String Matching and 1d Lattice Gases

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Abstract

We calculate the probability distributions for the number of occurrences \( n \) of a given \( l \) letter word in a random string of \( k \) letters. We consider the case in which the letters of the strings belong to an \( r \) letter alphabet. Analytical expressions for the distribution are known for the asymptotic regimes (i) \( k \gg r^l \gg 1 \) (Gaussian) and \( k, l \to \infty \) such that \( k/r^l \) is finite (Compound Poisson). However, it is known that these distributions do not work well in the intermediate regime \( k \gtrsim r^l \gtrsim 1 \). We show that the problem of calculating the string matching probability can be cast into a problem of determining the configurational partition function of a 1d lattice gas with interacting particles such that the string matching probability distribution becomes the grand-partition sum of the lattice gas, with the number of particles corresponding to the number of matches on the string. Using this analogy, we perform a virial expansion of the effective equation of state and thereby obtain the probability distribution function. Our result reproduces the behavior of the matching distribution in all regimes, i.e. the asymptotic as well as the intermediate regimes, rather well. We are also able to show analytically how the limiting distributions arise. Our analysis builds on the observation that the effective interactions between the particles consist of a relatively strong core of size \( l \), the word length, followed by a weak, exponentially decaying tail, whose overall strength decreases with increasing \( l \). We find that the asymptotic regimes correspond to the case where the tail of the interactions can be neglected, while in the intermediate regime the effects of the tail needs to be incorporated into the analysis. This is ultimately responsible for the failure of the asymptotic distributions in this regime. Our results are readily generalized to the case where the random strings are generated by more complicated stochastic process such as a non-uniform letter probability distribution or Markov chains. We show that by varying the parameters of the stochastic process, the tails of the effective interactions can be made even more dominant rendering thus the asymptotic approximations less accurate in such a regime.

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1 Introduction

The problem of determining the probability of encountering (matching) a given string of length \(l\) in another string of length \(k\), whose letters have been drawn randomly from an alphabet of \(r\) letters, has a variety of applications ranging from designing fast algorithms for pattern searching \([1,2]\), to problems in genetics such as assessing the likelihood of events such as the frequency of occurrence of DNA segments \([3,4]\), or that certain DNA segments align \([5,6]\). In each of these cases, the likelihood estimates for random sequences can be used as a benchmark against which one can evaluate the statistical significance of actually observed events.

The problem is non-trivial, because of the possibility of overlapping occurrences in the string, which introduce correlations that need to be dealt with. Guibas and Odlyzko \([8,9,10]\) derived the moment generating functions associated with the probability for not encountering a given set of words in a random string, whose letters were distributed independently and identically. The resulting distributions turn out to depend on a set of correlation functions that capture the overlap properties of the words with each other.

Building on the work of Guibas and Odlyzko, several authors have studied the probability distribution for the number of occurrences \(n\) of a given \(l\) letter word in a random string of \(k\) letters, under various assumptions on the distribution of random letters \([11,12,13,14,15,16,17,18]\): The cases where the letters of the random string are independently and identically distributed (i.i.d.) was treated by Fudos et al. \([13]\), whereas the case where the letter distribution follows the steady state distribution of a Markov process has been investigated by several authors \([11,12,14,16,18]\). All of these results have been obtained for asymptotic regimes \((k\) large + various assumptions on the length of the word \(l\)), where tools of statistics such as the central limit theorem \([13,14,4]\), theory of large deviations \([18,14]\), or (compound) poisson approximations for rare-events \([11,12,15,17]\) are applicable.

The regimes of applicability can be difficult to identify, however. It has been noted that, even in the case of i.i.d letters, when the length \(l\) of the word to be matched is fixed, and assuming the length of the random string to be large, the most accurate approximation to chose (gaussian or compound poisson) still depends on the word itself that is being matched \([10]\).

It is therefore desirable to come up with a single, explicit analytical expression for the probability distribution that is generally valid, and to obtain the asymptotic expressions, mentioned above, as special cases by taking certain limits. This is what we set out to do in this article. Besides the obvious advantage of having a single description, such an approach will naturally identify the regimes of application of the various asymptotic approximations, while also pointing out when and how they fail.

It turns out that all of these issues are present even in the simplest case where the letters of the random string are uniformly and independently distributed. For the sake of simplicity and clarity of presentation, we will perform the analysis for this case. However, we will point out in detail how these results carry over to the more general case of random letter distributions.

Our approach to this problem, which appears to be novel, can be summarized as
follows: We first show that the problem of calculating the probability distribution for the number of occurrences $n$ of a given $l$ letter word in a random string of $k$ letters, can be rigorously mapped into the problem of calculating the configurational part of the grand-canonical partition function of a 1d lattice gas. In this mapping the number of particles correspond to the number occurrences, the "volume" of the gas is the length of the random string, and the correlations between subsequent occurrences turn into pairwise interactions whose nature depends on properties of the word to be matched. It turns out that common to all interactions is a relatively strong and short-ranged segment of range $l$, followed by a weak and exponentially decaying tail.

With the help of the lattice gas analogy, and by using techniques of liquid theory, such as the virial expansion, we are able to obtain an analytical expression for the probability distribution that reproduces the known asymptotic limits. We show how the distribution crosses over into the asymptotic forms of the distribution, and thereby expose the conditions required for these limits to be applicable.

More importantly, our method allows us to analytically treat the intermediate regime of moderate string lengths, $k \gtrsim r^d \gtrsim 1$, as well. This regime is most relevant for biological applications and turns out to be the hardest one to tackle analytically, since in this regime the effects of the tail are strong and need to be kept in the analysis. This is also the reason behind the deviations of the asymptotic forms from the actual distribution, since, as we will show, these distributions are obtained by neglecting the tail of the interactions. These deviations become more pronounced for short words and small number of letters in the alphabet, small $l$ and $r$, respectively.

Our results are readily generalized to the broader class of letter distributions, such as non-uniformly distributed letters or letters generated by a Markov process. Such distributions give rise to a broader class of effective interactions. In particular, it turns out that these interactions can have stronger tails than can be achieved by a uniform letter distribution. This potentially renders our method of approach even more relevant to such letter distributions.

We would also like to note that our approach is in spirit similar to recent attempts at solving combinatorial problems, such as the k-SAT problem [20, 21, 22, 23], using ideas borrowed from statistical mechanics [24, 25].

The article is organized as follows: In Section II we introduce our notation and formalism, rederiving in this setting some of the relevant and known results. In Section III we establish the partition function analogy. We derive and study the properties of the effective particle interactions and then set up a virial expansion for the "equation of state". From the virial expansion we obtain the $n$-particle partition function, which in this analogy corresponds to the $n$-match probability distribution function. We show how the various known limits arise and discuss the underlying assumptions. Section IV discusses the generalization and implications of our approach to the more general class of letter distributions studied in the literature and we will discuss our results in Section V.
2 The matching probabilities

In this section we derive some of the known expressions for the matching and \( n \)-match probability, the probability of a at least one and precisely \( n \) occurrences of a given word, respectively. Besides providing a review of the relevant results, the main purpose of this section is to introduce our notation and provide the setting for the statistical mechanics approach to be taken up in the following Section.

2.1 Definitions

Assume that \( x \) and \( y \) are variables that take values from an \( r \) letter alphabet such that \( x, y \in \{0, \ldots, r-1\} \). Let \( x = (x_1, x_2, x_3, \ldots, x_l) \) and \( y = (y_1, y_2, y_3, \ldots, y_l) \), be two strings of \( l \) letters. Define the match indicator function \( \Phi(x, y) \) as

\[
\Phi(x, y) = \prod_{t=1}^{l} \delta(x_t, y_t)
\]  

(2.1)

So that we have

\[
\Phi(x, y) = \begin{cases} 
1, & \text{if } x = y \\
0, & \text{otherwise.}
\end{cases}
\]  

(2.2)

Let \( y = (y_1, y_2, \ldots, y_k) \) be a string of length \( k \geq l \) and denote by \( y_{a,l} = (y_{a+1}, y_{a+2}, \ldots, y_{a+l}) \) the substring of length \( l \) starting at position \( a \), \( a = 0, 1, \ldots, k - l \). Furthermore, let

\[
f_a(x, y) = \Phi(x, y_{a,l}).
\]  

(2.3)

We have

\[
f_a(x, y) = \begin{cases} 
1, & \text{if } x = y_{a,l} \\
0, & \text{otherwise.}
\end{cases}
\]  

(2.4)

In other words, \( f_a(x, y) = 1 \), if and only if \( x \) matches \( y \) at position \( a \), and zero otherwise.

2.2 The matching probability

Define \( p(m; x) \) to be the probability that a given word \( x \) of length \( l \) is contained at least once in a randomly drawn string \( y \) of length \( m + l \). We will refer to this as the matching probability. Let \( I_M(x, y) \) be the function that takes on the value one if the \( k \)-string \( y \) contains the given \( l \)-string \( x \) at least once, and zero otherwise. Using Eq. (2.3), we can write

\[
I_M(x, y) = 1 - \prod_{a=0}^{k-l} [1 - f_a(x, y)].
\]  

(2.5)

Since \( k \geq l \), it is convenient to define the excess length \( m = k - l \). We thus find

\[
p(m; x) = 1 - \frac{1}{r^{m+l}} \sum_y \prod_{a=0}^{m} [1 - f_a(x, y)],
\]  

(2.6)
where \( r^{m+l} \) is the number of distinct \( k = m + l \)-strings of \( r \)-letters, and the summation is over all such strings \( y \).

In [26], the products on the right hand side of Eq. (2.6) were expanded into a Mayer-like sum,

\[
p(m; x) = \frac{1}{r^k} \sum_y \sum_a f_a - \frac{1}{r^k} \sum_y \sum_{a < b} f_a f_b + \frac{1}{r^k} \sum_y \sum_{a < b < c} f_a f_b f_c - \ldots ,
\]

(2.7)

(arguments of \( f_a \) will be suppressed in what follows) and the terms in the sum where evaluated approximately. Here we will take a different approach.

The following algebraic identity will be of use in the following:

\[
1 - \prod_{a=0}^{m} (1 - f_a) = \sum_{b=0}^{m} f_b \prod_{a=0}^{b-1} (1 - f_a),
\]

(2.8)

with the convention that when \( b = 0 \), the product on the right hand side is set to one. Eq. (2.8) is readily proven by induction.

Using this identity, \( p(m; x) \) can be written as

\[
p(m; x) = \frac{1}{r^{m+l}} \sum_y \sum_{b=0}^{m} f_b \prod_{a=0}^{b-1} (1 - f_a).
\]

(2.9)

Note that for any given \( b \), the expression on the right hand side only involves the variables \( y_1, y_2, \ldots, y_{b+l} \). The sum over the remaining indices yields \( r^{m-b} \) and we find that

\[
p(m; x) = \sum_{b=0}^{m} \frac{1}{r^{b+l}} \sum_{y_1 \cdots y_{b+l}} f_b \prod_{a=0}^{b-1} (1 - f_a).
\]

(2.10)

Defining the correlator \( d(b; x) \) as

\[
d(b; x) = \sum_{y_1 \cdots y_{b+l}} f_b \prod_{a=0}^{b-1} (1 - f_a),
\]

(2.11)

\( p(m; x) \) can be therefore written as

\[
p(m; x) = \sum_{b=0}^{m} \frac{1}{r^{b+l}} d(b; x).
\]

(2.12)

We can obtain a recursion relation for \( d(b; x) \) by factoring out the \( a = 0 \) term in Eq. (2.11),

\[
d(b; x) = \sum_{y_1 \cdots y_{b+l}} f_b \prod_{a=1}^{b-1} (1 - f_a) - \sum_{y_1 \cdots y_{b+l}} f_0 f_b \prod_{a=1}^{b-1} (1 - f_a).
\]

(2.13)

The argument of the first sum does not contain the variable \( y_1 \), while the sum over the remaining variables yields \( d(b-1; x) \). Thus,

\[
d(b; x) = r d(b-1; x) - h(b; x),
\]

(2.14)
with the correlator \( h \) defined as

\[
h(b; x) = \sum_{y_1 \cdots y_{b+1}} f_0 \left[ \prod_{a=1}^{b-1} (1 - f_a) \right] f_b.
\]  

(2.15)

Note that for \( m = 0 \)

\[
p(0; x) = \frac{1}{r^l}.
\]  

(2.16)

Comparing with Eq. (2.12) this implies that

\[
d(0; x) = 1.
\]  

(2.17)

Since there are no constraints on \( h(0; x) \), we will define \( h(0; x) = 0 \).

