Weighted hyperprojective spaces and homotopy invariance in orbifold cohomology

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Abstract. We show that Chen-Ruan cohomology is a homotopy invariant in certain cases. We introduce the notion of a $T$-representation homotopy which is a stringent form of homotopy under which Chen-Ruan cohomology is invariant. We show that while hyperkähler quotients of $T^*\mathbb{C}^{n+1}$ by $S^1$ (here termed weighted hyperprojective spaces) are homotopy equivalent to weighted projective spaces, they are not $S^1$-representation homotopic. Indeed, we show that their Chen-Ruan cohomology rings (over $\mathbb{Q}$) are distinct.

Introduction

In the toric topology conference held at Osaka University in 2006, the author spoke about joint results in [GH] with M. Harada. In that paper, we compute the Chen-Ruan cohomology ring over $\mathbb{Q}$ defined in [CR2] for hypertoric varieties (similar results were also found by [JT] using entirely different techniques). Our results relied heavily on methods developed in [GHK] and also on results due to H. Konno [Ko] and T. Hausel and B. Sturmfels [HS] about the topology of these varieties. In this note, we review these and some of the results discussed at the conference. We then apply them to describe combinatorially the Chen-Ruan cohomology of the case of a hyperkähler reduction by a linear and hyperkähler $S^1$ action on a vector space, which we term a weighted hyperprojective space.

A secondary goal of this paper is to explain a sense in which Chen-Ruan cohomology is homotopy invariant, and to illustrate that Chen-Ruan cohomology fails to be invariant under a naïve notion of homotopy equivalence for global quotients. In general, homotopy invariance is tricky, since orbifolds are only locally defined as quotients. Any map between orbifolds must not only commute with the local group structure, but also preserve global topological properties (see [Ch] for a notion of homotopy groups, [Mo] for a groupoid treatment). Depending on the presentation of these orbifolds, it may be difficult to identify “good” maps between orbifolds, in the sense of [CR1]. However, even in the global quotient case, equivariant homotopies do not guarantee that the Chen-Ruan cohomology is preserved. We introduce the notion of a $T$-representation homotopy and show that Chen-Ruan cohomology (and inertial cohomology) is preserved under such a homotopy.

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The case of weighted hyperprojective spaces is particularly useful to describe the difficulty involved with homotopy equivalence. These spaces retract to weighted projective spaces \([BD]\), yet their Chen-Ruan cohomologies are not equal. This is the case even though the homotopy can be described “upstairs” on manifolds, before taking a quotient by \(S^1\). The existence of the homotopy implies that the (ordinary or equivariant) cohomology ring – even the integral cohomology ring – of a hyperprojective space is known to be the cohomology of the corresponding weighted projective space. While the groups occurring in Chen-Ruan cohomology are homotopy invariant because they are cohomology groups, the twisted product introduced in \([CR2]\) is not invariant under (ordinary or equivariant) homotopy.

Weighted projective spaces may be described by the quotient of \(S^{2n+1}\) by an appropriate \(S^1\) action. Let \(S^1\) act on \(S^{2n+1} = \{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} | |z_i|^2 = 1 \}\) by \(t \cdot (z_0, \ldots, z_n) = (t^{b_0}z_0, \ldots, t^{b_n}z_n)\) for some nonnegative integers \(b_i\). Then \(\mathbb{C}P^n_{(b)} := S^{2n+1}/S^1\) is the quotient space. The (ordinary) cohomology \(H^*(\mathbb{C}P^n_{(b)}; \mathbb{Z})\) is the cohomology of the underlying topological space, which was computed by Kawasaki \([Ka]\). In contrast, the cohomology of the orbifold \([\mathbb{C}P^n_{(b)}]\) is by definition \(H^*([\mathbb{C}P^n_{(b)}]; \mathbb{Z}) := H^*_S(S^{2n+1}; \mathbb{Z})\). The cohomology of the underlying space and that of the orbifold are isomorphic over \(\mathbb{Q}\), but not over \(\mathbb{Z}\). To describe this cohomology, one can use symplectic techniques and Kirwan surjectivity (see \([H0]\) for an explicit computation).

In sharp contrast to ordinary cohomology, the Chen-Ruan cohomology of \([M]\) is not isomorphic to that of its core \(C([M])\). Chen-Ruan cohomology is not a homotopy invariant, even over \(\mathbb{Q}\). In Section 3 we explicitly compute this twisted cohomology and note its disagreement with the Chen-Ruan cohomology of weighted projective spaces. In Section 4 we show that Chen-Ruan cohomology is invariant under \(T\)-representation homotopy, a stringent form of homotopy.

This article is organized as follows: inertial cohomology is defined in Section 1; its relation to Chen-Ruan cohomology of symplectic and hyperkähler (abelian) quotients is described in Section 2. Finally, the main results of the paper are found in Sections 3 and 4.

It was a particular pleasure to participate in the International Conference on Toric Topology last spring in Osaka. The talks were of unusually high quality, and the conference could not have been better organized. I would especially like to thank Megumi Harada, Taras Panov, Yael Karshon and Mikiya Masuda for providing such an inviting environment for mathematics.

1. Inertial cohomology

Here we briefly describe the inertial cohomology of a stably almost complex manifold \(Y\) with an action by a torus \(T\), introduced in \([GHK]\). When \(Y\) has a locally free action by \(T\) – i.e. \(T\) acts with finite isotropy on \(Y\) – the inertial cohomology \(H^*_T(Y)\) equals (as a ring) the Chen-Ruan cohomology of \([Y/T]\), as defined in \([CR2]\). On the other hand, when \(Y\) is Hamiltonian or hyperhamiltonian, its inertial cohomology has special properties that lend themselves to easy computations. In some circumstances the inertial cohomology of a (hyper)hamiltonian space \(Y\) surjects onto that of a level set of the moment map, which in turn is isomorphic the Chen-Ruan cohomology of the orbifold given by the level set quotiented by \(T\).

