Relative Entropy and Single Qubit
Holevo-Schumacher-Westmoreland Channel Capacity

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Abstract

The relative entropy description of Holevo-Schumacher-Westmoreland (HSW) classical channel capacities is applied to single qubit quantum channels. A simple formula for the relative entropy of qubit density matrices in the Bloch sphere representation is derived. The formula is combined with the King-Ruskai-Szarek-Werner qubit channel ellipsoid picture to analyze several unital and non-unital qubit channels in detail. An alternate proof is presented that the optimal HSW signalling states for single qubit unital channels are those states with minimal channel output entropy. The derivation is based on symmetries of the relative entropy formula, and the King-Ruskai-Szarek-Werner qubit channel ellipsoid picture. A proof is given that the average output density matrix of any set of optimal HSW signalling states for a (qubit or non-qubit) quantum channel is unique.

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1 Introduction

In 1999, Benjamin Schumacher and Michael Westmoreland published a paper entitled *Optimal Signal Ensembles* [1] that elegantly described the classical (product state) channel capacity of quantum channels in terms of a function known as the relative entropy. Building upon this view, we study single qubit channels, adding the following two items to the Schumacher-Westmoreland analysis.

I) A detailed understanding of the convex hull shape of the set of quantum states output by a channel. (The fact the set was convex has been known for some time, but the detailed nature of the convex geometry was unknown until recently.)

II) A useful mathematical representation (formula) for the relative entropy function, $D(\rho \parallel \phi)$, when both $\rho$ and $\phi$ are single qubit density matrices.

For single qubit channels, the work of King, Ruskai, Szarek, and Werner has provided a concise description of the convex hull set [2, 3]. In this paper, we derive a useful formula for the relative entropy between qubit density matrices. Combining this formula with the KRSW convexity information, we present from a relative entropy perspective several results, some previously known, and others new, related to the (product state) classical channel capacity of quantum channels. These include:

I) The average output density matrix for any optimal set of signalling states that achieves the maximum classical channel capacity for a quantum channel is unique. For single qubit unital channels, Donalds equality leads to a symmetry which tells us this average density matrix must be $\frac{1}{2} I$. This fact about the average density matrix allows us to conclude for unital qubit channels that the optimum signalling states are a subset of the states with minimum output von Neumann entropy, as previously shown in [2]. This symmetry also allows us to see why only two orthogonal signalling states are needed to achieve the optimum classical channel capacity for single qubit unital channels, and why the a priori probabilities for these two signalling states are $\frac{1}{2}$.

II) The single qubit relative entropy formula allows us to understand geometrically why the a priori probabilities for optimum signalling states for non-unital single qubit channels are not equal.

III) Examples of channels which require non-orthogonal signalling states to achieve optimal classical channel capacity are given. Such channels have been found before. Here these channels are presented in a geometrical fashion based on the relative entropy formula derived in Appendix A.

2 Background
2.1 Classical Communication over Classical and Quantum Channels

This paper discusses the transmission of classical information over quantum channels with no prior entanglement between the sender (Alice) and the recipient (Bob). In such a scenario, classical information is encoded into a set of quantum states $\psi_i$. These states are transmitted over a quantum channel. The perturbations encountered by the signals while transiting the channel are described using the Kraus representation formalism. A receiver at the channel output measures the perturbed quantum states using a POVM set. The resulting classical measurement outcomes represent the extraction of classical information from the channel output quantum states.

There are two common criteria for measuring the quality of the transmission of classical information over a channel, regardless of whether the channel is classical or quantum. These criteria are the (Product State) Channel Capacity$^4, 5, 6$ and the Probability of Error (Pe)$^7$. In this paper, we shall focus on the first criterion, the Classical Information Capacity of a Quantum Channel, $C$.

In determining the classical channel capacity, we typically have an input signal constellation consisting of classical signals $x_i$$^8$. The classical channel capacity $C$ is defined as $^8$:

$$C = \max \{all\ possible\ x_i\} \quad H(X) - H(X|Y)$$

Here $H(X)$ is the Shannon entropy for the discrete random variable $X$. $X \equiv \{ p_i = \text{prob}(x_i) \}, i = 1, \cdots, N$. The Shannon entropy $H(X)$ is defined as $H(X) = -\sum_{i=1}^{N} p_i \log(p_i)$. For conditional random variables, we denote the probability of the random variable $X$ given $Y$ as $p(X|Y)$. The corresponding conditional Shannon entropy is defined as $H(X|Y) = -\sum_{i=1}^{N_X} \sum_{j=1}^{N_Y} p(x_i, y_j) \log[p(x_i | y_j)]$. Our entropy calculations shall be in bits, so $\log_2$ is used.

Suppose we have $|X|$ linearly independent and equiprobable input signals $x_i$, and possible output signals $y_j$, with $|Y| \geq |X|$. If there is no noise in the channel, then $C = \log(|X|)$.  

\[ 4 \]
Noise in the channel increases the uncertainty in \( X \) given the channel output \( Y \), and thus noise increases \( H(X|Y) \), thereby decreasing \( C \) for fixed \( H(X) \). Geometrically, the presence of random channel noise causes the channel mapping \( x_i \rightarrow y_j \) to change from a noiseless one-to-one relationship, to a stochastic map. We say the possible channel mappings of \( x_i \) diffuse, occupying a region \( \theta_i \) instead of a single unique state \( y_j \). As long as the regions \( \theta_i \) have disjoint support, the receiver can use \( Y \) to distinguish which \( X \) was sent. In this disjoint support case, \( H(X|Y) \approx 0 \) and \( C \approx H(X) \). This picture is frequently referred to as sphere packing, since we view the diffused output signals as roughly a sphere around the point in the output space where the signals would have been deposited had the channel introduced no perturbations. The greater the channel noise, the greater the radius of the spheres. If these spheres can be packed into a specified volume without significant overlap, then the decoder can distinguish the input state transmitted by determining which output sphere the decoded signal falls into.

For sending classical information over a quantum channel, we adhere to the same picture. We seek to maximize \( H(X) \) and minimize \( H(X|Y) \), in order to maximize the channel capacity \( C \). We encode each classical input signal state \( \{x_i\} \) into a corresponding quantum state \( \psi_i \). Sending \( \psi_i \) through the channel, the POVM decoder seeks to predict which \( x_i \) was originally sent. Similar to the classical picture, the quantum channel will diffuse or smear out the density matrix \( \rho_i \) corresponding to the quantum state \( \psi_i \) as the quantum state passes through the channel. The resulting channel output density matrix \( \mathcal{E}(\rho_i) \) will have support over a subspace \( \phi_i \). As long as all the regions \( \phi_i \) have disjoint support, the POVM based decoder will be able to distinguish which quantum state \( \rho_i \) entered the channel, and hence \( H(X|Y) \approx 0 \), yielding \( C \approx H(X) \).

For the classical capacity of quantum channels, we encode classical binary data into quantum states. The product state classical capacity for a quantum channel maximizes channel throughput by encoding a long block of \( m \) classical bits \( x_i \) into a long block consisting of a tensor product of \( n \) single qubit quantum states \( \psi_j \) in an optimal manner which maximizes (product state) classical channel capacity.

\[
\{x_1, x_2, \cdots, x_m\} \rightarrow \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n
\]

It has been widely conjectured, but not proven, that the product state classical channel capacity of a quantum channel is the classical capacity of a quantum channel.

The Holevo-Schumacher-Westmoreland Theorem tells us that the classical product state channel capacity using the above encoding scheme is given by the Holevo quantity \( \chi \) of the output signal ensemble, maximized over a single copy of all possible input signal ensembles \( \{p_i, \rho_i\} \).

\[
c_1 = \text{Max}_{\{\text{all possible } p_i \text{ and } \rho_i\}} \chi_{\text{output}}
\]

\[
= \text{Max}_{\{\text{all possible } p_i \text{ and } \rho_i\}} \left( S\left( \mathcal{E}\left( \sum_i p_i \rho_i \right) \right) - \sum_i p_i S\left( \mathcal{E}\left( \rho_i \right) \right) \right)
\]
of the von Neumann entropy. The symbol $\mathcal{E}(\rho)$ represents the output density matrix obtained by presenting the density matrix $\rho$ at the channel input. Furthermore, the input signals $\rho_i$ can be chosen to be pure states without affecting the maximization\cite{6}. Hereafter we shall call $C_1$ defined above the Holevo-Schumacher-Westmoreland (HSW) channel capacity.

2.2 Relative Entropy and HSW Channel Capacity

An alternate, but equivalent, description of HSW channel capacity can be made using relative entropy\cite{1}. The relative entropy $D$ of two density matrices, $\rho$ and $\phi$, is defined as \cite{1, 9, 11, 12}:

$$D(\rho \parallel \phi) = \text{Tr} \left[ \rho \log(\rho) - \rho \log(\phi) \right]$$

Here Tr[.] is the trace operator. Klein’s inequality tells us that $D \geq 0$, with $D \equiv 0$ iff $\rho \equiv \phi$\cite{9}. Note that we shall usually take our logarithms to be base 2.

To see how to represent $\chi$ in terms of $D$, consider the optimal signalling state ensemble $\{p_k, \varphi_k = \mathcal{E}(\varphi_k)\}$. Define $\varrho$ as $\sum_k p_k \varphi_k$. Consider the following sum:

$$\sum_k p_k D(\varphi_k \parallel \varrho) = \sum_k \{p_k \text{Tr}[\varphi_k \log(\varphi_k)] - p_k \text{Tr}[\varphi_k \log(\varrho)]\}$$

$$= \sum_k \{p_k \text{Tr}[\varphi_k \log(\varphi_k)]\} - \text{Tr}\left[ \sum_k \{p_k \varphi_k \log(\varrho)\} \right]$$

$$= \sum_k \{p_k \text{Tr}[\varphi_k \log(\varphi_k)]\} - \text{Tr}[\varrho \log(\varrho)] = S(\varrho) - \sum_k p_k S(\varphi_k) = \chi$$

Thus, the HSW capacity $C_1$ can be written as

$$C_1 = \text{Max}_{\text{all possible } \{p_k, \varphi_k\}} \sum_k p_k D(\varphi_k \parallel \varrho)$$

where the $\varphi_k$ are the quantum states input to the channel and $\varphi = \sum_k p_k \varphi_k$. We call an ensemble of channel output states $\{p_k, \varphi_k = \mathcal{E}(\varphi_k)\}$ an optimal ensemble if this ensemble achieves $C_1$. Schumacher and Westmoreland proved the following five properties related to optimal ensembles\cite{1}.

I) $D(\varphi_k \parallel \varrho) = C_1 \forall \varphi_k$ in the optimal ensemble, and $\varrho = \sum p_k \varphi_k$.

II) $D(\xi \parallel \varrho) \leq C_1$ where $\{p_k, \varphi_k = \mathcal{E}(\varphi_k), \varrho = \sum p_k \varphi_k\}$ is an optimal ensemble, and $\xi$ is any permissible channel output density matrix.
III) There exists at least one optimal ensemble \( \{ p_k, \varrho_k = \mathcal{E}(\varphi_k) \} \) that achieves \( C_1 \).

IV) Let \( A \) be the set of possible channel output states for a channel \( \mathcal{E} \) corresponding to pure state inputs. Define \( B \) as the convex hull of the set of states \( A \). Then for \( \varrho \in A \) and \( \xi \in B \equiv \) the convex hull of \( A \), we have:

\[
C_1 = \min_\xi \max_\varrho D(\varrho \| \xi)
\]

V) For every \( \xi \) that satisfies the minimization in IV) above, there exists an optimum signalling ensemble \( \{ p_k, \rho_k \} \) such that \( \xi \equiv \sum_k p_k \rho_k \).

### 2.3 The King - Ruskai - Szarek - Werner Qubit Channel Representation

In this paper, we are primarily concerned with qubit channels, namely \( \mathcal{E}(\varphi) = \varrho \), where \( \varphi \) and \( \varrho \) are qubit density matrices. Several authors [2, 3] have developed a nice picture of single qubit maps. Recall that single qubit density matrices can be written in the Bloch sphere representation. Let the density matrices \( \varrho \) and \( \varphi \) have the respective Bloch sphere representations:

\[
\varphi = \frac{1}{2} (I + \vec{W}_\varphi \cdot \vec{\sigma}) \quad \text{and} \quad \varrho = \frac{1}{2} (I + \vec{W}_\varrho \cdot \vec{\sigma})
\]

The symbol \( \vec{\sigma} \) means the vector of \( 2 \times 2 \) Pauli matrices

\[
\vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} \quad \text{where} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

The Bloch vectors \( \vec{W} \) are real three dimensional vectors that have magnitude equal to one when representing a pure state density matrix, and magnitude less than one for a mixed (non-pure) density matrix.

The King - Ruskai et al. qubit channel representation describes the channel as a mapping of input to output Bloch vectors.

\[
\begin{bmatrix} \frac{1}{W_x} \\ \frac{1}{W_y} \\ \frac{1}{W_z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_x & \lambda_x & 0 & 0 \\ t_y & 0 & \lambda_y & 0 \\ t_z & 0 & 0 & \lambda_z \end{bmatrix} \begin{bmatrix} 1 \\ W_x \\ W_y \\ W_z \end{bmatrix}
\]

All qubit channels have such a representation. The representation is unique up to a unitary rotation, and hence requires a choice of basis. The \( t_k \) and \( \lambda_k \) are real parameters.[1]

---

1. This result was originally derived in [IV].
which must satisfy certain constraints in order to ensure the matrix above represents a completely positive qubit map. (Please see King - Ruskai for more details [2].)

