Coinduction Plain and Simple

FRANÇOIS BRY
Institute for Informatics, Ludwig-Maximilian University of Munich, Germany
(e-mail: bry@lmu.de)

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Abstract
Coinduction refers to both a technique for the definition of infinite streams, so-called codata, and a technique for proving the equality of coinductively specified codata. This article first reviews coinduction in declarative programming. Second, it reviews and slightly extends the formalism commonly used for specifying codata. Third, it generalizes the coinduction proof principle, which has been originally specified for the equality predicate only, to other predicates. This generalization makes the coinduction proof principle more intuitive and stresses its closeness with structural induction. The article finally suggests in its conclusion extensions of functional and logic programming with limited and decidable forms of the generalized coinduction proof principle.

KEYWORDS: Theory of computation, Semantics and reasoning, Programming logic

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1 Introduction
A data stream, short stream, is a possibly never ending sequence of data. A stream is observed when its data result from observations or measurements of natural or artificial systems. An observed data stream might for example convey temperatures, the reproduction rates of a disease, or energy consumption or traffic volumes. A stream is constructed when its data are synthetic. Constructed streams are useful as encoded models of observed streams. Using streams, an early detection of critical situations can be specified as the continuous comparison of the data of an observed stream with that of a constructed stream modelling some behavior.

Constructed streams are finitely defined from non-stream data and operations on such data.
Thus, two disjoint data types are considered in the following: Non-stream data the type of which is called \textit{data} and stream data the type of which is called \textit{codata}.

Constructive streams, or codata, are finitely defined for, otherwise, they could not be used in programming. Codata express repetitions by recursion. As a consequence, recursive functions like the Fibonacci function can be expressed as codata. The specification of recursive functions as codata is interesting in its own right, independently of stream-related applications, because recursive codata yield iterative computation processes like tail recursive functions do but without the recourse to accumulators.

Streams are related to processes. An observation process operating on an observed stream might be unpredictable, or only partly predictable, while a process operating on a constructed stream, even though it might never end, can be given a finite definition derived from that of the constructed stream it refers to. Coinduction in declarative programming has been used for the specification of both constructed streams and processes operating on constructed streams.

This article first reviews coinduction in declarative programming. Second, it reviews and slightly extends the formalism commonly used for specifying codata. Third, it generalizes the coinduction proof principle to other predicates than equality. The article finally suggests in its conclusion extensions of functional and logic programming with limited and decidable forms of the generalized coinduction proof principle.

\section*{2 Coinduction in Declarative Programming}

Coinduction was introduced in 1971 by Robin Milner in an investigation of the correctness of terminating or non-terminating imperative programs (Milner 1971). More precisely, the article (Milner 1971) introduced simulation relations and a precursor of the coinductive proof principle consisting in exhibiting finitely many pairs of related states and verifying that a simulation relation holds of each pair. David Park defined in 1981 bisimilarity as a greatest fixpoint, derived from fixpoint theory the bisimulation proof method (Park 1981) which is now also called coinductive proof principle. Bisimulation and the coinductive proof principle soon became cornerstones of process algebra (Hoare 1978; Fokkink 2007) and of the theory of communicating concurrent processes (Milner 1980). Davide Sangiorgi gives in (Sangiorgi 2009) an extensive account of the origins of coinduction, bisimulation and of related precursor concepts and methods in philosophical and mathematical logic, mathematics and computer science which, however, does not address coinduction in declarative programming and does not mention inductionless induction (Comon 2006).

Coinductive definitions, or codata, have been introduced in 1982 by Alain Colmerauer in logic programming under the name of \textit{rational trees} (Colmerauer 1982), “tree” meaning finite or infinite term. A rational tree is a finite or infinite tree the set of subtrees of which is finite. The name “rational tree” which, puzzlingly, is not explained in (Colmerauer 1982; Colmerauer 1984), was chosen in reference to the fact that a real number is a rational number if and only if its decimal expansion is finite or repeating (Colmerauer 1988). The name “rational tree” appropriately stresses that, by Georg Cantor’s diagonal argument (Cantor 1891), the set of (rational and non-rational) trees over a finite or denumerable alphabet is not denumerable, while the set of rational trees of course is denumerable.

\footnote{This fact challenges model theories for logic programming with observed streams.}
Following Courcelle (Courcelle 1983), rational trees are also known as regular trees. In the 1980s rational trees were easily expressed in Prolog because, for efficiency reasons at a time processors were slow, Prolog did not perform the occurs-check during unification: An infinite list of 1s could for example be finitely expressed as \( X = [1|X] \) or, redundantly, as \( X = [1,1|X] \). Colmerauer systematized this observation and designed a variant of Prolog, Prolog II (Colmerauer 1984, van Emde and Lloyd 1984), which supported rational trees. Michael Maher gave in 1988 in (Maher 1988) an axiomatization of the algebras of rational trees. Unification algorithms for rational trees were proposed in (Mukai 1983, Jaffar 1984, Martelli and Rossi 1984). A meta-interpreter for a logic programming language supporting rational (or regular) trees (or terms) is described in (Ancona 2013).

Coinductive definitions in logic programming are possible not only with rational trees defined using unification without occurs-check, but also through non-terminating SLD derivations based on unification with occurs-check. This approach has been first investigated in 1983 by Nait Abdallah in (Abdallah 1983), further investigated in (Abdallah 1984, van Emde and Abdallah 1985, Palamidessi et al. 1985) and popularized in 1987 by John Lloyd with the chapter on perpetual processes of the second edition of his Foundations of Logic Programming (Lloyd 1987). Each stage of a non-terminating SLD derivation like that resulting from the evaluation of \(?- p(Y) \) against the clause \( p(X) : - p(f(X)) \). is seen as a finite approximation of an atom containing an infinite term \( f(f(f(\ldots ))) \). Nait Abdallah formalized such infinite terms as follows.

The Herbrand universe \( H \) is first equipped with a metric:

\[
\begin{align*}
\delta(t_1,t_2) &= 0 & \text{if } t_1 = t_2, \\
\delta(t_1,t_2) &= 2^{-\inf\{|n|a_n(t_1) \neq a_n(t_2)\}} & \text{otherwise}
\end{align*}
\]

where \( a_n(t) \) is the “cut at depth \( n \)” of a term \( t \). Thus, \( \inf\{|n|a_n(t_1) \neq a_n(t_2)\} \) is the smallest depth at which \( t_1 \) and \( t_2 \) differ. Using this metric, Cauchy sequences of elements of \( H \) are defined as the standard equivalence relation on Cauchy sequences is considered and the complete Herbrand universe \( \mathcal{H} \) is defined as the set of equivalence classes of Cauchy sequences of elements of \( H \). Thus, \( H \subset \mathcal{H} \). It is then observed that, by a result of (Mycielski and Taylor 1976), \( \mathcal{H} \) is compact if its terms are defined from finitely many symbols. This is the case of the ground (or closed) terms of \( \mathcal{H} \) because programs are finite. Thus, an infinite sequence of elements of \( H \) like for example \( f^n(a) \) has a limit in \( \mathcal{H} \) which is denoted \( f(f(\ldots )) \) or \( f(\omega) \).

The complete Herbrand base \( \mathcal{B} \) is defined as the set of ground atoms built from terms in the complete Herbrand universe \( \mathcal{H} \), an immediate consequence operator \( T' \) is defined in reference to the complete Herbrand base \( \mathcal{B} \) in the same manner as the standard immediate consequence operator \( T \) is defined in reference to the Herbrand base \( B \) (van Emde and Kowalski 1976). The greatest fixpoint semantics of a definite (that is, negation-free) logic program is defined as the greatest fixpoint of \( T' \) for that program. The greatest fixpoint semantics of logic programs is extended

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2 The article does not clarify in which cases unification is performed with, respectively without, occurs-checks.
3 The “completion of a program” mentioned in (van Emde and Abdallah 1985) is Clark’s completion semantics of negation in logic programming (Clark 1978). Thus, it is not related to the metric-based completion of the Herbrand universe introduced in (Abdallah 1983).
4 A Cauchy sequence is such that the distance between two successive elements tends to zero when the rank tends to infinity.
5 Two Cauchy sequences are equivalent if the difference between their elements of same ranks tends to zero when the rank tends to infinity.
6 Like \( \mathbb{Q} \subset \mathcal{H} = \mathbb{R} \).
7 The compactness of \( \mathcal{H} \) means that every Cauchy sequence of elements of \( \mathcal{H} \) has a limit in \( \mathcal{H} \).
to programs with negation in (Jaffar and Stuckey 1986; Hein 1992). Refinements of the greatest
fixpoint semantics of logic programs are proposed in (Golson 1988; Levi and Palamidessi 1988;
Jaume 2002; Ancona et al. 2017; Komendantskaya and Li 2017; Li 2018).

