GENERALIZED LEIBNIZ’S RULE
ON A TWO-STEP NILPOTENT LIE GROUP

KRYSTIAN BEKALA

Abstract. For a function $f$ on $G$ and a multiindex $\alpha \in \mathbb{N}^{\dim G}$, let $T^{\alpha} f(x) = x^{\alpha} f(x)$. We find and prove a formula for $T^{\alpha} (f * g)$ for any $\alpha \in \mathbb{N}^{\dim G}$ and Schwartz functions $f, g$, where $*$ is the convolution on $G$. As an application we get the formula for $D^{\alpha} (f \# g)$, where $f \# g = (f^{\vee} * g^{\vee})^{\wedge}$ and $^{\wedge}$ is the Abelian Fourier Transform.

In the case of the Abelian group $\mathbb{R}^{d}$ we have $f \# g = fg$, so $D^{\alpha} (f \# g)$ (and by the Fourier transform also $T^{\alpha} (f * g)$) is given by the Leibniz rule.

1. Statement of the result

The problem of finding an explicit formula for the derivatives of the product of the two functions goes back to the beginning of the differential calculus. Let $f, g$ be real functions defined on an open interval of $\mathbb{R}$. If $f$ and $g$ are $k$ times differentiable, then

\begin{equation}
(fg)^{(k)} = \sum_{j=0}^{n} \binom{k}{j} f^{(j)} g^{(k-j)}.
\end{equation}

The formula (1.1) is known as the Leibniz rule in honour of its author (see [6]).

The Leibniz rule was generalized in many ways, e.g. for fractional derivatives (see [4], [9]), more than two functions and partial derivatives. If $f, g$ are Schwartz functions on $\mathbb{R}^{d}$ and $\alpha \in \mathbb{N}^{d}$ is a multiindex, then

\begin{equation}
D^{\alpha} (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\beta} f D^{\alpha-\beta} g.
\end{equation}

The (inverse) Fourier transform turn the product of function into the convolution product of them so,

\begin{equation}
T^{\alpha} (f *_{0} g) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} T^{\beta} f *_{0} T^{\alpha-\beta} g,
\end{equation}

where $T^{\alpha} f(x) = x^{\alpha} f(x)$ and $*_{0}$ is the standard convolution. On the other hand

\begin{equation}
x^{\alpha} = ((x - y) + y)^{\alpha} = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (x - y)^{\beta} y^{\alpha-\beta},
\end{equation}

so we can get (1.2) also by the binomial theorem.

2010 Mathematics Subject Classification. Primary 22E25; Secondary 22E15.

Key words and phrases. Leibniz’s rule, nilpotent Lie group, symbolic calculus, convolution.
Let $\mathfrak{g}$ be a nilpotent Lie algebra with a fixed scalar product. The dual vector space $\mathfrak{g}^*$ will be identified with $\mathfrak{g}$ by means of the scalar product. We shall also regard $\mathfrak{g}$ as a Lie group with the Baker-Campbell-Hausdorff multiplication (see [1])

$$x \circ y = x + y + r(x, y),$$

where $r(x, y)$ is the (finite) sum of commutator terms of order at least 2 in the Baker-Campbell-Hausdorff series for $\mathfrak{g}$. Suppose that the underlying manifold of the nilpotent Lie group is $\mathbb{R}^d$. Let $A = (a_{i,j,k})_{i,j,k}$ be the matrix of structural group constants given by

$$[X_i, X_j] = \sum_{k=1}^{d} a_{i,j,k}X_k, \quad 1 \leq i, j, k \leq d.$$  

The convolution $*$ on the nilpotent Lie group $\mathfrak{g}$ is associated with the multiplication $\circ$ and it is different from the standard convolution $*$ on the Abelian group $\mathbb{R}^d$. Consequently, the formula for $T^\gamma(f * g)$ is different. Glowacki in [2] showed that for every $f, g \in S(\mathfrak{g})$ and every multiindex $\gamma \neq 0$

$$T^\gamma(f * g) = T^\gamma f * g + f * T^\gamma g + \sum\limits_{d(\alpha)+d(\beta)=d(\gamma)} c_{\alpha\beta} T^\alpha f * T^\beta g,$$

for some real $c_{\alpha,\beta}$, where $d(\alpha)$ is the homogeneous length of multiindex $\alpha$. This formula does not give us exact values of $c_{\alpha,\beta}$ (in fact, most of them are equal to zero). The purpose of this note is to find the exact formula of $T^\gamma(f * g)$ on a nilpotent Lie group. We focus on the case of two-step nilpotent Lie group.

