Confinement Mechanism in Various Abelian Projections of $SU(2)$ Lattice Gluodynamics

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Abstract

We show that the monopole confinement mechanism in lattice gluodynamics is a particular feature of the maximal abelian projection. We give an explicit example of the $SU(2) \rightarrow U(1)$ projection (the minimal abelian projection), in which the confinement is due to topological objects other than monopoles. We perform analytical and numerical study of the loop expansion of the Faddeev–Popov determinant for the maximal and the minimal abelian projections, and discuss the fundamental modular region for these projections.

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1 Introduction

In his well known paper, ’t Hooft suggested a partial gauge fixing procedure for the $SU(N)$ gluodynamics which does not fix the $[U(1)]^{N-1}$ gauge group. Under the abelian transformations, the diagonal elements of the gluon field transform as gauge fields; the nondiagonal elements transform as matter fields. Due to the compactness of the $U(1)$ gauge group, the monopoles exist, and if they are condensed, the confinement of color can be explained in the framework of the classical equations of motion. The string between the colored charges is formed as the dual analogue of the Abrikosov string in a superconductor, the monopoles playing the role of the Cooper pairs.

Many numerical experiments (see e.g. the review) confirm the monopole confinement mechanism in the $U(1)$ theory obtained by the abelian projection from the $SU(2)$ lattice gluodynamics. The string tension $\sigma_{U(1)}$ calculated from the $U(1)$ Wilson loops (loops constructed only from the abelian gauge fields) coincides with the full $SU(2)$ string tension: the monopole currents satisfy the London equation for a superconductor. Recently it has been shown that the $SU(2)$ string tension is well reproduced by the contribution of the abelian monopole currents. Numerical study of the effective monopole action shows that the entropy of the monopole loops dominates over the energy, and therefore, there exists the monopole condensate in the zero temperature $SU(2)$ lattice gluodynamics. All these remarkable facts, however, have been obtained only for the so called maximal abelian (MaA) projection. Other abelian projections (such as the diagonalization of the plaquette matrix $U_{x,12}$) do not give evidence that the vacuum behaves as the dual superconductor. Below we give two relevant examples.

First, it turns out that the fractal dimensionality of the monopole currents extracted from the lattice vacuum by means of the maximal abelian projection is strongly correlated with the string tension. If monopoles are extracted by means of other projections, this correlation is absent (cf. Fig.2 and Fig.4 of ref.[12]). An other example is the temperature dependence of the monopole condensate measured on the basis of the percolation properties of the clusters of monopole currents. For the maximal abelian projection the condensate is nonzero below the critical temperature $T_c$ and vanishes above it. For the projection which corresponds to the diagonalization of $U_{x,12}$, the condensate is nonzero at $T > T_c$, and it is not the order parameter for the phase transition. The last result has been obtained by the authors of [13], but is unpublished.
In the present publication we discuss the dependence of the confinement mechanism on the type of the abelian projection. We find that the monopole confinement mechanism may be a particular property of the MaA projection (section 2), and we give an explicit example of the abelian projection \([1,4]\) in which confinement is due to topological defects which are not monopoles (section 3). We also study the effective \(U(1)\) action in the considered projections, and show that it is rather nonlocal (section 4).

## 2 Maximal Abelian Projection and Compact Electrodyamics

The MaA projection \([10,11]\) corresponds to the gauge transformation that makes the link matrices diagonal “as much as possible”. For the \(SU(2)\) lattice gauge theory, the matrices of the gauge transformation \(\Omega_x\) are defined by the following the maximization condition:

\[
\begin{align*}
\max_{\{\Omega_x\}} \sum_{x,\mu} \text{Tr}(U' x\mu \sigma_3 U' x\mu \sigma_3), \\
R(U') = \sum_{x,\mu} \text{Tr}(U' x\mu \sigma_3 U' x\mu \sigma_3), \quad U' x\mu = \Omega^+_x U x\mu \Omega^{-1}_x.
\end{align*}
\]

