Method of Contractions in Clifford Algebras

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Abstract. In this paper we consider some expressions (sums) in Clifford algebras which we call contractions or averaging.

We study full contractions, contractions by adjoint sets of multi-indices, simple contractions. We present the relation between simple contractions and projection operations onto fixed subspaces of Clifford algebras.

Using method of contractions we present solutions of system of commutator equations.

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1. Introduction

In this paper we consider expressions in Clifford algebras

\[ \sum_{A \in S} e_A U e^A, \quad e_A = (e^A)^{-1} \]

where \( e^A \) are basis elements and \( S \subseteq I \) is the subset of the set of all ordered multi-indices \( A \) of the length from 0 to \( n \). We call them contractions (or averaging) in Clifford algebras.

There is a relationship between contractions in Clifford algebras and representation theory of finite groups (see [1], [2]).

We study full contractions, contractions by adjoint sets of multi-indices, simple contractions. We present the relation between simple contractions and projection operations onto fixed subspaces of Clifford algebras.

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Using method of contractions we present solutions of system of commutator equations
\[
e^A X + \epsilon X e^A = q^A, \quad A \in S \subseteq I, \quad \epsilon \neq 0 \in \mathbb{C}
\]
for unknown element \(X \in \mathcal{C}(p, q)\) and known elements \(q^A \in \mathcal{C}(p, q)\).

2. Clifford algebras, ranks, projection operations

Consider complex Clifford algebra \(\mathcal{C}(p, q)\) (or real \(\mathcal{C}_R(p, q)\)) with \(p + q = n, n \geq 1\). The construction of Clifford algebra is discussed in details in [3] or [4].

Let \(e^a\), \(a = 1, \ldots, n\) be generators of the Clifford algebra \(\mathcal{C}(p, q)\),
\[
e^a e^b + e^b e^a = 2 \eta^{ab} e,
\]
where \(\eta = ||\eta^{ab}|| = ||\eta_{ab}||\) is the diagonal matrix with \(p\) pieces of \(+1\) and \(q\) pieces of \(-1\) on the diagonal. Elements
\[
e^{a_1 \ldots a_k} = e^{a_1} \ldots e^{a_k}, \quad a_1 < \ldots < a_k, \quad k = 1, \ldots, n,
\]
together with the identity element \(e\), form the basis of the Clifford algebra. The number of basis elements is equal to \(2^n\).

Let denote the set of ordered multi-indices of the length from 0 to \(n\) by
\[
I = \{-, 1, \ldots, n, 12, 13, \ldots, 1 \ldots n\}, \quad (2.1)
\]
where "-" is an empty multi-index. So, we have a basis of Clifford algebra \(\{e^A, A \in I\}\), where \(A\) is an arbitrary ordered multi-index. Let denote the length of multi-index \(A\) by \(|A|\).

Below we also consider different subsets of \(S \subseteq I:\)
\[
\begin{align*}
I_{\text{Even}} &= \{A \in I, |A| - \text{even}\}, \\
I_{\text{Odd}} &= \{A \in I, |A| - \text{odd}\}.
\end{align*}
\]

We have \(e_a = \eta_{ab} e^b, e^a = \eta^{ab} e_b\), where we use Einstein summation convection (there is a sum over index \(b\)). Also we have
\[
e^{a_1 \ldots a_k} = \eta_{a_1 b_1} \ldots \eta_{a_k b_k} e^{b_1} \ldots e^{b_k} = e_{a_1} \ldots e_{a_k} = (e^{a_1 \ldots a_k})^{-1}, \quad a_1 < \ldots < a_k.
\]

Any Clifford algebra element \(U \in \mathcal{C}(p, q)\) can be written in the form
\[
U = u e + u_a e^a + \sum_{a_1 < a_2} u_{a_1 a_2} e^{a_1 a_2} + \ldots + u_{1 \ldots n} e^{1 \ldots n} = u_A e^A, \quad (2.2)
\]
where we have a sum over ordered multi-index \(A\) and \(\{u_A\} = \{u, u_a, u_{a_1 a_2}, \ldots, u_{1 \ldots n}\}\) are complex (real) numbers.

