Screening and anti-screening in QED and in Weyl semimetals

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Distributions of charge near charged impurities in Weyl semimetals are considered with the help of relativistic Thomas-Fermi method in full analogy with the solutions previously found in QED. Screening and anti-screening, zero charge and asymptotic freedom solutions appearing in different physical situations are discussed.

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Introduction.– Existence of the Weyl semimetals, i.e. the materials with the points in Brillouin zone, where the completely filled valence and completely empty conduction bands meet with linear dispersion law, \( \omega = v_F p \), has been predicted indep.\(^1\). The order of value estimate of the Fermi velocity is \( v_F \sim 10^{-2}c \), where \( c \) is the speed of light in vacuum. Systems with relativistic dispersion law are likely to be realized in some doped silver chalcogenides,\(^2\) pyrochlore iridates,\(^3\) and in topological insulator multilayer structures.\(^4\) Weyl semimetals are 3-dimensional analogs of recently discovered graphene\(^5\), where the energy of excitations is also approximately presented by the linear function of the momentum but the electron subsystem is two dimensional one whereas the photon subsystem remains three dimensional. Even though the mass of excitations \( m = 0 \) for ideal graphene and Weyl semimetals without interactions, \( m \neq 0 \) can be induced in many ways resulting in the gapped dispersion relation\(^6\), \( \omega^2 = p^2 v_F^2 + m^2 v_F^2 \). To be specific and in order to avoid discussion of infrared divergence we consider below the gapped case. In difference with a weak value of the fine structure constant in QED, \( \alpha_{\text{QED}} = e^2/\hbar c = 1/137 \), the coupling in Weyl semimetals and in graphene is rather strong, \( \alpha = e^2/\hbar v_F \gtrsim 1 \). The effective coupling \( \alpha_{\text{eff}} = e^2/\hbar v_F \epsilon_0 \), where \( \epsilon_0 \) is the dielectric permittivity of the substance, can be as \( \gtrsim 1 \) as \( \ll 1 \) depending on the substance and both weak and strong coupling regimes are experimentally accessible. Thus Weyl semimetals and infinite stack of graphene layers make it possible to experimentally study various effects have been considered in 3+1 quantum electrodynamics (QED) for weak and effectively strong couplings.

Here we study the screening problem in Weyl semimetals, infinite stack of graphene layers and 3+1 QED. For that we use the relativistic Thomas-Fermi approach developed in QED and applied to find charge distributions near the extended charge source\(^7\) and the point-like charge source\(^8\). Recently Ref.\(^9\) argued for the zero charge effect for the impurity screening in Weyl semimetals in the limit of vanishing impurity size. We argue for another solutions.

Single-particle Coulomb problem. Falling to the center.– The energy of the electron levels in the Coulomb field \( V = -Z_0 e^2/r \) of a point charge \( Z_0 \) is found from the solution of the Dirac equation, see Ref.\(^10\).\(^1\)

\[
[k^2 - Q_0^2]^{1/2} - Q_0 \omega / \sqrt{m^2 c^4 - \omega^2} = n_r .
\]

Here \( \kappa = \mp(j + 1/2), \) \( j \) is the total angular momentum, \( n_r \) is the radial quantum number, \( n_r = 0, 1, 2... \) for \( \kappa < 0 \) and \( n_r = 1, 2,... \) for \( \kappa > 0 \), \( Q_0 = Z_0 \alpha_{\text{QED}} \) in QED case. Eq. (1) is symmetric under the simultaneous replacements \( \omega \to -\omega, Z_0 \to -Z_0 \). Thus, even for small \( Z_0 > 0 \) moreover the electron levels following from the upper continuum with \( \omega^{(1)} < mc^2 \) are levels with \( \omega_e^{(2)} = -\omega_e^{(1)} > -mc^2 \) following from the lower continuum. According to the traditional interpretation supported by experiments electron states with \( \omega_e^{(2)} > -mc^2 \) are interpreted as positron states for \( Z_0 < 0 \) with \( \omega_e^{(1)} = -\omega_e^{(2)} = \omega_e^{(1)} \).

