Profinite groups and the fixed points of coprime automorphisms

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Abstract. The main result of the paper is the following theorem. Let $q$ be a prime and $A$ an elementary abelian group of order $q^3$. Suppose that $A$ acts coprimely on a profinite group $G$ and assume that $C_G(a)$ is locally nilpotent for each $a \in A^\#$. Then the group $G$ is locally nilpotent.

1. Introduction

Let $A$ be a finite group acting on a finite group $G$. Many well-known results show that the structure of the centralizer $C_G(A)$ (the fixed-point subgroup) of $A$ has influence over the structure of $G$. The influence is especially strong if $(|A|, |G|) = 1$, that is, the action of $A$ on $G$ is coprime. Let $A^\#$ denote the set of non-identity elements of $A$. The following theorem was proved in [12].

Theorem 1.1. Let $q$ be a prime and $A$ an elementary abelian $q$-group of order at least $q^3$. Suppose that $A$ acts coprimely on a finite group $G$ and assume that $C_G(a)$ is nilpotent for each $a \in A^\#$. Then $G$ is nilpotent.

There are well-known examples that show that the above theorem fails if the order of $A$ is $q^2$. Indeed, let $p$ and $r$ be odd primes and $H$ and $K$ the groups of order $p$ and $r$ respectively. Denote by $A = \langle a_1, a_2 \rangle$ the noncyclic group of order four with generators $a_1, a_2$ and by $Y$ the semidirect product of $K$ by $A$ such that $a_1$ acts on $K$ trivially and $a_2$ takes every element of $K$ to its inverse. Let $B$ be the base group of the wreath product $H \wr Y$ and note that $[B, a_1]$ is normal in $H \wr Y$. Set

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$G = [B, a_1]K$. The group $G$ is naturally acted on by $A$ and $C_G(A) = 1$. Therefore $C_G(a)$ is abelian for each $a \in A^\#$. But, of course, $G$ is not nilpotent.

In [11] the situation of Theorem [1.1] was studied in greater detail and the following result was obtained.

**Theorem 1.2.** Let $q$ be a prime and $A$ an elementary abelian $q$-group of order at least $q^3$. Suppose that $A$ acts coprimely on a finite group $G$ and assume that $C_G(a)$ is nilpotent of class at most $c$ for each $a \in A^\#$. Then $G$ is nilpotent and the class of $G$ is bounded by a function depending only on $q$ and $c$.

Of course, the above results have a bearing on profinite groups. By an automorphism of a profinite group we always mean a continuous automorphism. A group $A$ of automorphisms of a profinite group $G$ is coprime if $A$ has finite order while $G$ is an inverse limit of finite groups whose orders are relatively prime to the order of $A$. Using the routine inverse limit argument it is easy to deduce from Theorem [1.1] and Theorem [1.2] that if $G$ is a profinite group admitting a coprime group of automorphisms $A$ of order $q^3$ such that $C_G(a)$ is pronilpotent for all $a \in A^\#$, then $G$ is pronilpotent; and if $C_G(a)$ is nilpotent for all $a \in A^\#$, then $G$ is nilpotent. Yet, certain results on fixed points in profinite groups cannot be deduced from corresponding results on finite groups. The purpose of the present paper is to establish the following theorem.

**Theorem 1.3.** Let $q$ be a prime and $A$ an elementary abelian $q$-group of order at least $q^3$. Suppose that $A$ acts coprimely on a profinite group $G$ and assume that $C_G(a)$ is locally nilpotent for each $a \in A^\#$. Then $G$ is locally nilpotent.

Recall that a group is locally nilpotent if every finitely generated subgroup is nilpotent. Though Theorem [1.3] looks similar to Theorems [1.1] and [1.2] in fact it cannot be deduced directly from those results. Moreover, the proof of Theorem [1.3] is very much different from those of Theorems [1.1] and [1.2]. In particular, unlike the other results, Theorem [1.3] relies heavily on the Lie-theoretical techniques created by Zelmannov in his solution of the restricted Burnside problem [14, 15]. The general scheme of the proof of Theorem [1.3] is similar to that of the result in [5].

