The method of nonlocal transformations:
Applications to singularly perturbed boundary-value problems

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Abstract. We present a method of numerical integration of singularly perturbed boundary-value problems with a small parameter, based on introducing a new nonlocal independent variable. As a result, we obtain more suitable problems that allow the application of standard fixed-step numerical methods. Two test boundary-value problems for second-order ODEs that have monotonic and non-monotonic exact or asymptotic solutions, expressed in elementary functions, are considered. Comparison of numerical, exact, and asymptotic solutions showed the high efficiency of the method based on generalized nonlocal transformations for solving singular boundary-value problems with a small parameter.

Keywords: singularly perturbed boundary-value problems, nonlocal transformations, boundary layers, nonlinear differential equations, numerical and exact solutions

1. Introduction

Singularly perturbed boundary-value problems with a small parameter, multiplying the highest derivative, are often encountered in applications (see, for example, [1–6]). An important qualitative feature of singularly perturbed boundary-value problems is that for the zero value of a small parameter the order of the differential equation under consideration decreases and some parts of the boundary conditions cannot be satisfied.

Solutions of singularly perturbed boundary-value problems with a small parameter have large gradients in the region of boundary layers, which leads to a loss of convergence of classical finite-difference schemes and makes them of little use or unsuitable for solving problems of this type. Various problems and special methods of the numerical integration for linear and nonlinear differential equations with a small parameter at the highest derivative are presented, for example, in [7–22]. For numerical solution of singularly perturbed boundary-value problems many authors use methods with a piecewise-uniform grid (two-grid methods) that is characterized by a small stepsize in the boundary layer and a large stepsize outside it (see, for example, [11, 17–20, 22]). It is important to note that in the methods based on the use of a piecewise-uniform grid, a priori information on the structure and rate of damping of asymptotic solutions in the boundary layer is explicitly or implicitly taken into account.
In this paper, for numerical integration of singularly perturbed boundary-value problems, we propose to apply (at the initial stage) nonlocal transformations that allow further integrate the reduced problem by standard numerical methods with uniform grid.

Remark 1. Nonlocal transformations were used in [23–27] for numerical integration of Cauchy problems with blow-up solutions having very large gradients in a neighborhood of a previously unknown point. Comparison of exact and numerical solutions of a number of test problems for differential equations with blow-up solutions having very large gradients in a neighborhood of a previously unknown point.

Remark 2. Nonlocal transformations of special type were used in [28–30] to obtain exact solutions, first integrals, and linearize some ordinary differential equations of the second order.

2. Qualitative features of boundary-layer type problems

2.1. An illustrating example of a linear boundary-value problem

We consider the boundary-value problem for the second-order linear differential equation with constant coefficients

\[ \varepsilon y'' + y' + y = 0 \quad (0 < x < 1); \quad y(0) = a, \quad y(1) = b, \]

in which, if \( \varepsilon \to 0 \), a boundary layer is formed near the point \( x = 0 \).

The exact solution of problem (1) is determined by the formulas

\[ y = \frac{ae^{\lambda_2} - b}{e^{\lambda_2} - e^{\lambda_1}} e^{\lambda_1 x} + \frac{b - ae^{\lambda_1}}{e^{\lambda_2} - e^{\lambda_1}} e^{\lambda_2 x}, \]

where \( \lambda_1 = \frac{1}{2} \varepsilon^{-1}(-1 - \sqrt{1 - 4\varepsilon}) \) and \( \lambda_2 = \frac{1}{2} \varepsilon^{-1}(-1 + \sqrt{1 - 4\varepsilon}) \). For small \( \varepsilon \), we have \( \lambda_1 \simeq -\varepsilon^{-1} \), \( \lambda_2 \simeq -1 \), and \( y'(0) \simeq \varepsilon^{-1}(eb - a) \), and the corresponding asymptotic solution of the problem (1) is written as follows:

\[ y_a = (a - eb)e^{-x/\varepsilon} + be^{1-x}. \]

For concreteness, we further assume that \( a \geq 0 \) and \( b \geq 0 \). If \( a > eb \), the function (3) decreases monotonically. If \( a < eb \), the function (3) increases monotonically (and very quickly) in a narrow region \( 0 \leq x < x_0 \), where \( x_0 = \varepsilon \ln \left( \frac{1}{\varepsilon} \left( 1 - \frac{a}{eb} \right) \right) \), \( y_0 = eb \), and in the remaining region, \( x_0 \leq x \leq 1 \), the solution decreases monotonically and changes slowly enough.

