Direct Solution of Initial Value Problems of Fourth Order Ordinary Differential Equations Using Modified Implicit Hybrid Block Method

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Authors’ contributions

This work was carried out in cooperation between all authors. Author SJK designed the study, wrote the concept of the article, taken care of formatting and provided finishing touch to this manuscript. Author MKD managed the experimental process and wrote the first draft of the manuscript. Author BB too managed the experimental process, along with the literature searches and analyses of the study. All authors read and approved the final manuscript.

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ABSTRACT

Our focus in this article is the derivation; analysis and implementation of a new modified implicit hybrid block method for the direct solution of initial value problems of fourth order ordinary differential equations. In the derivation of the method, we adopted the approach of collocation approximation to obtain the main scheme with continuous coefficients. From the main scheme, additional schemes were developed. The implementation strategy of the new method is by combining the main scheme and the additional schemes as

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simultaneous integrator to initial value problem of fourth order ordinary differential equations. As required of any numerical method, the properties analysis of the block was done and the result showed that it is consistent, convergent, zero stable and absolutely stable. We then test our method with numerical examples solved using existing method and were found to give better results.

**Keywords:** Interpolation; continuous coefficients; block method; numerical integration; fourth order ordinary differential equations.

### 1. INTRODUCTION

Some empirical problems and physical. Phenomena in science and engineering, such as mechanical systems without dissipation, celestial mechanics, control theory, computer aided designs when modeled result to higher order ordinary differential equations of the form:

\[ y^{(iv)} = f(x, y, y', y'', y'''), y(x_0) = \eta_0, \]
\[ y'(x_0) = \eta_1, y''(x_0) = \eta_2, y'''(x_0) = \eta_3 \]  

(1)

Conventionally, to solve (1) numerically, we first reduce it to system of first order ordinary differential equations and then apply any other existing first order method to solve it. Many literature abounds on this [1,2]. The drawback of this method is that it is time consuming, cumbersome to solve, and take much computer space. To circumvent these draw backs, many researchers have solved (1) directly, they include: [3,4,5,6] who developed block methods for numerical solution of fourth order ordinary differential equations. The works of [5,6] serve as improvement on the work of [4] who developed Linear multistep method for the solution of fourth order ordinary differential equations whose implementation is Predictor – Corrector mode.

We are motivated to advance the course of research work by continuing with the proposition of block method which have been shown to eliminate the drawbacks of Predictor corrector method as discussed in [5,7,8,9,10,11] in their works have proposed single Step hybrid methods for the direct numerical solution of initial value problems of second order and third order Ordinary differential equations respectively. In all Cases, their methods of implementation are block mode with the proposed methods being efficient, adequate and suitable towards catering for the class of problems for which they were designed.

Consequently, our motivation in this work is the success story of the adoption of single step method to solving higher order ordinary differential equations. Thus, in this work, we are proposing a single step method for the direct numerical solution of fourth order ordinary differential equations, which eliminates the use of predictors by providing sufficiently accurate simultaneous difference equations from a single continuous formula and its derivatives.

According to [12], the general block formula is given by:

\[ Y_m = c y_n + h^\mu df(y_n) + h^\mu bF(y_n) \]  

(2)
Where \( e \) is \( s \times s \) vector, \( d \) is \( r \times r \) vector and \( b \) is \( r \times r \) vector, \( s \) is the interpolation points and \( r \) is the collection points. \( F \) is a \( k \) – vector whose \( J \)th entry is \( f_{n+j} = f(t_{n+j}, y_{n+j}) \), \( \mu \) is the order of the differential equation.

Given a predictor equation in the form:

\[
Y_m^{(0)} = ey_n + h^\mu df(y_n).
\]

By Putting (3) in (2) we have:

\[
Y_m = ey_n + h^\mu df(y_n) + h^\mu bF(e y_n + h^\mu dfy_n).
\]

Equation (4) is called a self starting block-predictor-corrector method because the prediction equation is gotten directly from the block formula [13, 4].

Consequently, our focus in this paper is the proposition of an improved implicit continuous hybrid algorithm for the solution of initial value problems of fourth order ordinary differential equations.

