Classical instability of Kerr-AdS black holes and the issue of final state

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It is now established that small Kerr–Anti-de Sitter (Kerr-AdS) black holes are unstable against scalar perturbations, via superradiant amplification mechanism. We show that small Kerr-AdS black holes are also unstable against gravitational perturbations and we compute the features of this instability. We also describe with great detail the evolution of this instability. In particular, we identify its endpoint state. It corresponds to a Kerr-AdS black hole whose boundary is an Einstein universe rotating with the light velocity. This black hole is expected to be slightly oblate and to co-exist in equilibrium with a certain amount of outside radiation.

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I. INTRODUCTION

Black holes in an anti-de Sitter (AdS) background have attracted a great deal of attention over the last decade. One of the reasons for this intense study is the AdS/CFT correspondence [1, 2], stating that there is a duality between supergravity on $\text{AdS} \times M$ (with $M$ a compact manifold) and an appropriate conformal field theory (CFT) defined on the boundary of the AdS bulk. For instance, M-theory on $\text{AdS}_4 \times S^7$ is dual to a non-abelian superconformal field theory in three dimensions, and type IIB superstring theory on $\text{AdS}_5 \times S^5$ seems to be equivalent to a super Yang-Mills (SYM) theory in four dimensions [1, 2] (for a review see [3]). Of course the duality is not clearly and unambiguously understood, so much of the work on the subject involved building a catalog, or a dictionary taking us from one description to the other. Some of the most important issues to understand are the thermodynamic and classical stabilities of these AdS black holes.

Research on the thermodynamic stability of AdS black holes started two decades ago. It was found that Schwarzschild-AdS black holes are subjected to the Hawking-Page phase transition [4]: at low temperatures a thermal gas is a globally stable configuration in the AdS background (because in this background, as oppose to the flat one, a black hole can exist only for a temperature above a critical value) but when one increases the temperature above a critical value, the thermal bath becomes unstable and collapses to form a black hole. The interpretation of this transition in the dual CFT side was provided in [2]. The thermal AdS space at low temperature corresponds, in the dual picture, to a confined thermal SYM theory. When we heat up the thermal SYM it deconfines, i.e., quarks can be seen in isolation. It is then natural to argue that the Hawking-Page phase transition in the gravity side of the duality corresponds to a confinement/deconfinement transition in the CFT side [2, 5]. A number of other asymptotically AdS black holes have been studied, and their thermodynamical properties thoroughly investigated: charged [6, 8, 10, 11, 12, 13], rotating [12, 13, 14, 15], and non-spherical [16, 17, 18] black holes, and their interpretation in the AdS/CFT duality context.

On the other hand, the classical stability of Kerr-AdS black holes is not as completely understood. One might worry that any rotating black hole in AdS space might be unstable against superradiance: a rotating black hole amplifies
some incoming waves, and since AdS is a box-like spacetime, the amplified radiation would be reflected back, and so on, leading to a black hole bomb \[19, 21, 21\], or exponential growth of any perturbation. Two major results were reported:

(i) large \((r_+ > \ell)\) Kerr-AdS are stable \[20, 21, 22\] (where \(r_+\) is the horizon radius and \(\ell\) is the cosmological length). From a superradiance point of view, this is because the characteristic modes of large AdS holes are very large and do not satisfy the superradiant condition \(\omega < m\Omega_H\) (see \[21, 21, 23\] for details). Here \(\omega\) and \(m\) are the frequency and angular momentum of the mode, and \(\Omega_H\) is the horizon velocity of the black hole.

(ii) small \((r_+ < \ell)\) Kerr-AdS black holes are unstable against scalar perturbations \[21\], precisely by a black hole bomb mechanism. The value \(r_+ = r_+^c = \sqrt{a\ell}\), which obviously depends on the rotation parameter \(a\), plays a critical role. It was shown \[21\] that when, and only when, \(r_+ < r_+^c\), Kerr-AdS black holes exhibit a classical (superradiant) instability. One of the purposes of this paper is to show that small Kerr-AdS black holes are also unstable against gravitational perturbations and therefore the metric is itself unstable \(^1\). Now, the boundary of the Kerr-AdS black hole, where the dual CFT lives, is conformal to a rotating Einstein universe. Remarkably, a Kerr-AdS black hole whose parameters satisfy the condition \(r_+ = r_+^c = \sqrt{a\ell}\), is a black hole for which the corresponding Einstein universe on the boundary rotates precisely with a critical value equal to the velocity of light. Black holes with \(r_+ < r_+^c\) correspond to an Einstein universe that rotates faster than the speed of light, while for \(r_+ > r_+^c\), the Einstein universe on the boundary rotates with velocity smaller than the light velocity. We will be able to follow the black hole evolution induced by the classical instability, and to identify the precise final configuration that describes the equilibrium endpoint of the instability. In particular, we will show that a small Kerr-AdS black hole that starts with a given rotation and whose horizon radius is such that \(r_+ < r_+^c\), will lose angular momentum and increase its radius until its radius and rotation satisfy the relation \(r_+ = r_+^c = \sqrt{a\ell}\). At this point the system becomes classically stable. The endpoint configuration is therefore such that the Einstein universe that describes its boundary is rotating precisely with the speed of light.

