A mixed element scheme for the Helmholtz transmission eigenvalue problem for anisotropic media

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Abstract

In this paper, we study the Helmholtz transmission eigenvalue problem for inhomogeneous anisotropic media with the index of refraction $n(x) \equiv 1$ in two and three dimensions. Starting with the nonlinear fourth-order formulation established by Cakoni \textit{et al} 2009 \textit{J. Integral Equ. Appl.} 21 203–27, we present an equivalent mixed formulation for this problem with auxiliary variables, followed by finite element discretization. Using the proposed scheme, we rigorously show that the optimal convergence rate for the transmission eigenvalues on both convex and nonconvex domains can be expected. With this scheme, we obtain a sparse generalized eigenvalue problem whose size is too demanding, even with a coarse mesh that its smallest few real eigenvalues fail to be solved by the shift and invert method. We partially overcome this critical issue by deflating nearly all of the $\infty$ eigenvalues with huge multiplicity, resulting in a marked reduction in the matrix size without deteriorating the sparsity. Extensive numerical examples are reported to demonstrate the effectiveness and efficiency of the proposed scheme.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The transmission eigenvalue problem was first introduced by Colton and Monk [11] and Kirsch [18], and generally consists of a system of coupled eigenvalue problems that are defined on the support of scattering objects. Because transmission eigenvalues carry some important information about the material properties of the scattering object [13, 23, 25], the problem has been applied in many practical applications and is an important research in inverse scattering theory, particularly with acoustic and electromagnetic waves [3–5, 7].

In this paper, we consider the following transmission eigenvalue problem for inhomogeneous anisotropic media. Specifically, we find $k \in \mathbb{C} \setminus \{0\}$ and nontrivial $w$ and $v$ such that

\[
\begin{aligned}
\nabla \cdot A(x) \nabla w + k^2 n(x) w &= 0 \quad \text{in } \Omega, \\
\Delta v + k^2 v &= 0 \quad \text{in } \Omega, \\
w &= v \quad \text{on } \partial \Omega, \\
\nu \cdot A(x) \nabla w = \nu \cdot \nabla v \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded polygon or polyhedron with boundary $\partial \Omega$, $\nu$ is the unit outward normal vector of $\partial \Omega$, the matrix index of refraction $A(x) \in \mathbb{R}^{d \times d}$ is symmetric positive definite everywhere, and the index of refraction $n(x) > 0$ is bounded. The values $k$ for which the transmission eigenvalue problem (1.1) has nontrivial solutions $w$ and $v$ are called transmission eigenvalues. This problem arises from the electromagnetic wave scattering problem in two dimensions (2D) [6] and the acoustic wave scattering problem in three dimensions (3D) [10], where the smallest real transmission eigenvalue is essential in the reconstruction of the matrix index of refraction [4]. In this paper, we focus on the case of $n(x) \equiv 1$ on $\Omega$, which implies that the scatterer has the same permeability in 2D or sound speed in 3D as the external medium.

There have been a few algorithms for the numerical calculation of transmission eigenvalues for anisotropic media. Cakoni et al [4] provided a method to determine transmission eigenvalues by the far field equation. Ji and Sun [15] proposed a multi-level finite element method for the transmission eigenvalue problem. An and Shen [1] developed a spectral method to compute the transmission eigenvalues. Lechleiter and Peters [20] calculated transmission eigenvalues by the inside-outside duality technique. Xie and Wu [28] proposed a multi-level correction method based on finite elements to solve the transmission eigenvalue problem. Kleefeld and Pieronik [19] computed the transmission eigenvalues by the fundamental solutions method. Gong et al [14] transformed this problem into an eigenvalue problem of a holomorphic operator function, and then, Lagrange finite elements and the spectral projection were used to compute the transmission eigenvalues. Liu and Sun [22] proposed a Bayesian inversion approach for the transmission eigenvalue problem. We particularly note that, for the case $n(x) > 1$ or $n(x) < 1$, a theoretical foundation for corresponding numerical algorithms was established by Xie and Wu [28]. However, to our knowledge, few numerical results of 3D Helmholtz transmission eigenvalue problem for anisotropic media have been reported.

Cakoni et al [4] established the basic theory of the transmission eigenvalue problem for anisotropic media in different approaches depending on whether $n(x) \equiv 1$ or $n(x) \neq 1$. In particular, with $n(x) \equiv 1$, the primary technical novelty of [4] was to introduce $\mu := A \nabla w - \nabla v$.
and \( \lambda = k^2 \), and then recast (1.1) into the transmission eigenvalue problem as a fourth-order problem as follows

\[
(\nabla \nabla \cdot + \lambda A^{-1}) (A^{-1} - I)^{-1} (\nabla \nabla \cdot + \lambda) y = 0,
\]
(1.2)

where \( I \) is the identity matrix. To solve the transmission eigenvalue problem for isotropic media, theories and algorithms have been developed in [9, 12, 16, 17, 21, 27], etc. Among these studies, the fourth-order problem of the transmission eigenvalue problem was also constructed in [9, 16, 17, 27], but based on an auxiliary variable \( w - v \) rather than \( A \nabla w - \nabla v \) in the anisotropic case.

In this paper, we study the numerical method for (1.2), which is seldom discussed in the literature. Assuming that

\[
\kappa_* := \inf_{x \in \Omega} \inf_{\xi \in \mathbb{R}^d, ||\xi||_a = 1} (\xi^T A(x) \xi) \quad \text{and} \quad \kappa^* := \sup_{x \in \Omega} \sup_{\xi \in \mathbb{R}^d, ||\xi||_a = 1} (\xi^T A(x) \xi)
\]

satisfy either \( 0 < \kappa_* \leq \kappa^* < 1 \) or \( 1 < \kappa_* \leq \kappa^* < \infty \), by the spectral theory of the compact operator, it holds that [4]

(i) If \( \kappa^* < 1 \) or \( \kappa_* > 1 \), problem (1.2) has an infinite countable set of real transmission eigenvalues with \(+\infty\) as the only accumulation point.

