Non-Hermitian Hamiltonian beyond $\mathcal{PT}$-symmetry for time-dependant $SU(1,1)$ and $SU(2)$ systems — exact solution and geometric phase in pseudo-invariant theory

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Abstract

We investigate in this paper time-dependent non-Hermitian Hamiltonians, which consist respectively of $SU(1,1)$ and $SU(2)$ generators. The former Hamiltonian is PT symmetric but the latter one is not. A time-dependent non-unitary operator is proposed to construct the non-Hermitian invariant, which is verified as pseudo-Hermitian with real eigenvalues. The exact solutions are obtained in terms of the eigenstates of the pseudo-Hermitian invariant operator for both the $SU(1,1)$ and $SU(2)$ systems in a unified manner. Then, we derive the LR phase, which can be separated to the dynamic phase and the geometrical phase. The analytical results are exactly in agreement with those of corresponding Hermitian Hamiltonians in the literature.

Keywords: Non-Hermitian Hamiltonian, $su(1,1)$ and $su(2)$ Lie algebra, Pseudo-Hermitian invariant.

1 Introduction

In conventional quantum mechanics the Hamiltonian is always hermitian, this constrains the energy spectrum to be real. The non-hermiticity means that the usual arguments for the reality
of the spectrum cannot be used. Carl Bender et al [1, 2] interpreted the reality of this spectrum as being due to its $\mathcal{PT}$-symmetry. That is, if we simultaneously reflect in space and reverse time, the potential remains unchanged. Another idea that has been put forward as an extension of conventional hermitian quantum mechanics, is pseudo-hermiticity. The fact that it may be possible to find real eigenvalues in a Hamiltonian which is non-hermitian was a point of interest, particularly as the concept of $\mathcal{PT}$-symmetry appears to have a more physical interpretation than the very mathematical concept of hermiticity. Then came the idea of the pseudo Hermiticity, which was a generalisation of the conventional hermitian quantum mechanics. A Hamiltonian is said to be $\eta$ pseudo-hermitian [3] if

$$\hat{H}^\dagger = \hat{\eta} \hat{H} \hat{\eta}^{-1}.$$  

Mostafazadeh [4–7] associated the spectrum reality to a more general property than the $\mathcal{PT}$-symmetry: the pseudo-Hermiticity of the Hamiltonian. These new perspectives formed the foundation of a new research field based on the fact that the Hermiticity is a sufficient but unnecessary condition to have a real spectrum. Many concrete Hamiltonian systems can not be described by autonomous Hamiltonians $\hat{H}$, but require an explicit dependence on time $\hat{H}(t)$. In this work, we discuss how such type of systems can be treated consistently when $\hat{H}(t)$ is non-Hermitian [9–13]. While many researchers focused on studying time-independent non-Hermitian systems. Others directed their efforts to solving time dependent non-Hermitian systems in terms of the Lewis-Riesenfeld (LR) invariant theory [14], which presents the advantage of obtaining exact solutions. Invariants are important in modern theoretical physics and many theories are expressed in terms of their symmetries and invariants. In particular, the invariants are capable of finding the solution of the equation of motion. In the following, we recall the notion of the pseudo-hermitien invariants introduced in [15, 16] which have played a distinctive role in non hermitian quantum mechanics. In references [15–17], a particular attention was given to the special subset of pseudo-Hermitian invariant operators associated to time dependent non-Hermitian Hamiltonians, in which the reality of the eigenvalues of the invariant is guaranteed. Let’s review briefly the pseudo-Hermitian invariant theory. The invariant operator $\hat{I}(t)$ is said to be pseudo-Hermitian with respect to the metric operator $\hat{\eta}(t)$, if

$$\hat{I}^\dagger(t) = \hat{\eta}(t) \hat{I}(t) \hat{\eta}^{-1}(t),$$  

in which the metric operator is Hermitian. Thus the invariant $\hat{I}(t)$ can always be mapped to a Hermitian invariant operator $\hat{I}^h(t)$ by a similarity (Dyson) transformation $\hat{\rho}(t)$, such that

