On Planarity of Graphs in Homotopy Type Theory

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In this paper, we present a constructive and proof-relevant development of graph theory, including the notion of maps, their faces, and maps of graphs embedded in the sphere, in homotopy type theory. This allows us to provide an elementary characterisation of planarity for locally directed finite and connected multigraphs that takes inspiration from topological graph theory, particularly from combinatorial embeddings of graphs into surfaces. A graph is planar if it has a map and an outer face with which any walk in the embedded graph is walk-homotopic to another. A result is that this type of planar maps forms a homotopy set for a graph. As a way to construct examples of planar graphs inductively, extensions of planar maps are introduced. We formalise the essential parts of this work in the proof assistant Agda with support for homotopy type theory.

CCS Concepts: • Theory of computation → Constructive mathematics; Type theory; • Mathematics of computing → Graphs and surfaces.

Additional Key Words and Phrases: planarity, combinatorial maps, univalent mathematics, formalisation of mathematics

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1 INTRODUCTION

Topological graph theory studies the embedding of graphs into surfaces [4, 25, 37] such as: the plane, the sphere, the torus, etc. Even the simplest case, embedding graphs into the plane, has inspired a lot of interesting characterisations and mathematical results. Two such characterisations are Kuratowski’s theorem and the closely related Wagner’s theorem [15, 36]. Both theorems characterise planarity by excluding two sub graphs known as the forbidden minors, namely $K_5$ and $K_{3,3}$. Other approaches refer to algebraic methods as MacLane’s Theorem [31] and Schnyder’s theorem [12, §3.3].

One of the most powerful tools in topological graph theory is the combinatorial representation of graph embeddings, called rotation systems [25]. These representations encode what the embedding looks like around each node – characterising the embedding up to isotopy. Section 4 will give a more detailed description of rotation systems. For now, it suffices to know that for suitably general class of embeddings into closed surfaces – namely, the cellular ones – the embedding is characterised by the cyclic order of outgoing edges from each node as they lie around the node on the surface.

Homotopy type theory (HoTT)[39] is a variation of dependent type theory which emphasises the higher dimensional structure of types: Equalities within a type are seen as paths, and the type of all equalities between two elements – the identity type – is thought of as a path space. In this way, HoTT takes seriously the notion of proof-relevancy, and interesting questions arise when considering what the equality between two proofs are.

The goal of this paper is to develop a proof-relevant notion of planar graphs in homotopy type theory, based on rotation systems. In other words, planarity is a structure on a graph, not a mere...
Fig. 1. It is shown a graph along with three different planar embeddings, namely $e_1$, $e_2$, and $e_3$. We have shaded with different colours the three faces in each embedding.

property. Intuitively, a proof that a graph is planar is an embedding into the plane. The question is then, when are two such embeddings equal? One good answer is that the proofs ought to be equal when the embeddings are isotopic – i.e. can be deformed continuous to one another without crossing edges. This will be the case for the notion presented here. But in order to arrive at a type of graph embeddings where the identity type corresponds to isotopy, a lot of care has to be taken when defining embeddings, and planarity.

In short, a planar graph will be a graph with a combinatorial embedding into the sphere and a fixed face where to puncture, as in Figure 1. The intuition is that an embedding into the plane can be obtained from an embedding into the sphere by puncturing the sphere at a point symbolising infinity (in any direction) on the plane. Up to isotopy, the important data when choosing a point of puncture is which face the points lies in.

Our development here differs from other related works in the subject (see Section 6), essentially by adopting the Voevodsky’s Univalence axiom (UA) present in HoTT. One consequence is that the graphs maintain the structural identity principle, i.e. isomorphic graphs are equal and share the same structures and properties. Such a correspondence turns to be crucial for formalising mathematics, as in standard mathematical practice. For example, the identity type of a graph helps us to understand its symmetries. Any automorphism of a graph gives rise to an inhabitant of its identity type and vice versa. One can then describe the group structure of the set of automorphisms for a graph by studying its identity type [26, §11], see Section 3.6. We foresee an exciting opportunity to combine ideas and prove results in combinatorics, graph theory, and homotopy type theory.

Outline. Section 2 introduces the basic terminology and notation used throughout the paper. In Section 3, the category of graphs is described, along with a few relevant examples. In Section 4, we present different types for graph-theoretic concepts, including the type of maps, faces, and spherical maps, which allows us to define in HoTT the notion of planar maps, and consequently, planar graphs in Section 5. Additionally, to construct larger planar graphs, the planarity for cyclic graphs and graph extensions are proved. Section 6 addresses the connection of this work with other developments. A few concluding remarks and future work are discussed in Section 7.

Formalisation. One exciting feature to work with systems such as HoTT is producing machine-verified proofs [29]. To check the correctness of this work, we use the proof assistant Agda [38], a computer system with support for dependent type theories capable of managing the same abstraction level with which we reason our mathematical theories on paper. Formal machine-verified proofs can offer a window to new proofs and theorems [5]. Moreover, they also serve to find flaws and corner cases that human reasoning might not see. We must therefore, pay special
attention to definitions and theorems, as they are the primary input to these systems. The computer’s formalisation process is an exciting and challenging activity, full of details and technical issues [3, 23]. We use Agda v2.6.2 to type-check the formalisation [34] of the essential parts of this paper. To be compatible with HoTT, we use the flag without-K [13] and the flag exact-split to ensure all clauses in a definition are definitional equalities.

2 MATHEMATICAL FOUNDATION

In this paper, we work with homotopy type theory [24, 39]: a Martin-Löf intensional intuitionistic type theory extended with the Univalence Axiom [7, 21], proposed originally by Voevodsky [40], and some higher inductive types (HITs), such as propositional truncation. The presentation of our constructions is informal in the style of the HoTT book [39]. However, the essential constructions have been verified in the proof assistant Agda [34].

HoTT emphasises the rôle of the identity type as a path-type [6]. The intended interpretation is that elements, \( a, a' : A \), are points and that a witness of an equality \( p : a = a' \) is a path from \( a \) to \( a' \) in \( A \). Since the identity type again is a type, we can iterate the process, which gives each type the structure of an \( \infty \)-groupoid.

This may at first seem of little relevant when working with finite combinatorics, as one would expect only types with trivial path-types (sets) to show up in combinatorics. But we will see that types with nontrivial path types do indeed arise naturally in combinatorics – which is not surprise for someone familiar with the role of groups and groupoids in this field, such as Joyal’s work on combinatorial species [8, 43]– and that the paths in these types are often various forms of permutations.

2.1 Notation

- Definitions are introduced by (≡) while judgmental equalities use (=).
- The type \( \mathcal{U} \) is an univalent universe. The notation \( A : \mathcal{U} \) indicates that \( A \) is a type. To state that \( a \) is of type \( A \) we write \( a : A \).
- The equality sign of the identity type of \( A \) is denoted by (\( =_A \)). If the type \( A \) can be inferred from the context, we simply write (\( = \)). The equalities between \( x, y : A \) are of type \( x =_A y \).
- The type of non-dependent function between \( A \) and \( B \) is denoted by \( A \to B \).
- Type equivalences are denoted by (\( \simeq \)). The canonical map for types is \( \text{idtoequiv} : A = B \to A \simeq B \) and its inverse function is called \( \text{ua} \). Given \( e : A \simeq B \), the underlying function of type \( A \to B \) is denoted by \( e \) while \( \text{ua}(e) \) is denoted by \( \overline{e} \). The coercion along \( p : A = B \) is the function \( \text{coe} \) of type \( A \to B \).
- The point-wise equality for functions (also known as homotopy) is denoted by (\( \sim \)). The function happily is of type \( f = g \to f \sim g \) and its inverse function is called funext.
- The coproduct of two types \( A \) and \( B \) is denoted by \( A + B \). The corresponding data constructors are the functions \( \text{inl} : A \to A + B \) and \( \text{inr} : B \to A + B \).
- Dependent sum type (\( \Sigma \)-type) is denoted by \( \Sigma_{x:A}B(x) \) while dependent product type (\( \Pi \)-type) is denoted by \( \Pi_{x:A}B(x) \).
- The empty type and unit type are denoted by \( \emptyset \) and \( 1 \), respectively.
- The type \( x \neq y \) denotes the function type \( (x = y) \to \emptyset \).
- Natural numbers are of type \( \mathbb{N} \). The variable \( n \) is of type \( \mathbb{N} \), unless stated otherwise.
- The type with \( n \) points is denoted by \( [n] \).
- The universe \( \mathcal{U} \) is closed under the type formers considered above.
- The function transport/substitution is denoted by \( \text{tr} \). We denote by \( \text{tr}_2 \) the function of type \( \prod_{p:a_1 = a_2} (\text{tr}^B(p,b_1) = b_2) \to C(a_1, b_1) \to C(a_2, b_2) \), where \( A : \mathcal{U}, B : A \to \mathcal{U}, a_1, a_2 : A, b_1 : Ba_1, b_2 : Ba_2 \), and \( C : \prod_{x:A} Bx \to \mathcal{U} \).
For the sake of readability in the upcoming section, we will use variables $A, B$ and $X$ to denote types, unless stated otherwise.

### 2.2 Homotopy Levels

The following establishes a level hierarchy for types with respect to the nontrivial homotopy structure of the identity type. The first four homotopy levels are of special interest for this work. These four levels are enough for expressing the mathematical objects we want to construct.

**Definition 2.1.** Let $n$ be an integer such that $n \geq -2$. One states that a type $A$ is an $n$-type and that it has homotopy level $n$ if the proposition $\text{is-level}(n, A)$ holds.

\[
\text{is-level}(-2, A) \triangleq \sum_{(c: A)} \prod_{(x:A)} (c = x),
\]

\[
\text{is-level}(n + 1, A) \triangleq \prod_{(x,y:A)} \text{is-level}(n, x = y).
\]

For convenience of the presentation, one states that a type of homotopy level $(-2)$ is a contractible type, a level $(-1)$ is a proposition, a level $(0)$ is a set, and finally, the level $(1)$ is a 1-groupoid. We also define $\text{isProp}(A) \triangleq \text{is-level}(-1, A)$ and $\text{isSet}(A) \triangleq \text{is-level}(0, A)$. Types that are propositions are of type $\text{hProp}$ and similarly with the other levels. Additionally, it is possible to have an $n$-type out of any type $A$ for $n \geq 2$. This can be done using the construction of a higher inductive type called $n$-truncation $[39, \S 7.3]$ denoted by $\|A\|_n$. The case for $(−1)$-truncation is called propositional truncation (or reflection), and is often simply denoted by $\|A\|$.

**Definition 2.2.** Propositional truncation of a type $A$ denoted by $\|A\|_{-1}$ is the universal solution to the problem of mapping $A$ to a proposition $P$. The elimination principle of this construction gives rise a map of type $\|A\| \to P$ which requires a map $f : A \to P$ and a proof that $P$ is a proposition.

Propositional truncation allows us to model the mere existence of inhabitants of type $A$. We state that $x$ is merely equal to $y$ when $\|x = y\|$ for $x, y : A$. Then, we can express in HoTT by means of propositional truncation: logical conjunction$^1$, the disjunction$^2$, and existential$^3$ quantification.

**Definition 2.3.** Given $x : A$, the connected component of $x$ in $A$ is the collection of all $y : A$ that are merely equal to $x$, i.e. $\Sigma_{y:A}\|y = x\|$. If $\|x = y\|$, one says that $x$ is connected to $y$.

