The Dirac equation in Schwarzschild mass coupled to a Stationary Electromagnetic Field
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Abstract

We study the Dirac equation in a spacetime that represents the nonlinear superposition of the Schwarzschild solution to an external, stationary electromagnetic Bertotti-Robinson solution. We separate the Dirac equation into radial and angular equations using Newman–Penrose formalism. We obtain exact analytical solutions of the angular equations. We manage to obtain the radial wave equations with effective potentials. Finally, we study the potentials by plotting them as a function of radial distance and examine the effect of the twisting parameter and the frequencies on the potentials.

1 Introduction

The spacetime we are considering represents the metric of the Schwarzschild (S) mass coupled to an externally twisting electromagnetic (em) universe, namely, generalized Bertotti-Robinson (GBR) spacetime [1]. We shall refer to this solution as the SGBR. This metric is an example to coupling black hole solutions with the surroundings. Limiting case of this spacetime includes the case of a stationary em universe i.e. Bertotti-Robinson (BR) geometry. Recall that the BR solution [2-3], describes a non-singular, spherically symmetric universe filled with a static electric field. It is considered as a unique conformally flat solution of the source-free Einstein-Maxwell equations for non-null fields. This spacetime is the direct product of 2-sphere and 2-dimensional anti-de Sitter spacetime. Therefore BR solution has the symmetries of both these spaces with 6 parameter isometry group. In this present paper our interesting metric represents a black hole which is embedded into an em field. Similar properties of our interesting metric was found long time ago by Carter [4]. In contrast, the metric in [4] is given in terms of BR coordinates which are valid inside the throat region only. But, this metric uses S coordinates that refers to the entire universe.

Studying Dirac equation in different backgrounds has been extensively worked out and published in literature. For example, in S geometry [5-6] and Kerr spacetime [7-10]. BR geometry [11-12], where in [11] Silva-Ortigoza showed how the Dirac equation could be separated in the BR spacetime with cosmological constant. Whereas in [12], a more detailed study on the problem of the Dirac equation in the BR background was worked out. Dirac equation in rotating BR geometry [13], they have obtained exact solutions to both the radial equations and the angular massless neutrino equations. Where the exact solution of the radial equations is given in terms of hypergeometric functions. Recently, Dirac equation was examined in Kerr-Taub-NUT spacetime [14].

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the last reference, using Boyer-Lindquist coordinates they manage to separate
the Dirac equation into radial and angular parts. They obtained exact solution
of the angular equations for some special cases and obtained the radial wave
equations with an effective potentials and exposed the effect of the NUT param-
eter. Other similar studies with different background, Nutku helicoid spacetime
[15], Kerr–Newman–AdS black hole geometry [16] and in the background of the
Kerr–Newman family, which has been considered in several studies [17-18].

Here, we study the solution of the Dirac equation in the S mass coupled
to a GBR spacetime. The set of equations representing the Dirac equation in
the Newman–Penrose (NP) formalism is decoupled into a radial (function of
distance \( r \) only) and an angular parts (function of angle \( \theta \) only). The angular
equation is solved exactly in terms of associated Legendre functions. The radial
equations are discussed and the one dimensional Schrödinger-like wave equations
with effective potentials are obtained. Finally in order to understand and expose
the effect of the twisting parameter and frequencies on the potentials, curves
are plotted and discussed.

Our paper is organized as follows: in section 2, we present the Dirac equation
in the metric of SGBR solution and separate them into two parts. In section 3,
solutions of the angular and radial equations are presented. Finally, we discuss
and comment on the effect of the twist parameter and the frequency on the
potentials by plotting them as a function of radial distance.