Some approaches to coinduction in logic programming make use of ancestor subsumption
(Socher-Ambrosius 1992), a technique for avoiding redundant derivations in resolution theorem
proving: If a resolvent $R$ is subsumed (without occurs-check) by one of its ancestors $A$, then
further derivations from $R$ can be avoided without compromising completeness. Indeed, if $R$ is
subsumed (without occurs-check) by $A$, then every derivation from $R$ is also possible from $A$
and therefore redundant.

In the 1980s, as coinductive definitions began to be considered in logic programming, it was
folklore knowledge in the community that ancestor subsumption (without occurs-check) can be
used for preventing redundant non-terminating SLD derivations (with occurs-check) like in the
following examples:

Example 2.1

$$
\begin{align*}
    p(0). \\
    \text{allp}([H|T]) & : - p(H), \text{allp}(T). \\
    ?- X = [0|X], \text{allp}(X).
\end{align*}
$$

Example 2.2

$$
\begin{align*}
    p & : - p. \\
    q(X) & : - q(X). \\
    r(f(X)) & : - r(f(X)). \\
    s(X) & : - s(f(X)). \\
    ?- p. \\
    ?- q(Y). \\
    ?- r(Y). \\
    ?- s(Y).
\end{align*}
$$

Example 2.3

$$
\begin{align*}
    \text{member}(H, [H|\_]). \\
    \text{member}(X, [\_|T]) & : - \text{member}(X, T). \\
    ?- T = [0|T], \text{member}(1, T).
\end{align*}
$$

Avoiding redundant derivations by ancestor subsumption (without occurs-check) or any other
means does not mean resolving, though, as the examples given above illustrate.

In Example 2.1, $X = [0|X]$ refers to unification without occurs-check and ancestor sub-
sumption avoids that a Prolog system not performing the occurs-check repeatedly proves $p(0)$

\footnote{Ancestor subsumption was at the time a commonly considered redundancy elimination technique in resolution theorem proving.}
and tries to prove allp(X). Without reasoning by induction, the proof of the goal \( X = [0|X] \), allp(X) cannot be completed. By induction, it can of course be proved.

In the Example 2.2, ancestor subsumption (without occurs-check) avoids that a Prolog system (performing the occurs-check or not) endlessly attempts to prove \( p, q(Y), r(f(Y)) \), and \( s(f(...f(Y)...)) \). Even by induction, none of these four goals can be proved.

Example 2.3 is similar to Example 2.2 With ancestor subsumption (without occurs-check), a Prolog system not performing the occurs-check does not endlessly attempt to prove member(1, T). Even by induction, the goal member(1, T) cannot be proved because 1 does not occur in the cyclic list specified by \( T = [0|T] \).

Summing up, Examples 2.1, 2.2 and 2.3 illustrate that ancestor subsumption (without occurs-check) (Socher-Ambrosius 1992) is a pruning rule (that is, a rule for discarding redundant parts of a proof), not an inference rule (that is, a rule for deriving logical consequences).

The downside of ancestor subsumption is its cost what led Rolf Socher-Ambrosius to give in (Socher-Ambrosius 1992) syntactic characterizations of clause sets giving rise to ancestor subsumption. In (Socher-Ambrosius 1992), such clause sets are said to “roughly correspond to sets of logical equivalences”.

Luke Simon, Ajay Bansal, Ajay Mallya and Gopal Gupta proposed in 2006 and 2007 in (Simon et al. 2006; Simon et al. 2007) coinductive logic programming, a form of logic programming which is the first of its kind for two reasons: First, it combines the two kinds of codata so far proposed for logic programming, rational (or regular) terms like \( X = [0|X] \) based on unification without occurs-check, and infinite terms like \( f(\omega) \) defined as limits of an infinite sequence finite terms constructed during the steps of non-terminating SLDF derivations based, according to the greatest fixpoint semantics of logic programs (Abdallah 1983; Abdallah 1984; van Emden and Abdallah 1985; Lloyd 1987), on unification with occurs-check. Second, it introduces into logic programming a proof principle named coinductive hypothesis rule which resembles the coinduction proof principle. Applications of coinductive logic programming are presented in (Gupta et al. 2011). Implementations of coinductive logic programming languages are described in (Moura 2013; Mantadelis et al. 2014).

The declarative and operational semantics of coinductive logic programming given in (Simon et al. 2006; Simon et al. 2007) have confusing aspects. One of them is the claim that coinductive logic programming’s operational semantics is sound and complete with respect to the greatest fixpoint semantics of logic programs (Simon et al. 2006; Simon et al. 2007). No explanations are given of how this relates to the downward approximation sequence \( T_P,\downarrow \) of the immediate consequence operator \( T_P \), the limit of which specifies the greatest fixpoint semantics of a logic program (van Emden and Kowalski 1976), not reaching a fixpoint before \( \omega + 1 \) with some (finite and definite) programs \( P \) and even not before \( \omega^{CK} \) with some others (finite and definite) programs (Fitting 2002, p. 8). Another confusing aspect of coinductive logic programming is its “coinductive hypothesis rule of the form \( \nu(\alpha) \)” (Simon et al. 2006, p. 336) which is defined as follows in (Simon et al. 2007, p. 472): “The operational semantics is given in terms of the coinductive hypothesis rule which states that during execution, if the current resolvent \( R \) contains a call \( C' \) that unifies with a call \( C \) encountered earlier, then the call \( C' \) succeeds.”

\[ \omega \] is the ordinal defined as the set of all finite ordinals and thus the smallest limit ordinal and \( \omega^{CK} \), the Church-Kleene ordinal, is defined as the set of all recursive ordinals and thus a limit ordinal and the smallest non-recursive ordinal (Church and Kleene 1937; Church 1938; Kleene 1938).

The coinductive hypothesis rule and \( \nu(\alpha) \) do not seem to be defined in (Simon et al. 2006).
articles (Simon et al. 2006; Simon et al. 2007) do not explain how that coinductive hypothesis rule relates to the coinduction proof principle. The following examples cast a shadow over the coinductive hypothesis rule:

Example 2.4

\[ :- \text{coinductive } p/2. \]
\[ p(X, a) :- p(X, Z). \]
\[ p(X, b) :- p(s(X), b). \]
\[ ?- p(Y, a). \]

Example 2.5

\[ :- \text{coinductive } qc/2. \]
\[ qc(X, f(X)). \]
\[ ?- qc(f(Y), Y). \]

Example 2.6

\[ :- \text{inductive } q/2. \]
\[ qi(X, f(X)), r(X). \]
\[ ?- qi(f(Y), Y). \]

Example 2.7

\[ :- \text{inductive } pi/1. \]
\[ pi(_). \]
\[ :- \text{coinductive } pc/1. \]
\[ pc(_). \]
\[ ?- X = f(X), pi(X). \]
\[ ?- Y = f(Y), pc(Y). \]
\[ ?- Z = f(Z), pi(Z), pc(Z). \]

In Example 2.4 the SLD derivation from the goal ?- p(Y, a). is stopped at the subgoal ?- p(Y, Z) by the coinductive hypothesis rule what prevents the derivation of the subgoal p(s(Y), b) and therefore prevents that the rational term Y = s(Y) (which expresses s(ω)) be returned as answer. Example 2.4 suggests that the coinductive hypothesis rule of coinductive logic programming should not be based on ancestor unification (without occurs-check) but instead on ancestor subsumption (without occurs-check) (Socher-Ambrosius 1992).

