Sealing Quantum Message By Quantum Code

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Quantum error correcting code is a useful tool to combat noise in quantum computation. It is also an important ingredient in a number of unconditionally secure quantum key distribution schemes. Here, I am going to show that quantum code can also be used to seal a quantum message. Specifically, every one can still read the content of the sealed quantum message. But, any such attempt can be detected by an authorized verifier with an exponentially close to one probability.

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Introduction — We sometimes put an important document, such as a will, in an envelop sealed with molten wax so that others can open it only by breaking the wax. The sealed envelop, therefore, acts like a witness of whether the document has been open.

Clearly, it is meaningful and useful to extend the concept of physical wax seal to the digital world. Yet, no digital sealing scheme is unconditionally secure in the classical digital world as one can, in principle, copy all the bits without being caught.

Recently, Bechmann-Pasquinucci examined the possibility of sealing a classical digital signal using quantum mechanics. Specifically, he proposed a way to represent one bit of classical signal by three qubits out of which one of them is erroneous. Using single qubit measurement along the standard basis plus the classical \([3, 1, 3]_2\) majority vote code, everyone can obtain the original classical bit with certainty. And at the same time, an authorized verifier, who knows some extra information on the erroneous qubit, is able to check if someone has extracted the encoded classical bit with non-negligible probability \(\frac{1}{2}\) (see Ref. 1).

Nevertheless, Bechmann-Pasquinucci’s scheme is far from a good quantum seal as there is a non-negligible chance of knowing the original bit without being caught.\(^1\) For instance, one may randomly pick one of the qubits and measure its state along the standard basis. In this way, the chance of knowing the original classical bit correctly without being caught equals \(2/3 + (1/3)(1/2)(1/2) = 0.75\). Furthermore, it is not clear how to seal quantum information in Bechmann-Pasquinucci’s scheme.

In this Letter, I report a quantum seal scheme using quantum error correcting code. This scheme can seal both classical and quantum signals. I also prove that any person other than an authorized verifier has an exponentially small chance of knowing the original signal without being caught. This remains true even when the person has unlimited computational power.

The Quantum Seal — Let \(|\psi\rangle\) be the state of a qubit that Alice wants to make public so that anyone may use it. (Note that because of the no cloning theorem, Alice may know nothing about the state \(|\psi\rangle\).) To seal the state, Alice publicly announces a \([n, 1, d]_2\) Calderbank-Shor-Steane (CSS) code \(\mathcal{E}_L \otimes \mathcal{E}_N\) with \(d \geq 0.11n\) and uses it to encode \(|\psi\rangle\). (The existence of such a code is guaranteed by a theorem of Gottesman in Ref. 2.)

The encoded state is denoted by \(|\psi\rangle_L\). Alice further chooses a \([n', 1, 3]_2\) stabilizer code and uses it to encode \(t\) qubits used to represent \(|\psi\rangle_L\) and \(t\) qubits each from a separate copy of \(|0\rangle_L\). As depicted in Fig. 1, Alice randomly selects \((n - t)\) qubits from those used to represent \(|\psi\rangle_L\) and \(t\) qubits each from a separate copy of \(|0\rangle_L\). She makes these \(n\) selected qubits publicly accessible. At the same time, she keeps all the remaining \(n' t\) qubits (which consist of the \(t\) qubits used to encode \(|\psi\rangle_L\) and the remaining \((n' - 1)t\) qubits used to encode \(t\) copies of \(|0\rangle\)\) in a secure place. This marks the end of the quantum sealing phase. Clearly, the security of the quantum seal originates from the secrecy of which of the \((n - t)\) publicly available qubits are used to encode \(|\psi\rangle\).

To open the seal and obtain the original quantum state, Bob simply applies the standard quantum error correction procedure for the \([n, 1, d]_2\) CSS code to the \(n'\) publicly accessible qubits. This method works because there are only \(t\) out of the \(n\) qubits at unknown locations that are not used to encode \(|\psi\rangle\).