We next seek a recursion relation for \( h \). Using the identity, Eq. (2.8), we find from Eq. (2.15)

\[
h(b; x) = \sum_{y_1 \cdots y_{b+1}} \left\{ f_0 f_b - \sum_{c=1}^{b-1} f_0 \left[ \prod_{a=1}^{c-1} (1 - f_a) \right] f_c f_b \right\}.
\]  

(2.18)

Recall from the definition of \( f_b(x, y) \) that \( f_b \) is a product of Kronecker deltas, Eqs. (2.2) and (2.3). The Kronecker deltas enforce a transitive relation between their arguments, and we can write \( f_b(x, y; 0) = f_b(x, y; 0) f_b(x, \tilde{y}; 0) \), where we have introduced an auxiliary set of variables \( \tilde{y} \) over which a sum is to be performed. Thus Eq. (2.18) can be rewritten as

\[
\sum_{y_1 \cdots y_{b+1}} \sum_{c=1}^{b-1} f_0 \left[ \prod_{a=1}^{c-1} (1 - f_a) \right] f_c f_b = \sum_{c=1}^{b-1} \left\{ \sum_{y_1 \cdots y_{b+1}} f_0 \left[ \prod_{a=1}^{c-1} (1 - f_a) \right] f_c \right\} \left\{ \sum_{\tilde{y}_c \cdots \tilde{y}_{b+1}} f_c f_b \right\}.
\]  

(2.19)

Defining the correlator \( C(b; x) \) as

\[
C(b; x) = \sum_{y_1 \cdots y_{b+1}} f_0(x, y) f_b(x, y),
\]  

(2.20)

and substituting Eq. (2.19) into Eq. (2.18), we find

\[
h(b; x) = C(b; x) - \sum_{a=1}^{b-1} h(a; x) C(b - a; x).
\]  

(2.21)

Using Eq. (2.20), it can be easily shown that for \( b \geq l \)

\[
C(b; x) = r^{b-l}.
\]  

(2.22)

Denoting the values of \( C(b; x) \) for \( b < l \) by \( c_b(x) \), we have

\[
c_b(x) = \sum_{y_1 \cdots y_{b+1}} f_0(x, y; 0) f_b(x, y; 0), \quad 0 < b < l,
\]  

(2.23)
and thus
\[ C_b = \begin{cases} c_b, & 0 < b < l, \\ r^{b-i}, & b \geq l. \end{cases} \tag{2.24} \]

As evident from Eq. (2.23), the set of indices \( c_b(x) \in \{0, 1\} \), with \( b = 1, 2, \ldots, l-1 \) measure the auto-correlations of \( x \). They are referred to as the bit-vector \( c = (c_1, c_2, \ldots, c_{l-1}) \) associated with \( x \), and were studied by Harborth \[27\] and later in considerable detail by Guibas and Odlyzko \[8, 9, 10\].

### 2.3 Bit-vectors

From the definition, Eq. (2.23), it is clear that \( c_b = 1 \) if and only if the string \( x \) shifted by an amount \( b \) relative to itself coincides on the overlapping part. Conversely \( c_b = 0 \), if the overlapping part does not coincide. It turns out that the set of \( r^l \) possible words \( x \) of length \( l \) are partitioned into equivalence classes with respect to their bit-vectors \( c = (c_1, c_2, \ldots, c_{l-1}) \) and that the possible classes are independent of the number of letters \( r \) (as long as \( r \geq 2 \)) \[9\]. Tables 1 and 2 list the sets of possible bit-vectors up to \( l = 8 \) along with the number of elements in their respective equivalence classes for \( r = 2, 3, 4 \).

We see that the definition of \( c_b(x) \) imposes strong conditions on the possible values of the \( l-1 \) bits of a bit-vector and it turns out that the resulting bit-vectors have interesting properties \[8, 28\], of which we will mention only the most relevant ones.

For example, if \( c_p = c_q = 1 \) with \( p < q \) this implies that \( c_t = 1 \) for all \( t \) of the form \( t = p + i(q-p) \) with, \( i = 0, 1, 2, \ldots \) and \( t < l \). This is referred to as the forward propagation rule \[9\]. In particular, \( c_p = 1 \) implies that \( c_{ip} = 1 \) for all \( i, 1, 2, \ldots \) such that \( ip < l \). The latter result shows that \( p \) can be considered as a period. We define the fundamental period of a string \( x \), \( \chi(x) \), to be the smallest \( p \), with \( 0 < p < l \) such that \( c_p = 1 \). If \( x \) is such that its bit-vector is 000\( \cdots \)0 (all zeroes), we define \( \chi(x) = l \).

### 2.4 The \( n \)-match probability

Denote by \( p(n;m,x) \) the probability that that a randomly drawn \( k \)-string \( y \) contains a given \( l \)-string \( x \) precisely \( n \) times. We will refer to this as the \( n \)-match probability.

If we let the random variable \( N(x,y) \) denote the number of occurrences of \( x \) in \( y \), it follows that
\[ N(x,y) = \sum_{a=0}^{m} f_a. \tag{2.25} \]

Thus the average number of matches \( \langle n \rangle \) and its second moment \( \langle n^2 \rangle \) are readily obtained as
\[ \langle n \rangle = \frac{1}{r^{m+l}} \sum_y \sum_{a=0}^{m} f_a = \frac{m + 1}{r^l} \tag{2.26} \]
and
\[ \langle n^2 \rangle = \frac{1}{r^{m+l}} \sum_y \sum_{a<b} f_a f_b. \tag{2.27} \]
Table 1: Equivalence classes of bit-vectors and their number of elements. The table shows the bit-vectors $c = (c_1, c_2, \ldots, c_{l-1})$ associated with strings of length $l = 2, 3, 4, 5$ and 6 and the number of elements in these equivalence classes for $r = 2, 3$ and 4 letter alphabets.

|   | $r = 2$ | $r = 3$ | $r = 4$ |
|---|---------|---------|---------|
| 0 | 2       | 6       | 12      |
| 1 | 2       | 3       | 4       |
| 00| 4       | 18      | 48      |
| 01| 2       | 6       | 12      |
| 11| 2       | 3       | 4       |
| 000| 6      | 48      | 180     |
| 001| 6      | 24      | 60      |
| 010| 2      | 6       | 12      |
| 111| 2      | 3       | 4       |
| 0000| 12    | 144     | 720     |
| 0001| 10    | 66      | 228     |
| 0010| 4      | 18      | 48      |
| 0011| 2      | 6       | 12      |
| 0101| 2      | 6       | 12      |
| 1111| 2      | 3       | 4       |
| 00000| 20   | 414     | 2832    |
| 00001| 22    | 210     | 948     |
| 00010| 6      | 48      | 180     |
| 00011| 6      | 24      | 60      |
| 00100| 4      | 18      | 48      |
| 00101| 2      | 6       | 12      |
| 01010| 2      | 6       | 12      |
| 11111| 2      | 3       | 4       |

Table 2: Equivalence classes of bit-vectors and their number of elements. The table shows the bit-vectors $c = (c_1, c_2, \ldots, c_{l-1})$ associated with strings of length $l = 7$ and 8 and the number of elements in these equivalence classes for $r = 2, 3$ and 4 letter alphabets.

|   | $r = 2$ | $r = 3$ | $r = 4$ |
|---|---------|---------|---------|
| 000000| 40    | 1242    | 11328   |
| 000001| 38    | 606     | 3732    |
| 000010| 16    | 162     | 768     |
| 000011| 12    | 72      | 240     |
| 000100| 8     | 54      | 192     |
| 000101| 2     | 12      | 36      |
| 000111| 2     | 6       | 12      |
| 001001| 6     | 24      | 60      |
| 010101| 2     | 6       | 12      |
| 111111| 2     | 3       | 4       |
| 0000000| 74    | 3678    | 45132   |
| 0000001| 82    | 1866    | 15108   |
| 0000010| 26    | 462     | 3012    |
| 0000011| 22    | 210     | 948     |
| 0000100| 16    | 162     | 768     |
| 0000101| 8     | 54      | 192     |
| 0000111| 6     | 24      | 60      |
| 0001000| 6     | 48      | 180     |
| 0001001| 6     | 24      | 60      |
| 0010010| 4     | 18      | 48      |
| 0010011| 2     | 6       | 12      |
| 0101010| 2     | 6       | 12      |
| 1111111| 2     | 3       | 4       |
The latter expression can be worked out using Eqs. (2.20), (2.22) and (2.23) and we find for the variance

\[ \sigma_n^2 = \frac{m+1}{r^l} + \frac{1}{r^{2l}} [(m+1)(1-2l) + l(l-1)] + \frac{1}{r^l} \sum_{b=1}^{l-1} (m+1-b)c_b(x). \] (2.28)

Eq. (2.28) is a special case of a result due to Kleffe and Borodovsky [29], who considered general distributions of random letters.

Let \( I_{n,m}(a_1, a_2, \ldots, a_n; x, y) \) be the function that takes on the value 1 when \( n \) matches occur that are located at positions \( a_1, a_2, \ldots, a_n \), with \( 0 < a_1 < a_2 < \cdots < a_n < m \) and zero otherwise,

\[
I_{n,m}(a_1, a_2, \ldots, a_n; x, y) = \left[ \prod_{i=1}^{a_1-1} (1 - f_{i_1}) \right] f_{a_1} \left[ \prod_{i_2=a_1+1}^{a_2-1} (1 - f_{i_2}) \right] f_{a_2} \cdots \left[ \prod_{i_n=a_{n-1}+1}^{a_n-1} (1 - f_{i_n}) \right] f_{a_n} \left[ \prod_{i_{n+1}=a_n+1}^{m} (1 - f_{i_{n+1}}) \right].
\] (2.29)

In terms of \( I_{n,m}(a_1, a_2, \ldots, a_n; x, y) \) we can write \( p(n; m, x) \) as

\[
p(n; m, x) = \sum_{a_1 < a_2 < \cdots < a_n} \frac{1}{r^{m+l}} \sum_y I_{n,m}(a_1, a_2, \ldots, a_n; x, y). \] (2.30)

Analogously to the reasoning leading from Eq. (2.18) to Eq. (2.19), it can be shown that the sum over \( y \) factorizes \( I_{n,m}(a_1, a_2, \ldots, a_n; x, y) \) as

\[
\frac{1}{r^{m+l}} \sum_y I_{n,m}(a_1, a_2, \ldots, a_n; x, y) = \frac{1}{r^{m+l}} d(a_1) \left[ \prod_{i=1}^{n-1} h(a_{i+1} - a_i) \right] d(m-a_n), \] (2.31)

where \( d \) and \( h \) are as defined in Eqs. (2.11) and (2.15). Thus \( p(n; m, x) \) becomes

\[
p(n; m, c) = \sum_{a_1 < a_2 < \cdots < a_n} \frac{1}{r^{m+l}} d(a_1) \left[ \prod_{i=1}^{n-1} h(a_{i+1} - a_i) \right] d(m-a_n), \] (2.32)

where we have changed the argument of the distribution function to \( p(n; m, c) \), to emphasize that the distribution really depends on the bit-vector \( c \) only. It is readily seen that the sum over the positions \( a_i \) is an \( n+1 \) fold convolution of \( d \) and \( h \). To simplify the results as well as to be able to obtain asymptotic expressions, we next introduce generating functions.