2. DERIVATION OF THE METHODS

We take our basis function to be a power series of the form:

\[
y(x) = \sum_{j=0}^{r+s} a_j x^j
\]

(5)

The third derivative of (5) gives:

\[
y^{(iv)}(x) = \sum_{j=0}^{r+s} j(j-1)(j-2)(j-3)a_j x^{j-4}
\]

(6)

By putting (6) into (1) we have the differential system:

\[
\sum_{j=0}^{r+s} j(j-1)(j-2)(j-3)a_j x^{j-4} = f(x, y(x), y'(x), y''(x), y'''(x))
\]

(7)

Where \( a_j \) the parameters to be determined are, while \( r+s \) denotes the number of collocation and interpolation points. By collocating (7) at the mesh points \( x = x_{n+j}, j = 0(1/S)1 \), and interpolating (5) at \( x = x_{n+j}, j = 1/5, 2/5, 3/5, 4/5 \) yields a system of equations:
By putting these system of equations in matrix form and then solved to obtain the values of Parameters $a_j$’s, $j = 0, 1, 5, \ldots$ Which when substituted in (5), yields, after some manipulation, a hybrid linear method with continuous coefficients of the form:

$$y(x) = \sum_{j=0}^{1} \alpha_j y_{n+j}(x) + h^4 \sum_{j=0}^{1} \beta_j f_{n+j}(x)$$  \hspace{1cm} (10)

The coefficient of $\alpha_j(x)$ and $\beta_j$ are:

$$\alpha_0(t) = \frac{1}{3} t^3 + t^2 + t$$  \hspace{1cm} $\alpha_{\frac{1}{5}}(t) = -\frac{1}{5} t^3 + 15t^2 + 4t$$

$$\alpha_{\frac{1}{3}}(t) = \frac{1}{3} t^3 + 16t^2 + t$$  \hspace{1cm} $\alpha_{\frac{1}{5}}(t) = -\frac{1}{5} t^3 + 2t^2 + 15t + 2$$

$$\alpha_{\frac{1}{7}}(t) = \frac{1}{7} t^3 + t^2 + 3t + 2$$

$$\beta_0(t) = \frac{1}{450,000} (t^8 + 4t^7 - t^6 + 2t^5 - 3t^4 + t^3 - 4t - 1)$$

$$\beta_{\frac{1}{5}}(t) = \frac{1}{450,000} (3t^8 - 5t^7 - 124t^6 + 9t^5 - 3t^4 - 9t^3 + 5t)$$

$$\beta_{\frac{1}{3}}(t) = \frac{1}{450,000} (t^8 - 6t^7 - 2t^6 - 200t^5 - 274t^4 + 2t^3 - t - 6)$$

$$\beta_{\frac{1}{7}}(t) = \frac{1}{450,000} (4t^8 - 3t^7 + t^6 - 120t^5 - 4t^4 - t^3 + 3t - 4)$$

$$\beta_{\frac{1}{5}}(t) = \frac{1}{450,000} (t^8 + 9t^7 - 2t^6 + 7t^5 - 6t^4 - t^3 + 2t - 9)$$  \hspace{1cm} (11)

Where $t = \frac{x - x_n}{h}$
2.1 Derivation of the Block

The general block formula proposed by Awoyemi et al. [12], in the Normalized form is given by:

\[
A^{(0)}Y_m = ey_n + \mu^{\lambda-\lambda}dF(y_n) + \mu^{\lambda-\lambda}bF(y_m)
\]  

(12)

By evaluating (10) at \( t = 1 \); the first, second and the third derivative at \( x = x_{n+1}, i = 0(1/5)1 \) and substituting into (12) gives its coefficients as:

\[d = \begin{bmatrix}
49126 & 4264 & 25488 & 40448 & 3346 & 3229 & 317 & 5481 & 712 & 233 \\
1134000000 & 8859375 & 140000000 & 984 & 376 & 122 & 475 & 7875 & 56000 & 39375 & 8064 \\
2462 & 5680 & 984 & 376 & 122 & 475 & 7875 & 56000 & 39375 & 8064 & 19 \\
252000 & 252000 & 28000 & 7875 & 2016 & 7200 & 75600 & 800 & 255 & 288 \\
\end{bmatrix}^T\]