In Section \[III\] we show that small four dimensional Kerr-AdS black holes are unstable against gravitational perturbations, and we compute the instability timescale. From several arguments, presented in \[20, 21\] we expect higher dimensional Kerr-AdS black holes to be unstable as well. There is no known formalism to handle gravitational perturbations of this class of geometries, but in \[24\] the authors focused on a particular class of gravitational perturbations (which exist only for \(D \geq 5\)) and showed that they too lead to instabilities. Using the results in \[24\] we compute in Appendix \[A\] the instability timescale for higher dimensional geometries. We close with Section \[III\] where we discuss the possible final state of this instability.

### II. CLASICAL INSTABILITY OF KERR-ADAMS BLACK HOLES AND ITS ENDPOINT

#### A. Kerr-AdS black holes in four dimensions

The metric of a four-dimensional Kerr-AdS black hole is

\[
d s^2 = -\frac{\Delta_r}{\rho^2} \left( dt - \frac{a}{\Sigma} \sin^2 \theta \, d\phi \right)^2 + \frac{\rho^2}{\Delta_r} \, dr^2 + \frac{\rho^2}{\Delta_\theta} \, d\theta^2 + \frac{\Delta_\theta}{\rho^2} \sin^2 \theta \left( a \, dt - \frac{r^2 + a^2}{\Sigma} \, d\phi \right)^2 ,
\]

with

\[
\Delta_r = (r^2 + a^2) \left( 1 + \frac{r^2}{\ell^2} \right) - 2Mr, \quad \Sigma = 1 - \frac{a^2}{\ell^2} \]

\[
\Delta_\theta = 1 - \frac{a^2}{\ell^2} \cos^2 \theta, \quad \rho^2 = r^2 + a^2 \cos^2 \theta ,
\]

and \(\ell = \sqrt{3/\Lambda}\) is the cosmological length associated with the cosmological constant \(\Lambda\). This metric describes the gravitational field of the Kerr-AdS black hole, with mass \(M/\Sigma^2\), angular momentum \(J = aM/\Sigma^2\), and has an event horizon at \(r = r_+\) (the largest root of \(\Delta_r\)). A characteristic and important parameter of a Kerr black hole is the angular velocity of its event horizon given by

\[
\Omega_H = \frac{a}{r_+^c + a^2 \left( 1 - \frac{a^2}{\ell^2} \right)} .
\]

\(^1\) Recently, while this work was on its last stages, it was shown in \[24\] that for \(D \geq 5\), small rotating AdS black holes are unstable against a certain sector of gravitational perturbations.
The area of the black hole horizon is

\[ A = \frac{4\pi (r_+^2 + a^2)}{\Lambda}, \tag{2.4} \]

and its temperature is

\[ T = \frac{r_+ (1 + a^2 \ell^{-2} + 3r_+^2 \ell^{-2} - a^2 r_+^{-2})}{4\pi (r_+^2 + a^2)} . \tag{2.5} \]

In order to avoid singularities, the black hole rotation is constrained to be

\[ a < \ell . \tag{2.6} \]

### B. Wave equation for gravitational perturbations

Gravitational perturbations of rotating black holes are best dealt with using Teukolsky’s formalism \[25\]. The equation describing gravitational perturbations of Kerr-AdS black holes was obtained by Chambers and Moss \[26\] and also Giammatteo and Moss \[26\]. The result is again a system of two coupled ordinary differential equations, one radial and one angular.

To get the radial equation, start with Eq. (15) by Giammatteo and Moss \[26\], which can be written as

\[ r^4 \Delta_r \partial_r \left( \frac{\Delta_r}{r^2} \partial_r R \right) - r \Delta_r \left( -\frac{r}{\Delta_r} (K_r + i \Delta'_r)^2 - \frac{2\Delta_r}{r} + \Delta' + \lambda r - \frac{6r^3}{\ell^2} + 6ir^2\omega + irK'_r - r\Delta''_r \right) R = 0 , \tag{2.7} \]

where the primes stand for derivatives with respect to \( r \), and

\[ K_r = -am \left( 1 - \frac{a^2}{\ell^2} \right) + \omega(r^2 + a^2) . \tag{2.8} \]

Defining \( R = \frac{1}{\Delta_r} \Psi \) we get

\[ \Delta_r \Psi'' - \Delta'_r \Psi' - \left( -\frac{K_r^2 + 2iK_r \Delta'_r}{\Delta_r} + \lambda - \frac{6r^3}{\ell^2} + 6ir\omega + iK'_r \right) \Psi = 0 . \tag{2.9} \]

In the limit of asymptotically flat space (\( \ell \to \infty \)), Eq. (2.9) reduces to Teukolsky’s \[25\] radial equation with spin \( s = -2 \). On the other hand, for \( a = 0 \) the following transformation

\[ \Psi = \Delta_r \left( 1 + \frac{i\omega r^2}{\Delta_r} \right) \partial_r \left( 1 + \frac{i\omega r^2}{\Delta_r} \right) r X(r) , \tag{2.10} \]

yields the Regge-Wheeler equation for \( X(r) \), in Schwarzschild-AdS spacetimes \[27\]. We note that the decomposition used in \[26\] is of the form \( e^{i\omega t + i\mu \phi} \). We adopted here the most common form \( e^{i\omega t - i\mu \phi} \) so that waves co-rotating with the black hole have positive \( m \). This will be important in the sequel, to decide when and why there is an instability.