### 2.5 Generating functions

Define the generating function \( g(z) \) associated with a sequence \( g(b) \) by

\[
g(z) = \sum_{b=0}^{\infty} z^b g(b). \] (2.33)
From Eqs. (2.20), (2.22) and (2.23) we find

\[ C(z; \mathbf{x}) = c(z; \mathbf{x}) + \frac{z^l}{1 - zr}, \tag{2.34} \]

where \( c(z; \mathbf{x}) \) is a polynomial of degree \( l - 1 \),

\[ c(z; \mathbf{x}) = \sum_{b=1}^{l-1} z^b c_b(\mathbf{x}). \tag{2.35} \]

It is useful to also define the polynomial of degree \( l \), \( \lambda(z; \mathbf{c}) \), as

\[ \lambda(z; \mathbf{c}) = z^l + r^l (1 - z) \left[ 1 + c(z/r; \mathbf{x}) \right]. \tag{2.36} \]

Using Eqs. (2.14), (2.21), (2.22) and (2.36), we see that the generating function of \( h \) and \( d \) are given in terms of \( c(z/r; \mathbf{x}) \) as

\[ h(z/r; \mathbf{c}) = 1 - \frac{1}{1 + C(z/r; \mathbf{x})} = 1 - \frac{1}{1 + c(z; \mathbf{x}) + \frac{1}{r} \frac{z^l}{1 - z}}, \tag{2.37} \]

which can be written in terms of \( \lambda(z; \mathbf{c}) \) as

\[ h(z/r; \mathbf{c}) = 1 - r^l \frac{1 - z}{\lambda(z; \mathbf{c})}, \tag{2.38} \]

and likewise,

\[ d(z/r; \mathbf{c}) = \frac{1 - h(z/r; \mathbf{c})}{1 - z} = \frac{r^l}{\lambda(z; \mathbf{c})}. \tag{2.39} \]

The generating function for \( p(m; \mathbf{c}) \) thus becomes (8) [10]

\[ p(z; \mathbf{x}, 0) = \frac{1}{1 - z} \frac{1}{\lambda(z; \mathbf{c})}. \tag{2.40} \]

Turning next to the generating function of \( p(n; m, \mathbf{c}) \),

\[ p(n; z, \mathbf{c}) = \sum_{m=n}^{\infty} z^n p(n; m, \mathbf{c}), \tag{2.41} \]

we obtain (for \( n \geq 1 \))

\[ p(n; z, \mathbf{c}) = \frac{1}{r^n} d(z/r; \mathbf{c})^2 h(z/r; \mathbf{c})^{n-1}. \tag{2.42} \]

Eq. (2.42) is a special case of a more general result due to Régnier and Szpankowski [14] who consider a broader class of letter distributions, including inhomogeneous letter distributions as well as sequences of random letters generated from the steady-state of a Markov process.
As an aside, we can alternatively write \[ p(n; z, c) \] in terms of \( c(z/r) \) alone as

\[
p(n; z, c) = \frac{1}{r^l} \frac{[r^l(1-z)c(z/r) + z^l]^{n-1}}{[r^l(1-z)(1 + c(z/r)) + z^l]^{n+1}},
\]

(2.43)

or in terms of the matching probability \( p(z; c) \) as

\[
p(n; z, c) = r^l(1-z)^2p(z; c)^2 \left[ 1 - r^l(1-z)^2p(z; c) \right]^{n-1}.
\]

(2.44)

Note that from the last expression, we recover again Eq. (2.40) in terms of the generating functions,

\[
\sum_{n=1}^\infty p(n; z, c) = p(z; c).
\]

(2.45)

In fact for \( n = 0 \) we therefore have

\[
p(0; z, c) = \frac{1}{1-z} - p(m; c) = \frac{1}{1-z} \left[ 1 - \frac{1}{\lambda(z; c)} \right].
\]

(2.46)

### 2.6 Asymptotic behavior

Once the generating functions have been determined, the original functions can be obtained by an inverse transformation defined as follows: If \( f(z) \) is the generating function associated with \( f(b) \), then

\[
f(b) = \frac{1}{2\pi i} \oint_{\partial D} dz \frac{1}{z^{m+1}} f(z),
\]

(2.47)

where \( \partial D \) is the boundary of a domain \( D \) in the complex plane that includes the origin and on which \( f(z) \) is analytic.

Note that the generating functions of \( h(z; c) \), \( d(z; c) \), \( p(z; c) \), and \( p(n; z, c) \) are all rational functions, with their denominators involving \( \lambda(z; c) \) or its powers and that they all go to zero as \( \|z\| \to \infty \). For example, for the matching probability we have

\[
p(m, c) = \frac{1}{2\pi i} \oint_{\partial D} dz \frac{1}{z^{m+1}} \frac{1}{(1-z)\lambda(z; c)}.
\]

(2.48)

As we will show below, the behavior of \( p(m, c) \) for large \( m \) (and likewise for \( h \), \( d \) and \( p(n; m, c) \)) is dominated by the zeroes of \( \lambda(z; c) \) that are closest to the origin.

A numerical inspection of the zeroes of \( \lambda(z; c) \) for \( 2 < l < 10 \) and \( r = 2, 3, 4 \) shows that:

1. All zeroes \( z_i \) of \( \lambda \) are distinct,
2. The zero of smallest magnitude, \( z_1 \), is real, and greater but near 1 and
3. All other zeroes have magnitudes of the order \( \|z_i\| \sim r, i = 2, \ldots l \).

Fig. 1 shows a plot of the zeroes for \( l = 4, r = 2 \) and \( l = 8, r = 2, 3, 4 \). In fact, it can be rigorously proven that \( \lambda \) has a single (real) zero in a circular domain centered at \( z = 1 \) and of sufficiently small radius \( \epsilon \).

The asymptotic behavior of \( f(b) \) in Eq. (2.47) can be obtained by stretching the contour \( \partial D \) to infinity while circling around the zeroes of \( f(z) \) without including them.
Figure 1: Root Loci of the polynomial $\lambda(z;c)$, Eq. (2.36). The figures are for $(l,r)$-values (starting from the top left and going clockwise) $(4,2)$, $(8,2)$, $(8,3)$ and $(8,4)$. Plotted in each figure are the roots associated with the possible equivalence classes. For $l = 4$ these are $c = 000$ (+), $c = 001$ (*), $c = 010$ (diamonds) and $c = 111$ (triangles), while for the $l = 8$ cases they are $c = 0000000$ (+), $c = 0000001$ (*), $c = 0000010$ (diamonds), $c = 0000011$ (triangles) and we have shown the roots associated with the remaining classes as small dots. The dashed circles correspond to $\|z\| = 1$ and $\|z\| = r$ and have been inserted as a guide to the eye. All classes have a root near $z = (1,0)$. The remaining roots cluster around and beyond the circle $\|z\| = r$. 
The integral over the boundary at infinity turns out to yields no contribution since for the cases of interest \( f(z) \to 0 \) and the integrand is asymptotically of the order of at least \( 1/z^{m+1} \). This leaves the contributions from the zeroes of \((1 - z)\lambda(z; c)\), which are traversed counter-clockwise, if the contour at infinity is traversed clockwise.

Considering the matching probability, we find

\[
p(m, c) = -\sum_{i=0}^{N(\lambda)} \frac{1}{2\pi i} \oint_{\partial D_i} dz \frac{1}{z^{m+1}} \frac{1}{(1 - z)\lambda(z; c)},
\]

where \( \partial D_i \) is a clock-wise contour around the \( i \)th zero of \((1 - z)\lambda(z; c)\), \( N(\lambda) \) is the number of zeroes, and we assume that the zeroes are ordered such that \( z_0 = 1 < z_1 < \|z_2\| < \ldots < \|z_N\| \). Evaluating explicitly the residues for the first two poles we have,

\[
p(m, c) = 1 - A_1 \left( \frac{1}{z_1} \right)^{m+1} - \sum_{i=2}^{N(\lambda)} \frac{1}{2\pi i} \oint_{\partial D_i} dz \frac{1}{z^{m+1}} \frac{1}{(1 - z)\lambda(z; c)},
\]

with the residue \( A_1 \) given by

\[
A_1 = \frac{1}{\lambda(z_1; c)(1 - z_1)}.
\]

The remaining zeroes \( z_2, z_3, \ldots z_l \) of \( \lambda(z; c) \) are located near and beyond \( \|z\| \approx r \), so that in the limit of large \( m \), their relative contributions are smaller. We thus arrive at the asymptotic form

\[
p(m, c) \to 1 - A_1 \left( \frac{1}{z_1} \right)^{m+1}
\]

for large \( m \).

We can obtain approximate expressions for \( z_1 \) and thus an approximation of the asymptotic behavior as follows. With

\[
\lambda(z; c) = z^l + r^l (1 - z) \left[ 1 + c(z/r) \right]
\]

we see that when \( \|z\| \sim 1 \), the second term in the above equation is a large term, \( r^l \), multiplied with a term that will be small due to the \( z - 1 \) prefactor. The product of these two terms can be made of order 1, if \( z - 1 \sim 1/r^l \), which then can be made to cancel the first term \( z^l \) if \( z > 1 \). Using the Lagrange Inversion Formula, \( z - 1 \) can be expanded in a power series in \( 1/r^l \) [30]: Letting \( u = z - 1 \) and \( t = 1/r^l[1 + c(1/r)]^{-1} \), the equation \( \lambda(z; c) = 0 \) can be written in the form

\[
u = t \phi(u),
\]
where
\[ \phi(u) = (1 + u)^l \frac{1 + c \left( \frac{1}{r} \right)}{1 + c \left( \frac{1+u}{r} \right)}. \] (2.55)
is a formal power series in \( u \). Thus
\[ z_1 = 1 + u(t) = 1 + \sum_{i=1}^{\infty} u_i t^i, \] (2.56)
with
\[ u_i = \frac{1}{i!} \frac{d^{i-1} \phi^i}{du^{i-1}} \bigg|_{u=0}. \] (2.57)

One finds to leading and sub-leading order
\[ u_1 = 1, \] (2.58)
and
\[ u_2 = l - \frac{1}{r} \frac{c'(1/r)}{1 + c(1/r)}, \] (2.59)
with
\[ c'(1/r) = \sum_{i=1}^{l-1} ic_i \left( \frac{1}{r} \right)^{i-1} \] (2.60)
so that to leading order we have
\[ z_1 = 1 + \frac{1}{1 + c(1/r)} \frac{1}{r^l} \] (2.61)
The residue \( A_1 \) can be evaluated similarly, and we find to order \( 1/r^l \) that
\[ A_1 = 1 - \frac{1}{r^l} \frac{l c'(1/r)}{[1 + c(1/r)]^2}. \] (2.62)
Note that \( A_1 \) is of order one.

The asymptotic behavior of \( h(b; c) \) and \( d(b; c) \) can be worked out in a similar manner. For large \( b \) we find,
\[ h(b) \to h_{asy}(b) = \frac{1}{r^l} \frac{A_1}{z_1} \left[ r^l (z_1 - 1) \right]^2 \left( \frac{r}{z_1} \right)^b \] (2.63)
and
\[ d(b) \to d_{asy}(b) = \frac{A_1}{z_1} \left[ r^l (z_1 - 1) \right] \left( \frac{r}{z_1} \right)^b. \] (2.64)
where \( A_1 \) is given by Eq. (2.51), \( z_1 \) is the smallest root of \( \lambda(z; c) \) and from the expansion of \( z_1 \), Eq. (2.56), we see that the terms in square brackets are of order one.
Figure 2: Comparison of the asymptotic form, Eq. (2.52), with the exact matching probabilities \( p(m; c) \). The figure shows the matching probability for \( l = 4, r = 2, \) and \( 6 < k = m + l < 88 \). The open circles correspond to the numerically obtained matching probabilities for \( c = 000, 001, 010 \) and 111 (from top to bottom). The lines correspond to the asymptotic form, Eq. (2.52), with \( A_1 \) and \( z_1 \) calculated numerically. Inset: Plot of \( p(m; c) \) for intermediate values of \( m \). The symbols are as in the main figure. The equivalence classes are (from top to bottom): 000, 001, 010 and 111.

Taking the asymptotic forms of \( h \) and \( d \) to calculate the \( n \)-match distribution one finds

\[
p^{(1)}(n; m, c) = A_1 \left( \binom{m+n}{n} \right) \left[ A_1 r^l \left( 1 - \frac{1}{z_1} \right) \right]^n \left( \frac{1}{z_1} \right)^{m+1-n}.
\]  

(2.65)

Figure 2 shows a comparison of the asymptotic form, Eq. (2.52), with the exact matching probabilities \( p(m; c) \). The figure shows the matching probability for \( l = 4, r = 2, \) and \( 6 < k = m + l < 104 \). For \( l = 2 \) there are 4 equivalence classes: 000, 001, 010, and 111 (cf. Tables 1 and 2 for the case \( r = 2 \)). The open circles correspond to the numerically obtained matching probabilities for \( c = 000, 001, 010 \) and 111 (from top to bottom). For \( m \leq 20 \) (\( k \leq 24 \)), \( p(m; c) \) was obtained by direct enumeration of all possible strings and checking for matches. For \( m > 20 \) a sampling algorithm was used: for each value of \( k \), \( 10^6 \) strings of length \( k \) were generated randomly and the matching probability was obtained by counting the matching strings of the sample. The solid lines correspond to the asymptotic form, Eq. (2.52), with \( z_1 \)
and $A_1$ calculated numerically, from Eqs. (2.36) and (2.51) for each of the equivalence classes. The inset shows $p(m; c)$ for intermediate values of $m$, where we do not expect the asymptotic form to be very good.

The discrepancies become much more severe when we consider the $n$-match distribution. Fig. 3 shows the $n$-match distributions for a 4 letter binary string inside a random string of length $k = 256$ for the four possible equivalence classes $c = 000$ (top left), $c = 001$ (top right), $c = 010$ (bottom left), and $c = 111$ (bottom right). The solid circles are the exact matching probabilities that were obtained numerically using the algorithm described above. The dotted line corresponds to the approximation Eq. (2.65), normalized by an overall constant. The dashed line corresponds to the gaussian approximation of Kleffe and Borodovsky [29], while the dot-dashed line is the compound poisson approximation of Chrysaphinou and Papastavridis [11], Geske et al. [12], and Schbath [17].

