Rotational energy term in the empirical formula for the yrast energies in even-even nuclei

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We show that part of the empirical formula describing the gross features of the measured yrast energies of the natural parity even multipole states for even-even nuclei can be related to the rotational energy of nuclei. When the first term of the empirical formula, \( \alpha A^{-\gamma} \), is regarded as the rotational energy, we can better understand the results of the previous analyses of the excitation energies. We show that the values of the parameters \( \alpha \) and \( \gamma \) newly obtained by considering the \( \alpha A^{-\gamma} \) term as the rotational energy of a rigid rotor are remarkably consistent with those values extracted from the earlier ‘modified’ \( \chi^2 \) analyses, in which we use the logarithms of the excitation energies in defining the ‘modified’ \( \chi^2 \) values.

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In a series of works [1, 2, 3, 4, 5], an empirical formula was proposed to represent the gross features of the yrast energies \( E_x \) of the natural parity even multipole states including \( 2^+ \), \( 4^+ \), \( 6^+ \), \( 8^+ \), and \( 10^+ \) for even-even nuclei throughout the whole periodic table. This formula is expressed in terms of the mass number \( A \), the valence proton number \( N_p \), and the valence neutron number \( N_n \)

\[
E_x = \alpha A^{-\gamma} + \beta_p \exp(-\lambda_p N_p) + \beta_n \exp(-\lambda_n N_n),
\]

where six free parameters \( \alpha, \gamma, \beta_p, \lambda_p \), and \( \lambda_n(i = p, n) \) are fixed so that the experimental excitation energies can be fitted by Eq. (1) for each multipole state [3]. This empirical formula has been quite successful not only in explaining the main features of the measured excitation energies as a function of mass number \( A \) but also in reproducing the characteristic simple patterns observed in the \( N_p N_n \)-plot of the measured excitation energies [1, 2, 4, 5]. In spite of the success of Eq. (1), however, it was not clear how this simple formula could represent the overall features of the yrast energies so well. Very recently, it was suggested in Ref. [3] that Eq. (1) could be approximated as

\[
E_{\text{mid}} \approx \alpha A^{-\gamma}
\]

around the doubly mid-shell region, where \( N_p \) and \( N_n \) are quite large. In this region, the two exponential terms in Eq. (1), \( \beta_p \exp(-\lambda_p N_p) + \beta_n \exp(-\lambda_n N_n) \), are very small compared with the first term, \( \alpha A^{-\gamma} \). Ref. [3] also suggested that the two parameters, \( \alpha \) and \( \gamma \), carry the information about the effective moment of inertia. However, it was not explained how these parameters could be related to the effective moment of inertia.

In this Brief Report, we will first show that the first term \( \alpha A^{-\gamma} \) can be indeed expressed in terms of the effective moment of inertia. Then we will redo some of the previous analyses in Ref. [3] but with the first term \( \alpha A^{-\gamma} \) fixed as the rotational energy of a rigid rotor, and compare the new results with the previous ones.

It is well known that a nucleus near the doubly mid-shell region has a rotational band, which consists of different total angular momenta \( J \) but shares the same intrinsic state. The energy spectrum of the rotational band for \( J^z = 2^+ \), \( 4^+ \), \( 6^+ \), · · · with the intrinsic angular momentum \( K = 0 \) can be written as [8]

\[
E_{\text{rot}}(J^+) = \frac{J(J+1)\hbar^2}{2I},
\]

where \( I \) is the effective moment of inertia of the nucleus. Let us assume that a nucleus is a rigid body which has the axial symmetry about the intrinsic 3 axis. The moment of inertia \( I_{\text{rig}} \) of such a rigid body can be expressed as

\[
I_{\text{rig}} = \frac{2}{3}MR_{\text{rig}}^2(1 + \frac{\delta}{3}),
\]

where \( M \) is the mass of the nucleus given by \( M = Au \), \( u \) being the atomic mass unit. The distortion parameter \( \delta \approx (R_3 - R_\perp)/R_0 \) is typically 0.2 ~ 0.3 for nuclei with 150 \( \leq A \leq 188 \), where \( R_3 \), \( R_\perp \), and \( R_0 \) are the radius of a nucleus along the intrinsic 3 axis, in the direction perpendicular to it, and the mean radius, \( R_0 = 1.2A^{1/3}\text{fm} \), respectively [8]. Since the observed moments of inertia are smaller than the moment of inertia given by the simple form in Eq. (4) by roughly a factor of 2 for nuclei around the doubly mid-shell region with 150 \( \leq A \leq 188 \), we may introduce a factor \( k \) to take into account the difference between the effective moment of inertia \( I \) and \( I_{\text{rig}} \) of Eq. (4) so that

\[
I = kI_{\text{rig}}.
\]

