THE DYNAMICS OF SOLVABLE SUBGROUPS OF PSL(3, C)

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Abstract. In this work we propose to define elementary subgroups of PSL(3, C) as solvable groups, since they present simple dynamics as we shall show. We study and provide a full description of all solvable complex Kleinian subgroups of PSL(3, C). In particular, we show that solvable groups are virtually triangular and that non-commutative solvable groups can be decomposed in four layers, via the semi-direct product of four types of elements. It is also shown that solvable groups act properly and discontinuously on the complement of, either, a line, two lines, a line and a point outside of the line or a pencil of lines passing through a point. These results complete the study of elementary subgroups of PSL(3, C).

Introduction

Kleinian groups are discrete subgroups of PSL(2, C), the group of biholomorphic automorphisms of the complex projective line CP^1, acting properly and discontinuously on a non-empty region of CP^1. Kleinian groups have been thoroughly studied since the end of the 19th century by Lazarus Fuchs, Felix Klein and Henri Poincaré (who named them after Klein in [Poi83]). For a detailed study of Kleinian groups, see [Mas87], [MT98]. In [SV01], José Seade and Alberto Verjovsky introduced the notion of complex Kleinian groups, which are discrete subgroups of PSL(n + 1, C) acting properly and discontinuously on an open invariant subset of CP^n.

In the context of Kleinian groups, elementary groups are discrete subgroups of PSL(2, C) such that the limit set is a finite set, in which case the limit set has 0, 1, or 2 points. They are groups with simple dynamics. In complex dimension 2, the Kulkarni limit set is either a finite union of complex lines (1, 2 or 3) or it contains an infinite number of complex lines. On the other hand, the limit set either contains a finite number of lines in general position (1, 2, 3 or 4) or it contains infinitely many lines in general position (see [BCN16]). Therefore, one could define elementary groups in complex dimension 2 as discrete subgroups of PSL(3, C) such that its Kulkarni limit set contains a finite number of lines or discrete subgroups of PSL(3, C) such that its Kulkarni limit set contains a finite number of lines in general position. In bigger dimensions the situation is unknown. Another way one could define elementary subgroups of PSL(3, C) is to consider groups with reducible action, since they leave a proper subspace invariant. In this work, we propose to define elementary complex kleinian subgroups of PSL(3, C) as solvable groups. We will show that solvable groups present simple dynamics contrary to the
rich dynamics of strongly irreducible discrete subgroups of PSL (3, \( \mathbb{C} \)), which have been extensively studied (see [BCN11] or [CNS13]). The study of solvable groups will help to complete the classification and understanding of the dynamics of all complex Kleinian groups of PSL (3, \( \mathbb{C} \)). This work is an important step towards the implementation of a Sullivan’s dictionary between the theory of iteration of functions in several complex variables and the dynamics of complex Kleinian groups.

The main purpose of this paper is to provide a precise description of discrete solvable subgroups of PSL (3, \( \mathbb{C} \)) and their dynamics. This paper generalizes and builds upon the work done in [BCNS18], in which the authors study complex Kleinian groups made up only of parabolic elements. To make this generalizations, additional and new techniques were used in this work.

In this paper we show:

**Theorem 0.1.** Let \( \Gamma \subset \text{PSL} (3, \mathbb{C}) \) be a solvable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then, there exists a non-empty open region \( \Omega_\Gamma \subset \mathbb{CP}^2 \) such that

(i) \( \Omega_\Gamma \) is the maximal open set where the action is proper and discontinuous.

(ii) \( \Omega_\Gamma \) is homeomorphic to one of the following regions: \( \mathbb{C}^2 \), \( \mathbb{C}^2 \setminus \{0\} \), \( \mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-) \) or \( \mathbb{C} \times \mathbb{C}^* \).

(iii) \( \Gamma \) is finitely generated and \( \text{rank}(\Gamma) \leq 4 \).

(iv) The group \( \Gamma \) can be written as

\[
    \Gamma = \Gamma_p \rtimes \langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle \rtimes \langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle
\]

where \( \Gamma_p \) is the subgroup of \( \Gamma \) consisting of all the parabolic elements of \( \Gamma \). The elements \( \eta_1, \ldots, \eta_m \) are loxo-parabolic elements such that \( \lambda_{12}(\eta_1), \ldots, \lambda_{12}(\eta_m) \) generate the group \( \lambda_{12}(\text{Ker}(\lambda_{23})) \). The elements \( \gamma_1, \ldots, \gamma_n \) are strongly loxodromic and complex homotheties such that \( \lambda_{23}(\eta_1), \ldots, \lambda_{23}(\eta_m) \) generate the group \( \lambda_{23}(\Gamma) \).

(v) The group \( \Gamma \), up to a finite index subgroup, leaves a full flag invariant.

The paper is organized as follows: In Section 1 we give a brief background necessary for the next sections. In Section 2 we prove that solvable groups always contain a finite index triangularizable group. In Section 3 we give a full description of commutative discrete triangular subgroups of PSL (3, \( \mathbb{C} \)) and, in Section 4 we describe the non-commutative case. Finally, in Section 5 we prove Theorem 0.1.

1. **Notation and Background**

1.1. **Complex Kleinian groups.** The complex projective space \( \mathbb{CP}^n \) is defined as

\[
    \mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^* ,
\]

where \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) acts by the usual scalar multiplication. This is a compact connected complex \( n \)-dimensional manifold, equipped with the Fubini-Study metric \( d_n \). We denote the projectivization of the point \( x = (x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \) by \( [x] = [x_1 : \ldots : x_{n+1}] \). We denote by \( e_1, \ldots, e_{n+1} \) to the projectivization of the canonical base of \( \mathbb{C}^{n+1} \).
Let $\mathcal{M}_{n+1}(\mathbb{C})$ be the group of all square matrices of size $n + 1$ with complex coefficients and let $\text{SL}(n + 1, \mathbb{C}) \subset \mathcal{M}_{n+1}(\mathbb{C})$ be the subgroup of matrices with determinant equal to 1. It is clear that every element $g \in \text{SL}(n + 1, \mathbb{C})$ acts linearly on $\mathbb{C}^{n+1}$ and therefore, induce a biholomorphic automorphism $[g]$ on $\mathbb{CP}^n$. On the other hand, it is well known that every biholomorphism of $\mathbb{CP}^n$ arises in this way. Thus, the group of biholomorphic automorphisms of $\mathbb{CP}^n$ is given by

$$\text{PSL}(n + 1, \mathbb{C}) := \text{GL}(n + 1, \mathbb{C}) / (\mathbb{C}^*)^{n+1} \cong \text{SL}(n + 1, \mathbb{C}) / \mathbb{Z}_{n+1},$$

where $(\mathbb{C}^*)^{n+1}$ is being regarded as the subgroup of diagonal matrices with a single non-zero eigenvalue, and $\mathbb{Z}_{n+1}$ is being regarded as the group of the $n + 1$-roots of the unity. If $g \in \text{PSL}(3, \mathbb{C})$ (resp. $z \in \mathbb{CP}^2$), we denote by $g \in \text{SL}(3, \mathbb{C})$ to any of its lifts (resp. $z \in \mathbb{C}^3$). We denote by $\text{Fix}(g) \subset \mathbb{CP}^2$ the set of fixed points of an automorphism $g \in \text{PSL}(3, \mathbb{C})$.

Now we describe a useful construction used to reduce the action of a group $\Gamma \subset \text{PSL}(3, \mathbb{C})$ on $\mathbb{CP}^2$ to the action of a subgroup of $\text{PSL}(2, \mathbb{C})$ on a complex line in $\mathbb{CP}^2$, thus, simplifying the study of the dynamics of $\Gamma$. The details and proofs of this construction are given in chapter 5 of [CNS13]. Consider a subgroup $\Gamma \subset \text{PSL}(3, \mathbb{C})$ acting on $\mathbb{CP}^2$ with a global fixed point $p \in \mathbb{CP}^2$. Let $\ell \subset \mathbb{CP}^2 \setminus \{p\}$ be a projective complex line, we define the projection $\pi = \pi_{p,\ell} : \mathbb{CP}^2 \to \ell$, given by $\pi(x) = \ell \cap \overline{hp}$. This function is holomorphic, and it allows us to define the group homomorphism

$$\Pi = \Pi_{p,\ell} : \Gamma \to \text{Bihol}(\ell) \cong \text{PSL}(2, \mathbb{C})$$

given by $\Pi(g)(x) = \pi(g(x))$ for $g \in \Gamma$ (see Lemma 6.11 of [CSI14]). If we choose another line, $\ell' \subset \mathbb{CP}^2 \setminus \{p\}$, we obtain a projection $\pi' = \pi_{p,\ell'}$ and a group homomorphisms $\Pi' = \Pi_{p,\ell'}$. The homomorphisms $\Pi$ and $\Pi'$ are equivalent in the sense that there exists a biholomorphism $h : \ell \to \ell'$ inducing an automorphism $H$ of $\text{PSL}(2, \mathbb{C})$ such that $H \circ \Pi = \Pi'$. The line $\ell$ is called the horizon. To simplify the notation we will write $\text{Ker}(\Gamma)$ instead of $\text{Ker}(\Pi) \cap \Gamma$.

Let $M \in \mathcal{M}_3(\mathbb{C})$ and consider its kernel $\text{Ker}(M)$. Consider the projectivization of this set, $[\text{Ker}(M) \setminus \{0\}]$. Then $M$ induces a well defined map $[M] : \mathbb{CP}^2 \setminus [\text{Ker}(M) \setminus \{0\}] \to \mathbb{CP}^2$, given by $[M](z) = [Mz]$. We define in this way the quasi-projective maps $\text{QP}(3, \mathbb{C})$ as

$$\text{QP}(3, \mathbb{C}) = (\mathcal{M}_3(\mathbb{C}) \setminus \{0\}) / \mathbb{C}^*.$$  

This space is the closure of $\text{PSL}(3, \mathbb{C})$ in the space $\mathcal{M}_3(\mathbb{C})$ and therefore every sequence of elements in $\text{PSL}(3, \mathbb{C})$ converge to an element of $\text{QP}(3, \mathbb{C})$. For an element $T \in \text{QP}(3, \mathbb{C})$ we define its kernel as $\text{Ker}(T) = [\text{Ker}(T) \setminus \{0\}]$. For a more detailed overview of quasi-projective maps, see Section 3 of [CSI10] or Section 7.4 of [CNS13]. We will denote by $\text{Diag}(a_1, ..., a_{n+1})$ to the diagonal element of $\text{QP}(n + 1, \mathbb{C})$ with diagonal entries $a_1, ..., a_{n+1}$.

As in the case of automorphisms of $\mathbb{CP}^1$, we classify the elements of $\text{PSL}(3, \mathbb{C})$ in three classes: elliptic, parabolic and loxodromic. However, unlike the classical case, we consider several subclasses in each case. For a complete description of this classification, see Section 4.2 of [CNS13]. We will give a quick summary of the
Definition 1.1. An element \( g \in \text{PSL}(3, \mathbb{C}) \) is said to be:

- **Elliptic** if it has a diagonalizable lift in \( \text{SL}(3, \mathbb{C}) \) such that every eigenvalue has norm 1.
- **Parabolic** if it has a non-diagonalizable lift in \( \text{SL}(3, \mathbb{C}) \) such that every eigenvalue has norm 1.
- **Loxodromic** if it has a lift in \( \text{SL}(3, \mathbb{C}) \) with an eigenvalue of norm distinct of 1. Furthermore, we say that \( g \) is:
  - **Loxoparabolic** if it is conjugated to an element \( h \in \text{PSL}(3, \mathbb{C}) \) such that \( h = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}, |\lambda| \neq 1 \).
  - A **complex homothety** if it is conjugated to an element \( h \in \text{PSL}(3, \mathbb{C}) \) such that \( h = \text{Diag}(\lambda,\lambda,\lambda^{-2}) \), with \(|\lambda| \neq 1\).
  - A **rational (resp. irrational) screw** if it is conjugated to an element \( h \in \text{PSL}(3, \mathbb{C}) \) such that \( h = \text{Diag}(\lambda_1,\lambda_2,\lambda_3) \), with \(|\lambda_1|,|\lambda_2|,|\lambda_3|\) are pairwise different.
  - A **strongly loxodromic** if it is conjugated to an element \( h \in \text{PSL}(3, \mathbb{C}) \) such that \( h = \text{Diag}(\lambda_1,\lambda_2,\lambda_3) \), where the elements \(|\lambda_1|,|\lambda_2|,|\lambda_3|\) are pairwise different.

An element \( \gamma \in \Gamma \) is called a **torsion element** if it has finite order. The group \( \Gamma \) is a **torsion free group** if the only torsion element in \( \Gamma \) is the identity.

In the classic case of Kleinian groups, there are several characterizations for the concept of limit set and all of them coincide (see [MT98]). In the complex setting there is no right definition of limit set, there are several definitions (for a detailed discussion and examples see Section 3.1 of [CNS13]). In [Kul78], Kulkarni introduced a notion of limit set which works in a very general setting (see Section 3.3. of [CNS13]).

Definition 1.2. Let \( \Gamma \) be a discrete subgroup of \( \text{PSL}(n+1, \mathbb{C}) \) acting on \( \mathbb{C}P^n \). Let us define the following sets:

- Let \( L_0(\Gamma) \) be the closure of the set of points in \( \mathbb{C}P^n \) with infinite group of isotropy.
- Let \( L_1(\Gamma) \) be the closure of the set of accumulation points of orbits of points in \( \mathbb{C}P^n \setminus L_0(\Gamma) \).
- Let \( L_2(\Gamma) \) be the closure of the set of accumulation points of orbits of compact subsets of \( \mathbb{C}P^n \setminus (L_0(\Gamma) \cup L_1(\Gamma)) \).

We define the **Kulkarni limit set** of \( \Gamma \) as \( \Lambda_{\text{Kul}}(\Gamma) := L_0(\Gamma) \cup L_1(\Gamma) \cup L_1(\Gamma) \). The **Kulkarni region of discontinuity** of \( \Gamma \) is defined as \( \Omega_{\text{Kul}}(\Gamma) := \mathbb{C}P^n \setminus \Lambda_{\text{Kul}}(\Gamma) \).

The action of \( \Gamma \) on \( \Omega_{\text{Kul}}(\Gamma) \) is proper and discontinuous (see Section 3.2 of [CNS13]).

Definition 1.3. The **equicontinuity region** for a family \( \Gamma \) of automorphisms of \( \mathbb{C}P^n \), denoted \( \text{Eq}(\Gamma) \), is defined to be the set of points \( z \in \mathbb{C}P^n \) for which there is an open neighborhood \( U \) of \( z \) such that \( \Gamma \) restricted to \( U \) is a normal family.
The following propositions will be very useful to compute the Kulkarni limit set of a group in terms of the quasi-projective limits of sequences in the group.

**Proposition 1.4** (Proposition 7.4.1 of [CNS13]). Let $\{\gamma_k\} \subset \text{PSL}(n+1, \mathbb{C})$ a sequence of distinct elements, then there is a subsequence of $\{\gamma_k\}$, still denoted by $\{\gamma_k\}$, and a quasi-projective map $\gamma \in \text{QP}(n+1, \mathbb{C})$ such that $\gamma_k \rightarrow \gamma$ uniformly on compact sets of $\mathbb{CP}^n \setminus \text{Ker}(\gamma)$.

**Proposition 1.5** (Proposition 2.5 of [CLUP17]). Let $\Gamma \subset \text{PSL}(n+1, \mathbb{C})$ be a group, we say $\gamma \in \text{QP}(n+1, \mathbb{C})$ is a limit of $\Gamma$, in symbols $\gamma \in \text{Lim}(\Gamma)$, if there is a sequence $\{\gamma_m\} \subset \Gamma$ of distinct elements satisfying $\gamma_m \rightarrow \gamma$. Then, \[ \text{Eq}(\Gamma) = \mathbb{CP}^n \setminus \bigcup_{\gamma \in \text{Lim}(\Gamma)} \text{Ker}(\gamma). \]

**Proposition 1.6** (Corollary 2.6 of [CLUP17]). Let $\Gamma \subset \text{PSL}(n+1, \mathbb{C})$ be a discrete group, then $\Gamma$ acts properly discontinuously on $\text{Eq}(\Gamma)$. Moreover, $\text{Eq}(\Gamma) \subset \Omega_{\text{Kul}}(\Gamma)$.

### 1.2. Solvable groups.

**Definition 1.1.** Let $G$ be a group. The derived series $\{G^{(i)}\}$ of $G$ is defined inductively as $G^{(0)} = G$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. One says that $G$ is solvable if, for some $n \geq 0$, we have $G^{(n)} = \{\text{id}\}$. We call the minimum $n$ such that $G^{(n)} = \{\text{id}\}$, the solvability length of $G$.

We will consider the Zariski topology on $\text{GL}(n, \mathbb{C})$. If $\Gamma \subset \text{GL}(n, \mathbb{C})$ is a subgroup, we denote by $\overline{\Gamma}$ the Zariski closure of $\Gamma$. We recall the following properties of the Zariski closure:

1. $\overline{\Gamma}$ is an algebraic subgroup of $\text{GL}(n, \mathbb{C})$ with a finite number of connected components.
2. $\Gamma$ is solvable if and only if $\overline{\Gamma}$ is solvable.

The following are examples of solvable subgroups of $\text{PSL}(3, \mathbb{C})$.

**Example 1.2.**
- Cyclic groups are solvable.
- Denote by $\text{Rot}_{\infty}$ the group of all rotations around the origin, then the infinite dihedral group $\text{Dih}_{\infty} = \langle \text{Rot}_{\infty}, z \mapsto -z \rangle$ is solvable.
- The special orthogonal group, $\text{SO}(3) = \left\{ \begin{bmatrix} a & -c \\ c & a \end{bmatrix} \mid |a|^2 + |c|^2 = 1 \right\}$ is not solvable.
- Any triangular group is solvable, with solvability length at most 3.

The next theorem is known as the **topological Tits alternative** (see Theorem 1 of [Tit72] or Theorem 1.3 of [BG07]).

**Theorem 1.7.** Let $K$ be a local field and let $\Gamma \subset \text{GL}(n, K)$ be a subgroup. Then, either, $\Gamma$ contains an open solvable group or $\Gamma$ contains a dense free subgroup.

**Theorem 1.8** (Theorem 10.4 of [Bor91]). Let $G$ be a connected solvable group acting morphically on a non-empty complete variety $V$. Then $G$ has a fixed point in $V$. 
2. Solvable groups are triangularizable

In this section we prove that solvable groups have a finite index triangularizable subgroup. The proof of the following theorem is given for subgroups of PSL(3, \(\mathbb{C}\)), but it is also valid for subgroups of PSL(n + 1, \(\mathbb{C}\)).

Before stating the theorem, observe that \(G \subset GL(n + 1, \mathbb{C})\) is solvable (resp. triangularizable) if and only if \([G] \subset PSL(n + 1, \mathbb{C})\) is solvable (resp. triangularizable). This allows us to give the proof of the following theorem in terms of subgroups of GL(3, \(\mathbb{C}\)) instead of subgroups of PSL(3, \(\mathbb{C}\)).

**Theorem 2.1.** Let \(G \subset GL(3, \mathbb{C})\) be a discrete solvable subgroup then \(G\) is virtually triangularizable.

