On Ilmanen’s multiplicity-one conjecture for mean curvature flow with type-\(I\) mean curvature

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Abstract

In this paper, we show that if the mean curvature of a closed smooth embedded mean curvature flow in \(\mathbb{R}^3\) is of type-\(I\), then the rescaled flow at the first finite singular time converges smoothly to a self-shrinker flow with multiplicity one. This result confirms Ilmanen’s multiplicity-one conjecture under the assumption that the mean curvature is of type-\(I\). As a corollary, we show that the mean curvature at the first singular time of a closed smooth embedded mean curvature flow in \(\mathbb{R}^3\) is at least of type-\(I\).

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1 Introduction

In this paper, we study finite time singularities of closed smooth embedded mean curvature flow in $\mathbb{R}^3$. A one-parameter family of hypersurfaces $x(p, t) : \Sigma^n \to \mathbb{R}^{n+1}$ is called a mean curvature flow, if $x$ satisfies the equation

$$\frac{\partial x}{\partial t} = -Hn, \quad x(0) = x_0,$$

(1.1)

where $H$ denotes the mean curvature of the hypersurface $\Sigma_t := x(t)(\Sigma)$ and $n$ denotes the outward unit normal of $\Sigma_t$. In the previous paper [33], we proved that the mean curvature of (1.1) must blow up at the first finite singular time for a closed smooth embedded mean curvature flow in $\mathbb{R}^3$. This paper can be viewed as a continuation of [33], and we will develop the techniques in [33] further to study the finite time singularities of mean curvature flow.

1.1 Singularities of mean curvature flow

The mean curvature flow with convexity conditions has been well studied during the past several decades. In [36], Huisken proved that if the initial hypersurface is uniformly convex, then after rescaling the mean curvature flow exists for all time and converges smoothly to a round sphere. When the initial hypersurface is mean-convex or two-convex, there are a number of estimates for the mean curvature flow (c.f. Huisken-Sinestrari [38] [39], Haslhofer-Kleiner [34]), and these estimates are important to study the surgery of mean curvature flow (c.f. Huisken-Sinestrari [40], Brendle-Huisken [10], Haslhofer-Kleiner [35]). Moreover, for mean curvature flow with mean convex initial hypersurfaces, B. White gave some structural properties of the singularities in [58] [59], and B. Andrews also showed a noncollapsing estimate in [2].

However, all these results rely on convexity conditions of initial hypersurfaces, and it is very difficult to study general cases. For the curve shortening flow in the plane, following the work Gage [28] [29] and Gage-Hamilton [30] on convex curves Grayson [31] proved that any embedded closed curve in the plane evolves to a convex curve and subsequently shrinks to a point, and Andrews-Bryan [1] gave a direct proof of Grayson’s theorem without using the monotonicity formula or classification of singularities. In the higher dimensions, we know very little results without convexity conditions. Colding-Minicozzi studied the generic singularities of the mean curvature flow in [19] [20]. For the classification of self-shrinkers without convexity conditions, S. Brendle [9] proved that the round
sphere is the only compact embedded self-shrinkers in $\mathbb{R}^3$ with genus 0, and L. Wang [55] showed that each end of a noncompact self-shrinker in $\mathbb{R}^3$ of finite topology is smoothly asymptotic to either a regular cone or a self-shrinking round cylinder. However, it still remains wide open to understand the behavior of mean curvature flow at the singular time in the general cases.

1.2 The multiplicity-one conjecture and the main theorems

To study the singularities of mean curvature flow without convexity conditions, Ilmanen proposed a series of conjectures in [41],[42]. Suppose that the mean curvature flow (1.1) reaches a singularity at $(x_0, T)$ with $T < +\infty$. For any sequence $\{c_j\}$ with $c_j \to +\infty$, we rescale the flow (1.1) by

$$
\Sigma_j^t := c_j \left( \Sigma_{T + c_j^{-2} t} - x_0 \right), \quad t \in [-Tc_j^2, 0). 
$$

(1.2)

By Huisken’s monotonicity formula [37] and Brakke’s compactness theorem [3], a subsequence of $\Sigma_j^t$ converges weakly to a limit flow $\mathcal{T}_t$, which is called a tangent flow at $(x_0, T)$. In [41] Ilmanen showed that the tangent flow at the first singular time must be smooth for a smooth embedded mean curvature flow in $\mathbb{R}^3$, and he conjectured

**Conjecture 1.1.** (Ilmanen [41],[42], the multiplicity-one conjecture) For a smooth one-parameter family of closed embedded surfaces in $\mathbb{R}^3$ flowing by mean curvature, every tangent flow at the first singular time has multiplicity one.

Moreover, Ilmanen pointed out that the multiplicity-one conjecture implies a conjecture on the asymptotic structure of self-shrinkers in $\mathbb{R}^3$, and the latter conjecture has been confirmed recently by L. Wang [55]. If the initial hypersurface is mean convex or satisfies the Andrews condition, then the multiplicity-one conjecture holds (c.f. White [58], Haslhofer-Kleiner [34], Andrews [2]). Recently, A. Sun [52] proved that the generic singularity of mean curvature flow of closed embedded surfaces in $\mathbb{R}^3$ modelled by closed self-shrinkers with multiplicity has multiplicity one. In general the multiplicity-one conjecture is still wide open. In this paper, using the techniques from our previous work [33] we confirm the multiplicity-one conjecture under the assumption that the mean curvature is of type-I.

To state our result, we first introduce some notations. A hypersurface $x : \Sigma \to \mathbb{R}^{n+1}$ is called a self-shrinker, if $x$ satisfies the equation

$$
H = \frac{1}{2} \langle x, n \rangle.
$$

If $\Sigma$ is a self-shrinker, then we call $\Sigma_t := \sqrt{-t} \Sigma (t < 0)$ a self-shrinker flow. The main theorem of this paper is the following result.

**Theorem 1.2.** Let $x(t) : \Sigma^2 \to \mathbb{R}^3 (t \in [0, T))$ a closed smooth embedded mean curvature flow with the first singular time $T < +\infty$. If the mean curvature satisfies

$$
\max_{\Sigma_t} |H|(p, t) \leq \frac{\Lambda}{\sqrt{T - t}}, \quad \forall \ t \in [0, T),
$$

(1.3)

for some $\Lambda > 0$, then for any $a, b \in \mathbb{R}$ with $-\infty < a < b < 0$ and any sequence $c_j \to +\infty$ there exists a subsequence, still denoted by $\{c_j\}$, such that the flow $\{\Sigma_j^t, a < t < b\}$ defined by (1.2) converges smoothly to a self-shrinker flow with multiplicity one as $j \to +\infty$.

It is not hard to see that Theorem 1.2 is equivalent to the following result.
**Theorem 1.3.** Let \( \{\Sigma^2, x(t)\}, 0 \leq t < +\infty \) be a closed smooth embedded rescaled mean curvature flow

\[
\left( \frac{\partial x}{\partial t} \right)^\perp = -\left( H - \frac{1}{2} \langle x, n \rangle \right)
\]

satisfying

\[
d(\Sigma_t, 0) \leq D, \quad \text{and} \quad \max_{\Sigma_t} |H(p, t)| \leq \Lambda
\]

for two constants \( D, \Lambda > 0 \). Then for any \( t_i \to +\infty \) there exists a subsequence of \( \{\Sigma_{t_i+t}, -1 < t < 1\} \) such that it converges in smooth topology to a complete smooth self-shrinker with multiplicity one as \( i \to +\infty \).

In [33], we showed Theorem 1.3 under the assumption that the mean curvature decays exponentially to zero. In this special case, the flow (1.4) converges smoothly to a plane passing through the origin with multiplicity one. Theorem 1.3 means that under the assumption that the mean curvature is bounded for all time the flow (1.4) also converges smoothly to a self-shrinker with multiplicity one. In fact, Theorem 1.3 is not stated with the optimal condition. Checking the proof carefully, one can see that the conclusion of Theorem 1.3 still holds under the assumption that the mean curvature is uniformly bounded on any ball for all time:

\[
\max_{B_R(0) \cap \Sigma_t} |H(p, t)| \leq C_R,
\]

where \( C_R \) is a constant depending on \( R \). Note that if the flow (1.4) converges smoothly to a self-shrinker with multiplicity one, the condition (1.6) automatically holds by the self-shrinker equation. Thus, the condition (1.6) is also necessary for the smooth convergence of the flow (1.4). Therefore, we have reduced the solution of the multiplicity-one conjecture, i.e., Conjecture 1.1, to the examination of (1.5) or (1.6), which will be an interesting subject of study in the near future.

The multiplicity-one conjecture is closely related to the extension problem of mean curvature flow. Huisken [36] proved that if the flow (1.4) develops a singularity at time \( T < \infty \), then the second fundamental form will blow up at time \( T \). A natural question is whether the mean curvature will blow up at the finite singular time of a mean curvature flow. Toward this question, A. Cooper [24] proved that \( |A||H| \) must blow up at the singular time of the flow. In a series of papers [44]-[46] Le-Sesum systematical studied this problem and they proved that the extension problem is true if the multiplicity-one conjecture holds, or the second fundamental form is of type-I at the singular time

\[
\max_{\Sigma_t} |A| \leq \frac{C}{\sqrt{T-t}}, \quad \forall \ t \in [0, T).
\]

Furthermore, Le-Sesum [46] proved that the mean curvature is at least of type-I if (1.7) holds. Using Theorem 1.2 we can remove the type-I condition (1.7) of Le-Sesum’s result as follows, which can also be viewed as an improvement of the extension theorem in [33].

**Corollary 1.4.** If \( x(t) : \Sigma^2 \to \mathbb{R}^3(t \in [0, T)) \) is a closed smooth embedded mean curvature flow with the singular time \( T < +\infty \), then there is a constant \( \delta > 0 \) such that

\[
\max_{\Sigma_t} |H| \geq \frac{\delta}{\sqrt{T-t}}, \quad \forall \ t \in [0, T).
\]

Corollary 1.4. If \( x(t) : \Sigma^2 \to \mathbb{R}^3(t \in [0, T)) \) is a closed smooth embedded mean curvature flow with the singular time \( T < +\infty \), then there is a constant \( \delta > 0 \) such that

\[
\max_{\Sigma_t} |H| \geq \frac{\delta}{\sqrt{T-t}}, \quad \forall \ t \in [0, T).
\]

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1.3 Outline of the proof

Now we sketch the proof of Theorem 1.3. Assume that the mean curvature satisfies the type-I condition (1.3) along the flow (1.1) and the first singular time \( T < +\infty \). Then the mean curvature is uniformly bounded along the rescaled flow (1.4). We have to show that the flow (1.4) converges smoothly to a self-shrinker with multiplicity one. The strategy is similar to [33], we first show a weak-compactness theorem and obtain the flow convergence is smooth away from a singular set. Then we use stability argument to remove the singular set. However, the technique here is much more involved. The proof consists of three steps:

**Step 1. Convergence of the rescaled mean curvature flow with multiplicities.** In this step, since the mean curvature is uniformly bounded along the flow, we have the short-time pseudolocality theorem and the energy concentration property, and we can follow the arguments in [33] to develop the weak compactness theory of mean curvature flow. However, compared with [33], since the mean curvature doesn’t decay to zero, we have the following difficulties:

- No long time pseudolocality theorem;
- The space-time singularities in the limit don’t move along straight lines.

Because of lacking these results, we face a number of new technical difficulties to show the \( L \)-stability of the limit self-shrinker. These difficulties force us to use analysis tools to study the asymptotical behavior of the solution of the limit parabolic equation near the singular set.

**Step 2. Show that the multiplicity of the convergence is one for one subsequence.** As in [33], it suffices to show that the limit self-shrinker is \( L \)-stable. By the convergence of the flow away from the singular set, if every limit has multiplicity greater than one, we can renormalize the “height-difference” function to obtain a positive solution of the equation

\[
\frac{\partial w}{\partial t} = \Delta w - \frac{1}{2} \langle x, \nabla w \rangle + |A|^2 w + \frac{1}{2} w,
\]

away from the singular set. To show the \( L \)-stability of the limit self-shrinker, we have to show the following two estimates:

- For each time, the asymptotical behavior of \( w \) is “good” near the singular set.
- Uniform \( L^1 \) norm of \( w \) independent of time.

By its construction, \( w \) is defined on any compact set away from the singular set and we have no estimates near the singular set by the geometric method. However, we found that \( w \) satisfies many good properties from the PDE point of view. In [7], Kan-Takahashi studied similar problem for some semilinear parabolic equations along time-dependent singularities in the Euclidean spaces. Kan-Takahashi showed their result for one time-dependent singularity, and the solution of the equation looks like \( \log \frac{1}{r} \) in dimension 2, where \( r \) is the distance from any point \( x \) to the singularity. However, in our case the solution of (1.3) may have multiple singularities, and these singularities may coincide at one point. Thus, we cannot apply Kan-Takahashi’s result directly, and we need to develop their techniques to show that the solution \( w \) is in \( L^1 \) across the singularities and near the singular set the solution \( w \) roughly looks like

\[
w(x, t) \sim \sum_{k=1}^{l} c_k(t) \log \frac{1}{r_k(x, t)},
\]
where \( r_k(x, t) \) denotes the intrinsic distance from a point \( x \) to a singularity \( \xi_k(t) \) at time \( t \). Here the constant \( c_i \) may depend on \( t \). In general, the \( L^1 \) norm of \( w \) may tend to infinity as \( t \to +\infty \). In order to show uniform \( L^1 \) norm of \( w \), we refine the argument in [33] and also use the estimate of \( w \) near the singularities to choose a sequence of time slices \( \{t_i\} \), and then we show that for such a special sequence the corresponding function \( w \) has uniform \( L^1 \) bound independent of \( t \). Thus, for the special sequence \( t_i \), the auxiliary function \( w \) satisfies the two desired estimates. Then we can follow the argument in [33] to show that the convergence of (1.4) is smooth and of multiplicity one, for the special sequence \( \{t_i\} \).

**Step 3. Show the multiplicity-one convergence for each subsequence.** This step is a new difficulty beyond [33]. In [33], each limit, no matter what multiplicity it is, must be a flat plane passing through the origin. Therefore, up to rotation, different limits can be regarded as the same. By the monotonicity of the entropy, it is clear that if one limit is a multiplicity-one plane, then each limit must also be a multiplicity-one plane. However, in the current setting, each limit is only a self-shrinker and the limits may vary as the time sequences change. A priori, it is possible that one sequence converge to a multiplicity-one self-shrinker \( A \), and the other sequence converge to a multiplicity two self-shrinker \( B \neq A \). This possibility cannot be ruled out by only using the monotonicity of the entropy. To overcome this difficulty, we essentially use the smooth compactness theorem of self-shrinkers by Colding-Minicozzi [18]. Since the limit self-shrinkers form a compact set, we know that the local behavior of limit self-shrinkers are very close to that of planes on a fixed small scale. From this and the volume continuity, we derive an argument to show that the multiplicity is independent of the choice of subsequences. Therefore, every subsequence converges with multiplicity one.

It is interesting to know whether the above argument still works for the multiplicity-one conjecture without the mean curvature bound assumption (1.3). The main difficulty is the loss of pseudolocality result as in [33], since the points in the evolving surfaces may move drastically if the mean curvature is large. Furthermore, the loss of mean curvature bound also induces difficulties in applying PDE tools to analyze the singular set. However, as we discussed around (1.6), it is also logically possible to develop the estimate (1.6) directly.

### 1.4 Relation with other geometric flows

It is interesting to compare the mean curvature flow with the Ricci flow. The extension problem for Ricci flow has been extensively studied recently. Corollary [1.4] has a cousin theorem in the Ricci flow. In Theorem 1 of [54], it was shown that along the Ricci flow \( \{(M, g(t)), 0 \leq t < T\} \) with the singular time \( T < +\infty \), we have

\[
\max_M |\text{Ric}|_{g(t)} \geq \frac{\delta}{\sqrt{T-t}}, \quad t \in [0, T),
\]

which extends the famous Ricci extension theorem of N. Sesum [51]. Up to rescaling, the gap inequality (1.9) is equivalent to \( \max_M |\text{Ric}|_{g(t)} \geq \delta \) along the rescaled Ricci flow solution

\[
\partial_t g = -\text{Ric} + g, \quad t \in [0, \infty).
\]

Actually, we even believe that a gap for scalar curvature holds for a rescaled Ricci flow solution. In other words, along the rescaled Ricci flow (1.10) we should have

\[
\max_M |R|_{g(t)} \geq \delta.
\]
It is easy to see that the scalar extension conjecture of the Ricci flow will hold automatically if one can prove the above inequality along the rescaled Ricci flow (1.10), just like the extension theorem of mean curvature in [33] follows directly from Corollary 1.4.

The similarity between the regularity theory of rescaled mean curvature flow (1.4) and the rescaled Ricci flow (1.10) was noticed for a while. For example, such similarity was discussed in the introduction of [33]. Along the rescaled flows, the mean curvature bound condition (1.5) is comparable to the scalar curvature bound condition $|R| \leq C$. Note that the Fano Kähler-Ricci flow provides many examples of the global solutions of the rescaled Ricci flow (1.10) and Perelman showed that $|R| \leq C$ holds automatically. The boundedness of the scalar curvature is crucial to study the convergence of Kähler-Ricci flow to a limit flow (c.f. Theorem 1.5 of [14], with journal version [15] and [16]). For time-slice convergence, see Tian-Zhang [53] for example. Since (1.5) is the comparable condition of Perelman’s estimates, we can view Theorem 1.2 as the analogue of the convergence results in the Fano Kähler-Ricci flow. However, we have to confess that we do not know any non-trivial examples satisfying the condition (1.5). By non-triviality we mean that we do not have positivity assumption of $H$. It will be very interesting to find out such examples.

The rescaled mean curvature flow can also be compared with the Calabi flow. In [4] E. Calabi studied the gradient flow of the $L^2$-norm of the scalar curvature among Kähler metrics in a fixed cohomology class on a compact Kähler manifold, which is now well-known as the Calabi flow. X. X. Chen conjectured that the Calabi flow always exists globally for any initial smooth Kähler potential. Very recently, Chen-Cheng [5] proved that the Calabi flow exists as long as the scalar curvature is uniformly bounded. Therefore, to study the long time existence of Calabi flow, it is crucial to control the scalar curvature, which is similar to the mean curvature condition (1.5) for the rescaled mean curvature flow. Assuming the long time existence and the uniform boundedness of the scalar curvature, the current authors and K. Zheng showed the convergence of the Calabi flow in [43], just as Theorem 1.3 for rescaled mean curvature flow.

1.5 List of notations

In the following, we list the important notations in this paper.

- $d(x, y)$: the Euclidean distance from $x$ to $y$. Defined in Definition 2.7.
- $B_r(p)$: the open ball in $\mathbb{R}^3$ centered at $p$ with radius $r$. Defined in Definition 2.1.
- $d_g(x, y)$: the intrinsic distance of $(\Sigma, g)$ from $x$ to $y$. First appears in the beginning of Section 4.
- $B_r(p)$: the intrinsic geodesic ball in $(\Sigma, g)$ centered at $p$ with radius $r$. Defined in Definition 2.1.
- $C_x(B_r(p) \cap \Sigma)$: the connected component of $B_r(p) \cap \Sigma$ containing $x \in \Sigma$. Defined in Definition 2.1.
- $m(x, t)$: the multiplicity at $(x, t)$. Defined in (2.21).
- $\mathcal{S}$: the space-time singular set. Defined in Proposition 2.8.
- $\mathcal{S}_t = \{x \in \mathbb{R}^3 \mid (x, t) \in \mathcal{S}\}$: the singular set at time $t$. Defined in Proposition 2.8.
- $\xi(t)$: a Lipschitz singular curve in $\mathcal{S}$. First appears in Lemma 2.11.
• \( \rho : \mathbb{R}^+ \to \mathbb{R}^+ \): an increasing positive function. First appears in Definition 3.3.
• \( \Omega_{\epsilon,R}(t) \): a subset of the limit self-shrinker away from singularities. Defined in (3.31).
• \( S_I \): the union of the singular set on a time interval \( I \). Defined in (3.32).
• \( u_i \): the height difference function defined in (3.33).
• \( w_i \): the normalized difference function defined in (3.35).
• \( d_H \): the Hausdorff distance in the Euclidean space.
• \( r(x,t) \): the intrinsic distance function from \( x \) to the singular curve \( \xi(t) \). Defined in (4.22).
• \( r_k(x,t) \): the intrinsic distance function from \( x \) to the singular curve \( \xi_k(t) \). Defined in (3.125) and Theorem 4.2.
• \( E^k_t(\delta) \) and \( A^k_t(\delta, \rho) \): a subset around the singularities on the limit self-shrinker. Defined in (3.136) and (3.137).
• \( M_{k,m}(\rho, \Xi) \): a subset of a Riemannian manifold defined in Definition 4.1.
• \( \Gamma_{\xi,\bar{\xi}} \): the union of space-time singular curves. Defined in (4.19) and (4.60).
• \( Q_{r,\xi,\bar{\xi}} \) and \( \tilde{Q}_{r,\xi,\bar{\xi}} \): the neighborhood of the singular curves. Defined in and (4.20) and (4.60).
• \( \phi_\xi \): cutoff functions around the singular curves. Defined in Definition 4.7 and Definition 4.12.
• \( I_\xi \): a functional associated with a singular curve \( \xi \). Defined in Definition 4.12.
• \( \bar{v}(x,t) \): defined in Definition 4.12.
• \( H(z) \): a cutoff function defined in Definition 4.7. Note that the function \( H(z) \) is only used in Section 4. Since the mean curvature doesn’t appear in Section 4, we keep the same notation \( H(z) \) as in [7].

1.6 The organization

The organization of this paper is as follows. In Section 2 we recall some facts on the pseudolocality theorem and energy concentration property. Moreover, we will show the weak compactness of mean curvature flow under some geometric conditions and we show the multiplicity of the convergence is a constant. In Section 3 we show the rescaled mean curvature flow with bounded mean curvature converges smoothly to a self-shrinker with multiplicity one, under the assumption that the auxiliary function satisfies good growth properties at the singular set. In Section 4 we will show the estimates of the auxiliary function by developing Kan-Takahashi’s argument. Finally, we finish the proof of Theorem 1.2 in Section 5. In the appendices, we include two versions of the parabolic Harnack inequality and give the full details on the calculation of the linearized equation of rescaled mean curvature flow.

1.7 Acknowledgement

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2 Weak compactness of refined sequences

2.1 The pseudolocality theorem and energy concentration property

In this subsection, we recall some results in [33]. First, we have the following definition.

**Definition 2.1.** (1) We denote by \( B_r(p) \) the ball in \( \mathbb{R}^{n+1} \) centered at \( p \) with radius \( r \) with respect to the standard Euclidean metric, and \( B_r(p) \subset (M, g) \) the intrinsic geodesic ball on \( M \) centered at \( p \) with radius \( r \) with respect to the metric \( g \).

(2) For any \( r > 0, p \in \mathbb{R}^{n+1} \) and \( \Sigma^n \subset \mathbb{R}^{n+1} \), we denote by \( C_x(B_r(p) \cap \Sigma) \) the connected component of \( B_r(p) \cap \Sigma \) containing \( x \in \Sigma \).

We first recall the following result of Chen-Yin [23].

**Lemma 2.2.** (c.f. Lemma 7.1 of [23]) Let \( \Sigma^n \subset \mathbb{R}^{n+1} \) be properly embedded in \( B_{r_0}(x_0) \) for some \( x_0 \in \Sigma \) with

\[
|A|(x) \leq \frac{1}{r_0} \quad x \in B_{r_0}(x_0) \cap \Sigma.
\]

Let \( \{x^1, \ldots, x^{n+1}\} \) be the standard coordinates in \( \mathbb{R}^{n+1} \). Assume that \( x_0 = 0 \) and the tangent plane of \( \Sigma \) at \( x_0 \) is \( x^{n+1} = 0 \). Then there is a map

\[
u : \left\{ x' = (x^1, \ldots, x^n) \mid |x'| < \frac{r_0}{96} \right\} \to \mathbb{R}
\]

with \( \nu(0) = 0 \) and \( |\nabla \nu|(0) = 0 \) such that the connected component containing \( x_0 \) of \( \Sigma \cap \{(x', x^{n+1}) \mid |x'| < \frac{r_0}{96} \} \) can be written as a graph \( \{ (x', \nu(x')) \mid |x'| < \frac{r_0}{96} \} \) and

\[
|\nabla \nu(x')| \leq \frac{36}{r_0} |x'|.
\]

Using Lemma 2.2, we show that the local area ratio of the surface is very close to 1.

**Lemma 2.3.** (c.f. Lemma 3.3 of [33]) Suppose that \( \Sigma^n \subset B_{r_0}(p) \subset \mathbb{R}^{n+1} \) is a hypersurface with \( \partial \Sigma \subset \partial B_{r_0}(p) \) and

\[
\sup_{\Sigma} |A| \leq \frac{1}{r_0}.
\]

For any \( \delta > 0 \), there is a constant \( \rho_0 = \rho_0(r_0, \delta) \) such that for any \( r \in (0, \rho_0) \) and any \( x \in B_{r_0}(p) \cap \Sigma \) we have

\[
1 - \delta \leq \frac{\text{vol}_\Sigma(C_x(B_r(x) \cap \Sigma))}{\omega_n r^n} \leq 1 + \delta. \tag{2.1}
\]

**Proof.** By Lemma 2.2, for any \( x \in B_{r_0}(p) \cap \Sigma \) the component \( C_x(B_{\rho_0}(x) \cap \Sigma) \) with \( \rho_0 = \frac{r_0}{192} \) can be written as a graph of a function \( \overline{u} \) over the tangent plane at \( x \), which we assume to be \( P = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 0 \} \), with \( |\nabla \overline{u}(x')| \leq \frac{72}{r_0} |x'| \) where \( x' = (x_1, \ldots, x_n) \). Let \( r \in (0, \rho_0) \). We denote by \( \Omega_r \) the projection of \( C_x(B_r(x) \cap \Sigma) \) to the plane \( P \). Then for any \( x' \in \partial \Omega_r \), we have

\[
u(x')^2 + |x'|^2 = r^2. \tag{2.2}
\]

On the other hand, for any \( x' \in \Omega_{\rho_0} \) we have the inequality

\[
|u(x')| \leq |u(0)| + \max_{t \in [0,1]} |\nabla u|(tx') \cdot |x'| \leq \frac{72}{r_0} |x'|^2. \tag{2.3}
\]
Note that (2.2) and (2.3) imply that for any $x' \in \partial \Omega_r$,

$$|x'|^2 \leq r^2 = u(x')^2 + |x'|^2 \leq |x'|^2 \left(1 + \frac{5184}{r_0^2} \rho_0^2\right).$$

(2.4)

Thus, we have

$$\tilde{r} := \frac{r}{\sqrt{1 + \frac{5184}{r_0^2} \rho_0^2}} \leq |x'| \leq r, \quad \forall x' \in \partial \Omega_r,$$

(2.5)

which implies that

$$B_{\tilde{r}}(x) \cap P \subset \Omega_r \subset B_r(x) \cap P.$$  

(2.6)

Thus, the volume ratio of $C_x(B_r(x) \cap \Sigma)$ is bounded from above

$$\frac{\text{vol}_\Sigma(C_x(B_r(x) \cap \Sigma))}{\omega_n r^n} = \frac{1}{\omega_n r^n} \int_{\Omega_r} \sqrt{1 + |\nabla u|^2} \, d\mu \leq \frac{1}{\omega_n r^n} \int_{B_r(x) \cap P} \sqrt{1 + |\nabla u|^2} \, d\mu \leq \sqrt{1 + \frac{5184}{r_0^2} r^2},$$

(2.7)

where we used (2.3) and (2.6). Moreover, the volume ratio of $C_x(B_r(x) \cap \Sigma)$ is bounded from below

$$\frac{\text{vol}_\Sigma(C_x(B_r(x) \cap \Sigma))}{\omega_n r^n} \geq \frac{1}{\omega_n r^n} \int_{B_r(x) \cap P} \sqrt{1 + |\nabla u|^2} \, d\mu \geq \tilde{r}^n \geq \left(1 + \frac{5184}{r_0^2} \rho_0^2\right)^{-\frac{n}{2}}.$$  

(2.8)

Combining (2.7) with (2.8), for any $\delta > 0$ we can choose $\rho_0 = \rho_0(n, \delta, r_0)$ further small such that (2.1) holds. The lemma is proved.

Next we recall the two-sided pseudolocality theorem in [33]. If the initial hypersurface can be locally written as a graph of a single-valued function, then we have the pseudolocality type results of the mean curvature flow by Ecker-Huisken [26] [27], M. T. Wang [56], Chen-Yin [23] and Brendle-Huisken [10]. However, in our case we have to apply the pseudolocality theorem for the hypersurfaces which may converge with multiplicities. Thus, we use the boundedness of the mean curvature to get the two-sided pseudolocality theorem in [33].

**Theorem 2.4 (Two-sided pseudolocality).** (c.f. [33]) For any $r_0 \in (0, 1], \Lambda, T > 0$, there exist $\eta = \eta(n, \Lambda), \epsilon = \epsilon(n, \Lambda) > 0$ satisfying

$$\lim_{\Lambda \to 0} \eta(n, \Lambda) = \eta_0(n) > 0, \quad \lim_{\Lambda \to 0} \epsilon(n, \Lambda) = \epsilon_0(n) > 0$$

(2.9)

and the following properties. Let $\{(\Sigma^n, x(t)), -T \leq t \leq T\}$ be a closed smooth embedded mean curvature flow (7.1). Assume that

1. the second fundamental form satisfies $|A|(x, 0) \leq \frac{1}{r_0}$ for any $x \in C_p(B_{r_0}(p_0) \cap \Sigma_0)$ where $p_0 = x_0(p)$ for some $p \in \Sigma$;
(2) the mean curvature of \( \{ (\Sigma^n, x_t), -T \leq t \leq T \} \) is bounded by \( \Lambda \).

Then for any \( (x, t) \) satisfying
\[
x \in C_{p_t}(\Sigma_t \cap B_{\eta r_0}(p_0)), \quad t \in \left[ -\frac{\eta r_0^2}{2(\Lambda + \Lambda^2)}, \frac{\eta r_0^2}{2(\Lambda + \Lambda^2)} \right] \cap [-T, T]
\]
where \( p_t = x_t(p) \), we have the estimate
\[
|A|(x, t) \leq \frac{1}{\epsilon r_0}.
\]

Using the pseudolocality theorem, we have the energy concentration property.

**Lemma 2.5 (Energy concentration).** (c.f. [33]) For any \( \Lambda, K, T > 0 \), there exists a constant \( \epsilon(n, \Lambda, K, T) > 0 \) with the following property. Let \( \{ (\Sigma^n, x(t)), -T \leq t \leq T \} \) be a closed smooth embedded mean curvature flow (1.1). Assume that \( \max_{\Sigma_t \times [-T, T]} |H|(p, t) \leq \Lambda \). Then we have
\[
\int_{\Sigma_0 \cap B_{Q^{-1}}(q)} |A|^n \, d\mu_0 \geq \epsilon(n, \Lambda, K, T) \tag{2.10}
\]
whenever \( q \in \Sigma_0 \) with \( Q := |A|(q, 0) \geq K \).

A direct corollary of Lemma 2.5 is the following \( \epsilon \)-regularity of the mean curvature flow, which can be viewed as a generalization of the result of Choi-Schoen [12].

**Corollary 2.6 (\( \epsilon \)-regularity).** (c.f. [33]) There exists \( \epsilon_0(n) > 0 \) satisfying the following property. Let \( \{ (\Sigma^n, x(t)), -1 \leq t \leq 1 \} \) be a closed smooth embedded mean curvature flow (1.1). Suppose that the mean curvature satisfies \( \max_{\Sigma_t \times [-1, 1]} |H|(p, t) \leq 1 \). For any \( q \in \Sigma_0 \), if
\[
\int_{\Sigma_0 \cap B_r(q)} |A|^n \, d\mu_0 \leq \epsilon_0(n)
\]
for some \( r > 0 \), then we have
\[
\max_{B_{2r}(q) \cap \Sigma_0} |A| \leq \max\{1, \frac{2}{r}\}. \tag{2.11}
\]

### 2.2 Weak compactness

As in [33], we use the pseudolocality theorem and the energy concentration property to develop the weak compactness of the mean curvature flow. Here we will replace the zero mean curvature condition in [33] by the boundedness of the mean curvature in the definition of refined sequences. The name of refined sequence originates from [13].

**Definition 2.7 (Refined sequences).** Let \( \{ (\Sigma_i^2, x_i(t)), -1 < t < 1 \} \) be a one-parameter family of closed smooth embedded surfaces satisfying the mean curvature flow equation (1.1). It is called a refined sequence if the following properties are satisfied for every \( i \):

1. There exists a constant \( D > 0 \) such that \( d(\Sigma_{i,t}, 0) \leq D \) for any \( t \in (-1, 1) \), where \( d(\Sigma, 0) \) denotes the Euclidean distance from the point \( 0 \in \mathbb{R}^3 \) to the surface \( \Sigma \subset \mathbb{R}^3 \) and \( \Sigma_{i,t} = x_i(t)(\Sigma_i) \);
The measures $\nu_i$ satisfy
\begin{equation}
\max_{\Omega, t} |H| (p, t) \leq \Lambda_i,
\end{equation}
(2.12)

(3) There exists an increasing positive function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $R > 0$,
\begin{equation}
\int_{\Sigma_1 \cap B_R(0)} |A|^2 d\mu_{i,t} \leq \rho(R), \quad \forall \ t \in (-1, 1); \tag{2.13}
\end{equation}

(4) There is uniform $N > 0$ such that for all $r > 0$ and $p \in \mathbb{R}^3$ we have
\begin{equation}
\text{Area}_{g, i(t)}(B_r(p) \cap \Sigma_{i,t}) \leq N \pi r^2, \quad \forall \ t \in (-1, 1). \tag{2.14}
\end{equation}

(5) There exist uniform constants $\bar{r}, \kappa > 0$ such that for any $r \in (0, \bar{r}]$ and $p \in \Sigma_{i,t}$ we have
\begin{equation}
\text{Area}_{g, i(t)}(B_r(p) \cap \Sigma_{i,t}) \geq \kappa r^2, \quad \forall \ t \in (-1, 1). \tag{2.15}
\end{equation}

(6) There exists $T > 1$ such that
\begin{equation}
\lim_{i \to \infty} \int_{-1}^1 dt \int_{\Sigma_1} e^{-\frac{|x|^2}{4(T-t)}} |H_i - \frac{1}{2(T-t)} \langle x_\infty, n \rangle|^2 \ d\mu_{i,t} = 0. \tag{2.16}
\end{equation}

Following the arguments as in minimal surfaces (c.f. White [57], or Colding-Minicozzi [17]), we have the weak compactness for mean curvature flow.

**Proposition 2.8.** Let $\{(\Sigma_i^2, x_i(t)), -1 < t < 1\}$ be a refined sequence. Then there exists a subsequence, still denoted by $\{(\Sigma_i^2, x_i(t)), -1 < t < 1\}$, a smooth self-shrinker flow $\{(\Sigma_\infty, x_\infty(t)), -1 < t < 1\}$ satisfying
\begin{equation}
H = \frac{1}{2(T-t)} \langle x_\infty, n \rangle, \tag{2.17}
\end{equation}

for some $T > 1$, and a space-time singular set $S = \{(x, t) | t \in (-1, 1), x \in \mathbb{R}^3\}$ satisfying the following properties:

(1). The sequence $\{(\Sigma_i^2, x_i(t)), -1 < t < 1\}$ converges locally smoothly, possibly with multiplicity at most $N_0$, to $\{(\Sigma_\infty, x_\infty(t)), -1 < t < 1\}$ away from $S$;

(2). For each time $t \in (-1, 1)$ the singular set $S_t = \{x \in \mathbb{R}^3 | (x, t) \in S\}$ is locally finite in the sense that $S_t \cap B_R(0)$ consists of finite many points for any $R > 0$.

(3). The sequence in (1) also converges in extrinsic Hausdorff distance.

**Proof.** We first show that after taking a subsequence if necessary, $\Sigma_{i,0}$ converges locally smoothly to $\Sigma_{\infty,0}$ away from a locally finite set $S_0$. Fix large $R > 0$ and let $\Omega = B_R(0) \subset \mathbb{R}^3$. By Property (1) in Definition 2.7 we have $\Sigma_{i,t} \cap \Omega \neq \emptyset$ for large $R > 0$ and any $t \in (-1, 1)$. For any $U \subset \Omega$, we define the measures $\nu_i$ by
\begin{equation}
\nu_i(U) = \int_{U \cap \Sigma_{i,0}} |A_i|^2 d\mu_{i,0} \leq \rho(R),
\end{equation}
where we used (2.13) in the inequality. The general compactness of Radon measures implies that there is a subsequence, which we still denote by \( \nu_i \), converges weakly to a Radon measure \( \nu \) with \( \nu(\Omega) \leq \rho(R) \). We define the set

\[
S_0 = \{ x \in \Omega \ | \ \nu(x) \geq \epsilon_0 \},
\]

where \( \epsilon_0 \) is the constant in Corollary 2.6. It follows that \( S_0 \) contains at most \( \frac{\rho(R)}{\epsilon_0} \) points. Given any \( y \in \Omega \setminus S_0 \). There exists some \( s > 0 \) such that \( B_{10s}(y) \subset \Omega \) and \( \nu(B_{10s}(y)) < \epsilon_0 \). Since \( \nu_i \to \nu \), for \( i \) sufficiently large we have

\[
\int_{B_{10s}(y) \cap \Sigma_{i,0}} |A_i|^2 d\mu_{i,0} < \epsilon_0.
\]

Corollary 2.6 implies that for \( i \) sufficiently large we have the estimate

\[
\max_{B_{5s}(y) \cap \Sigma_{i,0}} |A|(x, 0) \leq \max\{1, \frac{1}{5s}\}, \quad \text{(2.18)}
\]

By Theorem 2.4 and (2.12), there exists \( \epsilon = \epsilon(s, n) > 0 \) such that

\[
\max_{B_{r_0}(y) \cap \Sigma_{i,t}} |A|(x, t) \leq \frac{1}{\epsilon r_0}, \quad \forall t \in [-\epsilon r_0^2, \epsilon r_0^2], \quad \text{(2.19)}
\]

where \( r_0 = 5s \). Therefore, for large \( i \) we have all higher order estimates of the second fundamental form at any point in \( \Sigma_{i,0} \setminus B_{2r_0}(S_0) \), where \( B_r(S_0) = \{ x \in \mathbb{R}^3 \ | \ d(x, S_0) \leq r \} \). Using a diagonal sequence argument and taking \( s \to 0 \) we can show that a subsequence of \( \Sigma_{i,0} \) converges in smooth topology, possibly with multiplicities, to a limit surface \( \Sigma_{\infty,0} \) away from the singular set \( S_0 \). The properties (2.14)-(2.15) imply that the multiplicity of the convergence is bounded by a constant \( N_0 \).

Note that by (2.19) the second fundamental form is uniformly bounded for any point \( (x, t) \in (\Sigma_{i,t} \setminus B_{2r_0}(S_0)) \times \{-\epsilon r_0^2, \epsilon r_0^2\} \cap (-1, 1) \). By compactness of mean curvature flow (c.f. Theorem 2.6 of [33]), the flow \{ \( \Sigma_{i,t} \setminus B_{2r_0}(S_0), t \in (-\epsilon r_0^2, \epsilon r_0^2) \cap (-1, 1) \} \) converges smoothly to a limit flow \{ \( \Sigma_{\infty,t} \setminus B_{2r_0}(S_0), t \in (-\epsilon r_0^2, \epsilon r_0^2) \cap (-1, 1) \} \) and by Property (6) in definition 2.7, \( \Sigma_{\infty,t} \setminus B_{2r_0}(S_0) \) satisfies the self-shrinker equation (2.17) for \( t \in (-\epsilon r_0^2, \epsilon r_0^2) \cap (-1, 1) \). We can also replace \( t = 0 \) by any other \( t_0 \in (-1, 1) \) and the above argument still works for the time interval \( (-\epsilon r_0^2 + t_0, \epsilon r_0^2 + t_0) \cap (-1, 1) \). Since \( r_0 = 5s > 0 \) is arbitrary small, by using a diagonal sequence argument and taking \( s \to 0 \) we have that \{ \( \Sigma_t, \Sigma_{\infty}(t) \), \(-1 < t < 1 \} \) converges locally smoothly to the flow \{ \( \Sigma_{\infty}, \Sigma_{\infty}(t) \), \(-1 < t < 1 \} \) away from \( S \) and \( \Sigma_{\infty,t} \) satisfies the equation (2.17). Note that \( \Sigma_{\infty} \) is a self-shrinker in \( \mathbb{R}^3 \) and it can be viewed as a minimal surface in \( (\mathbb{R}^3, g_{ij}) \) with \( g_{ij} = e^{-\frac{|x|^2}{4}} \delta_{ij} \). Thus, we can follow the argument in minimal surfaces (c.f. White [57], or Colding-Minicozzi [17]) to show that \( \Sigma_{\infty,t} \cup S_t \) is smooth and embedded and \( \Sigma_{i,t} \) converges to \( \Sigma_{\infty,t} \) in Hausdorff distance. The proposition is proved.

\[\square\]

As in [33], we show that the multiplicity in Proposition 2.8 is constant. To study the multiplicity, we define a function

\[
\Theta(x, r, t) := \lim_{i \to +\infty} \frac{\text{Area}_{S_i(t)}(\Sigma_{i,t} \cap B_r(x))}{\pi r^2}, \quad \forall (x, t) \in \Sigma_{\infty,t} \times (-1, 1), \quad \text{(2.20)}
\]

Then the multiplicity at \( (x, t) \in \Sigma_{\infty,t} \times (-1, 1) \) is defined by

\[
m(x, t) := \lim_{r \to 0} \Theta(x, r, t). \quad \text{(2.21)}
\]
It is clear that \( m(x, t) \) is an integer. In the following result, we show that \( m(x, t) \) is independent of \( x \) and \( t \). Note that in Lemma 3.14 of \cite{33} we proved the same result under the assumption that the mean curvature decays exponentially to zero. The first two steps of the proof here are similar to that of \cite{33} while the third step is different. We give all the details for completeness.

\textbf{Lemma 2.9.} \textit{Under the assumption of Proposition 2.8 the function } \( m(x, t) \) \textit{is a constant integer on } \( \Sigma_{\infty, t} \times (-1, 1) \). \textit{Namely, } \( m(x, t) \) \textit{is independent of } \( x \) \textit{and } \( t \).

\textit{Proof.} The proof can be divided into three steps.

