have
\[
\frac{d}{dx} f^{(a_1, b_1, \ldots, a_k, b_k)}(x)
= \sum_{j=1}^{k} \frac{\partial^{a_1+b_1} f}{(\partial t_1)^{a_1}(\partial t_2)^{b_1}}(x, y(x)) \cdot \frac{d}{dx} \left( \frac{\partial^{a_j+b_j} f}{(\partial t_1)^{a_j}(\partial t_2)^{b_j}}(x, y(x)) \right) \cdots \frac{\partial^{a_k+b_k} f}{(\partial t_1)^{a_k}(\partial t_2)^{b_k}}(x, y(x))
\]

where the second equality comes from (98); that is,

\[
\frac{d}{dx} f^{(a_1, b_1, \ldots, a_k, b_k)}(x) = \sum_{j=1}^{k} f^{(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j+1, b_j, a_{j+1}, \ldots, a_k, b_k)} + \sum_{j=1}^{k} f^{(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j+1, 0, 0, a_{j+1}, a_{j+2}, \ldots, a_k, b_k)}. \quad (99)
\]

Given (99), now we show that

\[
y^{(k)}(x) = \sum_{i=1}^{2^{k-1}(k-1)!} f^{(a_1, b_1, \ldots, a_k, b_k)}(x) \quad (100)
\]
satisfies the following properties: (1) the summation has at most \(2^{k-1}(k-1)\)! terms for each \(1 \leq k \leq \beta + 1\), and (2) \(\forall i, a_1 + b_1 + \ldots + a_k + b_k = k - 1\).

The base case is obvious. Notice that by (99), differentiating a term of the form \(f^{(a_1, b_1, \ldots, a_k, b_k)}(x)\) gives us \(2k\) terms of the form \(f^{(c_1, d_1, \ldots, c_m, d_m)}(x)\) where \(m = k\) or \(k + 1\), and \(c_1 + d_1 + \ldots + c_m + d_m = 1 + (a_1 + b_1 + \ldots + a_k + b_k) = 1 + (k - 1) = k\). So the total number of terms in \(y^{(k+1)} \leq 2k\) (number of terms in \(y^k\)) \(\leq 2k(k-1)!2^{k-1}\). This proves the induction hypothesis. By the assumption that \(|f(x, y(x))| \leq 1\), we have \(|y^{(k)}(x)| \leq 2^{k-1}(k-1)!\).

To show the second claim \(|y^{(\beta+1)}(x) - y^{(\beta+1)}(x')| \leq 2^{\beta+1}(\beta + 1)!|x - x'|\), we use (100). Note that

\[
|y^{(\beta+1)}(x) - y^{(\beta+1)}(x')| = \left| \sum_{i=1}^{2^{\beta}\beta!} f^{(a_1, b_1, \ldots, a_{\beta+1}, b_{\beta+1})}(x) + \sum_{i=1}^{2^{\beta}\beta!} f^{(a_1, b_1, \ldots, a_{\beta+1}, b_{\beta+1})}(x') \right|
\]

\[
\leq \sum_{i=1}^{2^{\beta}\beta!} \left| f^{(a_1, b_1, \ldots, a_{\beta+1}, b_{\beta+1})}(x) - f^{(a_1, b_1, \ldots, a_{\beta+1}, b_{\beta+1})}(x') \right|
\]

39
where

\[
\left| f(a_1^{(\beta)}, b_1^{(\beta)}, \ldots, a_{\beta+1}^{(\beta)}, b_{\beta+1}^{(\beta)}) (x) - f(a_1^{(\beta)}, b_1^{(\beta)}, \ldots, a_{\beta+1}^{(\beta)}, b_{\beta+1}^{(\beta)}) (x') \right|
\]

\[
= \sum_{j=1}^{\beta+1} \frac{\partial^{a_1^{(\beta)}+b_1^{(\beta)}} f}{(\partial t_1)^{a_1^{(\beta)}} (\partial t_2)^{b_1^{(\beta)}}} (x, y(x)) \cdot \ldots \cdot \left| \frac{\partial^{a_{j-1}^{(\beta)}+b_{j-1}^{(\beta)}} f}{(\partial t_1)^{a_{j-1}^{(\beta)}} (\partial t_2)^{b_{j-1}^{(\beta)}}} (x, y(x)) \right| \left| \frac{\partial^{a_j^{(\beta)}+b_j^{(\beta)}} f}{(\partial t_1)^{a_j^{(\beta)}} (\partial t_2)^{b_j^{(\beta)}}} (x, y(x)) - \frac{\partial^{a_j^{(\beta)}+b_j^{(\beta)}} f}{(\partial t_1)^{a_j^{(\beta)}} (\partial t_2)^{b_j^{(\beta)}}} (x', y(x')) \right| \left| \frac{\partial^{a_{j+1}^{(\beta)}+b_{j+1}^{(\beta)}} f}{(\partial t_1)^{a_{j+1}^{(\beta)}} (\partial t_2)^{b_{j+1}^{(\beta)}}} (x', y(x')) \right| \ldots \frac{\partial^{a_{\beta+1}^{(\beta)}+b_{\beta+1}^{(\beta)}} f}{(\partial t_1)^{a_{\beta+1}^{(\beta)}} (\partial t_2)^{b_{\beta+1}^{(\beta)}}} (x', y(x'))) \right| \leq 2|x - x'| \quad \text{by (93) and (95)}
\]

Hence,

\[
|y^{(\beta+1)}(x) - y^{(\beta+1)}(x')| \leq \left( 2^{\beta} \beta! \right) \left( 2 (\beta + 1) |x - x'| \right) \leq 2^{\beta+1} (\beta + 1)! |x - x'|.
\]

A.6.3 Lemma A.2

**Lemma A.2.** In terms of $S_{\beta+2}^+$, we have [87].

**Proof.** By Lemma A.1(ii), we have shown that $\mathcal{Y} \subseteq S_{\beta+2}^+$. The rest of argument modifies the original proof for smooth classes in Kolmogorov and Tikhomirov (1959). For every function $h \in S_{\beta+2}^+$ and $x, x + \Delta \in (0, 1)$, we have

\[
h(x + \Delta) = h(x) + \Delta h'(x) + \frac{\Delta^2}{2!} h''(x) + \cdots + \frac{\Delta^\beta}{\beta!} h^{(\beta)}(x) + \frac{\Delta^{\beta+1}}{(\beta + 1)!} h^{(\beta+1)}(x).
\]