(ii) If we denote \( \kappa_1(\Omega) \) as the first Dirichlet eigenvalue of \(-\Delta\) on \( \Omega \), let \( k \) be a real transmission eigenvalue and \( \lambda = k^2 \), and the following result will hold:

\[
\frac{\lambda}{\kappa_1(\Omega)} \geq \left\{ \begin{array}{ll}
||A^{-1}||_2^{-1}, & \kappa^* < 1, \\
1, & \kappa_* > 1.
\end{array} \right.
\]

Our key idea here is to rewrite the nonlinear eigenvalue problem (1.2) to an equivalent linear eigenvalue problem by introducing some auxiliary variables, followed up with the finite element discretization and calculation of the smallest few real transmission eigenvalues. Due to the huge kernel of the div operator, a direct introduction of auxiliary variables leads to an eigenvalue problem of an ill-defined operator. To avoid this issue, the primary ingredient of our approach is that we use the Helmholtz decomposition to deflate the kernel and work directly on \( H^1 \) and \( L^2 \) spaces. The final problem is a linear eigenvalue problem with a compact operator. The discretization scheme is easy to implement both on convex and nonconvex connected domains with firm theoretical support. On the other hand, the goal of computing a few smallest real transmission eigenvalues is hindered by the tremendous size of the sparse generalized eigenvalue problem (GEP) obtained by discretization, in that the \( LDL^T \) factorization cannot fit into the computer memory. We partially resolve this critical issue by deflating nearly all of the \( \infty \) eigenvalues with huge multiplicity, resulting in a drastic reduction in the matrix size without deteriorating the sparsity. Specifically, the ratio of the size of the original GEP to that of the GEP we finally solved is approximately 3 in 2D and 10 in 3D. Both the computational time and cost are markedly reduced, particularly in the 3D case. In a word, this paper provides a practical method for (1.2) and performs 3D numerical investigations of (1.1), arguably for the first time.

The remaining of this paper is organized as follows. In section 2, we present an equivalent linear formulation of the transmission eigenvalue problem for anisotropic media. In section 3, a mixed element scheme is proposed to discretize the mixed formulation. In section 4, we deflate nearly all of the huge eigenspace associated with the \( \infty \) eigenvalue of the GEP obtained by direct discretization before using any eigensolver. Numerical examples are presented in section 5. Finally, we draw some conclusions and discuss some future works in section 6.
To begin with, let $\rho_1$ represent the direct sum of the space or matrix, $\odot$ represent the Kronecker product of the matrix, and $(\cdot, \cdot)$ as $L^2$-scalar product. Define spaces

\[ L^2(\Omega) := (L^2(\Omega))^d, \quad L^0(\Omega) := \{ \tau \in L^2(\Omega) : (\tau, 1) = 0 \}, \]

\[ \tilde{H}_0(\Omega) := H_0(\Omega) \cap L^2(\Omega), \quad \tilde{H}^1(\Omega) := H^1(\Omega) \cap L^2(\Omega), \]

and

\[ H(\text{div}, \Omega) := \{ \tau \in L^2(\Omega) : \text{div}\tau \in L^2(\Omega) \}, \]

\[ H_0(\text{div}, \Omega) := \{ \tau \in H(\text{div}, \Omega) : \nu \cdot \tau = 0 \text{ on } \partial\Omega \}, \]

\[ H_0(\text{div}, \Omega) := \{ \tau \in H_0(\text{div}, \Omega) : \text{div}\tau = 0 \}, \]

\[ H_0^1(\text{div}, \Omega) := \{ \tau \in H_0(\text{div}, \Omega) : \text{div}\tau \in H_0^1(\Omega) \}. \]

Now we introduce the symbol $\preceq$ to denote an order of complex numbers. Let $c_k = \rho_k e^{i\theta_k}$, $k = 1, 2$ be two complex numbers, with $\rho_k \geq 0$ and $0 \leq \theta_k < 2\pi$. Then $c_1 \preceq c_2$ if and only if one of the items below holds:

1. $\rho_1 = \rho_2 = 0$;
2. $\rho_1 < \rho_2$;
3. $\rho_1 = \rho_2 \neq 0$ and $\theta_1 \geq \theta_2$.

It is evident that if $c_1 \subset c_2$ and $c_2 \subset c_3$, then $c_1 \subset c_3$. Coherently, we use the symbol $\succ$, whereas $c_2 \succ c_1$ if and only if $c_1 \subset c_2$.

2. Equivalent stable linear eigenvalue problems

Let $P = (A^{-1} - I)^{-1}$, as shown in [4], the variational form of (1.2) is to find $u \in H_0^1(\text{div}, \Omega)$, such that

\[ (P(\nabla \nabla \cdot u + \lambda_0 u), (\nabla \nabla \cdot v + \lambda_0^{-1} v)) = 0, \quad \forall v \in H_0^1(\text{div}, \Omega). \tag{2.1} \]

In this section, we rewrite the fourth-order problem (2.1) to equivalent stable linear eigenvalue problem. We discuss this problem in two cases: $\kappa^* < 1$ and $\kappa_+ > 1$.

2.1. Case 1: $\kappa^* < 1$

For an eigenpair $(\lambda, u)$ of (2.1), by introducing

\[ y = \lambda u, \quad p = P\nabla \nabla \cdot u + (I + P) y, \]

we have for any $v \in H_0^1(\text{div}, \Omega)$, $\xi \in L^2(\Omega)$ and $\tilde{q} \in L^2(\Omega),$

\[ \begin{cases} 
(P\nabla \nabla \cdot u, \nabla \nabla \cdot y) + (Py, \nabla \nabla \cdot y) &= \lambda(\nabla \cdot u, \nabla \cdot v) - \lambda(p, y) \\
(P\nabla \nabla \cdot u, \xi) + ((I + P) y, \xi) - (p, \xi) &= 0 \\
-(y, q) &= -\lambda(u, q). \end{cases} \tag{2.2} \]

Now, denote $\nabla \tilde{H}^1(\Omega) := \{ \nabla \tau : \tau \in \tilde{H}^1(\Omega) \}$ and

