A general U-block model-based design procedure for nonlinear polynomial control systems

Q. M. Zhu\textsuperscript{a,b}, D. Y. Zhao\textsuperscript{b} and Jianhua Zhang\textsuperscript{c}

\textsuperscript{a}Department of Engineering Design and Mathematics, University of the West of England, Bristol, UK; \textsuperscript{b}Department of Chemical Industrial Equipment and Control Engineering, College of Chemical Engineering, China University of Petroleum, Qingdao, China; \textsuperscript{c}Institute of Electrical Engineering, Hebei University of Science and Technology, Shijiazhuang, China

\textbf{ABSTRACT}

The proposition of U-model concept (in terms of ‘providing concise and applicable solutions for complex problems’) and a corresponding basic U-control design algorithm was originated in the first author’s PhD thesis. The term of U-model appeared (not rigorously defined) for the first time in the author’s other journal paper, which established a framework for using linear polynomial control system design approaches to design nonlinear polynomial control systems (in brief, linear polynomial approaches → nonlinear polynomial plants). This paper represents the next milestone work – using linear state-space approaches to design nonlinear polynomial control systems (in brief, linear state-space approaches → nonlinear polynomial plants). The overall aim of the study is to establish a framework, defined as the U-block model, which provides a generic prototype for using linear state-space-based approaches to design the control systems with smooth nonlinear plants/processes described by polynomial models. For analysing the feasibility and effectiveness, sliding mode control design approach is selected as an exemplary case study. Numerical simulation studies provide a user-friendly step-by-step procedure for the readers/users with interest in their ad hoc applications. In formality, this is the first paper to present the U-model-oriented control system design in a formal way and to study the associated properties and theorems. The previous publications, in the main, have been algorithm-based studies and simulation demonstrations. In some sense, this paper can be treated as a landmark for the U-model-based research from intuitive/heuristic stage to rigorous/formal/comprehensive studies.

\section{1. Introduction}

The main objective of control engineering is invariably to design a controlled system to best match the performance or design specifications. Standard procedures of control system design can be broadly classified into two groups according to whether the dynamic plant is linear or nonlinear. Linear control is a mature subject with a variety of powerful methodologies and a long history of successful industrial applications. Comparatively speaking nonlinear control is a more difficult subject, although some design schemes have been developed with ad hoc model structures (Slotine & Li, 1991). Generalisation of control system design for nonlinear dynamic plants has been a hot and challenging topic in research and applications. It has been the researcher’s belief that it is possible to use linear controller design methodologies to design a large class of smooth nonlinear control systems, and therefore to simplify and generalise nonlinear control system design. This is the insight of the study.

State-space model-based linear control system design approaches have been well studied in research and adopted in wide ranges of applications (Maciejowski, 1994; Ogata, 2009). The fundamental request is the linear state-space model. For linear polynomial models, there are many realisation approaches to convert them into state-space expressions, such as typical controllable and observable realisations. However, although a nonlinear polynomial model, such as the NARMAX (Nonlinear AutoRegressive Moving Average with eXogenous input) model (Billings, 2013), is a very practical and predominantly used model structure (Billings, 2013; Zhu, Wang, Zhao, Li, & Billings, 2015), it is very difficult to convert it into a state-space expression, and almost impossible to convert it into an equivalent linear state-space model. It should be made clear at the beginning of the study that piecewise linearisation and linear approximation approaches are not discussed in this study, simply because these are not the essence of nonlinear control system design problems.

There have been two main streams plus a less-attended research area in the related fields. The first one is to convert nonlinear state-space models into linear expressions. The approach is the feedback linearisation (input–state linearisation and input–output linearisation) (Isidori, 1995; Slotine & Li, 1991), in which the central idea of feedback linearisation is to convert, by coordinate transform, the non-linear model into a linear form, in order to cancel the nonlinear dynamics in the closed-loop. Then, linear state-space approaches to design the corresponding control systems are used. However, this is a case-by-case approach with a certain degree of skills in manipulating differential equations and selecting coordinates. Further shortage with the state-space linearisation approaches is that there is no way to deal with the nonlinear polynomial model-based control system design.
The second one is to treat nonlinear polynomial models as time-varying linear models. One of the typical approaches is the state-dependent parameter (SDP) transformation (Cimen, 2010; Taylor, Chotai, & Young, 2009), which in the main, reduces the closed-loop system to a linear transfer function with the specified (desired) poles. Hence, assuming pole assignability at each sample, global stability of the nonlinear system is guaranteed at the design stage. It is clear from these studies that a common strategy for the control of nonlinear systems involves attempting to bring the original system into a quasi-linear domain, before subsequently designing an appropriate linear control algorithm. In model structure, the parameters of this quasi-linear SDP are functionally dependent on other variables in the system (Young, 2000). Although this provides a bridge from the nonlinear polynomial model to the linear time-varying state-space expression, the transform is not unique and selection of SDP is subjective, personal and no clear rule to follow, which has been the main barrier for a wide range of users.

