THE ESCAPE OF HIGH-ENERGY PHOTONS FROM GAMMA-RAY BURSTS

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ABSTRACT

Eleven bright gamma-ray bursts (GRBs) detected by BATSE have also been seen at much higher energies by EGRET, six at energies above 10 MeV. Most distinctive among these is GRB 940217, which includes long duration, hard gamma-ray emission, and the most energetic GRB photon detection to date, around 18 GeV. Such observations imply that these bursts are optically thin to photon-photon pair production at all observed energies, for target photons both internal and external to the source. For bursts more than about 30 pc away, internal transparency can be achieved only if the source is moving with a relativistic bulk Lorentz factor \( \Gamma \gg 1 \), or if the radiation is highly beamed. Early calculations of \( \gamma \gamma \rightarrow e^+ e^- \) considerations for GRBs were limited to cases of a beam with opening half-angle \( \Theta_b \sim 1/\Gamma \), or expansions of infinitely thin spherical shells. This paper presents our extension of pair production optical depth calculations in relativistically expanding sources to more general geometries, including shells of finite thickness and arbitrary opening angle. The problem is reduced analytically to a single integral in the special, but quite broadly applicable, case of observing photons only along the axis of the expansion. We find that the minimum bulk Lorentz factor for the EGRET sources to be optically thin, i.e., display no spectral attenuation, is only moderately dependent on the shell thickness and virtually independent of its opening solid angle if \( \Theta_b \gtrsim 1/\Gamma \). This insensitivity to \( \Theta_b \) removes the commonly considered number problem for nonrepeating sources at cosmological distances, i.e., it is not necessary to invoke small \( \Theta_b \) to effect photon escape. The values of \( \Gamma \) obtained, typically of the order of 10 for halo bursts and \( \gtrsim 100 \) for sources of cosmological origin, depend somewhat on the choice of GRB timescale used to determine the expansion size. Our new limits on required velocity for given source geometries will aid in placing realistic constraints on GRB source models.

Subject headings: gamma rays: bursts — radiation mechanisms: nonthermal — relativity — stars: neutron

1. INTRODUCTION

Gamma-ray bursts (GRBs) are the brightest sources in the gamma-ray sky, and they may also be among the most distant sources in the universe. The discovery by the BATSE detector on the Compton Gamma Ray Observatory (CGRO) that the spatial distribution of GRBs is isotropic and inhomogeneous (Meegan et al. 1992, 1996) suggests that the sources are either in an extended Galactic halo or at cosmological distances. The level of isotropy of the GRB spatial distribution limits halo models to core radii of around 50–80 kpc (Hakkila et al. 1995); tighter constraints are expected for more recent data accumulations, and Briggs et al. (1996) suggest that a Galactic halo shell distribution must be at least 120 kpc distant. The observed average fluxes of GRBs at Earth therefore imply high luminosities for isotropically emitting sources: \( L \sim 10^{42}–10^{43} \) ergs s\(^{-1}\) at a distance of \( d = 100 \) kpc and \( L \sim 10^{50}–10^{51} \) ergs s\(^{-1}\) at a distance of \( d = 1 \) Gpc. In addition, rapid time variability (\( \Delta t \sim \) several milliseconds) is observed in GRB light curves, whose structural diversity is illustrated, for example, in the BATSE 1B catalog in Fishman et al. (1994). This variability implies a compact source size, which, in combination with the high luminosities, yields photon densities that are high enough to make Galactic halo or cosmological GRBs optically thick to photon-photon pair production by many orders of magnitude. One would then expect attenuation of the observed spectrum (perhaps as a quasi-exponential cutoff, trough, or shelf; examples are depicted in Baring & Harding 1997) around the pair production threshold of 1 MeV if the GRB sources are more distant than a few kiloparsecs and have quasi-isotropic radiation fields (Schmidt 1978; Epstein 1985).

Yet GRB spectra are observed to extend well beyond 1 MeV and into the GeV range. The Gamma Ray Spectrometer detector on the Solar Maximum Mission first measured emission in GRB spectra significantly above 1 MeV, often extending up to 10 MeV (e.g., Nolan et al. 1983), and in one case up to 80 MeV (Share et al. 1982). BATSE routinely observes GRB spectra extending up to and above 1 MeV. While most bursts exhibit spectral steepening at a variety of energies between 50 and a few hundred keV (Band et al. 1993; see also Schaefer et al. 1994 for the BATSE 1B spectroscopy catalog), a number of bursts display spectral breaks between 500 keV and about 2 MeV (Schaefer et al. 1992), but no cutoffs. The EGRET instrument, also on CGRO, has detected emission above 50 MeV from four of the brighter GRBs triggered by BATSE, a fifth up to 30 MeV, and another three up to a few MeV; all are consistent with power-law spectra extending to as high as 1.2 GeV in the case of GRB 930131 (Sommer et al. 1994) and 3.4 GeV in the case of GRB 940217 (Hurley et al. 1994). The GRB 940217 source is best known for exhibiting delayed or prolonged high-energy emission, detected 80–100 minutes (i.e., more than one full Earth orbit of CGRO) after the initial trigger, including a photon of energy 18 GeV (Hurley et al. 1994) that is not
markedly inconsistent with the extrapolation of the power-law continuum. In fact, some evidence for delayed high-energy emission predated GRB 940217, with the observation (Dingus et al. 1994) of a single 10 GeV photon that could have been associated with GRB 910503. It is clear that, in contrast to soft gamma-ray repeaters (SGRs), which are now believed to be a separate class of Galactic sources (although the classical GRB behavior reported by Fenimore, Klebesadel, & Laros 1996 in the 1979 March 5 outburst at early times can support proponents of GRB-SGR associations), there have been no attenuation-type turnovers or cutoffs observed in a GRB spectrum. Therefore, high-energy gamma-ray emission may be common in bursts, and the EGRET detection rate is consistent (Dingus 1995; although this inference is subject to poor statistics) with all bursts emitting above about 30 MeV. Observed GRB spectra are therefore in direct conflict with predicted pair production cutoffs for isotropic emission.

An obvious solution (e.g., Krolik & Pier 1991; Fenimore, Epstein, & Ho 1992) is to allow some anisotropy of the emission, so that the interaction angles $\theta_i$ of the photons are restricted. Therefore, the threshold for pair production, $e_i = 2[\epsilon_i(1 - \cos \theta_{i0})]$, where $\epsilon_I$ and $\epsilon_i$ are the energies of a test photon and an interacting photon, respectively, in units of $m_e c^2$, could be increased above the maximum observed energy. Beaming of the radiation can be achieved through relativistic motion: the radiation from a source that is isotropically emitting in the comoving frame will be beamed mostly within an angle of the order of $1/\Gamma$ in the observer’s frame, where $\Gamma$ is the bulk Lorentz factor. For the case of a small emitting blob moving relativistically, the pair production optical depth $\tau_{\gamma\gamma}$ is reduced by a factor $\Gamma^{-1(1+q)}$ below the optical depth for isotropic radiation, where $x$ is the photon spectral index (Krolik & Pier 1991; Baring 1993). The minimum bulk Lorentz factors required to make $\tau_{\gamma\gamma} < 1$ in the bright “superbowl” burst (GRB 930131) detected by EGRET (Sommer et al. 1994) up to an energy of ~1 GeV are $\Gamma \gtrsim 10^3$ at a distance of 1 Gpc and $\Gamma \gtrsim 10$ at 30 kpc (Harding 1994; Harding & Baring 1994). In this case of relativistic beaming within angle $1/\Gamma$, the required luminosity $L$ at the source is smaller because the observed flux, $F \sim \Gamma^2 L/4\pi d^2$, is enhanced by a solid-angle factor $\Gamma^2$ (Krolik & Pier 1991). However, the number of sources must be a factor $\Gamma^2$ higher in order to account for the observed number of GRBs. In the case of cosmological GRBs, this factor could be as high as $10^6$ for the above limits on $\Gamma$. This is unacceptably large for many of the proposed models, including neutron star–neutron star or neutron star–black hole mergers (Paczynski 1986; Eichler et al. 1989; Narayan, Piran, & Shemi 1991; Meszaros & Rees 1992), failed Type Ib supernovae (Woosley 1993), and the rapid spin-down of high-field millisecond pulsars (Usos 1992), and hence defines the so-called “number problem” for beamed cosmological bursts.

Source geometries with beaming angles larger than $1/\Gamma$ could ease this problem if the high-energy photons were able to escape. In fact, the radiation from GRB sources in the Galactic halo or at cosmological distances is expected to involve a wind or fireball expanding relativistically (Paczynski 1986; Goodman 1986; Piran & Shemi 1993). Fenimore, Epstein, & Ho (1993) have shown that a relativistically expanding, thin spherical shell will allow escape of high-energy gamma-rays, because a test photon on the surface of the shell will not be able to interact with all other emitted photons because of causality limits. This arises as a consequence of the transient nature of the emission, since then only photons emitted within a “look-back” surface around the test photon will interact and contribute to the pair production optical depth. The Fenimore et al. (1993) calculation was limited to the case of an infinitely thin shell.