Let us define

\[
R_h(x, \Delta) := h(x + \Delta) - h(x) - \Delta h'(x) - \frac{\Delta^2}{2!} h''(x) - \cdots - \frac{\Delta^\beta}{\beta!} h^{(\beta)}(x) - \frac{\Delta^{\beta+1}}{(\beta + 1)!} h^{(\beta+1)}(x)
\]

on $x + \Delta \in (0, 1)$. For every function $h \in S_{\beta+2}^+$ and $x, x + \Delta \in (0, 1)$, we have

\[
|R_h(x, \Delta)| \leq \frac{\Delta^{(\beta+1)}}{(\beta + 1)!} h^{(\beta+1)}(x).
\]

Hence,

\[
|y^{(\beta+1)}(x) - y^{(\beta+1)}(x')| \leq \left( 2^{\beta} \beta! \right) \left( 2 (\beta + 1) |x - x'| \right) \leq 2^{\beta+1} (\beta + 1)! |x - x'|.
\]
and we have
\[ |R_h(x, \Delta)| = \frac{\Delta^{\beta+1}}{(\beta + 1)!} |h^{(\beta+1)}(x) - h^{(\beta+1)}(z)| \leq (2|\Delta|)^{\beta+2}. \]

As a consequence, we obtain
\[ h(x + \Delta) = \sum_{k=0}^{\beta+1} \frac{\Delta^k}{k!} h^{(k)}(x) + R_h(x, \Delta) \quad \text{where} \quad |R_h(x, \Delta)| \leq (2|\Delta|)^{\beta+2}. \]

Let us define \( g = h^{(i)} \in S_{\beta+2-i}^i \) for \( 0 \leq i \leq \beta + 1 \) and the above implies that
\[ h^{(i)}(x + \Delta) = \sum_{k=0}^{\beta+1-i} \frac{\Delta^k}{k!} h^{(i+k)}(x) + R_h^{(i)}(x, \Delta) \quad \text{where} \quad |R_h^{(i)}(x, \Delta)| \leq (2|\Delta|)^{\beta+2-i}. \]  

(101)

To bound \( N_\infty(\delta, S_{\beta+2}^\dagger) \) from above, we fix \( \delta > 0 \) and \( x \in (0,1) \). Suppose that for some \( \delta_0, \ldots, \delta_{\beta+1} > 0 \), \( h, g \in S_{\beta+2}^\dagger \) satisfy
\[ |h^{(k)}(x) - g^{(k)}(x)| \leq \delta_k \quad \text{for all} \quad k = 0, \ldots, \beta + 1. \]

We can bound \( |h(x + \Delta) - g(x + \Delta)| \) from above for some \( \Delta \) such that \( x + \Delta \in (0,1) \) as follows:
\[ |h(x + \Delta) - g(x + \Delta)| = \left| \sum_{k=0}^{\beta+1} \frac{\Delta^k}{k!} (h^{(k)}(x) - g^{(k)}(x)) + R_h(x, \Delta) - R_g(x, \Delta) \right| \leq \sum_{k=0}^{\beta+1} \frac{|\Delta|^k \delta_k}{k!} + 2(2|\Delta|)^{\beta+2}. \]

If \( |\Delta| \leq \frac{(\frac{\delta}{5})^{\beta+2}}{2} \) and \( \delta_k = \left( \frac{\delta}{5} \right)^{1 - \frac{k}{\beta+2}} \), we then have
\[ |h(x + \Delta) - g(x + \Delta)| \leq \frac{\delta}{5} \left( \sum_{k=0}^{\beta+1} \frac{1}{k!} + 2 \right) \leq \delta. \]  

(102)

In other words, by considering a grid of points \( \left( \frac{\delta}{5} \right)^{\beta+2} \)-apart in \((0,1)\) and covering the \( k \)th derivative of functions in \( S_{\beta+2}^\dagger \) within \( \delta_k = \left( \frac{\delta}{5} \right)^{1 - \frac{k}{\beta+2}} \) at each point, we can then obtain a \( \delta \) cover in the sup-norm for \( S_{\beta+2}^\dagger \). Let \( x_1 < \cdots < x_s \) be a \( \left( \frac{\delta}{5} \right)^{\beta+2} \)-grid of points in \((0,1)\) with \( s \leq 2 \left( \frac{\delta}{5} \right)^{\beta+2} + 2 \). For each \( h_0 \in S_{\beta+2}^\dagger \), let us define
\[ \mathbb{H}(h_0) := \left\{ h \in S_{\beta+2}^\dagger : \left| \frac{h^{(k)}(x_i)}{\delta_k} \right| = \left| \frac{h_0^{(k)}(x_i)}{\delta_k} \right|, 1 \leq i \leq s, 0 \leq k \leq \beta + 1 \right\} \]

where \( |x| \) means the largest integer smaller than or equal to \( x \). Our earlier argument implies that the number of distinct sets \( \mathbb{H}(h_0) \) with \( h_0 \) ranging over \( S_{\beta+2}^\dagger \) bounds the \( \delta \)-covering number of \( S_{\beta+2}^\dagger \) from above. Note that for \( i = 1, \ldots, s \) and \( k = 0, \ldots, \beta + 1 \), \( \mathbb{H}(h_0) \) depends on \( \left| \frac{h_0^{(k)}(x_i)}{\delta_k} \right| \) only and the number of distinct sets \( \mathbb{H}(h_0) \) is bounded above by the cardinality of
\[ I := \left\{ \left[ \frac{h^{(k)} (x_i)}{\delta_k} \right], 1 \leq i \leq s \text{ and } 0 \leq k \leq \beta + 1 \right\} : h \in S_{\beta+2}^t \right\}. \] (103)

Starting from \( x_1 \), let us count the number of possible values of the vector

\[ \left( \left[ \frac{h^{(k)} (x_1)}{\delta_k} \right], 0 \leq k \leq \beta + 1 \right) \]

with \( h \) ranging over \( S_{\beta+2}^t \). Since \(|h(x_1)| \leq C, |h^{(1)} (x_1)| \leq 1, |h^{(k)} (x_1)| \leq 2^{k-1}(k - 1)! \) for \( 2 \leq k \leq \beta + 1 \), this number is at most

\[ \frac{C}{\delta_0 \delta_1 \delta_2 \ldots} \frac{2^{\beta-1}(\beta - 1)!}{\delta_\beta} \frac{2^\beta \beta!}{\delta_{\beta+1}} \leq \left( \frac{\delta}{5} \right)^{\frac{\beta+3}{2}} C \frac{2^{\beta+1} \beta!}{2} \prod_{i=0}^{\beta} i! \] (104)

