Derivation of a homogenized nonlinear plate theory from 3d elasticity

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Abstract We derive, via simultaneous homogenization and dimension reduction, the $\Gamma$-limit for thin elastic plates whose energy density oscillates on a scale that is either comparable to, or much smaller than, the film thickness. We consider the energy scaling that corresponds to Kirchhoff’s nonlinear bending theory of plates.

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1 Introduction

Kirchhoff’s nonlinear plate theory associates with a deformation $u : S \to \mathbb{R}^3$ of a two-dimensional stress-free reference configuration $S \subset \mathbb{R}^2$ the bending energy

$$\int_S Q_2(\Pi(x')) \, dx',$$  \hspace{1cm} (1)
where $Q_2$ is the quadratic form from linear elasticity, and $\mathbf{II}$ denotes the second fundamental form associated with $u$. The key condition on the admissible deformations $u$ is that they must satisfy the isometry constraint

$$\partial_\alpha u \cdot \partial_\beta u = \delta_{\alpha\beta}, \quad \alpha, \beta \in \{1, 2\}$$

where $\delta_{\alpha\beta}$ denotes the Kronecker delta. Physically, (1) describes the elastic energy stored in a deformed plate that can undergo large deformations but no shearing or stretching.

In [5] Kirchhoff’s nonlinear plate theory was rigorously derived as a zero-thickness $\Gamma$-limit from 3d nonlinear elasticity. In this article we combine that result with homogenization. We consider a plate with thickness $h \ll 1$ made of a composite material that periodically oscillates with period $\varepsilon \ll 1$ in in-plane directions. We derive a homogenized plate model via simultaneous homogenization and dimension reduction in the limit $(h, \varepsilon) \to 0$ when the material period $\varepsilon$ and the thickness $h$ are either comparable or behave as $\varepsilon \ll h$, see Theorem 2.4 below. The derived model is sensitive to the relative scaling of $h$ and $\varepsilon$. Our result generalizes recent results from [14] where the one-dimensional case is studied. Regarding plates, related results have been obtained for different energy scalings: In [3,4] the membrane regime has been considered. Recently, the energy scaling corresponding to the von-Kármán plate model was studied in [15] for plates and in [10] for shells.

This article is organized as follows. Section 2 introduces the general framework and discusses the main results. In Sect. 3 we recall the notion of two-scale convergence and characterize the two-scale limit of nonlinear strains. In Sect. 4 we prove our main result.

2 Setting and main result

From now on, $S \subset \mathbb{R}^2$ denotes a bounded Lipschitz domain whose boundary is piecewise $C^1$. The piecewise $C^1$-condition is necessary only for the proof of the upper bound and can be relaxed, cf. Sect. 4.2 for details.

We set $I = (-\frac{1}{2}, \frac{1}{2})$. For $h > 0$ we denote by $\Omega_h = S \times hI$ the reference configuration of the thin plate of thickness $h$. The elastic energy per unit volume associated with a deformation $v^h : \Omega_h \to \mathbb{R}^3$ is given by

$$\frac{1}{h} \int_{\Omega_h} W\left(\frac{z'}{\varepsilon}, \nabla v^h(z)\right) dz.$$  

Here and below $z' = (z_1, z_2)$ denotes the in-plane coordinates of a generic element $z = (z_1, z_2, z_3) \in \Omega_h$ and $W$ is the energy density that models the elastic properties of a periodic composite.

Assumption 2.1 We assume that

$$W : \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \to [0, \infty]$$

is Borel measurable, that $W(\cdot, F)$ is $[0, 1)^2$-periodic for all $F \in \mathbb{R}^{3 \times 3}$ and that

$$W(y, \cdot) \in C^0\left(\mathbb{R}^{3 \times 3}, [0, \infty]\right)$$

for almost every $y \in \mathbb{R}^2$.

Furthermore, we assume that

$$W(y, RF) = W(y, F) \quad \text{for all } F \in \mathbb{R}^{3 \times 3} \text{ and all } R \in SO(3),$$

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where $\delta_{\alpha\beta}$ denotes the Kronecker delta. Physically, (1) describes the elastic energy stored in a deformed plate that can undergo large deformations but no shearing or stretching.
and that there exist positive constants $c_1$, $c_2$, $\rho$ such that
\[
W(y, F) \geq c_1 \text{dist}^2(F, \text{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3},
\]
\[
W(y, F) \leq c_2 \text{dist}^2(F, \text{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3} \text{ with } \text{dist}^2(F, \text{SO}(3)) \leq \rho.
\]
(5)

We also assume that there exists a monotone function $r : [0, \infty) \to [0, \infty]$ with $r(t) \downarrow 0$ as $t \downarrow 0$ such that, for almost every $y \in \mathbb{R}^2$, there exists a quadratic form $Q(y, \cdot)$ on $\mathbb{R}^{3 \times 3}$ with
\[
|W(y, I + G) - Q(y, G)| \leq r(|G|)|G|^2 \quad \text{for all } G \in \mathbb{R}^{3 \times 3}.
\]
(6)

We define $\Omega := S \times I$. As in [5] we rescale the out-of-plane coordinate: for $y = (x', x_3) \in \Omega$ consider the scaled deformation $u^h(x', x_3) := v^h(x', hx_3)$. Then (3) equals
\[
\mathcal{E}^{h, \varepsilon}(u^h) := \int_{\Omega} W \left( \frac{x'}{\varepsilon}, \nabla_h u^h(x) \right) \, dx,
\]
(7)

where $\nabla_h u^h := (\nabla' u^h, \frac{1}{h}\partial_3 u^h)$ denotes the scaled gradient, and $\nabla' u^h := (\partial_1 u^h, \partial_2 u^h)$ denotes the gradient in the plane.

We recall some known results on dimension reduction in the homogeneous case when $W(y, F) = W(F)$. As explained in [6] a hierarchy of plate models can be derived from $\mathcal{E}^h := \mathcal{E}^{h, 1}$ in the zero-thickness limit $h \to 0$. The different limiting models are characterized by the relative scaling of the elastic energy with respect to the thickness. In [11] it is shown that the scaling $\mathcal{E}^h \sim 1$ leads to a membrane model, which is a fully nonlinear plate model for plates without resistance to compression. In the regime $\mathcal{E}^h \sim h^4$ finite energy deformations converge to rigid deformations and $h^{-4}\mathcal{E}^h$ converges to the von-Kármán model, cf. [6].

In this article we study the bending regime $\mathcal{E}^h \sim h^2$, which, as shown in [5], leads to Kirchhoff’s nonlinear plate model: as $h \to 0$ the energy $h^{-2}\mathcal{E}^h \Gamma$-converges to the functional (1), with $Q_2 : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ given by the relaxation formula
\[
Q_2(A) = \min_{d \in \mathbb{R}^3} Q \left( \iota(A) + d \otimes e_3 \right).
\]

Here, $Q$ denotes the quadratic form from (6), and $\iota$ denotes the natural injection of $\mathbb{R}^{2 \times 2}$ into $\mathbb{R}^{3 \times 3}$. Denoting the standard basis of $\mathbb{R}^3$ by $(e_1, e_2, e_3)$ it is given by
\[
\iota(A) := \sum_{\alpha, \beta=1}^2 A_{\alpha\beta}(e_\alpha \otimes e_\beta).
\]

We will see that in the inhomogeneous case considered here the effective quadratic form replacing $Q_2$ is determined by a relaxation formula that is more complicated and requires the solution of a corrector problem. In particular, our analysis shows that in-plane oscillations of the deformation couple with the behaviour in the out-of-plane direction. As a consequence, the effective behaviour depends on the relative scaling of the thickness $h$ with respect to the material period $\varepsilon$. To make this precise, we assume that $\varepsilon$ and $h$ are coupled as follows:

**Assumption 2.2** Let $\gamma \in (0, \infty]$ denote a constant which is fixed throughout this article. We assume that $\varepsilon = \varepsilon(h)$ is a monotone function from $(0, \infty)$ to $(0, \infty)$ such that $\varepsilon(h) \to 0$ and $\frac{h}{\varepsilon(h)} \to \gamma$ as $h \to 0$.

The effective behaviour of the homogenized plate with reduced dimension can be computed by means of a relaxation formula that we introduce next. We need to introduce some function spaces of periodic functions. From now on, $Y = [0, 1]^2$, and we denote by $\mathcal{Y}$ the set $Y$ endowed with the torus topology, so that functions on $\mathcal{Y}$ will be $Y$-periodic.
We write \( C(\mathcal{Y}) \), \( C^k(\mathcal{Y}) \) and \( C^\infty(\mathcal{Y}) \) for the Banach spaces of \( Y \)-periodic functions on \( \mathbb{R}^2 \) that are continuous, \( k \)-times continuously differentiable and smooth, respectively. Moreover, \( H^1(I \times \mathcal{Y}) \) denotes the closure of \( C^\infty(I, C^\infty(\mathcal{Y})) \) with respect to the norm in \( H^1(I \times Y) \) and we write \( \tilde{H}^1(I \times \mathcal{Y}) \) for the subspace of functions \( f \in H^1(S \times \mathcal{Y}) \) with \( \int_{I \times Y} f = 0 \). The definitions extend in the obvious way to vector-valued functions.

**Definition 2.3 (Relaxation formula)** Let \( Q \) be as in Assumption 2.1.

(a) For \( \gamma \in (0, \infty) \) we define \( Q_{2, \gamma} : \mathbb{R}^{2 \times 2}_{\text{sym}} \to [0, \infty) \) by

\[
Q_{2, \gamma}(A) := \inf_{B, \phi} \int_{I \times Y} Q(y, t(x_3 A + B) + (\nabla_y \phi, \frac{1}{\gamma} \partial_3 \phi)) \, dy \, dx_3,
\]

where the infimum is taken over all \( B \in \mathbb{R}^{2 \times 2}_{\text{sym}} \) and \( \phi \in H^1(I \times \mathcal{Y}, \mathbb{R}^3) \).

