JANTZEN FILTRATION AND STRONG LINKAGE PRINCIPLE
FOR MODULAR LIE SUPERALGEBRAS

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Abstract. In this paper, we introduce super Weyl groups, their distinguished elements and properties for basic classical Lie superalgebras. Then we formulate Jantzen filtration for baby Verma modules in graded restricted module categories of basic classical Lie superalgebras over an algebraically closed field of odd characteristic, and prove a sum formula in the corresponding Grothendieck groups. We finally obtain a strong linkage principle in such categories.

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1. INTRODUCTION

1.1. Recently, the study on representations of basic classical Lie superalgebras and the corresponding algebraic supergroups in prime characteristic becomes an increasing interest (cf. [2] [3] [25] [28] [29] [30] [35] [39] [40] [41] [42] [43] [44] [45], etc.), in the meanwhile the study on the counterparts over the complex number field is going on. In such a topic, it is a key point to understand irreducible modules through standard modules (baby Verma modules). So far, very few information on this has been known. Recall in the Bernstein-Gelfand-Gelfand categories of complex semisimple Lie algebras, and in (rational) representation categories of modular reductive algebraic group and of their reductive Lie algebras associated with specific tori of the corresponding algebraic groups, some kinds of artfully-constructed

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filtrations of standard modules introduced by Jantzen in [16] for complex semisimple Lie algebras, in [20, 19] for modular reductive algebraic groups and their Lie algebras, is a very powerful tool to understand character formulas of irreducible modules through the ones of standard modules (cf. [14, 16, 18, 19, 20], etc.). In this paper we exploit the theory of Jantzen filtration for reductive Lie algebras in prime characteristic to the super case.

1.2. We will focus on restricted representations of a basic classical Lie superalgebra $\mathfrak{g}$ over $\mathbf{k}$, an algebraically closed field of characteristic $p > 2$. In order to establish a satisfactory theory on characters for restricted irreducible modules in modular cases, a usual way is to refine the restricted module category $U_0(\mathfrak{g})\text{-mod}$ of $\mathfrak{g}$ by introducing $\mathbb{Z}$-graded structure. So we will actually work with a graded restricted module category $(U_0(\mathfrak{g}), \Xi)\text{-mod}$, via an additional $\Xi$-action for a maximal torus $\Xi$ of algebraic supergroup $G$ with $\text{Lie}(G) = \mathfrak{g}$ (see Definition 4.1).

A super Weyl group $\hat{W}$ is introduced here, which is a subgroup of the symmetric group of the set of all Borel subalgebras, generated by all real reflections and odd reflections. In our construction of Jantzen filtration for baby Verma modules in $(U_0(\mathfrak{g}), \Xi)\text{-mod}$, an important ingredient is to find a distinguished element $\hat{w}_0$ of the super Weyl group $\hat{W}$ for $\mathfrak{g}$ (see Theorem 3.10), which plays a role as important as the longest elements of usual Weyl groups.

In order to establish Jantzen filtration, apart from carrying some known information and methods by Jantzen in [19] forward to the Lie superalgebra from its even part, we need do more in seeking for and analysing homomorphisms between (twisted) Verma modules, resulted from odd roots. These (twisted) Verma modules are associated with twisted Borel subalgebras, corresponding different positive root sets arising from the reduced expression of $\hat{w}_0$. With the above, we finally obtain a satisfactory filtration and sum formula (see Theorem 6.4). As a consequence, we can read off a strong linkage principle from the sum formula (see Theorem 7.6 and Theorem 7.2).

1.3. One can compare some related results over complex numbers (cf. [32, Chapter 10], [33] and [38]).

2. Preliminaries

Throughout the article, the notations of vector spaces (resp. modules, subalgebras) means vector superspaces (resp. super modules, super subalgebras). For simplicity, we omit the adjunct word "super". For a superspace $V = V_0 + V_1$, the homogeneous element $v$ is assumed to have a parity $|v| \in \{\hat{1}, \hat{0}\}$. All vector spaces are defined over $\mathbf{k}$ for an algebraically closed field of characteristic $p > 2$.

2.1. Lie superalgebras and algebraic supergroups. In this section, we will recall some knowledge on basic classical Lie superalgebras along with the corresponding algebraic supergroups. We refer the readers to [6, 21, 32] for Lie superalgebras, [21, 31, 35] for algebraic supergroups.
2.1.1. Basic classical Lie superalgebras. Following [6 §1], [21 §2.3-§2.4], [22 §1] and [39 §2], we recall the list of basic classical Lie superalgebras over \( F \) for \( F = \mathbb{C} \) or \( F = k \) where \( k \) is an algebraically closed field of odd characteristic \( p \). These Lie superalgebras, with even parts being Lie algebras of reductive algebraic groups, are simple over \( F \) (the general linear Lie superalgebras, though not simple, are also included), and they admit an even non-degenerate supersymmetric invariant bilinear form in the following sense.

**Definition 2.1.** Let \( V = V_{\bar{0}} \oplus V_{\bar{1}} \) be a \( \mathbb{Z}_2 \)-graded space and \((\cdot, \cdot)\) be a bilinear form on \( V \).

1. If \((a, b) = 0\) for any \( a \in V_{\bar{0}}, b \in V_{\bar{1}}\), then \((\cdot, \cdot)\) is called even;
2. if \((a, b) = (-1)^{|a||b|}(b, a)\) for any homogeneous elements \( a, b \in V \), then \((\cdot, \cdot)\) is called supersymmetric;
3. if \([[a, b], c] = (a, [b, c])\) for any homogeneous elements \( a, b, c \in V \), then \((\cdot, \cdot)\) is called invariant;
4. if one can conclude from \((a, V) = 0\) that \( a = 0 \), then \((\cdot, \cdot)\) is called non-degenerate.

Note that when \( F = k \) is a field of characteristic \( p > 2 \), there are restrictions on \( p \); for example, as shown in [39, Table 1]. So we have the following list

*(Table 1): basic classical Lie superalgebras over \( k \)*

| \( \mathfrak{g}_k \) | \( \mathfrak{g}_{\bar{0}} \) | the restriction of \( p \) when \( F = k \) |
|-----------------|-----------------|------------------|
| \( \mathfrak{gl}(m|n) \) | \( \mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \) | \( p > 2 \) |
| \( \mathfrak{sl}(m|n) \) | \( \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus k \) | \( p > 2, p \nmid (m - n) \) |
| \( \mathfrak{osp}(m|n) \) | \( \mathfrak{so}(m) \oplus \mathfrak{sp}(n) \) | \( p > 2 \) |
| \( \mathfrak{D}(2,1,a) \) | \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \) | \( p > 3 \) |
| \( \mathfrak{F}(4) \) | \( \mathfrak{sl}(2) \oplus \mathfrak{so}(7) \) | \( p > 15 \) |
| \( \mathfrak{G}(3) \) | \( \mathfrak{sl}(2) \oplus G_2 \) | \( p > 15 \) |

2.1.2. Algebraic supergroups and restricted Lie superalgebras. For a given basic Lie superalgebra listed in §2.1.1, there is an algebraic supergroup \( G_F \) satisfying \( \text{Lie}(G_F) = \mathfrak{g}_F \) such that

1. \( G_F \) has a subgroup scheme \((G_F)_{ev}\) which is an ordinary connected reductive group with \( \text{Lie}((G_F)_{ev}) = (\mathfrak{g}_F)_0 \);
2. there is a well-defined action of \((G_F)_{ev}\) on \( \mathfrak{g}_F \), reducing to the adjoint action of \((\mathfrak{g}_F)_0 \).

The above algebraic supergroup can be constructed in the Chevalley group way, which we call an algebraic supergroup of Chevalley type (or, Chevalley supergroup in [7]). The pair \((G_F)_{ev}, \mathfrak{g}_F \) constructed in this way is called a Chevalley super Harish-Chandra pair (cf. [7 §5], [8 §3.3], [23] and [24]). Partial results on \( G_F \) and \((G_F)_{ev}\) can be found in [11 Part II, §2.2], [7], [8], [9], [24], etc. In the present paper, we will call \((G_F)_{ev}\) the purely even subgroup of \( G_F \). When the ground field \( F = k \) is of odd prime characteristic \( p \), one easily knows that \( \mathfrak{g}_k \) is a restricted Lie superalgebra (cf. [35, Definition 2.1] and [37]) in the following sense.
Definition 2.2. A Lie superalgebra \( \mathfrak{g}_k = (\mathfrak{g}_k)_0 \oplus (\mathfrak{g}_k)_1 \) over \( k \) is called a restricted Lie superalgebra, if there is a \( p \)-th power map \((\mathfrak{g}_k)_0 \rightarrow (\mathfrak{g}_k)_0, \) denoted as \((-)^{[p]}\), satisfying

(a) \((kx)^{[p]} = k^p x^{[p]}\) for all \( k \in k \) and \( x \in (\mathfrak{g}_k)_0 \);

(b) \([x^{[p]}, y] = (\text{ad}x)^p(y)\) for all \( x \in (\mathfrak{g}_k)_0 \) and \( y \in \mathfrak{g}_k \);

(c) \((x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)\) for all \( x, y \in (\mathfrak{g}_k)_0 \), where \( is_i(x, y) \) is the coefficient of \( \lambda^{-1} \) in \((\text{ad}(\lambda x + y))^{p-1}(x)\).

Let \( \mathfrak{g}_k \) be a restricted Lie superalgebra. For each \( x \in (\mathfrak{g}_k)_0 \), the element \( x^p - x^{[p]} \in U(\mathfrak{g}_k) \) is central by Definition 2.2 and all of which generate a central subalgebra of \( U(\mathfrak{g}_k) \). Let \( \{w_1, \ldots, w_c\} \) and \( \{w'_1, \ldots, w'_d\} \) be the basis of \( (\mathfrak{g}_k)_0 \) and \( (\mathfrak{g}_k)_1 \) respectively. For a given \( \chi \in (\mathfrak{g}_k)_0 \), let \( J_\chi \) be the ideal of the universal enveloping algebra \( U(\mathfrak{g}_k) \) generated by the even central elements \( \bar{w}^p - \bar{w}^{[p]} - \chi(\bar{w})p \) for all \( \bar{w} \in (\mathfrak{g}_k)_0 \). The quotient algebra \( U_\chi(\mathfrak{g}_k) := U(\mathfrak{g}_k)/J_\chi \) is called the reduced enveloping algebra with \( p \)-character \( \chi \). We often regard \( \chi \in \mathfrak{g}_k^* \) by letting \( \chi((\mathfrak{g}_k)_1) = 0 \). If \( \chi = 0 \), then \( U_0(\mathfrak{g}_k) \) is called the restricted enveloping algebra. It is a direct consequence from the PBW theorem that the \( k \)-algebra \( U_\chi(\mathfrak{g}_k) \) is of dimension \( p^{c_d} \), and has a basis

\[
\{w_1^{a_1} \cdots w_c^{a_c} (w'_1)^{b_1} \cdots (w'_d)^{b_d} | 0 \leq a_i < p, b_j \in \{0, 1\} \text{ for all } 1 \leq i \leq c, 1 \leq j \leq d\}.
\]

A supermodule \((\rho, V)\) for a restricted Lie superalgebra \((\mathfrak{g}_k, [p])\) is said to be restricted if \( \rho \) satisfies for all \( x \in (\mathfrak{g}_k)_0 \)

\[\rho(x)^p - \rho(x^{[p]}) = 0.\] (2.1)

All restricted modules of \( \mathfrak{g}_k \) constitute a full subcategory of the \( \mathfrak{g}_k \)-module category, which coincides with the \( U_0(\mathfrak{g}_k) \)-module category, denoted by \( U_0(\mathfrak{g}_k) \)-mod.

2.1.3. Root structural information. From now on, we will always assume, throughout the remaining of the paper, that all base fields are \( k \), which is algebraically closed and of odd characteristic \( p \) satisfying the condition listed in (Table 1). The notations involving the base field will be simplified. For example, an algebraic supergroups \( G_k \) and its Lie algebra \( \mathfrak{g}_k \) will be directly denoted by \( G \) and \( \mathfrak{g} \), with the subscript \( k \) being omitted.

For a given basic Lie superalgebra \( \mathfrak{g} \), by the arguments as above \( \mathfrak{g} \) is a Lie superalgebra of an algebraic supergroup \( G \) whose purely even group \( G_{ev} \) is reductive. Fix a standard maximal torus \( \mathfrak{T} \) of \( G_{ev} \) which consists of diagonal matrices in \( G_{ev} \), and set \( \mathfrak{h} = \text{Lie}(\mathfrak{T}) \). Then \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \). Let \( \Delta \) be a root system of \( \mathfrak{g} \) relative to \( \mathfrak{h} \) whose simple root system \( \Pi = \{\alpha_1, \ldots, \alpha_l\} \) is standard, corresponding to the standard diagrams in the sense of [22, §1.3] (cf. [22, Proposition 1.5]). Let \( \Delta^+ \) be the corresponding positive system in \( \Delta \), and put \( \Delta^- := -\Delta^+ \). Let \( \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \) be the corresponding triangular decomposition of \( \mathfrak{g} \). There is a canonical Borel subalgebra \( \mathfrak{b} := \mathfrak{h} + \mathfrak{n}^+ \). Furthermore, \( \mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 \), and \( \mathfrak{g}_0 = \mathfrak{b} + \sum_{\alpha \in \Delta_0} \mathfrak{g}_0 \) and \( \mathfrak{g}_1 = \sum_{\beta \in \Delta_1} \mathfrak{g}_\beta \). The set of even roots \( \Delta_0 \) is occasionally denoted by \( \Delta_{ev} \), and the set of odd roots \( \Delta_1 \) by \( \Delta_{odd} \) for clarification.
As discussed in §2.1.1, there exists a non-degenerate even invariant super-symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$, which restricts to a non-degenerate form $(\cdot,\cdot)$ on $\mathfrak{h}$ and on $\mathfrak{h}^*$. Especially, $(\cdot,\cdot)$ can be defined in $\Delta$. A root $\gamma \in \Delta$ is called isotropic if $(\alpha,\alpha) = 0$ (note an isotropic root is necessarily an odd root). Following [32] we set

$$\overline{\Delta}_0 := \{ \alpha \in \Delta_0 \mid \frac{\alpha}{2} \notin \Delta_1 \}, \quad \overline{\Delta}_1 := \{ \beta \in \Delta_1 \mid 2\beta \notin \Delta_0 \}.$$  

By Lemma 8.3.2 of [32], $\overline{\Delta}_1$ is just the set of isotropic roots. And then the set of nonisotropic roots is

$$\Delta_{\text{nonisotropic}} = \Delta_0 \cup (\Delta_1 \setminus \overline{\Delta}_1). \tag{2.2}$$

2.2. Chevalley bases and coroots. Under some mild condition (the one listed in (Table 1) is sufficient), we can write $\mathfrak{g} = \mathfrak{g}_Z \otimes \mathbb{Z} k$ where $\mathfrak{g}_Z$ is a $\mathbb{Z}$-form of $\mathfrak{g}$, and $\mathfrak{g}_Z = \mathfrak{h}_Z + \sum_{\alpha \in \Delta} (\mathfrak{g}_Z)_\alpha$ has a Chevalley basis $\{ X_\alpha \in (\mathfrak{g}_Z)_\alpha, Y_\alpha \in (\mathfrak{g}_Z)_{-\alpha} \mid \alpha \in \Delta^+ \} \cup \{ H_i \in \mathfrak{h}_Z \mid i = 1, \ldots, r := \dim \mathfrak{h} \}$ satisfying the Chevalley basis axioms as presented as in [7] §3.2] (see [7] Theorem 3.3.1], or [15] [35]. Especially, $\mathfrak{g}_Z = \mathfrak{h}_Z + \sum_{\alpha \in \Delta} (\mathfrak{g}_Z)_\alpha$ such that $(H_i \mid i = 1, \ldots, r)$ is a $\mathbb{Z}$-basis of $\mathfrak{h}_Z$, and $(\mathfrak{g}_Z)_\alpha = \mathbb{Z} X_\alpha$ and $(\mathfrak{g}_Z)_{-\alpha} = \mathbb{Z} Y_\alpha$ for $\alpha \in \Delta^+$ with $[X_\alpha, Y_\alpha] = H_\alpha$. Furthermore, associated with each root $\alpha \in \Delta^+$, there is a vector $H_\alpha \in \mathfrak{h}_Z$ satisfying $\mathfrak{h}_Z = \mathbb{Z}\text{-span}\{H_\alpha \mid \alpha \in \Delta^+\}$ and $(\alpha, \beta) = (H_\alpha, H_\beta) \in \mathbb{Z}$ for $\alpha, \beta \in \Delta^+$, and $\alpha(H_\alpha) = 2$ for any $\alpha \in \Delta^+ \setminus \overline{\Delta}_1$ (cf. [7] §3.1, §3.2]).