To formulate the main result we introduce notation and definitions. Some details will be explained in Section [2]. For the dimension $d$ of $\mathfrak{g}$, let $\overline{d} \geq d$ be a number of elements of the set

$$\overline{D} = \{1, 2, ..., d\} \cup \{(i, j) : a_{i,j,k} \neq 0, 1 \leq k \leq d\}.$$  

Note that we can index coordinates of a multiindex $\alpha \in \mathbb{N}^{\overline{d}}$ by numbers $1, 2, ..., \overline{d}$ or by elements of $\overline{D}$. Note also that we can consider a multiindex $\alpha \in \mathbb{N}^d$ as $\alpha = (\alpha_0, 0, ..., 0) \in \mathbb{N}^{\overline{d}}$. Let $\alpha \in \mathbb{N}^{\overline{d}}$ be a multiindex. We define the multiindices $\alpha[1], \alpha[2], \alpha[3] \in \mathbb{N}^d$ by

$$\alpha[1]_i = \alpha_i + \sum_{a_{i,j,k} \neq 0} \alpha(i,j), \quad \alpha[2]_j = \alpha_j + \sum_{a_{i,j,k} \neq 0} \alpha(i,j), \quad \alpha[3]_k = \alpha_k + \sum_{a_{i,j,k} \neq 0} \alpha(i,j).$$  

Note that if $\alpha \in \mathbb{N}^d$ is consider as multiindex from $\mathbb{N}^{\overline{d}}$, then $\alpha[1] = \alpha[2] = \alpha[3] = \alpha$. Let $\alpha, \beta \in \mathbb{N}^{\overline{d}}$. If $\alpha[3] \geq \beta[3]$ we define also the generalized multinomial coefficient

$$\binom{\alpha}{\beta} = \frac{\prod_{k=1}^{\overline{d}} \alpha_k!}{\prod_{k=1}^{\overline{d}} \beta_k! \prod_{k=1}^{d}(\alpha[3]_k - \beta[3]_k)!}.$$  

If $\alpha, \beta \in \mathbb{N}^d$ are consider as multiindices from $\mathbb{N}^{\overline{d}}$ and $\alpha \geq \beta$, then $\binom{\alpha}{\beta} = \binom{\alpha}{\beta}$. For $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^{\overline{d}}$ such that $\alpha \geq \beta[3]$, let $\alpha \beta^{-1}$ denote an element

$$(\alpha \beta^{-1})_k = \alpha_k - \beta[3]_k, \quad k = 1, 2, ..., d, \quad (\alpha \beta^{-1})_{(i,j)} = \beta_{(i,j)}, \quad (i, j) \in \overline{D}. $$
We will show afterwards that \( \alpha \beta^{-1} \) is the substraction of \( \alpha \) and \( \beta \) with respect to some operation \( \circ \). Moreover \( (\alpha \beta^{-1})[3] \leq \alpha \) and \( (\alpha \beta^{-1})[3] = (\alpha \beta^{-1})[3] \). Now, we can formulate the convolution rule on the two-step Lie group \( g \) as follows.

**Theorem 1.4.** For every \( f, g \in S(g) \) and every multiindex \( \gamma \in \mathbb{N}^d \)

\[
T^{\gamma}(f * g) = \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} c_{\delta, \gamma} T^{\delta[1]} f * T^{(\gamma[1]-1)[2]} g,
\]

where the constants \( c_{\delta, \gamma} \) are given by

\[
c_{\gamma, \delta} = 2^{-\sum_{i,j,k \neq 0} \delta(i,j,k)} \prod_{a_{i,j,k} \neq 0} a_{i,j,k}.
\]

As was explained above, the rule \( (1.5) \) is a natural generalization of \( (1.2) \). And if we use this result to the Abelian group \( \mathbb{R}^d \) we get \( (1.2) \).

Theorem \( (1.5) \) can be applied to the symbolic calculus on a nilpotent Lie group, which can be viewed as a higher order generalization of the Weyl calculus for pseudodifferential operators of Hörmander \[5\]. The calculus was created in \[8\] and developed in \[7\], \[3\]. The idea of such calculus consists in describing the product \( f \# g = (f \uparrow g \uparrow) \wedge \).

