Some notes on the Feigin-Losev-Shoikhet integral conjecture

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Abstract

Given a holomorphic vector bundle $\mathcal{E}$ on a smooth connected compact complex manifold $X$, [FLS] use a notion of completed Hochschild homology $\hat{\mathbb{H}}$ of $\text{Diff}(\mathcal{E})$ such that $\hat{\mathbb{H}}_0(\text{Diff}(\mathcal{E}))$ is isomorphic to $H^{2n}(X, \mathbb{C})$. On the other hand, they construct a linear functional on $\hat{\mathbb{H}}_0(\text{Diff}(\mathcal{E}))$. This therefore gives rise to a linear functional $I_\mathcal{E}$ on $H^{2n}(X, \mathbb{C})$. They show that this functional is $\int_X$ if $\mathcal{E}$ has non zero Euler characteristic. They conjecture that this functional is $\int_X$ for all $\mathcal{E}$.

These notes prove that $I_\mathcal{E} = I_\mathcal{F}$ for any pair $(\mathcal{E}, \mathcal{F})$ of holomorphic vector bundles on $X$. In particular, if $X$ has one vector bundle with nonzero Euler characteristic, $I_\mathcal{E} = \int_X$ for every vector bundle $\mathcal{E}$ on $X$.

[FLS] also use a notion of completed cyclic homology $\hat{\mathbb{C}}$ of $\text{Diff}(\mathcal{E})$ such that $\hat{\mathbb{C}}_{-i}(\text{Diff}(\mathcal{E})) \simeq H^{2n-i}(X, \mathbb{C}) \oplus H^{2n-i+2}(X, \mathbb{C}) \oplus \ldots$. The construction yielding $I_\mathcal{E}$ generalizes to yield linear functionals on $\hat{\mathbb{C}}_{-2i}(\text{Diff}(\mathcal{E}))$ for each $i \geq 0$. The linear functional thus obtained on $\hat{\mathbb{C}}_{-2i}(\text{Diff}(\mathcal{E}))$ thus yields a linear functional $I_{\mathcal{E}, 2i, 2k}$ on $H^{2n-2k}(X, \mathbb{C})$ for $0 \leq k \leq i$. [FLS] conjecture that $I_{\mathcal{E}, 2i, 0} = \int_X$ and make a further conjecture about $I_{\mathcal{E}, 2, 2}$.

These notes prove that $I_{\mathcal{E}, 2i, 0} = I_\mathcal{E}$ for all $i \geq 0$. In particular, if $X$ has at least one vector bundle with non zero Euler characteristic, then $I_{\mathcal{E}, 2i, 0} = \int_X$. We also prove that $I_{\mathcal{E}, 2i, 2k} = 0$ for $k > 0$. The latter is stronger than what [FLS] expect when $i = k = 1$.

Keywords: Hochschild cocycle, $A_\infty$-morphism; differential operators; Dolbeault complex; FLS functional; supertrace; completed Hochschild homology; completed cyclic homology; integral operator; kernel.
1 Introduction

Let $X$ be a smooth connected compact complex manifold such that $\dim_{\mathbb{C}} X = n$. Let $\mathcal{E}$ be a holomorphic vector bundle on $X$. Let $\text{Diff}(\mathcal{E})$ be the sheaf of holomorphic differential operators on $\mathcal{E}$. Let $\text{Diff}(\mathcal{E})$ be the algebra of global sections of $\text{Diff}(\mathcal{E})$. One is interested in understanding the Hochschild homology of $\text{Diff}(\mathcal{E})$. This is however, not an object that one can easily analyze.

Let $\text{Dolb}^\bullet(X, \mathcal{O}_X)$ denote the Dolbeaux resolution of the sheaf $\mathcal{O}_X$ of holomorphic functions on $X$. Let $\text{Diff}^\bullet(\mathcal{E}) = \text{Dolb}^\bullet(X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \text{Diff}(\mathcal{E})$. Following [FLS] we replace $\text{Diff}(\mathcal{E})$ by

$$\text{Diff}^\bullet(\mathcal{E}) := \Gamma(X, \text{Diff}^\bullet(\mathcal{E})).$$

Let $K^\bullet_{\mathcal{E}}$ denote the complex $\Gamma_{C^\infty}(X, \text{Dolb}^\bullet(X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{E})$ of smooth global sections of $\text{Dolb}^\bullet(X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{E}$. This is the Dolbeaux complex of $\mathcal{E}$. If $D$ is a holomorphic differential operator on $\mathcal{E}$, then $D$ gives rise to a $\mathbb{C}$-endomorphism of the complex $K^\bullet_{\mathcal{E}}$. It follows that $D$ induces an endomorphism $D_\ast : H^i(X, \mathcal{E}) \to H^i(X, \mathcal{E})$ for each $i$. The supertrace of $D$ is given by

$$\text{str}(D) := \sum_i (-1)^i \text{Tr}_{H^i(X, \mathcal{E})} D_\ast.$$

Note that $\text{Diff}(\mathcal{E})$ is a subalgebra of $\text{Diff}^\bullet(\mathcal{E})$. It follows that there is a natural map from the $i$th Hochschild homology of $\text{Diff}(\mathcal{E})$ to that of $\text{Diff}^\bullet(\mathcal{E})$ for all $i$. Also recall that if $\mathcal{A}$ is any associative $\mathbb{C}$-algebra concentrated in degree 0 then the 0-th Hochschild homology $\text{HH}_0(\mathcal{A})$ of $\mathcal{A}$ is the Abelianization of $\mathcal{A}$. It follows that any trace on $\mathcal{A}$ is a linear functional on $\text{HH}_0(\mathcal{A})$ and a Hochschild 0-cocycle of $\mathcal{A}$. Note that the supertrace described in the previous paragraph is a Hochschild 0-cocycle of $\text{Diff}(\mathcal{E})$.

In [FLS], a Hochschild 0-cocycle $\text{tr}$ on $\text{Diff}^\bullet(\mathcal{E})$ is constructed algebraically. This Hochschild cocycle is constructed in such a way that the following diagram commutes.

$$\begin{array}{ccc}
\text{HH}_0(\text{Diff}(\mathcal{E})) & \longrightarrow & \text{HH}_0(\text{Diff}^\bullet(\mathcal{E})) \\
\text{str} \downarrow & & \downarrow \text{tr} \\
\mathbb{C} & \xrightarrow{\text{id}} & \mathbb{C}
\end{array}$$
The top row of the above diagram is the natural map from $\mathrm{HH}_0(\text{Diff}(\mathcal{E}))$ to $\hat{\mathrm{HH}}_0(\text{Diff}^\bullet(\mathcal{E}))$ arising out of the natural inclusion of $\text{Diff}(\mathcal{E})$ as a subalgebra of $\text{Diff}^\bullet(\mathcal{E})$.

The theory that is more related to the cohomology of $X$ is however, that of the completed Hochschild homology of the sheaf $\text{Diff}(\mathcal{E})$ of holomorphic differential operators on $\mathcal{E}$. This is defined in this paper in Section 3. Let $\hat{\mathrm{HH}}_i(\text{Diff}(\mathcal{E}))$ denote the $i$-th completed Hochschild homology of $\text{Diff}(\mathcal{E})$. By a lemma (Lemma 3) that generalizes a Theorem of Brylinski [Bryl],

$$\hat{\mathrm{HH}}_{-i}(\text{Diff}(\mathcal{E})) \simeq H^{2n-i}(X, \mathbb{C}).$$

Call this isomorphism $\beta_{\mathcal{E}}$. This is the key property of $\hat{\mathrm{HH}}_i(\text{Diff}(\mathcal{E}))$ that relates it to the cohomology of $X$ with coefficients in $\mathbb{C}$. [FLS] also use a notion of completed Hochschild homology that has the same relation with $H^\ast(X, \mathbb{C})$. However, as hinted in [FLS] itself, understanding the isomorphism $\beta_{\mathcal{E}}$ is easier with the definition of completed Hochschild homology used here.

[FLS] shows that the Hochschild cocycle tr of $\text{Diff}^\bullet(\mathcal{E})$ "extends" to yield a linear functional $\hat{\text{tr}}$ on $\hat{\mathrm{HH}}_0(\text{Diff}(\mathcal{E}))$. In this paper, we are forced to redo this very step in Section 3 since we use a definition of $\hat{\mathrm{HH}}_i(\text{Diff}(\mathcal{E}))$ that is a priori different from that of [FLS]. Our definition of $\hat{\mathrm{HH}}_i(\text{Diff}(\mathcal{E}))$ implies that there is a natural map from $\mathrm{HH}_0(\text{Diff}(\mathcal{E}))$ to $\hat{\mathrm{HH}}_0(\text{Diff}(\mathcal{E}))$ (see Section 3 for more details). The outcome of our "extending" tr to a linear functional $\hat{\text{tr}}$ on $\hat{\mathrm{HH}}_0(\text{Diff}(\mathcal{E}))$ is the following commutative diagram.

$$
\begin{array}{ccc}
\mathrm{HH}_0(\text{Diff}(\mathcal{E})) & \longrightarrow & \hat{\mathrm{HH}}_0(\text{Diff}(\mathcal{E})) \\
\downarrow \text{str} & & \downarrow \hat{\text{tr}} \\
\mathbb{C} & \longrightarrow & \mathbb{C} \\
\end{array}
$$

The linear functional we construct on $\hat{\mathrm{HH}}_0(\text{Diff}(\mathcal{E}))$ still uses the basic construction of [FLS] and, as will be clear later in this paper, yields an algebraic construction of $\int_X$ when $X$ has at least one vector bundle with non-zero Euler characteristic.

Denote the composite map $\hat{\text{tr}} \circ \beta_{\mathcal{E}}^{-1} : H^{2n}(X, \mathbb{C}) \to \mathbb{C}$ by $I_{\mathcal{E}}$. $I_{\mathcal{E}}$ is a linear functional on $H^{2n}(X, \mathbb{C})$. 

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The following theorem was proven in [FLS]. Their proof goes through in our setup as well.

**Theorem 1 (FLS).** If the Euler characteristic $\chi_E$ of $E$ is nonzero, then $I_E = \int_X$.

Further, in [FLS], it was conjectured that $I_E = \int_X$ for all vector bundles $E$, even those of zero Euler characteristic. In these notes, we prove the following theorem.

**Theorem 2.** If $E$ and $G$ are two vector bundles on $X$, then $I_E = I_G$.

The following corollary is immediate from this and Theorem 1.

**Corollary 1.** If $X$ has a vector bundle $G$ whose Euler characteristic $\chi_G$ is nonzero, then for any vector bundle $E$ on $X$, $I_E = \int_X$.

In particular, if $X$ is a smooth separated scheme over $\mathbb{C}$, then the skyscraper sheaf supported at a point $p$ of $X$ has nonzero Euler characteristic. On the other hand, it is a coherent $\mathcal{O}_X$-module. It therefore, has a finite resolution by (holomorphic) vector bundles on $X$. It follows that at least one vector bundle in such a resolution has non-zero Euler characteristic. Thus, if $E$ is a vector bundle on a smooth compact complex manifold $X$ that is a smooth separated scheme over $\mathbb{C}$, then $I_E = \int_X$.

Note that if $\mathcal{D}$ is a global holomorphic differential operator on a vector bundle $E$ on $X$, then $\mathcal{D}$ is a 0-cycle of the Hochschild chain complex of $\text{Diff}(E)$. It follows that $\mathcal{D}$ yields an element in $\tilde{\text{HH}}_0(\text{Diff}(E))$, and hence, an element in $\text{H}^{2n}(X, \mathbb{C})$ which we will denote by $[\mathcal{D}]$. It follows from Corollary 1 that if $X$ admits at least one vector bundle with non-zero Euler characteristic, then

$$\text{str}(\mathcal{D}) = \int_X [\mathcal{D}].$$

The above statement is somewhat similar to a result (Corollary 5.6) in a paper by P.Schapira and J-P.Schneiders [S-S]. More recently, an entirely different approach has yielded a proof of a stronger version of this statement (the supertrace theorem) without the condition that $X$ admit a vector bundle with non-zero Euler characteristic [EnFe]. In a sequel to this paper, we shall develop our approach further to show that $I_E = \int_X$ for any vector bundle $E$ on any compact complex manifold $X$. This shall yield another
proof of the supertrace theorem different from the one in [EnFe].

One also defines the completed cyclic homology \( \widehat{HC}_*(\text{Diff}(\mathcal{E})) \) of \( \text{Diff}(\mathcal{E}) \). Further, as in [FLS], the construction of the linear functional on the 0-th completed Hochschild homology of \( \text{Diff}(\mathcal{E}) \) is extended to the construction of multiple linear functionals \( \hat{\text{tr}}_{2i} : \widehat{HC}_{-2i}(\text{Diff}(\mathcal{E})) \to \mathbb{C} \). On the other hand, it is shown that one has an isomorphism

\[
\widehat{HC}_{-2i}(\text{Diff}(\mathcal{E})) \simeq H^{2n}(X, \mathbb{C}) \oplus H^{2n-2}(X, \mathbb{C}) \oplus \ldots \oplus H^{2n-2i}(X, \mathbb{C}) .
\]

Denote the inverse of this isomorphism by \( I_{\mathcal{E},2i} \). Let \( I_{\mathcal{E},2i} = \text{tr}_{2i} \circ I_{\mathcal{E},2i} \). Then,

\[
I_{\mathcal{E},2i} : H^{2n}(X, \mathbb{C}) \oplus H^{2n-2}(X, \mathbb{C}) \oplus \ldots \oplus H^{2n-2i}(X, \mathbb{C}) \to \mathbb{C} .
\]

For \( 0 \leq k \leq i \), let \( I_{\mathcal{E},2i,2k} : H^{2n-2k}(X, \mathbb{C}) \to \mathbb{C} \) be the composition of \( I_{\mathcal{E},2i} \) with the inclusion of \( H^{2n-2k}(X, \mathbb{C}) \) in \( H^{2n}(X, \mathbb{C}) \oplus H^{2n-2}(X, \mathbb{C}) \oplus \ldots \oplus H^{2n-2i}(X, \mathbb{C}) \) as a direct summand. In [FLS] it was conjectured that \( I_{\mathcal{E},2i,0} = C \int_X \) where \( C \) is a constant independent of \( \mathcal{E} \). Further, there was a prediction in [FLS] that \( I_{\mathcal{E},2,2} \) is the integral over a cycle related to the Chern classes of \( \mathcal{E} \) and those of vector bundles intrinsic to \( X \). In this note, we prove the following theorem.

**Theorem 3.** \( I_{\mathcal{E},2i,0} = I_{\mathcal{E}} \). Further, \( I_{\mathcal{E},2i,2k} = 0 \) if \( k > 0 \).

The latter part is more than what [FLS] expected in their work, though a piece of circumstantial evidence for \( I_{\mathcal{E},2,2} \) to be 0 is provided there. By this theorem and Corollary 1, \( I_{\mathcal{E},2i,0} = \int_X \) whenever \( X \) has a vector bundle with nonzero Euler characteristic. In particular, this happens when \( X \) is a smooth separated scheme over \( \mathbb{C} \).

### 1.1 Outline of this note

The proofs of theorems 2 and 3 are simple and involve two easy "bookkeeping" lemmas (Lemma 1 and Lemma 4). These are done in Sections 3 and 4 respectively.

Section 2 recalls basic notions about Hochschild homology that we require. In addition, Section 2 recalls the definition of an \( A_\infty \)-morphism between two differential graded algebras. It shows that an \( A_\infty \)-morphism \( F \) between two dg-algebras \( A \) and \( B \) induces a map \( F_{\text{hoch}} \) of complexes from the complex
of Hochschild chains of $\mathcal{A}$ to that of $\mathcal{B}$ (Proposition 2). No material in this section is new. Proposition 2 is from [FLS].

The first part of Section 3 recalls the basic set up for this note from [FLS]. In particular, we recall the construction of the Hochschild 0-cocycle $\text{tr}$ of $\text{Diff}^\bullet(\mathcal{E})$ from [FLS]. The completed Hochschild homology $\hat{\text{HH}}_*(\text{Diff}(\mathcal{E}))$ is defined in Section 3. We show that the formula for the Hochschild cocycle $\text{tr}$ "extends" to a formula for a linear functional $\hat{\text{tr}}$ on $\hat{\text{HH}}_0(\text{Diff}(\mathcal{E}))$. [FLS] in [1] do something very similar. The definition of $\hat{\text{HH}}_*(\text{Diff}(\mathcal{E}))$ that is used in this paper is however, a priori somewhat different from that used in [FLS]. This forces us to redo the parts where they extend the Hochschild cocycle $\text{tr}$ of $\text{Diff}^\bullet(\mathcal{E})$ to a linear functional on $\hat{\text{HH}}_0(\text{Diff}(\mathcal{E}))$.

The last part of Section 3 is devoted to the "bookkeeping Lemma for $\hat{\text{tr}}$" (Lemma 1).

Section 4 is devoted to understanding the relation between $\hat{\text{HH}}_*(\text{Diff}(\mathcal{E}))$ and $H^*(X, \mathbb{C})$. In particular, we prove that

$$\hat{\text{HH}}_{-i}(\text{Diff}(\mathcal{E})) \simeq H^{2n-i}(X, \mathbb{C})$$

(Corollary 6). For $\mathcal{E} = \mathcal{O}_X$, this was proven by Brylinski [Bryl]. Let $\beta_{\mathcal{E}}$ denote the isomorphism between $\hat{\text{HH}}_{-i}(\text{Diff}(\mathcal{E}))$ and $H^{2n-i}(X, \mathbb{C})$. Section 4 also proves the "bookkeeping Lemma for $\beta$" (Lemma 4). Theorem 2 is an immediate consequence of Lemma 1 and Lemma 4.

Section 5 has four subsections. The first subsection recalls the basic notions of cyclic homology that we require. The second one carefully examines the linear functionals on the completed cyclic homology of $\text{Diff}(\mathcal{E})$. The third subsection examines the relation between the completed cyclic homology of $\text{Diff}(\mathcal{E})$ and $H^*(X, \mathbb{C})$. The final subsection writes down the final steps that prove Theorem 3.

**Note:** All complexes that appear in this paper are cochain complexes by convention. The term *Hochschild cocycle* of a dg-algebra shall only refer to cocycles of the complex of Hochschild cochains of that algebra as defined in Definition 2, Section 2.1.

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2 Basic results about Hochschild homology

Conventions used in this section

1. The term "dg-algebra" in this section refers to a differential graded C-algebra with unit.

2. All complexes of C-vector space are cochain complexes, i.e, the differential of any complex of C-vector spaces has degree +1.

2.1 The Hochschild chain complex and Bar complex of a dg-algebra

Definition 1: If \( \mathcal{A} \) is a differential graded C-algebra with differential \( \delta \), the complex of Hochschild chains \( C^\bullet(\mathcal{A}) \) is the cochain complex obtained by equipping the graded C-vector space \( \oplus_{i \geq 1} \mathcal{A}[1]^\otimes i[-1] \) with the Hochschild differential. The Hochschild differential \( d \) is given by the formula

\[
d(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{i=n-1} (-1)^{(d_0+\ldots+d_i+i+1)} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n
\]

\[
+(-1)^{(d_n+1)(d_0+\ldots+d_{n-1}+n-1)} a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}
\]

\[
+ \sum_{j=0}^{j=n} (-1)^{(d_0+\ldots+d_{j-1}+j)} a_0 \otimes \ldots \otimes \delta(a_j) \otimes \ldots \otimes a_n
\]

for homogenous elements \( a_0, \ldots, a_n \) of \( \mathcal{A} \) of degrees \( d_0, \ldots, d_n \) respectively.
Let \( \tau_n : A^\otimes n \to A^\otimes n \) denote the endomorphism given by the formula
\[
a_1 \otimes \ldots \otimes a_n \mapsto (-1)^{(d_n+1)(d_1+\ldots+d_{n-1}+n-1)} a_n \otimes a_1 \otimes \ldots \otimes a_{n-1}
\]
for homogenous elements \( a_1, \ldots, a_n \) of \( A \) of degrees \( d_1, \ldots, d_n \) respectively. Let \( \partial_1 : A^\otimes n \to A^\otimes n-1 \) denote the morphism given by the formula
\[
a_1 \otimes \ldots \otimes a_n \mapsto (-1)^{d_1+1} a_1 a_2 \otimes \ldots \otimes a_n
\]
for homogenous elements \( a_1, \ldots, a_n \) of \( A \) of degrees \( d_1, \ldots, d_n \) respectively. Then, the Hochschild differential on \( A^\otimes k \) is also given by the formula
\[
i = \sum_{i=1}^{\min(k, n)} (-1)^{d_1+\ldots+d_i+1} a_i a_{i+1} \otimes \ldots \otimes a_n
\]
Contrary to the standard practice, we will refer to a degree \( -n \) cocycle of \( C^*(A) \) as a Hochschild \( n \)-cycle of \( A \).

