THE AMALGAMATED PRODUCT OF FREE GROUPS AND RESIDUAL SOLVABILITY

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To the memory of Hanna Neumann and her mathematics

Abstract. In this paper we study residual solvability of the amalgamated product of two finitely generated free groups, in the case of doubles. We find conditions where this kind of structure is residually solvable, and show that in general this is not the case. However this kind of structure is always meta-residually-solvable.

1. Introduction and Motivation

The notion of residual properties was first introduced by Philip Hall in 1954 [11]. Let $X$ be a class [21] of groups. $G$ is residually-$X$ if and only if, for every non-trivial element $g$ in $G$ there is an epimorphism of $G$ in $X$ such that the element corresponding to $g$ is not the identity. There is an important method for constructing new groups from existing ones called an amalgamated product. These amalgamated products are special cases of groups acting on trees, embodied in what is now known as the Bass-Serre theory [23].

The question of being residually solvable can be simplified to whether the group is simple. Peter Neumann in [22] asks the following question: Is it possible that the free product $\{A \ast B; H = K\}$ with amalgamation, where $A$, $B$ are free groups of finite ranks, $H, K$ are finitely generated subgroups of $A, B$, respectively, is a simple group? For the case where the amalgamated subgroup is not finitely generated, Ruth Camm [7] constructed example of a simple free product $G = \{A \ast B; H = K\}$ where $A, B$ are free groups of finite rank and their subgroups $H, K$ have infinite rank. This example can be thought as a simple amalgamated product of two residually solvable groups where the amalgamation is

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not residually solvable. (i.e. where the amalgamated subgroup fails the maximal condition). In fact she showed that there exist continuously many non-isomorphic simple amalgamated products of two finitely generated free groups with non-finitely generated amalgamation.

For the case where amalgamated subgroup is finitely generated Burger and Mozes [6] constructed an infinite family of torsion free finitely presented simple groups. For every pair $(m, n)$ of sufficiently large even integers, they constructed a finite square complex whose universal covering is the product $T_m \times T_n$ of regular trees of respective degrees $m$ and $n$ and whose fundamental group $\Gamma_{m,n}$ enjoys the following properties: the group $\Gamma_{m,n}$ is simple, finitely presented and isomorphic to a free amalgamated product $\{F \ast F; G\}$ where $F$, $G$ are finitely generated free groups.

Note that a partial answer to the question of P. Neumann, is that under the same condition of the problem, $\{A \ast B; H = K\}$ is not simple provided either of indices $[A : H], [B : K]$ is infinite; (see [12], [16], [18]).

Clearly, doubles of free groups are never simple. In this paper, we study residual solvability of amalgamated products of finitely generated residually solvable groups. Since free groups are residually solvable, what we prove here is stronger than the corresponding result for the class of free groups. Indeed we focus attention on the class of residually solvable groups and show that the doubles of residually solvable groups are residually solvable if we impose the condition of solvable separability over the amalgamating subgroup (see Theorem 7). In general we show that doubles of residually solvable groups are not residually solvable (see Theorem 8 but meta-residually solvable (see Corollary 6).

Fine-Howie-Rosenberger [10] and Culler-Morgan [9] (see also [8]) showed that any one-relator group with torsion which has at least three generators can be decomposed, in a non-trivial way, as an amalgamated free product. Baumslag-Shalen [5] (see also [3]) showed that every one-relator group with at least four generators can be decomposed into a generalized free product of two groups where the amalgamated subgroup is proper in one factor and of infinite index in the other; this is so called Baumslag-Shalen decomposition.

We have recently used these results together with the results of [13] (see also [15]) to answer questions regarding residual solvability of one-relator groups [14]. Note that Baumslag [2] proved that positive one-relator groups are residually solvable (see [17] for exposition).
2. Preliminaries

In this section, we recall some facts and prove some lemma to be used later on.