Although not widely attended, the third one is the U-model methodology, which is a generic systematic approach to convert the nonlinear polynomial model into a controller output \( u(t-1) \)-based time-varying polynomial model. This has been studied (Du, Wu, & Zhu, 2012; Muhammad & Haseebiddon, 2005; Zhu, 1989; Zhu & Guo, 2002) for facilitating nonlinear control system design over the last decade. Consequently, linear polynomial model-based design approaches (such as pole placement and general predictive control) can be directly used to design such nonlinear control systems (Du et al., 2012; Zhu & Guo, 2002). The major contribution of the U-model-based design procedure can be listed in order.

1. In methodology, those well-known approaches developed from linear systems can be directly applied to nonlinear control system design, which significantly reduces the design complexity and effectively provides straightforward computational algorithms. It should be noted that this new approach does not request to linearise nonlinear plant models in design. It is just to use linear design approaches to directly design nonlinear plant model-based control systems.

2. In design, it obtains desired plant output first (comparing to designing desired controller output in classical framework) and then works out the controller output from the U-model in a relaxed root-resolving routine (comparing to resolving complex solutions from the inverse of the whole designed systems).

3. For linear control system design, it provides new insight and solutions within a more general and effective framework.

It should be noted that the U-model-associated publications are still in a very beginning stage. Although various algorithms have been developed and bench tested with simulations and applications, they have not had rigorous analytical description and top journal publication till today. It is the author’s belief that the next research progress should be the development to make the U-model approach available for using linear state-space-based design approaches to design the control of nonlinear polynomial models described plants. The key issue in the on-going study is how to systematically obtain an equivalent linear state-space model from its nonlinear polynomial model origin.

This paper takes the above described issues into study and proposes a general U-block model-based procedure for using linear state-space-based approaches to design the control systems with smooth nonlinear plants/processes described by polynomial models. It should be noted that this procedure is not the way of using linear approximation to obtain linear polynomials and then convert to the corresponding state-space realisations. For a nonlinear plant polynomial model, to design its control system with linear state-space-based approaches, the procedure can be ordered into the following step-by-step implementation.

1. Convert the nonlinear polynomial model into the U-model expression (Note, not linear approximation).

2. By pole assignor (a pole placement algorithm), convert the U-model into an input–output linear closed-loop model with assigned poles. This is defined as the U-block model, which is linear.

3. Convert the linear U-block model into its state-space realisation by commonly used realisation techniques. This is defined as the U-state-space model.

4. With reference to the U-state-space model, using linear state-space-based design approaches, design the control system. For feasibility test, this study selects sliding mode control (SMC) design. It should be noted, from the author’s knowledge, so far, there is no any existing approach to design SMC with nonlinear polynomial models. There are a few of representative publications using the terminology <SMC for nonlinear polynomial systems> (Basin & Calderon-Alvarez, 2010; Basin & Rodriguez-Ramirez, 2012), however, all the publications have used <state-space model with a nonlinear polynomial for each state variable>-. Simply it is a common basis that SMC need state-space models, no matter linear or nonlinear expressions.

The rest of the paper is organised into the following sections. In Section 2, the U-model is formally defined and the associated properties are presented to provide a proper description of the model structure and characteristics. Subsequently, remarks with an example are presented for explanation and understanding of the conversion from classical polynomials to the U-model. In Section 3, a U-model-based control system design procedure is presented with step-by-step implementation, which provides a foundation for the development of Section 4. Remarks are used to show the difference and novelty compared with classical linear control system design approaches. In Section 4, a U-block model is proposed to establish a framework for using linear state-space approaches to design nonlinear polynomial control systems. The related properties are analysed as well to provide assurance for applications. In Subsection 4.3, SMC design approach is selected as an exemplary case study to show the feasibility within the U-block model framework. In Section 5, a series of computational experiments are conducted to demonstrate numerically the theoretically developed results. In Section 6, a brief conclusion is drawn to summarise the results and the potential research in the future is briefly discussed as well.
2. U-model

2.1. Definition of smooth and analytic function (Isidori, 1995)

Let A be an open subset of \( R^n \) and \( f : A \rightarrow R \) a function. The value of \( f \) at \( x = (x_1 \ldots x_n) \) is denoted \( f(x) = f(x_1 \ldots x_n). \) The function \( f \) is said to be a function of class \( C^\infty \) (or simply \( C^\infty \), or, also smooth function) if its partial derivatives of any order with respect to \( x_1 \ldots x_n \) exist and are continuous. A function \( f \) is said to be analytic (sometimes denoted as \( C^\infty \)) if it is \( C^\infty \) and for each point \( x^0 \in A \) there exists a neighbourhood \( \chi \) of \( x^0 \), such that the Taylor series expansion of \( f \) at \( x^0 \) converges to \( f(x) \) for all \( x \in \chi. \)

Remark 1: This definition is well explained in classical calculus. In the contrast, nonsmooth systems are normally referred to those systems with switching or jumping functions, which are also called ‘hard nonlinearities’, such as Coulomb friction, saturation, dead-zones, backlash and hysteresis, and are frequently found in practical control systems (Slotine & Li, 1991).