**Proof.** Let \(G \subset GL(3, \mathbb{C})\) be as in the hypothesis, then \(\overline{G}\) is a connected solvable group acting morphically on \(\mathbb{CP}^2\). Denote by \(G_0\) the connected component of \(\overline{G}\) such that \(id \in G\), then \(G_0\) is a finite index connected solvable subgroup of \(\overline{G}\). By Theorem 1.8, \(G_0\) has a global fixed point in \(\mathbb{CP}^2\). Up to conjugation by an element of GL(3, \(\mathbb{C}\)), we can assume that this fixed point is \(e_1\) and therefore, every element of \(G_0\) has the form 

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{pmatrix}.
\]

Let \(a : G_0 \to a(G_0) \subset GL(2, \mathbb{C})\) be the group morphism given by

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{pmatrix} \mapsto \begin{pmatrix}
a_{22} & a_{23} \\
0 & a_{33}
\end{pmatrix}.
\]

Since \(G_0\) is solvable and \(a\) is a suprjective group morphism, then \(a(G_0)\) is solvable. Repeating the same argument we applied before to \(G\), we now have a connected solvable subgroup with finite index \(H_0 \subset H = a(G_0)\) acting with a fixed point on \(\mathbb{CP}^1\). Therefore, up to conjugation, every element of \(H_0\) has the form

\[
\begin{pmatrix}
a_{22} & a_{23} \\
0 & a_{33}
\end{pmatrix}.
\]

Taking the inverse image \(a^{-1}(H_0)\) we get an upper triangular finite index subgroup of \(G\). This completes the proof. \(\square\)

3. Commutative triangular groups

In this section we describe the commutative triangular groups of PSL(3, \(\mathbb{C}\)). We denote the upper triangular elements of PSL(3, \(\mathbb{C}\)) by

\[
U_+ = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \right| a_{11}a_{22}a_{33} = 1, \ a_{ij} \in \mathbb{C} \right\}.
\]

Let \(\lambda_{12}, \lambda_{23}, \lambda_{13} : (U_+, \cdot) \to (\mathbb{C}^*, \cdot)\) be the group morphisms given by

\[
\lambda_{12}([a_{ij}]) = a_{11}^{-1}a_{22}^{-1}, \quad \lambda_{23}([a_{ij}]) = a_{22}a_{33}^{-1}, \quad \lambda_{13}([a_{ij}]) = a_{11}a_{33}^{-1}.
\]

To simplify the notation we will write \(\text{Ker}(\lambda_{ij})\) instead of \(\text{Ker}(\lambda_{ij}) \cap \Gamma\) for subgroups \(\Gamma \subset U_+.\) We also define the projections \(\pi_{kl}([a_{ij}]) = a_{kl}^{-1}.)
Whenever we have a discrete subgroup $\Gamma \subset U_+$, we have a finite index torsion free subgroup $\Gamma' \subset \Gamma$ such that $\lambda_{12}(\Gamma')$ and $\lambda_{23}(\Gamma')$ are torsion free groups as well (see Lemma 5.8 of [BCNS18]). This subgroup and the original group satisfy $\Lambda_{Kul}(\Gamma) = \Lambda_{Kul}(\Gamma')$ (see Proposition 3.6 of [BCN16]). Therefore we can assume for the rest of this work that all discrete subgroups $\Gamma \subset U_+$ are torsion free.

The following immediate result will be used often.

**Proposition 3.1.** If $\Gamma \subset \text{PSL}(3, \mathbb{C})$ is a torsion free, commutative subgroup then $\Gamma \cong \mathbb{Z}^r$, where $r = \text{rank}(\Gamma)$.

The following lemma describe the form of the upper triangular commutative subgroups of $\text{PSL}(3, \mathbb{C})$.

**Lemma 3.2 (Lemma 5.11 of [BCNS18]).** Let $\Gamma \subset U_+$ be a commutative group, then $\Gamma$ is conjugated to a subgroup of one of the following Lie Groups:

1. $C_1 = \left\{ \begin{pmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C} \right\}$.
2. $C_2 = \left\{ \text{Diag} \left( \alpha, \beta, \alpha^{-1} \beta^{-1} \right) \middle| \alpha, \beta \in \mathbb{C}^* \right\}$.
3. $C_3 = \left\{ \begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \middle| \beta, \gamma \in \mathbb{C} \right\}$.
4. $C_4 = \left\{ \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \beta, \gamma \in \mathbb{C} \right\}$.
5. $C_5 = \left\{ \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \middle| \beta, \gamma \in \mathbb{C} \right\}$.

Observe that cases 3, 4 and 5 stated in Lemma 3.2 are purely parabolic, they have been already studied in [BCNS18]. Cases 1 and 2 can be purely parabolic or they can have loxodromic elements, therefore we will only study these Cases 1 and 2.

We will describe, in each of the 2 cases, how the groups should be in order to be commutative and discrete. We will also describe the Kulkarni limit set in each case.

**3.1. Case 1.**

**Proposition 3.3.** Let $\Gamma \subset U_+$ be a commutative subgroup such that each element of $\Gamma$ has the form given by (1) of Lemma 3.2. Then there exists an additive subgroup $W \subset (\mathbb{C}, +)$ and a group morphism $\mu : (W, +) \to (\mathbb{C}^*, \cdot)$ such that $\Gamma = \Gamma_{W, \mu} = \left\{ \begin{pmatrix} \mu(w)^{-2} & 0 & 0 \\ 0 & \mu(w) & w\mu(w) \\ 0 & 0 & \mu(w) \end{pmatrix} \middle| w \in W \right\}$.

**Proof.** Let $\zeta : (\Gamma, \cdot) \to (\mathbb{C}, +)$ be the group homomorphism given by $[\alpha_{ij}] \mapsto \alpha_{23}\alpha_{31}$. Clearly, $\text{Ker}(\zeta) = \{id\}$. Thus we can define the group homomorphism $\mu : (\zeta(\Gamma), +) \to (\mathbb{C}^*, \cdot)$ as $x \mapsto \pi_{22}(\zeta^{-1}(x))$. Define the additive group $W = \zeta(\Gamma)$. 

It is straightforward to verify that
\[
\Gamma = \left\{ \begin{bmatrix} \mu(w)^{-2} & 0 & 0 \\ 0 & \mu(w) & w\mu(w) \\ 0 & 0 & \mu(w) \end{bmatrix} \mid w \in W \right\}. \quad \Box
\]

For \( w \in W \), we will denote \( \gamma_w = \begin{bmatrix} \mu(w)^{-3} & 0 & 0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} \in \Gamma_{W,\mu} \).

Proposition 3.4. Let \( \Gamma = \Gamma_{W,\mu} \subset U^+ \) be a commutative subgroup with the form given by Proposition 3.3. If \( \text{rank}(\Gamma) = r \), then \( \text{rank}(W) = r \).

Proof. Let \( r = \text{rank}(\Gamma) \) and \( \gamma_1, \ldots, \gamma_r \in \Gamma \) such that \( \Gamma = \langle \gamma_1, \ldots, \gamma_r \rangle \). Let \( w_1, \ldots, w_r \in W \) such that \( \gamma_j = \gamma_{w_j} \), for \( j = 1, \ldots, r \). Let \( w \in W \) and consider \( \gamma_w \in \Gamma \), then there exist \( n_1, \ldots, n_r \in \mathbb{Z} \) such that \( \gamma_w = \gamma_1^{n_1} \cdots \gamma_r^{n_r} \). Comparing entries 23 of both sides yields \( w = n_1 w_1 + \cdots + n_r w_r \). This means that \( W = \langle w_1, \ldots, w_r \rangle \) and therefore, \( \text{rank}(W) \leq r \). To prove that \( \text{rank}(W) = r \), assume without loss of generality that \( w_r = n_1 w_1 + \cdots + n_{r-1} w_{r-1} \), then \( \gamma_r = \gamma_1^{n_1} \cdots \gamma_{r-1}^{n_{r-1}} \), contradicting that \( \text{rank}(\Gamma) = r \).

This completes the proof. \( \Box \)

The following lemma gives examples of non-discrete additive subgroups \( W \) such that \( \Gamma_{W,\mu} \) is discrete.

Lemma 3.5. If \( \alpha \) and \( \beta \) are two rationally independent real numbers then \( W = \text{Span}_\mathbb{Z}(\alpha, \beta) \) is a non-discrete additive subgroup of \( \mathbb{C} \).

Proof. Let \( h : (W, +) \to (S^1, \cdot) \) be a group homomorphism given by \( x \mapsto e^{2\pi i x/\alpha} \). Observe that \( \left\{ 1, \frac{\beta}{\alpha} \right\} \) are rationally independent, then \( h(W) \) is a sequence of distinct elements in \( S^1 \), therefore there is a subsequence, denoted by \( \{g_n\} \subset h(W) \), such that \( g_n = e^{2\pi i q_n \beta/\alpha} \to \xi \in S^1 \), for some \( \{q_n\} \subset \mathbb{Z} \) and some \( \xi \in S^1 \). Define the sequence \( \{h_n\} \subset S^1 \) by \( h_n = g_n g_{n+1}^{-1} \), then \( h_n \to 1 \). Denoting \( h_n = e^{2\pi ir_n \beta/\alpha} \), for some \( \{r_n\} \subset \mathbb{Z} \), and taking the logarithm of the sequence we have

\[
2\pi ir_n \frac{\beta}{\alpha} + 2\pi is_n \to 0
\]

for some logarithm branch defined by \( \{s_n\} \subset \mathbb{Z} \). As a consequence of (3.1), \( r_n \beta + s_n \alpha \to 0 \) and therefore, \( W \) is not discrete. \( \Box \)

Proposition 3.8 will provide the full description of discrete commutative subgroups of \( U^+ \) belonging to the case 1. In order to prove that proposition, we first need to determine the equicontinuity region for case 1 groups. To do this, consider the following table in which we list all of the possible quasi-projective limits of sequences in these groups.
Proposition 3.6. Let $\Gamma \subset PSL(3, \mathbb{C})$ be a commutative discrete group with the form given in Propositions 3.3 and 3.8 if we assume that $\Gamma$ contains loxodromic elements then $Eq(\Gamma) = \mathbb{CP}^2 \setminus (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\).$

Proof. We use Proposition 1.5 to determine $Eq(\Gamma)$. Let $\gamma_w \in \Gamma$ be a loxodromic element, then $|\mu(w)| \neq 1$. Let us suppose, without loss of generality, that $|\mu(w)| > 1$. Consider the sequence $\{\gamma_w^n\}_{n \in \mathbb{N}} \subset \Gamma$, then

$$\gamma_w^n \to \tau_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } \text{Ker}(\tau_1) = \tilde{e}_1, \tilde{e}_2.$$

Considering the sequence $\{\gamma_w^{-n}\}_{n \in \mathbb{N}} \subset \Gamma$ instead, we have $\gamma_w^{-n} \to \tau_2 = \text{Diag}(1, 0, 0)$, with $\text{Ker}(\tau_2) = \tilde{e}_2, \tilde{e}_3$. This, together with Proposition 1.5 and (3.2) imply that

$$\mathbb{CP}^2 \setminus (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3) \subset Eq(\Gamma).$$

Proposition 1.5 and the table in Figure 3.1 imply that $Eq(\Gamma) \subset \mathbb{CP}^2 \setminus (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$. This, together with (3.3) prove the proposition. $\square$

Observation 3.7. The previous proposition implies that, for a group $\Gamma$ of this first case, if we determine that $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \subset \Lambda_{Kul}(\Gamma)$ then, using Proposition 1.6 we have in fact,

$$\Lambda_{Kul}(\Gamma) = \tilde{e}_1, \tilde{e}_2, \tilde{e}_3.$$

This observation will be very useful when we determine the Kulkarni limit set.

Proposition 3.8. Let $\Gamma = \Gamma_{W, \mu} \subset U_+$ be a group as described in Proposition 3.3 $\Gamma$ is discrete if and only if $\text{rank}(W) \leq 3$, and the morphism $\mu$ satisfies the following condition:

(C) Whenever we have a sequence $\{w_k\} \in W$ of distinct elements such that $w_k \to 0$, either $\mu(w_k) \to 0$ or $\mu(w_k) \to \infty$. 

| Case | $\tau$ | Conditions | $\text{Ker}(\tau)$ | $\text{Im}(\tau)$ |
|------|--------|------------|---------------------|---------------------|
| (i)  | Diag $(1, 0, 0)$ | $w_n \to b \in \mathbb{C}$ and $\mu(w_n) \to 0$ or $w_n \to \infty$, $\mu(w_n) \to 0$ and $w_n \mu(w_n)^3 \to 0$ | $\tilde{e}_2, \tilde{e}_3$ | $\{e_1\}$ |
| (ii) | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ | $w_n \to b \in \mathbb{C}$ and $\mu(w_n) \to \infty$ | $\{e_1\}$ | $\tilde{e}_2, \tilde{e}_3$ |
| (iii) | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ | $w_n \to \infty$ and $\mu(w_n) \to \infty$ or $w_n \to \infty$ and $\mu(w_n) \to a \in \mathbb{C}$ or $w_n \to \infty$, $\mu(w_n) \to 0$ and $w_n \mu(w_n)^3 \to \infty$ | $\tilde{e}_1, \tilde{e}_2$ | $\{e_2\}$ |
| (iv) | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$ | $w_n \to \infty$, $\mu(w_n) \to 0$ and $w_n \mu(w_n)^3 \to b \in \mathbb{C}$ | $\{e_2\}$ | $\tilde{e}_1, \tilde{e}_2$ |

Table 1. Quasi-projective limits of sequences of distinct elements in $\Gamma$. 

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Proof. First, assume that \( \text{rank}(W) \leq 3 \) and that the group morphism \( \mu \) satisfies the condition (C). If \( W \) is discrete then, it is straightforward to see that \( \Gamma \) is discrete. Now, assume that \( W \) is not discrete. Suppose that \( \Gamma \) is not discrete, then let \( \{ \gamma_k \} \subset \Gamma \) be a sequence of distinct elements such that \( \gamma_k \to \text{id} \), denote \( \gamma_k = \gamma_{w_k} \). Then \( \mu(w_k) \) converges to some cubic root of the unity \( \gamma \). Since \( \mu \) satisfies condition (C) then \( \mu(w_k) \to 0 \) or \( \mu(w_k) \to \infty \) contradicting that \( \mu(w_k) \) converges to some cubic root of the unity. This contradiction proves that \( \Gamma \) is discrete.

Now assume that \( \Gamma \) is discrete, by Propositions \[1.6\] and \[3.6\], \( \Gamma \) acts properly and discontinuously on \( \mathbb{C} \times \mathbb{C} \). Let \( \tilde{\Gamma} \) be a covering group of \( \Gamma \) and \( \tilde{\Gamma} \) be covering groups of \( \Gamma \) and \( \Gamma \) respectively (see Theorem 9.1 of \[Bre72\]), let \( \tilde{\Gamma} = \Gamma_1 \). As a consequence of this, there is a group morphism, induced by \( \pi \) and still denoted by \( \pi \), given by

\[
\pi = (id, \exp) : \Gamma \cong \tilde{\Gamma} \times \tilde{\Gamma} \to \Gamma_1 \times \Gamma_2
\]

with the multiplicative group \( \Gamma_1 \) acting on \( \mathbb{C} \times \mathbb{C} \) and the additive group \( \Gamma_2 \) acting on \( \mathbb{C} \). Let \( \tilde{\Gamma} \) and \( \tilde{\Gamma} \) be covering groups of \( \Gamma \) and \( \Gamma \) respectively. Then, \( \tilde{\Gamma} \cong \mathbb{Z}^{k_1} \times \mathbb{Z}^{k_2} \), with \( k_1 = \text{rank}(\Gamma_1) \) and \( k_2 = \text{rank}(\Gamma_2) \). Observe that the kernel of \( \pi \) is given by \( \text{Ker}(\pi) = \text{Ker}(id) \times \ker(\exp) \cong \mathbb{Z} \times \mathbb{Z} \). Therefore,

\[
\text{rank}(\pi) = k + 1.
\]

On the other hand, since \( \Gamma \) acts properly and discontinuously on \( \mathbb{C} \times \mathbb{C} \), then \( \tilde{\Gamma} \) acts properly and discontinuously on \( \mathbb{C} \times \mathbb{C} \), which is simply connected. Then, by Theorem 4.24, \( \text{rank}(\tilde{\Gamma}) \leq 4 \). This, together with (3.4) yields \( \text{rank}(\Gamma) \leq 3 \). Using Proposition 3.4, we conclude that \( \text{rank}(W) \leq 3 \).

Now we will verify that \( \mu \) satisfies the condition (C). Let \( \{ w_k \} \subset W \) be a sequence of distinct elements such that \( w_k \to 0 \). Consider the sequence \( \{ \mu(w_k) \} \subset \mathbb{C}^* \), assume that it does not converge to 0 or \( \infty \), then there are open neighborhoods \( U_0 \) and \( U_\infty \) of 0 and \( \infty \) respectively such that \( \{ \mu(w_k) \} \subset \mathbb{CP}^1 \setminus (U_0 \cup U_\infty) \). Since \( \mathbb{CP}^1 \) is compact and \( \mathbb{CP}^1 \setminus (U_0 \cup U_\infty) \) is a closed subset of \( \mathbb{CP}^1 \), then \( \mathbb{CP}^1 \setminus (U_0 \cup U_\infty) \) is compact and therefore, there is a converging subsequence of \( \{ \mu(w_k) \} \), still denoted the same way. Let \( z \in \mathbb{C}^* \) such that \( \mu(w_k) \to z \), then \( \gamma_k \to \text{Diag}(z^{-3}, 1, 1) \), contradicting that \( \Gamma \) is discrete. This proves that \( \mu \) satisfies the condition (C).

We say that \( z \in \mathbb{C} \) is a rational rotation (resp. irrational rotation) if \( z = e^{2\pi i \theta} \) for some \( \theta \in \mathbb{Q} \) (resp. \( \theta \in \mathbb{R} \setminus \mathbb{Q} \)).

In order to describe the Kulkarni limit set, we will divide all groups of case 1 into the following subcases:


| Case | Conditions |
|------|-------------|
| C1.1 | $\mu(W)$ has rational rotations and $W$ is discrete. |
| C1.2 | $\mu(W)$ has rational rotations and $W$ is not discrete. |
| C1.3 | $\mu(W)$ has no rational rotations but has irrational rotations, $W$ is discrete. |
| C1.4 | $\mu(W)$ has no rational or irrational rotations, $W$ is discrete. |
| C1.5 | $\mu(W)$ has no rational rotations but has irrational rotations, $W$ is not discrete. |
| C1.6 | $\mu(W)$ has no rational or irrational rotations, $W$ is not discrete. |

We say that the commutative group $\Gamma = \Gamma_{W,\mu}$ satisfy the condition ($F$) if there is a sequence $\{w_k\} \subset W$ such that $w_k \to \infty$, $\mu(w_k) \to 0$ and $w_k\mu(w_k)^3 \to b \in \mathbb{C}^*$. 

**Observation 3.9.** If $\Gamma$ is cyclic then $W$ is cyclic (see Proposition 3.4), denote $W = \langle w \rangle$. The group $\Gamma$ is either generated by a loxo-parabolic or an ellipto-parabolic element depending on whether $|\mu(w)| \neq 1$ or $|\mu(w)| = 1$ respectively. According to Proposition 4.2.10 and 4.2.19 of [CNS13], $\Lambda_{\text{Kul}}(\Gamma) = \overline{\epsilon_1, e_2} \cup \overline{\epsilon_2, e_3}$ if $|\mu(w)| \neq 1$ and $\Lambda_{\text{Kul}}(\Gamma) = \overline{\epsilon_1, e_2}$ if $|\mu(w)| = 1$. Therefore, we can assume that the group $\Gamma$ is not cyclic. 

**Lemma 3.10.** Let $\Gamma \subset U_+$ be commutative discrete group with the form given in Proposition 3.8, then 

$$L_0(\Gamma) = \begin{cases} \{e_1, e_2\}, & \text{If } \mu(W) \text{ does not contain rational rotations} \\ \overline{\epsilon_1, e_2}, & \text{If } \mu(W) \text{ contains rational rotations} \end{cases}$$

Prove. In both cases $\{e_1, e_2\} \subset L_0(\Gamma)$ since both are global fixed points of $\Gamma$. A direct computation shows that if $X \in L_0(\Gamma)$ then $X \in \overline{\epsilon_1, e_2}$, that is, $L_0(\Gamma) \subset \overline{\epsilon_1, e_2}$. 

It is straightforward to verify that, if $\mu(W)$ contains rational rotations, then every point in $\overline{\epsilon_1, e_2}$ is in $L_0(\Gamma)$ and therefore $L_0(\Gamma) = \overline{\epsilon_1, e_2}$. Also, if $\mu(W)$ doesn’t contain rational rotations, then $L_0(\Gamma) = \{e_1, e_2\}$. \hfill \Box 

**Lemma 3.11.** Let $\Gamma = \Gamma_{W,\mu} \subset U_+$ be a commutative discrete group with the form given in Proposition 3.8, if $W$ is discrete and $\mu(W)$ contains rational rotations then $\Lambda_{\text{Kul}}(\Gamma) = \overline{\epsilon_1, e_2}$.