\textit{Step 1.} \textit{For each } \( t \in (-1, 1) \), \( m(x, t) \) \textit{is constant on } \( \Sigma_{\infty, t} \setminus S \). Fix \( t_0 \in (-1, 1) \) and \( x_0 \in \Sigma_{\infty, t_0} \setminus S_{t_0} \). Since \( x_0 \) is a regular point, there exists \( r_0 > 0 \) such that for large \( i \),

\[
|A|(x, t_0) \leq \frac{1}{r_0}, \quad \forall \ x \in B_{r_0}(x_0) \cap \Sigma_{\infty, t_0}.
\]  

(2.22)

By Lemma 2.2 we can assume \( r_0 \) small such that \( B_{r_0}(x_0) \cap \Sigma_{\infty, t_0} \) can be written as a graph over the tangent plane of \( \Sigma_{\infty, t_0} \) at \( x_0 \). Let \( r_1 = \frac{200}{\pi r} \). For any \( p \in B_{r_1}(x_0) \cap \Sigma_{i, t_0} \), we have \( B_{\frac{r_1}{2}}(p) \subset B_{r_0}(x_0) \). Thus, (2.22) implies that for large \( i \),

\[
|A|(x, t_0) \leq \frac{2}{r_0}, \quad \forall \ x \in B_{\frac{r_1}{2}}(p) \cap \Sigma_{i, t_0}.
\]  

(2.23)

By Lemma 2.3 for any \( \delta > 0 \) there exists \( \rho_0 = \rho_0(r_0, \delta) \in (0, \frac{r_0}{200}) \) such that for any \( r \in (0, \rho_0) \) and any \( p \in B_{r_1}(x_0) \cap \Sigma_{i, t_0} \) we have

\[
1 - \delta \leq \frac{\text{Area}_{g_{i}(t_0)}(C_{p}(B_{r}(p) \cap \Sigma_{i, t_0}))}{\pi r^2} \leq 1 + \delta.
\]  

(2.24)

Suppose that \( B_{r_1}(x_0) \cap \Sigma_{i, t_0} \) has \( m \) connected components, where \( m \) is an integer bounded by a constant independent of \( i \) by Proposition 2.8. After taking a subsequence of \( \{\Sigma_{i, t_0}\} \) if necessary, we can assume that \( m \) are the same integer denoted by \( m \) with \( m \geq 1 \). For any \( x \in B_{\frac{r_1}{2}}(x_0) \cap \Sigma_{\infty, t_0} \), we denote by \( \alpha_x \) the normal line passing through \( x \) of \( \Sigma_{\infty, t_0} \). Since each component of \( B_{r_1}(x_0) \cap \Sigma_{i, t_0} \) converges to \( B_{r_1}(x_0) \cap \Sigma_{\infty, t_0} \) smoothly and \( B_{r_1}(x_0) \cap \Sigma_{\infty, t_0} \) is a graph over the tangent plane of \( \Sigma_{\infty, t_0} \) at \( x_0 \), \( \alpha_x \) intersects transversally each component of \( \Sigma_{i, t_0} \) at exactly one point. Suppose that

\[
\alpha_x \cap \left( B_{r_1}(x_0) \cap \Sigma_{i, t_0} \right) = \{p_1^{(1)}, p_1^{(2)}, \ldots, p_1^{(m)} \}.
\]

Then (2.24) implies that for any integer \( j \) with \( 1 \leq j \leq m \) and any \( r \in (0, \rho_0) \),

\[
1 - \delta \leq \frac{\text{Area}_{g_{i}(t_0)}(C_{p_1^{(j)}}(B_{r}(p_1^{(j)}) \cap \Sigma_{i, t_0}))}{\pi r^2} \leq 1 + \delta.
\]  

(2.25)

After shrinking \( r_0 \) if necessary, we can assume that \( B_{r}(x) \cap \Sigma_{\infty, t_0} \) has only one component for any \( r \in (0, \frac{r_0}{200}) \) and any \( x \in B_{\frac{r_1}{2}}(x_0) \cap \Sigma_{\infty, t_0} \). Since for any \( 1 \leq j \leq m \) and \( r \in (0, \rho_0) \) we have \( p_1^{(j)} \to x \) and \( C_{p_1^{(j)}}(B_{r}(p_1^{(j)}) \cap \Sigma_{i, t_0}) \) converges smoothly to \( B_{r}(x) \cap \Sigma_{\infty, t_0} \) as \( i \to +\infty \), (2.25) implies that

\[
m(1 - \delta) \leq \lim_{i \to +\infty} \frac{\text{Area}_{g_{i}(t_0)}(B_{r}(x) \cap \Sigma_{i, t_0})}{\pi r^2} \leq m(1 + \delta).
\]  

(2.26)
In other words, for any \( x \in B_{\frac{1}{2}}(x_0) \cap \Sigma_{\infty, t_0} \) and any \( r \in (0, \rho_0) \) we have
\[
m(1 - \delta) \leq \Theta(x, r, t_0) \leq m(1 + \delta).
\] (2.27)

Taking \( r \to 0 \) in (2.27), we have
\[
m(x, t_0) = \lim_{r \to 0} \Theta(x, r, t_0) = m,
\] \( \forall x \in B_{\frac{1}{2}}(x_0) \cap \Sigma_{\infty, t_0}. \)

By the connectedness of \( \Sigma_{\infty, t_0} \setminus S_{t_0} \), we know that \( m(x, t_0) \) is constant on \( \Sigma_{\infty, t_0} \setminus S_{t_0}. \)

**Step 2.** For each \( t \in (-1, 1) \), \( m(x, t) \) is constant on \( \Sigma_{\infty, t}. \) Fix \( t_0 \in (-1, 1). \) It suffices to consider a singular point \( p_0 \in S_{t_0}. \) Suppose that \( B_r(p_0) \cap \Sigma_{\infty, t_0} \) has no other singular points except \( p_0 \) for any \( r \in (0, r_0). \) Then all points in \( (B_r(p_0) \setminus B_{\epsilon}(p_0)) \cap \Sigma_{\infty, t_0} \) are regular and \( (B_r(p_0) \setminus B_{\epsilon}(p_0)) \cap \Sigma_{i, t_0} \) has \( m \) connected components. Thus, we have
\[
\text{Area}_{g_i(t_0)}(\Sigma_{i, t_0} \cap B_r(p_0)) \leq \text{Area}_{g_i(t_0)}(\Sigma_{i, t_0} \cap (B_r(p_0) \setminus B_{\epsilon}(p_0))) + \text{Area}_{g_i(t_0)}(\Sigma_{i, t_0} \cap B_{\epsilon}(p_0))
\leq \text{Area}_{g_i(t_0)}(\Sigma_{i, t_0} \cap (B_r(p_0) \setminus B_{\epsilon}(p_0))) + N\epsilon^2,
\] (2.28)

where we used (2.14) in the last inequality. Since each component of \( \Sigma_{i, t_0} \cap (B_r(p_0) \setminus B_{\epsilon}(p_0)) \) converges to \( (B_r(p_0) \setminus B_{\epsilon}(p_0)) \cap \Sigma_{\infty} \) smoothly, we have
\[
\lim_{i \to +\infty} \text{Area}_{g_i(t_0)}(\Sigma_{i, t_0} \cap (B_r(p_0) \setminus B_{\epsilon}(p_0))) = m \text{Area}_{g_{\infty}(t_0)}(\Sigma_{\infty, t_0} \cap (B_r(p_0) \setminus B_{\epsilon}(p_0))).
\] (2.29)

Note that \( m \) is also the multiplicity at each regular point in \( \Sigma_{\infty, t_0} \) by Step 1. Combining (2.28) with (2.29), we have
\[
m \text{Area}_{g_{\infty}(t_0)}(\Sigma_{\infty, t_0} \cap (B_r(p_0) \setminus B_{\epsilon}(p_0))) \\
\leq \lim_{i \to +\infty} \text{Area}_{g_i(t_0)}(\Sigma_{i, t_0} \cap B_r(p_0)) \\
\leq m \text{Area}_{g_{\infty}(t_0)}(\Sigma_{\infty, t_0} \cap (B_r(p_0) \setminus B_{\epsilon}(p_0))) + N\epsilon^2.
\] (2.30)

Taking \( \epsilon \to 0 \) in (2.30), we have
\[
\lim_{i \to +\infty} \text{Area}_{g_i(t_0)}(\Sigma_{i, t_0} \cap B_r(p_0)) = m \text{Area}_{g_{\infty}(t_0)}(\Sigma_{\infty, t_0} \cap B_r(p_0)).
\] (2.31)

Thus, we have
\[
m(p_0, t_0) = \lim_{r \to 0} \frac{\text{Area}_{g_i(t_0)}(\Sigma_{i, t_0} \cap B_r(p_0))}{\pi r^2} \\
= m \lim_{r \to 0} \frac{\text{Area}_{g_{\infty}(t_0)}(\Sigma_{\infty, t_0} \cap B_r(p_0))}{\pi r^2} = m.
\]

This implies that the multiplicity of each singular point is the same as that of any regular point.

**Step 3.** \( m(x, t) \) is constant in \( t. \) Fix any \( t_0 \in (-1, 1) \) and \( x_0 \in \Sigma_{\infty, t_0} \setminus S_{t_0}. \) Since \( x_0 \) is a regular point, there exists \( r_0 > 0 \) such that for large \( i, \)
\[
|A|(x, t_0) \leq \frac{1}{r_0}, \quad \forall x \in B_{r_0}(x_0) \cap \Sigma_{\infty, t_0},
\] (2.32)
and for any \( r \in (0, r_0) \) the surface \( B_r(x_0) \cap \Sigma_{i,t_0} \) has only one component. Let \( m_0 = m(x_0, t_0) \) and \( r_1 = \frac{\rho_0}{2} \). For large \( i \), \( B_{r_1}(x_0) \cap \Sigma_{i,t_0} \) has \( m_0 \) connected components, which we denote by \( \Omega_{i,1}, \ldots, \Omega_{i,m_0} \). Since for each integer \( k \in [1, m_0] \) the component \( \Omega_{i,k} \) converges smoothly to \( \Sigma_{n_0} \cap B_{r_1}(x_0) \) as \( i \to +\infty \), similar to Step 1 we can find \( x_{i,k} \in \Omega_{i,k} \) such that \( \lim_{i \to +\infty} d(x_{i,k}, x_0) = 0 \). By the choice of \( r_1 \), we have
\[
B_{r_1}(x_{i,k}) \subset B_{r_0}(x_0).
\tag{2.33}
\]
Thus, (2.32) implies that for any integer \( k \in [1, m_0] \) and large \( i \),
\[
|A|(x, t_0) \leq \frac{1}{r_1}, \quad \forall x \in C_{x_{i,k}}(B_{r_1}(x_{i,k}) \cap \Sigma_{i,t_0}). \tag{2.34}
\]
By Lemma 2.2 for any \( \delta > 0 \) there exists \( r_0 = r_0(r_0, \delta) \in (0, \frac{\rho_0}{2}) \) such that for any \( r \in (0, \rho_0) \) we have
\[
1 - \delta \leq \frac{\text{Area}_{g_{\rho_0}}(C_{x_{i,k}}(B_r(x_{i,k}) \cap \Sigma_{i,t_0}))}{\pi r^2} \leq 1 + \delta. \tag{2.35}
\]
Note that by (2.33) and the definition of \( \Omega_{i,k} \), for any large \( i \) we have
\[
C_{x_{i,k}}(B_{2\rho_0}(x_{i,k}) \cap \Sigma_{i,t_0}) \neq C_{x_{i,k}}(B_{2\rho_0}(x_{i,k'}) \cap \Sigma_{i,t_0}), \quad \forall \ k \neq k'. \tag{2.36}
\]
Using (2.34) and the assumption that \( \max_{\Sigma_{i,t_0}} |H| \leq \Lambda \) by (2.12), Theorem 2.4 implies that there exists \( \eta(\Lambda) \) and \( \epsilon(\Lambda) > 0 \) such that
\[
|A|(x, t) \leq \frac{1}{\epsilon r_1}, \quad \forall x \in C_{x_{i,k,t}}(B_{r}(x_{i,k,t}) \cap \Sigma_{i,t}), \quad t \in [t_0 - \eta r_1^2, t_0 + \eta r_1^2], \tag{2.37}
\]
where \( x_{i,k,t} = x_t(x_{i,k}^{-1}(x_{i,k})) \). Similar to (2.35), there exists \( r_1 = r_1(r_0, \delta) \in (0, \frac{\rho_0}{10}) \) such that for any \( r \in (0, \rho_1) \) we have
\[
1 - \delta \leq \frac{\text{Area}_{g_{\rho_0}}(C_{x_{i,k,t}}(B_r(x_{i,k,t}) \cap \Sigma_{i,t}))}{\pi r^2} \leq 1 + \delta, \quad t \in [t_0 - \eta r_1^2, t_0 + \eta r_1^2]. \tag{2.38}
\]
We show that we can choose \( r_1 \) and \( \tau = \tau(r_0, \delta, \Lambda) \in (0, \eta r_1^2] \) small such that for any \( k \neq k' \)
\[
C_{x_{i,k,t}}(B_{r_1}(x_{i,k,t}) \cap \Sigma_{i,t}) \neq C_{x_{i,k',t}}(B_{r_1}(x_{i,k',t}) \cap \Sigma_{i,t}), \quad \forall t \in [t_0 - \tau, t_0 + \tau]. \tag{2.39}
\]
Suppose not, we can find \( \tau_0 \in (0, \eta r_1^2] \), a continuous curve \( \gamma_{\tau_0}(s)(s \in [0, 1]) \) connecting \( x_{i,k, t_0 + \tau_0} \) and \( x_{i,k', t_0 + \tau_0} \) with
\[
\gamma_{\tau_0} \subset B_{r_1}(x_{i,k, t_0 + \tau_0}) \cap \Sigma_{i,t_0 + \tau_0}, \quad \gamma_{\tau_0} \subset B_{r_1}(x_{i,k', t_0 + \tau_0}) \cap \Sigma_{i,t_0 + \tau_0}. \tag{2.40}
\]
Let \( \gamma_t = x_{t_0 + \tau}(x_{i,k}^{-1}(\gamma_{\tau_0})) \). Then \( \gamma_0(s)(s \in [0, 1]) \) is a curve connecting \( x_{i,k} \) and \( x_{i,k'} \). Since the mean curvature satisfies \( \max_{\Sigma_{i,t_0}} |H| \leq \Lambda \), we have
\[
|\dot{x}(p, t) - \dot{x}(q, t)| \leq |\dot{x}(p, s) - \dot{x}(q, s)| + 2\Lambda|t - s|. \tag{2.41}
\]
For small \( \tau_0, \tag{2.41} \) with (2.40) implies that
\[
\gamma_0 \subset B_{\rho_0}(x_{i,k}) \cap \Sigma_{i,t_0}, \quad \gamma_{\tau_0} \subset B_{\rho_0}(x_{i,k}) \cap \Sigma_{i,t_0}. \tag{2.42}
\]
which contradicts (2.36). Therefore, (2.39) holds.

Since \( x_{i,k} \in B_{r_1}(x_0) \) and the mean curvature is uniformly bounded, \( x_{i,k,t} \) lies in a bounded domain for any \( t \in [t_0 - \tau, t_0 + \tau] \). Thus, for each integer \( k \in [1, m_0] \) and any \( t \in [t_0 - \tau, t_0 + \tau] \) a subsequence of \( C_{x_{i,k,t}}(B_{r_1}(x_{i,k,t}) \cap \Sigma_{i,t}) \) converges to \( C_{x_t}(B_{r_1}(x_t) \cap \Sigma_{\infty,t}) \) smoothly, where \( x_t \in \Sigma_{\infty,t} \) is a limit point of \( \{x_{i,k,t}\}_{i=1}^{\infty} \). Then (2.38) and (2.39) imply that for any \( r \in (0, r_1) \) and \( t \in [t_0 - \tau, t_0 + \tau] \) we have

\[
\lim_{i \to +\infty} \frac{\text{Area}_{g_i(t)}(B_{r}(x_{i,t}) \cap \Sigma_{i,t})}{\pi r^2} \geq \lim_{i \to +\infty} \frac{\text{Area}_{g_i(t)}(C_{x_{i,t}}(B_{r}(x_{i,t}) \cap \Sigma_{i,t}))}{\pi r^2} \geq m_0(1 - \delta). \tag{2.43}
\]

Thus, we have

\[
m(x,t) \geq m_0 = m(x_0, t_0), \quad \forall \ t \in [t_0 - \tau, t_0 + \tau]. \tag{2.44}
\]

By Step 2, (2.44) implies that for any \( x \in \Sigma_{\infty,t} \) and \( y \in \Sigma_{\infty,t_0} \) we have

\[
m(x,t) \geq m(y, t_0), \quad \forall \ t \in [t_0 - \tau, t_0 + \tau]. \tag{2.45}
\]

Thus, the multiplicity \( m(x,t) \) is a constant independent of \( x \) and \( t \). The lemma is proved.

\[
\square
\]

**Lemma 2.10.** The same assumption as in Proposition 2.8. Fix any \( t_0 \in (-1, 1) \) and any \( \delta > 0 \).

(1) If \( x_0 \in \Sigma_{\infty,t_0} \setminus S_{t_0} \) and \( x_i \in \Sigma_{i,t_0} \) with \( x_i \to x_0 \), then there exists \( r' = r'(\delta, \Sigma_{\infty,t_0}) > 0 \) such that for any \( r \in (0, r') \) we have

\[
1 - \delta \leq \lim_{i \to +\infty} \frac{\text{Area}_{g_{i(t_0)}}(C_{x_i}(B_{r}(x_i) \cap \Sigma_{i,t_0}))}{\pi r^2} \leq 1 + \delta. \tag{2.46}
\]

(2) If \( x_0 \in S_{t_0} \) and \( x_i \in \Sigma_{i,t_0} \) with \( x_i \to x_0 \), then there exists \( r' = r'(\delta, \Sigma_{\infty,t_0}) > 0 \) such that for any \( r \in (0, r') \) we have

\[
\lim_{i \to +\infty} \frac{\text{Area}_{g_{i(t_0)}}(C_{x_i}(B_{r}(x_i) \cap \Sigma_{i,t_0}))}{\pi r^2} \geq 2(1 - \delta). \tag{2.47}
\]

**Proof.** (1). Since \( x_0 \) is a regular point, there exists \( r_0 = r_0(\Sigma_{\infty,t_0}) > 0 \) such that for large \( i \),

\[
|A|(x, t_0) \leq \frac{1}{r_0}, \quad \forall \ x \in B_{r_0}(x_0) \cap \Sigma_{\infty,t_0}. \tag{2.48}
\]

Let \( r_1 = \frac{r_0}{2} \). Then for large \( i \), we have

\[
|A|(x, t_0) \leq \frac{1}{r_1}, \quad \forall \ x \in B_{r_1}(x_i) \cap \Sigma_{i,t_0}. \tag{2.49}
\]

By Lemma 2.3, for any \( \delta > 0 \) there exists \( r' = r'(\delta, r_0) > 0 \) such that for any \( r \in (0, r') \) the area ratio of \( C_{x_i}(B_{r}(x_i) \cap \Sigma_{i,t_0}) \) is given by

\[
1 - \delta \leq \frac{\text{Area}_{g_{i(t_0)}}(C_{x_i}(B_{r}(x_i) \cap \Sigma_{i,t_0}))}{\pi r^2} \leq 1 + \delta. \tag{2.50}
\]
Thus, (2.46) holds.

(2). Let \( x_0 \in S_{t_0} \) and \( r_0 > 0 \) such that \( \Sigma_{\infty,t_0} \cap B_{r_0}(x_0) \) has only one component and no other singular points except \( x_0 \). Let \( Q_i := \max_{C_{x_i}(B_{r_0}(x_i) \cap \Sigma_{t_0})} |A| \rightarrow +\infty \). Since for any \( r \in (0, r_0) \) the surface \( C_{x_i}(B_{r_0}(x_i) \cap \Sigma_{t_0}) \) converges smoothly to \( \Sigma_{\infty,t_0} \cap (B_{r_0}(x_0) \setminus B_r(x_0)) \), for large \( i \), \( Q_i \) is achieved by some point \( x'_i \in C_{x_i}(B_r(x_i) \cap \Sigma_{t_0}) \) with \( x'_i \rightarrow x_0 \). As in Step 2 of the proof of Lemma 2.9 for any \( r \in (0, r_0) \) we have

\[
\lim_{i \rightarrow +\infty} \frac{\text{Area}_{g_i(t_0)}(C_{x'_i}(B_r(x'_i) \cap \Sigma_{t_0}))}{\pi r^2} = m \frac{\text{Area}_{g_\infty(t_0)}(B_r(x_0) \cap \Sigma_{\infty,t_0})}{\pi r^2},
\]

(2.51)

where \( m \) is a positive integer.

We next show that \( m = 1 \) cannot happen. For otherwise, (2.50) implies that for any \( \delta > 0 \) there exists \( r' = r'(\delta, \Sigma_{\infty,t_0}) > 0 \) such that

\[
\frac{\text{Area}_{g_\infty(t_0)}(B_r(x_0) \cap \Sigma_{\infty,t_0})}{\pi r^2} \leq 1 + \delta, \quad \forall r \in (0, r').
\]

(2.52)

Then (2.52) and (2.51) imply that for large \( i \),

\[
\frac{\text{Area}_{g_i(t_0)}(C_{x'_i}(B_r'(x'_i) \cap \Sigma_{t_0}))}{\pi r'^2} \leq 1 + 2\delta.
\]

(2.53)

We choose \( r' = r'(\delta, \Sigma_{\infty,t_0}, \Lambda) \) small such that \( (1 + 2\delta)e^{\Lambda r'} \leq 1 + 3\delta \). Since the mean curvature satisfies \( \max_{\Sigma_{t_0}} |H| \leq \Lambda \), by Lemma 3.5 of [33] for any \( r \in (0, r') \) we have

\[
\frac{\text{Area}_{g_i(t_0)}(C_{x'_i}(B_r(x'_i) \cap \Sigma_{t_0}))}{\pi r'^2} \leq e^{\Lambda r'} \frac{\text{Area}_{g_i(t_0)}(C_{x'_i}(B_r'(x'_i) \cap \Sigma_{t_0}))}{\pi r'^2} \leq 1 + 3\delta,
\]

(2.54)

where we used (2.53). We rescale the surface by

\[
\tilde{\Sigma}_{i,s} = Q_i(\Sigma_{t_0} + Q_{i2}s - x'_i), \quad \forall s \in (-1 + t_0)Q_{i2}^2, (1 - t_0)Q_{i2}^2).
\]

Then \( \{\tilde{\Sigma}_{i,s}, -1 < s < 1\} \) is a sequence of mean curvature flow with

\[
\max_{\tilde{\Sigma}_{i,s} \times (-1,1)} |H| \leq Q_i^{-1}\Lambda \rightarrow 0.
\]

By the choice of \( Q_i \) we have

\[
\sup_{C_{0}(\tilde{\Sigma}_{i,0} \cap B_{2Q_{i2}}(0))} |A| \leq 1.
\]

(2.55)

By Theorem 3.8 of [33], there exists a universal constant \( \epsilon \) such that

\[
\sup_{C_{0}(\tilde{\Sigma}_{i,s} \cap B_{2Q_{i2}}(0))} |A| \leq \frac{1}{\epsilon}, \quad \forall s \in (-1,1).
\]

Thus, by the compactness of mean curvature flow (c.f. Theorem 2.6 of [33]) the surface \( C_{0}(\tilde{\Sigma}_{i,0} \cap B_{2Q_{i2}}(0)) \) converges in smooth topology to a complete smooth minimal surface \( \tilde{\Sigma}_\infty \) with

\[
\sup_{\tilde{\Sigma}_\infty} |A| \leq 1, \quad |A|(0) = 1.
\]

(2.56)
Since (2.54) implies that
\[
\frac{\text{Area}_{g_i(0)}(C_0(B_r(0) \cap \Sigma_{i,0}))}{\pi r^2} \leq 1 + 3\delta, \quad \forall r \in (0, Q_i r'),
\] (2.57)
we have
\[
\frac{\text{Area}_{g_\infty(0)}(C_0(B_r(0) \cap \Sigma_{\infty}))}{\pi r^2} \leq 1 + 3\delta, \quad \forall r > 0.
\] (2.58)
By Lemma 3.6 of [33], there exists a universal constant \(\delta_0 > 0\) such that if we choose \(\delta = \frac{\delta_0}{4}\), then \(\Sigma_{\infty}\) must be a plane, which contradicts (2.56).

Therefore, we have \(m \geq 2\). By (2.50) we can find \(r' = r'(\delta, \Sigma_{\infty,0}) \in (0, r_0)\) such that for any \(r \in (0, r')\)
\[
\frac{\text{Area}_{g_{\infty,0}}(B_r(x_0) \cap \Sigma_{\infty,0})}{\pi r^2} \geq 1 - \delta.
\] (2.59)
Since the surface \(C_{x_i}(B_{r_0}(x_i) \cap \Sigma_{i,0}) \cap B_r(x_i)\) converges smoothly to \(\Sigma_{\infty,0} \cap (B_{r_0}(x_0) \setminus B_r(x_0))\) with multiplicity at least 2, we have
\[
\lim_{i \to +\infty} \frac{\text{Area}_{g_i(0)}(C_{x_i}(B_{r_i}(x_i) \cap \Sigma_{i,0}))}{\pi r^2} = m \frac{\text{Area}_{g_{\infty,0}}(B_r(x_0) \cap \Sigma_{\infty,0})}{\pi r^2} \geq 2(1 - \delta).\] (2.60)
The lemma is proved. \(\square\)

Using the boundedness of the mean curvature and Lemma 2.10 we show that the singular set \(S\) consists of locally finitely many Lipschitz curves.

**Lemma 2.11.** Fix large \(R > 0\). Under the assumption of Proposition 2.8 the singular set \(S\) is the union of locally finitely many space-time singular curves, i.e,
\[
S \cap \left( B_{R}(0) \cap (-1,1) \right) = \bigcup_{k=1}^{l} \left\{ (t, \xi_k(t)) \bigg| t \in (-1,1), \xi_k(t) \in B_{R}(0) \cap S_{l} \right\},
\]
where \(S_{l}\) is defined in Proposition 2.8 and \(\{\xi_k(t)\}_{k=1}^{l}\) are \(\Lambda\)-Lipschitz curves, i.e,
\[
|\xi_k(t_1) - \xi_k(t_2)| \leq \Lambda |t_1 - t_2|, \quad \forall t_1, t_2 \in (-1,1).
\] (2.61)
Here \(\Lambda\) is the constant in (2.12).

**Proof.** For any \(t_1 \in (-1,1)\) and any \(p_{t_1} \in S_{l} \cap B_{R}(0)\), we show that there exists a Lipschitz curve in \(S\) passing through \(p_{t_1}\). Since \(p_{t_1}\) is singular, by Lemma 2.10 we can find a sequence of points \(p_{i,t_1} \in \Sigma_{i,t_1}\) and \(r' = r'(\Sigma_{\infty,t_1}) > 0\) such that \(p_{i,t_1} \to p_{t_1}\) and for any \(r \in (0, r')\),
\[
\lim_{i \to +\infty} \frac{\text{Area}(C_{p_{i,t_1}}(B_r(p_{i,t_1}) \cap \Sigma_{i,t_1}))}{\pi r^2} \geq \frac{7}{4}.
\] (2.62)
Let \(r_1 \in (0, r')\). By Lemma 3.4 in Li-Wang [33] and (2.62), there exists \(\eta(r_1, \Lambda) > 0\) such that for any \(t_2 \in (t_1 - \eta, t_1 + \eta) \cap (-1,1)\) we have
\[
\frac{\text{Area}(C_{p_{i,t_2}}(B_{r_2}(p_{i,t_2}) \cap \Sigma_{i,t_2}))}{\pi r_2^2} \geq e^{-\Lambda^2 |t_2-t_1|} \left( 1 + \frac{2\Lambda}{r_1} |t_2 - t_1| \right)^{-2} \frac{\text{Area}(C_{p_{1,t_1}}(B_{r_1}(p_{1,t_1}) \cap \Sigma_{1,t_1}))}{\pi r_1^2}
\]
\[
\geq e^{-\Lambda^2 \eta} \left( 1 + \frac{2\Lambda \eta}{r_1} \right)^{-2} \frac{\text{Area}(C_{p_{1,t_1}}(B_{r_1}(p_{1,t_1}) \cap \Sigma_{1,t_1}))}{\pi r_1^2},
\] (2.63)
where \( r_2 = r_1 + 2\Lambda|t_2 - t_1| \). Taking the limit in (2.63) and choosing \( \eta(r_1, \Lambda) \) small, we have
\[
\lim_{i \to +\infty} \frac{\text{Area}(C_{p_i,t_2}(B_{r_2}(p_i,t_2) \cap \Sigma_{i,t_2}))}{\pi r_2^2} \geq \frac{3}{2},
\]
(2.64)
where \( p_{i,t_2} = x_{i,t_2}(x_{i,t_1}^{-1}(p_{i,t_1})) \). Since the mean curvature is uniformly bounded along the flow, all points \( \{p_{i,t_2}\}_{i=1}^{\infty} \) lie in a bounded ball at \( p_1 \). Thus, we can find a subsequence of \( \{p_{i,t_2}\}_{i=1}^{\infty} \) such that it converges to a point, which we denoted by \( p_{t_2} \).

Suppose that \( p_{t_2} \) is a regular point for some \( t_2 \in (t_1 - \eta, t_1 + \eta) \). Note that in Part (1) of Lemma 2.10 the constant \( r' \) depends only on the geometry of \( \Sigma_{\infty,t} \). Therefore, by Lemma 2.10 for \( \delta = \frac{1}{4} \) we can choose a uniform constant \( r' > 0 \) independent of \( t \in (t_1 - \eta, t_1 + \eta) \) such that
\[
\lim_{i \to +\infty} \frac{\text{Area}(C_{p_i,t_2}(B_{r}(p_{i,t_2}) \cap \Sigma_{i,t_2}))}{\pi r^2} \leq \frac{5}{4}, \quad \forall r \in (0, r').
\]
(2.65)
Now we choose \( t_2 \in (t_1 - \eta, t_1 + \eta) \) such that
\[
r_2 = r_1 + 2\Lambda|t_2 - t_1| < r'.
\]
Then (2.65) contradicts (2.64). Thus, \( p_{t_2} \) is a singular point. Moreover, we have
\[
|p_{i,t_1} - p_{i,t_2}| \leq \int_{t_1}^{t_2} |H| \, dt \leq \Lambda|t_1 - t_2|.
\]
By taking the limit \( i \to +\infty \), we have
\[
|p_{t_1} - p_{t_2}| \leq \Lambda|t_1 - t_2|.
\]
Therefore, \( p_{t_1} \) lies in a \( \Lambda \)-Lipschitz curve in \( S \). Since for any \( t \in (-1, 1) \) the set \( S_t \) is locally finite by Proposition 2.8 the singular curves are locally finite. The lemma is proved.

3 The rescaled mean curvature flow

In this section, we will show the smooth convergence of rescaled mean curvature flow under uniform mean curvature bound. As is pointed out in the introduction, we have no long-time pseudolocality of the flow and the singularities don’t move along straight lines. In order to show the \( L \)-stability of the limit self-shrinker, we need an estimate on the asymptotical behavior of the positive solution near the singular set (c.f. Lemma 3.21 and Lemma 3.28), and the proof of this estimate will be delayed to Section 4.

**Theorem 3.1.** Let \( \{ (\Sigma^2, x(t)), 0 \leq t < +\infty \} \) be a closed smooth embedded rescaled mean curvature flow
\[
\left( \frac{\partial x}{\partial t} \right)_\perp = -\left( H - \frac{1}{2} (x, n) \right)
\]
satisfying
\[
d(\Sigma_t, 0) \leq D, \quad \text{and} \quad \max_{\Sigma_t} |H(p,t)| \leq \Lambda
\]
for two constants \( D, \Lambda > 0 \). Then for any \( t_i \to +\infty \) there exists a subsequence of \( \{ \Sigma_{t_i+t}, -1 < t < 1 \} \) such that it converges in smooth topology to a complete smooth self-shrinker with multiplicity one as \( i \to +\infty \).
We sketch the proof of Theorem 3.1. First, we show the weak compactness for any sequence of the rescaled mean curvature flow in Lemma 3.4. Suppose that the multiplicity is at least two. By using the decomposition of spaces (c.f. Definition 3.5) we can select a special sequence \( \{t_i\} \) in Lemma 3.13 for each \( \epsilon > 0 \). This special sequence is needed to control the upper bound of the function \( w_i \) away from the singular set by using the parabolic Harnack inequality (c.f. Lemma 3.16). Then we can take the limit for the function \( w_i \) and obtain a positive function \( w \) with uniform bounds on any compact set away from the singular set (c.f. Lemma 3.17). The function \( w \) satisfies the linearized mean curvature flow equation. To study the growth behavior of \( w \) near the singular set, we take a sequence of \( \epsilon_i \to 0 \) and for each \( \epsilon_i \) we repeat the above process to get a sequence of functions \( \{w_{i,k}\}_{k=1}^\infty \). After choosing a diagonal sequence and taking the limit, we get a function \( w \) with good growth estimates near the singular set (c.f. Proposition 3.23) by assuming Theorem 4.2 in the next section. The bounds of \( w \) imply the \( L^1 \)-stability of the limit self-shrinker (c.f. Lemma 3.25), and this step also relies on Theorem 3.7. However, the limit self-shrinker is not \( L^1 \)-stable by Colding-Minicozzi’s theorem (c.f. Theorem 3.7) and we obtain a contradiction.

### 3.1 Convergence away from singularities

We recall Ilmanen’s local Gauss-Bonnet formula in [41] to control the \( L^2 \) norm of the second fundamental form. Let \( \Sigma \) be a smooth surface with smooth boundary \( \partial \Sigma \). We denote by \( e(\Sigma) \) the genus of \( \Sigma \) which is the genus of the closed surface obtained by capping off the boundary components of \( \Sigma \) by disks.

**Lemma 3.2.** (c.f. Ilmanen [41]) Let \( R > 1 \) and let \( \Sigma \) be a surface properly immersed in \( B_R(p) \). Then for any \( \epsilon > 0 \) we have

\[
(1 - \epsilon) \int_{\Sigma \cap B_1(p)} |A|^2 \, d\mu \leq \int_{\Sigma \cap B_R} |H|^2 \, d\mu + 8\pi e(\Sigma \cap B_R(p)) + \frac{24\pi R^2}{\epsilon(R-1)^2} \sup_{r \in [1,R]} \frac{\text{Area}(\Sigma \cap B_r(p))}{\pi r^2}.
\]

For simplicity, we introduce the following definition.

**Definition 3.3.** Let \( \rho : \mathbb{R}^+ \to \mathbb{R}^+ \) be an increasing positive function. For any \( D, N > 0 \), we denote by \( C(D, N, \rho) \) the space of all smooth embedded self-shrinkers \( \Sigma^2 \subset \mathbb{R}^3 \) satisfying the properties that for any \( r > 0 \) and \( p \in \Sigma \),

\[
\int_{\Sigma \cap B_r(0)} |A|^2 \leq \rho(r), \quad \text{Area}(B_r(p) \cap \Sigma) \leq \pi Nr^2, \quad \text{and} \quad d(0, \Sigma) \leq D.
\]

Note that the space \( C(D, N, \rho) \) is compact in the smooth topology by Colding-Minicozzi [18].

The following result shows that the rescaled mean curvature flow converges locally smoothly to a self-shrinker away from singularities.

**Lemma 3.4.** Under the assumption of Theorem 3.1, for any sequence \( t_i \to +\infty \), there is a smooth self-shrinker \( \Sigma_\infty \in C(D, N, \rho) \) and a space-time set \( S \subset \Sigma_\infty \times \mathbb{R} \) satisfying the following properties.

1. For any \( T > 1 \), there is a subsequence, still denoted by \( \{t_i\} \), such that \( \{\Sigma_{t_i+T}, -T < t < T\} \) converges in smooth topology, possibly with multiplicities, to \( \Sigma_\infty \) away from \( S \);
(2) For any \( R > 0, S \cap (B_R(0) \times (-T, T)) \) consists of finite many \( \sigma \)-Lipschitz curves with Lipschitz constant \( \sigma \) depending on \( T \) and \( R \);

(3) The convergence in part (1) is also in (extrinsic) Hausdorff distance;

(4) The limit self-shrinker \( \Sigma_\infty \) is independent of the choice of \( T \). In other words, for different \( T \) we can choose two different subsequences of \( \{ t_i \} \) such that the corresponding flows in part (1) have the same limit self-shrinker \( \Sigma_\infty \).

Proof. We divide the proof into the following steps.

Step 1. The area ratio along the flow (3.1) is uniformly bounded from above. In fact, we rescale the flow (3.1) by

\[
s = 1 - e^{-t}, \quad \tilde{\Sigma}_s = \sqrt{1 - s} \Sigma_{-\log(1-s)}
\]

such that \( \{ \tilde{\Sigma}_s, 0 \leq s < 1 \} \) is a mean curvature flow satisfying the equation (1.1). By Lemma 2.9 of Colding-Minicozzi [19] and Lemma 2.3 in Li-Wang [33], we have that the area ratio of (3.1) is uniformly bounded from above.

Step 2. For any large \( R \), the energy of \( \Sigma_t \cap B_R(0) \) is uniformly bounded along the flow (3.1). In fact, by Lemma 3.2 we have

\[
\int_{\Sigma_t \cap B_R(0)} |A|^2 \, d\mu_t \leq 2 \int_{\Sigma_t \cap B_{2R}(0)} |H|^2 \, d\mu_t + C(N, e(\Sigma)) \\
\leq 8\pi N \Lambda^2 R^2 + C(N, e(\Sigma)),
\]

where \( N \) denotes the upper bound of the area ratio. Therefore, for any \( t > 0 \) the energy of \( \Sigma_t \cap B_R(0) \) is bounded by a constant \( C(N, \Lambda, R, e(\Sigma)) \).

Step 3. For each sequence \( t_i \to +\infty \), we obtain a refined sequence converging to a limit self-shrinker. For any sequence \( t_i \to +\infty \), we rescale the flow \( \Sigma_t \) by

\[
s = 1 - e^{-(t-t_i)}, \quad \tilde{\Sigma}_{i,s} = \sqrt{1 - s} \Sigma_{t_i-\log(1-s)}
\]

such that for each \( i \) the flow \( \{ \tilde{\Sigma}_{i,s}, 1 - e^{t_i} \leq s < 1 \} \) is a mean curvature flow satisfying (1.1) with the following properties:

(a). For any small \( \lambda > 0 \), the mean curvature of \( \tilde{\Sigma}_{i,s} \) satisfies

\[
\max_{\tilde{\Sigma}_{i,s} \times [1-e^{t_i}, 1-\lambda]} |\tilde{H}_i|(p, s) \leq \tilde{A} := \frac{\Lambda}{\sqrt{\lambda}};
\]

(b). For any large \( R \), the energy of \( \tilde{\Sigma}_{i,s} \cap B_R(0) \) is uniformly bounded;

(c). The area ratio is uniformly bounded from above;

(d). The area ratio is uniformly bounded from below;

(e). There exists a constant \( D' > 0 \) such that \( d(\tilde{\Sigma}_{i,s}, 0) \leq D' \) for any \( i \);

(f). We have

\[
\lim_{i \to +\infty} \int_{-T}^{1-\lambda} dt \int_{\tilde{\Sigma}_{i,s}} e^{\frac{|\mathbf{x}_i|^2}{4(1-s)}} \left| \tilde{H}_i - \frac{1}{2(1-s)} \langle \mathbf{x}_i, \mathbf{n} \rangle \right|^2 d\tilde{\mu}_{i,s} = 0.
\]
In fact, Property (a) and (e) follow from the assumption (3.2), and Property (b) follows from (3.4). Property (c) follows from Step 1, and Property (d) follows from Lemma 3.5 in Li-Wang [33]. To prove Property (f), by Huisken’s monotonicity formula along the rescaled mean curvature flow (3.1) we have

\[ \frac{d}{dt} \int_{\Sigma_t} e^{-\frac{|x|^2}{4}} d\mu_t = - \int_{\Sigma_t} e^{-\frac{|x|^2}{4}} \left| H - \frac{1}{2} \langle x, n \rangle \right|^2 d\mu_t. \]  
(3.7)

This implies that

\[ \int_0^\infty dt \int_{\Sigma_t} e^{-\frac{|x|^2}{4}} \left| H - \frac{1}{2} \langle x, n \rangle \right|^2 d\mu_t < +\infty. \]

Let \( T, \lambda > 0 \) with \(-T < 1 - \lambda\). For any \( t_i \to +\infty \), we have

\[ \lim_{t_i \to +\infty} \int_{t_i - \log \lambda}^{t_i - \log(1 + T)} dt \int_{\Sigma_t} e^{-\frac{|x|^2}{4}} \left| H - \frac{1}{2} \langle x, n \rangle \right|^2 d\mu_t = 0. \]
(3.8)

Then (3.6) follows from (3.5) and (3.8). Therefore, by Definition 2.7 for any \( T > 0 \), small \( \lambda > 0 \) and any \( s_0 \in [-T + 1, -1) \) the sequence \( \{ \hat{\Sigma}_{s_0, t} \} \) is a refined sequence. By Proposition 2.8 a subsequence of \( \{ \hat{\Sigma}_{s_0, t} \} \) converges in smooth topology, possibly with multiplicities, to a self-shrinker flow \( \{ \Sigma_{s_0, t} \} \) away from a space-time, \( \Lambda \)-Lipschitz singular set \( \hat{S} \).

**Step 4.** Let \( t' = t - t_i \) and \( \Sigma_{t_i, t'} = \Sigma_{t_i + t'} \). Since \( \{ \Sigma_{s, t} \} \) converges locally smoothly to \( \{ \Sigma_{s_0, t} \} \) away from \( \hat{S} \), by (3.5) the flow \( \{ \Sigma_{s, t} \} \) also converges locally smoothly to a self-shrinker \( \Sigma_{s_0, t} \) satisfying

\[ H - \frac{1}{2} \langle x, n \rangle = 0 \]

away from a space-time singular set \( S \) with

\[ S_{t'} = \frac{1}{\sqrt{1 - s}} \hat{S}_s. \]
(3.9)

Here \( s = 1 - e^{-t'} \). Now we show the Lipschitz property of \( S \). By (3.5), for any curve \( \xi(t') \) of \( S \), we can find a curve \( \xi(s) \) of \( \hat{S} \) such that

\[ \xi(s) = \sqrt{1 - s} \xi(t'), \quad t' = -\log(1 - s). \]
(3.10)

Since \( \xi(s) \) is \( \Lambda \)-Lipschitz, we have

\[ |\xi(s_1) - \xi(s_2)| \leq \Lambda |s_1 - s_2|, \quad \forall \ s_1, s_2 \in (-T, 1 - \lambda), \]
(3.11)

which implies that

\[ |e^{-\frac{t_1}{2}} \xi(t_1) - e^{-\frac{t_2}{2}} \xi(t_2)| \leq \Lambda |e^{-\frac{t_1}{2}} - e^{-\frac{t_2}{2}}|. \]
(3.12)

Suppose that \( |\xi(t)| \leq R \). For any \( t_1', t_2' \) with \( |t_1' - t_2'| \leq 1 \) we have

\[ |\xi(t_1') - \xi(t_2')| = |\xi(t_1') - e^{\frac{t_1' - t_2'}{2}} \xi(t_2')| + |e^{\frac{t_1' - t_2'}{2}} - 1||\xi(t_2')| \]
\[ \leq \Lambda |1 - e^{\frac{t_1' - t_2'}{2}}| + |e^{\frac{t_1' - t_2'}{2}} - 1||\xi(t_2')| \]
\[ \leq (\Lambda + R)|t_1' - t_2'|, \]
(3.13)
where we used the inequality
\[ |e^x - 1| \leq 2|x|, \quad \forall x \in [-1, 1]. \]

Note that the Lipschitz constant in (3.13) is given by
\[ \sigma = \tilde{\Lambda} + R = \frac{\Lambda}{\sqrt{\Lambda}} + R. \]

Thus, if we consider the convergence of \( \{ \Sigma_{t, t+} \}, -T < t < T \) as in part (1), then \( S \cap (B_R(0) \times (-T, T)) \) consists of Lipschitz curves with Lipschitz constant \( \sigma = \Lambda e^T + R \). The convergence is also in extrinsic Hausdorff distance by Proposition 2.8 and the limit self-shrinker is independent of the choice of \( T \) by the argument of Claim 4.3 of [33]. The lemma is proved.

\[ \square \]

### 3.2 Decomposition of spaces

In this subsection, we follow the argument in [33] to decompose the space and define an almost “monotone decreasing” quantity, which will be used to select time slices such that the limit self-shrinker is \( L \)-stable. First, we decompose the space as follows.

**Definition 3.5.** Fix large \( R > 0 \) and small \( \epsilon > 0 \).

1. We define the set \( S = S(\Sigma_t, \epsilon, R) = \{ y \in \Sigma_t \mid |y| < R, |A|(y, t) > \epsilon^{-1} \} \).

2. The ball \( B_R(0) \) can be decomposed into three parts as follows:
   - the high curvature part \( H \), which is defined by
     \[ H = H(\Sigma_t, \epsilon, R) = \left\{ x \in \mathbb{R}^3 \mid |x| < R, d(x, S) < \frac{\epsilon}{2} \right\}. \]
   - the thick part \( TK \), which is defined by
     \[ TK = TK(\Sigma_t, \epsilon, R) = \left\{ x \in \mathbb{R}^3 \mid |x| < R, \text{there is a continuous curve } \gamma \subset B_R(0) \setminus (H \cup \Sigma_t) \text{ connecting } x \text{ and some } y \text{ with } B(y, \epsilon) \subset B_R(0) \setminus (H \cup \Sigma_t) \right\}. \]
   - the thin part \( TN \), which is defined by \( TN = TN(\Sigma_t, \epsilon, R) = B_R(0) \setminus (H \cup TK) \).

