Controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with delay and Poisson jumps

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Abstract: The current paper is concerned with the controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with infinite delay and Poisson jumps in Hilbert spaces. Using the theory of a strongly continuous cosine family of bounded linear operators, stochastic analysis theory and with the help of the Banach fixed point theorem, we derive a new set of sufficient conditions for the controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with infinite delay and Poisson jumps. Finally, an application to the stochastic nonlinear wave equation with infinite delay and Poisson jumps is given.

Subjects: Non-Linear Systems; Probability Theory & Applications; Stochastic Models & Processes

Keywords: controllability; impulsive neutral stochastic integro-differential equations; Poisson jumps; cosine functions of operators; infinite delay; Banach fixed point theorem

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1. Introduction

As one of the fundamental concepts in mathematical control theory, controllability plays an important role both in deterministic and stochastic control problems such as stabilization of unstable systems by feedback control. It is well known that controllability of deterministic equation is widely
used in many fields of science and technology, say, physics and engineering (e.g. see Ahmed, 2014a; Balachandran & Dauer, 2002; Coron, 2007; Curtain & Zwart, 1995; Zabczyk, 1992, and the references therein). Stochastic control theory is stochastic generalization of the classic control theory. The theory of controllability of differential equations in infinite dimensional spaces has been extensively studied in the literature, and the details can be found in various papers and monographs (Ahmed, 2014b; Aström, 1970; Balachandran & Dauer, 2002; Karthikeyan & Balachandran, 2013; Yang, 2001; Zabczyk, 1991, and the references therein). Any control system is said to be controllable if every state corresponding to this process can be affected or controlled in a respective time by some control signals. If the system cannot be controlled completely, then different types of controllability can be defined such as approximate, null, local null, and local approximate null controllabilities. On this matter, we refer the reader to Ahmed (2014c), Chang (2007), Karthikeyan and Balachandran (2009), Ntouyas and ØRegan (2009), Sakthivel, Mahmudov, and Lee (2009), and the references therein.

The theory of impulsive differential equations as much as neutral differential equations has been emerging as an important area of investigations in recent years, stimulated by their numerous applications to problems in physics, mechanics, electrical engineering, medicine biology, ecology, and so on. The impulsive differential systems can be used to model processes which are subject to abrupt changes, and which cannot be described by the classical differential systems (Lakshmikantham, Bainov, & Simeonov, 1989). Partial neutral integro-differential equation with infinite delay has been used for modeling the evolution of physical systems, in which the response of the system depends not only on the current state, but also on the past history of the system, for instance, for the description of heat conduction in materials with fading memory, we refer the reader to the papers of Gurtin and Pipkin (1968), Nunziato (1971), and the references therein related to this matter. Besides, noise or stochastic perturbation is unavoidable and omnipresent in nature as well as in man-made systems. Therefore, it is of great significance to import the stochastic effects into the investigation of impulsive neutral differential equations. As the generalization of the classic impulsive neutral differential equations, impulsive neutral stochastic integro-differential equations with infinite delays have attracted the researchers’ great interest. On the existence and the controllability for these equations, we refer the reader to (e.g. see Chang, 2007; Chang, Anguraj, & Arjunan, 2008; Karthikeyan & Balachandran, 2009, 2013; Park, Balachandran, & Annapoorani, 2009; Park, Balasubramaniam, & Kumaresan, 2007; Shen & Sun, 2012; Yan & Yan, 2013, and the references therein).

Recently, Park, Balachandran, and Arthi (2009) investigated the controllability of impulsive neutral integro-differential systems with infinite delay in Banach spaces using Schauder-type fixed point theorem. Arthi and Balachandran (2012) established the controllability of damped second-order impulsive neutral functional differential systems with infinite delay by means of the Sadovskii fixed point theorem combined with a noncompact condition on the cosine family of operators. Very recently, also using Sadovskii’s fixed point theorem, Muthukumar and Rojivanthi (2013) proved sufficient conditions for the approximate controllability of fractional order neutral stochastic integro-differential systems with nonlocal conditions and infinite delay.

By contrast, there has not been very much research on the controllability of second-order impulsive neutral stochastic functional differential equations with infinite delays, or in other words, the literature about controllability of second-order impulsive neutral stochastic functional differential equations with infinite delays is very scarce. To be more precise, Balasubramaniam and Muthukumar (2009) discussed on approximate controllability of second-order stochastic distributed implicit functional differential systems with infinite delay. Mahmudova and McKibben (2006) established the results concerning the global existence, uniqueness, approximate, and exact controllability of mild solutions for a class of abstract second-order damped McKean–Vlasov stochastic evolution equations in a real separable Hilbert space. More recently, using Holder’s inequality, stochastic analysis, and fixed point strategy, Sakthivel, Ren, and Mahmudov (2010) considered sufficient conditions for the approximate controllability of nonlinear second-order stochastic infinite dimensional dynamical systems with impulsive effects. And Muthukumar and Balasubramaniam (2010) investigated sufficient conditions for
the approximate controllability of a class of second-order damped McKean–Vlasov stochastic evolution equations in a real separable Hilbert space.

On the other hand, in recent years, stochastic partial differential equations with Poisson jumps have gained much attention since Poisson jumps not only exist widely, but also can be used to study many phenomena in real life. Therefore, it is necessary to consider the Poisson jumps into the stochastic systems. For instance, Luo and Liu (2008) studied the existence and uniqueness of mild solutions to stochastic partial functional differential equations with Markovian switching and Poisson jumps using the Lyapunov–Razumikhin technique. Ren, Zhou, and Chen (2011) investigated the existence, uniqueness, and stability of mild solutions for a class of time-dependent stochastic evolution equations with Poisson jumps. More specifically, just recently, there is an article on the complete controllability of stochastic evolution equations with jumps in a separable Hilbert space discussed by Sakthivel and Ren (2011) and in reference Ren, Dai, and Sakthivel (2013), Ren et al. studied the approximate controllability of stochastic differential systems driven by Teugels martingales associated with a Lévy process. For more details about the stochastic partial differential equations with Poisson jumps, one can see a recent monograph of Peszat and Zabczyk (2007) as well as papers of Cao (2005), Marinelli & Rockner (2010), Rockner and Zhang (2007), and the references therein.

To the best of our knowledge, there is no work reported on nonlocal second-order impulsive neutral stochastic functional integro-differential equations with infinite delay and Poisson jumps. To close the gap, motivated by the above works, the purpose of this paper is to study the controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with infinite delay and Poisson jumps in Hilbert spaces. More precisely, we consider the following form:

$$
\begin{align*}
\{ d[x'(t) - g(t, x_t, \int_0^t \sigma_1(t, s, x_s)ds)] &= [Ax(t) + f(t, x_t, \int_0^t \sigma_2(t, s, x_s)ds) + Bu(t)]dt \\
&+ \int_{-\infty}^0 \sigma_3(t, s, x_s)d\mu_s(t) + \int_{-\infty}^t \gamma(t, x(t-), v)N(dt, dv), \quad t_0 \leq t < T, \\
\Delta x(t_k) &= I_k^x(x(t_k)), \quad k = 1, \ldots, m = 1, m, \\
\Delta x'(t_k) &= I_k^x(x(t_k)), \quad k = 1, m, \\
x(0) &= x_0, \quad x_0 \in \mathbb{H}, \\
x(0) - q(x_{t_1}, x_{t_2}, \ldots, x_{t_m}) &= x_0 = \varphi \in B, \quad \text{for a.e. } s \in J_0 := (-\infty, 0],
\end{align*}
$$