The Hochschild homology \( \text{HH}_{-n}(A) \) of \( A \) is the \( -n \)th cohomology of the complex \( C^*(A) \). By the definition of the Hochschild complex, it is clear that if \( A \) is a \( C \)-algebra (i.e., \( A \) is concentrated in degree 0), then \( \text{HH}_0(A) = \frac{A}{[A, A]} \) where \([A, A]\) is the commutator of \( A \). Also, if \( A \) is a \( C \)-algebra, \( \text{HH}_{-n}(A) = 0 \) if \( n < 0 \). Also, if \( A \) is dg-algebra, then elements of \( \text{Ker}(\delta) \subset A^0 \) are 0-Hochschild cycles of \( C^*(A) \) where \( \delta \) denotes the internal differential on \( A \). It follows that there is a canonical map \( \text{Ker}(\delta) \to \text{HH}_0(A) \).

**Definition 2:** The complex of Hochschild cochains of \( A \) is the complex whose \( -i \)-th term is \( \text{Hom}_C(C^*(A), C) \) with differential induced by that on \( C^*(A) \). A Hochschild cocycle of \( A \) is a cocycle of the complex of Hochschild cochains of \( A \). Note that a Hochschild 0-cocycle of \( A \) induces a \( C \)-linear functional on \( \text{HH}_0(A) \).

**Definition 3:** The Bar complex \( \text{bar}^*(A) \) of \( A \) is the cochain complex obtained by equipping the graded \( C \)-vector space \( \bigoplus_{i \geq 1} A[i] \otimes [-1] \) with the Bar differential. The Bar differential is given by the formula
\[
d(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^{(d_0+\ldots+d_i+1)} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n
\]
\[
+ \sum_{j=0}^{n} (-1)^{(d_0+\ldots+d_{j-1}+j)} a_0 \otimes \ldots \otimes \delta(a_j) \otimes \ldots \otimes a_n
\]
for homogenous elements $a_0, ..., a_n$ of $A$ of degrees $d_0, ..., d_n$ respectively.

The Bar differential on $A^\otimes k$ is also given by the formula

$$i = k - 1 \sum_{i=1}^{i=k-1} \tau_{k-1} \circ \partial_1 \circ \tau_{k-i+1} + \sum_{i=1}^{i=k} \tau_{k-1} \circ (\delta \otimes id \otimes \ldots \otimes id) \circ \tau_{k-i+1}.$$ 

Recollection 1: If $s_n : \text{bar}^\bullet(A) \to \text{bar}^{\bullet-1}(A)$ is the map $a_0 \otimes \ldots \otimes a_n \mapsto -1 \otimes a_0 \otimes \ldots \otimes a_n$ then the maps $\{s_n\}, n \geq -1$ give a homotopy between $id : \text{bar}^\bullet(A) \to \text{bar}^\bullet(A)$ and $0 : \text{bar}^\bullet(A) \to \text{bar}^\bullet(A)$. It follows that $\text{bar}^\bullet(A)$ is acyclic.

If $V^\bullet$ is a finite dimensional graded $\mathbb{C}$-vector space then $\text{End}(V^\bullet) = \oplus_{i,j} \text{Hom}(V^i, V^j)$. Let $\pi_{i,j}$ denote the projection from $\text{End}(V^\bullet)$ to $\text{Hom}(V^i, V^j)$. If $M \in \text{End}(V^\bullet)$, the supertrace of $M$ is the alternating sum $\sum_i (-1)^i \text{tr}(\pi_{i,i}(M))$. We recall the following proposition from [FLS].

**Proposition 1.** Let $V^\bullet$ be a finite dimensional graded $\mathbb{C}$-vector space with zero differential.

1. $HH_i(\text{End}(V^\bullet)) = 0$ for $i \neq 0$
2. $HH_0(\text{End}(V^\bullet)) \simeq \mathbb{C}$
3. The composite $\text{End}(V^\bullet)^0 \to HH_0(\text{End}(V^\bullet)) \to \mathbb{C}$ takes an element of $\text{End}(V^\bullet)^0$ to its supertrace.
4. The isomorphism $HH_0(\text{End}(V^\bullet)) \simeq \mathbb{C}$ takes the class in $HH_0(\text{End}(V^\bullet))$ of a Hochschild 0-cycle in $\text{End}(V^\bullet)^\otimes k$ to 0 for all $k \geq 2$.

**Proof.** The proof of this proposition is a trivial modification of the proof of the Morita invariance of Hochschild homology for matrices in Loday [2](see Theorem 1.2.4 of [2]).

We may therefore, denote the isomorphism $HH_0(\text{End}(V^\bullet)) \simeq \mathbb{C}$ by $\text{str}$.

### 2.2 $A_\infty$-morphisms between dg-algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be two dg-algebras. Let $\mathcal{F}_i : A^\otimes i \to B[n_i]$ be $\mathbb{C}$-linear maps with $n_i \in \mathbb{Z}$ for each $i$. If $k_1, ..., k_l$ are positive integers such that $\sum_j k_j = k$ and if $\sum_j n_j = n$, then $\mathcal{F}_{k_1} \boxtimes \ldots \boxtimes \mathcal{F}_{k_l}$ will denote the $\mathbb{C}$-linear map from $A^\otimes k$ to $B^\otimes [n]$ such that

$$\mathcal{F}_{k_1} \boxtimes \ldots \boxtimes \mathcal{F}_{k_l}(a_1 \otimes \ldots \otimes a_k) = \mathcal{F}_{k_1}(a_1 \otimes \ldots \otimes a_{k_1}) \otimes \ldots \otimes \mathcal{F}_{k_l}(a_{k_1+\ldots+k_{l-1}+1} \otimes \ldots \otimes a_k).$$
Definition 2: An $A_\infty$-morphism $\mathcal{F}$ from a dg-algebra $A$ to a dg-algebra $B$ is a collection of maps
\[ F_k : A^{\otimes k} \rightarrow B[1-k] \]
for all $k \geq 1$ such that the map $F_{\text{bar}} : \text{bar}^* (A) \rightarrow \text{bar}^* (B)$ defined by
\[ F_{\text{bar}} (a_1 \otimes ... \otimes a_k) = \sum_{\{(k_1, ..., k_l) \mid l > 0 \text{ and } \sum j k_j = k\}} F_{k_1} \otimes ... \otimes F_{k_l} (a_1 \otimes ... \otimes a_k) \]
is a morphism of complexes from $\text{bar}^* (A)$ to $\text{bar}^* (B)$. The maps $F_k$ are called the Taylor components of $\mathcal{F}$.

The condition that $F_{\text{bar}}$ commutes with the differentials on the bar complexes of $A$ and $B$ respectively is equivalent to the condition that the maps $F_k$ satisfy the following relations.
\[
\begin{align*}
\pm F_{k-1} (a_1 a_2 \otimes a_3 \otimes ... \otimes a_k) & \mp F_{k-1} (a_1 \otimes a_2 a_3 \otimes ... \otimes a_k) \\
\pm F_{k-1} (a_1 \otimes a_3 \otimes ... \otimes a_{k-1} a_k) & \mp F_{k} (\delta (a_1 \otimes a_2 \otimes ... \otimes a_k)) \\
\pm \delta (F_k (a_1 \otimes ... \otimes a_k)) & \pm \sum_{l=1}^{l=k-1} F_l (a_1 \otimes ... \otimes a_l) \circ F_{k-l} (a_{l+1} \otimes ... \otimes a_k) = 0
\end{align*}
\]

$k \geq 1$

We recall the following proposition from [FLS].

Proposition 2. An $A_\infty$ morphism $\mathcal{F}$ from an associative dg-algebra $A$ to an associative dg-algebra $B$ induces a map $F_{\text{Hoch}}$ of complexes from the $C^* (A)$ to $C^* (B)$.

We recall the proof of this proposition from [FLS].

Proof. Let $\tau : A^{\otimes k} \rightarrow A^{\otimes k}$ be the map which takes $a_1 \otimes ... \otimes a_k$ to $(-1)^{(d_k+1)(d_1+...+d_{k-1}+k-1)} a_k \otimes a_1 \otimes ... \otimes a_{k-1}$ for homogenous elements $a_1, ..., a_k$ of $A$ of degrees $d_1, ..., d_k$ respectively. Consider the map $F_{\text{Hoch}} : C^* (A) \rightarrow C^* (B)$ defined by the formula
\[ F_{\text{Hoch}} (a_1 \otimes ... \otimes a_k) = \sum_{\{(k_1, ..., k_l) \mid l > 0 \text{ and } \sum j k_j = k\}} [F_{k_1} \otimes ... \otimes F_{k_l} (a_1 \otimes ... \otimes a_k)] \]
\[ + \sum_{j=1}^{j=k_l-1} \mathcal{F}_{k_l} \boxtimes \mathcal{F}_{k_l} \boxtimes \ldots \boxtimes \mathcal{F}_{k_l-1}(\tau^j(a_1 \otimes \ldots \otimes a_k)) \].

We leave the verification that \( \mathcal{F}_{\text{Hoch}} \) respects the Hochschild differential to the reader.

Let \( \mathcal{A} \) be a dg-algebra. Let \( V^\bullet \) be a finite dimensional graded \( \mathbb{C} \)-vector space with 0 differential. Suppose that \( \mathcal{F} \) is an \( A_\infty \)-morphism from \( \mathcal{A} \) to \( \text{End}(V^\bullet) \) with Taylor components \( \mathcal{F}_k \). Recall that for any \( k > 0 \), we have a map \( \tau : A^\otimes k \rightarrow A^\otimes k \) such that \( \tau(a_1 \otimes \ldots \otimes a_k) = (-1)^{(d_k+1)(d_1+\ldots+d_{k-1}+k-1)}a_k \otimes a_1 \otimes \ldots \otimes a_{k-1} \) for homogenous elements \( a_1, \ldots, a_k \) of \( \mathcal{A} \) of degrees \( d_1, \ldots, d_k \) respectively. By Proposition 1, \( \text{str} \) is a Hochschild 0-cocycle of \( \text{End}(V^\bullet) \).

We now have the following Corollary of Proposition 2.

**Corollary 2.** The supertrace on \( \text{End}(V^\bullet) \) pulls back to a Hochschild 0-cocycle \( \text{tr} \) of \( \mathcal{A} \). On Hochschild 0-cycles of \( \mathcal{A} \) that arise out of elements on degree \( k-1 \) in \( A^\otimes k \), the Hochschild cocycle \( \text{tr} \) is given by the map from \( A^\otimes k \) to \( \mathbb{C} \) given by

\[ a_1 \otimes \ldots \otimes a_k \mapsto \sum_{j=0}^{j=k-1} \text{str}(\mathcal{F}_k(\tau^j(a_1 \otimes \ldots \otimes a_k))) \].

**Proof.** The Hochschild cocycle \( \text{tr} \) is given by \( \text{tr}(x) = \text{str}(\mathcal{F}_{\text{hoch}}(x)) \) for any \( x \in C^0(\mathcal{A}) \). Note that by Proposition 2, \( \mathcal{F}_{\text{hoch}}(x) \in C^0(\text{End}(V^\bullet)) \). The exact formula for \( \text{tr} \) given in this corollary is now immediate from the formula for \( \mathcal{F}_{\text{hoch}} \) given in the proof of Proposition 2.

\[ \square \]

3 The completed Hochschild homology of \( \text{Diff}(\mathcal{E}) \) and a linear functional on \( \hat{\text{HH}}_0(\text{Diff}(\mathcal{E})) \).

3.1 The basic construction in [FLS].

Let \( K^\bullet_{\mathcal{E}} \) denote the Dolbeaux complex of \( \mathcal{E} \), as in the introduction to this paper. Then, \( K^\bullet_{\mathcal{E}} \) decomposes as the direct sum of a complex with zero differential and an acyclic complex i.e, \( K^\bullet_{\mathcal{E}} = K^0_{\mathcal{E}} \oplus K^1_{\mathcal{E}} \) where \( H^\bullet(K^0_{\mathcal{E}}) = H^\bullet(K^\bullet_{\mathcal{E}}) \), \( K^0_{\mathcal{E}} \) has 0 differential and \( K^1_{\mathcal{E}} \) is acyclic. This is
a consequence of Hodge theory (for instance, see Theorem 5.24 in [Vois]).

Recall that $\mathcal{D}iff^* (\mathcal{E}) = \text{Dolb}^* (X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{D}iff(\mathcal{E})$ and that

$$\mathcal{D}iff^* (\mathcal{E}) := \Gamma (X, \mathcal{D}iff^* (\mathcal{E})) .$$

The main construction in [FLS] is that of an $A_{\infty}$ morphism $F$ from $\mathcal{D}iff^*(\mathcal{E})$ to $\text{End}(K^*_{\mathcal{0}} \mathcal{E})$. Note that $K^*_{\mathcal{0}} \mathcal{E}$ is a finite dimensional $\mathbb{C}$-vector space with $0$ differential. As $\mathbb{C}$-vector spaces, $K^i_{\mathcal{0}} \mathcal{E} \simeq H^i (X, \mathcal{E})$. We may therefore apply Propositions 1, 2 and Corollary 2 of Section 2 with $A = \mathcal{D}iff^*(\mathcal{E})$ and $B = \text{End}(K^*_{\mathcal{0}} \mathcal{E})$. We obtain the following facts immediately.

**Fact 1:** By Proposition 1, $\text{HH}_i (\text{End}(K^*_{\mathcal{0}} \mathcal{E})) = 0$ for $i \neq 0$ and $\text{HH}_0 (\text{End}(K^*_{\mathcal{0}} \mathcal{E})) \simeq \mathbb{C}$. This isomorphism is induced by the map taking a degree $0$ element of $\text{End}(K^*_{\mathcal{0}} \mathcal{E})$ to its supertrace.

**Fact 2:** By Proposition 2, the supertrace on $\text{End}(K^*_{\mathcal{0}} \mathcal{E})$ pulls back to a Hochschild $0$-cocycle $\text{tr}$ on $\mathcal{D}iff^* (\mathcal{E})$. If $D$ is a Hochschild $0$-cycle of $\mathcal{D}iff^* (\mathcal{E})$ that arises out of a degree $k - 1$ element of $\mathcal{D}iff^*(\mathcal{E})^{\otimes k}$, and if $[D]$ denotes the class of $D$ in $\text{HH}_0 (\mathcal{D}iff^* (\mathcal{E}))$, then

$$\text{tr}([D]) = \sum_{j=0}^{j=k-1} \text{str}(\mathcal{F}_k (\tau^j (D))) .$$

**Notation:** We shall also denote the map

$$D \mapsto \sum_{j=0}^{j=k-1} \text{str}(\mathcal{F}_k (\tau^j (D)))$$

by $I_{\text{FLS}}$.

### 3.1.1 Construction of $F$

We now recall the construction of $F$ from [FLS]. Let $C_k$ denote the configuration space $\{ t_1 < ... < t_k \mid t_i \in \mathbb{R} \}/G^{(1)}$ where $G^{(1)}$ is the one dimensional group of shifts $(t_1, ..., t_k) \mapsto (t_1 + c, ..., t_k + c)$. This is a smooth $k - 1$ dimensional manifold though it is not compact if $k \geq 2$. Let $\tau_i = t_{i+1} - t_i$ for $1 \leq i \leq k - 1$. The map $(t_1, ..., t_k) \mapsto (\tau_1, ..., \tau_{k-1})$ is a diffeomorphism between $C_k$ and the product $\prod_{i=1}^{i=k-1} \{ \tau_i > 0 \}$. Let $\{ \tau_i > 0 \}$ denote the compactification of $\{ \tau_i \geq 0 \}$ by a point at infinity. The cube $\overline{C_k} := \prod_{i=1}^{i=k-1} \{ \tau_i > 0 \}$
is a compactification of $C_k$.

Let $D = D_1 \otimes ... \otimes D_k \in \text{Diff}(\mathcal{E})^\otimes k$. Recall that the $D_i$ yield endomorphisms of $K^\bullet \mathcal{E}$ as follows: Let $U \subset X$ be an open ball on which $\mathcal{E}$ is trivial. Let $z_1, ..., z_n$ be local holomorphic coordinates on $U$. A section of $K^\bullet \mathcal{E} |_U$ is a linear combination of sections of the form $s \otimes d\bar{z}_{i_1} \wedge ... \wedge d\bar{z}_{i_m}$. $D_i(s \otimes d\bar{z}_{i_1} \wedge ... \wedge d\bar{z}_{i_m}) = D_i(s) \otimes d\bar{z}_{i_1} \wedge ... \wedge d\bar{z}_{i_m}$.

Let $\bar{\partial} \mathcal{E}^*$ be the Hodge adjoint of $\bar{\partial} \mathcal{E}$. Let $\Delta \mathcal{E}$ denote the Laplacian of $\bar{\partial} \mathcal{E}^*$. We also note that $\bar{\partial} \mathcal{E}, \bar{\partial} \mathcal{E}^*$ and $\Delta \mathcal{E}$ yield endomorphisms of $K^\bullet \mathcal{E}$.

Let $\Omega_D$ denote the differential form on $C_k$ with values in $\text{End}(K^\bullet_0 \mathcal{E})$ given by the formula

$$\Omega_D = \Pi_{K^\bullet_0 \mathcal{E}} \circ D \circ \exp[-d(t_k - t_{k-1})\bar{\partial} \mathcal{E}^* - (t_k - t_{k-1})\Delta \mathcal{E}] \circ ... \circ D_1 \circ \mathcal{I}_{K^\bullet_0 \mathcal{E}}$$

where $\Pi_{K^\bullet_0 \mathcal{E}}$ and $\mathcal{I}_{K^\bullet_0 \mathcal{E}}$ denote the projection from $K^\bullet \mathcal{E}$ to $K^\bullet_0 \mathcal{E}$ and the inclusion from $K^\bullet_0 \mathcal{E}$ to $K^\bullet \mathcal{E}$ respectively.

As noted in [FLS], to write $\Omega_D$ this way, we require that $\Delta \mathcal{E}$ have discrete non negative spectrum (which is the case for a compact complex manifold). Further, as noted in [FLS], $\Omega_D$ extends to a $\text{End}(K^\bullet_0 \mathcal{E})$-valued (non-homogenous) differential form on $C_k$.

We define

$$\mathcal{F}_k(D) = \int_{C_k} \Omega_D = \int_{\overline{C_k}} \Omega_D .$$

By the integral over $\overline{C_k}$ of a non-homogenous differential form, we mean the integral over $\overline{C_k}$ of its component of top De-Rham degree. It helps to view the differential forms above as differential forms on $\overline{C_k}$ rather than on $C_k$ as that will ensure that the integrals defining the $\mathcal{F}_k$’s converge.

Proving that that $\mathcal{F}_k$ defined in this manner are the Taylor coefficients of an $A_\infty$ morphism is done in [FLS].

