MOMENT/SUM-OF-SQUARES HIERARCHY FOR COMPLEX POLYNOMIAL OPTIMIZATION

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Abstract. We consider the problem of finding the global optimum of a real-valued complex polynomial on a compact set defined by real-valued complex polynomial inequalities. It reduces to solving a sequence of complex semidefinite programming relaxations that grow tighter and tighter thanks to D’Angelo’s and Putinar’s Positivstellensatz discovered in 2008. In other words, the Lasserre hierarchy may be transposed to complex numbers. We propose an algorithm for exploiting sparsity and apply the complex hierarchy to problems with several thousand complex variables. They consist in computing optimal power flows in the European high-voltage transmission network.

Key words. Quillen property, Lasserre hierarchy, Shor relaxation, complex moment problem, sparse semidefinite programming, optimal power flow.

AMS subject classifications.

1. Introduction. Multivariate polynomial optimization where variables and data are complex numbers is a non-deterministic polynomial-time hard problem that arises in various applications such as electric power systems (Section 4), imaging science [8, 13, 29, 66], signal processing [1, 6, 18, 45, 48, 49], automatic control [70], and quantum mechanics [33]. Complex numbers are typically used to model oscillatory phenomena which are omnipresent in physical systems. Although complex polynomial optimization problems can readily be converted into real polynomial optimization problems, efforts have been made to find ad hoc solutions [35, 36, 67]. We observe that relaxing non-convex constraints and converting from complex to real numbers are two non-commutative operations. This leads us to transpose to complex numbers Lasserre’s moment/sum-of-squares hierarchy [41] for real polynomial optimization.

In 1968, Quillen [61] showed that a real-valued bihomogenous complex polynomial that is positive away from the origin can be decomposed as a sum of squared moduli of holomorphic polynomials when it is multiplied by \((\sum_{i=1}^{n}|z_i|^2)^r\) for some \(r \in \mathbb{N}\). The result was rediscovered by Catlin and D’Angelo [17] and ignited a search for complex analogues of Hilbert’s seventeenth problem [23, 24] and the ensuing Positivstellensätze [26, 58–60]. Notably, D’Angelo and Putinar [25] proved in 2008 that a positive complex polynomial on a sphere intersected by a finite number of polynomial inequality constraints can be decomposed as a weighted sum of the constraints where the weights are sums of squared moduli of holomorphic polynomials. Similar to Lasserre [41] and Parrilo [56], we use D’Angelo’s and Putinar’s Positivstellensatz to construct a complex moment/sum-of-squares hierarchy of semidefinite programs to solve complex polynomial optimization problems with compact feasible sets. To satisfy the assumption in the Positivstellensatz, we propose to add a slack variable \(z_{n+1} \in \mathbb{C}\) and a redundant constraint \(|z_1|^2 + \ldots + |z_{n+1}|^2 = R^2\) to the description of the feasible set when it is in a ball of radius \(R\). The complex hierarchy is more tractable.
than the real hierarchy yet produces potentially weaker bounds. Computational advantages are shown using the optimal power flow problem in electrical engineering. In addition to global convergence of the bounds, the complex hierarchy is endowed with sufficient conditions for extracting feasible points that are globally optimal.

The theoretical contributions of this paper regarding the complex hierarchy are:
1. its construction using real-valued Radon measures (Section 3) leading to a new notion of complex moment matrix and localization matrix (Remark 3.1) different from existing literature [21]; the Lasserre hierarchy [41] can thus be viewed as a special case of the proposed complex hierarchy (Figure 2);
2. a proof of global convergence (Proposition 3.2, Corollary 3.4); a sufficient condition for strong duality (Proposition 3.10); Karush-Kuhn-Tucker conditions involving complex sums-of-squares (Corollary 3.12); a multi-ordered hierarchy to exploit sparsity while preserving global convergence (Section 3.7);
3. a solution to a newly defined truncated complex moment problem (Theorem 3.8) different from existing literature [21, Theorem 5.1] which implies Curto and Fialkow’s solution of the real truncated moment problem (Corollary 3.9); as a result, sufficient conditions for extracting global solutions from the complex hierarchy (Proposition 3.5);
4. an invariant complex hierarchy whose convergence can be deduced from an invariant version of D’Angelo’s and Putinar’s Positivstellensatz (Proposition 3.13); in particular, an action of the torus in the complex plane (Proposition 3.14) and a subgroup of it (Proposition 3.15) are considered.

The paper is organized as follows. Section 2 uses Shor and second-order conic relaxations to motivate the complex moment/sum-of-squares hierarchy in Section 3. Using a sparsity-exploiting algorithm, numerical experiments on the optimal power flow problem are presented in Section 4. Section 5 concludes our work.

2. Motivation. Let $\mathbb{N}, \mathbb{N}^*, \mathbb{R}, \mathbb{R}_+$ and $\mathbb{C}$ denote the set of natural, positive natural, real, non-negative real, and complex numbers respectively. Also, let “$i$” denote the imaginary unit and $\mathbb{H}_n$ denote the set of Hermitian matrices of order $n \in \mathbb{N}^*$. Consider the subclass of complex polynomial optimization

\begin{equation}
\text{QCQP-}\mathbb{C} : \inf_{z \in \mathbb{C}^n} z^H H_0 z \quad \text{s.t.} \quad z^H H_i z \leq h_i, \quad i = 1, \ldots, m,
\end{equation}

where $m \in \mathbb{N}^*, H_0, \ldots, H_m \in \mathbb{H}_n, h_0, \ldots, h_m \in \mathbb{R}$, $(\cdot)^H$ denotes the conjugate transpose. The Shor [65] and second-order conic relaxations of QCQP-\(\mathbb{C}\) share the following property: it is better to relax non-convex constraints before converting from complex to real numbers rather than to do the two operations in the opposite order.

2.1. Shor Relaxation. For $H \in \mathbb{H}_n$ and $z \in \mathbb{C}^n$, the relationship $z^H H z = \text{Tr}(H zz^H)$ holds where $\text{Tr}(\cdot)$ denotes the trace $^1$ of a complex square matrix. Let $\succeq 0$ indicate positive semidefiniteness. Relaxing the rank of $Z = zz^H$ in (2.1) yields

\begin{align}
\text{SDP-}\mathbb{C} : \quad & \inf_{Z \in \mathbb{H}_n} \text{Tr}(H_0 Z) \\
\text{s.t.} \quad & \text{Tr}(H_i Z) \leq h_i, \quad i = 1, \ldots, m, \\
& Z \succeq 0,
\end{align}

\begin{footnote}
$^1$For all matrices $A, B \in \mathbb{C}^{n \times n}$, $\text{Tr}(AB) = \sum_{1 \leq i, j \leq n} A_{ij} B_{ji}$.
\end{footnote}
Let \( \text{Re}Z \) and \( \text{Im}Z \) denote the real and imaginary parts of the matrix \( Z \in \mathbb{C}^{n \times n} \) respectively. Consider the ring homomorphism \( \Lambda : (\mathbb{C}^{n \times n}, +, \times) \rightarrow (\mathbb{R}^{2n \times 2n}, +, \times) \)

\[
\Lambda(Z) := \begin{pmatrix} \text{Re}Z & -\text{Im}Z \\ \text{Im}Z & \text{Re}Z \end{pmatrix}.
\]

To convert SDP-\( \mathbb{C} \) into real numbers, real and imaginary parts of the complex matrix variable are identified using two properties: (1) a complex matrix \( Z \) is positive semidefinite if and only if the real matrix \( \Lambda(Z) \) is positive semidefinite, and (2) if \( Z_1, Z_2 \in \mathbb{H}_n \), then \( \text{Tr}[\Lambda(Z_1)\Lambda(Z_2)] = \text{Tr}[\Lambda(Z_1Z_2)] = 2\text{Tr}(Z_1Z_2) \). This yields

\[
\begin{align*}
(2.4a) & \quad \text{CSDP-} \mathbb{R} : \inf_{X \in \mathbb{S}_{2n}} \text{Tr}(\Lambda(H_0)X) \\
(2.4b) & \quad \text{s.t. } \text{Tr}(\Lambda(H_i)X) \leq h_i, \quad i = 1, \ldots, m, \\
(2.4c) & \quad X \succeq 0, \\
(2.4d) & \quad X = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \quad \& \quad A = C, \quad B^T = -B,
\end{align*}
\]

where \( \mathbb{S}_{2n} \) denotes the set of real symmetric matrices of order \( 2n \) and \((\cdot)^T\) indicates the transpose. Note that the set of matrices satisfying \( (2.4d) \) is isomorphic to \( \mathbb{C}^{n \times n} \).

A global solution to QCQP-\( \mathbb{C} \) can be retrieved from CSDP-\( \mathbb{R} \) if and only if \( \text{rank}(X) \in \{0, 2\} \) at optimality (proof in Appendix A). In order to convert QCQP-\( \mathbb{C} \) into real numbers, real and imaginary parts of the complex vector variable are identified. This is done by considering a new variable \( x = \left( (\text{Re}z)^T \ (\text{Im}z)^T \right)^T \) and observing that if \( H \in \mathbb{H}_n \), then \( z^THz = x^T\Lambda(H)x = \text{Tr}(\Lambda(H)xx^T) \). This gives rise to a problem which we will call QCQP-\( \mathbb{R} \). Relaxing the rank of \( X = xx^T \) yields

\[
\begin{align*}
(2.5a) & \quad \text{SDP-} \mathbb{R} : \inf_{X \in \mathbb{S}_{2n}} \text{Tr}(\Lambda(H_0)X) \\
(2.5b) & \quad \text{s.t. } \text{Tr}(\Lambda(H_i)X) \leq h_i, \quad i = 1, \ldots, m, \\
(2.5c) & \quad X \succeq 0.
\end{align*}
\]

A global solution to QCQP-\( \mathbb{C} \) can be retrieved from SDP-\( \mathbb{R} \) if and only if \( \text{rank}(X) \in \{0, 1\} \) or \( \text{rank}(X) = 2 \) and \( (2.4d) \) holds at optimality. We have \( \text{val(SDP-} \mathbb{C}) = \text{val(CSDP-} \mathbb{R}) = \text{val(SDP-} \mathbb{R}) \) where “val” is the optimal value of a problem (proof in Appendix B). The number of scalar variables of CSDP-\( \mathbb{R} \) is half that of SDP-\( \mathbb{R} \) due to constraint \((2.4d)\). This constraint also halves the possible ranks of the matrix variable, which must be an even integer in CSDP-\( \mathbb{R} \) whereas it can be any integer between 0 and \( 2n \) in SDP-\( \mathbb{R} \). The number of variables in SDP-\( \mathbb{R} \) can be reduced by a small fraction \( \frac{2}{2n+1} \) to be exact by setting a diagonal element of \( X \) to 0. This does not affect the optimal value (proof in Appendix C). See Figure 1 for a summary.

### 2.2. Second-Order Conic Relaxation

In SDP-\( \mathbb{C} \) of Section 2.1, assume that the semidefinite constraint \((2.2c)\) is relaxed to the second-order cones

\[
\begin{pmatrix} Z_{ii} & Z_{ij} \\ Z_{ij} & Z_{jj} \end{pmatrix} \succeq 0, \quad 1 \leq i \neq j \leq n.
\]

Equation \((2.6)\) is equivalent to constraining the determinant \( Z_{ii}Z_{jj} - Z_{ij}Z_{ij}^T \) and diagonal elements \( Z_{ii} \) to be non-negative. This yields SOCP-\( \mathbb{C} : \inf_{Z \in \mathbb{H}_n} \text{Tr}(H_0Z) \) s.t. \((2.2b)\), \(|Z_{ij}|^2 \leq Z_{ii}Z_{jj} \) for \( 1 \leq i \neq j \leq n \), and \( Z_{ii} \geq 0 \) for \( i = 1, \ldots, n \) where \(|\cdot|\) denotes
the complex modulus. Identifying real and imaginary parts of the matrix variable $Z$
leads to CSOCP-R : \( \inf_{X \in \mathbb{S}_n} \operatorname{Tr}(\Lambda(H_0)X) \) s.t. \( (2.4b), (2.4d), X_{ij}^2 + X_{n+i,j}^2 \leq X_{ii}X_{jj} \)
for \( 1 \leq i \neq j \leq n \), and \( X_{ii} \geq 0 \) for \( i = 1, \ldots, n \). In SDP-R of Section 2.1, assume that
the semidefinite constraint \( (2.5c) \) is relaxed to the second-order cones
\[
(2.7) \quad \begin{pmatrix} X_{ii} & X_{ij} \\ X_{ij} & X_{jj} \end{pmatrix} \succeq 0, \quad 1 \leq i \neq j \leq 2n.
\]
This leads to SOCP-R : \( \inf_{X \in \mathbb{S}_n} \operatorname{Tr}(\Lambda(H_0)X) \) s.t. \( (2.5b), X_{ij}^2 \leq X_{ii}X_{jj} \)
for \( 1 \leq i \neq j \leq 2n \), and \( X_{ii} \geq 0 \) for \( i = 1, \ldots, 2n \). We have \( \text{val}(\text{SOCP-R}) = \text{val}(\text{CSOCP-R}) \geq \text{val}(\text{SOCP-R}) \) (proof in Appendix D). The number of variables of CSOCP-R is half that
of SOCP-R due to constraint \( (2.4d) \). The number of second-order conic constraints in
CSOCP-R, equal to \( \frac{n(n-1)}{2} \), is roughly a fourth of that in SOCP-R, equal to \( \frac{2n(2n-1)}{2} \).

2.3. Exploiting Sparsity. The properties of chordal graphs enable sparsity
exploitation for the Shor relaxation [73]. Given an undirected graph \((V, \mathcal{E})\) where
\( V \subseteq \{1, \ldots, n\} \) and \( \mathcal{E} \subseteq V \times V \), define for all \( Z \in \mathbb{H}_n \)
\[
(2.8) \quad \Psi_{(V, \mathcal{E})}(Z)_{ij} := \begin{cases} Z_{ij} & \text{if } (i, j) \in \mathcal{E} \text{ or } i = j \in V, \\ 0 & \text{else}. \end{cases}
\]

We associate an undirected graph \( \mathcal{G} \) to QCQP-C whose nodes are \( \{1, \ldots, n\} \) and
that satisfies \( H_i = \Psi_{\mathcal{G}}(H_i) \) for \( i = 0, \ldots, m \). Let \( \mathbb{H}_n^+ \) denote the set of positive
semidefinite Hermitian matrices of size \( n \) and let “Ker” denote the kernel of a linear
application. Given the definition of \( \mathcal{G} \), constraint \( (2.2c) \) of SDP-C can be relaxed to
\( Z \in \mathbb{H}_n^+ + \text{Ker} \Psi_{\mathcal{G}} \) without changing its optimal value for any graph \( \mathcal{G} \) whose nodes
are \( \{1, \ldots, n\} \) and where \( \mathcal{G} \subset \mathcal{G} \). Consider a chordal extension \( \mathcal{G} \subset \mathcal{G}_{\text{ch}} \), that is
to say that all cycles of length four or more have a chord (edge between two non-
consecutive nodes of the cycle). Let \( \mathcal{C}_1, \ldots, \mathcal{C}_p \subset \mathcal{G}_{\text{ch}} \) denote the maximal cliques of
\( \mathcal{G}_{\text{ch}} \). (A clique is a subgraph where all nodes are linked to one another. The set of
maximally sized cliques of a chordal graph can be computed in linear time [68].) A
chordal extension has a useful property for exploiting sparsity [32]: for all \( Z \in \mathbb{H}_n \),
we have that \( Z \in \mathbb{H}_n^+ + \text{Ker} \Psi_{\mathcal{G}_{\text{ch}}} \) if and only if \( \Psi_{\mathcal{C}_i}(Z) \gg 0 \) for \( i = 1, \ldots, p \). Note that
\( \Psi_{\mathcal{C}_i}(Z) \gg 0 \) if and only if \( \Lambda \circ \Psi_{\mathcal{C}_i}(Z) \gg 0 \), where “\( \circ \)” is the composition of functions.
Given a graph \((\mathcal{V}, \mathcal{E})\), define for \(X \in \mathbb{S}_{2n}\)

\[
(2.9) \quad \hat{\Psi}_{(\mathcal{V}, \mathcal{E})}(X) := \begin{pmatrix}
\Psi_{(\mathcal{V}, \mathcal{E})}(A) & \Psi_{(\mathcal{V}, \mathcal{E})}(B) \\
\Psi_{(\mathcal{V}, \mathcal{E})}(B)^T & \Psi_{(\mathcal{V}, \mathcal{E})}(C)
\end{pmatrix},
\]

using the block decomposition in the left hand part of (2.4d). Notice that \(\Lambda \circ \Psi_{(\mathcal{V}, \mathcal{E})} = \hat{\Psi}_{(\mathcal{V}, \mathcal{E})} \circ \Lambda\). As a result, (2.4c) can be replaced by \(\hat{\Psi}_{\mathbb{C}}(X) \succcurlyeq 0\) for \(i = 1, \ldots, p\) without changing the optimal value of CSDP-\(\mathbb{R}\), with an analogous replacement for constraint \((2.5c)\) in SDP-\(\mathbb{R}\). If in SDP-\(\mathbb{R}\) we exploit the sparsity of matrices \(\Lambda(H_i)\) instead of that of \(H_i\), the resulting graph has twice as many nodes. Computing a chordal extension and maximal cliques is hence more costly. Sparsity in the second-order conic relaxations is exploited using the fact that applying constraints only for \((i, j)\) that are edges of \(\mathcal{G}\) does not change the optimal values of CSOCP-\(\mathbb{R}\) and SOCP-\(\mathbb{R}\).