The angular equation satisfies

\[ \Delta_\mu^2 \partial_\mu S + \Delta_\mu (\partial_\mu \Delta_\mu) \partial_\mu S + \left[ -K_\mu^2 - \Delta_\mu^2 + \Delta_\mu (\lambda - 2\mu^2 \Lambda + 6\mu \omega + 2K_\mu + \partial_\mu \Delta_\mu) \right] S = 0 , \tag{2.11} \]

where \( S = S(\mu) \) and

\[ \mu = a \cos \theta , \quad K_\mu = -\left( 1 - \frac{a^2}{\ell^2} \right) am + (a^2 - \mu^2) , \quad \Delta_\mu = (a^2 - \mu^2) \left( 1 - \frac{\mu^2}{\ell^2} \right) , \tag{2.12} \]

For small rotation and small cosmological constant, \( \lambda \to (l+1)(l+2) \). The primes stand for derivatives with respect to \( \mu \). Likewise, this angular equation reduces to the usual spin-weight 2 spherical harmonics when \( a \to 0 \) and it reduces to the spin-weight 2 spheroidal harmonics \[28\] when \( l \to \infty \).
1. Solution in the near-region

For small AdS black holes, $r_+ / \ell \ll 1$, in the near-region, $r - r_+ \ll 1/\omega$, we can neglect the effects of the cosmological constant, $\Lambda \sim 0$. Moreover, one has $r \sim r_+, r_+ \sim 2M$, and $\omega a \sim 0$ (since $\omega \ll M^{-1}$ and $a \ll M$), and $\Delta_r \sim \Delta$ with

$$\Delta = r^2 + a^2 - 2Mr = (r - r_+)(r - r_-).$$

(2.13)

The near-region radial wave equation can then be written as

$$\Delta_r \partial_r^2 \Psi - \Delta'_r \partial_r \Psi - \left( (l-1)(l+2) - \frac{K_r^2 + 2iK_r \Delta'_r}{\Delta_r} \right) \Psi = 0. \quad (2.14)$$

We find it convenient to express the last term in the following form, near the event horizon:

$$K_r^2 + 2iK_r \Delta'_r \sim [K_r + i(r_+ - r_-)]^2 + (r_+ - r_-)^2, \quad (2.15)$$

which using the explicit expression for $K_r$ can be shown to be, near $r_+$,

$$(K_r^2 + 2iK_r \Delta'_r)^2 \sim (r_+ - r_-)^2 [ (\omega + i)^2 + 1]. \quad (2.16)$$

We have defined the superradiant factor

$$\omega \equiv (\omega - m\Omega_H)\frac{r_+^2 + a^2}{r_+ - r_-}, \quad (2.17)$$

which will play an important role in the instability. For $\omega - m\Omega_H < 0$ this factor is negative, and we have superradiance (see [21] for more details). It turns out that it is also in this regime that the spacetime is unstable.

If one introduces a new radial coordinate,

$$z = \frac{r - r_+}{r_+ - r_-}, \quad 0 \leq z \leq 1, \quad (2.18)$$

with the event horizon being at $z = 0$, then the near-region radial wave equation can be written as

$$z(1-z)\partial_z^2 \Psi - (1+3z)\partial_z \Psi + \left[ (\omega + i)^2 + 1 \right] \frac{1-z}{z} - \frac{(l-1)(l+2)}{1-z} \Psi = 0, \quad (2.19)$$

Through the definition

$$\Psi = z^{\omega}(1-z)^{l-1} F, \quad (2.20)$$

the near-region radial wave equation takes the form

$$z(1-z)\partial_z^2 F + \left[ (-1 + i 2\omega) - [1 + 2l + i 2\omega] z \right] \partial_z F - (l+1) [ l + i 2\omega ] F = 0. \quad (2.21)$$

This wave equation is a standard hypergeometric equation [29], $z(1-z)\partial_z^2 F + [ c - (a + b + 1)z ] \partial_z F - abF = 0$, with

$$a = l - 1 + i 2\omega, \quad b = l + 1, \quad c = -1 + i 2\omega. \quad (2.22)$$

and its most general solution in the neighborhood of $z = 0$ is $A z^{-eF(a - c + 1, b - c + 1, 2 - c, z) + B F(a, b, c, z)}$. Using (2.20), one finds that the most general solution of the near-region equation is

$$\Psi = A z^{2-i\omega}(1-z)^{l-1} F(a - c + 1, b - c + 1, 2 - c, z) + B z^{i\omega}(1-z)^{l-1} F(a, b, c, z). \quad (2.23)$$

The first term represents an ingoing wave at the horizon $z = 0$, while the second term represents an outgoing wave at the horizon. We are working at the classical level, so there can be no outgoing flux across the horizon, and thus one sets $B = 0$ in (2.23). This boundary condition also follows from regularity requirements [25] since $R \sim (r - r_+)^{-1}$ at the horizon for outgoing waves.
One is now interested in the large \( r, z \to 1 \) behavior of the ingoing near-region solution. To achieve this aim one uses the \( z \to 1 - z \) transformation law for the hypergeometric function \[29\],

\[
F(a-c+1, b-c+1, 2-c, z) = (1-z)^{c-a-b} \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} F(1-a, 1-b, c-a-b+1, 1-z) \\
+ \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-c)\Gamma(1-b)} F(a-c+1, b-c+1, -c+a+b+1, 1-z),
\]

and the property \( F(a, b, c, 0) = 1 \). Finally, noting that when \( r \to \infty \) one has \( 1 - z = (r_+ - r_-)/r \), one obtains the large \( r \) behavior of the ingoing wave solution in the near-region,

\[
R \sim A \Gamma(3 - i 2\omega) \left( \frac{r_+ - r_-}{\Gamma(l)\Gamma(l + 2 - i 2\omega)} r^{l+2} + \frac{(r_+ - r_-)^{l-1}}{\Gamma(-l)\Gamma(2 - l - i 2\omega)} r^{1-l} \right).
\]

