Strong superadditivity of the entanglement of formation follows from its additivity

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The additivity of both the entanglement of formation and the classical channel capacity is known to be a consequence of the strong superadditivity conjecture. We show that, conversely, the strong superadditivity conjecture follows from the additivity of the entanglement of formation; this means that the two conjectures are equivalent and that the additivity of the classical channel capacity is a consequence of them.

I. INTRODUCTION

Entanglement, an exclusively quantum property, makes possible numerous new promising applications of quantum mechanics in computing, communication and cryptography. One says that there is entanglement between different parts of a quantum system if the state of the system cannot be represented as a product of states or a statistical mixture of products. One of the basic tasks of quantum information theory is to define the appropriate quantitative characteristics of how much a state is entangled. A simple and universal measure exists only for the case of two subsystems in a pure state. For a pure state \( \psi \) of a system composed of the subsystems \( A \) and \( B \) the entanglement \( E(\psi) \) is given by the entropy of the reduced density matrix:

\[
E(\psi) = S(\text{Tr}_B(\vert\psi\rangle \langle\psi\vert)) = S(\text{Tr}_A(\vert\psi\rangle \langle\psi\vert)),
\]

where \( S \) is the von Neumann entropy: \( S(\rho) = -\text{Tr} \rho \log_2 \rho \). Here and below the symbol Tr with subscripts means the partial trace over the corresponding subsystem (the subsystem \( B \) or \( A \) in this case) and we consider only systems with finite-dimensional Hilbert spaces. In contrast to the pure state case, for mixed states the different aspects of entanglement are characterized by different measures. For example, the entanglement cost, the quantity of entanglement required to prepare a given state, in general differs from the distillable entanglement, the quantity of entanglement which can be extracted from a given state. One of the most important and widely used measures is the entanglement of formation (EoF). It was introduced in [1] as the least expected entanglement of any ensemble of pure states realizing \( \rho \):

\[
E_F(\rho) = \min \sum_i p_i E(\psi_i),
\]

where an ensemble \( \{p_i, \psi_i\} \) realizes \( \rho \) if \( \rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| \), that is if a pure state \( \psi_i \) can be found in \( \rho \) with probability \( p_i \). We will call optimal ensemble of \( \rho \) an ensemble for which the minimum is attained.

The physical interpretation of the EoF depends on whether it is additive or not. Given two states \( \rho_1 \) and \( \rho_2 \) of two separate systems 1 and 2 (each being a bipartite system with the parts 1A, 1B and 2A, 2B respectively, we always consider entanglement between \( A \) and \( B \)), what is the EoF of the state \( \rho_1 \otimes \rho_2 \) of the composite system? It has been conjectured, that the EoF is additive:

\[
E_F(\rho_1 \otimes \rho_2) \geq E_F(\rho_1) + E_F(\rho_2),
\]

the EoF of the composite system is the sum of the EoF’s of its parts. This conjecture has been proved for some particular classes of states. Moreover the conjecture is supported by a number of numerical calculations and no counterexamples has been found. It is known [2], that the entanglement cost \( E_C \) of a state \( \rho \) is equal to the asymptotic ratio of the EoF of \( n \) copies of the state \( \rho \) to the number of copies \( n \), that is \( E_C = \lim_{n \to \infty} E_F(\rho^\otimes n)/n \). If the additivity conjecture is true, then the EoF gives us the entanglement cost \( (E_C = E_F) \), which would greatly simplify the problem of the practical calculation of \( E_C \). It is natural to consider the more general problem of comparing the EoF of a
system with the sum of the EoF’s of its subsystems, and it is conjectured \cite{2, 4} that the former is not less than the latter:

\[ E_F(\rho) \geq E_F(\text{Tr}_2 \rho) + E_F(\text{Tr}_1 \rho). \]  

(4)

This property is called strong superadditivity and it is not only interesting on its own, but also because it implies the additivity of the EoF \cite{2, 4}. It implies the additivity of the Holevo-Schumacher-Westmorland classical capacity of a quantum channel \cite{2} too. The problem of the additivity of this quantity is of considerable importance for the quantum communication theory \cite{6}, but it remains unresolved in the general case, though the additivity was proved for some particular classes of quantum channels. In this paper we uncover an even closer connection between the strong superadditivity of the EoF, the additivity of the EoF and the additivity of the classical channel capacity: we show that the additivity of the EoF implies the strong superadditivity and therefore that these two conjectures are equivalent and they imply the additivity of the classical channel capacity.