\[ (\nabla \tilde{H}^1(\Omega))^\perp := \{ \xi \in L^2(\Omega) : (\nabla w, \xi) = 0, \quad \forall w \in \tilde{H}^1(\Omega) \}. \tag{2.3} \]
Then \( \left( \nabla \hat{H}^l(\Omega) \right) \perp \subset \hat{H}_0(\text{div}, \Omega) \). As \( \nabla \hat{H}^l(\Omega) \) is closed in \( L^2(\Omega) \),
\[
L^2(\Omega) = \nabla \hat{H}^l(\Omega) \perp \nabla \hat{H}^l(\Omega) \perp .
\] (2.4)
Then, we have \( \mathcal{P} = \nabla r + (\nabla r)^\perp \) with \( r \in \hat{H}^l(\Omega) \) and \( (\nabla r)^\perp \in (\nabla \hat{H}^l(\Omega)) \perp \), and any \( q \) can be rewritten as \( q = \nabla s + (\nabla s)^\perp \) with \( s \in \hat{H}^l(\Omega) \) and \( (\nabla s)^\perp \in (\nabla \hat{H}^l(\Omega)) \perp \), then
\[
\begin{align*}
\begin{cases}
(P \nabla \cdot u, \nabla \cdot y) + (P_y \nabla \cdot y) &= \lambda (\nabla \cdot u, \nabla \cdot y) - \lambda (\nabla r + (\nabla r)^\perp, y) \\
(P \nabla \cdot u, \xi) + ((I + P) y, \xi) - (\nabla r + (\nabla r)^\perp, \xi) &= 0 \\
- (\chi, \nabla s + (\nabla s)^\perp) &= - \lambda (u, \nabla s + (\nabla s)^\perp).
\end{cases}
\end{align*}
\] (2.5)
Choosing \( v \in \hat{H}_0(\text{div}, \Omega) \) arbitrarily leads to that \( (\nabla r)^\perp = 0 \). It follows that
\[
\begin{align*}
\begin{cases}
(P \nabla \cdot u, \nabla \cdot y) + (P_y \nabla \cdot y) &= \lambda (\nabla \cdot u, \nabla \cdot y) + \lambda (r, \nabla \cdot y) \\
(P \nabla \cdot u, \xi) + ((I + P) y, \xi) - (\nabla r, \xi) &= 0 \\
- (\chi, \nabla s) &= \lambda (\nabla \cdot u, s).
\end{cases}
\end{align*}
\] (2.6)
Formally introducing \( \varphi = \nabla \cdot u \) and \( \psi = \nabla \cdot v \) leads us to the linear eigenvalue problem: find \( \lambda \in \mathbb{C} \) and \( (\varphi, \psi, r) \in V := \hat{H}_0^l(\Omega) \times L^2(\Omega) \times \hat{H}^l(\Omega) \), such that, for any \( (\psi, \varphi, s) \in V, \)
\[
\begin{align*}
\begin{cases}
(P \nabla \varphi, \nabla \psi) + (P_y \nabla \psi) &= \lambda ((\varphi, \psi) + (r, \psi)) \\
(P \nabla \varphi, \zeta) + ((I + P) \chi, \zeta) - (\nabla r, \zeta) &= 0 \\
- (\chi, \nabla s) &= \lambda (\varphi, s).
\end{cases}
\end{align*}
\] (2.7)
Let \( D := \hat{H}_0^l(\Omega) \times L^2(\Omega) \). Equip \( D \) and \( V \) with the norm
\[
\| (\varphi, \chi) \|_D = (\| \varphi \|^2_{l, \Omega} + \| \chi \|^2_{L^2(\Omega)})^{1/2}, \quad \| (\varphi, \chi, r) \|_V = (\| \varphi \|^2_{l, \Omega} + \| \chi \|^2_{L^2(\Omega)} + \| r \|^2_{l, \Omega})^{1/2},
\]
then \( D \) and \( V \) are both Hilbert spaces. Four bilinear forms are defined:
\[
a((\varphi, \chi), (\psi, \zeta)) := (P \nabla \varphi, \nabla \psi) + (P_y \nabla \psi) + (P \nabla \varphi, \zeta) + ((I + P) \chi, \zeta), \quad (\varphi, \chi, (\psi, \zeta)) \in D, \]
\[
b((\varphi, \chi), s) := - (\chi, \nabla s), \quad (\varphi, \chi) \in D, s \in \hat{H}^l(\Omega),
\]
and
\[
a_{\nu}((\varphi, \chi, r), (\psi, \zeta, s)) := (P \nabla \varphi, \nabla \psi) + (P_y \nabla \psi) + (P \nabla \varphi, \zeta) + ((I + P) \chi, \zeta) - (\nabla r, \zeta) - (\chi, \nabla s), \]
\[
b_{\nu}((\varphi, \chi, r), (\psi, \zeta, s)) := (\varphi, \psi) + (r, \psi) + (\varphi, s),
\]
where \( (\varphi, \chi, r), (\psi, \zeta, s) \in V \). Then, \( a(\cdot, \cdot), b(\cdot, \cdot), a_{\nu}(\cdot, \cdot) \) and \( b_{\nu}(\cdot, \cdot) \) are all symmetric, continuous and bounded. Associated with \( a_{\nu}(\cdot, \cdot) \) and \( b_{\nu}(\cdot, \cdot) \), we define an operator \( T_V : V \rightarrow V \) by
\[
a_{\nu}(T_V(\varphi, \chi, r), (\psi, \zeta, s)) = b_{\nu}((\varphi, \chi, r), (\psi, \zeta, s)), \quad \forall (\psi, \zeta, s) \in V.
\] (2.8)

**Lemma 2.1.** \( T_V \) is well-defined and compact on \( V \).

**Proof.** Denote \( P = P + \kappa^* I \), by the Poincaré inequality, the following two inequalities hold.
\[
a((\varphi, \chi), (\psi, \zeta)) = \| P^{-\frac{1}{2}} P \nabla \varphi + P_y \nabla \psi \|^2_{L^2(\Omega)} + (1 - \kappa^*) \| \chi \|^2_{L^2(\Omega)} + ((P - PP^{-1}P) \nabla \varphi, \nabla \varphi) \\
\geq (1 - \kappa^*) \| \chi \|^2_{L^2(\Omega)} + \kappa_s (1 - \kappa^*) ((1 - \kappa^*)(2 - \kappa^*)) \| \nabla \varphi \|^2_{L^2(\Omega)} \\
\geq C(\| \chi \|^2_{L^2(\Omega)} + \| \varphi \|^2_{L^2(\Omega)}),
\]
and given \( s \in \hat{H}^1(\Omega) \), set \( y = -\nabla s \), it follows that

\[
(\varphi, -\nabla s) = \|y\|_{0,\Omega}\|\nabla s\|_{0,\Omega} \geq C\|y\|_{0,\Omega}\|s\|_{1,\Omega}.
\]

The inf-sup condition holds as

\[
\inf_{\varphi \in \tilde{H}^1(\varphi, y) \in L^2_0} \sup_{s \in L^2} \frac{b((\varphi, y), s)}{\|s\|_{1,\Omega}(\|y\|_{0,\Omega} + \|\varphi\|_{1,\Omega})} \geq C > 0.
\]

By Brezzi theory for the following problem

\[
a((\varphi, y), (\psi, z)) + b((\psi, z), r) + b((\varphi, y), s) = f(\psi, z) + g(s),
\]

where \( f \in D' \) and \( g \in H^{-1}(\Omega) \), we can obtain

\[
\|T_V(\varphi, y, r)\|_V \leq C(\|\varphi\|_{-1,\Omega} + \|r\|_{-1,\Omega}).
\]

Now, let \( \{\varphi_i, y_i, r_i\} \) be a bounded sequence in \( V \), then there is a subsequence \( \{\varphi_j, y_j, r_j\} \), such that \( \{\varphi_j\} \) and \( \{r_j\} \) are two Cauchy sequences in \( L^2(\Omega) \). Therefore, \( \{T_V(\varphi_j, y_j, r_j)\} \) is a Cauchy sequence in \( V \), which, further, has a limit therein. The proof is completed. \( \square \)

The lemma below follows by the standard spectral theory of compact operators.