The energy of the ground state electron level (\( n_r = 0, \kappa = -1 \)) appearing from the upper continuum decreases with increase of \( Z_0 > 0, \omega_0 = mc^2 \sqrt{1 - Q_0^2} \), and vanishes for \( Z_c = 1/\alpha_{\text{QED}} \). For \( Z_0 > 1/\alpha_{\text{QED}} \) the single particle problem for a point nucleus loses its meaning because of the collapse to the center. Eq. (1) is also valid for Weyl semimetals after replacement \( c \to v_F \). Thus for semimetals the falling to the center in the single particle problem may occur already for \( Z_0 > Z_c = 1/\alpha \sim 1 \). For massless particles in the Coulomb field the bound state appears for \( Z_0 > Z_c \) and the critical charge for the falling to the center is the same as for the massive particles.

Finite size nucleus.— To resolve the problem Ref.\(^12\) suggested to consider the central charge as smeared out over a sphere of the finite radius \( R \). For the nucleus of the finite radius the same as that for ordinary atomic nuclei, i.e. \( r_0 A^{1/3} \) (for \( r_0 \approx 1.2 \) fm and atomic number \( A \approx 2Z_0 \)), the electron ground state level in the QED case crosses the upper boundary of the lower continuum at nuclear charge \( Z_0 \approx 170 \), see Ref.\(^13\). The energy of the level with \( \omega < -mc^2 \) (\( Z > Z_c \) and/or \( R < R_c \)) acquires a negative imaginary part\(^14\) \( \omega = Re\omega + i\Im\omega \), \( \Im\omega < 0 \), corresponding to a decay of quasi-stationary state. Two electrons penetrate from the lower continuum to the ground state level in upper continuum that corresponds to tunneling of two positrons (the holes for Weyl semimetals) to infinity. For fixed \( Z_0 > 1/\alpha \) the energy of the ground state electron level continues to drop with decrease of \( R \) and crosses the upper boundary of the lower continuum (valence band, \( \omega_- = -mc^2 \) for Weyl semimetals). It occurs at the impurity charge \( Z_0 \to 1/\alpha + 0 \) (\( 1/\alpha_{\text{eff}} \) if we included screening) for \( R \to 0 \), see Ref.\(^15\). As was believed during long time, at least for \( Z_0 \) close to
$Z_c$, owing to Im $\omega < 0$ for the overcritical level and the Pauli principle, smearing of the charge source even for $R \to 0$ allows to solve the problem. However in reality it is not the case, since at small distances and in a strong field there arises a strong polarization of the vacuum and the bare Coulomb potential $V = -Z_0e^2/r$ should be replaced by $V = -Z_0e^2/\epsilon(r)r$, where $\epsilon(r)$ is the dielectric permittivity. The later quantity decreases with decrease of $r$ that effectively corresponds to increase of the charge at small distances. On the other hand, with increase of $Z_0$ and decrease of $R$ many levels cross the boundary of the lower continuum (valence band). The level density, $\rho(\omega < -mv_F^2) = dn/d\omega \sim 1/y$ for $0 < y = R\sqrt{\omega^2 - m^2v_F^2}/\sqrt{Q_0^2 - 1} < 1$, is crowding toward the boundary of the valence band. After tunneling electrons occupy all available levels with $\omega < -mc^2$ (vacuum shell) in QED and $\omega < -mv_F^2$ in case of Weyl semimetals. Thus the problems of the screening of the point-like charge with $Z_0 > 1/\alpha$, as well as of an extended charge source with $Z_0 \gg 1/\alpha$, are actually many-particle problems.

**Many-particle problem. Electron condensation in upper continuum.** Consider a spherically symmetric charged impurity with the number density $n_{\text{imp}}(r)$ embedded into a semimetal. Let $V = -e\lambda_0$ is the self-consistent potential for the electron. Since many electron states condense (in the sense that electrons in accordance with the Pauli principle occupy all available energy levels) in presence of impurity with a sufficiently large charge, they have in average large angular momenta. Thereby spin effects (except Pauli principle) are inessential and the electron (Fermi) momentum can be found from the Klein-Gordon equation

$$h^2\Delta \psi + p^2(r)\psi = 0, \quad p^2(r) = (\omega - V)^2/v_F^2 - m^2v_F^2. \tag{2}$$