The rest of the paper is organized as follows: in section 2 we discuss the convexity property of the EoF and introduce the crucial notion of conjugate function. This and the majority of the other tools we use were introduced in \cite{5}. We also state in this section some related known facts. In section 3 we derive an equation which determines the optimal vectors for the conjugate function. This equation is used in section 4 to prove that the strong superadditivity of the EoF is equivalent to additivity. We conclude with a discussion of the possible implications and with some historical remarks.

II. CONVEXITY AND THE CONJUGATE FUNCTION

One of the most important properties of the EoF is its convexity. Convexity means that for any set of density matrices \( \rho_i \) and probabilities \( p_i \) the EoF of the average density matrix \( \rho = \sum_p p_i \rho_i \) is not greater than the average EoF:

\[ E_F(\rho) \leq \sum_i p_i E_F(\rho_i). \]  

(5)

To convince oneself that the convexity holds one can consider the state \( \rho \) as resulting from taking with probability \( p_i \) an index \( i \) and then preparing the system in a pure state from an optimal ensemble of \( \rho_i \) with the probability corresponding to that pure state in the ensemble. The expected entanglement for the resulting pure state ensemble is equal to the r.h.s. of Eq. (5), and the least expected entanglement \( E_F(\rho) \) is not greater than that.

Let us introduce following \cite{5} an indispensable notion of the conjugate function of the EoF. The transition from a function to its conjugate is a standard operation in convex analysis \cite{8}, and for the EoF we obtain the following function of a Hermitian matrix \( H \):

\[ E^*(H) = \max_{\rho} (\text{Tr}(\rho H) - E_F(\rho)), \]  

(6)

where maximization is performed over all density matrices \( \rho \). Instead, one can maximize over pure states only \cite{5}

\[ E^*(H) = \max_{\psi} (\langle \psi | H | \psi \rangle - E(\psi)), \]  

(7)

because the expression \( E_F(\rho) \) is the expected entanglement for an ensemble of pure states \( \psi_i \), and the whole r.h.s. of Eq. (6) is therefore also an average:

\[ \text{Tr}(\rho H) - E_F(\rho) = \sum_i p_i (\langle \psi_i | H | \psi_i \rangle - E(\psi_i)), \]  

(8)

and an average cannot be greater than all averaging numbers. Applying to a convex function the conjugation operation twice leaves it unchanged \cite{5, 8}. For the EoF it means that

\[ E_F(\rho) = \max_H (\text{Tr}(\rho H) - E^*(H)). \]  

(9)

Let us recall the statements of the strong superadditivity and additivity conjectures. Consider a system, composed of two bipartite subsystems 1 and 2: the subsystem 1 consists of parts 1A and 1B, and the subsystem 2 consists of parts 2A and 2B. We always consider the entanglement between the subsystems A and B. The following conjectured property is called strong superadditivity:

\[ E_F(\rho) \geq E_F(\rho_1) + E_F(\rho_2), \]  

(10)
where \( \rho_{1,2} = \Tr_{2,1}(\rho) \). Additivity is another conjectured property:

\[
E_F(\rho_1 \otimes \rho_2) \geq E_F(\rho_1) + E_F(\rho_2).
\]  

(11)

