Motivated by recent experiments, we investigate the excitation energy of a proximitized Rashba wire in the presence of a position dependent pairing. In particular, we focus on the spectroscopic pattern produced by the overlap between two Majorana bound states that appear for values of the Zeeman field smaller than the value necessary for reaching the bulk topological superconducting phase. The two Majorana bound states can arise because locally the wire is in the topological regime. We find three parameter ranges with different spectral properties: crossings, anticrossings and asymptotic reduction of the energy as a function of the applied Zeeman field. Interestingly, all these cases have already been observed experimentally. Moreover, since an increment of the magnetic field implies the increase of the distance between the Majorana bound states, the amplitude of the energy oscillations, when present, gets reduced. The existence of the different Majorana scenarios crucially relies on the fact that the two Majorana bound states have distinct $k$-space structures. We develop analytical models that clearly explain the microscopic origin of the predicted behavior.

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I. INTRODUCTION

Majorana fermions are fermionic particles which are their own antiparticles, i.e. $\gamma = \gamma^\dagger$. In condensed matter physics these particles arise as quasiparticle excitations in topological superconductors. Models for engineering topological superconductivity and its detection have been the matter of an extensive research over the last decade. The common ingredient in most of these models consists in proximitizing $s$-wave superconductivity into a system with strong spin-orbit interaction. Their interest is not only fundamental but also practical because they exhibit non-abelian statistics, and therefore, can potentially be used in protocols for fault-tolerant quantum computation. Experimentally, signatures of Majorana bound states (MBSs) have been found in conductance measurements, Shapiro steps, and Josephson radiation.

In the last years, the quality of spin-orbit coupled quantum wires substantially increased. Moreover, a new generation of proximitized Rashba wires were fabricated with longer coherence length and high superconducting gap. Some of these devices showed robust zero bias conductance peaks, and others allowed to explore excitation energy oscillations produced by an external magnetic field. Motivated by these experimental results, some theoretical approaches explain those deviations by introducing extra ingredients into the initial recipe: Adding Coulomb interactions between the electrons in the wire and the dielectric environment can explain the zero energy pinning feature. Including leakage current effects coming from the presence of a drain in the superconductor, this can explain the vanishing of conductance. Finally, the presence of decaying oscillations can be understood by taking into account wires with multiple occupied subbands, orbital effects, high temperature, and simultaneous presence of Andreev bound states and MBSs. We, instead, propose a simple scenario how topological decaying oscillations (see Fig. 1(b)) can appear: we introduce a finite coherence length in the superconducting pairing, that is,

$$\Delta(x) = \Delta_0 \tanh(x/\xi)$$

where $k_{F,\text{eff}}$, $L$, and $\chi$ are the effective Fermi wave vector, the length of the wire and the localization length of the MBS, respectively. Due to the fact that $k_{F,\text{eff}}$ and $\chi$ increase with the magnetic field, the resulting overlap, and hence the conductance, exhibit an oscillatory pattern with an increasing amplitude. Recent experiments show, however, clear deviations from this simple picture: For an increasing magnetic field, most samples experience, however, a decaying amplitude of the oscillations, resulting into crossings and anticrossings. On top of that, some samples feature that oscillations remain pinned at zero energy for a wide range of magnetic field ($\sim 40mT$). Furthermore, other samples manifest a vanishing conductance at high magnetic fields.

In this article, we will focus on the study of conductance oscillations that arise in the Majorana-Rashba wire. It is well established, that the origin of these oscillations resides in the spatial overlap between the MBSs typically located at the ends of the wire: The MBS wave functions exhibit an oscillatory exponential decay towards the center. In the limit of high magnetic fields, the energy resulting from the overlap between the modes is approximately given by

$$\Delta E \approx \frac{\hbar^2 k_{F,\text{eff}}}{m \chi} \cos(k_{F,\text{eff}} L) \exp\left(-\frac{2L}{\chi}\right),$$

where $\xi$ is the coherence length of the superconductor (see Fig. 1(a)). Under such a pairing potential, the critical field for observing MBSs reduces from $B_c = \sqrt{\Delta_0^2 + \mu^2}$ to $B_\mu = |\mu|$, with $\mu$ the chemical potential. When the wire is globally in the topological phase, the Majorana fermions are...
We initially study the discretized version of the model presented in Refs. [7, 8], given by the Hamiltonian

\[
H = (2t - \mu) \sum_{\sigma,x} c_{\sigma,x}^{\dagger} c_{\sigma,x} + B \sum_{\sigma,\sigma',x} c_{\sigma,x}^{\dagger} \sigma \sigma'_{x} c_{\sigma',x} + t \sum_{\sigma(x', x')} \langle \Delta(x) \rangle c_{\sigma, x}^{\dagger} c_{\sigma', x}^{\dagger} + h.c.,
\]