3. Complex Moment/Sum-of-Squares Hierarchy. We transpose [41] from real to complex numbers. Let \(z^\alpha\) denote the monomial \(z_1^{\alpha_1} \cdots z_n^{\alpha_n}\) where \(z \in \mathbb{C}^n\) and \(\alpha \in \mathbb{N}^n\) for some integer \(n \in \mathbb{N}^+\). Let \(|\alpha| := \alpha_1 + \cdots + \alpha_n\) and define \(\overline{w}\) as the conjugate of \(w \in \mathbb{C}\). Define \(\bar{z} := (\bar{z}_1, \ldots, \bar{z}_n)^T\) where \(z \in \mathbb{C}^n\). Consider the sets where \(d \in \mathbb{N}\)

\[
\begin{align*}
\mathbb{C}[z] &:= \{ p : \mathbb{C}^n \to \mathbb{C} | p(z) = \sum_{|\alpha| \leq l} p_\alpha z^\alpha, \ l \in \mathbb{N}, \ p_\alpha \in \mathbb{C} \}, \\
\mathbb{C}[\bar{z}, z] &:= \{ f : \mathbb{C}^n \to \mathbb{C} | f(z) = \sum_{|\alpha|, |\beta| \leq l} f_{\alpha, \beta} z^\alpha \bar{z}^\beta, \ l \in \mathbb{N}, \ f_{\alpha, \beta} \in \mathbb{C} \}, \\
\mathbb{R}[\bar{z}, z] &:= \{ f \in \mathbb{C}[\bar{z}, z] | \overline{f(z)} = f(z), \ \forall z \in \mathbb{C}^n \}, \\
\Sigma[z] &:= \{ \sigma : \mathbb{C}^n \to \mathbb{C} | \sigma = \sum_{j=1}^r |p_j|^2, \ r \in \mathbb{N}^+, \ p_j \in \mathbb{C}[z] \},
\end{align*}
\]

Note that the coefficients of a function \(f \in \mathbb{R}[\bar{z}, z]\) satisfy \(\overline{f_{\alpha, \beta}} = f_{\beta, \alpha}\) for all \(|\alpha|, |\beta| \leq l\) for some \(l \in \mathbb{N}\). The set of complex polynomials \(\mathbb{C}[\bar{z}, z]\) is a \(\mathbb{C}\)-algebra (i.e. commutative ring and vector space over \(\mathbb{C}\)) and the set of holomorphic polynomials \(\mathbb{C}[z]\) is a subalgebra of it (i.e. subspace closed under sum and product). The set of real-valued complex polynomials \(\mathbb{R}[\bar{z}, z]\) is an \(\mathbb{R}\)-algebra. The set of sums of squared moduli of holomorphic polynomials \(\Sigma[z]\) and the set \(\Sigma_d[z] \subset \mathbb{R}_d[z]\) are pointed cones (i.e. closed under multiplication by elements of \(\mathbb{R}_+\)) that are convex (i.e. \(tu + (1 - t)v\) with \(0 \leq t \leq 1\) belongs to them if \(u\) and \(v\) do). Let \(C(K, \mathbb{C})\) denote the Banach (i.e. complete) \(\mathbb{C}\)-algebra of continuous functions from a compact set \(K \subset \mathbb{C}^n\) to \(\mathbb{C}\) equipped with the norm \(\|\varphi\|_{\infty} := \sup_{z \in K} |\varphi(z)|\). Consider \(R_K : \mathbb{C}[\bar{z}, z] \to C(K, \mathbb{C})\) defined by \(f \mapsto f_{|K}\) where \(f_{|K}\) denotes the restriction of \(f\) to \(K\). \(R_K(\mathbb{C}[z, \bar{z}])\) is a unital subalgebra of \(C(K, \mathbb{C})\) (i.e. contains multiplicative unit) that separates points of \(K\) (i.e. \(u \neq v \in K \implies \exists \varphi \in R_K(\mathbb{C}[\bar{z}, z]) : \varphi(u) \neq \varphi(v)\)) and that is closed under complex conjugation. It is hence a dense subalgebra due to the Complex Stone-Weierstrass Theorem. Likewise, \(C(K, \mathbb{R}) := \{ \varphi \in C(K, \mathbb{C}) | \overline{\varphi(z)} = \varphi(z), \ \forall z \in \mathbb{C}^n \}\) is a Banach \(\mathbb{R}\)-algebra of which \(R_K(\mathbb{R}[\bar{z}, z])\) is a dense subalgebra. In other words, a continuous real-valued function of multiple complex variables can be approximated as close as desired by real-valued complex polynomials when restricted to a compact set. They are hence a powerful modeling tool in optimization. Speaking of which, let \(m \in \mathbb{N}^+\) and \(k, k_1, \ldots, k_m \in \mathbb{N}\). Consider \((f, g_1, \ldots, g_m) \in \mathbb{R}_k[\bar{z}, z] \times \mathbb{R}_{k_1}[\bar{z}, z] \times \cdots \times \mathbb{R}_{k_m}[\bar{z}, z]\) where
there exists $|\alpha| = k$ and $|\beta| \leq k$ such that $f_{\alpha,\beta} \neq 0$. In addition, for $i = 1, \ldots, m$, there exists $|\alpha| = k_i$ and $|\beta| \leq k_i$ such that $g_{i,\alpha,\beta} \neq 0$. Consider the problem

\begin{equation}
(3.3) \quad f_{\text{opt}} := \inf_{z \in \mathbb{C}^n} \int f(z) \quad \text{s.t.} \quad g_i(z) \geq 0, \quad i = 1, \ldots, m,
\end{equation}

where $f_{\text{opt}} := +\infty$ if the feasible set is empty. The feasible set $K := \{ z \in \mathbb{C}^n \mid g_i(z) \geq 0, \quad i = 1, \ldots, m \}$ is assumed to be compact. Let $K_{\text{opt}}$ denote the set of optimal solutions to (3.3) and $M(K)$ denote the Banach space over $\mathbb{R}$ of Radon measures on $K$. Since $K$ is compact, $M(K)$ may be identified with the set of linear continuous applications from $C(K, \mathbb{R})$ to $\mathbb{R}$ equipped with the operator norm (Riesz Representation Theorem). For $\varphi \in C(K, \mathbb{C})$, define $\int_K \varphi d\mu := \int_K \text{Re}(\varphi) d\mu + i \int_K \text{Im}(\varphi) d\mu$ \cite[1.31 Definition] {3}.

Consider the convex pointed cone $P$ to (3.7) $f$ $\frac{1}{2}$ $\int$ $\text{since, in the second case, we may consider}$ $f$ $\frac{1}{2}$ $\int$ $\text{to}$ $\text{consider}$ $f$ $\frac{1}{2}$ $\int$ $\text{the}$ $\text{Dirac}$ $\text{measure}$ $\delta_z$ $\text{is}$ $\text{a}$ $\text{feasible}$ $\text{point}$ $\text{of}$ $\text{(3.4)}$ $\text{for}$ $\text{which}$ $\text{the}$ $\text{objective}$ $\text{value}$ $\text{is}$ $\text{equal}$ $\text{to}$ $f(z)$. $\text{Hence}$ $\text{the}$ $\text{optimal}$ $\text{value}$ $\text{of}$ $\text{(3.4)}$ $\text{is}$ $\text{less}$ $\text{than}$ $\text{or}$ $\text{equal}$ $\text{to}$ $f_{\text{opt}}$. $\text{Conversly, if}$ $\mu$ $\text{is}$ $\text{a}$ $\text{feasible}$ $\text{point}$ $\text{of}$ $\text{(3.4)}$, $\text{then}$ $\int_K (f - f_{\text{opt}}) d\mu \geq 0$ $\text{and}$ $\text{hence}$ $\int_K f d\mu \geq f_{\text{opt}} \int_K d\mu = f_{\text{opt}}$.

**Proposition 3.1.** The set of optimal solutions to (3.4) is

\begin{equation}
(3.5) \quad \{ \mu \in M(K) \mid \mu(K_{\text{opt}}) = 1 \quad \& \quad \mu(K \setminus K_{\text{opt}}) = 0 \}\.
\end{equation}

As a consequence, if $K_{\text{opt}}$ is a finite set of $S \in \mathbb{N}^*$ points $z(1), \ldots, z(S) \in \mathbb{C}^n$, then the optimal solutions to (3.4) are \{ $\sum_{j=1}^S \lambda_j \delta_{z(j)} \mid \sum_{j=1}^S \lambda_j = 1 \quad \& \quad \lambda_1, \ldots, \lambda_S \in \mathbb{R}^+$ \}.

**Proof.** Consider $\mu$ an optimal solution to (3.4). It must be that $\int_K (f - f_{\text{opt}}) d\mu = 0$. Thus $\int_{K \setminus K_{\text{opt}}} (f - f_{\text{opt}}) d\mu = 0$ and $\mu(K \setminus K_{\text{opt}}) = 0$. Therefore $\int_{K \setminus K_{\text{opt}}} d\mu = \mu(K) - \mu(K \setminus K_{\text{opt}}) = 1$. Conversely, if $\mu$ belongs to the set in (3.5), then it is feasible for (3.4) and $\int_K (f - f_{\text{opt}}) d\mu = 0$.

Hence $\int_K f d\mu = \int_K f_{\text{opt}} d\mu = f_{\text{opt}} \int_K d\mu = f_{\text{opt}}$. \[ \square \]

In order to dualize the equality constraint in (3.4), consider the Lagrange function $L : M_{+}(K) \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $(\mu, \lambda) \mapsto \int_K f d\mu + \lambda (1 - \int_K d\mu)$. We have $L(\mu, \lambda) = \lambda + \int_K (f - \lambda) d\mu$ and

\begin{equation}
(3.6) \quad \inf_{\mu \in M_{+}(K)} \int_K (f - \lambda) d\mu = \begin{cases} 0 & \text{if } f(z) - \lambda \geq 0, \quad \forall z \in K, \\ -\infty & \text{else,} \end{cases}
\end{equation}

since, in the second case, we may consider $t\delta_z$ for a $z \in K$ such that $f(z) - \lambda < 0$ and $t \rightarrow +\infty$. This leads to the dual problem

\begin{equation}
(3.7) \quad f_{\text{opt}} = \sup_{\lambda \in \mathbb{R}} \lambda \quad \text{s.t.} \quad f(z) - \lambda \geq 0, \quad \forall z \in K.
\end{equation}

Primal problem (3.4) gives rise to the complex moment hierarchy in Section 3.1. Dual problem (3.7) gives rise to the complex sum-of-squares hierarchy in Section 3.2.

\[ ^2 \text{We wish to thank Bruno Nazaret for bringing this reference to our attention.} \]

\[ ^3 \text{The Dirac measure } \delta_z \text{ with } z \in K \text{ may be identified with the continuous linear application from} \]

\[ C(K, \mathbb{R}) \text{ to } \mathbb{R} \text{ defined by } \varphi \mapsto \varphi(z). \text{ This is one way to interpret the fact that } \int_K f d\delta_z = f(z). \]
3.1. Complex Moment Hierarchy. Let \( \mathcal{H} \) (respectively \( \mathcal{H}_d \)) denote the set of sequences of complex numbers \((y_{\alpha, \beta})_{\alpha, \beta \in \mathbb{N}^n} \) (respectively \((y_{\alpha, \beta})_{|\alpha|, |\beta| \leq d}\)) such that \( y_{\alpha, \beta} = y_{\beta, \alpha} \) for all \( \alpha, \beta \in \mathbb{N}^n \) (respectively \(|\alpha|, |\beta| \leq d\)). An element \( y \in \mathcal{H} \) is said to have a representing measure \( \mu \) on \( K \) if \( \mu \in M_+(K) \) and \( y_{\alpha, \beta} = \int_K \bar{z}^\alpha z^\beta d\mu \) for all \( \alpha, \beta \in \mathbb{N}^n \). When \( y \in \mathcal{H} \) has a representing measure on \( K \), the measure is unique because \( R_K(C[z, \bar{z}]) \) is dense in \( C(K, \mathbb{C}) \). The moment problem consists in characterizing the sequences that are representable by a measure on \( K \). For example, Atzmon [5, Theorem 2.1] proved that when \( K = \{ z \in \mathbb{C} \mid |z| = 1 \} \) the solutions are the sequences \( y \in \mathcal{H} \) such that \( \sum_{m,n,j,k \in \mathbb{N}} c_{m,j} \bar{z}_m y_{m+j,n+k} \geq 0 \) and \( \sum_{m,n \in \mathbb{N}} w_m \overline{w}_n (y_{m,n} - y_{m+1,n+1}) \geq 0 \) for all complex numbers \((c_j)_{j \in \mathbb{N}} \) and \((w_n)_{n \in \mathbb{N}} \) with only finitely many non-zero terms. Theorem 3.7 below generalizes this result.

Consider a feasible point \( \mu \) of (3.4) and the sequence \( y \in \mathcal{H} \) that has representation measure \( \mu \). Notice that \( \int_K f d\mu = \int_K \sum_{|\alpha|, |\beta| \leq d} f_{\alpha, \beta} \bar{z}^\alpha z^\beta d\mu = \sum_{|\alpha|, |\beta| \leq d} f_{\alpha, \beta} \int_K \bar{z}^\alpha z^\beta d\mu = \sum_{|\alpha|, |\beta| \leq d} f_{\alpha, \beta} y_{\alpha, \beta} = \langle L_{\mu} f, 1 \rangle \) and \( \int_K d\mu = \int_K \bar{z}^\alpha z^\beta d\mu = y_{0,0} \). For all \( p \in \mathbb{C}[z] \), we have \( |p|^2 g_i \geq 0 \) on \( K \). Since \( \mu \geq 0 \), this implies that \( \int_K |p|^2 g_i d\mu \geq 0 \). Naturally, we also have \( \int_K |p|^2 g_0 d\mu \geq 0 \) if we define \( g_0 := 1 \). Define \( k_0 := 0 \) and \( d_{\text{min}} := \max\{k, k_1, \ldots, k_m\} \). Consider \( d \geq d_{\text{min}} \), \( 0 \leq i \leq m \), and \( p \in \mathbb{C}_{d-k_1}[z] \). We have \( \int_K |p|^2 g_i d\mu = \int_K \sum_{|\alpha| \leq d-k_1} P_{\alpha} z^\alpha \bar{z}^\delta d\mu = \int_K \sum_{|\alpha|, |\beta| \leq d-k_1, \sum_{|\gamma|, |\delta| \leq d-k_1} P_{\alpha} P_{\beta} \bar{z}^\alpha z^\beta \bar{z}^\gamma z^\delta d\mu = \int_K \sum_{|\alpha|, |\beta| \leq d-k_1} P_{\alpha} P_{\beta} \bar{z}^\alpha z^\beta \bar{z}^\gamma z^\delta d\mu = M_{d-k_1}(g_i) \) where \( \rho := (p_0)_{|\alpha| \leq d-k_1} \) and \( M_{d-k_1}(g_i) \) is a Hermitian matrix indexed by \( |\alpha|, |\beta| \leq d - k_1 \). To sum up, \( y \) is a feasible point of

\[
\rho := \inf_{y \in \mathcal{H}} L_y(f) \quad \text{s.t.} \quad y_{0,0} = 1, \quad M_{d-k_1}(g_i) \geq 0, \quad i = 0, \ldots, m, \quad \forall d \geq d_{\text{min}},
\]

with same objective value as \( \mu \) in (3.4). Automatically, \( \rho \leq f_{\text{opt}} \). Consider the relaxation of (3.8) defined by

\[
\rho_d := \inf_{y \in \mathcal{H}_d} L_y(f) \quad \text{s.t.} \quad y_{0,0} = 1, \quad M_{d-k_1}(g_i) \geq 0, \quad i = 0, \ldots, m,
\]

which we name the complex moment relaxation of order \( d \) for reasons that will become clear with Theorem 3.7. In Section 3.2, we will introduce its dual counterpart.

