On the (absence of a) relationship between bound and scattering states in quantum mechanics. Application to $^{12}$C + α

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Using phase-equivalent supersymmetric partner potentials, a general result from the inverse problem in quantum scattering theory is illustrated, i.e., that bound-state properties cannot be extracted from the phase shifts of a single partial wave, as a matter of principle. In particular, recent R-matrix analyses of the $^{12}$C + α system, extracting the asymptotic normalization constant of the 2+ sub-threshold state, $C_{12}$, from the $\ell = 2$ elastic-scattering phase shifts and bound-state energy, are shown to be unreliable. In contrast, this important constant in nuclear astrophysics can be deduced from the simultaneous analysis of the $\ell = 0, 2, 4, 6$ partial waves in a simplified potential model. A new supersymmetric inversion potential and existing models give $C_{12} = 144.5 \pm 8.5 \times 10^3 \text{fm}^{-1/2}$.

I. INTRODUCTION

Can bound-state properties be deduced from scattering data? The answer to this question showed interesting evolutions throughout the history of quantum scattering theory (see Foreword by R. G. Newton in Ref. [1]), starting with Heisenberg’s conjecture in the early forties that the scattering (S) matrix contains all physical information, including that regarding bound states. This conjecture was later seriously weakened in the framework of the potential model: in 1949, Bargmann [2] constructed potentials that share the same S matrix for partial wave $\ell = 0$ (so-called phase-equivalent potentials because they share the same scattering phase shifts) but have different bound-state properties [different energies, or identical energies but different wave functions]. On the other hand, that same year, Levinson (see for instance Ref. [1]) proved that, for a given partial wave $\ell$, the number of bound states, $N_\ell$, can be deduced from the phase shifts. These results were then explained by the inverse-problem theory [1], which shows that for a given partial wave the potential is uniquely determined by (i) the scattering phase shifts at all energies, (ii) the bound-state energies and (iii) one additional real parameter for each bound-state wave function [e.g., the asymptotic normalization constant (ANC), see definition (1)].

This only applies to local central potentials displaying no singularity. Levinson’s theorem, however, was generalized by Swan in 1963 [4], who proved that $N_\ell$ cannot be deduced from the S matrix for potentials displaying an $r^{-2}$ repulsive singularity at the origin. Since then, the construction of phase-equivalent potentials for a given partial wave has known important progress [5, 6, 7, 8], thanks to the use of the algebraic formalism of supersymmetric quantum mechanics [10, 11, 12] used below (see Fig. 2).

As far as only one partial wave is concerned, the answer to the above question is thus rather clear-cut: neither the number of bound states, nor their energies nor their ANCs can be deduced from the S matrix alone. Moreover, these bound-state properties are independent of one another, which means that knowing, for instance, both the S matrix and a bound-state energy does not constrain the value of the ANC as long as no other physical information is available on this particular bound state (e.g., its radius or its lifetime) or on the considered system in general (e.g., the range of the interaction). This impossibility proof being established in the framework of the potential model, which is a particular case of more general models (e.g., microscopic or phenomenological models), it holds for these more general models too. However, several works [13, 14, 15] seem to ignore this argument and aim at extracting the ANC of a bound-state with known energy from the corresponding partial-wave S matrix. In Refs. [16, 17], for instance, the reaction- (R-)matrix formalism [18, 19] is used to extract the ANC of an $^{16}$O bound state from $^{12}$C + α elastic-scattering data. In the R-matrix phenomenological model, it is rather natural to assume that bound states have a measurable impact on the partial-wave S matrix, like elastic-scattering resonances, because both bound and resonant states are characterized by an energy and a reduced width. However, in scattering theory, bound and resonant states have very different natures, and the above impossibility argument seriously questions the ANC extraction of Refs. [16, 17]. In Sec. I.I I show how this misuse of the R-matrix formalism actually explains the inconsistency between the ANC values obtained in these references. Then in Sec. III I explicitly construct simple potential models which illustrate the independence of the bound-state ANC from the phase shifts for a single partial wave.