In this study, we have extended the calculation of the pair production optical depth in GRB sources to the full range of source geometries: opening angles from $1/\Gamma$ to a spherical expansion and shells of arbitrary thickness. The optical depth for test photons emitted within the expanding shell will be limited to interaction with other photons within a “look-back” volume. We present an analytic development and simplification of the pair production optical depth, make detailed numerical calculations, and derive analytic expressions in various limits. The intent is to provide a model-independent evaluation of the pair production opacity of a relativistically expanding, transient gamma-ray source whose emitted spectrum extends above observed energies. Using our results, we derive estimates for the minimum bulk Lorentz factors required for source transparency in those GRBs detected by EGRET at high energies. These limits are largely insensitive to the source opening angle, provided that it exceeds $1/\Gamma$. This reflects the strong impact of causality in determining the optical depth and clearly renders the number problem for cosmological source models a nonissue: the total negation of this number problem for a wide range of expansion geometries is a principal conclusion of this paper. A detailed description of the source geometry and the derivation of an analytic form for the pair production optical depth (and associated limiting cases) for infinite power-law source spectra are presented in § 2; there the general quintuple integral expression for $\tau_{\gamma\gamma}$ (see eq. [20]) is reduced expediently to a comparatively simple single integration in equation (37), a principal result of this research, rendering our developments quite amenable to various observational and theoretical applications. Section 3 is devoted to the application of these results to EGRET bursts and the estimation of minimum bulk Lorentz factors in these sources, including a discussion of the behavior of our results in relevant parameter spaces and various issues pertaining to our calculations. Readers more interested in the applications and implications of our calculations than in the detailed derivations presented in § 2 should note that equation (37) is the final form for the pair production optical depth, which should be used in conjunction with the normalization specified via the flux in equation (18).

2. PAIR PRODUCTION OPTICAL DEPTH

The generic picture of a gamma-ray burst that is considered here assumes the photon source (i.e., region of emission) to be expanding with constant and homogeneous bulk Lorentz factor $\Gamma$, with opening angle $2\Theta_b$ about some axis and thickness $\Delta R$ (constant throughout) in the observer’s frame, and with the initial condition (at time $t = 0$) that the source’s inner radius is $R_0$. The expansion therefore traces a conical volume and can assume a variety of geometries, such as solid cones, solid spheres, or spherical shells, at any given time, depending on the values of the input parameters $\Theta_b$ and $\Delta R/R_0$. The constancy of $\Gamma$ in time is a convenient assumption that is not strictly valid during early epochs of the expansion if $\Delta R/R_0$ is not much less than unity: when $\Delta R/R_0 \gtrsim 1$, the expansion initially resembles a quasi-isotropic and almost stationary radiation gas in the observer’s frame. The adiabatic redistribution of momenta that naturally occurs in expanding (and “inert”) photon gases is therefore
neglected. We also opt to ignore the consideration of possible dynamic acceleration or deceleration of the underlying plasma, since such dynamics are quite model dependent.

Let us suppose that a test (i.e., potentially observable) photon, with energy $\epsilon_i m_e c^2$, is emitted at time $t = 0$ from the inner radius $R_0$ of the shell and moves through the source to eventually escape and reach the observer. Throughout this paper, we opt for test photons originating at the back of the expansion; starting these photons closer to the outer surface will reduce the optical depth they encounter by a factor of order unity, so that the results we obtain will be qualitatively representative of general initial positions for test photons within the source. If the angle cosine between the test photon’s momentum and position vectors is $\mu$, then the radial distance $r_i$ of the test photon from the center of the expansion, at any time $t_i$ is

$$r_i = R_0 + \mu_i c t_i.$$  

(1)

The overall geometry of this source expansion is depicted in Figure 1a and is discussed in more detail below. Radial propagation of the test photon along the axis of the expansion corresponds to $\mu_i = 1$. An important assumption about the expansion that is made in this paper is that no photons are present prior to time $t = 0$. This “switch-on” stipulation restricts the photon population that can causally interact with the test photon at early times, and indeed it mimics burst temporal behavior; it is anticipated that the details of the switch-on will have only a quantitative, rather than a qualitative, influence on the results presented. The objective of this section is to derive an analytic expression for the pair production optical depth for this expanding source geometry. A consideration of the influence of the plasma that is present in the emission region on the photons it generates will be omitted from this analysis.

The optical depth for two-photon pair production $\gamma \gamma \rightarrow e^+ e^-$ can be obtained from well-known expressions for the reaction rate $R_{\gamma \gamma}$ for interactions of photons in a single population (e.g., see eq. [27] of Weaver 1976 or eq. [7] of Stepney & Guilbert 1983). In the case where one photon is a test photon of dimensionless energy $\epsilon_i$, the optical depth differential in the distance $r_i$ that the test photon travels is (e.g., see eq. [7] of Gould & Schreder 1967)

$$\frac{d\tau_{\gamma\gamma}(\epsilon_i)}{dr_i} = \frac{1}{2} \sigma_{\gamma\gamma}(\chi(1 - \mu_i)) n(\epsilon_i, \mu_i; r_i) d\epsilon_i d\mu_i.$$  

(2)

Here the subscripts $i$ denote quantities of the photon that interact with the test photon, $\chi = [\epsilon_i \epsilon_i (1 - \mu_i)]^{1/2}/2$ is the center-of-momentum (CM) frame energy scaled by $m_e c^2$, and

$$\sigma_{\gamma\gamma}(\chi) = \frac{\pi r_e^2}{\chi^2} [(2 \chi^4 + 2 \chi^2 - 1) \ln(\chi + \sqrt{\chi^2 - 1}) - \chi(1 + \chi) \sqrt{\chi^2 - 1}].$$  

(3)

is the Lorentz-invariant pair production cross section (e.g., see eqs. [13]–[40] of Jauch & Rohrlich 1980), where $r_e = e^2/m_e c^2$ is the classical electron radius. Hereafter, all photon energies will be assumed to be dimensionless, being scaled by $m_e c^2$. Also, $\mu_i = \cos \theta_i$ is the angle between the momentum vectors of the test and interacting photons, and $n(\epsilon_i, \mu_i; r_i)$ is the source photon density distribution at the position of the test photon. The factor of $\frac{1}{2}$ in equation (2) is the standard correction for double counting in interactions of identical particles; it is omitted in the calculations of Epstein (1985; see his eq. [2.8]) and Zdziarski (1984), who treat the test photons as a separate population from the interacting photons.

It is instructive to identify the typical energy $\epsilon_i$ of photons that interact with test photons at $\epsilon_i$, specifically for an expansion of bulk Lorentz factor $\Gamma$ that spawns power-law photon spectra, the conditions pertaining to the analysis of this paper. For

\[\text{FIG. 1.—(a) A depiction of the source geometry for the relativistic expansions of bulk Lorentz factor $\Gamma$ considered in this paper. The emission region is at all times a conical sector of a spherical shell with half-angle $\Theta_{\gamma\gamma}$. At some time $t$, the inner radius of the shell is given by $R$, while the thickness of the shell is always $\Delta R$. The test or observable photon is marked by $T$, and it can interact via $\gamma \gamma \rightarrow e^+ e^-$ with other (so-called interacting) photons at typical position $I$. The angles that their position vectors (from the origin $O$ of the expansion) make with the axis $OZ$ of the expansion are $\Theta_{\gamma\gamma}$ and $\Theta_r$. (b) A depiction of the spatial and angular variables, as defined in the text, that are relevant to the analysis of pair production in expanding burst sources. $OZ$ represents the axis of the expansion and generally is not coplanar with the plane ($OTI$) formed by the positions of the test ($T$) and interacting ($I$) photons and the origin ($O$) of the expansion. The two photon momentum vectors ($\text{thick lines}$) have an angle $\theta_i$ between them and define a third plane that is generally not coplanar with $OTI$. The “angles of nonradiality” of the test and interacting photons are $\theta_i$ and $\theta_r$, respectively, and $\Theta_u$ defines the angle subtended at the origin by the positions of the two photons. The labeled distances are $r_T = [OT]$, $r_i = [OT]$, and $r_u = [TI]$.} \]
such test photon energies, the minimum possible energy of the interacting photons, defined by the pair production threshold, is $1/\epsilon$, in the rest frame of the expansion, and of the order of $\Gamma^2/\epsilon$, in the stationary observer’s reference frame. The pair production cross section in equation (3) peaks not far above the threshold, and since the optical depth is a convolution of this cross section and the spectrum, which is a strongly decreasing function of energy, it is clear that the typical energy of an interacting photon is usually never far above the threshold, i.e., around $\Gamma^2/\epsilon$, in the observer’s frame. This result holds regardless of the source photon density, provided that the phase space near the threshold is accessible, which always is the case for infinite power-law spectra.