Remark 3.1. Given \( y \in \mathcal{H} \), the function \( L_y \) in this section can be formally be defined by the \( \mathbb{C} \)-linear operator \( L_y : \mathbb{C}[\bar{z}, z] \rightarrow \mathbb{C} \) such that \( L_y(z^n \bar{z}^m) = y_{n,m} \) for all \( n, m \in \mathbb{N} \) (i.e. Riesz functional). If \( \varphi \in \mathbb{C}[\bar{z}, z] \) and \( \varphi = \varphi_0 \), then \( L_y(\varphi) = L_y(\varphi_0) \).

Given \( l, d \in \mathbb{N} \) and \( \varphi \in \mathbb{R}[\bar{z}, z] \), the matrix \( M_d(\varphi) \) can be formally be defined as the Hermitian matrix indexed by \( |\alpha|, |\beta| \leq d \) such that \( M_d(\varphi)(\alpha, \beta) := L_y(\varphi(z) \bar{z}^\alpha z^\beta) = \sum_{|\gamma|, |\delta| \leq d} \varphi_{\gamma, \delta} y_{\alpha+\gamma, \beta+\delta} \). Notice that \( M_d(\varphi)(0,0) = L_y(\varphi) \). Lastly, define \( M_d(y) := M_d(g_i) \) which we refer to as the complex moment matrix of order \( d \).

3.2. Complex Sum-of-Squares Hierarchy. Given \( l \in \mathbb{N} \) and \( \varphi \in \mathbb{R}[\bar{z}, z] \), define \( \varphi := (\varphi_{\alpha, \beta})_{|\alpha|, |\beta| \leq l} \). This notation is well-defined due to the unicity of the coefficients of \( \varphi \). Notice that \( \varphi \in \Sigma[z] \) if and only if \( \varphi \geq 0 \). Also, define \( \langle A, B \rangle_{\mathcal{H}_d} := \) 4 The notation is ill-defined in the real case: if \( \varphi : x \in \mathbb{R}^n \rightarrow \sum_{|\alpha|, |\beta| \leq l} \varphi_{\alpha, \beta} x^\alpha \bar{x}^\beta \in \mathbb{R} \), then the coefficients \( \varphi_{\alpha, \beta} \) \( \in \mathbb{R} \) are not unique. Thus \( \sum_{|\alpha| \leq d} \sigma_\alpha x^\alpha \) is a real sum of squares if and only if
Tr(AB) where A, B ∈ ℱd. Given d ≥ dmin, consider the Lagrange function L_d : ℱd × ℜ×Σd−k,[z] → ℜ defined by (y, λ, σ_0, ..., σ_m) → Ly(f) + λ(1−y_0,0)−∑_{i=0}^{m}(M_{d−k,[i]}(g_i,y)), given σ_i = ∑_{j=0}^{m}p_j^i, i.e. σ_i = ∑_{j=0}^{m}p_j^i (p_j^i)^H, compute L_d(y, λ, σ_0, ..., σ_m) = λ + Ly(f) = Ly(f, f, f, ..., f) = λ + Ly(f, f, f, ..., f). Observe that

\[
\inf_{y \in H} L_y \left( f - \lambda - \sum_{i=0}^{m} \sigma_i g_i \right) = \begin{cases} 0 & \text{if } f(z) - \lambda - \sum_{i=0}^{m} \sigma_i g_i(z) = 0, \\ -\infty & \text{else} \end{cases}
\]

Indeed, in the second case, there exists z ∈ ℂ^n such that f(z) − λ − ∑_{i=0}^{m} σ_i g_i(z) ≠ 0. With (y_{α, β})_{α, β} ∈ ℂ^m, Ly(f) − ∑_{i=0}^{m} σ_i g_i → −∞ for either t → −∞ or t → +∞. The associated dual problem of (3.9) is thus

\[
\rho^*_d := \sup_{λ, \sigma} \lambda \\
\text{s.t. } f - λ - \sum_{i=0}^{m} \sigma_i g_i \\
λ ∈ ℜ, \ σ_i ∈ Σd−k,[z], \ i = 0, ..., m.
\]

which we name the complex sum-of-squares relaxation of order d. Consider

\[
\rho^* := \sup_{λ, \sigma} \lambda \\
\text{s.t. } f - λ - \sum_{i=0}^{m} \sigma_i g_i \\
λ ∈ ℜ, \ σ_i ∈ Σ[z], \ i = 0, ..., m.
\]

**Proposition 3.2.** We have ρ^*_d ≤ ρ^*_d for all d ≥ dmin and ρ^*_d → ρ^* ≤ ρ ≤ f^opt.

Proof. The sequence (ρ^*_d)_{d≥dmin} is non-decreasing and upper bounded by ρ^* ∈ ℜ∪{±∞}. Thus it converges towards some limit ρ^*_lim ∈ ℜ∪{±∞} such that ρ^*_lim ≤ ρ^*. If ρ^* = −∞, then ρ^*_d = −∞ for all d ≥ dmin and ρ^*_d → ρ^*. If not, by definition of the optimum ρ^*, there exists a sequence (λ^l, σ^l_0, ..., σ^l_m) of feasible points such that λ^l ≤ ρ^* and λ^l → ρ^*. To each l ∈ ℤ, we may associate an integer d(l) ∈ ℤ such that (λ^l, σ^l_0, ..., σ^l_m) is a feasible point of the complex sum-of-squares relaxation of order d(l). Thus λ^l ≤ ρ^*_d(l) ≤ ρ^*. As a result, ρ^*_lim = ρ^*. Moreover, (ρ_d)_{d≥dmin} is non-decreasing and upper bounded by ρ ∈ ℜ∪{±∞}. Thus it converges towards some limit ρ^lim ∈ ℜ∪{±∞} such that ρ^lim ≤ ρ. Moreover, weak duality implies that ρ^*_d ≤ ρ^lim ≤ ρ. It was shown in Section 3.1 that ρ ≤ f^opt. □

**Remark 3.2.** Problems (3.12) and (3.8) may be interpreted as a pair of primal-dual linear programs in infinite-dimensional spaces [4]. Consider the duality bracket ⟨., .⟩ defined from ℜ[z, z] × ℱ → ℜ by ⟨φ, y⟩ := Ly(φ). A sequence ⟨φ^n⟩_{n∈N} in ℜ[z, z] is said to converge weakly towards φ ∈ ℜ[z, z] if for all y ∈ ℱ, we have ⟨φ^n, y⟩ → ⟨φ, y⟩. Consider the weakly continuous ℜ-linear operator A : ℜ[z, z] → ℜ[z, z] defined by φ → φ − φ_0. Its dual A* : ℱ → ℱ is defined by y → y − y_0,0δ_0,0 where (δ_0,0,0,0) = 1 and (δ_0,α,β) = 0 if (α, β) ≠ (0, 0). Indeed, (Aφ, y) = ⟨φ, A* y⟩ for all (φ, y) ∈ ℜ[z, z] × ℱ. Consider the convex pointed cone defined by C := Σ[z|g_0 + ... + Σ[z|g_m] and its dual cone C* := {y ∈ ℱ | ∀φ ∈ C, ⟨φ, y⟩ ≥ 0}. If b := Af, then

\[
\begin{align*}
f_0,0 - ρ^* &= \inf_{φ ∈ ℜ[z, z]} ⟨φ, δ_0,0⟩ \\
f_0,0 - ρ &= \sup_{y ∈ ℱ} ⟨b, y⟩
\end{align*}
\]

there exists some real numbers ⟨φ_α, β⟩_{α, β} such that ∑_{|α|≤2} σ_α x^α = ∑_{|α|≤2} φ_α, β x^α x^β.
Let cl$(C)$ denote the weak closure of $C$ in $\mathbb{R}[z, \bar{z}]$. [2, 5.9.1 Bipolar Theorem\(^5\)] implies that cl$(C) = C^{**}$. Below, Theorem 3.3 and Theorem 3.7 provide a sufficient condition ensuring no duality gap in (3.13) and cl$(C) = \{ \varphi \in \mathbb{R}[z, \bar{z}] \mid \varphi|_K \geq 0 \}$ respectively.

3.3. Convergence of the Complex Hierarchy. We turn our attention to a result from algebraic geometry discovered in 2008.

**Theorem 3.3 (D’Angelo’s and Putinar’s Positivstellensatz [25]).** If one of the constraints that define $K$ is a sphere constraint $|z_1|^2 + \ldots + |z_n|^2 = 1$, and if $f|_K > 0$, then there exists $\sigma_0, \ldots, \sigma_m \in \Sigma[z]$ such that $f = \sum_{i=0}^m \sigma_ig_i$.

**Proof.** D’Angelo and Putinar wrote the theorem slightly differently. Say that constraints $g_{m-1}$ and $g_m$ are such that $g_{m-1} = s$ and $g_m = -s$ where $s(z) := 1 - |z_1|^2 - \ldots - |z_n|^2$. With the assumptions of Theorem 3.3, the authors of [25, Theorem 3.1] show that there exists $\sigma_0, \ldots, \sigma_{m-2} \in \Sigma[z]$ and $r \in \mathbb{R}[z, \bar{z}]$ such that $f(z) = \sum_{i=0}^{m-2} \sigma_i(z)g_i(z) + r(z)s(z)$ for all $z \in \mathbb{C}^n$. Thanks to [24, Proposition 1.2], there exists $\sigma_{m-1}, \sigma_m \in \Sigma[z]$ such that $r = \sigma_m - \sigma_m$ hence the desired result. \(\square\)

Theorem 3.3 can easily be generalized to any sphere $|z_1|^2 + \ldots + |z_n|^2 = R^2$ of radius $R > 0$. With scaled variable $w = \frac{z}{R} \in \mathbb{C}^n$, the sphere constraint has radius 1 and a monomial of $\binom{3.3}{3.3}$ with coefficient $c_{\alpha, \beta} \in \mathbb{C}$ reads $c_{\alpha, \beta}z^\alpha \bar{z}^\beta = c_{\alpha, \beta}(R \bar{w})^\alpha(R w)^\beta = R^{\alpha + \beta}c_{\alpha, \beta}\bar{w}^\alpha w^\beta$. With the scaled coefficients $R^{\alpha + \beta}c_{\alpha, \beta}$, Theorem 3.3 can then be applied. Reverting back to the old scale $z = Rw$ leads to the desired result.

Accordingly, we define the following statement which is true only when stated:

\[(3.14) \text{ Sphere Assumption:} \quad \text{One of the constraints of (3.3) is a sphere } |z_1|^2 + \ldots + |z_n|^2 = R^2 \text{ for some } R > 0.\]

**Corollary 3.4.** Under the sphere assumption (3.14), $\rho^*_d \to f^{opt}$ and $\rho_d \to f^{opt}$.

**Proof.** Theorem 3.3 implies that $\rho^* = f^{opt}$ because for all $\epsilon > 0$, function $f - (f^{opt} - \epsilon)$ is positive on $K$. The sequences $(\rho^*_d)_{d \geq d_{\min}}$ and $(\rho_d)_{d \geq d_{\min}}$ converge towards $f^{opt}$ due to Proposition 3.2. \(\square\)

To require a sphere constraint in a complex polynomial optimization problem seems very restrictive and irrelevant for many problems. But in fact, a sphere constraint can be applied to any complex polynomial optimization problem (3.3) with a feasible set contained in a ball $|z_1|^2 + \ldots + |z_n|^2 \leq R^2$ of known radius $R > 0$. Indeed, simply add a slack variable $z_{n+1} \in \mathbb{C}$ and the constraint $|z_1|^2 + \ldots + |z_{n+1}|^2 = R^2$. Let $\tilde{K}$ denote the feasible set of the problem in $n+1$ variables. If $(z_1, \ldots, z_{n+1}) \in \tilde{K}$, then $(z_1, \ldots, z_n) \in K$ and has the same objective value. Conversely, if $(z_1, \ldots, z_n) \in K$, then $(z_1, \ldots, z_n, z_{n+1}) \in \tilde{K}$ for all $z_{n+1} \in \mathbb{C}$ such that $|z_{n+1}|^2 = R^2 - |z_1|^2 - \ldots - |z_n|^2$. Again, the objective value is unchanged. To ensure a bijection between $K$ and $\tilde{K}$, add yet two more constraints $|z_{n+1}|^2 - \sigma_{n+1} = 0$ and $z_{n+1} + \sigma_{n+1} \geq 0$, thereby preserving the number of global solutions. In that case, the application from $K$ to $\tilde{K}$ defined by $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, \sqrt{R^2 - |z_1|^2 - \ldots - |z_n|^2})$ is a bijection. Adding the two extra constraints is optional and not required for convergence of optimal values.

As seen in Theorem 3.3, an equality constraint may be enforced via two opposite inequality constraints. Let $h_1, \ldots, h_e$ denote $e \in \mathbb{N}^*$ equality constraints in polynomial optimization problem (3.3). Putinar and Scheiderer [59, Propositions 6.6 and 3.2 (iii)] show that the sphere assumption in D’Angelo’s and Putinar’s Positivstellensatz may

\(^5\)We wish to thank Jean-Bernard Baillon for bringing this reference to our attention.
be weakened to the existence of $r_1, \ldots, r_n \in \mathbb{R}[\bar{z}, z], \sigma \in \Sigma[z]$, and $a \in \mathbb{R}$ such that
\begin{equation}
\sum_{j=1}^{r} r_j(z) h_j(z) = \sum_{i=1}^{n} |z_i|^2 + \sigma(z) + a, \quad \forall z \in \mathbb{C}^n.
\end{equation}

If the constraints include $|z_1|^2 - 1 = \ldots = |z_n|^2 - 1 = 0$, the assumption is satisfied by $r_1 = \ldots = r_n = 1, \sigma = 0$ and $a = -n$. In particular, there is no need to add a slack variable in the non-bipartite Grothendieck problem over the complex numbers [8].

**Example 3.1.** D’Angelo and Putinar [25] consider $\frac{1}{4} < a < \frac{4}{3}$ and problem
\begin{equation}
\inf_{z \in \mathbb{C}} \quad f(z) := 1 - \frac{4}{3}|z|^2 + a|z|^4
\end{equation}
s.t. \quad \begin{align*}
\text{g}(z) &:= 1 - |z|^2 \geq 0,
\end{align*}
whose set of global solutions is $K^{opt} = \{z \in \mathbb{C} \mid |z| = 1\}$ and $f^{opt} = a - \frac{1}{3} > 0$. They prove that the decomposition $f = \sigma_0 + \sigma_1 g$ such that $f^{opt} > 0$. Indeed, if $\rho_0 > 0$ for some order $d \geq d^{min}$, then there exists $\lambda \geq \frac{\rho_0}{d}$ and $\sigma_0, \sigma_1 \in \Sigma_d[z]$ such that $f - \lambda = \sigma_0 + \sigma_1 g$. Thus $f = \lambda + \sigma_0 + \sigma_1 g$ where $\lambda + \sigma_0 \in \Sigma_d[z]$, which is a contradiction. We suggest solving
\begin{equation}
\inf_{z_1, z_2 \in \mathbb{C}} \quad f^*(z_1, z_2) := 1 - \frac{4}{3}|z_1|^2 + a|z_1|^4
\end{equation}
s.t. \quad \begin{align*}
\text{g}^*(z_1, z_2) &:= 1 - |z_1|^2 - |z_2|^2 = 0.
\end{align*}

For all $\lambda < f^{opt}$, there exists $\delta_0 \in \Sigma[z_1, z_2]$ and $\hat{r} \in \mathbb{R}[\bar{z}_1, z_2, z_1, z_2]$ such that $f(z_1, z_2) - \lambda = \delta_0(z_1, z_2) + \hat{r}(z_1, z_2)g(z_1, z_2)$ for all $z_1, z_2 \in \mathbb{C}$. Plug in $z_1 = z$ and $z_2 = 0$ and obtain $f(z) - \lambda = \delta_0(z, 0) + \hat{r}(z, 0)g(z)$ for all $z \in \mathbb{C}$. While function $z \mapsto \delta_0(z, 0)$ belongs to $\Sigma[z]$, function $z \mapsto \hat{r}(z, 0)$ does not! Hence we do not contradict the fact that $f = \sigma_0 + \sigma_1 g$ such that $f^{opt} > 0$. Notice that $d^{min} = 2$ for (3.16) and (3.17). The complex relaxations of orders $2 \leq d \leq 3$ of (3.16) yield the value $-0.3333$. The complex relaxation of order 2 of (3.17) yields the value 0.0565 ($\approx f^{opt}$) and optimal polynomials $\sigma_0(z_1, z_2) = 0.2780|z_1|^2 + 0.2776|z_1z_2|^2 + 0.6667|z_2|^4$ and $\hat{r}(z_1, z_2) = 0.9444 - 0.3889|z_1|^2 + 0.6665|z_2|^2$.