**Lemma 2.2.** [24] The eigenvalues of \( T_V \) and (2.7), counting multiplicity, can be listed in a (finite or infinite) sequence as

\[
\mu_1 \geq \mu_2 \geq \cdots \geq 0 \quad \text{and} \quad 0 < \lambda_1 < \lambda_2 < \cdots,
\]

respectively. Moreover, for any \( i \in \mathbb{N}^+ \) such that \( \mu_i \neq 0 \), it holds \( \lambda_i \mu_i = 1 \).

**Theorem 2.3.** The eigenvalue problem (2.7) is equivalent to (2.1).

**Proof.** Let \((\lambda_i, (\varphi_i, y_i))\) be an eigenpair of (2.7), then we have \( y_i \in H_0(\text{div}, \Omega), \varphi = \nabla \cdot (y_i/\lambda) \) and \( \nabla r = P\nabla \varphi + (I + P)y_i \). Set \( y = y_i/\lambda \), and it follows that

\[
(P(\nabla \cdot u + \lambda y), (\nabla \cdot v + \lambda A^{-1} y)) = 0, \forall y \in H^1_0(\text{div}, \Omega).
\]

Namely \((\lambda, y/\lambda)\) is an eigenpair of (2.1).

On the other hand, if \((\lambda, u)\) is an eigenpair of (2.1), then choose a unique \( r \in \hat{H}^1(\Omega) \) such that \( \nabla r = P\nabla \cdot u + \lambda(I + P)u, (\nabla \cdot u, \lambda u, r) \in V \) and \((\lambda, (\nabla \cdot u, \lambda u, r))\) solves (2.7). The proof is completed. \( \square \)

2.2. case II: \( \kappa_s > 1 \)

Define \( Q = (A - I)^{-1} \). For an eigenpair \((\lambda, u)\) of (2.1), similar to case I, the following linear eigenvalue problem can be derived.

Find \((\lambda, (\varphi, y, r)) \in \mathbb{C} \times V \), such that, for \((\psi, z, s) \in V \),

\[
\begin{cases}
((I + Q)\nabla \varphi, \nabla \psi) + (Qy, \nabla \psi) = \lambda((\varphi, \psi) + (r, \psi)) \\
(Q\nabla \varphi, z) + (Qy, z) - (\nabla r, z) = 0 \\
-(\psi, \nabla s) = \lambda(\varphi, s).
\end{cases}
\] (2.9)
The bilinear form is defined on $V$:
\[
\hat{a}_V((\varphi, y, r), (\psi, z, s)) := ((I + Q)\nabla \varphi, \nabla \psi) + (Q \nabla \varphi, z) + (Q \nabla y, s),
\]
then $\hat{a}_V$ is symmetric, continuous and bounded. Associated with $\hat{a}_V(\cdot, \cdot)$ and $b_V(\cdot, \cdot)$, we define an operator $\hat{T}_V: V \to V$ by
\[
\hat{a}_V(T_V(\varphi, y, r), (\psi, z, s)) = b_V((\varphi, y, r), (\psi, z, s)), \quad \forall (\psi, z, s) \in V.
\]

**Lemma 2.4.** $T_V$ is well-defined and compact on $V$.

**Proof.** The proof is the same as that of lemma 2.1. \( \square \)

**Theorem 2.5.** The eigenvalue problem (2.9) is equivalent to (2.1).

**Proof.** The proof is the same as that of theorem 2.3. \( \square \)

### 3. The discretization scheme of the transmission eigenvalue problem for anisotropic media

For simplicity, only case I is discussed below, because case II is similar. Let $\{T_h\}$ be a shape regular triangular mesh in 2D or tetrahedral subdivision in 3D of $\Omega$, such that $\Omega = \cup_{K \in T_h} K$. Accordingly, the finite element spaces are defined as follows:

- $\mathcal{L}_h$ denotes the linear element space on $\Omega$, and we further define
  \[
  \tilde{\mathcal{L}}_h := \mathcal{L}_h \cap L^2(\Omega), \quad \mathcal{L}_{h0} := \mathcal{L}_h \cap H_0^1(\Omega) \quad \text{and} \quad \tilde{\mathcal{L}}_{h0} := \tilde{\mathcal{L}}_h \cap L^2(\Omega).
  \]
- $\mathcal{C}_h := (\mathcal{C}_h)^d$ is the vectorial piecewise constant finite element space on $\Omega$.

The discretized mixed eigenvalue problem takes the following form: find $\lambda_h \in \mathbb{C}$ and $(\varphi_h, y_h, r_h) \in V_h := \tilde{\mathcal{L}}_{h0} \times \mathcal{C}_h \times \tilde{\mathcal{L}}_h$, such that, for $(\psi_h, z_h, s_h) \in V_h$,
\[
\begin{aligned}
(P \nabla \varphi_h, \nabla \psi_h) + (P_y y_h, \nabla \psi_h) &= \lambda_h((\varphi_h, \psi_h) + (r_h, \psi_h)) \\
(P \nabla z_h, y_h) + ((I + P) y_h, z_h) - (\nabla r_h, z_h) &= 0
\end{aligned}
\quad (3.1)
\]

We propose the following lemma for the well-posedness of the discretized problem (3.1).

**Lemma 3.1.** There exists a constant $C$, uniformly with respect to $V_h$, such that
\[
\inf_{(\varphi_h, y_h, r_h) \in V_h} \sup_{(\psi_h, z_h, s_h) \in V_h} \frac{a_V((\varphi_h, y_h, r_h), (\psi_h, z_h, s_h))}{\|\varphi_h\|_{L^2(\Omega)}^2 + \|y_h\|_{L^2(\Omega)}^2 + \|r_h\|_{L^2(\Omega)}^2} \geq C > 0.
\quad (3.2)
\]

**Proof.** Again, by the Poincaré inequality, the following two inequalities can be derived.
\[
\begin{aligned}
a((\varphi_h, y_h), (\varphi_h, y_h)) &= \|P^{-\frac{1}{2}} P \nabla \varphi_h + \hat{P} \nabla \varphi_h\|_{L^2(\Omega)}^2 + \|P \nabla \varphi_h\|_{L^2(\Omega)}^2 + \|P - \hat{P} \nabla \varphi_h, \nabla \varphi_h\|_{L^2(\Omega)}^2 \\
&\geq (1 - \kappa^*)\|y_h\|_{L^2(\Omega)}^2 + \kappa_*(1 - \kappa^*)/(1 - \kappa_*)(2 - \kappa^*)\|\nabla \varphi_h\|_{L^2(\Omega)}^2 \\
&\geq C\|y_h\|_{L^2(\Omega)}^2 + \|\varphi_h\|_{L^2(\Omega)}^2.
\end{aligned}
\]
and given $s_h \in \tilde{\mathcal{L}}_h$, set $y_h = -\nabla s_h$, it follows that
\[
(\varphi_h, y_h, -\nabla s_h) = \|y_h\|_{L^2(\Omega)}\|\nabla s_h\|_{L^2(\Omega)} \geq C\|y_h\|_{L^2(\Omega)}\|s_h\|_{L^2(\Omega)},
\]

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The inf-sup condition holds as
\[
\inf_{s_h \in L_h} \sup_{(\varphi_h, \chi_h) \in L_h \times \mathbb{C}_h} \frac{b((\varphi_h, \chi_h), s_h)}{\|s_h\|_1.0.0 + \|\varphi_h\|_1.0.0} \geq C > 0.
\]

The proof is completed.