Consider a general single-input and single-output discrete time dynamic plant described by the following polynomial model:

\[
y(t) = f(P(\cdot), \Theta)
\]

\[
P(s) = P(y(t - 1), \ldots, y(t - n), u(t - 1), \ldots, u(t - n)),
\]

where \( y(t) \in R \) and \( u(t - 1) \in R \) are the output and input (also known as the controller output in the control system design) signals of the plant, respectively, at discrete time instant \( t(1, 2, \ldots, n) \) is the plant order. \( P(\cdot) = P(y(t - 1), \ldots, y(t - n), u(t - 1), \ldots, u(t - n)) \in R^{l+1} \) is the regression variable vector spanned from the delayed outputs and inputs and \( \Theta = [\theta_0 \ldots \theta_l] \in R^{l+1} \) is the associated parameter vector.

Function \( f(\cdot) \) is a smooth linear or nonlinear function. The model can be further expressed in terms of regression equation as

\[
y(t) = \sum_{l=0}^{L} p_l(t)\theta_l,
\]

where the regression terms \( p_l(t) \) are the products of past inputs and outputs, such as \( u(t - 1)y(t - 3), u(t - 1)u(t - 2), y^2(t - 1), \) and \( \theta_l \) are the associated parameters.

It should be noted that this model set has been widely accepted as NARMAX models (Billings, 2013).

2.2. Definition of the U-model

The U-model is defined as, under a U mapping from the above polynomial model, a controller output \( u(t - 1) \) oriented polynomial below,

\[
y(t) = U(t - 1)
\]

\[
U(t - 1) = (\text{const } u(t - 1) u(t - 1) \ldots u^M(t - 1)).
\]

Correspondingly, its regression equation is given as

\[
y(t) = \sum_{j=0}^{M} \lambda_j(t)u^j(t - 1).
\]

This is expanded from the above nonlinear function \( f(\cdot) \) as a polynomial with respect to \( u(t - 1) \), where \( M \) is the degree of model input (controller output) \( u(t - 1) \), the time varying parameter vector \( \lambda(t) = [\lambda_0(t) \ldots \lambda_M(t)] \in R^{M+1} \) is a function of past inputs and outputs \( u(t-2), \ldots, u(t-n), y(t-1), \ldots, y(t-n) \), and the parameters \( [\theta_0 \ldots \theta_l]. \)

Property 1: Let \( \varphi : R^{l+1} \rightarrow R^{M+1} \) be the map from the polynomial model to the U-model and its inverse be \( \varphi^{-1} \), that is

\[
f(P(\cdot), \Theta) \rightarrow \varphi f(\Upsilon(\cdot), U(t - 1)).
\]

Then, it has the following properties:
(1) the map is injective (one to one),
(2) the map is surjective (onto),
(3) therefore, the map is bijective as it is both injective and surjective,
(4) the map is invertible,
(5) the map does not change any both model characteristics, such as output response, stability, dynamics and statics.

Proof: In (2.2), the products of the term by parameter can be alternatively written as

\[
p_l(t)\theta_l = r_l(t)u^j(t - 1)\theta_l, \quad j = 0, 1, \ldots M.
\]

Accordingly, (2.2) can be expressed as

\[
y(t) = \sum_{l=0}^{L} p_l(t)\theta_l = \sum_{l=0}^{L} r_l(t)\theta_lu^j(t - 1)
\]

\[
= \sum_{j=0}^{M} \left( \sum_{l=0}^{L} r_l(t)\theta_l \right) u^j(t - 1), \quad l = 0, 1, \ldots L.
\]

For each \( u^j(t - 1), \) \( j = 0, 1, \ldots M, \) let \( \lambda_j(t) = \sum_{l=0}^{L} r_l(t)\theta_l \), then it gives the U-model expression (2.4).

Remark 2: With the U-model (2.4), there is nothing in change of the original polynomial model (2.2) properties, such as dynamic, stability and input and output external relationship. However, by such mapping (re-organisation) of the model expression, the U-model provides a generic prototype to facilitate control system design by using linear approaches. In model properties, the U-model and the NARMAX polynomial model are equivalent. In model expression, the U-model is time varying in parameters and convenient for the control systems design and the NARMAX polynomial model is time constant in parameters and convenient for system identification.

Remark 3: Actually the above proof provides a model conversion algorithm.

Remark 4: Such an example, from the polynomial model to the U-model conversion, is shown below.
The polynomial model is
\[ y(t) = 0.1y(t-1)y(t-2) - 0.5y(t-1)u^{2}(t-1) + 0.8u(t-1)u(t-2) \]  
\[ (2.8) \]
and the U-model can be determined in the notation of (2.4),
\[ y(t) = \lambda_{0}(t) + \lambda_{1}(t)u(t-1) + \lambda_{2}(t)u^{2}(t-1), \]
\[ (2.9) \]
where \( \lambda_{0}(t) = 0.1y(t-1)y(t-2), \lambda_{1}(t) = 0.8u(t-2), \) and \( \lambda_{2}(t) = -0.5y(t-1). \)

## 3. U-control – U-model-based control system design procedure

Hypothesis: It is possible to use linear controller design methodologies to directly design a large class of smooth nonlinear plant-based control systems, and therefore to simplify and generalise nonlinear control system design.