In Fig. 1, the exact solutions (2) of the problem (1) are shown by the solid lines for two sets of parameters: a) \( a = 1 \), \( b = 0 \), \( \varepsilon = 0.005 \) and b) \( a = 0 \), \( b = 1 \), \( \varepsilon = 0.005 \). For the second set of parameters, the solution in the region \( 0 \leq x \leq 0.02649 \) increases rapidly, and for \( 0.02649 \leq x \leq 1 \) decreases slowly: in this case the maximum difference between the asymptotic solution (3) and the exact solution (2) on the whole interval, \( 0 \leq x \leq 1 \), is 0.0127.

2.2. A relation between the derivatives

We note that the derivatives of the solution (3) on the left boundary are very large:

\[ (y_a)'|_{x=0} \simeq -\varepsilon^{-1}(a - eb), \quad (y_a)|_{x=0} \simeq \varepsilon^{-2}(a - eb). \]

For example, for \( a = 0 \), \( b = 1 \), \( \varepsilon = 0.005 \), we have \( (y_a)'|_{x=0} \simeq 543.656 \). Therefore, using direct numerical methods for solving similar problems with boundary layers, the shooting procedure should begin with large values of the derivative (of order \( \varepsilon^{-1} \)), which is a complicating factor.
Figure 1. Exact solutions (2) of the problem (1) (solid lines) and numerical solutions of the transformed problem (9) (points) with 
\( g = (1 + |z| + |f|)^{1/2} \) for two sets of numerical values of the determining parameters: 
\( a) \ a = 1, \ b = 0, \ \varepsilon = 0.005 \) and 
\( b) \ a = 0, \ b = 1, \ \varepsilon = 0.005. \)

Let \( |a - eb| = O(1) \). Eliminating \( \varepsilon \) from (4), we obtain the order relation
\[
|y''_{xx}| = O(|y'_x|^2).
\] (5)

The relation between the derivatives (5) is sufficiently general and is valid if, inside the boundary layer, the leading term of the asymptotic expansion of the solution as \( \varepsilon \to 0 \) has the form \( y_i = \varphi(x/\Delta) \), where \( \varphi = \varphi(z) \) is a smooth function having bounded and non-vanishing derivatives in some neighborhood of the point \( z = 0 \), and \( \Delta = \Delta(\varepsilon) \) is a function having the property \( \Delta \to 0 \) as \( \varepsilon \to 0 \).

3. Numerical solution of boundary-value problems based on nonlocal transformations

3.1. General description of the method

We consider two-point problems for second-order differential equations with boundary conditions of the first kind, which in dimensionless variables have the form
\[
y''_{xx} = f(x, y, y'_x) \quad (0 < x < 1); \quad y(0) = a, \quad y(1) = b,
\] (6)

where the function \( f \) can also depend on the small parameter \( \varepsilon > 0 \).

We introduce a new nonlocal independent variable \( \xi \) by means of the first-order differential equation and the initial condition [23–27]:
\[
\xi'_x = g(x, y, y'_x, \xi), \quad \xi(0) = 0.
\] (7)

Here \( g = g(x, y, y'_x, \xi) \) is a regularizing function that can vary.