Before we proceed further, we note that if $k = 1$, then $C_1$ is a point. The formula for $\mathcal{F}_k$ given here yields

$$\mathcal{F}_1(D) = \Pi_{K^\bullet_0 \mathcal{E}} \circ D \circ \mathcal{I}_{K^\bullet_0 \mathcal{E}}$$

for $D \in \text{Diff}(\mathcal{E})$. In particular, if $\mathcal{D} \in \text{Diff}(\mathcal{E})$ and if $\mathcal{D}_* \in \text{End}(H^\bullet(X, \mathcal{E}))$ is the endomorphism of $H^\bullet(X, \mathcal{E})$ induced by $\mathcal{D}$, then $\mathcal{F}_1(\mathcal{D}) = \mathcal{D}_*$. It follows
from Fact 2, Section 3.1 that tr([D]) = str(D). This proves that the following diagram commutes.

\[
\begin{array}{ccc}
HH_0(\text{Diff}(\mathcal{E})) & \longrightarrow & HH_0(\text{Diff}^\bullet(\mathcal{E})) \\
\downarrow \text{str} & & \downarrow \text{tr} \\
\mathbb{C} & \longrightarrow & \mathbb{C}
\end{array}
\]

3.1.2 Rewriting the formula for \( F_k \).

Given an endomorphism \( \varphi \) of \( K^\bullet \mathcal{E} \), let \([\varphi]_i\) denote the endomorphism \( id \otimes \ldots \otimes \varphi \otimes \ldots \otimes id \) of \( K^\bullet \mathcal{E}^{\otimes k} \) with \( \varphi \) acting on the \( i \)th factor from the right.

Let

\[
\Phi = [id]_k \circ [\exp[-d(t_k - t_{k-1}) \partial_{\mathcal{E}}^* - (t_k - t_{k-1}) \Delta_{\mathcal{E}}]]_{k-1} \circ \ldots
\]

\[
\ldots \circ [\exp[-d(t_2 - t_1) \partial_{\mathcal{E}}^* - (t_2 - t_1) \Delta_{\mathcal{E}}]]_1.
\]

This is a differential form on \( C_k \) with values in \( \text{End}(K^\bullet_{\mathcal{E}} \otimes^k) \) though it is not a differential operator. Similarly, if \( D = D_1 \otimes \ldots \otimes D_k \in \text{Diff}^\bullet(\mathcal{E})^{\otimes k} \), \( D \) yields an endomorphism \( D := [D_1]_k \circ \ldots \circ [D_k]_1 \) of \( \text{End}(K^\bullet_{\mathcal{E}} \otimes^k) \).

We have a composition map \( m : \text{End}(K^\bullet_{\mathcal{E}} \otimes^k) \rightarrow \text{End}(K^\bullet_{\mathcal{E}}) \). Identifying \( \text{End}(K^\bullet_{\mathcal{E}}^{\otimes k}) \) with \( \text{End}(K^\bullet_{\mathcal{E}} \otimes^k) \), we obtain a composition map \( m : \text{End}(K^\bullet_{\mathcal{E}} \otimes^k) \rightarrow \text{End}(K^\bullet_{\mathcal{E}}) \). We recall from [FLS] that the formula for \( \Omega_D \) can be rewritten as follows

\[
\Omega_D = \prod_{K^\bullet_{0, \mathcal{E}}} m(\Phi \circ D) \circ \mathcal{I}_{K^\bullet_{0, \mathcal{E}}}.
\]

Thus,

\[
\mathcal{F}_k(D) = \int_{C_k} \prod_{K^\bullet_{0, \mathcal{E}}} m(\Phi \circ D) \circ \mathcal{I}_{K^\bullet_{0, \mathcal{E}}}.
\]

3.2 Extending the supertrace-I.

Recall that \( \text{Diff}(\mathcal{E}) \) denotes the sheaf of holomorphic differential operators on \( \mathcal{E} \). Let \( \mathcal{D}(\mathcal{E})(U) \) denote \( \Gamma(U, \text{Diff}(\mathcal{E})) \) for any open \( U \subset X \). Let \( \mathcal{E}^{\otimes_k} \) denote the \( k \)-fold exterior tensor power of \( \mathcal{E} \) on \( X^k := X \times \ldots \times X \). We observe that the differential on the Hochschild complex \( C^\bullet(\text{Diff}(\mathcal{E})(U)) \) extends to a differential on the graded vector space \( \oplus_{k \geq 1} \text{Diff}(\mathcal{E}^{\otimes_k}(U^k))[k - 1] \). The
resulting complex is called the completed Hochschild complex of $\text{Diff}(\mathcal{E})(U)$ and denoted by $\hat{\mathcal{C}}^\bullet(\text{Diff}(\mathcal{E})(U))$.

**Definition 4:** The completed Hochschild complex of $\text{Diff}(\mathcal{E})$ is the sheaf of complexes associated to the presheaf $U \mapsto \hat{\mathcal{C}}^\bullet(\text{Diff}(\mathcal{E})(U))$ of complexes of $\mathbb{C}$-vector spaces.

The completed Hochschild complex of $\text{Diff}(\mathcal{E})$ is a sheaf of complexes of $\mathbb{C}$-vector spaces on $X$. It is denoted in this paper by $\hat{\text{hoch}}(\text{Diff}(\mathcal{E}))$.

**Definition 5:** The $i$th completed Hochschild homology $\hat{\text{HH}}^i(\text{Diff}(\mathcal{E}))$ of $\text{Diff}(\mathcal{E})$ is the hypercohomology $\mathbb{H}^i(X, \hat{\text{hoch}}(\text{Diff}(\mathcal{E})))$ of $\hat{\text{hoch}}(\text{Diff}(\mathcal{E}))$.

Recall that the $A_\infty$-map $\mathcal{F}$ whose construction we recalled in section 3.1 enables us to pull back the supertrace on $\text{End}(K^\bullet \mathcal{E})$ to a Hochschild 0-cocycle of $\text{Diff}^\bullet(\mathcal{E})$. Unfortunately, the $A_\infty$-map $\mathcal{F}$ does not automatically enable us to directly pull back the supertrace to a $\mathbb{C}$-linear functional on $\hat{\text{HH}}_0(\text{Diff}(\mathcal{E}))$.

This subsection is devoted to an important intermediate step that enables us to construct a $\mathbb{C}$-linear functional on $\hat{\text{HH}}_0(\text{Diff}(\mathcal{E}))$ which extends the supertrace on $\text{Diff}(\mathcal{E})$.

### 3.2.1 “Estimating” $I_{\text{FLS}}$.

We now use a homotopy very similar to the homotopy in Prop 3.1 [Tel] to show that the FLS functional of any 0-cycle in $C^\bullet(\text{Diff}^\bullet(\mathcal{E}))$ depends only on its component in $\text{Diff}^0(\mathcal{E})$.

**Construction 1:** Let $\phi : X \times X \to [0, \infty)$ be a Riemannian distance. Let $t$ be any positive real number. We can choose a finite cover of $X$ by open sets $U_i, 1 \leq i \leq m$ such that $\phi(x, y) < t$ for all $x, y \in U_i$ for any $i$. Choose a partition of unity $\{f_i\}$ by compactly supported (nonnegative valued) smooth functions subordinate to the cover $\{U_i\}$. Let $g_i$ be a compactly supported smooth function on $U_i$ with values in $[0, 1]$ that is identically 1 on the support of $f_i$. Then, $f := \sum_{i=1}^m f_i \otimes g_i$ is a smooth function on $X \times X$ whose restriction to the diagonal is identically 1. Also, $f$ vanishes outside the subset $\{\phi(x, y) \leq t\}$ of $X \times X$, and the maximum value of $f$ on $X \times X$ is 1.
Somewhat as in [Tel], let $E_f : \text{Diff}^\bullet(\mathcal{E})^\otimes k \to \text{Diff}^\bullet(\mathcal{E})^\otimes k+1$ be the map

$$D_1 \otimes \ldots \otimes D_k \mapsto - \sum_{i=1}^{i=m} f_i \otimes g_i D_1 \otimes D_2 \otimes \ldots \otimes D_k .$$

Let $d$ be the differential of the complex $C^\bullet(\text{Diff}^\bullet(\mathcal{E}))$. Then,

$$dE_f + E_f d = 1 - N_f$$

where

$$N_f(D_1 \otimes \ldots \otimes D_k) = \pm \sum_{i=1}^{i=m} \partial f_i \otimes g_i D_1 \otimes \ldots \otimes D_k \pm \sum_{i=1}^{i=m} f_i \otimes (\partial g_i) D_1 \otimes \ldots \otimes D_k$$

$$\pm \sum_{i=1}^{i=m} f_i \otimes g_i D_k D_1 \otimes \ldots \otimes D_{k-1} \pm \sum_{i=1}^{i=m} D_k f_i \otimes g_i D_1 \otimes \ldots \otimes D_{k-1} \quad \forall \ k \geq 2 . \quad (1)$$

**Basic argument.** Note that if $\alpha$ is a cycle in $C^\bullet(\text{Diff}^\bullet(\mathcal{E}))$, then $\alpha$ is homotopic to $N_f \alpha$. This is true for all $t > 0$. Also, the FLS linear functional $I_{\text{FLS}}$ is a Hochschild 0-cocycle of $\text{Diff}^\bullet(\mathcal{E})$. It follows that

$$I_{\text{FLS}}(\alpha) = I_{\text{FLS}}(N_f \alpha) .$$

Let $\alpha = \alpha_1 + \ldots + \alpha_p$ with $\alpha_i \in \text{Diff}^\bullet(\mathcal{E})^\otimes i$. We then show that

$$\sum_{i \geq 2} |I_{\text{FLS}}(N_f \alpha_i)| \leq C \epsilon(t)$$

where $C$ is a constant that only depends on $\alpha$ and $\epsilon(t)$, $\text{Vol}(X \times X)$ is the volume of the subset $\{ \phi(x, y) \leq t \}$ of $X \times X$. Similarly, we show that $|I_{\text{FLS}}(\alpha_1) - I_{\text{FLS}}(N_f(\alpha_1))| \leq C' \epsilon(t)$ where $C'$ depends only on $\alpha$. Since the construction of $f$ as in Construction 1 is possible for all positive $t$, letting $t$ approach 0 we see that $I_{\text{FLS}}(\alpha) = I_{\text{FLS}}(\alpha_1)$.

**Remark:** Recall that $K^\bullet_0 E$ can be identified with the kernel of the Laplacian $\Delta_{\mathcal{E}}$. Let $K^\bullet_{L^2 E}$ denote the Hilbert space of square integrable sections of $\text{Dolb}^\bullet(X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \text{Diff}(\mathcal{E})$. Then $I_{K^\bullet_0 E} \circ \Pi K^\bullet_0 E$ is an integral operator on $K^\bullet_{L^2 E}$ with smooth kernel that projects onto the image of $I_{K^\bullet_0 E}$ (see [BGV] Chapter 2). We will denote this operator by $\Pi_0 E$ or $\Pi_0$ when there is no confusion regarding the vector bundle being used. Also, one can check that if $M \in \text{End}(K^\bullet E)$, then $\Pi_0 M \Pi_0$ makes sense as a trace class operator on $K^\bullet_{L^2 E}$, and

$$\text{str}(\Pi_0 M \Pi_0) = \text{str}(\Pi K^\bullet_0 E \circ M \circ I_{K^\bullet_0 E}) .$$
Proposition 3. Let $D_1,\ldots,D_k \in \text{Diff}^\bullet(E)$. Then,

$$|I_{\text{FLS}}(N_f(D_1 \otimes \ldots \otimes D_k))| \leq C\epsilon(t) .$$

The constant $C$ above depends only on $D_1,\ldots,D_k$.

Proof. Part 1: The setup.

Recall that

$$N_f(D_1 \otimes \ldots \otimes D_k) = \pm \sum_{i=1}^{i=m} \tilde{\partial}f_i \otimes g_i D_1 \otimes \ldots \otimes D_k \pm \sum_{i=1}^{i=m} f_i \otimes (\tilde{\partial}g_i) D_1 \otimes \ldots \otimes D_k$$

$$\pm \sum_{i=1}^{i=m} f_i \otimes g_i D_k D_1 \otimes \ldots \otimes D_{k-1} \pm \sum_{i=1}^{i=m} D_k f_i \otimes g_i D_1 \otimes \ldots \otimes D_{k-1} \forall k \geq 2 .$$

We estimate the FLS functional of each of the summands on the RHS separately. Let $F_k$ denote the $k$-th Taylor component of the FLS $A_\infty$ map. Let $C_k$ denote the configuration space $\prod_{i=1}^{k-1} \{ \tau_i > 0 \}$. Then, if $\alpha_i \in \text{Diff}^\bullet(E)$,

$$F_k(\alpha_1 \otimes \ldots \otimes \alpha_k) = \int_{C_k} \Pi_0 \alpha_1 \tilde{\partial}^* e^{-\tau_1 \Delta} \alpha_2 \ldots e^{-\tau_{k-1} \Delta} \alpha_k \Pi_0 d\tau_1 \ldots d\tau_{k-1} .$$

Let $Q_\tau(x,y) \in \Gamma(X \times X, E \otimes \Omega^0 \otimes (\Omega^0)^* \otimes \mathcal{E}^*)$ denote the (smooth) kernel of the operator $\tilde{\partial}^* e^{-\tau \Delta}$. Let $p_\infty(x,y)$ denote the (smooth) kernel of $\Pi_0$. Then, the kernel of $F_k(\alpha_1 \otimes \ldots \otimes \alpha_k)$ is

$$\int_{C_k} \int_X p_\infty(y_0,x_1)\alpha_1(x_1,x_2)\ldots Q_{\tau_{k-1}}(x_1,x_2)\ldots \alpha_k(x_{k+1}) p_\infty(x_{k+1},y_0')|dx_1|\ldots|dx_k|d\tau_1\ldots d\tau_{k-1} .$$

Recall that

$$I_{\text{FLS}}(\alpha_1 \otimes \ldots \otimes \alpha_k) = \sum_{s=0}^{s=k-1} \str F_k((\sigma^s(\alpha))) .$$

Here $\sigma$ is a (signed) cyclic permutation. Each of the $k$ cyclic permutations of $\alpha_1 \otimes \ldots \otimes \alpha_k$ yields exactly one summand contributing towards $I_{\text{FLS}}(\alpha_1 \otimes \ldots \otimes \alpha_k)$. 

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Part 2: Estimating $I_{FLS}(\pm \sum_{i=1}^{i=m} \partial f_i \otimes g_i D_1 \otimes \ldots \otimes D_k)$.

The cyclic permutations of $\sum_{i=1}^{i=m} \partial f_i \otimes g_i D_1 \otimes \ldots \otimes D_k$ are as follows:

$$\pm \sum_{i=1}^{i=m} \partial f_i \otimes g_i D_1 \otimes \ldots \otimes D_k$$

$$\pm \sum_{i=1}^{i=m} D_{k-i+1} \otimes \ldots \otimes D_k \otimes \partial f_i \otimes g_i D_1 \otimes \ldots \otimes D_{k-i} \quad 1 \leq i \leq k - 1$$

$$\pm \sum_{i=1}^{i=m} g_i D_1 \otimes \ldots \otimes D_k \otimes \partial f_i \ .$$

We shall henceforth denote the kernel of an integral operator $T$ on $\Gamma(X, \mathcal{E} \otimes \Omega^{0,*}(X))$ by $\mathcal{T}$. Let $D^\circ$ denote the formal Adjoint of a differential operator $D$ on $\Gamma(X, \mathcal{E} \otimes \Omega^{0,*}(X))$. This is a differential operator on $\Gamma(X, \Omega^{0,*}(X) \otimes \mathcal{E}^*)$ since the bundle of densities on $X$ has a canonical trivialization (see [BGV] Chapter 2). With this in mind,

$$\mathcal{F}_{k+1}(\sum_{i=1}^{i=m} \partial f_i \otimes g_i D_1 \otimes \ldots \otimes D_k)$$

$$= \int_{C_{k+1}} \int_{X^{k+1}} \sum_{i=1}^{i=m} p_\infty(y_0, x_0)(\partial f_i(x_0)^e(x_0)Q_\tau_1(x_0, x_1)g_i(x_1)D_1, x_1 Q_\tau_1(x_1, x_2)\ldots$$

$$\ldots p_\infty(x_k, y_0')|dx_0|\ldots|dx_k|d\tau_1\ldots d\tau_k$$

$$= \pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{i=1}^{i=m} p_\infty(y_0, x_0)e_{x_0}f_i(x_0)Q_\tau_1(x_0, x_1)g_i(x_1)D_1, x_1 Q_\tau_1(x_1, x_2)\ldots$$

$$\ldots p_\infty(x_k, y_0')|dx_0|\ldots|dx_k|d\tau_1\ldots d\tau_k$$

$$= \pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{i=1}^{i=m} p_\infty(y_0, x_0)f_i(x_0)e_{x_0}Q_\tau_1(x_0, x_1)g_i(x_1)D_1, x_1 Q_\tau_1(x_1, x_2)\ldots$$

$$\ldots p_\infty(x_k, y_0')|dx_0|\ldots|dx_k|d\tau_1\ldots d\tau_k$$

$$= \pm \int_{C_{k+1}} \int_{X^{k+1}} [\partial^n_{x_0}p_\infty(y_0, x_0)]\sum_{i=1}^{i=m} f_i(x_0)g_i(x_1)[Q_\tau_1(x_0, x_1)][D_1, x_1 Q_\tau_1(x_1, x_2)]\ldots$$

$$\ldots p_\infty(x_k, y_0')|dx_0|\ldots|dx_k|d\tau_1\ldots d\tau_k$$
\[ \pm \int_{C_{k+1}} \int_{X^{k+1}} [p_{\infty}(y_0, x_0)] \left[ \sum_{i=1}^m f_i(x_0)g_i(x_1) \right] [\bar{\partial}x_0 Q_{\tau_1}(x_0, x_1)] [D_{1, \tau_1} Q_{\tau_1}(x_1, x_2)] \ldots \left[ \ldots p_{\infty}(x_k, y_0) \right] |dx_0| \ldots |dx_k| |d\tau_1| \ldots |d\tau_k \] 

(2)

The last equality in (2) is valid since \( X \) is compact. We also use the fact that if \( h(x, y) \in \Gamma(X \times X, \mathcal{E} \otimes \Omega^{0,0} \otimes \Omega^{0,0} \otimes \mathcal{E}^*) \), then \( a(x)h(x, y)b(y) = \pm a(x)b(y)h(x, y) \) for smooth functions \( a, b \) on \( X \). We now note that if \( p_{\tau} \) denotes the kernel of \( e^{-\tau \Delta} \) then

\[ Q_{\tau}(x, y) = \bar{\partial}^* p_{\tau}(x, y) = \bar{\partial}^* p_{(0,\infty)}(x)p_{\tau}(x, y)p_{(0,\infty)}(y) \]

where \( p_{(0,\infty)} \) is the projection to the span of the eigenvectors of \( \Delta \) corresponding to positive eigenvalues as in [BGV] Proposition 2.37. This is because \( \bar{\partial}^* \) kills the kernel of \( \Delta \). It follows from [BGV] Proposition 2.37 that

\[ ||Q_{\tau}(x, y)||_t \leq C ||.||_t e^{-\frac{1}{2} \lambda_1 \tau} \]

(3)

where \( ||.||_t \) is any \( C^l \) norm on \( \Gamma(X \times X, \mathcal{E} \otimes \Omega^{0,0} \otimes \Omega^{0,0} \otimes \mathcal{E}^*) \) and \( \lambda_1 \) is the first positive eigenvalue of \( \Delta \).