Note that while the approximation Eq. (2.65) performs very poorly, the gaussian and compound-poisson distributions approximate well the true distribution only for some equivalence classes $c$, but fail for others, as was noted by Robin and Schbath [19]. The solid line on the other hand, is the single analytical result of this article and agrees well with the actual distributions. We now turn to the description of the $n$-match probability in terms of the (configurational) partition function of a 1d lattice gas.

3 The $n$-match probability as the partition function of a 1d lattice gas

In this section we present the statistical mechanics approach to calculating the $n$-match distribution function. We first map the problem into one of calculating the (configurational) partition sum of a 1d-lattice gas. We next analyze the interaction emerging in such a description, then set up a virial expansion leading to an approximate evaluation of the partition function and finally discuss asymptotic limits.

3.1 The $n$-particle partition function

Our starting point is Eq. (2.32), which we reproduce below for convenience,

$$p(n; m, c) = \sum_{a_1 < a_2 < \cdots < a_n} \frac{1}{r^{m+l}d(a_1)} \left[ \prod_{i=1}^{n-1} h(a_{i+1} - a_i) \right] d(m - a_n).$$

(3.1)

with $d$ and $h$ as defined in Eqs. (2.11) and (2.15). The expression above for $p(n; m, c)$ already resembles the partition function of a gas of $n$ particles with particle boundary interactions proportional to $-\ln d$ and nearest neighbor particle-particle interactions proportional to $-\ln h$. In order to make this analogy work, we need to consider what we mean by the free-particle, i.e. no interaction limit.
Figure 3: The $n$-match distribution for matching a $l = 4$ letter binary string $x$ inside a random string of length $k = 256$, for $x = 0001$ (top left), $x = 1001$ (top right), $x = 1010$ (bottom left) and $x = 1111$ (bottom right). The circles are the exact probabilities, the dotted line corresponds to the approximation Eq. (2.65) (normalized by an overall constant) and the dashed and dashed-dotted lines correspond to the Gaussian and compound Poisson approximation (see text for details). The solid line is the analytical result of this paper.

Note that $d(b)$ and $h(b)$ are conditional matching weights. For example, $h(b)$ is the weight of the compound event: given a match at position $a$ what is the likelihood that the next match is at $a + b$. The asymptotic behavior of $d(b)$ and $h(b)$, Eqs. (2.64) and (2.63),
can be interpreted to correspond to the approximation when the correlations inherent in
the compound events are ignored. Thus the ratios $d(b)/d_{asy}(b)$ and $h(b)/h_{asy}(b)$ measure
the strength of the correlations in such events.

It is therefore natural to define the particle-boundary and particle-particle interactions,
$U^{bou\text{n}}(b)$ and $U(b)$, respectively as

$$e^{-\beta U^{bou\text{n}}(b)} = \frac{d(b)}{d_{asy}(b)}$$ \hspace{0.5cm} (3.2)

$$e^{-\beta U(b)} = \frac{h(b)}{h_{asy}(b)}.$$ \hspace{0.5cm} (3.3)

We thereby obtain meaningful physical interactions that vanish as $b \to \infty$. Note that since
the potentials do not have any characteristic scale, a temperature by itself is meaningless
and we will write ”energies” always with the pre-factor $\beta$, i.e. in dimension-less units.

The (configurational) partition function, Eq. (2.32) can now be written in terms of
these interactions as

$$p(n; m, c) = A_1^{m+1} \frac{\beta \mu}{z_1^{m+1}} \sum_{a_1 < a_2 < \ldots < a_n} e^{-\beta H_n(a_1, \ldots, a_n)},$$ \hspace{0.5cm} (3.4)

with

$$e^{\beta \mu} = A_1 \frac{r^l}{z_1} (z_1 - 1)^2$$ \hspace{0.5cm} (3.5)

and the Hamiltonian given by

$$H_n(a_1, \ldots, a_n) = U^{bou\text{n}}(a_1) + U^{bou\text{n}}(m - a_n) + \sum_{i=1}^{n-1} U(a_{i+1} - a_i)$$ \hspace{0.5cm} (3.6)

Eq. (2.65) corresponds to the free-particle limit ($U = U_b = 0$), which in the probability
language is the limit of all correlations suppressed. Before proceeding, it is instructive to
study these interactions in more detail.

### 3.2 Interactions

Consider the particle-particle interaction first. From Eqs. (2.63) and (3.3) we find that

$$e^{-\beta U(b)} = \left[ \frac{z_1}{A_1} \frac{1}{r^l(z_1 - 1)^2} \right] h(b) r^{l-b} z_1^{b},$$ \hspace{0.5cm} (3.7)

where the term in square brackets is of order one, with respect to the small parameter
$1/r^l$, cf. Eqs. (2.62) and (2.56).

Figure 4 shows the particle particle interactions for words of length $l = 4$ and $l = 6$
as parameterized by their associated equivalence classes $c$. The potentials are plotted
against distance measured in units of the word length $l$ and have been vertically offset for
clarity with the dashed lines representing $U = 0$. The crosses on the dashed line indicate
that the associated potential at that value is $+\infty$. We see that the potential have infinite
values only for $b \leq l$. Also, the values of the potential in the regime $b \leq l$ are generally much bigger than in the regime $b > l$, meaning that the potential is stronger in the former region. We will refer to the region $b \leq l$ and $b > l$ as the core and tail of the interaction, respectively. For a given length $l$ and depending on $c$, we also see that the interactions have different features. For $c = 0 \cdots 0$ the interaction has a hard-core of size $l$ followed by a repulsive tail, while for $c = 1 \cdots 1$ the interaction has a strongly attractive component at $b = 1$, followed by a hard-core region for $1 \leq b \leq l$, that goes over into an oscillatory but decaying tail. The potentials for the other values of $c$ seem to be a mixture of these two types of behavior.

Figure 5 shows the behavior of the potentials associated with the equivalence classes $c = 0 \cdots 0$ (left) and $c = 1 \cdots 1$ (right) in their dependence on the word length $l$. For both equivalence classes we see that the tail of the interaction becomes weaker as $l$ increases. When $c = 0 \cdots 0$, the core is hard-core and only the core-size $l$ changes. The situation is different for $c = 1 \cdots 1$. For the $c = 1 \cdots 1$ family of interactions we see that the attractive part of the core actually becomes stronger with increasing $l$. It turns out that the same is also true for the other equivalence classes, namely with increasing $l$, the cores of the interactions become stronger, while the tails become weaker.

In summary, Figs. 4 and 5 suggest the following generic features of the interactions: (i) a strong core $b \leq l$, followed by a weak tail for $b > l$, and, (ii) for a given family of interactions, as $l$ increases the core of the interaction tends to become stronger, while the tail of the interaction becomes weaker.

These observations can be readily proven from the small $b$ behavior of $h(b)$, which in turn can be extracted from the recursion for $h(b)$, Eqs. (2.21) and (2.24). Thus we find for $b < l$

$$h(b) = \begin{cases} c_b, & \text{if } \chi \text{ does not divide } b, \\ 1, & \text{if } b = \chi, \\ 0, & \text{otherwise}, \end{cases} \quad (3.8)$$

where $\chi$ is the fundamental period associated with $c$ that was defined at the end of Section 2.3. Recall that by definition, $h(0) = 0$.

Thus the interaction in the core region can be written as

$$\beta U(b) = -\ln h(b) + b \ln \left( \frac{r}{z_1} \right) - l \ln r + \beta U_0, \quad b \leq l \quad (3.9)$$

where

$$\beta U_0 = \ln \left( \frac{z_1}{A_1 [r^l(z_1 - 1)]^2} \right) \quad (3.10)$$

is a constant that is of order $1/r^l$, since the argument of the logarithm is of order 1 to the same order.
Figure 4: Plot of the effective potentials $\beta U(b)$, Eq. (3.7), associated with the equivalence classes $c$ of strings of lengths $l$. The potentials are plotted against distance measured in units of the word length $l$. Note that the potentials have been vertically offset for clarity. The dashed lines represent the $U = 0$ lines for each potential. The crosses on the line $U=0$ indicate that the associated potential at that point is $+\infty$. Left: Interparticle potentials associated with words of length $l = 4$, for which the possible equivalence classes are $c = 000, 010$ and $111$, as indicated in the figure. Right: same as left but for $l = 6$. Notice how the attractive part of the interaction emerges and grows stronger as the fundamental period of the string decreases to 1 ($c = 1 \cdots 1$). The tail of the interaction corresponds to the regime $b/l > 1$.

We see that the interaction becomes $+\infty$, whenever $h(b) = 0$. This is certainly the case for $b < \chi$. Furthermore, since $r/z_1 > 1$, in the core region finite values of $U(b)$ increase with increasing $b$, as clearly seen in Fig. 4.

The first finite value of $U(b)$ occurs at $b = \chi$. From Eq. (3.9) we obtain for $U(\chi)$

$$\beta U(\chi) = \chi \ln \left( \frac{r}{z_1} \right) - l \ln r + O \left( \frac{1}{r^l} \right).$$

(3.11)

Thus it is apparent that for fixed $\chi$, $\beta U(\chi)$ becomes more negative as either $l$ or $r$ increase. In fact we see that to leading order, the dependence of $\beta U(\chi)$ on $l$ is linear, while its dependence on $r$ is logarithmic. Also for $\chi < l$ we see that $\beta U(\chi)$ is negative.

The case when $\chi = l$, corresponding to $c = 00\ldots0$, is a little more complicated. In that case the $\ln r$ terms in Eq. (3.11) cancel and we are left with a term $l \ln z_1$ which is of order $1/r^l$ as well and thus $\beta U_0$ cannot be neglected anymore. However this means that the potential is of order $1/r^l$, which turns out to be the correct scale of the strength of the tail and indeed decreases as $r$ or $l$ increase (see Fig. 5). To be specific, for $\chi = l$, it
Figure 5: Plot of the effective potentials $\beta U(b)$, Eq. (3.7), associated with the equivalence classes $c = 0 \cdots 0$ and $1 \cdots 1$ and their dependence on the lengths $l$. The potentials are plotted against distance measured in units of the word length $l$. Note that the potentials have been vertically offset for clarity. The dashed line represents the $U = 0$ line for each potential. The crosses on the line indicate that the associated potential at that value is $+\infty$. Left: Interparticle potentials associated with the equivalence class $c = 0 \cdots 0$ for words of length $l = 3, 4, 6$ and 8. Note that the interactions have a hard-core of size $b/l = 1$ followed by a repulsive tail. The strength of the tail weakens with increasing $l$. Right: Interparticle potentials associated with the equivalence class $c = 1 \cdots 1$ for words of length $l = 3, 4, 6$ and 8. Note that the interactions have an attractive part at $b = 1$, followed by a hard-core for $b/l < 1$, and a weak, oscillatory decaying tail. Also note the opposite behavior of the strength of the core and the tail: With increasing $l$, the strength of the attractive part of the core is seen to increase, while the strength of the tail decreases.

can be readily checked that $h(b) = r^{b-l}$ for $l \leq b < 2l$ and the potential in this regime thus becomes

$$\beta U(b) = \frac{2l + 1 - b}{r^l} + O\left(\frac{1}{r^2}\right),$$

(3.12)

where we have substituted the expansions of $A_1$ and $z_1$, Eqs. (2.62) and (2.56), respectively, to lowest non-trivial order. Thus we see that for $\chi = l$ the characteristic energy scale of the tail of the interaction scales like $\sim l/r^l$, and decreases as $l$ or $r$ increase.

The case of general $\chi$ and $c$ is similar, but the calculation are tedious yet straightforward. Rather than doing this, we will motivate the result by considering the value of the
interaction at \( b = l + \chi \), which is readily worked out from

\[
h(l + \chi) = \begin{cases} 
  r^x - h(l) - 1, & \text{for } \chi < l/2, \\
  r^x - h(l) - 1, - \sum_{\tau=\chi+1}^{l-1} h(\tau)c(l + \chi - 1), & \text{for } l/2 \leq \chi < l, \\
  r^x - h(l), & \text{for } \chi = l.
\end{cases}
\] (3.13)

This means that

\[
h(l + \chi) = r^x (1 - \epsilon),
\] (3.14)

where \( \epsilon r^x \) is at most \( l - \chi + 1 \) and hence of order \( l \). Substituting this result into the expression for \( U(b) \) along with the expansions for \( z_1 \) and \( A_1 \), one finds that the result is of the form

\[
-\beta U(l + \chi) = \frac{\alpha}{r^l} + O\left(\frac{1}{r^{2l}}\right),
\] (3.15)

where \( \alpha \) is a \( c \) and \( l \) dependent constant of order one.

To conclude, we find that the characteristic energy of the core of the interaction scales like \( -(l - \chi) \ln r \) (\( \chi < l \)), while the energy of the tail goes to leading order like \( 1/r^l \). These results are consistent with the behavior observed in Figs. 4 and 5.