In order to use linear polynomial model-based design approaches, define the desired plant output as \( y_{d}(t) \), which is specified either by designers or customers in advance. Accordingly, the relationship between a specified plant output \( y_{d}(t) \) and the requested corresponding controller output \( u(t-1) \) can be expressed in terms of the U-model,
\[ y_{d}(t) = \sum_{j=0}^{M} \lambda_{j}(t)u^{j}(t-1). \]
\[ (3.1) \]
Accordingly, this establishes a prototype for proposing a new two-step design procedure.

**Step 1** (design \( y_{d}(t) \)): The first task of the design is to determine the desired plant output \( y_{d}(t) \) according to a specified performance index, for example,

Pole placement control (Zhu & Guo, 2002) has been designed in terms of
\[ R_{yd}(t) = Tw(t) - Sy(t), \]
\[ (3.2) \]
where \( y(t), y_{d}(t) \) and \( w(t) \) are the plant output, desired/plant output and reference input, respectively. The polynomials, \( R, S \) and \( T \) are used to specify the desired plant output \( y_{d}(t) \).

General predictive control (Du et al., 2012) has been designed in terms of
\[ J_{P} = E [(Y - W)^{T}Q_{1}(Y - W) + \Delta Y_{d}^{T}Q_{2}\Delta Y_{d}], \]
\[ (3.3) \]
where \( Y, Y_{d}(t) \) and \( W(t) \) are the vectors of \( y(t), y_{d}(t) \) and \( w(t) \), respectively. \( Q_{1} \) and \( Q_{2} \) are the corresponding weighting coefficient matrices.

**Step 2** (work out \( u(t-1) \)): Then, the remaining design task is to resolve one of the roots of (3.1) to obtain the controller output \( u(t-1) = \Psi \left[ y_{d}(t) - \sum_{j=0}^{M} \lambda_{j}(t)u^{j}(t-1) = 0 \right] \),
\[ (3.4) \]
where \( \Psi^{[*]} \) is a root-solving algorithm, such as Newton–Raphson algorithm (Chong & Zak, 2013). A detailed analysis on the root solving issues has been presented (Zhu & Guo, 2002).

**Remark 5:** It should be noted that any iterative algorithm takes time, however, in the framework, the designed closed-loop is stabilised, so that the controller output is established. By feedback, at each step \( u(t-1) \) magnitude changes small which makes the root solver initial searching point is closed to its solution. According to a general proposition (Chong & Zak, 2013; Nash, 2000), many iteration algorithms are converged with good convergent rate while its starting searching point is sufficiently close to its solution point. In regarding to Newton–Raphson algorithm, it requests the calculations of Jacobian and particularly inverse of Hassan matrix in iteration. If this involves multi-decision variable search, inverse of Hassan matrix is very time consuming. For the U-model case, it is only a single variable \( u(t-1) \) search, therefore, inverse of Hassan matrix is reduced to a single reciprocal calculation. A small number of iterations are sufficient to produce a ‘good’ step.

**Remark 6:** The U-model is a type of time-varying polynomials, the U-model-based controller is actually an online algorithm because its controller must be updated from the time-varying parameter vector \( \lambda(t) = [\lambda_{0}(t) \ldots \lambda_{M}(t)] \) in each sampling interval.

**Remark 7:** In methodology, the difference compared with linear control system design is the way to find the controller output \( u(t-1) \) from resolving a root with (3.1).

**Remark 8:** Comparison of classical design procedure with the new procedure, let design criterion or performance index
\[ J(C_{1}, C_{2}, \ldots C_{n}), \]
\[ (3.5) \]
where \( C_{1}, C_{2}, \ldots C_{n} \) are the polynomials specifying the control system performance, such a typical example is the pole placement scheme (Astrom & Wittenmark, 1995) with three polynomials \( R, S, T \) to specify the desired plant output, plant output and reference, respectively.

In classical design procedure (work out controller output \( u(t-1) \)), resolve the criterion function to obtain the controller output (that is the plant input) \( u(t-1) \). With this procedure, it, even for a linear system, requests the plant model to obtain the solution from inversion of the equations. The more difficult case is that of the nonlinear systems whose inversion becomes almost impossible most of the time.

In the new design procedure (work out desired plant output \( y_{d}(t) \) and then the controller output \( u(t-1) \)), first resolve the criterion function to obtain the designed/plant output...
y_d(t) (Note not the controller output) u(t − 1)!. Then, obtain the controller output u(t − 1) from the designed y_d(t) through the U-model (3.1), that is, resolve one of the roots from the U-model (3.1). With this procedure, it does not request the plant model (no matter line or nonlinear) in the first stage. It only uses the plant model, in the U-model expression, to obtain the controller output u(t − 1) in the second stage. For a nonlinear plant model, the calculation is merely to resolve one of the roots from the U-model (3.1). For a linear plant model, it has, from (3.1),

\[
u(t − 1) = \frac{y_d(t) − \lambda_0(t)}{\lambda_1(t)}.
\]

where \(\lambda_1(t)\) is the coefficient associated with \(u(t − 1)\) (for linear time invariant models, \(\lambda_1(t)\) is a constant); \(\lambda_0(t)\) (nonzero) is the summation of the rest of the terms in the linear model (Zhu & Guo, 2002).