where the index \( \sigma = \uparrow, \downarrow = 1, 2 \) and \( c_{\uparrow/\downarrow, x}^{\dagger} \) is the Fermi annihilation operator for an electron with spin \( \uparrow/\downarrow \) on the \( x \)-th site. Here, the sum over \( x \) runs from 1 up to \( L \), the total number of sites, and thus \( l = a_0 L \) the total length of the wire with \( a_0 \) the lattice spacing. The sum index \( \langle x,x' \rangle \) denotes hopping coefficients to nearest neighbours. In addition, the hopping energy \( t \) is determined by the effective mass \( m^* \) through \( t = \hbar^2/(2m^* a_0^2) \) (in the following we set \( \hbar = 1 \)). \( B = \frac{1}{2} g_\mu_B B_z \) is the Zeeman energy, originating from a magnetic field applied in the \( x \)-direction \( B_\parallel \) (\( B > 0 \) throughout the article), and \( \mu \) is the chemical potential. The coefficient \( t_{SO}^{\perp} = \pm \frac{1}{2} \alpha/a_0 \), where \( \alpha = 1/(m^* \lambda_{SO}) \) is the spin-orbit coupling and \( \lambda_{SO} \) is the spin-orbit coupling length. The pairing potential \( \Delta(x) \) is the discretized version of Eq. (2), \( \sigma_{x,y} \) are the first and second Pauli matrices.

III. MAIN RESULTS

In this section we characterize the effects that a finite coherence length introduce in the critical field and the oscillating pattern resulting from the hybridization of MBSs. To this aim, we diagonalize Eq. (3) and compare in Fig. 2 the lowest energy states in the parameter space \( (B, \mu) \) for different coherence lengths: \( \xi = 10 \text{nm}, 200 \text{nm}, \xi = 400 \text{nm} \) and \( \xi = 1 \mu \text{m} \). We can observe that for \( B > \sqrt{\Delta_0^2 + \mu^2} \equiv B_c \) (see blue curve in Fig. 2) the qualitative topological properties of the Rashba-wire are still present, i.e. MBSs localized close to the two ends of the wire arise and oscillate with increasing amplitude for increasing magnetic fields.

For \( B < B_c \), MBSs arise for an increasing \( \xi \), see Fig. 2(a)-(d). The reason for this can be understood if we consider a slowly varying pairing potential. In this situation, the critical condition \( B_c(x) = \sqrt{\Delta(x)^2 + \mu^2} \) can be satisfied locally, and thus, two MBSs arise: one placed close to the left end of the wire, \( x_B \sim 0 \), and another at \( x_B \), where the relation \( B = \sqrt{\Delta(x_B)^2 + \mu^2} \) is satisfied. Note that \( x_B \), and thus, the distance between the MBSs, increases for an increasing magnetic field. Roughly speaking, the requirement for having Majorana fermions is hence no longer \( B > B_c \), but becomes related to the existence of the point \( x_B \), that is guaranteed for \( B > |\mu| \equiv B_\mu \). This behavior is indeed what we observe in Fig. 2: for an increasing coherence length, zero.

FIG. 1. (a) Schematic of the system. The spin-orbit coupled wire is placed on top of a substrate and partially covered by a superconductor. We assume a space dependent proximity induced pairing amplitude \( \Delta(x) \). (b) Numerical tight-binding calculation of the lowest energy eigenvalues, as a function of the Zeeman energy \( B \) for \( L = 200, I = 2 \mu \text{m}, \alpha = -20.2 \text{meV/\mu m,} \Delta_0 = 1.26 \text{meV,} \mu = 0, \xi = 0.8 \mu \text{m.} \) (c) Density of the corresponding eigenfunctions for \( B = 0.9 B_c \), and \( d \) \( B = 1.1 B_c \) as a function of \( x \). In (c) and (d) the unspecified parameters have the same value as in (b).
energy states approach asymptotically $B = |\mu|$ (see black curves in Fig. 2). Interestingly, this means that MBSs are present whenever the system is in the quasi-helical regime of the spin-orbit coupled wire, that is, whenever the system in the absence of superconductivity is effectively spinless. We observe deviations from this behavior for shorter coherence lengths (see Fig. 2 (b)), and MBSs can arise even for $B < |\mu|$. Thus, extracting the parameters relevant for the quantum wire by imposing that the first appearance of a zero mode sets the topological phase transition can lead to a significant mistake.