The quasiparticle approximation holds at $\frac{d}{dr} \left( \frac{2}{p} \right) \ll 1$ that for the Coulomb field, $V = -Z_0e^2/r$, yields criterion $Z_0\alpha \gg 1$. However note that quasiclassical approximation reproduces exact Eq. (1) for purely Coulomb field for any $Z_0$ and has a good numerical accuracy also for finite size nucleus with $Z_0 \gtrsim 1/\alpha_{\text{ef}}$, see Eq. (5). E.g., with percent-age accuracy it allows to get expression for single-particle energy of electron levels below the upper boundary of the valence band,

$$\omega = -mv_F^2 \frac{R_c/R - (\kappa - 1/2)/2Q_0^2}{1 - (\kappa - 1/2)/2Q_0^2}, \tag{3}$$

with $Q_0 = Z_0\alpha_{\text{ef}}$, and critical radius $R_c = R_c(n_r, \kappa, Z_0)$. The classically accessible regions $p^2(r) > 0$ correspond to the upper and lower continua (conduction and valence bands), being curved in presence of the field and the region $p^2(r) < 0$ corresponds to the gap. The electron levels are in the upper continuum, if they drive to $\omega = mc^2$ ($\omega > mv_F^2$) with switching of the potential $V < 0$, and they would be in the lower continuum, if they reached $\omega = -mc^2$ ($\omega = -mv_F^2$) with switching of the potential. As we mentioned, in the later case following the traditional interpretation after the replacement $\omega \to -\omega$, $\kappa \to -\kappa$ the levels are interpreted not as electron levels in the lower continuum but as levels of positrons/holes (in the repulsive potential $-V$ for the electron) in the upper continuum. We return to this interpretation below.

Within the relativistic Thomas-Fermi approximation the chemical potential $\omega_-$ is determined by the last level filled by the electrons $-mv_F^2 \leq \omega_- \leq mc^2$. If there are no free electrons in the system, in the field $V$ only the levels of the vacuum shell $\omega < \omega_- = -mv_F^2$ are occupied by electrons penetrated from the valence band, that corresponds to $p^2(r) = (\omega_- - V)^2/v_F^2 - m^2v_F^2$. The holes (positrons in QED) go off to infinity. If there is sufficient amount of electrons in the conduction band (case of the neutral atom in QED), then $\omega_- = mv_F^2$.

The number density of filled electron states in upper continuum is given by

$$n_e(r) = gp^2\left(6\pi^2h^3\right)^{-1}, \tag{4}$$

where $g$ is the degeneracy factor, $g = 24$ for pyrochlore iridates and $g = 2$ in a topological insulator multilayer and for electrons in QED. With the Coulomb law the number of electrons in electron condensate $n_e = \int \rho_{\text{imp}}(r) dr \propto -\ln R$ diverges at $R \to 0$ showing that ”electron condensate” should significantly modify the Coulomb law at small distances.

The relativistic Thomas-Fermi equation for the description of the charge distribution is as follows, cf. Eq. (5):

$$\nabla(\rho(r)\nabla V(r)) = 4\pi e^2[n_{\text{imp}}(r) - n_e(r)], \tag{5}$$

with the boundary conditions at the ion radius $r_i$:

$$V(r_i) = \omega_- - mv_F^2 \geq -2mv_F^2, \tag{6}$$

$$V'(r_i) = (Z(r)e^2/\epsilon(r)r)^{1/3}_{r=\alpha R} = \frac{Z_0}{\alpha_0^{1/3}}. \tag{7}$$

The first condition follows from the requirement $n_e(r_i) = 0$. In case of the empty conduction band $\omega_- = -mv_F^2$, and for the metal $\omega_- = mv_F^2$, $r_i = \infty$. At distances, where the screening is most effective $|V| \gg mv_F^2$ and Eq. (5) simplifies as

$$\nabla(\nabla V) = 4\pi e^2[n_{\text{imp}} + gV^3/(6\pi^2h^3v_F^3)], \quad V < 0. \tag{7}$$