The natural problem is to find a similar formula on a general nilpotent Lie group and in the case of fractional derivatives.

## 2. Multindices on a two-step nilpotent Lie group

### 2.1. Two-step nilpotent Lie group. Let \( g \) be two-step nilpotent Lie algebra \( g = g_0 \oplus \mathfrak{g} \) and a Lie group with the Baker-Campbell-Hausdorff multiplication

\[
x \circ y = x + y + \frac{1}{2} [x, y].
\]

Let \( \dim g_0 = d_1 \), \( \dim g = d \). Suppose that \( X_1, \ldots, X_d \) is the base for Lie algebra \( g \). Then

\[
x \circ y = \sum_{j=1}^{d} x_j X_j \circ \sum_{i=1}^{d} y_i X_i = \sum_{k=1}^{d} (x_k + y_k) X_k + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} x_i y_j [X_i, X_j].
\]

Suppose that \( A = (a_{i,j,k})_{i,j,k} \) is a matrix given by

\[
[X_i, X_j] = \sum_{k=1}^{d} a_{i,j,k} X_k, \quad 1 \leq i, j, k \leq d.
\]

It is clear that \( a_{i,j,k} = -a_{j,i,k} \) and \( a_{i,j,k} = 0 \), if any of conditions \( i = j \), \( i \uparrow j \geq k \), \( i \uparrow j \geq d_1 \), \( k \leq d_1 \) is satisfied. Thus,

\[
x \circ y = \sum_{k=1}^{d} (x_k + y_k + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j,k} x_i y_j) X_k,
\]
and one can write the group multiplication in coordinates as
\[(x_1, x_2, ..., x_d) \circ (y_1, y_2, ..., y_d) = (x_1 + y_1, ..., x_{d_1} + y_{d_1}, x_{d_1+1} + y_{d_1+1} + r_{d_1+1}(x, y), ..., x_d + y_d + r_d(x, y)),\]
where for every \(k > d_1,\)
\[r_k(x, y) = \frac{1}{2} \sum_{i=1}^{d_1} \sum_{j=1}^{d_1} a_{i,j,k} x_i y_j.\]
As a consequence,
\[x \circ y^{-1} = (x_1 - y_1, ..., x_{d_1} - y_{d_1}, x_{d_1+1} - y_{d_1+1} - r_{d_1+1}(x, y), ..., x_d - y_d - r_d(x, y)).\]
Let
\[T_j f(x) = x_j f(x), \quad D_j f(x) = i f'(x) e_j,\]
and
\[T^\alpha f(x) = x^\alpha f(x), \quad D^\alpha f(x) = D_1^\alpha ... D_d^\alpha f(x).\]
The Schwartz space is denoted by \(S(\mathfrak{g}).\) Let Lebesgue measures \(dx, d\xi\) on \(\mathfrak{g}\) and \(\mathfrak{g}^*\) be normalized so that the relationship between a function \(f \in S(\mathfrak{g})\) and its Abelian Fourier Transform \(\hat{f} \in S(\mathfrak{g})\) is given by
\[(2.1) \quad \hat{f}(\xi) = \int_{X} e^{-ix\xi} f(x) dx, \quad f(x) = \int_{X^*} e^{ix\xi} \hat{f}(\xi) d\xi.\]
A normalized Lebesgue measure on the vector space \(\mathfrak{g}\) is the normalized Haar measure on the group \(\mathfrak{g}.\) The convolution \(*\) on \(\mathfrak{g}\) is given by
\[(2.2) \quad f * g(x) = \int_{\mathfrak{g}} f(x \circ y^{-1}) g(y) dy.\]

2.2. Multiindices. In the Section \(\text{[1]}\) we introduce some notation and definitions. Recall that we denoted the set \(\mathcal{D},\) the number \(d,\) which is depend from the group structure. For \(\alpha \in \mathbb{N}^d\) we also denoted multiindices \(\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}^d.\) Let us also recall that we defined the generalized multinomial coefficient \(\binom{\alpha}{\beta}_{\mathfrak{g}},\) which for \(\alpha, \beta \in \mathbb{N}^d,\) agrees with (ordinary) multinomial coefficient.

