ANOMALY FORMULAS FOR THE COMPLEX-VALUED ANALYTIC TORSION ON COMPACT BORDISMS

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Abstract. We extend the complex-valued analytic torsion, introduced by Burghelea and Haller on closed manifolds, to compact Riemannian bordisms. We do so by considering a flat complex vector bundle over a compact Riemannian manifold, endowed with a fiberwise nondegenerate symmetric bilinear form. The Riemannian metric and the bilinear form are used to define non-selfadjoint Laplacians acting on vector-valued smooth forms under absolute and relative boundary conditions. In the process to define the complex-valued analytic torsion, we study spectral properties associated to these generalized Laplacians. As main results, we obtain anomaly formulas for the complex-valued analytic torsion. Our reasoning takes into account that the coefficients in the heat trace asymptotic expansion associated to the boundary value problem under consideration, are locally computable. The anomaly formulas for the complex-valued Ray–Singer torsion are obtained by using the corresponding ones for the Ray–Singer metric, obtained by Brüning and Ma on manifolds with boundary, and an argument of analytic continuation. In odd dimensions, our anomaly formulas are in accord with the corresponding results of Su, without requiring the variations of the Riemannian metric and bilinear structures to be supported in the interior of the manifold.

Introduction

In this paper, we denote by $(M, \partial_+ M, \partial_- M)$ a compact Riemannian bordism. That is, $M$ is a compact Riemannian manifold of dimension $m$, with Riemannian metric $g$, whose boundary $\partial M$ is the disjoint union of two closed submanifolds $\partial_+ M$ and $\partial_- M$. For $E$ a flat complex vector bundle over $M$, we consider generalized Laplacians acting on the space $\Omega(M; E)$ of $E$-valued...
smooth differential forms on $M$ satisfying absolute boundary conditions on $\partial_+ M$ and relative boundary conditions on $\partial_- M$.

We study the complex-valued Ray–Singer torsion on $(M, \partial_+ M, \partial_- M)$. This torsion was introduced by Burghelea and Haller on closed manifolds, see [4] and [5], as a complex-valued version for the real-valued Ray–Singer torsion, originally studied by Ray and Singer in [21] for unitary flat vector bundles on closed manifolds. Our main results are Theorem 2 and Theorem 3. In Theorem 3 we provide so-called anomaly formulas providing a logarithmic derivative for the complex-valued analytic torsion on compact Riemannian bordisms and its proof is based on the work by Brüning and Ma in [8] for the real-valued Ray–Singer torsion on manifolds with boundary.

The classical (real-valued) Ray–Singer analytic torsion, see [21], [17], [19], and others, is defined in terms of a selfadjoint Laplacian $\Delta_{E,g,h}$, constructed by using a Hermitian metric on the bundle, the Riemannian metric $g$ and a flat connection $\nabla^E$ on $E$. In this paper $\Delta_{E,g,h}$ is referred as the Hermitian Laplacian. In [7], Bismut and Zhang interpreted the analytic torsion as a Hermitian metric in certain determinant line, and called it the Ray–Singer metric, see also [9]. In this paper, we also adopt this approach. The Ray–Singer metric on manifolds with boundary has been intensively studied by several authors, among them [21], [17], [19], [20], [17], [11], [8] and [9]. In particular, we are interested in the work of Brüning and Ma in [8], where the variation of the Ray–Singer metric, with respect to smooth variations on the underlying Riemannian and Hermitian metrics, was computed.

In order to define the complex-valued Ray–Singer torsion, we assume $E$ admits a fiberwise nondegenerate symmetric bilinear form $b$ and we proceed as in [4]. The bilinear form $b$ and the Riemannian metric $g$ induce a nondegenerate symmetric bilinear form on $\Omega(M; E)$ which is denoted by $\beta_{g,b}$. With this data, one constructs generalized Laplacians $\Delta_{E,g,b} : \Omega(M; E) \to \Omega(M; E)$, also referred as bilinear Laplacians. These generalized Laplacians are formally symmetric, with respect to $\beta_{g,b}$ on the space of smooth forms satisfying the boundary conditions specified above.

In Section 1 we use known theory on boundary value problems for differential operators to treat ellipticity, regularity and spectral properties for $\Delta_{E,g,b}$. In particular, under the specified elliptic boundary conditions, $\Delta_{E,g,b}$ extends to a not necessarily selfadjoint closed unbounded operator in the $L^2$-norm, it has compact resolvent and discrete spectrum, all its eigenvalues are of finite multiplicity, its (generalized) eigenspaces contain smooth differential forms only and the restriction of $\beta_{g,b}$ to each of these is also a nondegenerate bilinear form. Proposition 2 gives Hodge decomposition results in this setting, which are analog to the Hermitian situation, described for instance in [10], [19], [17] and more recently in [9]. Section 1 ends with Proposition
stating that the 0-generalized eigenspace of $\Delta_{E,g,b}$ still computes relative cohomology $H(M, \partial_- M; E)$, without necessarily being isomorphic to it.

In Section 2 we recall generalities on the coefficients of the heat kernel asymptotic expansion for an elliptic boundary value problem. These coefficients are spectral invariants and locally computable as polynomial functions in the jets of the symbols of the operators under consideration, see [14], [22], [23] and [24]. This fact provides the key ingredient in the proofs of Theorem 2 leading to Theorem 3. In [8], based on the computation of the coefficients of the constant terms in the heat trace asymptotic expansion for the Hermitian Laplacian under absolute boundary conditions, Brüning and Ma obtained anomaly formulas for the Ray–Singer metric. First, we use Poincaré duality in terms of Lemma 6 to infer from [8], the corresponding coefficients for the Hermitian Laplacian under relative boundary conditions and then we derive those corresponding to Hermitian Laplacian on the bordism $(M, \partial_+ M, \partial_- M)$ under absolute and relative boundary conditions, see Proposition 5 and Theorem 1. We point out here that the anomaly formulas for the Ray–Singer metric in Theorem 1 were also obtained by Brüning and Ma in [9] continuing their work in [8]. Next, in Lemma 10 we point out the holomorphic dependence of these coefficients on a complex parameter. Finally, an analytic continuation argument allows one to deduce the infinitesimal variation of these quantities for the bilinear Laplacian on the bordism $(M, \partial_+ M, \partial_- M)$ from those corresponding to the Hermitian one, see Theorem 2.

In Section 3 we use the results from Section 1 and Section 2 to define the complex-valued analytic torsion on a compact Riemannian bordism. Following the approach in [4], we obtain a nondegenerate bilinear form on the determinant line $\det(H(M, \partial_- M; E))$, denoted by $\tau_{E,g,b}(0)$ and induced by the restriction of $\beta_{g,b}$ to the generalized 0-eigenspace of $\Delta_{E,g,b}$. The (inverse square of) the complex-valued Ray–Singer torsion for manifolds with boundary is

$$\tau_{RS}^{E,g,b} := \tau_{E,g,b}(0) \cdot \prod_p \left( \det' (\Delta_{E,g,b,p}) \right)^{-1/p},$$

where the product above is, in this situation, a non zero complex number with $\det' (\Delta_{E,g,b,p})$ being the $\zeta$-regularized product of all non-zero eigenvalues of $\Delta_{E,g,b,p}$. For closed manifolds, the variation of the complex analytic Ray–Singer torsion, with respect to smooth changes on the metric $g$ and the bilinear form $b$, has been obtained in [1] Sections 7 and 8. Burghella and Haller obtained in [6] Theorem 4.2 a geometric invariant by introducing appropriate correction terms. In [25], by using techniques from [26], [27], [10] and [19], Su generalized the complex-valued analytic Ray–Singer torsion to the situation in which $\partial_+ M \neq \emptyset$ (or $\partial_- M \neq \emptyset$). Also in [25], Su proved that in odd dimensions, the complex-valued analytic torsion does depend neither
on smooth variations of the Riemannian metric nor on smooth variations of the bilinear form, as long as these are compactly supported in the interior of \( M \). This section ends with Theorem 3 which gives formulas for the variation of the complex-valued analytic Ray–Singer torsion with respect to smooth variations of the metric and the bilinear form. In analogy with the results in [4], the anomaly formulas for the complex-valued Ray–Singer torsion are obtained by using the results for the coefficients of the constant term in the heat trace asymptotic expansion for the bilinear Laplacian obtained in Section 2.

In the Appendix, see Section 4, for the reader’s convenience, we recall some formalism leading to the characteristic forms appearing in the anomaly formulas stated in Proposition 4, Proposition 5, Theorem 1, Theorem 2 and Theorem 3.

The anomaly formulas given in Theorem 3 generalize the ones obtained by Burghelea and Haller in the closed situation in [4], and also the ones in [25] by Su in odd dimensions: they do not longer require \( g \) and \( b \) to be constant in a neighborhood of the boundary and both kind of boundary conditions are considered at the same time.

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1. Bilinear Laplacians and Hodge decomposition on bordisms

1.1. Some background and notation. Let \((M, \partial_+ M, \partial_- M)\) be a compact Riemannian bordism of dimension \(m\). More precisely, \(M\) is a compact connected not necessarily orientable smooth manifold of dimension \(m\) with Riemannian metric \(g\), whose boundary \(\partial M\) is the disjoint union of two closed submanifolds, \(\partial_+ M\) and \(\partial_- M\), and it inherits the Riemannian metric from \(M\). We do not require the metric to satisfy any condition near the boundary. We denote by \(TM\) and \(T^* M\) (resp. \(T\partial M\) and \(T^* \partial M\)) the tangent and cotangent bundle of \(M\) (resp. \(\partial M\)) respectively. We denote by \(\varsigma\) the geodesic unit inwards pointing normal vector field on the boundary. Let \(\Theta_M\) (resp. \(\Theta_{\partial M}\)) be the orientation bundle of \(TM\) (resp. \(T\partial M\)), considered as the flat real line bundle \(\det(T^* M) \to M\) (resp. \(\det(T^* \partial M) \to \partial M\)) with transition functions \(\{\pm 1\}\), endowed with the unique flat connection specified by the de-Rham differential on (twisted) forms, see [3, page 88]. For the canonical embedding \(i: \partial M \hookrightarrow M\), we write \(\Theta_M|_{\partial M} := i^* \Theta_M\) and, as real line bundles over \(\partial M\), \(\Theta_M|_{\partial M}\) and \(\Theta_{\partial M}\) are identified as follows: over the boundary, a section \(\beta\) of \(\det(T^* \partial M)\) is identified with the section \(-\varsigma^\text{in} \wedge \beta\) of \(\det(T^* M)|_{\partial M}\), where \(\varsigma^\text{in} := g(\cdot, \varsigma_\text{in})\) is the 1-form dual to \(\varsigma_\text{in}\). For \(TM\) and \(T\partial M\), the corresponding
Levi–Cività connections are denoted by $\nabla$ and by $\nabla^\theta$ respectively. Recall the Hodge $*$-operator $*_{g,q} := *_{g,q} : \Omega^q(M) \to \Omega^{m-q}(M; \Theta_M)$, i.e., the linear isomorphism defined by $\alpha \wedge \star \alpha' = (\alpha, \alpha')_g vol_g(M)$, for $\alpha, \alpha' \in \Omega^q(M)$ and $0 \leq q \leq m$, where $vol_g(M) \in \Omega^m(M; \Theta_M)$ is the volume form of $M$.

In this paper, we consider a flat complex vector bundle $E$ over $M$, with a flat connection $\nabla^E$, and denote by $\Omega(M; E)$ be the space of $E$-valued smooth differential forms on $M$, endowed with the de-Rham differential $d_E := d_{\nabla^E}$. Moreover, assume $E$ is endowed with a fiber-wise nondegenerate symmetric bilinear form $b$. We denote by $E'$ the flat complex vector bundle dual to $E$ with the induced flat connection $\nabla^{E'}$ and bilinear form $b'$ dual to $\nabla^E$ and $b$ respectively. Recall that one is always able to fix a (positive definite) Hermitian structure on $E$ (in Section 2.3 we choose for instance a Hermitian structure compatible with the nondegenerate symmetric bilinear form). By choosing a Hermitian structure on $E$ and using the Riemannian metric on $M$, consider the induced $L^2$-norm on $\Omega(M; E)$ and denote by $L^2(M; E)$ its $L^2$-completion. Recall that $L^2(M; E)$ is independent the chosen Hermitian and Riemannian structures.

### 1.2. Generalized Laplacians on compact bordisms.

As a first step to define the complex-valued analytic torsion on a compact bordism, we recall certain generalized Laplacians which were introduced in [4] on closed manifolds. The nondegenerate symmetric bilinear form $b$ on $E$ and the Riemannian metric $g$ on $M$ permit to define a nondegenerate symmetric bilinear form on $\Omega(M; E)$ by

$$\beta_{g,b}(v, w) := \int_M Tr(v \wedge \star_b w)$$

where $Tr : \Omega(M, E \otimes E' \otimes \Theta_M) \to \Omega(M; \Theta_M)$ is the trace map, induced by the canonical pairing between the bundles $E$ and $E'$, and the map $\star_{b,q} := \star_q \otimes b : \Omega^q(M; E) \to \Omega^{m-q}(M; E' \otimes \Theta_M)$

is defined by using the Hodge $*$-operator $\star_q$ and the isomorphism of vector bundles between $E$ and $E'$, specified by the bilinear form $b$, also denoted by the same symbol. Thus, one defines $d^\beta_{E,g,b,q} : \Omega^q(M; E) \to \Omega^{q-1}(M; E)$ by

$$d^\beta_{E,g,b,q} := (-1)^q \star_{b,-1}^{-1} d_{E' \otimes \Theta_M, m-q} \star_{b,q},$$

where $\star^{-1}_{b,q}$ is the inverse of $\star_{b,q}$ and $d_{E' \otimes \Theta_M}$ is the de-Rham differential on $\Omega(M; E' \otimes \Theta_M)$ induced by the dual connection on $E'$. It can easily be checked that $d^\beta_{E,g,b}$ is a codifferential on $\Omega(M; E)$. In this way, the operator

$$\Delta_{E,g,b,q} := d_{E,q-1} d^\beta_{E,g,b,q} + d^\beta_{E,g,b,q+1} d_{E,q} : \Omega^q(M; E) \to \Omega^q(M; E),$$

for $q = 0, \ldots, m$. We denote by $\Delta_{E,g,b} := \Delta_{E,g,b,0}$ the analytic torsion on the bordism $M$.
is an operator of Laplace type, or generalized Laplacian in the sense that its principal symbol is a scalar positive real number, i.e., $\Delta_{E,g,b}$ is elliptic. For simplicity, the operator $\Delta_{E,g,b}$ in (2) will be called the bilinear Laplacian. A straightforward use of Stokes’ Theorem leads to the Green’s formulas:

\[
\begin{align*}
\beta_{g,b}(d_E v, w) - \beta_{g,b}(v, d_E w) &= \int_{\partial M} i^* (\text{Tr}(v \wedge \ast b w)), \\
\beta_{g,b}(\Delta_E v, w) - \beta_{g,b}(v, \Delta_E w) &= \int_{\partial M} i^* (\text{Tr}(d_E^2 v \wedge \ast b w)) - \int_{\partial M} i^* (\text{Tr}(v \wedge \ast b d_E w)) \\
&\quad - \int_{\partial M} i^* (\text{Tr}(d_E^2 v \wedge \ast b w)) + \int_{\partial M} i^* (\text{Tr}(v \wedge \ast b d_E w)).
\end{align*}
\]

\(v, w \in \Omega(M; E)\).

