EIGENVALUES FOR A NONLOCAL PSEUDO $p$–LAPLACIAN

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Abstract. In this paper we study the eigenvalue problems for a nonlocal operator of order $s$ that is analogous to the local pseudo $p$–Laplacian. We show that there is a sequence of eigenvalues $\lambda_n \to \infty$ and that the first one is positive, simple, isolated and has a positive and bounded associated eigenfunction. For the first eigenvalue we also analyze the limits as $p \to \infty$ (obtaining a limit nonlocal eigenvalue problem analogous to the pseudo infinity Laplacian) and as $s \to 1^-$ (obtaining the first eigenvalue for a local operator of $p$–Laplacian type). To perform this study we have to introduce anisotropic fractional Sobolev spaces and prove some of their properties.

1. Introduction

Our main goal is to introduce a nonlocal operator that is a nonlocal analogous to the local pseudo $p$–Laplacian, $\Delta_{p,x} u + \Delta_{p,y} u$ (here the subindexes $x$ and $y$ denote differentiation with respect to the $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ variables respectively). The local pseudo $p$–Laplacian appears naturally when one considers critical points of the functional $F(u) = \int_{\Omega} |\nabla_x u|^p + |\nabla_y u|^p \, dx \, dy$. See [5, 14, 25, 33, 34]. On the other hand, recently, it was introduced a nonlocal $p$–Laplacian that is given by

$$( - \Delta )^s v(x) = 2 \text{ P.V.} \int_{\mathbb{R}^k} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{k+ps}} \, dx,$$

the symbol P.V. stands for the principal value of the integral. We will omit it in what follows. For references involving this kind of operator we refer to [9, 16, 18, 23, 24, 26, 29, 30, 32, 31] and references therein.

Here, we introduce the following nonlocal operator that we will call the nonlocal pseudo $p$–Laplacian,

$$\mathcal{L}_{s,p}(u)(x,y) := 2 \int_{\mathbb{R}^n} \frac{|u(x,y) - u(z,y)|^{p-2}(u(x,y) - u(z,y))}{|x - z|^{n+sp}} \, dz$$

$$+ 2 \int_{\mathbb{R}^m} \frac{|u(x,y) - u(x,w)|^{p-2}(u(x,y) - u(x,w))}{|y - w|^{m+sp}} \, dw.$$

The natural space to consider when one deals with the operator $\mathcal{L}_{s,p}$ is given by

$$W^{s,p}(\mathbb{R}^{n+m}) := \left\{ u \in L^p(\mathbb{R}^{n+m}): [u]^p_{W^{s,p}(\mathbb{R}^{n+m})} < \infty \right\},$$

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where for \( p < +\infty \),

\[
[u]_{W^{s,p}(\mathbb{R}^{n+m})}^p := \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{|u(x,y) - u(z,y)|^p}{|x-z|^{n+sp}} \, dz \, dx \, dy
+ \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{|u(x,y) - u(x,w)|^p}{|y-w|^{m+sp}} \, dw \, dx \, dy
\]

and for \( p = +\infty \),

\[
[u]_{W^{s,\infty}(\mathbb{R}^{n+m})} := \max \left\{ \sup \left\{ \frac{|u(x,y) - u(z,y)|}{|x-z|^s} : (x,y) \neq (z,y) \right\} ; \sup \left\{ \frac{|u(x,y) - u(x,w)|}{|y-w|^s} : (x,y) \neq (x,w) \right\} \right\}.
\]

In this paper, we deal with the eigenvalue problem for this operator, that is, given a bounded domain \( \Omega \) we look for pairs \((\lambda, u)\) such that \( \lambda \in \mathbb{R} \) and \( u \in \tilde{W}^{s,p}(\Omega) \setminus \{0\} \) are such that \( u \) is a weak solution of

\[
\begin{aligned}
L_{s,p} u(x,y) &= \lambda |u(x,y)|^{p-2} u(x,y) \quad \text{in } \Omega, \\
u(x,y) &= 0 \quad \text{in } \Omega^c = \mathbb{R}^{n+m} \setminus \Omega.
\end{aligned}
\]

Here \( \tilde{W}^{s,p}(\Omega) = \{ u \in W^{s,p}(\mathbb{R}^{n+m}) : u \equiv 0 \text{ in } \Omega^c \} \). We will study the Dirichlet problem for this operator in a companion paper.

We impose the following assumptions on the data:

A1. \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^{n+m} \);
A2. \( s \in (0,1) \), and \( p \in (1, +\infty) \).

Under these conditions we have the following result.

**Theorem 1.1.** There exists a sequence of eigenvalues \( \lambda_n \) such that \( \lambda_n \to +\infty \) as \( n \to +\infty \). Moreover, every eigenfunction is in \( L^\infty(\mathbb{R}^{n+m}) \). The first eigenvalue (the smallest eigenvalue) is given by

\[
\lambda_1(s,p) := \inf \left\{ \frac{|u|^p_{W^{s,p}(\mathbb{R}^{n+m})}}{|u|^p_{L^p(\Omega)}} : u \in \tilde{W}^{s,p}(\Omega), u \neq 0 \right\}.
\]

This eigenvalue \( \lambda_1(s,p) \) is simple, isolated and an associated eigenfunction is strictly positive (or negative) in \( \Omega \).

Next, we analyze the limit as \( s \to 1^- \) of the first eigenvalue obtaining that there is a limit that is the first eigenvalue of a local operator that involve two \( p \)-Laplacians (one in the \( x \) variables and another one in \( y \) variables).

**Theorem 1.2.** Let \( \Omega \) is bounded domain in \( \mathbb{R}^{n+m} \) with smooth boundary, and fix \( p \in (1, \infty) \). Then

\[
\lim_{s \to 1^-} (1-s) \lambda_1(s,p) = \lambda_1(1,p)
\]

\[
:= \inf \left\{ \frac{K_{n,p} \|
abla_x u\|^p_{L^p(\Omega)}}{|u|^p_{L^p(\Omega)}} + K_{m,p} \|
abla_y u\|^p_{L^p(\Omega)} : u \in W^{1,p}_0(\Omega), u \neq 0 \right\},
\]

where the constant \( K_{n,p} > 0 \) depends only on \( n \) and \( p \), while \( K_{m,p} > 0 \) depends only on \( m \) and \( p \).
Observe that the limit value, \( \lambda_1(1, p) \), is the first eigenvalue of the following eigenvalue problem
\[
\begin{cases}
-K_{n,p} \Delta_{p,x} u - K_{m,p} \Delta_{p,y} u = \lambda |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Concerning the limit as \( p \to \infty \) (for a fixed \( s \)) for the first eigenvalue we have the following result.

**Theorem 1.3.** It holds that
\[
\lim_{p \to \infty} [\lambda_1(s, p)]^{1/p} = \Lambda_\infty(s)
\]
where
\[
\Lambda_\infty(s) := \inf \{ |u|_{W^{s,\infty}(\mathbb{R}^{n+m})} : u \in W^{s,\infty}(\mathbb{R}^{n+m}), \|u\|_{L^\infty(\Omega)} = 1, u = 0 \text{ in } \Omega^c \}.
\]
In addition, the eigenfunctions \( u_p \) normalized by \( \|u_p\|_{L^p(\Omega)} = 1 \) converge along subsequences \( p_n \to \infty \) uniformly to a continuous limit \( u_\infty \), that is a nontrivial viscosity solution to
\[
\begin{cases}
\max \{ A; C \} = \max \{-B; -D; \Lambda_\infty(s) u \} & \text{in } \Omega, \\
u = 0 & \text{in } \Omega^c,
\end{cases}
\]
with
\[
A = \sup_w \frac{u(x, w) - u(x, y)}{|y - w|^s}, \quad B = \inf_w \frac{u(x, w) - u(x, y)}{|y - w|^s},
\]
\[
C = \sup_z \frac{u(z, y) - u(x, y)}{|x - z|^s}, \quad D = \inf_z \frac{u(z, y) - u(x, y)}{|x - z|^s}.
\]

We can give a simple geometric characterization of the limit value \( \Lambda_\infty(s) \), this value is related to the maximum distance (measured in a way that involves the exponent \( s \)) from one point \((x, y)\) \( \in \Omega \) to the boundary. In fact,
\[
\Lambda_\infty(s) = \frac{1}{\max_{(x, y) \in \Omega} \min_{(z, w) \in \partial \Omega} \left( \frac{|x - z|^s + |y - w|^s}{|y - w|^s} \right)}.
\]

That the limit equation is verified in the viscosity sense and involve quotients of the form \( \frac{u(x, w) - u(x, y)}{|y - w|^s} \) is not surprising. In fact, viscosity solutions provide the right framework to deal with limits of \( p \)-Laplacians as \( p \to \infty \), see \([4, 6, 27]\), and quotients like the one mentioned above appeared in other related limits, see \([12, 23, 29]\). What is remarkable in the limit equation is that it involves the limit value \( \Lambda_\infty(s) \) and that the quotients that appear have perfectly identified the two groups of variables that are present in the fractional pseudo \( p \)-Laplacian that we introduced here.

Our results say that we can take the limits as \( s \to 1^- \) and as \( p \to \infty \) in the first eigenvalue. With the above notations we have the following commutative diagram
\[
\begin{array}{ccc}
(1 - s)\lambda_1(s, p)^{1/p} & \xrightarrow{s \to 1^-} & \lambda_1(1, p)^{1/p} \\
p \to \infty & \downarrow & \downarrow p \to \infty \\
\Lambda_\infty(s) & \xrightarrow{s \to 1^-} & \Lambda_\infty.
\end{array}
\]
Here
\[ \Lambda_\infty := \max_{(x,y) \in \Omega} \min_{(z,w) \in \partial \Omega} \frac{1}{(|x-z| + |y-w|)}. \]

The limit
\[ \lim_{p \to \infty} (\lambda_1(1,p))^{1/p} = \Lambda_\infty \]
can be obtained as in [27] using the variational characterization of \( \lambda_1(1,p) \) given in (1.1). We omit the details.

To end this introduction, let us comment on previous results. The limit as \( p \to \infty \) of the first eigenvalue \( \lambda_p^D \) of the usual local \( p \)-Laplacian with Dirichlet boundary condition was studied in [27, 28], (see also [5] for an anisotropic version). In those papers the authors prove that
\[ \lambda_p^D := \lim_{p \to +\infty} (\lambda_p(1,p))^{1/p} = \inf \left\{ \frac{\|\nabla v\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\Omega)}} : v \in W^{1,\infty}_0(\Omega), v \neq 0 \right\} = \frac{1}{R}, \]
where \( R \) is the largest possible radius of a ball contained in \( \Omega \). In addition, it was shown the existence of extremals, i.e. functions where the above infimum is attained. These extremals can be constructed taking the limit as \( p \to \infty \) in the eigenfunctions of the \( p \)-Laplacian eigenvalue problems (see [27]) and are viscosity solutions of the following eigenvalue problem (called the infinity eigenvalue problem in the literature)
\[ \min \{ |Du| - \lambda_\infty^D u, \Delta_\infty u \} = 0 \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial \Omega. \]
The limit operator \( \Delta_\infty \) that appears here is the \( \infty \)-Laplacian given by \( \Delta_\infty u = -(D^2 u Du, Du) \). Remark that solutions to \( \Delta_p v_p = 0 \) with a Dirichlet data \( v_p = f \) on \( \partial \Omega \) converge as \( p \to \infty \) to the viscosity solution to \( \Delta_\infty v = 0 \) with \( v = f \) on \( \partial \Omega \), see [4, 6, 13]. This operator appears naturally when one considers absolutely minimizing Lipschitz extensions in \( \Omega \) of a boundary data \( f \), see [2, 4]. Limits of \( p \)-Laplacians are also relevant in mass transfer problems, see [7, 19].

On the other hand, the pseudo infinity Laplacian is the second order nonlinear operator given by \( \Delta_\infty^\infty u = \sum_{i=1}^N (|u_{x_i}|^p - 2 u_{x_i})_{x_i} \), where the sum is taken over the indexes in \( I(\nabla u) = \{ i : |u_{x_i}| = \max_j |u_{x_j}| \} \). This operator, as happens for the usual infinity Laplacian, also appears naturally as a limit of \( p \)-Laplace type problems. In fact, any possible limit of \( u_p \), solutions to \( \Delta_p u = \sum_{i=1}^N (|u_{x_i}|^{p-2} u_{x_i})_{x_i} = 0 \), is a viscosity solution to \( \Delta_\infty^\infty u = 0 \). A proof of this fact is contained in [5], where are also studied the eigenvalue problem for this operator.

Concerning regularity, we mention [35] where it it proved that infinity harmonic functions, that is, viscosity solutions to \( -\Delta_\infty u = 0 \), are \( C^1 \) in two dimensions and [20, 21] where it is proved differentiability in any dimension. For the pseudo infinity Laplacian, we refer here to solutions to \( \Delta_\infty^\infty u = 0 \), the optimal regularity is Lipschitz continuity, see [34].

