BILINEAR OPTIMAL CONTROL FOR WEAK SOLUTIONS OF THE KELLER-SEGEL LOGISTIC MODEL IN 2D DOMAINS

P. BRAZ E SILVA, F. GUILLÉN-GONZÁLEZ, CILON F. PERUSATO, AND M.A. RODRÍGUEZ-BELLIDO

Abstract. An optimal control problem associated to the Keller-Segel with logistic reaction system will be studied in 2D domains. The control acts in a bilinear form only in the chemical equation. The existence of optimal control and a necessary optimality system are deduced. The main novelty is that control can be rather singular and the state (cell density $u$ and the chemical concentration $v$) remains only in a weak setting, which is not usual in the literature to solve optimal control problems subject to chemotaxis models (see e.g. [16]).

1. Introduction

1.1. The controlled model. In this work we study an optimal control problem for the (attractive or repulsive) Keller-Segel model in a 2D domain $\Omega \subset \mathbb{R}^2$ with logistic source term and bilinear control acting on the chemical equation:

$$
\begin{align*}
\partial_t u - \Delta u + \kappa \nabla \cdot (u \nabla v) &= r u - \mu u^2 \quad \text{in } \Omega \times (0, T), \\
\partial_t v - \Delta v + v &= u + f v 1_{\Omega_c} \quad \text{in } \Omega \times (0, T), \\
\partial_n u = \partial_n v &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(0, \cdot) &= u_0 \geq 0, \quad v(0, \cdot) &= v_0 \geq 0 \quad \text{in } \Omega.
\end{align*}
$$

Here, $f : Q_c := (0, T) \times \Omega_c \to \mathbb{R}$ is the control with $\Omega_c \subset \Omega \subset \mathbb{R}^2$ the control domain, and the states $u, v : Q := (0, T) \times \Omega \to \mathbb{R}_+^2$ are the cellular density and chemical concentration, respectively. Moreover, $r \in \mathbb{R}$ and $\mu > 0$ are coefficients of the...
logistic reaction, and \( \kappa \in \mathbb{R} \) is the chemotaxis coefficient \((\kappa > 0 \text{ models attraction and } \kappa < 0 \text{ repulsion})\). We are interested in the study of a control problem associated to the following weak solution concept of system (1.1). Hereafter, \( L^{2+} \) means \( L^{2+\varepsilon} \) for \( \varepsilon \) small enough.

**Definition 1.1.** Let \( f \in L^{2+}(Q_c) := L^{2+}(0,T;L^{2+}(\Omega_c)) \), \( u_0 \in L^2(\Omega) \), \( v_0 \in W^{1+,2+}(\Omega) \) with \( u_0 \geq 0 \) and \( v_0 \geq 0 \) a.e. in \( \Omega \). A pair \( (u,v) \) is called a weak solution of problem (1.1) in \((0,T)\), if

\[
\begin{align*}
 u \geq 0, & \quad v \geq 0 \quad \text{a.e. in } Q = (0,T) \times \Omega, \\
 u \in W_2 := \{ u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \}, & \quad \partial_t u \in L^2(0,T;H^1(\Omega)') \\
 v \in X_{2+} := \{ v \in C([0,T];W^{1+,2+}(\Omega)) \cap L^{2+}(0,T;W^{2+,2+}(\Omega)) \}, & \quad \partial_v \in L^{2+}(Q),
\end{align*}
\]

the equation (1.1) jointly boundary condition for \( u \) hold in a variational sense, while equation (1.1)2 and the boundary condition for \( v \) are satisfied pointwisely, and initial conditions (1.1)3 and (1.1)4 are satisfied in \( L^2(\Omega) \) and \( W^{1+,2+}(\Omega) \), respectively.

Notice that, since we are in 2D bounded domains, \( v \in C([0,T];W^{1+,2+}) \) implies \( v \in L^\infty(0,T;L^\infty) \), hence using that \( f \in L^{2+}(Q_c) \) one has \( fv \in L^{2+}(Q) \). That means that \( v \in X_{2+} \) is the maximal regularity which can be obtained. The previous weak regularity for \( u \in W_2 \) will be enough to solve the following optimal control problem.

\[
(1.2) \quad \left\{ \begin{array}{l}
\text{Find } (u,v,f) \in W_2 \times X_{2+} \times F \text{ minimizing the functional} \\
J(u,v,f) := \frac{\gamma_u}{2} \int_0^T \| u(t) - u_d(t) \|^2_{L^2(\Omega)} dt + \frac{\gamma_v}{2} \int_0^T \| v(t) - v_d(t) \|^2_{L^2(\Omega)} dt + \frac{\gamma_f}{2} \int_0^T \| f(t) \|^2_{L^{2+}(\Omega_c)} dt \\
\text{subject to } (u,v,f) \text{ be a weak solution of the PDE system (1.1),}
\end{array} \right.
\]

where \((u_d, v_d) \in L^2(Q)^2 \) represent the target states and the nonnegative numbers \( \gamma_u, \gamma_v \) and \( \gamma_f \) measure the cost of the states and control, respectively. With respect to the control constraint, we assume

\[
F \subset L^{2+}(Q_c) \quad \text{be a nonempty, closed and convex set.}
\]

The functional \( J \) defined in (1.2) describes the deviation of the cell density \( u \) and the chemical concentration \( v \) from a target cell density \( u_d \) and chemical concentration \( v_d \), respectively; plus the cost of the control \( f \) measured in the \( L^{2+}\)-norm.
1.2. Previous results. Before continuing our discussion, let us give some motivation for the study of (1.1). In the last decades, there has been a surge of activity on the study of the chemotaxis model which describes the movement of the cells directed by the concentration gradient of a chemical substance in their environment. Moreover, it is important to consider the biological situation where the bacterial population may proliferate according to a logistic law. An interesting feature in chemotaxis corresponds to the movement of cells directed by the gradient of the chemical signal which is produced by cells themselves. When one considers also the interactions between cells and the chemical signal with liquid environments, one gets the chemotaxis-fluid system, which is basically the chemotaxis model coupled with the Navier-Stokes equations. For more details, see the excellent review \[4\] and the references therein.

Let us recall some issues related to the uncontrolled equations (1.1), i.e., when $f \equiv 0$. Plenty of results have been obtained here. Amongst the many articles related to the uncontrolled system, let us mention those on existence of weak and strong solutions in $\mathbb{R}^2$. In this case, without considering logistic reaction (i.e. $r = \mu = 0$), the existence of global weak solutions was provided by Liu and Lorz \[24\]. In two-dimensional bounded convex domains, the existence of (global) classical solutions was obtained by M. Winkler \[33\]. In the presence of logistic source, the existence of global weak solutions (and the long time behavior) has been analyzed in \[21\] by J. Lankeit. In this case, the existence of global mild solutions was examined in \[12\]. For 3D domains, we also refer \[34\] and the references therein.

It is important to mention that remarkable progress has been made in mathematical and numerical analysis of optimal control problems for viscous flows described by the Navier-Stokes equations and other related models, see e.g., \[1, 7, 26\]. However, the literature related to optimal control for chemotaxis problems is still scarce. In \[27\], the authors study an optimal (distributed) control problem where the state problem is given by a stationary chemotaxis model coupled with the Navier-Stokes equations. We note that in \[8, 9\] the authors provide some results related to the controllability for the nonstationary Keller-Segel system and the nonstationary chemotaxis-fluid model with consumption of chemoattractant substance associated to a chemotaxis system, based on Carleman-type estimates for the solutions of the adjoint system. Recently, a bilinear optimal control problem associated to the chemotaxis-Navier-Stokes model (without logistic source) in bounded three-dimensional domains was examined in \[25\]. For the chemo-repulsion case, this problem was studied in \[16, 18\] for 2D and 3D domains respectively, and in \[17\] for 2D domains with a potential nonlinear production term, that is changing the production term $u$ in the $v$ equation by $u^p$, with $p > 1$. 
1.3. **Main contributions of the paper.** First of all, we will prove the existence and uniqueness of weak solutions of (1.1) and the continuous dependence of the weak solution \((u, v)\) respect the control \(f\).

**Theorem 1.2.** Let \(u_0 \in L^2(\Omega), v_0 \in W^{1,2+}(\Omega)\) with \(u_0 \geq 0\) and \(v_0 \geq 0\) in \(\Omega\), and \(f \in L^2(Q_c)\). There exists a unique weak solution \((u, v)\) of system (1.1) in sense of Definition 1.1. Moreover, there exists a positive constant
\[
\mathcal{K}_1 := K_1(r, \mu, \kappa, |\Omega|, T, \|u_0\|_{L^2}, \|v_0\|_{W^{1,2+}}, \|f\|_{L^2(Q_c)}),
\]

such that
\[
(1.3) \quad \|(u, v)\|_{W^2 \times X^2} \leq \mathcal{K}_1.
\]

where we denote
\[
\|(u, v)\|_{W^2 \times X^2} := \|(\partial_t u, \partial_t v)\|_{L^2(H^1') \times L^2(Q^2)} + \|(u, v)\|_{C(L^2 \times W^{1,2+})} + \|(u, v)\|_{L^2(H^1) \times L^2(0,T; W^{2,2})}
\]

Finally, for any \(r, \mu, \kappa, \Omega, T, u_0, v_0\), the constant \(\mathcal{K}_1\) is bounded if \(f\) is bounded in \(L^2(Q_c)\).