1.3. Boundary conditions. In order to study analytic and spectral properties of $\Delta_{E,g,b}$, we impose elliptic boundary conditions. We denote by $i_\pm : \partial_\pm M \mapsto M$ the canonical embedding of $\partial_\pm M$ into $M$ respectively.

For a form $w \in \Omega(M; E)$, we say that $w$ satisfies relative boundary conditions on $\partial_- M$ if $i^*_- w = 0$ and $i^*_- d_{E,g,b}^2 w = 0$ and $w$ satisfies absolute boundary conditions on $\partial_+ M$ if $i^*_+ w = 0$ and $i^*_+ d_{E,g,b}^2 w = 0$. The space of smooth forms satisfying relative boundary conditions on $\partial_- M$ and absolute boundary conditions on $\partial_+ M$ is

\[
(4) \quad \Omega(M; E)_{\mid B} := \left\{ w \in \Omega(M; E) \mid i^*_- w = 0, \quad i^*_+ d_{E,g,b}^2 w = 0 \right\}.
\]

For simplicity, a form satisfying boundary conditions in (4) will be referred as satisfying absolute/relative boundary conditions on $(M, \partial_+ M, \partial_- M)$. The integrants on the right of formulas in (3) vanish, on forms in $\Omega(M; E)_{\mid B}$. The boundary conditions in (4) are an example of mixed boundary conditions, which provide elliptic boundary conditions for operators of Laplace type, see (13).

Now we describe boundary operators implementing the boundary conditions in (4). Consider $E_\pm := i^*_\pm E$ and for $1 \leq q \leq m$ define

\[
(5) \quad B_{E,g,b} : \quad \Omega^q(M; E) \quad \longrightarrow \quad \Omega^{q-1}(\partial_+ M; E_+) \oplus \Omega^q(\partial_- M; E_-) \\
\quad \oplus \quad \Omega^q(\partial_- M; E_-) \oplus \Omega^{q-1}(\partial_+ M; E_+)
\]

\(w \mapsto (B_+ w, B_- w)\),

where the operators

\[
B_- : \quad \Omega^q(M; E) \quad \longrightarrow \quad \Omega^q(\partial_- M; E_-) \oplus \Omega^{q-1}(\partial_- M; E_-) \\
\quad w \mapsto (B_0^\circ w, B_1^\circ w)
\]

\[
B_+ : \quad \Omega^q(M; E) \quad \longrightarrow \quad \Omega^{q-1}(\partial_+ M; E_+) \oplus \Omega^q(\partial_+ M; E_+) \\
\quad w \mapsto (B_0^\circ w, B_1^\circ w)
\]
are respectively defined in terms of
\[ B^0_\ast w := i^\ast w, \quad B^1_\ast w := i^\ast d^g_{E,g,b} w, \]
(7)
\[ B^0_\ast w := \ast^b G M^{-1} (i^\ast \ast_b w), \quad B^1_\ast w := \ast^b G M^{-1} (i^\ast d^g_{E \otimes \Theta M,b \otimes b} \ast_b w). \]
A form \( w \) satisfies the boundary conditions, i.e., \( w \in \Omega(M;E)|_{\mathcal{B}} \), if and only if \( Bw = 0 \).

**Lemma 1.** For a subspace \( X \subseteq \Omega(M;E) \), denote by \( X|_{\mathcal{B}} := \{ w \in X|\mathcal{B}w = 0 \} \) the space of smooth forms in \( X \) which satisfy the boundary conditions specified by the vanishing of the operator \( \mathcal{B} \in \{ \mathcal{B}^0, \mathcal{B}^1, \mathcal{B}_\pm, \mathcal{B} \} \). Set
\[ X|_{\mathcal{B}^0} := X|_{\mathcal{B}^0} \cap X|_{\mathcal{B}^0} \]
Then the following assertions hold
(a) \( X|_{\mathcal{B}} = X|_{\mathcal{B}^0} \cap X|_{\mathcal{B}^1} \cap X|_{\mathcal{B}_\pm} \) and \( X|_{\mathcal{B}} \subset X|_{\mathcal{B}^0} \subset X|_{\mathcal{B}^0} \),
(b) \( d_E(\Omega(M;E)|_{\mathcal{B}^0}) \subset \Omega(M;E)|_{\mathcal{B}^0} \),
(c) \( d_E(\Omega(M;E)|_{\mathcal{B}}) \subset \Omega(M;E)|_{\mathcal{B}^0} \) and \( d^g_{E,g,b}(\Omega(M;E)|_{\mathcal{B}}) \subset \Omega(M;E)|_{\mathcal{B}^0} \),
(d) If \( v \in \Omega(M;E)|_{\mathcal{B}^0} \) and \( w \in \Omega(M;E)|_{\mathcal{B}} \) then \( \beta_{g,b}(d_E v, d^g_{E,g,b} w) = 0 \),
(e) If \( v, w \in \Omega(M;E)|_{\mathcal{B}^0} \), then \( \beta_{g,b}(d_E v, w) = \beta_{g,b}(v, d^g_{E,g,b} w) \),
(f) If \( v, w \in \Omega(M;E)|_{\mathcal{B}} \), then \( \beta_{g,b}(\Delta_{E,g,b} v, w) = \beta_{g,b}(v, \Delta_{E,g,b} w) \).

**Proof.** The first assertion is obvious. The remaining assertions follow from (8), (1), the Green’s formulas in (3) and straightforward manipulations coming from the definition of the operators and spaces above. \( \square \)

1.4. **Boundary conditions and Poincaré duality.** Consider the Riemannian bordism \((M, \partial_+ M, \partial_- M)\). The boundary value problem specified by the operator \( \Delta_{E,g,b} \) acting on the space \( \Omega(M;E)|_{\mathcal{B}} \) as defined by (1), will be denoted by
\[ [\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E,g,b}. \]
Let us denote by \((M, \partial_+ M, \partial_- M)' := (M, \partial_- M, \partial_+ M)\) the dual bordism to \((M, \partial_+ M, \partial_- M)\). Then, we are interested in \([\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E' \otimes \Theta M,b'}. \) the dual boundary value problem to (1), corresponding to the bilinear Laplacian \( \Delta_{E',g,b'} \) acting on \( E' \otimes \Theta M \)-valued forms (where the flat complex vector bundle \( E' \) is endowed with the dual connection \( \nabla^{E'} \) and dual bilinear form \( b' \) under the boundary conditions specified by the vanishing of the boundary operator \( \mathcal{B}' \), i.e., the same operator from (5) but associated to \((M, \partial_+ M, \partial_- M)\). The boundary value problem in (9) is naturally intertwined with its dual one.
by means of the Hodge $\star$-operator. Indeed, by the very definition of these operators, we have the equality

$$\star_b d^E_{E,g,b} d_E = d_{E'} \otimes \Theta_M d^E_{E'} \otimes \Theta_{M,g,b} \star_b$$

so that

$$\star_b \Delta_{E,g,b} = \Delta_{E'} \otimes \Theta_{M,g,b} \star_b,$$

and

$$w \in \Omega^q(M; E)|_B \iff \star_b w \in \Omega^{m-q}(M; E' \otimes \Theta_M)|_B'.$$

That is, the Hodge-$\star_b$-operator intertwines the roles of $\partial_+ M$ and $\partial_- M$ in (2) and its dual.

As a special case, if $\partial_+ M = \partial M$ and $\partial_- M = \emptyset$ (resp. $\partial_+ M = \emptyset$ and $\partial_- M = \partial M$), then $[\Delta, \mathcal{B}^E_{(M, \partial M, \emptyset)}]$ (resp. $[\Delta, \mathcal{B}^E_{(M, \emptyset, \partial M)}]$) is the boundary value problem where absolute (resp. relative) boundary conditions only are imposed on $\partial M$.

1.5. **Hermitian boundary value problems.** We recall some facts for the Hermitian situation. By using a Hermitian structure $h$ on $E$, instead of the bilinear form $b$, all over in the considerations above, one has $\ll v, w \gg_{g,h} := \int_M \text{Tr}(v \wedge \star_h w)$ a Hermitian product on $\Omega(M; E)$, where $\star_h$ is in this case a fiber-wise complex anti-linear isomorphism induced by $h$ and $\star_\circ$. Then, associated to this data, one considers a differential $d_E$, a codifferential $d^*_E$, and a Laplacian

$$\Delta_{E,g,h} := d_E d^*_E + d^*_E d_E : \Omega(M; E) \to \Omega(M; E),$$

which is formally selfadjoint with respect to $\ll v, w \gg_{g,h}$, under absolute/relative boundary conditions on $(M, \partial_+ M, \partial_- M)$. Let $\Omega(M; E)|^h_B$ be the space of $E$-valued smooth forms satisfying absolute/relative boundary conditions on $(M, \partial_+ M, \partial_- M)$ defined as in (1) but using instead the Hermitian form $h$. In order to distinguish this problem from the bilinear one, we refer to it as the **Hermitian boundary value problem**.

The Hermitian boundary value problem is an elliptic boundary value problem, see [12] and [13]. This permits one to consider $\Delta_{E,g,h}$, as an unbounded operator in the $L^2$-norm and extend it to a selfadjoint operator with domain of definition being the $H^2$-Sobolev closure of $\Omega(M; E)|^h_B$; see [17], [10], [19], [12] and [13]. In particular, in this Hermitian setting, there are well-known Hodge-decomposition results. For instance, if $\mathcal{H}^q_{\Delta g}(M; E)$ is the space of $q$-Harmonic forms satisfying boundary conditions, then [17] Theorem 1.10 (see also [19] page 239) states that for each $v \in \Omega^q(M; E)|^h_B$, there exist unique $v_0 \in \mathcal{H}^q_{\Delta g}(M; E)$, $v_1 \in d_E(\Omega^{q-1}(M; E)|^h_B)$ and $v_2 \in d^*_E(\Omega^{q+1}(M; E)|^h_B)$ such that $v = v_0 + v_1 + v_2$, where we have
used the notation suggested in (8) associated to \( h \). Moreover, the Hodge–De-Rham tells us that relative cohomology exactly coincides with the space of Harmonic forms of the Hermitian Laplacian:

\[
H^q_{\Delta_B}(M; E) \cong H^q(M, \partial_- M; E).
\]

In the bilinear setting, the isomorphism in (10) does no longer holds, but we have instead Proposition 3 below. One uses the isomorphism in (10) to define the Ray–Singer metric on manifolds with boundary, as a Hermitian metric on the determinant line in (relative) cohomology. This problem has been studied by many authors, see for instance [21], [17], [10], [19], [11], [8] and [9]. In particular, we are interested in the work by Brünning and Ma in [8], where the case \( \partial_- M = \emptyset \) was studied.

1.6. The spectrum of the bilinear Laplacian. Consider the boundary valued problem \([\Delta, B]^{E,g,b}_{(M,\partial_+,M,\partial_- M)}\). Here we denote by \( H_s(M; E) \) for \( s \geq 0 \), the corresponding Sobolev completions of \( \Omega(M; E) \) with respect to a Hermitian metric on \( E \). By [16, Section 20.1] and [1, Chapter 1], the operators \( \Delta_{E,g,b} \) and \( B_{E,g,b}^i \) extend as a linear bounded operators

\[
\Delta_{E,g,b} : H^2(M; E) \to L^2(M; E)
\]

and

\[
B_{E,g,b}^i : H^2(M, E) \to H^{\frac{1}{2}}(\partial M; E|_{\partial M}) \oplus H^{\frac{1}{2}-i}(\partial M, E|_{\partial M})
\]

respectively and again these are independent on the chosen Hermitian structure.

By the \( L^2 \)-realization of the bilinear Laplacian is understood the same operator in [11] but considered as the unbounded operator in \( L^2(M; E) \)

\[
\Delta_B : D(\Delta_B) \subset L^2(M; E) \to L^2(M; E)
\]

with domain of definition

\[
D(\Delta_B) := \Omega(M; E)|_B^{H^2}.
\]

The boundary value problem \([\Delta, B]^{E,g,b}_{(M,\partial_+,M,\partial_- M)}\) is elliptic with respect to the cone \( \mathbb{C} \setminus (0, \infty) \), see [13, Lemma 1.5.3]. Boundary ellipticity guarantees the existence of elliptic estimates, see [11, Theorem 6.3.1] and [16, Theorem 20.1.2]. Then, elliptic estimates permit one to conclude that the \( L^2 \)-realization of the bilinear Laplacian is a closed unbounded operator in \( L^2(M; E) \), which coincides with the \( L^2 \)-closure extension of

\[
\Delta_{E,g,b} : \Omega(M; E)|_B \subset L^2(M; E) \to \Omega(M; E) \subset L^2(M; E),
\]

regarded as unbounded operator in \( L^2(M; E) \).
Lemma 2. Let $\Delta_B$ be the unbounded operator with domain of definition $\mathcal{D}(\Delta_B)$ given in [17]. This operator is densely defined in $L^2(M;E)$, possesses a non-empty resolvent set, its resolvent is compact and its spectrum is discrete. More precisely, for every $\theta > 0$, there exists $R > 0$ such that $B_R(0)$, the closed ball in $\mathbb{C}$ centered at 0 and radius $R$, contains at most a finite subset of $\text{Spec}(\Delta_B)$ and the remaining part of the spectrum is entirely contained in the sector

$$\Lambda_{R,\theta} := \{ z \in \mathbb{C} \mid -\theta < \arg(z) < \theta \text{ and } |z| \geq R \}.$$ 

Furthermore, for every $\lambda \notin \Lambda_{R,\theta}$ large enough, there is $C > 0$, for which $\| (\Delta_B - \lambda)^{-1} \|_{L^2} \leq C/|\lambda|$.

