Analytical Exact Solution of Neural Functional-differential Equations

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Abstract In this paper, we introduce a well-known technique Differential Transform method (DTM) which is very effective to control the convergence region of the approximate solution. The Differential Transform Method is applied to neutral functional-differential equation with proportional delays. The DTM produces an approximate solution with few hand computations without rounding off the error. Obtained solution of Neutral functional-differential equation with proportional delays reveals that DTM is one of the efficient and accurate methods.

Keywords Differential Transformation Method, Neutral Functional-differential Equation with Proportional Delays, Differential Equations

1. Introduction

The rapid development of nonlinear sciences observes number of new analytical and numerical methods. Most of these introduced techniques are of some built deficiencies including complicated and lengthy calculation, divergent results, limited convergence, small parameter assumption and non-compatibility with the physical nature of the problems. Recently the Variational Iteration Method (VIM) [6-7-8-9-10] has been applied to handle the nonlinear problems. Several modifications are available for the variational iteration method by Herisanu and Marinca[11] Noor and Mohyud-Din worked on strongly nonlinear problems[14], Yilmaz and Mustafa Inc [15], Zayed and Rahman [16], Geng and Cui [3], several researchers include Yilmaz and Mustafa Inc [15] modification has been caught much attention. In Refs. [17-20] Adomian’s Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and Variation of Parameter Method (VPM) are successfully applied to obtain the exact solution of differential equations.

In this study, we use the new developed analytical differential transform method (DTM) to solve the Neutral functional-differential equation with proportional delay. Firstly Zhou [21] introduced Differential transform Method on different type of nonlinear differential equations and has shown various remarkable results of this method. Using differential transformation method, a closed form series solution or an approximate solution can be obtained. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor’s series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally expensive for large orders in terms of time. This method produced solutions in the form of polynomials and avoids large computational work and round off error. In present time, much nonlinear type of ODEs is easily solved by DTM .This method has been successfully applied to solve many types of nonlinear problems in science and engineering [22-24]. Recently, many adaptive numerical methods have been used which are very effective for these problems [26-29]. In this work the proposed DTM method is analytically applied to Neutral functional-differential equation with proportional delays. Several examples are given to verify the efficiency and compatibility of the proposed method.

Consider the following Neutral functional-differential equation with proportional delays:

\[(u(t) + a(t)u(p_n(t)))^{(n)} = \alpha u(t) + \sum_{i=0}^{n-1} b_i (t)u^{(i)}(p_i(t)) + g(t),t\] (1)

With initial conditions

\[\sum_{i=0}^{n-1} c_{ij}u^{(i)}(0) = \beta_j, j = 0,1,2 \ldots n - 1. \] (2)

Here, \(a\) and \(b_i (i = 0,1,2 \ldots n - 1)\) are given analytical functions, and \(\alpha, p_i, c_{ij}, \beta_j\) are the given constants with \(0 < p_i < 1 (k = 0,1,2 \ldots n).\)

Neutral functional-differential equations with proportional delays represent a particular class of delay differential equation. Such functional-differential equations play an important role in the mathematical modeling of real world phenomena [1].
Wang et al. obtained approximate solutions for neutral delay differential equations by continuous Runge-Kutta methods [13] and one leg methods [12-13] Chen and Wang [2] applied original variational iteration method to compute approximate solutions for neutral delay differential equations. Obviously, most of these equations cannot be solved exactly. It is therefore necessary to design efficient numerical methods to approximate their solutions. The purpose of this paper is to apply DTM, to find the analytical solutions of Neutral functional-differential equations with proportional delays (1) along with (2), by demonstrating its ability and efficiency.

2. Analysis of Differential Transform Method

Consider that the function \( f(x) \) is continuously differentiable function on the interval \((x_0 - r, x_0 + r)\). The Differential Transform of the function \( f(x) \) for the \( k \)th derivatives of the function is defined as

\[
F(k) = \frac{1}{k!} \left[ \frac{d^k f(x)}{dx^k} \right]_{x=x_0}
\]

(3)

Where \( f(x) \) is the original function and \( F(k) \) is the transformed function.

The inverse differential transform \( F(k) \) is defined as:

\[
f(x) = \sum_{k=0}^{\infty} (x-x_0)^k \frac{1}{k!} \left[ \frac{d^k f(x)}{dx^k} \right]_{x=x_0}
\]

(4)

The Basic operation of Differential Transformation can be constructed from Eq. (3) and Eq. (4) which is given below,