**Definition 2.4.** The type $A$ is called connected if $\|A\|$ holds and also, if all $x : A$ belong to the same connected component.

**Lemma 2.5.** Terms in the same connected component share the same propositional properties. If $P : A \to \text{hProp}$ and $x, y : A$ are connected in $A$ then one gets the equivalence $P(x) \simeq P(y)$.

### 2.3 Finite Types

The finiteness of a type $A$ is the existence of a bijection between $A$ and the type $[n]$ for some $n : \mathbb{N}$. However, this description is not a structure on $A$, providing it with a specific equivalence $A \simeq [n]$, but rather a property — a mere proposition. This ensures that the identity type on the total type of finite types is free to permute the elements, without having to respect a chosen equivalence.

**Definition 2.6.** Given $X : \mathcal{U}$, let $\text{isFinite}(X) : \mathcal{U}$ be given by

\[
\text{isFinite}(X) \triangleq \sum_{(n:\mathbb{N})} \|X \simeq [n]\|.
\]

$^1(P \lor Q) \triangleq \|P \lor Q\|,$

$^2(P \land Q) \triangleq \|P \land Q\|,$

$^3(\exists(x : A)P(x)) \triangleq \|\Sigma_{x:A}P(x)\|.$
Lemma 2.7. The type isFinite($X$) is a proposition.

Proof. Let $(n, p), (m, q) : \text{isFinite}(X)$, which we want to prove equal. Since $p$ and $q$ are elements of a family of propositions, it is sufficient to show that $n = m$. This equation is a proposition, so we can apply the truncation-elimination principle to get $X \simeq [n]$ and $X \simeq [m]$. Thus, from $[n] \simeq [m]$ follows that $n = m$ — by a well-known result on finite sets.

A type $X$ is finite if isFinite($X$) holds. The natural number $n$ is referred as the cardinal number of $X$. Now, since isFinite($X$) is a proposition, the total type of finite types, $\bigoplus_{X : \mathcal{U}} \text{isFinite}(X)$, has permutations as its identity type. This shows that Definition 2.6 is equivalent to the type $\exists_{n : \mathbb{N}} (X \simeq [n])$. However, the former definition allows us to easily obtain $n$ by projecting on the first coordinate. We find this distinction more practical than the latter for certain proofs, as in Lemma 2.11. Furthermore, one can show that any property on $[n]$, for example, "being a set" and "having decidable equality" can be transported to any finite type.

2.4 Cyclic Types

The notion of being a cycle for a type is not a property but a structure. The description of such a structure can be addressed in several ways. For example, one can state that a type is cyclic if it has a cycle order, i.e. there exists a (ternary) relation on the type for which the axioms of a cyclic ordering hold true [27]. In our approach, Definition 2.8, we establish that being cyclic for a type is a structure given by preserving the structure of cyclic subgroups of permutations on $[n]$.

As with natural numbers, we can define counterpart functions for the successor and the predecessor in $[n]$ if $n \geq 1$. The predecessor function pred, of type $[n] \to [n]$, can be defined as the mapping: $0 \mapsto (n - 1)$ and $(n + 1) \mapsto n$. The corresponding succ function is the inverse of pred, and they are therefore both equivalences. The pred function generates a cyclic subgroup (of order $n$) of the group of permutations on $[n]$. An equivalent cyclic subgroup can be defined by means of the succ function; however, pred is more straightforward to define than succ. Now, we want to mirror the structure of $[n]$ given by pred for any finite type $A$ along with an endomap $\varphi : A \to A$. This can be done by establishing a structure-preserving map between $(A, \varphi)$ and $([n], \text{pred})$ in the category of endomaps of sets. This is the idea behind Definition 2.8 and the fact that the condition structure-preserving can be attained.

Definition 2.8. The type of cyclic structures on a type $A$ is given by Cyclic($A$).

$$\text{Cyclic}(A) \equiv \sum_{(\varphi : A \to A)} \sum_{(n : \mathbb{N})} || \sum_{(e : A^{\simeq [n]})} e \circ \varphi = \text{pred} \circ e ||.$$  

A cyclic structure on a type $A$ is denoted by a triple $(A, f, n)$ where $(f, n, -) : \text{Cyclic}(A)$. Given such a triple, we can refer to $A$ as an $n$-cyclic type. By Lemmas 2.9 and 2.10, one can prove that the $n$-cyclic type $A$ is not only a finite set, but the function $f$ is a bijection. To avoid any confusions, we denote $f$ by $\text{pred}_A$ and the inverse by $\text{suc}_A$. One may also drop the previous sub-indices. To define such bijections as products of cycles, we use the same notation from group theory. For example, the permutation denoted by $(a)(bc)$ permutes $b$ and $c$ and fixes $a$.

Lemma 2.9. Let $P$ be a family of propositions of type $\prod_{X : \mathcal{U}} (X \to X) \to h\text{Prop}$ and an $n$-cyclic structure $(A, f, n)$. If $P([n], \text{pred})$, then $P(A, f)$.

Proof. It follows from Lemma 2.5. Note that being cyclic for a type is equivalent to saying $(A, f)$ and $([n], \text{pred})$ are connected in $(X : \mathcal{U}) \times (X \to X)$.
Lemma 2.10. Let \( P \) be a family of propositions of type \( \mathcal{U} \to \text{hProp} \) and an \( n \)-cyclic structure \( \langle A, f, n \rangle \). If \( P([n]) \), then \( P(A) \).

In any finite type, every element is searchable. In particular, given an \( n \)-cyclic type \( \langle A, f, n \rangle \), one can search any element by iterating at most \( n \) times, the function \( f \) on any other element.

Lemma 2.11. If \( A \) is an \( n \)-cyclic type, then there exists a unique number \( k \) with \( k < n \) such that \( \text{pred}_A^k(a) = b \) for all \( a, b : A \).

The total type, \( \sum_{A : \mathcal{U}} \text{Cyclic}(A) \), is the classifying type \([30, \S 4.6-7]\) of finite cyclic groups. In the remaining of this section we compute the identity type between two finite cyclic types that we use, for example, in Example 4.13 to enumerate the maps of the bouquet graph \( B_2 \).

Lemma 2.12. The type \( \text{Cyclic}(A) \) is a set.

Lemma 2.13. Given \( A \) and \( B \) defined by \( \langle A, f, n \rangle \) and \( \langle B, g, m \rangle \),

\[
\begin{align*}
(A = B) & \equiv \sum_{(a : A = B)} (\text{coe} (\alpha) \circ f = g \circ \text{coe} (\alpha)). \\
\end{align*}
\]

Proof. We show the equivalence by Calculation (2.1). In Equivalence (2.1a), we expand the cycle type definition for \( A \) and \( B \). Equivalence (2.1b) follows from the characterisation of the identity type between pairs in a \( \Sigma \)-type \([39, \S 3.7]\). Note that in Equivalence (2.1b), the type \( (n = m) \times (p = q) \) is a contractible type, i.e. equivalent to \( \mathbb{1} \), the one-point type. We can then simplify the inner \( \Sigma \)-type to its base in Equivalence (2.1c) to obtain by the equivalence \( \Sigma_{\mathbb{1} \times A} \mathbb{1} \simeq A \), Equivalence (2.1d). Finally, Equivalence (2.1e) is a consequence of transporting functions along the equality \( \alpha \). The conclusion is that the identity type \( A = B \) is equivalent to the type of equalities \( \alpha : A = B \) along with a proof that the structure of \( f \) is preserved in the structure of \( g \).

\[
\begin{align*}
(A = B) & \equiv ((A, (f, n, p)) = (B, (g, m, q))) \\
& \simeq \sum_{(a : A = B)} (n = m) \times (p = q) \\
& \simeq \sum_{(a : A = B)} (\beta_{A \times X \to X} (a, f) = g) \\
& \simeq \sum_{(a : A = B)} (\text{tr}_{A \times X \to X} (\alpha, f) = g) \\
& \simeq \sum_{(a : A = B)} (\text{coe} (\alpha) \circ f = g \circ \text{coe} (\alpha)). \\
\end{align*}
\]

\( \square \)

3 FUNDAMENTAL NOTIONS OF GRAPH THEORY

While graphs are ubiquitous in science, exactly what is referred to by the word “graph” depends upon the context. In some settings, graphs are undirected while in others, graphs are directed. Sometimes self-edges are allowed, sometimes not. Which notion is chosen in a given context depends on the application, e.g. power graphs in computational biology, quivers in category theory, and networks in network theory.

3.1 The Type of Graphs

In this work, a graph is a directed multigraph (self-edges are allowed). The type of graphs is the set-level structure of the notion of abstract graphs.
Definition 3.1. A graph is an object of type Graph. The corresponding data is a set of points in a space (also known as nodes) and a set for each pair of points (also known as edges).

\[
\text{Graph} \equiv \sum_{(N: \mathcal{U})} \sum_{(E: N \to \mathcal{U})} \text{isSet}(N) \times \prod_{(x,y:N)} \text{isSet}(E(x,y)).
\]

The initial graph examples are the trivial ones: the empty graph and the unit graph defined, respectively as \(K_0 \equiv (\emptyset, \lambda u.v. \emptyset)\) and \(K_1 \equiv (\{1\}, \lambda u.v. \{1\})\).

Given a graph \(G\), for brevity, the set of nodes and the family of edges are denoted by \(N_G\) and \(E_G\), respectively. In this way, the graph \(G\) is defined as \((N_G, E_G, (s_G, t_G))\) : Graph where \(s_G : \text{isSet}(N_G)\) and \(t_G : \prod_{x,y:N_G} \text{isSet}(E_G(x,y))\). We may refer to \(G\) only as the pair \((N_G, E_G)\), unless we require to show the remaining data, the mere propositions \(s_G\) and \(t_G\). Additionally, we will use the variables \(G\) and \(H\) to be graphs, and variables \(x, y\) and \(z\) to be nodes in \(G\), unless stated otherwise.

Definition 3.2. A graph homomorphism from \(G\) to \(H\) is a pair of functions \((\alpha, \beta)\) such that \(\alpha : N_G \to N_H\) and \(\beta : \prod_{x,y:N_G} E_G(x,y) \to E_H(\alpha(x), \alpha(y))\). We denote by \(\text{Hom}(G, H)\) the type of these pairs.

We denote by \(\text{id}_G\), for any graph \(G\), the identity graph homomorphism where the corresponding \(\alpha\) and \(\beta\) are the identity functions.

Lemma 3.3. The type \(\text{Hom}(G, H)\) forms a set.

3.2 The Category of Graphs

Graphs as objects and graph homomorphisms as the corresponding arrows form a small pre-category. In fact, the type of graphs is a small univalent category in the sense of the HoTT Book [39, §9.1.1]. This fact follows from Theorem 3.6 and, morally, because the \text{Graph} type is a set-level structure.

In a (pre-) category, an isomorphism is a morphism which has an inverse. In the particular case of graphs, this can be formulated in terms of the underlying maps being equivalences.

Lemma 3.4. Let \(h\) be a graph homomorphism given by the pair-function \((\alpha, \beta)\). The claim \(h\) is an isomorphism, denoted by \(\text{isIso}(h)\), is a proposition equivalent to stating that the functions \(\alpha\) and \(\beta(x,y)\) for all \(x, y : N_G\), are both bijections.