2 Dirac equation in the metric of SGBR

The metric we are dealing with represents the non-linear superposition of the S
solution and the GBR solution[1], is given by

\[
    ds^2 = \frac{r^2 - 2Mr}{r^2 f(r)} [dt - Mq(1 + a^2) \cos \theta d\phi] - \frac{r^2 f(r)}{r^2 - 2Mr} dr^2 - \frac{r^2 f(r)}{r^2 - 2Mr} (d\theta^2 + \sin^2 \theta d\phi^2). 
\]

where

\[
    r^2 f(r) = \frac{1}{2} r (r - 2M) \left[ p (1 + a^2) + a^2 - 1 + 2Mar + M^2 [p (1 + a^2) - 2a] \right] \tag{2}
\]

in which, \( M \) is the S mass, \( p = \sqrt{1 + q^2} \) is the twisting parameter of the external
em field and \( 0 \leq a \leq 1 \) is the interpolation parameter. It is seen that for \( a = 0 \) \( (q = 0, p = 1) \) metric (1) reduces to BR solution and for \( a = 1 \) \( (q = 0, p = 1) \) it reduces to S solution. One can justify the limiting cases by calculating the
NP curvatures using the null tetrad 1-form of NP formalism. we make the
choice of the following null tetrad basis 1-forms \( (l, n, m, \overline{m}) \) of the NP formalism
[19] in terms of the metric functions that satisfies the orthogonality conditions,
\( (l.n = -m.\overline{m} = 1) \). Where a bar over a quantity denotes complex conjugation.

We can write the covariant 1-forms as

\[
    \sqrt{2}l = H(r)(dt - Q \cos \theta d\phi) - \frac{dr}{H(r)}.
\]
\[ \sqrt{2}n = H(r)(dt - Q \cos \theta d\phi) + \frac{dr}{H(r)}, \]
\[ \sqrt{2}m = r \sqrt{f(r)}(d\theta + i \sin \theta d\phi), \]  
(3)

where
\[ H^2(r) = \frac{r^2 - 2Mr}{r^2 f(r)}, \quad Q = M \sqrt{p^2 - 1}(1 + a^2). \]

Using the above null tetrad (3), we obtain the non-zero Weyl and Ricci NP scalars [1]
\[ \Psi_2 = \frac{-M}{4R^6} \left( 2aa_0 r^3 + Ma_0 (4c_0 - b_0) r^2 + 2M^2 (b_0^2 - b_0 c_0 - 3a_0 c_0) r 
+ 2M^3 b_0 c_0 + iq \left( 1 + a^2 \right) \left( a_0 (3M - r) r^2 + 2M^2 \left( 1 - a^2 \right) r - 2c_0 M^3 \right) \right), \]
\[ \Phi_{11} = \frac{M^2 (1 - a^2)}{2R^4}, \]  
(5)

where
\[ R^2 = r^2 f(r), \]
\[ c_0 = p \left( 1 + a^2 \right) - 2a, \]
\[ b_0 = a_0 - 2a, \]
\[ a_0 = p \left( 1 + a^2 \right) + a^2 - 1. \]

This metric is a type-D metric whose conformal curvature is due to the twist parameter \( p \). It is seen that metric (1) satisfies all the required limits as boundary conditions:

\[ BR \text{ Solution} \rightarrow (a = 0, q = 0, M = 1) \rightarrow \Psi_2 = 0, \Phi_{11} = \frac{1}{2}, \]  
(7)
\[ SS \text{ Solution} \rightarrow (a = 1, q = 0) \rightarrow \Psi_2 = -\frac{M}{r^3}, \Phi_{11} = 0, \]
\[ GBR \text{ Solution} \rightarrow (a = 0, q \neq 0) \rightarrow \Psi_2 \neq 0 \neq \Phi_{11}. \]

The case \( GBR \text{ Solution} \rightarrow (a = 0, q \neq 0) \) has been studied in greater detail in [20], where its metric is an exact solution of Einstein-Maxwell equations of general relativity which represents a uniform em field in the rotating state.

