A group-theoretic characterisation of Taub-Nut spacetime

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We prove that any $G = SU(2) \times U(1)$ symmetric spacetime that is Ricci flat (i.e. solves the matter-free $\Lambda = 0$ Einstein equations) with non-null $G$-orbits is locally isometric to some maximally extended generalised Taub-NUT spacetime.

1 Introduction

This paper deals with Taub-NUT spacetime [19, 15] that is an exact solution to the matter-free (i.e. vacuum) equations of General Relativity without cosmological constant. This particular solution is known for its many surprising properties [12, 11, 13]. Also, it can be generalised in various ways and embedded into the wider Plebański–Demiański class of solutions; see, e.g., [3, chapter 12,16]. Here we are not interested in these generalisations and restrict attention to strict Taub-NUT only.

We prove a result that gives rise to a new group-theoretic characterisation of Taub-NUT spacetime, or rather some obvious topological generalisation of it. The main theorem will be presented and proved in Section 3. It states that any $SU(2) \times U(1)$ symmetric vacuum solution to Einstein’s equations with non-null group orbits is locally isometric to some maximally extended generalised Taub-NUT geometry, where the “generalisation” here consists in replacing the 3-sphere in the global $\mathbb{R} \times S^3$ topology with that of a lens space $L(n, 1)$. In that sense we can now say that (generalised) Taub-NUT can be characterised by its isometry group, together with a mild restriction on the group orbits.

This result may be seen as an (incomplete) analogue to the so-called Jebsen-Birkhoff theorem, going back in idea to [6, 7] and (without much proof) [1]; see [9] for more on its history and multiple discovery. In a modern formulation it states that any spherically
symmetric vacuum solution to Einstein’s equation is locally isometric to the maximally extended Schwarzschild-Kruskal manifold. Here “spherical symmetry” is defined by the existence of an isometric action of $SO(3)$ with spacelike $S^2$ or $\mathbb{R}P^2$ orbits. A modern proof can be found in [18, Chapter 4.10.1-2].

We call our result an incomplete analogue to the Jebsen-Birkhoff theorem because in the generalised Taub-NUT case we have several inequivalent (i.e. non globally isometric) maximal extensions, the precise classification of which we currently investigate.

There is another, different notion of generalised Taub-NUT introduced in [14], which relaxes the global isometry to be merely $U(1)$ (i.e. dropping the $SU(2)$ factor altogether) and requires the spacetime to contain a compact Cauchy horizon diffeomorphic to $S^3$ to which the $U(1)$ action restricts to a free action with lightlike orbits. Hence the $S^3$ Cauchy horizon is the total space of a $U(1)$ principal bundle with base $S^2$ (Hopf bundle) and lightlike fibres. The set of such “generalised Taub-NUT” spacetimes forms an infinite-dimensional proper submanifold within the set of all $U(1)$-symmetric vacuum spacetimes of “roughly half the dimensions” [14, p.108]. In this case there are uncountably many inequivalent maximal extensions.

In order to make our paper self contained, we will review the essential geometry and topology of Taub-NUT spacetime in Section 2, also providing a characterisation of the NUT-charge as “dual” to mass. Our main theorem concerning the group-theoretic characterisation will be stated and proved in Section 3. We end with a brief outlook in Section 4.

2 Taub-NUT space-time

In the following we will present some of the features of the Taub-NUT space-time with respect to the interpretation given by Misner [12]. In this interpretation the topology of space-time is $\mathbb{R} \times S^3$ and, using Euler coordinates, the metric is given by

$$g = -4l^2 f(r)(d\psi + \cos \theta d\varphi)^2 + \frac{1}{f(r)}dr^2 + (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.1)$$

with

$$f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2}. \quad (2.2)$$

The constant $m \in \mathbb{R}$ is interpreted as the mass and the constant $l \in \mathbb{R} \setminus \{0\}$ is referred to as the NUT parameter. The four-dimensional isometry group of this space-time is $SU(2)_L \times U(1)_R$ induced by the left-invariant vector field $\xi_0 = \partial_\psi$ (generating right-translations) and the right-invariant vector fields

$$\xi_1 = -\sin \varphi \partial_\theta - \cot \theta \cos \varphi \partial_\varphi + \csc \theta \cos \varphi \partial_\psi \quad (2.3a)$$

$$\xi_2 = \cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi + \csc \theta \sin \varphi \partial_\psi \quad (2.3b)$$

$$\xi_3 = \partial_\varphi. \quad (2.3c)$$
(generating left-translations) on the 3-sphere, which we identify with the group manifold of $SU(2)$. Note that the subscripts $L$ and $R$ on $SU(2)$ and $U(1)$, respectively, are meant to indicate that these groups act via left- and right-multiplication on $SU(2)$. The vector fields $\xi_0, \xi_1, \xi_2, \xi_3$ satisfy the commutation relations:

\[
[\xi_i, \xi_j] = -\varepsilon^k_{ij} \xi_k \quad (2.4a)
\]

\[
[\xi_0, \xi_i] = 0 \quad i, j, k = 1, 2, 3. \quad (2.4b)
\]

In terms of the left-invariant one-forms

\[
\sigma_z = \sin \psi d\theta - \sin \theta \cos \psi d\varphi \\
\sigma_y = \cos \psi d\theta + \sin \theta \sin \psi d\varphi \\
\sigma_z = d\psi + \cos \theta d\varphi 
\]

the metric can be written as

\[
g = -4l^2 f(r) \sigma_z^2 + \frac{1}{f(r)} dr^2 + (r^2 + l^2)(\sigma_x^2 + \sigma_y^2). \quad (2.6)
\]

The orbit generated by $\xi_1, \xi_2, \xi_3$ is three-dimensional, namely $S^3$, with the orbits generated by $\xi_0$ being subsets of it.

In the given coordinates the analytical expressions become singular at $r_{\pm} = m \pm \sqrt{m^2 + l^2}$, whereas all components of the Riemann tensor in an orthonormal tetrad, and hence in particular the Kretschmann scalar, are regular. This indicates that these singularities are, in fact, coordinate artefacts. They correspond to the Killing horizons of the Killing vector field $\partial_\psi$. A possible coordinate transformation removing these singularities is given by

\[
\psi' = \psi + \int \frac{1}{2lf(r)} dr \quad (2.7)
\]

such that the metric in these new coordinates is given by

\[
g = -4l^2 f(r)(d\psi' + \cos \theta d\varphi)^2 + 2(2l)(d\psi' + \cos \theta d\varphi)dr + (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.8)
\]

Another coordinate transformation would be

\[
\psi'' = \psi - \int \frac{1}{2lf(r)} dr, \quad (2.9)
\]

giving

\[
g = -4l^2 f(r)(d\psi'' + \cos \theta d\varphi)^2 - 2(2l)(d\psi'' + \cos \theta d\varphi)dr + (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.10)
\]
Written in terms of the left-invariant one-forms of $S^3$, it is immediate that the metrics are regular on the whole manifold $\mathbb{R} \times S^3$. Furthermore, it can be shown that both space-times are maximal [19]. In these coordinates both the stationary regions $r < r_-$ and $r > r_+$, the so-called NUT-regions, and the region $r_- < r < r_+$, called the Taub-region, are included. In particular, the hypersurfaces of $r = \text{const.}$ are 3-spheres being spacelike in the Taub-region, timelike in the NUT-regions, and lightlike at $r = r_{\pm}$.