The collection of all isomorphisms between \(G\) and \(H\) is denoted by \(G \equiv H\). If \(G \simeq H\), one says that \(G\) and \(H\) are isomorphic.

Lemma 3.5. The type \(G \equiv H\) forms a set.

We define a type to compare sameness in graphs in Lemma 3.4; the type of graph isomorphisms. In HoTT, the identity type (=) serves the same purpose, and one expects [14] the two notions coincide. In Theorem 3.6, we prove that they are in fact homotopy equivalent. The same correspondence for graphs also arises for many other structures [1, 2], for example, groups, and topological spaces.

Theorem 3.6 (Equivalence Principle). The canonical map \((G = H) \to (G \equiv H)\) is an equivalence for \(G, H : \text{Graph}\).

Proof. It is sufficient to show that \((G = H) \simeq (G \equiv H)\). Remember that being an equivalence for a function constitutes a proposition. The equivalence follows from Calculation (3.1). We consider the type families \(F_1(X) \equiv X \to X \to \mathcal{U}\) and \(F_2(X, R) \equiv \prod_{x,y:X} \text{isSet}(R(x,y))\) to shorten the presentation.

\[
(G = H) \equiv ((N_G, E_G, (s_G, t_G)) = (N_H, E_H, (s_H, t_H)))
\]
\[\sum \left( \alpha : N_{G} = N_{H} \right) \left( \beta : \text{tr}_{2}^{F_{2}}(\alpha, t_{G}) = t_{H} \right) \times (\text{tr}_{2}^{F_{2}}(\alpha, E_{G}) = E_{H}) \]

(3.1b)

\[\sum \left( \alpha : N_{G} = N_{H} \right) \left( \beta : \text{tr}_{2}^{F_{2}}(\alpha, E_{G}) = E_{H} \right) \]

(3.1c)

\[\sum \left( x, y : N_{G} \right) E_{G}(x, y) = E_{H}(\overline{\alpha}(x), \overline{\alpha}(y)) \]

(3.1e)

\[\sum \left( x, y : N_{G} \right) E_{G}(x, y) \cong E_{H}(\overline{\alpha}(x), \overline{\alpha}(y)) \]

(3.1f)

(3.1g)

We first unfold definitions in Equivalence (3.1a). Equivalence (3.1b) follows from the characterisation of the identity type between pairs in a \(\Sigma\)-type (Lemma 3.7 in HoTT book). Equivalence (3.1c) stems from the fact that being a set is a mere proposition and, thus, equations between proofs of such are contractible, similarly as in Lemma 2.13. To get Equivalence (3.1e), we apply function extensionality twice in the inner equality in Equivalence (3.1d). By the Univalence axiom, we replace in Equivalence (3.1f) equalities by equivalences. Finally, Equivalence (3.1g) follows from Lemma 3.4 completing the calculation from which the conclusion follows.

Lemma 3.7. The type of graphs is a 1-groupoid.

Proof. We want to show that the identity type \(G = H\) is a set for all \(G, H : \text{Graph}\). This follows since type equivalences preserve homotopy levels. The type \(G = H\) is equivalent to the set of isomorphisms, \(G \cong H\), by the equivalence principle, Theorem 3.6.

3.3 Graph Classes

Graphs can be collected in different classes. A class \(C\) of graphs is a collection of graphs that holds some given structure \(P : \text{Graph} \rightarrow \mathcal{U}\), i.e. a graph class is the total type of the corresponding predicate, \(C := \Sigma_{G : \text{Graph}} P(G)\). Examples of classes are simple graphs where the edge relation is propositional, or undirected graphs where the edge relation is symmetric. Definition 3.1 is the class of directed multigraphs, which leads us to more general statements about graphs and to write shorter proofs. It is more general, because one can see, for example that simple undirected graphs are instances of directed multigraphs.

Now, since any construction in HoTT respects the structure of its constituents, a graph class is invariant under graph isomorphisms. Specifically, given a graph isomorphism, we can transport any property on graphs along the equality obtained by Theorem 3.6. Graph properties provide a way to determine if negative statements attribute on certain graphs, for example, if two given graphs are not isomorphic.

Lemma 3.8 (Leibniz principle). Isomorphic graphs hold the same properties.

A related principle is equivalence induction [22, §3.15].

Lemma 3.9 (Equivalence induction). Given a graph \(G\) and a family of properties \(P\) of type \(\Sigma_{H : \text{Graph}} (G \cong H) \rightarrow \text{hProp}\), if the property \(P(G, \text{id}_{G})\) holds then the property also holds for any isomorphic graph \(H\) to \(G\), i.e. \(P(H, \varphi)\) holds for all \(\varphi : G \cong H\).
Lastly, of importance for this work are the class of connected finite graphs stated in Definition 3.10 and Section 3.3.2. We will assume any graph in the remaining of this paper, as connected and finite, unless stated otherwise.

3.3.1Finite graphs. A graph is finite if both the node set and each edge-set are finite sets. The corresponding graph property of finite graphs is stated in Definition 3.10. Similarly, as with finite types, a finite graph has associated a cardinal to the number of nodes and of the number of edges. Consequently, one can show that for finite graphs, the equality is decidable on the node set and on each edge-set.

Definition 3.10. A graph $G$ is finite if $\text{FiniteGraph}(G)$ holds.

$\text{FiniteGraph}(G) \equiv \text{isFinite}(N_G) \times \prod_{(x,y) \in N_G} \text{isFinite}(E_G(x, y))$.

3.3.2Connected Graphs. Among the many notions in graph theory, the concept of a walk plays a leading role for many computer algorithms. A walk in a graph is simply a sequence of edges that forms a chain of types stated in Definition 3.11. By considering the walks in a graph, one obtains an endofunctor $W$ in Graph by mapping a graph $G$ to a graph formed by the same node set in $G$, and the corresponding sets of walks in $G$ as the edge sets.

Definition 3.11. A walk between $x$ and $y$, is an element of the type $\text{Walk}(x, y)$, which is inductively generated by

- the trivial walk $\langle x \rangle$ when $x \equiv y$, and
- composite walks, $(e \odot w)$, where $e$ is an edge from $x$ to some $z$, and $w$ a walk from $z$ to $y$.

Lemma 3.12. The type of walks between $x$ to $y$ forms a set.

Definition 3.13. A graph $G$ is (strongly) connected when a walk merely exists in $G$ joining any pair of nodes. If $G$ is connected then the proposition $\text{Connected}(G)$ holds.

$\text{Connected}(G) \equiv \prod_{(x,y) \in N_G} \|E_W(G)(x,y)\|$.

Lemma 3.14. Being connected for a graph is a proposition.

3.4Graph Families

The path graph with $n$ nodes is the graph $P_n \equiv ([n], \lambda u v. N-\text{succ}(\text{toN}(u)) = \text{toN}(v))$. One alternative type of walks is the type of path graphs, where $n$ represents the length of the walk. In other words, a path in $G$ of length $n$ between nodes $a$ and $b$ is a graph homomorphism from $P_n$ to $G$ mapping 0 to $a$ and $n − 1$ to $b$. If $a$ and $b$ are equal, such a path is closed. A closed path induces a cycle graph of size of the same length of the path. The family of cycle graphs follows Definition 3.15.

Definition 3.15. An $n$-cycle graph denoted by $C_n$ is defined by $C_n \equiv ([n], \lambda u v. u = \text{pred}(v))$ if $n \geq 1$. Otherwise, $C_0$ is the one-point graph.

In the treatment of embeddings of graphs on surfaces, we find out that bouquet graphs, besides their simple structure, have nontrivial embeddings, see Section 4.5.
Fig. 2. The graph $K_{3,3}$. Each arrow in the picture represents a pair of edges, one in each direction.

**Definition 3.16.** The family of bouquet graphs $B_n$ consists of graphs obtained by considering a single point with $n$ self-loops.

$B_1$  $B_2$  $B_3$  $B_4$  $B_5$

**Definition 3.17.** A graph of $n$ nodes is called complete when every pair of distinct nodes is joined by an edge. In particular, we denote by $K_n$, the complete graph with node set $[n]$. For brevity, we use a double arrow in the pictures below to denote a pair of edges of opposite directions.

$K_1$  $K_2$  $K_3$  $K_4$  $K_5$

### 3.5 Other Structures on Graphs

#### 3.5.1 Cyclic Graphs.

Similarly, as for cyclic types, we introduce a class of graphs with a cyclic structure. A graph is cyclic when it is in the connected component of an $n$-cycle graph along with the corresponding automorphism, see Definition 2.3.

Now, let us consider the homomorphism $\text{rot} : \text{Hom}(C_n, C_n)$ that acts similarly as the function $\text{pred}$ in Definition 2.8. The cyclic structure for graphs can be defined as the property of preserving the structure in $C_n$ induced by the morphism $\text{rot}$. We will make use of the same notation as for cyclic sets to refer to cyclic graphs.

**Definition 3.18.** A graph $G$ is cyclic if it is of type $\text{CyclicGraph}(G)$.

\[
\text{CyclicGraph}(G) := \sum_{(\varphi : \text{Hom}(G,G))} \sum_{(n \in \mathbb{N})} \| (G, \varphi) = (C_n, \text{rot}) \| .
\]

#### 3.5.2 Graph Colourings.

An $n$-colouring of a graph $G$ is a homomorphism of type $\text{Hom}(G, K_n)$, where each node in $K_n$ represent a different colour for the nodes in $G$. Given an $n$-colouring of $G$, we say that $G$ is $n$-colourable or $n$-partite. Thus, a bipartite graph is a graph with a 2-colouring,
and a bipartite complete graph with six nodes is the graph $K_{3,3}$. The collection of all $n$-colourings of a graph forms a set by Lemma 3.3, and the collection of $n$-partite graphs forms a 1-groupoid. Since there are some $n$-partite graphs that are equal up to isomorphism, we have the following distinction. Two graph colourings of $G$, namely $f, g : \text{Hom}(G, K_n)$ are essentially different if a nontrivial isomorphism $\sigma : K_n \cong K_n$ exists and if the functions $f$ and $g \circ \sigma$ are equal. The type of essentially different colourings of a graph $G$ is Type (3.2).

$$\text{EssentiallyPartite}(n, G) := \sum_{(A : \text{Graph})} \text{Hom}(G, A) \times \|A \cong K_n\|.$$ (3.2)

**Example 3.19.** We compute the identity type of the essential different colourings of the path graph $P_3$ in Calculation (3.3). As we will see, there can only be two graph homomorphisms from $P_3$ to $K_2$, namely $\varphi_0$ and $\varphi_1$ as in Figure 3. Let $c_1$ and $c_2$ be of type $\text{EssentiallyPartite}(2, P_3)$.

$$(c_1 = c_2) \simeq \left((K_2, \varphi_0, !) = (K_2, \varphi_1, !)\right)$$ (3.3a)

$$\simeq \sum_{(\tau : K_2 = K_2)} (\text{tr}^{K_2, \text{Hom}(P_3, X)}(\tau, \varphi_0) = \varphi_1)$$ (3.3b)

$$\simeq \sum_{(\tau : K_2 = K_2)} (\tau \circ \varphi_0 = \varphi_1).$$ (3.3c)

In Equivalence (3.3b), the equality $\tau : K_2 = K_2$ is one of two alternatives: the trivial path or the path from the equivalence that swaps the only two nodes in $K_2$. Only the latter possibility, the equation $\tau \circ \varphi_0 = \varphi_1$ can hold.