REMARK 1.1. The space \(Y\) need not be honestly almost complex; the inertial cohomology of \(S^1\) acting appropriately on \(Y = S^3\) equals the Chen-Ruan orbifold
cohomology of the quotient, a (possibly weighted) projective space $S^3/S^1$. However, when $Y$ is stably almost complex, the normal bundles $\nu(Y^g \times Y^h) \subset Y^g$ are honestly complex, for all $g, h \in T$.

**Remark 1.2.** Clearly not all orbifolds may be expressed as a quotient $[Y/T]$ with $T$ an abelian Lie group. However, the set of orbifolds that can be presented this way include spaces that are not quotients by finite groups, such as weighted projective spaces.

As a module over $H^*_T(pt; \mathbb{Z})$, inertial cohomology is defined by

$$NH^*_T(Y; \mathbb{Z}) := \bigoplus_{g \in T} H^*_T(Y^g; \mathbb{Z}) = \bigoplus_{g \in T} H^*([Y^g/T]; \mathbb{Z}),$$

where $Y^g$ is the fixed point set of $g$ acting on $Y$, and $H^*_T(Y)$ is the *equivariant cohomology of $Y$*. The inertial cohomology is defined analogously for other coefficient rings, and we suppress the coefficient ring when it is irrelevant. See [GHK] for details on the grading.

The product on $NH^*_T(Y)$ is defined as follows. Choose $g, h \in T$, and let $H = (g, h)$ be the subgroup they generate. Then $Y^H$ is a submanifold of $Y$, and the normal bundle $\nu(Y^H)$ of $Y^H$ in $Y$ is naturally equipped with an $H$-action on the fibers. Let $X$ be a connected component of $Y^H$ and $\nu(X) := \nu(Y^H)|_X$ be the restriction of the normal bundle to $X$. We may decompose $\nu(X)$ into isotypic components with respect to the $H$-action:

$$\nu(X) = \bigoplus_{\lambda \in \hat{H}} I_{\lambda},$$

where $\hat{H}$ denotes the character group of $H$.

**Definition 1.3.** Let $\lambda \in \hat{H}$ and $t \in H$. The logweight of $t$ with respect to $\lambda$, denoted $a_{\lambda}(t)$, is the real number in $[0, 1)$ such that $\lambda(t) = e^{2\pi i a_{\lambda}(t)}$.

We need one other ingredient before defining the product.

**Definition 1.4.** The *obstruction bundle* is a vector bundle over each component $X$ of $Y^H$ given by

$$E|_X := \bigoplus_{\lambda \in \hat{H}} I_{\lambda}.$$  

We write $E \rightarrow Y^H$ to denote the union $E|_X$ over all connected components.

The dimension may vary over components. The *virtual fundamental class* $\epsilon \in H^*_T(Y^H)$ is given by

$$\epsilon := \sum_{X \in \pi_0(Y^H)} e(E|_X) \in H^*_T(Y^H),$$

where the sum is over connected components $X$ of $Y^H$ and $e(E|_X)$ is the $T$-equivariant Euler class of $E|_X$, considered as an element of $H^*_T(X)$.

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1. Let $ET$ be a contractible space with a free $T$-action on it, and let $Y_T := (Y \times ET)/T$ be the associated Borel homotopy quotient. Then $H^*_T(Y; \mathbb{Z}) := H^*(Y_T; \mathbb{Z})$ is the singular cohomology of $Y_T$.

2. Note that the sum $\lambda(g) + \lambda(h) + \lambda((gh)^{-1})$ is always 0, 1, or 2, and it is constant on a connected component of $Y^H$. 

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Finally, let \( e_1 : Y^H \to Y^g \), \( e_2 : Y^H \to Y^h \) and \( \tau_3 : Y^H \to Y^{gh} \) denote the natural inclusions. These induce pullbacks \( e_1^* : H^*_T(Y^g) \to H^*_T(Y^H) \) and \( e_2^* : H^*_T(Y^h) \to H^*_T(Y^H) \) and the pushforward \( (\tau_3)_* : H^*_T(Y^g) \to H^*_T(Y^{gh}) \). Let \( a \in NH^{*,g}_T(Y) \) and \( b \in NH^{*,h}_T(Y) \) be homogeneous classes in \( \circ \). Then we define

\[
eq 0
\]

(1.1)

where the products on the right hand side are computed in the usual product structure of \( H^*_T(Y) \). Note that the result lies in \( NH^{*,gh}_T(Y) \). By linear extension, the product is defined for any two classes \( a, b \in NH^{*,gh}_T(Y) \).

It is immediately clear from this definition that, for any subgroup \( \Gamma \subset T \), the product structure restricts to \( NH^*_T(\Gamma) := \oplus_{g \in \Gamma} NH^*_{T,g}(Y) \), making this set a ring as well. We call this the \( \Gamma \)-subring of the inertial cohomology.

**Remark 1.5.** In [GHK] we also introduce the product \( \star \) on \( NH^{*,g}_T(Y) \). In the case that \( Y \) is a Hamiltonian \( T \) space, \( \star \) and \( \sim \) coincide. The \( \star \) product has some computational advantages and makes associativity all but obvious. The corresponding combinatorics are used extensively in [GHK] and in [GH] to describe the cohomology of the corresponding quotient spaces as the inertial cohomology of \( Y \) modulo an ideal, all of which can be described using \( \star \).