For references concerning nonlocal fractional problems we refer to [18, 26, 29, 30, 31, 17] and references therein. For limits as \( p \to +\infty \) in nonlocal \( p \)-Laplacian problems and its relation with optimal mass transport we refer to [26] (eigenvalue problems were not considered there).
Finally, concerning limits as $p \to \infty$ in fractional eigenvalue problems, we mention [9, 23, 28]. In [28] the limit of the first eigenvalue for the fractional $p$–Laplacian is studied while in [23] higher eigenvalues are considered. We borrow ideas and techniques from these papers. In particular, when we prove the fact that there is a limit problem that is verified in the viscosity sense. For example, the fact that continuous weak solutions to our pseudo fractional $p$–Laplacian are viscosity solutions runs exactly as in [28] and hence we omit the details here.

The paper is organized as follows: In Section 2 we collect some preliminary results; in Section 3 we deal with our eigenvalue problem and prove Theorem 1.1; in Section 4 we analyze the limit as $s \to 1^-$, Theorem 1.2; finally, in Section 5 we study the limit as $p \to \infty$ proving Theorem 1.3.

2. Preliminaries

Throughout this section $s \in (0, 1)$, $p \in (1, +\infty)$, $\Omega$ is an open set of $\mathbb{R}^{n+m}$. We henceforth use the notation:

- $(x, y) = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (x_{n+1}, \ldots, x_{n+m}) \in \mathbb{R}^m$;
- $\Omega^2 = \Omega \times \Omega$;
- $\Omega_x = \{y \in \mathbb{R}^m : (x, y) \in \Omega\}$, and $\Omega_y = \{x \in \mathbb{R}^n : (x, y) \in \Omega\}$;
- $B_N(x, r)$ denotes the ball of $N$–ball of radius $r$ and center $x$, and $\omega_N$ denotes the $(N - 1)$–dimensional Hausdorff measure of the $N$–sphere of radius 1;
- $(a)^{p-1} = |a|^{p-2}a$.

Given a measurable function $u : \Omega \to \mathbb{R}$, we set for $p < +\infty$,

$$||u||^p_{L^p(\Omega)} : = \int_{\Omega} |u(x, y)|^p dxdy,$$

$$|u|^{p}_{W^{s,p}(\Omega)} = \int_{\Omega^2} \frac{|u(x, y) - u(z, w)|^p}{||(x, y) - (z, w)||^{n+m+sp}} dxdydzdw,$$

$$|u|^{p}_{W^{s,p}(\Omega)} = \int_{\Omega} \int_{\Omega_y} \frac{|u(x, y) - u(z, w)|^p}{|x - z|^{n+sp}} dzdxdy$$

$$+ \int_{\Omega} \int_{\Omega_x} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{m+sp}} dwdxdy$$

and for $p = +\infty$,

$$|u|^{\infty}_{W^{s,\infty}(\Omega)} = \sup \left\{ \frac{|u(x, y) - u(z, w)|}{||(x, y) - (z, w)||^s} : (x, y) \neq (z, w) \in \Omega \right\} = |u|^{C^{0,s}}_{\Omega},$$

$$|u|^{\infty}_{W^{s,\infty}(\Omega)} = \max \left\{ \sup \left\{ \frac{|u(x, y) - u(z, y)|}{|x - z|^s} : (x, y) \neq (z, y) \in \Omega \right\}, \sup \left\{ \frac{|u(x, y) - u(x, w)|}{|y - w|^s} : (x, y) \neq (x, w) \in \Omega \right\} \right\}.$$  

We denote by $W^{s,p}(\Omega)$ (here $p$ can be $+\infty$) the usual fractional Sobolev space, that is $W^{s,p}(\Omega) := \{ u \in L^p(\Omega) : |u|^{W^{s,p}(\Omega)} < +\infty \}$.
We introduce the space $W^{s,p}(\Omega)$ (again here $p$ can be $+\infty$) as follows:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)}^p < \infty \right\}.$$  

This space is a Banach space. We state this as a proposition but we omit its proof that is standard.

**Proposition 2.1.** The space $W^{s,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left( \|u\|_{L^p(\Omega)}^p + [u]_{W^{s,p}(\Omega)}^p \right)^{1/p}$$

is a Banach space. Moreover $W^{s,p}(\Omega)$ is separable for $1 \leq p < +\infty$ and it is reflexive for $1 < p < \infty$.

For $u: \Omega \to \mathbb{R}$ a measurable function, we set

$$u_+(x,y) = \max\{u(x,y),0\} \quad \text{and} \quad u_-(x,y) = \min\{-u(x,y),0\}.$$  

Observe that

$$|u_+(x,y) - u_+(z,w)| \leq |u(x,y) - u(z,w)|$$

for all $(x,y), (z,w) \in \Omega$. Therefore, we have

**Lemma 2.2.** Let $X = W^{s,p}(\Omega)$ or $W^{s,p}(\Omega)$. If $u \in X$ then $u_+, u_- \in X$.

For $1 \leq p < \infty$, we denote by $\tilde{W}^{s,p}(\Omega)$ the space of all $u \in W^{s,p}(\Omega)$ such that

$$\tilde{u} \in W^{s,p}(\mathbb{R}^{n+m})$$

where $\tilde{u}$ is the extension by zero of $u$.

The next result can be found in [1, 15].

**Theorem 2.3.** Under the assumptions A1 and A2 we have that

- If $sp < n + m$, then $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $1 \leq q < p^*_s = (n+m)p/(n+m-sp)$.
- If $sp = n + m$, then $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $1 \leq q < \infty$.
- If $sp > n + m$, then $W^{s,p}(\Omega)$ is compactly embedded in $C^{0,\lambda}(\overline{\Omega})$ with $\lambda < s - (n+m)/p$.

**Lemma 2.4.** Let $\Omega_1$ and $\Omega_2$ be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. If $\Omega = \Omega_1 \times \Omega_2$, and $p \in [1, +\infty)$, then $W^{s,p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$. Moreover, there exists a constant $C = C(n,m)$ such that

$$|u|_{W^{s,p}(\Omega)}^p \leq C[u]_{W^{s,p}(\Omega)}$$

for all $u \in W^{s,p}(\Omega)$.

**Proof.** Let $u \in W^{s,p}(\Omega)$. We have

$$|u|_{W^{s,p}(\Omega)}^p = \int_{\Omega_2} |u(x,y) - u(z,w)|^p \, dxdydzdw$$

$$\leq 2^{p-1} \int_{\Omega_2} \frac{|u(x,y) - u(z,w)|^p}{(x,y) - (z,w)^{n+m+sp}} \, dxdydzdw$$

$$+ 2^{p-1} \int_{\Omega_2} \frac{|u(z,y) - u(z,w)|^p}{(x,y) - (z,w)^{n+m+sp}} \, dxdydzdw$$

(2.1)
Now, we observe that

\[ I_1 = \int_{\Omega^2} \frac{|u(x, y) - u(z, y)|^p}{|x, y| - (z, w)|^{n+m+sp}^p} dxdzdw \]

\[ \leq \int_{\Omega} \int_{\Omega_2} |u(x, y) - u(z, y)|^p \int_{\mathbb{R}^m} \frac{|x - z|^{n+sp}}{|x, y| - (z, w)|^{n+m+sp}^p} dwdxdy \]

\[ \leq \int_{\Omega} \int_{\Omega_2} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} \int_{\mathbb{R}^m} \frac{|x - z|^{n+sp}}{(|x - z|^2 + |y - w|^2)^{\frac{n+m+sp}{2}}} dwdxdy \]

\[ = \omega_m \int_{\Omega} \int_{\Omega_2} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dwdxdy \int_0^{+\infty} \frac{r^{m-1}}{(1 + r^2)^{n+m+sp}} dr. \]

Since

\[ \int_0^{+\infty} \frac{r^{m-1}}{(1 + r^2)^{n+m+sp}} dr \leq \int_0^1 \frac{r^{m-1}}{(1 + r^2)^{n+m+sp}} dr + \int_1^{+\infty} \frac{1}{r^{n+sp+1}} dr = \frac{1}{m} + \frac{1}{n + sp} \]

we have that

\[ I_1 \leq 2\omega_m \int_{\Omega} \int_{\Omega_2} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dwdxdy. \]

One can also, in an analogous way, obtain

\[ I_2 \leq 2\omega_n \int_{\Omega} \int_{\Omega_1} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{m+sp}} dwdxdy. \]

By (2.1), (2.2) and (2.3), we get

\[ |u|_{W^{s,p}(\Omega)} \leq C(n, m)[u]_{W^{s,p}(\Omega)}. \]

This completes the proof. \( \square \)

**Remark 2.5.** If \( p = \infty \), it is straightforward to show that \( W^{s, \infty}(\Omega) \subset W^{s, \infty}(\Omega) \). Moreover, if \( \Omega = \Omega_1 \times \Omega_2 \) then \( W^{s, \infty}(\Omega) = W^{s, \infty}(\Omega) \).

**Lemma 2.6.** Let \( \Omega \) be an open subset of \( \mathbb{R}^{n+m} \) and \( p \in (1, \infty) \). If \( 0 < t < s < 1 \) then \( W^{t,p}(\Omega) \subset W^{s,p}(\Omega) \), and the embedding is continuous. Moreover

\[ |u|^p_{W^{s,p}(\Omega)} \leq |u|^p_{W^{t,p}(\Omega)} + \frac{2^p(\omega_n + \omega_m)}{tp} \| u \|_{L^p(\Omega)}^p \quad \forall u \in W^{s,p}(\Omega). \]

**Proof.** Let \( u \in W^{s,p}(\Omega) \). Observe that,

\[ \int_{\Omega} \int_{\Omega} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+tp}} dwdxdy \leq \int_{\Omega} \int_{A_v} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+tp}} dwdxdy \]

\[ + \int_{\Omega} \int_{A_v} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+tp}} dwdxdy \]
where \( A_y = \{ z \in \Omega_y : |z - x| < 1 \} \). Since \( t < s \), we have that
\[
\int_{\Omega_y} \int_{\Omega_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+tp}} dz dy \\
\leq \int_{\Omega_y} \int_{A_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dy + 2^{p-1} \int_{\Omega_y} \int_{A_y} \frac{|u(x, y)|^p + |u(z, y)|^p}{|x - z|^{n+tp}} dz dy \\
\leq \int_{\Omega_y} \int_{A_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dy + 2^p \int_{\Omega_y} \int_{A_y} \frac{|u(x, y)|^p}{|x - z|^{n+tp}} dz dy \\
\leq \int_{\Omega_y} \int_{A_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dy + \frac{2^p \omega_n}{tp} \int_{\Omega_y} |u(x, y)|^p dx dy.
\]
Similarly,
\[
\int_{\Delta_x} \int_{\Delta_x} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{n+tp}} dz dy \\
\leq \int_{\Delta_x} \int_{A_x} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dy + \frac{2^p \omega_n}{tp} \int_{\Delta_x} |u(x, y)|^p dx dy,
\]
where \( A_x = \{ w \in \Delta_x : |y - w| < 1 \} \). Therefore (2.4) holds. \( \square \)

Finally, we prove a Poincaré type inequality.

**Lemma 2.7.** Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^{n+m} \), \( s \in (0, 1) \) and \( p \in (1, \infty) \). Then there is a positive constant \( C \) such that
\[
\| u \|_{L^p(\Omega)} \leq C \| u \|_{\widetilde{W}^{s,p}(\mathbb{R}^{n+m})} \quad \forall u \in \widetilde{W}^{s,p}(\Omega).
\]

**Proof.** Let \( u \in \widetilde{W}^{s,p}(\Omega) \) and \( d = 2 \text{diam}(\Omega) \). It holds that
\[
[u]_{\widetilde{W}^{s,p}(\mathbb{R}^{n+m})} \geq \int_{\Omega} \frac{|u(x, y)|^p}{\int_{\mathbb{R}^{n+m} \setminus B^{n}(x, d)} \frac{d\tilde{z}}{|x - \tilde{z}|^{n+sp}}} \geq \frac{\omega_n d^{-sp}}{sp} \| u \|_{L^p(\Omega)}^p.
\]
\( \square \)

## 3. The first eigenvalue

Under assumptions A1 and A2, a natural definition of an eigenvalue is a real value \( \lambda \) for which there exists \( u \in \widetilde{W}^{s,p}(\Omega) \setminus \{ 0 \} \) such that \( u \) is a weak solution of
\[
\begin{align*}
\mathcal{L}_{s,p} u(x, y) & = \lambda (u(x, y))^{p-1} \quad \text{in } \Omega, \\
u(x, y) & = 0 \quad \text{in } \Omega^c,
\end{align*}
\]
that is
\[
\mathcal{H}_{s,p}(u, v) = \lambda \int_{\Omega} (u(x, y))^{p-1} v(x, y) dx dy \quad \forall v \in \widetilde{W}^{s,p}(\Omega).
\]
The function \( u \) is called a corresponding eigenfunction. Here
\[
\mathcal{H}_{s,p}(u, v) := \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n}} \frac{(u(x, y) - u(z, y))^{p-1}(v(x, y) - v(z, y))}{|x - z|^{n+sp}} dz dy \\
+ \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n}} \frac{(u(x, y) - u(x, w))^{p-1}(v(x, y) - v(x, w))}{|y - w|^{m+sp}} dw dy.
\]
Observe that
\[ H_{s,p}(u,v) = \int_{\Omega} |u(x,y)|^p \, dx \, dy. \]

Then, we have that
\[ \lambda = \frac{[u]_{W^{s,p}(\mathbb{R}^{n+m})}^p}{\|u\|_{L^p(\Omega)}^p} \geq 0. \]

By a standard compactness argument, we have the following result.