We denote by \(\mathcal{C}_k(p, q)\) the vector spaces that span over the basis elements \(e^{a_1 \ldots a_k}\). Elements of \(\mathcal{C}_k(p, q)\) are said to be elements of rank \(k\). We have
\[
\mathcal{C}(p, q) = \bigoplus_{k=0}^n \mathcal{C}_k(p, q). \quad (2.3)
\]
We consider projection operators on the vector subspaces \( \mathcal{C}_k(p, q) \)
\[
\pi_k : \mathcal{C}(p, q) \to \mathcal{C}_k(p, q), \quad \pi_k(U) = \sum_{a_1 < \ldots < a_k} u_{a_1 \ldots a_k} e^{a_1} \ldots e^{a_k}. \tag{2.4}
\]

Clifford algebra \( \mathcal{C}(p, q) \) is a superalgebra. It is represented as the direct sum of even and odd subspaces
\[
\mathcal{C}(p, q) = \mathcal{C}_{\text{Even}}(p, q) \oplus \mathcal{C}_{\text{Odd}}(p, q),
\mathcal{C}_{\text{Even}}(p, q) = \bigoplus_{k-\text{even}} \mathcal{C}_k(p, q), \quad \mathcal{C}_{\text{Odd}}(p, q) = \bigoplus_{k-\text{odd}} \mathcal{C}_k(p, q).
\]

3. Full contractions

We have the following well-known statement about center \( \text{cen}\mathcal{C}(p, q) = \{ U \in \mathcal{C}(p, q) \mid UV = VU \quad \forall V \in \mathcal{C}(p, q) \} \) of Clifford algebra.

**Theorem 3.1.** The center \( \text{cen}\mathcal{C}(p, q) \) of Clifford algebra \( \mathcal{C}(p, q) \) of dimension \( n = p + q \) is subspace \( \mathcal{C}_0(p, q) \) in the case of even \( n \) and subspace \( \mathcal{C}_0(p, q) \oplus \mathcal{C}_n(p, q) \) in the case of odd \( n \):
\[
\text{cen}\mathcal{C}(p, q) = \begin{cases} \mathcal{C}_0(p, q), & n \text{ is even;} \\ \mathcal{C}_0(p, q) \oplus \mathcal{C}_n(p, q), & n \text{ is odd.} \end{cases} \tag{3.1}
\]

Let consider the following contraction
\[
F(U) = \frac{1}{2^n} e_A U e^A,
\]
where we have a sum over all multi-indices \( A \in I \). We call this expression **full contraction**.

**Theorem 3.2.** We have
\[
F(U) = \frac{1}{2^n} e_A U e^A = \begin{cases} \pi_0(U), & \text{if } n \text{ is even;} \\ \pi_0(U) + \pi_n(U), & \text{if } n \text{ is odd}, \end{cases} \tag{3.2}
\]
where \( \pi_0 \) and \( \pi_n \) are projection operations (see (2.4)) onto the subspaces of fixed ranks. Operator \( F \) is a projector \( F^2 = F \) (on the center of Clifford algebra).

**Proof.** We have
\[
(e^a)^{-1} F(U) e^a = \sum_A (e^A e^a)^{-1} F(U) (e^A e^a) = \sum_B (e^B)^{-1} F(U) e^B = F(U).
\]
So, \( F(U) \) is in the center of Clifford algebra (see Theorem 3.1). For elements \( U \) of ranks \( k = 1, \ldots, n - 1 \) (and \( k = n \) in the case of even \( n \)) we have \( F(U) = 0 \). In other particular cases we have \( e_A e^A = 2^n e \) and (in the case of odd \( n \)) \( e_A e^{1 \ldots n} e^A = 2^n e^{1 \ldots n} \). It is also easy to verify that \( F^2 = F \). \( \square \)

Let consider system of \( 2^n \) commutator equations in Clifford algebra.
Theorem 3.3. Let unknown element $X \in \mathcal{C}(p,q)$ satisfy system of equations with known $q^A \in \mathcal{C}(p,q)$

$$e^A X + \epsilon X e^A = q^A \quad \forall A, \quad \epsilon \neq 0 \in \mathbb{C}. \quad (3.3)$$

If $\epsilon = -1$ (commutator case), then this system of equations has not got solutions or has unique solution up to element of center:

$$X = -\frac{1}{2^n} q^A e_A + Z, \quad Z \in \text{cen}\mathcal{C}(p,q). \quad (3.4)$$

If $\epsilon \neq -1$, then this system of equations has not got solutions or has unique solution

$$X = \begin{cases} 
\frac{1}{2^n \epsilon} (q^A e_A - \frac{1}{(\epsilon + 1)} \pi_0 (q^A e_A)), & \text{if } n \text{ is even,} \\
\frac{1}{2^n \epsilon} (q^A e_A - \frac{1}{(\epsilon + 1)} (\pi_0 (q^A e_A) + \pi_n (q^A e_A))), & \text{if } n \text{ is odd.} 
\end{cases} \quad (3.5)$$