Proof. This follows from boundary ellipticity with respect to the conical set $\mathbb{C} \setminus (0,\infty)$. For a detailed discussion on this result (which holds also in the more general setting of pseudodifferential boundary value problems for operators), we refer the reader to [15] Theorem 3.3.2, Corollary 3.3.3 and Remark 3.3.4 (see also [15] Section 1.5).

1.7. Generalized eigenspaces. By Lemma 2 $\text{Spec}(\Delta_B)$ is discrete and then, for each $\lambda \in \text{Spec}(\Delta_B)$, we choose $\gamma(\lambda)$ a closed counter-clock-wise oriented curve surrounding $\lambda$ as the unique point of $\text{Spec}(\Delta_B)$. Consider the corresponding Riesz or spectral projection:

$$P_{\Delta_B}(\lambda) : L^2(M;E) \to \mathcal{D}(\Delta_B) \subset L^2(M;E),$$

$$w \mapsto -(2\pi i)^{-1} \int_{\gamma(\lambda)} (\Delta_B - \mu)^{-1} w \, d\mu.$$ 

The integral above in (15) converges uniformly in the $L^2$-norm as the limit of Riemann sums, since the function $x \mapsto (\Delta_B - x)^{-1}$ is analytic in a neighborhood of $\gamma(\lambda)$. The image of $P_{\Delta_B}(\lambda)$ in $L^2(M;E)$ is denoted by

$$\Omega_{\Delta_B}(M;E)(\lambda) := P_{\Delta_B}(\lambda)(L^2(M;E)).$$

Since the resolvent of $\Delta_B$ is compact, the operator $P_{\Delta_B}(\lambda)$ is bounded on $L^2(M;E)$, and $\Omega_{\Delta_B}(M;E)(\lambda)$ is of finite dimension, see [18] Theorem 6.29]. The image of the complementary projection to $P_{\Delta_B}(\lambda)$ on $L^2(M;E)$ is denoted by

$$\text{Im}(\text{Id} - P_{\Delta_B}(\lambda)) := (\text{Id} - P_{\Delta_B}(\lambda))(L^2(M;E)).$$

Then the space $L^2(M;E)$ decomposes as a direct sum of Hilbert spaces compatible with the projections $P_{\Delta_B}(\lambda)$ and $(\text{Id} - P_{\Delta_B}(\lambda))$. More precisely, the following Lemma is a direct application of [18] Theorem 6.17.

Lemma 3. Consider the unbounded operator $(\Delta_B, \mathcal{D}(\Delta_B))$ from [18]. For $\lambda \in \text{Spec}(\Delta_B)$ consider the corresponding spectral projection $P_{\Delta_B}(\lambda)$. Then $\Delta_B$ commutes with $P_{\Delta_B}(\lambda)$; that is, for $u \in \mathcal{D}(\Delta_B)$, we have

$$P_{\Delta_B}(\lambda)u \in \mathcal{D}(\Delta_B) \text{ and } P_{\Delta_B}(\lambda)\Delta_B u = \Delta_B P_{\Delta_B}(\lambda)u.$$
The space $L^2(M; E)$ decomposes as

$$L^2(M; E) \cong \Omega_{\Delta_B}(M; E) \lambda \oplus \text{Im}(\text{Id} - P_{\Delta_B}(\lambda)),$$

such that

$$P_{\Delta_B}(\lambda)(\mathcal{D}(\Delta_B)) \subset \mathcal{D}(\Delta_B),$$

$$\Delta_B(\Omega_{\Delta_B}(M; E)(\lambda)) \subset \Omega_{\Delta_B}(M; E)(\lambda),$$

$$\Delta_B(\text{Im}(\text{Id} - P_{\Delta_B}(\lambda)) \cap \mathcal{D}(\Delta_B)) \subset \text{Im}(\text{Id} - P_{\Delta_B}(\lambda)).$$

The operator

$$\Delta_B|_{\Omega_{\Delta_B}(M; E)(\lambda)} : \Omega_{\Delta_B}(M; E)(\lambda) \rightarrow \Omega_{\Delta_B}(M; E)(\lambda),$$

is bounded on $\Omega_{\Delta_B}(M; E)(\lambda)$, $\text{Spec}(\Delta_B|_{\Omega_{\Delta_B}(M; E)(\lambda)}) = \{\lambda\}$ and the operator

$$\Delta_B - \lambda|_{\mathcal{D}(\Delta_B - \lambda)\text{Im}(\text{Id} - P_{\Delta_B}(\lambda))} : \mathcal{D}(\Delta_B - \lambda)\text{Im}(\text{Id} - P_{\Delta_B}(\lambda)) \rightarrow \text{Im}(\text{Id} - P_{\Delta_B}(\lambda)),$$

with domain of definition

$$\mathcal{D}(\Delta_B - \lambda)\text{Im}(\text{Id} - P_{\Delta_B}(\lambda)) := \text{Im}(\text{Id} - P_{\Delta_B}(\lambda)) \cap \mathcal{D}(\Delta_B) \subset L^2(M; E),$$

is invertible, i.e., the spectrum of $\Delta_B|_{\text{Im}(\text{Id} - P_{\Delta_B}(\lambda))}$ is exactly $\text{Spec}(\Delta_B) \setminus \{\lambda\}$.

The operator $\Delta_B|_{\Omega_{\Delta_B}(M; E)(\lambda)}$ in (16) being bounded, its spectrum containing $\lambda$ only and $\Omega_{\Delta_B}(M; E)(\lambda)$ being of finite dimension, the operator $(\Delta_B - \lambda)|_{\Omega_{\Delta_B}(M; E)(\lambda)}$ is nilpotent.

Commutativity of $P_{\Delta_B}(\lambda)$ with $\Delta_B$ on its domain $\mathcal{D}(\Delta_B)$, invariance of $\Omega_{\Delta_B}(M; E)(\lambda)$ under $\Delta_B$, and the (iterated) use of elliptic estimates with Sobolev embedding, one has $\Omega_{\Delta_B}(M; E)(\lambda) \subset \Omega(\Delta_B - \lambda)\text{Im}(\text{Id} - P_{\Delta_B}(\lambda)) \subset \Omega(\Delta_B - \lambda)\text{Im}(\text{Id} - P_{\Delta_B}(\lambda))$. Thus each $\lambda$-eigenspace can be described as

$$\Omega_{\Delta_B}(M; E)(\lambda) = \left\{ w \in \Omega(\Delta_B - \lambda)\text{Im}(\text{Id} - P_{\Delta_B}(\lambda)) : \exists n \in \mathbb{N} \text{ s.t. } (\Delta_{\Delta_B - \lambda})^n w = 0, \forall n \geq N \right\}.$$

**Lemma 4.** The space $\Omega_{\Delta_B}(M; E)(\lambda)$ is invariant under $d_E$ and $d_{E,g,b}^\sharp$.

**Proof.** We show that $\Omega_{\Delta_B}(M; E)(\lambda)$ is invariant under $d_E$ and $d_{E,g,b}^\sharp$. Since $\Omega_{\Delta_B}(M; E)(\lambda)$ contains smooth differential forms only, it suffices to show that $d_E w$ satisfies the boundary condition, whenever $w \in \Omega_{\Delta_B}(M; E)(\lambda)$. On $\partial_+ M$, the absolute part of the boundary, this immediately follows from $d_{E}^2 w = 0$. Let us turn to $\partial_- M$, the relative part of the boundary. But, we know that the Riesz projections are well defined as bounded operators and they commute with the Laplacian on its domain of definition. That is, $\Delta_{E,g,b} w$ lies in $\Omega_{\Delta_B}(M; E)(\lambda)$ as well; in particular, it satisfies relative boundary conditions on $\partial_- M$, so that $i_{\Delta_{E,g,b}}^\star (\Delta_{E,g,b} w) = 0$. Together with $i_{\Delta_{E,g,b}}^\star d_{E,g,b}^\sharp w = 0$, this implies $i_{\Delta_{E,g,b}}^\star d_{E,g,b}^\sharp d_E w = 0$, hence $d_E w$ also satisfies
relative boundary conditions. Finally, the corresponding statement for \( d_{E,g,b}^2 \) follows by the duality between the absolute and relative boundary operators.

\[ \square \]

1.8. Orthogonality and Hodge decomposition for smooth forms. We are interested in the space of smooth forms being in the complement image of \( P_E(\lambda) \), which is denoted by

\[ \Omega_{\Delta_B}(M;E)(\lambda)^c := \Omega(M;E) \cap \text{Im}(\text{Id} - P_{\Delta_B}(\lambda)). \]

Invertibility of the operator given in (17) and the existence of elliptic estimates imply that the restriction of \((\Delta_B - \lambda)\) to the space \( \Omega_{\Delta_B}(M;E)(\lambda)^c \) given in (18), satisfying boundary conditions provides, with the notation in display (8), the isomorphism

\[ (\Delta_B - \lambda)\big|_{\Omega_{\Delta_B}(M;E)(\lambda)^c|_{B}} : \Omega_{\Delta_B}(M;E)(\lambda)^c|_{B} \to \Omega_{\Delta_B}(M;E)(\lambda)^c. \]

**Lemma 5.** For \( \lambda \in \text{Spec}(\Delta_B) \) and \( v, w \in L^2(M;E) \), we have the formula \( \beta_{g,b}(P_{\Delta_B}(\lambda)v, w) = \beta_{g,b}(v, P_{\Delta_B}(\lambda)w) \).

**Proof.** Since \( \beta_{g,b} \) continuously extends to a nondegenerate bilinear form on \( L^2(M;E) \), it is enough to prove the statement on smooth forms. For \( v, w \in \Omega(M;E) \) and the definition of the spectral projection in (15), we have

\[ -2\pi i \beta_{g,b}(P_{\Delta_B}(\lambda)v, w) = \beta_{g,b}\left( \int_{\gamma_{\lambda}} (\Delta_B - \mu)^{-1}v \, d\mu, w \right) = \int_{\gamma_{\lambda}} \beta_{g,b}\left((\Delta_B - \mu)^{-1}v, w\right) \, d\mu, \]

where the last equality above holds, since \( \int_{\gamma_{\lambda}} \) converges uniformly in the \( L^2 \)-norm. Since \( \gamma_{\lambda} \cap \text{Spec}(\Delta_B) = \emptyset \), we have \( (\Delta_B - \mu)^{-1}w \in \mathcal{D}(\Delta_B) \) so that \( w = (\Delta_B - \mu)(\Delta_B - \mu)^{-1}w \) for each \( \mu \in \gamma_{\lambda} \). Now, from the isomorphism in (19), both \( (\Delta_B - \mu)^{-1}v \) and \( (\Delta_B - \mu)^{-1}w \) belong in fact to \( \Omega_{\Delta_B}(M;E)(\lambda)^c|_{B} \), so we can apply Lemma 4 and obtain

\[ \beta_{g,b}\left((\Delta_B - \mu)^{-1}v, w\right) = \beta_{g,b}\left((\Delta_{E,g,b} - \mu)(\Delta_B - \mu)^{-1}v, (\Delta_B - \mu)^{-1}w\right) \]

\[ = \beta_{g,b}\left((\Delta_{E,g,b} - \mu)(\Delta_B - \mu)^{-1}v, \left((\Delta_B - \mu)^{-1}w\right)\right); \]

that is, \( \beta_{g,b}(P_{\Delta_B}(\lambda)v, w) = -(-2\pi i)^{-1} \int_{\gamma_{\lambda}} \beta_{g,b}(v, (\Delta_B - \mu)^{-1}w) \, d\mu \) and hence the equality \( \beta_{g,b}(P_{\Delta_B}(\lambda)v, w) = \beta_{g,b}(v, P_{\Delta_B}(\lambda)w) \) holds. \( \square \)

**Proposition 1.** There is a \( \beta_{g,b} \)-orthogonal direct sum decomposition:

\[ \Omega(M;E) \cong \Omega_{\Delta_B}(M;E)(\lambda) \oplus \Omega_{\Delta_B}(M;E)(\lambda)^c. \]

If \( \lambda, \mu \in \text{Spec}(\Delta_B) \) with \( \lambda \neq \mu \), then \( \Omega_{\Delta_B}(M;E)(\mu) \perp_{\beta} \Omega_{\Delta_B}(M;E)(\lambda) \).

In particular, \( \beta_{g,b} \) restricts to each of these subspaces as a non degenerate
symmetric bilinear form. Furthermore, with the notation in Section 1.3, there is a \(\beta_{g,b}\)-orthogonal direct sum decomposition

\[
\Omega(M; E)|_{\mathcal{B}_0} \cong \Omega_{\Delta_B}(M; E)(\lambda) \oplus \Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}_0},
\]

which is invariant under \(d_E\).