Now we move to \( x_2 \). Given the values of \( \left( \left[ h^{(k)} (x_1) / \delta_k \right], 0 \leq k \leq \beta + 1 \right) \), we count the number of possible values of the vector

\[ \left( \left[ \frac{h^{(k)} (x_2)}{\delta_k} \right], 0 \leq k \leq \beta + 1 \right). \]

For each \( 0 \leq k \leq \beta + 1 \), we define

\[ A_k := \left[ \frac{h^{(k)} (x_1)}{\delta_k} \right] \text{ such that } A_k \delta_k \leq h^{(k)} (x_1) < (A_k + 1) \delta_k. \]

Let us fix \( 0 \leq i \leq \beta + 1 \). Applying (101) with \( x = x_1 \) and \( \Delta = x_2 - x_1 \) yields

\[ \left| h^{(i)} (x_2) - \sum_{k=0}^{\beta+1-i} \frac{\Delta_i}{k!} h^{(i+k)} (x_1) \right| \leq |2\Delta|^{\beta+2-i}. \]

Consequently we have

\[
\begin{align*}
& \left| h^{(i)} (x_2) - \sum_{k=0}^{\beta+1-i} \frac{\Delta_k}{k!} A_{i+k} \right| \\
& \leq \left| h^{(i)} (x_2) - \sum_{k=0}^{\beta+1-i} \frac{\Delta_k}{k!} h^{(i+k)} (x_1) \right| + \left| \sum_{k=0}^{\beta+1-i} \frac{\Delta_k}{k!} \left( h^{(i+k)} (x_1) - A_{i+k} \right) \right| \\
& \leq |2\Delta|^{\beta+2-i} + \sum_{k=0}^{\beta+1-i} \left| \frac{\Delta_k}{k!} \right| \delta_{i+k} \\
& \leq \left( \frac{\delta}{5} \right)^{1-\frac{i}{\beta+2}} + \left( \frac{\delta}{5} \right)^{1-\frac{i}{\beta+2}} = 2\delta_i
\end{align*}
\]

(recalling \(|\Delta| = |x_2 - x_1| = \left( \frac{\delta}{5} \right)^{\frac{\beta+2}{2}} \)). Therefore, given the values of \( \left( \left[ h^{(k)} (x_1) / \delta_k \right], 0 \leq k \leq \beta + 1 \right) \), \( h^{(i)} (x_2) \) takes values in an interval whose length is no greater than \( 2\delta_i \). Therefore, the number of
possible values of \((\frac{h(k)(x_k)}{\sigma_k}, 0 \leq k \leq \beta + 1)\) is at most 2. The same argument goes through when \(x_1\) and \(x_2\) are replaced with \(x_j\) and \(x_{j+1}\) for any \(j = 1, \ldots, s - 1\). This result along with (104) gives

\[
|I| \leq 2^s \left( \frac{\delta}{5} \right)^{-\frac{\beta+1}{2}} 2^\frac{(\beta+1)\delta}{2} \prod_{i=0}^{\beta} i!
\]

\[
\leq 4\left( \frac{\delta}{5} \right)^{-\frac{\beta+1}{2}} 2^\frac{(\beta+1)\delta}{2} \prod_{i=0}^{\beta} i!
\]

\[
\leq \exp \left[ \log \left( C \prod_{i=0}^{\beta} i! \right) + \frac{\beta + 3}{2} \log 2 - \frac{\beta + 3}{2} \log \frac{5}{3} + \left( \frac{\delta}{5} \right)^{-\frac{1}{\beta+2}} \log 21 + \log 4 \right]
\]

where \(I\) is defined in (103).

A.7 Theorem 2.2

**Proof.** Let us show the part \(T_1\) first. Since \(Y \subseteq S_{\beta+2}\), we let \(F = S_{\beta+2}\) in (77). Because we are working with \(F\) in (77) in terms of \(F, S_{\beta+2}\) all the coefficients \(2^k (k - 1)!\) associated with the derivatives for \(k = 1, \ldots, \beta + 1\), the coefficient \(2^{\beta+1} (\beta + 1)!\) associated with the Lipschitz condition, as well as the function value itself are multiplied by 2. Slight modifications of the proof for Lemma A.2 give

\[
|I| \leq 3^s \left( \frac{\delta}{7} \right)^{-\frac{\beta+1}{2}} 2^\frac{(\beta^2 + 3\beta + 2)}{2} \prod_{i=0}^{\beta} i!
\]

\[
\leq 9\left( \frac{\delta}{7} \right)^{-\frac{\beta+1}{2}} 2^\frac{(\beta^2 + 3\beta + 2)}{2} \prod_{i=0}^{\beta} i!
\]

\[
\leq \exp \left[ \log \left( C \prod_{i=0}^{\beta} i! \right) + \left( \frac{\beta^2 + 3\beta + 2}{2} \right) \log 2 - \frac{\beta + 3}{2} \log \frac{7}{5} + \left( \frac{\delta}{7} \right)^{-\frac{1}{\beta+2}} \log 21 + \log 9 \right].
\]

For every \(\gamma \in \{0, \ldots, \beta\}\), let \(\delta (\gamma) = c \left( \sqrt{\frac{\beta}{\pi \sigma^2}} \right)^{2(\gamma+2)+1} \frac{\sqrt{2\pi}}{\sqrt{\sigma}}\) and \(\gamma^* (\leq \beta)\) be the largest non-negative integer such that

\[
\log \left( \prod_{i=0}^{\gamma^*} i! \right) + \frac{\gamma^*}{2} + \frac{3}{2} \log 2 - \frac{\gamma^*}{2} + \frac{3}{2} \log 2 \leq \left( \frac{\delta (\gamma^*)}{7} \right)^{-\frac{1}{\gamma^*+1}} \log 21 + \max \left\{ 0, \log \left( \frac{36C}{\gamma^*} \right) \right\}.
\]

(105)