**Proof.** Since $\mu(W)$ contains rational rotations, $L_0(\Gamma) = \overline{\epsilon_1, e_2}$ (see Lemma 3.10). Since $W$ is discrete, then $\Gamma$ has no sequences with quasi-projective limit with the form (ii) in Table 3.1, this implies $\Lambda_{\text{Kul}}(\Gamma) \subset \overline{\epsilon_1, e_2}$ (see Proposition 1.6). All of this yields $\overline{\epsilon_1, e_2} = L_0(\Gamma) \subset \Lambda_{\text{Kul}}(\Gamma) \subset \overline{\epsilon_1, e_2}$. \hfill \Box 

**Lemma 3.12.** Let $\Gamma = \Gamma_{W,\mu} \subset U_+$ be a commutative discrete group with the form given in Proposition 3.8, if $\mu(W)$ contains rational rotations and $W$ is not discrete, then $\Lambda_{\text{Kul}}(\Gamma) = \overline{\epsilon_1, e_2} \cup \overline{\epsilon_2, e_3}$.

**Proof.** Since $W$ is not discrete, there is a sequence $\{w_n\} \subset W$ such that $w_n \to 0$ and therefore, either $\mu(w_n) \to \infty$ or $\mu(w_n) \to 0$ (Proposition 3.5). We can assume without loss of generality that the former happens (otherwise, consider the sequence $\{-w_n\}$). Then, the quasi-projective limit of the sequence $\{\gamma_{w_n}\}$ is $\tau = \text{Diag}(0, 1, 1)$. Let $z \in \mathbb{CP}^2 \setminus L_0(\Gamma)$, then $z \notin \text{Ker}(\tau) = \{e_1\}$ (see Lemma 3.10). Therefore, the set of accumulation points of points in $\mathbb{CP}^2 \setminus L_0(\Gamma)$ is $\text{Im}(\tau) = \overline{\epsilon_2, e_3}$. Then, $\overline{\epsilon_1, e_2} \cup \overline{\epsilon_2, e_3} \subset L_0(\Gamma) \cup L_1(\Gamma)$. Using Observation 3.7, we conclude that $\Lambda_{\text{Kul}}(\Gamma) = \overline{\epsilon_1, e_2} \cup \overline{\epsilon_2, e_3}$, whenever $W$ contains rational rotations and is not discrete. \hfill \Box
As an immediate consequence of the last lemma we conclude that groups $\Gamma$ belonging to the case $C1.2$ satisfy $\Lambda_{Kul}(\Gamma) = \overrightarrow{e_1, e_2} \cup \overrightarrow{e_2, e_3}$.

If $W$ is discrete, using Table 3.1 we conclude that $\overrightarrow{e_2, e_3}$ cannot be contained in $\Lambda_{Kul}(\Gamma)$. Then, if $\mu(W)$ contains no rational rotations, either $\Lambda_{Kul}(\Gamma) = \{e_1, e_2\}$ or $\Lambda_{Kul}(\Gamma) = \overrightarrow{e_1, e_2}$. This argument, together with Table 3.1 proves the following lemma, which describes cases $C1.3$ and $C1.4$.

**Lemma 3.13.** Let $\Gamma = \Gamma_{W, \mu} \subset U_+$ be a commutative discrete group with the form given in Proposition 3.8 if $\mu(W)$ contains no rational rotations and $W$ is discrete, then

$$\Lambda_{Kul}(\Gamma) = \begin{cases} \overrightarrow{e_1, e_2}, & \text{if } \Gamma \text{ satisfy condition (F)} \\ \{e_1, e_2\}, & \text{any other case} \end{cases}$$

**Lemma 3.14.** Let $\Gamma = \Gamma_{W, \mu} \subset U_+$ be a commutative discrete group with the form given in Proposition 3.8 if $\mu(W)$ contains no rational rotations and $W$ is not discrete, then

$$\Lambda_{Kul}(\Gamma) = \begin{cases} \overrightarrow{e_1, e_2}, & \text{if } \Gamma \text{ satisfy condition (F)} \\ \{e_1\} \cup \overrightarrow{e_2, e_3}, & \text{any other case} \end{cases}$$

**Proof.** Since $\mu(W)$ contains no rational rotations, $L_0(\Gamma) = \{e_1, e_2\}$ (see Lemma 3.10). Since $W$ is not discrete then there is a sequence $\{w_k\} \subset W$ such that $w_k \to 0$, and then $\mu(w_k) \to \infty$ or $\mu(w_k) \to 0$ (see Proposition 3.8). In the former case, we can conclude, using Table 3.1 that $\overrightarrow{e_2, e_3} \subset \Lambda_{Kul}(\Gamma)$. In the latter case we can consider the sequence $\{-w_k\}$ which satisfies $\mu(-w_k) \to \infty$ and we conclude again that $\overrightarrow{e_2, e_3} \subset \Lambda_{Kul}(\Gamma)$.

If the group $\Gamma$ satisfies condition (F), using Table 3.1 it follows that $\overrightarrow{e_1, e_2} \subset \Lambda_{Kul}(\Gamma)$ and using Observation 3.7 we conclude that $\Lambda_{Kul}(\Gamma) = \overrightarrow{e_1, e_2} \cup \overrightarrow{e_2, e_3}$. Analogously, if $\Gamma$ doesn't satisfy condition (F), $\Lambda_{Kul}(\Gamma) = \{e_1\} \cup \overrightarrow{e_2, e_3}$. \hfill $\square$

This previous lemma describes cases $C1.5$ and $C1.6$. Finally, Lemmas 3.11, 3.12, 3.13 and 3.14 together prove the following theorem.

**Theorem 3.15.** Let $\Gamma \subset PSL(3, \mathbb{C})$ be commutative discrete group with the form given in Proposition 3.8 then

$$\Lambda_{Kul}(\Gamma) = \begin{cases} \{e_1, e_2\}, & \text{Cases C1.3 or C1.4 with condition (F) not holding.} \\ \overrightarrow{e_1, e_2}, & \text{Cases C1.3 or C1.4, satisfying condition (F)} \\ \{e_1\} \cup \overrightarrow{e_2, e_3}, & \text{Case C1.1} \\ \overrightarrow{e_1, e_2} \cup \overrightarrow{e_2, e_3}, & \text{Cases C1.5 or C1.6 with condition (F) not holding.} \\ \overrightarrow{e_1, e_2} \cup \overrightarrow{e_2, e_3}, & \text{Cases C1.5 or C1.6, satisfying condition (F)} \\ \text{Case C1.2} \end{cases}$$

In Example 3.16 we give a group belonging to the case C1.6. This example is important because it was believed that only fundamental groups of Hopf surfaces had a Kulkarni limit set consisting of a line and a point. This is an example of a group with this limit set, which is not a fundamental group of a Hopf surface (since these groups are cyclic, see [Kat75]).
Example 3.16. Let \( W = \text{Span}_\mathbb{R} \{1, \sqrt{2}\} \) and let \( \mu : (W, +) \to (\mathbb{C}^*, \cdot) \) be the group homomorphism given by \( \mu(1) = e^{-1} \) and \( \mu(\sqrt{2}) = e^{\sqrt{2}} \), consider the commutative group \( \Gamma = \Gamma_{W, \mu} \). If \( \{w_n\} \subset W \) is a sequence of distinct elements such that \( w_n = p_n + q_n\sqrt{2} \to 0 \), then, without loss of generality assume that \( p_n, -q_n \to \infty \), then \( \mu(w_n) = \mu(1)^{p_n} \mu(\sqrt{2})^{q_n} = e^{-p_n + q_n\sqrt{2}} \to 0 \). Therefore \( \Gamma \) is discrete (Proposition 3.8).

On the other hand, if \( x \in \mu(W) \) then \( x = \mu(p + q\sqrt{2}) \) for some \( p, q \in \mathbb{Z} \) and then \( |x| = e^{-p + q\sqrt{2}} \). Observe that, \( |x| = 1 \) if and only if \(-p + q\sqrt{2} = 0 \) but this cannot happen because \( \{1, \sqrt{2}\} \) are rationally independent. This verifies that \( \Gamma \) belongs to the case C1.6. Using Theorem 3.15, if follows that \( \Lambda_{Kul}(\Gamma) = \{e_1\} \cup \{\varepsilon_2, \varepsilon_3\} \), unless condition (F) is satisfied. That is, unless there is a sequence \( \{w_n\} \subset W \) such that \( w_n \to \infty, \mu(w_n) \to 0 \) and \( w_n\mu(w_n)^{\frac{1}{3}} \to b \in \mathbb{C}^* \). Assume that this happens, since \( w_n \to \infty \), there are the following possibilities for the sequences \( \{p_n\}, \{q_n\} \subset \mathbb{Z} \):

1. If both sequences \( \{p_n\}, \{q_n\} \) are bounded, then there exists \( R > 0 \) such that \( |p_n|, |q_n| < R \) and then \( |p_n + q_n\sqrt{2}| < R(\sqrt{2} + 1) \). Therefore it’s impossible that \( w_n \to \infty \).
2. If \( \mu(w_n) \to 0 \) with \( p_n \to \infty \) and \( \{q_n\} \) bounded. Then
   \[
   w_n\mu(w_n)^{\frac{1}{3}} = p_n e^{-3p_n} \sqrt{2} q_n + q_n \sqrt{2} e^{-3p_n + 3q_n\sqrt{2}} \to 0
   \]
3. If \( \mu(w_n) \to 0 \) with \( q_n \to -\infty \) and \( \{p_n\} \) bounded. Then
   \[
   w_n\mu(w_n)^{\frac{1}{3}} = p_n e^{-3p_n + 3q_n\sqrt{2}} + q_n \sqrt{2} e^{3q_n\sqrt{2}} e^{-3p_n} \to 0.
   \]

Then condition (F) doesn’t hold. Therefore, \( \Lambda_{Kul}(\Gamma) = \{e_1\} \cup \{\varepsilon_2, \varepsilon_3\} \).

3.2. Case 2. Finally, we study the case where \( \Gamma \) is conjugate to a diagonal group. We start by describing the form of these groups.

Proposition 3.17. Let \( \Gamma \subset U_+ \) be a commutative subgroup such that each element of \( \Gamma \) has the form \( \text{Diag}(\alpha, \beta, \alpha^{-1}\beta^{-1}) \). Then there exist two multiplicative subgroups \( W_1, W_2 \subset (\mathbb{C}^*, \cdot) \) such that
\[
\Gamma = \Gamma_{W_1, W_2} = \{\text{Diag}(w_1, w_2, 1) \mid w_1 \in W_1, w_2 \in W_2\}.
\]

Proof. Let \( \Gamma \subset U_+ \) be a commutative group as in the hypothesis. Let \( \gamma = \text{Diag}(\alpha, \beta, \alpha^{-1}\beta^{-1}) \in \Gamma \), then \( \gamma = \text{Diag}(\alpha^2\beta, \alpha\beta^2, 1) \). Let \( W_1, W_2 \subset \mathbb{C}^* \) be the two multiplicative groups given by \( W_1 = \lambda_{13}(\Gamma) \) and \( W_2 = \lambda_{23}(\Gamma) \), then \( \gamma = \text{Diag}(w_1, w_2, 1) \), where \( w_1 := \alpha^2\beta = \lambda_{13}(\gamma) \in W_1 \) and \( w_2 := \alpha\beta^2 = \lambda_{23}(\gamma) \in W_2 \). Then \( \Gamma \) has the form given by (3.5).

Proposition 3.18. Let \( \Gamma \subset U_+ \) be a diagonal discrete group such that every element has the form \( \gamma = \text{Diag}(w_1, w_2, 1) \). Then \( \text{rank}(\Gamma) \leq 2 \).

Proof. Recall the group morphisms \( \lambda_{ij} \) defined in the beginning of Section 3. Let \( \mu : \Gamma \to \mathbb{R}^2 \) given by \( \mu(\gamma) = (\log|\lambda_{13}(\gamma)|, \log|\lambda_{23}(\gamma)|) \). Clearly, \( \mu \) is well defined and it’s a group homomorphism between \( \Gamma \) and the additive subgroup \( \mu(\Gamma) \subset \mathbb{R}^2 \). Furthermore, \( \text{Ker}(\mu) = \{\text{id}\} \) and then, \( \mu : \Gamma \to \mu(\Gamma) \) is a group isomorphism. Since \( \Gamma \) is discrete, then \( \mu(\Gamma) \) is discrete and therefore \( \text{rank}(\mu(\Gamma)) \leq 2 \), then \( \text{rank}(\Gamma) \leq 2 \).
Proposition 3.19. Let $\Gamma = \Gamma_{W_1, W_2}$ be a subgroup of $U_+$ with the form given by Proposition 3.17. Then $\Gamma \cong W_1 \oplus W_2$.

Proof. Let $H_1, H_2 \subset \Gamma$ be two subgroups given by $H_1 = \{\text{Diag}(w, 1, 1) \mid w \in W_1\}$ and $H_2 = \{\text{Diag}(w, 1, 2) \mid w \in W_2\}$. Observe that $H_1 \cong W_1$. Both $H_1$ and $H_2$ are normal in $\Gamma$ and $\Gamma = \langle H_1, H_2 \rangle$. Also, $H_1 \cap H_2 = \{id\}$, therefore $\Gamma = H_1 \oplus H_2 \cong W_1 \oplus W_2$.

The previous proposition, together with Proposition 3.18, imply that $\text{rank}(W_1) + \text{rank}(W_2) \leq 2$. If $\text{rank}(W_1) = 1$, $\text{rank}(W_2) = 0$ or $\text{rank}(W_1) = 0$, $\text{rank}(W_2) = 1$, then $\Gamma$ is cyclic, its Kulkarni limit set is described in Section 4.2 of [CNS13]. The cases $\text{rank}(W_1) = 2$, $\text{rank}(W_2) = 0$ and $\text{rank}(W_1) = 0$, $\text{rank}(W_2) = 2$ imply that $\Gamma$ is not discrete. Therefore we just have to describe the case:

$$\Gamma := \Gamma_{\alpha, \beta} = \{\text{Diag}(\alpha^n, \beta^m, 1) \mid n, m \in \mathbb{Z}\}.$$ 

for some $\alpha, \beta \in \mathbb{C}^*$ such that $|\alpha| \neq 1$ or $|\beta| \neq 1$.

Consider a sequence of distinct elements $\{\gamma_k\} \subset \Gamma$ given by $\gamma_k = \text{Diag}(\alpha^n, \beta^m, 1)$. In Table 2 we show all the possible quasi-projective limits of the sequences $\{\gamma_k\}$.

The following lemmas will be used to determine the Kulkarni limit set of the groups $\Gamma_{\alpha, \beta}$.

Lemma 3.20. Let $\Gamma_{\alpha, \beta} \subset U_+$ be a discrete group containing loxodromic elements then

$$L_0(\Gamma) = \begin{cases} \hat{e}_1, \hat{e}_2 \cup \{e_3\}, & \text{if } \alpha^n = \beta^m \text{ for some } n, m \in \mathbb{Z} \\ \{e_1, e_2, e_3\}, & \text{if there are no } n, m \in \mathbb{Z} \text{ such that } \alpha^n = \beta^m. \end{cases}$$

Proof. Let $\Gamma = \Gamma_{\alpha, \beta}$, suppose that $\alpha^n = \beta^m$ for some $n, m \in \mathbb{Z}$ and let $z = [z_1 : z_2 : z_3] \in L_0(\Gamma)$. Then $[\alpha^p z_1 : \beta^q z_2 : z_3] = [z_1 : z_2 : z_3]$ for an infinite number of $p, q \in \mathbb{Z}$.

If $z_3 \neq 0$ then, either $\alpha$ and $\beta$ are rational rotations or $z_1 = z_2 = 0$. If the former happens, then $\Gamma$ contains no loxodromic elements, contradicting the hypothesis. If the latter happens, then $z = e_3$. If $z_3 = 0$, we can assume without loss of generality that $z_1 \neq 0$. If $z_2 = 0$, then $z = e_2$. If $z_2 \neq 0$, then $\alpha^p = \beta^q$ for an infinite number of integers $p, q$ implies that $\alpha^j n = \beta^j m$ for any $j \in \mathbb{Z}$). Then $z = e_1, e_2$.

### Table 2. Quasi-projective limits in the diagonal case.

| Case | Conditions | Ker($\tau$) | Im($\tau$) |
|------|-------------|--------------|-------------|
| (i)  | $\alpha^n \to \infty$, $\beta^m \to \infty$ and $\alpha^n \beta^{-m} \to \infty$ | $\{e_1\}$ | |
| (ii) | $\alpha^n \to \infty$, $\beta^m \to \infty$ and $\alpha^{-n} \beta^{-m} \to \infty$ | $\{e_2\}$ | |
| (iii)| $\alpha^n \to 0$ and $\beta^m \to 0$ | $\{e_3\}$ | |
| (iv) | $\alpha^n \to b \in \mathbb{C}^*$ and $\beta^m \to 0$ | $\{e_1\} \cup \{e_3\}$ | $\{e_2\}$ |
| (v)  | $\alpha^n \to \infty$, $\beta^m \to \infty$ and $\alpha^n \beta^{-m} \to b \in \mathbb{C}^*$ | $\{e_1\}$ | $\{e_2\}$ |
| (vi) | $\alpha^n \to 0$ and $\beta^m \to b \in \mathbb{C}^*$ | $\{e_1\}$ | $\{e_2\}$ | $\{e_3\}$ |
If there are no \( n, m \in \mathbb{Z} \) such that \( \alpha^n = \beta^m \), then no point in \( \mathbb{H} \setminus \{e_1, e_2\} \) satisfies that \( [\alpha^p z_1 : \beta^q z_2 : z_3] = [z_1 : z_2 : z_3] \) for an infinite number of \( p, q \in \mathbb{Z} \). □

**Lemma 3.21.** Let \( \Gamma_{\alpha, \beta} \subset U_+ \) be a discrete group containing loxodromic elements, then

\[
L_1(\Gamma) = \begin{cases} 
\{e_1, e_2\}, & \text{if } |\alpha| > 1 > |\beta| \text{ or } |\alpha| < |\beta| < 1 \\
\{e_1, e_3\}, & \text{if } |\alpha| > |\beta| > 1 \text{ or } |\alpha| < |\beta| < 1 \\
\{e_1\} \cup \{e_2, e_3\}, & \text{if } \beta \text{ is an irrational rotation.}
\end{cases}
\]

**Proof.** Let \( \alpha, \beta \in \mathbb{C}^* \), since \( \Gamma = \Gamma_{\alpha, \beta} \) contains loxodromic elements, we can assume without loss of generality that \( |\alpha| \neq 1 \). Let \( \{\gamma_k\} \subset \Gamma \) be a sequence of distinct elements given by \( \gamma_k = \text{Ding}(\alpha^k, \beta^k, 1) \). The set \( L_0(\Gamma) \) is given by Lemma 3.20 and let \( z = [z_1 : z_2 : z_3] \in \mathbb{C}P^2 \setminus L_0(\Gamma) \). In any of the two possible outcomes for \( L_0(\Gamma) \) described in Lemma 3.20 we have

\[
\gamma_k \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \alpha^k z_1 \\ \beta^k z_2 \\ z_3 \end{bmatrix}.
\]

We have two essentially different sequences in \( \Gamma \), \( \{\gamma^k\} \) and \( \{\gamma^{-k}\} \). Since \( |\alpha| \neq 1 \), we can assume, without loss of generality that \( |\alpha| > 1 \), then \( \alpha^k \to \infty \) as \( k \to \infty \).

We have three cases:

1. \( |\beta| < 1 \), then \( \alpha^k \to 0 \) \( k \to \infty \), then \( \gamma_k z \to e_1 \) as \( k \to \infty \). Analogously, \( \gamma_k z \to e_2 \) as \( k \to -\infty \).

2. \( |\beta| = 1 \), \( \beta \) cannot be a rational rotation because \( \Gamma \) is torsion free, then \( \beta \) is an irrational rotation. Then, there are subsequences (still denoted by \( \{\gamma_k\}\)) converging to any \( b \in \mathbb{C} \), \( |b| = 1 \). Then \( \gamma_k z \to e_1 \) as \( k \to \infty \) and \( \gamma_k z \to [0 : b^{-1} z_2 : z_3] \) as \( k \to -\infty \).

