DEGREE 5 INVARIANT OF $E_8$

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1. Introduction

Throughout this paper, $k$ denotes a field of characteristic 0. We write $E_8$ for the split simple algebraic group with Killing-Cartan type $E_8$. The Galois cohomology set $H^1(k, E_8)$ classifies simple algebraic groups of type $E_8$ over $k$. One of the goals of the theory of algebraic groups over arbitrary fields is to understand the groups of type $E_8$, equivalently, understand the set $H^1(k, E_8)$.

The main tool is the Rost invariant $r_{E_8}: H^1(\ast, E_8) \rightarrow H^1(\ast, \mu_{60}^\otimes)$ discovered by Markus Rost and explained in Merkurjev's portion of the book [GMS]. It is a morphism of functors from the category of fields over $k$ to the category of pointed sets. We put

$$H^1(k, E_8)_0 := \{ \eta \in H^1(k, E_8) \mid r_{E_8}(\eta) \text{ has odd order} \}.$$ 

In [Sem08] Corollary 8.7, the second author proved that there is a morphism of functors:

$$u: H^1(\ast, E_8)_0 \rightarrow H^5(\ast, \mathbb{Z}/2\mathbb{Z}).$$

This is the degree 5 invariant from the title.

Let now $G$ be a group of type $E_8$. It corresponds with a canonical element of $H^1(k, E_8)$, so it makes sense to speak of “the Rost invariant of $G$”; we denote it by $r(G)$. Suppose now that $r(G)$ has odd order, so $G$ belongs to $H^1(k, E_8)_0$. The second author also proved in [Sem08]:

$$u(G) = 0 \text{ if and only if there is an odd-degree extension of } k \text{ that splits } G.$$ 

For example, the compact group $G$ of type $E_8$ over $\mathbb{R}$ has Rost invariant zero and $u(G) = (-1)^5$.

As an obvious corollary, we have:

$$\text{(1.2) If } k \text{ has cohomological dimension } \leq 2, \text{ then every } k\text{-group of type } E_8 \text{ is split by an odd-degree extension of } k.$$
Serre’s “Conjecture II” for groups of type $E_8$ is that in fact every group of type $E_8$ over such a field is split.

The purpose of this note is to calculate the value of $u(G)$ for certain $G \in H^1(k, E_8)_0$ and to give some applications of $u$.

2. Tits’s construction of groups of type $E_8$

2.1. There are inclusions of algebraic groups $G_2 \times F_4 \subset E_8$, where $G_2$ and $F_4$ denote split groups of those types. Furthermore, this embedding is essentially unique. Applying Galois cohomology gives a function $H^1(k, G_2) \times H^1(k, F_4) \to H^1(k, E_8)$. The first two sets classify octonion $k$-algebras and Albert $k$-algebras respectively, so this map gives a construction by Galois descent of groups of type $E_8$:

$$\text{octonion } k\text{-algebras} \times \text{Albert } k\text{-algebras} \to \text{groups of type } E_8$$

Jacques Tits gave concrete formulas on the level of Lie algebras for this construction in [T], see also [J]. This method of constructing groups of type $E_8$ is known as the Tits construction. (Really, Tits’s construction is more general and gives other kinds of groups as well. The variety of possibilities is summarized in Freudenthal’s magic square as in [Inv, p. 540]. However, the flavor in all cases is the same, and this case is the most interesting.)

Our purpose is to compute the value of $u$ on those groups of type $E_8$ with Rost invariant of odd order (so that it makes sense to speak of $u$) and arising from Tits’s construction. We do this in Proposition 2.6.

2.2. Following [Inv], we write $f_3(-)$ for the even component of the Rost invariant of an Albert algebra or an octonion algebra (equivalently, a group of type $F_4$ or $G_2$). We write $g_3(-)$ for the odd component of the Rost invariant of an Albert algebra; such algebras also have an invariant $f_5$ taking values in $H^5(k, \mathbb{Z}/2\mathbb{Z})$. An Albert algebra $A$ has $g_3(A) = 0$ and $f_3(A) = 0$ iff $A$ has a nonzero nilpotent, iff the group $\text{Aut}(A)$ is isotropic.

