Partial Identification and Inference for
Conditional Distributions of Treatment Effects*

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Abstract

This paper considers identification and inference for the distribution of treatment
effects conditional on observable covariates. Since the conditional distribution of treat-
ment effects is not point identified without strong assumptions, we obtain bounds on
the conditional distribution of treatment effects by using the Makarov bounds. We also
consider the case where the treatment is endogenous and propose two stochastic dom-
inance assumptions to tighten the bounds. We develop a nonparametric framework to
estimate the bounds and establish the asymptotic theory that is uniformly valid over
the support of treatment effects. An empirical example illustrates the usefulness of the
methods.

Keywords: treatment effects, conditional distribution, heterogeneity, partial identifi-
cation, uniform inference.

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1 Introduction

This paper considers identification and estimation of the conditional distribution of treatment effects. While the unconditional distribution of treatment effects has been considered extensively in the literature in both theoretical and applied econometrics (e.g., Heckman et al. (1997); Fan and Park (2010, 2012); Fan et al. (2017); Firpo and Ridder (2019); Frandsen and Lefgren (2021)), we focus on the conditional distribution of treatment effects to take into account the heterogeneity caused by differences in the value of covariates. Such heterogeneous treatment effects, if they exist, may help policy makers develop more effective policies. We use the Makarov bounds, which depend on the marginal distributions of potential outcomes while the dependence structure between them is left unspecified, to partially identify the conditional distribution of treatment effects (e.g., Makarov (1982), Rüschedendorf (1982), and Williamson and Downs (1990)). We then provide estimation and inference methods for the bounds.

We start with the case where the assumption of the conditional independence of the treatment and potential outcomes given covariates, which is called an unconfoundedness condition, is satisfied. However, there are many situations where such a conditional independence assumption fails to hold. We present identification results of the distribution of treatment effects when the treatment is endogenous without assuming the existence of an instrumental variable. We show that one can still obtain Makarov-type bounds on the distribution of treatment effects even with an endogenous treatment. When a treatment is endogenous, the bounds on the distribution of the treatment effect may be too wide to be informative. Motivated by the previous studies in the literature (e.g., Manski (1997); Manski and Pepper (2000); Blundell et al. (2007)), we consider two kinds of stochastic dominance between potential outcomes in this paper to tighten the bounds. These stochastic dominance assumptions are useful in many empirical situations as they are consistent with

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1The distribution of treatment effects has received significant attention, and its importance becomes greater when the benefit from the treatment is non-transferrable (Heckman and Vytlacil (2007)).

2For example, the effect of a mother’s smoking behavior on her child’s birth weight might differ across the mother’s age (e.g., Abrevaya et al. (2015)).

3This is also true for the case where we impose the unconfoundedness assumption. We can expect that, since we are interested in the distribution of treatment effects for some subgroup, the bounds on the conditional distribution may be narrower than those on the unconditional distribution. This is another motivation for considering the conditional distributions of treatment effects in this paper.
many economic theories. The resulting bounds under the stochastic dominance assumptions are easy to compute.

We construct nonparametric kernel-type estimators of the bounds on the conditional distribution of treatment effects and establish asymptotic theory for them that is uniformly valid over the support of treatment effects for some fixed subgroup of the population. They are useful for comparing bounds between two subpopulations defined in terms of different values of observable characteristics.\(^4\) We adapt the results on uniform inference for value functions that were developed recently by Firpo et al. (2021) and provide a set of conditions under which the estimated bounds are consistent for the true bounds. We propose a bootstrap procedure to mimic the asymptotic distributions of the estimated bounds and show its validity. The bootstrap scheme in this paper is based on the novel bootstrap procedure for Hadamard directionally differentiable functionals that was developed by Fang and Santos (2019).

The asymptotic theory developed in this paper relaxes some undesirable assumptions imposed in some existing studies in the literature. Specifically, the bounds on the distribution of treatment effects are defined as some functionals of marginal distributions of potential outcomes. These functionals involve the infimum and supremum over the supports of potential outcomes. The uniqueness of the infimum and supremum needs to be imposed to establish a pointwise asymptotic theory for those bounds, as in Fan and Park (2010, 2012).\(^5\) However, the uniqueness assumption may not hold for some data generating process (DGP), and the pointwise asymptotic results may not be valid in such cases (cf. Milgrom and Segal (2002), Firpo et al. (2021)). On the other hand, the inference results developed in this paper do not require such uniqueness assumptions.

One can use the novel approach developed by Chernozhukov et al. (2013) that yields confidence bands uniformly valid over the joint support of an outcome variable and covariates.\(^6\) Their inference methods for bounds defined by either supremum or infimum of some

\(^4\)One can perform statistical testing for global hypotheses, such as equality of or stochastic dominance between two bounds (e.g., Barrett and Donald (2003); Lee et al. (2009); Seo (2018)), and this requires asymptotic theory uniformly valid over the support of treatment effects.

\(^5\)Fan and Park (2010) impose Assumptions 3 and 4 to guarantee the uniqueness of the maximizer and minimizer involved in the bounds on the distribution of treatment effects. Fan and Park (2012) impose Assumptions A3 and A4 for the same purpose.

\(^6\)Chernozhukov et al. (2013) do not explicitly provide an inference result for nonparametric conditional
function are based on the strong Gaussian approximation using couplings. A key ingredient of their approach is the construction of argsup and/or arginf sets, and it is allowed to avoid imposing the uniqueness assumption on those sets. Our approach differs from theirs in that we rely on weak convergence of nonparametric estimators of the bounds to some fixed Gaussian process and the results on Hadamard directionally differentiable functionals. In addition, while our approach also requires to estimate argsup and arginf sets for the Hadamard directional derivatives, the construction of these sets is computationally easy in comparison to that of Chernozhukov et al. (2013). Therefore, the inference theory in this paper can be considered complementing the existing inference methods for the Makarov bounds.

The Monte Carlo simulation results show that the inference methods in this paper perform well in finite samples. We also provide an empirical example to illustrate the practical relevance of the methods proposed in this paper. We revisit the empirical question of the effect of 401(k) plans on net asset accumulations investigated by multiple papers in the literature (e.g., Abadie (2003); Chernozhukov and Hansen (2004); Wüthrich (2019); Sant’Anna et al. (2022)). We confirm that the identifying power of the stochastic dominance assumptions is substantial from the empirical application. Furthermore, we find evidence on heterogeneity in the treatment effect that is consistent with the results in the literature.

This paper contributes to several strands of the literature on treatment effects. First, this paper mainly contributes to the literature on identification and estimation of the distribution of treatment effects by providing nonparametric estimation and inference methods for the conditional distribution of treatment effects. The literature is too vast to list all the papers relevant to this point. Identification of the unconditional distribution of treatment effects has been studied by, for example, Heckman et al. (1997), Fan and Park (2010, 2012), Fan et al. (2017), Vuong and Xu (2017), Kim (2018a), and Firpo and Ridder (2019).

This paper is closely related to Kim (2018b) and Frandsen and Lefgren (2021). Kim (2018b) provides identification results under several distributional restrictions, together with distribution estimators that is uniformly valid over the joint support of an outcome variable and covariates. However, one can easily adapt their results in the literature to establish the strong Gaussian approximation of some nonparametric estimator of a conditional distribution function. This allows one to perform inference uniformly over the joint support of the outcome variable of interest and conditioning variables (e.g., Chernozhukov et al. (2014)).
with an instrumental variable, that help tighten the bounds on the distribution of treatment effects, when the treatment is endogenous. The restrictions considered by Kim (2018b) are general and closely related to the stochastic dominance assumptions in this paper that were proposed independently of the results in Kim (2018b). This paper differs from Kim (2018b) in that this paper focuses on estimation and inference for the conditional distribution of treatment effects with a motivation for treatment effect heterogeneity, whereas Kim (2018b) mainly considers identification of the unconditional distribution of treatment effects under some restrictions on the model. Moreover, it is much easier to compute and estimate the bounds proposed in this paper than those in Kim (2018b). Therefore, the bounds in this paper have great applicability to empirical research.

In comparison with Frandsen and Lefgren (2021), we focus on estimation and inference for the bounds on the conditional distribution of treatment effects with a motivation for heterogeneity across subgroups. On the other hand, Frandsen and Lefgren (2021) are mainly concerned with identification of the unconditional distribution of treatment effects under a novel restriction that is called the “stochastic increasingness” assumption. Frandsen and Lefgren (2021) also discuss how to incorporate covariates and estimate the bounds on the unconditional distribution of treatment effects. However, they do not develop the asymptotic theory for the conditional distribution of treatment effects, especially when the bounds on the conditional distribution are the Makarov bounds.

This paper also contributes to the literature on heterogeneity in treatment effects across subpopulations by considering the conditional distribution of treatment effects (e.g., Donald et al. (2012); Abrevaya et al. (2015); Chang et al. (2015); Hsu (2017); Shen (2019)). Most existing studies focus on average/quantile treatment effects or the marginal distributions of potential outcomes. The results of this paper complement them by considering conditional distributions of treatment effects.

The rest of this paper is organized as follows. Section 2 presents the model and identification results. Section 3 provides nonparametric estimators of the Makarov bounds on the conditional distribution of treatment effects and the bootstrap procedure. Section 4 develops the asymptotic theory for the estimated bounds on the conditional distribution of treatment effects. Section 5 presents the Monte Carlo simulation study, and Section 6
provides the empirical example considering the effect of 401(k) plan on net financial asset accumulations. We then conclude with Section 7. All mathematical proofs, technical expressions, and additional results are presented in Appendix.

Before proceeding, we introduce some notation that will be used throughout this paper. Two random variables $A$ and $B$ that are independent of each other are denoted by $A \perp B$, and $\mathbb{E}[]$ is the expectation operator. For a matrix $A$, $A^t$ is the transpose of $A$. For a set $A$, $l^\infty(A)$ denotes the set of uniformly bounded functions on $A$. For a sequence of random variables $(Z_n)$ and a random variable $Z$, we denote the weak convergence of $Z_n$ to $Z$ by $Z_n \Rightarrow Z$.\footnote{A formal definition of weak convergence can be found in, for example, Kosorok (2008, pp.107-108).}

For a set $A$, $\text{int}(A)$ is the interior of $A$.\footnote{A formal definition of weak convergence can be found in, for example, Kosorok (2008, pp.107-108).}

2 Model and Identification

2.1 Identification Under the Unconfoundedness Condition

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space. Let $D$ be a binary variable that indicates whether a person gets the treatment or not, in other words, $D = 1$ if the person gets the treatment, and $D = 0$ if the person does not get the treatment. For each $d \in \{0, 1\}$, let $Y_d$ denote the potential outcome when $D = d$, and the observed outcome is defined as $Y \equiv DY_1 + (1 - D)Y_0$.

We assume that $Y_d$ is a continuous random variable for each $d \in \{0, 1\}$. Let $X \in \mathbb{R}^{d_x}$ be a set of covariates and denote its support by $\mathcal{X}$. We can only observe $(Y, D, X^t)$ from the data. We denote the conditional distribution of $Y_d$ on $X = x$ by $F_{d|X}(\cdot|x)$ for each $d \in \{0, 1\}$, and let $p_0(x) \equiv \Pr(D = 1|X = x)$ for a given $x \in \mathcal{X}$. $F_X(\cdot)$ denotes the distribution function of $X$.

We begin with the models under the conditional independence of the treatment variable and impose the following assumptions.

**Assumption 2.1.** $(Y_1, Y_0) \perp D|X$.

**Assumption 2.2.** There exist $\underline{p}, \bar{p} \in (0, 1)$ such that $p_0(x) \in [\underline{p}, \bar{p}]$ uniformly in $x \in \mathcal{X}$.

Assumption 2.1 is the unconfoundedness assumption, which means that the treatment $D$ is independent of the potential outcomes conditional on $X$. Assumption 2.2 is an overlap
condition that is commonly imposed in the relevant literature (e.g., Imbens (2004)). This assumption implies that we can observe individuals with \( Y = Y_1 \) and those with \( Y = Y_0 \) for any value of \( x \in \mathcal{X} \). Under these assumptions, we have the following identification result:

**Lemma 2.1.** Let \( x \in \mathcal{X} \) be given. Suppose that Assumptions 2.1 and 2.2 are satisfied. For any measurable function \( G \) such that \( \mathbb{E}|G(Y_1)|X = x < \infty \) and \( \mathbb{E}|G(Y_0)|X = x < \infty \), we have

\[
\begin{align*}
\mathbb{E}[G(Y_1)|X = x] &= \frac{\mathbb{E}[D \cdot G(Y)|X = x]}{\mathbb{E}[D|X = x]}, \\
\mathbb{E}[G(Y_0)|X = x] &= \frac{\mathbb{E}[(1-D) \cdot G(Y)|X = x]}{\mathbb{E}[(1-D)|X = x]}.
\end{align*}
\]

Lemma 2.1 implies that we can identify the conditional distributions of \( Y_1 \) and \( Y_0 \) on \( X = x \) when we choose \( G(Y) \equiv 1(Y \leq y) \) for a given \( y \in \mathbb{R} \). This result is also closely related to the identification of the unconditional distributions of \( Y_1 \) and \( Y_0 \) that is considered by Donald and Hsu (2014).

The treatment effect \( \Delta \) is defined as the difference between \( Y_1 \) and \( Y_0 \) (i.e., \( \Delta \equiv Y_1 - Y_0 \)). The parameter of interest is the conditional distribution of treatment effects given \( X = x \) for a given value \( x \in \mathcal{X} \), and this conditional distribution is denoted by \( F_{\Delta|X}(\cdot | x) \).

The (unconditional) distribution of treatment effects has received a considerable amount of attention from the literature (e.g., Heckman et al. (1997), Fan and Park (2010, 2012), Fan et al. (2017), Firpo and Ridder (2019)). The distribution function is useful in the context of program evaluation, for example, to determine the proportion of people who benefit from being treated, namely, \( \Pr(\Delta \geq 0) = 1 - F_\Delta(0) \). One can refer to Abbring and Heckman (2007) for more discussion on the topic. The conditional distributions take potential heterogeneity across subpopulations into account and thus can provide much richer information on treatment effects. Such heterogeneity, if it exists, may deliver different policy implications for different subpopulations.

It is worth noting that even in a randomized experiment where one can point identify the conditional distributions of \( Y_1 \) and \( Y_0 \) given \( X \), the distribution of treatment effects is not point identified without additional structures of the model.\(^8\) In particular, Fan and

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\(^8\)When the conditional distributions of \( Y_1 \) and \( Y_0 \) given \( X \) are point identified, a sufficient condition for point identification of the joint distribution of \( Y_1 \) and \( Y_0 \) is the (conditional) rank invariance; that is, \( F_{1|X}(Y_1|X) = F_{0|X}(Y_0|X) \) almost surely.
Park (2010) provide sharp bounds on the distribution of treatment effects in randomized experiments that are based on Makarov (1982), and Firpo and Ridder (2019) improve the bounds of Fan and Park (2010) in the uniform sense. We do not focus on the uniform sharpness of the bounds but on pointwise sharp bounds of the conditional distribution of treatment effects and nonparametric estimation of the bounds. We recall the bounds on the conditional distribution of treatment effects $F_{\Delta|X}(\cdot|x)$:

**Proposition 2.1.** (Lemma 2.1 in Fan and Park (2010)) Let $x \in X$ be fixed. Then, for a given $\delta \in \mathbb{R}$, define

$$F_{\Delta|X}^L(\delta|x) = \max \left( \sup_{y \in \mathbb{R}} \{ F_{1|X}(y|x) - F_{0|X}(y - \delta|x) \} , 0 \right), \quad (2)$$

$$F_{\Delta|X}^U(\delta|x) = \min \left( \inf_{y \in \mathbb{R}} \{ F_{1|X}(y|x) - F_{0|X}(y - \delta|x) \} , 0 \right) + 1. \quad (3)$$

If Assumptions 2.1 and 2.2 are satisfied, we then have

$$F_{\Delta|X}(\delta|x) \in \left[ F_{\Delta|X}^L(\delta|x), F_{\Delta|X}^U(\delta|x) \right]. \quad (4)$$

Note that this result is essentially identical to the identification result in Fan and Park (2010). Without additional structures on the model, these bounds are sharp in the pointwise sense for given $x \in X$. It is also worth noting that one can derive bounds on the unconditional distribution of treatment effects by averaging the bounds on the conditional distribution. Specifically, if the potential outcomes are correlated with $X$, then the bounds on the unconditional distribution obtained from the conditional distributions of the potential outcomes are tighter than the those obtained from the unconditional distributions of the potential outcomes (cf. Firpo and Ridder (2019)).

### 2.2 Identification with an Endogenous Treatment

In this section, we discuss identification of the distribution of treatment effects with an endogenous binary treatment. When the treatment is endogenously determined, the conditional distributions of $Y_1$ and $Y_0$ given $X$ are generally partially identified without additional assumptions. The following theorem shows that even if the conditional distributions of $Y_1$ and $Y_0$ are only partially identified, the bounds on the conditional distribution of treatment effects can still be obtained.
and \( Y_0 \) given \( X \) are partially identified, we can still bound \( F_{\Delta|X}(\delta) \).

**Theorem 2.1.** Let \( x \in \mathcal{X} \) be fixed. Suppose that, for all \( y \in \mathbb{R} \), the identified sets of \( F_{1|X}(y|x) \) and \( F_{0|X}(y|x) \) are given by \([F_{1L|X}(y|x), F_{1U|X}(y|x)]\), and \([F_{0L|X}(y|x), F_{0U|X}(y|x)]\), respectively, where, for a given \( \delta \in \text{Supp}(\Delta|X=x) \), define

\[
F_{\Delta|X}^{e,L}(\delta|x) = \max \left( \sup_y \left\{ F_{1L|X}(y|x) - F_{0U|X}(y-\delta|x) \right\}, 0 \right), \tag{5}
\]
\[
F_{\Delta|X}^{e,U}(\delta|x) = \min \left( \inf_y \left\{ F_{1U|X}(y|x) - F_{0L|X}(y-\delta|x) \right\}, 0 \right) + 1. \tag{6}
\]

Then,

\[
F_{\Delta|X}^{e,L}(\delta|x) \leq F_{\Delta|X}(\delta|x) \leq F_{\Delta|X}^{e,U}(\delta|x).
\]

One example of the identified sets of \( F_{1|X} \) and \( F_{0|X} \) is Manski’s bounds (Manski (1990, 1994)) defined as follows:

\[
F_{1L|X}(y|x) \equiv \Pr(Y \leq y|D = 1, X = x) \Pr(D = 1|X = x),
\]
\[
F_{1U|X}(y|x) \equiv \Pr(Y \leq y|D = 1, X = x) \Pr(D = 1|X = x) + \Pr(D = 0|X = x), \tag{7}
\]
\[
F_{0L|X}(y|x) \equiv \Pr(Y \leq y|D = 0, X = x) \Pr(D = 0|X = x),
\]
\[
F_{0U|X}(y|x) \equiv \Pr(Y \leq y|D = 0, X = x) \Pr(D = 0|X = x) + \Pr(D = 1|X = x).
\]

The bounds presented in Theorem 2.1 based on Manski’s bounds are often too wide to be informative. Some model restrictions have been imposed to tighten bounds on the parameter of interest in the literature (e.g., Manski (1997); Manski and Pepper (2000); Blundell et al. (2007)). Motivated by these studies, we propose two stochastic dominance assumptions that may help tighten the bounds and be consistent with many economic theories in the absence of instrumental variables. The bounds on the distribution of treatment effects in Theorem 2.1 are based on the bounds on the conditional distributions of the potential outcomes, and therefore, we can tighten the bounds on \( F_{\Delta|X} \) once we have tighter bounds on the conditional distributions of the potential outcomes than (7).

Let \( F_{d,d'}(y|x) \equiv \Pr \left( Y_d \leq y|D = d', X = x \right) \) for given \( d,d' \in \{0,1\} \). The following assumption states that the potential outcome conditional on \( D = 1 \) first-order stochastically
dominates the potential outcome conditional on $D = 0$.

**Assumption 2.3.** Let $x \in \mathcal{X}$ be given. For all $d \in \{0, 1\}$ and $y \in \mathbb{R}$, $F_{d|1,X}(y|x) \leq F_{d|0,X}(y|x)$.

