Uniform Convergence of the Eigenfunction Expansions of Distributions Associated with the Polyharmonic Operator on Closed Domain

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Abstract. In the solving the boundary value problems of equations of mathematical physics in bounded domains it is important to adjust the solution on the closed domain. This leads to the investigation of convergence and summability problems near the boundary of the domain. In the present paper we study this problem for the eigenfunction expansions associated with the polyharmonic operator.

1. Introduction

In [1] Il’in studied uniform convergence of the spectral expansions $E_j$ associated with the Laplace operator in the Sobolev classes $W^{a, p}_{2\alpha}(\Omega)$ and found sufficient conditions of uniform convergence for the Riesz means $E^j_f$ as $a + s \geq \frac{N-1}{2}$, $pa > N$. Later, Alimov in [2] established the uniform convergence of the Riesz means for the functions from the Nikol'skii classes $H^{a, p}_{\alpha}(\Omega)$. The convergence of the spectral decompositions of the Laplace operator on closed domain firstly investigated by Il’in (see in [1]) and then Moiseev [3] proved uniformly convergence of the eigenfunction expansions of the functions from $W^{2, p}_{\frac{N+1}{2}}(\Omega)$ on closed domain $\overline{\Omega}$. Furthermore, in [4] Eskin considered the $2m$ order elliptic differential operator with the Lopatinsky boundary condition and proved uniformly convergent of the spectral expansions of the functions from $W^{\frac{N+1}{2}+1, p}_{\epsilon}(\Omega)$, $\epsilon > 0$ on closed domain $\overline{\Omega}$. The uniform convergence of the eigenfunction expansions of the Laplace operator in closed domain was considered by Rakhimov [5], where he showed that the conditions $a + s \geq \frac{N-1}{2}$, $pa = N$ secure the uniform convergence of the expansions in closed domain for the continuous functions from the Nikol'skii classes $H^{a, p}_{\alpha}(\Omega)$. Uniform convergence of the Riesz means of eigenfunction expansions of countinuous function in case of Laplace operator studied in [6] and for the general elliptic operator in [7]. The problems of the uniformly the Riesz summability on a closed domain for the Schredinger operator with the singular potential studied in [8] and [9].
2. The spaces of test functions

Let $\Omega \subset R^2$ be a domain with smooth boundary $\partial \Omega$. Let define the classes of the functions (see in [10]). We say that a function $f(x, y) \in L_p(\Omega)$ belongs to the $H^\alpha_\rho(\Omega)$, if for any $h = (h, k) \in R^2$ and for all integers $\alpha, \beta$ satisfying $\alpha + \beta = l$:

$$\left| \partial^\alpha_x \partial^\beta_y f(x + h, y + k) - 2 \partial^\alpha_x \partial^\beta_y f(x, y) + \partial^\alpha_x \partial^\beta_y f(x - h, y - k) \right|_{L_p(\Omega)} \leq C(h^2 + k^2)^\frac{\rho}{2}.$$

where $\alpha$ is written as $\alpha = l + \sigma$, $l$ - positive integer and $0 < \sigma \leq 1$. Using the notation $\Delta^2_h f(x, y) = f(x + h, y + k) - f(x, y) + f(x - h, y - k)$, a norm in $H^\alpha_\rho(\Omega)$ is defined by

$$\|f\|_{p, \alpha} = \|f\|_{L_p(\Omega)} + \sum_{\alpha + \beta = l} \sup_{h \in \Omega} (h^2 + k^2)^\frac{\rho}{2} \|\Delta^2_h \partial^\alpha_x \partial^\beta_y f(x, y)\|_{L_p(\Omega)}.$$

The closure of the space $C^\alpha_r(\Omega)$ in the norm of $H^\alpha_\rho(\Omega)$ denoted by $\dot{H}^\alpha_\rho(\Omega)$.

Let $a \geq 0$ an integer and a function $\gamma(t)$ defined and continuous for $1 \leq t \leq \infty$, positive and vanishing in $t \to \infty$. Consider an operator

$$l_{r, \gamma} f = F^{-1} \left[ (1 + |\omega|^2)^{-a} \gamma \left( \frac{1}{1 + |\omega|^2} \right) \cdot Ff \right]$$

Then the generalized Besov spaces $B^\alpha_\rho(\Omega)$ defined as

$$B^\alpha_\rho(\Omega) = l_{r, \gamma} B^0_{p, \rho}(\Omega)$$

Here $F$ and $F^{-1}$ direct and inverse Fourier transformations in $J'(R^N)$.