Suppose now that $G \in H^1(k, E_8)$ is the image of an octonion algebra $O$ and an Albert algebra $A$. It follows from a twisting argument as in the proof of Lemma 5.8 in [GQ] that

$$r(G) = r_{G_2}(O) + r_{F_4}(A).$$

In particular, $G$ belongs to $H^1(k, E_8)_0$ if and only if $f_3(O) + f_3(A) = 0$ in $H^3(k, \mathbb{Z}/2\mathbb{Z})$, i.e., if and only if $f_3(O) = f_3(A)$.

2.3. Definition. Define

$$t: H^1(*, F_4) \to H^1(*, E_8)_0$$
by sending an Albert $k$-algebra $A$ to the group of type $E_8$ constructed from $A$ and the octonion algebra with norm form $f_3(A)$, via Tits’s construction from 2.1. By the preceding paragraph, $r(G) = g_3(A) \in H^3(k, \mathbb{Z}/3\mathbb{Z})$, so $G$ does indeed belong to $H^1(k, E_8)_0$.

2.4. Example. If $A$ has a (nonzero) nilpotent element, then the group $t(A)$ is split. Indeed, $g_3(A)$ is zero so $t(A)$ is in the kernel of the Rost invariant. Also, $t(A)$ is isotropic because it contains the isotropic subgroup $\text{Aut}(A)$, hence $t(A)$ is split by, e.g., [Ga Prop. 12.1(1)].

2.5. Example. In case $k = \mathbb{Q}$, there are exactly three Albert algebras up to isomorphism. All have $g_3 = 0$; they are distinguished by the values of $f_3$ and $f_5$.

$$
\begin{array}{|c|c|c|}
\hline
f_3(A) & f_5(A) & t(A) \\
\hline
0 & 0 & \text{split by Example 2.4} \\
(-1)^3 & 0 & \text{split by Example 2.4} \\
(-1)^3 & (-1)^5 & \text{anisotropic by [J, p. 118]} \\
\hline
\end{array}
$$

It follows from Chernousov’s Hasse Principle for groups of type $E_8$ [PR] that for every number field $K$ with a unique real place, the set $H^1(K, E_8)_0$ has two elements: the split group and the anisotropic group constructed as in the last line of the table.

2.6. Proposition. For every Albert $k$-algebra $A$, we have:

$$u(t(A)) = f_5(A) \in H^5(k, \mathbb{Z}/2\mathbb{Z}).$$

Proof. The composition $ut$ is an invariant $H^1(*, F_4) \to H^5(*, \mathbb{Z}/2\mathbb{Z})$, hence is given by

$$ut(A) = \lambda_5 + \lambda_2 \cdot f_3(A) + \lambda_0 \cdot f_5(A)$$

for uniquely determined elements $\lambda_i \in H^i(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$, see [GMS p. 50].

We apply this formula to each of the three lines in the table from Example 2.5. Obviously $u$ of the split $E_8$ is zero, so the first line gives:

$$0 = u(\text{split } E_8) = \lambda_5 \in H^5(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}).$$

Applying this to the second line gives:

$$0 = u(\text{split } E_8) = \lambda_2 \cdot (-1)^3 \in H^5(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}).$$

For the last line, $u$ of the compact $E_8$ is $(-1)^5$ by (1.1), see the end of [Sem08] for details. We find:

$$(-1)^5 = u(\text{compact } E_8) = \lambda_0 \cdot (-1)^5,$$

so $\lambda_0$ equals 1 in $H^0(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

To show that $\lambda_2 = 0$ we proceed as follows. Consider the pure transcendental extension $F = \mathbb{Q}(x, y, z, a, b)$ and let $H$ be the group of
type $F_4$ with $f_3(H) = (x, y, z)$, $f_5(H) = f_3(H) \cdot (a, b)$ and $g_3(H) = 0$. Then $ut(H) = f_5(H) + f_3(H) \cdot \lambda_2$.

Let $K$ be a generic splitting field for the symbol $f_5(H)$. Since $H_K$ is isotropic, the resulting group $t(H)$ of type $E_8$ is isotropic over $K$, and, since it has trivial Rost invariant, it splits over $K$ [Ga, Prop. 12.1]. Obviously, $ut(H)$ is killed by $K$. Therefore $f_3(H) \cdot \lambda_2$ is zero over $K$. If $f_3(H) \cdot \lambda_2$ is zero over $F$, then by taking residues we see that $\lambda_2$ is zero in $H^2(Q(a, b), \mathbb{Z}/2\mathbb{Z})$, hence also in $H^2(Q, \mathbb{Z}/2\mathbb{Z})$. Otherwise, $f_3(H) \cdot \lambda_2$ is equal to $f_5(H)$ by [OViVo, Theorem 2.1], and again completing and taking residues with respect to the $x$-, $y$-, and $z$-adic valuations, we find that $\lambda_2 = (a, b) \in H^2(Q(a, b), \mathbb{Z}/2\mathbb{Z})$. But this is impossible because $\lambda_2$ is defined over $Q$. This proves that $\lambda_2 = 0$. □