However, the \( \star \) product obscures the ring map to the inertial cohomology of a level set \( L \) of the hyperkähler moment map. Indeed, the product is 0 on the inertial cohomology of any space with no fixed points. In particular, it does not coincide with the product introduced by Chen and Ruan on the quotient orbifold.

In the case of an \( S^1 \)-action, this inertial cohomology can be computed as follows. We follow notation introduced in [Ho].

**Main Example 1.6.** Let \( S^1 \) act on \( \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1} \) by

\[
i \cdot (z_0, \ldots, z_n, w_0, \ldots, w_n) = (t^{b_0}z_0, \ldots, t^{b_n}z_n, t^{-b_0}w_0, \ldots, t^{-b_n}w_n),
\]

where \( b_i \geq 0 \). Let \( \Gamma \) be the group generated by the finite stabilizers occurring under this action. Then \( \Gamma \cong \mathbb{Z}/\ell \mathbb{Z} \) with \( \ell := \text{lcm}(b_0, \ldots, b_n) \), and the \( \Gamma \)-subring of the inertial cohomology is given by

\[
NH^*_S(\Gamma)(\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}; \mathbb{Z}) = \mathbb{Z}[u, a_0, \ldots, a_{\ell-1}]/\mathcal{I},
\]

with

\[
\mathcal{I} = \langle \alpha_g \sim \alpha_h - \alpha_{[g+h]} \prod_{i=0}^n (b_iu)^{\lceil g_i \rceil + h_i - \lfloor b_i(g+h) \rfloor} \rangle
\]

\[
\cdot (b_iu)^{\lfloor -b_i\lfloor g_i \rfloor - h_i - \lfloor b_i(g+h) \rfloor} \rangle
\]

for all \( g, h \in \Gamma \), where \( |m| \) is the smallest nonnegative integer congruent \( m \) modulo \( \ell \). Note that, for any \( i \), these exponents are equal when \( b_i(g+h) = 0 \), and otherwise only one of the two exponents will be nonzero.

**Proof.** Note that the group \( \Gamma \) is given by the \( \ell \text{th} \) roots of unity, where \( \ell = \text{lcm}(b_0, \ldots, b_n) \). We identify \( \Gamma \cong \mathbb{Z}/\ell \mathbb{Z} \) with these roots. Then as a module over \( H^*_S(pt; \mathbb{Z}) \),

\[
NH^*_S(\Gamma)(\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}; \mathbb{Z}) = \bigoplus_{g \in \Gamma} H^*_S((\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})_g; \mathbb{Z}).
\]

For any \( g \in \Gamma \), we note that \( (\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})_g = \oplus_{i \in S_g} C_i \oplus \mathbb{C}_{n+1+i} \), where \( S_g = \{ i \in \{0, \ldots, n\} \mid b_ig = 0 \text{ in } \mathbb{Z}/\ell \mathbb{Z} \} \) and \( C_i \) indicates the \( i \text{th} \) copy of \( \mathbb{C} \) in \( \mathbb{C}^{n+1} = \oplus_{i=0}^n C_i \).
In particular, all the fixed points sets are equivariantly homotopic to a point. Thus $H^*_S((\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})^g; \mathbb{Z}) \cong H^*_S(pt) \cong \mathbb{Z}[u]$ for each $g \in \Gamma$. Thus as a $\mathbb{Z}[u]$-module, $NH^*_S((\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})^\ell)$ is free with one generator for each element in $\Gamma$. We denote the $\mathbb{Z}[u]$-module generators by $\alpha_0, \ldots, \alpha_{\ell-1}$. In other words, $\alpha_{[g]}$ is the symbol we use to denote “1” in $H^*_S((\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})^g)$.

A fiber of the normal bundle to the $g$-fixed point set is isomorphic to $\oplus_{i \in (S_i)^c} \mathbb{C}_i \oplus \mathbb{C}_{n+1+i}$. For any $g, h \in \Gamma$, the obstruction bundle restricted to $(\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})^{g,h}$ consists of lines $\mathbb{C}_n$ such that

\begin{equation}
\frac{[b_i g]}{\ell} + \frac{[b_i h]}{\ell} + \frac{[b_i ((-g+h))]}{\ell} = 2,
\end{equation}

as well as those $\mathbb{C}_{n+1+i}$ such that

\begin{equation}
\frac{[-b_i g]}{\ell} + \frac{[-b_i h]}{\ell} + \frac{[b_i ((g+h))]}{\ell} = 2.
\end{equation}

Using the language of logweights, $\frac{[b_i g]}{\ell} = a_b(g)$. These are the lines whose equivariant Euler classes contribute to the virtual fundamental class. Notice that in order for the sum to be 2, the first two summands must have sum strictly greater than 1. In particular, none of the summands are 0. Note also that when $[bg] \neq 0$, we have $[-b_i g] = \ell - [b_i g]$. This implies that either Equation (1.2) or (1.3) holds, but not both (and possibly neither).