For the standard parametrization of the \(SU(2)\) link matrix, we have

\[
\begin{align*}
U_{11}^{11} &= \cos \phi_{x\mu} e^{i\theta_{x\mu}}, \\
U_{12}^{12} &= \sin \phi_{x\mu} e^{i\chi_{x\mu}}, \\
U_{21}^{21} &= U_{11}^{11*}, \\
U_{22}^{22} &= U_{21}^{21*}.
\end{align*}
\]

\(0 \leq \phi \leq \pi/2, -\pi < \theta, \chi \leq \pi\); condition (1) has the form:

\[
\begin{align*}
\max_{\{\Omega_x\}} \sum_{x,\mu} \cos 2\phi'_{x\mu}.
\end{align*}
\]

The \(U(1)\) gauge transformations, which leave invariant the gauge conditions \([1], [3]\), show that after the abelian projection \(\theta\) becomes the abelian gauge field and \(\chi\) is the vector goldstone field, which carry charge two in the continuum limit:

\[
\begin{align*}
\theta_{x\mu} &\rightarrow \theta_{x\mu} + \alpha_x - \alpha_{x+\mu}, \\
\chi_{x\mu} &\rightarrow \chi_{x\mu} + \alpha_x + \alpha_{x+\mu}.
\end{align*}
\]

It is instructive to consider the plaquette action in terms of the angles \(\phi, \theta\) and \(\chi\):
\[ S_P = \frac{1}{2} \text{Tr} U_1 U_2 U_3^+ U_4^+ = S^a + S^n + S^i, \quad (6) \]

where

\[ S^a = \cos \theta_P \cos \phi_1 \cos \phi_2 \cos \phi_3 \cos \phi_4, \]
\[ S^n = -\cos(\theta_3 + \theta_4 - \chi_1 + \chi_2) \cos \phi_3 \cos \phi_4 \sin \phi_1 \sin \phi_2 + \cos(\theta_2 + \theta_4 - \chi_1 + \chi_3) \cos \phi_2 \cos \phi_4 \sin \phi_1 \sin \phi_3 + \cos(\theta_1 - \theta_4 + \chi_2 - \chi_3) \cos \phi_1 \cos \phi_4 \sin \phi_2 \sin \phi_3 + \cos(\theta_2 - \theta_3 - \chi_1 + \chi_4) \cos \phi_2 \cos \phi_3 \sin \phi_1 \sin \phi_4 + \cos(\theta_1 + \theta_3 + \chi_2 - \chi_4) \cos \phi_1 \cos \phi_3 \sin \phi_2 \sin \phi_4 - \cos(\theta_1 + \theta_2 + \chi_3 - \chi_4) \cos \phi_1 \cos \phi_2 \sin \phi_3 \sin \phi_4, \]
\[ S^i = \cos \chi_\tilde{P} \sin \phi_1 \sin \phi_2 \sin \phi_3 \sin \phi_4; \]

here we have set:

\[ \theta_P = \theta_1 + \theta_2 - \theta_3 - \theta_4, \quad (8) \]
\[ \chi_\tilde{P} = \chi_1 - \chi_2 + \chi_3 - \chi_4, \quad (9) \]

and the subscripts 1, ..., 4 correspond to the links of the plaquette: 1 → \{x, x + \hat{\mu}\}, ..., 4 → \{x, x + \hat{\nu}\}. Note that \( S^a \) is proportional to the Wilson plaquette action of compact electrodynamics for the “gauge” field \( \theta \); the corresponding action \( S^i \) for the “matter” field \( \chi \) contains the unusual combination \( \chi_\tilde{P} \) \( (9) \), which is invariant under the gauge transformations \( (4) \). Action \( S^n \) describes the interaction of the fields \( \theta \) and \( \chi \).