2.7. Corollary. Let $G$ be as in Proposition 2.6. Let $\text{Kill}_-$ denote the Killing form of $-$ and $E_8$ the split group. Then

$$\langle 60 \rangle (\text{Kill}_G - \text{Kill}_{E_8}) = 2^3 \cdot u(G) \in I^8(k).$$

Proof. Follows from [Ga, 13.5 and Example 15.9] and Proposition 2.6. □

2.8. Example. Whatever field $k$ of characteristic zero one starts with, there is an extension $K/k$ that supports an anisotropic 5-Pfister quadratic form $q_5$—one can adjoin 5 indeterminates to $k$, for example. Let $q_3$ be a 3-Pfister form dividing $q_5$ and let $A$ be the Albert $K$-algebra with $f_d(A) = e_d(q_d)$ for $d = 3, 5$. The group $G := t(A)$ of type $E_8$ over $K$ has Rost invariant zero yet $u(G) = f_5(A)$ nonzero by Proposition 2.6. In particular, $G$ is not split, hence is anisotropic by [Ga, Prop. 12.1(1)].

Example 15.9 in [Ga] produced anisotropic groups of type $E_8$ in a similar manner, but used the Killing form to see that the resulting groups were anisotropic; that method does not work if $-1$ is a square in $k$. Roughly, Example 2.8 above exhibits more anisotropic groups because $u$ is a finer invariant than the Killing form.

3. Invariants of $H^1(\ast, \text{Spin}_{16})_0$

Recall from [Invpp. 436, 437] that the Rost invariant of a class $\eta \in H^1(k, \text{Spin}_{16})$ is given by the formula

$$r_{\text{Spin}_{16}}(\eta) = e_3(q_\eta) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$$

where $q_\eta$ is the 16-dimensional quadratic form in $I^3k$ corresponding to the image of $\eta$ in $H^4(k, SO_{16})$ and $e_3$ is the Arason invariant. It follows that $\eta$ belongs to the kernel of the Rost invariant if and only if $q_\eta$ belongs to $I^4k$. 

We can quickly find some invariants of the kernel $H^1(k, \text{Spin}_{16})_0$ of the Rost invariant. For $\eta$ in that set, $q_\eta$ is $(\alpha_\eta)\gamma$ for some $\alpha_\eta \in k^\times$ and some 4-Pfister quadratic form $\gamma$ [Lam X.5.6]. (One can take $\alpha_\eta$ to be any element of $k^\times$ represented by $q_\eta$ [Lam X.1.8].) We define invariants $f_d: H^1(\ast, \text{Spin}_{16})_0 \to H^d(\ast, \mathbb{Z}/2\mathbb{Z})$ for $d = 4, 5$ via:

$$f_4(\eta) := e_4(q_\eta) \quad \text{and} \quad f_5(\eta) := (\alpha_\eta) \cdot e_4(q_\eta),$$

where $e_4$ is the usual additive map $I^4(\ast) \to H^4(\ast, \mathbb{Z}/2\mathbb{Z})$. If $q_\eta$ is isotropic, then $q_\eta$ is hyperbolic and $e_4(q_\eta)$ is zero, so the value of $f_5(\eta)$ depends only on $\eta$ and not on the choice of $\alpha_\eta$, see [Ga09 10.2].

