CRITICAL VALUE OF THE QUANTUM ISING MODEL ON
ST AR-LIKE GRAPHS

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Abstract. We present a rigorous determination of the critical value of the
ground-state quantum Ising model in a transverse field, on a class of planar
graphs which we call star-like. These include the star graph, which is a junction
of several copies of \( Z \) at a single point. Our approach is to use the graphical,
or FK-, representation of the model, and the probabilistic and geometric tools
associated with it.

1. Introduction

The Hamiltonian of the quantum Ising model with transverse field on a finite
graph \( G = (V, E) \) is the operator

\[
H = -\frac{1}{2} \lambda \sum_{e=xy \in E} \sigma_x^{(3)} \sigma_y^{(3)} - \delta \sum_{y \in V} \sigma_x^{(1)}
\]
on the Hilbert space \( \mathcal{H} = \bigotimes_{x \in V} \mathbb{C}^2 \). Here the Pauli spin-1/2 matrices

\[
\sigma_x^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
and we use as basis for each copy of \( \mathbb{C}^2 \) in \( \mathcal{H} \) the vectors \( |+\rangle_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( |\rangle_x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \); also, \( \lambda, \delta > 0 \) are the spin-coupling and external-field intensities, respectively. Let
\( \beta \geq 0 \) denote the inverse temperature, and define the positive temperature states

\[
\rho_{G, \beta}(Q) = \frac{1}{Z_{G}(\beta)^{\text{tr}(e^{-\beta H} Q)}},
\]
where \( Z_{G}(\beta) = \text{tr}(e^{-\beta H}) \) and \( Q \in \mathbb{C}^{2 \times 2} \). Also define the ground state to be the limit
\( \rho_G \) of \( \rho_{G, \beta} \) as \( \beta \to \infty \). If \( G_n \) is an increasing sequence of graphs tending to an infinite
graph \( S \), then we may also speak of infinite-volume limits \( \rho_{S, \beta} = \lim_{n \to \infty} \rho_{G_n, \beta} \) and \( \rho_S = \lim_{n \to \infty} \rho_{G_n} \). The existence of these limits is discussed in [2].

In this article we will use the FK- or random-cluster representation of the ground
state, see for example [12] and references therein. Details will be provided in
the next section, but roughly speaking the FK-representation may be considered as a
limit of “discrete time” random-cluster models on \( S \times (\varepsilon \mathbb{Z}) \) as \( \varepsilon \downarrow 0 \). This is related
to the well-known mapping of the quantum Ising model onto the classical Ising
model in one dimension higher [17], and the FK-representation of that model [8].
The relevance of this representation is that it relates the occurrence of long range order
in the ground state to the existence of infinite percolation paths in \( S \times \mathbb{R} \);
here we say that the model exhibits long range order if for all \( x \), the correlation function

\[
G(x, y) = \rho_S(\sigma_x^{(3)} \sigma_y^{(3)})
\]
is bounded below by a positive function of \( x \). There is a critical value of the ratio \( \lambda/\delta \) above which the model exhibits long range order, and below which it does not.

The main result of this article is a rigorous determination of the critical ratio for a certain class of planar graphs \( S \) (see Definition 2). This extends the calculation for the graph \( S = \mathbb{Z} \), to, amongst other graphs, the star graph, which is the junction of several copies of \( \mathbb{Z} \) at a single point. See Figure 1. A special case of our main result (Theorem 8) is therefore the following.

**Theorem 1.** The critical ratio for the ground state quantum Ising model on the star graph is \( \lambda/\delta = 2 \).

![Figure 1. The star graph has a central vertex of degree \( k \geq 3 \) and \( k \) infinite arms, on which each vertex has degree 2. In this illustration, \( k = 4 \).](image-url)

In other words the critical ratio is the same for the star as for \( \mathbb{Z} \); this is to be expected since the star is only *locally* different from \( \mathbb{Z} \). We emphasise, however, that the class of graphs for which we prove this result contains many more graphs than just the star.

