SYMMETRIES IN INTEGER PROGRAMS

KATRIN HERR AND RICHARD BÖDI

Abstract. The notion of symmetry is defined in the context of Linear and Integer Programming. Symmetric integer programs are studied from a group theoretical viewpoint. We investigate the structure of integer solutions of integer programs and show that any integer program on \( n \) variables having an alternating group \( A_n \) as a group of symmetries can be solved in linear time in the number of variables.

1. Introduction

This paper continues to investigate symmetries of linear and integer programs which we have started in \([3]\). For the sake of completeness, we will briefly summarize the definitions and results from our previous paper.

In practice, highly symmetric integer programs often turn out to be particularly hard to solve. The problem is that branch-and-bound or branch-and-cut algorithms, which are commonly used to solve integer programs, work efficiently only if the bulk of the branches of the search tree can be pruned. Since symmetry in integer programs usually entails many equivalent solutions, the branches belonging to these solutions cannot be pruned, which leads to a very poor performance of the algorithm.

Only in the last few years first efforts were made to tackle this irritating problem. In 2002, Margot presented an algorithm that cuts feasible integer points without changing the optimal value of the problem, compare \([6]\). Improvements and generalizations of this basic idea can be found in \([7,8]\). In \([9,10]\), Linderoth et al. concentrate on improving branching methods for packing and covering integer problems by using information about the symmetries of the integer programs. Another interesting approach to these kinds of problems has been developed by Kaibel and Pfetsch. In \([5]\), the authors introduce special polyhedra, called orbitopes, which they use in \([4]\) to remove redundant branches of the search tree. Friedman’s fundamental domains in \([2]\) are also aimed at avoiding the evaluation of redundant solutions. For selected integer programs like generalized bin-packing problems there exists a completely different idea how to deal with symmetries, see e.g. \([1]\). Instead of eliminating the effects of symmetry during the branch-and-bound process, the authors exclude symmetry already in the formulation of the problem by choosing an appropriate representation for feasible packings.

Date: August 23, 2009.

Key words and phrases. symmetry, symmetry group, orbit, group action, alternating group, linear programming, integer program.
In this paper we will examine symmetries of integer programs in their natural environment, the field of group theory.

2. Preliminaries

The main object of our studies are linear or integer programs, LP or IP for short:

$$\begin{align*}
\max & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b, \quad x \in \mathbb{R}^n,
\end{align*}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n \setminus \{0\}$. We are especially interested in points that are candidates for solutions of an LP.

**Definition.** A point $x \in \mathbb{R}^n$ is **feasible** for an LP if $x$ satisfies all constraints of the LP. The LP itself and any set of points is **feasible** if it has at least one feasible point.

Hence, the set of feasible points $X$ of (1) is given by

$$X := \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$  

**Convention.** We call $X$ the **feasible region**, $c$ the **utility vector** and $n$ the **dimension** of $\Lambda$. The map $x \mapsto c^T x$ is called the **utility function**, and the value of the utility function with respect to a specific $x \in \mathbb{R}^n$ is called the **utility value** of $x$.

We can interpret the feasible region of an LP in a geometric sense. The following definition is adopted from [11], p. 87.

**Definition.** A **polyhedron** $P \subseteq \mathbb{R}^n$ is the intersection of finitely many affine half-spaces, i.e.,

$$P := \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

for a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$.

Note that every row of the system $Ax \leq b$ defines an affine half-space. Obviously, the set $X$ is a polyhedron. Since every affine half-space is convex, the intersection of affine half-spaces – hence, any polyhedron – is convex as well. Therefore, we can now state the convexity of $X$.

**Remark 1.** The feasible region of an LP is convex.

Whenever we consider linear programs, we are particularly interested in points with maximal utility values that satisfy all the constraints.

**Definition.** A **solution** of an LP is an element $x^* \in \mathbb{R}^n$ that is feasible and maximizes the utility function.

If we additionally insist on integrality of the solution, we get a so-called integer program, IP for short. According to the LP formulation in (1), the appropriate formulation for the related IP is given by

$$\begin{align*}
\max & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b, \quad x \in \mathbb{Z}^n,
\end{align*}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n \setminus \{0\}$.

Analogously, the set of feasible points $X_I$ of (2) is given by

$$X_I := \{x \in \mathbb{R}^n \mid Ax \leq b, \quad x \in \mathbb{Z}^n\} = X \cap \mathbb{Z}^n.$$
3. Symmetries

In [3] symmetries of linear and integer programs have been defined as elements of $O_n(\mathbb{Z})$, the group of all orthogonal matrices with integral entries, that leave invariant the inequality system and the utility vector of the problem. Taking into account the usual linear and integer programming constraint $x \in \mathbb{R}_{\geq 0}^n$, which forces non-negativity of the solutions, the set of possible symmetries shrinks from $O_n(\mathbb{Z})$ to the group of permutation matrices $P_n \leq O_n(\mathbb{Z})$.

We can always think of symmetry groups of linear or integer programs as subgroups of $S_n$ by Remark 2.

**Remark 2.** A group $G \leq S_n$ acts on the linear space $\mathbb{R}^n$ via the $G$-equivariant mapping

$$\beta : \{1, \ldots, n\} \to B : i \mapsto e_i,$$

where $B$ is the set of the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$.