Due to condition (3), in the MaA projection the angle \( \phi \) fluctuates about zero, and we can expect that the largest contribution to the total action \( (6) \) comes from \( S^a \), and that \( S^a > S^n > S^i \). This conjecture is confirmed by numerical calculations. We use the standard heat bath method to simulate \( SU(2) \) gluodynamics on the \( 10^4 \) lattice, at 15 values of \( \beta \); at each value of \( \beta \) we used 15 field configurations separated by 100 of heat bath sweeps. To get obtain the MaA projection, we performed 800 gauge fixing sweeps through the lattice for each field configuration. It occurs that \( < S^a > \) is close to the total action, the maximal difference between \( < S_P > \) and \( < S^i > \) is at \( \beta \approx 2.2 \), where \( < S^a > \approx 0.82 < S_P >; \) \( S^i \) is unacceptably small: \( < S^i > \approx -0.001 \pm 0.0004 \).
at $\beta = 2.2$, at other values of $\beta$ the absolute value of $< S^i >$ is even smaller. It is clear that if we neglect the fluctuations of the angle $\phi$, as well as the Faddeev-Popov determinant, the $SU(2)$ action in the maximal abelian gauge is well approximated by the $U(1)$ action: $S_P \approx \cos \theta_P$, with the renormalized constant $\bar{\beta} = \beta \cos^4 \phi$.

Since in the compact electrodynamics the confinement is due to the monopole condensation, it is not surprising that in numerical experiments the vacuum of gluodynamics behaves in the MaA projection as the dual superconductor. Of course, this is only an intuitive argument. The confinement in the $U(1)$ theory exists in the strong coupling region, in which the rotational invariance is absent. Therefore, in order to explain the confinement at large values of $\beta$ in $SU(2)$ gluodynamics, we have to study in detail some special features of the gauge fixing procedure (such as the Faddeev-Popov–determinant, fluctuations of the angle $\phi$, etc.). We discuss some of these questions in the section 4.

The fact that $< S^a >$ is close to $< S_P >$ is very interesting; it means that in the MaA projection there is a small parameter in the $SU(2)$ lattice gluodynamics, which is $\varepsilon = \frac{< S_P > - < S^a >}{< S_P >}$; at all values of $\beta$, we have $\varepsilon \leq 0.18$. The meaning of this parameter is simple: it is the natural measure of closeness between the diagonal matrices and the link matrices after the gauge projection.

### 3 $SU(2)$ Gluodynamics in the Minimal Abelian Projection

The minimal abelian (MiA) projection \cite{14} is defined similarly to the MaA projection \cite{1} by

$$
\min_{\{\Omega_\mu\}} R(U') , \tag{10}
$$

where $R(U')$ is defined by \cite{2}. In this projection the largest part of the plaquette action \cite{3} is $S^i$, and the term which is most important for the dynamics is $\cos \chi_P$ (rather than $\cos \theta_P$ as it is in the MaA projection). The fields in the MiA projection can be transformed into the fields in the MaA projection by the following gauge transformation:

$$
\Omega(x) = -i\sigma_2 \cdot \frac{(-1)^{x_1 + x_2 + x_3 + x_4} + 1}{2} + 1 \cdot \frac{(-1)^{x_1 + x_2 + x_3 + x_4} - 1}{2} . \tag{11}
$$
Thus $\Omega(x)$ is equal to the unity in the “odd” sites of the lattice, and to $-i\sigma_2$ in the “even” sites; this gauge transformation becomes singular in the continuum limit. The angles $\phi$, $\theta$ and $\chi$, which parametrize the link matrix $U_l$, transform under this gauge transformation in the following way. If the link starts at an even point, ($(-1)^{x_1+x_2+x_3+x_4} = 1$), then $U_l \rightarrow (-i\sigma_2)U_l$ and

$$\phi \rightarrow \frac{\pi}{2} - \phi, \quad \theta \rightarrow -\chi, \quad \chi \rightarrow (\pi - \theta) \text{ mod } 2\pi.$$  \hspace{1cm} (12)

If the link starts at an odd point, then $U_l \rightarrow U_l^*(-i\sigma_2)U_l^*$ and

$$\phi \rightarrow \frac{\pi}{2} - \phi, \quad \theta \rightarrow (\pi + \chi) \text{ mod } 2\pi, \quad \chi \rightarrow \theta.$$  \hspace{1cm} (13)

Since $Tr(U'_{x\mu}\sigma_3U'^*_{x\mu}\sigma_3) = \cos 2\phi'$, it follows that under this gauge transformation $Tr(U'_{x\mu}\sigma_3U'^*_{x\mu}\sigma_3) \rightarrow -Tr(U'_{x\mu}\sigma_3U'^*_{x\mu}\sigma_3)$, and the fields in the MaA projection are transformed into fields in the MiA projection (and vice versa). Moreover, the monopoles extracted from the field $\theta$ in the MaA projection turn, in the MiA projection, into some topological defects constructed from the “matter” fields $\chi$. We call these topological defects “minopoles”.