**Proposition 3.5.** Assume that the sphere assumption (3.14) holds, that $n > 1$, and that $y \in \mathcal{H}_d$ is an optimal solution to the complex moment relaxation of order $d \geq d^{min}$. With $d_K := \max_{1 \leq i \leq m} k_i (k_i \text{ is defined above (3.3)})$ and $d^{min} \leq t \leq d$, if
\begin{enumerate}
\item rank $M_t(y) = \text{rank} M_{t-d_K}(y) (=: S),$
\item
\begin{equation}
\begin{pmatrix}
M_{t-d_K}(y) & M_{t-d_K}(\bar{z}_j y) & M_{t-d_K}(\bar{z}_i) \\
M_{t-d_K}(\bar{z}_j y) & M_{t-d_K}(|z|^2 y) & M_{t-d_K}(\bar{z}_j z_i y) \\
M_{t-d_K}(\bar{z}_i) & M_{t-d_K}(\bar{z}_j z_i y) & M_{t-d_K}(|z|^2 y)
\end{pmatrix} > 0, \quad \forall 1 \leq i < j \leq n,
\end{equation}
\end{enumerate}
then $\rho_d = f^{opt}$ and complex polynomial problem (3.3) has at least $S$ global solutions.

**Proof.** Thanks to Theorem 3.8 below, $y \in \mathcal{H}_d$ can be represented by a measure $\mu$ on $K$ (i.e. $y_{\alpha, \beta} = \int_K z^{\alpha} \bar{z}^\beta d\mu, \forall |\alpha|, |\beta| \leq t$) and can thus be extended to $y \in \mathcal{H}$. The same theorem implies that $\mu = \sum_{j=1}^{S} \lambda_j z^{(j)}$ for some $S$ different point $z(1), \ldots, z(S)$ in $K$ and some $\lambda_1, \ldots, \lambda_S > 0$. In addition, $y_{0,0} = \int_K z^0 d\mu = \sum_{j=1}^{S} \lambda_j = 1$ and thus $f^{opt} \geq \rho_d = L_y(f) = \int_K f d\mu = \sum_{j=1}^{S} \lambda_j f(z(j)) \geq \sum_{j=1}^{S} \lambda_j f^{opt} = f^{opt}$. We simultaneously deduce that $\rho_d = f^{opt} = f(z(1)) = \ldots = f(z(S))$. \(\blacksquare\)

\textsuperscript{6}MATLAB 2013a, YALMIP 2015.06.26 [46], and MOSEK are used for the numerical experiments.
In particular, if $S = 1$ in Proposition 3.5, then Point 2 in Proposition 3.5 need not be checked for (see comment under (3.25)) and $y_{\alpha, \beta} = \int_K \bar{z}^\alpha \bar{z}^\beta d\bar{z} = \bar{z}^\alpha \bar{z}^\beta$, $\forall |\alpha|, |\beta| \leq d^{\min}$ for some $z \in K^\text{opt}$. A global solution can be read from $y$ because $z = (y_{\alpha, \beta})_{|\beta|=1}$.

Example 3.2. Putinar and Scheiderer [60] consider parameters $0 < a < \frac{1}{2}$ and $C > \frac{1}{1-2a}$, and problem

$$
\inf_{z \in \mathbb{C}} f(z) := C - |z|^2
$$

subject to

$$
g(z) := |z|^2 - a z^2 - a \bar{z}^2 - 1 = 0,
$$

whose set of global solutions is $K^\text{opt} = \left\{ \pm \frac{1}{\sqrt{1-2a}} \right\}$ and $f^\text{opt} = C - \frac{1}{1-2a} > 0$. They prove that the decomposition of Theorem 3.3 does not hold. Since the feasible set is included in the Euclidean ball of radius $\sqrt{C}$, we suggest solving

$$
\inf_{z_1, z_2 \in \mathbb{C}} f(z_1, z_2) := C - |z_1|^2
$$

subject to

$$
g_1(z_1, z_2) := |z_1|^2 - a z_1^2 - a \bar{z}_1^2 - 1 = 0,
$$

$$
g_2(z_1, z_2) := |z_2|^2 - a z_2^2 - a \bar{z}_2^2 = 0,
$$

$$
\tilde{g}_3(z_1, z_2) := i z_2 - i \bar{z}_2 = 0,
$$

$$
\tilde{g}_4(z_1, z_2) := z_2 + \bar{z}_2 \geq 0.
$$

Consider $a = \frac{1}{4}$ and $C = 3$ so that $f^\text{opt} = 1$. Notice that $d^{\min} = 2$ for (3.18) and (3.19). The complex relaxations of orders $2 \leq d \leq 3$ of (3.18) are unbounded. The complex relaxation of order 2 of (3.19) yields the value 0.6813. That of order 3 yields 1.0000, rank $M_3(y) = \text{rank } M_1(y) = 2$, and Point 2 in Proposition 3.5. Thus $f^\text{opt} \approx 1.000$ and there exists at least 2 global solutions to (3.19), and hence to (3.18).

We next transpose [38, Lemma 3] from real to complex numbers.

Lemma 3.6. Define $s(z) := R^2 - |z_1|^2 - \ldots - |z_n|^2$. Given $d \in \mathbb{N}^*$ and $y \in \mathcal{H}_d$, if $M_d(y) \succ 0$ and $M_{d-1}(s y) = 0$, then $\text{Tr}(M_d(y)) \leq y_{0,0} \sum_{l=0}^d R^{2l}$.

Proof. Given $1 \leq l \leq d$, we have $\text{Tr}(M_{l-1}(s y)) = \sum_{|\alpha| \leq l} M_{l-1}(s y)(\alpha, \alpha) = \sum_{|\alpha| \leq l-1} L_y(s(z)z^\alpha) = \sum_{|\alpha| \leq l-1} \sum_{|\gamma| \leq l-1} s_\gamma y_{\gamma+\alpha, \gamma+\alpha} = \sum_{|\alpha| \leq l-1, |\gamma| = 0} s_\gamma y_{\gamma+\alpha, \gamma+\alpha} + \sum_{|\alpha| \leq l-1, |\gamma| = 1} s_\gamma y_{\gamma+\alpha, \gamma} = \sum_{|\alpha| \leq l-1, |\gamma| = 0} \sum_{|\alpha| \leq l-1, |\gamma| = 1} R^2 y_{\alpha, \alpha} - \sum_{|\alpha| \leq l-1, |\gamma| = 1} y_{\gamma+\alpha, \gamma+\alpha}. \text{ We have } M_{d-1}(s y) = 0 \text{ so } M_{d-1}(s y) = 0 \text{ for all } 1 \leq l \leq d \text{ and hence } \text{Tr}(M_{l-1}(s y)) = 0.

In addition, $\sum_{|\alpha| \leq l} y_{\alpha, \alpha} = \sum_{|\alpha| \leq l-1, |\gamma| = 0} y_{\gamma+\alpha, \gamma+\alpha} \leq y_{0,0} + \sum_{|\alpha| \leq l-1} y_{\alpha, \alpha}$ for $1 \leq l \leq d$, which proves the lemma. \(\square\)

Theorem 3.7 (Putinar and Scheiderer [59]). Under assumption (3.14), $y \in \mathcal{H}$ has a representing measure on $K$ if and only if $M_d(y) \succ 0$, $i = 0, \ldots, m, \forall \alpha \in \mathbb{N}$.

Proof. We provide an alternative proof using Lemma 3.6. The “only if” part is a consequence of Section 3.1. Concerning the “if” part, if $y_{0,0} = 0$, then Lemma 3.6 implies that $y = 0$ which can be represented by $\mu = 0$ on $K$. Otherwise $y_{0,0} > 0$ and $y_{0,0}$ is a feasible point of problem (3.8) whose optimal value is $f^\text{opt}$ for all $f \in \mathbb{R}[z, \bar{z}]$ according to Corollary 3.4. If moreover $f_{1K} \geq 0$, then $L_{y_{0,0}}(f) \geq f^\text{opt} \geq 0$. In particular, if $f_{1K} = 0$, then $L_{y_{0,0}}(f) = 0$. We may therefore define $L_{y_{0,0}}(f) := M_{y_{0,0}}(f)$ (similar to Schweighöfer [64, Proof of Theorem 2]). If $\varphi \in R_K(\mathbb{R}[z, \bar{z}])$, then $L_{y_{0,0}}(\|\varphi\|) \geq 0$ and $L_{y_{0,0}}(\varphi) \leq \|\varphi\|$. Linearity implies that $|L_{y_{0,0}}(\varphi)| \leq \|\varphi\|$. As a result, for all $\varphi \in R_K(C[z, \bar{z}])$, we have $|L_{y_{0,0}}(\varphi)| = |L_{y_{0,0}}(\text{Re}(\varphi) + i\text{Im}(\varphi))| = |L_{y_{0,0}}(\text{Re}(\varphi)) + iL_{y_{0,0}}(\text{Im}(\varphi))| \leq |L_{y_{0,0}}(\text{Re}(\varphi))| + |L_{y_{0,0}}(\text{Im}(\varphi))| \leq \|\text{Re}(\varphi)\| + \|\text{Im}(\varphi)\| \leq 2\|\varphi\|$. Moreover, $R_K(C[z, \bar{z}])$ is dense in $C(K, \mathbb{C})$. Therefore $L_{y_{0,0}}$ may be extended to a continuous linear functional on $C(K, \mathbb{C})$ (we preserve the same
name for the extension). K is compact thus the Riesz Representation Theorem implies that there exists a unique Radon measure \( \mu \) such that \( \int_K \varphi \, d\mu \) for all \( \varphi \in C(K, \mathbb{C}) \) and \( \mu \geq 0 \) because \( \varphi \in \mathcal{P}(K) \) implies that \( \int_K \varphi \, d\mu \) (density argument). Finally, if \( \alpha, \beta \in \mathbb{N}^n \), \( y_{\alpha, \beta} = L_{y, 0, 0}(z^\alpha z^\beta) \) (Remark 3.1) so \( y \) has representing measure \( y_{0, 0} \) on \( K \).

**Theorem 3.8.** Let \( n > 1 \) and \( y \in \mathcal{H}_d \) with \( d \geq d_K = \max_{1 \leq i \leq m} k_i \) (\( k_i \) is defined above (3.3)). Assume that \( K \) contains the constraints \( |z_k|^2 \leq R_k^2 \), \( k = 1 \ldots n \), for some radii \( R_k \geq 0 \) or the constraint \( \sum_{k=1}^n |z_k|^2 \leq R^2 \) for some radius \( R > 0 \) (where \( \leq \) is an equality or an inequality). Then there exists a positive rank\( M_{d-d_K}(y) \)-atomic measure \( \mu \) supported on \( K \) such that:

\[
y_{\alpha, \beta} = \int_{\mathbb{C}^n} z^\alpha z^\beta \, d\mu, \quad \text{for all } |\alpha|, |\beta| \leq d
\]

if and only if:

1. \( M_d(y) \geq 0 \) and \( M_{d-d_K}(g_i) \geq 0 \), \( i = 1 \ldots m \);
2. \( \text{rank} M_d(y) = \text{rank} M_{d-d_K}(y) \);
3. 
   \[
   \begin{pmatrix}
   M_{d-d_K}(y) & M_{d-d_K}(\bar{z} y) \\
   M_{d-d_K}(\bar{z} y) & M_{d-d_K}(\bar{z} z y)
   \end{pmatrix}
   \geq 0, \quad \forall 1 \leq i < j \leq n.
   \]

Moreover, for each \( 1 \leq i \leq m \), the measure \( \mu \) has exactly \( \text{rank} M_d(y) - \text{rank} M_{d-d_K}(g_i) \) atoms that are zeros of \( g_i \).

**Proof.** \( \Leftarrow \) Point 1 implies that \( (y_{\alpha, \beta})_{|\alpha|, |\beta| \leq d} \geq 0 \). Thus there exists a complex matrix \( x \) of the same size as \( (y_{\alpha, \beta})_{|\alpha|, |\beta| \leq d} \) such that we have the Cholesky factorization \( (y_{\alpha, \beta})_{|\alpha|, |\beta| \leq d} = x^H x \). Let \( x_{\alpha} \) denote the columns of \( x \). Also, \( C_d \equiv \text{the column space and consider the inner product } (u, v)_{C_d} \equiv u^H v \) and its induced norm \( \| \cdot \|_{C_d} \). We have \( y_{\alpha, \beta} = (x_{\alpha}, x_\beta)_{C_d} \) for all \( |\alpha|, |\beta| \leq d \). Let \( V := \text{span}(x_{\alpha})_{|\alpha| \leq d} \subset C_d \). Point 2 implies that \( V = \text{span}(x_{\alpha})_{|\alpha| \leq d-1} \). Given \( 1 \leq k \leq n \), define the \( \mathbb{C} \)-linear operator \( T_k : V \to V \) such that \( T_k x_{\alpha} = x_{\alpha + e_k} \) for all \( |\alpha| \leq d - 1 \) where \( e_k \) is the row vector of size \( n \) that contains only zeros apart from 1 in position \( k \). This shift operator is well defined because each element of \( V \) has a unique image by \( T_k \). Indeed, consider some complex numbers \( (u_{\alpha})_{|\alpha| \leq d-1} \). The assumption on \( K \) in the case of multiple constraints and Point 1 imply that \( M_{d-d_K}([R_k^2 - |z_k|^2] y) = 0 \). Thus \( \| \sum_{|\alpha| \leq d-1} u_{\alpha} x_{\alpha + e_k} \|_{C_d}^2 = \sum_{|\alpha|, |\beta| \leq d-1} (x_{\alpha + e_k}, x\beta + e_k)_{C_d} u_{\alpha} u_{\beta} = \sum_{|\alpha|, |\beta| \leq d-1} y_{\alpha, \beta} u_{\alpha} u_{\beta} = R_k^2 \sum_{|\alpha|, |\beta| \leq d-1} (x_{\alpha}, x\beta)_{C_d} u_{\alpha} u_{\beta} = R_k^2 \sum_{|\alpha| \leq d-1} u_{\alpha} x_{\alpha} \|_{C_d}^2 \). Thus \( T_k \) is well-defined and bounded by \( R_k \). The assumption on \( K \) in the case of a single constraint and Point 1 imply that \( M_{d-d_K}([R^2 - \sum_{|\beta| \leq d-1} |z_\beta|^2] y) = 0 \). Thus \( \| \sum_{|\alpha| \leq d-1} u_{\alpha} x_{\alpha + e_k} \|_{C_d}^2 = \sum_{|\alpha| \leq d-1} u_{\alpha} x_{\alpha} \|_{C_d}^2 \). Hence \( T_k \) is well-defined and bounded by \( R \).

Clearly, \( (T_1, \ldots, T_n) \) is a pair-wise commuting tuple of operators on \( V \). Let’s now prove that \( (T_1^*, \ldots, T_n^*, T_1, \ldots, T_n) \) is a pair-wise commuting tuple of operators, which reduces to showing that \( T_i T_j - T_j T_i = 0 \) for all \( 1 \leq i < j \leq n \) (where \( \cdot^* \) stands for adjoint). To do so, consider \( 1 \leq i < j \leq n \) and \( u, v, w \in V \). Point 2 implies that \( V = \text{span}(x_{\alpha})_{|\alpha| \leq d-K} \). Thus there exists some complex numbers \( (u_{\alpha})_{|\alpha| \leq d-K} \), \( (v_{\alpha})_{|\alpha| \leq d-K} \), and \( (w_{\alpha})_{|\alpha| \leq d-K} \) such that \( u = \sum_{|\alpha| \leq d-K} u_{\alpha} x_{\alpha}, \quad v = \sum_{|\alpha| \leq d-K} v_{\alpha} x_{\alpha}, \quad w = \sum_{|\alpha| \leq d-K} w_{\alpha} x_{\alpha} \). Given \( k \in \mathbb{N} \) and \( \varphi \in \mathbb{C}[z^\gamma, z^\delta] \), notice that \( (u, \varphi(T)v)_{C_d} = \sum_{|\gamma|, |\delta| \leq d-K} \varphi_{\gamma, \delta}(T^\gamma x_{\alpha}, T^\delta x_{\beta}) c_{\gamma} c_{\delta} u_{\alpha} v_{\beta} = \sum_{|\gamma|, |\delta| \leq d-K} \sum_{|\gamma|, |\delta| \leq d-K} \varphi_{\gamma, \delta}(x_{\alpha + \gamma}, x_{\beta + \delta}) c_{\gamma} c_{\delta} u_{\alpha} v_{\beta} = \ldots \)
Thus |σ|, |β| ≤ d−dK(Σ|σ|, |β| ≤ k φγ,δyαγ,β+δ)ραβ = \bar{u}^H M_{d−dK}(ϕy)\bar{u}. As a result,

\begin{equation}
\begin{pmatrix}
T_\lambda^* T_i & T_j^* T_i \\
T_j & T_j^* T_i \\
T_j & T_j^* T_i \\
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w \\
\end{pmatrix}
= \ldots
\end{equation}

Point 3 implies that

\begin{equation}
\begin{pmatrix}
T_\lambda^* T_i & T_j^* T_i \\
T_j & T_j^* T_i \\
T_j & T_j^* T_i \\
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w \\
\end{pmatrix}
= 0
\end{equation}

which is equivalent to the fact that Schur complement satisfies

\begin{equation}
\begin{pmatrix}
T_\lambda^* T_i & T_j^* T_i \\
T_j & T_j^* T_i \\
T_j & T_j^* T_i \\
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w \\
\end{pmatrix}
= 0.
\end{equation}

Thus \( T_\lambda^* T_i \) and \( T_j^* T_i \) are pair-wise commuting operators, it follows that they are commonly diagonalizable. In other words, there exists a diagonal matrix \( D \) such that \( D T_i D^{-1} \) and \( D T_j D^{-1} \) are both diagonalizable. Hence \( T_\lambda^* T_i \) and \( T_j^* T_i \) are pair-wise commuting operators. It follows that there exists a diagonal matrix \( D \) such that \( D T_i D^{-1} \) and \( D T_j D^{-1} \) are both diagonalizable. Hence \( T_\lambda^* T_i \) and \( T_j^* T_i \) are pair-wise commuting operators.
Hence \( \dim \ker g_i(T) = p - \text{rank } g_i(T) \) due to the rank-nullity theorem) is equal to number of atoms that are zeros of \( g_i \). To conclude, notice that rank \( g_i(T) = \text{rank} (\langle x, g_i(T)x \rangle c_d)_{|\alpha|,|\beta| \leq d-d_K} = \text{rank} M_{d-d_K}(g_iy) \). \( \square \)

In the univariate case \( n = 1 \), Theorem 3.8 holds when Point 3 is replaced by

\[
(3.25) \quad \begin{pmatrix} M_{d-d_K}(y) & M_{d-d_K}(\bar{z} y) \\ M_{d-d_K}(\bar{z} y) & M_{d-d_K}(|z|^2 y) \end{pmatrix} \succ 0.
\]

In Theorem 3.8, if we assume that \( y_{0,0} > 0 \), then Point 2 and Point 3 may be replaced by \( \text{rank} M_d(y) = 1 \). Indeed, in that case, the shift operators act on a one dimensional space, so they and their adjoints must commute pair-wise. For previous work on the link between linear functionals that are nonnegative on a quadratic module and bounded operators that admit a cyclic vector, see [57] and [22, Theorem 2.3].