1. **Limit of strong coupling.** Let the bare charge distribution in impurity is $n_{\text{imp}} = n_0\Theta(R - r)$, $n_0 = Z_0/(4\pi R^3/3) = \text{const}$, $\Theta(x) = 1$ for $x > 0$ and $0$ for $x < 0$ is the step function. Assume $R \gg l$. Then at typical distances $x = (r - R)/l \sim 1$ the geometry is reduced to the flat one. Also assume that the dielectric permittivity $\epsilon$ varies at distances $l'_c \gg l$, where $l'$ is the typical size of the change of the electric field. Then at distances $l_c \sim l$ our interest in the given case may not vary $\epsilon$. Replacing $V = -V_0\chi(x)$ in Eq. (7) we find, cf. Eq. (8):

$$\chi'' = \chi^3 - \Theta(-x), \quad V_0 = h\chi_0(6\pi^2n_0/g)^{1/3}, \tag{8}$$

$$l'^2 = \frac{\epsilon}{\epsilon_0\alpha_{\text{ef}}} \left(\frac{3}{32\pi g n_0}\right)^{1/3} = \frac{\epsilon R^2}{3^{2/3}\epsilon_0(AZ_0^2)^{1/3}}, \tag{8}$$
where $\lambda = \frac{2 e^2}{\alpha^2 a_0^4}$. Inequality $V_0 \gg \hbar v_F^2$ yields $n_0 \gg g(mv_F/\hbar)^2/(6\pi^2)$. The used condition $R > l$ is satisfied for $(\lambda Z_0^2)^{1/6} \gg (\epsilon/\epsilon_0)^{1/2}$ for arbitrary $R$, even for $R \to 0$.

Solutions of (8) are as follows:

$$
\chi(x) = 1 - \frac{3}{1 + 2^{-1/2} \sinh(a - x/\sqrt{3})}, \quad x < 0, \quad (9)
$$

$$
\chi(x) = \sqrt{2}/(x + b), \quad x > 0. \quad (10)
$$

Matching of these solutions with derivatives at $x = 0$ yields values of constants: $a = 11\sqrt{2}$ and $b = 4\sqrt{2}/3$.

Note that the solution (10) would have a pole at $r = r_{\text{pole}}$, if it were analytically continued in the region $x < 0$. In reality the pole does not manifest since $R > r_{\text{pole}}$. The solutions (9, 10) demonstrate that for $(\lambda Z_0^2)^{1/6} \gg 1$ (at $\epsilon(R) \simeq \epsilon_0$), i.e. in the strong coupling limit the impurity interior is neutral, whereas the charge is repelled to a narrow (of the length $l \ll R$) layer near the droplet surface. The total charge $Z(R)$ situated inside the impurity is found with the help of the Gauss theorem, cf. Ref. Z, $Z(R)/Z_0 = 3^{1/3}.2^{-3/2}(\lambda Z_0^2)^{-1/6}$. Solution (10) becomes invalid at $r \gg R$ when the charge $Z(r)$ decreases up to values $Z(r) \sim 1/\lambda^{1/2}$. For still larger $r$ the system is described in the effectively weak coupling regime: $\lambda Z^2(r) \ll 1$. In case $\omega_- = mv_F^2$ the charge tends to zero but since this residual screening occurs at $r \gg R$, the solution does not manifest the Landau zero charge effect, i.e. full screening of any bare charge at $r \lesssim R$, for $R \to 0$. For the gapless case $(m = 0)$ the screening stops for $Z(r) < 1/\alpha_{\text{eff}}$, since for such $Z$ there are already no electron levels in the Coulomb field.

2. Limit of weak coupling. Existence of a minimal radius. Introduce convenient variables

$$
V = Q_1(r)/r = -Z(r)e^2/r, \quad t = -\ln(r/a). \quad (11)
$$

Although $a$ is an arbitrary constant it is convenient to make a specific choice, e.g. $a = r_{\text{pole}}$, $a = h/(\hbar v_F)^{1/3}$, or $a = R^{10}$.