The existence of MBSs below $B_c$ is not the only interesting effect of a finite coherence length. The dependence that the lowest energy level has as a function of the applied magnetic field is also very interesting. Since the distance between the two Majorana fermions increases when the magnetic field is increased, the resulting overlap decreases, see Figs. 1 (b) and 3 (a)-(b). This feature is often observed in experiments and is not always easy to interpret. However, in the context of a finite coherence length, decaying oscillations for $B < B_c$ appear very naturally. In this scenario, decaying oscillations are, remarkably, just one of the possible behaviours. It is worth to notice that decaying oscillations (Fig. 3 (a)) can evolve into anticrossings (Fig. 3 (c)) and finally into an monotonic decay to zero (Fig. 3 (d)) as the chemical potential or the spin-orbit coupling are increased. In order to understand the microscopic mechanisms that originate the different patterns, we analyze the physical properties of the lowest energy BdG wavefunctions.

The Majorana fermion around $x_B$ is expected to be a non-oscillating function of $x$, in particular for $\mu = 0$, a Gaussian. An oscillating hybridization energy can hence only emerge from an oscillating wavefunction for the Majorana fermion located at $x_B$. For a qualitative discussion of the wavefunction around $x_B$, we start by considering the case of a constant superconducting pairing. In this picture, MBSs mainly have dominant contributions from momenta $k$ around $k \approx k_0 = 0$ and $k = \pm k_F$. Thus, the MBS wave function can be expressed as the linear combination $\psi \sim \psi_{k_0} + \psi_{k_F} + \psi_{-k_F}$. All these contributions have a spinor structure and decay exponentially, with a typical localization length related to the corresponding direct energy gap, that is,

$$\psi_j \propto \exp(-\Delta_j x + ik_j x),$$

where the index $j = k_0, k_F$. Around $k_0$ the gap is given by $\Delta_{k_0} = \sqrt{B^2 - \mu^2} - \Delta_0$, and does not depend on $\alpha$ and $\mu$. In contrast, at $k = \pm k_F$, the gap is given by a fraction of the bare induced superconducting coupling $\Delta_{k_F} = a_\Delta(\alpha, \mu, B)\Delta_0$. The behavior of $a_\Delta(\alpha, \mu, B)$ as a function of $\alpha$, for different values of $\mu$ is depicted in Fig. 4 (a). We can observe that as $\alpha$ or $\mu$ are decreased, $a(\alpha, \mu, B)$ and hence $\Delta_{\pm k_F}$ decrease. Note in passing that the contributions $\psi_{\pm k_F}$ oscillate with the wave vector $k_F$. Therefore, whenever $\Delta_{\pm k_F}/\Delta_{k_0} < 1$, the wave function will exhibit a spatial oscillatory pattern.

In the limit of a slowly varying pairing potential, i.e. $k_F \gg 1/\xi$, it is possible to find similar expressions as

![FIG. 2. Numerical results for the lowest energy eigenvalues of the spin-orbit coupled wire as a function of Zeeman energy $B$ and chemical potential $\mu$ in meV. The calculations are done for $L = 200, I = 2 \mu m, \alpha = -20.2 \text{ meV} \mu m, \Delta_0 = 0.63 \text{ meV}$ and different values for the coherence length $\xi$: (a) $\xi = 10$ nm, (b) $\xi = 200$ nm, (c) $\xi = 400$ nm and (d) $\xi = 1 \mu m$. We highlight the lines $B = |\mu|$ and $\mu = \pm \sqrt{B^2 - \Delta^2}$ in black and blue, respectively.](image)

![FIG. 3. Numerical results of the low energy eigenvalues of a spin-orbit coupled wire of total length $L = 200, I = 2 \mu m$ with $\alpha = -22.7 \text{ meV} \mu m, \Delta_0 = 0.76 \text{ meV}, \xi = 0.2 \mu m$ and different chemical potential: (a)-(b) $\mu = 0.63 \text{ meV}$, (c) $\mu = 0.76 \text{ meV}$, (d) $\mu = 1.01 \text{ meV}$. (a) and (b) have the same parameters with different scaling of the axes. The vertical dashed line in (b) represents $B = |\mu|$.

![Diagram](image)
in the constant pairing case (see Sec. IV). Due to the position dependent pairing, the exponents are replaced by

$$\Delta_j(x) \rightarrow \int_0^x \Delta_j(x')dx',$$  

(5)

where $\Delta_j(x)$ is the result of substituting $\Delta(x)$ into the direct gap expressions. An oscillatory pattern of the hybridization energy is observed when the oscillating contribution of the wavefunction located at $x_j$ is dominant around the location of the second Majorana ($x = x_B$). This condition is indeed fulfilled whenever

$$\frac{\int_0^{x_B} \Delta_k(x) dx}{\int_0^\infty \Delta_k(x) dx} < 1.$$  

(6)

As shown in Fig. 4 (b) (see Eq. (32) for more details), the relation is satisfied for small values of $a_\Delta(\alpha, \mu, B)$, i.e. weak spin-orbit coupling and/or small $\mu$ close to the topological phase transition. For strong spin-orbit coupling and large $\mu$, however, the relation does not hold anymore and the oscillations disappear. A deeper analysis of the wavefunctions, going beyond simple scaling arguments, is presented in the next section.