4. U-block model and its state-space realisation

4.1. U-block model

The U-block model is defined as a general linearised model from the nonlinear polynomial model (2.2) through the conversion to the U-model (2.4) and then assigned with required poles through a linear feedback control algorithm. In this study, the pole assignor is formulated by a revised U-model-based pole placement controller design algorithm. Figure 2 shows the U-block model structure.

4.1.1. Pole assignor – design of the pole placement controller

A standard reference (Astrom & Wittenmark, 1995) has been used to develop the following pole placement controller (Zhu & Guo, 2002). Consider the U-model of (2.4), a general controller can be structured by

\[
Ru(t) = Tw(t) − Sy(t),
\]

where \(w(t)\) is the reference for output target and \(R, S\) and \(T\) are the polynomials of the forward shift operator \(q\), which are described by

\[
R = q^n + r_1 q^{n−1} + \cdots + r_n, \\
T = t_0 q^m + t_1 q^{m−1} + \cdots + t_m, \\
S = s_0 q^l + s_1 q^{l−1} + \cdots + s_l,
\]

(4.2)

where \(q\) is the forward operator, \(n, m\) and \(l\) are the orders of the polynomials \(R, T\) and \(S\), respectively. A designed controller must satisfy the causality conditions \(\text{order}(S) < \text{order}(R)\) \(\text{order}(T) \leq \text{order}(R)\), that is \(l < n\) and \(m \leq n\).

The control law of (4.1) represents a negative feedback with the transfer function \(S/R\) and a feedforward with the transfer function \(T/R\). It thus has two degree of freedom.

For the proposed design procedure, as shown in Figure 1, rewrite (4.1) as

\[
Ry_d(t) = Tw(t) − Sy(t).
\]

(4.3)

By letting \(y(t) = y_d(t)\), the designed U-block model can be linked to the reference \(w(t)\) as

\[
y_d(t) = \frac{T}{R + S}w(t) = \frac{T}{A_c}w(t),
\]

(4.4)

where polynomial \(A_c\) is the closed-loop characteristic equation and specified in advance, that is

\[
R + S = A_c.
\]

(4.5)

To cancel the possible output offset in steady state, i.e., to make steady state error equal to zero at the controlled output, polynomial \(T\) is specified with

\[
T = A_c(1).
\]

(4.6)

The key idea of the design is to specify the desired closed-loop characteristic polynomial \(A_c\), then resolve the polynomials \(R\) and \(S\) through a Diophantine equation (Zhu & Guo 2002). After the desired plant output \(y_d(t)\) is designed, the controller output \(u(t − 1)\) can be determined by resolving one of the root of the U-model (3.1), which the two step design procedure has formulated in Section 3.

**Theorem 1:** The U-model based pole placement design procedure does not depend on the plant model. Only the solution of the designed controller output involves in the plant model.

![Figure 1. U-model-based pole placement control system.](image1)

![Figure 2. U-block model.](image2)
The U-state-space model is defined as the state-space realisation

$$AR + BS = A_c,$$  
(4.7)

where $A$ and $B$ are the numerator polynomial and the denominator polynomial of a plant model, respectively, which indicate the classical design depending on the plant model.

**Remark 9:** Actually Theorem 1 provides a new design framework, using the U-control design procedure once, then it can be applied to many different plant models, which only calculates the corresponding controller output each time from a given plant model.

### 4.1.2 Pole assignability

For linear systems this has been well studied (Taylor et al., 2009; Young, Behzadi, Wang, & Chotai 1987). In principle, for linear case, model input variable (that is the controller output) should have some influence on the system output. For the nonlinear case, in terms of the SDP model, Taylor et al. (2009) derived similar condition. In regarding to the U-model structure, the condition of the pole assignability can be stated in the following proposition.

**Proposition:** For the U-model (3.1), pole assignment (placement) algorithm described by (4.1) through to (4.6), can be resolved if and only if

$$\exists \alpha_j(t) \neq 0, j \in [1 \ldots M] \Rightarrow u(t-1) \text{ has some influence on the system output}.$$  

This is a direct inference of the condition for SDP-based pole assignment algorithm (Taylor et al., 2009).

### 4.2 U-state-space model

The U-state-space model is defined as the state-space realisation of the U-block model. Let the assigned poles as $\alpha_1 \ldots \alpha_k$, then its corresponding characteristic equation is

$$A_c = (q - \alpha_1) \ldots (q - \alpha_k) = q^k + a_1q^{k-1} + \cdots + a_k = 0.$$  
(4.8)

With reference to (4.4) and (4.6), the assigned numerator polynomial of the U-block model is

$$T = A_c(1).$$  
(4.9)

Correspondingly, the U-block model transfer function is

$$\frac{Y(q)}{W(q)} = \frac{A_c(1)}{q^k + a_1q^{k-1} + \cdots + a_k}.$$  
(4.10)