As in Colding-Minicozzi [18], we define the \( L \)-stability of a self-shrinker.

**Definition 3.6.** For any \( R > 0 \), a complete smooth self-shrinker \( \Sigma^n \subset \mathbb{R}^{n+1} \) is called \( L \)-stable in the ball \( B_R(0) \), if for any function \( \varphi \in W_0^{1,2}(B_R(0)) \), we have
\[ \int_{\Sigma} -\varphi L_{\Sigma} \varphi e^{-\frac{|x|^2}{4}} \geq 0, \quad (3.14) \]
where \( L_{\Sigma} \) is the operator on \( \Sigma \) defined by
\[ L_{\Sigma} = \Delta - \frac{1}{2} \langle x, \nabla (\cdot) \rangle + |A|^2 + \frac{1}{2}. \]

The subindex \( \Sigma \) in \( L_{\Sigma} \) will be omitted when it is clear in the context. We say \( \Sigma \) is not \( L \)-stable in the ball \( B_R(0) \) if (3.14) doesn’t hold for some \( \varphi \in W_0^{1,2}(B_R(0)) \). We call that \( \Sigma \) is \( L \)-stable in \( \mathbb{R}^{n+1} \) if \( \Sigma \) is \( L \)-stable in the ball \( B_R(0) \) of \( \mathbb{R}^{n+1} \) for any \( R > 0 \).
Recall Colding-Minicozzi’s result:

**Theorem 3.7.** (c.f. [18],[19]) There are no $L$-stable smooth complete self-shrinkers without boundary and with polynomial volume growth in $\mathbb{R}^{n+1}$.

As a corollary of Theorem 3.7, we have the following result.

**Lemma 3.8.** Let $D, N > 0$ and $\rho$ an increasing positive function. There exists $R_0 = R_0(D, N, \rho) > 0$ such that any self-shrinker $\Sigma \in \mathcal{C}(D, N, \rho)$ is not $L$-stable in the ball $B_{R_0}(0)$.

**Proof.** For otherwise, we can find a sequence $R_i \to +\infty$ and self-shrinkers $\Sigma_i \in \mathcal{C}(D, N, \rho)$ such that $\Sigma_i$ is $L$-stable in the ball $B_{R_i}(0)$. By smooth compactness of $\mathcal{C}(D, N, \rho)$ in [18], a subsequence of $\{\Sigma_i\}$ converges smoothly to a self-shrinker $\Sigma_{i\infty} \in \mathcal{C}(D, N, \rho)$. By Theorem 3.7, $\Sigma_{i\infty}$ is not $L$-stable in a ball $B_{R_0}(0)$ for some $R_0 > 0$. This implies that there exists a smooth function $\phi_{i\infty} \in C^\infty_0(\Sigma_{i\infty} \cap B_{R_0}(0))$ such that

$$\int_{\Sigma_{i\infty}} -\phi_{i\infty} L_{\Sigma_i} \phi_{i\infty} e^{-\frac{|x|^2}{4}} < 0. \tag{3.15}$$

Since $\Sigma_i$ converges smoothly to $\Sigma_{i\infty}$, we define the map $f_i : \Sigma_{i\infty} \cap B_{R_0+1}(0) \to \Sigma_i$ by

$$f_i(x) = x + u_i(x) n(x), \quad \forall x \in \Sigma_{i\infty} \cap B_{R_0+1}(0),$$

where $n(x)$ denotes the normal vector field of $\Sigma_{i\infty}$ and $u_i(x)$ is the graph function of $\Sigma_i$ over $\Sigma_{i\infty}$. Let $\Omega = \Sigma_{i\infty} \cap B_{R_0+1}(0)$ and $\Omega_i = f_i(\Omega) \subset \Sigma_i$. We assume that $i$ is large such that $\Omega_i \subset \Sigma_i \cap B_{R_0+2}(0)$. Note that $f_i$ converges smoothly to the identity map on $\Omega$ as $i \to +\infty$ and for large $i$ its inverse map $f_i^{-1} : \Omega_i \to \Omega$ exists and is also smooth. Moreover, $f_i^{-1}$ also converges smoothly to the identity map on $\Omega$ as $i \to +\infty$. We define the function $\phi_i := (f_i^{-1})^* \phi_{i\infty} \in C^\infty_0(\Omega_i)$ and we can extend $\phi_i$ to $\Sigma_i$ such that $\phi$ is zero on $\Sigma_i \setminus \Omega_i$. Then by (3.15), the function $\phi_i \in C^\infty_0(\Sigma_i)$ satisfies

$$\lim_{i \to +\infty} \int_{\Sigma_i} -\phi_i L_{\Sigma_i} \phi_i e^{-\frac{|x|^2}{4}} = \int_{\Sigma_{i\infty}} -\phi_{i\infty} L_{\Sigma_i} \phi_{i\infty} e^{-\frac{|x|^2}{4}} < 0. \tag{3.16}$$

Thus, for large $i$ we have

$$\int_{\Sigma_i} -\phi_i L_{\Sigma_i} \phi_i e^{-\frac{|x|^2}{4}} < 0. \tag{3.16}$$

Note that $\text{Supp}(\phi_i) \subset \Omega_i \subset \Sigma_i \cap B_{R_0+2}(0)$ for large $i$. Thus, the inequality (3.16) contradicts our assumption that $\Sigma_i$ is $L$-stable in the ball $B_{R_i}(0)$ and $R_i \to +\infty$. The lemma is proved.

□

**Lemma 3.9.** Let $R, D, N > 0$ and $\rho$ an increasing positive function. For any $\Sigma \in \mathcal{C}(D, N, \rho)$ and $x \in \Sigma$, we define $r_\Sigma(x)$ the supreme of the radius $r$ such that

$$B_r(x + r n(x)) \cap \Sigma = \emptyset, \quad B_r(x - r n(x)) \cap \Sigma = \emptyset, \tag{3.17}$$

where $n(x)$ denotes the normal vector of $\Sigma$ at $x$. Then there exists $\epsilon_0(R, D, N, \rho) > 0$ such that for any $\Sigma \in \mathcal{C}(D, N, \rho)$ and $x \in \Sigma \cap B_R(0)$ we have

$$r_\Sigma(x) \geq \epsilon_0.$$
Proof. We divide the proof into several steps.

Step 1. For otherwise, we can find a sequence of $\Sigma_i \in C(D, N, \rho)$ and points $x_i \in \Sigma_i \cap B_R(0)$ with $\delta_i := r_{\Sigma_i}(x_i) \rightarrow 0$. By the smooth compactness of $C(D, N, \rho)$, there is a subsequence of $\{\Sigma_i\}$ converging smoothly to a self-shrinker $\Sigma_\infty$ in $C(D, N, \rho)$. We assume that $x_i \rightarrow x_\infty \in \Sigma_\infty \cap B_{R+1}(0)$. By the embeddedness of $\Sigma_\infty$, we have $\delta := r_{\Sigma_\infty}(x_\infty) > 0$. Since $\Sigma_\infty$ is smooth and embedded, there exists $r' > 0$ such that $B_{r'}(x_\infty) \cap \Sigma_\infty$ has only one component and

$$\inf_{y \in B_{r'}(x_\infty) \cap \Sigma_\infty} r_{\Sigma_\infty}(y) \geq \frac{\delta}{2}. \quad (3.18)$$

Moreover, we choose $r'$ sufficiently small such that $B_{r'}(x_\infty) \cap \Sigma_\infty$ is almost flat by Lemma 2.2. Let

$$\Omega(r', \frac{\delta}{2}) := \bigcup_{y \in B_{r'}(x_\infty) \cap \Sigma_\infty} \left( B_\frac{\delta}{2}(y + \frac{\delta}{2} n_{\Sigma_\infty}(y)) \cup B_\frac{\delta}{2}(y - \frac{\delta}{2} n_{\Sigma_\infty}(y)) \right). \quad (3.19)$$

Then $(3.18)$ implies that $\Omega(r', \frac{\delta}{2}) \cap \Sigma_\infty = \emptyset$. By the smooth convergence of $\Sigma_i$ to $\Sigma_\infty$, for large $i$ we have

$$\Omega(r', \frac{\delta}{4}) \cap (\Sigma_i \setminus B_{r'}(x_\infty)) = \emptyset. \quad (3.20)$$

By the construction of $\Omega(r', \frac{\delta}{4})$, we have

$$B_\frac{\delta}{4}(y) \subset \Omega(r', \frac{\delta}{4}) \cup (\Sigma_\infty \cap B_{r'}(x_\infty)), \quad \forall y \in B_{r'}(x_\infty) \cap \Sigma_\infty. \quad (3.21)$$

Step 2. Since $x_i \rightarrow x_\infty$, we can choose $r'$ sufficiently small such that for all large $i$ the projection of $x_i$ to $\Sigma_\infty$ lie in the ball $B_{r'}(x_\infty)$. This can be done since $B_{r'}(x_\infty) \cap \Sigma_\infty$ is almost flat. We denote by $y_i$ the projection of $x_i$ to $\Sigma_\infty$ and we have $y_i \in B_{r'}(x_\infty) \cap \Sigma_\infty$. Let $s_i \in \mathbb{R}$ such that $y_i + s_i n_{\Sigma_\infty}(y_i) = x_i$. Combining this with $x_i \rightarrow x_\infty$, we have

$$B_{2\delta_i}(x_i + 2\delta_i n_{\Sigma_i}(x_i)) \subset B_{4\delta_i}(x_i) \subset B_{4\delta_i + |s_i|}(y_i). \quad (3.22)$$

On the other hand, $|s_i| \rightarrow 0$ and for large $i$ we have

$$B_{4\delta_i + |s_i|}(y_i) \subset B_{\frac{\delta}{4}}(y_i). \quad (3.23)$$

Combining $(3.21)$-$(3.23)$, we have

$$B_{2\delta_i}(x_i + 2\delta_i n_{\Sigma_i}(x_i)) \subset \left( \Omega(r', \frac{\delta}{4}) \cup (\Sigma_\infty \cap B_{r'}(x_\infty)) \right). \quad (3.24)$$

Step 3. We show that

$$B_{2\delta_i}(x_i + 2\delta_i n_{\Sigma_i}(x_i)) \cap \Sigma_i = \emptyset. \quad (3.25)$$

Let $\Sigma_i = \Sigma_i^{(1)} \cup \Sigma_i^{(2)}$, where $\Sigma_i^{(1)}$ and $\Sigma_i^{(2)}$ are defined by

$$\Sigma_i^{(1)} = \Sigma_i \cap B_{r'}(x_\infty), \quad \Sigma_i^{(2)} = \Sigma_i \setminus B_{r'}(x_\infty). \quad (3.26)$$
By the smooth convergence of $\Sigma_i$ to $\Sigma_\infty$ and the choice of $r'$ such that $B_{r'}(x_\infty) \cap \Sigma_\infty$ is almost flat, we have that for large $i$ $B_{r'}(x_\infty) \cap \Sigma_i$ is also almost flat. Consequently, for large $i$ we have

$$B_{2\delta_i}(x_i \pm 2\delta_i n_{\Sigma_i}(x_i)) \cap \Sigma_i^{(1)} = \emptyset.$$  

(3.27)

On the other hand, (3.20) and (3.24) imply that

$$B_{2\delta_i}(x_i \pm 2\delta_i n_{\Sigma_i}(x_i)) \cap \Sigma_i^{(2)} \subset \left( \frac{\Omega(r', \delta)}{2} \cup (\Sigma_\infty \cap B_{r'}(x_\infty)) \right) \cap \Sigma_i^{(2)} = \emptyset,$$

(3.28)

where we used the fact that $\Sigma_i^{(2)} \cap B_{r'}(x_\infty) = \emptyset$. Thus, (3.25) follows from (3.27)-(3.28). Note that (3.25) contradicts the definition of $\delta_i = r_{\Sigma_i}(x_i)$. The lemma is proved.

A direct corollary of Lemma 3.9 is the following result.

**Lemma 3.10.** Let $R, D, N > 0$, and an increasing positive function $\rho$. Then there exists a constant $\epsilon_0(R, D, N, \rho) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ we have

$$|\text{TN}(\Sigma_\epsilon, R)| = 0, \quad \forall \Sigma \in C(D, N, \rho).$$

(3.29)

**Proof.** We choose $\epsilon_0$ the same constant in Lemma 3.9. Thus, (3.29) follows from Lemma 3.9 and the definition of $\text{TN}$.

Using Lemma 3.10, we show that the quantity $|\text{TN}|$ along the flow will tend to zero.

**Lemma 3.11.** Fix $R, D, N > 0$, and an increasing positive function $\rho$. Under the assumption of Theorem 3.1, there exists a constant $\epsilon_0(R, D, N, \rho) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, we have

$$\lim_{t \to \infty} |\text{TN}(\Sigma_t, \epsilon, R)| = 0.$$

**Proof.** By Lemma 3.4 for any $t_i \to \infty$ there exists a subsequence, still denoted by $\{t_i\}$, such that it converges locally smoothly to a limit self-shrinker $\Sigma_\infty \in C(D, N, \rho)$ away from the singular set $S_0 \subset \mathbb{R}^3$. For any $\epsilon > 0$, by Definition 3.5 we have

$$\text{TN}(\Sigma_{t_i}, \epsilon, R) \to \text{TN}(\Sigma_\infty, \epsilon, R) \setminus B_{\frac{\Omega}{2}}(S_0),$$

where $B_\epsilon(S_0) = \cup_{p \in S_0} B_\epsilon(p)$. Therefore, by Lemma 3.10 we have

$$\lim_{t_i \to +\infty} |\text{TN}(\Sigma_{t_i}, \epsilon, R)| \leq \lim_{t_i \to +\infty} |\text{TN}(\Sigma_\infty, \epsilon, R)| = 0,$$

where $\epsilon \in (0, \epsilon_0)$ and $\epsilon_0$ is the constant in Lemma 3.10. The lemma is proved.

As in Lemma 4.7 of [33], we have

**Lemma 3.12.** Fix $D, R > 0$ and $\tau \in (0, 1)$. Let $\{t_i\}$ be any sequence as in Lemma 3.4. If the multiplicity of the convergence in Lemma 3.4 is more than one, then for any $\epsilon > 0$, there exists $i_0 > 0$ such that for any $i \geq i_0$ we have

$$\inf_{t \in [t_i - \tau, t_i]} |\text{TN}(\Sigma_t, \epsilon, R)| > 0.$$
\textbf{Proof.} Since \( \Sigma_t \) is embedded and \( \{ \Sigma_{t+t}, -\tau \leq t \leq \tau \} \) converges locally smoothly to the limit self-shrinker \( \Sigma_\infty \), all components of \( (\Sigma_t \cap B_\epsilon(0)) \setminus H(\epsilon, \Sigma_t, R) \) with \( t \in [t_1, t_1] \) lie in the \( \frac{\tau}{2} \)-neighborhood of the plane \( \Sigma_\infty \). By the definition of \( \mathbf{TN} \), for any \( t \in [t_1 - \tau, t_1] \) the quantity \( \mathbf{TN}(\epsilon, \Sigma_t, R) \) is nonempty and we have \( |\mathbf{TN}(\epsilon, \Sigma_t, R)| > 0 \).

Using Lemma 3.11 and Lemma 3.12 we have the following result as in Lemma 4.8 of \cite{33}.

\textbf{Lemma 3.13.} Let \( R, \epsilon, \tau > 0 \) and \( f(t, \epsilon) = \inf_{s \in [t-\tau, t]} |\mathbf{TN}(\Sigma_s, \epsilon, R)| \). For any \( t_0 > 0 \) and \( l > 0 \), we can find a sequence \( \{ t_i \} \) with \( t_{i+1} > t_i + l \) such that for any \( i \in \mathbb{N} \),

\begin{equation}
\sup_{t \in [t_i, t_{i+l}]} f(t, \epsilon) \leq 2 f(t_i, \epsilon),
\end{equation}

then we denote \( t_1 = s_1^{(1)} \) and stop the searching process. Otherwise, choose a time \( s_1^{(k+1)} \) such that \( f(s_1^{(k)}, \epsilon) > 2 f(t_1, \epsilon) \). Inductively, after we find \( s_1^{(k)} \), we have

\begin{equation}
\sup_{t \in [s_1^{(k)}, s_1^{(k+1)}]} f(t, \epsilon) \leq 2 f(s_1^{(k)}, \epsilon),
\end{equation}

then we denote \( t_1 = s_1^{(k)} \) and stop the searching process. Otherwise, choose a time \( s_1^{(k+1)} \) such that \( f(s_1^{(k)}, \epsilon) > 2 f(t_1, \epsilon) \). Note that

\begin{equation*}
f(s_1^{(k)}, \epsilon) \geq 2^{k} f(s_1^{(1)}, \epsilon) \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty.
\end{equation*}

Since \( \lim_{t \rightarrow +\infty} f(t, \epsilon) = 0 \) by Lemma 3.11, this process must stop in finite steps, and we can find \( k_1 \) such that

\begin{equation*}
\sup_{t \in [s_1^{(k_1)}, s_1^{(k_1+1)}]} f(t, \epsilon) \leq 2 f(s_1^{(k_1)}, \epsilon).
\end{equation*}

We denote by \( t_1 = s_1^{(k_1)} \). After we find \( t_1 \), set \( s_2^{(0)} = t_1 + l + 1 \) and continue the previous process to find time in \([s_2^{(0)}, s_2^{(0)} + l]\) such that \( f(t, \epsilon) > 2 f(s_2^{(0)}, \epsilon) \). Similarly, for some \( k \) we have

\begin{equation}
\sup_{t \in [s_2^{(k)}, s_2^{(k+1)}]} f(t, \epsilon) \leq 2 f(s_2^{(k)}, \epsilon).
\end{equation}

Then we define \( t_2 = s_2^{(k)} \). Inductively, after we find \( t_i \), we set \( s_i^{(0)} = t_i + l + 1 \). Then we start the process to search time in \([s_i^{(0)}, s_i^{(0)} + l]\) with \( f(t, \epsilon) > 2 f(s_i^{(0)}, \epsilon) \). This process is well defined. Repeating this process and we can find a sequence of times \( \{ t_i \} \) such that for any \( t_i \) the inequality (3.30) holds. The lemma is proved.

\[\boxed{}
\]

\textbf{3.3 Construction of auxiliary functions}

In this subsection, we construct functions which will be used to show the \( L \)-stability of the limit self-shrinker. We fix \( R, T > 1 \) in this section. For any sequence \( t_i \rightarrow +\infty \), by Lemma 3.4 a subsequence of \( \{ \Sigma_{t_i}, -T < t < T \} \) converges in smooth topology to a self-shrinker \( \Sigma_\infty \) away from a locally finite, \( \sigma \)-Lipschitz singular set \( S \subset \mathbb{R}^3 \times (-T, T) \). We denote by \( S_t = \{ x \in \mathbb{R}^3 \mid (x, t) \in S \} \) the singular set at \( t \). By Lemma 2.9, we assume that the multiplicity of the convergence is a constant \( N_0 \geq 2 \). As in \cite{33}, we construct some functions as follows:
(1). Let $\epsilon > 0$ and large $R > 0$. We define

$$\Omega_{\epsilon,R}(t) = (\Sigma_\infty \cap B_R(0)) \setminus B_\epsilon(S_t)$$

(3.31)

and for any time interval $I \subset (-T, T)$ we define

$$\Omega_{\epsilon,R}(I) = \cap_{t \in I} \Omega_{\epsilon,R}(t), \quad S_I = \cup_{t \in I} S_t,$$

(3.32)

For any $\epsilon > 0$, the surface $\Sigma_{i,t} \cap B_R(0)$ is a union of graphs over the set $\Omega_{\epsilon,R}(t)$ for large $t_i$ and any $t \in (-T, T)$.

(2). Let $u_i^+(x, t)$ and $u_i^-(x, t)$ be the graph functions representing the top and bottom sheets (which we denote by $\Sigma_i^+$ and $\Sigma_i^-$ respectively) over $\Sigma_\infty \cap B_R(0)$. The readers are referred to \[33\] for the details on the construction of $u_i^+(x, t)$ and $u_i^-(x, t)$. By the convergence property of the flow \{(\Sigma_{i,t}, x_i(t))\}, $-T < t < T$, for any $\epsilon > 0$ and large $R$ there exists $i_0 > 0$ such that for any $i \geq i_0$ and any $t \in (-T, T)$ the functions $u_i^+(x, t)$ and $u_i^-(x, t)$ are well-defined on $\Omega_{\epsilon,R}(t)$. By the calculation in Appendix [C] the function

$$u_i(x, t) = u_i^+(x, t) - u_i^-(x, t),$$

(3.33)

which we call the height difference function of $\Sigma_{i,t}$ over $\Sigma_\infty$, satisfies the equation

$$\frac{\partial u_i}{\partial t} = \Delta_0 u_i - \frac{1}{2} \langle x, \nabla u_i \rangle + |A|^2 u_i + \frac{u_i}{2} + a_i^{pq} u_i u_{pq} + b_i^p u_{ip} + c_i u_i$$

(3.34)

for any $(x, t) \in \Omega_{\epsilon,R}(I) \times I$. Here $\Delta_0$ denotes the Laplacian operator on $\Sigma_\infty$. The coefficients $a_i^{pq}, b_i^p$ and $c_i$ are small on $\Omega_{\epsilon,R}(I) \times I$ as $t_i$ large and tend to zero as $t_i \to + \infty$.

(3). Fix a point $x_0 \in (\Sigma_\infty \cap B_R(0)) \setminus S_1$. We choose a sequence of points \{\(x_i\)\}_{i=1}^\infty \subset (\Sigma_\infty \setminus S_1) \cap B_R(0)$ with $x_i \to x_0$. Then for sufficiently small $\epsilon > 0$ we have $x_0 \in \Omega_{\epsilon,R}(1)$ and \{\(x_i\)\}_{i=1}^\infty \subset \Omega_{\epsilon,R}(1)$. For any $t \in (-T, T)$ and $x \in \Omega_{\epsilon,R}(t)$ we define the normalized height difference function

$$w_i(x, t) = \frac{u_i(x, t)}{u_i(x_i, 1)},$$

(3.35)

Then $w_i(x, t)$ is a positive function with $w_i(x_i, 1) = 1$ and by (3.34) $w_i(x, t)$ satisfies the equation on $\Omega_{\epsilon,R}(I) \times (I)$ for any $I \subset (-T, T)$

$$\frac{\partial w_i}{\partial t} = \Delta_0 w_i - \frac{1}{2} \langle x, \nabla w_i \rangle + |A|^2 w_i + \frac{w_i}{2} + a_i^{pq} w_i w_{pq} + b_i^p w_{ip} + c_i w_i.$$  

(3.36)

Note that the construction of the function $w_i$ is slightly different from that of [33]. In (3.35) we choose a sequence of points \{\(x_i\)\} $\subset \Sigma_\infty \setminus S_1$ to normalize the function $u_i$, while in [33] we choose a fixed point $x_0$. The reason why we choose such a normalization is that we need the inequality (3.73) in Lemma 3.19 below.

As in [33], we have the following result which implies that for large $t_i$ the integral of $u_i$ is comparable to the set $|TN|$. Note that (3.37) doesn’t hold as $\epsilon \to 0$ since the function $u_i$ is not defined near the singularities.
Lemma 3.14. (c.f. [33]) Fix $\epsilon, R$ and $T$ as above. For any sequence $\{t_i\}$ chosen in Lemma 3.14 there exists $t_T > 0$ such that for any $t \in (-T, T)$ and $t_i > t_T$ we have
\[
\int_{\Omega_{\epsilon, t_i}(t)} u_i(x, t) \, d\mu_\infty \leq |TN(\Omega_{\epsilon, t_i}, \epsilon, R)| \leq \int_{\Omega_{\epsilon, t_i}(t)} u_i(x, t) \, d\mu_\infty,
\]  
(3.37)

where $d\mu_\infty$ denotes the volume form of $\Sigma_\infty$.

Since $w_i$ satisfies the parabolic equation (3.36), we have the following parabolic Harnack inequality by using Theorem A.5 in Appendix A.

Lemma 3.15. For any $-T < a < s < t < b < T$, $\epsilon > 0$, $x \in \Omega_{\epsilon, R}(s)$ and $y \in \Omega_{\epsilon, R}(t)$, there exists a constant $C = C(\epsilon, R, s - a, t - s, \Sigma_\infty, S_{\{a, b\}})$ such that
\[
w_i(x, s) \leq C w_i(y, t).
\]  
(3.38)

Proof. We divide the proof into several steps:

Step 1. Since $S_i \cap B_R(0)$ consists of finitely many points, we can choose sufficiently small $\delta_0(\Sigma_\infty, S_{\{a, b\}}) > 0$ such that for any $s \in [a, b]$,
\[
\Omega_{2\epsilon, R}(s) \subset \Omega_{\epsilon, R}(t), \quad \Omega_{\frac{\epsilon}{100}, R+2}(s) \subset \Omega_{\frac{\epsilon}{100}, R+2}(t), \quad \forall \, t \in [s - \delta_0, s + \delta_0] \cap [a, b].
\]  
(3.39)

Let $N$ be a positive integer satisfying
\[
N > \max \{ \frac{5(b - a)}{\delta_0}, \frac{b - a}{s - a}, \frac{5(b - a)}{t - s} \}.
\]  
(3.40)

Set
\[
\tau_k = a + \frac{b - a}{N} k, \quad \forall \, k \in \{0, 1, \cdots, N\}.
\]  
(3.41)

Then $\tau_0 = a$ and $\tau_N = b$. By (3.40) we have $s \geq \tau_1$. Note that (3.39) and (3.41) imply that for any $k = 1, 2, \cdots, N - 1$ we have
\[
\Omega_{\frac{\epsilon}{100}, R+2}(\tau_k) \subset \Omega_{\frac{\epsilon}{100}, R+2}(t), \quad \forall \, t \in [\tau_{k-5}, \tau_{k+5}] \cap [a, b].
\]  
(3.42)

Step 2. Let
\[
\Omega' := \Omega_{\epsilon, R}(\tau_k), \quad \Omega'' := \Omega_{\frac{\epsilon}{100}, R+1}(\tau_k), \quad \Omega := \Omega_{\frac{\epsilon}{100}, R+2}(\tau_k).
\]  
(3.43)

Then we have $\Omega' \subset \Omega'' \subset \Omega$. Clearly, $\Omega''$ has a positive distance $\delta = \delta(\epsilon)$ away from the boundary of $\Omega$. Since $\Omega'$ is compact, we can cover $\Omega'$ by finite many balls contained in $\Omega''$ with radius $r = \frac{\epsilon}{100}$ and the number of these balls is bounded by a constant depending only on $\epsilon, R$ and $\Sigma_\infty$. Since $w_i$ satisfies the parabolic equation (3.36), applying Theorem A.5 in Appendix A for the function $w_i$, the domains $\Omega', \Omega''$, $\Omega$ and the interval $[\tau_{k-1}, \tau_{k+1}]$, we have
\[
w_i(x, \tau_k) \leq C w_i(y, \tau_{k+1}), \quad \forall \, x, y \in \Omega_{\epsilon, R}(\tau_k),
\]  
(3.44)

where $C = C(\epsilon, R, b - a, N, \Sigma_\infty, S_{\{a, b\}})$ is a constant independent of $i$. Moreover, since $S_{\{a, b\}} \cap B_R(0)$ consists of finitely many Lipschitz curves, there exists a sequence of points $\{z_k\}$ such that
\[
z_k \in \Omega_{2\epsilon, R}([\tau_{k-1}, \tau_k]) \cap \Omega_{2\epsilon, R}([\tau_k, \tau_{k+1}) \neq \emptyset.
\]  
(3.45)
Step 3. For \( s, t \in (a, b) \) with \( s < t \), there exist integers \( k_s \) and \( k_t \) such that \( s \in [\tau_{k_s}, \tau_{k_s+1}] \) and \( t \in (\tau_{k_t}, \tau_{k_t+1}] \). Note that (3.40) implies

\[
t - s \geq \frac{5(b-a)}{N}.
\]

On the other hand, (3.41) implies that

\[
t - s \leq \tau_{k_t+1} - \tau_{k_s} = \frac{b-a}{N} (k_t + 1 - k_s).
\]

Combining (3.47) with (3.46), we have

\[
k_t - k_s \geq 4.
\]

Thus, (3.44), (3.48) and (3.45) implies that

\[
w_i(z_{k_s+2}, \tau_{k_s+2}) \leq C w_i(z_{k_s+3}, \tau_{k_s+3}) \leq \cdots \leq C N w_i(z_{k_t-1}, \tau_{k_t-1}),
\]

where \( C = C(\epsilon, R, b - a, N, \Sigma_{\infty}, S_{[a,b]}) \).

Step 4. Set

\[
\Omega' = \Omega_{\epsilon, R}(s), \quad \Omega'' = \Omega_{\frac{3}{2} \epsilon, R+1}(s), \quad \Omega = \Omega_{\frac{1}{2} \epsilon, R+2}(s).
\]

Then by (3.39) we have

\[
\Omega' \subset \Omega'' \subset \Omega = \Omega_{\frac{1}{2} \epsilon, R+2}(s) \subset \Omega_{\frac{3}{4} \epsilon, R+2}(s'), \quad \forall s' \in [\tau_{k_s-2}, \tau_{k_s+2}],
\]

where we used the fact that \([\tau_{k_s-2}, \tau_{k_s+2}] \subset [s - \delta_0, s + \delta_0] \). Note that by (3.45) and (3.39), we have

\[
z_{k_s+2} \in \Omega_{2 \epsilon, R}(\tau_{k_s+2}) \subset \Omega_{\epsilon, R}(s).
\]

As in Step 2, \( \Omega'' \) has a positive distance \( \delta = \delta(\epsilon) \) from the boundary of \( \Omega \), and we can cover \( \Omega' \) by finite many balls contained in \( \Omega'' \) with radius \( r = \frac{\epsilon}{10} \) and the number of these balls is bounded by a constant depending only on \( \epsilon, R \) and \( \Sigma_{\infty} \). Applying Theorem A.5 for such \( \Omega', \Omega'', \Omega \) and the interval \([\tau_{k_s-2}, \tau_{k_s+2}] \) and using (3.52), we have

\[
w_i(x, s) \leq C w_i(z_{k_s+2}, \tau_{k_s+2}), \quad \forall x \in \Omega_{\epsilon, R}(s),
\]

where \( C = C(\epsilon, R, b - a, N, \Sigma_{\infty}, S_{[a,b]}) \).

Step 5. Set

\[
\Omega' = \Omega_{\epsilon, R}(t), \quad \Omega'' = \Omega_{\frac{3}{4} \epsilon, R+1}(t), \quad \Omega = \Omega_{\frac{1}{2} \epsilon, R+2}(t).
\]

Then by (3.39) we have

\[
\Omega' \subset \Omega'' \subset \Omega = \Omega_{\frac{1}{2} \epsilon, R+2}(t) \subset \Omega_{\frac{3}{4} \epsilon, R+2}(t'), \quad \forall t' \in [\tau_{k_t-2}, \tau_{k_t+2}],
\]

where we used the fact that \([\tau_{k_t-2}, \tau_{k_t+2}] \subset [t - \delta_0, t + \delta_0] \). Note that by (3.45) and (3.39), we have

\[
z_{k_t-1} \in \Omega_{2 \epsilon, R}(\tau_{k_t-1}) \subset \Omega_{\epsilon, R}(t).
\]
Applying Theorem \ref{thm:approximation} as in Step 4 for such \( \Omega', \Omega'', \Omega \) and the interval \([\tau_{k_l-2}, \tau_{k_l+2}]\) and using \eqref{eq:intermediate_4}, we have
\[
 w_i(z_{k_l-1}, \tau_{k_l-1}) \leq C w_i(y, t), \quad \forall \ y \in \Omega_{\epsilon,R}(t),
\]
where \( C = C(\epsilon, R, b - a, N, \Sigma_{\infty}, S_{[\alpha,\beta]}) \). Combining this with \eqref{eq:intermediate_4} and \eqref{eq:intermediate_5}, we have
\[
 w_i(x, s) \leq C w_i(y, t), \quad \forall \ x \in \Omega_{\epsilon,R}(s), \quad y \in \Omega_{\epsilon,R}(t),
\]
where \( C = C(\epsilon, R, b - a, N, \Sigma_{\infty}, S_{[\alpha,\beta]}) \). The lemma is proved. \( \square \)

For any fixed \( \epsilon, R \) and \( T \), the following result shows that we can find a sequence \( \{t_i\} \) such that the functions \( w_i \) are uniformly bounded on a compact set away from singularities. Note that we have no estimates of \( w_i \) near the singularities.

**Lemma 3.16.** Fix \( \epsilon, \tau \in (0, \frac{1}{2}) \) and \( R, T \) large. Let \( \{t_i\} \) be the sequence chosen in Lemma \ref{lem:approximation} for such \( \epsilon, \tau, R \) and \( l = T \). For any time interval \( I = [a, b] \subset [-1, T - 2] \) and a compact set \( K \subset \subset (\Sigma_{\infty} \cap B_R(0)) \setminus S_I \), there exists a constants \( C = C(K, \Sigma_{\infty}, S_{[-2, b+2]}) > 0 \) such that the function \( w_i \) defined by \eqref{eq:sequence} satisfies
\[
 0 < w_i(x, t) < C, \quad \forall \ (x, t) \in K \times I.
\]
Moreover, if \( a \in [2, T - 2] \) there exists \( C' = C'(K, \Sigma_{\infty}, S_{[0, a+1]}) > 0 \) independent of \( b \) such that
\[
 w_i(x, a) \geq C'.
\]
**Proof.** By the assumption, we can assume that \( K \subset \Omega_{\epsilon',R}(I) \) and \( \{x_i\} \subset \Omega_{\epsilon',R}(1) \) for some \( \epsilon' \in (0, \epsilon) \), where \( \{x_i\} \) is the sequence in \eqref{eq:sequence}. Note that \( w_i(x, 1) = 1 \). We divide the rest of the proof into several steps.

**Step 1.** \( w_i \) is bounded on \( K \times I \) for the time interval \( I = [-1, \frac{1}{2}] \) and any \( K \) above. Applying Lemma \ref{lem:approximation} for \( a = -2 \) and \( b = 2 \) we have
\[
 w_i(x, t) \leq C(\epsilon', R, \Sigma_{\infty}, S_{[-2, 2]}) w_i(x, 1) = C(\epsilon', R, \Sigma_{\infty}, S_{[-2, 2]}), \quad \forall \ (x, t) \in K \times I.
\]

**Step 2.** \( w_i \) is bounded from above on \( K \times I \) for any \( I = [a, b] \subset (0, T - 2) \) and \( K \) above. For any \( t \in [a, b] \subset (0, T - 2) \), we have \( t' := t + 1 \in (1, T - 1) \). By Lemma \ref{lem:approximation} and Lemma \ref{lem:approximation} for large \( i \) we have
\[
 \inf_{s \in [t' - \tau, t']} \int_{\Omega_{\epsilon,R}(s)} w_i(x, s) \, d\mu \leq \frac{1}{w_i(x_i, 1)} \inf_{s \in [t' - \tau, t']} |TN(\Sigma_{t_i+s}, \epsilon, R)| \\
 \leq \frac{2}{w_i(x_i, 1)} \inf_{s \in [-\tau, 0]} |TN(\Sigma_{t_i+s}, \epsilon, R)| \\
 \leq 2 \inf_{s \in [-\tau, 0]} \int_{\Omega_{\epsilon,R}(s)} w_i(x, s) \, d\mu.
\]
Moreover, by \eqref{eq:intermediate_6} we have
\[
 w_i(x, 0) \leq C(\epsilon, R, \Sigma_{\infty}, S_{[-2, 2]}), \quad \forall \ x \in \Omega_{\epsilon,R}(0),
\]
which implies that
\[
\int_{\Omega_{\epsilon,R}(0)} w_i(x,0) \leq C(\epsilon, R, \Sigma_\infty, S_{[-2,2]}).
\] (3.63)

Combining (3.63) with (3.62), we have
\[
\inf_{s \in [t-\tau,t]} \int_{\Omega_{\epsilon,R}(s)} w_i(x,s) \, d\mu \leq C(\epsilon, R, \Sigma_\infty, S_{[-2,2]}).
\] (3.64)

This implies that for any \( t \in (1, T-1) \) there exists \( s(t) \in [t-\tau, t] \) such that
\[
\int_{\Omega_{\epsilon,R}(s(t))} w_i(x,s(t)) \, d\mu \leq C(\epsilon, R, \Sigma_\infty, S_{[-2,2]}).
\] (3.65)

On the other hand, Lemma 3.15 implies that for any \( x \in K \subset \Omega_{\epsilon',R}([a,b]), t \in [a,b], \) and \( y \in \Omega_{\epsilon,R}(s(t+1)) \subset \Omega_{\epsilon',R}(s(t+1)) \) we have
\[
w_i(x,t) \leq C(\epsilon', R, \Sigma_\infty, S_{[a-1,b+2]})w_i(y, s(t+1)),
\] (3.66)

where we used the fact that \( \tau \in (0, \frac{1}{2}) \) and
\[
s(t+1) \geq t+1-\tau \geq t + \frac{1}{2}.
\]

Integrating the right-hand side of (3.66) and using (3.65), we have
\[
w_i(x,t) \leq C(\epsilon', R, \Sigma_\infty, S_{[a-1,b+2]}) \int_{\Omega_{\epsilon,R}(s(t+1))} w_i(y, s(t+1)) \leq C(\epsilon', R, \Sigma_\infty, S_{[-2,b+2]}), \quad \forall t \in [a,b].
\]

**Step 3.** \( w_i(x,t) \) is bounded from below on \( K \times I \) for any \( I = [a,b] \subset [2, T-2] \) and \( K \) above.

By Lemma 3.15 for any \( (x,t) \in K \times I \) we have
\[
w_i(x,t) \geq C(\epsilon', R, \Sigma_\infty, S_{[0,t+1]})w_i(x_i, 1).
\] (3.67)

In particular, for \( t = a \) the constant in (3.67) depends only on \( \epsilon', R, \Sigma_\infty \) and \( S_{[0,a+1]} \). Thus, the lemma is proved.

\[ \square \]

**Lemma 3.17.** The same assumption as in Lemma 3.16 As \( t_i \to +\infty \), we can take a subsequence of the functions \( w_i(x,t) \) such that it converges in \( C^2 \) topology on any compact subset \( K \subset \subset (\Sigma_\infty \cap B_R(0)) \setminus S_I \), where \( I = [a,b] \subset [-1, T-2] \), to a positive function \( w(x,t) \) with \( w(x_0,1) = 1 \) and satisfying
\[
\frac{\partial w}{\partial t} = \Delta_0 w + |A|^2 w - \frac{1}{2} \langle x, \nabla w \rangle + \frac{1}{2} w, \quad \forall (x,t) \in K \times I.
\] (3.68)

**Proof.** Since \( w_i \) is positive by definition and \( w_i \) is uniformly bounded from above by Lemma 3.16 by the interior estimates of the parabolic equation we have the space-time \( C^{2,\alpha} \) estimates of \( w_i (\text{c.f. } \text{Theorem 4.9 of [50]}) \), and the estimates are independent of \( i \). Therefore, as \( i \to +\infty \), the function \( w_i \) converges to a limit function \( w \) in \( C^2 \) topology on \( K \times [a,b] \) with \( w(x_0,1) = 1 \) and \( w \) is positive by the strong maximal principle. The lemma is proved.

\[ \square \]
3.4 The auxiliary functions near the singular set

In this subsection, we show that there exists a refined sequence such that the limit auxiliary function has uniform estimates across the singular set. Recall that by Lemma 3.17 the function \( w \) is uniformly bounded on any compact set away from the singular set and \( w \) has no estimates near the singularities. In this section, we will use Lemma 3.13 repeatedly for a sequence bounded on any compact set away from the singular set and \( w \) has uniform estimates across the singular set. Recall that by Lemma 3.17 the function \( w \) is uniformly bounded on any compact set away from the singular set and \( w \) has no estimates near the singularities. In this section, we will use Lemma 3.13 repeatedly for a sequence bounded on any compact set away from the singular set and \( w \) has uniform estimates across the singular set.

**Lemma 3.18.** Let \( R > 1, \tau \in (0, \frac{1}{2}) \), and \( \{\epsilon_i\} \) be a sequence of positive numbers with \( \epsilon_i \to 0 \). For any \( i \in \mathbb{N} \), there exists a sequence \( \{t_{i,k}\}_{k=1}^{\infty} \) with \( t_{i,k+1} > t_{i,k} + i \) satisfying the following properties.

1. For any \( k \in \mathbb{N} \)
   \[
   \sup_{s \in [0, i]} f(t_{i,k} + s, \epsilon_i) \leq 2f(t_{i,k}, \epsilon_i),
   \]
   where \( f(t, \epsilon) = \inf_{s \in [t-\tau, t]} |\text{TN}(\Sigma_s, \epsilon, R)| \).

2. For any \( T > 0 \), \( \{\Sigma_{t_{i,k}+s}, -T < s < T\} \) converges locally smoothly to a self-shrinker \( \Sigma_{i,\infty} \in \mathcal{C}(D, N, \rho) \) away from the space-time singular set \( S_i \) as \( k \to +\infty \).

3. For large \( k \) the surface \( \Sigma_{t_{i,k}+s} \) can be written as a union of graphs over \( \Sigma_{i,\infty} \) away from the singular set \( S_i \). We denote by \( \tilde{u}_{i,k}^+(x, s), \tilde{u}_{i,k}^-(x, s) \) the graph functions of the top and bottom sheets of \( \Sigma_{t_{i,k}+s} \) over \( \tilde{\Omega}_{i,\epsilon,R}(s) \), where
   \[
   \tilde{\Omega}_{i,\epsilon,R}(s) = (\Sigma_{i,\infty} \cap B_R(0)) \setminus B_\epsilon(S_i, s).
   \]
   Let \( \tilde{u}_{i,k}(x, s) = \tilde{u}_{i,k}^+(x, s) - \tilde{u}_{i,k}^-(x, s) \) be the height difference function of \( \Sigma_{t_{i,k}+s} \) over \( \tilde{\Omega}_{i,\epsilon,R}(s) \). These functions are constructed as in Section 3.3. By Lemma 3.14 we can choose \( k_i \) large such that for any \( k \geq k_i \),
   \[
   \int_{\tilde{\Omega}_{i,\epsilon,R}(s)} \tilde{u}_{i,k}(x, s) \leq \left| \text{TN}(\Sigma_{t_{i,k}+s}, \epsilon_i, R) \right| \leq \int_{\tilde{\Omega}_{i,\epsilon,R}(s)} \tilde{u}_{i,k}(x, s), \quad \forall s \in (-T, T).
   \]
   (3.71)

4. By the smooth compactness of \( \mathcal{C}(D, N, \rho) \) in [18], we assume that \( \Sigma_{i,\infty} \) in item (2) converges smoothly to \( \Sigma_\infty \in \mathcal{C}(D, N, \rho) \).

5. For any \( i \in \mathbb{N} \), there exists \( k_i > 0 \) such that for any \( \{s_i\}_{i=1}^{\infty} \) with \( s_i > k_i \), \( \{\Sigma_{t_{i,s_i}+s}, -T < s < T\} \) converge locally smoothly to the same self-shrinker \( \Sigma_\infty \) as in item (4) away from the space-time singular set \( S_\infty \). Moreover, the singular set \( S_i \) in item (2) converges to \( S_\infty \) in Hausdorff distance.

**Proof.** Applying Lemma 3.13 for \( \epsilon_i \) and \( l = i \), we have (3.69). Item (2) follows from Lemma 3.4 and item (3) follows from Lemma 3.14. It is clear that item (4) follows from Colding-Minicozzi’s compactness theorem [18].