From the King - Ruskai et al. qubit channel representation, we see that \( \tilde{W}_k = t_k + \lambda_k W_k \) or

\[
W_k = \frac{\tilde{W}_k - t_k}{\lambda_k}
\]

It has been shown that \( C_1 \) can always be achieved using only pure input states [1]. Therefore, all input signalling Bloch vectors obey \( \| \vec{W} \| = 1 \). Thus \( \| \tilde{W} \|^2 = 1 = W_x^2 + W_y^2 + W_z^2 \) implies

\[
\left( \frac{\tilde{W}_x}{\lambda_x} - t_x \right)^2 + \left( \frac{\tilde{W}_y}{\lambda_y} - t_y \right)^2 + \left( \frac{\tilde{W}_z}{\lambda_z} - t_z \right)^2 = 1
\]

The set of possible channel output states we shall be interested in is the set of channel outputs corresponding to pure state channel inputs. This set of states was defined as \( \mathcal{A} \) in section 2.2, and is the surface of the ellipsoid shown above. The convex hull of the set of states \( \mathcal{A} \) is the solid ellipsoid defined as \( \tilde{W} \) such that

\[
\left( \frac{\tilde{W}_x}{\lambda_x} - t_x \right)^2 + \left( \frac{\tilde{W}_y}{\lambda_y} - t_y \right)^2 + \left( \frac{\tilde{W}_z}{\lambda_z} - t_z \right)^2 \leq 1
\]

3 Relative Entropy In The Bloch Sphere Representation

The key formula we shall use extensively is the relative entropy in the Bloch sphere representation. Here \( \rho \) and \( \phi \) have the respective Bloch sphere representations:

\[
\rho = \frac{1}{2} (I + \tilde{W} \cdot \vec{\sigma}) \quad \phi = \frac{1}{2} (I + \tilde{V} \cdot \vec{\sigma})
\]

We define \( \cos(\theta) \) as:

\[
\cos(\theta) = \frac{\tilde{W} \cdot \tilde{V}}{r q} \quad \text{where} \quad r = \sqrt{\tilde{W} \cdot \tilde{W}} \quad \text{and} \quad q = \sqrt{\tilde{V} \cdot \tilde{V}}.
\]

In Appendix A, we prove the following formula for the relative entropy \( \mathcal{D}(\rho \| \phi) \) of two single qubit density matrices \( \rho \) and \( \psi \) with Bloch sphere representations given above.

\[
\mathcal{D}(\rho \| \phi) = \frac{1}{2} \log_2 \left( 1 - r^2 \right) + \frac{r}{2} \log_2 \left( \frac{1 + r}{1 - r} \right) - \frac{1}{2} \log_2 \left( 1 - q^2 \right) - \frac{\tilde{W} \cdot \tilde{V}}{2 q} \log_2 \left( \frac{1 + q}{1 - q} \right)
\]

\[
= \frac{1}{2} \log_2 \left( 1 - r^2 \right) + \frac{r}{2} \log_2 \left( \frac{1 + r}{1 - r} \right) - \frac{1}{2} \log_2 \left( 1 - q^2 \right) - \frac{r \cos(\theta)}{2} \log_2 \left( \frac{1 + q}{1 - q} \right)
\]
where $\theta$ is the angle between $\vec{W}$ and $\vec{V}$, and $r$ and $q$ are as defined above.

When $\phi$ in $\mathcal{D}(\rho \parallel \phi)$ is the maximally mixed state $\phi = \frac{1}{2} \mathcal{I}$, we have $q = 0$, and $\mathcal{D}(\rho \parallel \phi)$ becomes the radially symmetric function

$$\mathcal{D}(\rho \parallel \phi) = \mathcal{D} \left( \rho \parallel \left. \frac{1}{2} \mathcal{I} \right) \right) = \frac{1}{2} \log_2 (1 - r^2) + \frac{r}{2} \log_2 \left( \frac{1 + r}{1 - r} \right) = 1 - S(\rho).$$

It is shown in Appendix A that $\mathcal{D} \left( \rho \parallel \frac{1}{2} \mathcal{I} \right) = 1 - S(\rho)$, where $S(\rho)$ is the von Neumann entropy of $\rho$. In what follows, we shall often write $\mathcal{D}(\rho \parallel \phi)$ as $\mathcal{D}(\vec{W} \parallel \vec{V})$, where $\vec{W}$ and $\vec{V}$ are the Bloch sphere vectors for $\rho$ and $\phi$ respectively.

In what follows, we shall graphically determine the HSW channel capacity from the intersection of contours of constant relative entropy with the channel ellipsoid. To that end, and to help build intuition regarding channel parameter tradeoffs, it is advantageous to obtain a rough idea of how the contours of constant relative entropy $\mathcal{D}(\rho \parallel \phi)$ behave, for fixed $\phi$, as $\rho$ is varied. Furthermore, it will turn out that due to symmetries in the relative entropy, we frequently will only need to understand the relative entropy behavior in a plane of the Bloch sphere, which we choose to be the Bloch X-Y plane. In Figure 2, we plot a few contour lines for $\mathcal{D}(\rho \parallel \phi = \frac{1}{2} \mathcal{I})$ in the X-Y Bloch sphere plane. In the figures that follow, we shall mark the location of $\phi$ with an asterisk. The contour values for $\mathcal{D}(\rho \parallel \phi)$ are shown in the plot title. The smallest value of $\mathcal{D}(\rho \parallel \phi)$ corresponds to the contour closest to the location of $\phi$. The largest value of $\mathcal{D}(\rho \parallel \phi)$ corresponds to the outermost contour. For $\phi = \frac{1}{2} \mathcal{I}$, the location of $\phi$ is the Bloch sphere origin.
q = 0.0, \ D(\rho\parallel\phi) = 0.01, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0

Figure 2: Contours of constant relative entropy $\mathcal{D}(\rho\parallel\phi)$ as a function of $\rho$ in the Bloch sphere $X$-$Y$ plane for the fixed density matrix $\phi = \frac{1}{2} \mathcal{I}$.

As an example of how these contour lines change as $\phi$ moves away from the maximally mixed state $\phi = \frac{1}{2} \mathcal{I}$, or equivalently as $q$ becomes non-zero, we give contour plots below for $q \neq 0$. We let $\phi = \frac{1}{2} \{ \mathcal{I} + q \sigma_y \}$ with corresponding Bloch vector $\vec{V} = \begin{bmatrix} 0 \\ q \\ 0 \end{bmatrix}$.

The asterisk in these plots denotes the location of $\vec{V}$. The dashed outer contour is a radius equal to one, indicating where the pure states lie.
Figure 3: Contours of constant relative entropy $D(\rho||\phi)$ as a function of $\rho$ in the Bloch sphere X-Y plane for the fixed density matrix $\phi = \frac{1}{2} \{ I + 0.1 \sigma_y \}$.

Figure 4: Contours of constant relative entropy $D(\rho||\phi)$ as a function of $\rho$ in the Bloch sphere X-Y plane for the fixed density matrix $\phi = \frac{1}{2} \{ I + 0.2 \sigma_y \}$.
Figure 5: Contours of constant relative entropy $\mathcal{D}(\rho||\phi)$ as a function of $\rho$ in the Bloch sphere X-Y plane for the fixed density matrix $\phi = \frac{1}{2} \{ I + 0.3 \sigma_y \}$.

Figure 6: Contours of constant relative entropy $\mathcal{D}(\rho||\phi)$ as a function of $\rho$ in the Bloch sphere X-Y plane for the fixed density matrix $\phi = \frac{1}{2} \{ I + 0.4 \sigma_y \}$.
Figure 7: Contours of constant relative entropy $D(\rho\|\phi)$ as a function of $\rho$ in the Bloch sphere X-Y plane for the fixed density matrix $\phi = \frac{1}{2} \{ I + 0.5 \sigma_y \}$.

Figure 8: Contours of constant relative entropy $D(\rho\|\phi)$ as a function of $\rho$ in the Bloch sphere X-Y plane for the fixed density matrix $\phi = \frac{1}{2} \{ I + 0.6 \sigma_y \}$. 
Figure 9: Contours of constant relative entropy $D(\rho\|\phi)$ as a function of $\rho$ in the Bloch sphere X-Y plane for the fixed density matrix $\phi = \frac{1}{2} \{ \mathcal{I} + 0.7 \sigma_y \}$.

Figure 10: Contours of constant relative entropy $D(\rho\|\phi)$ as a function of $\rho$ in the Bloch sphere X-Y plane for the fixed density matrix $\phi = \frac{1}{2} \{ \mathcal{I} + 0.8 \sigma_y \}$.
Figure 11: Contours of constant relative entropy $D(\rho\|\phi)$ as a function of $\rho$ in the Bloch sphere X-Y plane for the fixed density matrix $\phi = \frac{1}{2} \{ I + 0.9 \sigma_y \}$.

The two dimensional plots of $D(\rho\|\phi)$ shown above tell us about the three dimensional nature of $D(\rho\|\phi)$. To see why, first note that we can always rotate the Bloch sphere X-Y-Z axes to arrange for $\phi \equiv \vec{V} \rightarrow \vec{q}$ to lie on the Y axis, as the density matrices $\phi$ are shown in Figures 2 through 11 above.

Second, recall that our description of $D(\rho\|\phi)$ is a function of the three variables $\{ r, q, \theta \}$ only, which were defined above as the length of the Bloch vectors corresponding to the density matrices $\rho$ and $\phi$ respectively, and the angle between these Bloch vectors.

$$D(\rho\|\phi) \equiv f(r, q, \theta)$$

This means the two dimensional curves of constant $D(\rho\|\phi)$ can be rotated about the Y axis as surfaces of revolution, to yield three dimensional surfaces of constant $D(\rho\|\phi)$. (In these two and three dimensional plots, the first argument of $D(\rho\|\phi)$, $\rho$, is being varied, while the second argument, $\phi$, is being held fixed at a point on the Y axis.)

Our two dimensional plots above give us a good idea of the three dimensional behavior of the surfaces of constant relative entropy about the density matrix $\phi$ occupying the second slot in $D(\cdots\|\cdots)$. A picture emerges of slightly warped "eggshells" nested like Russian dolls inside each other, roughly centered on $\phi$. A mental picture of the behavior of $D(\rho\|\phi)$ is useful because in what follows we shall superimpose the KRSW channel ellipsoid(s) onto Figures 2 through 11 above. By moving $\vec{V}$ (the asterisk) around in these pictures, we
shall adjust the contours of constant $\mathcal{D}(\rho \parallel \phi)$, and thereby graphically determine the HSW channel capacity, optimum (output) signalling states, and corresponding a priori signalling probabilities. The resulting intuition we gain from these pictures will help us understand channel parameter tradeoffs.

### 4 Linear Channels

Recall the KRSW specification of a qubit channel in terms of the six real parameters $\{t_x, t_y, t_z, \lambda_x, \lambda_y, \lambda_z\}$ as defined on page 7 of this paper. A linear channel is one where $\lambda_x = \lambda_y = 0$, but $\lambda_z \neq 0$. The shift quantities $t_k$ can be any real number, up to the limits imposed by the requirement that the map be completely positive. For more details on the complete positivity requirements of qubit maps, please see [3].

A linear channel is a simple system that illustrates the basic ideas behind our graphical approach to determining the HSW channel capacity $C_1$. Recall the relative entropy formulation for $C_1$.

$$C_1 = \max_{\{all\ possible\ \{p_k, \varphi_k\}\}} \sum_k p_k \mathcal{D}(\mathcal{E}(\varphi_k) \parallel \mathcal{E}(\varphi))$$

where the $\varphi_k$ are the quantum states input to the channel and $\varphi = \sum_k p_k \varphi_k$. We call an ensemble of states $\{p_k, \varphi_k = \mathcal{E}(\varphi_k)\}$ an optimal ensemble if this ensemble achieves $C_1$.

As discussed on page 6, Schumacher and Westmoreland showed the above maximization to determine $C_1$ is equivalent to the following min-max criterion:

$$C_1 = \min_{\phi} \max_{\varrho_k} \mathcal{D}(\varrho_k \parallel \phi)$$

where $\varrho_k$ is a density matrix on the surface of the channel ellipsoid, and $\phi$ is a density matrix in the convex hull of the channel ellipsoid. For the linear channel, the channel ellipsoid is a line segment of length $2 \lambda_z$ centered on $\{t_x, t_y, t_z\}$. Thus, both $\varrho_k$ and $\phi$ must lie somewhere along this line segment. Furthermore, Schumacher and Westmoreland tell us that $\phi$ must be expressible as a convex combination of the $\varrho_k$ which satisfy the above min-max[1].

To graphically implement the min-max criterion, we overlay the channel ellipsoid on the contour plots of relative entropy previously found. We wish to determine the location of the optimum $\phi$ and the optimum relative entropy contour that achieves the min-max. The generic overlap scenarios are shown below, labeled Cases 1 - 5. From our plots of relative entropy, we know that contours of relative entropy are roughly circular about $\phi$. We denote the location of $\phi$ below by an asterisk ($*$). The permissible $\varrho_k$ are those density matrices at the intersection of the relative entropy contour and the channel ellipsoid, here a line segment.

Let us examine the five cases shown below, seeking the optimum $\phi$ and the relative entropy contour corresponding to $C_1$ (the circles below), by eliminating those cases which do not make sense in light of the minimization-maximization above.
Figure 12: Scenarios for the intersection of the optimum relative entropy contour with a linear channel ellipsoid.

Case 1 is not an acceptable configuration because $\phi$ does not lie inside the channel ellipsoid, meaning for the linear channel, $\phi$ does not lie on the line segment. Case 2 is not acceptable because there are no permissible $g_k$, since the relative entropy contour does not intersect the channel ellipsoid line segment anywhere. Case 3 is not acceptable because Schumacher and Westmoreland tell us that $\phi$ must be expressible as a convex combination of the $g_k$ density matrices which satisfy the above min-max requirement. There is only one permissible $g_k$ density matrix in Case 3, and since, as seen in the diagram for Case 3, $\phi \neq g_1$, we do not have an acceptable configuration. Case 4 at first appears acceptable. However, here we do not achieve the maximization in the min-max relation, since we can do better by using a relative entropy contour with a larger radius. Case 5 is the ideal situation. The relative entropy contour intersects both of the line segment endpoints. Taking a larger radius relative entropy contour does not give us permissible $g_k$, since we would obtain Case 2 with a larger radii. For Case 5, if we moved $\phi$ as we increased the relative entropy contour, we would obtain Case 3, again an unacceptable configuration. In Case 5, using the two $g_k$ that lie at the intersection of the relative entropy contour and the channel ellipsoid line segment, we can form a convex combination of these $g_k$ that equals $\phi$. Case 5 is the best we can do, meaning Case 5 yields the largest radius relative entropy contour which satisfies the Schumacher-Westmoreland requirements. The value of this largest radii relative entropy contour is the HSW channel capacity we seek, $C_1$.

We now restate Case 5 in Bloch vector notation. We shall associate the Bloch vector $\vec{V}$ with $\phi$, and the Bloch vectors $\vec{W}_k$ with the $g_k$ density matrices. For the linear channel, from our analysis above which resulted in Case 5, we know that $\vec{V}$ must lie on the line segment between the two endpoint vectors $\vec{W}_+$ and $\vec{W}_-$. (Note that from here on, we shall drop the tilde $\tilde{}$ we were previously using to denote channel output Bloch vectors, as almost all the Bloch vectors we shall talk about below are channel output Bloch vectors. The few instances when this is not the case shall be obvious.)

For a general linear channel, the KRSW ellipsoid channel parameters satisfy

\[
\{ t_x \neq 0 , t_y \neq 0 , t_z \neq 0 , \lambda_x = 0 , \lambda_y = 0 , \lambda_z \neq 0 \}.
\]

Thus, we can explicitly determine the Bloch vectors $\vec{W}_+$ and $\vec{W}_-$, which we write below.

\[
\rho_+ \to \vec{W}_+ = \begin{bmatrix} t_x \\ t_y \\ t_z + \lambda_z \end{bmatrix}, \quad \text{and} \quad \rho_- \to \vec{W}_- = \begin{bmatrix} t_x \\ t_y \\ t_z - \lambda_z \end{bmatrix}.
\]
Note that the \( t_k \) and \( \lambda_z \) are real numbers, and any of them may be negative.