**Corollary 3.9.** Let \( y \in \mathcal{H}_d \) be a Hankel matrix (i.e. \( y_{\alpha,\beta} = y_{\gamma,\delta} \) for all \( |\alpha|, |\beta|, |\gamma|, |\delta| \leq d \) such that \( \alpha + \beta = \gamma + \delta \)). Then there exists a positive \( \text{rank} M_{d-d_K}(y) \)-atomic measure \( \mu \) supported on \( K \) such that:

\[
(3.26) \quad y_{\alpha,\beta} = \int_{C^n} z^\alpha \bar{z}^\beta d\mu, \quad \text{for all } |\alpha|, |\beta| \leq d
\]

if and only if:

1. \( M_d(y) \succ 0 \) and \( M_{d-d_K}(y) \succ 0 \), \( i = 1 \ldots m \);
2. \( \text{rank} M_{d-d_K}(y) = \text{rank} M_{d-d_K}(g_iy) \).

Moreover, for each \( 1 \leq i \leq m \), the measure \( \mu \) has exactly \( \text{rank} M_d(y) - \text{rank} M_{d-d_K}(g_iy) \) atoms that are zeros of \( g_i \).

**Proof.** (\( \Rightarrow \)) Same as in proof of Theorem 3.8. (\( \Leftarrow \)) The Hankel property implies that the shifts in the proof of Theorem 3.8 are well-defined and self-adjoint. Indeed, consider \( 1 \leq k \leq n \) and \( u, v \in V \). According the Point 2, there exists some complex numbers \( (u_{\alpha})_{|\alpha| \leq d-1} \) and \( (\bar{u}_{\alpha})_{|\alpha| \leq d-1} \) such that \( u = \sum_{|\alpha| \leq d-1} u_{\alpha} x_\alpha \) and \( v = \sum_{|\alpha| \leq d-1} \bar{u}_{\alpha} x_\alpha \). If \( \sum_{|\alpha| \leq d-1} u_{\alpha} x_\alpha = 0 \), then for all \( |\beta| \leq d-1 \), we have \( \langle \sum_{|\alpha| \leq d-1} u_{\alpha} x_\alpha + e_\alpha, x_\beta \rangle c_d = \langle \sum_{|\alpha| \leq d-1} \bar{u}_{\alpha} x_\alpha + e_\alpha, x_\beta \rangle c_d = 0 \), hence \( \sum_{|\alpha| \leq d-1} u_{\alpha} x_\alpha + e_\alpha = 0 \). \( T_k \) is thus well defined. Moreover, we have \( T_k^* = T_k \) because \( \langle T_k u, v \rangle c_d = \langle \sum_{|\alpha| \leq d-1} u_{\alpha} x_\alpha + e_\alpha, \sum_{|\alpha| \leq d-1} \bar{u}_{\alpha} x_\alpha + e_\alpha \rangle c_d = \sum_{|\alpha| \leq d-1} u_{\alpha} \bar{u}_{\alpha} + v_{\alpha} \bar{v}_{\alpha} \).

A Hermitian matrix that is a Hankel matrix is real symmetric. Hence the atoms in Corollary 3.9 lie in \( K \cap \mathbb{R}^n \). Corollary 3.9 is thus the same as [42, Theorem 3.11] due to Curto and Fialkow [21, Theorem 1.1]. This observation leads to Figure 2.

Next we transpose [38, Theorem 1] from real to complex numbers.

**Proposition 3.10.** Under assumption (3.14), \( \rho_d = \rho_d \in \mathbb{R} \cup \{+\infty\}, \forall d \geq d_{\min} \).

**Proof.** Given \( A \in \mathcal{H}_d \), consider the operator norm \( \| A \| \), the largest eigenvalue of \( A \) in absolute value, and the Frobenius norm \( \| A \|_F := \sqrt{\langle A, A \rangle_{H_d}} \). Consider \( d \geq d_{\min} \). Two cases can occur. Case 1: the feasible set of the complex moment relaxation of order \( d \) is non-empty. All norms are equivalent in finite dimension so there exists a constant \( C_d \in \mathbb{R} \) such that for all feasible points \( (y_{\alpha,\beta})_{|\alpha|,|\beta| \leq d} \) we have \( \sqrt{\sum_{|\alpha|,|\beta| \leq d} |y_{\alpha,\beta}|^2} = \| M_d(y) \|_F \leq C_d \| M_d(y) \| \leq C_d \sum_{l=0}^d R_l^2 \) according to Lemma 3.6. As a result, the feasible set of the complex moment relaxation of order \( d \) is a non-empty compact set and so is its image by \( \Lambda \) (defined in (2.3)). We can thus apply Trnovská’s result [71] which states that in a semidefinite program in real numbers, if the primal feasible set is non-empty and compact, then there exists
a dual interior point and there is no duality gap. Case 2: the feasible set of the complex moment relaxation of order $d$ is empty, i.e. $\rho_d = +\infty$. It must be strongly infeasible because it cannot be weakly infeasible (see [27, Section 5.2] for definitions). Indeed, if it is weakly infeasible, then there exists a sequence $(y^i_j)_{j \in \mathbb{N}}$ of elements of $\mathcal{H}$ such that for all $j \in \mathbb{N}$, we have $|y^i_j|_{i=0}^{|i|} \leq \frac{\sqrt{c}}{i+1}$ and $\lambda_{\min}(M_{d-k_i}(y^i_j)) \geq -\frac{\sqrt{c}}{i+1}$ where $i = 0, \ldots, m$. Define $c := (n + d)!/(n!d!)$. We now mimic the computations in Lemma 3.6 using $y^i_{0,0} \leq 1 + \frac{1}{i+1} \leq 2$ and $|\text{Tr}(M_{d-1}(sy^i))| \leq \frac{c}{i+1} \leq c$ if $1 \leq l \leq d$. Consider $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ and $1 \leq l \leq d$, we have $\sum_{|\alpha| \leq l-1,|\gamma|=1} y^i_{l,a,\alpha} \leq -1$. The concluding equation in the proof of Lemma 3.6 then becomes $\sum_{|\alpha| \leq l} y^i_{l,a,\alpha} \leq 2 + R^2 \left( \sum_{|\alpha| \leq l-1} y^i_{l,a,\alpha} \right) + c + 1$.

As a result, $\text{Tr}(M_d(y^i)) = \sum_{|\alpha| \leq d} y^i_{d,a,\alpha} \leq (3 + c) \sum_{i=0}^d R^{2d}$, which, together with $\lambda_{\min}(M_d(y^i)) \geq -\frac{1}{i+1} \geq -1$, yields $\lambda_{\max}(M_d(y^i)) \leq (3 + c) \sum_{i=0}^d R^{2d} + c - 1$. Hence for all $j \geq j_0$, the spectrum of $M_d(y^i)$ is lower bounded by $-1$ and upper bounded by $B_d := (3 + c) \sum_{i=0}^d R^{2d} + c - 1 \geq 1$. We therefore have $\sqrt{\sum_{|\alpha| \leq d} |y^i_{d,a,\alpha}|^2} \leq C_d \|M_d(y^i)\| \leq C_d \times B_d$. The sequence $(y^i_j)_{j \geq j_0}$ is thus included in a compact set. Hence there exists a subsequence that converges towards a limit $y^{\lim}$ which satisfies $y^{\lim}_{0,0} = 1$ and the constraints $\lambda_{\min}(M_{d-k_i}(y^{\lim}_i)) \geq 0, i = 0, \ldots, m$. Therefore $y^{\lim}$ is a feasible point of the complex moment relaxation of order $d$, which is a contradiction.

Strong infeasibility means that the dual feasible set contains an improving ray [27, Definition 5.2.2]. Moreover, $\inf_{y \in \mathcal{H}_d} L_y(f)$ subject to $y_{0,0} = 1$, $M_d(y) \succ 0$, and $M_{d-1}(sy) = 0$ is a semidefinite program with a non-empty compact feasible set hence the dual feasible set contains a point $(\lambda, \sigma_0, \sigma_1)$. As result $(\lambda, \sigma_0, \sigma_1, 0, \ldots, 0)$ is a feasible point of the complex sum-of-squares relaxation of order $d$. Together with the improving ray, this means that $\rho_d^* = +\infty$. To conclude, $\rho_d^* = \rho_d$ in both cases.

**Proposition 3.11.** Assume that (3.3) satisfies (3.15) and has a global solution $z^{\text{opt}} \in K^{\text{opt}}$. In addition, assume that $(\sigma_0^{\text{opt}}, \ldots, \sigma_m^{\text{opt}}) \in \Sigma[z]^{m+1}$ is an optimal solution to the sum-of-squares problem (3.12). Then $(z^{\text{opt}}, \sigma_1^{\text{opt}}, \ldots, \sigma_m^{\text{opt}})$ is a saddle point of $\phi : \mathbb{C}^n \times \Sigma[z]^{m} \rightarrow \mathbb{R}$ defined by $(z, \sigma) \mapsto f(z) - \sum_{i=1}^m \sigma_i(z)g_i(z)$.

**Proof.** The optimality of $(\sigma_0^{\text{opt}}, \ldots, \sigma_m^{\text{opt}})$ means that $f - f^{\text{opt}} = \sum_{i=0}^m \sigma_i^{\text{opt}} g_i$. With $f(z^{\text{opt}}) - f^{\text{opt}} = \sum_{i=0}^m \sigma_i^{\text{opt}} g_i(z^{\text{opt}}) = 0, \sigma_i^{\text{opt}}(z^{\text{opt}}) \geq 0$, and $g_i(z^{\text{opt}}) \geq 0$, we
have $\sigma_i^{\text{opt}}(z^{\text{opt}})g_i(z^{\text{opt}}) = 0$ for $i = 0, \ldots, m$. It follows that $\phi(z^{\text{opt}}, \sigma) \leq \phi(z^{\text{opt}}, \sigma^{\text{opt}})$ for all $\sigma \in \Sigma[z]$. For all $z \in \mathbb{C}^n$, $\phi(z^{\text{opt}}, \sigma^{\text{opt}}) \leq \phi(z, \sigma)$ because $f(z) - f^{\text{opt}} - \sum_{i=1}^{m} \sigma_i^{\text{opt}}(z)g_i(z) = \sigma_0^{\text{opt}}(z) > 0$. \[ \Box \]

Given an application $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$, define $\check{\varphi} : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ by $(x, y) \mapsto \varphi(x + iy)$. If $\check{\varphi}$ is $\mathbb{R}$-differentiable at point $(x, y) \in \mathbb{R}^{2n}$, consider the Wirtinger derivative \cite{75} defined by $\nabla \varphi(x + iy) := \frac{1}{2}(\nabla_x \check{\varphi}(x, y) - i \nabla_y \check{\varphi}(x, y)) \in \mathbb{C}^n$.

**Corollary 3.12.** With the same assumptions as in Proposition 3.11, we have

\begin{equation}
\nabla f(z^{\text{opt}}) = \sum_{i=1}^{m} \sigma_i^{\text{opt}}(z^{\text{opt}}) \nabla g_i(z^{\text{opt}}), \\
\sigma_i^{\text{opt}}(z^{\text{opt}})g_i(z^{\text{opt}}) \geq 0, \quad i = 1, \ldots, m, \\
\sigma_i^{\text{opt}}(z^{\text{opt}})g_i(z^{\text{opt}}) = 0, \quad i = 1, \ldots, m.
\end{equation}

**Proof.** $z^{\text{opt}}$ is a minimizer of $z \in \mathbb{C}^m \rightarrow \phi(z, \sigma)$ thus $\nabla z \phi(z^{\text{opt}}, \sigma^{\text{opt}}) = \nabla f(z^{\text{opt}}) - \sum_{i=1}^{m} \nabla \sigma_i^{\text{opt}}(z^{\text{opt}})g_i(z^{\text{opt}}) - \sum_{i=1}^{m} \sigma_i^{\text{opt}}(z^{\text{opt}})\nabla g_i(z^{\text{opt}}) = 0$. Consider $1 \leq i \leq m$. Since $\sigma_i^{\text{opt}}(z^{\text{opt}}) = 0$ and $\sigma_i^{\text{opt}} \in \Sigma[z]$, it must be that $|z_k - z_k^{\text{opt}}|^2$ divides $\sigma_{i,k}^{\text{opt}} : z_k \in \mathbb{C} \rightarrow \sigma_i^{\text{opt}}(z^{\text{opt}})$, $\sigma_i^{\text{opt}}(z_{k-1}^{\text{opt}})$, $\sigma_i^{\text{opt}}(z_{k+1}^{\text{opt}}), \ldots, \sigma_i^{\text{opt}}(z_{m}^{\text{opt}})$. With $z_k^{\text{opt}} = x_k^{\text{opt}} + iy_k^{\text{opt}}$, the real number $x_k^{\text{opt}}$ is a root of multiplicity 2 of $x_k \in \mathbb{R} \rightarrow \sigma_{i,k}^{\text{opt}}(x_k + iy_k^{\text{opt}})$, with an analogous remark for $y_k^{\text{opt}}$. Thus $\nabla \sigma_i^{\text{opt}}(z^{\text{opt}}) = 0$ which leads to the desired result. \[ \Box \]

![Fig. 3. Comparison of Real and Complex Hierarchies](image-url)

### 3.4. Comparison of Real and Complex Hierarchies

In Figure 3 where $p_{i,\alpha}^j, p_{i,\alpha,\beta}^j \in \mathbb{C}$, the real sum-of-squares hierarchy is artificially written using squares of moduli of complex polynomials. It thus yields bounds superior or equal to the complex hierarchy. For example, at order 2, the real hierarchy yields 1.0000 while the...
complex hierarchy yields 0.6813 for (3.19). However, the size of the largest semidefinite constraint in the complex hierarchy when converted to real numbers, i.e. $2 \times \text{card} \{ \alpha \in \mathbb{N}^n \text{ s.t. } |\alpha| \leq d \} = 2(n + d)!/(n!d!)$, is far inferior to that of the real hierarchy, i.e. $\text{card} \{ \alpha, \beta \in \mathbb{N}^n \text{ s.t. } |\alpha + \beta| \leq d \} = (2n + d)!/((2n)!d!)$. At fixed $d$, the size reduction converges towards $2^{d-1}$ as $n \to \infty$. Further reduction is possible (Section 3.5).

