STRONGLY KOSZUL EDGE RINGS

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ABSTRACT. We classify the finite connected simple graphs whose edge rings are strongly Koszul. From the classification, it follows that if the edge ring is strongly Koszul, then its toric ideal possesses a quadratic Gröbner basis.

INTRODUCTION

Edge rings, edge polytopes as well as toric ideals arising from finite simple graphs ([7], [8], [9]) are fashionable in the current trend of combinatorics and commutative algebra. The purpose of the present paper is to classify the finite connected simple graphs for which the edge ring $K[G]$ of $G$ is strongly Koszul (Theorem 2.8). From the classification, it follows that if $K[G]$ is strongly Koszul, then its toric ideal $I_G$ possesses a quadratic Gröbner basis (Corollary 2.10). It is unclear whether the toric ideal of a strongly Koszul toric ring possesses a quadratic Gröbner basis.

It seems that the class of strongly Koszul toric rings is rather small. In fact, the classification of strongly Koszul algebras has been achieved for (i) edge rings of bipartite graphs ([3, Theorem 4.5]), (ii) toric rings arising from finite distributive lattices ([3, Theorem 3.2]), and (iii) toric rings of stable set polytopes ([5, Theorem 5.1]). We refer the reader to, e.g., [1] and [2] for the background of Koszul algebras.

1. Strongly Koszul algebra

Let $K$ be a field and let $m = R_+$ be the homogeneous maximal ideal of a graded $K$-algebra $R$.

Definition 1.1 ([3]). A graded $K$-algebra $R$ is said to be strongly Koszul if $m$ has a minimal system of generators $\{u_1, \ldots, u_t\}$ which satisfies the following condition:

For all subsequences $u_{i_1}, \ldots, u_{i_r}$ of $\{u_1, \ldots, u_t\}$ ($i_1 \leq \cdots \leq i_r$) and for all $j = 1, \ldots, r - 1$, $(u_{i_1}, \ldots, u_{i_{j-1}}) : u_{i_j}$ is generated by a subset of elements of $\{u_1, \ldots, u_t\}$.

A graded $K$-algebra $R$ is called Koszul if $K = R/m$ has a linear resolution. By the following proposition ([3, Theorem 1.2]), we can see that a strongly Koszul algebra is Koszul.

Proposition 1.2. If $R$ is strongly Koszul with respect to the minimal homogeneous generators $\{u_1, \ldots, u_t\}$ of $m = R_+$, then for all subsequences $u_{i_1}, \ldots, u_{i_r}$ of $\{u_1, \ldots, u_t\}$, $R/(u_{i_1}, \ldots, u_{i_r})$ has a linear resolution.

2010 Mathematics Subject Classification. 13P20, 16S37.
Key words and phrases. strongly Koszul algebra, finite graph, edge ring.
Let $R = K[t_1, \ldots, t_n]$ be the polynomial ring in $n$ variables over $K$. Let $A$ be a homogeneous affine semigroup ring generated by monomials belonging to $R$. If $T$ is a nonempty subset of $[n] = \{1, \ldots, n\}$, then we write $RT$ for the polynomial ring $K[t_j : j \in T]$ with the restricted variables. A subring of $A$ of the form $A \cap RT$ with $\emptyset \neq T \subset [n]$ is called a combinatorial pure subring of $A$. See [3] for details. The following is known to be true [3, Corollary 1.6].

**Proposition 1.3.** Let $A$ be a homogeneous affine semigroup ring generated by monomials belonging to $R$ and let $A \cap RT$ be a combinatorial pure subring of $A$. If $A$ is strongly Koszul, then so is $A \cap RT$.

This fact is very useful. For example, we can determine when the $d$-th squarefree Veronese subring $R_{n,d}$ of polynomial ring $K[t_1, \ldots, t_n]$ is strongly Koszul. Note that $R_{n,2}$ is the edge ring of the complete graph $K_n$ of $n$ vertices (which will be defined later).

**Proposition 1.4.** Let $2 \leq d < n$ be integers. Then the following conditions are equivalent:

(i) $R_{n,d}$ is strongly Koszul.