2. Preparatory work

Throughout the paper we use without special references the well-known properties of coprime actions:
**Lemma 2.1.** If a group $A$ acts coprimely on a finite group $G$, then $C_{G/N}(A) = C_G(A)N/N$ for any $A$-invariant normal subgroup $N$.

**Lemma 2.2.** If $A$ is a noncyclic abelian group acting coprimely on a finite group $G$, then $G$ is generated by the subgroups $C_G(B)$, where $A/B$ is cyclic.

The above results both easily extend to the case of coprime automorphisms of profinite groups (see for example [9, Lemma 3.2]). Let $x, y$ be elements of a group, or a Lie algebra. We define inductively

$$[x, 0y] = x \text{ and } [x, n y] = [[x, n-1 y], y] \text{ for } n \geq 1.$$  

Let $L$ be a Lie algebra. An element $a \in L$ is called ad-nilpotent if there exists a positive integer $n$ such that $[x, n a] = 0$ for all $x \in L$. Let $X \subseteq L$ be any subset of $L$. By a commutator in elements of $X$ we mean any element of $L$ that can be obtained as a Lie product of elements of $X$ with some system of brackets. The next theorem is due to Zelmanov (see [16] or [17]).

**Theorem 2.3.** Let $L$ be a Lie algebra generated by finitely many elements $a_1, a_2, \ldots, a_m$ such that each commutator in these generators is ad-nilpotent. If $L$ satisfies a polynomial identity, then $L$ is nilpotent.

An important criterion for a Lie algebra to satisfy a polynomial identity is the following theorem.

**Theorem 2.4 (Bahturin-Linchenko-Zaicev).** Assume that a finite group $A$ acts on a Lie algebra $L$ by automorphisms in such a manner that $C_L(A)$, the subalgebra formed by fixed elements, satisfies a polynomial identity. Assume further that the characteristic of the ground field is either 0 or prime to the order of $A$. Then $L$ satisfies a polynomial identity.

The above theorem was first proved by Bahturin and Zaicev in the case where $A$ is soluble [1] and later extended by Linchenko to the general case [7]. In the present paper we only require the case where $A$ is abelian.

Let $G$ be a (profinite) group. A series of subgroups

$$G = G_1 \geq G_2 \geq \ldots$$  

is called an $N$-series if it satisfies $[G_i, G_j] \leq G_{i+j}$ for all $i, j \geq 1$. Here and throughout the paper when dealing with a profinite group we consider only closed subgroups. Obviously any $N$-series is central, i.e. $G_i/G_{i+1} \leq Z(G/G_{i+1})$ for any $i$. Let $p$ be a prime. An $N$-series is called $N_p$-series if $G_i^p \leq G_{pi}$ for all $i$. Given an $N$-series $(*)$, let
$L^*(G)$ be the direct sum of the abelian groups $L^*_i = G_i / G_{i+1}$, written additively. Commutation in $G$ induces a binary operation $[.,.]$ in $L$. For homogeneous elements $xG_{i+1} \in L^*_i$, $yG_{j+1} \in L^*_j$ the operation is defined by

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in L^*_{i+j}$$

and extended to arbitrary elements of $L^*(G)$ by linearity. It is easy to check that the operation is well-defined and that $L^*(G)$ with the operations $+$ and $[,]$ is a Lie ring. If all quotients $G_i / G_{i+1}$ of an $N$-series ($\ast$) have prime exponent $p$ then $L^*(G)$ can be viewed as a Lie algebra over $\mathbb{F}_p$, the field with $p$ elements. In the important case where the series ($\ast$) is the $p$-dimension central series (also known under the name of Zassenhaus-Jennings-Lazard series) of $G$ we write $L_p(G)$ for the subalgebra generated by the first homogeneous component $G_1 / G_2$ in the associated Lie algebra over the field with $p$ elements. Observe that the $p$-dimension central series is an $N_p$-series (see [3] p. 250) for details.