Note that the LHS of (105) is a strictly increasing function of \(\gamma^*\) (since \(\frac{\delta (\gamma^*)}{7} \in (0, 1)\) and the larger \(\gamma^*\) is, the more negative \(\log (\gamma^*)\) is) and the RHS is a strictly decreasing function of \(\gamma^*\), and the LHS is smaller than the RHS for any \(\frac{\delta (\gamma^*)}{7} \in (0, 1)\) when \(\gamma^* = 0\) (to see this, note that \(LHS = 3 \log \left( \frac{\delta}{5} \right)^{-\frac{1}{2}}\) and \(RHS \geq \left( \frac{\delta}{\gamma} \right)^{-\frac{1}{2}} \log 21 \geq 3 \left( \frac{\delta}{\gamma} \right)^{-\frac{1}{2}}\)). Therefore, the largest non-negative solution \(\gamma^* (\leq \beta)\) to (105) exists (i.e., \(\gamma^*\) is well defined).
Observe that the choice \( \bar{r}_n = \delta (\gamma^*) =: \delta^* \) solves (80) because
\[
\frac{1}{\sqrt{n}} \int_{\frac{\delta^*}{2}}^{\delta^*} \sqrt{\log N_n (\delta, \Omega (\delta^*; \bar{F}))} \, d\delta \leq \frac{1}{\sqrt{n}} \int_{0}^{\delta^*} \sqrt{\log N_\infty (\delta, \bar{F})} \, d\delta \\
\leq \frac{1}{\sqrt{n}} \int_{0}^{\delta^*} \sqrt{\left( \frac{\delta}{7} \right)^{\frac{1}{7+1}} \, d\delta \\
\leq \frac{1}{\sqrt{n}} \left( \frac{2}{\sqrt{1+7}} \delta + 4 \right) \leq 2 \gamma + 4 \sqrt{\delta^* + 1} \\
\leq \frac{\delta^2}{\sigma}.
\]

The second line is argued as follows: When \( \delta = \delta^* \), the critical smoothness parameter is \( \gamma^* \) and \( \log N_\infty (\delta, \bar{F}) \lesssim \left( \frac{\delta}{7} \right)^{\frac{1}{7+1}} \); as \( \delta \) decreases from \( \delta^* \), by (35) in Assumption G, the term \( \left( \frac{\delta}{7} \right)^{\frac{1}{7+1}} \) in the right-hand-side of (105) increases at a faster rate than the term \(- \left( \frac{\gamma+3}{2} \right) \log \frac{\gamma}{\delta} \) in the left-hand-side. Now the left-hand-side can accommodate for a new critical smoothness parameter \( \gamma \) such that \( \gamma \geq \gamma^* \). As a result, \( \log N_\infty (\delta, \bar{F}) \lesssim \left( \frac{\delta}{7} \right)^{\frac{1}{7+1}} \) for any \( \delta \leq \delta^* \).

By Theorem 13.5 in Wainwright (2019), in terms of (31), we have
\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \lesssim \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\gamma^*+2)+1}}
\]
with probability at least \( 1 - \exp \left( \frac{-n\delta^2}{2\sigma^2} \right) \). Integrating the tail bound yields
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \right] \lesssim \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma^*+2)}{2(\gamma^*+2)+1}} + \epsilon \exp \left\{ -c_2 n \sigma^{-2} [\delta (\gamma^*)]^2 \right\}.
\]

We now show the part \( T_2 \). For a given \( \delta > 0 \), let us consider the smallest \( \frac{\delta}{2C_{\max}} \)-covering \( \{ f^1, \ldots, f^N \} \) (w.r.t. the sup-norm) of \( S_{\beta+1,2} (1, \Xi) \) and the smallest \( \frac{\delta}{2C_{\max}} \)-covering \( \{ y_{0,1}, \ldots, y_{0,N'} \} \) for the interval \([-C_0, C_0]\) where the initial value lies. By Theorem IV and arguments similar to those in the proof for Theorem 2.1(i), for any \( f, \bar{f} \in S_{\beta+1,2} (1, \Xi) \) and initial values \( y_0, \bar{y}_0 \in [-C_0, C_0] \), we can find some \( f^i, f^j \in \{ f^1, \ldots, f^N \} \) and \( y_{0,i'}, y_{0,j'} \in \{ y_{0,1}, \ldots, y_{0,N'} \} \) such that
\[
\left| y (x) - \bar{y} (x) - (y_{i,i'} (x) - y_{j,j'} (x)) \right| \\
\leq \left| y (x) - y_{i,i'} (x) \right| + \left| \bar{y} (x) - y_{j,j'} (x) \right| \\
\leq \delta
\]
where \( y, \bar{y}, y_{i,i'} \), and \( y_{j,j'} \) are solutions to the ODE associated with \( \{ f, y_0 \}, \{ \bar{f}, \bar{y}_0 \}, \{ f^i, y_{0,i'} \} \) and \( \{ f^j, y_{0,j'} \} \), respectively. Thus, we obtain forms a \( \delta \)-cover of \( \bar{F} \) in terms of \( F = \mathcal{Y} \).
For $\beta > 0$, we have
\[
\frac{1}{\sqrt{n}} \int_0^{\tilde{r}_n} \sqrt{\log N_n(\delta, \Omega(\tilde{r}_n; \mathcal{F}))} d\delta \leq \frac{1}{\sqrt{n}} \int_0^{\tilde{r}_n} \sqrt{\log N_\infty(\delta, \Omega(\tilde{r}_n; \mathcal{F}))} d\delta \\
\leq \frac{1}{\sqrt{n}} \int_0^{\tilde{r}_n} \left( \frac{\delta}{2C_{\max}} \right)^{\frac{2}{\beta+1}} d\delta \\
= \sqrt{\frac{2}{n}} \left( 2C_{\max} \right)^{\frac{\beta+1}{\beta+2}} \frac{1}{\beta} \tilde{r}_n^{\frac{\beta}{\beta+1}}
\] (106)
\[
\tilde{r}_n \asymp \frac{\sigma}{\sqrt{n}}.
\]

Setting $\sqrt{\frac{2}{n}} \left( 2C_{\max} \right)^{\frac{\beta+1}{\beta+2}} \frac{1}{\beta} \tilde{r}_n^{\frac{\beta}{\beta+1}} \asymp \frac{\sigma^2}{\beta}$ yields
\[
\tilde{r}_n \asymp C_{\max} \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{\beta+1}{\beta+2}}.
\]

(Note that (106) follows from (36) in Assumption G.) Consequently, in terms of (31), we have
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \hat{y}(x_i) - y^*(x_i) \right)^2 \lesssim C_{\max}^{\frac{2}{\beta+2}} \left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\beta+2}}
\]
with probability at least $1 - \exp \left( -\frac{n}{2\sigma^2} C_{\max}^{\frac{2}{\beta+2}} \left( \frac{\sigma}{\sqrt{n}} \right)^{\frac{\beta+1}{\beta+2}} \right)$. Integrating the tail bound yields
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} (\hat{y}(x_i) - y^*(x_i))^2 \right] \lesssim C_{\max}^{\frac{2}{\beta+2}} \left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\beta+2}} + c' \exp \left\{ -c'' n \sigma^2 \left( \frac{\sigma^2}{n} \right)^{\frac{\beta+1}{\beta+2}} \right\}.
\]

A.8 Proposition 2.4

Proof. When $f \in S_{1,2}(1, \Xi)$, the critical smoothness parameter $\gamma^* = 0$. Hence, (43) is a standard oracle inequality (see, e.g., Theorem 13.13 in Wainwright, 2019). The terms $E_1(x_i)$ and $E_2(x_i)$ come from Theorem IV. Note that Theorem IV does not require $g$ to satisfy a Lipschitz condition.