**Theorem 3.1.** Under the assumptions A1 and A2, the first eigenvalue is given by
\[ \lambda_1(s,p) := \inf \left\{ \frac{[u]_{W^{s,p}(\mathbb{R}^{n+m})}^p}{\|u\|_{L^p(\Omega)}^p} : u \in \tilde{W}^{s,p}(\Omega), u \not\equiv 0 \right\}. \]

**Proof.** Consider a minimizing sequence \( u_n \) normalized according to \( \|u_n\|_{L^p(\Omega)} = 1 \). Then, as \( u_n \) in bounded in \( \tilde{W}^{s,p}(\Omega) \), by Lemma 2.4 and Theorem 2.3, there is a subsequence such that \( u_{n_j} \rightharpoonup u \) weakly in \( \tilde{W}^{s,p}(\Omega) \) and \( u_{n_j} \rightarrow u \) strongly in \( L^p(\Omega) \). Therefore, \( u \) is a nontrivial minimizer to the variational problem defining \( \lambda_1(s,p) \).

The fact that this minimizer is a weak solution to (3.1) is straightforward and can be obtained from the arguments in [29].

To finish the proof we just observe that any other eigenfunction associated with an eigenvalue \( \lambda \) verifies
\[ \lambda = \frac{[u]_{W^{s,p}(\mathbb{R}^{n+m})}^p}{\|u\|_{L^p(\Omega)}^p} \geq \lambda_1(s,p), \]
and then we get that \( \lambda_1(s,p) \) is the first eigenvalue.

Now we observe that using a topological tool (the genus) we can construct an unbounded sequence of eigenvalues.

**Theorem 3.2.** Assume A1 and A2. There is a sequence of eigenvalues \( \lambda_n \) such that \( \lambda_n \rightarrow +\infty \) as \( n \rightarrow +\infty \).

**Proof.** We follow ideas from [22] and hence we omit the details. Let us consider
\[ M_\alpha = \{ u \in \tilde{W}^{s,p}(\Omega) : [u]_{W^{s,p}(\mathbb{R}^{n+m})} = p\alpha \} \]
and
\[ \varphi(u) = \frac{1}{p} \int_{\Omega} |u(x,y)|^p \, dx \, dy. \]
We are looking for critical points of \( \varphi \) restricted to the manifold \( M_\alpha \) using a minimax technique. We consider the class

\[
\Sigma = \{ A \subset \widetilde{N}^{s,p}(\Omega) \setminus \{0\} : A \text{ is closed, } A = -A \}.
\]

Over this class we define the genus, \( \gamma : \Sigma \to \mathbb{N} \cup \{\infty\} \), as

\[
\gamma(A) = \min \{ k \in \mathbb{N} : \text{there exists } \phi \in C(A, \mathbb{R}^k - \{0\}), \phi(x) = -\phi(-x) \}.
\]

Now, we let \( C_k = \{ C \subset M_\alpha : C \text{ is compact, symmetric and } \gamma(C) \leq k \} \) and let

\[
\beta_k = \sup_{C \in C_k} \min_{u \in C} \varphi(u).
\]

Then \( \beta_k > 0 \) and there exists \( u_k \in M_\alpha \) such that \( \varphi(u_k) = \beta_k \) and \( u_k \) is a weak eigenfunction with \( \lambda_k = \alpha/\beta_k \).

The following lemma shows that the eigenfunctions are bounded.

**Lemma 3.3.** Under assumptions A1 and A2, if \( u \) is an eigenfunction associated to some eigenvalue \( \lambda \) then \( u \in L^\infty(\mathbb{R}^{n+m}) \).

**Proof.** In this proof we follow ideas form [23].

If \( ps > n + m \), by Lemma 2.4 and Theorem 2.3, then the assertion holds. From now on, we suppose that \( sp \leq n + m \).

We will show that if \( \|u_+\|_{L^p(\Omega)} \leq \delta \) then \( u_+ \) is bounded, where \( \delta > 0 \) is some small constant to be determined. Let \( k \in \mathbb{N}_0 \), we define the function \( u_k \) by

\[
u_k(x, y) := (u(x, y) - 1 + 2^{-k})_+.
\]

Observe that, \( u_0 = u_+ \) and for any \( k \in \mathbb{N}_0 \) we have that \( u_k \in \widetilde{W}^{s,p}(\Omega) \) verifies

\[
\begin{align*}
&u_{k+1} \leq u_k \text{ a.e. } \mathbb{R}^{n+m}, \\
u_+ &< (2^{k+1} - 1)u_k \text{ in } \{u_{k+1} > 0\}, \\
&\{u_{k+1} > 0\} \subset \{u_k > 2^{-(k+1)}\}.
\end{align*}
\]

Now, for any function \( v : \mathbb{R}^{n+m} \to \mathbb{R} \), it holds that

\[
|v_+(x, y) - v_+(z, w)|^p \leq |v(x, y) - v(z, w)|^p - 1(v_+(x, y) - v_+(x, w))
\]

for all \( (x, y), (z, w) \in \mathbb{R}^{n+m} \). Then

\[
[u_{k+1}]_{W^{s,p}(\mathbb{R}^{n+m})}^p \leq \mathcal{H}_{s,p}(u, u_{k+1}) = \lambda \int_\Omega (u(x, y))^{p-1}u_{k+1}(x, y)\,dxdy
\]

for all \( k \in \mathbb{N}_0 \). Hence, by (3.2) and Hölder’s inequality, we get

\[
[u_{k+1}]_{W^{s,p}(\mathbb{R}^{n+m})}^p \leq \lambda \int_\Omega (u(x, y))^{p-1}u_{k+1}(x, y)\,dxdy \leq (2^{k+1} - 1)^{p-1}\lambda\|u_k\|_{L^p(\Omega)}^p
\]

for all \( k \in \mathbb{N}_0 \).

On the other hand, in the case \( sp < n + m \), using Hölder’s inequality, Lemma 2.4 and Theorem 2.3, the formulas in (3.2), and Chebyshev’s inequality, for any \( k \in \mathbb{N}_0 \)
we have that
\[
\|u_{k+1}\|_{L^p(\Omega)}^p \leq \|u_{k+1}\|_{W^{s,p}(\mathbb{R}^{n+m})}^p \{u_{k+1} > 0\}^{\frac{sp}{n+m}}
\]
\[
\leq C\|u_{k+1}\|_{W^{s,p}(\mathbb{R}^{n+m})}^p \{u_k > 2^{-(k+1)}\}^{\frac{sp}{n+m}}
\]
\[
\leq C\|u_{k+1}\|_{W^{s,p}(\mathbb{R}^{n+m})}^p \left(2^{(k+1)p}\|u_k\|_{L^p(\Omega)}^p\right)^{\frac{sp}{n+m}},
\]
where \(C\) is a constant independent of \(k\). Then, by (3.3) and (3.4), for any \(k \in \mathbb{N}_0\) we obtain
\[
\|u_{k+1}\|_{L^p(\Omega)}^p \leq C \left(2^{(k+1)p}\|u_k\|_{L^p(\Omega)}^p\right)^{1+\frac{sp}{n+m}},
\]
where \(C\) is a constant independent of \(k\) and \(\alpha = \frac{sp}{n+m} > 0\).

Arguing similarly, in the case \(sp = n + m\), taking \(r > p\) and proceeding as in the previous case, \(sp < n + m\) (with \(r\) in place of \(p^*_s\)), we obtain (3.5) holds with \(\alpha = 1 - \frac{p}{r} > 0\).

Therefore, if \(sp \leq n + m\), there exist \(\alpha > 0\) and a constant \(C > 1\) such that
\[
\|u_{k+1}\|_{L^p(\Omega)}^p \leq C^k \left(\|u_k\|_{L^p(\Omega)}^p\right)^{1+\alpha},
\]
for any \(k \in \mathbb{N}_0\). Hence, if \(\|u_m\|_{L^p(\Omega)}^p = \|u_1\|_{L^p(\Omega)}^p \leq C^{-1/\alpha} =: \delta^p\) then \(u_k \to 0\) strongly in \(L^p(\Omega)\). But \(u_k \to (u - 1)^+\) a.e in \(\mathbb{R}^{n+m}\), hence we conclude that \((u - 1)^+ \equiv 0\) in \(\mathbb{R}^{n+m}\). Therefore, \(u_+\) is bounded.

Taking \(-u\) in place of \(u\) we have that \(u_-\) is bounded if \(\|u_-\|_{L^p(\Omega)} < \delta\).

Hence, as we can multiply an eigenfunction \(u\) by a small constant in order to obtain \(\|u_+\|_{L^p(\Omega)}\) and \(\|u_-\|_{L^p(\Omega)} < \delta\), we conclude that \(u\) is bounded. \(\square\)

Our next goal is to show that if \(u\) is a eigenfunction associated with \(\lambda_1(s,p)\) then \(u\) does not change sign. Before showing this result we need the following two technical lemmas.

**Lemma 3.4.** Assume A1 and A2. If \(u \in \overline{W}^{s,p}(\Omega)\) is such that
\[
\mathcal{H}_{s,p}(u,v) \geq 0 \quad \forall v \in \overline{W}^{s,p}(\Omega), v \geq 0 \text{ in } \Omega.
\]
and \(u \geq 0\) in \(B^n(x_0,R) \times B^m(y_0,R) \subset \subset \Omega\) for some \(R > 0\) then for any \(d > 0\) and \(0 < 2r < R\) there holds
\[
\int_{B^p R} \int_{B^p R} \frac{1}{|x - z|^{n+sp}} \left[\log \left(\frac{u(x,y) + d}{u(z,y) + d}\right)\right]^p dxdy 
\]
\[
+ \int_{B^p R} \int_{B^p R} \int_{B^p R} \frac{1}{|y - w|^{m+sp}} \left[\log \left(\frac{u(x,y) + d}{u(x,w) + d}\right)\right]^p dwdxdy 
\]
\[
\leq C \left(\frac{\rho}{d^{p-1} r^m} \int_{(B^p R)^c} \int_{(B^p R)^c} \frac{u_-(x,y)^p}{|x - x_0|^{n+sp}} dxdy 
\]
\[
+ \frac{\rho}{d^{p-1} r^m} \int_{(B^p R)^c} \int_{(B^p R)^c} \frac{u_-(x,y)^p}{|y - y_0|^{m+sp}} dydx + 1\right}\}
\]
where \(B^p_R = B^n(x_0, \rho) \times B^m(y_0, \rho)\) and \(C = C(n, m, p, s) \geq 0\) is a constant.
Proof. Let \(d > 0, r \in (0, R/2)\),

\[
\begin{align*}
\phi & \in C^\infty_0(B^n_{3r/2}), \quad 0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } B^n_r, \quad |D_x \phi| < \frac{c}{r} \text{ in } B^n_{3r/2}, \text{ and} \\
\psi & \in C^\infty_0(B^m_{3r/2}), \quad 0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ in } B^m_r, \quad |D_x \psi| < \frac{c}{r} \text{ in } B^m_{3r/2}.
\end{align*}
\]

Taking \(v(x, y) = \phi^p(x)\psi^p(y)(u(x, y) + d)^{1-p}\) as test function in (3.6) and following the proof of Lemma 1.3 in [16], we get (3.7). \(\square\)

**Lemma 3.5.** Assume A1 and A2. If \(\Omega\) is connected and \(u \in \tilde{W}^{s,p}(\Omega)\) is such that

\[
H_{s,p}(u, v) \geq 0 \quad \forall v \in \tilde{W}^{s,p}(\Omega), v \geq 0 \text{ in } \Omega,
\]

\(u \geq 0 \text{ in } \Omega\) and \(u \equiv 0 \text{ in } \Omega\) then \(u > 0 \text{ in } \Omega\).

**Proof.** In this proof we borrow ideas from [8]. Since \(\Omega\) is a bounded connected open set, it is enough to prove that \(u > 0 \text{ in } K\) for any \(K \subset \subset \Omega\) a connected compact set such that \(u \equiv 0 \text{ in } K\).