Associated with \(a_V(\cdot, \cdot)\) and \(b_V(\cdot, \cdot)\), we define operator \(T_{V_h} : V \to V_h\), for \((\psi_h, \tilde{z}_h, s_h) \in V_h\),
\[
a_V(T_{V_h}(\varphi, \chi, r), (\psi_h, \tilde{z}_h, s_h)) = b_V((\varphi, \chi, r), (\psi_h, \tilde{z}_h, s_h)),
\]
and operator \(S_{V_h} : V \to V_h\), for \((\psi_h, \tilde{z}_h, s_h) \in V_h\),
\[
a_V(S_{V_h}(\varphi, \chi, r), (\psi_h, \tilde{z}_h, s_h)) = a_V((\varphi, \chi, r), (\psi_h, \tilde{z}_h, s_h)).
\]

Here, we present some fundamental results on the spectral approximation of compact operators by the standard theory of finite element methods [2, 26].

**Lemma 3.2.** With the stable condition (3.2),

1. \(S_{V_h}\) is a well-defined idempotent operator from \(V\) onto \(V_h\);

2. The approximation holds:
\[
\|S_{V_h}(\varphi, \chi, r) - (\varphi, \chi, r)\|_V \leq C \inf_{(\psi_h, \tilde{z}_h, s_h) \in V_h} \|((\varphi, \chi, r) - (\psi_h, \tilde{z}_h, s_h))\|_V;
\]

3. If \(\|S_{V_h}(\varphi, \chi, r) - (\varphi, \chi, r)\|_V \to 0\) as \(h \to 0\) for any \((\varphi, \chi, r) \in V\), then \(\|T_{V_h} - T_V\|_V \to 0\) as \(h \to 0\);

4. The operator \(T_{V_h}\) is well-defined and compact on \(V_h \subset V\).

**Lemma 3.3.** Let \(\{T_{V_h}\}\) be a family of compact operators on \(V\) such that \(\|T_{V_h} - T_V\|_V \to 0\) as \(h \to 0\). For any \(i \in \mathbb{N}^+\), it follows that the eigenvalues of \(T_{V_h}\) can be ordered as
\[
\mu_{1,h} \geq \mu_{2,h} \geq \cdots \geq 0 \text{ with } \lim_{h \to 0} \mu_{i,h} = \mu_i,
\]
and the eigenvalues of (3.1) can be ordered as
\[
0 \leq \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \text{ with } \lim_{h \to 0} \lambda_{i,h} = \lambda_i.
\]

Moreover, if \(\mu_{i,h} \neq 0\), it holds that \(\lambda_{i,h} = \lambda_i = 1\).

Define \(W := L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)\) and let \(\lambda_1 \leq \lambda_2 \leq \cdots\) be the eigenvalues of (2.7) and \(\lambda_{1,h} \leq \lambda_{2,h} \leq \cdots\) be the eigenvalues of (3.1), respectively. We use \(N(\lambda_i)\) and \(N_h(\lambda_{i,h})\) to denote the eigenspaces of (2.7) and (3.1) that are associated with \(\lambda_i\) and \(\lambda_{i,h}\), respectively. The gap between \(N(\lambda_i)\) and \(N_h(\lambda_{i,h})\) is defined as
\[
\delta(N(\lambda_i), N_h(\lambda_{i,h})) := \max(\delta(N(\lambda_i), N_h(\lambda_{i,h})), \delta(N_h(\lambda_{i,h}), N(\lambda_i))),
\]
with \(\delta(N(\lambda_i), N_h(\lambda_{i,h})) := \sup_{\zeta \in N(\lambda_i), \|\zeta\|_W = 1} \text{dist}(\zeta, N_h(\lambda_{i,h}))\). An abstract estimation holds as below.

**Lemma 3.4.** For any \(i \in \mathbb{N}^+\), there exists a constant \(C\) independent of \(h\), such that for a sufficiently small \(h\),
\[
\delta(N(\lambda_i), N_h(\lambda_{i,h})) \leq C \|\langle I_V - S_{V_h}\rangle_{N(\lambda_i)}\|_W^*,
\]
where \(I_V\) is the identity operator on \(V\).
For any $i$, we only have to note that (3.4) are equivalent. The result follows from the standard theory of finite element methods.

In this paper, only a few smallest real eigenvalues of GEP (3.1) is taken into consideration by introducing Lagrangian multipliers $\lambda, \mu, \nu$. Substituting (3.3) and (3.4) multiplied by $\sigma$ and $\zeta$ into (3.1), the discretization gives rise to a GEP

$$Kz = \lambda Mz,$$

(3.5)

where

$$K = \begin{bmatrix} K_P & F_P & O & \alpha & O \\ F_P^T & M_P & -G & O & O \\ O & -G^T & O & O & \beta \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} X & O & Y & O & O \\ O & O & O & O & O \\ Y^T & O & O & O & O \end{bmatrix} \quad \text{and} \quad \zeta = \begin{bmatrix} \omega \\ \eta \\ \gamma \\ \sigma \\ \zeta \end{bmatrix}.$$

In this paper, only a few smallest real eigenvalues of GEP (3.5) are desired as the approximate transmission eigenvalues.

Let $H^1(\Omega) := (H^1(\Omega))^d$, the discretization (3.1) can be performed with $V_h$, and a first-order accuracy for eigenfunctions and second-order accuracy for eigenvalues can be expected.

**Theorem 3.5.** For any $i \in \mathbb{N}^+$, if $N(\lambda_i) \subset (H^2(\Omega) \times H^2(\Omega)) \cap V$, then

$$\delta(N(\lambda_i), N_i(\lambda_i, h)) \leq C(\lambda_i)h \quad \text{and} \quad |\lambda_i - \lambda_i,h| \leq Ch^2.$$

**Proof.** We only have to note that $N(\lambda_i)$ is of finite (fixed) dimension, and any two norms on that are equivalent. The result follows from the standard theory of finite element methods.