We represent the second-order equation (6) in the form of an equivalent system of two equations of the first order
\[
y'_x = z, \quad z'_x = f(x, y, z).
\] (8)

Using (7), we pass from \( x \) to the new independent variable \( \xi \) in (8) and (6). As a result, the boundary-value problem (6) is transformed to the following problem for the system of three equations:
\[
x'_\xi = \frac{1}{g(x, y, z, \xi)}, \quad y'_\xi = \frac{z}{g(x, y, z, \xi)}, \quad z'_\xi = \frac{f(x, y, z)}{g(x, y, z, \xi)} \quad (0 < \xi < \xi_1);
\] (9)
\[
x(0) = 0, \quad y(0) = a, \quad y(\xi_1) = b,
\] \( \xi_1 = \Delta(\varepsilon) \).
where the value $\xi_1$ is determined in the process of calculations according to the condition $x(\xi_1) = 1$.

If $g \equiv 1$, the first equation of the system (9), taking into account the initial condition $x(0) = 0$, gives $\xi = x$ and numerical integration of the remaining two equations is equivalent to the integration of the original problem (6). Successfully selected the regularizing function $g = g(x, y, z, \xi)$ itself will determine the position and the density of points of integration with respect to the original variables $x$ and $y$ and allow more accurately (for a given number of grid points) to solve the problem (9) applying the shooting method and using standard fixed-step numerical methods with respect to $\xi$ [31–34].

3.2. Conditions to be satisfied by regularizing functions. Examples of regularizing functions

For numerical solution of boundary-value problems, as well as for solving Cauchy problems, it is reasonable to use regularizing functions of the form [23–27]:

$$g = G(|z|, |f|) \equiv G(|y_x'|, |y'_{x\xi}|),$$

(10)

where $f = f(x, y, z)$ is the right-hand side of the equation (6) and $z = y'_x$. We impose the following conditions on the function $G = G(u, v)$:

$$G > 0; \quad G_u \geq 0, \quad G_v \geq 0; \quad G \to \infty \text{ as } u + v \to \infty; \quad G(0, 0) = 1,$$

(11)

where $u \geq 0, v \geq 0$. The last relation in (11) is the normalization condition and is not mandatory.

For singularly perturbed boundary-value problems (6) with a small parameter at the highest derivative, for which the right-hand side of the equation (6) has the form $f = \varepsilon^{-1}F(x, y, z)$, when choosing regularizing functions, in addition to conditions (10)–(11), some other considerations should be taken into account. For $g = 1$ (in this particular case, nonlocal transformations are not applied) and $\varepsilon \to 0$, the right-hand sides of the last two equations of the system (9) will tend to infinity (since $|z| \to \infty$ and $|f| \to \infty$) in the boundary-layer region; in addition, the order relation $|f| = O(z^2)$ is valid, which follows from the formula (5). This circumstance considerably complicates numerical integration of the problem under consideration and leads to the need to proportionally refine the grid spacing as $\varepsilon$ decreases.

It is possible to avoid refining the grid as $\varepsilon \to 0$ and to work with a fixed stepsize with respect to the nonlocal variable $\xi$ by using regularizing functions satisfying the condition

$$|z|/g = O(1) \quad \text{as} \quad \varepsilon \to 0$$

(12)

(in this case, the right-hand side of the second equation of the system (9) will not have singularities for small $\varepsilon$, and the third equation of this system in the boundary-layer region will have a substantially smaller singularity than for $g = 1$). In particular, we can choose regularizing functions having the asymptotics $g = O(|z|)$ as $|z| \to \infty$ or $g \to O(|f|^{1/2})$ as $|f| \to \infty$.

It is advisable to use, for example, regularizing functions of the form

$$g = (k_1 + k_2|z| + k_3|f|)^{1/2} \quad \text{or} \quad g = (k_1 + k_2z^2 + k_3|f|)^{1/2},$$

(13)

which satisfy the condition (12). The formulas (13) include the three constants $k_1 \geq 0, k_2 \geq 0, \text{ and } k_3 \geq 0$ ($k_2 + k_3 \neq 0$), which can vary.