3.5. Invariant Hierarchy. We generalize and transpose to complex numbers the work in [62] (see also [19]). Let $(G, \times)$ denote a compact group whose unit we denote 1. First, consider the continuous action of $G$ on $\mathbb{C}^n$ via $\mathcal{A} : G \times \mathbb{C}^n \to \mathbb{C}^n$ such that $\mathcal{A}(1, z) = z$, $\mathcal{A}(g_1 \times g_2, z) = \mathcal{A}(g_1, \mathcal{A}(g_2, z))$ for all $z \in \mathbb{C}^n$ and $g_1, g_2 \in G$. Second, consider the action of $G$ on $\mathbb{R}^2$ via $\mathcal{A}' : G \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathcal{A}'(g, \varphi) := \varphi(\mathcal{A}(g, z))$. Third, consider the action of $G$ on the set $\mathcal{B}(K)$ of Borel subsets of $K$ via $\mathcal{A}'' : G \times \mathcal{B}(K) \to \mathcal{B}(K)$ defined by $\mathcal{A}''(g, B) := \{ z \in K \mid \mathcal{A}(g, z) \in B \}$. Last, consider the action of $G$ on $\mathcal{M}(K)$ via $\mathcal{A}''' : G \times \mathcal{M}(K) \to \mathcal{M}(K)$ defined by $\mathcal{A}'''(g, \mu)(z) := \mu(\mathcal{A}(g, z))$. Given a set $S$ on which $G$ is acting via $\mathcal{T}$ and $Y \subset S$, let $Y^G := \{ y \in Y \mid \forall g \in G, \mathcal{T}(g, y) = y \}$. If $f_1, g_1, \ldots, g_m \in \mathbb{R}^2$, then:

$$f^\text{opt} = \inf_{\mu \in \mathcal{M}(K)^G} \int_K f d\mu \text{ s.t. } \int_K d\mu = 1 \text{ } \& \text{ } \mu \geq 0.$$  

If $\mu$ is feasible for (3.28), then $\int_K |p|^2 d\mu \geq 0$ for all $d \in \mathbb{N}$ and $p \in \mathbb{C}_d[z]^G$ such that $|p|^2 \in \Sigma_d[z]^G$. Given $A \in \mathcal{H}_d$, let $A \geq 0$. If $f_1, \ldots, g_m \in \mathbb{R}^2$ such that $|p|^2 \in \Sigma_d[z]^G$. This yields a $G$-invariant hierarchy for all $d \geq d^\text{min}$

$$\rho_d^G := \inf_{y \in \mathcal{H}_d} L_y(f) \text{ s.t. } y_{0,0} = 1, M_{d-k,0}(g_i, y) \geq 0, \text{ } i = 0, \ldots, m,$$

whose convergence we now discuss. Assume that the first 2e ($e \in \mathbb{N}^*$) constraint functions $g_1, \ldots, g_m$ form equality constraints (i.e. $g_2, \ldots, g_m = h_i$, $i = 1 \ldots e$). Define $S := \Sigma[z]^G + \sum_{i=1}^e \mathbb{R}[z, z]^{G} h_i$ and $S := \Sigma[z]^G$ if there are no equality constraints.

**Proposition 3.13.** Assume that $f, g_1, \ldots, g_m \in \mathbb{R}^2$ and that $\mathbb{R}^2[z]^G = S + \mathbb{R}$. If $f_{K} > 0$, then there exists $\lambda_0, \ldots, \lambda_m \in \Sigma[z]^G$ such that $f = \sum_{i=0}^m \lambda_i g_i$.

**Proof.** Assume that $A, \mathbb{R}^2[z]^G$ is an $S$-module of $\mathbb{R}^2[z]^G$. As a result, $S$ is a semiring of $\mathbb{R}^2[z]^G$ (i.e. contains $\mathbb{R}_+$ and is closed in $\mathbb{R}^2[z]^G$ under taking sums and products) and $M := S + \sum_{i=2e+1}^m \Sigma[z]^G g_i$ is an $S$-module of $\mathbb{R}^2[z]^G$ (i.e. contains 1 and satisfies $M + M \subset M$ and $SM \subset M$). The conclusion then follows from [60, Theorem 2.6].

**Proposition 3.14.** The torus $G = \mathbb{T}$ in $\mathbb{C}$ with the action $\mathcal{A}(g, z) := g_2$ satisfies $\rho_d^\mathbb{T} = \rho_d$ and $(\rho_d^\mathbb{T})^* = \rho_d^\mathbb{T}$ for all $d \geq d^\text{min}$ if $f_1, g_1, \ldots, g_m \in \mathbb{R}^2[z]^\mathbb{T}$.

**Proof.** $\rho_d^\mathbb{T} = \rho_d$ and $(\rho_d^\mathbb{T})^* = \rho_d^\mathbb{T}$ for all $d \geq d^\text{min}$ if $f_1, g_1, \ldots, g_m \in \mathbb{R}^2[z]^\mathbb{T}$.

Finally, $\varphi \in \mathbb{R}^2[z]^\mathbb{T}$ if and only if $\forall \alpha, \beta \in \mathbb{N}^n, |\alpha - \beta| \varphi_{\alpha, \beta} = 0$. Indeed, for all $\theta \in \mathbb{R}$ and $z \in \mathbb{C}^n, \varphi(\epsilon^{i\theta} z) = \sum_{\alpha, \beta \in \mathbb{N}^n} \varphi_{\alpha, \beta} (\epsilon^{i\theta} z)^{\alpha} \beta \varphi(\epsilon^{i\theta} z)^{\alpha+\beta} = \sum_{\alpha, \beta \in \mathbb{N}^n} \varphi_{\alpha, \beta} |\alpha|^\theta |\beta|^\theta = \sum_{l \in \mathbb{N}} \sum_{i=1}^r |\alpha|= |\beta|= i |\alpha|= |\beta|= 1 \text{ s.t. } \sum_{|\alpha|= |\beta|= i} \varphi_{\alpha, \beta} (\epsilon^{i\theta} z)^{\alpha} \beta = \sum_{l \in \mathbb{N}} \sum_{i=1}^r |\alpha|= |\beta|= i |\alpha|= |\beta|= 1 \text{ s.t. } \sum_{|\alpha|= |\beta|= i} \varphi_{\alpha, \beta} (\epsilon^{i\theta} z)^{\alpha} \beta = \mathbb{E}(\varphi(\epsilon^{i\theta} z)^{\alpha} \beta)$.}

Thus $(\rho_d^\mathbb{T})^* = \rho_d$. Similarly, if $y$ is feasible for (3.29), then $(y_\alpha, \delta_{|\alpha|=|\beta|} |\alpha|, |\beta| \leq d)$ is feasible for (3.9) (where $\delta$ is the Kronecker symbol). Hence $\rho_d^\mathbb{T} = \rho_d$. 

If \( f, g_1, \ldots, g_m \in \mathbb{R}[\bar{z}, z]^T \), then the minimum order \( d_{\min} \) of the complex hierarchy, i.e. \( \max\{|\alpha|, |\beta|\} \) s.t. \( |f| + |g_1, \ldots, g_m| \neq 0 \), is equal to that of the real hierarchy, i.e. \( \max\{|\alpha| + |\beta|/2\} \) s.t. \( |f| + |g_1, \ldots, g_m| \neq 0 \), where \([.]\) denotes the ceiling of a real number.

**Proposition 3.15.** The subgroup \( G = \{-1, 1\} \) of \( T \) with \( A(g, z) := gz \) satisfies \( \rho_d^{(-1,1)} = \rho_d \) and \( (\rho_d^{(-1,1)})^* = \rho_d^* \) for all \( d \geq d_{\min} \) if \( f, g_1, \ldots, g_m \in \mathbb{R}[\bar{z}, z]^{-1,1} \).

**Proof.** Firstly, \( \varphi \in \mathbb{R}[\bar{z}, z]^{-1,1} \) if and only if \( \forall |\alpha| + |\beta| \) odd, \( \varphi_{\alpha, \beta} = 0 \). Secondly, if \( \sigma \in \Sigma[z] \), i.e. \( \sigma = \sum_{j=1}^{|\sigma|} \langle p_j \rangle^2 \), then \( \sum_{|\alpha| + |\beta| \text{ even}} \sigma_{\alpha, \beta} z^\alpha \bar{z}^\beta = \sum_{j=1}^{|\sigma|} |\sum_{|\alpha| \text{ even}} \sigma_{\alpha, \beta} z^\alpha \bar{z}^\beta|^2 + \sum_{|\alpha| \text{ odd}} |\sum_{|\alpha| \text{ even}} \sigma_{\alpha, \beta} z^\alpha \bar{z}^\beta|^2 \in \Sigma[z]^{-1,1} \). Thirdly, if \( (\lambda, \sigma_0, \ldots, \sigma_m) \) is feasible for (3.11), then \( (\lambda, \sum_{|\alpha| + |\beta| \text{ even}} \sigma_{\alpha, \beta} z^\alpha \bar{z}^\beta, \cdots, \sum_{|\alpha| + |\beta| \text{ even}} \sigma_{m, \alpha, \beta} z^\alpha \bar{z}^\beta) \) is feasible for (3.30). Lastly, if \( y \) is feasible for (3.29), then \( (y_{\alpha, \beta}, \sigma_{\alpha, \beta} \text{ even}) |_{|\alpha|, |\beta| \leq d} \) is feasible for (3.9). \( \square \)

A problem with \( T \)-invariance in complex numbers converts in real numbers to a problem with \( \{-1, 1\}\)-invariance. If \( \sigma \in \Sigma_d[z]^T \), then \( (\sigma_{\alpha, \beta})_{|\alpha|, |\beta| \leq d} \) has a \( (d + 1) \)-block-diagonal structure, whereas if \( \sigma \in \Sigma_d[x, y]^{-1,1} \) with \( z := x + iy \), then it has a \( 2 \)-block-diagonal structure (after permutation) whose 2 blocks are much bigger.

### 3.6. Multi-Ordered Relaxation

We generalize and transpose to complex numbers the work in [53]. The idea is to associate a relaxation order to each constraint. In addition, we consider the coupling of the variables induced by the monomials present in the optimization problem, to which we add the coupling of the variables induced by only some constraints (those with a “high-order”). For instance, the coupling induced by the monomials in \( g_1(z_1, z_2, z_3) := \mathbb{R}[z_1(z_2 + z_3)] \geq 0 \) is \( \{(1, 2), (1, 3), (2, 3)\} \), while the coupling induced by the constraint is \( \{(1, 2), (1, 3), (2, 3)\} \).

Given \( \alpha \in \mathbb{N} \) with \( n > 1 \), let \( \text{supp}(\alpha) := \{1 \leq s \leq n \mid \alpha_s \neq 0\} \). Consider the coupling induced by monomials defined by \( \mathcal{E}_{\text{mono}} := \{(l, m) \mid l \neq m \text{ s.t. } \exists \alpha, \beta \in \mathbb{N} \text{ s.t. } \{l, m\} \subset \text{supp}(\alpha + \beta) \text{ and } |f_{\alpha, \beta}| + |g_{l, \alpha, \beta} + \cdots + |g_{m, \alpha, \beta}| \neq 0\} \). Given \( I \subset \{1, \ldots, n\} \), let \( z(I) := \{z_i \mid i \in I\} \) if \( I \neq \emptyset \), else \( z(C) := 1 \). Given \( y \in \mathbb{H} \), \( d \in \mathbb{N} \), and \( \varphi \in \mathbb{R}[z(I), z(I)] \), let \( M_d(\varphi, y, I) := \left( M_d(\varphi(y)(\alpha, \beta) |_{\text{supp}(\alpha), \text{supp}(\beta) \subset I} \right) \).

Let \( G_1, \ldots, G_m \subset \{1, \ldots, n\} \) denote the minimal sets in terms of inclusion such that \( (g_1, \ldots, g_m) \in \mathbb{R}[k_1, z(G_1), z(G_1)] \times \cdots \times \mathbb{R}[k_m, z(G_m), z(G_m)] \).

Let \( (d_1, \ldots, d_m) \in \mathbb{N}^m \) be such that \( d_i - k_i \geq 0 \) for all \( 1 \leq i \leq m \). Consider the coupling induced by monomials and high-order constraints defined by \( \mathcal{E}_{\text{con}} := \mathcal{E}_{\text{mono}} \cup \bigcup_{d > k_1} \{(l, m) \mid l \neq m \text{ s.t. } \{l, m\} \subset G_1\} \). Let \( C_1, \ldots, C_p \subset \{1, \ldots, m\} \) denote the maximal cliques of a chordal extension of \( \{(1, \ldots, n), \mathcal{E}_{\text{con}}\} \). Given \( 1 \leq i \leq m \), let \( I_i := \bigcup_{l \in L} L_i^l \) where \( L \in \text{argmin}\{\sum_{l \in G_1} |C_l| \mid G_i \subset \bigcup_{l \in L} L_i^l, L \subset \{1, \ldots, m\}\} \). For \( i \) such that \( d_i > k_i \), \( L \) is a singleton due to the definition of \( \mathcal{E}_{\text{con}} \). Define \( (d_1^l, \ldots, d_p^l) \in \mathbb{N}^p \) such that \( d_i^l := \min\{d_1, \ldots, d_m\} \) if \( C_l \neq I_i \) for all \( 1 \leq i \leq m \); if not, let \( d_i^l := \max\{d_i \mid I_i = C_l\} \). Define the relaxation of order \( d_1, \ldots, d_m \) by

\[
\rho_{d_1, \ldots, d_m} := \inf_{y \in \mathcal{H}} \mathbb{E}(f)
\text{ s.t.}
\begin{array}{l}
y_{0,0} = 1, \\
M_{d_i^l}(y, C_i) \geq 0, \quad l = 1, \ldots, p, \\
M_{d_i - k_i}(y, I_i) \geq 0, \quad i = 1, \ldots, m,
\end{array}
\]

(3.31)

\[
\rho_{d_1, \ldots, d_m}^{\lambda, \sigma} := \sup_{\lambda, \sigma} \lambda
\text{ s.t.}
\begin{array}{l}
f - \lambda = \sum_{l=1}^p (\sigma_{0, l} + \sum_{C_i \in L_l} \sigma_{g_l}), \\
\sigma_{0, l} \in \Sigma_{d_i^l}(z(C_l)), \quad l = 1, \ldots, p, \\
\sigma_{g_l} \in \Sigma_{d_i - k_i}(z(I_i)), \quad i = 1, \ldots, m,
\end{array}
\]

(3.32)
3.7. Multi-Ordered Hierarchy. Given $H : \mathbb{N} \rightarrow [k_1, +\infty] \times \ldots \times [k_m, +\infty]$ such that $\min_{d \rightarrow +\infty} H(d) = +\infty$, consider the sequence indexed by $d \in \mathbb{N}$ of relaxations of order $H(d)$. We refer to such a sequence as multi-ordered hierarchy.

The uniform case where $H(d) := (d + d_{\min}^0, \ldots, d + d_{\min}^m)$ is a special case of the sparse real hierarchy of [74] when transposed to complex numbers. The hierarchy of [74] converges to the global value of a real polynomial optimization problem if a ball constraint is added for each clique of a chordal extension of the sparsity pattern [42, equation (2.29)]. The same holds in the complex case if a slack variable and redundant sphere constraint is added for each clique. The proof is the same as in the real case [42, Lemma B.13 and 4.10.2 Proof of Theorem 4.7] once the real vector spaces on which measures are defined are replaced by complex vector spaces. (For other proofs of [42, Theorems 2.28 and 4.7], see [31] and [40].) For $d \in \mathbb{N}$ great enough, i.e. once $d_i > k_i$ for all $1 \leq i \leq m$, the relaxation of order $H(d) = (d_1, \ldots, d_m)$ is at least as tight as the complex sparse relaxation of [74] of order $\min H(d)$. Any multi-ordered hierarchy thus globally converges (if a slack variable and sphere constraint is added for each clique).

3.8. Example of Multi-Ordered Hierarchy: the Mismatch Hierarchy. Conceptually, the mismatch hierarchy is defined by the following procedure. Until a measure can be extracted from a solution $y$ to the multi-ordered relaxation,

1. Compute a solution $y$ to the moment relaxation of order $(d_1, \ldots, d_m)$;
2. Find a closest measure $\mu$ to $y$ not necessarily supported on $K$:

$$\arg \min_{\mu \, \text{Dirac}} \left\| \left( y_{\alpha,\beta} - \int_{\mathbb{C}^n} z^\alpha \overline{z}^\beta \, d\mu \right)_{|\alpha|,|\beta|=1} \right\|_F$$

3. Increment $d_i = d_i + 1$ at the highest mismatch, that is to say:

$$\arg \max_{1 \leq i \leq m} \left| \sum_{\alpha,\beta} g_{i,\alpha,\beta} \left( y_{\alpha,\beta} - \int_{\mathbb{C}^n} z^\alpha \overline{z}^\beta \, d\mu \right) \right|.$$ 

Strictly speaking, we refer to the mismatch hierarchy as the following recursively defined multi-ordered hierarchy $H$. It depends on 3 parameters: a mismatch tolerance $\epsilon > 0$; the number $h \in \mathbb{N}^*$ of highest mismatches considered at each iteration; and an upper bound $\Delta_{\min h}^*$ on the difference between maximum and minimum orders, i.e. $\{\max H(d) - \min H(d) \mid d \in \mathbb{N}\}$.