It follows from (3) that the sup-norm of each term within a square bracket in each integrand that involves a \( Q_{\tau_1} \) is bounded above by \( C_i e^{-\tau_1 \frac{\lambda_k}{k}} \) for some constant \( C_i \) depending only on \( D_1 \otimes \ldots \otimes D_k \). The remaining terms have finite sup-norm. The sup-norm of \( \sum_{i=1}^m f_i(x_0)g_i(x_1) \) is 1. Hence, there exists a positive constant \( C \) depending only on \( D_1 \otimes \ldots \otimes D_k \) such that

\[ ||\mathcal{F}_{k+1}(\sum_{i=1}^m \bar{\partial}f_i \otimes g_i D_1 \otimes \ldots \otimes D_k)||_0 \leq C \int_{C_{k+1}} e^{-\tau_1 \frac{\lambda_k}{k}}. \]

\[ \ldots e^{-\tau_k \frac{\lambda_k}{k}} \epsilon(t) \text{Vol}(X)^{k+1} d\tau_1 \ldots d\tau_k = \frac{2^k}{\lambda_1} C \epsilon(t) \text{Vol}(X)^{k+1}. \]

(4)

where \( \epsilon(t) \text{Vol}(X \times X) \) is the volume of the support of \( \{ \phi(x, y) \leq t \} \) in \( X \times X \). Now,

\[ \text{str}(\mathcal{F}_{k+1}(\pm \sum_{i=1}^m \bar{\partial}f_i \otimes g_i D_1 \otimes \ldots \otimes D_k)) \]

\[ = \pm \int_X \text{str}(\mathcal{F}_{k+1}(\sum_{i=1}^m \bar{\partial}f_i \otimes g_i D_1 \otimes \ldots \otimes D_k)(y_0, y_0)(y_0, y_0) |dy_0| \]

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It follows from (4) that

$$|\text{str}(\mathcal{F}_{k+1}(\pm \sum_{i=1}^{i=m} \bar{\partial}f_i \otimes g_i D_1 \otimes \ldots \otimes D_k))| \leq C_1 \epsilon(t)$$

for some constant $C_1$ depending on $D_1 \otimes \ldots \otimes D_k$ only.

The same method is used to estimate the other summands that contribute to $I_{\text{FLS}}(\pm \sum_{i=1}^{i=m} \bar{\partial}f_i \otimes g_i D_1 \otimes \ldots \otimes D_k)$ and show that those contributions are at most $C' \epsilon(t)$ as well - in fact the FLS functionals of the other summands of $N_f(D_1 \otimes \ldots \otimes D_k)$ are estimated in the same way. We however, present the detailed calculations in the next 2 parts to be extra careful.

**Part 3: Estimating $I_{\text{FLS}}(\pm \sum_{i=1}^{i=m} \bar{\partial}f_i \otimes g_i D_1 \otimes \ldots \otimes D_k)$ - II.**

We show the equivalent of the calculation (2) for the other summands contributing to $I_{\text{FLS}}(\pm \sum_{i=1}^{i=m} \bar{\partial}f_i \otimes g_i D_1 \otimes \ldots \otimes D_k)$. At the end of each calculation, the integrand in the integral computing the kernel of the operator whose supertrace we need is written as a product of terms marked by square brackets. The argument in Part 2 then works almost word for word to show that the corresponding contribution to $I_{\text{FLS}}(N_f(D_1 \otimes \ldots \otimes D_k))$ is bounded by a constant depending on the $D_i$’s times $\epsilon(t)$.

\[
\overline{F}_{k+1}(\sum_{i=1}^{i=m} D_{k-i+1} \otimes \ldots \otimes D_k \otimes \bar{\partial}f_l \otimes g_l D_1 \otimes \ldots \otimes D_{k-i})
\]

\[
= \int_{C_{k+1}} \int_{X^{k+1}} \sum_{i=1}^{i=m} p_\infty(y_0, x_0) D_{k-i+1, x_0} Q_{r_i}(x_0, x_1) \ldots
\]

\[
\ldots D_{k, x_{l-1}} Q_{r_l}(x_{l-1}, x_l) (\bar{\partial}f_l)(x_l) Q_{r_{l+1}}(x_l, x_{l+1}) g_l(x_{l+1}) D_{1, x_{l+1}} \ldots
\]

\[
\ldots p_\infty(x_k, y_0') |dx_0| \ldots |dx_k| d\tau_1 \ldots d\tau_k
\]

\[
= \pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{i=1}^{i=m} p_\infty(y_0, x_0) D_{k-i+1, x_0} Q_{r_i}(x_0, x_1) \ldots
\]

\[
\ldots D_{k, x_{l-1}} Q_{r_l}(x_{l-1}, x_l) \bar{\partial}_x f_l(x_l) Q_{r_{l+1}}(x_l, x_{l+1}) g_l(x_{l+1}) D_{1, x_{l+1}} \ldots
\]

\[
\ldots p_\infty(x_k, y_0') |dx_0| \ldots |dx_k| d\tau_1 \ldots d\tau_k
\]
\[
\pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} \rho(y_0, x_0) D_{k-x_{l+i+1}, x_0} Q_{\tau_l}(x_0, x_1) \ldots \\
\ldots D_{k-x_{i-1}, x_i} f_i(x_i) \bar{D}_{x_i} Q_{\tau_{i+1}}(x_i, x_{i+1}) g_i(x_{i+1}) \partial D_{1, x_{i+1}} \ldots \\
\ldots p(x_k, y_0') [dx_0] [dx_k] d\tau_1 \ldots d\tau_k
\]

\[
= \pm \int_{C_{k+1}} \int_{X^{k+1}} \left[ \rho(y_0, x_0) \right] \left[ D_{k-x_{l+i+1}, x_0} Q_{\tau_l}(x_0, x_1) \right] \ldots \\
\ldots \left[ D_{k-x_{i-1}, x_i} \partial D_{x_i} Q_{\tau_{i+1}}(x_i, x_{i+1}) \right] \left[ D_{1, x_{i+1}} \right] \ldots \\
\ldots \left[ p(x_k, y_0') \right] [dx_0] [dx_k] d\tau_1 \ldots d\tau_k
\]

\[
\begin{align*}
\ldots & \left[ D_{k-x_{i-1}, x_i} \partial D_{x_i} Q_{\tau_{i+1}}(x_i, x_{i+1}) \right] \left[ \sum_{l=1}^{l=m} f_i(x_i) g_i(x_{i+1}) \right] \left[ D_{1, x_{i+1}} \right] \ldots \\
\ldots & \left[ p(x_k, y_0') \right] [dx_0] [dx_k] d\tau_1 \ldots d\tau_k \\
\end{align*}
\]

\[
\mathcal{F}_{k+1} = \sum_{l=1}^{l=m} g_l D_l \otimes \ldots \otimes D_k \otimes \bar{D} f_i \\
= \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} \rho(y_0, x_0) g_l(x_0) D_{1-x_l, x_0} Q_{\tau_l}(x_0, x_1) \ldots \\
\ldots D_{k} Q_{\tau_k}(x_{k-1}, x_k) (\bar{D} f_i)(x_k) p(x_k, y_0') [dx_0] [dx_k] d\tau_1 \ldots d\tau_k \\
= \pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} \rho(y_0, x_0) g_l(x_0) D_{1-x_l, x_0} Q_{\tau_l}(x_0, x_1) \ldots \\
\ldots D_{k} Q_{\tau_k}(x_{k-1}, x_k) \bar{D}_{x_k} f_i(x_k) p(x_k, y_0') [dx_0] [dx_k] d\tau_1 \ldots d\tau_k \\
\pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} \rho(y_0, x_0) g_l(x_0) D_{1-x_l, x_0} Q_{\tau_l}(x_0, x_1) \ldots \\
\ldots D_{k} Q_{\tau_k}(x_{k-1}, x_k) f_i(x_k) \bar{D}_{x_k} p(x_k, y_0') [dx_0] [dx_k] d\tau_1 \ldots d\tau_k \\
= \pm \int_{C_{k+1}} \int_{X^{k+1}} [\rho(y_0, x_0)] \left[ \sum_{l=1}^{l=m} f_i(x_i) g_l(x_i) \right] [D_{1-x_l, x_0} Q_{\tau_l}(x_0, x_1)] \ldots
\]

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\[\ldots [D_k \partial_{x_k} Q \tau_k(x_{k-1}, x_k)] [p_{\infty}(x_k, y'_0)] |dx_0| \ldots |dx_k| d\tau_1 \ldots d\tau_k\]

\[\pm \int_{C_k} \int_{X_k+1} [p_{\infty}(y_0, x_0)] [\sum_{l=1}^{l=m} f_l(x_0) g_l(x_0)] [D_{1,x_0} Q \tau_1(x_0, x_1)] \ldots \]

\[\ldots [D_k Q \tau_k(x_{k-1}, x_k)] [\partial_{x_k} p_{\infty}(x_k, y'_0)] |dx_0| \ldots |dx_k| d\tau_1 \ldots d\tau_k\]  

(6)

Part 4: Estimating other summands contributing to \(I_{FLS}(N_f(D_1 \otimes \ldots \otimes D_k))\).

Recall that

\[N_f(D_1 \otimes \ldots \otimes D_k) = \pm \sum_{i=1}^{i=m} \partial f_i \otimes g_i D_1 \otimes \ldots \otimes D_k \pm \sum_{i=1}^{i=m} f_i \otimes (\partial g_i) D_1 \otimes \ldots \otimes D_k\]

\[\pm \sum_{i=1}^{i=m} f_i \otimes g_i D_k D_1 \otimes \ldots \otimes D_{k-1} \pm \sum_{i=1}^{i=m} D_k f_i \otimes g_i D_1 \otimes \ldots \otimes D_{k-1} \quad \forall \ k \geq 2.\]

The cyclic permutations of \(\sum_{l=1}^{l=m} f_l \otimes g_l D_k D_1 \otimes \ldots \otimes D_{k-1}\) that contribute to its FLS functional are as follows:

\[\sum_{l=1}^{l=m} f_l \otimes g_l D_k D_1 \otimes \ldots \otimes D_{k-1}\]

\[D_{k-2} \otimes \ldots \otimes D_{k-1} \otimes \sum_{l=1}^{l=m} f_l \otimes g_l D_k D_1 \otimes \ldots \otimes D_{k-1-2}\] for \(1 \leq i \leq k - 2\)

\[\sum_{l=1}^{l=m} g_l D_k D_1 \otimes \ldots \otimes D_{k-1} \otimes f_l\]

As in earlier parts,

\[\mathcal{F}_k \left( \sum_{i=1}^{i=m} f_l \otimes g_l D_k D_1 \otimes \ldots \otimes D_{k-1} \right)\]

\[= \int_{C_k} \int_{X_k} [p_{\infty}(y_0, x_1)] [\sum_{l=1}^{l=m} f_l(x_1) g_l(x_2)] [Q \tau_1(x_1,x_2)] [D_{k,x_2} D_1,x_2 \ldots]\]

\[\ldots [D_{k-1} p_{\infty}(x_k, y'_0)] |dx_1| \ldots |dx_k| d\tau_1 \ldots d\tau_{k-1}.\]  

(7)
\[ F_k(D_{k-1} \otimes \ldots \otimes D_{k-1} \otimes \sum_{l=1}^{l=m} f_l \otimes g_l D_k D_1 \otimes \ldots \otimes D_{k-i-1}) \]

\[ = \int_{C_k} \int_{X^n} [p_\infty(y_0, x_1)][D_{k-i,x_1}Q_{\tau_1}(x_1, x_2)] \]

\[ \ldots \sum_{l=1}^{l=m} f_l(x_{i+1})g_l(x_{i+2})[Q_{\tau_{i+1}}(x_{i+1}, x_{i+2})][D_{k,x_{i+2}}D_{1,x_{i+2}}] \ldots \]

\[ [..p_\infty(x_k, y'_0)]dx_1|\ldots|dx_k|d\tau_1|\ldots d\tau_{k-1} \quad (8) \]

\[ \mathcal{F}_k(\sum_{l=1}^{l=m} g_l D_k D_1 \otimes \ldots \otimes D_{k-1} \otimes f_l) \]

\[ = \int_{C_k} \int_{X^n} [p_\infty(y_0, x_1)]\left[\sum_{l=1}^{l=m} f_l(x_k)g_l(x_1)[D_{k,x_1}D_1,x_1, Q_{\tau_1}(x_1, x_2)]\right] \]

\[ \ldots [p_\infty(x_k, y'_0)]dx_1|\ldots|dx_k|d\tau_1|\ldots d\tau_{k-1} \quad (9) \]

For each of the summands dealt with by equations (7), (8) and (9) respectively, one can argue exactly as we did after (2).

The cyclic permutations of \( \sum_{l=1}^{l=m} D_k f_l \otimes g_l D_1 \otimes \ldots \otimes D_{k-1} \) that contribute to its FLS functional are

\[ \sum_{l=1}^{l=m} D_k f_l \otimes g_l D_1 \otimes \ldots \otimes D_{k-1} \]

\[ \sum_{l=1}^{l=m} D_{k-i} \otimes \ldots \otimes D_{k-1} \otimes D_k f_l \otimes g_l D_1 \otimes \ldots \otimes D_{k-i-1} \quad \text{for} \ 1 \leq i \leq k - 2 \]

\[ \sum_{l=1}^{l=m} g_l D_1 \otimes \ldots \otimes D_k f_l \]

\[ \mathcal{F}_k(D_k f_l \otimes g_l D_1 \otimes \ldots \otimes D_{k-1}) \]

\[ = \int_{C_k} \int_{X^n} \sum_{l=1}^{l=m} p_\infty(y_0, x_1)D_{k,x_1} f_l(x_1)Q_{\tau_1}(x_1, x_2)g_l(x_2)D_1,x_2 \ldots \]

\[ \ldots D_{k-1,x_k} p_\infty(x_k, y'_0)|dx_1|\ldots|dx_k|d\tau_1|\ldots d\tau_{k-1} \]
For each of the summands dealt with by equations (10), (11) and (12), respectively, one can argue exactly as we did after (2).
The cyclic permutations of \( \sum_{l=1}^{l=m} f_l \otimes (\bar{\partial} g_l) D_1 \otimes \ldots \otimes D_k \) that contribute to its FLS functional are as follows:

\[
\sum_{l=1}^{l=m} f_l \otimes (\bar{\partial} g_l) D_1 \otimes \ldots \otimes D_k
\]

\[
\sum_{l=1}^{l=m} D_{k-i} \otimes \ldots \otimes D_k \otimes f_l \otimes (\bar{\partial} g_l) D_1 \otimes \ldots \otimes D_{k-i-1} \text{ for } 0 \leq i \leq k - 2
\]

\[
\sum_{l=1}^{l=m} (\bar{\partial} g_l) D_1 \otimes \ldots \otimes D_k \otimes f_l.
\]

\[
F_{k+1}(\sum_{l=1}^{l=m} f_l \otimes (\bar{\partial} g_l) D_1 \otimes \ldots \otimes D_k)
\]

\[
\int_{C_{k+1}} \int_{X_{k+1}}^{l=m} \sum_{l=1}^{l=m} p_\infty(y_0, x_0) f_l(x_0) Q_{\tau_1}(x_0, x_1)(\bar{\partial} g_l)(x_1) D_{1,x_1} \ldots
\]

\[
\ldots p_\infty(x_k, y'_0)|dx_0|\ldots|dx_k|d\tau_1\ldots d\tau_k
\]

\[
= \pm \int_{C_{k+1}} \int_{X_{k+1}}^{l=m} \sum_{l=1}^{l=m} p_\infty(y_0, x_0) f_l(x_0) Q_{\tau_1}(x_0, x_1)\bar{\partial}_{x_1} g_l(x_1) D_{1,x_1} \ldots
\]

\[
\ldots p_\infty(x_k, y'_0)|dx_0|\ldots|dx_k|d\tau_1\ldots d\tau_k
\]

\[
\pm \int_{C_{k+1}} \int_{X_{k+1}}^{l=m} \sum_{l=1}^{l=m} p_\infty(y_0, x_0) f_l(x_0) Q_{\tau_1}(x_0, x_1) g_l(x_1)\bar{\partial}_{x_1} D_{1,x_1} \ldots
\]

\[
\ldots p_\infty(x_k, y'_0)|dx_0|\ldots|dx_k|d\tau_1\ldots d\tau_k
\]

\[
= \pm \int_{C_{k+1}} \int_{X_{k+1}}^{l=m} [p_\infty(y_0, x_0)] \sum_{l=1}^{l=m} f_l(x_0) g_l(x_1)] [\bar{\partial}_{x_1}^\alpha Q_{\tau_1}(x_0, x_1)] [D_{1,x_1} \ldots].
\]

\[
\ldots [\ldots p_\infty(x_k, y'_0)]|dx_0|\ldots|dx_k|d\tau_1\ldots d\tau_k
\]

\[
\pm \int_{C_{k+1}} \int_{X_{k+1}}^{l=m} [p_\infty(y_0, x_0)] \sum_{l=1}^{l=m} f_l(x_0) g_l(x_1)] [Q_{\tau_1}(x_0, x_1)] [\bar{\partial}_{x_1} D_{1,x_1} \ldots] \ldots
\]

\[
\ldots [\ldots p_\infty(x_k, y'_0)]|dx_0|\ldots|dx_k|d\tau_1\ldots d\tau_k
\]

\[
F_{k+1}(\sum_{l=1}^{l=m} D_{k-i} \otimes \ldots \otimes D_k \otimes f_l \otimes (\bar{\partial} g_l) D_1 \otimes \ldots \otimes D_{k-i-1})
\]

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\[
\int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} p_{\infty}(y_0, x_0) \ldots Q_{\tau_1}(x_{i-1}, x_i) f_l(x_i) Q_{\tau_{i+1}}(x_i, x_{i+1}) (\overline{\partial g_l})(x_{i+1}) D_{1,x_{i+1}} \ldots \\
\quad \ldots p_{\infty}(x_k, y_0') |dx_0| \ldots |dx_k| |d\tau_1| ... d\tau_k
\]

\[
= \pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} p_{\infty}(y_0, x_0) \ldots Q_{\tau_1}(x_{i-1}, x_i) f_l(x_i) Q_{\tau_{i+1}}(x_i, x_{i+1}) \overline{\partial x_{i+1}} g_l(x_{i+1}) D_{1,x_{i+1}} \ldots \\
\quad \ldots p_{\infty}(x_k, y_0') |dx_0| \ldots |dx_k| |d\tau_1| ... d\tau_k
\]

\[
= \pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} p_{\infty}(y_0, x_0) \ldots Q_{\tau_1}(x_{i-1}, x_i) f_l(x_i) Q_{\tau_{i+1}}(x_i, x_{i+1}) [\overline{\partial g_l} Q_{\tau_{i+1}}(x_i, x_{i+1})] \\
\quad [D_{1,x_{i+1}} \ldots \ldots p_{\infty}(x_k, y_0') |dx_0| \ldots |dx_k| |d\tau_1| ... d\tau_k]
\]

\[
= \pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} p_{\infty}(y_0, x_0) \ldots Q_{\tau_1}(x_{i-1}, x_i) [\sum_{l=1}^{l=m} f_l(x_i) g_l(x_{i+1})] [Q_{\tau_{i+1}}(x_i, x_{i+1})] \\
\quad [\overline{\partial x_{i+1}} D_{1,x_{i+1}} \ldots \ldots p_{\infty}(x_k, y_0') |dx_0| \ldots |dx_k| |d\tau_1| ... d\tau_k] \quad (14)
\]

\[
\mathcal{F}_{k+1}[\sum_{l=1}^{l=m} (\overline{\partial g_l}) D_1 \otimes ... \otimes D_k \otimes f_l)]
\]

\[
= \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} p_{\infty}(y_0, x_0) (\overline{\partial g_l}) (x_0) D_{1,x_0} Q_{\tau_1}(x_0, x_1) \ldots \\
\quad \ldots f_l(x_k) p_{\infty}(x_k, y_0') |dx_0| \ldots |dx_k| |d\tau_1| ... d\tau_k
\]

\[
= \pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} p_{\infty}(y_0, x_0) \overline{\partial x_0} g_l(x_0) D_{1,x_0} Q_{\tau_1}(x_0, x_1) \ldots \\
\quad \ldots f_l(x_k) p_{\infty}(x_k, y_0') |dx_0| \ldots |dx_k| |d\tau_1| ... d\tau_k
\]

\[
\pm \int_{C_{k+1}} \int_{X^{k+1}} \sum_{l=1}^{l=m} p_{\infty}(y_0, x_0) g_l(x_0) \overline{\partial x_0} D_{1,x_0} Q_{\tau_1}(x_0, x_1) \ldots \\
\quad \ldots f_l(x_k) p_{\infty}(x_k, y_0') |dx_0| \ldots |dx_k| |d\tau_1| ... d\tau_k
\]

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\[= \pm \int_{C^{k+1}} \int_{X^{k+1}} \left[\partial_{x_0} p_\infty(y_0, x_0) \right] \left[ \sum_{l=1}^{l=m} f_l(x_k) g_l(x_0) [D_{1, x_0} Q_{1, l}(x_0, x_1)] \right] \ldots \left[ \partial_{x_0} D_{1, x_0} Q_{1, l}(x_0, x_1) \right] \ldots \left[ p_\infty(x_k, y'_0) \right] dx_0 \ldots dx_k |d\tau_1 \ldots d\tau_k \]

\[\pm \int_{C^{k+1}} \int_{X^{k+1}} \left[ p_\infty(y_0, x_0) \right] \left[ \sum_{l=1}^{l=m} f_l(x_k) g_l(x_0) [D_{1, x_0} Q_{1, l}(x_0, x_1)] \right] \ldots \left[ \partial_{x_0} D_{1, x_0} Q_{1, l}(x_0, x_1) \right] \ldots \left[ p_\infty(x_k, y'_0) \right] dx_0 \ldots dx_k |d\tau_1 \ldots d\tau_k \]

(15)

For each of the summands dealt with by equations (13), (14) and (15) respectively, one can argue exactly as we did after (2).