Now, considering several partial waves simultaneously leads to a less clear situation. The inverse-problem theory implies that a regular interaction potential is uniquely determined by phase shifts of a given partial wave with angular momentum high enough so that the partial wave contains no bound state. Using inversion techniques [1], a unique potential can thus in principle...
be deduced from such phase shifts; this potential can in turn be used for lower partial waves, for which it may sustain bound states. In that case, bound state properties (of low partial waves) are totally determined by scattering phase shifts (of a high partial wave). However, this program presents three difficulties. First, it assumes that the potential is local and energy independent, which, in general, is not a realistic approximation when the interacting particles have an internal structure (see the discussion in Secs. IV and V for the particular case of $^{12}$C + $\alpha$). Second, it requires that the interaction be independent of the partial wave, which is not always the case: scattering phase shifts of several partial waves are not necessarily compatible with one unique potential. Third, it is based on the knowledge of the phase shifts of a high-angular-momentum partial wave at all energies, which are generally not available: at high energy, the complexity of multichannel effects increases, which makes the data difficult to analyze; and at low energy, the phase shifts of high-angular-momentum partial waves are very small because of the centrifugal barrier. Hence, considering several partial waves at the same time may help deducing bound-state properties from scattering data, but the practical applicability of the method could strongly depend on the considered system. In Sec. IV I show that for the particular case of $^{12}$C + $\alpha$ encouraging results are obtained by considering several partial waves at the same time, in the framework of a simplified model in which $^{12}$C and $\alpha$ are considered as rigid nuclei in their $0^+$ ground state, interacting through a local energy- and angular-momentum-independent potential. These results are then compared with more sophisticated models available in the literature. Conclusions and perspectives for the $^{12}$C + $\alpha$ system are given in Sec. V.

II. R-MATRIX ANALYSES

The $^{12}$C + $\alpha$ system is particularly important in nuclear astrophysics since the $^{12}$C ($\alpha, \gamma$) $^{16}$O capture is a key reaction in the helium-burning phase of red giant stars: by competing with the triple $\alpha$ reaction (leading to $^{12}$C), it determines the ratio of carbon and oxygen which results from stellar nucleosynthesis. Experimentally, the Coulomb repulsion between $^{12}$C and $\alpha$ makes the direct measurement of the capture reaction impossible at the very low energies of astrophysical interest (around 300 keV in the center-of-mass frame). Hence, available experimental results at higher energies ($E_{c.m.} > 1$ MeV) have to be extrapolated to these low energies. This extrapolation, generally performed with the R-matrix model (the only model to date able to precisely fit all the available data), is itself problematic because of the presence of two $^{16}$O bound states just below the $^{12}$C + $\alpha$ threshold. Whereas the influence of the $E_{c.m.} = -45.1 \pm 0.1$ keV state is well understood [21, 22], the influence of the $E_{c.m.} = -244.9 \pm 0.6$ keV state is still the major source of uncertainty on the reaction rate [10, 17, 22]. The unknown key quantity is its ANC (the energies of both states are well known), which is related to its $\alpha$ reduced width in the R-matrix formalism (I do not use reduced widths here since they depend on the R-matrix radius, which makes comparison between different models difficult).

The $-245$ keV subthreshold state has a positive parity and a $J = 2$ angular momentum ($2^+$ state); microscopic cluster models (see Ref. 22 and references therein) suggest that its structure is dominated by a $^{12}$C cluster and an $\alpha$ cluster, both in their fundamental $0^+$ state, with a relative motion of angular momentum $\ell = 2$. It is thus on the phase shifts of this partial wave that one could naively expect to see the influence of the $2^+$ state, and it is from these phase shifts that one could try to extract its ANC, $C_{12}$ (where the second index refers to $\ell = 2$ and the first index means that this state is the lowest one for this partial wave). This constant is defined by the asymptotic behavior of the radial part, $u_{12}(r)$, of the relative wave function between the $^{12}$C and $\alpha$ nuclei for this particular bound state:

$$u_{12}(r) \sim r_{\to \infty} C_{12} W_{-6.615,5/2}(0.3751 r),$$

where $r$ is the radial coordinate (in fm) and $W$ is the Whittaker function 23 (a decreasing exponential function in the absence of Coulomb interaction).

In Ref. 16, the $\ell = 2$ phase shifts from Ref. 22 are analyzed by the R-matrix model, leading to a minimum in the $\chi^2$ (see Fig. 1 of Ref. 16) for $C_{12} = 402 \times 10^3$ fm$^{-1/2}$. However, this extraction is very delicate since the influence of the subthreshold state has to be disentangled from the background phase shifts in the energy region where experimental data are available; this influence is smaller (see Fig. 2 of Ref. 16) than the main components of this background, i.e., the hard-sphere and high-energy-pole phase shifts. As a consequence, this minimum strongly depends on the range of data fitted (see Fig. 1 of Ref. 16). Moreover, I have checked that the minimum is highly sensitive to the high-energy background-pole energy, which is fixed at 10 MeV in Ref. 16. Without this restriction, very different values of $C_{12}$ are obtained; hence, the error on $C_{12}$, which is not estimated in Ref. 16, is actually very large.