where $0 < t_1 < t_2 < \cdots < t_n < T$, $n \in \mathbb{N}$; $x(\cdot)$ is a stochastic process taking values in a real separable Hilbert space $\mathbb{H}$; $A: D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous cosine family on $\mathbb{H}$. The history $x_t: J_0 \to \mathbb{H}, x_t(\theta) = x(t + \theta)$ for $t \geq 0$, belongs to the phase space $B$, which will be described in Section 2. Assume that the mappings $f, g: J \times \mathbb{B} \to \mathbb{H}$, $\sigma: J \times \mathbb{R} \to \mathbb{L}^2(\Omega; \mathbb{H})$, $\gamma: J \times \mathbb{H} \times \mathbb{U} \to \mathbb{H}$ are appropriate functions to be specified later. The control function $u(\cdot)$ takes values in $L^2(J, U)$ of admissible control functions for a separable Hilbert space $U$ and $B$ is a bounded linear operator from $U$ into $\mathbb{H}$. Furthermore, let $0 = t_0 < t_1 < \cdots < t_m = T$ be prefixed points, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump of the function $x$ at time $t_k$ with $I_k^x$ determining the size of the jump, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Similarly, $x'(t_k^+)$ and $x'(t_k^-)$ denote, respectively, the right and left limits of $x'(t)$ at $t_k$. Let $q(t) \in L^2(\Omega, \mathbb{H})$ and $x_0(t) \in \mathbb{H}$-valued $\mathcal{F}_t$-measurable random variables independent of the Wiener process $\{w(t)\}$ and the Poisson point process $\mathcal{P}_t(\cdot)$ with a finite second moment.

The main techniques used in this paper include the Banach contraction principle and the theories of a strongly continuous cosine family of bounded linear operators.

The structure of this paper is as follows: in Section 2, we briefly present some basic notations, preliminaries, and assumptions. The main results in Section 3 are devoted to study the controllability for the system (1.1) with their proofs. An example is given in Section 4 to illustrate the theory. In Section 5, concluding remarks are given.
2. Preliminaries

In this section, we briefly recall some basic definitions and results for stochastic equations in infinite dimensions and cosine families of operators. For more details on this section, we refer the reader to Da Prato and Zabczyk (1992), Fattorini (1985), Protter (2004), and Travis and Webb (1978).

Let \( (\mathbb{H}, \| \cdot \|_{\mathbb{H}}, \langle \cdot, \cdot \rangle_{\mathbb{H}}) \) and \((\mathbb{K}, \| \cdot \|_{\mathbb{K}}, \langle \cdot, \cdot \rangle_{\mathbb{K}})\) denote two real separable Hilbert spaces, with their vectors, norms, and their inner products, respectively. We denote by \( \mathcal{L}(\mathbb{K}; \mathbb{H}) \) the set of all linear bounded operators from \( \mathbb{K} \) into \( \mathbb{H} \), which is equipped with the usual operator norm \( \| \cdot \| \). In this paper, we use the symbol \( \| \cdot \| \) to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete filtered probability space satisfying the usual condition (i.e. it is right continuous and \( \mathcal{F}_t \) contains all \( \mathbb{P} \)-null sets). Let \( \mathbb{W} = \left\{ \mathbb{W}(t) \right\}_{t \geq 0} \) be a \( Q \)-Wiener process defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the covariance operator \( Q \) such that \( \text{Tr}(Q) < \infty \). We assume that there exists a complete orthonormal system \( \{ e_k \}_{k \geq 1} \) in \( \mathbb{K} \), a bounded sequence of nonnegative real numbers \( \lambda_k \) such that \( Q e_k = \lambda_k e_k, \ k = 1, 2, \ldots \), and a sequence of independent Brownian motions \( \{ \beta_k \}_{k \geq 1} \) such that

\[
\langle \mathbb{W}(t), e \rangle_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \langle e, e_k \rangle_{\mathbb{K}} \beta_k(t), \quad e \in \mathbb{K}, \ t \geq 0.
\]

Let \( \mathcal{L}_2^Q = L_2(\Omega; \mathbb{K}'; \mathbb{H}) \) be the space of all Hilbert–Schmidt operators from \( Q \)-\( \mathbb{K} \) into \( \mathbb{H} \) with the inner product \( \langle \Psi, \phi \rangle_{\mathcal{L}_2^Q} = \text{Tr}(\Psi^* Q \phi) \), where \( \phi^* \) is the adjoint of the operator \( \phi \). Let \( \mathcal{P} = \mathcal{P}(t), \ t \in \mathbb{D}_p \) \( ( \text{the domain of } \mathcal{P}(t) ) \) be a stationary \( \mathcal{F}_t \)-Poisson point process taking its value in a measurable space \( (\mathbb{U}, \mathcal{B}(\mathbb{U})) \) with a \( \mathcal{F}_0 \)-finite intensity measure \( \lambda(\mathcal{D}) \) by \( N(dt, \mathcal{D}) \) the Poisson counting measure associated with \( \mathcal{P} \), that is,

\[
N(t, \mathcal{U}) = \sum_{s \in \mathcal{U}, t \leq s} \mathbb{1}_\mathcal{U}(\mathcal{P}(s))
\]

for any measurable set \( \mathcal{U} \in \mathcal{B}(\mathbb{K} - \{ 0 \}) \), which denotes the Borel \( \sigma \)-field of \( (\mathbb{K} - \{ 0 \}) \). Let

\[
\tilde{N}(dt, \mathcal{D}) := N(dt, \mathcal{D}) - \lambda(\mathcal{D})dt
\]

be the compensated Poisson measure that is independent of \( \mathbb{W}(t) \). Denote by \( \mathcal{T}^2(J \times \mathcal{U}^*; \mathbb{H}) \) the space of all predictable mappings \( \gamma: J \times \mathcal{U}^* \rightarrow \mathbb{H} \) for which

\[
\int_{0}^{t} E\| \gamma(t, \mathcal{V}) \|_{\mathbb{H}}^2 \lambda(d\mathcal{V})dt < \infty.
\]

We may then define the \( \mathbb{H} \)-valued stochastic integral \( \int_{0}^{t} \int_{\mathcal{U}^*} \gamma(t, \mathcal{V}) \tilde{N}(dt, \mathcal{D}) \), which is a centered square-integrable martingale. For the construction of this kind of integral, we can refer to Protter (2004).

The collection of all strongly measurable, square-integrable \( \mathbb{H} \)-valued random variables, denoted by \( \mathcal{L}_2(\Omega, \mathbb{H}) \), is a Banach space equipped with norm \( \| x \|_{\mathcal{L}_2} = \left( E\| x \|^2 \right)^{1/2} \). Let \( C(\Omega, \mathcal{L}_2(\Omega, \mathbb{H})) \) be the Banach space of all continuous maps from \( J \) to \( \mathcal{L}_2(\Omega, \mathbb{H}) \), satisfying the condition \( \sup_{t \in J} E\| x(t) \|^2 < \infty \). An important subspace is given by \( \mathcal{L}_2^0(\Omega, \mathbb{H}) = \{ f \in \mathcal{L}_2(\Omega, \mathbb{H}) : f \text{ is } \mathcal{F}_0 \text{-measurable} \} \). Further, let \( \mathcal{L}_2^1(\Omega, T; \mathbb{H}) = \{ g: J \times \Omega \rightarrow \mathbb{H} : g(\cdot) \text{ is } \mathcal{F}_t \text{-progressively measurable and } E(\int_{J} \| g(t) \|_{\mathbb{H}}^2 dt) < \infty \} \).