IV. ANALYTICAL ANALYSIS

A. Full continuum model

At low energies and length scales larger than the lattice discretization, the system can be described by the continuum Hamiltonian $H_c = \frac{1}{2} \int_0^l dx \Psi^\dagger(x) \mathcal{H}(x) \Psi(x)$, (7)

with

$$\mathcal{H}(x) = \begin{pmatrix} -\frac{\partial^2}{2m^*} - \mu & \tau_z \otimes \sigma_0 \\ -i\alpha \partial_x \tau_z \otimes \sigma_z + B \tau_z \otimes \sigma_x + \Delta(x) \tau_x \otimes \sigma_z, \end{pmatrix}.$$  

(8)

where $x$ is now a continuous variable, $\Psi^\dagger(x) = (\psi_\uparrow(x), \psi_\downarrow(x), \psi_\uparrow(x), \psi_\downarrow(x))^T$ with the annihilation operators $\psi_\uparrow, \psi_\downarrow$ of an $\uparrow / \downarrow$ particle at position $x$ and the Pauli matrices $\sigma_i$, $\tau_i$ with $i \in \{x, y, z\}$ acting in spin- and particle-hole-space, respectively. In order to derive simple relations explaining the behavior observed by means of the numerical solution, we follow Ref. 43 and we simplify the model in two regimes: strong and weak spin orbit coupling. Differently from Ref. 43, we will then solve the models in the presence of a non-uniform pairing potential.

B. Effective model for strong spin-orbit coupling

In the strong spin-orbit coupling regime, $m^*\alpha^2 \gg B$, $\Delta_0$, we complement the continuum model with the assumptions of a slowly varying superconducting pairing potential $m^*\alpha \gg 1/\xi$. Then the continuum Hamiltonian can be further simplified to the effective (linear) Hamiltonians around zero average momentum i) and momentum $k_F \approx 2m^*\alpha$ ii) (see also Fig. 5 (a)-(b)):

$$i) \langle -i\partial_x \rangle \approx 0, \quad ii) \langle -i\partial_x \rangle \approx \pm k_F.$$  

(9)

For case i), we are allowed to neglect the quadratic part of Eq. (7), resulting in the long-wave Hamiltonian

$$H_l = \int_0^l dx \Psi^\dagger(x) \begin{pmatrix} -i\alpha \partial_x \tau_z \otimes \sigma_z \\ -\mu \tau_z \otimes \sigma_0 + B \tau_z \otimes \sigma_x + \Delta(x) \tau_x \otimes \sigma_z \end{pmatrix} \Psi(x).$$  

(10)
In case of ii) we can perform a spin-dependent gauge transformation

$$\Psi(x) = e^{-2im^*\alpha x(\tau_0 \otimes \sigma_z)} \tilde{\Psi}(x),$$

(11)

where $\tilde{\Psi}(x)$ is a slowly varying function of $x$ at low energy. After plugging Eq. (11) into Eq. (7) and linearizing, the transformed Hamiltonian becomes

$$H_E = \int_0^l dx \tilde{\Psi}^\dagger(x) \left[ i\alpha \partial_x \tau_z \otimes \sigma_z - \mu \tau_z \otimes \sigma_0 + a_\Delta(\alpha, \mu, B) \Delta(x) \tau_x \otimes \sigma_z \right] \tilde{\Psi}(x).$$

(12)

where the fast oscillating terms are integrated out.

$a_\Delta(\alpha, \mu, B) \in \{0, 1\}$ is determined by the dispersion relation of the lowest energy eigenvalue of Eq. (8) (assuming constant superconducting pairing $\Delta_0$). In the strong spin orbit regime, we obtain $a_\Delta(\alpha, \mu, B) \rightarrow 1$. The full behavior of $a_\Delta(\alpha, \mu, B)$ as a function of $\alpha$ is illustrated in Fig. (4) (a). For small $\alpha$, we find

$$a_\Delta(\alpha, \mu, B) = \frac{\sqrt{\Delta_0^2 + B^2} - B}{\Delta_0} + \frac{4m^2(B + \mu)}{\Delta_0} \alpha^2 + O(\alpha^3).$$

(13)

The rational behind the approximation scheme leading to Eqs. (10) and (12) is that, for strong spin-orbit coupling and for weak translational symmetry breaking by the applied superconducting pairing, we expect that the main effect of superconductivity is to renormalize the helical gap close to zero momentum and open a gap close to $k_F$, as schematically shown in Fig 5. Any low-energy eigenstate is then evaluated as linear combination of eigenstates of $H_I$ and $H_E$.