This finally proves Proposition 3.

\[\square\]

**Proposition 4.** Let \( \beta = \beta_1 + \ldots + \beta_k \) be a 0-cycle in \( C^\ast(\text{Diff}^\ast(\mathcal{E})) \) with \( \beta_k \in \text{Diff}^\ast(\mathcal{E}) \otimes^k \). Then,

\[ I_{\text{FLS}}(\beta) = I_{\text{FLS}}(\beta_1) . \]

**Proof.** Pick any \( t > 0 \). If \( f \) is as in Construction 1, then the cycle \( \beta \) is homotopic to the cycle \( N_f(\beta) \). Since \( I_{\text{FLS}} \) is a Hochschild 0-cocycle of \( \text{Diff}^\ast(\mathcal{E}) \),

\[ I_{\text{FLS}}(\beta) = I_{\text{FLS}}(N_f(\beta)) = I_{\text{FLS}}(N_f(\beta_1)) + \ldots + I_{\text{FLS}}(N_f(\beta_k)) . \]

By proposition 4,

\[ \sum_{j=2}^{j=k} \left| I_{\text{FLS}}(N_f(\beta_j)) \right| \leq C\epsilon(t) \]

for some constant \( C \) dependent on \( \beta \) only. Also,

\[ N_f(\beta_1) = \beta_1 \pm \sum_{l=1}^{l=m} (\partial f_l) \otimes g_l \beta_1 \pm \sum_{l=1}^{l=m} f_l \otimes (\partial g_l) \beta_1 . \]

The proof of Proposition 4 can also be used to verify that

\[ \left| I_{\text{FLS}}(\sum_{l=1}^{l=m} (\partial f_l) \otimes g_l \beta_1 \pm \sum_{l=1}^{l=m} f_l \otimes (\partial g_l) \beta_1) \right| \leq C'\epsilon(t) \]
for some constant $C'$ dependent only on $\beta_1$. It follows that

$$|I_{\text{FLS}}(\beta) - I_{\text{FLS}}(\beta_1)| = |I_{\text{FLS}}(N_f(\beta) - I_{\text{FLS}}(\beta_1))| \leq (C + C')\epsilon(t).$$

Now, $t$ may be chosen to be arbitrarily small. In this case, $\epsilon(t)$ becomes arbitrarily small as well. As $C + C'$ depends only on $\beta$, the desired proposition follows.

Proposition 3 implies the following (surprising) statement.

**Proposition 5.** The linear functional

$$D \mapsto \text{str}(\Pi_0 D \Pi_0)$$

vanishes on 0-chains of $\text{Diff}^\bullet(\mathcal{E})$ that are commutators of elements in $\text{Diff}^\bullet(\mathcal{E})$.

**Proof.** Let $D_1, D_2 \in \text{Diff}^0(\mathcal{E})$. This proposition is immediate from Proposition 4 and the following facts.

1. $I_{\text{FLS}}(d(D_1 \otimes D_2)) = 0$ since $I_{\text{FLS}}$ is a Hochschild 0-cocycle.

2. The component of $d(D_1 \otimes D_2)$ in $\text{Diff}^\bullet(\mathcal{E})$ is precisely $-D_1D_2 + D_2D_1$.

Of course, when $D$ is the commutator of two purely holomorphic differential operators on $\mathcal{E}$, the above proposition is standard. The above proposition in general, is however very counterintuitive and the author does not see any other way of proving it.

### 3.3 Extending the supertrace II.

Let $\mathcal{D}\text{iff}^\bullet(\mathcal{E})(U)$ denote $\Gamma(U, \mathcal{D}\text{iff}^\bullet(\mathcal{E}))$. Note that the differential on $C^\bullet(\mathcal{D}\text{iff}^\bullet(\mathcal{E})(U))$ extends to a differential on the graded vector space $\oplus_k \mathcal{D}\text{iff}^\bullet(\mathcal{E}^{\otimes k})(U^k)[k - 1]$. We denote the resulting complex by $\widehat{C}^\bullet(\mathcal{D}\text{iff}^\bullet(\mathcal{E})(U))$.

Let $\text{hoch}(\mathcal{D}\text{iff}(\mathcal{E}))$ denote the sheaf associated to the presheaf

$$U \mapsto \widehat{C}^\bullet(\mathcal{D}\text{iff}^\bullet(\mathcal{E})(U))$$
of complexes of \( C \)-vector spaces. Note that \( \widetilde{\text{hoch}}(\text{Diff}(\mathcal{E})) \) is a complex whose terms are modules over the sheaf of smooth functions on \( X \). It follows that

\[
\mathbb{H}^\bullet(\Gamma(X, \widetilde{\text{hoch}}(\text{Diff}(\mathcal{E})))) \cong \mathbb{H}^\bullet(X, \widetilde{\text{hoch}}(\text{Diff}(\mathcal{E}))).
\]

The following proposition follows from Proposition 4.

**Proposition 6.** \( I_{FLS} \) extends to a linear functional on \( \Gamma(X, \widetilde{\text{hoch}}(\text{Diff}(\mathcal{E}))) \) that vanishes on the image of the differential \( d \) of \( \Gamma(X, \widetilde{\text{hoch}}(\text{Diff}(\mathcal{E}))) \).

**Proof.**

**Part 1: Constructing \( \hat{\text{str}} \):**

Let \( \widetilde{\text{hoch}}^k(\text{Diff}(\mathcal{E})) \) denote the sheaf associated to the presheaf

\[
U \mapsto \text{Diff}^\bullet(\mathcal{E}^\otimes k)(U^k)[k-1]
\]

of graded \( C \)-vector spaces. Then, any 0-cycle in \( \Gamma(X, \widetilde{\text{hoch}}^2(\text{Diff}(\mathcal{E}))) \) is given by \( \beta_1 + \ldots + \beta_k \) where \( \beta_i \in \Gamma(X, \widetilde{\text{hoch}}^i(\text{Diff}(\mathcal{E}))) \). Note that \( \widetilde{\text{hoch}}^1(\text{Diff}(\mathcal{E})) = \text{Diff}^\bullet(\mathcal{E}) \). It follows that \( \beta_1 \) is an element of \( \text{Diff}^0(\mathcal{E}) \).

Let \( \hat{\text{str}} \) be the linear functional on \( \Gamma(X, \widetilde{\text{hoch}}(\text{Diff}(\mathcal{E}))) \) that vanishes on \( p \)-chains whenever \( p \neq 0 \). For a zero chain \( \beta = \beta_1 + \ldots + \beta_k \) with \( \beta_i \in \Gamma(X, \widetilde{\text{hoch}}^i(\text{Diff}(\mathcal{E}))) \), set

\[
\hat{\text{str}}(\beta) = I_{FLS}(\beta_1).
\]

This is a well defined linear functional on the space of 0-chains of \( \Gamma(X, \widetilde{\text{hoch}}(\text{Diff}(\mathcal{E}))) \).

**Part 2:**

We now need to show that \( \hat{\text{str}} \) vanishes on the image of \( d \).

**Claim A:** The natural map from \( \text{Diff}^\bullet(\mathcal{E}^\otimes 2)[1] \) to \( \Gamma(X, \widetilde{\text{hoch}}^2(\text{Diff}(\mathcal{E}))) \) is a surjection of graded \( C \)-vector spaces.

We postpone the proof of Claim A for the next part of the proof. Let Note that the differential on \( C^\bullet(\text{Diff}^\bullet(\mathcal{E})) \) extends to a differential \( d \) on
\[ \oplus_k \text{Diff}^\bullet(\mathcal{E}^{\leq k})[k-1]. \] This differential is a sum of the differentials \( d_H : \text{Diff}^\bullet(\mathcal{E}^{\leq k}) \to \text{Diff}^\bullet(\mathcal{E}^{\leq k-1}) \) and \( \tilde{\partial} \). By Proposition 4 and Claim 1, to show that \( \hat{\text{str}} \) vanishes on the image of \( d \), it suffices to show that the map \( \text{Diff}^0(\mathcal{E}) \to \mathbb{C} \)

\[ D \mapsto \text{Str}(\Pi_0 \text{Diff}^0) \]

vanishes on the image of \( d_H : \text{Diff}^0(\mathcal{E}^{\leq 2}) \to \text{Diff}^0(\mathcal{E}) \).

Let \( \text{Diff}^{\leq k, \bullet}(\mathcal{E}) \) denote \( \Gamma(X, \text{Dolb}^\bullet(X, \mathcal{O}_X) \otimes \text{Diff}^{\leq k}(\mathcal{E})) \) where \( \text{Diff}^{\leq k, \bullet}(\mathcal{E}) \) denotes the sheaf of differential operators on \( \mathcal{E} \) of order \( \leq k \). Equip \( \text{Diff}^{\leq k, \bullet}(\mathcal{E}) \) with the topology generated by the family of seminorms \( \{ || . ||_{K, s} | K \subset X \text{ compact }, s \in \Gamma(K, \mathcal{E} \otimes \Omega^\bullet) \} \) given by

\[ || D ||_{K, s} = \text{Sup} \{ || D(s)(x) || | x \in K \}. \]

The topology on \( \text{Diff}^\bullet(\mathcal{E}) \) is the direct limit of the topologies on the \( \text{Diff}^{\leq k, \bullet}(\mathcal{E}) \). The topology on \( \text{Diff}^\bullet(\mathcal{E}^{\leq 2}) \) is defined analogously.

Note that \( \text{Diff}^\bullet(\mathcal{E})^{\otimes 2} \) is dense in \( \text{Diff}^\bullet(\mathcal{E}^{\leq 2}) \). Further, \( d_H : \text{Diff}^\bullet(\mathcal{E}^{\leq 2}) \to \text{Diff}^\bullet(\mathcal{E}) \) is continuous. The restriction of \( d_H \) to \( \text{Diff}^0(\mathcal{E})^{\otimes 2} \) is just the map

\[ D_1 \otimes D_2 \mapsto D_2 D_1 - D_1 D_2. \]

The desired proposition now follows from Proposition 5.

### Part 3: Proof of Claim A

Given any element \( \alpha \) of \( \Gamma(X, \text{hoch}^2(\text{Diff}(\mathcal{E}))) \), pick a finite cover \( X = \bigcup_i U_i \) of open sets such that \( \alpha|_{U_i} \) is an element of \( \text{Diff}^\bullet(\mathcal{E}^{\leq 2})(U_i \times U_i)[1] \). Pick a partition of unity \( \{ f_i \} \) by compactly supported smooth functions subordinate to the cover \( \bigcup_i U_i \). Let \( g_i \) be a compactly supported smooth function supported on a subset of \( U_i \) that is identically 1 on the support of \( f_i \). Then,

\[ \sum_i (f_i \otimes g_i) \alpha|_{U_i} \]

is an element of \( \text{Diff}^\bullet(\mathcal{E}^{\leq 2})[1] \) whose image in \( \Gamma(X, \text{hoch}^2(\text{Diff}(\mathcal{E}))) \) is \( \alpha \).

Next, we note that we have a natural (degree preserving) map

\[ \gamma : \text{hoch}(\text{Diff}(\mathcal{E})) \to \text{hoch}(\hat{\text{Diff}}(\mathcal{E})) \]

of complexes of sheaves of \( \mathbb{C} \)-vector spaces.
on $X$. This induces a map $\gamma_* : \mathbb{H}^\bullet(X, \hat{\text{hoch}}(\text{Diff}(\mathcal{E}))) \rightarrow \mathbb{H}^\bullet(X, \hat{\text{hoch}}(\text{Diff}(\mathcal{E})))$. Recall that $\mathbb{H}^0(\Gamma(X, \hat{\text{hoch}}(\text{Diff}(\mathcal{E})))) = \mathbb{H}^0(X, \hat{\text{hoch}}(\text{Diff}(\mathcal{E})))$. The following corollary of Proposition 6 is immediate.

**Corollary 3.** *(Corollary to Proposition 6)* The Hochschild cocycle $\text{str} : HH_0(\text{Diff}^\bullet(\mathcal{E})) \rightarrow \mathbb{C}$ extends to a $\mathbb{C}$-linear functional $\hat{\text{str}} : \mathbb{H}^0(X, \hat{\text{hoch}}(\text{Diff}(\mathcal{E}))) \rightarrow \mathbb{C}$.

It is then immediate that $\hat{\text{str}} \circ \gamma_*$ gives us a linear functional on $\mathbb{H}^0(X, \hat{\text{hoch}}(\text{Diff}^\bullet(\mathcal{E})))$. We state this as a corollary for emphasis.

**Corollary 4.** The linear functional $\hat{\text{str}}$ in the previous corollary gives us a linear functional $\hat{\text{tr}} : \mathbb{H}^0(X, \hat{\text{hoch}}(\text{Diff}(\mathcal{E}))) \rightarrow \mathbb{C}$.

We denote $\mathbb{H}^{-i}(X, \hat{\text{hoch}}(\text{Diff}(\mathcal{E})))$ by $\hat{\mathbb{H}}^{-i}(\text{Diff}(\mathcal{E}))$.

Now consider a global holomorphic differential operator $D$ on $\mathcal{E}$. $D$ can be thought of as an element of $\Gamma(X, \text{Diff}^\bullet(\mathcal{E}))$ as well. Thus, $D$ gives us a 0-cocycle of the cochain complex $\Gamma(X, \text{D}(\mathcal{E})^{\otimes \bullet})$. Let $\hat{D}$ denote this cocycle. One the other hand, $D$ can be thought of as a global section of the degree 0 term of the complex $\hat{\text{hoch}}(\text{Diff}(\mathcal{E}))$. It follows that $D$ yields a 0-cycle in the bi-complex computing $\mathbb{H}^\bullet(X, \hat{\text{hoch}}(\text{Diff}(\mathcal{E})))$. Call this 0-cycle $\tilde{D}$. It is easy to check that $\gamma(\tilde{D}) = \hat{D}$. We have thus, proven the following proposition.

**Proposition 7.** The following diagram commutes.

\[
\begin{array}{ccc}
HH_0(\text{Diff}(\mathcal{E})) & \longrightarrow & \hat{HH}_0(\text{Diff}(\mathcal{E})) \\
\downarrow^{\hat{\text{str}}} & & \downarrow_{\hat{\text{tr}}} \\
\mathbb{C} & \longrightarrow & \mathbb{C}
\end{array}
\]

**3.4 The first bookkeeping lemma**

Consider $\mathcal{E}$ as a sub bundle of $\mathcal{E} \oplus \mathcal{F}$ where $\mathcal{F}$ is another vector bundle on $X$. We then have a map $\iota : \text{Diff}(\mathcal{E}) \rightarrow \text{Diff}(\mathcal{E} \oplus \mathcal{F})$ of sheaves of $\mathbb{C}$-vector spaces whose restriction to $U \iota : \Gamma(U, \text{Diff}(\mathcal{E})) \rightarrow \Gamma(U, \text{Diff}(\mathcal{E} \oplus \mathcal{F}))$ is an injection that preserves addition and multiplication for each open $U \subset X$. This also induces an map $\iota : \text{Diff}^\bullet(\mathcal{E}) \rightarrow \text{Diff}^\bullet(\mathcal{E} \oplus \mathcal{F})$ of sheaves of $\mathbb{C}$-vector spaces.
whose restriction to $U \tilde{i} : \text{Diff}^\bullet(E)(U) \to \text{Diff}^\bullet(E \oplus F)(U)$ is an injection for each open $U \subset X$.

$\tilde{i}$ preserves addition and multiplication though it does not preserve the identity. $\tilde{i}$ therefore induces a map from $\text{hoch}(\text{Diff}(E))$ to $\text{hoch}(\text{Diff}(E \oplus F))$, which we will denote by $\tilde{\iota}$. Let $\tilde{\iota}_\ast$ denote the map from $H^\ast(\Gamma(X, \text{hoch}(\text{Diff}(E))))$ to $H^\ast(\Gamma(X, \text{hoch}(\text{Diff}(E \oplus F))))$ induced by $\tilde{i}$. The following "bookkeeping lemma" holds.

**Lemma 1.** The following diagram commutes:

$$
\begin{array}{ccc}
H^0(\Gamma(X, \text{hoch}(\text{Diff}(E)))) & \xrightarrow{\tilde{\iota}_\ast} & H^0(\Gamma(X, \text{hoch}(\text{Diff}(E \oplus F)))) \\
\downarrow{sir} & & \downarrow{sir} \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C}
\end{array}
$$

**Proof.** By Proposition 5 and the proof of Proposition 6, it is enough to show that for any $D \in \text{Diff}^0(E)$,

$$
\text{str}(\Pi_K^\bullet \cdot_0 E D \mathcal{I}_K^\bullet \cdot_0 \varepsilon) = \text{str}(\Pi_K^\bullet \cdot_0 E \mathcal{I}_K^\bullet \cdot_0 \varepsilon(D) \mathcal{I}_K^\bullet \cdot_0 E \varepsilon \mathcal{I}_F).
$$

The above equality holds since the following diagrams commute:

$$
\begin{array}{ccc}
K^\bullet \varepsilon & \xrightarrow{D} & K^\bullet E \mathcal{I}_F \\
\downarrow{\Pi_K^\bullet \cdot_0 \varepsilon \Pi_K^\bullet \cdot_0 E \varepsilon \mathcal{I}_F} & & \downarrow{\text{str}(\Pi_K^\bullet \cdot_0 E \mathcal{I}_K^\bullet \cdot_0 \varepsilon(D) \mathcal{I}_K^\bullet \cdot_0 E \varepsilon \mathcal{I}_F)} \\
K^\bullet \varepsilon & \xrightarrow{\Pi_K^\bullet \cdot_0 \varepsilon} & K^\bullet \varepsilon \mathcal{I}_F \\
\downarrow{\mathcal{I}_K^\bullet \cdot_0 \varepsilon \mathcal{I}_K^\bullet \cdot_0 E \varepsilon \mathcal{I}_F} & & \downarrow{\text{str}(\Pi_K^\bullet \cdot_0 E \mathcal{I}_K^\bullet \cdot_0 \varepsilon(D) \mathcal{I}_K^\bullet \cdot_0 E \varepsilon \mathcal{I}_F)} \\
K^\bullet \varepsilon & \xrightarrow{\mathcal{I}_K^\bullet \cdot_0 \varepsilon} & K^\bullet \varepsilon \mathcal{I}_F
\end{array}
$$


\end{proof}
The map $\iota$ induces a map $\hat{\iota} : \text{hoch}(\text{Diff}(\mathcal{E})) \to \text{hoch}(\text{Diff}(\mathcal{E} \oplus \mathcal{F}))$. Let $\hat{\iota}_*$ denote the map induced by $\hat{\iota}$ from $\tilde{\text{HH}}_*(\text{Diff}(\mathcal{E}))$ to $\tilde{\text{HH}}_*(\text{Diff}(\mathcal{E} \oplus \mathcal{F}))$. The following corollary now follows.