Let \(K \subset \subset \Omega\) be a connected compact set such that \(u \equiv 0 \text{ in } K\). Then there exists \(r > 0\) such that

\[
K \subset \left\{ (x, y) \in \Omega : \max_{(z, w) \in \partial \Omega} \{|z - x|, |w - y|\} > 2r \right\}.
\]

Since \(K\) is compact, there exists \(\{(x_j, y_j)\}_{j=1}^k \subset K\) such that

\[
K \subset \bigcup_{j=1}^k B^n_j \times B^m_j, \quad \text{and} \quad |(B^n_j \times B^m_j) \cap (B^n_{j+1} \times B^m_{j+1})| > 0
\]

for any \(j \in \{1, \ldots, k - 1\}\), where \(B^n_j = B^n(x_j, r/2)\) and \(B^m_j = B^m(y_j, r/2)\).

To obtain a contradiction, suppose that \(|\{(x, y) : u(x, y) = 0\} \cap K| > 0\) then there exists \(j \in \{1, \ldots, k\}\) such that

\[
Z = \{(x, y) : u(x, y) = 0\} \cap (B^n_j \times B^m_j)
\]

has positive measure.

Given \(d > 0\), we define

\[
F_d : B^n_j \times B^m_j \rightarrow \mathbb{R} \quad \text{by} \quad F_d(x, y) = \log \left(1 + \frac{u(x, y)}{d}\right).
\]
Then, for any \((x, y) \in B^n(x_j, r/2) \times B^m(y_j, r/2)\) and \((z, w) \in Z\) we have

\[
F_d(z, w) = 0
\]

\[
|F_d(x, y)|^p = |F(x, y) - F(z, w)|^p
\]

\[
\leq 2^{p-1} |F(x, y) - F(z, y)|^p \frac{|z - x|^{n+sp}}{|z - x|^{n+sp}}
\]

\[
+ 2^{p-1} |F(z, y) - F(z, w)|^p \frac{|w - y|^{m+sp}}{|w - y|^{m+sp}}
\]

\[
\leq 2^{p-1} |F(x, y) - F(z, w)|^p \frac{1}{|z - x|^{n+sp}}
\]

\[
+ 2^{p-1} |F(z, y) - F(z, w)|^p \frac{1}{|w - y|^{m+sp}}.
\]

Therefore,

\[
|Z||F_d(x, y)|^p = \int_Z \int_{B^n_y} |F_d(x, y)|^p dwdz
\]

\[
\leq c_1 r^{n+m+sp} \int_{B^n_x} \left( \frac{u(x, y) + d}{u(z, y) + d} \right)^p \frac{dz}{|z - x|^{n+sp}}
\]

\[
+ 2^{p-1} r^{m+sp} \int_{B^n_y} \int_{B^m_{x_j}} \left( \frac{u(z, y) + d}{u(z, w) + d} \right)^p dwdz.
\]

for any \((x, y) \in B^n(x_j, r/2) \times B^m(y_j, r/2)\). Here \(c_1 = c_1(m, p) > 0\) is a constant. Then

\[
\int_{B^n_y} \int_{B^m_{x_j}} |F_d(x, y)|^p dx dy
\]

\[
\leq c_1 r^{n+m+sp} \frac{|Z|}{|Z|} \int_{B^n_y} \int_{B^m_{x_j}} \int_{B^n_x} \left( \frac{u(x, y) + d}{u(z, y) + d} \right)^p \frac{dzdxdy}{|z - x|^{n+sp}}
\]

\[
+ 2^{p-1} r^{m+sp} \frac{|Z|}{|Z|} \int_{B^n_y} \int_{B^m_{x_j}} \int_{B^n_x} \left( \frac{u(z, y) + d}{u(z, w) + d} \right)^p \frac{dwdxdy}{|w - y|^{m+sp}}.
\]

Thus, by Lemma 3.4 and since \(u \geq 0\) in \(\Omega\), we get

\[
\int_{B^n_y} \int_{B^m_{x_j}} |F_d(x, y)|^p dx dy \leq C r^{2n+2m} |Z|,
\]

where \(C = C(n, m, s, p) > 0\) is a constant. Taking \(d \to 0\) in the last inequality, we get that \(u \equiv 0\) in \(B^n_y \times B^m_{x_j}\).

By (3.8), there exists \(i \in \{1, \ldots, k\}\) such that \(i \neq j\) and

\[
|(B^n_i \times B^m_i) \cap \{(x, y): u(x, y) = 0\}| > 0.
\]

Then, we can repeat the previous argument for \(B^n_i \times B^m_i\) and obtain \(u \equiv 0\) in \(B^n_i \times B^m_i\). In this way we conclude that \(u \equiv 0\) in \(K\) which contradicts the fact that \(u \neq 0\) in \(K\). Thus \(|\{(x, y): u(x, y) = 0\} \cap K| = 0. \)

\[\square\]
Now, we are ready to prove that the eigenfunctions associated to $\lambda_1(s,p)$ do not change sign.

**Theorem 3.6.** Assume A1 and A2. If $u$ is an eigenfunction associated to $\lambda_1(s,p)$ then $|u| > 0$ in $\Omega$.

**Proof.** We start by showing that if $u$ is an eigenfunction corresponding to $\lambda_1(s,p)$ then $|u| \neq 0$ in all connected components of $\Omega$. Our proof is by contradiction. We therefore assume that there is a connected component $A$ of $\Omega$ such that $|u| \equiv 0$. Since $u$ is an eigenfunction corresponding to $\lambda_1(s,p)$ then so is $|u|$. Then

$$0 = \lambda_1(s,p) \int_{\Omega} |u(x,y)|^{p-1} \phi(x,y) \, dx \, dy = \mathcal{H}_{s,p}(|u|, \phi)$$

$$= -2 \int_{A} \int_{A_y} \frac{|u(x,y)|^{p-1} \phi(x,y)}{|x-z|^{n+sp}} \, dx \, dy - 2 \int_{A} \int_{A_y} \frac{|u(x,y)|^{p-1} \phi(x,w)}{|y-w|^{m+sp}} \, dw \, dx$$

for all $\phi \in C_0^\infty(A)$, which is a contradiction.

Therefore, if $A$ connected components $C$ of $\Omega$ then $|u| \not\equiv 0$ in $A$ and

$$\mathcal{H}_{s,p}(|u|, v) = \lambda_1(s,p) \int_{\Omega} |u(x,y)|^{p-1} v(x,y) \, dx \, dy \geq 0 \quad \forall v \in \widetilde{W}^{s,p}(A).$$

Then, by Lemma 3.5, $|u| > 0$ in $A$. Therefore $|u| > 0$ in $\Omega$. \qed

Our next result show that $\lambda_1(s,p)$ is simple.

**Theorem 3.7.** Assume A1 and A2. Let $u$ be a positive eigenfunction corresponding to $\lambda_1(s,p)$. If $\lambda > 0$ is such that there exists a non-negative eigenfunction $v$ of (3.1) with eigenvalue $\lambda$, then $\lambda = \lambda_1(s,p)$ and there exists $k \in \mathbb{N}$ such that $v = kv$ a.e. in $\Omega$.

**Proof.** Since $\lambda_1(s,p)$ is the first eigenvalue we have that $\lambda_1(s,p) \leq \lambda$. Let $k \in \mathbb{N}$ and define $v_k := v + 1/k$.

We begin proving that $w_k := u_p/v_k^{p-1} \in \widetilde{W}^{s,p}(\Omega)$. It is immediate that $w_k = 0$ in $\Omega^c$ and $w_k \in L^p(\Omega)$, due to the fact that $u \in L^\infty(\Omega)$, see Lemma 3.3.

On the other hand

$$|w_k(x,y) - w_k(z,w)|$$

$$= \left| \frac{u(x,y)^p - u(z,w)^p}{v_k(x,y)^{p-1}} + \frac{u(z,w)^p (v_k(z,w)^{p-1} - v_k(x,y)^{p-1})}{v_k(x,y)^{p-1}v_k(z,w)^{p-1}} \right|$$

$$\leq k^{p-1} \left| \frac{u(x,y)^p - u(z,w)^p}{v_k(x,y)^{p-1}} \right| + \left| \frac{u(z,w)^p (v_k(z,w)^{p-1} - v_k(x,y)^{p-1})}{v_k(x,y)^{p-1}v_k(z,w)^{p-1}} \right|$$

$$\leq 2\|u\|_{L^\infty(\Omega)}^{p-1} k^{p-1} |u(x,y) - u(z,w)|$$

$$+ \|u\|_{L^\infty(\Omega)}^{p-1} (p-1) \frac{v_k(x,y)^{p-2} + v_k(z,w)^{p-2}}{v_k(x,y)^{p-1}v_k(z,w)^{p-1}} |v_k(x,y) - v_k(z,w)|$$

$$\leq 2\|u\|_{L^\infty(\Omega)}^{p-1} k^{p-1} |u(x,y) - u(z,w)|$$

$$+ \|u\|_{L^\infty(\Omega)}^{p-1} (p-1) k^{p-1} \left( \frac{1}{v_k(x,y)} + \frac{1}{v_k(z,w)} \right) |v(y) - v(x)|$$

$$\leq C(k, p, \|u\|_{L^\infty(\Omega)}) (|u(x,y) - u(z,w)| + |v(x,y) - v(z,w)|)$$
for all \((x, y), (z, w) \in \mathbb{R}^{n+m}\). Hence, we have that \(w_k \in \mathcal{W}^{s,p}(\Omega)\) for all \(k \in \mathbb{N}\) since \(u, v \in \mathcal{W}^{s,p}(\Omega)\).

Set
\[
L(u, v_k)(x, y, z, w) = |u(x, y) - u(w, z)|^p
\]
\[
- (v_k(x, y) - v_k(w, z))^{p-1} \left( \frac{u(x, y)^p}{v_k(x, y)^{p-1}} - \frac{u(z, w)^p}{v_k(z, w)^{p-1}} \right).
\]

Then, by [2, Lemma 6.2] and since \(u, v\) are two positive eigenfunctions of problem (3.1) with eigenvalues \(\lambda_1(s, p)\) and \(\lambda\) respectively, we have
\[
0 \leq \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} L(u, v_k)(x, y, z, y) |x-z|^{n+sp} \, dz \, dx \, dy + \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} L(u, v_k)(x, y, x, w) |y-w|^{m+sp} \, dw \, dx \, dy
\]
\[
- \mathcal{H}_{s,p}(v, w_k)
\]
\[
\leq \lambda_1(s, p) \int_{\Omega} u(x, y)^p \, dx \, dy - \lambda \int_{\Omega} v(x, y)^{p-1} u_k(x, y) \, dx \, dy
\]
\[
= \lambda_1(s, p) \int_{\Omega} u(x, y)^p \, dx \, dy - \lambda \int_{\Omega} v(x, y)^{p-1} \frac{u(x, y)^p}{v_k(x, y)^{p-1}} \, dx \, dy.
\]

By Fatou’s lemma and the dominated convergence theorem we obtain
\[
\int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} L(u, v)(x, y, z, y) |x-z|^{n+sp} \, dz \, dx \, dy + \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} L(u, v)(x, y, x, w) |y-w|^{m+sp} \, dw \, dx \, dy = 0
\]
due to \(\lambda_1(s, p, h) \leq \lambda\). Then \(L(u, v)(x, y, z, y) = L(u, v)(x, y, x, w) = 0\) a.e. Hence, again by Lemma 6.2 in [2], \(u(x, y) = \ell_1(y)v(x, y)\) and \(u(x, y) = \ell_2(x)v(x, y)\) for all \((x, y) \in \mathbb{R}^{n+m}\). Then, we conclude that \(u = \ell v\) for some constant \(\ell > 0\).

Finally we will prove that \(\lambda_1(s, p)\) is isolated.

**Theorem 3.8.** Assume A1 and A2. Then \(\lambda_1(s, p)\) is isolated.

**Proof.** We split the proof into two steps.

**Step 1.** If \(u\) is an eigenfunction associated to some eigenvalue \(\lambda > \lambda_1(s, p)\) then there is a positive constant \(C\) such that
\[
(1/C\lambda)^{1/(r-p)} \leq |\Omega_\pm|
\]
for all \(p < r < p_*^s\). Here \(\Omega_\pm = \{(x, y) : u_\pm \neq 0\}\), and
\[
p_*^s = \begin{cases} \frac{(n+m)p}{n+m-sp} & \text{if } sp < n+m, \\ \infty & \text{if } sp \geq n+m. \end{cases}
\]

Let \(r \in (p, p_*^s)\). By Theorem 2.3, Lemmas 2.7 and 2.4 and Hölder inequality, we have
\[
\|u_+\|_{L^r(\Omega)}^r \leq C\|u_+\|^p_{\mathcal{W}^{s,p}(\Omega)} \leq C\mathcal{H}_{s,p}(u, u_+) = C\lambda\|u_+\|^p_{L^r(\Omega)} |\Omega_+|^{(r-p)/r}.
\]

Then
\[
(1/C\lambda)^{1/(r-p)} \leq |\Omega_+|.
\]
In order to prove the inequality for $|\Omega_\cdot|$, it suffices to proceed as above, using the function $-u$ instead of $u$.