C. Effective model for weak spin-orbit coupling

In case of weak spin-orbit coupling, $m^* \alpha^2 \ll B$, we additionally assume $\sqrt{m^*B} \gg 1/\xi$. To develop effective linear models for our purposes, in this case, we explicitly distinguish two regimes in parameter space: deep inside the topological phase and close to the phase transition. Far away from any boundary, the latter case is described within the linear Hamiltonian of Eq. (10) (see Fig. 5 (d)), while, close to the boundaries, the contribution around $k = \pm k_F$ is still important. Deep inside the topological phase, the gap opened at $k = 0$ is large compared to the gap opened at $k = \pm k_F$ given by $a_\Delta(\alpha, \mu, B) \Delta_0$ (see Fig. 5 (c)). For weak spin-orbit coupling, $a_\Delta(\alpha, \mu, B) \ll 1$, the low energy physics is described around the points $k = \pm k_F$. An appropriate linear model has to take into account that spins are not (quasi-) helical in the weak spin-orbit regime but acquire a spin-tilting. To implement this requirement in the linear model we, demand an artificial $\epsilon, \gamma$ and $\nu$, acting as magnetic field, chemical potential and Fermi velocity. The $k$ space Hamiltonian in the absence of superconductivity is given in the spin-resolved basis

$$H_{\text{lin}}^0(k) = \begin{pmatrix} \nu k - \gamma & \epsilon \\ \epsilon & -\nu k - \gamma \end{pmatrix}.$$ 

(14)

Unlike former linear models of this paper, here we require $\gamma \geq \epsilon$ to make the model appropriate. To connect those parameters to the physical parameters of the spin-orbit coupled wire, we demand three conditions to hold: (i) Fermi-surface, (ii) velocity at the Fermi points, and (iii) spin-tilting at the Fermi surface have to coincide in both models. The demands (i)-(iii) result in the following equations

$$k_{F,\text{lin}} = k_{F,\text{SOC}} = k_F,$$

$$v_{F,\text{lin}} = v_{F,\text{SOC}} = v_F,$$

$$\nu k_F + \sqrt{\nu^2 k_F^2 + \epsilon^2} \cdot \frac{\epsilon}{\nu} = k_F \alpha - \sqrt{B^2 + k_F^2 \alpha^2},$$

(15)

where the last equation is built from the eigenvectors of Eq. (8) with $\Delta(x) = 0$ and Eq. (14). The equation system (15) has the unique solution

$$\nu = \frac{(1 + \kappa^2)v_F}{-1 + \kappa^2}, \quad \epsilon = \frac{2\kappa(1 + \kappa^2)v_F k_F}{(-1 + \kappa^2)^2},$$

(16)

with the replacements

$$\kappa = \frac{k_F \alpha - \sqrt{B^2 + k_F^2 \alpha^2}}{B},$$

$$v_F = 4m^*k_F - \frac{k_F \alpha^2}{\sqrt{B^2 + k_F^2 \alpha^2}},$$

$$k_F = \sqrt{2} \sqrt{m^* \alpha^2 + m^* \mu + m^* \sqrt{B^2 + m^* \alpha^4 + 2m^* \alpha^2 \mu}}.$$ 

(17)

A feature that does not coincides in both models is the spin rotation length along the dispersion relation. However, this feature only plays a minor role for spectroscopic properties. The direction of the spin rotation along the dispersion, however, is equal in both models. Including superconducting pairing, the linearized model then becomes

$$H_{\text{lin}} = \int_0^l dx \Psi^\dagger(x) \left[ i\alpha \partial_x \tau_z \otimes \sigma_z - \gamma \tau_z \otimes \sigma_0 + \epsilon \tau_z \otimes \sigma_x + a_\Delta(\alpha, \mu, B) \Delta(x) \tau_x \otimes \sigma_z \right] \Psi(x).$$

(18)