To check if the seal is opened, we need an authorized verifier who has access to the remaining \(n't\) qubits and knows the locations of the \(t\) publicly accessible qubits that are not used to encode \(|\psi\rangle\). Clearly, if no one has touched the \(n'\) publicly available qubits, the \((n + n')\) qubits accessible by the verifier should be in the state \(|\psi\rangle_L \otimes |0\rangle_L^{\otimes t}\). Therefore, the verifier accepts the seal as unbroken only if the \([n, 1, d]_2\) and the \([n', 1, 3]_2\) stabilizer code error syndrome measurements reveal that all the \((n + n')t\) qubits are error-free.

Security Of The Quantum Seal — Now, I am going to
The revised scheme can be reduced to a classical equivalent of the original scheme, and this reduction can be done in a way that preserves the security of the quantum sealing protocol. Specifically, for any security parameters $(\epsilon_p, \epsilon_f)$, there exists a quantum sealing scheme using a sufficiently long quantum codeword length $n$ such that whenever Bob (who is not an authorized verifier) applies a cheating strategy whose probability of success is at least $\epsilon_p$, his information on the state $|\psi\rangle$ is less than $\epsilon_f$. Moreover, this is true even if he has unlimited computational power.

I prove the unconditional security of the above scheme by reduction. First, I consider a revised quantum sealing scheme and show that it is as secure as the original one introduced above. The encoding procedure of the revised scheme is the same as the original one. But in the verification procedure of the revised scheme, the verifier also measures the encoded spin flip operator for the $[[n, 1, d]]_2$ code and the $[[n', 1, 3]]_2$ stabilizer codes in addition to the error syndrome measurement. Yet, the verifier accepts the quantum seal as unbroken based only on the result of the error syndrome measurement in exactly the same way as in the original scheme. Note that the encoded spin flip operations commute with the stabilizer and the acceptance criterion of the revised scheme does not depend on the result of the encoded spin flip measurement. Consequently, any cheating strategy will have an equal chance to pass the verification test and will reveal an equal amount of information on the sealed quantum state when applied to the original and the revised schemes. In this respect, the two schemes are equally secure; and it suffices to prove the security of the revised scheme.

Second, I reduce every quantum cheating strategy for the revised scheme to a corresponding classical cheating strategy. Observe that there are $(n + n't)$ independent stabilizers and encoded spin flip operations, all the $(n + n't)$ qubits are measured in the revised scheme during the verification phase. Hence, following the argument of Lo and Chau in Ref. [8], any quantum cheating strategy for the revised scheme can be reduced to a classical probabilistic one. More precisely, all cheating strategies can be generated by the classical deterministic strategies $\otimes_{i=1}^n s_i$ acting on the $n$ qubits accessible by Bob, where $s_i \in \{I, \sigma_x, \sigma_y, \sigma_z\}$.

Third, I analyze what type of classical cheating strategy reveals the information on the locations of the $t$ publicly accessible qubits that are not used to encode the sealed state $|\psi\rangle$. As the $t$ qubits taken from $|0\rangle_L^t$ are unentangled with the $(n - t)$ qubits taken from $|\psi\rangle_L$, the entropy and hence the information obtained from the measurement of those qubits coming from $|\psi\rangle_L$ and $|0\rangle_L^t$ using any classical strategy are independent. Since the code $[[n, 1, d]]_2$ used is publicly known and all the qubits used to encode $|\psi\rangle$ do not suffer from any quantum error, any measurement of the $(n - t)$ publicly available qubits taken from $|\psi\rangle_L$, only reveals information on the sealed quantum state $|\psi\rangle$. Similarly, any measurement of the $t$ qubits taken from $|0\rangle_L^t$ only reveals information on the location and the nature of the “quantum error” by regarding the $n$ publicly available qubits as a $[[n, 1, d]]_2$ codeword.