2.1. Source Geometry

Before developing the calculation of the pair production optical depth, it is both necessary and elucidating to elaborate on the details of the expansion geometry and to define useful spatial variables. The general picture of the expansion at any instant is given in Figure 1a; the definitions of the test and interacting photons’ spatial and angular variables are depicted in Figure 1b and are now enunciated. The radius of the test photon at any time is given by equation (1), and the angle between the radius vector $(\mathbf{OT})$ in Figure 1b) to the test photon and the cone axis $(\mathbf{OZ})$ in Figure 1b), which bisects $2\Theta_{\beta}$, is defined as $\Theta_{\beta}$ ($\leq \Theta_{\beta}$). If the expansion has an inner radius $R$ at time $t$, and a thickness $\Delta R$ that is constant in time, then the test photon remains within the expanding volume only when

$$R = R_0 + \beta ct \leq r_i \leq R_0 + \Delta R + \beta ct,$$

(4)

where $\beta = (1 - 1/\Gamma^2)^{1/2}$ is the bulk velocity of the expansion (in units of $c$). In general, the angle between the test photon’s position and momentum vectors is $\theta_i = \arccos \mu_i$; however, unless otherwise stated, in § 2.1 and subsequent portions of the paper, we will assume that $\theta_i$ is zero so that $r_i = R_0 + ct$; this will be a specialization to the most salient case of radial propagation of test photons. Equation (4) leads to the determination of the time $t_e$ the test photon takes to escape the expanding plasma:

$$t_e = \frac{\Delta R}{c(1 - \beta)}.$$

(5)

Of course, calculation of the pair production optical depth will involve an integration over times $0 \leq t < \infty$, including when the test photon has escaped the expanding plasma.

The test photon interacts with photons at positions within some causally connected look-back volume. Detailed considerations of such look-back regions for relativistic expansions are presented in Rees (1966) and Fenimore et al. (1993). Let us suppose that a typical interacting photon is located at a radius $r_i$ with the angle between its position vector $(\mathbf{OI})$ and the expansion axis $(\mathbf{OZ})$ being $\Theta_{\beta}$ (see Fig. 1b). Such an interacting photon was emitted at time $t_i$ ($0 < t_i < t$) and at a distance $r_{ii} = c(t - t_i)$ from the test photon. Let $\Theta_{\beta}$ be the angle between the radius vectors $(\mathbf{OT}$ and $\mathbf{OI}$ in Figure 1b) of the test and interacting photons. Furthermore, let the angle between the momentum vectors of the test and interacting photons be $\theta_{ii}$. If the line $(\mathbf{TI}$ in Figure 1b) between the positions of the test and interacting photons makes an angle $\theta_i$ with the radius vector of the interacting photon, then simple geometrical analysis gives

$$r_{ii}^2 = r_i^2 + r_{ii}^2 - 2r_i r_{ii} \cos \Theta_{\beta}, \quad r_i^2 = r_i^2 + r_{ii}^2 - 2r_i r_{ii} \cos (\theta_i - \Theta_{\beta}), \quad r_{ii}^2 = r_i^2 + r_{ii}^2 + 2r_i r_{ii} \cos \theta_i.$$

(6)

These relationships will be used to develop the integrations in the expression for the optical depth that is derived in the following subsection. In general, if $\phi$ is the angle between the planes defined by the test photon momentum and position vectors and the momentum vectors of the test and interacting photons, then spherical trigonometry yields

$$\cos (\theta_i - \Theta_{\beta}) = \cos \theta_i \cos \Theta_{\beta} - \sin \theta_i \sin \Theta_{\beta} \cos \phi.$$

(7)

However, when specializing to the case of radial propagation of test photons, the photon momenta lie in the $\mathbf{OTI}$ plane so that $\phi = \pi$ and $\theta_i = \theta_{ii} + \Theta_{\beta}$; this simplification will be used in subsequent sections.

The geometry of the source defined above restricts the values of the variables $r_{ii}$ and $\theta_{ii}$ that prescribe the position of the interacting photons. These restrictions arise because the interacting photons can only be emitted from the portion of the region that the expanding plasma occupied at the time of emission, which is causally connected to the test photon. In the radial direction, the volume that the emission region occupies at time $t = t_i$ is specified simply by

$$R_i \leq r_i \leq R_i + \Delta R, \quad R_i = R_0 + \beta ct_i,$$

(8)

where the relation $r_{ii} = c(t - t_i)$ implies that $R_i = (1 - \beta)R_0 + \beta(r_i - r_{ii})$. With the aid of equation (7), this radial constraint becomes

$$\frac{r_i^2 + r_{ii}^2 - (R_i + \Delta R)^2}{2r_i r_{ii}} \leq \cos (\theta_i - \Theta_{\beta}) \leq \frac{r_i^2 + r_{ii}^2 - R_i^2}{2r_i r_{ii}},$$

(9)

which, for the case of radially propagating test photons (i.e., when $\theta_i = \Theta_{\beta}$, is a compact representation of the limits to the $\mu_i$ integration in equation (2). The values of $r_{ii}$ that are achievable are further constrained by the causality condition

$$0 \leq r_{ii} \leq ct = \frac{r_i - R_0}{\mu_i}.$$

(10)
In fact, this restriction automatically guarantees that \((r_i^2 + r_{i0}^2 - R_i^2)/(2r_i r_{i0}) \geq 1\) and therefore that the right-hand inequality of equation (9) is always satisfied. Physically, this occurs because interacting photons emitted at the rear of the expanding volume can never catch the test photon.

The constraints that the emission volume places on angles are simply enunciated. The requirement that the interacting photon be within the expanding shell imposes no restriction on the radial variables, but it does constrain \(\Theta_i\) according to

\[
0 \leq \Theta_i \leq \Theta_B.
\]

A similar condition limits the values of \(\Theta_i\) at \(t = 0\). Equation (11) restricts the allowable azimuthal angles of the interacting photon for off-axis propagation of test photons; a discussion of this restriction is deferred to § 2.2.2, specifically equation (25). In addition, the range of \(\Theta_i\) is clearly bounded by the expansion geometry. Inspection of Figure 1b reveals that the maximum possible value of \(\Theta_i\) is \(\Theta_B + \Theta_i^*\); with the aid of equation (7), this becomes the global \(\Theta_i\) constraint

\[
\cos \Theta_i = \frac{1 - \rho \mu_i}{\sqrt{1 - 2 \rho \mu_i + \rho^2}} \geq \zeta = \cos \left( \min \{\pi, \Theta_B + \Theta_i^*\} \right),
\]

where \(\rho = r_{i0}/r_i\) is a scaling of \(r_{i0}\) that proves convenient in the algebraic manipulations of this paper (see immediately below). This can be inverted to find the ranges of acceptable values of \(\mu_i = \cos \Theta_{i*}\), as is outlined in equations (31a) and (31b). This concludes the presentation of the general forms for the constraints the source geometry places on the spatial variables defined; specific developments in subsequent sections are made according to algebraic need.

It shall prove convenient to define three dimensionless variables that will facilitate the algebraic developments of the optical depth that are performed in this paper:

\[
\rho = \frac{r_{i0}}{r_i}, \quad \psi = \frac{r_i}{R_0}, \quad s = \frac{1}{1 - \rho} \sqrt{1 - 2 \rho \mu_i + \rho^2}.
\]

The first two of these are scaling transformations that define the coupling of length scales in the expansion; they are used to reduce the number of integrations in the optical depth over spatial variables by one (see eq. [28]). Note that the causality condition in equation (10) yields \(0 \leq \rho \leq 1\) for \(\mu_i \geq 1\). The definition of \(s\) is effectively an alternative to the angular variable \(\mu_i\) that proves convenient in reducing the integration of interacting photon angles analytically (see § 2.3). These three variables will be referred to extensively in subsequent equations.

Hereafter, this paper addresses the special case of radial propagation of test photons along the axis of the expansion, so that \(\mu_i = \cos \Theta_i = 1\) and \(\Theta_i = 0\). While this choice is motivated by the simplifications it introduces to the analysis, it is concordant with the goal of obtaining representative estimates of source bulk Lorentz factors that are consistent with GRB observations. Relativistically expanding sources contribute most of their observable emission along the direction of motion, corresponding to radial and on-axis propagation of the test photons. Off-axis (i.e., nonradial) emission will mostly be outside the peak of beamed radiation, and therefore it will form only a minor part of the observable flux of gamma rays. Hence, we expect that photons produced somewhat off-axis will contribute minimally to the observed flux and therefore will be largely irrelevant to the determination of minimum bulk Lorentz factors. A brief discussion of this specialization, in the light of the results obtained, is presented in § 4. Note that while “limb” photons move, on average, at large angles to the beamed photon population in their locale, the phase space that connects them causally to the remaining photon population is small; it is not clear whether or not limb photons will have enhanced pair production optical depths relative to line-of-sight radiation.

