Connection between the ideals generated by traces and by supertraces in the superalgebras of observables of Calogero models

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If $G$ is a finite Coxeter group, then symplectic reflection algebra $H := H_1, \eta(G)$ has Lie algebra $\mathfrak{sl}_2$ of inner derivations and can be decomposed under spin: $H = H_0 \oplus H_1/2 \oplus H_1 \oplus H_3/2 \oplus \ldots$ We show that if the ideals $\mathcal{I}_i$ ($i = 1, 2$) of all the vectors from the kernel of degenerate bilinear forms $B_i(x, y) := sp_i(x \cdot y)$, where $sp_i$ are (super)traces on $H$, do exist, then $\mathcal{I}_1 = \mathcal{I}_2$ if and only if $\mathcal{I}_1 \cap H_0 = \mathcal{I}_2 \cap H_0$.

1. Preliminaries and notation

Let $\mathcal{A}$ be an associative superalgebra with parity $\pi$. All expressions of linear algebra are given for homogenous elements only and are supposed to be extended to inhomogeneous elements via linearity.

**Definition 1.1.** A linear function $\text{str}$ on $\mathcal{A}$ is called a supertrace if

$$\text{str}(f \cdot g) = (-1)^{\pi(f)\pi(g)} \text{str}(g \cdot f) \quad \text{for all } f, g \in \mathcal{A}. $$

**Definition 1.2.** A linear function $\text{tr}$ on $\mathcal{A}$ is called a trace if

$$\text{tr}(f \cdot g) = \text{tr}(g \cdot f) \quad \text{for all } f, g \in \mathcal{A}. $$

We will use the notation “$\text{sp}$” and the term “(super)trace” to denote both cases, traces and supertraces, simultaneously.
2. The superalgebra of observables

Let \( V = \mathbb{R}^N \) be endowed with a positive definite symmetric bilinear form \((\cdot, \cdot)\). For any nonzero \( \vec{v} \in V \), define the reflections \( r_{\vec{v}} \) as follows:

\[
 r_{\vec{v}} : \vec{x} \mapsto \vec{x} - 2 \frac{\langle \vec{x}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} \quad \text{for any} \; \vec{x} \in V.
\] (2.1)

A finite set of non-zero vectors \( \mathcal{R} \subset V \) is said to be a root system and any vector \( \vec{v} \in \mathcal{R} \) is called a root if the following conditions hold:

i) \( \mathcal{R} \) is \( r_{\vec{w}} \)-invariant for any \( \vec{w} \in \mathcal{R} \),

ii) if \( \vec{v}_1, \vec{v}_2 \in \mathcal{R} \) are proportional to each other, then either \( \vec{v}_1 = \vec{v}_2 \) or \( \vec{v}_1 = -\vec{v}_2 \).

The Coxeter group \( G \subset O(N, \mathbb{R}) \subset \text{End}(V) \) generated by all reflections \( r_{\vec{v}} \) with \( \vec{v} \in \mathcal{R} \) is finite.

We do not apply any conditions on the scalar products of the roots because we want to consider both crystallographic and non-crystallographic root systems, e.g., \( I_2(n) \) (see Theorem 4.1).

Let \( \eta \) be a complex-valued \( G \)-invariant function on \( \mathcal{R} \), i.e., \( \eta(\vec{v}) = \eta(\vec{w}) \) if \( r_{\vec{v}} \) and \( r_{\vec{w}} \) belong to one conjugacy class of \( G \).

We consider here the Symplectic Reflection (Super)algebra over complex numbers (see [6]) \( H := H_{1, \eta}(G) \) and call it the superalgebra of observables of Calogero model based on root system \( \mathcal{R} \).\(^a\)

This algebra consists of noncommuting polynomials in \( 2N \) indeterminates \( a^\alpha_i \), where \( \alpha = 0, 1 \) and \( i = 1, \ldots, N \), with coefficients in \( \mathbb{C}[G] \) satisfying the relations (see [6] Eq. (1.15))\(^b\)

\[
 [a^\alpha_i, a^\beta_j] = \varepsilon^{\alpha\beta} \left( \delta_{ij} + \sum_{\vec{v} \in \mathcal{R}} \eta(\vec{v}) \frac{v_i v_j}{\langle \vec{v}, \vec{v} \rangle} r_{\vec{v}} \right),
\] (2.2)

and

\[
 r_{\vec{v}} a^\alpha_i = \sum_{j=1}^N \left( \delta_{ij} - 2 \frac{v_i v_j}{\langle \vec{v}, \vec{v} \rangle} \right) a^\alpha_j r_{\vec{v}}.
\] (2.3)