Furthermore, with respect to the $U(1)$ right multiplication the space-time can be considered to be a principal fibre bundle analogous to the Hopf bundle. Since the $r = \text{const.}$ hypersurfaces are 3-spheres, there exist no equal-time hypersurfaces intersecting these 3-spheres in two-spheres along which we could evaluate the Komar integral for mass in the usual form. However, we can use the structure of $S^3$ as $U(1)$ principle fibre bundle over the base $S^2$, which has a natural connection given by the distribution of orthogonal complements to the fibre in each tangent space where the generating vector field of $U(1)$ is non-null. It is then possible to uniquely identify horizontal and right-invariant $k$-forms with $k$-forms on the base $S^2$. Thus, considering the NUT regions, admitting the timelike Killing vector field $\partial_\psi$ (generating the right-$U(1)$ translation) the Komar mass of the space-time can be calculated. We will be using the orthonormal tetrad

\begin{equation}
\vartheta^0 = 2lf^{1/2}(r)(d\psi + \cos \theta d\varphi) \quad (2.11a)
\end{equation}

\begin{equation}
\vartheta^1 = f^{-1/2}(r)dr \quad (2.11b)
\end{equation}

\begin{equation}
\vartheta^2 = \left(r^2 + l^2\right)^{1/2}d\theta \quad (2.11c)
\end{equation}

\begin{equation}
\vartheta^3 = \left(r^2 + l^2\right)^{1/2} \sin \theta d\varphi. \quad (2.11d)
\end{equation}

For $\lim_{r \to \infty} f(r) = 1$, we will calculate the Komar mass with respect to the Killing vector field $k := -\frac{1}{2} \partial_\psi$ which is normalised at infinity $r \to \infty$. The metric-dual one-form of the timelike Killing vector field is then given by

\begin{equation}
k^\flat = 2lf(r)(d\psi + \cos \theta d\varphi) \quad (2.12)
\end{equation}

and hence

\begin{equation}
dk^\flat = 2lf'(r)dr \wedge (d\psi + \cos \theta d\varphi) - 2lf(r) \sin \theta d\theta \wedge d\varphi \quad (2.13a)
\end{equation}

\begin{equation}
= -f'(r) \vartheta^0 \wedge \vartheta^1 - 2lf(r) \frac{r^2}{r^2 + l^2} \vartheta^2 \wedge \vartheta^3, \quad (2.13b)
\end{equation}

where we used the standard notation that denotes the one-form image of the vector $k$ under the metric isomorphism by $k^\flat =: g(k, \cdot)$. Now, picking the orientation defined by $\omega = \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$, we get

\begin{equation}
s*dk^\flat = f'(r) \vartheta^2 \wedge \vartheta^3 - 2lf(r) \frac{r^2}{r^2 + l^2} \vartheta^0 \wedge \vartheta^1 \quad (2.14a)
\end{equation}

\begin{equation}
= f'(r) \left(r^2 + l^2\right) \sin \theta d\theta \wedge d\varphi + 4l^2 \frac{f(r)}{r^2 + l^2} dr \wedge (d\psi + \cos \theta d\varphi). \quad (2.14b)
\end{equation}
Then with respect to an arbitrary hypersurface \( r_0 = \text{const.} \), the two-form is given by
\[
*dk^3 = -f'(r_0)(r_0^2 + l^2)d\sigma_z. \tag{2.15}
\]

Since \( \frac{1}{2}\sigma_z \) is a connection one-form for the principal fibre bundle, \( *dk^3 \) can be considered as a multiple of the curvature form and hence is horizontal and right-invariant, as well as closed. Thus, we can identify it with a closed two-form on the base space \( S^2 \), such that using the formula for the Komar mass, we have
\[
-\frac{1}{8\pi} \int_{S^2_\infty} *dk^3 = \lim_{r \to \infty} -\frac{1}{8\pi} f'(r)(r^2 + l^2) \int_{S^2} \sin \theta d\theta \wedge d\varphi = -m. \tag{2.16}
\]

Therefore \( m \) can be interpreted as the Komar mass of the space-time. Moreover, considering \( dk^3 \) instead of \( *dk^3 \), the same line of argument can be applied to give
\[
-\frac{1}{8\pi} \int_{S^2_\infty} dk^3 = \lim_{r \to \infty} \frac{1}{8\pi} 2lf(r) \int_{S^2} \sin \theta d\theta \wedge d\varphi = l. \tag{2.17}
\]

So the constants \( m \) and \( l \) are related by Hodge duality.

Duality also arises in the description of the dual-Bondi-mass of space-times which are asymptotically empty and flat at null infinity and with vanishing Bondi news. It can be shown that in this case null infinity, for space-times having a non-vanishing dual-Bondi-mass, is topologically a Lens space \( L(n,1) \) and a principal fibre bundle \( (L(n,1), \pi, S^2; S^1) \), with the dual-Bondi-mass being proportional to the number of twists, \( n \), in the bundle. Conversely, if null infinity is a non-trivial \( S^1 \) principal fibre bundle over \( S^2 \), the news tensor field vanishes and there exists an infinitesimal translation such that the dual-Bondi-mass with respect to it is non-zero. In particular, the Taub-NUT space-time can be shown to be asymptotically empty and flat at null infinity, with null infinity being a 3-sphere. The dual-Bondi-mass with respect to the infinitesimal translation induced by the Killing vector field \( -\frac{1}{2l} \partial_t \) can be computed to be the NUT parameter \( l \) [16].

3 Main theorem

In this section we intend to give a unique characterisation of the Taub-NUT space-time in terms of the isometry group and its orbits. In particular, the Taub-NUT space-time can be seen to be the universal cover of a family of space-times admitting \( SU(2) \times U(1) \) as an isometry group such that the group orbits of \( SU(2) \times U(1) \) and \( SU(2) \) are three-dimensional and non-null.

Since the metric of the Taub-NUT space-time induces a \( SU(2)_L \times U(1)_R \) invariant metric on the hypersurfaces \( r = \text{const.} \), being diffeomorphic to \( SU(2) \), we will begin by studying special metrics on \( SU(2) \). For Lorentz metrics on \( SU(2) \) we have
Lemma 3.1. Let $G = SU(2)$ and $g$ a Lorentz metric on $G$ such that it is $SU(2)$ left-invariant and $U(1)$ right-invariant, whereby $U(1)_R$ is considered as a subgroup of the $SU(2)_R$. Then the orbits of the $U(1)$ right-multiplication are timelike curves.

Proof. The left-action $^{[1]} SU(2)_L \times U(1)_R \subset S(U)_L \times SU(2)_R$ is simply obtained by restricting the standard left-action of $S(U)_L \times SU(2)_R$ on $SU(2)$:

$$((SU(2)_L \times U(1)_R) \times SU(2) \to SU(2)$$

$$(h, h') \mapsto hgh'^{-1}. \quad (3.1a)$$

Now let $e \in G$ be the identity, then we have for $h \in SU(2)_L$ and $h' \in U(1)_R$

$$((h, h'), e) \mapsto hhe'^{-1} = hh'^{-1}. \quad (3.2)$$

Hence the isotropy group at $e$ is the diagonal $U(1)$ subgroup in $SU(2)_L \times U(1)_R$, denoted by $C_h$. Then

$$(dC_h)_e : T_e G \to T_e G \quad (3.3)$$

induces the adjoint representation

$$Ad : U(1) \to GL(T_e G)$$

$$Ad(h) = (dC_h)_e. \quad (3.4a)$$

Since the tangent space is three-dimensional, the action induced by this $U(1)$ on the tangent space is given by a $U(1)$ subgroup of the three-dimensional Lorentz group. Furthermore, because the $U(1)$ subgroups of the three-dimensional Lorentz group consist of rotations acting by orthogonal transformations in a spacelike plane, such that the corresponding orthogonal timelike direction is invariant, the three-dimensional tangent space

$^{[1]}$The reader should be aware of the conceptual difference between left/right-multiplication on groups and left/right-action of groups on sets: A left-action of a group $G$ on a set $S$ is simply a homomorphism $\Phi : G \to \text{Bij}(S)$, from the group $G$ into the group of bijections of $S$, with group multiplication of the latter just being composition of maps. This means that $\Phi : g \mapsto \Phi_g$ is such that $\Phi_g \circ \Phi_h = \Phi_{gh}$. In contrast, a right-action is an anti-homomorphism $\Phi : G \to \text{Bij}(S)$ which satisfies $\Phi_g \circ \Phi_h = \Phi_{hg}$. A left-action can be turned into a right-action (and vice versa) if we compose it with the group inversion $I : G \to G, g \mapsto I(g) := g^{-1}$ which is an anti-homomorphims, i.e. $I(gh) = I(h)I(g)$. Then $\tilde{\Phi} := \Phi \circ I$ is a right-action if $\Phi$ is a left-action. Now, if $S = G$, there are two natural actions of $G$ on itself, called $L$ and $R$ and given by left- and right-multiplication respectively: $L_g(p) := gp$ and $R_g(p) := pg$. Associativity of group multiplication implies that these two actions commute (as maps): $L_g \circ R_h = R_h \circ L_g$ for all $g, h \in G$. Written in this way $L$ is a left- and $R$ is a right-action which as such do not combine to any action, left or right. However, the right-multiplication can be turned into a left-action by composing $R$ with $I$. In this way we get two different and commuting left-actions of $G$ on itself, one by left-multiplication with $g \in G$ and one by right-multiplication with $g^{-1}$. Together they define a left-action of $G \times G$ on $G$, given by $\Phi_{(g, h)} := L_g \circ R_{h^{-1}}$, that is $\Phi_{(g, h)}(p) = gh^{-1}$. In order to distinguish the group $G$ that acts by left-multiplication from the one that acts by right-multiplication (with the inverse) we distinguish them notationally and call them $G_L$ and $G_R$, respectively. Restricting this to the diagonal subgroup $G_\Delta := \{(g, g) : G \in G\} \subset G_L \times G_R$ gives the left-action of $G$ on itself that is usually referred to as “conjugation”.