In Ref. 17, the $^{12}$C + $\alpha$ elastic scattering has been re-measured with very high precision. An R-matrix analysis with free high-energy-pole energies leads to a $\chi^2$ minimum (see Fig. 2 (a) of Ref. 17) for $C_{12} = 154 \pm 18 \times 10^3$ fm$^{-1/2}$. The error bar on $C_{12}$ is estimated by following the guideline 21, 22:

$$\chi^2 < \chi^2_{\text{min}} + 9\chi^2_{\text{min}}/\nu,$$

where $\chi^2_{\text{min}}$ is the minimal $\chi^2$ and $\nu$ is the number of degrees of freedom. In the present case, one has $\chi^2_{\text{min}} = 18941$ (the $\chi^2$ of Fig. 2 in Ref. 17) have to be divided by the 32 reference angles), for 11392 data points and 65 parameters (32 R-matrix parameters and 33 experimental parameters), which leads to $\nu = 11327$.  

Following the theory of $\chi^2$ minimization \cite{25}, one finds that the probability of having $\chi^2 \geq 18941$ for $\nu = 11327$ is of the order of $10^{-391}$, which probably indicates that some systematic errors have been underestimated in Ref. \cite{17}. Consequently, Eq. (2) cannot be used to define error bars on parameters since this equation only takes into account statistical errors and requires that the fit be good in the sense of $\chi^2$ theory. A less rigorous way of checking the validity of a fit is to use the “chi-by-eye” approach \cite{25}. In the case of Ref. \cite{17}, the fit looks good (see Fig. 1 of this reference), which is related to the fact that $\chi^2_{\text{min}}/\nu = 1.672$ is not that far from 1. However, the fits obtained with $C_{12}$ = 136 or $172 \times 10^3$ fm$^{-1/2}$, which correspond to $\chi^2/\nu = 1.674$, look probably equally good. I thus consider that the error bar on $C_{12}$ obtained in Ref. \cite{17} is underestimated.

In summary, the values of $C_{12}$ obtained in both Refs. \cite{16} and \cite{17} are most probably unreliable, for different reasons. This explains why, despite the fact that the $\ell = 2$ phase shifts of Refs. \cite{24} and \cite{17} are compatible with one another, their R-matrix analyses give totally incompatible results for $C_{12}$. One might think that this inconsistency is due to the fact that in Ref. \cite{17} all relevant partial waves are taken into account whereas in Ref. \cite{16} only $\ell = 2$ is considered. However, since the R-matrix formalism is based on a partial-wave decomposition of the cross section, $C_{12}$ is actually independent of the $\ell \neq 2$ parameters (in Ref. \cite{17}, the only influence of the $\ell \neq 2$ phase shifts on $\ell = 2$ comes through the choice of a common R-matrix radius for all partial waves; however, this does not seem to strongly constrain the fit of the $\ell = 2$ phase shifts, as discussed in the next paragraph). I argue that this incompatibility rather illustrates the impossibility, well-known in quantum scattering theory, of deducing a bound-state ANC when only the bound-state energy and the elastic-scattering phase shifts of the corresponding partial wave are known.

Let us however notice that, in the case of the R-matrix phenomenological model, this impossibility is probably not as strong as in general scattering theory. As shown in the next Section (see Fig. 2), the range of a $^{12}$C + $\alpha$ interaction that reproduces both the $\ell = 2$ phase shifts and the bound-state energy increases with the value of $C_{12}$. But the R-matrix formalism is only valid when the nuclear interaction is negligible above the R-matrix radius. In the R-matrix \textit{theory}, where the number of R-matrix poles is infinite \cite{14}, this does not cause any problem: the results are independent of the R-matrix radius, which can thus be chosen as large as necessary (in other words, it would be possible to calculate an R-matrix for any of the potentials plotted in Fig. 2 but the R-matrix radius should be larger than the potential range). In the R-matrix \textit{phenomenological model}, where the number of poles is limited to the number of bound states and resonances in the energy interval considered, plus one possible high-energy pole to help simulating the background, a constraint appears on the R-matrix radius: in Ref. \cite{27}, a value of $5.5-6$ fm is found, based on the $\ell = 1$ partial wave; in Ref. \cite{17} [see Fig. 2 (b)] $5.5$ fm leads to the best fit (I have checked that this constraint is mainly due to the $\ell = 0$ partial wave, for which the background is maximal); in Ref. \cite{16}, the radius is fixed at $6.5$ fm. Hence, very large values of $C_{12}$ can probably be excluded by an R-matrix fit; very small values, on the other hand, cannot be excluded. I could actually find very good fits of the $\ell = 2$ phase shifts with $C_{12} = 0$ for R-matrix radii in the range $4.5-6.5$ fm. This confirms the doubts expressed above on the analyses of Refs. \cite{16,17} and it shows that, in this particular case, the quality of the fit does not strongly depend on the precise value of the R-matrix radius.