To prove item (5), we first note that the convergence in item (2) is also in Hausdorff distance by Lemma 3.4 for any \( i \) there exists \( k_i > 0 \) such that for any \( k \geq k_i \) and \( s \in (-2, 2) \) we have
   \[
   d_H \left( \Sigma_{t_{i,k}+s} \cap B_R(0), \Sigma_{i,\infty} \cap B_R(0) \right) \leq \frac{1}{l}, \quad d_H \left( S(\Sigma_{t_{i,k}+s}, \epsilon_i, R), S_i, s \cap B_R(0) \right) \leq \frac{1}{l}, \quad (3.72)
   \]

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where \( d_H \) denotes the Hausdorff distance. By item (4), we assume that \( \Sigma_{i,\infty} \) converges smoothly to \( \Sigma_\infty \in C(D, N, \rho) \). By Lemma 3.14 for any sequence of times \( \{s_i\}_{i=1}^\infty \) with \( s_i > k_i \) the surfaces \( \{\Sigma_{i,s_i} + s, -T < s < T\} \) converge locally smoothly to a self-shrinker, which is denoted by \( \Sigma_\infty \), away from a singular set \( S_x \subset \Sigma_\infty \) as \( i \to +\infty \). Moreover, as \( i \to +\infty \),

\[
\begin{align*}
& d_H(\Sigma_\infty \cap B_R(0), \Sigma_\infty \cap B_R(0)) \\
& \leq d_H(\Sigma_\infty \cap B_R(0), \Sigma_{i,s_i} + s \cap B_R(0)) + d_H(\Sigma_{i,s_i} + s \cap B_R(0), \Sigma_{i,\infty} \cap B_R(0)) \\
& + d_H(\Sigma_{i,\infty} \cap B_R(0), \Sigma_\infty \cap B_R(0)) \\
& \leq d_H(\Sigma_\infty \cap B_R(0), \Sigma_{i,s_i} + s \cap B_R(0)) + \frac{1}{i} + d_H(\Sigma_{i,\infty} \cap B_R(0), \Sigma_\infty \cap B_R(0)) \\
& \to 0,
\end{align*}
\]

where we used \((3.72)\). Thus, \( \Sigma_\infty \) coincides with \( \Sigma_\infty \). Moreover, since \( S(\Sigma_{i,s_i} + s, \epsilon_i, R) \) converges to \( S_x \cap B_R(0) \) as \( i \to +\infty \), we have

\[
\begin{align*}
& d_H(S_{i,s} \cap B_R(0), S_x \cap B_R(0)) \\
& \leq d_H(S(\Sigma_{i,s_i} + s, \epsilon_i, R), S_x \cap B_R(0)) + d_H(S(\Sigma_{i,s_i} + s, \epsilon_i, R), S_x \cap B_R(0)) \\
& \leq d_H(S(\Sigma_{i,s_i} + s, \epsilon_i, R), S_x \cap B_R(0)) + \frac{1}{i} \\
& \to 0,
\end{align*}
\]

where we used \((3.72)\). Thus, \( S_{i,s} \cap B_R(0) \) converges to \( S_\infty \cap B_R(0) \) as \( i \to +\infty \). The lemma is proved.

\[\square\]

**Lemma 3.19.** Under the same assumptions as in Lemma 3.18, we can choose \( x_0 \in (\Sigma_\infty \setminus S_1) \cap B_R(0) \) and \( \{x_{i,k}\} \subset (\Sigma_{i,\infty} \setminus S_{i,1}) \cap B_R(0) \) satisfying the following properties.

1. \( x_{i,k} \to x_0 \) as \( i \to +\infty \) and \( k \to +\infty \).
2. For each \( i \), there exists \( k_i > 0 \) such that for any \( k \geq k_i \),

\[
\hat{u}_{i,k}(x_{i,k}, 1) \leq 2u_{i,k}(x_0, 1), \quad \forall \ k \geq k_i. \tag{3.73}
\]

Here \( u_{i,k} \) denotes the height difference function of \( \Sigma_{i,s_i} \) over \( \Sigma_\infty \).

**Proof.** Choose \( x_0 \in (\Sigma_\infty \setminus S_1) \cap B_R(0) \) and we denote by \( l_{x_0} \) the normal line of \( \Sigma_\infty \) passing through the point \( x_0 \). Then the set \( \Sigma_{i,k} \cap l_{x_0} \) is nonempty for large \( i \) and \( k \). Since \( \Sigma_{i,k} \) can be viewed as a union of multiple graphs over \( \Sigma_{i,\infty} \) away from singularities, we assume that \( l_{x_0} \) intersects with the bottom sheet of \( \Sigma_{i,k} \) at the point \( y_{i,k} \), and the projection of \( y_{i,k} \) on \( \Sigma_{i,\infty} \) is \( x_{i,k} \in \Sigma_{i,\infty} \). We denote by \( l_{x_{i,k}} \) the normal line of \( \Sigma_{i,\infty} \) passing through the point \( x_{i,k} \). Since \( x_0 \notin S_1 \), we have \( x_{i,k} \notin S_{i,1} \) for large \( i \) and \( k \). By the construction of \( x_{i,k} \), it is clear that \( x_{i,k} \) converges to \( x_0 \) as \( i \to +\infty \) and \( k \to +\infty \).

Fix \( \theta_0 \in (0, \frac{\pi}{2}) \). Since \( \Sigma_{i,\infty} \) converges smoothly to \( \Sigma_\infty \), the angle between the two lines \( l_{x_0} \) and \( l_{x_{i,k}} \) will lie in \([0, \theta_0]\) for large \( i \) and there is a uniform \( r_0 > 0 \) independent of \( i \) such that

\[
|A|(x) \leq \frac{1}{r_0}, \quad \forall \ x \in \Sigma_{i,\infty} \cap B_{r_0}(x_0).
\]

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We assume that $\Sigma'$ is $\Sigma_{i,\infty}$, $\Sigma'_{u_1}$ is the top sheet of $\Sigma_{i,k}$, $\Sigma'_{u_2}$ is the bottom sheet of $\Sigma_{i,k}$ and $P$ is the point $x_{i,k}$ as above. Then we apply Lemma 3.24 below for such $\Sigma'$, $\Sigma'_{u_1}$, $\Sigma'_{u_2}$ and the point $P$ and we can get that the functions $u_{i,k}$ and $u_{i,k}$ satisfy (3.73) for large $k$. The lemma is proved.

By Lemma 3.17 for each $i$ the function $\hat{w}_{i,k}$ converges in $C^2$ to the limit function $\hat{w}_{i,\infty}$ on any $K \times I$ with $I \subset [-1, T-2]$ and $K \subset (\Sigma_{i,\infty} \cap B_R(0)) \setminus \Sigma_{i,t}$, and $x_{i,k} \to x_{i,0}$ as $k \to +\infty$. Moreover, $\hat{w}_{i,\infty}(x_{i,0},1) = 1$. Note that $\hat{w}_{i,\infty}$ satisfies the equation (3.68), and the function $\hat{w}_i = \hat{w}_{i,\infty} e^{-\frac{|x|^2}{4}}$ satisfies the equation

$$
\frac{\partial \hat{w}_i}{\partial t} = \Delta \hat{w}_i + \left(|A|^2 + \frac{3}{4} - \frac{1}{16}|x|^2\right) \hat{w}_i, \quad \forall (x,t) \in K \times I.
$$

(3.74)

We would like to show that $\hat{w}_i$ satisfies the parabolic Harnack inequality with uniform constants independent of $i$. Note that here we need to use Theorem B.3 in Appendix B instead of Theorem A.5 in Appendix A. The reason is that $\hat{w}_i$ are functions defined on subdomains in $\Sigma_{i,\infty}$, which varies when $i$ is different. The constants in the Harnack inequality of Theorem A.5 depend on the manifold $\Sigma_{i,\infty}$ and it is difficult to show that the constants are independent of $i$. However, we can use Theorem B.3 to avoid this difficulty since the constants can be explicitly written down by Theorem B.1. We note that Theorem B.3 cannot be used for the equation (3.76) of $\hat{w}_i$ and we have to use Theorem A.5 in the proof of Lemma 3.15.

**Lemma 3.20.** Let $\hat{w}_i = \hat{w}_{i,\infty} e^{-\frac{|x|^2}{8}}$. For any $-T < a < s < t < b < T$, any $\epsilon > 0$, $x \in \Omega_{i,\epsilon,R}(s)$ and $y \in \Omega_{i,\epsilon,R}(t)$, there exists a constant $C = C(\epsilon, R, s-a, t-s, \Sigma_{i,\infty}, S_{[a,b]})$ independent of $i$ such that

$$
\hat{w}_i(x,s) \leq C \hat{w}_i(y,t).
$$

(3.75)

**Proof.** The lemma follows from the combination of the proof of Lemma 3.15 and Theorem B.3. For the readers’ convenience, we give the detailed proof here.

By Lemma 3.18 $S_t$ converges to $S$ in the Hausdorff topology. Since $S_t \cap B_R(0)$ consists of finitely many points, we can choose $\delta_0(\Sigma_{i,\infty}, S_{[a,b]}) > 0$ small such that for any $s \in [a, b],$

$$
\Omega_{\frac{1}{\epsilon},R}(s) \subset \Omega_{\epsilon,R}(t), \quad \Omega_{\frac{1}{\epsilon},R+2}(s) \subset \Omega_{\epsilon,R+2}(t), \quad \forall t \in [s - \delta_0, s + \delta_0] \cap [a, b].
$$

(3.76)

Thus, for large $i$ we have

$$
\Omega_{i,2\epsilon,R}(s) \subset \Omega_{i,\epsilon,R}(t), \quad \Omega_{i,\frac{1}{\epsilon},R+2}(s) \subset \Omega_{i,\frac{1}{\epsilon},R+2}(t), \quad \forall t \in [s - \delta_0, s + \delta_0] \cap [a, b].
$$

(3.77)

Let $N$ be a positive integer satisfying

$$
N > \max \left\{ \frac{5(b-a)}{\delta_0}, \frac{b-a}{s-a}, \frac{5(b-a)}{t-s} \right\}.
$$

(3.78)

Set

$$
\tau_k = a + \frac{b-a}{N} k, \quad \forall k \in \{0, 1, \ldots, N\}.
$$

(3.79)

Then $\tau_0 = a$ and $\tau_N = b$. By (3.78) we have $s \geq \tau_1$. Note that (3.76) and (3.79) imply that for any $k = 1, 2, \ldots, N - 1$ we have

$$
\Omega_{i,\frac{1}{\epsilon},R+2}(\tau_k) \subset \Omega_{i,\frac{1}{\epsilon},R+2}(t), \quad \forall t \in [\tau_{k-5}, \tau_{k+5}] \cap [a, b].
$$

(3.80)
Let
\[
\Omega'_i := \Omega_{i,\epsilon,R}(\tau_k), \quad \Omega''_i := \Omega_{i,\frac{\epsilon}{2},R+1}(\tau_k), \quad \Omega_i := \Omega_{i,\frac{\epsilon}{2},R+2}(\tau_k).
\] (3.81)

Then we have \(\Omega'_i \subset \Omega''_i \subset \Omega_i\). By Lemma 3.15, \(\Sigma_{i,\infty}\) converges smoothly to \(\Sigma_\infty\), \(S_i\) converges to \(S\), the domains \(\Omega'_i, \Omega''_i, \Omega_i\) converge to \(\Omega', \Omega'', \Omega\) respectively, where
\[
\Omega' := \Omega_{\epsilon,R}(\tau_k), \quad \Omega'' := \Omega_{\frac{\epsilon}{2},R+1}(\tau_k), \quad \Omega := \Omega_{\frac{\epsilon}{2},R+2}(\tau_k).
\] (3.82)

Note that \(\hat{w}_i\) satisfies the equation (3.74), which is exactly the same as the equation (B.9) in the appendix. Thus, we can apply Theorem B.3 in appendix B for the function \(w_i\), the domains \(\Omega', \Omega'', \Omega\) and the interval \([\tau_{k-1}, \tau_{k+1}]\) to obtain
\[
\hat{w}_i(x, \tau_k) \leq C\hat{w}_i(y, \tau_{k+1}), \quad \forall \ x, y \in \Omega_{\epsilon,R}(\tau_k),
\] (3.83)

where \(C = C(\epsilon, R, b - a, N, \Sigma_\infty, S_{[a,b]}\) is a constant independent of \(i\). Moreover, there exists a sequence of points \(\{z_k\}\) such that
\[
z_k \in \Omega_{i,2\epsilon,R}([\tau_{k-1}, \tau_k]) \cap \Omega_{i,2\epsilon,R}([\tau_k, \tau_{k+1}]) \neq \emptyset.
\] (3.84)

For \(s, t \in (a, b)\) with \(s < t\), there exist integers \(k_s\) and \(k_t\) such that \(s \in [\tau_{k_s}, \tau_{k_s+1})\) and \(t \in (\tau_{k_t-1}, \tau_{k_t+1}]\). Then we have
\[
k_t - k_s \geq 4.
\] (3.85)

as in Lemma 3.15 Set
\[
\Omega'_i = \Omega_{i,\epsilon,R}(s), \quad \Omega''_i = \Omega_{i,\frac{\epsilon}{2},R+1}(s), \quad \Omega = \Omega_{i,\frac{\epsilon}{2},R+2}(s).
\] (3.86)

Applying Theorem B.3 for such \(\Omega', \Omega'', \Omega\) and the interval \([\tau_{k_s-2}, \tau_{k_s+2}]\) as in Lemma 3.15 we have
\[
\hat{w}_i(x, s) \leq C\hat{w}_i(z_{k_s+2}, \tau_{k_s+2}), \quad \forall \ x \in \Omega_{i,\epsilon,R}(s).
\] (3.87)

Moreover, (3.83) and (3.84) implies that
\[
\hat{w}_i(z_{k_s+2}, \tau_{k_s+2}) \leq C\hat{w}_i(z_{k_s+3}, \tau_{k_s+3}) \leq \cdots \leq C^N\hat{w}_i(z_{k_t-1}, \tau_{k_t-1}),
\] (3.88)

where we used (3.85). Similar to the proof of (3.87), we have
\[
\hat{w}_i(z_{k_t-1}, \tau_{k_t-1}) \leq C\hat{w}_i(y, t), \quad \forall \ y \in \Omega_{i,\epsilon,R}(t).
\] (3.89)

Combining this with (3.87)-(3.89), we have
\[
\hat{w}_i(x, s) \leq C\hat{w}_i(y, t), \quad \forall \ x \in \Omega_{i,\epsilon,R}(s), \quad y \in \Omega_{i,\epsilon,R}(t).
\] (3.90)

The constants \(C\) in (3.87)-(3.90) depend on \(\epsilon, R, b - a, N, \Sigma_\infty\) and \(S_{[a,b]}\). The lemma is proved.

The next result shows that the normalized height difference function \(\hat{w}_{i,k}\) has uniformly \(L^1\) estimate away from the singular set near \(t = 0\), and the estimate doesn’t depend on \(i\). The proof of this result relies on the growth estimates of \(\hat{w}_{i,\infty}\) near the singular set, which is given in Theorem 3.2 in the next section.
Lemma 3.21. Fix $\tau \in (0, \frac{1}{2})$. Under the same assumptions as in Lemma 3.18 for each $i$ we can choose $k_i$ sufficiently large such that for any $k \geq k_i$, the normalized height difference function

$$\tilde{w}_{i,k}(x, s) = \frac{\tilde{u}_{i,k}(x, s)}{\tilde{u}_{i,k}(x_i, 1)},$$  \hspace{1cm} (3.91)$$

where the points $\{x_{i,k}\}$ are chosen as in Lemma 3.19 satisfies the inequality

$$\inf_{s \in [-\tau, 0]} \int_{\Sigma_{i,\infty} \cap \tilde{\Omega}_i} \tilde{w}_{i,k}(x, s) \leq 2W_0. $$  \hspace{1cm} (3.92)$$

Here $W_0$ is a constant independent of $i$.

Proof. Fix large $R > 0$. Since $\Sigma_{i,\infty}$ converges to $\Sigma_\infty$ smoothly, there exist uniform constants $\rho_0, \Xi_0 > 0$ such that for any large $i$ we have $B_R(0) \cap \Sigma_{i,\infty} \in \mathcal{M}_{k_0, 2}(\rho_0, \Xi_0)$. Here the set $\mathcal{M}_{k_0, 2}(\rho_0, \Xi_0)$ is defined in Definition 4.1. Note that by Lemma 3.17 for each $i$ the function $\tilde{w}_{i,k}$ converges in $C^1$ to the limit function $\tilde{w}_{i,\infty}$ away from $S_i$ and $x_{i,k} \to x_{i,0}$ as $k \to +\infty$. Applying Theorem 4.2 to the function $\tilde{w}_i = \tilde{w}_{i,\infty}e^{-\frac{|x|^2}{2}}$, we obtain that there exist uniform constants $C = C(\rho_0, \Xi_0, R)$ and $r_1(\rho_0, \Xi_0, R) > 0$ such that

$$\|\tilde{w}_{i,\infty}\|_{L^1((\Sigma_{i,\infty} \cap B_R(0)) \times [-\frac{1}{2}, 0])} \leq C(R, \rho_0, \Xi_0)\|\tilde{w}_{i,\infty}\|_{L^1(K_i)}, $$  \hspace{1cm} (3.93)$$

where $K_i$ is a compact set defined by

$$K_i := \left\{ (x, t) \in (\Sigma_{i,\infty} \cap B_{R+1}(0)) \times \left[ -1, \frac{1}{2} \right] \mid \min_{p \in S_i \cap r(B_{R+1}(0))} d_{g_i}(x, p) \geq r_1 \right\}, $$  \hspace{1cm} (3.94)$$

where $d_{g_i}$ denotes the intrinsic distance function of $(\Sigma_{i,\infty}, g_i)$. For any $t \in (-T, T)$, we define

$$K_i,r(t) := \left\{ x \in \Sigma_{i,\infty} \mid \min_{p \in S_i \cap r(B_{R+1}(0))} d_{g_i}(x, p) \geq r \right\}. $$

Since $(\Sigma_{i,\infty}, g_i)$ converges smoothly to $(\Sigma_\infty, g_\infty)$ and $S_i$ converges to $S_\infty$ by Lemma 3.18 for any $t \in (-T, T)$ $K_i,r(t)$ converges smoothly to a limit set, which we denote by $K_{\infty,r}(t) \subset \Sigma_\infty$. By part (5) of Lemma 3.18 $K_{\infty,r}(t) \cap S_t = \emptyset$. Note that $K_{\infty,r}(t)$ is defined with respect to the metric $g_\infty$ while $\Omega_{r,R}(t)$ is with respect to the Euclidean metric in $\mathbb{R}^3$. Let

$$r'_1 := \frac{1}{2} \min \left\{ d(x, p) \mid x \in K_{\infty,r_1}(t), p \in S_t, t \in [-2, 2] \right\} > 0, $$  \hspace{1cm} (3.95)$$

where $d(x, p)$ denotes the Euclidean distance in $\mathbb{R}^3$. Thus, we have

$$K_{\infty,r_1}(t) \subset \Omega_{r'_1,R+1}(t) \subset \Sigma_\infty, \hspace{1cm} \forall t \in [-2, 2]. $$  \hspace{1cm} (3.96)$$

Since $K_{i,r_1}(t)$ and $\tilde{\Omega}_{i,r'_1,R+1}(t)$ converge to $K_{\infty,r_1}(t)$ and $\Omega_{r'_1,R+1}(t)$ respectively for each $i$, for large $i$ we have

$$K_{i,r_1}(t) \subset \tilde{\Omega}_{i,\frac{1}{2}r'_1,R+2}(t), \hspace{1cm} \forall t \in [-2, 2]. $$  \hspace{1cm} (3.97)$$

Applying Lemma 3.20 for $\tilde{\Omega}_{i,\frac{1}{2}r'_1,R+2}(t)$ and $[-2, 2]$, we have

$$\tilde{w}_{i,\infty}(x, t) \leq C(r'_1, R, S_{[-2,2]})\tilde{w}_{i,\infty}(x_{i,0}, 1) = C(r'_1, R, \Sigma_\infty, S_{[-2,2]}), \hspace{1cm} \forall (x, t) \in K_i, $$  \hspace{1cm} (3.98)$$
where we used the fact that $\tilde{w}_{i,\infty}(x_{i,0},1) = 1$. Integrating both sides of (3.98) on $K_i$, we have
\[ \|\tilde{w}_{i,\infty}\|_{L^1(K_i)} \leq C\left(r'_i, R, \Sigma_{\infty}, S_{[-2,2]}\right) \text{Area}_g\left(\Sigma_{i,\infty} \cap B_{R+1}(0)\right) \leq C\left(r'_i, R, \Sigma_{\infty}, S_{[-2,2]}, N\right), \tag{3.99} \]
where we used the upper bound of area ratio in Lemma 3.4 in the last inequality. Combining (3.93) with (3.99), we have
\[ \|\tilde{w}_{i,\infty}\|_{L^1((\Sigma_{i,\infty} \cap B_R(0)) \times [-\frac{1}{2},0])} \leq C\left(R, \Sigma_{\infty}, S_{[-2,2]}, N, \rho_0, \Xi_0\right). \tag{3.100} \]
Thus, the $L^1$ norm of $\tilde{w}_{i,\infty}$ is uniformly bounded. Since $\tilde{w}_{i,k}$ converges to $\tilde{w}_{i,\infty}$ on any compact set away from singularities as $k \to +\infty$, we can choose $k_i$ large such that for any $k \geq k_i$,
\[ \inf_{s \in [-\tau,0]} \int_{\Sigma_{i,\infty} \cap \tilde{\Omega}_{i,\infty,0}(s)} \tilde{w}_{i,k}(x, s) \leq 2 \inf_{s \in [-\tau,0]} \int_{\Sigma_{i,\infty} \cap \tilde{\Omega}_{i,\infty,0}(s)} \tilde{w}_{i,\infty}(x, s) \leq 2 \int_{-\tau}^{0} dt \int_{\Sigma_{i,\infty} \cap B_R(0)} \tilde{w}_{i,\infty}(x, t) \leq C\left(R, \Sigma_{\infty}, S_{[-2,2]}, N, \rho_0, \Xi_0, \tau\right). \]
where we used the inequality (3.100). Thus, the inequality (3.92) is proved.

Combining Lemma 3.18, Lemma 3.19 with Lemma 3.21 we have the following result.

**Lemma 3.22.** Let $R > 0$ and $\tau \in (0, \frac{1}{4})$. There is a sequence of times $t_i \to \infty$, a self-shrinker $\Sigma_{\infty} \in C(D, N, \rho)$, a locally finite singular set $S$, and a constant $W$ satisfying the following properties.

1. For any $T > 1$, there exists a subsequence $\{t_{i_k}\}_{k=1}^{\infty}$ of $\{t_i\}$ such that $\{\Sigma_{t_{i_k}+s}, -T < s < T\}$ converges locally smoothly to $\Sigma_{\infty} \in C(D, N, \rho)$ away from $S$.
2. Let $x_0 \in \Sigma_{\infty} \setminus S_1$. We define the functions $u_i$ as in (3.33) and $w_i$ by
\[ w_i(x, t) = \frac{u_i(x, t)}{u_i(x_0, 1)}. \tag{3.101} \]
For any $\epsilon > 0$ and large $t_i$, we have the inequality
\[ \inf_{s \in [t-\tau, t]} \int_{\Omega_{i,R}(s)} w_i(x, s) \leq W, \quad \forall \ t \in [2, T), \tag{3.102} \]
where $W$ is a constant independent of $i$ and $T$.
3. For any $I = [a, b] \subset [-1, T - 2]$ and $K \subset (\Sigma_{\infty} \cap B_R(0)) \setminus S_1$, there exists a constant $C = C(\epsilon, K, S_1, a, b)$ such that
\[ 0 < w_i(x, t) < C, \quad \forall \ (x, t) \in K \times I. \tag{3.103} \]
Moreover, if $a \in [2, T - 2]$ there exists $C' = C'(K, \Sigma_{\infty}, S_{[0, a+1]}$ independent of $b$ such that
\[ w_i(x, a) \geq C', \quad \forall \ x \in K. \tag{3.104} \]
Proof. Fix a sequence of $\epsilon_i \to 0$. We choose $t_i = t_i, k_i$ with $k_i$ large such that Lemma 3.19 and Lemma 3.21 hold. Note that $u_i(x, s) = u_i, k_i(x, s)$ is the height difference function of $\Sigma_{t_i, k_i + s}$ over $\Sigma_\infty$. Then for any $T > 0$ the sequence $\{\Sigma_{t_i, k_i}, -T < s < T\}$ converges locally smoothly to $\Sigma_\infty \in C(D, N, \rho)$ away from $S$. Note that the limit self-shrinker $\Sigma_\infty$ is independent of the choice of $T$ by Lemma 3.4. For any $\epsilon > 0$, we have $\epsilon_i \in (0, \epsilon)$ for large $i$. Moreover, for large $t_i$ we have

$$\inf_{s \in [t-\tau, t]} \int_{\Omega_{x, R}(s)} w_i(x, s) \leq \frac{1}{u_i(x_0, 1)} \inf_{s \in [t-\tau, t]} \left| \text{TN}(\Sigma_{t_i, k_i}, \epsilon_i, R) \right| \leq \frac{2}{u_i(x_0, 1)} \inf_{s \in [-\tau, 0]} \left| \text{TN}(\Sigma_{t_i, k_i}, \epsilon_i, R) \right|,$$

where we used Lemma 3.14 in the first inequality and (3.69) in the second inequality. Note that (3.71) implies that

$$\left| \text{TN}(\Sigma_{t_i, k_i}, \epsilon_i, R) \right| \leq \int_{\bar{\Omega}_{1/4} R} \tilde{u}_{i, k_i}(x, s).$$

Thus, we have

$$\inf_{s \in [t-\tau, t]} \int_{\Omega_{x, R}(s)} w_i(x, s) \leq \frac{2\tilde{u}_{i, k_i}(x_{i, k_i}, 1)}{u_i(x_0, 1)} \cdot \frac{1}{\tilde{u}_{i, k_i}(x_{i, k_i}, 1)} \inf_{s \in [-\tau, 0]} \int_{\bar{\Omega}_{1/4} R} \tilde{w}_{i, k_i}(x, s)$$

$$= \frac{2\tilde{u}_{i, k_i}(x_{i, k_i}, 1)}{u_i(x_0, 1)} \cdot \inf_{s \in [-\tau, 0]} \int_{\bar{\Omega}_{1/4} R} \tilde{w}_{i, k_i}(x, s) \leq 4W_0 \cdot \frac{\tilde{u}_{i, k_i}(x_{i, k_i}, 1)}{u_i(x_0, 1)} \leq 8W_0,$$

where we used (3.92) and (3.73) in the fourth inequality and (3.73) in the last inequality. As in the proof of Lemma 3.16 (3.106) implies a uniform upper bound of $w_i$ on $K$, and we also have the lower bounds (3.103)-(3.104) of $w_i$. The lemma is proved.

\[\square\]

Proposition 3.23. Under the same assumption as in Lemma 3.22, $w_i$ converges in $C^2$ to a positive function $w(x, t)$ satisfying the equation (3.68) on $(\Sigma_\infty \times (0, \infty)) \setminus S$ with $w(x_0, 1) = 1$ and

$$\inf_{s \in [t-\tau, t]} \int_{\Sigma_{\infty} \cap B_R(0)} w(x, s) \leq W, \quad \forall t \in [1, \infty).$$

Moreover, for any $a \in [2, \infty)$ there exists a constant $C = C(a, \Sigma_\infty, S_{[a+1]}, K) > 0$ such that the function $w(x, t)$ satisfies

$$\int_{\Sigma_{\infty} \cap B_R(0)} w(x, a) \geq C.$$

Proof. For any $I \subset [1, T-2]$ and $K \subset (\Sigma_\infty \cap B_R(0)) \setminus S$, by Lemma 3.22 and the interior estimates of the parabolic equations (c.f. Theorem 4.9 of [50]), we have the space-time $C^{2, \alpha}$ estimates of $w_i$ on $K \times I$. Taking the limit $i \to +\infty$, $w_i$ converges in $C^2$ to a limit function $w(x, t)$ on $K \times I$ with the estimate (3.103)-(3.104). Moreover, (3.107) holds on $I$ by (3.102) and (3.108) holds on $I \cap [2, \infty)$ by (3.104). Since $\Sigma_\infty$ is independent of the choice of $T$ and the estimates of $w$ are independent of $T$, by
taking $T \to +\infty$ we obtain a function, still denoted by $w$, on $(\Sigma_\infty \times (0, \infty)) \setminus S$ with the estimates (3.107)-(3.108). The proposition is proved.

The following result was used in the proof of Lemma 3.19.

**Lemma 3.24.** Let $\Sigma \subset \mathbb{R}^3$ be a surface properly embedded in $B_{r_0}(x_0)$ with

\[ |A|(x) \leq \frac{1}{r_0} \quad x \in B_{r_0}(x_0) \cap \Sigma. \]  

(3.109)

Assume that $\Sigma_u_i$ is the graph of a functions $u_i$ over $\Sigma$ for $i = 1, 2$ and $\Sigma_u_1 \cap \Sigma_u_2 = \emptyset$. Let $P \in \Sigma, l_P$ the normal of $\Sigma$ at the point $P$, $G = l_P \cap \Sigma_u_1$ and $Q = l_P \cap \Sigma_u_2$. For any $\theta \in (0, \frac{\pi}{2})$, we denote by $l_\theta$ the line which passes through $Q$ and has angle $\theta$ with the line $l_P$. Let $B = \Sigma_u_1 \cap l_\theta$. Then there are two constants $\epsilon \in (0, 1)$ and $\theta_0 > 0$ both depending only on $r_0$ such that if $\theta \in (0, \theta_0)$ and

\[ \|u_1\|_{C^1(\Sigma \cap B_{r_0}(x_0))} + \|u_2\|_{C^1(\Sigma \cap B_{r_0}(x_0))} \leq \epsilon, \]  

(3.110)

then we have

\[ |GQ| \leq 2|BQ|. \]  

(3.111)

**Proof of Lemma 3.24** Without loss of generality, we assume that the tangent plane of $\Sigma$ at $P$ is the plane $\pi := \{(x_1, x_2, 0) | x_1, x_2 \in \mathbb{R}\}$ and the point $P$ is the origin $O$ of $\mathbb{R}^3$. Let $\hat{B}_{\delta_0}(0) = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 < \delta_0^2 \}$. By Lemma 2.2 there exists $\delta_0 = \delta_0(r_0) > 0$ such that $\Sigma \cap \hat{B}_{\delta_0}(0)$ can be written as a graph of a function $f$ over the plane $\pi$,

\[ \Sigma \cap \hat{B}_{\delta_0}(0) = \{(x_1, x_2, f(x_1, x_2)) | |x| < \delta_0\}, \]  

(3.112)

where $x = (x_1, x_2)$, and the graph function $f$ satisfies

\[ f(0) = 0, \quad Df(0) = 0, \quad |\nabla f|(y) \leq C_0|y|. \]  

(3.113)

Here $C_0$ depends only on $r_0$. Note that the coordinates of $G$ and $Q$ are given by $G = (0, 0, u_1(0))$ and $Q = (0, 0, u_2(0))$ respectively. For the point $B \in \Sigma_u_1 \cap l_\theta$, we define the point $E \in \Sigma$ the projection of $B$ onto $\Sigma$, which means

\[ \overrightarrow{OE} + u_1(E)\mathbf{n}(E) = \overrightarrow{OB}, \]  

(3.114)

where $\mathbf{n}(E)$ is the unit normal vector of $\Sigma$ at $E$.

We claim that there exist $\epsilon_0 = \epsilon_0(\delta_0) \in (0, 1)$ and $\theta_0 = \arctan 3 > 0$ such that if $\theta \in (0, \theta_0)$ and (3.110) holds for some $\epsilon \in (0, \epsilon_0)$, then $E \in \Sigma \cap \hat{B}_{\delta_0}(0)$. In fact, we assume that $\theta_0 = \arctan \frac{\delta_0}{\pi r_0}$. Then for any $\theta \in (0, \theta_0)$, we have $|\overrightarrow{OB}| \leq 2 \epsilon + \frac{\delta_0}{\pi}$. Combining this with (3.114) we have

\[ |\overrightarrow{OE}| \leq |\overrightarrow{OB}| + |u_1(E)| \leq 3 \epsilon + \frac{\delta_0}{2} \leq \frac{3}{4} \delta_0, \]

where we choose $\epsilon \in (0, \frac{1}{12} \delta_0)$. Therefore, by (3.112) we have $E \in \Sigma \cap \hat{B}_{\delta_0}(0)$. The claim is proved.

Assume that $E = (y, f(y)) \in \Sigma \cap \hat{B}_{\delta}(0)$ with $y = (y_1, y_2)$ and $\delta \in (0, \delta_0)$. Note that the normal vector at $E$ is given by

\[ \mathbf{n}(E) = \frac{(-\partial_{y_1} f(y), -\partial_{y_2} f(y), 1)}{\sqrt{1 + |\nabla f(y)|^2}}, \]
Thus, if we write \( u \) for simplicity. Since \( B_1^2 + B_2^2 = |QB|^2 \sin^2 \theta \), using (3.115)-(3.116) we have

\[
y_1^2 + y_2^2 + \frac{u_1(y)^2|\nabla f(y)|^2}{1 + |\nabla f(y)|^2} - 2 \frac{u_1(y)|y, \nabla f(y)|}{\sqrt{1 + |\nabla f(y)|^2}} = |QB|^2 \sin^2 \theta,
\]

where \( \langle y, \nabla f(y) \rangle = y_1 \partial_{y_1} f(y) + y_2 \partial_{y_2} f(y) \). Combining this with (3.113), we have

\[
|QB|^2 \sin^2 \theta \geq y_1^2 + y_2^2 - 2 \frac{u_1(y)|y, \nabla f(y)|}{\sqrt{1 + |\nabla f(y)|^2}} \geq (1 - 2C_0|u_1(y)|)(y_1^2 + y_2^2) \geq (1 - 2C_0\epsilon)(y_1^2 + y_2^2).
\]

Thus, if \( \epsilon \in (0, \frac{1}{2C_0}) \), we have

\[
|y|^2 = y_1^2 + y_2^2 \leq \frac{|QB|^2 \sin^2 \theta}{1 - 2C_0\epsilon}.
\]

Since \( l_\theta \) has the angle \( \theta \) with the line \( l_P \), we assume that the unit direction vector of \( l_\theta \) is \( \vec{v} = (v_1, v_2, \cos \theta) \). Thus, we have

\[
|QB| = |\overrightarrow{QB}, \vec{v}| = |B_1v_1 + B_2v_2 + (B_3 - u_2(0)) \cos \theta| \geq |(B_3 - u_2(0)) \cos \theta| - |B_1v_1| - |B_2v_2|.
\]

Note that by (3.113)

\[
|B_3 - u_2(0)| = |f(y) + \frac{u_1(y)}{\sqrt{1 + |\nabla f(y)|^2}} - u_2(0)|
\geq |u_1(0) - u_2(0)| - |u_1(0) - u_1(y)| - |u_1(y)| \cdot \left| \frac{1}{\sqrt{1 + |\nabla f(y)|^2}} - 1 \right| - |f(y)|
\geq |u_1(0) - u_2(0)| - \max_{B_3(0)} |\nabla u_1| \cdot |y| - C_0(1 + \max_{B_4(0)} |u_1|)|y|^2
\geq |u_1(0) - u_2(0)| - C_1|y|,
\]

where \( C_1 = \epsilon + C_0(1 + \epsilon)\delta_0 \), and by (3.113), (3.115)-(3.116) we have

\[
|B_1| \leq (1 + C_0\epsilon)|y|, \quad |B_2| \leq (1 + C_0\epsilon)|y|.
\]

Combining (3.118)-(3.121) we have

\[
|QB| \geq |u_1(0) - u_2(0)| \cos \theta - C_1|y|
\geq |u_1(0) - u_2(0)| \cos \theta - C_1 \sin \theta |QB|.
\]

This implies that

\[
\frac{|GQ|}{|QB|} = \frac{|u_1(0) - u_2(0)|}{|QB|} \leq 1 + C_1 \frac{\sin \theta}{\cos \theta} \leq 2,
\]

if we choose \( \theta \) sufficiently small. Thus, the lemma is proved. \( \Box \)
3.5 The $L$-stability of the limit self-shrinker

In this subsection, we show that the limit self-shrinker is $L$-stable. The rough idea is similar to that of [33], but the details are much more complicated. Compared with [33], the singularities here no longer move along straight lines, we cannot choose time large enough such that a given compact set doesn’t contain the singularities (c.f. Lemma 4.13 of [33]). Therefore, we have to choose a cutoff function near the singularities and analyze the asymptotical behavior of the positive solution near the singular set. The analysis of the asymptotical behavior is very difficult and we delay the arguments in the next section.

The main result in this subsection is the following lemma.

**Lemma 3.25.** Fix $R > 1$. Let $\{t_i\}$ be the sequence of times, and a self-shrinker $\Sigma_\infty \in C(D, N, \rho)$ in Lemma 3.22. Then we have

$$-\int_{\Sigma_\infty} (\psi L\psi)e^{-\frac{|x|^2}{4}} \geq 0,$$

for any smooth function $\psi \in C^\infty_0(\Sigma_\infty, R)$.

For any time interval $I = [a, b] \subset (0, \infty)$ and any compact set $K \subset \Omega_{e,R}(I)$, the function $v := \log w$ satisfies the equation

$$\frac{\partial v}{\partial t} = \Delta_0 v + \frac{1}{2} - \frac{1}{2} \langle x, \nabla v \rangle + |\nabla v|^2, \quad \forall (x,t) \in K \times I.$$

We assume that $\phi(x,t)$ is a function satisfying the properties that for any $t \in I$ we have

$$\phi(\cdot, t) \in W^{1,2}_0(\Sigma_\infty, R), \quad \text{Supp}(\phi(\cdot, t)) \cap S_t = \emptyset.$$  \hspace{0.5cm} (3.123)

Then for any $t \in I$, we have

$$0 = \int_{\Sigma_\infty} \text{div} \left( \phi^2 e^{-\frac{|x|^2}{4}} \nabla v \right)$$

$$= \int_{\Sigma_\infty} \left( 2\phi \langle \nabla \phi, \nabla v \rangle + \left( \frac{\partial v}{\partial t} - \frac{1}{2} \langle A \rangle^2 - |\nabla v|^2 \right) \phi^2 \right) e^{-\frac{|x|^2}{4}}$$

$$\leq \int_{\Sigma_\infty} \left( |\nabla \phi|^2 - \frac{1}{2} \phi^2 - |\langle A \rangle \phi^2 + \frac{\partial v}{\partial t} \phi^2 \right) e^{-\frac{|x|^2}{4}}.$$  

This implies that for any $t \in I$,

$$-\int_{\Sigma_\infty} (\phi L\phi)e^{-\frac{|x|^2}{4}} \geq -\int_{\Sigma_\infty} \frac{\partial v}{\partial t} \phi^2 e^{-\frac{|x|^2}{4}}$$

$$= -\frac{d}{dt} \int_{\Sigma_\infty} v \phi^2 e^{-\frac{|x|^2}{4}} + \int_{\Sigma_\infty} 2v \phi \frac{\partial \phi}{\partial t} e^{-\frac{|x|^2}{4}}.$$  \hspace{0.5cm} (3.124)

Assume that $I = (a, b) \subset (0, \infty)$. Integrating both sides, we have

$$-\int_a^b \int_{\Sigma_\infty} (\phi L\phi)e^{-\frac{|x|^2}{4}}$$

$$\geq \int_{\Sigma_\infty} v \phi^2 e^{-\frac{|x|^2}{4}} \big|_{t=a} - \int_{\Sigma_\infty} v \phi^2 e^{-\frac{|x|^2}{4}} \big|_{t=b} + \int_a^b \int_{\Sigma_\infty} 2v \phi \frac{\partial \phi}{\partial t} e^{-\frac{|x|^2}{4}}.$$
To get the inequality (3.122), the main difficulty is to estimate the last term of (3.124). Using a cutoff function inspired by (33), we will see that the last term of (3.124) depends on the asymptotical behavior of $w$ near the singular set.