6
decomposes into an orthogonal sum of a two-dimensional spacelike subspace and a one-dimensional timelike subspace. Choosing any normed vector $v$ in the one-dimensional timelike subspace, we define the left-invariant vector field $X \in \text{Lie}(G)$ by

$$X(g) = \left. \frac{d}{dt} \right|_{t=0} (g \exp(tv)).$$

(3.5)

The right action on this left-invariant vector field is determined by the adjoint of $v \in T_e G$ with respect to $h^{-1} \in U(1),$

$$(dR_h)_g(X_g) = \left. \frac{d}{dt} \right|_{t=0} (gh \exp(tAd(h^{-1})(v))).$$

(3.6a)

(3.6b)

(3.6c)

Since the timelike direction is invariant with respect to the adjoint representation, we obtain

$$(dR_h)_g(X_g) = \left. \frac{d}{dt} \right|_{t=0} (gh \exp(tAd(h^{-1})(v))).$$

(3.7a)

(3.7b)

(3.7c)

Therefore the left-invariant vector field is also $U(1)_R$–invariant. Furthermore, because it is a left-invariant vector field, it generates a $U(1)_R$-action, considered as $U(1)_R' \subset SU(2)_R$. By being also invariant under $U(1)_R \subset SU(2)_R$, the two $U(1)$ right actions have to commute, hence $U(1)_R = U(1)_R'$. Therefore, the orbits of the $U(1)$ right action coincide with the orbits of $X$ and thus are timelike. \hfill \Box

Next we will prove that a $SU(2)_L \times U(1)_R$–invariant metric on $SU(2)$ can be put into a canonical form:

**Lemma 3.2.** Let $G = SU(2)$ and $g$ a non-degenerate, symmetric bilinear form on $G$ which is $SU(2)_L \times U(1)_R$–invariant. Then $g$ can be written as

$$g = A\sigma_z^2 + B(\sigma_x^2 + \sigma_y^2),$$

(3.8)

where $\sigma_x, \sigma_y, \sigma_z$ are left–invariant one-forms on $G$.

**Proof.** Let $Z$ be a fundamental vector field associated to the $U(1)_R$-action and any element $ix \in \text{Lie}(U(1)) = i\mathbb{R},$

$$Z(g) = \left. \frac{d}{dt} \right|_{t=0} (g \exp(tx)), \quad g \in G.$$
Then the vector field $Z$ is left-invariant. We will complete it to a basis for $\text{Lie}(G)$ by choosing two linearly independent left-invariant vector fields in the orthogonal complement of $Z$, so $X, Y \in \text{Lie}(G)$ such that $X, Y \perp Z$. Then, denoting the basis as $e_1 = Z, e_2 = X, e_3 = Y$ and their dual one forms by $\omega_1, \omega_2, \omega_3$, $g$ can be written as

$$g = \lambda(\omega^1)^2 + \mu(\omega^2)^2 + \nu(\omega^3)^2 + \kappa \omega^2 \omega^3.$$  \hfill (3.10)

Now since $g$ is $U(1)_R$–invariant, we have

$$L_{e_1} g = 0.$$  \hfill (3.11)

If the structure constants are given by

$$[e_i, e_j] = c_{ij}^k e_k$$  \hfill (3.12)

we have for their dual one-forms

$$d\omega^k = -\sum_{i<j} c_{ij}^k \omega^i \wedge \omega^j.$$  \hfill (3.13)

Thus, using Cartan’s magic formula, we obtain

$$L_{e_1} \omega^1 = i_{e_1} d\sigma^1 + d(i_{e_1} \sigma^1)$$  \hfill (3.14a)

$$= -i_{e_1} \left( \sum_{i<j} c_{ij}^1 \omega^i \wedge \omega^j \right)$$  \hfill (3.14b)

$$= - \sum_{i<j} c_{ij}^1 \left( (i_{e_1} \omega^j) \wedge \omega^i - \omega^j \wedge (i_{e_1} \omega^i) \right)$$  \hfill (3.14c)

$$= - (c_{12}^1 \omega^2 + c_{13}^1 \omega^3)$$  \hfill (3.14d)

and similarly

$$L_{e_1} \omega^2 = - (c_{12}^2 \omega^2 + c_{13}^2 \omega^3)$$  \hfill (3.15a)

$$L_{e_1} \omega^3 = - (c_{12}^3 \omega^2 + c_{13}^3 \omega^3).$$  \hfill (3.15b)

Because

$$L_V (T \otimes S) = (L_V T) \otimes S + T \otimes (L_V S),$$  \hfill (3.16)

for any vector field $V$ and tensor fields $T, S$, we see that

$$L_{e_1} g = 0 \implies c_{12}^1 = c_{13}^1 = 0,$$  \hfill (3.17)

by noting that terms of the form $\omega^1 \omega^2$ and $\omega^1 \omega^3$ can only be obtained by $L_{e_1} \omega^1$. Thus we have

$$[e_1, e_2] \in \text{span}\{e_2, e_3\}$$  \hfill (3.18a)

$$[e_1, e_3] \in \text{span}\{e_2, e_3\}.$$  \hfill (3.18b)
Defining a vector space endomorphism \( F: \text{Lie}(SU(2)) \to \text{Lie}(SU(2)) \) by

\[
F(e_1) = [e_2, e_3] \quad F(e_2) = [e_3, e_1] \quad F(e_3) = [e_1, e_2],
\]

the matrix of \( F \) with respect to \( \{e_1, e_2, e_3\} \) is given by

\[
F = \begin{pmatrix}
c_{23}^2 & 0 & 0 \\
c_{23}^2 & c_{31}^2 & c_{12}^2 \\
c_{23}^2 & c_{31}^2 & c_{12}^2 \\
\end{pmatrix}.
\]

Since \( SU(2) \) is a unimodular group, we have: \( \text{tr} \text{ad}(x) = 0 \ \forall x \in \text{Lie}(SU(2)) \), therefore implying \( c_{23}^2 = c_{23}^3 = 0 \). Since we also have \( c_{12}^2 = c_{31}^2 \), \( F \) is self-adjoint with respect to \( g \) and hence we can find an orthogonal transformation diagonalizing \( F \). It is an orthogonal transformation keeping \( e_1 \) fixed and transforming in its orthogonal complement. Thus, by the appropriate transformation, we get a new basis \( \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\} \) with \( \tilde{e}_1 = e_1 \) satisfying

\[
[\tilde{e}_1, \tilde{e}_2] = c_{12}^3 \tilde{e}_3 \quad \text{(3.21a)}
\]

\[
[\tilde{e}_3, \tilde{e}_1] = c_{31}^2 \tilde{e}_2 \quad \text{(3.21b)}
\]

\[
[\tilde{e}_2, \tilde{e}_3] = c_{23}^1 \tilde{e}_1 \quad \text{(3.21c)}
\]

and by rescaling, we obtain the basis \( \{e'_1, e'_2, e'_3\} \) with the commutation relations

\[
[e'_1, e'_2] = e'_3 \quad \text{(3.22a)}
\]

\[
[e'_3, e'_1] = e'_2 \quad \text{(3.22b)}
\]

\[
[e'_2, e'_3] = e'_1 \quad \text{(3.22c)}
\]

Now, denoting the dual one-forms of this basis by \( \{\sigma_z, \sigma_x, \sigma_y\} \) respectively, \( g \) can be written as

\[
g = A\sigma_z^2 + B\sigma_x^2 + C\sigma_y^2 + D\sigma_x\sigma_y \quad \text{(3.23)}
\]

and we obtain

\[
L_{e'_1}\sigma_z = 0 \quad \text{(3.24a)}
\]

\[
L_{e'_1}\sigma_x = \sigma_y \quad \text{(3.24b)}
\]

\[
L_{e'_1}\sigma_y = -\sigma_x. \quad \text{(3.24c)}
\]

Then, since \( e'_1 \) is just a scalar multiple of \( e_1 \), the \( U(1) \) right invariance implies

\[
0 = L_{e'_1}g = B(\sigma_y \otimes \sigma_x + \sigma_x \otimes \sigma_y) + C(-\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x) + D(\sigma_y \otimes \sigma_y - \sigma_x \otimes \sigma_x) \quad \text{(3.25a)}
\]

\[
= (B - C)(\sigma_y \otimes \sigma_x + \sigma_x \otimes \sigma_y) + D(\sigma_y \otimes \sigma_y - \sigma_x \otimes \sigma_x). \quad \text{(3.25b)}
\]