By inserting Eq. (5) into Eq. (3), we immediately get

\[
E_{\text{rot}}(J^+) = \alpha' A^{-\gamma'}
\]

where

\[
\alpha' = \alpha_0 J(J+1) \text{ and } \gamma' = \frac{5}{3}
\]

with

\[
\alpha_0 = \frac{5\hbar^2}{4\pi k 1.2^2(1 + \frac{\delta}{3})}.
\]
Table I: Six parameters $\alpha$, $\gamma$, $\beta_i$, and $\lambda_i(i = p, n)$ in the empirical formula of Eq. (1) are listed for three cases. In the upper part, four parameters $\beta_i$ and $\lambda_i$ determined by using fixed $\gamma = \gamma’ = 5/3$ and $\alpha = \alpha_0J(J + 1)$ with $\alpha_0 = 65.96$ MeV are listed: case i). In the middle part, $\alpha$, $\beta_i$, and $\lambda_i$ determined with fixed $\gamma = 1.40$ are listed: case ii). In the lower part, the parameters previously extracted are quoted from Table 2 of Ref. [3]: case iii). $N_0$ refers to the number of data points for each multipole state. The values of the ‘modified’ $\chi^2$ are also listed.

| $J^P$ | $\gamma$ | $\alpha$(MeV) | $\alpha_0$(MeV) | $\beta_p$(MeV) | $\beta_n$(MeV) | $\lambda_p$ | $\lambda_n$ | $\chi^2$ | $N_0$ |
|-------|---------|-------------|----------------|--------------|--------------|-----------|-----------|--------|------|
| $2^+$ | 1.67    | 305.76      | 65.96         | 0.79         | 1.00         | 0.42      | 0.29      | 0.157  | 557  |
| $4^+$ | 1.67    | 1319.20     | 65.96         | 1.12         | 1.54         | 0.34      | 0.24      | 0.094  | 430  |
| $6^+$ | 1.67    | 2770.32     | 65.96         | 1.31         | 1.46         | 0.32      | 0.18      | 0.086  | 375  |
| $8^+$ | 1.67    | 4749.12     | 65.96         | 1.27         | 1.34         | 0.26      | 0.17      | 0.060  | 309  |
| $10^+$| 1.67    | 7255.60     | 65.96         | 1.30         | 1.46         | 0.23      | 0.18      | 0.040  | 265  |
| $2^+$ | 1.40    | 89.89       | 14.98         | 0.82         | 1.15         | 0.41      | 0.28      | 0.126  | 557  |
| $4^+$ | 1.40    | 297.87      | 14.89         | 1.20         | 1.67         | 0.33      | 0.23      | 0.071  | 430  |
| $6^+$ | 1.40    | 654.71      | 15.59         | 1.40         | 1.64         | 0.31      | 0.18      | 0.069  | 375  |
| $8^+$ | 1.40    | 1155.90     | 16.05         | 1.34         | 1.50         | 0.26      | 0.15      | 0.053  | 309  |
| $10^+$| 1.40    | 1702.79     | 15.48         | 1.34         | 1.64         | 0.22      | 0.15      | 0.034  | 265  |
| $2^+$ | 1.34    | 68.37       | 11.40         | 0.83         | 1.17         | 0.42      | 0.28      | 0.126  | 557  |
| $4^+$ | 1.38    | 268.04      | 13.40         | 1.21         | 1.68         | 0.33      | 0.23      | 0.071  | 430  |
| $6^+$ | 1.38    | 598.17      | 14.24         | 1.40         | 1.64         | 0.31      | 0.18      | 0.069  | 375  |
| $8^+$ | 1.45    | 1438.59     | 19.98         | 1.34         | 1.50         | 0.26      | 0.15      | 0.053  | 309  |
| $10^+$| 1.47    | 2316.85     | 21.06         | 1.36         | 1.65         | 0.21      | 0.14      | 0.034  | 265  |