(b) We define \( Q_{2, \infty} : \mathbb{R}^{2 \times 2}_{\text{sym}} \to [0, \infty) \) by

\[
Q_{2, \infty}(A) := \inf_{B, \phi, d} \int_{I \times Y} Q(y, t(x_3 A + B) + (\nabla_y \phi, d)) \, dy \, dx_3,
\]

where the infimum is taken over all \( B \in \mathbb{R}^{2 \times 2}_{\text{sym}}, \phi \in L^2(I, H^1(\mathcal{Y}, \mathbb{R}^3)) \) and \( d \in L^2(I, \mathbb{R}^3) \).

Kirchhoff’s plate model is defined for pure bending deformations of \( S \) into \( \mathbb{R}^3 \); precisely:

\[
W^{2,2}_{\delta}(S, \mathbb{R}^3) := \left\{ u \in W^{2,2}(S, \mathbb{R}^3) : u \text{ satisfies (2) a.e. in } S \right\}.
\]

With each \( u \in W^{2,2}_{\delta}(S) \) we associate its normal \( n := \partial_1 u \wedge \partial_2 u \), and we define its second fundamental form \( \Pi : S \to \mathbb{R}^{2 \times 2}_{\text{sym}} \) by defining its entries as

\[
\Pi_{\alpha\beta} = \partial_\alpha u \cdot \partial_\beta n = -\partial_\alpha \partial_\beta u \cdot n.
\]

Note that this is the negation of the usual definition of the second fundamental form adopted in geometry.

We write \( \Pi^h \) and \( n^h \) for the second fundamental form and normal associated with some \( u^h \in W^{2,2}_{\delta}(S, \mathbb{R}^3) \). The \( \Gamma \)-limit is a functional of the form (1): for \( \gamma \in (0, \infty) \) define \( \mathcal{E}_\gamma : L^2(\Omega, \mathbb{R}^3) \to [0, \infty] \) by

\[
\mathcal{E}_\gamma(u) := \left\{ \int_{S} Q_{2, \gamma}(\Pi(x')) \, dx', \text{ if } u \in W^{2,2}_{\delta}(S, \mathbb{R}^3) \right\} + \infty, \quad \text{otherwise.}
\]

Here and elsewhere we tacitly identify functions on \( S \) with their trivial extension to \( \Omega = S \times I \): above \( u \in W^{2,2}_{\delta}(S, \mathbb{R}^3) \) means that \( u(x', x_3) = \overline{u}(x') := \int_I u(x', z) \, dz \) for almost every \( x_3 \in I \), and \( \overline{u} \in W^{2,2}_{\delta}(S, \mathbb{R}^3) \). Our main result is the following:

**Theorem 2.4** Suppose that Assumptions 2.1 and 2.2 are satisfied. Then:

(i) (Lower bound). If \( (u^h)_h \geq 0 \) is a sequence with \( u^h \to u \) in \( L^2(\Omega, \mathbb{R}^3) \), then

\[
\liminf_{h \to 0} h^{-2} \mathcal{E}^{h, e(h)}(u^h) \geq \mathcal{E}_\gamma(u).
\]

(ii) (Upper bound). For every \( u \in W^{2,2}_{\delta}(S, \mathbb{R}^3) \) there exists a sequence \( (u^h)_h \geq 0 \) with \( u^h \to u \) strongly in \( W^{1,2}(\Omega, \mathbb{R}^3) \) such that

\[
\lim_{h \to 0} h^{-2} \mathcal{E}^{h, e(h)}(u^h) = \mathcal{E}_\gamma(u).
\]
This theorem is complemented by the following compactness result from [5], which in particular shows that the family of functionals $(E_h^{\varepsilon}(u^h))$ is equicoercive on $L^2(\Omega, \mathbb{R}^3)$.

**Theorem 2.5** ([5, Theorem 4.1]) Suppose a sequence $u^h \in H^1(\Omega, \mathbb{R}^3)$ has finite bending energy, that is

$$\limsup_{h \to 0} \frac{1}{h^2} \int_{\Omega} \text{dist}^2(\nabla_h u^h(x), \text{SO}(3)) \, dx < \infty.$$  

Then there exists $u \in W^{2,2}_0(S, \mathbb{R}^3)$ such that

$$u^h - \int_{\Omega} u^h \, dx \to u, \quad \text{strongly in } L^2(\Omega, \mathbb{R}^3),$$

$$\nabla_h u^h \to (\nabla' u, n) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^3 \times 3),$$

as $h \to 0$ after passing to subsequences and extending $u$ and $n$ trivially to $\Omega$.

**Remark 1** Theorem 2.4 can be extended to materials that are layered in $x_3$-direction, i.e., to situations when (3) is replaced by energies of the form

$$\frac{1}{h} \int_{\Omega_h} W \left( \frac{z_3}{h}, \frac{z'}{\varepsilon}, \nabla u^h(z) \right) \, dz,$$

where $W : [-1, 1] \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \to [0, +\infty]$ is Borel measurable, $[0, 1)^2$-periodic in the second variable, and the map $W(x_3, y, \cdot)$ is continuous and satisfies (4)–(6) for almost every $x_3$ and $y$. In that setting, the quadratic form $Q$ and the coefficients in the cell formulae (see Definition 2.3) will also depend on $x_3$.

With the same basic ideas, it is also possible to treat yet more general situations when $W$ also depends on the macroscopic in-plane variable $x' \in S$ (compare [15]).

In the particular case when $W(y, F) = W(F)$ is homogeneous, Theorem 2.4 reduces to the result in [5]. The proof of our main result emulates their argument as far as possible.

We now explain our approach. The bending regime is a borderline case in the hierarchy of plate models. On one hand it allows for large deformations, while on the other hand it corresponds to small strains: By Theorem 2.5 a sequence $(u^h)$ with finite bending energy in general converges to a non-trivial deformation. However, the associated non-linear strain

$$\sqrt{(\nabla_h u^h)^T (\nabla_h u^h) - I}$$

converges to zero. Indeed, let

$$E^h := \frac{\sqrt{(\nabla_h u^h)^T (\nabla_h u^h) - I}}{h}$$

denote the scaled non-linear strain associated with $u^h$. Then due to the elementary inequality

$$\left| \sqrt{F^T F} - I \right| \leq \text{dist}(F, \text{SO}(3))$$

we find that $(E^h)$ is bounded in $L^2$ when $(u^h)$ has finite bending energy.

The smallness of the nonlinear strain is crucial for our extension to simultaneous homogenization and dimension reduction: By (6) the elastic energy is related to the nonlinear strain in a quadratic way – indeed, we formally have

$$\frac{1}{h^2} E^{h, \varepsilon}(u^h) \approx \int_{\Omega} Q \left( \frac{x'}{\varepsilon}, E^h(x) \right) \, dx.$$  

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Heuristically, the right-hand side is obtained by linearizing the stress-strain relation, while preserving the geometric non-linearity.

Due to the convexity of the right-hand side in (11) only oscillations of $E^h$ that emerge precisely on the scale $\varepsilon$ are relevant for homogenization. A tool to describe such oscillations is two-scale convergence. In Sect. 3 we characterize (partially) the possible two-scale limits of $E^h$. This is the main ingredient for the lower bound in Theorem 2.4.

Assume $u^h$ converges to some bending deformation with second fundamental form $II$. Then any two-scale accumulation point of $\{E^h\}$ can be written in the form

$$\iota\left(x_3II(x')\right) + \tilde{E}(x, y)$$

(12)

where $\tilde{E} : \Omega \times Y \to \mathbb{R}^{3\times 3}_{\text{sym}}$ is a relaxation field that captures oscillations and is a priori “unknown”. In Proposition 3.2 we prove that $\tilde{E}$ has to be of a particular form. The $\Gamma$-limit of $h^{-2}E^{h,\varepsilon}$ is then obtained by relaxation:

$$\inf_{\tilde{E}} \int \int_{\Omega} Q\left(y, \iota\left(x_3II(x')\right) + \tilde{E}(x, y)\right) \, dy \, dx,$$

where the infimum is taken over all $\tilde{E}$ of the specific form given in Proposition 3.2.

We conclude this section by commenting on the dependency of our limiting model on the parameter $\gamma$, which describes the relative scaling between $h$ and $\varepsilon$.

The relaxed quadratic form $Q_{2,\gamma}$ depends continuously on $\gamma$. In fact, with [15, Lemma 5.2] at hand one can easily identify the limits

$$\lim_{\gamma \to 0} Q_{2,\gamma}(A) \quad \text{and} \quad \lim_{\gamma \to \infty} Q_{2,\gamma}(A) \quad (A \in \mathbb{R}^{2\times 2}),$$

which yield proper quadratic forms on $\mathbb{R}^{2\times 2}$ that vanish on skew-symmetric matrices and are positive definite on symmetric matrices. In particular, the limit for $\gamma \to \infty$ coincides with $Q_{2,\infty}$. The limit for $\gamma \to 0$ can be identified as well: We introduce the dimension reduced quadratic form $Q_2(y, A)$ for all $A \in \mathbb{R}^{2\times 2}$ via

$$Q_2(y, A) = \min_{d \in \mathbb{R}^3} Q\left(y, \iota(A) + d \otimes e_3\right).$$

Then $Q_{2,\gamma}(A)$ converges for $\gamma \to 0$ to

$$Q_{2,0}(A) := \inf_{B, \xi, \varphi} \int I \times Y Q_2\left(y, x_3A + B + \text{sym}(\nabla_y\xi) + x_3\nabla^2_y\varphi\right) \, dy \, dx_3$$

$$= \frac{1}{12} \inf_{\varphi} \int Y Q_2\left(y, A + \nabla^2_y\varphi\right) \, dy,$$

where the infimum is taken over all $B \in \mathbb{R}^{2\times 2}_{\text{sym}}, \xi \in H^1(Y, \mathbb{R}^2)$ and $\varphi \in H^2(Y)$.