Let $n_e, n_t, n_c$ be the dimensions of $\mathcal{L}_{h0}, \mathcal{C}_h$ and $\mathcal{L}_h$, respectively, and $\{\phi_1, \phi_2, \ldots, \phi_{n_c}\}$, $\{\chi_1, \chi_2, \ldots, \chi_{n_c}\}$ and $\{\phi_1, \phi_2, \ldots, \phi_{n_t}\}$ be the finite element basis of $\mathcal{L}_{h0}, \mathcal{C}_h$ and $\mathcal{L}_h$, respectively. Then, we have

$$\varphi_h = \sum_{i=1}^{n_c} \omega_i \phi_i, \quad \chi_h = \sum_{i=1}^{n_c} \eta_i \chi_i \quad \text{and} \quad r_h = \sum_{i=1}^{n_c} \gamma_i \phi_i,$$

(3.3)

and $\omega = [\omega_1, \omega_2, \ldots, \omega_{n_c}]^\top$, $\eta = [\eta_1, \eta_2, \ldots, \eta_{n_c}]^\top$ and $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_{n_c}]^\top$. We further specify the stiffness, mass and convection matrices in Table 1. Moreover, in the discrete setting zero mean value on $\varphi_h$ and $r_h$ means that

$$\langle \varphi_h, 1 \rangle = \sum_{i=1}^{n_c} \omega_i (\phi_i, 1) = \alpha^\top \omega = 0 \quad \text{and} \quad \langle r_h, 1 \rangle = \sum_{i=1}^{n_c} \gamma_i (\phi_i, 1) = \beta^\top \gamma = 0,$$

(3.4)

where

$$\alpha = [\phi_1, 1, \phi_2, 1, \ldots, \phi_{n_c}, 1]^\top \quad \text{and} \quad \beta = [(\phi_{n_c}, 1, \phi_{n_c}, 2, \ldots, \phi_{n_t}, 1]^\top.$$

The constraint (3.4) is taken into consideration by introducing Lagrangian multipliers $\sigma$ and $\zeta$. Substituting (3.3) and (3.4) multiplied by $\sigma$ and $\zeta$ into (3.1), the discretization gives rise to a GEP

Table 1. Stiffness, mass and convection matrices.

| Matrix | Dimension | Definition |
|--------|-----------|------------|
| $K_P$  | $n_e \times n_e$ | $(K_P)_{ij} = (P \nabla \phi_i, \nabla \phi_j)$ |
| $F_P$  | $n_e \times n_t$ | $(F_P)_{ij} = (P \chi_j, \nabla \phi_i)$ |
| $M_P$  | $n_t \times n_t$ | $(M_P)_{ij} = ((I + P) \chi_j, \chi_j)$ |
| $G$    | $n_t \times n_e$ | $(G)_{ij} = (\nabla \phi_i, \chi_j)$ |
| $X$    | $n_e \times n_e$ | $(X)_{ij} = (\phi_i, \phi_j)$ |
| $Y$    | $n_e \times n_e$ | $(Y)_{ij} = (\phi_i, \phi_j)$ |
4. Preprocessing GEP (3.5)

Let \( \hat{n} = n_i / d \), by definition, \( M_p = (I + P) \otimes I_{\hat{n}} \) is a block diagonal matrix with \( M_p^{-1} = (I + P)^{-1} \otimes I_{\hat{n}} \). Define the invertible matrix

\[
\mathcal{W} = \begin{bmatrix}
I & -F_pM_p^{-1} & 0 & 0 & 0 \\
0 & G^\top M_p^{-1} & I & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix},
\]

then the GEP (3.5) is equivalent to the GEP

\[
(\mathcal{W}\hat{K}\mathcal{W}^\top)(\mathcal{W}^{-\top}z) = \lambda(\mathcal{W}\hat{M}\mathcal{W}^\top)(\mathcal{W}^{-\top}z),
\]

(4.1)

where

\[
\mathcal{W}\hat{K}\mathcal{W}^\top = \begin{bmatrix}
\hat{K} & \hat{F} & \alpha & 0 \\
\hat{F}^\top & \hat{G} & 0 & \beta \\
\alpha^\top & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} := \hat{K} \oplus M_p
\]

with

\[
\hat{K} = K_p - F_pM_p^{-1}F_p^\top, \quad \hat{F} = F_pM_p^{-1}G \quad \text{and} \quad \hat{G} = -G^\top M_p^{-1}G.
\]

(4.2)

\[
\mathcal{W}\hat{M}\mathcal{W}^\top = \begin{bmatrix}
X & Y & O & O & O \\
Y^\top & 0 & O & O & 0 \\
0 & 0 & O & O & 0 \\
0 & 0 & 0 & O & 0 \\
0 & 0 & 0 & 0 & O
\end{bmatrix} := \hat{M} \oplus O.
\]

Now, we consider the following GEP

\[
\hat{K} \hat{z} = \lambda \hat{M} \hat{z},
\]

(4.3)

where \( \hat{z} = [\omega, \gamma, \sigma, \varsigma]^\top \). Based on the analysis above, the following theorem can be derived immediately.

**Theorem 4.1.** The GEPs (3.5) and (4.3) have the same non-infinite eigenvalues, and

\[
\text{Spec}(\hat{K}, \hat{M}) = \text{Spec}(K, M) \setminus \{\infty\}_{k=1}^n,
\]

where \( \text{Spec}(\hat{K}, \hat{M}) \) denotes the set of eigenvalues of the linear pencil \( \hat{K} - \lambda \hat{M} \).

**Proof.** GEP (3.5) has the same eigenvalues as GEP (4.1), and according to the block structure of GEP (4.1) and the definition of GEP (4.3), this theorem holds obviously. \( \square \)

The deflation shown above markedly reduces the size of the GEP under consideration. Specifically, one can see that in the 2D case, the ratio of the number of triangles to that of nodes is approximately 2, while in the 3D case, the ratio of the number of tetrahedrons to that of nodes...
is approximately 6. In other words, \( n_i/n_e \sim 4 \) in the 2D case and \( n_i/n_e \sim 18 \) in the 3D case. Therefore, a drastic reduction in the matrix size is expected.

Unlike several Schur complement based techniques where additional linear systems should be solved, here, the inverse of \( E_P \) is rather trivial due to the block diagonal structure of \( M_P \). Even more interesting, we show instantly that \( K, F \) and \( G \) in (4.2) can be directly assembled without carrying out any matrix multiplications in their formal definitions.