Let us note that additivity holds when the states \( \rho_1 \) and \( \rho_2 \) are pure. One of the reasons why the strong superadditivity conjecture is interesting, is that it implies the additivity of the EoF \[4, 7\]. This is easy to see if we consider an optimal decomposition for \( \rho_1 \):

\[
\rho_1 = \sum_i p_i (|\psi_i^{(1)}\rangle\langle\psi_i^{(1)}|), \quad E_F(\rho_1) = \sum_i p_i E(\psi_i^{(1)})
\]

and an analogous optimal decomposition for \( \rho_2 \). We have a decomposition of the tensor product

\[
\rho_1 \otimes \rho_2 = \sum_{ij} p_i p_j (|\psi_i^{(1)}\psi_j^{(2)}\rangle\langle\psi_i^{(1)}\psi_j^{(2)}|).
\]

(13)

The mean EoF of this decomposition cannot exceed \( E_F(\rho_1 \otimes \rho_2) \):

\[
E_F(\rho_1 \otimes \rho_2) \leq \sum_{ij} p_i p_j E(|\psi_i^{(1)}\rangle|\psi_j^{(2)}\rangle)
\]

(14)

where we used the additivity of the EoF for pure states. The inequalities Eqs. (14) and (11) combined give Eq. (11).

Eq. (11) holds for all states \( \rho \) if and only if it holds for pure states, that is if for all pure states \( \psi \):

\[
E(\psi) \geq E_F(\Tr(|\psi\rangle|\psi\rangle)) + E_F(\Tr(|\psi\rangle|\psi\rangle)).
\]

(15)

Indeed, let us consider an optimal decomposition of \( \rho \):

\[
\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad E_F(\rho) = \sum_i p_i E(\psi_i).
\]

(16)

If Eq. (15) holds for these pure states \( \psi_i \) then

\[
E_F(\rho) = \sum_i p_i E(\psi_i) \geq \sum_i p_i (E_F(\Tr(|\psi_i\rangle|\psi_i\rangle)) + E_F(\Tr(|\psi_i\rangle|\psi_i\rangle)))
\]

Using the linearity of the trace: \( \sum_i p_i \Tr(|\psi_i\rangle|\psi_i\rangle) = \Tr \rho \) and the same for the subsystem 2, and using the convexity of the EoF Eq. (4), we obtain Eq. (4) for the state \( \rho \).

The strong superadditivity conjecture can be restated in terms of the conjugate function \( E^*(H) \). For this purpose, let us substitute Eq. (4) in the r.h.s. of Eq. (15):

\[
E(\psi) \geq \max_{H} (\Tr(|\psi\rangle|\psi\rangle)_1 - E^*(H_1)) \quad \text{and} \quad \max_{H_2} (\Tr(|\psi\rangle|\psi\rangle)_2 - E^*(H_2))
\]

An equivalent statement is that for all \( \psi, H_1 \) and \( H_2 \)

\[
E(\psi) \geq \langle \psi | (H_1 \otimes 1 + 1 \otimes H_2)|\psi\rangle - E^*(H_1) - E^*(H_2).
\]

(18)

One can further rewrite it as follows

\[
\langle \psi | (H_1 \otimes 1 + 1 \otimes H_2)|\psi\rangle - E(\psi) \leq E^*(H_1) + E^*(H_2).
\]

(19)

The inequality above is true for all \( \psi \) if and only if it is true for the maximal value of the l.h.s:

\[
\max_{\psi} (\langle \psi | (H_1 \otimes 1 + 1 \otimes H_2)|\psi\rangle - E(\psi)) \leq E^*(H_1) + E^*(H_2),
\]

(20)

that is \[3\]:

\[
E^*(H_1 \otimes 1 + 1 \otimes H_2) \leq E^*(H_1) + E^*(H_2).
\]

(21)

On the other hand, let us consider vectors \( \psi_1 \) and \( \psi_2 \), optimal (in the sense of the definition of the conjugate function Eq. (3)) for \( H_1 \) and \( H_2 \) respectively. Using their product \( |\psi_1\rangle|\psi_2\rangle \) as a trial function for finding \( E^*(H_1 \otimes 1 + 1 \otimes H_2) \) we have
\[ E^*(H_1 \otimes 1 + 1 \otimes H_2) \geq \langle \psi_2 | (H_1 \otimes 1 + 1 \otimes H_2) | \psi_1 \rangle |\psi_2 \rangle - E(|\psi_1 \rangle |\psi_2 \rangle) \]
\[ = \langle \psi_1 | H_1 |\psi_1 \rangle - E(|\psi_1 \rangle |\psi_1 \rangle) + \langle \psi_2 | H_2 |\psi_2 \rangle - E(|\psi_2 \rangle |\psi_2 \rangle) = E^*(H_1) + E^*(H_2). \tag{22} \]