We shall now write the Dirac equation in the SGBR spacetime. The Dirac equation in the NP formalism [8] is given by
\[ (D + \epsilon - \rho) F_1 + (\delta + \pi - \alpha) F_2 - i\mu_0 G_1 = 0, \]
\[ (\Delta + \mu - \gamma) F_2 + (\delta + \beta - \tau) F_1 - i\mu_0 G_2 = 0, \]
\[ (D + \bar{\epsilon} - \bar{\rho}) G_2 - (\delta + \pi - \bar{\alpha}) G_1 - i\mu_0 F_2 = 0, \]
\[(\Delta + \overline{\mu} - \overline{\gamma}) G_1 - (\delta + \beta - \overline{\gamma}) G_2 - i \mu_0 F_1 = 0, \tag{8}\]

where \(\mu_0\) is the mass of the Dirac particle.

Let us write the corresponding directional derivatives as

\[
\sqrt{2} \mathcal{D} = \frac{1}{H} \partial_t + H \partial r, \\
\sqrt{2} \Delta = \frac{1}{H} \partial_t - H \partial r, \\
\sqrt{2} \delta = -\frac{1}{R} \left[ \partial_\theta + i \frac{1}{\sin \theta} \partial_\phi \right], \\
\sqrt{2} \overline{\delta} = -\frac{1}{R} \left[ \partial_\theta - i \frac{1}{\sin \theta} \partial_\phi \right], \tag{9}\]

Using the above tetrad we determine the nonzero NP complex spin coefficients [19] as,

\[
\rho = \mu = -\frac{R'H}{\sqrt{2}R} + i \frac{HQ}{2\sqrt{2}R^2}, \\
\epsilon = \gamma = \frac{H'}{2\sqrt{2}} + i \frac{HQ}{4\sqrt{2}R^2}, \\
\alpha = -\beta = \frac{\cot \theta}{2\sqrt{2}R}. \tag{10}\]

The form of the Dirac equation suggests that we assume [8],

\[
F_1 = F_1 (r, \theta) e^{i(kt+n\phi)}, \\
F_2 = F_2 (r, \theta) e^{i(kt+n\phi)}, \\
G_1 = G_1 (r, \theta) e^{i(kt+n\phi)}, \\
G_2 = G_2 (r, \theta) e^{i(kt+n\phi)}. \tag{11}\]

We consider the corresponding Compton wave of the Dirac particle as in the form of \(F = F (r, \theta) e^{i(kt+n\phi)}\), where \(k\) is the frequency of the incoming wave and \(n\) is the azimuthal quantum number of the wave. Substituting the appropriate spin coefficients (10) and the spinors (11) into the Dirac equation (8), we obtain

\[
\left( H \mathcal{D} - i \frac{HQ}{4R^2} \right) F_1 - \frac{1}{R} \mathbf{L} F_2 - i \mu_0 G_1 = 0, \\
\left( -H \mathcal{D}^\dagger + i \frac{HQ}{4R^2} \right) F_2 - \frac{1}{R} \mathbf{L}^\dagger F_1 - i \mu_0 G_2 = 0, \\
\left( H \mathcal{D} + i \frac{HQ}{4R^2} \right) G_2 + \frac{1}{R} \mathbf{L}^\dagger G_1 - i \mu_0 F_1 = 0, \\
\left( H \mathcal{D} + i \frac{HQ}{4R^2} \right) G_1 - \frac{1}{R} \mathbf{L} G_2 = 0.
\]
\[
\left(-HD^\dagger - i\frac{HQ}{4R^2}\right)G_1 + \frac{1}{R}LG_2 - i\mu_0 F_1 = 0,
\] (12)

where the radial and the angular operators are

\[
D = \frac{d}{dr} + \frac{H'}{2H} + \frac{R'}{R} + i\frac{k}{H^2}
\]
\[
D^\dagger = \frac{d}{dr} + \frac{H'}{2H} + \frac{R'}{R} - i\frac{k}{H^2}
\]
\[
L = \frac{d}{d\theta} + \frac{n}{\sin^2 \theta} + \frac{\cot \theta}{2}
\]
\[
L^\dagger = \frac{d}{d\theta} - \frac{n}{\sin^2 \theta} + \frac{\cot \theta}{2}
\]