Next, to be able to access controllability for the system (1.1), we need to introduce the theory of cosine functions of operators and the second-order abstract Cauchy problem.
**Definition 2.1** Let the one-parameter family \( \{ C(t) \}_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H}) \) be a strongly continuous cosine family if the following hold:

(i) \( C(0) = I \), \( I \) is the identity operators in \( \mathbb{H} \);

(ii) \( C(t)x \) is continuous in \( t \) on \( \mathbb{R} \) for any \( x \in \mathbb{H} \); and

(iii) \( C(t + s) + C(t - s) = 2C(t)C(s) \) for all \( t, s \in \mathbb{R} \).

(2) The corresponding strongly continuous sine family \( \{ S(t) \}_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H}) \), associated to the given strongly continuous cosine family \( \{ C(t) \}_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H}) \) is defined by

\[
S(t) = \int_0^t C(s)ds, \quad t \in \mathbb{R}, x \in \mathbb{H}.
\]

(3) The infinitesimal generator \( A : \mathbb{H} \to \mathbb{H} \) of \( \{ C(t) \}_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H}) \) is given by

\[
Ax = \frac{d^2}{dt^2} C(t)x \bigg|_{t=0}, \quad \text{for all } x \in D(A) = \{ x \in \mathbb{H} : C(\cdot) \in C^2(\mathbb{R}, \mathbb{H}) \}
\]

It is well known that the infinitesimal generator \( A \) is a closed, densely defined operator on \( \mathbb{H} \), and the following properties hold (see Travis & Webb, 1978).

**Proposition 2.1** Suppose that \( A \) is the infinitesimal generator of a cosine family of operators \( \{ C(t) \}_{t \in \mathbb{R}} \). Then, the following hold:

(i) There exist a pair of constants \( M_A \geq 1 \) and \( a \geq 0 \) such that \( \| C(t) \| \leq M_A e^{at} \), and hence \( \| S(t) \| \leq M_A e^{at} \);

(ii) \( A \int_0^r S(u)du = [C(r) - C(s)]x \), for all \( 0 \leq s \leq r < \infty \); and

(iii) There exists \( N \geq 1 \) such that \( \| S(s) - S(r) \| \leq N \| e^{at}ds \|, 0 \leq s \leq r < \infty \).

Thanks to the Proposition 2.1 and the uniform boundedness principle that we see a direct consequence that both \( \{ C(t) \}_{t \in \mathbb{R}} \) and \( \{ S(t) \}_{t \in \mathbb{R}} \) are uniformly bounded by \( \tilde{M} = M_A e^{at} \).

The existence of solutions for the second-order linear abstract Cauchy problem

\[
\begin{cases}
  x''(t) = Ax(t) + h(t), & t \in J, \\
  x(0) = z, & x'(0) = w,
\end{cases}
\]

where \( h : J \to \mathbb{H} \) is an integrable function that has been discussed in Travis and Webb (1977). Similarly, the existence of solutions of the semilinear second-order abstract Cauchy problem has been treated in Travis and Webb (1978).

**Definition 2.2** The function \( x(\cdot) \) given by

\[
x(t) = C(t)z + S(t)w + \int_0^t S(t-s)h(s)ds, \quad t \in J,
\]

is called a mild solution of (2.1), and that when \( z \in \mathbb{H} \), \( x(\cdot) \) is continuously differentiable and

\[
x'(t) = AS(t)z + C(t)w + \int_0^t C(t-s)h(s)ds, \quad t \in J.
\]
For additional details about cosine function theory, we refer the reader to Travis and Webb (1977, 1978).

Since the system (1.1) has impulsive effects, the phase space used in Balasubramaniam and Ntouyas (2006) and Park et al. (2007) cannot be applied to these systems. So, we need to introduce an abstract phase space $B$, as follows:

Assume that $l: J_0 \rightarrow (0, +\infty)$ is a continuous function with $l_0 = \int_{J_0} l(t) dt < \infty$. For any $a > 0$, we define

$$B = \left\{ {\psi: J_0 \rightarrow H: \mathbb{E} \left\| \psi(t) \right\|^2}^{\frac{1}{2}} \right\}$$

is a bounded and measurable function on $[-a, 0]$ and

$$\int_{J_0} l(s) \sup_{\psi \in B} (\mathbb{E} \left\| \psi(t) \right\|^2)^{\frac{1}{2}} ds < +\infty.$$ 

If $B$ is endowed with the norm

$$\left\| \psi \right\|_B = \int_{J_0} l(s) \sup_{\psi \in B} (\mathbb{E} \left\| \psi(t) \right\|^2)^{\frac{1}{2}} ds, \quad \forall \psi \in B,$$

then, it is clear that $(B, \left\| \cdot \right\|_B)$ is a Banach space (Hino, Murakami, & Naito, 1991).

Let $J_T = (-\infty, T]$. We consider the space

$$B_{J_T} = \left\{ x: J_T \rightarrow \mathbb{R} \mid x \in C(J_t, \mathbb{R}) \text{ and there exist } x(t^+_k) \text{ with } x(t^+_k) = x(t_k^+), x(0) = q(x_t, x_{t^+_1}, \ldots, x_{t^+_k}) = \varphi \in B, k = \overline{1, m} \right\},$$

where $x_{t_k}$ is the restriction of $x$ to $J_k = (t_k, t_{k+1})$. Let $| \cdot |_T$ be a seminorm in $B_{J_T}$ defined by

$$|x|_T = \| \varphi \|_B + \sup_{s \in J_T} (\mathbb{E} \left\| x(s) \right\|^2)^{\frac{1}{2}}, \quad x \in B_{J_T}.$$

Now, we recall the following useful lemma that appeared in Chang (2007).

LEMMA 2.1 (Chang, 2007) Assume that $x \in B_{J_T}$, then for $t \in J$, $x_t \in B$. Moreover,

$$l_0 (\mathbb{E} \left\| x(t) \right\|^2)^{\frac{1}{2}} \leq \| x_t \|_B \leq \| x_0 \|_B + l_0 \sup_{s \in (0,t)} (\mathbb{E} \left\| x(s) \right\|^2)^{\frac{1}{2}}.$$ 

Next, we give the definition of mild solution for (1.1).