The second main result of this paper will be the existence of a global optimal solution of (1.2):

**Theorem 1.3.** Let \((u_0, v_0) \in L^2(\Omega) \times W^{1,2+}(\Omega)\) with \(u_0 \geq 0\) and \(v_0 \geq 0\) in \(\Omega\). Assuming that either \(\gamma_f > 0\) or \(\mathcal{F}\) is bounded in \(L^2(Q_c)\), then the bilinear optimal control problem (1.2) has at least one global optimal solution \((\tilde{u}, \tilde{v}, \tilde{f})\).

Finally, we obtain the existence and uniqueness of Lagrange multipliers associated to the optimal control (1.2):

**Theorem 1.4.** Let \(\hat{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in S_{ad}\) be a local optimal solution for the control problem (1.2). Then, there exists a unique Lagrange multiplier \((\lambda, \eta) \in X_2 \times W_2\) satisfying the optimality system
\[
(1.4) \quad \begin{cases}
- \partial_t \lambda - \Delta \lambda + \kappa \nabla \lambda \cdot \nabla \tilde{v} - \eta = \gamma_u (\tilde{u} - u_d) & \text{in } Q, \\
- \partial_t \eta - \Delta \eta - \kappa \nabla \cdot (\tilde{u} \nabla \lambda) + \eta - \tilde{f} \eta 1_{\Omega_c} = \gamma_v (\tilde{v} - v_d) & \text{in } Q, \\
\lambda(T) = 0, \quad \eta(T) = 0 & \text{in } \Omega, \\
\frac{\partial \lambda}{\partial n} = 0, \quad \frac{\partial \eta}{\partial n} = 0 & \text{on } (0,T) \times \partial \Omega, \\
\int_0^T \int_{\Omega_c} (\gamma_f \text{sgn} \tilde{f} |\tilde{f}|^{1+} + \tilde{v} \eta)(f - \tilde{f}) \geq 0 & \forall f \in \mathcal{F}.
\end{cases}
\]

**Remark 1.5.** If \(\gamma_f > 0\) and there is no convexity constraint on the control, that is, \(\mathcal{F} = L^2(Q_c)\), then optimality condition (1.4)_5 becomes
\[
\gamma_f \text{sgn} \tilde{f} |\tilde{f}|^{1+} 1_{\Omega_c} + \tilde{v} \eta 1_{\Omega_c} = 0.
\]
The rest of the paper is organized as follows. In Section 2 some Preliminary results which will be used later are introduced. The proofs of Theorems 1.2, 1.3 and 1.4 are given in Sections 3, 4 and 5 respectively.

2. Preliminary results

Along this manuscript the following result on $L^p$ regularity will be considered.

**Theorem 2.1** ([13], page 344). For $\Omega \in C^2$, let $1 < p < 3$, $u_0 \in W^{2-2/p,p}(\Omega)$ and $g \in L^p(Q)$. Then the problem

$$
\begin{aligned}
\partial_t u - \Delta u &= g \quad \text{in } Q, \\
u(0, \cdot) &= u_0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } (0,T) \times \partial \Omega,
\end{aligned}
$$

admits a unique solution $u$ such that

$$u \in C([0,T]; W^{2-2/p,p}(\Omega)) \cap L^p(W^{2,p}), \quad \partial_t u \in L^p(Q).$$

Moreover, there exists a positive constant $C := C(p,\Omega,T)$ such that

$$
\|u\|_{C(W^{2-2/p,p})} + \|\partial_t u\|_{L^p(Q)} + \|\partial_t u\|_{L^p(W^{2,p})} \leq C(\|g\|_{L^p(Q)} + \|u_0\|_{W^{2-2/p,p}}).
$$

The existence of solutions of some initial and boundary-value problems will be proven by means of:

**Theorem 2.2** (Leray-Schauder fixed-point Theorem). Let $\mathcal{X}$ a Banach space and $T: \mathcal{X} \to \mathcal{X}$ a continuous and compact operator. If the set

$$\{x \in \mathcal{X} : x = \alpha Tx \quad \text{for some } 0 \leq \alpha \leq 1\}$$

is bounded, then $T$ has (at least) a fixed point.

In this paper, the following two compactness results will be applied.

**Theorem 2.3** (Aubin-Lions lemma). (See [22, Théorème 5.1, p. 58].) Let $\mathcal{X}$, $\mathcal{B}$ and $\mathcal{Y}$ be reflexive Banach spaces such that $\mathcal{X} \subset \mathcal{B} \subset \mathcal{Y}$, with compact embedding $\mathcal{X} \hookrightarrow \mathcal{B}$ and continuous embedding $\mathcal{B} \hookrightarrow \mathcal{Y}$. It is defined

$$W = \{w : w \in L^{p_0}(0,T;\mathcal{X}'), \partial_t w \in L^{p_1}(0,T;\mathcal{Y})\}$$

for a finite $T > 0$ and $p_0, p_1 \in (1, +\infty)$. Then, the injection of $W$ into $L^{p_0}(0,T;\mathcal{B})$ is compact.
Theorem 2.4 (Simon’s compactness result). (See [30, Corollary 4].) Let \( X, B \) and \( Y \) be Banach spaces such that \( X \subset B \subset Y \), with compact embedding \( X \hookrightarrow B \) and continuous embedding \( B \hookrightarrow Y \). Let \( F \) be a bounded set in \( L^\infty(0,T;X) \) such that the set \( \partial_t F = \{ \frac{df}{dt}; f \in F \} \) is bounded in \( L^r(0,T;Y) \) for some \( r > 1 \). Then \( F \) is relatively compact in \( C([0,T];B) \).

3. Proof of Theorem 1.2

The proof of Theorem 1.2 will be made in the next two subsections. For the existence, we use the Leray-Schauder fixed point theorem. The uniqueness is get by a comparison argument.

3.1. Existence. Let us introduce the spaces

\[
X_u := L^{4-}(Q) \quad \text{and} \quad X_v := L^\infty(Q),
\]

and the operator \( R : X_u \times X_v \to W_2 \times X_2+ \to X_u \times X_v \) defined by \( R(\bar{u}, \bar{v}) = (u, v) \) is the solution of the decoupled linear problem

\[
\begin{align*}
\int_0^T (\partial_t u, \varphi) + \int_0^T \int_\Omega \nabla u \cdot \nabla \varphi + \mu \bar{u} + u \varphi &= \int_0^T \int_\Omega (\bar{u} + f \bar{v}) + \kappa \int_0^T \int_\Omega \bar{v} + \nabla v \cdot \nabla \varphi, \quad \forall \varphi \in L^2(H^1),
\end{align*}
\]

(3.2)

\[
\begin{align*}
\partial_t v - \Delta v + v = \bar{u} + f \bar{v} + 1_{\Omega}, & \quad \text{in } Q, \\
u(0) = u_0, \quad v(0) = v_0, & \quad \text{in } \Omega, \\
\frac{\partial v}{\partial n} = 0, & \quad \text{on } (0,T) \times \partial \Omega,
\end{align*}
\]

where \( \bar{u}_+ := \max\{\bar{u}, 0\} \geq 0, \bar{v}_+ := \max\{\bar{v}, 0\} \geq 0 \). In fact, first we compute \( v \) and after \( u \). In the following lemmas we will prove the hypotheses of Leray-Schauder fixed point theorem.

Lemma 3.1. The operator \( R : X_u \times X_v \to X_u \times X_v \) is well defined and compact.

Proof. Since \( f \in L^{2+}(Q) \) and \( \bar{v} \in L^\infty(Q) \) implies \( f \bar{v} \in L^{2+}(Q) \), hence there exists a unique \( v \in X_{2+} \) solution of the \( v \)-problem in (3.2). By considering the linear parabolic \( u \)-problem in (3.2), one has \( u \in W_2 \) owing to \( v \in X_{2+} \), hence \( \nabla v \in L^{4}(Q) \) and then \( \bar{u} + \nabla v \in L^2(Q) \). Finally, since \( R \) maps bounded sets of \( X_u \times X_v \) into bounded sets of \( W_2 \times X_{2+} \), then \( R \) is compact from \( X_u \times X_v \) to itself.