D. Wave function at $x_B$ and new critical field

The existence of zero energy MBS in the trivial phase can be fully understood within an analytical approach. We first focus on the large $\xi$ limit. In this regime, we can apply the linear approximations of Eqs. (10), (12)
and (18). As a starting point, we concentrate on a system with no boundaries and demand the existence of one point in space, \( x = x_B \), which satisfies, as mentioned before, the relation \( \Delta^2(x_B) = B^2 - \mu^2 \). Around this point, the low-energy physics (for strong and weak spin-orbit coupling) is described within the linear approximation of Eq. (10) only, since it becomes gapless. We, therefore, search for zero-energy solutions \( \mathcal{H}_I(x) \Phi(x) = 0 \), where we demand \( \Phi(x) \) to be of the form \( \Phi(x) = U \chi(x) \), with

\[
U = \frac{1}{\sqrt{2}} (\tau_0 \otimes \sigma_0 - i r_x \otimes \sigma_y). \tag{19}
\]

This unitary transformation reorganizes the Hamiltonian in the Majorana basis. For \( \chi(x) \) we further assume a solution of the form

\[
\chi(x) = (a, b, c, d)^T \exp[f(x)]. \tag{20}
\]

After plugging in the ansatz we obtain the solution

\[
f(x) = \pm \int dx \frac{\Delta(x) \pm \sqrt{B^2 - \mu^2}}{\alpha}. \tag{21}
\]

In the case of linear behavior of \( \Delta(x) \) and \( \mu = 0 \), we restore the groundstate solution of the displaced harmonic oscillator, solved in Ref. 8. With the solutions of Eq. (21) we obtain the corresponding spinor structure of \( \Phi(x) \)

\[
(u, v, \tilde{v}, \tilde{u})^T = U(a, b, c, d)^T = \frac{1}{\sqrt{2}} (\pm e^{i\varphi} + i, ie^{i\varphi} \pm 1, -ie^{i\varphi} \pm 1, \pm e^{i\varphi} - i)^T, \tag{22}
\]

where \( \varphi = \arccos(B/\mu) \). To fulfill the Majorana condition \( \tilde{u} = u^* \) and \( \tilde{v} = v^* \), we need \( \exp[i\varphi] \in \mathbb{R} \), which is only true for

\[
B \geq |\mu|. \tag{23}
\]

For a slowly varying \( \Delta(x) \), the latter relation represents a bound for the formation of Majorana zero-modes. The hand-waving argument given in the previous section can hence be put on a formal basis. Compiling Eqs. (20), (21) and (22) and imposing that the wavefunction is normalizable and centered around \( x_B \), we explicitly obtain

\[
\Phi(x) = \frac{1}{\sqrt{N}} \begin{pmatrix}
  e^{i\varphi} + i \\
  e^{i\varphi} + 1 \\
  -ie^{i\varphi} + 1 \\
  e^{i\varphi} - i
\end{pmatrix} e^{-\int_0^x dx' \frac{1}{\alpha} (\Delta(x') - \sqrt{B^2 - \mu^2})}, \tag{24}
\]

with the normalization constant \( N \).

E. Wavefunction at \( x_\nu \) in the strong spin-orbit coupling regime

At the left end of the wire, where the proximity induced pairing decreases to zero, we have to distinguish between effective models of strong and weak spin-orbit coupling. For strong spin-orbit coupling, the low-energy physics is captured by a linear combination of eigenstates of Eqs. (10) and (12)

\[
\Psi(x) \simeq a_1 \Psi_I(x) + b_1 e^{-2im_\nu \alpha x (\tau_0 \otimes \sigma_2)} \Psi_E(x) \tag{25}
\]

with the coefficients \( a_1 \) and \( b_1 \) to be derived by the boundary conditions, and \( \Psi_I(x) \), \( I \in I, E \), satisfying

\[
\mathcal{H}_I(x) \Psi_I(x) = 0. \tag{26}
\]

The solution for \( \Psi_I(x) \) is constructed by means of Eqs. (20), (21) and (22). The solution for \( \mathcal{H}_E(x) \) can be found in an analogous way after multiplying Eq. (26) (with \( I = E \)) from the left with \( \tau_z \otimes \sigma_z \). Using the properties of Pauli matrices, especially \( [\tau_y \otimes \sigma_0, \tau_0 \otimes \sigma_z] = 0 \), where \([,\] denotes the commutator, integration yields the solution

\[
\Psi_E(x) = \exp \left[ \int_0^x dx' \frac{i}{\alpha} \left( \Delta(x') (\tau_y \otimes \sigma_0) - \mu (\tau_0 \otimes \sigma_z) \right) \right] \Psi_0. \tag{27}
\]