Turning to the particle boundary interactions, note that Eq. (2.14), which can be conveniently written as

\[
\frac{d(b)}{r^b} = 1 - \sum_{a=1}^{b} \frac{h(a)}{p^a}
\] (3.16)

relates the properties of \( h \) to those of \( d \). We thus see that analogous results can be obtained for the boundary interaction \( U^{bound}(b) \) and we leave the details to the interested reader.

### 3.3 The Hamiltonian

The results of the previous section allow us to obtain approximate expressions for the probability \( p(n; m, c) \), by first approximating the effective Hamiltonian \( H_n \) and then carrying out the configurational sums. This is most easily done using generating functions.

Define the generating functions associated with Eqs. (3.2) and (3.3) as

\[
D(z) = \sum_{b=0}^{\infty} z^b e^{-\beta U(b)};
\] (3.17)

\[
H(z) = \sum_{b=0}^{\infty} z^b e^{-\beta U(b)}.
\] (3.18)

It is not difficult to show that in terms of the generating functions of \( d(b) \) and \( h(b) \), \( D(z) \) and \( H(z) \) are given by

\[
D(z) = e^{-\beta \mu} \left( z_{z_1} - 1 \right) d \left( \frac{zz_{z_1}}{r}; c \right)
\] (3.19)

\[
H(z) = e^{-\beta \mu} h \left( \frac{zz_{z_1}}{r}; c \right)
\] (3.20)
Using the convolution property, Eq. (3.21) can be written in terms of the generating functions $D(z)$ and $H(z)$ as

$$p(n; m, c) = \frac{A_1}{z_1^{m+1}} e^{\beta \mu} \frac{1}{2\pi i} \oint_{\partial D} dz \frac{1}{z_1^{m+1}} D^2(z) H^{n-1}(z),$$  \hspace{1cm} (3.21)

where the contour is again the boundary of a domain enclosing the origin inside of which $D^2(z) H^{n-1}(z)$ is analytic. Eq. (3.21) is the lattice analog of the partition function of a 1d gas with pairwise nearest neighbor interactions. The 1d continuum case has been treated in detail by Gürsey [33] (see also Fisher [34]).

Next, define the truncated generated functions $D_{\Lambda}(z)$ and $H_{\Lambda}(z)$ as

$$D_{\Lambda}(z) = \sum_{b=0}^{\Lambda-1} z^b e^{-\beta U_{\Lambda}(b)},$$  \hspace{1cm} (3.22)

$$H_{\Lambda}(z) = \sum_{b=0}^{\Lambda-1} z^b e^{-\beta U(b)}.$$  \hspace{1cm} (3.23)

It is readily seen that these generating functions are associated with the Boltzmann factor of an interaction that has been cut-off at $b \geq \Lambda$. The idea is that since, by construction, the interactions decay to zero at large distances, introducing a finite cut-off $\Lambda$ will introduce only a small and controllable error in the overall calculation. In what follows, we will use this to set up a perturbation expansion of the probability distribution. We need to note however that since the result has to be a normalized distribution, setting the potential to zero beyond the cut-off will destroy the normalization of the distribution. Indeed there are at least two ways to handle the interaction beyond the cut-off: (i) we can either set the interaction to a constant $U_{\Lambda}$ for $b \geq \Lambda$ and eventually choose $U_{\Lambda}$ such that the distribution is normalized, or (ii) we take the interaction beyond $\Lambda$ to be rapidly decaying. It turns out that the calculation can be done for either of the cases. The approximation by a constant potential beyond the cut-off lends itself readily for obtaining error bounds, as we will sketch below. On the other hand, it turns out that the tail of the actual interactions does asymptotically decay exponentially. Thus letting the interaction decay exponentially beyond the cut-off turns out to be a very good approximation and we will calculate the probability distributions in this way.

Consider the case of a constant potential beyond the cut-off first and define the approximate interaction $\hat{U}(b)$ as

$$\hat{U}_{\Lambda}(b) = \begin{cases} U(b), & b < \Lambda \\ U_{\Lambda}, & b \geq \Lambda, \end{cases}$$  \hspace{1cm} (3.24)

with the corresponding generating function given by

$$\hat{H}_{\Lambda}(z) = \sum_{b=0}^{\infty} z^b e^{-\beta \hat{U}_{\Lambda}(b)} = H_{\Lambda}(z) + e^{-\beta U_{\Lambda}} \frac{z^\Lambda}{1-z},$$  \hspace{1cm} (3.25)
Since \( d(z) \) is related to \( h(z) \) via Eq. (2.39), this implies a corresponding boundary interaction which can be worked out as

\[
\hat{D}_\Lambda(z) = \sum_{b=0}^{\infty} z^b e^{-\beta \hat{U}_\Lambda(b)} = D_\Lambda(z) + e^{-\beta U_\Lambda} \frac{z^\Lambda}{1 - z}. \tag{3.26}
\]

Define the approximation to \( p(n; m, c) \), Eq. (3.21), as

\[
\hat{p}(n; U_\Lambda, m, c) = \frac{A_1 e^{\beta \mu}}{z_1^{m+1}} \frac{1}{2\pi i} \oint_{\beta D} dz \frac{1}{z^{m+1}} \hat{D}_\Lambda^2(z) \hat{H}^{n-1}(z), \tag{3.27}
\]

It is clear that as \( \Lambda \to \infty \) we must have \( U_\Lambda \to 0 \), since an increasingly larger part of the true interactions is kept. By using the definition of \( U_\Lambda(b) \) and writing \( \hat{p}(n; U_\Lambda, m, c) \) in the partition sum form of Eq. (2.32), it can readily be verified that if

\[
U_- \leq U_\Lambda \leq U_+
\]

this implies that

\[
\hat{U}_-(b) \leq \hat{U}_\Lambda(b) \leq \hat{U}_+(b)
\]

for all values of \( b \), which in turn implies that

\[
\hat{p}(n; U_+, m, c) \leq \hat{p}(n; U_\Lambda, m, c) \leq \hat{p}(n; U_-, m, c).
\]

Thus by choosing \( U_+ \) and \( U_- \) as

\[
U_+ = \max_{b \geq \Lambda} \left\{ U(b), U^{\text{boun}}(b) \right\}, \tag{3.31}
\]
\[
U_- = \min_{b \geq \Lambda} \left\{ U(b), U^{\text{boun}}(b) \right\} \tag{3.32}
\]

one could in principle obtain error bounds on the approximate distribution, which will become tighter as \( \Lambda \to \infty \). We will not pursue this any further in the present article, but instead perform the calculation with an exponentially decaying interaction beyond the cut-off \( \Lambda \).

Recall that the tail of the true interaction is due to the other zeroes of \( \lambda(z; c) \), which are located a distance \( \sim r \) from the origin, (see Fig. 1). Thus superposed on the asymptotic behavior of \( h(b) \), which we have shown to fall-off like \( z_1^{-b} \), there will be terms that decay more rapidly and roughly as \( r^{-b} \), since \( z_1 < r \). In fact it is the latter that are responsible for the asymptotic behavior of the interactions. For \( b \) large, we therefore take approximately

\[
h(b) \approx e^{\beta \mu z_1^{-b}} + \gamma e^{\beta \mu r^{-b}} \tag{3.33}
\]

which upon taking logarithms and factoring out the first terms implies that asymptotically

\[
\beta U(b) \approx \beta \mu b \ln z_1 - \gamma \left( \frac{z_1}{r} \right)^b, \tag{3.34}
\]
where we have neglected higher order terms $\gamma^k(z_1/r)^k$. Of course, with increasing cut-off $\Lambda$, the residual tail will be less important.

This suggest taking the following approximate interactions:

$$\hat{U}_\Lambda(b) = \begin{cases} U(b), & b < \Lambda \\ -\gamma \left( \frac{z_1}{r} \right)^b, & b \geq \Lambda, \end{cases}$$

(3.35)

with the corresponding approximate generating function given by

$$\hat{H}_\Lambda(z) = \sum_{b=0}^{\infty} z^b e^{-\beta U_\Lambda(b)} = H_\Lambda(z) + \gamma \left( \frac{z_1}{r} \right)^\Lambda \frac{z^\Lambda}{1 - \frac{z_1}{r} z} + \frac{z^\Lambda}{1 - z}. \quad (3.36)$$

Since $d(z)$ is related to $h(z)$ via Eq. (2.39), this implies a corresponding approximate interaction for the boundary interaction, which can be worked out,

$$\hat{D}_\Lambda(z) = \sum_{b=0}^{\infty} z^b e^{-\beta U_\Lambda(b)} = D_\Lambda(z) + \gamma \left( \frac{z_1}{r} \right)^\Lambda \frac{z^\Lambda}{1 - \frac{z_1}{r} z} + \frac{z^\Lambda}{1 - z}. \quad (3.37)$$

Denoting the generating function of the approximate tail of the interaction as

$$\Gamma(z) = \gamma \left( \frac{z_1}{r} \right)^\Lambda \frac{z^\Lambda}{1 - \frac{z_1}{r} z}, \quad (3.38)$$

$\hat{p}(n; m, c)$ becomes

$$\hat{p}(n; \gamma, m, c) = A_{1} e^{\beta \mu n} \frac{1}{2\pi i} \oint_{\partial D} dz \frac{1}{z^{m+1}} \hat{D}_\Lambda^2(z) \hat{H}_\Lambda^{n-1}(z). \quad (3.39)$$

What therefore remains to be done is to evaluate the contour integral, Eq. (3.39), which can be carried out by the method of stationary phase, which in the context of generating functions is also known as Hayman’s method [30]:

### 3.4 Distributions

Write the integral in Eq. (3.39) as

$$I = \frac{1}{2\pi i} \oint_{\partial D} dz \frac{1}{z^{m+1}} f(z). \quad (3.40)$$

Then for large $m$, the value of the integral is given approximately by

$$I \approx \left( \frac{1}{u_m} \right)^m \sqrt{2\pi b_m} f(u_m), \quad (3.41)$$

where $u_m$ is the smallest positive real root of the equation

$$m = u \frac{d}{du} \ln f(u) \quad (3.42)$$
and $b_m$ is given by

$$b_m = u \frac{d}{du} \ln f(u) + u^2 \frac{d^2}{du^2} \ln f(u). \quad (3.43)$$

Applying Hayman’s method to the integral, Eq. (3.39), we let

$$f(u) = \hat{D}_A^2(u) \hat{H}_A^{n-1}(u) \quad (3.44)$$

and find after a little bit of algebra

$$m = u \frac{d}{du} \ln f(u)$$

$$= \frac{2}{x} \left[ 1 + \Lambda x + x^2(1 + x)^{\Lambda-2} \left( \hat{D}_A' \left( \frac{1}{1+x} \right) + \Gamma' \left( \frac{1}{1+x} \right) \right) \right]$$

$$+ \frac{n-1}{x} \left[ 1 + \Lambda x + x^2(1 + x)^{\Lambda-2} \left( \hat{H}_A' \left( \frac{1}{1+x} \right) + \Gamma' \left( \frac{1}{1+x} \right) \right) \right]$$

$$= \frac{2}{1 + \Lambda x + x^2(1 + x)^{\Lambda-2} \left[ \hat{D}_A' \left( \frac{1}{1+x} \right) + \Gamma' \left( \frac{1}{1+x} \right) \right]}$$

$$+ \frac{n-1}{1 + \Lambda x + x^2(1 + x)^{\Lambda-2} \left[ \hat{H}_A' \left( \frac{1}{1+x} \right) + \Gamma' \left( \frac{1}{1+x} \right) \right]}$$

$$+ \frac{2}{x} \left[ \hat{D}_A \left( \frac{1}{1+x} \right) + \Gamma \left( \frac{1}{1+x} \right) \right]$$

$$+ \frac{n-1}{x} \left[ \hat{H}_A \left( \frac{1}{1+x} \right) + \Gamma \left( \frac{1}{1+x} \right) \right]$$

where we have parameterized $u$ as

$$u = \frac{1}{1 + x}. \quad (3.46)$$

Since we are interested in solutions for large $m$, it is clear from the above that to leading order $x \propto 1/m$. Multiplying both sides of the above equation by $x$ and expanding the fractions in a power series around $x = 0$, we obtain

$$mx = (n + 1) \{ 1 + \epsilon_1 x + \epsilon_2 x^2 + \ldots \} \quad (3.47)$$

The first two orders can be readily worked out, yielding

$$\epsilon_1 = \Lambda - \tilde{\gamma} \xi - \frac{2 \hat{D}_A(1) + (n - 1) \hat{H}_A(1)}{n + 1} \quad (3.48)$$

and

$$\epsilon_2 = \frac{1}{n + 1} \left\{ 2 \hat{D}_A^2(1) + (n - 1) \hat{H}_A^2(1) + 2 \left[ 2 \hat{D}_A(1) + (n - 1) \hat{H}_A(1) \right] \right\}$$

$$- \left[ 2(\Lambda - \tilde{\gamma} \xi) - 1 \right] \left[ 2 \hat{D}_A(1) - (n - 1) \hat{H}_A(1) \right], \quad (3.49)$$

where

$$\xi = \frac{1}{1 - \frac{2}{r}}, \quad (3.50)$$

and $\tilde{\gamma} = \gamma (1 - 1/\xi)^\Lambda$.