**Step 2.** By definition, $\lambda_1(s,p)$ is left-isolated. To prove that $\lambda_1(s,p)$ is right-isolated, we argue by contradiction. We assume that there is a sequence of eigenvalues \( \{\lambda_k\}_{k \in \mathbb{N}} \) such that \( \lambda_k \searrow \lambda_1(s,p) \) as \( k \to \infty \). Let \( u_k \) be an eigenfunction associated to \( \lambda_k \) such that \( \|u_k\|_{L^p(\Omega)} = 1 \). Then \( \{u_k\}_{k \in \mathbb{N}} \) is bounded in \( \mathcal{W}^{s,p}(\Omega) \) and therefore we can extract a subsequence (that we still denoted by \( \{u_k\}_{k \in \mathbb{N}} \)) such that

\[
 u_k \to u \text{ weakly in } \mathcal{W}^{s,p}(\Omega), \quad u_k \to u \text{ strongly in } L^p(\Omega).
\]

Then \( \|u\|_{L^p(\Omega)} = 1 \) and

\[
 [u]_{W^{s,p}(\mathbb{R}^{n+m})}^p \leq \liminf_{k \to \infty} [u_k]_{W^{s,p}(\mathbb{R}^{n+m})}^p = \lim_{k \to \infty} \lambda_k = \lambda_1(s,p).
\]

Then \( u \) is an eigenfunction associated to \( \lambda_1(s,p) \). Therefore \( u \) has constant sign.

Now, proceeding as in the proof of [3, Theorem 2], we arrive to a contradiction. In fact, by Egoroff’s theorem we can find a subset \( A_\delta \) of \( \Omega \) such that \( |A_\delta| < \delta \) and \( u_k \to u \) uniformly in \( \Omega \setminus A_\delta \). From (3.9) we get that \( u \) and the uniform convergence in \( \Omega \setminus A_\delta \) we obtain that \(|\{u > 0\}| > 0 \) and \(|\{u > 0\}| < 0 \). This contradicts the fact that an eigenfunction associated with the first eigenvalue does not change sign. \( \square \)

### 4. The limit as \( s \to 1^- \)

In this section, our goal is to show that

\[
\lim_{s \to 1^-} (1 - s)\lambda_1(s,p) = \lambda_1(1,p)
\]

(4.1)

\[
= \inf_{u \in W^{1,p}(\Omega), u \not= 0} \left\{ \frac{K_{n,p} \int_{\Omega} |\nabla u(x,y)|^p dxdy + K_{m,p} \int_{\Omega} |\nabla_y u(x,y)|^p dxdy}{\|u\|^p_{L^p(\Omega)}} \right\}
\]

where \( K_{n,p} \) is a constant that depends only on \( n \) and \( p \), and \( K_{m,p} \) depends only on \( m \) and \( p \). Before proving (4.1), we need some technical results.

**Lemma 4.1.** Let \( \Omega \) be an open subsets of \( \mathbb{R}^{n+m} \) with smooth boundary and \( p \in (1, \infty) \). For all \( s \in (0,1) \) we have that \( W^{1,p}(\Omega) \) is continuity embedded in \( W^{s,p}(\Omega) \).

**Proof.** In this proof, we follow the ideas of the proof of [11, Theorem 1]. Let \( u \in W^{1,p}(\Omega) \). By an extension argument, we can assume that \( u \in W^{1,p}(\mathbb{R}^{n+m}) \). We have that

\[
\int_{\mathbb{R}^{n+m}} |u(x+h,y) - u(x,y)|^p dxdy \leq |h|^p \int_{\mathbb{R}^{n+m}} |\nabla u(x,y)|^p dxdy,
\]

(4.2)

\[
\int_{\mathbb{R}^{n+m}} |u(x,y+h) - u(x,y)|^p dxdy \leq |h|^p \int_{\mathbb{R}^{n+m}} |\nabla u(x,y)|^p dxdy.
\]

The proof of this fact can be carried out as that of Proposition XI.3 in [10] and is omitted.
Then, by (4.2), we have
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+m}} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} \, dx \, dy \, dz \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+m}} \frac{|u(x + h, y) - u(x, y)|^p}{|h|^{n+sp}} \, dx \, dy \, dh \\
\leq \int_{\{|h| \leq 1\}} \frac{dh}{|h|^{(s-1)p + n}} \int_{\mathbb{R}^{n+m}} |\nabla_x u(x, y)|^p \, dx \, dy \\
+ 2 \int_{\{|h| > 1\}} \frac{dh}{|h|^{sp + n}} \int_{\mathbb{R}^{n+m}} |u(x, y)|^p \, dx \, dy \\
\leq \frac{\omega_n}{(1 - s)p} \int_{\mathbb{R}^{n+m}} \left| \nabla_x u(x, y) \right|^p \, dx \, dy + \frac{2\omega_n}{sp} \int_{\mathbb{R}^{n+m}} |u(x, y)|^p \, dx \, dy,
\]
which completes the proof. 

Remark 4.2. Proceeding as in the proof of previous lemma and using the Poincaré inequality, we have that
\[
(1 - s)|u|^p_{W^{s,p}(\mathbb{R}^{n+m})} \leq C \left( 1 + \frac{1}{s} \right) \int_{\Omega} |\nabla u|^p \, dx \, dy \quad \forall u \in W^{1,p}_0(\Omega)
\]
where $C$ is a constant independent of $s$.

Lemma 4.3. Let $\Omega$ be an open subset of $\mathbb{R}^{n+m}$ with smooth boundary and $p \in (1, \infty)$. If $u \in W^{1,p}_0(\Omega)$ then
\[
(1 - s)|u|^p_{W^{s,p}(\mathbb{R}^{n+m})} \to K_{n,p} \int_{\Omega} |\nabla_x u|^p \, dx \, dy + K_{m,p} \int_{\Omega} |\nabla_y u|^p \, dx \, dy
\]
as $s \to 1$.

Proof. We split the proof into two cases.

Case 1. First we prove the lemma for $\phi \in C_0^\infty(\Omega)$. Let $B_1$ and $B_2$ be two open balls in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively such that $\Omega \subset B_1 \times B_2$.

Given $y \in B_2$, we have that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left| \phi(x, y) - \phi(z, y) \right|^p}{|x - z|^{n+sp}} \, dx \, dz = \int_{B_1} \int_{B_1} \frac{\left| \phi(x, y) - \phi(z, y) \right|^p}{|x - z|^{n+sp}} \, dx \, dz \\
+ 2 \int_{B_1} \int_{B_1} \frac{\left| \phi(x, y) \right|^p}{|x - z|^{n+sp}} \, dx \, dz.
\]
By [11, Theorem 1], there is a constant $K_{n,p}$ (that depends only the $n$ and $p$) such that
\[
(1 - s) \int_{B_1} \int_{B_1} \frac{\left| \phi(x, y) - \phi(z, y) \right|^p}{|x - z|^{n+sp}} \, dx \, dz \to K_{n,p} \int_{B_1} |\nabla_x \phi(x, y)|^p \, dx.
\]
as \( s \to 1^- \). On the other hand, since \( \text{supp}(\varphi) \subset \subset \Omega \subset B_1 \times B_2 \), there exists \( \delta > 0 \) such that \( |x-z| > \delta \) for all \( z \in B_1^c \) and \( x \in \{ t \in B_1: (t, y) \in \text{supp}(\varphi) \} \). Thus

\[
(4.5) \quad (1-s) \int_{B_1} \int_{B_1^c} \frac{|\phi(x,y)|^p}{|x-z|^{n+sp}} \,dx \,dz \leq (1-s) \frac{\omega_n}{s \delta^{sp}} \|\phi(\cdot,y)\|_{L^p(B_1)}^p \to 0
\]
as \( s \to 1^- \). Then by (4.3), (4.4), and (4.5) we have that

\[
(4.6) \quad (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(x,y) - \phi(z,y)|^p}{|x-z|^{n+sp}} \,dx \,dz \to K_{n,p} \int_{B_1} |\nabla_x \phi(x,y)|^p \,dx
\]
as \( s \to 1^- \). Proceeding as in the proof of Lemma 4.1, we have that

\[
(1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(x,y) - \phi(z,y)|^p}{|x-z|^{n+sp}} \,dx \,dz \leq \frac{\omega_n}{p} \int_{\mathbb{R}^n} |\nabla_x \phi(x,y)|^p \,dx \,dy
\]

\[
+ (1-s) \frac{2\omega_n}{s \delta^p} \int_{\mathbb{R}^n} |\phi(x,y)|^p \,dx \,dy.
\]

Thus, (4.6) and the dominated convergence theorem imply

\[
(1-s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|\phi(x,y) - \phi(z,y)|^p}{|x-z|^{n+sp}} \,dz \,dy \to K_{n,p} \int_{\mathbb{R}^m} \int_{B_1} |\nabla_x \phi(x,y)|^p \,dx \,dy,
\]
as \( s \to 1^- \), that is,

\[
(1-s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|\phi(x,y) - \phi(z,y)|^p}{|x-z|^{n+sp}} \,dz \,dy \to K_{n,p} \int_{\Omega} |\nabla_x \phi(x,y)|^p \,dx \,dy,
\]
as \( s \to 1^- \).

In the same manner we can see that there exists a constant \( K_{m,p} \) (that depends only the \( m \) and \( p \)) such that

\[
(1-s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{|\phi(x,y) - \phi(x,w)|^p}{|y-w|^{m+sp}} \,dw \,dy \to K_{m,p} \int_{\Omega} |\nabla_y \phi(x,y)|^p \,dx \,dy,
\]
as \( s \to 1^- \).

Then, we have

\[
(1-s)[\phi]_{W^{s,p}(\mathbb{R}^{n+m})} \to K_{n,p} \int_{\Omega} |\nabla_x \phi|^p \,dx \,dy + K_{m,p} \int_{\Omega} |\nabla_y \phi|^p \,dx \,dy,
\]
as \( s \to 1^- \).

**Case 2.** Now we prove the general case. Given \( u \in W^{1,p}(\Omega) \), we define

\[
F_s^u(x,y,z) = (1-s)^{1/p} \frac{|u(x,y) - u(z,y)|}{|x-z|^{n/p+s}},
\]

\[
G_s^u(x,y,z) = (1-s)^{1/p} \frac{|u(x,y) - u(x,w)|}{|y-w|^{m/p+s}}
\]
and we want to show that

\[
\|F_s^u\|_{L^p(\mathbb{R}^{2n+m})} \to K_{n,p}^1 \|\nabla_x u\|_{L^p(\Omega)}, \quad \|G_s^u\|_{L^p(\mathbb{R}^{n+2m})} \to K_{m,p}^1 \|\nabla_y u\|_{L^p(\Omega)},
\]
as \( s \to 1^- \).

Given \( \varepsilon > 0 \) there is \( \phi \in C_0^\infty(\Omega) \) such that

\[
\|\nabla u - \nabla \phi\|_{L^p(\Omega)} < \varepsilon.
\]
Thus
\[
\|\nabla x u\|_{L^p(\Omega)} - \|\nabla x \phi\|_{L^p(\Omega)} < \varepsilon \quad \text{and} \quad \|\nabla x u\|_{L^p(\Omega)} - \|\nabla x \phi\|_{L^p(\Omega)} < \varepsilon.
\]

By case 1, there exists \( s_0 \in (0, 1) \) such that
\[
\|F_{s_0}^\phi\|_{L^p(R^{2n+m})} - K_{n,p}^s \|\nabla x \phi\|_{L^p(\Omega)} < \varepsilon,
\]
(4.8)
\[
\|G_{s_0}^\phi\|_{L^p(R^{n+2m})} - K_{m,p}^s \|\nabla y \phi\|_{L^p(\Omega)} < \varepsilon,
\]
for all \( s \in (s_0, 1) \).

On the other hand, using Remark 4.2, we have that
\[
\|F_{s}^u\|_{L^p(R^{2n+m})} - \|F_{s}^\phi\|_{L^p(R^{2n+m})} \leq C \|\nabla u - \nabla \phi\|_{L^p(\Omega)} < C\varepsilon,
\]
(4.9)
\[
\|G_{s}^u\|_{L^p(R^{n+2m})} - \|G_{s}^\phi\|_{L^p(R^{n+2m})} \leq C \|\nabla u - \nabla \phi\|_{L^p(\Omega)}, < C\varepsilon,
\]
where \( C \) is a constant independent of \( s \).