**Definition 2.3** An $T$-adapted càdlàg stochastic process $x: J \rightarrow H$ is called a mild solution of (1.1) on $J$ if $x(0) = q(x_t, x_{t^+_1}, \ldots, x_{t^+_k}) = x_0 = \varphi \in B$ and $x'(0) = x_1 \in H$, satisfying $q, x_t, q \in \mathcal{L}_2^2(\Omega, H)$; the functions $C(t - s)g(s, x_s, \int_0^s \sigma_1(s, r, x_r) dr)$ and $S(t - s)f(s, x_s, \int_0^s \sigma_2(s, r, x_r) dr)$ are integrable on $[0, T]$ such that the following conditions hold:

(i) $(x_t : t \in J)$ is a $B$-valued stochastic process;

(ii) For arbitrary $t \in J$, $x(t)$ satisfies the following integral equation:

$$x(t) = C(t)\varphi(0) + q(x_t, x_{t^+_1}, \ldots, x_{t^+_k}(0)) + S(t)(x_1 - g(0, x_0, 0))$$

$$+ \int_0^t C(t - s)g(s, x_s, \int_0^s \sigma_1(s, r, x_r) dr) ds + \int_0^t S(t - s)f(s, x_s, \int_0^s \sigma_2(s, r, x_r) dr) ds + \int_0^t S(t - s)b u(s) ds$$

$$+ \int_0^t S(t - s)\int_{\mathbb{R}} \sigma_3(s, r, x_r) w(\nu) \nu(0) d\nu$$

$$+ \int_0^t S(t - s)\int_{\mathbb{R}} \sigma_4(s, r, x_r) w(\nu) \nu(0) d\nu$$

$$+ \int_0^t S(t - s)\int_{\mathbb{R}} \sigma_5(s, r, x_r) w(\nu) \nu(0) d\nu$$

$$+ \sum_{0 < t_k < T} C(t - t_k)^2 x_{t_k}^2(t) + \sum_{0 < t_k < T} S(t - t_k)^2 x_{t_k}^2(t)$$

and

$$(2.2)$$

$$(2.2)$$
(iii) $\Delta x(t_k) = I_k^j(x_{t_k}), \Delta x'(t_k) = I_k^j(x_{t_k}), k = \overline{1,m}$.

Definition 2.4 The system (1.1) is said to be controllable on the interval $J_s$, if for every initial stochastic process $\phi \in \mathcal{B}$ defined on $J_s$, $x'(0) = x_1 \in \mathcal{H}$ and $y_1 \in \mathcal{H}$; there exists a stochastic control $u \in L^2(J,\mathcal{U})$ which is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ such that the solution $x(\cdot)$ of the system (1.1) satisfies $x(T) = y_1$, where $y_1$ and $T$ are the preassigned terminal state and time, respectively.

To prove our main results, we list the following basic assumptions of this paper.

**H1** There exists positive constants $M_C, M_o$, and $M_{\delta_1}$ such that for all $s, t \in J, x, y \in \mathcal{B}$

$$
\| C(t) \|^2 \leq M_C, \quad \| S(t) \|^2 \leq M_o;
$$

$$
\mathbb{E} \| \int_0^t [\sigma_1(t,s,x) - \sigma_1(t,s,y)] ds \|_2^2 \leq M_{\delta_1} \| x - y \|_{\mathcal{B}}^2.
$$

**H2** The function $g: J \times B \times \mathbb{H} \rightarrow \mathbb{H}$ is continuous and there exists a positive constant $M_g$ such that for all $t \in J, x_1, x_2 \in B, y_1, y_2 \in L^2(\Omega,\mathcal{H})$

$$
\mathbb{E} \| g(t, x_1, y_1) - g(t, x_2, y_2) \|^2 \leq M_g (\| x_1 - x_2 \|_H^2 + \mathbb{E} \| y_1 - y_2 \|_2^2).
$$

**H3** For each $(t, s) \in J \times J$, the function $\sigma_2: J \times J \times B \rightarrow \mathcal{H}$ is continuous and there exists a positive constant $M_{\sigma_2}$ such that for all $t, s \in J, x, y \in B$

$$
\mathbb{E} \| \int_0^t [\sigma_2(t,s,x) - \sigma_2(t,s,y)] ds \|_2^2 \leq M_{\sigma_2} \| x - y \|_{\mathcal{B}}^2.
$$

**H4** The function $f: J \times B \times \mathcal{H} \rightarrow \mathbb{H}$ is continuous and there exists a positive constant $M_f$ such that for all $t \in J, x_1, x_2 \in B, y_1, y_2 \in L^2(\Omega,\mathcal{H})$

$$
\mathbb{E} \| f(t, x_1, y_1) - f(t, x_2, y_2) \|^2 \leq M_f (\| x_1 - x_2 \|_B^2 + \mathbb{E} \| y_1 - y_2 \|_2^2).
$$

**H5** The functions $I_1^k, I_2^k \in C(B, \mathcal{H})$, $k = \overline{1,m}$ and there exist positive constants $M_{I_1^k}, \overline{M}_{I_1^k}$ $M_{I_2^k}, \overline{M}_{I_2^k}$ such that for all $x, y \in B$

$$
\mathbb{E} \| I_1^k(x) \|^2 \leq M_{I_1^k}, \quad \mathbb{E} \| I_2^k(x) \|^2 \leq M_{I_2^k};
$$

$$
\mathbb{E} \| I_1^k(x) - I_1^k(y) \|^2 \leq \overline{M}_{I_1^k} \| x - y \|_B^2, \quad \mathbb{E} \| I_2^k(x) - I_2^k(y) \|^2 \leq \overline{M}_{I_2^k} \| x - y \|_B^2.
$$

**H6** For each $\phi \in \mathcal{B}, h(t) = \lim_{c \rightarrow \infty} \int_c^0 \sigma(t, s, \phi) dW(s)$ exists and continuous. Further, there exists a positive constant $M_h$ such that

$$
\mathbb{E} \| h(t) \|^2 \leq M_h.
$$

**H7** The function $\sigma: J \times J \times B \rightarrow \mathcal{L}(\mathbb{H}, \mathbb{H})$ is continuous and there exists positive constants $M_{\sigma}, \overline{M}_{\sigma}$ such that for all $s, t \in J$ and $x, y \in B$

$$
\mathbb{E} \| \sigma(t, s, x) \|^2 \leq M_{\sigma};
$$

$$
\mathbb{E} \| \sigma(t, s, x) - \sigma(t, s, y) \|^2 \leq \overline{M}_{\sigma} \| x - y \|_B^2.
$$

**H8** The function $q: B^m \rightarrow B$ is continuous and there exist positive constants $M_q, \overline{M}_q$ such that for all $x, y \in B$, $t \in J_0$

$$
\mathbb{E} \| q(x_{t_1}, x_{t_2}, \cdots, x_{t_m}) \|^2 \leq M_q.
$$
\( E \| q(x, t) - q(y, t) \|_2^2 \leq M_q \| x - y \|_2^2 \)

\((H9)\) The linear operator \( W: L^2(J, U) \to L^2(\Omega, H) \) defined by

\[ Wv = \int_{J} S(T - s)Bv(s)\,ds \]

has an induced inverse \( W^{-1} \) which takes values in \( L^2(J, U) / Ker W \) (see Carimichel & Quinn, 1984) and there exist two positive constants \( M_B \) and \( M_W \) such that

\[ \| B \|_2^2 \leq M_B \quad \text{and} \quad \| W^{-1} \|_2^2 \leq M_W. \]

\((H10)\) The function \( \gamma: J \times H \times L^2 \to H \) is a Borel measurable function and satisfies the Lipschitz continuity condition, the linear growth condition, and there exists positive constants \( M_\gamma, M_\gamma^* \) such that for any \( x, y \in L^2(0, T; H), t \in J \)

\[
E \left( \int_{0}^{t} \| \gamma(t, x(s), v) - \gamma(t, y(s), v) \|_{H_2}^2 \lambda(ds) \right) \geq E \left( \int_{0}^{t} \| \gamma(t, x(s), v) \|_{H_2}^2 \lambda(ds) \right)^{1/2}
\]

\[ \leq M_\gamma E \int_{0}^{t} (1 + \| x(s) \|_{H_2}^2) \, ds; \]

\[ E \left( \int_{0}^{t} \| \gamma(t, x(s), v) - \gamma(t, y(s), v) \|_{H_2}^2 \lambda(ds) \right) \]

\[ \lor E \left( \int_{0}^{t} \| \gamma(t, y(s), v) \|_{H_2}^2 \lambda(ds) \right)^{1/2} \leq M_\gamma^* E \int_{0}^{t} \| x(s) - y(s) \|_{H_2}^2 \, ds. \]

3. Main results

In this section, we shall investigate the controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with infinite delay and Poisson jumps in Hilbert spaces.

