ON THE DIRICHLET PROBLEM

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Abstract. Using, as main tool, the convergence theorem for discrete martingales and the mean value property of harmonic functions we solve, a particular case of, Dirichlet problem.

1. Introduction

Let \((\mathbb{R}^d, || \cdot ||)\) be the normed Euclidean space. If \(A \subset \mathbb{R}^d\), we denote by \(\overline{A}\) and \(\partial A\) the closure and frontier (or boundary) of \(A\), respectively. Let us fix first the object of study in this paper.

The Dirichlet problem (DP): Given a non-empty, bounded, and open set \(V \subset \mathbb{R}^d\) and a continuous function \(f : \partial V \rightarrow \mathbb{R}\), the Dirichlet problem consist in finding a unique continuous function \(f : \overline{V} \rightarrow \mathbb{R}\) such that
\[
h(x) = f(x), \; \forall x \in \partial V,
\]
and having in \(V\) continuous partial derivatives of second order which satisfy Laplace’s differential equation, i.e.,
\[
\Delta h(x) = \sum_{k=1}^{d} \frac{\partial^2 h(x)}{\partial x_k^2} = 0, \; \forall x \in V.
\]

The Dirichlet problem has a long history in pure and applied mathematics (see [6], [5]), and it is the basis for more elaborated problems (see [10], [5])). Such problems can be approached in many different ways. In fact, they can be solved using techniques from differential equations, Monte Carlo methods, stochastic differential equations, potential theory, etc.

The Monte Carlo method, introduced by Metropolis and Ulam, is a proceeding for solving physical problems by a method which essentially depends on a statistical sampling technique. There are many studies of (DP) using Monte Carlo techniques (see, for example, [9], [10] and the references there in). In some sense such methods were the motivation to introduce a random sequence \((X^n_v)_{n}, \; X^v_1 = v \in V\), that converges almost surely to a point \(X^v_\infty\) belonging to \(\partial V\) (see Subsection 2.1). Using this convergence we are going to deduce that a solution of (DP) has a specific representation, and of course this expression implies uniqueness of (DP). Actually, this interplay between partial differential equations and stochastic methods was incited by Kakutani [6] who give a probabilistic representation of the solution to (DP) in terms of certain functional of Brownian motion (see [1] or [7]).

In the one-dimensional case (DP) always has a solution, in fact it is piecewise-linear. But for \(d \geq 2\), Zaremba [11] observes that (DP) was not always solvable.

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Hence, the existence is the difficult part of (DP). However, restricting our attention to certain regions $V$ we get existence. More specifically, we introduce the Poincaré’s regularity of $\partial V$ and we proof that the expression given in the uniqueness is well defined and it is a solution to (DP).

So, the present paper is a nice application of some elementary results of martingale theory to a classical subject in mathematics (pure and applied), as it is the Dirichlet problem.

In the next section we begin remembering a characterization of harmonic functions, and we use this and the martingale convergence theorem to prove uniqueness, in Subsection 2.1, and existence, in Subsection 2.2, of (DP).

2. Solving the Dirichlet problem

Before we deal with the Dirichlet problem (DP) we introduce an important class of differentiable functions which are close related with it.

As usual, by $d(x, A)$ we design the distance from the point $x \in \mathbb{R}^d$ to the set $A \subset \mathbb{R}^d$, to be precise

$$d(x, A) = \inf \{||x - y|| : y \in A\}.$$ 

Let $B_r(x) = \{y \in \mathbb{R}^d : ||x - y|| < r\}$ be the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. We also define the sphere, $S_r(x) = \partial B_r(x)$.

Definition 2.1. Let $A \subset \mathbb{R}^d$ be a non empty open set and $h : V \to \mathbb{R}$. We say that

(i) $h$ is harmonic in $A$ if

$$\Delta h(x) = 0, \quad \forall x \in V.$$ 

(ii) $h$ has the mean value property in $A$ if for each $x \in A$ and $r > 0$, such that $B_r(x) \subset A$,

$$h(x) = \frac{1}{\sigma(S_r(x))} \int_{S_r(x)} h(z)\sigma(dz),$$ 

where $\sigma(dz)$ is the Lebesgue (area) measure on $S_r(x)$ and $\sigma(S_r(x)) = cr^{d-1}$, here $c > 0$ is a constant.