The Bloch vector \( \vec{V} \) however requires more work. We parameterize the Bloch sphere vector \( \vec{V} \) corresponding to \( \phi \) by the real number \( \alpha \), specifying a position for \( \vec{V} \) along the line segment between \( \vec{W}_+ \) and \( \vec{W}_- \).

\[
\phi \rightarrow \vec{V} = \begin{bmatrix} t_x \\ t_y \\ t_z + \alpha \lambda_z \end{bmatrix}
\]

Here \( \alpha \in [-1, 1] \). Now recall that the Schumacher-Westmoreland maximal distance property (see property \# I in Section 2.2) tells us that \( D(\rho_+||\phi) = D(\rho_-||\phi) \). To find \( \vec{V} \), we shall apply the formula we have derived for relative entropy in the Bloch representation to \( D(\rho_+||\phi) = D(\rho_-||\phi) \), and solve for \( \alpha \). The details are in Appendix B.

### 4.1 A Simple Linear Channel Example

To illustrate the ideas presented above, we take as a simple example the linear channel with channel parameters: \{ \( t_x = 0 \), \( t_y = 0 \), \( t_z = 0.2 \), \( \lambda_x = 0 \), \( \lambda_y = 0 \), \( \lambda_z = 0.4 \) \}. Because the channel is linear with \( t_x = t_y = 0 \), we shall be able to easily solve for \( \vec{V} \) and \( \mathcal{C}_1 \).

We define the real numbers \( r_+ \) and \( r_- \) as the Euclidean distance in the Bloch sphere from the Bloch sphere origin to the Bloch vectors \( \vec{W}_+ \) and \( \vec{W}_- \). That is, \( r_+ \) and \( r_- \) are the magnitudes of the Bloch vectors \( \vec{W}_+ \) and \( \vec{W}_- \) defined above. For the channel parameter numbers given, we find \( r_+ = \| 0.4 + 0.2 \| = 0.6 \) and \( r_- = \| 0.2 - 0.4 \| = 0.2 \). We similarly define \( q \) to be the magnitude of the Bloch vector \( \vec{V} \).

To find \( \vec{V} \), we shall apply the formula we have derived for relative entropy in the Bloch representation to \( D(\rho_+||\phi) = D(\rho_-||\phi) \), or in Bloch sphere notation, \( D(\vec{W}_+||\vec{V}) = D(\vec{W}_-||\vec{V}) \). The formula for relative entropy derived in Appendix A is:

\[
D(\rho_k \| \phi) = \frac{1}{2} \log_2 \left( 1 - r_k^2 \right) + \frac{r_k}{2} \log_2 \left( \frac{1 + r_k}{1 - r_k} \right) - \frac{1}{2} \log_2 \left( 1 - q^2 \right) - \frac{\vec{W}_k \cdot \vec{V}}{2q} \log_2 \left( \frac{1 + q}{1 - q} \right)
\]

\[
= \frac{1}{2} \log_2 \left( 1 - r_k^2 \right) + \frac{r_k}{2} \log_2 \left( \frac{1 + r_k}{1 - r_k} \right) - \frac{1}{2} \log_2 \left( 1 - q^2 \right) - \frac{r_k \cos(\theta_k)}{2} \log_2 \left( \frac{1 + q}{1 - q} \right)
\]

where \( \theta_k \) is the angle between \( \vec{W}_k \) and \( \vec{V} \). Intuitively, one notes that the nearly circular relative entropy contours about \( \phi \equiv \vec{V} \) tells us that \( \vec{V} \approx \frac{\vec{W}_+ + \vec{W}_-}{2} \). Given the channel parameter numbers, this fact about \( \vec{V} \), together with the linear nature of the channel ellipsoid, tell us that \( \theta_+ = 0 \) and \( \theta_- = \pi \), so that \( \cos(\theta_+) = 1 \) and \( \cos(\theta_-) = -1 \). Using this information about the \( \theta_k \), and the identity

\[
\tanh^{-1}(x) = \frac{1}{2} \log \left( \frac{1 + x}{1 - x} \right)
\]

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the relative entropy equality relation between the two endpoints of the linear channel can be solved for q.

\[
q_{\text{optimum}} = \tanh \left[ \frac{1}{2} \ln \left( \frac{1 - r_+^2}{1 - r_-^2} \right) + r_+ \tanh^{-1}(r_+) - r_- \tanh^{-1}(r_-) \right] = 0.2125.
\]

Thus,

\[
\vec{W}_+ = \begin{bmatrix} 0 \\ 0 \\ 0.6 \end{bmatrix}, \quad \vec{W}_- = \begin{bmatrix} 0 \\ 0 \\ -0.2 \end{bmatrix}, \quad \text{and} \quad \vec{V} = \begin{bmatrix} 0 \\ 0 \\ 0.2125 \end{bmatrix}
\]

The corresponding density matrices are:

\[
\rho_+ = \frac{1}{2} \left( I + \vec{W}_+ \cdot \sigma \right), \quad \rho_- = \frac{1}{2} \left( I + \vec{W}_- \cdot \sigma \right), \quad \phi = \frac{1}{2} \left( I + \vec{V} \cdot \sigma \right)
\]

This yields \( D(\rho_+ \parallel \phi) = D(\rho_- \parallel \phi) = 0.1246 \). Thus, the HSW channel capacity \( C_1 \) is 0.1246. The location of the two density matrices \( \rho_+ \) and \( \rho_- \) are shown in Figure 13 below as \( \mathbf{O} \).

Furthermore, the Schumacher-Westmoreland analysis tells us that the two states \( \rho_+ \) and \( \rho_- \) must average to \( \phi \), in the sense that if \( p_+ \) and \( p_- \) are the a priori probabilities of the two output signal states, then \( p_+ \rho_+ + p_- \rho_- = \phi \). In our Bloch sphere notation, this relationship becomes \( p_+ \vec{W}_+ + p_- \vec{W}_- = \vec{V} \). The asterisk (\( * \)) in Figure 13 below shows the position of \( \vec{V} \).

Another relation relating the a priori probabilities \( p_+ \) and \( p_- \) is \( p_+ + p_- = 1 \). Using these two equations, we can solve for the a priori probabilities \( p_+ \) and \( p_- \). For our example,

\[
p_+ \vec{W}_+ + p_- \vec{W}_- = p_+ \begin{bmatrix} 0 \\ 0 \\ 0.6 \end{bmatrix} + p_- \begin{bmatrix} 0 \\ 0 \\ -0.2 \end{bmatrix} = \vec{V} = \begin{bmatrix} 0 \\ 0 \\ 0.2125 \end{bmatrix}
\]

Solving for \( p_+ \) and \( p_- \) yields \( p_+ = 0.5156 \) and \( p_- = 0.4844 \).

Note that here we have found the optimum output signal states \( \rho_+ \) and \( \rho_- \). From these one can find the optimum input signal states by finding the states \( \varphi_+ \) and \( \varphi_- \) which map to the respective optimum output states \( \rho_+ \) and \( \rho_- \). In our example above, these are

\[
\varphi_+ \rightarrow \vec{W}_+^{\text{Input}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \varphi_- \rightarrow \vec{W}_-^{\text{Input}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\]
For the general linear channel, where any or all of the $t_k$ can be non-zero, we can reduce the capacity calculation to the solution of a single, one dimensional transcendental equation. (Please see Appendix B for the full derivation.)

Define

$$r_+^2 = t_x^2 + t_y^2 + (t_z + \lambda_z)^2$$

$$q^2 = t_x^2 + t_y^2 + (t_z + \beta \lambda_z)^2$$

$$r_-^2 = t_x^2 + t_y^2 + (t_z - \lambda_z)^2$$

The two quantities $r_+$ and $r_-$ are the Euclidean distances from the Bloch sphere origin to the signalling states $\rho_+ \equiv \hat{W}_+$ and $\rho_- \equiv \hat{W}_-$ respectively. The quantity $q$ is the Euclidean distance from the Bloch sphere origin to the density matrix $\phi \equiv \hat{V}$. We define the three Bloch vectors $\vec{r}_+, \vec{q}$ and $\vec{r}_-$ in Figure 14 below, and refer to their respective magnitudes as $r_+, q$, and $r_-$. 

Figure 13: The intersection in the Bloch sphere X-Z plane of a linear channel ellipsoid and the optimum relative entropy contour. The optimum output signal states are shown as $O$.
Figure 14: Definition of the Bloch vectors $\vec{r}_+$, $\vec{q}$, and $\vec{r}_-$ used in the derivation below.

We solve the transcendental equation below for $\beta$.

$$\frac{4 \lambda_z (t_z + \beta \lambda_z)}{q} \, \tanh^{-1}(q) = 2 r_+ \, \tanh^{-1}(r_+) - 2 r_- \, \tanh^{-1}(r_-) + \ln(1 - r_+^2) - \ln(1 - r_-^2)$$

Note that $q$ is a function of $\beta$, while $r_+$ and $r_-$ are not. Thus, the right hand side remains constant while $\beta$ is varied. The smooth nature of the functions of $\beta$ on the left hand side allow a solution for $\beta$ to be found fairly easily.

As in our simpler linear channel example above, we have

$$\mathcal{W}_+ = \begin{bmatrix} t_x \\ t_y \\ t_z + \lambda_z \end{bmatrix}, \quad \mathcal{W}_- = \begin{bmatrix} t_x \\ t_y \\ t_z - \lambda_z \end{bmatrix}, \quad \text{and} \quad \vec{V} = \begin{bmatrix} t_x \\ t_y \\ t_z + \beta \lambda_z \end{bmatrix}.$$ 

where $\beta \in (-1, 1)$. The corresponding density matrices are:

$$\rho_+ = \frac{1}{2} (\mathcal{I} + \mathcal{W}_+ \cdot \vec{\sigma}), \quad \rho_- = \frac{1}{2} (\mathcal{I} + \mathcal{W}_- \cdot \vec{\sigma}), \quad \phi = \frac{1}{2} (\mathcal{I} + \vec{V} \cdot \vec{\sigma})$$

The channel capacity $C_1$ is found from the relations

$$D(\rho_+ || \phi) = D(\rho_- || \phi) = \chi_{optimum} = C_1.$$ 

The a priori signaling probabilities are found by solving the simultaneous probability equations $p_+ + p_- = 1$, and

$$p_+ \mathcal{W}_+ + p_- \mathcal{W}_- = p_+ \begin{bmatrix} t_x \\ t_y \\ t_z + \lambda_z \end{bmatrix} + p_- \begin{bmatrix} t_x \\ t_y \\ t_z - \lambda_z \end{bmatrix} = \vec{V} = \begin{bmatrix} t_x \\ t_y \\ t_z + \beta \lambda_z \end{bmatrix}.$$ 

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This leads to a second probability equation of $p_+ - p_- = \beta$, yielding:

\[
p_+ = \frac{1 + \beta}{2} \quad \text{and} \quad p_- = \frac{1 - \beta}{2}
\]

### 4.2 A More General Linear Channel Example

In the simple linear channel example above, we used \( \{ t_x = t_y = 0 , \lambda_x = \lambda_y = 0 \} \). This choice yielded a rotational symmetry about the \( Z \)-axis which assured us the location of the optimum average output density matrix \( \rho = p_+ \rho_+ + p_- \rho_- \) was on the \( Z \)-axis. We used this fact to advantage in predicting the angles \( \theta_{\{+,\}+} \), where \( \theta_{\{+,\}+} \) was the angle between \( \vec{W}_{\{+,\}+} \) and \( \vec{V} \). Since we knew \( \vec{W}_{\{+,\}+} \) lay on the \( Z \)-axis, we found \( \theta_+ = 0 \) and \( \theta_- = \pi \), simplifying the \( \cos(\theta_{\{+,\}+}) \) terms in the relative entropy expressions for \( D(\rho_+||\phi) \) and \( D(\rho_-||\phi) \). In general, we do not have values of \( \pm 1 \) for \( \cos(\theta_{\{+,\}+}) \), and this complicates finding a solution for the linear channel relation \( D(\rho_+||\phi) = D(\rho_-||\phi) \).

A more general linear channel example is one where the parameters \( \{ t_x , t_y , t_z \} \) are all non-zero. Consider the parameter set \( \{ t_x = 0.1 , t_y = 0.2 , t_z = 0.3 , \lambda_x = 0 , \lambda_y = 0 , \lambda_z = 0.4 \} \). Solving the transcendental equation derived in Appendix B yields \( \beta = 0.0534 \) and \( \vec{V} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3214 \end{bmatrix} \). Using the density matrix \( \phi \) calculated from the Bloch vector \( \vec{V} \) gives us a HSW channel capacity \( C_1 = 0.1365 \).

As discussed above, \( p_+ + p_- = 1 \), and \( p_+ - p_- = \beta \). Solving for \( p_+ \) and \( p_- \) yields \( p_+ = 0.5267 \) and \( p_- = 0.4733 \).

The optimum input Bloch vectors are:

\[
\phi_+ \rightarrow \vec{W}_{\text{Input}}^{\phi_+} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \phi_- \rightarrow \vec{W}_{\text{Input}}^{\phi_-} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.
\]

The optimum output Bloch vectors are:

\[
\rho_+ = \mathcal{E}(\phi_+) \rightarrow \vec{W}_{\text{Output}}^{\rho_+} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix} \quad \text{and} \quad \rho_- = \mathcal{E}(\phi_-) \rightarrow \vec{W}_{\text{Output}}^{\rho_-} = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.1 \end{bmatrix}.
\]

Below we show in Figure 15 and Figure 16 the \( \{x,z\} \) and \( \{y,z\} \) slices of the linear channel ellipsoid. One can see that the relative entropy curve \( D(\rho||\phi) = C_1 = 0.1365 \) touches the ellipsoid at two locations in both cross sections. (The \( \{x,y\} \) cross section is trivial.)
Figure 15: The intersection in the Bloch sphere X-Z plane of a linear channel ellipsoid and the optimum relative entropy contour. The optimum output signal states are shown as O.

Figure 16: The intersection in the Bloch sphere Y-Z plane of a linear channel ellipsoid and the optimum relative entropy contour. The optimum output signal states are shown as O.
5 Planar Channels

A planar channel is a quantum channel where two $\lambda_k$ are non-zero, and one $\lambda_k$ is zero. For a planar channel, the $\{t_k\}$ can have any values allowed by complete positivity. A planar channel restricts the possible output density matrices to lie in the plane in the Bloch sphere which is specified by the non-zero $\lambda_k$. In comparison to the linear channels discussed above, the planar channels additional output degree of freedom (planar has two non-zero $\lambda_k$ versus a single linear non-zero $\lambda_k$) means a slightly different approach to determining $C_1$ than that discussed for linear channels must be developed. As for linear channels, we seek to find the optimum density matrix $\phi \equiv \vec{V}$ interior to the ellipsoid which minimizes the distance to the most ”distant”, in a relative entropy sense, point(s) on the ellipsoid surface. We shall find the optimum $\vec{V}$ in two ways : graphically and iteratively. Both approaches utilize the following theorem from Schumacher and Westmoreland[1].