Eqs. 21 and 22 taken together allow one to restate the strong superadditivity conjecture as the following additivity conjecture for conjugate functions \[ \tilde{f} \]:
\[ E^*(H_1 \otimes 1 + 1 \otimes H_2) \geq E^*(H_1) + E^*(H_2). \tag{23} \]

## III. PROPERTIES OF THE OPTIMAL VECTORS

Let us consider in a bipartite system \( A - B \) a Hermitian operator \( H \) and an optimal (in the sense of the definition of \( E^*(H) \), Eq. 24) vector \( \tilde{\psi} \) for it:
\[ E^*(H) = \langle \tilde{\psi} | \tilde{\psi} \rangle - E(\tilde{\psi}). \tag{24} \]

Let us denote by \( f(\psi) \) the function, the maximum of which is \( E^*(H) \):
\[ f(\psi) = \langle \psi | H |\psi \rangle - E(\psi). \tag{25} \]

The necessary condition for \( f(\psi) \) to have a maximum at the point \( \psi \) is the vanishing of its derivatives: \( \delta f(\psi) = 0 \).

To compute the derivatives we need to return to the definition of \( E(\psi) \) and rewrite it more explicitly in terms of the components \( \psi_{ij} \) of the vector \( |\psi\rangle \), where the first index refers to the subsystem \( A \) and the second index refers to the subsystem \( B \). One can consider \( \psi_{ij} \) as components of a matrix \( \psi \). In terms of this matrix the definition of \( E(\psi) \) becomes
\[ E(\psi) = - \text{Tr}(\psi^\dagger \log_2 (\psi^\dagger \psi)) = - \text{Tr}(\rho \log_2 (\rho)), \quad \rho = \psi^\dagger. \tag{26} \]

One has also \( \langle \psi | H |\psi \rangle = \sum_{ijkl} \psi_{ij}^* H_{ijkl} \psi_{kl} \).

Let us note that because the trace of a product of matrices is invariant under cyclic permutations we have \( \delta \text{Tr}(F(\rho)) = \text{Tr}(F'(\rho) \delta \rho) \) for any function of one variable \( F(x) \) and its derivative \( F'(x) \). To prove this one can use Taylor series expansion of \( F(x) \). In our case \( F(x) = - x \log_2 x \) and \( F'(x) = - \log_2 x - 1 / \ln 2 \), which gives
\[ \delta E(\psi) = - \text{Tr}(\rho \log_2 (\rho + 1 / \ln 2) \delta \rho) = - \text{Tr}(\delta \rho \log_2 \rho). \tag{27} \]

Substituting here \( \rho = \psi^\dagger \) we obtain
\[ \delta E(\psi) = - \text{Tr}(\psi^\dagger \log_2 (\psi^\dagger \psi) \delta \psi + \log_2 (\psi^\dagger \psi) \psi \delta \psi^\dagger). \tag{28} \]

For the variation of \( f(\psi) \) we have now
\[ \delta f(\psi) = \sum_{ijkl} (\delta \psi_{ij}^* H_{ijkl} \psi_{kl} + \psi_{ij}^* H_{ijkl} \delta \psi_{kl}) - \delta E(\psi). \tag{29} \]