Assumption 2.3 is identical to the stochastic dominance assumption of Blundell et al. (2007) that is used to tighten the bounds on the wage distribution of the whole population in the presence of sample selection. Okumura and Usui (2014) generalize this stochastic dominance assumption to the case where $D$ is not binary. Since Assumption 2.3 implies that $E[Y_d|D = 1, X = x] \geq E[Y_d|D = 0, X = x]$, this assumption is a sufficient condition for the monotone treatment selection (MTS) assumption in Manski and Pepper (2000).

Assumption 2.3 is consistent with some economic theories in empirical studies. As mentioned earlier, Blundell et al. (2007) utilize this assumption in conjunction with positive selection of labor force participation from the standard labor supply model that wage and probability of labor force participation are positively correlated. As another example, we can consider sorting models for return to education, where potential employees signal their ability to employers by using their education level. Considering return to college education (i.e., $Y$ is wage, and $D$ is college entrance), it is likely that the more capable people are, the more likely it is for them to complete a college education. As a result, we may anticipate that people with college degrees are more likely to have higher learning ability than those who did not complete a college education (e.g., Bedard (2001)). Many studies in the labor economics literature consider learning ability as an important factor affecting wage (Lang and Kropp (1986)), and thus, we can assume that people with college degrees have higher wages than those without degrees. As a consequence, the sorting hypothesis is consistent with Assumption 2.3.

The following theorem gives identification results for the conditional distributions of the potential outcomes under Assumption 2.3.

**Theorem 2.2.** Let $x \in \mathcal{X}$ be fixed. Suppose that Assumption 2.3 holds. For a given $y \in \mathbb{R}$,
Define

\[ F_{1|X}^{L,FSD}(y|x) \equiv \Pr(Y \leq y | D = 1, X = x), \]

\[ F_{1|X}^{U,FSD}(y|x) \equiv \Pr(Y \leq y | D = 1, X = x) \Pr(D = 1 | X = x) + \Pr(D = 0 | X = x), \]

\[ F_{0|X}^{L,FSD}(y|x) \equiv \Pr(Y \leq y | D = 0, X = x), \]

\[ F_{0|X}^{U,FSD}(y|x) \equiv \Pr(Y \leq y | D = 0, X = x). \]

Then,

\[
F_{1|X}(y|x) \in \left[ F_{1|X}^{L,FSD}(y|x), F_{1|X}^{U,FSD}(y|x) \right], \tag{8}
\]

\[
F_{0|X}(y|x) \in \left[ F_{0|X}^{L,FSD}(y|x), F_{0|X}^{U,FSD}(y|x) \right]. \tag{9}
\]

Comparing the bounds on the conditional distributions of the potential outcomes in Theorem 2.2 with those in equation (7), we can find that the upper bound on \( F_{1|X}(y|x) \) and lower bound on \( F_{0|X}(y|x) \) in Theorem 2.2 are identical to their counterparts in (7). Since Assumption 2.3 designates only one direction of the monotonicity of the distribution functions, it is impossible to improve the lower bound on \( F_{0|X}(y|x) \) and the upper bound on \( F_{1|X}(y|x) \). Nevertheless, the bounds on the conditional distributions of the potential outcomes provided in Theorem 2.2 are narrower than those in (7).

We now introduce another stochastic dominance assumption. The following stochastic dominance assumption may be regarded as an assumption corresponding to the monotone treatment response (MTR) assumption in Manski (1997).

**Assumption 2.4.** Let \( x \in \mathcal{X} \) be given. For all \( d \in \{0, 1\} \) and \( y \in \mathbb{R} \), \( F_{1|d,X}(y|x) \leq F_{0|d,X}(y|x) \).

Note that Assumption 2.4 implies that \( \mathbb{E}[Y_1 | X = x] \geq \mathbb{E}[Y_0 | X = x] \). To see this, recall that \( F_{1|1,X}(y|x) = F_{1|1,X}(y|x)p_0(x) + F_{1|0,X}(y|x)(1 - p_0(x)) \). Under Assumption 2.4, we have \( F_{1|1,X}(y|x) \leq F_{0|1,X}(y|x) \) and \( F_{1|0,X}(y|x) \leq F_{0|0,X}(y|x) \), and therefore, we have \( F_{1|X}(y|x) \leq F_{0|X}(y|x) \) and \( \mathbb{E}[Y_1 | X = x] \geq \mathbb{E}[Y_0 | X = x] \).

Assumption 2.4 can be regarded as a distributional generalization of, but is weaker
than, the MTR assumption of Manski (1997). This assumption is also applicable to many empirical studies as it is compatible with some economic theories. Recalling the example of return to college education, Assumption 2.4 may be consistent with the human capital theory in which education increases productivity through a human capital production function and return to education reflects the increased productivity (e.g., Mincer (1974)). For example, Assumption 2.4 can be translated into that people who have completed their college degrees are likely to be paid higher wages than when they have not, and this argument can be justified by the human capital theory in the labor economics.

It is worth noting that Assumption 2.4 is a special case of conditional negative quadrant dependence, which is a dependence concept considered by Lehmann (1966). Kim (2018b) proposes a concept of conditional quadrant dependence, as well as a negative stochastic dominance, in the presence instrumental variables. He provides tighter bounds on the distribution of treatment effects than bounds using (7) under these assumptions with assuming the existence of an instrumental variable. A difference between Kim (2018b) and this paper is that this paper does not impose additional assumptions on the data generating process, such as monotonicity in structural functions for $Y_1$ and/or $Y_0$ and the availability of instrumental variables. Furthermore, this paper focuses on estimation of the bounds on conditional distribution of treatment effects, whereas Kim (2018b) focuses on identification of the unconditional distribution of treatment effects.

The following theorem shows that we can tighten the bounds on the conditional distributions of $Y_1$ and $Y_0$ given $X$ under Assumption 2.4:

**Theorem 2.3.** Let $x \in X$ be fixed. Suppose that Assumption 2.4 holds. For a given $y \in \mathbb{R}$, define

\[
F^{L,FSD2}_{1|X}(y|x) = \Pr(Y \leq y | D = 1, X = x) \Pr(D = 1 | X = x),
\]

\[
F^{U,FSD2}_{1|X}(y|x) = \Pr(Y \leq y | X = x),
\]

\[
F^{L,FSD2}_{0|X}(y|x) = F^{U,FSD2}_{1|X}(y|x),
\]

\[
F^{U,FSD2}_{0|X}(y|x) = \Pr(Y \leq y | D = 0, X = x) \Pr(D = 0 | X = x) + \Pr(D = 1 | X = x).
\]
Then,

\[
\begin{align*}
F_{1|X}(y|x) & \in \left[ F_{1|X}^{L,FSD}(y|x), F_{1|X}^{U,FSD}(y|x) \right], \\
F_{0|X}(y|x) & \in \left[ F_{0|X}^{L,FSD}(y|x), F_{0|X}^{U,FSD}(y|x) \right].
\end{align*}
\]

(10)

As Assumption 2.3 is not enough to improve \( F_{1|X}(y|x) \) and \( F_{0|X}(y|x) \) in (7), we can see that the lower bound on \( F_{1|X}(y|x) \) and the upper bound on \( F_{0|X}(y|x) \) in Theorem 2.3 remain the same as those in (7).

If both Assumptions 2.3 and 2.4 hold, it can be shown that one can further improve the bounds on the conditional distributions of the potential outcomes. This result is summarized in the following corollary:

**Corollary 2.4.** Let \( x \in \mathcal{X} \) be fixed. Suppose that Assumptions 2.3 and 2.4 hold. For given \( y \in \mathbb{R} \),

\[
\begin{align*}
F_{1|X}(y|x) & \in \left[ F_{1|X}^{L,FSD1}(y|x), F_{1|X}^{U,FSD2}(y|x) \right], \\
F_{0|X}(y|x) & \in \left[ F_{0|X}^{L,FSD2}(y|x), F_{0|X}^{U,FSD1}(y|x) \right].
\end{align*}
\]

(11)

It is straightforward to see that the identified sets for conditional distribution functions of \( Y_1 \) and \( Y_0 \) on \( X = x \) presented in Corollary 2.4 are connected and that the intersection of these sets is the boundary of each set, that is \( F_{1|X}^{U,FSD2}(y|x) = F_{0|X}^{L,FSD2}(y|x) \) for all \( y \in \mathbb{R} \).

It is worth mentioning that the stochastic dominance relationships in Assumptions 2.3 and 2.4 can be reversed, depending on the empirical context. For example, one can impose a condition that for all \( d \in \{0, 1\} \) and \( y \in \mathbb{R} \), \( F_{d|1,X}(y|x) \geq F_{d|0,X}(y|x) \) instead of Assumption 2.3. Similarly, one can consider a stochastic dominance relationship that for all \( d \in \{0, 1\} \) and \( y \in \mathbb{R} \), \( F_{1|d,X}(y|x) \geq F_{0|d,X}(y|x) \) instead of Assumption 2.4. These stochastic dominance relationships can be motivated by, for example, the empirical study of Angrist (1990), who considers the long-term effect of veteran status on earnings. As pointed out by Angrist (1990), it would be likely that men with fewer civilian opportunities served in the armed force, and this may yield a negative selection in the sense that the potential earnings of veterans are likely to be smaller than those of non-veterans (i.e., ...
In addition, since military service may hinder human capital accumulation, the potential earnings they would have received when they served in the armed force are likely to be smaller than the potential earnings they would have received when they did not (i.e., \( F_{1|d,X}(y|x) \geq F_{0|d,X}(y|x) \)).

There are other ways to tighten the bounds on the distribution of treatment effects. For example, one can utilize support restrictions (e.g., Kim (2018a)), consider a weaker condition than the conditional independence, such as \( c\)-independence proposed by Masten and Poirier (2018), or impose some dependence structure as in Frandsen and Lefgren (2021). In addition, although we do not assume the availability of (monotone) instrumental variables in this paper, one can consider the monotone instrumental variable (MIV) assumption when there exists such a variable, as in Manski and Pepper (2000) and Kim (2018b). A related but stronger assumption on instrumental variables is the stochastically MIV (SMIV) assumption considered by Mourifie et al. (2020). The SMIV assumption imposes a stochastic dominance ordering on the joint distribution of \( Y_0 \) and \( Y_1 \) across values of an instrumental variable, and the conditional joint distribution of the potential outcomes given a value of the instrumental variable, say \( z \), (first-order) stochastically dominates the conditional joint distribution given a value of the instrumental variable that is smaller than \( z \). These restrictions help tighten the bounds on the conditional distributions of the potential outcomes and/or the bounds on \( F_{\Delta|X} \), but estimation and (uniform) inference for such bounds is not trivial. Therefore, we leave it for future study.

### 3 Estimation of Bounds and Confidence Bands

#### 3.1 Estimation and Bootstrap Procedure

We consider estimation of the bounds and construction of confidence bands for them in this section. To this end, we assume that the conditional distributions of the potential outcomes are identified as follows: for all \( y \in \mathbb{R} \) and \( x \in \mathcal{X} \),

\[
F_{1|X}(y|x) \in \left[ LB_{1|X}(y|x), UB_{1|X}(y|x) \right], \\
F_{0|X}(y|x) \in \left[ LB_{0|X}(y|x), UB_{0|X}(y|x) \right].
\]

(12)
We denote nonparametric estimators of $LB_{1|X}$, $UB_{1|X}$, $LB_{0|X}$, and $UB_{0|X}$ by $\hat{LB}_{1|X,n}$, $\hat{UB}_{1|X,n}$, $\hat{LB}_{0|X,n}$, and $\hat{UB}_{0|X,n}$, respectively. In this paper, we consider kernel-type estimators of the bounds. Let

\[
\begin{align*}
F_{Y|X}((y_{11}, y_{1u}, y_{0l}, y_{0u})|x) &\equiv (LB_{1|X}(y_{11}|x), UB_{1|X}(y_{1u}|x), LB_{0|X}(y_{0l}|x), UB_{0|X}(y_{0u}|x))^t, \\
\hat{F}_{Y|X,n}((y_{11}, y_{1u}, y_{0l}, y_{0u})|x) &\equiv (\hat{LB}_{1|X,n}(y_{11}|x), \hat{UB}_{1|X,n}(y_{1u}|x), \hat{LB}_{0|X,n}(y_{0l}|x), \hat{UB}_{0|X,n}(y_{0u}|x))^t.
\end{align*}
\]

Suppose that there exists a positive real sequence $(r_n)_n$ such that $r_n \to \infty$ and that

\[
r_n \left( \hat{F}_{Y|X,n}(\cdot|x) - F_{Y|X}(\cdot|x) \right) \Rightarrow \mathcal{G}(\cdot) \text{ in } (l^\infty(\mathcal{Y}))^4,
\]

where $\mathcal{G}(\cdot)$ is a four-dimensional centered Gaussian process. This weak convergence result can be derived under a set of mild regularity conditions when using standard nonparametric estimators. When we use kernel-type estimators, we have $r_n = \sqrt{nh_n^2}$, where $h_n$ is a bandwidth.

Define

\[
\Pi_L (F_{Y|X})(y, \delta|x) \equiv LB_{1|X}(y|x) - UB_{0|X}(y - \delta|x),
\]

\[
\Pi_U (F_{Y|X})(y, \delta|x) \equiv UB_{1|X}(y|x) - LB_{0|X}(y - \delta|x),
\]

and

\[
\phi_L (F_{Y|X}) \equiv \sup_{\delta \in \mathcal{D}} \left| \sup_{y \in \mathcal{Y}} \Pi_L (F_{Y|X})(y, \delta|x) - F^{L,0}_{\Delta|X}(\delta|x) \right| \equiv \sup_{\delta} \left| F^{L}_{\Delta|X}(\delta|x) - F^{L,0}_{\Delta|X}(\delta|x) \right|,
\]

\[
\phi_U (F_{Y|X}) \equiv \sup_{\delta \in \mathcal{D}} \left| \inf_{y \in \mathcal{Y}} \left\{ \Pi_U (F_{Y|X})(y, \delta|x) + 1 \right\} - F^{U,0}_{\Delta|X}(\delta|x) \right| \equiv \sup_{\delta} \left| F^{U}_{\Delta|X}(\delta|x) - F^{U,0}_{\Delta|X}(\delta|x) \right|,
\]

where $F^{L,0}_{\Delta|X}$ and $F^{U,0}_{\Delta|X}$ are some fixed bounds that are true under the hypotheses $F^{L}_{\Delta|X}(\cdot|x) = F^{L,0}_{\Delta|X}(\cdot|x)$ and $F^{U}_{\Delta|X}(\cdot|x) = F^{U,0}_{\Delta|X}(\cdot|x)$. The functionals in (13) and (14) can be considered as the Kolmogorov-Smirnov (KS) type statistics, and they are useful to construct two-sided

15
uniform confidence bands for the lower and upper bounds on the conditional distribution of treatment effects.\footnote{One can consider one-sided test statistics to construct one-sided uniform confidence bands. However, we do not examine such test statistics in this paper as the main purpose is to obtain uniform confidence sets for the identified region and the two-sided uniform confidence bands are enough to provide confidence sets for the identified region.}

Let \( \mathbb{H} \equiv (h_1, h_2, h_3, h_4)^t \) be a four-dimensional vector-valued function and \( f \) be a real-valued function. We denote the Hadamard directional derivatives of \( \phi_L \) and \( \phi_U \) at \( f \) in direction \( \mathbb{H} \) by \( \phi_L'(f; \mathbb{H}) \) and \( \phi_U'(f; \mathbb{H}) \), respectively. We also let \( \hat{\phi}_L(f; \mathbb{H}, a_n) \) and \( \hat{\phi}_U(f; \mathbb{H}, a_n) \) denote estimators of \( \phi_L'(f; \mathbb{H}) \) and \( \phi_U'(f; \mathbb{H}) \), respectively, where \( (a_n) \) is a positive real sequence decreasing to zero. We provide the forms of the Hadamard directional derivatives and their estimators in Appendix A. The main goal is to establish the limiting distributions of \( r_n \left( \phi_L \left( \hat{F}_{Y|X,n} \right) - \phi_L \left( F_{Y|X} \right) \right) \) and \( r_n \left( \phi_U \left( \hat{F}_{Y|X,n} \right) - \phi_U \left( F_{Y|X} \right) \right) \). The limiting distributions allow one to construct confidence bands for the bounds on the conditional distribution of treatment effects.

As will be shown below, the limiting distributions are nonstandard. We propose to use a multiplier bootstrap to mimic the asymptotic distributions of \( r_n \left( \phi_L \left( \hat{F}_{Y|X,n} \right) - \phi_L \left( F_{Y|X} \right) \right) \) and \( r_n \left( \phi_U \left( \hat{F}_{Y|X,n} \right) - \phi_U \left( F_{Y|X} \right) \right) \). Specifically, let

\[
\hat{\psi}_i((y_{11}, y_{1u}, y_{01}, y_{0u})|x) \equiv \left( \hat{\psi}_{1, i|X,n}(y_{11}|x), \hat{\psi}_{1u, i|X,n}(y_{1u}|x), \hat{\psi}_{0l, i|X,n}(y_{0l}|x), \hat{\psi}_{0u, i|X,n}(y_{0u}|x) \right)^t
\]

be an estimated influence function for the \( i \)-th observation of \( \hat{F}_{Y|X,n}((y_{11}, y_{1u}, y_{0l}, y_{0u})|x) \) and \( \{B_i : i = 1, 2, \ldots, n\} \) be \( n \) random draws from \( B \) that is independent of the data and satisfies some moment conditions (cf. Assumption 4.6). Define

\[
\hat{F}_{Y|X,n}((y_{11}, y_{1u}, y_{0l}, y_{0u})|x) \\
\equiv \left( \sum_i B_i \hat{\psi}_{1, i|X,n}(y_{11}|x), \sum_i B_i \hat{\psi}_{1u, i|X,n}(y_{1u}|x), \sum_i B_i \hat{\psi}_{0l, i|X,n}(y_{0l}|x), \sum_i B_i \hat{\psi}_{0u, i|X,n}(y_{0u}|x) \right)^t.
\]

Then, one can implement the following bootstrap procedure to obtain confidence bands for \( F_{\Delta|X}^L(\cdot|x) \) and \( F_{\Delta|X}^U(\cdot|x) \).

**Bootstrap procedure**
1. Estimate the conditional distribution functions of the potential outcomes or bounds on them, and construct

\[
\Pi_L(\hat{F}_Y|X,n)(y, \delta|x) = LB_1|X,n(y|x) - UB_0|X,n(y - \delta|x),
\]

\[
\Pi_U(\hat{F}_Y|X,n)(y, \delta|x) = UB_1|X,n(y|x) - LB_0|X,n(y - \delta|x).
\]

2. Repeat the following procedure for \(m\) times, where \(m\) denotes the number of bootstrap iterations.

(a) Generate the bootstrap weights \(\{B_i : i = 1, 2, \ldots n\}\) from the random variable \(B\) in Assumption 4.6.

(b) Using the estimated conditional distribution functions of the potential outcomes or bounds on them, compute \(r_n\hat{F}_Y^*(\cdot|x)\).

(c) For each iteration \(b = 1, 2, \ldots, m\), compute

\[
\tilde{\phi}_L^{(b)}\left(\Pi_L(\hat{F}_Y|X,n)(y, \delta|x); r_n\hat{F}_Y^*|X,n(\cdot|x), a_n\right),
\]

\[
\tilde{\phi}_U^{(b)}\left(\Pi_U(\hat{F}_Y|X,n)(y, \delta|x) + 1; r_n\hat{F}_Y^*|X,n(\cdot|x), a_n\right).
\]

3. For a given significance level \(\alpha \in (0, 1)\), find the \((1 - \frac{\alpha}{2})\)-th quantiles of

\[
\left\{\tilde{\phi}_L^{(b)}\left(\Pi_L(\hat{F}_Y|X,n)(y, \delta|x); r_n\hat{F}_Y^*|X,n(\cdot|x), a_n\right) : b = 1, 2, \ldots, m\right\},
\]

\[
\left\{\tilde{\phi}_U^{(b)}\left(\Pi_U(\hat{F}_Y|X,n)(y, \delta|x) + 1; r_n\hat{F}_Y^*|X,n(\cdot|x), a_n\right) : b = 1, 2, \ldots, m\right\}.
\]

We denote the quantiles by \(c_{L,\frac{\alpha}{2}}^1\) and \(c_{U,\frac{\alpha}{2}}^1\), respectively.