2. Far-region wave equation and solution

In the far-region, \( r - r_+ \gg M \), the effects induced by the black hole can be neglected (\( a \approx 0, M \approx 0, \Delta_r \approx r^2[1 + r^2/\ell^2] \)) and the radial wave equation reduces to the wave equation of a graviton in pure AdS background. We will first find the solution to the Regge-Wheeler equation and then use \[2.10\] to find the Teukolsky function itself. Regge-Wheeler’s equation in pure AdS is \[27\]

\[
(r^2 + 1)^2 \partial_r^2 \Psi_{\text{RW}} + 2r(r^2 + 1) \partial_r \Psi_{\text{RW}} + \left( \omega^2 - \frac{l(l+1)}{r^2} \right) \Psi_{\text{RW}} = 0.
\]

An exact solution to this equation was found in \[27\] (see their Appendix):

\[
\Psi_{\text{RW}} = \sqrt{x - 1} x^{(l+1)/2} \left[ CF(a, \beta, \gamma, x) + Dx^{1-\gamma}F(1 + \alpha - \gamma, 1 + \beta - \gamma, 2 - \gamma, x) \right],
\]

where

\[
x = \frac{r^2}{r^2 + 1}, \quad \alpha = 2 + l - \omega, \quad \beta = 2 + l + \omega, \quad \gamma = 3/2 + l.
\]

Near infinity \( x \sim 1 \), we have, using known transformation laws for the hypergeometric functions,

\[
CF(a, \beta, \gamma, x) \sim C(1 - x)^{-l-5/2} \frac{\Gamma[\gamma] \Gamma[\alpha + \beta - \gamma]}{\Gamma[\alpha] \Gamma[\beta]},
\]

\[
Dx^{1-\gamma}F(1 + \alpha - \gamma, 1 + \beta - \gamma, 2 - \gamma, x) \sim D(1 - x)^{-l-5/2} \frac{\Gamma[2 - \gamma] \Gamma[\alpha + \beta - \gamma]}{\Gamma[\alpha - \gamma + 1] \Gamma[\beta - \gamma + 1]}.
\]

The boundary conditions appropriate for gravitational fields in AdS space have been discussed by a number of authors \[30\]. We will adopt Dirichlet boundary conditions here, although we suspect the results hold for more general boundary conditions (for instance, QNMs in AdS are weakly affected by boundary conditions \[30\], since there is a direct relation between the instability and QNMs in AdS \[21\], we conjecture that the instability itself is weakly sensitive to these). These “reflective” boundary conditions imply that

\[
C = \frac{\Gamma[2 - \gamma] \Gamma[\alpha] \Gamma[\beta]}{\Gamma[\gamma] \Gamma[\alpha - \gamma + 1] \Gamma[\beta - \gamma + 1]} = \frac{\Gamma[1/2 - l] \Gamma[2 + l - \omega]}{\Gamma[3/2 + l] \Gamma[3/2 - \omega]}.
\]

For small values of \( r \), we have

\[
\Psi_{\text{RW}} \sim iC r^{l+1} + iDr^{-l}.
\]

We can now use \[2.10\], which for small values of \( r \) takes the form

\[
\Psi \sim l(l+1)r \Psi_{\text{RW}} + 2r^2 \Psi'_{\text{RW}}.
\]