3. \( |\beta| > 1 \), then \( \alpha^k \to \infty \) as \( k \to \infty \). We have three subcases:
   - \( \alpha^k \beta^{-k} \to \infty \) as \( k \to \infty \), then \( \gamma_k z \to e_1 \) as \( k \to \infty \). Also, \( \gamma_k z \to e_3 \) as \( k \to -\infty \).
   - \( \alpha^k \beta^{-k} \to b \in \mathbb{C}^* \) as \( k \to \infty \), then \( \gamma_k z \to [b : z_2] : 0 \) as \( k \to \infty \), contradicting that \( \Gamma \) is discrete.
   - \( \alpha^k \beta^{-k} \to 0 \) as \( k \to \infty \), \( \gamma_k z \to e_2 \) as \( k \to \infty \) and \( \gamma_k z \to e_2 \) as \( k \to -\infty \), and \( \gamma_k z \to e_3 \) as \( k \to -\infty \).

\( \square \)
Proposition 3.22. Let $\Gamma_{\alpha, \beta} \subset U_+$ be a discrete group containing loxodromic elements, then

(i) $\Lambda_{Kul}(\Gamma) = \left\{ e_1, e_2 \right\}$ in Cases [D1] and [D2].
(ii) $\Lambda_{Kul}(\Gamma) = \left\{ e_1, e_2, e_3 \right\}$ in Cases [D3] and [D4].
(iii) $\Lambda_{Kul}(\Gamma) = \left\{ e_1, e_2 \right\}$ in Case [D5].

Proof. We consider a sequence $\gamma$ in $\Gamma$. We will determine the quasi-projective limits of this sequence using Table 2 and then, using Proposition 1.5, we determine the set $L_2(\Gamma)$ and therefore, $\Lambda_{Kul}(\Gamma)$.

- In Case [D1], $\alpha^k \to \infty$ and $\beta^k \to 0$ and the quasi-projective limit is given by (i) of Table 2. Therefore, the orbits of compact subsets of $\mathbb{C}P \setminus L_0(\Gamma) \cup L_1(\Gamma)$ accumulate on $\{ e_1 \}$ and $\{ e_2 \}$ under the sequences $\{ \gamma^k \}$ and $\{ \gamma_k \}$ respectively. Then $\Lambda_{Kul}(\Gamma) = \left\{ e_1, e_2 \right\}$.
- In Case [D2], $\alpha^k \to \infty$, $\beta^k \to \infty$ and $\alpha^k \beta^{-k} \to \infty$, then the quasi-projective limit is also given by (i) of Table 2. We have again, $\Lambda_{Kul}(\Gamma) = \left\{ e_1, e_2 \right\}$.
- In Cases [D3] and [D4], the orbits of compact subsets of $\mathbb{C}P \setminus L_0(\Gamma) \cup L_1(\Gamma)$ accumulate on $\{ e_1 \}$ and (accumulate on $\{ e_1 \}$) under the sequence of $\{ \gamma^k \}$.
- In Case [D5], we have $|\beta| = 1$ and we can assume without loss of generality that $|\alpha| > 1$. Then, the quasi-projective limit of the sequence $\{ \gamma_k \}$ is $\tau = \text{Diag}(1, 0, 0)$ and therefore, the orbits of compact subsets of $\mathbb{C}P^2 \setminus \text{Ker}(\tau) = \mathbb{C}P^2 \setminus \xi_1, \xi_2, \xi_3$ accumulate on $\text{Im}(\tau) = \{ e_1 \}$. If $K \subset \mathbb{C}P^2 \setminus \left\{ \xi_1, \xi_2, \xi_3 \right\}$, then $K \subset \mathbb{C}P^2 \setminus \xi_1, \xi_2, \xi_3$. Therefore $L_2(\Gamma) = \{ e_1 \}$, we finally conclude that $\Lambda_{Kul}(\Gamma) = \left\{ e_1, e_2 \right\}$. $\square$

4. Non-commutative triangular groups

In this section, we describe the non-commutative upper triangular discrete subgroups of $\text{PSL}(3, \mathbb{C})$.

4.1. Restrictions on the elements of a non-commutative group. In this subsection we study the restrictions that complex homotheties, irrational screws and irrational ellipto-parabolic elements impose on the groups they belong. These propositions generalize the results presented in Chapter 5 of [GU18] and will be useful in Subsection 4.3.

For the sake of simplicity, let us denote

$$h_{x,y} = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \quad x, y \in \mathbb{C}.$$  

We start with complex homotheties. For the rest of the paper we will say that $\gamma \in \text{PSL}(3, \mathbb{C})$ is a type I complex homothety if, up to conjugation, $\gamma = \text{Diag}(\lambda^{-2}, \lambda, \lambda)$ for some $\lambda \in \mathbb{C}^*$ with $|\lambda| \neq 1$. Analogously, we say that $\gamma$ is a type III complex homothety if, up to conjugation, $\gamma = \text{Diag}(\lambda, \lambda, \lambda^{-2})$ for some $|\lambda| \neq 1$.

Proposition 4.1. Let $\gamma \in \text{PSL}(3, \mathbb{C})$ be a type I complex homothety. Let $\alpha \in U_+ \setminus \langle \gamma \rangle$ such that $\alpha$ is neither a complex homothety or a screw, then the group $\langle \alpha, \gamma \rangle$ is discrete if and only if $\alpha$ leaves $\Lambda_{Kul}(\gamma)$ invariant.
**Proof.** We have that $\Lambda_{Kul}(\gamma) = \{ e_1 \} \cup \hat{e}_2, e_3$ (see Proposition 4.2.23 of [CNS13]). Denote $\alpha = [\alpha_{ij}]$, a straightforward calculation shows that $\alpha$ leaves $\hat{e}_2, e_3$ invariant if and only if $\alpha_{12} = \alpha_{13} = 0$.

First we prove that if $\langle \alpha, \gamma \rangle$ is discrete then $\alpha$ leaves $\Lambda_{Kul}(\gamma)$ invariant. Assume that $\alpha$ doesn’t leave $\Lambda_{Kul}(\gamma)$ invariant, this means that $|\alpha_{12}| + |\alpha_{13}| \neq 0$. Let $\{g_n\} \subset \langle \alpha, \gamma \rangle$ be the sequence of distinct elements given by $g_n := \gamma^n \alpha \gamma^{-n}$. Clearly,

\[ g_n \rightarrow \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & \alpha_{33} \end{bmatrix} \in \text{PSL}(3, \mathbb{C}). \]

Therefore $\langle \alpha, \gamma \rangle$ is not discrete.

Now we prove that if $\alpha$ leaves $\Lambda_{Kul}(\gamma)$ invariant then $\langle \alpha, \gamma \rangle$ is discrete. We rewrite $\gamma = \text{Diag}(\lambda^{-3}, 1, 1)$. Suppose that $\langle \alpha, \gamma \rangle$ is not discrete, then there exists a sequence of distinct elements $\{w_k\} \subset \langle \alpha, \gamma \rangle$ such that $w_k \rightarrow \text{id}$. Since $\alpha$ leaves $\Lambda_{Kul}(\gamma)$ invariant then $[\alpha, \gamma] = \text{id}$. Then the sequence $w_k$ can be expressed as

\[ w_k = \gamma^i \alpha^j. \]

Observe that $h \gamma h^{-1} = \gamma$ for any $h \in \text{PSL}(3, \mathbb{C})$ with the form

\[ h = \begin{bmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & h_{23} \\ 0 & 0 & h_{33} \end{bmatrix}. \]

Furthermore, there exists an element $h \in \text{PSL}(3, \mathbb{C})$ with the previous form such that $h \alpha h^{-1}$ has one of the following forms:

1. \[ h \alpha h^{-1} = \text{Diag} (\alpha_{11}^{-1} \alpha_{33}, \alpha_{22}, \alpha_{33}) \]
   if $\Pi(\alpha)$ is loxodromic. In this case, from the previous equation and (4.1) it follows $w_k = \text{Diag} (\lambda^{-3i} \alpha_{22}^{-j} \alpha_{33}^{-j}, \alpha_{22}, \alpha_{33}) \rightarrow \text{id}$. Therefore $\alpha_{22}^j, \alpha_{33}^j \rightarrow 1$, but then we cannot have $\lambda^{-3i} \alpha_{22}^{-j} \alpha_{33}^{-j} \rightarrow 1$, since $|\lambda| \neq 1$.

2. \[ h \alpha h^{-1} = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \]
   if $\Pi(\alpha)$ is parabolic. Then we have

\[ w_k = \begin{bmatrix} \lambda^{-3i} \alpha_{11}^j & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{id}. \]

But this is clearly impossible.

3. \[ h \alpha h^{-1} = \text{Diag} (\alpha_{11}, \alpha_{22}, \alpha_{22}), \] with $|\alpha_{22}| = 1$, if $\Pi(\alpha)$ is elliptic. But in this case $\alpha$ is a complex homothety or a screw.

Therefore neither of the three cases can occur and thus, $\langle \alpha, \gamma \rangle$ is discrete. \qed
From the proof of the previous proposition, we have the following immediate consequence.

**Corollary 4.2.** If $\Gamma \subset U_+$ is a discrete subgroup and $\Gamma$ contains a type I complex homothety as in the previous proposition, then every element of $\Gamma$ has the form

\[
\alpha = \begin{bmatrix}
\alpha_{11} & 0 & 0 \\
0 & \alpha_{22} & \alpha_{23} \\
0 & 0 & \alpha_{33}
\end{bmatrix}.
\]

**Proposition 4.3.** Let $\Gamma \subset U_+$ be a discrete group containing a type I complex homothety, if the control group $\Pi(\Gamma)$ is discrete then

1. $\Pi(\Gamma)$ is purely parabolic or purely loxodromic.
2. $\Gamma$ is commutative.

**Proof.** Using Corollary 4.2 we know that every element of $\Gamma$ has the form given by (4.2). Denote the control group by $\Sigma = \Pi(\Gamma)$:

1. If $\Sigma$ contains an elliptic element $\Pi(\alpha) = \text{Diag}(e^{2\pi i \theta_1}, e^{2\pi i \theta_2})$, then $\alpha = \text{Diag}(e^{-2\pi i (\theta_1 + \theta_2)}, e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) \in \Gamma$ is elliptic. If $\alpha$ has infinite order, this contradicts that $\Gamma$ is discrete and if $\alpha$ has finite order, this contradicts that $\Gamma$ is torsion free. Therefore, $\Sigma$ has no elliptic elements.

2. If $\Sigma$ has a parabolic element $\Pi(\alpha)$ then $\Sigma$ is purely parabolic. Otherwise, there exists $\beta \in \Gamma$ such that $\Pi(\beta)$ is loxodromic and then, using the Jørgensen inequality, it would follow that $\langle A, B \rangle \subset \Sigma$ is not discrete. Then $\Gamma$ is commutative, since every element of $\Gamma$ has the form

\[
\alpha = \begin{bmatrix}
\alpha_{11} & 0 & 0 \\
0 & 1 & \alpha_{23} \\
0 & 0 & 1
\end{bmatrix}.
\]

3. If $\Sigma$ has a loxodromic element $\Pi(\alpha)$, then $\Sigma$ is purely loxodromic (see the previous case). Let $\Pi(\beta) \in \Sigma \setminus \{\alpha\}$ be a loxodromic element, $\Pi(\alpha)$ and $\Pi(\beta)$ share at least one fixed point, since they both are upper triangular elements. We have two cases:

   a. If $\Pi(\alpha)$ and $\Pi(\beta)$ share exactly one fixed point then they have the form

\[
\Pi(\alpha) = \begin{bmatrix}
\alpha_{22} & \alpha_{23} \\
0 & \alpha_{33}
\end{bmatrix}, \quad \Pi(\beta) = \begin{bmatrix}
\beta_{22} & \beta_{23} \\
0 & \beta_{33}
\end{bmatrix}
\]

with $\alpha_{23}, \beta_{23} \neq 0$ and $[\alpha_{23} : \alpha_{33} - \alpha_{22}] \neq [\beta_{23} : \beta_{33} - \beta_{22}]$. Then $[\Pi(\alpha), \Pi(\beta)] \neq id$ is parabolic, contradicting that $\Sigma$ is purely loxodromic.

   b. If $\text{Fix}(\Pi(\alpha)) = \text{Fix}(\Pi(\beta))$, then every element of $\Gamma$ has the form

\[
\alpha = \begin{bmatrix}
\alpha_{11} & 0 & 0 \\
0 & 1 & \alpha_{23} \\
0 & 0 & 1
\end{bmatrix},
\]

with $\text{Fix}(\alpha_1) = \{e_1, e_2\}$ and $\text{Fix}(\alpha_2) = \{e_2, p\}$ for some point $p$. Then, analogously to the proof of proposition 4.26, $\Gamma$ is commutative.

**Proposition 4.4.** Let $\Gamma \subset U_+$ be a discrete group such that $\Sigma = \Pi(\Gamma)$ is not discrete and $\Lambda_G(\Sigma) = S^1$. Then $\Gamma$ cannot contain a type I complex homothety.
Proof. Suppose that $\Gamma$ contains such a complex homothety. Then, by Corollary 4.2, each element in $\Gamma$ has the form given in (4.2). Since $\Lambda_{Gr}(\Sigma) = S^1$, then $\Gamma$ is a non-elementary, non-discrete group and since $\Lambda_{Gr}(\Sigma)$ is the closure of fixed points of loxodromic elements of $\Sigma$, then, there are an infinite number of loxodromic elements in $\Sigma$ sharing exactly one fixed point. Let $f \in \Sigma$ such that $\Pi(f)$ is loxodromic and $f = \text{Diag}(\alpha^{-1}, \alpha, 1)$, with $|\alpha| < 1$. Let $\gamma_1, \gamma_2 \in \Gamma$ be two elements such that $\Pi(\gamma_1), \Pi(\gamma_2)$ are loxodromic and share exactly one fixed point, also $\gamma_i \neq f$, for $i = 1, 2$. It follows that $\gamma := [\gamma_1, \gamma_2] \neq \text{id}$ is a parabolic element. Let $\{f^k \gamma f^{-k}\} \subset \Gamma$ be the sequence given by $f^k \gamma f^{-k} = h_{0,y}$, where $y = \alpha^k b_0$ and $b_0$ is the entry 23 of $\gamma$. Then $f^k \gamma f^{-k} \to \text{id}$, contradicting that $\Gamma$ is discrete. If $|\alpha| > 1$, we consider the sequence $\{f^{-k} \gamma f^k\}$ instead. This concludes the proof. □

Proposition 4.5. Let $\Gamma \subset U_+$ be a non-commutative discrete group such that $\Sigma = \Pi(\Gamma)$ is not discrete and $|\Lambda_{Gr}(\Sigma)| = 2$. Then $\Gamma$ cannot contain a type I complex homothety.

Proof. Suppose that $\Gamma$ contains such a complex homothety. Then, by Corollary 4.2, each element in $\Gamma$ has the form given by (4.2). If $|\Lambda_{Gr}(\Sigma)| = 2$, then, up to conjugation, every element of $\Sigma$ has the form $\text{Diag}(\beta, \delta)$ for some $\beta, \delta \in \mathbb{C}^*$. Then, using (4.2), it follows that each element in $\Gamma$ is diagonal and hence, $\Gamma$ would be commutative. □

Proposition 4.6. Let $\Gamma \subset U_+$ be a non-commutative discrete group such that $\Sigma = \Pi(\Gamma)$ is not discrete and $|\Lambda_{Gr}(\Sigma)| = 1$. Then $\Gamma$ cannot contain a type I complex homothety.

Proof. Suppose that there exists such a complex homothety $\gamma \in \Gamma$. Then, by Corollary 4.2, each element in $\Gamma$ has the form given by (4.2). Since $\Sigma = \Pi(\Gamma)$ is not discrete and $|\Lambda_{Gr}(\Sigma)| = 1$, then $\Sigma$ is a dense subgroup of $\text{Epa}(\mathbb{C})$ containing parabolic elements (see Theorem 2.14 of [CS14]). Then, every element of $\Sigma$ has the form $\Pi(\mu) = \begin{bmatrix} a & b & 0 \\ 0 & a^{-1} & 0 \end{bmatrix}$, for $|a| = 1$ and $b \in \mathbb{C}^*$.

This implies that every element of $\Gamma$ has the form

$$\mu = \begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha \alpha^{-1} & \alpha \alpha^{-1} \end{bmatrix},$$

for some $\alpha \in \mathbb{C}^*$.

We have two possibilities:

1. If every element of $\Sigma$ is parabolic, then $\Gamma$ would be commutative, since every element of $\Gamma \setminus \{\text{id}\}$ has the form

$$\mu = \begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & \alpha \alpha^{-1} \end{bmatrix},$$

for some $\alpha \in \mathbb{C}^*$.

2. If there is an elliptic element $\Pi(\gamma) \in \Sigma$ then $\Pi(\gamma) = \text{Diag}(e^{2\pi i \theta}, e^{-2\pi i \theta})$, for some $\theta \notin \mathbb{Z}$. Then

$$\gamma = \text{Diag}(\beta^{-2}, \beta e^{2\pi i \theta}, \beta e^{-2\pi i \theta}),$$

for some $\beta \in \mathbb{C}^*$.

Observe that, if $\theta \in \mathbb{Q}$ then $\lambda_{23}(\gamma) = e^{4\pi i \theta}$ would be a torsion element in $\lambda_{23}(\Gamma)$. This contradictions implies that, if there exists an elliptic element in $\Sigma$, then
\( \theta \in \mathbb{R} \setminus \mathbb{Q} \). On the other hand, since \( \Sigma \) is not discrete, there is a sequence of distinct elements \( \{ \Pi(\mu_k) \} \subset \Sigma \) such that \( \Pi(\mu_k) \to \text{id} \). We have two cases:

(i) If \( \{ \Pi(\mu_k) \} \) contains an infinite number of parabolic elements, then, considering an adequate subsequence, we can assume that \( \{ \Pi(\mu_k) \} \) is a sequence of distinct parabolic elements and we denote

\[
\Pi(\mu_k) = \begin{bmatrix} 1 & b_k \\ 0 & 1 \end{bmatrix}
\]

where \( \{ b_k \} \subset \mathbb{C}^* \) is a sequence of distinct elements such that \( b_k \to 0 \). Then,

\[
\mu_k = \begin{bmatrix} \alpha_k^{-2} & 0 & 0 \\ 0 & \alpha_k & \alpha_k b_k \\ 0 & 0 & \alpha_k \end{bmatrix}, \text{ for some } \{ \alpha_k \} \subset \mathbb{C}^*.
\]

Let \( \{ \xi_k \} \subset \Gamma \) be the sequence of distinct elements given by \( \xi_k = [\gamma, \mu_k] = h_{0,y} \), where \( y = b_k \left( 1 - e^{-4\pi i \theta} \right) \) and \( \gamma \) is an elliptic element in \( \Sigma \) given by (4.3). Then \( \xi_k \to \text{id} \), contradicting that \( \Gamma \) is discrete.

(ii) If \( \{ \Pi(\mu_k) \} \) contains only a finite number of parabolic elements, then we can assume that the whole sequence \( \{ \Pi(\mu_k) \} \) is made up of irrational elliptic elements. We denote,

\[
\Pi(\mu_k) = \begin{bmatrix} e^{2\pi i \theta_k} & b_k \\ 0 & e^{-2\pi i \theta_k} \end{bmatrix},
\]

with \( b_k \to 0 \). Since \( \{ \theta_k \} \subset \mathbb{R} \setminus \mathbb{Q} \), we can pick an adequate subsequence of \( \{ \Pi(\mu_k) \} \), still denoted in the same way, such that \( \theta_k \to 0 \) by distinct elements \( \{ \theta_k \} \). Let \( \{ \Pi(\sigma_k) \} \subset \Sigma \) the sequence of distinct elements given by \( \Pi(\sigma_k) = [\Pi(\mu_k), \Pi(\mu_{k+1})] \). This sequence is made up of distinct parabolic elements of \( \Sigma \), and since \( \theta_k \to 0 \), we have that \( \Pi(\sigma_k) \to id \). Applying the same argument as in the last case (i), we get a contradiction.

Together, Propositions 4.3, 4.4, 4.5, 4.6 and Lemma 4.14 imply the following conclusion.

**Corollary 4.7.** Let \( \Gamma \subset U_+ \) be a non-commutative, torsion-free discrete subgroup, then \( \Gamma \) cannot contain a type I complex homothety.