Lemma 3.2. The set

\[
T_\alpha = \{(u,v) \in W_2 \times X_{2+} : (u,v) = \alpha R(u,v) \text{ for some } \alpha \in [0,1]\}
\]

is bounded in \( X_u \times X_v \) (independently of \( \alpha \in [0,1] \)). In fact, \( T_\alpha \) is also bounded in \( W_2 \times X_{2+} \), because there exists

\[
M = M(r, \mu, \kappa, |\Omega|, T, \|u_0\|_{L^2}, \|v_0\|_{W^{1+,2+}}, \|f\|_{L^{2+}(Q_\cdot)}) > 0,
\]

(3.4)
with $M$ independent of $\alpha$, such that all pairs of functions $(u, v) \in T_\alpha$ for $\alpha \in [0, 1]$ satisfy
\begin{equation}
\|(u, v)\|_{W_2 \times X_2^+} \leq M.
\end{equation}

**Proof.** Let $(u, v) \in T_\alpha$ for $\alpha \in (0, 1]$ (the case $\alpha = 0$ is trivial). Then, owing to Lemma 3.1, $(u, v) \in W_2 \times X_2^+$ and satisfies the following problem:
\begin{equation}
\begin{cases}
\int_0^T (\partial_t u, \varphi) + \int_0^T \int_\Omega \nabla u \cdot \nabla \varphi + \mu u_+ u \varphi & = \alpha \int_0^T \int_\Omega r u_+ \varphi - \kappa \int_0^T \int_\Omega u_+ \nabla v \cdot \nabla \varphi, \quad \forall \varphi \in L^2(\Omega), \\
\partial_t v - \Delta v + v = \alpha u_+ + \alpha f v_+ 1_{\Omega_e} \quad \text{a.e. in } Q,
\end{cases}
\end{equation}
endowed with the corresponding initial and boundary conditions. Therefore, it suffices to look for a bound of $(u, v)$ in $W_2 \times X_2^+$ independent of $\alpha$. This bound is carried out into six steps:

**Step 1:** Non-negativity: $u, v \geq 0$.

By taking in (3.6)$_1$ $\varphi = u_- := \min \{u, 0\} \leq 0$ (that is possible because $u \in L^2(\Omega)$), and considering that $u_- = 0$ if $u \geq 0$, $\nabla u_- = \nabla u$ if $u \leq 0$, and $\nabla u_- = 0$ if $u > 0$, we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|u_-\|^2 + \|\nabla u_-\|^2 = \kappa (u_+ \nabla v, \nabla u_-) + \alpha r (u_+, u_-) - \mu ((u_+)^2, u_-) = 0,
\end{equation}
thus $u_- \equiv 0$ and, consequently, $u \geq 0$. Similarly, testing (3.6)$_2$ by $v_- := \min \{v, 0\} \leq 0$ we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|v_-\|^2 + \|\nabla v_-\|^2 + \|v_-\|^2 = \alpha (u_+, v_-) + \alpha (f v_+, v_-) 1_{\Omega_e} \leq 0,
\end{equation}
which implies $v_- \equiv 0$, then $v \geq 0$. Therefore $(u_+, v_+) = (u, v)$. In particular, $(u, v, f)$ is also the solution of problem (3.6) changing $u_+$ by $u$ and $v_+$ by $v$. Therefore, fixed-point of $R$ are in particular weak solutions of problem (1.1).

**Step 2:** Boundedness of $\int_\Omega u(x, t) \, dx$.

By taking $\varphi = 1$ in (3.6)$_1$, we obtain:
\begin{equation}
\frac{d}{dt} \int_\Omega u(x, t) \, dx + \mu \int_\Omega u^2(x, t) \, dx = \alpha r \int_\Omega u(x, t) \, dx
\end{equation}
Using Cauchy-Schwartz inequality, we have:
\begin{equation}
\int_\Omega u(x, t) \, dx \leq |\Omega|^{1/2} \left( \int_\Omega u^2(x, t) \, dx \right)^{1/2},
\end{equation}
which from (3.7) let us deduce that:
\begin{equation}
\frac{d}{dt} \int_\Omega u(x, t) \, dx + \frac{\mu}{|\Omega|} \left( \int_\Omega u(x, t) \, dx \right)^2 \leq \alpha r \int_\Omega u(x, t) \, dx.
\end{equation}
Using the change of variable $y(t) = \int_\Omega u(x, t) \, dx$, (3.8) becomes:

$$y'(t) + \frac{\mu}{|\Omega|} y(t)^2 \leq \alpha r y(t) \leq r y(t) \tag{3.9}$$

which is related to a Bernoulli ODE. Making use of $z(t) = y(t)^{-1}$, we can deduce that:

$$-z'(t) + \frac{\mu}{|\Omega|} \leq r z(t),$$

and thus

$$z'(t) + r z(t) \geq \frac{\mu}{|\Omega|}. \tag{3.10}$$

This inequality is equivalent to:

$$\frac{d}{dt} \left(e^{rt} z(t)\right) \geq \frac{\mu}{|\Omega|} e^{rt} = \frac{\mu}{|\Omega|} \frac{d}{dt} \left(e^{rt}\right)$$

and therefore

$$z(t) \geq z(0) e^{-rt} + \frac{\mu}{r|\Omega|} \left(1 - e^{-rt}\right) = \frac{\mu}{r|\Omega|} + \left(z(0) - \frac{\mu}{r|\Omega|}\right) e^{-rt}. \tag{3.11}$$

Now, we consider two cases:

- if $z(0) \geq \frac{\mu}{r|\Omega|}$ (i.e., $y(0) \leq \frac{r|\Omega|}{\mu}$), then:

$$z(t) \geq \frac{\mu}{r|\Omega|},$$

which implies that:

$$y(t) \leq \frac{r|\Omega|}{\mu}, \quad \forall t \geq 0 \quad \text{(independently of } \alpha\text{).} \tag{3.12}$$

- if $z(0) \leq \frac{\mu}{r|\Omega|}$ (i.e., $y(0) \geq \frac{r|\Omega|}{\mu}$), then from (3.11) we can deduce that:

$$z(t) \geq z(0) e^{-rt} + \frac{\mu}{r|\Omega|} \left(1 - e^{-rt}\right) \geq z(0) e^{-rt} + z(0) \left(1 - e^{-rt}\right) = z(0)$$

and therefore

$$y(t) \leq y(0) = \int_\Omega u_0(x) \, dx = m_0, \quad \forall t \geq 0 \quad \text{(independently of } \alpha\text{).} \tag{3.13}$$

As a conclusion, from (3.12) and (3.13), we arrive at the bound

$$y(t) = \int_\Omega u(x, t) \, dx \leq \max\left\{ m_0, \frac{r|\Omega|}{\mu} \right\} := K_1, \quad \forall t \geq 0. \tag{3.14}$$

**Step 3:** Bound of $u$ in $L^2(0, T; L^2(\Omega))$

Integrating directly in $(0, T)$ for a fixed $T$ in (3.7), and using (3.14), we obtain that:

$$\int_0^T \int_\Omega u^2(x, t) \, dx \, dt \leq \frac{m_0 + \alpha r K_1 T}{\mu} \leq \frac{m_0 + r K_1 T}{\mu} := K_2(T), \tag{3.15}$$
which implies that

(3.16) \[ \|u\|_{L^2(Q)}^2 \leq K_2(T). \]

**Step 4:** Bound of \(v\) in \(L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))\)

Taking \(v\) as test function in (3.6)_2, we obtain:

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|v\|_{H^1}^2 = \alpha \int_{\Omega} u v \, dx + \alpha \int_{\Omega} f v^2 \, dx
\]

because \((\alpha \in (0,1))\)

\[
\leq \|u\|_{L^2} \|v\|_{L^2} + \|f\|_{L^2} \|v\|_{L^2}^2,
\]

\[
\leq \delta \|v\|_{H^1}^2 + C_\delta \left( \|u\|_{L^2}^2 + \|f\|_{L^2}^2 \|v\|_{L^2}^2 \right)
\]

where we used the following standard inequality in 2D domains

\[ \|u\|_{L^2} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2}, \quad \forall u \in H^1(\Omega). \]

Therefore, by taking \(\delta\) small enough,

(3.18) \[ \frac{d}{dt} \|v\|_{L^2}^2 + \|v\|_{H^1}^2 \leq C \left( \|u\|_{L^2}^2 + \|f\|_{L^2}^2 \|v\|_{L^2}^2 \right) \]

From Gronwall’s lemma, and thanks to the boundedness of \(u\) and \(f\) in \(L^2(Q)\), one has \(v\) bounded in \(L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))\).

By taking \(-\Delta v\) as test function in (3.6)_2, we obtain:

\[
\frac{1}{2} \frac{d}{dt} \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\nabla v\|_{H^1}^2 = -\alpha \int_{\Omega} u \Delta v \, dx - \alpha \int_{\Omega} f \Delta v \, dx
\]

(3.19)

\[ \leq \|u\|_{L^2} \|\Delta v\|_{L^2} + \|f\|_{L^2} \|\Delta v\|_{H^1} \|\Delta v\|_{L^2} \]

\[ \leq \delta \left( \|\Delta v\|_{L^2}^2 + \|v\|_{H^1}^2 \right) + C_\delta \left( \|u\|_{L^2}^2 + \|f\|_{L^2}^2 \|v\|_{H^1}^2 \right) \]

Adding (3.17) to (3.19), we obtain:

(3.20) \[ \frac{d}{dt} \|v\|_{H^1}^2 + \|v\|_{H^2}^2 \leq C \left( \|u\|_{L^2}^2 + \|f\|_{L^2}^2 + \|f\|_{L^2(\Omega)}^2 \right) \|v\|_{H^1}^2 \]

From Gronwall’s lemma, one has \(v\) bounded in \(L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))\).