It is now apparent that Eqs. (13) can be separated by implying the separability ansatz

\[
F_1 = f_1(r) A_1(\theta),
\]
\[
F_2 = f_2(r) A_2(\theta),
\]
\[
G_1 = f_2(r) A_1(\theta),
\]
\[
G_2 = f_1(r) A_2(\theta).
\]

With this ansatz, Eqs. (13) become

\[
\begin{bmatrix}
\left(RH D - i\frac{HQ}{4R}\right) f_1 - i\mu_0 R f_2 \\
\left(-RH D^\dagger + i\frac{HQ}{4R}\right) f_2 - i\mu_0 R f_1 \\
\left(RH D + i\frac{HQ}{4R}\right) f_1 - i\mu_0 R f_2 \\
\left(-RH D^\dagger - i\frac{HQ}{4R}\right) f_2 - i\mu_0 R f_1
\end{bmatrix} A_1 - [LA_2] f_2 = 0,
\]
\[
\begin{bmatrix}
\left(RH D - i\frac{HQ}{4R}\right) f_1 - i\mu_0 R f_2 \\
\left(-RH D^\dagger + i\frac{HQ}{4R}\right) f_2 - i\mu_0 R f_1 \\
\left(RH D + i\frac{HQ}{4R}\right) f_1 - i\mu_0 R f_2 \\
\left(-RH D^\dagger - i\frac{HQ}{4R}\right) f_2 - i\mu_0 R f_1
\end{bmatrix} A_2 - [L^\dagger A_1] f_1 = 0,
\]
\[
\begin{bmatrix}
\left(RH D - i\frac{HQ}{4R}\right) f_1 - i\mu_0 R f_2 \\
\left(-RH D^\dagger + i\frac{HQ}{4R}\right) f_2 - i\mu_0 R f_1 \\
\left(RH D + i\frac{HQ}{4R}\right) f_1 - i\mu_0 R f_2 \\
\left(-RH D^\dagger - i\frac{HQ}{4R}\right) f_2 - i\mu_0 R f_1
\end{bmatrix} A_2 + [L^\dagger A_1] f_2 = 0,
\]
\[
\begin{bmatrix}
\left(RH D - i\frac{HQ}{4R}\right) f_1 - i\mu_0 R f_2 \\
\left(-RH D^\dagger + i\frac{HQ}{4R}\right) f_2 - i\mu_0 R f_1 \\
\left(RH D + i\frac{HQ}{4R}\right) f_1 - i\mu_0 R f_2 \\
\left(-RH D^\dagger - i\frac{HQ}{4R}\right) f_2 - i\mu_0 R f_1
\end{bmatrix} A_1 + [LA_2] f_1 = 0.
\] (15)

These equations (15) imply that

\[
LA_2 = \lambda_1 A_1,
\]
\[
L^\dagger A_1 = \lambda_2 A_2,
\]
\[
L^\dagger A_1 = \lambda_3 A_2,
\]
\[
LA_2 = \lambda_4 A_1,
\]
\[
RH D - i\frac{HQ}{4R} f_1 - i\mu_0 R f_2 = \lambda_1 f_2,
\] (16)
\[-RHD + i\frac{HQ}{4R} \] \[ f_2 - i\mu_0 R f_1 = \lambda_2 f_1, \]
\[ (RHD + i\frac{HQ}{4R}) f_1 - i\mu_0 R f_2 = -\lambda_3 f_2, \]
\[ (-RHD^\dagger - i\frac{HQ}{4R}) f_2 - i\mu_0 R f_1 = -\lambda_4 f_1, \] \tag{17}