The main result of this section is the following theorem.

**Theorem 3.1** Assume that the assumptions \((H1) \rightarrow (H10)\) hold. If \( \Xi < 1 \) and \( \Theta < 1 \), then the system \((1.1)\) is controllable on \( J_T \), where

\[
\Xi = 32 \left( 1 + 9T^2M_BM_WM_W \right) \left\{ 2T^2 \left[ M_cM_s(1 + 2M_s) + M_WM_s(1 + 2M_s) \right] + TM_cC \right\},
\]

\[
\Theta = \left\{ 96T^2M_WM_cM_sM_WM_W \left[ T^2M_cM_s(1 + M_s) \right] + 144T^2M_WM_cM_WM_W \left[ T^2M_cM_s(1 + M_s) \right] + T^2M_cM_WM_cTr(\Omega) + \frac{TMC_c}{2l_0^2} + mM_c \sum_{k=1}^{m} M_{H_k} + mM_c \sum_{k=1}^{m} M_{H_k} \right\}.
\]

**Proof** Using the assumption \((H9)\), for an arbitrary function \( x(\cdot) \), we define the control process
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http://dx.doi.org/10.1080/23319196.2015.1065585

\[ u_i(t) = W^{-1} \left\{ y_1 - C(T)(\varphi(0) + q(x_{i_1}, x_{i_2}, \ldots, x_{i_k})_0) - S(T)(x_{i_1} - g(0, x_0, 0)) \right\} \]

\[
= \int_0^T C(T-s)g(s, x_t) \int_0^t \sigma(s, \tau, x_t) d\tau ds - \sum_{0 \leq t \leq T} C(T-t)x_i^1(x_i) \\
- \int_0^T S(T-s)f(s, x_t) \int_0^t \sigma(s, \tau, x_t) d\tau ds - \sum_{0 \leq t \leq T} S(t-t)x_i^2(x_i) \\
- \int_0^T S(T-s)\left[ h(s) + \int_0^s \sigma(s, \tau, x_t) dw(\tau) \right] ds \\
- \int_0^T S(T-s) \left\{ \gamma(t, x(t-), v) \tilde{N}(dt, dv) \right\} (t) .
\]

We transform (1.1) into a fixed point problem. Consider the operator \( \Pi: \mathcal{B}_r \to \mathcal{B}_r \) defined by

\[ \Pi x(t) = \varphi(t) + q(x_{i_1}, x_{i_2}, \ldots, x_{i_k})(t), \quad t \in J_0; \]

\[ \Pi x(t) = C(t)(\varphi(0) + q(x_{i_1}, x_{i_2}, \ldots, x_{i_k})_0) + S(t)(x_{i_1} - g(0, x_0, 0)) \]

\[ + \int_0^T C(t-s)g(s, x_t) \int_0^t \sigma(s, \tau, x_t) d\tau ds \]

\[ + \int_0^T S(t-s)f(s, x_t) \int_0^t \sigma(s, \tau, x_t) d\tau ds + \int_0^T S(t-s)Bu_i(s) ds \]

\[ + \int_0^T S(t-s) \left[ h(s) + \int_0^s \sigma(s, \tau, x_t) dw(\tau) \right] ds \]

\[ + \sum_{0 \leq t \leq T} S(t-t)x_i^2(x_i) \]

\[ \text{for a.e. } t \in J. \]

In what follows, we shall show that using the control \( u_i^1(\cdot) \), the operator \( \Pi \) has a fixed point, which is then a mild solution for system (1.1).

Clearly, \( \Pi \varphi(T) = y_1 \).

For \( \varphi \in \mathcal{B}_r \), we defined \( \tilde{\varphi} \) by

\[ \tilde{\varphi}(t) = \left\{ \begin{array}{ll}
\varphi(t) + q(x_{i_1}, x_{i_2}, \ldots, x_{i_k})(t) & \text{if } t \in J_0, \\
C(t)(\varphi(0) + q(x_{i_1}, x_{i_2}, \ldots, x_{i_k})_0) & \text{if } t \in J,
\end{array} \right. \]

then \( \tilde{\varphi} \in \mathcal{B}_r \).

Set \( x(t) = z(t) + \tilde{\varphi}(t) \), \( t \in J_r \). It is easy to see that \( x \) satisfies (2.2) if and only if \( z \) satisfies \( z_0 = 0 \), \( x^*(0) = x_t = z^*(0) = z_1 \) and

\[ z(t) = S(t)(z_1 - g(0, \tilde{z}_0, 0)) + \int_0^T C(t-s)g(s, z_t + \tilde{z}_0, \tilde{\varphi}_s) \int_0^t \sigma(s, \tau, z_t + \tilde{\varphi}_s) d\tau ds \]

\[ + \int_0^T S(t-s)\left[ f(s, z_t + \tilde{\varphi}_s) \int_0^s \sigma(s, \tau, z_t + \tilde{\varphi}_s) d\tau ds + \int_0^s S(t-s)Bu_i(s) ds \right] \]

\[ + \int_0^T S(t-s) \left[ h(s) + \int_0^s \sigma(s, \tau, z_t + \tilde{\varphi}_s) dw(\tau) \right] ds \]

\[ + \int_0^T S(t-s) \left\{ \gamma(t, z(t-), \tilde{\varphi}(t-), v) \tilde{N}(dt, dv) \right\} (t) \]

\[ + \sum_{0 \leq t \leq T} S(t-t)x_i^2(z_t + \tilde{\varphi}_t) + \sum_{0 \leq t \leq T} S(t-t)x_i^2(z_t + \tilde{\varphi}_t), \quad t \in J, \]

\[ \text{for a.e. } t \in J. \]
Let $\mathcal{B}_r = \{ y \in \mathcal{B}; \|y\|_r = 0 \in \mathcal{B} \}$. For any $y \in \mathcal{B}_r$, we have

$$\|y\|_r = \|y_0\|_H + \sup_{s \in J} \|E(||y(s)||^2)\|_r^2 = \sup_{s \in J} \|E(||y(s)||^2)\|_r^2,$$

and thus $(\mathcal{B}_r, \| \cdot \|_r)$ is a Banach space. Set

$$B_r = \{ y \in \mathcal{B}_r; \|y\|_r \leq r \} \quad \text{for some } r \geq 0,$$

then $B_r \subseteq \mathcal{B}_r$ is uniformly bounded, and for $u \in B_r$, by Lemma 2.1, we have

$$\|u \|_r \leq 2(\|u\|_r + \|\nu\|_r) \leq 4(\|u\|_r + \|\nu\|_r) \leq 4(\|u\|_r + \|\nu\|_r + \|\nu\|_r + \|\nu\|_r) \leq 4(\|u\|_r + \|\nu\|_r) \leq 4(\|u\|_r + \|\nu\|_r) \leq 4\|u\|_r,$$

where $u(t) \in \mathcal{B}_r$ is obtained from (3.1) by replacing $x_1 = z_1 + \nu_1$.