2.2. Analytic Reduction of the Optical Depth

The differential optical depth in equation (2) can be developed once the photon distribution function \(n(\epsilon_i, \mu_i, r_i)\) is known, and eventually an integration over \(r_i\) will be performed to obtain the total optical depth \(\tau_{\gamma y}(\epsilon_i)\). The form that \(n(\epsilon_i, \mu_i, r_i)\) takes depends on basic assumptions about the expanding photon gas. In this paper, the rate of photon emission is taken to be constant in time and space after time \(t = 0\) (following, for example, Fenimore et al. 1993, but in contrast to the uniform density choice made by Harding & Baring 1994), but it is zero for \(t < 0\) and isotropic only in the comoving frame of the expansion. This simplifying assumption is made for its convenience; it is unrealistic since it may be acausal for some initial conditions (e.g., the instantaneous “switch-on” over a finite volume). Observed temporal behavior in individual sources is somewhat chaotic (e.g., see the BATSE 1B catalog; Fishman et al. 1994), and so the assumption of constant emissivity is not truly accurate, depending on the timescale of specific consideration. However, in conjunction with the source geometry prescribed here, a constant emissivity may be able to produce approximately the global properties of bursts, such as longer average decay times than rise times (Nemiroff et al. 1994; Norris et al. 1994; Mitrofanov 1995). Our assumption of a temporally invariant emissivity after switch-on is an appropriate approximation for the objectives of this paper, since only estimates of the bulk relativistic motion in gamma-ray bursts are at present obtainable since the origin of bursts is still uncertain. Note that the rate of emission might be expected to decline in time because of expansion effects such as adiabatic cooling and a decrease in plasma density.

An immediate consequence of this approximation is that the photon distribution in the comoving frame of the expansion is anisotropic even for isotropic photon emission, because of radiative transfer effects in finite-source volumes. This assumption is clearly different from the premise of Gould & Schreder (1967), from which the bulk motion analyses of Krolik & Pier (1991) and Baring (1993) are derived, who all effectively assumed that the photon distribution is everywhere and at all times isotropic (Krolik & Pier 1991 actually invoked the equivalent assumption of isotropy of the photon intensity in the comoving frame). Isotropy of the photon distribution is an even more elementary approximation: it is perhaps less realistic than the assumption of isotropic emission rate that is made here, given that isotropic radiation fields are usually best generated in optically
Here traces out the photon path through the look-back volume, and is the rate of photon emission in the observer’s frame, per unit solid angle, at the position of the interacting photon (the point labeled $I$ in Fig. 1b). The arguments of the emission rate in the integrand are implicitly functions of the spatial variables relating to the test photon position, i.e., $\mu_i = \cos \theta_i = \mu(r_i, t_i, \mu_i)$ and $r_i = r(r_i, t_i, \mu_i)$ are defined by equation (7) and determined by the geometry in Figure 1b. Here $\mu_i = \cos \theta_i$. Throughout the following analysis, it is assumed that $\theta_i = 0$. The angular dependence of the distribution and emission rate has been retained because of the highly anisotropic conditions encountered in this calculation. The azimuthal angle $\phi_{ti}$ is defined as the angle between the OZT and OZI planes in Figure 1b. Since the emission rate is independent of position within the expanding volume, no azimuthal dependence appears in the arguments of $\hat{n}(\epsilon_i, \mu_i; r)$. Note that Harding & Baring (1994) combine equations (2) and (14) into a single expression for the optical depth in their equation (2); their result is mildly erroneous, being a factor of $2/\pi$ too small.

If the source generates isotropic radiation in the comoving frame of the expansion with a power-law emission spectrum $\hat{n}_i(\epsilon_i, \mu_i) \propto \epsilon_i^{-\gamma}$, where $\epsilon_i$ and $\mu_i$ are the photon energy and emission angle in the comoving frame, respectively, then it follows that the photon emission rate in the observer’s frame takes the form

$$\hat{n}(\epsilon, \mu, r) = \mathcal{N} \epsilon^{-\gamma} (1 - \beta \mu)^{-(1 + \gamma)}, \quad \epsilon_- \leq \epsilon \leq \epsilon_+,$$

(15)

where $\epsilon_\pm$ define the bounds to the observed source spectrum. This form is derived from the Lorentz transformation relationships $\epsilon_\mp = \Gamma \epsilon(1 - \beta \mu)$ and $\mu_\mp = (\mu \mp \beta)/(1 - \beta \mu)$ and their associated Jacobian $d\epsilon_\mp / d\epsilon_\mp = \Gamma(1 - \beta \mu)^{-1}$, given that the total photon number $\hat{n}(\epsilon, \mu, r) d\mu dV dt$ is a Lorentz invariant. Note that in equation (15), a factor of $1/\Gamma$ has been absorbed in the definition of $\mathcal{N}$.

The value of the coefficient $\mathcal{N}$ in equation (15) can be determined by computing the photon flux $\mathcal{F}$ at large distances from the source and equating the result to the observed flux in individual GRBs. Specifically, the flux at test photon energy $\epsilon = 1$ (i.e., 511 keV) is

$$\mathcal{F} = c \int d\mu_i \int \epsilon_i(\epsilon_i, \mu_i = 1, \mu_i; r_i) \mathcal{N} \int \frac{\mu_i}{(1 - \beta \mu)^{\gamma+1}} d\mu_i d\mu_i d\epsilon_i,$$

(16)

after integrating equation (14) over azimuthal angles. The units of $\mathcal{F}$ are photons per square centimeter per second. For the moment, let’s assume that there are no angular restrictions to the phase space, i.e., $\Theta_\mu = \pi/2$. Furthermore, note that in this integral, the test photon acts purely as a position marker and can be taken to be on-axis without loss of generality, i.e., $\Theta_\mu = 0$. The angle cosine $\mu_i$ in the distribution is given by equation (23), and the limits on the integrals are defined by the radial and causality constraints in equations (9) and (10). At large distances from the source, these restrictions imply that $r_i/\rho \approx 1$ and $\mu_i \approx 1$, as is obvious from the description of the geometry in Figure 1a. In fact, by defining $\psi = r_i/\rho_i$, then equations (9) and (10) can be expressed as $\psi_0 \leq \psi \leq \psi_\infty$, where $\psi_\pm$ are given in equation (29). The evaluation of the integrals in equation (16) can be facilitated by changing variables thusly: the $\mu_i$ integration is performed using the variable $s = (1 - 2\rho \mu_i + \rho^2)^{1/2}/(1 - \rho)$ for $\rho = r_i/\rho_i$ ($\approx 1$), and the $\rho$-integration is calculated using the variable $t = \psi(1 - \rho)$. Then, for $\psi > 1$, $\mu_i \approx 1/s$. Reversing the order of integration yields $1 \leq t \leq (1 + \Delta R/R_0)/(s - \beta)$ and a trivial result for the $t$ integral, so that

$$\mathcal{F} = \frac{\mathcal{N} R_0^3}{3} \int_0^t \frac{(s^2 + 2s ds) \left[ (1 - \beta + \Delta R/R_0)^3 - (s - \beta)^3 \right]}{(s - \beta)^{\alpha+4}}, \quad s_\infty = 1 + \Delta R/R_0,$$

(17)

Here $r_i$ is set equal to the distance $d$ between the source and the observer, and the flux naturally obeys an inverse-square law: $\mathcal{F} \propto d^{-2}$. Note that the algebraic manipulations here closely resemble those applied to the expression for the optical depth; for this reason, detail is minimized here and is deferred to § 2.2.2.

In general, the result in equation (17) can be expressed in terms of a hypergeometric function of two variables; however, it is simple to evaluate it directly by numerical integration. In the special cases where the expansion is a thin spherical shell with $\Delta R/R_0 \ll 1 - \beta$ in the comoving frame, or a filled shell with $\Delta R/R_0 \gg 1 - \beta$, it becomes analytically tractable, giving

$$\mathcal{F} \approx \frac{\mathcal{N}}{2} \frac{(\Delta R)^2}{d^2} \left\{ \begin{array}{ll}
\frac{R_0}{(1 - \beta)^{\alpha + 1}}, & \frac{\Delta R}{R_0} \ll 1 - \beta, \\
\frac{2\Delta R}{3(\alpha + 3)\beta (1 - \beta)^{\alpha + 3} - 1}, & \frac{\Delta R}{R_0} \gg 1 - \beta.
\end{array} \right.$$

(18)
The quadratic dependence of $\mathcal{F}$ in $\Delta R$ when $\Delta R/R_0 \ll 1$ reflects the two dimensions of the integration in equation (16). At the same time, $\mathcal{F}$ is independent of $R_0$ when $R_0 \ll \Delta R$ since the inner radius contributes negligibly to the source volume. The value of $\mathcal{N}$ is therefore determined by equating the flux in equation (17) to that observed at 511 keV for sources with unbroken power-law spectra, or by a power-law extrapolation of the high-energy spectrum down to 511 keV for those sources with spectral breaks above this energy (e.g., GRB 940217; see the discussion in § 3).