(ii) Either $(n, d) = (4, 2)$ or $n = d + 1$.

**Proof.** It is known [3, Example 1.6 (3)] that $R_{n,2}$ is strongly Koszul if and only if $n \leq 4$. If $n = d + 1$, then $R_{d+1,d}$ is isomorphic to a polynomial ring. In particular, it is strongly Koszul.

If $(n, d) = (6, 3)$, then the semigroup ring $A$ generated by $\{t_it_j : 2 \leq i < j \leq 6\}$ over $K$ is a combinatorial pure subring of $R_{6,3}$. Since $A$ is isomorphic to $R_{5,2}$, $R_{6,3}$ is not strongly Koszul. By a similar argument, $R_{n,d}$ with $n \geq d + 3$ is not strongly Koszul. In general, it is known that $R_{n,d} \simeq R_{n,n-d}$. If $d \geq 3$ and $n = d + 2$, then $R_{d+2,d} \simeq R_{d+2,2}$ is not strongly Koszul since $d + 2 \geq 5$. □

We call a semigroup ring $A$ trivial if, starting with polynomial rings, $A$ is obtained by repeated applications of Segre products and tensor products. If $A$ is trivial, then it is strongly Koszul. See [3].

**Remark 1.5.** We note that $R_{4,2}$ is a non-trivial strongly Koszul complete intersection semigroup ring. Indeed, if we assume that $R_{4,2}$ is trivial, then there exists a 3-poset $P$ such that the Hibi ring (see [4]) $\mathcal{R}_K[P]$ is isomorphic to $R_{4,2}$ since all of the trivial rings can be constructed as Hibi rings. However, there exists no 3-poset $P$ such that $\mathcal{R}_K[P]$ is a complete intersection and its embedding dimension is 6. Hence, $R_{4,2}$ is not trivial.

## 2. Strongly Koszul edge rings

Let $G$ be a finite connected simple graph on the vertex set $V(G) = [n] = \{1, 2, \ldots, n\}$. Let $E(G) = \{e_1, \ldots, e_d\}$ be its edge set. Recall that a finite graph is simple if it possesses no loops or multiple edges. Let $K[X] = K[X_1, \ldots, X_n]$ be the polynomial ring in $n$ variables over a field $K$. If $e = \{i, j\} \in E(G)$, then $X^e$ stands for the quadratic monomial $X_iX_j \in K[X]$. The edge ring of $G$ is the subring $K[G] = K[X^{e_1}, \ldots, X^{e_d}]$ of $K[X]$. Let $K[Y] = K[Y_1, \ldots, Y_d]$ denote the polynomial ring in $d$ variables over $K$ with each $\deg Y_i = 1$ and define the surjective
ring homomorphism $\pi : K[Y] \to K[G]$ by setting $\pi(Y_i) = X^e$ for each $1 \leq i \leq d$. The toric ideal $I_G$ of $G$ is the kernel of $\pi$. It is known [22] Corollary 4.3 that $I_G$ is generated by binomials of the form $u - v$, where $u$ and $v$ are monomials of $K[Y]$ with $\deg u = \deg v$, such that $\pi(u) = \pi(v)$. In this section, we determine graphs $G$ such that $K[G]$ is strongly Koszul.

By Proposition 2.3 we have the following fact concerning edge rings.

**Corollary 2.1.** Let $G_W$ be an induced subgraph of a graph $G$ on the vertex set $W$. If $K[G]$ is strongly Koszul, then so is $K[G_W]$.

Let $G$ be a connected simple graph. Then $G$ is said to be 2-connected if $G_{[n] \setminus v}$ is also connected for all $v \in [n]$. Maximal 2-connected subgraphs of $G$ are called 2-connected components of $G$. If $G$ is bipartite, then the following characterization is known [3] Theorem 4.5:

**Proposition 2.2.** Let $G$ be a connected simple bipartite graph. Then $K[G]$ is strongly Koszul if and only if any 2-connected component of $G$ is complete bipartite.

On the other hand, all complete multipartite graphs $G$ such that $K[G]$ is strongly Koszul are classified in [10] Proposition 3.6.