Any automorphism of $G$ in the natural way induces an automorphism of $L^*(G)$. If $G$ is profinite and $\alpha$ is a coprime automorphism of $G$, then the subring (subalgebra) of fixed points of $\alpha$ in $L^*(G)$ is isomorphic with the Lie ring associated to the group $C_G(\alpha)$ via the series formed by intersections of $C_G(\alpha)$ with the terms of the series ($\ast$) (see [10] for more details).

Let $w = w(x_1, x_2, \ldots, x_k)$ be a group-word. Let $H$ be a subgroup of a group $G$ and $g_1, g_2, \ldots, g_k \in G$. We say that the law $w \equiv 1$ is satisfied on the cosets $g_1H, g_2H, \ldots, g_kH$ if $w(g_1h_1, g_2h_2, \ldots, g_kh_k) = 1$ for all $h_1, h_2, \ldots, h_k \in H$. Wilson and Zelmanov showed in [13] that if a profinite group $G$ has an open subgroup $H$ and elements $g_1, g_2, \ldots, g_k$ such that the law $w \equiv 1$ is satisfied on the cosets $g_1H, g_2H, \ldots, g_kH$, then $L_p(G)$ satisfies a polynomial identity for each prime $p$. More precisely, the proof in [13] shows that whenever a profinite group $G$ has an open subgroup $H$ and elements $g_1, g_2, \ldots, g_k$ such that the law $w \equiv 1$ is satisfied on the cosets $g_1H, g_2H, \ldots, g_kH$, the Lie algebra $L^*(G)$ satisfies a multilinear polynomial identity for any prime $p$ and any $N_p$-series ($\ast$) in $G$.

**Lemma 2.5.** For any locally nilpotent profinite group $G$ there exist a positive integer $n$, elements $g_1, g_2 \in G$ and an open subgroup $H \leq G$ such that the law $[x, n, y] \equiv 1$ is satisfied on the cosets $g_1H, g_2H$.

**Proof.** Since any finitely generated subgroup of $G$ is nilpotent, for every pair of elements $g_1, g_2$ there exists a positive number $j$ such that
Since the sets $S_i$ are closed in $G \times G$ and have union $G \times G$, by Baire category theorem \[4, p. 200\] at least one of these sets has a non-empty interior. Therefore we can find an open subgroup $H$ in $G$, elements $g_1, g_2 \in G$ and an integer $n$ with the required property. \hfill $\square$

The following proposition is now straightforward.

**Proposition 2.6.** Assume that a finite group $A$ acts coprimely on a profinite group $G$ in such a manner that $C_G(A)$ is locally nilpotent. Then for each prime $p$ the Lie algebra $L_p(G)$ satisfies a multilinear polynomial identity.

**Proof.** Let $L = L_p(G)$. In view of Theorem \[2.3\] it is sufficient to show that $C_L(A)$ satisfies a polynomial identity. We know that $C_L(A)$ is isomorphic with the Lie algebra associated with the central series of $C_G(A)$ obtained by intersecting $C_G(A)$ with the $p$-dimension central series of $G$. Since $C_G(A)$ is locally nilpotent, Lemma \[2.5\] applies. Thus, the Wilson-Zelmanov result \[13, Theorem 1\] tells us that $C_L(A)$ satisfies a polynomial identity. \hfill $\square$

We will also require the following lemma that essentially is due to Wilson and Zelmanov (cf \[13, Lemma in Section 3\]).

**Lemma 2.7.** Let $G$ be a profinite group and $g \in G$ an element such that for any $x \in G$ there exists a positive $n$ with the property that $[x, n g] = 1$. Let $L^*(G)$ be the Lie algebra associated with $G$ using an $N_p$-series $(\ast)$ for some prime $p$. Then the image of $g$ in $L^*(G)$ is ad-nilpotent.

Finally, we quote a useful lemma from \[5\].

**Lemma 2.8.** Let $L$ be a Lie algebra and $H$ a subalgebra of $L$ generated by $m$ elements $h_1, \ldots, h_m$ such that all commutators in the generators $h_i$ are ad-nilpotent in $L$. If $H$ is nilpotent, then we have $[L, H, \ldots, H]_d = 0$ for some number $d$.