In Example 2.5 since the predicate qc is declared coinductive, the goal ?- qc(f(Y), Y). is unified (without occurs-check) with the fact qc(X, f(X)) yielding as an answer the rational term Y = f(Y) even though no clauses in the program specify non-terminating SLD derivations.

Example 2.6 is like Example 2.5 except that the predicate, now called qi instead of qc, is
declared inductive instead of coinductive. Since \( q_c \) is declared inductive, a unification with
occurs-check of \(- q_1(f(Y), Y). \) with \( q_1(X, f(X)) \), is attempted which fails because of
the occurs-check. The articles (Simon et al. 2006, Simon et al. 2007) give no clues about the in-
tentions behind treating differently cases like Examples 2.5 and 2.6.

In Example 2.7, the goal \(- X = f(X), \pi_1(X). \) probably fails because \( \pi_1 \) is declared in-
ductive. Indeed, to the best of the author’s understanding, that implies that unifications involv-
ing the variable \( X \) perform the occurs-check. In contrast, the goal \(- Y = f(Y), \pi_c(X). \) probably
succeeds returning the answer \( Y = f(Y) \) because \( \pi_c \) is declared inductive what, as the author
understands, means that unifications involving \( Y \) do not perform the occurs-check. Note that, ac-
cording to the examples of (Simon et al. 2007) Section 2.5, in coinductive logic programming
the system predicate = is both inductive and coinductive. The three goals of Example 2.7 are
therefore licit. Example 2.7 suggests that whether a unification involving a term \( t \) performs the
occurs-check or not should depend on the type of \( t \), data or codata, not on a type inductive or
coinductive of predicates in which \( t \) occurs. Example 2.7 raises the question whether with coin-
ductive logic programming as defined in (Simon et al. 2006, Simon et al. 2007) a same term \( t \)
might occur as an argument of both inductive or coinductive predicates, in which case the third
goal \(- Z = f(Z), \pi_1(Z), \pi_c(Z). \) is licit but does not seem to be covered by the semantics
given in (Simon et al. 2006, Simon et al. 2007), or whether inductive and coinductive predicates
(except =) cannot share arguments, in which case coinductive logic programming as defined in
(Simon et al. 2006, Simon et al. 2007) would be a rather limited approach.

Coinductive definitions have entered functional programming with lazy evaluation (Henderson and
Morris 1976, Friedman and Wise 1976). The infinite list of Fibonacci numbers can for example be coded in
Haskell as \((f 0 1)\) and the finite list of its 43 first elements as \( \text{take} 43 (f 0 1) \) where the
function \( f \) is coded as \( f\ a\ b = a : f\ b\ (a+b) \). A type system for coinductively defined data
objects, or codata, has been proposed in (Jeannin et al. 2017). The use of the coinduction proof
principle for the analysis of functional programs with codata is discussed in (Gordon 1994). The
lazy evaluation of infinite objects has entered logic programming through logic-functional lan-
guages (Giovannetti et al. 1991, Hanus 1994).

Coinductive definitions have been elegantly formalized in 2005 by Jan Rutten as behavioural
differential equations (Rutten 2005). Rutten’s behavioural differential equations are further used in
(Kupke et al. 2011, Hansen et al. 2017). In their term-based form, behavioural differential
equations are close to, and extend, the aforementioned representation of rational trees in Pro-
log II. In the following, coinductive definitions are given in the term-based form of Rutten’s
behavioural differential equations.

The coinduction proof principle relates to mathematical induction[11] because it relates to struc-
tural induction which itself derives from mathematical induction. Structural induction has been
formalized in 1959 East of the Iron Curtain by Rózsa Péter in (Péter 1961) and in 1969 West of
it by Rod M. Burstall (Burstall 1969)[12].

Proof by (mathematical or structural) induction cannot be completely automated because
finding a suitable induction hypothesis is neither decidable, nor semi-decidable: “In contrast
to first-order inference, inductive inference is incomplete in the sense that any axiomatization

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[11] Mathematical induction goes back to the Greek mathematics of the antiquity.

[12] The idea was, however, already “in the air”: Gödel’s article (Gödel 1931) on the incompleteness theorems bearing
his name among others sketches a proof by mathematical induction on degrees of primitive recursive functions which
resembles a proof by structural induction.
which includes a non-trivial form of induction, there are formulas both true and unprovable” (Bundy 2006) which follows from Kurt Gödel’s first incompleteness theorem (Gödel 1931).

Proofs by induction have nonetheless be (incompletely) automated (Boyer and Moore 1988; Bundy 2006; Comon 2006; Avenhaus et al. 2003) in two manners: Explicitly (Bundy 2006) and implicitly after the approach called “inductive completion” or “inductionless induction” (Comon 2006). The later approach consists in assuming the conjecture to prove, in attempting to derive an inconsistency using the Knuth-Bendix completion algorithm (Knuth and Bendix 1967), and if no inconsistency is derived to consider the conjecture proved. Observe that inductive completion (or inductionless induction) resembles the coinduction proof principle.13

Proofs by induction are supported by proof assistants (Geuvers 2009), that is, proof systems requiring the participation of their human users. Coinduction as a proof method is usually described as dual of induction, a description referring to the coalgebras of category theory (Jacobs and Rutten 1997). Like proofs by induction and for the same reason, proofs by coinduction cannot be completely automated. Coinduction as a proof method has been formalized in higher-order logic and automated with the proof assistant Isabelle as reported in (Paulson 1997) and in terms of Labelled Transition Systems (LTSs) in (Pous and Sangiorgi 2011). Like proofs by induction, proofs by coinduction have been partly automated in proof assistants (Paulson 1997).

A proof method which resembles the coinduction proof principle has been proposed for logic programming in 2008 by Joxan Jaffar, Andrew Santos and Răzvan Voicu in (Jaffar et al. 2008). The authors stress in this article that the proof method they propose refers to the least fixpoint semantics, not to the greatest fixpoint semantics of of logic programs: “we would like to clarify that the use of the term coinduction pertains to the way the proof rules are employed for a proof obligation \( G \models H \), and has no bearing on the greatest fixed point of the underlying logic program \( P \). In fact, our proof method, when applied successfully, proves that \( G \) is a subset of \( H \) wrt. the least fixpoint of (the operator associated with) the program” (Jaffar et al. 2008). In contrast to standard induction and coinduction proof methods, the proof method proposed in (Jaffar et al. 2008) can be completely automated: “a search-based method […] automatically discovers the opportunity of applying induction instead of the user having to specify some induction schema”, “our method is amenable to automation” and “the unfolding process and the application of the coinduction principle require no manual intervention” (Jaffar et al. 2008). The generalized coinduction proof principle (see Section 7 below) precises how the proof method of (Jaffar et al. 2008) relate to the coinduction proof principle.

Jan Rutten gives in (Jacobs and Rutten 1997; Rutten 2000; Rutten 2005) coalgebraic treatments of coinduction.

Davide Sangiorgi’s book (Sangiorgi 2012) is a comprehensive introduction to coinduction, bisimulation and the coinduction proof principle which, however, does not address coinduction in declarative programming and does not mention inductionless induction (Comon 2006).

3 Coinductive Definitions

This section introduces terms of type codata, short codata terms after one of the formalisms which have been proposed for the finite representation of constructed streams (Rutten 2005; Kupke et al. 2011). Codata terms rely on corecursive definitions which are introduced in the next section.

13 This resemblance seems to have so far remained unnoticed.
Two disjoint data types are considered in the following: *Data* for the representation of non-stream objects and *codata* for the representation of streams the elements of which are of type data. Considering only the two types data and codata is a convenient simplification. In practice, the type data consists of several pairwise disjoint types like integer, floating-point number, character, etc. and structured types like finite list of integers, finite list of floating-point numbers, finite list of characters, etc. Streams of streams could be considered, though, since such streams might make sense for example in specifying stream processing operations but such streams and the issues they raise are not addressed in this article.