Finally, by (4.7), (4.8), and (4.9), we obtain that
\[
\|F_{s}^u\|_{L^p(R^{2n+m})} - K_{n,p}^s \|\nabla x u\|_{L^p(\Omega)} < C\varepsilon,
\]
\[
\|G_{s}^u\|_{L^p(R^{n+2m})} - K_{m,p}^s \|\nabla y u\|_{L^p(\Omega)} < C\varepsilon,
\]
and the proof is complete. \( \square \)

**Corollary 4.4.** Let \( \Omega \) be an open subset of \( \mathbb{R}^{n+m} \) with smooth boundary and \( p \in (1, \infty) \). If \( u \in W^{1,p}_0(\Omega) \) then
\[
(1 - s)[u]^p_{W^{s,p}(\Omega)} \to K_{n,p} \int_\Omega |\nabla x u|^p \, dx \, dy + K_{p,m} \int_\Omega |\nabla y u|^p \, dx \, dy
\]
as \( s \to 1^- \).

**Proof.** By Lemma 4.3, we only need to proof that if \( u \in W^{1,p}_0(\Omega) \) then
\[
(1 - s) \left( [u]^p_{W^{s,p}(\Omega)} - [u]^p_{W^{p,p}(\Omega)} \right) \to 0
\]
as \( s \to 1^- \). First we prove the result for \( \phi \in C_0^\infty(\Omega) \). We have
\[
\left( [\phi]^p_{W^{s,p}(\Omega)} - [\phi]^p_{W^{p,p}(\Omega)} \right) = 2 \int_{\text{supp}(\phi)} \int_{\Omega_y} \frac{|\phi(x, y)|^p}{|x - z|^{n + sp}} \, dz \, dx \, dy
\]
(4.10)
\[
+ 2 \int_{\text{supp}(\phi)} \int_{\Omega_y} \frac{|\phi(x, y)|^p}{|y - w|^{m + sp}} \, dw \, dx \, dy.
\]
Since \( \text{supp}(\phi) \subset \Omega \) is compact, there exists \( \delta > 0 \) such that \( |x - z| > \delta \) and \( |y - w| > \delta \) for all \( (x, y) \in \text{supp}(\phi), z \in \Omega_y^n, w \in \Omega_x^n \). Then
\[
\int_{\text{supp}(\phi)} \int_{\Omega_y} \frac{|\phi(x, y)|^p}{|x - z|^{n + sp}} \, dz \, dx \, dy \leq \frac{\omega_n}{sp\delta^{sp}} \int_\Omega |\phi(x, y)|^p \, dx \, dy,
\]
\[
\int_{\text{supp}(\phi)} \int_{\Omega_y} \frac{|\phi(x, y)|^p}{|y - w|^{m + sp}} \, dw \, dx \, dy \leq \frac{\omega_m}{sp\delta^{sp}} \int_\Omega |\phi(x, y)|^p \, dx \, dy.
\]
Therefore, using (4.10), we have that
\[
(1 - s) \left( [\phi]^p_{W^{s,p}(\Omega)} - [\phi]^p_{W^{p,p}(\Omega)} \right) \to 0
\]
as \( s \to 1^- \).
The argument for the general case is analogous to the one performed in case 2 in the proof of Lemma 4.3.

For the proof of the following lemma, see [11, Lemma 2].

Lemma 4.5. Let \( \delta > 0 \) and \( g, h : (0, \delta) \to (0, +\infty) \). Assume that \( g(t) \leq g(t^{1/2}) \) and that \( h \) is non-increasing. Then
\[
\int_0^\delta t^{N-1} g(t) h(t) \, dt \geq \frac{N}{(2\delta)^N} \int_0^\delta t^{N-1} g(t) \, dt \int_0^\delta t^{N-1} h(t) \, dt
\]
for all \( N > 0 \).

Lemma 4.6. Let \( 0 < s_0 < s \) and \( u \in \widetilde{W}^{s,p}(\Omega) \). Then
\[
\frac{(1 - s_0)[u]^p_{\tilde{W}^{s,p}(\Omega)}}{2(1-s_0)p \text{diam}(\Omega)^{(s-s_0)p}} \leq (1 - s)[u]^p_{\tilde{W}^{s,p}(\mathbb{R}^{n+m})}
\]

Proof. Let \( B_1 \) and \( B_2 \) be two balls in \( \mathbb{R}^n \) and \( \mathbb{R}^m \) respectively such that \( \Omega \subset B_1 \times B_2 \) and \( \text{diam}(B_1) = \text{diam}(B_2) = \text{diam}(\Omega) \). Then
\[
\int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|u(x,y) - u(z,y)|^p}{|x - z|^{n+sp}} \, dz \, dx \, dy \geq
\]
\[
\int_{\mathbb{R}^m} \int_{S^{n-1}} \int_{\mathbb{R}^{n-1}} |u(x+tw,y) - u(x,y)|^p \frac{dxdtdy}{t^{1+sp}}
\]
\[
\geq \int_{\mathbb{R}^m} \int_{\text{diam}(\Omega)} \int_{S^{n-1}} |u(x+tw,y) - u(x,y)|^p \frac{dxdtdy}{t^{1+sp}}
\]
\[
\geq \int_{\mathbb{R}^m} \int_{\text{diam}(\Omega)} \int_{S^{n-1}} |u(x+y,0) - u(x,0)|^p \frac{dxdy}{t^{1+sp}}
\]

Taking \( N = (1 - s_0)p, \delta = \text{diam}(\Omega) \), we get
\[
g(t) = \int_{S^{n-1}} \int_{\mathbb{R}^m} |u(x+tw,y) - u(x,y)|^p \frac{dxd\sigma}{tp}
\]
and \( h(t) = \frac{1 - s}{t^{1+sp}} \).

By Lemma 4.5, we have that
\[
(1 - s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|u(x,y) - u(z,y)|^p}{|x - z|^{n+sp}} \, dz \, dx \, dy \geq
\]
\[
\frac{(1 - s_0)p}{2(1-s_0)p \text{diam}(\Omega)^{(s-s_0)p}} \int_{\mathbb{R}^m} \int_0^\delta |(1-s_0)p-1 g(t) dt \int_0^\delta |t^{1-s_0}p-1 h(t) dt
\]
\[
\geq \frac{(1 - s_0)p}{2(1-s_0)p \text{diam}(\Omega)^{(s-s_0)p}} \int_{\mathbb{R}^m} \int_0^\delta |(1-s_0)p-1 g(t) dt \int_0^\delta |(1-s_0)p-1 dt
\]
\[
\geq \frac{(1 - s_0)p}{2(1-s_0)p \text{diam}(\Omega)^{(s-s_0)p}} \int_{\mathbb{R}^m} \int_0^\delta |u(x+tw,y) - u(x,y)|^p \frac{dxdtdy}{t^{1+sp}}
\]
\[
\geq \frac{(1 - s_0)p}{2(1-s_0)p \text{diam}(\Omega)^{(s-s_0)p}} \int_\Omega \int_{\Omega_y} |u(x,y) - u(z,y)|^p \frac{dxdy}{|x - z|^{n+sp}}
\]

Similarly
\[
(1 - s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|u(x,y) - u(z,w)|^p}{|y - w|^{m+sp}} \, dz \, dx \, dy \geq
\]
\[
\frac{(1 - s_0)}{2(1-s_0)p \text{diam}(\Omega)^{(s-s_0)p}} \int_\Omega \int_{\Omega_y} |u(x,y) - u(z,y)|^p \frac{dxdy}{|y - w|^{m+sp}}
\]
This concludes the proof. □

We can now show the main result of this section.

**Theorem 4.7.** Let $\Omega$ be bounded domain in $\mathbb{R}^{n+m}$ with smooth boundary, $s \in (0, 1)$ and $p \in (1, \infty)$. Then

$$
\lim_{s \to 1^-} (1 - s)\lambda_1(s, p) = \lambda_1(1, p).
$$

**Proof.** First, we observe that, from Lemma 4.1, if $u \in W_0^{1,p}(\Omega)$ then $u \in \tilde{W}_s^{s,p}(\Omega)$. Then

$$(1 - s)\lambda_1(s, p) \leq \frac{\|u\|_{W_0^{s,p}(\mathbb{R}^{n+m})}^p}{\|u\|_{L_p(\Omega)}^p}$$

for all $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. Therefore, by Lemma 4.3, we have that

$$\limsup_{s \to 1^-} (1 - s)\lambda_1(s, p) \leq \frac{K_{n,p} \int_{\Omega} |\nabla x u(x, y)|^p dx dy + K_{m,p} \int_{\Omega} |\nabla y u(x, y)|^p dx dy}{\|u\|_{L_p(\Omega)}^p}$$

for all $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. Then

$$\limsup_{s \to 1^-} (1 - s)\lambda_1(s, p) \leq \lambda_1(1, p).$$

To finish the proof, we have to show that

$$\liminf_{s \to 1^-} (1 - s)\lambda_1(s, p) \geq \lambda_1(1, p).$$

Let $\{s_k\}_{k \in \mathbb{N}} \subset (0, 1)$ be such that $s_k \to 1$ as $k \to \infty$.

$$\lim_{k \to \infty} (1 - s_k)\lambda_1(s_k, p) = \liminf_{s \to 1^-} (1 - s)\lambda_1(s, p).$$

For each $k \in \mathbb{N}$, we let $u_k$ be an eigenfunction corresponding to $\lambda_1(s_k, p)$ such that $\|u_k\|_{L_p(\Omega)} = 1$. By (4.12), there is a positive constant $C$ such that

$$(1 - s_k)\|u_k\|_{W_0^{s_k,p}(\mathbb{R}^{n+m})}^p \leq C \quad \forall k \in \mathbb{N}.$$

Then, by Lemma 2.4, there is a positive constant $C$ such that

$$(1 - s_k)\|u_k\|_{W_0^{s_k,p}(\mathbb{R}^{n+m})}^p \leq C \quad \forall k \in \mathbb{N}.$$

Thus, by [11, Corollary 7], up to a subsequence, $\{u_k\}_{k \in \mathbb{N}}$ converges in $L^p(\Omega)$ to some $u \in W_0^{1,p}(\Omega)$. Moreover, for all $\delta > 0$, $u_k \to u$ strongly in $W^{1-\delta,p}(\Omega)$. Therefore $\|u\|_{L_p(\Omega)} = 1$.

Let $s_0 \in (0, 1)$. Since $s_k \to 1$, there exists $k_0 \in \mathbb{N}$ such that $s_0 < s_k$ for all $k \geq k_0$. Then, by Lemma 4.6, we have that

$$\frac{(1 - s_0)\|u_k\|_{W_0^{s_0,p}(\Omega)}^p}{2^{(1-s_0)p}} \leq \text{diam}(\Omega)^{(s_k-s_0)p}(1 - s_k)\|u_k\|_{W_0^{s_k,p}(\mathbb{R}^n)}^p$$

Thus, by (4.12) and Fatou’s lemma, we get

$$\frac{(1 - s_0)\|u\|_{W_0^{s_0,p}(\Omega)}^p}{2^{(1-s_0)p}} \leq \text{diam}(\Omega)^{(1-s_0)p} \liminf_{s \to 1^-} (1 - s)\lambda_1(s, p).$$
By Corollary 4.4, it holds that
\[ K_{n,p} \int_{\Omega} |\nabla x u(x,y)|^p \, dx \, dy + K_{m,p} \int_{\Omega} |\nabla y u(x,y)|^p \, dx \, dy = \lim_{s_0 \to 1^-} \frac{(1 - s_0)[u]_{W^{\gamma,0,p}(\Omega)}}{2(1 - s_0)^p} \leq \lim \inf_{s \to 1^-} (1 - s)\lambda_1(s,p). \]

Then
\[ \lambda_1(1, p) \leq \lim \inf_{s \to 1^-} (1 - s)\lambda_1(s, p). \]

Therefore, by (4.11),
\[ \lambda_1(1, p) = \lim_{s \to 1^-} (1 - s)\lambda_1(s, p), \]
as we wanted to prove. \( \square \)

5. THE LIMIT AS \( p \to \infty \)

Now we want to pass to the limit as \( p \to \infty \) in the first eigenvalue \( \lambda_1(s, p) \). Our goal now is to show that
\[ [\lambda_1(s, p)]^\frac{1}{p} \to \Lambda_\infty(s) \]
where
\[ \Lambda_\infty(s) = \inf \{ [u]_{W^{\gamma,0}(\mathbb{R}^{n+m})} : u \in W^{s,\infty}(\mathbb{R}^{n+m}), \|u\|_{L^\infty(\Omega)} = 1, u = 0 \text{ in } \Omega_c \}. \]

Observe that, by Arzela-Ascoli’s theorem, the previous infimum is attained.

We first prove a geometric characterization of \( \Lambda_\infty(s) \).