\section{Inversion of one partial wave}

Up to now, the discussion is based on general arguments from scattering theory. Let me now construct simplified two-body potential models that illustrate these arguments in the particular case of $^{12}$C + $\alpha$, using the supersymmetric-quantum-mechanics \cite{10,11,12} inversion technique of Refs. \cite{28,29}. Starting from a phase-shift analysis \cite{26} of the purely elastic data ($E_{\text{c.m.}} < 4.966$ MeV) of Ref. \cite{17}, I have removed, using the R-matrix formalism, resonances that have a more complicated structure than $^{12}$C + $\alpha$ in their $0^+$ ground states (such resonances cannot be explained by the present simplified model). For $\ell = 2$, this means removing the narrow resonance at $2.683$ MeV and the wide resonance at $4.358$ MeV \cite{30}, which leads to the phase shifts plotted in Fig. 3 (the spread of the data above $4.2$ MeV is due to the difficulty of performing a reliable phase-shift analysis in the vicinity of a resonance). By solving the so-called “singular inverse problem” defined in Ref. \cite{28}, I have constructed a potential that reproduces these phase shifts with very high precision. This potential has an $r^{-2}$ singularity at the origin and satisfies Swan’s theorem \cite{4}; to simplify further developments, its nuclear part has been approxi-
dashed-line potentials are its phase-equivalent supersymmetric partners with one bound state at $-245$ keV, for $C_{12} = 20$, 200 and $2000 \times 10^3$ fm$^{-1/2}$.

where $V = 43.40$ MeV fm$^2$ and $R = 5.091$ fm. The corresponding $\ell = 2$ effective potential (centrifugal barrier plus Coulomb and nuclear interactions) is represented in Fig. 2 (solid line) and its phase shifts are shown in Fig. 4. Surprisingly, a potential of the form (3) is also able to reproduce the $\ell = 0$ background phase shifts with satisfactory precision, which suggests an alternative way of parameterizing the $^{12}$C + $\alpha$ background in future R-matrix analyses (in the spirit of the “hybrid” R-matrix model of Refs. [31, 32, 33]). The fit of the $\ell = 2$ scattering phase shifts is very good, despite the fact that potential has, by construction, no bound state (it is purely repulsive); this is a first proof that, from the point of view of scattering theory, the $\ell = 2$ background phase shifts are totally disconnected from the $2^+$ subthreshold bound state.

Now, since this state is known to exist and to have a structure such that it could be well simulated by the present simplified model, it is physically meaningful to perform a further step in the inversion of the $\ell = 2$ phase shifts, i.e., to add a bound state to the above potential without modifying its phase shifts [22]. This phase-equivalent bound-state addition may be carried out through a further use of supersymmetric quantum mechanics [6]; it introduces two arbitrary parameters: the bound-state energy and its ANC. In the present case, the energy is known experimentally but not the ANC; hence, there is a one-parameter family of $\ell = 2$ effective potentials that share identical phase shifts (Fig. 1) and have a bound-state at $-245$ keV. Such potentials are represented by dashed lines for three arbitrary values of $C_{12}$ in Fig. 2, they constitute generalizations (to $\ell \neq 0$ and in the presence of a Coulomb interaction) of Bargmann’s potentials (see Fig. 1 in Ref. [6]). The corresponding bound-state wave functions are plotted in Fig. 3 (dashed lines), where their very different asymptotic behaviors are clearly seen. This proves that, from the $\ell = 2$ phase shifts and bound-state energy only, there is no way of telling what is the best value for $C_{12}$, in contrast with R-matrix attempts described above (see however the discussion of the R-matrix phenomenological model in the previous Section).