![Fig. 3. Two graph homomorphisms $\varphi_0$ and $\varphi_1$ from $P_3$ to $K_2$. The dashed arrows represent how $\varphi_0$ and $\varphi_1$ map the nodes of $P_3$ into $K_2$. We represent the colours of the 2-coloring of $P_3$ by the nodes black and white in $K_2$.](image)

### 3.6 The Identity Type on Graphs

For any element, $x$ of a groupoid type, $X$, the type $\text{Aut}_X(x) := (x = x)$ has a group structure given by path composition. Applying this definition to the groupoid of graphs, the equivalence principle of Theorem 3.6 gives that for any graph $G$, we identify $\text{Aut}(G)$ with its automorphisms, $G \cong G$. This allows us to compute $\text{Aut}(G) = G \cong G$ in the examples which follow.

1. $\text{Aut}(B_2)$ is the group of two elements. With only two edges in $B_2$ and one node, we can only have, besides the identity function, the function that swaps the two edges. In general, the identity type $B_n = B_n$ is equivalent to the group $S_n$, the group which contains the permutations of $n$ elements.

2. $\text{Aut}(K_{3,3})$ is the subgroup $\mathbb{Z}_2 \times S_3 \times S_3$ in $S_6$, since the nodes of $K_{3,3}$ can be partitioned into two sets of three, which can be permuted independently. Additionally, the two partitions are interchangeable.

3. Any isomorphism in $\text{Aut}(C_n)$ is completely determined by how it acts on a fixed node in $C_n$. 


LEMMA 3.20. Given the function \( \text{rot} \) as in Definition 3.18 and \( k < n \), the function \( \text{rot}^k \) is a bijection between \([n]\) and \((C_n \cong C_n)\). Moreover, if \( k_1 < n \) for \( i = 1, 2 \) and \( \text{rot}^{k_1} = \text{rot}^{k_2} \) then \( k_1 = k_2 \).

4 GRAPH EMBEDDINGS

Graphs are commonly represented by their drawings on a surface like the plane. In topology, such a drawing — also called graph embedding — can be represented as an embedding of the topological realization of the abstract graph on some given surface [37]. Not all finite graphs can be drawn in the plane, but all finite graphs can be drawn on some orientable surface. If \( G \) denotes the abstract graph, then we denote by \(|G|\) the topological realization of \( G \).

Given a graph embedding in the surface \( S \), say \( f : |G| \hookrightarrow S \), one can consider the space \( S - f(G) \), the surface with the image of the graph removed. This space consists of a collection of connected components. Such a component is called a face if it is homeomorphic to an open disk. If all connected components are faces, one says that the graph embedding is cellular [25, §3.1.4].

Cellular embeddings are interesting because they can be characterised combinatorially — up to isotopy — by the cyclic order which they induce on the set of nodes around each node.

4.1 Locally Finite Graphs

A graph \( G \) is locally finite if the set of incident edges at every node \( x \), also called the star of \( x \) in \( G \), is a finite set. The valency of a star in a locally finite graph is the cardinal number of the corresponding set of edges. Now if we consider the graph \( U(G) \) as the symmetrisation of \( G \), with \( N_G \) as the node set, and edges between \( x \) and \( y \), of the type \( E_G(x, y) + E_G(y, x) \), then the star at \( x \) is Type (4.1).

\[
\text{Star}_G(x) := \sum_{(y \in N_G)} \text{Edge}_{U(G)}(x, y).
\]

LEMMA 4.1. The stars at each node forms a set.

PROOF. It follows since the base type of Type (4.1) is the set of edges in the graph, and each of the fibers of the \( \Sigma \)-type is a coproduct of sets, which we know they form sets [39, Exer. 3.2]. \( \square \)

It is proven that if the graph has at least one node of infinite valency, such a graph can not have a cellular embedding into any surface [32, Proposition §3.2]. In this work, we therefore only consider locally finite graphs, and we also assume that all graphs are connected, see Definition 3.13. One can prove that if the graph \( G \) is connected, then \( U(G) \) is connected.

4.2 The Type of Combinatorial Maps

Cellular embeddings are interesting because they can be characterised combinatorially — up to isotopy — by the cyclic order which they induce on the set of nodes around each node. We illustrate in Figure 4 (b) with a graph such a cyclic order. Since all embeddings in the plane are cellular, we will only work with this kind of embeddings. We forsake the topological definition of the embedding to work with their combinatorial characterisation, Definition 4.2.

Definition 4.2. A map for a graph \( G \), of type Map\((G)\), is a local rotation system at each node.

\[
\text{Map}(G) := \prod_{(x \in N_G)} \text{Cyclic} \left( \text{Star}_G(x) \right).
\]

LEMMA 4.3. The type of maps for a graph forms a set.

A graph with a map is a locally finite graph. Note that the stars of any map are finite sets, because being cyclic for a type implies that the type is a finite set.
Fig. 4. We show in (a) the drawing of a graph $G$ with edge crossings. A representation of the graph $G$ embedded in the sphere is shown in (b). The graph embedded $U(G)$ serves as the symmetrisation of the graph $G$. Recall an edge $e$ in $G$ from $x$ to $y$ induces an edge in $U(G)$ from $x$ to $y$ and an edge from $y$ to $x$. For brevity, we only draw a segment representing such a pair of related edges. The corresponding faces of the graph embedding shaded in (b) are named $F_i$ for $i$ from 1 to 6. It is shown in (c) with fuchsia colour the incident edges at the node $a$ in $U(G)$. The rotation system at $a$, i.e. the cyclic set denoted by $(baaxad)$, is shown in green colour. The dashed lines represent edges not visible to the view.

**Lemma 4.4.** The type of maps for a finite graph is a finite set.

For brevity, we will use from now the variable $M$ to denote a map of the graph $G$.

### 4.3 The Type of Faces

Combinatorially, a face consists of a cyclic walk in the embedded graph where there are no edges on the inside of the cycle, and no node occur twice. Definition 4.5 is our attempt to make this intuition formal. The criterion “no edges on the inside” for a face — called map-compatibility below — is captured by the fact that each pair of sequential edges on the face is a successor–predecessor pair in the cyclic order of the edges around their common node. Also, note that our graphs are directed, and therefore, the underlying graph of a face can consist of edges in any direction. Then, faces of a map of a graph $G$ are related to the graph $U(G)$.

**Definition 4.5.** A face of $M$, of type $\text{Face}(G, M)$, consists of:

1. a cyclic graph $A$, and
2. a graph homomorphism $h$ given by $(\alpha, \beta)$ of type $\text{Hom}(A, U(G))$, which is
   (a) edge-injective, as in Definition 4.6,
   (b) star-compatible, if the condition in Type (4.2) holds, and
   
   $\prod_{(x:G)} \|\text{Star}_G(\alpha x)\| \to \|\text{Star}_A(x)\|$, \hspace{1cm} (4.2)

   (c) corner-compatible, which is the evidence that $h$ is compatible with the edge-ordering given by the map $M$ at the node $\alpha(x)$ and the edge-ordering coming from the star at that node $x$ in $A$. The corresponding type is Type (4.3).

   $M(\alpha(x)) \preceq \langle (\alpha(\text{pred}(x)), \text{flip}(\beta(\text{pred}(x), x, a)) \rangle$

   $=\text{Star}_G(\alpha(x)) \cdot (\alpha(\text{suc}(x)), \beta(x, \text{suc}(x), a^+) \rangle$. \hspace{1cm} (4.3)

   The previous edge at $x$ is the edge $a : E_{N_A}(\text{pred}(x), x)$, and the edge after $a$ is the edge denoted by $a^+$ of type $E_{N_A}(x, \text{suc}(x))$, see Figure 5.
Fig. 5. On the right side, we shade the face $F_1$ of the graph $G$ embedded in the sphere given in Figure 4. We have the cycle graph $C_3$ and $h : \text{Hom}(C_3, U(G))$ given by $(\alpha, \beta)$ on the left side. $C_3$ and $h$ can be used to define the face $F_1$ using $C_3$ as the graph $A$ in Definition 4.5.

Given a face, we refer to ismapcomp(h)(x) to the witness conditions of Types (4.2) and (4.3). Figure 5 illustrates part of the required data to define a face for the map of $G$ given in Figure 4 (b).

**Definition 4.6.** A graph homomorphism $h$ from $G$ to $H$ given by $(\alpha, \beta)$ is edge-injective, denoted by isedgeinj(h), if the function $f$ defined below is an embedding.

$$f : \sum_{(x,y : N_G)} E_G(x, y) \rightarrow \sum_{(x,y : N_H)} E_H(x, y)$$

$$f(x, y, e) : = (\alpha(x), \alpha(y), \beta(x, y, e)).$$

**Lemma 4.7.** For a graph homomorphism, (1) being edge-injective is a proposition and (2) being map-compatible is a proposition.

We devote the rest of this section to proving that the type of faces forms a set in Lemma 4.9. This claim rests on the fact that (i) the type of cyclic graphs forms a set, (ii) the type of graph homomorphisms forms a set, and (iii) the conditions, edge-injective and map-compatible in Definition 4.5 are mere proposition. One might suspect this type forms a homotopy 1-groupoid from the previous facts. However, the edge-injectivity property of the underlying graph homomorphism of each face suffices to show that the type of faces is a set.

**Lemma 4.8.** Let $f$ and $g$ be edge-injective graph homomorphisms from $C_n$ to a graph $G$ and $n > 0$. Then the type $\Sigma_{e : C_n = C_n} (\text{tr}^{\lambda X. \text{Hom}(X,G)}(e, f) = g)$ is a mere proposition.

**Proof.** The result follows from proving that the $\Sigma$-type in question is equivalent to a proposition. The corresponding equivalence is given by Calculation (4.4), in which we use some known results about Univalence and Lemma 3.20, as in the very last step.

$$\sum_{e : C_n = C_n} (\text{tr}^{\lambda X. \text{Hom}(X,G)}(e, f) = g) \simeq \sum_{e : C_n = C_n} (f = g \circ \text{coe} (e))$$

$$\simeq \sum_{e : C_n = C_n} (f = g \circ e)$$

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It remains to show that the last equivalent type is a proposition. Let $(k_1, p_1)$, and $(k_2, p_2)$ be of type $\Sigma_{k_1}[n](f = g \circ \text{rot}^k)$. We must show that $(k_1, p_1) = (k_2, p_2)$. Since $\text{Hom}(C_n, G)$ is a set, we only need to prove that $k_1$ is equal to $k_2$. To show that, Lemma 3.20 is used in the proof. Let us consider the equality $(p_1^{-1} \cdot p_2)$ of type $(g \circ \text{rot}^{k_1} = g \circ \text{rot}^{k_2})$. Then, by computing the identity type of graph isomorphisms, one can obtain $(p_1^{-1} \cdot p_2)$ is equivalent to having two equalities, namely $p : \alpha(g \circ \text{rot}^{k_1}) = \alpha(g \circ \text{rot}^{k_2})$ and $q$ of type

$$\text{tr}^{\lambda e. \Pi(x,y,N_{C_n})E_{C_n}(x,y) \rightarrow E_G(e(x),e(y))}(p, g \circ \text{rot}^{k_1}) = \beta(g \circ \text{rot}^{k_2}).$$

By the characterisation of $\Sigma$-types and with the previous equalities, $p$ and $q$, one can get another equality $r$ of Type (4.5) for $x, y : N_{C_n}$ and $e : E_{C_n}(x, y)$.

$$\begin{align*}
(\alpha(g \circ \text{rot}^{k_1})(x), & \quad \alpha(g \circ \text{rot}^{k_1})(y), \quad \beta(g \circ \text{rot}^{k_1})(x, y, e)) \\
= (\alpha(g)(\alpha(\text{rot}^{k_1})(x)), & \quad \alpha(g)(\alpha(\text{rot}^{k_1})(y)), \quad \beta(g)(\beta(\text{rot}^{k_1})(x, y, e))).
\end{align*}$$

(4.5)

Now since the graph homomorphism $g$ is edge-injective, applying Definition 4.6 to the equality $r$, one gets an equality $r'$ of Type (4.6). By applying Lemma 3.20 to $r'$, we conclude that $k_1$ is equal to $k_2$ from where the required conclusion follows.

$$\begin{align*}
(\alpha(\text{rot}^{k_1})(x), & \quad \alpha(\text{rot}^{k_1})(y), \quad (\beta(\text{rot}^{k_1})(x, y, e))) \\
= (\alpha(\text{rot}^{k_2})(x), & \quad \alpha(\text{rot}^{k_2})(y), \quad (\beta(\text{rot}^{k_2})(x, y, e))).
\end{align*}$$

(4.6)

\textbf{LEMMA 4.9.} The faces of a map forms a set.