A similar behaviour has been observed in [14,15] where also the case $\gamma = 0$ is considered (for rods and von-Kármán plates, respectively). In the von-Kármán case (see [15]) it turns out that in the regime $h \ll \varepsilon$ the limit $\gamma \to 0$ of the quadratic energy density indeed recovers the energy density obtained via $\Gamma$-convergence.
2.1 Effective energy density for a simple case of layered isotropic composites

In this section we compute and discuss the effective energy density \( Q_{2,y} \) for \( y \in (0, \infty) \) in the particular case when

\[
Q(y, G) = E(y_1)|\text{sym } G|^2,
\]

where \( E : \mathbb{R} \to \mathbb{R} \) denotes a measurable, \([0, 1]-periodic function with \( 0 < \text{ess inf}_{y_1} E \leq \text{ess sup}_{y_1} E < \infty \). This corresponds to a composite that is layered in \( y_1 \)-direction, and is composed of isotropic materials with (non-degenerate) Young’s modulus \( E \), but vanishing Poisson’s ratio.

In this section, we denote by \( \mathcal{Y}_1 \) the set \([0, 1]\) endowed with the torus topology. We define the spaces \( H^1(\mathcal{Y}_1) \) and \( H^1(I \times \mathcal{Y}_1) \) in analogy to \( H^1(\mathcal{Y}) \) and \( H^1(\mathcal{Y} \times I) \), respectively. Moreover, we identify functions on \( \mathcal{Y} \) which are constant in the \( y_2 \)-direction with functions on \( \mathcal{Y}_1 \).

We claim that

\[
Q_{2,y}(a_1 A^{(1)} + a_2 A^{(2)} + a_3 A^{(3)}) = a_1^2 E^{(\infty)} + a_2^2 E^{(y)} + a_3^2 E^{(0)},
\]

where

\[
A^{(1)} := e_1 \otimes e_1, \quad A^{(2)} := \frac{1}{\sqrt{2}} (e_1 \otimes e_2 + e_2 \otimes e_1), \quad A^{(3)} := e_2 \otimes e_2
\]

consitutes a basis of \( \mathbb{R}^{2 \times 2}_{\text{sym}} \), and \( E^{(y)} \) denotes the relaxation of \( E \) defined for \( 0 \leq \delta < \infty \) as

\[
E^{(\delta)} := \begin{cases} 
\frac{1}{12} \left( \int_{(0,1)} E(y_1)^{-1} dy_1 \right)^{-1} & \text{for } \delta = \infty, \\
\min_{\varphi \in H^1(I \times \mathcal{Y}_1)} \int_{I \times (0,1)} E(y_1) \left( (x_3 + \partial_{y_1} \varphi)^2 + \left( \frac{1}{\delta} \partial_{y_3} \varphi \right)^2 \right) dx_3 dy_1 & \text{for } \delta = 0.
\end{cases}
\]

The relaxed Young’s moduli \( E^{(0)} \) and \( E^{(\infty)} \) are the average and the harmonic mean of \( E \). Moreover, \( E^{(\delta)} \) lies in between \( E^{(\infty)} \) and \( E^{(0)} \), and one can check that

\[
\lim_{\delta \downarrow 0} E^{(\delta)} = E^{(0)}, \quad \lim_{\delta \uparrow \infty} E^{(\delta)} = E^{(\infty)}.
\]

Before we sketch the argument for (14), let us give a heuristic explanation. First, let us comment on the assumption of vanishing Poisson’s ratio. An isotropic material that is stretched in one direction typically tends to shrink in perpendicular directions. This phenomenon (the Poisson effect) is measured by Poisson’s ratio. Hence, in our case, strain in one direction does not induce stress in perpendicular directions. In particular, when the plate is bent about an axis, say \( e \), then the induced stresses are aligned with the in-plane direction orthogonal to \( e \).

Let us discuss (14). In the case \( a_1 \neq 0 = a_2 = a_3 \), which corresponds to a bending of the plate about the \( e_2 \)-axis, the material above and below the mid-plane is stretched perpendicular to the direction of the layers. Therefore, on the microscale the plate adapts to the layered microstructure leading to the formation of oscillations, i.e. homogenization occurs. These oscillations are described by the function \( \varphi^{(\infty)} \) defined as the unique minimizer to

\[
H^1(\mathcal{Y}_1) \ni \varphi \mapsto \int_{(0,1)} E(y_1)(1 + \partial_{y_1} \varphi)^2 dy_1 \quad \text{with} \quad \int_{(0,1)} \varphi dy_1 = 0.
\]
Note that the minimum of the integral functional above is precisely the harmonic mean \( E(\infty) \).

The case \( a_3 \neq 0 = a_1 = a_2 \) corresponds to a bending of the plate about the \( e_1 \)-axis. In that case stresses are parallel to the layers, and we do not expect the formation of oscillations. Hence, the effective modulus is given by the average \( E(0) \). When the plate is bent about an axis other than any the coordinate directions, the coefficient \( a_2 \) becomes relevant and a correction emerges that is captured by the function \( \varphi^{(\gamma)} \in H^1(I \times Y) \) defined as the unique minimizer to

\[
H^1(I \times Y) \ni \varphi \mapsto \int_{I \times (0,1)} E(y_1) \left( (x_3 + \partial_{y_1} \varphi)^2 + \left( \frac{1}{\gamma} \partial_3 \varphi \right)^2 \right) dy_1 dx_3
\]

with \( \int_I \varphi(y_1, x_3) dx_3 = 0 \) for almost every \( y_1 \).

(16)

The minimum of the integral functional above is precisely \( E(\gamma) \) and depends in a non-trivial way on the coupling parameter \( \gamma \).

We finally sketch the argument for (14). By (13), the relaxation formula simplifies to

\[
Q_{2, \gamma}(A) = \min_{B \in \mathbb{R}^{2 \times 2}} \left( \int_I \int_{(0,1)} E(y_1) \times \left| \text{sym} \left( \iota(x_3 A + B) + (\partial_{y_1} \phi, 0, 1 / \gamma \partial_3 \phi) \right) \right|^2 dy_1 dx_3 \right) \]

(17)

We introduce the correctors

\[
\phi^{(1, \gamma)}(y_1, x_3) := \left( x_3 \varphi^{(\infty)}(y_1), 0, -1 / \gamma \int_0^{y_1} \varphi^{(\infty)}(s) ds \right),
\]

\[
\phi^{(2, \gamma)}(y_1, x_3) := \left( 0, \sqrt{2} \varphi^{(\gamma)}(y_1), 0 \right), \quad \phi^{(3, \gamma)}(y_1, x_3) := 0,
\]

where \( \varphi^{(\infty)} \) and \( \varphi^{(\gamma)} \) are defined as minimizers of (15) and (16), respectively. Standard arguments show that the right-hand side in (17) with \( A = A^{(\alpha)} \), \( \alpha = 1, 2, 3 \), is minimized for \( B = 0 \) and \( \phi = \phi^{(\alpha, \gamma)} \). Furthermore, a straightforward calculation shows that

\[
Q_{2, \gamma}(A^{(\alpha)}) = Q(A^{(\alpha)}, \phi^{(\alpha, \gamma)}) = \begin{cases} E(\infty) & \text{for } \alpha = 1, \\ E(\gamma) & \text{for } \alpha = 2, \\ E(0) & \text{for } \alpha = 3, \end{cases}
\]

where

\[
Q(A, \phi) := \int_{I \times (0,1)} E(y_1) \left| \text{sym} \left( \iota(x_3 A + \varphi, 0, 1 / \gamma \partial_3 \varphi) \right) \right|^2 dx_3 dy_1.
\]

Since the minimization problem corresponds to a linear Euler-Lagrange equation, we deduce from the minimality of \( \phi^{(\alpha)} \) that

\[
Q_{2, \gamma} \left( \sum_{\alpha=1}^3 a_\alpha A^{(\alpha)} \right) = Q \left( \sum_{\alpha=1}^3 a_\alpha A^{(\alpha)}, \sum_{\alpha=1}^3 a_\alpha \phi^{(\alpha)} \right) = \sum_{\alpha=1}^3 a_\alpha^2 Q(A^{(\alpha)}, \phi^{(\alpha)})
\]

and the argument for (14) is complete.
3 Two-scale limits of the nonlinear strain

Two-scale convergence was introduced in [1,16] and has been extensively applied to various problems in homogenization. In this article we work with the following variant of two-scale convergence which is adapted to dimension reduction.

Definition 3.1 (two-scale convergence) We say a bounded sequence \((f^h)_{h>0} \subset L^2(\Omega)\) two-scale converges to \(f \in L^2(\Omega \times Y)\) and we write \(f^h \overset{2, Y}{\rightharpoonup} f\), if

\[
\lim_{h \to 0} \int_{\Omega} f^h(x) \psi(x, x') \frac{dx'}{\varepsilon(h)} = \int_{\Omega \times Y} f(x, y) \psi(x, y) \, dy \, dx
\]

for all \(\psi \in C_0^\infty(\Omega, C(\mathcal{Y}))\). When \(\|f^h\|_{L^2(\Omega)} \to \|f\|_{L^2(\Omega \times Y)}\) in addition, we say that \(f^h\) strongly two-scale converges to \(f\) and write \(f^h \overset{2, Y}{\rightharpoonup} f\). For vector-valued functions, two-scale convergence is defined componentwise.

Since we identify functions on \(S\) with their trivial extension to \(\Omega\), the definition above contains the standard notion of two-scale convergence on \(S \times Y\) as a particular case. Indeed, when \((f^h)\) is a sequence in \(L^2(S)\), then \(f^h \overset{2, Y}{\rightharpoonup} f\) is equivalent to

\[
\lim_{h \to 0} \int_S f^h(x') \psi(x', x') \frac{dx'}{\varepsilon(h)} = \int_S f(x', y) \psi(x', y) \, dy \, dx'
\]

for all \(\psi \in C_0^\infty(S, C(\mathcal{Y}))\).

The main ingredient in the proof of the lower bound part of Theorem 2.4 is the following characterization of the possible two-scale limits of nonlinear strains.