The proofs of the previous propositions can be repeated in an analogously way to prove the following corollary.

**Corollary 4.8.** Let \( \Gamma \subset U_+ \) be a non-commutative, torsion-free discrete subgroup, then \( \Gamma \) cannot contain a type III complex homothety.

**Lemma 4.9.** Let \( \Sigma \) be a non-discrete, upper triangular subgroup of \( \text{PSL}(2, \mathbb{C}) \) such that \( \Lambda_{Gr}(\Sigma) = S^1 \). Then the parabolic part of \( \Sigma \) is a non-discrete group.

**Proof.** Let \( \Sigma_p \) be the parabolic part of \( \Sigma \). Let \( g \in \Sigma \) be a loxodromic element such that \( g = \text{Diag}(\alpha, \alpha^{-1}) \), for some \( |\alpha| < 1 \). Let \( h_1, h_2 \in \Sigma \) be two loxodromic elements such that \( \text{Fix}(h_1) \neq \text{Fix}(h_2) \) and \( \text{Fix}(h_1) \neq \text{Fix}(g) \). Let \( h = [h_1, h_2] \), then \( h \neq \text{id} \) is a parabolic element in \( \Sigma \). Hence, \( f_k := g^k h g^{-k} \to \text{id} \) and therefore \( \Sigma \) is not discrete. The fact that one can take the elements \( g, h_1, h_2 \) is a consequence of \( \Lambda_{Gr}(\Sigma) = S^1 \). \( \square \)
We now look at the case of irrational screws.

**Proposition 4.10.** Let $\gamma \in U_+$ be an irrational screw given by
\[
\gamma = \text{Diag} (\beta^{-2}, \beta e^{4\pi i \theta}, \beta e^{2\pi i \theta}),
\]
for some $|\beta| \neq 1$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $\alpha \in U_+ \setminus \langle \gamma \rangle$, if $\langle \alpha, \gamma \rangle$ is discrete then $\alpha$ is diagonal.

**Proof.** Let $\gamma, \alpha \in U_+$ be as in the statement of the proposition, denote $\alpha = [\alpha_{ij}]$ and assume, without loss of generality that $|\beta| > 1$. Suppose that $\alpha$ is not diagonal. Since $\theta \in \mathbb{R} \setminus \mathbb{Q}$, there is a subsequence, still denoted by the index $k$, such that $e^{2k\pi i \theta} \to 1$. If $|\beta| > 1$, consider the sequence $\{\gamma - k\alpha \gamma^k\} \subset \langle \alpha, \gamma \rangle$. Since $\alpha_{12} \neq 0$, $\alpha_{13} \neq 0$ or $\alpha_{23} \neq 0$, this is a sequence of distinct elements such that $\gamma - k\alpha \gamma^k \to \text{Diag} (\alpha_{11}, \alpha_{22}, \alpha_{33})$. Then $\langle \alpha, \gamma \rangle$ is not discrete. \hfill \Box

We have the following immediate consequence of Proposition 4.10.

**Corollary 4.11.** Let $\Gamma \subset U_+$ be a discrete subgroup and $\gamma \in \Gamma$ an irrational screw as in proposition 4.10, then $\Gamma$ is commutative.

**Corollary 4.12.** Let $\Gamma \subset U_+$ be a discrete subgroup such that $\Sigma = \Pi(\Gamma)$ is non-discrete and $\Sigma = \text{Rot}_\infty$. Then $\Gamma$ is commutative.

**Proof.** The elements of $\Sigma$ have the form $\Pi(\gamma) = \text{Diag} (e^{2\pi i \theta}, e^{-2\pi i \theta})$. Then every element of $\Gamma \setminus \{id\}$ is diagonalizable and therefore, it is an irrational screw. Then $\Gamma$ contains only screws. If there were a rational screw $\gamma \in \Gamma$ then $\lambda_{23}(\Gamma)$ would not be a torsion free group. Then $\Gamma$ contains only irrational screws, it follows from corollary 4.11 that $\Gamma$ is commutative. \hfill \Box

The following is an immediate consequence of Lemma 5.10 of [BCNS18].

**Proposition 4.13.** Let $\Gamma \subset U_+$ be a discrete group containing an irrational ellipto-parabolic element $\gamma$ with one of the following two forms:
\[
\gamma = \begin{bmatrix}
  e^{-4\pi i \theta} & \beta & \gamma \\
  0 & e^{2\pi i \theta} & \mu \\
  0 & 0 & e^{2\pi i \theta}
\end{bmatrix}, \quad \text{or} \quad \gamma = \begin{bmatrix}
  e^{2\pi i \theta} & \beta & \gamma \\
  0 & e^{2\pi i \theta} & \mu \\
  0 & 0 & e^{-4\pi i \theta}
\end{bmatrix},
\]
with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $\Gamma$ is commutative.

**Lemma 4.14.** Let $\Gamma \subset U_+$ be a torsion free, non-commutative, discrete group such that $\Sigma = \Pi(\Gamma)$ is not discrete. Then $\Lambda_{Gr}(\Sigma) \neq \emptyset$.

**Proof.** Let us suppose that $\Lambda_{Gr}(\Sigma) = \emptyset$. Then, according to Theorem 1.14 of [CSI14], we have two possibilities:

(i) $\Sigma = \text{SO}(3)$. Since $\Gamma$ is solvable, $\Sigma$ is solvable. However $\text{SO}(3)$ is not solvable.

(ii) $\Sigma = \text{Dih}_\infty$ or $\Sigma = \text{Rot}_\infty$. Observe that $\Sigma \cap \text{Rot}_\infty \neq \emptyset$, otherwise $\Sigma$ would be discrete. Let $\Pi(\gamma) \in \Sigma \cap \text{Rot}_\infty$, then $\Pi(\gamma) = \text{Diag} (e^{2\pi i \theta}, e^{-2\pi i \theta})$ and, since $\lambda_{23}(\Gamma)$ is a torsion free group, it follows $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and then, $\gamma$ is an irrational screw. Conjugating $\gamma$ by a suitable element of $\text{PSL}(3, \mathbb{C})$ and applying Corollary 4.12, $\Gamma$ would be commutative.

These contradictions verify the lemma. \hfill \Box
We have shown that, whenever a discrete subgroup $\Gamma \subset U_+$ contains a complex homothety, an irrational screw or an irrational ellipto-parabolic element then $\Gamma$ has to be commutative.

4.2. The core of a group. In this subsection we define and study an important purely parabolic subgroup of $\Gamma$ which determines the dynamics of $\Gamma$. Let us define

$$\text{Core}(\Gamma) = \text{Ker}(\Gamma) \cap \text{Ker}(\lambda_{12}) \cap \text{Ker}(\lambda_{23}).$$

We denote the elements of $\text{Core}(\Gamma)$ by

$$g_{x,y} = \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We will use this notation for the rest of the work. It is straightforward to verify that $\Lambda_{\text{Kul}}(\text{Core}(\Gamma)) = \bigcup_{g_{x,y} \in \text{Core}(\Gamma)} e_1, [0 : -y : x]$. We denote this pencil of lines by $C(\Gamma) = \Lambda_{\text{Kul}}(\text{Core}(\Gamma))$.

**Proposition 4.15.** Let $\Gamma \subset U_+$ be a discrete group, then every element of $\Gamma$ leaves $C(\Gamma)$ invariant.

**Proof.** For $g_{x,y} \in \text{Core}(\Gamma)$, denote by $\ell_{x,y} = e_1, [0 : -y : x] \subset C(\Gamma)$ the line determined by the element $g_{x,y}$. Let $\gamma = [\gamma_{ij}] \in \Gamma$, observe that

$$\gamma g_{x,y} \gamma^{-1} = g_{\gamma_{11} x, \gamma_{22} y - \gamma_{23} x} \in \text{Core}(\Gamma).$$

This element determines the line $\ell_{\gamma_{33} x, \gamma_{22} y - \gamma_{23} x}$. Therefore, this is a line in $C(\Gamma)$. On the other hand, a direct calculation shows that $\gamma(\ell_{x,y}) = \ell_{\gamma_{33} x, \gamma_{22} y - \gamma_{23} x}$. This proofs that $\gamma$ leaves $C(\Gamma)$ invariant. \qed

We have shown that every element $\gamma \in \Gamma$ moves the line $\ell_{x,y}$ to the line $\gamma(\ell_{x,y})$ according to the proof of the last proposition, also, the line $\ell_1, e_2$ is fixed by every element of $\Gamma$. In particular, loxodromic elements leave $C(\Gamma)$ invariant and, in turn, this invariance imposes strong restrictions on these loxodromic elements.

We say that the discrete group $\Gamma \subset U_+$ is conic if $\overline{C(\Gamma)}$ is a cone homeomorphic to the complement of $\mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-)$. If $\mathbb{C}(\Gamma)$ is a line, we say that $\Gamma$ is non-conic. In [BCNS18], conic groups are called irreducible and non-conic groups, reducible. The next proposition follows immediately from Propositions 4.3 and 4.10.

**Proposition 4.16.** Let $\Gamma \subset U_+$ be a discrete torsion free group. If $\text{Ker}(\Gamma)$ is finite, then $\text{Ker}(\Gamma) = \{id\}$.

**Proposition 4.17.** Let $\Gamma \subset U_+$ be a non-commutative discrete group such that one of the following hypothesis hold:

- Its control group $\Sigma = \Pi(\Gamma)$ is discrete.
- $\Sigma$ is not discrete and $\Lambda_{\text{Gr}}(\Sigma) = \mathbb{S}^1$.
- $\Sigma$ is not discrete and $|\Lambda_{\text{Gr}}(\Sigma)| = 2$.

Then $\text{Ker}(\Gamma) = \text{Core}(\Gamma)$. 
Proof. We only have to prove that \( \text{Ker}(\Gamma) \subseteq \text{Core}(\Gamma) \). Let \( \gamma \in \text{Ker}(\Gamma) \), then

\[
\gamma = \begin{bmatrix}
\alpha^{-2} & \gamma_{12} & \gamma_{13} \\
0 & \alpha & 0 \\
0 & 0 & \alpha
\end{bmatrix}, \text{ for some } \alpha \in \mathbb{C}^*
\]

- If \( |\alpha| \neq 1 \) then \( \gamma \) is a complex homothety. Using the hypotheses and Propositions 4.3 and 4.5, \( \Gamma \) would be commutative.
- If \( |\alpha| = 1 \) but \( \alpha \neq 1 \), then \( \alpha = e^{2\pi i \theta} \) for some \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) (otherwise, \( \lambda_{12}(\Gamma) \) would not be a torsion free group). Then \( \gamma \) is an irrational ellipto-parabolic element, and then \( \Gamma \) would be commutative (Proposition 4.13).

This contradictions imply that \( \alpha = 1 \) and therefore, \( \gamma \in \text{Core}(\Gamma) \).

The proof of the following corollary is similar to the proof of the previous proposition.

Corollary 4.18. Under the hypotheses of the previous proposition, if

\[
\gamma = \begin{bmatrix}
\alpha^{-2} & \gamma_{12} & \gamma_{13} \\
0 & \alpha & \gamma_{23} \\
0 & 0 & \alpha
\end{bmatrix} \in \Gamma,
\]

then \( \alpha = 1 \).

Proposition 4.19. Let \( \Gamma \subset U_+ \) be a non-commutative, discrete group such that \( |\text{A}_{Gr}(\Pi(\Gamma))| = 2 \). Let \( \ell \) be a line passing through \( e_1 \) such that \( \ell \neq \overrightarrow{e_1, e_2}, \ell \neq \overrightarrow{e_1, e_3} \) and \( \ell \subset \mathcal{C}(\Gamma) \), then \( \Gamma \) is conic.

Proof. If \( |\text{A}_{Gr}(\Pi(\Gamma))| = 2 \) then, up to conjugation, every element of \( \Sigma \) has the form

\[
\gamma = \text{Diag}(\beta, \delta), \text{ where } \beta \neq \delta.
\]

Then, every element of \( \Gamma \) has the form

\[
(4.4) \quad \gamma = \begin{bmatrix}
\gamma_{22}^{-1} \gamma_{33}^{-1} & \gamma_{12} & \gamma_{13} \\
0 & \gamma_{22} & 0 \\
0 & 0 & \gamma_{33}
\end{bmatrix}, \text{ with } |\gamma_{22}| \neq |\gamma_{33}|.
\]

Let us assume that \( \Gamma \) is not conic, with \( \mathcal{C}(\Gamma) = \ell \). Let \( [0 : -y : x] \in \overrightarrow{e_2, e_3} \) such that \( \ell = \ell_{x,y} \), then, by hypothesis, \( x \neq 0 \) and \( y \neq 0 \). Since \( |\text{Ker}(\Gamma)| = \infty \), using Proposition 4.17 it follows that \( \text{Ker}(\Gamma) = \text{Core}(\Gamma) \). Since \( \text{Ker}(\Gamma) \) is a normal subgroup of \( \Gamma \), so is \( \text{Core}(\Gamma) \). Therefore, if \( \gamma = [\gamma_{ij}] \in \Gamma \), then \( \gamma g_{x,y} \gamma^{-1} = h_{x', y'} \in \text{Core}(\Gamma) \) where \( x' = x \gamma_{22}^{-1} \gamma_{33}^{-1} \) and \( y' = y \gamma_{22}^{-1} \gamma_{33}^{-1} \). This element \( \gamma g_{x,y} \gamma^{-1} \) determines the same line \( \ell_{x', y'} \) in \( \mathcal{C}(\Gamma) \). Then

\[
[-y \gamma_{22}^{-1} \gamma_{33}^{-2} : x \gamma_{22}^{-2} \gamma_{33}^{-1}] = [-y : x],
\]

which means that \( \gamma_{22} = \gamma_{33} \), contradicting (4.4). This contradiction proves the proposition.

Proposition 4.20. Let \( W \subset \mathbb{C}^2 \) be a non-empty and \( \mathbb{R} \)-linearly independent set, consider \( \ell = \overline{\text{Span}_\mathbb{R}(W) \setminus \{0\}} \). Then:

(i) If \( W \) has exactly one point or has exactly two points, which are \( \mathbb{C} \)-linearly dependent, then \( \ell \) is a single point.
(ii) If \( W \) has exactly two points, such that they are \( \mathbb{C} \)-linearly independent, then \( \ell \) is a real line in \( \mathbb{C}\mathbb{P}^1 \).
(iii) If \( W \) contains more than two points, then \( \ell = \mathbb{C}\mathbb{P}^1 \).
Proof. If $|W| = 1$ or $W$ has exactly two points, which are $\mathbb{C}$-linearly dependent, it follows trivially that $\ell$ is a single point. If $W$ has exactly two points, such that they are $\mathbb{C}$-linearly independent, we can assume that $W = \{(1,0),(0,1)\}$, then $[\text{Span}_\mathbb{C}(W) \setminus \{0\}] \cong \mathbb{Q}$ and therefore $\ell = \mathbb{R}$, which is a line in $\mathbb{CP}^1$. Finally, if $W$ contains more than two points, we can assume that $W = \{(1,0),(0,1),(w_1,w_2)\}$, where $w_1, w_2 \neq 0$. Then

$$\ell = \{[k + nw_1 : m + nw_2] | k, m, n \in \mathbb{Z}\} = \{[r + sw_1 : t + sw_2] | r, s, t \in \mathbb{R}\}.$$ 

Clearly $[0 : 1] \in \ell$. Let $z \in \mathbb{C}$ such that $\text{Im}(z) \neq 0$, denoting

$$r_0 = \frac{\text{Im}(w_2) - \text{Re}(z) \text{Im}(w_1)}{\text{Im}(z)}, \quad s_0 = 1, \quad t_0 = z(r + w_1) - w_2,$$

it follows trivially that $[1 : z] = [r_0 + s_0w_1 : t_0 + s_0w_2]$. If $\text{Im}(z) = 0$, clearly, $[1 : z] \in \ell$. Therefore $\ell = \mathbb{CP}^1$. \hfill $\square$

### 4.3. Decomposition of non-commutative discrete groups of $U_+$. In this subsection we state and prove the main theorem of this section. For the sake of clarity, we will divide the theorem in two parts (Theorems 4.21 and 4.25).

**Theorem 4.21.** Let $\Gamma \subset U_+$ be a non-commutative, torsion free, complex Kleinian group, then $\Gamma$ can be written in the following way

$$\Gamma = \text{Core}(\Gamma) \rtimes \langle \xi_1 \rangle \rtimes \ldots \rtimes \langle \xi_r \rangle \rtimes \langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle \rtimes \langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle$$

where

$$\lambda_{23}(\Gamma) = \langle \lambda_{23}(\gamma_1), \ldots, \lambda_{23}(\gamma_n) \rangle, \quad n = \text{rank} (\lambda_{23}(\Gamma)),$$

$$\lambda_{12}(\text{Ker}(\lambda_{23})) = \langle \lambda_{12}(\eta_1), \ldots, \lambda_{12}(\eta_m) \rangle, \quad m = \text{rank} (\lambda_{12}(\text{Ker}(\lambda_{23}))),$$

$$\Pi(\text{Ker}(\lambda_{12}) \cap \text{Ker}(\lambda_{23})) = \langle \Pi(\xi_1), \ldots, \Pi(\xi_r) \rangle, \quad r = \text{rank} (\Pi(\text{Ker}(\lambda_{12}) \cap \text{Ker}(\lambda_{23}))).$$

**Proof.** We will divide the proof in three parts.

**Part I. Decomposition of $\Gamma$ in terms of $\text{Ker}(\lambda_{23})$.** Let $\Gamma \subset U_+$ be a torsion free complex Kleinian group. Since $\Gamma$ is triangular, it is finitely generated (see [Aus60]) and therefore, $\lambda_{23}(\Gamma)$ is finitely generated. Let $n = \text{rank} (\lambda_{23}(\Gamma))$ and let $\{\gamma_1, \ldots, \gamma_n\} \subset \mathbb{C}^*$ be a generating set for $\lambda_{23}(\Gamma)$. We choose elements $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\gamma_i \in \lambda_{23}^{-1}(\gamma_i)$. Observe that

$$\Gamma = \langle \text{Ker}(\lambda_{23}), \gamma_1, \ldots, \gamma_n \rangle. \quad (4.5)$$

Furthermore, we will prove that

$$\Gamma = \langle (\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle) \rtimes \ldots \rangle \rtimes \langle \gamma_n \rangle. \quad (4.6)$$

Since $\lambda_{23}$ is a group homomorphism, $\text{Ker}(\lambda_{23})$ is a normal subgroup of $\Gamma$ and therefore, it is a normal subgroup of $\langle (\text{Ker}(\lambda_{23}), \lambda_1) \rangle$. Now, assume that $\text{Ker}(\lambda_{23}) \cap \lambda_1$ is not trivial, then there exist $p \in \mathbb{Z}$ such that $\gamma_i^p \in \text{Ker}(\lambda_{23})$. If $\gamma_1 = [a_{ij}]$ then the previous assumption means that either $a_{22} = a_{33}$ or $a_{22}^p = a_{33}^p$. In the latter case, this means, without loss of generality that

$$a_{22}a_{33}^{-1} = \omega, \quad (4.7)$$

where $\omega$ is a $p$-th root of the unity, with $p > 1$. On the other hand, $\lambda_{23}(\Gamma) \subset \mathbb{C}^*$ is a torsion free group, however, it follows from (4.7) that $a_{22}a_{33}^{-1}$ is a torsion element of $\lambda_{23}(\Gamma)$; thus $a_{22} = a_{33}$. But if this is the case then $\gamma_1 \in \text{Ker}(\lambda_{23})$ contradicting...
that $\lambda_{23}(\gamma_1) = \bar{\gamma}_1$ belongs to a generating set for $\lambda_{23}(\Gamma)$.

Thus, we can form the semi-direct product $\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle$. Now we verify that we can form the semi-direct product $((\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle)) \rtimes \langle \gamma_2 \rangle$.