The vector variation \( |\delta \hat{\psi} \rangle \) is orthogonal to \( |\hat{\psi} \rangle \) due to the normalization condition \( \langle \psi |\psi \rangle = 1 \), but otherwise arbitrary. Instead of the real and imaginary parts of its components \( \Re(\delta \psi_{ij}) \) and \( \Im(\delta \psi_{ij}) \), one can consider as independent their complex linear combinations \( \delta \psi_{ij}^* \) and \( \delta \psi_{ij} \). Then the necessary condition for maximum reads
\[ \sum_{kl} H_{ijkl} \tilde{\psi}_{kl} + \log_2 (\tilde{\psi}^\dagger \tilde{\psi}) \delta \psi_{ij} = C \tilde{\psi}_{ij}. \tag{30} \]

By taking the scalar product of both sides of this equation with \( \tilde{\psi} \) we find that \( C = E^*(H) \). Taking this into account we finally have
\[ \sum_{kl} H_{ijkl} \tilde{\psi}_{kl} = - \langle \log_2 (\tilde{\psi}^\dagger \tilde{\psi}) \rangle_{ij} + E^*(H) \tilde{\psi}_{ij}. \tag{31} \]

This equation determines how the operator \( H \) acts on the optimal vectors and therefore it determines how it acts on any linear combination of them. The Hermiticity of \( H \) requires that for any pair of optimal vectors \( \tilde{\psi}_\alpha \) and \( \tilde{\psi}_\beta \) the following condition holds:
\[ \text{Tr} \left[ \tilde{\psi}_\alpha \tilde{\psi}_\beta^\dagger (\log_2 (\tilde{\psi}_\alpha \tilde{\psi}_\beta^\dagger) - \log_2 (\tilde{\psi}_\beta \tilde{\psi}_\alpha^\dagger)) \right] = 0. \tag{32} \]
IV. CONNECTION BETWEEN THE ADDITIVITY AND THE STRONG SUPERADDITIVITY

The following theorem links the additivity and the strong superadditivity of the entanglement of formation.

**Theorem:** For an arbitrary state of the whole system (consisting of 4 parts: 1A, 1B, 2A and 2B) with the corresponding density matrix \( \rho \), let us compute its partially reduced density matrices \( \rho_1 = \text{Tr}_2(\rho) \) and \( \rho_2 = \text{Tr}_1(\rho) \). If for these two density matrices \( \rho_1 \) and \( \rho_2 \) the EoF is additive, that is if

\[
E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2)
\]

then the EoF is strongly superadditive for the state \( \rho \):

\[
E_F(\rho) \geq E_F(\rho_1) + E_F(\rho_2).
\]

**Proof:** Let us consider a Hermitian matrix \( H \), optimal for \( \rho_1 \otimes \rho_2 \) in the sense of Eq. (39), that is

\[
E_F(\rho_1 \otimes \rho_2) = \text{Tr}[H(\rho_1 \otimes \rho_2)] - E^*(H).
\]

From the definition of the conjugate function (Eqs. (35) and (36)) we have also:

\[
E^*(H) \geq \langle \psi | H | \psi \rangle - E(\psi), \quad E^*(H) \geq \text{Tr}(H\rho') - E_F(\rho'),
\]

for all pure states \( \psi \) and all density matrices \( \rho' \). Let

\[
\rho_1 = \sum_m p_m^{(1)} |\psi_m^{(1)}\rangle \langle \psi_m^{(1)}|, \quad p_m^{(1)} > 0, \quad E_F(\rho_1) = \sum_m p_m^{(1)} E(\psi_m^{(1)})
\]

be an optimal decomposition for \( \rho_1 \) and let

\[
\rho_2 = \sum_n p_n^{(2)} |\psi_n^{(2)}\rangle \langle \psi_n^{(2)}|, \quad p_n^{(2)} > 0, \quad E_F(\rho_2) = \sum_n p_n^{(2)} E(\psi_n^{(2)})
\]

be an optimal decomposition for \( \rho_2 \). Then for all \( m \) and \( n \) the products \( |\psi_m^{(1)}\rangle |\psi_n^{(2)}\rangle \) are optimal pure states for \( H \) in the sense of Eq. (7):

\[
E^*(H) = \langle \psi_m^{(1)} | \langle \psi_n^{(2)} | H | \psi_m^{(1)} \rangle | \psi_n^{(2)} \rangle - E(\langle \psi_m^{(1)} | \psi_n^{(2)} \rangle).
\]