We can identify two more (candidates for) invariants of $H^1(\ast, \text{Spin}_{16})_0$. The split $E_8$ has a subgroup isomorphic to $H\text{Spin}_{16}$, the nontrivial quotient of $\text{Spin}_{16}$ that is neither adjoint (i.e., not $\text{PSO}_{16}$) nor $\text{SO}_{16}$. Further, the composition

$$H^1(\ast, \text{Spin}_{16}) \to H^1(\ast, H\text{Spin}_{16}) \to H^1(\ast, E_8) \xrightarrow{r_{E_8}} H^3(\ast, \mathbb{Z}/60\mathbb{Z}(2))$$

is the Rost invariant of $\text{Spin}_{16}$ [Ga (5.2)]. We find a morphism of functors

$$H^1(\ast, \text{Spin}_{16})_0 \to H^1(\ast, E_8)_0.$$

Composing this with the invariant $u$ gives an invariant

$$u_5: H^1(\ast, \text{Spin}_{16})_0 \to H^5(\ast, \mathbb{Z}/2\mathbb{Z}).$$

(Roughly speaking, we have used the invariant $u$ of $H^1(\ast, E_8)_0$ to get an invariant of $H^1(\ast, \text{Spin}_{16})_0$ in the same way that Rost used the $f_5$ invariant of $H^1(\ast, F_4)$ to get an invariant of $H^1(\ast, \text{Spin}_9)$, see [Rost] or [Ga09 18.9].)

The purpose of this section is to prove:

3.1. **Proposition.** The invariants $H^1(\ast, \text{Spin}_{16})_0 \to H^\ast(\ast, \mathbb{Z}/2\mathbb{Z})$ form a free $H^\ast(k, \mathbb{Z}/2\mathbb{Z})$-module with basis

$$1, f_4, f_5, u_5, u_6,$$

where the invariant $u_6$ is given by the formula $u_6(\eta) := (\alpha_\eta) \cdot u_5(\eta)$.

We first replace $\text{Spin}_{16}$ with a more tractable group. The first author described in [Ga §11] a subgroup of $H\text{Spin}_{16}$ isomorphic to $\text{PGL}_2^{\times 4}$. Examining the root system data for this subgroup given in Tables 7B and 11 of that paper, we see that the inverse image of this subgroup in $\text{Spin}_{16}$ is a subgroup $H$ obtained by modding $\text{SL}_2^{\times 4}$ out by the subgroup generated by $(-1, -1, 1, 1), (-1, 1, -1, 1)$, and $(-1, 1, 1, -1)$. Indeed, in the notation of that paper, each copy of $\text{PGL}_2^{\times 4}$ lifts to a copy of
SL\(_2\) with a maximal torus the image of one of the four elements of the root lattice of \(D_8\):

\[
\begin{align*}
\delta_1 + 2\delta_2 + 3\delta_3 + 4\delta_4 + 3\delta_5 + 2\delta_6 + \delta_7, \\
\delta_1 + \delta_3 - \delta_5 - \delta_7, \\
\text{or } \delta_1 + 2\delta_2 + \delta_3 - \delta_5 - 2\delta_6 - \delta_7.
\end{align*}
\]

(Here \(\delta_i\) denotes the simple root of \(D_8\) that is denoted \(\alpha_i\) in \([B]\).) The center of each copy of SL\(_2\) has nontrivial element

\[
h_{\delta_1}(-1) h_{\delta_3}(-1) h_{\delta_5}(-1) h_{\delta_7}(-1)
\]

in the notation of \([S1]\). This defines a homomorphism \(\mu_2 \to H\) that gives a short exact sequence:

\[
1 \to \mu_2 \to H \to \text{PGL}_2^{\times 4} \to 1.
\]

The image of \(H^1(k, H)\) in \(H^1(k, \text{PGL}_2)^{\times 4}\) consists of quadruples \((Q_1, Q_2, Q_3, Q_4)\) of quaternion algebras so that \(Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4\) is split.

Let \(\varphi\) map the Klein four-group \(V := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) into \((\text{SL}_2 \times \text{SL}_2)/\mu_2\) via

\[
\varphi(1, 0) := ((1, 0), (-1, 0)) \quad \text{and} \quad \varphi(0, 1) := ((0, 1), (0, -1)).
\]

This defines a homomorphism. Twisting \((\text{SL}_2 \times \text{SL}_2)/\mu_2\) by a pair \((a, b) \in k^\times/k^\times \times k^\times/k^\times = H^1(k, V)\) gives \((\text{SL}(Q) \times \text{SL}(Q))/\mu_2\), where \(Q\) denotes the quaternion algebra \((a, b)\). (Of course, composing \(\varphi\) with either of the projections \((\text{SL}_2 \times \text{SL}_2)/\mu_2 \to \text{PGL}_2\) sends \((a, b)\) to the same quaternion algebra \(Q\).) The composition

\[
V \times V \xrightarrow{\varphi \times \varphi} (\text{SL}_2 \times \text{SL}_2)/\mu_2 \times (\text{SL}_2 \times \text{SL}_2)/\mu_2 \to H
\]
gives a map whose image does not meet the center of \(\text{Spin}_{16}\), which we denote by \(Z\). This gives a homorphism \(Z \times V \times V \to \text{Spin}_{16}\).