The type codata encompasses codata terms and their names. The following illustrates on the Fibonacci function the concept of codata term and of name, base cases and defining expression of a codata term:

\[
\text{fib as } [0, 1 | \text{fib}^0 + \text{fib}^1]
\]

This expression is a codata term. Its name is fib, its base cases are 0 and 1 and its defining expression is fib^0 + fib^1. In contrast to (Rutten 2005), the keyword as is used instead of = for avoiding a confusion with the the Prolog fashion’s use of = in the former section of this article for denoting unification. Codata terms as illustrated above and defined below slightly generalise those of (Rutten 2005) by allowing compound codata names which are needed for expressing mutual corecursion.

It is convenient to think of the codata term fib as a definition of an infinite list, a view which is justified below in Section 5 fib^0 denotes fib and for all natural number n, fib^(n+1) denotes the tail of fib^n(n). Thus, fib^1 denotes the tail of fib.

If f and g respectively represent

\[
[f(0), f(1), ..., f(n), ...] \\
[g(0), g(1), ..., g(n), ...]
\]

then f+g represents

\[
[f(0)+g(0), f(1)+g(1), ..., f(n)+g(n), ...]
\]
(see below Section 5). Thus, the definition

\[
\text{fib as } [0, 1 | \text{fib}^0 + \text{fib}^1]
\]

can be step-wise expanded yielding

\[
[0, 1 | \text{fib}^0 + \text{fib}^1] \\
[0, 1, 1 | \text{fib}^1 + \text{fib}^2] \\
[0, 1, 1, 2 | \text{fib}^2 + \text{fib}^3] \\
[0, 1, 1, 2, 3 | \text{fib}^3 + \text{fib}^4] \\
[0, 1, 1, 2, 3, 5 | \text{fib}^4 + \text{fib}^5]
\]
etc.

Even though the codata term fib is a recursive definition of the Fibonacci function, it specifies an iterative computation, as does a tail recursive definition (Friedman and Wise 1974; Steele 1977), but it does not require accumulators, in contrast to a tail recursive definition. This is an essential aspect of coinductive definitions which can be expressed in logic programming parlance as follows: Codata are meant to express forward chaining inferences as defined by the immediate consequence operator T (van Emden and Kowalsi 1976).

In the following, Data denotes the set of terms of type data and Codata denotes the set of terms of type codata. In Section 5 below, it is shown that a codata term s as [s_a, s_b, ..., s_n | E] with s_a a constant induces the definition of a function:
This justifies to view $s$ as a definition of the infinite list

$[s(0), s(1), s(2), \ldots, s(n), \ldots]$  

and the following definitions.

**Definition 3.1 (Codata names)**  
A *codata name* is either a codata constant (called an *atomic codata name*), or a tuple of codata constants (called a *compound codata name*).

If $(s_1, \ldots, s_m)$ with $m \geq 2$ is a compound codata name and if $n \in \mathbb{N} \setminus \{0\}$, then $(s_1, \ldots, s_m)(n)$ denotes $(s_1(n), \ldots, s_m(n))$.

**Definition 3.2 (Head, tail, n-tail, nth element of a codata term)**  
Let $s$ be a codata name.

- $s(n)$ denotes the *nth element* of $s$.
- $s(0)$ is the *head* of $s$.
- $s^0 = s$.
- $s^1$ is the *tail* of $s$ which expresses the infinite list
  $[s(1), s(2), \ldots, s(n), \ldots]$.
- for all $n \in \mathbb{N}$, $s^n(n) = s^{n-1}$
- for all $n \in \mathbb{N}$, $s^n$ is the *nth-tail* of $s$

It follows from Definition 3.2 by mathematical induction that for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $s^n \cdot m = s^{n+m}$.

**Definition 3.3 (Codata term)**  
A *codata term* or *codata definition* is an expression of the form $s \text{ as } [s_0, s_1, \ldots, s_n \mid E]$ where:

- $s$ is an atomic or compound codata name.
- $n \geq 0$.
- the $s_i$, $0 \leq i \leq n$, are terms of sort data called the *base cases*.
- $E$ is a *codata expression*.

$s \text{ as } [s_0, s_1, \ldots, s_n \mid E]$ is a *definition* of $s$ and $s$ is called the *name* of the codata term.

Codata expressions are defined below in Section 4.

**Example 3.1**  
The following mutually recursive functions

\[
\begin{align*}
  f(0) &= 0 \\
  g(0) &= 1 \\
  f(n) &= g(n - 1) \quad \text{for } n \geq 1 \\
  g(n) &= 2 \times f(n - 1) \quad \text{for } n \geq 1
\end{align*}
\]

can be expressed by the following codata term:

$(f, g) \text{ as } [(0, 1) \mid (g, 2 \ast f)]$

Defining expressions like $2 \ast f$ are defined below in Section 4.
Definition 3.4 (Well-formed codata term)

A codata term

\[ s \text{ as } [s_0, s_1, \ldots, s_n \mid E] \]

is well-formed if

1. For all \( i = 1, \ldots, n \), \( s_i \) and \( s \) are either both atomic or tuples of the same size.
2. If \( s \) is atomic, then \( E \) defines an atomic codata, else \( E \) defines a tuple of the size of \( s \).
3. All atomic names occurring in the defining expression \( E \) also occur in the codata name \( s \).
4. If the defining expression \( E \) refers to a \( k \)th tail \( t^k \), then \( n \geq k + 1 \).

The first two conditions ensure that the name, the base cases and the defining expression of a codata term are consistent in their sizes. The third conditions ensures that every name referred to in a codata term is defined in that codata term. The fourth and last condition ensures that the number of base cases in a codata term and the references to tails in that codata term’s defining expression are consistent.

Example 3.2

Using \( \text{nat} \), a codata term expressing the sequence of natural numbers, Douglas Hofstadter’s female and male sequences (Hofstadter 1979):

\[
\begin{align*}
  f(0) &= 1 \\
  m(0) &= 0 \\
  f(n) &= n - m(f(n-1)) \quad \text{for } n \geq 1 \\
  m(n) &= n - f(m(n-1)) \quad \text{for } n \geq 1
\end{align*}
\]

can be expressed by the following codata term:

\[(\text{nat}, f, m) \text{ as } [(0,1,0) \mid (1+\text{nat}, \text{nat-m}^f, \text{nat-f}^m)]\]

This example shows that codata terms can express non-linear recurrence relations. Such a definition requires a more involved notion of well-formedness than that given above in Definition 3.4 though. This more involved notion of well-formedness is out of the scope of this article.

4 Corecursive Definitions

The qualifier “inductive” (“coinductive”, respectively) refers to data (codata, respectively) definitions while the qualifier “recursive” (“corecursive”, respectively) refers to the definitions of functions or predicates on data (codata, respectively). In most cases, the pedantic distinction (co)inductive/(co)recursive can be ignored. Furthermore, this distinction is blurred in the presence of codata which can be seen both as infinite lists and function definitions. The distinction is nonetheless used in the titles of this section and of the former section because it provides a convenient structuring of the exposition.

In the following, pointwise means element-wise and refers to the interpretation of codata terms as infinite lists which is justified below in Section 5. In the following, all corecursive definitions refer to natural numbers. Example of corecursive referring to other data types (like Boolean, characters, strings of characters, etc.) can easily be derived from the following “number-based” examples.