Thus, implying \( B = C, D = 0 \), such that

\[
g = A\sigma_z^2 + B(\sigma_x^2 + \sigma_y^2) \quad \text{(3.26)}
\]

\]
Combining the results of Lemma 3.1 and 3.2, we see that a $SU(2)_L \times U(1)_R$-invariant Riemannian/Lorentz metric $g$ on $SU(2)$ can always be written as

$$g = \varepsilon A^2 \sigma_z^2 + B^2 (\sigma_x^2 + \sigma_y^2),$$

(3.27)

where $\varepsilon = 1$ corresponds to the Riemannian and $\varepsilon = -1$ to the Lorentzian case. Using Euler-angle coordinates the metric is given by

$$g = \varepsilon A^2 (d\psi + \cos \theta d\phi)^2 + B^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(3.28)

An essential observation regarding the orbits of the isometry group $SU(2) \times U(1)$ of the Taub-NUT space-time is that the orbit corresponding to the $SU(2)$-action are three-dimensional and the orbit with respect to the $U(1)$-action is a subset of it. Thus simply requiring the group orbits of a general space-time with isometry group $SU(2) \times U(1)$ to be three-dimensional, does not exclude the possibility that the action of $SU(2)$ generates two-dimensional orbits and the action of $U(1)$ transversal one-dimensional orbits. For that matter we will in the following study the implications of three-dimensional orbits generated by a $SU(2)$-action on a space-time.

Let $M$ be a manifold admitting a $SU(2)$ left-action in such a way that the group orbits are three-dimensional. Since $SU(2)$ is a compact Lie group the action is proper and thus each group orbit is a closed subset of $M$ and each isotropy group is compact. Let $O(p), I_p$ denote the orbit and isotropy group of $p$ respectively. Then since $I_p$ is a closed subgroup and the action of $SU(2)$ on its group orbits is transitive, we have

$$\dim O(p) = \dim SU(2) - \dim I_p.$$  

(3.29)

Thus the isotropy group is a discrete subgroup of $SU(2)$ and since it is compact it has to be finite. The orbit $O(p)$ is a homogeneous $SU(2)$-space and we have

$$O(p) \cong SU(2)/I_p.$$  

(3.30)

Since for a connected Lie group $G$ and a discrete subgroup $\Gamma$ the quotient is a manifold and the quotient map is a (normal) covering map, we get a fibration with base space $O(p) \cong SU(2)/I_p$, discrete fibers $I_p$ and total space $SU(2)$

$$I_p \longrightarrow SU(2)$$

$$\downarrow$$

$$O(p) \cong SU(2)/I_p.$$  

For $SU(2)$ is connected and simply connected it is the universal cover. By the use of the long exact sequence of homotopy groups for the fibration we obtain

$$\cdots \rightarrow \pi_1(I_p) \rightarrow \pi_1(SU(2)) \rightarrow \pi_1(O(p)) \rightarrow \pi_0(I_p) \rightarrow \pi_0(SU(2)).$$  

(3.31)
Noting that $SU(2) \cong S^3$, $\pi_1(SU(2))$ is trivial like $\pi_0(SU(2))$. Furthermore, we have $\pi_0(I_p) \cong I_p$ and thus obtain the short exact sequence

$$0 \to \pi_1(O(p)) \to I_p \to 0,$$

(3.32)

implying $\pi_1(O(p)) \cong I_p$. Summarizing, we see that the group orbits are closed three-dimensional manifolds with finite fundamental group. But then by Thurston’s elliptisation conjecture (now proven) the group orbits have to be elliptic 3-manifolds. These have been classified to be of the form $M = S^3/\Gamma$, with $\pi_1(M) = \Gamma$ being a finite subgroup of $SO(4)$, acting freely and orthogonally on $M$ in the standard fashion. Out of these, the only ones admitting $SU(2) \times U(1)$ as an isometry group are the Lens spaces $L(n, 1)$ [5].

As already proven a $SU(2)_L \times U(1)_R$–invariant metric on $SU(2) \cong S^3$ can be put into a canonical form. Now we want to study the case for $L(n, 1)$. Considering $S^3 \subset \mathbb{C}^2$ the left action of $SU(2)$ on $S^3$ is the natural action of $SU(2)$ on $\mathbb{C}^2$ and the $\Gamma = \mathbb{Z}_n$-action on $S^3$ for $L(n, 1)$ is given by

$$(z_0, z_1) \mapsto (e^{2\pi i/n}z_0, e^{2\pi i/n}z_1), \quad (z_0, z_1) \in S^3.$$  

(3.33)

Now we can define a left action of $SU(2)$ on $L(n, 1)$ by

$$SU(2) \times L(n, 1) \to L(n, 1)$$

(3.34a)

$$A, \pi(p)) \mapsto \pi(Ap),$$

(3.34b)

where $\pi : S^3 \to L(n, 1)$ is the projection map, so the covering map. This induces a well-defined $SU(2)$ left action on $L(n, 1)$. Similarly we have a well-defined induced $U(1)$ right action.

Moreover, given a $SU(2)_L \times U(1)_R$–invariant metric it is also invariant with respect to the $\mathbb{Z}_n$-action [3,33] and hence the following construction is well-defined:

Let $g$ be a $\mathbb{Z}_n$–invariant metric on $S^3$. We define a metric on $L(n, 1)$ pointwise by

$$g'_q : T_q L(n, 1) \times T_q L(n, 1) \to \mathbb{R}$$

(3.35a)

$$g'_q(X', Y') := g_p(X, Y), \quad X', Y' \in T_q L(n, 1), \quad X, Y \in T_p S^3$$

(3.35b)

where $\pi(p) = q$ and $d\pi_p(X) = X'$, $d\pi_p(Y) = Y'$. The $\mathbb{Z}_n$-invariance of $g$ implies the independence of the choice of a representative $p$ and since the covering map is a local diffeomorphism its differential at any point is a linear isomorphism. Therefore this definition makes sense. The metric $g'$ is in fact smooth, since given any smooth local section $s : U \subset L(n, 1) \to S^3$ and smooth local vector fields $X', Y'$ on $U$ we have on $U$

$$g'(X', Y') = g(ds(X'), ds(Y')).$$

(3.36)

This linear map is a bijection, because conversely given any metric $g'$ on $L(n, 1)$ we can define a metric $g = \pi^*g'$ on $S^3$, being just the preimage of the preceding construction. Thus, we see that the $\mathbb{Z}_n$–invariant metrics on $S^3$ are in bijection to metrics on
Since a $SU(2)_L \times U(1)_R$-invariant metric on $S^3$ is also invariant with respect to the $\mathbb{Z}_n$-action \((3.33)\), by definition of the induced action and the constructed bijection above, it is immediate that the $SU(2)_L \times U(1)_R$-invariant metrics on $S^3$ are in one-to-one correspondence to the metrics on $L(n,1)$ which are invariant under the induced $SU(2)_L \times U(1)_R$-action.

Using Euler coordinates and Lemma \(3.2\), a Lorentz or Riemannian metric on $L(n,1)$ invariant under $SU(2)_L \times U(1)_R$ can always be written as

$$g = \varepsilon A^2 (d\psi + \cos \theta d\varphi)^2 + B^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad \varepsilon = \pm 1 \quad (3.37)$$

where $\theta$ and $\varphi$ ranging from 0 to $\pi$ and 0 to $2\pi$ respectively and $\psi$ being $4\pi/n$-periodic.

In particular, based on the preceding results, we see that the metric \((2.1)\) is invariant with respect to the $\mathbb{Z}_n$-action \((3.33)\) and hence there exists a well-defined metric on any Lens space $L(n,1)$. Thus the space-time can be generalized to $(\mathbb{R} \times L(n,1), g)$ with

$$g = -4l^2 f(r)(d\psi + \cos \theta d\varphi)^2 + \frac{1}{f(r)} dr^2 + (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (3.38)$$

which will be called the generalized Taub-NUT space-time. The space-time $(\mathbb{R} \times L(n,1), g')$, with

$$g' = -4l^2 f(r)(d\psi' + \cos \theta d\varphi)^2 + 2(2l)(d\psi' + \cos \theta d\varphi) dr + (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\varphi^2)$$

will be called a maximal extension of the generalized Taub-NUT space-time.