Example 2.3. Let \(\mathfrak{h}\) be a \(2n + 1\)-dimensional Euclidean space which is Lie algebra and simultaneously a group with the multiplication
\[(x_1, ..., x_{2n+1}) \circ (y_1, ..., y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2} \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)).\]
The group is a model of the Heisenberg group. Thus, the matrix \(A\) is given by
\[a_{i,n+i,2n+1} = 1, \quad a_{n+i,i,2n+1} = -1, \quad i = 1, ..., n\]
and \(a_{i,j,k} = 0\) otherwise. We have
\[\mathcal{D} = \overline{2N + 1} = \{1, ..., 2n + 1, (1, n + 1), ..., (n, 2n), (n + 1, 1), ..., (2n, n)\},\]
and then $d = 2n + 1 = 4n + 1$. Let $\alpha \in \mathbb{N}^{2n+1}$. Then $\alpha[1], \alpha[2], \alpha[3]$ are given by

$$\alpha[1] = (\alpha_1 + \alpha_{1(n+1)}, \ldots, \alpha_{2n} + \alpha_{(2n,n), \alpha_{2n+1}}),$$

$$\alpha[2] = (\alpha_1 + \alpha_{(n+1,1)}, \ldots, \alpha_{2n} + \alpha_{(n,2n), \alpha_{2n+1}}),$$

$$\alpha[3] = (\alpha_1, \ldots, \alpha_{2n}, \alpha_{2n+1} + \sum_{i=1}^{n} (\alpha_{(i,n+1)} + \alpha_{(n+i,i)})).$$

**Definition 2.4.** Let $\alpha, \beta \in \mathbb{N}^d$. We define the operation $\alpha \circ \beta$ by

$$(\alpha \circ \beta)_k = \alpha_k + \beta_k, \quad k = 1, \ldots, d_1,$$

$$(\alpha \circ \beta)_k = \alpha_k + \beta_k + \sum_{a_{i,j,k} \neq 0} \min (\alpha_{(i,j)}, \beta_{(i,j)}), \quad k = d_1 + 1, \ldots, d,$$

$$\alpha \circ \beta_{(i,j)} = \alpha_{(i,j)} + \beta_{(i,j)} - 2 \min (\alpha_{(i,j)} + \beta_{(i,j)}), \quad (i, j) \in \overline{D}.$$ 

It is easy to see that $\circ$ is commutative, but not associative and $(0, \ldots, 0)$ is the neutral element. Note that for $\alpha, \beta \in \mathbb{N}^d$, then $\alpha \circ \beta = \alpha + \beta$. If $\alpha \in \mathbb{N}^d$ and $\alpha \geq \beta[3]$ we can define the subtraction $a / \beta^{-1}$ with respect to $\circ$.

**Proposition 2.5.** Suppose that $\beta \in \mathbb{N}^d$, $\alpha \in \mathbb{N}^d$ and $\beta[3] \leq \alpha$. There exists unique $\gamma \in \mathbb{N}^d$ such that

$$(2.6) \quad \beta \circ \gamma = \gamma \circ \beta = \alpha.$$ 

Such element will be denoted by $\alpha / \beta^{-1}$ and is given by

$$(2.7) \quad (\alpha / \beta^{-1})_k = \alpha_k - \beta[3]_k, \quad k = 1, 2, \ldots, d, \quad (\alpha / \beta^{-1})_{(i,j)} = \beta_{(i,j)}, \quad (i, j) \in \overline{D}.$$ 

Moreover $(\alpha / \beta^{-1})[3] \leq \alpha$ and $((a / \beta)_{0} = (\alpha / \alpha_{\beta^{-1}})$.