The quantum Ising model on \( \mathbb{Z} \) has been thoroughly studied, and the critical ratio \( \lambda/\delta = 2 \) has been computed for this model in for example [16]. See also [17] and references therein. These calculations have relied on matrix methods and techniques such as Jordan–Wigner transformation. Recently, in [5], sharpness of the phase transition, and hence exponential decay of correlations below the critical point, was established rigorously for \( G = \mathbb{Z}^d \) with any \( d \geq 1 \), using graphical methods similar to the corresponding proof [1] for the classical Ising model. Combining this result with duality arguments analogous to the classical two-dimensional random-cluster model [8], this gives another proof that the critical ratio is \( \lambda/\delta = 2 \) for this model, using only tools from stochastic geometry (see [5] for details). One aim of this paper is to extend and illustrate the graphical methods, and show how they can be applied to a wider range of structures than just \( \mathbb{Z} \). The Ising model on the star-graph has also recently arisen in the study of boundary effects in the two-dimensional classical Ising model, see for example [14, 15]. Similar geometries have also arisen in different problems in quantum theory, such as transport properties of quantum wire systems, see [6, 11, 13].

### 2. Background and notation

In this article we will let \( G = (V, E) \) be a *star-like graph*:

**Definition 2.** A star-like graph is a countably infinite connected planar graph, in which all vertices have finite degree and only finitely many vertices have degree larger than two.
A star-like graph $G$ (left) and its line-hypergraph $H$ (right). Any vertex of degree $\geq 3$ in $G$ is associated with a “polygonal” hyperedge in $H$.

Such a graph is illustrated in Figure 2; note that the graph of Theorem 1 is an example in which exactly one vertex has degree at least three. Fix a planar embedding $G$ of $G$, and denote $X = G \times \mathbb{R}$; also let $X = G \times \mathbb{R} := (V \times \mathbb{R}, E \times \mathbb{R})$. We will sometimes use $X$ and $Y$ interchangeably. Let $O$ be a fixed but arbitrary vertex of $G$ of degree two or more, which we think of as the origin.

Recall that a hypergraph is a set $W$ together with a collection $F$ of subsets of $W$, called edges; a graph is a hypergraph in which all edges contain two elements. In our analysis we will use a suitably defined hypergraph “dual” of $X$: let $H = (W, F)$ be the “line-hypergraph” of $G$, where $W = E$ and the set $\{e_1, \ldots, e_n\} \subseteq E = W$ is in $F$ if and only if $e_1, \ldots, e_n$ are all the edges adjacent to some particular vertex of $G$. Note that only finitely many edges of $H$ have size larger than two. There is a natural planar embedding of $H$ defined via the embedding $G$, in which an edge of size more than two is represented as a polygon. See Figure 2. Let $Y = H \times \mathbb{R}$ and $Y = H \times \mathbb{R}$.

Our configuration space $\Omega$ will be the set of pairs $\omega = (B, D)$ where $B \subseteq E \times \mathbb{R}$ and $D \subseteq V \times \mathbb{R}$ are locally finite, which is to say that $B \cap (\{e\} \times [-n, n])$ and $D \cap (\{v\} \times [-n, n])$ are finite sets for all $e \in E$, $v \in V$ and $n \in \mathbb{N}$. We think of $B$ as a set of bridges and $D$ as a set of deaths or cuts. There is a natural embedding of any $\omega \in \Omega$ into $X$, where deaths are represented as missing points and bridges as “horizontal” lines connecting two “vertical” lines. See Figure 3 for an illustration of this when $G = \mathbb{Z}$. Often we will identify $\omega \in \Omega$ with its embedding. Denote by $d(\cdot, \cdot)$ the graph distance in $G$, and let $\Lambda_n$ denote the set of points $(v, t)$ and $(e, t)$ where $v \in V$ has $d(v, O) \leq n$, $e \in E$ has at least one endpoint at distance at most $n$ from $O$, and $|t| \leq n$. For each $\omega \in \Omega$, we will employ two restricted embeddings $\omega_n^1$ and $\omega_n^0$, one “wired” and one “free”. The free embedding $\omega_n^0$ is simply the intersection of (the natural embedding of) $\omega$ with $\Lambda_n$. The wired embedding $\omega_n^1$ is defined by

\begin{equation}
\omega_n^1 = \omega_n^0 \cup \{(v, t) : d(v, O) = n + 1, |t| \leq n\} \cup \{(e, \pm n) : e \in G_n\},
\end{equation}
where we have taken the liberty to identify \( v \in V \) and \( e \in E \) with their embeddings in \( G \). In words, \( \omega_n^1 \) is obtained by tying together the top and bottom of \( \omega_n^0 \), as well as all bridges protruding from its "sides". We let the functions \( k_0^0, k_1^1 : \Omega \to \mathbb{N} \) count the number of connected components of \( \omega_n^0 \) and \( \omega_n^1 \), respectively.

