Mathematical modeling of three-layer flows with evaporation based on exact solutions

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Abstract. The stationary flow in the "liquid-liquid-gas" system in a horizontal channel with solid impermeable upper and lower walls is investigated. Mathematical modeling in each of the layers of the system is based on exact solutions of a special type of Navier-Stokes equations in the Boussinesq approximation. The processes of vapor evaporation or condensation at the liquid-gas interface are modeled using the boundary conditions of the problem. In the upper layer the thermal diffusion effect and the effect of diffusional thermal conductivity are taken into account. Examples of three-layer flows for the "silicone oil - water - air" system are given. The influence of the thermal regime at the boundaries of the system and the thickness of the upper layer on the longitudinal velocity and temperature distribution is considered.

1. Introduction

The problems of studying convective flows have a rather long history. At present, they do not lose their relevance due to the development of new science-intensive technologies that require the study of issues related to the description of the processes of heat and mass transfer. The construction of exact solutions describing convective flows seems to be quite important due to the possibility of studying physicochemical factors affecting the nature of flows and predicting the results of experiments. The works [1-3] are devoted to the modeling of heat and mass transfer processes at the interfaces in two-layer systems using exact solutions of a special type, as well as to the study their stability.

Modeling convective flows is often complicated by the presence of additional effects. The work [4] is devoted to the study of the influence of the processes of thermal diffusion and diffusional thermal conductivity on the character of flows. An important issue in modeling flows accompanied by additional effects at the boundaries is the formulation of boundary conditions [5, 6].

In this paper, a three-layer flow in the "liquid-liquid-gas" system is considered. The process is complicated by evaporation at the thermocapillary liquid-gas interface and taking into account the Soret and Dufour effects in the gas-vapor layer. The purpose of this work is to construct an exact solution of a special form to the system of Navier-Stokes equations in the Boussinesq approximation. This solution has a group nature, which makes it especially valuable, and allows one to study the influence of various physicochemical factors on the nature of the flow.

2. Problem statement and construction of exact solutions

The stationary flow in a horizontal channel in the “liquid-liquid-gas” system is investigated taking into account evaporation or condensation of liquid of the middle layer (see figure 1). The upper and lower walls of the channel are solid, impermeable and subject to non-uniform heating. Thermocapillary "liquid-liquid" and "liquid-gas" interfaces are assumed to be non-deformable. The gravity vector is...
directed opposite to the Oy axis \((\mathbf{g} = (0, -g))\). The effects of Soret (thermal diffusion) and Dufour (diffusional thermal conductivity) are taken into account in the gas-vapor layer.

The mathematical model describing the flow in each of the layers of the system is based on the Boussinesq approximation of the Navier-Stokes system of equations. This system in a two-dimensional case has the form [7]

\begin{align}
    uu_x + vu_y &= -\frac{1}{\rho} p_x + v\Delta u, \quad (1) \\
    u\nu_x + v\nu_y &= -\frac{1}{\rho} p_y + v\Delta\nu + g(\beta T + \gamma C), \quad (2) \\
    u_x + v_y &= 0, \quad (3) \\
    uT_x + vT_y &= \chi(\Delta T + \delta\Delta C), \quad (4) \\
    uC_x + vC_y &= D(\Delta C + \alpha\Delta T). \quad (5)
\end{align}

Here \(u\) and \(v\) are the longitudinal and transverse velocities, \(\rho\) is the density, \(p\) is the modified pressure (deviation from hydrostatic), \(\nu\) is the coefficient of kinematic viscosity, \(\beta\) is the thermal expansion coefficient, \(T\) is the temperature, \(\gamma\) is the concentration density coefficient, \(C\) is the vapor concentration, \(\chi\) is the coefficient of thermal diffusivity, \(D\) is the diffusion coefficient, \(\alpha\) and \(\delta\) determine the Soret and Dufour effects. Equation (5) and the terms containing the vapor concentration function in equations (2) and (4) are taken into account only when modeling the flow in the upper gas-vapor layer.