IV. INVERSION OF SEVERAL PARTIAL WAVES

The general discussion from the introduction suggests to look at other partial waves simultaneously. The $2^+$ subthreshold state is actually a member of a rotational band, together with the $0^+$ bound state at $-1113 \pm 1$ keV and the $4^+$ and $6^+$ resonances at $3194 \pm 3$ keV (width: $26 \pm 3$ keV) and $9113 \pm 7$ keV (width: $420 \pm 20$ keV) respectively [30]. This is indicated by cluster-model calculations (see Ref. [22] and references therein), which give a fair description of these four states in a single-channel calculation where both the $^{12}$C and $\alpha$ nuclei are in their $0^+$ ground state. This suggests to ask the following question: could all these states be reproduced by a unique $\ell$-independent potential? Of course, in principle, the internal structure of the nuclei could play a role and a satisfactory potential could be non local and energy dependent. It is however worth testing the simplest hypothesis, i.e., that of an energy- and $\ell$-independent local potential, because if such a potential were sufficient, it would be strongly constrained by the data of the four partial waves considered simultaneously, which could help solving the $C_{12}$ indefiniteness.

I could not find such a potential within the above one-parameter potential family. This is not very surprising since general arguments from the microscopic cluster model show that $\ell$-independent interaction potentials between nuclei should have low-energy non-physical bound states that simulate the Pauli principle between the constituting nucleons [because of these so-called Pauli

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**Fig. 2**: Effective $^{12}$C + $\alpha$ interaction potentials for $\ell = 2$. The solid-line potential corresponds to Eq. (3) while the dashed-line potentials are its phase-equivalent supersymmetric partners with one bound state at $-245$ keV, for $C_{12} = 20$, 200 and $2000 \times 10^3$ fm$^{-1/2}$.

**Fig. 3**: Radial part of the $2^+$ subthreshold state $^{12}$C + $\alpha$ relative wave functions. The dashed lines correspond to the potentials of Fig. 2, the solid line to the deep inversion potential and the dash-dotted line to the potential of Ref. [37].
forbidden states (PFSs), such potentials are deep, in contrast with shallow potentials that only sustain physical bound states. For the \( \ell = 2 \) partial wave of \( ^{12}\text{C} + \alpha \), the expected number of PFSs is two \( 2 \).

In the context of inversion, such bound states complicate things a lot since each of them introduces two additional arbitrary parameters (neither their energy nor their ANC is known experimentally). Following Ref. \[29\], I introduce these states phenomenologically by regularizing the above inversion potential with the expression \( V_{\text{reg}} = -a r^2 + b r^3 \) for \( r \leq r_{\text{reg}} \), where \( V_{\text{reg}} \) and \( r_{\text{reg}} \) are fitting parameters (\( a \) and \( b \) are calculated to make the potential and its first derivative continuous in \( r_{\text{reg}} \)). Since this regularization slightly alters the phase shifts, parameters \( V \) and \( R \) in Eq. \[\text{3}\] have to be adapted to maintain the fit of Fig. \[1\]. Altogether, there are thus five parameters \((V, R, C_{12}, V_{\text{reg}} \) and \( r_{\text{reg}} \)) which can be varied to fit all the \( \ell = 0, 2, 4, 6 \) data at the same time. The best fit I have obtained is for \( V = 63.81 \text{ MeV fm}^2 \), \( R = 4.925 \text{ fm} \), \( V_{\text{reg}} = -107.9241 \text{ MeV} \), \( r_{\text{reg}} = 4.1 \text{ fm} \). By construction, this inversion potential perfectly reproduces the \( 2^+ \) state energy. The wave function is represented in Fig. \[4\] (solid line); its two nodes are due to its orthogonality to the PFSs at \(-46\) and \(-18\) MeV, which confirms that the potential is deep. The corresponding value for \( C_{12} \) is \( 137 \times 10^3 \text{ fm}^{-1/2} \).