Initialize by $H(0) := k_1 \times \ldots \times k_m$ and let’s define $H(d + 1)$ in function of $H(d)$. We distinguish two cases. Case 1: if there exists no solution to the moment relaxation of order $H(d)$, then let $H(d + 1) := H(d) + (1, \ldots, 1)$. Case 2: if not, consider a solution $y$. For $1 \leq l \leq p$, consider some complex numbers $(u(l)_j)_{j \in C_l}$ such that $u(l) u(l)^H$ is the closest rank 1 matrix to $y(l) := (y_{\alpha,\beta})_{\alpha=|\beta|=1} \subseteq C_l$ with respect to the Frobenius norm. Let $\lambda_l(1) \geq \lambda_l(2) \geq 0$ respectively denote the first and second largest eigenvalues of $y(l)$. Let $\theta \in \mathbb{R}^p$ be a minimizer of $\sum_{l=1}^p \sum_{j \in C_l \cap C_m} (y_{\alpha,\beta}) u(l)_j + \theta_1 - \lambda_l(2)/\theta_m)^2$ s.t. $\theta \in [0, 2\pi]^p$. Let $z \in \mathbb{C}^n$ be a minimizer of $\sum_{l=1}^p \lambda_l(1)/\lambda_l(2)lz(C_l) - u(l)e^{i\theta} \|z\|^2_2 + 2 \max \lambda_l(1)/\lambda_l(2) \lambda_l(2) \neq 0 \times \sum_{l=1}^p \lambda_l(1)lz(C_l) - u(l)e^{i\theta} \|z\|^2_2$. We distinguish 3 cases:

- Case 2.1: $\mathcal{M} := \{1 \leq i \leq m \mid |L_\theta(g_i) - g_i(z)| > \epsilon$ and $H_i(d) < \max H(d)\} \neq \emptyset$
- Case 2.2: $\mathcal{M} = \emptyset$ and $\mathcal{M}' := \{i \in S \mid |L_\theta(g_i) - g_i(z)| > \epsilon\} \neq \emptyset$
- Case 2.3: $\mathcal{M} = \mathcal{M}' = \emptyset$
In Case 2.1, let $H_j(d + 1) := H_j(d) + 1$ if $I_j \subset I_i$ and $i$ has one of the $h$ highest mismatches $|l_v(g_i) - g_i(z)|$ among $i \in \mathcal{M}$. For all other $1 \leq j \leq m$, let $H_j(d + 1) := H_j(d)$ unless the bound $\Delta_{\min}^{\max}$ is violated, in which case for all $1 \leq j \leq m$ such that $H_j(d) = \min H(d)$, let $H_j(d + 1) := H_j(d) + 1$. In Case 2.2, apply instructions of Case 2.1 where $\mathcal{M}$ is replaced by $\mathcal{M}'$. In Case 2.3, let $H(d + 1) := H(d) + (1, \ldots, 1)$. Observe that $\min H(d) \geq \max H(d) - \Delta_{\min}^{\max} \to +\infty$ as $d \to +\infty$.

4. Application to Electric Power Systems. The optimal power flow is a central problem in power systems introduced half a century ago in [14]. While many non-linear methods [16, 77] have been developed to solve this difficult problem, there is a strong motivation for producing more reliable tools. Since 2006, the ability of the Shor and second-order conic relaxations to find global solutions [3, 7, 20, 34, 47, 52, 69] has been studied. Some relaxations are presented in real numbers [43, 54] and some in complex numbers [9, 10, 76]. However, in all numerical applications, standard solvers such as SeDuMi, SDPT3, and MOSEK are used which currently handle only real numbers. Modeling languages such as YALMIP and CVX do handle inputs in complex numbers, but the data is transformed into real numbers before calling the solver [11, Example 4.42]. We use the European network to illustrate that it is beneficial to relax non-convex constraints before converting from complex to real numbers.

4.1. Optimal Power Flow. A transmission network can be modeled using an undirected graph $G = (\mathcal{B}, \mathcal{L})$ where buses $\mathcal{B} = \{1, \ldots, n\}$ are linked to one another via lines $\mathcal{L} \subset \mathcal{B} \times \mathcal{B}$. Power flows are governed by the admittance matrix $Y \in \mathbb{C}^{n \times n}$ whose extra diagonal terms $(l, m) \in \mathcal{L}$ are equal to $y_{lm}/(\rho_{lm}^T\rho_{lm})$ and whose diagonal terms $(l, l)$ are equal to $\sum_{(l,m) \in \mathcal{L}}(y_{lm} + y_{ml})/|\rho_{lm}|^2$. All other terms are equal to zero. Here, $y_{lm} \in \mathbb{C}$ denotes the mutual admittance between buses $(l, m) \in \mathcal{L}$, $y_{lm}^T \in \mathbb{C}$ denotes the admittance-to-ground at end $l$ of line $(l, m) \in \mathcal{L}$, and $\rho_{lm} \in \mathbb{C}$ denotes the ratio of the ideal phase-shifting transformer at end $l$ of line $(l, m) \in \mathcal{L}$.

Each bus injects power $p_k^\text{gen} + jq_k^\text{gen}$ into the network with capacity limits $p_k^{\min}$, $p_k^{\max}$, $q_k^{\min}$, $q_k^{\max}$ (potentially all equal to 0) and extracts power demand $p_k^\text{dem} + jq_k^\text{dem}$ from the network. Each bus operates at a voltage $v_k \in \mathbb{C}$. Finding power flows that minimize active power loss is a problem that can be cast as an instance of QCQP-

\[
\inf_{v \in \mathbb{C}^n} \frac{v^H Y^H + Y v}{2}
\]

\[
\text{s.t.} \quad \forall k \in \mathcal{B}, \quad p_k^{\min} - p_k^{\text{dem}} \leq v^H H_k v \leq p_k^{\max} - p_k^{\text{dem}},
\]

\[
\forall k \in \mathcal{B}, \quad q_k^{\min} - q_k^{\text{dem}} \leq v^H \tilde{H}_k v \leq q_k^{\max} - q_k^{\text{dem}},
\]

\[
\forall k \in \mathcal{B}, \quad (v_k^{\min})^2 \leq v^H e_k^T e_k v \leq (v_k^{\max})^2,
\]

where $H_k := \frac{Y^H e_k e_k^T + e_k^T Y}{2}$ and $\tilde{H}_k := \frac{Y^H e_k e_k^T - e_k^T Y}{2}$ are Hermitian and $e_k$ is the $k^{th}$ column of the identity matrix. In Section 4.2, power flows are computed that seek to minimize either power loss or generation costs $\sum_{k \in \mathcal{B}} a_k (p_k^{\text{gen}})^2 + b_k (p_k^{\text{gen}} + c_k$ where $a_k, b_k, c_k \in \mathbb{R}$, $a_k \geq 0$, and $p_k^{\text{gen}} = v^H H_k v + p_k^{\text{dem}}$. In the case of generation costs, new real variables $(t_k)_{k \in \mathcal{B}}$ are introduced, objective (4.1) is replaced by $\sum_{k \in \mathcal{B}} t_k$, and new constraints are added for all $k \in \mathcal{B}$: $a_k (v^H H_k v + p_k^{\text{dem}})^2 + b_k (v^H H_k v + p_k^{\text{dem}}) + c_k \leq t_k$. In Section 4.2 apparent power flow limits $|v_{\text{lim}}|^2 \leq s_{\text{lim}}^{\max}$ are enforced where $v_{\text{lim}} := \frac{v^H F_{\text{lim}} v}{2}$ and $F_{\text{lim}} := a_{\text{lim}}^H e_k e_k^T + b_{\text{lim}}^H e_m e_m^T$, with $a_{\text{lim}} := (y_{\text{lim}} + y_{\text{lim}}^H)/|\rho_{\text{lim}}|^2$ and $b_{\text{lim}} := -y_{\text{lim}}/(\rho_{\text{lim}}^T \rho_{\text{lim}})$. These can be written for all $(l, m) \in \mathcal{L}$: $(v^H F_{\text{lim}} v + v_{\text{lim}}^2)^2 \leq (s_{\text{lim}}^{\max})^2$. Note that generation cost and line flow constraints yield
second-order conic constraints for all the relaxations considered in this paper as well as
semidefinite constraints for higher orders of the moment/sum-of-squares hierarchies.
The optimal power flow problem is invariant under the action of the torus (Section 3.5)
due to alternating current. We thus implement invariant hierarchies in Section 4.2.3.

4.2. Numerical Results. We consider large test cases representing portions of
European power systems: Great Britain (GB) [72], Poland (PL) [77], and systems
from the PEGASE project [28, 37]. They were preprocessed (see Table 1) to remove
low-impedance lines in order to improve the solver’s numerical convergence, which
is a typical procedure in power system analysis. A $1 \times 10^{-3}$ per unit low-impedance
line threshold was used for all test cases except for PEGASE-1354 and PEGASE-2869
which use a $3 \times 10^{-3}$ per unit threshold. Table 1 includes the at-least-locally-optimal
objective values obtained from the interior point solver in MATPOWER [77] for the
problems after preprocessing. Note that the PEGASE systems specify generation
costs that minimize active power losses, so the objective values in both columns
are the same. Implementations use YALMIP 2015.06.26 [46], Mosek 7.1.0.28, and
MATLAB 2013a on a computer with a quad-core 2.70 GHz processor and 16 GB of
RAM. The results do not include the typically small formulation times.

| Test Case Name | Number of Complex Variables | Number of Edges in Graph | MATPOWER Solution [77] |
|----------------|-----------------------------|--------------------------|------------------------|
|                | Gen. Cost ($/hr)            | Loss Min. (MW)           |
| GB-2224        | 2,053                       | 2,581                    | 1,942,260              | 60,614 |
| PL-2383wp      | 2,177                       | 2,651                    | 1,868,350              | 24,991 |
| PL-2736sp      | 2,182                       | 2,675                    | 1,307,859              | 18,336 |
| PL-2737sop     | 2,183                       | 2,675                    | 777,617                | 11,397 |
| PL-2746wop     | 2,189                       | 2,708                    | 1,208,257              | 19,212 |
| PL-2746wp      | 2,192                       | 2,686                    | 1,631,737              | 25,269 |
| PL-3012wp      | 2,292                       | 2,805                    | 2,592,462              | 27,646 |
| PL-3120sp      | 2,314                       | 2,835                    | 2,142,720              | 21,513 |
| PEGASE-89      | 70                          | 185                      | 5,819                  | 5,819  |
| PEGASE-1354    | 983                         | 1,526                    | 74,043                 | 74,043 |
| PEGASE-2869    | 2,120                       | 3,487                    | 133,945                | 133,945 |
| PEGASE-9241    | 7,154                       | 12,292                   | 315,749                | 315,749 |
| PEGASE-9241R\(^7\) | 7,154                     | 12,292                   | 315,785                | 315,785 |

4.2.1. Shor Relaxation. Table 2 shows the results of applying SDP-R and
SDP-C. They yield global decision variables and the global objective value for the
cases marked an asterisk (*) in Table 2. For those cases, the eigenvector associated to
the largest eigenvalue is feasible up to 0.005 p.u. at voltage constraints and 1 MVA at
all other constraints, and the objective evaluated in the eigenvector matches the bound
within 0.05% relative to the bound. The lower bounds in Table 2 suggest that the
corresponding MATPOWER solutions in Table 1 are at least very close to being globally
optimal. The gap between the MATPOWER solutions and the lower bounds from
SDP-C for the generation cost minimizing problems are less than 0.72% for GB-2224,

\(^7\)PEGASE-9241 contains negative resistances to account for generators at lower voltage levels.
In PEGASE-9241R these are set to 0.
0.29% for the Polish systems, and 0.02% for the PEGASE systems with the exception of PEGASE-9241. The non-physical negative resistances in PEGASE-9241 result in weaker lower bounds, yielding a gap of 1.64%. In accordance with Appendices B

| Case Name | SDP-R Val. ($/hr) | Time (sec) | SDP-C Val. ($/hr) | Time (sec) |
|-----------|-------------------|------------|-------------------|------------|
| GB-2224   | 1,928,194         | 10.9       | 1,928,444         | 6.2        |
| PL-2383wp | 1,862,979         | 48.1       | 1,862,985         | 23.0       |
| PL-2736sp*| 1,307,749         | 35.7       | 1,307,764         | 22.0       |
| PL-2737sop*| 777,505         | 41.7       | 777,539           | 19.5       |
| PL-2746wop*| 1,208,168        | 51.1       | 1,208,182         | 22.8       |
| PL-2746wp | 1,631,589         | 43.8       | 1,631,655         | 20.0       |
| PL-3012wp | 2,588,249         | 52.8       | 2,588,259         | 24.3       |
| PL-3120sp | 2,140,568         | 64.4       | 2,140,605         | 25.5       |
| PEGASE-89*| 5,819             | 1.5        | 5,819             | 0.9        |
| PEGASE-1354| 74,035           | 11.2       | 74,035            | 5.6        |
| PEGASE-2869| 133,936          | 38.2       | 133,936           | 20.6       |
| PEGASE-9241| 310,658          | 369.7      | 310,662           | 136.1      |
| PEGASE-9241R| 315,848         | 317.2      | 315,731           | 95.9       |

and C, all objective values in Table 2 match within 0.037%. SDP-C is faster (between a factor of 1.60 and 3.31) than SDP-R. Exploiting the isomorphic structure of complex matrices in SDP-C is thus better than eliminating a row and column in SDP-R.

4.2.2. Second-Order Conic Relaxation. Table 3 shows the results of applying SOCP-R and SOCP-C. Unlike the Shor relaxation, they do not yield the global solution to any of the test cases.8 SOCP-C provides better lower bounds and is faster than SOCP-R. Lower bounds from SOCP-C are between 0.87% and 3.96% larger and solver times are faster by between a factor of 1.24 and 6.76 than those from SOCP-R.

4.2.3. Moment/Sum-of-Squares Hierarchy. The real hierarchy globally solves a broad class of optimal power flow problems [30, 39, 51, 53] by first converting them to real numbers. The dense real and complex hierarchies solve problems up to 10 buses while the sparse ones solve problems with up 40 buses. In order to solve large-scale instances, we apply the mismatch hierarchy of Section 3.8 with the following parameters: \( \epsilon := 1 \text{ MVA}; h := 2; \) and \( \Delta_{\text{min}} := 2. \) See Appendix E for a small example. The mismatches are taken to be the modulus of the complex number whose real part is the mismatch for constraint \( k \) in (4.2) and whose imaginary part is the mismatch for constraint \( k \) in (4.3). In other words, apparent power mismatches are considered rather than active and reactive power separately. To improve numerics, \( |y_{\alpha,\beta} + \frac{y_{\alpha,\beta}}{\|y_{\alpha,\beta}\|} - 2(v_{\alpha} + \beta)\) and \( |y_{\alpha,\beta} - \frac{y_{\alpha,\beta}}{\|y_{\alpha,\beta}\|} - 2(v_{\alpha} + \beta)\) are added to the complex hierarchy and \( |y_{\alpha}| \leq (v_{\alpha})^{\alpha + \beta} \) is added to the real hierarchy for all \( |\alpha|, |\beta| \leq \max H(d) \) where \( v_{\alpha} := (v_{\alpha}^{\max}, \ldots, v_{\alpha}^{\max}) \) (see (4.4)). A similar procedure can be found in [74].

In Tables 4 and 5, the mismatch hierarchy is applied until the solution obtained is feasible up to 0.005 p.u. at voltage constraints and 1 MVA at all other constraints9.

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8SOCP-C generally does not provide a global solution with the exception of radial systems when certain non-trivial technical conditions are satisfied [47].

9Typical violations are smaller than 1 MVA. For instance, with the complex hierarchy PL-3012wp
Table 3
Real and Complex SOCP (Generation Cost Minimization)

| Case Name  | SOCP-\text{R} | SOCP-\text{C} |
|------------|---------------|---------------|
|            | Val. ($/hr)  | Time (sec)    | Val. ($/hr)  | Time (sec)    |
| GB-2224    | 1,855,393    | 3.5           | 1,925,723    | 1.4           |
| PL-2383wp  | 1,776,726    | 8.5           | 1,849,906    | 2.4           |
| PL-2736sp  | 1,278,926    | 4.8           | 1,303,958    | 1.7           |
| PL-2737sop | 765,184      | 5.5           | 775,672      | 1.6           |
| PL-2746wop | 1,180,352    | 5.1           | 1,203,821    | 1.7           |
| PL-2746wp  | 1,586,226    | 5.5           | 1,626,418    | 1.7           |
| PL-3012wp  | 2,499,097    | 5.9           | 2,571,422    | 2.0           |
| PL-3120sp  | 2,080,418    | 6.2           | 2,131,258    | 2.2           |
| PEGASE-89  | 5,744        | 0.5           | 5,810        | 0.4           |
| PEGASE-1354| 73,102       | 3.4           | 73,999       | 1.5           |
| PEGASE-2869| 132,520      | 9.0           | 133,869      | 2.7           |
| PEGASE-9241| 306,050      | 35.3          | 309,309      | 10.0          |
| PEGASE-9241R| 312,682      | 36.7          | 315,411      | 5.4           |

and until the objective evaluated in the solution matches the bound within 0.05% relative to the bound. The optimal values in the two tables match to at least 0.007%, which is within the expected solver tolerance. Further, they match the optimal values for the loss minimizing problems in Table 1 to within 0.013%, further proving that they are globally optimal. However, local solvers do not always globally solve the optimal power flow [12,15,50,53]. Though both hierarchies solve many small- and medium-size test cases which minimize generation cost, the mismatch hierarchy requires too many higher-order constraints for larger generation-cost-minimizing test cases.