We now construct the cutoff function near the singular set. Let $\{\xi_1(t), \xi_2(t), \cdots, \xi_l(t)\} (t \in I)$ be $\sigma$-Lipschitz curves on $\Sigma_\infty$. We denote by

$$\Gamma_k = \{(\xi_k(t), t) \mid t \in I\} \subset \Sigma_\infty \times I, \quad \Gamma = \bigcup_{k=1}^l \Gamma_k.$$ Choose $0 < \delta < \rho < 1$. We define the function on $\mathbb{R}$

$$\eta(s) = \begin{cases} \frac{\log \frac{\rho}{s}}{\log \rho}, & 0 < |s| < \rho, \\ \log |s|, & |s| \geq \rho \end{cases}$$

and the function $\beta(s) \in C^\infty(\mathbb{R})$ such that $\beta(s) = 0$ for $|s| < \frac{\delta}{2}$, $\beta(s) = 1$ for $|s| \geq \delta$, $0 \leq \beta(s) \leq 1$ and $|\nabla \beta| \leq \frac{3}{\delta}$. We define the function on $\Sigma_\infty \times I$,

$$f_{\delta, \rho}(x, t) = \prod_{k=1}^l \left( \eta(r_k(x, t)) \beta(r_k(x, t)) \right) \in W^{1,2}( (\Sigma_\infty \times I) \setminus \Gamma ),$$

where

$$r_k(x, t) = d_\sigma(x, \xi_k(t)). \quad (3.125)$$

For any $\psi(x) \in C_0^\infty(\Sigma_\infty, R)$, we define

$$\phi(x, t) = \psi(x) f_{\delta, \rho}(x, t). \quad (3.126)$$

Then $\phi(x, t)$ satisfies the properties (3.123). With loss of generality, we assume that $\sup_{\Sigma_\infty} |\psi| \leq 1$. Then we have

**Lemma 3.26.** For any small $\epsilon > 0$ we have

$$- \int_{\Sigma_\infty} \left( \phi L \phi \right) e^{-\frac{|x|^2}{4}} \leq \int_{\Sigma_\infty} \psi L(\psi)e^{-\frac{|x|^2}{4}} + \Psi(\epsilon, \rho, \delta \mid \Sigma_\infty, R), \quad (3.127)$$

where $\Psi$ depends on $\rho, \delta, \epsilon$ and the geometry of $\Sigma_\infty, R$ and satisfies

$$\lim_{\epsilon \to 0} \lim_{\rho \to 0} \lim_{\delta \to 0} \Psi(\epsilon, \rho, \delta \mid \Sigma_\infty, R) = 0. \quad (3.128)$$

**Proof.** Since the function $\phi(x, t) = \psi(x) f_{\delta, \rho}(x, t)$ satisfies

$$|\nabla \phi|^2 \leq \left(1 + \epsilon\right) f_{\delta, \rho}^2 |\nabla \psi|^2 + \left(1 + \frac{1}{\epsilon}\right) \psi^2 |\nabla f_{\delta, \rho}|^2,$$

we have

$$- \int_{\Sigma_\infty} \phi L(\phi) e^{-\frac{|x|^2}{4}} \leq \int_{\Sigma_\infty} \left( |\nabla \phi|^2 - \left( \frac{1}{2} + |A|^2 \right) \phi^2 \right) e^{-\frac{|x|^2}{4}}$$

$$\leq \int_{\Sigma_\infty} \left( |\nabla \psi|^2 - \left( \frac{1}{2} + |A|^2 \right) \psi^2 \right) e^{-\frac{|x|^2}{4}} + \int_{\Sigma_\infty} \left( (1 + \epsilon) f_{\delta, \rho}^2 - 1 \right) |\nabla \psi|^2 e^{-\frac{|x|^2}{4}}$$

$$+ \int_{\Sigma_\infty} \left( \frac{1}{2} + |A|^2 \right) (1 - f_{\delta, \rho}^2) \psi^2 e^{-\frac{|x|^2}{4}} + \left(1 + \frac{1}{\epsilon}\right) \int_{\Sigma_\infty} \psi^2 |\nabla f_{\delta, \rho}|^2 e^{-\frac{|x|^2}{4}}$$

$$:= I_0 + I_1 + I_2 + I_3. \quad (3.129)$$
We next estimate $I_3$. Let $f_k(x, t) = \eta(r_k(x, t))\beta(r_k(x, t))$. We define

$$\Xi_R := \inf \left\{ \Xi > 0 \mid \frac{1}{\Xi} \delta_{ij} \leq g_{ij}(x) \leq \Xi \delta_{ij}, \forall x \in B_R(0) \cap \Sigma_\infty \right\},$$

where $g_{ij}$ is the induced metric on $\Sigma_\infty$. Note that

$$\int_{\Sigma_\infty} |\nabla f_k|^2 e^{-\frac{|x|^2}{4}} \leq 2 \int_{\Sigma_\infty} \left( \beta^2 |\nabla \eta|^2 + \eta^2 |\nabla \beta|^2 \right) e^{-\frac{|x|^2}{4}}. \tag{3.131}$$

We estimate

$$\int_{\Sigma_\infty} \beta(r_k)^2 |\nabla(\eta(r_k))|^2 e^{-\frac{|x|^2}{4}} \leq \int_{\frac{4}{3} \leq r_k \leq \rho} (\eta'(r_k))^2 \leq C \int_{\frac{4}{3}}^{\rho} \frac{(\log \rho)^2}{s(\log s)^4} ds \leq C \left( \frac{1}{\log \rho} + \frac{(\log \rho)^2}{|\log \frac{\rho}{\delta}|^3} \right), \tag{3.132}$$

where $C$ is a constant depending on the metric $g$. Moreover,

$$\int_{\Sigma_\infty} \eta^2 |\nabla \beta|^2 e^{-\frac{|x|^2}{4}} \leq \int_{\frac{4}{3} \leq r_k \leq \delta} \eta(r_k)^2 (\beta'(r_k))^2 |\nabla r_k|^2 \leq C \int_{\frac{4}{3}}^{\delta} \frac{(\log \rho)^2}{(\log s)^2} \cdot \frac{4}{\delta^2} \cdot s \ ds \leq C \int_{\frac{4}{3}}^{\delta} \frac{(\log \rho)^2}{s(\log s)^2} ds \leq C \frac{(\log \rho)^2}{|\log \delta|}, \tag{3.133}$$

where $C$ is a constant depending on the metric $g$. Combining (3.132), (3.132), with (3.131), we have

$$\int_{\Sigma_\infty} |\nabla f_k|^2 e^{-\frac{|x|^2}{4}} \leq 2 \int_{\Sigma_\infty} \left( \beta^2 |\nabla \eta|^2 + \eta^2 |\nabla \beta|^2 \right) e^{-\frac{|x|^2}{4}} \leq C \left( \frac{1}{\log \rho} + \frac{|\log \rho|^2}{|\log \delta|^3} \right).$$

Since $|\psi| \leq 1$ and $|f_k| \leq 1$, we have

$$\int_{\Sigma_\infty} \psi^2 |\nabla f_{\delta, \rho}|^2 e^{-\frac{|x|^2}{4}} \leq l \int_{\Sigma_\infty} \sum_{k=1}^{l} |\nabla f_k|^2 e^{-\frac{|x|^2}{4}} \leq C(l, g) \left( \frac{1}{\log \rho} + \frac{|\log \rho|^2}{|\log \delta|^3} \right).$$

Therefore, we have

$$\lim_{\rho \to 0} \lim_{\delta \to 0} I_3 = 0. \tag{3.134}$$

Combining (3.130), (3.134) with (3.129), we have (3.127) and (3.128).
Lemma 3.27. For the function \( \phi \) defined by (3.126), we have

\[
\int_{\Sigma_{\infty}} 2v \phi \frac{\partial \phi}{\partial t} e^{-|\psi|^2/4} \geq -2\sigma (\log \rho)^2 \sum_{k=1}^{l} \left( \int_{A_t^{(k)}(\delta, \rho) \cap F_t^{(k)}(\delta/2)} \frac{3|v|}{|r_k| \log |r_k|^2} \right) + \int_{A_t^{(k)}(\delta, \rho) \cap F_t^{(k)}(\delta/2)} \frac{|v|}{|r_k| \log |r_k|^2}. \tag{3.135}
\]

where \( F_t^{(k)}(\delta) \) and \( A_t^{(k)}(\delta, \rho) \) are defined by

\[
F_t^{(k)}(\delta) = \bigcap_{i \neq k} \{ x \in \Sigma_{\infty} | r_i(x, t) \geq \delta \}, \tag{3.136}
\]

\[
A_t^{(k)}(\delta, \rho) = \{ x \in \Sigma_{\infty} | \delta < r_k(x, t) < \rho \}. \tag{3.137}
\]

Proof. We use the same notations as in the proof of Lemma 3.26. Direct calculation shows that

\[
\left| \int_{\Sigma_{\infty}} 2v \phi \frac{\partial \phi}{\partial t} e^{-|\psi|^2/4} \right| \leq \sum_{k=1}^{l} \int_{\Sigma_{\infty} \cap F_t^{(k)}(\delta/2)} 2|v| |f_k| \frac{\partial f_k}{\partial t} e^{-|\psi|^2/4}
\]

\[
\leq \sum_{k=1}^{l} \int_{\Sigma_{\infty} \cap F_t^{(k)}(\delta/2)} 2|v| \psi^2 \phi(r_k) \beta(r_k) \left( |\eta'(r_k)| \beta(r_k) + |\beta'(r_k)| \eta(r_k) \right) |\partial r_k/\partial t| e^{-|\psi|^2/4}. \tag{3.138}
\]

Note that for a.e. \( t \in I \), \( |\partial r_k/\partial t| \leq |\xi_k'(t)| \leq \sigma \), and we assumed that \( \sup_{\Sigma_{\infty}} |\psi| \leq 1 \). Therefore, using the definition of \( \eta \) and \( \beta \) we have

\[
\int_{\Sigma_{\infty} \cap F_t^{(k)}(\delta/2)} 2|v| \psi^2 \eta(r_k) \beta(r_k) |\partial r_k/\partial t| e^{-|\psi|^2/4}
\]

\[
\leq 2\sigma \int_{\Sigma_{\infty} \cap F_t^{(k)}(\delta/2)} \eta(r_k) \beta(r_k)^2 |v| \cdot |\eta'(r_k)|
\]

\[
\leq 2\sigma \int_{A_t^{(k)}(\delta, \rho) \cap F_t^{(k)}(\delta/2)} |v| \frac{(\log \rho)^2}{\log |r_k|^2}. \tag{3.139}
\]

and

\[
\int_{\Sigma_{\infty} \cap F_t^{(k)}(\delta/2)} 2|v| \psi^2 \eta(r_k)^2 \beta(r_k) |\beta'| \cdot |\partial r_k/\partial t| e^{-|\psi|^2/4}
\]

\[
\leq 2\sigma \int_{\Sigma_{\infty} \cap F_t^{(k)}(\delta/2)} |v| \eta^2(r_k) \gamma(r_k) \beta'(r_k)|
\]

\[
\leq \frac{6\sigma}{\delta} \int_{A_t^{(k)}(\delta, \rho) \cap F_t^{(k)}(\delta/2)} |v| \frac{(\log \rho)^2}{\log |r_k|^2}. \tag{3.139}
\]

Combining (3.138)-(3.139), we have (3.135).
Lemma 3.28. For any $t > 0$, we have
\[
\lim_{\rho \to 0} \lim_{\delta \to 0} (\log \rho)^2 \int_a^b dt \int_{A_t^{(k)}(\delta, \rho) \cap F_t^{(k)}(\frac{\delta}{2})} \frac{|v|}{r_k |\log r_k|^2} = 0, \hspace{1cm} (3.140)
\]
\[
\lim_{\rho \to 0} \lim_{\delta \to 0} \frac{(\log \rho)^2}{\delta} \int_a^b dt \int_{A_t^{(k)}(\frac{\delta}{2}, \rho) \cap F_t^{(k)}(\frac{\delta}{2})} \frac{|v|}{r_k |\log r_k|^3} = 0. \hspace{1cm} (3.141)
\]

Proof. Since $w(x, t)$ satisfies (3.68) away from the singular set, the function $f(x, t) = w(x, t)e^{-\frac{|x|^2}{4}}$ satisfies the equation
\[
\frac{\partial f}{\partial t} = \Delta f + \left( |A|^2 + \frac{3}{4} - \frac{1}{16} |x|^2 \right) f.
\]
By Theorem 4.2, we have
\[
\lim_{\rho \to 0} \lim_{\delta \to 0} \frac{1}{\delta} \int_a^b dt \int_{A_t^{(k)}(\frac{\delta}{2}, \rho) \cap F_t^{(k)}(\frac{\delta}{2})} f = 0.
\]
Since near the singular curve $\xi(t)$, the function $w$ is large and we have $\nu = \log w \leq w$. Thus, (3.140)-(3.141) also hold for $\nu$, and this directly implies (3.140)-(3.141). The lemma is proved.

Combining the above results, we can show Lemma 3.25.

Proof of Lemma 3.25. Combining Lemma 3.26 Lemma 3.27 with the inequality (3.124), we have
\[
-(b - a) \int_{\Sigma_{\infty}} \psi L(\psi) e^{-\frac{|x|^2}{4} + \Psi(\epsilon, \rho, \delta | \Sigma_{\infty}, R)}(b - a)
\geq \int_{\Sigma_{\infty}} \nu \psi^2 e^{-\frac{|x|^2}{4}} \bigg|_{t=a} - \int_{\Sigma_{\infty}} \nu \psi^2 e^{-\frac{|x|^2}{4}} \bigg|_{t=b}
-2\sigma (\log \rho)^2 \sum_{k=1}^t \left( \int_a^b \int_{A_t^{(k)}(\frac{\delta}{2}, \rho) \cap F_t^{(k)}(\frac{\delta}{2})} \frac{3|v|}{\delta |\log r_k|^2} \right)
+ \int_a^b \int_{A_t^{(k)}(\delta, \rho) \cap F_t^{(k)}(\frac{\delta}{2})} \frac{|v|}{r_k |\log r_k|^2}.
\]
(3.144)
Taking $\delta \to 0$ and next $\rho \to 0$, and then $\epsilon \to 0$ in (3.144), we get
\[
-\int_{\Sigma_{\infty}} \psi L(\psi) e^{-\frac{|x|^2}{4}} \geq \frac{1}{b - a} \left( \int_{\Sigma_{\infty}} \nu \psi^2 e^{-\frac{|x|^2}{4}} \bigg|_{t=a} - \int_{\Sigma_{\infty}} \nu \psi^2 e^{-\frac{|x|^2}{4}} \bigg|_{t=b} \right).
\]
(3.145)
Note that by Proposition 3.23 $w(x, t)$ is a function on $(\Sigma_{\infty} \times (0, \infty)) \setminus S$ with uniform estimates (3.107) and (3.108). Thus, there is a sequence $b_i \to +\infty$ such that
\[
\int_{\Sigma_{\infty}} \nu \psi^2 e^{-\frac{|x|^2}{4}} d\mu_{\infty} \bigg|_{t=b_i} \leq \int_{\Sigma_{\infty}} w d\mu_{\infty} \bigg|_{t=b_i} \leq W
\]
for a constant $W$. Therefore, by taking $b_i \to +\infty$ and $a = 2$ in (3.145) we get (3.122). The lemma is proved.

□
3.6 Proof of Theorem 3.1

In this subsection, we show Theorem 3.1. First, using Lemma 3.9 and the compactness result of Colding-Minicozzi [18] we have the following result.

Lemma 3.29. Let $R, D, N > 0$ and $\rho$ an positive constant. For any $\delta > 0$, there exists a constant $\xi = \xi(R, D, N, \rho, \delta) > 0$ such that for any $\Sigma \in \mathcal{C}(D, N, \rho)$ and any $r \in (0, \xi)$ we have

$$1 - \delta \leq \frac{\text{Area}(\Sigma \cap B_r(x))}{\pi r^2} \leq 1 + \delta, \quad \forall \, x \in B R(0) \cap \Sigma.$$  \hspace{1cm} (3.146)

Proof. We show that there exists a constant $C_R = C(R, D, N, \rho) > 0$ such that for all $\Sigma \in \mathcal{C}(D, N, \rho)$ we have $\sup_{\Sigma \cap B_{R+1}(0)} |A| \leq C_R$. For otherwise, we can find a sequence $\Sigma_i \in \mathcal{C}(D, N, \rho)$ such that

$$\sup_{\Sigma_i \cap B_{R+1}(0)} |A| \to +\infty.$$  \hspace{1cm} (3.147)

On the other hand, by the compactness theorem of Colding-Minicozzi [18], $\Sigma_i$ converges smoothly to $\Sigma_\infty \in \mathcal{C}(D, N, \rho)$, which has bounded $|A|$ on any compact set. This contradicts (3.147).

Since $\sup_{\Sigma \cap B_{R+1}(0)} |A|$ is uniformly bounded by $C_R$, (3.146) follows directly from Lemma 2.3.

The lemma is proved.

Using the uniform upper bound of the area ratio and Lemma 3.29, we have the following result.

Lemma 3.30. Under the same assumption as in Lemma 3.29, if $\{\Sigma_i\}$ converges locally smoothly to $\Sigma_\infty$ with multiplicity $m \in \mathbb{N}$ away from a locally finite singular set $S_0$, then for any $x_i \in \Sigma_i \cap B_{R_0}(0)$ with $x_i \to x_\infty \in \Sigma_\infty \cap B_{R+1}(0)$ and $r > 0$ we have

$$\lim_{i \to +\infty} \text{Area}(\Sigma_i \cap B_r(x_i)) = m \cdot \text{Area}(\Sigma_\infty \cap B_r(x_\infty)).$$  \hspace{1cm} (3.148)

Proof. Since $S_0$ is locally finite, without loss of generality we assume that $B_r(x_\infty) \cap \Sigma_\infty$ consists of only one singular point $y_\infty$. For any $\epsilon > 0$ by the smooth convergence of $\Sigma_i \cap (B_r(x_i) \setminus B_{\epsilon}(y_\infty))$ we have

$$\lim_{i \to +\infty} \text{Area}(\Sigma_i \cap (B_r(x_i) \setminus B_{\epsilon}(y_\infty))) = m \cdot \text{Area}(\Sigma_\infty \cap (B_r(x_\infty) \setminus B_{\epsilon}(y_\infty))).$$  \hspace{1cm} (3.149)

Since the area ratio is uniformly bounded from above along the rescaled mean curvature flow, we have

$$\text{Area}(\Sigma_i \cap B_{\epsilon}(y_\infty)) \leq N \pi \epsilon^2 \to 0$$

as $\epsilon \to 0$. Taking $\epsilon \to 0$ in both sides of (3.149), we have (3.148). The lemma is proved.

Combining Lemma 3.29, Lemma 3.30 with Lemma 2.9 we show that the area ratio is always close to an integer after a fixed time.

Lemma 3.31. Fix large $R$ and small $\delta_0 \in (0, \frac{1}{4})$. Under the same assumption as in Lemma 3.29 there exists $t_0 > 0$ such that for any $t > t_0$ we have

$$m(1 - 2\delta_0) \leq \frac{\text{Area}(\Sigma \cap B_t(x))}{\pi x^2} \leq m(1 + 2\delta_0), \quad \forall \, x \in B R(0) \cap \Sigma_t,$$  \hspace{1cm} (3.150)

where $m$ is a positive integer independent of $x$ and $t$. Here $\xi = \xi(R+1, D, N, \rho, \delta_0)$ is the constants in Lemma 3.29 with $D, N$ and $\rho$ determined as in the assumption 3.2 and Lemma 3.4.
Proof. We divide the proof into several steps.

Step 1. We show that there exists $t_0 > 0$ such that for any $t > t_0$ (3.150) holds for some integer $m(x, t)$, which may depend on $x$ and $t$. For otherwise, there exist a sequence $t_i \to +\infty$ and $x_i \in B_R(0) \cap \Sigma_{t_i}$ such that

$$\left| \frac{\text{Area}(\Sigma_{t_i} \cap B_\xi(x_i))}{\pi m \xi^2} - 1 \right| \geq 2\delta_0, \quad \forall \ m \in \mathbb{N} \cap [1, N_0].$$

(3.151)

By Proposition 2.8, by taking a subsequence if necessary we assume that $m(t_i)$ converges locally smoothly to a self-shrinker $\Sigma_\infty \in C(D,N,\rho)$ with multiplicity $m_0 \in \mathbb{N}$ and $x_i \to x_\infty \in \Sigma_\infty \cap B_{R+1}(0)$. By the convergence of $\{\Sigma_{t_i}\}$ and Lemma 3.30 we have

$$\lim_{i \to +\infty} \frac{\text{Area}(\Sigma_{t_i} \cap B_\xi(x_i))}{\pi \xi^2} = m_0 \frac{\text{Area}(\Sigma_\infty \cap B_\xi(x_\infty))}{\pi \xi^2}.$$  

(3.152)

Lemma 3.29 implies that

$$1 - \delta_0 \leq \frac{\text{Area}(\Sigma_\infty \cap B_\xi(x_\infty))}{\pi \xi^2} \leq 1 + \delta_0.$$  

(3.153)

Combining (3.152) with (3.153), for large $t_i$ we have

$$\left| \frac{\text{Area}(\Sigma_{t_i} \cap B_\xi(x_i))}{\pi m_0 \xi^2} - 1 \right| \leq \frac{3\delta_0}{2},$$  

(3.154)

which contradicts (3.151).

Step 2. We show that $m(x, t)$ is independent of $x$ and we can write $m(t)$ for short. For otherwise, we can find a sequence $t_i \to +\infty$ and $x_i, y_i \in \Sigma_{t_i}$ with $m(x_i, t_i) \neq m(y_i, t)$. Since $m(x, t) \in [1, N_0]$, by taking a subsequence if necessary we assume that $m(x_i, t_i) = m_1$ for all $i$. Thus, for any $i$ we have

$$m(y_i, t_i) \neq m_1.$$  

(3.155)

By Proposition 2.8 by taking a subsequence if necessary we assume that $\Sigma_{t_i}$ converges locally smoothly to a self-shrinker $\Sigma_\infty \in C(D,N,\rho)$ with multiplicity $m_0 \in \mathbb{N}$, and

$$x_i \to x_\infty, \quad y_i \to y_\infty, \quad x_\infty, y_\infty \in \Sigma_\infty \cap B_{R+1}(0).$$

By (3.154), we have $m(x_i, t_i) = m_0 = m(y_i, t_i)$, which contradicts (3.155).

Step 3. We show that $m(t)$ is independent of $t$. It suffices to show that for any $s \in (-\frac{1}{2}, \frac{1}{2})$, we have $m(t) = m(t + s)$. For otherwise, we can find a sequence $t_i \to +\infty$ and $s_i \in (-\frac{1}{2}, \frac{1}{2})$ such that for all $i$,

$$m(t_i) \neq m(t_i + s_i).$$  

(3.156)

We follow the same argument as in Step 2. Since $m(t_i)$ is uniformly bounded, by taking a subsequence if necessary we can assume that $m(t_i) = m_1$ for all $i$. By (3.156), for all $i$ we have

$$m(t_i + s_i) \neq m_1.$$  

(3.157)

Note that $m(t_i + s_i)$ is also bounded, we can assume that a subsequence of $\{m(t_i + s_i)\}$ converges to an integer $m_2$ with

$$m_2 \neq m_1$$  

(3.158)
by (3.157). By Proposition 2.8 by taking a subsequence if necessary we assume that \( \{\Sigma_{t_i+s}, -1 < s < 1\} \) converges locally smoothly to a self-shrinker \( \Sigma_\infty \in C(D,N,\rho) \) with multiplicity \( m_0 \in \mathbb{N} \). The inequality (3.154) implies that \( m_0 = m_1 \). Since the multiplicity \( m_0 \) of the convergence is independent of time by Lemma 2.9 we have \( m_0 = m_2 \). Thus, we have \( m_1 = m_2 \), which contradicts (3.158).

Using Lemma 3.31 and the results in previous sections, we show Theorem 3.1.

**Proof of Theorem 3.1** Fix large \( R > R_0 \), where \( R_0 \) is the constant chosen in Lemma 3.8. We choose a sequence \( t_i \to +\infty \) as in Lemma 3.22 Then there is a self-shrinker \( \Sigma_\infty \in C(D,N,\rho) \) such that for any \( T > 1 \) we can find a subsequence, still denoted by \( \{t_i\} \), such that \( \{\Sigma_{t_i+t}, -T < t < T\} \) converges in smooth topology, possibly with multiplicities at most \( N_0 \), to \( \Sigma_\infty \) away from a singular set \( \mathcal{S} \). If the multiplicity of the convergence is greater than one, Lemma 3.25 shows that the limit self-shrinker \( \Sigma_\infty \) is \( L \)-stable in the ball \( B_R(0) \). This contradicts Lemma 3.8. Therefore, the multiplicity is one and the convergence is smooth.

We next show that for any sequence of \( s_i \to +\infty \) there exists a subsequence such that the multiplicity of the convergence is also one. For otherwise, there exists a sequence \( s_i \to +\infty \) such that \( \Sigma_{s_i} \) converges locally smoothly to a self-shrinker \( \Sigma'_\infty \in C(D,N,\rho) \) with multiplicity \( m' > 1 \). By Lemma 3.31 there exists \( t_0 > 0 \) such that for any \( t > t_0 \) we have

\[
m(1 - 2\delta_0) \leq \frac{\text{Area}(\Sigma_t \cap B_\xi(x))}{\pi \xi^2} < m(1 + 2\delta_0), \quad \forall \, x \in B_R(0) \cap \Sigma_t,
\]

where \( m \) is a positive integer independent of \( x \) and \( t \). By taking \( t = t_i \to +\infty \) in (3.159), we have \( m = m' > 1 \), which is a contradiction. Thus, the theorem is proved.

### 4 Estimates near the singular set

In this section, we will study the asymptotical behavior of the function \( w \) near the singular set. These estimates are used in the proof of Lemma 3.21 and Lemma 3.28. In [7], Kan-Takahashi studied time-dependent singularities in semilinear parabolic equations along one singular curve. Here we develop Kan-Takahashi’s techniques to estimate the solution when the singular sets consists of multiple singular curves.

First, we introduce the following notations. Throughout this section, we denote by \( B_r(p) \) the (intrinsic) geodesic ball centered at \( p \) in \( (M,g) \) and \( d_g(x,y) \) the distance from \( x \) to \( y \) with respect to the metric \( g \).

**Definition 4.1.** Let \( (M,g) \) be a complete Riemannian manifold of dimension \( m \). For any \( k \in \mathbb{N}, \rho, \Xi > 0 \), we define \( M_{k,m}(\rho,\Xi) \) the set of all subsets \( A \subset (M,g) \) such that

1. for any \( p \in A \), the harmonic radius at \( p \) satisfies \( r_h(p) \geq \rho \);

2. For any \( p \in A \), the ball \( B_\rho(p) \) has harmonic coordinates \( \{x_1,x_2,\cdots,x_m\} \) such that the metric tensor \( g_{ij} \) in these coordinates satisfies

\[
\Xi^{-1} \delta_{ij} \leq g_{ij} \leq \Xi \delta_{ij}, \quad \left| \frac{\partial^\alpha g_{ij}}{\partial x^\alpha} \right| \leq \Xi, \quad \text{on} \quad B_\rho(p),
\]

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for any multi-index $\alpha$ with $1 \leq |\alpha| \leq k$.

The following theorem is the main result in this section, which gives the asymptotical behavior of a positive solution of a parabolic equation near a time-dependent singular set.

**Theorem 4.2.** Let $(\Sigma^2, g)$ be a two-dimensional complete surface and $\{\xi_1, \xi_2, \cdots, \xi_l\}$ with $\xi_k : [T_1, T_2] \to \Sigma$ be $\alpha$-Lipschitz curves in $\Sigma$. Assume that $u(x, t) \in L^1_{\text{loc}}((\Sigma \times (T_1, T_2)) \setminus \bigcup_{k=1}^l \Gamma_k)$ is a nonnegative solution of the equation:

$$
\frac{\partial u}{\partial t} = \Delta_g u + c(x, t)u,
$$

where $c(x, t) \in L^\infty(\Sigma \times [T_1, T_2])$ and $\Gamma_k = \{(\xi_k(t), t)\} \subset \Sigma \times [T_1, T_2]$. Assume that for any $k \in \{1, 2, \cdots, l\}$ and any $t \in [T_1, T_2]$ the ball $B_1(\xi_k(t))$ is in $\mathcal{M}_{k_0,2}(\rho_0, \Xi_0)$, where $k_0$ is an integer chosen as in Corollary 4.4. Then we have

(1) $u \in L^1_{\text{loc}}((\Sigma \times (T_1, T_2))$. More precisely, for any $(t_1, t_2) \subset (T_1, T_2)$, there exists a constant $r_1 = r_1(\rho_0, \Xi_0, l, t_1, t_2, T_1, T_2) > 0$ such that

$$
\|u\|_{L^1(Q_{r_1, t_1, t_2})} \leq C \|u\|_{L^1(K)},
$$

where $C$ is a constant depending on $\|c\|_{L^\infty(Q_{1, t_1, t_2})}$, $\rho_0, \Xi_0, \sigma, t_1, t_2, T_1, T_2$ and $K$ is defined by $K = \bigcup_{l=1}^l Q_{r_1, t_1, t_2} \setminus Q_{r_1, t_1, t_2}$. Here $Q_{r, t_1, t_2}$ is defined by

$$
Q_{r, t_1, t_2} = \bigcup_{k=1}^l \{(x, t) \in \Sigma \times \mathbb{R}, x \in B_r(\xi_k(t)) \subset \Sigma, t \in (t_1, t_2)\}.
$$

(2) For any $(t_1, t_2) \subset (T_1, T_2)$, we have

$$
\lim_{R \to 0} \lim_{\delta \to 0} \int_{t_1}^{t_2} \frac{1}{R} \int_{A^k_t(\xi_k, \frac{t}{R}) \cap F^k_t(\xi_k, \frac{t}{R})} \frac{u}{r_k \log r_k} \omega \ d\omega = 0,
$$

and

$$
\lim_{\delta \to 0} \int_{t_1}^{t_2} \frac{1}{\delta} \int_{A^k_t(\xi_k, \frac{t}{\delta}) \cap F^k_t(\xi_k, \frac{t}{\delta})} u \ d\omega < +\infty,
$$

where $A^k_t(\xi_k)$ and $F^k_t(\xi_k)$ are defined by (3.136)- (3.137) and $r_k(x, t) = d(x, \xi_k(t))$.

We sketch the proof of Theorem 4.2. First, we show an asymptotical formula for the heat kernel on a Riemannian manifold in Theorem 4.3. Using this formula, we construct a special function $U_k(x, t)$ for each singular curve $\xi_k$ and a measure $\nu$, and show that $U_k(x, t)$ behaves like $\log \frac{1}{r_k(x, t)}$ when $x$ is near $\xi_k$ and $\nu$ is the Lebesgue measure in Lemma 4.5. Moreover, $U_k(x, t)$ satisfies the growth estimates (4.4)-(4.5) by Lemma 4.6 and we use $U_k(x, t)$ to construct a function $v_k$ in Lemma 4.5 which satisfies the backward heat equation. The function $v_k$ is important to construct some cutoff functions (c.f. Definition 4.12). When the singular curves are disjoint, using these cutoff functions we can show (4.3) directly in Lemma 4.9. When the singular curves are not disjoint, we show the finiteness of a functional $I$ and use the functional $I$ to show the $L^1$ norm of $u$ (4.3) in Lemma 4.13. By using the functional $I$, we get a positive linear functional $\mu_k$ for each singular curve $\xi_k$ in Lemma 4.15 and by Lemma 4.16 $\mu_k$ is uniformly bounded even if the singular curves are not disjoint. Finally, we use $\mu_k$ to construct $U_k$ and show that $u$ is controlled by $U_k$. By the properties of $U_k$, we have that $u$ satisfies the growth estimates (4.4)-(4.5).
4.1 Properties of the heat kernel

In this subsection, we will give the expansion of the heat kernel on Riemannian manifolds. Let \( (M, g) \) be a complete Riemannian manifold (without boundary) of dimension \( m \). Suppose that \( p(x, y, t) \) is the heat kernel. Then \( p(x, y, t) \) has the following asymptotical formula (c.f. Theorem 11.1 of [6])

\[
p(x, y, t) \sim (4\pi t)^{-\frac{m}{2}} e^{-\frac{d^2(x, y)}{4t}}
\]

for \( t \to 0 \) and \( d_g(x, y) \to 0 \). The next result gives more estimates on the asymptotical formula.

**Theorem 4.3.** (c.f. Theorem 11.1 of [6], or Theorem 2.30 of [11]) Let \( \rho_0, \Xi_0 > 0 \) and integers \( m \geq 2, k \geq 0 \). There exists an integer \( k_0 = k_0(k) \) depending only on \( k \) satisfying the following property. Let \( (M, g) \) be a complete Riemannian manifold of dimension \( m \) and \( x_0 \in M \) with \( B_{\rho_0}(x_0) \subset M_{k_0, m}(\rho_0, \Xi_0) \). There exists a sequence of smooth functions \( \{u_i(x, y)\} \) with \( u_0(x, x) = 1 \) such that for any \( x, y \in B_{\rho_0}(x_0) \) and \( t \in (0, 1) \) we have

\[
|p(x, y, t) - (4\pi t)^{-\frac{m}{2}} e^{-\frac{d^2(x, y)}{4t}} \sum_{i=0}^{k} u_i(x, y) t^i| \leq C(\rho_0, \Xi_0, m) t^{k+1-\frac{m}{2}},
\]

\[
|\nabla_x p(x, y, t) - \nabla_x \left((4\pi t)^{-\frac{m}{2}} e^{-\frac{d^2(x, y)}{4t}} \sum_{i=0}^{k} u_i(x, y) t^i\right)| \leq C(\rho_0, \Xi_0, m) t^{k-\frac{m}{2}}.
\]

**Proof.** We follow the argument in Theorem 11.1 of [6] to prove (4.7)-(4.8). Define the function

\[
G(x, y, t) = (4\pi t)^{-\frac{m}{2}} e^{-\frac{d^2(x, y)}{4t}} \sum_{i=0}^{k} u_i(x, y) t^i.
\]

Direct calculation shows that

\[
\left(\Delta_y - \frac{\partial}{\partial t}\right) G = (4\pi t)^{-\frac{m}{2}} e^{-\frac{d^2(x, y)}{4t}} \times \left(\left( -\frac{r\Delta_y}{2} + \frac{m-1}{2}\right) \sum_{i=0}^{k-1} u_i t^i - \frac{r}{2} \sum_{i=1}^{k-1} \left(\nabla r, \nabla u_{i+1}\right) t^i + \sum_{i=0}^{k-1} (\Delta_y u_i) t^i - \sum_{i=0}^{k-1} (i+1) u_{i+1} t^i\right).
\]

For fixed \( x \) and \( y \in B_{\rho_0}(x_0) \), there exists a sequence of function \( \{u_i(x, y)\} \) satisfying

\[
\left(\frac{r\Delta_y}{2} - \frac{m-1}{2}\right) u_0 + r \left(\nabla r, \nabla u_0\right) = 0,
\]

\[
\left(\frac{r\Delta_y}{2} - \frac{m-1}{2}\right) u_{i+1} + r \left(\nabla r, \nabla u_{i+1}\right) + (i+1) u_{i+1} = \Delta_y u_i, \quad 0 \leq i \leq k-1.
\]

This implies that

\[
\left(\Delta_y - \frac{\partial}{\partial t}\right) G = (4\pi t)^{-\frac{m}{2}} e^{-\frac{d^2(x, y)}{4t}} \Delta_y u_k t^k.
\]

As in the proof of Theorem 11.1 of [6], we have that

\[
u_0(x, y) = C d_g(x, y)^{\frac{m+1}{2}} J^{-\frac{1}{2}},
\]

\[
u_{i+1} = d_g(x, y)^{\frac{m+3-2i}{2}} J^{-\frac{1}{2}} \int_0^r s^{\frac{2i+3-m}{2}} J^{\frac{1}{2}} \Delta u_i ds,
\]

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where $C$ is a constant such that $u_0(x, x) = 1$ and $J(y)$ is the area element of the sphere $d_g(x, y)$ at the point $y$. There exists integer $k_0$ depending only on $k$ such that under the assumption $B_{\rho_0}(x_0) \in M_{k_0, m}(\rho_0, \Xi_0)$, for any integer $i \in [0, k]$ we have

$$|u_i(x, y)| + |\nabla_y u_i(x, y)| + |\nabla_x \nabla_y u_i(x, y)| \leq C(\rho_0, \Xi_0, m), \quad \forall x, y \in B_{\rho_0}(x_0). \quad (4.10)$$

Let $\rho = \frac{\rho_0}{4}$. Now we choose a cutoff function $\eta(r)$ with $0 \leq \eta \leq 1$ such that $\eta(r) = 1$ when $r \leq \rho$ and $\eta(r) = 0$ when $r \geq 2\rho$. Define $\chi(x, y) = \eta(d_g(x, y))$ and $F(x, y, t) = \chi(x, y)G(x, y, t)$. If $d_g(x, y) \leq \rho$ and $t \leq 1$, the identity (4.9) gives that

$$\left| \left( \Delta_y - \frac{\partial}{\partial t} \right) F \right| = \left| \left( \Delta_y - \frac{\partial}{\partial t} \right) G \right| \leq C(\rho_0, \Xi_0) t^{k-\frac{m}{2}} e^{-\frac{d_g^2(x, y)}{4t}}$$

and

$$\left| \left( \Delta_y - \frac{\partial}{\partial t} \right) \nabla_x F \right| = \left| \left( \Delta_y - \frac{\partial}{\partial t} \right) \nabla_x G \right|$$

$$= \left| \left( \Delta_y - \frac{\partial}{\partial t} \right) \nabla_x G \right| = \left| \left( 4\pi t \right)^{-\frac{m}{2}} t^{k-\frac{m}{2}} e^{-\frac{d_g^2(x, y)}{4t}} \left( -\frac{1}{2t} d\nabla_x d\Delta_y u_k + \nabla_x \Delta_y u_k \right) \right|$$

$$\leq C(\rho_0, \Xi_0) t^{k-1-\frac{m}{2}} e^{-\frac{d_g^2(x, y)}{4t}},$$

where we used (4.10) in the last inequality. Similarly, for $\rho \leq d_g(x, y) \leq 2\rho$ we can also check that

$$\left| \left( \Delta_y - \frac{\partial}{\partial t} \right) F \right| \leq C(\rho_0, \Xi_0) t^{-\frac{m}{2}-1} e^{-\frac{d_g^2(x, y)}{4t}},$$

$$\left| \left( \Delta_y - \frac{\partial}{\partial t} \right) \nabla_x F \right| \leq C(\rho_0, \Xi_0) t^{-2-\frac{m}{2}} e^{-\frac{d_g^2(x, y)}{4t}}.$$

Combining the above estimates, we have

$$\left| F(x, y, t) - p(x, y, t) \right|$$

$$= \left| \int_0^t ds \int_M p(z, y, t-s) \left( \Delta_z - \frac{\partial}{\partial s} \right) F(x, z, s) dz \right|$$

$$\leq C(\rho_0, \Xi_0) \int_0^t s^{k-\frac{m}{2}} ds \int_{B_{\rho}(x)} p(z, y, s) dz$$

$$+ C(\rho_0, \Xi_0) \int_0^t s^{-\frac{m}{2}-1} e^{-\frac{\tau^2}{4t}} ds \int_{B_{2\rho}(x) \setminus B_{\rho}(x)} p(z, y, s) dz$$

$$\leq C(\rho_0, \Xi_0, m) t^{k+1-\frac{m}{2}} \quad (4.11),$$

where we used the fact that $\int_M p(x, y, t) d\nu_y \leq 1$. Thus, (4.11) gives (4.7). Similarly, we can show that

$$\left| \nabla_x F(x, y, t) - \nabla_x p(x, y, t) \right|$$

$$= \left| \int_0^t ds \int_M p(z, y, t-s) \left( \Delta_z - \frac{\partial}{\partial s} \right) \nabla_x F(x, z, s) dz \right|$$

$$\leq C(\rho_0, \Xi_0) \int_0^t s^{k-\frac{m}{2}} ds \int_{B_{\rho}(x)} p(z, y, s) dz$$

$$+ C(\rho_0, \Xi_0) \int_0^t s^{-\frac{m}{2}-1} e^{-\frac{\tau^2}{4t}} ds \int_{B_{2\rho}(x) \setminus B_{\rho}(x)} p(z, y, s) dz$$

$$\leq C(\rho_0, \Xi_0, m) t^{k-\frac{m}{2}}. \quad (4.12)$$
Thus, (4.12) implies (4.8). The theorem is proved. □

As a corollary, we have the following result in dimension two.

**Corollary 4.4.** Fix \( \rho_0, \Xi_0 > 0 \) and an integer \( k_0 = k_0(0) \) chosen as in Theorem 4.3 for \( k = 0 \). Let \((\Sigma^2, g)\) be a complete surface and \( x_0 \in \Sigma \) with \( B_1(x_0) \in \mathcal{M}_{k_0,2}(\rho_0, \Xi_0) \), there exists a constant \( C(\rho_0, \Xi_0) > 0 \) such that for any \( x, y \in B_{\rho_0}(x_0) \) and \( t \in (0, 1] \) we have

\[
p(x, y, t) \leq \left( 1 + C(\rho_0, \Xi_0) d_g(x, y) \right) p_0(x, y, t) + C(\rho_0, \Xi_0),
\]

(4.13)

\[
p(x, y, t) \geq \left( 1 - C(\rho_0, \Xi_0) d_g(x, y) \right) p_0(x, y, t) - C(\rho_0, \Xi_0),
\]

(4.14)

\[
|\nabla_x p(x, y, t)| \leq \left( 1 + C(\rho_0, \Xi_0) d_g(x, y) \right) |\nabla_x p_0(x, y, t)| + \frac{C(\rho_0, \Xi_0)}{t},
\]

(4.15)

\[
|\nabla_x p(x, y, t)| \geq \left( 1 - C(\rho_0, \Xi_0) d_g(x, y) \right) |\nabla_x p_0(x, y, t)| - \frac{C(\rho_0, \Xi_0)}{t},
\]

(4.16)

where \( p_0(x, y, t) = \frac{1}{4\pi t} e^{-\frac{d_g(x,y)^2}{4t}} \).

**Proof.** By (4.10), for any \( x, y \in B_{\rho_0 T}(x_0) \) we have

\[
|u_0(x, y)| - 1 | \leq \sup_{B_{\rho_0 T}(x_0)} |\nabla_x u_0| \cdot d_g(x, y) \leq C(\rho_0, \Xi_0) d_g(x, y).
\]

(4.17)

Applying Theorem 4.3 for \( k = 0 \) and using (4.17), we have (4.13)-(4.16). The corollary is proved. □

### 4.2 Properties of a solution with time-dependent singularities

In this subsection, we follow the arguments in Section 3 of [7] to discuss a solution of the linear equation on \((\Sigma, g)\)

\[
\frac{\partial U}{\partial t} = \Delta U + \delta_{\xi(t)} \otimes \nu,
\]

(4.18)

where \( \Sigma \) is a complete two-dimensional surface. Here we assume that \( \xi : (T, \bar{T}) \to \Sigma \) is a \( \sigma \)-Lipschitz curve with \(-\infty < T < \bar{T} < +\infty \) and \( \nu \in (C_0(\{T, \bar{T}\}))' \). For \( 0 < r < +\infty \) and \( \bar{T} \leq t < \bar{t} \leq \bar{T} \), we set

\[
\Gamma_{\bar{T} \bar{t}} = \{(\xi(t), t) \in \Sigma \times \mathbb{R}, t \in (\bar{T}, \bar{t})\},
\]

(4.19)

\[
Q_{r \bar{T} \bar{t}} = \{(x, t) \in \Sigma \times \mathbb{R}, x \in B_r(\xi(t)) \subset \Sigma, t \in (\bar{T}, \bar{t})\}.
\]

(4.20)

We define the function \( U(x, t) \) by

\[
U(x, t) = \int_T^t p(x, \xi(s), t-s) \, dv(s),
\]

(4.21)

where \( p(x, y, t) \) is the heat kernel of \((\Sigma, g)\). Then \( U(x, t) \) satisfies (4.18) for any \((x, t) \in Q_{1, \bar{T} \bar{t}}\). Moreover, we define

\[
\Phi(x, y) = \frac{1}{2\pi} \log \frac{1}{d_g(x, y)}, \quad r(x, t) = d_g(x, \xi(t)).
\]

(4.22)

Following the argument in [7] and using Theorem 4.3 we have
Lemma 4.5. (c.f. [7]) Let $\xi : (\bar{T}, \bar{T}) \to \Sigma$ be a $\sigma$-Lipschitz curve and $\bar{T} < t < \bar{t} < \bar{T}$. Assume that for any $t \in (\bar{T}, \bar{t})$ the ball $B_1(\xi(t))$ is in $M_{k_0,2}(\rho_0, \Xi_0)$ as in Corollary 4.4. Then we have

(1). Assume that $v$ is the Lebesgue measure. For any $\varepsilon > 0$, there exists $r_0 = r_0(\varepsilon, \sigma, \Xi_0, \rho_0, \bar{T}, \bar{t})$ such that if $r(x, t) \leq r_0$ and $t \in (\bar{t}, \bar{t})$, then we have

\[
(1 - \varepsilon)\Phi(x, \xi(t)) \leq U(x, t) \leq (1 + \varepsilon)\Phi(x, \xi(t)),
\]
\[
(1 - \varepsilon)|\nabla \Phi(x, \xi(t))| \leq |\nabla U(x, t)| \leq (1 + \varepsilon)|\nabla \Phi(x, \xi(t))|.
\] (4.23) (4.24)

(2). For any $\gamma \in (\frac{1}{5}, 1)$, there exist constants $r_0 = r_0(\rho_0, \Xi_0, \sigma, \bar{T}, \bar{t}, \gamma) \in (0, 1)$ and a function $v \in C^\infty(Q_{r_0, \bar{t}} \setminus \Gamma_{\bar{t}})$ satisfying

\[
\frac{\partial v}{\partial t} + \Delta v = 0, \quad \text{in} \quad Q_{r_0, \bar{t}} \setminus \Gamma_{\bar{t}}
\]

such that for all $(x, t) \in Q_{r_0, \bar{t}} \setminus \Gamma_{\bar{t}}$ the following inequalities hold:

\[
\gamma \log \frac{1}{r(x, t)} \leq v(x, t) \leq \log \frac{1}{r(x, t)},
\]
\[
\gamma \frac{r(x, t)}{r(x, t) - 1} \leq |\nabla v(x, t)| \leq \frac{r(x, t)}{r(x, t) - 1}.
\] (4.26) (4.27)

Proof. The proof is almost the same as that of Proposition 3.1, Proposition 3.3 and Lemma 4.1 in [7], and we sketch some details here. For $r > 0$, $\beta > 0$ and $\delta > 0$, we define

\[
S_{\beta}(r) = \pi^{-1} \int_0^\delta (4s)^{-\beta} e^{-\frac{s^2}{4}} ds.
\]

Since $\xi$ is $\sigma$-Lipschitz continuous, we have

\[
|r(x, t) - r(x, s)| \leq \sigma |t - s|.
\] (4.28)

Thus, for any $c > 0$ we have

\[
|r(x, s)|^2 \leq (1 + c)|r(x, t)|^2 + \left(1 + \frac{1}{c}\right)\sigma^2 |t - s|^2.
\] (4.29)

This implies that

\[
|r(x, s)|^2 \geq \frac{1}{1 + c}|r(x, t)|^2 - \frac{1}{c}\sigma^2 |t - s|^2.
\] (4.30)

Combining this with Corollary 4.4, we have

\[
\int_{t-\delta}^t p(x, \xi(s), t - s) ds \leq \left(1 + C(\rho_0, \Xi_0)(r(x, t) + \sigma\delta)\right) e^{-\frac{k_0^2}{4} \int_{t-\delta}^t \frac{1}{4\pi(t - s)} ds} e^{-\frac{\sigma^2 t^2}{4\pi(t - s)}} + C(\rho_0, \Xi_0, \delta)
\]
\[
= \left(1 + C(\rho_0, \Xi_0)(r(x, t) + \sigma\delta)\right) e^{-\frac{k_0^2}{4} S_2 \left(\frac{r(x, t)}{\sqrt{1 + c}}\right)} + C(\rho_0, \Xi_0, \delta).
\]

Choosing the constant $c = \sqrt{\delta}$, we have

$$U(x, t) = \left( \int_{t-\delta}^{t} + \int_{t}^{t+\delta} \right) p(x, \xi(s), t-s) ds$$

$$\leq \left( 1 + C(\rho_0, \Xi_0)(r(x, t) + \sigma \delta) \right) \frac{a^2}{4} S_2 \left( \frac{r(x, t)}{1 + \sqrt{\delta}} \right) + C(\rho_0, \Xi_0, \delta, \tilde{T}, \tilde{T}).$$

Note that $\lim_{r \to 0} (\log \frac{1}{r})^{-1} S_2(r) = \frac{1}{2\pi}$. Therefore, for any $\epsilon > 0$ there exists $r_0 = r_0(\epsilon, \sigma, \Xi_0, \rho_0, \tilde{T})$ such that for any $x$ with $r(x, t) \leq r_0$ we have

$$\frac{U(x, t)}{\Phi(x, \xi(t))} \leq 1 + \epsilon.$$

Similarly, we can show that $\frac{U(x, t)}{\Phi(x, \xi(t))} \geq 1 - \epsilon$ when $r(x, t)$ is small. Thus, (4.23) is proved. Similarly, we can use (4.15) and (4.16) of Corollary 4.4 to estimate $|\nabla U|$.