Theorem :

$$C_1 = \min_{\phi} \max_{\rho} D(\rho \parallel \phi)$$

The maximum is taken over the surface of the ellipsoid, and the minimum is taken over the interior of the ellipsoid. In order to apply the min max formula above for $C_1$ for planar channels, we need a result about the uniqueness of the average output ensemble density matrix $\rho = \sum_k p_k \rho_k$ for different optimal ensembles $\{p_k, \rho_k\}$.

5.1 Uniqueness Of The Average Output Ensemble Density Matrix

The question we address is if there exists two optimum signalling ensembles, $\{p_k, \rho_k\}$ and $\{p'_k, \rho'_k\}$ of channel output states, whether the two resulting average density matrices, $\rho = \sum_k p_k \rho_k$ and $\rho' = \sum_k p'_k \rho'_k$ are equal.

Theorem : The density matrix $\phi$ which achieves the minimum in the min-max formula above for $C_1$ is unique.

Proof :

From property V in Section 2.2, we know the optimum $\phi$ which attains the minimum above must correspond to the average of a set of signal states of an optimum signalling ensemble. We shall prove the uniqueness of $\phi$ by postulating there are two optimum signal ensembles, with possibly different average density matrices, $\sigma$ and $\xi$. We will then prove that $\sigma$ must equal $\xi$, thereby implying $\phi$ is unique.

Let $\{\alpha_i, \rho_i\}$ be an optimum signal ensemble, with probabilities $\alpha_i$ and density matrices $\rho_i$, where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. Define $\sigma = \sum_i \alpha_i \rho_i$. By property I in Section 2.2, we know that $D(\rho_i \parallel \sigma) = \chi_{optimum} = C_1 \forall i$.

Now consider a second, optimum signal ensemble $\{\beta_j, \phi_j\}$ differing in at least one density matrix $\rho_i$ and/or one probability $\alpha_i$ from the optimum ensemble $\{\alpha_i, \rho_i\}$. Define $\xi = \sum_j \beta_j \phi_j$. Consider the quantity $\sum_i \alpha_i D(\rho_i \parallel \xi)$. Let us apply Donald’s equality, which is discussed in Appendix C.
\[
\sum_i \alpha_i D(\rho_i \| \xi) = D(\sigma \| \xi) + \sum_i \alpha_i D(\rho_i \| \sigma)
\]

Since \(D(\rho_i \| \sigma) = \chi_{\text{optimum}} \quad \forall \ i\), and \(\sum_i \alpha_i = 1\), we obtain:
\[
\sum_i \alpha_i D(\rho_i \| \xi) = D(\sigma \| \xi) + \chi_{\text{optimum}}
\]

From property II in Section 2.2, since \(\xi\) is the average of a set of optimal signal states \(\{\beta_j, \phi_j\}\), we know that \(D(\rho_i \| \xi) \leq \chi_{\text{optimum}} \quad \forall \ i\). Thus \(\sum_i \alpha_i D(\rho_i \| \xi) \leq \chi_{\text{optimum}}\). Combining this inequality constraint on \(\sum_i \alpha_i D(\rho_i \| \xi)\) with what we know about \(\sum_i \alpha_i D(\rho_i \| \xi)\) from Donald’s equality, we obtain the two relations:
\[
\sum_i \alpha_i D(\rho_i \| \xi) = D(\sigma \| \xi) + \chi_{\text{optimum}} \quad \text{and} \quad \sum_i \alpha_i D(\rho_i \| \xi) \leq \chi_{\text{optimum}}
\]

From Klein’s inequality, we know that \(D(\sigma \| \xi) \geq 0\), with equality iff \(\sigma \equiv \xi\). Thus, the only way the equation
\[
\sum_i \alpha_i D(\rho_i \| \xi) = D(\sigma \| \xi) + \chi_{\text{optimum}}
\]
can be satisfied is if we have \(\sigma \equiv \xi\), for then \(D(\sigma \| \xi) = 0\) and we have
\[
\sum_i \alpha_i D(\rho_i \| \xi) = D(\sigma \| \xi) + \chi_{\text{optimum}} = \chi_{\text{optimum}}
\]

and
\[
\sum_i \alpha_i D(\rho_i \| \xi) = \sum_i \alpha_i D(\rho_i \| \sigma) = \sum_i \alpha_i \chi_{\text{optimum}} = \chi_{\text{optimum}}
\]

Therefore, only in the case where \(\sigma \equiv \xi\) is Donald’s equality satisfied. Since \(\sigma\) and \(\xi\) were the average output density matrices for two different, but arbitrary optimum signalling ensembles, we conclude the average density matrices of all optimum signalling ensembles must be equal, thereby implying \(\phi\) is unique.

\(\triangle\ - \text{End of Proof}\).

Note that although we are primarily concerned with qubit channels in this paper, only generic properties of the relative entropy were used in the above proof of uniqueness, and therefore the result holds for all channels.

5.2 Graphical Channel Optimization Procedure

We shall now describe a graphical technique for finding \(\phi_{\text{optimum}} \equiv \vec{\mathcal{V}}_{\text{optimum}}\). Recall the contour surfaces of constant relative entropy for various values of \(\vec{\mathcal{V}}\) shown previously. We
seek to adjust the location of \( \vec{V} \) inside the channel ellipsoid such that the largest possible contour value \( D_{\text{max}} = D(\vec{W} \parallel \vec{V}) \) touches the ellipsoid surface, and the remainder of the \( D_{\text{max}} \) contour surface lies entirely outside the channel ellipsoid. Our linear channel example illustrated this idea. In that example, the \( D_{\text{max}} \) contour intersects the "ellipsoid" at \( r_+ \) and \( r_- \), and otherwise lies outside the line segment between \( r_+ \) and \( r_- \) representing the convex hull of \( \mathcal{A} \). (Recall from the discussion of the Schumacher and Westmoreland paper in Section 2.2 that the points on the ellipsoid surface were defined as the set \( \mathcal{A} \), and the interior of the ellipsoid, where \( \vec{V} \) lives, is the convex hull of \( \mathcal{A} \).)

A good place to start is with \( \vec{V}_{\text{initial}} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \). We then "tweak" \( \vec{V} \) as described above to find \( \vec{V}_{\text{optimum}} \). Note that \( \vec{V}_{\text{optimum}} \) should be near \( \vec{V}_{\text{initial}} \) because of the almost radial symmetry of \( D \) about \( \vec{V} \) as seen in Figures 2 through 11.

This technique is graphically implementing property IV in Section 2.2. In Bloch sphere notation, we have:

\[
c_1 = \min_{\vec{V}} \max_{\vec{W}} D(\vec{W} \parallel \vec{V})
\]

where \( \vec{W} \) is on the channel ellipsoid surface and \( \vec{V} \) is in the interior of the ellipsoid. Moving \( \vec{V} \) from the optimum position described above will increase \( \max_{\vec{W}} D(\vec{W} \parallel \vec{V}) \), since a larger contour value of \( D \) would then intersect the channel ellipsoid surface, thereby increasing \( \max_{\vec{W}} D(\vec{W} \parallel \vec{V}) \). Yet \( \vec{V} \) should be adjusted to minimize \( \max_{\vec{W}} D(\vec{W} \parallel \vec{V}) \).

### 5.3 Iterative Channel Optimization Procedure

For the iterative treatment, we outline an algorithm which converges to \( \vec{V}_{\text{optimum}} \). First, we need a lemma.

**Lemma:** Let \( \vec{V} \) and \( \vec{W} \) be any two Bloch sphere vectors. Define a third Bloch sphere vector \( \vec{U} \) as:

\[
\vec{U} = (1 - \alpha) \vec{W} + \alpha \vec{V}
\]

where \( \alpha \in (0, 1) \). Then

\[
D(\vec{W} \parallel \vec{U}) < D(\vec{W} \parallel \vec{V})
\]

**Proof:** By the joint convexity property of the relative entropy [3]:

\[
D(\{ \alpha \rho_1 + (1 - \alpha) \rho_2 \} \| \{ \alpha \phi_1 + (1 - \alpha) \phi_2 \}) \leq \alpha D(\rho_1 \| \phi_1) + (1 - \alpha) D(\rho_2 \| \phi_2)
\]

where \( \alpha \in (0, 1) \). Let \( \rho_1 = \rho_2 \equiv \vec{W}, \phi_1 \equiv \vec{V} \) and \( \phi_2 \equiv \vec{W} \) with \( \vec{U} = (1 - \alpha) \vec{W} + \alpha \vec{V} \). We obtain:

\[
D(\vec{W} \parallel \vec{U}) = D(\vec{W} \parallel \alpha \vec{V} + (1 - \alpha) \vec{W}) \leq \alpha D(\vec{W} \parallel \vec{V}) + (1 - \alpha) D(\vec{W} \parallel \vec{W})
\]

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But $\mathcal{D}(\tilde{W} \parallel \tilde{V}) = 0$, by Klein’s inequality\[9\]. Thus,

$$\mathcal{D}(\tilde{W} \parallel \tilde{U}) \leq \alpha \mathcal{D}(\tilde{W} \parallel \tilde{V}) < \mathcal{D}(\tilde{W} \parallel \tilde{V})$$

since $\alpha \in (0, 1)$.

$\triangle$ - End of Proof.

We use the lemma above to guide us in iteratively adjusting $\tilde{V}$ to converge towards $\tilde{V}_{optimal}$. Consider $\mathcal{D}(\tilde{W} \parallel \tilde{V})$, where $\tilde{W} \in \mathcal{A}$ and $\tilde{V} \in \mathcal{B} \equiv$ the convex hull of $\mathcal{A}$. We seek to find $C_1$ in an iterative fashion. We do this by holding $\tilde{V}$ fixed, and finding one of the $\tilde{W}' \in \mathcal{A}$ which maximizes $\mathcal{D}(\tilde{W} \parallel \tilde{V})$. From our lemma above, if we now move $\tilde{V}$ towards $\tilde{W}'$, we shall cause $\mathcal{D}_{max}(\tilde{V}) = Max_{\tilde{W}} \mathcal{D}(\tilde{W} \parallel \tilde{V})$ to decrease. We steadily decrease $\mathcal{D}_{max}(\tilde{V})$ in this manner until we reach a point where any movement of $\tilde{V}$ will increase $\mathcal{D}_{max}(\tilde{V})$. Our uniqueness theorem above tells us there is only one $\tilde{V}_{optimal}$. Our lemma above tells us we cannot become stuck in a local minima in moving towards $\tilde{V}_{optimal}$. Thus, when we reach the point where any movement of $\tilde{V}$ will increase $\mathcal{D}_{max}(\tilde{V})$, we are done and have found $\tilde{V}_{final} = \tilde{V}_{optimum}$.

To summarize, we find the optimum $\tilde{V}$ using the following algorithm.

1) Generate a random starting point $\tilde{V}_{initial}$ in the interior of the ellipsoid ($\in \mathcal{B}$). (In actuality, since the contour surfaces of constant relative entropy are roughly spherical about $\tilde{V}$, a good place to start is $\tilde{V}_{initial} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$.)

2) Determine the set of points $\{ \tilde{W}' \}$ on the ellipsoid surface most distant, in a relative entropy sense, from our $\tilde{V}$. This maximal distance is $\mathcal{D}_{max}(\tilde{V})$ defined above as $\mathcal{D}_{max}(\tilde{V}) = Max_{\tilde{W}} \mathcal{D}(\tilde{W} \parallel \tilde{V})$.

3) Choose at random one Bloch sphere vector from our maximal set of points $\{ \tilde{W}' \}$. Call this selected point $\tilde{W}'$. In the 3 real dimensional Bloch sphere space, make a small step from $\tilde{V}$ towards the surface point vector, $\tilde{W}'$. That is, update $\tilde{V}$ as follows:

$$\tilde{V}_{new} = (1 - \epsilon) \tilde{V}_{old} + \epsilon \tilde{W}'$$

4) Loop by going back to step 2) above, using our new, updated $\tilde{V}_{new}$, and continue to loop until $\mathcal{D}_{max}$ is no longer changing.

This algorithm converges to $\phi_{optimum} \equiv \tilde{V}_{optimum}$, because we steadily proceed downhill minimizing $Max_{\tilde{W}} \mathcal{D}(\tilde{W} \parallel \tilde{V})$, and our lemma above tells us we can never get stuck in a local minima.

5.4 Planar Channel Example

We demonstrate the iterative algorithm above with a planar channel example. Let $\{ t_x = 0.3 , t_y = 0.1 , t_z = 0 , \lambda_x = 0.4 , \lambda_y = 0.5 , \lambda_z = 0 \}$. The
iterative algorithm outlined above yields $\vec{V} = \begin{bmatrix} 0.3209 \\ 0.1112 \\ 0 \end{bmatrix}$ and a HSW channel capacity $C_1 = D_{\text{optimum}} = 0.1994$. Shown below in Figure 17 is a plot of the planar channel ellipsoid (the inner curve), and the curve of constant relative entropy $D(\rho \parallel \phi) = D_{\text{optimum}}$ centered at $\vec{V}$, which is marked with an asterisk *. One can see that the $D_{\text{max}}$ curve intersects the ellipsoid curve at two points, marked with O, and these two points are the optimum channel output signals $\rho_i$.

The optimum input and output signalling states for this channel were determined as described in Appendix E and are:

$$P_1 = 0.4869, \quad \vec{W}_{1\text{Input}} = \begin{bmatrix} -0.0207 \\ -0.9998 \\ 0 \end{bmatrix}, \quad \vec{W}_{1\text{Output}} = \begin{bmatrix} 0.2917 \\ -0.3999 \\ 0 \end{bmatrix}.$$ $$P_2 = 0.5131, \quad \vec{W}_{2\text{Input}} = \begin{bmatrix} 0.1215 \\ 0.9926 \\ 0 \end{bmatrix}, \quad \vec{W}_{2\text{Output}} = \begin{bmatrix} 0.3486 \\ 0.5963 \\ 0 \end{bmatrix}.$$ 

These signal states yield an average channel output Bloch vector $\vec{V}$ of

$$\vec{V} = P_1 \cdot \vec{W}_{1\text{Output}} + P_2 \cdot \vec{W}_{2\text{Output}} = \begin{bmatrix} 0.3209 \\ 0.1113 \\ 0 \end{bmatrix}.$$ 

Figure 17 below shows the location of the channel ellipsoid (the inner dashed curve), the contour of constant relative entropy (the solid curve) for $D = 0.1994$, the location of the two optimum input pure states $\rho_i^{\text{Input}}$, (the two O states on the circle of radius one), and the two optimum output signal states $\rho_i^{\text{Output}}$, also denoted by O, on the channel ellipsoid curve. Note that the optimum input signalling states are non-orthogonal.
Figure 17: The intersection in the Bloch sphere X-Y plane of a planar channel ellipsoid (the inner dashed curve) and the optimum relative entropy contour (the solid curve). The two optimum input signal states (on the outer bold dashed Bloch sphere boundary curve) and the two optimum output signal states (on the channel ellipsoid and the optimum relative entropy contour curve) are shown as O.