3.2. **Lemma.** For every extension \(K/k\), the map \(Z \times V \times V \to \text{Spin}_{16}\) defined above induces a map \(H^1(K, Z \times V \times V) \to H^1(K, \text{Spin}_{16})\) that is a surjection.

**Proof.** Fix \(\nu \in H^1(K, Z \times V \times V)\); we write \(\zeta\) for its image in \(H^1(K, Z)\) and \(Q_1, Q_2\) respectively for its images under the two projections \(H^1(K, Z \times V \times V) \to H^1(K, V) \to H^1(K, \text{PGL}_2)\). It follows from the description of the subgroup \(\text{PGL}_2^{\times 4}\) in \(\text{HSpin}_{16}\) and the map \(H^1(K, \text{HSpin}_{16}) \to H^1(K, \text{PSO}_{16})\) in \([G]\), §4] that the image of \(\nu\) in \(H^1(K, \text{PSO}_{16})\) is the same as the image of the quadratic form \(q_1 \otimes q_2\) under \(H^1(K, \text{SO}_{16}) \to H^1(K, \text{PSO}_{16})\), where \(q_i\) denotes the norm form of \(Q_i\). As the Rost invariant of \(\text{Spin}_{16}\) “factors through” \(\text{SO}_{16}\) to give the Arason invariant, the image of \(\nu\) in \(H^1(K, \text{Spin}_{16})\) is in the kernel of the Rost invariant.

As for surjectivity, if \(\eta\) is in \(H^1(K, \text{Spin}_{16})\) then \(q_\eta = (\alpha)q_1 q_2\) where \(q_1, q_2\) are 2-Pfister forms. We define \(\nu\) to be \((0, Q_1, Q_2)\) where \(Q_i\) is a
quaternion algebra with norm $q_i$. Then the image of $\nu$ in $H^1(K, \text{Spin}_{16})$ is $\zeta \cdot \eta$ for some $\zeta \in H^1(K, Z)$, and we deduce that $\zeta \cdot \nu$ maps to $\eta$. \hfill $\square$

It follows from the lemma (using \cite[Ga09, 5.3]{Ga}) that the module of invariants $H^1(\ast, \text{Spin}_{16})$ injects into the module of invariants $H^1(\ast, Z \times V \times V) \rightarrow H^\ast(\ast, Z/2Z)$. But we know this larger module by \cite[6.7]{GMS}; it is spanned by products $\pi_1\pi_2\pi_3$ for $\pi_1 : H^1(\ast, Z) \rightarrow H^\ast(\ast, Z/2Z)$ and $\pi_2, \pi_3$ invariants $H^1(\ast, V) \rightarrow H^\ast(\ast, Z/2Z)$ composed with projection on the 2nd or 3rd term in the product.

For the rest of this section, we use the notation $(\zeta, Q_1, Q_2)$ for elements of $H^1(K, Z \times V \times V)$ as in the proof of Lemma \ref{lem:3.2}.

3.3. Lemma. Fix $(\zeta, Q_1, Q_2) \in H^1(K, Z \times V \times V)$. If $[Q_1] \cdot [Q_2] = 0$ in $H^1(K, Z/2Z)$, then the image of $(\zeta, Q_1, Q_2)$ in $H^1(K, \text{Spin}_{16})$ is zero.

Proof. The image $\eta$ of $(\zeta, Q_1, Q_2)$ has $q_\eta = \langle \alpha_\eta \rangle q_1q_2$ where $q_\eta$ is the norm of $Q_\eta$. If $[Q_1] \cdot [Q_2]$ is zero, then $q_1 \otimes q_2$ is in $I^5(k)$ and so is hyperbolic. It follows that the image of $\eta$ in $H^1(K, \text{PSO}_{16})$ is zero, hence $\eta$ is in the image of $H^1(K, Z)$. But $\text{Spin}_{16}$ is split semisimple, so the image of $H^1$ of the center in $H^1$ of the group is zero. \hfill $\square$