Proof. Let multiply each equation by $e_A$ on the right and add them (see Theorem 3.2):

$$e^A X e_A + \epsilon X e^A e_A = q^A e_A \quad \Rightarrow \quad 2^n \pi_{\text{centr}}(X) + \epsilon X 2^n = q^A e_A,$$

where $\pi_{\text{centr}}$ is the projection on the center of Clifford algebra. Using $X = \sum_{k=0}^n \pi_k(X)$ and Theorem 3.1 we obtain statement of the theorem. \(\square\)

Note that we have solution or have not solution of system of commutator equations. It depends on elements $q^A$ (it suffices to substitute solution in equation and check the equality).

4. Adjoint sets of multi-indices

We call ordered multi-indices $a_1 \ldots a_k$ and $b_1 \ldots b_l$ adjoint multi-indices if they have no common indices and they form multi-index $1 \ldots n$ of the length $n$. We write $b_1 \ldots b_l = a_\sim \sim a_k$ and $a_1 \ldots a_k = b_\sim \sim b_l$. We call corresponding basis elements $e^{a_1 \ldots a_m}$, $e^{b_1 \ldots b_l} \sim$ adjoint and write $e^{b_1 \ldots b_l} = e^{a_1 \ldots a_m}$, $e^{a_1 \ldots a_m} = e^{b_1 \ldots b_l}$. We can write also that $e^{a_1 \ldots a_m} e^{b_1 \ldots b_l} = \pm e^{1 \ldots n}$ and $e^{a_1 \ldots a_m} = \pm e^{b_1 \ldots b_l}$, where $\star$ is Hodge operator. We denote the sets of corresponding multi-indices by $I_{\text{Adj}}$ and $\widetilde{I}_{\text{Adj}} = I - I_{\text{Adj}}$. We have

$$\{ e^A \mid A \in I \} = \{ e^A \mid A \in I_{\text{Adj}} \} \cup \{ e^A \mid A \in \widetilde{I}_{\text{Adj}} \}. \quad (4.1)$$

For Clifford algebra $\mathcal{C}(p,q)$ of dimension $n = p + q$ we have $2^{2^n - 1}$ different partitions of the form $\{ e^A \mid A \in I \}$. For example,

$$I_{\text{Adj}} = I_{\text{First}}, \quad \widetilde{I}_{\text{Adj}} = I - I_{\text{First}} = I_{\text{Last}},$$

\cite{1}

\footnote{\text{It is the analogue of Hodge operator in Clifford algebra } \star U = U^\sim e^{1 \ldots n}, \text{ where } \sim \text{ is the standard reverse operation in Clifford algebra.} }
where $I_{\text{First}}$ consist of the first (in the order) $2^{n-1}$ multi-indices of the set $I$. In the case of odd $n$ we can write

$$I_{\text{First}} = \{ A \in I, \ |A| \leq \frac{n-1}{2} \}, \quad I_{\text{Last}} = \{ A \in I, \ |A| \geq \frac{n+1}{2} \}.$$

In the case of odd $n$ we can consider the following adjoint sets

$$I_{\text{Adj}} = I_{\text{Even}}, \quad \overline{I}_{\text{Adj}} = I_{\text{Odd}}.$$

5. Commutative properties of basis elements

**Theorem 5.1.** Consider real or complex Clifford algebra $C\ell(p,q)$ and the set of basis elements $e^A = \{e^{b_1\ldots b_m}\}$. Let denote arbitrary element of this set by $e^{a_1\ldots a_k}$.

Then element $e^{a_1\ldots a_k}$ (if it is not $e$ or $e^{1\ldots n}$) commutes with $2^{n-2}$ even ($|A|\text{-even}$) elements of the set, commutes with $2^{n-2}$ odd elements of the set, anticommutes with $2^{n-2}$ even elements of the set and anticommutes with $2^{n-2}$ elements of the set $e^A$. Element $e$ commutes with all elements of the set $e^A$.

1. if $n$ - even, then $e^{1\ldots n}$ commutes with all $2^{n-1}$ even elements of the set and anticommutes with all $2^{n-1}$ odd elements of the set $e^A$;
2. if $n$ - odd, then $e^{1\ldots n}$ commutes with all $2^n$ elements of the set $e^A$.

