Tetrahedral Symmetry $A_4$ in Anti-SU(5) GUT

Jihn E. Kim$^{(1,2)}$, Se-Jin Kim$^{(1)}$, Soonkeon Nam$^{(1)}$

$^{(1)}$Department of Physics, Kyung Hee University, 26 Gyunghoedaeoro, Dongdaemun-Gu, Seoul 02447, Republic of Korea
$^{(2)}$Department of Physics and Astronomy, Seoul National University, I Gwanakro, Gwanak-Gu, Seoul 08826, Republic of Korea

We construct a flavor model in an anti-SU(5) GUT with a tetrahedral symmetry $A_4$. We choose a basis where $Q_{em} = -\frac{1}{3}$ quarks and charged leptons are already mass eigenstates. This choice is possible from the $A_4$ symmetry. Then, matter representation $\overline{10}_{\text{matter}}^{\text{em}}$ contains both a quark doublet and a heavy neutrino $N$, which enables us to use the $A_4$ symmetry to both $Q_{em} = +\frac{2}{3}$ quark masses and neutrino masses (through the see-saw via $N$). This is made possible because the anti-SU(5) breaking is achieved by the Higgs fields transforming as anti-symmetric representations of SU(5), $\overline{10}_{\text{em}}^H \otimes 10_{\text{em}}^H$, reducing the rank-5 anti-SU(5) group down to the rank-4 standard model group SU(3)$_C \times$SU(2)$_W \times U(1)_Y$. For possible mass matrices, the $A_4$ symmetry predictions on mass matrices at field theory level are derived. Finally, an illustration from string compactification is presented.

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I. INTRODUCTION

Recently, we pointed out analytically how the tetrahedral discrete symmetry $A_4$ results from the permutation symmetry $S_4$ [1]. The $A_4$ discrete symmetry [2][11] in connection with the tri-bimaximal form of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton mixing matrix [12][14] has been observed long time ago. The underlying permutation symmetry is useful in model building and furthermore it can be accommodated to string compactification. In string compactification, chiral fields can arise from fixed points also [15]. The multiplicity $N$ in a fixed point should respect permutation symmetry $S_N$ because the chiral fields at that fixed point are not distinguished. In this paper, we will use a specific grand unified theory (GUT) anti-SU(5) [16][17].

Georgi and Glashow’s GUT SU(5) [18] is an important prototype in the consideration of GUTs. An initial success was attributed to the $b - \tau$ unification [19]. However, there may be two issues against the GG model when one tries to include it in an ultraviolet completed theory. The rank of the GG group is 4 which is identical to that of the Standard Model (SM) gauge group SU(3)$_C \times$SU(2)$_W \times U(1)_Y$. Therefore, string compactification, an ultraviolet completion of the GG SU(5), needs an adjoint representation for breaking the GG SU(5) down to the SM gauge group without changing the rank. Firstly, in string compactification, it is not possible to obtain an adjoint representation at the level-1 construction [20]. Second, the $Q_{em} = - \frac{1}{3}$ Georgi–Jarlskog quark mass relations [21] need another representation 45 beyond a quintet of Higgs fields. The need for this additional representation makes it difficult for it to be realized in the string compactification. Of course, one may argue that 45 may arise from non-renormalisable interactions, which needs another fine-tuning.

Therefore, the anti-SU(5) or flipped SU(5) is preferred in string compactification. Barr commented that flipped-SU(5) is a subgroup of SO(10) [16] but here we consider it an independent GUT since string compactification may not go through an intermediate SO(10) which also needs an adjoint representation for spontaneous symmetry breaking to obtain Barr’s flipped SU(5). On the other hand, For breaking anti-SU(5), we use a vectorlike representation $\overline{10}_{-1}$ and $\overline{10}_{+1}$ (the subscripts are $X$ charges) which are anti-symmetric tensor representations of SU(5) and hence it is called ‘anti-SU(5)’ in [17]. This generalization for spontaneous symmetry breaking by anti-symmetric representations in string compactification stops at SU(7) [22].

Since the anti-SU(5) gauge group SU(5)$_X \times U(1)_X$ is rank-5, one can use anti-symmetric representations to reduce rank 1 to arrive at the rank-4 SM gauge group via the Higgs fields,

$$\overline{10}_H \supseteq \{ (3,1)_L, (3,2)_L, B_{45} \}_{-1} ; \quad 10_H \supseteq \{ (3,1)_L, (3,2)_L, B_{45} \}_{+1} ,$$

(1)
we use the $X$ definition given in Ref. [15]. The vacuum expectation values (VEVs) of neutral singlets $B^{45}$ and $B_{45}^5$ (in $\text{10}_{\pm 1}$ and $\text{10}_{e 1}$) break the anti-SU(5) down to the SM gauge group. But, there is no $b - \tau$ unification in this anti-SU(5).