Equip \( \Omega \) with the Skorokhod topology and the associated \( \sigma \)-algebra; the details of their definitions are not immediately important, but may be found in \([3]\) or \([4]\). Fix \( \lambda, \delta > 0 \) and let \( \mu = \mu_{\lambda, \delta} \) be the probability measure on \( \Omega \) governed by a collection of independent Poisson processes \( B_e \) on \( \{e\} \times \mathbb{R} \), for \( e \in E \), and \( D_v \) on \( \{v\} \times \mathbb{R} \), for \( v \in V \). Here each \( B_e \) has intensity \( \lambda \), each \( D_v \) has intensity \( \delta \), and \( B = \cup_{e \in E} B_e, D = \cup_{v \in V} D_v \). This \( \mu \) is the space-time (or "continuum") percolation measure of \([9]\).

We may now define the random-cluster probability measures.

**Definition 3.** The random-cluster measure \( \Phi_n^b \) on \( \Lambda_n \) with parameters \( \lambda, \delta, q > 0 \) and boundary condition \( b \in \{0, 1\} \) is the probability measure on \( \Omega \) given by

\[
\frac{d\Phi_n^b}{d\mu}(\omega) \propto q^{k_n^b(\omega)}, \quad \omega \in \Omega.
\]

Let

\[
\theta^b = \Phi^b((\mathcal{O}, 0) \text{ lies in an unbounded component}).
\]

The following basic facts may be proved in a conventional manner, as in \([8, \text{Theorem 5.5}]\); details for this particular model may be found in \([4]\).

**Proposition 4.** Let \( q \geq 1 \). The weak limits \( \Phi^b := \lim_{n \to \infty} \Phi_n^b \) exist, and enjoy a phase transition in the sense that there is \( \rho_c = \rho_c(q) \in (0, \infty) \), depending only on \( q \) (and \( G \)), such that \( \theta^0 = 0 \) if \( \lambda/\delta < \rho_c \) and \( \theta^b > 0 \) if \( \lambda/\delta > \rho_c \). We call \( \rho_c \) the critical value of the random-cluster model on \( G \times \mathbb{R} \).

The relevance of the space-time random-cluster measures to the quantum Ising (or more generally quantum Potts) model is explained in \([2]\); in particular the ground state quantum Potts model on \( G \) exhibits long-range-order iff the corresponding random-cluster model has \( \theta^b > 0 \). Hence, to investigate the phase-diagram of the
quantum Ising model we will set $q = 2$ and focus on finding the critical value $\rho_c$ above which percolation occurs.

Let us say a few more words about the “dual” $Y$ of $X$. Given any configuration $\omega \in \Omega$, one may associate with it a dual configuration on $Y$ by placing a death wherever $\omega$ has a bridge, and a (hyper)bridge wherever $\omega$ has a death. This is illustrated in Figure 4. More precisely, we let $\Omega_d$ be the set of pairs of locally finite subsets of $F \times \mathbb{R}$ and $W \times \mathbb{R}$, and for each $\omega = (B, D) \in \Omega$ we define its dual to be $\omega_d := (D, B)$. As before, we may identify $\omega_d$ with its embedding in $Y$, noting that some bridges may be embedded as polygons. We let $\Psi^b_n$ and $\Psi^b$ denote the laws of $\omega_d$ under $\Phi^{1-b}_n$ and $\Phi^{1-b}$ respectively.

The case when $G = \mathbb{Z}$ is particularly important, and for this case we use the lower case symbols $\phi$ and $\psi$ in place of $\Phi$ and $\Psi$, respectively. When $G = \mathbb{Z}$, the dual space $Y$ is isomorphic to $X$, and we have the following result. Again the proof is similar to that for the discrete random-cluster model on $\mathbb{Z}^2$, but details for our model may be found in [4].

**Lemma 5.** If $\phi^b_n, \phi^b$ have parameters $q, \lambda$ and $\delta$, then the dual measures $\psi^{1-b}_n, \psi^{1-b}$ are random cluster measures with parameters $q' = q, \lambda' = q\delta$ and $\delta' = \lambda/q$, and boundary condition $1 - b$.