Minopoles can be extracted from a given configuration of gauge fields similarly to monopoles: from the angles $\chi$ we construct gauge invariant plaquette variables $\chi_{\tilde{P}} = \tilde{d}\chi \text{ mod } 2\pi$, where $\tilde{d}\chi$ is defined by (9). From these plaquette variables we construct the variables attached to the elementary cubes $^{*}\!j = \frac{1}{2\pi}\tilde{d}\chi_{\tilde{P}}$; for $^{*}\!j \neq 0$ the link dual to the cube carries the minopole current. We use the notation $\tilde{d}$ (instead of $d$), since the gauge transformations of $\chi$ given by (3) differ from the gauge transformations of $\theta$ given by (4), and the construction of the plaquette variable from the link variables and that of the cube variable from the plaquette variables differ in an obvious way from the standard construction. For example, $d\theta$ is defined by (8) and $\tilde{d}\chi$ is defined by (9). In Fig.1 we illustrate the standard construction of the monopoles from the field $\theta$, and the construction of the minopoles from the field $\chi$.

Since monopoles, which exist in the MaA projection become minopoles in the MiA projection, then if in the MaA projection the confinement phenomenon is due to condensation of monopoles (constructed from the field $\theta$), then in the MiA projection the confinement is due to other topological objects (minopoles), constructed from the “matter” field $\chi$. We thus conclude that in the MiA projection the confinement is not due to monopoles and the vacuum is not an analogue of the dual superconductor. It should be stressed that monopoles still exist in the MiA projection; they can be extracted from...
the fields $\theta$ in the usual way, but they are not at all related to the dynamics. To illustrate this simple fact we plot in Fig. 2 the space–time asymmetry of the monopole currents \cite{13, 16} for the $SU(2)$ gauge theory on the $10^3 \times 4$ lattice for the MiA projection. In the same figure we also show the asymmetry of the minopole currents in the MiA projection. These results have been obtained by averaging over 10 statistically independent field configurations for each value of $\beta$, and 500–800 of gauge fixing sweeps have been performed for each configuration.

It is clearly seen that the asymmetry of the minopole currents is the order parameter for the temperature phase transition, while the asymmetry of the monopole currents is not. Since the monopole currents and the minopole currents are interchanged when the fields are transformed from the MiA to the MaA projection, Fig. 2 also shows that for the MaA projection the asymmetry of the monopole currents is the order parameter, whereas the asymmetry of minopole currents is not an order parameter.

Minopoles are to some extend the lattice artifacts, since the gauge fields in the MaA and the MiA projections are related by the gauge transformation, which becomes singular in the continuum limit. We discuss minopoles, since they clearly illustrate the dependence of the confinement mechanism on the lattice, upon the type of the abelian projection.

4 MaA and MiA Projections and the Lattice Path Integral

There are a lot of facts (see Introduction) that in the MaA projection the confinement is due to the monopole condensation. The confinement mechanism is in some sense the same in the MiA projection, since the MiA projection is related to the MaA projection by the global gauge transformation (11). Therefore it is important to study the effective $U(1)$ action in the MaA (and in the MiA) projection and in this section we discuss the MiA and the MaA projections in the lattice path integral. It is easy to find that the Faddeev–Popov determinant is the same for the MaA and the MiA projections, the difference between them being in the region of integration over the gauge fields; this region is called the “Fundamental Modular Region” (FMR) \cite{18}. In Appendix 1 the definition of this asymmetry is obvious: $A = <(J_t - J^S)/J_t >$, where $J^S = (J_x + J_y + J_z)/3$, $J_\mu$ is the monopole current in the direction $\mu$. 