I consider this prediction as reliable because all the even phase shifts in the elastic region are also rather well fitted (for example, the \( 4^+ \) resonance is at \( 3196 \) keV with width \( 22 \text{ keV} \)). However, this potential underestimates the \( 0^+ \) bound-state energy \((-1696 \text{ keV})\), as well as the \( 6^+ \) resonance energy \((8985 \text{ keV})\) and width \((259 \text{ keV})\). These inaccuracies are probably due to the extreme simplicity of the present model (energy- and \( \ell \)-independent local potential) and it is worth comparing it with more sophisticated models of the literature, which better take into account many-body effects. The two non-local \( \ell \)-dependent potentials of Ref. \[33\], which also fit the physical properties of the members of the rotational band, lead to similar wave functions for the \( 2^+ \) subthreshold state (2 nodes). The Gaussian potential provides \( C_{12} = 144 \times 10^3 \text{ fm}^{-1/2} \) while the Woods-Saxon potential provides \( 153 \times 10^3 \text{ fm}^{-1/2} \). In Ref. \[34\], a simple local potential is given, which precisely reproduces the energies of the members of the rotational band, as well as the resonance widths, provided a slight \( \ell \) dependence is allowed. Although the number of PFSs is three for this potential, which adds a node in the subthreshold-bound-state wave function (see dash-dotted line in Fig. \[4\]), it provides a value of \( C_{12} \) very similar to the inversion potential: \( 136 \times 10^3 \text{ fm}^{-1/2} \).

V. CONCLUSION

I have illustrated on \( ^{12}\text{C} + \alpha \) two general results from the inverse problem in quantum scattering theory: (i) for a given partial wave, the scattering phase shifts, the number of bound states, their energies and their ANCs are independent of one another; (ii) when several partial waves can be described by the same theoretical model, these quantities may be related to one another. This is reasonably the case for \( ^{12}\text{C} + \alpha \), for which I have deduced the ANC value of the \( 2^+ \) subthreshold state from the simultaneous analysis of the \( \ell = 0, 2, 4, 6 \) partial waves, with the help of a local energy- and angular-momentum-independent potential constructed by supersymmetric inversion.

Since this potential lies on rather restrictive hypotheses and is not able to exactly fit all the data, I have compared it with results from more sophisticated potential models available in the literature. All theoretical predictions lie in a rather limited range, \( C_{12} = 144.5 \pm 8.5 \times 10^3 \text{ fm}^{-1/2} \), which confirms somehow their overall validity. The error bar can thus be considered as a sensible estimate of the theoretical uncertainty on \( C_{12} \) in a \( ^{12}\text{C} + \alpha \) potential model. The fact that both local (present work and Ref. \[34\]) and non-local \( 33 \) potentials give consistent results is an indication that non-local effects are not very important for this particular state. This confirms that its structure is dominated by rigid \( ^{12}\text{C} \) and \( \alpha \) clusters with \( \ell = 2 \) relative angular momentum. A similar smallness of non-local effects has been found in Ref. \[32\] for the \( ^{16}\text{O} + \alpha \) system.

All the potential models discussed above neglect multichannel effects, which means that the predicted value of \( C_{12} \) might be overestimated. This would be consistent with the experimental value obtained from transfer reactions \[21\], \( C_{12} = 114 \pm 10 \times 10^3 \text{ fm}^{-1/2} \), and with the theoretical estimate from the microscopic multiconfiguration cluster model of Ref. \[22\], \( C_{12} = 134 \times 10^3 \text{ fm}^{-1/2} \). Whereas compatible with the unreliable value from Ref. \[17\], the above theoretical estimate excludes that from Ref. \[14\], as well as the cascade-transition value \( C_{12} = 228^{+33}_{-25} \times 10^3 \text{ fm}^{-1/2} \) from Ref. \[27\]. A new measurement of the cascade transitions is planned at TRIUMF, which hopefully should clarify the situation. In the meanwhile, R-matrix fits should reflect the uncertainty on \( C_{12} \).

Let me end with a general discussion of possible theoretical models for \( ^{12}\text{C} + \alpha \) and similar systems relevant to nuclear astrophysics. Ideally, \textit{ab initio} calculations should be performed; however, despite the success of the microscopic cluster model, such calculations are still too primitive to provide precise predictions of all available data. Hence, phenomenological models still have an important role to play. Among them, the potential model is probably the most physical and intuitive one. However, simulating the full complexity of the many-body nuclear system requires a non-local \( 33 \) multichannel \( 38 \) treatment. This considerably complicates the model and a good quality fit of all available data with the potential model is not available to date. In contrast, the R-matrix model allows such a fit because it remains rather simple even in the presence of many channels. The price to pay for that is a large number of parameters and a very phe-
nomenological description, in particular of background terms. An interesting approach which combines the advantages of both the potential and R-matrix models is that of “hybrid” models [31, 32, 33]. The potentials constructed in the present work renew the interest for these models. In particular, purely repulsive potentials of the form (3) could be used as a new description of the background, a possibility which will be explored in a future work.