Recall that the character group $X(\Xi)$ of $\Xi$ is a free abelian group of rank $r := \dim \Xi$, identified with $\mathbb{Z}^r$. Associated with $\alpha \in \Delta^+$, by the construction of Chevalley supergroups we have a one-parameter multiplicative supersubgroup $\chi_\alpha$ such that $\chi_\alpha(t) \in \Xi$ for $t \in k^\times := k \setminus \{0\}$ and $\lambda(\chi_\alpha(t)) = t^{\lambda(H_\alpha)}$ for $\lambda \in X(\Xi)$ (cf. [7] §5.2). By the same way as the theory of algebraic groups ([13] §16.1] or [18] §II. 1.3), we can assign the set of pairs of $\lambda, \chi_\alpha$ to $\mathbb{Z}$ via $\langle \lambda, \chi_\alpha \rangle = \lambda(H_\alpha)$. Usually, $\chi_\alpha$ is called a coroot corresponding to $\alpha$ (cf. [18] §II. 1.3]). One can describe precisely $X(\Xi)$ for basic classical cases (see [3] and [33] for type $A, B, C$ and $D$). The following general fact is clear.

**Lemma 2.3.** $\Pi \subset X(\Xi)$.

**Proof.** It follows from [7] §5.2].

2.3. There is a standard involution $\tau_0$ in Aut($\mathfrak{g}$) which interchanges $X_\alpha$ with $Y_\alpha$ for all $\Phi^+$ and stabilizes $\mathfrak{h}$ with $\tau|_{\mathfrak{h}} = -\text{id}_{\mathfrak{h}}$. Actually, this $\tau_0$ is the differential of an automorphism $\tau \in \text{Aut}(G)$ which satisfies $\tau(t) = t^{-1}$ for all $t \in \Xi$. This $\tau_0$ can be realized as $\tau_0(X) = -X^{\text{st}}$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$ where $\text{st}$ means super transpose (see [32] 5.6.9, page 128]), and $\tau$ can be realized as $\tau(x) = (x^{-1})^{\text{st}}$ for $G = \text{GL}(m|n)$ and $x \in G(A)$ for any super commutative $k$-algebra $A$. Obviously, $\tau$ induces $-\text{id}$ on $X(\Xi)$.

3. Super Weyl groups and their distinguished elements

Keep the notations and assumptions as in [2.1.3] In particular, let $\mathfrak{g}$ be an any given basic Lie superalgebra over $k$, with Lie($G$) = $\mathfrak{g}$ and Chevalley super
Harish-Chandra pair \((G_{ev}, \mathfrak{g})\), and \(\mathfrak{T}\) be a fixed maximal torus \(G_{ev}\) which consists of diagonal matrices in \(G_{ev}\), and \(\mathfrak{h} = \text{Lie}(\mathfrak{T})\).

3.1. Odd and real reflections. Beside real reflections associated to even roots as for semisimple Lie algebras, one can define odd reflections associated to isotropic odd simple roots. Both real and odd reflections permute the fundamental systems of a root system. There is a fundamental fact on an odd simple root as following.

**Lemma 3.1.** (\[^{[6]}\text{Lemma 1.30}\]) Let \(\mathfrak{g}\) be a basic Lie superalgebra and maintain the notations as in \[^{[2.1.2]}\]. For a given isotropic simple root \(\gamma\) in \(\Pi\), set \(\Delta_\gamma^+ := \{-\gamma\} \cup \Delta^+ \setminus \{\gamma\}\). Then \(\Delta_\gamma^+\) is a new positive root system whose corresponding fundamental root system \(\Pi_\gamma\) is given by

\[
\Pi_\gamma = \{\alpha \in \Pi \mid (\alpha, \gamma) = 0, \alpha \neq \gamma\} \cup \{\alpha + \gamma \mid \alpha \in \Pi, (\alpha, \gamma) \neq 0\} \cup \{-\gamma\}.
\]

For an isotropic simple root \(\gamma\) with respect to the simple root system \(\Pi(\mathfrak{B})\) of a given Borel subalgebra \(\mathfrak{B}\) (and the corresponding positive system \(\Delta(\mathfrak{B})^+\)), we can write \(\mathfrak{B} = \mathfrak{h} + \sum_{\theta \in \Delta(\mathfrak{B})^+} \mathfrak{g}_\theta\). The above lemma is actually to give an operator which is to transform \(\mathfrak{B}\) to a new Borel subalgebra

\[
\mathfrak{B}^\gamma := \mathfrak{h} + \sum_{\theta \in \Delta(\mathfrak{B}^\gamma)^+} \mathfrak{g}_\theta
\]

with \(\Delta(\mathfrak{B}^\gamma)^+ = \Delta(\mathfrak{B})_\gamma^+\) and \(\Pi(\mathfrak{B}^\gamma) = \Pi(\mathfrak{B})_\gamma\). Such operation is called odd reflection (with respect to \(\gamma\)), which is denoted by \(r_\gamma\). Obviously, \(r_{-\gamma}(\mathfrak{B}^\gamma) = \mathfrak{B}\). We will identify \(r_\gamma\) with \(r_{-\gamma}\) when we deal with such a pair \(\{\mathfrak{B}, \mathfrak{B}^\gamma\}\), which will be helpful to introduce so-called super Weyl groups (see Definition \(^{[3.3]}\)), not making any confusion.

For a non-isotropic root, by \(^{[2.2]}\) we know that it is either an even root \(\alpha\), or an odd root \(\beta\) with \(2\beta \in \Delta_0\). So for a given non-isotropic root \(\theta\) we can set \(s_\theta(\lambda) = \lambda - 2(\lambda, \theta)/(\theta, \theta)\theta\), which becomes a linear map on \(\mathfrak{h}^*\). Note that \(s_\beta = s_{2\beta}\) for any \(\beta \in \Delta_1 \setminus \Sigma_1\). All of them are called real reflections. The Weyl group \(W\) for \(\mathfrak{g}_0\) is just generated by those real reflections \(s_\theta\) for all \(\theta \in \Sigma_0^+ \cup (\Delta_1^+ \setminus \Sigma_1^+)\) (note that \(s_\theta = s_{-\theta}\), as a subgroup of \(\text{GL}(\mathfrak{h}^*)\), which stabilize \(\Delta, \Delta_0\) and \(\Delta_1\) respectively.

In the sequent arguments, we will unify the notations of odd and real reflections, by using \(\hat{r}_\theta\) respect to the root \(\theta\). We first have an observation that if \(\theta \in \Pi\) then \(\hat{r}_\theta \Delta^+\) and \(\Delta^+\) have differences by at most two roots, i.e. \(#(\hat{r}_\theta \Delta^- \cap \Delta^+) \leq 2\). Actually, if either \(\theta \in \Pi \cap \Sigma_0\), or \(\theta \in \Pi \cap \Sigma_1\), then \(\hat{r}_\theta \Delta^- \cap \Delta^+ = \{\theta\}\). If \(\theta \in \Delta_1 \setminus \Sigma_1\), then \(\hat{r}_\theta \Delta^- \cap \Delta^+ = \{\theta, 2\theta\}\).

Thus we have a further observation which is a strengthened version of \(^{[6]}\text{Proposition 1.32}\)).

**Lemma 3.2.** Let \(\Pi(i), i = 1, 2\) are two different simple root systems of \(\mathfrak{g}\). There exist a series of real and odd reflections \(\hat{r}_1, \hat{r}_2, \cdots, \hat{r}_n\) such that \(\hat{w} := \hat{r}_n \cdots \hat{r}_2 \hat{r}_1\) with \(\hat{w}(\Pi(1)) = \Pi(2)\). Moreover, for any \(i \in \{1, 2, \cdots, n\}\) there are at most two different roots between \(\hat{r}_{i+1} \cdots \hat{r}_2 \hat{r}_1(\Delta^+_{(1)})\) and \(\hat{r}_i \cdots \hat{r}_2 \hat{r}_1(\Delta^+_{(1)})\).
Proof. Set $\Delta_{(i)}^{\pm}$ to be the positive (resp. negative) root system corresponding to $\Pi_{(i)}$ for $i = 1, 2$. Due to the difference between $\Pi_{(1)}$ and $\Pi_{(2)}$, we have $\Delta_{(2)}^- \cap \Pi_{(1)} = \emptyset$. We verify the statement by induction on #$\{\Delta_{(2)}^- \cap \Pi_{(1)}\}$, say, $\{1, \ldots, n\}$. Consider $\hat{r}_1(\Pi_{(1)})$ for $\hat{r}_1 := \hat{r}_{0_1}$. Denote by $\Pi_{(2)}$ to be the new simple root system $\hat{r}_1(\Pi_{(1)})$. By the definition of real and odd reflections and the choice of $\theta_1$, we can verify in a moment

$$
#(\Delta_{(2)}^- \cap \Pi_{(1)}) - #(\Delta_{(2)}^- \cap \Pi_{(2)}) = 1.
$$

(3.1)

Now we begin to verify the above formula, taking different cases of $\hat{r}_1$ into account. Note that $\{\theta_1, \ldots, \theta_t\} \subset \Delta_{(2)}^-$ while $\{\theta_{t+1}, \ldots, \theta_{t+s}\} \subset \Delta_{(2)}^+$. When $\hat{r}_1$ is a real reflection, $\Pi_{(2)} = \{\hat{r}_1(\theta_1), \ldots, \hat{r}_1(\theta_t), \hat{r}_1(\theta_{t+1}), \ldots, \hat{r}_1(\theta_{t+s})\}$. Note that the row associated with $\theta_1$ of the Cartan matrix corresponding to $\Pi_1$ are non-negative integers except the diagonal entry which is $2$ (cf. [22] Page 604-605). So the property that $\Pi_1$ is a fundamental root system implies that $\{\hat{r}_1(\theta_2), \ldots, \hat{r}_1(\theta_t)\} \subset \Delta_{(2)}^-$ while $\{\hat{r}_1(\theta_{t+1}), \ldots, \hat{r}_1(\theta_{t+s})\} \subset \Delta_{(2)}^+$. In this case, the formula (3.1) is proved. When $\hat{r}_1$ is an odd reflection, or to say, $\theta_1$ is an isotropic simple odd root in $\Pi_{(1)}$, we can verify in a moment

$$
\hat{r}_1(\Pi_{(1)}) = \{-\theta_1\} \cup \{\theta_i \mid (\theta_i, \theta_1) = 0\} \cup \{\theta_j + \theta_1 \mid (\theta_j, \theta_1) \neq 0\}.
$$

The property that $\Pi_1$ is a fundamental root system implies that $\{\theta_2, \ldots, \theta_t\} \subset \Delta_{(2)}^-$ while $\{\theta_{t+1}, \ldots, \theta_{t+s}\} \subset \Delta_{(2)}^+$ where $\theta_i = \theta_i$ if $(\theta_i, \theta_1) = 0$ or $\theta_i + \theta_1$ if $(\theta_i, \theta_1) \neq 0$. So in this case, the formula (3.1) is proved. Thus, we complete the verification of (3.1).

By the inductive assumption, we can get a series of real and odd reflections $\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n$ such that $\hat{r}_n \cdots \hat{r}_2 \hat{r}_1(\Pi_{(1)}) = \Pi_{(2)}$. As to the second statement, it follows from our inductive construction and the observation mentioned before the lemma. \hfill \Box

3.2. Super Weyl groups. Denote by $\mathcal{B}$ the set of all Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$. By the definition of real and odd reflections, all of them can be regarded as transforms of the set $\mathcal{B}$. Actually, for a given $\hat{r}_{0_0}$, if it is a real reflection, i.e. $\theta_0 \in \Delta_0^+ \cup (\Delta_0^- \setminus \Delta_1^+)$, then $\hat{r}_{0_0}(B) = B + \sum_{\theta \in \Delta_{(B)}^+} \mathfrak{g}_{\theta_{0_0}}(\theta)$ for any given Borel subalgebra $B = B + \sum_{\theta \in \Delta_{(B)}^+} \mathfrak{g}_{\theta_{0_0}}(\theta)$. If $\hat{r}_{0_0}$ is an odd reflection, then either $\theta_0$ or $-\theta_0$ is a simple isotropic odd root of some Borel subalgebra $B$. Denote by $\mathcal{B}_{\theta_0}$ the set of such Borel subalgebras. Then $\mathcal{B}_{\theta_0}$ is a subset of $\mathcal{B}$ containing even number Borel subalgebras, say, $\mathcal{B}_{\theta_0} = \{B_1, B_1', \ldots, B_t, B_t'\}$ such that the operation of $\hat{r}_{0_0}$ makes exchanged between $B_i$ and $B_i'$, $i = 1, \ldots, t$. For any $B \in \mathcal{B} \setminus \mathcal{B}_{\theta_0}$, we define an assignment of $\hat{r}_{0_0}$ which sends $B$ to itself. Thus an odd reflection $\hat{r}_{\theta}$ really can be

\footnote{Associating to any generalized root systems in Serganova’s sense, Sergeev and Veselov introduced in [34] a notion of super Weyl groupoid. Heckenberger and Yamane introduced in [11] a groupoid related to basic classical Lie superalgebras motivated by Serganova work and in [10] the notion of the Weyl groupoid for Nichols algebras. Our super Weyl group has no direct relation with their notions.}
regarded an transformation of $\mathcal{B}$. Notice that for any isotropic root $\theta \in \Delta_1^+$, there exists $w \in W$ such that $w(\theta) \in \Pi$ (cf. [6, Lemma 1.29]). Thereby any isotropic $\theta$ always gives rise to an odd reflection.

**Definition 3.3.** A subgroup of the transform group of $\mathcal{B}$ generated by all real and odd reflections $\hat{r}_\theta$ is called the super Weyl group of $g$, which is denoted by $\hat{W}$.

Proposition 1.32 of [6] shows an important property of $\hat{W}$ that any two fundamental systems of $g$ are conjugate under $\hat{W}$.

It is readily seen that $\hat{W}$ contains the usual Weyl group $W$ of $g_0$, as a subgroup. Actually, for any $w \in W$, we regard $w$ as an element in $\hat{W}$, denoted by $\hat{w}$. Then the map $\varphi : w \mapsto \hat{w}$ defines an group injective homomorphism from $W$ to $\hat{W}$. We only need to check that $\varphi$ is injective. This can be known that if $\hat{w}$ is an identity transform of $\mathcal{B}$, then $\sigma(\Pi) = \Pi$. Owing to [12, Theorem 10.3(e)], $w = 1$. So $\hat{W}$ is generated by $W$ and $\hat{r}_\theta$ for all $\theta \in \Delta_1^+$.

We introduce a so-called extended fundamental root system $\hat{\Pi}$ for $g$ except type $A(m\mid n)$, $B(0\mid n)$, and $C(n)$. For those exceptional cases $A(m\mid n)$, $B(0\mid n)$ and $C(n)$, we set $\hat{\Pi} = \Pi$. Recall the standard fundamental root system $\Pi$ for a basic classical Lie algebras is as follows (cf. [21, §2.5] or [6, §1.3]):

\[
\begin{align*}
\text{spo}(2n\mid 2m + 1), m > 0 : & \quad \Pi = \{\delta_1 - \delta_2, \ldots, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\}; \\
\text{spo}(2n\mid 2m), m \geq 2 : & \quad \Pi = \{\delta_1 - \delta_2, \ldots, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m + \varepsilon_m\}; \\
F(4) : & \quad \Pi = \{\frac{1}{2} (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta), -\varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}; \\
G(3) : & \quad \Pi = \{\delta + \varepsilon_1, \varepsilon_2, \varepsilon_3 - \varepsilon_2\}; \\
D(2, 1, \alpha) : & \quad \Pi = \{\varepsilon_1 + \varepsilon_2 + \varepsilon_3, -2\varepsilon_2, -2\varepsilon_3\}. \\
\end{align*}
\]

(3.2)

We define $\hat{\Pi}$ for $g$ listed in the above table:

\[
\begin{align*}
\text{spo}(2n\mid 2m + 1), m > 0 : & \quad \hat{\Pi} = \{\alpha_0 := -2\delta_1\} \cup \Pi; \\
\text{spo}(2n\mid 2m), m \geq 2 : & \quad \hat{\Pi} = \{\alpha := -2\delta\} \cup \Pi; \\
F(4) : & \quad \hat{\Pi} = \{\alpha_0 := -\delta\} \cup \Pi; \\
G(3) : & \quad \hat{\Pi} = \{\alpha_0 := -2\delta\} \cup \Pi; \\
D(2, 1, \alpha) : & \quad \hat{\Pi} = \{\alpha_0 := -2\varepsilon_1\} \cup \Pi. \quad (3.3)
\end{align*}
\]

Then for the above non-exceptional cases, one has an extended standard Dynkin diagram corresponding to $\hat{\Pi}$ which is such a diagram extending the standard Dynkin diagram associated with $\Pi$, by adding a vertex $\alpha_0$ connected to the first vertex of the standard Dynkin diagram.