Indeed, substituting the decompositions Eq. (37) and Eq. (38) in the Eqs. (35) and (36) we have

\[
E^*(H) = \sum_{mn} p_m^{(1)} p_n^{(2)} (\langle \psi_m^{(1)} | \langle \psi_n^{(2)} | H | \psi_m^{(1)} \rangle | \psi_n^{(2)} \rangle - E(\langle \psi_m^{(1)} | \psi_n^{(2)} \rangle)),
\]

\[
E^*(H) \geq \langle \psi_m^{(1)} | \langle \psi_n^{(2)} | H | \psi_m^{(1)} \rangle | \psi_n^{(2)} \rangle - E(\langle \psi_m^{(1)} | \psi_n^{(2)} \rangle),
\]

with all probabilities strictly positive:

\[
p_m^{(1)} p_n^{(2)} > 0, \quad \sum_{mn} p_m^{(1)} p_n^{(2)} = 1.
\]

Clearly, this is possible only if Eq. (39) holds for all \( m \) and \( n \).

Let us denote by \( V_1 \) the subspace spanned by the vectors \( \psi_m^{(1)} \), its orthogonal complement by \( V_1^\perp \), and by \( V_2 \) and \( V_2^\perp \) the analogous subspaces for the subsystem 2. Let us note, that the state \( \rho \) must be an ensemble of linear combinations of the optimal optimal vectors \( |\psi_m^{(1)}\rangle |\psi_n^{(2)}\rangle \), that is an ensemble of pure states from \( V_1 \oplus V_2 \):

\[
\rho = \sum_k \phi^k |\phi^k\rangle, \quad \phi^k \in V_1 \oplus V_2.
\]

Indeed, we have \( \phi^k \in [(V_1^\perp \oplus V_2) \oplus (V_1^\perp \oplus V_2^\perp)]^\perp \), because for any vector \( |v\rangle \in V_1^\perp \) the orthogonality relation

\[
\sum_i \phi_{ij}^k v_i = 0 \quad \text{(the first index } i \text{ corresponds here to the subsystem 1 and the second index } j \text{ corresponds to the}
\]

subsystem 2) follows from $\sum_{jk} |\sum_i \phi_{ij}^k v_i|^2 = \langle \psi | \rho_1 | \psi \rangle = 0$. Analogously, one has $\phi^k \in [(V_1 \otimes V_2^\perp) \oplus (V_1^\perp \otimes V_2^\perp)]^\perp$, and then

$$\phi^k \in [(V_1^\perp \otimes V_2) \oplus (V_1 \otimes V_2^\perp) \oplus (V_1^\perp \otimes V_2^\perp)]^\perp = V_1 \otimes V_2. \quad (43)$$

Now, let us show that for the matrix $H$ from $(44)$ one has

$$\text{Tr}[(\rho_1 \otimes \rho_2)H] = \text{Tr}(\rho H). \quad (44)$$

For this purpose one needs to know only the matrix elements of $H$ between states from $V_1 \otimes V_2$ (only such matrix elements are present in $(44)$). One can find these elements from Eq. $(31)$, writing it down for an optimal vector $|\psi_s^{(1)}\rangle|\psi_t^{(2)}\rangle$ and taking the scalar product of both sides of the equation with an optimal vector $\langle \psi_s^{(1)}|\psi_n^{(2)}\rangle$:

$$\langle \psi_s^{(1)}|\langle \psi_n^{(2)}|H|\psi_s^{(1)}\rangle|\psi_t^{(2)}\rangle = -\text{Tr} \left[ \psi_s^{(1)}\psi_m^{(1)}^{\dagger} \log_2(\psi_s^{(1)}\psi_m^{(1)}\rangle\langle 1_1) \text{Tr}(\psi_t^{(2)}\psi_n^{(2)}\rangle\langle 1_1) - \text{Tr} \left[ \psi_t^{(2)}\psi_m^{(2)}^{\dagger} \log_2(\psi_t^{(2)}\psi_m^{(2)}\rangle\langle 1_1) \text{Tr}(\psi_s^{(1)}\psi_n^{(1)}\rangle\langle 1_1) ight] \right] \text{Tr}(\psi_s^{(1)}\psi_n^{(1)}\rangle\langle 1_1) \text{Tr}(\psi_t^{(2)}\psi_n^{(2)}\rangle\langle 1_1). \quad (45)$$