**Proof.** Let $i$ be the number of coincident indices in multi-indices $a_1\ldots a_k$ and $b_1\ldots b_m$. Then for any $i$ number of sets $b_1\ldots b_m$ for fixed set $a_1\ldots a_k$ equals $C^i_kC^{m-i}_{n-k}$, where $C^n_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ is binomial coefficient (we have $C^n_k = 0$ for $k > n$). When we swap element $e^{a_1\ldots a_k}$ with element $e^{b_1\ldots b_m}$ we obtain coefficient $(-1)^{km-i}$.

If $k$ is even and does not equal $0$ and $n$, then number of even and odd elements $e^{b_1\ldots b_m}$ that commutes with $e^{a_1\ldots a_k}$ ($km-i$ is even, and, so $i$ is even) respectively equals

$$\sum_{m-\text{even}} \sum_{i-\text{even}} C^i_kC^{m-i}_{n-k} = 2^{n-2}, \quad \sum_{m-\text{odd}} \sum_{i-\text{even}} C^i_kC^{m-i}_{n-k} = 2^{n-2}.$$

If $k$ is odd and does not equal $n$, then number of even and odd elements $e^{b_1\ldots b_m}$ that anticommutes with $e^{a_1\ldots a_k}$ ($m-i$ is even) respectively equals

$$\sum_{m-\text{even}} \sum_{i-\text{even}} C^i_kC^{m-i}_{n-k} = 2^{n-2}, \quad \sum_{m-\text{odd}} \sum_{i-\text{odd}} C^i_kC^{m-i}_{n-k} = 2^{n-2}.$$

The cases $k = 0$ and $k = n$ are trivial (see Theorem 3.1). $\square$

Also we have the following theorem about adjoint sets of multi-indices.

**Theorem 5.2.** Consider real or complex Clifford algebra $C\ell(p,q)$ and the set of basis elements $e^A = \{e^{b_1\ldots b_m}\}$. Let the set of basis elements is a sum of $2$ adjoint sets of multi-indices $I = I_{\text{Adj}} \cup \overline{I}_{\text{Adj}}$. 


If $n$ is even then any even (not odd!) basis element (if it is not $e$) commutes with $2^{n-2}$ basis elements from $\{e^A | A \in I_{\text{Adj}}\}$, anticommutes with $2^{n-2}$ basis elements from $\{e^A | A \in \tilde{I}_{\text{Adj}}\}$, commutes with $2^{n-2}$ basis elements from $\{e^A | A \in I_{\text{Adj}}\}$ and anticommutes with $2^{n-2}$ basis elements from $\{e^A | A \in \tilde{I}_{\text{Adj}}\}$.

If $n$ is odd then any basis element (if it is not $e$ and $e^1...n$) commutes with $2^{n-2}$ basis elements from $\{e^A | A \in I_{\text{Adj}}\}$, anticommutes with $2^{n-2}$ basis elements from $\{e^A | A \in \tilde{I}_{\text{Adj}}\}$, commutes with $2^{n-2}$ basis elements from $\{e^A | A \in I_{\text{Adj}}\}$ and anticommutes with $2^{n-2}$ basis elements from $\{e^A | A \in \tilde{I}_{\text{Adj}}\}$.

Note that in the case of odd $n$ we can take $I_{\text{Adj}} = I_{\text{Even}}$, $\tilde{I}_{\text{Adj}} = I_{\text{Odd}}$ and obtain the statement from the Theorem 5.1.

**Proof.** If $n$ is odd then $e^1...n$ is in the center of Clifford algebra. So if basis element commutes with some basis element, then it commutes with adjoint basis element. But we know from Theorem 5.1 that basis elements (except $e$ and $e^1...n$) commutes with $2^{n-1}$ basis elements and anticommutes with $2^{n-1}$ basis elements. So we obtain the statement of theorem for the case of odd $n$.

If $n$ is even then even (not odd) basis element commutes with $e^1...n$. So if even basis element commutes with some basis element, then it commutes with adjoint basis element. \QED

Let represent the commutative property of basis elements in the following tables. At the intersection of two basis elements is a sign “+” if they commute and the sign “–” if they anticommute. For small dimensions we have the following tables:

| $n = 1$ | $e$ | $e^1$ |
|---------|-----|-------|
| $e$     | +   | +     |
| $e^1$   | +   | +     |

| $n = 2$ | $e$ | $e^1$ | $e^2$ | $e^{12}$ |
|---------|-----|-------|-------|---------|
| $e$     | +   | +     | +     | +       |
| $e^1$   | +   | +     | -     | -       |
| $e^2$   | +   | -     | +     | -       |
| $e^{12}$| +   | -     | -     | +       |