Proposition 3.2 Let \((u^h)\) be a sequence of deformations with finite bending energy, let \(u \in W_0^{2,2}(S, \mathbb{R}^3)\) with second fundamental form \(\Pi\), and assume that

\[
\begin{align*}
&u^h - \int_\Omega u^h \, dx \rightharpoonup u \quad \text{strongly in } L^2(\Omega, \mathbb{R}^3), \\
&E^h := \sqrt{(\nabla_h u^h)^T \nabla_h u^h - I} \overset{2, Y}{\rightharpoonup} E \quad \text{weakly two-scale}
\end{align*}
\]

for some \(E \in L^2(\Omega \times Y; \mathbb{R}^{3 \times 3})\).

(a) If \(\gamma \in (0, \infty)\) then there exist \(B \in L^2(S, \mathbb{R}^{2 \times 2}_{\text{sym}})\), and \(\phi \in L^2(S, \hat{H}^1(I \times \mathcal{Y}, \mathbb{R}^3))\) such that

\[
E(x, y) = \iota \left( x_3 \Pi(x') + B(x') \right) + \text{sym} \left( \nabla_y \phi(x, y), \frac{1}{\gamma} \partial_3 \phi(x, y) \right). \tag{18}
\]

(b) If \(\gamma = \infty\) then there exist \(B \in L^2(S, \mathbb{R}^{2 \times 2}_{\text{sym}}), \phi \in L^2(\Omega, \hat{H}^1(\mathcal{Y}, \mathbb{R}^3)),\) and \(d \in L^2(\Omega, \mathbb{R}^3)\) with

\[
E(x, y) = \iota \left( x_3 \Pi(x') + B(x') \right) + \text{sym} \left( \nabla_y \phi(x, y), d(x) \right). \tag{19}
\]

Remarks (i) Proposition 3.2 still only yields an incomplete characterization of the possible structure of the two-scale limiting strain \(E\): it is not true that every \(E\) of the form in (18) (resp. 19) can be recovered as a two-scale limit of a sequence of nonlinear strains. For instance, when \(u\) is affine, i.e. \(\Pi = 0\), then not every two-scale limiting strain of the form (18) with \(B\) arbitrary and \(\phi = 0\) can emerge. An important observation is...
that, in spite of not giving an exhaustive characterization of limiting strains, the result of Proposition 3.2 is just sharp enough to obtain the optimal lower bound for \( h^{-2}E^{h,ε(h)} \).

In contrast, for rods and von-Kármán plates, exhaustive characterizations were obtained in [14, Theorem 3.5] and [15, Proposition 3.3].

(ii) A key technical ingredient in the proof of Proposition 3.2 is Lemma 3.8 below. It allows us to work with piecewise constant \( SO(3) \)-valued approximations of the deformation gradient, as opposed to smooth \( SO(3) \)-valued approximations. The latter were used in the proof of the 1d case given in [14]. In the 2d case studied here, the use of such a smooth \( SO(3) \)-valued approximation would require \textit{small} limiting energy, cf. [6, Remark 5]. Thanks to Lemma 3.8, our result is not restricted to small limiting energy. Incidentally, the use of this lemma also simplifies the proof of the convergence statement in the 1d case.

The starting point of the proof of the previous Proposition is [6, Theorem 6], which we combine with the last remark in [5, Section 3] in order to allow for \( γ_0 < 1 \).

**Lemma 3.3** Let \( γ_0 ∈ (0, 1] \) and let \( h, δ > 0 \) with \( γ_0 ≤ \frac{h}{δ} ≤ \frac{1}{γ_0} \). There exists a constant \( C \), depending only on \( S \) and \( γ_0 \), such that the following is true: if \( u ∈ H^1(Ω, \mathbb{R}^3) \) then there exists a map \( R : S → SO(3) \) which is piecewise constant on each cube \( x + δY \) with \( x ∈ δZ^2 \) and there exists \( \tilde{R} ∈ H^1(S, \mathbb{R}^{3×3}) \) such that

\[
∥∇_h u - R∥_{L^2(Ω)}^2 + ∥R - \tilde{R}∥_{L^2(S)}^2 + h^2∥∇_h \tilde{R}∥_{L^2(S)}^2 ≤ C∥\text{dist}(∇_h u, SO(3))∥_{L^2(Ω)}^2.
\]

Let us recall some well-known properties of two-scale convergence. We refer to [1,12,19] for proofs in the standard two-scale setting and to [13] for the easy adaption to the notion of two-scale convergence considered here.

**Lemma 3.4** (i) Any sequence that is bounded in \( L^2(Ω) \) admits a two-scale convergent subsequence.

(ii) Let \( \tilde{f} ∈ L^2(Ω × Y) \) and let \( f^h ∈ L^2(Ω) \) be such that \( f^h \overset{2,Y}{\rightharpoonup} \tilde{f} \). Then \( f^h → \int_Y \tilde{f}(., y) \, dy \) weakly in \( L^2(Ω) \).

(iii) Let \( f^0, f^h ∈ L^2(Ω) \) be such that \( f^h → f^0 \) weakly in \( L^2(Ω) \). Then (after passing to subsequences) we have \( f^h \overset{2,Y}{\rightharpoonup} f^0 + \tilde{f} \) for some \( \tilde{f} ∈ L^2(Ω × Y) \) with \( ∫_Y \tilde{f}(., y) \, dy = 0 \) almost everywhere in \( S \).

(iv) Let \( f^0, f^h ∈ H^1(Ω) \) be such that \( f^h → f^0 \) strongly in \( L^2(Ω) \). Then \( f^h \overset{2,Y}{\rightharpoonup} f^0 \), where we extend \( f^0 \) trivially to \( Ω × Y \).

(v) Let \( f^0, f^h ∈ H^1(S) \) be such that \( f^h → f^0 \) weakly in \( H^1(S) \). Then there exists \( φ ∈ L^2(S, H^1(Y)) \) such that (after passing to subsequences)

\[
∇' f^h \overset{2,Y}{\rightharpoonup} ∇' f^0 + ∇_Y φ.
\]

The following lemma is [13, Theorem 6.3.3].

**Lemma 3.5** Let \( u^0, u^h ∈ H^1(Ω, \mathbb{R}^3) \) be such that \( u^h → u^0 \) weakly in \( H^1(Ω, \mathbb{R}^3) \), and assume that \( ∥∇_h u^h∥_{L^2(Ω)} \) is uniformly bounded. Then:

(a) If \( γ ∈ (0, ∞) \) then there exists \( φ ∈ L^2(S, H^1(I × Y, \mathbb{R}^3)) \) such that (after passing to subsequences)

\[
∇_h u^h \overset{2,Y}{\rightharpoonup} (∇' u^0, 0) + (∇_Y φ, \frac{1}{γ} ∂_3 φ).
\]
(b) If $\gamma = \infty$ then there exist $\phi \in L^2(\Omega, \tilde{H}^1(\mathcal{Y}, \mathbb{R}^3))$ and $d \in L^2(\Omega, \mathbb{R}^3)$ such that (after passing to subsequences)$$\nabla_h u^h \overset{2,\gamma}{\rightharpoonup} (\nabla' u^0, 0) + (\nabla_y \phi, d).$$

3.1 Renormalized two-scale convergence

At several places in our proof we will need to make sense of a two-scale limit for sequences which might be unbounded in $L^2$, but which nevertheless have controlled oscillations on the scale $\varepsilon$. In order to capture these oscillations, we ‘renormalize’ the sequence by ignoring the (divergent) part which does not oscillate on the scale $\varepsilon$. (For bounded sequences, this latter part gives rise to the weak limit, but the point here is that our sequences may be unbounded.) Equivalently, we weaken the notion of two-scale convergence by restricting the admissible test functions to functions with vanishing cell average.

For a sequence $(f^h) \subset L^2(\Omega)$ and $\tilde{f} \in L^2(\Omega \times \mathcal{Y})$ with $\int_{\mathcal{Y}} \tilde{f}(\cdot, y) \, dy = 0$ almost everywhere in $\Omega$, we write

$$f^h \overset{osc,\gamma}{\rightharpoonup} \tilde{f}$$

if

$$\lim_{h \to 0} \int_{\Omega} f^h(x) \varphi(x) g(\frac{x'}{\varepsilon(h)}) \, dx = \int_{\Omega \times \mathcal{Y}} \tilde{f}(x, y) \varphi(x) g(y) \, dy \, dx$$

for all $\varphi \in C^\infty_0(\Omega)$ and $g \in C^\infty(\mathcal{Y})$ with $\int_{\mathcal{Y}} g \, dy = 0$. (20)

Of course this notion of convergence is not restricted to thin films, as $h$ plays no role here. Nevertheless we stick to the notation of the rest of the paper. General results are obtained by replacing $\varepsilon(h)$ by $\varepsilon$ and $f^h$ by $f^\varepsilon$, and considering the limits as $\varepsilon \to 0$.

The proof of the following lemma is straightforward.

**Lemma 3.6** Let $f^0, f^h \in L^2(\Omega)$ be such that $f^h \rightharpoonup f^0$ weakly in $L^2(\Omega)$ and $f^h \overset{osc,\gamma}{\rightharpoonup} \tilde{f}$. Then $f^h \overset{2,\gamma}{\rightharpoonup} f^0 + \tilde{f}$ weakly two-scale.

For the proof of Proposition 3.2 we have to identify the oscillatory part of two-scale limits for renormalized functions of the form $\frac{1}{\varepsilon(h)} f^h$ where $f^h$ is either a sequence bounded in $H^1(S)$ or piecewise affine with respect to the lattice $\varepsilon(h)\mathbb{Z}^2$.

