Differentiation on spaces of triangulations and optimized triangulations.

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Abstract. We describe a smooth structure, called Frölicher space, on CW complexes and spaces of triangulations. This structure enables differential methods for e.g. minimization of functionnals. As an application, we exhibit how an optimized triangulation can be obtained in order to get an optimized $H^1$ approximation in a prescribed a class of triangulations.

Introduction

1. Frölicher structures on CW complexes

1.1. Frölicher spaces

Definition 1.1

• A Frölicher space is a triple $(X, F, C)$ such that
  - $C$ is a set of paths $\mathbb{R} \to X$,
  - A function $f : X \to \mathbb{R}$ is in $F$ if and only if for any $c \in C$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$;
  - A path $c : \mathbb{R} \to X$ is in $C$ (i.e. is a contour) if and only if for any $f \in F$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.

• Let $(X, F, C)$ et $(X', F', C')$ be two Frölicher spaces, a map $f : X \to X'$ is differentiable (=smooth) if and only if $F' \circ f \circ C \subseteq C^\infty(\mathbb{R}, \mathbb{R})$.

Any family of maps $F_g$ from $X$ to $\mathbb{R}$ generate a Frölicher structure $(X, F, C)$, setting [6, 7]:

- $C(F_g) = \{ c : \mathbb{R} \to X \text{ such that } F_g \circ c \subseteq C^\infty(\mathbb{R}, \mathbb{R}) \}$
- $F(C(F_g)) = \{ f : X \to \mathbb{R} \text{ such that } f \circ C \subseteq C^\infty(\mathbb{R}, \mathbb{R}) \}$

A Frölicher space carries a natural topology, which is the pull-back topology of $\mathbb{R}$ via $F$, see e.g. [1]. In the case of a finite dimensional differentiable manifold, the underlying topology of the Frölicher structure is the same as the manifold topology. In the infinite dimensional case, these two topologies differ very often.

The classical properties of a tangent space are not automatically checked in the case of Frölicher spaces following [3]. There are two possible tangent spaces:

- the internal tangent space, defined by the derivatives of smooth paths,
- the external tangent space, made of derivations on $\mathbb{R}$--valued smooth maps.

The internal tangent space is not necessarily a vector space, were as the external tangent space is. For example, consider the diffeological subspace of $\mathbb{R}^2$ made of the two lines $y = x$ and $y = -x$. They cross at the origin, the external tangent space is of dimension 2, the internal tangent space is made of two directions that cannot be combined by addition. This is why some authors sometimes call the internal tangent space by “tangent cone”.

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Proposition 1.2 Let $X$ be a Frölicher space equipped with an equivalence relation $\mathcal{R}$. The quotient map $\pi : X \to X/\mathcal{R}$ defines by push-forward a diffeology on $X/\mathcal{R}$.

We can even state the same results in the case of infinite products, in a very trivial way by taking the cartesian products of the contours. Let us now give the description of what happens for projective limits of Frölicher spaces.

1.2. Topological gluing of Frölicher spaces
Let us assume that $X$ is a topological space, and that there is a collection $\{(X_i, \mathcal{F}_i, C_i)\}_{i \in I}$ of Frölicher spaces, together with continuous maps $f_i : X_i \to X$. Then we can define a Frölicher structures on $X$ setting

$$F_{I,0} = \{ f \in C^0(X, \mathbb{R}) | \forall i \in I, f \circ f_i \circ C_i \subset C^\infty(\mathbb{R}, \mathbb{R}) \},$$

wa define $C_I$ the contours generated by the family $F_{I,0}$, and $F_I = F(C_I)$.

Example: Frölicher structure on CW complexes.
A CW-complex is a topological space built by induction:
- 1st step: gluing a family of intervals of the type $[0; 1]$ along their border, in order to have a graph. We get a 1–CW complex.
- Assume that we have obtained a $n$–CW complex $X_n$. We glue topologicaly some $(n + 1)$ disks

$$D_{n+1} = \{ x \in \mathbb{R}^{n+1} | ||x|| \leq 1 \}$$

identifying the plots of the border $S^n$ with some plots of $X_n$.

Since at each step we glue together some Frölicher space, the obtained CW complex is a Frölicher space.

1.3. Frölicher structure of a triangulation
Let $M$ be a smooth manifold for dimension $n$. Let

$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_+ | x_0 + \ldots + x_n = 1 \}$$

be the $n$– simplex. a triangulation of $M$ is a family $\sigma = (\sigma_i)_{i \in I}$ where $I \subset \mathbb{N}$ is a set of indexes, finite or infinite, each $\sigma_i$ is a smooth map $\Delta_n \to M$, and such that:

(i) $\forall i \in I, \sigma_i$ is an embedding.
(ii) $\bigcup_{i \in I} \sigma_i(\Delta_n) = M$. (open covering)
(iii) $\forall (i,j) \in I^2, \sigma_i(\Delta_n) \cap \sigma_j(\Delta_n) \subset \sigma_i(\partial \Delta_n) \cap \sigma_j(\partial \Delta_n)$. (intersection along the borders)
(iv) $\forall (i,j) \in I^2$ such that $D_{i,j} = \sigma_i(\Delta_n) \cap \sigma_j(\Delta_n) \neq \emptyset$, for each $(n – 1)$-face $F$ of $D_{i,j}$, the “transition maps” $\sigma_j^{-1} \circ \sigma_i : \sigma_i^{-1}(F) \to \sigma_j^{-1}(F)$ are affine maps.