As in [3] we formulate the definition of symmetries of linear programs and the corresponding integer programs simultaneously. Consider an LP of the form

$$\begin{align*}
\text{max} & \quad c^t x \\
\text{s.t.} & \quad Ax \leq b, \quad x \in \mathbb{R}^n_{\geq 0},
\end{align*}$$

and the corresponding IP given by

$$\begin{align*}
\text{max} & \quad c^t x \\
\text{s.t.} & \quad Ax \leq b, \quad x \in \mathbb{R}^n_{\geq 0}, \quad x \in \mathbb{Z}^n,
\end{align*}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n \setminus \{0\}$.

Note that the LP [3] and the IP [4] have the additional constraint $x \in \mathbb{R}^n_{\geq 0}$.

**Notation.** An LP of the form [3] is denoted by $\Lambda$.

Apparently, applying a permutation to the matrix $A$ according to Remark 2 translates into permuting the columns of $A$. Since the ordering of the inequalities does not affect the object they describe, we need to allow for arbitrary row permutations of the matrix $A$. The following definition takes these thoughts into account.

**Definition.** A symmetry of a matrix $A \in \mathbb{R}^{m \times n}$ is an element $g \in S_n$ such that there exists a row permutation $\sigma \in S_m$ with

$$P_\sigma AP_g = A,$$

where $P_\sigma$ and $P_g$ are the permutation matrices corresponding to $\sigma$ and $g$. The full symmetry group of a matrix $A \in \mathbb{R}^{m \times n}$ is given by

$$\{g \in S_n \mid \exists \sigma \in S_m : P_\sigma AP_g = A\}.$$

A symmetry of a linear inequality system $Ax \leq b$, where $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$, is a symmetry $g \in S_n$ of the matrix $A$ via a row permutation $\sigma \in S_m$ which satisfies $b^\sigma = b$. 

A symmetry of an LP \( \Lambda \) or its corresponding IP is a symmetry of the linear inequality system \( Ax \leq b \) that leaves the utility vector \( c \) invariant. The \textit{full symmetry group} of \( \Lambda \) and the corresponding IP is given by

\[ \{ g \in S_n \mid c^g = c, \exists \sigma \in S_m : (b^\sigma = b \land P_{\sigma}AP_\sigma = A) \} \]

This is a definition of symmetry as it can be found in literature as well, see e.g. [7].

4. Symmetries in Integer Programming

Due to [3], Corollary 19, we notice that in the LP case, transitivity of the group action already implies a one-dimensional set of fixed points, giving rise to a one-dimensional linear program, which is the best possible result we can obtain. In this section, it will turn out that the assumption of transitivity is not strong enough in the IP case to lead to satisfying results. Moreover, we will see that not only the decomposition into orbits but also the detailed structure of the symmetry group influences the complexity of integer programs. The algorithm we are going to develop in this chapter builds on our approach for the linear case.

We start with the consideration of the integer program corresponding to the LP given by

\[ c^t x = x_1 + x_2 \]

subject to

\[ x_1 \leq 2.5 \]
\[ x_2 \leq 2.5 \]
\[ x_1 + x_2 \leq 3.7 \]

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[step=1,very thin,gray!50] (0,0) grid (3,3);
\draw[thick] (0,0) -- (3,0) -- (3,3) -- (0,3) -- cycle;
\draw[dashed] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw[thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\node at (1.5,1.5) {$c^t x = z$};
\end{tikzpicture}
\caption{The set \( X \) of feasible points}
\end{figure}

Since we only have to handle two dimensions, we can solve this LP in a graphical way. By pushing the blue line towards the feasible region \( X \), we continuously decrease the utility value \( z \). The first non-empty intersection of the line \( c^t x = z \) and \( X \) then represents the set of all solutions of the LP, marked as a bold line. Mathematically, the act of ”pushing the dashed utility line” translates into looking at the affine hyperplanes

\[ H_{c,t} := \ker(x \mapsto c^t x) + t \cdot c \]
for decreasing \( t \in \mathbb{R} \). For every \( x \in H_{c,t} \), there exists a vector \( x' \in \ker(y \mapsto c^t y) \) such that

\[
x = x' + t \cdot c .
\]

The computation of

\[
 c^t x = c^t (x' + t \cdot c) = c^t x' + t \cdot c^t c = t \|c\|^2
\]

proves that all points of an affine hyperplane \( H_{c,t} \) have the same utility value \( t \|c\|^2 \).

**Remark 3.** Given \( t \in \mathbb{R} \) and a utility vector \( c \), the utility value is constant on the affine hyperplane \( H_{c,t} \).

The family \( (H_{c,t})_{t \in \mathbb{R}} \) consists of all affine hyperplanes that are orthogonal to \( c \), thus they are parallel to each other. Therefore, every point is contained in exactly one affine hyperplane \( H_{c,t} \) for a specific utility vector \( c \).

**Lemma 4.** Given a point \( x \in \mathbb{R}^n \) and a vector \( c \in \mathbb{R}^n \\setminus \{0\} \), the point \( x \) is contained in the affine hyperplane \( H_{c,t} \) for \( t = \frac{c^t x}{\|c\|^2} \).