**Proof.** It is easy to check that the condition $\alpha / \beta^{-1}$ given by (2.7) satisfies (2.6). We show that there is exactly one such element. Suppose that

$$\beta \circ \gamma^1 = \beta \circ \gamma^2 = \alpha,$$

for $\gamma_1 \neq \gamma_2 \in \mathbb{N}^d$. For $k = 1, \ldots, d_1$, we have

$$\beta_k + \gamma^1_k = \beta_k + \gamma^2_k = \alpha_k,$$

so then $\gamma^1_k = \gamma^2_k = \alpha_k - \beta_k$. For every $(i, j) \in \overline{D}$ the condition

$$\beta_{(i,j)} + \gamma^1_{(i,j)} - 2 \min (\beta_{(i,j)}, \gamma^1_{(i,j)})$$

implies that $\gamma^1_{(i,j)} = \beta_{(i,j)}$. Similarly, $\gamma^2_{(i,j)} = \beta_{(i,j)}$. For $k = d_1 + 1, \ldots, d$ we have

$$\beta_k + \gamma^1_k + \sum_{a_{i,j,k} \neq 0} \min (\beta_{(i,j)}, \gamma^1_{(i,j)}) = \beta_k + \gamma^2_k + \sum_{a_{i,j,k} \neq 0} \min (\beta_{(i,j)}, \gamma^2_{(i,j)}) = \alpha_k,$$

so with $\gamma^1_{(i,j)} = \gamma^2_{(i,j)}$ we get $\gamma^1_k = \gamma^2_k = \alpha_k - \sum_{a_{i,j,k} \neq 0} \beta_{(i,j)}$. 

LEIBNIZ'S RULE ON A TWO-STEP NILPOTENT LIE GROUP 5
For \( k = 1, \ldots, d \) it follows that \( \beta_{[3],k} + (\alpha \beta^{-1})_k = \alpha_k \) and \((\alpha \beta^{-1})_{[3],k} + \beta_k = \alpha_k \). Thus, \((\alpha \beta^{-1})_{[3]} \leq \alpha \). With \( \beta_{(i,j)} = (\alpha \beta^{-1})_{(i,j)} \) we get

\[
\left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right)_g = \prod_{k=1}^d \frac{\alpha_k!}{\beta_k! \prod_{(i,j) \in \mathcal{D}} \beta_{(i,j)}! \prod_{k=1}^d (\alpha_k - \beta_{[3],k})!} = \prod_{k=1}^d (\alpha \beta^{-1})_k! \prod_{(i,j) \in \mathcal{D}} (\alpha \beta^{-1})_{(i,j)}! \prod_{k=1}^d (\alpha_k - (\alpha \beta^{-1})_{[3],k})! = \left( \begin{array}{c}
\alpha \\
(\alpha \beta^{-1})
\end{array} \right).
\]

This completes the proof. \( \square \)

In other words, the subtraction of \( \alpha \in \mathbb{N}^d \) and \( \beta \in \mathbb{N}^d \) is well-defined if \( \alpha \geq \beta_{[3]} \) and it agrees with generalized multinomial coefficient.

3. Leibniz’s Rule on a Two-Step Nilpotent Lie Group

3.1. The proof of Theorem 1.5. Let us assume for a moment that we are in the Heisenberg group \( h \) case. It is directly checked that for \( j = 1, \ldots, 2n \)

\[ T_j(f \ast g) = T_jf \ast g + f \ast T_jg, \]

and

\[ T_{2n+1}(f \ast g) = T_{2n+1}f \ast g + f \ast T_{2n+1}g + \frac{1}{2} \sum_{j=1}^n (T_jf \ast T_{n+j}g - T_{n+j}f \ast T_jg). \]

The proof of the general case is as follows.

Proof. Let \( \gamma \in \mathbb{N}^d \). We are going to find the analogue of (1.3) on the two-step nilpotent Lie group. We have

\[
(3.1) \quad x^\gamma = \prod_{k=1}^d x_k^{\gamma_k} = \prod_{k=1}^{d_1} ((x \circ y^{-1})_k + y_k)^{\gamma_k} \prod_{k=d_1+1}^d ((x \circ y^{-1})_k + y_k + r_k(x, y))^{\gamma_k} \\
= \prod_{k=1}^{d_1} \sum_{\delta_k \leq \gamma_k} \frac{\gamma_k}{\delta_k} (x \circ y^{-1})_k^\delta y_k^{\gamma_k - \delta_k} \prod_{k=d_1+1}^d \sum_{\delta_{k,1} = \gamma_k} \left( \frac{\gamma_k}{\delta_{k,1}} \right)(x \circ y^{-1})_k^{\delta_{k,1}} y_k^{\delta_{k,2}} r_k(x, y)^{\delta_{k,3}} \\
= \sum_{\delta_k \leq \gamma_k} \sum_{\sum_{k=1}^3 \delta_{k,i} = \gamma_k} \prod_{k=d_1+1}^{d_1} \left( \frac{\gamma_k}{\delta_k} \right) \prod_{k=d_1+1}^d \left( \frac{\gamma_k}{\delta_{k,1}} \right) \prod_{k=d_1+1}^d y_k^{\delta_{k,2}} \prod_{k=d_1+1}^d r_k(x, y)^{\delta_{k,3}}.
\]