Rewriting Eq. (3.47) in a form suitable for Lagrange’s Inversion Formula,

$$x = \frac{n + 1}{m - (n + 1) \epsilon_1} \left\{ 1 + \epsilon_2 x^2 + \ldots \right\}, \quad (3.51)$$
We obtain an expansion of $x$ in terms of $(n+1)/(m-(n-1)\epsilon_1)$ and the coefficients $\epsilon_i$ as

$$x = \frac{n+1}{m-(n-1)\epsilon_1} + \left[ \frac{n+1}{m-(n-1)\epsilon_1} \right]^3 \epsilon_2 \ldots .$$  \hspace{1cm} (3.52)

The term $b_m$ can be worked out in a similar manner and we find

$$b_m = m + \frac{n+1}{x^2} - (n+1)(\epsilon_1 + \epsilon_2) + \ldots ,$$  \hspace{1cm} (3.53)

where the omitted terms are of order $x$ and higher.

Combining Eqs. (3.39), (3.41), (3.44), with Eqs. (3.52) and (3.53) we finally obtain

$$\hat{p}(n; \gamma, m, c) \approx A_1 e^{\beta \mu n} \frac{1}{z_{m+1}^\lambda} (1+x)^m D_\lambda^2 \left( \frac{1}{1+x} \right) H_{\lambda^{-1}} \left( \frac{1}{1+x} \right) \frac{1}{\sqrt{2\pi b_m}}.$$  \hspace{1cm} (3.54)

The strength of the tail, $\tilde{\gamma}$, is still underdetermined and we will determine it by fitting the approximate tail to the actual interaction in the interval $b \in [\Lambda, \Lambda + l - 1]$. Note that this way there are no adjustable parameters and since the tail is only approximate, the normalization is not perfect and is found to vary by a few percent. Alternatively, one can choose $\tilde{\gamma}$ such that normalization is achieved. In either of the cases the distributions do not vary significantly, meaning that for a certain range of $\tilde{\gamma}$ values, the shape of the distribution is robust.

The solid lines in Fig. 3 show the approximate distribution, Eq. (3.54), for the four equivalence classes associated with words of length $l = 4$ and with $r = 2$, $k = 256$. We will refer to this approximation as the liquid theory approximation. In this and all the other results that we will present, the cut-off $\Lambda$ was chosen as $\Lambda = 3l$ and $x$ was expanded to 2nd order. The dashed lines in Fig. 3 are the Gaussian approximation of Kleffe and Borodovsky (KB) [29] with the distribution mean and variance given by Eqs. (2.26) and (2.28). The dot-dashed lines are the compound poisson (CP) approximation of Chrysaphinou and Papastavridis [11], Geske et al. [12] and Schbath [17].

The variation between actual and approximate distributions, $p(n)$ and $\hat{p}(n)$, can be quantified by the total variational distance [31] between the two distributions and is defined as

$$d_{TV}(p, \hat{p}) = \frac{1}{2} \sum_n \| \hat{p}(n) - p(n) \| .$$  \hspace{1cm} (3.55)

Table 3 shows the variational distances between the actual and approximate distributions depicted in Fig. 3 ($l = 4$) and $k = 256$.

We see that the (un-normalized) liquid theory approximation, Eq. (3.54) (L), as well as the liquid theory approximation normalized by an overall constant (NL) perform better than the compound poisson (CP) and gaussian approximation (KB). Note that for $c = 000$, none of the approximations captures the height of the peak of the distribution accurately and we will remark on this shortly.

Tables 4 and 5 show the total variational distances between the actual and approximate distributions for word lengths $l = 3, 4, 5, 6, 7$ and $l = 8$ and string lengths $k$ chosen such that $k/r^l = 16$, i.e. the distributions have approximately the same mean. Overall,
Table 3: Total variational distance between the actual distribution and the various approximate distribution for the case \( r = 2, k = 256 \): liquid theory approximation (L), Eq. (3.54), the liquid theory approximation normalized by an overall constant (NL), the compound poisson approximation (CP) and the gaussian approximation (KB).

| \( c \) | \( d_{TV}^L \) | \( d_{TV}^{NL} \) | \( d_{TV}^{CP} \) | \( d_{TV}^{KB} \) |
|-------|----------|----------|----------|----------|
| 000   | 0.052    | 0.053    | 0.189    | 0.052    |
| 001   | 0.035    | 0.031    | 0.079    | 0.075    |
| 010   | 0.011    | 0.003    | 0.108    | 0.071    |
| 111   | 0.032    | 0.021    | 0.047    | 0.148    |

The liquid theory approximation, Eq. (3.54) (L), as well as the liquid theory approximation normalized by an overall constant (NL) perform better than or as well as the compound poisson approximation (CP) and gaussian approximation (KB) taken by themselves. The CP approximation gives a better approximation for \( c = 11 \cdots 1 \) and for some of the low and high \( \chi \) equivalence classes associated with \( l = 7 \) and \( l = 8 \). Also note that for \( l \geq 6 \) the CP approximation performs generally better than the KB approximation, as was noted before by Robin and Schbath [19]. The poor performance of the liquid theory approximation for the case \( l = 3 \) and \( c = 00 \) turns out to be due to the fact that the expansion of \( x \) and \( b_m \) to second order is not adequate. Upon calculating \( x \) (and \( b_m \)) more accurately, the agreement with the actual distributions turns out to be nearly perfect.

Regarding the robustness of the liquid theory approximations (L) and (NL), we have checked that going to a higher cut-off does not improve the distributions very much. Also, it turns out that for large \( \chi \) and \( l \), the first order expression for \( x \) is often sufficient, however it is almost always insufficient for small \( \chi \) and in particular when \( \chi = 1 \), i.e. \( x \) belongs to the equivalence class \( c = 11 \cdots 1 \).

Fig. 6 shows the \( n \) match distributions for \( l = 4 \) and with a string length that has been increased to \( k = 4096 \). Comparing with the case \( k = 256 \), Fig. 3, the distributions for small \( \chi \) are more symmetric around their mean. The total variational distances are given in the table below. Note that they are comparable with the values that we obtained for \( k = 256 \), Table 3.

The discrepancy between actual and approximate distributions for \( c = 000 \) is persistent: it does not improve with increasing \( \Lambda \), or going to third order in the expansion of \( x \), or by taking the stationary phase approximation to higher order (which turns out to be a \( 1/n \) expansion). The discrepancy for \( c = 000 \) does not seem to be a finite-size effect either as can be seen by comparing Figs. 6 and 3.

On the other hand, increasing \( r \), does reduce the total variations. Fig. 7 shows the \( n \)-match distribution for \( l = 4, m = 4092 \) and strings whose letters come from a 4 letter alphabet. Notice that the total variation of the approximate distributions, are overall much smaller and all three approximations yield similar results. In particular the
Table 4: Total variational distance between the actual distribution and the various approximate distribution for the case $r = 2$ and $(l, k) (3, 128), (4, 256), (5, 512), (6, 1024), (7, 2048) \text{ and } (8, 4096)$: liquid theory approximation (L), Eq. (3.54), the liquid theory approximation normalized by an overall constant (NL), the compound poisson approximation (CP) and the gaussian approximation (KB).
Table 5: Total variational distance between the actual distribution and the various approximate distribution for the case $r = 2$ and $(l, k)$ values of $(7, 2048)$ and $(8, 4096)$; liquid theory approximation (L), Eq. (3.54), the liquid theory approximation normalized by an overall constant (NL), the compound poisson approximation (CP) and the gaussian approximation (KB).

| c   | $d^{L}_{TV}$ | $d^{NL}_{TV}$ | $d^{CP}_{TV}$ | $d^{KB}_{TV}$ |
|-----|--------------|---------------|---------------|---------------|
| 0000000 | 0.025 | 0.024 | 0.004 | 0.037 |
| 0000010 | 0.003 | 0.003 | 0.028 | 0.028 |
| 0000110 | 0.004 | 0.002 | 0.023 | 0.031 |
| 0001110 | 0.005 | 0.006 | 0.031 | 0.031 |
| 0010100 | 0.004 | 0.002 | 0.023 | 0.038 |
| 0010101 | 0.004 | 0.003 | 0.029 | 0.037 |
| 0011110 | 0.004 | 0.003 | 0.029 | 0.042 |
| 0101001 | 0.011 | 0.012 | 0.033 | 0.041 |
| 0101010 | 0.023 | 0.013 | 0.026 | 0.067 |
| 1111111 | 0.052 | 0.022 | 0.015 | 0.146 |

Table 6: Total variational distance between the actual distribution and the various approximate distribution for the case $r = 2$, $l = 4$ and $k = 4096$; liquid theory approximation (L), Eq. (3.54), the liquid theory approximation normalized by an overall constant (NL), the compound poisson approximation (CP) and the gaussian approximation (KB).

| c   | $d^{L}_{TV}$ | $d^{NL}_{TV}$ | $d^{CP}_{TV}$ | $d^{KB}_{TV}$ |
|-----|--------------|---------------|---------------|---------------|
| 000 | 0.061 | 0.060 | 0.197 | 0.060 |
| 001 | 0.035 | 0.035 | 0.076 | 0.075 |
| 010 | 0.011 | 0.004 | 0.108 | 0.065 |
| 111 | 0.045 | 0.023 | 0.038 | 0.140 |
deviations for $c = 000$ have disappeared now. Table 7 gives the corresponding variational distances: Comparing with Table 3, we see indeed that for $r = 4$ the total variational distances are overall smaller.

It seems that for the case $c = 000$ and $r = 2$, the stationary phase approximation around the single point $u \approx 1$ is not capturing all the contributions to the probability
Finally, we would like to remark that the expansion of $x$, Eq. (3.52) is in fact the virial expansion of the equation of state for the (discrete) lattice gas. The parameter $x$ is related to $z$ as $x = 1/z - 1$, Eq. (3.46). In the continuous 1d gas of $n$ particles in a "volume" $L$ and nearest-neighbor interactions, the partition function can be written as

$$Q(n, L) = \frac{1}{2\pi i} \oint ds e^{sL} D^2(s) H^{n-1}(s)$$  \hspace{1cm} (3.56)$$

where $D(s)$ and $H(s)$ are the Laplace transforms of the Boltzmann factor for the particle-boundary and particle-particle interactions, and Eq. (3.56) is the inverse Laplace transform with an appropriately chosen contour. For physical interactions and in the thermodynamical limit, it turns out that the integral in the above equation can be evaluated by a saddle point expansion around the point $s_0$ [34] and as a result, it turns out that $s_0 = \beta P$, where $\beta$ is the Boltzmann factor and $P$ is the pressure [33, 34]. Comparing with Eq. (3.39) we see that upon discretizing the length of the container by letting $L = m \Delta$, and assuming that the interactions vary slowly with respect to $\Delta$, Eq. (3.39) can be recovered under the identification

$$e^{-s_0 \Delta} = u = \frac{1}{1 + x},$$  \hspace{1cm} (3.57)$$

which for small $\Delta$ implies that $x = s_0 \Delta = \beta P \Delta$. We thus see that the virial expansion Eq. (3.52) leads to a van der Waals type equation of state [35]. Indeed as can be seen from Eq. (3.48), $\epsilon_1$ is the effective hard-core size and the term $(n - 1) \epsilon_1$ is the total excluded "volume" due to the interaction (core + tail).

Fig. 8 shows the "$P - V$ isotherms" of the lattice gas with $l = 4, r = 2$ and fixed particle number $n = 15$ for the four equivalence classes $c = 000, 001, 010$ and 111 (from top to bottom). The thick solid line is the "ideal gas" law $x = n/m$. The data points have been obtained from numerically solving Eq. (3.45). Using the approximate equation

$$c d_{TV} | d_{TV}^L | d_{TV}^CP | d_{TV}^KB$$

| c   | $d_{TV}^L$ | $d_{TV}^CP$ | $d_{TV}^KB$ |
|-----|------------|-------------|-------------|
| 000 | 0.008      | 0.016       | 0.030       |
| 001 | 0.005      | 0.004       | 0.034       |
| 010 | 0.006      | 0.014       | 0.036       |
| 111 | 0.028      | 0.005       | 0.080       |

Table 7: Total variational distance between the actual distribution and the various approximate distribution for the case $r = 4, l = 4$ and $k = 4096$: liquid theory approximation (L), Eq. (3.54), the liquid theory approximation normalized by an overall constant (NL), the compound poisson approximation (CP) and the gaussian approximation (KB).
Figure 7: The $n$-match distribution for matching a $l = 4$ letter 4-ary string $x$ inside a random string of length $k = 4096$, for $x = 0001$ (top left), $x = 1001$ (top right), $x = 1010$ (bottom left) and $x = 1111$ (bottom right). The circles are the exact probabilities, the dashed and dashed-dotted lines correspond to the Gaussian and compound poisson approximation (see text for details). The solid line is the analytical Eq. (3.54).

of state, Eq. (3.47) give similar results but with increasing deviations at high densities.
Figure 8: The “P-V diagram” of the lattice gas with $l = 4$, $r = 2$ and fixed particle number $n = 15$ for the four possible interactions $c = 000, 001, 010$ and $111$ (from top to bottom). The thick solid line corresponds to the “ideal gas” law $x = n/m$ (refer to text for details).