If \( a_d \) is a data constant, then the codata constant \( a_c \) is defined by:

\[ a_c \text{ as } [a_d \mid a_c] \]
The sum of a data constant \( a_d \) and a codata \( s \) is defined by:
\[
a_d + s \text{ as } [a_d + s(0) \mid a_d + s^1]\]

The pointwise product of codata \( s \) and \( t \) is defined by:
\[
s \cdot t \text{ as } [s(0) \cdot t(0) \mid s^1 \cdot t^1]\]

The codata \( \text{nat} \) (natural numbers) is defined by:
\[
\text{nat} \text{ as } [0 \mid 1_d + \text{nat}].
\]

The pointwise sum \( s + t \) of codata \( s \) and \( t \) is defined by:
\[
s + t \text{ as } [s(0) + t(0) \mid s^1 + t^1]\]

The codata \( \text{nat} \) (natural numbers) can also be defined by:
\[
\text{nat} \text{ as } [0 \mid 1_c + \text{nat}].
\]

The product of a data constant \( a_d \) and a codata \( s \) is defined by:
\[
a_d \cdot s \text{ as } [a_d \cdot s(0) \mid a_d \cdot s^1]\]

The pointwise pairing of codata \( s \) and \( t \) is defined by:
\[
(s, t) \text{ as } [(s(0), t(0)) \mid (s^1, t^1)]
\]

The pointwise application of data term constructor \( f \) to codata \( s \) is defined by:
\[
f(s) \text{ as } [f(s(0)) \mid f(s^1)]
\]

Right-side sums and products with data constants are defined similarly to the left-side sums and products with data constants specified above. Pointwise \( n \)-groupings for \( n \geq 3 \) of codata are defined similarly to the pairing of codata.

Further examples of corecursive definitions (Rutten 2005), where \( s \) and \( t \) are codata names:
\[
even(s) \text{ as } [s(0) \mid \text{even}(s^2)]
\]
\[
odd(s) \text{ as } [s^1(0) \mid \text{odd}(s^2)]
\]
\[
\text{split}(s) \text{ as } (\text{even}(s), \text{odd}(s))
\]
\[
\text{zip}(s, t) \text{ as } [s(0) \mid \text{zip}(t, s^1)]
\]

The codata \( \text{fact} \) (factorial numbers) is defined by:
\[
\text{fact} \text{ as } [1 \mid \text{nat}^1 \cdot \text{fact}]
\]

**Definition 4.1 (Codata expression)**

A codata expression is a codata \( n \)-th tail (including a codata name) or the application of a corecursive function to codata expression(s).

Let \( s \) as \([s_0, s_1, \ldots, s_n \mid E]\) be a codata term with an atomic name. \( s \) immediately depends on every codata constant and every codata name occurring in the codata expression \( E \). Depends is used for the transitive closure of directly depends.

Immediate dependency extends component-wise to compound codata names.

**Definition 4.2 (Structural codata expression)**

A codata expression is structural if it contains neither sums, nor products.

More generally, a function definition is structural if its does not involve expressions determining values of which requires reductions (that is, computations).

Observe that the corecursive function definitions given above are all pointwise. Observe also that the codata constants, the codata pairing and grouping and the pointwise data term application are the only structural cases of structural corecursion given above.
5 Interpretation of Codata as Infinite Streams

This section shows that a codata term $s$ as $[s_0, s_1, \ldots, s_n \mid E]$ specifies an infinite list in the sense that it induces the definition of a function $s : \mathbb{N}^m \to \text{Data}$ where $m$ is the dimension of $s$, that is, $m = 1$ if $s$ is atomic, and $m$ is the tuple size of $s$ otherwise. The overloading of the symbol $s$ used both as a codata name and a function name is intentional and justified by Definition 3.2 of Section 3.

**Definition 5.1**

Let $s$ as $[d_1, \ldots, d_k \mid E]$ be a codata term.

The notation $E[s_0, \ldots, s_n]$ expresses that the set of atomic codata names occurring in expression $E$ is $\{s_0, \ldots, s_n\}$, that is, for all $0 \leq i \leq n$, $s_i = s(i)$.

$E^0$ denotes $E$ and if $k \in \mathbb{N} \setminus \{0\}$, then $E^k$ denotes the expression obtained from expression $E$ by replacing every $n$th-tail $t^n$ in $E$ by $t^n_k$.

Thus, by Definition 5.1 if $k \in \mathbb{N}$, then

$E[s_0, \ldots, s_i, \ldots, s_n]^k$ denotes $E[s_0]^k, \ldots, s_i^k, \ldots, s_n^k]$.

**Definition 5.2 (nth expansion of a codata term)**

Let $s$ as $[s_0, s_1, \ldots, s_k \mid E]$ be a codata term and let $n \geq k$ be a natural number. The $nth$ expansion of $s$ is the codata term $s[n]$ as $[s(0), \ldots, s(k), \ldots, s(n) \mid E^{(n-k)}]$ obtained from $s$ by repeatedly applying the codata term defining expression $E$. Recall that for all $0 \leq i \leq k$, $s_i = s(i)$.

Section 3 above gives examples of expansions of the codata term fib. The following shows that the applications mentioned in Definition 5.2 terminate if the codata term considered is well-formed.

**Proposition 5.1 (Expansion Theorem)**

For all well-formed codata terms $s$ as $[s_0, s_1, \ldots, s_k \mid E]$ and all $n \in \mathbb{N}$ such that $n \geq k$, the $n$th expansion $s[n]$ of $s$ is uniquely defined.

**Proof**

First observe that if $s$ as $[s_0, s_1, \ldots, s_k \mid E]$ is a well-formed codata term, then $s$ (directly or indirectly) depends on finitely many codata names all occurring in $s$. Second, observe that $[s_0, s_1, \ldots, s_k \mid E]$ is its $k$th expansion of $s$. The result follows by structural induction. \qed
Definition 5.3 (nth element of a codata term)

Let

\[ s \equiv [s_0, s_1, \ldots, s_n | E] \]

be a well-formed codata term and let \( m \in \mathbb{N} \setminus \{0\} \) be the dimension of \( s \). The codata term \( s \) induces a function

\[ s : \mathbb{N}^m \to \text{Data} \]

defined by: for \( n \in \mathbb{N} \) \( s(n) \) is the head of the \( n \)th expansion of \( s \).

By the Expansion Theorem, the function \( s : \mathbb{N}^m \to \text{Data} \) referred to in Definition 5.3 is well-defined. As a consequence, a well-formed codata term \( s \) can be viewed as a definition of the infinite list

\[ [s(0), s(1), s(2), \ldots, s(n), \ldots] \]

and the concepts of Definition 5.2 from Section 3 above are well-defined.

Corollary 5.1 (tail computation)

Let \( s = (s_0, s_1, \ldots, s_k) \) with \( k \geq 1 \) be the name of a well-formed codata term.

1. For all \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \) \( s^* n^m = s^*(n+m) = s^* n(m) = s^* m(n) \)
2. If \( s \equiv [d_1, \ldots, d_k | E[s_0, \ldots, s_k]] \), then for all \( n \in \mathbb{N} \), if \( n \geq k \) then
\[ s^* n = E[s_0^* (n-k), \ldots, s_k^* (n-k)] \]

Proof

By structural induction and the Expansion Theorem 5.1.

A few remarks are worthwhile.

First, even though the set of infinite lists on a finite alphabet is not enumerable\(^{14}\) the set of codata terms on a finite alphabet is recursively enumerable. Indeed, like a data term, codata terms are finite expressions which can be inductively defined. Thus, using the terminology of the introduction, not all observed streams can be specified as constructed streams.

Second, two distinct codata terms might specify the same infinite stream as is the case for example with \( s_1 = [1 | s_1] \) and \( s_2 = [1, 1 | s_2] \) which both specify the same codata constant specifying the infinite list of 1s: \([1, 1, 1, \ldots]\).

Third, let \( s \) and \( t \) be the names of two well-formed codata terms. Whether for all \( n \in \mathbb{N} \) \( s(n) = t(n) \) (that is, \( s \) and \( t \) specify the same infinite list) is neither decidable nor semi-decidable. Algorithms are possible, though, which in some cases will decide whether, or recognize that, two codata terms specify a same infinite list or whether all elements of the infinite list specified by a codata term has a certain property.