On the other hand, the fiber of the normal bundle to $(\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})^{g,h}$ in $(\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})^{g+h}$ consists of those pairs $\mathbb{C}_i \oplus \mathbb{C}_{i+n+1}$ that are fixed by $g + h$ but not by both $g$ and $h$. This implies $g$ and $h$ satisfy

\begin{equation}
[b_i g] \neq 0, \quad [b_i h] \neq 0, \quad [b_i (g+h)] = 0, \quad \text{implying} \quad [b_i g] + [b_i h] = \ell.
\end{equation}

These are the planes contributing to the pushforward map $(\mathcal{P}_3)^*$. We let $R_{g,h} := \{i \in \{0, \ldots, n\} | \mathbb{C}_i \text{ occurs in } E \}$ and $R'_{g,h} := \{i \in \{0, \ldots, n\} | \mathbb{C}_{i+n+1} \text{ occurs in } E \}$. Then $R_{g,h} \cup R'_{g,h}$ are the set of indices that occur in the obstruction bundle. Let $R''_{g,h}$ be the set of $i \in \{0, \ldots, n\}$ satisfying Equations (1.4). Note that $(R_{g,h} \cup R'_{g,h}) \cap R''_{g,h} = \emptyset$.

For any pair $g, h$, the equivariant Euler class of the obstruction bundle is \(\prod_{g,h} \in R''_{g,h} \ (b_i u)(-b_i u)\). On the other hand, the pushforward in Formula (1.4) contributes an equivariant Euler class $\prod_{i \in R''_{g,h}} -b_i^2 u^2$. Thus,

\[\alpha_g \sim \alpha_h = \left( \prod_{\substack{i \in R''_{g,h}, j \in R_{g,h}, k \in R'_{g,h}}} -b_i^2 u^2(b_j u)(-b_k u) \right) \alpha_{[g+h]}..\]

A case-by-case analysis shows that this expression agrees with the ideal $\mathcal{I}$ above. When $i$ satisfies $[b_i (g + h)] = 0$, the exponents appearing in $\mathcal{I}$ are equal and either 0 or 1, depending on whether $i \in R''_{g,h}$ or not. For $[b_i (g + h)] \neq 0$, we have

\[ [b_i g] + [b_i h] - [b_i (g+h)] = [b_i g] + [b_i h] - (\ell - [b_i (g+h)]) = [b_i g] + [b_i h] + [-b_i (g+h)] - \ell.\]

\footnote{Note that in general $[b_i ((-g + h))] \neq -[b_i (g + h)]$, as the number on the left is always nonnegative. Equations (1.2) and (1.3) and the exponents in the description of $\mathcal{I}$ are not taken modulo $\ell$.}
This sum is \( \ell \) if and only if \([b_i g] + [b_i h] + [-b_i (g + h)] = 2\ell\), or \( i \in R^*_{g,h} \). Similarly, the sum \([-b_i g] + [b_i h] - [-b_i (g + h)] = \ell \) if and only if \([-b_i g] + [b_i h] + [b_i (g + h)] = 2\ell\), or \( i \in R^*_{g,h} \).

Lastly we note that \( \alpha_0 = 1 \) because it is the generator of \( H^*_{\text{GHK}} ((\mathbb{C}^n \oplus \mathbb{C}^n)^{id}) \); for any class \( \alpha \in NH^*_{\text{GHK}} (Y) \), we have \( \alpha \sim \alpha_0 = \alpha \) (note that \( R^*_{g, id} \cup R^*_{g, id} \cup R^*_{g, id} = \emptyset \) for any \( g \)).

2. Chen-Ruan cohomology of global \( T \)-quotients

In the case that \( T \) acts on a space \( Z \) locally freely, the corresponding inertial cohomology of \( Z \) the equals the Chen-Ruan cohomology of the orbifold \([Z/T]\), i.e.

\[
NH^*_{\text{CR}} (Z) \cong H^* \text{CR} ([Z/T])
\]

where here the \( * \)-grading is the same for these two rings \([\text{GHK}]\). Indeed, since the \( T \) action has only finite stabilizers, \( NH^*_{\text{CR}} (Z) = NH^*_{T} (Z) \), where \( \Gamma \) is the subgroup of \( T \) generated by the isotropy. In contrast, the Chen-Ruan cohomology is not defined for Hamiltonian (or hyperhamiltonian) \( T \)-spaces, as these spaces always have fixed points. For this reason, inertial cohomology is a good tool to use in the symplectic and hyperkähler categories. Let \( Y \) be a Hamiltonian \( T \) space with moment map \( \Phi : Y \rightarrow \mathfrak{t}^* \). The symplectic reduction \( Y//T \) at a regular value \( \alpha \) is defined by

\[
Y//T (\alpha) := \Phi^{-1}(\alpha)/T,
\]

where we suppress \( \alpha \) when it is understood. We say \( Y \) is a proper Hamiltonian \( T \)-space if for some \( \xi \in \mathfrak{t} \), \( \langle \Phi, \xi \rangle \) is a proper function on \( Y \).

Inertial cohomology – like Chen-Ruan cohomology – is not in general functorial. A map - even an equivariant map - between spaces does not necessarily induce a ring map in the other direction on inertial or Chen-Ruan cohomology. The inclusion \( \Phi^{-1}(\alpha) 
xrightarrow{Y} \) does not \emph{a priori} induce a map of rings in inertial cohomology; however, we proved it does in this case.

\textbf{Theorem 2.1 (Goldin, Holm, Knutson).} Let \( Y \) be a proper Hamiltonian \( T \)-space, with moment map \( \Phi : Y \rightarrow \mathfrak{t}^* \). Let \( \alpha \) be a regular value of \( \Phi \). Then the inclusion \( \Phi^{-1}(\alpha) \xrightarrow{Y} \) induces a surjection of rings

\[
\mathcal{K} : NH^*_{T} (Y; \mathbb{Q}) \twoheadrightarrow H^* \text{CR} ([\Phi^{-1}(\alpha)/T]; \mathbb{Q}).
\]

Furthermore, the kernel of \( \mathcal{K} \) is given in \([\text{GHK}]\).