From the condition \( a_{i,j,k} = -a_{j,i,k} \),

\[
r_k(x, y) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{i,j,k} x_i y_j = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{i,j,k} (x_i - y_i) y_j,
\]

\[
r_k(x, y) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{i,j,k} x_i y_j = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{i,j,k} (x_i - y_i) y_j,
\]

\[
r_k(x, y) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{i,j,k} x_i y_j = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{i,j,k} (x_i - y_i) y_j,
\]
and then

\[
(3.2) \quad r_k(x, y)^{\delta_{k,3}} = \left( \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j,k}(x_i - x_j)y_j \right)^{\delta_{k,3}}
\]

\[
= 2^{-\delta_{k,3}} \sum_{(i,j)} \delta_{k,3,i,j} \prod_{(i,j)} (a_{i,j,k}(x \circ y^{-1})_i y_j)^{\delta_{k,3,i,j}}
\]

\[
= 2^{-\delta_{k,3}} \sum_{(i,j)} \delta_{k,3,i,j} \prod_{(i,j)} a_{i,j,k} \prod_{(i,j)} (x \circ y^{-1})_i \prod_{(i,j)} y_j^{\delta_{k,3,i,j}}.
\]

By using (3.2), the expression from (3.1) is equal to

\[
(3.3) \quad \sum \prod_{k=1}^{d_1} \left( \frac{\gamma_k}{\delta_k} \right) \prod_{k=d_1+1}^{d} \left( \frac{\gamma_k}{\delta_{k,1}\delta_{k,2}\delta_{k,3}} \right) \prod_{(i,j)} \left( \delta_{k,3} \right)^{\delta_{k,3,i,j} \cdots} = \prod_{k=1}^{d} \left( \frac{\gamma_k!}{\prod_{k=1}^{d_1} \delta_{k,1}! \prod_{k=1}^{d} \delta_{k,3}! (\gamma_k - \delta_{[3],k})!} \right)^{\gamma \delta \delta_{[3]} \gamma \delta}.
\]

For simplify the notation we denote for \( k > d_1 \), \( \delta_k = \delta_{k,1}, \delta_{(i,j)} = \delta_{k,3,i,j} \) (for a given \( (i,j) \), the number \( k \) is uniquely determinate). Notice that \( \delta_{k,2} = \gamma_k - \delta_{k,1} - \delta_{k,3} = \gamma_k - \delta_{[3],k} \). We obtain

\[
\prod_{k=1}^{d_1} \left( \frac{\gamma_k}{\delta_k} \right) \prod_{k=d_1+1}^{d} \left( \frac{\gamma_k}{\delta_{k,1}\delta_{k,2}\delta_{k,3}} \right) \prod_{(i,j)} \left( \delta_{k,3} \right)^{\delta_{k,3,i,j} \cdots} = \prod_{k=1}^{d} \left( \frac{\gamma_k!}{\prod_{k=1}^{d_1} \delta_{k,1}! \prod_{k=1}^{d} \delta_{k,3}! (\gamma_k - \delta_{[3],k})!} \right)^{\gamma \delta \delta_{[3]} \gamma \delta}.
\]

Moreover, the conditions \( \delta_k \leq \gamma_k, k = 1, 2, \ldots, d_1 \) and \( \sum_{l=1}^{3} \delta_{k,l} = \gamma_k, \sum_{(i,j)} \delta_{k,3,i,j} = \delta_{k,3}, k = d_1, \ldots, d \) we can simply write as \( \delta_{[3],k} \leq \gamma_k, so \delta_{[3]} \leq (\gamma, 0, \ldots, 0) \). Set also

\[
c_{\gamma, \delta} = 2^{-\sum_{a_{i,j,k} \neq 0} \delta_{(i,j)}} \prod_{a_{i,j,k} \neq 0} \delta_{(i,j)}.
\]