The solution to the system of equations (1)-(5) is defined as a generalization of the Ostroumov-Birikh solution [8, 9]. It is assumed that the transverse component of the velocity is equal to zero, and the temperature and vapor concentration functions depend linearly on the longitudinal coordinate \(x\). The functions of longitudinal velocities in each of the layers of the system are determined by differentiating equation (2) with respect to the variable \(y\), substituting the result in (1), and further sequential integration of the resulting expression over the \(y\) coordinate. Further, using equation (3), one can determine the temperature functions in liquids. The pressure function for the lower and middle layers is determined by integrating equation (1) and substituting the result into equation (2). To determine the temperature and vapor concentration in the upper layer, it is necessary to solve the system of equations (4), (5). The pressure function in the gas-vapor layer is calculated using equations (1), (2), when the functions \(u_3, T_3\) and \(C\) are defined. Then the required functions take the following form:

![Flow scheme in a horizontal channel](image.png)
\[ u_i = \frac{L_2 y^4}{24} + \frac{L_3 y^3}{6} + \frac{y^2}{2} c_i^j + \gamma c_i^j + c_i^j, \]  
\[ T_i = (a_i^j + a_i^2 y)x + N_i^6 y^7 + N_i^5 y^6 + N_i^4 y^5 + N_i^3 y^4 + N_i^2 y^3 + N_i^1 y^2 + \gamma y c_i^j + c_i^5, \]  
\[ p_i = \rho_i y_i (d_i^j + d_i^2 y + c_i^j)x + k_i^j y^7 + k_i^j y^6 + k_i^j y^5 + k_i^j y^4 + k_i^j y^3, \]  
\[ C = (b_i^j + b_i^2 y)x + S_i^6 y^7 + S_i^5 y^6 + S_i^4 y^5 + S_i^3 y^4 + S_i^2 y^3 + S_i^1 y^2 + y c_i^j + c_i^5, \]  

where index \( i \) defines the belonging of functions, parameters and constants to the corresponding layer: \( i = 1 \) is the lower, \( i = 2 \) is the middle, \( i = 3 \) is the upper layer of the system. Coefficients \( L_i^j, N_i^j, d_i^j, k_i^j, \) and \( S_i^j \) are expressed in terms of the physicochemical characteristics of the system and the coefficients of the longitudinal gradients of temperature and vapor concentration. Parameters \( a_i^j, a_i^2, b_i^j, b_i^2, \) and \( c_i^j \) are the constants of integration arising in the course of solving the system of differential equations (1) - (5).

The no-slip conditions for the longitudinal velocity are satisfied on the upper and lower channel walls

\[ u_i(-h_i) = u_i(h_i) = 0; \]  
the temperature is distributed linearly:

\[ T_i \big|_{y=-h_i} = A_1 x + \mathcal{G}^- \ , \ T_i \big|_{y=h_i} = A_3 x + \mathcal{G}^+, \]  

where \( \mathcal{G}^- \) and \( \mathcal{G}^+ \) are the coefficients set based on the physical parameters of the problem, \( A_1 \) and \( A_3 \) are the longitudinal temperature gradients on the lower and upper walls, respectively. The vapor concentration at the upper wall satisfies the condition of zero vapor flow taking into account the Soret effect:

\[ C_y + \alpha T_{3y} = 0 \]  

The conditions of continuity of velocities and temperatures are assumed to be satisfied at the boundaries \( y = 0 \) and \( y = h_2 \):

\[ u_i = u_{i+1}, \ T_i = T_{i+1}, i = 1, 2 \]

Kinematic conditions at the interface boundaries are performed automatically. The projections of the dynamic condition on the liquid-liquid interface taking into account the Antonov rule (see [10]) have the form:

\[ p_i = p_2, \ p_{1y} u_{1y} = p_{2y} u_{2y} + (\sigma_{T1} - \sigma_{T2}) \mathcal{F}_x, \]  

where \( \sigma_{T1}, \sigma_{T2} \) are the temperature coefficients of surface tension of liquids in the lower and middle layers, respectively. Also the heat transfer condition is satisfied at the boundary \( y = 0 \):
The unknown constants of integration arising in the course of constructing exact solutions, the mass of evaporating liquid, as well as the relations that determine the dependences of the main parameters of the problem, can be found with the help of the boundary conditions of the problem presented above. It was found that only one of the longitudinal gradients of the temperature at the boundaries can be specified in an arbitrary way, while others depend on it and are calculated by formulas that are consequences of the boundary conditions (11), (12), the second of relations in formula (13), and the relation that determines the given gas flow in the upper layer (20).

\[
\kappa_y T_{1y} = \kappa_y T_{2y}
\]  
(15)