Here we have written the scalar products as the traces of matrix products, for example $\langle \psi_s^{(1)}|\psi_n^{(2)}\rangle = \text{Tr}(\psi_s^{(1)}\psi_n^{(2)}\rangle\langle 1_1)$, and have used the fact that the logarithm of a tensor product of matrices is the sum of logarithms of the matrices: $\log_2(X \otimes Y) = \log_2(X) \otimes 1 + 1 \otimes \log_2(Y)$. We have used also the multiplicativity of trace operation: $\text{Tr}(X \otimes Y) = \text{Tr}(X) \text{Tr}(Y)$. It is easy to see from Eq. $(45)$, that the matrix elements of $H$ between the states from $V_1 \otimes V_2$ have the form:

$$\langle \psi'|H|\psi\rangle = \langle \psi'|\left( H_1 \otimes 1 + 1 \otimes H_2 \right)|\psi\rangle, \quad (46)$$

for some matrices $H_1$ and $H_2$. Then Eq. $(44)$ follows from this formula applied to the expectation value $\text{Tr}(H\rho)$.

Now, we have all necessary means to prove the theorem statement. Replacing $\rho'$ with $\rho$ in the second inequality in Eq. $(45)$ we obtain the following inequality:

$$E^*(H) \geq \text{Tr}(H\rho) - E_F(\rho), \quad (47)$$

If one uses Eqs. $(45)$ and $(46)$ to find $E^*(H)$ this inequality takes the form:

$$\text{Tr} \left[ H(\rho_1 \otimes \rho_2) \right] - E_F(\rho_1) - E_F(\rho_2) \geq \text{Tr}(H\rho) - E_F(\rho). \quad (48)$$

Finally, taking into account Eq. $(44)$ one obtains the inequality $(44)$ which is the statement of the theorem.

The theorem above states that the strong superadditivity of EoF holds for a state $\rho$ if additivity holds for its reduced density matrices $\rho_1$ and $\rho_2$. Then it is clear that the strong superadditivity of EoF for all states of the system follows from the additivity of EoF for all states $\rho_1$ and $\rho_2$ of the subsystems 1 and 2. If the additivity conjecture is not true in general, the above theorem will be still useful, because it connects the strong superadditivity of a state with the additivity for its reduced density matrices $\rho_1$ and $\rho_2$ only and does not require the additivity for all states.

V. CONCLUDING REMARKS

The conjectures stating that the entanglement of formation and the Holevo-Schumacher-Westmorland classical capacity of a quantum channel are additive have not been proved in general case, but they are supported by a number of numerical calculations and they were proved in some particular cases. No counterexamples has been found. It was shown in $(41)$, that both conjectures are true if EoF has the strong superadditivity property. The purpose of the present paper was to deepen this connection by establishing that the strong superadditivity of EoF follows from its additivity and thus the two conjectures are equivalent. This fact makes more important the further study of both of them. The strong superadditivity conjecture which until now seemed rather speculative becomes as plausible as the additivity conjecture. It becomes clear that it is not by chance that all known proofs of additivity of EoF for particular subspaces of quantum states $(41)$ are based on proofs of the strong superadditivity conjecture for these subspaces, and finding a counterexample to the former would immediately give a counterexample for the latter. And the study of the EoF’s additivity problem becomes even more important than it was before, because now its proof would automatically give a proof of the additivity of the classical channel capacity.

After this work has been completed, the preprint $(41)$ appeared, containing among other results a proof of the equivalence of the additivity and strong superadditivity of the EoF, which is the main result of the present work. Our proof is analogous but uses a different language and thus can be useful for some readers.
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