This theorem relies heavily on the result \([K1]\) due to Kirwan that the inclusion \( \Phi^{-1}(\alpha) \xrightarrow{Y} \) induces a surjection of rings

\[
H^*_{T} (Y; \mathbb{Q}) \twoheadrightarrow H^* ([\Phi^{-1}(\alpha)/T]; \mathbb{Q}).
\]

This property is often referred to as Kirwan surjectivity. We are specifically interested in the toric cases, when the quotient spaces arise as the symplectic or hyperkähler reduction of a linear torus action on an affine vector space \( Y \). These quotients have large residual torus action on them, making them toric varieties in the Kähler case, and hypertoric varieties in the hyperkähler case.

When \( T \cong S^1 \), the reduced space is a weighted projective space \( \mathbb{C}P^h_b \), where \( (b) \) indicates the set of weights specified by the \( S^1 \) action on \( Y \). Theorem 2.1 thus provides a new proof and formula for the Chen-Ruan cohomology of toric varieties -
including weighted projective spaces - over \( \mathbb{Q} \). See also [BCS] (and an explanation of the equivalence in [GHK]).

The difficulty in proving a theorem analogous to Theorem 2.1 over \( \mathbb{Z} \) is that Kirwan surjectivity (Equation (2.1)) does not hold over \( \mathbb{Z} \). However, it does in certain cases: for an effective \( S^1 \) action on a vector space, it can be shown that the critical set of \( \Phi \) and also of \( \|\Phi\|^2 \) are torsion-free. This implies the following corollary:

**Corollary 2.2** (Corollary 9.3 [GHK]). Let \( S^1 \) act on \( \mathbb{C}^{n+1} \) with positive weights. Then the ring homomorphism

\[
\mathcal{K}: NH_{S^1}^*(\mathbb{C}^{n+1}; \mathbb{Z}) \longrightarrow H_{CR}^*([\mathbb{C}P^1]; \mathbb{Z})
\]

is a surjection.

In the hyperkähler case, \( Y \) has three Kähler structures; if the \( T \) action is hyperhamiltonian, then there are three moment maps, one for each Kähler structure. They may be combined into the two maps \( \Phi_Y: Y \to \mathfrak{t} \) and \( \Phi_C: Y \to \mathfrak{t} \oplus \mathbb{C} \), as explained in [Ko]. The hyperkähler reduction at a regular value \( (\nu, 0 + 0t) \) is given by \( \Phi^{-1}_R(\nu) \cap \Phi^{-1}_C(0)/T \), and is denoted \( X///T \), sometimes with a subscript to indicate the point of reduction.

For hyperkähler quotients, there is also a well-defined Kirwan map induced from the inclusion \( \Phi^{-1}_R(\nu) \cap \Phi^{-1}_C(0) \to Y \), though the induced map

\[
H^*_T(Y; \mathbb{Q}) \to H^*_T(\Phi^{-1}_R(\nu) \cap \Phi^{-1}_C(0); \mathbb{Q})
\]

is not known to be surjective. It also has not been proven that a ring map exists from the inertial cohomology of a hyperkähler manifold \( Y \) to the Chen-Ruan cohomology of its hyperkähler reduction. However, in the case that \( Y = T^* \mathbb{C}^{n+1} \) with a linear \( T \) action, surjectivity of Equation (2.2) is known to hold [Ko]. This allowed the author and M. Harada to prove the following:

**Theorem 2.3** (Goldin-Harada). Let \( Y = T^* \mathbb{C}^{n+1} \) with a \( T \) action given by acting on \( \mathbb{C}^{n+1} \) and by its inverse on the fiber directions, as specified in [Ko]. Let \( \Phi_R \oplus \Phi_C \) be a hyperhamiltonian moment map for this action. Then the inclusion

\[
\Phi^{-1}_R(\alpha) \cap \Phi^{-1}_C(0) \to Y
\]

induces a surjection

\[
NH^*_T(Y; \mathbb{Q}) \longrightarrow H^*_T(\Phi^{-1}_R(\alpha) \cap \Phi^{-1}_C(0)/T; \mathbb{Q})
\]

The ring \( NH^*_T(Y) \) and the kernel of (2.3) are computed in [GH].

This theorem led the authors to a combinatorial description of the Chen-Ruan cohomology of hypertoric varieties. Another description was independently discovered by [JT] using the language of stacks and fans.

### 3. The Chen-Ruan cohomology of weighted hyperprojective spaces

Let \( S^1 \) act on \( T^* \mathbb{C}^{n+1} \cong \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1} \) with weights \( b_0, \ldots, b_n \in \mathbb{Z} \) on the first copy of \( \mathbb{C}^{n+1} \) and with weights \( -b_0, \ldots, -b_n \) on the second copy. Consider the homomorphism \( \phi: S^1 \to T^*(\mathbb{C}^{n+1}) \cong S^1 \times \cdots \times S^1 \) given by \( t \to (t^{b_0}, \ldots, t^{b_n}) \). This induces an inclusion of Lie algebras \( \iota: \mathfrak{t} \to \mathfrak{t}^{n+1} \). Let \( \{\varepsilon_i\} \) be a basis for \( \mathfrak{t}^{n+1} \), and \( \{u_i\} \) a basis for its dual \( (\mathfrak{t}^{n+1})^* \). Let \( T^n \cong T^{n+1}/S^1 \). We define the vectors \( \{a_i\} \) by the image of \( \{\varepsilon_i\} \) in the exact sequence

\[
0 \longrightarrow \iota = \mathfrak{t} \longrightarrow \mathfrak{t}^{n+1} \stackrel{\beta}{\longrightarrow} T^n \longrightarrow 0,
\]