Eq. (6) shows that the first term $\alpha A^{\gamma}$ of the empirical formula in Eq. (1) can be derived and identified as the rotational energy of a deformed nucleus. However, this interpretation of the first term $\alpha A^{\gamma}$ is valid only for nuclei near the doubly mid-shell region, but not for closed shell nuclei or for light nuclei with $A \leq 30$, most of which do not have rotational bands. For closed shell nuclei, on the other hand, the first term $\alpha A^{\gamma}$ is negligible compared to the other two exponential terms, which will be shown later in Fig.3. Thus if we accept this interpretation to this extent, Eq. (6) shows that $\gamma$ in Eq. (1) can be identified as a constant $\gamma’$ close to 1.67 for all of the even multipole states. Note that the values of $\gamma$ in Eq. (1) extracted earlier and shown in Table 2 of Ref. [3] range from 1.34 to 1.47. These previous values of $\gamma$ are listed again in the lower part of Table I. The average of these previous $\gamma$ values is 1.40 which is remarkably close to $\gamma’ = 5/3$, in spite of the fact that the values of $\gamma$ were extracted earlier without any constraints on $\gamma$ or consideration presented in Eqs. (6) ~ (11).

Let us repeat the $\chi^2$ analyses as was done in Ref. [3] but with a constant average value of $\gamma = 1.40$. In the fitting procedure we use as in Ref. [3] the logarithm of the excitation energies $E_x^{\text{cal}}(i)$ and $E_x^{\text{exp}}(i)$ since the excitation energies are spread over a wide range. This ‘modified’ $\chi^2$ analysis is different from the conventional $\chi^2$ analysis in that the small values of the excitation energies are emphasized. By defining $R_E(i) = \log [E_x^{\text{cal}}(i)] - \log [E_x^{\text{exp}}(i)]$, the five parameters $\alpha$, $\beta_p$ and $\lambda_i$ are fixed to minimize $\chi^2 = \frac{1}{N_0} \sum_{i=1}^{N_0} [R_E(i)]^2$, where $N_0$ is the total number of the data points considered for the corresponding multipole state. $\alpha$, $\beta_p$, and $\lambda_i$ extracted in this way are given in the middle part of Table I. The values of $\alpha$ and $\alpha_0 = \alpha/J(J + 1)$ determined with $\gamma = 1.40$ are somewhat different from those of $\alpha$ and $\alpha_0$ in the lower part of Table I. On the other hand, the values of $\beta_p$ and $\lambda_i$ in the middle part of Table I are still very close to those values in the lower part for all the multipole states. This shows that the two exponential terms $\beta_p \exp(-\lambda_p N_p) + \beta_n \exp(-\lambda_n N_n)$ remain rather robust even if $\alpha$ and $\gamma$ change and thus that the two exponential terms can be well separated from the rotational energy term.

Let us now fix $\alpha$ and $\gamma$ as given by Eq. (7) for all $J$ and redo the ‘modified’ $\chi^2$ analyses for $2^+$ to $10^+$ states. In the analyses we use $k = 1/2$ and the distortion parameter $\delta = 0.3$ determined for $^{174}$Yb ($N_p = 12, N_n = 22$) which is a nucleus in the doubly mid-shell region [8]. Then the value of $\alpha_0$ becomes 65.96 MeV. In Table I we show that the ‘modified’ $\chi^2$ values obtained by using these fixed values of $\alpha’$ and $\gamma’$ of Eq. (7) turn out to be larger than those previously obtained [3] by 25% (for $2^+$), 32% (for $4^+$), 25% (for $6^+$), 13% (for $8^+$), and 18% (for $10^+$). Nevertheless, the excitation energies calculated by using fixed $\alpha_0 = 65.96$ MeV and $\gamma’ = 5/3$ appear almost identical to those obtained earlier in [3].

To compare the different values of $\alpha$ listed in Table I we plot in Fig. 1 the values of $\alpha$ and $\alpha_0$ against $J$ for three cases: i) $\gamma = \gamma’ = 1.67$ (circles), ii) $\gamma = 1.40$ (squares), and iii) $\gamma$ from Ref. [3] (triangles). $\alpha$’s for all three cases
FIG. 1: (Color online) The values of $\alpha$ and $\alpha_0$ in Table I are plotted for the following three cases: i) $\gamma = 1.67$ (circles), ii) $\gamma = 1.40$ (squares), and iii) $\gamma$ from Ref. [3] (triangles).