where \(\lambda_1, \lambda_2, \lambda_3\) and \(\lambda_4\) are four constants of separation. However, let us assume \((\lambda_1 = \lambda_4 = \lambda, \lambda_2 = \lambda_3 = -\lambda)\), and adding equations, then we obtain the radial and the angular pair equations

\[ RH (D f_1) = (\lambda + i\mu_0 R) f_2, \] \tag{18}
\[ RH (D^\dagger f_2) = (\lambda - i\mu_0 R) f_1, \] \tag{19}

\[ L A_2 = \lambda A_1, \] \tag{20}
\[ L^\dagger A_1 = -\lambda A_2. \] \tag{21}

## 3 Solution of angular and radial equations

Angular Eqs. (20) and (21) we can be written as

\[ \frac{dA_1}{d\theta} + \left( \cot \frac{\theta}{2} - \frac{n}{\sin \theta} \right) A_1 = -\lambda A_2, \] \tag{22}
\[ \frac{dA_2}{d\theta} + \left( \cot \frac{\theta}{2} + \frac{n}{\sin \theta} \right) A_2 = \lambda A_1. \] \tag{23}

The structure of the angular equations admits the solution of \(A_1\) is similar to \(A_2\). Thus, it is enough to decouple the angular equations for \(A_1\).

First, we affect the transformation

\[ A_1 (\theta) = \cos \left( \frac{\theta}{2} \right) S_1 + \sin \left( \frac{\theta}{2} \right) S_2 \]
\[ A_2 (\theta) = -\sin \left( \frac{\theta}{2} \right) S_1 + \cos \left( \frac{\theta}{2} \right) S_2 \] \tag{24}

Then, let \(x = \cos \theta\) and one can write Eq. (22) into second order differential equation

\[ (1 - x^2) \frac{d^2 S_1}{dx^2} - 2x \frac{dS_1}{dx} + \left[ \lambda (\lambda + 1) - \left( n + \frac{1}{2} \right)^2 \right] S_1 = 0, \] \tag{25}
with \((n + \frac{1}{2})^2 \leq \lambda^2\). The solution can be obtained in terms of the associated Legendre functions

\[
P^n_\lambda(x) = (1 - x^2)^{\lambda/2} \frac{d^n}{dx^n} P_\lambda(x).
\] (26)

For the radial equations (18) and (19) can be rearranged as

\[
\frac{df_1}{dr} + \left( \frac{R^2 H' H}{2H^2 R^2} + \frac{RR'H^2}{H^2 R^2} + i \frac{kR^2}{H^2 R^2} \right) f_1 = \frac{1}{RH} (i\mu_0 R + \lambda) f_2,
\] (27)

\[
\frac{df_2}{dr} + \left( \frac{R^2 H H'}{2H^2 R^2} + \frac{RR'H^2}{H^2 R^2} - i \frac{kR^2}{H^2 R^2} \right) f_2 = \frac{1}{RH} (\lambda - i\mu_0 R) f_1.
\] (28)

Our aim now is to put the radial equations (27) and (28) in the form of one dimensional wave equations and obtain the effective potentials. Therefore, to achieve our aim we follow the method applied by Chandrasekhar’s book [8]. We will starts with the transformations

\[
P_1 = RH f_1, \quad P_2 = RH f_2.
\] (29)

and next the scaling

\[
T_1 = P_1 \exp \left( \int \frac{H'}{2H} dr \right), \quad T_2 = P_2 \exp \left( \int \frac{H'}{2H} dr \right)
\] (30)

Then, Eqs. (27) and (28) become

\[
\frac{dT_1}{dr} + i \frac{kR^2}{H^2 R^2} T_1 = \frac{1}{RH} (i\mu_0 R + \lambda) T_2,
\] (31)

\[
\frac{dT_2}{dr} - i \frac{kR^2}{H^2 R^2} T_2 = \frac{1}{RH} (\lambda - i\mu_0 R) T_1.
\] (32)

Assume

\[
\frac{du}{dr} = \frac{1}{H^2}.
\] (33)