Obviously, all real and odd reflections $\hat{r}_\theta$ satisfy $\hat{r}_\theta^2 = \text{id}$. All of them are called super reflections. A super reflection is called a simple one if it arises from a simple root of $\hat{\Pi}$. We have further that $\hat{W}$ is generated by all simple super reflections, in the same way as in the classical Lie algebras case.
Lemma 3.4. The super Weyl group $\hat{W}$ is generated by the simple super reflections, this to say $\hat{W} = \langle \hat{r}_g | \theta \in \hat{\Pi} \rangle$.

Proof. We have known that $\hat{W}$ is generated by $W$ and $\hat{r}_g$ for $\theta \in \hat{\Delta}$. Note that $W$ is generated by the simple reflections corresponding to the fundamental system of $\mathfrak{g}_0$ which is contained in the extended standard simple root system $\hat{\Pi}$. Hence $W \subset \langle \hat{r}_g | \theta \in \hat{\Pi} \rangle$. On the other hand, checking the standard Dynkin diagrams listed in [6] §1.3 (cf. [22] Proposition 1.5), we know that there is only one isotropic simple root in the standard simple root system $\Pi$. So the remaining thing is to check that any two isotropic positive roots $\gamma_i$, $i = 1, 2$ are conjugate by some $w \in W$. By [6] Lemma 1.29, there exist $w_i \in W$ such that $w_i(\gamma_i) \in \Pi$. So $w_1(\gamma_1) = w_2(\gamma_2)$. Hence $\gamma_i$ are conjugate under $W$. The proof is completed. $\square$

Thus $\hat{W}$ is actually a quotient of some Coxeter group associated with the generator system associated with $\hat{\Pi}$. So one can study the super Weyl group by the Coxeter group theory. One can study the presentation of $\hat{W}$ (cf. [4]), the reduced expressions and lengths of any elements, and the relation between the corresponding Hecke algebras, and representations of $\mathfrak{g}$, which will be done somewhere else.

Example 3.5. In the case $\mathfrak{g} = \mathfrak{gl}(1|2)$, partially using the notations from [6] we have $\Delta_0 = \{\pm(\epsilon_1 - \epsilon_2)\}$, and $\Delta_1 = \{\pm(\delta_1 - \epsilon_1), \pm(\delta_1 - \epsilon_2)\}$. There are 6 Borel subalgebras corresponding to 6 systems of simple roots $\{\Pi_i | i = 1, \cdots, 6\}$:

$$\Pi_1 = \{\delta_1 - \epsilon_1, \epsilon_1 - \epsilon_2\}, \quad \Pi_2 = \{\epsilon_1 - \delta_1, \delta_1 - \epsilon_2\}, \quad \Pi_3 = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \delta_1\}$$

$$\Pi_4 = \{\delta_1 - \epsilon_2, \epsilon_2 - \epsilon_1\}, \quad \Pi_5 = \{\epsilon_2 - \delta_1, \delta_1 - \epsilon_1\}, \quad \Pi_6 = \{\epsilon_2 - \epsilon_1, \epsilon_1 - \delta_1\}$$

With the numbers indexing those fundamental root systems, the super Weyl group can be regarded as a subgroup of $S_6$ presented precisely as follows:

$$\hat{W} = \langle \hat{r}_{\delta_1 - \epsilon_1} = (12)(56), \hat{r}_{\epsilon_1 - \epsilon_2} = ((14)(25)(36)) \rangle$$

which is the Coxeter group of type $G_2$.

3.2.1. Standard reduced expressions and the standard lengths of elements in the super Weyl groups. For a given element $\hat{w}$ of $\hat{W}$, set $\Pi' = \hat{w}(\Pi)$. Note that even if $\hat{w} \neq \text{id}$, it could happen that $\Pi' = \Pi$. And $\Pi' \neq \Pi$ if only if $\Phi^- \cap \Pi \neq \emptyset$, where $\Phi^-$ means the negative root system corresponding to $\Pi'$. Two elements $\hat{w}_i$, $i = 1, 2$ are called isogenous if $\hat{w}_1(\Pi) = \hat{w}_2(\Pi)$. Denote by the isogeny class $\hat{W}(\Pi, \Pi')$ of $\hat{w}$ which consists of all $\hat{\sigma}$ sending $\Pi$ into $\Pi' = \hat{w}(\Pi)$. In $\hat{W}(\Pi, \Pi')$, by Lemma 3.4 we can take a representative $\hat{\sigma}$ such that

1. $\hat{\sigma} = \hat{r}_n \cdots \hat{r}_1$ for $\hat{r}_i = \hat{r}_{\theta_i}$.
2. $\theta_i \in \Pi_{i-1}$ where $\Pi_{i-1} = \hat{r}_{i-1} \cdots \hat{r}_1(\Pi)$ for $i = 1, \cdots, n$, and $\Pi_n = \Pi'$.

Denote by $\ell_{\Pi}(\hat{w})$ the smallest one among all possible $n$ as above, called the standard length of $\hat{w}$. The expression of $\hat{\sigma} = \hat{r}_n \cdots \hat{r}_1$ is called standard reduced if it matches the above (1) (2) with $n = \ell_{\Pi}(\hat{w})$. In this case, we call $\sigma$ is a standard representative element of $\hat{W}(\Pi, \Pi')$. Generally, $\hat{W}(\Pi, \Pi')$ is not necessarily to be a subgroup. But $\hat{W}(\Pi, \Pi)$ is a subgroup of $\hat{W}$. 
For any given simple root system $\Pi'$ different from $\Pi$, we will introduce the distance between $\Pi'$ and $\Pi$, denoted by $d(\Pi, \Pi')$.

**Definition 3.6.** Define $d(\Pi, \Pi') = \#(\Delta^- \cap \Delta^+) - \#((\Delta^- \cap \Delta^+) \cap \Delta_i^+) \setminus ((\Delta^- \cap \Delta^+) \cap \Delta_i^+) )$.

Obviously, $d(\Pi, \Pi') = d(\Pi', \Pi)$. Furthermore, $d(\Pi, \Pi') = 0$ if only if $\Delta^- \cap \Delta^+ = \emptyset$, i.e. $\Delta^+ = \Delta^+$, or to say $\Pi = \Pi'$. And $d(\Pi, \Pi') = 1$ if and only if there is one simple super reflection $\hat{r}_{\theta}$ with $\theta \in \Delta^- \cap \Pi$ such that $\Pi' = \hat{r}_{\theta}(\Pi)$ for $\theta \in \Pi$. If $\Pi' \neq \Pi$, then $d(\hat{r}_{\theta}(\Pi), \Pi') = d(\Pi, \Pi') - 1$ if only if and $\theta \in \Delta^- \cap \Pi$.

**Proposition 3.7.** For any given $\hat{w} \in \hat{W}$ and $\Pi' = \hat{w}(\Pi)$, $\ell_{\Pi}(\hat{w}) = d(\Pi, \Pi')$.

Hence the expression of $\hat{w}$ in Lemma 3.2 is standard reduced, while $\hat{w}$ is a standard representative element.

**Proof.** By Lemma 3.2, $\ell_{\Pi}(\hat{w}) \leq d := d(\Pi, \Pi')$. Take a standard representative element $\hat{\sigma} \in \hat{W}(\Pi, \Pi')$, and its reduced expression $\hat{\sigma} = \hat{r}_n \cdots \hat{r}_1$ for $\hat{r}_i = \hat{r}_{\theta_i}$ and $n = \ell_{\Pi}(\hat{w})$. We now prove the statement by induction on $n$. If $n = 0$ then $\hat{\sigma} = \text{id}$, and $\Pi = \Pi'$. We don’t need do anything. Assume that $n > 0$, and that the statement holds for the case when less than $n$. Notice that $\Pi' = \hat{\sigma}(\Pi) = \hat{r}_n \cdots \hat{r}_1(\Pi) = \hat{r}_n(\Pi_{n-1})$. By the inductive hypothesis, $n - 1 \geq d(\Pi, \Pi_{n-1})$. And $d(\Pi, \Pi') = d(\Pi, \hat{r}_{\theta_n}(\Pi_{n-1})) = d(\Pi, \Pi_{n-1}) + 1$ if and only if $\theta_n \in \Delta^+ \cap (-\Pi')$. In the other case, the standard reduced property of $\hat{\sigma} = \hat{r}_n \cdots \hat{r}_1$ implies that $d(\Pi, \Pi') = d(\Pi, \hat{r}_{\theta_n}(\Pi_{n-1})) = d(\Pi, \Pi_{n-1}) - 1$. So in any case, $n - 1 \geq d(\Pi, \Pi_{n-1}) = d - 1$, or $d + 1$, thereby $n \geq d$. Combining the above arguments, we complete the proof for the case equal to $n$, thereby the proof of the relation $\ell_{\Pi}(\hat{w}) \geq d(\Pi, \Pi')$.

**Lemma 3.8.** Maintain the notations as in Lemma 3.2. Then, for $0 < j < i \leq n$, $\hat{r}_i$ is different from $\hat{r}_j$ where $n = d(\Pi, \Pi')$.

**Proof.** For convenience we denote $\Pi = \Pi_0$, $\Pi_{i-1} = \hat{r}_{i-1} \cdots \hat{r}_2 \hat{r}_1(\Pi)$, one can choose $\theta_1 \in (\Delta^-) \cap \Pi_0, \theta_i \in (\Delta^-) \cap \Pi_{i-1}$. Because for any integral number $0 < i \leq n$ $\theta_{i-1}, \cdots, \theta_1 \in \Delta^-_{i-1}$ but $\theta_i \in \Pi_{i-1} \subset \Delta^+_{i-1}$ so we can have that for any integral number $0 < i \leq n$ $\hat{r}_i$ is different with $\hat{r}_j$ for any integral number $0 < j < i$.

### 3.3. Distinguished elements.

**Definition 3.9.** An element $\hat{\sigma}$ of $\hat{W}$ is called a distinguished element if $\hat{\sigma}$ is a standard representative element of $\hat{W}(\Pi, -\Pi)$.

**Theorem 3.10.** Keep the notations as above. Then there exists a distinguished element $\hat{w}_0 = \hat{r}_N \cdots \hat{r}_1 \in \hat{W}$ with $\hat{r}_i = \hat{r}_{\theta_i}$ for a series of positive roots $\theta_i, i = 1, \cdots, N$, satisfying the following axioms:

1. $\theta_i$ is in the simple root system of the Borel subalgebra corresponding to $\hat{\sigma}_{i-1}(\Delta^+)$. Here $\hat{\sigma}_0 := \text{id}$, $\hat{\sigma}_{i-1} = \hat{r}_{i-1} \cdots \hat{r}_1$ ($i > 0$). And $\hat{r}_{i-1}(\Delta^+)$ stands for the positive root system after a series of odd and real reflections $\hat{r}_1, \cdots, \hat{r}_{i-1}$.
(2) \( \hat{w}_0(\Pi) = -\Pi \) and \( \hat{w}_0(\Delta^+) = -\Delta^+ \).
(3) For \( \hat{\sigma}_i = \hat{r}_i \cdots \hat{r}_1 \), the following holds
\[
\hat{\sigma}_i(\Delta^+) = \begin{cases} 
(\hat{\sigma}_{i-1}(\Delta^+) \setminus \{\theta_i\}) \cup \{-\theta_i\}, & \text{if } \theta_i \in \hat{\sigma}_{i-1}(\Delta^+) \cap \bar{\Delta}_0; \\
(\hat{\sigma}_{i-1}(\Delta^+) \setminus \{\theta_i\}) \cup \{-\theta_i\}, & \text{if } \theta_i \in \hat{\sigma}_{i-1}(\Delta^+) \cap \bar{\Delta}_1; \\
(\hat{\sigma}_{i-1}(\Delta^+) \setminus \{2\theta_i, \theta_i\}) \cup \{-2\theta_i, -\theta_i\}, & \text{if } \theta_i \in \hat{\sigma}_{i-1}(\Delta^+) \cap (\Delta_1 \setminus \bar{\Delta}_1). 
\end{cases}
\]
(4) \( N = \#\Delta^+ - \#(\Delta_1^+ \setminus \bar{\Delta}_1) \).
(5) \( \Delta^+ = \{\theta_i \mid i = 1, 2, \ldots, N\} \cup \{2\theta_i \mid \theta_i \in (\Delta_1^+ \setminus \bar{\Delta}_1) \cap \sigma_{i-1}(\Delta^+), i = 1, \ldots, N\} \).

**Proof.** We apply Lemma 3.2 to the case when \( \Pi' = -\Pi \). By Lemma 3.8 we know that \( \hat{r}_{\theta_i} \neq \hat{r}_{\theta_j} \) whenever \( i \neq j \). Furthermore, both \( \theta_i \) and \( 2\theta_i \) never occur simultaneously in \( \{\theta_1, \theta_2, \cdots\} \) if \( \theta_i \in \Delta_1^+ \setminus \bar{\Delta}_1^+ \). So, the choice of those \( \theta_i \) from the arguments in the proof of Lemma 3.2 implies that all positive roots have to be involved when we finish the operation of changing \( \Pi \) into \( -\Pi \) (note that we identify the reflections arising from \( \bar{\Delta}_0 \) with the ones from \( \Delta_1^+ \setminus \bar{\Delta}_1^+ \)). From the above arguments, the statements (1)-(4) follow. Furthermore, combining with Proposition 3.7 we know \( \hat{w}_0 \) is a distinguished element. The proof is completed. \( \square \)

**Remark 3.11.** By Proposition 3.7 and Theorem 3.10 we know that the longest standard length of \( \hat{W} \) is \( N \), and \( \hat{w}_0 \) is of the longest standard length \( N \). This element plays a role as important as in the longest element \( \omega_0 \) in the usual Weyl group \( W \) of a classical Lie algebra. Be careful that the longest standard length may not coincide with the longest length in the corresponding Coxeter system.

### 4. Graded restricted module categories

Keep the notations and assumptions as in the previous two sections. Set \( \Pi \Xi \) to be the free abelian group spanned by \( \Pi \). Then \( g = \mathfrak{h} + \sum_{\alpha \in \Delta} g_\alpha \) is naturally a \( \Pi \Xi \)-graded Lis superalgebra. We can consider a so-called \( \Pi \Xi \)-graded \( g \)-module category which consists of the objects satisfying the \( \Pi \Xi \)-graded structure compatible with \( \Pi \Xi \)-graded structure of \( g \), i.e. for \( M = \sum_{\lambda \in \Pi \Xi} M_\lambda \), \( g_\alpha \cdot M_\lambda \subset M_{\alpha + \lambda} \).

Set
\[
Y := X(\Xi).
\]
By Lemma 2.3 we can extend a \( \Pi \Xi \)-graded module category of \( g \) to some \( Y \)-graded module category of \( g \).