\textbf{Proof.} Let $F_1$ and $F_2$ be two faces of a map $\mathcal{M}$. We will show that the type $F_1 = F_2$ is a mere proposition in Calculation (4.7), with the following conventions.

- $\mathcal{A}$ is the cyclic graph related to the face $F_1$, $\mathcal{A} := (A, (\varphi_A, n, \text{iscyclic}(A, \varphi_A, n)))$,
- $\mathcal{B}$ is the cyclic graph related to the face $F_2$, $\mathcal{B} := (B, (\varphi_B, m, \text{iscyclic}(B, \varphi_B, m)))$.

We first unfold definitions of $F_1$ and $F_2$ in (4.7a), and simplifying propositions in Equivalence (4.7b), namely isedgeinj, ismapcomp, and iscyclic. Then, by expanding the definitions of $\mathcal{A}$ and $\mathcal{B}$ in (4.7c), and simplifying the propositions terms such as being a cyclic graph, one gets Equivalence (4.7d). Next, we reorder in Equivalence (4.7d) the tuple equalities to create an opportunity for path induction towards the application of Lemma 4.8. Now, since we want to prove that the type of faces is a set, and that itself is a proposition, the truncation elimination is applied to the propositions iscyclic($A, \varphi_A, n)$ and iscyclic($A, \varphi_A, n$). Then, the graphs $A$ and $B$ become respectively $C_n$ and $C_m$ in Equivalence (4.7e). Equivalence (4.7f) follows from the characterisation of the identity type between tuples in a nested $\Sigma$-type.

\begin{align*}
(F_1 = F_2) & \equiv ((\mathcal{A}, f, \text{isodeinj}(f), \text{ismapcomp}(f)) = (\mathcal{B}, g, \text{isodeinj}(g), \text{ismapcomp}(g))) \quad \text{(4.7a)} \\
& \approx ((\mathcal{A}, f) = (\mathcal{B}, g)) \quad \text{(4.7b)} \\
& \equiv (((A, (\varphi_A, n, \text{iscyclic}(A, \varphi_A, n))), f) = ((B, (\varphi_B, m, \text{iscyclic}(B, \varphi_B, m))), g)) \quad \text{(4.7c)} \\
& \approx (((A, (\varphi_A, n)), f) = ((B, (\varphi_B, m)), g)) \quad \text{(4.7d)} \\
& \approx ((n, ((C_n, f, \varphi_{C_n}) = (m, ((C_m, g, \varphi_{C_m}))) \approx \sum_{(p,n=m)} \sum_{(e':e=n)} \text{tr}_{\mathcal{F}.\text{Hom}}(X,X') \Pi(e, e'=g)). \quad \text{(4.7f)}
\end{align*}
It only remains to show that Equivalence (4.7f) is a mere proposition. We show this by proving that each type in Equivalence (4.7f) is a proposition. First, we unfold the cyclic graph definition for \( C_n \) and \( C_m \), using Definition 3.18. Secondly, a case analysis on \( n \) and \( m \) is performed. This approach creates four cases, where \( n \) and \( m \) can be zero or positive. However, we only keep the cases where \( n \) and \( m \) are structurally equal. One can show the other cases are impossible with an equality between \( n \) and \( m \).

1. If both, \( n \) and \( m \), are zero, then, by definition, \( C_n \) and \( C_m \) are the one-point graph. In this case, the conclusion easily follows. The base type \( n = m \) of the total space in Equivalence (4.7f) is a proposition because \( \mathbb{N} \) is a set. The type \( C_0 = C_0 \) is a proposition since it is contractible. The identity graph homomorphism is the unique automorphism of \( C_0 \). Lastly, because \( \text{Hom}(C_n, C_n) \) is a set, the remaining type of the \( \Sigma \)-type is a mere proposition, completing the proof obligations.

2. If \( n \) and \( m \) are positive, we reason similarly. The type \( n = m \) is a proposition. By path induction on \( p : n = m \), the second base type of the \( \Sigma \)-type becomes Type (4.8).

\[
\sum_{(cC_n = C_n)} (tr^{\lambda X. \text{Hom}(X, U(G))} (e, f) = g).
\]

Type (4.8) is a proposition by Lemma 4.8. The remaining type of the \( \Sigma \)-type is a mere proposition, because \( \text{Hom}(C_n, C_n) \) is a set. Therefore, the \( \Sigma \)-type in Equivalence (4.7f) is a proposition as required.

\[\square\]

4.4 Face Boundary

Each face \( F \) of a map \( M \) given by \( \langle A, h \rangle \) is bounded by a closed walk in \( U(G) \) induced by the non-empty cyclic graph \( A \) through \( h \). We refer to such a walk as the boundary of \( F \), and it is denoted by \( \partial F \). The degree of \( F \) is the length of \( \partial F \), which is the number of nodes in \( A \). The boundary \( \partial F \) can be walked in two directions with respect to the orientation given by its map.

As illustrated by Figure 6, given two different nodes \( x \) and \( y \) in \( \partial F \), we can connect \( x \) to \( y \) using the walk in the clockwise direction, \( \text{cw}_F(x, y) \). Similarly, one can connect \( x \) to \( y \) using the walk in the counter-clockwise direction, \( \text{ccw}_F(x, y) \). Such walks are induced by the walks in the cyclic graph \( A \), see Lemma 4.11.

**Lemma 4.10.** Supposing \( x, y : N_{C_n} \), the following claims hold for the cycle graph \( C_n \).

1. The type \( E_{C_n}(x, y) \) is a proposition.
2. There exists an edge of type \( E_{C_n}(\text{pred}(x), x) \) and an edge of type \( E_{C_n}(x, \text{succ}(x)) \).
3. There exists a walk going in the clockwise direction denoted by \( \text{cw}_{C_n}(x, y) \) from \( x \) to \( y \).

**Lemma 4.11.** Supposing \( x, y : N_{C_n} \), the following claims hold for the cyclic graph \( U(C_n) \).

1. If \( n > 2 \), then the type \( E_{U(C_n)}(x, y) \) is a proposition.
2. There exists an edge of type \( E_{C_n}(\text{pred}(x), x) \) and of type \( E_{C_n}(x, \text{succ}(x)) \).
3. There exist two quasi-simple\(^4\) walks from \( x \) to \( y \):
   a. There is a walk in the clockwise direction denoted by \( \text{cw}_{U(C_n)}(x, y) \).
   b. If \( x \neq y \), then the walk denoted by \( \text{ccw}_{U(C_n)}(x, y) \) is the walk in the counter-clockwise direction from \( x \) to \( y \). Otherwise, if \( x = y \), the walk \( \text{ccw}_{U(C_n)}(x, y) \) is the trivial walk \( \langle x \rangle \).

**Lemma 4.12.** Supposing \( x, y : N_{C_n} \), the following claims hold for the cyclic graph \( U(C_n) \).

1. If \( x \neq y \), then \( n \) is the length of each walk in Item (3) in Lemma 4.11.

\(^4\)In a quasi-simple walk no repetitions of nodes occur, except its head, see Definition 4.7 in [35].
Fig. 6. It is shown a face $F$ given by $⟨A, f⟩$ for the graph embedding $U(G)$ given in Figure 4. There are two quasi-simple walks in the underlying cyclic graph $A$ between two different nodes $x$ and $y$. Such walks are the clockwise and counter-clockwise closed walks in $U(G)$, denoted by $\text{cw}_F(x, y)$ and $\text{ccw}_F(x, y)$, respectively.

Fig. 7. The six possible maps of the bouquet $B_2$.

(2) Otherwise, the walk $\text{cw}_{U(C_n)}(x, x)$ is of a length $n$, and the walk $\text{ccw}_{U(C_n)}(x, x)$ is of length 0.

Additionally, one can prove that $n$ is the maximum length of a quasi-simple walk in the cyclic graph $U(C_n)$, as in Lemma 4.13 in [35]. Lastly, as illustrated in Figure 6 for the face $F$, the graph $U(C_n)$ is completely covered by the walks $\text{ccw}_{U(C_n)}(x, y)$ and $\text{cw}_{U(C_n)}(x, y)$.

4.5 Examples of Graph Maps

In this subsection, we examine the combinatorial maps for the bouquet graph $B_2$ and for the family of cycle graphs $C_n$ from Section 3.4. We also give a map for the graph $K_{3,3}$ mentioned in Section 3.5.2.

Let us introduce some notation for readability. An edge $a : E_G(x, y)$ induces the edge $a^{-}$ in $U(G)$ from $x$ to $y$. Similarly, the edge $a : E_G(y, x)$ induces the edge $a^{-}$ in $U(G)$ from $x$ to $y$. Lastly, a face of a map is given by the corresponding cyclic graph in Definition 4.5.