Recall that there is a partial order on $\Omega$ given by $(B', D') = \omega' \geq \omega = (B, D)$ if $B' \supseteq B$ and $D' \subseteq D$, and that an event $A$ is called increasing if whenever $\omega \in A$ and $\omega' \geq \omega$ then also $\omega' \in A$. Also recall that $A$ is called a cylinder event if it only depends on a bounded region of $X$, which is to say that there is a bounded set $\Lambda \subseteq X$ such that if $\omega = \omega'$ on $\Lambda$ then $\omega \in A$ if and only if $\omega' \in A$.

**Definition 6.** Let $\kappa$ be a probability measure on $\Omega$.

- We say that $\kappa$ is positively associated if for $A, B$ any increasing cylinder events, $\kappa(A \cap B) \geq \kappa(A) \kappa(B)$.
- Another probability measure $\kappa_1$ on $\Omega$ stochastically dominates $\kappa$ if for all increasing cylinder events $A$, we have $\kappa_1(A) \geq \kappa(A)$. We write $\kappa_1 \geq \kappa$.  

![Figure 4. Part of a configuration $\omega$ (solid) and its dual $\omega_d$ (dashed with grey crosses) in the special case when $G = \mathbb{Z}$.

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We say that $\kappa$ has the positivity property if for all $\varepsilon > 0$ there exists a constant $0 < c = c(\varepsilon) < 1$ such that for all $e \in E, v \in V, t \in \mathbb{R}$,

$$c < \kappa(\text{no bridges in } \{e\} \times [t, t + \varepsilon]) < 1 - c$$

and

$$c < \kappa(\text{no deaths in } \{v\} \times [t, t + \varepsilon]) < 1 - c.$$ 

**Proposition 7.** Let $q \geq 1$. The measures $\Phi^b_n, \Phi^b, \Psi^b, \Psi^b (b = 0, 1)$ are positively associated and have the positivity property. Moreover, $\Phi^1 \geq \Phi^0$ and $\Psi^1 \geq \Psi^0$.

The proof of this is similar to the discrete random-cluster model and is omitted; full details may be found in [4].

### 3. The Critical Value

We assume henceforth that $q = 2$. It is known that, if $G = \mathbb{Z}$, the critical value $\rho_c(2) = 2$. The following is the main result of this paper.

**Theorem 8.** Let $G$ be any star-like graph. Then the critical value on $G \times \mathbb{R}$ is $\rho_c(2) = 2$.

In other words, the critical value for any star-like graph is the same as for $\mathbb{Z}$. Simpler arguments than those presented here can be used to establish the analogous result when $q = 1$, namely that $\rho_c(1) = 1$. Also, the same arguments can be used to calculate the critical value of the discrete graphs $G \times \mathbb{Z}$ when $q = 1, 2$.

Here is a brief outline of the proof of Theorem 8. First we make the straightforward observation that $\rho_c(2) \leq 2$. Second, we use exponential decay and the GHS inequality to establish the existence of certain infinite paths in the dual model when $\lambda/\delta < 2$. Finally, we show how to put these paths together to form “blocking circuits” in $\Xi$, which prevent the existence of infinite paths in $\Xi$ when $\lambda/\delta < 2$. Parts of the argument are inspired by [7].

**Lemma 9.** For $G$ any star-like graph, $\rho_c(2) \leq 2$.

**Proof.** Any star-like graph $G$ contains an isomorphic copy of $\mathbb{Z}$ as a subgraph. Let $Z$ be such a subgraph; we may assume that $O \in Z$. Also we let $\phi^b_n, \phi^b$ denote the random-cluster measures on $Z \times \mathbb{R}$. For each $n \geq 1$, let $C_n$ be the event that in $\Lambda_n$ there are no bridges between $Z \times \mathbb{R}$ and its complement. Clearly each $C_n$ is a decreasing event. It follows from a standard property of random-cluster measures, sometimes called the DLR-property, that $\Phi^b_n(\cdot \mid C_n) = \phi^b_n(\cdot)$. The proof of this uses standard techniques [8]; details for this model may be found in [4]. If $A$ is an increasing cylinder event, this means that

$$\phi^b_n(A) = \Phi^b_n(A \mid C_n) \leq \Phi^b_n(A),$$

i.e. $\phi^b_n \leq \Phi^b_n$ for all $n$. Letting $n \to \infty$ it follows that $\phi^b \leq \Phi^b$. If $\lambda/\delta > 2$ then $\phi^b((O, 0) \leftrightarrow \infty) > 0$ so then also