Let $G$ be a graph on the vertex set $[n]$. A walk of length $q$ of $G$ connecting $v_1 \in V(G)$ and $v_{q+1} \in V(G)$ is a finite sequence of the form

$(1) \quad \Gamma = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_{q+1}\})$, \]

where $\{v_k, v_{k+1}\} \in E(G)$ for all $k$. A walk $\Gamma$ is called a path if $v_i \neq v_j$ for all $1 \leq i < j \leq q + 1$. An even (resp. odd) walk is a walk of even (resp. odd) length. A walk $\Gamma$ of the form (1) is called closed if $v_{q+1} = v_1$. A cycle is a closed walk of the form

$(2) \quad C = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\})$, \]

where $q \geq 3$ and $v_i \neq v_j$ for all $1 \leq i < j \leq q$. A chord of a cycle (2) is an edge $e \in E(G)$ of the form $e = \{v_i, v_j\}$ for some $1 \leq i < j \leq q$ with $e \notin E(C)$. If a cycle (2) is even, then an even-chord (resp. odd-chord) of (2) is a chord $e = \{v_i, v_j\}$ with $1 \leq i < j \leq q$ such that $j - i$ is odd (resp. even). A minimal cycle of $G$ is a cycle having no chords. If $C_1$ and $C_2$ are cycles of $G$ having no common vertices, then a bridge between $C_1$ and $C_2$ is an edge $\{i, j\}$ of $G$ with $i \in V(C_1)$ and $j \in V(C_2)$.

A graph $G$ is called almost bipartite if there exists a vertex $v$ such that any odd cycle of $G$ contains $v$. Note that the induced subgraph $G_{[n] \setminus v}$ is bipartite. Let $V_1 \cup V_2$ be a bipartition of $G_{[n] \setminus v}$. Then we define the bipartite graph $G(v, V_1, V_2)$ on the vertex set $[n + 1]$ together with the edge set

$E(G_{[n] \setminus v}) \cup \{\{i, v\} \in E(G) \mid i \in V_1\} \cup \{\{i, n + 1\} \mid i \in V_2, \{i, v\} \in E(G)\}$.

Note that $G$ is obtained from $G(v, V_1, V_2)$ by identifying two vertices $v$ and $n + 1$. In [11] Proposition 5.5, the following is shown.

**Proposition 2.3.** Assume the same notation as above. Then we have $K[G] \simeq K[G(v, V_1, V_2)]$.

In order to prove the main theorem of this paper, we need the following lemmata.

**Lemma 2.4.** Suppose that $K[G]$ is strongly Koszul. Then $G$ satisfies the following:
Lemma 2.5. Suppose that \( K[G] \) is strongly Koszul. Then at most one 2-connected component of \( G \) is not complete bipartite.
Proof. Assume that \( G \) has two distinct non-complete bipartite 2-connected components \( G_1, G_2 \). Then these are not even bipartite from Lemma 2.4 (i), and hence \( G_1 \) (resp. \( G_2 \)) has an odd cycle \( C_1 \) (resp. \( C_2 \)). By Lemma 2.4 (ii), \( C_1 \) and \( C_2 \) have at least two common vertices. This contradicts that \( C_1 \) and \( C_2 \) belong to distinct 2-connected components. \( \square \)

**Lemma 2.6.** Suppose that \( K'[G] \) is strongly Koszul. If \( G \) has a 2-connected component \( G' \) containing \( K_4 \), then \( G' \) is the complete graph \( K_4 \).

Proof. Assume that \( G' \supseteq K_4 \). Let \( W = \{a,b,c,d\} \subset V(G) \) such that \( G'_W = K_4 \). Since \( G' \) is connected and \( G' \neq G'_W \), there exists \( e \notin W \) such that \( \{a,e\} \in E(G') \). Moreover since \( G' \) is 2-connected, there exists a path \( P : e \rightarrow b \) that does not contain the vertex \( a \). If \( P \) contains either \( c \) or \( d \), then we replace \( b \) by it. Thus, we may assume that \( P \) contains neither \( c \) nor \( d \). Then \( C = a \rightarrow e \xrightarrow{P} b \rightarrow a \) is a cycle such that \( c, d \notin V(C) \).