#### 4.1. \((U_0(g), \Xi)\)-module categories

By definition, a rational \( \Xi \)-module \( V \) means a \( Y \)-graded vector space admissible with rational \( \Xi \)-action, i.e. \( V = \sum_{\mu \in Y} V_\mu \), where \( V_\mu = \{v \in V \mid t \cdot v = \mu(t)v, \forall t \in \Xi\} \). Recall that the automorphism group of \( g \) contains a closed subgroup \( \hat{G}_{ev} \). For \( \mu \in Y \), its differential \( d\mu : \mathfrak{h} \rightarrow k \), is a homomorphism of restricted Lie algebras and satisfies \( d\mu(H^{[p]}) = (d\mu(H))^{[p]} \). This means that \( d\mu \in \Lambda := \{\lambda \in \mathfrak{h}^* \mid \chi(H)^p = \lambda(H^{[p]}), \forall H \in \mathfrak{h}\} \). The map \( \phi : \mu \mapsto d\mu \) has kernel \( pY \). And this induces a bijection \( Y/pY \cong \Lambda \). We may identify \( Y/pY \) with \( \Lambda \). Sending \( \mu \in Y \) to \( \bar{\mu} \in \Lambda = Y/pY \), we write \( d\mu(H) \) directly as \( \bar{\mu}(H) \).
without any confusion, and call them restricted weights. (Sometimes, $d\mu$ and $\mu$ are not discriminated in use if no confusion happens in context.)

Naturally, $U_0(\mathfrak{g})$ and its canonical subalgebras which will be used later become rational $\mathfrak{T}$-modules with the action denoted by $\text{Ad}(T)a$ for $T \in \mathfrak{T}$ and $a \in U_0(\mathfrak{g})$.

Let us introduce the full subcategory $(U_0(\mathfrak{g}), \mathfrak{T})\text{-mod}$ of the $U_0(\mathfrak{g})$-module category $U_0(\mathfrak{g})\text{-mod}$:

**Definition 4.1.** The category $(U_0(\mathfrak{g}), \mathfrak{T})\text{-mod}$ is defined as such a category whose objects are finite-dimensional $\mathfrak{k}$-superspaces endowed with both $U_0(\mathfrak{g})$-module and rational $\mathfrak{T}$-module structure satisfying the following compatibility conditions for $V \in (U_0(\mathfrak{g}), \mathfrak{T})\text{-mod}$:

1. The action of $U_0(\mathfrak{h})$ coincides with the action of $\text{Lie}(\mathfrak{T})$ induced from $\mathfrak{T}$.
2. For $a \in U_0(\mathfrak{g})$, $T \in \mathfrak{T}$, and $v \in V$: $T(av) = (\text{Ad}T(a))Tv$.

The morphisms of $(U_0(\mathfrak{g}), \mathfrak{T})\text{-mod}$ are defined to be linear maps of $\mathfrak{k}$-superspaces acting as both $U_0(\mathfrak{g})$-module homomorphisms, and rational $\mathfrak{T}$-module homomorphisms.

**Remark 4.2.** (1) Let $u$ be a Hopf subalgebra of $U_0(\mathfrak{g})$ (or $U(\mathfrak{g})$) with adjoint $\mathfrak{h}$-module structure. We can define a category of $(u, \mathfrak{T})\text{-mod}$ in the same way as above, of which each object is simply called a $\hat{u}$-module.

(2) The $\hat{u}$-module category can be realized a module category of the precise Hopf algebra $\hat{u} = u\#\text{Dist}(\mathfrak{T})$, where $\text{Dist}(\mathfrak{T})$ denote the distribution algebra of $\mathfrak{T}$ as defined in [27].

4.1.1. Baby Verma modules. Naturally, $U_0(\mathfrak{g})$, $U_0(\mathfrak{b})$ become $\mathfrak{T}$-modules, compatible with the restricted $\mathfrak{h}$-module structure. For stressing such structure, we denote them by $\hat{U}_0(\mathfrak{g})$ and $\hat{U}_0(\mathfrak{b})$ respectively. Hence for any $\mu \in \mathcal{Y}$, one naturally has a one-dimensional $\hat{U}_0(\mathfrak{b})$-module $k_\mu$ with trivial $\mathfrak{n}^+$-action, $\mathfrak{h}$-action of multiplication via $d(\mu)$, and $\mathfrak{T}$-action of multiplication via $\mu$. Therefore, we can endow with $\hat{U}_0(\mathfrak{g})$-module structure on

$$\hat{Z}(\mu) = \hat{U}_0(\mathfrak{g}) \otimes_{\hat{U}_0(\mathfrak{b})} k_\mu.$$ 

Once the parity of the one-dimensional space $k_\mu$ is given

say $\epsilon \in \mathbb{Z}_2 = \{0, 1\}$, then the super-structure of $\hat{Z}(\mu)$ is determined by the super structure of $\hat{U}(\mathfrak{n}^-) = \hat{U}(\mathfrak{n}^-)_{\delta+\epsilon} = \hat{U}_0(\mathfrak{n}^-)_{\delta} \otimes k_\mu$ for $\delta \in \mathbb{Z}_2$.

Note that $Y$ is a poset with partial order ”$\leq$”: $\lambda \leq \mu$ if and only if $\mu - \lambda \in \mathbb{Z}_{\geq 0} \Pi$. By the definition, $\hat{Z}(\mu)$ is a highest weight module with highest weight $\mu$. As

---

With the same spirit, there are other formulations different from the category $(U_0(\mathfrak{g}), \mathfrak{T})\text{-mod}$ used here, such as the $G_1\mathfrak{T}$-module category for the first Frobenius kernel (cf. [17] and [20]), the $\mathbb{Z}$-graded $U_0(\mathfrak{g})$-module category (cf. [18, §11]), and the $u(\mathfrak{g})\#\text{Dist}(\mathfrak{T})$-module category (cf. [27], see Remark [12, 2]). Here we follow the formulation used in [36].

For the case of restricted representation categories or $(u, \mathfrak{T})$-module categories, one can give parities for weight spaces similar to [5, §6]
$U_0(\mathfrak{g})$-modules, $Z(\mu)$ is isomorphic to $Z(\bar{\mu}) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} k_{\bar{\mu}}$. Both of them share the same super space structure.

Generally, we can define for any $\lambda \in \Lambda$,

$$Z(\lambda) = U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} k_{\lambda},$$
called a baby Verma module. So there is a natural forgetful functor $\mathcal{F}$ from $(U_0(\mathfrak{g}), \Sigma)$-mod to $U_0(\mathfrak{g})$-mod which sends $Z(\mu)$ to $Z(\mu)$ for any $\mu \in Y$.

Following the same arguments as in [18, Proposition 10.2] or [36, Lemma 2.1], we can easily obtain

**Lemma 4.3.** In the categories $U_0(\mathfrak{g})$-mod and $(U_0(\mathfrak{g}), \Sigma)$-mod, the following statements hold.

1. Both of $Z(\lambda)$ and $Z(\mu)$ for any $\lambda \in \Lambda$ and $\mu \in Y$ admit unique maximal submodules respectively. The unique irreducible quotients are denoted by $L(\lambda)$ and $\hat{L}(\mu)$ respectively.
2. The above $\hat{L}(\mu)$ belongs to $(U_0(\mathfrak{g}), \Sigma)$-mod.

### 4.1.2. Isomorphism classes of irreducible modules in $U_0(\mathfrak{g})$-mod and $(U_0(\mathfrak{g}), \Sigma)$-mod.

**Proposition 4.4.** (1) The iso-classes of irreducible modules in $(U_0(\mathfrak{g}), \Sigma)$-mod are in one-to-one correspondence with $Y$. Precisely, each simple objects in $(U_0(\mathfrak{g}), \Sigma)$-mod is isomorphic to $\hat{L}(\mu)$ for $\mu \in Y$.

(2) $Z(\mu)|_{U_0(\mathfrak{g})} \cong Z(\bar{\mu})$, and $\hat{L}(\mu)|_{U_0(\mathfrak{g})} \cong L(\bar{\mu})$. Furthermore, sending $\mu \in Y$ to $\bar{\mu} \in \Lambda$ gives rise to the surjective map $\hat{L}(\mu) \mapsto L(\bar{\mu})$ from the set of iso-classes of simple objects of $(U_0(\mathfrak{g}), \Sigma)$-mod and to those of $U_0(\mathfrak{g})$-mod.

By the above proposition, good understanding of $(U_0(\mathfrak{g}), \Sigma)$-mod can provide us sufficient information on restricted simple modules of $\mathfrak{g}$.

In this paper, we will focus on $(U_0(\mathfrak{g}), \Sigma)$-mod. For the simplicity of notations, we simply write $\mathcal{C}$ for this category in the most time of the rest of the paper.

### 4.1.3. Base change of $\mathcal{C}$. Keep the above notations. In particular, let $\mathfrak{g}$ be a given basic classical Lie superalgebra over $k$ with $\text{Lie}(G) = \mathfrak{g}$ for an algebraic supergroup $G$ of Chevalley type, $Y = X(\Sigma)$. Let $A$ be a commutative $k$-algebra with unity element. We will extend the category $(U_0(\mathfrak{g}), \Sigma)$-mod over $k$ to the category $(U_0(\mathfrak{g}) \otimes A, \Sigma)$-mod in the same way as in [19], simply denoted by $\mathcal{C}_A$ over $A$, which satisfies the same axioms as in Definition 4.4(i)(ii) together with a given $k$-algebra homomorphism $U(\mathfrak{h}) \rightarrow A$. For this we first introduce $U$ which denotes the quotient of $U(\mathfrak{g})$ by the ideal generated by $x^p_\alpha - x^{[p]}_\alpha$ for $x_\alpha \in \mathfrak{g}_\alpha$, $\forall \alpha \in \Delta_0$. Then we have an isomorphism of vector spaces

$$U_0(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U_0(\mathfrak{n}^+) \rightarrow U.$$

We shall write $U^0$ for the image of $U(\mathfrak{h})$ in $U$. This image is isomorphic to $U(\mathfrak{h})$.

**Definition 4.5.** For a given $k$-algebra homomorphism $\pi : U^0 \rightarrow A$, we define the extended category $\mathcal{C}_A$ of $\mathcal{C}$ which consists of the objects satisfy the condition (1) (2) and (3) as below: for $M = \sum_{\mu \in Y} M_\mu \in \text{obj}(\mathcal{C}_A)$
(1) $M$ is a $\hat{U}_0(g) \otimes A$ module with $Y$-graded structure admissible with $\mathfrak{so}$-action, as shown as formulated in Definition 4.1(i)(ii).

(2) $A \cdot M_\mu \subset M_\mu$.

(3) The action of $\mathfrak{h}$ on each graded component $M_\mu$ is diagonalisable and compatible with $Y$-graded structure and $\pi$, this is to say for $Z := \text{Hom}_k(\mathfrak{h}, A)$,

$$M_\mu = \bigoplus_{\nu \in Z} M_\mu^{(\nu)}$$

satisfying that $M_\mu^{(\nu)} \neq 0$ implies $\nu = \pi + d(\mu)$.

Given a commutative $k$-algebra $A$, a $k$-algebra homomorphism $\pi : U^0 \to A$, and $\lambda \in Y$, by Remark 4.2 we can define a baby Verma module $\hat{Z}_A(\lambda)$ in $\mathcal{C}_A$

$$\hat{Z}_A(\lambda) = \hat{U} \otimes \hat{U}^0 A_\lambda$$

where $A_\lambda = A \otimes k_\lambda$ is endowed with a $\hat{U}^0 \hat{U}^0(\mathfrak{n}^+)$-module structure with $H$-action for $H \in \mathfrak{h}$ as multiplication by $\pi(H) + d(\lambda)(H)$ and trivial $\mathfrak{n}^+$-action, and $A_\lambda$ admits a trivial $\mathfrak{n}^+$-module structure and shares the same parity structure with $k_\lambda$.

5. Baby Verma modules and their twists in $\mathcal{C}_A$

Keep the notations as in the previous sections. For $\sigma \in \hat{W}$, we have a new Borel subalgebra $\sigma(b) = \mathfrak{h} \oplus \sigma(\mathfrak{n}^+)$ where $\sigma(\mathfrak{n}^+) := \sum_{\alpha \in \sigma(\Delta_+)} \mathfrak{g}_\alpha$. Obviously, $\sigma(\mathfrak{n}^+)$ is still a restricted Lie super subalgebra of $g$. Hence we have a twisted baby Verma module for $\lambda \in Y$

$$\hat{Z}_A^\sigma(\lambda) := \hat{U} \otimes \hat{U}^0 A_\lambda$$

In this section, we will make some preparation for constructing Jantzen filtration for baby Verma modules in the category $\mathcal{C}$. More precisely speaking, we need to describe (twisting) homomorphisms in $\mathcal{C}_A$ between a given twisted Verma modules $\hat{Z}_A(\lambda)$ and some of its twisted ones via super reflection $\hat{Z}_A^{\hat{r}_\alpha}(\lambda)$ (see 5.1 for $\lambda$). For this, we only need to understand such homomorphisms from a standard baby Verma modules $\hat{Z}(\lambda)$ to its super-simple-reflection twist $\hat{Z}(\lambda)$. We will exploit the arguments of constructing homomorphisms between twisted baby Verma modules in [19] to the super case, for which we need to deal with extra complicated situations arising from odd roots and odd reflections.

5.1. Homomorphisms between baby Verma modules and their twists via super reflections. Now we begin with $\hat{Z}_A(\lambda)$. For a given super simple reflection $\hat{r} := \hat{r}_\alpha$ for $\alpha \in \Pi$, we will define different twisting homomorphisms $\varphi$ from $\hat{Z}_A(\lambda)$ to $\hat{Z}_A^{\hat{r}_\alpha}(\lambda)$, according to the different type for $\alpha$ (the definition of $\lambda^{\hat{r}_\alpha}$ is given in 5.1). We construct those homomorphisms, according to three different cases of super simple reflections.

Recall the Chevalley bases introduced in §2.2 and the Weyl vector $\rho = \rho_0 - \rho_1$ where $\rho_0 = \frac{1}{2} \sum_{\theta \in \Delta_0^+} \theta$ and $\rho_1 = \frac{1}{2} \sum_{\theta \in \Delta_1^+} \theta$. We set $\hat{r}_0 \rho_0 := \frac{1}{2} \sum_{\theta \in \hat{r}_0(\Delta)_0^+} \theta$
and \( \hat{r}_\alpha \rho_1 := \frac{1}{2} \sum_{\theta \in \sigma_1(\Delta)^+} \theta \) for \( \hat{r}_\alpha(\Delta)^+ = \hat{r}_\alpha(\Delta)^+ \cup \hat{r}_\alpha(\Delta)^+ \) the positive root set associated with the simple root system \( \hat{r}_\theta(\Pi) \). Then for \( \lambda \in Y \), we set

\[
\lambda^{\hat{r}_\alpha} := (\lambda - (p-1)(\rho_0 - \hat{r}_\alpha \rho_0) - (\rho_1 - \hat{r}_\alpha \rho_1)).
\]

In this section, we will give and investigate a homomorphism

\[
\varphi : \hat{Z}_A(\lambda) \to \hat{Z}_A^{\hat{r}_\alpha}(\lambda^{\hat{r}_\alpha})
\]

whose exact definition will be presented sequentially, according to the different types for \( \alpha \).

5.1.1. (Case 1): \( \alpha \in \Pi \cap \Delta_0 \) an even simple root, but half of which is not a root.
In this case, \( \lambda^{\hat{r}_\alpha} = \lambda - (p-1)\alpha \). We can construct a homomorphism \( \varphi \) by the same way as in [19, Lemma 3.4]:

\[
\varphi : \hat{Z}_A(\lambda) \to \hat{Z}_A^{\hat{r}_\alpha}(\lambda - (p-1)\alpha))
\]
given by \( \varphi(1 \otimes 1) = X_\alpha^{p-1} \otimes 1 \), and a converse homomorphism

\[
\varphi' : Z_A^{\hat{r}_\alpha}(\lambda - (p-1)\alpha) \to Z_A(\lambda)
\]
given by \( \varphi'(1 \otimes 1) = Y_\alpha^{p-1} \otimes 1 \). Here and thereafter \( 1 \otimes 1 \) stands for the highest weight vector in its baby Verma module in \( C_A \).

Then we have the following result by the same arguments as in modular representations of reductive Lie algebras, which is due to Jantzen.

**Lemma 5.1.** ([19, Lemma 3.5]) Assume \( A = k \) and \( \pi(\mathfrak{h}) = 0 \). Take \( d \in \{0, 1, \cdots, p-1\} \) such that \( d = \langle \lambda, \chi_\alpha \rangle (\text{mod } p) \) where \( \alpha \) is as in (Case 1). Then the following statements hold.