**Proof.** We define a vector \( x' \in \mathbb{R}^n \) by

\[
x' = x - t_x \cdot c .
\]

The computation of \( c^t x' \) yields

\[
c^t x' = c^t (x - \frac{c^t x}{\|c\|^2} \cdot c) = c^t x (1 - \frac{c^t c}{\|c\|^2}) = 0 .
\]

That is, the vector \( x' \) is an element of \( \ker(y \mapsto c^t y) \). We conclude that the point \( x = x' + t_x \cdot c \) is contained in

\[
\ker(y \mapsto c^t y) + t_x \cdot c = H_{c,t_x} .
\]

\[\square\]

Given a symmetry group \( G \), we know by Remark 12 of [3] that the line \( l \) through the origin spanned by \( c \) is invariant under \( G \). Hence, this is also true for its orthogonal complement \( \ker(x \mapsto c^t x) \). Since the line \( l \) is even pointwise fixed by \( G \), we finally obtain the invariance of the affine hyperplanes \( H_{c,t} \) under \( G \). Referring to Theorem 5 of [3], we may add the constraint \( x \in H_{c,t} \) without losing symmetry.

**Remark 5.** Given an LP with utility vector \( c \) and a symmetry group \( G \), the intersection of the feasible region and an affine hyperplane \( H_{c,t} \) is invariant under \( G \). In particular, the orbit \( x^G \) is contained in the same affine hyperplane \( H_{c,t} \) as \( x \).

Back to our example, we now want to find a solution of the corresponding IP. Typically, the set of all solutions of the LP does not contain any integer solutions. Therefore, we have to push the line further on to the closest integer point. In our example, this procedure leads to the following situation.

Obviously, both accentuated points solve the IP. Furthermore, we observe that in this case, hardly any of the affine hyperplanes \( H_{c,t} \) contain integer points. Therefore, we introduce a special term for affine hyperplanes that contain integer points.
5. INTEGER-LAYERS

Definition. A c-layer is an affine hyperplane $H_{c,t}$ that contains at least one integer point.

The definition of c-layers immediately raises the following questions:

How many c-layers do we find, and what are the corresponding parameters $t$?

To give a detailed answer to these questions, we need to distinguish two different types of utility vectors $c$.

Definition. A utility vector is called projectively rational if it is a real multiple of a rational vector, and hence also of an integer vector. Otherwise, it is called projectively irrational. The coprime multiple of a projectively rational utility vector $c$ is a real multiple $c' \in \mathbb{Z}^n$ of $c$ whose entries $c'_1$ to $c'_n$ are coprime.

We force uniqueness of the coprime multiple by demanding the first non-zero entry of $c'$ to be positive. For example, the utility vector

$$(-2\sqrt{2}, 2\sqrt{2}, 4\sqrt{2}, 6\sqrt{2})^t = \sqrt{2}(-2, 2, 4, 6)^t$$

is projectively rational with coprime multiple $c' = (1, -1, -2, -3)^t$, whereas the vector

$$(\sqrt{2}, \sqrt{6}, 2\sqrt{2}, 3\sqrt{2})^t = \sqrt{2}(1, \sqrt{3}, 2, 3)^t$$

is projectively irrational.

Considering a utility vector $c$ and an arbitrary real multiple $c' \neq 0$ of $c$, we observe that

$$\ker(x \mapsto c'x) = \ker(x \mapsto (c')^t x).$$

Therefore the sets $(H_{c,t})_{t \in \mathbb{R}^n}$ and $(H_{c',t})_{t \in \mathbb{R}^n}$ of affine hyperplanes are equal. In particular, this is also true for the corresponding layers.

Remark 6. Given a vector $c \in \mathbb{R}^n \setminus \{0\}$, the set of c-layers is equal to the set of $(r \cdot c)$-layers for every $r \in \mathbb{R} \setminus \{0\}$.

We first want to study the configuration of c-layers for projectively rational utility vectors $c$. In this case, the c-layers are arranged in a very clear way.

Theorem 7. Given a projectively rational utility vector $c \neq 0$, the family of c-layers is given by $(H_{c',k\|c'\|^{-2}})_{k \in \mathbb{Z}}$, where $c'$ is the coprime multiple of $c$. 

\[c'x = z\]
Given a projectively rational utility vector \(c\) with coprime multiple \(c'\), the number of \(c\)-layers between \(mc\) and \((m + 1)c\) for any \(m \in \mathbb{Z}\) is equal to the squared Euclidean norm \(\|c'\|^2\) of the coprime multiple \(c'\). These \(c\)-layers are the affine hyperplanes \(H_{c',k\|c'\|^2}\), where

\[
k \in \{m\|c'\|^2, \ldots, (m + 1)\|c'\|^2 - 1\}.
\]

To put it precisely, the number of \(c\)-layers includes the layer through \(mc'\) but excludes the layer through \((m + 1)c'\). The representation of the \(c\)-layers in Corollary 8 allows us not only to count the layers but also to access every single layer directly by its characteristic parameter \(k\).

**Convention.** Given a projectively rational utility vector \(c\) with coprime multiple \(c'\), the \(c\)-layer \(H_{c',k\|c'\|^2}\) is called the \(k\)-th \(c\)-layer.

Note that we always refer to the coprime multiple \(c'\) of a utility vector \(c\) when we talk about \(c\)-layers. Figure 3 and Figure 4 give a graphical impression of the arrangement of \(c\)-layers for two different utility vectors. In both figures, the outer two \(c\)-layers contain the two integer points \(mc'\) and \((m + 1)c'\) on the line spanned by \(c'\).
by $c$. In contrast to the situation in Figure 3, we notice that in Figure 4, the two layers between the outer two layers do not cover all integer points.