If \( C \) is an odd cycle, then two odd cycles \( (a,c,d) \) and \( C \) have exactly one common vertex, a contradiction. Hence \( C \) is an even cycle. Since \( G' \) has an even cycle \( a \rightarrow e \xrightarrow{P} b \rightarrow d \rightarrow c \rightarrow a \) of length \( \geq 6 \), we have \( \{d,e\} \in E(G) \) by Lemma 2.4 (i). Hence two cycles \( (a,b,c) \) and \( (a,d,e) \) have exactly one common vertex, but this is a contradiction. Therefore we have \( G' = K_4 \). \( \square \)

![Figure 1](image-url)

**Lemma 2.7.** If a graph \( G \) is strongly Koszul and not almost bipartite, then the complete graph \( K_4 \) is a subgraph of \( G \).

Proof. Since \( G \) is not almost bipartite, there exist at least two minimal odd cycles \( C_1 \) and \( C_2 \). By Lemma 2.4 (ii), \( \# \{V(C_1) \cap V(C_2)\} \geq 2 \). Then \( C_1 \) is divided into several paths of the form \( P = (a_0, a_1, \ldots, a_{t-1}, a_t) \) in \( C_1 \) satisfying that \( a_0, a_t \in V(C_2) \) and \( a_i \notin V(C_2) \) \( (1 \leq i \leq t - 1) \).

We will show that \( 1 \leq t \leq 2 \). Suppose that \( t \geq 3 \) and \( t \) is odd. Since \( C_1 \) is minimal, the length of a path \( a_t \xrightarrow{\text{odd in } C_2} a_0 \) is not one. Thus, \( C = a_0 \xrightarrow{P} a_t \xrightarrow{\text{odd in } C_2} a_0 \) is an even cycle of length \( \geq 6 \). By Lemma 2.4 (i), \( C \) has all possible even-chords. This contradict that \( C_1 \) is minimal. Suppose that \( t \geq 4 \) and \( t \) is even. Then \( C' = a_0 \xrightarrow{P} a_t \xrightarrow{\text{even in } C_2} a_0 \) is an even cycle of length \( \geq 6 \). By Lemma 2.4 (i), \( C' \) has all possible even-chords, and hence either \( C_1 \) or \( C_2 \) has a chord. This is a contradiction.

If \( t = 1 \), then the length of a path \( a_t \xrightarrow{\text{odd in } C_2} a_0 \) is one since \( C_2 \) is minimal. If \( t = 2 \), then the length of a path \( a_t \xrightarrow{\text{even in } C_2} a_0 \) is two by the same argument as
above. Thus, \( C_1 \) and \( C_2 \) have a common edge and the lengths of \( C_1 \) and \( C_2 \) are equal. Let \( \ell \) be the length of any minimal odd cycle of \( G \).

Suppose that \( \ell \geq 5 \). Let \( C_1 = (1, 2, \ldots, \ell) \) and \( \{2, 3\} \) be an edge of \( C_2 \). Since \( G \) is not almost bipartite, there exists a minimal odd cycle \( C_3 \) such that \( 2 \notin V(C_3) \). Then \( C_3 \) has edges \( \{1, i\} \) and \( \{i, 3\} \) such that \( i \notin V(C_1) \). Again, since \( G \) is not almost bipartite, there exists a minimal odd cycle \( C_4 \) such that \( 3 \notin V(C_4) \). Then \( C_4 \) has edges \( \{2, j\} \) and \( \{j, 4\} \) such that \( j \notin V(C_1) \). If \( i = j \), then \( G \) has a cycle \( (2, 3, i) \). This contradicts that \( \ell \geq 5 \). Hence, \( i \neq j \). It then follows that \( (1, 2, j, 4, 3, i) \) is a cycle of length 6. By Lemma 2.3 (i), this cycle has even-chord \( \{1, 4\} \). This contradicts that \( C_1 \) is minimal.