1. **Theorem 1**: If \( f(x) = r(x) \pm p(x) \), then \( F(k) = R(k) \pm P(k) \).
2. **Theorem 2**: If \( f(x) = ar(x) \), then \( F(k) = aR(k) \).
3. **Theorem 3**: If \( f(x) = \frac{d^r(x)}{dx} \), then \( F(k) = (k+1)R(k+1) \).
4. **Theorem 4**: If \( f(x) = \frac{d^r(x)}{dx^2} \), then \( F(k) = (k+1)(k+2)R(k+2) \).
5. **Theorem 5**: If \( f(x) = \frac{d^r(x)}{dx^3} \), then \( F(k) = (k+1)(k+2)(k+3)R(k+3) \).
6. **Theorem 6**: If \( f(x) = r(x)p(x) \), then \( F(k) = \sum_{i=0}^{\infty} P(i)R(k-i) \).
7. **Theorem 7**: If \( f(x) = x^n \), then \( F(k) = \delta(k-n) \) where \( \delta(k-n) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases} \).
8. **Theorem 8**: If \( f(x) = e^{\beta(x)} \), then \( F(k) = \frac{(\beta)^k}{k!} \).
9. **Theorem 9**: if \( f(x) = (1+x)^n \), then \( F(k) = \frac{n(n-1) \cdots (n-k+1)}{k!} \).
10. **Theorem 10**: If \( f(x) = \sin(jx + \alpha) \), then \( F(k) = \frac{(j)^k}{k!} \left( \sin \left( \frac{\pi k}{2} + \alpha \right) \right) \).

11. **Theorem 11**: If \( f(x) = \cos(jx + \beta) \), then \( F(k) = \frac{(j)^k}{k!} \left( \cos \left( \frac{\pi k}{2} + \beta \right) \right) \).
12. **Theorem 12**: if \( f(x) = r \left( \frac{x}{a} \right) \), then \( F(k) = \frac{1}{a^k} R(k) \).

3. Numerical Applications

In this section we consider examples that show the efficiency of Differential Transform Method for solving Neutral functional-differential equations with proportional delays.

**Example 3.1**

Consider the first-order neutral functional-differential equation with proportional delay

\[
u'(t) = -u(t) + 0.1u(0.8t) + 0.5u'(0.8t) + (0.32t - 0.5)e^{-2t} + e^{-t},
\]

(6)

with initial condition

\[
u(0) = 0.
\]

(7)

Taking the Differential Transform of equation (6) and (7),

\[
(k+1) \left( 1 - \frac{0.5}{(0.8)^2} \right) U(k+1) = -U(k) + \frac{0.1}{(0.8)^2} U(k) + 0.32 \sum_{m=0}^{k} \frac{(-0.8)^m}{m!} \delta(k-m) - 0.5 \left( \frac{(-0.8)^k}{k!} + \frac{(-1)^k}{k!} \right)
\]

(8)

\[
U(0) = 0
\]

(9)

The inverse differential Transform of \( U(k) \) is defined as

\[
u(t) = \sum_{k=0}^{\infty} U(k)t^k,
\]

(10)

Now substituting Eq. (9) into Eq. (8), we get the following values,

\[
U(1) = 1, U(2) = -1, U(3) = \frac{1}{2!}, U(4) = -\frac{1}{3!}, U(5) = \frac{1}{4!}...
\]

Substitute all the values of \( U(k) \) into Eq. (10), the following series solution will be obtained,

\[
u(t) = \sum_{k=0}^{\infty} U(k)t^k = 0 + t - t^2 + \frac{t^3}{2!} - \frac{t^4}{3!} + \frac{t^5}{4!} - \cdots
\]

(11)

The closed form solution is \( u(t) = te^{-t} \).

**Example 3.2**

Consider the first-order homogenous neutral functional-differential equation with proportional delay

\[
u'(t) + u(t) - \frac{1}{2} u \left( \frac{t}{2} \right) - \frac{1}{2} u' \left( \frac{t}{2} \right) = 0,
\]

(11)
With initial condition
\[ u(0) = 1. \] (12)

Taking the Differential Transform of Eq. (11) and (12),
\[ (k + 1) \left( 1 - \frac{1}{2\pi k} \right) U(k + 1) = -U(k) + \frac{1}{2\pi k} U(k), \] (13)
with initial condition
\[ U(0) = 1, \] (14)
The inverse differential Transform of \( U(k) \) is defined as
\[ u(t) = \sum_{k=0}^{\infty} U(k) t^k, \] (15)
Now substituting Eq. (14) into Eq. (13), we get the following values,
\[ U(1) = -1, U(2) = \frac{1}{2}, U(3) = -\frac{1}{3!}, U(4) = \frac{1}{4!} \ldots \]
and so on. Substituting all the values of \( U(k) \) into Eq. (15), the following series solution will be obtained,
\[ u(t) = \sum_{k=0}^{\infty} U(k) t^k = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots \]
The closed form solution is \( u(t) = e^{-t}. \)