**Figure 3.** Some integer layers for $c' = (1, 1, 1)^t$

**Figure 4.** Some integer layers for $c' = (1, 1, 2)^t$

Compared to the clear structure in the rational case, the arrangement of the $c$-layers for projectively irrational utility vectors $c$ is rather complicated. In particular, we lose finiteness of the number of $c$-layers between the origin and the point $c$. 
Theorem 9. Given a projectively irrational utility vector \( c \), there exist infinitely many \( c \)-layers between the origin and the point \( c \).

Proof. Being projectively irrational, the vector \( c \) is a real multiple of a vector \( c' \) of the form

\[
c' = \left( 1, c'_2, \ldots, c'_{j-1}, r, c'_{j+1}, \ldots, c'_n \right)^t,
\]

where \( r \in \mathbb{R} \setminus \mathbb{Q} \), and \( c'_i \in \mathbb{R} \) for \( i \in \{2, \ldots, j-1, j+1, \ldots, n\} \). We define a sequence of integer points by

\[
\left( x^{(k)} \right)_{k \in \mathbb{N}} := \left( x_1^{(k)}, \ldots, x_n^{(k)} \right)_{k \in \mathbb{N}}, \quad x_i^{(k)} = \begin{cases} 
-\lfloor rk \rfloor & \text{if } i = 1 \\
 k & \text{if } i = j \\
 0 & \text{otherwise} 
\end{cases}
\]

Since \( r \) is irrational, the equation

\[
y_1 + ry_2 = y'_1 + ry'_2
\]

implies \( y_1 = y'_1 \) and \( y_2 = y'_2 \) for arbitrary integers \( y_1, y_2, y'_1, y'_2 \). Hence, the utility values

\[
c^t x^{(k)} = -\lfloor rk \rfloor + r \cdot k
\]

are pairwise distinct for different \( k \in \mathbb{N} \). Furthermore, we observe that

\[
0 \leq rk - \lfloor rk \rfloor \leq 1 \leq \|c\|^2.
\]

Therefore, every integer point of the sequence \( \left( x^{(k)} \right)_{k \in \mathbb{N}} \) is contained in a separate \( c' \)-layer \( H_{c', t^{(k)}} \), with \( 0 \leq t^{(k)} \leq 1 \), see Lemma 4. Referring to Remark 6, this shows that we have infinitely many \( c \)-layers between the origin and the point \( c \). \( \square \)

Apparently, stepwise sifting through \( c \)-layers is practicable only if the number of \( c \)-layers is not too large but at least finite. Therefore, we will only follow up this method with respect to projectively rational utility vectors.

The basic idea of our approach is to generalize the graphical method we studied in the beginning of this chapter. By Remark 3 we know that the utility value is constant on every \( c \)-layer. Hence, we start looking for a feasible integer point on the \( c \)-layer next to an LP solution, which we can access directly because of the characterization given in Corollary 8. If there is no feasible integer point on this layer, we go on to the \( c \)-layer with the next smaller utility value, again accessible due to Corollary 8. The first feasible integer point we find by this method then is a solution to the IP problem.

Of course, it is not clear yet how to test feasibility of infinitely many integer points that are contained in every single \( c \)-layer. Even if the IP has the additional assumption of positive solutions, it is still not practicable to test a possibly exponentially large number of integer points. Furthermore, we would like to detect infeasibility of an IP problem without exhaustive testing of all possible integer points.

We will tackle these problems by means of symmetry. In contrast to the LP case, transitivity of the group action in the IP case is not the end of the line but the initial assumption for our analysis.
6. Transitive Actions

Consider an LP with a symmetry group $G$ acting transitively on the standard basis. Then the utility vector is a real multiple of the integer vector $\overline{c} := (1, \ldots, 1)^t$, see Corollary 9 of [3]. Since the utility value is constant on any $c$-layer, compare Remark 3, we obtain a useful characterization of the points on the $k$-th $c$-layer by applying Lemma 4.

**Remark 10.** Given the utility vector $c = (\gamma, \ldots, \gamma)^t$, an integer point $x$ is contained in the $k$-th $c$-layer if and only if the sum of its coordinates is equal to $k$.

Further, we proved in Theorem 14 of [3] that the set of fixed points $\text{Fix}_G(\mathbb{R}^n)$ is one-dimensional. Hence, it only consists of multiples of the utility vector, compare Remark 12 of [3]. If we solve the LP according to the substitution algorithm we discussed in the previous section, we get a solution of the LP of the form $x^*_{\text{fix}} = (a, \ldots, a)^t \in \text{Fix}_G(\mathbb{R}^n)$.

Therefore, the $c$-layer to start with in the transitive case is the $c$-layer next to this solution given by $H_{\overline{c}, k}\mathbb{Z} = \mathbb{Z}$, where $k = \lfloor na \rfloor$. If the IP is feasible, we stop as soon as we find a feasible point. But what could be a reasonable stopping criterion if the IP does not have any solutions? In general, the problem to decide whether an IP is feasible or not, is NP-complete, see e.g. [11], p. 245. We are now going to study this problem for transitive actions.

**Detecting Infeasibility.** We start by defining a certain point of reference for every $c$-layer that shows an exceptional property in the transitive case.

**Definition.** The center of a $c$-layer is the intersection point of the $c$-layer and the line spanned by the utility vector $c$.