1. If \( d = p-1 \), then \( \varphi \) and \( \varphi' \) are isomorphisms.
2. If \( d < p-1 \), \( \ker \varphi = \text{im} \varphi' \cong \text{coker} \varphi \) and \( \ker \varphi' = \text{im} \varphi \cong \text{coker} \varphi' \).

Furthermore, we have an exact sequence

\[
\cdots \to \hat{Z}(\lambda - (p+d+1)\alpha) \to \hat{Z}(\lambda - p\alpha) \to \hat{Z}(\lambda - (d+1)\alpha) \to \ker \varphi \to 0.
\]

5.1.2. (Case 2): \( \alpha \in \Pi \cap \Delta_1 \) a simple isotropic odd root. In this case, \( \lambda^{\hat{r}_\alpha} = \lambda - \alpha \).
We can construct a homomorphism

\[
\varphi : \hat{Z}_A(\lambda) \to \hat{Z}_A^{\hat{r}_\alpha}(\lambda - \alpha) \quad \text{and} \quad \varphi' : Z_A^{\hat{r}_\alpha}(\lambda - \alpha) \to Z_A(\lambda)
\]
given by \( \varphi(1 \otimes 1) = X_\alpha \otimes 1 \) and \( \varphi'(1 \otimes 1) = Y_\alpha \otimes 1 \) respectively.

Set \( v_0 = 1 \otimes 1, v_1 = Y_\alpha \otimes 1 \in \hat{Z}_A(\lambda) \) and \( v_0' = 1 \otimes 1, v_1' = X_\alpha \otimes 1 \in \hat{Z}_A^{\hat{r}_\alpha}(\lambda - \alpha) \).

By a straightforward calculation we have

\[
\varphi(v) = \begin{cases} 
  v_1', & v = v_0; \\
  (\pi(H_\alpha) + \lambda(H_\alpha))v_0', & v = v_1;
\end{cases}
\]

\[
\varphi'(v') = \begin{cases} 
  v_1', & v' = v_0'; \\
  (\pi(H_\alpha) + \lambda(H_\alpha))v_0, & v' = v_1'.
\end{cases}
\]

We have the following lemma.
Lemma 5.2. Assume that \( \alpha \) is as in (Case 2), \( A = k \) and \( \pi(\mathfrak{h}) = 0 \). Still take \( d \in \{0, 1, \ldots, p - 1\} \) such that \( d = \langle \lambda, \chi_\alpha \rangle (\mod p) \). The following statements hold.

1. If \( d \neq 0 \), i.e. \( \langle \lambda, \chi_\alpha \rangle \not\equiv 0(\mod p) \), then \( \varphi \) and \( \varphi' \) are isomorphisms.
2. If \( d = 0 \), i.e. \( \langle \lambda, \chi_\alpha \rangle \equiv 0(\mod p) \), then
   \[
   \ker \varphi = \text{im}\varphi' \cong \text{coker}\varphi; \quad \ker \varphi' = \text{im}\varphi \cong \text{coker}\varphi'.
   \]

Furthermore, we have an exact sequence
   \[
   \cdots \rightarrow \hat{Z}(\lambda - 2\alpha) \rightarrow \hat{Z}(\lambda - \alpha) \rightarrow \ker \varphi \rightarrow 0
   \]
in \( \mathfrak{c} \).

Proof. (1) Set
   \[
   m := \bigoplus_{\beta \in \Delta^+, \beta \neq \alpha} \mathfrak{g}_\beta.
   \]
Then \( m \) is a restricted Lie super subalgebra of \( \mathfrak{n}^- \). Take a basis \( B \) of \( U_0(m) \). Then \( \{mv_i \mid i = 0, 1; m \in B\} \) constitute a basis of \( \hat{Z}(\lambda) \). And \( \{mv'_i \mid i = 0, 1; m \in B\} \) constitute a basis of \( \hat{Z}^r_\alpha(\lambda - \alpha) \). Thus, by \( \text{[5.2]} \) we know that \( \varphi \) and \( \varphi' \) are isomorphisms when \( \langle \lambda, \chi_\alpha \rangle \not\equiv 0(\mod p) \). The statement (1) is proved.

(2) Now suppose \( \langle \lambda, \chi_\alpha \rangle \equiv 0(\mod p) \). Then by \( \text{[5.2]} \) again, we have
   \[
   \ker \varphi = \text{k-span}\{Bv_1\}, \quad \text{im}\varphi = \text{k-span}\{Bv'_1\},
   \]
   \[
   \ker \varphi' = \text{k-span}\{Bv'_1\}, \quad \text{im}\varphi' = \text{k-span}\{Bv_1\}.
   \]
So we have
   \[
   \text{coker}\varphi = \frac{\hat{Z}^\psi_\alpha(\lambda - \alpha)}{\text{im}\varphi} = \frac{\hat{Z}^r_\alpha(\lambda - \alpha)}{\ker \varphi'} \cong \text{im}\varphi'.
   \]
Similarly, \( \text{coker}\varphi' \cong \text{im}\varphi \). The statement (2) is proved.

(3) We continue to consider homomorphisms \( \varphi_1 : \hat{Z}(\lambda - 2\alpha) \rightarrow \hat{Z}(\lambda - \alpha) \) via \( \varphi_1(1 \otimes 1) = Y_\alpha \otimes 1 \); and \( \varphi_2 : \hat{Z}(\lambda - \alpha) \rightarrow \hat{Z}(\lambda) \) via \( \varphi_2(1 \otimes 1) = Y_\alpha \otimes 1 \). Then we have an exact sequence
   \[
   \cdots \rightarrow \hat{Z}(\lambda - 2\alpha) \xrightarrow{\varphi_2} \hat{Z}(\lambda - \alpha) \xrightarrow{\varphi_1} \ker \varphi \rightarrow 0.
   \]
We complete the proof. \( \square \)

5.1.3. (Case 3): \( \alpha \in \Pi \cap (\Delta_1 \setminus \overline{\Delta}_1) \). In this case, \( 2\alpha \in \Delta_0 \), and \( \hat{r}_\alpha = \hat{r}_{2\alpha} \). Then \( \lambda^{\hat{r}_\alpha} = \lambda - (2p - 1)\alpha \). We can define homomorphisms
   \[
   \varphi : \hat{Z}_A(\lambda) \rightarrow \hat{Z}^{\hat{r}_\alpha}_A(\lambda - (2p - 1)\alpha)
   \]
via \( \varphi(1 \otimes 1) = X^{2p-1}_\alpha \otimes 1 \) and
   \[
   \varphi' : \hat{Z}^{\hat{r}_\alpha}_A(\lambda - (2p - 1)\alpha) \rightarrow \hat{Z}_A(\lambda)
   \]
via \( \varphi'(1 \otimes 1) = Y^{2p-1}_\alpha \otimes 1 \).
For $0 \leq i \leq 2p - 1$, set $v_0 := 1 \otimes 1_\lambda$, $v_i := Y^i_\alpha \otimes 1 \in \hat{Z}_A(\lambda)$ and $v'_0 := 1 \otimes 1$, $v'_i = X^i_\alpha \otimes 1 \in \hat{Z}_A^r(\lambda - (2p - 1)\alpha)$ for $i = 1, \cdots, 2p - 1$. Simple calculation show for $i > 0$ that

$$X^i_\alpha.Y^i_\alpha \otimes 1 = \begin{cases} -iY^{i-1}_\alpha \otimes 1, & \text{when } i \text{ is even} \\ (\pi(H) + (i - 1))Y^{i-1}_\alpha \otimes 1, & \text{when } i \text{ is odd} \end{cases}$$

in $\hat{Z}_A(\lambda)$ and

$$Y^i_\alpha.X^i_\alpha \otimes 1 = \begin{cases} iX^{i-1}_\alpha \otimes 1, & \text{when } i \text{ is even} \\ (\pi(H) + i + 1)X^{i-1}_\alpha \otimes 1, & \text{when } i \text{ is odd} \end{cases}$$

in $\hat{Z}_A^r(\lambda - (2p - 1)\alpha)$. Thus we can get the following by induction on $i$:

$$\varphi(v_i) = \begin{cases} \prod_{j=1}^2(\pi(H) + (i - 2(p - j))v'_{2p-1-i}, & \text{when } i \text{ is even} \\ \prod_{j=1}^2(\pi(H) + (i - 2(p - j)))v'_{2p-1-i}, & \text{when } i \text{ is odd} \end{cases}$$

$$\varphi'(v'_i) = \begin{cases} (-1)^{\frac{i}{2}}\prod_{j=1}^2(\pi(H) + (2(p - j))v'_{2p-1-i}, & \text{when } i \text{ is even} \\ (-1)^{\frac{i+1}{2}}\prod_{j=1}^2(\pi(H) + (2(p - j)))v'_{2p-1-i}, & \text{when } i \text{ is odd} \end{cases}$$

(5.4)

Then we have the following

**Lemma 5.3.** Assume that $A = k$ and $\pi(\mathfrak{h}) = 0$. Take $d \in \{0, 1, \cdots, p - 1\}$ such that $\langle \lambda, \chi_\alpha \rangle \equiv d(\text{mod } p)$ where $\alpha$ is as in (Case 3).

1. $\ker \varphi = \text{im} \varphi' \cong \text{coker} \varphi$ and $\ker \varphi' = \text{im} \varphi \cong \text{coker} \varphi'$.

2. Furthermore, we have an exact sequence

   (i) When $d$ is an odd number,

   $$\cdots \to \hat{Z}(\lambda - (3p + d + 1)\alpha) \to \hat{Z}(\lambda - 2p\alpha) \to \hat{Z}(\lambda - (p + d + 1)\alpha) \to \ker \varphi \to 0;$$

   (ii) When $d$ is an even number,

   $$\cdots \to \hat{Z}(\lambda - (2p + d + 1)\alpha) \to \hat{Z}(\lambda - 2p\alpha) \to \hat{Z}(\lambda - (d + 1)\alpha) \to \ker \varphi \to 0;$$

**Proof.** Still consider the restricted Lie super subalgebra of $\mathfrak{n}^-$:

$$\mathfrak{m} = \bigoplus_{\beta \in \Delta, \beta \notin \Lambda_\alpha} \mathfrak{g}_{-\beta}$$

where $\mathbb{N}$ stands for the positive integer set. Take a basis $B$ of $U_0(\mathfrak{m})$. Then $\{mv_i | i = 0, 1, \cdots, 2p - 1; m \in B\}$ constitute a basis of $\hat{Z}(\lambda)$. And $\{mv'_i | i = 0, 1, \cdots, 2p - 1; m \in B\}$ constitute a basis of $\hat{Z}_A^r(\lambda - \alpha)$.

Under the assumption, the formulas in (5.4) become

$$\varphi(v_i) = \begin{cases} \prod_{j=1}^2(d + 2j)\prod_{j=1}^2(2(p - j))v'_{2p-1-i}, & \text{when } i \text{ is even} \\ \prod_{j=1}^2(d + 2j)\prod_{j=1}^2(2(p - j))v'_{2p-1-i}, & \text{when } i \text{ is odd} \end{cases}$$
and
\[
\varphi'(v_i') = \begin{cases}
(-1)^i \prod_{j=1}^{i} (d + 2j) \prod_{j=1}^{\frac{i}{2}} (2(p-j)) v_{2p-1-i}, & \text{when } i \text{ is even} \\
(-1)^{\frac{i-1}{2}} \prod_{j=1}^{\frac{i+1}{2}} (d + 2j) \prod_{j=1}^{\frac{i+1}{2}} (2(p-j)) v_{2p-1-i}, & \text{when } i \text{ is odd}.
\end{cases}
\]

From the above formula, by a direct computation we have the following observations
(i) \(\text{im}\varphi'\) is spanned by \(\{mv_i' \mid i \geq p + d + 1, m \in B\}\) if \(d\) is odd, and spanned by \(\{mv_i' \mid i = 1, \ldots, 2p-1, m \in B\}\) if \(d = 0\); by \(\{mv_i' \mid i \geq d + 1, m \in B\}\) if \(d\) is a positive even number;
(ii) \(\ker\varphi'\) is spanned by \(\{mv_i' \mid i \geq p - d - 1, m \in B\}\) if \(d\) is odd; and by \(mv_{2p-1}'\) if \(d = 0\); and by \(\{mv_i' \mid i \geq 2p - d - 1, m \in B\}\) if \(d\) is a positive even number;
(iii) \(\ker\varphi\) is spanned by \(\{mv_i \mid i \geq p + d + 1, m \in B\}\) if \(d\) is odd, and by \(\{mv_i \mid i = 1, \ldots, 2p-1, m \in B\}\) if \(d = 0\); spanned by \(\{mv_i \mid i \geq d + 1, m \in B\}\) if \(d\) is a positive even number.
(iv) \(\text{im}\varphi\) is spanned by \(\{mv_i' \mid i \geq p - d - 1, m \in B\}\) if \(d\) is odd, and by \(mv_{2p-1}'\) if \(d = 0\), and by \(\{mv_i' \mid i \geq 2p - d - 1, m \in B\}\) if \(d\) is a positive even number.

Then we have
\[
\ker\varphi' = \text{im}\varphi = \begin{cases}
\sum_{m \in B} \sum_{i=p-d+1}^{2p-1} mv_i', & \text{when } d \text{ odd}, \\
\sum_{m \in B} \sum_{i=2p-d}^{2p-1} mv_i', & \text{when } d \text{ even}.
\end{cases}
\]

Therefore, we have
\[
coker\varphi = \hat{Z}^{\hat{r}_A}(\lambda - (2p-1)\alpha)/\text{im}\varphi \\
= \hat{Z}^{\hat{r}_A}(\lambda - (2p-1)\alpha)/\ker\varphi' \\
\cong \text{im}\varphi' = \ker\varphi.
\]

Similarly, \(\text{coker}\varphi' \cong \ker\varphi'\). We complete the proof of (1).

For the proof of (2), we divide our argument into two cases. When \(d\) is an odd number, \(\ker\varphi\) is generated by \(v_{p+d+1}\). Thus, we can construct the following sequence of homomorphisms in \(C\)
\[
\cdots \xrightarrow{\varphi_5} \hat{Z}(\lambda - 4\alpha \alpha) \xrightarrow{\varphi_4} \hat{Z}(\lambda - (3p + d + 1)\alpha) \\
\xrightarrow{\varphi_3} \hat{Z}(\lambda - 2\alpha \alpha) \xrightarrow{\varphi_2} \hat{Z}(\lambda - (p + d + 1)\alpha) \xrightarrow{\varphi_1} \ker\varphi \longrightarrow 0
\]
where \(\varphi_i, i = 1, 2, 3, \ldots\) are defined via \(\varphi_{1}(1 \otimes 1) = v_{p+d+1}\), \(\varphi_{2j}(1 \otimes 1) = Y_{\alpha}^{p-d-1} \otimes 1\), and \(\varphi_{2j+1}(1 \otimes 1) = Y_{\alpha}^{p+d+1} \otimes 1, j = 1, 2, \ldots\) respectively. It is easy to check that this sequence is exact.

When \(d\) is an even, \(\ker\varphi\) is generated by \(v_{d+1}\). We construct the following sequence of homomorphisms in \(C\)
\[
\cdots \xrightarrow{\varphi_5} \hat{Z}(\lambda - 4\alpha \alpha) \xrightarrow{\varphi_4} \hat{Z}(\lambda - (2p + d + 1)\alpha) \\
\xrightarrow{\varphi_3} \hat{Z}(\lambda - 2\alpha \alpha) \xrightarrow{\varphi_2} \hat{Z}(\lambda - (d + 1)\alpha) \xrightarrow{\varphi_1} \ker\varphi \longrightarrow 0
\]
where \( \varphi_1 \) is defined via \( \varphi_1(1 \otimes 1) = \nu_{d+1} \), furthermore, \( \varphi_i, i = 2, 3, \cdots \) are defined via \( \varphi_{2j}(1 \otimes 1) = Y_{\alpha}^{2p-d-1} \otimes 1 \), and \( \varphi_{2j+1}(1 \otimes 1) = Y_{\alpha}^{d+1} \otimes 1 \), \( j = 1, 2, \cdots \) respectively. It is easy to check that this sequence is exact. We complete proof.

\[ \square \]

**Remark 5.4.** The lemma implies that both (twisted) baby Verma modules arising from \( \varphi \) define the same class in the Grothendieck group of \( \mathcal{C} \).