**Corollary 5.** The following diagram commutes.

$$
\begin{array}{ccc}
\tilde{\text{HH}}_0(\text{Diff}(\mathcal{E})) & \xrightarrow{\hat{\iota}_*} & \tilde{\text{HH}}_0(\text{Diff}(\mathcal{E} \oplus \mathcal{F})) \\
\downarrow \hat{\iota}_r & & \downarrow \hat{\iota}_r \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C}
\end{array}
$$

**Proof.** For this we only need to recall that by the definition of $\iota$ at the beginning of this section the following diagram commutes.

$$
\begin{array}{ccc}
\Gamma(U^k, \text{Diff}(\mathcal{E}^{\boxtimes k})) & \xrightarrow{\iota} & \Gamma(U^k, \text{Diff}(\mathcal{E} \oplus \mathcal{F})^{\boxtimes k}) \\
\downarrow \gamma & & \downarrow \gamma \\
\Gamma(U^k, \text{Diff}^\bullet(\mathcal{E}^{\boxtimes k})) & \xrightarrow{\tilde{\iota}} & \Gamma(U^k, \text{Diff}^\bullet((\mathcal{E} \oplus \mathcal{F})^{\boxtimes k}))
\end{array}
$$

Taking hypercohomology, we see that $\gamma_* \circ \hat{\iota}_* = \tilde{\iota}_* \circ \gamma_*$. The corollary now follows immediately from Lemma 1.

\[ \square \]

### 4 The completed Hochschild homology of $\text{Diff}(\mathcal{E})$ and the cohomology of $X$

#### 4.1 Preliminaries

Recall the definition of $\text{hoch}(\text{Diff}(\mathcal{E}))$ from Section 3.2. Let $\text{hoch}(\text{Diff}(X))$ denote $\text{hoch}(\text{Diff}(\mathcal{O}_X))$. Let $\mathbb{C}$ denote the constant sheaf on $X$ such that $\Gamma(U, \mathbb{C}) \simeq \mathbb{C}$ for every open $U \subset X$ and whose restriction maps are all identity.

**Lemma 2** (Bryl). $\text{hoch}(\text{Diff}(X))$ is quasiisomorphic to $\mathbb{C}[2n]$ where $n$ is the dimension of $X$. 

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Proof. (Sketch of the Proof recalled from [Bryl]) It is enough to check the above fact for any open ball \( U \subset X \), at the level of presheaves. In other words, it is enough to show that the complex \( \text{Diff}(U^\bullet) \) with Hochschild differential is quasiisomorphic to \( \mathbb{C}[2n] \) as complexes of \( \mathbb{C} \)-vector spaces.

We filter \( \text{Diff}(U) \) by degree. More precisely, set \( F^{-n}(\text{Diff}(U)) := \text{Diff}^{\leq n}(U) \). The associated graded \( \text{gr}(\text{Diff}(U)) \) is the space of functions \( \mathcal{O}_{T^*U} \) on the cotangent bundle \( T^*U \) that are algebraic (polynomial) along the fibres.

This filtration of \( \text{Diff}(U) \) is exhaustive. It yields a filtration of the complex \( \text{Diff}(U^\bullet) \) equipped with Hochschild differential. This filtration yields a spectral sequence converging to the cohomology of the complex \( \text{Diff}(U^\bullet) \).

\[ E_1^{p, m} := \bigoplus_{p+q=m} E_1^{p, q} \] term is \( m \) th cohomology of the completed Hochschild complex of \( \mathcal{O}_{T^*U} \). This is isomorphic to the space of \(-m\)-holomorphic forms on \( T^*U \) that are algebraic along the fibres (see [Bryl]). Given local coordinates \( z_1, \ldots, z_n \) on \( U \) and \( y_1, \ldots, y_n \) on the fibre of \( T^*U \), defining the weight of \( dz_i \) to be 0 and that of \( dy_i \) to be 1 for all \( i \) enables us to define the notion of the weight of a holomorphic form on \( T^*U \). In the spectral sequence of this proof-sketch, \( E_1^{p, m-p} \) is simply the space \(-m\)-forms on \( T^*U \) of weight \(-p\). Therefore, \( E_1^{-n, -n} \) is the only nonzero summand of \( E_1^{-2n} \). By Theorem 3.1.1 and corollary 2.2.2 of [Bryl], the \( E_2^m = E_2^m \) term of this spectral sequence is the \( 2n + m \) th De-Rham cohomology of \( T^*U \). This is 0 if \( m \neq -2n \) and \( \mathbb{C} \) otherwise. This proves the desired lemma. Moreover, \( E_2^{-n, -n} \) is the only nontrivial summand of \( E_2^{-2n} \). Therefore, the \(-2n\)th cohomology of this complex can be identified with \( E_2^{-n, -n} \).

Lemma 3. \( \hat{\text{hoch}}(\text{Diff}(\mathcal{E})) \) is quasiisomorphic to \( \mathbb{C}[2n] \) where \( n \) is the dimension of \( X \).

Proof. This is again something that needs to be verified locally. We imitate the proof of the Morita invariance of Hochschild homology in [Loday] Section 1.2 here.

Part 1: Recalling the proof of Morita invariance of the Hochschild homology of a \( \mathbb{C} \)-algebra:

Recall from [Loday] that if \( A \) is any \( \mathbb{C} \)-algebra, and if \( M_r(A) \) denotes the algebra of \( r \times r \) matrices with entries in \( A \), then we have a map \( \text{tr} \) from the
Hochschild complex of $M_r(A)$ to that of $A$. The Hochschild chain $M_1 \otimes \ldots \otimes M_k$ is mapped to $\text{tr}(M_1 \otimes \ldots \otimes M_k)$ where $\otimes : M_r(A) \otimes M_r(B) \to M_r(A \otimes B)$ is an exterior multiplication. There is also a map of complexes $\text{inc}$ in the opposite direction which is induced by the inclusion of $A$ in $M_r(A)$ taking an element $a$ of $A$ to the matrix with $a.E_{11}$. $\text{tr} \circ \text{inc} = \text{id}$ and there is a pre simplicial homotopy $h$ from $\text{inc} \circ \text{tr}$ to $\text{id}$. This homotopy is given by $h = \sum_i (-1)^i h_i$ where $h_i : M_r(A)^{\otimes k} \to M_r(A)^{\otimes k+2}$ is defined by the formula

$$h_i(\alpha^0 \otimes \ldots \otimes \alpha^k) = \sum E_{j_1}(\alpha^0_{r_1}) \otimes E_{11}(\alpha^1_{r_1}) \otimes \ldots \otimes E_{11}(\alpha^i_{pq}) \otimes E_{i+1}(1) \otimes \alpha^{i+1} \otimes \ldots \otimes \alpha^k$$

The sum here is over all possible tuples of indices $(j, r, ..., p, q)$. 

Let $U$ be an open ball contained in $X$.

**Part 2: Morita invariance of the completed Hochschild homology of $\text{Diff}(U)$**

Recall that the completed Hochschild complex $C^\bullet(\widetilde{\text{Diff}(U)})$ is obtained by equipping the graded vector space $\oplus_{k \geq 1} \text{Diff}(U^k)[k-1]$ with the Hochschild differential. In this complex, $\text{Diff}(U^k)$ should be viewed as the $k$-th completed tensor power of $\text{Diff}(U)$. With this in mind, the $k$-th completed tensor power of $M_r(\text{Diff}(U))$ will be $M_r(\mathbb{C})^{\otimes k} \otimes \text{Diff}(U^k)$. One may verify that the Hochschild differential on $C^\bullet(M_r(\widetilde{\text{Diff}(U)}))$ extends to a differential on the graded vector space $\oplus_{k \geq 1} M_r(\mathbb{C})^{\otimes k} \otimes \text{Diff}(U^k)[k-1]$. The resulting complex is the completed Hochschild complex $C^\bullet(M_r(\widetilde{\text{Diff}(U)}))$.

One can verify without much difficulty that $\text{tr}$ and $\text{inc}$ extend to maps of complexes $\text{tr} : C^\bullet(M_r(\widetilde{\text{Diff}(U)})) \to C^\bullet(\text{Diff}(U))$ and $\text{inc} : C^\bullet(\widetilde{\text{Diff}(U)}) \to C^\bullet(M_r(\widetilde{\text{Diff}(U)}))$ respectively such that $\text{tr} \circ \text{inc} = \text{id}$. It is also useful for us to note that explicitly, if $m_1, ..., m_k \in M_r(\mathbb{C})$, and if $\alpha \in \text{Diff}(U^k)$, then

$$\text{tr}(m_1 \otimes \ldots \otimes m_k \otimes \alpha) = \text{tr}(m_1 \circ \ldots \circ m_k) \alpha .$$

Moreover $h$ extends to a map $h : C^k(M_r(\widetilde{\text{Diff}(U)})) \to C^{k+1}(M_r(\widetilde{\text{Diff}(U)}))$ for all $k$ with $dh + hd = \text{id} - \text{inc} \circ \text{tr}$. Thus,

$$\text{tr} : C^\bullet(M_r(\widetilde{\text{Diff}(U)})) \to C^\bullet(\widetilde{\text{Diff}(U)})$$

is a quasiisomorphism. It follows from Lemma 2 that $C^\bullet(M_r(\widetilde{\text{Diff}(U)}))$ is quasiisomorphic to $\mathbb{C}[2n]$. 

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Part 3: Proof of the Lemma

Let \( r \) be the rank of \( \mathcal{E} \). If \( U \) is an open ball of \( X \), on which \( \mathcal{E} \) is trivial, then \( \text{Diff}(\mathcal{E}|_U) \) is isomorphic to \( M_r(\text{Diff}(U)) \) as topological algebras. The actual isomorphism depends on the choice of (holomorphic) trivialization of \( \mathcal{E} \). For a holomorphic trivialization \( \phi \) of \( \mathcal{E} \) over \( U \), let \( \phi_\circ \) denote the isomorphism between \( \text{Diff}(\mathcal{E}|_U) \) and \( M_r(\text{Diff}(U)) \). Then, \( \phi_\circ \) induces an isomorphism of complexes \( \mathbb{C}^*(\text{Diff}(\mathcal{E})) \to \mathbb{C}^*(M_r(\text{Diff}(U))) \). This extends to an isomorphism of complexes \( \phi_\circ \): \( \widehat{\mathbb{C}}^*(\text{Diff}(\mathcal{E}|_U)) \to \widehat{\mathbb{C}}^*(M_r(\text{Diff}(U))) \). We however, know that \( \widehat{\mathbb{C}}^*(M_r(\text{Diff}(U))) \) is quasiisomorphic to \( \mathbb{C}[2n] \).

The desired lemma will follow provided that we check that the quasiisomorphism \( \text{tr} \circ \phi_\circ \) is independent of the choice of trivialization \( \phi \). We will show that if \( \psi \) is another holomorphic trivialization of \( \mathcal{E} \) over \( U \), then the maps \( \text{tr} \circ \psi_\circ = \text{tr} \circ \phi_\circ \) of complexes from \( \widehat{\mathbb{C}}^*(\text{Diff}(\mathcal{E}|_U)) \) to \( \widehat{\mathbb{C}}^*(\text{Diff}(U)) \) induce the same map on cohomology.

Note that \( \phi_\circ \circ \psi_\circ^{-1} : \mathbb{C}^*(M_r(\text{Diff}(U))) \to \mathbb{C}^*(M_r(\text{Diff}(U))) \) is induced by the map \( M_r(\text{Diff}(U)) \to M_r(\text{Diff}(U)) \) taking an element of \( M_r(\text{Diff}(U)) \) to its conjugate by an \( r \times r \) matrix \( N \) of holomorphic functions on \( U \). Denote this map of completed Hochschild complexes by \( c_N \). We only need to show that the maps \( \text{tr} \circ c_N \) and \( \text{tr} \) of complexes of \( \mathbb{C} \)-vector spaces induce the same map on cohomology.

Recall that \( \text{Diff}(U) \) is a filtered algebra with \( F^{-m}(\text{Diff}(U)) = \text{Diff}^{-m}(U) \), the space of (holomorphic) differential operators on \( U \) of order at most \( m \). This yields us a filtration on \( M_r(\text{Diff}(U)) \) with \( F^{-m}(M_r(\text{Diff}(U))) = M_r(\text{Diff}^{-m}(U)) \). We have \( \text{gr}(M_r(\text{Diff}(U))) = M_r(\text{gr}(\text{Diff}(U))) \).

The above filtration gives us a spectral sequence with \( E_1^{n} = \mathbb{H}^{n-}\text{gr}(M_r(\text{Diff}(U))) \) and \( E_\infty^{n} = \mathbb{H}^{n-}\text{gr}(M_r(\text{Diff}(U))) \). The endomorphism induced by the endomorphism \( \alpha \sim N \alpha N^{-1} \) where \( N \) is a matrix of holomorphic functions preserves the filtration on \( M_r(\text{Diff}(U)) \) and thus induces an endomorphism of a spectral sequence on the spectral sequence described above.

Note that \( \text{gr}(M_r(\text{Diff}(U))) = M_r(\text{gr}(\text{Diff}(U))) \) and that the endomorphism induced by conjugation by \( N \) is still conjugation by \( N \). Denote this endomorphism by \( c_N \) as well. We now claim that the following diagram commutes...
upto cohomology.

\[
\begin{array}{ccc}
C^\bullet(M_\mathbb{R}(\text{gr}(\text{Diff}(U)))) & \xrightarrow{c_N} & C^\bullet(M_\mathbb{R}(\text{gr}(\text{Diff}(U)))) \\
\downarrow \text{tr} & & \downarrow \text{tr} \\
C^\bullet(\text{gr}(\text{Diff}(U))) & \xrightarrow{id} & C^\bullet(\text{gr}(\text{Diff}(U)))
\end{array}
\]

To see this, note that if \( \alpha \in C^k(\text{gr}(\text{Diff}(U))) \) is a cocycle, \( \alpha = \text{tr}(\frac{1}{r}\text{id} \otimes \ldots \otimes \text{id} \otimes \alpha) \). In other words, \( \alpha \) is the trace of a "scalar" matrix whose diagonal elements are \( \alpha \) up to a scalar factor. Now,

\[
c_N(\text{id} \otimes \ldots \otimes \text{id} \otimes \alpha) = (N \otimes \ldots \otimes N)(\text{id} \otimes \ldots \otimes \text{id} \otimes \alpha)(N^{-1} \otimes \ldots \otimes N^{-1}) = (\text{id} \otimes \ldots \otimes \text{id} \otimes \alpha).
\]

It follows that \( (\text{tr} \circ c_N)_*(\text{id} \otimes \ldots \otimes \text{id} \otimes \alpha) = \text{tr}_*(\text{id} \otimes \ldots \otimes \text{id} \otimes \alpha) \) where \( \text{id} \otimes \ldots \otimes \text{id} \otimes \alpha \) denotes the class in cohomology of the cocycle \( \text{id} \otimes \ldots \otimes \text{id} \otimes \alpha \).

Therefore, the map of spectral sequences induced by \( \text{tr} \circ c_N \) coincides with that induced by \( \text{tr} \) at the \( E_1 \) level, and hence at the \( E_2 = E_\infty \) level. Finally, by part 2 of this proof and by the proof-sketch for Lemma 2, \( E_2^{-n,-n} \) is the only nonzero \( E_2 \) term in this spectral sequence, and \( E_2^{-n,-n} \simeq \mathbb{C} \). It follows that the endomorphisms on cohomology induced by \( \text{tr} \) and \( \text{tr} \circ c_N \) are indeed the endomorphisms they induce on \( E_2^{-n,-n} \). We have just shown that the maps induced \( \text{tr} \) and \( \text{tr} \circ c_N \) on \( E_r \) terms coincide for any \( r \geq 1 \). This proves the desired lemma.

\[\square\]

We can now state the following immediate corollary to Lemma 3.

**Corollary 6.**

\[\widehat{\text{HH}}_{-i}(\text{Diff}(\mathcal{E})) \simeq H^{2n-i}(X, \mathbb{C}).\]

We denote the isomorphism described in this corollary by \( \beta_\mathcal{E} \).

### 4.2 The second "bookkeeping" lemma

We once more look at the situation where \( \mathcal{E} \) is a direct summand of \( \mathcal{E} \oplus \mathcal{F} \). Notation is as in Section 3.4 of this paper. We have a map \( \iota : \text{Diff}(\mathcal{E}) \rightarrow \)
\( \text{Diff}(\mathcal{E} \oplus \mathcal{F}) \). This induces a map, denoted by \( \hat{\iota} \) from \( \hat{\text{hoch}}(\text{Diff}(\mathcal{E})) \) to \( \hat{\text{hoch}}(\text{Diff}(\mathcal{E} \oplus \mathcal{F})) \). The following lemma holds.

**Lemma 4.** The following diagram commutes.

\[
\begin{array}{ccc}
\hat{\text{HH}}_{-i}(\text{Diff}(\mathcal{E})) & \xrightarrow{\hat{i}_*} & \hat{\text{HH}}_{-i}(\text{Diff}(\mathcal{E} \oplus \mathcal{F})) \\
\downarrow{\beta_{\mathcal{E}}} & & \downarrow{\beta_{\mathcal{E} \oplus \mathcal{F}}} \\
H^{2n-i}(X, \mathbb{C}) & \xrightarrow{id} & H^{2n-i}(X, \mathbb{C})
\end{array}
\]

*Proof. Step 1:*

Let \( \text{D}(\text{Sh}_{\mathbb{C}}(X)) \) denote the derived category of sheaves of \( \mathbb{C} \)-vector spaces on \( X \).

Recall that in the proof of Lemma 3 we showed that the complex \( \hat{\text{hoch}}(\text{Diff}(\mathcal{E})) \) of sheaves of \( \mathbb{C} \)-vector spaces was quasiisomorphic to \( \hat{\text{hoch}}(\text{Diff}(\mathcal{O}_X)) \). Denote this quasiisomorphism by \( i_\mathcal{E} \). \( \hat{\text{hoch}}(\text{Diff}(\mathcal{O}_X)) \) is quasiisomorphic to \( \mathbb{C}[2n] \). Let \( i \) denote this quasiisomorphism for this proof.

It suffices to verify that the following diagram commutes in \( \text{D}(\text{Sh}_{\mathbb{C}}(X)) \).

\[
\begin{array}{ccc}
\hat{\text{hoch}}(\text{Diff}(\mathcal{E})) & \xrightarrow{i} & \hat{\text{hoch}}(\text{Diff}(\mathcal{E} \oplus \mathcal{F})) \\
\downarrow{i_\mathcal{E}} & & \downarrow{i_{\mathcal{E} \oplus \mathcal{F}}} \\
\mathbb{C}[2n] & \xrightarrow{id} & \mathbb{C}[2n]
\end{array}
\]  \hspace{1cm} \text{(16)}

Since a sheaf of \( \mathbb{C} \)-vector spaces is injective iff it is flasque (see [Riet], Lemma 3.3), the constant sheaf \( \mathbb{C} \) is an injective object in the category of sheaves of \( \mathbb{C} \)-vector spaces on \( X \). It follows from this that

\[
\text{Hom}_{\text{D}(\text{Sh}_{\mathbb{C}}(X))}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}.
\]

The diagram \text{(16)} therefore, commutes in \( \text{D}(\text{Sh}_{\mathbb{C}}(X)) \) upto a scalar factor. Checking that that scalar factor is one can be "done locally". It therefore, suffices to verify that there exists a neighbourhood \( U \) of every point in \( X \) such that the following diagram commutes in \( \text{D}(\text{Sh}_{\mathbb{C}}(U)) \).
Step 2: Verifying (17).

It suffices to verify (17) at the level of pre-sheaves. We must therefore, prove that the following diagram commutes in the category of complexes of \( \mathbb{C} \)-vector spaces up to cohomology.

\[
\begin{array}{c}
\widehat{\text{hoch}}(\text{Diff}(\mathcal{E}))|_U \\
\downarrow \text{id}|_U \end{array} \longrightarrow \begin{array}{c}
\widehat{\text{hoch}}(\text{Diff}(\mathcal{E} \oplus \mathcal{F}))|_U \\
\downarrow \text{id}|_U \end{array}
\]

(18)

To verify that the above diagram commutes up to cohomology, it suffices to verify that the diagram below commutes up to cohomology in the category of complexes of \( \mathbb{C} \)-vector spaces.

\[
\begin{array}{c}
\widehat{C}^\bullet(\text{Diff}(\mathcal{E})(U)) \\
\downarrow \text{id}|_U \end{array} \longrightarrow \begin{array}{c}
\widehat{C}^\bullet(\text{Diff}(\mathcal{E} \oplus \mathcal{F})(U)) \\
\downarrow \text{id}|_U \end{array}
\]

(19)

Most of the hard work necessary for this step has been done already. Let \( \phi_\mathcal{E} \) and \( \phi_\mathcal{F} \) denote holomorphic trivializations over \( U \) of \( \mathcal{E} \) and \( \mathcal{F} \) respectively. Then, \( \phi := \phi_\mathcal{E} \oplus \phi_\mathcal{F} \) is a holomorphic trivialization of \( \mathcal{E} \oplus \mathcal{F} \) over \( U \).