**Step 5:** \(u\) is bounded in \(L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))\).

By testing (3.6)_1 by \(u\), after a few computations, we get,

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \mu \|u\|_{L^2}^2 + \|u\|_{L^2}^2
\]

\[ \leq \kappa \|u\|_{L^s} \|\nabla v\|_{L^s} \|\nabla u\|_{L^2} + (r_+ \alpha + 1) \|u\|_{L^2}^2
\]

\[ \leq C \|u\|_{L^2}^2 \|\nabla v\|_{L^s}^4 + \frac{1}{2} \|u\|_{H^1}^2 + (r_+ + 1) \|u\|_{L^2}^2,
\]
Then, we arrive at
\begin{equation}
\frac{d}{dt} \|u\|_{L^2}^2 + \|u\|_{H^1}^2 \leq C \|\nabla v\|_{L^2}^2 + 2 (r_+ + 1) \|u\|_{L^2}^2
\end{equation}
Therefore, applying the Gronwall lemma and using Step 4, we obtain that \(u\) is bounded in \(L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))\).

**Step 6:** \(v\) is bounded in \(L^\infty(0, T; W^{1,2+}(\Omega)) \cap L^{2+}(0, T; W^{2,2+}(\Omega))\). From **Step 4** we can deduce that \(v \in L^\infty(0, T; W^{1,2+}(\Omega))\) and, from **Step 5**, we can deduce that \(u \in L^4(\Omega)\). Therefore, \(u + f v \in L^{2+}(Q)\). Then, heat regularity result in Theorem 2.1, allows us to deduce that \(v \in X_{2+}\) and the corresponding bound on \(X_{2+}\) depending on \(\|v_0\|_{W^{1,2+}(\Omega)}\) and the bound of \(u + f v\) in \(L^{2+}(Q)\).

This finishes Lemma 3.2. \(\square\)

**Lemma 3.3.** The operator \(R : \mathcal{X}_u \times \mathcal{X}_v \to \mathcal{X}_u \times \mathcal{X}_v\), defined in (3.2), is continuous.

The proof is similar to Lemma 3.4 in [16].

Consequently, from Lemmas 3.1, 3.2 and 3.3, it follows that the operator \(R\) satisfy the conditions of the Leray-Schauder fixed-point theorem. Thus, we conclude that the map \(R(\bar{u}, \bar{v})\) has at least a fixed point, \(R(u, v) = (u, v)\), which is a weak solution to system (1.1) in \((0, T)\).

Finally, we observe that estimate (1.3) follows the same steps giving in the proof of Lemma 3.2 (now for the case \(\alpha = 1\)).

**3.2. Uniqueness of solution.** This proof follows the same argument than in [16], but it is included here for reader convenience. Let \((u_1, v_1), (u_2, v_2) \in W_2 \times X_2\) two weak solutions of system (1.1). Substracting equations (1.1) for \((u_1, v_1)\) and \((u_2, v_2)\), and denoting \(u := u_1 - u_2\) and \(v := v_1 - v_2\), we obtain the following system
\begin{equation}
\begin{cases}
\partial_t u - \Delta u + \kappa \nabla \cdot (u_1 \nabla v + u \nabla v_2) = r u - \mu u (u_1 + u_2) \text{ in } Q, \\
\partial_t v - \Delta v + v = u + f v 1_{\Omega_+} \text{ in } Q, \\
u(0, \cdot) = 0, v(0, \cdot) = 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial n} = 0, \frac{\partial v}{\partial n} = 0 \text{ on } (0, T) \times \partial \Omega.
\end{cases}
\end{equation}

Testing (3.22)_1 by \(u \in L^2(\Omega^1)\) and (3.22)_2 by \(-\Delta v \in L^2(Q)\) we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|u\|^2 + \|\nabla v\|^2 \right) + \|\nabla u\|^2 + \|\Delta v\|^2 + \|\nabla v\|^2 + \mu \int_\Omega u^2 (u_1 + u_2) dx = r \|u\|^2 + \kappa (u_1 \nabla v + u \nabla v_2, \nabla u) + (u + f v, -\Delta v).
\end{equation}

The term \(\mu \int_\Omega u^2 (u_1 + u_2) dx\) has the good sign.
Applying the Hölder and Young inequalities, we obtain
\[
(u_1 \nabla v, \nabla u) \leq ||u_1||_{L^4} ||\nabla v||_{L^4} ||\nabla u|| \leq C ||u_1||_{L^4} ||\nabla v||^{1/2} ||\nabla v||^{1/2} ||\nabla u||
\]
(3.24)
\[
(u \nabla v_2, \nabla u) \leq ||u||_{L^4} ||\nabla v_2||_{L^4} ||\nabla u|| \leq C ||u||^{1/2} ||u||^{1/2} ||\nabla v_2||_{L^4} ||\nabla u||
\]
(3.25)
\[
(u, -\Delta v) \leq \delta ||\Delta v||^2 + C_\delta ||u||^2,
\]
(3.26)
\[
(f v, -\Delta v) \leq ||f||_{L^2} ||v||_{H^2} ||\Delta v||_{L^2} \leq \delta ||v||^2_{H^2} + C_\delta ||f||^2_{L^2} ||v||^2_{H^1}.
\]
(3.27)
Adding (3.17) to (3.23), and using (3.24)-(3.27), we obtain:
\[
\frac{d}{dt} (||u||^2 + ||\nabla v||^2) + ||\nabla u||^2 + ||\nabla v||^2_{H^1}
\leq C \left(||u||^2 + ||u_1||^2_4 ||\nabla v||^2 + ||\nabla v_2||^2_4 ||u||^2 + ||f||^2_4 ||v||^2_{H^1}\right)
\]
(3.28)
In order to consider the completed norm for \(v\), we take \(v\) as test function in (3.22), we obtain:
\[
\frac{1}{2} \frac{d}{dt} (||v||^2) + ||v||^2_{H^1} = \int_{\Omega} u v \, dx + \int_{\Omega_c} f v^2 \, dx
\]
which implies:
\[
\frac{d}{dt} ||v||^2_{L^2} + ||v||^2_{H^1} \leq C \left(||u||^2_{L^2} + ||f||^2_{L^2} ||v||^2_{L^2}\right)
\]
(3.29)
Adding (3.28) to (3.29), we obtain:
\[
\frac{d}{dt} (||u||^2 + ||v||^2_{H^1}) + ||u||^2_{H^1} + ||v||^2_{H^2}
\leq C \left(||u||^2 + ||u_1||^2_4 ||\nabla v||^2 + ||\nabla v_2||^2_4 ||u||^2 + ||f||^2_4 ||v||^2_{H^1}\right)
\]
(3.30)
Since \(||u_1||^2_4 + ||\nabla v_2||^2_4 + ||f||^2_{L^2} \in L^1(0,T)\) and \(u_0 = v_0 = 0\), then Gronwall lemma implies uniqueness.
Thus, the proof of Theorem 1.2 is finished.

4. PROOF OF THEOREM 1.3

The admissible set for the optimal control problem (1.2) is defined by
\[
\mathcal{S}_{ad} = \{ s = (u, v, f) \in W_2 \times X_{2+} \times \mathcal{F} : s \text{ is a weak solution of } (1.1) \text{ in } (0,T) \}.
\]
From Theorem 1.2 one has \(\mathcal{S}_{ad} \neq \emptyset\). Let \(\{ s_m \}_{m \in \mathbb{N}} := \{ (u_m, v_m, f_m) \}_{m \in \mathbb{N}} \subset \mathcal{S}_{ad}\) be a minimizing sequence of \(J\), that is, \(\lim_{m \rightarrow +\infty} J(s_m) = \inf_{s \in \mathcal{S}_{ad}} J(s)\). Then, by definition of \(\mathcal{S}_{ad}\), for each \(m \in \mathbb{N}\), \(s_m\) satisfies system (3.7)_1 variationally in \(L^2((H^1)')\) and (3.7)_2 a.e. \((t,x) \in Q\).
From the definition of \(J\) and the assumption \(\gamma_f > 0\) or \(\mathcal{F}\) is bounded in \(L^{2+}(Q_c)\), it follows that
\[
\{ f_m \}_{m \in \mathbb{N}} \text{ is bounded in } L^{2+}(Q_c)
\]
From (3.4)-(3.5) there exists a positive constant $C$, independent of $m$, such that
\begin{align}
(4.2) \quad \|(u_m, v_m)\|_{W^2 \times X_{2+}} \leq C.
\end{align}