\[e = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5/5 & 5/5 & 5/5 & 2/2 & 3/3 & 4/4 & 1/1 & 1/1 & 1/1 & 1/1 & 1/1 & 0/0 & 0/0 & 0/0 & 1/1 \\
50/25 & 25/25 & 5/5 & 5/5 & 5/5 & 5/5 & 8/8 & 2/2 & 3/3 & 4/4 & 1/1 & 1/1 & 1/1 & 0/0 & 0/0 & 1/1 \\
1/1 & 2/2 & 9/9 & 4/4 & 32/32 & 1/1 & 1/1 & 1/1 & 1/1 & 1/1 & 1/1 & 1/1 & 1/1 & 1/1 \\
750/375 & 375/375 & 250/250 & 250/250 & 250/250 & 250/250 & 250/250 & 5/5 & 5/5 & 5/5 & 5/5 & 5/5 & 5/5 & 5/5 & 5/5 & 5/5 & 5/5 \\
\end{bmatrix}^T\]

\[A^0 = 20 \times 20 \text{ identity matrix}\]

\[B = \begin{bmatrix}
2780 & 15104 & 34668 & 114688 & 2140 & 21248 & 67068 & 139264 \\
30965760 & 7741440 & 3440640 & 3870720 & 1290240 & 1290240 & 1290240 & 1290240 \\
30965760 & -8472 & -10107 & -11492 & -15675 & -2365 & -8809 & -9877 & 5015 \\
30965760 & 716 & 2816 & 4860 & 16384 & 516 & 3328 & 8100 & 24576 \\
30965760 & -131 & -512 & -891 & -2560 & -94 & -608 & 1458 & -2560 \\
30965760 & 7741440 & 3440640 & 3870720 & 1290240 & 1290240 & 1290240 & 1290240 \\
45900 & 32256 & 156 & 24 & -264 & 2356 & 102 & 32 \\
1958400 & 322560 & 2560 & 90 & 2880 & 6840 & 320 & 90 \\
-23970 & -6735 & 18 & 6 & -264 & 456 & 72 & 12 \\
1958400 & 322560 & 2560 & 90 & 2880 & 6840 & 320 & 90 \\
9860 & 3584 & 20 & 8 & 106 & 76 & 42 & 32 \\
1958400 & 322560 & 2560 & 90 & 2880 & 6840 & 320 & 90 \\
-1792 & -672 & -3 & 0 & -19 & -18 & -3 & 7 \\
1958400 & 322560 & 2560 & 90 & 2880 & 6840 & 320 & 90 \\
\end{bmatrix}^T\]
3. ANALYSIS OF THE PROPERTIES OF THE BLOCK

In this section we carry out the analysis of the Basic properties of the new method.

3.1 Order of the Method

The linear operator of the block (12) is defined as:

\[ L\{y(x): h\} = Y_m - e y_m + h^{\mu-\lambda} df \left( y_m \right) + h^{\mu-\lambda} b F \left( y_m \right) \]  \hspace{1cm} (13)

By expanding \( y(x_n + ih) \) and \( f(x_n + jh) \) in Taylor series, (12) becomes:

\[ L\{y(x): h\} = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_p h^p y^{(p)}(x) + \]  \hspace{1cm} (14)

The block (11) and associated linear operator are said to have order \( p \) if

\[ C_0 = C_1 = \ldots = C_{p+1} = 0, C_{p+2} \neq 0. \]

The term \( C_{p+2} \) is called the error constant and implies that the local truncation error is given by:

\[ t_{n+k} = C_{p+2} h^{(p+2)} y^{(p+2)}(x_n) + O(h^{(p+3)}) \]  \hspace{1cm} (15)

Hence the block (12) has order 8 with error constant:

\[
C_{p+2} = \begin{bmatrix}
1355 & 1129 & 1129 & 71131 \\
6193152 & 2580480 & 1720320 & 81283860 \\
6117 & 2417 & 5227 & 6226 \\
671420 & 321560 & 1638200 & 31180194 \\
2104 & 1835 & 1292 & 2155 \\
891019 & 287923 & 298142 & 893287 \\
5105 & 4221 & 1942 & 28182 \\
1935152 & 2819456 & 1834231 & 341940
\end{bmatrix}^{T}
\]

3.2 Zero Stability of the Block

The block (11) is said to be Zero stable if the roots \( z_s = 1, 2, \ldots, N \) of the characteristic polynomial \( \rho(z) = \det(z A - E) \), satisfies \( |z| \leq 1 \) and the root \( |z| = 1 \) has multiplicity not exceeding the order of the differential equation. Moreover as \( h^\mu \rightarrow 0, \rho(z) = z^{-\mu} (\lambda - 1) \),
Where $\mu$ is the order of the differential equation, for the block (11), $r = 16, \mu = 4$

$$
\rho(z) = \lambda^{12} (\lambda - 1)^4 = 0
\Rightarrow \lambda = 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1
$$

Hence our method is Zero stable.