$$\hat{I}^h(t) = \hat{\rho}(t) \hat{I}(t) \hat{\rho}^{-1}(t) = \left(\hat{I}^h(t)\right)^+, $$  

with $\hat{\eta}(t) = \hat{\rho}^\dagger(t) \hat{\rho}(t)$. The exact solution of a $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian was presented recently for the periodically driven $SU(1, 1)$ generators [18, 19]. We emphasize that the spectrum reality of a non-Hermitian Hamiltonian is not confined to the $\mathcal{PT}$-symmetry. In this paper, we follow the pseudo-Hermitian invariant theory to solve the Schrödinger equation for a non-Hermitian Hamiltonian consisting of time-dependent $SU(1, 1)$ and $SU(2)$ generators. The $SU(1, 1)$ Hamiltonian is $\mathcal{PT}$-symmetric but the $SU(2)$ Hamiltonian is not. The paper is organized as follows: in Sec.II we put forward a non-Hermitian Hamiltonian consisting of periodically driven $SU(1, 1)$ and $SU(2)$ generators. We propose a non-unitary transformation operator $\hat{R}(t)$
to construct the pseudo-Hermitian invariant. A non-unitary transformation transforming the time-dependent Schrödinger equation of the free particle into that for the quantum harmonic oscillator was considered in [20]. In Sec. III, exact solutions of the Schrödinger equation are found along with the LR phase and non-adiabatic Berry phase, which reduces to the adiabatic phase in slowly varying limit [21]. The conclusion is given in Sec. IV.

2 Non-Hermitian Hamiltonian and pseudo-Hermitian invariant

The considered system is described by the following time dependent Hamiltonian

$$\hat{H}(t) = \Omega \hat{K}_0 + G \left( \hat{K}_+ \exp(i\phi(t)) - \hat{K}_- \exp(-i\phi(t)) \right),$$  \hspace{1cm} (1)

with \(\Omega, G\) and \(\phi(t)\) are real parameters: \(\Omega\) being the driving frequency, \(G\) a coupling parameter and \(\phi(t) = \omega t\) the periodicity parameter \(\phi(t) = \phi(t + T)\).

\(\hat{K}_0\) is Hermitian, while \(\hat{K}_-^\dagger = \hat{K}_+\). These operators are \(SU(1,1)\) and \(SU(2)\) generators that satisfy these commutation relations:

$$\begin{align*}
[\hat{K}_0, \hat{K}_\pm] &= \pm \hat{K}_\pm, \\
[\hat{K}_+, \hat{K}_-] &= D \hat{K}_0, 
\end{align*}$$  \hspace{1cm} (2)

where \(D = \pm 2\) respectively for the \(SU(2)\) and \(SU(1,1)\) Lie algebras.

It is obvious that the Hamiltonian is periodic \(\hat{H}(t) = \hat{H}(t + T)\) but non Hermitian

$$\hat{H}^+(t) = \Omega \hat{K}_0 - G \left( \hat{K}_+ \exp(i\phi(t)) - \hat{K}_- \exp(-i\phi(t)) \right) \neq \hat{H}(t).$$  \hspace{1cm} (3)

The non-Hermitian Hamiltonian Eq.(3) is \(\mathcal{PT}\) symmetric for \(SU(1,1)\) system but is asymmetric for \(SU(2)\). The time dependent Schrödinger equation for this Hamiltonian is given by

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle. $$  \hspace{1cm} (4)

Since the time-dependent Hamiltonian is not a conserved quantity, we solve the time-dependent Schrödinger equation (4) with the help of the pseudo-Hermitian invariant scheme. The total time-derivative of the invariant \(\hat{I}(t)\) must be zero,

$$i \frac{d\hat{I}(t)}{dt} = i \frac{\partial}{\partial t} \hat{I}(t) + \left[ \hat{I}(t), \hat{H}(t) \right] = 0.$$  \hspace{1cm} (5)