| $n = 3$ | $e$ | $e^1$ | $e^2$ | $e^3$ | $e^{12}$ | $e^{13}$ | $e^{23}$ | $e^{123}$ |
|---------|-----|-------|-------|-------|---------|---------|---------|---------|
| $e$     | +   | +     | +     | +     | +       | +       | +       | +       |
| $e^1$   | +   | +     | -     | -     | +       | +       | +       | +       |
| $e^2$   | +   | -     | +     | -     | +       | +       | +       | +       |
| $e^3$   | +   | -     | -     | +     | +       | +       | +       | +       |
| $e^{12}$| +   | -     | -     | +     | -       | +       | +       | +       |
| $e^{13}$| +   | -     | +     | -     | +       | +       | +       | +       |
| $e^{23}$| +   | +     | -     | -     | +       | +       | +       | +       |
| $e^{123}$| + | + | + | + | + | + | + | + |
6. Simple contractions

We denote the corresponding square symmetric matrices of size \(2^n\) from the previous section (with elements 1 and \(-1\), see tables) by \(M_n = ||m_{AB}||\). For arbitrary element of these matrices we have \(m_{AB} = (e^A)^{-1}e^B e^A(e^B)^{-1}\), \(A, B \in I, e \equiv 1\). We have

\[
m_{AB} = m_{BA} = \begin{cases} 
1, & \text{if } [e^A, e^B] = 0; \\
-1, & \text{if } \{e^A, e^B\} = 0, 
\end{cases} \tag{6.1}
\]

In the case of odd \(n\) we also consider symmetric matrix \(L\) of size \(2^{n-1}\)
\(L_n = ||l_{AB}||, l_{AB} = m_{AB}, A, B \in I_{\text{First}} = \{A \in I, \ |A| \leq \frac{n-1}{2}\}\).

**Theorem 6.1.** Matrix \(M_n\) is invertible in the case of even \(n\) and \(M_n^{-1} = \frac{1}{2^n} M_n\). Matrix \(M_n\) is not invertible in the case of odd \(n\).

Matrix \(L_n\) is invertible in the case of odd \(n\) and \(L_n^{-1} = \frac{1}{2^{n-1}} L_n\).

**Proof.** Matrices are symmetric \(M_n^T = M_n, N_n^T = N_n\) by definition. Let multiply matrix \(M_n\) by itself. For two arbitrary rows we have

\[
\sum_B m_{AB} m_{CB} = \sum_B ((e^A)^{-1}e^B e^A(e^B)^{-1}e^C e^B(e^C)^{-1}) = \\
= (e^A)^{-1}(\sum_B (e^B)^{-1}e^A e^C e^B(e^C)^{-1})(e^C)^{-1}.
\]

In the last expression sum is equal zero if \(A \neq C\) and (in the case of odd \(n\)) \(A, C\) are not adjoint multi-indices (see below), because contraction \(e_B U e^B\) is projection onto the center of Clifford algebra (see Theorem 3.2). In the other cases the last expression equals \(2^n\). In the case of odd \(n\) we must use matrix \(L_n\), because we does not have adjoint multi-indices in this matrix. \(\square\)

Let consider simple contractions \(F_{e^A}(U) = (e^A)^{-1} U e^A\).

**Theorem 6.2.** For simple contraction \(F_{e^A}(U) = (e^A)^{-1} U e^A\) we have

\[
F_{e^A}(U) = \sum_B m_{AB} \pi_{e^B}(U), \tag{6.2}
\]

where \(\pi_{e^B}\) is a projection onto subspace spanned over element \(e^B\). We have \(F_{e^A}(F_{e^A}(U)) = U\).

**Proof.** The statement follows from the definition of matrix \(M_n = ||m_{AB}||\) and definition of simple contraction. \(\square\)

Fixed multi-index \(A\) divides the set \(I\) into 2 sets \(I = I_{\{A\}} \cup I_{\{A\}}\), where \(e^B, B \in I_{\{A\}}\) commutes with \(e^A\), and \(e^B, B \in I_{\{A\}}\) anticommutes with \(e^A\). Denote the corresponding subspaces of Clifford algebra by \(\mathcal{C}_{\{A\}}(p, q)\) and corresponding projection operations by \(\pi_{\{A\}}\) and \(\pi_{\{A\}}\). We have \(\mathcal{C}(p, q) = \mathcal{C}_{\{A\}}(p, q) \oplus \mathcal{C}_{\{A\}}(p, q)\) and

\[
F_{e^A}(U) = (e^A)^{-1} U e^A = \pi_{\{A\}}(U) - \pi_{\{A\}}(U), \quad \forall A.
\]
Theorem 6.3. For arbitrary Clifford algebra element $U$ we have
\[ \pi_{\{A\}}(U) = \frac{1}{2}(U + (e^A)^{-1}Ue^A), \quad \pi_{[A]}(U) = \frac{1}{2}(U - (e^A)^{-1}Ue^A). \]