Another useful picture is how the relative entropy changes as we make our way around the channel ellipsoid. We consider the Bloch X-Y plane in polar coordinates \( \{ r, \theta \} \), where we measure the angle \( \theta \) with respect to the origin of the Bloch X-Y plane axes. (Note that \( \theta \) only fully ranges over \([0, 2\pi]\) when the origin of the Bloch sphere lies inside the channel ellipsoid.) The horizontal line at the top of the plot is the channel capacity \( C_1 = 0.1994 \). Note that the two relative entropy peaks correspond to the locations of the two output optimum signalling states.
Figure 18: The change in $D(\rho \parallel \phi) \equiv \ast$ as we move $\rho$ around the channel ellipsoid. The angle theta is with respect to the Bloch sphere origin.

For this channel, the optimum channel capacity is achieved using an ensemble consisting of only two signalling states. Davies theorem tells us that for single qubit channels, an optimum ensemble need contain at most four signalling states. Using the notation of [14], we call $C_2$ the optimum output $C_1$ HSW channel capacity attainable using only two input signalling states, $C_3$ is the optimum output $C_1$ HSW channel capacity attainable using only three input signalling states, and $C_4$ is the optimum output $C_1$ HSW channel capacity attainable using only four input signalling states. Thus, for this channel, we see that $C_2 = C_3 = C_4$. That is, for this channel, allowing more than two signalling states in your optimal ensemble does not yield additional channel capacity over an optimal ensemble with just two signalling states.

6 Unital Channels

Unital channels are quantum channels that map the identity to the identity: $\mathcal{E}(I) = I$. Due to this behavior, unital channels possess certain symmetries. In the ellipsoid picture, King and Ruskai [2] have shown that for unital channels, the $\{t_k\}$ are zero. This yields
an ellipsoid centered at the origin of the Bloch sphere. The resulting symmetry of such an ellipsoid will allow us to draw powerful conclusions.

First, recall that we know there exists at least one optimal signal ensemble, \( \{ p_i, \rho_i \} \), which attains the HSW channel capacity \( \mathcal{C}_1 \). (See property III in Section 22.) Now consider the symmetry evident in the formula we have derived for the relative entropy for two single qubit density operators. We have:

\[
\mathcal{D}(\rho \| \phi) = \mathcal{D}(\tilde{\mathcal{W}} \| \tilde{V}) = f(r, q, \theta)
\]

where \( r = \| \tilde{W} \| , q = \| \tilde{V} \| , \) and \( \theta \) is the angle between \( \tilde{W} \) and \( \tilde{V} \). Thus, if \( \rho_i \in \mathcal{A} \) and \( \phi \in \mathcal{B} \), with \( \mathcal{D}(\rho_i \| \sigma) = \mathcal{D}(\tilde{W}_i \| \tilde{V}) = \chi_{optimum} = \mathcal{C}_1 \), then acting in \( \mathcal{R}^3 \), reflecting \( \rho_i \equiv \tilde{W}_i \) and \( \sigma \equiv \tilde{V} \) through the Bloch sphere origin to obtain \( \rho'_i \equiv \tilde{W}'_i \) and \( \sigma' \equiv \tilde{V}' \), yields elements of \( \mathcal{A} \) and \( \mathcal{B} \) respectively. Furthermore, these transformed density matrices will also satisfy \( \mathcal{D}(\rho'_i \| \sigma') = \mathcal{D}(\tilde{W}'_i \| \tilde{V}') = \chi_{optimum} = \mathcal{C}_1 \), because \( r, q, \) and \( \theta \) remain the same when we reflect through the Bloch sphere origin. That is, the symmetry of the unital channel ellipsoid about the Bloch sphere origin, corresponding to the density matrix \( \frac{1}{2} I \), together with the symmetry present in the qubit relative entropy formula yields a symmetry for the optimal signal ensemble \( \{ p_i, \rho_i \} \), where \( \sigma = \sum_i p_i \rho_i \), or equivalently \( \tilde{V} = \sum_i p_i \tilde{W}_i \). This symmetry indicates that for every optimal signal ensemble \( \{ p_i, \rho_i \} \), there exists another ensemble, \( \{ p'_i, \rho'_i \} \), obtained by reflection through the Bloch sphere origin. Since we know there exists at least one optimal signal ensemble, we must conclude that if \( \sigma = \sum_i p_i \rho_i \neq \frac{1}{2} I \), then two optimal ensembles exist with \( \sigma \neq \sigma' \). However, by our uniqueness proof above, we are assured that \( \sigma = \sum_i p_i \rho_i \) is a unique density matrix, regardless of the states \( \{ p_i, \rho_i \} \) used, as long as the states \( \{ p_i, \rho_i \} \) are an optimal ensemble. Thus we must conclude that \( \sigma = \sum_i p_i \rho_i \equiv \frac{1}{2} I \), since only the density matrix \( \frac{1}{2} I \) maps into itself upon reflection through the Bloch sphere origin. Summarizing these observations, we can state the following.

**Theorem:**

For all unital qubit channels, and all optimal signal ensembles \( \{ p_i, \rho_i \} \), the average density matrix \( \sigma = \sum_i p_i \rho_i \equiv \frac{1}{2} I \).

In Appendix A, it is shown that

\[
\mathcal{D} \left( \rho \| \frac{1}{2} I \right) = 1 - S(\rho)
\]

where \( S(\rho) \) is the von Neumann entropy of the density matrix \( \rho \). Thus, our relation for the HSW channel capacity \( \mathcal{C}_1 \) becomes:

\[
\mathcal{C}_1 = \sum_i p_i \mathcal{D} \left( \rho_i \| \frac{1}{2} I \right) = 1 - \sum_i p_i S(\rho_i)
\]

To maximize \( \mathcal{C}_1 \), we seek to minimize the \( \sum_i S(\rho_i) \), subject to the constraint that the \( \rho_i \) satisfy \( \sum_i p_i \rho_i = \frac{1}{2} I \), for some set of a priori probabilities \( \{ p_i \} \). Recall that
$S(\rho) \equiv S(r)$ is a strictly decreasing function of $r$, where $r$ is the magnitude of the Bloch vector corresponding to $\rho$. (Please see the plot below of $S(\rho) \equiv S(r)$.)

![Figure 19: The Von Neumann entropy $S(\rho)$ for a single qubit $\rho$ as a function of the Bloch sphere radius $r \in [0, 1]$.](image)

Thus we seek to find a set of $\rho_i$ which lie most distant, in terms of Euclidean distance in $\mathbb{R}^3$, from the ellipsoid origin, and for which a convex combination of these states equals the Bloch sphere origin.

Let us examine a few special cases. For the unital channel ellipsoid, consider the case where the major axis is unique in length, and has total length $2\lambda^{\text{major axis}}$. Let $\rho_+$ and $\rho_-$ be the states lying at the end of the major axis. By the symmetry of the ellipsoid, we have

$$\frac{1}{2} \rho_+ + \frac{1}{2} \rho_- = \frac{1}{2} I.$$

Furthermore, the magnitude of the corresponding Bloch sphere vectors $r_+ = \|\vec{W}_+\|$ and $r_- = \|\vec{W}_-\|$ are equal, $r_+ = r_- = 1 - \|\lambda^{\text{major axis}}\|$. Above, we use $\|\cdot\|$ around $\lambda^{\text{major axis}}$ because $\lambda^{\text{major axis}}$ can be a negative quantity in the King - Ruskai et al. formalism. Using this value of $r = r_+ = r_-$ yields for $\mathcal{C}_1$:

$$\mathcal{C}_1 = 1 - 2 \left( \frac{1}{2} S(r) \right) = 1 - S \left( \|\lambda^{\text{major axis}}\| \right).$$

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If the major axis is not the unique axis of maximal length, then any set of convex probabilities and states \( \{ p_i, \rho_i \} \) such that the states lie on the major surface and \( \sum_i p_i \rho_i \equiv \frac{1}{2} I \) will suffice.

Thus we reach the same conclusion obtained by King and Ruskai in an earlier paper\(^1\). Summarizing, we can state the following.

**Theorem:**
The optimum output signalling states for unital qubit channels correspond to the minimum output von Neumann entropy states.

Furthermore, we can also conclude:

**Theorem:**
For unital qubit channels, the channel capacities consisting of signal state ensembles with two, three and four signalling states are equal. Furthermore, the optimum HSW channel capacity can be attained with a, possibly non-unique, pair of equiprobable \( (p_1 = p_2 = \frac{1}{2}) \) signalling states arranged opposite one another with respect to the Bloch sphere origin.

**Proof:**
Using the notation above, \( C_2 = C_3 = C_4 \). From the geometry of the centered channel ellipsoid, we can always use just two signalling states with the minimum output entropy to convexly reach \( \frac{1}{2} I \). Thus, utilizing more than two signaling states will not yield any channel capacity improvement beyond using two signalling states. The equiprobable nature of the two signalling states derives from the symmetry of the signalling states on the channel ellipsoid, in that one signalling state being the reflection of the other signalling state through the Bloch sphere origin means the states may be symmetrically added to yield an average state corresponding to the Bloch sphere origin. It is this reflection symmetry which makes the two signalling states equiprobable.

\[ \triangle - \text{End of Proof.} \]

The last three theorems were previously proven by King and Ruskai in section 2.3 of \(^2\). Here we have merely shown their results in the relative entropy picture.

### 6.1 The Depolarizing Channel

The depolarizing channel is a unital channel with \( \{ t_k = 0 \} \) and \( \{ \lambda_k = \frac{4x-1}{3} \} \), as discussed in more detail in Appendix D. The parameter \( x \in [0, 1] \). Using the analysis above, we can conclude that:

\[
C_1 = 1 - 2 \left( \frac{1}{2} S(r) \right) = 1 - S \left( \| \lambda_{\text{major axis}} \| \right) = 1 - S \left( \left| \frac{4x-1}{3} \right| \right)
\]

\[
= \frac{1 + \left| \frac{4x-1}{3} \right|}{2} \log_2 \left( 1 + \left| \frac{4x-1}{3} \right| \right) + \frac{1 - \left| \frac{4x-1}{3} \right|}{2} \log_2 \left( 1 - \left| \frac{4x-1}{3} \right| \right)
\]
We plot $C_1$ below.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{The Holevo-Schumacher-Westmoreland classical channel capacity for the depolarizing channel as a function of the depolarizing channel parameter $x$.}
\end{figure}

6.2 The Two Pauli Channel

The Two Pauli channel is a unital channel with $\{t_k = 0\}$ and $\{\lambda_x = \lambda_y = x\}$, and $\{\lambda_z = 2x - 1\}$, as discussed in more detail in Appendix D. The parameter $x \in [0, 1]$. The determination of the major axis/surface is tricky due to the need to take into account the absolute value of the $\lambda_k$. We plot below the absolute value of the $\lambda_k$. The dotted curve below corresponds to the absolute value of $\lambda_x$ and $\lambda_y$. The V-shaped solid curve corresponds to the absolute value of $\lambda_z$. 
Figure 21: Calculating the length of the major axis of the channel ellipsoid for the two pauli channel as a function of the two pauli channel parameter $x$.

The intersection point occurs at $x = \frac{1}{3}$. Thus $\lambda_z$ is the major axis for $x \leq \frac{1}{3}$ and the $\{\lambda_x, \lambda_y\}$ surface is the major axis surface for $x \geq \frac{1}{3}$. The Bloch sphere radius corresponding to the minimum entropy states is $1 - 2x$ for $x \leq \frac{1}{3}$ and $x$ for $x \geq \frac{1}{3}$.

Using our analysis above, we can conclude that for $x \leq \frac{1}{3}$, we have:

$$ C_1 = 1 - 2 \left( \frac{1}{2} S(r) \right) = 1 - S(|\lambda_z|) = 1 - S(1 - 2x) $$

$$ = 1 + x \log_2(x) + (1 - x) \log_2(1 - x) $$

while for $x \geq \frac{1}{3}$, we have:

$$ C_1 = 1 - 2 \left( \frac{1}{2} S(r) \right) = 1 - S(|\lambda_x|) = 1 - S(x) $$

$$ = \frac{1 + x}{2} \log_2(1 + x) + \frac{1 - x}{2} \log_2(1 - x) $$

We plot $C_1$ below, using the appropriate function in their allowed ranges of $x$. 

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Figure 22: The Holevo-Schumacher-Westmoreland classical channel capacity for the two Pauli channel as a function of the two Pauli channel parameter $x$.

Note the symmetry evident in the plots. Examining our graph above for $C_1$, one sees that for $0 \leq \alpha \leq \frac{1}{3}$, we have $C_1(\frac{1}{3} - \alpha) \equiv C_1(\frac{1}{3} + 2\alpha)$. This symmetry is also readily seen from the relations for $C_1$ in the two allowed ranges of $x$ (less than and greater than $\frac{1}{3}$).

For $x \leq \frac{1}{3}$, setting $x = \frac{1}{3} - \alpha$,

$$C_1^-(\alpha) = 1 + \frac{1-3\alpha}{3} \log_2 \left( \frac{1-3\alpha}{3} \right) + \frac{2+3\alpha}{3} \log_2 \left( \frac{2+3\alpha}{3} \right)$$

For $x \geq \frac{1}{3}$, setting $x = \frac{1}{3} + 2\alpha$,

$$C_1^+(\alpha) = \frac{4+6\alpha}{6} \log_2 \left( \frac{4+6\alpha}{3} \right) + \frac{2-6\alpha}{6} \log_2 \left( \frac{2-6\alpha}{3} \right)$$

$$= \left( \frac{4+6\alpha}{6} + \frac{2-6\alpha}{6} \right) + \frac{4+6\alpha}{6} \log_2 \left( \frac{2+3\alpha}{3} \right) + \frac{2-6\alpha}{6} \log_2 \left( \frac{1-3\alpha}{3} \right)$$

$$= 1 + \frac{2+3\alpha}{3} \log_2 \left( \frac{2+3\alpha}{3} \right) + \frac{1-3\alpha}{3} \log_2 \left( \frac{1-3\alpha}{3} \right) = C_1^-(\alpha)$$

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7 Non-Unital Channels

Non-unital channels are generically more difficult to analyze due to the fact that one or more of the \( \{ t_k \} \) can be non-zero. This allows the average density matrix \( \rho = \sum_i p_i \rho_i \) for an optimal signal ensemble \( \{ p_i, \rho_i \} \) to move away from the Bloch sphere origin \( \rho = \frac{1}{2} I \equiv \vec{V} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). However, there still remains the symmetry present in the qubit form of the relative entropy formula, namely that \( D(\rho \parallel \phi) = D(\vec{W} \parallel \vec{V}) = f(r, q, \theta) \), where \( r = \| \vec{W} \|, q = \| \vec{V} \|, \) and \( \theta \) is the angle between \( \vec{W} \) and \( \vec{V} \). The fact that the qubit relative entropy depends only on \( r, q, \) and \( \theta \) yields a symmetry which can be used to advantage in analyzing non-unital channels, as our last example will demonstrate.