Note that in the transitive case, the center $m_k$ of the $k$-th $c$-layer is given by

$$m_k = \left(\frac{k}{n}, \ldots, \frac{k}{n}\right)^t.$$

If we consider two feasible points $x_1$ and $x_2$, the convexity of the feasible region $X$ guarantees that the segment between $x_1$ and $x_2$ is feasible. Conversely, if only $x_1$ is feasible, then no point beyond $x_2$ on the ray from $x_1$ to $x_2$ can be feasible. In particular, we can apply this reasoning to the solution $(a, \ldots, a)^t$ of the LP and the center of any $c$-layer. In the transitive case, the line through $(a, \ldots, a)^t$ and a center $m_k$ is equal to the line generated by the utility vector $c$. Hence, we get the following statement.

**Remark 11.** Let $(a, \ldots, a)^t$ be a solution of an LP with a symmetry group acting transitively on the standard basis. If the center of the $k$-th $c$-layer is infeasible for some $k \leq \lfloor na \rfloor$, then the center of the $l$-th $c$-layer is infeasible for any $l \leq k$.

Note that the following statement holds for any affine hyperplane $H_{c,t}$, where $c = (\gamma, \ldots, \gamma)^t$ and $t \in \mathbb{R}$. However, we are interested in the result only in relation to $c$-layers.

**Theorem 12.** Given an LP with a symmetry group $G$ acting transitively on the standard basis, a $c$-layer is feasible if and only if its center is feasible.
Proof. We cut down the feasible region of the LP to a feasible c-layer. The feasibility of the c-layer assures that the substitution algorithm yields a solution to the resulting LP. By Remark 5 we know that G still is a symmetry group of the resulting LP. Hence, both LP problems share the same one-dimensional set of fixed points consisting of the line l spanned by \( c = (\gamma, \ldots, \gamma)^t \), compare Remark 12 of [3]. Therefore, the solution to the resulting LP provided by the substitution algorithm is the intersection point of l and the c-layer, i.e., the center. Hence, in particular, the center is feasible for the resulting LP, and therefore also for the original LP. □

Conversely, we conclude that there is no feasible point on a c-layer whose center is not feasible. Hence, referring to Remark 11 there are no feasible points in c-layers that have smaller utility values than the c-layer with the first infeasible center. Therefore, we only need to search the layers beginning with the \( \lfloor na \rfloor \)-th c-layer down to the \( (n\lfloor a \rfloor) \)-th c-layer. If the center of one of the layers is infeasible, we already know that the IP is infeasible. Otherwise, we arrive at the last layer and test the feasibility of the center \( (\lfloor a \rfloor, \ldots, \lfloor a \rfloor)^t \). Since the center is integral, we then either have found a solution or we conclude that the IP is infeasible. Thus, the algorithm stops after having searched at most n c-layers, see Corollary 8.

Corollary 13. Let \((a, \ldots, a)^t\) be a solution of an LP with a symmetry group acting transitively on the standard basis. Then stepwise sifting through the \( \lfloor na \rfloor \)-th c-layer down to the \((n\lfloor a \rfloor)\)-th c-layer either leads to a solution of the corresponding IP or reveals its infeasibility. The algorithm stops after at most n steps.

Corollary 13 discloses that the complexity of the infeasibility problem only depends on the efficiency of the search algorithm that is used to sift through a single c-layer. Therefore, we will now focus on the searching of a c-layer, i.e., on the problem how to check the IP-feasibility of a c-layer without testing every single integer point on the layer.

Reducing to Neighbors. The main idea is to define an appropriate set of integer points – the set of neighbors – such that the feasibility of any exterior integer point implies the feasibility of an integer point in the same c-layer that belongs to the set of neighbors. In this case, it suffices to test the feasibility of the neighbors. Unfortunately, transitivity is not strong enough to be able to reduce the problem to the center which is not necessarily integral. In contrast to the LP case, compare Corollary 19 of [3], the Purkiss Principle of symmetric solutions of symmetric problems is not suitable for IP problems. Therefore, the following definition leads to the smallest possible set of neighbors with respect to the Euclidean distance.

Definition. Given a c-layer, a neighbor is an integer point on the c-layer that has minimal Euclidean distance from the center of the c-layer.

Due to the simple structure of the utility vector in the transitive case (see [3]), we can easily describe the corresponding set of neighbors by using Remark 10.

Remark 14. Given the utility vector \( c = (\gamma, \ldots, \gamma)^t \), and an integer \( k = dn + r \), where \( d \in \mathbb{Z} \) and \( r \in \{0, \ldots, n - 1\} \), the set of neighbors \( \mathcal{N}_k \) in the k-th c-layer consists of all integer points that have \( r \) entries equal to \( d + 1 \) and \( n - r \) entries equal to \( d \). The number of neighbors in the k-th c-layer is given by

\[
|\mathcal{N}_k| = \binom{n}{r}.
\]
Consider two points $x$ and $y$ on the same hyperplane $H_{c,t}$ satisfying

$$\|x\|^2 \geq \|y\|^2.$$ 

Because of the orthogonality of $H_{c,t}$ and the line spanned by $c$, the Pythagorean theorem yields that in this case, we also have

$$\|x - m\|^2 \geq \|y - m\|^2,$$

where $m$ is the center of $H_{c,t}$. Therefore, we only need to determine the distance between the origin and two different points on the same $c$-layer in order to decide which of them is closer to the center.