3. **Proof**

As usual, for a profinite group $G$ we denote by $\pi(G)$ the set of prime divisors of the orders of finite continuous homomorphic images of $G$. We say that $G$ is a $\pi$-group if $\pi(G) \subseteq \pi$ and $G$ is a $\pi'$-group if $\pi(G) \cap \pi = \emptyset$. If $m$ is an integer, we denote by $\pi(m)$ the set of prime divisors of $m$. If $\pi$ is a set of primes, we denote by $O_\pi(G)$ the
maximal normal $\pi$-subgroup of $G$ and by $O_\pi'(G)$ the maximal normal $\pi'$-subgroup.

We are ready to embark on the proof of Theorem 1.3.

**Proof of Theorem 1.3** Recall that $q$ is a prime and $A$ an elementary abelian group of order $q^n$ acting coprimely on a profinite group $G$ in such a manner that $C_G(a)$ is locally nilpotent for all $a \in A^\#$. We wish to show that $G$ is locally nilpotent. In view of Ward’s Theorem 1.1 for any $G$ the group $G$ is pronilpotent and therefore $G$ is the Cartesian product of its Sylow subgroups.

Choose $a \in A^\#$. By Lemma 2.5 $C_G(a)$ contains an open subgroup $H$ and elements $u, v$ such that for some $n$ the law $[x, n y] \equiv 1$ is satisfied on the cosets $uH, vH$. Let $[C_G(a) : H] = m$ and let $\pi_1 = \pi(m)$. Denote $O_{\pi_1}((C_G(a)))$ by $T$. Since $T$ is isomorphic to the image of $H$ in $C_G(a)/O_{\pi_1}(C_G(a))$, it is easy to see that $T$ satisfies the law $[x, n y] \equiv 1$, that is, $T$ is $n$-Engel. By the result of Burns and Medvedev [2] the subgroup $T$ has a nilpotent normal subgroup $U$ such that $T/U$ has finite exponent, say $e$. Set $\pi_2 = \pi(e)$. Of course, the sets $\pi_1$ and $\pi_2$ depend on the choice of $a \in A^\#$ so strictly speaking they should be denoted by $\pi_1(a)$ and $\pi_2(a)$. For each such choice let $\pi_a = \pi_1(a) \cup \pi_2(a)$.

We repeat this argument for every $a \in A^\#$. Set $\pi = \cup_{a \in A^\#} \pi_a$ and $K = O_\pi(G)$. Since all sets $\pi_1(a)$ and $\pi_2(a)$ are finite, so is $\pi$. The choice of the set $\pi$ guarantees that $C_K(a)$ is nilpotent for every $a \in A^\#$. Thus, by Theorem 1.2 the subgroup $K$ is nilpotent. Let $p_1, p_2, \ldots, p_r$ be the finitely many primes in $\pi$ and let $P_1, P_2, \ldots, P_r$ be the corresponding Sylow subgroups of $G$. Then $G = P_1 \times P_2 \times \cdot \cdot \cdot \times P_r \times K$ and therefore it is sufficient to show that each subgroup $P_i$ is locally nilpotent. Thus, from now on without loss of generality we assume that $G$ is a pro-$p$ group for some prime $p$. Since every finite subset of $G$ is contained in a finitely generated $A$-invariant subgroup, we can further assume that $G$ is finitely generated.

Let $A_1, A_2, \ldots, A_s$ be the distinct maximal subgroups of $A$. We denote by $D_j = D_j(G)$ the terms of the $p$-dimension central series of $G$. Set $L = L_p(G)$ and $L_j = L \cap (D_j/D_{j+1})$, so that $L = \oplus L_j$. The group $A$ naturally acts on $L$. Since each subgroup $A_i$ is noncyclic, by Lemma 2.2 we have $L = \sum_{a \in A_i^\#} C_L(a)$ for every $i \leq s$.

