(P, Q)-Lucas polynomial coefficient inequalities of the bi-univalent function class

Arzu AKGÜL*
Department of Mathematics, Faculty of Arts and Science, Kocaeli University, Kocaeli, Turkey

Received: 23.01.2019 • Accepted/Published Online: 10.07.2019 • Final Version: 28.09.2019

Abstract: Recently, Lucas polynomials and other special polynomials gained importance in the field of geometric function theory. In this study, by connecting these polynomials, subordination, and the Al-Oboudi differential operator, we introduce a new class of bi-univalent functions and obtain coefficient estimates and Fejete–Szegö inequalities for this new class.

Key words: (P, Q)-Lucas polynomials, coefficient bounds, bi-univalent functions

1. Introduction
Let \( A \) denote the class of functions of the form

\[
u(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}\]

which are analytic in the open unit disk \( U = \{ z : |z| < 1 \} \), and let \( S = \{ u \in A : u \text{ is univalent in } U \} \).

The Koebe one-quarter theorem [3] states that the range of every function \( u \in S \) contains the disc of radius \( \{ w : |w| < \frac{1}{4} \} \). Thus, every such function \( u \in S \) has an inverse \( u^{-1} \), which satisfies

\[
u^{-1}(u(z)) = z \quad (z \in U)
\]

and

\[
u \left( u^{-1}(w) \right) = w \left( |w| < r_0(u) , \ r_0(u) \geq \frac{1}{4} \right),
\]

where

\[
u^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \tag{2}\]

If both \( u \) and \( u^{-1} \) are univalent in \( U \), then a function \( u \in A \) is said to be bi-univalent in \( U \). We say that \( u \) is in the class \( \Sigma \) for such functions.

For analytic functions \( u \) and \( v \), \( u \) is said to be subordinate to \( v \), denoted

\[
u(z) \prec v(z), \tag{3}\]

*Correspondence: akgul@kocaeli.edu.tr
2010 AMS Mathematics Subject Classification: 30C45

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if there is an analytic function \( w \) such that \( w(0) = 0 \), \( |w(z)| < 1 \), and \( u(z) = v(w(z)) \).

For a function \( u(z) \in A \), Al-Oboudi [1] defined the following differential operator, named the Al-Oboudi differential operator:

\[
D_0^\delta u(z) = u(z),
\]

\[
D_1^\delta u(z) = (1 - \delta)u(z) + \delta u'(z) = D_\delta u(z), \delta \geq 0,
\]

\[
D_n^\delta u(z) = D_\delta (D_{\delta}^{n-1} u(z)), \quad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},
\]

If \( u \) is given by (1), then from (4) and (5) we see that

\[
D_n^\delta u(z) = z + \sum_{k=2}^{n} [1 + (k - 1) \delta]^n a_k z^k,
\]

with \( D_n^0 u(0) = 0 \). When \( \delta = 1 \), we get Salagean’s differential operator [9].

**Definition 1** [6] Let \( P(x) \) and \( Q(x) \) be polynomials with real coefficients. The \( (P, Q) \)-Lucas polynomials \( L_{P, Q, n}(x) \) are defined by the recurrence relation

\[
L_{P, Q, n}(x) = P(x)L_{P, Q, n-1}(x) + Q(x)L_{P, Q, n-2}(x) \quad (n \geq 2),
\]

from which the first few Lucas polynomials can be found as follows:

\[
L_{P, Q, 0}(x) = 2,
\]

\[
L_{P, Q, 1}(x) = P(x),
\]

\[
L_{P, Q, 2}(x) = P^2(x) + 2Q(x),
\]

\[
L_{P, Q, 3}(x) = P^3(x) + 3P(x)Q(x).
\]

**Definition 2** [6] Let \( G_{\{L_n(x)\}}(z) \) be the generating function of the \( (P, Q) \)-Lucas polynomial sequence \( L_{P, Q, n}(x) \). Then

\[
G_{\{L_n(x)\}}(z) = \sum_{n=0}^{\infty} L_{P, Q, n}(x) z^n = \frac{2 - P(x)z}{1 - P(x)z - Q(x)z^2}.
\]

2. The class \( Q^{\Sigma, \delta}(\zeta, n; x) \) and the Fekete–Szegö inequality

We begin this section by defining the class \( Q^{\Sigma, \delta}(\zeta, n; x) \) and by finding the estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in this class.