Therefore, from (4.1), (4.2), and taking into account that $F$ is a closed convex subset of $L^{2+}(Q)$, (hence is weakly closed in $L^{2+}(Q)$), it is deduced that there exists $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in W_2 \times X_{2+} \times F$ such that, for some subsequence of $\{s_m\}_{m \in \mathbb{N}}$, still denoted by $\{s_m\}_{m \in \mathbb{N}}$, the following convergences hold, as $m \to +\infty$:
\begin{align}
(4.3) \quad u_m & \to \tilde{u} \quad \text{weakly in } L^2(H^1) \text{ and weakly}^* \text{ in } L^\infty(L^2), \\
(4.4) \quad v_m & \to \tilde{v} \quad \text{weakly in } L^{2+}(W^{2,2+}) \text{ and weakly}^* \text{ in } L^\infty(W^{1+2+}), \\
(4.5) \quad \partial_t u_m & \to \partial_t \tilde{u} \quad \text{weakly in } L^2((H^1)\prime), \\
(4.6) \quad \partial_t v_m & \to \partial_t \tilde{v} \quad \text{weakly in } L^{2+}(Q), \\
(4.7) \quad f_m & \to \tilde{f} \quad \text{weakly in } L^{2+}(Q), \quad \text{and } \tilde{f} \in F.
\end{align}

From (4.3)-(4.6), and using Sobolev embedding and Aubin-Lions compactness results, one has
\begin{align}
(4.8) \quad (u_m, v_m) & \to (\tilde{u}, \tilde{v}) \quad \text{strongly in } C^0([0, T]; ((H^1(\Omega))\prime \times L^2(\Omega)) \\
(4.9) \quad v_m & \to \tilde{v} \quad \text{strongly in } L^\infty(Q). \\
(4.10) \quad (u_m, \nabla v_m) & \to (\tilde{u}, \nabla \tilde{v}) \quad \text{strongly in } L^4(Q) \times L^{4+}(Q).
\end{align}

In particular, using (4.7), (4.9) and (4.10) the limit of the nonlinear terms of (3.7) can be controlled as follows
\begin{align}
(4.11) \quad u_m \cdot \nabla v_m & \to \tilde{u} \cdot \nabla \tilde{v} \quad \text{strongly in } L^2(Q), \\
(4.12) \quad f_m v_m 1_{\Omega_c} & \to \tilde{f} \tilde{v} 1_{\Omega_c} \quad \text{weakly in } L^{2+}(Q).
\end{align}

Moreover, from (4.8), $(u_m(0), v_m(0))$ converges to $(\tilde{u}(0), \tilde{v}(0))$ in $H^1(\Omega) \times L^2(\Omega)$, and since $u_m(0) = u_0$, $v_m(0) = v_0$, it is deduced that $\tilde{u}(0) = u_0$ and $\tilde{v}(0) = v_0$. Thus, $\tilde{s}$ satisfies the initial conditions given in (1.1). Therefore, considering the convergences (4.3)-(4.12), and taking the limit in equation (3.6) replacing $(u, v, f)$ by $(u_m, v_m, f_m)$, as $m$ goes to $+\infty$, it is possible to conclude that $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f})$ is a weak solution of the system (1.1), that is, $\tilde{s} \in \mathcal{S}_{ad}$. Therefore,
\begin{align}
(4.13) \quad \lim_{m \to +\infty} J(s_m) = \inf_{s \in \mathcal{S}_{ad}} J(s) \leq J(\tilde{s}).
\end{align}

Additionally, since $J$ is lower semicontinuous on $\mathcal{S}_{ad}$, one has $J(\tilde{s}) \leq \liminf_{m \to +\infty} J(s_m)$, which jointly to (4.13), implies that $\tilde{s}$ is a global optimal control.

5. Proof of Theorem 1.4

5.1. A generic Lagrange multipliers theorem.

We introduce a Lagrange multipliers theorem given by J. Zowe and S. Kurcyusz [35] (see also [32, Chapter 6], for more details) that we will apply to get first-order
necessary optimality conditions for a local optimal solution $(\hat{u}, \hat{v}, \hat{f})$ of problem (1.2). First, we consider the following (generic) optimization problem:

\[(5.1) \quad \min_{s \in \mathbb{M}} J(s) \quad \text{subject to} \quad G(s) = 0,\]

where $J : \mathbb{X} \to \mathbb{R}$ is a functional, $G : \mathbb{X} \to \mathbb{Y}$ is an operator, $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces, and $\mathbb{M}$ is a nonempty closed and convex subset of $\mathbb{X}$. The corresponding admissible set for problem (5.1) is

$$\mathcal{S} = \{ s \in \mathbb{M} : G(s) = 0 \}.$$ 

**Definition 5.1.** *(Lagrangian)* The functional $\mathcal{L} : \mathbb{X} \times \mathbb{Y}' \to \mathbb{R}$, given by

\[(5.2) \quad \mathcal{L}(s, \xi) = J(s) - \langle \xi, G(s) \rangle_{\mathbb{Y}'},\]

is called the Lagrangian functional related to problem (5.1).

**Definition 5.2.** *(Lagrange multiplier)* Let $\hat{s} \in \mathcal{S}$ be a local optimal solution for problem (5.1). Suppose that $J$ and $G$ are Fréchet differentiable in $\hat{s}$. Then, any $\xi \in \mathbb{Y}'$ is called a Lagrange multiplier for (5.1) at the point $\hat{s}$ if

\[(5.3) \quad \mathcal{L}'(\hat{s}, \xi)[r] = J'(\hat{s})[r] - \langle \xi, G'(\hat{s})[r] \rangle_{\mathbb{Y}'} \geq 0 \quad \forall r \in \mathcal{C}(\hat{s}),\]

where $\mathcal{C}(\hat{s}) = \{ \theta(\hat{s} - s) : s \in \mathbb{M}, \theta \geq 0 \}$ is the conical hull of $\hat{s}$ in $\mathbb{M}$.

**Definition 5.3.** Let $\hat{s} \in \mathcal{S}$ be a local optimal solution for problem (5.1). It will be said that $\hat{s}$ is a regular point if

$$G'(\hat{s})[\mathcal{C}(\hat{s})] = \mathbb{Y}.$$ 

**Theorem 5.4.** *(\cite{32}, Theorem 6.3, p.330, \cite{35}, Theorem 3.1)* Let $\hat{s} \in \mathcal{S}$ be a local optimal solution for problem (5.1). Suppose that $J$ is Fréchet differentiable in $\hat{s}$, and $G$ is continuous Fréchet-differentiable in $\hat{s}$. If $\hat{s}$ is a regular point, then there exists Lagrange multipliers for (5.1) at $\hat{s}$.

5.2. **Application of the Lagrange multiplier theory.** Now, in order to reformulate the optimal control problem (1.2) in the abstract setting (5.1), we introduce the Banach spaces

$$\mathbb{X} := \hat{W}_2 \times \hat{X}_{2+} \times L^{2+}(Q), \quad \mathbb{Y} := L^2(\mathbb{H}^1)' \times L^{2+}(Q),$$

where

$$\hat{W}_2 = \{ u \in W_2 : u(0) = 0 \}, \quad \hat{X}_{2+} = \{ v \in X_{2+} : v(0) = 0, \partial_n v|_{\partial \Omega} = 0 \}$$

and the operator $G = (G_1, G_2) : \mathbb{X} \to \mathbb{Y}$, where

$$G_1 : \mathbb{X} \to L^2(\mathbb{H}^1)', \quad G_2 : \mathbb{X} \to L^{2+}(Q)$$
are defined at each point \( s = (u, v, f) \in \mathbb{X} \) by

\[
\begin{aligned}
\langle G_1(s), \varphi \rangle &= \langle \partial_t u, \varphi \rangle_{L^2(H^1)} + (\nabla u - \kappa u \nabla v, \nabla \varphi)_{L^2} \\
&= +(r^* u + \mu u^2, \varphi)_{L^2} \quad \forall \varphi \in L^2(H^1) \\
G_2(s) &= \partial_t v - \Delta v + v - u - f v 1_{\Omega_c} \quad \text{in } L^2(\Omega).
\end{aligned}
\]

Thus, the optimal control problem (1.2) is reformulated as follows

\[
(5.4) \quad \min_{s \in \mathbb{M}} J(s) \quad \text{subject to} \quad G(s) = 0,
\]

where

\[
M := (\hat{u}, \hat{v}, 0) + \hat{W}_2 \times \hat{X}_2 \times F,
\]

with \((\hat{u}, \hat{v})\) the global weak solution of (1.1) without control \((\hat{f} = 0)\) and \(F\) is defined in (1.1).