Proof of Proposition \ref{prop:3.7}. The invariant $f_4$ sends $(\zeta, Q_1, Q_2)$ to $[Q_1] \cdot [Q_2]$. Lemma \ref{lem:3.3} combined with arguments like those in \cite[pp. 43, 44]{GMS} shows that every invariant of $H^1(\ast, \text{Spin}_{16})$ restricts to one of the form $\lambda + \phi \cdot f_4$ for uniquely determined $\lambda \in H^\ast(k, Z/2Z)$ and $\phi : H^1(\ast, Z) \rightarrow H^\ast(\ast, Z/2Z)$, i.e., is given by the formula

$$ (\zeta, Q_1, Q_2) \mapsto \lambda + \phi(\zeta) \cdot [Q_1] \cdot [Q_2]. $$

The collection of such invariants of $H^1(\ast, Z \times V \times V)$ forms a free $H^\ast(k, Z/2Z)$-module with basis

$$ 1, f_4, \chi_v \cdot f_4, \chi_h \cdot f_4, \chi_v \cdot \chi_h \cdot f_4, $$

where $\chi_v$ and $\chi_h$ denote the maps $H^1(\ast, Z) \rightarrow H^1(\ast, Z/2Z)$ defined by restricting to $Z$ the vector representation $\text{Spin}_{16} \rightarrow \text{SO}_{16}$ and the half-spin representation $\text{Spin}_{16} \rightarrow \text{HSpin}_{16}$ implicit in our root-system description above. Obviously, $f_5$ restricts to be $\chi_v \cdot f_4$.

At this point, it suffices to prove:

$$ (3.4) \quad u_5 \text{ restricts to be } \chi_h \cdot f_4. $$

Indeed, this statement implies that $u_5$ is zero when $q_\eta$ is isotropic, hence by \cite[10.2]{Ga} the formula for $u_6$ gives a well-defined invariant; it obviously restricts to $\chi_v \cdot u_5 = \chi_v \cdot \chi_h \cdot f_4$ on $H^1(\ast, Z \times V \times V)$. Spanning and linear independence follow from the previous paragraph.
We now prove (3.4). The restriction of $u_5$ sends zero to zero, so it is $\phi \cdot f_4$ for some $\phi: H^1(\ast, \mathbb{Z}) \to H^1(\ast, \mathbb{Z}/2\mathbb{Z})$ that itself sends zero to zero. Therefore (by [GMS, p. 40]) $\phi$ is induced by some homomorphism $\chi: \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. As $u_5$ is defined by pulling back along the map $\text{Spin}_{16} \to E_8$, one quickly sees that $\chi$ must be zero or $\chi_h$. As $\chi_h$, the zero invariant, and $\chi_h$ are all defined over $\mathbb{Q}$, it suffices to prove that $\chi$ is not the zero invariant in the case where $k = \mathbb{Q}$. Example 15.1 of [Ga] gives a class $\nu \in H^1(\mathbb{R}, \mathbb{Z} \times V \times V)$ whose image in $H^1(\mathbb{R}, E_8)$ is the compact $E_8$, on which $u$ is nonzero. Hence the restriction of $u_5$ to $\mathbb{Z} \times V \times V$ is not zero over $\mathbb{Q}$ and must be $\chi_h \cdot f_4$. □

4. Essential dimension of $H^1(\ast, \text{Spin}_{16})_0$

The following is a corollary of the previous section:

4.1. Corollary. The essential dimension of the functor $H^1(\ast, \text{Spin}_{16})_0$ over every field of characteristic zero is 6.

Proof. The existence of the nonzero invariant $u_6: H^1(\ast, \text{Spin}_{16})_0 \to H^0(\ast, \mathbb{Z}/2\mathbb{Z})$ implies that the essential dimension is at least 6 by [RY, Lemma 6.9]; this is the interesting inequality. One can deduce that the essential dimension is at most 6 by, for example, the surjectivity in Lemma 3.2. □

By way of contrast, the essential dimension of the functor $H^1(\ast, \text{Spin}_{16})$ (without restricting to the kernel of the Rost invariant) is 24 by [BRV, Remark 3.9].

5. Galois descent for representations of finite groups

In this section, we restate some observations of Serre from [Serre00] and [GR98] regarding projective embeddings of simple groups in exceptional algebraic groups. Combining these results with the $u$-invariant for $E_8$ gives some new embeddings results, see Example [5.5] below.