One family in the anti-SU(5) in terms of left-handed (L-handed) fields is

$$\text{10}_{-1} = \left\{ (d^\alpha)_L, Q^\alpha_L, N^\alpha_L \right\}, \quad \text{5}_{+3} = \left\{ (u^\alpha)_L, \ell_L \right\}, \quad \text{1}_- = e^e_L, \quad (2)$$

where

$$Q^\alpha_L = \begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_L, \quad \ell_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L. \quad (3)$$

Note that all SU(2) singlets are with superscript $c$. So, the singlet neutrino $N^\alpha_L$ is in $\text{10}_{-1}$ and $N_R$ has $X = +1$. To break the SM gauge group to $U(1)_{\text{em}}$, we need a Higgs quintet(s) $\text{5}_0^H$ and $\text{5}_2^H$.

The family problem or the flavor problem consists of two parts. Firstly, why are there three families which have exactly the same gauge interactions. Second, why do these families have different Yukawa couplings? In GUTs, the first problem was formulated by Georgi [23] which was applied in extended GUTs [24, 25]. In string theory, three family models have been searched in various compactification schemes [26–52]. The second problem is usually talked in terms of flavor symmetry. The flavor symmetry is designed to calculate the CKM and PMNS matrices. Permutation symmetry $S_3$ has been started to calculate the CKM matrix [53, 54] but permutation symmetries blossomed recently in fitting the PMNS matrix [55].

In Sec. II, we summarize the results of Ref. [1]. In Sec. III, we discuss the $A_4$ symmetry at field theory level for three families in the anti-SU(5) GUT. We obtain possible forms of mass matrices of quarks and leptons, which are related by the anti-SU(5) representations. In Sec. IV, we present an example for possible quark and lepton mass matrices in a string derived spectra presented in Ref. [15]. Finally, a brief conclusion is given in Sec. V.

II. $A_4$ FROM $S_4$

The permutation symmetry $S_3$ has been used in the leptonic sector for a bimaximal PMNS matrix in the late 1990s [57, 58], and the $A_4$ symmetry has been started in the early 2000s [3]. The flavor symmetry in the PMNS matrix of a tri-bimaximal form

$$V \sim \begin{pmatrix} \times & \times & \times \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \quad (4)$$

has led to an $A_4$ symmetry, as shown analytically in [1]. The key points of Ref. [1] are

1. We choose the bases such that $Q_{\text{em}} = -\frac{1}{3}$ quarks and $Q_{\text{em}} = -1$ leptons are mass eigenstates.

2. $\ell_L$ in $5$ of Eq. (2) is a triplet under the tetrahedral group $A_4$.

3. All quark states in Eq. (2) are singlets under $A_4$.

4. The Higgs doublet(s) is a singlet under the permutation symmetry group $A_4$.

There are four representations in $A_4$: $3, 1, 1'$, and $1''$. Let us remark first that Item 1 evades the problem encountered in the Georgi-Jarlskog relation. We choose the needed mass values in the definition of the $Q_{\text{em}} = -\frac{1}{3}$ quark masses. Item 4 requires that the Higgs quintet $\text{5}_{+3}^H$ is a tetrahedral group singlet. Then, Items 2 and 3 dictate to assign $u^c_L$ in the triplet representation $3$ of $A_4$ since both $\ell_L$ and $u^c_L$ belongs to the same representation $\text{5}_{+3}$.

The tensor product of two $3'$s of $A_4$ is

$$3 \otimes 3 = 2 \otimes 3 \oplus 1 \oplus 1' \oplus 1'' . \quad (5)$$
We use the representations where three $Q_{em} = -\frac{1}{3}$ quarks of each chirality form a representation 3 of $A_4$, so do charged leptons. Then, the tensor product Eq. (5) allows three parameters, viz. three singlets, for three $Q_{em} = -\frac{1}{3}$ quark masses and choosing the diagonal basis for $Q_{em} = -\frac{1}{3}$ quarks is guaranteed from $A_4$. The same applies to charged leptons also.