A.9 Proposition 2.5

Proof. Like in Proposition 2.3, we write $\hat{y}_{R+1}(x_i; \hat{f}) - y^*(x_i; f^*) = \sum_{j=1}^{3} T_j(x_i)$ where
\[
T_1(x_i) = \hat{y}_{R+1}(x_i; \hat{f}) - \hat{y}_{R+1}(x_i; f^*), \quad \text{estimation error due to } \hat{f}
\]
\[
T_2(x_i) = \hat{y}_{R+1}(x_i; f^*) - y^*_R(x_i; f^*), \quad \text{estimation error due to } \hat{y}_0
\]
\[
T_3(x_i) = y^*_R(x_i; f^*) - y^*(x_i; f^*), \quad \text{estimation error due to the finite iterations},
\]
where
\[
\begin{align*}
y_0^* &= y_0, \\
y_1^* (x; f^*) &= y_0^* + \int_0^x f (s, y_0^*; f^*) \, ds, \\
y_2^* (x; f^*) &= y_0^* + \int_0^x f (s, y_1^* (s; f^*); f^*) \, ds, \\
&\vdots \\
y_{R+1}^* (x; f^*) &= y_0^* + \int_0^x f (s, y_R^* (s; f^*); f^*) \, ds.
\end{align*}
\]

As a result, we have
\[
\left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i \left( \hat{y}_{R+1} (x_i; \hat{f}) - y^* (x_i; f^*) \right) \right| \lesssim \sqrt{\frac{1}{n} \sum_{i=1}^n |T_1 (x_i)|} + \sigma \sqrt{\frac{1}{n} \sum_{i=1}^n |T_2 (x_i)|} + \sigma \sqrt{\frac{1}{n} \sum_{i=1}^n |T_3 (x_i)|}.
\]

Following the argument in the proof for Proposition 2.3, we have
\[
\sqrt{\frac{1}{n} \sum_{i=1}^n |T_2 (x_i)|} \leq \frac{1}{1 - \bar{\alpha}} |\hat{y}_0 - y_0^*|,
\]
\[
\sqrt{\frac{1}{n} \sum_{i=1}^n |T_3 (x_i)|} \lesssim \bar{\alpha}^{R+1} \max \{C_0, \bar{\alpha} \}.
\]

It remains to analyze \( \frac{1}{n} \sum_{i=1}^n \varepsilon_i T_1 (x_i) \) in (107). In (107), let
\[
\mathcal{F} = \{ g_f (x) = \hat{y}_{R+1} (x; f) : f \in \mathcal{S}_{\beta+1,2} (1, \Xi), x \in [0, \bar{\alpha}] \}
\]
where \( \hat{y}_{R+1} (s; f) \) is constructed in the following fashion:
\[
\begin{align*}
\hat{y}_0 &= \hat{y}_0, \\
\hat{y}_1 (x; f) &= \hat{y}_0 + \int_0^x f (s, \hat{y}_0) \, ds, \\
\hat{y}_2 (x; f) &= \hat{y}_0 + \int_0^x f (s, \hat{y}_1 (s; f)) \, ds, \\
&\vdots \\
\hat{y}_{R+1} (x; f) &= \hat{y}_0 + \int_0^x f (s, \hat{y}_R (s; f)) \, ds.
\end{align*}
\]

Consider another \( f \in \mathcal{S}_{\beta+1,2} (1) \) and let \( \hat{y}_1 (x; \hat{f}) , \ldots , \hat{y}_{R+1} (x; \hat{f}) \) constructed according to the fashion above. At the beginning, we have
\[
\hat{y}_1 (x; f) - \hat{y}_1 (x; \hat{f}) = \int_0^x \left[ f (s, \hat{y}_0) - \hat{f} (s, \hat{y}_0) \right] \, ds \\
\leq \bar{\alpha} |f - \hat{f}| \quad \forall x \in [0, \bar{\alpha}] .
\]

46
For the second and third iterations, we have

\[
\begin{align*}
\left| \hat{y}_2 (x; f) - \hat{y}_2 (x; \bar{f}) \right| & \leq \left| \int_0^x f (s, \hat{y}_1 (s; f)) \, ds - \int_0^x f (s, \hat{y}_1 (s; \bar{f})) \, ds \right| \\
& \quad + \left| \int_0^x f (s, \hat{y}_1 (s; \bar{f})) \, ds - \int_0^x \bar{f} (s, \hat{y}_1 (s; \bar{f})) \, ds \right| \\
& \leq \bar{\alpha} \sup_{s \in [0, \bar{\alpha}]} \left| \hat{y}_1 (s; f) - \hat{y}_1 (s; \bar{f}) \right| + \bar{\alpha} \left| f - \bar{f} \right|_\infty \\
& \leq \bar{\alpha}^2 \left| f - \bar{f} \right|_\infty + \bar{\alpha} \left| f - \bar{f} \right|_\infty \quad \forall x \in [0, \bar{\alpha}]
\end{align*}
\]

and

\[
\begin{align*}
\left| \hat{y}_3 (x; f) - \hat{y}_3 (x; \bar{f}) \right| & \leq \left| \int_0^x f (s, \hat{y}_2 (s; f)) \, ds - \int_0^x f (s, \hat{y}_2 (s; \bar{f})) \, ds \right| \\
& \quad + \left| \int_0^x f (s, \hat{y}_2 (s; \bar{f})) \, ds - \int_0^x \bar{f} (s, \hat{y}_2 (s; \bar{f})) \, ds \right| \\
& \leq \bar{\alpha} \sup_{s \in [0, \bar{\alpha}]} \left| \hat{y}_2 (s; f) - \hat{y}_2 (s; \bar{f}) \right| + \bar{\alpha} \left| f - \bar{f} \right|_\infty \\
& \leq \bar{\alpha}^3 \left| f - \bar{f} \right|_\infty + \bar{\alpha}^2 \left| f - \bar{f} \right|_\infty + \bar{\alpha} \left| f - \bar{f} \right|_\infty \quad \forall x \in [0, \bar{\alpha}].
\end{align*}
\]