Thus,

\[
\Psi \sim iC(l + 1)(l + 2)r^{l+2} + iD(l - 1)r^{-l+1}.
\]
Matching (2.34) with (2.25) we get
\[
\frac{(r_+ - r_-)^{-2l-1} \Gamma[2l + 1] \Gamma[-l] \Gamma[2 - l - 2i\varpi]}{\Gamma[l] \Gamma[l + 2 - 2i\varpi] \Gamma[-2l - 1]} = -\frac{(l + 1)(l + 2) \Gamma[1/2 - l] \Gamma[2 + l - \omega] \Gamma[2 + l + \omega]}{l(l - 1) \Gamma[3/2 + l] \Gamma[3/2 - \omega] \Gamma[3/2 + \omega]}. \tag{2.35}
\]
This is an eigenvalue equation for \( \omega \). To show that instabilities are possible, we will expand this last equation around the pure AdS value \( \omega_{\text{AdS}} \) which is
\[
\omega_{\text{AdS}} = \frac{2n + l + 2}{\ell}, \tag{2.36}
\]
with \( n \) an integer. We thus set (see [20, 21] for a more elaborate discussion on this)
\[
\omega = \frac{2n + l + 2}{\ell} + i \delta. \tag{2.37}
\]
First, we can use the following identities
\[
\frac{\Gamma[-2l - 1]}{\Gamma[-l]} = (-1)^{l+1} \frac{l!}{(2l + 1)!}, \tag{2.38}
\]
\[
\frac{\Gamma[l + 2 - 2i\varpi]}{\Gamma[2 - l - 2i\varpi]} \sim 2i(-1)^{l+1} \frac{l + 1}{l(l - 1)} \varpi \prod_{k=1}^{l} (k^2 + 4\varpi^2), \tag{2.39}
\]
to simplify the LHS of (2.35) to
\[
\text{LHS}^{-1} = \left( \frac{(l - 1)!}{(2l)!} \right)^2 2i(l + 1)(r_+ - r_-)^{-2l-1} \frac{\Gamma[1/2 - l] \Gamma[2 + l - \omega] \Gamma[2 + l + \omega]}{(l - 1)(2l + 1)} \prod_{k=1}^{l} (k^2 + 4\varpi^2) \equiv iP_1 \varpi. \tag{2.40}
\]
Here \( P_1 \) is obviously positive. The RHS of (2.35) can be simplified by noting that \( \Gamma[-n + \epsilon] \sim \frac{(-1)^n}{n^\epsilon}, \ \epsilon \to 0 \). We then have
\[
\text{RHS} = -\frac{(l + 1)(l + 2)}{l(l - 1)} \frac{\Gamma[1/2 - l] \Gamma[2 + l - \omega] \Gamma[2 + l + \omega]}{\Gamma[3/2 + l] \Gamma[3/2 - \omega] \Gamma[3/2 + \omega]} = \frac{\Gamma[1/2 - l] \Gamma[-2n - i\delta] \Gamma[2n + 2l + 4 + i\delta]}{\Gamma[3/2 + l] \Gamma[-2n - l - 1/2 - i\delta] \Gamma[2n + l + 7/2 + i\delta]} \cong iP_2 \frac{1}{\delta}, \tag{2.41}
\]
with \( P_2 \) a positive constant,
\[
P_2 = -\frac{(l + 1)(l + 2)}{l(l - 1)} \frac{\Gamma[2n + 2l + 4] \Gamma[1/2 - l]}{(2n)! \Gamma[3/2 + l] \Gamma[2n + l + 7/2] \Gamma[-2n - l - 1/2]}. \tag{2.42}
\]
Finally, equating the RHS and LHS of (2.35), i.e., the inverse of (2.40) with (2.41) we get
\[
\delta = -P_1 P_2 \varpi. \tag{2.43}
\]
Clearly, there is an instability whenever \( \varpi \) is negative (superradiant regime), since \( \Psi \propto e^{i\omega t} \propto e^{-\delta t} \).

**III. THE ENDPOINT STATE OF THE CLASSICAL INSTABILITY**

It is in general impossible to predict the endpoint of an unstable system but, at least in the Kerr-AdS case, one can actually evolve the system perturbatively, or at least make some educated guesses. The black hole is evolving while the instability is acting: the angular momentum of the hole is decreasing since superradiant amplification of modes outside of the horizon will transfer angular momentum to the exterior fields. The energy release through gravitational waves (or any other field \( \Psi \sim \Psi_0 e^{i\omega t} \sin \text{Re} [\omega] t \), goes like \( dE/dt \sim |\Psi|^2 \) and therefore between \( t = 0 \) and \( t = T \) we have a radiated energy of
\[
E \sim |\Psi_0|^2 e^{2T \text{Im} [\omega]} \text{Re} [\omega]^2 \frac{\text{Re} [\omega]^2}{2 \text{Im} [\omega]} \sim |\Psi(T)|^2 \text{Re} [\omega]^2 \frac{\text{Re} [\omega]^2}{2 \text{Im} [\omega]}. \tag{3.1}
\]
Now, an upper limit estimate for the total energy radiated can be obtained by equating to zero the final angular momentum and assuming an adiabatic process. Under these conditions the first law yields, for \( a, r_+ \ll \ell \),

\[
\Delta M \sim \frac{a^2}{2r_+}.
\]  

We thus conclude that for

\[
\frac{a^2 \Im[\omega]}{r_+ \Re[\omega]^2} \ll 1,
\]

the amplitude \( \Psi \) of the field is never large, i.e., that we can trust the perturbative regime at all times, since the instability stops for sufficiently low rotation. It is of course, possible that the instability timescale is much lower than the other timescales at stake (like for instance, a rotation period) that the instability is effectively unimportant, but we shall assume otherwise. Working in regime (3.3) we should then be able to follow the evolution of the instability in some detail, as we now explain.

From the results of the last section and from [21] one has that the real part of the frequencies that can propagate in the Kerr-AdS background are

\[
\omega = \frac{2n + l + c_s}{\ell},
\]

where \( c_s \) is a constant that depends on the spin of the particular wave we are considering; e.g., for scalar modes one has \( c_s = 3 \) [21], and for gravitational modes one has \( c_s = 2 \) [see (2.36)]. The condition that must be satisfied in order to have superradiance is \( \omega \leq m \Omega_H \). Use of (3.4) with large \( l = m \), together with \( a \leq \ell \), yields that superradiance holds while

\[
r_+ \lesssim r_+^c \quad \text{with} \quad r_+^c = \sqrt{a \ell}.
\]  

The superradiant amplification of the wave is fed by the rotational energy of the black hole and thus, as the superradiant scattering proceeds, the angular parameter \( a \) decreases. Classically, when \( a \) decreases the radius of the horizon, \( r_+ \), increases. Indeed, the variation of the horizon area (2.4) yields

\[
dA = \frac{8\pi}{\Sigma} \left[ r_+ dr_+ + \frac{r_+^2 + \ell^2/\Sigma}{\Sigma} ada \right],
\]

and the classical constraint \( dA \geq 0 \) implies that \( dr_+ > 0 \) when \( da < 0 \). This fact, together with condition (3.5), allows us to clearly identify the endpoint of the superradiant classical instability. We start with a small black hole with small \( r_+ \) and an angular parameter \( a \) satisfying the inequality displayed in (3.5). As the superradiant scattering continues, \( a \) decreases and \( r_+ \) increases. But as \( a \) decreases the critical radius \( r_+^c = \sqrt{a \ell} \) also decreases, and thus there is a critical minimum value of \( a \) for which the radius of the black hole reaches the value of the critical radius \( r_+^c \). This critical point describes the endpoint of the superradiant classical instability. We end up with a Kerr-AdS black hole whose horizon radius is larger than the initial black hole and with smaller rotation. This black hole is in equilibrium with rotating radiation that rotates in the same sense as the black hole. This radiation is trapped between the black hole horizon and the effective AdS wall.