This yields the following commutative diagram, all of whose morphisms are continuous. \( r \) and \( s \) denote the ranks of \( \mathcal{E} \) and \( \mathcal{F} \) respectively.

\[
\begin{array}{c}
\widehat{\text{Diff}}(\mathcal{E})|_U \\
\downarrow \phi_\mathcal{E} \end{array} \longrightarrow \begin{array}{c}
\widehat{\text{Diff}}(\mathcal{E} \oplus \mathcal{F})|_U \\
\downarrow \phi \end{array}
\]

\[
\begin{array}{c}
\text{M}_r(\text{Diff}(U)) \\
\downarrow \text{id} \end{array} \longrightarrow \begin{array}{c}
\text{M}_{r+s}(\text{Diff}(U)) \\
\downarrow \text{id} \end{array}
\]

This yields the following commutative diagram.
\[ \hat{C}\cdot(D\hat{i}ff(E)(U)) \xrightarrow{i_{\ast}} \hat{C}\cdot(D\hat{i}ff(E \oplus F)(U)) \]
\[ \downarrow \phi_{E,\ast} \quad \quad \quad \quad \quad \quad \downarrow \phi_{\ast} \]
\[ \hat{C}\cdot(M_{r}(D\hat{\hat{i}}ff(U))) \xrightarrow{i_{r,s,\ast}} \hat{C}\cdot(M_{r+s}(D\hat{\hat{i}}ff(U)))) \]

\(i_{r,s}\) denotes the embedding from \(M_{r}(D\hat{\hat{i}}ff(U))\) to \(M_{r+s}(D\hat{\hat{i}}ff(U))\) which takes a matrix \(\alpha \in M_{r}(D\hat{\hat{i}}ff(U))\) to \(\tilde{\alpha} \in M_{r+s}(D\hat{\hat{i}}ff(U))\) where \(\tilde{\alpha}_{ij} = \alpha_{ij}\) if \((i, j) \in \{1, ..., r\} \times \{1, ..., r\}\) and \(\tilde{\alpha}_{ij} = 0\) otherwise.

Now, we proved that \(i_{E}|_{U} = tr_{r} \circ \phi_{E,\ast}\) in part 3 of the proof Lemma 3. Here, \(tr_{r}\) is the map \(tr : \hat{C}\cdot(M_{r}(D\hat{\hat{i}}ff(U))) \rightarrow \hat{C}\cdot(D\hat{\hat{i}}ff(U))\) described in the proof of Lemma 3. Similarly, \(i_{(E \oplus F)|_{U}} = tr_{r+s} \circ \phi_{\ast}\).

It therefore, suffices to prove that the following diagram commutes.

\[ \hat{C}\cdot(M_{r}(D\hat{\hat{i}}ff(U))) \xrightarrow{i_{r,s,\ast}} \hat{C}\cdot(M_{r+s}(D\hat{\hat{i}}ff(U)))) \]
\[ \downarrow tr_{r} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow tr_{r+s} \]
\[ \hat{C}\cdot(D\hat{\hat{i}}ff(U))) \xrightarrow{id} \hat{C}\cdot(D\hat{\hat{i}}ff(U))) \]

This follows immediately from the explicit formula for \(tr_{r}\) recalled in Part 2 of the proof of Lemma 3. This finally verifies (19), thus proving the desired lemma.

\[ \square \]

### 4.3 Proof of theorem 2

**Proof.** (Proof of theorem 2)

Corollary 5 states that the following diagram commutes.

\[ \hat{\hat{H}}_{0}(D\hat{\hat{i}}ff(E)) \xrightarrow{i_{\ast}} \hat{\hat{H}}_{0}(D\hat{\hat{i}}ff(E \oplus F)) \]
\[ \downarrow tr \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \tilde{u} \]
\[ \mathbb{C} \xrightarrow{id} \mathbb{C} \]

Lemma 4 for \(i = 0\) now states that the following diagram commutes.
It follows that $\hat{\text{tr}} \circ \beta_E^{-1} = \hat{\text{tr}} \circ \beta_{E \oplus F}^{-1}$. In the notation of Theorem 2, this says that $I_E = I_{E \oplus F}$. A symmetric argument shows that $I_F = I_{E \oplus F}$. This completes the proof of Theorem 2.

\[ \square \]

5 The completed Cyclic homology of $\text{Diff}(\mathcal{E})$

5.1 Recollections on Cyclic homology

5.1.1 Tsygan’s double complex

Let $A$ be a dg-$\mathbb{C}$-algebra such that $A^n = 0$ for almost all $n$. Let $d_{\text{bar}}$ and $d_{\text{hoch}}$ denote the differentials of $\text{bar}^*(A)$ and $C^*(A)$ respectively.

Let $\tau : A^\otimes k \rightarrow A^\otimes k$ denote the map $a_1 \otimes \ldots \otimes a_k \mapsto (-1)^{(d_k+1)(d_1+\ldots+d_{k-1})+k-1}a_k \otimes a_1 \otimes \ldots \otimes a_{k-1}$ for homogenous elements $a_1, \ldots, a_k$ of $A$ of degrees $d_1, \ldots, d_k$ respectively. Let $N : A^\otimes k \rightarrow A^\otimes k$ be the map $N = 1 + \tau + \ldots + \tau^{k-1}$.

$\tau$ and $N$ induce maps from $A^\otimes k$ to $A^\otimes k$ for any $k$. It follows that $\tau$ and $N$ induce maps from $\text{bar}^{-n}(A)$ to $C^{-n}(A)$ and $C^{-n}(A)$ to $\text{bar}^{-n}(A)$ respectively for any $n$.

Consider the following double complex the degree of whose non-zero columns is non positive.
The horizontal differential \( d \) is odd and negative and

\[ \text{Definition} \quad \text{The double complex } CC^{p,q} = CC^{p,q}(A) \text{ is even and non-positive, } CC^{p,q} = bar^q(A) \text{ if } p \text{ is odd and negative, and } CC^{p,q}(A) = 0 \text{ otherwise.} \]

The vertical differential \( d_{\text{hoch}} : CC^{p,q} \rightarrow CC^{p,q+1} \) is \( d_{\text{hoch}} \) if \( p \) is even and non positive, and \( -d_{\text{bar}} \) if \( p \) is odd and negative.

The horizontal differential \( d_{\text{hoch}} : CC^{p,q} \rightarrow CC^{p+1,q} \) is given by \( id - \tau \) if \( p \) is odd and negative and \( N \) if \( p \) is even and negative.

The double complex \( CC^{p,q}(A) \) is called the Tsygan’s double complex of \( A \).

**Definition** The Cyclic complex of \( A \) is the total complex of \( CC^{p,q}(A) \). It is denoted by \( \text{Cycl}^*(A) \).

**Definition** The cyclic homology \( HC_{-i}(A) \) is the \( -i \)th cohomology of \( \text{Cycl}^*(A) \). As in Section 2.1, contrary to the standard practice, we refer to a \( -i \)th cocycle of \( \text{Cycl}^*(A) \) as an \( i \)th-cycle in \( \text{Cycl}^*(A) \). The following standard propositions are important to us.

**Proposition 8.** (i) If \( B = \text{End}(V^\bullet) \) where \( V^\bullet \) is a finite dimensional graded \( \mathbb{C} \)-vector space with zero differential, then,

\[ HC_{-2i}(B) \cong \mathbb{C} \]
for all \( i \geq 0 \). All other cyclic homologies of \( \mathcal{B} \) vanish.

(ii) Further, the map \( \text{tr}_{2i} \) yielding the above isomorphism, is obtained by mapping the class in \( \text{HC}_{-2i}(\mathcal{B}) \) of a \( 2i \)-cycle in \( \text{Cycl}^\bullet(\mathcal{B}) \) given by a tuple \((b_{2i}, b_{2i-1}, \ldots, b_0)\) to \( \text{str}(b_0) \) where \( b_k \in \text{CC}^{-2i+k,-k}(\mathcal{B}) \).

(iii) Still further, the map \( \text{tr}_{2i} \) yielding the above isomorphism, maps the class in \( \text{HC}_{-2i}(\mathcal{B}) \) of a \( 2i \)-cycle in \( \text{Cycl}^\bullet(\mathcal{B}) \) given by a tuple \((b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j})\) \((j > 0, b_k \in \text{CC}^{-2i+k,-k}(\mathcal{B}))\) to \( \text{str}(b_0) \).

**Proof.** There is a spectral sequence converging to \( \text{HC}_\bullet(\mathcal{B}) \) such that \( E_1^{p,q} = \mathcal{H}^q(\text{CC}^p\bullet(\mathcal{B})) \). This is the spectral sequence that arises out of the filtration of the double complex \( \text{CC}^\bullet\bullet(\mathcal{B}) \) by columns. Now, since \( \mathcal{B} \) has a unit \( \text{bar}^\bullet(\mathcal{B}) \) is acyclic by recollection 1 of Section 2.1. \( \text{C}^\bullet(\mathcal{B}) \) is quasiisomorphic to \( \mathcal{C} \) concentrated in degree 0 by Proposition 1. It follows that \( E_1^{2i,0} \simeq \mathcal{C} \) for all \( i \geq 0 \) and \( E_1^{p,q} = 0 \) for all other \((p,q)\). This spectral sequence therefore collapses at \( E_1 \). It follows that \( \text{HC}_{-2i}(\mathcal{B}) \simeq E_1^{-2i,0} \simeq \mathcal{C} \) for all \( i \geq 0 \). This proves part (i).

Let \( F^\bullet \) be the filtration on \( \text{Cycl}^\bullet(\mathcal{B}) \) yielding the spectral sequence in the proof of part (i) of this proposition. Consider a tuple \((b_{2i}, b_{2i-1}, \ldots, b_0)\) with \( b_k \in \text{CC}^{-2i+k,-k}(\mathcal{B}) \) that yields a cyclic cycle. Then, \((b_{2i}, b_{2i-1}, \ldots, b_0) \in F^{-2i}\text{Cycl}^\bullet(\mathcal{B}) \). It follows that the image of \((b_{2i}, b_{2i-1}, \ldots, b_0)\) in \( \frac{F^{-2i}\text{Cycl}^\bullet(\mathcal{B})}{F^{-2i+1}\text{Cycl}^\bullet(\mathcal{B})} \simeq \text{C}^\bullet(\mathcal{B}) \) is the Hochschild 0-cycle \( b_0 \). Note that \( E_1^{-2i+j,-j} = 0 \) for \( j \neq 0 \) and \( \text{HH}_0(\mathcal{B}) \). It follows that the image of the tuple \((b_{2i}, b_{2i-1}, \ldots, b_0)\) in \( E_1^{-2i,0} \), and therefore in \( \text{HC}_{-2i}(\mathcal{B}) \) is the image of \( b_0 \) in \( \text{HH}_0(\mathcal{B}) \). By proposition 1, this is precisely \( \text{str}(b_0) \). This proves that \( \text{tr}_{2i}((b_{2i}, \ldots, b_0)) = \text{str}(b_0) \) where \((b_{2i}, \ldots, b_0)\) is the class in \( \text{HC}_{-2i}(\mathcal{B}) \) of the cycle obtained from \((b_{2i}, b_{2i-1}, \ldots, b_0)\). This proves part (ii).

To prove part (iii), note that if \((b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j})\) \((j > 0)\) is a cyclic cycle, then \( b_{-2j} \) is a Hochschild \(-2j\)-cycle. It follows from proposition 1 that \( b_{-2j} = d_{\text{hoch}}c_{-2j+1} \) for some \( c_{-2j+1} \in \text{Cycl}^{-2j-1}(\mathcal{B}) \). Consider \( c_{-2j+1} \in \text{CC}^{-2i-2j,2j-1}(\mathcal{B}) \) as an element of \( \text{Cycl}^{-2i-1}(\mathcal{B}) \). Then, the cycle \((b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j}) - d_{\text{cycl}}c_{-2j+1} \) arises out of the cycle \((b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j+1} + Nc_{-2j+1}, 0)\) of \( \text{CC}^\bullet(\mathcal{B}) \). Therefore, the class of \((b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j})\) in \( \text{HC}_{-2i}(\mathcal{B}) \) is the same as the class of \((b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j+1} + Nc_{-2j+1}, 0)\) in \( \text{HC}_{-2i}(\mathcal{B}) \). Since the bar complex of \( \mathcal{B} \) is acyclic, \( b_{-2j+1} + Nc_{-2j+1} = d_{\text{bar}}c_{-2j+2} \) for some element.
c_{−2j+2} of \text{bar}^{2j−2}(\mathcal{B}). The previous step can be repeated to show that the class of \((b_{2i}, b_{2i−1}, ..., b_0, b_{−1}, ..., b_{−2j})\) in \(\text{HC}_{−2i}(\mathcal{B})\) is the same as the class of \((b_{2i}, b_{2i−1}, ..., b_0, b_{−1}, ..., b_{−2j+2} + (\text{id} − \tau) c_{−2j+2}, 0, 0)\) in \(\text{HC}_{−2i}(\mathcal{B})\). This process can be continued to show that the class of \((b_{2i}, b_{2i−1}, ..., b_0, b_{−1}, ..., b_{−2j})\) in \(\text{HC}_{−2i}(\mathcal{B})\) is the same as the class of \((b_{2i}, b_{2i−1}, ..., b_0 + (\text{id} − \tau) c_0, 0, 0)\) in \(\text{HC}_{−2i}(\mathcal{B})\) for some element \(c_0\) of \(\text{bar}^0(\mathcal{B})\). By part (ii) of this proposition, this is equal to \(\text{str}(b_0 + (\text{id} − \tau) c_0)\). But \(\text{str((id − \tau)c_0)} = 0\) for any element \(c_0\) of \(\text{bar}^0(\mathcal{B})\). This proves (iii).

\[\square\]

If \(\mathcal{A}\) is a (unital) dg-\(C\)-algebra, let \(\text{CC}^{(2)}\bullet\bullet(\mathcal{A})\) denote the bicomplex consisting of the columns \(\text{CC}^{−1}\bullet(\mathcal{A})\) and \(\text{CC}^0\bullet(\mathcal{A})\) with differentials as in \(\text{CC}^\bullet\bullet(\mathcal{A})\).

We recall that we have an exact sequence of complexes

\[0 \longrightarrow \text{Tot}(\text{CC}^{(2)}\bullet(\mathcal{A})) \overset{I}{\longrightarrow} \text{Cycl}^\bullet(\mathcal{A}) \overset{S}{\longrightarrow} \text{Cycl}^{\bullet+2}(\mathcal{A}) \longrightarrow 0\]

We recall that \(\text{CC}^1\bullet(\mathcal{A})\) is acyclic (Recollection 1, Section 2.1). It follows that \(\text{CC}^{(2)}\bullet(\mathcal{A})\) is quasiisomorphic to \(\text{C}^\bullet(\mathcal{A})\). This quasiisomorphism is realized by the map of complexes taking \(a_k \in \text{C}^{-k}(\mathcal{A})\) to \((a_k, 0) \in \text{C}^{-k}(\mathcal{A}) \oplus \text{bar}^{1−k}(\mathcal{A})\). We denote the composite of \(I\) with this quasiisomorphism by \(I\). The map \(S\) is obtained by projection to the double complex obtained from \(\text{CC}^\bullet\bullet(\mathcal{A})\) by truncating the columns \(\text{CC}^i\bullet(\mathcal{A})\) for \(i = 0, −1\). We now obtain the following proposition.

**Proposition 9.** If \(\mathcal{B} = \text{End}(\mathcal{V}^\bullet)\) where \(\mathcal{V}^\bullet\) is a finite dimensional graded \(\mathbb{C}\)-vector space with zero differential, then, the following diagrams commute.

\[
\begin{array}{ccc}
\text{HC}_{−2i}(\mathcal{B}) & \overset{S}{\longrightarrow} & \text{HC}_{−2i+2}(\mathcal{B}) \\
\downarrow_{\text{tr}_{2i}} & & \downarrow_{\text{tr}_{2i−2}} \\
\mathbb{C} & \overset{\text{id}}{\longrightarrow} & \mathbb{C} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{HH}_0(\mathcal{B}) & \overset{I}{\longrightarrow} & \text{HC}_0(\mathcal{B}) \\
\downarrow_{\text{str}} & & \downarrow_{\text{tr}_0} \\
\mathbb{C} & \overset{\text{id}}{\longrightarrow} & \mathbb{C} \\
\end{array}
\]

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Proof. Let \((b_{2i}, b_{2i-1}, \ldots, b_0)\), \(b_j \in \text{CC}^{-2i+j}(-B)\) be a tuple yielding a cyclic cycle. We already demonstrated while proving part (iii) of proposition 8 that any class in \(\text{HC}_{-2i}(B)\) can be represented by a cycle coming from a tuple of this form. Let 
\((b_{2i}, b_{2i-1}, \ldots, b_0)\) denote the class of \((b_{2i}, b_{2i-1}, \ldots, b_0)\) in \(\text{HC}_{-2i}(B)\). Then, 
\(\text{tr}_{2i}((b_{2i}, b_{2i-1}, \ldots, b_0)) = \text{str}(b_0)\) by Proposition 8.

Let \(\bar{b}_0\) denote the class of \(b_0\) in \(\text{HH}_0(B)\). Then, \(\text{tr}_0(\bar{b}_0) = \text{str}(b_0)\) by Proposition 8. This proves that the first diagram commutes.

Further, if \(F : \mathcal{A} \to \mathcal{B}\) is an \(A_\infty\) morphism with Taylor components \(F_k\), we have the following proposition.

**Proposition 10.** The map \(F_{\text{Hoch}}\) mentioned in Proposition 2 extends to a map \(F_{\text{cycl}}\) of complexes from \(\text{Cycl}(\mathcal{A})\) to \(\text{Cycl}(\mathcal{B})\).

Proof. It suffices to check that \(F_{\text{Hoch}}\) extends to a map of \(F_{\text{tsyg}}\) bicomplexes from \(\text{Cycl}(\mathcal{A})\) to \(\text{Cycl}(\mathcal{B})\).

Let \(F_{\text{tsyg}} : \text{CC}^{p,q}(\mathcal{A}) \to \text{CC}^{p,q}(\mathcal{B})\) be \(F_{\text{Hoch}}\) if \(p\) is even and non positive and \(F_{\text{bar}}\) if \(p\) is odd and negative.

By Proposition 2, \(F_{\text{Hoch}} : \text{CC}^{p,\bullet}(\mathcal{A}) \to \text{CC}^{p,\bullet}(\mathcal{B})\) is a map of complexes of \(\mathbb{C}\)-vector spaces if \(p\) is even and non positive. By the definition of an \(A_\infty\)-morphism, \(F_{\text{bar}} : \text{CC}^{p,\bullet}(\mathcal{A}) \to \text{CC}^{p,\bullet}(\mathcal{B})\) is a map of complexes if \(p\) is odd and negative.

The following verifications, which we leave to the reader, complete the proof that \(F_{\text{tsyg}}\) is a map of bi-complexes, and thus yields a map \(F_{\text{cycl}} : \text{Cycl}(\mathcal{A}) \to \text{Cycl}(\mathcal{B})\) of complexes.

\[(i) \ (id - \tau) \circ F_{\text{bar}} = F_{\text{Hoch}} \circ (1 - \tau) .\]
(ii) $N \circ \mathcal{F}_{\text{Hoch}} = \mathcal{F}_{\text{Bar}} \circ N$.