**Proof.** Remark that \(\Omega_{\Delta_B}(M; E)(\lambda) = P_{\Delta_B}(\lambda)(\Omega(M; E))\). Therefore the decomposition in (20) follows from the direct sum decomposition of \(L^2(M; E)\) stated in Lemma 3. We show that \(\Omega_{\Delta_B}(M; E)(\lambda)\) is \(\beta_{g,b}\)-orthogonal to \(\Omega_{\Delta_B}(M; E)(\lambda)^c\), by taking \(v \in \Omega_{\Delta_B}(M; E)(\lambda)\) and \(w \in \Omega_{\Delta_B}(M; E)(\lambda)^c\) and noticing that

\[
\beta_{g,b}(v, w) = \beta_{g,b}(P_{\Delta_B}(\lambda)v, w) = \beta_{g,b}(v, P_{\Delta_B}(\lambda)w) = 0,
\]

where the second equality above follows from Lemma 5 and the last one is true because \(w\) is in the image of the complementary projection of \(P_{\Delta_B}(\lambda)\). Since \(\Omega_{\Delta_B}(M; E)(\lambda)\) is contained in the space \(\Omega(M; E)|_{\mathcal{B}_0}\), the decomposition in (20) implies directness and \(\beta_{g,b}\)-orthogonality for the one in (21). By Lemma 4, \(\Omega_{\Delta_B}(M; E)(\lambda)\) is invariant under both \(d_E\) and \(d_{E,g,b}^*\). But the space \(d_E(\Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}_0})\) is contained in \(\Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}_0}\) as well, as it can be checked by using the Green’s formulas from Lemma 3 that \(d_{E,g,b}^*\) leaves invariant \(\Omega_{\Delta_B}(M; E)(\lambda)\) and \(\beta_{g,b}\)-orthogonality of (21). \(\square\)

**Corollary 1.** For \(\lambda \in \text{Spec}(\Delta_B)\) and with the notation in (3), consider the space \(\Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}_0}\). Then, the spaces \(d_E(\Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}_0})\) and \(d_{E,g,b}^*(\Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}_0})\) are \(\beta_{g,b}\)-orthogonal to \(\Omega_{\Delta_B}(M; E)(\lambda)\).

**Proof.** If \(u \in \Omega_{\Delta_B}(M; E)(\lambda)\) and \(v \in \Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}_0}\), then, by using Lemma 1, invariance of \(\Omega_{\Delta_B}(M; E)(\lambda)\) under \(d_{E,g,b}^*\) (see also Lemma 4 and Proposition 1 above), we have \(\beta_{g,b}(u, d_E v) = \beta_{g,b}(d_{E,g,b}^* u, v) = 0\). The proof for \(d_{E,g,b}^*\) is analog. \(\square\)

**Corollary 2.** (Hodge decomposition) We have the \(\beta_{g,b}\)-orthogonal decomposition \(\Omega(M; E) \cong \Omega_{\Delta_B}(M; E)(0) \oplus \Delta_{E,g,b}(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}_0})\).

**Proof.** This follows from Proposition 1 and the isomorphism in (12). \(\square\)

Compare the following result with [6, Proposition 2.1].
Proposition 2. The following are $\beta_{g,b}$-orthogonal direct sum decompositions.

\[
\Omega(M; E) \cong \Omega_{\Delta_b}(M; E)(0) \oplus d_E(d_{E,g,b}(\Omega_{\Delta_b}(M; E)(0)^c |_{B}))
\]

(22)

\[
\Omega(M; E)|_{B_0} \cong \Omega_{\Delta_b}(M; E)(0) \oplus d_E(\Omega_{\Delta_b}(M; E)(0)^c |_{B_0})
\]

(23)

\[
\Omega(M; E)|_{B_0} \cong \Omega_{\Delta_b}(M; E)(0) \oplus d_E(\Omega_{\Delta_b}(M; E)(0)^c |_{B})
\]

(24)

Moreover, the restriction of $\beta_{g,b}$ to each of the spaces appearing above is nondegenerate.

Proof. We prove (22). From Corollary 2, every $u \in \Omega(M; E)$ can be written as $u = u_0 + d_E(d_{E,g,b} u) + d_{E,g,b}(d_E u)$, with $u_0 \in \Omega_{\Delta_b}(M; E)(0)$ and $u \in \Omega_{\Delta_b}(M; E)(0)^c |_{B}$. That

\[
d_E(d_{E,g,b}^{\sharp}(\Omega_{\Delta_b}(M; E)(0)^c |_{B})) \perp d_{E,g,b} d_E(\Omega_{\Delta_b}(M; E)(0)^c |_{B}),
\]

follows from Lemma 1 and $d_E^2 = 0$. To see that (22) is a direct sum, we check that the intersection of the last two spaces on the right of (22) is trivial. So, take $u \in \Omega_{\Delta_b}(M; E)(0)^c$, and suppose there are $v, w \in \Omega_{\Delta_b}(M; E)(0)^c |_{B}$ with $u = d_E(d_{E,g,b}^{\sharp} v) = d_{E,g,b} d_E w$. Remark obviously that $d_E d_{E,g,b} u = 0$ but also that $u \in \Omega_{\Delta_b}(M; E)(0)$, since

(a) $i^*_u v = d_E(i^*_u d_{E,g,b} v) = 0$, as $v$ satisfies boundary conditions,

(b) $i^*_u d_{E,g,b} u = i^*_u d_{E,g,b} d_E v = 0$,

(c) $i^*_u \ast_b u = \pm d_E(i^*_u d_{E,g,b} \ast_b w) = 0$; as $w$ satisfies boundary conditions,

(d) $i^*_u d_{E,g,b} \ast_b u = \pm i^*_u \ast_b d_E d_{E,g,b}^\sharp v = 0$;

therefore, from Proposition 1 $u$ must vanish, so that the sum in (22) is direct. This decomposition is clearly $\beta_{g,b}$-orthogonal. The decompositions in (23) and (24) follow from that in (22). Lemma 1 the isomorphism in (19) and the definition of boundary conditions as we have proceeded to prove the statement (22); we omit the details. Now, since $d_E(\Omega_{\Delta_b}(M; E)(0)^c |_{B}) \subset d_E(\Omega_{\Delta_b}(M; E)(0)^c |_{B_0})$, directness of decomposition (24) follows from that of (23). To check directness in (23), firstly observe that by Proposition 1 we have $d_E(\Omega_{\Delta_b}(M; E)(0)^c |_{B_0}) \subset \Omega_{\Delta_b}(M; E)(0)^c |_{B_0}$ and therefore the intersection of the space $\Omega_{\Delta_b}(M; E)(0)$ with $d_E(\Omega_{\Delta_b}(M; E)(0)^c |_{B_0})$ is trivial. Secondly, from the inclusion $\Omega_{\Delta_b}(M; E)(0)^c |_{B} \subset \Omega_{\Delta_b}(M; E)(0)^c |_{B_0}$, Corollary 1 and Proposition 1 the intersection of $\Omega_{\Delta_b}(M; E)(0)$ with the space $d_{E,g,b} d_E(\Omega_{\Delta_b}(M; E)(0)^c |_{B})$ is also trivial. Thirdly, the intersection between
1.9. Cohomology. Recall the notation suggested in Lemma 4. The space \( \Omega(M; E)|_{\mathcal{E}_0^\partial} \) endowed with the differential \( d_E \) is a cochain complex, which computes De-Rham cohomology of \( M \) relative to \( \partial_M \) with coefficients on \( E \), see for instance [3]. For \( \lambda \in \text{Spec}(\Delta_B) \), consider \( \Omega_{\Delta_B}(M; E)(\lambda) \) as a cochain subcomplex of \( \Omega(M; E)|_{\mathcal{E}_0^\partial} \). From Lemma 3, Lemma 4 and the isomorphism in [19], every generalized eigenspace corresponding to a non-zero eigenvalue is acyclic, i.e., \( H(\Omega_{\Delta_B}(M; E)(\lambda)) = 0 \) whenever \( \lambda \neq 0 \). For \( \lambda = 0 \), we have the following.

**Proposition 3.** The inclusion \( \Omega_{\Delta_B}(M; E)(0) \hookrightarrow \Omega(M; E)|_{\mathcal{E}_0^\partial} \) induces an isomorphism in cohomology: \( H^*(\Omega_{\Delta_B}(M; E)(0)) \cong H^*(M, \partial_M, E) \).

**Proof.** Since \( \Omega_{\Delta_B}(M; E)(0) \subset \Omega(M; E)|_{\mathcal{E}_0^\partial} \), the space \( \Omega(M; E)|_{\mathcal{E}_0^\partial} \) admits a decomposition compatible with the one in Corollary 2 and therefore it decomposes as

\[
\Omega(M; E)|_{\mathcal{E}_0^\partial} \cong \Omega_{\Delta_B}(M; E)(0) \oplus \Delta_{E,g,b}(\Omega_{\Delta_B}(M; E)(0)|_{\mathcal{E}_0^\partial}),
\]

where \( \Delta_{E,g,b}(\Omega_{\Delta_B}(M; E)(0)|_{\mathcal{E}_0^\partial}) \) is also a cochain subcomplex, because of Proposition 4 and that \( \Omega(M; E)|_{\mathcal{E}_0^\partial} \) is invariant under the action of \( d_E \). Thus the assertion is true, if the corresponding cohomology groups vanish; that is, if every closed form \( w \) in \( \Delta_{E,g,b}(\Omega_{\Delta_B}(M; E)(0)|_{\mathcal{E}_0^\partial}) \) is exact. By Proposition 2 (23), there exist \( w_1 \in \Omega_{\Delta_B}(M; E)(0)|_{\mathcal{E}_0^\partial} \) and \( w_2 \in \Omega_{\Delta_B}(M; E)(0)|_{\mathcal{E}_0^\partial} \) such that \( w = d_E w_1 + d_{E,g,b} w_2 \). First, we claim that \( \beta_{g,b}(d_{E,g,b} w_2, v_1) = 0 \), for all \( v_1 \in \Omega_{\Delta_B}(M; E)(0)|_{\mathcal{E}_0^\partial} \), see (8); indeed, from Proposition 2 (22), there exist \( v_2, u_2 \in \Omega_{\Delta_B}(M; E)(0)|_{\mathcal{E}_0^\partial} \), such
that $v_1 = d_E v_2 + d_{E,g,b}^2 w_2$ and hence $\beta_{g,b}(d_{E,g,b}^2 w_2, d_E v_2 + d_{E,g,b}^2 u_2) = 0$
where we have used that $d_{E,g,b}^2 w_2, d_E v_2$ and $d_{E,g,b}^2 w_1 \in \Omega_{\Delta_B}(M;E) (0)^c |_{B^0}$,
Lemma 1 $(d_{E,g,b}^2)^2 = 0$ and that $\beta_{g,b}(d_E d_{E,g,b}^2 w_2, u_2)$ vanishes, because
$w$ being close implies $d_E d_{E,g,b}^2 w_2 = 0$. Finally, since $d_{E,g,b}^2 w_2$ belongs
to $\Omega_{\Delta_B}(M;E) (0)^c |_{B^0}$ as well, and that $\beta_{g,b}$ restricted to this sub-space
is also nondegenerate, see Proposition 2 from the claim above, we have
$d_{E,g,b}^2 w_2 = 0$. That is, $w$ is exact in $\Delta_{E,g,b}(\Omega_{\Delta_B}(M;E) (0)^c |_{B^0}) |_{B^0}$.

\[ \square \]

2. Heat trace asymptotic expansion and anomaly formulas

2.1. Heat trace asymptotics for an elliptic boundary value problem.
Let $(\mathcal{D}, \mathcal{B})$ be a boundary value problem, where $\mathcal{D}$ is an operator of Laplace
\text{type and $\mathcal{B}$ is a boundary operator specifying absolute/relative boundary}
\text{conditions, (or more generally mixed boundary conditions, see [13]) and denote by $\mathcal{D}_B$ its $L^2$-realization, see Section 1.6.}
\text{Then, by [13] Theorem 1.4.5, for $t > 0$ the heat kernel $\exp(-t \mathcal{D}_B)$ is a smoothing operator, of trace class
in $L^2$-norm and for $t \to 0$, there is a complete asymptotic expansion:}

\[
\text{Tr}_{L^2} (\psi \exp(-t \mathcal{D}_B)) \sim \sum_{n=0}^{\infty} a_n(\psi, \mathcal{D}, \mathcal{B}) t^{(n-m)/2},
\]

where $\psi$ is a bundle endomorphism. The coefficients $a_n(\psi, \mathcal{D}, \mathcal{B})$, the heat trace asymptotic coefficients associated to $\psi$ and the boundary value problem $(\mathcal{D}, \mathcal{B})$, are given by the formula

\[(25) \quad a_n(\psi, \mathcal{D}, \mathcal{B}) = \int_M \text{Tr} (\psi \cdot \mathcal{e}_n(\mathcal{D})) \text{vol}_g(M) + \sum_{k=0}^{n-1} \int_{\partial M} \text{Tr} (\nabla_{\mathcal{v}_n^k} \psi \cdot \mathcal{e}_{n,k}(\mathcal{D}, \mathcal{B})) \text{vol}_g(\partial M), \]

where $\nabla_{\mathcal{v}_n^k}$ denotes the $k$-covariant derivative along the inwards pointing
godesic unit vector field normal to $\partial M$, computed with respect to the Levi–Civitá connection on $\Lambda^*(T^* M)$ and an auxiliary connection on the bundle.
The quantities $\mathcal{e}_n(x, \mathcal{D})$ and $\mathcal{e}_{n,k}(y, \mathcal{D}, \mathcal{B})$ in (25) are invariant endomorphism-valued
forms locally computable as polynomials in the jets of the symbol of $\mathcal{D}$ and $\mathcal{B}$, see [14, 22, 23] and [24]. By using Weyl’s theory of invariants,
these endomorphism invariants can be expressible as universal polynomials
in locally computable tensorial objects, see [13] Sections 1.7 and 1.8 (see also [12 Sections 1.7, 1.9 and 4.8]) and [13] Section 3.1.8.

We are interested in the coefficient of the constant term in the heat asympotic expansion in (25) corresponding to $n = \dim(M) = m$, which in
accord with the notation in [2], we denote by

\[(26) \quad \lim_{t \to 0} (\text{Tr}_{L^2} (\psi \exp(-t \mathcal{D}_B))) := a_m(\psi, \mathcal{D}, \mathcal{B}). \]
2.2. Heat trace asymptotics for the Hermitian Laplacian. Brüning and Ma studied in [8] the Hermitian Laplacian on a manifold with boundary under absolute boundary conditions and obtained anomaly formulas for the associated Ray–Singer analytic metric. They do so by computing the coefficient of the constant term in certain heat trace asymptotic expansion associated to the Hermitian boundary value problem.