7.1 The Amplitude Damping Channel

The amplitude damping channel is a non-unital channel with \( \{ t_x = t_y = 0 \} \) and \( \{ t_z = 1 - \xi \} \). The \( \lambda_k \) are \( \{ \lambda_x = \lambda_y = \sqrt{\xi} \} \), and \( \{ \lambda_z = \xi \} \), where \( \xi \) is the channel parameter, \( \xi \in [0, 1] \). The amplitude damping channel is discussed in more detail in Appendix D. The determination of the major axis/surface reduces to an analysis in either the X-Y or X-Z Bloch sphere plane because of symmetries of the channel ellipsoid and the relative entropy formula for qubit density matrices. Since the relative entropy formula depends only on the \( r, q \) and \( \theta \) quantities which were defined above, by examining contour curves of relative entropy in the X-Z plane, we can create a surface of constant relative entropy in the three dimensional X-Y-Z Bloch sphere space by the solid of revolution technique. That is, we shall revolve our X-Z contour curves about the axis of symmetry, here the Z-axis. Now the channel ellipsoid in this case is also rotationally symmetric about the Z-axis, because \( t_x = t_y = 0 \) and \( \lambda_x = \lambda_y \). Thus optimum signal points (points on the channel ellipsoid surface which have maximal relative entropy distance from the average signal density matrix), in the X-Z plane, will become circles of optimal signals in the full three dimensional Bloch sphere picture after the revolution about the Z-axis is completed. Therefore, due to the simultaneous rotational symmetry about the Bloch sphere Z axis of the relative entropy formula (for qubits) and the channel ellipsoid, a full three dimensional analysis of the amplitude damping channel reduces to a much easier, yet equivalent, two dimensional analysis in the Bloch X-Z plane.

To illustrate these ideas, we take a specific instance of the amplitude damping channel with \( \xi = 0.36 \). Then \( \{ t_x = t_y = 0 \} \) and \( \{ t_z = 0.64 \} \). The \( \lambda_k \) are \( \{ \lambda_x = \lambda_y = 0.6 \} \), and \( \{ \lambda_z = 0.36 \} \). In this case \( C_1 = 0.3600 \) is achieved with two equiprobable signalling states. The optimum average density matrix has Bloch vector \( \vec{V} = \begin{bmatrix} 0 \\ 0.7126 \end{bmatrix} \), and is shown with an asterisk in the plots below.

In the first plot, we show the X(horizontal)-Z(vertical) Bloch sphere plane. The outer bold dotted ring is the pure state boundary, with Bloch vector magnitude equal to one. The inner dashed circle is the channel ellipsoid. The middle solid contour is the curve of
constant relative entropy, equal to 0.3600, and centered at $\vec{V}$. This relative entropy contour in the X-Z plane contacts the channel ellipsoid at two *symmetrical* points, indicated in the plot as O. Note that these two contact points, and the location of $\vec{V}$, all lie on a perfectly horizontal line. The fact that the line is horizontal is due to the fact that the two optimum signalling states in the X-Z plane are symmetric about the Z axis. The point $\vec{V}$ is simply the two optimal output signal points average. The corresponding optimal input signals are shown as O’s on the outer bold dotted pure state boundary semicircular curve.

Figure 23: The intersection in the Bloch sphere X-Z plane of the amplitude damping channel ellipsoid (the inner dashed curve) and the optimum relative entropy contour (the solid curve). The two optimum input signal states (on the outer bold dashed Bloch sphere boundary curve) and the two optimum output signal states (on the channel ellipsoid and the optimum relative entropy contour curve) are shown as O.

Note that the optimum input signalling states are nonorthogonal. Furthermore, this analysis tells us that $C_2 = C_3 = C_4$. For the amplitude damping channel, there is no advantage to using more than two signals in the optimum signalling ensemble.

The following is a picture similar to those we have done for the planar channels we examined. We plot the magnitude of the relative entropy as one moves around the channel ellipsoid in the X-Z plane. The angle $\theta$ is with respect to the Bloch sphere origin (i.e.: the X-Z plane origin).
Thus, the rotational symmetry about the Z-axis of the relative entropy formula, coupled with the same Z-axis rotational symmetry of the amplitude damping channel ellipsoid, yields a complete understanding of the behavior of the amplitude damping channel with just a simple two dimensional analysis.

8 Summary and Conclusions

In this paper, we have derived a formula for the relative entropy of two single qubit density matrices. By combining our relative entropy formula with the King-Ruskai et al. ellipsoid picture of qubit channels, we can use the Schumacher-Westmoreland relative entropy approach to classical HSW channel capacity to analyze unital and non-unital single qubit channels in detail.

The following observation also emerges from the examples and analyses above. In numerical simulations by this author and others, it was noted that the a priori probabilities of the optimum signalling states for non-unital qubit channels were in general, approximately, but not exactly, equal. For example, consider the case of linear channels, where the optimum HSW channel capacity is achieved with two signalling states. In our first linear...
channel example, one signalling state had an a priori probability of 0.5156 and the other
signalling state had an a priori probability of 0.4844. Similarly, in our second linear channel
example, the respective a priori probabilities were 0.5267 and 0.4733. These asymmetries
in the a priori probabilities are due to the fact that $\mathcal{D}$ is not purely a radial function of
distance from $\vec{V}_{\text{optimum}}$. The relative entropy contours shown in Figures 2 through 11 are
moderately, but not exactly, circular about $\vec{V}_{\text{optimum}}$. This slight radial asymmetry leads
to a priori signal probabilities that are approximately, but not exactly, equal. Thus, a
graphical estimate of the a priori signal probabilities can be made by observing the degree
of asymmetry of the optimum relative entropy contour about $\vec{V}_{\text{optimum}}$.

In conclusion, the analysis above yields a geometric picture which we hope will lead to
future insights into the transmission of classical information over single qubit channels.

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A Appendix A - The Derivation Of The Bloch Sphere Relative
Entropy Formula

The relative entropy of two density matrices $\rho$ and $\psi$ is defined to be

$$
\mathcal{D}(\rho \parallel \psi) = Tr[\rho (\log_2(\rho) - \log_2(\psi))] 
$$

Our main interest is when both $\rho$ and $\psi$ are qubit density operators. In that case, $\rho$
and $\psi$ can be written using the Bloch sphere representation.

$$
\rho = \frac{1}{2} \left( I + \vec{W} \cdot \vec{\sigma} \right) \quad \psi = \frac{1}{2} \left( I + \vec{V} \cdot \vec{\sigma} \right)
$$

To simplify notation below, we define

$$
r = \sqrt{\vec{W} \cdot \vec{W}} \quad \text{and} \quad q = \sqrt{\vec{V} \cdot \vec{V}}
$$

We shall also define $\cos(\theta)$ as :

$$
\cos(\theta) = \frac{\vec{W} \cdot \vec{V}}{rq}
$$
where \( r \) and \( q \) are as above.

The symbol \( \vec{\sigma} \) means the vector of 2 x 2 Pauli matrices

\[
\vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} \quad \text{where} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

The Bloch vectors \( \vec{W} \) and \( \vec{V} \) are real, three dimensional vectors which have magnitude equal to one when representing a pure state density matrix, and magnitude less than one for a mixed (non-pure) density matrix.

The density matrices for \( \rho \) and \( \psi \) in terms of their Bloch vectors are:

\[
\rho = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} w_3 & \frac{1}{2} w_1 - \frac{1}{2} i w_2 \\ \frac{1}{2} w_1 + \frac{1}{2} i w_2 & \frac{1}{2} - \frac{1}{2} w_3 \end{bmatrix}
\]

\[
\psi = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} v_3 & \frac{1}{2} v_1 - \frac{1}{2} i v_2 \\ \frac{1}{2} v_1 + \frac{1}{2} i v_2 & \frac{1}{2} - \frac{1}{2} v_3 \end{bmatrix}
\]

We shall prove the following formula in two ways, an algebraic proof and a brute force proof. We conclude Appendix A with some alternate representations of this formula.

\[
\mathcal{D}(\rho \| \psi) = \mathcal{D}_1 - \mathcal{D}_2 = \frac{1}{2} \log_2 \left( 1 - r^2 \right) + \frac{r}{2} \log_2 \left( \frac{1 + r}{1 - r} \right) - \frac{1}{2} \log_2 \left( 1 - q^2 \right) - \frac{\vec{W} \cdot \vec{V}}{2 q} \log_2 \left( \frac{1 + q}{1 - q} \right)
\]

\[
= \frac{1}{2} \log_2 \left( 1 - r^2 \right) + \frac{r}{2} \log_2 \left( \frac{1 + r}{1 - r} \right) - \frac{1}{2} \log_2 \left( 1 - q^2 \right) - \frac{r \cos(\theta)}{2} \log_2 \left( \frac{1 + q}{1 - q} \right)
\]

where \( \theta \) is the angle between \( \vec{W} \) and \( \vec{V} \).

\[\text{A.1 Proof I : The Algebraic Proof}\]

\[
\mathcal{D}(\rho \| \psi) = Tr[ \rho ( \log_2(\rho) - \log_2(\psi) ) ]
\]

Recall the following Taylor series, valid for \( \| x \| \leq 1 \).

\[
\ln(1 + x) = - \sum_{n=1}^{\infty} \frac{(-x)^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots
\]
\[ \ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \cdots \]

Combining these two Taylor series yields another Taylor expansion we shall be interested in:

\[
\frac{1}{2} \left\{ \ln(1 + x) - \ln(1 - x) \right\} = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right)
\]

\[
= -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} - \left( -\sum_{n=1}^{\infty} \frac{x^n}{n} \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots
\]

A different combination of the first two Taylor series above yields yet another Taylor expansion we shall be interested in:

\[
\frac{1}{2} \left\{ \ln(1 + x) + \ln(1 - x) \right\} = \frac{1}{2} \ln \left[ 1 - x^2 \right]
\]

\[
= -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} + \left( -\sum_{n=1}^{\infty} \frac{x^n}{n} \right) = -\frac{x^2}{2} - \frac{x^4}{4} - \frac{x^6}{6} - \frac{x^8}{8} - \cdots
\]

Consider \(\log(\varrho)\) with the Bloch sphere representation for \(\varrho\).

\[ \varrho = \frac{1}{2} \left( \mathbb{I} + \mathbf{W} \cdot \mathbf{\sigma} \right) \]

We obtain, using the expansion given above for \(\log(1 + x)\),

\[ \log(\varrho) = \log \left[ \frac{1}{2} \left( \mathbb{I} + \mathbf{W} \cdot \mathbf{\sigma} \right) \right] = \log \left[ \frac{1}{2} \right] + \log \left[ \mathbb{I} + \mathbf{W} \cdot \mathbf{\sigma} \right] = \log \left[ \frac{1}{2} \right] - \sum_{n=1}^{\infty} \frac{(-\mathbf{W} \cdot \mathbf{\sigma})^n}{n} \]

Recall that \(\left( \mathbf{W} \cdot \mathbf{\sigma} \right)^2 = r^2\), where \(r = \sqrt{\mathbf{W} \cdot \mathbf{W}}\). Thus we have for even \(n\), \(\left( \mathbf{W} \cdot \mathbf{\sigma} \right)^n = r^n\), while for odd \(n\) we have \(\left( \mathbf{W} \cdot \mathbf{\sigma} \right)^n = r^{n-1} \mathbf{W} \cdot \mathbf{\sigma}\). The expression for \(\log(\varrho)\) then becomes

\[ \log(\varrho) = \log \left[ \frac{1}{2} \right] - \sum_{n=1}^{\infty} \frac{(-\mathbf{W} \cdot \mathbf{\sigma})^n}{n} \]
\[= \log \left[ \frac{1}{2} \right] + \vec{W} \cdot \vec{\sigma} - \frac{r^2}{2} + \frac{r^2}{3} \vec{W} \cdot \vec{\sigma} - \frac{r^4}{4} + \frac{r^4}{5} \vec{W} \cdot \vec{\sigma} - \frac{r^6}{6} + \frac{r^6}{7} \vec{W} \cdot \vec{\sigma} - \ldots \]

\[= \log \left[ \frac{1}{2} \right] + \frac{\vec{W} \cdot \vec{\sigma}}{r} \left( r + \frac{r^3}{3} + \frac{r^5}{5} + \frac{r^7}{7} + \ldots \right) + \left( -\frac{r^2}{2} - \frac{r^4}{4} - \frac{r^6}{6} - \frac{r^8}{8} - \frac{r^{10}}{10} - \ldots \right) \]

\[= \log \left[ \frac{1}{2} \right] + \frac{\vec{W} \cdot \vec{\sigma}}{2r} \log \left[ \frac{1 + r}{1 - r} \right] + \frac{1}{2} \log \left[ 1 - r^2 \right] \]

To evaluate \( Tr[\rho \log(\rho)] \) we again use the Bloch sphere representation for \( \rho \).

We write

\[\rho = \frac{1}{2} \left( I + \vec{W} \cdot \vec{\sigma} \right)\]

Using our results above,

\[\frac{1}{2} Tr[I \cdot \log(\rho)] = \log \left[ \frac{1}{2} \right] + \frac{1}{2} \log \left[ 1 - r^2 \right] \]

since \( Tr[I] = 2 \) and \( Tr[\sigma_x] = Tr[\sigma_y] = Tr[\sigma_z] = 0 \).

Similarly,

\[Tr \left[ \left( \vec{W} \cdot \vec{\sigma} \right) \log(\rho) \right] = \frac{\left( \vec{W} \cdot \vec{\sigma} \right)^2}{r} \log \left[ \frac{1 + r}{1 - r} \right] = r \log \left[ \frac{1 + r}{1 - r} \right] \]

where we again used the fact \( Tr[I] = 2 \) and \( Tr[\sigma_x] = Tr[\sigma_y] = Tr[\sigma_z] = 0 \).