**Theorem 4.2.** Entries of the matrices \( \hat{K}, \hat{F} \) and \( \hat{G} \) in (4.2) are explicitly given as follows:

\[
\begin{align*}
(\hat{K})_{ij} &= (P\nabla \phi_i, \nabla \phi_j) - (P(I + P)^{-1}P\nabla \phi_i, \nabla \phi_j) = (A\nabla \phi_i, \nabla \phi_i), \quad i,j = 1,2,\ldots,n_e, \\
(\hat{F})_{ij} &= (P(I + P)^{-1}\nabla \phi_j, \nabla \phi_i) = (A\nabla \phi_j, \nabla \phi_i), \quad i = 1,2,\ldots,n_e, \quad j = 1,2,\ldots,n_e, \\
(\hat{G})_{ij} &= -((I + P)^{-1}\nabla \phi_j, \nabla \phi_j) = ((A - I)\nabla \phi_j, \nabla \phi_j), \quad i,j = 1,2,\ldots,n_e.
\end{align*}
\]

By theorem 4.2, the sparsity of \( \hat{K}, \hat{F} \) and \( \hat{G} \) is similar to that of \( K_P \), which implies that GEP (4.3) is still sparse in addition to the significant reduction in the matrix size, this characteristic is extremely helpful in practical calculations. In the numerical examples in the 3D case presented in section 5, solving GEP (4.3) takes much less memory and time than solving GEP (3.5) to calculate the smallest few real transmission eigenvalues with the same mesh.

### 5. Numerical experiments on the mixed element discretization scheme

In this section, we present some numerical results of transmission eigenvalue problems (2.7) and (2.9) on convex and nonconvex domains. All calculations are performed using MATLAB 2020a installed on a workstation with memory 768G and Intel(R) Xeon(R) Gold 6244 CPU @ 3.60GHz.

Unless otherwise specified, the mesh size in the numerical examples below is set to \( h_l = 2^{-3-l}, l = 1,2,3,4,5,6 \) in 2D and \( l = 0,1,2,3 \) in 3D. Given a mesh size \( h_l \), we obtain a sequence of eigenvalues \( \lambda_{i_l,h_l}, i_l \in \mathbb{N}^+ \). DoF1 and DoF2 represent the degrees of freedom of the GEP (3.5) and (4.3), respectively. The rate of convergence of the transmission eigenvalue is computed by

\[
\log_2 \left( \frac{\lambda_{i_l,h_{l+1}} - \lambda_{i_l,h_l}}{\lambda_{i_l,h_{l+2}} - \lambda_{i_l,h_{l+1}}} \right), \quad l = 1,2,3,4 \text{ in 2D and } l = 0,1 \text{ in 3D}.
\]

#### 5.1. 2D examples

Here, we are concerned with four domains in 2D [8, 15] that are shown in figure 1, of which two are the convex domains and the rest are the nonconvex domains. Specifically, these domains are \( \Omega_1 = \{(x,y)|0 \leq x^2 + y^2 \leq 1/4\} \), \( \Omega_2 = [0,1]^2 \), \( \Omega_3 = [-0.5,0.5] \times [-0.5,0.5] \times [0,0.5] \times [-0.5,0] \) and \( \Omega_4 = \{(x,y)|1/16 \leq x^2 + y^2 \leq 1/4\} \). A in (2.1) can be one of the following 2-by-2 matrices

\[
A_1 = \frac{1}{4}I_2, \quad A_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/8 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/8 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0.1372 & 0.0189 \\ 0.0189 & 0.1545 \end{bmatrix}.
\]

We first verify the proposed scheme by comparing the smallest real transmission eigenvalue of

**Example 1** \( \Omega_1 \) with the anisotropic case \( A_1 \) calculated by our MATLAB implementation with the known exact result, and then we investigate the following examples:

**Example 2** \( \Omega_2 \) with the anisotropic cases \( A_i, i = 1,2,3,4 \), respectively.
Example 3 $\Omega_3$ with the anisotropic cases $A_i, i = 1, 2, 3, 4$, respectively.

Example 4 $\Omega_4$ with the anisotropic cases $A_i, i = 1, 2, 3, 4$, respectively.

We compute the first six real transmission eigenvalues $(k_i, h_i = \sqrt{\lambda_i})$ of Example 1 on the mesh level $h_1$ to $h_6$ and the smallest real transmission eigenvalues of Examples 2, 3 and 4 on the mesh level $h_6$. The results are listed in tables 2 and 3. $Dof1/Dof2$ is approximately 3, as expected. In particular, the first real transmission eigenvalue of Example 1 calculated by our method is 5.8053, which is consistent with the exact transmission eigenvalue of 5.8 in [8, 15]. In Example 1, it takes approximately 4.3 GB of memory with 101 seconds to solve the GEP (4.3) on the mesh level $h_6$. Moreover, as the mesh is refined, the calculated real eigenvalue sequence monotonically decreases to the exact results. We show the convergence rate of eigenvalues in section 5.3.

5.2. 3D examples

Here, we are concerned with four domains in 3D illustrated in figure 2, of which two are the convex domains and the rest are the nonconvex domains. Specifically, these domains are $\Omega_5 = B(0, 1)$, $\Omega_6 = [0, 1]^3$, $\Omega_7 = [0, 1]^3 \setminus (0.25, 0.75)^3$ and $\Omega_8 = [0, 1]^3 \\{ (x, y, z) | (x - 1/2)^2 + (y - 1/2)^2 < 1/16, 0 \leq z \leq 1 \}$. $A$ in (2.1) can be one of the following 3-by-3 matrices
Table 2. The first six real transmission eigenvalues of Example 1.

| Mesh | Dof1  | Dof2  | \(k_1, h_1\) | \(k_2, h_1\) | \(k_3, h_1\) | \(k_4, h_1\) | \(k_5, h_1\) | \(k_6, h_1\) |
|------|-------|-------|--------------|--------------|--------------|--------------|--------------|--------------|
| \(l = 1\) | 2116  | 708   | 5.8978       | 6.9404       | 6.9541       | 7.8375       | 7.8555       | 7.8845       |
| \(l = 2\) | 8056  | 2688  | 5.8301       | 6.8386       | 6.8392       | 7.6392       | 7.6405       | 7.6747       |
| \(l = 3\) | 31792 | 10600 | 5.8117       | 6.8108       | 6.8109       | 7.5850       | 7.5851       | 7.6239       |
| \(l = 4\) | 132796| 44268 | 5.8067       | 6.8031       | 6.8031       | 7.5705       | 7.5705       | 7.6107       |
| \(l = 5\) | 536656| 178888| 5.8056       | 6.8013       | 6.8013       | 7.5671       | 7.5671       | 7.6076       |
| \(l = 6\) | 2107438| 702482| 5.8053      | 6.8009       | 6.8009       | 7.5663       | 7.5663       | 7.6069       |

Table 3. The smallest real transmission eigenvalues of Example 2, 3 and 4.