**Lemma 3.7** Let $f^0, f^h \in H^1(S)$ be such that $f^h \rightharpoonup f^0$ weakly in $H^1(S)$ and assume that$$\nabla' f^h \overset{2,\gamma}{\rightharpoonup} \nabla' f^0 + \nabla_y \phi$$

for some $\phi \in L^2(S, H^1(\mathcal{Y}))$ with $\int_{\mathcal{Y}} \phi(\cdot, y) \, dy = 0$ almost everywhere in $S$. Then

$$f^h \overset{osc,\gamma}{\rightharpoonup} \phi.$$ 

**Proof** Let $A \in C^1(\mathcal{Y}, \mathbb{R}^2)$ be such that

$$g = \text{div} A.$$
One may, for instance, take $A = \nabla G$, where $G \in C^2(\mathcal{Y})$ satisfies $\Delta G = g$. After subtracting a constant, we may assume without loss of generality that $\int_Y A = 0$. We write $\epsilon = \epsilon(h)$ and compute:

$$\frac{1}{\epsilon} \int_S f^h(x') \psi(x') g \left( \frac{x'}{\epsilon} \right) dx' = \frac{1}{\epsilon} \int S f^h(x') (\text{div} A) \left( \frac{x'}{\epsilon} \right) \psi(x') dx'$$

$$= \int S f^h(x') \text{div} \left( A \left( \frac{x'}{\epsilon} \right) \right) \psi(x') dx'$$

$$= - \int (\nabla' f^h)(x') \cdot A \left( \frac{x'}{\epsilon} \right) \psi(x') dx' - \int S f^h(x') A \left( \frac{x'}{\epsilon} \right) \cdot \nabla \psi(x') dx'$$

$$\rightarrow - \int S \nabla_y \phi(x', y) \cdot A(y) \psi(x') dx'dy$$

$$= - \int S \nabla_y \phi(x', y) \cdot A(y) \psi(x') dx'dy.$$

We used twice that $A$ has zero average over $Y$. An integration by parts with respect to $y$ yields the claim. $\square$

The following lemma is crucial, as it allows us to work with piecewise constant maps instead of smooth ones.

**Lemma 3.8** Let $f^0, f^h \in L^\infty(S)$ be such that $f^h \overset{\alpha}{\rightharpoonup} f^0$ weakly-* in $L^\infty(S)$. Assume that $f^h$ is constant on each square $x + \epsilon(h) Y$ with $x \in \epsilon(h) \mathbb{Z}^2$. Then we have

$$\frac{1}{\epsilon(h)} \int S f^h(x') \psi(x') g \left( \frac{x'}{\epsilon(h)} \right) dx' \rightarrow \int S f^0(x') \nabla' \psi(x') dx' \cdot \int Y g(y) dy$$

(21)

for all $g \in C(\mathcal{Y})$ with $\int g = 0$ and all $\psi \in C_0^\infty(S)$. In particular, if $f^0 \in W^{1,2}(S)$ we have

$$\frac{1}{\epsilon(h)} f^h \overset{\alpha \text{sc}, \mathcal{Y}}{\rightharpoonup} -(y \cdot \nabla') f^0.$$  

(22)

Here we write

$$(y \cdot \nabla') f^0(x') = \sum_{\alpha = 1,2} y_\alpha \partial_\alpha f^0(x').$$

**Proof** We first check that (21) combined with $f^0 \in W^{1,2}(S)$ implies (22). In fact, since $f^h$ is independent of $x_3$ it suffices to consider test functions $g$ and $\psi$ as in (21). Now the statement simply follows from the observation that the right-hand side of (21) becomes

$$- \int S \nabla_y f^0(x') \psi(x') g(y) dy dx'$$

by an integration by parts.

We now prove (21). For simplicity we write $\epsilon$ instead of $\epsilon(h)$. We denote by $\tilde{\psi}^h$ an approximation of $\psi$ that is constant on each of the cubes $\xi + \epsilon Y, \xi \in \epsilon \mathbb{Z}^2$, say $\tilde{\psi}^h(x) := \psi(\xi_x)$ where $\xi_x \in \epsilon \mathbb{Z}^2$ is the center of the cube $\xi_x + \epsilon Y$ containing $x$. Then we have

$$\int S f^h(x) \psi(x) g \left( \frac{x}{\epsilon} \right) dx = \int S f^h(x) \psi(x) \frac{\tilde{\psi}^h(x)}{\epsilon} \left( \frac{x}{\epsilon} \right) g \left( \frac{x}{\epsilon} \right) dx$$

(23)
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because

\[ \int_S f^h(x) \tilde{\psi}^h(x) g \left( \frac{x}{\varepsilon} \right) \, dx = 0, \]

since \( g \) has zero average over \( Y \), and \( f^h \) and \( \tilde{\psi}^h \) are both piecewise constant. Let us compute the right-hand side of (23). As \( \xi_x \in \mathbb{Z}^2 \) and \( g \in C(Y) \), we have

\[ g \left( \frac{x - \xi_x}{\varepsilon} \right) = g \left( \frac{x}{\varepsilon} \right). \]

Hence (after extending \( f^h \) to \( \mathbb{R}^2 \) by zero)

\[
\int_S f^h(x) \frac{\psi(x) - \tilde{\psi}^h(x)}{\varepsilon} g \left( \frac{x}{\varepsilon} \right) \, dx
\]

\[
= \sum_{\xi \in \mathbb{Z}^d} f^h(\xi) \int_{\xi + \varepsilon Y} \left( \int_0^1 (\nabla' \psi(\xi + t(x - \xi)) \cdot \frac{x - \xi}{\varepsilon}) g \left( \frac{x}{\varepsilon} \right) \, dx \right) 
\]

\[
= \sum_{\xi \in \mathbb{Z}^d} f^h(\xi) \int_{\xi \varepsilon Y} \left( \int_0^1 \nabla' \psi(\xi + tx) \cdot \frac{x}{\varepsilon} \right) g \left( \frac{x}{\varepsilon} \right) \, dx 
\]

\[
= \sum_{\xi \in \mathbb{Z}^d} f^h(\xi) \epsilon^2 \int_Y \left( \int_0^1 \nabla' \psi(\xi + t \varepsilon y) \cdot y \right) g(y) \, dy 
\]

\[
= \sum_{\xi \in \mathbb{Z}^d} f^h(\xi) \epsilon^2 \int_Y \nabla' \psi(\xi) \cdot yg(y) \, dy + \sum_{\xi \in \mathbb{Z}^d} f^h(\xi) \epsilon^2 \int_Y \nabla' \psi(\xi + t \varepsilon y) - \nabla' \psi(\xi) \cdot y \right) g(y) \, dy 
\]

The first term on the right-hand side converges to zero as \( h \to 0 \) because

\[ |\nabla' \psi(\xi + t \varepsilon y) - \nabla' \psi(\xi)| \leq C \varepsilon \]

for all \( t \in [0, 1] \), because \( \nabla' \psi \) is Lipschitz.

Hence it remains to compute:

\[
\sum_{\xi \in \mathbb{Z}^d} \epsilon^2 f^h(\xi) \nabla' \psi(\xi) \cdot \int_Y yg(y) \, dy 
\]

\[
= \sum_{\xi \in \mathbb{Z}^d} \epsilon^2 \left( f^h(\xi) \nabla' \psi(\xi) - \int_{\xi + \varepsilon Y} f^h(z) \nabla' \psi(z) \, dz \right) \cdot \int_Y yg(y) \, dy 
\]

\[
+ \sum_{\xi \in \mathbb{Z}^d} \epsilon^2 \int_{\xi + \varepsilon Y} f^h(z) \nabla' \psi(z) \, dz \cdot \int_Y yg(y) \, dy 
\]
\[
= \sum_{\xi \in \mathbb{Z}^d} \varepsilon^2 \left( f^h(\xi) \nabla' \psi(\xi) - \int_{\xi + \varepsilon Y} f^h(z) \nabla' \psi(z) \, dz \right) \cdot \int_Y yg(y) \, dy \\
+ \int_{\mathbb{R}^2} f^h(x) \nabla' \psi(x) \, dx \cdot \int_Y yg(y) \, dy.
\]

Since spt $\psi \subset S$, the last term equals
\[
\int_S f^h(x) \nabla' \psi(x) \, dx \cdot \int_Y yg(y) \, dy.
\]

The claim follows because $f^h \rightharpoonup f^0$ and because
\[
\sum_{\xi \in \mathbb{Z}^d} \varepsilon^2 \left( f^h(\xi) \nabla' \psi(\xi) - \int_{\xi + \varepsilon Y} f^h(z) \nabla' \psi(z) \, dz \right) \cdot \int_Y yg(y) \, dy \to 0
\]
as $h \to 0$. To see this, we compute recalling that $f^h(x) = f^h(\xi)$ for all $x \in \xi + \varepsilon Y$:
\[
f^h(\xi) \nabla' \psi(\xi) - \int_{\xi + \varepsilon Y} f^h(z) \nabla' \psi(z) \, dz = f^h(\xi) \left( \nabla' \psi(\xi) - \int_{\xi + \varepsilon Y} \nabla' \psi(z) \, dz \right) \leq C \varepsilon,
\]
again because $\nabla' \psi$ is Lipschitz. \hfill $\Box$

3.2 Compactness

**Proof of Proposition 3.2** Case $\gamma \in (0, \infty)$ **Step 1.** Without loss of generality we assume that all $u^h$ have average zero. Theorem 2.5 implies that
\[
\nabla h u^h \rightharpoonup R := (\nabla' u, n) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^{3 \times 3})
\]
where $n$ denotes the normal to $u$. Let $R^h$, $\tilde{R}^h$ be the maps obtained by applying Lemma 3.3 to $u^h$ with $\delta(h) = \varepsilon(h)$. Due to the uniform bound on $\nabla' \tilde{R}^h$ given by Lemma 3.3, $R^h$ and $\tilde{R}^h$ are precompact in $L^2(S, \mathbb{R}^{3 \times 3})$. Hence, combining (24) with
\[
\|R^h - \nabla h u^h\|_{L^2} \leq C h,
\]
which also follows from Lemma 3.3, we see that $R^h$ and $\tilde{R}^h$ converge strongly in $L^2(S, \mathbb{R}^{3 \times 3})$ to $R$. Following [5], we introduce the approximate strain
\[
G^h(x) = \frac{(R^h)'^T \nabla h u^h(x) - I}{h}.
\]
We set $\overline{u^h}(x') = \int_I u^h(x', x_3) \, dx_3$ and define $z^h \in H^1(\Omega, \mathbb{R}^3)$ via
\[
u^h(x', x_3) = \overline{u^h}(x') + h x_3 \tilde{R}^h(x') e_3 + h z^h(x', x_3).
\]
Then clearly $\int_I z^h(x', x_3) dx_3 = 0$ and we compute
\[
\frac{\nabla h u^h - R_h}{h} = \left( \frac{\nabla' \overline{u^h} - (R^h)'}{h} + x_3 \nabla' \tilde{R}^h e_3, \frac{1}{h} (\tilde{R}^h e_3 - R^h e_3) \right) + \nabla h z^h.
\]
For a given matrix $M \in \mathbb{R}^{3 \times 3}$, we denote by $M'$ the $3 \times 2$-matrix obtained by deleting the third column. We will use the notation $(y \cdot \nabla')R(x') := y_1 \partial_1 R(x') + y_2 \partial_2 R(x')$.