Putting all the pieces together yields :

\[Tr[\rho \log(\rho)] = \frac{1}{2} Tr[I \cdot \log(\rho)] + \frac{1}{2} Tr \left[ \left( \vec{W} \cdot \vec{\sigma} \right) \log(\rho) \right] \]

\[= \log \left[ \frac{1}{2} \right] + \frac{1}{2} \log \left[ 1 - r^2 \right] + \frac{r}{2} \log \left[ \frac{1 + r}{1 - r} \right] \]

To evaluate \( Tr[\rho \log(\psi)] \), we follow a similar path and use the Bloch sphere representation for \( \psi \) of

\[\psi = \frac{1}{2} \left( I + \vec{V} \cdot \vec{\sigma} \right)\]
The expression for \( \log(\psi) \) then becomes

\[
\log(\psi) = \log \left[ \frac{1}{2} \right] + \frac{\vec{V} \cdot \vec{\sigma}}{2q} \log \left[ \frac{1 + q}{1 - q} \right] + \frac{1}{2} \log \left[ 1 - q^2 \right]
\]

Using our results above,

\[
\frac{1}{2} Tr \left[ I \cdot \log(\psi) \right] = \log \left[ \frac{1}{2} \right] + \frac{1}{2} \log \left[ 1 - q^2 \right] = -\log(2) + \frac{1}{2} \log \left[ 1 - q^2 \right]
\]

\[
Tr \left[ \left( \vec{W} \cdot \vec{\sigma} \right) \log(\psi) \right] = \frac{\vec{W} \cdot \vec{V}}{q} \log \left[ \frac{1 + q}{1 - q} \right] = r \cos(\theta) \log \left[ \frac{1 + q}{1 - q} \right]
\]

where we again used the fact \( Tr[I] = 2 \) and \( Tr[\sigma_x] = Tr[\sigma_y] = Tr[\sigma_z] = 0 \). We also used the fact that

\[
\left( \vec{V} \cdot \vec{\sigma} \right) \left( \vec{W} \cdot \vec{\sigma} \right) = \left( \vec{V} \cdot \vec{W} \right) I + \left( \vec{V} \times \vec{W} \right) \cdot \vec{\sigma}
\]

and therefore

\[
Tr \left[ \left( \vec{V} \cdot \vec{\sigma} \right) \left( \vec{W} \cdot \vec{\sigma} \right) \right] = Tr \left[ \left( \vec{V} \cdot \vec{W} \right) I \right] + Tr \left[ \left( \vec{V} \times \vec{W} \right) \cdot \vec{\sigma} \right]
\]

\[
= \left( \vec{V} \cdot \vec{W} \right) Tr[I] + \left( \vec{V} \times \vec{W} \right) \cdot Tr[\vec{\sigma}] = 2 \vec{V} \cdot \vec{W}
\]

Assembling the pieces:

\[
Tr \left[ \varrho \log(\psi) \right] = \frac{1}{2} Tr \left[ I \cdot \log(\psi) \right] + \frac{1}{2} Tr \left[ \left( \vec{W} \cdot \vec{\sigma} \right) \log(\psi) \right]
\]

\[
= \log \left[ \frac{1}{2} \right] + \frac{1}{2} \log \left[ 1 - q^2 \right] + \frac{r}{2} \cos(\theta) \log \left[ \frac{1 + q}{1 - q} \right]
\]

Using these pieces, we obtain our final formula:

\[
D(\varrho || \psi) = Tr \left[ \varrho \left( \log_2(\varrho) - \log_2(\psi) \right) \right]
\]

\[
= \log_2 \left[ \frac{1}{2} \right] + \frac{1}{2} \log_2 \left[ 1 - r^2 \right] + \frac{r}{2} \log_2 \left[ \frac{1 + r}{1 - r} \right]
\]

\[
- \log_2 \left[ \frac{1}{2} \right] - \frac{1}{2} \log_2 \left[ 1 - q^2 \right] - \frac{r}{2} \cos(\theta) \log_2 \left[ \frac{1 + q}{1 - q} \right]
\]

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\[ \frac{1}{2} \log_2 \left[ 1 - r^2 \right] + \frac{r}{2} \log_2 \left[ \frac{1 + r}{1 - r} \right] - \frac{1}{2} \log_2 \left[ 1 - q^2 \right] - \frac{r}{2} \cos(\theta) \log_2 \left[ \frac{1 + q}{1 - q} \right] \]

which is our desired formula.

\[ \triangle \quad \text{- End of Proof I}. \]

### A.2 Proof II: The Brute Force Proof

The density matrices for \( \rho \) and \( \psi \) in terms of their Bloch vectors are:

\[
\rho = \begin{bmatrix}
\frac{1}{2} + \frac{1}{2} w_3 & \frac{1}{2} w_1 - \frac{i}{2} w_2 \\
\frac{1}{2} w_1 + \frac{1}{2} i w_2 & \frac{1}{2} - \frac{1}{2} w_3
\end{bmatrix}
\]

\[
\psi = \begin{bmatrix}
\frac{1}{2} + \frac{1}{2} v_3 & \frac{1}{2} v_1 - \frac{i}{2} v_2 \\
\frac{1}{2} v_1 + \frac{1}{2} i v_2 & \frac{1}{2} - \frac{1}{2} v_3
\end{bmatrix}
\]

The eigenvalues of these two density matrices are:

\[
\lambda^{(1)}_\rho = \frac{1}{2} + \frac{1}{2} \sqrt{w_2^2 + w_3^2 + w_1^2} = \frac{1 + r}{2}
\]

\[
\lambda^{(2)}_\rho = \frac{1}{2} - \frac{1}{2} \sqrt{w_2^2 + w_3^2 + w_1^2} = \frac{1 - r}{2}
\]

\[
\lambda^{(1)}_\psi = \frac{1}{2} + \frac{1}{2} \sqrt{v_2^2 + v_3^2 + v_1^2} = \frac{1 + q}{2}
\]

\[
\lambda^{(2)}_\psi = \frac{1}{2} - \frac{1}{2} \sqrt{v_2^2 + v_3^2 + v_1^2} = \frac{1 - q}{2}
\]

We shall also be interested in the two eigenvectors of \( \psi \). These are:

\[
|e_1 \rangle = N_1 \left[ -2 \left( \frac{1}{2} + \frac{i}{2} \sqrt{w_2^2 + w_3^2 + w_1^2} \right) w_1 - 2i \left( \frac{1}{2} + \frac{1}{2} \sqrt{w_2^2 + w_3^2 + w_1^2} \right) w_2 + w_2 + iw_2 + iw_3 w_1 + iw_3 w_2 \right]
\]

where \( N_1 \) is the normalization constant given below.

\[
N_1 = \sqrt{2 \frac{w_1^2 + \sqrt{w_2^2 + w_3^2 + w_1^2} w_2 + w_2 + i w_2 + i w_3 w_1 + i w_3 w_2}{w_1^2 + w_2^2}}
\]

Similarly,
\[ |e_2\rangle = N_2 \begin{pmatrix} \frac{1}{2} - \frac{i}{2} \sqrt{w_2^2 + w_3^2 + w_1^2} \rangle \langle e_1 \mid - 2i \left( \frac{1}{2} - \frac{i}{2} \sqrt{w_2^2 + w_3^2 + w_1^2} \right) \langle e_2 \mid w_2 - w_1 + iw_2 + w_3 w_1 - iw_3 w_2 \end{pmatrix} \]

\[
N_2 = \sqrt{\frac{2}{w_1^2 + w_2^2 + w_3^2 + w_1^2 w_3 + w_2^2 + w_3^2}}
\]

We wish to derive a formula for \( D(\rho \parallel \psi) \) in terms of the Bloch sphere vectors \( \vec{W} \) and \( \vec{V} \). We do this by breaking \( D(\rho \parallel \psi) \) up into two terms, \( D_1 \) and \( D_2 \).

\[
D(\rho \parallel \psi) = D_1 - D_2
\]

We expand \( D_1 \) using our knowledge of the eigenvalues of \( \rho \).

\[
D_1 = Tr[ \rho \log_2(\rho) ] = \lambda_\psi^{(1)} \log_2(\lambda_\psi^{(1)}) + \lambda_\psi^{(2)} \log_2(\lambda_\psi^{(2)})
\]

\[
= \left( \frac{1 + r}{2} \right) \log_2 \left( \frac{1 + r}{2} \right) + \left( \frac{1 - r}{2} \right) \log_2 \left( \frac{1 - r}{2} \right)
\]

\[
= -1 + \left( \frac{1 + r}{2} \right) \log_2 (1 + r) + \left( \frac{1 - r}{2} \right) \log_2 (1 - r)
\]

One notes that \( D_1 = -S(\rho) \), where \( S(\rho) \) is the von Neumann entropy of the density matrix \( \rho \). The second term, \( D_2 \), is \( D_2 = Tr[ \rho \log_2(\psi) ] \). We evaluate \( D_2 \) in the basis which diagonalizes \( \psi \).

\[
D_2 = Tr[ \rho \log_2(\rho) ] = \log_2(\lambda_\psi^{(1)}) Tr[ \rho |e_1\rangle \langle e_1| ] + \log_2(\lambda_\psi^{(2)}) Tr[ \rho |e_2\rangle \langle e_2| ]
\]

We use the Bloch sphere representation for \( \rho \) in the expression for \( D_2 \).

\[
\psi = \frac{1}{2} ( I + \vec{V} \cdot \vec{\sigma} )
\]

\[
D_2 = Tr[ \rho \log_2(\psi) ] = \frac{1}{2} \log_2(\lambda_\psi^{(1)}) \left[ Tr[ |e_1\rangle \langle e_1| ] + \sum_i w_i Tr[ \sigma_i |e_1\rangle \langle e_1| ] \right] +
\]

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\[
\frac{1}{2} \log_2(\lambda_\psi^{(2)}) \left[ \text{Tr}[|e_2\rangle\langle e_2|] + \sum_i w_i \text{Tr}[\sigma_i |e_2\rangle\langle e_2|] \right]
\]

First note that \( \text{Tr}[|e_1\rangle\langle e_1|] = \text{Tr}[|e_2\rangle\langle e_2|] = 1 \) since the \(|e_j\rangle\) are projection operators.

Next define

\[
\alpha_i^{(j)} = \text{Tr}[\sigma_i |e_j\rangle\langle e_j|] = \langle e_j | \sigma_i |e_j\rangle
\]

Evaluating these six (\( i = 1,2,3 \) and \( j = 1,2 \)) constants yields:

\[
\begin{align*}
\alpha_1^{(1)} &= \frac{v_1 \left( \sqrt{v_2^2 + v_3^2 + v_1^2} + v_3 \right)}{v_1^2 + \sqrt{v_2^2 + v_3^2 + v_1^2} v_3 + v_2^2 + v_3^2} = \frac{v_1 (q + v_3)}{q^2 + q v_3} = \frac{v_1}{q} \\
\alpha_2^{(1)} &= \frac{v_2 \left( \sqrt{v_2^2 + v_3^2 + v_1^2} + v_3 \right)}{v_1^2 + \sqrt{v_2^2 + v_3^2 + v_1^2} v_3 + v_2^2 + v_3^2} = \frac{v_2 (q + v_3)}{q^2 + q v_3} = \frac{v_2}{q} \\
\alpha_3^{(1)} &= \frac{\left( \sqrt{v_2^2 + v_3^2 + v_1^2} + v_3 \right) v_3}{v_1^2 + \sqrt{v_2^2 + v_3^2 + v_1^2} v_3 + v_2^2 + v_3^2} = \frac{v_3 (q + v_3)}{q^2 + q v_3} = \frac{v_3}{q} \\
\alpha_1^{(2)} &= \frac{v_1 \left( \sqrt{v_2^2 + v_3^2 + v_1^2} - v_3 \right)}{-v_1^2 + \sqrt{v_2^2 + v_3^2 + v_1^2} v_3 - v_2^2 - v_3^2} = -\frac{v_1 (q - v_3)}{q^2 - q v_3} = -\frac{v_1}{q} \\
\alpha_2^{(2)} &= \frac{v_2 \left( \sqrt{v_2^2 + v_3^2 + v_1^2} - v_3 \right)}{-v_1^2 + \sqrt{v_2^2 + v_3^2 + v_1^2} v_3 - v_2^2 - v_3^2} = -\frac{v_2 (q - v_3)}{q^2 - q v_3} = -\frac{v_2}{q} \\
\alpha_3^{(2)} &= \frac{\left( \sqrt{v_2^2 + v_3^2 + v_1^2} - v_3 \right) v_3}{-v_1^2 + \sqrt{v_2^2 + v_3^2 + v_1^2} v_3 - v_2^2 - v_3^2} = -\frac{v_3 (q - v_3)}{q^2 - q v_3} = -\frac{v_3}{q}
\end{align*}
\]

Putting it all together yields:

\[
\mathcal{D}_2 = \text{Tr}[\varrho \log_2(\psi)]
\]

\[
= \frac{1}{2} \log_2 \left( \lambda_\psi^{(1)} \right) \left[ 1 + \sum_i w_i \alpha_i^{(1)} \right] + \frac{1}{2} \log_2 \left( \lambda_\psi^{(2)} \right) \left[ 1 + \sum_i w_i \alpha_i^{(2)} \right]
\]
\[
\begin{align*}
\mathcal{D}_2 &= Tr[\rho \log_2(\psi)] \\
&= \frac{1}{2} \left[ 1 + \frac{\vec{W} \cdot \vec{V}}{q} \right] \log_2 \left( \frac{1 + q}{2} \right) + \frac{1}{2} \left[ 1 - \frac{\vec{W} \cdot \vec{V}}{q} \right] \log_2 \left( \frac{1 - q}{2} \right) \\
&= \frac{1}{2} \log_2(1 - q^2) - 1 + \frac{\vec{W} \cdot \vec{V}}{2q} \log_2 \left( \frac{1 + q}{1 - q} \right)
\end{align*}
\]

Plugging in for the eigenvalues \(\lambda_1(\psi)\) and \(\lambda_2(\psi)\) which we found above yields:

\[
\mathcal{D}_2 = D_1 - D_2 = \frac{1}{2} \log_2(1 - r^2) + \frac{r}{2} \log_2 \left( \frac{1 + r}{1 - r} \right) - \frac{1}{2} \log_2(1 - q^2) - \frac{\vec{W} \cdot \vec{V}}{2q} \log_2 \left( \frac{1 + q}{1 - q} \right)
\]

Putting all the pieces together to obtain \(\mathcal{D}(\rho \parallel \psi)\), we find

\[
\mathcal{D}(\rho \parallel \psi) = \mathcal{D}_1 - \mathcal{D}_2 = \frac{1}{2} \log_2(1 - r^2) + \frac{r}{2} \log_2 \left( \frac{1 + r}{1 - r} \right) - \frac{1}{2} \log_2(1 - q^2) - \frac{\vec{W} \cdot \vec{V}}{2q} \log_2 \left( \frac{1 + q}{1 - q} \right)
\]

where \(\theta\) is the angle between \(\vec{W}\) and \(\vec{V}\).

\(\triangle - \text{End of Proof II.}\)

Ordinarily, \(\mathcal{D}(\rho \parallel \phi) \neq \mathcal{D}(\phi \parallel \rho)\). However, when \(r = q\), we can see from the above formula that \(\mathcal{D}(\rho \parallel \phi) = \mathcal{D}(\phi \parallel \rho)\).