Define the map $\Pi : \mathcal{B}_r \to \mathcal{B}_r$ defined by $\Pi(z) = 0$, for $t \in J_0$ and

$$\Pi(z)(t) = \int_0^t C(t-s)g(s, z_s + \nu_s, \int_0^s \sigma(s, r, z_r + \nu_r)dr)ds$$

$$+ \int_0^t \int_0^s \sigma(s, r, z_r + \nu_r)dr)ds + \int_0^t S(t-s)Bu(t)ds$$

$$+ \int_0^t \int_0^s \sigma(s, r, z_r + \nu_r)dr)ds + \int_0^t S(t-s)\gamma(t, z(t-), \nu)\tilde{N}(dt, dv)$$

$$+ \sum_{0 \leq t \leq T} C(t-t_-)z(t_- + \nu(t_-)) + \sum_{0 < t \leq T} S(t-t_-)u(t_-), \quad t \in J.$$

Obviously, the operator $\Pi$ has a fixed point which is equivalent to prove that $\Pi$ has a fixed point. Note that, by our assumptions, we infer that all the functions involved in the operator are continuous, therefore $\Pi$ is continuous.

Let $z, \bar{z} \in \mathcal{B}_r$. From (3.1), by our assumptions, Hölder’s inequality, the Doob martingale inequality, and the Burkholder–Davis–Gundy inequality for pure jump stochastic integral in Hilbert space (see Luo & Liu, 2008), Lemma 2.1, and in view of (3.2), for $t \in J$, we obtain the following estimates.

$$\mathbb{E}[\|u(t)\|^2] \leq 9\mathcal{M}_W \left\{ \mathbb{E}[\|y_0\|^2] + 2M_2[\mathbb{E}[\|\nu(0)||^2 + M_2] + 2M_2[\mathbb{E}[\|x_1||^2 + 2M_2\|\nu\|^2] + C_2] \right\}$$

$$+ 2T^2M_2 \left[ M_2[(1 + 2M_2)\mathbb{E}[\|\nu(t)||^2 + 2C_1] + C_2] \right] + 2T^2M_2 \left[ M_2[(1 + 2M_2)\mathbb{E}[\|\nu(t)||^2 + 2C_1] + C_2] \right]$$

$$+ 2T^2M_2(M_h + TTr(OM)) + TM_r(1 + \frac{r^*}{l_0}) + m\mathcal{M}_E \sum_{k=1}^m M_k + m\mathcal{M}_E \sum_{k=1}^m M_k = \mathcal{E},$$

and
\[ E[|u_{1,\xi}^2(t) - u_{2,\xi}^2(t)|^2] \leq 14\rho_2^2 M_0 \left( M_1 M_0 + T^2 M_1 M_2 (1 + M_{1,1}) + T^2 M_1 M_2 (1 + M_{1,1}) \right. \\
\left. + T^2 M_1 M_2 \sum_{k=1}^{m} M_{1,1}^k + M_{1,2} \sum_{k=1}^{m} M_{1,2}^k \right) \sup_{t \in I} E[|z(t) - \tilde{z}(t)|^2], \]

where \( \tilde{c} > 0 \) is a positive constant and

\[ C_s := T \sup_{(t,s) \in J} \sigma_s^2(t,s,0), \quad C_c := \sup_{t \in J} \|g(t,0,0)\|^2, \]

\[ C_1 := T \sup_{(t,s) \in J} \sigma_1^2(t,s,0), \quad C_{\sigma} := \sup_{t \in J} \|f(t,0,0)\|^2. \]

**Lemma 3.1**  
*Under the assumptions of Theorem 3.1, there exists \( r > 0 \) such that \( B_r \subseteq B_s.\)

**Proof**  
If this property is false, then for each \( r > 0, \) there exists a function \( z'(\cdot) \in B_s \) but \( \tilde{z}' \not\in B_s, \) i.e. \( \|\tilde{z}'(t)\| > r \) for some \( t \in J. \) However, by our assumptions, Hölder’s inequality and the Burkholder–Davis–Gundy inequality, we have

\[ r < E[\|\tilde{z}'(t)\|^2] \]

\[ \leq 8 \left[ 2M_2 \mathbb{E}[|x_1|^2 + 2M_2 \mathbb{E}[|\tilde{u}||\tilde{\sigma}_m|^2 + C_2] \right] + 2T^2 M_1 \left[ M_1 (1 + 2M_{1,1}) r^* + 2C_1 + C_s \right] \\
+ 2T^2 M_1 \left[ M_1 (1 + 2M_{1,1}) r^* + 2C_1 + C_s \right] + T^2 M_1 r^* \]

\[ + 2T^2 M_1 (M_2 + T \sigma t) M_2 + T M_1 \tilde{c} \left( 1 + \frac{T^2}{T_{0}} \right) + M_{1,2} \sum_{k=1}^{m} M_{1,1}^k + M_{1,2} \sum_{k=1}^{m} M_{1,2}^k \right], \]

\[ \leq M^{**} + 8 \left[ 1 + 9T^2 M_2 M_3 M_{1,0} \right] \left[ 2T^2 \left( M_1 M_2 (1 + 2M_{1,1}) + M_1 M_2 (1 + 2M_{1,1}) \right) + \frac{T M_1 \tilde{c}}{T_{0}} \right] r^*, \]

where

\[ M^{**} := 72 \left( 1 + 9T^2 M_2 M_3 M_{1,0} \right) \mathbb{E}[|y_1|^2 + 2M_2 \mathbb{E}[|\tilde{u}||\tilde{\sigma}_m|^2 + M_{1,1}] \mathbb{E}[|\tilde{u}||\tilde{\sigma}_m|^2 + M_{1,1}] + 8(1 + 9T^2 M_2 M_3 M_{1,0}) \times \left[ 2M_2 \mathbb{E}[|x_1|^2 + 2M_2 \mathbb{E}[|\tilde{u}||\tilde{\sigma}_m|^2 + C_2] \right] + 2T^2 M_2 (2M_2 C_1 + C_s) + 2T^2 M_2 (2M_2 C_1 + C_s) \right. \\
+ 2T^2 M_1 (M_2 + T \sigma t) M_2 + T M_1 \tilde{c} + m M_{1,2} \sum_{k=1}^{m} M_{1,1}^k + m M_{1,2} \sum_{k=1}^{m} M_{1,2}^k \right]. \]

Dividing both sides of (3.3) by \( r \) and noting that

\[ r^* = 4\rho_2^2 \left( r + 2M_2 \mathbb{E}[|\tilde{u}||\tilde{\sigma}_m|^2 + M_{1,1}] \right) + 4\|\tilde{\sigma}_m\|_\alpha^{\text{ess}} \to \infty \]

and taking the limit as \( r \to \infty, \) we obtain

\[ 1 \leq \Xi \]

which contradicts our assumption. Thus, for some positive number \( r, \) \( B_r \subseteq B_s. \) This completes the proof of Lemma 3.1.