Proof. Using
\[ (e^A)^{-1}Ue^A = \pi_{\{A\}}(U) - \pi_{\{A\}}(U), \quad U = \pi_{\{A\}}(U) + \pi_{[A]}(U) \]
we obtain the statement of theorem. \( \square \)

For empty multi-index $A = -$ we have $m_{-B} = 1$ for all $B$, $I = I_{[A]}$, $I_{\{A\}} = \emptyset$. For multi-index $A = 1 \ldots n$ we have $m_{1 \ldots n,B} = 1$ for all $B$ in the case of odd $n$ and
\[ m_{1 \ldots n,B} = \begin{cases} 1, & \text{if } B \text{ is even;} \\ -1, & \text{if } B \text{ is odd;} \end{cases} \tag{6.3} \]
and $e_{1 \ldots n}Ue^{1 \ldots n} = \pi_{\text{Even}}(U) - \pi_{\text{Odd}}(U)$ in the case of even $n$, where $\pi_{\text{Even}}$ and $\pi_{\text{Odd}}$ are projection operations onto the even and odd subspaces of Clifford algebra. In the other cases (when $A$ is not empty and in not $1 \ldots n$) we have $2^{n-1}$ elements in each of the sets $I_{[A]}$, $I_{\{A\}}$ (see Theorem 5.1) i.e. we have $\dim \mathcal{C} \{A\}(p,q) = \dim \mathcal{C} \{A\}(p,q) = 2^{n-1}$ in these cases.

In particular case we obtain the following statement (for $A = 1 \ldots n$): in the case of even $n$ we have
\[ \pi_{\text{Even}}(U) = \frac{1}{2}(U + e_{1 \ldots n}Ue^{1 \ldots n}), \quad \pi_{\text{Odd}}(U) = \frac{1}{2}(U - e_{1 \ldots n}Ue^{1 \ldots n}). \]

Let consider commutator equation. We have the following theorem.

Theorem 6.4. Let unknown element $X \in \mathcal{C}(p,q)$ satisfy the following equation with known element $q^A \in \mathcal{C}(p,q)$
\[ e^A X + \epsilon X e^A = q^A, \quad \epsilon \neq 0 \in \mathbb{C}. \tag{6.4} \]

If $\epsilon \neq \pm 1$, then
\[ X = \sum_B \frac{1}{1 + \epsilon m_{AB}} \pi_B((e^A)^{-1}q^A). \]

If $\epsilon = -1$ (commutator case), then:
- if $A$ is empty or $A = 1 \ldots n$ (in the case of odd $n$) and $q^A \neq 0$, then there is no solutions,
- if $A$ is empty or $A = 1 \ldots n$ (in the case of odd $n$) and $q^A = 0$, then solution is arbitrary Clifford algebra element,
- in other cases we have solution
\[ \frac{1}{2}\pi_{\{A\}}((e^A)^{-1}q^A) + \pi_{[A]}(U), \]
where $U$ is arbitrary Clifford algebra element.

If $\epsilon = 1$ (anticommutator case), then we have solution
\[ \frac{1}{2}\pi_{[A]}((e^A)^{-1}q^A) + \pi_{\{A\}}(U), \]
where $U$ is arbitrary Clifford algebra element.
Proof. Multiply equation on the left by \((e^A)^{-1} = e_A\) and use Theorem 6.2
\[ X + \epsilon (e^A)^{-1} X e^A = (e^A)^{-1} q^A \Rightarrow X + \epsilon \sum_B m_{AB} \pi_{e^B}(X) = (e^A)^{-1} q^A. \]

Using \(X = \sum_B \pi_{e^B}(X)\) we obtain
\[ \sum_B (1 + \epsilon m_{AB}) \pi_{e^B}(X) = \sum_B \pi_B((e^A)^{-1} q^A). \]

In the case \(\epsilon \neq \pm 1\) we obtain the statement of the theorem for this case.

Let \(\epsilon = -1\). If \(A\) is empty or \(A\) is 1...\(n\) in the case of odd \(n\), then \(m_{AB} = 1\) for all \(B\). We has not got solution if \(q^A \neq 0\), or solution is arbitrary element if \(q^A = 0\). In the other cases \((1 + \epsilon m_{AB}) = (1 - m_{AB})\) equals 0 or 2 and we obtain formula from the statement.