Subsequently, the spinor \( \Psi_0 \) has to be chosen such that the wavefunction satisfies the Majorana condition \( \Psi(x) = [u(x), v(x), v^*(x), u^*(x)]^T \), which results in four possible solutions. Furthermore, assuming a semi-infinite system (\( x > 0 \)), we have to satisfy the boundary condition \( \Psi(0) = 0 \). Moreover, it has to decay away from \( x = 0 \). The first condition implies that the spinors \( \Psi_I(0) \) and \( \Psi_E(0) \) are linearly dependent. Hence, from Eq. (27) we select the solutions for \( \Psi_E(x) \) which decay away from \( x = 0 \) and combine them with their linearly dependent counterparts \( \Psi_I(x) \). This leads to the only physical solution

\[
\Psi(x) = \frac{1}{\sqrt{N}} \begin{pmatrix}
  i - e^{i\varphi} \\
  ie^{i\varphi} - 1 \\
  -ie^{i\varphi} - 1 \\
  -e^{i\varphi} - i
\end{pmatrix} e^{\int_0^x dx' \frac{1}{\alpha} (\Delta(x') - \sqrt{B^2 - \mu^2})} - \frac{1}{\sqrt{N}} e^{-i(2m_\nu \alpha + \frac{\pi}{2})x (\tau_0 \otimes \sigma_\nu)} \begin{pmatrix}
  i - e^{i\varphi} \\
  ie^{i\varphi} - 1 \\
  -ie^{i\varphi} - 1 \\
  -e^{i\varphi} - i
\end{pmatrix} e^{-\int_0^x dx' \frac{\Delta(x')}{\alpha}} \tag{28}
\]

with normalization constant \( N \).

F. Wavefunction at \( x_\nu \) in the weak spin-orbit coupling regime

The wave function at \( x_\nu \) for the case of weak spin-orbit coupling (\( a_\Delta(\alpha, \mu, B) \ll 1 \)) is given by linear combination of eigenstates of Eq. (18), which indeed have the same form as the eigenstates of Eq. (10) with the replacements \( B \rightarrow \epsilon, \mu \rightarrow \gamma \) and \( \alpha \rightarrow -\nu \). A consequence of
neglecting all other contributions in this linear approach is that we can only accomplish the boundary condition $\Psi(0) = 0$ if we neglect the contribution of the spin orbit coupling in the spinors. If so, the only reasonable wave function is obtained by

$$\Psi(x) = \frac{1}{\sqrt{N}} (-1 + i, i - 1, -i - 1, -1 - i)^T \sin (k_F x) \exp \left[ - \int_0^x dx' \alpha_{\Delta} (\alpha, \mu, B) \Delta (x') \right].$$

(29)

G. Overlap of wavefunctions

The analysis of the latter sections is done for isolated Majorana fermions in (semi-)infinite space with a spatial variation of the superconducting pairing. However, since Majorana fermions always appear in pairs and our system is finite, there can be a finite hybridization energy between them. The hybridization energy is directly related to the overlap of the two wavefunctions. For $\Delta(x)$ defined in Eq. (2), in the regime where $B < B_c$, we can approximate the solution at the left end of the wire, where the proximity induced pairing decreases to zero, by the wavefunction of Eqs. (28), (29), when spin-orbit coupling is strong/weak. On the other hand, around $x = x_B$ we have the wavefunction of Eq. (24). For the strong spin orbit coupling regime, Eq. (28) has an oscillatory and a non-oscillatory part, while Eq. (24) is non-oscillatory. Therefore, the hybridization energy will also be constituted by an oscillatory and a non-oscillatory part. Which of them is dominant is strongly dependent on the corresponding decay length and the relative position of the states. The hybridization energy is expected to show an oscillatory nature if the oscillatory part of Eq. (28) is dominant when $x = x_B$ is approached, which is the case if

$$\int_0^{x_B} dx (1 + \alpha_{\Delta} (\alpha, \mu, B)) \Delta (x) - \sqrt{B^2 - \mu^2} < 0.$$  

(30)

For $\Delta(x)$, following Eq. (2) with $B^2 < \Delta_0^2 + \mu^2$, $x_B$ is determined by

$$x_B = \xi \text{arctanh} \left( \frac{\sqrt{B^2 - \mu^2}}{\Delta_0} \right).$$

(31)

After performing the integration, we obtain

$$R(\alpha, \mu, B) = [1 + \alpha_{\Delta} (\alpha, \mu, B)] \ln[\cosh(\text{arctanh}(\eta))] - \eta \text{arctanh}(\eta) < 0,$$

(32)

with $\eta = \sqrt{B^2 - \mu^2}/\Delta_0$. $R(\alpha, \mu, B)$ is illustrated for different values of $\alpha_{\Delta} (\alpha, \mu, B)$ in Fig. 4. If $\alpha_{\Delta} (\alpha, B, \mu) \rightarrow 1$, which is the case in the strong spin-orbit regime, Eq. (32) can not be fulfilled. Hence, the long wave contribution will always dominate the behavior of the wavefunction at $x = x_B$ and the hybridization energy will show a non-oscillatory behavior, which is coherent with numerical results (Fig. 6 (e), (f)). For very large $\alpha$, similar arguments hold for the hybridization energy in the topological phase. Since the decay length of the wavefunction in the strong spin-orbit regime is proportional to $\alpha$, the overlap of the different wavefunctions is large, resulting in suppression of zero-modes before the topological phase for large $\alpha$ (see Fig. 7).