$$\Phi^b((O, 0) \leftrightarrow \infty) > 0,$$

which is to say that $\rho_c(2) \leq 2$. \qed
Infinite paths in the half-plane. Let us now establish some facts about the random-cluster model on $\mathbb{Z}^+ \times \mathbb{R}$ which will be useful later. Our notation is as follows: for $n \geq 1$,
\begin{equation}
S_n = \{(a, t) \in \mathbb{Z} \times \mathbb{R} : -n \leq a \leq n, |t| \leq n\}
\end{equation}
\begin{equation}
S_n(m, s) = S_n + (m, s) = \{(a + m, t + s) \in \mathbb{Z}^+ \times \mathbb{R} : (a, t) \in S_n\}.
\end{equation}

For brevity write $T_n = S_n(n, 0)$; also let $\partial$ denote the boundary,
\begin{equation}
\partial S_n = \{(a, t) \in \mathbb{Z} \times \mathbb{R} : a = \pm n \text{ or } t = \pm n\}
\end{equation}
and $\partial S_n(m, s) = \partial S_n + (m, s)$. For $b = 0, 1$ and $\Delta$ one of $S_n, T_n$, we let $\phi^b_\Delta$ denote the $q = 2$ random-cluster measure on $\Delta$ with boundary condition $b$ and parameters $\lambda, \delta$. Note that
\begin{equation}
\phi^b = \lim_{n \to \infty} \phi^b_{S_n}, \quad \psi^b = \lim_{n \to \infty} \psi^b_{S_n}.
\end{equation}

We will also be using the limits
\begin{equation}
\phi^w = \lim_{n \to \infty} \phi^w_{T_n}, \quad \psi^f = \lim_{n \to \infty} \psi^0_{T_n}.
\end{equation}
These are measures on configurations $\omega$ on $\mathbb{Z}^+ \times \mathbb{R}$; but according to our definition they cannot be random-cluster measures since the regions $T_n$ do not tend to the whole of $\mathbb{Z} \times \mathbb{R}$. However, standard arguments let us deduce all the properties of $\phi^w, \psi^f$ that we need. In particular $\psi^f$ and $\phi^w$ are mutually dual (with the obvious interpretation of duality) and they enjoy the positive association and positivity properties of Definition 6.

Let $W$ be the “wedge”
\begin{equation}
W = \{(a, t) \in \mathbb{Z}^+ \times \mathbb{R} : 0 \leq t \leq a/2 + 1\},
\end{equation}
and write $0$ for the origin $(0, 0)$.

**Lemma 10.** Let $\lambda/\delta < 2$. Then
\begin{equation}
\psi^f(0 \leftrightarrow \infty \text{ in } W) > 0.
\end{equation}

Here is some intuition behind the proof of Lemma 10. The claim is well-known with $\psi^0$ in place of $\psi^f$, by standard arguments using duality and exponential decay. However, $\psi^f$ is stochastically smaller than $\psi^0$, so we cannot deduce the result immediately. Instead we pass to the dual $\phi^w$ and establish directly a lack of blocking paths. The problem is the presence of the infinite “wired side”; we get the required fast decay of two-point functions by using the following result of [10], adapted to our model.

**Proposition 11.** Let $\lambda/\delta < 2$. There is $\alpha > 0$ such that for all $n$,
\begin{equation}
\phi^1_{S_n}(0 \leftrightarrow \partial S_n) \leq e^{-\alpha n}.
\end{equation}

In words, the two-point function decays exponentially also in finite volume. Higuchi [10] proves a more general result for the discrete Ising model, but attributes to Aizenman the simpler result for that model. It was pointed out to us by Grimmett (personal communication) that the original proof may be shortened by using the Lieb inequality in place of the GHS inequality, and we present the full proof for our model using the Lieb inequality here. Apart from the Lieb inequality, the proof uses another fact known for the $q = 2$ Ising case but not for the general case $q \geq 1$, namely exponential decay in the infinite volume subcritical Gibbs state.
Proof. Let \( \overline{S}_n \supset S_n \) denote the “tall” box
\[
(19) \quad \overline{S}_n = \{(a, t) \in \mathbb{Z} \times \mathbb{R} : -n \leq a \leq n, |t| \leq n + 1\}.
\]
We will use a variant of the random-cluster measure on \( \overline{S}_n \) which has non-constant intensities for bridges and deaths, and also a process of ghost-bonds. To this end we create a new site \( g \), which we think of as a “point at infinity”, and let \( \delta(\cdot), \gamma(\cdot) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \) and \( \lambda(\cdot) : (\mathbb{Z} + 1/2) \times \mathbb{R} \to \mathbb{R} \) be bounded, nonnegative and measurable functions. Given independent Poisson processes of bridges and deaths of rates \( \lambda(\cdot) \) and \( \delta(\cdot) \), respectively, and of links to \( g \) of rate \( \gamma(\cdot) \), we may define random-cluster measures as in Definition 3, where now any components connected to \( g \) are to be counted as the same.