Here \( \varepsilon^{\alpha\beta} \) is the antisymmetric tensor such that \( \varepsilon^{01} = 1 \), and \( v_i \) (\( i = 1, \ldots, N \)) are the coordinates of the vector \( \vec{v} \). The commutation relations (2.2), (2.3) suggest to define the parity \( \pi \) by setting:

\[
 \pi(a^\alpha_i) = 1 \quad \text{for any} \; \alpha, \; i; \quad \pi(r_{\vec{v}}) = 0 \quad \text{for any} \; \vec{v} \in \mathcal{R}.
\] (2.4)

and we can consider the algebra \( H \) as a superalgebra as well.

\(^a\)This algebra has a faithful representation via Dunkl differential-difference operators \( D_i \), see [5], acting on the space of \( G \)-invariant smooth functions on \( V \), namely \( \tilde{D}^\alpha_i = \frac{1}{2} (x_i + (-1)^\alpha D_i) \), see [1, 14]. The Hamiltonian of the Calogero model based on the root system \([2-4, 13]\) is the operator \( \tilde{H}^{01} \) defined in (3.2) (see [1]). The wave functions are obtained in this model via the standard Fock procedure with the Fock vacuum \( |0\rangle \) such that \( \tilde{D}^0_i |0\rangle = 0 \) for all \( i \) by acting on \( |0\rangle \) with \( G \)-invariant polynomials of the \( \tilde{D}^1_i \).

\(^b\)The sign and coefficient of the sum in the rhs of Eq. (2.2) is chosen for obtaining the Calogero model in the form [1], Eq. (1), Eq. (5), Eq. (9), Eq. (10) when \( \mathcal{R} \) is of type \( A_{N-1} \).
3. \(\mathfrak{sl}_2\)

Observe an important property of the superalgebra \(H\): the Lie (super)algebra of its inner derivations contains the Lie subalgebra \(\mathfrak{sl}_2\) generated by operators

\[ D^{\alpha\beta} : f \mapsto D^{\alpha\beta} f = [T^{\alpha\beta}, f], \tag{3.1} \]

where \(\alpha, \beta = 0, 1\), and \(f \in H\), and polynomials \(T^{\alpha\beta}\) are defined as follows:

\[ T^{\alpha\beta} := \frac{1}{2} \sum_{i=1}^{N} \left( a_i^\alpha a_i^\beta + a_i^\beta a_i^\alpha \right). \tag{3.2} \]

These operators satisfy the following relations:

\[ [D^{\alpha\beta}, D^{\gamma\delta}] = \varepsilon^{\alpha\gamma} D^{\beta\delta} + \varepsilon^{\alpha\delta} D^{\beta\gamma} + \varepsilon^{\beta\gamma} D^{\alpha\delta} + \varepsilon^{\beta\delta} D^{\alpha\gamma}, \tag{3.3} \]

since

\[ [T^{\alpha\beta}, T^{\gamma\delta}] = \varepsilon^{\alpha\gamma} T^{\beta\delta} + \varepsilon^{\alpha\delta} T^{\beta\gamma} + \varepsilon^{\beta\gamma} T^{\alpha\delta} + \varepsilon^{\beta\delta} T^{\alpha\gamma}. \]

It follows from Eq. (3.3) that the operators \(D^{00}, D^{11}\) and \(D^{01} = D^{10}\) constitute an \(\mathfrak{sl}_2\)-triple:

\[ [D^{01}, D^{11}] = 2D^{11}, \quad [D^{01}, D^{00}] = -2D^{00}, \quad [D^{11}, D^{00}] = -4D^{01}. \]

The polynomials \(T^{\alpha\beta}\) commute with \(\mathbb{C}[G]\), i.e., \([T^{\alpha\beta}, r_v] = 0\), and act on the \(a_i^{\alpha\beta}\) as on vectors of the irreducible 2-dimensional \(\mathfrak{sl}_2\)-modules:

\[ D^{\alpha\beta} a_i^\gamma = [T^{\alpha\beta}, a_i^\gamma] = \varepsilon^{\alpha\gamma} a_i^\beta + \varepsilon^{\beta\gamma} a_i^\alpha, \quad \text{where } i = 1, \ldots, N. \tag{3.4} \]

We will denote this \(\mathfrak{sl}_2\) thus realized by the symbol \(\text{SL}2\).