Therefore, the above equations (31) and (32) in terms of the new independent variable \(u\), become

\[
\frac{dT_1}{du} + \frac{H}{R} (i\mu_0 R + \lambda) T_2 = \frac{H}{R} (i\mu_0 R + \lambda) T_2,
\] (34)

\[
\frac{dT_2}{du} - \frac{H}{R} (\lambda - i\mu_0 R) T_1 = \frac{H}{R} (\lambda - i\mu_0 R) T_1.
\] (35)
where
\[ u = \frac{1}{2} \left[ p \left(1 + a^2\right) + a^2 - 1\right] r + \frac{1}{2} \sqrt{\frac{M}{2r}} \left[ p \left(1 + a^2\right) + a^2 - 4a - 1\right] r - \left[ p \left(1 + a^2\right) - 2a\right] M \arctan\left(\sqrt{\frac{r}{2M}}\right). \] (36)

Let us apply another transformation, namely the new functions
\[ T_1 = \phi_1 \exp\left(-\frac{i}{2} \arctan\left(\frac{\mu_0 R}{\lambda}\right)\right), \quad T_2 = \phi_2 \exp\left(\frac{i}{2} \arctan\left(\frac{\mu_0 R}{\lambda}\right)\right), \] (37)

with these transformations, Eqs. (34) and (35) take the form
\[ \frac{d\phi_1}{du} + ik\left(1 - \frac{H^2}{2k} \left[ \frac{\mu_0 \lambda R'}{\lambda^2 + \mu_0^2 R^2} \right]\right) \phi_1 = \frac{H}{R} \sqrt{\lambda^2 + \mu_0^2 R^2} \phi_2, \] (38)
\[ \frac{d\phi_2}{du} - ik\left(1 - \frac{H^2}{2k} \left[ \frac{\mu_0 \lambda R'}{\lambda^2 + \mu_0^2 R^2} \right]\right) \phi_2 = \frac{H}{R} \sqrt{\lambda^2 + \mu_0^2 R^2} \phi_1. \] (39)

Again change the variable \( u \) into \( \hat{r} \) as
\[ \hat{r} = u - \frac{1}{2k} \arctan\left(\frac{\mu_0 R}{\lambda}\right). \] (40)

Then we can write Eqs.(38) and (39) in the alternative forms
\[ \frac{d\phi_1}{d\hat{r}} + W \phi_2 = -ik \phi_1, \] (41)
\[ \frac{d\phi_2}{d\hat{r}} - W \phi_2 = -ik \phi_1, \] (42)

where
\[ W = \frac{2k \left(\lambda^2 + \mu_0^2 R^2\right)^{3/2} H}{2k \left(\lambda^2 + \mu_0^2 R^2\right) R - \lambda \mu_0 R' RH^2}. \] (43)

Finally, in order to put the above equations (41) and (42) into one dimensional Schrödinger-like wave equations, we define
\[ 2\phi_1 = \psi_1 + \psi_2, \quad 2\phi_2 = \psi_1 - \psi_2. \] (44)

Then Eqs.(41) and (42) become
\[ \frac{d\psi_1}{d\hat{r}} - W \psi_1 = -ik \psi_2, \] (45)
\[ \frac{d\psi_2}{d\hat{r}} + W \psi_2 = -ik \psi_1. \]
Which can be cast into
\[ \frac{d^2 \psi_1}{d\hat{r}^2} + k^2 \psi_1 = V_+ \psi_1, \]
\[ \frac{d^2 \psi_2}{d\hat{r}^2} + k^2 \psi_2 = V_- \psi_2, \] (46)
where the effective potentials can be obtained from
\[ V_\pm = W^2 \pm \frac{dW}{d\hat{r}}. \] (47)