where

\[
\beta(\varepsilon_i) = a_i \]
Theorem 3.1. The (rational) Chen-Ruan cohomology of the weighted hyperprojective space \( M = T^* \mathbb{C}^{n+1} \ldots \mathbb{C}^{n+1} / / \mathbb{C}^1 \) for any regular value \( \nu \) is
\[
H^*_{CR}(M) \cong \mathbb{Q}[u_0, \ldots, u_n, \alpha_0, \ldots, \alpha_{\ell-1}]/I + J + \mathcal{K},
\]
where \( I, J \) and \( \mathcal{K} \) are given by
\[
I = \langle \alpha_g - \alpha_h - \alpha_{[g+h]} \prod_{i=0}^{n} \{b_i u\}^{[b_i g + [b_i h] - [b_i (g+h)]]} \rangle \quad \text{and}
\[
J = \langle \text{image}(\beta^*) \rangle
\]
\[
\mathcal{K} = \sum_{g \in \Gamma} \langle \alpha_g \prod_{c_i \neq 0} b_i u_i | \text{there exist } c_j \in \mathbb{R} \text{ such that } \sum_{j=0}^{n} c_j \varepsilon_j \in \text{image}(\iota - 0) \rangle,
\]
where \( \nu \) is given in by the inclusion in \([3.7]\) and \( \beta^* \) is the dual map \((T^*)^* \rightarrow (T^{n+1})^* \) to \( \beta \). See [Ko] for more on the construction of hypertoric varieties.

Proof. Recall that \( \Gamma \cong \mathbb{Z}/\ell \mathbb{Z} \), where \( \ell = \text{lcm}(b_0, \ldots, b_n) \). The \( \Gamma \)-subring of the inertial cohomology \( N H^*_{Sl}((T^* \mathbb{C}^{n+1})^\nu) \) is given by \( \mathbb{Q}[u, \alpha_0, \ldots, \alpha_{\ell-1}]/I \), as shown in Example [L6]. Note that this ring is isomorphic to \( \mathbb{Q}[u_0, \ldots, u_n, \alpha_0, \ldots, \alpha_{\ell-1}]/I + J \) where \( J \) consists of the linear relations among the \( u_i \) given by killing \( \langle \text{image}(\beta^*) \rangle \).

We need only compute the kernel of the surjective map described in Theorem [2.4]. By [GH], the kernel is generated by the kernel of \( H_\Gamma^*((T^* \mathbb{C}^{n+1})^\nu) \rightarrow H^*(((T^* \mathbb{C}^{n+1})^\nu / / / T) \) for each \( g \in \Gamma \). This computation is done in [HS], where they found that the kernel for each \( g \in \Gamma \) is given by the product of those \( u_i \) such that the intersection of the corresponding hyperplanes perpendicular to \( a_i \) is empty. Because the torus may act noneffectively on \( (T^* \mathbb{C}^{n+1})^\nu \), this is equivalent to the ideal given in [GH], Equation 5.6). Here we express this condition as Konno does in [Ko], by the product of those \( u_i \) such that the nonzero sum of the corresponding vectors is in \( \text{image}(\iota) \).

Remark 3.2. The Chen-Ruan cohomology of a weighted hyperprojective space is different from that of its core, a weighted projective space – even over \( \mathbb{Q} \). For example, in the case that \( S^1 \) acts on \( T^* \mathbb{C}^3 \) with \( (b) = (2, 1, 1) \), the core \( C(M) \) is a weighted \( \mathbb{CP}^2 \) with exactly one singular point whose isotropy is \( \mathbb{Z}/2 \mathbb{Z} \). The Chen-Ruan cohomology of \([\mathbb{CP}^2(2, 1, 1)]\) according to [GHK] is \( \mathbb{Q}[u, \alpha]/\langle u^3, u\alpha, \alpha^2 - u^2 \rangle \) with degree \( \deg u = 2 \) and \( \deg \alpha = 2 \). In contrast, the Chen-Ruan cohomology of \([M] \) is \( \mathbb{Q}[u, \gamma]/\langle u^3, \gamma^2, u\gamma \rangle \) with degree \( \deg u = 2 \) and \( \deg \gamma = 4 \). This computation is done in [GH], §6.

Remark 3.3. A natural (open) question is whether the Chen-Ruan cohomology of weighted hyperprojective spaces can be computed over \( \mathbb{Z} \). Indeed, Theorem 3.1 suggests what the answer should be. The inertial cohomology of \( T^* \mathbb{C}^n \) computed in Example [L6] is over \( \mathbb{Z} \), and the ideals in Theorem 3.1 are expressed with integers arising from the weights. Indeed, a proof of such a formula would follow from the results in [GH] if Kirwan surjectivity holds over \( \mathbb{Z} \) for \( S^1 \) quotients of vector spaces. It is possible that Konno’s techniques could be sufficient to prove this result, if one generalizes the line bundles he constructs over hypertoric manifolds to orbibundles over hypertoric orbifolds. It may also be possible to analyze directly the Morse theory in this case, to prove that torsion does not obstruct surjectivity.

[4] The numbers \( b_i \) are not relevant to the ideal \( \mathcal{K} \); here they serve only as a reminder of a conjectural answer to the question posed in Remark 3.2.
4. Homotopy Invariance of Inertial Cohomology

Inertial cohomology, like Chen-Ruan cohomology, is not a cohomology theory. An equivariant map of $T$-spaces, $f : X \to Y$ does not generally induce a map (of rings) $NH_T^*(Y) \to NH_T^*(X)$. Of course, a map in equivariant cohomology $H_T^*(Y^g) \to H_T^*(X^g)$ exists for all $g \in T$, so there is a map of graded vector spaces in inertial cohomology. It is the product structure that fails to respect functoriality.