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[1] K. Chadan and P. C. Sabatier, Inverse Problems in Quantum Scattering Theory (Springer, New York, 1977).
[2] V. Bargmann, Phys. Rev. 75, 301 (1949).
[3] V. Bargmann, Rev. Mod. Phys. 21, 488 (1949).
[4] P. Swan, Nucl. Phys. 46, 669 (1963).
[5] D. Baye, Phys. Rev. Lett. 58, 2738 (1987).
[6] D. Baye, J. Phys. A 20, 5529 (1987).
[7] L. U. Ancarani and D. Baye, Phys. Rev. A 46, 206 (1992).
[8] D. Baye, Phys. Rev. A 48, 2040 (1993).
[9] D. Baye and J.-M. Sparenberg, Phys. Rev. Lett. 73, 2789 (1994).
[10] E. Witten, Nucl. Phys. B188, 513 (1981).
[11] C. V. Sukumar, J. Phys. A 18, 2917 (1985).
[12] C. V. Sukumar, J. Phys. A 18, 2937 (1985).
[13] Z. R. Iwinski, L. Rosenberg, and L. Spruch, Phys. Rev. C 29, 349 (1984).
[14] L. D. Blokhintsev, A. M. Mukhamedzhanov, and A. N. Safronov, Fiz. Elem. Chastits At. Yadra 15, 1296 (1984) [Sov. J. Part. Nucl. 15, 580 (1984)].
[15] L. D. Blokhintsev, V. I. Kukulin, A. A. Sakharuk, D. A. Savin, and E. V. Kuznetsova, Phys. Rev. C 48, 2390 (1993).
[16] C. Angulo and P. Descouvemont, Phys. Rev. C 61, 064611 (2000).
[17] P. Tischhauser, R. E. Azuma, L. Buchmann, R. Detwiler, U. Giesen, J. Görres, M. Heil, J. Hinnefeld, F. Käppeler, J. J. Kolata, et al., Phys. Rev. Lett. 88, 072501 (2002).
[18] E. P. Wigner and L. Eisenbud, Phys. Rev. 72, 29 (1947).
[19] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30, 257 (1958).
[20] R. E. Azuma, L. Buchmann, F. C. Barker, C. A. Barnes, J. M. D’Auria, M. Dombsky, U. Giesen, K. P. Jackson, J. D. King, R. G. Korteling, et al., Phys. Rev. C 50, 1194 (1994).
[21] C. R. Brune, W. H. Geist, R. W. Kavanagh, and K. D. Veal, Phys. Rev. Lett. 83, 4025 (1999).
[22] P. Descouvemont, Nucl. Phys. A470, 309 (1987).
[23] M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions (Dover, 1965).
[24] R. Plaga, H. W. Becker, A. Redder, C. Rolfs, H. P. Trautvetter, and K. Langanke, Nucl. Phys. A 465, 291 (1987).
[25] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Numerical Recipes. The Art of Scientific Computing (FORTRAN Version) (Cambridge University, New York, 1989).
[26] L. Buchmann, private communication (2003).
[27] E. Vogt, Phys. Lett. B 389, 637 (1996).
[28] J.-M. Sparenberg and D. Baye, Phys. Rev. C 55, 2175 (1997).
[29] J.-M. Sparenberg, Phys. Rev. Lett. 85, 2661 (2000).
[30] D. R. Tilley, H. R. Weller, and C. M. Cheves, Nucl. Phys. A564, 1 (1993).
[31] C. H. Johnson, Phys. Rev. C 7, 561 (1973).
[32] S. E. Koonin, T. A. Tombrello, and G. Fox, Nucl. Phys. A220, 221 (1974).
[33] K. Langanke and S. E. Koonin, Nucl. Phys. A439, 384 (1985).
[34] B. Buck and J. A. Rubio, J. Phys. G 10, L209 (1984).
[35] H. Friedrich, Phys. Rep. 74, 209 (1981).
[36] D. Baye and P. Descouvemont, Ann. Phys. (N. Y.) 165, 115 (1985).
[37] L. Buchmann, Phys. Rev. C 64, 022801(R) (2001).
[38] R. A. Baldock, B. Buck, and J. A. Rubio, Nucl. Phys. A426, 222 (1984).