The feasible set of the optimal power flow problem is included in the ball of radius \( \sum_{k \in B} (v_{i_{\text{max}}})^2 \) so a slack variable and a sphere constraint may be added as suggested in Section 3.3. In order to preserve sparsity, a slack variable and a sphere constraint may be added for each maximal clique of the chordal extension of the network graph. However, it tends to introduce numerical convergence challenges in problems with several thousand buses, resulting in higher-order constraints at more buses and correspondingly longer solver times. Interestingly, the results in Table 5 were obtained without the slack variables and sphere constraints. A potential way to account for this would be to compute the Hermitian complexity [26] of the ideal generated by the polynomials associated with equality constraints. A step in that direction would be to assess the greatest number of distinct points (possibly infinite) \( v^i \in \mathbb{C}^n \), \( 1 \leq i \leq p \), such that \( (v^i)^H (H_k + i\tilde{H}_k)v^j = -p_{\text{dem}}^k - q_{\text{dem}}^k \) for all buses not connected to a generator and for all \( 1 \leq i,j \leq p \). The Hermitian complexity of the ideal generated by \( \sum_{i=1}^n |z_i|^2 + \sigma(z) + a \) as defined in (3.15) with \( a < 0 \) is equal to 1.

Tables 4 and 5 show that the complex hierarchy has advantages over the real hierarchy. In all cases except PEGASE-1354, there is a speedup factor in solver time of between 1.31 and 21.42. The most significant improvements are seen for cases (e.g., PL-2383wp and PL-2746wop whose biggest maximal clique has 19 nodes) where the higher-order constraints account for a large portion of the solver times. This is due to fewer terms in the higher-order constraints. There is also a speedup in solver has over 99% of the buses with less than 0.02 MVA violation, and only 0.09% of the buses with greater than 0.1 MVA violation. Maximum line flow violation is 0.0006 MVA.
time of between 2.0 and 5.9 for 7 out of the 8 small- to moderate-size generation-cost-minimizing test cases in [53], the exception being case39Q due to numerical difficulties. For those 7 cases, the maximum violation for the complex hierarchy is 0.08 MVA, with the remaining case (case118Q) having a maximum violation of 0.32 MVA.

PL-3012wp, PL-3120sp, PEGASE-1354, and PEGASE-2869 require more iterations in the complex case than the real one. However, the improved speed per iteration results in faster overall solution times for all of these test cases except for PEGASE-1354, for which 6 additional iterations result in a factor of 2.78 slower solver time. Interestingly, the dense versions of the real and complex hierarchies yield the same bounds at each order for small test cases (\( \leq 10 \) buses) from [12,44,50,55].

### 5. Conclusion

We construct a complex moment/sum-of-squares hierarchy for complex polynomial optimization and prove convergence toward the global optimum. Theoretical and experimental evidence suggest that relaxing non-convex constraints before converting from complex to real numbers is better than doing the operations in the opposite order. We conclude with the question: is it possible to gain efficiency by transposing convex optimization algorithms from real to complex numbers?

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**Table 4**

Real Moment/Sum-of-Squares Hierarchy (Active Power Loss Minimization)

| Case Name     | Num. Iter. | Global Obj. Val. (MW) | Max. Viol. (MVA) | Solver Time (sec) |
|---------------|------------|-----------------------|------------------|------------------|
| PL-2383wp     | 3          | 24,990                | 0.25             | 583.4            |
| PL-2736sp     | 1          | 18,334                | 0.39             | 44.0             |
| PL-2737sop    | 1          | 11,397                | 0.45             | 52.4             |
| PL-2746wop    | 2          | 19,210                | 0.28             | 2,662.4          |
| PL-2746wp     | 1          | 25,267                | 0.40             | 45.9             |
| PL-3012wp     | 5          | 27,642                | 1.00             | 318.7            |
| PL-3120sp     | 7          | 21,512                | 0.77             | 386.6            |
| PEGASE-1354   | 5          | 74,043                | 0.85             | 406.9            |
| PEGASE-2869   | 6          | 133,944               | 0.63             | 921.3            |

**Table 5**

Complex Moment/Sum-of-Squares Hierarchy (Active Power Loss Minimization)

| Case Name     | Num. Iter. | Global Obj. Val. (MW) | Max. Viol. (MVA) | Solver Time (sec) |
|---------------|------------|-----------------------|------------------|------------------|
| PL-2383wp     | 3          | 24,991                | 0.10             | 53.9             |
| PL-2736sp     | 1          | 18,335                | 0.11             | 17.8             |
| PL-2737sop    | 1          | 11,397                | 0.07             | 25.7             |
| PL-2746wop    | 2          | 19,212                | 0.12             | 124.3            |
| PL-2746wp     | 1          | 25,269                | 0.05             | 18.5             |
| PL-3012wp     | 7          | 27,644                | 0.91             | 141.0            |
| PL-3120sp     | 9          | 21,512                | 0.27             | 193.9            |
| PEGASE-1354   | 11         | 74,042                | 1.00             | 1,132.6          |
| PEGASE-2869   | 9          | 133,939               | 0.97             | 700.8            |
Appendix A. Rank-2 Condition. It is proven here that a Hermitian matrix $Z$ is positive semidefinite and has rank 1 if and only if $\Lambda(Z)$ is positive semidefinite and has rank 2.

(\(\Rightarrow\)) Say $Z = zz^H$ where real and imaginary parts are defined by $z = x + ix_2$ and $(x_1, x_2) \neq (0, 0)$. Then

\[
\Lambda(Z) = \begin{pmatrix}
(x_1 x_1^T + x_2 x_2^T) & (x_1 x_2^T - x_2 x_1^T) \\
(x_2 x_1^T - x_1 x_2^T) & (x_1 x_1^T + x_2 x_2^T)
\end{pmatrix}
\]

The rank of $\Lambda(Z)$ is equal to 2 since $(x_1^T x_1^T)^T$ and $(x_2^T x_1^T)^T$ are non-zero orthogonal vectors.

(\(\Leftarrow\)) Say $\Lambda(Z) = xx^T + yy^T$ where $x$ and $y$ are non-zero real vectors. Consider the block structure $x = (x_1^T x_2^T)^T$ and $y = (y_1^T y_2^T)^T$. For $i = 1, \ldots, n$, it must be that

\[
x_{1i}^2 + y_{1i}^2 = x_{2i}^2 + y_{2i}^2,
\]

(\(A.2a\))

\[
x_{1i} x_{2i} + y_{1i} y_{2i} = 0.
\]

(A.2b)

Two cases can occur. The first is that $x_{1i} x_{2i} \neq 0$ in which case there exists a real number $\lambda_i \neq 0$ such that

\[
\begin{cases}
y_{1i} = -\lambda_i x_{2i}, \\
y_{2i} = \frac{1}{\lambda_i} x_{1i}.
\end{cases}
\]

(A.3)

Equation (A.2a) implies that $(1 - \lambda_i^2 x_{1i}^2) = (1 - \lambda_i^2) x_{2i}^2$ thus $(1 - \lambda_i^2)(1 - \frac{1}{\lambda_i^2}) \geq 0$ and $\lambda_i = \pm 1$. The second case is that $x_{1i} x_{2i} = 0$. Then, according to (A.2b), $y_{1i} y_{2i} = 0$. If either $x_{1i} = y_{1i} = 0$ or $x_{2i} = y_{2i} = 0$, then (A.2a) implies that $x_{1i} = x_{2i} = y_{1i} = y_{2i} = 0$. If $x_{1i} = y_{2i} = 0$, then (A.2a) implies that $y_{1i} = \pm x_{2i}$. If $x_{2i} = y_{1i} = 0$, then (A.2a) implies that $y_{2i} = \pm x_{1i}$.

In any case, there exists $\epsilon_i = \pm 1$ such that

\[
\begin{cases}
y_{1i} = -\epsilon_i x_{2i}, \\
y_{2i} = \epsilon_i x_{1i}.
\end{cases}
\]

(A.4)

For $i, j = 1, \ldots, n$ it must be that

\[
(1 - \epsilon_i \epsilon_j)(x_{1i} x_{1j} - x_{2i} x_{2j}) = 0,
\]

(A.5a)

\[
(1 - \epsilon_i \epsilon_j)(x_{1i} x_{2j} + x_{1j} x_{2i}) = 0.
\]

(A.5b)

Moreover

\[
\begin{cases}
x_{1i} x_{1j} + y_{1i} y_{1j} = x_{1i} x_{1j} + \epsilon_i \epsilon_j x_{2i} x_{2j}, \\
x_{1i} x_{2j} + y_{1i} y_{2j} = x_{1i} x_{2j} - \epsilon_i \epsilon_j x_{2i} x_{1j}.
\end{cases}
\]

(A.6)

It will now be shown that

\[
\begin{cases}
x_{1i} x_{1j} + y_{1i} y_{1j} = x_{1i} x_{1j} + x_{2i} x_{2j}, \\
x_{1i} x_{2j} + y_{1i} y_{2j} = x_{1i} x_{2j} - x_{2i} x_{1j}.
\end{cases}
\]

(A.7)
It is obvious if \( \epsilon_i \epsilon_j = 1 \). If \( \epsilon_i \epsilon_j = -1 \), then (A.5a)–(A.5b) imply

\[
\begin{align*}
(A.5a) & \quad x_{1i} x_{1j} - x_{2i} x_{2j} = 0, \\
(A.5b) & \quad x_{1j} x_{2i} + x_{1i} x_{2j} = 0.
\end{align*}
\]

If \( x_{1i} x_{1j} x_{2i} x_{2j} = 0 \), it can be seen that (A.7) holds. If not, (A.8a) implies that there exists a real number \( \mu_{ij} \neq 0 \) such that

\[
(A.8a) \quad \left\{ \begin{array}{l}
x_{2i} = \mu_{ij} x_{1i}, \\
x_{2j} = \frac{1}{\mu_{ij}} x_{1j}.
\end{array} \right.
\]

Further, (A.8b) implies that \((\mu_{ij} + \frac{1}{\mu_{ij}}) x_{1j} x_{2i} = 0\). This is impossible \((\mu_{ij} + \frac{1}{\mu_{ij}} \neq 0 \) and \( x_{1j} x_{2i} \neq 0 \)). Thus, (A.7) holds.

With the left hand side corresponding to \( \Lambda(Z) = xx^T + yy^T \) and the right hand side corresponding to (A.1b), equation (A.7) implies that \( \Lambda(Z) \) is equal to (A.1b). Since the function \( \Lambda \) is injective, it must be that \( Z = (x_1 + ix_2)(x_1 + ix_2)^H \).

**Appendix B. Invariance of Shor Relaxation Bound.** We have \( \text{val}(\text{CSDP-}R) \geq \text{val}(\text{SDP-}R) \) since the feasible set is more tightly constrained due to (2.4d). To prove the opposite inequality, define \( \tilde{\Lambda}(X) := (A + C)/2 + i(B - B^T)/2 \) for all \( X \in \mathbb{S}_{2n} \) using the block decomposition in the left hand part of (2.4d). It is proven here that if \( X \) is a feasible point of \( \text{SDP-}R \), then \( \Lambda \circ \tilde{\Lambda}(X) \) is a feasible point of \( \text{CSDP-}R \) with same objective value as \( X \). Firstly, \( \Lambda \circ \tilde{\Lambda}(X) \) satisfies (2.4d) because \( \tilde{\Lambda}(X) \) is a Hermitian matrix. Secondly, in order to show that \( \Lambda \circ \tilde{\Lambda}(X) \) satisfies (2.4c), notice that if \( x = (x_1^T x_2^T)^T \) then

\[
(B.1) \quad \left( \begin{array}{c}
x_1 \\
x_2
\end{array} \right)^T \left( \begin{array}{cc}
C & -B \\
-B^T & A
\end{array} \right) \left( \begin{array}{c}
x_1 \\
x_2
\end{array} \right) = \left( \begin{array}{c}
-x_2 \\
x_1
\end{array} \right)^T \left( \begin{array}{cc}
A & B^T \\
B & C
\end{array} \right) \left( \begin{array}{c}
-x_2 \\
x_1
\end{array} \right).
\]

Hence \( \Lambda \circ \tilde{\Lambda}(X) \) is equal to the sum of two positive semidefinite matrices. Finally, to prove that \( \Lambda \circ \tilde{\Lambda}(X) \) satisfies (2.4d) and has same objective value as \( X \), notice that if \( H \in \mathbb{H}_n \) and \( Y \in \mathbb{S}_{2n} \), then \( \text{Tr}[\Lambda(H)Y] = \sum_{1 \leq i, j \leq 2n} \Lambda(H)_{ij} Y_{ij} = \sum_{1 \leq i,j \leq 2n} \Lambda(H)_{ij} Y_{ij} = \sum_{1 \leq i,j \leq n} \text{Re}(H)_{ij} A_{ij} + i \text{Im}(H)_{ij} B_{ij} \). Completing the proof, for all \( H \in \mathbb{H}_n \), \( \text{Tr}[\Lambda(H) \Lambda \circ \tilde{\Lambda}(X)] = 2 \text{Tr}[\Lambda \tilde{\Lambda}(X)] = \text{Tr}(\Lambda(H)X) \).

**Appendix C. Invariance of SDP-\( R \) Relaxation Bound.** We assume that \( X \) is a feasible point of \( \text{SDP-}R \) and construct a feasible point of \( \text{SDP-}R \) with same objective value and first diagonal entry equal to 0. Consider the eigenvalue decomposition

\[
X = \sum_{k=1}^p x_k x_k^T
\]

for some \( x_k \in \mathbb{R}^{2n} \) and \( p \in \mathbb{N} \). For all \( \theta \in \mathbb{R} \), define

\[
(C.1) \quad R_\theta := \Lambda[\cos(\theta)I_n + i \sin(\theta)I_n] = \begin{pmatrix}
\cos(\theta)I_n & -\sin(\theta)I_n \\
\sin(\theta)I_n & \cos(\theta)I_n
\end{pmatrix}.
\]

For \( k = 1, \ldots, p \), define \( \theta_k \in \mathbb{R} \) such that \( x_{k,n+1} + ix_{k,1} = : \sqrt{x_{k,n+1}^2 + x_{k,1}^2} e^{i \theta_k} \). Construct \( \tilde{X} := \sum_{k=1}^p (R_{\theta_k} x_k)(R_{\theta_k} x_k)^T \geq 0 \) whose first diagonal entry is equal to 0.
where the text in brackets indicates the origin of the constraint: Clique by solid lines in Fig. 3, connecting variables that appear in the same monomial in any of the constraint equations or objective function. The supergraph (\{1, \ldots, 5\}, \mathcal{E}^{\text{con}}) has edges \mathcal{E}^{\text{mon}} comprised of \mathcal{E}^{\text{mono}} (solid lines in Fig. 4) augmented with edges connecting all variables within each constraint with \(d_i > 1\) (dashed lines in Fig. 4). In this case, the supergraph is already chordal, so there is no need to form a chordal extension \(\mathcal{G}^{\text{ch}}\). The maximal cliques of the supergraph are \(C_1 = \{1, 2, 3\}\) and \(C_2 = \{2, 3, 4, 5\}\). Clique \(C_2\) is the minimal covering clique for all second-order constraints \(g_i(z) \geq 0, \forall i \in \{7, 8, 9, 10, 17, 18, 19, 20\}\). The order associated with \(C_2\) is two (\(d_i^2 = 2\)) since the highest order \(d_i\) among all constraints for which \(C_2\) is the minimal covering clique is two. Clique \(C_1\) is not the minimal covering clique for any constraints with \(d_i > 1\), so \(d_i^1 = 1\). The globally optimal objective value obtained from the complex
Fig. 4. Graph Corresponding to Equations (E.1) from Five-Bus System in [12]

hierarchy specified above is 946.8 with corresponding decision variable \( z = (1.0467 + 0.0000i, 0.9550 − 0.0578i, 0.9485 − 0.0533i, 0.7791 + 0.6011i, 0.7362 + 0.7487i)^T \).

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