Now we can prove the following statement:

**Theorem 3.3.** Every $(C^2)$-solution to the vacuum Einstein field equations admitting $SU(2) \times U(1)$ as an isometry group, such that $SU(2) \times U(1)$ and $SU(2)$ both have three-dimensional non-null orbits in an open subset $U$, is locally isometric to a maximal extension of the generalized Taub-NUT space-time.

**Proof.** Let $p \in M$ be an arbitrary point of the space-time $(M, g)$ and $O(p)$ be the three-dimensional orbit of $p$ with respect to the action of $SU(2)$. We define the orthogonal complement of the tangent space of the orbit to be: $N_p := T_p O(p)^\perp$. Then the induced distributions

$$N := \cup_{p \in M} N_p, \quad O := \cup_{p \in M} T_p O(p) \quad (3.40)$$

are both integrable, for $N$ is a one-dimensional distribution, which is always integrable, and $O$ is by construction integrable, with its integral manifolds being the orbits. So we have a involutive three-dimensional distribution, spanned by Killing vector fields and the involutive one-dimensional normal bundle with $N \cap O = 0$, since the orbits are non-null. Thus, it is possible to introduce local coordinates $\{x^\mu\} = \{r, x^1, x^2, x^3\}$ such that

$$g = g_{rr} dr^2 + g_{ab}(x^\mu) dx^a dx^b \quad (3.41)$$
where \( r = \text{const.} \) are the integral manifolds of \( O \), the orbits, which are homogeneous spaces. Since the three-dimensional orbits of a \( SU(2) \)-action admitting \( SU(2) \times U(1) \) as isometry group have to be topologically the Lens spaces \( L(n, 1) \) and the \( SU(2)_L \times U(1)_R \)-invariant Lorentz or Riemannian metrics on \( L(n, 1) \) can always be put into a canonical form, the metric can be written as

\[
g = -\varepsilon A^2(r) dr^2 + \varepsilon B^2(r)(d\psi + \cos \theta d\varphi)^2 + R^2(r)(d\theta^2 + \sin^2 \theta d\varphi^2), \tag{3.42}
\]

where the case \( \varepsilon = 1 \) represents spacelike orbits and \( \varepsilon = -1 \) timelike orbits.

Now to solve the field equations, it is necessary to calculate the corresponding Ricci tensor. We will calculate them using an orthonormal tetrad and the Cartan structure equations.

First we will consider the case \( \varepsilon = 1 \), so spacelike orbits. The orthonormal tetrad we will be using is

\[
\begin{align*}
\vartheta^0 &= A(r) dr \\
\vartheta^1 &= B(r)(d\psi + \cos \theta d\varphi) \\
\vartheta^2 &= R(r)d\theta \\
\vartheta^3 &= R(r) \sin \theta d\varphi.
\end{align*}
\tag{3.43}
\]

In the following the argument of the functions \( A, B, R \) will be omitted and a prime indicates the derivative with respect to \( r \). Then exterior differentiation and expressing the results in terms of the tetrad leads to

\[
\begin{align*}
d\vartheta^0 &= 0 \quad \tag{3.44a}

d\vartheta^1 &= \frac{B'}{AB}\vartheta^0 \wedge \vartheta^1 - \frac{B}{R^2}\vartheta^2 \wedge \vartheta^3 \quad \tag{3.44b}

d\vartheta^2 &= \frac{R'}{AR}\vartheta^0 \wedge \vartheta^2 \quad \tag{3.44c}

d\vartheta^3 &= \frac{R'}{AR}\vartheta^0 \wedge \vartheta^3 + \cot \theta \frac{R}{R}\vartheta^2 \wedge \vartheta^3. \quad \tag{3.44d}
\end{align*}
\]

Since the tetrad is orthonormal we have \( \omega_{\mu\nu} + \omega_{\nu\mu} = 0 \). Now using the first structure equation with an ansatz for every connection one-form of the form \( a_\mu \vartheta^\mu \), the unique
solution is given by

\[
\begin{align*}
\omega^0_1 &= \omega^1_0 = \frac{B'}{A'B} \vartheta^1 \\
\omega^0_2 &= \omega^2_0 = \frac{R'}{AR} \vartheta^2 \\
\omega^0_3 &= \omega^3_0 = \frac{R'}{AR} \vartheta^3 \\
\omega^1_2 &= -\omega^1_1 = -\frac{B}{2R^2} \vartheta^3 \\
\omega^1_3 &= -\omega^3_1 = \frac{B}{2R^2} \vartheta^2 \\
\omega^2_3 &= -\omega^3_2 = \frac{B}{2R^2} \vartheta^1 - \cot \frac{\theta}{R} \vartheta^3.
\end{align*}
\] (3.45a-f)

Then using the second structure equations, the curvature 2-form can be calculated:

\[
\begin{align*}
\Omega^0_1 &= d\omega^0_1 + \omega^0_2 \wedge \omega^2_1 + \omega^0_3 \wedge \omega^3_1 \\
&= \left( \frac{B''}{AB} - \frac{B'A'}{A^2B} - \frac{B'^2}{AB^2} \right) dr \wedge \vartheta^1 + \frac{B'}{AB} d\vartheta^1 + \frac{BR'}{2AR^3} \vartheta^2 \wedge \vartheta^3 - \frac{BR'}{2AR^3} \vartheta^3 \wedge \vartheta^2 \\
&= \left( \frac{B''}{A^2B} - \frac{B'A'}{A^3B} - \frac{B'^2}{A^2B^2} \right) \vartheta^0 \wedge \vartheta^1 + \frac{B'^2}{A^2B^2} \vartheta^0 \wedge \vartheta^1 - \frac{B'}{AR^2} \vartheta^2 \wedge \vartheta^3 + \frac{BR'}{AR^3} \vartheta^2 \wedge \vartheta^3 \\
&= \left( \frac{B''}{A^2B} - \frac{B'A'}{A^3B} \right) \vartheta^0 \wedge \vartheta^1 + \left( \frac{BR'}{AR^3} - \frac{B'}{AR^2} \right) \vartheta^2 \wedge \vartheta^3 \\
&= \left( \frac{\varrho'}{A^2R} - \varrho' \right) \vartheta^0 \wedge \vartheta^2 + \frac{\varrho'^2}{A^2R^2} \vartheta^0 \wedge \vartheta^2 - \frac{B'}{2AR^2} \vartheta^1 \wedge \vartheta^3 - \frac{BR'}{2AR^3} \vartheta^3 \wedge \vartheta^1 \\
&= \left( \frac{\varrho'}{A^2R} - \varrho' \right) \vartheta^0 \wedge \vartheta^2 - \frac{1}{2} \left( \frac{B'}{AR^2} - \frac{BR'}{AR^3} \right) \vartheta^1 \wedge \vartheta^3 \\
&= \left( \frac{\varrho'}{A^2R} - \varrho' \right) \vartheta^0 \wedge \vartheta^2 + \frac{\varrho'^2}{A^2R^2} \vartheta^0 \wedge \vartheta^2 + \frac{\varrho'^2}{A^2R^2} \vartheta^0 \wedge \vartheta^2 + \frac{B'}{2AR^2} \vartheta^1 \wedge \vartheta^2 \\
&= \left( \frac{\varrho'}{A^2R} - \varrho' \right) \vartheta^0 \wedge \vartheta^3 + \frac{1}{2} \left( \frac{B'}{AR^2} - \frac{BR'}{AR^3} \right) \vartheta^1 \wedge \vartheta^2.
\end{align*}
\] (3.46a-c)
The components of the Ricci tensor are
\[
\Omega_{12} = d\omega^{1}_{2} + \omega^{1}_{0} \wedge \omega^{0}_{2} + \omega^{1}_{3} \wedge \omega^{3}_{2} \\
\Omega_{13} = d\omega^{1}_{3} + \omega^{1}_{0} \wedge \omega^{0}_{3} + \omega^{1}_{2} \wedge \omega^{2}_{3} \\
\Omega_{23} = d\omega^{2}_{3} + \omega^{2}_{0} \wedge \omega^{0}_{3} + \omega^{2}_{1} \wedge \omega^{1}_{3} \\
\]
Using \( \Omega^{\mu \nu} = \frac{1}{2} R_{\alpha \beta \gamma}^{\mu} \theta^{\alpha} \wedge \theta^{\beta} \wedge \theta^{\gamma} \), the non-vanishing components of the Riemann tensor, up to symmetry, are
\[
R_{0101}^{0} = \frac{B''}{A^2 B} - \frac{B' A'}{A B^2} \\
R_{123}^{0} = 2 R_{023}^{0} = -2 R_{0312}^{0} = \frac{B R'}{A^2 R^3} - \frac{B'}{A^2 R^3} \\
R_{020}^{0} = R_{303}^{0} = \frac{R''}{A^2 R} - \frac{R' A'}{A^3 R} \\
R_{121}^{1} = R_{313}^{1} = \frac{B' R'}{A^2 B R} + \frac{B^2}{4 R^4} \\
R_{232}^{0} = \frac{1}{R^2} + \frac{R'^2}{A^2 R^2} - \frac{3 B^2}{4 R^4} \\
\]
The components of the Ricci tensor are
\[
R_{00} = -R_{0101}^{0} - 2 R_{020}^{0} \\
R_{11} = R_{0101}^{0} + 2 R_{121}^{1} \\
R_{22} = R_{33} = R_{020}^{0} + R_{121}^{1} + R_{232}^{0}, \\
\]
with all other components being zero. The Ricci scalar is then
\[ R = -R_{00} + R_{11} + R_{22} + R_{33} \]
\[ = 2 \left( R^0_{\; 101} + 2R^1_{\; 212} + 2R^0_{\; 202} + R^2_{\; 323} \right). \]  
(3.49d)
Thus, we obtain the following non-vanishing components of the Einstein tensor
\[ G_{00} = 2R^1_{\; 212} + R^2_{\; 323} \]  
(3.49f)
\[ G_{11} = -2R^0_{\; 202} - R^2_{\; 323} \]  
(3.49g)
\[ G_{22} = G_{33} = -R^0_{\; 202} - R^1_{\; 212} - R^0_{\; 101}. \]  
(3.49h)