Consequently, the U-state-space model, in controllable realisation, can be expressed as

$$x(t + 1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t),$$  
(4.11)

where the state equation

$$\begin{bmatrix}
    x_1(t + 1) \\
    x_2(t + 1) \\
    \vdots \\
    x_{k-1}(t + 1) \\
    x_k(t + 1)
\end{bmatrix}
= \begin{bmatrix}
    0 & 1 & 0 & \ldots & 0 \\
    0 & 0 & 1 & \ldots & 0 \\
    \vdots \\
    -a_k & -a_{(k-1)} & -a_{(k-2)} & \ldots & -a_1 \\
\end{bmatrix}
\times
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    \vdots \\
    x_{k-1}(t) \\
    x_k(t)
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    0 \\
    1
\end{bmatrix} w(t).$$  
(4.12)

and the output equation

$$y(t) = \begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    \vdots \\
    x_{k-1}(t) \\
    x_k(t)
\end{bmatrix}.$$  
(4.13)

**Remark 10:** The state variable measurement can be obtained by standard observers (Ogata, 2009) from the available input $w(t)$ and output $y(t)$, while the U-state-space model is assigned with the poles to make it completely observable.

**Remark 11:** The whole framework in using linear state-space approaches to design control systems with smooth nonlinear polynomial plant models is shown in Figure 3.

### 4.3 Feasibility studies – design of sliding mode control (SMC)

As this is the first time to establish such framework for using linear state-space approaches to design control systems with nonlinear polynomial plant models, it should be made clear at the beginning of this section that selecting SMC design approach is, in the main as an example, to demonstrate the feasibility of the developed design framework. Subsequently, the associated comprehensive studies on the SMC design will be described in the future work.

**Theorem 2** (feasibility theorem): It is feasible to design SMC for nonlinear polynomial plant models, while all the following conditions are satisfied.

(C1) A corresponding U-block model existed.
(C2) A completely observable observer existed.
(C3) An asymptotic stable sliding manifold (mode) existed.

**Proof:** (C1) provides a necessary control-oriented model structure. Otherwise, SMC cannot be used without state-space model. The existence of a U-block model can be achieved from the assignability presented in Section 4. (C2) provides a channel to obtain the necessary state variable signal for design. Otherwise, SMC cannot be used in practice. The existence can be achieved from assigning characteristic equation of the U-block model. (C3) provides stable control functionality. Otherwise, simply a designed system is out of control. The existence can be achieved from the following step-by-step algorithm (Utkin, Guldner, & Shi, 2009).
Step 1: With reference to the U-state-space model of (4.11) and (4.12), design a discrete time sliding mode (SM). By choosing a matrix $\hat{C} \in \mathbb{R}^{k \times 1}$ ($k$ is the dimension of the dynamic matrix (4.11) or the order of the characteristic equation of the U-block model (4.9), assign a stable manifold

$$s(t) = \hat{C}x(t), \quad (4.14)$$

where $x(t)$ is the so-called SM.

Step 2: Design an equivalent control $w_{eq}(t)$,

$$w_{eq}(t) = -\left(\hat{C}B\right)^{-1}s(t) - \left(\hat{C}B\right)^{-1}(\hat{C}A - \hat{C})x(t). \quad (4.15)$$

Step 3: Design SMC

$$w(t) = \begin{cases} w_{eq}(t) & \|w_{eq}(t)\| \leq w_0, \\ \frac{w_{eq}(t)}{\|w_{eq}(t)\|}w_0 & \|w_{eq}(t)\| > w_0, \end{cases} \quad (4.16)$$

where $w_0$ is the controller output bound, that is, makes $\|w(t)\| \leq w_0$.

It should be noted that the SM controller $w(t)$ is a standard form and the stability analysis can be examined by using $\|s(t+1)\| < \|s(t)\|$ (Utkin et al., 2009).

5. Case studies – design and simulation

Consider a nonlinear dynamic plant described by the following Hammerstein model,

$$\begin{align*}
y(t) &= 0.5y(t-1) + x(t-1) + 0.1x(t-2) \\
x(t) &= 1 + u(t) - u^2(t) + 0.2u^3(t).
\end{align*} \quad (5.1)$$

5.1. Design the U-block model

The first step is to covet the Hammerstein model into the U-model, with reference to (2.7), it gives

$$\begin{align*}
y(t) &= \lambda_0(t) + \lambda_1(t)u(t-1) + \lambda_2(t)u^2(t-1) \\
&\quad + \lambda_3(t)u^3(t-1),
\end{align*} \quad (5.2)$$

where

$$\begin{align*}
\lambda_0(t) &= 0.5y(t-1) + 1 + 0.3x(t-2) \\
\lambda_1(t) &= \alpha_1(t) = 1 \\
\lambda_2(t) &= -1 \\
\lambda_3(t) &= \alpha_3(t) = 0.2.
\end{align*} \quad (5.3)$$