Under these (well-known) conditions, we can apply section 1.2, and equip the triangulated manifold $(M, \sigma)$ with a Frölicher structure $(\mathcal{F}, \mathcal{C})$, generated by the smooth maps $\sigma_i$. The following result is obtained from the construction of $\mathcal{F}$ and $\mathcal{C}$:

Theorem 1.3 The inclusion $(M, \mathcal{F}, \mathcal{C}) \to M$ is smooth.
2. Smooth structures on the space of triangulations

2.1. The Frölicher structure

Now, let us fix the set of indexes $I$ and fix a so-called model triangulation $\sigma$. We note by $\mathcal{T}_\sigma$ the set of triangulations $\sigma'$ of $M$ such that the domains $D_{i,j} = \sigma_i(\Delta_n) \cap \sigma_j(\Delta_n)$ and $D'_{i,j} = \sigma'_i(\Delta_n) \cap \sigma'_j(\Delta_n)$ are diffeomorphic (in the Frölicher category). Based on the properties described in [6], since $\mathcal{T}_\sigma \subset C^\infty(\Delta_n, M)^I$, we can equip $\mathcal{T}_\sigma$ with the trace Frölicher structure, in other words, the Frölicher structure on $\mathcal{T}_\sigma$ whose generating family of contours $\mathcal{C}$ are the contours in $C^\infty(\Delta_n, M)^I$ which lie in $\mathcal{T}_\sigma$. In practice, as we shall see below, we often consider a subspace of $\mathcal{T}_\sigma$.

2.2. Application: optimized approximations

We wish here to describe applications to optimization of $H^1$—approximation of functions and differential forms, in the sense of [9], which is commonly used in numerical analysis. $M$ is assumed Riemannian. In the two examples, we consider the Sobolev space $H^1(M, \mathbb{R})$ or $H^1(I^\ast (M, \mathbb{R}))$, both equipped with self-adjoint positive injective pseudo-differential operator $A$ of order 2. With this setting, it is well-known that $(A, \cdot)_{L^2}$ is a $H^1$—scalar product, where $(\cdot, \cdot)_{L^2}$ is the $L^2$—scalar product. For example, we can assume that $A = \Delta$ is the Laplace-Beltrami operator on $H^1(M, \mathbb{R})$, or the Hodge Laplacian on $H^1(\Omega^\ast (M, \mathbb{R}))$. These approximations are the base of the finite elements method.

2.2.1. Approximation of functions in $H^1(M, \mathbb{R})$

For a fixed map $f \in C^0(M, \mathbb{R})$, the approximation $f_{\sigma'}$ of $f$ with respect to the triangulation $\sigma' \in \mathcal{T}_\sigma$ is given by the following constraints:

- $f_{\sigma'} = f$ at the 0-vertex of the triangulation
- $\forall \sigma' \in \sigma$, $f_{\sigma'} \circ \sigma_i$ is an affine map $\Delta_n \to \mathbb{R}$.

**Definition 2.1** An optimized $\mathcal{T}_\sigma$ triangulation $\sigma_{op}(f)$ is a triangulation that minimize

$$\Phi : \sigma' \mapsto (A(f_{\sigma'} - f), f_{\sigma'} - f)_{L^2}.$$  

This is a smooth map, and the condition $D\Phi = 0$ is necessary at the minimizing point $\sigma_{op}$.

2.2.2. Approximation of differential forms in $H^1(\Omega^\ast (M, \mathbb{R}))$

Let $\alpha \in \Omega^k(M, \mathbb{R})$. Following [9], for any triangulation $\sigma' \in \mathcal{T}_\sigma$, there exists a discretization of $\alpha$ with respect to $\sigma'$ which can be realized as a $H^1$-differential form. Then, one can get, the same way, an optimized triangulation $\sigma_{op}$.

3. Optimized triangulations in $H^1$ approximation

If one considers the whole space $\mathcal{T}_\sigma$, one quickly see that the minimization functional furnished only a reparametrization of the function $f$.

Let us consider the following example: let $f(x) = \frac{x + x^2}{2}$ defined on $[0; 1]$, and let us fix the model triangulation as the trivial parametrization $[0; 1] \to [0; 1]$. An optimal triangulation $\sigma_{op} : [0; 1] \to [0; 1]$ is then a smooth map such that

$$\sigma_{op} = f(x).$$

This shows that, if we wish one minimize $\Phi$ on $\mathcal{T}_\sigma$,

- we have got here an optimized triangulation $\sigma_{op} = f(x)$, which is a toy example of what can happen when, for example, the first derivative of $f$ is Lipschitz and when $M$ is a compact domain.
- defining an optimal triangulation leads to choose a function $\sigma_{op}$ in a too large class of functions.

In the finite elements method, finding a base triangulation is not the main feature, and one needs to get quickly a “not so bad” triangulation. This is why one usually considers, when $M$ is a bounded domain of an Euclidian space, the space of affine triangulations:

$$\mathcal{A}ff T_\sigma = \{ \sigma' \in T_\sigma | \forall i, \sigma'_i \text{ is affine} \}.$$

4. Conclusion and perspectives

Under the lights of this approach, which was not fully justified till the definition of the Frölicher structure of a triangulation in [8] and the remark that the space of triangulations could be equipped with a smooth structure in the present communication, we can sketch two promising directions:

- the systematic review of existing procedures for choosing a triangulation, which can be very sophisticated, but often justified filling technical gaps by intuition. The present setting may furnish a technical tool to fully justify existing methods.

- the exploration of the variation of functional integrals in the space of connections, which are often defined via cylindrical integrals over $H^1$ approximations with respect to a sequence of triangulations.

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