Example 4.13. In Figure 7, we show the six combinatorial maps for the bouquet graph $B_2$. It is important to note that combinatorial maps are equal up to isomorphism. The equality of rotation systems at each node is given by Lemma 2.13. Thus, the bouquet $B_2$ has only three distinct maps called $M_a$, $M_b$, and $M_c$. The pairs of equal maps are respectively $(a, d)$, $(b, e)$, and $(c, f)$ in Figure 7.

\[
M_a := (0 \mapsto (a^{-}a^{-}b^{-}b^{-})).
\]

\[
M_b := (0 \mapsto (a^{-}a^{-}b^{-}b^{-})).
\]

\[
M_c := (0 \mapsto (a^{-}b^{-}a^{-}b^{-})).
\]

The surface arising from the maps $M_a$ and $M_b$ is the two-dimensional plane. For the map $M_c$, the surface is the topological torus. We recall that the torus is a surface homeomorphic to the cartesian product of a circle with itself, as the embeddings $c$ and $f$ in Figure 7.
Example 4.14. For cycle graphs $C_n$, only one combinatorial map exists. To show this, we consider the equivalence given by the function $f_x$ for all $x : C_n$ in Equation (4.9). Then, any star in $C_n$ consists of two different edges.

$$f_x : \text{Star}_{C_n}(x) \to [2]$$

$$f_x(y, \text{inl}(p)) : \equiv 0,$$

$$f_x(y, \text{inr}(p)) : \equiv 1.$$  \hspace{1cm} (4.9)

If we consider the cyclic structures of the two-point type, $c_1 : \equiv ([2], \text{pred}, 2)$ and $c_2 : \equiv ([2], \text{succ}, 2)$, then they induce precisely the maps of $C_n$. In other words, one can obtain a map $M$ using $c_1$ by Equivalence (4.10) and $(\text{pred}, 2, ([\text{ideq}, \text{refl}_{\text{pred}}])) : \text{Cyclic}([2])$. By Lemma 2.13 implies that the map induced by $c_2$ and the map $M$ are equal, by function extensionality.

$$\text{Map}(C_n) \equiv \prod_{(x : [n])} \text{Cyclic}(|\text{Star}_{C_n}(x)|)$$

$$\simeq \prod_{(x : [n])} \text{Cyclic}([2]).$$  \hspace{1cm} (4.10)

Example 4.15. We below define for $K_{3,3}$ a map $M$ and its faces $F_1, F_2, F_3$. The corresponding surface to $M$ is that of the torus, as illustrated by its polygonal schema in Figure 8.

$$M : \equiv (0 \mapsto ((03) (04) (05)), 1 \mapsto ((13) (15) (14))),$$

$$2 \mapsto ((24) (25) (23)), 3 \mapsto ((32) (31) (30)),$$

$$4 \mapsto ((40) (41) (42)), 5 \mapsto ((51) (50) (52)).$$

$$F_1 : \equiv ((30) (04) (41) (13)).$$

$$F_2 : \equiv ((14) (42) (25) (51)).$$

$$F_3 : \equiv ((03) (32) (24) (40) (05) (52) (23) (31) (15) (50)).$$

5 PLANAR EMBEDDINGS

In this section, we examine the class of graphs with an embedding in the two-dimensional plane. Such embeddings are called planar embeddings or planar maps. A graph is planar if it has a planar embedding and the graph embedded is called a plane graph. To discuss the notion of planar embeddings, we take inspiration from topological graph theory [25, §3]. Then one can work with combinatorial maps that represent graph embeddings into a surface — up to isotopy. In the following, we focus on describing embeddings of graphs in the sphere called spherical maps. These maps are used later to establish the type of planar embeddings for a given graph.
5.1 Spherical Maps

Any graph embedding gives rise to an implicit surface. For planar embeddings, this surface is a space homeomorphic to the sphere. In particular, any embedding in the sphere induces an embedding in the plane. To see this, for a graph embedded in the sphere, one can puncture the sphere at some distinguished point, and subsequently, apply the stereographic projection to it.

The sphere in topology has two main invariants: path-connectedness and simply-connectedness. The former states that a path connects any pair of points in the sphere, and the latter states that any two paths with the same endpoints in the sphere can be deformed into one another.

If we now consider a walk as a path in the corresponding space induced by the map, then the path-connectedness property coincides with being connected for the graph embedded. However, if we want to address simply-connectedness for the surface induced by a graph-embedding, then we need to have an equivalent notion to saying how a pair of walks can be deformed into one the other. One proposal of such a notion is homotopy for walks in directed multigraphs [35].

5.1.1 Homotopy for Walks. Given a map $M$ for a graph $G$, we consider the relation $\sim_M$ on the set of walks defined in [35]. This relation is a congruence relation on the category of objects induced by the endofunctor $W$. The relation $\sim_M$ states in which way one can transform one walk into another considering adjacent faces. Then the relation strictly depends on the map $M$.

Definition 5.1. Let $w_1, w_2$ be two walks from $x$ to $y$ in $U(G)$. The expression $w_1 \sim_M w_2$ denotes that one can deform $w_1$ into $w_2$ along the faces of $M$. We acknowledge the evidence of this deformation as a walk-homotopy between $w_1$ and $w_2$, of type $w_1 \sim_M w_2$. The relation $\sim_M$ has four constructors as follows. The first three constructors are functions to indicate that homotopy for walks is an equivalence relation, i.e. $\text{hrefl}$, $\text{hsym}$, and $\text{htrans}$. The fourth constructor, illustrated in Figure 9, is the $\text{hcollapse}$ function that establishes the homotopy

$$(w_1 \cdot \text{ccw}_F(a,b) \cdot w_2) \sim_M (w_1 \cdot \text{cw}_F(a,b) \cdot w_2),$$

supposing one has the following,

(i) a face $F$ given by $(A,f)$ of the map $M$,
(ii) a walk $w_1$ of type $W_{U(G)}(x, f(a))$ for $x: N_G$ with one node $a: N_A$, and
(iii) a walk $w_2$ of type $W_{U(G)}(f(b), y)$ for $b: N_A$ and $y: N_G$.

![Fig. 9.](image)

Fig. 9. Given a face $F$ of a map $M$, we illustrate here $\text{hcollapse}$, one of the four constructors of the homotopy relation on walks in Definition 5.1. The arrow (⇓) represents a homotopy of walks.

One consequence of Definition 5.1 is that, in each face $F$, there is a walk-homotopy between $\text{ccw}_F(x, y)$ and $\text{cw}_F(x, y)$ using the constructor $\text{hcollapse}$.

5.1.2 The Type of Spherical Maps. As a property of maps, we can now state under which conditions the surface arising from a map is simply-connected. We call such maps as spherical maps.
Definition 5.2. A map $M$ of a graph $G$ is spherical, of type $\text{Spherical}(M)$, if any pair of walks sharing the same endpoints are merely walk-homotopic.

$\text{Spherical}(M) \equiv \biglor_{(x,y) \in \text{N}(G)} \left\| w_1 \sim_M w_2 \right\|.$

Lemma 5.3. Being spherical for a map is a proposition.

Lemma 5.4. The collection of all spherical maps for a (finite) graph is a (finite) set.

5.2 The Type of Planar Maps

Definition 5.5. A planar map $M$ of a connected and locally finite graph $G$ is of type $\text{Planar}(G)$.

$\text{Planar}(G) \equiv \sum_{(M: \text{Map}(G))} \text{Spherical}(M) \times \text{Face}(G, M) | \{ \text{outer face} \}.$

Theorem 5.6. The type of all planar maps of a graph forms a set.

Proof. The type of planar embeddings in Definition 5.5 is not a proposition. It encompasses two sets: the set of combinatorial maps, see Lemma 4.3, and the set of faces, see Lemma 4.9. Since being spherical for a map is a mere proposition, one concludes that the $\Sigma$-type collecting all planar maps of a graph forms a set. □

Example 5.7. Let us prove that there exists a planar map for $C_n$ with $n > 0$. Consequently, there exists a planar map for every cyclic graph. Beside their simple structure, cyclic graphs are building blocks in a few relevant constructions in formal systems related to the study of planarity of graphs, as planar triangulations using $C_3$, or a characterisation of all 2-connected planar graphs.

The graph $C_n$ is connected and locally finite, which mostly follows from Lemma 4.10. We must then show that $C_n$ has at least one spherical embedding and one outer face. As described in Example 4.14, there is only one such map that we denote here by $M$. This map gives rise to two faces, $F_1$ and $F_2$, the inner face and the outer face, respectively. As the cycle graph $C_n$ is a finite graph, it is only required to consider the finite set of quasi-simple walks to show that $M$ is spherical, see Lemma 5.8 in [35]. The set of such walks is precisely given in Lemma 4.11. We must now show that any pair of such walks are homotopic, from where one can conclude that the map $M$ is spherical, and consequently planar with outer face $F_2$.

(1) If $n = 1$, the only walk to consider is the trivial walk, which is homotopic to itself.
(2) If $n > 1$ and $x \neq y$, then we only need to consider the quasi-simple walks $\text{ccw}_{U(C_n)}(x, y)$ and $\text{cw}_{U(C_n)}(x, y)$. Such walks are homotopic by $\text{hcollapse}(F_1, x, y, x, y, \langle x \rangle, \langle y \rangle)$.
(3) Otherwise, if $n > 1$ and $x = y$, the only walks to consider are the trivial walk at $x$ and $\text{cw}_{U(C_n)}(x, x)$. Remember that the $\text{ccw}_{U(C_n)}(x, y)$ is by definition $\langle x \rangle$. Similarly, as in the previous case, these two walks are homotopic by the constructor $\text{hcollapse}$.}

5.3 Planar Extensions

In this subsection, we describe how to construct a planar map from another planar map. The characterisation of 2-connected graphs [41], ear decompositions [9, §5.3], reliable networks, and planar graph extensions for undirected graphs [26, §5.2,7.3] are related constructions. In the current section, $G$ is a locally connected finite graph with decidable equality on the set of nodes. For brevity, the variables $p$ and $p_i$ will represent finite path graphs of a positive length.
5.3.1 Path Additions. An internal node of a path is any node that is not an endpoint of the path. A simple path addition to $G$ is the graph formed by adding $p$ to $G$ between two existing nodes of $G$, such that the edges and internal nodes of $p$ are not in $G$. A simple cyclic addition is the addition of a cyclic graph to $G$ with exactly one node in common with $G$. A non-simple path addition is the path addition of the graph $U(q)$ to $G$ for a path graph $q$. Similarly, one can define non-simple cyclic additions. The simple and non-simple addition of $p$ to $G$ are denoted by $G \bullet p$ and $G \bullet \overline{p}$, respectively, and are referred as graph extensions. The operator $(\bullet)$ is regarded as a left-associative operator.

Hereinafter, the path $p$ in the addition $G \bullet p$ proceeds from $u$ to $v$ and its length is $n + 1$. The construction of $G \bullet p$ is equivalent to adding a path $p'$ with two distinguished edges $\hat{u}$ and $\hat{v}$, as illustrated in Figure 10b. If $n = 0$, the edges $\hat{u}$ and $\hat{v}$ are equal. Or else, we have one edge from $u : N_G$ to $0 : N_{p'}$, and another edge from $n - 1 : N_{p'}$ to $v$. The graph $G \bullet p$ is formed by the set of nodes $N_G + N_{p'}$, and the corresponding edges, i.e. the set of edges in $G$, $\hat{u}$, $\hat{v}$, and the set of edges in $p'$.

Fig. 10. The figure (a) shows the planar map for $G$ given in Figure 4 (b) with three different graph extensions: a path addition of $p$, a cyclic addition of $q$, and a spike addition of $r$. The additions of $p$ and $q$ replace/divide the faces $F_2$ and $F_3$, we have in Figure 4, by two new faces for each addition. The addition of $r$ replaces $F_4$ with a face of a greater degree. The figure (b) shows the path addition discussed in the proof of Lemma 5.9.

**Lemma 5.8.** If $G$ is connected, then $G \bullet p$ and $G \bullet \overline{p}$ are connected.

From now on, we will assume the existence of a planar map $M$ of $G$. We will only consider the addition of $p$ to $G$ in a fixed face $\mathcal{F}$ of $M$ if the endpoints of $p$, $u$ and $v$, belong to the boundary walk of $\mathcal{F}$. Under these considerations, one can prove that the addition of $p$ to $G$ has a planar map.

