Repeated Angles in the Plane for Angles with Algebraic Tangents

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Abstract

We construct a set of points with $\Omega(n^2 \log n)$ triples determining an angle $\theta$ whenever $\tan(\theta)$ is algebraic over $\mathbb{Q}$, matching the upper bound of Pach and Sharir. This improves upon the original construction, which was optimal only for $\tan(\theta) = a\sqrt{m}/b$ with $a, b, m$ positive integers.

1 Introduction

Pach and Sharir [4] proved that, for all $0 < \theta < \pi$, any set of $n$ points in the Euclidean plane contains at most $O(n^2 \log n)$ triples which determine an angle with measure $\theta$. They also showed that, for $\tan(\theta) = a\sqrt{m}/b$ with $a, b, m$ positive integers, this upper bound is tight, and that there exists a configuration of $n$ points in the plane with $\Omega(n^2 \log n)$ triples determining angle $\theta$. However, for other $\theta$, it remains open whether the upper bound is asymptotically tight (see problem 6, Chapter 6.2 in [1]).

We shall construct a configuration of $n$ points with $\Omega(n^2 \log n)$ triples determining angle $\theta$ for any $0 < \theta < \pi$ where $\tan(\theta)$ is algebraic over $\mathbb{Q}$. This class contains many angles of geometric interest; in particular, this gives a construction with $\Omega(n^2 \log n)$ triples if $\theta$ is any rational multiple of $\pi$, as well as for any constructible angle $\theta$.

The question remains open in the case that $\tan(\theta)$ is transcendental. We note the similar problem of determining the maximum number of subsets similar to a desired set $S$; Laczkovich and Ruzsa [3] showed that there exist configurations with $\Omega(n^2)$ copies, matching the trivial upper bound, if and only if the cross-ratio of every quadruple in $S$ is algebraic. Their construction builds upon that of Elekes and Erdős [2], which finds similar copies within “pseudo-grids,” which are grid-like configurations of points made from generalized arithmetic progressions. Our construction makes use of a similar idea, generalizing the original approach of Pach and Sharir [4] from rectangle lattice grids to similar grid-like sets.

2 Proof

Theorem 2.1. If $\tan(\theta)$ is algebraic over $\mathbb{Q}$, then for $n \geq 3$, there exist arrangements of $n$ points in the Euclidean plane with $\Omega(n^2 \ln n)$ triples determining angle $\theta$.

Proof. Let $\tan(\theta)$ be a algebraic. Then we can write $\tan(\theta) = \frac{a}{b}$, where $b \in \mathbb{Z}$ and $a$ is a positive real algebraic integer. Then we can write $a$ as the root of a monic, irreducible polynomial of degree $d$ in $\mathbb{Z}[x]$. So $\mathbb{Z}[a]$ is a $\mathbb{Z}$-module with basis $1, a, \ldots, a^{d-1}$. 
We shall represent the Euclidean plane as $\mathbb{C}$. Note that $b + i\alpha$ will have argument $\theta$, so we seek triples of the form $z, z + \lambda_1 v, z + (b + i\alpha)\lambda_2 v$, with $z, v \in \mathbb{C}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Fix some positive integer $t$ (to be determined later), and define $K_t$ and $G_t$ as

$$K_t := \left\{ \sum_{k=0}^{d-1} a_k \alpha^k : a_k \in \mathbb{Z}, |a_k| \leq t \right\}$$

$$G_t := K_t + iK_t = \{ u_1 + iu_2 : u_1, u_2 \in K_t \}$$

Note that $G_t$ is the cartesian product of two real generalized arithmetic progression, while the pseudo-grids of Elekes and Erdős [2] are one complex generalized arithmetic progression. Clearly, $|K_t| = (2t + 1)^d$ and $|G_t| = (2t + 1)^{2d}$.

First, suppose $\mu \in K_s$ and $\lambda \in K_t$; clearly, $\mu + \lambda \in K_{s+t}$. We shall now consider $\mu\lambda$. Since $\alpha$ is an algebraic integer with minimal polynomial of degree $d$, we can write $\alpha^d = c_0 + c_1 \alpha + \cdots + c_{d-1} \alpha^{d-1}$ for $c_i \in \mathbb{Z}$, let $C_1 = 1 + \max(|c_0|, |c_1|, \ldots, |c_{d-1}|)$. By writing $\lambda = a_0 + a_1 \alpha + \cdots + a_{d-1} \alpha^{d-1}$ with $a_i \in \mathbb{Z}$, $|a_i| \leq t$, we see that $\alpha\lambda = (a_0 \alpha + \cdots + a_{d-2} \alpha^{d-1}) + a_{d-1}(c_0 + \cdots + c_{d-1} \alpha^{d-1}) \in K_{C_1t}$. Applying this observation repeatedly, we see that $\alpha^k \lambda \in K_{C_1^k t}$. Hence, if we write $\mu = b_0 + b_1 \alpha + \cdots + b_{d-1} \alpha^{d-1}$, we see that $\mu\lambda = b_0 \lambda + b_1 \alpha \lambda + \cdots + b_{d-1} \alpha^{d-1} \lambda \in K_{dC_1^{d-1}t}$.

Now suppose $u = (u_1 + iu_2) \in G_s$ and $v = (v_1 + iv_2) \in G_t$. Then $u + v = (u_1 + u_2) + i(v_1 + v_2) \in G_{s+t}$, and $uv = (u_1 + iu_2)(v_1 + iv_2) = (u_1 v_1 - u_2 v_2) + i(u_1 v_2 + u_2 v_1) \in G_{2dC_1^{d-1}t}$. Letting $C_2 = 2dC_1^{d-1}$, we have that $uv \in G_{C_2t}$.