Fourth, I show that the verification test put a stringent limit on the information obtained by any classical cheating strategy. Recall that only one qubit for each $[[n', 1, 3]]_2$ encoded $|0\rangle$ are publicly accessible. Consequently, amongst the deterministic cheating strategies $\otimes_{i=1}^n s_i$ only those that cause no error in all the $t$ qubits taken from $|0\rangle_L^t$ can pass the verification test. So, in order to have at least $\epsilon_p$, chance of passing the verification test, Bob cannot put too much weight on those cheating strategies that cause error in these $t$ qubits. Similarly, only those errors in the $(n - t)$ qubits taken from $|\psi\rangle_L$ that commute with the stabilizer of $[[n, 1, d]]_2$ can pass the test. Recall that operations commuting with the stabilizer of $[[n, 1, d]]_2$ are generated by the stabilizer and the encoded operations. Besides, an error in the stabilizer of a code simply permutes the stabilizer itself. Therefore, effectively the types of action on $|\psi\rangle$, that can pass the verification test are those generated by the encoded operations of $[[n, 1, d]]_2$.

By the Holevo theorem [9], Bob’s information $I_e$ on the locations of the $t$ publicly accessible qubits that are not used to encode $|\psi\rangle$ is upper bounded by the entropy of the reduced density matrix of the $n't$ qubits originally used to encode $|0\rangle_L^t$. Therefore, $I_e$ is upper bounded by the entropy of the reduced density matrix $\text{diag}(a, (1 - a)/(2^t - 1), (1 - a)/(2^t - 1), \ldots, (1 - a)/(2^t - 1))$. In other words,

$$I_e \leq -a \log_2 a - (1 - a) \log_2((1 - a)/(2^t - 1)) < H(a) + t(1 - a),$$

(1)

where $H(a) \equiv -a \log_2 a - (1 - a) \log_2(1 - a)$ and $a$ is the probability of choosing a deterministic strategy that causes no error in all the $t$ qubits taken from $|0\rangle_L^t$.

If the strategy passes the verification test with probability at least $\epsilon_p$, we demand $\epsilon_p \leq a \leq 1$. Provided that
Alice chooses a sufficiently long code \([n, 1, d]_2\) in such a way that \(t \equiv \lfloor (d - 1)/2 \rfloor\) satisfies
\[
epsilon_p > 1/(1 + 2^d), \tag{2}\]
Bob’s information on the locations of those \(t\) qubits in Eq. (1) is upper bounded by
\[
I_e < H(\epsilon_p) + (1 - \epsilon_p)t \equiv I_{\text{bound}}(\epsilon_p). \tag{3}\]
Consequently, for any cheating strategy that passes the verification test with probability at least \(\epsilon_p\), the locations of at most \(I_{\text{bound}}(\epsilon_p)\) out of the \(t\) publicly accessible qubits taken from \(|0\rangle^{\otimes t}\) are known to Bob.

Finally, I am ready to bound the amount of information on the sealed quantum state revealed by a cheating strategy that passes the verification test with probability at least \(\epsilon_p\). But before I do so, let me summarize the situation after Bob passes the verification test. Bob knows the locations of at most \(I_{\text{bound}}(\epsilon_p)\) out of the \(n\) publicly accessible qubits that are not used to encode \(|\psi\rangle\). Denote \(\rho\) the covering radius of the \([n, 1, d]_2\) quantum code. That is to say, \(\rho\) is the smallest integer such that \(\mathbb{F}_4^\rho\) equals the union of spheres of radius \(\rho\) centered at the vectors spanned by the stabilizers and encoded operations of the \([n, 1, d]_2\) CSS code over \(\mathbb{F}_4\). In simple terms, \(\rho\) is nothing but the maximum number of errors the quantum code \([n, 1, d]_2\) can handle. Thus, in order to recover the sealed quantum state \(|\psi\rangle\), Bob has to determine the locations of at least \((n - \rho)\) out of the remaining \(|n - I_{\text{bound}}(\epsilon_p)|\) qubits that are used to encode \(|\psi\rangle\). More importantly, he has to do so in the presence of at least \(|t - I_{\text{bound}}(\epsilon_p)|\) qubits chosen from \(|0\rangle^{\otimes t}\) without affecting them.