Now we will solve the vacuum Einstein field equations. Suppose \( \langle dR, dR \rangle = 0 \) in \( U \). Then, if \( R \) is constant in \( U \), \( R = R_0 > 0 \), we have
\[ 0 = G_{00} = \frac{B^2}{2R_0^4} + \frac{1}{S^2} - \frac{3B^2}{4R_0^4} \]  
(3.50a)
\[ \iff B^2 = 4R_0^2 \text{ is const.} \]  
(3.50b)
Inserting this condition in \( 0 = G_{11} \) we arrive at the following contradiction:
\[ 0 = G_{11} = -\frac{3B^2}{4R_0^4} = \frac{2}{R_0^2}. \]  
(3.51)

On the other hand, \( dR \) can not be non-zero and lightlike, since the group orbits are non-null. Therefore, considering \( \langle dR, dR \rangle \neq 0 \) in \( U \), we will choose coordinates such that \( R(r) = r \). Next, to solve the field equations, we will consider
\[ 0 = G_{00} + G_{11} = 2 \left( R^1_{\; 212} - R^0_{\; 202} \right) \]  
(3.52a)
\[ \iff 0 = r^3(AB' + A'B) + \frac{1}{4}A^3B^3 \]  
(3.52b)
\[ = r^3(AB)' + \frac{1}{4}(AB)^3. \]  
(3.52c)
Defining \( D(r) = A(r)B(r) \), we get the first order ordinary differential equation for \( D \)
\[ r^3D' + \frac{1}{4}D^3 = 0, \]  
(3.53)
which can be integrated to give
\[ D^2(r) = \frac{4r^2c_0}{r^2 - c_0}, \]  
(3.54)
with \( c_0 > 0 \). Thus we need to have \( r^2 > c_0 \). Now to solve
\[ 0 = G_{00} = 2 \frac{B'}{A^2Br} - \frac{B^2}{4r^4} + \frac{1}{r^2} + \frac{1}{A^2r^2} \]  
(3.55)
we use $D^2 = A^2B^2$ and multiply the equation by $4c_0r^4$ to get

$$0 = r(r^2 - c_0)2BB' - c_0B^2 + 4r^2c_0 + (r^2 - c_0)B^2$$

$$= r(r^2 - c_0)(B^2)' + (r^2 - 2c_0)B^2 + 4r^2c_0. \quad (3.56a)$$

Then, by introducing $F(r) := B^2(r)$, we obtain an inhomogeneous first order linear differential equation

$$0 = F' + \frac{r^2 - 2c_0}{r(r^2 - c_0)}F + \frac{4rc_0}{r^2 - c_0}. \quad (3.57)$$

To solve this differential equation, we will first solve the corresponding homogeneous equation:

$$0 = F'_h + \frac{r^2 - 2c_0}{r(r^2 - c_0)}F_h$$

$$= F'_h + \frac{2(r^2 - c_0) - r^2}{r(r^2 - c_0)}F_h$$

$$= F'_h + \left(2 - \frac{r}{r^2 - c_0}\right)F_h. \quad (3.58c)$$

Thus, by integrating, the solution is given by

$$\ln(F_h) = - \left(\ln(r^2) - \ln\left(\sqrt{r^2 - c_0}\right)\right) + c$$

$$\iff F_h = c' \sqrt{r^2 - c_0}, \quad c' = e^c. \quad (3.59b)$$

Now to obtain the general solution, we will multiply the inhomogeneous equation by $c'/F_h$:

$$0 = \left(\sqrt{r^2 - c_0}\right)' + \frac{4r^3c_0}{(r^2 - c_0)^{3/2}}$$

$$= \left(\sqrt{r^2 - c_0}\right)' + \frac{4r^3c_0}{(r^2 - c_0)^{3/2}}. \quad (3.60b)$$

Hence, the solution is given by

$$F = c_1 \sqrt{r^2 - c_0} \frac{r^2}{r^2 - c_0} - \sqrt{r^2 - c_0} \frac{r^2}{r^2 - c_0} \int \frac{4r^3c_0}{(r^2 - c_0)^{3/2}}dr. \quad (3.61a)$$

To solve the integral we will use the substitution $u = \sqrt{r^2 - c_0}$ with $dr = \frac{u}{\sqrt{u^2 + c_0}}du$ such that

$$\int \frac{4(u^2 + c_0)^{3/2}c_0}{u^3} \frac{u}{\sqrt{u^2 + c_0}}du$$

$$= \int 4c_0 + \frac{4c_0^2}{u^2} du$$

$$= 4c_0u - \frac{4c_0^2}{u}. \quad (3.62c)$$
Therefore, the solution of the inhomogeneous differential equation is
\[ F = B^2 = \frac{-4c_0 r^2 + c_1 \sqrt{r^2 - c_0} + 8c_0^2}{r^2}. \]  \hspace{1cm} (3.63)

Thus, we have solved \( G_{00} = G_{11} = 0 \) and obtained functions \( B^2(r) \) and \( A^2(r) = \frac{D^2(r)}{B^2(r)} \) implying
\[ R_{1212} = R_{0202}, \] \hspace{1cm} (3.64a)
\[ 2R_{2121} = -R_{3232}. \] \hspace{1cm} (3.64b)

One can check that these functions satisfy \( R_{0101} = R_{2323} \), implying \( G_{22} = G_{33} = 0 \).

Inserting the functions in the metric, we see that there is a singularity at \( r^2 = c_0 \). To check if the space-time has a curvature singularity, we will compute the Kretschmann scalar
\[ K = R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}. \] \hspace{1cm} (3.65)

Using the symmetry of the Riemann tensor and the implications of the field equations, the Kretschmann scalar is given by
\[ K = 4(R^{0101})^2 - 8(R^{0123})^2 - 8(R^{0231})^2 - 8(R^{0^231})^2 + 4(R^{0202})^2 \\
+ 4(R^{0303})^2 + 4(R^{1212})^2 + 4(R^{1313})^2 + 4(R^{2323})^2 \\
= 12 \left( (R^{0101})^2 - (R^{0202})^2 \right) \\
= \frac{3}{4}c_0^{-2}r^{-12} \left( 2048c_0^6 - 3072c_0^5r^2 - 18c_0c_1^2r^4 + c_1^2r^6 + 128c_0^4 \left( 9r^4 + 4c_1 \sqrt{r^2 - c_0} \right) \\
+ 48c_0^2c_1r^2 \left( c_1 + 2r^2 \sqrt{r^2 - c_0} \right) - 32c_0^3 \left( c_1^2 + 2r^6 + 16c_1r^2 \sqrt{r^2 - c_0} \right) \right). \] \hspace{1cm} (3.66)

observing that it is regular at \( r^2 = c_0 \), thus indicating that the singularity is due to a poor choice of coordinates. To solve the coordinate singularity, we will use the following coordinate transformation:
\[ r' = \frac{1}{2\sqrt{c_0}} \int D(r)dr = \sqrt{r^2 - c_0}, \quad dr' = \frac{1}{2\sqrt{c_0}} D(r)dr. \] \hspace{1cm} (3.68)