The modification to the expression in equation (17) for the flux that is induced by reduction of $\Theta_\beta$ below $\pi/2$ is straightforward. The considerations of angular constraints in § 2.1 lead to the simple expression for the restriction of the $(\mu_\beta, \rho)$ phase space in equation (12). Since the flux is observed at infinity, $\mu_\beta = 1$, and we take $\Theta_\gamma = 0$ for the flux calculation. Then equation (12) implies immediately that $s \leq 1/\zeta$, for $\zeta$ defined in equation (19), and it quickly follows that subspherical expansions generate a flux given by equation (17) but with

$$s_+ = \min \left\{ 1 + \frac{\Delta R}{R_0}, \frac{1}{\zeta}, \zeta = \cos (\min \{\pi, \Theta_\beta\}) \right\}$$

(19)

substituted as the upper limit to the integral. Clearly then, the angular restrictions play no role in determining the flux until the solid angle $[2\pi(1 - \cos \Theta_\beta)]$ of the expansion becomes comparable to the fractional shell thickness $\Delta R/R_0$.

2.2.2. Optical Depth for Radial Propagation of Test Photons

The differential optical depth in equation (2) can now be expressed in more explicit form using equation (14) and the form of the photon emission rate in equation (15), evaluated at the position of the interacting photon:

$$\frac{d\tau}{dt} = \frac{s_+}{4\pi c} \int \sigma_{\gamma\gamma}(\chi)e^{-s} \frac{1 - \mu_\beta}{(1 - \beta\mu_\beta)^{n+1}} d\zeta d\mu_\mu d\nu d\phi_\nu.$$  

(20)

Here $\mathcal{N}$ has been removed from the integration because it is assumed to depend only on $\Theta_\beta$ and be independent of the position within the source. Hereafter, it will be assumed that the emission spectrum in equation (15) has a large or infinite range $(\epsilon_+ \gg \epsilon_-)$, for which it is possible to evaluate the $\epsilon_i$ integration separately and analytically. Specifically, it is permissible to change variables in equation (20) to the CM frame energy variable $\chi = [\epsilon_i \epsilon(1 - \mu_\beta)/2]^{1/2}$ in $4\chi d\chi = \epsilon(1 - \mu_\beta) d\epsilon_i$, following the procedure of Gould & Schreder (1967; see also Baring 1993), and to perform the integration of the cross section separately. Consequently, the differential optical depth assumes the form

$$\frac{d\tau}{dt} = \frac{s_+}{4\pi c} \frac{\sigma_{\gamma\gamma}(\chi)}{2^{n+2}} \int (1 - \mu_\beta)^{n+1} d\epsilon_\mu d\nu d\phi_\nu.$$  

(21)

where the integration of the cross section over $\chi$ is

$$\mathcal{N}(\chi) = \frac{4}{\sigma_T} \int_1^{\infty} y^{-1} \sigma_{\gamma\gamma}(\chi) d\chi \approx \frac{7}{6\chi^{7/3}}.$$  

(22)

The approximation in equation (22) was obtained (see Baring 1993) from equation (B6) of Svensson (1987), who also gave the exact analytic expression for the integral; it is accurate to better than 1% for $1 < \chi < 7$. The angle cosine $\mu_\beta = \cos \theta_\beta$ of the interacting photon that appears in equation (21) can be determined explicitly from the geometry in Figure 1b using equation (7), eliminating the variables $r_i$ and $\Theta_\nu$ gives (for $\theta_\nu = 0$)

$$\mu_\beta = \frac{\mu_\mu - \rho}{\sqrt{1 - 2\rho\mu_\mu + \rho^2}}.$$  

(23)

The scaling variable $\rho = r_i/r_i$ ($< 1$ for $\mu_\mu = 1$) will be of use in the development of the optical depth integration.

In applications where the emission spectrum is of a finite energy range, the modification for performing the $\epsilon_i$ integration has been developed by Gould & Schreder (1967) and Krolik & Pier (1991). Low-energy spectral turnovers or cutoffs are unlikely to be influential in pair production, continuum attenuation calculations applied to gamma-ray bursts that have bulk motions with large Lorentz factors $\Gamma$ and maximum energies under 100 MeV (Baring 1994). While turnovers are observed as photon energies drop into the BATSE range, sharp cutoffs can presumably be in the soft X-ray range only, where photons interact with gamma rays of energy much more than 1 GeV to produce pairs. Therefore, any suppression of $\gamma \gamma \rightarrow e^+ e^-$ continuum attenuation that is introduced by low-energy depletion is unlikely to be observed by EGRET. However, for observations in the super-GeV range, spectral structure in the BATSE range becomes quite relevant to opacity determinations (Baring & Harding 1997) and is discussed briefly at the end of § 3. The introduction of high-energy cutoffs to the spectrum of interacting photons is also largely irrelevant to this investigation because the most energetic EGRET source photons interact predominantly with photons at energies considerably below the maximum detected.

The azimuthal integration in equation (21) can be performed after establishing the restrictions the source geometry places on $\phi_\nu$. For the interacting photon to be in the expanding "conical shell," the radial restriction in equation (9) is independent of $\phi_\nu$. In contrast, the angular constraint $0 \leq \Theta_\nu \leq \Theta_\beta$ in equation (11), which is independent of time, does restrict the allowable azimuthal angles. Assuming that the plasma emits uniformly at any one time, this angular constraint results in an analytic determination of the azimuthal integration, because at a given radius, each azimuthal angle within the cone of expansion contributes equally. Since the azimuthal angle $\phi_\nu$ is defined as the angle between the planes $OZT$ and $OTI$ (see Fig.
1b), then considerations of spherical trigonometry yield
\[
\cos \Theta_i = \cos \Theta_t \cos \Theta_{ii} + \sin \Theta_t \sin \Theta_{ii} \cos \phi_{ii} \geq \cos \Theta_B
\]  
(24)
when \(|\Theta_i \pm \Theta_{ii}| \leq \Theta_t\). When this condition is not satisfied, the interacting photon is always within the cone defined by the expansion, and all values \(0 \leq \phi_{ii} \leq 2\pi\) are permitted. It follows that the \(\phi_{ii}\) integration has limits defined by \(|\cos \phi_{ii}| \leq \eta_t\), where
\[
\eta = \begin{cases} 
\frac{\cos \Theta_B - \cos \Theta_t \cos \Theta_{ii}}{\sin \Theta_t \sin \Theta_{ii}}, & \Theta_B - \Theta_t \leq \Theta_{ii} \leq \Theta_B + \Theta_t, \\
-1, & 0 \leq \Theta_{ii} \leq \Theta_B - \Theta_t.
\end{cases}
\]  
(25)

The restriction \(\Theta_{ii} > \Theta_B - \Theta_t\) is necessary to achieve \(\eta > 1\), and as \(\Theta_{ii} \rightarrow \Theta_B + \Theta_t\), then \(\eta \rightarrow 1\) and the permitted \(\phi_{ii}\) phase space shrinks to zero. When \(\Theta_t \rightarrow 0\) (i.e., the test photon is on the axis of the expansion), \(\eta \rightarrow -1\) for all permissible \(\Theta_{ii}\). This simple, special case will be assumed throughout subsequent sections of the paper. The \(\phi_{ii}\) integration in equation (21) is then trivially evaluated to give 2 arcsec \(\eta\). Equation (21) can then be integrated over the test photon position \(r_t\) to give the total optical depth:
\[
\tau_{\gamma_\lambda}(\epsilon_t) = \mathcal{N} \frac{\sigma_T}{\pi c} \epsilon_t^{-1} \frac{\mathcal{H}(x)}{2x + 1} \int \arccos \eta \frac{(1 - \mu_i)^2}{(1 - \beta \mu_i)^{2+s}} \, d\mu_i \, d\tau_{ii} \, d\tau_t,
\]  
(26)
and the value for \(\eta\) when \(|\Theta_i \pm \Theta_{ii}| \leq \Theta_B\) becomes (for \(\theta_t = 0\))
\[
\eta = \frac{\cos \Theta_B \sqrt{1 - 2\rho \mu_i + \rho^2 - \cos \Theta_B (1 - \rho \mu_i)}}{\rho \sin \Theta_t \sqrt{1 - \frac{\rho^2}{\mu_i^2}}},
\]  
(27)
where the substitution for \(\Theta_{ii}\) has been effected using equation (7), the variable \(\rho\) is defined in equation (13), and the sine rule is applied to triangle OPT in Figure 1b.