Since the locations of these \(|t - I_{\text{bound}}(\epsilon_p)|\) qubits are unknown to Bob, this problem is equivalent to picking \((n - \rho)\) good balls out of an urn of \(|n - I_{\text{bound}}(\epsilon_p)|\) balls with \(|t - I_{\text{bound}}(\epsilon_p)|\) of them being bad. The probability of correctly picking the good balls is given by
\[
p = \frac{n - \rho - 1}{n - I_{\text{bound}}(\epsilon_p) - i} \prod_{i=0}^{n-\rho-1} \frac{n - t - i}{n - I_{\text{bound}}(\epsilon_p) - i} \leq \left(\frac{n - t}{n - I_{\text{bound}}(\epsilon_p)}\right)^{n-\rho}. \tag{4}\]
The \([n, 1, d]_2\) CSS code can be regarded as a classical \((n, [n + 1]/2)_2\) code over \(\mathbb{F}_4\). Applying the redundancy bound \(\alpha\) to this classical code, I conclude that
\[
\rho \leq n - \left\lfloor \frac{n + 1}{2} \right\rfloor = \left\lfloor \frac{n - 1}{2} \right\rfloor. \tag{5}\]
Recall that Bob may obtain some information on \(|\psi\rangle\) only when he makes use of at least \((n - \rho)\) out of the \(n\) publicly accessible qubits. Therefore, from Eqs. (1) and (5), Bob’s information \(I_\psi\) on \(|\psi\rangle\) is upper bounded by
\[
I_\psi \leq \left[\frac{n - t}{n - I_{\text{bound}}(\epsilon_p)}\right]^{(n+1)/2}. \tag{6}\]
Remember that one can always choose \(t/n\) to be at least 0.5 \(H^{-1}(1/2) \approx 0.055\). Hence, Eq. (3) implies that \(I_{\text{bound}}(\epsilon_p)\) increases linearly with \(t\). Since \(I_{\text{bound}}(\epsilon_p) < t\) whenever Eq. (2) holds, I conclude that \((n - t)/n - I_{\text{bound}}(\epsilon_p))\) is upper bounded by a positive number \(\alpha(\epsilon_p) < 1\) for sufficiently large \(n\). More importantly, \(\alpha(\epsilon_p)\) is independent of \(n\). (Actually, \(\alpha(0) \leq (1 - 0.055)/(1 + 0.055) < 0.896\).) Therefore, by choosing
\[
n \geq 2\log\alpha(\epsilon_p)\left(\frac{1}{\epsilon_1}\right), \tag{7}\]
Bob’s information on the sealed quantum state \(|\psi\rangle\) is less than \(\epsilon_1\).

In conclusion, I show that for any security parameters \((\epsilon_p, \epsilon_1)\), there is an unconditionally secure quantum seal with \(n\) satisfying Eqs. (2) and (7) such that whenever Bob uses a cheating strategy that passes with probability at least \(\epsilon_p\), his information on the sealed quantum state \(|\psi\rangle\) is less than \(\epsilon_1\).

**Discussions** — Three remarks are in place. First, although I focus the discussion on sealing a pure quantum state, the analysis above applies equally well to mixed state. Hence, the quantum sealing scheme reported here is also valid for sealing mixed state.

Second, since the verification test is nothing but an error syndrome measurement procedure, the quantum sealing scheme works equally well if the encoded state \(|0\rangle^{\otimes t}\) is replaced by \(\bigotimes_{i=1}^{t} |\phi_i\rangle_{L'}\) for some pure states \(|\phi_i\rangle\) \((i = 1, 2, \ldots, t)\).

Third, all versions of quantum sealing schemes introduced in the Letter uses two stabilizer codes. It is more efficient to seal a quantum state by using just one stabilizer code. That is to say, one encodes the quantum state to be sealed by a \([n, 1, d]_2\) code and randomly replace \(t \equiv \lfloor (d - 1)/2 \rfloor\) qubits in the codeword by some randomly chosen pure states similar to the Bechmann-Pasquinucci scheme [1]. It is instructive to investigate the unconditional security of this scheme by suitably bounding the information on the locations of the \(t\) replaced qubits.

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