But we can identify another important property of this final configuration. First note that the coordinate transformation

\[
\begin{align*}
\varphi &= \phi - \frac{a}{\ell} t \\
\varphi \cos \Theta &= r \cos \theta \\
\varphi^2 &= \frac{1}{\Sigma} \left( r^2 \Delta_\theta + a^2 \sin^2 \theta \right),
\end{align*}
\]

takes the Kerr-AdS metric (2.11) into a form in which the asymptotic AdS nature of the Kerr-AdS solution is manifest. In particular, in these coordinates the angular velocity of the horizon is

\[
\Omega = \frac{a(1 + r_+^2/\ell^2)}{r_+^2 + a^2},
\]  

(3.7)
and the angular velocity of the spacetime as \( \rho \to \infty \) vanishes. The relation between the angular velocities at the horizon in the two coordinate systems, \((t, \rho, \Theta, \varphi)\) and \((t, r, \theta, \phi)\), is \( \Omega \equiv \Omega_{H} + a/\ell^{2} \). In these new coordinates the \( M = 0 \) Kerr AdS metric, that describes the asymptotic behavior of the Kerr AdS metric, takes the standard form,

\[
\begin{align*}
\text{ds}^{2} &= -\left(1 + \frac{\rho^{2}}{\ell^{2}}\right)dt^{2} + \left(1 + \frac{\rho^{2}}{\ell^{2}}\right)^{-1}d\rho^{2} + \rho^{2}(d\Theta^{2} + \sin^{2}\Theta d\varphi).
\end{align*}
\]

(3.9)

A hypersurface of constant large radius \( \rho \) in the Kerr AdS has therefore a metric that is conformal to a three dimensional Einstein universe,

\[
\begin{align*}
\text{ds}^{2}_{\text{boundary}} &= \rho^{2}\left(-\ell^{-2}dt^{2} + d\Theta^{2} + \sin^{2}\Theta d\varphi\right),
\end{align*}
\]

(3.10)

The coordinates \( t \) and \( \varphi \) in (3.10) are constrained to an identification required by regularity. Indeed, when we consider the Euclidean section of the Kerr-AdS black hole by analytically continuing the time coordinate and the angular parameter \( a \), one finds that to avoid a conical singularity at \( \rho_{+} \), one must identify the Euclidean time coordinate with period \( \beta = 1/T \) where \( T \) is the black hole temperature (2.5). But this identification must also be accompanied by a rotation in \( \varphi \), i.e., we must identify the points \((t, \rho, \Theta, \varphi) \sim (t + i\beta, \rho, \Theta, \varphi + i\beta \Omega)\), with \( \Omega \) given by (3.8). Now, the boundary solution (3.10) inherits the above identifications from the bulk geometry. In particular this means that this boundary Einstein universe is rotating with an angular velocity \( \Omega \) given by (3.8). Moreover, this solution is the boundary of a bulk AdS background whose typical radius is \( R_{\text{AdS}} = \ell \), and thus the linear velocity of the rotating Einstein universe is \( v = \Omega \ell \). Therefore, for \( \Omega > 1/\ell \) (\( \Omega < 1/\ell \)), the Einstein universe on the boundary rotates faster (slower) than the speed of light. At the critical angular velocity \( \Omega = 1/\ell \) the boundary rotates exactly with light velocity. Use of (3.8) allows us to write the equivalent condition

\[
\begin{align*}
\Omega = \frac{1}{\ell} \iff \rho_{+} = \sqrt{a\ell}.
\end{align*}
\]

(3.11)

But this value of \( \rho_{+} \) coincides with the critical value \( r^{c}_{+} \) defined in (3.4).

To conclude, we find that the endpoint of the classical instability is described by a Kerr-AdS black hole whose boundary is an Einstein universe rotating with the light velocity. This black hole should be slightly oblate since the most unstable modes are those with \( l = m = 2 \) and it should co-exist in equilibrium with a certain amount of outside radiation.

IV. CONCLUSION

We have shown that small Kerr-AdS black holes rotating with a velocity higher than the velocity of light are unstable: any small perturbation is exponentially amplified, via a superradiant mechanism, and the system behaves as a black hole bomb [20]. We have computed the growing timescales and oscillation frequencies of the corresponding unstable modes. More importantly, we have shown that for a large class of these geometries one can follow the evolution of the system, as the perturbative regime is always valid. The endpoint of the instability corresponds to a Kerr-AdS black hole whose boundary is an Einstein universe rotating with the light velocity. This black hole is expected to be slightly oblate and to co-exist in equilibrium with a certain amount of outside radiation. It is thus conceivable that new solutions to Einstein’s equations exist which describe such a geometry. In fact, the perturbative approach followed here already gives such a geometry to first order in the “oblateness” parameter, i.e., in deviation from the Kerr-AdS black hole solution.