Note that the charged currents (CCs) in the SM are given by

$$g \sqrt{2} \left( \bar{u}_L^{(0)} \gamma^\mu d_L^{(mass)} + \bar{\nu}_L^{(0)} \gamma^\mu e_L^{(mass)} \right) W_{\mu}^+ + \text{h.c.}$$

where

$$u_L^{(0)} = \begin{pmatrix} u^{(0)}_e \\ e^{(0)}_d \\ t^{(0)}_d \end{pmatrix}_L, \quad \nu_L^{(0)} = \begin{pmatrix} \nu^{(0)}_e \\ \nu^{(0)}_d \\ \nu^{(0)}_t \end{pmatrix}_L.$$

With the anti-SU(5) representations of (2), these CC’s are included in

$$g \left( \bar{10} -_1 \gamma^\mu T^5_{10} \bar{10} -_1 + 5_{+3} \gamma^\mu T_5 5_{+3} \right) W_{\mu}^+ + \text{h.c.}$$

where

$$T_5^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $T_{10}^-$ changes $d^a$ to $u^a$. Three families are

$$\mathbf{T} = \left( \mathbf{10}^d_{-1}, \mathbf{10}^e_{-1}, \mathbf{10}^d_{-1} \right), \quad \mathbf{F} = \left( 5^e_{+3}, 5^b_{+3}, 5^e_{+3} \right)$$

where $d, s, b$ and $e, \mu, \tau$ are family indices. In terms of mass eigenstates $Q_{em} = +\frac{2}{3}$ quarks ($u, c, t$) and neutrinos ($\nu_1, \nu_2, \nu_3$), the weak eigenstates of (7) are related by L-sector unitary matrices $U$ and R-sector unitary matrices $\mathcal{U}$ by

$$U^{(u)} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_L, \quad \nu^{(0)}_L = \begin{pmatrix} \nu^{(0)}_1 \\ \nu^{(0)}_2 \\ \nu^{(0)}_3 \end{pmatrix}_L, \quad N^{(1)}_L = \begin{pmatrix} N^{(1)}_e \\ N^{(1)}_\mu \\ N^{(1)}_t \end{pmatrix}_L, \quad \nu^{(\nu)} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L.$$

Now, Eq. (5) reads for three families as

$$g \left( \mathbf{T}^\gamma \mu \mathbf{T}^5_{10} \mathbf{T} + \mathbf{F}^\gamma \mu T_5^- \mathbf{F} \right) W_{\mu}^+ + \text{h.c.}$$

The CKM and PMNS matrices are given by

$$V^{(\text{CKM})} = U^{(u)} U^{(d)} \dagger = U^{(u)}, \quad V^{(\text{PMNS})} = U^{(\nu)} U^{(\nu)} \dagger = U^{(\nu)}.$$
there remains to choose $\nu^{(0)}$. Thus, the $A_4$ symmetric property of $\mathbf{F} \otimes \mathbf{F}$ is $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus 2 \cdot \mathbf{3}$, from which we choose $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$ for Eq. (12) to be $A_4$ symmetric. Thus, $\mathbf{F}$ can be chosen as
\[
\mathbf{F}^{(0)} \ni a\nu_e^{(0)}, b\nu_{\mu}^{(0)}, c\nu_{\tau}^{(0)},
\]
which are matched with charged leptons $e, \mu, \tau$.

In the quark sector, quarks are treated as singlets $\mathbf{1}, \mathbf{1}'$ and $\mathbf{1}''$. So, the first term of Eq. (12) is $A_4$ symmetric. With these CC couplings, the question to discuss next is how the quark and lepton Yukawa couplings are given.

### III. YUKAWA COUPLINGS

To realise $A_4$ symmetry, we assign the Yukawa couplings such that the flavor indices of $i$ respect the $A_4$ symmetry requirements. Since the $A_4$ symmetry was suggested from the PMNS matrix, let us first discuss the L violating neutrino masses. Since $\mathbf{F}$ is complex, it can have a global U(1) phase which is not violated by Eq. (12). The charged lepton in $\mathbf{F}$ obtains mass by the Yukawa coupling to $\bar{\nu}_c$. For consistency, we require no global anomaly. So, $\mathbf{F}_{++3}$ should carry a vanishing global charge which can be (B–L). $\mathbf{F}_{++3}$ couples to $\mathbf{T}^{-1}$ by $\mathbf{T}^{-1}C^{-1}\mathbf{F}_{++3}\mathbf{5}_{H_2}^H$. Since $\mathbf{5}_{H_2}^H$ is just the inverse of the mass matrix of $\mathbf{4}$. What is the $A_4$ representation of $\mathbf{T}^{-1}$? To write $\frac{m_N}{2}(N_R^2)$, $\mathbf{T}^{-1}$ transforms as a singlet(s) or 3 of $A_4$. These L violating heavy neutrino masses are contained in
\[
(T^{-1})_{ij}(H)_{ij}(T^{-1})_{jj},
\]
where $(H)$ is the heavy neutrino mass matrix. Since we do not introduce any triplet in the Higgs or fermion sectors, our neutrino mass matrix will be a Type 1 see-saw. The Dirac neutrino mass is given by
\[
\mathbf{F}_{++3}, Y_{ij} \mathbf{T}^{-1}_{i,j}\mathbf{5}_{H_2}^H
\]
where $(Y)$ is the Yukawa coupling matrix. In Fig. 1 we show the tree diagram for the Type 1 see-saw mechanism. In Fig. 1 the chiralities of $\nu$ and $N$ are L and R, respectively. This diagram depends on the $A_4$ property of $N$. With these diagrams, we obtain the effective Weinberg operators,
\[
h_{ij}\frac{v_u^2}{m_N}\bar{\nu}_i^{(0)}C^{-1}\nu_j^{(0)}.
\]