For the moment, let us consider 4\pi steradian expansions, where \(\Theta_B = \pi\) and angular restrictions to the interaction phase space do not enter the analysis. Evaluation of the triple integral in equation (26) can be facilitated by changing variables via the scaling transformations defined in equation (13), so that \(\mu_i\) and \(\eta\) are rendered independent of \(\tau_t\) (see eqs. [23] and [27]). Therefore, a complete set of dimensionless variables has been chosen, and reversing the order of integration so that the \(\psi\)-integration is performed first yields an analytic reduction of the optical depth to a double integral:
\[
\tau_{\gamma_\lambda}(\epsilon_t) = \mathcal{N} \frac{\sigma_T}{\pi c} \epsilon_t^{-1} \frac{\mathcal{H}(x)}{2x + 1} \int_0^1 \, d\rho \int_{\mu_{\text{MIN}}}^{\mu_{\text{MAX}}} \, d\mu_i \, (\psi_+ - \psi_-) \, \arccos \eta \frac{(1 - \mu_i)^2}{(1 - \beta \mu_i)^{2+s}}.
\]  
(28)
The value of \(\mu_{\text{MIN}}\) is given in equation (30). Here the \(\psi\) are the limits of the \(\psi\)-integration (\(\psi_- \leq \psi \leq \psi_+\)) and are determined from the radial constraint in equation (9) and the causality condition in equation (10):
\[
\psi_- = \frac{1}{1 - \rho}, \quad \psi_+ = \frac{1 - \beta + \Delta R/R_0}{\sqrt{1 - 2\rho \mu_i + \rho^2 - \beta(1 - \rho)}},
\]  
(29)
where \(0 \leq \rho \leq 1\) defines a “look-back” volume. If the expansion is at least fully hemispherical (\(\Omega \geq 2\pi\)), then \(\eta = -1\), and the \(\mu_{ii}\) integration is over a range determined by the condition \(\psi_+ \geq \psi_-\). It follows from equation (29) that the range of \(\mu_{ii}\) permitted in equation (28) is \(\mu_{\text{MIN}} \leq \mu_{ii} \leq 1\), where, for any \(\beta < 1\),
\[
\mu_{\text{MIN}} = \begin{cases} 
\frac{1}{2\rho} \left[ 1 + \rho^2 - (1 - \rho)^2 \left(1 + \frac{\Delta R}{R_0}\right)^2 \right], & \frac{\Delta R}{2R_0 + \Delta R} < \rho \leq 1, \\
-1, & \text{otherwise}.
\end{cases}
\]  
(30)
Clearly, \(\mu_{\text{MIN}} \leq 1\) is always true. This restriction indicates that the look-back volume is not spherical because of the presence of edges to the expansion in the radial direction. The form of the pair production optical depth in equation (28) is not generally reducible to a simpler form, and is ready for numerical evaluation in cases where \(\Theta_B \geq \pi/2\). However, analytic development is possible in the special case of the propagation of test photons along the axis of the expansion (i.e., \(\Theta_t = 0\)), which will be treated in the next subsection.

Let us now consider the additional restrictions on the integration phase space due to a reduction in the expansion opening angle \(\Theta_B\). The way the expansion has been defined automatically precludes any necessity to treat cases where \(\Theta_B > \pi/2\), since they reduce to the \(\Theta_B = \pi/2\) case. This arises because switching on the expansion at \(t = 0\) implies that interacting photons from the back hemisphere \(\Theta_B > \pi/2\) (for \(\Theta_t = 0\)) can never reach the test photon originating in the forward hemisphere, and therefore they cannot contribute to the optical depth. This switch-on stipulation is a reasonable approximation to a real burst, and the contribution to the optical depth from photons originating in the back hemisphere is expected to be strongly suppressed because of the relativistic nature of the expansion. When \(\Theta_B < \pi\) both \(\mu_{ii}\) and the azimuthal angle \(\phi_{ii}\) are restricted. The constraint on \(\Theta_{ii}\) in equation (12) can be inverted to find the ranges of acceptable values of \(\mu_{ii}\) in terms of \(\rho\), which defines how the reduction of the opening angle of the expansion restricts the \((\mu_{ii}, \rho)\) phase space:
\[
-1 \leq \mu_{ii} \leq \mu_- , \quad \mu_+ \leq \mu_{ii} \leq 1 ,
\]  
(31a)
where

\[
\mu_{\pm} = \begin{cases} 
\frac{1}{\rho} \left( 1 - \zeta^2 \pm |\zeta| \sqrt{\rho^2 + \zeta^2 - 1} \right), & \rho \geq \sqrt{1 - \zeta^2}, \\
\sqrt{1 - \zeta^2}, & \rho < \sqrt{1 - \zeta^2}.
\end{cases}
\]  

(31b)

Therefore, when \( \rho < (1 - \zeta^2)^{1/2} \), all values of \( \mu_{ji} \) are permitted, since then the angular boundary to the expansion lies outside the look-back volume defined by \( \rho \). The azimuthal restrictions are reflected in the value of \( \eta \) in equation (27), and by a similar analysis, the boundary where \( \eta \) increases above \(-1\) is also defined by equations (31a) and (31b), but with \( \zeta \to \cos(\Theta_0 - \Theta) \). Of course, the ranges in equations (31a) and (31b) are subject to the \( \mu_{\min} \leq \mu_{ii} \) limitation that is imposed by radial considerations.

In the case where the test photon propagates radially along the axis of the expansion (i.e., \( \Theta_0 \), \( \Theta \)), which will be the focal point of all subsequent developments in this paper, the double integral expression for the optical depth in equation (28) can be manipulated into a form that is more convenient for numerical computation, where one integral can be expressed in terms of familiar hypergeometric functions. The analytical approach is similar to the derivation of the photon flux in equation (17). The \( \mu_{ii} \) integration is expressed in terms of the variable \( s \), defined in equation (13), and the \( \rho \)-integration can be performed first using the variable \( t = 1 - \rho \). This change of variables leads to the range

\[
1 \leq s \leq 1 + \frac{\Delta R}{R_0},
\]

(32)

for the variable \( s \), which can be deduced easily from the requirement that \( \mu_{ii} < 1 \) and the condition that \( \psi_+ > \psi_- \) in equation (29). For specific \( s \) within this range, an inversion of the restriction in equation (30) leads to the upper bound \( t_+ = 2/(1 + s) \) for \( t \), which can also be obtained equivalently from the radial constraint in equation (9). The lower bound to \( t \) can be derived directly from the angular constraint in equation (12); the result gives

\[
\max \left\{ 0, \frac{2(\zeta s - 1)}{s^2 - 1} \right\} = t_- \leq t \leq t_+ = \frac{2}{1 + s},
\]

(33)

where \( \zeta \) is defined in equation (19). Since \( \zeta \leq 1 \), it follows that \( t_+ \geq t_- \) so that only one range arises for the \( t \)-integration. This simplicity does not arise if the \( s \)-integration is performed first since \( t_- \) has a maximum of \( 1 - (1 - \zeta^2)^{1/2} \) that can then yield two integration ranges for \( s \) for some \( t \). Note that \( t_- \) exceeds zero only when \( s > 1/\zeta \). Hence, the angular constraint only impacts the calculation of the optical depth when \( \zeta > R_0/(R_0 + \Delta R) \), a situation identical to that arising in the treatment of the source flux. Remembering that \( \arccos \eta = \pi \) for this case of radial and axial propagation of test photons, the optical depth in equation (28) develops into the form

\[
\tau_{\gamma\gamma}(\epsilon_i) = \frac{\sigma_T}{c} R_0^2 \epsilon_i^{-1} \frac{\mathcal{H}(x)}{2^{x+1}} \int_1^{1+\delta} ds \left[ \frac{1 - \beta - \delta^2}{s - \beta} - 1 \right] s^{2+\delta}(s^2 - 1)^x \int_{t_-}^{t_+} dt \frac{t^{2s}}{s^{1/2}}, \quad \mathcal{D} = 2(s - \beta) + t[(1 + s^2)\beta - 2s],
\]

(34)

for fractional shell thickness \( \delta = \Delta R/R_0 \).