Proposition 4. (Brüning–Ma) Recall the remarks and the notation from Section 1.4. Let $(M, \partial M, \emptyset)$ be a compact Riemannian bordism. Consider $[\Delta, B]^{E;g,h}_{(M, \partial M, \emptyset)}$ the Hermitian boundary value problem and denote by $\Delta_{\text{abs}, h}$ its $L^2$-realization. Let here $\text{STr}$ stand for supertrace. For $\phi \in \Gamma(M, \text{End}(E))$ we have

\begin{equation}
\text{LIM}_{t \to 0} \{ \text{STr}(\phi \exp(-t \Delta_{\text{abs}, h})) \} = \int_{\partial M} \text{Tr}(\phi) e(M, g) - (-1)^{m} \int_{\partial M} i^* \text{Tr}(\phi) e_b(\partial M, g).
\end{equation}

Moreover, for $\xi \in \Gamma(M, \text{End}(TM))$ a symmetric endomorphism with respect to the metric $g$, and $D^* \xi \in \Gamma(M, \text{End}(\Lambda^* T^* M))$ its extension as a derivation on $\Lambda^*(T^* M)$, set

\begin{equation}
\Psi := D^* \xi - \frac{1}{2} \text{Tr}(\xi).
\end{equation}

If $\tau \in \mathbb{R}$ is taken small enough so that $g + \tau g \xi$ is a nondegenerate symmetric metric on $TM$, then we have

\begin{equation}
\text{LIM}_{t \to 0} \{ \text{STr}(-\Psi \exp(-t \Delta_{\text{abs}, h})) \} = -2 \int_{\partial M} \left. \frac{d}{d\tau} \right|_{\tau = 0} e(M, g + \tau g \xi) \wedge \omega(\nabla^E, h) + \frac{\text{rank}(E)}{2} \int_{\partial M} \left. \frac{d}{d\tau} \right|_{\tau = 0} B(\partial M, g + \tau g \xi),
\end{equation}

where $\omega(\nabla^E, h) := -\frac{1}{2} \text{Tr}(h^{-1} \nabla^E h)$ is a real valued closed one-form.

Proof. We prove formula (27). First, each $\phi \in \Gamma(M, \text{End}(E))$ can be uniquely written as $\phi = \phi^e + i \phi^m$ where $\phi^e, \phi^m$ are selfadjoint elements. Thus, it is enough to prove (27) for $\phi$ selfadjoint. First, suppose that $\phi_u := h_u^{-1} \frac{d}{dh_u} \in$
\[ \Gamma(M, \text{End}(E)), \text{ where } h_u \text{ is a smooth one real parameter family of Hermitian forms on } E \text{ with } h_0 = h. \] Then, (27) exactly is the infinitesimal version of Brüning and Ma’s formulas, see \cite{8} Theorem 4.6 and \cite{8} expression (5.72). Next, suppose \( \phi \in \Gamma(M, \text{End}(E)) \) to be an arbitrary selfadjoint element.

Then, for \( u \) small enough, the family \( h_u := h + uh\phi \) is a smooth family of Hermitian forms on \( E \) and \( h_u^{-1}\frac{\partial h_u}{\partial u} = h_u^{-1}h\phi \) defines a smooth family of selfadjoint elements in \( \Gamma(M, \text{End}(E)) \). Therefore, we apply Brüning and Ma’s formulas for \( h_0^{-1}\left( \frac{\partial h_0}{\partial u} \right|_{u=0} \right) = \phi \) so that the proof of (27) is complete. We now prove (29). Let \( g_u \) be a smooth family of Riemannian metrics on \( TM \) with \( g_0 = g \) and denote by \( *_u \) the Hodge *-operator corresponding to \( g_u \). First, consider the case where \( \xi_u := g_u^{-1}\frac{\partial g_u}{\partial u} \in \Gamma(M; \text{End}(TM)) \) so that, by (28), we obtain \( \Psi_u = D^*(g_u^{-1}\frac{\partial g_u}{\partial u}) - \frac{1}{2} \text{Tr}(g_u^{-1}\frac{\partial g_u}{\partial u}) = -*_u^{-1}\frac{\partial g_u}{\partial u}, \) see \cite{7} Proposition 4.15, considered as a smooth family in \( \Gamma(M, \text{End}(\Lambda^*T^*M)) \). Then, (29) is the infinitesimal version of Brüning and Ma’s formulas, see \cite{8} Theorem 4.6 and \cite{8} expressions (5.74) and (5.75). In the general case, take a symmetric \( \xi \in \Gamma(M; \text{End}(TM)) \). Then, for \( u \) small enough the formula \( g_u := g + u\xi \) defines a smooth family of nondegenerate metrics on \( TM \) and hence \( g_u^{-1}\frac{\partial g_u}{\partial u} = g^{-1}g\xi \) a smooth family of symmetric elements in \( \Gamma(M, \text{End}(TM)) \).

Hence we obtain a smooth family of symmetric endomorphisms \(-*_u^{-1}\frac{\partial g_u}{\partial u} \) in \( \Gamma(M, \text{End}(\Lambda^*T^*M)) \), for which we can use again Brüning and Ma’s formulas. In particular, they must hold for \( u = 0 \) for which we have \( g_0^{-1}(\frac{\partial g_0}{\partial u}|_{u=0}) = \xi \), so that \( \Psi_0 = D^*(\xi) - \frac{1}{2} \text{Tr}(\xi) = -*_0^{-1}(\frac{\partial g_0}{\partial u}|_{u=0}) \). That is, (29) holds. \( \Box \)

**Lemma 6.** Let \( \bar{E}' \) be the dual of the complex conjugated vector bundle of \( E \), endowed with the dual flat connection and dual Hermitian form to those on \( E \). Consider the compact Riemannian bordisms \( (M, \emptyset, \partial M) \) together with its dual \( (M, \emptyset, \partial M)' := (M, \partial M, \emptyset) \). Let \( \Delta_{\text{rel,}h} \) be the \( L^2 \)-realization associated to the Hermitian boundary value problem \( [\Delta, \mathcal{B}]_{(M, \emptyset, \partial M)}^{E, g, h} \) and \( \Delta_{\text{rel,}h}' \) the one associated to \( [\Delta, \mathcal{B}]_{(M, \emptyset, \partial M)'}^{E', \Theta_M, g, h'} \). If \( \phi \), \( \xi \) and \( \Psi \) are as in Proposition 4.5, then

\[
\text{LIM}_{t \to 0} (\text{STr} (\phi \exp (-t\Delta_{\text{rel,}h}))) = (-1)^m \text{LIM}_{t \to 0} (\text{STr} (\phi^* \exp (-t\Delta_{\text{rel,}h}'))),
\]

where \( \phi^* := h\phi h^{-1} \), and

\[
\text{LIM}_{t \to 0} (\Psi \exp (-t\Delta_{\text{rel,}h}')) = (-1)^{m+1} \text{LIM}_{t \to 0} (\text{STr} (\Psi \exp (-t\Delta_{\text{rel,}h}'))).
\]

**Proof.** Consider \( h \in \Omega^0(M; \text{End}(E, \bar{E}')) \) the complex vector bundle isomorphism between \( E \) and \( \bar{E}' \) provided by the Hermitian metric on \( E \) (see for instance \cite{3} page 286), and its covariant derivative \( \nabla^E h \in \Omega^1(M; \text{End}(E, \bar{E}')) \) computed by using the induced connection on \( \text{End}(E, \bar{E}'). \) With the Hermitian metric on \( E \) and the Riemannian metric on \( M, \) we have a **complex linear isomorphism** \( *_h := * \otimes h : \Omega(M; E) \to \Omega(M; \bar{E}' \otimes \Theta_M), \) which is used
to define
\[ d^E_{E,g,h} := (-1)^q \ast_h^{-1} d_{E' \otimes \Theta_M} \ast_h : \Omega^q(M; E) \to \Omega^{q-1}(M; E); \]
being the formal adjoint to \( d_E \) with respect to the Hermitian product on \( \Omega(M; E) \). Remark here that the formula
\[ d_{E' \otimes \Theta_M} d^E_{E' \otimes \Theta_M, g,h} \ast_h = \ast_h d^E_{E,g,h} d_E \]
holds and therefore
\[ \ast_h \Delta_{E,g,h} = \Delta_{E' \otimes \Theta_M, g,h} \ast_h. \]
As in Section 1.4, the operator \( \ast_h \) intertwines \( E \)-valued forms satisfying relative (resp. absolute) boundary conditions with \( E' \)-valued forms satisfying absolute (resp. relative) boundary conditions. That is,
\[ \Delta_{\text{rel},h} = \ast_h^{-1} \Delta_{\text{abs},h} \ast_h \]
and therefore \( \phi \exp(-t \Delta_{\text{rel},h}) = \ast_h^{-1} \phi^* \exp(-t \Delta_{\text{abs},h'}) \ast_h \), where \( \phi^* := h \phi h' \). Thus, since the supertrace vanishes on supercommutators of graded complex-linear operators and the degree of \( \ast_{h,q} \) is \( m - q \), we obtain the formula
\[ \text{STr} (\phi \exp(-t \Delta_{\text{rel},h})) = (-1)^m \text{STr} (\phi^* \exp(-t \Delta_{\text{abs},h'})) \]
and hence (30). We now turn to formula (31). First, remark that
\[ \ast_q \left( d^s \xi - \frac{1}{2} \Tr(\xi) \right) \ast_q^{-1} = -D^s \xi + \frac{1}{2} \Tr(\xi). \]
We prove (33), by pointwise computing \( \ast_q D^s \xi \ast_q^{-1} \). Since \( \xi \) is a symmetric complex endomorphism of \( T_x M \), we may choose an orthonormal frame \( \{e_i\}_{i=1}^m \) such that \( \xi e_i = \lambda_i e_i \). Then, for \( \{e^{i_1} \wedge \cdots \wedge e^{i_q}\}_{1 \leq i_1 < \cdots < i_q \leq m} \) a positive definite oriented frame for \( \Lambda^q T^*_x M \), the Hodge \( \ast \)-operator is given by
\[ \ast_q \left( e^{i_1} \wedge \cdots \wedge e^{i_q} \right) = e^{j_1} \wedge \cdots \wedge e^{j_{m-q}} \in \Lambda^{m-q} T^*_x M, \]
where the ordered indices \( (j_1, \ldots, j_{m-q}) := (1, \ldots, \hat{i_1}, \ldots, \hat{i_q}, \ldots, m) \) with \( 1 \leq j_1 < \cdots < j_{m-q} \leq m \), are obtained as the unique possible choice of ordered indices complementary to \( \leq \hat{i_1} < \cdots < \hat{i_q} \). Therefore
\[ \ast_q D^s \xi \ast_q^{-1} (e^{i_1} \wedge \cdots \wedge e^{i_{m-q}}) = \ast_q D^s \xi (e^{i_1} \wedge \cdots \wedge e^{i_q}) \]
\[ = \ast_q \sum_{i=1}^m \lambda_i (e^{i_1} \wedge \cdots \wedge e^{i_{m-q}}) \]
\[ = \sum_{i=1}^m \lambda_i (e^{i_1} \wedge \cdots \wedge e^{i_{m-q}}) \]
\[ = \sum_{i=1}^m \lambda_i (e^{i_1} \wedge \cdots \wedge e^{i_{m-q}}) - \sum_{i=1}^{m-q} \lambda_i (e^{j_1} \wedge \cdots \wedge e^{j_{m-q}}) \]
\[ = \left( \Tr \xi - D^s \xi \right) (e^{i_1} \wedge \cdots \wedge e^{i_{m-q}}) \]
and we obtain (33), which in turn allows us to conclude
\[
\Psi(\ast_h) - 1 = ((D^* - \frac{1}{2} Tr(\xi)) \ast_h) - 1
\]
(34)
\[
= (\ast_h - 1)((D^* - \frac{1}{2} Tr(\xi)) \ast_h - 1)
\]
\[
= -(\ast_h - 1)((D^* - \frac{1}{2} Tr(\xi)) \ast_h)
\]
\[
= -(\ast_h - 1)\Psi.
\]
Finally, we use (34) to pass to the complex conjugated; hence with (32) and duality between these boundary value problems we obtain
\[
\Psi \exp(-t\Delta_{rel,h}) = \Psi \ast_h^{-1} \exp(-t\Delta'_{abs,h}) \ast_h = -\ast_h^{-1} \Psi \exp(-t\Delta'_{abs,h}) \ast_h
\]
thus, as for (30), we have
\[
\text{STr}(\Psi \exp(-t\Delta_{rel,h})) = -(-1)^m \text{STr}(\Psi \exp(-t\Delta'_{abs,h}))
\]
\[
\square
\]

**Proposition 5.** For the Riemannian bordism \((M, \emptyset, \partial M)\), consider the Hermitian boundary value problem \([\Delta, B]_{E,g,h}^{\emptyset, \emptyset, \partial M}\) with its \(L^2\)-realization denoted by \(\Delta_{rel,h}\). If \(\phi, \xi\) and \(\Psi\) are as in Proposition 4 then
\[
\lim_{t \to 0} \left( \text{STr}(\phi \exp(-t\Delta_{rel,h})) \right) = \int_M \text{Tr}(\phi e(M,g) - \int_{\partial M} i^* \text{Tr}(\phi e_b(\partial M,g))
\]
\[
\lim_{t \to 0} \left( \text{STr}(\Psi \exp(-t\Delta_{rel,h})) \right) = -2 \int_M \frac{\partial}{\partial t} \text{Tr} \mid_{t=0} (\partial(M,g,g+\beta) \wedge \partial(M,g,g+\beta) / (E,\xi))
\]
\[
+ 2(-1)^{m+1} \int_{\partial M} \frac{\partial}{\partial t} \text{Tr} \mid_{t=0} (\partial(M,g,g+\beta) \wedge \partial(M,g,g+\beta) / (E,\xi))
\]
\[
+ (-1)^{m+1} \text{rank}(E) \int_{\partial M} \frac{\partial}{\partial t} \text{Tr} \mid_{t=0} (\partial(M,g,g+\beta) / (E,\xi)).
\]

**Proof.** A form \(w \in \Omega^*(M; E)\) satisfies relative boundary conditions if and only if the smooth form \(\ast_h w \in \Omega^{m-*}(M; E' \otimes \Theta_M)\) satisfies absolute boundary conditions on \(\partial M\). Hence, the first formula in the statement follows from formula (30) in Lemma 6 and the results from Brüning and Ma for the Hermitian Laplacian stated in Proposition 4. The second formula follows from Lemma formula (31) in 6 Proposition 4 and \(\omega(E,h) = -\omega(E',h')\), see for instance 4 Section 2.4.

\[
\square
\]

**Lemma 7.** For \((M, \partial M, \emptyset), (M, \emptyset, \partial M)\) and \(M, \partial M, M, \partial M\) let us consider \([\Delta, B]_{E,g,h}^{\emptyset, \emptyset, \partial M}\), \([\Delta, B]_{E,g,h}^{\emptyset, \emptyset, \partial M}\) and \([\Delta, B]_{E,g,h}^{\emptyset, \emptyset, \partial M}\) the corresponding Hermitian boundary value problems, together with their \(L^2\)-realizations \(\Delta_{abs,h}\), \(\Delta_{rel,h}\) and \(\Delta_{B,h}\), respectively. Let \(\psi_{\pm} \in \Gamma(M; L^2(M, E))\) be chosen in such a way that \(\text{supp}(\psi_{\pm}) \cap \partial M = \emptyset\), then
\[
\lim_{t \to 0} \left( \text{STr}(\psi_+ \exp(-t\Delta_{B,h})) \right) = \lim_{t \to 0} \left( \text{STr}(\psi_+ \exp(-t\Delta_{abs,h})) \right)
\]
\[
\lim_{t \to 0} \left( \text{STr}(\psi_- \exp(-t\Delta_{B,h})) \right) = \lim_{t \to 0} \left( \text{STr}(\psi_- \exp(-t\Delta_{rel,h})) \right).
\]
Proof. This is a immediate consequence of \( \partial_+ M \) and \( \partial_- M \) being mutually disjoint and that the coefficients in the heat kernel asymptotic expansion are computable as universal polynomials in terms of finite order derivatives of the symbols expressed in local coordinates around each point of \( M \), see Section 2.1.