Let $A$ be an abstract finite group and $G$ a split semisimple linear algebraic group defined over $\mathbb{Q}$. Fix a faithful representation $G \to \text{GL}_N$ defined over $\mathbb{Q}$. Further on fix a monomorphism $\varphi: A \to G(\overline{k})$, where $\overline{k}$ is an algebraically closed field of characteristic zero.

5.1. Definition. Let $F$ be a field extension of $\mathbb{Q}$. We say that the character of a representation $A \to G(\overline{k}) \to \text{GL}_N(\overline{k})$ is defined over $F$, if all its values belong to $F$.

Let $\chi$ be the character of the representation

$$\varphi: A \to G(\overline{k}) \to \text{GL}_N(\overline{k})$$
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and $F$ its field of definition. Assume additionally that $Z_{G(E)}(A) = 1$ and that there is exactly one $G(E)$-conjugacy class of homomorphisms $A \to G(E)$ with character $\chi$.

The following theorem can be extracted from Serre’s paper [Serre00, 2.5.3]:

5.2. Theorem. In the above notation there exists a twisted form $G_0$ of $G$ defined over $F$ together with a monomorphism $A \to G_0(F)$. Moreover, for a field extension $K/F$ there is a representation $A \to G(K)$ with character $\chi$ iff $G \simeq G_0$ over $K$.

Proof. Let

$$P = \{ \alpha : A \to G \mid \alpha \text{ is a representation with character } \chi \}.$$ 

Then $G$ acts on $P$ by conjugation. By assumptions on $A$ and $G$ this action is transitive. Moreover, the condition on the centralizer guarantees that this action is simply transitive, i.e., for any $\alpha, \beta \in P(K)$ there exists a unique $g \in G(K)$ with $\beta = \alpha g$. Thus, $P$ is a $G$-torsor. Since all values of the character $\chi$ belong to the field $F$, the torsor $P$ is defined over $F$.

Let $\eta \in H^1(F,G)$ be the 1-cocycle corresponding to the torsor $P$. Then $\sigma \cdot \varphi = \eta^{-1}_\sigma \varphi \eta_\sigma$ for all $\sigma$ in the absolute Galois group $\text{Gal}(\overline{F}/F)$. Define now $G_0$ as the twisted form of $G$ over $F$ by the torsor $P$. The group $G_0$ is defined out of $G(F)$ by a twisted Galois action:

$$\sigma \ast g = \eta_\sigma(\sigma \cdot g)\eta_\sigma^{-1}.$$ 

Now it is easy to see that the homomorphism $\varphi : A \to G(F)$ is an $F$-defined homomorphism $A \to G_0(F)$.

Let $K/F$ be a field extension. If there is a representation $A \to G(K)$ with character $\chi$, then obviously $G$ and $G_0$ are isomorphic over $K$. Conversely, if $G$ and $G_0$ are isomorphic over $K$, then the image of the cocycle $\eta$ in $H^1(K,\text{Aut}(G))$ is zero. Since the centralizer of $A$ in $G$ is trivial, the group $G$ is adjoint. Therefore $\eta$ is already zero in $H^1(K,G)$.

To characterize the isomorphism criterion of Theorem 5.2 we need the following proposition.

5.3. Proposition. For each Killing-Cartan type $\Phi$ in the table

\[
\begin{array}{c|ccc}
\text{Type } \Phi & F_4 & G_2 & E_8 \\
\hline
n & 3 & 3 & 5
\end{array}
\]

there is a unique algebraic group $G_0$ of type $\Phi$ that is compact at every real place of every (equivalently, a particular) number field; it is defined
over $\mathbb{Q}$. For every field $K$ of characteristic zero and $n$ as in the table, the following are equivalent:

1. $G_0 \otimes K$ is split.
2. $(-1)^n = 0 \in H^n(K, \mathbb{Z}/2)$.
3. $-1$ is a sum of $2^{n-1}$ squares of the field $K$.

Proof. The first sentence is a standard part of the Kneser-Harder-Chernousov Hasse principle. The group $G_0$ is split at every finite place. For the second claim, all cases but $E_8$ are well-known. For $E_8$, if $G_0 \otimes K$ is split, then $(-1)^5$ is zero by the existence of $u$; see [1.1]. For the converse, $G_0$ equals $t(A)$ where $A$ is the unique Albert $\mathbb{Q}$-algebra with no nilpotents (see Example 2.5). If $(-1)^5$ — i.e., $f_5(A)$ — is zero in $H^5(K, \mathbb{Z}/2\mathbb{Z})$, then $A \otimes K$ has nilpotents and $G_0 \otimes K$ is split by Example 2.4. □

In the following examples we denote as $\text{Alt}_l$ the alternating group of degree $l$ and as $\zeta_l = e^{2\pi i/l}$ a primitive $l$-th root of unity.