**Remark 15.** Given two points $x, y \in H_{c,t}$, then $\|x\|^2 \geq \|y\|^2$ implies

$$\|x - m\|^2 \geq \|y - m\|^2,$$

where $m$ is the center of $H_{c,t}$. In particular, all elements of an orbit with respect to a group $G \leq S_n$ have the same distance to the center.

The following lemma describes a method to approach the set of neighbors without leaving the feasible region. In the proof, Remark 15 helps us to avoid technical difficulties.

**Lemma 16.** Given an LP with symmetry group $G$ and a feasible point $x$ in the $k$-th $c$-layer, any interior point of the segment determined by $x$ and a point $x^g \neq x$, where $g \in G$, is feasible and closer to the center of the $k$-th $c$-layer than $x$.

**Proof.** Note that $x$, $x^g$ and – since $c$-layers are affine hyperplanes – any convex combination

$$y := px + (1 - p)x^g$$

are in the same $c$-layer, compare Remark 6. Furthermore, the feasibility of $x$ implies the feasibility of $x^g$, see Remark 6 of [3], and therefore also the feasibility of $y$ due to the convexity of the feasible region $X$. Hence, referring to Remark 15, we only need to show that the squared Euclidean norm of $y$ is smaller than $\|x\|^2$ for any $p \in ]0; 1[$. Since $\|x\|^2$ is equal to $\|x^g\|^2$, we can write

$$\|x\|^2 = p^2\|x\|^2 + (1 - p)^2\|x^g\|^2 + 2p(1 - p)\|x\|^2 =$$

$$= p^2\|x\|^2 + (1 - p)^2\|x^g\|^2 + p(1 - p)\|x\|^2 + (1 - p)\|x^g\|^2,$$

and therefore

$$\|x\|^2 - \|y\|^2 = (p^2\|x\|^2 + (1 - p)^2\|x^g\|^2 + p(1 - p)\|x\|^2 + p(1 - p)\|x^g\|^2) -$$

$$- \left( p^2\|x\|^2 + 2p(1 - p)\sum_{i=1}^{n} x_ix_{i^g} + (1 - p)^2\|x^g\|^2 \right) =$$

$$= p(1 - p)\|x\|^2 + p(1 - p)\|x^g\|^2 - 2p(1 - p)\sum_{i=1}^{n} x_ix_{i^g} =$$

$$= p(1 - p)(\|x - x^g\|^2) > 0,$$

since $x \neq x^g$ and $p \in ]0; 1[$. \qed

Hence, we can approach the set of neighbors by considering convex combinations of two elements of the same orbit. Since we are interested in solutions to IP problems, the convex combinations should not only be feasible with respect to the LP.
but also with respect to the corresponding IP, i.e., they should be integral in addition. The next theorem shows that we can find such integral convex combinations for any feasible integer point as long as the degree of transitivity of the symmetry group is large enough.

**Theorem 17.** Let $G \leq S_n$ be a symmetry group of an LP acting $(\lfloor \frac{n}{2} \rfloor + 1)$-transitively on the standard basis, and $n \geq 2$. If an integer point $x$ is feasible and not a neighbor, then there exists a feasible integer point in the same $c$-layer that is closer to the center of the $c$-layer than $x$.

**Proof.** Due to the invariance of the standard lattice $\mathbb{Z}^n$ under translation by integer vectors, we only need to prove the statement for $c$-layers between the origin and the point $(1, \ldots, 1)^t$. Let $x$ be a feasible integer point on the $k$-th $c$-layer, where $k \in \{0, \ldots, n-1\}$. If all coordinates of $x$ are equal, the point $x$ is an element of the line spanned by $c = (\gamma, \ldots, \gamma)^t$, and therefore the center of the $k$-th $c$-layer, thus a neighbor. Otherwise, there exist at least two different coordinates $x_i, x'_i$ of $x$. We split the set of indices into the two sets

$$\{i \mid x_i \equiv 0 \mod 2\}, \quad \{i \mid x_i \equiv 1 \mod 2\}.$$

Then one of the two sets – denoted by $I$ – contains at least $\lfloor \frac{n+1}{2} \rfloor$ indices, while the other set $J$ has at most $\lfloor \frac{n}{2} \rfloor$ elements. Therefore, we will use the $\lfloor \frac{n}{2} \rfloor$-transitivity of $G$ to control $J$, and the additional degree of transitivity to produce two different feasible integer points. We distinguish the following two cases:

1) Suppose that $x$ has two different coordinates $x_i \neq x'_i$ of the same congruence class modulo 2, that is, the corresponding indices $i, i'$ are contained in the same set $I$ or $J$. Note that this condition is always satisfied if $x$ has more than two different coordinates. By the $(\lfloor \frac{n}{2} \rfloor + 1)$-transitivity of $G$, we then find a permutation $g \in G$ such that

$$i'^g = i, \quad J^g = J,$$

which implies $I^g = I$. These assignments do not contradict each other since we assumed that $i' \in J$ if and only if $i \in J$. Note that we do not require non-emptiness of $J$ in this case. By construction, all pairs of coordinates $(x_j, x_j^g)$ are in the same congruence class modulo 2 for all $j = 1, \ldots, n$, but $x^g$ is different from $x$ due to the assumption $x_i \neq x'_i$. Hence, the convex combination

$$y = \frac{1}{2}(x + x^g)$$

is an interior integer point of the segment determined by $x$ and $x^g$. Applying Lemma 16 to $x, x^g$ and $y$, we conclude that $y$ is a feasible integer point that is closer to the center of the $k$-th $c$-layer than $x$.