For $\alpha_{\Delta} (\alpha, \mu, B) < 1$, on the other hand, Eq. (32) can be fulfilled for some values of $B$ within $B^2 < \Delta_0^2 + \mu^2$ (see Fig. 4 (b)). The regime of dominant oscillatory behavior is amplified for small $\alpha_{\Delta} (\alpha, \mu, B)$, i.e. weak spin-orbit coupling, a behavior explaining the numerical results (see Fig. 6 (a), (b)). In this regime, the wave function at $x = x_B$ provides strongly oscillatory character (Eq. (29)) leading to an oscillatory behavior of the hybridization energy (see Fig. 6 (a)-(b)). In the region between strong and weak spin-orbit coupling, instead it is difficult to determine the analytic
FIG. 7. Numerical results for the lowest energy eigenvalue as a function of Zeeman energy \( B \) and chemical potential \( \mu \) in meV. The calculations are done for: \( \Delta_0 = 0.63 \text{ meV} \), \( \xi = 0.5 \mu \text{m} \), \( l = 2 \mu \text{m} \) and different spin-orbit coupling: (a) \( \alpha = -10.1 \text{ meVnm} \), (b) \( \alpha = -20.2 \text{ meVnm} \), (c) \( \alpha = 50.4 \text{ meVnm} \) and (d) \( \alpha = -100.8 \text{ meVnm} \).

form of the wave function around \( x_B \). However, it will be a mixture of Eqs. (28) and (29). This is the regime, where we witness anticrossings in the hybridization energy, as the oscillatory and non-oscillatory contribution to the wave function at \( x = x_B \) have similar decay lengths (Fig. 6 (c)-(d)).

H. Role of the chemical potential

In the intermediate \( \alpha \) regime, which ranges around \( \alpha \sim 10 - 50 \text{ meVnm} \), the chemical potential \( \mu \) plays a crucial role since, especially as \( \mu \to B \), \( a_\Delta(\alpha, \mu, B) \) is a strongly asymmetric function with respect to \( \mu \to -\mu \) (Eq. (13), Fig. 4 (a)). Since \( a_\Delta(\alpha, \mu, B) \) controls the gap size at \( k = \pm k_F \), oscillations are more prominent in the negative \( \mu \) regime, which is indeed again coherent with numerical results (see Fig. 2 (d)). This allows us to witness low energy eigenvalues with oscillatory, anticrossing or monotonous convergence to zero in dependence of the value of \( \mu \).

For experimentally relevant values of \( \alpha \sim 20 \text{ meVnm} \), we indeed expect to be in the transition regime between strong and weak spin-orbit coupling. Interestingly, the tendency to an asymmetric behavior with respect to \( \mu \) is maintained qualitatively even for experimentally relevant coherence length.

With respect to our findings, the signature of a low energy conductance measurement could give an indication to the magnitude of the spin-orbit coupling as well as of the chemical potential inside the wire.

V. CONCLUSION

Majorana fermions in a spin-orbit coupled quantum wire can exist even when the wire is not in the topological phase. The requirement is a finite coherence length \( \xi \) of the proximity induced superconducting pairing amplitude. For slowly varying \( \Delta(x) \) (i.e. large coherence length), we have given analytical and numerical demonstrations that the existence of zero-modes is possible in the whole \( B \geq |\mu| \) region. More importantly, in this article, we have demonstrated that the \( k \) space decompositions of the two Majorana fermions that can form before the topological phase transition are profoundly different. The one located at the end of the wire has an oscillating wavefunction, while the one located at the end of the locally topological region of the wire has a non-oscillating structure. This particular behavior implies a rich scenario for the hybridization energy. As a function of spin-orbit coupling and chemical potential, different behaviours can be obtained, ranging from a pattern of decaying oscillations, to anticrossings, and to a monotonous decay to zero energy. Oscillations are favoured by weak spin-orbit coupling and tendentially small chemical potential, and their amplitude decays as a function of the magnetic field because the two Majorana fermions get separated in space. Stronger spin-orbit coupling and higher chemical potential favour, on the other hand, anticrossings and monotonous decay. We have interpreted the results by means of effective Dirac-like models, which allowed us to interpret them as a consequence of the different decay lengths characterizing the various \( k \) components of the Majorana fermion wavefunctions.

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