We calculate the potentials as
\[ V_\pm = \frac{4k^2 L^{3/2} RH}{D^2} \left[ L^{3/2} RH \pm 3\mu_0^2 R^3 \beta^2 \pm L \left( R' RH + RH' \right) \right] + \frac{2kLR^2H^2}{I} \left( 2\mu_0^2 R^4 \beta' - \frac{\lambda \mu_0}{2k} B + \frac{2RR'}{k} L \right), \] (48)
where
\[ L = (\lambda^2 + \mu_0^2 R^2), \quad D = 2kL^2 R^2 - \lambda \mu_0 R^2 \beta^2, \]
\[ B = R' R^2 \beta^2 + 2RR' \left( R' RH + RH' \right). \] (49)

Let us note that for the case of massless Dirac particle \( \mu_0 = 0 \), the potentials take the form
\[ V_\pm = \frac{\lambda H}{R^3} \left[ \lambda RH \pm (R' RH + RH') + \frac{2H^2 RRR'}{k} \right]. \] (50)

Now we will find the complete solution of Eq. (46). First we can rewrite Eq. (46) as
\[ \frac{d^2 \psi_1}{d\hat{r}^2} + \left( k^2 - V_+ \right) \psi_1 = 0, \]
\[ \frac{d^2 \psi_2}{d\hat{r}^2} + \left( k^2 - V_- \right) \psi_2 = 0. \] (51)
This is nothing but the one dimensional Schrödinger wave equations with total energy of the wave \( k^2 \). Eq. (51) can be solved by WKB approximation method [21,22]. The solution is given by
\[ \psi_1 = \sqrt{T_1} \left[ \omega_1 (\hat{r}) \right] e^{+iy_1} + \sqrt{R_1} \left[ \omega_1 (\hat{r}) \right] e^{-iy_1}, \]
\[ \psi_2 = \sqrt{T_2} \left[ \omega_2 (\hat{r}) \right] e^{+iy_2} + \sqrt{R_2} \left[ \omega_2 (\hat{r}) \right] e^{-iy_2}, \] (52)
where
\[ \omega_1 (\hat{r}) = \sqrt{(k^2 - V_+)}, \quad \omega_2 (\hat{r}) = \sqrt{(k^2 - V_-)} \] (53)
\[ y_1(\hat{r}) = \int \omega_1(\hat{r}) \, d\hat{r} + \text{constant}, \quad y_2(\hat{r}) = \int \omega_2(\hat{r}) \, d\hat{r} + \text{constant} \quad (54) \]

with

\[ T_1(r) + R_1(r) = 1, \quad T_2(r) + R_2(r) = 1 \quad \text{instantaneously.} \quad (55) \]

Where \( \omega \) is the wave number of the incoming wave and \( y \) is the eikonal. \( T_{1,2} \) and \( R_{1,2} \) are the instantaneous transmission and reflection coefficients \([5]\) respectively. Let us note that the above solution is valid when \( (1/\omega)(d\omega/d\hat{r}) \ll \omega \).

4 Discussion

In this section, we are going to expose the effect of the twisting parameter \( p \) on the effective potentials by plotting the potentials as a function of radial distance with varying \( p \). Next, we will study the behavior of the potentials by drawing potentials curves for some different value of frequency \( k \). From the potentials Eqs. (48) and (50), it is very clear that the potentials depend strictly on the twisting parameter \( p \) and the interpolation parameter \( a \) via the functions \( R \) and \( H \). We must notice that the expressions becomes singular when \( D = 0 \) for the case \( \mu_0 \neq 0 \), and when \( R = 0 \) for the case \( \mu_0 = 0 \). Also we notice that the potentials have local extrema when \( \frac{dV}{dr} = 0 \), which is a very complicated equation to be solved since the potentials are rather involved.