Continuing with this pattern until the \((R + 1)\)th iteration, we obtain

\[
\left| \hat{y}_{R+1} (x; f) - \hat{y}_{R+1} (x; \bar{f}) \right| \leq \frac{\bar{\alpha}}{1 - \bar{\alpha}} \left| f - \bar{f} \right|_\infty \quad \forall x \in [0, \bar{\alpha}]. \tag{109}
\]

In particular, \(109\) holds for \(x \in \{x_1, x_2, \ldots, x_n\}\). Consequently,

\[
\left\{ \frac{1}{n} \sum_{i=1}^n \left| \hat{y}_{R+1} (x; f) - \hat{y}_{R+1} (x; \bar{f}) \right|^2 \right\}^{\frac{1}{2}} \leq \frac{\bar{\alpha}}{1 - \bar{\alpha}} \left| f - \bar{f} \right|_\infty.
\]

Let \(\tilde{b} = \frac{\bar{\alpha}}{1 - \bar{\alpha}}\). For a given \(\delta > 0\), let us consider the smallest \(\tilde{b} \cdot \frac{\delta}{20}\)-covering \(\{f^1, \ldots, f^N\}\) (with respect to the sup-norm) of \(S_{\beta+1, 2} (1, \Xi)\), and by \((109)\), for any \(f, \bar{f} \in S_{\beta+1, 2} (1, \Xi)\), we can find some \(f^i\) and \(f^j\) from the covering set \(\{f^1, \ldots, f^N\}\) such that

\[
\begin{align*}
& \left| \hat{y}_{R+1} (x; f) - \hat{y}_{R+1} (x; \bar{f}) \right| - \left| \left( \hat{y}_{R+1} (x; f^i) - \hat{y}_{R+1} (x; f^j) \right) \right| \\
& \leq \left| \hat{y}_{R+1} (x; f) - \hat{y}_{R+1} (x; f^i) \right| + \left| \hat{y}_{R+1} (x; f^i) - \hat{y}_{R+1} (x; f^j) \right| \\
& \leq \delta.
\end{align*}
\]

Thus, \(\{g_{f^1}, g_{f^2}, \ldots, g_{f^N}\} \times \{g_{f^1}, g_{f^2}, \ldots, g_{f^N}\}\) forms a \(\delta\)–cover of \(\tilde{F}\) in terms of \(F\) defined in \((108)\). For \(\beta > 0\), we have

\[
\frac{1}{\sqrt{n}} \int_0^{\hat{x} n} \sqrt{\log N_n (\delta, \Omega (\hat{r}_n; \bar{F}))} \, d\delta \leq \frac{1}{\sqrt{n}} \int_0^{\hat{x} n} \sqrt{\log N_\infty (\delta, \Omega (\hat{r}_n; \bar{F}))} \, d\delta \leq \frac{1}{\sqrt{n}} \int_0^{\hat{x} n} \sqrt{2 \left( \frac{\delta}{2b} \right)^{\frac{\beta - 1}{\beta + 1}}} \, d\delta \leq \sqrt{\frac{2}{n}} \left( \frac{\beta + 1}{\beta} \frac{\beta - 1}{\beta + 1} \right) \frac{\hat{x} n}{\bar{r}_n} \frac{\beta}{\sigma}.
\]

47
Setting $\sqrt{2/n} \left(2\tilde{b}\right)^{\frac{1}{\beta+1}} \times \beta + 1 \tilde{r}_{n}^{\beta} \times \tilde{r}_{n}^{\beta} \times \tilde{r}_{n}^{\beta} \times \tilde{r}_{n}^{\beta}$ yields $\tilde{r}_{n} \sim \tilde{b}^{\frac{1}{\beta+1}}$ and therefore,

$$\left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} T_{1}(x_{i}) \right| \lesssim \tilde{b}^{\frac{2}{\beta+2}} \left( \frac{\sigma^{2}}{n} \right)^{\frac{\beta+1}{\beta+2}}$$

with probability at least $1 - c_{1} \exp \left( -c_{2} n \sigma^{2} \tilde{b}^{\frac{2}{\beta+2}} \left( \frac{\sigma^{2}}{n} \right)^{\frac{\beta+1}{\beta+2}} \right)$.  

### A.10 Theorems 2.3-2.4, Proposition 2.6

With slight modifications, the proofs for Theorems 2.3-2.4 and Proposition 2.6 are almost identical to those for Theorems 2.1-2.2 and Proposition 2.5. Because most of these modifications are straightforward, we only point out the main differences in the proofs for Theorems 2.3 and 2.4. As in the proof for part(ii) of Theorem 2.1, the proof for part (ii) of Theorem 2.3 consists of three steps.

**Step 1:** In Lemma A.1(i) (subsection A.6.2), we have established $\mathcal{Y} \in \mathcal{A} \mathcal{S}_{\beta+2}^{\dagger}$ where $\mathcal{A} \mathcal{S}_{\beta+2}^{\dagger}$ denotes the class of functions such that any function $h \in \mathcal{S}_{\beta+2}^{\dagger}$ satisfies the following properties:

- $h$ is continuous on $[0, 1]$ and differentiable $\beta + 1$ times;
- $|h(x)| \leq C_{\beta}$, and $|h^{(k)}(x)| \leq (k - 1)!$ for all $k = 1, ..., \beta + 1$ and $x \in [0, 1]$;$
- |h^{(\beta+1)}(x) - h^{(\beta+1)}(x')| \leq |x - x'|$ for all $x, x' \in [0, 1]$.