**Step 2.** Let us for the moment take for granted that there exist $B' \in L^2(S, \mathbb{R}^{3 \times 2})$, $\zeta \in L^2(S, H^1(I \times \mathcal{Y}, \mathbb{R}^3))$, $\tilde{v}$, $\tilde{w} \in L^2(S, H^1(\mathcal{Y}, \mathbb{R}^3))$ and $w^0 \in L^2(S, \mathbb{R}^3)$, such that, after passing to a subsequence,

\[
\nabla h z^h \rightharpoonup_{\gamma} (\nabla_y \tilde{z}, \frac{1}{\gamma} \partial_3 \tilde{z}),
\]

\[
\frac{\nabla' \tilde{w}^h - (R^h)' \gamma}{h} \rightharpoonup_{\gamma} B'(x') + \frac{1}{\gamma} (y \cdot \nabla') R'(x') + \nabla_y \tilde{v}(x', y),
\]

\[
x_3 \nabla' \tilde{R}^h e_3 \rightharpoonup_{\gamma} x_3 \nabla' R(x') e_3 + x_3 \nabla_y \tilde{w}(x', y),
\]

\[
\frac{1}{h} (\tilde{R}^h e_3 - R^h e_3) \rightharpoonup_{\gamma} \frac{1}{\gamma} (y \cdot \nabla') R(x') e_3 + \frac{1}{\gamma} (w^0(x') + \tilde{w}(x', y)).
\]

We claim that the proposition follows from these convergences. In fact, first notice that it suffices to identify the symmetric part of the two-scale limit $G$ of the sequence $G^h$. Indeed, since $(I + h F)^T(I + h F)$ differs from $I + h \text{sym } F$ only by terms of higher order, the convergence $G^h \rightharpoonup_{\gamma} G$ implies $E = \text{sym } G$ (see e.g. [14, Lemma 4.4] for a proof).

We now identify $\text{sym } G$. By combining (29)–(32) with identity (28), we find that $R^h G^h$ weakly two-scale converges to

\[
(B', 0) + \left( x_3 \nabla' R(x') e_3, 0 \right) + \left( \nabla_y \tilde{\phi}, \frac{1}{\gamma} \partial_3 \tilde{\phi} \right) + \frac{1}{\gamma} (y \cdot \nabla') R(x')
\]

where

\[
\tilde{\phi}(x, y) := \zeta(x, y) + \tilde{v}(x', y) + x_3 \tilde{w}(x', y) + x_3 w^0(x').
\]

Due to the strong convergence $R^h \to R$, we deduce that $G^h$ weakly two-scale converges to (33) multiplied with $R^T$ from the left. The first and second terms yield

\[
\begin{pmatrix}
\tilde{B}(x') + x_3 \Pi(x') & 0 \\
b_1(x') & b_2(x')
\end{pmatrix},
\]

where $\tilde{B}$ denotes the $2 \times 2$-matrix obtained by deleting the third column of $R^T B'$ and $(b_1, b_2)$ are defined as the entries of the third row of $R^T B'$. Upon left multiplication by $R^T$, the last term in (33) yields a skew-symmetric term. Thus we have shown:

\[
\text{sym } G = \iota (\text{sym } \tilde{B} + x_3 \Pi) + \text{sym } \left( \nabla_y \phi, \frac{1}{\gamma} \partial_3 \phi \right)
\]

where

\[
\phi(x, y) := R^T(x') \tilde{\phi}(x, y) + \gamma x_3 \begin{pmatrix} b_1(x') \\ b_2(x') \\ 0 \end{pmatrix}.
\]

**Step 3.** It remains to prove (29)–(32). Since $\nabla h z^h$ is uniformly bounded in $L^2$ and since $\int_I z^h \, dx_3 = 0$ by construction, (29) directly follows from Lemma 3.5.
Next we prove (30). By (25) and by Lemma 3.4 (i), there exists $V \in L^2(S \times Y, \mathbb{R}^{3 \times 2})$ such that (after taking subsequences)

$$\frac{\nabla' \tilde{u}^h - (R^h)_{2,\gamma}'}{h} \rightharpoonup V.$$  \hfill (34)

Let us verify that

$$V(x', y) = B'(x') - \frac{1}{\gamma} (y \cdot \nabla') R'(x') + \nabla_y \tilde{v}(x', y),$$

where $B' := \int_Y V(\cdot, y) \, dy$ and $\tilde{v} \in L^2(S, H^1(Y))$. This is equivalent to showing that

$$\int_{S \times Y} V(x', y) : (\nabla^\perp_y \hat{G})(y) \psi(x') \, dy \, dx' = -\frac{1}{\gamma} \int_{S \times Y} (y \cdot \nabla') R'(x') : (\nabla^\perp_y \hat{G})(y) \psi(x') \, dy \, dx'$$

for all $\hat{G} \in \check{C}^1(Y, \mathbb{R}^3)$ and all $\psi \in C_0^\infty(S)$. Here and below $\nabla^\perp_y := (-\partial_{y_2}, \partial_{y_1})$.

To prove (35), note that since $\int_S \nabla' \tilde{u}^h : \nabla'^\perp (\hat{G}^h \psi) = 0$, we have

$$\int_S \nabla' \tilde{u}^h(x') : (\nabla^\perp_y \hat{G}) \left( \frac{x'}{\varepsilon(h)} \right) \psi(x') \, dx' = -\varepsilon(h) \int_S \nabla' \tilde{u}^h : \hat{G} \left( \frac{x'}{\varepsilon(h)} \right) \otimes \nabla'^\perp \psi(x') \, dx'.$$

The right-hand side converges to 0, since $\varepsilon(h) \nabla' \tilde{u}^h$ is strongly compact in $L^2$ and $\hat{G} \left( \frac{\cdot}{\varepsilon(h)} \right) \rightharpoonup 0$ weakly in $L^2$. On the other hand, Lemma 3.8 yields

$$\frac{R^h}{h} = \frac{\varepsilon(h)}{\varepsilon(h)} \frac{1}{\varepsilon(h)} R^h \overset{osc, Y}{\rightharpoonup} \frac{1}{\gamma} (y \cdot \nabla') R(x'),$$

and thus

$$\int_S \nabla' \tilde{u}^h(x') - (R^h)_{2,\gamma}'(x') : (\nabla^\perp_y \hat{G}) \left( \frac{x'}{\varepsilon(h)} \right) \psi(x') \, dx' \rightharpoonup - \int_{S \times Y} \frac{1}{\gamma} (y \cdot \nabla') R'(x') : \nabla^\perp_y \hat{G}(y) \psi(x') \, dy \, dx'.$$

The left-hand side converges to

$$\int_{S \times Y} V(x', y) : \nabla^\perp_y \hat{G}(y) \psi(x') \, dy \, dx'.$$

Hence, (35) and thus (30) follows.

We prove (31) and (32). By Lemma 3.3 the left-hand side of (32) is uniformly bounded in $L^2(S, \mathbb{R}^3)$ and thus we have (after passing to subsequences)

$$\frac{(\tilde{R}^h - R^h)e_3}{h} \rightharpoonup \frac{1}{\gamma} w^0 \text{ weakly in } L^2(S, \mathbb{R}^3).$$  \hfill (37)
for some $w^0 \in L^2(S, \mathbb{R}^3)$. Since $\tilde{R}^h e_3 \rightharpoonup R e_3$ weakly in $H^1(S, \mathbb{R}^3)$, we know from Lemma 3.4 (v) that there exists $\widetilde{w} \in L^2(S, H^1(\mathcal{V}, \mathbb{R}^3))$ such that

$$\nabla' \tilde{R}^h e_3 \overset{2,\gamma}{\rightharpoonup} \nabla' R e_3 + \nabla_y \widetilde{w}. \quad (38)$$

This implies (31). The combination of (38) with Lemma 3.7 yields $\tilde{R}^h e_3 \overset{osc,\gamma}{\rightharpoonup} \gamma^{-1} \tilde{w}$. Together with (36) we get

$$\frac{(\tilde{R} - R^h)e_3}{h} \overset{osc,\gamma}{\rightharpoonup} \frac{1}{\gamma} \widetilde{w}(x', y) + \frac{1}{\gamma} (y \cdot \nabla') R(x') e_3. \quad (39)$$

By (37) and (39), Lemma 3.6 implies (32).

**Proof of Proposition 3.2** case $\gamma = \infty$ The argument is similar to the case $\gamma \in (0, \infty)$. We only indicate the required modifications. Step 1 of the proof for $\gamma \in (0, \infty)$ is the same, except for the following change: As a difference to $\gamma \in (0, \infty)$, now we set

$$\delta(h) := \lceil \frac{h}{\epsilon(h)} \rceil \epsilon(h),$$

where $[s]$ denotes the smallest positive integer larger or equal to $s$. By construction $\delta(h)$ is an integer multiple of $\epsilon(h)$ and we have $\delta(h) \sim h$. Hence, Lemma 3.3 yields maps $R^h$ and $\tilde{R}^h$ with bounds uniform in $h$, and moreover $R^h$ is constant on each cube $x + \delta(h) Y$ with $x \in \delta(h) \mathbb{Z}^2$, hence also on each cube $x + \epsilon(h) Y$ with $x \in \epsilon(h) \mathbb{Z}^2$.