2) Otherwise, the point $x$ has exactly two different coordinates $x_i, x'_i$, where $i \in I$ and $i' \in J$, that is, $x_k = x_i$ for all $k \in I$, and $x_j = x'_i$ for all $j \in J$. In this case, transitivity of $G$ is sufficient to guarantee the existence of an appropriate permutation $g \in G$ satisfying $i^g = i$. Then $x$ and $x^g$ are distinct elements of the orbit $x^G$. Consider an interior point

$$y = px + (1 - p)x^g, \quad p \in (0, 1),$$

where $p$ and $1 - p$ are chosen to be integral.
of the segment defined by $x$ and $x^g$. We want to determine a parameter $p$ such that $y$ is integral. Obviously, the coordinates $y_l$ of $y$ can only take the values

$$y_l = px_i + (1-p)x_i = x_i$$

(5)

$$y_l = px_i' + (1-p)x_i' = x_i'$$

(6)

$$y_l = px_i + (1-p)x_i' = p(x_i - x_i') + x_i'$$

(7)

$$y_l = px_i' + (1-p)x_i = p(x_i' - x_i) + x_i$$

(8)

Since $x$ is integral, the coordinates of $y$ of type (5) and type (6) are integral for any $p \in (0, 1)$. The coordinates of type (7) and (8) are integral if $p - 1$ divides the absolute value $|x_i - x_i'|$. We may assume that $x$ is not a neighbor. Since the sum of all coordinates is between 0 and $n - 1$, compare Remark 10, we therefore conclude that either the coordinates $x_i$ and $x_i'$ have different signs or one of the coordinates is equal to 0 and the other one is greater or equal than 2. In any case, we get $|x_i - x_i'| \geq 2$. Hence, the choice

$$p = \frac{1}{|x_i - x_i'|}$$

is well-defined, and it guarantees that $y$ is an interior integer point of the segment defined by $x$ and $x^g$. Again, we apply Lemma 16 to $x, x^g$ and $y$ in order to conclude that $y$ is a feasible integer point that is closer to the center of the $k$-th $c$-layer than $x$.

\[ \square \]

Iterated application of Theorem 17 demonstrates that the set of neighbors is approachable over a sequence of feasible integer points starting from any integer point within the feasible region. Therefore, we deduce the following statement.

**Corollary 18.** Let $G \leq S_n$ be a symmetry group of an LP acting $\lfloor \frac{n}{2} \rfloor + 1$-transitively on the standard basis. Then the $k$-th $c$-layer is feasible if and only if the set of neighbors $N_k$ is feasible.

Thus, we may reduce the problem of looking for a feasible integer point on the whole $c$-layer to just testing the set of neighbors, hence at most $\lfloor \frac{n}{2} \rfloor + 1$ points on that layer, given $(\lfloor \frac{n}{2} \rfloor + 1)$-transitivity of the symmetry group. In this respect, the set of neighbors is representative for its layer. As a last step, we therefore concentrate on how to test the set of neighbors in an efficient way.

**Testing Neighbors.** Once more, we want to exploit our knowledge about the symmetries of an IP problem. To this end, we consider the set of neighbors in the transitive case as described in Remark 14. Obviously, this set is invariant under the action of any group $H \leq S_n$. Hence, we can study the action of a symmetry group $G \leq S_n$ of an IP not only on the IP itself but also on the set of neighbors in each layer. In particular, we are interested in the decomposition of the set of neighbors into orbits. Since $G$ leaves invariant the feasible region of the IP, the infeasibility of one neighbor implies the infeasibility of any neighbor in the same orbit, compare Remark 6 of [3]. Hence, we only need to test one neighbor in each orbit. Of course, we would like to have a small number of orbits, preferably one orbit only, which is the more likely the more symmetries we have. Due to the simple structure of neighbors, we can relax assumptions on transitivity to assumptions on homogeneity, which is weaker in principle. The following theorem provides an upper
bound on the degree of homogeneity of the action on the standard basis that suffices to guarantee transitivity of $G$ on the set of neighbors.

**Theorem 19.** Let $G \leq S_n$ be a symmetry group of an LP acting $k$-homogeneously on the standard basis, where $k \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \}$. Then the group $G$ acts transitively on the set of neighbors $\mathcal{N}_{r+n\mathbb{Z}}$ for any integer

$$r \in \{0, \ldots, k\} \cup \{n-k, \ldots, n-1\}.$$  

**Proof.** Let $r \in \{0, \ldots, k\}$ and $d \in \mathbb{Z}$. By Remark 14 we know that any neighbor $N \in \mathcal{N}_{r+nd}$ has exactly $r$ coordinates of value $d+1$, and $n-r$ coordinates of value $d$. We denote the corresponding sets of indices by $I_{d+1}^r N$ and $I_d N$. Due to the $k$-homogeneity of $G$ on the standard basis, there exists a permutation $g \in G$ that maps the $r \leq k$ elements of $I_{d+1}^r N$ of a neighbor $N \in \mathcal{N}_{r+nd}$ to the $r$ coordinates in $I_{d+1}^r N$ of any other neighbor $N' \in \mathcal{N}_{r+nd}$. Since $g$ is a bijection, the remaining set of indices $I_d^r N$ is automatically mapped to $I_d^r N'$. Hence, we find a permutation $g \in G$ with $N g = N'$ for any two neighbors $N, N' \in \mathcal{N}_{r+nd}$, i.e., the group $G$ acts transitively on the set of neighbors $\mathcal{N}_{r+nd}$. For $r \in \{n-k, \ldots, n-1\}$, we switch the roles of $d$ and $d+1$ and apply the same reasoning. \qed

Thus, the assumption of a symmetry group acting $\lfloor \frac{n}{2} \rfloor$-homogeneously on the standard basis is sufficient to assure transitivity on the set of neighbors in every layer. Combining Remark 6 of [3] and Theorem 19, we conclude that in case of a $(\lfloor \frac{n}{2} \rfloor + 1)$-transitive action, we only need to test one neighbor in any $c$-layer in order to decide its IP-feasibility.