**Remark 5.5.** From Definition 5.1, the Lagrangian associated to optimal control problem (5.4) is the functional \(\mathcal{L} : \mathbb{X} \times L^2(H^1) \times L^2(\Omega) \to \mathbb{R}\) given by

\[
\mathcal{L}(s, \lambda, \eta) = J(s) - \langle \lambda, G_1(s) \rangle_{L^2(H^1)} - \langle \eta, G_2(s) \rangle_{L^2(\Omega)}.
\]

The set \(\mathbb{M}\) defined in (5.5) is a closed convex subset of \(\mathbb{X}\) and the admissible set of control problem (5.4) is

\[
(5.6) \quad \mathcal{S}_{ad} = \{ s = (u, v, f) \in \mathbb{X} : G(s) = 0 \}.
\]

Concerning to the differentiability of the functional \(J\) and the constraint operator \(G\), one has the following results.

**Lemma 5.6.** The functional \(J : \mathbb{X} \to \mathbb{R}\) is Fréchet differentiable and the Fréchet derivative of \(J\) in \(\hat{s} = (\hat{u}, \hat{v}, \hat{f}) \in \mathbb{X}\) in the direction \(r = (U, V, F) \in \mathbb{X}\) is

\[
(5.7) \quad J'(\hat{s})[r] = \gamma_u \int_0^T \int_\Omega (\hat{u} - u_d) U + \gamma_v \int_0^T \int_\Omega (\hat{v} - v_d) V \\
+ \gamma_f \int_0^T \int_{\Omega_c} \text{sgn}(\hat{f}) |\hat{f}|^{1+} F.
\]

**Lemma 5.7.** The operator \(G : \mathbb{X} \to \mathbb{Y}\) is continuous-Fréchet differentiable and the Fréchet derivative of \(G\) in \(\hat{s} = (\hat{u}, \hat{v}, \hat{f}) \in \mathbb{X}\) in the direction \(r = (U, V, F) \in \mathbb{X}\) is the linear operator \(G'(\hat{s})[r] = (G'_1(\hat{s})[r], G'_2(\hat{s})[r])\) defined by

\[
(5.8) \quad \begin{cases}
\langle G'_1(\hat{s})[r], \varphi \rangle = \langle \partial_t U, \varphi \rangle + (\nabla U - \kappa U \nabla \hat{v} - \kappa \hat{u} \nabla V, \nabla \varphi) \\
+ (-r U + 2\mu \hat{u} U, \varphi), \quad \forall \varphi \in L^2(H^1) \\
G'_2(\hat{s})[r] = \partial_t V - \Delta V + V - U - \hat{f} V 1_{\Omega_c} - F \hat{v} 1_{\Omega_c}.
\end{cases}
\]
Remark 5.8. From Definition 5.3 one has that \( \tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in S_{ad} \) is a regular point if for any \((g_u, g_v) \in Y\) there exists \( r = (U, V, F) \in \tilde{W}_2 \times \tilde{X}_{2+} \times C(f) \) such that

\[
G'(\tilde{s})[r] = (g_u, g_v),
\]

where \( C(f) := \{ \theta(f - \tilde{f}) : \theta \geq 0, f \in F \} \) is the conical hull of \( \tilde{f} \) in \( F \).

5.3. The linearized problem (5.8) is surjective.

Lemma 5.9. Let \( \tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in S_{ad} \) (\( S_{ad} \) defined in (5.6)), then \( \tilde{s} \) is a regular point.

Proof. Fixed \((\tilde{u}, \tilde{v}, \tilde{f}) \in S_{ad}, \) let \((g_u, g_v) \in L^2((H^1)' \times L^{2+}(Q)).\) Since \( 0 \in C(\tilde{f}) = \{ \theta(f - \tilde{f}) : \theta \geq 0, f \in F \}, \) it suffices to show the existence of \((U, V) \in \tilde{W}_2 \times \tilde{X}_{2+}\) solving the linear problem

\[
\begin{aligned}
\langle \partial_t U, \varphi \rangle + (\nabla U - \kappa U \nabla \tilde{v} - \kappa \tilde{u} \nabla V, \nabla \varphi) \\
+ (-r U + 2 \mu \tilde{u} U, \varphi) = \langle g_u, \varphi \rangle \quad \forall \varphi \in L^2(H^1), \\
\partial_t V - \Delta V + V - \tilde{f} V 1_{\Omega_c} = g_v \quad \text{in } L^{2+}(Q).
\end{aligned}
\]

For this, we will use the Leray-Schauder fixed-point Theorem, for the operator

\[
S : (U, V) \in L^4(Q) \times L^\infty(Q) \mapsto (U, V) \in \tilde{W}_2 \times \tilde{X}_{2+}
\]

where \((U, V)\) is the solution of the decoupled problem (first computing \(V\) and after \(U)\)

\[
\begin{aligned}
\langle \partial_t U, \varphi \rangle + (\nabla U - \kappa U \nabla \tilde{v} - \kappa \tilde{u} \nabla V, \nabla \varphi) \\
= (r U - 2 \mu \tilde{u} U, \varphi) + \langle g_u, \varphi \rangle \quad \forall \varphi \in L^2(H^1), \\
\partial_t V - \Delta V + U = \tilde{f} V 1_{\Omega_c} + g_v \quad \text{in } Q.
\end{aligned}
\]

Step 1 (\(S\) is well-defined and bounded): Prove that operator \(S\) defined in (5.10) maps bounded sets in \(L^4(Q) \times L^\infty(Q)\) in bounded sets in \((U, V) \in \tilde{W}_2 \times \tilde{X}_{2+}.\)

For this, one first bound \(V\) and later bound \(U.\) Indeed, since \((U, V) \in L^4(Q) \times L^\infty(Q)\) implies \(f V \in L^{2+}(Q),\) then by applying \(L^{2+}\)-regularity to the heat equation (5.11) (Theorem 2.1), it is deduced that \(V \in X_{2+}\) and

\[
\|V\|_{X_{2+}} \leq C \left( \|U\|_{L^2} + \|f V 1_{\Omega_c}\|_{L^{2+}(Q)} + \|g_v\|_{L^{2+}(Q)} \right)
\]

(5.12)

By taking \(\varphi = U\) in (5.11)1, we arrive at

\[
\|U\|_{L^2}^2 + \|U\|_{L^2}^2 \leq C_1 (1 + \|\nabla \tilde{v}\|_{L^4}^4) \|U\|_{L^2}^2
\]

\[
+ C_2 \left( \|\tilde{u}\|_{L^4}^4 + (1 + \|\tilde{u}\|_{L^4}^4) \|f V 1_{\Omega_c}\|_{L^2}^2 + \|g_v\|_{L^{2+}(Q)} \right)
\]

(5.13)

Finally, using 2D interpolation estimates,

\[
\|\tilde{u}\|_{L^4(Q)}^2 \|\nabla V\|_{L^4(Q)}^2 \leq \|\tilde{u}\|_{L^2}^2 \|V\|_{X_2}^2
\]
Then, using (5.12), the Gronwall Lemma applied to (5.13) guarantees the bound for U in W₂.

**Step 2 (compactness):** By using that W₂ × X₂⁺ is compactly embedded in L⁴⁻(Q) × L∞(Q), then operator S is compact.

**Step 3 (continuity):** In particular, using Steps 1 and 2, it is not difficult to prove the continuity of S from L⁴⁻(Q) × L∞(Q) to itself.

**Step 4 (boundedness of possible fixed-points):** Now, the aim is to show that the set of the possible fixed-points of αS with α ∈ [0, 1] defined as Sα := { (U, V) ∈ ¯W₂ × ¯X₂⁺ : (U, V) = αS(U, V) for some α ∈ [0, 1] } is bounded in L⁴⁻(Q) × L∞(Q) (with respect to α). Indeed, if (U, V) ∈ Sα, then (U, V) ∈ ¯W₂ × ¯X₂⁺ and solves the coupled linear problem

\[
\begin{align*}
\partial_t U - \Delta U &= \alpha(rU - 2\mu \bar{u}U, \varphi) + \alpha(g_u, \varphi) & \forall \varphi \in L^2(H^1), \\
\partial_t V - \Delta V &= \alpha U + \alpha \tilde{f}V1_{\Omega_e} + \alpha g_v & \text{in } Q.
\end{align*}
\]

Then, taking φ = U in (5.14), one obtains (see (5.13)):

\[
\frac{d}{dt} \|U\|_{H^1}^2 + \|\nabla U\|_{L^2}^2 + 2\alpha \mu \int_{\Omega_e} \bar{u} U^2 \leq C (\alpha + \|\nabla \bar{u}\|_{L^4}^4) \|U\|_{H^1}^2 + C \|\bar{u}\|_{L^4}^4 \|\nabla V\|_{L^2}^2 + \alpha^2 \|g_u\|_{L^4}^4.
\]

Now, testing (5.14)₂ by V − ΔV ∈ L²⁺(Q), it holds:

\[
\frac{d}{dt} \|V\|_{H^1}^2 + \|V\|_{H^2}^2 \leq C \alpha^2 \|f\|_{L^2}^2 \|V\|_{H^1}^2 + \alpha^2 \left( \|g_v\|_{L^4}^2 + \|U\|_{L^2}^2 \right).
\]