![FIG. 1: The Type 1 see-saw diagram.](image-url)
where $m$ is a constant (or matrix). This dictates that $10^{H}_{+1}$ (fermion) of Eq. (19) transforms as $3$ of $A_4$. In Fig. 2, we draw a schematic Feynman diagram generating the heavy neutrino masses from the anti-SU(5) × $A_4$ symmetry. The mass matrix $M$ transforms, under $A_4$, as $3 \otimes 3, 3 \otimes 1, 1 \otimes 3$, or 1’s, where the left factor combines with $N^c_\ell$ and the right factor combines with $N^c_\nu$. For each case, we study the L-violating neutrino masses.

Before discussing each neutrino mass matrix, we present the $Q_{em} = +\frac{2}{3}$ quark masses from the anti-SU(5) coupling which depends only on the coupling given in Eq. (16). The $Q_{em} = +\frac{2}{3}$ quark Yukawa couplings are determined from the $A_4$ tensor product $3 \otimes 3 = 1 \oplus Y \oplus 1'' \oplus 2 \cdot 3$. There are three independent singlets, which are three independent Yukawa couplings. $Y_{ij}$ in Eq. (16) are matrix elements. There is only on class for matrices which have Det$= 1$ and Tr$= -1$ for entries with $\pm 1,$

$$
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

(20)

For example, the matrix

$$
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
$$

satisfies the required conditions but changing the indices $1 \leftrightarrow 2$ gives the form Eq. (20). Similarly, all the other cases can be reduced to the form (20). For Eq. (20), the Yukawa couplings are defined as $Y_{11} = h_1, Y_{22} = h_2, Y_{33} = h_3$, and all the rest are zeros.

Now let us proceed to discuss each class of $M$ on neutrino masses.

**A.** $M \sim 3 \otimes 3$

In this case, three values are the same for the left and right factors. In the matrix form,

$$
M = \begin{pmatrix}
M & M & M \\
M & M & M \\
M & M & M
\end{pmatrix}
$$

(22)

which has eigenvalues of $3M$, 0, and 0. The above is a democratic form suggested in Refs. [68, 69]. The heavy neutrino mass components are

$$
M_{mn} = \frac{m^2}{M}.
$$

(23)

In this case $Y_{ij}$ of Eq. (16) is $Y_{ij} = h_0 \delta_{ij}$. Then, the SM neutrinos obtain masses through Fig. 1

$$
m_{ij} = \frac{h^2_m v^2_u}{m^2}.
$$

(24)

Identifying $10^{H}_{+1}$’s of Eqs. (18) and (19), we may be led to introduce supersymmetry.
The above universal mass matrix is diagonalized by
\[
\begin{pmatrix}
\sqrt{3}-1 & -\sqrt{3}-1 & 1 \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\
-\sqrt{3}-1 & \sqrt{3}-1 & 1 \\
\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\]
(25)
which is tri-maximal.

B. \( M \sim 3 \otimes 1 \)

The left factors give the same value and the right factors give three different values.

\[
M = \begin{pmatrix}
M_1 & M_2 & M_3 \\
M_1 & M_2 & M_3 \\
M_1 & M_2 & M_3
\end{pmatrix}
\]
(26)
which has eigenvalues of \( M_1 + M_2 + M_3 \), 0, and 0. All the heavy neutrinos have the same mass,

\[
M_{mn} = \frac{m^2}{M_n}. 
\]
(27)

In this case \( Y_{ij} \) of Eq. (16) is \( Y_{ij} = h\delta_{ij} \) at the LHS vertex and \( h_j \) at the RHS vertex. Thus, the SM neutrinos obtain masses through Fig. 1 as

\[
m_{ij} = hh_j \frac{M_j v^2}{m^2},
\]
(28)
which is proportional to

\[
m \propto \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}
\]
(29)
whose eigenvalues are 0, 0, and \( h_1 + h_2 + h_3 \). Three column vectors of \( m^T \) with eigenvalues 0, 0, and \( h_1 + h_2 + h_3 \) are

\[
\psi_1 \sim \begin{pmatrix} h_2 \\ -h_1 \\ 0 \end{pmatrix}, \psi_2 \sim \begin{pmatrix} h_1 \\ h_2 \\ -h^2_h \end{pmatrix}, \psi_3 \sim \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.
\]
(30)