The use of the regularizing functions (13) allows us to suppress the unbounded growth of the right-hand side of the second equation of the system (9) as $\varepsilon \to 0$ and to reduce (in comparison with $g = 1$) the right-hand side of the third equation.
The maximum absolute error of the numerical solutions of problem (1) for \( a = 1, b = 0 \)

| No. | Regularizing function | Stepsize 0.1 | Stepsize 0.05 | Stepsize 0.01 |
|-----|-----------------------|--------------|--------------|--------------|
| 1   | \( g = 1 + |z| \)     | 0.017119347  | 0.006702741  | 0.000137030  |
| 2   | \( g = (1 + |f|)^{1/2} \) | 0.000707586  | 0.000160259  | 0.000001602  |
| 3   | \( g = (1 + |z| + |f|)^{1/2} \) | 0.000611528  | 0.000146118  | 0.000001741  |
| 4   | \( g = (1 + z^2 + |f|)^{1/2} \) | 0.000900004  | 0.000204128  | 0.000001775  |
| 5   | \( g = 1 \)         | process diverges | process diverges | 0.193331173 |

The maximum absolute error of the numerical solutions of problem (1) for \( a = 0, b = 1 \)

| No. | Regularizing function | Stepsize 0.1 | Stepsize 0.05 | Stepsize 0.01 |
|-----|-----------------------|--------------|--------------|--------------|
| 1   | \( g = 1 + |z| \)     | 0.047029578  | 0.013710597  | 0.000713696  |
| 2   | \( g = (1 + |f|)^{1/2} \) | 0.000824707  | 0.000249922  | 0.000001663  |
| 3   | \( g = (1 + |z| + |f|)^{1/2} \) | 0.000570299  | 0.000115649  | 0.000000554  |
| 4   | \( g = (1 + z^2 + |f|)^{1/2} \) | 0.000559160  | 0.000109360  | 0.000000180  |
| 5   | \( g = 1 \)         | process diverges | process diverges | 0.528189578  |

Table 1. Comparison of the efficiency of various regularizing functions for the transformed problem (9) used for numerical solution of the original problem (1) by nonlocal transformations with \( \varepsilon = 0.005 \).

4. Singularly perturbed boundary-value problems. Comparison of numerical and exact solutions

4.1. Linear problems. Numerical solutions obtained using various regularization functions

In Fig. 1, the results of numerical solutions of the transformed problem (9), used for solving the linear problem (1), for \( f = -\varepsilon^{-1}(z+y) \) with the regularizing function \( g = (1 + |z| + |f|)^{1/2} \), which are obtained by the shooting method (from the point \( x = 0 \)) with the fixed stepsize \( h = 0.01 \) by using Maple [34], are shown by points for two sets of numerical values of the defining parameters: \( a = 1, b = 0, \varepsilon = 0.005 \) (monotonic solution) and \( a = 0, b = 1, \varepsilon = 0.005 \) (non-monotonic solution). It can be seen that there is a good coincidence between the numerical solutions and the corresponding exact solutions, which are determined by the formula (2) and are represented by solid lines.

Table 1 shows the maximum absolute errors of numerical solutions of the transformed problem (9) used for numerical integration of the original problem (1) with \( \varepsilon = 0.005 \) by the method based on nonlocal transformations for \( a = 1, b = 0 \) and \( a = 0, b = 1 \) for three stepsizes \( h \) and five different regularizing functions \( g \). For comparison, similar data are also indicated for the case \( g = 1 \), which corresponds to the direct numerical solution (without using transformations) with the same stepsize with respect to \( x \). It can be seen that the three regularizing functions (Nos. 2–4) make it possible to obtain numerical solutions in the entire region with high accuracy even with a sufficiently large stepsize \( h = 0.1 \) (with respect to \( \xi \)).