5.2. **Base change of exact sequences.** For given two commutative \( \mathfrak{k} \)-algebras \( A' \) and \( A'' \), and a \( \mathfrak{k} \)-algebra homomorphism \( f : A' \to A'' \), we have a functor from \( \mathcal{C}_{A'} \) to \( \mathcal{C}_{A''} \) with \( f(M) = M \otimes_{A'} A'' \). Especially, for (twisted) baby Verma modules we have \( \hat{Z}^\varphi_{A''}(\lambda) = \hat{Z}^\varphi_{A'}(\lambda) \otimes_{A'} A'' \).

5.2.1. From now on, we will always set \( A \) to be the localization of the polynomial ring \( \mathfrak{k}[t] \) in indeterminant \( t \) at the maximal ideal generated by \( t \) (unless other statement). Then \( A = \mathfrak{k} + At \). There is a natural homomorphism \( p : A \to A/At \cong \mathfrak{k} \). So we get a functor from \( \mathcal{C}_A \) to \( \mathcal{C}_k \). Here \( \mathcal{C} = \mathcal{C}_k \) is actually the category \((U(\mathfrak{g}), \mathfrak{T})\)-mod.

**Lemma 5.5.** Assume that \( A \) as above, and \( \pi(H_\alpha) = ct \) for \( c \in \mathfrak{k}^\times \). The following statements hold.

(1) The diagrams below of homomorphisms in \( \mathcal{C}_A \) and \( \mathcal{C} \) corresponding to different cases (Cases 1-3) listed in the previous subsection are commutative:

in (Case 1),

\[
\begin{array}{c}
\hat{Z}_A(\lambda) \xrightarrow{\varphi} \hat{Z}^\varphi_{A'}(\lambda-(p-1)\alpha) \quad \text{coker} \varphi \quad \to \quad 0 \\
\downarrow \quad \downarrow \\
\hat{Z}(\lambda) \xrightarrow{\overline{\varphi}} \hat{Z}^\varphi_{A'}(\lambda-(p-1)\alpha) \quad \text{coker} \overline{\varphi} \quad \to \quad 0; \\
\end{array}
\]  

(5.7)

in (Case 2),

\[
\begin{array}{c}
\hat{Z}_A(\lambda) \xrightarrow{\varphi} \hat{Z}^\varphi_{A'}(\lambda-\alpha) \quad \text{coker} \varphi \quad \to \quad 0 \\
\downarrow \quad \downarrow \\
\hat{Z}(\lambda) \xrightarrow{\overline{\varphi}} \hat{Z}^\varphi_{A'}(\lambda-\alpha) \quad \text{coker} \overline{\varphi} \quad \to \quad 0; \\
\end{array}
\]  

(5.8)

in (Case 3),

\[
\begin{array}{c}
\hat{Z}_A(\lambda) \xrightarrow{\varphi} \hat{Z}^\varphi_{A'}(\lambda-(2p-1)\alpha) \quad \text{coker} \varphi \quad \to \quad 0 \\
\downarrow \quad \downarrow \\
\hat{Z}(\lambda) \xrightarrow{\overline{\varphi}} \hat{Z}^\varphi_{A'}(\lambda-(2p-1)\alpha) \quad \text{coker} \overline{\varphi} \quad \to \quad 0 \\
\end{array}
\]  

(5.9)

where \( \overline{\varphi} \) has the same meaning as the \( \varphi \) from Lemma 5.3, the homomorphisms in the first two columns arises from base change.

(2) The third vertical homomorphisms in each case are bijective.

**Proof.** (I) For (Case 1), the same arguments can be done as in [19]. For the remaining cases, we easily shows the existence of the diagram because we can identify both \( \hat{Z}(\lambda) \) and \( \hat{Z}^\varphi_{A'}(\mu) \) with \( \hat{Z}_A(\lambda) \otimes_A (A/At) \) and \( \hat{Z}^\varphi_{A'}(\mu) \otimes_A (A/At) \) respectively. So we only need to verify that the induced maps between cokernels are bijective for the remaining cases.
Remark 5.7. There exists a $\mathbf{k}$-algebra homomorphism $\pi : U^0 \to \mathbf{k}[t]$ such that $\pi(H_\alpha) = c_\alpha t$, $\forall \alpha \in \Delta^+$ for some $c_\alpha \in \mathbf{k}^\times$. Actually, $\mathbf{k}$ is infinite while $\Delta^+$ is finite. So we can find a linear function $\pi_0 : \mathfrak{h} \to \mathbf{k}$ such that $\pi_0(H_\alpha) \neq 0$ for all $\alpha \in \Delta^+$.  

(II) We begin with (Case 2). By (5.2) we have

$$\varphi(Amv) = \begin{cases} Amv_1', & v = v_0, \\ A(ct + d)mv_0', & v = v_1; \end{cases}$$

$$\varphi'(Amv') = \begin{cases} Amv_1, & v' = v_0', \\ A(ct + d)mv_0, & v' = v_1', \end{cases}$$

(5.10)

where $m$, $v$ and $v'$ are the same as in the proof of Lemma 5.2. When $\langle \lambda, \chi_\alpha \rangle \neq 0 (\mod p)$, $ct + d$ is a unit in $A$. In this case both $\varphi$ and $\varphi'$ are isomorphisms. Hence $\coker \varphi = 0 = \coker \varphi'$. Assume $\langle \lambda, \chi_\alpha \rangle \equiv 0 (\mod p)$, then $\coker \varphi$ is spanned by all $(A/At)mv_0'$ with $m \in B$. Comparing with (5.5), we have $\coker \varphi$ is mapped bijectively to $\coker \varphi'$.

(III) We now discuss (Case 3). By the arguments in (5.1.3) we have for any $0 \leq i \leq 2p - 1$, $m \in B$

$$\varphi(mv_i) = \begin{cases} \prod_{j=1}^i (ct + d + 2 - 2j) \prod_{j=1}^{2p-i} (2(p - j)) mv_{2p-1-i}', & \text{when } i \text{ is even} \\ \prod_{j=1}^i (ct + d + 2 - 2j) \prod_{j=1}^{2p-i} (2(p - j)) mv_{2p-1-i}', & \text{when } i \text{ is odd}, \end{cases}$$

(5.11)

where $B$ is a basis of $U_0(m)$ as described in the proof of Lemma 5.3 and $d = \langle \lambda, \chi_\alpha \rangle (\mod p)$, $0 \leq i \leq 2p - 1$. In the above formula, $ct + d + 2 - 2j$ is a unit in $A$ if $d + 2 - 2j \neq 0 (\mod p)$, otherwise it is a unit times $t$ in $A$. We continue our arguments into three cases according to the possibilities of $d$.

(i) When $d$ is an odd number, we have

$$\varphi(Amv_i) = \begin{cases} Amv_{2p-1-i}', & 0 \leq i < p + d + 1, \\ Amv_{2p-1-i}', & p + d + 1 \leq i \leq 2p - 1, \end{cases}$$

(5.12)

therefore $\coker \varphi$ is spanned by all $(A/At)mv_i'$ with $m \in B$ and $i \leq p - d - 2$. Comparing with (5.5), we have $\coker \varphi \cong \coker \varphi'$.

(ii) When $d$ is an even number, we have

$$\varphi(Amv_i) = \begin{cases} Amv_{2p-1-i}', & 0 \leq i < d + 1, \\ Amv_{2p-1-i}', & d + 1 \leq i \leq 2p - 1, \end{cases}$$

(5.13)

therefore $\coker \varphi$ is spanned by all $(A/At)mv_i'$ with $m \in B$ and $i \leq 2p - d - 2$. Comparing with (5.5), we have $\coker \varphi$ is mapped bijectively to $\coker \varphi'$.

Combining the above, we complete the proof. 

Remark 5.6. In the above lemma, we can regard $\mathcal{C}$ as a full subcategory of $\mathcal{C}_A$ (see the arguments in the beginning paragraph of the next section). Hence all maps of the diagrams are morphisms in $\mathcal{C}_A$. In particular, $\coker(\varphi) \xrightarrow{\cong} \coker(\varphi')$ in $\mathcal{C}_A$. 

Remark 5.7. There exists a $\mathbf{k}$-algebra homomorphism $\pi : U^0 \to \mathbf{k}[t]$ such that $\pi(H_\alpha) = c_\alpha t$, $\forall \alpha \in \Delta^+$ for some $c_\alpha \in \mathbf{k}^\times$. Actually, $\mathbf{k}$ is infinite while $\Delta^+$ is finite. So we can find a linear function $\pi_0 : \mathfrak{h} \to \mathbf{k}$ such that $\pi_0(H_\alpha) \neq 0$ for all $\alpha \in \Delta^+$. 

Extend \( \pi_0 \) to be a \( k \)-algebra homomorphism \( \pi \) from \( U^0 \) to \( k[t] \) by mapping \( H_\alpha \) to \( \pi_0(H_\alpha)t \). Then \( \pi \) is desired.

5.2.2. We look at some inverse images of \( \varphi \). In Case 1, \( \alpha \in \Delta_0^+ \). By the same arguments as reducive Lie algebra case ([19], §3.6) we have for \( m > 0 \),
\[
\varphi^{-1}(t^m \hat{Z}_A^\alpha(\lambda - (p - 1)\alpha) \subset t^{m-1} \hat{Z}_A(\lambda)),
\]
and we can say more that \( \varphi^{-1}(t^m \hat{Z}_A^\alpha(\lambda - (p - 1)\alpha) = t^m \hat{Z}_A(\lambda) \) once \( d = p - 1 \) for \( d \) as in Lemma 5.1.

In Case 2, \( \alpha \in \Delta_1^+ \). By (5.10), we have for \( m > 0 \), \( \varphi^{-1}(t^m \hat{Z}_A^\alpha(\lambda - \alpha) \subset t^{m-1} \hat{Z}_A(\lambda)) \), and we can say more that \( \varphi^{-1}(t^m \hat{Z}_A^\alpha(\lambda - \alpha) = t^m \hat{Z}_A(\lambda) \) once \( d \neq 0 \).

In Case 3, \( \alpha \in \Delta_1^+ \setminus \Delta_1^+ \). By (5.12) and (5.13) we have for \( m > 0 \), \( \varphi^{-1}(t^m \hat{Z}_A^\alpha(\lambda - (2p - 1)\alpha) \subset t^{m-1} \hat{Z}_A(\lambda)) \).

6. JANTZEN FILTRATION AND SUM FORMULA

In this section, we will construct a Jantzen filtration and formulate a sum formula for baby Verma modules on \( \mathcal{C} \). For this, we first make some preparations on \( \mathcal{C}_A \).

6.1. Throughout this section, we still set \( A \) to be the localization of \( k[t] \) at the maximal ideal generated by \( t \). Let \( Q(A) \) stand for the fractional field of \( A \). Note that there is a canonical \( k \)-algebra embedding \( k[t] \hookrightarrow A \). We can choose a homomorphism \( \pi : U^0 \to A \) satisfying the condition in Remark 5.7. This is to say, \( \pi(h) \subset A t \) satisfy \( \pi(H_\alpha) = c_\alpha t \), \( c_\alpha \in k^x \). Then each objects in \( \mathcal{C} \) can be regarded as an object in \( \mathcal{C}_A \) via the surjection \( A \) onto \( A/tA \cong k \). So we can regard \( \mathcal{C} \) as a full subcategory of \( \mathcal{C}_A \) consisting of all objects \( M \) with \( tM = 0 \). It is obviously that the simple objects of \( \mathcal{C} \) is also simple ones in \( \mathcal{C}_A \). Conversely, if \( M \in \text{obj}(\mathcal{C}_A) \) is simple in \( \mathcal{C}_A \) then \( tM = 0 \) or \( tM = M \). However we cannot have \( M = tM \) by Nakayama lemma because \( M \neq 0 \) and \( M \) is finitely generated over \( A \). So we have \( tM = 0 \), and \( M \) is an object from \( \mathcal{C} \), which means that \( M \) is also simple in \( \mathcal{C} \).

Next we introduce a new category \( \text{trsn}(\mathcal{C}_A) \) as in the reductive Lie algebra case (cf. [19]), which is the subcategory of \( \mathcal{C}_A \) and consist with torsion modules in \( \mathcal{C}_A \). The category \( \text{trsn}(\mathcal{C}_A) \) can be described as the full subcategory of all objects of finite length in \( \mathcal{C}_A \) (cf. [19], §3.7). Denote by \( K(\text{trsn}(\mathcal{C}_A)) \) the Grothendieck group of the category \( \text{trsn}(\mathcal{C}_A) \).

6.2. Let \( M \) and \( M' \) be torsion free modules over \( A \). If \( \varphi : M \to M' \) satisfies
\[
\varphi \otimes \text{id}_{Q(A)} : M \otimes Q(A) \to M' \otimes Q(A)
\]
is an isomorphism. Then
\[
\text{coker} \varphi \in \text{obj}(\mathcal{C}_A).
\]

Define
\[
\nu(\varphi) := [\text{coker} \varphi] \in K(\text{trsn}(\mathcal{C}_A)).
\]
If \( \psi : M' \to M'' \) is another homomorphism of torsion free \( A \)-modules satisfying (6.1), then we have
\[
\nu(\psi \circ \varphi) = \nu(\psi) + \nu(\varphi).
\]
The injectivity of $\psi$ shows that
$$0 \to \text{coker} \varphi \to \text{coker}(\psi \circ \varphi) \to \text{coker} \psi \to 0$$
is an exact sequence.

We note that $\overline{\varphi} : \overline{M} \to \overline{M}'$ is homomorphism induced by $\varphi$, where $\overline{M} = M/tM$ and $\overline{M}' = M'/tM'$. We denote the natural functor from $\mathcal{C}_A$ to $\mathcal{C}$ by $p$, i.e. $p(M) = M$. We can define
$$M^i = \{m \in M \mid \varphi(m) \in t^i M'\}$$
and set $\overline{M}'$ to be the image of $M^i$ in $\overline{M}$. Then $\overline{M} = \overline{M}' = \overline{M}$. It is obvious that $\overline{M}^i$ is the kernel of $\overline{\varphi} : \overline{M} \to \overline{M}'$. Hence
$$\overline{M}/\overline{M}^i \cong \text{im}(\overline{\varphi}).$$

We turn to other terms in the filtration $\{\overline{M}^i\}$. In view of $M \in \text{obj}trsn(\mathcal{C}_A)$, there exists $n$ such that $t^n M = 0$. And $M^i$ satisfy that $M = M^0 \supset M^1 \supset \cdots \supset M^n = 0$.

so the filtration $\{\overline{M}^i\}$ of $\overline{M} = M/tM$ is of finite length.