**Lemma 3.2**  
*Under the assumptions of Theorem 3.1, \( \Pi : B_0 \to B_0 \) is a contraction mapping.*

**Proof**  
Let \( z, \tilde{z} \in B_0. \) Then, by our assumptions, Hölder’s inequality, Burkholder–Davis–Gundy’s inequal-
ity, Lemma 2.1, and since \( \|z_0\|_2^2 = 0 \) and \( \|\overline{z}_0\|_2^2 = 0 \), for each \( t \in J \), we see that

\[
E\|\overline{Tz}(t) - (\overline{Tz}(t))\|^2 \leq 14\|c\|_{\infty} \left\{ T^2M_2M_q(1 + M_{e_1}) + T^2M_2M_q(1 + M_{e_1}) + T^2M_2M_q Tr(Q) + \frac{\overline{TMC}}{2}\right\}^2
\]

\[
+ mM_c \sum_{k=1}^{m} \overline{M}_{1k} + mM_s \sum_{k=1}^{m} \overline{M}_{1k}\sup_{s \in J} E\|z(t) - \overline{z}(t)\|^2 + 7T^2M_3M_q E\|u_{\overline{z}_0}(t) - u_{\overline{z}_0}(t)\|^2
\]

\[
\leq \left\{ 98\|c\|_{\infty} T^2M_2M_q(1 + M_{e_1}) + T^2M_2M_q(1 + M_{e_1}) + T^2M_2M_q Tr(Q) + \frac{\overline{TMC}}{2}\right\}^2
\]

\[
\times \left\{ T^2M_2M_q(1 + M_{e_1}) + T^2M_2M_q(1 + M_{e_1}) + T^2M_2M_q Tr(Q) + \frac{\overline{TMC}}{2}\right\}^2
\]

\[
+ mM_c \sum_{k=1}^{m} \overline{M}_{1k} + mM_s \sum_{k=1}^{m} \overline{M}_{1k}\sup_{s \in J} E\|z(t) - \overline{z}(t)\|^2.
\]

Taking the supremum over \( t \), we obtain

\[
\|\overline{Tz} - (\overline{Tz})\|_2^2 \leq \Theta\|z - \overline{z}\|_2^2.
\]

By our assumption, we conclude that \( \overline{T} \) is a contraction on \( K^2 \). Thus, we have completed the proof of Lemma 3.2.

On the other hand, by Banach fixed point theorem, there exists a unique fixed point \( x(\cdot) \in K^2 \) such that \( (\overline{T}x)(t) = x(t) \). This fixed point is then the mild solution of the system (1.1). Clearly, \( x(T) = (\overline{T}x)(T) = y_T \).

Thus, the system (1.1) is controllable on \( J_r \). The proof for Theorem 3.1 is thus complete.

Now, let us consider a special case for the system (1.1).

If \( \gamma(x, t \rightarrow -, y) \equiv 0 \), the system (1.1) becomes the following nonlocal second-order impulsive neutral stochastic functional integro-differential equations with infinite delay without Poisson jumps:

\[
\begin{align*}
\begin{cases}
  \frac{d}{dt} [x(t)] - g(t, x, \int_0^t \sigma_1(t, s, x(s))ds)] + A[x(t)] + f(t, x(t), \int_0^t \sigma_2(t, s, x(s))ds) + Bu(t)dt \\
  + \int_0^t \sigma(t, s, x(s))dw(s), & t \neq t_k = [0, T], \\
  \Delta x(t_k) = I_k(x(t_k)), & k = 1, \ldots, m,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  \Delta x(t_k) = I_k(x(t_k)), & k = 1, m,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  x(0) = x_1 \in H_1, \\
  x(0) - q(x_1, x_2, \ldots, x_m) = x_0 = \varphi \in B, \\
  \text{for a.e. } s \in J_r = (-\infty, 0],
\end{cases}
\end{align*}
\]

**Corollary 3.1** Assume that all assumptions of Theorem 3.1 hold except that \((\mathbf{H11})\) and \( \Xi, \Theta \) replaced by \( \Xi, \Theta \) such that

\[
\Xi = 56\|c\|_{\infty} T^2(1 + 8T^2M_2M_qM_0) \left[ M_cM_q(1 + 2M_{e_1}) + M_2M_q(1 + 2M_{e_1}) \right] ,
\]

and

\[
\Theta = \left\{ 72\|c\|_{\infty} T^2M_2M_qM_0M_q + 12\|c\|_{\infty} (1 + 6T^2M_2M_qM_0) \left[ T^2M_2M_q(1 + M_{e_1}) + T^2M_2M_q Tr(Q) + mM_s \sum_{k=1}^{m} \overline{M}_{1k} + mM_s \sum_{k=1}^{m} \overline{M}_{1k} \right] \right\}.
\]

If \( \Xi < 1 \) and \( \Theta < 1 \), then the system (3.4) is controllable on \( J_r \).
4. Application

In this section, the established previous results are applied to study the controllability of the stochastic nonlinear wave equation with infinite delay and Poisson jumps. Specifically, we consider the following controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with infinite delay and Poisson jumps of the form:

\[
\begin{align*}
\frac{\partial}{\partial t} [ & y(t, \xi) - \int_0^t \delta_1(t, \xi, s - t)P_1(y(s, \xi))ds - \int_0^t \int_{-\infty}^s b_1(s - \tau)P_2(y(\tau, \xi))d\tau ds ] \\
& = \int_0^t \frac{\partial}{\partial t} [ y(t, \xi) + \int_{-\infty}^s \delta(t, \xi, s - t)G_1(y(s, \xi))ds + \int_0^s \int_{-\infty}^\tau b_2(s - \tau)G_2(y(\tau, \xi))d\tau ds ] \\
& + b(\xi)u(t)dt + \int_{-\infty}^t \delta(s - t)y(t, \xi)d\beta(s) + \int_{-\infty}^t y(t, \xi)\nu N(dt, dv), \quad t \neq J, \xi \in [0, 1], \\
\Delta y(t, \xi) = & \int_{-\infty}^0 \eta_k(t_k - s)(y(s, \xi)ds, \quad k = 1, m, \xi \in [0, 1], \\
\Delta y'(t_k(\xi)) = & \int_{-\infty}^0 \rho_j(t_k - s)(y(s, \xi)ds, \quad k = 1, m, \xi \in [0, 1], \\
y(t, 0) = & y(t, \pi) = 0, \quad t \in J, \\
\frac{\partial}{\partial t} y(0, \xi) = & \lambda_1(x(t), \xi, \xi) \in [0, 1], \\
y(t, \xi) - & \sum_{i=1}^{\infty} p_i(\xi, \xi)y(t, \xi)\xi = \varphi(t, \xi), \quad t \in J, \xi \in [0, 1],
\end{align*}
\]

where \( \beta(t) \) is a standard one-dimensional Wiener process in \( \mathbb{H} \), defined on a stochastic basis \( (\Omega, \mathcal{F}, \mathbb{P}) \), \( \mathcal{F} = \{ \mathcal{F}_t \}_{t \geq 0} \); \( \nu = \mathbb{P} \{ 0, \mathbb{P} > \mathbb{P} \} \), let \( N(ds, dv) = N(ds, dv) - \lambda(dv)ds \), with the characteristic measure \( \lambda(dv) \) on \( \mathcal{F} \). Assume that

\[
\int_{\mathcal{F}} v^2 \lambda(dv) < \infty \quad \text{and} \quad \int_{\mathcal{F}} v^4 \lambda(dv) < \infty.
\]

To rewrite (4.1) into the abstract from of (1.1), we consider the space \( \mathcal{H} = L^2([0, 1]) \) with the norm \( \| \cdot \| \). Let \( e_n(\xi) = \sqrt{\frac{2}{\pi_2}} \sin n\xi, \quad n = 1, 2, 3, \ldots \) denote the completed orthogonal basics in \( \mathcal{H} \) and \( \beta(t) = \sum_{n=1}^{\infty} \sqrt{1/2} \beta_n(t)e_n, \quad t \geq 0, \lambda_n = 0, \) where \( \{ \beta_n(t) \}_{n=0} \) are one-dimensional standard Brownian motions mutually independent on a usual complete probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \).