Let $L_{ij} = C_{L_j}(A_i)$. Again by Lemma 2.2 for any $j$ we have

$$L_j = \sum_{1 \leq i \leq s} L_{ij}.$$  

In view of Lemma 2.1 for any $l \in L_{ij}$ there exists $x \in D_j \cap C_G(A_i)$ such that $l = xD_{j+1}$. Therefore, by Lemma 2.7 the element $l$ is ad-nilpotent.
in $C_L(a)$ for every $a \in A_i^\#$. Since $L = \sum_{a \in A_i^\#} C_L(a)$, we conclude that any element $l$ in $L_{ij}$ is ad-nilpotent in $L$. 

Let $\omega$ be a primitive $q$th root of unity and $\mathcal{L} = L \otimes \mathbb{F}_p[\omega]$. We can view $\mathcal{L}$ both as a Lie algebra over $\mathbb{F}_p$ and that over $\mathbb{F}_p[\omega]$. It is natural to identify $\mathcal{L}$ with the $\mathbb{F}_p$-subalgebra $\mathcal{L} \otimes 1$ of $\mathcal{L}$. We note that if an element $x \in L$ is ad-nilpotent of index $r$, say, then the “same” element $x \otimes 1$ is ad-nilpotent in $L$ of the same index $r$. Put $L_{ij} = L_{ij} \otimes \mathbb{F}_p[\omega]$; then $L = \langle L_1 \rangle$, since $L = \langle L_1 \rangle$, and $\mathcal{L}$ is the direct sum of the homogeneous components $L_{ij}$.

The group $A$ acts naturally on $L$, and we have $L_{ij} = C_{\mathcal{L}_j}(A_i)$, where $L_{ij} = L_{ij} \otimes \mathbb{F}_p[\omega]$. Let us show that any element $y \in L_{ij}$ is ad-nilpotent in $L$. 

Since $L_{ij} = L_{ij} \otimes \mathbb{F}_p[\omega]$, we can write

$$y = x_0 + \omega x_1 + \omega^2 x_2 + \cdots + \omega^{q-2} x_{q-2}$$

for some $x_0, x_1, x_2, \ldots, x_{q-2} \in L_{ij}$, so that each of the summands $\omega^t x_t$ is ad-nilpotent by (**) 

A commutator of weight $k$ in the elements $\omega^t x_t$ has the form $\omega^s x$ for some $x$ that belongs to $L_{im}$, where $m = kj$. By (**) the element $x$ is ad-nilpotent and so such a commutator must be ad-nilpotent.

Proposition 2.6 tells us that the Lie algebra $L$ satisfies a multilinear polynomial identity. The multilinear identity is also satisfied in $\mathcal{L}$ and so it is satisfied in $J$. Hence by Theorem 2.3 $J$ is nilpotent. Lemma 2.8 now says that $[L_j, J, \ldots, J] = 0$ for some $d$. This establishes (***).

Since $A$ is abelian and the ground field is now a splitting field for $A$, every $L_j$ decomposes in the direct sum of common eigenspaces for $A$. In particular, $L_1$ is spanned by finitely many common eigenvectors for $A$. Hence $\mathcal{L}$ is generated by finitely many common eigenvectors for $A$ from $L_1$. Every common eigenspace is contained in the centralizer $C_{\mathcal{L}}(A_i)$ for some $i \leq s$, since $A$ is of order $q^3$. We also note that any commutator in common eigenvectors is again a common eigenvector. Thus, if $l_1, \ldots, l_r \in L_1$ are common eigenvectors for $A$ generating $\mathcal{L}$ then any commutator in these generators belongs to some $L_{ij}$ and therefore, by (***), is ad-nilpotent.

As we have seen, $\mathcal{L}$ satisfies a polynomial identity. It follows from Theorem 2.3 that $\mathcal{L}$ is nilpotent. We now deduce that $L$ is nilpotent as well.
According to Lazard [6] the nilpotency of $L$ is equivalent to $G$ being $p$-adic analytic. The Lubotzky-Mann theory [8] now tells us that $G$ is of finite rank, that is, all closed subgroups of $G$ are finitely generated. In particular, we conclude that $C_G(a)$ is finitely generated for every $a \in A^\#$. It follows that the centralizers $C_G(a)$ are nilpotent. Theorem [1,2] now tells us that $G$ is nilpotent. The proof is complete. □

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