**Lemma 5.9.** There exists an extended planar map of $M$ for $G \bullet p$.

Given a planar map $M$ for $G$, we label $E(M, p)$ for the map given by Lemma 5.9. The extended map $E(M, p)$ is called the face division of the face $\mathcal{F}$ by $p$, assuming that $p$ is embedded in $\mathcal{F}$. To outline, the proof of Lemma 5.9 contains the following stages. First, one should define a map that extends $M$ for the nodes in $p$. Second, as illustrated in Figure 10, one should define two faces, both induced by collocating $p$ on $\mathcal{F}$. Finally, considering the new walks in $U(G \bullet p)$, given by walk-compositions with $p$, we can prove that $E(M, p)$ is planar.

**Proof of Lemma 5.9.** For brevity, let $H$ be the graph $G \bullet p$ as constructed above, and $p$ be the walk $\hat{u} \cdot p' \cdot \hat{v}$; a walk from $u$ to $v$. Let $\mathcal{F}$ be a face such that its boundary contains $u$ and $v$. We will define a specific map $M'$ for $H$ that extends the given planar map $M$ of $G$. In this way, one embeds $p$ in $\mathcal{F}$. By Definition 4.5, in $\partial \mathcal{F}$, one has the previous edge at $u$, denoted by $a : E_G(\text{pred}(u), u)$,
and an edge after $a$, denoted by $a^+ : E_G(u, \text{succ}(u))$. Similarly, for $v$, we have $b : E_G(\text{pred}(v), v)$ and $b^+ : E_G(v, \text{succ}(v))$, as illustrated by Figure 10.

If $x = u$ then $M'(x)$ is the cycle $M(x)$ with the insertion of $\hat{u} : E_H(u, 0)$ between $a$ and $a^+$, i.e. $M'(u)$ is $(\cdots a \hat{u} a^+ \cdots)$. Similarly, if $x = v$, the cycle $M'(v)$ is $(\cdots b \hat{b} b^+ \cdots)$. If $x$ is a node in $p'$, i.e. $x = i$ for $i$ from 0 to $n - 1$, then $M'(i)$ is $(e_i e_{i+1})$, where $e_i : E_{p'}(i, i + 1)$. Otherwise, $x$ is in $G$, and $M'(x)$ is $M(x)$.

The path $p$ splits $\mathcal{F}$ in two faces, $F_1$ and $F_2$. Let $\mathcal{F}$ be given by $(A, h)$ of degree $m$, as in Definition 4.5, and $k$ be the length of the walk $\text{cw}_F(u, v)$ from $u$ to $v$ in $\partial \mathcal{F}$. For brevity, let $n_1, n_2$ be $k + (n + 1)$ and $(m - k) + (n + 1)$, respectively. Let $F_1, F_2$ be the faces $(C_{n_1}, h_1)$ and $(C_{n_2}, h_2)$, where $h_1$ and $h_2$ are of type $\text{Hom}(C_{n_1}, U(H))$ for $i = 1, 2$. We will define $h_1$ and $h_2$ in a way that their boundary walks are $\text{cw}_F(u, v) \cdot U(p)$ and $\text{ccw}_F(u, v) \cdot U(p)$, respectively.

Let $h_1$ be $(\alpha_1, \beta_1)$. If $i < k$ for $i : N_{C_{n_1}}$, one puts the node $i$ in $\text{cw}_F(u, v)$, i.e. $\alpha_1(i)$ is $\alpha(i)$ and consequently, $\beta_1(i, i + 1, e)$ is $\beta(i, i + 1, e)$ for $e : E_{C_{n_1}}(i, i + 1)$. Otherwise, if $k \leq i \leq n_1$, then one puts the node $i$ in $U(p)$, i.e. $\alpha_1(i)$ is $n - i$, and consequently, $\beta_1(i, i + 1, e)$ is the edge $\text{inl}(e)$ in $U(H)$. It is clear that $h_1$, and similarly, $h_2$, is an edge-injective and map-compatible graph homomorphism with $M'$, since their properties are inherited from $h$ and $p$.

Let us now prove that $M'$ is spherical. Since this is a proposition by Lemma 5.3, we can use the elimination of the propositional truncation applied to the evidence that $M$ is spherical. This permits us to freely obtain a walk homotopy for any pair of walks sharing endpoints in $U(G)$. On the other hand, one must observe that a pair of homotopic walks in $U(G)$ that deforms along different faces than $\mathcal{F}$, remain homotopic in $U(H)$. Hence, we only need to consider for our goal, (i) the set of walks in $U(G)$ deforming along $\mathcal{F}$ and (ii) the set of walks created by compositions of $p$ with existing walks in $U(G)$.

If both walks belong to the former set, then we know that they are homotopic by vertical composition of the homotopies along $F_1$ and $F_2$ [35, §5].

![Fig. 11. The figure shows a part of the graph $U(G \bullet p)$ embedded in the sphere. As constructed in the proof of Lemma 5.9, the faces, $F_1$ and $F_2$, of the map $M'$ are given by a face division of $\mathcal{F}$ by the path $p$. Such gives rise to new walk homotopies, as $h_{F_1}$ and $h_{F_2}$ in the picture. The walk $U(p)$ from $u$ to $v$ is the walk composition of $p_1$, a walk from $u$ to $y$, and $p_2$, a walk from $y$ to $v$. The walks $\delta_1$ and $\delta_2$ are walks in $U(G)$ from $x$ to $z$.](image)

Otherwise, if both walks, $w_1$ and $w_2$ from $x$ to $z$, belong to the latter set, then we proceed by case analysis to show that $w_1 \sim_{M'} w_2$. For $w_1$ and $w_2$, it is only required to consider walks without inner loops, by Lemma 5.8 in [35]. In the following, the variables $p_1, p_2, \delta_1$, and $\delta_2$ denote walks, as in Figure 11.
• If $w_1$ is $\delta_1 \cdot U(p) \cdot \delta_2$, and $p$ is not a subwalk of $w_2$, then one can obtain the following walk homotopy.

$$w_1 \equiv \delta_1 \cdot U(p) \cdot \delta_2 \equiv \delta_1 \cdot \text{ccw}_{F_i}(u, v) \cdot \delta_2 \quad \text{(By the construction of $F_i$)}$$

$$\sim_M \delta_1 \cdot \text{cw}_{F_i}(u, v) \cdot \delta_2 \quad \text{(By the construction of $F_i$)}$$

$$\equiv \delta_1 \cdot \text{cw}_{F_i}(u, v) \cdot \delta_2 \quad \text{(By the construction of $F_i$)}$$

Therefore, $M$ is spherical with the outer face $F_i$.

- If $w_1$ is $\delta_1 \cdot p_1$ and $w_2$ is $\delta \cdot p_1$ for a walk $\delta$ from $x$ to $u$, then, by right whiskering of walk homotopies, one gets that $\delta_1 \cdot p_1 \sim_{M'} \delta \cdot p_1$. By assumption, $M$ is spherical, and then $\delta_1 \sim_{M'} \delta$, which implies by definition of $M'$ that $\delta_1 \sim_{M'} \delta$. Similarly, by left whiskering, one can also prove that if $w_1$ is $p_2 \cdot \delta_2$ and $w_2$ is $p_2 \cdot \delta$ where, $\delta$ is a walk from $v$ to $z$, then there is a walk homotopy such that $p_2 \cdot \delta_2 \sim_{M'} p_2 \cdot \delta$.

For the remainder cases of $w_1$ and $w_2$, one can similarly construct the required walk homotopy. Therefore, $M'$ is spherical, and is also a planar map of $H$ with the outer face $F_i$.

If $G$ is a finite graph with a map $M$, then the Euler’s characteristic of $G$ by $M$, denoted by $\chi_M$, is the number relating the cardinal of the set of nodes ($n$), edges ($e$), and faces ($f$).

$$\chi_M :\equiv n - e + f. \quad \text{(5.2)}$$

One can prove that if $M$ is a planar map then $\chi_M$ and $\chi_{E(M,G)}$ are equal. As described in the proof of Lemma 5.9 to construct $E(M, G)$, a path addition of $p$ of length $k + 1$ to $G$ increases $n$ by $k$, $e$ by $k + 1$, and $f$ by one. A major result is the characterisation of connected and finite planar graphs by Euler’s formula, which states that $\chi_M$ is two. Using the development in this section, one could show Euler’s formula for graph extensions, and for the class of biconnected graphs as described in Section 5.3.2. However, it is still unclear how to verify Euler’s formula, when it is not given the cardinal of the set of faces, i.e. elements of type Face($G, M$), see Definition 4.5.

$$\chi_{E(M,p)} :\equiv (n + k) - (e + k + 1) + (f + 1) = \chi_M.$$  

![Fig. 12.](image)

The figure is a planar synthesis of the construction of a planar map for $K_4$ from a planar map of $C_3$. One first divides the face $\mathcal{F}$ into $F_1$ and $F_2$. Then one splits $F_1$ into $F_3$ and $F_4$.

There are several methods to construct graphs inductively, as the construction of $K_4$ in Figure 12. Whitney-Robbins synthesis and an ear decomposition of a graph are some related methods. Inspired by these constructions and Lemma 5.9, we define the construction of larger planar graphs using graph extensions, in a way that we never leave the class of planar graphs.

**Definition 5.10.** A synthesis of a graph $G$ from a graph $H$ is a sequence of graphs $G_0, G_1, \ldots, G_n$ where $G_0$ is $H$, $G_n$ is $G$, and $G_i$ is the addition of $p_i$ to $G_{i-1}$ for $i$ from $1$ to $n$. If the sequence only contains simple additions, then it is called a simple synthesis. Else, if the sequence only contains
non-simple additions, the sequence is called a non-simple synthesis. If the graph $G$ is only obtained by a sequence of path additions, then the sequence is called a Whitney synthesis.

**Lemma 5.11.** In a synthesis from a connected graph, every graph in the sequence is connected.

**Definition 5.12.** Given a planar map $M$ of the graph $G$, a planar synthesis of $H$ from a graph $G$ is a sequence $(G_0, M_0), (G_1, M_1) \ldots, (G_n, M_n)$, where $n$ represents the length of the synthesis, $(G_0, M_0)$ is $(G, M)$, and $(G_n, M_n)$ is $(H, E(M_{n-1}, p_{n-1}))$. The graph $G_i$, for $i$ from 1 to $n$, is the graph $G \circ p_i$ and the map $M_i$ is $E(M_{i-1}, p_{i-1})$, for $i$ from 1 to $n$.

**Lemma 5.13.** In a planar synthesis, every graph in the sequence is planar. 

**Proof.** By induction on the synthesis length and Lemma 5.9. □

Lemma 5.9 can be further extended to consider non-simple additions, and consequently, one could extend Lemma 5.13 to define non-simple planar syntheses. Given a map $M$ for $G$, the corresponding planar map for $G \circ p$ is denoted by $E(M, \overline{p})$. Similarly, as with path additions, by extending the map by non-simple additions, new faces show up. Let $k + 1$ be the length of the path added to $G$. Then the map $E(M, \overline{p})$ induce $k + 2$ new faces, and one gets that $\chi_{E(M, \overline{p})}$ and $\chi_M$ are equal.

As illustrated by Figure 10, using spike additions, larger planar graphs can be constructed. A spike addition to $G$ is the addition of a path that only has one node in common with $G$. Given a map for $G$, a simple addition of a spike $p$ to $G$ induces a new face of a greater degree than the face where the spike is inserted. In contrast to simple additions, the number of faces of a map extended by non-simple spike additions vary as new faces arise between pairs of edges that share their endpoints.