From (5.15) and (5.16) and using that α ≤ 1:

\[
\frac{d}{dt} (\|U\|_{H^1}^2 + \|V\|_{H^2}^2) + \|U\|_{H^1}^2 + \|V\|_{H^2}^2 \leq C (1 + \|\nabla \bar{u}\|_{L^4}^4) \|U\|_{H^1}^2 + C \left( \|f\|_{L^2}^2 + \|\bar{u}\|_{L^4}^4 \right) \|V\|_{H^1}^2 + C \left( \|g_u\|_{L^4}^2 + \|g_v\|_{L^4}^2 \right).
\]

Using that U(0) = V(0) = 0 and \|g_u\|_{L^2(L^4)}^2, \|g_v\|_{L^2(L^2)}, \|f\|_{L^2}, \|\bar{u}\|_{L^4(\Omega)} and \|\nabla \bar{u}\|_{L^4(\Omega)} are constant finite values, then the Gronwall Lemma implies that

\[
\|(U, V)\|_{W₂ × X₂} \leq C.
\]

Finally, by applying L²⁺-regularity provided by Theorem 2.1 related to the parabolic-Neumann problem, one has

\[
\|V\|_{X₂⁺} \leq C.
\]
Step 5: Conclusion: By applying Leray-Schauder fixed-point theorem (Theorem 2.2), one has the existence of \((U, V) \in W_2 \times X_{2+}\) a solution of problem (5.9). The uniqueness of solution is directly deduced from the linearity of problem (5.9). □

5.4. **Existence of Lagrange multipliers.** Now, the existence of Lagrange multiplier for problem (1.2) associated to any local optimal solution \(\hat{s} = (\hat{u}, \hat{v}, \hat{f}) \in \mathcal{S}_{ad}\) will be shown.

**Theorem 5.10.** Let \(\hat{s} = (\hat{u}, \hat{v}, \hat{f}) \in \mathcal{S}_{ad}\) be a local optimal solution for the control problem (1.2). Then, there exists a Lagrange multiplier \(\xi = (\lambda, \eta) \in L^2(H^1) \times L^2(Q)\) such that for all \((U, V, F) \in \widehat{W}_2 \times \widehat{X}_{2+} \times \mathcal{C}(\hat{f})\)

\[
\begin{align*}
\gamma_u \int_0^T \int_{\Omega} (\hat{u} - u_d)U &+ \gamma_v \int_0^T \int_{\Omega} (\hat{v} - v_d)V + \gamma_f \int_0^T \int_{\Omega_c} \text{sgn}(\hat{f})|\hat{f}|^{1+}F \\
- \int_0^T \int_{\Omega} \langle \partial_t U, \lambda \rangle - \int_0^T \int_{\Omega} (\nabla U - \kappa U\nabla \hat{v} - \kappa \hat{u}\nabla V, \nabla \lambda) + (-rU + 2\mu \hat{u}U, \lambda) &-
\end{align*}
\]

(5.18) \[- \int_0^T \int_{\Omega} \left( \partial_t V - \Delta V + V - \hat{f}V1_{\Omega_c} \right) \eta + \int_0^T \int_{\Omega_c} F\hat{v}1_{\Omega_c} \eta \geq 0.
\]

**Proof.** From Lemma 5.9, \(\hat{s} \in \mathcal{S}_{ad}\) is a regular point, then from Theorem 5.4 there exists a Lagrange multiplier \(\xi = (\lambda, \eta) \in L^2(H^1) \times L^2(Q)\) such that by (5.3) and Remark 5.5 one must satisfy

\[
(5.19) \quad \mathcal{L}'_2(s, \lambda, \eta)[r] = J'(\hat{s})[r] - \langle \lambda, G'_1(\hat{s})[r] \rangle_{L^2(H^1), L^2(H^1)} - \langle \eta, G'_2(\hat{s})[r] \rangle_{L^2} \geq 0,
\]

for all \(r = (U, V, F) \in \widehat{W}_2 \times \widehat{X}_{2+} \times \mathcal{C}(\hat{f}).\) The proof follows from (5.7), (5.8), and (5.19). □

From Theorem 5.10, an optimality system for problem (1.2) can be derived.

**Corollary 5.11.** Let \(\hat{s} = (\hat{u}, \hat{v}, \hat{f}) \in \mathcal{S}_{ad}\) be a local optimal solution for the control problem (1.2). Then any Lagrange multiplier \((\lambda, \eta) \in L^2(H^1) \times L^2(Q),\) provided by Theorem 5.10, satisfies the system

\[
\begin{align*}
\int_0^T \langle \partial_t U, \lambda \rangle + \int_0^T \int_{\Omega} (\nabla U - \kappa U\nabla \hat{v}) \cdot \nabla \lambda + (-rU + 2\mu \hat{u}U, \lambda) - \int_0^T \int_{\Omega} U \eta \\
= \gamma_u \int_0^T \int_{\Omega} (\hat{u} - u_d)U, \quad \forall U \in \widehat{W}_2, \\
\int_0^T \int_{\Omega} \left( \partial_t V - \Delta V + V \right) \eta - \int_0^T \int_{\Omega_c} \hat{f}V \eta + \kappa \int_0^T \int_{\Omega_c} \hat{u}\nabla V \cdot \nabla \lambda \\
= \gamma_v \int_0^T \int_{\Omega} (\hat{v} - v_d)V, \quad \forall V \in \widehat{X}_{2+},
\end{align*}
\]

(5.20) \[(5.21)
\]

and the optimality condition

\[
\int_0^T \int_{\Omega_c} (\gamma_f \text{sgn}(\hat{f})|\hat{f}|^{1+} + \hat{v}\eta)(f - \hat{f}) \geq 0, \quad \forall f \in \mathcal{F}.
\]

(5.22)
Proof. From (5.18), taking \((V, F) = (0, 0)\), and using that \(\widehat{W}_2\) is a vectorial space, (5.20) holds. Similarly, taking \((U, F) = (0, 0)\) in (5.18), and taking into account that \(\overline{X}_{2+}\) is a vectorial space, (5.21) is deduced. Finally, taking \((U, V) = (0, 0)\) in (5.18) it holds
\[
\gamma_f \int_0^T \int_{\Omega_c} \text{sgn}(\tilde{f})\overline{\tilde{f}}^{1+} + \int_0^T \int_{\Omega_c} \tilde{\nu} \eta F \geq 0, \quad \forall F \in C(\tilde{f}).
\]
Thus, choosing \(F = \theta(f - \tilde{f}) \in C(\tilde{f})\) for all \(f \in F\) and \(\theta \geq 0\), (5.22) is deduced. \(\square\)

Remark 5.12. A pair \((\lambda, \eta) \in L^2(H^1) \times L^{2-}(Q)\) satisfying (5.20)-(5.21) corresponds to the concept of very weak solution (at least for the \(\eta\)-variable) of the linear problem
\[
\begin{align*}
-\partial_t \lambda - \Delta \lambda - \kappa \nabla \lambda \cdot \nabla \tilde{\nu} - \eta - r \lambda + 2 \mu \tilde{\nu} \lambda &= \gamma_u(\tilde{u} - u_d) \quad \text{in } Q, \\
-\partial_t \eta - \Delta \eta + \eta - \kappa \nabla \cdot (\tilde{u} \nabla \lambda) - \tilde{\nu} \eta 1_{\Omega_c} &= \gamma_v(\tilde{v} - v_d) \quad \text{in } Q, \\
\lambda(T) = 0, \quad \eta(T) &= 0 \quad \text{in } \Omega, \\
\frac{\partial \lambda}{\partial n} = 0, \quad \frac{\partial \eta}{\partial n} &= 0 \quad \text{on } (0, T) \times \partial \Omega.
\end{align*}
\]
\[\text{(5.23)}\]

5.5. Regularity of Lagrange multipliers.

Theorem 5.13. Let \(\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}\) be a local optimal solution for the problem (1.2). Then the problem (5.23) has a unique solution \((\lambda, \eta)\) such that
\[
(\lambda, \eta) \in X_2 \times W_2
\]
\[\text{(5.24)}\]