2.1. Subgroups of amalgamated products. We will use a theorem by Hanna Neumann \cite{20} extensively in this paper. With regard to abstract groups, Hanna Neumann showed in the 1950s that, in general, subgroups of amalgamated products are no longer amalgamated products, but generalized free products, indeed she proved the following: let $K$ be a subgroup of $G = \{A \ast B; C\}$, then $K$ is an HNN-extension of a tree product in which the vertex groups are conjugates of subgroups of either $A$ or $B$ and the edge groups are conjugates of subgroups of $C$. The associated subgroups involved in the HNN-extension are also conjugates of subgroups of $C$. As a corollary, if $K$ misses the factors $A$ and $B$ (i.e. $K \cap A = \{1\} = K \cap B$), then $K$ is free; and if $K$ misses the amalgamated subgroup $C$ (i.e. $K \cap C = \{1\}$), then $K = \prod_{i \in I} X_i \ast F$, where the $X_i$ are conjugates of subgroups of $A$ and $B$ and $F$ is free (see \cite{4} for more information).

Let us mention that later a description was given by the Bass-Serre theory \cite{23}, with groups acting on graphs to give geometric intuition: the fundamental group of a graph of groups generalizes both amalgamated products, HNN-extensions and tree products.

2.2. Some Lemmas. Here we prove some lemmas to be used in proving the main results of this paper.

Lemma 1. If $A$ is a group, $C$ is a subgroup of $A$, $\phi$ is an isomorphic mapping of $A$ onto a group $B$, and $D$ is the amalgamated product of $A$ and $B$ amalgamated $C$ with $C\phi$, that is

$$D = \{A \ast B; C = C\phi\},$$

then there is a homomorphism, $\psi$, from $D$ onto one of the factors, and the kernel of $\psi$, $K$, is:

$$K = gp(a(a\phi)^{-1} | a \in A).$$

Furthermore this map injects to each factor.

Proof. Let $\alpha$ be the homomorphism from $A$ onto itself, and $\beta$ be the homomorphism from $B$ onto the inverse of the isomorphic copy of $A$, i.e. $\beta = \phi^{-1}$. These homomorphisms can be extended to a homomorphism from $D$ onto $A$, \cite{19, 4}. By the way that $\alpha$ and $\beta$ are defined, it follows that this homomorphism is one-to-one restricted to either $A$ or $B$. \hfill \square
Lemma 2. Let $A$, $B$, $C$, $D$, $K$ and $\phi$ be as in Lemma 1. Then $K$ is not central in $D$, in other words, $[K, D] \neq \{1\}$.

Proof. First, note that the center of $D$ is $\xi D = \xi A \cap \xi B \cap C$. By Lemma 1, $K \cap A = \{1\}$, and hence $\xi A \cap K = \{1\}$. This implies that $[K, D] \neq \{1\}$. \hfill \qed

Lemma 3. Let $A$, $B$, $C$, $D$, $K$ and $\phi$ be as in Lemma 1. Furthermore, let $C$ be normal in $A$. Then $K$ commutes with $C$, i.e. $[C, K] = 1$.

Proof. Let $c, c' \in C$, and $a \in A$. Since $C \triangleleft A$ and $C \phi \triangleleft B$, and $\phi$ is an isomorphism, we can do the following computation:

$$a^{-1}ca = c' = c'\phi = (a^{-1}\phi)(c\phi)(a\phi) = (a^{-1}\phi)c(a\phi).$$

Note that $a(a\phi)^{-1} \in K$, so

$$ca(a\phi)^{-1} = a(a\phi)^{-1}c;$$

that is $[C, K] = \{1\}$. \hfill \qed

Corollary 4. Let $A$, $B$, $C$, $D$, $K$ and $\phi$ be as above. Then $K$ is free.

Proof. Since the homomorphism $\psi$ is one-to-one restricted to either $A$ or $B$, then

$$K \cap A = \{1\} = K \cap A\phi.$$ 

So, by the theorem of Hanna Neummann mentioned in subsection 2.1, $K$ is free. \hfill \qed

2.3. The filtration approach to residual solvability. In this section we provide some background for filtration approach which we will use later.

A family $(A_\lambda|\lambda \in \Lambda)$ of normal subgroups of $A$ is termed a solvable filtration of $A$ if $A/A_\lambda$ is solvable for every $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} A_\lambda = \{1\}$. We shall say that $H$ is solvably separable in $A$ if $\bigcap_{\lambda=1}^{\infty} HA_\lambda = H$.