4. Let

\[
\tilde{CI}\left(F_{\Delta|X}(\delta; x), 1 - \alpha\right) = \left[\hat{F}_{\Delta|X,n}(\delta|x) - c_{L,\frac{\alpha}{2}}^1, \hat{F}_{\Delta|X,n}(\delta|x) + c_{L,\frac{\alpha}{2}}^1\right],
\]

\[
\tilde{CI}\left(F_{\Delta|X}(\delta; x), 1 - \alpha\right) = \left[\hat{F}_{\Delta|X,n}(\delta|x) - c_{U,\frac{\alpha}{2}}^1, \hat{F}_{\Delta|X,n}(\delta|x) + c_{U,\frac{\alpha}{2}}^1\right].
\]

The \((1 - \alpha) \times 100\%\) confidence bands of \(F_{\Delta|X}^L(\cdot|x)\) and \(F_{\Delta|X}^U(\cdot|x)\) can be constructed.
The difference between $\bar{CI}$ and $CI$ in the above bootstrap procedure is that $CI$’s are confidence bands obtained to impose the logical bounds when necessary. Chen et al. (2021) show that one advantage of applying such operators to the original confidence bands is that the resulting confidence bands have greater coverage than the original confidence bands. In addition, this procedure is very simple and easy to implement in practice.

The above bootstrap procedure allows us to construct the pointwise confidence set of the identified set, where the identified set is $[F_{\Delta|X}(\delta; x), F_{\Delta|X}(\delta; x)]$ for given $\delta \in \text{Supp}(\Delta|X = x)$. Specifically, we can show that

$$
\lim \inf_{n \to \infty} \Pr \left( F_{\Delta|X}(\delta|x) \in \left[ \hat{F}_{\Delta|X,n}^L(\delta|x) - c_{1-\frac{\alpha}{2}}, \hat{F}_{\Delta|X,n}^U(\delta|x) + c_{1-\frac{\alpha}{2}} \right] \right) \geq 1 - \alpha,
$$

and this can be used as a $(1 - \alpha) \times 100\%$ confidence set for the identified set. This confidence set does not require the uniqueness of the infimum and supremum in the Makarov bounds. As pointed out by Firpo et al. (2021), however, this confidence set is likely to be conservative.

### 3.2 Nonparametric Estimators of the Bounds on Conditional Distributions of Potential Outcomes

We propose to use kernel-type estimators of bounds on the conditional distributions of the potential outcomes in (12). Let $K(\cdot) : \mathbb{R}^{d_x} \to \mathbb{R}$ be a $d_x$-dimensional kernel function and $h_n$ be a bandwidth such that $h_n \to 0$, $nh_n^{d_x} \to \infty$, and $nh_n^{d_x+4} \to 0$ as $n \to \infty$. We start with the case where Assumptions 2.1 and 2.2 hold. In this case, the conditional distributions of the potential outcomes are point identified (i.e., $LB_{j|X}(y|x) = UB_{j|X}(y|x) = F_{j|X}(y|x)$ for
those used to estimate $F$. Note that the kernel function and bandwidth used to estimate $F$ and define $F_1(y) = F_0(y) + \frac{1}{2} \left( F_1(y) - F_0(y) \right)$ as estimators of $F_1$ and $F_0$, respectively. We then define

$$\hat{F}_Y(Y; x) \equiv \left( \hat{F}_1(Y; x), \hat{F}_0(Y; x) \right)^t.$$

(15)

Note that the kernel function and bandwidth used to estimate $F_1$ can be different from those used to estimate $F_0$. In addition, we have $\Pi_L(F_1(Y; x))(y, \delta|x) = \Pi_U(F_1(Y; x))(y, \delta|x) = F_1(Y; x) - F_0(Y; x - \delta|x)$ and can use the bootstrap procedure with $r_n = \sqrt{n h_n^d}$ under a set of regularity conditions.

We now consider the case where the treatment is endogenous and that Assumptions 2.3 and 2.4 hold. Recall that $F_{d|d', x}(y|x) = \Pr(Y_d \leq y|D = d', X = x)$ for given $d, d' \in \{0, 1\}$, and the conditional distribution of $Y$ given $X = x$ is denoted by $F_{Y|x}(|x)$. By corollary 2.4, we can construct kernel estimators of these objects as

$$\hat{F}_{1|X,n}(y_1|x) \equiv \frac{\sum_n 1(Y_i \leq y_1) D_i K_1(X_i - x)}{\sum_n D_i K_1(X_i - x)},$$

$$\hat{F}_{0|X,n}(y_0|x) \equiv \frac{\sum_n 1(Y_i \leq y_0)(1 - D_i) K_1(X_i - x)}{\sum_n (1 - D_i) K_1(X_i - x)},$$

$$\hat{F}_{Y|X,n}(y|x) \equiv \frac{\sum_n 1(Y_i \leq y) K_1(X_i - x)}{\sum_n K_1(X_i - x)},$$

and define

$$\hat{F}_{Y|X,n}((y_1, y_1, y_0, y_0)|x) \equiv \left( \hat{F}_{1|X,n}(y_1|x), \hat{F}_{Y|X,n}(y_1|x), \hat{F}_{Y|X,n}(y_0|x), \hat{F}_{0|X,n}(y_0|x) \right)^t.$$

(16)
Then, we have

\[ \Pi_L(\hat{F}_Y|X,n)(y,\delta|x) = \hat{F}_{1|1X,n}(y|x) - \hat{F}_{0|0X,n}(y-\delta|x), \]

\[ \Pi_U(\hat{F}_Y|X,n)(y,\delta|x) = \hat{F}_{Y|X,n}(y|x) - \hat{F}_{Y|X,n}(y-\delta|x). \]

We provide a description on how to estimate the influence functions in Appendix B.

4 Asymptotic Theory

4.1 Inference under the Unconfoundedness Assumption

We first develop the asymptotic theory for the kernel estimators in (15).

Assumption 4.1. \( \{W_i \equiv (Y_i, D_i, X_i^i) : i = 1, 2, ..., n \} \) is a random sample.

Assumption 4.2. (i) The support of \( X \), \( \mathcal{X} \), is a compact subset of \( \mathbb{R}^{d_x} \); (ii) the distribution of \( X \) admits its density \( f_X(\cdot) \) on \( \mathcal{X} \) such that \( 0 < \inf_{x \in \mathcal{X}} f_X(x) < \sup_{x \in \mathcal{X}} f_X(x) < \infty \). The density function \( f_X(\cdot) \) is twice continuously differentiable and \( \sup_{x \in \mathcal{X}} |f_X^{(1)}(x)| \) and \( \sup_{x \in \mathcal{X}} |f_X^{(2)}(x)| \) are bounded.

Assumption 4.3. (i) The propensity score function \( p_0(x) \) is twice continuously differentiable and all derivatives are uniformly bounded over \( \mathcal{X} \); (ii) For each \( j \in \{0, 1\} \), \( F_j|X(\cdot|x) \) is twice continuously differentiable with respect to \( x \) and all derivatives are uniformly bounded.

Assumption 4.4. The kernel function \( K(\cdot) \) is a product of a univariate bounded kernel function \( k(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+ \) such that \( \int k(u)du = 1 \), \( \int uk(u)du = 0 \), and \( \int u^2k(u)du \equiv k_2 < \infty \). The support of the univariate kernel function is compact.

Assumption 4.5. The bandwidth \( h_n \) satisfies the following conditions: (i) \( h_n \rightarrow 0 \); (ii) \( nh_n^{d_x} \rightarrow \infty \); (iii) \( nh_n^{d_x+4} \rightarrow 0 \) as \( n \rightarrow \infty \).

Assumption 4.1 is an i.i.d assumption on the sample. This can be relaxed at a cost of more complicated proofs of the theoretical results. Assumption 4.2 imposes some degree of smoothness on the distribution of \( X \). This assumption is standard in the literature on kernel estimation. Assumption 4.3 requires that the conditional distribution functions of
$Y_1$ and $Y_0$ and the propensity score functions be smooth enough. This condition, together with Assumptions 4.4 and 4.5, allows to eliminate the bias of estimators of conditional distribution functions of $Y_1$ and $Y_0$. Assumption 4.4 is also standard in the literature, and there are several kernel functions that satisfy this assumption (e.g., Epanechnikov, biweight, triweight kernels). Assumption 4.5 restricts the rate of bandwidth. The last condition $nh_n^{d_4+4} \to 0$ is required to eliminate the asymptotic bias of the kernel estimators. Note that when the dimension of $X$ is large, one can use a higher-order kernel to handle the bias term of kernel estimators.

We first establish the weak convergence of the kernel estimators under the unconfoundedness assumption. For given $x \in \mathcal{X}$, define $F_{Y|x}(\cdot|x) \equiv (F_{1|x}(\cdot|x), F_{0|x}(\cdot|x), F_{0|x}(\cdot|x))^t$ and $\hat{F}_{Y|x,n}(\cdot|x) \equiv (\hat{F}_{1|x,n}(\cdot|x), \hat{F}_{0|x,n}(\cdot|x), \hat{F}_{0|x,n}(\cdot|x))^t$.

**Theorem 4.1.** Suppose that Assumptions 2.1 and 2.2 hold. Let $x \in \text{int}(\mathcal{X})$ be given. If Assumptions 4.1–4.5 hold, then,

$$\sqrt{nh_n^d} \left( \hat{F}_{Y|x,n}(\cdot|x) - F_{Y|x}(\cdot|x) \right) \Rightarrow \mathcal{G}_x(\cdot) \equiv (\mathcal{G}_{1,x}(\cdot), \mathcal{G}_{1,x}(\cdot), \mathcal{G}_{0,x}(\cdot), \mathcal{G}_{0,x}(\cdot))^t \text{ in } (l^\infty(Y))^t,$$

where $\mathcal{G}_{1,x}(\cdot)$ and $\mathcal{G}_{0,x}(\cdot)$ are mean zero Gaussian processes with covariance kernels $H_{1,x}(s,t)$ and $H_{0,x}(s,t)$, respectively, whose the forms are given in Appendix E, and $\mathcal{G}_x((y_{11}, y_{1u}, y_{01}, y_{0u})) = (\mathcal{G}_{1,x}(y_{11}), \mathcal{G}_{1,x}(y_{1u}), \mathcal{G}_{0,x}(y_{01}), \mathcal{G}_{0,x}(y_{0u}))^t$.

Theorem 4.1 is useful not only for deriving the asymptotic distribution of the estimated bounds on $F_{\Delta|x}(\cdot|x)$, but also for conducting uniform inference for some policy-relevant parameter (e.g., quantile treatment effects).

We now establish the asymptotic theory for confidence bands for the identified set of $F_{\Delta|x}(\cdot|x)$. To this end, we consider the following two hypotheses: $F_{\Delta|x}^L(\delta|x) = F_{\Delta|x}^{L_0}(\delta|x)$ and $F_{\Delta|x}^U(\delta|x) = F_{\Delta|x}^{U_0}(\delta|x)$, where $F_{\Delta|x}^{L_0}(\delta|x)$ and $F_{\Delta|x}^{U_0}(\delta|x)$ are some fixed lower and upper bounds on the conditional distribution of treatment effects, respectively.

The main challenge to constructing uniform confidence bands, however, is that the mappings $\phi_L$ and $\phi_U$ are not Hadamard differentiable. To overcome this difficulty, we use Theorem 3.2 in Firpo et al. (2021) that shows that these functionals are Hadamard
directionally differentiable. Based on this result, we employ the inference method of Fang and Santos (2019) to establish the asymptotic theory.

Recall that $\Pi_L(F_Y|X)(y,\delta|x) = F_{1|X}(y|x) - F_{0|X}(y-\delta|x)$. The following theorem establishes the limiting theory.

**Theorem 4.2.** Suppose that Assumptions 2.1 and 2.2 hold. Let $x \in \text{int}(\mathcal{X})$ be given. If Assumptions 4.1–4.5 hold, then,

$$
\sqrt{n h_n} \left( \phi_L \left( \hat{F}_{Y|X,n}(\cdot|x) \right) - \phi_L \left( F_{Y|X}(\cdot|x) \right) \right) \Rightarrow \phi'_L \left( \Pi_L(F_Y|X)(y,\delta|x); G_x(\cdot) \right) \text{ in } l^\infty (Y^4),
$$

$$
\sqrt{n h_n} \left( \phi_U \left( \hat{F}_{Y|X,n}(\cdot|x) \right) - \phi_U \left( F_{Y|X}(\cdot|x) \right) \right) \Rightarrow \phi'_U \left( \Pi_U(F_Y|X)(y,\delta|x) + 1; G_x(\cdot) \right) \text{ in } l^\infty (Y^4),
$$

where and $G_x(\cdot)$ is the Gaussian process defined in Theorem 4.1. The forms of $\phi'_L$ and $\phi'_U$ are provided in Appendix A.

It is worth pointing out that the development of weak convergence of the estimated bounds does not rely on some pointwise asymptotic theory when establishing the asymptotic theory for the estimated Makarov bounds. Instead, we first derive the weak convergence of the standard kernel estimators for a fixed conditioning value and consider the double-supremum as an operator to establish the weak convergence of the estimated bounds, as in Firpo et al. (2021). In doing so, we can avoid imposing the uniqueness of $\text{argsup}$ and $\text{arginf}$ that is needed for pointwise inference.

Theorem 4.2 is a direct consequence of Theorem 2.1 of Fang and Santos (2019). The limiting distribution presented in Theorem 4.2 is non-standard, and therefore we need to rely on some resampling method to mimic the limiting distribution and conduct inference. Since the functionals $\phi_L$ and $\phi_U$ are not Hadamard differentiable but only Hadamard directionally differentiable, the standard bootstrap fails (cf. Theorem 3.1 of Fang and Santos (2019)). To resolve this problem, we utilize the result of bootstrap validity that was proposed by Firpo et al. (2021). The next assumption imposes conditions on the bootstrap weight $B$:

**Assumption 4.6.** Let $B$ be a random variable that is independent of the data $\mathcal{W}$ such that $E[B] = 0$, $\text{Var}(B) = 1$, and $\int_0^\infty \sqrt{\Pr(|B| > x)} dx < \infty$.

The last condition in Assumption 4.6 is satisfied if $E \left[ |B|^{2+\epsilon} \right] < \infty$ for some $\epsilon > 0$. One
can use a standard normal random variable as the bootstrap weight $B$.

We employ the approach of Fang and Santos (2019) to approximate the limiting distribution presented in Theorem 4.2, which was also considered by Firpo et al. (2021). Note that it is required to consistently estimate the Hadamard directional derivatives in (A.1) and that the Hadamard directional derivatives are defined in terms of the limit operator. To this end, we consider a tuning sequence $(a_n)$ that satisfies the conditions in the following assumption:

**Assumption 4.7.** Let $(a_n)$ be a sequence of positive real numbers such that $a_n \downarrow 0$ and $a_n \sqrt{nh_{dx}} \to \infty$.

Since the limiting distributions in Theorem 4.2 are nonstandard, we use a bootstrap to mimic the limiting distributions. The next theorem demonstrates that one can use the bootstrap procedure in Section 3 to approximate the asymptotic distributions of $\hat{\phi}'_L \left( \Pi_L(\mathbb{F}_{Y|X})(y, \delta|x); \mathbb{G}_x(\cdot) \right)$ and $\hat{\phi}'_U \left( \Pi_U(\mathbb{F}_{Y|X})(y, \delta|x) + 1; \mathbb{G}_x(\cdot) \right)$. Let $\hat{\mathbb{F}}_{Y|X,n}(\cdot|x)$ denote the vector of simulated stochastic processes using estimated influence functions in (B.1) in Appendix B.

**Theorem 4.3.** Suppose that Assumptions 2.1 and 2.2 hold. Let $x \in \text{int}(\mathcal{X})$ be given and Assumptions 4.1–4.5, 4.6, and 4.7 hold. Then, we have

\[
\hat{\phi}'_L \left( \Pi_L(\hat{\mathbb{F}}_{Y|X,n})(y, \delta|x); \sqrt{nh_{dx}} \hat{\mathbb{F}}_{Y|X,n}(\cdot|x), a_n \right) \Rightarrow \phi'_L \left( \Pi_L(\mathbb{F}_{Y|X})(y, \delta|x); \mathbb{G}_x(\cdot) \right),
\]

\[
\hat{\phi}'_U \left( \Pi_U(\hat{\mathbb{F}}_{Y|X,n})(y, \delta|x) + 1; \sqrt{nh_{dx}} \hat{\mathbb{F}}_{Y|X,n}(\cdot|x), a_n \right) \Rightarrow \phi'_U \left( \Pi_U(\mathbb{F}_{Y|X})(y, \delta|x) + 1; \mathbb{G}_x(\cdot) \right),
\]

in $l^\infty(\mathcal{Y}^4)$, conditional on data. The forms of $\hat{\phi}'_L$ and $\hat{\phi}'_U$ are provided in Appendix A.

### 4.2 Inference with an Endogenous Treatment

We now develop the asymptotic theory when the treatment is endogenous. Our asymptotic theory for an endogenous treatment focuses on the situation where Assumptions 2.3 and 2.4 hold, and thus, one can use the kernel estimators in Section 3. To be concrete, define, for given $x \in \mathcal{X}$, $\mathbb{F}_{Y|X}(\cdot|x) \equiv (F_{1|1X}(\cdot|x), F_Y|X(\cdot|x), F_{Y|X}(\cdot|x), F_{0|0X}(\cdot|x))^t$ and $\hat{\mathbb{F}}_{Y|X,n}(\cdot|x) \equiv \left( \hat{F}_{1|1X,n}(\cdot|x), \hat{F}_{Y|X,n}(\cdot|x), \hat{F}_{Y|X,n}(\cdot|x), \hat{F}_{0|0X,n}(\cdot|x) \right)^t$, where each component
of \( \hat{F}_{Y|X,n}(\cdot|x) \) is given in (16). Let \( \Pi_L(Y_{X,n})(y,\delta|x) \equiv F_{1|X}(y|x) - F_{0|0,X}(y-\delta|x) \) and \( \Pi_U(Y_{X,n})(y,\delta|x) \equiv F_{Y|X}(y|x) - F_{Y|X}(y-\delta|x) \).

**Assumption 4.8.** \( F_{1|X}(y|x) , F_{0|0,X}(y|x) , \) and \( F_{Y|X}(y|x) \) are twice continuously differentiable with respect to \( x \) and all derivatives are uniformly bounded.

**Theorem 4.4.** Suppose that Assumptions 2.3 and 2.4 hold. Let \( \phi_{Y|X,n}^{*}(\cdot|x) \) denote the vector of simulated stochastic processes using estimated influence functions in (B.2) in Appendix B. The following theorem shows the validity of the bootstrap procedure:

**Theorem 4.5.** Suppose that conditions in Theorem 4.4 are satisfied. In addition, if Assumptions 4.6 and 4.7 also hold, then, we have

\[
\hat{\phi}_L' \left( \Pi_L(\hat{F}_{Y|X,n})(y,\delta|x); \sqrt{nh_n^{d_{x}}} \hat{F}_{Y|X,n}(\cdot|x), a_n \right) \Rightarrow \phi_L' \left( \Pi_L(F_{Y|X})(y,\delta|x); G_x^e(\cdot) \right),
\]

\[
\hat{\phi}_U' \left( \Pi_U(\hat{F}_{Y|X,n})(y,\delta|x) + 1; \sqrt{nh_n^{d_{x}}} \hat{F}_{Y|X,n}(\cdot|x), a_n \right) \Rightarrow \phi_U' \left( \Pi_U(F_{Y|X})(y,\delta|x) + 1; G_x^e(\cdot) \right),
\]

in \( l^\infty (\mathcal{Y}^4) \), conditional on data. The forms of \( \hat{\phi}_L' \) and \( \hat{\phi}_U' \) are provided in Appendix A.
We provide several extensions of the main results in this section in Appendix. We extend the results of Abrevaya et al. (2015) who consider estimating conditional average treatment effects on a subset of covariates to the conditional distribution of treatment effects (Appendix C). This result may be practically relevant when the number of covariates is large. We also discuss how to adopt the results to conduct global hypothesis testing (Appendix D).