To prove part (2), we denote by $U(x, t; \xi, \nu)$ the function (4.21) constructed by $\xi(t)$ and the measure $\nu$. We define $\xi(t) = \xi(\tilde{T} + t - \tilde{T})$ and let $\nu$ be the Lebesgue measure. Then the function $v(x, t) = kU(x, T + \tilde{T} - t; \xi, \nu)$ satisfies the properties in part (2) by choosing some $k > 0$. See Lemma 4.1 of [7] for details.

Using Corollary 4.4 we have the following result.

**Lemma 4.6.** The same assumption as in Lemma 4.5. Let $\nu \in (C_0((\tilde{T}, \tilde{T}))')$ and $U(x, t)$ be the function defined by (4.27). Then for $\tilde{T} < t_1 < t_2 < \tilde{T}$ we have

$$\lim_{R \to 0} \lim_{\delta \to 0} \int_{t_1}^{t_2} dt \int_{A_1(\delta, R)} \frac{U(x, t)}{r(x, t) \log r(x, t)} = 0,$$  

(4.31)

$$\lim_{\delta \to 0} \lim_{R \to 0} \frac{1}{\delta} \int_{t_1}^{t_2} dt \int_{A_1(\frac{4}{\delta}, R)} U(x, t) = 0.$$  

(4.32)

where $A_1(\delta, R) = \{ x \in \Sigma \mid \delta < r(x, t) < R \}$.

**Proof.** We follow the arguments in the proof of Proposition 3.3 in [7]. Without loss of generality, we can assume that the curve $\xi(s) (s \in (\tilde{T}, \tilde{T}))$ is contained in $B_{1/\tilde{p}_0}(x_0)$ for some $x_0 \in \Sigma$ and $B_{\tilde{p}_0}(x_0) \in \mathcal{M}_{k_0, 2}(\rho_0, \Xi_0)$. Corollary 4.4 gives that for any $x \in B_{1/\tilde{p}_0}(x_0)$ and $t \in (\tilde{T}, \tilde{T})$,

$$U(x, t) \leq \int_{\tilde{T}}^{t} \left( C(\rho_0, \Xi_0)p_0(x, \xi(s), t-s) + C(\rho_0, \Xi_0) \right) ds$$

$$= C(\rho_0, \Xi_0)U_0(x, t) + C(\rho_0, \Xi_0)(\tilde{T} - \tilde{T}),$$

where $U_0$ is defined by

$$U_0(x, t) = \int_{\tilde{T}}^{t} p_0(x, \xi(s), t-s) ds.$$

Thus, it suffices to show (4.31) - (4.32) for $U_0(x, t)$.

For $t \in (\tilde{T}, \tilde{T})$ with $|D\nu| < +\infty$ we write

$$\nu((s, \tilde{T}) = D\nu(t)(t - s) - G(s), \quad T_1 < s < t,$$
where \( G(s) \) satisfies \( \lim_{s \to t} \frac{G(s)}{t-s} = 0 \) for a.e. \( t \in (T, \bar{T}) \). Let \( \lambda \in (0, t-\delta) \). Note that \( U_0 \) can be written as

\[
U_0(x, t) = \int_{T}^{t-\lambda} p_0(x, \xi(s), t-s) \, d\nu(s) + D\nu(t) \int_{t-\lambda}^{t} p_0(x, \xi(s), t-s) \, ds
\]

\[
+ \int_{t-\lambda}^{t} p_0(x, \xi(s), t-s) \, dG(s)
\]

\[= I_1 + I_2 + I_3.\]

By Theorem 4.3, \( I_1 \) satisfies

\[
I_1 \leq \frac{1}{4\pi \lambda} \nu((T, \bar{T} - \lambda)) < +\infty. \]

Thus, we have

\[
\int_{t_1}^{t_2} dt \int_{A_\lambda(\delta, R)} \frac{I_1(x, t)}{r(x,t)| \log r(x,t)|} \, d\text{vol} \\
\leq \frac{1}{4\pi \lambda} \nu((T, \bar{T})) \int_{t_1}^{t_2} dt \int_{A_\lambda(\delta, R)} \frac{1}{r(x,t)| \log r(x,t)|} \, d\text{vol} \\
= \frac{1}{2\lambda} (t_2 - t_1) \nu((T, \bar{T})) \int_{\delta}^{R} \frac{1}{| \log r |} \, dr \\
\leq \frac{1}{2\lambda} (t_2 - t_1) \nu((T, \bar{T}))(R - \delta),
\]

(4.33)

where we assumed that \( R \) is small such that \( | \log r | \geq 1 \) for any \( r \in (0, R) \). Moreover, we have

\[
\frac{1}{\delta} \int_{t_1}^{t_2} dt \int_{A_{\frac{\delta}{2}}(\delta, \delta)} I_1(x, t) \, d\text{vol} \leq \frac{C_1}{\lambda} (t_2 - t_1) \nu((T, \bar{T})) \delta,
\]

(4.34)

where \( C_1 \) is a universal constant. Next, we estimate \( I_2 \). Using Corollary 4.3 and integration by parts we have

\[
\int_{t-\lambda}^{t} p_0(x, \xi(s), t-s) \, ds \\
\leq \int_{0}^{\lambda} \frac{1}{4\pi \tau} e^{\frac{r(x, \xi)^2}{4\tau}} \, d\tau \leq e^{\frac{c^2}{4\lambda}} \int_{0}^{\lambda} \frac{1}{4\pi \tau} e^{\frac{r(x, z)^2}{4(1+c)\tau}} \, d\tau \\
\leq \frac{1}{4\pi} e^{\frac{c^2}{4\lambda}} \left( \log \frac{4(1+c)\lambda}{r(x, t)^2} e^{-\frac{r^2}{4(1+c)\lambda} + \int_{r(x, t)^2}^{\infty} e^{-z \log z} \, dz} \right) \\
\leq C_2 | \log r(x, t) | + C_2,
\]

(4.35)

where we can choose \( c = 1 \) and \( C_2 \) is a constant depending on \( \sigma \) and \( \lambda \). Therefore, we have

\[
\int_{t_1}^{t_2} dt \int_{A_\lambda(\delta, R)} \frac{I_2(x, t)}{r(x,t)| \log r(x,t)|} \, d\text{vol} \\
\leq C_2 \int_{t_1}^{t_2} d\nu(t) \int_{\delta}^{R} \frac{1}{r| \log r |} \left( | \log r | + 1 \right) r \, dr \\
\leq 2C_2 \cdot (R - \delta) \nu((t_1, t_2))
\]

(4.36)

and

\[
\frac{1}{\delta} \int_{t_1}^{t_2} dt \int_{\frac{\delta}{2} < r(x, t) < \delta} I_2(x, t) \, d\text{vol} \leq C_2 \cdot (\delta + \delta | \log \delta |) \nu((t_1, t_2)).
\]

(4.37)
Finally, we estimate $I_3$. Using the inequality (4.31) for $c = 1$ and integration by parts, we have

$$\int_{t-\lambda}^{t} p_0(x, \xi(t), t-s) d|G(s)|$$

$$\leq \frac{1}{4\pi} e^{\frac{x^2}{2}} \int_{t-\lambda}^{t} \frac{1}{t-s} e^{\frac{r(x,t)^2}{8(t-s)^2}}$$

$$\leq \frac{1}{4\pi} e^{\frac{x^2}{2}} \left( - \frac{1}{\lambda} e^{\frac{r(x,t)^2}{8\lambda}} + \int_{t-\lambda}^{t} \frac{G(s)}{t-s} \left( \frac{1}{(t-s)^2} + \frac{r(x,t)^2}{8(t-s)^2} \right) e^{-\frac{r(x,t)^2}{8(t-s)^2}} ds \right)$$

$$\leq \frac{1}{4\pi} e^{\frac{x^2}{2}} \sup_{(t-\lambda, t)} \frac{|G(s)|}{t-s} \int_{t-\lambda}^{t} \left( \frac{1}{(t-s)^2} + \frac{r(x,t)^2}{8(t-s)^2} \right) e^{-\frac{r(x,t)^2}{8(t-s)^2}} ds$$

$$\leq C_3 \sup_{(t-\lambda, t)} \frac{|G(s)|}{t-s} |\log r(x,t)|,$$

where $C_3$ depends on $\sigma$ and $\lambda$. Thus, we have

$$\int_{t_1}^{t_2} dt \int_{A_t(\delta, R)} \frac{|I_3(x, t)|}{r(x, t) |\log r(x, t)|} d\text{vol}$$

$$\leq C(\sigma, \lambda) \sup_{(t-\lambda, t)} \frac{|G(s)|}{t-s} \int_{t_1}^{t_2} dt \int_{A_t(\delta, R)} \frac{1}{r(x, t)} d\text{vol}$$

$$\leq C(\sigma, \lambda) \sup_{(t-\lambda, t)} \frac{|G(s)|}{t-s} (R - \delta)(t_2 - t_1)$$

(4.38)

and

$$\frac{1}{\delta} \int_{t_1}^{t_2} dt \int_{\delta < r(x,t) < \delta} |I_3(x, t)| d\text{vol} \leq C(\sigma, \lambda) \sup_{(t-\lambda, t)} \frac{|G(s)|}{t-s} (t_2 - t_1) |\log \delta| \delta.$$ (4.39)

Combining (4.33)-(4.39), we have (4.31)-(4.32).

\[\square\]

### 4.3 Estimates of the solution with disjoint singularities

In this subsection, we follow Section 4.1 of [7] to construct some cutoff functions and show the integrability of the solution across the singular set when the singular curves are disjoint. First, we construct some cutoff functions.

**Definition 4.7.** (c.f. Section 4.1 of [7])

(1) Let $t_3 < t_1 < t_2 < t_4$ and $0 < \delta < r_1$. Define $\zeta = \zeta(t; t_1, t_2, t_3, \delta, r_1) \in C^\infty(\mathbb{R})$ such that

$$\zeta(t) = \delta, \ (t \in [t_1, t_2]), \quad \zeta(t) = r_1 \ (t \in (-\infty, t_3] \cup [t_4, \infty) ),$$

$$0 \leq \left| \frac{\partial \zeta}{\partial t} \right| \leq 2r_1 \left( \frac{1}{t_1 - t_3} + \frac{1}{t_4 - t_2} \right).$$ (4.40)

(2) Let $\eta$ be a smooth function on $\mathbb{R}$ satisfying

$$\eta(z) = \begin{cases} 0, & (z \leq 0), \\ 1, & (z \geq 1), \quad 0 < \eta'(z) \leq 2 \ (0 < z < 1), \end{cases}$$ (4.41)
and define \( H(z) = \int_0^z \eta(t) \, dt \). Then \( H(z) \) satisfies the inequality
\[
0 \leq zH'(z) - H(z) \leq H'(z).
\] (4.42)

We keep the same notation \( H(z) \) as in [7]. Throughout this section, \( H \) always denotes the function as above and it should not be confused with the mean curvature.

(3). Let \( 0 < \bar{r} < \tilde{r} < 1, T_1 < \bar{T} < \tilde{T} < T_2 \) and \( \xi : [T_1, T_2] \rightarrow \Sigma \) be a continuous curve. We define \( \phi_\xi = \phi_\xi(x, t; \bar{r}, \tilde{r}, \bar{T}, \tilde{T}, T_1, T_2) \in C^\infty(Q_1(T_1, T_2)) \) satisfying
\[
0 \leq \phi_\xi \leq 1, \quad \phi_\xi = \begin{cases} 1, & \text{on } Q_{\bar{r}, \bar{T}, T_1, T_2} \setminus Q_{T_1, T_2}, \\ 0, & \text{on } Q_{T_1, T_2} \setminus Q_{\bar{r}, \bar{T}, T_1, T_2}, \end{cases} \quad \nabla_x \phi_\xi = 0 \text{ in } Q_{\bar{r}, T_1, T_2}. \tag{4.43}
\]

A direct corollary of Lemma 4.5 is the following result.

**Lemma 4.8.** Under the assumption of Lemma 4.5, we define \( V(x, t) = e^{-2v(x, t)} \in C^\infty(Q_{\bar{r}, \bar{T}, T_1, T_2}) \). Then \( V(x, t) \) satisfies
\[
\frac{\partial V}{\partial t} + \Delta V = 4e^{-2v}|\nabla v|^2, \quad \text{in } Q_{\bar{r}, \bar{T}, T_1, T_2} \setminus \Gamma_{\bar{T}, \bar{T}} \tag{4.44}
\]
By using the inequalities (4.25)-(4.27), for all \( (x, t) \in Q_{r_0, \bar{T}, T_1, T_2} \setminus \Gamma_{\bar{T}, \bar{T}} \) the following inequalities hold:
\[
\begin{align*}
\rho(x, t)^2 & \leq V(x, t) \leq \rho(x, t)^{2\gamma}, \\
1 & \leq V(x, t)^{-1}|\nabla V(x, t)|^2 \leq 4\rho(x, t)^{2\gamma - 2}, \\
1 & \leq \frac{\partial V}{\partial t} + \Delta V \leq 4\rho(x, t)^{2\gamma - 2},
\end{align*}
\] (4.45)-(4.47)

where \( \gamma \in (\frac{1}{2}, 1) \).

Consider the case that there is only one singular curve. We show that the solution of (4.2) is in \( L^1 \) across the singular set. The argument is the same as that of [7] and we give all the details for the readers’ convenience.

**Lemma 4.9.** (c.f. Lemma 4.2 of [7]) Fix \( \gamma \in (\frac{1}{2}, 1) \). Under the same assumption as in Theorem 4.2 if there is only one singular curve \( \xi : [T_1, T_2] \rightarrow \Sigma \), then for any \( (t_1, t_2) \subset (T_1, T_2) \) there exists \( r_1 = r_1(\rho_0, \Xi_0, \sigma, t_1, t_2, T_1, T_2, \gamma) > 0 \) such that
\[
\|u\|_{L^1(Q_{r_1, t_1, t_2})} \leq C\|u\|_{L^1(K)}, \tag{4.48}
\]
where \( C \) is a constant depending on \( \|c\|_{L^\infty(Q_{T_1, T_2})}, \gamma, \rho_0, \Xi_0, \sigma, t_1, t_2, T_1, T_2 \) and \( K \) is defined by \( K = Q_{2r_1, T_1, T_2} \setminus Q_{r_1, T_1, T_2} \).

**Proof.** Let \( T_1 < t_5 < t_3 < t_1 < t_2 < t_4 < t_6 < T_2 \), \( \gamma \in (\frac{1}{2}, 1) \) and \( r_0 = r_0(\rho_0, \Xi_0, \sigma, t_5, t_6, \gamma) > 0 \) as in Lemma 4.5. Let \( 0 < \delta < r_1 < \frac{r_0}{2} \). We construct the function
\[
\phi(x, t) = \phi(x, t; r_1, 2r_1, t_3, t_4, t_5, t_6, T_1, T_2)
\]
satisfying (4.43), and the function \( v \in C^\infty(Q_{\rho_0, t_5, t_6} \setminus \Gamma_{t_5, t_6}) \) satisfying (4.25) with the properties (4.26)-(4.27). Moreover, we define
\[
V(x, t) = e^{-2v(x, t)}, \quad w(x, t) = \zeta(t)^{-1}V(x, t) - 1 \tag{4.49}
\]
and

\[ \varphi(x, t) = \phi(x, t) \zeta(t)(H \circ w)(x, t), \tag{4.50} \]

where \( \zeta = \zeta(t; t_1, t_2, t_3, t_4, \delta, r_1) \) and \( H \) are given in Definition 4.7. Note that \( H \circ w = 0 \) near \( \Gamma \) in \( Q_{r_0, t_5, t_6} \). This implies that \( \varphi \in C_0^\infty(Q_{r_0, t_5, t_6} \setminus \Gamma_{t_5, t_6}) \). By (4.2) we have

\[ - \int_{Q_{r_0, t_5, t_6}} w \left( \frac{\partial \varphi}{\partial t} + \Delta \varphi \right) = \int_{Q_{r_0, t_5, t_6}} cu \varphi. \tag{4.51} \]

Note that (4.42) and (4.49) imply that

\[ \zeta H \circ w \leq \zeta w H' \circ w \leq VH' \circ w, \tag{4.52} \]

we have \( \varphi \leq \phi VH' \circ w \). Thus, the right hand side of (A.3) can be estimated by

\[ \int_{Q_{r_0, t_5, t_6}} cu \varphi \geq - \|cV\|_{L^\infty(Q_{r_0, t_5, t_6})} \int_{Q_{r_0, t_5, t_6}} w \phi H' \circ w. \tag{4.53} \]

On the other hand, direct calculation shows that

\[ \frac{\partial \varphi}{\partial t} + \Delta \varphi = \phi A + B \]

where

\[
A = (\partial_t V + \Delta V) H' \circ w - \partial_t \zeta \left( (w + 1)H' \circ w - H \circ w \right) + \zeta^{-1}|\nabla V|^2 H'' \circ w, \\
B = (\partial_t \phi + \Delta \phi) H \circ w + 2(\nabla \phi, \nabla V) H' \circ w.
\]

By (4.42) and (4.47) we have

\[ A \geq (\partial_t V + \Delta V) H' \circ w - 2|\partial_t \zeta| H' \circ w \geq (1 - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}) H' \circ w. \tag{4.54} \]

Note that

\[
\text{Supp}(B) \subset \text{Supp}(|\nabla \phi| + |\partial_t \phi|) \cap \{(x, t) \in Q_{2r_1, t_5, t_6} \mid w \geq 0\} \\
\subset \{(x, t) \in Q_{2r_1, t_5, t_6} \setminus Q_{r_1, t_3, t_4} \mid w \geq 0\} \\
\subset \{(x, t) \in Q_{2r_1, t_5, t_6} \setminus Q_{r_1, t_3, t_4} \mid r(x, t) \geq \zeta(t)^{\frac{1}{2}}\} \\
\subset \{(x, t) \in Q_{2r_1, t_5, t_6} \mid r(x, t) \geq r_1\} =: K,
\]

where we used the construction of \( \zeta(t) \) in Definition 4.7. Thus, we have

\[ |B| \leq \left( \|\partial_t \phi + \Delta \phi\|_{L^\infty(K)} \|V\|_{L^\infty(K)} + 2\|\nabla \phi\|_{L^\infty(K)} \|\nabla V\|_{L^\infty(K)} \right) \chi K \leq C(c_{\phi, K}, \gamma, r_1) \chi K,
\]

where \( c_{\phi, K} = \sup_K (|\partial_t \phi| + |\Delta \phi| + |\nabla \phi|) \). Combining the above estimates, we have

\[ \frac{\partial \varphi}{\partial t} + \Delta \varphi \geq (1 - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}) \phi H' \circ w - C(c_{\phi, K}, \gamma, r_1) \chi K. \tag{4.54} \]
Combining (4.53) (4.54) with (4.51), we have
\[
(1 - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})} - \|cV\|_{L^\infty(Q_{r_0, t_5, t_6})}) \int_{Q_{r_0, t_5, t_6}} u \phi H' \circ w \leq C(c_{\phi,K}, \gamma, r_1) \int_K u.
\]
Taking \( r_0 \) sufficiently small and using the assumption that \( c(x, t) \) is locally bounded, we have
\[
1 - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})} - \|cV\|_{L^\infty(Q_{r_0, t_5, t_6})} \geq \frac{1}{2}.
\]
Therefore, by the definition of \( \phi \) we have
\[
\int_{Q_{r_1, t_1, t_2}} u H' \circ w \leq \int_{Q_{r_0, t_5, t_6}} u \phi H' \circ w \leq C(c_{\phi,K}, \gamma, r_1) \int_K u. \tag{4.55}
\]
Note that the function \( H' \circ w \) converges to 1 on \( Q_{r_1, t_1, t_2} \setminus \Gamma_{t_1, t_2} \) as \( \delta \to 0 \). Thus, taking \( \delta \to 0 \) in (4.55), we have that \( u \) is integrable on \( Q_{r_1, t_1, t_2} \). The lemma is proved.

\[
\square
\]

4.4 Estimates of the solution with multiple singularities

In this subsection, we consider the case that the solution of (4.2) has multiple singular curves. When any two singular curves don’t coincide at any time, we can use Lemma 4.9 for each singular curve to show that the solution is \( L^1 \) across the singularities. Otherwise, we need to develop more techniques to estimate the solution.

First, we introduce the following definition.

**Definition 4.10.** Let \( \{\xi_1, \xi_2, \cdots, \xi_l\}(t \in [T_1, T_2]) \) be continuous curves in \( \Sigma \), and \( I \subset [T_1, T_2] \). We say that \( \{\xi_1(t), \cdots, \xi_i(t)\} \) are disjoint on \( I \), if for any time \( t_0 \in I \), we have
\[
\xi_i(t_0) \neq \xi_j(t_0), \quad \forall i \neq j.
\]
Let \((x_0, t_0)\) be a point in the singular set. By Lemma 2.11, there exists finitely many singular curves passing through \((x_0, t_0)\). There are two cases for the singular curves:

(A) There exists \((t_1, t_2)\) with \( t_0 \in (t_1, t_2) \) and singular curves \( \{\xi_1(t), \cdots, \xi_i(t)\}(t \in (t_1, t_2)) \) such that \( \{\xi_1(t), \cdots, \xi_i(t)\} \) are disjoint on \((t_1, t_2) \setminus \{t_0\}) and
\[
\xi_1(t_0) = \xi_2(t_0) = \cdots = \xi_i(t_0). \tag{4.56}
\]

(B) There exists \((t_1, t_2)\) with \( t_0 \in (t_1, t_2) \), singular curves \( \{c_1(t), \cdots, c_k(t)\}(t \in (t_1, t_0]) \) and \( \{\tilde{c}_1(t), \cdots, \tilde{c}_l(t)\}(t \in [t_0, t_2)) \) such that
\[ (a) \ \{c_1(t), \cdots, c_k(t)\} \) are disjoint on \((t_1, t_0)\);
\[ (b) \ \{\tilde{c}_1(t), \cdots, \tilde{c}_l(t)\} \) are disjoint on \((t_0, t_2)\);
\[ (c) \ The \ singular \ curves \ coincide \ at \ t_0:\n\]
\[
c_1(t_0) = \cdots = c_k(t_0) = \tilde{c}_1(t_0) = \cdots = \tilde{c}_l(t_0) = x_0. \tag{4.57}
\]
If $k = l$, then this is just the case (A). Note that the union of two Lipschitz curves is still a Lipschitz curve. Thus, for $k < l$ we can construct the curves

$$
\xi_i(t) = \begin{cases} 
q_i(t), & \forall \ t \in (t_1, t_0], \\
\eta_i(t), & \forall \ t \in (t_0, t_2), 
\end{cases} \quad \text{for} \quad 1 \leq i \leq k, \quad (4.58)
$$

and

$$
\xi_i(t) = \begin{cases} 
q_i(t), & \forall \ t \in (t_1, t_0], \\
\eta_i(t), & \forall \ t \in (t_0, t_2), 
\end{cases} \quad \text{for} \quad k < i \leq l. \quad (4.59)
$$

Then $\{\xi_1(t), \cdots, \xi_l(t)\}(t \in (t_1, t_2))$ are Lipschitz curves. For $k > l$ we can also construct similar curves $\{\xi_1(t), \cdots, \xi_k(t)\}(t \in (t_1, t_2))$.

Summarizing the above discussion, we define

**Definition 4.11.** Let $I = [t_1, t_2]$ or $(t_1, t_2)$ where $t_1 < t_0 < t_2$. We call that the singular curves $\{\xi_1, \xi_2, \cdots, \xi_l\}(t \in I)$ are around $(x_0, t_0)$ on $I$, if the curves satisfy the conditions in Case (A) or are constructed as in Case (B) on $I$.

We construct some cutoff functions when the singular curves are not disjoint.

**Definition 4.12.** Let $0 < \underline{r} < \bar{r}_i < 1, T_1 < \underline{r} < \bar{r} < \bar{T} < T_2$ and $\{\xi_1, \xi_2, \cdots, \xi_l\}(t \in [T_1, T_2])$ be $\sigma$-Lipschitz curves. We assume that $\{\xi_1(t), \cdots, \xi_l(t)\}$ are around $(x_0, t_0)$ on $(T_1, T_2)$ for some $t_0 \in (t, \bar{t})$.

1. For each $\xi_k$ and $(t_1, t_2) \subset [T_1, T_2]$, we define the notations $Q_{r, t_1, t_2}^{(k)}$ and $\Gamma_{t_1, t_2}^{(k)}$ as in (4.19)-(4.20), and we define

$$
Q_{r, t_1, t_2} = \bigcup_{k=1}^{l} Q_{r, t_1, t_2}^{(k)}, \quad \Gamma_{t_1, t_2} = \bigcup_{k=1}^{l} \Gamma_{t_1, t_2}^{(k)}, \quad \hat{Q}_{r, t_1, t_2} = \bigcap_{k=1}^{l} Q_{r, t_1, t_2}^{(k)}. \quad (4.60)
$$

2. For each $\xi_k$ we define the function $\phi_{\xi_k}(x, t; r, \bar{r}, \bar{t}, \bar{T}, T_1, T_2) \in C^{\infty}(Q_{1, T_1, T_2})$ as in (4.43). Then the function

$$
\phi(x, t; r, \bar{r}, \bar{t}, \bar{T}) = 1 - (1 - \phi_{\xi_1})(1 - \phi_{\xi_2}) \cdots (1 - \phi_{\xi_l}) \in C^{\infty}(\hat{Q}_{1, T_1, T_2}) \quad (4.61)
$$

satisfies the properties:

$$
0 \leq \phi \leq 1, \quad \phi = \begin{cases} 
1, & \text{on} \ \hat{Q}_{1, T_1, T_2} \cap \overline{Q_{r, t, \bar{T}}}, \\
0, & \text{on} \ \hat{Q}_{1, T_1, T_2} \setminus \overline{Q_{r, t, \bar{T}}}. 
\end{cases} \quad (4.62)
$$

Moreover, $\phi$ satisfies the properties

$$
\text{Supp}(\phi) \cap \hat{Q}_{1, T_1, T_2} \subset \overline{Q_{r, t, \bar{T}}}, \quad \text{Supp}(\nabla \phi + |\partial_t \phi|) \cap \hat{Q}_{1, t, \bar{T}} \subset \overline{Q_{r, t, \bar{T}}} \setminus \overline{Q_{r, t, \bar{T}}}, \quad (4.63)
$$

and

$$
\text{Supp}(\nabla \phi + |\partial_t \phi|) \cap \hat{Q}_{1, t, \bar{T}} \subset \overline{Q_{r, t, \bar{T}}} \setminus \overline{Q_{r, t, \bar{T}}}
\subset \bigcup_{k=1}^{l} \{(x, t) \in \hat{Q}_{1, t, \bar{T}} \mid r_k(x, t) \leq \bar{r}, \ r_i(x, t) \geq \bar{r}, \forall i \neq k\}. \quad (4.64)
$$

Here we assumed that $Q_{r, t, \bar{T}} \subset \hat{Q}_{1, T_1, T_2}$ by shrinking the interval $[T_1, T_2]$ if necessary.
(3) Fix $\gamma \in (\frac{1}{2}, 1)$. For each $\xi_k$, we define $v_k \in C^\infty(\overline{Q}^{(k)}_{1, T} \setminus \Gamma^{(k)}_T)$ as in (2) of Lemma 4.3 and let $v_0^{(k)}$ the constant in (2) of Lemma 4.3 such that the inequalities (4.26)–(4.27) hold for $(x, t) \in Q^{(k)}_{r_0^2, T} \setminus \Gamma^{(k)}_T$. Set
\[ r_0 := \min\{r_0^{(1)}, r_0^{(2)}, \ldots, r_0^{(l)}\}. \] (4.65)
After shrinking the interval $[T_1, T_2]$ if necessary, we can assume that
\[ \Gamma_{T_1, T_2} \subset \hat{Q}_{r_0, T_1, T_2}. \] (4.66)
By (4.63)-(4.66), we know that the inequalities (4.26)–(4.27) hold for all functions $v_k$ and all $(x, t) \in \hat{Q}_{r_0, T} \setminus \Gamma_T$.

(4) For any $\epsilon > 0$ and $(x, t) \in \hat{Q}_{r_0, T} \setminus \Gamma_T$, we define
\[ \tilde{v}(x, t) = \sum_{k=1}^l v_k(x, t), \quad \tilde{w}_k(x, t) = 2 - \tilde{v}(x, t), \quad \tilde{\varphi}_k(x, t) = (H \circ \tilde{w}_k)(x, t), \] (4.67)
where $H$ is defined in (2) of Definition 4.7. Note that $H \circ \tilde{w}_k = 0$ near each $\xi_k$ and this implies that $\tilde{\varphi}_k$ vanishes near $\Gamma_T$. Moreover, for any $(x, t) \in \hat{Q}_{r_0, T} \setminus \Gamma_T$ we have
\[ \lim_{\epsilon \to 0} \tilde{\varphi}_k(x, t) = H(2), \quad \lim_{\epsilon \to 0} |\nabla \tilde{\varphi}_k|(x, t) = 0, \] (4.68)
\[ \frac{\partial \tilde{\varphi}_k}{\partial t} + \Delta \tilde{\varphi}_k = H'' \circ \tilde{w}_k |\nabla \tilde{w}_k|^2. \] (4.69)
Let
\[ \tilde{r}(x, t) = e^{-\tilde{v}(x,t)}. \] (4.70)
Then the inequalities (4.26) imply that for any $(x, t) \in \hat{Q}_{r_0, T} \setminus \Gamma_T$
\[ r_1 r_2 \cdots r_l \leq \tilde{r}(x, t) \leq (r_1 r_2 \cdots r_l)^{\gamma}. \] (4.71)

(5) Under the above assumptions, for $\rho > 0$ and $h \in L^1(Q_{r_0, T})$ we define
\[ I(\rho; \tilde{r}, h, r_0) = \int_{Q_{r_0, T} \cap \{\rho \leq \tilde{r}(x,t) \leq 1\}} \frac{h |\nabla \tilde{v}|^2}{|\log |\rho|^2}, \]
where $\tilde{r}(x,t)$ and $\tilde{v}(x,t)$ are the function defined in (4) above.

(6) Assume that $\{\xi_1(t), \cdots, \xi_l(t)\}$ are disjoint on $[T_1, T_2]$. We choose $\bar{\rho} > 0$ such that $Q^{(i)}_{\bar{\rho}, T_1, T_2} \cap Q^{(j)}_{\bar{\rho}, T_1, T_2} = \emptyset$ for any $1 \leq i \neq j \leq l$. For any $\rho \in (0, \bar{\rho})$, $T_1 \leq t \leq T_2$ and $h \in L^1(Q^{(k)}_{1, T})$, we define
\[ I_{\xi_k}(\rho; \tilde{r}, h, \bar{\rho}) = \frac{1}{|\log |\rho|^2} \int_{Q^{(k)}_{1, T}} \int_{\rho \leq r_k(x,t) \leq \rho} \frac{h}{r_k(x,t)^2}. \] (4.72)
The next result gives the $L^1$ estimate of the solution near the singularities when the singular curves are not disjoint.
Lemma 4.13. Under the same assumption as in Theorem 4.2 for any \((t_1, t_2) \subset (T_1, T_2)\) there exists \(r_1 = r_1(\rho_0, \Xi_0, t, t_1, t_2, T_1, T_2, \gamma) > 0\) such that

\[
\|u\|_{L^1(Q_{r_1, t_1, t_2})} \leq C\|u\|_{L^1(K)},
\]

(4.73)

where \(C\) is a constant depending on \(\|c\|_{L^\infty(Q_{t_1, t_2})}, \rho_0, \Xi_0, \sigma, t_1, t_2, T_1, T_2\) and \(K\) is defined by \(K = \overline{Q_{2r_1, T_1, T_2} \setminus Q_{r_1, T_1, T_2}}\). Moreover, we have

\[
\sup_{\rho \in (0, \frac{1}{2})} I(\rho; t_1, t_2, u, r_0) < +\infty.
\]

(4.74)

Proof. We divide the proof into several steps:

Step 1. Without loss of generality, we can assume that \(c(x, t) \geq 0\) on \(Q_{1, T_1, T_2}\). In fact, let \(u(x, t)\) be a solution of (4.2). Then for any \(k \in \mathbb{R}\) the function \(\tilde{u}(x, t) = u(x, t)e^{kt}\) satisfies the equation

\[
\frac{\partial \tilde{u}}{\partial t} = \Delta \tilde{u} + (c + k)\tilde{u}, \quad \forall (x, t) \in Q_{1, T_1, T_2} \setminus \Gamma_{T_1, T_2}.
\]

Since \(c\) is locally bounded by the assumption, we can choose \(k\) large such that \(c + k \geq 0\) on \(Q_{1, T_1, T_2}\). Thus, it suffices to show Lemma 4.13 for \(c(x, t) \geq 0\).

Step 2. Assume that \(\{\xi_1(t), \cdots, \xi_l(t)\}\) are around \((x_0, t_0)\) on \([T_1, T_2]\). Let \(T_1 < t_5 < t_3 < t_1 < t_0 < t_2 < t_4 < t_6 < T_2\). We construct \(v_k, r_0, \tilde{w}_\epsilon\) and \(\tilde{\phi}_\epsilon\) as in Definition 4.12 by setting

\[
T = t_5, \quad \tilde{T} = t_6, \quad \underline{t} = t_3, \quad \tilde{t} = t_4.
\]

Assume that (4.66) holds. Let \(0 < \delta < r_1 < \frac{r_0}{2}\) and set \(r = r_1, \tilde{r} = 2r_1\). After shrinking \(r_1\) and the interval \([T_1, T_2]\) if necessary, we assume that \(Q_{2r_1, T_1, T_2} \subset \tilde{Q}_{r_0, T_1, T_2}\). We choose \(t_7, t_8\) such that \(T_1 < t_7 < t_5 < t_6 < t_8 < T_2\) and define the function

\[
\phi = \phi(x, t; r_1, 2r_1, t_5, t_6, t_7, t_8, T_1, T_2)
\]

(4.75)

as in Definition 4.12. Then by (4.64) the function \(\phi\) satisfies the properties

\[
\supp(|\nabla \phi| + |\partial_t \phi|) \cap Q_{2r_1, t_5, t_6} \subset Q_{2r_1, t_5, t_6} \setminus Q_{r_1, t_5, t_6}
\]

\[
\subset \bigcup_{k=1}^{l} \left\{ (x, t) \in Q_{2r_1, t_5, t_6} \mid r_1 \leq r_k(x, t) \leq 2r_1, \quad r_1(x, t) \geq r_1, \quad \forall i \neq k \right\}.
\]

(4.76)

Moreover, we define the following functions on \(\tilde{Q}_{r_0, t_5, t_6} \setminus \Gamma_{t_5, t_6}\)

\[
V_k(x, t) = e^{-2v_k(x, t)}, \quad V(x, t) = \sum_{k=1}^{l} V_k(x, t),
\]

\[
w(x, t) = \zeta(t)^{-1}V(x, t) - 1, \quad \varphi_0(x, t) = \phi(x, t)\zeta(t)(H \circ \varphi)(x, t),
\]

where \(\zeta = \zeta(t; t_1, t_2, t_3, t_4, \delta, r_1)\) is the function defined in (1) of Definition 4.7. By using the prop-
vertices (4.45)-(4.47), for any \((x, t) \in \mathring{Q}_{r_0, t_5, t_6} \setminus \Gamma_{t_5, t_6}\) we have
\[
\sum_{k=1}^{l} r_k(x, t)^2 \leq V(x, t) \leq \sum_{k=1}^{l} r_k(x, t)^{2\gamma}, \tag{4.77}
\]
\[
l \leq \frac{\partial V}{\partial t} + \Delta V \leq 4 \sum_{k=1}^{l} r_k(x, t)^{2\gamma-2}, \tag{4.78}
\]
\[
|\nabla V| \leq 2 \sum_{k=1}^{l} r_k(x, t)^{2\gamma-1}. \tag{4.79}
\]

Note that the function \(\varphi_0(x, t)\) vanishes near the point \((\xi_1(t_0), t_0)\), but \(\varphi_0(x, t)\) may not be zero on \(\Gamma_{t_5, t_6}\). The function \(\tilde{\varphi}_\epsilon\) defined in Definition 4.12 vanishes near \(\Gamma_{t_5, t_6}\), but it doesn’t satisfy the inequality (4.78) and the inequality (4.54). Therefore, the argument of Lemma 4.9 doesn’t work any more.

**Step 3.** Direct calculation as in the proof of Lemma 4.9 for any \((x, t) \in Q_{2r_1, t_5, t_6}\) we have
\[
\tilde{\varphi}_\epsilon(x, t) \leq \tilde{\varphi}_\epsilon(x, t) V(x, t) H' \circ w, \tag{4.80}
\]
\[
\frac{\partial \varphi_0}{\partial t} + \Delta \varphi_0 \geq \left(l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}\right) \phi H' \circ w - C(c_{\phi, K, \gamma, r_1}) \chi_K, \tag{4.81}
\]
where \(K\) and \(c_{\phi, K}\) are defined by
\[
K = \text{Supp}(|\nabla \phi| + |\partial_t \phi|) \cap Q_{2r_1, t_5, t_6}, \tag{4.82}
\]
\[
c_{\phi, K} = \sup_K (|\partial_t \phi| + |\Delta \phi| + |\nabla \phi|). \tag{4.83}
\]

Let \(\varphi = \varphi_0 \tilde{\varphi}_\epsilon \in C^\infty_0(Q_{2r_1, t_5, t_6} \setminus \Gamma_{t_5, t_6})\). Then we have
\[
\frac{\partial \varphi}{\partial t} + \Delta \varphi = \left(\frac{\partial \varphi_0}{\partial t} + \Delta \varphi_0\right) \tilde{\varphi}_\epsilon + \left(\frac{\partial \tilde{\varphi}_\epsilon}{\partial t} + \Delta \tilde{\varphi}_\epsilon\right) \varphi_0 + 2(\nabla \tilde{\varphi}_\epsilon, \nabla \varphi_0)
\]
\[
= \left(\frac{\partial \varphi_0}{\partial t} + \Delta \varphi_0\right) \tilde{\varphi}_\epsilon + \tilde{\varphi}_\epsilon H' \circ \tilde{\omega}_\epsilon |\nabla \tilde{\omega}_\epsilon|^2 \varphi_0 + 2 \phi H' \circ \tilde{\omega}_\epsilon \nabla V + 2 \zeta H' \circ \tilde{\omega}_\epsilon \nabla \omega_\epsilon \nabla V
\]
\[
+ 2 \zeta H' \circ \tilde{\omega}_\epsilon \nabla V \cdot \nabla \omega_\epsilon |\chi_{\tilde{\omega}_\epsilon > 0} - 2V H' \circ \tilde{\omega}_\epsilon |\nabla \tilde{\omega}_\epsilon| \cdot |\nabla \phi|,
\]
where we used (4.52) (4.69) (4.81) and the definition of \(w\). Combining this with (4.51) and using the assumption \(c(x, t) \geq 0\), we have
\[
0 \leq \int_{Q_{2r_1, t_5, t_6}} cu \varphi = -\int_{Q_{2r_1, t_5, t_6}} u \left(\frac{\partial \varphi}{\partial t} + \Delta \varphi\right)
\]
\[
\leq -\left(l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}\right) \int_{Q_{2r_1, t_5, t_6}} u \phi \tilde{\varphi}_\epsilon H' \circ w + C(c_{\phi, K, \gamma, r_1}) \int_K u \tilde{\varphi}_\epsilon
\]
\[
+ 2\|\nabla V\|_{L^\infty(Q_{2r_1, t_5, t_6})} \int_{Q_{2r_1, t_5, t_6}} u \phi H' \circ w |\nabla \omega_\epsilon| \chi_{\omega_\epsilon > 0}
\]
\[
+ 2\|V\|_{L^\infty(Q_{2r_1, t_5, t_6})} \int_{Q_{2r_1, t_5, t_6}} u H' \circ w H' \circ \tilde{\omega}_\epsilon |\nabla \tilde{\omega}_\epsilon| \cdot |\nabla \phi|.
\]

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Therefore, we have
\[
(1 - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})}) \int_{Q_{2r_1, t_5, t_6}} u \phi H' \circ \omega \varphi_c \\
\leq 2\|\nabla V\|_{L^\infty(Q_{2r_1, t_5, t_6})} \int_{Q_{2r_1, t_5, t_6}} u \phi H' \circ \omega |\nabla \omega| \chi(\omega_c > 0) \\
+ C(c_\phi, K, \gamma, r_1) \int_K u \varphi_c + 2\|V\|_{L^\infty(Q_{2r_1, t_5, t_6})} \int_{Q_{2r_1, t_5, t_6}} u |\nabla \phi| \cdot |\nabla \omega| \\
\leq 2\|\nabla V\|_{L^\infty(Q_{2r_1, t_5, t_6})} \int_{Q_{2r_1, t_5, t_6}} u \phi H' \circ \omega \chi(\omega_c > 0) \\
+ 2\|\nabla V\|_{L^\infty(Q_{2r_1, t_5, t_6})} \int_{Q_{2r_1, t_5, t_6}} u \phi H' \circ \omega |\nabla \omega|^2 \chi(\omega_c > 0) \\
+ C(c_\phi, K, \gamma, r_1) \int_K u \varphi_c + 2\|V\|_{L^\infty(Q_{2r_1, t_5, t_6})} \int_{Q_{2r_1, t_5, t_6}} u |\nabla \phi| \cdot |\nabla \omega| \quad (4.84)
\]

The main difficulty is to estimate the integral
\[
\int_{Q_{2r_1, t_5, t_6}} u \phi H' \circ \omega |\nabla \omega|^2 \chi(\omega_c > 0) \quad (4.85)
\]
on the right hand side of (4.84).