Now we define the generalized Sobolev spaces in the finite domain $\Omega$. Unless whole space $R^N$ in the finite domain the definition not involving the Fourier transformation.

For any square integrable in the domain $\Omega$ function $u$ and for any positive number $r$ by $\omega(u, r)$ we denote the second order modulus of continuity in the metric of the space $L_2(\Omega)$

$$\omega(u, r) = \sup_{|h| < r} \|u(x + h) - 2u(x) + u(x + h)\|_{L_2(\Omega)}$$

where $\Omega_h = \{ y \in \Omega: \text{dist}(y, \partial \Omega) < h \}$.

Let $a > 0$, $a = \mu + \varepsilon$, where $\mu \geq 0$ integer number and $0 < \varepsilon \leq 1$. Let $\beta(\delta) = \ln \delta$, $1 \leq \delta < \infty$. Then by $W^{\alpha, \beta}_{2}(\Omega)$ we denote the space of functions from $L_2(\Omega)$ with the finite norm

$$\|u\|_{a, \beta} = \|u\|_0 + \sum_{|\alpha| = \mu} \sup_{0 < r < 1} \frac{\omega(D^\alpha u, r)}{\delta^\varepsilon \beta(\frac{1}{r})}$$

here $\alpha$ is multiindex, i.e. $N$ - dimensional vector with non negative integer components $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ called length of multiindex. Then $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$ denote the generalized partial derivative.
3. Eigenfunction expansion associated with the polyharmonic operator

Let consider the polyharmonic operator \((-\Delta)^p\) with the domain 
\[ D_B = \{ u \in W_2^{2p}(\Omega) : u \big|_{\partial\Omega} = \Delta u \big|_{\partial\Omega} = \ldots = \Delta^{p-1} u \big|_{\partial\Omega} = 0 \}. \]
The polyharmonic operator \((-\Delta)^p\) with the domain of definition 
\[ D_B \] is symmetric and nonnegative. By Fredrichs Theorem it can be extended to 
\[ L_2(\Omega) \] as self-adjoint operator, which again denote by \((-\Delta)^p\). It is well known that self-adjoint polyharmonic operator \((-\Delta)^p\) has a set of eigenfunctions \(\{u_{mm}(x,y)\}\), that is complete in 
\[ L_2(\Omega). \]
We denote by \(\{\lambda_{nm}\}\) the set of eigenvalues of polyharmonic operator in 
\(\Omega\) then we have
\[ (-\Delta)^p u_{mm}(x,y) - \lambda_{nm} u_{mm}(x,y) = 0, \quad (x,y) \in \Omega, \]
with the boundary conditions
\[ u_{mm} |_{\partial\Omega} = \Delta u_{mm} |_{\partial\Omega} = \ldots = \Delta^{p-1} u_{mm} |_{\partial\Omega} = 0. \]

Let \(E_s^\lambda f\) be the Riesz means of the eigenfunction expansions associated with the operator above
\[ E_s^\lambda f(x,y) = \sum_{\lambda_{nm} = \lambda} (1 - \frac{\lambda_{nm}}{\lambda})^s f_{nm} u_{nm}(x,y), \quad s \geq 0, \]
where \(f_{nm}\) is the Fourier coefficients of the function \(f\):
\[ f_{nm} = \iint_{\Omega} f(x,y) u_{nm}(x,y) \, dx \, dy, \quad n,m = 1,2,\ldots. \]

The following estimation is established evaluation for the package of eigenfunction on closed domain with respect to the \(\lambda \to \infty\) (Anvarjon, Siti Nor Aini and Gafurjan Ibragimov).

For the eigenfunctions \(\{u_{mm}(x,y)\}\) and eigenvalues \(\lambda_{nm}\) of the polyharmonic operator \((-\Delta)^p\), corresponding to the boundary conditions \(u_{mm} |_{\partial\Omega} = \Delta u_{mm} |_{\partial\Omega} = \ldots = \Delta^{p-1} u_{mm} |_{\partial\Omega} = 0\) we have
\[ \sum_{|\lambda_{nm} - \lambda| \leq \lambda} u_{nm}^2(x,y) \leq O(1) \lambda^{N-1} \ln^2 \lambda, \quad \lambda > 1, \quad (1) \]
uniformly for all \((x,y) \in \overline{\Omega} = \Omega \cup \partial \Omega\).

From this it follows that if \(a + s > \frac{1}{2}, pa > 2, p \geq 1\), then the Riesz means \(E_s^\lambda f(x,y)\) uniformly convergence for the functions from Nikolskii classes \(H^a_p\) in closed domain.