Fourth and finally, it is worth stressing that one of the interests of codata terms is to (corecursively) define functions \( f : \mathbb{N} \to \text{Data} \) without having to give an equation on \( n \) for \( f(n) \). There are indeed cases in which this is desirable. The Fibonacci function for example has a simple recursive definition, \( \text{fib} \equiv [0, 1 | \text{fib} + \text{fib}^* 1] \), but a significantly more complicated definition as an equation on \( n \):

\(^{14}\) This can be proven by the diagonalisation argument Georg Cantor used for proving that the set of decimal numbers is not countable (Cantor 1891).
fib(n) = \left\lfloor \frac{1 + \sqrt{5}n}{2} \right\rfloor

In other cases, however, an equation on \( n \) is simpler than a recursive definition: For all \( n \in \mathbb{N} \) \( f(n) = n \) is for example simpler than:

\[
\text{nat as } [0 \mid 1 + \text{nat}]
\]

6 First Generalized Coinduction Theorem

In this section and the following, a predicate is called a data predicate if all its arguments are of type data and a predicate is called a codata predicate if all its arguments are of type codata, data predicates are denoted by identifiers beginning with lower case characters and codata predicates are denoted by identifiers beginning with upper case characters, codata terms are implicitly assumed to be well-formed and the phrase “the codata term \( s \)” refers to a (well-formed) codata term of the form \( s \ as \ [d_1, \ldots, d_k \mid E] \) with \( k \geq 1 \).

Definition 6.1 (Pointwise-defined and hereditary codata predicate)

Let \( p \) be a unary data predicate and \( s \) a codata name.

The unary codata predicate \( P_p \) defined by:

For all codata term \( s \), \( P_p(s) \) iff for all \( n \in \mathbb{N} \) \( p(s(n)) \)

is called the pointwise-derived from \( p \).

A unary codata predicate \( H \) which fulfills the following condition is said to be hereditary with respect to \( p \), short \( p \)-hereditary:

If \( H(s) \) holds of some codata \( s \), then

1. \( p(s(0)) \) holds.
2. \( H(s^*1) \) holds.

Lemma 6.1

Let \( p \) be a unary data predicate. The pointwise-derived \( P_p \) from \( p \) is \( p \)-hereditary.

Proof

Let \( p \) be a unary data predicate and let \( s \) be a codata name. By definition of the pointwise-derived \( P_p \) from \( p \), \( P_p(s) \) iff for all \( n \in \mathbb{N} \) \( p(s(n)) \). Thus, if \( P_p(s) \) holds, then \( p(s(0)) \) holds and \( P_p(s^*1) \) holds.

Observe that the concept of \( p \)-hereditary codata predicate generalises that of bisimulation relation (Park 1981, Rutten 2005, Sangiorgi 2012) since a bisimulation relation \( B \) is defined as a binary relation such that \( s_1 B s_2 \) if

1. \( s_1(0) = s_2(0) \)
2. \( s_1^*1 B s_2^*1 \)

More precisely, a bisimulation relation \( B \) is an \( = \)-hereditary predicate applying of pair codata \( \langle s_1, s_2 \rangle \). Thanks to codata pairing defined above in Section 4 a binary codata predicate can be seen as a unary codata predicate the argument of which is a pair of codata.

Observe also that if \( p \) is a unary data predicate and \( P_1 \) and \( P_2 \) are \( p \)-hereditary codata predicates, then the codata predicates \( A \) and \( O \) defined as follows are also \( p \)-hereditary:

For all codata names \( s \):

- \( A(s) \) iff \( (P_1(s) \land P_2(s)) \)
- \( O(s) \) iff \( (P_1(s) \lor P_2(s)) \)
Definition 6.2 (Maximal p-hereditary codata predicate)
Let \( p \) be a unary data predicate. The maximal p-hereditary codata predicate \( \text{Max}_p \) is defined by:

If \( H \) is a p-hereditary codata predicate and \( s \) is as codata name such that \( H(s) \) holds, then \( \text{Max}_p(s) \) holds.

Observe that \( \text{Max}_p \) is the greatest p-hereditary predicate in the sense that it expresses the union of all unary relations on codata which are defined by unary p-hereditary codata predicates. Observe also that \( \text{Max}_p \) generalizes the bisimilarity relation (Park 1981; Rutten 2005; Sangiorgi 2012) which, in the terminology of Definition 6.2, is \( \text{Max}_\infty \).

Theorem 6.1 (First Generalized Coinduction Theorem)
Let \( s \) be the name of a codata term and let \( p \) a unary data predicate. The following statements are equivalent:

1. There is a unary p-hereditary codata predicate \( H \) such that \( H(s) \) holds.
2. There is a unary p-hereditary codata predicate \( H \) such that: \( \forall n \in \mathbb{N} \, H(s^n) \)
3. \( \forall n \in \mathbb{N} \, p(s(n)) \)

Proof
Let \( s \) a codata term, \( p \) a unary data predicate and \( H \) a unary codata predicate.

(1) \( \Rightarrow \) (2): Assume \( H \) is p-hereditary and \( H(s) \) holds. The proof is by induction. Since by hypothesis \( H \) is p-hereditary and \( H(s) \) holds, \( H(s^1) \) holds. Let \( n \in \mathbb{N} \) and assume that \( H(s^n) \) holds. Since by hypothesis \( H \) is p-hereditary and \( H(s^n) \) holds, \( H(s^{n+1}) \) holds.

(2) \( \Rightarrow \) (3): Since by assumption for all \( n \in \mathbb{N} \, H(s^n) \) holds and \( H \) is p-hereditary, for all \( n \in \mathbb{N} \, p(s(n)) = p(s(n)) \) holds.

(3) \( \Rightarrow \) (1): If for all \( n \in \mathbb{N} \, p(s(n)) \), then \( P_p(s) \) holds where \( P_p \) is the pointwise-derived from \( p \) defined in Definition 6.1. By Lemma 6.1, \( P_p \) is p-hereditary. \( \square \)

7 Generalized Coinduction Proof Principle
Recall that all codata terms considered are implicitly assumed to be well-formed.

Corollary 7.1 (Generalized Coinduction Proof Principle)
Let \( p \) be a unary data predicate and let \( s \) be the name of a codata term. For proving \( \forall n \in \mathbb{N} \, p(s(n)) \)

1. it suffices to exhibit a unary codata predicate \( H \) such that:
   (a) \( H(s) \) holds.
   (b) \( H \) is p-hereditary.
   (That is, for all codata names \( t \), \( p(t(0)) \) and \( H(t^1) \) holds.)
2. it suffices to exhibit a unary p-hereditary codata predicate \( H \) such that \( \forall n \in \mathbb{N} \, H(s^n) \).

Proof
(1) is (1) \( \Rightarrow \) (3) of Theorem 6.1 and (2) is (2) \( \Rightarrow \) (3) of Theorem 6.1. \( \square \)

The generalized coinduction proof principle extends to every unary data predicate the coinduction proof principle formulated in (Park 1981; Rutten 2005; Sangiorgi 2012) for equality.

If several codata terms mutually depend on each other (in the sense of Definition 4.1, that is, are mutually corecursive), then a proof by strong coinduction referring to one of these codata
names $s$ must, of course, also refer to the codata names $s$ depends on. This stresses the necessity for the codata terms considered to be well-formed.

For a better understanding, the following examples of proofs by (generalized) coinduction are detailed (and possibly over-detailed).

**Example 7.1**

$c_1$ as $[1 \mid 1]$  

The singleton set consisting of this codata term is well-defined. Let $P$ be the codata predicate defined by: For all codata name $s$, $P(s)$ if $s(0) > 0$ and $s^1_1 = s$. Thus, $P$ is $(>0)$-hereditary and $P(1_c)$ holds. By the generalized coinduction proof principle, $\forall n \in \mathbb{N} \ 1_c(n) > 0$.

**Example 7.2**

$\text{fib} = [0, 1 \mid \text{fib} + \text{fib}^1]$  

The singleton set consisting of this co-data term is well-defined. Proof by generalized coinduction of $\forall n \in \mathbb{N} \ (n > 0 \Rightarrow \text{fib}(n) > 0)$.  