3.5 Asymptotics

We now consider the asymptotic form of the $n$-match distributions in the limit that the length $k = m + l$ of the random string is large. It turns out that this is most readily done using generating functions. We define the generating function $p(\zeta, z; c)$ of $p(n, m; c)$ as

$$p(\zeta; m, c) = \sum_{n=0}^{\infty} p(n; m, c)\zeta^n$$  (3.58)
From Eq. (3.21) we thus find that
\[
p(\zeta; m, c) = \frac{A_1}{z_1^{m+1}} + \frac{A_1}{z_1^{m+1}} \sum_{n=1}^{\infty} (\zeta e^{\beta \mu})^n \frac{1}{2\pi i} \oint_{\partial D} \frac{dz}{z^{m+1}} \ D^2(z) H^{n-1}(z),
\]
where we have used the asymptotic form \( p(0; m, c) = A_1/z_1^{m+1} \) for the \( n = 0 \) term, since \( m \) is assumed to be large. The order of summation and integration can be exchanged if the integrand is uniformly converging in the region of integration. It is not hard to show that this can be achieved for example by a circular path \( \|z\| = R \), with a suitably chosen \( R < 1 \). Thus carrying out the sum first, we obtain
\[
p(\zeta; m, c) = \frac{A_1}{z_1^{m+1}} + \frac{A_1}{z_1^{m+1}} \zeta e^{\beta \mu} \frac{1}{2\pi i} \oint_{\partial D} \frac{dz}{z^{m+1}} \left( 1 - \zeta e^{\beta \mu} H(z) \right). \tag{3.60}
\]
Substituting the approximate forms for \( D(z) \) and \( H(z) \), Eqs. (3.37) and (3.37), we find
\[
\hat{p}(\zeta; \gamma, m, c) = \frac{A_1}{z_1^{m+1}} \zeta e^{\beta \mu} \left( \frac{1}{2\pi i} \oint_{\partial D} \frac{dz}{z^{m+1}} \left( \frac{1}{z^{m+1}} - \zeta e^{\beta \mu} H(z) \right) \right).
\tag{3.61}
\]
Denote the expression in the denominator by \( \tilde{\lambda}(z; \zeta, c) \),
\[
\tilde{\lambda}(z; \zeta, c) = (1 - z) \left[ 1 - \zeta e^{\beta \mu} (H_A(z) + \Gamma(z)) \right] - \zeta e^{\beta \mu} z^A. \tag{3.62}
\]
Since \( \exp(\beta \mu) \), is of order \( 1/r^4 \) it follows that \( \tilde{\lambda}(z; \zeta, c) \) has a root near \( z = 1 \). It turns out again that this is the root closest to the origin and that all other roots are of order \( \|z\|^4 \zeta \exp(\beta \mu) \sim 1 \). Denoting the root of smallest magnitude by \( \tilde{z}_1 \), and using the method of Section 2.6, a series expansion of \( \tilde{z}_1 \) can be made. One finds to lowest order that
\[
\tilde{z}_1 = 1 - \frac{\zeta e^{\beta \mu}}{1 - \zeta e^{\beta \mu} H_A(1) - \zeta e^{\beta \mu} \Gamma(1)}. \tag{3.63}
\]
The integrand in Eq. (3.61) has therefore two dominant poles at \( z = 1 \) and \( z = \tilde{z}_1 \). For large \( m \), the contour integral can again be evaluated approximately by pushing the countour out to infinity and keeping only the residues from the dominant poles (which are traversed counter-clockwise), as explained in Section 2.6.

We find
\[
\hat{p}(\zeta; \gamma, m, c) = \frac{A_1}{(z_1 \tilde{z}_1)^{m+1}} \frac{\zeta e^{\beta \mu}}{1 - \tilde{z}_1} \left( \frac{1}{\lambda'((z; \zeta, c))} \left[ (1 - \tilde{z}_1) (D_A(\tilde{z}_1) + \Gamma(\tilde{z}_1)) + \tilde{z}_1 A \right] \right)^2. \tag{3.64}
\]
Notice that the \( m \) dependence is entirely confined to the term \( 1/(z_1 \tilde{z}_1)^{m+1} \). Thus this term alone is responsible for the large \( m \) behavior. The term in the square brackets is the effect due to the boundaries of the string. When \( m \) is large boundary effects should
not matter and we will set this term to 1. Alternatively, we can assume that the random string is circular and in this case the boundary term will not arise.

Apart from the cut-off assumption on the behavior of the tails, and the assumption of large \(m\) leading to the \(m\)-asymptotic expression, Eq. (3.64), we have not made any assumptions on \(r\) or \(l\) so far. To proceed further, we will assume that \(1/r^l \ll 1\) so that the lowest order expressions for \(\bar{z}_1\) and \(z_1\) will provide the leading order approximation to Eq. (3.64).

Substituting the lowest order expression for \(\bar{z}_1\), Eq. (3.63) and noting that to this order \(-\bar{\lambda}'(\zeta, c) = 1 - \zeta \exp(\beta \mu) H_\Lambda(1) - \zeta \exp(\beta \mu) \Gamma(1)\), the result simplifies to

\[
\hat{p}(\zeta; \gamma, m, c) = \frac{A_1}{(z_1 \bar{z}_1)^{m+1}} \tag{3.65}
\]

The compound poisson distribution arises in the limit when \(m \to \infty\) and \(\langle n \rangle\) is finite. From Eq. (2.26) this implies that the word length \(l\) scales as \(l \sim \log r (m+1)\). From the properties of the interactions that were derived in Section 3.2, we see that the tails are very weak and of order \(1/m\), while the core is relatively strong and of order \(\log m\). Thus it is permissible to set \(\Lambda = l\) and ignore the tails (\(\Gamma = 0\)). Note that in this limit \(1/r^l \sim 1/m\) and thus to lowest order \(A_1 = 1\), and

\[
e^{\beta \mu} = \frac{1}{r^l} \frac{1}{[1 + c(1/r)]^2} \tag{3.66}
\]

We thus obtain

\[
\hat{p}(\zeta; \gamma, m, c) = \left[1 + \frac{1}{1 + c(1/r)} \frac{1}{r^l} \left(1 - \frac{\zeta}{r^l} \frac{1}{[1 + c(1/r)]^2} \frac{1}{1 - \zeta e^{\beta \mu} H_1(1)}\right)\right]^{-(m+1)} \tag{3.67}
\]

Further simplifications occur, noting that from Eq. (2.37) to order \(1/r^l\) we have

\[
\frac{1}{1 + c(1/r)} = 1 - h \left(\frac{1}{r}; c\right) \tag{3.68}
\]

while from Eq. (3.20) we find that

\[
e^{\beta \mu} H_1(1) = h \left(\frac{1}{r}; c\right) \tag{3.69}
\]

Multiplying out the product in Eq. (3.67) and keeping only terms to order \(1/r^l \sim 1/m\), we thus obtain

\[
\hat{p}(\zeta; \gamma, m, c) = \left[1 + \frac{1}{r^l} \left(1 - h \left(\frac{1}{r}; c\right)\right)^2 \left(1 - \frac{1}{1 - h \left(\frac{1}{r}; c\right)} - \frac{\zeta}{1 - \zeta h \left(\frac{1}{r}; c\right)}\right)\right]^{-(m+1)} \tag{3.70}
\]

Taking now the limit \(m \to \infty\) such that \((m+1)/r^l = \langle n \rangle\) is finite, the expression is readily brought to the form

\[
\hat{p}(\zeta; \gamma, m, c) = e^{-\sum_{j=1}^{\infty} (1-\zeta) \lambda_j} \tag{3.71}
\]
with

\[ \hat{\lambda}_j = \langle n \rangle \left[ 1 - h \left( \frac{1}{r}; c \right) \right]^2 h \left( \frac{1}{r}; c \right)^j. \]  

(3.72)

Eq. (3.71) is the generating function of a compound poisson distribution \[32\] and precisely the result derived by various other methods by Chrysaphinou and Papastavridis [11], Geske et al. [12], and Schbath [17] in the special case of uniformly i.i.d letters. Also note that the CP distribution is normalized, \( \hat{p}(1; \gamma, m, c) = 1 \).

Note that setting the tails \( b \geq l \) of the interactions to zero means that given the next match is a distance at least \( l \) away, it can occur with equal probability at any \( b \geq l \). Since nearest neighbor match separations \( b < l \) define an overlapping cluster, this means that the location of the clusters themselves, \( b \geq l \), are distributed like the arrivals of a poisson process \[11, 18, 16\]. We therefore see that the liquid theory description in terms of interactions along with the separation of cores and tails provides an alternative and very simple explanation of this property. Conversely, strong tails mean that the positions of the clusters themselves are correlated and deviate from a poisson process (meaning that the probability of initiating a new cluster depends on the distance from the last cluster).

We now consider the limit \( m \to \infty \) and \( n \to \infty \) such that in this limit the number density \( n/(m+1) = 1/r^l \) remains constant and is small. In this limit the tails of the interaction are also small, and we obtain (to lowest order in \( 1/r^l \))

\[ \hat{p}(\zeta; \gamma, m, c) = \left[ \left( 1 + \frac{1}{1 + c(1/r)} \frac{1}{r^l} \right) \frac{1}{1 - \zeta h(1/r; c)} \right]^{-(m+1)}, \]  

(3.73)

Notice that if \( \Lambda = \infty \), there would be nothing left for the remaining tail and thus \( \Gamma \) would be zero and we would obtain

\[ \hat{p}(\zeta; 0, m, c) = \left[ \left( 1 + \frac{1}{1 + c(1/r)} \frac{1}{r^l} \right) \frac{1}{1 - \zeta h(1/r; c)} \right]^{-(m+1)}, \]  

(3.74)

The normalization is given by \( \hat{p}(1; 0, m, c) = 1 \), and using the relation Eq. \[3.68\] it is readily seen that the distribution is normalized to order \( 1/r^l \). This observation immediately gives us a way to estimate \( \Gamma(1) \), which must be chosen such that the distribution is normalized to that order. We have

\[ e^{\beta \mu} \Gamma(1) = e^{\beta \mu} H_l(1) - h \left( \frac{1}{r}; c \right) \]  

(3.75)

and the normalized distribution becomes

\[ \hat{p}(\zeta; m, c) = \left[ \left( 1 + \left[ 1 - h \left( \frac{1}{r}; c \right) \right] \frac{1}{r^l} \right) \left( 1 - \frac{1}{r^l} \left[ 1 - h \left( \frac{1}{r}; c \right) \right]^2 \right) \right]^{-(m+1)}. \]  

(3.76)
The large $n$ limit can again be obtained using Hayman’s method introduced in the previous sub-section. Choosing $\zeta_0$ such that

$$n = \left. \left( \zeta \frac{d}{d\zeta} \ln \hat{\rho}(\zeta; m, c) \right) \right|_{\zeta = \zeta_0}$$

we find to order $1 - \langle n \rangle / n$

$$\zeta_0 = 1 + \frac{1}{2} \frac{1 - h(1/r; c)}{1 + h(1/r; c)} \left( 1 - \frac{\langle n \rangle}{n} \right),$$

(3.78)

where $\langle n \rangle$ is as defined in Eq. (2.26). Using this approximation for $\zeta_0$, we find after a little bit of algebra that the distribution of $n$ around its mean is Gaussian distributed,

$$\hat{\rho}(\zeta; m, c) \frac{1}{\sqrt{2\pi \hat{\sigma}_n^2}} \exp \left( -\frac{(n - \langle n \rangle)^2}{2\hat{\sigma}_n^2} \right),$$

(3.79)

with

$$\hat{\sigma}_n^2 = \langle n \rangle \frac{1 - h(1/r; c)}{1 + h(1/r; c)}.$$

(3.80)

In concluding this section we would like to point out that our derivation of the CP and Gaussian asymptotic forms rests on determining the dominant root of $\bar{\lambda}(z; \zeta, c)$, Eq. (3.62), which in turn emerges as a result of introducing a cut-off $\Lambda$ and approximating the interactions beyond $\Lambda$. In a sense, it is the presence of the cut-off that simplifies the analytical treatment of the problem, since it makes explicit the separation of small and therefore negligible terms from the dominant ones.