**Step 4.** We estimate the integral (4.85). For any \( \rho \in (0, \frac{1}{2}) \), we define the functions
\[
\bar{w}_\rho(x, t) = \frac{1}{3} \left( 2 - \frac{\tilde{v}(x, t)}{10 \log \left( \frac{1}{\rho} \right)} \right), \quad \psi(x, t) = \phi(x, t) H \circ \bar{w}_\rho \in C^\infty_0(Q_{2r_1, t_5, t_6} \setminus \Gamma_{t_5, t_6}),
\]
where \( \tilde{v} \) is the function defined in (4.67). Note that \( \bar{w}_\rho(x, t) \) satisfies \( \partial_t \bar{w}_\rho + \Delta \bar{w}_\rho = 0 \). Direct calculation shows that
\[
\frac{\partial \psi}{\partial t} + \Delta \psi = \phi |\nabla \bar{w}_\rho|^2 H'' \circ \bar{w}_\rho + \left( \frac{\partial \phi}{\partial t} + \Delta \phi \right) H \circ \bar{w}_\rho + 2(\nabla \phi, \nabla \bar{w}_\rho) H' \circ \bar{w}_\rho.
\]
Since \( u \) satisfies
\[
- \int_{Q_{2r_1, t_5, t_6}} u \left( \frac{\partial \psi}{\partial t} + \Delta \psi \right) = \int_{Q_{2r_1, t_5, t_6}} cu \psi \geq 0,
\]
we have
\[
\int_{Q_{2r_1, t_5, t_6}} u \phi |\nabla \bar{w}_\rho|^2 H'' \circ \bar{w}_\rho \\
\leq - \int_{Q_{2r_1, t_5, t_6}} u \left( \left( \frac{\partial \phi}{\partial t} + \Delta \phi \right) H \circ \bar{w}_\rho + 2(\nabla \phi, \nabla \bar{w}_\rho) H' \circ \bar{w}_\rho \right). \quad (4.86)
\]
We estimate each term of (4.86). Note that
\[
\{ (x, t) \in Q_{2r_1, t_5, t_6} \mid H'' \circ \bar{w}_\rho \geq \min_{\frac{1}{4} \leq z \leq \frac{5}{4}} H''(z) \} \\
\supset \{ (x, t) \in Q_{2r_1, t_5, t_6} \mid \frac{1}{3} \leq \bar{w}_\rho \leq \frac{2}{3} \} \\
= \{ (x, t) \in Q_{2r_1, t_5, t_6} \mid 1 \leq \bar{w}_\rho \leq 2 \} \\
= \{ (x, t) \in Q_{2r_1, t_5, t_6} \mid \rho \leq \bar{r}(x, t) \leq 1 \}, \quad (4.87)
\]
where $\tilde{\omega}_\rho$ is defined in (4.67) and $\tilde{r}(x, t) = e^{-\tilde{v}(x, t)}$. Thus, the left-hand side of (4.86) satisfies the inequality

$$\int_{Q_{2r_1, t_5, t_6}} u\phi|\nabla \tilde{w}_\rho|^2 H'' \circ \tilde{w}_\rho \geq C \int_{Q_{2r_1, t_5, t_6}} u\phi|\nabla \tilde{v}|^2 \chi_{\{1 \leq \tilde{\omega}_\rho \leq 2\}} \left|\log \rho\right|^2$$

$$= C \int_{Q_{2r_1, t_5, t_6}} u\phi|\nabla \tilde{\omega}_\rho|^2 \chi_{\{1 \leq \tilde{\omega}_\rho \leq 2\}}, \quad (4.88)$$

where $C$ is a universal constant. We choose $2r_1 < 1$ and by (4.71) we have $\tilde{r}(x, t) < 1$ on $\text{Supp}(\phi) \cap Q_{2r_1, t_5, t_6}$. Thus, on $\text{Supp}(\phi) \cap Q_{2r_1, t_5, t_6}$ we have

$$\tilde{\omega}_\rho = \frac{1}{3} \left(2 - \frac{\tilde{v}(x, t)}{\log(\rho)}\right) \leq \frac{2}{3}, \quad H \circ \tilde{\omega}_\rho \leq \tilde{\omega}_\rho \leq \frac{2}{3}. \quad (4.89)$$

Combining this with (4.86), the first term of the right-hand side of (4.86) satisfies the inequality

$$-\int_{Q_{2r_1, t_5, t_6}} u \left(\frac{\partial \phi}{\partial t} + \Delta \phi\right) H \circ \tilde{w}_\rho \leq C(\phi, K) \int_K u, \quad (4.90)$$

where $K$ and $c_{\phi, K}$ are given by (4.82)-(4.83). Note that by (4.64) for any $i$ we have $r_i(x, t) \geq r_1$ on $\text{Supp}(|\nabla \phi|) \cap Q_{2r_1, t_5, t_6}$. Combining this with (4.27), we have

$$|\nabla \tilde{v}|^2 \leq l \sum_{k=1}^l |\nabla v_k|^2 \leq l \sum_{k=1}^l \frac{1}{r_k^2} \leq \frac{l^2}{r_1^2}, \quad \forall \ (x, t) \in \text{Supp}(|\nabla \phi|) \cap Q_{2r_1, t_5, t_6}. \quad (4.91)$$

Thus, when $\rho \in (0, \frac{1}{2})$ we have

$$|\nabla \tilde{\omega}_\rho| = \frac{2}{3 \log \frac{1}{\rho}} \leq \frac{2l^2}{3 \log 2 \ r_1^2}, \quad \forall \ (x, t) \in \text{Supp}(|\nabla \phi|) \cap Q_{2r_1, t_5, t_6}. \quad (4.91)$$

This implies that the second term of the right-hand side of (4.86) satisfies

$$-\int_{Q_{2r_1, t_5, t_6}} 2u \langle \nabla \phi, \nabla \tilde{w}_\rho \rangle H'' \circ \tilde{w}_\rho \leq C(\phi, K, r_1, l) \int_K u. \quad (4.92)$$

Let $\rho = e^2$. Note that $\tilde{r}(x, t) \leq 1$ on $\text{Supp}(\phi) \cap Q_{2r_1, t_5, t_6}$. By (4.87) we have

$$\{(x, t) \in Q_{2r_1, t_5, t_6} | \tilde{\omega}_\rho > 0\} \cap \text{Supp}(\phi)$$

$$= \{(x, t) \in Q_{2r_1, t_5, t_6} | \tilde{r}(x, t) > 1\} \cap \text{Supp}(\phi)$$

$$= \{(x, t) \in Q_{2r_1, t_5, t_6} | \rho \leq \tilde{r}(x, t) \leq 1\} \cap \text{Supp}(\phi),$$

$$= \{(x, t) \in Q_{2r_1, t_5, t_6} | 1 \leq \tilde{\omega}_\rho \leq 2\} \cap \text{Supp}(\phi). \quad (4.93)$$

Combining (4.90) (4.92) with (4.86), we have

$$\int_{Q_{2r_1, t_5, t_6}} u\phi|\nabla \tilde{w}_\rho|^2 H'' \circ \tilde{w}_\rho \leq C(\phi, K, r_1, l) \int_K u. \quad (4.94)$$
This together with (4.88) implies that
\[ \int_{Q_{2r_1,t_5,t_6}} u\phi |\nabla \tilde{\omega}_\rho|^2 \chi_{\{1 \leq \tilde{\omega}_\rho \leq 2\}} \leq C \int_{Q_{2r_1,t_5,t_6}} u\phi |\nabla \tilde{\omega}_\rho|^2 H'' \circ \tilde{w}_\rho \]

\[ \leq C(c_\phi,K,r_1,l) \int_K u. \quad (4.95) \]

Thus, by (4.95) and (4.93) we have
\[ \int_{Q_{2r_1,t_5,t_6} \cap \{ \rho < \rho \leq 1\}} u\phi |\nabla \tilde{\omega}_\rho|^2 = \int_{Q_{2r_1,t_5,t_6}} u\phi |\nabla \tilde{\omega}_\rho|^2 \chi_{\{1 \leq \tilde{\omega}_\rho \leq 2\}} \leq C(c_\phi, r_1, l) \int_K u. \quad (4.96) \]

Moreover, we have the estimate for the integral (4.85)
\[ \int_{Q_{2r_1,t_5,t_6}} u\phi H' \circ w |\nabla \tilde{\omega}_\epsilon|^2 \chi_{\{\tilde{\omega}_\epsilon > 0\}} \leq \int_{Q_{2r_1,t_5,t_6} \cap \{\tilde{\omega}_\epsilon > 0\}} \frac{u\phi |\nabla \tilde{v}|^2}{|\log \epsilon|^2} \]

\[ = 4 \int_{Q_{2r_1,t_5,t_6} \cap \{1 \leq \tilde{\omega}_\rho \leq 2\}} u\phi |\nabla \tilde{\omega}_\rho|^2 \leq C(c_\phi, K, r_1, l) \int_K u. \quad (4.97) \]

**Step 5.** Now we turn back to the inequality (4.84). Moreover, by (4.91) we have
\[ \int_{Q_{2r_1,t_5,t_6}} u |\nabla \phi| : |\nabla \tilde{\omega}_\epsilon| = 2 \int_{Q_{2r_1,t_5,t_6}} u |\nabla \phi| : \frac{|\nabla \tilde{v}|}{\log \frac{1}{\rho}} \leq C(c_\phi,K,l,r_1) \int_K u. \quad (4.98) \]

Combining (4.84), (4.98) with (4.97), we have
\[ (l - 2 \|\partial_t \zeta\|_{L^\infty(\mathbb{R})}) \int_{Q_{2r_1,t_5,t_6}} u\phi H' \circ w \tilde{\phi}_\epsilon \leq 2 \|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} \int_{Q_{2r_1,t_5,t_6}} u\phi H' \circ w \chi_{\{\tilde{\omega}_\epsilon > 0\}} + C(c_\phi,K,r_1,l) \left( \|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} \right) \int_K u \]

\[ + C(c_\phi,K,\gamma,r_1) \int_K u \tilde{\phi}_\epsilon. \quad (4.99) \]

Since all singular curves are disjoint on $Q_{2r_1,t_5,t_6} \cap \{ w \geq 0\}$ by our assumption, by Lemma 4.9 $u$ is integrable on $Q_{2r_1,t_5,t_6} \cap \{ w \geq 0\}$. Taking $\epsilon \to 0$ in (4.99) and using the dominated convergence theorem, we have
\[ (l - 2 \|\partial_t \zeta\|_{L^\infty(\mathbb{R})}) \int_{Q_{2r_1,t_5,t_6}} u\phi H' \circ w \]

\[ \leq 2 \|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} \int_{Q_{2r_1,t_5,t_6}} u\phi H' \circ w + C(c_\phi,K,\gamma,r_1) \int_K u \]

\[ + C(c_\phi,K,r_1,l) \left( \|\nabla V\|_{L^\infty(Q_{2r_1,t_5,t_6})} + \|V\|_{L^\infty(Q_{2r_1,t_5,t_6})} \right) \int_K u. \quad (4.100) \]
It follows that
\[
\left( l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})} - 2\|\nabla V\|_{L^\infty(\mathbb{R})} \right) \int_{Q_{2r_1, t_5, t_6}} u \phi H' \circ w \leq C(c_\phi, K, \gamma, r_1) \int_K u. \tag{4.101}
\]

By (4.40) and (4.79), we choose \( r_1 \) small such that
\[
l - 2\|\partial_t \zeta\|_{L^\infty(\mathbb{R})} - 2\|\nabla V\|_{L^\infty(\mathbb{R})} > \frac{1}{2} l.
\]
Combining this with (4.101), we have
\[
\int_{Q_{2r_1, t_5, t_6}} u \phi H' \circ w \leq C(c_\phi, K, \gamma, r_1) \int_K u. \tag{4.102}
\]

Note that the function \( H' \circ w \) converges to 1 on \( Q_{r_1, t_1, t_2} \) as \( \delta \to 0 \). Thus, taking \( \delta \to 0 \) in (4.102) we have
\[
\int_{Q_{r_1, t_1, t_2}} u \leq C(c_\phi, K, \gamma, r_1) \int_K u, \tag{4.103}
\]
which implies (4.74). Note that (4.96) implies (4.74) since
\[
I(\rho; t_1, t_2, u, r_0) = \int_{Q_{r_0, t_1, t_2}} u |\nabla \rho| \log \rho \leq \int_{Q_{r_1, t_5, t_6}} u |\nabla \rho| \log \rho + \int_{Q_{r_0, t_5, t_6} \setminus Q_{r_1, t_5, t_6}} u |\nabla \rho| \log \rho \\
\leq \int_{Q_{2r_1, t_5, t_6}} u |\nabla \rho| \log \rho + C(l, l) \int_{Q_{r_0, t_5, t_6} \setminus Q_{r_1, t_5, t_6}} u \leq C(c_\phi, r_1, l) \int_K u + C(l, r_1) \int_{Q_{r_0, t_5, t_6} \setminus Q_{r_1, t_5, t_6}} u < +\infty. \tag{4.104}
\]

The lemma is proved.

As a byproduct of the above proof, we have the following result.

**Lemma 4.14.** Under the assumption of Lemma 4.9 for the singular curve \( \xi : [T_1, T_2] \to \Sigma \) we have
\[
\sup_{\rho \in (0, \frac{1}{2})} I(\rho; t_1, t_2, u, r_1) < +\infty. \tag{4.105}
\]

**Proof.** (4.105) follows directly from the inequality (4.104) and Step 4 of the proof of Lemma 4.13 by choosing \( l = 1 \).

By using Lemma 4.9 Lemma 4.14 and following the same arguments as in [7], we have the following results when the singular curves are disjoint.

**Lemma 4.15.** (c.f. [7]) Under the same assumption as in Theorem 4.2 if we assume that \( \{\xi_1(t), \cdots, \xi_l(t)\} \) are disjoint on \( [T_1, T_2] \) and \( \bar{\rho} \) is the constant in (6) of Definition 4.12 then we have
Lemma 4.16. The same assumption as in Theorem 4.2. Suppose that the singular curves blow up as where the right-hand side is finite by (4.74). Here the right-hand side is finite by (4.109). Since the proof is exactly the same as in [7], we omit the details here. (3) follows from Lemma 5.2 of [7] and (4) follows from the non-negativity of the right-hand side of Lemma 4.4 of [7]. Part (2) follows from the proof of Theorem 2.1 of [7] (See Page 7303 of [7]), (3) follows from Lemma 5.2 of [7] and (4) follows from the non-negativity of the right-hand side of Lemma 4.14. Since the proof is exactly the same as in [7], we omit the details here.

(1). For each \( \xi_k \) and \( (t_1, t_2) \subset [T_1, T_2] \), the mapping \( J_k : C^\infty_0(Q^{(k)}_{\hat{p}, t_1, t_2}) \to \mathbb{R} \)

\[
J_k(f) = \int_{Q^{(k)}_{\hat{p}, t_1, t_2}} u \left( -\frac{\partial f}{\partial t} - \Delta f \right) \psi \, dxdt
\]
defines a distribution whose support is contained in \( \Gamma^{(k)}_{t_1, t_2} \), and satisfies

\[
|J_k(f)| \leq C \left( \sup_{\rho \to 0} |f| \right) \liminf_{\rho \to 0} I_{\xi_k}(\rho; t_1, t_2, u, \hat{p}),
\]

where \( C \) is a universal constant. Here \( I_{\xi_k}(\rho; t_1, t_2, u, \hat{p}) \) is defined in (6) of Definition 4.12 and it is finite by Lemma 4.14.

(2). There exists linear functionals \( \{\mu_1, \cdots, \mu_l\} \) with each \( \mu_k \in (C_0((T_1, T_2)))' \) such that for all \( \varphi \in C^\infty_0(Q_{T_1, T_2}) \),

\[
\int_{Q_{T_1, T_2}} u \left( -\frac{\partial \varphi}{\partial t} - \Delta \varphi \right) = \int_{Q_{T_1, T_2}} cu \varphi + \sum_{k=1}^l \int_{(T_1, T_2)} \varphi(\xi_k(t), t) \, d\mu_k(t).
\]
The identity (4.107) can be rewritten as

\[
\frac{\partial u}{\partial t} - \Delta u = cu + \sum_{k=1}^l \delta_{\xi_k} \otimes \mu_k, \quad \text{in} \quad \mathcal{D}'(Q_{T_1, T_2}).
\]

(3). Let \( \mu_k \) be one of the measures in (2). For any \( \psi \in C^\infty_0((T_1, T_2)) \) with \( \text{Supp}(\psi) \subset (t_1, t_2) \), we have

\[
\int_{T_1}^{T_2} \psi \, d\mu_k = 2 \lim_{\rho \to 0} \frac{1}{|\log \rho|^2} \int_{Q^{(k)}_{\hat{p}, T_1, T_2}} |\nabla \psi|^2 \chi_{\{v_{\rho} \leq |\log \rho|\}} \psi u.
\]

(4). Each measure \( \mu_k \) obtained in (3) is positive.

Proof. Since \( \{\xi_1(t), \cdots, \xi_l(t)\} \) are disjoint on \( [t_1, t_2] \), we can consider each \( \xi_k \) as in [7]. After replacing the function \( g \) in (4.21) of [7] by the function \( c(x, t) \), we know that (1) follows directly from Lemma 4.4 of [7]. Part (2) follows from the proof of Theorem 2.1 of [7] (See Page 7303 of [7]), (3) follows from Lemma 5.2 of [7] and (4) follows from the non-negativity of the right-hand side of 4.109. Since the proof is exactly the same as in [7], we omit the details here.

When the singular curves are around \( (x_0, t_0) \), the measures \( \mu_k \) constructed in Lemma 4.15 may blow up as \( t \to t_0 \). The next result shows that \( \mu_k \) is actually bounded when \( t \) is close to \( t_0 \).

**Lemma 4.16.** The same assumption as in Theorem 4.2. Suppose that the singular curves \( \{\xi_1(t), \cdots, \xi_l(t)\} \) are around \( (x_0, t_0) \) on \( [t_1, t_2] \) as in Definition 4.11. Define the measure \( \mu \) on \( (t_1, t_2) \) by

\[
\int_{t_1}^{t_2} \psi \, d\mu = \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{Q_{t_1, t_2}} |\nabla \tilde{v}|^2 \chi_{\{\tilde{v} \leq |\log \rho|\}} \psi u \, d\text{vol} \, dt,
\]

where the right-hand side is finite by 4.74. Here \( \tilde{v} \) is the function defined by 4.67. Then \( \mu \in (C_0((t_1, t_2)))' \) and for each \( \xi_k \) the measure \( \mu_k \) obtained by Lemma 4.15 satisfies

\[
0 \leq \gamma^4 \mu_k(t) \leq \mu(t), \quad \forall \, t \in (t_1, t_0) \cup (t_0, t_2),
\]

where \( \gamma \in (\frac{1}{2}, 1) \) is the constant chosen in Lemma 4.5.
Proof. Since \( \{\xi_1(t), \cdots, \xi_l(t)\} \) are around \((x_0, t_0)\) on \((t_1, t_2)\), by Definition 4.11 we can assume that \( \{\xi_1(t), \cdots, \xi_{l'}(t)\} \) are disjoint for some \( l' \leq l \) on \([t_1, t_0)\) and

\[
\xi_{l'}(t) = \xi_{l'+1}(t) = \cdots = \xi_l(t), \quad \forall t \in (t_1, t_0).
\] (4.111)

Let \( r_0 \) be the constant defined by (4.65). After shrinking \((t_1, t_2)\) if necessary, we can assume that \( Q_{r_0, t_1, t_2} \subset \tilde{Q}_{r_0, t_1, t_2} \) for some \( r' > 0 \). Let \( \rho_1 > 0 \) be the constant such that for any \((x, t) \in Q_{r_0, t_1, t_2} \) and \( 1 \leq i \leq l' \) we have \( r_i(x, t) \leq \rho_1 \). Since for any \( \delta > 0 \) the curves \( \{\xi_1(t), \cdots, \xi_{l'}(t)\} \) are disjoint on \([t_1, t_0 - \delta]\), we define

\[
d_\delta := \min\{d_g(\xi_i(t), \xi_j(t)) \mid 1 \leq i \neq j \leq l', t \in [t_1, t_0 - \delta]\} > 0.
\] (4.112)

Let \( \alpha_0 = \frac{d_\delta}{2} \) and \((x, t) \in Q_{r_0, t_1, t_0 - \delta}\). By the choice of \( \alpha_0 \), if \( r_k(x, t) < \alpha_0 \), then we have

\[
\text{r}_i(x, t) \geq \alpha_0, \quad \forall i \neq k.
\] (4.113)

For any \( \rho_2 > 0 \), we can find some integer \( k \in [1, l'] \) such that if \( t \in [t_1, t_0 - \delta] \) and \( r_1 r_2 \cdots r_{l'} \leq \rho_2 \) we have

\[
r_i \geq \alpha_0, \quad \forall i \neq k, \quad \text{and} \quad r_k \leq \frac{\rho_2}{\alpha_0^{l-1}}.
\] (4.114)

We choose \( \rho_2 \) such that

\[
\frac{\rho_2}{\rho_1^{l-1}} = \alpha_0.
\] (4.115)

By (4.114) for any \( k \in \{1, 2, \cdots, l'\} \) and \( \rho \in (0, \rho_2) \),

\[
\left\{(x, t) \in Q_{r'_0, t_1, t_0 - \delta} \mid \bar{v}(x, t) \leq \log \frac{1}{\rho}\right\} \supset \left\{(x, t) \in Q_{r'_0, t_1, t_0 - \delta} \mid r_1 r_2 \cdots r_{l'} \geq \rho\right\}
\]

\[
\supset \left\{(x, t) \in Q_{r'_0, t_1, t_0 - \delta} \mid \rho_2 \geq r_1 r_2 \cdots r_{l'} \geq \rho\right\}
\]

\[
= \bigcup_{k=1}^{l'} \Omega_{k, \rho},
\]

where \( \Omega_{k, \rho} \) is defined by

\[
\Omega_{k, \rho} := \left\{(x, t) \in Q_{r'_0, t_1, t_0 - \delta} \mid \alpha_0 \leq r_i \leq \rho, \quad \forall i \neq k, \quad \frac{\rho}{\alpha_0^{l-1}} \leq r_k \leq \frac{\rho_2}{\rho_{1}^{l-1}}\right\}
\]

\[
= \left\{(x, t) \in Q_{r'_0, t_1, t_0 - \delta} \mid \alpha_0 \leq r_i \leq \rho, \quad \forall i \neq k, \quad \frac{\rho}{\alpha_0^{l-1}} \leq r_k \leq \alpha_0\right\}.
\] (4.116)

Note that we used (4.115) in the equality of (4.116). By the definition of \( r'_0 \) and Lemma 4.5 for any \((x, t) \in \Omega_{k, \rho} \) with \( 1 \leq k \leq l' - 1 \) we have

\[
|\nabla \bar{v}|^2 \geq |\nabla v_k|^2 - \sum_{i \neq k} |\nabla v_i|^2
\]

\[
\geq \frac{\gamma^2}{r_k^2} - \sum_{i \neq k} \frac{1}{r_i^2} \geq \frac{\gamma^2}{r_k^2} - \frac{l - 1}{\alpha_0^2},
\]

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and for any \((x, t) \in \Omega_{k, \rho}\) with \(l' \leq k \leq l\) we have

\[
|\nabla \tilde{v}|^2 \geq (l - l' + 1)|\nabla v_k|^2 - \sum_{i=1}^{l'-1} |\nabla v_i|^2 \geq (l - l' + 1)\frac{\gamma^2}{r_k^2} - \sum_{i=1}^{l'-1} \frac{1}{r_i^2} \geq (l - l' + 1)\frac{\gamma^2}{r_k^2} - \frac{l'-1}{\alpha_0^2}.
\]

Consequently, by (4.110) for any \(k \in \{1, 2, \ldots, l'-1\}\) we have

\[
\int_{t_1}^{t_{0-\delta}} \psi d\mu \geq \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k, \rho}} |\nabla \tilde{v}|^2 \psi u \geq \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k, \rho}} \left(\frac{\gamma^2}{r_k^2} - \frac{l'-1}{\alpha_0^2}\right) \psi u = \gamma^2 \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k, \rho}} \frac{\psi u}{r_k^2}.
\]

(4.117)

and for \(k \in \{l', \ldots, l\}\) we have

\[
\int_{t_1}^{t_{0-\delta}} \psi d\mu \geq (l - l' + 1)\gamma^2 \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k, \rho}} \frac{\psi u}{r_k^2} \geq \gamma^2 \lim_{\rho \to 0} \frac{2}{|\log \rho|^2} \int_{\Omega_{k, \rho}} \frac{\psi u}{r_k^2}.
\]

(4.118)

On the other hand, taking \(\rho_0^{-1} = \frac{\rho}{\alpha_0}\) and using (4.109) we have

\[
\int_{t_1}^{t_{0-\delta}} \psi d\mu_k = \lim_{\rho \to 0} \frac{1}{|\log \rho|^2} \int_{Q_{\rho_0, t_1, t_{0-\delta}}} |\nabla v_k|^2 \chi_{\{v_k \leq |\log \rho|\}} \psi u \leq \lim_{\rho \to 0} \frac{1}{|\log \rho|^2} \int_{Q_{t_1, t_{0-\delta}}} |\nabla v_k|^2 \chi_{\{v_k \leq |\log \rho|\}} \psi u \leq \lim_{\rho \to 0} \frac{1}{|\log \rho|^2} \int_{\Omega_{k, \rho}} \frac{\psi u}{r_k^2}.
\]

(4.119)

where we used the fact that \(\{v_k \leq |\log \rho|\} \subseteq \{\rho_0^{-1} \leq r_k\}\). Note that

\[
Q_{t_1, t_{0-\delta}}^{(k)} \cap \{\rho_0^{-1} \leq r_k\} = \left\{(x, t) \in Q_{t_1, t_{0-\delta}} \mid \rho_0^{-1} \leq r_k \leq \alpha_0\right\} = \left\{(x, t) \in Q_{t_1, t_{0-\delta}} \mid \rho_0^{-1} \leq r_k \leq \alpha_0, \ for \ i \neq k\right\} = \Omega_{k, \rho},
\]

(4.120)

where we used (4.113) and (4.116). Combining (4.120) with (4.119), we have

\[
\int_{t_1}^{t_{0-\delta}} \psi d\mu_k \leq \frac{2}{\gamma^2} \lim_{\rho \to 0} \frac{1}{|\log \rho|^2} \int_{\Omega_{k, \rho}} \frac{\psi u}{r_k^2}.
\]

(4.121)
The inequalities (4.117)-(4.118) and (4.121) implies that
\[
\int_{t_1}^{t_0-\delta} \psi \, d\mu \geq \gamma^4 \int_{t_1}^{t_0-\delta} \psi \, d\mu_k.
\] (4.122)

Thus, we have
\[
0 \leq \gamma^4 \mu_k \leq \mu, \quad \forall \ t \in (t_1, t_0).
\] (4.123)

Similarly, we can consider the case when \( \{\xi_1(t), \cdots, \xi_l(t)\} \) are disjoint for some \( l' \leq l \) on \( (t_0, t_2) \).

The lemma is proved.

\[\square\]

4.5 Proof of Theorem 4.2

In this subsection we show Theorem 4.2. Part (1) of Theorem 4.2 follows from (4.48) and (4.73). For part (2), the proof divides into the following steps.

Step 1. Without loss of generality, we can assume that \( c(x,t) \leq 0 \). In fact, let \( u(x,t) \) be a solution of (4.2). Then for any \( k \in \mathbb{R} \) the function \( \tilde{u}(x,t) = u(x,t)e^{kt} \) satisfies the equation
\[
\frac{\partial \tilde{u}}{\partial t} = \Delta \tilde{u} + (c + k)\tilde{u}.
\]

Therefore, for any compact set \( K \) in \( \Sigma \times [T_1, T_2] \) we can choose \( k \) such that the function \( \tilde{c} := c + k \) is nonpositive on \( K \). Thus, it suffices to show Theorem 4.2 for \( c(x,t) \leq 0 \).

Step 2. Suppose that the curves \( \{\xi_1(t), \cdots, \xi_l(t)\} \) are disjoint on \( [T_1, T_2] \). Let \( T_1 < t_1 < t_2 < T_2 \). Lemma 4.9 implies that \( u \) is in \( L^1 \). For any \( (x,t) \in \Sigma \times (t_1, t_2) \), we define
\[
w_k(x,t) = \int_{t_1}^{t} \int_{\Sigma} p(x,y,t-s) \tilde{g}_k(y,s) \, d\mu_k(s), \tag{4.124}
\]
\[
\tilde{g}_k(x,t) = c(x,t) u(x,t) x^{(k)}_Q t_1, t_2, \tag{4.125}
\]
\[
\tilde{U}_k(x,t) = \int_{(t_1,t)} p(x, \xi_k(s), t-s) \, d\mu_k(s), \tag{4.126}
\]

where \( \mu_k \) is the measure obtained in Lemma 4.15 and \( \delta_0 > 0 \) is a constant chosen such that \( t_1 - \delta_0 > T_1 \). Then \( u - \sum_{k=1}^{l} (\tilde{U}_k + w_k) \) satisfies the heat equation in \( D'(Q_{(1/2)}^{1}, t_1, t_2) \), which implies that \( u - \sum_{k=1}^{l} (\tilde{U}_k + w_k) \) is bounded in \( Q_{(1/2)}^{1}, t_1, t_2 \). Since \( c(x,t) \leq 0 \), we have \( w_k(x,t) \leq 0 \) and
\[
u(x,t) \leq \sum_{k=1}^{l} \tilde{U}_k(x,t) + f(x,t)
\]

where \( f(x,t) \) is a bounded function on \( Q_{(1/2)}^{1}, t_1, t_2 \). Therefore, by Lemma 4.6 \( u \) satisfies the inequalities (4.4)-(4.5).

Step 3. In general, the singular curves may not be disjoint. In this case, we assume that the curves \( \{\xi_1(t), \cdots, \xi_l(t)\} \) are around \( (x_0,t_0) \) on \( (t_1, t_2) \). Consider the interval \( (t_1, t_0) \). By Definition 4.11 we can find an integer \( l' \in [1, l] \) such that \( \{\xi_1(t), \cdots, \xi_{l'}(t)\} \) are disjoint on \( (t_1, t_0) \). By Lemma 4.15 we get positive measures \( \mu_k \in (C_0((t_1, t_0)))^l \) for each \( \xi_k \) with \( k \in [1, l'] \), and by Lemma 4.16 we have
\[
0 \leq \gamma^4 \mu_k(t) \leq \mu(t), \quad \forall t \in (t_1, t_0).
\] (4.127)
For each $k$, we define $U_k$ as in (4.126). Using the same argument as in (2), for any $t \in (t_1, t_0)$ we have
\[ u(x, t) \leq \sum_{k=1}^{l'} U_k(x, t) + f(x, t), \quad \forall \ t \in (t_1, t_0), \] (4.128)
where $f(x, t)$ is a bounded function. By (4.127), we have
\[ u(x, t) \leq \frac{1}{l} \sum_{i=1}^{l'} \int_{t_1}^{t} p(x, \xi_k(s), t-s) \, d\mu + f(x, t), \quad \forall \ t \in (t_1, t_0). \] (4.129)
Similarly, we can prove that (4.129) also holds for $t \in (t_0, t_2)$. Therefore, by Lemma 4.6 $u$ satisfies the inequalities (4.4)-(4.5). The theorem is proved.

5 Proof of main theorems

In this section, we prove Theorem 1.2 and Corollary 1.4.

Proof of Theorem 1.2. Suppose that the mean curvature flow (1.1) reaches a singularity at $(x_0, T)$ with $T < +\infty$. Then Corollary 3.6 of [25] implies that for all $t < T$ we have
\[ d(\Sigma_t, x_0) \leq 2\sqrt{T-t}. \] (5.1)

We rescale the flow by
\[ s = -\log(T-t), \quad \tilde{\Sigma}_s = e^s \left( \Sigma_{T-e^{-s}} - x_0 \right). \] (5.2)
such that the flow $\{(\tilde{\Sigma}_s, \tilde{x}(p, s)), -\log T \leq s < +\infty\}$ satisfies the following properties:

1. $\tilde{x}(p, s)$ satisfies the equation
   \[ \left( \frac{\partial \tilde{x}}{\partial s} \right)_{\perp} = -\left( \tilde{H} - \frac{1}{2} \langle \tilde{x}, n \rangle \right); \] (5.3)

2. the mean curvature of $\tilde{\Sigma}_s$ satisfies $|\tilde{H}(p, s)| \leq \Lambda_0$ for some $\Lambda_0 > 0$;

3. $d(\tilde{\Sigma}_s, 0) \leq 2$.

Fix $\tau > 0$. By Theorem 3.1, for any sequence $s_i \to +\infty$ there exists a subsequence, still denoted by $\{s_i\}$, such that the flow $\{\Sigma_{s_i+t}, -\tau < s < \tau\}$ converges smoothly to a self-shrinker with multiplicity one. In other words, taking $c_j = e^{2s_i}$ the flow $\{\tilde{\Sigma}_j, -\tau < s < \tau\}$ where $\tilde{\Sigma}_j := c_j e^{2s_i} (\Sigma_{T-e^{-c_j^{-2}e^{-s}} - x_0})$ converges smoothly to a self-shrinker with multiplicity one as $j \to +\infty$. Consider the corresponding flow
\[ \tilde{t} = -e^{-s}, \quad \Sigma_{\tilde{t}} := \sqrt{-\tilde{t}} \tilde{\Sigma}_j \mid_{-\log(-\tilde{t})} = c_j (\Sigma_{T+c_j^{2}e^{-t}} - x_0). \]

Therefore, for fixed $\tau > 0$ the flow $\{\Sigma_{\tilde{t}}, -e^{-\tau} < \tilde{t} < -e^{-\tau}\}$ converges smoothly to a smooth self-shrinker flow with multiplicity one as $j \to +\infty$. Theorem 1.2 is proved.
Proof of Corollary 1.4. We follow the argument in the proof of Theorem 1.2. Suppose that
\[ \delta_0 := \sup_{\Sigma \times [0,T)} \left( \sqrt{T - t} \cdot |H|(p,t) \right) < +\infty. \] (5.4)

Then the rescaled mean curvature flow (5.2) satisfies \( |\tilde{H}| \leq \delta_0 \). There exists a sequence of times \( s_i \to +\infty \) such that for any fixed \( \tau > 0 \) the flow \( \{\tilde{\Sigma}_{s_i + s}, -\tau < s < \tau\} \) converges smoothly to a self-shrinker \( \Sigma_{\infty} \in \mathcal{C}(2, N, \rho) \) with multiplicity one. Moreover, the mean curvature of the limit self-shrinker satisfies \( \sup_{\Sigma_{\infty}} |H| \leq \delta_0 \). On the other hand, we have

Lemma 5.1. For any \( D, N > 0 \) and any increasing function \( \rho \), there exists a constant \( \delta(D, N, \rho) > 0 \) such that any self-shrinker \( \Sigma \in \mathcal{C}(D, N, \rho) \) with \( |H| \leq \delta \) must be a plane passing through the origin.

Proof of Lemma 5.1. For otherwise, there exists a sequence of non-flat self-shrinkers \( \Sigma_i \in \mathcal{C}(D, N, \rho) \) with \( \sup_{\Sigma_i} |H| \leq \delta_i \to 0 \). By the smooth compactness result of self-shrinkers in [18], we can assume that \( \Sigma_i \) converges smoothly to a self-shrinker \( \Sigma_{\infty} \in \mathcal{C}(D, N, \rho) \) with multiplicity one. Since the convergence is smooth, the limit self-shrinker \( \Sigma_{\infty} \) has zero mean curvature and by Corollary 2.8 of [19] it must be a plane passing through the origin.

For any hypersurface \( \Sigma^n \subset \mathbb{R}^{n+1} \), the F-functional of \( \Sigma \) is defined by Colding-Minicozzi [19]
\[ F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|x|^2}{4(T-t)}} d\mu. \] (5.5)

Moreover, the entropy \( \lambda \) of \( \Sigma \) in [19] is defined by
\[ \lambda(\Sigma) = \sup_{x_0, t_0} F_{x_0, t_0}(\Sigma) = \sup_{x_0, t_0} (4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} d\mu, \] (5.6)

where the supremum is taken over all \( t_0 > 0 \) and \( x_0 \in \mathbb{R}^{n+1} \). By Lemma 7.10 of [19], the entropy \( \lambda \) is achieved by \( (x_0, t_0) = (0, 1) \) for a self-shrinker with polynomial volume growth. Thus, we have
\[ \lambda(\Sigma) = F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|x|^2}{4(T-t)}} d\mu. \] (5.7)

Since \( \Sigma_i \) converges smoothly to a plane \( \Sigma_{\infty} \) with multiplicity one, by (5.7) we have
\[ \lim_{i \to +\infty} \lambda(\Sigma_i) = \lambda(\Sigma_{\infty}) = 1. \]

Thus, for any \( \epsilon > 0 \) and large \( i \) we have \( \lambda(\Sigma_i) < 1 + \epsilon \). By Theorem 1.2 of C. Bao [8] \( \Sigma_i \) must be a plane, or we can use Guang-Zhu’s rigidity result in [32]. Note that for any large \( R \), \( \Sigma_i \cap B_R(0) \) is graphic over \( \Sigma_{\infty} \cap B_R(0) \) for large \( i \). Together with \( \lambda(\Sigma_i) < 1 + \epsilon < 2 \), we have that \( \Sigma_i \) is a plane by Theorem 0.1 of [32]. This contradicts our assumption that \( \Sigma_i \) is non-flat. The lemma is proved.

Therefore, by Lemma 5.1 the limit self-shrinker \( \Sigma_{\infty} \) must be a plane passing through the origin. Consider the Heat kernel function
\[ \Phi_{(x_0, T)}(x, t) = \frac{1}{4\pi(T-t)} e^{-\frac{|x-x_0|^2}{4(T-t)}}, \quad \forall (x, t) \in \Sigma_t \times [0, T). \]
Thus, Huisken’s monotonicity formula (c.f. Theorem 3.1 in [37]) implies that
\[
\Theta(\Sigma, x_0, T) := \lim_{t \to T} \int_{\Sigma_t} \Phi(x_0, t)(x, t) \, d\mu_t = \lim_{s_i \to +\infty} \frac{1}{4\pi} \int_{\tilde{\Sigma}_{s_i}} e^{-\frac{|x|^2}{4}} \, d\tilde{\mu}_{s_i} = 1,
\]
which implies that \((x_0, T)\) is a regular point by Theorem 3.1 of White [60]. Thus, the flow \(\{\Sigma, x(t)\}, 0 \leq t < T\) cannot blow up at \((x_0, T)\). The corollary is proved.

**Appendix A  Krylov-Safonov’s parabolic Harnack inequality**

In this appendix, we include the parabolic Harnack inequality from Krylov-Safonov [48]. First, we introduce some notations. Let \(x = (x^1, x^2, \cdots, x^n) \in \mathbb{R}^n\). Denote
\[
|x| = \left(\sum_{i=1}^{n}(x^i)^2\right)^{\frac{1}{2}}, \quad B_R(x) = \{y \in \mathbb{R}^n \mid |x - y| < R\}, \quad Q(\theta, R) = B_R(0) \times (0, \theta R^2).
\]
Consider the parabolic operator
\[
Lu = -\frac{\partial u}{\partial t} + a^{ij}(x, t)u_{ij} + b^i(x, t)u_i - c(x, t)u,
\]
where the coefficients are measurable and satisfy the conditions
\[
\mu|\xi|^2 \leq a^{ij}(x, t)\xi_i\xi_j \leq \frac{1}{\mu}|\xi|^2, \quad |b(x, t)| \leq \frac{1}{\mu}, \quad 0 \leq c(x, t) \leq \frac{1}{\mu}.
\]
Here \(b(x, t) = (b^1(x, t), \cdots, b^n(x, t))\). Then we have

**Theorem A.1.** (Theorem 1.1 of [48]) Suppose the operator \(L\) in (A.1) satisfies the conditions (A.2)-(A.4). Let \(\theta > 1, R \leq 2, u \in W^{1,2}_{n+1}(Q(\theta, R)), u \geq 0 \text{ in } Q(\theta, R),\) and \(Lu = 0 \text{ on } Q(\theta, R).\) Then there exists a constant \(C,\) depending only on \(\theta, \mu\) and \(n,\) such that
\[
u(0, R^2) \leq C u(x, \theta R^2), \quad \forall x \in B_{\frac{R}{\theta}}(0).
\]
Moreover, when \(\frac{1}{\theta-1}\) and \(\frac{1}{\mu}\) vary within finite bounds, \(C\) also varies within finite bounds.

Note that in our case the equation (3.36) doesn’t satisfy the assumption that \(c(x, t) \geq 0\) in (A.4). Therefore, we cannot use Theorem A.1 directly. The following result shows that the Harnack inequality still works when \(c(x, t)\) is bounded.
Theorem A.2. Let $\theta > 1, R \leq 2$. Suppose that $u(x, t) \in W^{1,2}_{n+1}(Q(\theta, R))$ is a nonnegative solution to the equation
\[
Lu = -\frac{\partial u}{\partial t} + a^{ij}(x, t)u_{ij} + b^i(x, t)u_i + c(x, t)u = 0, \tag{A.6}
\]
where the coefficients $a^{ij}(x, t)$ and $b^i(x, t)$ satisfy (A.2)-(A.3), and $c(x, t)$ satisfies
\[
|c(x, t)| \leq \frac{1}{\mu}, \quad \forall (x, t) \in Q(\theta, R) \tag{A.7}
\]
Then there exists a constant $C$, depending only on $\theta, \mu$ and $n$, such that
\[
u(0, R^2) \leq C\nu(x, \theta R^2), \quad \forall |x| < \frac{1}{2}R. \tag{A.8}
\]

Proof. Since $u(x, t)$ is a solution of (A.6) and $c(x, t)$ satisfies (A.7), the function $v(x, t) = e^{-\frac{1}{\mu}t}u$ satisfies
\[
-\frac{\partial v}{\partial t} + a^{ij}(x, t)v_{ij} + b^i(x, t)v_i + \tilde{c}(x, t) = 0. \tag{A.9}
\]
where
\[
-\frac{2}{\mu} \leq \tilde{c}(x, t) = c(x, t) - \frac{1}{\mu} \leq 0. \tag{A.10}
\]
Applying Theorem (A.1) to the equation (A.9), we have
\[
v(0, R^2) \leq C\nu(x, \theta R^2), \quad \forall |x| < \frac{1}{2}R,
\]
where $C$ depends only on $\theta, \mu$ and $n$. Thus, for any $x \in B_{\frac{R}{2}}(0)$ we have
\[
u(0, R^2) \leq Ce^{-k(\theta-1)R^2}u(x, \theta R^2) \leq \nu(x, \theta R^2),
\]
where $C'$ depends only on $\theta, \mu$ and $n$. Here we used $R \leq 2$ by the assumption. The theorem is proved.

We generalize Theorem A.2 to a general bounded domain in $\mathbb{R}^n$.

Theorem A.3. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Suppose that $u(x, t) \in W^{1,2}_{n+1}(\Omega \times (0, T))$ is a nonnegative solution to the equation
\[
Lu = -\frac{\partial u}{\partial t} + a^{ij}(x, t)u_{ij} + b^i(x, t)u_i + c(x, t)u = 0, \tag{A.11}
\]
where the coefficients $a^{ij}(x, t)$ and $b^i(x, t)$ satisfy (A.2)-(A.3), and $c(x, t)$ satisfies (A.7) for a constant $\mu > 0$. For any $s, t$ satisfying $0 < s < t < T$ and any $x, y \in \Omega$ with the following properties

1. $x$ and $y$ can be connected by a line segment $\gamma$ with the length $|x - y| \leq \ell$;

2. Each point in $\gamma$ has a positive distance at least $\delta > 0$ from the boundary of $\Omega$;

3. $s$ and $t$ satisfy $T_1 \leq t - s \leq T_2$ for some $T_1, T_2 > 0$;

we have
\[
u(y, s) \leq C\nu(x, t), \tag{A.12}
\]
where $C$ depends only on $n, \mu, \min\{s, \delta^2\}, \ell, T_1$ and $T_2$. 

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Proof. Let \( \gamma \) be the line segment with the property (1) and (2) connecting \( x \) and \( y \). We set

\[
p_0 = y, \quad p_N = x, \quad p_i = p_0 + \frac{x - y}{N} i \in \gamma
\]

for any \( 0 \leq i \leq N \). Here we choose \( N \) to be the smallest integer satisfying

\[
N > \max \left\{ \frac{2(t - s)}{s}, \frac{l}{\min \left\{ \sqrt{s}, \delta \right\}} \right\}.
\]

(A.13)

We define

\[
R = \frac{2l}{N}, \quad \theta = 1 + \frac{t - s}{R^2 N}.
\]

(A.14)

We can check that \( R \leq \frac{\delta}{2} \). For any \( s, t \in (0, T) \), we choose \( \{t_i\}_{i=0}^N \) such that \( t_0 = s, t_N = t \) and

\[
t_i - t_{i-1} = \frac{t - s}{N} \quad \text{for all integers} \quad 1 \leq i \leq N.
\]

(A.15)

for all integers \( 1 \leq i \leq N \). Note that (A.13)-(A.15) imply that for any \( 0 \leq i \leq N - 1 \),

\[
t_{i+1} - \theta R^2 \geq s - \theta R^2 = s - R^2 - \frac{t - s}{N} \geq \frac{s}{4} > 0
\]

and

\[
|p_{i+1} - p_i| = \frac{|x - y|}{N} \leq \frac{l}{N} = \frac{R}{2}.
\]

Therefore, for any \( 0 \leq i \leq N - 1 \) we have \( (t_{i+1} - \theta R^2, t_{i+1}) \subset (0, T) \) and \( p_{i+1} \in B_{\frac{R}{2}}(p_i) \). Applying Theorem A.2 on \( B_R(p_i) \times (t_{i+1} - \theta R^2, t_{i+1}) \subset \Omega \times (0, T) \), we have

\[
u(p_i, t_i) \leq C u(p_{i+1}, t_{i+1}),
\]

(A.16)

where \( C \) depends only on \( c, n, \mu \) and \( \frac{1}{\theta - 1} = \frac{R^2 N}{t - s} \). Here we used the fact that \( t_i = (t_{i+1} - \theta R^2) + R^2 \). Therefore,

\[
u(y, s) = u(p_0, t_0) \leq C^N u(p_N, t_N) = C' u(x, t)
\]

(A.17)

where the constant \( C' \) in (A.17) depends only on \( c, n, \mu, \min\{s, \delta^2\}, l, T_1 \) and \( T_2 \). The theorem is proved.