We assume that the invariant \(\hat{I}(t)\) can be generated from the operator \(\hat{K}_0\) such that

$$\hat{I}(t) = \hat{R}(t) \hat{K}_0 \hat{R}^{-1}(t), $$  \hspace{1cm} (6)

where \(\hat{R}(t)\) is a non-unitary transformation operator defined as

$$\hat{R}(t) = \exp\left[ \frac{\varepsilon}{2} \left( \hat{K}_+ \exp(i\phi(t)) + \hat{K}_- \exp(-i\phi(t)) \right) \right], $$  \hspace{1cm} (7)
\[ \hat{R}^{-1}(t) = \exp \left[ -\frac{\varepsilon}{2} \left( \hat{K}_+ \exp (i\phi(t)) + \hat{K}_- \exp (-i\phi(t)) \right) \right]. \]  

(8)

And \( \varepsilon \) is a real parameter to be determined.

Using the following transformations [37–39]:

\[
\begin{align*}
\hat{R}(t)\hat{K}_+\hat{R}^{-1}(t) & = \hat{K}_+ \cosh^2 \left( \frac{\alpha}{2} \right) - \hat{K}_- \exp (-2i\phi(t)) \sinh^2 \left( \frac{\alpha}{2} \right) - \hat{K}_0 \exp (-i\phi(t)) \sqrt{\frac{D}{2}} \sinh (\alpha) , \\
\hat{R}(t)\hat{K}_-\hat{R}^{-1}(t) & = \hat{K}_- \cosh^2 \left( \frac{\alpha}{2} \right) - \hat{K}_+ \exp (2i\phi(t)) \sinh^2 \left( \frac{\alpha}{2} \right) + \hat{K}_0 \exp (i\phi(t)) \sqrt{\frac{D}{2}} \sinh (\alpha) , \\
\hat{R}(t)\hat{K}_0\hat{R}^{-1}(t) & = \hat{K}_0 \cosh (\alpha) - \frac{1}{\sqrt{2D}} \sinh (\alpha) \left( \hat{K}_+ \exp (i\phi(t)) - \hat{K}_- \exp (-i\phi(t)) \right) , \\
i\hat{R}^{-1}(t)\frac{\partial}{\partial t} \hat{R}(t) & = -2\omega \hat{K}_0 \sinh^2 \left( \frac{\alpha}{2} \right) - \frac{\omega}{\sqrt{2D}} \sinh (\alpha) \left( \hat{K}_+ \exp (i\phi(t)) - \hat{K}_- \exp (-i\phi(t)) \right) , 
\end{align*}
\]

in which

\[ \alpha = \varepsilon \sqrt{\frac{D}{2}}, \]  

(10)

we can obtain the invariant \( \hat{I}(t) \) written as

\[ \hat{I}(t) = \hat{K}_0 \cosh (\alpha) - \frac{1}{\sqrt{2D}} \sinh (\alpha) \left( \hat{K}_+ \exp (i\phi(t)) - \hat{K}_- \exp (-i\phi(t)) \right) . \]  

(11)

The invariant is obviously non-Hermitian,

\[ \hat{I}^+(t) = \hat{K}_0 \cosh (\alpha) + \frac{1}{\sqrt{2D}} \sinh (\alpha) \left( \hat{K}_+ \exp (i\phi(t)) - \hat{K}_- \exp (-i\phi(t)) \right) \neq \hat{I}(t). \]  

(12)

Substituting the invariant into the equation (5) we have

\[ \left[ \hat{I}(t), \hat{H}(t) \right] = \left[ \frac{\Omega}{\sqrt{2D}} \sinh (\alpha) + G \cosh (\alpha) \right] \left( \hat{K}_+ \exp (i\phi) + \hat{K}_- \exp (-i\phi) \right) , \]

and

\[ i\frac{\partial}{\partial t} \hat{I}(t) = -\frac{\omega}{\sqrt{2D}} \sinh (\alpha) \left( \hat{K}_+ \exp (i\phi) + \hat{K}_- \exp (-i\phi) \right) . \]

The equation (5) is fulfilled under the auxiliary condition:

\[ G \cosh (\alpha) = -\frac{\omega + \Omega}{\sqrt{2D}} \sinh (\alpha) , \]  