5.4. Example (type $G_2$). Let $G$ denote the split group of type $G_2$, $A = G(\mathbb{F}_2)$ (resp. $PSL(2, 8)$, $PSL(2, 13)$), and $K$ a field of characteristic zero. Then there is an embedding $A \to G(K)$ iff $-1$ is a sum of 4 squares of $K$ and $\zeta_9 + \bar{\zeta}_9 \in K$ (for $PSL(2, 8)$), resp. $\sqrt{13} \in K$ (for $PSL(2, 13)$).

Indeed, fix the minimal fundamental representation $G \to GL_7$. By [A87, Theorem 9(3,4,5)] there is a representation $\varphi: A \to G(\bar{K})$ whose character $\chi$ is defined over $F = \mathbb{Q}$ (resp. $F = \mathbb{Q}(\zeta_9 + \bar{\zeta}_9)$, $F = \mathbb{Q}(\sqrt{13})$). Moreover, $G$ acts transitively on the homomorphisms $A \to G(\bar{K})$ with character $\chi$ (see [A87] and [Griess, Cor. 1 and 2]).

By [A87, 9.3(1)] the representation $\varphi$ is irreducible. Therefore $Z_{G(\mathbb{F})}(A) = 1$. Thus, all conditions of Theorem 5.2 are satisfied. Therefore there is a twisted form $G_0$ of $G$ defined over $F$ and an embedding $A \to G_0$.

In particular, there is an embedding $A \to G_0(\mathbb{R})$. Since any finite subgroup of a Lie group is contained in its maximal compact subgroup, it is easy to see that $G_0 \otimes_F \mathbb{R}$ is compact for all embeddings of $F$ into $\mathbb{R}$. Moreover, by Theorem 5.2 we have an embedding $A \to G(K)$ iff $G_0$ and $G$ are isomorphic over $K$. By Proposition 5.3 the latter occurs iff $-1$ is a sum of 4 squares of $K$.

(Thus, we have recapitulated the argument from [Serre00, 2.5.3]).

5.5. Example (type $E_8$). Let $G$ denote the split group of type $E_8$, $A = PGL(2, 31)$ (resp. $A = SL(2, 32)$), and $K$ a field of characteristic zero. We view $G$ as a subgroup of $GL_{248}$ via the adjoint representation. There is an embedding $A \to GL_2$ if $-1$ is a sum of 16 squares and $\zeta_{11} + \bar{\zeta}_{11} \in K$ (for $SL(2, 32)$).
Indeed, by [GR 98, Theorem 2.27 and Theorem 3.25] there exists an embedding \( A \to G(\mathbb{F}) \) whose character is defined over \( F = \mathbb{Q} \) (resp. \( F = \mathbb{Q}(\zeta_{11} + \bar{\zeta}_{11}) \)). Using [GR 98] one can check all conditions of Theorem 5.2 (cf. Example 5.4).

It follows by Theorem 5.2 that there is an embedding \( A \to G_0(\mathbb{F}) \) for some twisted form \( G_0 \) of \( G \). Again as in Example 1 one can see that \( G_0 \) is the unique group such that \( G_0 \otimes_F \mathbb{R} \) is compact for all embeddings of \( F \) into \( \mathbb{R} \). Finally by Proposition 5.3 \( G \) and \( G_0 \) are isomorphic over a field extension \( K/F \) iff \(-1\) is a sum of 16 squares in \( K \).

Roughly speaking, we have added the facts about the compact \( E_8 \) contained in the proof of Proposition 5.3 (which uses the existence of the \( u \)-invariant) to Serre’s appendix [GR 98, App. B].

In the same way one can get the following example:

5.6. Example (type \( A_1 \)). Let \( G = \text{PGL}_2 \), \( A = \text{Alt}_4 \) (resp. \( \text{Alt}_5 \)), and \( K \) a field of characteristic zero. Then there is an embedding \( A \to G(K) \) iff \(-1\) is a sum of 2 squares and for \( \text{Alt}_5 \) additionally \( \sqrt{5} \in K \) (see [Serre 80]).

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