Recall that the twisting parameter \( p \) has been discussed in details \([23]\), indeed they interpret it as the twist parameter of the em universe. If there is no em field this parameter represents the twist of the vacuum spacetime. Therefore, in this case it couples to S mass and generates a NUT parameter given by \( l = \pm M \sqrt{p - \frac{1}{p}} \). To examine the effect of \( p \) on the potentials we obtain two dimensional plots of Eq. (48) for the case \( \mu_0 \neq 0 \). Let us note that, the physical regions for our potentials start from \( r > 2M \) in all figures. In fig. 1, we exhibit the behaviour of the potentials \( V_\pm \) for different values of twisting parameter \( p \), the case \( (p = 1, q = 0) \) is excluded because it lead us to a special case in the metric, where we have chosen the rest mass to be \( \mu_0 = 0.12 \), and fixed value of \( k = 0.2 \) such that \( \mu_0 < k \). It is seen from fig.1 that, potentials have sharp peaks in the physical distance of \( r \), and when the twisting parameter \( p \) increases the values of the sharp peaks decrease. We conclude that for large value of \( p \), a massive Dirac particle moving in the physical region faces a small potential barriers whereas for small value of \( p \) encounters large potentials barriers. Therefore, as the twisting parameter increases the kinetic energy of the particle increases and advances without strong retarding potentials. Anyhow, potentials are bounded, regardless of the value of \( p \), which means that as \( (r \to \infty) \), the potentials \( V_\pm \to \text{constant} \) as seen from all figures. To examine the behaviour of the potentials for some values of frequencies we
obtain figure 2 for various value of $k$ and fixed value of $p = 10$. It is seen from fig. 2 that, the general behavior of the potentials are not changed, they still have sharp peaks and behave similar for large distances. However, we notice that at low frequencies the sharp peaks are very clear, but for high frequencies the levels of sharp peaks decrease. In the case of massless Dirac particle, we would remark that in the two dimensional plots the behavior of the potentials are similar to the case $\mu_0 \neq 0$.

Now, the three dimensional plot of the potentials with respect to the twisting parameter and the radial distance $r$ is given in Figure 3. It shows the effect of the twisting parameter on the the potentials explicitly for the massive Dirac particle ($\mu_0 = 0.12$ and $k = 0.2$). It is seen from Fig. 3 that, high potential barriers are observed for small value of twisting parameter whereas for small value of $p$ potential barriers decrease. Again for large distances, potentials levels decrease and asymptote behaviour is manifested. Note that, the behaviour of the $V_+$, which is given in figure 4, is similar as in Figure 3. The three dimensional plots of potentials with respect to frequency, for fixed twisting parameter ($p = 10$ and $\mu_0 = 0.12$), and the radial distance is given in figure 5. We observe in Fig. 5 that, sharp peaks are clear for low frequencies only. Now, for massless Dirac particle ($\mu_0 = 0$), three dimensional plots are given in figures 6 and 7. The general behaviour of the potentials for massless Dirac particle shows similar behavior as for massive case. However, the change of the sharp peaks and the levels for potential barriers as the twisting parameter increases is weak compared with the massive case. Therefore, the influence of the twisting parameter on the massless case is not strong as in the massive one.

### 5 Conclusion

In this paper, we have considered the Dirac equation in a background that represents the coupling of S mass and the GBR spacetime, using NP null tetrad formalism. By employing an axially symmetric ansatz for the Dirac spinor, we managed to separate the equation into radial and angular parts. We were able to obtain an exact solution for the angular equations in terms of associated Legendre functions. We studied the radial equations and obtained the radial wave equations with effective potentials. Finally, we studied the behaviour of the potentials by varying the twisting parameter and the frequency, by plotting two and three dimensional plots. We showed that, as the twisting parameter increases, the potentials barriers decrease. However, for low frequencies sharp peaks of potentials are more observed than high frequencies. In the case of massless Dirac particle, it is seen that the effect of twisting parameter is weaker than the massive one. Our main motivation was to give an analytical expression of the solution which could be useful for further study of the thermodynamical properties of the spinor field in SGBR background. For future work, the study of the charged Dirac particles in this spacetime may reveal more information compared to the present case.

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