Theorem 1. For \( (M, \partial_+ M, \partial_- M) \), consider the Hermitian boundary value problem \( [\Delta, B]_{(E,g,h)}^{E,g,h} \) with its corresponding \( L^2 \)-realization \( \Delta_{E,h} \). If \( \phi, \xi \) and \( \Psi \) are as in Proposition 4 then

\[
\text{LIM}_{t \to 0} (\text{STr} \{ \phi \exp(-t \Delta_{E,h}) \}) = \int_M \text{Tr} (\phi) e(M,g) + (-1)^{m-1} \int_{\partial_+ M} \text{Tr}(\phi) i^* e_b(\partial M,g) - \int_{\partial_- M} \text{Tr}(\phi) i^* e_b(\partial M,g).
\]

and

\[
\text{LIM}_{t \to 0} (\text{STr} \{ -\Psi \exp(-t \Delta_{E,h}) \}) = -2 \int_M \frac{\partial}{\partial \tau} \bigg|_{\tau=0} \tilde{e}(\Omega_M g + \tau \xi) \wedge \omega(\nabla^E_h) - 2 \int_{\partial_+ M} \frac{\partial}{\partial \tau} \bigg|_{\tau=0} i^* \tilde{e}_b(\partial M, g + \tau \xi) \wedge \omega(\nabla^E_h) + \text{rank}(E) \int_{\partial_+ M} \frac{\partial}{\partial \tau} \bigg|_{\tau=0} i^* B(\partial M, g + \tau \xi) - 2(-1)^m \int_{\partial_- M} \frac{\partial}{\partial \tau} \bigg|_{\tau=0} i^* \tilde{e}_b(\partial M, g + \tau \xi) \wedge \omega(\nabla^E_h) + (-1)^{m+\text{rank}(E)} \int_{\partial_- M} \frac{\partial}{\partial \tau} \bigg|_{\tau=0} i^* B(\partial M, g + \tau \xi).
\]

Proof. This follows from Proposition 4 (Brüning and Ma), Proposition 5 and Lemma 7. More recently, Brüning and Ma gave also a proof of this statement, see [9, Theorem 3.2], based on the methods developed in [8].

2.3. Involutions, bilinear and Hermitian forms. We fix a Hermitian structure compatible with the bilinear one as follows. Since \( E \) is endowed with a bilinear form \( b \), there exists an anti-linear involution \( \nu \) on \( E \) satisfying

\[ b(\nu e_1, \nu e_2) = b(e_1, e_2) \quad \text{and} \quad b(\nu e, e) > 0 \quad \text{for all} \quad e_1, e_2, e \in E \quad \text{with} \quad e \neq 0, \]

see for instance the proof of [11, Theorem 5.10]. In this way, we obtain a (positive definite) Hermitian form on \( E \) given by

\[ h(e_1, e_2) := b(e_1, \nu e_2). \]

Remark that \( \nabla^E \nu = 0 \) is not required so that

\[ h^{-1}(\nabla^E h) = \nu^{-1}(h^{-1}(\nabla^E b)) \nu + \nu^{-1}(\nabla^E \nu). \]

Therefore, this yields a Hermitian form on \( \Omega(M; E) \) compatible with \( \beta_{g,h} \) in the sense that \( \ll v, w \gg_{g,h} = \beta_{g,h}(v, \nu w) \), for \( v, w \in \Omega(M; E) \). In [26] and [25], given a bilinear form \( b \), this involution has been exploited to study the bilinear Laplacian in terms of the Hermitian one associated to the compatible Hermitian form in [36], in both cases with and without boundary. However, our approach is a little different since we do not use a Hermitian form globally.
compatible with $\beta_{g,b}$ on $\Omega(M;E)$, but instead a local compatibility only, see section 2.3 below.

We now study the situation where $\nu$ is parallel with respect to $\nabla^E$.

**Lemma 8.** Let us consider $(M, \partial_+, M, \partial_- M)$ the compact Riemannian bordism together with the complex flat vector bundle $E$ as above. Suppose $E$ admits a nondegenerate symmetric bilinear form. Moreover, suppose there exists a complex anti-linear involution $\nu$ on $E$, satisfying the conditions in (35) and $\nabla^E \nu = 0$. Let $h$ be the (positive definite) Hermitian form on $E$ compatible with $b$ defined by (36). Then, $\Delta_{E,g,b} = \Delta_{E,g,h}$ and $\mathcal{B}_{E,g,b} = \mathcal{B}_{E,g,h}$.

**Proof.** Consider $\ll \cdot, \cdot \gg_{g,h}$ the Hermitian product on $\Omega(M;E)$, compatible with the bilinear form, and $d^*_{E,g,h}$; the formal adjoint to $d_E$ with respect to this product, which in terms of the Hodge $*$-operator can be written up to a sign as $d^*_{E,g,h} = \pm \nu^{-1} d_E \star_h$. Remark that $\nabla^E \nu = 0$ implies that $d_E \nu = \nu d_E$; hence, with $*_{\nu} = \nu \circ *_b$, we have

\begin{equation}
(37) \quad d_{E,g,b}^* = \pm \nu^{-1} d_E \star_h = \pm \nu^{-1} \nu^{-1}_b \nu = \pm \nu^{-1} d_E \star_b = d_{E,g,b}^*.
\end{equation}

and therefore the Hermitian and bilinear Laplacians coincide. We turn to the assertion for the corresponding boundary operators. On the one hand, the assertion is clear for $\mathcal{B}_{-,E,g,b} = \mathcal{B}_{-,E,g,h}$, because of (37) and (7). On the other hand, for a form $v \in \Omega^p(M;E)$ and $\omega_m$, the interior product with respect to the dual form corresponding to $\omega_m$, the identity $\nu_{\omega_m}^M \nu_{\omega_m}^* v = i_* \nu_{\omega_m}^M \nu_{\omega_m}^* v$ holds; therefore the operator specifying absolute boundary can be written, independently of the Hermitian or bilinear forms, as $\mathcal{B}_{+,E,g,b}^p v = (i_* \nu_{\omega_m}^M \nu_{\omega_m}^*(d_E v)) = \mathcal{B}_{+,E,g,h}^p v$. That finishes the proof. \hfill \Box

**Lemma 9.** Let $(M, g)$ be a compact Riemannian manifold and $E$ a flat complex vector bundle over $M$. Assume $E$ is endowed with a fiberwise nondegenerate symmetric bilinear form $b$. For each $x \in M$ there exists an open neighborhood $U$ of $x$ in $M$, a parallel anti-linear involution $\nu$ on $E|_U$ and a symmetric bilinear form $\tilde{b}$ on $E$ such that, for $z \in \mathbb{C}$, the family of fiberwise symmetric bilinear forms

\begin{equation}
(38) \quad b_z := b + \tilde{z}b,
\end{equation}

has the following properties.

(i) $b_z$ is fiberwise nondegenerate for all $z \in \mathbb{C}$ with $|z| \leq \sqrt{2}$,

(ii) $b_z^{-1}(v_e, v_e) = b_z^{-1}(e, e)$, for all $s \in \mathbb{R}$ and $e \in E|_U$,

(iii) $b_z^{-1}(e, v_e) > 0$ for all $s \in \mathbb{R}$, $|s| \leq 1$ and $0 \neq e \in E|_U$.

**Proof.** Since flat vector bundles are locally trivial, there exists a neighborhood $V$ of $x$ and a parallel complex anti-linear involution $\nu$ on $E|_V$. Moreover, since $b$ is nondegenerate and $\nu$ an involution, we can assume without loss of
generality that \( \nu \) can be chosen to be compatible with \( b \) at the fiber \( E_x \) over \( x \), such that
\[
b_x(\nu e_1, \nu e_2) = b_x(e_1, e_2) \quad \text{for all} \quad e_i \in E_x
\]
and
\[
b_x(\nu e, e) > 0 \quad \text{for all} \quad 0 \neq e \in E_x.
\]
Consider
\[
b^{\text{Re}}(e_1, e_2) := \frac{1}{2} \left( b(e_1, e_2) + b(\nu e_1, \nu e_2) \right),
\]
\[
b^{\text{Im}}(e_1, e_2) := \frac{1}{2i} \left( b(e_1, e_2) - b(\nu e_1, \nu e_2) \right),
\]
as symmetric bilinear forms on \( E|_V \). In particular, note that by construction
\[
(39) \quad b|_V = b^{\text{Re}} + i b^{\text{Im}} \quad \text{with} \quad b^{\text{Im}}|_{E_x} = 0,
\]
(40) \quad \overline{b^{\text{Re}}(\nu e_1, \nu e_2)} = b^{\text{Re}}(e_1, e_2) \quad \text{and} \quad \overline{b^{\text{Im}}(\nu e_1, \nu e_2)} = b^{\text{Im}}(e_1, e_2),
for all \( e_i \in E|_V \). Now, choose an open neighborhood \( U \subset V \) of \( x \) and a compactly supported smooth function \( \lambda : V \to [0,1] \) such that \( \lambda|_U = 1 \).
Thus, by extending \( \lambda \) by zero to \( M \), we set
\[
(41) \quad \tilde{b} := \lambda b^{\text{Im}},
\]
as a globally defined symmetric bilinear form on \( E \). Using
\[
b_{s-1}|_U = \left( b + (s - i)\tilde{b} \right)|_U = b|_U + (s - i)b^{\text{Im}}|_U = b^{\text{Re}}|_U + sb^{\text{Im}}|_U
\]
and (40) we immediately obtain (ii). In turn, (ii) implies
\[
\overline{b_{s-1}(\nu e, e)} = b_{s-1}(\nu e, e)
\]
and hence \( b_{s-1}(\nu e, e) \) is real for all \( s \in \mathbb{R} \) and \( e \in E|_U \). Finally, by the formula (38) defining \( b_z \) at \( x \), we have \( b^{\text{Im}}|_x = 0 \) and therefore
- \( b_z|_x \) is nondegenerate,
- \( b_{s-1}|_x(\nu e, e) = b_z(\nu e, e) > 0 \) for all \( 0 \neq e \in E_x \),
from which (i) (resp. (iii)) follows by taking \( |z| \leq \sqrt{2} \) (resp. \( |s| \leq 1 \)) and then choosing the support of \( \lambda \) small enough around \( x \).

The following Proposition provides the key argument in the proof of Theorem \( \Box \) below.

**Proposition 6.** Let \( [\Delta, \mathcal{B}]^{\text{E,g,b}}_{(M, \partial_\pm M, \partial_\pm M)} \) be the bilinear boundary value problem under absolute and relative boundary conditions on \( (M, \partial_+ M, \partial_- M) \). Then, for each \( x \in M \), there exist \( \{b_z\}_{z \in \mathbb{C}} \) a family of fiberwise symmetric bilinear forms on \( E \), and \( \{h_s\}_{s \in \mathbb{R}} \) a family of fiberwise sesquilinear Hermitian forms on \( E \) such that

(i) \( b_z \) is fiberwise nondegenerate for all \( z \in \mathbb{C} \) such that \( |z| \leq \sqrt{2} \).