We also have the next lemma

\textbf{Lemma 6.1.} $\sum_{i=1}^n [\overline{M}^i] = [\nu(\varphi)].$

\textit{Proof.} It can be proved by the same arguments as in \cite{[19]} Lemma 3.8. \hfill \Box

\section{Jantzen filtration for baby Verma modules.}

\subsection{By Theorem [3.10] there is a distinguished element $\hat{w}_0 = \hat{r}_N \cdots \hat{r}_2 \hat{r}_1$. Still denote $\hat{\sigma}_0 = \text{id}$, and $\hat{\sigma}_i = \hat{r}_i \cdots \hat{r}_1$ for $i = 1, \cdots, N$ ($\hat{\sigma}_N = \hat{w}_0$). For $\lambda \in Y$, we extend the notation $\lambda^{\hat{\sigma}_0}$ to $\lambda^{\hat{\sigma}_i}$
$$\lambda^{\hat{\sigma}_i} := \lambda - (p - 1)(1 - \hat{\sigma}_i)(\rho_0) - (1 - \sigma_i)(\rho_1)$$
where $\hat{\sigma}_i \rho_0 := \frac{1}{2} \sum_{\alpha \in \hat{\sigma}_i(\Delta)_0^+} \alpha$ and $\sigma_i \rho_1 := \frac{1}{2} \sum_{\alpha \in \hat{\sigma}_i(\Delta)_1^+} \alpha$ for $\hat{\sigma}_i(\Delta)^+ = \hat{\sigma}_i(\Delta)_0^+ \cup \hat{\sigma}_i(\Delta)_1^+$ the positive root set associated with the simple root system $\hat{\sigma}_i(\Pi)$. Then we have
$$\lambda^{\hat{\sigma}_i} = \lambda^{\hat{\sigma}_i} - \mu \text{ for any } \mu \in Y.$$

We will often denote $\lambda^{\hat{\sigma}_i}$ by $\lambda_i$ for simplicity. Next we can describe the inductive relation between those $\lambda_{i-1}$ and $\lambda_i$:
$$\lambda_i = \begin{cases} 
\lambda_{i-1} - (p - 1)\theta_i, & \text{if } \theta_i \in \overline{\Delta}_0^+; \\
\lambda_{i-1} - \theta_i, & \text{if } \theta_i \in \overline{\Delta}_1^+; \\
\lambda_{i-1} - (2p - 1)\theta_i, & \text{if } \theta_i \in \Delta_1^+ \setminus \overline{\Delta}_1^+.
\end{cases}$$
The inductive calculation shows that $\lambda_N = \lambda - 2(p - 1)\rho_0 - 2\rho_1$. In the sequent arguments, we still speak of (Case 1) for $\theta_i \in \overline{\Delta}_0^+$ and (Case 2) for $\theta_i \in \overline{\Delta}_1^+$ and (Case 3) for $\theta_i \in \Delta_1^+ \setminus \overline{\Delta}_1^+$.
By the same construction as in §5, of $\varphi : \hat{Z}_A(\lambda) \rightarrow \hat{Z}_A^{\hat{\varphi}_0}(\lambda_1)$, we can construct a series of homomorphisms
$$\Xi_{i-1} : \hat{Z}_A^{\hat{\varphi}_{i-1}}(\lambda_{i-1}) \rightarrow \hat{Z}_A^{\hat{\varphi}_i}(\lambda_i), i = 1, 2, \cdots, N$$
which share the same properties as $\varphi$ presented in §5, and therefore we have a sequence of homomorphisms
$$\hat{Z}_A(\lambda) \xrightarrow{\Xi_1} \hat{Z}_A^{\hat{\varphi}_1}(\lambda_1) \xrightarrow{\Xi_2} \hat{Z}_A^{\hat{\varphi}_2}(\lambda_2) \xrightarrow{\Xi_3} \cdots \xrightarrow{\Xi_{N-1}} \hat{Z}_A^{\hat{\varphi}_{N-1}}(\lambda_{N-1}) \xrightarrow{\Xi_N} \hat{Z}_A^{\hat{\varphi}_N}(\lambda_N).$$
Denote the composition of these sequence of homomorphisms by $\Xi$, i.e.
$$\Xi = \Xi_{N-1} \circ \cdots \circ \Xi_1 \circ \Xi_0 : \hat{Z}_A(\lambda) \rightarrow \hat{Z}_A^{\hat{\varphi}_0}(\lambda_N).$$

### 6.3.2
By the construction of all $\Xi_i$'s, we can show that $p(\Xi)$ is a non-zero homomorphism from $\hat{Z}(\lambda)$ to $\hat{Z}^{\hat{\varphi}_0}(\lambda_N)$. Actually, by a straightforward computation
$$\Xi(1 \otimes 1) = X_1^{r_1} \cdots X_N^{r_N} \otimes 1 \in \hat{Z}_A^{\hat{\varphi}_0}(\lambda_N)$$
(6.6)
with $X_i = X_{\theta_i}$ and $r_i = p - 1$ for $\theta_i$ in (Case 1), 1 for $\theta_i$ in (Case 2), and $2p - 1$ for $\theta_i$ in (Case 3) respectively. Hence $p(\Xi)$ contains the nonzero vector $X_1^{r_1} \cdots X_N^{r_N} \otimes 1 \in \hat{Z}_A^{\hat{\varphi}_0}(\lambda_N)$.

There is another interesting observation which will be useful for the sequel arguments. Set
$$v = Y_N^{r_N} \cdots Y_1^{r_1} \otimes 1 \in \hat{Z}_A(\lambda)$$
with $Y_i = Y_{\theta_i}$ and $r_i$ as above for $i = 1, \cdots, N$. Then by (5.11) we have
$$\Xi_i(Y_i^{r_i} \otimes 1) = b_i t \otimes 1$$
for some unit $b_i$ in $A$ whenever $\theta_i$ is in (Case 3). As to the other two cases, by [19 §3.6] and (5.2) we have
$$\Xi_i(Y_i^{r_i} \otimes 1)$$
$$= \begin{cases} \begin{aligned} &tb_i \otimes 1 \text{ if } m_j < p \text{ when } \theta_i \text{ in (Case 1), or if } m_i = p \text{ when } \theta_i \text{ in (Case 2)}; \\ &b_i \otimes 1 \text{ if } m_j = p \text{ when } \theta_i \text{ in (Case 1), or if } m_i \neq p \text{ when } \theta_i \text{ in (Case 2)} \end{aligned} \end{cases}$$
where $m_i \in \{1, \cdots, p\}$ with $m_i \equiv (\lambda_{i-1}, \chi_{\theta_i})(\text{mod } p)$, and $b_i$ is some unit in $A$. Set
$$N_\lambda := \#\{\theta_i \in \Phi_0^+ | m_i < p - 1\} + \#\{\theta_i \in \Phi_1^+ | m_i = p\} + \#\Phi_1^+ \setminus \Phi_1^+.$$ 
By an inductive calculation, we finally have that there is a unit $b \in A$ such that
$$\Xi(v) = t^{N_\lambda} b \otimes 1.$$ (6.7)

**Remark 6.2.** At the first glance, the expression of $N_\lambda$ is dependent on the inductive steps. However, from the proof of Theorem [6.4] we will read off a concise expression independent of any inductive steps, which is as follows:
$$N_\lambda = \#\{\alpha \in \Delta_0^+ | m_\alpha < p\} + \#\{\gamma \in \Delta_1^+ | m_\gamma = p\} + \#\Delta_1^+ \setminus \Delta_1^+,$$ (6.8)
where $m_\theta \in \{1, \cdots, p\}$ with $m_\theta \equiv (\lambda + \rho, \chi_{\theta})(\text{mod } p)$ for $\theta \in \Delta^+$. 
6.3.3. Now we investigate more about both modules involved from $\Xi$ in the category $\mathcal{C}$. Before that, consider linear dual module $M^*$ for an superspace $M = M_1 + M_0$ in $\mathcal{C}_A$. The module structure on the superspace $M^* = (M_1)^* + (M_0)^*$ is given via $X \cdot \phi = (-1)^{|X||\phi|+1} \phi \circ X$ for homogeneous elements $X \in \mathfrak{g}_X$ and $\phi \in M^*_{|\phi|}$. It is readily shown that $M^*$ is still in the category $\mathcal{C}$. Actually, for $M = \sum_{\mu \in Y} M_\mu$, we have $M^* = \sum_{\mu \in Y} (M^*)_\mu$ with $(M^*)_\mu = (M_\mu)^*$. So it is easy to check

$$\hat{Z}_0(\lambda)^* \cong \hat{Z}_0(-\lambda + (p-1)2\rho_0 + 2\rho_1)$$

by comparing the highest weight of $\hat{Z}_0(\lambda)^*$ and $\hat{Z}_0(-\lambda + (p-1)2\rho_0 + 2\rho_1)$.

**Lemma 6.3.** The socle of $\hat{Z}_0^\lambda(\lambda_N)$ is isomorphic to $\hat{L}(\lambda)$.

**Proof.** By the arguments in §2.3 there is a standard involution $\tau_0 \in \text{Aut}(\mathfrak{g})$ (and a standard involution $\tau \in \text{Aut}(G)$). This $\tau_0$ or $\tau$ induces a self-equivalence functor on $\mathcal{C}$, sending $M$ to $\hat{M}$ where $\hat{M}$ is $M$ itself as a superspace, with action as $X \cdot m = \tau^{-1}(X)m$ for $X \in \mathfrak{g}$ and $m \in M$. It is easily known that for $M = \sum_{\lambda \in Y} M_\lambda \in \mathcal{C}$

$$\hat{M} = \sum_{\lambda \in Y} (\hat{M})_\lambda \text{ with } (\hat{M})_\lambda = M_{-\lambda}.$$Especially, for $\hat{Z}(\lambda) \cong \hat{U}_0(\mathfrak{g}) \otimes_{U_0(b+n-)} k_{-\lambda} = \hat{Z}_0^\lambda(-\lambda)$. Thus we have

$$\hat{Z}_0(\lambda)^* \cong \hat{Z}_0^\lambda(\lambda_N).$$

Note that linear duality changes a head into a socle. On the other hand, $\hat{L}(\lambda)^*$ is still a simple object in $\mathcal{C}$ with highest weight $\lambda$, thereby must be isomorphic to $\hat{L}(\lambda)$. So (6.3) implies that $\text{Soc}(\hat{Z}_0^\lambda(\lambda_N)) \cong \hat{L}(\lambda)$. We complete the proof. \hfill \Box

6.3.4. We return to the homomorphism in $\mathcal{C}_A$

$$\Xi : \hat{Z}_A(\lambda) \longrightarrow \hat{Z}_A^\lambda(\lambda_N).$$

Remark 5.4 is suitable to each $\Xi_i$. So we have an important consequence that all $\hat{Z}_A^\lambda_i(\lambda_i)$ define the same class in the Grothendieck group of $\mathcal{C}$ as $\hat{Z}(\lambda)$, i.e.

$$[\hat{Z}_A^\lambda_i(\lambda_i)] = [\hat{Z}(\lambda)], \ i = 0, 1, \ldots, N.$$ (6.10)

which will be used later. By the definition, $\hat{Z}_A(\lambda)$ and $\hat{Z}_A^\lambda(\lambda_N)$ are torsion free over $A$. Furthermore, Formula (5.5) and Formulas (5.7)-(5.8) along with [19, §3.6(2)] are suitable to all $\Xi_i : \hat{Z}_A^\lambda_i(\lambda_i) \rightarrow \hat{Z}_A^{\lambda_{i+1}}(\lambda_{i+1}), \ i = 0, 1, \ldots, N - 1$. This implies that if we work over the fractional field $Q(A)$ by base change, then for $i$

$$\Xi_i \otimes \text{id}_{Q(A)} : \hat{Z}_A^\lambda_i(\lambda_i) \otimes Q(A) \rightarrow \hat{Z}_A^{\lambda_{i+1}}(\lambda_{i+1}) \otimes Q(A)$$

become isomorphisms. Hence $\Xi \otimes \text{id}_{Q(A)}$ becomes an isomorphism. So $\Xi$ satisfies the condition (6.1). Hence we can construct through $\Xi$ the filtration as introduced in §6.2. By application of general filtration construction in §6.2 to the baby Verma module case, we can get a filtration $\{\hat{Z}_A(\lambda)^i\}$ and then $\{\hat{Z}(\lambda)^i\}$, where

$$\hat{Z}_A(\lambda)^i = \{v \in \hat{Z}_A(\lambda) \mid \Xi(v) \in \iota^i \hat{Z}_0^\lambda(\lambda_N)\}$$
and
\[ \hat{Z}(\lambda)^i \cong \tilde{Z}_A(\lambda)^i. \]

6.4. Jantzen sum formula.

**Theorem 6.4.** For the above filtration \{\(\hat{Z}(\lambda)^i\)\} of \(\hat{Z}(\lambda)\), the following statements hold.

(1) There is a sum formula in the Grothendieck group of \((U_0(g), \Sigma)\)-mod

\[
\sum_{i>0} [\hat{Z}(\lambda)^i] = \sum_{\alpha \in \Delta_0^+} \sum_{i \geq 0} (\sum_{\gamma \in \Sigma^+} [\hat{Z}(\lambda - (ip + m_\alpha)\alpha)] - \sum_{i>0} [\hat{Z}(\lambda - ip\alpha)])
\]

\[ + \sum_{\gamma \in \Sigma^+} (\sum_{p \geq 0} [\hat{Z}(\lambda - (2i + 1)\gamma)] - \sum_{i>0} [\hat{Z}(\lambda - 2i\gamma)])
\]

\[ + \sum_{\beta \in \Delta_i^+ \setminus \Delta_1^+} (\sum_{i \geq 0} [\hat{Z}(\lambda - ((2i + \delta_{1,(-1)^{m_\beta-1}})p + m_\beta)\beta)] - \sum_{i>0} [\hat{Z}(\lambda - (2ip)\beta)])
\]

where \(\delta_{i,j} := 1\) if \(i = j\), \(\delta_{i,j} := 0\) if \(i \neq j\); and \(m_\theta \in \{1, \cdots, p - 1, p\}\) with \(m_\theta \equiv (\lambda + p, \chi_\theta)(\text{mod } p)\) for \(\theta \in \Delta^+\).

(2) \(\hat{Z}(\lambda)^i = 0\) if and only if \(i > N_\lambda\) where \(N_\lambda\) is as in (6.8).

(3) \(\hat{Z}(\lambda)/\hat{Z}(\lambda)^1 \cong L(\lambda)\).

**Proof.** (1) By Lemma 6.1 and the formula 6.2, we know that

\[
\sum_{i>0} [\hat{Z}(\lambda)^i] = \sum_{j=0}^{N-1} [\text{coker}\Xi_j]. \quad (6.11)
\]

Thanks to Lemma 5.5 along with Remark 5.6, we know \([\text{coker}(\Xi_j)] = [\text{coker}(\Xi_{j+1})]\) for all \(j = 0, \cdots, N - 1\) where \(\Xi_j : \hat{Z}^{\sigma_j}(\lambda_j) \to \hat{Z}^{\sigma_{j+1}}(\lambda_{j+1})\), a homomorphism in \(\mathcal{C}\) which is gotten from \(\Xi_j\) by base change arising from the projection \(A \to k \cong A/tA\).

For the further arguments, we have to divide them into three cases, according to the situations for \(\theta_{j+1}\).

(i) Suppose \(\theta_{j+1} \in \Delta_1^+ \setminus \Delta_1^+\). Lemma 5.3 is suitable to \(\Xi_j\). Replace \(d\) in Lemma 5.3 by \(m_j\), i.e., \(m_j \in \{0, 1, \cdots, p - 1\}\) with \(m_j \equiv (\lambda_j, \chi_{\theta_{j+1}})(\text{mod } p)\). So we have

\[
[\text{coker}\Xi_j] = \sum_{i \geq 0} [\hat{Z}^{\sigma_j}(\lambda_j - ((2i + \delta_{1,(-1)^{m_j}})p + m_j + 1)\theta_{j+1})\theta_{j+1})] - \sum_{i>0} [\hat{Z}^{\sigma_j}(\lambda_j - (2ip\theta_{j+1}))].
\]

According to (6.10), we know

\[
[\hat{Z}^{\sigma_j}(\lambda_j)] = [\hat{Z}(\lambda)].
\]

Replacing \(\lambda\) by \(\lambda - 2ip\theta_{j+1}\) or \(\lambda - ((2i + \delta_{1,(-1)^{m_j}})p + m_j + 1)\theta_{j+1}\), then by the definition of \(\lambda_j = \lambda^{\sigma_j}\) (6.3) and its property (6.5) we have

\[
[\hat{Z}^{\sigma_j}(\lambda_j - 2ip\theta_{j+1})] = [\hat{Z}(\lambda - 2ip\theta_{j+1})]
\]

and

\[
[\hat{Z}^{\sigma_j}(\lambda_j - ((2i + \delta_{1,(-1)^{m_j}})p + m_j + 1)\theta_{j+1})] = [\hat{Z}(\lambda - ((2i + \delta_{1,(-1)^{m_j}})p + m_j + 1)\theta_{j+1})]
\].
So we have
\[ [\text{coker} \Xi_j] = \sum_{i>0} [\hat{Z}(\lambda - (2i + \delta_1,(-1)^{2}p + m_j + 1)\theta_{j+1})] - \sum_{i>0} [\hat{Z}(\lambda - (2i)\theta_{j+1})]. \]

(ii) Suppose \( \theta_{j+1} \in \Delta^+_1 \). By the same arguments as above, with application of Lemma 5.2 we get for \( m_j = 0 \)
\[ [\text{coker} \Xi_j] = \sum_{i>0} [\hat{Z}(\lambda - (2i + 1)\theta_{j+1})] - \sum_{i>0} [\hat{Z}(\lambda - 2i\theta_{j+1})]. \]

As to the case when \( m_j \neq 0 \), \( \text{coker} \Xi_j = \text{coker} \Xi_j = 0 \) by Lemma 5.2

(iii) Suppose \( \theta_{j+1} \in \Delta^+_1 \). By the same arguments as the above (i), with application of Lemma 5.1 we get for the case when \( m_j < p - 1 \)
\[ [\text{coker} \Xi_j] = \sum_{i>0} [\hat{Z}(\lambda - (ip + m_j + 1)\theta_{j+1})] - \sum_{i>0} [\hat{Z}(\lambda - ip\theta_{j+1})]. \]

where \( m_j \)'s are in the same sense as the \( d \) from Lemma 5.1 i.e. \( m_j \in \{0, 1, \ldots, p - 1\} \) with \( m_j \equiv (\lambda_j, \chi_{\theta_{j+1}})(\mod p) \). The above formula is also suitable to the case when \( m_j = p - 1 \) because in the case, \( \text{coker} \Xi_j = 0 \) by Lemma 5.1 and the sum on the right-hand side becomes 0.