**5.3.2 Biconnected Planar Graphs.** One can look at how much a graph is connected by examining its node-connectivity or edge-connectivity. Some graphs, yet after removing parts of them, preserve one or both connectivity measures. In this subsection, we want to characterise how to construct the class of 2-connected planar graphs. A graph is $k$-connected if it cannot be disconnected by removing less than $k$ nodes. There exist, depending on $k$, different ways to construct the class of $k$-connected graphs. For example, it is known that one can construct any undirected $(2)$-connected graph, if one applies path additions to a proper cyclic graph [15, §3].

**Definition 5.14.** A graph $G$ is 2-connected or biconnected if the graph formed by removing from $G$ a node $x$, denoted by $G - x$, is connected. If $G$ is 2-connected then the proposition Biconnected$(G)$ holds. Precisely, $G - x$ is the graph formed by the set of nodes, $\Sigma_{y \in N_G} (x \neq y)$, and the corresponding edges in $G$.

**Biconnected(G) := \prod_{x \in N_G} Connected(G - x).**

**Lemma 5.15.** If $G$ is a cyclic graph, then $U(G)$ is 2-connected.

The 2-connectedness of a graph is not preserved by simple path additions. Clearly, removing a node from the added path $p$ disconnects $G \circ p$. However, using non-simple path additions, we can preserve and enlarge 2-connected graphs.

**Lemma 5.16.** Suppose $G$ is a 2-connected graph, then the following claims hold.

1. Every node in $G$ has degree of minimum two.
2. There exists a cyclic graph $H$ and an injective morphism from $U(H)$ to $G$.
3. The graphs $G \circ \overline{p}, U(G \circ p)$, and $U(G) \circ \overline{p}$ are all 2-connected.
**Lemma 5.17.** In a non-simple Whitney synthesis of $G$ of length $n$ from a 2-connected cyclic graph $H$, every graph $G_i$ in the sequence is a 2-connected planar graph.

**Proof.** By induction on $n$. If $n = 0$, the graph $H$ is 2-connected by hypothesis and is planar by a similar construction as in Example 5.7. Assuming that the claim holds for a sequence of length $n$, then $G_n$ is a 2-connected planar graph. By Item (iii) in Lemma 5.16, we get that $G_n \cdot \overline{p_i}$ is 2-connected. Using a similar construction as in the proof of Lemma 5.9, one defines a planar map for $G_n \cdot \overline{p_i}$, from where the conclusion follows. □

The converse of Lemma 5.17 can be proved by closely following the informal proof of Lemma 3 and Proposition 4 for undirected 2-connected planar graphs in [42]. We must formalise several notions before considering such a proof in our formalism, including, the notion of maximal sub graphs, adjacent faces, and deletion of edge sequences. One understands, therefore, that the class of 2-connected planar graphs is completely determined by all non-simple Whitney syntheses [15, §3]. Any planar graph, in the sense of Definition 5.5, and 2-connected, as in Definition 5.14, can be inductively generated from a cycle graph and iterative additions of non-simple paths.

Further investigation to study of other graph extensions to generate planar graphs, as graph amalgamations, graph appendages, deletions, contractions and subdivisions should be considered [26, §7.3].

### 6 RELATED WORK

One can find the study of planar graphs and more general graph theoretic topics in relevant projects and big libraries formalised in Coq [17] and Isabelle/HOL [33]. For example, the formal proof of the Four-Colour Theorem (FCT) in Coq by Gonthier [23], the proof of the discrete form of the Jordan Curve Theorem in Coq by Dufourd [20], and the proof of the Kepler’s Conjecture in HOL by Bauer et al. [28] are a few of such notable projects in the subject.

Different approaches have been proposed to address planarity of graphs in formal systems. These works use different mathematical objects depending on the system. We use combinatorial maps in this work, but other related constructions are, for example, root maps defined in terms of permutations by Dubois et al. [18], and hypermaps by Dufourd and Gonthier [19, 20, 23], among others. In particular, one can see that the notion of a hypermap is a generalisation of a combinatorial map for undirected finite graphs. Such a concept is one fundamental construction to formalise mathematics of graph embeddings amongst in theorem provers, along with the computer-checked proof of FCT. Additionally, Dufourd states and proves the Euler’s polyhedral formula and the Jordan Curve Theorem using an inductive characterisation of hypermaps [19, 20]. Recently, for a more standard representation of finite graphs, Doczkal proved that, according to his notion of a plane map based on hypermaps, every $K_{3,3}$-free graph and $K_5$-free graph without isolating vertices is planar, a direction of Wagner’s theorem [16].

An alternative approach for planarity using combinatorial maps is the iterative construction of certain kind of planar graphs. For example, Yamamoto et al. [42] showed that every biconnected and finite planar graph can be decomposed as a finite set of cycle graphs, where every face is the region bounded by a closed walk [26, §5.2, §7.3]. Such construction defines an inductive data type that begins with a cycle graph $C_n$ serving as the base case, and by repeatedly merging new instances of cycle graphs, one gets the final planar graph. Bauer formalises in Isabelle/HOL a similar construction of planar graphs from a set of faces [10, 11]. A related approach in our setting is of the treatment of planar graph extensions, as described in Section 5.3.

However, to the best of our knowledge, in type theory, related work to the planarity of graphs has been done in a different formal system and for different classes of graphs. These studies mostly define planarity for undirected finite graphs, in contrast, our definition considers the more general
class of connected and locally finite directed multigraphs. Our work is closer to the foundations of mathematics, specifically, to the formalisation of mathematics in HoTT, than to more practical aspects of graph theory. This approach forces us to propose new constructions, even sometimes, for the most fundamental and basic concepts in the theory.

7 CONCLUDING REMARKS

This document is a case study of graph-theoretic concepts in constructive mathematics using homotopy type theory. An elementary characterisation of planarity of connected and locally finite directed multigraphs is presented in Section 5.2. We collected all the maps of a graph in the two-dimensional plane—identified up to isotopy—in a homotopy set, see Theorem 5.6. The type of these planar maps displays some of our main contributions, e.g. the type of spherical maps stated in Definition 4.2 and the type of faces for a given map in Definition 4.5. As far as we know, the presentation of these types in a dependent type theory like HoTT is novel. For example, besides its rather technical definition, we believe the type of faces encodes in a better combinatorial way the essence of the topological intuition behind it, rather than, being defined as simply cyclic lists of nodes, as by other authors [11, 23, 42], see Section 4.3.

Additionally, as a way to construct planar graphs inductively, we presented extensions for planar maps. We demonstrated that any cycle graph is planar, and by means of planar extensions like path additions, one can construct larger planar graphs, e.g. to illustrate this approach, a planar map for $K_4$ using simply path additions from a planar map of $C_3$ is illustrated in Figure 12. Other relevant notions to this work are cyclic types, cyclic graphs, homotopy for walks [35], and spherical maps.

We chose HoTT as the reasoning framework to directly study the symmetry of our mathematical constructions. Many of the proofs supporting our development could only be constructed by adopting the Univalence Axiom, a main principle in HoTT. A primary example of using Univalence in this paper is the structural identity principle for graphs, as stated in Theorem 3.6.

Another contribution of this work include the (computer-checked) proofs. The major results in this document have been formalised in the proof assistant Agda, in a development fully self-contained way, which does not depend on any library [34]. However, for technical reasons, the formalisation of Example 5.7 and further in depth studies on the main results in Section 5.3 like Lemmas 5.9 and 5.17 will be conducted in future.

This work can serve as a starting point for further developments of graph theory in HoTT or related dependent type theories. We expect further research to provide other interesting results as the equivalences between different characterisations of planarity for graphs, e.g., the Kuratowski’s and Wagner’s characterisations for planar graphs.

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REFERENCES

[1] Benedikt Ahrens and Paige Randall North. 2019. Univalent Foundations and the Equivalence Principle. Springer International Publishing, Cham, 137–150. https://doi.org/10.1007/978-3-030-15655-8_6
[28] Thomas Haless, Mark Adams, Gertrud Bauer, Tat dat Dang, John Harrison, Le truong Hoang, Cezary Kaliszyk, Victor Magron, Sean Mclaughlin, Tat Thang Nguyen, and et al. 2017. A Formal Proof Of The Kepler Conjecture. Forum of Mathematics, Pi 5 (2017), e2. https://doi.org/10.1017/fmp.2017.1

[29] John Harrison. 2008. Formal Proof –Theory and Practice. Notices of the American Mathematical Society 55 (2008), 1395–1406.

[30] The Homotopy Type Theory and Univalent Foundations CAS project. 2019. Symmetry Book. http://github.com/UniMath/SymmetryBook

[31] Saunders MacLane. 1937. A combinatorial condition for planar graphs.

[32] Bojan Mohar. 1988. Embeddings of infinite graphs. Journal of Combinatorial Theory, Series B 44, 1 (1988), 29–43. https://doi.org/10.1016/0095-8956(88)90094-9

[33] Lars Noschinski. 2015. Formalizing Graph Theory and Planarity Certificates. Ph.D. Dissertation. Technischen Universität München, Germany. https://d-nb.info/1104933624/34

[34] Jonathan Prieto-Cubides. 2019. Investigations on graph-theoretical constructions in Homotopy type theory – Agda formalisation. https://doi.org/10.5281/zenodo.577570 Work-in-progress.

[35] Jonathan Prieto-Cubides. 2021. On homotopy of walks and spherical maps in homotopy type theory. (2021). https://doi.org/10.1145/3497775.3503671

[36] Md. Saidur Rahman. 2017. Planar Graphs. Springer International Publishing, Cham, 77–89. https://doi.org/10.1007/978-3-319-49475-3_6

[37] Saul Stahl. 1978. The embeddings of a graph–A survey. Journal of Graph Theory 2, 4 (1978), 275–298. https://doi.org/10.1002/jgt.3190020402

[38] The Agda Development Team. 2021. Agda 2.6.1 documentation. https://agda.readthedocs.io/en/v2.6.1.3/

[39] The Univalent Foundations Program. 2013. Homotopy Type Theory: Univalent Foundations of Mathematics. https://homotopytypetheory.org/book, Institute for Advanced Study.

[40] Vladimir Voevodsky. 2010. The equivalence axiom and univalent models of type theory. (Talk at CMU on February 4, 2010). , 1–11 pages. https://arxiv.org/abs/1002.5556

[41] Hassler Whitney. 1932. Non-separable and planar graphs. Trans. Amer. Math. Soc. 34, 2 (Feb. 1932), 339–339. https://doi.org/10.1090/s0002-9947-1932-1501641-2

[42] Mitsuharu Yamamoto, Shin-ya Nishizaki, Masami Hagiya, and Yozo Toda. 1995. Formalization of Planar Graphs. In Higher Order Logic Theorem Proving and Its Applications, 8th International Workshop, Aspen Grove, UT, USA, September 11-14, 1995, Proceedings (Lecture Notes in Computer Science), E. Thomas Schubert, Phillip J. Windley, and Jim Alves-Foss (Eds.), Vol. 971. Springer, Ut, Usa, 369–384. https://doi.org/10.1007/3-540-60275-5_77

[43] Brent Abraham Yorgey. 2014. Combinatorial Species And Labelled Structures. Ph.D. Dissertation. University of Pennsylvania, Pa, Usa. https://www.cis.upenn.edu/~sweirich/papers/yorgey-thesis.pdf