Proof. Let \(s = T - t\), with \(t \in (0, T)\) and \(\lambda(s) = \lambda(t), \tilde{\eta}(s) = \eta(t)\). Then, system (5.23) is equivalent to
\[
\begin{align*}
\partial_t \tilde{\lambda} - \Delta \tilde{\lambda} - \kappa \nabla \tilde{\lambda} \cdot \nabla \tilde{\nu} - \tilde{\eta} - r \tilde{\lambda} + 2 \mu \tilde{\nu} \tilde{\lambda} &= \gamma_u(\tilde{u} - u_d) \quad \text{in } Q, \\
\partial_t \tilde{\eta} - \Delta \tilde{\eta} + \tilde{\eta} - \kappa \nabla \cdot (\tilde{u} \nabla \tilde{\lambda}) - \tilde{\nu} \tilde{\eta} \eta 1_{\Omega_c} &= \gamma_v(\tilde{v} - v_d) \quad \text{in } Q, \\
\tilde{\lambda}(0) = 0, \quad \tilde{\eta}(0) &= 0 \quad \text{in } \Omega, \\
\frac{\partial \tilde{\lambda}}{\partial n} = 0, \quad \frac{\partial \tilde{\eta}}{\partial n} &= 0 \quad \text{on } (0, T) \times \partial \Omega.
\end{align*}
\]
\[\text{(5.25)}\]

In order to prove the existence of a solution for (5.25), the Leray-Schauder fixed-point Theorem can be applied as before, now over the operator
\[
\widehat{T}: (\tilde{\lambda}, \tilde{\eta}) \in L^\infty \times L^{4-} \mapsto (\lambda, \eta) \in X_2 \times W_2
\]
where \((\lambda, \eta) = \widehat{T}(\tilde{\lambda}, \tilde{\eta})\) solves the decoupled problem (first computing \(\lambda\) and after \(\mu\):
\[
\begin{align*}
\partial_t \lambda - \Delta \lambda - \kappa \nabla \lambda \cdot \nabla \tilde{\nu} &= \tilde{\eta} + r \tilde{\lambda} + 2 \mu \tilde{\nu} \tilde{\lambda} + \gamma_u(\tilde{u} - u_d) \quad \text{in } Q, \\
\partial_t \eta - \Delta \eta + \eta - \kappa \nabla \cdot (\tilde{u} \nabla \lambda) &= \tilde{\nu} \eta \eta 1_{\Omega_c} + \gamma_v(\tilde{v} - v_d) \quad \text{in } Q, \\
\lambda(0) = 0, \quad \eta(0) &= 0 \quad \text{in } \Omega, \\
\frac{\partial \lambda}{\partial n} = 0, \quad \frac{\partial \eta}{\partial n} &= 0 \quad \text{on } (0, T) \times \partial \Omega,
\end{align*}
\]
\[\text{(5.27)}\]
The proof follows the same lines and it will be omitted. Indeed, the key point is to show that the set of possible fixed-points

$$\tilde{T}_\alpha := \{(\lambda, \eta) \in X_2 \times W_2 : (\lambda, \eta) = \alpha \tilde{T}(\lambda, \eta) \text{ for some } \alpha \in [0, 1] \}$$

is bounded in $X_2 \times W_2$ (with respect to $\alpha$). In fact, if $(\lambda, \eta) \in \tilde{T}_\alpha$, then $(\lambda, \eta) \in X_2 \times W_2$ and solves the coupled linear problem:

$$\begin{cases}
\partial_s \lambda - \Delta \lambda + \kappa \nabla \lambda \cdot \nabla \tilde{v} - \alpha r \lambda + 2\alpha \mu \tilde{u} \lambda - \alpha \eta = \alpha \gamma_u (\tilde{u} - u_d) & \text{in } Q, \\
\partial_s \eta - \Delta \eta + \eta - \tilde{f} \eta 1_{\Omega_e} - \kappa \nabla \cdot (\tilde{u} \nabla \lambda) = \alpha \gamma_v (\tilde{v} - v_d) & \text{in } Q, \\
\lambda(0) = 0, \eta(0) = 0 & \text{in } \Omega, \\
\frac{\partial \lambda}{\partial n} = 0, \frac{\partial \eta}{\partial n} = 0 & \text{on } (0, T) \times \partial \Omega.
\end{cases}$$

(5.28)

Now, by taking $\lambda - \Delta \lambda \in L^2(Q)$ as test function in (5.28)$_1$ and $\eta \in L^2(H^1)$ as test function in (5.28)$_2$, then the following bound is obtained via the Gronwall Lemma

$$\|(\lambda, \eta)\|_{X_2 \times W_2} \leq C(\|\tilde{u}\|_{W_2}, \|\tilde{v}\|_{X_2}, \|	ilde{f}\|_{L^2(Q)}, \|u_d\|_{L^2(Q)}, \|v_d\|_{L^2(Q)}).$$

Therefore, by applying Leray-Schauder fixed-point theorem, the existence of a solution of problem (5.23), $(\lambda, \eta) \in X_2 \times W_2$, is obtained. The uniqueness of solution is directly deduced from the linearity of problem (5.23).

Theorem 5.14. Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in S_{ad}$ be a local optimal solution for the control problem (1.2). Then the Lagrange multiplier, provided by Theorem 5.10, is unique and satisfies $(\lambda, \eta) \in X_2 \times W_2$.

Proof. Let $(\lambda, \eta) \in L^2(H^1) \times L^2(\Omega)$ be a Lagrange multiplier given in Theorem 5.10, which is a very weak solution of problem (5.23). In particular, $(\lambda, \eta)$ satisfies (5.20)-(5.21). On the other hand, from Theorem 5.13, system (5.23) has a unique solution $(\tilde{\lambda}, \tilde{\eta}) \in X_2 \times W_2$. Then, it suffices to identify $(\lambda, \eta)$ with $(\tilde{\lambda}, \tilde{\eta})$.

With this objective, for any $(U, V) \in \tilde{W}_2 \times \tilde{X}_{2+}$, we write (5.23) for $(\tilde{\lambda}, \tilde{\eta})$ (instead of $(\lambda, \eta)$), testing the first equation by $U$, and the second one by $V$, and integrating by parts in $\Omega$, it is obtained

$$\begin{align*}
\int_0^T \langle \partial_t U, \tilde{\lambda} \rangle + \int_0^T \int_\Omega (\nabla U - \kappa U \nabla \tilde{v}) \cdot \nabla \tilde{\lambda} + (-rU + 2\mu \tilde{u} U) \tilde{\lambda} - U \tilde{\eta} &= \gamma_u \int_0^T \int_\Omega (\tilde{u} - u_d) U, \\
\int_0^T \int_\Omega (\partial_t V - \Delta V + V - \tilde{f} V 1_{\Omega_e}) \tilde{\eta} + \kappa \tilde{u} \nabla V \cdot \nabla \tilde{\lambda} &= \gamma_v \int_0^T \int_\Omega (\tilde{v} - v_d) V.
\end{align*}$$

(5.29) (5.30)
Making the difference between (5.20) for \((\lambda, \eta)\) and (5.29) for \((\overline{\lambda}, \overline{\eta})\), and between (5.21) and (5.30), and then adding the respective equations, since the right-hand side terms vanish, it can be deduced
\[
\int_0^T \langle \partial_t U, \lambda - \overline{\lambda} \rangle_{(H^1)'} + \int_0^T \int_\Omega (\nabla U - \kappa U \nabla \hat{v} - \kappa \hat{u} \nabla V) \cdot \nabla (\lambda - \overline{\lambda}) \\
+ \int_0^T \int_\Omega (-\tau U + 2\mu \hat{u} U, \lambda - \overline{\lambda}) \\
+ \int_0^T \int_\Omega \left( \partial_t V - \Delta V + V - U - \hat{f} V 1_{\Omega_c} \right) (\eta - \overline{\eta}) = 0.
\]

Then, if \((U, V) \in \tilde{W}^1 \times \tilde{X}^2_{2+}\) is the unique solution of linear system (5.9) associated to any \((g_u, g_v) \in L^2((H^1)') \times L^2(Q)\) (given by Lemma 5.9), we arrive at
\[
(5.31) \int_0^T \langle g_u, \lambda - \overline{\lambda} \rangle_{(H^1)'} + \int_0^T \int_\Omega g_v (\eta - \overline{\eta}) = 0.
\]

By density arguments, it is easy to deduce that \(\lambda - \overline{\lambda} = 0\) and \(\eta - \overline{\eta} = 0\), which implies that \((\lambda, \eta) = (\overline{\lambda}, \overline{\eta})\). As a consequence of the regularity of \((\overline{\lambda}, \overline{\eta})\), it holds that \((\lambda, \eta) \in X_2 \times W_2\).

All previous arguments of Section 5 prove Theorem 1.4.

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(P. Braz e Silva) Departamento de Matemática, Universidade Federal de Pernambuco, CEP 50740-560, Recife - PE, Brazil

Email address: pablo.braz@ufpe.br

(F. Guillén-González) Dpto. de Ecuaciones Diferenciales y Análisis Numérico and IMUS, Universidad de Sevilla, Sevilla, Spain

Email address: guillen@us.es

(C. F. Perusato) Departamento de Matemática, Universidade Federal de Pernambuco, CEP 50740-560, Recife - PE, Brazil

Email address: cilon.perusato@ufpe.br

(M.A. Rodríguez-Bellido) Dpto. de Ecuaciones Diferenciales y Análisis Numérico and IMUS, Universidad de Sevilla, Sevilla, Spain

Email address: angeles@us.es