**Example 3.3**

Consider the second order homogenous neural functional-differential equation with proportional delay.
\[ u''(t) = \frac{3}{4} u(t) - u \left( \frac{t}{2} \right) - \frac{1}{2} u' \left( \frac{t}{2} \right) + t^2 + t - 1 = 0, \quad 0 \leq t \leq 1 \] (16)
with initial conditions
\[ u(0) = 0, u'(0) = 0. \] (17)
Taking the Differential Transform of Eq. (16) and (17),
\[ (k + 1)(k + 2) \left( 1 - \frac{1}{2\pi k} \right) U(k + 2) = \frac{3}{4} U(k) + \frac{1}{2\pi k}(k + 1)U(k + 1) - \delta(k - 2) - \delta(k - 1) + \delta(k), \] (18)
with initial conditions
\[ U(0) = 0, U(1) = 0, \] (19)
The inverse differential Transform of \( U(k) \) is defined as
\[ u(t) = \sum_{k=0}^{\infty} U(k) t^k, \] (20)
Now substituting Eq. (19) into Eq. (18), we get the following values,
\[ U(2) = 1, U(3) = 0, U(4) = 0 \]
and so on. Substituting all the values of \( U(k) \) into Eq. (19), the following series solution will be obtained,
\[ u(t) = \sum_{k=0}^{\infty} U(k) t^k = 0 + 0t + t^2 + 0t^3 + 0t^4 \ldots \]
The exact solution is \( u(t) = t^2. \)

**Example 3.4**

Consider the third order neutral functional-differential equation with proportional delay.
\[ u'''(t) = u(t) + u' \left( \frac{t}{3} \right) + \frac{1}{2} u'' \left( \frac{t}{3} \right) - t^4 - \frac{t^3}{2} - \frac{4}{3} t^2 + 21t, \] (21)
with initial conditions
\[ u(0) = 0, u'(0) = 0, u''(0) = 0. \] (22)
Taking the Differential Transform of Eq. (21) and (22),
\[ (k + 1)(k + 2)(k + 3) \left( 1 - \frac{1}{2\pi k} \right) U(k + 3) = U(k) + \frac{1}{2\pi k}(k + 1)U(k + 1) + \frac{1}{3!}(k + 1)(k + 2)U(k + 2) - \delta(k - 4) - \frac{1}{2} \delta(k - 3) - \frac{4}{3} \delta(k - 2) + 21 \delta(k - 1), \] (23)
with initial conditions,
\[ U(0) = 0, U(1) = 0, U(2) = 0, \] (24)
The inverse differential Transform of \( U(k) \) is defined as
\[ u(t) = \sum_{k=0}^{\infty} U(k) t^k, \] (25)
Now substituting Eq. (24) into Eq. (23), we get the following values,
\[ U(3) = 0, U(4) = 1, U(5) = 0 U(6) = 0 \]
and so on. Substituting all the values of \( U(k) \) into Eq. (25), the following series solution will be obtained,
\[ u(t) = \sum_{k=0}^{\infty} U(k) t^k = 0 + 0t + 0t^2 + 0t^3 + t^4 \ldots \]
The exact solution is \( u(t) = t^4. \)

**Example 3.5**

Consider the Logistic equation with proportional delay of the form [25] is
\[ u'(t) = \rho u(t) \left( 1 - u(t) - r \right) t > 0, \rho > 0, \] (26)
with initial conditions
\[ u(0) = u_0 = 0.85, \rho = 0.5, r = 0. \] (27)
Taking the Differential Transform of equation (26) and (27)
\[ (k + 1)U(k + 1) = 0.5 U(k) - 0.5 \sum_{l=0}^{k} U(l) U(k - l), \] (28)
with initial conditions
\[ U(0) = 0.85, \] (29)
The inverse differential Transform of \( U(k) \) is defined as
\[ u(t) = \sum_{k=0}^{\infty} U(k) t^k, \] (30)
Now substituting Eq. (29) into Eq. (28), we get the following values,
\[ U(1) = 0.064, U(2) = -0.112, U(3) = -0.0068 \]
and so on. Substituting all the values of \( U(k) \) into Eq. (30), the following series solution will be obtained,
approximate solution of neutral functional-differential equation. By applying DTM, we lucratively obtained the analytical solutions. The obtained exact solutions revealed that the proposed method is easy to implement in finding the analytical solutions, accurate, fast and it reduce the size of the computational involvement. There is no need for calculating multiple integrals or derivatives, polynomials and less computational work is demanded compared to other popular methods. Differential Transform Method offer excellent opportunity for the future research.

4. Conclusions

In this study, Differential Transform Method is extended successfully to solve analytical the neutral functional-differential equation with proportional delays. By applying DTM, we lucratively obtained the analytical approximate solution of neutral functional-differential equation with proportional delays. The obtained exact solutions revealed that the proposed method is easy to implement in finding the analytical solutions, accurate, fast and it reduce the size of the computational involvement. There is no need for calculating multiple integrals or derivatives, polynomials and less computational work is demanded compared to other popular methods. Differential Transform Method offer excellent opportunity for the future research.

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