In the new variables Eq. (8) becomes for $r > R$

$$
\frac{d}{dt} \left[ \frac{\epsilon(t)Z(t)}{\epsilon_0} \right] + \frac{d}{dt} \left[ \frac{\epsilon(t) dZ(t)}{\epsilon_0} \right] = \lambda Z^3(t) \xi^2 \Theta(t), \quad (12)
$$

$$
\xi = 1 + \frac{2\omega_-e_0 a e^{-t}}{Z(t)e^2} + \frac{\omega_-^2a^2 e^{-2t}}{Z^2(t)e^4}. \quad (13)
$$

2.1. Approximation $\epsilon = \epsilon_0 = \text{const}$. The spatial dependence of $\epsilon$ can be disregarded for rather extended charged objects. With $\epsilon = \epsilon_0$, Eq. (12) reduces to

$$
\frac{dZ(t)}{dt} + \frac{d^2 Z(t)}{dt^2} = \lambda Z^3(t) \xi^{3/2} \Theta(t), \quad r > R. \quad (13)
$$

In QED the solution of this equation has been found in Ref. Z and matched with the Coulomb law for $r = r_i$.

As has been shown in Ref. Z, Eq. (13) has the pole solution

$$
Z(t) = \frac{\sqrt{2/\lambda}}{t_{\text{pole}} - t} \left[ 1 + \frac{1}{6}(t_{\text{pole}} - t) + \ldots \right], \quad (14)
$$

for $t_{\text{pole}} - t \ll 1$. In the strong coupling limit the first term transforms to the solution (10) after change of variable $t$.

Near the pole the second term in the l.h.s. of Eq. (13) is the dominant one, i.e. $|\frac{d^2 Z(t)}{dt^2}| \ll |\frac{d Z(t)}{dt}|$, whereas the first term determines the correction in brackets of (14).

The value $t_{\text{pole}}(\lambda)$ can be obtained by numerical solution of Eq. (13) with the boundary condition (10). For $\omega_- = -mv_F^2$, $v_F = c$, $\epsilon_0 = 0$, this solution was found in Ref. Z. In the weak coupling limit, $(\lambda Z(r^2)^{1/6} \ll 1$, the value $t_{\text{pole}} = 1/8\mu - \ln D$, where $\ln D(\mu)$ is a smooth function of $\mu = \lambda Z^2(t = 0)/(2g)$, e.g., $D \simeq 0.2\mu^{1/2}$ for $0.4 < \mu < 1$, see Fig. 4 in Ref. Z. This result approximately (due to a smooth logarithmic dependence of $t_{\text{pole}}$ on $D$) holds also for $\omega_- \neq -mv_F^2, v_F \neq c$. Thus in the weak coupling limit, for any value of the charge at large distances $Z(r > r_i) = Z_\infty$ the solution Eq. (13) matched with the purely Coulomb law for $r > r_i$ exists only if the impurity has the radius $R > r_{\text{pole}}$.

In the limit $\frac{|dZ(t)|}{d^2 Z(t)/dt^2} \gg \frac{|d Z(t)|}{d^2 Z(t)/dt^2}$, for $r \ll r_i$, i.e. for $\xi \simeq 1$, Eq. (13) also has analytic solution

$$
Z(t) = Z(t = 0)/\sqrt{1 - 2\lambda Z^2(t = 0)t}, \quad (15)
$$

where $Z(t = 0) = Z(r = a)$. Comparing values $|\frac{dZ(t)}{dt}|$ and $|\frac{d^2 Z(t)}{dt^2}|$ we see that solution (13) holds only for $\lambda Z^2(t = 0)/(1 - 2\lambda Z^2(t = 0)t) \ll 1$. As we see, for $a = r_i$ (then $t \geq 0$) this inequality holds only in the weak coupling limit $\lambda Z^2(t = 0) \ll 1$ and at distances when $1 - 2\lambda Z^2(t = 0) \gg \lambda Z^2(t = 0)$. Taking $a = r_i, r > R < r_{\text{pole}}$ in (15) we recover the relation between the observable charge $Z_\infty = Z(r \geq r_i)$ and the charge $Z(r = R)$ at the surface of impurity

$$
Z_\infty \simeq Z(R)[1 - \lambda Z^2(R) \ln \frac{r_i}{R}], \quad (15)
$$

where $\lambda Z^2(R) \ln \frac{r_i}{R} \ll 1$. Summation in LLA recovers solution (16). Note that (15) has the square-root singularity for $t \to t_{\text{LLA}} \simeq \lambda Z^2(\epsilon_0) / \lambda Z^2(t = 0)$, whereas the real solution has pole for $t \to t_{\text{pole}} < t_{\text{LLA}}$. This is because the LLA becomes invalid in the region where $1 - \lambda Z^2(t = 0) \ll \lambda Z^2(t = 0)$. In this region sub-leading corrections $\mu^{2n-1}$ omitted in the derivation should be incorporated.