With respect to \( r' \) we have
\[ B^2(r') = -4c_0 \frac{r'^2 - c_1}{r'^2 + c_0}. \] \hspace{1cm} (3.69)
Defining $l^2 := c_0 > 0$, $m := \frac{c_1}{8c_0}$, the metric takes the form

\[ g = -\frac{4l^2}{B^2(r')} dr'^2 + B^2(r')(d\psi + \cos \theta d\varphi)^2 + (r'^2 + l^2) (d\theta^2 + \sin^2 \theta d\varphi^2) \] (3.70a)

\[ = \frac{r'^2 + l^2}{r'^2 - 2mr - l^2} dr'^2 - 4l^2 \frac{r'^2 - 2mr - l^2}{r'^2 + l^2} (d\psi + \cos \theta d\varphi)^2 + (r'^2 + l^2) (d\theta^2 + \sin^2 \theta d\varphi^2). \] (3.70b)

So we see that we obtain the generalized Taub-NUT space-time and since we assumed that the orbits are spacelike, we get in fact the Taub-region. Now if we consider the case $\varepsilon = -1$, so timelike orbits, we have

\[ g = A^2(r) dr^2 - B^2(r)(d\psi + \cos \theta d\varphi)^2 + R^2(r)(d\theta^2 + \sin^2 \theta d\varphi^2). \] (3.71)

Using the orthonormal tetrad

\[ \vartheta^0 = B(r)(d\psi + \cos \theta d\varphi) \] (3.72a)

\[ \vartheta^1 = A(r) dr \] (3.72b)

\[ \vartheta^2 = R(r) d\theta \] (3.72c)

\[ \vartheta^3 = R(r) \sin \theta d\varphi. \] (3.72d)

we obtain

\[ d\vartheta^0 = \frac{B'}{AB} \vartheta^1 \wedge \vartheta^0 - \frac{B}{AR^2} \vartheta^2 \wedge \vartheta^3 \] (3.73a)

\[ d\vartheta^1 = 0 \] (3.73b)

\[ d\vartheta^2 = \frac{R'}{AR} \vartheta^1 \wedge \vartheta^2 \] (3.73c)

\[ d\vartheta^3 = \frac{R'}{AR} \vartheta^1 \wedge \vartheta^3 + \frac{\cot \theta}{R} \vartheta^2 \wedge \vartheta^3. \] (3.73d)

The connection one-forms are then given by

\[ \omega^0_1 = \frac{B'}{AB} \vartheta^0 \] (3.74a)

\[ \omega^0_2 = -\frac{B}{2R^2} \vartheta^3 \] (3.74b)

\[ \omega^0_3 = \frac{B}{2R^2} \vartheta^2 \] (3.74c)

\[ \omega^1_2 = -\frac{R'}{AR} \vartheta^2 \] (3.74d)

\[ \omega^1_3 = -\frac{R'}{AR} \vartheta^3 \] (3.74e)

\[ \omega^2_3 = -\frac{B}{2R^2} \vartheta^0 - \frac{\cot \theta}{R} \vartheta^3. \] (3.74f)
Due to their strong resemblance to the case of spacelike orbits we will see that solving the vacuum Einstein equations is analogous. Using the second structure equations the curvature 2-form can be calculated

\[
\Omega^0_1 = - \left( \frac{B''}{A^2 B} - \frac{B'A'}{A^3 B} \right) \theta^0 \wedge \theta^1 + \left( \frac{BR'}{AR^3} - \frac{B'}{AR^2} \right) \theta^2 \wedge \theta^3 \]  

(3.75a)

\[
\Omega^0_2 = \frac{1}{2} \left( \frac{BR'}{AR^3} - \frac{B'}{AR^2} \right) \theta^1 \wedge \theta^3 - \left( \frac{B'R'}{A^2 BR} + \frac{B^2}{4R^4} \right) \theta^0 \wedge \theta^2 \]  

(3.75b)

\[
\Omega^0_3 = \frac{1}{2} \left( \frac{B'}{AR^2} - \frac{BR'}{AR^3} \right) \theta^1 \wedge \theta^2 - \left( \frac{B'R'}{A^2 BR} + \frac{B^2}{4R^4} \right) \theta^0 \wedge \theta^3 \]  

(3.75c)

\[
\Omega^1_2 = - \left( \frac{R''}{AR^2} - \frac{R'A'}{AR^3} \right) \theta^1 \wedge \theta^2 - \frac{1}{2} \left( \frac{B'}{AR^2} - \frac{BR'}{AR^3} \right) \theta^0 \wedge \theta^3 \]  

(3.75d)

\[
\Omega^1_3 = - \left( \frac{R''}{AR^2} - \frac{R'A'}{AR^3} \right) \theta^1 \wedge \theta^3 - \frac{1}{2} \left( \frac{B'}{AR^2} - \frac{BR'}{AR^3} \right) \theta^0 \wedge \theta^2 \]  

(3.75e)

\[
\Omega^2_3 = \left( \frac{B'}{AR^2} - \frac{BR'}{AR^3} \right) \theta^0 \wedge \theta^1 + \left( \frac{1}{R^2} - \frac{R'^2}{A^2 R^2} + \frac{3B^2}{4R^4} \right) \theta^2 \wedge \theta^3. \]  

(3.75f)

Then using \( \Omega^\mu_\nu = \frac{1}{2} R^\mu_{\nu\alpha\beta} \theta^\alpha \wedge \theta^\beta \), the non-vanishing components of the Riemann tensor, up to symmetry, are

\[
R^0_{101} = - \frac{B''}{A^2 B} + \frac{B'A'}{A^3 B} \]  

(3.76a)

\[
R^0_{123} = 2R^0_{213} = -2R^0_{312} = \frac{BR'}{AR^3} - \frac{B'}{AR^2} \]  

(3.76b)

\[
R^0_{202} = R^0_{303} = - \frac{B'R'}{A^2 BR} + \frac{B^2}{4R^4} \]  

(3.76c)

\[
R^1_{212} = R^1_{313} = - \frac{R''}{AR^2} + \frac{R'A'}{A^3 R} \]  

(3.76d)

\[
R^2_{323} = \frac{1}{R^2} - \frac{R'^2}{A^2 R^2} + \frac{3B^2}{4R^4}. \]  

(3.76e)

Analogously, we have the following non-vanishing components of the Einstein tensor

\[
G_{00} = 2R^1_{212} + R^3_{323}, \]  

(3.77a)

\[
G_{11} = -2R^0_{202} - R^2_{323}, \]  

(3.77b)

\[
G_{22} = G_{33} = -R^0_{202} - R^1_{212} - R^0_{101}. \]  

(3.77c)

Now we will solve the vacuum Einstein field equations. Supposing \( \langle dR, dR \rangle = 0 \) in \( U \) leads analogously to a contradiction. Therefore, we have \( \langle dR, dR \rangle \neq 0 \) in \( U \) and we can choose coordinates such that \( R(r) = r \). Then to solve the field equations we will consider again \( 0 = G_{00} + G_{11} = 2 \left( R^1_{212} - R^0_{202} \right) \), but since \( R^1_{212} \) and \( R^0_{202} \) correspond to curvature components \( -R^0_{202} \) and \( -R^1_{212} \), respectively, for the case \( \varepsilon = 1 \) we see that they satisfy the same differential equation. Hence defining \( D(r) = A(r)B(r) \), we
have

\[ D^2(r) = \frac{4r^2c_0}{r^2 - c_0}, \]  

(3.78)

with \( c_0 > 0 \) and \( r^2 > c_0 \). Now we will solve

\[ 0 = G_{11} = 2 \frac{B'}{A^2 Br} - \frac{B^2}{4r^4} - \frac{1}{r^2} + \frac{1}{A^2 r^2}. \]  

(3.79)

Using \( D^2 = A^2 B^2 \) and multiplying the equation by \( 4c_0 r^4 \) we obtain

\[ 0 = r(r^2 - c_0)2BB' - c_0B^2 - 4r^2c_0 + (r^2 - c_0)B^2 \]  

\( (3.80a) \)

\[ = r(r^2 - c_0)(B^2)' + (r^2 - 2c_0)B^2 - 4r^2c_0. \]  

(3.80b)

Then again introducing \( F(r) := B^2(r) \) we have the inhomogeneous first order linear differential equation

\[ 0 = F' + \frac{r^2 - 2c_0}{r(r^2 - c_0)}F - \frac{4rc_0}{r^2 - c_0}. \]  

(3.81)