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we briefly describe how the FMR appears in the gauge fixing procedure. The gauge fixing actions $S_{gf}$ for the MaA and the MiA projections (1), (10) are given by:

$$S_{gf}^a(U) = 2 - R(U), S_{gf}^i(U) = 2 + R(U),$$

and $R(U)$ is defined by (4). It is clear that $S_{gf}^a$ and $S_{gf}^i$ transform into each other by the local gauge transformation (11). From the definition of the FMR, given by eq.(A.6), we conclude that all fields from the FMR for the MaA projection transform, by this transformation, into fields which lie in the FMR for the MiA projection and vice versa. Moreover, the determinants of the matrices $D$ (A.3) coincide for both projections; and the final expression for the partition function for the fixed projection (A.5) differs by the FMR. It is remarkable that the difference in the integration regions leads to different confinement mechanisms in these projections.

There exist a natural loop expansion of the Faddeev–Popov determinant; below we calculate two leading terms of this expansion. For definiteness, consider the MaA projection. Expanding the MaA gauge fixing action (14) in powers of $\omega$ (see eq.(A.3)) we obtain the stationary point condition (A.4):

$$C^a_x(W) = \sum_{b=1,2} \sum_{\mu=1}^4 \sum_{\nu=1} \sum_{\sigma=1}^[[\tilde{U}_{x,\mu}^a \sigma^3 \tilde{U}_{x,\nu}^a + \tilde{U}_{x,\mu}^a \sigma^3 \tilde{U}_{x,\nu}^a]_{b}^{A} = 0.$$ (15)

The corresponding gauge fixing equations in the continuum limit are (10):

$$\sum_{\mu}(A^a_{\mu} \pm i \partial_{\mu}) A^a_{\pm} = 0, A^a_{\pm} = A^a_{\mu} \pm i A^a_{\nu}.$$ (16)

The Faddeev–Popov operator $\mathcal{D}$, which enters the expansion (A.3), is given by (up to irrelevant constant):

$$\mathcal{D}^a_{ab} = 2D K^a_{ab}(U) + B^a_{ab}(U),$$ (17)

where

$$K^a_{ab}(U) = \tilde{K}_{x} \delta_{xy} (\delta^{ab} - \delta^{a3} \delta^{b3}), \quad \tilde{K}_{x} = \frac{1}{2D} \sum_{\mu=-D}^{D} \cos(2\phi_{x,x+\mu});$$ (18)

here $D$ is the dimension of the space–time; the angle $\phi_{x,x}$ is one of the parameters, defining the link matrix (see eq.(4)); and

$$B^a_{ab}(U) = Tr \left[ \sigma^a U_{x,x+\mu} \sigma^b U_{x,x+\mu}^a \right] \epsilon^{a3} \epsilon^{b3}$$ (19)

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The indices \(x, y\) correspond to sites of the lattice and \(a, b\) are the color indices. The matrix \(K_{ab}^{xy}\) is diagonal with respect to the spatial indices \(x, y\); the sum in (17) is taken over all links connected to the site \(x\). The matrix \(B_{xy}^{ab}\) is nonzero if the points \(x\) and \(y\) belong to the beginning point and to the end–point of a link.

The determinant of the matrix \(D\) can be represented as the exponential function of the sum over all closed graphs \(\{L\}\) of length \(L\):

\[
\text{Det} \frac{1}{2} \{D(U)\} = \text{const.} \exp \left\{ \sum_x \ln(|\tilde{K}_x|) - \sum_{L=2}^{\infty} \frac{1}{2L} \left( -\frac{1}{2D} \right)^L \sum_{l \in \{L\}} \text{Tr} \prod_l K^{-1} B \right\},
\]

where the product \(\prod_l\) is along the path \(l\). Since any closed path consists of the even number of links, this expansion is in powers \(1/D^2\). We define the effective action as:

\[
e^{-S} \text{Det} \frac{1}{2} \{D(U)\} = e^{-S_{\text{eff}}}.
\]

The loop expansion of the effective action is \(S_{\text{eff}} = \beta \sum_P S_P + S^{(0)} + S^{(1)} + S^{(2)} + O(1/D^3)\). Here \(S^{(k)}\) corresponds to the loop of length \(2k\), and has the order \(O(1/D^k)\):

\[
S^{(0)} = -\sum_x \ln(|\tilde{K}_x|),
\]

\[
S^{(1)} = \sum_{x, \mu} \frac{3 + \cos 4\phi_{x\mu}}{4D^2 \tilde{K}_x \tilde{K}_{x+\mu}}.
\]