The angular constraint in equation (12) is unimportant only when \( t_- = 0 \). In general, this is not so, and the \( t \)-integration in equation (34) can be written as the difference between integrations over the ranges \([0, t_+]\) and \([0, t_-]\). Therefore, two terms appear, each of which can be manipulated in a similar fashion. First, let us consider the integration over \([0, t_-]\): the \( t \)-integration can be rewritten via the substitution \( t = t_- (1 - q) \), leading to the transformation

\[
s^{2+\delta}(s^2 - 1)^x \int_0^{t_-} dt \frac{t^{2s}}{s^{1/2}} = \frac{2^s\sigma^{s+1}}{(1 + \beta)^{s+1}} \frac{s}{s - 1} \left( \frac{\zeta s - 1}{s - 1} \right)^s \int_0^{(1 - q)^2} dq \frac{(1 - q)^{2s}}{(1 - \sigma q)^{s+1}}.
\]

(35)

Here \( \lambda = \lambda(s, \beta) \) and \( \sigma = \sigma(s, \beta, \zeta) \) are given by

\[
\lambda = \frac{(1 + s^2)\beta - 2s}{s(s - 1)(1 + \beta)}, \quad \sigma = \frac{\zeta s - 1}{(s - 1)(1 - \zeta s)\delta}.
\]

(36)

It can be shown that \( \lambda \) is a monotonically decreasing function of \( s \) with the range \(-\infty < \lambda < \beta/(1 + \beta) \), and furthermore that \( 0 \leq \sigma \leq 1 \). When \( \zeta \to 1 \) (i.e., \( \Theta_0 \to 0 \)), then \( t_- \to t_+ \) and \( \sigma \to 1 \), so that the result for \( t \)-integration over the range \([0, t_-]\) is recovered. Putting the two terms together and defining the Heaviside step function \( \mathcal{H}(x) \) to be unity when \( x > 0 \) and zero otherwise, the optical depth can therefore be written in the form (for \( \delta = \Delta R/R_0 \))

\[
\tau_{\gamma\gamma}(\epsilon_i) = \frac{\sigma_T}{c} R_0^2 \epsilon_i^{-1} \frac{\mathcal{H}(x)}{2^{x+1}} \int_1^{1+\delta} ds \frac{s}{s - 1} \left[ \frac{1 - \beta - \delta^2}{s - \beta} - 1 \right] \mathcal{H}(s; \beta, \zeta),
\]

\[
\mathcal{H}(s; \beta, \zeta) \equiv \mathcal{H}(\lambda) - \mathcal{H}(\zeta s - 1) \left( \frac{\zeta s - 1}{s - 1} \right)^s \sigma^{s+1} \mathcal{H}(\sigma \lambda).
\]

(37)
The function $\mathcal{G}(z)$ is just the integral that appears in equation (35) and is expressible in terms of the standard hypergeometric function $F(\alpha, \beta; \gamma; z)$ using the identity 3.197.3 in Gradshteyn & Ryzhik (1980):

$$\mathcal{G}(z) = \int_0^1 dq \frac{(1-z)^2 q}{(1-z q)^{1+\gamma}} = \frac{1}{1+2z} F(\alpha+1, 1; 2x+2; z). \quad (38)$$

The numerical evaluation of $\mathcal{G}$ is straightforward and is described in the Appendix. An alternative form for the optical depth can be derived by leaving the $t$-integration in equation (34) as one integral over the range $[t_-, t_+]$ and rescaling the integration variable. This second form yields equation (37), but with an alternative representation of the function $\mathcal{J}(s; \beta, \gamma)$:

$$\mathcal{J}(s; \beta, \gamma) = \int_0^{s+} dq \frac{(1-z)^2 q}{(1-z q)^{1+\gamma}}, \quad q_+ = \min \left\{ 1, \frac{s(1-\gamma)}{s-1} \right\}. \quad (39)$$

While slightly less convenient than equation (37) for the numerical evaluation of the optical depth, this second form is useful when obtaining results in the limiting case of small opening angles, treated in §2.3.3. Computationally, if the series representation of $\mathcal{G}$ described in the Appendix is used, equation (37) is a single integral that is relatively simple to evaluate. Note that the integrand does not diverge at $s = 1$ because of the behavior of $\mathcal{G}(z)$ there (see the Appendix).

It is important to emphasize that the optical depth in equation (37) was obtained under the assumptions that the test photon originates at the rear of the expansion and propagates radially outward. With this specification, equation (37) is intended to approximate a variety of possibilities for test photon initial conditions. Nonradial test photon motion will increase the optical depth above that in equation (37), primarily because of increased angles with interacting photons; however, an observer's unique perspective will strongly bias against such situations for relativistic expansions. On the other hand, test photons can plausibly originate closer to the surface of the expansion than diminishing the optical depth accordingly. Let us suppose that the test photon starts at radius $R_0 + \Delta R$ at time $t = 0$ and then all interacting photons inside this radius are always causally disconnected from the test photon because it propagates radially. Hence, the region interior to $R_0 + \Delta R$ is irrelevant to the optical depth calculation, and a new initial condition can be defined, with $R_0 + \Delta R$ and $R_0 + \Delta R$ denoting the relevant inner and outer radius of the expansion, respectively. The test photon is now at the rear of this section of the conical shell, and the optical depth computation can be repeated entirely with the aid of the substitution

$$R_0 \to R_0 + \Delta R, \quad \Delta R \to (1 - \nu)\Delta R, \quad (40)$$

without any additional manipulation. This elementary transformation propagates all the way through the development so that the optical depth for test photons starting at arbitrary positions in the expansion is simply from equation (37) by the substitution $\delta \to (1 - \nu)\delta/(1 + \nu)$, a very attractive scheme of generalization. Note that the normalizing flux in equation (17) is unaffected by these considerations. This concludes the analytic development of the optical depth formula, which is ready for numerical computation and certainly much more amenable than the quintuple integral in equation (20). Before presenting such computations (in §2.4.2), it is instructive to examine the optical depth for some limiting cases of the source geometry.

### 2.3. Approximations in Limiting Cases

There are four special cases where it is both possible and useful to obtain analytic limits to the pair production optical depth: these correspond to the thin-shell limit, thick-shell or filled-sphere expansions, narrow beams, and a stationary photon gas.

#### 2.3.1. Thin-Shell Limit

The expression in equation (37) is in suitable form for the derivation of the optical depth in certain special cases. The first of these is the limit of a thin, spherical shell for the expansion, where $\delta = \Delta R/R_0 \ll 1 - \beta$ (i.e., the shell is also thin in the comoving frame of the expansion) and $\delta < 1 - \zeta$ so that the angular constraints are immaterial. The $\mathcal{G}(\sigma \lambda)$ term in equation (37) is therefore absent. As noted in the Appendix, in the $s \to 1$ limit, $(1 + 2x)\mathcal{G}(\lambda)$ approaches $F(\alpha + 1, 1; 2x + 2; 1)/(1 - \lambda)$, which, with the aid of identity 9.122 of Gradshteyn & Ryzhik (1980), leads to the limit $\mathcal{G}(\lambda) \to 1/[2(1 - \lambda)] \approx (s - 1)/(1 + \beta)/(2x(1 - \beta))$. It is then elementary to derive the result $\tau_{\gamma;i}(\epsilon) \propto \mathcal{N}(\Delta R)^2$. The two powers of $\Delta R$ are due to the thin shell severely restricting the spatial extent of the $\tau$ and $\tau_{\gamma;i}$ integrations (e.g., see eq. [20]). Such behavior is largely meaningless until the dependence of the formula for the flux is factored in. Remembering that in this limit, for a fixed observed flux, equation (18) yields $\mathcal{N} \propto 1/(\Delta R)^2$, the optical depth is virtually independent of $\Delta R$, as is expected. Explicitly, we obtain

$$\tau_{\gamma;i}(\epsilon) \approx \frac{\sigma T}{2c R_0} \mathcal{F} \frac{\mathcal{G}(\alpha)}{\alpha} \frac{1 - \beta}{1 + \beta} \epsilon_i^{-1}, \quad \frac{\Delta R}{R_0} \ll \min \{ 1 - \beta, 1 - \zeta \}. \quad (41)$$

This thin-shell limit displays a strong inverse dependence on the bulk Lorentz factor $\Gamma = 1/(1 - \beta^2)^{1/2}$ of the expansion, as is evident from Figure 2: typically, $\alpha \sim 2 - 3$ for EGRET bursts (e.g., see Table 2 below). This dependence is enhanced by one or two powers of $\Gamma$ that appear in the determination of $R_0$. This case is closest to the work of Fenimore et al. (1993), who treat test photons coming from an entire spherical shell (i.e., including the limb regions). As argued in §2.1.4, the major contribution to the optical depth comes from test photons originating in near-axis environs so that the differences between conclusions made using a formula like equation (41) and those using the work of Fenimore et al. (1993) are marginal.