\( \kappa_1, \kappa_2 \) are the thermal conductivity coefficients of liquids. The projections of the dynamic condition at the boundary \( y = h_2 \) have the form

\[
p_1 = p_2, \quad p_2 V_2 u_{2y} = p_1 V_1 u_{3y} + \sigma_x T_x,
\] 
(16)

where \( \sigma_x \) is the temperature coefficient of surface tension. The heat transfer condition takes into account the Dufour effect and the vapor mass flow through the interface:

\[
\kappa_2 T_{2y} - \kappa_3 T_{3y} - \delta \kappa_y C_y = -\lambda M
\] 
(17)

\( \kappa_3 \) is the thermal conductivity coefficient of gas, \( \lambda \) is the amount of heat generated by evaporation, \( M \) is the mass of liquid evaporating per unit surface area per unit time. Also, the mass balance condition taking into account the Soret effect

\[
M = -D \rho_3 (C_y + \varepsilon T_y)
\] 
(18)

and the equation that determines the concentration of saturated vapor

\[
C = C_s (1 + \varepsilon T_y)
\] 
(19)

at the liquid-gas interface is assumed to be satisfied [1, 11]. Here \( \varepsilon = \lambda \mu (RT_0^2) \), \( C_s \) is the saturated vapor concentration at zero temperature, \( T_0 = 20^\circ C \), \( \mu \) is the molecular weight of an evaporating liquid, \( R \) is the universal gas constant. Gas flow rate is set in the upper layer:

\[
Q = \int_{h_2}^{h_1} \rho_3 u_y dy.
\] 
(20)
3. Examples of three-layer flows of the "silicone oil - water - air" system

The system "silicone oil - water - air" is considered as an example of a three-layer flow. The values of the physicochemical parameters of the problem are given in [11, 12]. Figures 2 and 3 show the profiles of the velocity and temperature distributions in the case when the layer thicknesses of the system are assumed to be the same and equal to $0.5 \times 10^{-2}$ m. The values on the abscissa axis determine at the same time the values of the longitudinal coordinate for the temperature distribution and the value of the longitudinal velocity, which does not depend on the $x$ coordinate. The value of the gas flow rate in the upper gas-vapor layer $Q$ is chosen equal to $10^{-4}$ kg/(m $\cdot$ s). The figures demonstrate the effect of longitudinal temperature gradients on the flow pattern. In the case when the temperature gradient on the lower wall $A_1$ is 0.5 K/m (see Figure 2), the temperature function obtains its minimum values at the boundary of the "liquid-gas" system. In this case, the process of liquid evaporation of the middle layer takes place. The opposite flow pattern is observed when the value of the longitudinal gradient $A_1 = -0.5$ K/m (Figure 3). The temperature function reaches its maximum value at the liquid-gas interface; the minimum temperature is observed on the walls of the channel. In this case, the flow in the system is accompanied by the process of condensation of the liquid filling the middle layer from the gas-vapor layer. Let us note also that the velocity profiles in the examples under consideration vary only slightly.

**Figure 2.** Velocity profile and temperature distribution at $h_1 = h_2 = h_3 = 0.5 \times 10^{-2}$ m, $A_1 = 0.5$ K/m.

**Figure 3.** Velocity profile and temperature distribution at $h_1 = h_2 = h_3 = 0.5 \times 10^{-2}$ m, $A_1 = -0.5$ K/m.

Figures 4 and 5 demonstrate the effect of the upper layer thickness on the flow character at the same values of the longitudinal temperature gradients at the system boundaries. In the case when the value of $h_3$ decreases to $0.3 \times 10^{-2}$ m, there is a significant increase in the velocity of the gas-vapor mixture (see Figure 4). The maximum value of the longitudinal velocity function decreases with an increase in the thickness of the gas-vapor layer (Figure 5). In addition, a decrease in the values of the temperature function is observed. However, in both cases, the maximum values of the temperature function are achieved at the liquid-gas interface, which corresponds to the process of liquid evaporation.

**Figure 4.** Velocity profile and temperature distribution at $h_1 = h_2 = 0.5 \times 10^{-2}$ m, $h_3 = 0.3 \times 10^{-2}$ m, $A_1 = 0.5$ K/m.

**Figure 5.** Velocity profile and temperature distribution at $h_1 = h_2 = 0.5 \times 10^{-2}$ m, $h_3 = 0.8 \times 10^{-2}$ m, $A_1 = 0.5$ K/m.
The exact solution of the convection equations proposed in the work allows modeling three-layer convective flows of liquid with evaporation or condensation at the interface between the liquid and gas-vapor mixture. The influence of such parameters as the thickness of the gas-vapor layer and the heat load at the system boundaries on the velocity profiles and temperature distribution is considered by the example of the "silicone oil - water - air" system.

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