Similar to Step 2 of the proof for $\gamma \in (0, \infty)$, the statement of the proposition can be reduced to showing that (up to subsequences)

$$\nabla \xi^h \overset{2,\gamma}{\rightharpoonup} (\nabla_y \tilde{z}, d'), \quad (40)$$

$$\frac{\nabla' \tilde{u} - (R^h)' \overset{2,\gamma}{\rightharpoonup}}{h} B'(x') + \nabla_y \tilde{u}(x', y), \quad (41)$$

$$x_3 \nabla' \tilde{R}^h e_3 \overset{2,\gamma}{\rightharpoonup} x_3 \nabla' R(x') e_3 + x_3 \nabla_y \tilde{w}(x', y), \quad (42)$$

$$\frac{1}{h} (\tilde{R}^h e_3 - R^h e_3) \overset{2,\gamma}{\rightharpoonup} w^0(x'), \quad (43)$$

where, as before, $R$ is the strong $L^2$-limit of $R^h$, and $B' \in L^2(S, \mathbb{R}^3 \times \mathbb{R}^3)$, $\tilde{z} \in L^2(\Omega, H^1(\mathcal{V}, \mathbb{R}^3))$, $d' \in L^2(\Omega, \mathbb{R}^3)$, $\tilde{u}, \tilde{w} \in L^2(S, H^1(\mathcal{V}, \mathbb{R}^3))$ and $w^0 \in L^2(S, \mathbb{R}^3)$.

Indeed, as in the case $\gamma \in (0, \infty)$, inserting (40) through (43) into (28) we see that

$$\text{sym} G(x, y) = \iota \left( \text{sym} \tilde{B}(x') + x_3 II(x') \right) + \text{sym} \left( \nabla_y \phi(x, y), d(x) \right),$$

where

$$\phi(x, y) := R^T (\tilde{z} + \tilde{v} + x_3 \tilde{w}), \quad d = R^T d' + R^T w^0 + \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix},$$

and $\tilde{B}$ and $(b_1, b_2)$ are defined as in the case $\gamma \in (0, \infty)$.

The proof of (40) through (43) is similar to Step 3 of the proof for the case $\gamma \in (0, \infty).$
4 Proof of Theorem 2.4

4.1 Lower bound

As a preliminary step we need to establish some continuity properties of the quadratic form appearing in (6) and its relaxed version introduced in Definition 2.3. For the proof we refer to [14, Lemma 2.7].

Lemma 4.1 Let $W$ be as in Assumption 2.1 and let $Q$ be the quadratic form associated with $W$ via (6). Then

\begin{align*}
(Q1) & \quad Q(\cdot, G) \text{ is } Y\text{-periodic and measurable for all } G \in \mathbb{R}^{3 \times 3}, \\
(Q2) & \quad \text{for almost every } y \in \mathbb{R}^2 \text{ the map } Q(y, \cdot) \text{ is quadratic and satisfies}
\end{align*}

\begin{equation}
\begin{aligned}
c_1 |\text{sym } G|^2 & \leq Q(y, G) = Q(y, \text{sym } G) \leq c_2 |\text{sym } G|^2 \forall G \in \mathbb{R}^{3 \times 3}.
\end{aligned}
\end{equation} (44)

Lemma 4.2 (a) Let $\gamma \in (0, \infty)$. For all $A \in \mathbb{R}^{2 \times 2}_{\text{sym}}$ there exist unique $B \in \mathbb{R}^{2 \times 2}_{\text{sym}}$ and $\phi \in \dot{H}^1(I \times \mathcal{Y}, \mathbb{R}^3)$, such that:

$$Q_{2,\gamma}(A) = \int_{I \times Y} Q\left(y, x_3 A + B + (\nabla_y \phi, \frac{1}{\gamma} \partial_3 \phi)\right) dy dx_3$$

The map $A \mapsto (B, \phi)$ is linear.

(b) Let $\gamma = \infty$. For all $A \in \mathbb{R}^{2 \times 2}_{\text{sym}}$ there exist unique $B \in \mathbb{R}^{2 \times 2}_{\text{sym}}$, $\phi \in L^2(I, \dot{H}^1(\mathcal{Y}, \mathbb{R}^3))$ and $d \in L^2(I, \mathbb{R}^3)$ such that

$$Q_{2,\infty}(A) = \int_{I \times Y} Q\left(y, x_3 A + B + (\nabla_y \phi, d)\right) dy dx_3.$$ 

The map $A \mapsto (B, \phi, d)$ is linear.

Proof We only prove part (a); part (b) is similar. After possibly rescaling and changing the length of $I$, we may assume without loss of generality that $\gamma = 1$. We denote by $\nabla$ the derivative on $Y \times I$. By (44) and by Korn’s inequality on $Y \times I$, for each $A \in \mathbb{R}^{2 \times 2}_{\text{sym}}$ the functional

$$\psi \mapsto \int_{Y \times I} Q(y, x_3 A + \nabla \psi) dy dx_3$$

is elliptic on the closed linear subspace of $H^1(Y \times I; \mathbb{R}^3)$ given by

$$X := \left\{(B y, 0)^T + \phi(y, x_3) : B \in \mathbb{R}^{2 \times 2}_{\text{sym}}, \phi \in \dot{H}^1(Y \times I; \mathbb{R}^3)\right\}.$$ 

Hence it admits a unique minimizer $\psi \in X$. Its Euler-Lagrange equation is a linear elliptic equation depending linearly on the inhomogeneity $A$, so in particular $\psi$ depends linearly on $A$. \qed

Proof of Theorem 2.4 (lower bound) Without loss of generality we may assume that $\int_{\Omega} u_h^h dx = 0$ and $\limsup_{h \to 0} h^{-2} E_h^\infty(u_h) < \infty$. We only consider the case $\gamma \in (0, \infty)$. The proof in the case $\gamma = \infty$ is similar. In view of (5), the sequence $u_h$ has finite bending energy and the sequence $E_h^\infty$, see (10), is bounded in $L^2(\Omega, \mathbb{R}^{3 \times 3})$. Hence, from Theorem 2.5

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we deduce that $u \in W_\delta^{2,2}(S, \mathbb{R}^3)$. By Lemma 3.4 (i) and Proposition 3.2 (i) we can pass to a subsequence such that, for some $E \in L^2(\Omega \times Y; \mathbb{R}^{3 \times 3})$,

$$E^h \rightharpoonup E,$$

where $E$ has the form (18). Using condition (6) together with the upper bound in (5), arguing as in the proof of [5, Theorem 6.1 (i)] and combining this with the lower semi-continuity of convex integral functionals with respect to weak two-scale convergence (see e.g. [20, Proposition 1.3]), we obtain the lower bound

$$\liminf_{h \to 0} \frac{1}{h^2} E^{h, e(h)}(u^h) \geq \int_{\Omega \times Y} Q(y, E(x, y)) \, dy \, dx \geq \int_{\Omega \times Y} Q\left(y, \epsilon(x_3 \Pi(x') + B(x')) + \left(\nabla \phi(x, y), \frac{1}{\gamma} \partial_3 \phi(x, y)\right)\right) \, dy \, dx.$$

Here we have used (18). Minimization over $B \in L^2(S, \mathbb{R}^{2 \times 2})$ and $\phi \in L^2(S, H^1(I \times Y, \mathbb{R}^3))$ yields

$$\liminf_{h \to 0} \frac{1}{h^2} E^{h, e(h)}(u^h) \geq \int_S Q_{2, \gamma}(\Pi(x')) \, dx' = \mathcal{E}_{\gamma}(u).$$

$$\Box$$

4.2 Upper bound

It remains to prove the upper bound. As in [18] and other related results, the key ingredient here is the density result for $W^{2,2}$ isometric immersions established in [8, 9] (cf. also [17] for an earlier result in this direction). It is the need for the results in [8] that forces us to consider domains $S$ which are not only Lipschitz but also piecewise $C^1$. More precisely, we only need that the outer unit normal be continuous away from a subset of $\partial S$ with length zero.

For a given $u \in W_\delta^{2,2}(S)$ and for a displacement $V \in W^{2,2}(S, \mathbb{R}^3)$, as in [10] we introduce the quadratic form $q_V$ defined by

$$q_V = \text{sym} \left( (\nabla u)^T (\nabla V) \right).$$

We denote by $A(S)$ the set of all $u \in W_\delta^{2,2}(S, \mathbb{R}^3) \cap C_{\infty}(\overline{S}, \mathbb{R}^3)$ with the property that

$$\begin{cases} B \in C_{\infty}(\overline{S}, \mathbb{R}_{\text{sym}}^{2 \times 2}) : B = 0 \text{ in a neighborhood of } \{x' \in S : \Pi(x') = 0\} \\ \subset \left\{ q_V : V \in C_{\infty}(\overline{S}; \mathbb{R}^3) \right\}. \end{cases}$$

In other words, if $u \in A(S)$ and $B \in C_{\infty}(\overline{S}, \mathbb{R}_{\text{sym}}^{2 \times 2})$ is a matrix field which vanishes in a neighborhood of $\{\Pi = 0\}$, then there exists a displacement $V \in C_{\infty}(\overline{S}; \mathbb{R}^3)$ such that $q_V = B$. The key ingredient in the proof of the upper bound is the following lemma.

**Lemma 4.3** The set $A(S)$ is dense in $W_\delta^{2,2}(S)$ with respect to the strong $W^{2,2}$ topology.

**Proof** By [9, Theorem 3] every $W^{2,2}$ isometric immersion $u$ can be approximated (strongly in $W^{2,2}$) by $W^{2,2}$ isometric immersions whose gradient is finitely developable in the sense of [9, Definition 3].
In [8] it is shown that each $W^{2,2}$ isometric immersion with finitely developable gradient can, in turn, be approximated by smooth (up to the boundary) isometric immersions with finitely developable gradient.

This shows that the arguments in [18] leading to his Lemma 3.3 (which require finite developability) indeed apply to a $W^{2,2}$-dense set of smooth isometric immersions. That lemma asserts that every matrix field $B \in C^\infty (\bar{\Omega}, \mathbb{R}^{2\times 2}_{\text{sym}})$ which vanishes in a neighborhood of $\{\Pi = 0\}$ can be written in the form $\nabla g + \alpha \Pi$ for smooth functions $g_1$, $g_2$ and $\alpha$. Setting $V = g_i \partial_i u + \alpha n$ yields the desired displacement.