4.2. A nonlinear boundary-value problem. Exact, asymptotic, and numerical solutions

We consider the boundary-value problem with quadratic nonlinearity

\[ \varepsilon y''_{xx} + (x+y)y'_{x} + x + y = 0 \quad (0 < x < 1); \quad y(0) = a, \quad y(1) = b. \quad (14) \]

The general solution of this equation has the form

\[ y = c \frac{1 - Ae^{-cx}/\varepsilon}{1 + Ae^{-cx}/\varepsilon} - x. \quad (15) \]
Figure 2. Exact solution (15) of the original problem (14) (solid line) and numerical solution of the transformed problem (9) (circles) obtained by nonlocal transformations with the regularizing function 

\[ g = (1 + z^2 + |f|)^{1/2} \]

for \( a = b = 1, \varepsilon = 0.005 \).

| No. | Regularizing function | Stepsize 0.1 | Stepsize 0.05 | Stepsize 0.01 |
|-----|-----------------------|--------------|---------------|---------------|
| 1   | \( g=1+|z| \)        | 0.137389203  | 0.053399823   | 0.000857913   |
| 2   | \( g=(1 + |f|)^{1/2} \) | 0.000937303  | 0.000228167   | 0.000005030   |
| 3   | \( g=(1 + |z| + |f|)^{1/2} \) | 0.000786873  | 0.000196698   | 0.00003249    |
| 4   | \( g=(1 + z^2 + |f|)^{1/2} \) | 0.000607467  | 0.000154641   | 0.000003261   |
| 5   | \( g=1 \)            | process diverges | process diverges | process diverges |

Table 2. Comparison of the efficiency of various regularizing functions for the transformed problem (9) used for numerical solution of the original problem (14) by nonlocal transformations with \( a = b = 1, \varepsilon = 0.005 \) and three stepsizes \( h \).

The constants of integration \( A \) and \( c \) are determined from the transcendental system of equations

\[ \begin{align*}
  c \frac{1 - A}{1 + A} &= a, \\
  c \frac{1 - A e^{-c/\varepsilon}}{1 + A e^{-c/\varepsilon}} - 1 &= b,
\end{align*} \tag{16} \]

obtained by substituting the expression (15) into the boundary conditions (14). As \( \varepsilon \to 0 \) and \( b > -1 \), the asymptotic solution of the system (16) is given by the formulas

\[ \begin{align*}
  A &= \frac{b - a + 1}{b + a + 1}, \\
  c &= b + 1. \tag{17}
\end{align*} \]

We note that the asymptotic solution (17) exactly satisfies the first the equation of the system (16), and the discrepancy of the second equation of this system has the order \( e^{-(b+1)/\varepsilon} \) as \( \varepsilon \to 0 \).

In Fig. 2, the exact solution of the problem (14) is shown by solid line for \( a = b = 1, \varepsilon = 0.005 \), which is determined by the formula (15) for \( A = 3, c = -2 \) (in this case the difference between the asymptotic and exact solutions is far beyond the limits of accuracy of our calculations). The circles represent the results of numerical solution of the corresponding transformed problem (9) with the regularizing function \( g = (1 + z^2 + |f|)^{1/2} \), which are obtained by the shooting method (from the point \( x = 0 \)) with the stepsize \( h = 0.01 \) by using Maple. The maximum modulus of the difference between the exact and numerical solution is equal to 0.000003261.

Table 2 shows the maximum absolute errors of numerical solutions of the transformed problem (9) used for numerical integration of the original problem (14) for \( a = b = 1, \varepsilon = 0.005 \) with...
three stepsizes $h$ and four different regularizing functions $g$. It can be seen that the functions Nos. 2–4 allow one to obtain numerical solutions in the entire region with high accuracy even with a sufficiently large stepsize (with respect to $\xi$) equal to $h = 0.1$.

5. Brief conclusions

We offer a new method of numerical integration of singularly perturbed boundary-value problems for second-order ODEs of the form $\varepsilon y''_{xx} = F(x, y, y'_{x})$ based on introducing a nonlocal independent variable $\xi$, which is related to original variables $x$ and $y$ by the auxiliary differential equation $\xi'_{x} = g(x, y, y'_{x}, \xi)$. With a suitable choice of the regularizing function $g$, the proposed method leads to more convenient problems that allow the application of standard numerical methods with the fixed stepsize with respect to $\xi$. Comparison of numerical and exact solutions of several singularly perturbed boundary-value problems showed the high efficiency of this method.

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