Thus, (3.3) is equal to

\[
\sum_{N \geq \delta_{[3]} \leq \gamma} \left( \frac{\gamma}{\delta} \right) c_{\gamma, \delta} \prod_{k=1}^{d} (x \circ y^{-1})_k^{\delta_{k}} \prod_{(i,j)} (x \circ y^{-1})_i^{\delta_{(i,j)}} \prod_{k=1}^{d} y_k^{\gamma_k - \delta_k} \prod_{j=1}^{d} y_j^{\delta_{(i,j)}}
\]

\[
= \sum_{N \geq \delta_{[3]} \leq \gamma} \left( \frac{\gamma}{\delta} \right) c_{\gamma, \delta} \prod_{i=1}^{d} (x \circ y^{-1})_i^{\delta_i + \sum_{a_{i,j,k} \neq 0} \delta_{(i,j)}} \prod_{j=1}^{d} y_j^{\gamma_j - \delta_j + \sum_{a_{i,j,k} \neq 0} \delta_{(i,j)}}
\]

\[
= \sum_{\delta_{[3]} \leq \gamma} \left( \frac{\gamma}{\delta} \right) c_{\delta, \gamma} (x \circ y^{-1})_{\delta(1)} y^{(\gamma \delta^{-1})[2]}.
\]

By the definition of the convolution (2.2), we get the thesis. \( \Box \)
3.2. **Leibniz’s rule for the product** $f \# g$. The result from the previous section can be applied to the symbolic calculus on a nilpotent Lie group. The idea of such calculus consists in describing the product $f \# g = (f^\ast * g^\ast)^\wedge$ In the case of Abelian group $\mathbb{R}^d$ we have $f \# g = fg$ and $D^\alpha(fg)$ is given by the Leibniz rule. By applying the inverse Fourier transform to the Theorem 1.5 we get a formula for derivatives for the product $f \# g$.

**Corollary 3.4.** On two-step nilpotent Lie group $\mathfrak{g} \cong \mathbb{R}^d$

$$D^\alpha(f \# g) = \sum_{\delta \leq \gamma} \left(\begin{array}{c} \gamma \\ \delta \end{array}\right)_{\mathfrak{g}} c_{\delta, \gamma} (D^{\delta_{[1]} f})(\#)(D^{(\gamma_{[1]} - \delta_{[1]})_{[2]} g}).$$

where the constants $c_{\delta, \gamma}$ are given by

$$c_{\gamma, \delta} = 2^{-\sum_{\alpha_{i,j,k} \neq 0} \delta(i,j)} \prod_{\alpha_{i,j,k} \neq 0} \delta_{i,j,k}.$$ 

**Acknowledgements.** The author is grateful for conversations on the subject of paper to P.Głowacki and M.Preisner.

**References**

[1] L. Corwin, F.P. Greenleaf *Representations of nilpotent Lie groups and their applications. Part 1: Basic theory and examples*, Cambridge University Press, Cambridge 1990.

[2] P. Głowacki *The algebra of Calderon-Zygmund Kernels on a Homogeneous Group is inverse-closed*, Preprint (2014).

[3] P. Głowacki *The Melin calculus for general homogeneous groups*, Arkiv mat. 45 (2007), 31-48.

[4] A. K. Grünwald *Über Begrenzte Derivation - und deren Anwendung*, Zeitschrift für Mathematik und Physik 12 (1867), 441-480.

[5] L. Hörmander *The Weyl calculus of pseudodifferential operators*, Comm. Pure Appl. Math. 32 (1979), 359-443.

[6] G.W. Leibniz *Pro methodo tangenitum universa et aliis tetragonisticis specimina et inventa*, Infinitesimalmathematik (1675), 356-371.

[7] D. Manchon *Formule de Weyl pour les groupes de Lie nilpotents*, J. reine angew. Math. 418 (1991), 77-129.

[8] A. Melin *Parametrix constructions for right-invariant differential operators on nilpotent Lie groups*, Ann. Glob. Anal. Geom. 1 (1983), 79-130.

[9] T.J. Osler *A Further Extension of the Leibniz Rule to Fractional Derivatives and Its Relation to Parseval’s Formula*, SIAM J. Math. Anal. 3(1) (1972), 1-16.

Institute of Mathematics, University of Wrocław, 50-384 Wrocław, Poland

E-mail address: krystian.bekala@math.uni.wroc.pl