Proposition 2.2. Let $A \subset \mathbb{R}^d$ be a non empty open set. A function $h : A \to \mathbb{R}$ is harmonic in $A$ if and only if it has the mean value property in $A$.

Proof. See Theorem 2 in Section 4.3 of [1], or [2].

In what follows we are going to consider $(V, f)$ as in (DP). That is, $V \subset \mathbb{R}^d$ is a set non-empty, open and bounded and $f : \partial V \to \mathbb{R}$ is a continuous function.

2.1. Uniqueness. Let $\vartheta_1, \vartheta_2, \ldots$ be a sequence of random variables (r.v.) defined on the same probability space $(\Omega, \mathcal{F}, P)$. Such r.v. are independently and identically distributed with uniform distribution on the unitary sphere, $S_1(0) \subset \mathbb{R}^d$.

We denote by $E[\cdot]$ the expectation with respect to $P$.

Let $v \in V$ and $0 < r < 1$ be arbitrary and fix. Define the random sequence

$$(2.1) \quad X^v_r(1) = v, \quad X^v_r(n + 1) = X^v_r(n) + rd(X^v_r(n), \partial V)\vartheta_n, \quad n \geq 1.$$ 

The basic connection between harmonic functions and martingales is given by:
Proposition 2.3. Let \( h : \mathbf{V} \rightarrow \mathbb{R} \) be continuous and harmonic in \( V \), then the sequence \( (h(X^\nu_r(n)))_n \) is a martingale with respect \( \mathcal{F}_n = \sigma(X^\nu_r(1), \ldots, X^\nu_r(n)) \), the minimal \( \sigma \)-algebra such that \( X^\nu_r(1), \ldots, X^\nu_r(n) \) are measurable.

Proof. From the definition (2.1) of \( (\vartheta_r)_r \) it follows immediately that \( X^\nu_r(n) \in V \), for each \( n \in \mathbb{N} \). Since \( V \) is bounded, then \( \mathbf{V} \) is compact, therefore \( h \) continuous in \( \mathbf{V} \) implies that \( (h(X^\nu_r(n))) \) is an integrable sequence. On the other hand, since \( \vartheta_n \) is independent of \( \mathcal{F}_n = \sigma(\vartheta_1, \ldots, \vartheta_{n-1}) \) and \( X^\nu_r(n) \) is \( \mathcal{F}_n \) measurable we get (see Example 1.5 in Chapter 4 of [4])

\[
E[h(X^\nu_r(n+1)|\mathcal{F}_n] = E[h(x + rd(x, \partial V)\vartheta_n)|x = X^\nu_r(n)] = \frac{1}{\sigma(S_{rd(x, \partial V)}(x))} \int_{S_{rd(x, \partial V)}(x)} h(z) \sigma(dz) \mathbb{E}_x = h(X^\nu_r(n)).
\]

Observe that we have used Proposition 2.2 in the third equality.

In particular, for each \( j \in \{1, \ldots, d\} \) consider the function \( h_j : \mathbf{V} \rightarrow \mathbb{R} \) defined by \( h_j(x) = x_j \) where \( x = (x_1, \ldots, x_d) \). By the harmonicity of \( h_j \) the preceding result implies that \( (h_j(X^\nu_r(n)))_n \) is a bounded martingale, then the martingale convergence theorem (see Theorem 2.10 in Chapter 4 of [4]) yields that 

\[
\lim_{n \to \infty} h_j(X^\nu_r(n)) = X^\nu_r. \quad \text{a.s.}
\]

In this way,

\[
\lim_{n \to \infty} X^\nu_r(n) = X^\nu_r(\infty) := (X^\nu_r,1(\infty), \ldots, X^\nu_r,d(\infty)) \quad \text{a.s.}
\]

Proposition 2.4. Under the preceding notation, we have \( X^\nu_r(\infty) \in \partial V, \text{ a.s.} \)

Proof. Suppose the contrary, this means that there exits a measurable set \( \Omega' \subset \Omega \) with positive probability such that \( \lim_{n \to \infty} X^\nu_r(n, \omega) = X^\nu_r(\infty, \omega) \notin \partial V \), for each \( \omega \in \Omega' \). Because \( \partial V \) is a closed set we have, \( d(X^\nu_r(\infty, \omega), \partial V) > 0 \). Then, there exists \( n_0 \in \mathbb{N} \) for which

\[
||X^\nu_r(n, \omega) - X^\nu_r(\infty, \omega)|| < \frac{r}{4} d(X^\nu_r(\infty, \omega), \partial V), \quad \forall n \geq n_0.
\]