In the case \(\epsilon = 1\) proof is similar. \(\square\)

We have the following relation between projection operations onto subspaces of fixed basis element and simple contractions.

**Theorem 6.5.** In the case of even \(n\) we have
\[ \pi_{e^A}(U) = \frac{1}{2^n} \sum_B m_{AB}(e^B)^{-1} U e^B. \] (6.5)

In the case of odd \(n\) we have
\[ \pi_{e^A, \tilde{e}^A}(U) = \pi_{e^A}(U) + \pi_{e^A}(U) = \frac{1}{2^{n-1}} \sum_{B \in \text{First}} l_{AB}(e^B)^{-1} U e^B. \] (6.6)

Note that we can use instead \(\text{I}_{\text{First}}\) any adjoint set \(\text{I}_{\text{Adj}}\).

Proof. From (6.2) we obtain
\[
\begin{pmatrix}
F_e(U) \\
F_{e^1}(U) \\
\vdots \\
F_{e^1...n}(U)
\end{pmatrix} = M_n
\begin{pmatrix}
\pi_e(U) \\
\pi_{e^1}(U) \\
\vdots \\
\pi_{e^1...n}(U)
\end{pmatrix}.
\]

Using Theorem 6.1 in the case of even \(n\) we obtain
\[
\begin{pmatrix}
\pi_e(U) \\
\pi_{e^1}(U) \\
\vdots \\
\pi_{e^1...n}(U)
\end{pmatrix} = \frac{1}{2^n} M_n
\begin{pmatrix}
F_e(U) \\
F_{e^1}(U) \\
\vdots \\
F_{e^1...n}(U)
\end{pmatrix}.
\]

In the case of odd \(n\) we have
\[ F_{e^A}(U) = (e^A)^{-1} U e^A = \sum_B m_{AB} \pi_{e^B}(U) = \sum_{B \in \text{I}_{\text{First}}} l_{AB} \pi_{e^B, \tilde{e}^B}(U). \] (6.7)

and use Theorem 6.1 \(\square\)
Let’s give some examples. In the case $n = 1$ we have

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad N_1 = (1), \quad F_e(U) = F_e^1(U) = U, \quad \pi_{e,e^1}(U) = U.$$  

In the case $n = 2$ we have $M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, $F_e(U) = U$, $F_e^1(U) = \pi_e(U) + \pi_{e^1}(U) - \pi_{e^2}(U) - \pi_{e^12}(U)$, $F_e^2(U) = \pi_e(U) - \pi_{e^1}(U) + \pi_{e^2}(U) - \pi_{e^12}(U)$, $F_e^{12}(U) = \pi_e(U) - \pi_{e^1}(U) - \pi_{e^2}(U) + \pi_{e^12}(U)$.

$$\pi_e(U) = \frac{1}{4}(e_A U e^A),$$  

$$\pi_{e^1}(U) = \frac{1}{4}(e U e + (e^1)^{-1} U e^1 - (e^2)^{-1} U e^2 - (e^{12})^{-1} U e^{12}),$$  

$$\pi_{e^2}(U) = \frac{1}{4}(e U e - (e^1)^{-1} U e^1 + (e^2)^{-1} U e^2 - (e^{12})^{-1} U e^{12}),$$  

$$\pi_{e^{12}}(U) = \frac{1}{4}(e U e - (e^1)^{-1} U e^1 - (e^2)^{-1} U e^2 + (e^{12})^{-1} U e^{12}).$$  

In the case $n = 3$ we have $N_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, $\pi_{e,e^{123}}(U) = \frac{1}{4}(e U e + (e^1)^{-1} U e^1 + (e^2)^{-1} U e^2 + (e^{12})^{-1} U e^{12})$, $\pi_{e^1,e^{23}}(U) = \frac{1}{4}(e U e + (e^1)^{-1} U e^1 - (e^2)^{-1} U e^2 - (e^{12})^{-1} U e^{12})$, $\pi_{e^2,e^{13}}(U) = \frac{1}{4}(e U e - (e^1)^{-1} U e^1 + (e^2)^{-1} U e^2 - (e^{12})^{-1} U e^{12})$, $\pi_{e^3,e^{12}}(U) = \frac{1}{4}(e U e - (e^1)^{-1} U e^1 - (e^2)^{-1} U e^2 + (e^{12})^{-1} U e^{12})$.