On the other hand, solution (15) coincides with the one obtained in Ref. Z for $a = R$ ($t \leq 0$):

$$
Z(r \geq R) = Z(R)/\sqrt{1 + 2\lambda Z^2(R) \ln \frac{r}{R}. \quad (18)}
$$
Now the criterion \( |d^2 \varphi | \gg |d \varphi | \) holds for \( \lambda Z^2(R)/(1 + 2 \lambda Z^2(R) \ln \frac{R}{R_0}) \ll 1 \), i.e. for all \( t \leq 0 \), but only for \( \lambda Z^2(R) \ll 1 \), i.e. in the weak coupling limit. Contrary, Ref.\(^{10}\) used solution \( \text{(18)} \) for any \( r > R \) at arbitrary small \( R \) in the strong coupling limit \( \lambda Z^2(R) \gg 1 \) pointing on the zero charge behavior of the charge distribution, i.e. dropping of the charge \( Z(r) \) to zero in the narrow vicinity of \( r = R \). However \( \text{(15)} \) does not hold in this strong coupling limit at \( r \sim R \). Also, for gapless substances, as considered in Ref.\(^{19}\), the relativistic Thomas-Fermi approximation holds only for \( Z(r) \alpha_{\text{ct}} \gg 1 \) and thus solution \( \text{(18)} \) is not correct for \( r \gg R \) when \( Z(r) \) becomes \( \ll 1/\alpha_{\text{ct}} \). Thus we do not support conclusion of Ref.\(^{19}\) on the zero charge behavior of the solution of the relativistic Thomas-Fermi equation for the point charge source.

Concluding above discussion, the receipt to consider the point source of charge as smeared one for \( R \to 0 \) to resolve the problem of the falling to the center, which worked in the single-particle problem, does not work in the many-particle problem in the weak coupling limit. Repeating the results\(^{12}\) we stress that for \( \varepsilon = \text{const} \) and \( \lambda Z^2 \ll 1 \) relevant solutions of Eq.\( \text{(12)} \) reproduce the Coulomb law at large distances, but only for extended charged droplets with \( R > r_{\text{pole}} \).

2.2. Taking into account spatial dispersion of \( \varepsilon \). The dielectric permittivity of the 3+1 QED vacuum in electric field can be found with the help of the interpolation formula, valid with logarithmic accuracy\(^{9,17,12}\):

\[
e_{\text{QED}}(r) = 1 - \frac{\alpha_{\text{QED}}}{3\pi} \ln \left[ \frac{eE}{m^2c^2} + \frac{\hbar^2}{r^2m^2c^2} + 1 \right], \tag{19}
\]

where \( eE = -V'(r) = Q(r)/r^2 \) is the electric field tension. From this equation in case of the strong homogeneous electric field we recover the result of Heisenberg and Euler\(^{12}\). In case of the strong inhomogeneous electric field, for \( Q(r) \gg 1 \), Eq.\( \text{(19)} \) yields the solution\(^{12}\) that generalizes the Heisenberg and Euler result. In case of a weak field we recover the Uehling and Serber correction at \( r < \hbar/mc \) (see second term \( \text{(19)} \)). Also Eq.\( \text{(19)} \) correctly transforms to the result derived in LLA\(^{11}\).

Note that the latter result is derived at the condition \( \ln(h/rmc) \gg 1 \) being formally valid as for \( \varepsilon > 0 \) as for \( \varepsilon < 0 \) (for \( |\varepsilon| \gg \alpha_{\text{QED}}/3\pi \)).

At small distances the field becomes strong independently of whether the observed charge \( Z_{\infty} \) is large or small. Indeed, in absence of the electron condensation \( Z(r) = Z_{\infty}/\varepsilon(r) \) and dielectric permittivity decreases as \( r \) decreases. Thus, at small distances any case we have \( Z(r) > 1/\alpha_{\text{QED}} \) and the electron condensation occurs at levels in the upper continuum, which have crossed the boundary \( \omega_\ast = -mc^2 \).