In Fig. 1 the values of $\alpha$ change as a quadratic function of $J$. Thus we plot in Fig. 1(b) $\alpha_0 = \alpha/J(J + 1)$. Figure 1(b) shows that $\alpha_0$'s are indeed constant for all three cases. It is remarkable to see that the values of $\alpha_0$ (triangles) obtained from those of $\alpha$ extracted previously in Ref. [3] turn out to be also almost constant. Figure 1 shows that the values of $\alpha$ for the cases ii) and iii) are very close to each other but are smaller than those for the case i) roughly by a factor 4. This factor is to compensate for the changes in $\alpha A^{-\gamma}$ due to the changes in $\gamma$ from about 1.40 to 1.67. Note that the ratio $A^{-1.40}/A^{-1.67} = A^{0.27}$ is about 4 for $^{174}$Yb.

In Fig. 2 the value of $\alpha A^{-\gamma}$ term with $\alpha$ and $\gamma$ as given in Ref. [3] is plotted by the solid curve, and that with $\gamma = 1.40$ is plotted by the dashed curve. They agree very well. The values of $E_{\text{rot}}(J^+)$ of Eq. (6) with $\alpha'$ and $\gamma'$ given in Eq. (7) are plotted by the dotted curve, which also agree with the other curves quite well for larger $A$. This shows that the values of $\alpha$ compensate for the changes in the values of $\gamma$ so that $\alpha A^{-\gamma}$ remains more or less the same. Note, however, that the dotted curve deviates from the other two curves for small values of $A$. It is because we use the logarithm of energies rather than the energies themselves for the 'modified' $\chi^2$ calculations. From the $\chi^2$ analyses with $\alpha_0 = 65.96$ MeV and $\gamma = 1.67$ we obtain new parameters $\beta_i$ and $\lambda_i$ listed in the upper part of Table II. The new parameters of $\beta_i$ and $\lambda_i$ are still very close to those values in the middle and lower parts. As mentioned earlier, the parameters of $\beta_i$ and $\lambda_i$ are almost the same for three cases of i) $\sim$ iii), implying that the two exponential terms are rather independent of the rotational energy term $\alpha A^{-\gamma}$.

In Fig. 2 we show the yrast energies for each state of $2^+_1$ (a), $4^+_1$ (b), $6^+_1$ (c), $8^+_1$ (d), and $10^+_1$ (e) against the mass number $A$. For each $J^+$ state the experimental yrast energies are plotted in the upper part and are compared with the calculated values in the lower part obtained by using the parameters given in the upper part of Table II. The measured excitation energies of the first 2$^+$ states, because the ratios for the other states are given in Fig. 1. We plot the ratios only for the first $4^+ \sim 10^+$ are extracted from Ref. [10]. The data points are connected by the solid line along the isotopic chains. Figure 2 shows that the dashed curve is still very close to those values in the middle and lower parts. As mentioned earlier, the parameters of $\beta_i$ and $\lambda_i$ are almost the same for three cases of i) $\sim$ iii), implying that the two exponential terms are rather independent of the rotational energy term $\alpha A^{-\gamma}$.
in Fig. 2) overshoots $\alpha A^{-\gamma}$ with $\gamma = 1.40$ (dashed curve in Fig. 2) for small values of $A$. Finally we show in the bottom part of Fig. 3(a) the separation of the calculated energies into the first term $\alpha A^{-\gamma}$ and the two exponential terms. The first term $\alpha A^{-\gamma}$ (the solid curve) is the major contribution in the yrast energies in the doubly mid-shell region while it is negligible compared to the two exponential terms (circles) in the closed shell region.

In summary, we have shown that the term $\alpha A^{-\gamma}$ can be obtained by considering the moment of inertia of a deformed nucleus. The yrast energies calculated with constant $\alpha'$ and $\gamma'$ can describe the main features of the data. It is remarkable that the values of $\alpha A^{-\gamma}$ extracted earlier in Ref. [3] agree well with the values of $\alpha' A^{-\gamma'}$ obtained from the rotor model and that the previous values of $\alpha$ divided by $J(J+1)$ are almost constant as expected from the rotor model. It is also seen that the parameters $\beta_i$ and $\lambda_i$ newly extracted with constant $\alpha'$ and $\gamma'$ in this work are very consistent with those parameters previously obtained in Ref. [3]. It shows that the empirical formula can be well separated into the rotational energy term $\alpha A^{-\gamma}$ and the two exponential terms which are thought to be related to the shell effect.

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FIG. 4: The ratios of the calculated excitation energies to the measured energies for the $2_1^+$ and $8_1^+$, $E_{x}^{\text{cal}}/E_{x}^{\text{exp}}$, are plotted against the mass number $A$.

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