7. Contractions by adjoint set

Consider contractions by adjoint set

$$F_{\text{Adj}}(U) = \frac{1}{2^{n-1}} \sum_{A \in \text{Adj}} e_A U e^A.$$  

**Theorem 7.1.** Consider an arbitrary Clifford algebra element $U$. Let we have 2 adjoint sets $1 = I_{\text{Adj}} \cup \overline{I_{\text{Adj}}}$. In the case of arbitrary $n$ we have

$$F_{\text{Adj}}(U) = F(U).$$
Proof. If $n$ is odd, then $e^{1\ldots n}$ is in the center of Clifford algebra,
$$(e^{a_1\ldots a_m})^{-1}Ue^{a_1\ldots a_m} = e^{1\ldots n}(e^{1\ldots n})^{-1}(e^{a_1\ldots a_m})^{-1}Ue^{a_1\ldots a_m} = (e^{a_1\ldots a_m})^{-1}Ue^{a_1\ldots a_m},$$
and
$$e_AUe^A = 2 \sum_{A \in I_{Adj}} e_AUe^A. \quad (7.1)$$

If $n$ is even, then $e^{1\ldots n}$ anticommutes with all odd basis elements and commutes with all even basis elements (see Theorem 5.1). So if $U = U_0 + U_1$, $U_0 \in \mathcal{C}_\text{Even}(p, q)$, $U_1 \in \mathcal{C}_\text{Odd}(p, q)$, then for $k = 0, 1$ we have
$$(e^{a_1\ldots a_m})^{-1}U_ke^{a_1\ldots a_m} = e^{1\ldots n}(e^{1\ldots n})^{-1}(e^{a_1\ldots a_m})^{-1}U_ke^{a_1\ldots a_m} =$$
$$= (-1)^{2m+k}(e^{a_1\ldots a_m})^{-1}(e^{1\ldots n})^{-1}U_ke^{1\ldots n}e^{a_1\ldots a_m} = (-1)^k(e^{a_1\ldots a_m})^{-1}U_ke^{a_1\ldots a_m},$$
and we obtain (7.1) again. \hfill \square

So we can use contraction $F_{\text{Adj}}$ (with $2^{n-1}$ summands) instead of full contraction $F(U)$ (with $2^n$ summands) in all calculations.

8. Conclusion

In the present paper we consider full contractions, simple contractions and contractions by adjoint sets of multi-indices. We can also consider another contractions (for other subsets $S \subseteq I$).

In [4] we consider generator contractions in Clifford algebra $\mathcal{C}(p, q)$
$$F_1(U) = e_aUe^a$$
and prove that $e_AUe^a = \sum_{k=0}^n (-1)^k(n - 2k)\pi_k(U)$.

In [10] we present relation between generator contractions and projections onto subspaces of fixed ranks. We use this relation to present new class of gauge invariant solutions of Yang-Mills equations.

We can consider contractions $\sum_{A \in S} e_AUe^A$ by another subsets $S \subseteq I$:
$$I_{\text{Even}} = \{ A \in I, |A| - \text{ even} \}, \quad I_{\text{Odd}} = \{ A \in I, |A| - \text{ odd} \},$$
$$I_k = \{ A \in I, |A| = k \}, \quad k = 0, 1, \ldots, n,$$
$$I_m = \{ A \in I, |A| = m \mod 4 \}, \quad m = 0, 1, 2, 3.$$

We call them even and odd contractions, contractions of ranks $k$ (in particular case $k = 1$ we obtain generator contraction) and contractions of quaternion types $m$. There is a relation between these contractions and projective operations onto fixed subspaces of Clifford algebras. This is a subject for further research.

Note that in all theorems of this paper we can consider not basis elements of Clifford algebra elements but arbitrary set of Clifford algebra elements $\gamma^a \in \mathcal{C}(p, q)$ that satisfy conditions $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}e$. This set may generate another basis of Clifford algebra $\mathcal{C}(p, q)$ (but in some cases of odd dimension $n$ this set does not generate basis of Clifford algebra element $\mathcal{C}(p, q)$, see [11]).
In [11] we consider generalized contractions $\sum_{\alpha} A^\alpha U_{\beta}^\alpha$ when we have 2 such different sets $\gamma^\alpha, \beta^\alpha$ in Clifford algebra. We use these contractions to prove generalized Pauli’s theorem and some other problems about spin groups (see [5], [6], [7], [8], [9]).

The results of this article (especially about relation between projection operations and contractions; solving commutator equations) may be used in computer calculations.

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