Here, $N$ and $\tau$ are as in the definitions of $CC^{\bullet,\bullet}(A)$ and $CC^{\bullet,\bullet}(B)$. \hfill \Box

We also state the following consequence of Propositions 10 and 8 as a proposition.

**Proposition 11.** Let $A$ be a dg-$\mathbb{C}$ algebra. Let $B$ be as in Proposition 9. Suppose that $F$ is an $A_\infty$ morphism from $A$ to $B$. Let $\mathcal{F}_{\text{cycl}}$ denote the map from $HC^{\bullet}(A)$ to $HC^{\bullet}(B)$ induced by $\mathcal{F}_{\text{cycl}}$.

1. $tr_{2i} \circ \mathcal{F}_{\text{cycl}}((a_{2i}, ..., a_0, a_{-1}, ..., a_{-l})) = str(F_{\text{Hoch}}(a_0))$.

2. If $a_0$ is a Hochschild 0- cycle arising out of a degree $k-1$ element of $A^{\otimes k}$ then,

$$tr_{2i} \circ \mathcal{F}_{\text{cycl}}((a_{2i}, ..., a_0, a_{-1}, ..., a_{-l})) = \sum_{s=0}^{s=k-1} str(F_k(\tau^s(a_0)))$$

where $\tau$ is as in Corollary 2.

**Proof.** By the proof of Proposition 10,

$$\mathcal{F}_{\text{tsyg}}(a_{2i}, ..., a_0, a_{-1}, ..., a_{-l}) =$$

$$(\mathcal{F}_{\text{Hoch}}(a_{2i}), \mathcal{F}_{\text{Bar}}(a_{2i-1}), ..., \mathcal{F}_{\text{Bar}}(a_1), \mathcal{F}_{\text{Hoch}}(a_0), ..., \mathcal{F}_{\text{Bar/hoch}}(a_{-l}))$$

where $\mathcal{F}_{\text{Bar/hoch}}(a_{-l}) = \mathcal{F}_{\text{Hoch}}(a_{-l})$ if $l$ is even and $\mathcal{F}_{\text{Bar/hoch}}(a_{-l}) = \mathcal{F}_{\text{Bar}}(a_{-l})$ if $l$ is odd. Part 1 of this proposition now follows immediately from Proposition 8, part (iii). Part 2 of this proposition is immediate from Part 1 and Corollary 2.

\hfill \Box

### 5.2 The complex $\mathcal{Cycl}^{\bullet}(\text{Diff}(\mathcal{E}))$

Let $\mathbb{C}^\infty$ denote the sheaf of smooth functions on $X$. For each open $U \subset X$, we can consider the completed Hochschild complex of $\text{Diff}^\bullet(\mathcal{E})(U)$ as in
Section 3.3. Denote this complex by $\mathsf{C}^\bullet(\mathsf{Diff}^\bullet(\mathcal{E}))(U)$. Consider the sheafification of the presheaf $U \rightsquigarrow \mathsf{C}^\bullet(\mathsf{Diff}^\bullet(\mathcal{E}))(U)$. Recall that this sheaf of complexes of $\mathbb{C}^\infty$-modules was denoted by $\mathsf{hoch}(\mathsf{Diff}(\mathcal{E}))$ in Section 3.3. Unlike in Section 3.3, let $\mathsf{hoch}^n(\mathsf{Diff}(\mathcal{E}))$ denote the degree $n$ component of $\mathsf{hoch}(\mathsf{Diff}(\mathcal{E}))$.

One can also consider the completed Bar complex $\mathsf{bar}^\bullet(\mathsf{Diff}^\bullet(\mathcal{E}))(U))$. This is defined as in Section 3.3 - the underlying graded $\mathbb{C}$-vector space of the complex $\mathsf{bar}^\bullet(\mathsf{Diff}^\bullet(\mathcal{E}))(U))$ is the same as that of $\mathsf{C}^\bullet(\mathsf{Diff}^\bullet(\mathcal{E}))(U)$ but the differential on $\mathsf{bar}^\bullet(\mathsf{Diff}^\bullet(\mathcal{E}))(U))$ is the bar differential. We will denote the sheafification of the presheaf $U \rightsquigarrow \mathsf{bar}^\bullet(\mathsf{Diff}^\bullet(\mathcal{E}))(U))$ by $\mathsf{bar^\bullet(hoch(\mathsf{Diff}(\mathcal{E}))}$. We have the following proposition.

**Proposition 12.** The differentials in Tsygan’s double complex for $\mathsf{Diff}^\bullet(\mathcal{E}))(U)$ extend to yield differentials for the following double complex of $\mathbb{C}^\infty$-modules on $X$.

\[
\begin{array}{cccccccc}
\vdots & \rightarrow & \vdots & \rightarrow & \vdots & \rightarrow & \vdots & \rightarrow & \vdots \\
\mathsf{d}_{\text{hoch}} & \uparrow & \mathsf{d}_{\text{bar}} & \downarrow & \mathsf{d}_{\text{hoch}} & \uparrow & \mathsf{d}_{\text{bar}} & \downarrow & \mathsf{d}_{\text{hoch}} \\
\vdots & \rightarrow & \mathsf{bar}^0(\mathsf{Diff}(\mathcal{E})) & \rightarrow & \mathsf{hoch}^0(\mathsf{Diff}(\mathcal{E})) & \rightarrow & \mathsf{bar}^0(\mathsf{Diff}(\mathcal{E})) & \rightarrow & \mathsf{hoch}^0(\mathsf{Diff}(\mathcal{E})) \\
\mathsf{d}_{\text{hoch}} & \uparrow & \mathsf{d}_{\text{bar}} & \downarrow & \mathsf{d}_{\text{hoch}} & \uparrow & \mathsf{d}_{\text{bar}} & \downarrow & \mathsf{d}_{\text{hoch}} \\
\vdots & \rightarrow & \mathsf{bar}^1(\mathsf{Diff}(\mathcal{E})) & \rightarrow & \mathsf{hoch}^1(\mathsf{Diff}(\mathcal{E})) & \rightarrow & \mathsf{bar}^1(\mathsf{Diff}(\mathcal{E})) & \rightarrow & \mathsf{hoch}^1(\mathsf{Diff}(\mathcal{E})) \\
\mathsf{d}_{\text{hoch}} & \uparrow & \mathsf{d}_{\text{bar}} & \downarrow & \mathsf{d}_{\text{hoch}} & \uparrow & \mathsf{d}_{\text{bar}} & \downarrow & \mathsf{d}_{\text{hoch}} \\
\vdots & \rightarrow & \mathsf{bar}^2(\mathsf{Diff}(\mathcal{E})) & \rightarrow & \mathsf{hoch}^2(\mathsf{Diff}(\mathcal{E})) & \rightarrow & \mathsf{bar}^2(\mathsf{Diff}(\mathcal{E})) & \rightarrow & \mathsf{hoch}^2(\mathsf{Diff}(\mathcal{E})) \\
\mathsf{d}_{\text{hoch}} & \uparrow & \mathsf{d}_{\text{bar}} & \downarrow & \mathsf{d}_{\text{hoch}} & \uparrow & \mathsf{d}_{\text{bar}} & \downarrow & \mathsf{d}_{\text{hoch}} \\
\vdots & \rightarrow & \vdots & \rightarrow & \vdots & \rightarrow & \vdots & \rightarrow & \vdots \\
\end{array}
\]

*Proof.* This only needs to be checked at the level of presheaves. The double complex mentioned in this proposition is the sheafification of the presheaf $U \rightsquigarrow \mathsf{CC}^\bullet\bullet(\mathsf{Diff}^\bullet(\mathcal{E}))(U))$. Here, $\mathsf{CC}^\bullet\bullet(\mathsf{Diff}^\bullet(\mathcal{E}))(U))$ is the double complex obtained by replacing the Hochschild and bar complexes that make up the columns of Tsygan’s double complex of $\mathsf{Diff}^\bullet(\mathcal{E}))(U)$ with their completed
versions. It is also clear that \( id - \tau \) and \( N \) extend to horizontal differentials on the double complex in this proposition.

We denote the total complex of this double complex by \( \tilde{\text{Cycl}}^\bullet(D\text{iff}(\mathcal{E})) \). Since it is a complex of \( \mathbb{C}^\infty \) modules, its Hypercohomology is computed by the complex \( \Gamma(X, \tilde{\text{Cycl}}^\bullet(D\text{iff}(\mathcal{E}))) \).

The following cyclic analog of Proposition 6 follows from Proposition 11 and the fact that \( \sum_s \tau^s(id - \tau) = 0 \). We denote the \(-2i\)th cohomology of \( \Gamma(X, \tilde{\text{cycl}}^\bullet(D\text{iff}(\mathcal{E}))) \) by \( \tilde{\text{HC}}_{-2i}(D\text{iff}(\mathcal{E})) \).

**Proposition 13.** The formula described in Proposition 11 extends to yield us a \( \mathbb{C} \)-linear functional \( \tilde{\text{tr}}_{-2i} : \tilde{\text{HC}}_{-2i}(D\text{iff}(\mathcal{E})) \to \mathbb{C} \).

**Proof.** The fact that the formula makes sense follows directly from Proposition 11 and Proposition 6. To show that it vanishes on coboundaries, we recall that Proposition 6 also tells us that it vanishes on the image of the \( d_{\text{hoch}} \) differential. We only need to verify that it vanishes on the image of the \( id - \tau \) differential. This is a consequence of the fact that \( \sum_s \tau^s(id - \tau) = 0 \).

Recall that we had a map \( S : \text{Cycl}^\bullet(D\text{iff}(\mathcal{E})) \to \text{Cycl}^{\bullet+2}(D\text{iff}(\mathcal{E})) \). It is easy to verify that this yields us a map \( S : \tilde{\text{Cycl}}^\bullet(D\text{iff}(\mathcal{E})) \to \tilde{\text{Cycl}}^{\bullet+2}(D\text{iff}(\mathcal{E})) \). Similarly, \( I \) can be seen to extend to a map of complexes \( I : \text{hoch}(D\text{iff}(\mathcal{E})) \to \text{Cycl}^\bullet(D\text{iff}(\mathcal{E})) \). A direct consequence of the formula in Proposition 11 is the following proposition.

**Proposition 14.** The following diagrams commute.

\[
\begin{array}{ccc}
\tilde{\text{HC}}_{-2i}(D\text{iff}(\mathcal{E})) & \xrightarrow{S} & \tilde{\text{HC}}_{-2i+2}(D\text{iff}(\mathcal{E})) \\
\downarrow \tilde{\text{tr}}_{2i} & & \downarrow \tilde{\text{tr}}_{2i-2} \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C} \\
\end{array}
\]

\[
\begin{array}{ccc}
H^0(\Gamma(X, \text{hoch}(D\text{iff}(\mathcal{E})))) & \xrightarrow{I} & \tilde{\text{HC}}_0(D\text{iff}(\mathcal{E})) \\
\downarrow \text{str} \tilde{\text{tr}} & & \downarrow \tilde{\text{tr}}_0 \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C} \\
\end{array}
\]
5.3 The completed cyclic homology of \( \mathcal{D}iff(\mathcal{E}) \) and the cohomology of \( X \)

5.3.1 The completed cyclic homology of \( \mathcal{D}iff(\mathcal{E}) \)

We can define the completed Bar complex of \( \mathcal{D}iff(\mathcal{E}) \). This is the sheafification of the presheaf \( U \to \text{bar}^\bullet (\mathcal{D}iff(\mathcal{E}))(U) \). The terms of this complex are the same as in \( \text{hoch}(\mathcal{D}iff(\mathcal{E})) \) but the differential is the bar differential. We denote this by \( \hat{\text{bar}}(\mathcal{D}iff(\mathcal{E})) \).

We can obtain the completed Tsygan’s double complex of \( \mathcal{D}iff(\mathcal{E}) \). This is a double complex of sheaves of \( \mathbb{C} \)-vector spaces on \( X \), whose non-positive even columns are \( \text{hoch}^\bullet (\mathcal{D}iff(\mathcal{E})) \) and whose negative odd columns are \( \hat{\text{bar}}(\mathcal{D}iff(\mathcal{E})) \). The horizontal differentials in Tsygan’s double complex extend to this situation to give us well defined horizontal differentials. We will denote the total complex of this double complex by \( \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) \).

Before we proceed, we note that as before, we have a (degree preserving) natural map of complexes \( \gamma : \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) \to \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) \). This induces a map \( \gamma_* : \mathbb{H}^\bullet (X, \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E}))) \to \mathbb{H}^\bullet (X, \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E}))) \). Note that the map \( S \) extends to a map of complexes \( \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) \to \hat{\text{Cycl}}^{\bullet +2}(\mathcal{D}iff(\mathcal{E})) \). Similarly, the map \( I \) extends to a map of complexes \( \mathcal{I} : \text{hoch}(\mathcal{D}iff(\mathcal{E})) \to \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) \). Further, we have the following commutative diagrams.

\[
\begin{array}{ccc}
\text{hoch}(\mathcal{D}iff(\mathcal{E})) & \xrightarrow{\mathcal{I}} & \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
\text{hoch}(\mathcal{D}iff(\mathcal{E})) & \xrightarrow{\mathcal{I}} & \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) \\
\hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) & \xrightarrow{S} & \hat{\text{Cycl}}^{\bullet +2}(\mathcal{D}iff(\mathcal{E})) \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
\hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) & \xrightarrow{S} & \hat{\text{Cycl}}^{\bullet +2}(\mathcal{D}iff(\mathcal{E}))
\end{array}
\]

We denote by \( \hat{\mathbb{H}}\mathcal{C}_{-j}(\mathcal{D}iff(\mathcal{E})) \) the hypercohomology \( \mathbb{H}^{-j}(X, \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E}))) \).

Again we note that the complex \( \hat{\text{Cycl}}^\bullet (\mathcal{D}iff(\mathcal{E})) \) is a complex of \( \mathbb{C}^\infty \) modules. Thus, its hypercohomology is precisely \( \hat{\mathbb{H}}\mathcal{C}_\bullet (\mathcal{D}iff(\mathcal{E})) \). It follows that we have the following corollaries of Propositions 13 and 14 respectively.
Corollary 7. (Corollary to Proposition 13) There exist traces $\hat{\text{tr}}_{2i}: \overline{HC}_{-2i}(\text{Diff}(E)) \to \mathbb{C}$.

Proof. $\hat{\text{tr}}_{2i} = \tilde{\text{tr}}_{2i} \circ \gamma_{s}$. 

Corollary 8. (Corollary to Proposition 14) The following diagrams commute.

\[
\begin{array}{ccc}
\widehat{HH}_0(\text{Diff}(E)) & \xrightarrow{\iota} & \widehat{HC}_0(\text{Diff}(E)) \\
\downarrow \iota_r & & \downarrow \iota_0 \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C} \\
\end{array}
\]

\[
\begin{array}{ccc}
\widehat{HC}_{-2i}(\text{Diff}(E)) & \xrightarrow{S} & \widehat{HC}_{-2i+2}(\text{Diff}(E)) \\
\downarrow \hat{\text{tr}}_{2i} & & \downarrow \hat{\text{tr}}_{2i-2} \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C} \\
\end{array}
\]

Proof. This is an immediate consequence of Proposition 14 and the commutative diagrams shown immediately before Corollary 7.

We have the following cyclic homology analog of Lemma 3.

Lemma 5. $\widehat{\text{Cycl}}^{\bullet}(\text{Diff}(E))$ is quasiisomorphic to $\mathbb{C}[2n] \oplus \mathbb{C}[2n+2] \oplus \ldots$.

Proof. It suffices to verify this assertion locally at the level of pre sheaves. In other words, we need to look at the completed Tsygan’s double complex for $\text{Diff}(E)(U)$ where $U$ is an open set so that $E$ is trivial on $U$. Of course, all tensor products here are completed tensor products.

The completed bar complex of $\text{Diff}(E)(U)$ is acyclic since the homotopy of recollection 1, Section 2.1 between the identity endomorphism and the 0 endomorphism of the bar complex of $\text{Diff}(E)(U)$ can be shown to "extend to" a homotopy between the identity and 0 endomorphisms of the completed bar complex of $\text{Diff}(E)(U)$. The complex $\text{hoch}(\widehat{\text{Diff}}(E)(U))$ is quasiisomorphic to $\mathbb{C}[2n]$ by Lemma 3. It follows that the spectral sequence converging to the
completed cyclic homology of $\text{Diff}(\mathcal{E})(U)$ that arises out of the filtration of the completed Tsygan’s double complex for $\text{Diff}(\mathcal{E})(U)$ by columns satisfies

$$E_1^{-2i,-2n} \cong \mathbb{C} \quad \forall i \geq 0$$

$$E_1^{p,q} = 0 \text{ otherwise.}$$

The desired lemma follows immediately from this.

\[\square\]

**Corollary 9.** $\widehat{HC}_{-2i}(\text{Diff}(\mathcal{E})) = \mathbb{H}^{2n}(X, \mathbb{C}) \oplus \mathbb{H}^{2n-2}(X, \mathbb{C}) \oplus \ldots \oplus \mathbb{H}^{2n-2i}(X, \mathbb{C})$.

Denote the isomorphism in the above corollary by $J_{2i}$. Then, following the proof of lemma 5, we obtain the following proposition.

**Proposition 15.** The following diagrams commute.

\[
\begin{array}{ccc}
\widehat{HC}_{-2i}(\text{Diff}(\mathcal{E})) & \xrightarrow{J_{2i}} & \mathbb{H}^{2n}(X, \mathbb{C}) \oplus \mathbb{H}^{2n-2}(X, \mathbb{C}) \oplus \ldots \oplus \mathbb{H}^{2n-2i}(X, \mathbb{C}) \\
\downarrow S & & \downarrow \\
\widehat{HC}_{-2i+2}(\text{Diff}(\mathcal{E})) & \xrightarrow{J_{2i-2}} & \mathbb{H}^{2n}(X, \mathbb{C}) \oplus \mathbb{H}^{2n-2}(X, \mathbb{C}) \oplus \ldots \oplus \mathbb{H}^{2n-2i+2}(X, \mathbb{C}) \\
\end{array}
\]

\[
\begin{array}{ccc}
\widehat{HH}_0(\text{Diff}(\mathcal{E})) & \xrightarrow{I} & \widehat{HC}_0(\text{Diff}(\mathcal{E})) \\
\downarrow \beta_{\mathcal{E}} & & \downarrow J_0 \\
\mathbb{H}^{2n}(X, \mathbb{C}) & \xrightarrow{id} & \mathbb{H}^{2n}(X, \mathbb{C}) \\
\end{array}
\]

The vertical arrow in the right column of the first diagram is the obvious projection.

5.4 Proof of theorem 3

**Proof.** (Proof of Theorem 3)

We apply the second diagram in Corollary 8 $i$ times to obtain the following diagram.
We apply the first diagram Proposition 15 $i$ times to obtain the following diagram.

In the notation used to state Theorem 3, this tells us that $I_{E,2i,0} = I_{E,0,0}$ and $I_{E,2i,2k} = 0$ for $k > 0$. Lastly, the second diagram of Proposition 15 and the first diagram of corollary 8 together imply that $\hat{\text{tr}} \circ \beta^{-1}_E = \hat{\text{tr}}_0 \circ J^{-1}_0$. This shows that $I_{E,0,0} = I_{E}$, thereby completing the proof of Theorem 3.

Remark 1: Our notion of completed Hochschild and cyclic homologies seems to differ in detail from that used in [FLS]. This forced us to rework some steps of [FLS], in particular, Proposition 6 and related matters, in our situation.

Remark 2: The key step in the proof of Theorem 1 in [FLS] consists of showing on one hand that the linear functional defined in [FLS] applied to the operator id in Diff($E$) is indeed its supertrace i.e, the Euler characteristic of $E$, and showing that the image of id in $\hat{\text{HH}}_0(Diff(E))$ gives the class $\text{ch}(E).\text{td}(T_X)_{2n}$ (this part is done by citing [NT1] and [NT2]) after passing to $H^{2n}(X,\mathbb{C})$ and then applying the Hirzebruch R-R theorem. In spite of the differences in detail between our construction of completed Hochschild homology and that of [FLS] it may be checked that these two key steps go through, thus maintaining Theorem 1 in this situation.
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