**Step 2:** Following the argument in the proof for Lemma A.2, we establish a crude bound as follows:

$$N_{\infty}(\delta, \mathcal{A} \mathcal{S}_{\beta+2}^{\dagger}) \leq \exp \left[ \log \left( \frac{\beta}{\prod_{i=0}^{\beta} i!} \right) - \frac{\beta + 3}{2} \log \delta + \left( \frac{\delta}{5} \right)^{\frac{1}{(\beta+2)\log 21 + \log 4}} \right].$$

(111)

**Step 3:** The third step refines the crude bound (111). For a given $\frac{\delta}{5} \in (0, 1)$, let $\beta^{*}(\delta) = \gamma$ be the largest non-negative integer such that

$$\log \left( \prod_{i=0}^{\gamma} i! \right) + \gamma^{*} \log 2 - \frac{\gamma^{*} + 3}{2} \log \frac{\delta}{5} \leq \left( \frac{\delta}{5} \right)^{\frac{1}{\gamma+2} \log 21 + \max \{0, \log (4C) \}}.$$  

(112)

As for Theorem 2.4, we replace (34) with

$$\log \left( \prod_{i=0}^{\gamma} i! \right) + \gamma^{*} \log 2 - \frac{\gamma^{*} + 3}{2} \log \frac{\delta}{7} \leq \left[ \frac{\delta}{7} \right]^{\frac{1}{\gamma+2} \log 21 + \max \{0, \log (36C) \}}.$$  

### A.11 Proposition 2.7

**Proof.** Let $L_{\text{max}} = \sup_{x \in [0, a]} \left\{ \exp \left( x \sqrt{L_{m}^{2} + 1} \right) \left[ 1 + \int_{0}^{x} \exp \left( -s \sqrt{L_{m}^{2} + 1} \right) ds \right] \right\}$. For a given $\delta > 0$, we consider the smallest $\frac{\delta}{L_{\text{max}}}$—covering of $\mathcal{F}$ with respect to the sup-norm. We also consider the smallest $\frac{\delta}{L_{\text{max}}}$—covering of $\mathcal{B}_{2}(C_{0}) := \{ \theta \in \mathbb{R}^{m} : |\theta|_{2} \leq C_{0} \}$ (where the initial values lie) with respect to the $l_{2}$—norm. Note that the standard volumetric argument yields

$$\log N_{2} \left( \frac{\delta}{L_{\text{max}}}, \mathcal{B}_{2}(C_{0}) \right) \leq m \log \left( \frac{2C_{0}L_{\text{max}}}{\delta} + 1 \right).$$  

48
By Theorem B.1, for any \( y \in \mathcal{Y} \) with \( Y_0 \in \mathbb{B}_2 (C_0) \), we can find an element (indexed by \( i \)) from the smallest \( \frac{\delta}{L_{\max}} \)-covering of \( \mathcal{F} \) and an element (indexed by \( i' \)) from the smallest \( \frac{\delta}{L_{\max}} \)-covering of \( \mathbb{B}_2 (C_0) \) such that
\[
\left| y(x) - y_{(i,i')} (x) \right| \leq \frac{\delta}{L_{\max}} \exp \left( x \sqrt{L_{m}^2 + 1} \right) \left[ 1 + \int_0^x \exp \left( -s \sqrt{L_{m}^2 + 1} \right) ds \right] \leq \delta \quad \forall x \in [0, \alpha]
\]
where \( y_{(i,i')} \) is a solution to the ODE associated with \( f_i \) and the initial value \( Y_{0,i} \) from the covering sets. Consequently, we obtain a \( \delta \)-cover of \( \mathcal{Y} \). We conclude that
\[
\log N_\infty (\delta, \mathcal{Y}) \leq \log N_\infty \left( \frac{\delta}{L_{\max}}, \mathcal{F} \right) + m \log \left( \frac{2C_0L_{\max}}{\delta} + 1 \right).
\]

**B. Additional technical results and proofs**

**B.1 Gronwall inequalities**

**Theorem IV** (Gronwall inequality for first order ODEs). Consider the following pair of ODEs:
\[
y' (x) = f (x, y (x)), \quad y (0) = y_0,
\]
and
\[
z' (x) = g (x, z (x)), \quad z (0) = z_0,
\]
with \( |y_0|, |z_0| \leq C_0 \), and \((x, y(x)), (x, z(x)) \in \Lambda := [0, 1] \times [-C_0 - b, C_0 + b] \). Suppose \( f \) and \( g \) are continuous on \( \Lambda \); for all \((x, y), (x, \tilde{y}) \in \Lambda\),
\[
|f(x, y) - f(x, \tilde{y})| \leq L_1 |y - \tilde{y}|.
\]
Assume there is a continuous function \( \varphi : [0, 1] \to [0, \infty) \) such that
\[
|f(x, y(x)) - g(x, y(x))| \leq \varphi(x).
\]
Then we have
\[
|y(x) - z(x)| \leq \exp (L_1 x) \int_0^x \exp (-L_1 s) \varphi(s) ds + \exp (L_1 x) |y_0 - z_0|
\]
for \( x \in \left[ 0, \min \left\{ 1, \frac{\alpha}{M} \right\} \right] \) where \( M = \max \left\{ \max_{(x, y) \in \Lambda} |f(x, y)|, \max_{(x, z) \in \Lambda} |g(x, z)| \right\} \).

**Remark.** Theorem IV is a slight modification of Theorem 2.1 in Howard (1998), which gives a variant of the famous Gronwall inequality (Gronwall, 1919).

In the following result, we extend Theorem IV to higher order ODEs. Let
\[
Y (x) = \begin{bmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(m-1)}(x) \end{bmatrix} \quad \text{and} \quad Z (x) = \begin{bmatrix} z(x) \\ z'(x) \\ \vdots \\ z^{(m-1)}(x) \end{bmatrix}
\]
with
\[
Y_0 := \begin{bmatrix} y(0) \\ y'(0) \\ \vdots \\ y^{(m-1)}(0) \end{bmatrix} \quad \text{and} \quad Z_0 := \begin{bmatrix} z(0) \\ z'(0) \\ \vdots \\ z^{(m-1)}(0) \end{bmatrix}.
\]
In addition, let
\[ \bar{\Gamma} := \{(x, Y) : x \in [0, 1], |Y|_2 \leq b + C_0 \} . \]