In matter at large distances \( r \gg l_\ast \) one has \( \varepsilon \simeq \text{const} = \varepsilon_0 \). But at much smaller distances and/or in presence of very strong electric field the screening should occur similar to that in vacuum. Thus the interpolation formula describing both large and small distances is as follows

\[
\varepsilon(r) = \varepsilon_0 \left( 1 - \frac{g_{\text{ct}}}{6\pi} \ln \left[ \frac{eEh}{m^2c^3} + \frac{\hbar^2}{r^2m^2c^2} + 1 \right] \right), \tag{20}
\]

Assume \( \left| \frac{d\varepsilon}{dt} \right| \gg \left| \frac{dZ}{dt} \right| \). Then disregarding second term in the l.h.s. of Eq.\( \text{(12)} \), setting \( \xi = 1 \) in the r.h.s. and using that at the above condition the charge is a smooth function, \( Q \gg Q_1 \), we arrive at the solution

\[
Z^2(r) = (-2\alpha_{\text{ct}}^2 + Cc^2)^{-1}. \tag{21}
\]

Constant \( C \) can be found from interpolation of the solution to the Coulomb law at large distances, i.e. from that \( \varepsilon \to \varepsilon_0 \), \( Z \to Z_{\infty} \) for \( r \to \infty \). Thus we obtain

\[
Z^2(r) = -2\alpha_{\text{ct}}^2 Z_{\infty}^2 (1 + 2\alpha_{\text{ct}}^2 Z_{\infty}^2) e^2(r)/\varepsilon_0^2. \tag{22}
\]

This equation has the inflexion point at \( t = t_{\text{infl}} \) at which \( dZ/dt = \infty \). However already before reaching this point, i.e. for \( \varepsilon = -\sqrt{2\alpha_{\text{ct}}^2 Z_{\infty}^2 (1 + 2\alpha_{\text{ct}}^2 Z_{\infty}^2)} \sim g_{\text{ct}}/6\pi \), inequality \( \left| \frac{d\varepsilon}{dt} \right| \gg \left| \frac{dZ}{dt} \right| \) becomes invalid. \( Z(r) \) continues to grow with decrease of \( r \). For smaller \( r \) inequality \( \left| \frac{d\varepsilon}{dt} \right| \ll \left| \frac{dZ}{dt} \right| \) is fulfilled and the pole-like solution is generated. Using that \( \varepsilon \) is still a smooth function of \( t \), except very near the pole, we obtain

\[
Z(r) \simeq \sqrt{\frac{2\varepsilon_0}{\lambda\varepsilon_0}} \frac{1}{\sqrt{t - t_{\text{pole}}}}, \tag{23}
\]

similar to \( \text{(14)} \). This solution ceases to be useful only for \( \varepsilon \gtrsim g_{\text{ct}}/6\pi \), since the condition of smoothness of variation of \( \varepsilon \) becomes invalid. In this region, before occurring of the pole but very near the pole, the solution\(^{23}\) has a point of inflection \( t = t_{\text{infl}} \). Thus Eq.\( \text{(12)} \) has a solution that falls off with increase of \( r \) and reduces to the Coulomb solution only if the radius of the source is \( R > r_{\text{pole}} > r_{\text{infl}} \). This is how the falling to the center manifests in the many-particle problem provided spatial dispersion of \( \varepsilon \) and electron condensation on the levels in the conduction band (upper continuum) are included. We see that the initial problem of the falling to the centre became more severe than in the single particle case where any smearing of the charge source (even for \( R \to 0 \)) was sufficient to overcome difficulties.

Impurity of an arbitrary small size. Hypothesis of electron condensation in lower continuum.— In Ref.\(^{12}\) a solution of the problem for the charge distribution near a source of the radius \( R < r_{\text{pole}} \) was proposed in case of QED. One observes that the dielectric permittivity given by Eqs.\( \text{(19)} \) and \( \text{(20)} \) becomes negative for \( r < r_\ast \), \( r_\ast \) is the point where \( \varepsilon = 0 \). Thereby attraction in the original potential (for \( Z_0 > 0 \)) is replaced by an effective repulsion to the electron at small distances, \( V > 0 \). On the other hand, in the repulsive potential either positrons (holes) are accumulated in the upper continuum (this case after replacement \( Z \to -Z \))
yields the same solution, as we have found for electrons in the attractive field, being valid only for $r > r_{\text{pole}}$ or the electrons are condensed right in the lower continuum (without any tunneling) as it is allowed by the symmetry $\omega \rightarrow -\omega$, $V \rightarrow -V$ of Eq. (1). Thereby now consider possibility of the electron condensation on the levels in the lower continuum in the repulsive potential.