As explained in Remark 6, the U-model-based design procedure does not request the plant model (no matter line or nonlinear) in the first stage. To assign the obtained U-model into its U-block model, specify the closed-loop characteristic equation with

$$A_c = q^2 - 1.3205q + 0.4966. \quad (5.4)$$

Therefore, the closed-loop poles are a complex conjugate pair of $-0.6603 \pm j0.2463$. This design specification corresponds to a natural frequency of 1 rad/sec and a damping ratio of 0.7. To achieve zero steady state error, specify

$$T = A_c(1) = 1 - 1.3205 + 0.4966 = 0.1761. \quad (5.5)$$

For the polynomials $R$ and $S$, specify

$$R = q^3 + r_1q + r_2 \quad S = s_0q + s_1. \quad (5.6)$$

Substitute the specification of (5.5) into the Diophantine equation of (4.5), to formulate the coefficients in polynomials $R$ and $S$ (Zhu & Guo, 2002) with

$$\begin{align*}
r_2 + s_1 &= 0.4966 \\
r_1 + s_0 &= -1.3205. \quad (5.7)
\end{align*}$$

To guarantee the computation convergence of the sequence $U(t)$, that is to keep the difference equation with stable dynamic, let $r_1 = -0.9, r_2 = 0.009$. This assignment corresponds to the characteristic equation of $U(t)$ as $(q - 0.89)(q - 0.01) = 0$. Then, the coefficients in polynomial $S$ can be determined from the Diophantine equation of (5.7)

$$s_0 = -0.4205, s_1 = 0.4876. \quad (5.8)$$

Substituting the coefficients of the polynomials $R$ and $S$ into the controller of (3.1) give rise to

$$\begin{align*}
y_d(t+1) &= 0.9y_d(t) - 0.009y_d(t-1) + 0.1761w(t-1) \\
&\quad + 0.4205y(t) - 0.4876y(t-1).
\end{align*} \quad (5.9)$$

Therefore, the controller output $u(t)$ can be determined from resolving (3.5).

The output response of the designed U-block model with assigned poles and steady state property is shown in Figure 4(a).
5.2. Convert the U-block model into its controllable U-state-space model

With reference to the designed U-block model (5.4) (the numerator of the transfer function) and (5.5) (the denominator of the transfer function), the controllable form in terms of the state-space model is

\[ x(t + 1) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

(5.10)

where

\[ A = \begin{bmatrix} 0 & 1 \\ -0.4966 & 1.3205 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1761 & 0 \end{bmatrix}. \]

(5.11)

As the original polynomial plant model only has input and output available for design, a full-order state observer (Ogata, 2009) with two poles at \([-0.1, -0.2]\) is designed to accommodate the state-space type controller, which is given by

\[ \dot{x}(t + 1) = L_{obs}\hat{x}(t) + M_{obs}u(t) + N_{obs}y(t), \]

(5.12)

where \(\hat{x}(t)\) is the estimated state and

\[ L_{obs} = \begin{bmatrix} -1.0205 & 1 \\ -1.3676 & 1.3205 \end{bmatrix}, \quad M_{obs} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad N_{obs} = \begin{bmatrix} 5.7950 \\ 4.9459 \end{bmatrix}. \]

(5.13)

5.3. Design SMC

Step 1: Design a time SM. By choosing a matrix \(\hat{C} \in \mathbb{R}^{1 \times 2} = [1 \ 2]\) to make the SM be Hurwitz polynomial. Therefore, this choice assigns a stable manifold

\[ s(t) = \hat{C}x(t),\]

(5.14)

where \(s(t)\) is the so-called SM.

Step 2: Design an equivalent control \(w_{eq}(t)\),

\[ w_{eq}(t) = -(\hat{C}B)^{-1}s(t) - (\hat{C}B)^{-1}(\hat{C}A - \hat{C})x(t). \]

(5.15)

Step 3: Design SMC,

\[ w(t) = \begin{cases} w_{eq}(t) \quad \|w_{eq}(t)\| \leq w_0 \\ w_{eq}(t) \quad \|w_{eq}(t)\| > w_0 \end{cases} \]

(5.16)

where \(w_0 = 0.05\) is the bound to specify the controller output saturation, that is, makes \(\|w(t)\| \leq w_0\).

The designed performances with the initial state \(x_0 = [1 \ 0.5]^T\) are shown in Figure 5.

5.4. Discussion of the case studies

This is the first study with such proposed framework and procedure, the main purpose of the case study is to demonstrate the feasibility and effectiveness of the developed methodology. For classical comparative demonstrations with the other approaches will be conducted in the future studies. The second motivation in the case study is to give an example for the potential users to expand their studies in this new research domain.

In regarding to Section 5.1, the plant has the first-order dynamic with nonlinear linear input. To design its U-block model, a linear closed-loop transfer function was specified with second order, decade oscillatory response. The assignability is satisfied to confirm the existence of the U-block model. An algorithm for resolving the Diophantine equation was used to obtain the requested polynomial coefficients linked to the designed poles. To reduce the computational burden, the algorithm could be programed, for example, MATLAB coded. Inspection of Figure 4(a) and 4(b), the U-block model does behave with the requested polynomial coefficients linked to the designed poles.

In regarding to Section 5.2, by checking through the U-state-space model (5.10), it is completely controllable and observable, this provides an assurance to design the observer (5.13).