(ii) \( h_s \) is fiberwise positive definite Hermitian form for \( s \in \mathbb{R} \) with \( |s| \leq 1 \).
(iii) For each \( s \in \mathbb{R} \) with \(|s| \leq 1\), consider \([\Delta, \Omega_B]_{(M, \partial_+ M, \partial_- M)}^{E,g,h_s}\) the corresponding Hermitian boundary value problem. Then, there exists a neighborhood \( U \) of \( x \) such that
\[
\Delta_{E,g,b_{s-1}}|U = \Delta_{E,g,h_s}|U \quad \text{and} \quad \mathcal{B}_{E,g,b_{s-1}}|U = \mathcal{B}_{E,g,h_s}|U.
\]

Proof. By Lemma 6(i), for each \( x \in M \), there exists a globally defined fiberwise symmetric bilinear form \( \tilde{b} \) on \( E \) such that the formula \( b_z := b + \tilde{b} \) in (38) defines a family of fiberwise nondegenerate symmetric bilinear forms on \( E \), satisfying the required property in (i). In addition, we know that for each \( x \in M \), there exist an open neighborhood \( V \) of \( x \) and a parallel complex anti-linear involution \( \nu \) on \( E|_V \). By Lemma 6(i)-(ii), we also know that we can find \( U \subset V \) a small enough open neighborhood of \( x \), such that \( b_{s-1} \) satisfies the conditions (i) and (ii) on \( E|_U \), for \(|s| \leq 1\). Hence, by using the formula in (38), we obtain a fiberwise positive definite Hermitian form compatible with \( b_{s-1} \) on \( E|_U \) given by \( h^U_s = b_{s-1}(e_1, e_2) \). Now we extend \( h^U_s \) to a (positive definite) Hermitian form on \( E \) as follows. We take any arbitrary Hermitian form on \( E \) and consider the finite open covering \( \{U_0', U_1', \ldots, U_N'\} \) of \( M \), with \( U_0' := U \), together with a subordinate partition of unity \( \{f_j\}_{j=1}^N \). If \( h_j' = h'|_{U_j} \), then \( h_s := f_0 h^U_s + \sum_{j=1}^N f_j h'_j \) globally defines a fiberwise positive definite Hermitian form on \( E \), as the space of Hermitian forms on \( E \) is a convex space. This proves (ii). Then, (iii) follows from Lemma 8. \( \square \)

2.4. Heat trace asymptotics for bilinear boundary value problems.

Lemma 10. Let \( O \) be an open connected subset in \( \mathbb{C} \) and \( \{z \mapsto b_z\}_{z \in U} \) a holomorphic family of fiberwise nondegenerate symmetric bilinear forms on \( E \). For the bordism \((M, \partial_+ M, \partial_- M)\) consider \( \{[\Delta, \Omega_B]^{E,g,b_z}_{(M, \partial_+ M, \partial_- M)}\}_{z \in O} \), the family of boundary value problems corresponding to bilinear Laplacians under absolute/relative boundary conditions, together with their \( L^2 \) realizations denoted by \( \Delta_{E,b_z} \). Then, for each \( \psi \in \text{End}(\Lambda^t M \otimes E) \), the map
\[
z \mapsto \lim_{t \to 0} ([\text{STr} (\psi \exp(-t\Delta_{E,b_z})))
\]
is holomorphic on \( O \).

Proof. By compactness, we may assume without loss of generality that \( \psi \) is compactly supported in the interior of a sufficiently small open set \( U \) in \( M \).

Remark that the function \( z \mapsto b_z^{-1} \) is holomorphic, since \( z \mapsto b_z \) is a holomorphic family of fiberwise nondegenerate bilinear forms in \( z \in O \). Then, as it can directly be checked by construction of the bilinear Laplacian in (2), and the boundary operators in (5), the assignments \( z \mapsto \Delta_{E,g,b_z} \) and \( z \mapsto \mathcal{B}_{E,g,b_z} \) respectively define holomorphic functions in \( z \in O \). Therefore, the coefficients of the symbols of \( \Delta_{E,g,b_z} \) and \( \mathcal{B}_{E,g,b_z} \) are holomorphic functions in \( z \in O \).
$O$. Now, the expression $\text{LIM}_{t \to 0}(\text{STr}(\psi \exp(-t\Delta_{B,b})))$ is computed with the formula (23), by integrating the complex-valued function $\text{STr}(\psi \cdot \epsilon_m(\Delta_{E,g,b})))$ over $U$, and the complex-valued function $\text{STr}(\nabla_x^k \psi \cdot \epsilon_{m,k}(\Delta_{E,g,b}, \mathcal{B}_{E,g,b}))$ over $U \cap \partial M$. Since $\epsilon_m(\Delta_{E,g,b})$ are locally computable endomorphism invariants, the value of $\text{STr}_x(\psi_x \cdot \epsilon_m(\Delta_{E,g,b}))$ can be computed inductively by using explicit formulas as a universal polynomial in terms of (finite number of the derivatives of) the coefficients of the symbol of $\Delta_{E,g,b}$, whenever these are given in local coordinates around at $x \in M$, see [24, Theorem 3] and [23, formulas (3)-(6) and Lemma 1], see also [14, Section 2.6]. In the same token, since $\epsilon_{m,k}(\Delta_{E,g,b}, \mathcal{B}_{E,g,b})$ are locally computable endomorphism invariants on the boundary, the value of $\text{STr}_y(\nabla_{\omega}^k \psi)_y \cdot \epsilon_{m,k}(\Delta_{E,g,b}, \mathcal{B}_{E,g,b})$ is expressible, by inductively solving certain systems of ordinary differential equations, as a universal polynomial in terms of (finite number of the derivatives of) the coefficients of the symbols of $\Delta_{E,g,b}$ and $\mathcal{B}_{E,g,b}$, whenever these are given in local coordinates around at $y \in \partial M$, see [24, Theorem 3] and [23, formulas (9)-(14) and Lemma 2], see also [14, Section 2.6]. Thus the mappings $z \mapsto \text{STr}_x(\epsilon_m(\Psi, \Delta_x)_x)$ and $z \mapsto \text{STr}_x(\epsilon_{m,k}(\Psi, \Delta_x, \mathcal{B}_x)_x)$ are holomorphic on $O$ for each $x \in U$. Finally, for Morera’s Theorem, the integral of a function depending holomorphically on a parameter $z$, also depends holomorphically on $z$, that is, the function $z \mapsto \text{LIM}_{t \to 0}(\text{STr}(\psi \exp(-t\Delta_{B,b})))$ depends holomorphically on $z \in O$.

**Theorem 2.** For $(M, \partial_+ M, \partial_- M)$ consider the bilinear boundary value problem $[\Delta, \mathcal{B}]^{E,g,b}_{(M, \partial_+ M, \partial_- M)}$, together with its $L^2$-realization $\Delta_{B,b}$. If $\phi$, $\xi$ and $\Psi$ are as in Proposition 4, then

$$\text{LIM}_{t \to 0}(\text{STr}(\phi \exp(-t\Delta_{B,b}))) = f_\partial M \text{Tr}(\phi) e(M,g) + (-1)^{m-1} f_\partial_+ M \text{Tr}(\phi)^* e_b(\partial M,g) - f_\partial_- M \text{Tr}(\phi)^* e_b(\partial M,g),$$

(42)

and

$$\text{LIM}_{t \to 0}(\text{STr}(\psi \exp(-t\Delta_{B,b}))) = -2 f_\partial M \frac{\partial}{\partial x} \bigg|_{x=0} \delta(\partial M,g + \tau \xi) \wedge (\nabla^E, b)$$

$$- 2 f_\partial_+ M \frac{\partial}{\partial x} \bigg|_{x=0} i^* \delta_b(\partial M,g + \tau \xi) \wedge (\nabla^E, b)$$

$$+ \text{rank}(E) f_\partial_+ M \frac{\partial}{\partial x} \bigg|_{x=0} i^* B(\partial M,g + \tau \xi)$$

$$- 2(-1)^m f_\partial_- M \frac{\partial}{\partial x} \bigg|_{x=0} i^* \delta_b(\partial M,g + \tau \xi) \wedge (\nabla^E, b)$$

$$+ (-1)^{m+1} \text{rank}(E) f_\partial_- M \frac{\partial}{\partial x} \bigg|_{x=0} i^* B(\partial M,g + \tau \xi).$$

(43)

**Proof.** By compactness of $M$, it suffices to show that each point $x \in M$ admits a neighborhood $U$ so that the formulas above hold for all $\phi$ with $\text{supp}(\phi) \subset U$ and $\xi$ with $\text{supp}(\xi) \subset U$. For each $x \in M$, choose $b_x = b + \xi$, $h_x$ and $U$ as in Proposition 6, with $\text{supp}(\phi) \subset U$. By Proposition 6 (iii), we
obtain $\text{LIM}_{t \to 0} \text{STr}(\phi \exp(-t \Delta_{B,b_{s-1}})) = \text{LIM}_{t \to 0} \text{STr}(\phi \exp(-t \Delta_{B,b_{s}}))$, for all $|s| \leq 1$, for these quantities depend on the geometry over $U$ only. From Theorem 1 we have

$$\text{LIM}_{t \to 0} \text{STr}(\phi \exp(-t \Delta_{B,b_{s-1}})) = \int_{M} \text{Tr}(\phi)_{e(M,g)} + (-1)^{m-1} \int_{\partial_{+} M} \text{Tr}(\phi)^{e_{b}(\partial M,g)}$$

for all $|s| \leq 1$. Now, since the function $z \mapsto \text{LIM}_{t \to 0} \text{STr}(\phi \exp(-t \Delta_{B,b_{z}}))$ depends holomorphically on $z$ (see Lemma 10), that the right hand side of the equality above is constant in $z$, and that the domain of definition of $z$ contains an accumulation point, these formulas are extended by analytic continuation to

$$\text{LIM}_{t \to 0} \text{STr}(\phi \exp(-t \Delta_{B,b_{z}})) = \int_{M} \text{Tr}(\phi)_{e(M,g)} + (-1)^{m-1} \int_{\partial_{+} M} \text{Tr}(\phi)^{e_{b}(\partial M,g)}$$

for all $|z| \leq \sqrt{2}$. After setting $z = 0$ we obtain the desired identity in (42). We now show (43). Similarly take $\xi$ with $\text{supp}(\xi) \subset U$, using Proposition 6 (iii), we obtain

$$\text{LIM}_{t \to 0} \text{STr}(-\Psi \exp(-t \Delta_{B,b_{s-1}})) = \text{LIM}_{t \to 0} \text{STr}(-\Psi \exp(-t \Delta_{B,b_{s}}))$$

for all $|s| \leq 1$, for these quantities depend on the geometry over $U$ only. Then, we apply Theorem 1 to the right hand side of the equality in (44) we conclude

$$\text{LIM}_{t \to 0} \text{STr}(-\Psi \exp(-t \Delta_{B,b_{s-1}})) = -2 \int_{M} \frac{\partial}{\partial t}_{t=0} \tilde{e}(M,g_{g} + t\omega) \wedge (\nabla_{E} b_{s-1})$$

$$-2 \int_{\partial_{+} M} \frac{\partial}{\partial t}_{t=0} \tilde{e}_{b}(\partial M,g_{g} + t\omega) \wedge (\nabla_{E} b_{s-1})$$

$$+ \text{rank}(E) \int_{\partial_{+} M} \frac{\partial}{\partial t}_{t=0} \tilde{e}_{b}(\partial M,g_{g} + t\omega) \wedge (\nabla_{E} b_{s-1})$$

$$-2(-1)^{m} \int_{\partial_{+} M} \frac{\partial}{\partial t}_{t=0} \tilde{e}_{b}(\partial M,g_{g} + t\omega) \wedge (\nabla_{E} b_{s-1})$$

$$+ (-1)^{m+1} \text{rank}(E) \int_{\partial_{+} M} \frac{\partial}{\partial t}_{t=0} \tilde{e}_{b}(\partial M,g_{g} + t\omega)$$

for all $|s| \leq 1$. Now, the function $z \mapsto \text{LIM}_{t \to 0} \text{STr}(\phi \exp(-t \Delta_{B,b_{z}}))$ on the left of (45) depends holomorphically on $z$ see Lemma 10. On the other hand the long expression on the right hand side of the equality above in (45) is also a holomorphic function in $z \in \mathbb{C}$ with $|z| \leq \sqrt{2}$, since it can be formally considered as the composition of constant functions (in $z$) and the function $z \mapsto \omega(\nabla_{E}, b_{z}) = -\frac{i}{2} \text{Tr}(b_{z}^{-1} \nabla_{E} b_{z})$, which is holomorphic, since by Proposition 6 the bilinear form $b_{z}$ in (38) is fiberwise nondegenerate for $|z| \leq \sqrt{2}$. Then the identity in (45) can be analytically extended to
\[
\lim_{t \to 0} \text{Str} \left( -\Psi \exp(-t\Delta_{E,g,b}) \right) = -2 \int_M \left. \frac{\partial}{\partial \tau} \right|_{\tau=0} \tilde{e}(M,g,g+\tau g) \wedge \omega(\nabla^E,b_{z-1}) \\
+ \text{rank}(E) \int_{\partial_- M} \left. \frac{\partial}{\partial \tau} \right|_{\tau=0} i^* B(\partial(M,g+\tau g) \\
- 2(-1)^m \int_{\partial_- M} \left. \frac{\partial}{\partial \tau} \right|_{\tau=0} i^* \tilde{e}(\partial M,g+\tau g) \wedge \omega(\nabla^E,b_{z-1}) \\
+ (-1)^{m+1} \text{rank}(E) \int_{\partial_- M} \left. \frac{\partial}{\partial \tau} \right|_{\tau=0} i^* B(\partial(M,g+\tau g)).
\]

for \( z \in \mathbb{C} \) with \( |z - i| \leq \sqrt{2} \). Finally (43) follows from setting \( z = i \) into (46) and then \( b_0 = b \) follows from (38).

\[\square\]

3. Complex-valued analytic torsion on compact Bordisms

Let \((M,\partial_+ M,\partial_- M)\) be a Riemannian bordism and \( E \) be complex flat vector bundle over \( M \) endowed with a nondegenerate symmetric bilinear form. Consider \( \Delta_B \) the \( L^2 \)-realization of the bilinear Laplacian acting on \( E \)-valued smooth forms satisfying absolute boundary conditions on \( \partial_+ M \) and relative ones on \( \partial_- M \).