| \(\Omega\) | Dof1  | Dof2  | \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) |
|----------|-------|-------|--------|--------|--------|--------|
| \(\Omega_2\) | 2722144| 907384| 5.2987 | 4.3867 | 3.5816 | 3.6105 |
| \(\Omega_3\) | 2041984| 680664| 6.7284 | 5.9350 | 4.3026 | 4.3514 |
| \(\Omega_4\) | 1605368| 535124| 11.3526| 11.6678| 7.1936 | 7.1943 |

Figure 2. Domains of 3D examples.
Table 4. The first six real transmission eigenvalues of Example 5.

| Mesh | Dof1 (λ) | Dof2 (λ) | \( k_{1,h} \) | \( k_{2,h} \) | \( k_{3,h} \) | \( k_{4,h} \) | \( k_{5,h} \) | \( k_{6,h} \) |
|------|----------|----------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( l = 0 \) | 77093 | 7796 | 5.2964 | 5.9969 | 6.2199 | 7.4322 | 8.6910 | 9.7576 |
| \( l = 1 \) | 532126 | 53500 | 5.2281 | 5.9160 | 6.1348 | 7.2947 | 8.4790 | 9.4481 |
| \( l = 2 \) | 3805418 | 382271 | 5.2097 | 5.8936 | 6.1113 | 7.2562 | 8.4193 | 9.3611 |
| \( l = 3 \) | 28970976 | 2907867 | 5.2052 | 5.8882 | 6.1057 | 7.2469 | 8.4048 | 9.3402 |

Table 5. The first six real transmission eigenvalues of Example 6, 7 and 8.

| \( \Omega \) | A | Dof1 | Dof2 | \( k_{1,h} \) | \( k_{2,h} \) | \( k_{3,h} \) | \( k_{4,h} \) | \( k_{5,h} \) | \( k_{6,h} \) |
|------------|---|-------|-------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( \Omega_6 \) | \( A_6 \) | 5.2602 | 5.9167 | 5.9168 | 5.9168 | 6.3506 | 6.3506 |
| \( \Omega_7 \) | \( A_7 \) | 7201086 | 722772 | 4.7106 | 5.0600 | 5.5420 | 5.7049 | 6.2222 | 6.3625 |
| \( \Omega_8 \) | \( A_8 \) | 9.4155 | 9.5137 | 9.5139 | 9.5182 | 9.5311 | 9.5965 |

We first verify the proposed scheme by comparing the first four real transmission eigenvalues of Example 5 with the anisotropic case \( A_5 \) calculated by our MATLAB implementation with the known exact results in [20], and then we investigate the following examples:

**Example 6** \( \Omega_6 \) with the anisotropic cases \( A_i, i = 6, 7, 8 \), respectively.

**Example 7** \( \Omega_7 \) with the anisotropic cases \( A_i, i = 6, 7, 8 \), respectively.

**Example 8** \( \Omega_8 \) with the anisotropic cases \( A_i, i = 6, 7, 8 \), respectively.

We compute the first six real eigenvalues \((k_{i,h} = \sqrt{\lambda_{i,h}})\) of Example 5 on the mesh level \( h_0 \) to \( h_3 \) and Examples 6, 7 and 8 on the mesh level \( h_3 \). The results are listed in tables 4 and 5. Dof1/Dof2 is approximately 10, as expected. The first four transmission eigenvalues of Example 5 calculated by our method are 5.2052, 5.886, 6.1057 and 7.2469, as shown in table 4, which are consistent with the exact eigenvalues 5.204, 5.886, 6.104 and 7.244 in [20]. Moreover, as the mesh is refined, the calculated real eigenvalue sequence monotonically decreases to the exact results.

In Example 5, approximately 2.6, 9.3 and 178 GB of memory with 5.7, 326 and 16943 seconds are required to calculate the smallest real transmission eigenvalue by solving the GEP (3.5) on the mesh level \( h_0 \), \( h_1 \) and \( h_2 \), respectively. Correspondingly solving the GEP (4.3) only requires 2.3, 5.5 and 46.4 GB of memory with 4.3, 168 and 5845 seconds, respectively. That is, both the computational cost and time are markedly lower after preprocessing as described in section 4.

5.3. **Convergence rate of transmission eigenvalues**

Figure 3 demonstrates that the discretization (3.1) can be performed with \( V_h \) for the 2D and 3D cases, and a second-order accuracy for eigenvalues can be obtained both on convex and
Figure 3. The convergence rates for 2D and 3D examples. X-axis means the size of mesh and Y-axis means $|\lambda_{i,h_l+1} - \lambda_{i,h_l}|$. 

(a) $\Omega_1$ with the anisotropic case $A_1$. 
(b) $\Omega_2$ with the anisotropic case $A_2$. 
(c) $\Omega_3$ with the anisotropic case $A_3$. 
(d) $\Omega_4$ with the anisotropic case $A_4$. 
(e) $\Omega_5$ with the anisotropic case $A_5$. 
(f) $\Omega_6$ with the anisotropic case $A_6$. 
(g) $\Omega_7$ with the anisotropic case $A_7$. 
(h) $\Omega_8$ with the anisotropic case $A_8$. 

Figure 3. The convergence rates for 2D and 3D examples. X-axis means the size of mesh and Y-axis means $|\lambda_{i,h_l+1} - \lambda_{i,h_l}|$. 

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nonconvex domains. Therefore, the optimal convergence rate for eigenvalues can be obtained by this discrete mixed element scheme, which is consistent with theorem 3.5.

6. Conclusions and discussions

In this paper, we discuss the Helmholtz transmission eigenvalue problem for anisotropic inhomogeneous media with the index of refraction \( n(x) \equiv 1 \) in 2D and 3D, and its discretization using a mixed finite element scheme. Our key theoretical result of this paper is that the proposed mixed formulation is equivalent to (2.1) without introducing any spurious eigenvalues, and the proposed discretization scheme is easy to implement with firm theoretical support. In practice, the goal of computing a few smallest real transmission eigenvalues is hindered by the tremendous size of the resulting GEP, in that the \( LDL \) factorization cannot fit into the computer memory. We partially resolve this critical issue by the carefully designed preprocessing shown in section 4. With preprocessing, nearly all of the huge eigenspace associated with the infinite eigenvalue is deflated, and the size of the GEP is markedly reduced without deteriorating the sparsity, hence, the \( LDL \) factorization becomes feasible. Both the computational time and cost are also markedly reduced, particularly in the 3D case.

Numerical examples on the convex and nonconvex connected domains both in 2D and 3D confirm that the optimal convergence rate of the transmission eigenvalues can be achieved by the proposed scheme, which is consistent with the theoretical prediction.

In the future, we plan to generalize this scheme to other types of transmission eigenvalue problems, such as elastic waves and Maxwell transmission eigenvalue problems. Further studies should be performed so that the memory cost can be controlled as the mesh becomes finer in 3D domains.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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