From the triangle inequality we obtain

\[
||X^\nu_r(n_0 + 1, \omega) - X^\nu_r(n_0, \omega)|| < \frac{r}{4} d(X^\nu_r(\infty, \omega), \partial V).
\]

On the other hand, (2.1) implies

\[
||X^\nu_r(n_0, \omega) - X^\nu_r(n_0 + 1, \omega)|| = rd(X^\nu_r(n_0, \omega), \partial V).
\]

Using the inequality (2.3) we get

\[
d(X^\nu_r(\infty, \omega), \partial V) \leq ||X^\nu_r(\infty, \omega) - X^\nu_r(n_0, \omega)|| + d(X^\nu_r(n_0, \omega), \partial V)
\]

\[
\leq \frac{r}{4} d(X^\nu_r(\infty, \omega), \partial V) + d(X^\nu_r(n_0, \omega), \partial V),
\]

then (2.3) yields

\[
(1 - \frac{r}{4}) d(X^\nu_r(\infty, \omega), \partial V) \leq d(X^\nu_r(n_0, \omega), \partial V)
\]

\[
(2.6) \quad = \frac{1}{r} ||X^\nu_r(n_0, \omega) - X^\nu_r(n_0 + 1, \omega)||.
\]

From (2.6) and (2.4) we conclude that \( 1 - \frac{r}{4} < \frac{1}{4} \), which is a contradiction to \( 0 < r < 1 \). \( \square \)
Now we are ready to deal with the uniqueness of the Dirichlet problem.

**Theorem 2.5.** There is at most one solution to (DP).

**Proof.** Let $h : \overline{V} \to \mathbb{R}$ be a solution to (DP). By the continuity of $h$ in $\overline{V}$ we get

$$\lim_{n \to \infty} h(X_r^n(n)) = f(X_r^n(\infty)), \quad \text{a.s.}$$

Now, due to $(h(X_r^n(n)))_n$ is a martingale (see Proposition 2.3) and dominated convergence theorem implies

$$h(v) = E[h(X_r^n(1))] = \lim_{n \to \infty} E[h(X_r^n(n))] = E[f(X_r^n(\infty))].$$

Therefore, $h(v) = E[f(X_r^n(\infty))]$, for each $v \in V$. This equality implies the uniqueness of (DP). □

2.2. **Existence.** Let $0 < r < 1$ be fix. Define $h : \overline{V} \to \mathbb{R}$ as

$$h(v) = \begin{cases} f(v), & v \in \partial V, \\ E[f(X_r^n(\infty))], & v \in V, \end{cases}$$

where $X_r^n(\infty)$ is given by (2.2).

From the uniqueness argument we see that such function should be the solution of (DP), moreover it also suggest that $h$ does not depend of $r$. We begin verifying that this is the case.

**Proposition 2.6.** The function $h$ given in (2.7) is well define.

**Proof.** By the Tietze-Urysohn theorem (see (4.5.1) in [3]) there exists a continuous function $\bar{f} : V_1 := \{ x \in \mathbb{R}^d : d(x, \overline{V}) < 1 \} \to \mathbb{R}$ such that $\bar{f}|_{\partial V} = f$. It is easy to see that $\bar{f} \in L^2(V_1)$, then there exists a sequence $(f_{\varepsilon})_{\varepsilon > 0}$ of harmonic functions in $V_1$ such that (see Proposition 21.2c in [2])

$$\lim_{\varepsilon \downarrow 0} f_{\varepsilon}(x) = \bar{f}(x), \quad \text{uniformly in } \overline{V}.$$

Let $r, s \in (0, 1)$. The Proposition 2.4 and dominated convergence theorem yields

$$E[f(X_r^n(\infty))] = E[\bar{f}(X_r^n(\infty))] = \lim_{\varepsilon \downarrow 0} E[f_{\varepsilon}(X_r^n(\infty))] = \lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} E[f_{\varepsilon}(X_r^n(n))] = \lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} f_{\varepsilon}(v) = E[f(X_s^n(\infty))].$$

In the last equality we have used Proposition 2.6. □

As we will see the harmonicity of $h$ is the easy part of (DP).