Hence, we see that the corresponding homogeneous equation coincides with the one for spacelike orbits. Thus, we have

\[ F_h = c' \frac{\sqrt{r^2 - c_0}}{r^2}. \]  

(3.82)

Now to obtain the general solution we will multiply the inhomogeneous equation by \( c' / F_h \):

\[ 0 = \frac{r^2}{\sqrt{r^2 - c_0}}F' + \frac{r^3 - 2rc_0}{(r^2 - c_0)^{3/2}}F - \frac{4r^3 c_0}{(r^2 - c_0)^{3/2}} \]  

\( (3.83a) \)

\[ = \left( \frac{r^2}{\sqrt{r^2 - c_0}}F \right)' - \frac{4r^3 c_0}{(r^2 - c_0)^{3/2}}. \]  

(3.83b)

Hence, the solution is given by

\[ F = B^2 = c_1 \frac{\sqrt{r^2 - c_0}}{r^2} + \frac{\sqrt{r^2 - c_0}}{r^2} \int \frac{4r^3 c_0}{(r^2 - c_0)^{3/2}} dr \]  

\( (3.84a) \)

\[ = 4c_0 r^2 + c_1 \sqrt{r^2 - c_0} - 8c_0^2 \]  

\( (3.84b) \)

Now with the same line of argument we use the coordinate transformation

\[ r' = \frac{1}{2\sqrt{c_0}} \int D(r) dr = \sqrt{r^2 - c_0}, \quad dr' = \frac{1}{2\sqrt{c_0}} D(r) dr. \]  

(3.85)
such that
\[ B^2(r') = 4c_0 \frac{r'^2 + \frac{c_1}{c_0} r' - c_0}{r'^2 + c_0}. \] (3.86)

Then defining \( l^2 := c_0 > 0, m := -\frac{c_1}{8c_0} \) we again obtain the generalized Taub-NUT metric
\[
g = \frac{4l^2}{B^2(r')} dr'^2 - B^2(r') (d\psi + \cos \theta d\varphi)^2 + (r'^2 + l^2) (d\theta^2 + \sin^2 \theta d\varphi^2) \] (3.87a)
\[
= \frac{r'^2 + l^2}{r'^2 - 2mr - l^2} dr'^2 - 4l^2 \frac{r'^2 - 2mr - l^2}{r'^2 + l^2} (d\psi + \cos \theta d\varphi)^2 + (r'^2 + l^2) (d\theta^2 + \sin^2 \theta d\varphi^2), \] (3.87b)

however describing the NUT-regions. If the orbits are not everywhere space- or timelike, we can join them smoothly along the null hypersurfaces by an extension of the form described in the last chapter.

Thus, with respect to the constants, \( m, l \) and \( n \), we have a three parameter family of vacuum space-times admitting \( SU(2) \times U(1) \) as an isometry group, such that \( SU(2) \times U(1) \) and \( SU(2) \) both have three-dimensional non-null orbits. The Taub-NUT spacetime is then the unique universal cover. As in the case of the Taub-NUT space-time the generalized space-time can be considered to be a principal fibre bundle with respect to the \( U(1) \) right action with its first chern class being the constant \( n \). Furthermore, recalling the remarks in the last section, the constant \( m \) can be considered to be the Komar mass of the space-time. In particular, in this case null infinity is the Lens space \( L(n, 1) \) and the NUT parameter \( l \), being the dual-Bondi-mass with respect to the infinitesimal translation induced by the Killing vector field \( -\frac{1}{2} \partial_\psi \), is proportional to \( n \). Moreover, being a non-trivial \( S^1 \) principal fibre bundle over \( S^2 \) implies that the NUT parameter is non-zero.

4 Outlook

Taub-NUT is a very peculiar spacetime in many respects, not only mathematically, but also concerning its possible physical interpretation. Yet it is frequently regarded for possible applications in astrophysics and cosmology, thereby suggesting that it may be taken as an adequate model for some astrophysical object. Geodesic motions, shadows, and lensing in NUT-spacetime have been investigated in detail; see, e.g., [10, 8, 4]. The question of whether and how NUT-spacetime could be regarded as the exterior geometry produced by some star made of ordinary matter, like, e.g., a perfect fluid, is an old one with partially controversial claims, in particular regarding the physical interpretation of the NUT charge. So far no compelling physical insight seems to exists as to what known properties of ordinary matter could source a non-zero NUT charge. Perfect-fluid solutions with radially pointing vorticity fields have been constructed for that end, but the solutions established in [2] are singular, as has been discussed in [17].
In view of this mismatch between hypothetical physical applications eventually leading to measurements of the NUT parameter on one hand, and the lacking of a proper physical understanding of what might possibly be a matter source (if any) of it on the other, it seems a viable strategy to first characterise the solution as uniquely as possible by its symmetry properties. This is what we attempted and achieved in this work. The physical problem proper clearly remains open for the time being. Also, the mathematical problem of classifying the inequivalent maximal extensions of generalised Taub-NUT should be addressed, which we plan to do in a future publication.
References

[1] George David Birkhoff. *Relativity and Modern Physics*. Harvard University Press, Harvard, Massachusetts, 1923.

[2] Michael Bradley, Gyula Fodor, László Gergely, Mattias Marklund, and Zoltan Perjés. Rotating perfect fluid sources of the NUT metric. *Classical and Quantum Gravity*, 16(6):667–1675, 1999.

[3] Jerry B. Griffiths and Jiří Podolský. *Exact Space-Times in Einstein’s General Relativity*. Cambridge University Press, Cambridge, 2009.

[4] Mourad Halla and Volker Perlick. Application of the Gauss–Bonnet theorem to lensing in the NUT metric. *General Relativity and Gravitation*, 52:112 (1–19), 2020.

[5] Sungnok Hong, John Kalliongis, Darryl McCullough, and J. Hyam Rubinstein. *Diffeomorphisms of Elliptic 3-Manifolds*. Springer Verlag, Berlin, 2012.

[6] Jørg Tofte Jebsen. Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen im Vakuum. *Arkiv för Matematik, Astronomi och Fysik*, 15(18):1–9, 1921.

[7] Jørg Tofte Jebsen. On the general spherically symmetric solutions of einstein’s gravitational equations in vacuo. *General Relativity and Gravitation*, 37(12):2253–2259, 2006. (English translation and reprint as ‘Golden Oldie’ of [6]).

[8] Paul Jefremov and Volker Perlick. Circular motion in NUT space-time. *Classical and Quantum Gravity*, 33(24):245014 (1–24), 2016. Corrigendum: Class. Quantum Grav. 35 (2018) 179501 (2pp).

[9] Nils Voje Johansen and Finn Ravndal. On the discovery of Birkhoff’s theorem. *General Relativity and Gravitation*, 38(3):537–540, 2006.

[10] Valeria Kagramanova, Jutta Kunz, Eva Hackmann, and Claus Lämmerzahl. Analytic treatment of complete and incomplete geodesics in Taub-NUT space-times. *Physical Review D*, 81(12):124044 (1–17), 2010.

[11] Charles Misner. Taub-NUT as a counterexample to almost anything. In Jürgen Ehlers, editor, *Relativity and Astrophysics*, volume 8 of *Lectures in Applied Mathematics*, pages 160–169. American Mathematical Society, Providence, Rhode Island, 167.

[12] Charles Misner. The flatter regions of Newman, Unti, and Tamburino’s generalized Schwarzschild space. *Journal of Mathematical Physics*, 4(7):924–937, 1963.

[13] Charles W. Misner and Abraham H. Taub. A singularity-free empty universe. *Soviet Physics JETP*, 28(1):122–133, 1969.
[14] Vincent Moncrief. The space of (generalized) Taub-NUT spacetimes. *Journal of Geometry and Physics*, 1(1):107–130, 1984.

[15] Ezra Theodore Newman, Lois A. Tamburino, and Theodore W.J. Unti. Empty-space generalization of the Schwarzschild metric. *Journal of Mathematical Physics*, 4(7):915–923, 1963.

[16] Sriram Ramaswamy and Amitabha Sen. Dual-mass in general relativity. *Journal of Mathematical Physics*, 22(11):2612–2619, 1981.

[17] Wajahat Rana. Interpretation von Sternmodellen aus idealen Flüssigkeiten mit NUT-Ladung. Master’s thesis, Leibniz University of Hannover, 6 2019.

[18] Norbert Straumann. *General Relativity*. Graduate Texts in Physics. Springer Verlag, Dordrecht, 2 edition, 2013.

[19] Abraham Haskel Taub. Empty space-times admitting a three parameter group of motions. *Annals of Mathematics*, 53(3):472–490, 1951.