Lemma 5.1. Let \( R_s = \max_{(x,y) \in \Omega} \min_{(z,w) \in \partial \Omega} (|x - z|^s + |y - w|^s) \), then
\[ \Lambda_\infty(s) = \frac{1}{R_s}. \]

Proof. Let \( u \in W^{s,\infty}(\mathbb{R}^{n+m}) \), such that \( \|u\|_{L^\infty(\Omega)} = 1 \), \( u = 0 \) in \( \Omega_c \) and \( \Lambda_\infty(s) = [u]_{W^{s,\infty}(\mathbb{R}^{n+m})} \). Then, let \( (x_0, y_0) \in \Omega \) be such that \( u(x_0, y_0) = 1 \). If \( (z, w) \in \partial \Omega \) we have
\[ |u(x_0, y_0) - u(z, y_0)| \leq \Lambda_\infty(s)|x_0 - z|^s \]
and
\[ |u(z, y_0) - u(z, w)| \leq \Lambda_\infty(s)|y_0 - w|^s. \]

Then
\[ |u(x_0, y_0) - u(z, w)| \leq \Lambda_\infty(s)(|x_0 - z|^s + |y_0 - w|^s). \]

Therefore,
\[ 1 \leq \Lambda_\infty(s) \min_{(z,w) \in \partial \Omega} (|x_0 - z|^s + |y_0 - w|^s), \]
and then, we get
\[ \Lambda_\infty(s) \geq \frac{1}{\min_{(z,w) \in \partial \Omega} (|x_0 - z|^s + |y_0 - w|^s)} \geq \frac{1}{R_s}. \]

Now, we choose \( (x_0, y_0) \) that solves the geometric maximization problem
\[ R_s = \max_{(x,y) \in \Omega} \min_{(z,w) \in \partial \Omega} (|x - z|^s + |y - w|^s), \]
and consider the function
\[ u(x, y) = \left(1 - \frac{|x_0 - x|^s + |y_0 - y|^s}{R_s}\right)^+. \]

Observe that, \( \|u\|_{L^\infty(\Omega)} = 1 \). On the other hand, since for any \( s \in (0, 1] \)
\[ |a^s - b^s| \leq |a - b|^s \quad \forall a, b \in [0, \infty), \]
we have that \[ [u]_{W^{s,\infty}(\mathbb{R}^{n+m})} \leq 1/R_s. \] Hence, using this functions as a test function in the variational problem defining \( \Lambda_\infty(s) \) we get
\[ (5.2) \quad \Lambda_\infty(s) \leq \frac{1}{R_s}. \]

From (5.1) and (5.2) we obtain the desired result. □

**Lemma 5.2.** Let \( u_p \) be a positive eigenfunction for \( \lambda_1(s, p) \) normalized according to \( \|u_p\|_{L^p(\Omega)} = 1 \). Then, there exists a sequence \( p_j \to \infty \) such that
\[ u_j \to u \]
uniformly in \( \mathbb{R}^N \). This limit function \( u \) belongs to the space \( W^{s,\infty}(\Omega) \) and is a solution to the variational problem
\[ \Lambda_\infty(s) = \min \{ [u]_{W^{s,\infty}(\Omega)} : u \in W^{s,\infty}(\Omega), \|u\|_{L^\infty(\Omega)} = 1, u = 0 \text{ on } \partial\Omega \}. \]

In addition, it holds that
\[ (\lambda_1(s, p))^{1/p} \to \Lambda_\infty(s). \]

**Proof.** Let \( \alpha > 1 \) and
\[ R_{\alpha} = \max_{(x,y) \in \Omega} \min_{(z,w) \in \partial\Omega} (|x - z|^{\alpha} + |y - w|^{\alpha}). \]

We first claim that
\[ (R_s)^\alpha \leq \frac{(R_s)^\alpha}{2^{\alpha-1}} \leq R_{\alpha} \]
for any \( \alpha > 1 \). To this end, let \( (x_0, y_0) \in \Omega \) such that
\[ R_s = \min_{(z,w) \in \partial\Omega} (|x_0 - z|^s + |y_0 - w|^s). \]

Then for all \( (z, w) \in \partial\Omega \) we have
\[ (R_s)^\alpha \leq (|x_0 - z|^s + |y_0 - w|^s)^\alpha \leq 2^{\alpha-1} (|x_0 - z|^{\alpha} + |y_0 - w|^{\alpha}) \]
\[ \leq 2^{\alpha-1} R_{\alpha}, \]
that is, (5.3). On the other hand, it is clear that if \( s\alpha < 1 \) we have that
\[ u_\alpha(x, y) = \left(1 - \frac{|x - x_0|^{\alpha s} + |y - y_0|^{\alpha s}}{R_{\alpha}}\right)^+. \]

belongs to \( \tilde{W}^{s,p}(\Omega) \) for all \( p > 1 \). Then
\[ (\lambda_1(s, p))^{1/p} \leq \frac{[u_\alpha]_{W^{s,p}(\mathbb{R}^{n+m})}}{[u_\alpha]_{L^p(\Omega)}} \]
for all \( p > 1 \) and \( 1 < \alpha < 1/s \). Therefore
\[ \limsup_{p \to \infty} (\lambda_1(s, p))^{1/p} \leq \frac{[u_\alpha]_{W^{s,\infty}(\Omega)}}{[u_\alpha]_{L^\infty(\Omega)}} \quad \forall \alpha \in (1/s). \]
Observe that if $\alpha \in (1, 1/s)$, by (5.3), we have
\[
\frac{|u_{\alpha}(x, y) - u_{\alpha}(z, y)|}{|x - z|^s} \leq \frac{|x - z|^{(\alpha - 1)s}}{R_{s\alpha}} \leq 2^{\alpha - 1} \frac{\text{diam}(\Omega)^{(\alpha - 1)s}}{(R_s)^{\alpha}}
\]
for all $(x, y) \neq (z, y) \in \bar{\Omega}$, and
\[
\frac{|u_{\alpha}(x, y) - u_{\alpha}(x, w)|}{|y - w|^s} \leq \frac{|y - w|^{(\alpha - 1)s}}{R_{s\alpha}} \leq 2^{\alpha - 1} \frac{\text{diam}(\Omega)^{(\alpha - 1)s}}{(R_s)^{\alpha}},
\]
for all $(x, y) \neq (z, y) \in \bar{\Omega}$, that is
\[
[u_{\alpha}]_{W^{s, \infty}(\Omega)} \leq 2^{\alpha - 1} \frac{\text{diam}(\Omega)^{(\alpha - 1)s}}{(R_s)^{\alpha}}.
\]
Then, by (5.4) we get
\[
\limsup_{p \to \infty} (\lambda_1(s, p))^{1/p} \leq 2^{\alpha - 1} \frac{\text{diam}(\Omega)^{(\alpha - 1)s}}{(R_s)^{\alpha}} \quad \alpha \in (1, 1/s),
\]
since $\|u_{\alpha}\|_{L^{\infty}(\Omega)} = 1$. Therefore, passing to the limit as $\alpha \to 1$ in the previous inequality we get
\[
(5.5) \quad \limsup_{p \to \infty} (\lambda_1(s, p))^{1/p} \leq \frac{1}{R_s} = \Lambda_\infty(s).
\]

Our next goal is to show that
\[
\Lambda_\infty(s) \leq \liminf_{p \to \infty} (\lambda_1(s, p))^{1/p}.
\]
Let $p_j > 1$ be such that
\[
\liminf_{p \to \infty} (\lambda_1(s, p))^{1/p} = \lim_{j \to \infty} (\lambda_1(s, p_j))^{1/p_j}.
\]
By (5.5), without of loss of generality, we can assume
\[
(\lambda_1(s, p_j))^{1/p_j} = [u_{p_j}]_{W^{s, p_j}((\mathbb{R}^n + m)} \leq \Lambda_\infty(s) + \epsilon \quad \forall j \in \mathbb{N},
\]
where $u_{p_j}$ is an eigenfunction for $\lambda_1(s, p_j)$ normalized according to $\|u_{p_j}\|_{L^{p_j}(\Omega)} = 1$ and $\epsilon$ is any positive number. Then, by Lemma 2.4, we have that there exists a constant $C$, independent of $j$, such that
\[
|u_{p_j}|_{W^{s, p_j}(\Omega)} \leq C \quad \forall j \in \mathbb{N}.
\]
Therefore, for any $j \in \mathbb{N}$ there exists a constant $C$ independent of $j$, such that
\[
(5.6) \quad \|u_{p_j}\|_{W^{s, p_j}(\Omega)} \leq C.
\]

On the other hand, given $q > 1$ such that $sq > 2(n + m)$ and taking $t = s - n + m/q$, by Hölder’s inequality, for any $p_j > q$ we have that
\[
\|u_{p_j}\|_{L^q(\Omega)} \leq |\Omega|^{1 - \frac{s}{p_j}} \|u_{p_j}\|_{L^p(\Omega)}^{q/p} = |\Omega|^{1 - \frac{s}{p_j}}.
\]
and

\[ |u_{p_j}|_{W^{r,q}(\Omega)}^q = \int_{\Omega^2} \frac{|u_{p_j}(x,y) - u_{p_j}(z,w)|^q}{dxdydzdw} \]

\[ \leq |\Omega|^{2(\frac{1}{p_q} - \frac{1}{q})} \left( \int_{\Omega^2} \frac{|u_{p_j}(x,y) - u_{p_j}(z,w)|^{p_j}}{|(x,y) - (z,w)|^{p_j}} dxdydzdw \right)^{\frac{q}{p_j}} \]

\[ \leq |\Omega|^{2(\frac{1}{p_q} - \frac{1}{q})} \max \left\{ 1, \text{diam}(\Omega)^{(n+m)\frac{q}{p_j}} \right\} |u_{p_j}|_{W^{r,q}((\Omega)^2)}^{\frac{q}{p_j}}. \]

Hence, by (5.6), for \( j \) large there exists a constant \( C \), independent of \( j \), such that

\[ |u_{p_j}|_{W^{r,q}(\Omega)} \leq C \max \left\{ |\Omega|^{\frac{1}{q} - \frac{1}{p_k}}, |\Omega|^{2(\frac{1}{q} - \frac{1}{p_k})}, |\Omega|^{2(\frac{1}{q} - \frac{1}{p_k})} \text{diam}(\Omega)^{\frac{n+m}{p_k}} \right\}, \]

that is, there exists \( j_0 > 1 \) such that \( \{u_{p_j}\}_{j>0} \) is bounded in \( W^{r,q}(\Omega) \). Then, since \( tq > n + m \), by Theorem 2.3, there exists a subsequence \( \{u_{p_k}\}_{k\in\mathbb{N}} \) of \( \{u_{p_j}\}_{j>0} \) and a function \( u \in C^{\alpha,\gamma}(\Omega) \) \((0 < \gamma < t - (n+m)/q)\) such that \( u_{p_k} \to u \) uniformly in \( \Omega \).

Thus, if \( q > 1 \) there exists \( k_0 \in \mathbb{N} \) such that \( p_k > q \) if \( k > k_0 \) and therefore, by Hölder’s inequality, for any \( k > k_0 \) we have

\[ \left( \int_{\Omega} \int_{\Omega} \frac{|u_k(x,y) - u_k(z,w)|^q}{dxdydzdw} \right)^{\frac{1}{q}} \]

\[ \leq C^{\frac{1}{q} - \frac{1}{p_k}} \max \left\{ 1, \text{diam}(\Omega)^{\frac{n}{p_k}} \right\} \left( \int_{\Omega} \int_{\Omega} \frac{|u_k(x,y) - u_k(z,w)|^{p_k}}{|x-z|^{p_k+n}} dxdydzdw \right)^{\frac{1}{p_k}} \]

\[ \leq C^{\frac{1}{q} - \frac{1}{p_k}} \max \left\{ 1, \text{diam}(\Omega)^{\frac{n}{p_k}} \right\} \left[ u_k \right]_{W^{r,p_k}(\Omega)}, \]

and similarly

\[ \left( \int_{\Omega} \int_{\Omega} \frac{|u_k(x,y) - u_k(z,w)|^q}{dxdydzdw} \right)^{\frac{1}{q}} \]

\[ \leq C^{\frac{1}{q} - \frac{1}{p_k}} \max \left\{ 1, \text{diam}(\Omega)^{\frac{n}{p_k}} \right\} \left[ u_k \right]_{W^{r,p_k}(\Omega)}. \]