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APPENDIX A: THE INSTABILITY IN AN ARBITRARY NUMBER OF DIMENSIONS

In this appendix we will briefly sketch how to extend the results in the body of the paper to higher dimensions. In particular, we consider the subset of gravitational perturbations considered in [24]. There it is shown that a special subset of gravitational perturbations obeys the equation:

\[-rf \frac{d}{dr} \left( \frac{rf d\Psi}{h} \right) + V \Psi = 0,\]  

(A1)

where

\[V = \frac{r^2 f \sqrt{h}}{h^{2N+1}} \frac{d}{dr} \left[ \frac{r^2 f}{h^2} \mu^2 - (\omega - m\Omega)^2 + \frac{f}{h^2} \left[ l(l+2N) - m^2 \left( 1 - \frac{r^2}{h^2} \right) + 4(1 - \sigma) \left( \frac{h^2}{r^2} - 1 \right) \right] \right],\]  

(A2)

and

\[h = r \sqrt{1 + \frac{2Ma^2}{r^{2N+2}}}, \quad f = 1 + r^2 - \frac{2M(1 - a^2)}{r^{2N}} + \frac{2Ma^2}{r^{2N+2}}.\]  

(A3)

The quantity \(\sigma = \pm 1\), but it will have no bearing on the final result, since it drops out in the regime we focus on. We will set the cosmological radius \(\ell = 1\) and so that all quantities are dimensionless. We again consider the regime \(r_+ \ll 1\) and \(r_+^2 \ll a \ll r_+\). In this regime we have

\[2M \sim r_+^{2N}, \quad h \sim r, \quad \Omega \sim \frac{a}{r^2_+} \gg 1.\]  

(A4)  

(A5)  

(A6)

The wave equation thus reduces to

\[f \frac{d}{dr} \left( f \frac{d\Psi}{dr} \right) + \left[ (\omega - m\Omega)^2 - f \left( \frac{l(l+2N)}{r^2} + \frac{(2N+1)}{4r^2} \left[ (2N - 1)f + 2rf' \right] + \mu^2 \right) \right] \Psi = 0,\]  

(A7)

If we change wavefunction by defining \(\Psi = r^{\beta_0/2}Z(r)\), with \(\beta_0 = 2N + 1\), we get

\[\frac{f}{r^{\beta_0}} \partial_r \left( f r^{\beta_0} \partial_r Z \right) + \left[ (\omega - m\Omega)^2 - f \left( \frac{l(l+2N)}{r^2} + \mu^2 \right) \right] Z = 0.\]  

(A8)

1. Solution in the near-region

In the near region, \(f \sim 1 - \frac{2M}{r_+^{2N}}\) so it is natural to introduce the following change of variables

\[v = 1 - \frac{2M}{r_+^{2N}}.\]  

(A9)

Then, Eq. (A8) reads

\[(1 - v)^2 v^2 \partial_v^2 Z + v(1-v)^2 \partial_v Z + \left( (\omega - m\Omega_H)^2 \frac{r_+^2}{4N^2} - \frac{v}{4N^2} \left[ l(l+2N) + \mu^2 r_+^2 \right] \right) Z = 0,\]  

(A10)

This equation can be put in a standard hypergeometric form by setting

\[Z = v^{c_1} (1 - v)^{c_2} F,\]  

(c1)  

(c2)  

(a)  

(b)  

(A11)  

(A12)  

(A13)  

(A14)  

(A15)
The result is

\[ v(1 - v)\partial_v^2 F + [\gamma - v(1 + \alpha + \beta)] \partial_v F - \alpha \beta F = 0, \]  
(A16)

where

\[ \gamma = 1 - 2ia, \quad \alpha = \beta = c_1 + c_2. \]  
(A17)

The most general solution of Eq. (A16) in the neighborhood of \( v = 0 \) is

\[ F = A F(\alpha, \beta, \gamma, v) + B v^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, v). \]  
(A18)

Since the second term describes an outgoing wave near the horizon, we set \( B = 0 \). The asymptotic behavior of the near-horizon solution is

\[ Z \sim r^l (2M)^{-l/(2N)} \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} + r^{-2N-l}(2M)^{1+l/(2N)} \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}, \]  
(A19)

where we have used the property of the hypergeometric functions:

\[ F(\alpha, \beta, \gamma, v) = (1-v)^{-\gamma-\alpha-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} F(\gamma - \alpha - \beta + 1, 1-v) + \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\alpha, \beta - \gamma, \gamma - \alpha - \beta + 1, 1-v). \]  
(A20)

2. Solution in the far-region

In the far-region, \( r \gg M \), the effects induced by the black hole can be neglected \( (a \sim 0, M \sim 0, \Delta_r \sim r^2[1 + r^2/l^2]) \) and the radial wave equation (A8) reduces to the wave equation of a scalar field of frequency \( \omega \) and angular momentum \( \ell \) in a pure AdS background. The wave equation can be written in a standard hypergeometric form. First we introduce a new radial coordinate,

\[ x^{-1} = 1 + \frac{r^2}{\ell^2}, \quad 0 \leq x \leq 1, \]  
(A21)

with the origin of the AdS space, \( r = 0 \), being at \( x = 1 \), and \( r = \infty \) corresponds to \( x = 0 \). Then the radial wave equation (A8) can be written as

\[ x(x-1)\partial_x^2 Z + (x + N) \partial_x Z - \left[ \omega^2 + \frac{l(l + 2N)}{4(x-1)} \right] Z = 0. \]  
(A22)