The subalgebra

\[ H_0 := \{ f \in H \mid D^{\alpha\beta} f = 0 \text{ for any } \alpha, \beta \} \subset H \tag{3.5} \]

is called the subalgebra of singlets.

Introduce also the subspaces \(H_s := \bigoplus_{s=0}^{\infty} H_s\), which is the direct sum of all irreducible \(\text{SL}2\)-modules \(H_s\) of spin \(s\), for \(s = 0, 1/2, 1, \ldots\). It is clear that \(H_0\) is the defined above subalgebra of singlets.

The (super)algebra \(H\) can be decomposed in the following way

\[ H = H_0 \oplus H_{\text{rest}}, \quad \text{where } H_{\text{rest}} := H_{1/2} \oplus H_1 \oplus H_{3/2} \oplus \ldots. \]

Then each element \(f \in H\) can be represented in the form \(f = f_0 + f_{\text{rest}}\), where \(f_0 \in H_0\) and \(f_{\text{rest}} \in H_{\text{rest}}\).

Note, that since \(\text{SL}2\) is generated by inner derivations and \(T^{\alpha\beta}\) are even elements, each two-sided ideal \(\mathcal{I} \subset H\) can be decomposed in an analogous way: \(\mathcal{I} = \mathcal{I}_0 \oplus \mathcal{I}_{1/2} \oplus \ldots\).

Since \(T^{\alpha\beta}\) are even elements of the superalgebra \(H\), we have \(\text{sp}(D^{\alpha\beta} f) = 0\) for any (super)trace \(\text{sp}\) on \(H\), and hence the following proposition takes place:

**Proposition 3.1.** \(\text{sp}(f) = \text{sp}(f_0)\) for any \(f \in H\) and any (super)trace \(\text{sp}\) on \(H\).

\[\text{This elementary fact is known for a long time, see, eg. [12].}\]
Proof. If \( s \neq 0 \), then the elements of the form \( D^{\alpha\beta} f \), where \( \alpha, \beta = 0, 1 \), and \( f \in H^i \), \( f \neq 0 \), span the irreducible \( SL_2 \)-module \( H^i \). This implies \( \text{sp} f = 0 \) for any (super)trace on \( H \) and any \( f \in H_{\text{rest}} \).

\[ \square \]

4. The (super)traces on \( H \)

It is shown in [9, 10, 12] that the algebra \( H \) has a multitude of independent (super)traces. For the list of dimensions of the spaces of the (super)traces on \( H_{1,n}(M) \) for all finite Coxeter groups \( M \), see [8]. In particular, there is an \( m \)-dimensional space of traces and an \((m + 1)\)-dimensional space of supertraces on \( H_{1,2}(I_2(2m + 1)) \).

Every (super)trace \( \text{sp}(\cdot) \) on any associative (super)algebra \( \mathcal{A} \) generates the following bilinear form on \( \mathcal{A} \):

\[
B^{\text{sp}}(f, g) = \text{sp}(f \cdot g) \quad \text{for any} \quad f, g \in \mathcal{A}.
\]

It is obvious that if such a bilinear form \( B^{\text{sp}} \) is degenerate, then the kernel of this form (i.e., the set of all vectors \( f \in \mathcal{A} \) such that \( B^{\text{sp}}(f, g) = 0 \) for any \( g \in \mathcal{A} \)) is the two-sided ideal \( \mathcal{I}^{\text{sp}} \subset \mathcal{A} \).

The ideals of this sort are found, for example, in [11, Theorem 9.1] (generalizing the results of [15, 16] and [7] for the two- and three-particle Calogero models).