It is natural to subdivide the loops which contribute to \(S^{(2)}\) into two types \((S^{(2)} = S^{(2,1)} + S^{(2,2)})\): the one–dimensional loops passing through the points \(x, x+\mu, x, x+\nu\) and returning to the point \(x\), and the loops which correspond to plaquettes. The contribution of the one–dimensional loops is:

\[
S^{(2,1)} = \sum_x \sum_{\mu, \nu = -D}^{D} \frac{1}{64D^4 \tilde{K}_x^2 \tilde{K}_{x+\mu} \tilde{K}_{x+\nu}} \left[ (3 + \cos 4\phi_{x\mu})(3 + \cos 4\phi_{x\nu}) + 4 \sin^2 2\phi_{x\mu} \sin^2 2\phi_{x\nu} \cos(2(\chi_{x\mu} - \chi_{x\nu} + \theta_{x\mu} - \theta_{x\nu})) \right].
\]

Note that in this expression we have \(\mu = 1, \ldots, 4\) and \(\nu = -4, \ldots, -1, 1, \ldots, 4\). If \(\nu = \mu\), then loop belongs to a single link; if \(\nu = -\mu\), then the loop corresponds to the straight line; and if \(|\nu| \neq \mu\), then the loop corresponds to two neighboring links, perpendicular to each other.
The effective potential of the field \( \phi \), which corresponds to \( S^{(0)} + S^{(1)} \), has its minimum at the points \( \phi = 0 \) (\( \phi = \pi/2 \)), and it tends to infinity as \( \phi \) approaches \( \pi/4 \). Thus the field \( \phi \) fluctuates about the value \( \phi = 0 \) (\( \phi = \pi/2 \)) for the MaA (MiA) projection.

The term corresponding to the plaquette loop in the expansion (19) defines the correction to the plaquette action (6):

\[
S^{(2,2)} = \delta S^a + \delta S^n + \delta S^i,
\]

where

\[
\begin{align*}
\delta S^a &= C_P(\phi) \cos 2\theta_P \cos^2 \phi_1 \cos^2 \phi_2 \cos^2 \phi_3 \cos^2 \phi_4, \\
\delta S^n &= C_P(\phi) \left[ \cos(2(\theta_3 + \theta_4 - \chi_1 + \chi_2)) \cos^2 \phi_3 \cos^2 \phi_4 \sin^2 \phi_1 \sin^2 \phi_2 \\
&+ \cos(2(\theta_2 + \theta_4 - \chi_1 + \chi_3)) \cos^2 \phi_3 \cos^2 \phi_2 \sin^2 \phi_1 \sin^2 \phi_3 \\
&+ \cos(2(\theta_1 - \theta_4 + \chi_2 - \chi_3)) \cos^2 \phi_1 \cos^2 \phi_4 \sin^2 \phi_2 \sin^2 \phi_3 \\
&+ \cos(2(\theta_2 - \theta_3 - \chi_1 + \chi_4)) \cos^2 \phi_2 \cos^2 \phi_3 \sin^2 \phi_1 \sin^2 \phi_4 \\
&+ \cos(2(\theta_1 + \theta_3 + \chi_2 - \chi_4)) \cos^2 \phi_1 \cos^2 \phi_3 \sin^2 \phi_2 \sin^2 \phi_4 \\
&+ \cos(2(\theta_1 + \theta_2 + \chi_3 - \chi_4)) \cos^2 \phi_1 \cos^2 \phi_2 \sin^2 \phi_3 \sin^2 \phi_4 \right], \\
\delta S^i &= C_P(\phi) \cos(2\chi_P) \sin^2 \phi_1 \sin^2 \phi_2 \sin^2 \phi_3 \sin^2 \phi_4;
\end{align*}
\]

\( \theta_P, \chi_P \) and \( \tilde{K}_x(\phi) \) are given by eqs. (8), (9) and (17);

\[
C_P(\phi) = \frac{2}{D^4} \frac{1}{K_{x_1}(\phi) K_{x_2}(\phi) K_{x_3}(\phi) K_{x_4}(\phi)}.
\]

As in (14), the subscripts 1,...,4 of the angles \( \phi, \theta, \chi \) correspond to four links of the plaquette under consideration, and the points \( x_1...x_4 \) are the corners of this plaquette. Note that \( S_{eff} \) is invariant under the \( U(1) \) gauge transformations (14), (15), as it should be. In addition to the \( U(1) \) action for the fundamental representation \( S^a \sim \cos \theta_P \) the adjoint action \( \delta^{(1)} S^a \sim \cos 2\theta_P \) exists in the effective action.