Let $P$ be the codata predicate defined by: For all codata names $s$, $P(s)$ holds if $s(0) > 0$ and $\forall n \in \mathbb{N} \ ((s(n) > 0 \land s(n+1) > 0) \Rightarrow s(n+2) > 0)$. Thus, $P$ is $(>0)$-hereditary. Let $s$ be a codata name. If $P(s)$ holds, then by definition of $P$, $s(0) > 0$ and $\forall n \in \mathbb{N} \ ((s(n) > 0 \land s(n+1) > 0) \Rightarrow s(n+2) > 0)$. As a consequence, $s(1) > 0$ and $\forall n \in \mathbb{N} \setminus \{0\} \ ((s(n) > 0 \land s(n+1) > 0) \Rightarrow s(n+2) > 0)$, that is, $P(s^1)$ holds.

Second, it is proved that $P(\text{fib}^1)$ holds. $\text{fib}^1(0) = \text{fib}(1) = 1 > 0$ and $\text{fib}^1(1) = \text{fib}(2) = 2 > 0$. Let $n \in \mathbb{N}$. From $\text{fib} = [0, 1 \mid \text{fib} + \text{fib}^1]$ it follows by Corollary [5.1] that  

$\begin{align*} 
\text{fib}^1(2) &= \text{fib}^1 + \text{fib}^1^1 \\
\text{fib}^1(3) &= \text{fib}^1^2 + \text{fib}^1^1 \\
\text{fib}^1(n+2) &= \text{fib}^1(n) + \text{fib}^1(n+1) \\
\text{fib}^1(n) &= \text{fib}^1(n+1) \\
\end{align*}$

Thus, for all $n \in \mathbb{N}$:

$\begin{align*} 
\text{fib}^1(n+2) &= \text{fib}^1(n) + \text{fib}^1(n+1) \\
\text{fib}^1(n) &= \text{fib}^1(n+1) \\
\end{align*}$

It follows that if $\text{fib}^1(n) > 0$ and $\text{fib}^1(n+1) > 0$, then $\text{fib}^1(n+2) > 0$. Thus $P(\text{fib}^1)$ holds.

The result follows by the generalized coinduction principle.

**Example 7.3**

This example is adapted from (Rutten 2005).

$\text{even}(s) \text{ as } [s(0) \mid \text{even}(s^2)]$

$\text{odd}(s) \text{ as } [s^1(0) \mid \text{odd}(s^2)]$

$\text{zip}(s, t) \text{ as } [s(0) \mid \text{zip}(t, s^1)]$

This set of codata terms is well-defined. Proof by coinduction for all codata name $s$ and $t$ $\text{even}(\text{zip}(s, t)) = s$.

Let $P$ be the binary codata defined by: 

$(\ast)$ for all codata names $s$ and $t$, $P(\text{even}(\text{zip}(s, t)), s)$.

$P$ is $(=)$-hereditary, that is, if $P(u, v)$ holds, then

1. $u(0) = v(0)$
Francois Bry

2. $P(u^1, v^1)$

(1) follows from the fact that $(*)$ implies $\text{even}(\text{zip}(s, t))(0) = s(0)$ what is proven as follows:

$\text{even}(\text{zip}(s, t))(0) = \text{by def. of even}$

$\text{zip}(s, t)(0) = \text{by def. of zip}$

$s(0)$

(2) follows from the fact that $(*)$ implies that for all codata names $s$ and $t$

$P(\text{even}(\text{zip}(s, t))^1, s^1)$ holds, as it is now shown. Let $s$ and $t$ be codata names.

$\text{even}(\text{zip}(s, t))^1 = \text{even}(\text{zip}(s, t)^2) = \text{even}(\text{zip}(s, t)^1) = \text{even}(\text{zip}(t, s^1)^1) = \text{even}(\text{zip}(s^1, t^1))$

Thus, $P(\text{even}(\text{zip}(s, t))^1, s^1)$ holds iff:

$P(\text{even}(\text{zip}(s^1, t^1), s^1)$

holds. By $(*)$, $P(\text{even}(\text{zip}(s^1, t^1), s^1)$

holds, that is,

$P(\text{even}(\text{zip}(s, t))^1, s^1)$

holds. By the coinduction proof principle, it follows that:

$\text{even}(\text{zip}(s, t)) = s$

The proof by coinduction of $\text{zip}(\text{even}(s), \text{odd}(s)) = s$ of Example 7.3 given above stresses an aspect of some proofs by coinduction. A codata predicate $P$ is constructed as the closure over tails of codata, what is often described as the computation of a greatest fixpoint (Rutten 2005; Sangiorgi 2012). This view is recalled in the following.

Let $p$ be a unary data predicate and consider the following operator $\Phi_p$ on the unary codata relations:

$\Phi_p : \mathcal{P}($Codata$) \rightarrow \mathcal{P}($Codata$)$

$R \rightarrow \{s \in \text{Codata} | p(s(0)) \land R(s^1)\}$

If $s$ is a codata name, a proof by generalized coinduction of

$\forall n \in \mathbb{N} \; p(n)$

consists in establishing the existence of a unary $p$-hereditary predicate $R$, which defines a unary relation on $\mathcal{P}($Codata$)$, such that $R(s)$ holds, that is, that $R \subseteq \Phi_p(R)$ that means that $R$ is a post-fixpoint of $\Phi_p$. By a theorem of Knaster (Knaster 1928; Tarski 1930; Tarski 1955), commonly called the Knaster-Tarski theorem, the greatest fixpoint of $\Phi_p$ is the unary relation on Codata characterized by the predicate $\text{Max}_p$ defined above in Definition 6.2.

15 The greatest fixpoint $\text{Max}_p$ of $\Phi_p$ is unrelated to the greatest fixpoint semantics of logic programs (Abdallah 1983; Abdallah 1984; van Emden and Abdallah 1985; Lloyd 1987) mentioned in Section 5.
The need for a strengthening of the definition of the predicate to be proven \( p \)-hereditary in a proof by coinduction, as in the proof given in Example 7.3 above, is not specific of coinduction proofs. Such strengthening are common in induction proofs where they are called “induction loading”. A traditional example of induction loading is that a proof by induction of

\[
\forall n \in \mathbb{N} \setminus \{0\} \sum_{i=1}^{n} \frac{1}{i^2} < 2
\]

which is impossible without strengthening that statement for example into:

\[
\forall n \in \mathbb{N} \setminus \{0\} \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}
\]

**Example 7.4**

Proof by coinduction of:

\[
(\dagger) \quad \forall n \in \mathbb{N} \sum_{i=0}^{n} i = \frac{n \times (n+1)}{2}
\]

Consider the following well-formed codata terms:

(nat, sum1) as \([(0, 0)| (1+nat, 1+nat+sum1)]

(nat, sum2) as \([(0, 0)| (1+nat, (1/2)*nat*(1+nat))]\)

First, a codata predicate \( P \) is defined as follows: For all codata names \( s \) and \( t \), \( P(s,t) \) if

1. \( s(0) = t(0) \)
2. \( \forall n \in \mathbb{N} (s(n) = t(n) \Rightarrow s(n+1) = t(n+1)) \)

Second, it is proved that \( P \) is \( = \)-hereditary. Let \( s \) and \( t \) be codata names such that \( P(s,t) \) holds. By (1) of the definition of \( P \), \( s(0) = t(0) \). If follows now from (2) of the definition of \( P \) that \( s(1) = t(1) \). If also follows from (2) of the definition of \( P \) that \( \forall n \in \mathbb{N} 0 (s(n) = t(n) \Rightarrow s(n+1) = t(n+1)) \). This means that \( P(s^*1,t^*1) \) what completes the proof that \( P \) is \( = \)-hereditary.