It is worth noting that Firpo and Ridder (2019) provide formal definitions of pointwise and uniform sharpness of bounds on distribution functions and show that the Makarov bounds are not uniformly sharp. The inference methods developed in this paper are based on pointwise sharp bounds on the conditional distribution of treatment effects and are valid uniformly over the support of treatment effects. Inference for uniformly sharp bounds on the distribution of treatment effects is left as future research.

5 Monte Carlo Simulation

We conduct a set of simulations to investigate the performance of the bootstrap in finite samples. To this end, the following DGP is considered:

\[
Y_1 = \mu_1 + X\beta_1 + (\phi_1 + X\gamma_1)U_1, \\
Y_0 = \mu_0 + X\beta_0 + (\phi_0 + X\gamma_0)U_0, \\
D = \mathbb{1}(X\alpha \geq V),
\]

where \(X = 2\tilde{X} - 1\) with \(\tilde{X}\) being an uniform a random variable on \([0, 1]\), \(U \equiv (U_1, U_0)^t \sim N \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\) and \(V \sim N(0, 1)\). The conditional treatment effect on \(X = x\) is defined as \((\mu_1 - \mu_0) + x(\beta_1 - \beta_0) + ((\phi_1 + X\gamma_1)U_1 - (\phi_0 + X\gamma_0)U_0)\). The parameter values are set as follows: \(\beta_1 = \gamma_1 = 1, \beta_0 = \gamma_0 = 0.9, \phi_1 = \phi_0 = 1,\) and \(\alpha = 1\). We consider various values for parameter vector \((\mu_1, \mu_0)^t\) to investigate the performance of a KS test statistic under the null and alternative hypotheses. Specifically, to investigate the performance of
the KS test under null hypotheses, we consider the following hypothesis:

\[ H_0 : F_{\Delta|X}^L(\delta|x = 0) = \left(2 \cdot \Phi \left(\frac{\delta}{2}\right) - 1\right) \cdot 1(\delta \geq 0) \text{ for all } \delta, \]

where \( \Phi(\cdot) \) is the standard normal distribution function.

The lower bound in the null hypothesis is one derived by Frank et al. (1987), and the null hypothesis is true if and only if \( \mu_1 = \mu_0 = 0 \). The KS statistic for this null is constructed as follows:

\[ KS_n = \sup_\delta \sqrt{n h_n} \left| F_{\Delta|X}^L(\delta|x = 0) - \left(2 \cdot \Phi \left(\frac{\delta}{2}\right) - 1\right) \cdot 1(\delta \geq 0) \right|. \]

To investigate the performance of the KS test under some alternative hypotheses, we consider the cases where \((\mu_1, \mu_0)^t = (\mu, 0)^t\) for \( \mu \in \{-1, 1\} \). The bandwidth is chosen to be \( h_n = 1.06 \times s.d(X) \times n^{-1/6} \), where \( s.d(X) \) denotes the (sample) standard deviation of \( X \).

It is worth emphasizing that the asymptotic distribution of the KS test statistic is nonstandard and may not be uniformly valid with respect to the DGP. For this reason, it is important to investigate whether the KS statistic performs well with various choices for \((a_n)\). Specifically, we consider several rates at which \( a_n \) grows to the infinity: (i) \( a_n = c \log \left(\log (nh_n)\right) / \sqrt{nh_n} \), (ii) \( a_n = c \log \left(\log (nh_n)\right) / \sqrt{nh_n} \), and (iii) \( a_n = c (nh_n)^{1/6} / \sqrt{nh_n} \) for some \( c > 0 \). These choices of \((a_n)\) satisfy Assumption 4.7. We consider various values of \( c \) ranging from 0.1 to 0.5 to see whether the finite-sample performance of the KS test is sensitive to the choice of \( c \).

The sample size \( n \) is set to be 500, and the number of bootstrap iterations is 500. The bootstrap weight \( B \) is drawn from the standard normal distribution, and all simulation results are obtained from 500 iterations. The nominal level is set to be 0.05.

Table 1 presents the simulation results for rejection probabilities. We find that the rejection probability under \( H_0 \) tends to decrease as \( c \) increases, except for the case of \( \sqrt{nh_n}a_n \propto (nh_n)^{1/6} \). The KS statistic performs well in finite samples for various values of

\(^{10}\text{Firpo et al. (2021) suggest using } c = 0.2 \text{ with } a_n = c \times \log \left(\log (nh_n)\right) / \sqrt{nh_n} \text{ in our context.} \)

\(^{11}\text{We also considered larger values of } c \text{ than 0.5, but they are not reported here. The simulation results with those values of } c \text{ suggest that the resulting confidence bands would be too conservative (i.e., the rejection probabilities under the null hypothesis are relatively smaller than the nominal rate). As a result,}

26
(a_n) in the sense that the rejection probability under H_0 is close to the nominal probability and that the rejection probabilities under H_1 are large in general.

### Table 1: Rejection Probabilities, n = 500, B = 500, x = Q_X(0.5)

| c \(\alpha_n\) | \(c \cdot \log (\log (n\mu_n)) / \sqrt{n\mu_n}\) | \(c \sqrt{\log (n\mu_n)} / \sqrt{n\mu_n}\) | \(c (n\mu_n)^{1/6} / \sqrt{n\mu_n}\) |
|-----------------|----------------------------------|----------------------------------|----------------------------------|
| \(\mu = 0\) | \(\mu = -1\) | \(\mu_1 = 1\) | \(\mu = 0\) | \(\mu = -1\) | \(\mu_1 = 1\) | \(\mu = 0\) | \(\mu = -1\) | \(\mu_1 = 1\) |
| 0.1 | 0.078 | 0.998 | 0.824 | 0.068 | 0.996 | 0.816 | 0.044 | 0.996 | 0.816 |
| 0.2 | 0.066 | 0.986 | 0.806 | 0.060 | 0.992 | 0.782 | 0.064 | 0.992 | 0.782 |
| 0.3 | 0.060 | 0.994 | 0.778 | 0.058 | 0.986 | 0.756 | 0.050 | 0.992 | 0.754 |
| 0.4 | 0.058 | 0.994 | 0.734 | 0.044 | 0.992 | 0.748 | 0.056 | 0.988 | 0.746 |
| 0.5 | 0.050 | 0.988 | 0.772 | 0.040 | 0.998 | 0.704 | 0.058 | 0.984 | 0.738 |

Note: The nominal level is set to be 0.05. The case of \(\mu = 0\) is where the null hypothesis is true.

### 6 Empirical Application: The Effect of 401(k) Plans on Net Financial Assets

In this section, we provide an empirical example to illustrate the usefulness of the methods proposed in this paper. We revisit the empirical question on the effect of participation in 401(k) plans on net financial assets investigated by many studies in the literature (e.g., Abadie (2003); Chernozhukov and Hansen (2004); Wüthrich (2019); Sant’Anna et al. (2022)). Our main goals in this empirical application are twofold. First, we empirically show that the stochastic dominance assumptions (Assumptions 2.3 and 2.4) can provide considerable identifying power. Second, we complement the existing results that there is substantial heterogeneity in treatment effects across income groups by estimating the bounds on the conditional distribution of treatment effects on income levels without an instrumental variable. In doing so, we complement the empirical results on the effect of 401(k) plans on net financial assets documented in the literature by providing estimation results on the distribution of the treatment effect. Since our focus is on verifying the identifying power of the stochastic dominance assumptions and potential heterogeneity in the

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we do not recommend using a too large value of \(c\), based on our simulation results. It would be an interesting question how to choose \(c\) or the rate of \((a_n)\) in a data-dependent way, but this is far beyond the scope of this paper. We therefore leave this interesting and important question for future research.
treatment effect across different subpopulations, we do not report confidence bands for clear
illustration.

The U.S. introduced several tax-deferred retirement plans, including 401(k) plans and
individual retirement accounts (IRAs), in the early 1980s. These retirement plans can be
used as a way to accumulate individual assets. Many papers in the literature have considered
how those tax-deferred retirement plans affect asset accumulation or savings. The main
challenge with identifying and estimating the causal effect of 401(k) participation on assets
is that participation in 401(k) plans is endogenously determined. Furthermore, the effect
of 401(k) plans on net financial assets is heterogeneous across income levels, as shown by
Chernozhukov and Hansen (2004). Motivated by the empirical results of Chernozhukov
and Hansen (2004), we focus on the distribution of treatment effects conditional on an
individual’s income. Moreover, while it is common to use the eligibility for 401(k) as an
instrumental variable to point identify some distributional effects (e.g., quantile treatment
effects), our identification and estimation strategies do not rely on such an instrumental
variable.

We use the data from Sant’Anna et al. (2022) for this empirical analysis. The original
dataset contains 9,910 households from the 1991 Survey of Income and Program Participa-
tion. The dependent variable is the amount of net financial assets measured in ten thousand
dollars. We exclude observations with a value of the dependent variable higher than the
0.99 sample quantile and lower than the 0.01 sample quantile of the net financial assets.
This results in a sample of 9,712 households. The treatment variable is a binary variable
indicating whether a household participates in 401(k) plans. To investigate the potential
heterogeneity across income levels, we use the income variable as the covariate of interest.
For stability of estimation, we standardize the original variable of income and consider the
sample mean and various quantiles of income. Table 2 reports the summary statistics of
the data.
Table 2: Summary Statistics of the Data

| Variables          | Mean   | Median | S.D.   | Min    | Max    | Obs. |
|--------------------|--------|--------|--------|--------|--------|------|
| Net financial assets | 1.4506 | 0.1499 | 3.1991 | -2.350 | 21.995 | 9,712|
| Treatment          | 0.26   | 0      | 0.4387 | 0      | 1      | 9,712|
| Income             | 3.661  | 3.120  | 2.379  | 0.003  | 19.299 | 9,712|
| Age                | 40.969 | 40     | 10.311 | 25     | 64     | 9,712|

Note: The net financial assets and income are measured in $10,000.

We first discuss the validity of Assumptions 2.3 and 2.4 in this empirical example. It is well known that the preference for saving is heterogeneous in that some people have a stronger preference for saving than others. This leads to the nonrandom selection into participation in 401(k) plans or other tax-deferred retirement plans (e.g., Chernozhukov and Hansen (2004)). Based on this observation, it is likely that people who participate in 401(k) plans would have a stronger preference for saving than those who do not participate. Therefore, the net financial assets of people with a strong preference for saving tend to be larger than those of people with a weak preference for saving, suggesting that it may be plausible to impose Assumption 2.3 on the model.

Assumption 2.4 is consistent with the empirical example as most tax-deferred retirement plans, including 401(k) plans, by themselves increase the amount of assets that an individual possesses. Therefore, regardless of whether an individual participates in 401(k) plans or not, the potential net financial assets that one would have had if she participated in 401(k) plans are likely to be larger than those she would have had if she did not participate in 401(k). As a result, it is plausible to impose Assumption 2.4 on the model.

For estimation, we set $h_n = 1.06 \times n^{-1/(5+d_x)}$ and $a_n = 0.2 \times \log \left( \log \left( n h_n^{d_x} \right) \right) / \sqrt{n h_n^{d_x}}$ with $d_x = 1$. We denote the $\tau$-th (sample) quantile of income by $Q_{\text{income}}(\tau)$.

We note that when Assumptions 2.3 and 2.4 are not imposed, the estimated bounds are the logical ones, regardless of the conditioning value of the income. On the other hand, the estimated bounds under Assumptions 2.3 and 2.4 are informative in the sense that they are not the logical bounds. This indicates that the stochastic dominance assumptions have considerable identifying power.

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12 By the logical bounds, we mean that the lower and upper bounds are equal to 0 and 1, respectively, for all $\delta \in \text{Supp}(|X-x|)$. 
We find substantial heterogeneity in the distribution of treatment effects across different values of the income. Figure F.1 compares the estimated bounds on the conditional distribution of treatment effects at the 0.2 and 0.8 quantiles of the income. Specifically, the star-marked lines are the bounds on the conditional distribution of treatment effects given the 0.2 quantile of the income. The circle-marked lines are the bounds on the conditional distribution of treatment effects given the 0.8 quantile of income. We find that the lower bound conditional on the 0.2 quantile of the income is larger than the lower bound conditional on the 0.8 quantile of the income. Conversely, the upper bound conditional on the 0.8 quantile of the income is larger than the upper bound conditional on the 0.2 quantile of the income for all $\delta \in [-24, 24]$ (i.e., from -$240,000 to $240,000). When considering the bounds on $\Pr(Y_1 - Y_0 \leq 1 | X = x)$ for $x \in \{Q_{income}(0.2), Q_{income}(0.8)\}$, the estimation results show that $\Pr(Y_1 - Y_0 \leq 1 | X = Q_{income}(0.2)) \in [0.5804, 1]$ and that $\Pr(Y_1 - Y_0 \leq 1 | X = Q_{income}(0.8)) \in [0.0638, 1]$. These estimated bounds suggest that the proportion of individuals who experience a positive treatment effect larger than $10,000 among those with the income being equal to the 0.2 sample quantile is at most 41.96%. The proportion among those with the income being equal to the 0.8 sample quantile is at most 93.62%. As a result, there may be a possibility that the proportion of individuals who experience a certain level of positive treatment effect is larger when considering a higher level of income. This argument is in part consistent with the finding of Chernozhukov and Hansen (2004) that the quantile treatment effects of 401(k) plans on net financial wealth tend to increase as the income level increases (see Figure 2 in Chernozhukov and Hansen (2004)).

Figure F.2 compares the estimated bounds at a specific quantile level with those at the mean of income under Assumptions 2.3 and 2.4. When considering a low quantile level, e.g., $\tau \in \{0.1, 0.2, 0.3\}$, we find that the lower bound conditional on $X = Q_{income}(\tau)$ is larger than that conditional on the mean of income. The upper bound conditional on $X = Q_{income}(\tau)$ is smaller than that conditional on the mean of income over the potential support of the treatment effect. However, when considering a high quantile level, e.g., $\tau \in \{0.7, 0.8, 0.9\}$, the estimation results are the opposite. These estimation results indicate that the treatment effect of 401(k) plans on net financial assets is likely to be heterogeneous.
across income levels, which is consistent with the finding of Chernozhukov and Hansen (2004).

7 Conclusion

This paper considers identification and estimation of bounds on the conditional distribution of treatment effects. The conditional distribution may provide evidence on potential heterogeneity in treatment effects across subpopulations that are defined in terms of values of covariates, and therefore, is of practical importance in many empirical studies. We show that when the treatment is endogenously determined, one can tighten the bounds by imposing stochastic dominance assumptions. These assumptions are consistent with many economic theories and easy to interpret, and the resulting bounds on the distribution of treatment effects are easy to compute. We propose nonparametric estimators of the bounds and establish the uniform asymptotic theory based on the novel approach of Fang and Santos (2019) and Firpo et al. (2021). The asymptotic theory in this paper is useful for constructing uniform confidence bands and conducting statistical tests for global hypotheses. We then provide an empirical application of the methodology proposed in this paper to illustrate its relevance to empirical research.

There are several interesting directions for future research. First, one can consider inference that is uniformly valid regardless of whether the (conditional) distributions of the potential outcomes are point identified or partially identified, as in, for example, Imbens and Manski (2004), Stoye (2009), and Andrews and Soares (2010). Second, one can consider testing global hypotheses, such as stochastic dominance. Although we cannot directly test stochastic dominance between two conditional distribution of treatment effects as they are not point identified, one can provide weak evidence by using similar arguments to those in Firpo et al. (2021). Third, it would be fruitful to develop inference methods that are uniformly valid in values of covariates. In work in progress, we consider a semiparametric approach to uniform inference over the support of treatment effects and covariates. It is expected to help resolve many interesting questions in empirical analysis that cannot be answered by the framework proposed in this paper. Lastly, it is worth considering inference
in the presence of instrumental variables.

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A Hadamard Directional Derivatives

To derive the Hadamard directional derivatives of $\phi_L$ and $\phi_U$, we introduce some additional notations that are used in Firpo et al. (2021). For a set-valued map (or correspondence) from $A$ to the collection of subsets of $B$, $S$, $gr(S)$ is the graph of $S$ in $A \times B$. Recall that for $F \in \mathcal{Y}|X((y_{1l}, y_{1u}, y_{0l}, y_{0u})|x) \equiv \begin{pmatrix} LB_{1|X}(y_{1l}|x) \\ UB_{1|X}(y_{1u}|x) \\ LB_{0|X}(y_{0l}|x) \\ UB_{0|X}(y_{0u}|x) \end{pmatrix}$, we have defined

$$
\Pi_L(F \in \mathcal{Y}|X)(y, \delta|x) \equiv LB_{1|X}(y|x) - UB_{0|X}(y - \delta|x),
$$

$$
\Pi_U(F \in \mathcal{Y}|X)(y, \delta|x) \equiv UB_{1|X}(y|x) - LB_{0|X}(y - \delta|x).
$$

Let $S^+: D \rightrightarrows \mathcal{Y}$ be a set-valued map such that for each $\delta \in D$,

$$
S^+(\delta) \equiv \{ y \in \mathcal{Y}: y = \text{arg sup} \Pi_L(F \in \mathcal{Y}|X)(y, \delta|x) \}.
$$

Similarly, we define a set-valued map $S^- : D \rightrightarrows \mathcal{Y}$ such that for each $\delta \in D$,

$$
S^-(\delta) \equiv \{ y \in \mathcal{Y}: y = \text{arg inf}(\Pi_U(F \in \mathcal{Y}|X)(y, \delta|x) + 1) \}.
$$

For given $\epsilon > 0$, $\delta \in D$, and a set-valued correspondence $S : D \rightrightarrows \mathcal{Y}$, we define the set of $\epsilon$-maximizers as

$$
\Lambda_f(\delta, \epsilon; S) \equiv \left\{ y \in S(\delta): f(\delta, y) \geq \sup_{y \in \mathcal{Y}} f(\delta, y) - \epsilon \right\}.
$$

Let $H \equiv (h_1, h_2, h_3, h_4)^t$ be a four-dimensional vector-valued function. After some
algebra, one can show that the Hadamard directional derivatives of $\phi_L$ and $\phi_U$ are

$$
\phi'_L(f; H) \equiv \max \left\{ \lim_{\epsilon \to 0^+} \sup_{\delta \in \mathcal{D}} \sup_{y \in \Lambda_f(\delta; \epsilon; S^+)} (h_1(y) - h_4(y - \delta)), \lim_{\epsilon \to 0^+} \sup_{\delta \in \mathcal{D}} \inf_{y \in \Lambda_f(\delta; \epsilon; S^+)} -(h_2(y) - h_3(y - \delta)) \right\},
$$

(A.1)

$$
\phi'_U(f; H) \equiv \max \left\{ \lim_{\epsilon \to 0^+} \sup_{\delta \in \mathcal{D}} \sup_{y \in \Lambda_f(\delta; \epsilon; S^-)} -(h_1(y) - h_4(y - \delta) + 1), \lim_{\epsilon \to 0^+} \sup_{\delta \in \mathcal{D}} \inf_{y \in \Lambda_f(\delta; \epsilon; S^-)} (h_2(y) - h_3(y - \delta) + 1) \right\}.
$$

The form of $\phi'_L(f; H)$ can be found in the second part of Theorem 3.2. of Firpo et al. (2021). Here we derive the Hadamard directional derivative of $\phi_U$. Recall that

$$
\phi_U(f) = \sup_{\delta \in \mathcal{D}} \inf_{y \in \mathcal{Y}} f(\delta, y) = \sup_{\delta \in \mathcal{D}} \sup_{y \in \mathcal{Y}} (-f(\delta, y))
$$

since $\inf f = -\sup(-f)$.

$$
\phi_U(f + t_n h) - \phi_U(f) = \sup_{\delta \in \mathcal{D}} \sup_{y \in \mathcal{Y}} (-f(\delta, y) - t_n h(\delta, y)) \sup_{\delta \in \mathcal{D}} \sup_{y \in \mathcal{Y}} (-f(\delta, y))
$$

$$
= \sup_{\delta \in \mathcal{D}} \sup_{y \in \mathcal{Y}} (-f(\delta, y) + t_n (-h(\delta, y))) \sup_{\delta \in \mathcal{D}} \sup_{y \in \mathcal{Y}} (-f(\delta, y))
$$