Here \( C \) is a constant independent of \( k \). Then passing to the limit as \( k \to \infty \) and using Fatou’s lemma we have that

\[ \left( \int_{\Omega} \int_{\Omega} \frac{|u(x,y) - u(z,w)|^q}{dxdydzdw} \right)^{\frac{1}{q}} \leq C^{\frac{1}{q}} \liminf_{k\to\infty} [u_k]_{W^{r,p_k}(\Omega)} \]

\[ \leq C^{\frac{1}{q}} \liminf_{p\to\infty} (\lambda_1(s,p))^{1/p}, \]

\[ \left( \int_{\Omega} \int_{\Omega} \frac{|u(x,y) - u(z,w)|^q}{dxdydzdw} \right)^{\frac{1}{q}} \leq C^{\frac{1}{q}} \liminf_{k\to\infty} [u_k]_{W^{r,p_k}(\Omega)} \]

\[ \leq C^{\frac{1}{q}} \liminf_{p\to\infty} (\lambda_1(s,p))^{1/p} \]

for all \( q > 1 \). Now passing to the limit as \( q \to \infty \) we obtain

\[ \sup \left\{ \frac{|u(x,y) - u(z,y)|}{|x-z|^s} : (x,y) \neq (z,y) \in \Omega \right\} \leq \liminf_{p\to\infty} (\lambda_1(s,p))^{1/p}, \]

\[ \sup \left\{ \frac{|u(x,y) - u(x,w)|}{|x-z|^s} : (x,y) \neq (x,w) \in \Omega \right\} \leq \liminf_{p\to\infty} (\lambda_1(s,p))^{1/p}, \]
that is
\begin{equation}
\|u\|_{W^{r,\infty}(\Omega)} \leq \liminf_{p \to \infty}(\lambda_1(s, p))^{1/p}.
\end{equation}

To conclude we need to show that \(\|u\|_{L^\infty(\Omega)} = 1\). For all \(q > 1\) there exists \(k_0 \in \mathbb{N}\) such that \(p_k > q\) if \(k > k_0\) and therefore, by Hölder’s inequality, for any \(k > k_0\) we get
\[\|u_k\|_{L^q(\Omega)} \leq |\Omega|^{1/q} \|u\|_{L_p(\Omega)}^q = |\Omega|^{1/q - 1/p}.\]

Then passing to the limit as \(k \to \infty\) and using that \(u_k \to u\) uniformly in \(\Omega\), \(\|u\|_{L^q(\Omega)} \leq 1\) for all \(q > 1\). Hence \(\|u\|_{L^\infty(\Omega)} \leq 1\). On the other hand, for all \(k\) we have \(1 = \|u_k\|_{L^p_k(\Omega)} \leq |\Omega|^{1/p_k} \|u\|_{L^\infty(\Omega)}\). Then, since \(u_k \to u\) uniformly in \(\Omega\), we get \(1 \leq \|u\|_{L^\infty(\Omega)}\). Hence \(\|u\|_{L^\infty(\Omega)} = 1\). Thus, by (5.7), we get
\[\Lambda_\infty(s) \leq \liminf_{p \to \infty}(\lambda_1(s, p))^{1/p},\]
and by (5.5) we conclude that
\[\Lambda_\infty(s) = \lim_{p \to \infty}(\lambda_1(s, p))^{1/p}.\]

This ends the proof. \(\square\)

Using the geometric characterization given in Lemma 5.1 we can compute \(\Lambda_\infty(s)\) in some concrete examples.

**Example 1.** When \(\Omega = B_R\) is a ball of radius \(R\) we have
\[\Lambda_\infty(s) = \frac{1}{R^s}.\]

**Example 2.** When \(\Omega = (-R, R) \times (-L, L)\) is a rectangle in \(\mathbb{R}^2\) we have
\[\Lambda_\infty(s) = \frac{1}{\min\{R^s, L^s\}}.\]

**Remark 5.3.** One can consider two different powers \(r\) and \(s\) in the definition of the pseudo \(p\)-Laplacian. In this case we get that,
\[\Lambda_\infty(r, s) = \max_{(x, y) \in \Omega, (z, w) \in \partial \Omega} (|x - z|^{r} + |y - w|^{s}).\]

**Viscosity solutions.** To obtain an eigenvalue problem that is satisfied by the limit of the eigenfunctions \(u_p\) when \(p \to \infty\), we need to introduce the definition of viscosity solutions. This is a notion of solution different from the weak one considered before. We refer to [13] for an introduction to the subject of viscosity solutions. In the theory of viscosity solutions the equation is evaluated for test functions at points where they touch the graph of a solution. Viscosity solutions are assumed to be continuous and the fractional Sobolev space is absent from the definition (no derivatives of a solutions are needed).

**Definition 5.4.** (Viscosity solutions). Suppose that the function \(u\) is continuous in \(\mathbb{R}^{n+m}\) and that \(u = 0\) in \(\Omega^c\). We say that \(u\) is a viscosity supersolution of the equation \(-\mathcal{L}_{s,p} u + \lambda |u|^{p-2} u = 0\) if the following holds: whenever \(x_0 \in \Omega\) and \(\varphi \in C^0_0(\mathbb{R}^{n+m})\) (the test function) are such that \(\varphi(x_0) = u(x_0)\) and \(\varphi(x) \leq u(x)\) for every \(x \in \mathbb{R}^{n+m}\), then we have
\[-\mathcal{L}_{s,p} \varphi(x_0) + \lambda |\varphi(x_0)|^{p-2} \varphi(x_0) \leq 0.\]
The requirement for being a viscosity subsolution is symmetric: the test function is touching from above and the inequality is reversed.

Finally, a viscosity solution is defined as being both a viscosity supersolution and a viscosity subsolution.

For our eigenvalue problem, we have that a continuous weak solution is a viscosity solution. For the proof we refer to [29].

**Theorem 5.5.** An eigenfunction $u \in C(\Omega)$ (in the weak sense) is a viscosity solution of the equation $-\mathcal{L}_{s,p}u + \lambda |u|^{p-2}u = 0$ in the sense of Definition 5.4.

We will also use the following lemmas.

**Lemma 5.6.** Assume that

\[
(A_p)^{1/p} \to A, \quad (B_p)^{1/p} \to -B, \\
(C_p)^{1/p} \to C, \quad (D_p)^{1/p} \to -D,
\]

and that

\[
\theta_p \to \Theta,
\]

as $p \to \infty$. If

\[
2^{1/p}(A_p + C_p)^{1/p} \geq (B_p + D_p + \theta_p^{p-1})^{1/p}
\]

for every $p$ large enough, then, passing to the limit, it holds that

\[
\max\{A; C\} \geq \max\{-B; -D; \Theta\}.
\]

**Proof.** First, assume that $A > C$ and $-B > \max\{-D; \Theta\}$. Then for $p$ large enough we have $A_p \geq C_p$, $-B_p \geq -D_p$ and $-B_p \geq (\theta_p)^p$. Then taking $p \to \infty$ in

\[
(A_p)^{1/p}2^{1/p} \left(1 + \frac{C_p}{A_p}\right)^{1/p} \geq (B_p)^{1/p} \left(1 + \frac{D_p}{B_p} + \frac{\theta_p^{p-1}}{B_p}\right)^{1/p}
\]

we get

\[
A \geq -B.
\]

The rest of the cases ($A = C$, $A < C$, etc) can be handled in an analogous way. \qed

**Lemma 5.7.** For a smooth test function $\phi$ let

\[
A_p = \int_{\mathbb{R}^n} \frac{\phi(x_p, y_p) - \phi(z, y_p)|^{p-2}(\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}} dz.
\]

If $x_p \to x_0$, $y_p \to y_0$ as $p \to \infty$, then

\[
(A_p)^{1/p} \to A = \sup_z \frac{\phi(x_0, y_0) - \phi(z, y_0)}{|x_0 - z|^{s}}.
\]

**Proof.** We just have to observe that

\[
(A_p)^{1/p} = \left(\int_{\mathbb{R}^n} \frac{\phi(x_p, y_p) - \phi(z, y_p)|^{p-2}(\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}} dz\right)^{1/p}.
\]
The integrand satisfies
\[
\frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2}(\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}} \sim \frac{|\phi(x_0, y_0) - \phi(z, y_0)|^{p-2}(\phi(x_0, y_0) - \phi(z, y_0))^+}{|x_0 - z|^{n+sp}}
\]
and hence the result follows from the fact that \((\int f^p)^{1/p} \to \|f\|_\infty\). \hfill \Box

**Lemma 5.8.** Any uniform limit of \(u_p\) a sequence of eigenfunctions for \(\lambda_1(s, p)\) normalized according to \(\|u_p\|_{L^p(\Omega)} = 1\), \(u\) is a nontrivial solution to
\[
\begin{align*}
\max\{A; C\} &= \max\{-B; -D; \Lambda_\infty(s)u\} & \text{in } \Omega, \\
u &= 0 & \text{in } \Omega^c,
\end{align*}
\]
in the viscosity sense. Here
\[
A = \sup_{w} \frac{u(x, w) - u(x, y)}{|y - w|^s}, \quad B = \inf_{w} \frac{u(x, w) - u(x, y)}{|y - w|^s}, \\
C = \sup_{z} \frac{u(z, y) - u(x, y)}{|x - z|^s}, \quad D = \inf_{z} \frac{u(z, y) - u(x, y)}{|x - z|^s}.
\]

**Proof.** We call \(u_p\) a sequence of solutions to \(-\mathcal{L}_{s,p}u + \lambda|u|^{p-2}u = 0\) that converges uniformly to \(u\). That \(u = 0\) in \(\Omega^c\) follows since \(u_p = 0\) in \(\Omega^c\) and we have uniform convergence.

Let \(\phi \in C^1_0(\mathbb{R}^{n+m})\) be such that \(u - \phi\) has a strict minimum at \((x_0, y_0)\) \(\in \Omega\). Since \(u_p\) converges uniformly to \(u\) we have that there exist \((x_p, y_p) \in \Omega\) such that \(u_p - \phi\) has a minimum at \((x_p, y_p)\) and \((x_p, y_p) \to (x_0, y_0)\) as \(p \to \infty\). Since \(u_p\) is a viscosity solution to \(-\mathcal{L}_{s,p}v(x, y) + \lambda_1(s, p)v(x, y)^{p-1} = 0\) in \(\Omega\), we obtain
\[
((\lambda_1(s, p))^{1/(p-1)}u_p(x_p, y_p))^{p-1} \leq 2 \int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2}(\phi(x_p, y_p) - \phi(z, y_p))}{|x_p - z|^{n+sp}}dz
\]
\[
+ 2 \int_{\mathbb{R}^m} \frac{|\phi(x_p, y_p) - \phi(x_p, w)|^{p-2}(\phi(x_p, y_p) - \phi(x_p, w))}{|y_p - w|^{m+sp}}dw
\]
\[
= 2(A_p - B_p + C_p - D_p),
\]
where
\[
A_p = \int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2}(\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}}dz,
\]
\[
B_p = \int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2}(\phi(x_p, y_p) - \phi(z, y_p))^-}{|x_p - z|^{n+sp}}dz,
\]
\[
C_p = \int_{\mathbb{R}^m} \frac{|\phi(x_p, y_p) - \phi(x_p, w)|^{p-2}(\phi(x_p, y_p) - \phi(x_p, w))^+}{|y_p - w|^{m+sp}}dw,
\]
\[
D_p = \int_{\mathbb{R}^m} \frac{|\phi(x_p, y_p) - \phi(x_p, w)|^{p-2}(\phi(x_p, y_p) - \phi(x_p, w))^-}{|y_p - w|^{m+sp}}dw.
\]
We observe that
\[(A_p)^{1/p} \to A, \quad (B_p)^{1/p} \to -B,\]
\[(C_p)^{1/p} \to C, \quad (D_p)^{1/p} \to -D,\]
and
\[
(\lambda_1(s, p))^{1/(p-1)} u_p(x_p, y_p) \to \Lambda_\infty u(x_0, y_0).
\]
Hence, taking limit as \(p \to \infty\) in (5.8), from Lemma 5.6, we get
\[
\max\{-B; -D; \Lambda_\infty(s)u(x_0, y_0)\} \leq \max\{A; C\}.
\]

Now, if \(\psi\) is such that \(u - \psi\) has a strict minimum at \((x_0, y_0)\) in \(\Omega\). Since \(u_p\) converges uniformly to \(u\) we have that there exist \((x_p, y_p)\) in \(\Omega\) such that \(u_p - \psi\) has a minimum at \((x_p, y_p)\) and \((x_p, y_p) \to (x_0, y_0)\) as \(p \to \infty\). Since \(u_p\) is a solution to
\[-L_{s,p}v(x, y) + \lambda v(x, y)^{p-1} = 0\]
in \(\Omega\) we obtain
\[
((\lambda_{1,p})^{1/(p-1)} u_p(x_p, y_p))^{p-1} \geq
\]
\[
\geq \left(\int_{\mathbb{R}^n} \frac{|\psi(x_p, y_p) - \psi(z, y_p)|^{2(p-2)}|\psi(x_p, y_p) - \psi(z, y_p)|}{|x_p - z|^{n+sp}} dz\right)
\]
\[
\quad + \left(\int_{\mathbb{R}^n} \frac{|\psi(x_p, y_p) - \psi(x_p, w)|^{2(p-2)}|\psi(x_p, y_p) - \psi(x_p, w)|}{|y_p - w|^{m+sp}} dw,\right.
\]
and, arguing as before, we obtain
\[
\max\{A; C\} \geq \max\{-B; -D; \Lambda_\infty(s)u(x_0, y_0)\}.
\]

\[\square\]

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