In regarding to Section 5.3, again it is not a comprehensive comparative study. SMC has been a well-matured topic. There
is nothing new in this SMC design, therefore, it has no need to compare with the other leading SMC results. However, its significance is the demonstration of the feasibility in using linear state-space approaches to design control systems with nonlinear polynomial plants, which had no other results reported before even though the bottle neck issue has been commonly widely recognised for long time. Figure 5(a)–(d) show the designed performance.

In summary, with the simulation demonstration or so-called computational experiment tests, the case study supports the following insight.

1. It is feasible to combine the linear SMC design with the U-model structures to analytically design control systems nonlinear polynomial plant models. In the other terms, the plant is nonlinear (Hammerstein nonlinear polynomial model), the designs are linear approaches (pole placement and sliding mode) and the bridge to connect is the U-model (direct re-expression of the original nonlinear model) and the U-block model (designed into linear state-space model).

2. It has demonstrated the effectiveness of the idea by using simple exiting linear state-space control design approaches for nonlinear polynomial plant models, which could be expanded to cover the other strong nonlinear systems (such as those represented by nonlinear rational models (Zhu et al., 2015)) design which is very challenging or unsolved systematically by using exiting control design approaches.

3. The algorithms developed could be conveniently converted into online recursive implementation for the control of those complex nonlinear systems in process industry.

4. In some sense, a complex problem is split into a simple step-by-step procedure for solutions.

6. Conclusions and future directions

As a natural research progression from the U-model-based controller design for nonlinear polynomial plants using linear polynomial based approaches, this study establishes a U-block model framework to facilitate using linear state-space-based design approaches for nonlinear polynomial models. The concept is novel and design procedure is straightforward and effective without complicated mathematical manipulation. On the other hand, this study reflects the deep understanding and comprehensive skills in dealing with control-oriented model structure and control system design, which is a good example to provide easy and general solutions to complicated nonlinear dynamic problems. It is believed that this study will bring forward a new direction in using linear approaches to design nonlinear control systems.

For future directions, some of potential studies associated with this work are recommended below.

1. Even though the bottle neck problem has been resolved in converting nonlinear polynomial into state-space model, the SMC selected in the study is just for feasibility test, further comprehensive studies and simulation bench tests should be conducted in the future with reference to many leading research results (Bartolini & Punta, 2012; Kalsi, Lian, Hui, & Zak, 2010; Moreno & Osorio, 2012).
Within such U-model framework, proper stability analysis methods and stabilisation control should be considered in conjunction with Lyapunov stability analysis theorems (Johansen, 2000; Li & Khalil, 2012; Zhou, Zheng, & Duan, 2011).

With U-block realisation, nonlinear polynomial models can be easily converted into state-space models, therefore, this provides a framework for robust control system design using linear matrix inequality and MATLAB software. Surely some of the exiting well-known results in robust control (ELBsat & Yaz, 2013; Zemouche & Boutayeb, 2013) will give useful reference for the new development.

New topics for iterative learning control (ILC), from the author understanding, ILC with linear dynamic plants has been well studied into maturity. However, for nonlinear dynamic plants, ILC are still predominantly using linearisation approaches (such as Newton algorithms and scant algorithms) in design, which are not accurate and nonlinear plant model structure are not properly accommodated. The U-model framework will obviously reduce the complexity in nonlinear type ILC schemes, particularly in the proof of convergent rate and widening of range of learning (Meng, Jia, Du, & Zhang, 2014; Tan, Dai, Huang, & Xu, 2012; Xu & Tan, 2003; Xu & Xu, 2013).

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Notes on contributors
Quanmin Zhu is a professor in control systems at Faculty of Environment and Technology, University of the West of England, Bristol, UK. He obtained his MSc in Harbin Institute of Technology, China in 1983 and PhD in Faculty of Engineering, University of Warwick, UK in 1989. His main research interest is in the area of nonlinear system modelling, identification, and control. His other research interest is in investigating electrodynamics of acupuncture points and sensory stimulation effects in human body, modelling of human meridian systems, and building up electro-acupuncture instruments. Currently professor Zhu is acting as a member of editorial committee of Chinese Journal of Scientific Instrument, editor (and founder) of International Journal of Modelling, Identification and Control, and editor of International Journal of Computer Applications in Technology.

Dongya Zhao received his BEng from Shandong University, Jinan, China in 1998, MSc from Tianhua Institute of Chemical Machinery & Automation, Lanzhou, China in 2002 and PhD from Shanghai Jiao Tong University, Shanghai, China in 2009. He was a research fellow in Nanyang Technological University during July 2011 to July 2012. Since 2002, he has been with College of Chemical Engineering, China University of Petroleum, where he is currently an associate professor. His research interests include robot control, sliding mode control, process modeling and control, nonlinear system control and analysis.

Jianhua Zhang was born in Yanji, Jilin province, P.R. China in 1980. He received his MS degree and PhD degree from Yanshan University in 2006 and 2011, respectively. From 2012 to 2013, he did his post-doctoral research at University of the West of England (UWE), Bristol, UK. He is currently working at Institute of Electrical Engineering, Hebei University of Science and Technology, China. His research interests include adaptive nonlinear control, neural network control, and control applications.

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