First we verify that $\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle$ is a normal subgroup of $((\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle)) \rtimes \langle \gamma_2 \rangle$. Let $g \in \text{Ker}(\lambda_{23})$ and consider arbitrary elements $g\gamma_1^n \in \text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle$ and $g\gamma_2^m \in \langle \gamma_2 \rangle$, then

$$\lambda_2^m \, (g\lambda_1^n) \lambda_2^{-m} = \left(\lambda_2^m g\lambda_2^{-m}\right) \lambda_2^m \lambda_1^n \lambda_2^{-m} = g_1 \cdot \left(\lambda_2^m \lambda_1^n \lambda_2^{-m}\right)$$

where $g_1 = \lambda_2^m g\lambda_2^{-m} \in \text{Ker}(\lambda_{23})$. Since $\lambda_{23} \left(g_1 \lambda_2^m \lambda_1^n \lambda_2^{-m}\right) = \lambda_{23} \left(\lambda_1^n\right)$, we know that there exists $g_2 \in \text{Ker}(\lambda_{23})$ such that $g_1 \lambda_2^m \lambda_1^n \lambda_2^{-m} = g_2 \lambda_1^n$. Thus

$$\lambda_2^m \, (g\lambda_1^n) \lambda_2^{-m} = g_2 \lambda_1^n \in \text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle$$

Therefore $\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle$ is a normal subgroup of $((\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle)) \rtimes \langle \gamma_2 \rangle$.

Now we verify that $((\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle)) \rtimes \langle \gamma_2 \rangle$ is a normal subgroup of $((\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle)) \rtimes \langle \gamma_2 \rangle)$. Assume that there exist $p,q \in \mathbb{Z}$ such that $\gamma_2^p = g\lambda_1^q$ for some $g \in \text{Ker}(\lambda_{23})$ with $\gamma_2^p \neq \text{id}$, then

$$\gamma_2^p \gamma_1^{-q} \in \text{Ker}(\lambda_{23})$$

If we write $\lambda_1 = [\alpha_{ij}]$ and $\lambda_2 = [\beta_{ij}]$, then (4.8) means that $\left(\beta_{22} \alpha_{22}^{-\frac{3}{2}}\right)^p = \left(\beta_{33} \alpha_{33}^{-\frac{3}{2}}\right)^p$, this yields

$$\beta_{22} \alpha_{22}^{-\frac{3}{2}} \left(\beta_{33} \alpha_{33}^{-\frac{3}{2}}\right)^{-1} = \omega,$$

where $\omega$ is a $p$-th root of the unity. As before, $\lambda_{23}(\Gamma) \subset \mathbb{C}^*$ is a torsion free group, but (4.9) gives a torsion element in $\lambda_{23}(\Gamma)$. This contradiction proves that

$$((\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle)) \rtimes \langle \gamma_2 \rangle) = \text{id}.$$ 

Therefore we can define the semi-direct product $((\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle)) \rtimes \langle \gamma_2 \rangle)$. In the same way we can form the semi-direct product $((((\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle)) \rtimes \langle \gamma_2 \rangle) \rtimes \langle \gamma_3 \rangle) \rtimes \langle \gamma_4 \rangle) \rtimes \langle \gamma_5 \rangle)$. For clarity we will just write $\text{Ker}(\lambda_{23}) \rtimes \langle \gamma_1 \rangle \rtimes \langle \gamma_2 \rangle \rtimes \langle \gamma_3 \rangle \rtimes \langle \gamma_4 \rangle \rtimes \langle \gamma_5 \rangle$ instead. Using (4.5) we have proven (4.6).

Part II. Decompose $\text{Ker}(\lambda_{23})$ in terms of $\text{Ker}(\lambda_{12})$. Now consider the restriction $\lambda_{12} : \text{Ker}(\lambda_{23}) \rightarrow \mathbb{C}^*$ still denoted by $\lambda_{12}$. Again, $\lambda_{12}(\text{Ker}(\lambda_{23}))$ is finitely generated and let $m$ be its rank. Let $\{\tilde{\eta}_1, ..., \tilde{\eta}_m\}$ be a generating set, we choose elements $\eta_i \in \lambda_{12}^{-1}(\tilde{\eta}_i)$. Denote $A = \text{Ker}(\lambda_{12}) \cap \text{Ker}(\lambda_{23})$ and observe that $\text{Ker}(\lambda_{23}) = \langle A, \eta_1, ..., \eta_m \rangle$. Every element of $A$ has the form

$$\begin{bmatrix}
1 & x & y \\
0 & 1 & z \\
0 & 0 & 1
\end{bmatrix}.$$

Since $A$ is the kernel of the morphism $\lambda_{12}$, $A$ is a normal subgroup of $\langle A, \eta_1 \rangle$.

Now suppose that $A \cap \langle \eta_1 \rangle$ is not trivial, let $\eta_1^p \in A$ with $\eta_1^p \neq \text{id}$. Denote $\eta_1 = [a_{ij}]$, since $\eta_1^p \in A$ it must hold $a_{11}^p = a_{22}^p = a_{33}^p$, which means that $a_{11} a_{22}^{-1} = \omega$, which is a contradiction.
where either $\omega \neq 1$ is a $p$-th root of unity or $\omega = 1$. In the former case, $a_{11}a_{22}^{-1}$ is a torsion element in $\lambda_{12}(\ker(\lambda_{23}))$, which is a torsion free group. If $\omega = 1$ then $a_{11} = a_{22}$ and since $\eta_1 \in \ker(\lambda_{23})$, $a_{22} = a_{33}$ it follows that $a_{11} = a_{22} = a_{33}$ and thus, $\eta_1 \in A$ which contradicts that $\eta_1$ is part of the generating set. All this proves that $A \cap \langle \eta_1 \rangle = \emptyset$. This guarantees that we can form the semi-direct product $A \rtimes \langle \eta_1 \rangle$.

Now we verify that we can make the semi-direct product with $\langle \eta_2 \rangle$. The same argument used in part I to prove normality when we added $\gamma_2$ to $\ker(\lambda_{23}) \rtimes \langle \gamma_1 \rangle$ can be applied in the same way now to prove that $A \rtimes \langle \eta_1 \rangle$ is a normal subgroup of $A \rtimes \langle \eta_1, \eta_2 \rangle$ and that $(A \rtimes \langle \eta_1 \rangle) \cap \langle \eta_2 \rangle$ is trivial. Using this argument for $\eta_3, ..., \eta_m$ we get,

$$\ker(\lambda_{23}) = A \rtimes \langle \eta_1 \rangle \times \cdots \times \langle \eta_m \rangle.$$

Part III. Decompose $A$ in terms of $\ker(\Gamma)$, Consider the restriction $\Pi : A \to \text{PSL}(2, \mathbb{C})$. As before, $\text{Core}(\Gamma)$ is a normal subgroup of $\Gamma$. Since $A$ is solvable, it is finitely generated and so is $\Pi(A) \subset \text{PSL}(2, \mathbb{C})$, denote by $r$ its rank. Let $\tilde{\xi}_1, ..., \tilde{\xi}_r \subset \Pi(A)$ be a generating set for $\Pi(A)$, choose $\xi_1 \in \tilde{\xi}_i^{-1} \subset A$, then $A = \langle \text{Core}(\Gamma), \xi_1, ..., \xi_r \rangle$. Observe that $\text{Core}(\Gamma)$ is a normal subgroup of $\langle \text{Core}(\Gamma), \xi_1 \rangle$. Assume that $\text{Core}(\Gamma) \cap \xi_1$ is not trivial, then $\xi_1^p \in \text{Core}(\Gamma)$ for some $p \in \mathbb{Z} \setminus 0$. Since $\xi_1 \in A$, we write

$$\xi_1 = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\xi_1^p = \begin{bmatrix} 1 & x' & y' \\ 0 & 1 & pz \\ 0 & 0 & 1 \end{bmatrix},$$

for some $x', y' \in \mathbb{C}$.

Observe that $\xi_1^p \in \text{Core}(\Gamma)$ if and only if $z = 0$, which contradicts that $\tilde{\xi}_i$ is a generator of $\Pi(A)$. Then we can form the semi-direct product $\text{Core}(\Gamma) \rtimes \langle \xi_1 \rangle$. Observe that $\Pi(A)$ is a commutative subgroup of $\text{PSL}(2, \mathbb{C})$, then we can apply the same argument we used when we formed the semi-direct product with $\lambda_\gamma$ to conclude that $\text{Core}(\Gamma) \times \langle \xi_1 \rangle$ is a normal subgroup of $\langle \text{Core}(\Gamma) \times \langle \xi_1 \rangle, \xi_2 \rangle$. This concludes the proof of the first part of the theorem.

Before proving Theorem 4.25, we state some results needed in the proof.

**Proposition 4.22** (Chapter 2 of [Wal94]). The closed subgroup $H \subset \mathbb{R}^n$ is additive if and only if

$$H \cong \mathbb{R}^p \oplus \mathbb{Z}^q$$

with non-negative integers $p, q$ such that $p + q \leq n$.

**Theorem 4.23** (Theorem 1 of [BKK02]). Let $\Gamma$ be a group acting properly and discontinuously on a contractible manifold of dimension $m$, then $\text{obdim}(\Gamma) \leq m$.

In the statement of the previous Theorem, $\text{obdim}(\Gamma)$ is called the obstructor dimension. We can determine its value using the hypotheses of Theorem 4.21 and the following properties (see Corollary 27 of [BKK02] and [BF02] respectively):

- If $\Gamma = H \rtimes Q$ with $H$ and $Q$ finitely generated and $H$ weakly convex, then $\text{obdim}(\Gamma) \geq \text{obdim}(H) + \text{obdim}(Q)$. 


• If $\Gamma = \mathbb{Z}^n$, then $\text{obdim}(\mathbb{Z}^n) = n$.

Under the hypotheses of Theorem 4.21 and using Corollary 4.28, the previous properties imply the following reformulation of Theorem 4.23.

**Theorem 4.24.** Let $\Gamma \subset U_+$ be a non-commutative, torsion free, complex Kleinian group acting properly and discontinuously on a simply connected domain $\Omega \subset \mathbb{C}\mathbb{P}^2$, then $k + r + m + n \leq 4$.

The strategy to prove Theorem 4.25 will be to find a simply connected domain $\Omega \subset \mathbb{C}\mathbb{P}^2$ where $\Gamma$ acts properly and discontinuously and then apply Theorem 4.24. In some cases, we will obtain the explicit decomposition of $\Gamma$ and verify that $\text{rank}(\Gamma) \leq 4$.

**Theorem 4.25.** Let $\Gamma \subset U_+$ be a non-commutative, torsion free, complex Kleinian group, then $\text{rank}(\Gamma) \leq 4$. Using the notation of Theorem 4.21 it holds $k + r + m + n \leq 4$, where $k = \text{rank}(\text{Core}(\Gamma))$.

**Proof.** We denote $\Sigma = \Pi(\Gamma)$. If $k, r, m, n$ are defined as in the proof of Theorem 4.21 then

$$\text{(4.11)} \quad \text{rank}(\Gamma) \leq k + r + m + n.$$  

We will divide the proof in the following cases:

(i) $\Sigma$ is discrete and $\text{Ker}(\Gamma)$ is finite.

(ii) $\Sigma$ is discrete and $\text{Ker}(\Gamma)$ is infinite.

(iii) $\Sigma$ is not discrete.

Before dealing with each case, consider the diagram in Figure 1. This diagram summarizes each subcase we will be considering. Each subcase is listed with the same name it appears on the proof.

(i) Assume that $\Sigma$ is **discrete** and $\text{Ker}(\Gamma)$ is **finite**. Then by Theorem 5.8.2 of [CNS13] we know that $\Gamma$ acts properly and discontinuously on

$$\Omega = \left( \bigcup_{z \in \Omega(\Sigma)} \frac{z}{z_1, z_2} \right) \setminus \{ e_1 \},$$  

where $\Omega(\Sigma) = \mathbb{C}\mathbb{P}^1 \setminus \Lambda(\Sigma)$ denotes the discontinuity set of $\Sigma$. If $|\Lambda(\Sigma)| = 0, 1, \infty$ then each connected component of $\Omega$ is simply connected, since they are respectively homeomorphic to $\mathbb{C}\mathbb{P}^2$, $\mathbb{C}^2$ or $\mathbb{C} \times \mathbb{H}$. Using Theorem 4.24 it follows $k + r + m + n \leq 4$.

If $|\Lambda(\Sigma)| = 2$, then $\Sigma$ is elementary and therefore it is a cyclic group generated by a loxodromic element and therefore, $\Sigma \cong \mathbb{Z}$. On the other hand, since $\Gamma$ is torsion free and $\text{Ker}(\Gamma)$ is finite, $\text{Ker}(\Gamma) = \{ \text{id} \}$ (Proposition 4.16) and therefore, $\Pi : \Gamma \to \Sigma$ is a group isomorphism and then, $\Gamma \cong \Sigma$. It follows that $\Gamma \cong \mathbb{Z}$ and therefore, $\text{rank}(\Gamma) = 1$. Then it holds trivially, $k + r + m + n \leq 4$.

(ii) Now, let us assume that $\text{Ker}(\Gamma)$ is **infinite** and $\Sigma$ is **discrete**. Observe that $\text{Core}(\Gamma)$ is infinite (Proposition 4.17), which means that there exist elements $g_{x,y} \in \text{Core}(\Gamma)$, with $g_{x,y} \neq \text{id}$. Denote $B(\Gamma) = \pi(\text{C}(\Gamma) \setminus \{ e_1 \})$, then
$B(\Gamma) \cong S^1$ or it is a single point (Proposition 4.20).

On the other hand, consider a sequence of distinct elements $\{\mu_k\} \subset \Gamma$. Since $\Sigma$ is discrete, the sequence $\{\Pi(\mu_k)\}$ is either constant or converges, by distinct elements, to a quasi-projective map $\sigma \in QP(2, \mathbb{C})$. Let us assume
first that \( \{\Pi(\mu_k)\} \) diverges, then \( \sigma \) has one of the following forms

\[
\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_4 = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}.
\]

with \( a \in \mathbb{C}^* \). Since \( \Sigma \) is triangular, it follows that \( \Sigma \) is a solvable discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \) and therefore, it is elementary. Using Theorem 1.6 of [Ser05], we have three possibilities:

- **[e1]** \( \Sigma = \langle h \rangle \), with \( h \) a loxodromic element. Then, the quasi-projective limit of sequences in \( \Sigma \) can only be \( \sigma_1 \) or \( \sigma_3 \).
- **[e2]** \( \Sigma = \langle h \rangle \), with \( h \) a parabolic element.
- **[e3]** \( \Sigma = \langle g, h \rangle \), with \( g, h \) parabolic elements with different translation directions.

Now, considering whether \( \Gamma \) is conic or not, we have the two possible cases:

- **[con]** \( \Gamma \) is conic, then \( B(\Gamma) \cong \mathbb{S}^1 \). Define
  \[
  \Omega(\Gamma) = \mathbb{CP}^1 \setminus \overline{C(\Gamma)}.
  \]
  We will verify that \( \Gamma \) acts properly and discontinuously on \( \Omega(\Gamma) \). Let \( K \subset \Omega(\Gamma) \) be a compact set, denote \( R = \{ \gamma \in \Gamma \mid \gamma (K) \cap K \neq \emptyset \} \) and assume that \( |R| = \infty \). Then, we can write \( R = \{ \gamma_1, \gamma_2, \ldots \} \). Consider the sequence of distinct elements \( \{\gamma_k\} \subset \Gamma \), and the sequence \( \{\Pi(\gamma_k)\} \subset \Sigma \). Let \( \sigma \in \text{QP}(2, \mathbb{C}) \) be the quasi-projective limit of \( \{\Pi(\gamma_k)\} \), then \( \text{Ker}(\sigma), \text{Im}(\sigma) \subset B(\Gamma) \).
  By definition, \( \pi(K) \cap B(\Gamma) = \emptyset \) and then, by Proposition [4.14] \( \pi(K) \) accumulates on \( e_1 \) (observe that we are considering this \( e_1 \) as a point in \( \mathbb{CP}^1 \approx \mathbb{P}^1 \mathbb{C} \), but we are actually referring to \( \pi(e_2) \)). On the other hand, since \( |R| = \infty \) then \( |\{ \alpha \in \Sigma \mid \alpha(\pi(K)) \cap \pi(K) \neq \emptyset \}| = \infty \). This contradicts the fact that \( \Pi(\gamma_k)(\pi(K)) \) accumulates on \( e_1 \). Therefore, \( |R| < \infty \) and then, \( \Gamma \) acts properly and discontinuously on each connected component of \( \Omega(\Gamma) \). Since \( \Omega(\Gamma) \cong \mathbb{C} \times (\mathbb{H} \cup \mathbb{H}^{-}) \), each connected component of \( \Omega(\Gamma) \) is simply connected. Theorem [4.24] yields \( k + r + m + n \leq 4 \).

- **[n-con]** \( \Gamma \) is not conic, then \( B(\Gamma) = \{ p \} \) for some \( p \in \mathbb{CP}^1 \). We can assume that \( p = e_1 \) or \( p = e_2 \) (Proposition [4.19]). We have the following cases:
  - **[n-con-1]** If \( \Sigma \) has the form **[e2]** or **[e3]** then \( \Lambda(\Sigma) = \{ e_1 \} \). Let
    \[
    \Omega(\Gamma) = \mathbb{CP}^1 \setminus \{ e_1, e_2 \}.
    \]
    The same argument used in the previous case **[con]** shows that \( \Gamma \) acts properly and discontinuously on \( \Omega(\Gamma) \cong \mathbb{C}^2 \). Since \( \Omega(\Gamma) \) is simply connected, then \( k + r + m + n \leq 4 \) (Theorem [4.24]).
  - **[n-con-2]** If \( \Sigma \) has the form **[e1]** then \( \Sigma \) is cyclic and therefore, \( \Sigma \cong \mathbb{Z} \). Since \( \pi \) is a group morphism, then \( \Gamma \cong \text{Ker}(\Gamma) \rtimes \text{Im}(\Gamma) = \text{Core}(\Gamma) \times \Sigma \). Now, since \( \text{Core}(\Gamma) \) is not conic, then \( \text{Core}(\Gamma) \cong \mathbb{Z} \) and Therefore \( \Gamma \cong \mathbb{Z} \times \mathbb{Z} \). Then \( \text{rank}(\Gamma) \leq 4 \).

Now, let us assume that \( \{\Pi(\mu_k)\} \) is a constant sequence. Then, there exists a sequence \( \{g_k\} \subset \text{ker}(\Gamma) \) such that \( \gamma_k = \gamma_0 g_k \); it follows that \( \{g_k\} \subset \text{Core}(\Gamma) \) (Proposition [4.17]). Let \( \tau \in \text{QP}(3, \mathbb{C}) \) be the quasi-projective limit of \( \{g_k\} \), since \( \text{Im}(\tau) \not\subset \text{Ker}(\tau) \) then \( \gamma_0 g_k = \gamma_k \to \gamma_0 \tau \). It is straightforward to verify that \( \text{Im}(\tau) = \text{Im}(\gamma_0 \tau) \) and \( \text{Ker}(\tau) = \text{Ker}(\gamma_0 \tau) \). We now have two possibilities:
[C-con] Γ is conic, then Core(Γ) = \langle g_{x_1,y_1}, g_{x_2,y_2} \rangle for some \( x_1, x_2, y_1, y_2 \in \mathbb{C} \). Denote \( g_k = g_{n_k x_1 + m_k x_2, n_k y_1 + m_k y_2} \), for some sequences \( \{n_k\}, \{m_k\} \subset \mathbb{Z} \). Then
\[
\tau = \begin{bmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ for some } x, y \in \mathbb{C}, \ |x| + |y| \neq 0,
\]
and then Ker(τ) = \( \ell_{x,y} \in \mathcal{C}(\Gamma) \). Let \( \Omega = \mathbb{CP}^2 \setminus \mathcal{C}(\Gamma) \cong \mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-) \).

Let \( K \subset \Omega \) be a compact set, then \( K \subset \mathbb{CP}^2 \setminus \ker(\tau) \). By Lemma 1.4, \( \Gamma K \) accumulates on \( \text{Im}(\tau) = \{e_1\} \not\in \Omega \). This means that \( \Gamma \) acts properly and discontinuously on each component of \( \Omega \). Since each component of \( \Omega \) is simply connected, Theorem 4.24 implies \( k + r + m + n \leq 4 \).