The charge distribution is then determined by

$$\nabla(e\nabla V) = 4\pi e^2 [n_{\text{imp}} - gV^3/(6\pi^2 h^3 v_F^3)], \quad V > 0. \quad (24)$$

For a point charge $Z_0$ we have $n_{\text{imp}} = Z_0 \delta(r)$. Notice change of the sign in the second term in the r.h.s. compared to that in Eq. (1) since this term corresponds now to electrons condensed on levels of the lower continuum in the field $V > 0$. For $r > 0$ Eq. (24) reduces to

$$\frac{d}{dt} \left( \frac{\epsilon(t)}{\epsilon_0} Z(t) \right) = -\lambda Z^3(t). \quad (25)$$

Assuming $\left| \frac{d\epsilon}{dt} \right| \gg \left| \frac{d\epsilon Z}{dt} \right|$ we now obtain

$$Z^2(r) = (2\alpha e^2 + C_1 \epsilon^2)^{-1}. \quad (26)$$

Constant $C_1$ can be found from the condition $Z^2(r \rightarrow 0) \rightarrow Z_0^2$ since then $\epsilon^2 \rightarrow \infty$. Thus we obtain

$$Z^2(r) = \frac{Z_0^2}{2\alpha e^2 Z_0^2 + \epsilon^2}. \quad (27)$$

The solution is similar to that found in QCD with taking into account of the quark condensation in glu-electric field. A similarity and difference between QED and QCD solutions were analyzed in Ref. [13]. Solution (27) corresponds to asymptotically free regime at extremely small distances $Z(r \rightarrow R \rightarrow 0) \rightarrow -Z_0/|\epsilon(R)| \rightarrow 0$. Notice difference in the sign of $Z(r)$ and $Z_0$. $Z^2(r)$ grows with $r$ and reaches maximum at $\epsilon = 0$ (for $r = r_0$). Then $Z^2(r)$ decreases with subsequent increase of $r$. At $r \gg r_0$, $\epsilon \rightarrow \epsilon_0$ and $Z(r) \rightarrow Z_{\infty} = -Z_0/\sqrt{2\alpha e^2 Z_0^2 + \epsilon_0}$. For $Z_0 \ll 1/\alpha e^2$, $Z_{\infty} \simeq -Z_0/\epsilon_0$. In QED such a solution proves to be consistent with the renormalization relation between the bare and physical charges that argues in favor of consistency of QED as theory with point interaction. It would be very interesting to experimentally check this peculiar possibility, e.g. measuring the field of a nucleus embedded in the Weyl semimetal.

Concluding, in the strong coupling limit, $Z_0 \gg e^{-3/2}$, owing to the screening the interior of the impurity embedded in a Weyl semimetal becomes electrically neutral even for $R \rightarrow 0$ and the charge is repelled in a narrow layer near the surface, $\Delta r \ll R$. With taking into account of the electron condensation on the levels in the upper continuum (conduction band) and $\epsilon(r) > 0$ we also found the charge distribution for impurity with a radius $R > r_{\text{pole}}$ in the weak coupling limit, $\alpha^{-1}_e \ll Z_0 \ll e^{-3/2}$. For the charge source with the radius $R < r_{\text{pole}}$ there also exists a solution demonstrating asymptotic freedom, provided negativeness of the dielectric permitivity at small distances and the electron condensation on the levels in the lower continuum (in valence band). Thus Weyl semimetals give intriguing possibility to experimentally check validity of these solutions, in particular to verify the hypothesis of the electron condensation on the levels of the lower continuum, possibilities of $\epsilon < 0$ and asymptotic freedom at small distances.

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20. More detailed derivation see in, the relativistic quasiclassics for the Dirac equation was developed in [16].
21. Putting $R \rightarrow 0$, in reality we require only that $R \ll r_{\text{pole}}$. 

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