**Theorem B.1** (Gronwall inequality for higher order ODEs). Consider the following pair of ODEs:
\[
\begin{align*}
\dot{y}^{(m)}(x) &= f\left(x, y(x), y'(x), \ldots, y^{(m-1)}(x)\right), \\
y(0) &= y(0), \quad y'(0) = y(1), \quad \ldots, \quad y^{(m-1)}(0) = y(m-1),
\end{align*}
\]  \hspace{1cm} (115)
and
\[
\begin{align*}
\dot{z}^{(m)}(x) &= g\left(x, z(x), z'(x), \ldots, z^{(m-1)}(x)\right), \\
z(0) &= z(0), \quad z'(0) = z(1), \quad \ldots, \quad z^{(m-1)}(0) = z(m-1),
\end{align*}
\]  \hspace{1cm} (116)
with \(|Y_0|_2, |Z_0|_2 \leq C_0\) and \((x, Y(x)), (x, Z(x)) \in \bar{\Gamma}\). Suppose \(f, g\) are continuous on \(\bar{\Gamma}\); and
\[
|f(x, Y) - f(x, \tilde{Y})| \leq L_m |Y - \tilde{Y}|_2
\]  \hspace{1cm} (117)
for all \((x, Y) := (x, y, \ldots, y^{(m-1)})\) and \((x, \tilde{Y}) := (x, \tilde{y}, \ldots, \tilde{y}^{(m-1)})\) in \(\bar{\Gamma}\). Assume there is a continuous function \(\varphi : [0, 1] \to [0, \infty)\) such that
\[
|f(x, Y(x)) - g(x, Y(x))| \leq \varphi(x).
\]  \hspace{1cm} (118)

Then we have
\[
|y(x) - z(x)| \leq \exp\left(x\sqrt{L_m^2 + 1}\right) \int_0^x \exp\left(-s\sqrt{L_m^2 + 1}\right) \varphi(s) \, ds + \exp\left(x\sqrt{L_m^2 + 1}\right) |Y_0 - Z_0|_2
\]
for \(x \in \left[0, \min\left\{1, \frac{b}{M}\right\}\right]\) where 
\[
M = \max \left\{\max_{(x, Y) \in \Gamma} |f(x, Y)|, \max_{(x, Z) \in \Gamma} |g(x, Z)|\right\}.
\]

**Proof.** Let \(W = (w_j)_{j=0}^{m-1}\) and
\[
F(x, W) := \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{m-1} \\ f(x, W) \end{bmatrix}.
\]
We transform (115) and (116) into
\[
\begin{align*}
Y'(x) &= F(x, Y(x)), \\
Z'(x) &= G(x, Z(x)),
\end{align*}
\]
where
\[
F(x, Y(x)) := \begin{bmatrix} y'(x) \\ y^{(2)}(x) \\ \vdots \\ y^{(m-1)}(x) \\ f\left(x, y(x), y'(x), \ldots, y^{(m-1)}(x)\right) \end{bmatrix}.
\]
and
\[ G(x, Z(x)) := \begin{bmatrix} z'(x) \\ z''(x) \\ \vdots \\ z^{(m-1)}(x) \\ g(x, z(x), z'(x), ..., z^{(m-1)}(x) \end{bmatrix}. \]

The following arguments extend those in Howard (1998). The inequality \( \frac{d}{ds} |Y(s)|_2 \leq |Y'(s)|_2 \), (117) and (118) yield
\[
\frac{d}{ds} |Y(s) - Z(s)|_2 \leq |Y'(s) - Z'(s)|_2
= |F(s, Y(s)) - G(s, Z(s))|_2
\leq |G(s, Z(s)) - F(s, Z(s))|_2 + |F(s, Z(s)) - F(s, Y(s))|_2
\leq \varphi(s) + \left[ L_m^2 |Y(s) - Z(s)|^2_2 + \sum_{k=1}^{m-1} \left( y^{(k)}(s) - z^{(k)}(s) \right)^2 \right]^\frac{1}{2}
\leq \varphi(s) + \sqrt{L_m^2 + 1} |Y(s) - Z(s)|_2,
\]
which is equivalent to
\[
\frac{d}{ds} |Y(s) - Z(s)|_2 - \sqrt{L_m^2 + 1} |Y(s) - Z(s)|_2 \leq \varphi(s).
\]
Multiplying both sides above by \( \exp \left( -s \sqrt{L_m^2 + 1} \right) \) gives
\[
\frac{d}{ds} \left[ \exp \left( -s \sqrt{L_m^2 + 1} \right) |Y(s) - Z(s)|_2 \right] \leq \exp \left( -s \sqrt{L_m^2 + 1} \right) \varphi(s).
\]
By a higher order version of Theorem I, there exist solutions \( y \) and \( z \) to (115) and (116), respectively, on \([0, \min \{1, \frac{h}{M}\}]\). Then integrating both sides of the inequality above from 0 to \( x \) gives
\[
\exp \left( -x \sqrt{L_m^2 + 1} \right) |Y(x) - Z(x)|_2 - |Y_0 - Z_0|_2 \leq \int_0^x \exp \left( -s \sqrt{L_m^2 + 1} \right) \varphi(s) ds
\]
for all \( x \in \left[0, \min \{1, \frac{h}{M}\}\right] \). The above implies that
\[
|y(x) - z(x)| \leq \exp \left( x \sqrt{L_m^2 + 1} \right) \int_0^x \exp \left( -s \sqrt{L_m^2 + 1} \right) \varphi(s) ds + \exp \left( x \sqrt{L_m^2 + 1} \right) |Y_0 - Z_0|_2
\]
for all \( x \in \left[0, \min \{1, \frac{h}{M}\}\right] \).

**Proposition B.1.** (i) Let \( f \) be a continuous and non-decreasing function on \([0, 1]\). Then, \( F(x) = c + \int_0^x f(t)dt \) is convex on \([0, 1]\). (ii) Let \( f \) be a continuous and non-increasing function on \([0, 1]\). Then, \( F(x) = c + \int_0^x f(t)dt \) is concave on \([0, 1]\).

**Proof.** We show (i) and the argument for (ii) is almost identical. Letting \( 0 \leq x_1 < x_2 \), we have
\[
F(x_2) - F \left( \frac{x_1 + x_2}{2} \right) = \int_{x_1 + x_2}^{x_2} f(t) dt
\]
51
and
\[ F \left( \frac{x_1 + x_2}{2} \right) - F(x_1) = \int_{x_1}^{x_1 + x_2} f(t) dt. \]

Since \( f(x) \) is non-decreasing, we have
\[ \int_{x_1 + x_2}^{x_2} f(t) dt \geq \int_{x_1}^{\frac{x_1 + x_2}{2}} f(t) dt \]
and therefore,
\[ F(x_2) - F \left( \frac{x_1 + x_2}{2} \right) \geq F \left( \frac{x_1 + x_2}{2} \right) - F(x_1) \]
which implies that
\[ \frac{F(x_1) + F(x_2)}{2} \geq F \left( \frac{x_1 + x_2}{2} \right). \]  \hspace{1cm} (119)

Since \( f \) is continuous, (119) implies that \( F(x) \) is convex.