Third, it is proved that \( P(sum1, sum2) \) holds. By definition of \( sum1 \) and \( sum2 \), \( sum1(0) = sum2(0) \). Let \( n \in \mathbb{N} \) and assume that \( sum1(n) = sum2(n) \). By definition of \( sum1 \):

\[
sum1(n+1) = sum1^*(n) + (n+1)
\]

By definition of \( sum2 \):

\[
sum2(n+1) = \frac{(n+1) \times (n+2)}{2} = \frac{n \times (n+1)}{2} + (n+1) = sum2(n) + (n+1)
\]

Thus, \( \forall n \in \mathbb{N} (sum1(n) = sum2(n) \Rightarrow sum1(n+1) = sum2(n+1)) \). This completes the proof that \( P(sum1, sum2) \) holds and, by the generalized coinduction proof principle, that of (\( \dagger \)).

Under certain notational assumption, the proof of \( \forall n \in \mathbb{N} (n > 0 \Rightarrow fib(n) > 0) \) given in Example 7.2 can be shortened as follows. From

\[\text{This example assumes that the product of codata with data constants defined above in Section 4 is extended to rational numbers.}\]

\[\text{The definitions of sum1 and sum2 differ in a significant aspect: While sum1 is corecursively defined, sum2 is not corecursively defined.}\]
\[ \text{fib} = [0, 1 \mid \text{fib} + \text{fib}^1] \]

it follows that
\[ \text{fib}^1 = [1, 1 \mid \text{fib}^1 + \text{fib}^2] \]

Thus, if \( \text{fib}^1 > 0 \) and \( \text{fib}^2 > 0 \), then \( \text{fib}^1 + \text{fib}^2 > 0 \). Since \( \text{fib}^1(0) = \text{fib}^1(1) = 1 \), the result holds by the generalized coinduction proof principle.

The notational assumption giving sense to the above proof is that if \( p \) denotes a data predicate, then the pointwise-derived from \( p \), \( P_p \), defined in [6.1] is also denoted \( p \), the context allowing to disambiguate. The first sentence above states that the expression defining \( \text{fib}^1 \), \( \text{fib}^1 + \text{fib}^2 \), preserves the pointwise-derived of \( (> 0) \), \( P_{(>0)} \). This is indeed the case because so does the “down-lifted” of that expression to the data: If \( s \) and \( t \) are data names such that \( s > 0 \) and \( t > 0 \), then \( s + t > 0 \). The second sentence states that each of the finitely many initial values defining \( \text{fib}^1 \) does satisfy the data predicate \( (> 0) \).

Under the same notational assumption, the proof can also be expressed as follows. Assume \( \text{fib}^1 > 0 \). \( \text{fib}^1(0) = 1 \) and \( \text{fib}^2(1) = 1 \) and since \( \text{fib}^1 > 0 \), \( \text{fib}^1 + \text{fib}^2 > 0 \). Since there are no contradictions, the assumption \( \text{fib}^1 > 0 \) follows by the generalized coinduction proof principle.

It is similar shortened bisimulation proofs that have made the coinduction proof principle "seem a bit magical" ([Kozen and Silva 2016] p. 2).

Example [7.4] suggests the following theorem.

**Theorem 7.1 (Second Generalized Coinduction Theorem)**
The principle of mathematical induction and the generalized coinduction proof principle follow from each other.

**Proof**

Since the First Generalized Coinduction Theorem [5.1] from which generalized coinduction proof principle is derived, is proved above by induction, it suffices to show that the principle of mathematical induction follows from the generalized coinduction proof principle.

Assume that \( p \) is a unary data predicate such that \( p(0) \) holds and \( \forall n \in \mathbb{N} \ (p(n) \Rightarrow p(n+1)) \).

We show that the generalized coinduction proof principle implies that \( \forall n \in \mathbb{N} \ p(n) \).

Let \( P \) be a codata predicate defined by:

For all codata names \( s \), \( P(s) \) holds if

1. \( p(s(0)) \)
2. \( \forall n \in \mathbb{N} \ p(s(n)) \Rightarrow p(s(n+1)) \)

Let \( s \) be a codata name. Assume \( P(s) \) holds. it follows from the definition of \( P \) that \( p(s(1)) \) holds and that \( \forall n \in \mathbb{N} \ \{0\} \ p(s(n)) \Rightarrow p(s(n+1)) \), that is, by definition of \( P \), \( P(s^1) \) holds. Thus, \( P \) is \( p \)-hereditary. By the generalized coinduction proof principle, \( \forall n \in \mathbb{N} \ p(n) \), what completes the proof. \( \square \)

### 8 Perspectives and Conclusion

An observed stream is a possibly never ending sequence of observations or measurements of natural or artificial systems like temperatures, the reproduction rates of a disease, and energy consumption or traffic volumes. Observed streams are compared with behavioral models, typically for detecting situations of interest like critical raises of a stream’s data.
Behavioral models of observed streams can be expressed as constructed, or synthetic, streams specified as codata terms (as defined in Sections 3 and 4). Functions or predicate on codata then give rise to express properties of such models which can be established using the generalized coinduction proof. A proof similar to that of Example 7.1 for example establishes that \( s_1 = s_2 \) if \( s_1 \) and \( s_2 \) are defined by:

\[
\begin{align*}
  s_1 & \text{ as } [1 \mid s_1] \\
  s_2 & \text{ as } [1, 1 \mid s_2]
\end{align*}
\]

For a less trivial example, consider the following codata terms which specify repeating oscillations over respectively 2, 3 and 6 states denoted 0 to 5:

\[
\begin{align*}
  o_2 & \text{ as } [0, 1 \mid o_2] \\
  o_3 & \text{ as } [0, 1, 2 \mid o_3] \\
  o_6 & \text{ as } [0, 1, 2, 3, 4, 5 \mid o_6]
\end{align*}
\]

**Definition 8.1**
The codata \( s \) and \( t \) as **finitely equivalent**, denoted \( s \sim_f t \), if for some \( n \in \mathbb{N} \):

1. there is a bijection \( b : \{s(0), \ldots, s(n)\} \rightarrow \{t(0), \ldots, t(n)\} \)
   \[ s(i) \rightarrow t(i) \]
2. for all \( m \in \mathbb{N} \) \( s(n + m) = s(n) \) and \( t(n + m) = t(n) \)

Recall that \((o_2, o_3)\) defines (see Section 4) the stream:
\[(0, 0), (1, 1), (0, 2), (1, 0), (0, 1), (1, 2), (0, 0), \ldots \]
A proof similar to that of Example 7.1 establishes that \((o_2, o_3) \sim_f o_6\)
that is, \((o_2, o_3)\) specifies a stream endlessly oscillating over 6 states.

Consider now the codata term
\( n \text{ as } [0 \mid s(n)] \)
which specifies the stream:
\[ [0, s(0), s(s(0)), s(s(s(0))), \ldots] \]
Recall the following codata term from Section 4 which specifies the stream of natural numbers:
\( \text{nat as } [0 \mid 1 + \text{nat}] \)
A proof by generalized coinduction that \( n \sim_f \text{nat} \) does not essentially differ from the proof given in Example 7.1 and from the above-mentioned proof of \((o_2, o_3) \sim_f o_6\).

The aforementioned proofs by generalized coinduction have in common that they refer either to structurally defined codata (in the sense of Definition 4.2) or, for the last one, to a decidable first-order theory, Presburger arithmetic (Presburger 1929). As a consequence, such proofs can be automated. More precisely, proofs of the finite equivalence (in the sense of \( \sim_f \) defined above) of structurally defined codata terms (in the sense of Definition 4.2) and of codata terms defined in terms of decidable theories can be completely automated and therefore supported by a declarative, functional or logic, programming language.
A declarative programming language providing automated proofs by generalized coinduction for codata terms defined in terms of decidable theories would be a useful tool for processing observed streams.

This article has first given an extensive review of coinduction in declarative programming. Second, it has reviewed and slightly extended the codata formalism. Third, it has generalized to any predicate the coinduction proof principle which was originally specified for the equality predicate only. This generalization has been shown to make the coinduction proof principle more intuitive by making better visible its closeness with structural induction. The article has finally suggested to extend declarative, functional or logic, programming with a limited, decidable form of the generalized of the coinduction proof principle based upon structural codata definitions or upon codata definitions referring to decidable theories.

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