If \( \Omega_{\Delta_B}(0) \) is the 0-generalized eigenspace of \( \Delta_B \), consider the restriction of \( \beta_{g,b} \) to \( \Omega_{\Delta_B}(0) \); this is a non degenerate symmetric bilinear form in view of Proposition 1. By \([4, \text{Lemma 3.3}]\) we obtain a nondegenerate bilinear form on \( \det(\Omega_{\Delta_B}(0)) \), which in turn, by Proposition 3, induces a bilinear form on \( \det(H(M,\partial_- M;E)) \), which we denote by \( \tau(0)_{E,g,b} \). Let us denote by

\[\Delta^g_{E,q} := \Delta_B|_{\Omega^q_{\Delta_B}(M;E)(0)^c}|_B\]

the restriction of \( \Delta_B \) to \( \Omega^q_{\Delta_B}(M;E)(0)^c|_B \), i.e., the space of smooth differential forms of degree \( q \) which are not in \( \Omega_{\Delta_B}(M;E)(0) \) but satisfy boundary conditions. Lemma 2 permits us to choose a non-zero Agmon angle avoiding the spectrum of \( \Delta^g_{E,q} \) so that complex powers of the bilinear Laplacian can be defined. Then, the function \( s \mapsto (\Delta^g_{E,q})^{-s} \) associates to each \( s \in \mathbb{C} \), with \( \text{Re}(s) > \dim(M)/2 \), an operator of Trace class and it extends to a meromorphic function on the complex plane which is holomorphic at 0, see \([14], [22], [23] \) and \([24] \) or more generally, for pseudo-differential boundary value problems, see \([15] \text{Chapter 4} \). The \( \zeta \)-regularized determinant of \( \Delta^g_{E,q} \) is defined as

\[\det'(\Delta^g_{E,q}) := \exp\left(-\frac{\partial}{\partial s}\right|_{s=0} \text{Tr}((\Delta^g_{E,q})^{-s})).\]

From Lemma 2 this determinant does not depend on the choice of the Agmon’s angle. By using \([4, \text{Lemma 3.3}] \), the complex-valued Ray–Singer torsion on the bordism \((M,\partial_+ M,\partial_- M)\) is defined as the bilinear form on the
Proof. The method described in [4, Section 6] leading to the infinite simal $M$ supported on a compactly supported in the interior of $LIM \ (54)$], the problem of computing this infinitesimal variation boils down to with boundary; this was also used in [25]. In particular, by [4, formula variation of the torsion in the closed situation also holds in the situation where $\omega$ Theorem 3. $E$ generalizes the formulas obtained in [4] in the case without boundary and they are based on the corresponding ones for the Ray–Singer determinant line $M$. Consider $g_u$ a smooth one-parameter family of Riemannian metrics on $M$ and $b_u$ a smooth one-parameter family of a fiber wise nondegenerate symmetric bilinear forms on $E$ and denote by $g_t$ and $b_t$ their corresponding infinitesimal variations. Let $\tau_{E,g_t,b_t}$ the associated family of complex valued analytic torsions. Then, we have the following logarithmic derivative

$$\frac{\partial}{\partial \omega}igg|_u \left( \tau_{E,g_t,b_t} \right)^2 = E(b_u,g_u) + \bar{E}(b_u,g_u) + B(g_u),$$

where $\omega(\nabla E,b) := -\frac{1}{2} \text{Tr}(b^{-1} \nabla E b)$ is the Kamber–Tondeur form, see [4] Section 2.4 and

$$E(b_u,g_u) := \int_M \text{Tr}(b^{-1} b_u)e(M,g)+(-1)^{m-1} \int_{\partial_- M} \text{Tr}(b_u^{-1} b_u) e_b(\partial M,g_u)$$

$$- \int_{\partial_- M} \text{Tr}(b^{-1} b_u) e_b(\partial M,g_u).$$

$$\bar{E}(b_u,g_u) := -2 \int_M \frac{\partial}{\partial \omega} |_{\omega=0} e(M,g_u+g_u-t_b u) \wedge \omega(\nabla E,b_u)$$

$$-2 \int_{\partial_- M} \frac{\partial}{\partial \omega} |_{\omega=0} i^{*}_b e_b(\partial M,g_u+g_u-t_b u) \wedge \omega(\nabla E,b_u)$$

$$-2(-1)^m \int_{\partial_- M} \frac{\partial}{\partial \omega} |_{\omega=0} i^{*}_b e_b(\partial M,g_u+g_u-t_b u) \wedge \omega(\nabla E,b_u),$$

$$B(g_u) := \text{rank}(E) \int_{\partial_- M} \frac{\partial}{\partial \omega} |_{\omega=0} i^{*}_b B(\partial M,g_u+g_u-t_b u)$$

$$+(-1)^{m+1} \text{rank}(E) \int_{\partial_- M} \frac{\partial}{\partial \omega} |_{\omega=0} i^{*}_b B(\partial M,g_u+g_u-t_b u),$$

Proof. The method described in [4, Section 6] leading to the infinitesimal variation of the torsion in the closed situation also holds in the situation with boundary; this was also used in [25]. In particular, by [4] formula (54), the problem of computing this infinitesimal variation boils down to computing $\text{LIM}_{t \to 0}(\text{STr}(\phi \exp(-t \Delta_B)))$ and $\text{LIM}_{t \to 0}(\text{STr}(-\Psi \exp(-t \Delta_B)))$ associated to $\Delta_B$ with $\phi = b_u^{-1} g_u$ and $\xi = g_u^{-1} \dot{g}_u$ respectively given by [42] and [43] in Theorem 2. □
The material in this section, entirely contained in §3, summarizes the background needed to understand the characteristic forms appearing in the anomaly formulas in Sections 2 and 3.

4.1. The Berezin integral and Pfaffian. For $A$ and $B$ two $\mathbb{Z}_2$ graded unital algebras, $A \otimes B$ denotes their $\mathbb{Z}_2$-graded tensor product. We write $A := A \otimes I, \hat{B} := I \otimes B$ and $\land := \otimes$, so that $A \land \hat{B} = A \otimes B$.

For $W$ and $V$ finite dimensional vector spaces of dimension $n$ and $l$ respectively, where $W$ is endowed with a Hermitian product $\langle \cdot , \cdot \rangle$ and $V'$ the dual of $V$, the Berezin integral $\int^B : \Lambda V' \land \Lambda(W') \to \Lambda V' \otimes \Theta_W$ associates to each element $\alpha \land \hat{\beta}$ in the $\mathbb{Z}_2$-graded tensor product $\Lambda V' \land \Lambda(W')$ the element $C_B \beta^b, (w_1, \ldots, w_n)$ in $\Lambda V' \otimes \Theta_W$, where $\{w_i\}_{i=1}^n$ is an orthonormal basis of $W$, $\Theta_W$ is the orientation bundle of $W$ and the constant $C_B := \lambda n (n+1)/2 \pi - n/2$. Now, each antisymmetric endomorphism $K$ of $W$ can be identified with the unique element $K := \langle \cdot , K \cdot \rangle$ in $\Lambda(W')$ given by

$$K := \frac{1}{2} \sum_{1 \leq i,j \leq n} \langle w_i, K w_j \rangle \hat{w}_i \land \hat{w}_j,$$

where $\{w^i\}_{i=1}^n$ is the corresponding dual basis in $W'$. Then, $\text{Pf}(K/2\pi)$, the Pfaffian of $K/2\pi$, is defined by

$$\text{Pf} (K/2\pi) := \int^B \exp(K/2\pi).$$

Remark that $\text{Pf}(K/2\pi) = 0$, if $n$ is odd. By standard fiberwise considerations the map $\text{Pf}$ is extended for vector bundles over $M$.

4.2. Certain characteristic forms on the boundary. We denote by $g := g^T_M$ (resp. $g^0 := g^{T,0}_M$) the Riemannian metric on $TM$ (resp. on $T\partial M$ and induced by $g$), by $\nabla$ (resp. $\nabla^0$) the corresponding Levi-Civita connection and by $R^T_M$ (resp. $R^{T,0}_M$) its curvature. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame of $TM$ with the property that near the boundary, $e_m = \varsigma_m$, i.e., the inwards pointing geodesic unit normal vector field on the boundary. The corresponding induced orthonormal local frame on $T\partial M$ will be denoted by $\{e_\alpha\}_{\alpha=1}^{m-1}$. As usual, the metric is used to fix $\{e_i\}_{i=1}^m$ (resp. $\{e_\alpha\}_{\alpha=1}^{m-1}$) the corresponding dual frame of $T^*M$ (resp. $T^*\partial M$).

With the notation in Section 4.1 a smooth section $w$ of $\Lambda T^*M$ is identified with the section $w \hat{\circ} 1$ of $\Lambda T^*M \otimes \Lambda T^*M$, whereas $\hat{w}$ is in one-to-one correspondence with the section $1 \hat{\circ} \hat{w}$ of $\Lambda T^*M \otimes \Lambda T^*M$. Here one considers the Berezin integrals $\int^{BM} : \Gamma(M; \Lambda T^*M \land \Lambda T^*M) \to \Gamma(M; \Lambda T^*M \otimes \Theta_M)$ and $\int^{B\partial M} : \Gamma(\partial M; \Lambda T^*\partial M \land \Lambda(T^*\partial M)) \to \Gamma(\partial M; \Lambda T^*\partial M \otimes \Theta_{\partial M})$ which can
be compared under the taken convention for the induced orientation bundle on the boundary discussed in Section [1].

The curvature $R^{TM}$ associated to $\nabla$, considered as a smooth section of $\Lambda^2(T^*M) \wedge \Lambda^2(T^*M) \to M$, can be expanded in terms of the frame above as

$$R^{TM} := \frac{1}{2} \sum_{1 \leq k, l \leq m} g^{TM}(e_k R^{TM} e_l) e^k \wedge e^l \in \Gamma(M; \Lambda^2(T^*M) \wedge \Lambda^2(T^*M)).$$

In the same way, one first sets

$$e(M, \nabla^{TM}) := \int_{BM} \exp(-\frac{1}{2} R^{TM}),$$

$$e(\partial M, \nabla^{T\partial M}) := \int_{B\partial M} \exp(-\frac{1}{2} R^{T\partial M}),$$

$$e_b(\partial M, \nabla^{TM}) := (-1)^{m-1} \int_{B\partial M} \exp(-\frac{1}{2} (R^{TM} |_{\partial M})) \sum_{k=0}^{\infty} \frac{s^k}{2^k (\frac{k}{2} + 1)},$$

$$B(\partial M, \nabla^{TM}) := -\int_{0}^{1} \frac{du}{u} \int_{B\partial M} \exp(-\frac{1}{2} R^{T\partial M} - u^2 S^2) \sum_{k=1}^{\infty} \frac{(uS)^k}{2^k (\frac{k}{2} + 1)}.$$

### 4.3. Secondary characteristic forms.

Now, given $\{g_s := g_s^{TM}\}_{s \in \mathbb{R}}$ (resp. $\{g_s^{\partial M} := g_s^{T\partial M}\}_{s \in \mathbb{R}}$) a smooth family of Riemannian metrics on $TM$ (resp. the induced family of metrics on $T\partial M$), we sketch the construction given in [8] for the (secondary) Chern–Simons forms $e(M, g_0, g_s) \in \Omega^{m-1}(M, \Theta^M)$ and $e_b(\partial M, g_0, g_s) \in \Omega^{m-2}(\partial M, \Theta^M)$ (see also [7, (4.53)]).

Let $\nabla := \nabla^{TM}_g$ and $R := R^{TM}_g$ (resp. $\nabla^{T\partial M}_g$ and $R^{T\partial M}_g$) be the Levi-Civita connections and curvatures on $TM$ (resp. $T\partial M$) associated to the metrics $g$ (resp. $g^{\partial M}$). Consider the deformation spaces $\tilde{M} := M \times \mathbb{R}$ (resp. $\tilde{M} = \partial M \times \mathbb{R}$) with $\pi_{\tilde{M}} : \tilde{M} \to \mathbb{R}$ and $p_{\tilde{M}} : \tilde{M} \to M$, its canonical projections (resp. $\pi_{\tilde{\partial M}} : \tilde{\partial M} \to \mathbb{R}$ and $p_{\tilde{\partial M}} : \tilde{\partial M} \to \partial M$). If $\tilde{i} := i \times d\tilde{r}$ : $\tilde{M} \to \tilde{M}$ is the natural embedding induced by $i : \partial M \to M$, then $\pi_{\tilde{\partial M}} = \pi_{\tilde{M}} \circ \tilde{i}$. The vertical bundle of the fibration $\pi_{\tilde{M}} : \tilde{M} \to \mathbb{R}$ (resp. $\pi_{\tilde{\partial M}} : \tilde{\partial M} \to \mathbb{R}$) is given as the pull-back of the tangent bundle $TM \to M$ along $p_{\tilde{M}} : \tilde{M} \to M$ (resp. the pull-back of $T\partial M \to \partial M$ along $p_{\tilde{\partial M}} : \tilde{\partial M} \to \partial M$), i.e.,

$$\mathcal{T}M := p_{\tilde{M}}^*TM \to \tilde{M}, \quad (\text{resp. } \mathcal{T}\partial M := p_{\tilde{\partial M}}^*T\partial M \to \tilde{\partial M})$$


and it is considered as a subbundle of $T\tilde{M}$ (resp. $T\partial M$). The bundle $\mathcal{T}M$ (resp. $\mathcal{T}\partial M$) in (49) is naturally endowed with a Riemannian metric $g_{\mathcal{T}M}$ which coincides with $g_s$ (resp. $g^{\partial}_{\mathcal{T}M}$) at $M \times \{s\}$ (resp. $\partial M \times \{s\}$), and for which there exists a unique natural metric connection $\nabla_{\mathcal{T}M}$ (resp $\nabla_{\mathcal{T}\partial M}$) and the corresponding curvature tensor is denoted by $R_{\mathcal{T}M}$ (resp $R_{\mathcal{T}\partial M}$). For more details, see [8, Section 1.5, (1.44) and Definition 1.1], and also [7, (4.50) and (4.50)]. Near the boundary, consider orthonormal frames of $\mathcal{T}M$ such that $e_{\alpha}(y,s) = \varsigma$ in for each $y \in \partial M$ with respect to the metric $g_s$. Finally, by using the formalism described above associated to $R_{\mathcal{T}M}$ and $R_{\mathcal{T}\partial M}$ to define (48), if $\text{incl}_s : M \to \tilde{M}$ is the inclusion map given by $\text{incl}_s(x) = (x,s)$ for $x_0 \in M$ and $s \in \mathbb{R}$, then, one defines

$$\bar{e}(M,g_0,g_r) := \int_0^r \text{incl}_s^*(\iota(\frac{\partial}{\partial s}) e(\tilde{M},\nabla_{\mathcal{T}M})) \, ds$$

$$\bar{e}_b(\partial M,g_0,g_r) := \int_0^r \text{incl}_s^*(\iota(\frac{\partial}{\partial s}) e_b(\partial \tilde{M},\nabla_{\mathcal{T}M})) \, ds,$$

where $\iota(X)$ indicates the contraction with respect to the vector field $X$.

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