Theorem 9.1 from [11] may be shortened to the following theorem:

**Theorem 4.1.** Let \( m \in \mathbb{Z} \), where \( m \geq 1 \) and \( n = 2m + 1 \). Then

1) The associative algebra \( H_{1,n}(I_2(n)) \) has nonzero traces \( tr^i \) such that the symmetric invariant bilinear form \( B_{tr^i}(x, y) = tr^i(x \cdot y) \) is degenerate if and only if \( \eta = \frac{z}{n} \), where \( z \in \mathbb{Z} \setminus n\mathbb{Z} \). For each such \( z \), all nonzero degenerate traces on \( H_{1,z/n}(I_2(n)) \) are proportional to each other.

2) The associative superalgebra \( H_{1,n}(I_2(n)) \) has nonzero supertraces \( \text{str}^i \) such that the supersymmetric invariant bilinear form \( B_{\text{str}^i}(x, y) = \text{str}^i(x \cdot y) \) is degenerate if \( \eta = \frac{z}{n} \), where \( z \in \mathbb{Z} \setminus n\mathbb{Z} \). For each such \( z \), all nonzero degenerate supertraces on \( H_{1,z/n}(I_2(n)) \) are proportional to each other.

3) The associative superalgebra \( H_{1,n}(I_2(n)) \) has nonzero supertraces \( \text{str}^i \) such that the supersymmetric invariant bilinear form \( B_{\text{str}^i}(x, y) = \text{str}^i(x \cdot y) \) is degenerate if \( \eta = z + \frac{1}{2} \), where \( z \in \mathbb{Z} \). For each such \( z \), all nonzero degenerate supertraces on \( H_{1,z+1/2}(I_2(n)) \) are proportional to each other.

4) For all other values of \( n \), all nonzero traces and supertraces are nondegenerate.

Theorem 4.1 implies that if \( z \in \mathbb{Z} \setminus n\mathbb{Z} \), then there exists the degenerate trace \( tr_z \) generating the ideal \( \mathcal{I}^{tr_z} \) consisting of the kernel of the degenerate form \( B_{tr_z}(f, g) = tr_z(f \cdot g) \), and simultaneously the degenerate supertrace \( \text{str}_z \) generating the ideal \( \mathcal{I}^{\text{str}_z} \) consisting of the kernel of the degenerate form \( B_{\text{str}_z}(f, g) = \text{str}_z(f \cdot g) \).

A question arises: is it true that \( \mathcal{I}^{tr_z} = \mathcal{I}^{\text{str}_z} \)?

Answer to this and other similar questions can be considerably simplified by considering only the singlet parts of these ideals.

The following theorem justifies this method:

**Theorem 4.2.** Let \( \text{sp}_1 \) and \( \text{sp}_2 \) be degenerate (super)traces on \( H \). They generate the two-sided ideals \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) consisting of the kernels of bilinear forms \( B_1(f, g) = \text{sp}_1(f \cdot g) \) and \( B_2(f, g) = \text{sp}_2(f \cdot g) \), respectively.
Then $\mathcal{I}_1 = \mathcal{I}_2$ if and only if $\mathcal{I}_1 \cap H_0 = \mathcal{I}_2 \cap H_0$.

Proof. It suffices to prove that if $\mathcal{I}_1 \cap H_0 = \mathcal{I}_2 \cap H_0$, then $\mathcal{I}_1 = \mathcal{I}_2$.

Consider any non-zero element $f \in \mathcal{I}_1$. For any $g \in H$, we have $sp_1(f \cdot g) = 0$, $f \cdot g \in \mathcal{I}_1$ and $(f \cdot g)_0 \in \mathcal{I}_1$. So $(f \cdot g)_0 \in \mathcal{I}_1 \cap H_0$. Due to hypotheses of this Theorem, $(f \cdot g)_0 \in \mathcal{I}_2 \cap H_0$, and hence $sp_2((f \cdot g)_0) = 0$. Proposition 3.1 gives $sp_2(f \cdot g) = sp_2((f \cdot g)_0)$ which implies $sp_2(f \cdot g) = 0$. Therefore, $f \in \mathcal{I}_2$. □

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