By using the same argument of the proof of Theorem 3.2. of Firpo et al. (2021), we can obtain that

$$
\phi'_U(f; h) \equiv \max \left\{ \lim_{\epsilon \to 0^+} \delta \in \mathcal{D} \sup_{y \in \Lambda_f(\delta; \epsilon; S^-)}, \lim_{\epsilon \to 0^+} \delta \in \mathcal{D} \inf_{y \in \Lambda_f(\delta; \epsilon; S^-)} h(\delta, y) \right\}.
$$

Let $a_n$ be a positive real sequence satisfying the conditions in Assumption 4.7. Then, one can estimate $\phi'_L(f; H)$ and $\phi'_U(f; H)$ as follows:
\[ \hat{\phi}_L^\prime (f; \mathbb{H}, a_n) \equiv \max \left\{ \sup_{\delta \in \mathcal{D}} \sup_{y \in \Lambda_f(\delta, a_n; S^+)} (h_1(y) - h_4(y - \delta)), \right. \\
\left. \quad \sup_{\delta \in \mathcal{D}} \inf_{y \in \Lambda_f(\delta, a_n; S^+)} -(h_2(y) - h_3(y - \delta)) \right\}, \]

\[ \hat{\phi}_U^\prime (f; \mathbb{H}, a_n) \equiv \max \left\{ \sup_{\delta \in \mathcal{D}} \sup_{y \in \Lambda_f(\delta, a_n; S^+)} -(h_1(y) - h_4(y - \delta)), \right. \\
\left. \quad \sup_{\delta \in \mathcal{D}} \inf_{y \in \Lambda_f(\delta, a_n; S^-)} (h_2(y) - h_3(y - \delta) + 1) \right\}. \] (A.2)

\section*{B Estimation of Influence Functions}

When the unconfoundedness and overlap assumptions hold, the conditional distributions of the potential outcomes are point identified and they can be estimated as

\[ \hat{F}_{1|X,n}(y_1|x) = \sum_i^n 1(Y_i \leq y_1) D_i K\left(\frac{X_i-x}{h_n}\right) \sum_i^n D_i K\left(\frac{X_i-x}{h_n}\right), \]

\[ \hat{F}_{0|X,n}(y_0|x) = \sum_i^n 1(Y_i \leq y_0)(1 - D_i) K\left(\frac{X_i-x}{h_n}\right) \sum_i^n (1 - D_i) K\left(\frac{X_i-x}{h_n}\right). \]

Then, one can estimate the influence functions of \( \hat{F}_{1|X,n}(y|x) \) and \( \hat{F}_{0|X,n}(y|x) \) for the \( i \)-th observation by

\[ \hat{\psi}_{1,i}(y|x) = \frac{\{1(Y_i \leq y) - \hat{F}_{1|X,n}(y|x)\} \cdot D_i \cdot K\left(\frac{X_i-x}{h_n}\right)}{\sum_j^n D_j \cdot K\left(\frac{X_j-x}{h_n}\right)}, \]

\[ \hat{\psi}_{0,i}(y|x) = \frac{\{1(Y_i \leq y) - \hat{F}_{0|X,n}(y|x)\} \cdot (1 - D_i) \cdot K\left(\frac{X_i-x}{h_n}\right)}{\sum_j^n (1 - D_j) \cdot K\left(\frac{X_j-x}{h_n}\right)}, \] (B.1)

respectively.

We now assume that Assumptions 2.3 and 2.4 hold. Then, the bounds on the conditional
distributions of the potential outcomes can be estimated by using

\[
\hat{F}_{1|X,n}(y_1|x) = \frac{\sum^n \mathbf{1}(Y_i \leq y_1)D_i K(\frac{X_i-x}{h_n})}{\sum^n D_i K(\frac{X_i-x}{h_n})},
\]

\[
\hat{F}_{0|X,n}(y_0|x) = \frac{\sum^n \mathbf{1}(Y_i \leq y_0)(1 - D_i)K(\frac{X_i-x}{h_n})}{\sum^n (1 - D_i) K(\frac{X_i-x}{h_n})},
\]

\[
\hat{F}_{Y|X,n}(y|x) = \frac{\sum^n \mathbf{1}(Y_i \leq y)K(\frac{X_i-x}{h_n})}{\sum^n K(\frac{X_i-x}{h_n})},
\]

Then, one can estimate the influence functions of \(\hat{F}_{1|X,n}(y_1|x)\), \(\hat{F}_{0|X,n}(y_0|x)\), and \(\hat{F}_{Y|X,n}(y|x)\) for the \(i\)-th observation by

\[
\hat{\psi}_{11,i}(y|x) = \frac{\mathbf{1}(Y_i \leq y) - \hat{F}_{1|X,n}(y_1|x)) \cdot D_i \cdot K(\frac{X_i-x}{h_n})}{\sum^n D_j K(\frac{X_j-x}{h_n})},
\]

\[
\hat{\psi}_{00,i}(y|x) = \frac{\mathbf{1}(Y_i \leq y) - \hat{F}_{0|X,n}(y_0|x)) \cdot (1 - D_i) \cdot K(\frac{X_i-x}{h_n})}{\sum^n (1 - D_j) K(\frac{X_j-x}{h_n})},
\]

\[
\hat{\psi}_{Y,i}(y|x) = \frac{\mathbf{1}(Y_i \leq y) - \hat{F}_{Y|X,n}(y|x)) K(\frac{X_i-x}{h_n})}{\sum^n K(\frac{X_i-x}{h_n})},
\]

respectively.

## C Semiparametric Estimation of Conditional Distributions of Treatment Effects on a Subset of \(X\)

We may be interested in the conditional distribution of treatment effects within some subpopulation characterized by a subset of \(X\). For example, suppose we are interested in the effect of smoking on birth weight. There are many potential factors that affect the birth weight, such as mother’s age and education level, family income, and baby’s gender. Among those factors, the focus may be on the heterogeneity in the treatment effect across mother’s age (we call this variable \(X_1\)). Note that Assumption 2.1 does not necessarily imply that \((Y_1, Y_0) \perp D|X_1\), and Abrevaya et al. (2015) develop approaches to estimating conditional average treatment effects to capture heterogeneity in some subpopulation. We complement Abrevaya et al. (2015) by providing a way to identify and estimate the distribution of
treatment effects of some subpopulation.

Let \( x_1 \in X_1 \equiv \text{Supp}(X_1) \subseteq \mathbb{R}^{d_1} \) be given. Then, from Lemma 2.1 and the law of iterated expectations, it is straightforward to see that

\[
F_{1|X_1}(y|x_1) = \mathbb{E} \left[ \frac{D(Y \leq y)}{p_0(X)} \bigg| X_1 = x_1 \right],
\]

\[
F_{0|X_1}(y|x_1) = \mathbb{E} \left[ \frac{(1 - D)1(Y \leq y)}{1 - p_0(X)} \bigg| X_1 = x_1 \right],
\]

(C.1)

where \( F_{d|X_1}(y|x_1) \equiv \text{Pr}(Y_d \leq y|X_1 = x_1) \) for each \( d \in \{0, 1\} \). We focus on a semiparametric approach that uses parametric estimation for the propensity score but nonparametric kernel-smoothing estimation for the second-step, which was proposed by Abrevaya et al. (2015). We use parametric estimators of the propensity score mainly for practical reasons. First, we can easily incorporate discrete regressors when estimating the propensity score. Second, we can avoid the curse of dimensionality when the dimension of \( X \) is very large. Lastly, the semiparametric approach is expected to be less sensitive to tuning parameters as the number of tuning parameters required for estimation is fewer for the semiparametric approach than fully nonparametric approaches. While the semiparametric approach using a parametric specification for the propensity has some advantages over the fully nonparametric approach, it can lead to model misspecification. To mitigate the issues about potential model misspecification, one may employ the fully nonparametric approach in Abrevaya et al. (2015).

We assume that the propensity score function is parameterized by a finite-dimensional parameter \( \theta_0 \): \( p_0(x) = p(x; \theta_0) \) for all \( x \in \mathcal{X} \). Let \( \hat{\theta}_n \) be an estimator of \( \theta_0 \), then the propensity score function can be estimated by \( p(x; \hat{\theta}_n) \). Let \( K_1(\cdot) : \mathbb{R}^{d_1} \rightarrow \mathbb{R} \) be a kernel function that is symmetric around zero. The identification results in equation (C.1) suggest the following semiparametric estimators of the conditional distributions of \( Y_1 \) and \( Y_0 \) on \( X_1 = x_1 \):

\[
\hat{F}_{1|X_1,n}(y|x_1) \equiv \sum_i^n \frac{D_i \cdot 1(Y_i \leq y)}{p(X_i; \hat{\theta}_n)} \cdot K_1\left(\frac{X_{1i} - x_1}{h_{1n}}\right) / \sum_i^n K_1\left(\frac{X_{1i} - x_1}{h_{1n}}\right),
\]

(C.2)

\[
\hat{F}_{0|X_1,n}(y|x_1) \equiv \sum_i^n \frac{(1 - D_i) \cdot 1(Y_i \leq y)}{(1 - p(X_i; \hat{\theta}_n))} \cdot K_1\left(\frac{X_{1i} - x_1}{h_{1n}}\right) / \sum_i^n K_1\left(\frac{X_{1i} - x_1}{h_{1n}}\right),
\]

(C.3)
where \( h_{1n} \) is a bandwidth. Let \( \hat{F}_{Y|X_1,n}((y_1, y_0)|x_1) \equiv \left( \hat{F}_{1|X_1,n}(y_1|x_1), \hat{F}_{0|X_1,n}(y_0|x_1) \right)^t \). With these estimators of \( F_{1|X_1} \) and \( F_{0|X_1} \), the bounds in Lemma 2.1 can be estimated as follows:

\[
\hat{F}_{\Delta|X_1,n}(\delta|x_1) \equiv \max \left( \sup_y \left\{ \hat{F}_{1|X_1,n}(y|x_1) - \hat{F}_{0|X_1,n}(y - \delta|x_1) \right\}, 0 \right), \quad \text{(C.4)}
\]

\[
\hat{F}_{\Delta|X_1,n}(\delta|x_1) = \min \left( \inf_y \left\{ \hat{F}_{1|X_1,n}(y|x) - \hat{F}_{0|X_1,n}(y - \delta|x_1) \right\}, 0 \right) + 1. \quad \text{(C.5)}
\]

One can use the same bootstrap procedure to approximate the distributions of the estimated bounds on \( F_{\Delta|X_1}(\cdot|x_1) \). Define

\[
\hat{F}_{Y|X_1,n}(y|x; B_1) = \left( \begin{array}{c} F_{1|X_1,n}(y|x_1; B_1) \\ F_{0|X_1,n}(y|x_1; B_1) \end{array} \right) = \left( \frac{\sum_i B_{1i} \hat{\psi}_{11,i}(y|x_1)}{\sum_i B_{1i} \hat{\psi}_{10,i}(y|x_1)} \right),
\]

where

\[
\hat{\psi}_{11,i}(y|x_1) \equiv \frac{\left( D_i \cdot \mathbf{1}(Y_i \leq y) - \hat{F}_{1|X_1,n}(y|x_1) \right)}{\hat{p}_n(X_i; \theta_n)} \cdot K_1 \left( X_{1i} - x_1 \right) / \sum_j K_1 \left( X_{1j} - x_1 \right),
\]

\[
\hat{\psi}_{10,i}(y|x_1) \equiv \frac{\left( 1 - D_i \right) \cdot \mathbf{1}(Y_i \leq y) - \hat{F}_{0|X_1,n}(y|x_1) \right)}{1 - \hat{p}_n(X_i; \theta_n)} \cdot K_1 \left( X_{1i} - x_1 \right) / \sum_j K_1 \left( X_{1j} - x_1 \right),
\]

and \( B_1 \) is a random variable independent of the data. We consider the following set of assumptions to prove the validity of the bootstrap in this case.

Assumption C.1. (i) The support of \( X_1, X_1 \), is a compact subset of \( \mathbb{R}^{d_1} \); (ii) the distribution of \( X_1 \) admits its density \( f_{X_1}(\cdot) \) on \( X_1 \) such that \( 0 < \inf_{x_1 \in X_1} f_{X_1}(x_1) < \sup_{x_1 \in X_1} f_{X_1}(x_1) < \infty \). The density function \( f_{X_1}(\cdot) \) is twice continuously differentiable and \( \sup_{x_1 \in X_1} |f_{X_1}^{(1)}(x_1)| \) and \( \sup_{x_1 \in X_1} |f_{X_1}^{(2)}(x_1)| \) are bounded.

Assumption C.2. For given \( j \in \{0, 1\} \) and \( x_1 \in X_1 \), \( F_{j|X_1,n}(y|x_1) \) is continuously differentiable with respect to \( x \) and the derivative is uniformly bounded.

Assumption C.3. Let \( \hat{\theta}_n \) be an estimator of \( \theta_0 \in \Theta \subseteq \mathbb{R}^{d_0} \), where propensity score function \( p(x; \theta_0) \) is the propensity score function. \( \hat{\theta}_n \) satisfies \( \sup_{x \in X} |p(x; \hat{\theta}_n) - p(x; \theta_0)| = O_p(n^{-1/2}) \).
Assumption C.4. The kernel function $K_1(\cdot)$ is a $d_1$-dimensional product kernel with a univariate bounded kernel function $k_1(\cdot) : \mathbb{R} \to \mathbb{R}_+$ such that $\int k_1(u)du = 1$, $\int uk_1(u)du = 0$, and $\int u^2k_1(u)du < \infty$. The support of the univariate kernel function is compact.

Assumption C.5. (i) $h_{1n} \to 0$; (ii) $nh_{1n}^{d_1} \to \infty$; and (iii) $nh_{1n}^{d_1+4} \to 0$.

Assumption C.6. Let $B_1$ be a random variable that is independent of the data $W$ such that $\mathbb{E}[B_1] = 0$, $\mathbb{V}ar(B_1) = 1$, and $\int_0^\infty \sqrt{\mathbb{V}ar(|B_1| > x)}dx < \infty$.

Assumption C.7. Let $a_{1n}$ be a sequence of positive real numbers such that $a_{1n} \downarrow 0$ and $a_{1n}\sqrt{nh_{1n}^{d_1}} \to \infty$.

Define $||K_1||_2^2 \equiv \int K_1^2(u)du$, $G_{1|X_1}(y|x_1) = \mathbb{E}\left[\frac{F_{1|X,Y}(y|x_1)}{p_0(x_1)}|X_1 = x_1\right]$, and $G_{0|X_1}(y|x_1) = \mathbb{E}\left[\frac{F_{0|X,Y}(y|x_1)}{1-p_0(x_1)}|X_1 = x_1\right]$. The following theorem establishes the asymptotic distributions of the estimated bounds on the conditional distribution of treatment effects and the bootstrap validity: $\Pi_L(\mathbb{F}_{Y|X_1,n})(y, \delta|x_1) = \Pi_U(\mathbb{F}_{Y|X_1})(y, \delta|x_1) = F_{1|X_1}(y|x_1) - F_{0|X_1}(y - \delta|x_1)$.

Theorem C.1. Let $x_1 \in \text{int}(\mathcal{X}_1)$ be given. Suppose that Assumptions 2.1, 2.2, 4.1, and C.1–C.5 hold. Then,

\[
\sqrt{nh_{1n}^{d_1}} \left( \phi_L \left( \hat{\mathbb{F}}_{Y|X_1,n}(\cdot|x_1) \right) - \phi_L \left( \mathbb{F}_{Y|X_1}(\cdot|x_1) \right) \right) \Rightarrow \phi_L' \left( \Pi_L(\mathbb{F}_{Y|X_1})(y, \delta|x_1); \tilde{G}(\cdot) \right) \quad \text{in } l^\infty(\mathcal{Y}^2),
\]

\[
\sqrt{nh_{1n}^{d_1}} \left( \phi_U \left( \hat{\mathbb{F}}_{Y|X_1,n}(\cdot|x_1) \right) - \phi_U \left( \mathbb{F}_{Y|X_1}(\cdot|x_1) \right) \right) \Rightarrow \phi_U' \left( \Pi_U(\mathbb{F}_{Y|X_1})(y, \delta|x_1) + 1; \tilde{G}(\cdot) \right) \quad \text{in } l^\infty(\mathcal{Y}^2),
\]

(C.6)

where $\mathbb{F}_{Y|X_1}((y_1, y_0)|x_1) \equiv (F_{1|X_1}(y_1|x_1), F_{0|X_1}(y_0|x_0))^t$ and $\tilde{G}((\cdot, \cdot)) \equiv (\tilde{G}_{1}(\cdot), \tilde{G}_{0}(\cdot))^t$ is a two-dimensional Gaussian process with mean zero and covariance kernels

\[
\tilde{H}_1(y_1, y_2) \equiv \left\{ G_{1|X_1}(\min(y_1, y_2)|x_1) - F_{1|X_1}(y_1|x_1)F_{1|X_1}(y_2|x_1) \right\} \frac{||K_1||_2^2}{f_{X_1}(x_1)}
\]

and

\[
\tilde{H}_0(y_1, y_2) \equiv \left\{ G_{0|X_1}(\min(y_1, y_2)|x_1) - F_{0|X_1}(y_1|x_1)F_{0|X_1}(y_2|x_1) \right\} \frac{||K_1||_2^2}{f_{X_1}(x_1)},
\]

respectively.
If, in addition, Assumptions C.6–C.7 hold, then, conditional on data,

\[
\hat{\phi}_L \left( \Pi_L(\hat{F}_{Y|X_1,n})(y, \delta|x_1); \sqrt{nh_{1n}} \hat{F}_{Y|X_1,n}(|x_1; B_1), a_{1n} \right) \Rightarrow \phi'_L \left( \Pi_L(F_{Y|X_1})(y, \delta|x_1); \hat{G}(\cdot) \right),
\]

\[
\hat{\phi}_U \left( \Pi_U(\hat{F}_{Y|X_1,n})(y, \delta|x_1) + 1; \sqrt{nh_{1n}} \hat{F}_{Y|X_1,n}(|x_1; B_1), a_{1n} \right) \Rightarrow \phi'_U \left( \Pi_U(F_{Y|X_1})(y, \delta|x_1) + 1; \hat{G}(\cdot) \right),
\]

in \( l^\infty(Y^2) \).

D Global Hypotheses Testing

In this section, we briefly discuss global hypotheses testing for the bounds on the distribution of treatment effects to illustrate the usefulness of our approach. We focus on the case where we compare lower bounds between two groups, but the result can easily be generalized to other cases.

We first consider testing lower bounds on the distribution of treatment effects between two groups, where each group is defined in terms of the value of the covariate. Suppose that we consider two values \( x_A, x_B \in \text{int}(\mathcal{X}) \). The resulting lower bounds are denoted by \( L_A(\delta) \) and \( L_B(\delta) \), respectively (i.e. \( L_A(\delta) = \sup_y \Pi_y(F_{Y|X})(y, \delta|x_A) \) and \( L_B(\delta) = \sup_y \Pi_y(F_{Y|X})(y, \delta|x_B) \)), and the null hypothesis is as follows:

\[
H_0 : L_A(\delta) = L_B(\delta), \quad \forall \delta \in \mathcal{D}.
\]

Let \( \mu \) be the Lebesgue measure on \( \mathcal{D} \) and define

\[
\theta_{L,p,e} \equiv \left( \int_{\mathcal{D}} \left| L_A(\delta) - L_B(\delta) \right|^p d\mu(\delta) \right)^{1/p}
\]

for some \( 1 \leq p < \infty \). Replacing \( L_A(\delta) \) and \( L_B(\delta) \) with their estimators, we obtain a uniform test statistic \( \hat{\theta}_{L,p,e,n} \). Define \( \rho(f)(\delta|x) \equiv \sup_{y \in Y} f(y, \delta|x) \), then the Hadamard directional derivative of \( \rho(f)(\delta|x) \) for directions \( h \) at \( f \) is

\[
\rho'_f(h)(\delta|x) = \lim_{\epsilon \to 0} \sup_{y \in \Lambda_f(\delta, \epsilon; S^+)} h(y, \delta|x)
\]
(cf. Firpo et al. (2021)). We also define $G_{(x_A,x_B)}((y_1,y_2,y_3,y_4)) \equiv (G_{x_A}((y_1,y_2)), G_{x_B}((y_3,y_4)))^t$. 