4. Spectral expansions associated with the biharmonic operator

Denote by \(W_2^{\alpha,\beta}(\Omega)\) the space of the linear continuous functionals on \(W_2^{\alpha,\beta}(\Omega)\). The main results of this paper are the proof of the following theorems.
**Theorem 1.** Let a distribution \( f \in W_2^{a,b}(\Omega) \), \( a > 0 \), has a compact support in the two dimensional bounded domain. If \( s \geq \frac{N-1}{2} + a \). Then the relation
\[
\lim_{\lambda \to \infty} E_{\lambda}^{s} f(x,y) = 0
\]
holds uniformly in any compact set \( K \subset \bar{\Omega} \setminus \text{supp} f \).

Note that from the embedding \( W_2^{a}(\Omega) \subset W_2^{a,\beta}(\Omega) \subset W_2^{a,-\varepsilon}(\Omega) \), \( \varepsilon > 0 \), it follows that the condition \( s \geq \frac{N-1}{2} + a \) is precise (see [1]). Moreover from this embedding and the Theorem 1 we get the following theorem for the Sobolev spaces

**Theorem 2.** Let \( f \in \mathcal{E}'(\Omega) \cap W_2^{-a}(\Omega) \), \( a > 0 \), and \( s > \frac{N-1}{2} + a \). Then the relation
\[
\lim_{\lambda \to \infty} E_{\lambda}^{s} f(x,y) = 0
\]
holds uniformly in any compact set \( K \subset \bar{\Omega} \setminus \text{supp} f \).

Let \( f \in C_0^\infty(\Omega) \). Then introduce a function
\[
\beta_d^a(f)(x) = \sum_{n=1}^{\infty} \lambda_n^{-a} \ln \lambda_n u_n(x)
\]
which is infinitely differentiable on the closed domain \( \bar{\Omega} \).

**Lemma 1.** For any function \( f \in C_0^\infty(\Omega) \) the following inequality is valid
\[
\sum_{n=1}^{\infty} \lambda_n^{-a} \ln \lambda_n u_n^2 \leq c \| f \|_{-(a,\beta)}
\]
with the constant in depended from \( f \).

The Lemma 1 can be proven using formula (1). From this we get the following:

**Lemma 2.** Let a function \( \varphi(t) \) defined for \( t \in [1,\infty) \) and \( \sum_{n=1}^{\infty} \varphi^2(n) \leq \text{const} \). Moreover, let \( \varphi(t+s) \leq c \varphi(t) \), \( 0 \leq s \leq 1 \), \( t \geq 1 \). Then the estimate
\[
\sum_{n=1}^{\infty} \varphi(\sqrt{\lambda_n}) \lambda_n^{-2a-\frac{N-1}{2}} u_n(x) \leq c \| f \|_{-(a,\beta)}, \quad a > 0.
\]
is valid uniformly in any compact set \( K \subset \bar{\Omega} \) and for any function \( f \in C_0^\infty(\Omega) \).

Finally from the Lemma 1 and 2, we obtain that if \( f \in C_0^\infty(\Omega) \) and \( s = \frac{N-1}{2} + a \), then
\[
E_{\lambda}^{s} f(x) = O(1) \| f \|_{-(a,\beta)}
\]
is valid on an arbitrary compact set \( K \subset \bar{\Omega} \setminus \text{supp} f \).
Then the statement of the Theorem 1 follows from the fact that a distribution has a compact support and can be approximated by the functions from the space \( C_0^\infty(\Omega) \) in the norm of corresponding generalized Sobolev spaces.

Note that, in the paper [11] it is proved necessary conditions for the summability of the eigenfunction expansions of the distributions from the generalized Sobolev spaces. These questions also discussed in [12]. In the papers [13] and [16] spectral expansions associated with elliptic differential operators in the negative Sobolev spaces are studied. In case of the Hilbert spaces in [13] sharp conditions for the localization are established. In [14], [15] and [17] special methods of summations associated with the elliptic differential operators are studied. In [18] the uniform convergence of the Fourier series studied on a closed domain.

5. Summary

In this paper the uniform convergence of eigenfunction expansions associated with the polyharmonic operator for the distributions from the generalized Sobolev spaces is studied and sharp theorems on the localization on the closed domain proved. In this we studied expansions in eigenfunction with the Navier boundary conditions. Note, that study of the problem for the first and second boundary problems remains open even for the bi-harmonic operator. Moreover, the problem is still open even for the Laplace operator in the case of second and third boundary conditions.

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6. References

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