The particular intensities we use are these. Fix \( n \), and fix \( m \geq 0 \), which we think of as large. Let \( \lambda(\cdot), \delta(\cdot) \) and \( \gamma_m(\cdot) \) be given by
\[
\begin{align*}
\delta(a, t) &= \begin{cases} 
\delta, & \text{if } (a, t) \in S_n \\
0, & \text{otherwise}, \end{cases} \\
\lambda(a + 1/2, t) &= \begin{cases} 
\lambda, & \text{if } (a, t) \in S_n \text{ and } (a + 1, t) \in S_n \\
0, & \text{otherwise}, \end{cases} \\
\gamma_m(a, t) &= \begin{cases} 
\lambda, & \text{if exactly one of } (a, t) \text{ and } (a + 1, t) \text{ is in } S_n \\
m, & \text{if } (a, t) \in \overline{S}_n \setminus S_n \\
0, & \text{otherwise}. \end{cases}
\end{align*}
\]
In words, the intensities are as usual “inside” \( S_n \) and in particular there is no external field in the interior; on the left and right sides of \( S_n \), the external field simulates the wired boundary condition; and on top and bottom, the external field simulates an approximate wired boundary (as \( m \to \infty \)). We introduce another parameter \( r \in [0, 1] \), and let \( \tilde{\phi}^r_{m, n} \) denote the random-cluster measure on \( \overline{S}_n \) with intensities \( \lambda(\cdot), \delta(\cdot), r\gamma_m(\cdot) \). Note that \( \tilde{\phi}^0_{m, n} \) and \( \phi_{S_n}^0 \) agree on events defined on \( S_n \), for any \( m \).

Let \( X \) denote \( \overline{S}_n \setminus S_n \) together with the left and right sides of \( S_n \). By the Lieb inequality, proved for the space-time Ising formulation of the present model in [5] (see also [4]), we have that
\[
(21) \quad \tilde{\phi}^1_{m, n}(0 \leftrightarrow g) \leq e^{8\delta} \int_X dx \, \tilde{\phi}^0_{m, n}(0 \leftrightarrow x) \tilde{\phi}^1_{m, n}(x \leftrightarrow g) \leq e^{8\delta} \int_X dx \, \phi_{m, n}^0(0 \leftrightarrow x),
\]
since \( X \) separates 0 from \( g \). Therefore, by stochastic domination by the infinite-volume measure,
\[
(22) \quad \tilde{\phi}^1_{m, n}(0 \leftrightarrow g) \leq e^{8\delta} \int_X dx \, \phi^0(0 \leftrightarrow x).
\]
All the points \( x \in X \) are at distance at least \( n \) from the origin. By exponential decay in the infinite volume, as proved in [5] using similar methods to the discrete case [1], there is an absolute constant \( \tilde{\alpha} > 0 \) such that
\[
(23) \quad \tilde{\phi}^1_{m, n}(0 \leftrightarrow g) \leq e^{8\delta} |X| e^{-\tilde{\alpha}n} = e^{8\delta} (8n + 2) e^{-\tilde{\alpha}n}.
\]
Now let \( C \) be the event that all of \( \overline{S}_n \setminus S_n \) belongs to the connected component of \( g \), which is to say that all points on \( \overline{S}_n \setminus S_n \) are linked to \( g \). Then by the DLR-property of random-cluster measures the conditional measure \( \tilde{\phi}^1_{m, n}(\cdot \mid C) \) agrees with \( \phi_{S_n}^0(\cdot) \)
on events defined on $S_n$. Therefore
\[ \phi^1_{S_n} (0 \leftrightarrow \partial S_n) = \phi^1_{m,n} (0 \leftrightarrow \partial S_n \mid C) = \phi^1_{m,n} (0 \leftrightarrow g \mid C) \]
(24)
\[ \leq \frac{\phi^1_{m,n} (0 \leftrightarrow g)}{\phi^1_{m,n} (C)} \leq \frac{e^{8\delta}}{\phi^1_{m,n} (C)} \cdot (8n + 2)e^{-\alpha n}. \]
Since $\phi^1_{m,n} (C) \to 1$ as $m \to \infty$ we conclude that
\[ \phi^1_{S_n} (0 \leftrightarrow \partial S_n) \leq e^{8\delta} (8n + 2)e^{-\alpha n}. \]
(25)
Since each $\phi^1_{S_n} (0 \leftrightarrow \partial S_n) < 1$ it is a simple matter to tidy this up to get the result claimed. \( \square \)