Note that Eq. (29) has a freedom to choose the scale. We fix such that the unitarity matrix results. The unitarity matrix diagonalizing \( m^T \) is

\[
U = \frac{1}{\sqrt{h^2(1+h^2)}} \begin{pmatrix}
h_2\sqrt{1+h^2} & h_1 & h_1h \\
-h_1\sqrt{1+h^2} & h_2 & h_2h \\
0 & -h^2_h & h
\end{pmatrix},
\]
(31)
where we choose

\[
h_3 = 1, \\
h^2 \equiv h_1^2 + h_2^2.
\]
(32)

Then, the diagonalized states and matrix are expressed in terms of the original ones as

\[
\psi^{(\text{diag})} = U\psi_0, \\
m^{(\text{diag})} = UmU^\dagger.
\]
(33)
C. $\mathbf{M} \sim 1 \otimes 3$

The left factors give three different value and and the right factors give the same values.

$$\mathbf{M} = \begin{pmatrix} M_1 & M_1 & M_1 \\ M_2 & M_2 & M_2 \\ M_3 & M_3 & M_3 \end{pmatrix}$$  \hspace{1cm} (34)

which has eigenvalues of $3M, 0, 0$. All the heavy neutrinos have the same mass,

$$\mathbf{M}_{mn} = \frac{m^2}{M_m}. \hspace{1cm} (35)$$

In this case $Y_{ij}$ of Eq. (16) is $Y_{ij} = h_i$ at the LHS vertex and $h\delta_{ij}$ at the RHS vertex. Thus, the SM neutrinos obtain masses through Fig. 1 as

$$m_{ij} = hh_i \frac{M_{ij}v^2_u}{m^2}. \hspace{1cm} (36)$$

As in Case B, we obtain the following

$$U = \frac{1}{\sqrt{h^2(1 + h^2)}} \begin{pmatrix} h_2 \sqrt{1 + h^2} & -h_1 \sqrt{1 + h^2} & 0 \\ h_1 & h_2 & -h^2 \\ h_1 h & h_2 h & h \end{pmatrix}, \hspace{1cm} (37)$$

where

$$\psi^{(\text{diag})} = U\psi_0,$$

$$m^{(\text{diag})} = U m U^\dagger. \hspace{1cm} (38)$$

D. $\mathbf{M} \sim 1$’s

In this case, both the left and right factors give three different values.

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \hspace{1cm} (39)$$

which in general gives three different nonzero eigenvalues. All the heavy neutrinos have the same mass,

$$\mathbf{M}_{ij} = \frac{m^2}{M_{ij}}. \hspace{1cm} (40)$$

In this case $Y_{ij}$ of Eq. (16) is $Y_{ij} = h_i$ at the LHS vertex and $h_j$ at the RHS vertex. Thus, the SM neutrinos obtain masses through Fig. 1 as

$$m_{ij} = h_i h_j \frac{M_{ij}v^2_u}{m^2}, \hspace{1cm} (41)$$

which is general enough to obtain any unitarity matrix $U$.

E. Allowed matrices for $\mathbf{M}$ from effective neutrino masses

In the above subsections, the heavy heavy neutrino mass matrix $\mathbf{M}$ of Fig. 2 leading to the heavy neutrino masses of $N$ in Eq. (2) were given. On the other hand, the effective neutrino mass operator of Weinberg [61],

$$\sim \ell_i^T C^{-1} \ell_j \hspace{1cm} (42)$$
is symmetric on the exchange $i \leftrightarrow j$. But, Cases B and C allow asymmetric neutrino masses. Therefore, the heavy neutrino mass matrix $M$ can take only Cases A and D. Since Case D is not very much predictive at this stage, we present the $A_4$ from the anti-SU(5) prediction given in Eq. (25).

$$
\begin{pmatrix}
\frac{-\sqrt{3}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{\sqrt{3}-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{-\sqrt{3}-1}{2\sqrt{3}}
\end{pmatrix}
$$ (43)

where $|(U^\dagger)_{13}| = \frac{\sqrt{3}-1}{2\sqrt{3}} \approx 0.211$. The best fit [55] gives 0.147 and $3\sigma$ range is $0.138 - 0.156$. Therefore, Case A is ruled out. Only, Case D, which is general enough, is a viable mass pattern of the heavy heavy neutrinos.