**Theorem D.1.** Suppose that Assumptions 2.1 and 2.2 hold and that $\mu(D) < \infty$. Let $x_A, x_B \in \text{int}(X)$ be given and Assumptions 4.1–4.5 hold. Then, under $H_0$,

$$\sqrt{n} \frac{d_n}{n} \hat{\theta}_{L,P,e,n} \Rightarrow \theta'_{L,P,e}(G_{(x_A,x_B)}((\cdot,\cdot,\cdot))),$$

where, for $H_A(\cdot,\cdot) = (h_1(\cdot|x_A), h_2(\cdot|x_A))^t$, $H_B(\cdot,\cdot) = (h_3(\cdot|x_B), h_4(\cdot|x_B))^t$,

$$\theta'_{L,P,e}(H_A, H_B) \equiv \left( \int_D |\hat{\rho}'_{\Pi(\hat{F}_{Y,Y}|X)(y, \delta|x_A)}(\Pi(H_A)(y, \delta|x_A)) - \hat{\rho}'_{\Pi(\hat{F}_{Y,Y}|X,n)(y, \delta|x_B)}(\Pi(H_B)(y, \delta|x_B))|^p d\mu \right)^{1/p}.$$

If Assumptions 4.6 and 4.7 additionally hold, then, conditional on data,

$$\hat{\theta}_{L,P,e}^*(F_{Y|X,n}^*(\cdot|x_A), F_{Y|X,n}^*(\cdot|x_B)) \Rightarrow \theta'_{L,P,e}(G_{(x_A,x_B)}((\cdot,\cdot,\cdot)))$$,

where $\hat{\rho}'(h)(\delta|x) \equiv \sup_{y \in \Lambda_f(\delta,a_n;S^+)} h(y, \delta|x)$ and

$$\hat{\theta}_{L,P,e}^*(H_A, H_B) \equiv \left( \int_D |\hat{\rho}'_{\Pi(\hat{F}_{Y,Y}|X,n)(y, \delta|x_A)}(\Pi(H_A)(y, \delta|x_A)) - \hat{\rho}'_{\Pi(\hat{F}_{Y,Y}|X,n)(y, \delta|x_B)}(\Pi(H_B)(y, \delta|x_B))|^p d\mu \right)^{1/p}.$$

Theorem D.1 establishes the limiting distribution of the normalized test statistic and the validity of the bootstrap. Although it is interesting to investigate the performance of this uniform test, it is beyond the scope of this paper. Therefore, we leave this interesting topic for future research.
E.1 Proof of Lemma 2.1

Proof. We only prove the first result, as the second result can be proven in a similar way.

Observe that
\[
\mathbb{E}[D \cdot G(Y)|X = x] = \mathbb{E}[D | G(Y)|X = x, D = 1] \Pr(D = 1|X = x)
\]
\[
= \mathbb{E}[G(Y_1)|X = x, D = 1] \Pr(D = 1|X = x)
\]
\[
= \mathbb{E}[G(Y_1)|X = x] \cdot \Pr(D = 1|X = x).
\]

where the last holds by Assumption 2.1. In addition, Assumption 2.2 implies that \(\Pr(D = 1|X = x) > 0\). Therefore,
\[
\frac{\mathbb{E}[D \cdot G(Y)|X = x]}{\mathbb{E}[D|X = x]} = \frac{\mathbb{E}[G(Y_1)|X = x] \cdot \Pr(D = 1|X = x)}{\Pr(D = 1|X = x)} = \mathbb{E}[G(Y_1)|X = x],
\]
and this completes the proof.

E.2 Proof of Theorem 2.1

Proof. To see this, recall that from Proposition 2.1, we have
\[
\max\left(\sup_y \left\{ F_{1|X}(y|x) - F_{0|X}(y - \delta|x) \right\} , 0 \right) \leq F_{\Delta|X}(\delta|x) \leq \min\left(\inf_y \left\{ F_{1|X}(y|x) - F_{0|X}(y - \delta|x) \right\} , 0 \right) + 1.
\]

Since the conditional distributions of the potential outcomes given \(X = x\) are partially identified by the hypothesis and the functions \(\max[\cdot, \cdot]\) and \(\min[\cdot, \cdot]\) are non-decreasing, we obtain that
\[
F_{\Delta|X}^{e,L}(\delta|x) \leq \max\left(\sup_y \left\{ F_{1|X}(y|x) - F_{0|X}(y - \delta|x) \right\} , 0 \right)
\]
and that
\[
F_{\Delta|X}^{e,U}(\delta|x) \geq \min\left(\inf_y \left\{ F_{1|X}(y|x) - F_{0|X}(y - \delta|x) \right\} , 0 \right) + 1.
\]

Replacing the bounds in Theorem 2.1 with \(F_{\Delta|X}^{e,L}(\delta|x)\) and \(F_{\Delta|X}^{e,U}(\delta|x)\) yields the result.
E.3 Proof of Theorem 2.2

Proof. To prove the results in Theorem 2.2, recall that $F_{1|X}(y|x) = F_{1|1X}(y|x)\Pr(D = 1|X = x) + F_{1|0X}(y|x)\Pr(D = 0|X = x)$. Since $F_{1|0X}(y|x) \geq F_{1|1X}(y|x)$ for all $y$, replacing $F_{1|0X}(y|x)$ with $F_{1|1X}(y|x)$ in the above decomposition of $F_{1|X}(y|x)$ results in that $F_{1|X}^{L,FSD1}(y|x) = \Pr(Y \leq y|D = 1, X = x) = \Pr(Y_1 \leq y|D = 1, X = x)$ and that $F_{1|X}(y|x) = \Pr(Y_1 \leq y|X = x) \geq F_{1|X}^{L,FSD1}(y|x)$. Note that $F_{1|X}^{U,FSD1}(y|x)$ and $F_{0|X}^{L,FSD1}(y|x)$ are identical to $F_{1|X}^{U}(y|x)$ and $F_{0|X}^{L}(y|x)$ in (7) and hence, they are valid. Since $F_{0|X}(y|x) = F_{0|X}^{L,FSD1}(y|x) + \Pr(D = 1|X = x) \cdot F_{0|1X}(y|x)$, for all $y \in \mathbb{R}$, we obtain that $F_{0|X}^{U,FSD1}(y|x) \leq \Pr(Y_0 \leq y|D = 0, X = x) = \Pr(Y \leq y|D = 0, X = x)$ = $F_{0|X}^{U,FSD1}(y|x)$.

E.4 Proof of Theorem 2.3

Proof. The proof proceeds as follows. Since we have, for all $y \in \mathbb{R}$, $\Pr(Y_1 \leq y|D = 0, X = x) \leq \Pr(Y_0 \leq y|D = 0, X = x)$, we obtain the upper bound on $F_{1|X}(y|x)$ by replacing $\Pr(Y_1 \leq y|D = 0, X = x)$ with $\Pr(Y_0 \leq y|D = 0, X = x)$. Similarly, we have $\Pr(Y_1 \leq y|D = 1, X = x) \leq \Pr(Y_0 \leq y|D = 1, X = x)$, and this inequality is used to establish the lower bound on $F_{0|X}(y|x)$. Note that $\Pr(Y \leq y|D = d, X = x) = \Pr(Y_d \leq y|D = d, X = x)$ for $d \in \{0, 1\}$ and that $\Pr(Y \leq y|D = 1, X = x)\Pr(D = 1|X = x) + \Pr(Y \leq y|D = 0, X = x)\Pr(D = 0|X = x) = \Pr(Y \leq y|X = x)$, and this completes the proof.

E.5 Proof of Corollary 2.4

Proof. The results in Corollary 2.4 are directly implied by Theorems 2.2 and 2.3. Recall that

$$F_{1|X}^{U,FSD2}(y|x) = F_{1|1X}(y|x)\Pr(D = 1|X = x) + F_{0|0X}(y|x)\Pr(D = 0|X = x)$$

$$\leq F_{1|1X}(y|x)\Pr(D = 1|X = x) + \Pr(D = 0|X = x) = F_{1|X}^{U,FSD1}(y|x)$$

47
and

\begin{align*}
F_{1|X}^{L,FSD1}(y|x) &= \Pr(Y \leq y|D = 1, X = x) 
\geq F_{1|X}(y|x) \Pr(D = 1|X = x) = F_{1|X}^{L,FSD2}(y|x),
\end{align*}

and therefore these establish the bounds on \( F_{1|X}(y|x) \). One can use a similar argument for the bounds on \( F_{0|X}(y|x) \).

For any real sequences \((a_n)\) and \((b_n)\), \(a_n \preceq b_n\) means that there is a constant \(C\), not depending on \(n\), such that \(|a_n| \leq C \cdot |b_n|\) for all \(n \in \mathbb{N}\). For a set \(A \subseteq \mathbb{R}\), \(l^\infty(A)\) denotes the set of uniformly bounded functions on the set \(A\). Let \((Z_n)\) be a sequence of random variables and \(Z\) be a random variable. \(Z_n \xrightarrow{P} Z\) means that \(Z_n\) converges in probability to \(Z\). We also denote the weak convergence of \(Z_n\) to \(Z\) by \(Z_n \Rightarrow Z\). We abbreviate Vapnik-Červonenkis to VC. For a class of functions, \(F\), a probability measure \(Q\) and \(\epsilon > 0\), \(N(\epsilon, F, L_r(Q))\) denotes the covering number which is the minimum number of \(L_r(Q)\) \(\epsilon\)-balls that cover \(F\), where \(r \geq 1\). For a pseudo-metric space \((T, \rho)\), the diameter of \(T\) is \(\sup_{s,t \in T} \rho(s, t)\). Throughout this section, I sometimes use the same notation but for some possibly different object.

E.6 Proof of Theorem 4.1

We first provide the forms of covariance kernels:

\begin{align*}
H_{1,x}(s, t) &= \int K^2(u)du \left\{ F_{1|X}(\min(s, t)|x) - F_{1|X}(s|x) \cdot F_{1|X}(t|x) \right\} \frac{1}{p_0(x) \cdot f_X(x)}, \\
H_{0,x}(s, t) &= \int K^2(u)du \left\{ F_{0|X}(\min(s, t)|x) - F_{0|X}(s|x) \cdot F_{0|X}(t|x) \right\} \frac{1}{(1 - p_0(x)) \cdot f_X(x)}.
\end{align*}

Proof. I only consider the weak convergence of \(\sqrt{nh_n^d} (\hat{F}_{1|X,n}(\cdot|x) - F_{1|X}(\cdot|x))\), and for the other objects one can use a similar argument. Note that
\[
\hat{F}_{1|X_n}(y|x) - F_{1|X}(y|x) = \frac{\sum_{i}^{n} 1(Y_i \leq y) \cdot D_i \cdot K\left(\frac{X_i-x}{h_n}\right)}{\sum_{i}^{n} D_i K\left(\frac{X_i-x}{h_n}\right)} - F_{1|X}(y|x)
\]

\[
= \frac{1}{p_0(x) \cdot f_X(x) + o_p(1)} \cdot \frac{1}{nh_n^{d_x}} \sum_{i}^{n} \left\{ \{1(Y_i \leq y) - F_{1|X}(y|x)\} D_i \cdot K\left(\frac{X_i-x}{h_n}\right) \right\}
\]

\[
= \frac{1}{p_0(x) \cdot f_X(x) + o_p(1)} \cdot \frac{1}{nh_n^{d_x}} \sum_{i}^{n} \left\{ \{1(Y_i \leq y) - F_{1|X}(y|X_i)\} D_i \cdot K\left(\frac{X_i-x}{h_n}\right) \right\}
\]

under Assumption 4.2, 4.3, 4.4, and 4.5. To show the weak convergence, I verify the conditions for the functional central limit theorem in Pollard (1990, Theorem 10.6). We first consider the second term in the above equation. Note that

\[
\frac{1}{nh_n^{d_x}} \sum_{i}^{n} \mathbb{E} \left[ \left\{ \{1(Y_i \leq y) - F_{1|X}(y|X_i)\} D_i \cdot K\left(\frac{X_i-x}{h_n}\right) \right\} \right]
\]

\[
= \frac{1}{h_n^{d_x}} \int \{F_{1|X}(y|t) - F_{1|X}(y|x)\} p_0(t) K\left(\frac{t-x}{h_n}\right) f_X(t) dt
\]

\[
= \int \left\{ F_{1|X}^{(1)}(y|x) h_n \sum_{j=1}^{d_x} u_j + \frac{1}{2} F_{1|X}^{(2)}(y|x) h_n^2 \sum_{j=1}^{d_x} u_j^2 \right\}
\]

\[
\times \left\{ p_0(x) + \frac{1}{h_n} \sum_{j=1}^{d_x} u_j + \frac{1}{2} p_0^{(2)}(x) h_n^2 \sum_{j=1}^{d_x} u_j \right\}
\]

\[
\times \left\{ f_X(x) + f_X^{(1)}(x) h_n \sum_{j=1}^{d_x} u_j + \frac{1}{2} f_X^{(2)}(x) h_n^2 \sum_{j=1}^{d_x} u_j \right\} K(u) du,
\]

where \( \hat{x}, \tilde{x}, \tilde{x} \) are some values between \( x + th_n \) and \( x \), under Assumptions 4.2, 4.3 and 4.4. Since this bias term is \( O\left(h_n^2\right) \) and \( \sqrt{nh_n^{d_x+1}} \rightarrow 0 \) by Assumption 4.5, this term is \( o\left(\left(nh_n^{d_x}\right)^{-1/2}\right)\).

Now we consider the term

\[
\frac{1}{\sqrt{nh_n^{d_x}}} \sum_{i}^{n} \left\{ \{1(Y_i \leq y) - F_{1|X}(y|X_i)\} D_i \cdot K\left(\frac{X_i-x}{h_n}\right) \right\}.
\]

For all \( \omega \in \Omega \), define \( f_{n\omega}(\omega, y|x) \equiv \frac{1}{\sqrt{nh_n^{d_x}}} \sum_{i}^{n} \left\{ \{1(Y_i \leq y) - F_{1|X}(y|X_i)\} D_i \cdot K\left(\frac{X_i-x}{h_n}\right) \right\} \).

Then, we have \( \mathbb{E}[f_{n\omega}(\omega, y|x)] = 0 \) by the law of iterated expectations. Define an envelope function

\[
F_{n\omega}(x) \equiv \frac{1}{\sqrt{nh_n^{d_x}}} K\left(\frac{X_i-x}{h_n}\right), \text{ where } p \equiv \inf_{x \in \mathcal{X}} p_0(x) > 0 \text{ and } f \equiv \inf_{x \in \mathcal{X}} f_X(x) > 0.
\]
Let $\mathcal{C} = \{(-\infty, y] : y \in \mathcal{Y}\}$ and define

$$
\mathcal{F}_{1n} \equiv \left\{ \frac{1}{p_0(x) \cdot f_X(x) \sqrt{nh_n^d}} K\left( \frac{X - x}{h_n} \right) \cdot D \cdot 1_{C} : C \in \mathcal{C} \right\},
$$

$$
\mathcal{F}_{2n} \equiv \left\{ -\frac{1}{p_0(x) \cdot f_X(x) \sqrt{nh_n^d}} K\left( \frac{X - x}{h_n} \right) \cdot D \cdot F_{1|X}(y|X) : y \in \mathcal{Y} \right\},
$$

and $\mathcal{F}_n \equiv \{f_n(\omega, y; x) : y \in \mathcal{Y}\} = \mathcal{F}_{1n} + \mathcal{F}_{2n}$. The VC index of $\mathcal{C}$ is 2, and thus it follows from Theorem 9.2 and Lemma 9.9 in Kosorok (2008) that $N(\epsilon, \mathcal{F}_{1n}, L_2(P)) \lesssim \epsilon^{-2}$ for any probability measure $P$. Since $F_{1|X}(\cdot|X)$ is a monotone increasing function almost all $X$, applying Lemmas 9.9 and 9.10 in Kosorok (2008) implies that the VC index of $\mathcal{F}_{2n}$ is equal to 2, and therefore $N(\epsilon, \mathcal{F}_{2n}, L_2(P)) \lesssim \epsilon^{-2}$ for any probability measure $P$. By the proof of Theorem 3 in Andrews (1994), one obtains that

$$
\sup_Q \log N(\epsilon, \mathcal{F}_n, L_2(Q)) \leq \sup_Q \log N\left( \frac{\epsilon}{2}, \mathcal{F}_{1n}, L_2(Q) \right) + \sup_Q \log N\left( \frac{\epsilon}{2}, \mathcal{F}_{2n}, L_2(Q) \right) \lesssim \log \epsilon^{-2},
$$

where the supremum is taken over all finite probability measures $Q$. Therefore,

$$
\int_0^1 \sqrt{\sup_Q \log N(\epsilon, \mathcal{F}_n, L_2(Q))} d\epsilon \lesssim \int_0^1 \sqrt{\log(1/\epsilon^2)} d\epsilon = \int_0^\infty u^{1/2} \exp(-u/2) du < \infty,
$$

which implies that $\mathcal{F}_n$ satisfies Pollard’s entropy condition. Since Pollard’s entropy condition implies manageability (see Andrews (1994, p.2284)), condition (i) of Theorem 10.6 in Pollard (1990) is satisfied.

Let $Z_n(\omega, t; x) \equiv \sum_{i=1}^{n} f_{ni}(\omega, t; x)$ and consider $\lim_{n \to \infty} \mathbb{E}[Z_n(\omega, s; x)Z_n(\omega, t; x)]$ for any
\[ s, t \in \mathcal{X} \]. Then,

\[
\lim_{n \to \infty} \mathbb{E}[Z_n(\omega, s; x)Z_n(\omega, t; x)] = \lim_{n \to \infty} \mathbb{E} \left[ \sum_{i} f_{ni}(\omega, s; x) \cdot \sum_{j} f_{nj}(\omega, t; x) \right]
\]

(by i.i.d assumption)

\[
= \lim_{n \to \infty} \mathbb{E} \left[ \sum_{i} f_{ni}(\omega, s; x) f_{ni}(\omega, t; x) \right] \quad \text{(by i.i.d assumption)}
\]

is well-defined under Assumptions 4.3, 4.4, and 4.5. Therefore, condition (ii) of Theorem 10.6 in Pollard (1990) is satisfied.

To verify condition (iii) of Theorem 10.6 in Pollard (1990), we can show that

\[
\sum_i \mathbb{E}[F_{ni}^2(x)] \lesssim f_X(x) + O(h_n^{2d_x}) < \infty,
\]

which implies condition (iii) of Theorem 10.6 in Pollard (1990).