\( \square \)

A direct corollary of Theorem A.3 is the following result.

**Theorem A.4.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Suppose that \( u(x, t) \in W^{1,2}_{n+1}(\Omega \times (0, T)) \) is a nonnegative solution to the equation

\[
Lu = -\frac{\partial u}{\partial t} + a^{ij}(x, t)u_{ij} + b^j(x, t)u_i + c(x, t)u = 0,
\]

(A.18)

where the coefficients \( a^{ij}(x, t) \) and \( b^j(x, t) \) satisfy (A.2)-(A.3), and \( c(x, t) \) satisfies (A.7) for a constant \( \mu > 0 \). Suppose that \( \Omega', \Omega'' \) are subdomains in \( \Omega \) satisfying the following properties:

1. \( \Omega' \subset \Omega'' \subset \Omega \), and \( \Omega'' \) has a positive distance \( \delta > 0 \) from the boundary of \( \Omega \);
2. \( \Omega' \) can be covered by \( k \) balls with radius \( r \), and all balls are contained in \( \Omega'' \).
Then for any \( s, t \) satisfying \( 0 < s < t < T \) and any \( x, y \in \Omega' \), we have
\[
u(y, s) \leq C \nu(x, t),
\] (A.19)
where \( C \) depends only on \( n, \mu, \min\{s, \delta^2\}, t - s, r \) and \( k \).

**Proof.** By the assumption, we can find finite many points \( A = \{q_1, q_2, \ldots, q_k\} \) such that
\[
\Omega' \subset \bigcup_{q_i \in A} B_r(q) \subset \Omega''.
\] (A.20)

For any \( x, y \in \Omega' \), there exists two points in \( A \), which we denote by \( q_1 \) and \( q_2 \), such that \( x \in B_r(q_1) \) and \( y \in B_r(q_2) \). Then \( x \) and \( y \) can be connected by a polygonal chain \( \gamma \), which consists of two line segments \( \overline{qq_1}, \overline{qq_2} \) and a polygon chain with vertices in \( A \) connecting \( q_1 \) and \( q_2 \). Clearly, the number of the vertices of \( \gamma \) is bounded by \( k + 2 \) and the total length of \( \gamma \) is bounded by \( (k + 2)r \). Moreover, by the assumption we have \( \gamma \subset \Omega'' \) and each point in \( \gamma \) has a positive distance at least \( \delta > 0 \) from the boundary of \( \Omega \).

Assume that the polygonal chain \( \gamma \) has consecutive vertices \( \{p_0, p_1, \ldots, p_N\} \) with \( p_0 = y, p_N = x \) and \( 1 \leq N \leq k + 2 \). We apply Theorem A.3 for each line segment \( \overline{p_ip_{i+1}} \) and the interval \( [t_i, t_{i+1}] \), where \( \{t_i\} \) is chosen as in (A.15). Note that
\[
\frac{t - s}{k + 2} \leq t_{i+1} - t_i = \frac{t - s}{N} \leq t - s.
\]
Thus, for any \( 0 \leq i \leq N - 1 \) we have
\[
u(p_i, t_i) \leq C\nu(p_{i+1}, t_{i+1}),
\] (A.21)
where \( C \) depends only on \( c, n, \mu, \min\{s, \delta^2\}, r, k \) and \( t - s \), and (A.21) implies (A.23). This finishes the proof of Theorem A.4.

Theorem A.4 can be generalized to Riemannian manifolds by using the partition of unity. Here we omit the proof since the argument is standard. Note that the constant in (A.23) depends on the geometry of \((M, g)\).

**Theorem A.5.** Let \((M, g)\) be a Riemannian manifold with boundary \( \partial M \) and \( \Omega \subset M \) a bounded domain which doesn’t intersect with \( \partial M \). Suppose that \( u(x, t) \in W^{1,2}_{n+1}(\Omega \times (0, T)) \) is a nonnegative solution to the equation
\[
Lu = \frac{\partial u}{\partial t} + a^{ij}(x, t)\nabla_i \nabla_j u + b^i(x, t)\nabla_i u + c(x, t)u = 0,
\] (A.22)
where the coefficients \( a^{ij}(x, t) \) and \( b^i(x, t) \) satisfy (A.2)-(A.3), and \( c(x, t) \) satisfies (A.7) for a constant \( \mu > 0 \). Suppose that \( \Omega', \Omega'' \) are subdomains in \( \Omega \) satisfying the following properties:

1. \( \Omega' \subset \Omega'' \subset \Omega \), and \( \Omega'' \) has a positive distance \( \delta > 0 \) from the boundary of \( \Omega \);
2. \( \Omega' \) can be covered by \( k \) balls with radius \( r \), and all balls are contained in \( \Omega'' \).

Then for any \( s, t \) satisfying \( 0 < s < t < T \) and any \( x, y \in \Omega' \), we have
\[
u(y, s) \leq C \nu(x, t),
\] (A.23)
where \( C \) depends only on \( c, n, \mu, \min\{s, \delta^2\}, t - s, r, k \) and \((M, g)\).
Appendix B  Li-Yau’s parabolic Harnack inequality

In this appendix, we include Li-Yau’s parabolic Harnack inequality in \cite{49}. Compared with the Harnack inequality in appendix A, Li-Yau’s result gives explicit dependence of the constants on the geometric quantities of the metric. Thus, we can apply Li-Yau’s result to a class of Riemannian manifolds and we obtain uniform bounds of the constants in the Harnack inequality.

**Theorem B.1.** (c.f. Theorem 2.1 of \cite{49}) Let \( M \) be a Riemannian manifold with boundary \( \partial M \). Assume \( p \in M \) and let \( B_{2R}(p) \) be a geodesic ball of radius \( 2R \) centered at \( p \) which does not intersect \( \partial M \). We denote \(-K(2R)\), with \( K(2R) \geq 0 \), to be a lower bound of the Ricci curvature on \( B_{2R}(p) \). Let \( q(x,t) \) be a function defined on \( M \times [0,T] \) which is \( C^2 \) in the \( x \)-variable and \( C^1 \) in the \( t \)-variable. Assume that

\[
\Delta q \leq \theta(2R), \quad |\nabla q| \leq \gamma(2R) \tag{B.1}
\]
on \( B_{2R}(p) \times [0,T] \) for some constants \( \theta(2R) \) and \( \gamma(2R) \). If \( u(x,t) \) is a positive solution of the equation

\[
\left( \Delta - q - \frac{\partial}{\partial t} \right) u(x,t) = 0 \tag{B.2}
\]
on \( M \times (0,T) \), then for any \( \alpha > 1 \), \( 0 < t_1 < t_2 \leq T \), and \( x, y \in B_R(p) \), we have the inequality

\[
u(x,t_1) \leq u(y,t_2)\left( \frac{t_2}{t_1} \right)^\frac{\alpha}{2} e^{A(t_2-t_1) + \rho_{A,R}(x,y,t_2-t_1)}, \tag{B.3}
\]
where

\[
A = C \left( \alpha R^{-1}\sqrt{K} + \alpha^3(\alpha - 1)^{-1}R^{-2} + \gamma^2(\alpha - 1)^1\alpha^{-1} + (\alpha \theta)^1 + \alpha(\alpha - 1)^{-1}K \right) \tag{B.4}
\]
and

\[
\rho_{A,R}(x,y,t_2-t_1) = \inf_{\gamma \in \Gamma(R)} \left( \frac{\alpha}{4(t_2-t_1)} \int_0^1 |\gamma|^2 + (t_2-t_1) \int_0^1 q(\gamma(s),(1-s)t_2+st_1) \, ds \right), \tag{B.5}
\]
with \( \inf \) taken over all paths in \( B_R(p) \) parametrized by \([0,1]\) joining \( y \) to \( x \).

A direct corollary of Theorem B.1 is the following result.

**Theorem B.2.** The same assumptions as in Theorem B.1 on \( M, B_{2R}(p) \) and the function \( q(x,t) \). If \( u(x,t) \) is a positive solution of the equation

\[
\left( \Delta - q - \frac{\partial}{\partial t} \right) u(x,t) = 0 \tag{B.6}
\]
on \( \Omega \times (0,T) \), where \( \Omega \) is a connected open subset of \( B_R(p) \). Let \( \Omega', \Omega'' \) are connected open subsets of \( \Omega \) satisfying the following properties, which we called \((\delta, k, r)\) property:

1. \( \Omega' \subset \Omega'' \subset \Omega \), and \( \Omega'' \) has a positive distance \( \delta > 0 \) from the boundary of \( \Omega \);
2. \( \Omega' \) can be covered by \( k \) geodesic balls with radius \( r \), and all balls are contained in \( \Omega'' \).

Then for any \( 0 < t_1 < t_2 \leq T \) and \( x, y \in \Omega' \), we have the inequality

\[
u(x,t_1) \leq Cu(y,t_2), \tag{B.7}
\]
where \( C \) depends only on \( n, K(2R), \theta(2R), \gamma(2R), t_1, t_2 - t_1, k, \delta \) and \( r \).
Proof. By the assumption on $\Omega', \Omega''$ and $\Omega$, $x$ and $y$ can be connected by a path $\gamma$ in $\Omega''$ with bounded length and every point in $\gamma$ has a distance at least $\delta$ from the boundary of $\Omega$. Thus, the theorem follows directly from Theorem B.1 by choosing $R = \delta$ and $\alpha = 2$. □

In the proof of Lemma 3.21, we need to use Theorem B.2 to a class of surfaces with bounded geometry. In order to show that the constants in the Harnack inequality is uniformly bounded, we have the following result.

**Theorem B.3.** Fix $R > 0$. We assume that

1. $\Sigma^2 \subset \mathbb{R}^3$ is a sequence of complete surfaces which converges smoothly to a complete surface $\Sigma$ in $\mathbb{R}^3$;

2. The Ricci curvature of $\Sigma \cap B_R(0)$ is bounded by a constant $-K$ with $K \geq 0$. Here $B_R(0) \subset \mathbb{R}^3$ denotes the extrinsic ball centered at 0 with radius $R$;

3. $\Omega_i, \Omega_i', \Omega_i''$ are bounded domains in $\Sigma_i \cap B_{\frac{R}{2}}(0)$ with $\Omega_i' \subset \Omega_i'' \subset \Omega_i$, and $\Omega_i, \Omega_i', \Omega_i''$ converges smoothly to $\Omega, \Omega', \Omega''$ with $\Omega' \subset \Omega'' \subset \Omega \subset \Sigma \cap B_{\frac{R}{2}}(0)$ respectively. Here the smooth convergence of $\Omega_i$ to $\Omega$ means that for any $\epsilon > 0$ and sufficiently large $i$, there exists a smooth function $u_i$ on $\Omega$ with $|u_i|_{C^2(\Omega)} \leq \epsilon$ such that $\Omega_i$ can be written as a normal exponential graph of $u_i$ over $\Omega$;

4. $\Omega''$ has a positive geodesic distance $\delta > 0$ from the boundary of $\Omega$;

5. $\Omega'$ can be covered by $k$ geodesic balls with radius $r \in (0, \delta)$, and all balls are contained in $\Omega''$;

6. $q_i(x,t)$ is a function defined on $\Sigma_i \times [0,T]$ which is $C^2$ in the $x$-variable and $C^1$ in the $t$-variable. Assume that

$$\Delta_{g_i} q_i \leq \theta, \quad |\nabla q_i|_{g_i} \leq \theta \quad \text{(B.8)}$$

on $\Omega_i \times [0,T]$ for some constant $\theta$.

If $f_i(x,t)$ are positive functions satisfying

$$\left(\Delta_{g_i} - q_i(x,t) - \frac{\partial}{\partial t}\right)f_i(x,t) = 0 \quad \text{(B.9)}$$

on $\Omega_i \times (0,T)$, where $q_i(x,t) \in C^2(\Sigma_i \times [0,T])$, then for any $0 < t_1 < t_2 \leq T$ and $x, y \in \Omega_i'$, we have the inequality

$$f_i(x,t_1) \leq Cf_i(y,t_2), \quad \text{(B.10)}$$

where $C$ depends only on $n, K, \theta, t_1, t_2 - t_1, k, \delta$ and $r$.

Proof. It suffices to show that $\Omega_i', \Omega_i''$ and $\Omega_i$ satisfy the $(\delta', k', r')$ property of Theorem B.2 with uniform constants $\delta', k'$ and $r'$. By the smooth convergence of $\Omega_i$ to $\Omega$, we define the map $\varphi_i : \Omega \to \Omega_i$ by

$$\varphi_i(x) = x + u_i(x)n(x), \quad \forall x \in \Omega, \quad \text{(B.11)}$$

where $u_i(x)$ is the graph function of $\Omega_i$ over $\Omega$ and $n(x)$ denotes the normal vector of $\Sigma$ at $x$. Note that $\varphi_i(\Omega) = \Omega_i$, and $\varphi_i$ converges in $C^2$ to the identity map on $\Omega$ as $i \to +\infty$. By the assumption (5), there exists $k$ points $\{p_{\alpha}\}_{\alpha=1}^k \subset \Omega'$ and $\epsilon > 0$ such that

$$\Omega' \subset \bigcup_{\alpha=1}^k B_r(p_{\alpha}), \quad B_r(p_{\alpha}) \subset \Omega''_i, \quad \text{(B.12)}$$
where \( \Omega''_{4\epsilon} = \{ x \in \Omega'' \mid d_{\Sigma}(x, \partial \Omega'') \geq 4\epsilon \} \). Therefore, we have

\[
\Omega'_i = \varphi_i(\Omega') \subset \varphi_i \left( \bigcup_{\alpha=1}^{k} \mathcal{B}_r(p_\alpha) \right) = \bigcup_{\alpha=1}^{k} \varphi_i \left( \mathcal{B}_r(p_\alpha) \right).
\]

(B.13)

Since the \( C^l \) norms of \( u_i \) in (B.11) are small, for large \( i \) we have

\[
\varphi_i(\mathcal{B}_r(p_\alpha)) \subset \mathcal{B}_{r+\epsilon}(\varphi_i(p_\alpha)) \subset \Omega''_{r,2\epsilon},
\]

(B.14)

where \( \Omega''_{r,2\epsilon} = \{ x \in \Omega'' \mid d_{\Sigma}(x, \partial \Omega'') \geq 2\epsilon \} \) and \( \mathcal{B}_{r}(p) \) denotes the geodesic ball of \( \Sigma_i \) centered at \( p \) with radius \( r \). Combining (B.13) with (B.14), we have

\[
\Omega_i' \subset \bigcup_{\alpha=1}^{k} \mathcal{B}_{r+\epsilon}(\varphi_i(p_\alpha)) \subset \Omega''_{r,2\epsilon} \subset \Omega''_i.
\]

(B.15)

Therefore, \( \Omega_i' \) can be covered by \( k \) geodesic balls with radius \( r + \epsilon \), and all balls are contained in \( \Omega''_i \). It is clear that \( \Omega''_i \) has a positive geodesic distance \( \frac{\delta}{2} > 0 \) from the boundary of \( \Omega_i \) for large \( i \). Thus, \( \Omega'_i, \Omega''_i \) and \( \Omega_i \) satisfy the \((\frac{\delta}{2}, k, r + \epsilon)\) property and the theorem follows directly from Theorem B.2.

\[
\square
\]

Appendix C  The linearized equation of rescaled mean curvature flow

In this appendix, we follow the calculation in Appendix A of Colding-Minicozzi [21] to show (3.34). See also Appendix A of Colding-Minicozzi [22]. Let \( \Sigma \) be a hypersurface in \( \mathbb{R}^{n+1} \) and \( \Sigma_u \) the graph of a function \( u \) over \( \Sigma \). Then \( \Sigma_u \) is given by

\[
\Sigma_u = \{ x + u(x)n(x) \mid x \in \Sigma \},
\]

(C.1)

where \( n(x) \) denotes the normal vector of \( \Sigma \) at \( x \). We assume that \( |u| \) is small. Let \( e_{n+1} \) be the gradient of the signed distance function to \( \Sigma \) and \( e_{n+1} \) equals \( n \) on \( \Sigma \). We define

\[
\nu_u(p) = \sqrt{\frac{\det g_{ij}^u(p)}{\det g_{ij}(p)}}, \quad w_u(p) = \langle e_{n+1}, n_u \rangle, \quad \eta_u(p) = \langle p + u(p)n(p), n_u \rangle
\]

(C.2)

where \( g_{ij} \) denotes the metric on \( \Sigma \) at \( p \), \( g_{ij}^u \) is the induced metric on \( \Sigma_u \) and \( n_u \) is the normal to \( \Sigma_u \).

**Lemma C.1.** (Lemma A.3 of [27]) There are functions \( w, \nu, \eta \) depending on \( (p, s, y) \in \Sigma \times \mathbb{R} \times T_p \Sigma \) that are smooth for \( |s| \) less than the normal injectivity radius of \( \Sigma \) so that

\[
w_u(p) = w(p, s, y) = \sqrt{1 + |B^{-1}(p, s)(y)|^2}, \quad \nu_u(p) = \nu(p, s, y) = w(p, s, y) \det(B(p, s)), \quad \eta_u(p) = \eta(p, s, y) = \frac{\langle p, n(p) \rangle + s - \langle p, B^{-1}(p, s)(y) \rangle}{w(p, s, y)}
\]

(C.3, C.4, C.5)

where the linear operator \( B(p, s) = Id - sA(p) \). Finally, we have

(1). \( w \) satisfies

\[
w(p, s, 0) = 1, \quad \partial_s w(p, s, 0) = 0, \quad \partial_{y_\alpha} w(p, s, 0) = 0, \quad \partial_{y_\alpha} \partial_{y_\beta} w(p, 0, 0) = \delta_{\alpha\beta}.
\]

(C.6, C.7)
(2). \( \nu \) satisfies
\[
\nu(p, 0, 0) = 1, \quad \partial_s \nu(p, 0, 0) = H(p),
\]
\[
\partial_{p_j} \partial_s \nu(p, 0, 0) = H_j(p), \quad \partial_{y_\alpha} \partial_{y_\beta} \nu(p, 0, 0) = \delta_{\alpha\beta},
\]
\[
\partial_s^2 \nu(p, 0, 0) = H^2(p) - |A|^2(p).
\]

(3). \( \eta \) satisfies
\[
\eta(p, 0, 0) = \langle p, n \rangle, \quad \partial_s \eta(p, 0, 0) = 1,
\]
\[
\partial_{y_\alpha} \eta(p, 0, 0) = -p_\alpha.
\]

(4). Furthermore, we have
\[
\partial_{y_i} \nu(p, 0, 0) = 0, \quad \partial_{p_j} \partial_{y_i} \nu(p, 0, 0) = 0,
\]
\[
\partial_s \partial_{p_j} \partial_{y_i} \nu(p, 0, 0) = 0, \quad \partial_{y_k} \partial_{p_j} \partial_{y_i} \nu(p, 0, 0) = 0.
\]

\textbf{Proof.} Part (1)-(3) and (C.3)-(C.5) follow directly from Lemma A.3 of [21]. It suffices to show Part (4). Following the notations in the proof of Lemma A.3 of [21], we assume that \((p, s)\) is the Fermi coordinates on the normal tubular neighborhood of \(\Sigma\) so that \(s\) measures the signed distance to \(\Sigma\). We define
\[
B(p, s) \equiv (\text{Id} - sA(p)) : T_p\Sigma \to T_p\Sigma.
\]
Let \(B(p, s) = \det(B(p, s))\) and \(J(p, s) = B^{-1}(p, s)\). Then we have
\[
B(p, 0) = 1, \quad \partial_s B(p, 0) = H(p),
\]
\[
\partial_{y_i} B(p, s) = 0, \quad \partial_{p_j} B(p, 0) = -s\partial_{p_j} A|_{s=0} = 0,
\]
\[
\partial_{p_j} B(p, 0) = B(p, 0) \cdot \text{tr}(\partial_{p_j} B(p, 0)) = 0.
\]

Since \(J = B^{-1}\), we have
\[
\partial_{p_j} J B + J \partial_{p_j} B = 0.
\]
This implies that
\[
\partial_{p_j} J(p, 0) = -J(p, 0) \cdot \partial_{p_j} B(p, 0) \cdot J(p, 0) = 0.
\]
Note that by (C.3), \(w\) can be rewritten as
\[
w(p, s, y) = \sqrt{1 + J_{\alpha\beta} J_{\alpha\gamma} y_\beta y_\gamma}.
\]

It follows immediately that
\[
\partial_{y_i} w = \frac{1}{2w} J \ast J \ast y,
\]
\[
\partial_{p_i} w = \frac{1}{w} \partial_{p_i} J \ast J \ast y \ast y,
\]
\[
\partial_s \partial_{y_i} w = -\frac{1}{2w^2} \partial_s w \cdot J \ast J \ast y + \frac{1}{w} \partial_s J \ast J \ast y,
\]
\[
\partial_{p_j} \partial_{y_i} w = -\frac{1}{2w^2} \partial_{p_j} w \cdot J \ast J \ast y + \frac{1}{w} \partial_{p_j} J \ast J \ast y,
\]
where the notation "\(*\)" denotes the multiplication of two matrices. Furthermore, we calculate

\[
\partial_s \partial_p \partial_y \omega = \frac{w^3}{2w^2} \partial_s \omega \partial_p \omega \cdot \frac{1}{w^2} \partial_s \partial_p \omega \cdot J \ast J \ast y - \frac{1}{w^3} \partial_s \partial_p \omega \cdot \frac{1}{w^2} \partial_s \partial_p \omega \cdot J \ast J \ast y
\]

\[
-1 \partial_s \partial_p \omega \cdot \partial_y J \ast J \ast y \ast y - \frac{1}{w^2} \partial_s \partial_p \omega \cdot J \ast J \ast y,
\]

\[
\partial_y \partial_p \partial_y \omega = \frac{w^3}{2w^2} \partial_y \omega \partial_p \omega \cdot \frac{1}{w^2} \partial_y \partial_p \omega \cdot J \ast J \ast y - \frac{1}{w^3} \partial_y \partial_p \omega \cdot \frac{1}{w^2} \partial_y \partial_p \omega \cdot J \ast J \ast y
\]

\[
-1 \partial_y \partial_p \omega \cdot \partial_y J \ast J \ast y \ast y - \frac{1}{w^2} \partial_y \partial_y \partial_p \omega \cdot J \ast J \ast y
\]

\[
+ \frac{1}{w} \partial_y \partial_p J \ast \partial_y J \ast J + \frac{1}{w} \partial_y \partial_y J \ast J.
\]

Combining the above identities with (C.6) (C.7) and (C.19), we have

\[
\partial_y \omega(p, 0, 0) = 0, \quad \partial_p \omega(p, 0, 0) = 0,
\]

\[
(C.21)
\]

\[
\partial_y \partial_y \omega(p, 0, 0) = 0, \quad \partial_p \partial_y \omega(p, 0, 0) = 0,
\]

\[
(C.22)
\]

\[
\partial_s \partial_y \partial_y \omega(p, 0, 0) = 0, \quad \partial_y \partial_p \partial_y \omega(p, 0, 0) = 0.
\]

\[
(C.23)
\]

Moreover, we calculate the derivatives of the function \( \nu(p, s, y) = w(p, s, y)B(p, s) \)

\[
\partial_y \omega = \partial_y \omega B + w \partial_y B,
\]

\[
\partial_p \partial_y \omega = \partial_p \partial_y \omega B + \partial_y \partial_p \omega B + \partial_p w \partial_y \partial_y B + w \partial_p \partial_y \omega B,
\]

\[
\partial_y \partial_y \partial_y \omega = \partial_y \partial_y \partial_y \omega B + \partial_p \partial_y \partial_y \omega B + \partial_y \partial_y \partial_p \omega B + \partial_y \partial_y \partial_y \partial_y B + \partial_y \partial_y \partial_y \omega B + \partial_y \partial_y \partial_y \partial_y B + \partial_y \partial_y \partial_y \partial_y B + \partial_y \partial_y \partial_y \partial_y B + \partial_y \partial_y \partial_y \partial_y B.
\]

Combining this with (C.17)–(C.18), (C.21)–(C.23), we have (C.13)–(C.14). The lemma is proved.

\[
\square
\]

We have the following expression for the mean curvature of \( \Sigma_u \).

**Lemma C.2.** (Corollary A.30 of 8214) The mean curvature \( H_u \) of \( \Sigma_u \) is given by

\[
H_u(p) = \frac{w}{v} \left( \partial_y \omega - \partial_y \partial_y \omega - (\partial_y \partial_y \partial_y \omega) u_{\alpha}(p) - (\partial_y \partial_y \partial_y \partial_y \omega) u_{\alpha\beta}(p) \right),
\]

where \( w, \nu \) and their derivatives are all evaluated at \( (p, u(p), \nabla u(p)) \).

Combining Lemma [C.1] with Lemma [C.2] we can show (3.34).

**Lemma C.3.** The function \( u_i = u_i^+ - u_i^- \) satisfies the following parabolic equations on \( \Omega_{\epsilon,R}(I) \times I \)

\[
\partial u_i \partial = \Delta u_i - \frac{1}{2} (x, \nabla u_i) + |A|^2 u_i + \frac{u_i}{2} + a_{pq} u_{i,pq} + b_p u_{i,p} + c_i u_i,
\]

where \( \Delta \) denotes the Laplacian operator on \( \Sigma_\infty \) with respect to the induced metric, and the coefficients \( a_{pq}, b_p \) and \( c_i \) are small and tend to zero as \( u_i^+ \) and \( u_i^- \) tend to zero.
Proof. We divide the proof into several steps.

Step 1. We calculate the difference of the mean curvature of $\Sigma_{u_1^+}$ and $\Sigma_{u_i^-}$. Let $u = u_i^+ - u_i^-$ and $\tilde{u}_\tau = u_i^+ + \tau u$ for $\tau \in [0, 1]$. Thus, we have $\tilde{u}_0 = u_i^-$ and $\tilde{u}_1 = u_i^+$. Note that

$$H_{u_1^+}(p) - H_{u_i^-}(p) = \int_0^1 \partial_\tau(H_{\tilde{u}_\tau}(p)) d\tau. \quad (C.26)$$

For any function $f(p, s, y)$, we calculate the derivative with respect to $\tau$

$$\partial_\tau(f(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau)) = (\partial_s f)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) \cdot u + (\partial_y f)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) \cdot u_\alpha, \quad (C.27)$$

where $u_\alpha = \partial_{x_\alpha} u$. Therefore, we have

$$\partial_\tau(\partial_s \nu) = \partial_s^2 \nu \cdot u + \partial_{y_1} \partial_s \nu \cdot u_i,$$

$$\partial_\tau(\partial_{p_\alpha} \partial_{y_\alpha} \nu) = \partial_s \partial_{p_\alpha} \partial_{y_\alpha} \nu \cdot u + \partial_{y_1} \partial_{p_\alpha} \partial_{y_\alpha} \nu \cdot u_i,$$

$$\partial_\tau\left(\left(\partial_s \partial_{y_\alpha} \nu \tilde{u}_{\tau,\alpha}\right)\right) = (\partial_s \partial_{y_\alpha} \nu) u_\alpha + (\partial_s^2 \partial_{y_\alpha} \nu) u \tilde{u}_{\tau,\alpha} + (\partial_{y_1} \partial_{y_\alpha} \nu) u_i \tilde{u}_{\tau,\alpha},$$

$$\partial_\tau\left(\left(\partial_{y_\beta} \partial_{y_\alpha} \nu \tilde{u}_{\tau,\alpha\beta}\right)\right) = (\partial_{y_\beta} \partial_{y_\alpha} \nu) u_\alpha + (\partial_{y_\beta}^2 \partial_{y_\alpha} \nu) u \tilde{u}_{\tau,\alpha\beta} + (\partial_{y_1} \partial_{y_\beta} \partial_{y_\alpha} \nu) u_i \tilde{u}_{\tau,\alpha\beta},$$

where $\tilde{u}_{\tau,\alpha} = \partial_{x_\alpha} \tilde{u}$ and $\tilde{u}_{\tau,\alpha\beta} = \partial_{x_\alpha} \partial_{x_\beta} \tilde{u}$. By Lemma [C.2] we have

$$\partial_\tau(H_{\tilde{u}_\tau}(p))$$

$$= \left(\partial_s \left(\frac{w}{\nu}\right) u + \partial_{y_1} \left(\frac{w}{\nu}\right) u_i\right) \cdot \left(\partial_s \nu - \partial_{y_1} \partial_{y_\alpha} \nu \tilde{u}_{\tau,\alpha} - (\partial_{y_\beta} \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha\beta}\right)$$

$$+ \left(\frac{w}{\nu}\right) \cdot \partial_\tau\left(\partial_s \nu - \partial_{p_\alpha} \partial_{y_\alpha} \nu - (\partial_s \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha} - (\partial_{y_\beta} \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha\beta}\right)$$

$$= E u + F_\alpha u_\alpha + G_{\alpha\beta} u_{\alpha\beta}, \quad (C.28)$$

where $E, F$ and $G$ are given by

$$E(p, \tilde{u}) = \partial_s \left(\frac{w}{\nu}\right) \left(\partial_s \nu - \partial_{p_\alpha} \partial_{y_\alpha} \nu - (\partial_s \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha} - (\partial_{y_\beta} \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha\beta}\right)$$

$$\quad + \left(\frac{w}{\nu}\right) \cdot \left(\partial_s^2 \nu - \partial_{y_1} \partial_{y_\alpha} \nu - (\partial_s \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha} - (\partial_{y_\beta} \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha\beta}\right), \quad (C.29)$$

$$F_{\gamma}(p, \tilde{u}) = \partial_{y_\gamma} \left(\frac{w}{\nu}\right) \left(\partial_s \nu - \partial_{p_\alpha} \partial_{y_\alpha} \nu - (\partial_s \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha} - (\partial_{y_\beta} \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha\beta}\right)$$

$$\quad + \left(\frac{w}{\nu}\right) \left(\partial_{y_\gamma} \partial_{y_\alpha} \nu - \partial_{y_1} \partial_{y_\alpha} \nu - (\partial_s \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha} - (\partial_{y_\beta} \partial_{y_\alpha} \nu) \tilde{u}_{\tau,\alpha\beta}\right)$$

$$\quad - \partial_{y_\gamma} \partial_{y_\alpha} \partial_{y_\alpha} \nu \cdot \tilde{u}_{\tau,\alpha\beta} \quad (C.30)$$

$$G_{\alpha\beta}(p, \tilde{u}) = - \left(\frac{w}{\nu}\right) \cdot \partial_{y_\beta} \partial_{y_\alpha} \nu. \quad (C.31)$$

In view of \[C.29-31\], we define the functions depending on $(p, s, y, Q) \in \Sigma \times \mathbb{R} \times T_p(\Sigma) \times \Sigma$.\]
\(GL(2, \mathbb{R})\) such that

\[
E(p, s, y, Q) = \partial_s \left( \frac{w}{\nu} \right) (p, s, y) \left( \partial_s \nu - \partial_p \partial_y \nu - (\partial_s \partial_y \nu) y_\alpha - (\partial_y \partial_y \nu) Q_{\alpha\beta} \right) + \left( \frac{w}{\nu} \right) (p, s, y) \cdot \left( \partial_s^2 \nu - \partial_s \partial_p \partial_y \nu - (\partial_s^2 \partial_y \nu) y_\alpha - \partial_s \partial_y \partial_y \nu \cdot Q_{\alpha\beta} \right),
\]

\[
F_\gamma(p, s, y, Q) = \partial_{y_\gamma} \left( \frac{w}{\nu} \right) (p, s, y) \cdot \left( \partial_{y_\gamma} \nu - \partial_p \partial_y \nu - (\partial_s \partial_y \nu) y_\alpha - (\partial_y \partial_y \nu) Q_{\alpha\beta} \right) + \left( \frac{w}{\nu} \right) \left( \partial_{y_\gamma} \partial_s \nu - \partial_{y_\gamma} \partial_p \partial_y \nu - \partial_s \partial_y \nu - (\partial_s \partial_y \partial_y \nu) y_i \right) - \partial_{y_\gamma} \partial_y \partial_y \nu \cdot Q_{\alpha\beta},
\]

\[
G_{\alpha\beta}(p, s, y) = - \left( \frac{w}{\nu} \right) (p, s, y) \cdot \partial_{y_\beta} \partial_y \nu.
\]

Let \(\hat{u}_\lambda = \lambda \tilde{u}_r\) for \(\lambda \in [0, 1]\). Then we have

\[
E(p, u_r) = E(p, 0) + \int_0^1 \partial_\lambda (E(p, \hat{u}_\lambda)) \, d\lambda,
\]  
(C.32)

\[
F_\gamma(p, u_r) = F_\gamma(p, 0) + \int_0^1 \partial_\lambda (F_\gamma(p, \hat{u}_\lambda)) \, d\lambda,
\]  
(C.33)

\[
G_{\alpha\beta}(p, u_r) = G_{\alpha\beta}(p, 0) + \int_0^1 \partial_\lambda (G_{\alpha\beta}(p, \hat{u}_\lambda)) \, d\lambda.
\]  
(C.34)

Note that

\[
\partial_\lambda (E(p, \hat{u}_\lambda)) = \left( \partial_s E \right) \cdot \tilde{u}_r + \left( \partial_y E \right) \cdot \tilde{u}_{r,i} + \left( \partial_{Q_{\alpha\beta}} E \right) \cdot \tilde{u}_{r,\alpha\beta},
\]  
(C.35)

\[
\partial_\lambda (F_\gamma(p, \hat{u}_\lambda)) = \left( \partial_y F_\gamma \right) \cdot \tilde{u}_r + \left( \partial_{y_r} F_\gamma \right) \cdot \tilde{u}_{r,i} + \left( \partial_{Q_{\alpha\beta}} F_\gamma \right) \cdot \tilde{u}_{r,\alpha\beta},
\]  
(C.36)

\[
\partial_\lambda (G_{\alpha\beta}(p, \hat{u}_\lambda)) = \left( \partial_s G_{\alpha\beta} \right) \cdot \tilde{u}_r + \left( \partial_y G_{\alpha\beta} \right) \cdot \tilde{u}_{r,i},
\]  
(C.37)

where the right-hand sides of (C.35)-(C.37) are evaluated at \((p, s, y, Q) = (p, \hat{u}_\lambda, \nabla \hat{u}_\lambda, \nabla^2 \hat{u}_\lambda)\). By Lemma [C.1] we have

\[
E(p, 0) = -|A|^2,  
\]  
(C.38)

\[
F_\gamma(p, 0) = 0,  
\]  
(C.39)

\[
G_{\alpha\beta}(p, 0) = -\delta_{\alpha\beta}.  
\]  
(C.40)

Combining (C.32)-(C.35) with (C.38), we have

\[
E(p, u_r) = -|A|^2 + \tilde{u}_r \int_0^1 \left( \partial_s E \right)(p, \tilde{u}_r, \nabla \tilde{u}_r, \nabla^2 \tilde{u}_r) \, d\lambda
\]

\[+ \tilde{u}_{r,i} \int_0^1 \left( \partial_y E \right)(p, \tilde{u}_r, \nabla \tilde{u}_r, \nabla^2 \tilde{u}_r) \, d\lambda
\]

\[+ \tilde{u}_{r,\alpha\beta} \int_0^1 \left( \partial_{Q_{\alpha\beta}} E \right)(p, \tilde{u}_r, \nabla \tilde{u}_r, \nabla^2 \tilde{u}_r) \, d\lambda.
\]  
(C.41)
Similar, we have

\[ F_\gamma(p, u_\tau) = \tilde{u}_\tau \int_0^1 (\partial_s F_\gamma)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau, \nabla^2 \tilde{u}_\tau) \, d\lambda + \tilde{u}_\tau, i \int_0^1 (\partial_y F_\gamma)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau, \nabla^2 \tilde{u}_\tau) \, d\lambda + \tilde{u}_\tau, \alpha \beta \int_0^1 (\partial_{Q_{\alpha\beta}} F_\gamma)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau, \nabla^2 \tilde{u}_\tau) \, d\lambda \]  

(C.42)

and

\[ G_{\alpha\beta}(p, u_\tau) = -\delta_{\alpha\beta} + \tilde{u}_\tau \int_0^1 (\partial_s G_{\alpha\beta})(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) \, d\lambda + \tilde{u}_\tau, i \int_0^1 (\partial_y G_{\alpha\beta})(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) \, d\lambda \]  

(C.43)

Combining (C.41)-(C.43), (C.26) with (C.28), we have

\[ H_{u_i^+}(p) - H_{u_i^-}(p) = \int_0^1 \partial_\tau (H_{u_i}(p)) \, d\tau = -|\lambda|^2 u - \Delta u + a_{1}^{\alpha\beta} u_{\alpha\beta} + b_1^i u_{\gamma} + c_1 u, \]

(C.44)

where the coefficients \(a_{1}^{\alpha\beta}, b_1^i\) and \(c_1\) are small, and tend to zero as \(u_i^+\) and \(u_i^-\) tend to zero.

**Step 2.** We calculate the difference of \(\eta_{u_i^+}\) and \(\eta_{u_i^-}\). Note that

\[ \eta_{u_i^+}(p) = \eta_{u_i^-}(p) + \int_0^1 \partial_\tau (\eta_{\tilde{u}_\tau}(p)) \, d\tau. \]  

(C.45)

By (C.27), we have

\[ \partial_\tau (\eta_{\tilde{u}_\tau}(p)) = (\partial_s \eta)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) \cdot u + (\partial_y \eta)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) \cdot u_{\alpha}, \]  

(C.46)

where the function \(\eta\) of the right-hand side is defined by (C.5). Let \(\tilde{u}_\lambda = \lambda \tilde{u}_\tau\) as in Step 1. Then we have

\[ (\partial_s \eta)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) = (\partial_s \eta)(p, 0, 0) + \tilde{u}_\tau \int_0^1 (\partial_s^2 \eta)(p, \tilde{u}_\lambda, \nabla \tilde{u}_\lambda) \, d\lambda + \tilde{u}_\tau, \alpha \int_0^1 (\partial_{\tilde{y}_\alpha} \partial_s \eta)(p, \tilde{u}_\lambda, \nabla \tilde{u}_\lambda) \, d\lambda \]  

(C.47)

Similarly, we have

\[ (\partial_y \eta)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) = (\partial_y \eta)(p, 0, 0) + \tilde{u}_\tau \int_0^1 (\partial_s \partial_y \eta)(p, \tilde{u}_\lambda, \nabla \tilde{u}_\lambda), d\lambda + \tilde{u}_\tau, \beta \int_0^1 (\partial_{\tilde{y}_\beta} \partial_y \eta)(p, \tilde{u}_\lambda, \nabla \tilde{u}_\lambda) \, d\lambda. \]  

(C.48)

Combining (C.45)-(C.48) with Part (3) of Lemma C.1, we have

\[ \eta_{u_i^+}(p) = \eta_{u_i^-}(p) + u - \langle p, \nabla u \rangle + b_{2}^i u_{i} + c_2 u, \]  

(C.49)
where $c_2$ and $b_2^i$ are small and tend to zero as $u_i^+$ and $u_i^-$ tend to zero.

**Step 3.** We calculate the difference of $\phi_{u_i^+}(p)$ and $\phi_{u_i^-}(p)$ where $\phi_u = H_u - \frac{1}{2} \langle x_u, n_u \rangle$. Combining (C.49) with (C.44), we have

$$\phi_{u_i^+}(p) - \phi_{u_i^-}(p) = -Lu + a_{\alpha\beta}^3 u_{\alpha\beta} + b_{\beta}^3 u_{\gamma} + c_3 u,$$

where $a_{\alpha\beta}^3$, $b_{\beta}^3$ and $c_3$ are small and tend to zero as $u_i^+$ and $u_i^-$ tend to zero. Note that

$$(\partial_\tau x_{u_i^+})^\perp = \langle \partial_\tau x_{u_i^+}, n_{u_i^+} \rangle = \partial_\tau u_{i_+}^+ \langle n, n_{u_i^+} \rangle = \partial_\tau u_{i_+}^+ w_{u_i^+},$$

where $w_{u_i^+}$ is defined by (C.4), and $x_{u_i^-}$ satisfies a similar equation as (C.51). Moreover, we have

$$\partial_\tau u_{i_+}^+ w_{u_i^+} - \partial_\tau u_{i_-}^- w_{u_i^+} = \int_0^1 \partial_\tau (\partial_\tau \tilde{u}_\tau w_{\tilde{u}_\tau}) d\tau.$$

As in (C.27), we have

$$\partial_\tau (\partial_\tau \tilde{u}_\tau w_{\tilde{u}_\tau}) = \partial_\tau u w_{\tilde{u}_\tau} + \partial_\tau \tilde{u}_\tau \partial_\tau w_{\tilde{u}_\tau} = \partial_\tau u w_{\tilde{u}_\tau} + \partial_\tau \tilde{u}_\tau (\langle \partial_\tau w \rangle(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) \cdot u + (\partial_{\gamma^i} w)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) \cdot u_{\gamma^i} \rangle).$$

Since $w(p, 0, 0) = 1$ by (C.6), we have

$$w_{\tilde{u}_\tau}(p) = 1 + \int_0^1 (\partial_\lambda w_{\tilde{u}_\lambda})(p) d\lambda = 1 + \tilde{u}_\tau \int_0^1 (\partial_\lambda w)(p, \tilde{u}_\tau, \nabla \tilde{u}_\tau) d\lambda + \tilde{u}_{\gamma^i} \int_0^1 (\partial_{\gamma^i} w)(p, \tilde{u}_\lambda, \nabla \tilde{u}_\lambda) d\lambda.$$

Combining the above identities, we have

$$\partial_\tau u_{i_+}^+ w_{u_i^+} - \partial_\tau u_{i_-}^- w_{u_i^+} = \partial_\tau u (1 + b_4^i \tilde{u}_{\gamma^i} + c_4 u_{\gamma^i}) + b_5^i u_{\gamma^i} + c_5 u,$$

where $c_4, c_5, b_4^i$ and $b_5^i$ are small and tend to zero as $u_i^+$ and $u_i^-$ tend to zero. Combining (C.54) (C.50) with the equation of rescaled mean curvature flow, we have

$$\frac{\partial u}{\partial t} = \frac{1}{1 + b_4^i \tilde{u}_{\gamma^i} + c_4 u_{\gamma^i}} \left( \partial_\tau u_{i_+}^+ w_{u_i^+} - \partial_\tau u_{i_-}^- w_{u_i^+} - b_5^i u_{\gamma^i} - c_5 u \right)$$

$$= \frac{1}{1 + b_4^i \tilde{u}_{\gamma^i} + c_4 u_{\gamma^i}} \left( Lu + a_{\alpha\beta}^6 u_{\alpha\beta} + b_6^i u_{\gamma^i} + c_6 u \right)$$

$$= Lu + a_{\alpha\beta}^7 u_{\alpha\beta} + b_7^i u_{\gamma^i} + c_7 u,$$

where $a_{\alpha\beta}^6$, $b_6^i$, $c_6$, $a_{\alpha\beta}^7$, $b_7^i$ and $c_7$ are small and tend to zero as $u_i^+$ and $u_i^-$ tend to zero. The lemma is proved. \hfill \square
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