[C-n-con] Γ is not conic, then Core(Γ) = \( \langle g_{x_0,y_0} \rangle \) for some \( x_0, y_0 \in \mathbb{C} \). Now the quasi-projective limit \( \tau \) has the form
\[
\tau = \begin{bmatrix} 0 & x_0 & y_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]
then Ker(τ) = \( \ell_{x_0,y_0} \in \mathcal{C}(\Gamma) \). We define \( \Omega = \mathbb{CP}^2 \setminus \ell_{x_0,y_0} \) and use the same argument as in the previous case [con] to conclude that \( k + r + m + n \leq 4 \).

(iii) Assume that \( \Sigma \) is not discrete. Let \( \{\gamma_k\} \subset \Gamma \) be a sequence of distinct elements, denote \( \gamma_k = \left[ \begin{smallmatrix} \gamma_{11}^{(k)} \\ \gamma_{12}^{(k)} \\ \gamma_{13}^{(k)} \end{smallmatrix} \right] \). A direct calculations shows that
\[
\gamma_k^{-1} = \begin{bmatrix} \gamma_{11}^{(k)} & \frac{\gamma_{12}^{(k)} - \gamma_{13}^{(k)} \gamma_{22}^{(k)}}{\gamma_{22}^{(k)}} \\ 0 & \gamma_{22}^{(k)^{-1}} \\ 0 & 0 \end{bmatrix}.
\]
(4.12)

Considering the sequence \( \{\Pi(\gamma_k)\} \subset \Sigma \), we have two possibilities:

[Conv] The sequence \( \{\Pi(\gamma_k)\} \subset \Sigma \) converges to some \( \alpha \in \text{PSL}(2, \mathbb{C}) \) where
\[
\alpha = \begin{bmatrix} \gamma_{22} & \gamma_{23} \\ 0 & 1 \end{bmatrix}, \text{ for some } \gamma_{22} \in \mathbb{C}^*.
\]
(4.13)

such that \( \gamma_{22}^{(k)} (\gamma_{33}^{(k)})^{-1} \rightarrow \gamma_{22} \) and \( \gamma_{23}^{(k)} (\gamma_{33}^{(k)})^{-1} \rightarrow \gamma_{23} \). Consider the sequence \( \{\gamma_{33}^{(k)}\} \subset \mathbb{C}^* \), we have three cases:

(a) \( \gamma_{33}^{(k)} \rightarrow \gamma_{33} \in \mathbb{C}^* \). Since \( \gamma_{11}^{(k)} \gamma_{12}^{(k)} \gamma_{13}^{(k)} = 1 \), and considering (4.13), it follows that \( \gamma_{11}^{(k)} (\gamma_{33}^{(k)})^{-1} \rightarrow \gamma_{12} \gamma_{33}^{-1} \).

\[\text{If } \gamma_{12}^{(k)} (\gamma_{33}^{(k)})^{-1} \rightarrow \gamma_{12} \in \mathbb{C} \text{ and } \gamma_{13}^{(k)} (\gamma_{33}^{(k)})^{-1} \rightarrow \gamma_{13} \in \mathbb{C} \text{ then } \gamma_k \text{ would converge to an element of } \text{PSL}(3, \mathbb{C}), \text{ contradicting that } \Gamma \text{ is discrete. Then, } \gamma_{12}^{(k)} (\gamma_{33}^{(k)})^{-1} \rightarrow \infty \text{ or } \gamma_{13}^{(k)} (\gamma_{33}^{(k)})^{-1} \rightarrow \infty. \text{ Thus, } \gamma_k \rightarrow \tau_{a,b} := \begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ for some } a, b \in \mathbb{C}, |a| + |b| \neq 0.\]
Observe that
\[ \text{Ker}(\tau_{a,b}) = \left\{ e_1, [0 : -b : a] \right\} \quad \text{and} \quad \text{Im}(\tau_{a,b}) = \{ e_1 \}. \]

For each quasi-projective limit \( \tau_{a,b} \in \text{QP}(3, \mathbb{C}) \) of sequences \( \{ \gamma_k \} \subset \Gamma \) like the ones studied in this case, we consider the point \([0 : -b : a] \in e_2, e_4\) and the horocycle determined by this point and \( e_2 \), then we consider the pencil of lines passing through \( e_1 \) and each point of the horocycle. Denote by \( \Omega \) to the complement of this pencils of lines in \( \mathbb{C}P^2 \). Each connected component of \( \Omega \) is simply connected.

As a consequence of Proposition 4.4 and (4.14), the accumulation points of orbits of compact subsets of \( \Omega \) is \( \{ e_1 \} \) and therefore, the action of \( \Gamma \) in \( \Omega \) is proper and discontinuous. Theorem 4.24 implies \( k + r + m + n \leq 4 \).

(b) \( \gamma_{33}^{(k)} \to \infty \). From (4.13), it follows that \( \gamma_{22}^{(k)} \to \infty \). Given that \( \gamma_{11}^{(k)} \gamma_{22}^{(k)} \gamma_{33}^{(k)} = 1 \) for all \( k \in \mathbb{N} \), then \( \gamma_{11}^{(k)} \to 0 \). Then \( \gamma_{11}^{(k)} \left( \gamma_{33}^{(k)} \right)^{-1} \to 0 \) and then

\[ \gamma_k \to \begin{bmatrix} 0 & \gamma_{12} & \gamma_{13} \\ 0 & \gamma_{22} & \gamma_{23} \\ 0 & 0 & 1 \end{bmatrix}, \]

where \( \gamma_{12} \left( \gamma_{33}^{(k)} \right)^{-1} \to \gamma_{12} \in \mathbb{C} \) and \( \gamma_{13} \left( \gamma_{33}^{(k)} \right)^{-1} \to \gamma_{13} \in \mathbb{C} \).

If \( \gamma_{12} \left( \gamma_{33}^{(k)} \right)^{-1} \to \infty \) or \( \gamma_{13} \left( \gamma_{33}^{(k)} \right)^{-1} \to \infty \), then the quasi-projective limit \( \tau \) has the form (4.20) with \( a = 0 \) and \( b, c \in \mathbb{C} \). We will deal with this case in case (c). Then, assume that \( \tau \) has the form (4.15).

We can conjugate \( \tau \) by an adequate element of \( \text{PSL}(3, \mathbb{C}) \) in such way that

\[ \tau = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma_{22} & \gamma_{23} \\ 0 & 0 & 1 \end{bmatrix}. \]

Using Lemma 4.14, we have the following possibilities for the Greenberg limit set of \( \Sigma \):

\begin{itemize}
  \item [(LS1)] \( \Lambda_{Gr}(\Sigma) = S^1 \).
  \item [(LS2)] \( \Lambda_{Gr}(\Sigma) = \mathbb{C}P^1 \).
  \item [(LS3)] \( |\Lambda_{Gr}(\Sigma)| = 1 \).
  \item [(LS4)] \( |\Lambda_{Gr}(\Sigma)| = 2 \).
\end{itemize}

In the case (LS1) there are four possibilities for the convergence of \( \{ \Pi(\gamma_k) \} \):

\begin{itemize}
  \item [(LS1.1)] The sequence \( \{ \Pi(\gamma_k) \} \) converges to \( \text{id} \in \Sigma \).
  \item [(LS1.2)] The sequence \( \{ \Pi(\gamma_k) \} \) converges to a elliptic element in \( \Sigma \).
  \item [(LS1.3)] The sequence \( \{ \Pi(\gamma_k) \} \) converges to a parabolic element in \( \Sigma \).
  \item [(LS1.4)] The sequence \( \{ \Pi(\gamma_k) \} \) converges to a loxodromic element in \( \Sigma \).
\end{itemize}
Consider the case (LS1.1), then \( \gamma_{22} = 1 \) and \( \gamma_{23} = 0 \), denote the quasi-projective limit of \( \{ \gamma_k \} \) by \( \tau_{id} = \text{Diag}(0, 1, 1) \). Since \( \Lambda_{Gr}(\Sigma) = S^1 \), we can guarantee the existence of parabolic elements in \( \Sigma \) (see the proof of Proposition 4.4). Let \( g = [g_{ij}] \in \Gamma \) be such that \( \Pi(g) \) is parabolic, then \( g_{22} = g_{33} = \lambda \in \mathbb{C}^* \) and \( g_{23} \neq 0 \) and therefore \( g_{11} = \lambda^{-2} \). A straight-forward calculation shows that

\[
\gamma_k g \gamma_k^{-1} \rightarrow \begin{bmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda & g_{23} \\ 0 & 0 & \lambda \end{bmatrix} \in \text{PSL}(3, \mathbb{C}) ,
\]

which contradicts that \( \Gamma \) is discrete, unless the sequence \( \{ \gamma_k g \gamma_k^{-1} \} \) is eventually constant. Without loss of generality, we can assume that this sequence is constant in that case. It can be verified that this yields \( \gamma_k = \begin{bmatrix} \xi_k & \gamma_{12}^{(k)} & \gamma_{13}^{(k)} \\ 0 & \xi_k & \gamma_{33}^{(k)} \\ 0 & 0 & \xi_k \end{bmatrix} \). Corollary 4.18 then implies \( \xi_k = 1 \) for all \( k \), but then it is not possible that \( \gamma_k \rightarrow \tau_{id} \). Then subcase (LS1.1) cannot happen.

Consider the case (LS1.2). Let us assume that \( \Pi(\gamma_k) \) converges to an elliptic element of \( \Sigma \) with the form \( \text{Diag}(e^{2\pi i \theta}, 1) \). As before, using the adequate conjugation we can assume that

\[
\gamma_k \rightarrow \tau_{\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{2\pi i \theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} .
\]

A straight-forward calculation shows that \( \gamma_k^n \rightarrow \tau_{\theta}^n \) for any \( n \in \mathbb{N} \). If \( \theta \in \mathbb{Q} \), then there exist \( p \in \mathbb{Z} \) such that \( \gamma_k^p \rightarrow \tau_{\theta}^p = \tau_{id} \), which cannot happen, as we have already proven. If \( \theta \in \mathbb{R} \setminus \mathbb{Q} \), and since \( S^1 \) is compact, there is a subsequence \( \{ e^{2\pi imj \theta} \} \) such that \( e^{2\pi imj \theta} \rightarrow 1 \). Then \( \tau_{\theta}^n \rightarrow \tau_{id} \) as \( j \rightarrow \infty \). Consider the diagonal sequence \( \{ \gamma_{nk} \} \subset \Gamma \), then \( \gamma_k \rightarrow \tau_{id} \) which cannot happen. This dismisses case (LS1.2).

Next, consider the case (LS1.3). Assume that

\[
\gamma_k \rightarrow \tau_b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} .
\]

Therefore, the sequence \( \{ \Pi(\gamma_k) \} \) converges to the parabolic element

\[
\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} .
\]

Since \( \Sigma \) is not discrete, it follows from lemma 4.9 that there exists a sequence \( \{ h_k \} \subset \Gamma \) such that \( \Pi(h_k) \) is parabolic for all \( k \), and

\[
\Pi(h_k) = \begin{bmatrix} 1 & \varepsilon_k \\ 0 & 1 \end{bmatrix} \rightarrow \text{id} ,
\]
for some sequence \( \varepsilon_k \to 0 \). Then, using the same reasoning as in the case (LS1.1), we know that

\[
h_k = \begin{bmatrix}
1 & h_{12} & h_{13} \\
0 & 1 & \varepsilon_k \\
0 & 0 & 1
\end{bmatrix}.
\]

A direct calculation shows that \( \gamma_k h \gamma_k^{-1} h_k^{-1} \to g_{h_{12},0} \in \text{PSL}(3, \mathbb{C}) \) contradicting that \( \Gamma \) is discrete. Then the case (LS1.3) cannot occur.

Finally, we deal with the case (LS1.4). Let us suppose that \( \gamma_k \to \tau_\alpha = \text{Diag}(0, \alpha, 1) \). Since \( \Lambda_{G_{\Sigma}}(\Sigma) = S^1 \), there is an element \( h = [h_{ij}] \in \Gamma \) such that \( h_{23} \neq 0 \) and

\[
h = \begin{bmatrix}
1 & h_{12} & h_{13} \\
0 & \lambda & h_{23} \\
0 & 0 & \lambda^{-1}
\end{bmatrix},
\]

with \( |\lambda| \neq 1 \). A direct calculation yields

\[
f_k := h \gamma_k h^{-1} \gamma_k^{-1} \to \begin{bmatrix}
1 & \lambda^{-1} h_{12} & h_{13}(\lambda - h_{12} \alpha) \\
0 & 1 & \lambda h_{23}(1 - \alpha) \\
0 & 0 & 1
\end{bmatrix},
\]

contradicting that \( \Gamma \) is discrete. We have dismissed the case (LS1).

Observe that the case (LS2) cannot happen, otherwise the action of \( \Gamma \) would be nowhere proper and discontinuous.

Consider the case (LS3). It follows that \( \Sigma \) is conjugated to a subgroup of \( \text{Epa}(\mathbb{C}) \) (Theorem 2.14 of [CS14]). Let \( h \in \Gamma \) such that \( \Pi(h) \) is parabolic, denote

\[
\Pi(h) = \begin{bmatrix}
1 & h_{23} \\
0 & 1
\end{bmatrix}, \quad h_{23} \neq 0.
\]

Then

\[
h = \begin{bmatrix}
\lambda^{-2} & h_{12} & h_{13} \\
0 & \lambda & h_{23} \\
0 & 0 & \lambda
\end{bmatrix},
\]

for some \( \lambda \in \mathbb{C}^* \) and \( h_{12}, h_{13} \in \mathbb{C} \). A straight-forward computation shows that the sequence of distinct elements \( \{ \eta_k := h \gamma_k h^{-1} \gamma_k^{-1} \} \subset \Gamma \) converges to an element of \( \text{PSL}(3, \mathbb{C}) \), contradicting that \( \Gamma \) is discrete. This proves that the case (LS3) cannot happen.

Now, finally, consider the case (LS4). Then \( \Sigma \) is conjugated to a subgroup of \( \text{Aut}(\mathbb{C}^*) \) (Theorem 2.14 of [CS14]). This means that, conjugating by a suitable element of \( \text{PSL}(3, \mathbb{C}) \), every element of \( \Sigma \) has the form \( \Pi(g) = \text{Diag}(\alpha, \alpha^{-1}) \), for some \( \alpha \in \mathbb{C}^* \). Then,

\[
g = \begin{bmatrix}
\lambda^{-2} & g_{12} & g_{13} \\
0 & \lambda \alpha & 0 \\
0 & 0 & \lambda \alpha^{-1}
\end{bmatrix}.
\]
Let \( g_1, g_2 \in \Gamma \) be two elements such that \([g_1, g_2] \neq id\), then

\[
h = [g_1, g_2] = \begin{bmatrix}
1 & h_{12} & h_{13} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

with \(|h_{12}| + |h_{13}| \neq 0\). The sequence \( \{\gamma_k\} \) has the same form described in (4.19). Then the quasi-projective limit \( \tau \), given in (4.16), has the form \( \tau = \text{Diag}(0, \gamma_{22}, 1) \). Consider the sequence \( \{f_k\} \in \Gamma \) given by

\[
f_k = h_{\gamma_k} h_{\gamma_k}^{-1} h_{\gamma_k}^{-1},
\]

it can be directly verified that \( f_k \to \begin{bmatrix}
1 & h_{12} & h_{13} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \in \text{PSL}(3, \mathbb{C}),
\]

contradicting that \( \Gamma \) is discrete. Therefore the case (LS4) cannot happen either. This dismisses the case (b) altogether.

(c) \( \gamma_{33}^{(k)} \to 0 \). From this, it follows that \( \gamma_{22}^{(k)} \to 0 \), otherwise \( \alpha \notin \text{PSL}(2, \mathbb{C}) \). Since \( \gamma_{11}^{(k)} \gamma_{22}^{(k)} \gamma_{33}^{(k)} = 1 \), we have that \( \gamma_{11}^{(k)} \to \infty \). All of this yields

\[
(4.20) \quad \gamma_k \to \tau = \begin{bmatrix}
a & b & c \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

with \( a \neq 0 \), unless \( \gamma_{11}^{(k)} \gamma_{12}^{(k)} \to 0 \) or \( \gamma_{11}^{(k)} \gamma_{13}^{(k)} \to 0 \), but then we would be in case (b). Then, \( \gamma_{12}^{(k)} \gamma_{11}^{(k)} \to b \) and \( \gamma_{13}^{(k)} \gamma_{11}^{(k)} \to c \), for some \( b, c \in \mathbb{C} \). Then, considering the sequence \( \{\gamma_k\}^{-1}\) and dividing each entry by \( \gamma_{33}^{(k)} \to 0 \), we have

\[
\gamma_k^{-1} \to \mu = \begin{bmatrix}
0 & -\frac{b}{\gamma_{22}} & \frac{b \gamma_{23}^2 - c}{\gamma_{22}} \\
0 & \frac{1}{\gamma_{22}} & \frac{-\gamma_{23}^2}{\gamma_{22}} \\
0 & 0 & 1
\end{bmatrix}.
\]

This means that, if we consider the sequence of the inverses instead, this case is the same as case (b).

[Div] The sequence \( \{\Pi(\gamma_k)\} \subset \Sigma \) diverges. Then, one of the following cases occur:

[Div-1] \( \Sigma = \text{SO}(3) \). This case cannot happen since \( \Sigma \) is solvable but \( \text{SO}(3) \) is not solvable.

[Div-2] and [Div-3], then \( \Sigma = \text{Rot}_\infty \) or \( \Sigma = \text{Dih}_\infty \). This is not possible, otherwise \( \Lambda_{Gr}(\Sigma) = \emptyset \), contradicting Lemma 4.14.

[Div-4] \( \Sigma = \text{PSL}(2, \mathbb{C}) \). This case cannot happen since \( \text{PSL}(2, \mathbb{C}) \) is not solvable (it contains Schottky groups, see Theorem 1.7).

[Div-5] The group \( \Sigma \) is a subgroup of the affine group \( \text{Epa}(\mathbb{C}) \). There cannot be elliptic elements in \( \Sigma \), otherwise, \( \lambda_{23}(\Gamma) \) would have a torsion element or \( \Gamma \) would contain an irrational screw and therefore,
it would be commutative (Corollary 4.11). Then every element of \( \Sigma \) has the form
\[
\Pi(\gamma) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix},
\]
where \( a \in A \), for some additive group \( A \subset (\mathbb{C}, +) \). Since \( \Sigma \) is not discrete, \( A \) is not discrete. Then, either \( A \cong \mathbb{R} \) or \( A \cong \mathbb{R} \oplus \mathbb{Z} \) (Proposition 4.22). Let us define the following union of pencils of lines passing through \( e_1 \),
\[
\Lambda = \{ e_1 \} \cup \left( \bigcup_{p \in A} \pi(p)^{-1} \right),
\]
and \( \Omega = \mathbb{CP}^2 \setminus \Lambda \). The \( \Sigma \)-orbits of compact subsets of \( \hat{\mathbb{C}} \setminus A \) accumulate on \( \infty \), which correspond to the point \( \pi(e_2) \). Then, the \( \Sigma \)-orbits of compact subsets of \( \Omega \) accumulate on \( \{ e_1 \} \), and therefore, the action of \( \Gamma \) on each connected component of \( \Omega \) is proper and discontinuous. Since each connected component of \( \Sigma \) is simply connected, this concludes the case [Div-5].