A few special cases of \(\mathcal{D}(\rho \parallel \phi)\) are worth examining. Consider the case when \(\phi = \frac{1}{2} \mathcal{I}\). In this case, \(q = 0\), and

\[
\mathcal{D}(\rho \parallel \phi) = \frac{1}{2} \log_2 \left( 1 - r^2 \right) + \frac{r}{2} \log_2 \left( \frac{1 + r}{1 - r} \right)
\]

\[
= \frac{1}{2} \log_2 \left( \frac{1 + r}{2} \right) + \frac{1 + r}{2} - \frac{1 - r}{2} \log_2 \left( \frac{1 - r}{2} \right) + \frac{1 - r}{2} = 1 - \mathcal{S}(\rho)
\]

Thus, \(\mathcal{D}(\rho \parallel \frac{1}{2} \mathcal{I}) = 1 - \mathcal{S}(\rho)\), where \(\mathcal{S}(\rho)\) is the von Neumann entropy of \(\rho\), the first density matrix in the relative entropy function.
B Appendix B - The Derivation Of The Linear Channel Transcendental Equation

In this appendix, we derive the transcendental equation for determining the optimum position of the average density matrix for a linear channel. The picture of the quantities we shall define shortly is below.

Figure 25: Definition of the Bloch vectors \( \vec{r}_+, \vec{q}, \) and \( \vec{r}_-\) used in the derivation below.

We assume that in general all the \( \{ t_k \neq 0 \}\). We also assume the linear channel is oriented in the z direction, so that \( \lambda_x = \lambda_y = 0, \) but \( \lambda_z \neq 0. \) We define

\[
A = t_x^2 + t_y^2 + (t_z + \lambda_z)^2 = r_+^2
\]

\[
B = t_x^2 + t_y^2 + (t_z + \beta \lambda_z)^2 = q(\beta)^2
\]

\[
C = t_x^2 + t_y^2 + (t_z - \lambda_z)^2 = r_-^2
\]

The three quantities above refer respectively to the distance from the Bloch sphere origin to \( r_+ \), the optimum point \( q \) we seek, and \( r_- \). We define the three Bloch vectors \( \vec{r}_+, \vec{q} \) and \( \vec{r}_- \) in Figure 25 above, and refer to their respective magnitudes as \( r_+, \) \( q, \) and \( r_- \). Here \( \beta \in [-1, 1], \) so that \( q \) can range along the entire line segment between \( r_+ \) and \( r_- \).

As discussed in the Linear Channels section of this paper, the condition on \( q \) is that \( \mathcal{D}(r_+ \parallel q) = \mathcal{D}(r_- \parallel q). \)

Now recall that

\[
\mathcal{D}(r \parallel q) = \frac{1}{2} \log_2(1 - r^2) + \frac{r}{2} \log_2 \left( \frac{1 + r}{1 - r} \right) - \frac{r}{2} \log_2(1 - q^2) - \frac{r \cos(\theta)}{2} \log_2 \left( \frac{1 + q}{1 - q} \right)
\]
where \( \theta \) is the angle between \( r \) and \( q \). To determine \( \theta \), we use the law of cosines. If \( \theta \) is the angle between sides \( a \) and \( b \) of a triangle with sides \( a, b \) and \( c \), then we have:

\[
\cos(\theta) = \frac{a^2 + b^2 - c^2}{2ab}
\]

Our condition \( D(r_+ \parallel q) = D(r_- \parallel q) \) becomes:

\[
\frac{1}{2} \log(1 - r_+^2) + \frac{r_+}{2} \log \left( \frac{1 + r_+}{1 - r_+} \right) - \frac{r_+ \cos(\theta_+)}{2} \log \left( \frac{1 + q}{1 - q} \right)
\]

\[
= \frac{1}{2} \log(1 - r_-^2) + \frac{r_-}{2} \log \left( \frac{1 + r_-}{1 - r_-} \right) - \frac{r_- \cos(\theta_-)}{2} \log \left( \frac{1 + q}{1 - q} \right)
\]

where we canceled the term which was identically a function of \( q \) from both sides, and converted all logs from base 2 to natural logs by multiplying both sides by \( \log(2) \).

Determining \( \theta_+ \) and \( \theta_- \), we find:

\[
\cos(\theta_+) = \frac{r_+^2 + q^2 - ((1 - \beta)\lambda_z)^2}{2qr_+}
\]

\[
\cos(\theta_-) = \frac{r_-^2 + q^2 - ((1 + \beta)\lambda_z)^2}{2qr_-}
\]

Next, recall the identity

\[
\tanh^{-1}[x] = \frac{1}{2} \log \left( \frac{1 + x}{1 - x} \right)
\]

Using this identity for \( \text{arctanh} \), our relative entropy equality relation between the two endpoints of the linear channel becomes:

\[
\frac{1}{2} \log(1 - A) + \sqrt{A} \tanh^{-1} \left( \sqrt{A} \right) - \frac{\sqrt{A}(A + B - ((1 - \beta)\lambda_z)^2)}{2\sqrt{AB}} \tanh^{-1} \left( \sqrt{B} \right)
\]

\[
= \frac{1}{2} \log(1 - C) + \sqrt{C} \tanh^{-1} \left( \sqrt{C} \right) - \frac{\sqrt{C}(C + B - ((1 + \beta)\lambda_z)^2)}{2\sqrt{BC}} \tanh^{-1} \left( \sqrt{B} \right)
\]

We can cancel several terms to obtain

\[
\frac{1}{2} \log(1 - A) + \sqrt{A} \tanh^{-1} \left( \sqrt{A} \right) - \frac{(A + 2\beta\lambda_z^2)}{2\sqrt{B}} \tanh^{-1} \left( \sqrt{B} \right)
\]

\[
= \frac{1}{2} \log(1 - C) + \sqrt{C} \tanh^{-1} \left( \sqrt{C} \right) - \frac{(C - 2\beta\lambda_z^2)}{2\sqrt{B}} \tanh^{-1} \left( \sqrt{B} \right)
\]
which in turn becomes:

\[
\frac{1}{2} \log(1 - A) + \sqrt{A} \tanh^{-1}(\sqrt{A}) - \frac{(A - C + 4 \beta \lambda^2_z)}{2 \sqrt{B}} \tanh^{-1}\left(\sqrt{B}\right)
\]

\[
= \frac{1}{2} \log(1 - C) + \sqrt{C} \tanh^{-1}\left(\sqrt{C}\right)
\]

Using our definitions above for A and C, we find that A - C = 4 \lambda_z t_z. Substituting this into the relation immediately above yields:

\[
\frac{1}{2} \log(1 - A) + \sqrt{A} \tanh^{-1}(\sqrt{A}) - \frac{4 \lambda_z t_z + 4 \beta \lambda^2_z}{2 \sqrt{B}} \tanh^{-1}\left(\sqrt{B}\right)
\]

\[
= \frac{1}{2} \log(1 - C) + \sqrt{C} \tanh^{-1}\left(\sqrt{C}\right)
\]

which we adjust to our final answer:

\[
\frac{4 \lambda_z (t_z + \beta \lambda_z)}{\sqrt{B}} \tanh^{-1}\left(\sqrt{B}\right)
\]

\[
= \log(1 - A) - \log(1 - C) + 2 \sqrt{A} \tanh^{-1}\left(\sqrt{A}\right) - 2 \sqrt{C} \tanh^{-1}\left(\sqrt{C}\right)
\]

Note that \( B \) is a function of \( \beta \), so the entire functionality of \( \beta \) lies to the left of the equality sign in the expression above. All terms on the right hand side are functions of the \( \{t_k\} \) and \( \{\lambda_z\} \), so the right hand side is a constant while we vary \( \beta \). Since all the functions of \( \beta \) on the left hand side are smooth functions, the search for the optimum \( \beta \equiv q \), although transcendental, is well behaved and fairly easy.

C Appendix C - Donald’s Equality

We prove Donald’s Equality below\[13\]. Let \( \rho_i \) be a set of density matrices with a priori probabilities \( \alpha_i \), so that \( \alpha_i \geq 0 \) and \( \sum_i \alpha_i = 1 \). Let \( \phi \) be any density matrix, and define \( \sigma = \sum_i \alpha_i \rho_i \). Then:

\[
\sum_i \alpha_i D(\rho_i \| \phi) = D(\sigma \| \phi) + \sum_i \alpha_i D(\rho_i \| \sigma)
\]

Proof:
\[
\sum_i \alpha_i D(\rho_i \parallel \phi) = \sum_i \alpha_i \{ Tr[\rho_i \log(\rho_i)] - Tr[\rho_i \log(\phi)] \} \\
= \sum_i \alpha_i \{ Tr[\rho_i \log(\rho_i)] \} - Tr[\sigma \log(\phi)] \\
= \{ Tr[\sigma \log(\sigma)] - Tr[\sigma \log(\sigma)] \} - Tr[\sigma \log(\phi)] + \sum_i \alpha_i Tr[\rho_i \log(\rho_i)] \\
= D(\sigma \parallel \phi) - Tr[\sigma \log(\sigma)] + \sum_i \alpha_i Tr[\rho_i \log(\rho_i)] \\
= D(\sigma \parallel \phi) + \sum_i \alpha_i \{ Tr[\rho_i \log(\rho_i)] - Tr[\rho_i \log(\sigma)] \} \\
= D(\sigma \parallel \phi) + \sum_i \alpha_i D(\rho_i \parallel \sigma)
\]

$\Delta$ - End of Proof.

D Appendix D - Quantum Channel Descriptions

The Kraus quantum channel representation is given by the set of Kraus matrices \( \mathcal{A} = \{ A_i \} \) which represent the channel dynamics via the relation:

\[
\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger
\]

The normalization requirement for the Kraus matrices is:

\[
\sum_i A_i^\dagger \rho A_i = I
\]

A channel is unital if it maps the identity to the identity. This requirement becomes, upon setting \( \rho = I \):

\[
\sum_i A_i \rho A_i^\dagger = \sum_i A_i A_i^\dagger = I
\]

Each set of Kraus operators, \( \{ A_i \} \) can mapped to a set of King-Ruskai-Szarek-Werner ellipsoid channel parameters \( \{ t_k, \lambda_k \} \), where \( k = 1, 2, 3 \).
The Two Pauli Channel Kraus Representation

\[
A_1 = \begin{bmatrix} \sqrt{x} & 0 \\ 0 & \sqrt{x} \end{bmatrix} \quad A_2 = \sqrt{\frac{1-x}{2}} \sigma_x = \begin{bmatrix} 0 & \sqrt{\frac{1-x}{2}} \\ \sqrt{\frac{1-x}{2}} & 0 \end{bmatrix} \\
A_3 = -i \sqrt{\frac{1-x}{2}} \sigma_y = \begin{bmatrix} 0 & -\sqrt{\frac{1-x}{2}} \\ \sqrt{\frac{1-x}{2}} & 0 \end{bmatrix}
\]

In words, the channel leaves the qubit transiting the channel alone with probability \(x\), and does a \(\sigma_x\) on the qubit with probability \(\frac{1-x}{2}\) or does a \(\sigma_y\) on the qubit with probability \(\frac{1-x}{2}\). The Two Pauli channel is a unital channel. The corresponding King-Ruskai-Szarek-Werner ellipsoid channel parameters are \(t_x = t_y = t_z = 0\), and \(\lambda_x = \lambda_y = x\), while \(\lambda_z = 2x - 1\). [2] Here \(x \in [0, 1]\).

The Depolarization Channel Kraus Representation

\[
A_1 = \begin{bmatrix} \sqrt{x} & 0 \\ 0 & \sqrt{x} \end{bmatrix} \quad A_2 = \sqrt{\frac{1-x}{3}} \sigma_x = \begin{bmatrix} 0 & \sqrt{\frac{1-x}{3}} \\ \sqrt{\frac{1-x}{3}} & 0 \end{bmatrix} \\
A_3 = -i \sqrt{\frac{1-x}{3}} \sigma_y = \begin{bmatrix} 0 & -\sqrt{\frac{1-x}{3}} \\ \sqrt{\frac{1-x}{3}} & 0 \end{bmatrix} \\
A_4 = \sqrt{\frac{1-x}{3}} \sigma_z = \begin{bmatrix} \sqrt{\frac{1-x}{3}} & 0 \\ 0 & -\sqrt{\frac{1-x}{3}} \end{bmatrix}
\]

In words, the channel leaves the qubit transiting the channel alone with probability \(x\), and does a \(\sigma_x\) on the qubit with probability \(\frac{1-x}{3}\) or does a \(\sigma_y\) on the qubit with probability \(\frac{1-x}{3}\). or does a \(\sigma_z\) on the qubit with probability \(\frac{1-x}{3}\). The Depolarization channel is a unital channel. The corresponding King-Ruskai-Szarek-Werner ellipsoid channel parameters are \(t_x = t_y = t_z = 0\), and \(\lambda_x = \lambda_y = \lambda_z = \frac{4x-1}{3}\). [2] Again \(x \in [0, 1]\).

The Amplitude Damping Channel Kraus Representation

\[
A_1 = \begin{bmatrix} \sqrt{x} & 0 \\ 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ \sqrt{1-x} & 0 \end{bmatrix}
\]

In this scenario, the channel leaves untouched a spin down qubit. For a spin up qubit, with probability \(x\) it leaves the qubit alone, while with probability \(1 - x\) the channel flips the spin from up to down. Thus, when \(x = 0\), every qubit emerging from the channel is in the spin down state. The Amplitude Damping channel is not a unital channel. The corresponding King-Ruskai-Szarek-Werner ellipsoid channel parameters are \(t_x = 0\), \(t_y = 0\), \(t_z = 1 - x\), \(\lambda_x = \sqrt{x}\), \(\lambda_y = \sqrt{x}\), and \(\lambda_z = x\). [2] Again \(x \in [0, 1]\).
Appendix E - Numerical Analysis Of Optimal Signal Ensembles Using MAPLE and MATLAB

The iterative, relative entropy based algorithm outlined above was implemented in MAPLE, and provided the plots and numbers cited in this paper. In addition, numerical answers were verified using a brute force algorithm based on MATLAB’s Optimization Toolbox. The MATLAB optimization criterion was the channel output Holevo $\chi$ quantity. Input qubit ensembles of two, three and four states were used. After channel evolution, the output ensemble Holevo $\chi$ was calculated. With this function specified as to be maximized, the MATLAB Toolbox varied the parameters for the ensemble qubit input pure states and the states corresponding a priori probabilities. Pure state qubits were represented as :

$$|\psi\rangle = \left[ \frac{\alpha}{\sqrt{1 - \alpha^2}} e^{i\theta} \right]$$

thereby requiring two parameters, $\{\alpha, \theta\}$, for each input qubit state. Thus a two state input qubit ensemble required an optimization over a space of dimension five, when the a priori probabilities are included. Three and four state ensembles required optimization over spaces of dimension eight and eleven respectively.

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