### F. The CKM matrix

For Case D, let us consider the CKM matrix. In Ref. [1], we argued that the CKM matrix is close to the identity because of the huge ratio of $m_t/m_c$. So, the mass matrix is of the form,

$$
\sim \begin{pmatrix}
a \varepsilon^2 & b \varepsilon^2 & c \varepsilon \\
b \varepsilon^2 & e \varepsilon & f \varepsilon^2 \\
c \varepsilon & e \varepsilon & 1
\end{pmatrix}
$$ (44)

where $\varepsilon$ is $O(\frac{m_c}{m_t}) \approx 0.007$. The determinent of the above matrix is $D = (ae - bd + bfg + cdh - afh - ceg)\varepsilon^3$. Choose $e \approx 1$ such that trace is almost $m_t + m_c$. $D = (a - bd - cg + bfg + cdh - afh)\varepsilon^3$, and hence $m_u \approx m_c(m_c/m_t)^2(a - bd - cg + bfg + cdh - afh)\varepsilon^3 \approx 2.5\text{MeV}$ leading to $(a - bd - cg + bfg + cdh - afh) \approx 43$. Since we follow Case D, all these coefficients $a \sim g$ are arbitrary. Let us take a real symmetric matrix, choosing simple numbers just for an illustration,

$$
a = -32.1615, b = 1, d = c = g = (43)^{1/3}, f = h = 1.42,
$$ (45)

where $a$ is chosen to satisfy $(a - bd - cg + bfg + cdh - afh) \approx 43$. In this case, the mass matrix is

$$
M \sim \begin{pmatrix}
-0.00157596, & 0.00205181, & 0.0245238 \\
0.00205181, & 0.007, & 0.118806 \\
0.0245238, & 0.118806, & 1
\end{pmatrix}
$$ (46)

Then, eigenvalues of $M$ are

$$
-0.00203125, -0.00715645, 1.01461,
$$ (47)

where the first term can be corrected more by higher dimensional operators. Here, $m_c/m_t \approx 0.007$, and the diagonalizing matrix, $UMU^\dagger = \text{diagonal}$, is

$$
V^{(\text{CKM})} = U^{(u)} = \begin{pmatrix}
0.985959 & -0.166933 & -0.00433802 \\
-0.165227 & -0.978988 & 0.119506 \\
0.0241964 & 0.117111 & 0.992824
\end{pmatrix}
$$ (48)

which gives the Cabibbo angle $|\theta_C| \approx 9.61^\circ$, roughly $3.4^\circ$ smaller than the needed one. Note however that we neglected the CP phase $\delta$ and other higher dimensional contributions. Most importantly, it is for a specific set of parameters in Eq. (45). In general, the mass matrix is complex which can be diagonalized by bi-unitary matrices, by $U$ and $U$. In sum, we tried to show there can be a reasonable set of parameters fitting all the flavor data for Case D.

### IV. STRING COMPACTIFICATION

To discuss flavor symmetry from string compactification, one needs a compactification model where details of the SM field assignment is presented. In doing so, the key SM phenomenologies are automatically included, i.e. it is not ruled
Also, any SM Yukawa couplings at the renormalizable level. But, at the level of dimension-5 there appear the SM Yukawa from Table II, we note that the doublet representation of permutation symmetry which is obtained from the Z_{12}–1 compactification of the E_{8} × E_{8} heterotic string. Anyway, for a detail study of flavor physics, one has to specify every aspect of the flavors for which we do not find any reference except Ref. [15]. So, we show an example of A_{4} symmetry based on an anti-SUY(5) GUT of [13] based on the model [66]. Here, we just cite the needed information from Refs. [13] [66]. In string compactification, the needed Yukawa couplings arise by satisfying all the selection criteria. The anti-SUY(5) GUT of [15] does not allow any SM Yukawa couplings at the renormalizable level. But, at the level of dimension-5 there appear the SM Yukawa couplings which are proportional to the VEVs of \(10_{-1}^{H} = \left(\bar{10}_{-1}^{u}\right)^{T}\). Since these VEVs are near the string scale, we obtain top quark mass at the order the electroweak scale. Since we are not attempting to discuss details of models in string compactification, we only pay attention to the multiplicities of the needed chiral fields.