**Corollary 20.** Let $G \leq S_n$ be a symmetry group of an LP acting $(\lfloor \frac{n}{2} \rfloor + 1)$-transitively on the standard basis, and $n \geq 2$. Then the set of neighbors $\mathcal{N}_k$ is feasible if and only if any neighbor $N \in \mathcal{N}_k$ is feasible.

Now we are ready to bring together all the results of this section in order to deduce an applicable algorithm.

**A Linear Algorithm for the Alternating and the Symmetric Group.** If the number of dimensions $n$ is greater or equal than 5, the assumption of $(n-2)$-transitivity implies $(\lfloor \frac{n}{2} \rfloor + 1)$-transitivity. Hence, we can apply our results to any IP corresponding to an LP whose full symmetry group is isomorphic to $A_n$ or $S_n$, where $n \geq 5$. For these problems, Corollary 13 describes which $c$-layers need to be tested and shows that we can stop after at most $n$ layers. Corollary 18 allows for reducing the problem of testing the feasibility of every single point on a layer to simply testing the set of neighbors. Finally, Corollary 20 guarantees that we only need to check the feasibility of one neighbor per layer. Therefore, the following algorithm works correctly, and it is linear in the number of dimensions $n$.

**Corollary 21.** Let $n \geq 5$, and $(a, \ldots, a)^t$ be a solution of an LP with a symmetry group isomorphic to $A_n$ or $S_n$. Then testing the feasibility of one neighbor on every $c$-layer beginning with the $[na] \cdot c$-layer down to the $(n[a]) \cdot c$-layer either leads to a solution of the corresponding IP or reveals its infeasibility. The algorithm stops after at most $n$ steps.
7. Conclusion

At the end of this chapter, we want to summarize the results and insights we gained. From an application-oriented point of view, the most promising result seems to be the knowledge about the existence and the configuration of $c$-layers, as it may contribute to a more systematic search for integral solutions. If we consider also the decline in the utility value for descending $c$-layers, the check for maximality becomes redundant, and the testing of the feasibility of integer points comes to the fore.

Certainly, in practice, we cannot expect symmetry groups of integer programs that act $(\lfloor n/2 \rfloor + 1)$-transitively on the standard basis, not even transitively. Hence, the results of Section 6 should be regarded as an abstract approach to the question which role is played by symmetries in integer programs. On the one hand, we proved that the complexity of integer programs with extremely large symmetry groups like the alternating or the symmetric group is linear. On the other hand, we also notice that the number of orbits of neighbors, thus the number of points to be tested, can get exponentially large as soon as we consider integer problems with smaller symmetry groups. But in any case, knowledge about symmetries helps us to reduce the number of points we need to check. Therefore, symmetry in integer programs should not be demonized but seized in all its potential.

References

1. Sándor P. Fekete and Jörg Schepers, A combinatorial characterization of higher-dimensional orthogonal packing, Mathematics of Operations Research 29 (2004), 353–368.
2. Eric J. Friedman, Fundamental domains for integer programs with symmetries, Combinatorial Optimization and Applications, First International Conference, COCOA 2007, Proceedings, 2007, pp. 146–153.
3. Katrin Herr and Richard Bödi, Symmetries in linear and integer programs, Available at http://arxiv.org/pdf/0908.3329, 2009.
4. Volker Kaibel, Matthias Peinhardt, and Marc E. Pfetsch, Orbitopal fixing, Integer Programming and Combinatorial Optimization, 12th International Conference, IPCO 2007, Proceedings, 2007, pp. 74–88.
5. Volker Kaibel and Marc Pfetsch, Packing and partitioning orbitopes, Math. Program. 114 (2008), no. 1, 1–36.
6. François Margot, Pruning by isomorphism in branch-and-cut, Math. Program. 94 (2002), no. 1, 71–90.
7. Claude Leupold and Peter M. Pardalos, Exploiting orbits in symmetric ILP, Math. Program. 98 (2003), no. 1-3, 3–21.
8. Francisco Dalfo, Symmetric ILP: Coloring and small integers, Discrete Optim. 4 (2007), no. 1, 40–62.
9. Robert J. Thomas, Jeff Linderott, Fabrizio Rossi, and Stefano Smriglio, Orbital branching, Integer Programming and Combinatorial Optimization, 12th International Conference, IPCO 2007, Proceedings, 2007, pp. 104–118.
10. Claude Leupold and Peter M. Pardalos, Constraint orbital branching, Integer Programming and Combinatorial Optimization, 13th International Conference, IPCO 2008, Proceedings, 2008, pp. 225–239.
11. Alexander Schrijver, Theory of linear and integer programming. Repr., Chichester: John Wiley & Sons. XI, 471 p., 1998.