A simple example of this failure is as follows. Let $S^1$ act on $X = \mathbb{C}$ with weight 2 and let $f : X \to pt$ map every point in $X$ to $Y = pt$. As vector spaces,
\[
NH_T^*(X) = \bigoplus_{g=\pm 1} H_T^*(\mathbb{C}) \oplus \bigoplus_{g \in T, g \neq \pm 1} H_T^*(\{0\}) \quad \text{and} \quad NH_T^*(Y) = \bigoplus_{g \in T} H_T^*(pt),
\]
with the natural map induced in equivariant cohomology $f^* : H_T^*(Y^g) \to H_T^*(X^g)$ given by the identity map for each $g \in T$. The product on $NH_T^{\circ \circ}(Y)$ is just the usual product on $H_T^*(pt)$ shifted by the grading of the group element. Thus in $NH_T^{\circ \circ}(Y)$, $1_g \sim 1_h = 1_{gh}$ for all $g, h \in S^1$, where the subscript indicates the $T$-grading. In $NH_T^{\circ \circ}(X)$, however, an obstruction bundle may play a role. Choose $g = h = (gh)^{-1} = e^{2\pi i}$. These elements fix $\{0\}$ alone. Since the action on the normal bundle $\mathbb{C}$ is with weight 2, we have $a(g) = a(h) = a((gh)^{-1}) = \frac{2}{3}$, implying that $E = \mathbb{C}$ is the obstruction bundle. The equivariant Euler class of this bundle is $2u$, where $u$ is the (positive) generator of $H_T^*(pt; \mathbb{Z})$. It follows that $1_g \sim 1_h = (2u)_{gh}$. Thus $f^*$ does not induce a ring map on the inertial cohomology. It is clear that requiring the homotopy to be $T$-equivariant does not fix the problem.

We can avoid this “change in obstruction bundle” by insisting that maps preserve (in an appropriate sense) the normal bundles to the fixed point sets. Such maps induce ring maps in inertial cohomology. When the maps are also homotopy equivalences, we obtain an isomorphism in inertial cohomology.

We say that the $T$-spaces $X$ and $Y$ are $T$-equivariantly homotopic if there exist equivariant maps $f : X \to Y$ and $e : Y \to X$ such that $f \circ e$ is $T$-equivariantly homotopic to $id_Y$ and $e \circ f$ is $T$-equivariantly homotopic to $id_X$. In this case, the map $f : X \to Y$ is said to be an equivariant homotopy equivalence. Note that if $X$ and $Y$ are $T$-equivariantly homotopic, then so are $X^g$ and $Y^g$ for all $g \in T$.

Lastly, we recall some notions of equivalence of vector bundles. Let $f : X \to Y$ be a smooth map, and $E$ a vector bundle over $Y$. Let $f^*(E) \to X$ denote the pullback of $E$ to $X$. We say that a $T$-bundle $E \to Y$ is isomorphic to a $T$-bundle $E' \to X$ if there exists a $T$-equivariant homotopy equivalence $f : X \to Y$ such that $f^*(E) \cong E'$ as $T$-bundles over $X$.

DEFINITION 4.1. Let $T$ act on manifolds $X$ and $Y$, and let $F : [0,1] \times X \to Y$ be a smooth $T$-equivariant homotopy between smooth equivariant maps $f : X \to Y$ and $f' : X \to Y$, such that $F(s,X)$ is a submanifold of $Y$ for all $s \in [0,1]$. We say that $F$ is a $T$-representation homotopy if, for all $g, h \in T$ and all $k \in \langle g, h \rangle$, the $T$-bundles $\nu(F(s,X)^{g,h}) \subset F(s,X)^k$ are isomorphic for all $s \in [0,1]$. In this case we write $f \sim_T f'$. We say that $X$ and $Y$ are $T$-representation homotopic spaces if they exist smooth equivariant maps $f : X \to Y$ and $e : Y \to X$ such that $f \circ e \sim_T id_Y$ and $e \circ f \sim_T id_X$.

REMARK 4.2. Note that, for $S^1$ acting on $X = \mathbb{C}$ with weight 2 and on $Y = pt$ trivially, the contraction map is an $S^1$-equivariant homotopy from $X$ to $Y$, but

\footnote{Note that, for all $s \in [0,1]$, $F(s,X)^g$ is a smooth submanifold of $F(s,X)$.}
there is no \( T \)-representation homotopy from \( X \) to \( Y \). On the other hand, if \( X = \mathbb{C} \) with a weight 1 action, and \( Y = \mathbb{C}^3 \) with a weight 1 action on one copy of \( \mathbb{C} \) and a trivial action on a copy of \( \mathbb{C}^2 \), then \( X \) and \( Y \) are \( S^1 \)-representation homotopic, though they are not (equivariantly) homeomorphic. Similarly, if \( X = \mathbb{R}^2 - \{0\} \) with a rotating \( S^1 \) action, and \( Y = S^1 = \{ x \in X : |x|^2 = 1 \} \) with \( S^1 \) acting by the restriction of how it acts on \( X \), then there is a \( T \)-representation homotopy equivalence from \( X \) to \( Y \), though the spaces are not homeomorphic.

**Theorem 4.3.** Let \( X \) and \( Y \) be \( T \)-representation homotopic spaces. Then there is a ring isomorphism

\[
NH_*^T(X; \mathbb{Z}) \cong NH_*^T(Y; \mathbb{Z}).
\]

Indeed, if the homotopy equivalence is given by the maps \( f : X \to Y \) and \( f' : Y \to X \), then \( f^*: NH_*^T(Y; \mathbb{Z}) \to NH_*^T(X; \mathbb{Z}) \) induced by \( f \) is an isomorphism.