**Proposition 2.7.** The function $h$ defined in (2.7) is harmonic in $V$.

**Proof.** Let $v \in V$ and $0 < s$ such that $B_s(v) \subset V$. This implies that

$$r := \frac{s}{d(v, \partial V)} < 1.$$

Using the notation of Proposition 2.6 we have for $n = 2, 3, \ldots$, by Proposition 2.3 that

$$E[\bar{f}(X_r^n(n))|X_r^n(2)] = E[\bar{f}(X_r^n(n - 1))|_{\partial = X_r^n(2)}].$$
The dominated convergence theorem, for conditional expectations, guaranties that
\[ E[\bar{f}(X^x_r(\infty))|X^x_r(2)] = E[\bar{f}(X^x_{\infty}(\infty))]|_{\theta=X^x_r(2)}. \]
Therefore, by (2.8),
\[ E[f(X^x_r(\infty))] = E[E[f(X^x_r(\infty))|X^x_r(2)] = E \left[ \frac{1}{\sigma(S_0(v))} \int_{S_0(v)} E[f(X^z_r(\infty))] \sigma(dz) \right]. \]
The result follows from Proposition 2.2.

For \( d \geq 2 \), as we have already mention, it is needed to impose some regularity condition on the frontier of \( V \) in order to get the continuity of \( h \) in \( V \).

**Definition 2.8.** We say that \( v \in \partial V \) is a regular point for \((V, f)\) if
\[ \lim_{x \to v} E[f(X^x_r(\infty))] = f(v). \]

**Remark 2.9.** If each point of \( \partial V \) is regular, then (2.7) is the solution to (DP).

As we have observed there exist \((V, f)\) such that (DP) does not have a solution. Hence it is convenient to have a sufficient condition to analyze the regularity of frontier points of \( V \). This is the reason why we introduce the following concept.

**Definition 2.10.** Let \( v \in \partial V \). A continuous function \( q_v : V \to \mathbb{R} \) is called a barrier at \( v \) if \( q_v \) is harmonic in \( V \), \( q_v(v) = 0 \) and
\[ q_v(x) > 0, \quad \forall x \in \overline{V}\setminus\{v\}. \]

**Theorem 2.11.** Let \( v \in \partial V \) be a point with a barrier \( q_v \), then it is regular.

**Proof.** Let \( M = \sup\{|f(x)| : x \in \partial V\} \). Let \( \varepsilon > 0 \), then there exists \( \delta > 0 \) such that
\[ x \in \partial V, \quad ||x - v|| < \delta \Rightarrow |f(x) - f(v)| < \varepsilon. \]
On the other hand, from (2.8)
\[ K := \inf\{q_v(z) : ||z - v|| \geq \delta, z \in \overline{V}\} > 0, \]
hence
\[ K^{-1}q_v(z) \geq 1, \quad \forall z \in \overline{V}, \quad ||z - v|| \geq \delta. \]
Therefore
\[ |f(x) - f(v)| < \varepsilon + 2M \leq \varepsilon + (2MK^{-1})q_v(x), \quad \forall x \in \partial V. \]
Let \((v_k)\) be an arbitrary sequence in \( V \) such that \( \lim_{k \to \infty} v_k = v \). Define \( X^{v_k}_r(\infty) \) as we did in (2.2). Proposition 2.3 implies,
\[ |E[f(v)] - E[f(X^{v_k}_r(\infty))]| \leq E[|f(v) - f(X^{v_k}_r(\infty))|] \leq \varepsilon + (2MK^{-1})E[q_v(X^{v_k}_r(\infty))]| \]
\[ = \varepsilon + (2MK^{-1})q_v(v_k). \]
From the continuity of \( q_v \) we are done. \( \square \)
As an application we get a classical condition for regularity of the points in the frontier of $V$. A point $v \in \partial V$ satisfies Poincaré’s condition if we have a ball $B_s(u)$ such that $\overline{V} \cap B_s(u) = \{v\}$.

**Proposition 2.12.** If $v \in \partial V$ satisfies Poincaré’s condition, then it is regular.

**Proof.** Merely observe that $q_v : \overline{V} \to \mathbb{R}$, defined as,

$$q_v(x) = \begin{cases} 
\log \left( \frac{||x-u||}{s} \right), & d = 2, \\
\frac{s^2-d - ||x-u||^2}{d}, & d \geq 3,
\end{cases}$$

is a barrier at $v$. $\square$

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