Summing up, Formula (6.11) becomes
\[
\sum_{i>0} [\hat{Z}(\lambda)^i] = \sum_{\theta_{j+1} \in \Delta^+_0} (\sum_{i>0} [\hat{Z}(\lambda - (ip + m_j + 1)\theta_{j+1})] - \sum_{i>0} [\hat{Z}(\lambda - ip\theta_{j+1})])
+ \sum_{\theta_{j+1} \in \Delta^+_0, m_j = 0} (\sum_{i>0} [\hat{Z}(\lambda - (2i + 1)\theta_{j+1})] - \sum_{i>0} [\hat{Z}(\lambda - 2i\theta_{j+1})])
+ \sum_{\theta_{j+1} \in \Delta^+_1 \setminus \Delta^+_1} (\sum_{i>0} [\hat{Z}(\lambda - (2i + \delta_1,(-1)^{2}p + m_j + 1)\theta_{j+1})]
- \sum_{i>0} [\hat{Z}(\lambda - (2i)\theta_{j+1})]).
\]

Set \( m_j \in \{0, 1, \ldots, p - 1\} \) with \( m_j \equiv (\lambda_j, \chi_{\theta_{j+1}})(\mod p) \). In the first and third summands on the right hand side of the above formula,
\[
m_j + 1 \equiv (\lambda_j, \chi_{\theta_{j+1}}) + 1(\mod p)
= (\lambda - (p - 1)(\rho_0 - \sigma_j\rho_0) - (\rho_1 - \sigma_j\rho_1), \chi_{\theta_{j+1}}) + 1(\mod p)
= (\lambda + (\rho_0 - \rho_1), \chi_{\theta_{j+1}}) - (\sigma_j(\rho_0) - \sigma_j(\rho_1), \chi_{\theta_{j+1}}) + 1(\mod p).
\]

Note that by the definition, \( \sigma_j(\rho_0) - \sigma_j(\rho_1) \) is the Weyl vector with respect to \( \sigma_j(\Delta^+) \) the positive root set corresponding to the fundamental system \( \sigma_j(\Pi) \), and \( \theta_{j+1} \) is a simple root of \( \sigma_j(\Pi) \) (Theorem 3.10(1)), and non-isotropic. By (i) Proposition 1.33 and (ii) §3.1, we have \( \langle \sigma_j(\rho_0) - \sigma_j(\rho_1), \chi_{\theta_{j+1}} \rangle = \frac{1}{2} \langle \theta_{j+1}, \chi_{\theta_{j+1}} \rangle = 1 \). Hence, modulo \( p \) we can replace \( m_j + 1 \) by \( \langle \lambda + \rho_1, \chi_{\theta_{j+1}} \rangle \) with \( \theta_{j+1} \) running over \( \Delta^+_0 \cup (\Delta^+_1 \setminus \Delta^+_1) \).
For \( \theta_{i+1} \in \Sigma_1^+ \), by the same arguments as above we have \( \langle \lambda_j, \chi_{\theta_{j+1}} \rangle \equiv \langle \lambda + \rho, \chi_{\theta_{j+1}} \rangle \pmod{p} \) because \( \langle \theta_{j+1}, \theta_{j+1} \rangle = 0 \). So modulo \( p \) we can regard \( m_j = \langle \lambda + \rho, \chi_{\theta_{j+1}} \rangle \).

Thus, we finally get the desired sum formula as expressed in the theorem. We prove the statement (1).

(2) Now we apply the arguments on some inverse images of \( \varphi \) in \([6.2.2]\) to all \( \Xi \), we can describe an inverse image of \( \varphi \) for \( m > 0 \):

\[
\Xi^j_{-1}(t^m \hat{Z}_A^j(\lambda_j+1)) \subset t^{m-1}\hat{Z}_A^j(\lambda_j)
\]

when \( \theta_{j+1} \in \Delta^+ \setminus \Delta_1^+ \), i.e. in (Case 3) as called in \([6.3.1]\) As to the other two cases ((Case 1) for \( \Delta^+_0 \) and (Case 2) for \( \Delta^+_1 \), we have

\[
\Xi^j_{-1}(t^m \hat{Z}_A^j(\lambda_j+1)) = \begin{cases} 
\subset t^{m-1}\hat{Z}_A^j(\lambda_j) & \text{if } m_j < p - 1 \text{ when } \theta_{j+1} \text{ in (Case 1), or if } m_j = 0 \text{ when } \theta_{j+1} \text{ in (Case 2)}; \\
= t^{m}\hat{Z}_A^j(\lambda_i) & \text{if } m_j = p - 1 \text{ when } \theta_{j+1} \text{ in (Case 1), or if } m_j \neq 0 \text{ when } \theta_{j+1} \text{ in (Case 2)}.
\end{cases}
\]

So we have

\[
\Xi^{-1}(t^{N^+N\lambda+1}\hat{Z}_A^\varphi(\lambda_N)) \subset t\hat{Z}_A(\lambda)
\]

which implies that \( \hat{Z}(\lambda)^{N\lambda+1} = 0 \). On the other hand, by \([6.7]\) we have known that there is a distinguished vector \( v = Y_1^{r_1} \cdots Y_N^{r_\lambda} \otimes 1 \in \hat{Z}_A(\lambda)^{N\lambda} \). We can think that the nonzero vector \( v \) is contained in \( \hat{Z}(\lambda)^{N\lambda} \). The statement (2) is proved.

(3) From \([6.3]\), it follows that

\[
\hat{Z}(\lambda)/\hat{Z}(\lambda) \cong \text{im}(\Xi).
\]

Note that \( \hat{Z}(\lambda) \) has a unique simple quotient \( \hat{L}(\lambda) \), and \( \Xi \) is a nontrivial homomorphism (see the arguments around \([6.6]\)). Hence \( \text{im}(\Xi) \) has a simple head isomorphic \( \hat{L}(\lambda) \). From \([\hat{Z}(\lambda) : \hat{L}(\lambda)] = 1 \), it follows that \([\text{im}(\Xi) : \hat{L}(\lambda)] = 1 \). On the other hand, by Lemma \([6.3]\) we know that \( \hat{Z}_A^\varphi(\lambda_N) \) has a simple socle isomorphic to \( \hat{L}(\lambda) \), thereby the submodule \( \text{im}(\Xi) \) naturally has a simple socle isomorphic to \( \hat{L}(\lambda) \). Hence \( \text{im}(\Xi) \cong \hat{L}(\lambda) \). We complete the proof of the statement (3). \( \square \)

7. Strong linkage principle

In the concluding section, we will use Theorem \([6.4]\) to get a strong linkage principle in \( \mathcal{E} \).

7.1. Set \( \Delta^+_c = \Sigma_0^+ \cup (\Delta^+_1 \setminus \Delta_1^+) \). Then the super reflection \( \hat{r}_\alpha \) is a real reflection, simply written as \( r_\alpha \). Denote by \( r_{\alpha,n} \) for \( n \in \mathbb{Z} \) the affine reflection given by given by \( r_{\alpha,n}(\lambda) = r_\alpha(\lambda) + n\alpha \) for any \( \lambda \in Y \). Denote by \( W_p \) the affine Weyl group generated by all \( r_{\alpha,np} \) with \( n \in \mathbb{Z} \) and \( \alpha \in \Delta^+_c \). Define \( w \cdot \lambda = w(\lambda + \rho) - \rho \) for \( w \in W_p \).

**Definition 7.1.** Let \( \lambda, \mu \in Y \).
Furthermore, we get for
\[ i > 0, \quad \lambda_i \uparrow \lambda \text{ if either } \mu = \lambda, \text{ or there are affine reflections } r_1, \ldots, r_s \in W_p \]
such that
\[ \lambda \geq r_1 \cdot \lambda \geq r_2 r_1 \cdot \lambda \geq \cdots \geq (r_s \cdots r_2 r_1) \cdot \lambda = \mu. \]

(2) Call \( \mu \uparrow \lambda \) if \( \mu \leq \lambda \) and there exist \( \mu', \lambda' \in Y \) such that \( \mu' \uparrow \lambda' \) with \( \mu = \mu' \mod \mathbb{Z}\Delta^+_i \) and \( \lambda = \lambda' \mod \mathbb{Z}\Delta^+_i \). Here \( \mathbb{Z}\Delta^+_i \) means the free abelian group spanned by \( \Delta^+_i \).

Note that the relation \( \uparrow \) is transitive. So, it is easy to see that the relation \( \uparrow \uparrow \) is transitive, this is to say, if \( \mu \uparrow \lambda \) and \( \kappa \uparrow \mu \) then \( \kappa \uparrow \lambda \).

**Theorem 7.2.** Let \( \lambda, \mu \in Y \). If \( \tilde{Z}(\lambda), \tilde{L}(\mu) \neq 0 \), then \( \mu \uparrow \lambda \).

We will leave the proof of the theorem in \[7.3\].

**Remark 7.3.** One can compare the strong linkage principle formulated here with some related results from \[28, 30\].

7.2. For \( \lambda \in Y \) and \( \theta \in \Delta^+_c \cup \Delta^+_i \), keep the notation \( m_\theta \) as in Theorem 6.4. We can write \( (\lambda + \rho, \lambda_\theta) = np + m_\theta \) for some \( n \in \mathbb{Z} \), and make the following convention:

- When \( \theta \) is in (Case 1), set for \( i \geq 0 \)
  \[ \lambda_{2i} = \lambda - ip\theta, \quad \lambda_{2i+1} = \lambda - (ip + m_\theta)\theta; \]

- When \( \theta \) is in (Case 2), set for \( i \geq 0 \)
  \[ \lambda_{2i} = \lambda - 2i\theta, \quad \lambda_{2i+1} = \lambda - (2i + 1)\theta; \]

- When \( \theta \) is in (Case 3), set for \( i \geq 0 \)
  \[ \lambda_{2i} = \lambda - 2ip\theta, \quad \lambda_{2i+1} = \lambda - ((2i + \delta_{1,(-1)m_\theta})p + m_\theta)\theta \]

Then the summand corresponding to \( \theta \) on the right hand side of the sum formula in Theorem 6.3(1) becomes

\[ \sum_{i \geq 0} \tilde{Z}(\lambda_{2i+1}) - \sum_{i > 0} \tilde{Z}(\lambda_{2i}). \]

**Lemma 7.4.** For given \( \lambda \in Y \) and \( \theta \in \Delta^+_c \cup \Delta^+_i \), \( \lambda_{i+1} \uparrow \lambda_i \) for all \( i \geq 0 \).

**Proof.** We prove this lemma by case-by-case arguments.

(i) When \( \theta \) is in (Case 1), from \( \lambda \geq \lambda_1 = r_\theta_{np} \cdot \lambda \) it follows that \( \lambda_1 \uparrow \lambda_0 = \lambda \). Furthermore, we get for \( i > 0 \), \( \lambda_{2i-1} \geq \lambda_{2i} = r_{\alpha, (n-2i-1)} \cdot \lambda_{2i-1} \) and \( \lambda_{2i} \geq \lambda_{2i+1} = r_{\theta, (n-2i)} \cdot \lambda_{2i} \). In this case, the claim is true.

(ii) When \( \theta \) is in (Case 2), the claim is obviously true.

(iii) When \( \theta \) is in (Case 3), set \( \epsilon = \delta_{1,(-1)m_\theta} \). From \( \lambda \geq \lambda_1 = r_{\theta, (n-2(\epsilon+1))} \cdot \lambda \) it follows that \( \lambda_1 \uparrow \lambda_0 = \lambda \). Furthermore, we get for \( i > 0 \), \( \lambda_{2i-1} \geq \lambda_{2i} = r_{\theta, (n-(4i-2)-\epsilon)} \cdot \lambda_{2i-1} \) and \( \lambda_{2i} \geq \lambda_{2i+1} = r_{\theta, (n-4i-\epsilon)} \cdot \lambda_{2i} \). In this case, the claim is true.

Summing up, we complete the proof. \( \square \)
7.3. Now we prove Theorem 7.2. If \( \hat{Z}(\lambda), \hat{L}(\mu) \neq 0 \), then \( \mu \leq \lambda \). Hence we can prove the theorem by induction on the height of \( \lambda - \mu \) which is equal to \( \sum_{\alpha \in \Pi} c_{\alpha} \) for \( \lambda - \mu = \sum_{\alpha \in \Pi} c_{\alpha} \alpha \), \( c_{\alpha} \in \mathbb{Z}_{\geq 0} \).

When \( \mu = \lambda \), the claim is obvious. We suppose \( \mu < \lambda \). By Theorem 6.4(3), \( \hat{L}(\mu) \) has to be a composition factor of the first term \( \hat{Z}(\lambda) \) of the Jantzen filtration. Hence we can prove the theorem by induction on the height of \( \lambda - \mu \) which is equal to \( \sum_{\alpha \in \Pi} c_{\alpha} \alpha \) for \( \lambda - \mu = \sum_{\alpha \in \Pi} c_{\alpha} \alpha \), \( c_{\alpha} \in \mathbb{Z}_{\geq 0} \).

When \( \mu = \lambda \), the claim is obvious. We suppose \( \mu < \lambda \). By Theorem 6.4(3), \( \hat{L}(\mu) \) has to be a composition factor of the first term \( \hat{Z}(\lambda) \) of the Jantzen filtration. Hence we can prove the theorem by induction on the height of \( \lambda - \mu \) which is equal to \( \sum_{\alpha \in \Pi} c_{\alpha} \alpha \) for \( \lambda - \mu = \sum_{\alpha \in \Pi} c_{\alpha} \alpha \), \( c_{\alpha} \in \mathbb{Z}_{\geq 0} \).

Owing to Theorem 6.4(1), there exists \( \theta \in \Delta^{+} + c_{\alpha} \bigcup \Delta^{+} \) such that \( \hat{L}(\mu) \) is a composition factor of \( \hat{Z}(\lambda_{2i+1}) \) for some \( i \geq 0 \), using the notation from §7.2. By induction we get \( \mu \uparrow \lambda_{2i+1} \). Combining with Lemma 7.4 and the transitive property of the relation \( \uparrow \), we complete the proof of the theorem.

7.4. By the same arguments as above (with checking more in (Case 2)), we can give another version of strong linkage principle, which can be regarded as an optimal version.

**Proposition 7.5.** Let \( \lambda, \mu \in Y \). If \( \hat{Z}(\lambda), \hat{L}(\mu) \neq 0 \), then there exist \( s \in \mathbb{N} \) and a series of weights \( \lambda_{i} \), \( i = 0, 1, \ldots, s \) satisfying

\[
\begin{align*}
\lambda &= \lambda_{0} \geq \lambda_{1} \geq \cdots \geq \lambda_{s} - 1 \geq \lambda_{s} = \mu, \text{ such that} \\
\text{either} \ &\lambda_{i} = r_{\alpha_{i}, n_{i}} \cdot \lambda_{i-1} \text{ for some } \alpha_{i} \in \Delta^{+}, n_{i} \in \mathbb{Z} \\
\text{or} \ &\lambda_{i} = \lambda_{i-1} - \gamma_{i} \text{ for some } \gamma_{i} \in \Delta^{+} \text{ with } p | \langle \lambda_{i-1} + \rho, \chi_{i} \rangle. \quad (7.1)
\end{align*}
\]

We further reformulate the statement in the proposition concisely by defining \( \mu \uparrow \lambda \) if either \( \lambda = \mu \) or they satisfy (7.1)

**Theorem 7.6.** Let \( \lambda, \mu \in Y \). If \( \hat{Z}(\lambda), \hat{L}(\mu) \neq 0 \), then \( \mu \uparrow \lambda \).

**Remark 7.7.** One can expect to have Jantzen filtration and sum formula for Weyl modules of algebraic supergroups (cf. [26]).

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