(13)

from which the parameter \( \varepsilon \) is determined. It is easy to check that the invariant \( \hat{I}(t) \) is pseudo-Hermitian with respect to the metric operator \( \hat{\eta} \)

\[ \hat{I}(t) = \hat{\eta} \hat{I}(t) \hat{\eta}^{-1} , \]  

(14)

where

\[ \hat{\eta} = \left( \hat{R}^{-1} \right)^\dagger \hat{R}^{-1} = \hat{R}^{-2} , \]  

(15)

for the Hermitian operator \( \hat{R}^\dagger = \hat{R} \).
3 Exact solution and geometric phase

As one of the most important results of the LR invariant theory, geometrical phases attracted considerable interests in both theoretical and experimental physics. The first general treatment of geometric phases is due to Berry \cite{21} who have considered Hermitian Hamiltonians undergoing adiabatic changes. The instantaneous eigenstates returns to the same ray in the Hilbert space, but acquires a phase factor consisting of a dynamical and a geometrical part. Berry’s phase knew many generalizations \cite{22}- \cite{25}. There have also been attempts to extend geometric phases to systems described by non-Hermitian Hamiltonians \cite{26}- \cite{36}.

Assuming that the pseudo-Hermitian invariant possesses a set of non-degenerate eigenstates,

$$\hat{I}(t) |n(t)\rangle = \lambda_n |n(t)\rangle,$$  \hspace{1cm} (16)

with the orthogonality condition

$$\langle n(t) | \hat{\eta}(t) | m(t) \rangle = \delta_{nm}.$$

The general solution of the time-dependent Schrödinger equation (4) is the superposition of the eigenstates of the pseudo-Hermitian invariant $\hat{I}(t)$,

$$|\psi(t)\rangle = \sum_n C_n e^{i\alpha_n(t)} |n(t)\rangle,$$  \hspace{1cm} (17)

where $C_n$ are time independent coefficients and $\alpha_n(t)$ is the LR phase. Substituting the general solution Eq.(17) into the Schrödinger equation Eq.(4) yields the LR phase

$$\alpha_n(t) = \int_0^t dt' \langle n(t') | \eta \left[ i \frac{\partial}{\partial t'} - \hat{H}(t') \right] | n(t') \rangle.$$  \hspace{1cm} (18)

Using the transformation relations (9) and the auxiliary condition (13) we obtain the LR phase

$$\alpha_n(t) = -\lambda_n \int_0^t dt' \left( \Omega + 2\sqrt{D_G} \sinh(\alpha) + 2 (\Omega + \omega) \sinh^2\left(\frac{\alpha}{2}\right) \right),$$  \hspace{1cm} (19)

in which

$$\sinh^2\left(\frac{\alpha}{2}\right) = -\frac{1}{2} \pm \frac{(\omega + \Omega)}{2\sqrt{(\omega + \Omega)^2 - 2D_G^2}}.$$  \hspace{1cm} (20)

The first term of LR phase $\alpha_n(t)$ gives rise to the geometrical phase or Berry phase, which can be evaluated in one period of driving field $T = 2\pi/\omega$ as

$$\gamma_n(T) = i \int_0^T dt' \langle n(t') | \eta \frac{\partial}{\partial t'} | n(t') \rangle = -2\lambda_n \int \sinh^2\left(\frac{\alpha}{2}\right) d\phi.$$  \hspace{1cm} (21a)

Substituting the parameter of Eq.(20) into the Eq.(21a) yields the non-adiabatic Berry phase suitable for both $SU(2)$ and $SU(1, 1)$ systems

$$\gamma_n(T) = -4\pi\lambda_n \left( -\frac{1}{2} \pm \frac{(\omega + \Omega)}{2\sqrt{(\omega + \Omega)^2 - 2D_G^2}} \right).$$  \hspace{1cm} (22)
In the adiabatic approximation \( T \to \infty, \phi = \omega = 0 \), the exact Berry-phase reduces to the well known adiabatic form \[\gamma_n(T) = -4\pi\lambda_n \left(-\frac{1}{2} \pm \frac{\Omega}{2\sqrt{\Omega^2 - 2DG^2}}\right). \tag{23}\]