of Lemma 10. Let $T = \{(a, a/2 + 1) : a \in \mathbb{Z}_+\}$ be the “top” of the wedge $W$. We claim that
\[ \sum_{n \geq 1} \phi^w ((n, 0) \leftrightarrow T \text{ in } W) < \infty. \]
(26)

Once this is proved, it follows from the Borel–Cantelli lemma that with probability one under $\phi^w$, at most finitely many of the points $(n, 0)$ are connected to $T$ inside $W$. Hence under the dual measure $\psi^f$ there is an infinite path inside $W$ with probability one, and by the positivity- and positive association properties it follows that
\[ \psi^f (0 \leftrightarrow \infty \text{ in } W) > 0, \]
(27)
as required.

To prove the claim we note that, if $n$ is larger than some constant, then the event “$(n, 0) \leftrightarrow T \text{ in } W$” implies the event “$(n, 0) \leftrightarrow \partial S_{n/3} (n, 0)$”. The latter event, being increasing, is more likely under the measure $\phi^1_{S_{n/3} (n, 0)}$ than under $\phi^w$. But by Proposition 11,
\[ \phi^1_{S_{n/3} (n, 0)} ((n, 0) \leftrightarrow \partial S_{n/3} (n, 0)) = \phi^1_{S_{n/3} (0 \leftrightarrow \partial S_{n/3})} \leq e^{-\alpha n/3}, \]
which is clearly summable. \( \square \)

3.2. Proof of the main result. We prove one more lemma about the half-plane before going on to the main result.

Lemma 12. Let $\lambda/\delta < 2$. There exists $\varepsilon > 0$ such that for each $n$,
\[ \psi^f ((0, 2n + 1) \leftrightarrow (0, -2n - 1) \text{ off } T_n) \geq \varepsilon. \]
(29)

Proof. Let $L_n = \{(a, n) : a \geq 0\}$ be the horizontal line at height $n$, and let $\varepsilon > 0$ be such that $\psi^f (0 \leftrightarrow \infty \text{ in } W) \geq \sqrt{\varepsilon}$. We claim that
\[ \psi^f ((0, -2n - 1) \leftrightarrow L_{2n+1} \text{ off } T_n) \geq \sqrt{\varepsilon}. \]
(30)

Clearly $\psi^f$ is invariant under reflection in the $x$-axis, and standard arguments [8, Theorem 4.19] imply that it is also invariant under vertical translation. Thus once the claim is proved we get that
\[ \psi^f ((0, 2n + 1) \leftrightarrow (0, -2n - 1) \text{ off } T_n) \geq \psi^f ((0, -2n - 1) \leftrightarrow L_{2n+1} \text{ off } T_n) \]
\[ \geq \psi^f ((0, -2n - 1) \leftrightarrow L_{-2n-1} \text{ off } T_n) \]
and $(0, 2n + 1) \leftrightarrow L_{-2n-1} \text{ off } T_n) \geq (\sqrt{\varepsilon})^2,$
as required. See Figure 5.
The claim follows if we prove that

\[ \psi^f(0 \leftrightarrow \infty \text{ in } R) = 0, \]

where \( R \) is the strip

\[ R = \{(a, t) : a \geq 0, -2n - 1 \leq t \leq 2n + 1 \}. \]