### 3.3 Convergence

The necessary and sufficient condition for a numerical method to be convergent is for it to be Zero stable and has order $p \geq 1$. Since our method has been shown to be zero stable and has order 8, it satisfies the above condition, thus our method is convergent.

### 4. NUMERICAL EXPERIMENTS

To test the accuracy, workability and suitability of the method, we adopted our method to solving some initial value problems of fourth order ordinary differential equations.

**Test problem 1.**

We consider a non linear fourth order problem:

$$
y^{(iv)} = \left(y'\right)^2 - y(y') - 4x^2 + e\left(1 - 4x + x^2\right) \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 3, \quad y'''(0) = 1,
$$

$$
h = \frac{1}{32}
$$

Whose exact solution is given by: $y(x) = x^2 + e^x$. The result is as shown in Table 1.

| XVAL     | ERC       | NRC       | ERR           |
|----------|-----------|-----------|---------------|
| 0.103125 | 1.1192647474787591900 | 1.11926474787634518 | 4.261834E-14  |
| 0.206250 | 1.2715994931980483000 | 1.27159949319779782 | 7.314820E-13  |
| 0.306250 | 1.4521109070651220000 | 1.45211090706637521 | 1.625321E-12  |
| 0.406250 | 1.6662168625001208000 | 1.666216862506568972 | 6.568972E-12  |
| 0.506250 | 1.9153471099209134000 | 1.915347109923762547 | 2.849147E-12  |
| 0.603125 | 2.2010817679089656000 | 2.201081767913196057 | 4.231457E-12  |
| 0.703125 | 2.5144402933700900000 | 2.514440293349477520 | 1.246852E-11  |
| 0.803125 | 2.8775163877466183000 | 2.87751638778097928 | 5.146142E-11  |
| 0.903125 | 3.2829361588051174000 | 3.282936158837654230 | 3.765423E-11  |
| 1.003125 | 3.7330495114952011000 | 3.733049511556738124 | 7.218014E-11  |

**Test problem 2.**

We consider special fourth order problem:

$$
y^{iv} = x; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0, \quad h = 0.1
$$
Whose exact solution is:  
\[ y(x) = \frac{x^5}{120} + x \]

Our method was used to solve the problem and result compared with [14]. The result is as shown in Table 2.

### Table 2. Showing results for problem 2

| XVAL | ERC             | NRC             | ERR in [14] | ERR   |
|------|-----------------|-----------------|-------------|-------|
| 0.1  | 0.10000000083333334000 | 0.10000000083333351720 | 1.832E-13   | 7.000E-10 |
| 0.2  | 0.200002666666666900 | 0.200002666667150250 | 4.835E-12   | 8.999E-10 |
| 0.3  | 0.300020250000000004 | 0.30002025000721480 | 7.214E-12   | 2.999E-09 |
| 0.4  | 0.400085333333333333 | 0.4000853340160457 | 4.835E-11   | 5.100E-09 |
| 0.5  | 0.500260416666666665 | 0.50026041674083458 | 7.416E-11   | 7.799E-09 |
| 0.6  | 0.600648000000000007 | 0.60064800002714565 | 2.714E-11   | 1.180E-08 |
| 0.7  | 0.701400583333333344 | 0.70140058361478378 | 2.815E-10   | 1.240E-08 |
| 0.8  | 0.802730666666666700 | 0.80273066700848838 | 3.412E-10   | 1.410E-08 |
| 0.9  | 0.904920750000000005 | 0.90492075019356814 | 1.936E-10   | 1.880E-08 |
| 1.0  | 1.008333333333333300 | 1.0083333361984509 | 2.865E-10   | 2.600E-08 |

4.1 Numerical Results

We make use of the following Notations in the table of results:

- **XVAL**: Value of the independent variable where numerical value is taken.
- **ERC**: Exact result at XVAL
- **NRC**: Our Numerical result at XVAL
- **ERR**: Error of our result at XVAL

5. CONCLUSION

In this paper, we have proposed a modified Implicit Hybrid Block algorithm for the numerical solution of initial value problems of fourth order ordinary differential equations. For better performance of the method, step size is chosen within the stability interval. The results of our new method when compared with the block method proposed by [14] showed that our method is more accurate.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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