In order to study the effective \( U(1) \) action, we calculated numerically the quantum averages \( \langle S^{(0)} \rangle, \langle S^{(1)} \rangle \) and \( \langle S^{(2)} \rangle \) in the MaA projection. The dependence of these quantities on \( \beta \) is shown in Fig.3. These results have been obtained by averaging over 10 statistically independent field configurations for each value of \( \beta \), 500–800 of the gauge fixing sweeps have been
performed for each configuration. The asymptotics of $S^{(i)}$, shown in this figure, are given by the formulae:

\[
< S^{(0)} > = \frac{1}{12\beta} + O\left(\frac{1}{\beta^2}\right) \tag{26}
\]

\[
< S^{(1)} > = \frac{1}{24} \left(1 + \frac{1}{2\beta}\right) + O\left(\frac{1}{\beta^2}\right) \tag{27}
\]

\[
< S^{(2)} > = \frac{1}{96} \left(\frac{7}{4} + \frac{1}{\beta}\right) + O\left(\frac{1}{\beta^2}\right) \tag{28}
\]

These expressions can be easily found from (20)–(24) by the standard low temperature expansion technique. For their derivation we have used the fact that $\langle S_n \rangle = O\left(\frac{1}{\beta^2}\right)$ (we can not prove this fact analytically but our numerical data clearly confirm it).

From Fig.3 we conclude that the higher loops in the effective action are not strongly suppressed; the same conclusion was maid in [4], [19]. In these papers the effective $U(1)$ action integrated over $\phi$ and $\chi$ was studied numerically.

5 Conclusions and Acknowledgments

If monopoles are responsible for the confinement in MaA projection and minopoles are responsible for the confinement in the MiA projection, what are the important topological excitations in a general abelian projection? If both diagonal and nondiagonal gluons are not suppressed, then string-like topological defects can also be important for the dynamics of the system [20]. The idea is: nondiagonal gluons transform under the $U(1)$ gauge transformations as matter fields, diagonal gluons transform as gauge fields, and an analogue of the Abrikosov–Nielsen–Olesen strings exists in gluodynamics after the abelian projection. Between strings made of condensed nondiagonal gluons (which carry the $U(1)$ charge 2) and the test quark of the charge 1, there exists topological interaction [21, 22], which is the analogue of the Aharonov–Bohm effect. Thus, in the effective $U(1)$ action of the $SU(2)$ gluodynamics there probably exists a very specific topological interaction. We describe an analytical and numerical study of this interaction in a separate publication.

The topological defects discussed above may be a reflection of some $SU(2)$ gauge field configuration. For example, monopoles and minopoles may be the
abelian projection of $SU(2)$ monopoles [14]. In ref. [23] it is found that the “extended monopoles” [12] may be important for the confinement mechanism in different abelian projections of the 3D $SU(2)$ gluodynamics. Finally, we note that it was found recently [24] that the contribution of the Dirac sheets to the abelian Polyakov loops plays the role of the order parameter for finite temperature lattice gluodynamics; it is interesting that this fact holds not only for the MaA projection but also for others unitary gauges. Note that in the MiA projection monopoles are substituted by minopoles, and in this projection we expect that the contribution of the minopole “Dirac sheets” is correlated with the expectation values of the Polyakov loops.