Defined \( \mathcal{A}: \mathcal{H} \to \mathcal{H} \) by \( \mathcal{A} = \frac{\partial}{\partial t} \), with domain \( \mathcal{D}(\mathcal{A}) = \mathcal{H}^2 \cap \mathcal{H}^0 \) and

\[
\mathcal{H}^0(0, 1) = \{ w \in L^2([0, 1]): \frac{\partial w}{\partial z} \in L^2([0, 1]), w(0) = w(1) = 0 \}
\]

and

\[
\mathcal{H}^2(0, 1) = \{ w \in L^2([0, 1]): \frac{\partial w}{\partial z}, \frac{\partial^2 w}{\partial z^2} \in L^2([0, 1]) \}.
\]

Then,

\[
\mathcal{A} x = -\sum_{n=1}^{\infty} p_n(x, e_n)e_n, \quad x \in \mathcal{D}(\mathcal{A}),
\]

(see Travis & Webb, 1987, Example 5.1). Using (4.2), one can easily verify that the operators \( \mathcal{C}(t) \) defined by

\[
\mathcal{C}(t) = \sum_{n=1}^{\infty} \cos(nt)(x, e_n)e_n, \quad t \in \mathbb{R},
\]
from a cosine function on \( H \), with associated sine function
\[
S(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle x, e_n \rangle e_n, \quad t \in \mathbb{R}.
\]

It is clear that (see Travis & Webb, 1977), for all \( x \in H, t \in \mathbb{R}, C(t)x \) and \( S(t)x \) are periodic functions with \( \|C(t)\| \leq 1 \) and \( \|S(t)\| \leq 1 \). Thus, (H1) is true.

Now, we give a special \( B \)-space. Let \( l(s) = e^{2is}, s \leq 0 \), then \( l_0 = \int_{j_0} l(s)ds = \frac{1}{2} \) and define
\[
\|\psi\|_B = \int_{j_0} l(s) \sup_{\theta \in [0,1]} (E\|\psi(\theta)\|)^{\frac{1}{2}} ds, \quad \forall \psi \in B.
\]

It follows from Hino et al. (1991) that \( (B, \| \cdot \|_B) \) is a Banach space. Hence, for \( (t, \psi) \in J \times B \), where \( \psi(\theta)x = \psi(\theta, x), (\theta, x) \in J_0 \times [0, \pi] \). Let \( y(t, \xi) = y(t, \xi). \)

To study the system (4.1), we assume that the following conditions hold:

(i) Let \( B \in L(\mathbb{R}, H) \) be defined as
\[
Bu(\xi) = b(\xi)u, \quad 0 \leq \xi \leq \pi, \quad u \in \mathbb{R}, \quad b(\xi) \in L^2([0, \pi]).
\]

(ii) The linear operator \( W: L^2(J, U) \rightarrow H \) defined by
\[
Wu = \int_{j_0} S(T-s)b(\xi)u(s)ds
\]
is a bounded linear operator but not necessarily one-to-one. Let \( KerW = \{ u \in L^2(J, U): Wu = 0 \} \) be null space of \( W \) and \( KerW^+ \) be its orthogonal complement in \( L^2(J, U) \). Let \( W^*: KerW^+ \rightarrow Range(W) \) be the restriction of \( W \) to \( KerW^+ \). \( W^* \) is necessarily one-to-one operator. The inverse mapping theorem says that \( (W^*)^{-1} \) is bounded since \( KerW^+ \) and \( Range(W) \) are Banach spaces. Since the inverse operator \( W^{-1} \) is bounded and takes values in \( L^2(J, U)/KerW \), the assumption (H9) is satisfied.

(iii) The functions \( p_i: [0, \pi] \times 0, \pi \rightarrow \mathbb{R} \) are C\(^2\)-functions, for each \( i = 1, \ldots, n \).

(iv) The functions \( \eta_1, \rho_1 \in C(\mathbb{R}, \mathbb{R}) \) such that for \( k = 1, \ldots, m \),
\[
\overline{M}_1 = \int_{j_0} l(s)\eta_1^2(s)ds < \infty, \quad \overline{M}_2 = \int_{j_0} l(s)\rho_1^2(s)ds < \infty.
\]

We define the functions \( g, f: J \times B \times H \rightarrow H, \sigma: J \times J \times B \rightarrow L_2^0, \gamma: J \times H \times U \rightarrow \mathbb{H} \) and \( I_k^1, I_k^2: B \rightarrow \mathbb{H}, \)
\( k = 1, \ldots, m \) by
\[
g(t, \psi, V_1\psi)(\xi) = \int_{j_0} \delta_1(t, \xi, \theta)p_1(\psi(\theta), \xi)\theta d\theta + V_1\psi(\xi),
\]
\[
f(t, \psi, V_2\psi)(\xi) = \int_{j_0} \delta_2(t, \xi, \theta)p_1(\psi(\theta), \xi)\theta d\theta + V_2\psi(\xi),
\]
\[
\sigma(t, s, \psi)(\xi) = \int_{j_0} \delta(\theta)\psi(\theta, \xi)\theta d\theta, \quad \gamma(t, \psi(\xi), \nu) = \psi(\xi)\nu,
\]
\[
I_k^1(t, \psi)(\xi) = \int_{j_0} \eta_k(-s)\psi(\theta, \xi)\theta d\theta, \quad k = 1, \ldots, m,
\]
\[
I_k^2(t, \psi)(\xi) = \int_{j_0} \rho_k(-s)\psi(\theta, \xi)\theta d\theta, \quad k = 1, \ldots, m,
\]
where
\[ V_1\psi(\xi) = \int_0^t \int_{\mathbb{R}^n} b_1(s-\theta)P_1(\psi(\theta)(\xi))d\theta ds, \quad V_2\psi(\xi) = \int_0^t \int_{\mathbb{R}^n} b_2(s-\theta)G_2(\psi(\theta)(\xi))d\theta ds. \]

Then, the system (4.1) can be written in the abstract form as the system (1.1). Further, we can impose some suitable conditions on the above-defined functions as those in the assumptions (H1) – (H10). Therefore, by Theorem 3.1, we can conclude that the system (4.1) is controllable on \( J_t \).

5. Conclusion
In this paper, we have studied the controllability for a class of nonlocal second-order impulsive neutral stochastic functional integra-differential equations with infinite delay and Poisson jumps in Hilbert spaces, which is new and allows us to develop the controllability of the second-order stochastic partial differential equations. Using the Banach fixed point theorem combined with theories of a strongly continuous cosine family of bounded linear operators, and stochastic analysis theory, the controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with infinite delay and Poisson jumps is obtained. In addition, an application is provided to illustrate the effectiveness of the controllability results obtained. The results in our paper extend and improve the corresponding ones announced by Arthi and Balachandran (2012), Balasubramaniam and Muthukumar (2009), Muthukumar and Rajivganthi (2013), Park, Balachandran, and Annapoorni (2009), Park, Balachandran, and Arthi (2009), Travis and Webb (1978), and some other results.
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