First consider \(10_{-1}^{H}\) and \(\bar{10}_{-1}^{u}\) needed for breaking anti-SU(5). In the \(T_{3}^{3}\) twisted sector, chiral fields are contructed in Eqs. (23) and (24) of Ref. [15],

\[
\begin{align*}
\text{s} & \quad \text{Multiplicity} & \quad P \cdot V \\
\oplus & \quad (++) & 2, & \quad \frac{1}{4} (\Sigma_{1}^{+}) \\
\ominus & \quad (--) & 1, & \quad \frac{1}{4} (\Sigma_{2}^{\prime})
\end{align*}
\]

\[
\begin{align*}
\text{s} & \quad \text{Multiplicity} & \quad P \cdot V \\
\oplus & \quad (++) & 1, & \quad \frac{1}{4} (\Sigma_{1}^{+}) \\
\ominus & \quad (--) & 2, & \quad \frac{1}{4} (\Sigma_{2}^{\prime})
\end{align*}
\]

where \(\oplus\) and \(\ominus\) denote L-handed and R-handed chiral fields respectively. So, here we consider only the number of chiral fields at the same fixed points. We cited only chirality and multiplicity. \(\Theta_{1}\) in Table I and \(P \cdot V\) in Eqs. (49) and (50) are used to calculate the multiplicity. From Eqs. (49) and (50), note that there appear three L-handed fields \(\Sigma_{2}\), and three R-handed fields \(\Sigma_{1}^{\prime}\). These chiral fields at the same fixed points are not distinguished. Thus, the L-handed fields \(\Sigma_{2}\) has the representation \(3\) of \(A_{4}\), so do the R-handed fields \(\Sigma_{1}^{\prime}\). \(\Sigma_{2}\) is the one for \(10_{-1}^{u}\) of Eq. (49). But \(T_{-1}\) of Eq. (19) belongs to the matter fields in Table I of [15]. Two matter \(\bar{10}_{-1}\)'s appear in \(T_{4}^{2}\), viz. Table I. But it is better to check all \(\bar{10}_{-1}\)'s before removing vectorlike representations, for which we go back to Ref. [66]. In fact, there was no vectorlike representations of \(\bar{10}_{-1} \oplus 10_{+1}\)'s removed in Ref. [66]. So, from our string model, \(\bar{10}_{-1}\) is a doublet \(2\) of permutation symmetry \(S_{3}\). We do not realize the coupling of Eq. (19).

From Table II, we note that the doublet representation \(2\) of the permutation group \(S_{3}\) can be obtained from \(3\) of \(S_{4}\). Also, \(3\) of \(A_{4}\) is from \(3\) of \(S_{4}\).

In Eq. (19), \(\bar{T}_{-1}\) transforms as \(2\) under the permutation group \(S_{3}\) and \(10_{+1}^{H}\) transforms as \(3\) of \(A_{4}\). Note that \(2\) and \(3\) of \(S_{4}\) produce \(2\) of \(S_{3}\). Out of two \(1\)'s of \(S_{3}\), we restrict to only \(1\). Let us consider the relevant tensor products of

| State \((P + kV_{3})\) | \(\Theta_{1}\) | \(R_{X}\) (Sect.) |
|---------------------|----------|------------------|
| \(\xi_{1}\) \((++--=--;--+)\) | 0 \(\bar{10}_{-1}(U_{3})\) | |
| \(\eta_{3}\) \((-++--=++;--+)\) | 0 \(5_{+3}(U_{3})\) | |
| \(\tau^{c}\) \((++--=--;--+)\) | \(\frac{1}{2} T_{1}^{1}\) | |
| \(\xi_{2}\) \((-++--=--;--+)\) | \(\frac{1}{2} T_{1}^{1}\) | |
| \(\eta_{2}\) \((-++--=--;--+)\) | \(\frac{1}{2} T_{1}^{1}\) | |
| \(\mu^{c}\) \((-++--=--;--+)\) | \(\frac{1}{2} T_{1}^{1}\) | |
| \(\xi_{1}\) \((-++--=--;--+)\) | \(\frac{1}{2} T_{1}^{1}\) | |
| \(\eta_{1}\) \((-++--=--;--+)\) | \(\frac{1}{2} T_{1}^{1}\) | |
| \(\epsilon^{c}\) \((-++--=--;--+)\) | \(\frac{1}{2} T_{1}^{1}\) | |
| \(H_{aL}\) \((+10000; 000)(0^{5}; \frac{1}{2} 1 \frac{1}{2} 0)\) | \(\frac{1}{2} 2.5_{-2}(T_{6})\) | |
| \(H_{dL}\) \((-10000; 000)(0^{5}; \frac{1}{2} 1 \frac{1}{2} 0)\) | \(\frac{1}{2} 2.5_{+2}(T_{6})\) | |
TABLE II: Branching of $S_4$ representations $1, 1', 2, 3$ and $3'$ into the $A_4$ and $S_3$ representations [59].