Ungar [5] proved that a set of $N$ points forms has at least $N - 1$ pairs of points each forming a distinct direction. By applying this to the set $G_t$, we see that $G_t$ contains at least $(2t + 1)^{2d} - 1 \geq (2t)^{2d}$ pairs giving distinct directions. Since $G_t - G_t \subseteq G_{2t}$, this means that the nonzero elements of $G_{2t}$ determine at least $(2t)^{2d}$ distinct arguments (directions) from $[0, \pi)$.

Now, for each $1 \leq k \leq t$, pick $T_k$ as $(2k)^{2d} - (2(k - 1))^{2d} \geq (2k)^{2d-1}$ elements from $G_{2k}$ whose angles are all distinct from each other and those of $T_1, \ldots, T_{k-1}$.

For each choice of $v \in \bigcup_{k=1}^{t} T_k$ and each $0 < \lambda \in K_{|t/k|}$, the element $\lambda v$ is distinct, since each $v \in T_k$ has a distinct angle and each $\lambda$ is a unique positive real number. Moreover, if $v \in T_k \subseteq G_{2k}$ and $\lambda \in K_{|t/k|}$, $\lambda v \in G_{2k}|t/k| \subseteq G_{2C_2t}$. Therefore, for any choice of $1 \leq k \leq t$ and any $z \in G_t$, $v \in T_k \subseteq G_{2k}$, and $\lambda_1, \lambda_2 \in K_{|t/k|}$ with $\lambda_1, \lambda_2 > 0$, the triple $$(z, z + \lambda_1 v, z + (b + i\alpha)\lambda_2 v)$$
is a unique triple determining angle $\theta$. Observe that $z + \lambda_1 v \in G_{t+2C_2t}$ and (since $b + i\alpha \in G_{|b|}$) $z + (b + \alpha i)\lambda_2 v \in G_{t+2C_2t}$, so for all choices of $z, v, \lambda_1, \lambda_2$, we have $z, z + \lambda_1 v, z + (b + i\alpha)\lambda_2 v \in G_{(1+2|b|C_2)^t}$. Since the elements of $K_{|t/k|}$ are equally distributed between positive and negative, with one element 0, the number of choices for $\lambda_1$ and $\lambda_2$ is $\left( \frac{|K_{|t/k|}| + 1}{2} \right) \geq \left( \frac{t}{k} \right)^{2d}$. Hence the total number of triples chosen in this way is

$$\left| G_t \right| \sum_{k=1}^{t} |T_k| \left( \frac{k}{t} \right)^{2d} \geq (2t + 1)^{2d} \sum_{k=1}^{t} (2k)^{d-1} \left( \frac{t}{2k} \right)^{2d} \geq t^{2d} \sum_{k=1}^{t} k^{d-1} \left( \frac{t}{k} \right)^{2d} \geq t^{4d} \sum_{k=1}^{t} \frac{1}{k} \geq t^{4d} \ln t$$
Hence, there are at least \( t^{4d} \ln t \) triples which determine angle \( \theta \) with elements chosen from \( G_{(1+2|b|C^2_2)t} \). This set contains \((2(1 + 2|b|C^2_2)t + 1)^{2d} \leq (4|b|C^2_2 + 3)^{2d}t^{2d}\), so if we let \( C_3 = (4|b|C^2_2 + 3)^{2d} \), we have a set of at most \( C_3t^{2d} \) points whose elements form at least \( t^{4d} \ln t \) triples.

Now, for an arbitrary \( n \), pick \( t \) so that \( C_3t^{2d} < n \leq C_3(t+1)^{2d} \). If \( n \geq C_3 \), then we have a set with \( C_3t^{2d} \leq n \) points. Moreover, if \( t \geq 1 \), then \( n \leq C_3(2t)^{2d} \), so \( t \geq \frac{(n/C_3)^{1/2d}}{2} \), so the number of triples determining angle \( \theta \) is at least \( t^{4d} \ln t \geq \frac{(n/C_3)^2}{2^{2d}} \ln \left( \frac{(n/C_3)^{1/2d}}{2} \right) = \frac{n^2}{2^{2d} C_3^{1/2d}} \ln \left( \frac{n}{2^{2d} C_3} \right) \). Since \( d \) and \( C_3 \) depend only on our angle \( \theta \) and not on \( n \), we have a construction of at most \( n \) points with \( \Omega_\theta(n^2 \log n) \) triples determining angle \( \theta \), as desired.

\( \Box \)

References

[1] P. Brass, W. Moser, and J. Pach. Research problems in discrete geometry. 2005.

[2] G. Elekes and P. Erdős. Similar configurations and pseudo grids. Intuitive geometry (Colloquia mathematica Societatis János Bolyai), pages 85–104, 1991.

[3] M. Laczkovich and I. Z. Ruzsa. The number of homothetic subsets. The Mathematics of Paul Erdős II, pages 294–302, 1997.

[4] J. Pach and M. Sharir. Repeated angles in the plane and related problems. Journal of Combinatorial Theory, Ser. A, 59:12–22, 1992.

[5] P. Ungar. \( 2N \) noncollinear points determine at least \( 2N \) directions. Journal of Combinatorial Theory, Ser. A, 33:343–347, 1982.