A result similar to [18, Lemma 3.3] was recently re-derived in [7] in the context of developable shells.

Thanks to Lemma 4.3 it will be enough to construct recovery sequences for limiting deformations belonging to $\mathcal{A}(S)$. First, however, we present a construction assuming additional information about the limit.

**Lemma 4.4** Let $u \in W^{2,2}_b(S, \mathbb{R}^3) \cap W^{2,\infty}(S, \mathbb{R}^3)$ and let $V \in W^{2,\infty}(S, \mathbb{R}^3)$.

(a) Let $\gamma \in (0, \infty)$ and $\phi \in C^\infty_c(S, C^\infty(I \times \gamma, \mathbb{R}^3))$. Then there exists a sequence $(u^h) \subset H^1(\Omega, \mathbb{R}^3)$ such that $u^h \to u$ and $\nabla u^h \to (\nabla u, n)$ uniformly in $\Omega$ and

$$
\lim_{h \to 0} \frac{1}{h^2} \mathcal{E}^{h, e(h)}(u^h) = \int_{\Omega \times \gamma} Q \left( y, \iota(x_3 \Pi(x')) + q_V(x') \right) \, dy \, dx_3 \, dx'.
$$

(b) Let $\gamma = \infty$, $\phi \in C^\infty_c(\Omega, C^\infty(\gamma, \mathbb{R}^3))$ and $d \in C^\infty_c(\Omega, \mathbb{R}^3)$. Then there exists a sequence $(u^h) \subset H^1(\Omega, \mathbb{R}^3)$ such that $u^h \to u$, $\nabla u^h \to (\nabla u, n)$ uniformly in $\Omega$ and

$$
\lim_{h \to 0} \frac{1}{h^2} \mathcal{E}^{h, e(h)}(u^h) = \int_{\Omega \times \gamma} Q \left( y, \iota(x_3 \Pi(x')) + q_V(x') \right) \, dy \, dx_3 \, dx'.
$$

**Proof** As in [10], we start with the Kirchhoff-Love ansatz, augmented by its linearization induced by the displacement $V$:

$$
v^h(x) := u(x') + hx_3 n(x') + h \left(V(x') + hx_3 \mu(x')\right),
$$

where $\mu$ is given by

$$
\mu = (I - n \otimes n)(\partial_1 V \wedge \partial_2 u + \partial_1 u \wedge \partial_2 V).
$$

We set $R(x') = (\nabla' u(x'), n(x'))$. A straightforward computation shows that

$$
\nabla h v^h = R + h \left((\nabla V, \mu) + x_3 (\nabla n, 0)\right) + h^2 x_3 (\nabla \mu, 0).
$$

The actual recovery sequence $u^h$ is obtained by adding to $v^h$ the oscillating correction given by $\phi$, regarded as a map in $C^\infty(\Omega \times \gamma)$:

$$
u^h(x) := v^h(x) + h \varepsilon(h) R(x') \phi \left(x, \frac{x'}{\varepsilon(h)}\right).
$$
By the regularity of $V$, the uniform convergence of $u^h$ and $\nabla_h u^h$ is immediate. Equation (47) implies
\[
R^T \nabla_h u^h = I + h \left( (\nabla u)^T (\nabla V) + x_3 \Pi \right) + h \left( (\mu \cdot \nabla u) \otimes e_3 + e_3 \otimes (n \cdot \nabla V) \right) \\
+ h \left( \nabla_y \phi, \frac{\epsilon(h)}{h} \partial_3 \phi \right) + h^2 x_3 R^T (\nabla \mu, 0) \\
+ h \epsilon(h) (R^T \nabla R) \phi + h \epsilon(h) (\nabla \phi, 0);
\]
the argument of $\nabla_y \phi$ and of $\partial_3 \phi$ is $(x, x'/\epsilon(h))$. Defining
\[
G^h = \frac{1}{h} \left( R^T \nabla_h u^h - I \right)
\]
and using the fact that $n \cdot \nabla V + \mu \cdot \nabla u = 0$, we deduce from (49) that
\[
\text{sym } G^h = \iota (q_V + x_3 \Pi) + \text{sym} \left( \nabla_y \phi, \frac{\epsilon(h)}{h} \partial_3 \phi \right) \\
+ \text{sym} \left( h x_3 R^T (\nabla \mu, 0) + \epsilon(h) (R^T \nabla R) \phi + \epsilon(h) (\nabla \phi, 0) \right).
\]
Hence the sequence $G^h$ strongly two-scale converges in $L^2$ to a limit $\tilde{G} \in L^2(\Omega \times Y, \mathbb{R}^{3 \times 3})$ with
\[
\text{sym } \tilde{G} = \iota (x_3 \Pi + q_V) + \text{sym} \left( \nabla_y \phi(x, y), \frac{1}{\gamma} \partial_3 \phi(x, y) \right).
\]
Properties (4), (6) and (5) yield
\[
\limsup_{h \to 0} \frac{1}{h^2} \mathcal{E}^{h, \epsilon(h)}(u^h) - \int_{\Omega} Q\left( \frac{x'}{\epsilon(h)}, G^h(x) \right) \, dx = 0.
\]
Hence, by (44) and by strong two-scale convergence of $G^h$, we can pass to the limit in the second term in (51). Since $Q(\gamma, F)$ only depends on the symmetric part of $F$, the claim follows from (50).

The proof for $\gamma = \infty$ is similar to the above reasoning. Essentially, we only need to replace (48) by
\[
u^h(x) = v^h(x) + h \epsilon(h) R(x') \phi(x, \frac{x'}{\epsilon(h)}) + h^2 \bar{d}(x),
\]
where
\[
\bar{d}(x) = R(x') \int_{-1/2}^{x_3} d(x', t) \, dt.
\]

Proof of Theorem 2.4 (Upper bound) We only consider the case $\gamma \in (0, \infty)$; the argument for the case $\gamma = \infty$ is similar. We may assume that $\mathcal{E}_\gamma(u) < \infty$, so $u \in W^{2,2}_\delta(S, \mathbb{R}^3)$. Moreover, since $Q_{2,\gamma}$ is quadratic (by Lemma 4.2), it suffices to prove the statement for a dense subset of $W^{2,2}_\delta(S, \mathbb{R}^3)$. Hence, by Lemma 4.3, we may assume without loss of generality that $u \in \mathcal{A}(S)$. ☐
By Lemma 4.2 there exist $B \in L^2(S, \mathbb{R}^{2 \times 2})$ and $\phi \in L^2(S, \tilde{H}^1(I \times \gamma, \mathbb{R}^3))$ such that

$$E^\gamma (u) = \int_{\Omega \times Y} Q(y, \iota(x_3 \Pi + B) + \text{sym}(\nabla_y \phi, \frac{1}{Y} \partial_3 \phi)) \, dy \, dx. \quad (52)$$

Since $B(x')$ depends linearly on $\Pi(x')$, we know in addition that $B(x') = 0$ for almost every $x' \in \{ \Pi = 0 \}$.

By a density argument it suffices to show the following: There exists a doubly indexed sequence $u_{h, \delta} \in H^1(\Omega, \mathbb{R}^3)$ such that

$$\lim \sup_{\delta \to 0} \lim \sup_{h \to 0} \| u_{h, \delta} - u \|_{H^1(\Omega, \mathbb{R}^3)} = 0, \quad (53)$$

$$\lim \sup_{h \to 0} \left| \frac{1}{h^2} E^{h, \varepsilon(h)}(u_{h, \delta}) - E^\gamma (u) \right| < \delta. \quad (54)$$

Indeed, if this is the case then we obtain the desired sequence by extracting a diagonal sequence (cf. [2, Corollary 1.16]).

We construct $u_{h, \delta}$ as follows: By density, for each $\delta > 0$ there exist maps $B^\delta \in C^\infty(\overline{S}, \mathbb{R}^{2 \times 2})$ and $\phi^\delta \in C^\infty_c(S, C^\infty(I \times \gamma, \mathbb{R}^3))$ such that

$$\| B^\delta - B \|_{L^2(S)} + \| \nabla_y \phi^\delta - \nabla_y \phi \|_{L^2(\Omega \times Y)} + \| \partial_3 \phi^\delta - \partial_3 \phi \|_{L^2(\Omega \times Y)} \leq \delta^2, \quad (55)$$

$$B^\delta = 0 \text{ in a neighborhood of } \{ \Pi = 0 \}. \quad (56)$$

In fact, to construct the maps $B^\delta$, one multiplies $B$ with the characteristic function of the set where

$$\text{dist}(\Pi = 0) \geq \delta \quad (57)$$

and then mollifies the resulting map on a scale $\delta/4$. Since $\{ \Pi = 0 \}$ is relatively closed in $S$, the characteristic function of the set where (57) is satisfied indeed converges (boundedly in measure) to that of the set $\{ \Pi \neq 0 \}$. So $B^\delta \to B$ because $B$ vanishes almost everywhere on $\{ \Pi = 0 \}$.

Since $u \in A(S, \mathbb{R}^3)$ and due to (56), for each $\delta > 0$ there exists a smooth displacement $V^\delta$ such that

$$B^\delta = q V^\delta. \quad (58)$$

We apply Lemma 4.4 to $u$ and $V^\delta$ to obtain a sequence $u_{h, \delta}$ that converges uniformly to $u$ as $h \to 0$. Hence (53) is satisfied. Lemma 4.4 also ensures that

$$\lim_{h \to 0} \frac{1}{h^2} \mathcal{E}^{h, \varepsilon(h)}(u_{h, \delta}) = \int_{\Omega \times Y} Q(y, \iota(x_3 \Pi + B^\delta) + \text{sym}(\nabla_y \phi^\delta, \frac{1}{Y} \partial_3 \phi^\delta)) \, dy \, dx. \quad (59)$$

By continuity of the functional on the right-hand side, combined with (55) and (52), the bound (54) follows.

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