Through the definition

\[ Z = x^{N+1}(1 - x)\ell/2 \Upsilon, \]  
(A23)

the radial wave equation becomes

\[ x(1-x)\partial_x^2 \Upsilon + \left[ (2 + N) - (3 + l + 2N+) \right] \partial_x \Upsilon \]
\[ - \frac{1}{4} (2 + l + 2N - \omega)(2 + l + 2N + \omega) \Upsilon = 0. \]  
(A24)

This wave equation is a standard hypergeometric equation \( 29 \), \( x(1-x)\partial_x^2 F + [\gamma - (\alpha + \beta + 1)x] \partial_x F - \alpha \beta F = 0, \) with

\[ \alpha = \frac{2 + l + 2N - \omega}{2}, \quad \beta = \frac{2 + l + 2N + \omega}{2}, \quad \gamma = 2 + N. \]  
(A25)

The most general solution in the neighborhood of \( x = 0 \) is

\[ \Upsilon = A F(\alpha, \beta, \gamma, x) + B x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x). \]  
(A26)
Since $F(a, b, c, 0) = 1$, as $x \to \infty$ this solution behaves as $Z \sim A r^{-2N-2} + B$. But the AdS infinity behaves effectively as a wall, and thus the scalar field must vanish there which implies that we must set $B = 0$ in (A26). We are now interested in the small $r, x \to 1$, behavior of (A26). We note that $\gamma$ is an integer and that $\gamma = \alpha + \beta - (l + N)$ and thus some care must be exercised in expanding this function. We get

$$Z \sim C \left[ \frac{\Gamma(1 + N)\Gamma(N + 2)}{\Gamma\left(\frac{2 + l + 2N + \omega}{2}\right)\Gamma\left(\frac{2 + l + 2N + \omega}{2} - l - N\right)} r^{-l-2N} + \frac{(-1)^{l+N}\Gamma(N + 2)}{\Gamma(l + N + 1)\Gamma\left(\frac{2 + l + 2N + \omega}{2} - l - N\right)\Gamma\left(\frac{2 + l + 2N + \omega}{2} - l - N\right)} r^l \right].$$

This can be matched to (A10), and the instability timescale can be obtained using the procedure explained in the main body of the paper.

[1] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).
[2] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998).
[3] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Physics Reports 323, 183 (2000).
[4] S. W. Hawking and D. N. Page, Commun. Math. Phys. 87, 577 (1983).
[5] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, M. Van Raamsdonk, Phys. Rev. D71, 125018 (2005).
[6] J. Louko and S. N. Winters-Hilt, Phys. Rev. D54 (1996) 2647.
[7] C. S. Peça and P. S. José Lemos, Phys. Rev. D 70 (1999) 124007.
[8] P. Mitra, Phys. Lett. B441 (1998) 89.
[9] P. Mitra, Phys. Lett. B459 (1999) 119.
[10] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Phys. Rev. D60 (1999) 064018.
[11] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Phys. Rev. D60 (1999) 104026.
[12] M. M. Caldarelli, G. Cognola, D. Klemm, Class. Quantum Grav. 17 (2000) 399.
[13] S. W. Hawking, and H. S. Reall, Phys. Rev. D61 024014 (2000).
[14] D. S. Berman, M. K. Parikh, Phys. Lett. B463 (1999) 168.
[15] S. W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, Phys. Rev. D59, 064005 (1999).
[16] D. Birmingham, Class. Quant. Grav. 16 (1999) 1197.
[17] M. M. Caldarelli and D. Klemm, Nucl. Phys. B555 (1999) 157.
[18] R. Emparan, JHEP 06 (1999) 036.
[19] W. H. Press and S. A. Teukolsky, Nature 238, 211 (1972).
[20] V. Cardoso, O. J. C. Dias, J. P. S. Lemos and S. Yoshida, Phys. Rev. D 70, 044039 (2004); Erratum-ibid.D 70, 049903 (2004).
[21] V. Cardoso and O. J. C. Dias, Phys. Rev. D 70, 084011 (2004).
[22] S. W. Hawking and H. S. Reall, Phys. Rev. D 61, 024014 (1999).
[23] V. Cardoso and J. P. S. Lemos, Phys. Lett. B 621, 219 (2005); V. Cardoso and S. Yoshida, JHEP 0507, 009 (2005); O. J. C. Dias, Phys. Rev. D 73, 124035 (2006).
[24] H. K. Kunduri, J. Lucietti, H. S. Reall, hep-th/0606076.
[25] S. A. Teukolsky, Astrophys. J. 185, 635 (1973).
[26] C. M. Chambers and I. G. Moss, Class. Quant. Grav. 11, 1035 (1994); M. Giammatteo and I. G. Moss, Class. Quant. Grav. 22, 1803 (2005).
[27] V. Cardoso and J. P. S. Lemos, Phys. Rev. D 64, 084017 (2001); V. Cardoso, R. Konoplya and J. P. S. Lemos, Phys. Rev. D 68, 044024 (2003).
[28] E. Berti, V. Cardoso and M. Casals, Phys. Rev. D 73, 024013 (2006); Erratum-ibid. D 73, 109902 (2006).
[29] M. Abramowitz and A. Stegun, Handbook of mathematical functions, (Dover Publications, New York, 1970).