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**Appendix A  The Gauge Fixing Procedure And The Fundamental Modular Region**

Here we briefly describe how the Fundamental Modular Region [18] arises in the gauge fixing procedure. The standard method of gauge fixing is to substitute the “Faddeev – Popov unity”

$$1 = \Delta_{FP}(U) \int [d\Omega] \exp\{-\lambda S_{gf}(\Omega U\Omega^+)\}$$  \hspace{1cm} (A.1)

into the path integral

$$Z = \int [dU] \exp\{-S(U)\},$$ \hspace{1cm} (A.2)

the limit $\lambda \to +\infty$ is implied. The gauge fixing actions $S_{gf}$ for the MaA and the MiA projections are given by (14). Since $\lambda \to +\infty$ in (A.1), we can use the saddle point approximation to calculate $\Delta_{FP}(U)$. To this end, we parametrize $\Omega$ in (A.1) in the following way: $\Omega(\omega) = \Omega^0 \exp\left[\frac{ig_a}{2} \omega^a_x\right]$, where $\Omega^0$ is such that the fields $\tilde{U} = \Omega^0 U_{x,\bar{x}+\bar{\mu}} \Omega^0_{x,\bar{x}+\bar{\mu}}$ correspond to the absolute minimum of $S_{gf}$, so that $\Omega^0 = \Omega^0(U)$. At the saddle point we can restrict ourselves to small values of $\omega = \omega^a_x$, and
\[ S_{gf}(\Omega(\omega)U\Omega^+(\omega)) = S_{gf}(\tilde{U}) + C\omega + \frac{1}{2}\omega D\omega + O(\omega^3), \]  
(A.3)

where \( C = C^a_a(\tilde{U}) \) and \( D = D_{ab}^{xy}(\tilde{U}) \). Since \( \tilde{U} \) corresponds to the absolute minimum of \( S_{gf} \), we have

\[ C^a_a(\tilde{U}) = 0. \]  
(A.4)

Substituting (A.3) into (A.1) and integrating over \( \omega \), we get the standard expression for the Faddeev – Popov determinant: \( \Delta_{FP}(\tilde{U}) = \text{const.} \exp\{\lambda S_{gf}(\tilde{U})\} \text{Det}^{\frac{1}{2}}\{D(\tilde{U})\} \). Substituting (A.1) into the path integral (A.2) and using the gauge invariance of \( S(U) \), we get the product of the group volume \( \int[\text{d}\Omega] \) and the partition function for the fixed gauge:

\[ Z = \int[\text{d}U] \exp\{-S(U) + \lambda [S_{gf}(\tilde{U}) - S_{gf}(U)]\} \text{Det}^{\frac{1}{2}}\{D(U)\}. \]  
(A.5)

Using once again the fact that \( \lambda \to +\infty \), we see that the nonzero contribution to this path integral is given by \( U \) belonging to the global minima of \( S_{gf}(U) \):

\[ \Lambda = \left\{ U : S_{gf}(U) \leq S_{gf}(\Omega U\Omega^+) \; \forall \; \Omega \right\}. \]  
(A.6)

By definition, in this region we have \( \Omega^0(U) = 1 \) and \( S_{gf}(\tilde{U}) \equiv S_{gf}(U) \). The global minimum of the gauge fixing action is usually called the Fundamental Modular Region (FMR) [18]. In order to restrict the integration over the gauge fields to this region, the step function

\[ \Gamma_{FMR}(U) = \begin{cases} 1 & \text{if } U \in \Lambda; \\ 0 & \text{otherwise}, \end{cases} \]  
(A.7)

should be inserted [18]; into the path integral (A.5) and we thus obtain the following form of the partition function for the fixed gauge:

\[ Z = \int[\text{d}U] \exp\{-S(U)\} \text{Det}^{\frac{1}{2}}\{D(U)\} \Gamma_{FMR}(U). \]  
(A.8)

To get this expression we have used the fact that \( \Omega^0(U) = 1 \) if \( U \) lies in the FMR.
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Figure captions

Fig.1. Construction of a monopole from the field $\theta$ and minopole from the field $\chi$.

Fig.2. Asymmetry of the monopole currents (circles) and the minopole currents (crosses) in the minimal abelian projection.

Fig.3. The corrections to the plaquette action, $S^{(i)}$, versus $\beta$. The asymptotics shown in this figure, are given by the formulae (27). Estimated statistical errors are of the size of the graphical symbols.
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Fig. 1
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Fig. 3