**Proof.** It is immediate that \( f : X \to Y \) induces a homotopy equivalence of spaces \( f^*: H_*^T(Y; \mathbb{Z}) \to H_*^T(X; \mathbb{Z}) \) for all \( g \in T \), since \( f \) is a \( T \)-equivariant homotopy equivalence and equivariant cohomology is a homotopy invariant. This immediately implies that \( f \) induces an isomorphism of rings. Fix \( g, h \) and \( k \in \langle g, h \rangle \), and let \( \nu_X := \nu(X^{g, h} \subset X^k) \), and \( \nu_Y := \nu(Y^{g, h} \subset Y^k) \). We identify \( \nu_X \) with the complement of the tangent bundle \( T(X^{g, h}) \) in \( T^*(X^k) \) via a \( T \)-invariant metric, and similarly for \( \nu_Y \). Since \( f' \circ f \sim_T \text{id}_X \) is a \( T \)-representation homotopy, the map \( df|_{\nu_X} : \nu_X \to TY \) is injective. By equivariance of \( f \), the image of \( df|_{\nu_X} \) lies in \( \nu_Y \) restricted to \( f(X) \). It now follows that \( \nu_X \subset f^* \nu_Y \), i.e. \( \nu(X^{g, h} \subset X^k) \) is isomorphic to a subbundle of \( \nu(Y^{g, h} \subset Y^k) \) restricted to \( f(X) \). Similarly, \( f \circ f' \sim_T \text{id}_Y \) implies that \( \nu(Y^{g, h} \subset Y^k) \) occurs as a subbundle of \( \nu(X^{g, h} \subset X^k) \) restricted to \( f'(X) \) for all \( g, h \in T \) and all \( k \in \langle g, h \rangle \). Since \( f|_{X^{g, h}} : X^{g, h} \to Y^{g, h} \) and \( f'|_{Y^{g, h}} : Y^{g, h} \to X^{g, h} \) are homotopy equivalences, \( \nu(X^{g, h} \subset X^k) \cong \nu(Y^{g, h} \subset Y^k) \) for all \( k \in \langle g, h \rangle \). We assume that \( Y^{g, h} \) is connected (and otherwise make the same argument for connected components). The obstruction bundle \( E_Y|_{Y^{g, h}} \) is a subbundle of \( \nu(Y^{g, h} \subset Y = Y^1) \) composed of those isotypic components \( I_\lambda \) for which \( a\lambda(g) + a\lambda(h) + a\lambda((gh)^{-1}) = 2 \).

Similarly, the obstruction bundle \( E_X|_{X^{g, h}} \) in the \( (g, h) \)-equivariantly isomorphic vector bundle \( \nu(X^{g, h} \subset X) \) consists of those isotypic components \( I_\nu \) for which \( a\nu(g) + a\nu(h) + a\nu((gh)^{-1}) = 2 \). Since the representations of \( (g, h) \) on each fiber are isomorphic, the obstruction bundles are isomorphic. Note that these bundles are \( T \)-equivariantly isomorphic, because the total normal bundles are. Denote the equivariant Euler class of \( E_X|_{X^{g, h}} \) by \( \epsilon_X \) and that of \( E_Y|_{Y^{g, h}} \) by \( \epsilon_Y \).

The isomorphism of equivariant bundles implies that, under the map \( f^*: H_*^T(Y^{g, h}; \mathbb{Z}) \to H_*^T(X^{g, h}; \mathbb{Z}) \), we have \( f^*(\epsilon_X) = \epsilon_Y \). This in turn implies that for all \( a \in NH_*^{g, h}(Y) \) and \( b \in NH_*^{g, h}(Y) \), we have

\[
f^*(a \cdot b) = f^*[\langle e_3, Y \rangle^* (e_{1,Y}^* (a) \cdot e_{2,Y}^* (b) \cdot \epsilon_Y)]
= \langle e_3, X \rangle^* [f^*[e_{1,Y}^* (a) \cdot e_{2,Y}^* (b) \cdot \epsilon_Y]]
= \langle e_3, Y \rangle^* [e_{1,Y} (f^*(a)) \cdot e_{2,Y} (f^*(b)) \cdot f^*(\epsilon_Y)]
= \langle e_3, X \rangle^* [e_{1,X} (f^*(a)) \cdot e_{2,X} (f^*(b)) \cdot (\epsilon_X)]
= f^*(a) \cdot f^*(b),
\]

as desired. \( \square \)
Chen-Ruan cohomology is not obviously a representation homotopy invariant in the same way that inertial cohomology is. A generic orbifold is described locally by isotropy groups and their representations at particular points in the orbifold (see [CR2]). However, these groups are not naturally subgroups of one larger group. To define a similar concept of maps between orbifolds, one must work with local charts (upstairs) and their gluing maps. However, a map between orbifolds \( f : [X] \to [Y] \) may not induce a well-defined pull-back orbibundle \( f^*E \) for an orbibundle \( E \) over \([Y]\). See [CR1] for more details on good maps. However, one immediate consequence of Theorem 4.3 is the following corollary.

**Corollary 4.4.** Let \( T \) act on a stably complex spaces \( X \) and \( Y \) with finite isotropy. If \( X \) and \( Y \) are \( T \)-representation homotopic, then \( H^*_CR([X/T]; \mathbb{Z}) \cong H^*_CR([Y/T]; \mathbb{Z}) \).

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