| $S_4$ | $A_4$ | $S_3$ |
|-------|-------|-------|
| 1     | 1     | 1     |
| 1'    | 1     | 1'    |
| 2     | 1' $\oplus 1''$ | 2 |
| 3     | 3     | 1 $\oplus 2$ |
| 3'    | 3     | 1' $\oplus 2$ |

$S_4$,

Tensor products in $S_4$

\[
\begin{align*}
3 \times 3 &= 1 \oplus 2 \oplus 3 \oplus 3' \\
3 \times 3' &= 1' \oplus 2 \oplus 3 \oplus 3' \\
3' \times 3' &= 1 \oplus 2 \oplus 3 \oplus 3' \\
2 \times 3 &= 3 \oplus 3'
\end{align*}
\]

(51)

where the last line does not produce a singlet. The other three lines produce $S_4$ singlets and we consider the first line, $3 \otimes 3$. The other cases can be equivalent to this by redefining the origin of $3$ of $A_4$. Then, $\mathbf{T}_{-1}$ and $10^H_{-1}$ can be traced back to $3$ of $S_4$.

\[
R(S_4) \otimes R(A_4) : (1, 2) \otimes 3 \\
\rightarrow (1 \otimes 3) \oplus (2 \otimes 3) \\
\rightarrow 1 \otimes 3 \oplus 1' \otimes 3 + 1'' \otimes 3
\]

(52)

where the first line is the fourth line of Table II written as $S_4$ and $A_4$ subgroups. In the 2nd line, $1 \otimes 3$ is the $S_4$ product and can be interpreted as the $A_4$ triplet. In the 2nd line, $2 \otimes 3$ is $S_4$ product from the 3rd and 4th lines of Table II. In terms of $A_4$, it produces $1' \otimes 3 + 1'' \otimes 3$, i.e. two independent $3$'s. In total, there are three independent $3$’s. Therefore, $M$ of Fig. 2 is

\[
M = \begin{pmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{pmatrix}
\]

(53)

which is Case D of Sec. III which is allowed from the neutrino mass data.

So far we paid attention to the heavy neutrino mass in $\mathbf{T}_{-1}$. Now, let us check how this representation containing a quark doublet predicts on the Yukawa couplings through Eq. (16) with a R-handed $Q_{em} = \frac{2}{3}$ quark in $F + 3$. Both $\mathbf{T}_{-1}$ and $F + 3$ are doublets under $S_3$. It belongs to the third row of Table II. The $S_4$ tensor product is $2 \otimes 2 = 2 \oplus 1 \oplus 1'$ which becomes $2 \cdot 1 \oplus 1' \oplus 1''$ under $A_4$. Thus, there are three independent couplings\(^2\) which can be of the form in the $2 \times 2$ subspace (due to doublets in the $T_0^4$ twisted sector),

\[
\begin{pmatrix}
0 & 0 \\
0 & a & b \\
0 & c & a
\end{pmatrix},
\]

(54)

which is general enough to allow the mixing between top and charm quarks. With higher dimensional operators [15], the 0 entries will be supplied with small numbers and may fulfill the needed $3 \times 3$ matrix for the $Q_{em} = \frac{2}{3}$ quark matrix.

The above illustration from a compactification model was intended to show a possibility. To study the flavor problem from string compactification, one needs an explicit model locating all the SM fields in the sectors of the compactification as shown in this section.

\(^2\) Two 1’s are counted as the same entry.
V. CONCLUSION

We constructed quark and lepton mass matrices in an anti-SU(5) GUT with a tetrahedral symmetry \( A_4 \). In the previous paper [1], we showed the hint of the \( A_4 \) from the PMNS matrix form with one entry being zero. In this paper, for a convenience of presentation we chose a basis where \( Q_{em} = -\frac{1}{3} \) quarks and charged leptons are already diagonalised. Then, matter representation \( T_{-1} \) contains both a quark doublet and a heavy neutrino \( N \). For \( Q_{em} = +\frac{2}{3} \) quark masses \( T_{-1} \) coupling to \( F_{+3} \) is used, and for neutrino masses the Weinberg operator of \((F_{+3})^2\) is used through the see-saw of \( T_{-1} \). In this sense, the quark and neutrino masses are related by the symmetry \( A_4 \). One notable feature is the anti-SU(5) breaking achieved by the Higgs fields transforming as anti-symmetric representations of SU(5), \( 10_{-1}^H \oplus 10_{+1}^H \). This set reduce the rank-5 anti-SU(5) group down to the rank-4 standard model group SU(3)\(_C \) × SU(2)\(_W \) × U(1)\(_Y \). Finally, a string compactification example is presented. As illustrated in this example, the definite assignments of the SM fields in the twisted sectors are needed to compare with the CKM and PMNS data.

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