**Definition 3** The function \( u \) is said to be in the class \( Q^{\Sigma, \delta}(\zeta, n; x) \) if the following conditions are satisfied:

\[
(1 - \zeta) \frac{D_n^\delta u(z)}{z} + \zeta \left( D_n^\delta u(z) \right)' \prec G_{\{L_{P, Q, n}(x)\}}(z) - 1
\]

and
where the function \( D^\alpha \) is the Al-Oboudi differential operator and \( v = u^{-1} \) is given by (2).

**Theorem 4** Let \( u \) given by (1) be in the class \( Q^{E,\delta}(\zeta, n; x) \). Then,

\[
|a_2| \leq \frac{|P(x)|}{\sqrt{|P(x)|}} \sqrt{\left\{(1 + 2\delta)^n (1 + 2\zeta) - (1 + \delta)^{2n} (1 + \zeta)^2 \right\} P^2(x) - 2(1 + \delta)^{2n} (1 + \zeta)^2 Q(x)} \tag{12}
\]

and

\[
|a_3| \leq \frac{P^2(x)}{(1 + \delta)^{2n} (1 + 2\zeta)} + \frac{|P(x)|}{(1 + 2\delta)^n (1 + 2\zeta)}. \tag{13}
\]

**Proof** Let \( u \in Q^{E,\delta}(\zeta, n; x) \). Then, from Definition 2, for some analytic functions \( \Omega, \Lambda \) such that \( \Omega(0) = \Lambda(0) = 0 \) and \( |\Omega(z)| < 1, |\Lambda(w)| < 1 \) for all \( z, w \in U \), we can write

\[
(1 - \zeta) \frac{D_\delta^\alpha u(z)}{z} + \zeta (D_\delta^\alpha u(z))' = G_{(L_D, Q, n)(x)}(\Omega(z)) - 1 \tag{14}
\]

and

\[
(1 - \zeta) \frac{D_\delta^\alpha v(w)}{w} + \zeta (D_\delta^\alpha v(w))' = G_{(L_D, Q, n)(x)}(\Lambda(w)) - 1, \tag{15}
\]

or equivalently

\[
(1 - \zeta) \frac{D_\delta^\alpha u(z)}{z} + \zeta (D_\delta^\alpha u(z))' = -1 + L_{D, Q, 0}(x) + L_{D, Q, 1}(x)\Omega(z) + L_{D, Q, 2}(x)\Omega^2(z) + \cdots \tag{16}
\]

and

\[
(1 - \zeta) \frac{D_\delta^\alpha v(w)}{w} + \zeta (D_\delta^\alpha v(w))' = -1 + L_{D, Q, 0}(x) + L_{D, Q, 1}(x)\Lambda(w) + L_{D, Q, 2}(x)\Lambda^2(w) + \cdots \tag{17}
\]

From equalities (16) and (17), we obtain that

\[
(1 - \zeta) \frac{D_\delta^\alpha u(z)}{z} + \zeta (D_\delta^\alpha u(z))' = 1 + L_{D, Q, 1}(x)l_1 z + \left[ L_{D, Q, 1}(x)l_2 + L_{D, Q, 2}(x)l_1^2 \right] z^2 + \cdots \tag{18}
\]

and

\[
(1 - \zeta) \frac{D_\delta^\alpha v(w)}{w} + \zeta (D_\delta^\alpha v(w))' = 1 + L_{D, Q, 1}(x)r_1 w + \left[ L_{D, Q, 1}(x)r_2 + L_{D, Q, 2}(x)r_1^2 \right] w^2 + \cdots \tag{19}
\]

It is already known that if for \( z, w \in U \),

\[
\Omega(z) = \sum_{i=1}^{n} l_iz^i < 1
\]

and

\[
\Lambda(w) = \sum_{i=1}^{n} r_iw^i < 1,
\]
then

$$\Omega(z) = |l_i| < 1$$

and

$$\Lambda(w) = |r_i| < 1$$

where $i \in \mathbb{N}$.