Take any \( \epsilon > 0 \), then,

\[
\mathbb{E}[F_{ni}^{2+\epsilon}] \lesssim \frac{1}{(nh_n^{d_x})^{1+\epsilon/2}} \int K^{2+\epsilon}(\frac{t-x}{h_n}) f_X(t) dt
\]

by using the standard arguments for kernel estimation. Note that the second line holds due to the kernel function being bounded. Therefore, \( \sum_i \mathbb{E}[F_{ni}^2(x)] \lesssim f_X(x) + O(h_n^{2d_x}) < \infty \),

which implies condition (iii) of Theorem 10.6 in Pollard (1990).
under Assumptions 4.2, 4.4, and 4.5. Therefore, one can show that for any \( \epsilon_0 > 0 \), we have

\[
\sum_i \mathbb{E} \left[ F_{n_i}^2 1(F_{n_i} > \epsilon_0) \right] = \sum_i \mathbb{E} \left[ \frac{F_{n_i}^{2+\epsilon} + F_{n_i}^{-\epsilon}}{\epsilon} 1(F_{n_i} > \epsilon_0) \right] \\
\leq \sum_i \frac{1}{\epsilon^2} \mathbb{E} \left[ F_{n_i}^{2+\epsilon} \right] \leq \frac{1}{\epsilon^2} \cdot o(1) = o(1),
\]

and thus condition (iv) of Theorem 10.6 in Pollard (1990) is satisfied. Lastly, define

\[
\rho^2_n(s, t) \equiv \sum_i \mathbb{E} |f_{n_i}(\cdot, s; x) - f_{n_i}(\cdot, t; x)|^2
\]

for any \( s, t \in \mathcal{Y} \). Without loss of generality, we assume that \( s \geq t \). Then,

\[
\sum_i \mathbb{E} |f_{n_i}(\cdot, s; x) - f_{n_i}(\cdot, t; x)|^2 \\
= \frac{1}{nh_n^d \cdot p_0(x)^2 \cdot f_X(x)^2} \sum_i \mathbb{E} \left| K \left( \frac{X_i - x}{h_n} \right) \cdot \left( \left\{ 1(\mathcal{Y}_i \leq s) - 1(\mathcal{Y}_i \leq t) \right\} - \{F_{1|X}(s|X_i) - F_{1|X}(t|X_i)\} \right)^2 \right| \\
= \frac{1}{nh_n^d \cdot p_0(x)^2 \cdot f_X(x)^2} \sum_i \mathbb{E} \left[ p_0(X_i) \cdot K^2 \left( \frac{X_i - x}{h_n} \right) \cdot \left\{ F_{1|X}(s|X_i) + F_{1|X}(t|X_i) - 2F_{1|X}(t|X_i) - (F_{1|X}(s|X_i) - F_{1|X}(t|X_i))^2 \right\} \right] \\
= \int \frac{p_0(z) \cdot K^2 \left( \frac{z - x}{h_n} \right) \cdot \left\{ F_{1|X}(s|z) + F_{1|X}(t|z) - 2F_{1|X}(t|z) - (F_{1|X}(s|z) - F_{1|X}(t|z))^2 \right\} dz}{p_0(x)^2 \cdot f_X(x)^2 \cdot h_n^d} \\
= \int \frac{K^2(u) du}{p_0(x) \cdot f_X(x)} \cdot \left\{ F_{1|X}(s|x) + F_{1|X}(t|x) - 2F_{1|X}(t|x) - (F_{1|X}(s|x) - F_{1|X}(t|x))^2 \right\} + O \left( h_n^d \right) \\
= \int \frac{K^2(u) du}{p_0(x) \cdot f_X(x)} \left\{ (F_{1|X}(s|x) - F_{1|X}(t|x)) \cdot (1 - (F_{1|X}(s|x) - F_{1|X}(t|x))) \right\} + O \left( h_n^d \right),
\]

(E.1)

where the term \( O \left( h_n^d \right) \) holds uniformly in \( s \) and \( t \). Letting \( \rho(s, t) \equiv \lim_{n \to \infty} \rho_n(s, t) \), we have

\[
\rho^2(s, t) = \int \frac{K^2(u) du}{p_0(x) \cdot f_X(x)} \left\{ (F_{1|X}(s|x) - F_{1|X}(t|x)) \cdot (1 - (F_{1|X}(s|x) - F_{1|X}(t|x))) \right\}
\]

for \( s \geq t \). Take any \( \{s_n\} \) and \( \{t_n\} \) such that \( \rho(s_n, t_n) \to 0 \). Since the term \( O \left( h_n^d \right) \) in (E.1) holds uniformly in \( s \) and \( t \) and the conditional distribution is continuous, \( \rho_n(s_n, t_n) \to 0 \), and this establishes that condition (v) of Theorem 10.6 in Pollard (1990) is met.

In all, all conditions of Theorem 10.6 in Pollard (1990) are satisfied, and thus applying Theorem 10.6 in Pollard (1990) results in that

\[
\sqrt{nh_n^d} \left( \hat{F}_{1|X,n}(|x) - F_{1|X}(\cdot|x) \right) \Rightarrow \mathcal{G}_{1,x}(\cdot)
\]

52
in $l^\infty(\mathcal{Y})$, where $\mathbb{G}_{1,x}$ is a zero mean Gaussian process with covariance kernel $H_{1,x}(\cdot, \cdot)$.

Similarly, we can show that $\sqrt{n}h_n^{\mathcal{G}_1, x} \left( \hat{F}_{0|X,n}(\cdot|x) - F_{0|X}(\cdot|x) \right) \Rightarrow \mathbb{G}_{0,x}(\cdot)$ in $l^\infty(\mathcal{Y})$ with covariance kernel

$$H_{0,x}(s, t) \equiv \int K^2(u) du \left\{ F_{0|X}(\min(s, t)|x) - F_{0|X}(s|x) \cdot F_{0|X}(t|x) \right\} \frac{1}{(1 - p_0(x)) \cdot f_X(x)}$$

Since each component weakly converges to a Gaussian process in $l^\infty(\mathcal{Y})$, the vector consisting of these components also weakly converges to $\mathbb{G}_x(\cdot)$ (cf. Van der Vaart (1998, p.270)), and this ends the proof.

\section*{E.7 Proof of Theorem 4.2}

\textit{Proof.} The weak convergence is a direct result of Theorem 2.1 of Fang and Santos (2019). Specifically, Theorem 4.1 implies that Assumptions 1 and 2 of Fang and Santos (2019) hold with $\phi L(\cdot)$ and $\phi U(\cdot)$, and therefore this ends the proof.

\section*{E.8 Proof of Theorem 4.3}

We verify the sufficient conditions for Theorem 4.3 of Firpo et al. (2021) to prove Theorem 4.3. To this end, we first provide useful lemmas.

\textbf{Lemma E.1.} Suppose that all of the conditions in Theorem 4.1 hold and let $x \in \text{int}(\mathcal{X})$ be fixed. Then, the weak convergence presented in Theorem 4.1 is uniform in underlying probability measures.

\textit{Proof.} We have already shown that $\int_0^1 \sqrt{\sup_Q \log N(\epsilon, \mathcal{F}_n, L_2(Q))} d\epsilon < \infty$ in the proof Theorem 4.1. Since the diameter of $\mathcal{F}_n$ is finite for every $n$, this implies that

$$\int_0^\infty \sqrt{\sup_Q \log N(\epsilon, \mathcal{F}_n, L_2(Q))} d\epsilon < \infty.$$ 

In addition, the condition on the envelope function that is assumed in Theorem 2.8.3 of van der Vaart and Wellner (1996) holds. Therefore, the weak convergence is uniform in the underlying probability measure by Theorem 2.8.3 of van der Vaart and Wellner (1996). \hfill $\blacksquare$
The following lemma establishes the validity of the bootstrap for the kernel estimators of the conditional distributions of the potential outcomes given $X$.

**Lemma E.2.** Suppose that all of the conditions in Theorem 4.3 hold. Let $x \in \text{int}(X)$ be fixed. Then, we have, conditional on the data,

$$
\sqrt{n h_n^{d_x}} \mathbb{E}_{Y \mid X, n}^* \Rightarrow \mathcal{G}_x(\cdot)
$$

in $(l^\infty(\mathcal{Y}))^\mathbb{R}$.

**Proof.** I only consider $\frac{1}{\sqrt{n h_n^{d_x}}} \sum_i B_i \hat{\psi}_{1,i}(\cdot|x) \Rightarrow \mathcal{G}_{1,x}(\cdot)$ conditional on the data, and one can prove the results for other components by using a similar argument. In this proof, we let $\mathbb{E}_B[\cdot]$ denote the conditional expectation operator on the data. Recall that

$$
\frac{1}{\sqrt{n h_n^{d_x}}} \sum_i B_i \cdot \hat{\psi}_{1,i}(y|x) = \frac{1}{\sqrt{n h_n^{d_x}}} \sum_i B_i \{1(Y_i \leq y) - \hat{F}_{1|X,n}(y|x)\} \cdot D_i \cdot K\left(\frac{X_i - x}{h_n}\right)
$$

$$
= \frac{1}{\hat{p}_n(x) \cdot f_X(x)} \frac{1}{\sqrt{n h_n^{d_x}}} \sum_i B_i \{1(Y_i \leq y) - \hat{F}_{1|X,n}(y|x)\} \cdot D_i \cdot K\left(\frac{X_i - x}{h_n}\right)
$$

$$
= \frac{1}{\hat{p}_n(x) \cdot f_X(x)} \frac{1}{\sqrt{n h_n^{d_x}}} \sum_i B_i \{F_{1|X}(y|x) - \hat{F}_{1|X,n}(y|x)\} \cdot D_i \cdot K\left(\frac{X_i - x}{h_n}\right)
$$

where $\hat{f}_X(x) = \frac{1}{n h_n^{d_x}} \sum_i K\left(\frac{X_i - x}{h_n}\right)$. I first show that $\frac{1}{\hat{p}_n(x) \cdot f_X(x)} \frac{1}{\sqrt{n h_n^{d_x}}} \sum_i B_i \{F_{1|X}(y|x) - \hat{F}_{1|X,n}(y|x)\} \cdot D_i \cdot K\left(\frac{X_i - x}{h_n}\right)$ converges to a zero process conditional on sample path by verifying conditions (i)–(v) of Theorem 10.6 in Pollard (1990). Define

$$
f_i^u(y|x) \equiv \frac{1}{\hat{p}_n(x) \cdot f_X(x) \sqrt{n h_n^{d_x}}} B_i \{F_{1|X}(y|x) - \hat{F}_{1|X,n}(y|x)\} \cdot D_i \cdot K\left(\frac{X_i - x}{h_n}\right),
$$

$$
F_i^u(x) \equiv \frac{B_i}{\hat{p}_n(x) \cdot f_X(x) \sqrt{n h_n^{d_x}}} K\left(\frac{X_i - x}{h_n}\right).
$$

Since $\hat{F}_{1|X,n}(\cdot|x)$ and $F_{1|X}(\cdot|x)$ are monotone increasing functions and bounded by 1, $\{\hat{F}_{1|X,n}(y|x) : y \in \mathcal{Y}\}$ and $\{F_{1|X}(y|x) : y \in \mathcal{Y}\}$ satisfy the Pollard’s entropy condition by the same argument in the proof of Theorem 4.1. Therefore, these classes are manageable.

For the second condition of Theorem 10.6 in Pollard (1990), note that $\mathbb{E}_B[f_i^u(y|x)] = 0$.
since $E[B] = 0$ and $B$ is independent of data. In consequence, we have, for any $y_1, y_2 \in \mathcal{Y}$,

$$
\left| E_B \left[ \sum_i^n f_i^u(y_1|x) \sum_j^n f_j^u(y_2|x) \right] \right| = E_B \left[ \sum_i^n f_i^u(y_1|x) \cdot f_i^u(y_2|x) \right] \leq \frac{1}{(\hat{p}_n(x)\hat{f}_X(x))^2} \sup_y \left| F_{1|X}(y|x) - \hat{F}_{1|X,n}(y|x) \right|^2 \cdot \frac{1}{n h_n^{d_x}} \sum_i^n K^2 \left( \frac{X_i - x}{h_n} \right),
$$

conditional on sample path. We have $\sqrt{nh_n^{d_x}} \left( F_{1|X}(\cdot|x) - \hat{F}_{1|X,n}(\cdot|x) \right) \Rightarrow G_{1,x}(\cdot)$ in $l^\infty(\mathcal{Y})$ from Theorem 4.1, and therefore $\sup_y |F_{1|X}(y|x) - \hat{F}_{1|X,n}(y|x)| \overset{p}{\rightarrow} 0$. We also have $\hat{f}_X(x) \overset{p}{\rightarrow} f_X(x) > 0$, $\hat{p}_n(x) \overset{p}{\rightarrow} p_0(x)$, and $\frac{1}{n h_n^{d_x}} \sum_i^n K^2 \left( \frac{X_i - x}{h_n} \right) \overset{p}{\rightarrow} \int K^2(u) du$ under the set of assumptions. In all, we obtain that

$$
H^B_{1,x}(y_1, y_2) \equiv E_B \left[ \sum_i^n f_i^u(y_1|x) \sum_j^n f_j^u(y_2|x) \right] \overset{p}{\rightarrow} 0,
$$

and thus condition (ii) of Theorem 10.6 in Pollard (1990) is satisfied with $H^B_1(y_1, y_2)$.

Recall that

$$
\sum_i^n E_B \left[ F_i^u(x)^2 \right] = \frac{1}{(\hat{p}_n(x)\hat{f}_X(x))^2} \frac{1}{n h_n^{d_x}} \sum_i^n K^2 \left( \frac{X_i - x}{h_n} \right)
$$

conditional on sample path. Therefore, $\sum_i^n E_B \left[ F_i^u(x)^2 \right] \overset{p}{\rightarrow} \frac{1}{(p_0(x)\tilde{f}_X(x))^2} \int K^2(u) du < \infty$, which implies that condition (iii) is satisfied. In addition, it can be easily shown that

$$
\sum_i^n E_B \left[ F_i^u(x)^2 \cdot 1(F_i^u(x) > \epsilon) \right] \overset{p}{\rightarrow} 0
$$

for each $\epsilon > 0$, so condition (iv) is also satisfied.
Lastly, note that for any \( s, t \in \mathcal{Y} \) such that \( s \geq t \),

\[
\rho_n(s, t)^2 \equiv \sum_i \mathbb{E}_B |f_i^n(s|x) - f_i^n(t|x)|^2
\]

\[
= \left\{ (F_{1|X}(s|x) - \hat{F}_{1|X,n}(s|x)) - (F_{1|X}(t|x) - \hat{F}_{1|X,n}(t|x)) \right\}^2
\]

\[
\times \frac{1}{\hat{p}_n(x)\hat{f}_X(x)} \frac{1}{nh_n^d} \sum_i D_i K^2 \left( \frac{X_i - x}{h_n} \right)
\]

\[
\overset{p}{\Rightarrow} \int \frac{K^2(u)du}{p_0(x) \cdot f_X(x)} \left\{ (F_{1|X}(s|x) - F_{1|X}(t|x)) \cdot (1 - (F_{1|X}(s|x) - F_{1|X}(t|x))) \right\}
\]

\[
= \rho_n(s, t)
\]

by the same argument above. Hence, \( \rho_n(s, t) \) is well-defined, and if \( \rho_n(s_n, t_n) \to 0 \), then

\( \rho_n(s_n, t_n) \to 0 \). Therefore, condition (v) of Theorem 10.6 in Pollard (1990) is met. Applying Theorem 10.6 in Pollard (1990) leads to that, conditional on data,

\[
\frac{1}{\hat{p}_n(x)\hat{f}_X(x)} \frac{1}{nh_n^d} \sum_i B_i \{ F_{1|X}(\cdot|x) - \hat{F}_{1|X,n}(\cdot|x) \} \cdot D_i \cdot K \left( \frac{X_i - x}{h_n} \right)
\]

converges to a zero process in \( l^\infty(\mathcal{Y}) \).

By applying Theorem 10.4 in Kosorok (2008) and Theorem 4.1, we have

\[
\frac{1}{\hat{p}_n(x)\hat{f}_X(x)} \frac{1}{nh_n^d} \sum_i B_i \{ 1(Y_i \leq \cdot) - F_{1|X}(\cdot|x) \} \cdot D_i \cdot K \left( \frac{X_i - x}{h_n} \right) \Rightarrow \mathcal{G}_{1,x}(\cdot) \text{ in } l^\infty(\mathcal{Y})
\]

conditional on sample path. Therefore,

\[
\frac{1}{\sqrt{nh_n^d}} \sum_i B_i \hat{\psi}_{1,i}(\cdot|x) \Rightarrow \mathcal{G}_{1,x}(\cdot) \text{ in } l^\infty(\mathcal{Y})
\]

conditional on the data. Using a similar argument, we can show that

\[
\frac{1}{\sqrt{nh_n^d}} \sum_i B_i \hat{\psi}_{0,i}(\cdot|x) \Rightarrow \mathcal{G}_{0,x}(\cdot) \text{ in } l^\infty(\mathcal{Y})
\]

conditional on the data. In all, \( \sqrt{nh_n^d} \hat{\mathbb{E}}_{Y,X,n}(\cdot|x; B) \Rightarrow \mathcal{G}_x(\cdot) \) conditional on the data. \( \blacksquare \)

**Proof of Theorem 4.3**

*Proof.* We verify the assumption in Theorem 4.3 of Firpo et al. (2021) (i.e., Assumptions A1-A3). Theorem 4.1 implies Assumption A1 with \( r_n = \sqrt{nh_n^d} \). Assumption A2 in
Firpo et al. (2021) is satisfied because Lemma E.1 shows that the weak convergence in Theorem 4.1 is uniform in underlying probability measures. Lastly, Lemma E.2 establishes the bootstrap validity. Lemma E.2, together with Lemma A.2 of Linton et al. (2010), implies that Assumption A3 in Firpo et al. (2021) is met. In all, applying Theorem 4.3 of Firpo et al. (2021) ends the proof.

E.9 Proof of Theorem 4.4

We first present the forms of covariance kernels in the theorem:

\[ H_{1,x}(s,t) \equiv \int K^2(u)du \left\{ F_{1|1X}(\min(s,t)|x) - F_{1|1X}(s|x) \cdot F_{1|1X}(t|x) \right\} \frac{1}{p_0(x) \cdot f_X(x)}, \]

\[ H_{0,x}(s,t) \equiv \int K^2(u)du \left\{ F_{0|0X}(\min(s,t)|x) - F_{0|0X}(s|x) \cdot F_{0|0X}(t|x) \right\} \frac{1}{(1 - p_0(x)) \cdot f_X(x)}, \]

\[ H_Y(s,t) \equiv \int K^2(u)du \left\{ F_Y|X(\min(s,t)|x) - F_Y|X(s|x) \cdot F_Y|X(t|x) \right\} \frac{1}{f_X(x)}. \]

Proof. Note that

\[ \sqrt{nh^{-d}_{11}} \left( \hat{F}_{1|1X,n}(y|x) - F_{1|1X}(y|x) \right) \]

\[ = \frac{1}{\sqrt{nh^{-d}_{11}}} \sum_{n} \{ 1(Y_i \leq y) - F_{1|1X}(y|x) \} D_i K \left( \frac{X_i - x}{h_n} \right) \]

\[ = \frac{1}{p_0(x) f_X(x) + o_p(1)} \frac{1}{\sqrt{nh^{-d}_{11}}} \sum_{n} \{ 1(Y_i \leq y) - F_{1|1X}(y|x) \} D_i K \left( \frac{X_i - x}{h_n} \right) \]

\[ + \frac{1}{p_0(x) f_X(x) + o_p(1)} \frac{1}{\sqrt{nh^{-d}_{11}}} \sum_{n} \{ F_{1|1X}(y|X_i) - F_{1|1X}(y|x) \} D_i K \left( \frac{X_i - x}{h_n} \right) \]

Using the same argument for the proof of Theorem 4.1,

\[ \sqrt{nh^{-d}_{11}} \left( \hat{F}_{1|1X,n}(\cdot|x) - F_{1|1X}(\cdot|x) \right) \Rightarrow \mathcal{G}^{c}_1(\cdot) \text{ in } l^\infty(Y), \]
where $G^{e}_{1,x}(\cdot)$ is a Gaussian process with zero mean and covariance kernel

$$H^{e}_{1,x}(s,t) \equiv \int K^2(u)du \left\{ F_{1|1X}(\min(s,t)|x) - F_{1|1X}(s|x) \cdot F_{1|1X}(t|x) \right\} \frac{1}{p_0(x) \cdot f_X(x)}.$$  

Similarly,

$$\sqrt{nh_n}d_x (\hat{F}_{0|0X,n}(\cdot|x) - F_{0|0X}(\cdot|x)) \Rightarrow G^{e}_{0,x}(\cdot) \text{ in } l^\infty(Y),$$

where $G^{e}_{0}(\cdot)$ is a Gaussian process with zero mean and covariance kernel

$$H^{e}_{0,x}(s,t) \equiv \int K^2(u)du \left\{ F_{0|0X}(\min(s,t)|x) - F_{0|0X}(s|x) \cdot F_{0|0X}(t|x) \right\} \frac{1}{(1 - p_0(x)) \cdot f_X(x)}.$$  

Lastly, I prove the weak convergence of that $\sqrt{nh_n}d_x (\hat{F}_{Y|X,n}(\cdot|x) - F_{Y|X}(\cdot|x))$. Note that $\hat{F}_{Y|X,n}(\cdot|x) - F_{Y|X}(\cdot|x)$ is given by

$$\hat{F}_{Y|X,n}(\cdot|x) = \frac{1}{nh_n} \sum_{i=1}^{n}(1(Y_i \leq y) - F_{Y|X}(y|X_i)) K(X_i-x_{nh}) f_X(X_i) + o_p(1).$$  

The latter term is $o_p((nh_n)^{-1/2})$ under the conditions in the theorem. We then focus on the first term. The Pollard’s entropy condition is satisfied for

$$F^{e}_n \equiv \left\{ \frac{1}{f_X(x)}(1(Y \leq y) - F_{Y|X}(y|X)) K\left(\frac{X-x}{h_n}\right) : y \in Y \right\}$$

by the same argument in the proof of Theorem 4.1. For given $\omega \in \Omega$, let

$$f_{ni}(\omega; y; x) \equiv \frac{1}{\sqrt{nh_n}d_x f_X(x)} (1(Y_i \leq y) - F_{Y|X}(y|X_i)) K\left(\frac{X_i-x}{h_n}\right).$$
and $Z_n(\omega,y;x) \equiv \sum_i^n f_{ni}(\omega,y;x)$. Then, for any $s,t \in \mathcal{Y}$,

$$
\lim_{n \to \infty} \mathbb{E}[Z_n(\omega,s;x)Z_n(\omega,t;x)] = \lim_{n \to \infty} \mathbb{E} \left[ \sum_i^n f_{ni}(\omega,s;x)f_{ni}(\omega,t;x) \right] = \lim_{n \to \infty} \int K^2(u)du \left\{ F_{Y|X}(\min(s,t)|x) - F_{Y|X}(s|x) \cdot F_{Y|X}(t|x) + O\left(h_n^{d_x}\right) \right\} \frac{1}{f_X(x)} = \int K^2(u)du \left\{ F_{Y|X}(\min(s,t)|x) - F_{Y|X}(s|x) \cdot F_{Y|X}(t|x) \right\} \frac{1}{f_X(x)} \equiv H_Y(s,t)
$$

by using the same argument for the proof of Theorem 4.1. The remaining conditions can be verified by the same way as before, and thus

$$
\sqrt{nh_n^{d_x}} \left( \hat{F}_{|X,n}(\cdot|x) - F_{Y|X}(\cdot|x) \right) \Rightarrow \mathbb{G}_{Y,x}(\cdot) \text{ in } l^\infty(\mathcal{Y}),
$$

where $\mathbb{G}_{Y,x}(\cdot)$ is a Gaussian process with zero mean and covariance kernel $H_Y(s,t)$.