4 The case of general random letter strings

All the calculations and results presented so far, have been worked out for the case of uniformly and i.i.d letters of the random string. However for many applications this requirement is too restrictive. Letter distributions that have been considered in the literature are non-uniform i.i.d letters and letter sequences generated by a Markov process. For either of the cases asymptotic results in the form of large deviations, Gaussian and compound poisson distributions exist [11, 12, 13, 14, 15, 17, 4, 18, 16].

In this section we show that the $n$-match probability associated with a broader class of letter distributions can be worked out using the lattice gas description introduced in the previous section. The essential insights gained from this approach are not changed by this generalization. The problem to be solved is still that of calculating the partition function of a 1d lattice gas of $n$ particles with nearest-neighbor interactions among themselves and the boundaries. The only difference is that the interactions and hence the calculations become more involved.

The required generating functions have been already derived by Régnier and Szpankowski [14] and we will adopt their results to our notation. Let again $y = (y_1, y_2, \ldots, y_k)$ be the letters of the random string and let $x = (x_1, x_2, \ldots, x_l)$ be the word to be matched.
Régnier and Szpankowski consider the case of i.i.d letters with arbitrary letter distribution (Bernoulli Model) and letter sequences generated by a one-step Markov process with transition matrix $P$, such that $P_{ij}$ is the transition probability $P\{y_{a+1} = i|y_a = j\}$, $\pi = (\pi_1, \pi_2, \ldots, \pi_r)$ is the stationary letter distribution satisfying $\pi P = \pi$, and the stationary matrix $\Pi$ is the matrix whose $r$ rows are $\pi$ (Markov Model).

Given any subsequence of letters $y_{a+1}, y_{a+2}, \ldots, y_{a+l}$, denote by $p(y_{a,l})$ the probability of encountering $y_{a,l}$, without any conditions on the letters preceding or following it. Likewise, denote by $p(x)$ the probability of generating the word $x$. The generating function of the $n$-match probability is given by [14]:

$$p(n; z, c) = p(x)\tilde{d}(z; c)\tilde{h}^{n-1}(z; c),$$

(4.1)

with

$$\tilde{d}(z; c) = \frac{1 - \tilde{h}(z; c)}{1 - z} = \frac{1}{p(x)} \frac{1}{\lambda(z; c)},$$

(4.2)

$$\tilde{h}(z; c) = 1 - \frac{1}{p(x)} \frac{1 - z}{\lambda(z; c)},$$

(4.3)

and

$$\lambda(z; c) = z^l + \frac{1}{p(x)}(1 - z) \left[ 1 + \tilde{c}(z) + \frac{p(x)}{\pi(x_1)} T(z) z^l \right],$$

(4.4)

In the last equation $\pi(x_1)$ is the steady state probability of encountering the letter $x_1$ and $T(z)$ is the generating function for the steady-state transition probability from the end of one word match to the beginning of the next word match as a function of the gap length between the two words (for the Bernoulli Model $T(z) = 0$). The generating function $\tilde{c}(z)$ is defined as

$$\tilde{c}(z) = \sum_{b=1}^{l-1} c_b p(x_{1,b}) z^b,$$

(4.5)

where $c_b(x)$ are the bit-vectors associated with the word $x$. Note that an overall factor of $z^l$ in the definition of $p(n; z, c)$ in [14] is absent, since the generating function $p(n; z, c)$, as defined above, corresponds in our case to $p(n; m, c)$, where $m = k - l$ is the effective length of the string.

Comparing with the corresponding equations of the uniformly distributed random letter case, Eqs. (2.32), (2.38), (2.39) and (2.36), we see that the form of the equations as well as the relationships between the generating functions are identical.

In particular, all recursions can be recovered by making the replacements $h_a / r^a \rightarrow \tilde{h}_a$, $d_a / r^a \rightarrow \tilde{d}_a$, and $c_a / r^a \rightarrow \tilde{c}_a$ so that $h(z/r) \rightarrow \tilde{h}(z)$ etc.

The Markov property introduces the additional complication that one has to propagate the end of one word match at $a_i$ to the beginning of the next match at $a_{i+1}$ through the $(a_{i+1} - a_i - l)$-step steady-state transition probability.

Régnier and Szpankowski have also proven that the polynomial $\lambda(z; c)$ has at least one real root and that all roots have $\|z\| \geq 1$, as in the case of uniform letter distributions. The asymptotic behavior of $\tilde{h}$ and $\tilde{d}$ is again due to the root closest to $z = 1$. 


As can be seen from Eq. (4.4), for \( p(x) \) small, the root closest to \( z = 1 \) is located at roughly
\[
z_1 \approx 1 + \frac{p(x)}{1 + \tilde{c}(1) + \frac{p(x)}{\pi(z_1)} T(1)}
\]
and all other roots are roughly located at \( \|z\| \sim \frac{1}{p(x)} \). Recall that in the case of uniformly distributed letters, \( p(x) = \frac{1}{\ell \leq \frac{1}{2}} \). For the general letter distributions, both the distribution as well as the word \( x \) can be chosen arbitrarily and thus there is no constraint on the values that \( 0 \leq p(x) \leq 1 \) can take. This means in particular that there is a broader class of possible interactions.

Defining again the effective particle-particle interaction as
\[
e^{-\beta U(b)} = \frac{\tilde{h}(b)}{\tilde{h}_{asy}(b)},
\]
\( \tilde{h}_{asy}(b) \) is readily worked out as
\[
\tilde{h}_{asy}(b) = p(x) \frac{A_1}{z_1} \left( \frac{z_1 - 1}{p(x)} \right)^2 \left( \frac{1}{z_1} \right)^b = e^{\beta \mu} \left( \frac{1}{z_1} \right)^b
\]
where \( A_1(z_1 - 1) = -1/\lambda'(z_1) \), cf. Eq. (2.51). We thus obtain for the particle-particle interaction
\[
\beta U(b) = -\ln \tilde{h}(b) - b \ln z_1 + \ln p(x) + \beta U_0,
\]
where
\[
\beta U_0 = \ln \left[ \frac{A_1}{z_1} \left( \frac{z_1 - 1}{p(x)} \right)^2 \right].
\]
If \( p(x) \ll 1 \), \( \beta U_0 \) is a constant of order \( p(x) \), since the argument of the logarithm is of order 1 to the same order.

In the core-region \( b < \ell \), the non-zero values of \( \tilde{h} \) are still determined by the bit-vector \( c \) associated with \( x \) and we find analogous to Eq. (3.8) that
\[
\tilde{h}(b) = \begin{cases} 
  c_b p(x_{1,b}), & \text{if } \chi \text{ does not divide } b, \\
  p(x_{1,\chi}), & \text{if } b = \chi, \\
  0, & \text{otherwise},
\end{cases}
\]
where \( \chi \) is the fundamental period associated with \( c \) that was defined at the end of Section 2.3 and by definition, \( \tilde{h}(0) = 0 \).

We see that the interaction is \( +\infty \), whenever \( c_b = 0 \). This is certainly the case for \( b \ll \chi \). The interaction in the core-region is given by
\[
\beta U(b) = -\ln c_b - b \ln \left[ z_1 p^{1/b}(x_{1,b}) \right] + \ln p(x) + \beta U_0,
\]
Comparing the above with the uniform letter distribution case, Eq. (3.9), we see similarities as well as differences: the argument of the logarithm in square brackets is no longer necessarily smaller than one, but can depend on the subtle interplay of the overall
word matching probability $p(x)$ (which determines $z_1$) with the (smaller) probabilities of matching the subwords $p(x_{1,b})$. Thus such cases require more care. Assuming that $p(x)$ is sufficiently small so that the expression in square brackets is smaller than one, it is again the case that the energy of the core region is of order $\ln p(x)$ and increases with $b$. Under the same assumptions, the characteristic energy of the tail region can be worked out and one finds that it goes like $p(x)$ similarly to the line of reasoning in Section 3.2.

Thus we see that by suitably choosing the set of probabilities $p(x_{1,b})$, for $b = 1, 2, \ldots, l$ the strengths of the core and tail of the interactions can be varied and can possibly move the distribution functions into a regime where approximations ignoring the contributions from the tails (such as the compound poisson approximation) are inappropriate.

Also note that by letting $p(x)$ to be arbitrarily close to one, the difference of magnitudes between the root closest to the origin $z_1$ and the other roots can be made to vanish. Since the attenuation length of the tail of the interactions depends on the separation of these roots, we see that in this limit $z_1$ ceases to be the dominant root. For the interactions this means an increasingly more slowly decaying tail as the two roots approach each other. In such a regime the tails of the interactions should become very important and thus cannot be neglected.

It is possible that for certain distributions and choices of words, this can cause a break-down of the liquid theory approach, which essentially is a perturbation theory and it would be interesting to find out if and how this can happen. Strong tails will certainly affect the quality of approximations such as the compound poisson distribution, which was based on the assumption that tails can be ignored, which turned out to be equivalent to assuming a poissonian distribution of cluster locations, as explained in Section 2.6.

We will further discuss these points in the Discussion section below.

5 Discussion

We have presented a new approach to calculating the probability distribution for the number of matches of a given word inside a random string of letters. Our approach rests on the observation that the exact expression for such a distribution can be interpreted as the partition function of an $n$-particle system on a linear lattice, with pairwise nearest neighbor interactions. By exploiting this analogy and focusing on the generic properties of the interaction, we have been able to set up a virial expansion for the equation of state of this lattice gas and thereby obtained an analytical expression for the $n$-match probability distribution, which besides extrapolating between the known asymptotic forms, also provides a good approximation in the intermediate regimes.

The identification and subsequent analysis of the effective interactions in the lattice gas description turns out to be key in our solution of this problem. The interactions are characterized by a strong core-region of the size of the word-length followed by a relatively weak and exponentially decaying tail. Although we have carried out the detailed analysis for the special case of uniform letter distributions, we showed in Section IV, that our method is readily extended to the broader class of distributions, such as non-uniform letter distributions and random letter sequences generated by a Markov process. Regardless
of the underlying stochastic process for the random string, the generic feature of the interactions are still the same, namely a relatively strong core and a weaker tail and our approach should be readily applicable to these types of problems as well.

We should also point out that our method of approach bears some similarity with the work of Régnier and Szpankowski [14], who also use generating functions in their approach to this problem. Our approach is however distinct in at least one crucial point: In the cited work, upon deriving the generating functions for the \( n \)-match distribution, Eqs. (4.1), (4.2), (4.3) and (4.4), the authors perform a Laurent expansion of the generating function around its dominant \( n+1 \) order pole at \( z_1 \). Such an expansion is asymptotic in the interactions and runs the risk of capturing more accurately the tail of the interaction rather than its core (or at least many terms must be kept in order to capture the core part to a sufficient degree of accuracy [30]). The approximation scheme presented here precisely avoids this by introducing a cut-off distance \( \Lambda \) and keeping the exact interaction upto \( \Lambda \), while approximating the interactions only beyond \( \Lambda \). As we have shown, this is easily done, since the structure of the core of the interaction \((b < l)\) directly follows from the overlap properties of the string to be matched. Our analysis also shows that since the core part of the interaction is typically stronger than the exponentially decaying tail, keeping the core is crucial in determining the global properties of the distribution. Moreover, our approach allows us to understand approximations such as the compound distribution as being applicable in a regime where the tails of the interaction can be neglected and only the core is kept. This also highlights the relative importance of the core part of the interaction with respect to its tail.

Lastly, we would like to remark that our treatment of interactions, by separating out its strong and short-ranged core from its weak tail, is actually not new. Interactions with a strong core and an exponentially decaying tail are known as Kac potentials, named after M. Kac, who along with co-workers studied one-dimensional particle systems with such interactions (continuum and lattice version) in considerable detail, as part of an effort to understand the liquid-gas transition in the context of the van der Waals equation of state [36, 37, 38] (for an overview, see the review article by Hemmer and Lebowitz [38]).

Such systems are interesting, since they lead to phase transitions in the limit when the characteristic decay length of the interaction tends to infinity [36, 37, 38] (for an overview of phase transitions in one dimensions see, the review article by Griffiths [39]). The similarity of such systems with the string matching problem is at hand, since one can make the interactions to decay as slowly as one wishes by choosing a suitable random letter distribution and string \( x \) to be matched such that the dominant poles of the generating function of the \( n \)-match distribution function become arbitrarily close to each other. It would therefore be of interest to see whether the distribution functions in this regime can be calculated using the more sophisticated techniques, such as integral equations and operator methods, which have been introduced particularly for the purpose of dealing with such types of interactions [38].

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