Thus, comparing the corresponding coefficients in (18) and (19), we get

\begin{align*}
(1 + \zeta)(1 + \delta)^n a_2 &= L_{P, Q, 1}(x) l_1, \quad (20) \\
(1 + 2\zeta)(1 + 2\delta)^n a_3 &= L_{P, Q, 1}(x) l_2 + L_{P, Q, 2}(x) l_1^2, \quad (21) \\
-(1 + \zeta)(1 + \delta)^n a_2 &= L_{P, Q, 1}(x) r_1, \quad (22) \\
(1 + 2\zeta)(1 + 2\delta)^n (2a_2^2 - a_3) &= L_{P, Q, 1}(x) r_2 + L_{P, Q, 2}(x) r_1^2, \quad (23)
\end{align*}

From (20) and (22),

$$l_1 = -r_1, \quad (24)$$

\begin{align*}
2(1 + \zeta)^2 (1 + \delta)^{2n} a_2^2 &= L_{P, Q, 1}^2(x) (l_1^2 + r_1^2). \quad (25)
\end{align*}

Adding (21) and (23), we get

$$2(1 + 2\zeta)(1 + 2\delta)^n a_3^2 = L_{P, Q, 1}(x) (l_2 + r_2) + L_{P, Q, 2}(x) (l_1^2 + r_1^2). \quad (26)$$

By using (25) in (26), we have

$$2 \left[ L_{P, Q, 1}^2(x)(1 + 2\zeta)(1 + 2\delta)^n - L_{P, Q, 2}(x)(1 + \zeta)^2(1 + \delta)^{2n} \right] a_2^2 = L_{P, Q, 1}^3(x) (l_2 + r_2), \quad (27)$$

which gives

$$|a_2| \leq \frac{|P(x)| \sqrt{|P(x)|}}{\sqrt{\left| \left( (1 + 2\delta)^n (1 + 2\zeta) - (1 + \delta)^{2n} (1 + \zeta)^2 \right) \frac{P^2(x)}{2} - 2(1 + \delta)^{2n} (1 + \zeta)^2 Q(x) \right|}}.$$
2.1. Corollaries

For the special choices of parameters \( \delta, \zeta \), and \( n \) in Theorem 4, we obtain the following:

**Corollary 5** Let \( u \in Q^{\Sigma,1}(\zeta, n; x) = Q^\Sigma(\zeta, n; x) \). Then,

\[
|a_2| \leq \frac{|P(x)| \sqrt{|P(x)|}}{\sqrt{\left\{ 3^n (1 + 2\zeta) - 2^{2n} (1 + \zeta)^2 \right\} P^2(x) - 2^{2n+1} (1 + \zeta)^2 Q(x)}} \tag{29}
\]

and

\[
|a_3| \leq \frac{P^2(x)}{2^{2n} (1 + \zeta)^2} + \frac{|P(x)|}{3^n (1 + 2\zeta)}. \tag{30}
\]

**Corollary 6** Let \( u \in Q^{\Sigma,\delta}(\zeta, 0; x) = Q^{\Sigma,\delta}(\zeta; x) \). Then,

\[
|a_2| \leq \frac{|P(x)| \sqrt{|P(x)|}}{\sqrt{\zeta^2 P^2(x) + 2 (1 + \zeta)^2 Q(x)}} \tag{31}
\]

and

\[
|a_3| \leq \frac{P^2(x)}{(1 + \zeta)^2} + \frac{|P(x)|}{(1 + 2\zeta)}. \tag{32}
\]

**Corollary 7** Let \( u \in Q^{\Sigma,\delta}(1, 0; x) = Q^{\Sigma,\delta}(x) \). Then,

\[
|a_2| \leq \frac{|P(x)| \sqrt{|P(x)|}}{\sqrt{|P^2(x) + 4Q(x)|}} \tag{33}
\]

\[
|a_3| \leq \frac{P^2(x)}{4} + \frac{|P(x)|}{3}. \tag{34}
\]

The next theorem gives us the Fekete–Szegö inequality:

**Theorem 8** Let \( u \) given by (1) be in the class \( Q^{\Sigma,\delta}(\zeta, n; x) \). Then,

\[
|a_3 - \lambda a_2^2| \leq \begin{cases} 
\frac{|P(x)|}{(1 + 2\zeta)(1 + 2\delta)^n}, & 0 \leq |t(\lambda; x)| < \frac{3}{2(1 + 2\zeta)(1 + 2\delta)^n}, \\
2 |P(x)| |t(\lambda; x)|, & |t(\lambda; x)| \geq \frac{3}{2(1 + 2\zeta)(1 + 2\delta)^n}, 
\end{cases} \tag{35}
\]

where

\[
t(\lambda; x) = \frac{(1 - \lambda) L_{P,Q,1}(x)}{2 \left[ L_{P,Q,1}(x)(1 + 2\zeta)(1 + 2\delta)^n - L_{P,Q,3}(x)(1 + \zeta)^2 (1 + \delta)^{2n} \right]}.
\]

**Proof** From equations (27) and (28), we get

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By using equalities in (9), we complete the proof.