[Div-6] The group \( \Sigma \) is a subgroup of the group \( \text{Aut} \left( \mathbb{C}^* \right) \). Then \( \Sigma \) is a purely loxodromic group and then, up to conjugation, each element has the form \( \Pi(\gamma) = \text{Diag} (\alpha, 1) \), for some \( |\alpha| \neq 1 \). Let \( G = \lambda_{23}(\Gamma) \), then \( G \cong \Sigma \), and since \( \Sigma \) is not discrete, \( G \) is not discrete. Let us write each \( \alpha \in G \) as
\[
\alpha = re^{i\theta}, \quad r \in \mathbb{R}^+, \quad r \in A \subset (\mathbb{C}^*, \cdot) \quad \theta \in \mathbb{R}, \quad \theta \in B \subset (\mathbb{C}, +),
\]
for some multiplicative group \( A \) and some additive group \( B \). Then \( G \cong A \times B \) and, since \( G \) is not discrete, then \( A \) is not discrete or \( B \) is not discrete (it is not possible that both \( A \) and \( B \) are not discrete, otherwise the action of \( \Gamma \) would be nowhere proper and discontinuous). Therefore we have two possibilities:

* \( A \) is discrete and \( B \), not discrete. Since \( A \) is a discrete multiplicative subgroup of \( \mathbb{C} \), then \( \text{rank}(A) = 1 \) and therefore, \( A = \{ r^n | n \in \mathbb{Z} \} \) for some \( r \in \mathbb{C}^* \). Hence,
\[
\overline{G} = \overline{\lambda_{23}(\Gamma)} = \left\{ r^n e^{i\theta} \mid \theta \in [0, 2\pi) \right\} \subset \overline{\lambda_{23}(\Gamma)}.
\]
In particular, \( \{ r^n e^{i\theta} \mid \theta \in [0, 2\pi) \} \subset \overline{\lambda_{23}(\Gamma)} \). This means that there exists an element \( \gamma \in \Gamma \) such that \( \lambda_{23} = e^{i\theta} \) for some \( \theta \in [0, 2\pi) \). If \( \theta \in \mathbb{Q} \), then \( \lambda_{23}(\gamma) \) is a torsion element, contradicting that \( \lambda_{23}(\Gamma) \) is torsion free. If \( \theta \in \mathbb{R} \setminus \mathbb{Q} \), then \( \gamma \) is an irrational screw, contradicting that \( \Gamma \) is non-commutative (see Corollary 4.11). This dismisses this possibility.

* \( A \) is not discrete and \( B \), discrete. Since \( B \) is discrete, then \( B \) is finite and \( B \cong \mathbb{Z}_k \), for some \( k \in \mathbb{N} \). We know there is an homomorphism \( \varphi : \hat{\mathbb{Z}}_2, e^3 \rightarrow \hat{\mathbb{C}} \). For the sake of simplifying notation, we will denote indistinctly a point \( x \in \overline{G} \subset \mathbb{C} \) and
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\[
\begin{bmatrix}
1 & x & y \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha & x & y \\
0 & \beta & z \\
0 & 0 & \gamma
\end{bmatrix}
\begin{bmatrix}
\alpha & x & y \\
0 & \beta & z \\
0 & 0 & \gamma
\end{bmatrix}
\]

\[z \neq 0, \alpha \neq \beta, z \neq 0, \beta \neq \gamma\]

Loxo-parabolic
Loxo-parabolic
Complex homothety
Strongly loxodromic

Core(Γ)
A \ Ker(Γ)
Ker(λ_{23}) \ A

Table 3. The decomposition of a non-commutative subgroup of \( U_+ \) in four layers.

The point \( \varphi^{-1}(x) \in \overrightarrow{e_2, e_3} \). Let \( \Lambda \subset \mathbb{C}P^2 \) be given by

\[\Lambda = \{ e_1 \} \cup \left( \bigcup_{p \in \tilde{\mathbb{C}}} \pi(p)^{-1} \right),\]

and \( \Omega = \mathbb{C}P^2 \setminus \Lambda \). Since the orbits of compact subsets of \( \mathbb{C} \setminus \tilde{\mathbb{C}} \) accumulate on \( \pi(e_2) \) and \( \pi(e_3) \), then the orbits of compact subsets of \( \Omega \) accumulate on \( \{ e_1 \} \). This means that the action of \( \Gamma \) on \( \Omega \) is proper and discontinuous. Besides, each connected component of \( \Omega \) is simply connected. This concludes the case [Div-6].

[Div-7] The group \( \Sigma \) is a subgroup of the group PSL(2, \( \mathbb{R} \)). Then, \( \Lambda_{Gr}(\Sigma) \cong \tilde{\mathbb{R}} \) (Theorem 2.14 of [CS14]). Then, up to a suitable conjugation, the orbits of compact subsets of \( \mathbb{C}P^1 \setminus \Lambda_{Gr}(\Sigma) \) accumulate on \( \tilde{\mathbb{R}} \), we regard the points \( \pi(e_2) \) and \( \pi(e_3) \) as the points 0 and \( \infty \) in this euclidean circle. We define the pencil of lines passing through \( e_1 \),

\[\Lambda = \{ e_1 \} \cup \left( \bigcup_{p \in \tilde{\mathbb{R}}} \pi(p)^{-1} \right) .\]

Then its complement \( \Omega = \mathbb{C}P^2 \setminus \Lambda \) is homeomorphic to a cone \( \mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-) \) and therefore, each of its connected components are simply connected. The action of \( \Gamma \) on each of these connected components is proper and discontinuous, this completes this case [Div-7].

This concludes the proof. \( \square \)

4.4. Consequences of the theorem of decomposition of non-commutative groups. Theorem 4.21 gives a decomposition of the group \( \Gamma \) in four layers, the first two layers are made of parabolic elements and the last two layers are made of loxodromic elements (see Corollary 4.29 and Table 3). The description of these four layers are summarized in Table 3.
Let $F_1, F_2, F_3 \subset U_+$ be the pairwise disjoint subsets defined as

\[
F_1 = \left\{ \alpha = \begin{bmatrix} \alpha_{11} & 0 & \alpha_{13} \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} \mid \alpha \in U_+ \right\}
\]

\[
F_2 = \left\{ \alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} \mid \alpha_{12} \neq 0, \alpha \in U_+ \right\}
\]

\[
F_3 = \left\{ \alpha = \begin{bmatrix} \alpha_{11} & 0 & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & \alpha_{33} \end{bmatrix} \mid \alpha_{23} \neq 0, \alpha \in U_+ \right\}
\]

\[
F_4 = \left\{ \alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & \alpha_{33} \end{bmatrix} \mid \alpha_{12}, \alpha_{23} \neq 0, \alpha \in U_+ \right\}
\]

These four subsets classify the elements of $U_+$ depending on whether they have zeroes in positions 12 and 23. We need this classification because, as we will see in Proposition 4.26, a necessary condition for two elements of $U_+$ to commute is that they both have the same form given by the sets $F_1, F_2, F_3$. And this is equivalent to say that the two $2 \times 2$ diagonal sub-blocks of one element share the same fixed points with the corresponding sub-block of the other element. This argument will be key to prove Corollary 4.28.

**Proposition 4.26.** Let $\Gamma \subset U_+$ be a subgroup and let $\alpha = [\alpha_{ij}], \beta = [\beta_{ij}] \in \Gamma \setminus \{id\}$. If $[\alpha, \beta] = id$ then $\alpha, \beta \in F_i \text{ for some } i = 1, 2, 3, 4$.

**Proof.** Let $\alpha = [\alpha_{ij}]$ and $\beta = [\beta_{ij}]$ two elements in $\Gamma$, suppose that they commute. Denote by $\alpha_1$ and $\alpha_2$ (resp. $\beta_1$ and $\beta_2$) the upper left and bottom right $2 \times 2$ blocks of $\alpha$ (resp. $\beta$)

\[
\alpha_1 = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{33} \end{bmatrix}.
\]

A direct calculation shows that, since $\alpha$ and $\beta$ commute, $\alpha_i$ and $\beta_i$ commute, for $i = 1, 2$. Considering $\alpha_i$ and $\beta_i$ as elements of $\text{PSL}(2, \mathbb{C})$ we observe that

\[
\text{Fix}(\alpha_1) = \{e_1, [\alpha_{12} : \alpha_{22} - \alpha_{11}]\}, \quad \text{Fix}(\alpha_2) = \{e_1, [\alpha_{23} : \alpha_{33} - \alpha_{22}]\}.
\]

Similar expressions hold for $\text{Fix}(\beta_i)$. It follows that

\[
[\alpha_1, \beta_1] = id \iff \alpha_{11}\beta_{12} + \alpha_{12}\beta_{22} = \alpha_{12}\beta_{11} + \alpha_{22}\beta_{12} \iff \text{Fix}(\alpha_1) = \text{Fix}(\beta_1).
\]

If $\alpha_{12} = 0$ and $\alpha_{11} \neq \alpha_{22}$, then $\text{Fix}(\alpha_1) = \{e_1, e_2\}$. In this case, the previous calculation shows that $\beta_{12} = 0$, then $\text{Fix}(\beta_1) = \{e_1, e_2\} = \text{Fix}(\alpha_1)$. If $\alpha_{11} = \alpha_{22}$ and $\alpha_{12} \neq 0$, then $\text{Fix}(\alpha_1) = \{e_1\}$. Again, the previous calculation shows that $\beta_{11} = \beta_{22}$, then $\text{Fix}(\beta_1) = \{e_1\} = \text{Fix}(\alpha_1)$. All this shows that $\alpha_1$ and $\beta_1$ commute if and only if $\text{Fix}(\alpha_1) = \text{Fix}(\beta_1)$.

A similar conclusion holds for $\alpha_2$ and $\beta_2$. From this we can conclude that, if $\alpha$ and $\beta$ commute then $\alpha_{12}$ and $\beta_{12}$ are zero or non-zero simultaneously, the same holds for $\alpha_{23}$ and $\beta_{23}$. This proves the proposition. \(\square\)

**Proposition 4.27.** Let $\alpha, \beta \in F_i, i = 1, 2, 3$. Then, the relation defined by $\alpha \sim \beta$ if and only if $[\alpha, \beta] = id$, is an equivalence relation on each subset $F_i, i = 1, 2, 3$. 
Proof. It is direct to verify that the relation is reflexive and symmetric. We now verify that it is also transitive, let \( \alpha, \beta, \gamma \in F_i \), for some \( i = 1, 2, 3 \) such that \([\alpha, \beta] = id\) and \([\beta, \gamma] = id\). Denote \( \alpha = [\alpha_{ij}] \), \( \beta = [\beta_{ij}] \) and \( \gamma = [\gamma_{ij}] \). Then \([\alpha, \beta] = id\) if and only if
\[
(4.21) \quad \beta_{i3}(\alpha_{11} - \alpha_{33}) + \alpha_{13}(\beta_{11} - \beta_{33}) = \alpha_{23}\beta_{12} - \alpha_{12}\beta_{23}.
\]
If both \( \alpha, \beta \in F_i \), \( i = 1, 2, 3 \), then \( \alpha_{12} = \beta_{12} = 0 \) or \( \alpha_{23} = \beta_{23} = 0 \), which implies that the right side of (4.21) is zero and then \( \beta_{i3}(\alpha_{11} - \alpha_{33}) + \alpha_{13}(\beta_{11} - \beta_{33}) = 0 \). From this, analogously to the proof of Proposition 4.26, it follows that \( \text{Fix}(\alpha) = \text{Fix}(\beta) \).

Analogously, \([\beta, \gamma] = id\) implies that \( \text{Fix}(\beta) = \text{Fix}(\gamma) \), then \( \text{Fix}(\alpha) = \text{Fix}(\gamma) \) and this implies that \([\alpha, \gamma] = id\). This proves that the relation is transitive and thus, it is an equivalence relation on \( F_i \), \( i = 1, 2, 3 \).

In [AA10] the author proves a version of Corollary 4.28 for groups with real entries. In [BCNS18], another version of this corollary is proven, this time for purely parabolic groups. In this sense, Corollary 4.28 is a generalization of these previous results.

Propositions 4.26 and 4.27 state that, unlike the purely parabolic case (see Lemma 6.8 of [BCNS18]), commutativity does not define an equivalence relation on \( U_+ \), this equivalence relation only happens on each \( F_i \), \( i = 1, 2, 3 \) separately. On \( F_4 \), commutativity does not define an equivalence relation. The following corollary simplifies the decomposition described in Theorem 4.21.

**Corollary 4.28.** Under the same hypothesis and notation of Theorem 4.21 the group \( \Gamma \) can also be written as
\[
\Gamma \cong \mathbb{Z}^{r_0} \times \cdots \times \mathbb{Z}^{r_m},
\]
for integers \( r_0, \ldots, r_m \geq 1 \) satisfying \( r_0 + \cdots + r_m \leq 4 \).

**Proof.** Using the notation of Theorem 4.21 and Corollary 4.29 we know that the group \( A = \text{Core}(\Gamma) \times (\xi_1) \times \cdots \times (\xi_r) \) is purely parabolic and therefore, by Lemma 6.10 of [BCNS18] we can write
\[
(4.22) \quad A \cong \mathbb{Z}^{k_0} \times \cdots \times \mathbb{Z}^{k_{n_1}}.
\]
For some integers \( k_0, \ldots, k_{n_1} \) such that \( k_0 + \cdots + k_{n_1} \leq 4 \). We denote
\[
(4.23) \quad r_i = k_i, \quad \text{for } i = 0, \ldots, n_1.
\]

Let us re-order the elements \( \{\eta_1, \ldots, \eta_m\} \) in the third layer, in such way that if \( i < j \) then \( \eta_i \in F_{s_1}, \eta_j \in F_{s_2} \) with \( s_1 \leq s_2 \). We re-order the elements \( \{\gamma_1, \ldots, \gamma_n\} \) in the same way. Define the relation in
\[
(4.24) \quad \Gamma_1 := \langle \eta_1, \ldots, \eta_m \rangle \cap (F_1 \cup F_2 \cup F_3)
\]
given by \( \alpha \sim \beta \) if and only if \([\alpha, \beta] = id\), this is an equivalence relation (Proposition 4.27). Denote by \( A_1, \ldots, A_{n_2} \) the equivalence classes in \( \Gamma_1 \). Let \( B_i = \langle A_i \rangle \), clearly \( B_i \) is a commutative and torsion free group, denoting \( p_i = \text{rank}(B_i) \) and using Proposition 3.1 we have \( B_i \cong \mathbb{Z}^{p_i} \). Then
\[
(4.24) \quad \Gamma_1 \cong \mathbb{Z}^{p_1} \times \cdots \times \mathbb{Z}^{p_{n_2}}.
\]
Denote by \( \bar{\eta_i} \) the remaining elements of the third layer. That is,
\[
\langle \eta_1, \ldots, \eta_m \rangle \cap F_4 = \{\bar{\eta_1}, \ldots, \bar{\eta_{n_3}}\}.
\]
Then, it follows from (4.24),
\begin{equation}
\langle \eta_1, ..., \eta_m \rangle \cong \mathbb{Z}^{p_1} \times \cdots \times \mathbb{Z}^{p_{n_2}} \times \langle \eta_{p_1} \rangle \times \cdots \times \langle \eta_{p_{n_3}} \rangle.
\end{equation}

Let us denote
\begin{equation}
r_{n_1+i} = p_i, \quad \text{for } i = 1, ..., n_2.
\end{equation}

\begin{equation}
r_{n_2+n_1+i} = 1, \quad \text{for } i = 1, ..., n_3.
\end{equation}

Applying the same argument to the elements of the fourth layer \{\gamma_1, ..., \gamma_n\} we have
\begin{equation}
\langle \gamma_1, ..., \gamma_n \rangle \cong \mathbb{Z}^{q_1} \times \cdots \times \mathbb{Z}^{q_{n_4}} \times \langle \gamma_{q_1} \rangle \times \cdots \times \langle \gamma_{q_{n_5}} \rangle.
\end{equation}

Again, we denote
\begin{equation}
r_{n_1+n_2+n_3+i} = q_i, \quad \text{for } i = 1, ..., n_4.
\end{equation}

\begin{equation}
r_{n_4+n_1+n_2+n_3+i} = 1, \quad \text{for } i = 1, ..., n_5.
\end{equation}

Putting together (4.22), (4.25) and (4.27) we prove the corollary. The indices \( r_0, ..., r_m \) are given by (4.23), (4.26) and (4.28) and \( m = n_1 + ... + n_5 \). □

The following corollary describes the type of elements found in each layer of the decomposition.

**Corollary 4.29.** Let \( \Gamma \subset U_+ \) be a non-commutative discrete subgroup. Consider the decomposition in four layers described in the proof of Theorem 4.21, and summarized in Table 3.

The first two layers \( \text{Core}(\Gamma) \) and \( A \setminus \text{Core}(\Gamma) \) are purely parabolic and the last two layers \( \text{Ker}(\lambda_{23}) \setminus A \) and \( \Gamma \setminus \text{Ker}(\lambda_{23}) \) are made up entirely of loxodromic elements. Furthermore,
\begin{enumerate}
  \item The third layer \( \text{Ker}(\lambda_{23}) \setminus A \) contains only loxo-parabolic elements.
  \item The fourth layer \( \Gamma \setminus \text{Ker}(\lambda_{23}) \) contains only loxo-parabolic and strongly loxodromic elements or complex homotheties with the form \( \text{Diag}(\lambda, \lambda^{-2}, \lambda) \).
\end{enumerate}

**Proof.** From the definition of \( \text{Core}(\Gamma) \) and \( A = \text{Ker}(\lambda_{12}) \cap \text{Ker}(\lambda_{23}) \) is clear that this two subgroups are purely parabolic. Now we deal with the third layer \( \text{Ker}(\lambda_{23}) \setminus A \).

If there were a elliptic element \( \gamma \) in this layer, we have two cases:
\begin{itemize}
  \item If \( \gamma \) has infinite order then \( \Gamma \) cannot be discrete.
  \item If \( \gamma \) has finite order \( p > 0 \), but we are assuming that \( \Gamma \) is torsion free.
\end{itemize}

If there were a parabolic element \( \gamma \) in this layer, then it must have exactly two repeated eigenvalues (if it had 3, then \( \gamma \in A \)). Furthermore, all of its eigenvalues must be unitary (they can’t be 1, because then \( \gamma \in A \)). Then,
\[
\gamma = \begin{bmatrix}
ed^{-4\pi i \theta} & x & y \\
0 & e^{2\pi i \theta} & z \\
0 & 0 & e^{2\pi i \theta}
\end{bmatrix}
\]
with \( z \neq 0 \) and \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) (since \( \lambda_{12}(\Gamma) \) is a torsion free group). This means that \( \gamma \) is an irrational ellipto-parabolic element, and by Proposition 4.13 \( \Gamma \) would be commutative. All of these arguments prove that the third layer \( \text{Ker}(\lambda_{23}) \setminus A \) is purely loxodromic. Finally, since \( \Gamma \) is not commutative, it cannot contain a complex homothety as a consequence of Corollary 4.7.
Now we deal with the fourth layer $\Gamma \setminus \text{Ker}(\lambda_{23})$. Using the same argument as in the third layer, there cannot be elliptic elements. Now assume that there is a parabolic element $\gamma \in \Gamma \setminus \text{Ker}(\lambda_{23})$. In the same way as before, $\gamma$ must have exactly two distinct eigenvalues and neither of them are equal to 1. Since $\gamma \notin \text{Ker}(\lambda_{23})$ then

$$
\gamma = \begin{bmatrix}
  e^{2\pi i \theta} & x & y \\
  0 & e^{2\pi i \theta} & z \\
  0 & 0 & e^{-4\pi i \theta}
\end{bmatrix}
$$

with $x \neq 0$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $\gamma$ is an irrational ellipto-parabolic element, and by Proposition 4.13 $\Gamma$ is commutative. Then the fourth layer $\Gamma \setminus \text{Ker}(\lambda_{23})$ is purely loxodromic. Inspecting the form of these elements, they can be strongly loxodromic or complex homotheties of the form $\text{Diag}(\lambda, \lambda^{-2}, \lambda)$ (see Corollaries 4.7 and 4.8).

5. Proof of the Main Theorem

In this section we prove Theorem 0.1.

Proof. In each subcase of the proof of Theorem 4.25 for every divergent sequence $a := \{\gamma_k\} \subset \Gamma$, we have constructed an open subset $U_a \subset \mathbb{CP}^2$, such that the orbits of every compact set $K \subset U_a$ accumulate on $\mathbb{CP}^2 \setminus U_a$. Thus, we can define a limit set for the action of $\Gamma$ in the following form

$$
\Lambda_{\Gamma} := \bigcup_a (\mathbb{CP}^2 \setminus U_a).
$$

This limit set describes the dynamics of non-commutative upper triangular subgroups of $\text{PSL}(3, \mathbb{C})$ and the open region $\Omega_{\Gamma} = \mathbb{CP}^2 \setminus \Lambda_{\Gamma}$ verifies (i) and (ii).

$\Gamma$ is finitely generated (see [Aus60]) and we have proven in Theorem 4.25 that $\text{rank}(\Gamma) \leq 4$. This verifies (iii).

Conclusion (iv) follows immediately from Theorem 4.24 and Corollary 4.29.

Finally, conclusion (v) follows from Theorem 2.1.

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