Hence \( \ell = 3 \). Let \( C_1 = (1, 2, 3) \) and \( C_2 = (1, 2, 4) \) be cycles of \( G \). Since \( G \) is not almost bipartite, there exists a cycle \( C_3 \) of length 3 with \( 1 \notin V(C_3) \). By \( \#\{V(C_1) \cap V(C_3)\} \geq 2 \) and \( \#\{V(C_2) \cap V(C_3)\} \geq 2 \), we have \( C_3 = (2, 3, 4) \). Thus, \( C_1 \cup C_2 \cup C_3 = K_4 \), as desired. \( \square \)

We now generalize Proposition 2.2 to an arbitrary graph. The main theorem of this paper is as follows.

**Theorem 2.8.** Let \( G \) be a connected simple graph and let \( G_1, \ldots, G_s \) be the 2-connected components of \( G \). Then \( K[G] \) is strongly Koszul if and only if by a permutation of the indices, \( G_1, \ldots, G_{s-1} \) are complete bipartite and \( G_s \) is of the following types:

(i) A complete bipartite.

(ii) An almost bipartite graph such that each 2-connected component of \( G_s(v, V_1, V_2) \) is complete bipartite where \( V_1 \cup V_2 \) is a bipartition of the vertex set of the bipartite graph \( (G_s)_{|\setminus|v} \).

(iii) The complete graph \( K_4 \).

**Proof.** ("If") It is enough to show that \( K[G_s] \) is strongly Koszul. If \( G_s \) satisfies one of (i) and (ii), then \( K[G_s] \) is trivial and hence, in particular, strongly Koszul. Suppose that \( G_s \) satisfies (iii). By Proposition 1.4, \( R_{4,2} = K[K_4] \) is strongly Koszul.

("Only if") Suppose that \( K[G] \) is strongly Koszul and has 2-connected components \( G_1, \ldots, G_s \). By Lemma 2.3, we may assume that \( G_1, \ldots, G_{s-1} \) is complete bipartite. Suppose that \( G_s \neq K_4 \) and \( G_s \) is not complete bipartite. If \( G_s \) is not almost bipartite, then \( K_4 \) is a subgraph of \( G_s \) by Lemma 2.7. Then by Lemma 2.6, \( G_s = K_4 \), a contradiction. Thus, \( G_s \) is almost bipartite. Then by Proposition 2.3, the bipartite graph \( G_s(v, V_1, V_2) \) satisfies \( K[G_s] \simeq K[G_s(v, V_1, V_2)] \). By Proposition 2.2, any 2-connected component of \( G_s(v, V_1, V_2) \) is complete bipartite. Therefore, \( G_s \) satisfies condition (ii), as desired. \( \square \)

Since a trivial edge ring is strongly Koszul, such an edge ring satisfies one of the conditions in Theorem 2.8. As stated in Remark 1.5, \( K[K_4] = R_{4,2} \) is nontrivial. Hence we have the following.

**Corollary 2.9.** Let \( G \) be a connected simple graph and let \( G_1, \ldots, G_s \) be the 2-connected components of \( G \). Then \( K[G] \) is trivial if and only if by a permutation of the indices, \( G_1, \ldots, G_{s-1} \) are complete bipartite and \( G_s \) is of the following types:

(i) A complete bipartite.
(ii) An almost bipartite graph such that each 2-connected component of $G_s(v, V_1, V_2)$ is complete bipartite where $V_1 \cup V_2$ is a bipartition of the vertex set of the bipartite graph $(G_s)[n]\setminus v$.

It is known (e.g., [12]) that the toric ideals of both $K_4$ and a complete bipartite graph have a quadratic Gröbner basis. We have the following from Theorem 2.8.

**Corollary 2.10.** If $K[G]$ is strongly Koszul, then the toric ideal $I_G$ of $K[G]$ possesses a quadratic Gröbner basis.

**Acknowledgement.** This research was supported by the JST (Japan Science and Technology Agency) CREST (Core Research for Evolutional Science and Technology) research project Harmony of Gröbner Bases and the Modern Industrial Society in the framework of the JST Mathematics Program “Alliance for Breakthrough between Mathematics and Sciences.”

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