**Corollary 9** Let $u \in Q^{\Sigma_1}(\zeta, n; x) = Q^{\Sigma}(\zeta, n; x)$ and $\lambda \in R$. Then,

$$|a_3 - \lambda a_2^2| \leq \left\{ \begin{array}{ll} \frac{|P(x)|}{2|P(x)||t(\lambda; x)|} & , \quad 0 \leq |t(\lambda; x)| < \frac{1}{2(1+2\zeta)n^2}, \\ \frac{1}{2(1+2\zeta)n^2} & , \quad |t(\lambda; x)| \geq \frac{1}{2(1+2\zeta)}, \end{array} \right.$$  \hspace{1cm} (36)

where

$$t(\lambda; x) = \frac{(1 - \lambda) L^2_{P,Q_1}(x)}{2 L^2_{P,Q_1}(x)(1 + 2\zeta)(1 + 2\delta)} - L_{P,Q_1}(x)(l_2 - r_2).$$

**Corollary 10** Let $u \in mathcal{Q}^{\Sigma,\delta}(\zeta, 0; x) = Q^{\Sigma,\delta}(\zeta; x)$ and $\lambda \in R$. Then,

$$|a_3 - \lambda a_2^2| \leq \left\{ \begin{array}{ll} \frac{|P(x)|}{2|P(x)||t(\lambda; x)|} & , \quad 0 \leq |t(\lambda; x)| < \frac{1}{2(1+2\zeta)}, \\ \frac{1}{2(1+2\zeta)} & , \quad |t(\lambda; x)| \geq \frac{1}{2(1+2\zeta)} \end{array} \right.$$  \hspace{1cm} (37)

where

$$t(\lambda; x) = \frac{(1 - \lambda) L^2_{P,Q_1}(x)}{2 L^2_{P,Q_1}(x)(1 + 2\zeta) - L_{P,Q_2}(x)(1 + \zeta)^2 2^{2n}}.$$  \hspace{1cm} (38)

**Corollary 11** Let $u \in Q^{\Sigma,\delta}(1, 0; x) = Q^{\Sigma,\delta}(x)$ and $\lambda \in R$. Then,

$$|a_3 - \lambda a_2^2| \leq \left\{ \begin{array}{ll} \frac{|P(x)|}{2|P(x)||t(\lambda; x)|} & , \quad 0 \leq |t(\lambda; x)| < \frac{1}{6}, \\ \frac{1}{6} & , \quad |t(\lambda; x)| \geq \frac{1}{6}, \end{array} \right.$$  \hspace{1cm} (39)

where

$$t(\lambda; x) = \frac{(1 - \lambda) L^2_{P,Q_1}(x)}{2 [3 L^2_{P,Q_1}(x) - 4 L_{P,Q_2}(x)].}$$  \hspace{1cm} (40)

Choosing $\lambda = 1$ in Theorem 8, we have the following corollaries:
Corollary 12 Let \( u \in Q^{\Sigma,1}(\zeta, n; x) = Q^{\Sigma}(\zeta, n; x) \). Then,
\[
|a_3 - a_2^2| \leq \frac{|P(x)|}{(1 + 2\zeta)^3n}.
\]
(39)

Corollary 13 Let \( u \in Q^{\Sigma,\delta}(\zeta, 0; x) = Q^{\Sigma,\delta}(\zeta; x) \). Then,
\[
|a_3 - a_2^2| \leq \frac{|P(x)|}{(1 + 2\zeta)}.
\]
(40)

Corollary 14 Let \( u \in Q^{\Sigma,\delta}(1, 0; x) = Q^{\Sigma,\delta}(x) \). Then,
\[
|a_3 - a_2^2| \leq \frac{|P(x)|}{3}
\]
(41)

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