E.10 Proof of Theorem 4.5

Proof. The only thing to show is the bootstrap validity, but this can be shown by using the same way of the proof of Theorem 4.3, together with Theorem 4.4. Therefore, we omit the proof of this theorem.

E.11 Proof of Theorem C.1

Let $x_1 \in \text{int}(X_1)$ and define $||K_1||_2^2 \equiv \int K_1^2(u)du$, $G_{1|X_1}(y|x_1) \equiv \mathbb{E} \left[ \frac{F_{1|X}(y|X)}{p_0(X)} \bigg| X_1 = x_1 \right]$, and $G_{0|X_1}(y|x_1) \equiv \mathbb{E} \left[ \frac{F_{0|X}(y|X)}{1-p_0(X)} \bigg| X_1 = x_1 \right]$.

Lemma E.3. Suppose that conditions in Theorem C.1 hold. Then, for any given $x_1 \in \text{int}(X_1)$,

$$
\sqrt{nh_n^{d_x}} \left( \hat{F}_{1|X_1,n}(\cdot|x_1) - F_{1|X_1}(\cdot|x_1) \right) \Rightarrow \mathbb{G}_1(\cdot) \text{ in } l^\infty(\mathcal{Y}),
$$

$$
\sqrt{nh_n^{d_x}} \left( \hat{F}_{0|X_1,n}(\cdot|x_1) - F_{0|X_1}(\cdot|x_1) \right) \Rightarrow \mathbb{G}_0(\cdot) \text{ in } l^\infty(\mathcal{Y}),
$$

59
where \( \tilde{G}_1 \) and \( \tilde{G}_0 \) are Gaussian processes with mean zero and covariance kernels

\[
\tilde{H}_1(y_1, y_2) \equiv \min \left( G_{1|X_1}(y_1|x_1), G_{1|X_1}(y_2|x_1) \right) - F_{1|X_1}(y_1|x_1)F_{1|X_1}(y_2|x_1) \frac{||K_1||_2^2}{f_{X_1}(x_1)},
\]

\[
\tilde{H}_0(y_1, y_2) \equiv \min \left( G_{0|X_1}(y_1|x_1), G_{0|X_1}(y_2|x_1) \right) - F_{0|X_1}(y_1|x_1)F_{0|X_1}(y_2|x_1) \frac{||K_1||_2^2}{f_{X_1}(x_1)},
\]

respectively.

**Proof.** Pick any \( x_1 \in \text{int}(X_1) \) and define

\[
\Psi^1(d, y, t, p) \equiv \frac{d \cdot 1(y \leq t)}{p},
\]

\[
\Psi^0(d, y, t, p) \equiv \frac{(1 - d) \cdot 1(y \leq t)}{1 - p}.
\]

We also denote the first-order partial derivative of \( \Psi^j(d, y, t, p) \) with respect to \( p \) by \( \Psi_p^j(d, y, t, p) \) for given \( j \in \{0, 1\} \). For simplicity of notation, let \( p_0(x) \equiv p(x; \theta_0) \) and \( \hat{p}_n(x) \equiv p(x; \hat{\theta}_n) \).

Under Assumptions C.2, C.4, and C.5, we have \( \frac{1}{nh_{1n}^d} \sum_i K_1(\frac{X_{1i} - x_1}{h_{1n}}) = f_{X_1}(x_1) + o_p(1) \). Hence,

\[
\sqrt{nh_{1n}^d} \left( \hat{F}_{1|X_1, n}(y|x_1) - F_{1|X_1}(y|x_1) \right)
= \frac{1}{\sqrt{nh_{1n}^d}} \sum_i \left\{ \frac{D_i \cdot 1(Y_i \leq y)}{\hat{p}_n(X_i)} - F_{1|X_1}(y|x_1) \right\} \cdot K_1(\frac{X_{1i} - x_1}{h_{1n}}) / \frac{1}{nh_{1n}^d} \sum_i K_1(\frac{X_{1i} - x_1}{h_{1n}}),
\]

\[
= \frac{1}{\sqrt{nh_{1n}^d}} \sum_i \left\{ \Psi^1(D_i, Y_i, y, \hat{p}_n(X_i)) - F_{1|X_1}(y|x_1) \right\} K_1(\frac{X_{1i} - x_1}{h_{1n}}) / f_{X_1}(x_1) + o_p(1)
\]

\[
= \frac{1}{\sqrt{nh_{1n}^d}} \sum_i \left\{ \Psi^1(D_i, Y_i, y, \hat{p}_n(X_i)) - F_{1|X_1}(y|X_{1i}) \right\} K_1(\frac{X_{1i} - x_1}{h_{1n}}) / f_{X_1}(x_1)
\]

\[
+ \frac{1}{\sqrt{nh_{1n}^d}} \sum_i \left\{ F_{1|X_1}(y|X_{1i}) - F_{1|X_1}(y|x_1) \right\} K_1(\frac{X_{1i} - x_1}{h_{1n}}) / f_{X_1}(x_1) + o_p(1).
\]

The latter term is \( o_p(1) \), because, by the standard argument in the literature on kernel
estimation, one can show that

\[
\mathbb{E} \left[ \frac{1}{\sqrt{nh_1^{d_1}}} \sum_{i}^{n} \left\{ F_{1|X_1}(y|X_{1i}) - F_{1|X_1}(y|x_1) \right\} K_1(\frac{X_{1i} - x_1}{h_1}) \right]
\]

= \sqrt{nh_1^{d_1}} \cdot \int \left\{ F_{1|X}(y|x_1 + uh_1) - F_{1|X}(y|x_1) \right\} K_1(u) \cdot f_{X_1}(x_1 + uh_1) du

= O \left( \sqrt{nh_1^{d_1+4}} \right) = o(1).

Now we consider the first term. By a Taylor approximation of \( \Psi^1(d,y,t,p) \) around at \( p(X_i; \theta_0) \), we have

\[
\frac{1}{\sqrt{nh_1^{d_1}}} \sum_{i}^{n} \left\{ \Psi^1(D_i, Y_i, y, \hat{p}_n(X_i)) - F_{1|X_1}(y|X_{1i}) \right\} K_1(\frac{X_{1i} - x_1}{h_1})
\]

= \frac{1}{\sqrt{nh_1^{d_1}}} \sum_{i}^{n} \left\{ \Psi^1(D_i, Y_i, y, p_0(X_i)) - F_{1|X_1}(y|X_{1i}) \right\} K_1(\frac{X_{1i} - x_1}{h_1})

+ \frac{1}{\sqrt{nh_1^{d_1}}} \sum_{i}^{n} \left\{ \Psi_p(D_i, Y_i, y, \hat{p}(X_i)) \cdot (\hat{p}_n(X_i) - p_0(X_i)) \right\} K_1(\frac{X_{1i} - x_1}{h_1})

(E.2)

where \( \hat{p}(x) \equiv p(x; \hat{\theta}_n) \) and \( \hat{\theta}_n \) lies between \( \hat{\theta}_n \) and \( \theta_0 \). Note that the second term in (E.2) is \( o_p(1) \) uniformly in \( y \) under Assumption C.3 because

\[
\left| \frac{1}{\sqrt{nh_1^{d_1}}} \sum_{i}^{n} \left\{ \Psi_p(D_i, Y_i, y, \hat{p}(X_i)) \cdot (\hat{p}_n(X_i) - p_0(X_i)) \right\} K_1(\frac{X_{1i} - x_1}{h_1}) \right|
\]

\leq \sup_{x \in X} \sqrt{nh_1^{d_1}} (\hat{p}_n(x) - p_0(x)) \cdot \frac{1}{\sqrt{nh_1^{d_1}}} \sum_{i}^{n} \left| \Psi_p(D_i, Y_i, y, \hat{p}(X_i)) \cdot K_1(\frac{X_{1i} - x_1}{h_1}) \right|

= o_p(1) \cdot O_p(1) = o_p(1).

We now establish the limiting process (with respect to \( y \)) of the first term (equation (E.2)). To this end, we verify the conditions of the functional central limit theorem in Pollard (1990, p.53, Theorem 10.6).

For given \( \omega \in \Omega \), let \( f_{ni}(\omega, y) \equiv \frac{1}{f_{X_1}(x_1) \sqrt{nh_1^{d_1}}} \left\{ \Psi^1(D_i, Y_i, y, p_0(X_i)) - F_{1|X_1}(y|X_{1i}) \right\} \cdot K_1(\frac{X_{1i} - x_1}{h_1}) \) and \( F_n^1 \equiv \{ f_{ni}(\omega, y) : y \in \mathbb{R} \} \). By using the same argument in the proof of Theorem 4.1, we can show that \( F_n^1 \) satisfies Pollard’s entropy condition. Since Pollard’s
entropy condition implies manageability (see Andrews (1994, p.2284)), condition (i) of Theorem 10.6 in Pollard (1990) is satisfied.

Let $Z_n(\omega, y) \equiv \sum_i^n f_{ni}(\omega, y)$ and $y_1, y_2 \in \mathbb{R}$ be given. By the law of iterated expectations, we have

$$
\mathbb{E}[Z_n(\omega, y_1) \cdot Z_n(\omega, y_2)]
= \mathbb{E}\left[\left\{\Psi^1(D_i, Y_i, y_1, p_0(X_i)) - F_{1|X}(y_1|X_i)\right\} \cdot \left\{\Psi^1(D_i, Y_i, y_2, p_0(X_i)) - F_{1|X}(y_2|X_i)\right\} \cdot K_1^2\left(\frac{X_i - x_1}{h_{1n}}\right)\right]
\times \frac{1}{f_{X_1}(x_1)^2h_{1n}^4}
= \mathbb{E}\left[K_1^2\left(\frac{X_i - x_1}{h_{1n}}\right) \cdot \frac{1}{f_{X_1}(x_1)^2h_{1n}^4} \left\{G_{1|X_1}(\min(y_1, y_2)|X_i) - F_{1|X_1}(y_1|X_i)F_{1|X_1}(y_2|X_i)\right\}\right].
$$

Under Assumptions C.1, C.2, and C.5, it follows that

$$
\mathbb{E}[Z_n(\omega, y_1) \cdot Z_n(\omega, y_2)] = \left\{G_{1|X_1}(\min(y_1, y_2)|x_1) - F_{1|X_1}(y_1|x_1)F_{1|X_1}(y_2|x_1) + O\left(h_{1n}^d\right)\right\} \frac{\|K_1\|^2_2}{f_{X_1}(x_1)}
$$

by using the standard arguments for kernel estimators. Therefore,

$$
\tilde{H}_1(y_1, y_2) \equiv \lim_{n \to \infty} \mathbb{E}[Z_n(\omega, y_1) \cdot Z_n(\omega, y_2)]
= \left\{G_{1|X_1}(\min(y_1, y_2)|x_1) - F_{1|X_1}(y_1|x_1)F_{1|X_1}(y_2|x_1)\right\} \frac{\|K_1\|^2_2}{f_{X_1}(x_1)}
$$

is well-defined, and hence condition (ii) of Theorem 10.6 in Pollard (1990) is satisfied.

Let $F_{ni} \equiv \frac{1}{f_{X_1}(x_1)^2h_{1n}^4}K_1(\frac{X_i - x_1}{h_{1n}})$ be an envelope function, where $f_{X_1} \equiv \inf_{x_1 \in X_1} f_{X_1}(x_1) > 0$. Since the kernel function is uniformly bounded and symmetric around zero, we have, for any $n \in \mathbb{N}$,

$$
\lim_{n} \sum_{i}^n \mathbb{E}[F_{ni}^2] = \lim_{n} \frac{1}{f_{X_1}(x_1)^2h_{1n}^4} \int \frac{1}{f_{X_1}(x_1)^2h_{1n}^4} K_1^2\left(\frac{t - x_1}{h_{1n}}\right) \cdot f_{X_1}(t)dt
\leq \lim_{n} \frac{1}{h_{1n}^{2d_1}} \int K_1\left(\frac{t - x_1}{h_{1n}}\right) \cdot f_{X_1}(t)dt = \lim_{n} \left\{f_{X_1}(x_1) + O\left(h_{1n}^{2d_1}\right)\right\} = f_{X_1}(x_1) < \infty
$$

under Assumptions C.4 and C.5. Therefore, condition (iii) of Theorem 10.6 in Pollard
the covariance kernel is \( \tilde{\rho} \), where

\[
\sum_{i} \mathbb{E} \left[ F_{ni}^{2 + \eta} \right] \lesssim \sum_{i} \frac{1}{(nh_{1n}^{d_{1}})^{1+\eta/2}} \int K_1^{2+\eta}(\frac{t-x_1}{h_{1n}}) f_{X_1}(t) dt
\]

\[
\lesssim \frac{1}{(nh_{1n}^{d_{1}})^{\eta/2}} \int u^2 K_1(u) du = o(1) \cdot O(1) = o(1).
\]

Hence, for any \( \epsilon > 0 \),

\[
\sum_{i} \mathbb{E} \left[ F_{ni}^{2+\eta} 1(F_{ni} > \epsilon) \right] = \sum_{i} \mathbb{E} \left[ F_{ni}^{2+\eta} F_{ni}^{-\eta} 1(F_{ni} > \epsilon) \right] \leq \frac{1}{\epsilon^{\eta}} \sum_{i} \mathbb{E} \left[ F_{ni}^{2+\eta} \right] \lesssim \frac{1}{\epsilon^{\eta}} \cdot o(1) = o(1),
\]

which implies condition (iv) of Theorem 10.6 in Pollard (1990) is met.

For \( y_1, y_2 \in \mathbb{R} \), define \( \rho_n(y_1, y_2) \equiv \left( \sum_{i} \mathbb{E}[|f_{ni}(\cdot, y_1) - f_{ni}(\cdot, y_2)|^2] \right)^{1/2} \) and \( \rho(y_1, y_2) \equiv \lim_{n} \rho_n(y_1, y_2) \). Then,

\[
\rho_n(y_1, y_2)^2 = \left\{ (G_{1|X_1}(y_1|x_1) - F_{1|X_1}(y_1|x_1)^2) + (G_{1|X_1}(y_2|x_1) - F_{1|X_1}(y_2|x_1)^2) + G_{1|X_1}(\min(y_1, y_2)|x_1) - F_{1|X_1}(y_1|x_1)F_{1|X_1}(y_2|x_1) + o(1) \right\} \cdot \frac{\|K_1\|_2^2}{f_{X_1}(x_1)}. \]

Therefore, \( \rho(y_1, y_2) \) is well-defined for any \( y_1, y_2 \in \mathbb{R} \). Since the components in \( \rho(y_1, y_2) \) are continuous in \( (y_1, y_2) \), this leads to condition (v) of Theorem 10.6 in Pollard (1990).

In all, by Theorem 10.6 in Pollard (1990), we have

\[
\frac{1}{f_{X_1}(x_1)} \sqrt{n h_{1n}^{d_{1}}} \sum_{i} \left\{ \Psi^1(D_{i}, Y_{i}, \cdot, p_0(X_{i})) - F_{0|X_{i}}(\cdot|x_{i}) \right\} \cdot K_1(\frac{X_{i} - x_1}{h_{1n}}) \Rightarrow \tilde{G}_1(\cdot) \text{ in } L^\infty(\mathcal{Y}),
\]

where \( \tilde{G}_1(\cdot) \) is a mean zero Gaussian process with covariance kernel \( \tilde{H}_1(y_1, y_2) \). One can prove the weak convergence of \( \sqrt{n h_{1n}^{d_{1}}} \left( \hat{F}_{0|X_{i}, n}(\cdot|x_{i}) - F_{0|X_{i}}(\cdot|x_{i}) \right) \) by a similar way, and the covariance kernel is

\[
\tilde{H}_0(y_1, y_2) = \left\{ G_{0|X_{i}}(\min(y_1, y_2)|x_{i}) - F_{0|X_{i}}(y_1|x_{i})F_{0|X_{i}}(y_2|x_{i}) \right\} \frac{\|K_1\|_2^2}{f_{X_1}(x_1)}.
\]

Since the Cartesian product of two Donsker classes is Donsker by Van der Vaart (1998, 1998).
Lemma E.4. Let $x_1 \in \text{int}(X_1)$ be given. Suppose that the conditions in Theorem C.1 hold. Then,

$$\sqrt{n h_1^{d_1} F_{1|X_1,n}^* (\cdot|x_1)} \Rightarrow \tilde{c}_1(\cdot),$$

$$\sqrt{n h_1^{d_1} F_{0|X_1,n}^* (\cdot|x_1)} \Rightarrow \tilde{c}_0(\cdot),$$

conditional on data in $l^\infty(Y)$.

Proof. One can use the similar arguments of the proof of Lemma E.2, together with Lemma E.3, and thus we omit the proof.

Proof of Theorem C.1

Proof. Note that the functionals $\phi_L$ and $\phi_U$ are Hadamard directionally differentiable. Therefore, the weak convergence result follows from, together with Lemma E.3, the same argument for the proof of Theorem 4.2. The bootstrap validity can be shown by using the same argument of the proof of Theorem 4.3 with letting $r_n = \sqrt{n h_1^{d_1}}$, the weak convergence result, and Lemma E.4.

E.12 Proof of Theorem D.1

Proof. Under $H_0$, $\theta_{L,p,e} = 0$. Therefore, extending part 3 of Corollary 4.2 in Firpo et al. (2021) and using the chain rule establishes the result. The proof of the validity of the bootstrap is almost identical to the proof of Theorem 4.3.
F Figures

Figure F.1: Heterogeneity in the treatment effects across income groups

Note: The blue and red lines are estimated lower and upper bounds on the conditional distributions of treatment effects, respectively. The star-marked lines are the bounds on the conditional distribution of treatment effects given the 0.2 quantile of the income. The circle-marked lines are the bounds on the conditional distribution of treatment effects given the 0.8 quantile of income. All of these bounds are obtained under Assumptions 2.3 and 2.4. When the assumptions are not imposed, the resulting bounds are the logical ones.
Figure F.2: The effect of 401k on net financial assets - Comparison of bounds across different income groups

0.1 quantile of income

0.2 quantile of income

0.3 quantile of income

0.4 quantile of income

0.5 quantile of income

0.6 quantile of income

0.7 quantile of income

0.8 quantile of income

0.9 quantile of income

Note: Estimated bounds at various quantile levels and at the mean of income are reported. These bounds are obtained under Assumptions 2.3 and 2.4. When the assumptions are not imposed, the resulting bounds are the logical ones.