However, (32) follows from the positivity property of Definition 6 and the Borel–Cantelli lemma, since the event “no bridges between \( \{k\} \times [-2n - 1, 2n + 1] \) and \( \{k + 1\} \times [-2n - 1, 2n + 1] \)” must happen for infinitely many \( k \) with \( \psi^f \)-probability one. To see this we can compare \( \psi^f \) with an independent percolation measure, as in the proof of Proposition 11. We have that \( \psi^f \leq \mu \), where \( \mu \) has parameters \( \lambda, \delta \); under \( \mu \) the events above are independent, so

\[ \psi^f(0 \leftrightarrow \infty \text{ in } R) \leq \mu(0 \leftrightarrow \infty \text{ in } R) = 0. \]

\[ \square \]

of Theorem 8. We may assume that \( G \neq \mathbb{Z} \), since the case \( G = \mathbb{Z} \) is known. Let \( \lambda/\delta < 2 \), and recall that \( G \) consists of finitely many infinite “arms”, where each vertex has degree two, together with a “central” collection of other vertices. On each of the arms, let us fix one arbitrary vertex (of degree two) and call it an exit point. Let \( U \) denote the set of exit points of \( G \).

Given an exit point \( u \in U \), call its two neighbours \( v \) and \( w \); we may assume that they are labelled so that only \( v \) can reach the origin \( O \) without passing \( u \). If the edge \( uv \) were removed from \( G \), the resulting graph would consist of two components, where we denote by \( J_u \) the component containing \( w \). Let \( \hat{\Phi}^b_n, \hat{\Phi}^b \) denote the marginals of \( \Phi^b_n, \Phi^b \) on \( X_u := J_u \times \mathbb{R} \); similarly let \( \hat{\Psi}^b_n, \hat{\Psi}^b \) denote the marginals of the dual measures. Of course \( X_u \) is isomorphic to the half-plane graph considered in the previous subsection. By positive association and the DLR-property of random-cluster measures, \( \hat{\Phi}^0_n \leq \phi^0_{T_n(u)} \), so letting \( n \to \infty \) also \( \hat{\Phi}^0 \leq \phi^w \). Passing to the dual, it follows that \( \hat{\Psi}^1 \geq \psi^f \). The (primal) edge \( uv \) is a vertex in the
line-hypergraph; denoting it still by $uv$ we therefore have by Lemma 12 that there is an $\varepsilon > 0$ such that for all $n$,

\[(35) \quad \Psi((uv, -2n - 1) \leftrightarrow (uv, 2n + 1) \text{ off } T_n(u) \text{ in } X_u) \geq \varepsilon.\]

Here $T_n(u)$ denotes the copy of the box $T_n$ contained in $X_u$. Letting $A$ denote the intersection of the events above over all exit points $u$, and letting $A_1 = A(n)$ be the dual event $A_1 = \{\omega_l : \omega \in A\}$, it follows from positive association that $\Phi^0(A_1) \geq \varepsilon$, where $k = |U|$ is the number of exit points. Note that $A_1$ is a decreasing event in the primal model. The intuition is that on $A_1$, no point in $T_n(u)$ can reach $\infty$ without passing the line $\{u\} \times [-2n - 1, 2n + 1]$, since there is a dual blocking path in $X_u$.

Next let $I$ denote the (finite) subgraph of $G$ spanned by the complement of all the $J_u$ for $u \in U$, and let $A_2 = A_2(n)$ denote the event that for all vertices $v \in I$, the intervals $\{v\} \times [2n + 1, 2n + 2]$ and $\{v\} \times [-2n - 1, -2n - 2]$ all contain at least one death and the endpoints of no bridges (in the primal model). By the positivity property, there is $\eta > 0$ independent of $n$ such that $\Phi^0(A_2) \geq \eta$. So by positive association $\Phi^0(A_1 \cap A_2) \geq \eta \varepsilon^k > 0$. On the event $A_1 \cap A_2$, no point inside the union of $I \times [-n, n]$ with $\cup_{u \in U} T_n(u)$ can lie on an infinite path. See Figure 6. Taking the intersection of the $A_1(n) \cap A_2(n)$ over all $n$, it follows that

\[(36) \quad \Phi^0(\text{there is no unbounded connected component}) \geq \eta \varepsilon^k.\]

The event that there is no unbounded connected component is a tail event. All infinite-volume random-cluster measures are tail-trivial (see [8, Theorem 4.19] or [4]), so it follows, whenever $\lambda/\delta < 2$, that

\[(37) \quad \Phi^0(0 \not\leftrightarrow \infty) = 1.\]

In other words, $\rho_c(2) \geq 2$. Combined with the opposite bound in Lemma 9, this gives the result. \qed
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