Now let $H \leq A$, then $(A_\lambda|\lambda \in \Lambda)$ is called an $H$-filtration of $H$ if $\bigcap_{\lambda \in \Lambda} HA_\lambda = H$. Two equally indexed filtrations $(A_\lambda|\lambda \in \Lambda)$ and $(B_\lambda|\lambda \in \Lambda)$ of $A$ and $B$ respectively are termed $(H, K, \phi)$–compatible if $(A_\lambda \cap H)\phi = B_\lambda \cap K$ (\forall $\lambda \in \Lambda$). The following Proposition of Baumslag will help us to prove one of the results: let $(A_\lambda|\lambda \in \Lambda), (B_\lambda|\lambda \in \Lambda)$ be solvable $(H, K, \phi)$–compatible filtrations of the residually solvable groups $A$ and $B$ respectively. Suppose $(A_\lambda|\lambda \in \Lambda)$ is an $H$–filtration of $A$ and $(B_\lambda|\lambda \in \Lambda)$ is a $K$–filtration of $B$. If, for every $\lambda \in \Lambda$,

$$\{A/A_\lambda \ast B/B_\lambda; HA_\lambda/A_\lambda = KB_\lambda/B_\lambda\},$$

is residually solvable, then so is $G = \{A \ast B; H = K\}$. 

3. Doubles of residually solvable groups

In this section we prove the theorems concerning the doubles of residually solvable groups. By doubles we mean the amalgamated product of two groups where the factors are isomorphic and under the same isomorphism the amalgamated subgroups are identified.

3.1. Meta-residual-solvability. Here we prove that in general the amalgamated products of doubles of residually solvable groups are meta-residually-solvable.

Proposition 5. Let $A$ be a residually solvable group, $C$ be a subgroup of $A$, and $\overline{\phi}$ be an isomorphic mapping of $A$ onto $A$. Then the generalized free product of $A$ and $\overline{A}$ amalgamated $C$ with $\overline{C}$,

$$G = \{A \ast \overline{A}; C = \overline{C}\}$$

is an extension of a free group by a residually solvable group.

Proof. Let $\phi : G \rightarrow A$; then $K = \ker \phi = gp(a\overline{a}^{-1} | a \in A)$. $K$ is free by the theorem of Hanna Neumann mentioned in subsection 2.1 since

$$A \cap K = \{1\} = A \cap \ker \phi.$$

Therefore $G$ is an extension of a free group by a residually solvable group. 

Corollary 6. $G$ is meta-residually-solvable.

Proof. Since free groups are residually solvable, then by Proposition 5, $G$ is residually solvable-by-residually solvable. That is to say that $G$ is meta-residually-solvable.

3.2. Effect of solvably separability on the amalgamated subgroup and residual solvability. We now prove that if we impose the solvable separability condition on the amalgamated subgroup of doubles of residual solvable groups then the resulting group is residually solvable.

Theorem 7. Let $\overline{\phi}$ be an isomorphism from a group $A$ onto itself, $C$ be a subgroup of $A$, and $G$ be the amalgamated product of $A$ and $\overline{A}$ amalgamated $C$ with $\overline{C}$:

$$G = \{A \ast \overline{A}; C = \overline{C}\}.$$

If $A$ is residually solvable then $G$ is also residually solvable, provided that $C$ is solvably separable in $A$. 

Proof. Assuming $C$ is solvably separable in $C$ and $A$ is residually solvable, we want to show that $G$ is residually solvable. That is we must show that for every non-trivial element $(1 \neq) d \in G$, there exists a homomorphism, $\phi$, from $G$ onto a solvable group $S$, $\phi : G \to S$, such that $d\phi \neq 1$. We consider two cases:

Case 1: Let $1 \neq d \in A$. There exists an epimorphism $\phi$ from $G$ onto $A$, so that $d\phi = d$. Since $A$ is residually solvable, there exists $\lambda \in \mathbb{N}$, such that $d \notin \delta_\lambda A$, where $\delta_\lambda A$, is the $\lambda$-th derived group of $A$. Now put $S = A/\delta_\lambda A$, a solvable group of derived length at most $\lambda$. Note that the canonical homomorphism, $\theta$, from $A$ onto $S$, maps $d$ onto a non-trivial element in $S$. Now consider the composition of these two epimorphism, $\theta \circ \phi$, which maps $G$ onto $S$. The image of $d$ in $S$ is non-trivial:

$$\theta \circ \phi(d) = d\theta = d\delta_\lambda A \neq 1.$$ 

Case 2: Let $1 \neq d \notin A$ but $d \in G$. Now $d$ can be expressed as follows:

$$d = a_1 b_1 a_2 b_2 \cdots a_n b_n \ (a_i \in A - C \ b_i \in \bar{A} - \bar{C}).$$

Since the equally indexed filtrations, $\{\delta_\lambda A\}_{\lambda \in \mathbb{N}}$, and $\{\delta_\lambda \bar{A}\}_{\lambda \in \mathbb{N}}$ of $A$ and $\bar{A}$ are compatible, we can form $G_\lambda$:

$$G_\lambda = \{A/\delta_\lambda A \ast \bar{A}/\delta_\lambda \bar{A}; C \delta_\lambda A/\delta_\lambda A = \bar{C} \delta_\lambda \bar{A}/\delta_\lambda \bar{A}\}.$$ 

Note that $G_\lambda$ is residually solvable (by mapping it to one of the factors and noting that the kernel of the map is free). Consider the canonical homomorphism $\theta$ from $G$ onto $G_\lambda$. Since $C$ is solvably separable in $A$, i.e.

$$\bigcap_{\lambda \in \mathbb{N}} C \delta_\lambda A = C,$$

$\lambda \in \mathbb{N}$ can be so chosen that

$$a_i \notin C \delta_\lambda A, \ a_i \notin \bar{C} \delta_\lambda \bar{A} \ (\text{for } i = 1, \cdots, n).$$

Hence

$$a_1 \delta_\lambda A b_1 \delta_\lambda \bar{A} \cdots a_n \delta_\lambda A b_n \delta_\lambda \bar{A} \neq 1.$$ 

This completes the proof of theorem by using Baumslag’s Proposition [1], we recalled in Section 2.2.

3.3. **Solvable separability is a sufficient condition for residual solvability.** Note that the condition of solvable separability of the amalgamated subgroup in the factors, in the case of doubles, is necessary. The following theorem shows that the amalgamated product of doubles is not residually solvable where the factors are residually solvable groups.
Theorem 8. Let $A$ be a finitely generated residually solvable group, and $C$ a normal subgroup of $A$, such that $A/C$ is perfect. Let $\bar{\phi}$ be an isomorphic mapping of $A$ onto itself. Then

$$D = \{A \ast \bar{A}; C = \bar{C}\}$$

is meta-residually solvable, but not residually solvable.

Proof. $D$ is meta-residually solvable by Corollary $6$. By Lemma $1$ there exists an epimorphism from $D$ onto $A$. Let $K$ be the kernel of this epimorphism. Since $C$ is normal in $A$, by using Lemma $3$.

$$[C, K] = 1. \quad (*)$$

Now we want to show that $D$ is not residually solvable. We proceed by contradiction. Suppose $D$ is residually solvable. Let $d$ be a non-trivial element in $[K, D]$. The existence of such an element is guaranteed by Lemma $2$. Now assume $\mu$ is a homomorphism of $D$ onto a solvable group $S$, so that $d\mu \neq S 1$. Since $\mu$ is an epimorphism, $C\mu$ is a normal subgroup of $S$, and by $(*)$,

$$[C\mu, K\mu] = 1.$$ 

If we can show that $S = C\mu$, then $[K\mu, S] = 1$, which implies that $d\mu = S 1$, a contradiction. We now need to show that $S = C\mu$. We have that $D \rightarrow S$ induces a homomorphism from $D/C$ onto $S/C\mu$. Since $A/C$ and $\bar{A}/\bar{C}$ each have a perfect subgroup, this induces a homomorphism from $A/C$ to $1$. So, $S/C\mu = 1$ and hence $C\mu = S$. \qed

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