For \( SU(1,1) \) system with \( D = -2 \), the generators of Lie algebra can be expressed by boson creation and annihilation operators such that
\[
\hat{K}_0 = \frac{1}{2} \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad \hat{K}_+ = \frac{1}{2} (\hat{a}^\dagger)^2, \quad \hat{K}_- = \frac{1}{2} (\hat{a})^2. \tag{24}\]

The non-Hermitian Hamiltonian is \( \mathcal{PT} \)-symmetric. The eigenstates of \( \hat{K}_0 \) in this case are Fock states \( \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle \) with the eigenvalues \( \lambda_n = \frac{1}{2} (n + \frac{1}{2}) \),
\[
\hat{K}_0 |n\rangle = \frac{1}{2} \left(n + \frac{1}{2}\right) |n\rangle. \tag{25}\]

The LR phase (19) is then
\[
\alpha_n(t) = -\frac{1}{2} \left(n + \frac{1}{2}\right) \int_0^t dt' \left( \Omega - G \sin \varepsilon + 2 (\Omega + \omega) \sin \frac{\varepsilon}{2} \right), \tag{26}\]
in which
\[
\sin^2 \frac{\varepsilon}{2} = -\frac{1}{2} \pm \frac{(\omega + \Omega)}{2\sqrt{(\omega + \Omega)^2 + 4G^2}}.
\]
And the Berry phase in the adiabatic approximation (23) is given by
\[
\gamma_n(T) = -\pi \left(n + \frac{1}{2}\right) \left(-1 \pm \frac{\Omega}{\sqrt{\Omega^2 + 4G^2}}\right). \tag{27}\]

For \( D = 2 \) the generators of \( SU(2) \) are simply spin operators. Since the spin operators \( \hat{K}_0, \hat{K}_\pm \) change respectively to \( -\hat{K}_0, -\hat{K}_\pm \) under \( \mathcal{PT} \) transformation, the non-Hermitian Hamiltonian is \( \mathcal{PT} \)-symmetric. The eigenstates and the eigenvalues of \( \hat{K}_0 \) are defined in this case as
\[
\hat{K}_0 |j, n\rangle = n |j, n\rangle. \tag{28}\]

The LR phase (19) in this case is,
\[
\alpha_n(t) = -n \int_0^t dt' \left( \Omega + G \sinh \varepsilon + 2 (\Omega + \omega) \sinh^2 \frac{\varepsilon}{2} \right), \tag{29}\]
where
\[
\sinh^2 \frac{\varepsilon}{2} = -\frac{1}{2} \pm \frac{(\omega + \Omega)}{2\sqrt{(\omega + \Omega)^2 + 4G^2}}.
\]
And the Berry phase in the adiabatic approximation (23) is given by
\[
\gamma_n(T) = -2\pi n \left(-1 \pm \frac{\Omega}{\sqrt{\Omega^2 - 4G^2}}\right). \tag{30}\]
4 Conclusion

It is well known that non-Hermitian Hamiltonians with $\mathcal{PT}$-symmetry can possess a real spectrum [1, 2]. However, the spectrum reality is not restricted to the $\mathcal{PT}$-symmetry only. If a pseudo-Hermitian invariant exists for a non-Hermitian Hamiltonian, the real spectrum is guaranteed. We solve the time-dependent non-Hermitian Hamiltonian consisting of $SU(1,1)$ and $SU(2)$ generators with the help of a pseudo-Hermitian invariant without considering the $\mathcal{PT}$-symmetry property. We propose a non-unitary but a Hermitian transformation operator $\hat{R}(t)$ to construct the non-Hermitian invariant operator $\hat{I}(t)$, which is proved to be pseudo-Hermitian in regards to the metric operator given by $\hat{\eta} = \hat{R}^{-2}$. This invariant operator $\hat{I}(t)$ possesses real eigenvalues for both the $SU(1,1)$ and $SU(2)$ systems. Exact solutions are obtained in terms of its eigenstates. We obtain the LR and the Berry phases, which are in agreement with those of the corresponding Hermitian Hamiltonians in the literature [38, 39].

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