#### 2.3.2. Filled-Sphere Expansions

From the point of view of the radial dimension, the other extreme class of expansions initially contains filled spheres, i.e., thick shells (in the comoving frame) with $\Delta R/R_0 \gg 1 - \beta$. Again, we shall ignore the impact of narrowing the solid angle down
and demand $\Theta_B = \pi/2$ here, so that only one term in equation (37) contributes to the optical depth. In this limit, inspection of equation (37) soon reveals that the dominant contribution to the integral is for $s - 1 \leq 1 - \beta$. Then it follows that $\lambda \approx -(1 - \beta)/(s - 1)$, since relativistic expansions with $\beta \approx 1$ are considered here. Choosing $\lambda/(\lambda - 1)$ as the integration variable and using the transformation of the hypergeometric function in equation (A2) yield a result that is proportional to the integral in equation (A4). As in the thin-shell limit, here $\tau_{\gamma}(\epsilon_i) \propto (\Delta R)^2$, reflecting the dimensionality of the integrations. The filled sphere limit of equation (18) indicates that an observed flux gives a volume-determined photon injection rate $N(\lambda \ll 1/(\Delta R))$, so that the overall expression for the optical depth is a declining function of the expansion thickness: for $\beta \approx 1$, the asymptotic result

$$\tau_{\gamma}(\epsilon_i) \approx \frac{3sq_2d^2}{4c}\Delta R(1 - \beta)^{y+1}\epsilon_i^{-1}(\alpha + 3)\frac{\mathcal{W}(\alpha)}{2}\left[\frac{1}{\alpha} + \frac{2}{\alpha - 1}\right](\psi(2\alpha) - \psi(\alpha) - 1)$$

(42)

is derived, where $\psi(\alpha)$ is the derivative of the logarithm of the gamma function, defined in equation (A4). Again, a strong dependence on the bulk Lorentz factor of the expansion is evident. The moderate decline of $\tau_{\gamma}(\epsilon_i)$ with $\Delta R$ reflects the fact that large regions are less compact for a given source luminosity. Note that extremely filled spheres with $\Delta R > R_s$ are not really discussed in this paper; these seem unlikely to be realized in bursts and require an alternative coupling of length scale to time variability, via $\Delta R = c\Delta t$, as is mentioned below.

2.3.3. Narrow Beam Expansions

The case of small opening angles of the expansion is of interest also. Since the emphasis here is on the axial-viewing perspective, this limiting case corresponds to $\zeta \equiv \cos \Theta_B \approx 1$. Specifically, this narrow beam limit satisfies $1 - \zeta < 1 - \beta$, so that the reduction in opening angle dominates the causality limitations, and also $1 - \zeta \ll \delta \equiv \Delta R/R_0$. However, such an identification with small solid angles is not sufficient to define narrow beam cases; as will be evident shortly, the size of the opening angle itself is also quite pertinent. The most suitable form of the optical depth for development here is the use of $N(1/\Delta R) \ll 1$ (1\Delta R)\ll 1$, and this contributes an order of 1

$$N(1/\Delta R) \ll 1$$

Such a solid-angle limited range differs from the requirements imposed by equation (37), thereby complicating the consideration of the narrow beam limit. To aid in understanding this limit, the various dependences of equations (17) and (37), and the resulting behavior of the overall optical depth, as functions of $\Theta_B$ and $\delta$, are listed in Table 1. In this table, four parameter regimes with $1 - \zeta<1 - \beta$ are
identified, depending on $\delta$. Three of these regimes are strictly narrow beam limits, while the fourth, for $\delta \ll 1 - \zeta$, corresponds to the thin-shell limit in equation (41) and is independent of $\Theta_{B}$. For the first two regimes in Table 1, the developments of equations (17) and (37) just mentioned lead to an approximate overall optical depth that can be written as one expression:

$$
\tau_{\gamma}(\epsilon_{i}) \approx \frac{3\sigma_{T}}{2c R_{0}} d^{2} \mathcal{F} \sqrt{\pi} \frac{\Gamma(\alpha + 1/2)}{\Gamma(\alpha + 1)} \frac{\mathcal{H}(\alpha)[2(1 - \beta) + \delta]}{3(1 - \beta)^{2} + 3(1 - \beta) \delta + \delta^{2}} \frac{(1 - \beta)^{\delta + 1/2}}{(1 + \beta)^{1/2}} e_{i}^{i-1} \frac{1}{\sqrt{1 - \zeta}}, \quad \sqrt{1 - \zeta} \ll \min \left\{ 1, \frac{\Delta R}{R_{0}} \right\}, \quad (43)
$$

for $\delta = \Delta R/R_{0}$. This formula encompasses both thin-shell narrow beam $[(1 - \zeta)^{3/2} \ll \delta \ll 1 - \beta]$ and thick-shell narrow beam $[(1 - \zeta)^{3/2} \ll 1 - \beta \ll \delta]$ regimes. Surprisingly, the calculated optical depth actually increases when the opening angle closes down, reflecting the explicit dependence (see eqs. [17] and [19]) of the observed flux on the solid angle of the expansion, combined with the pair production rate depending explicitly on the angle between the test and interacting photon momenta only. Essentially, the photon density in the source increases for the constant observed flux as the expansion opening angle is reduced. The resulting optical depth varies only with the thickness of the shell ($\propto 1/\Delta R$) in the regime of thick-shell expansions, i.e., for $\Delta R/R_{0} \gg 1 - \beta$. The third regime in Table 1 requires the use of the thin-shell evaluation of $\tau_{\gamma}/\mathcal{N}$ and yields the asymptotic approximation

$$
\tau_{\gamma}(\epsilon_{i}) \approx \frac{\sigma_{T}}{4c R_{0}} d^{2} e_{i}^{i-1} \frac{\mathcal{H}(\alpha)}{\alpha} \frac{1 - \beta}{1 + \beta} \frac{\delta}{1 - \zeta}, \quad 1 - \zeta \ll \delta \ll \sqrt{1 - \zeta} \ll 1 - \beta. \quad (44)
$$

This bears an even stronger dependence on the opening angle $\Theta_{B} \approx [(1 - \zeta)^{1/2} \ll \delta \ll 1 - \beta]$, again rising with increased narrowness of the beam, a property that corresponds to an enhanced mean density of radiation in the expansion. In this case, the phase space for pair production is not restricted by the opening angle but rather by the thinness of the shell only, while the flux is still solid-angle limited.

Finally, in concluding the consideration of narrow beam cases, let us observe that equation (43) approximately reproduces the thin-shell and filled-sphere forms in equations (41) and (42) when the beam is opened up to $1 - \zeta \sim 1 - \beta$. For this intermediate (or critical) regime of opening angles, a domain common to all three of the limiting cases discussed so far is achieved when $\Delta R/R_{0} \sim 1 - \beta$. This locality in phase space corresponds to the so-called “blob” scenario of earlier work (e.g., Krolik & Pier 1991; Baring 1993; Baring & Harding 1993) on pair production transparency constraints in gamma-ray bursts, a situation that is discussed in § 3.

2.3.4. Stationary Radiation Gas

The remaining limiting case of the optical depth is for a $\beta = 0$ or nonrelativistic expansion. This is mostly of academic interest, as a check on the numerical evaluations that follow, and does not have great physical import for the problem considered in this paper. In fact, the result in equation (37) does not aptly model stationary gases since we have neglected the limb contributions to the optical depth; these become significant in nonrelativistic expansions. Set $\Theta_{B} = \pi/2$ for simplicity. In the $\beta \to 0$ limit, $\lambda \to -2/(\delta - 1)$, which simplifies equation (37) somewhat. However, analytic development of the subsequent result is not fruitful, and so it is convenient to consider separately the thin- and thick-shell cases. The limiting result in equation (41) was derived without restriction on $\beta$, and so the limit $\beta \to 0$ can be taken to obtain the optical depth for stationary, thin-shell sources. For $\Delta R/R_{0} \gg 1 - \beta$, a derivation alternative to that in § 2.3.2 is requisite. Then the dominant contribution to the integration comes from $s - 1 \ll \delta$. The transformation relation in equation (A2) can be used, together with a change of variable, to yield a result proportional to both $\Delta R^{2}$ and the integral in equation (A5). The flux is simply evaluated when $\beta = 0$, so that the optical depth for a thick stationary gas is

$$
\tau_{\gamma}(\epsilon_{i}) \approx \frac{3\sigma_{T}}{4c \Delta R} d^{2} \mathcal{F} \mathcal{H}(\alpha) \left[ 1 + \frac{\delta}{2(\alpha + 1)} \right]^{\gamma/\beta} e_{i}^{i-1}, \quad \frac{\Delta R}{R_{0}} \gg 1 - \beta. \quad (45)
$$

Note that this estimate differs somewhat from results derived for isotropic photons (e.g., Schmidt 1978; Epstein 1985; in particular, those that use the formalism of Gould & Schrader 1967), because in this paper we have assumed isotropic injection in the comoving frame (in this particular limit, the observer’s frame), which is not equivalent to radiation isotropy because of the radiative transfer in the sphere.
2.4. Numerical Computation of the Optical Depth

The various limiting cases just explored guide the technique for numerical determination of the optical depth and further act as checks on computational accuracy. The numerics are generally straightforward, and it is expedient to use \( s - 1 \) as an integration variable in equations (17) and (37), and scale quantities in terms of \( 1 - \beta \) to maintain good accuracy for large Lorentz factors. Since the range of integration variables contributing significantly to the two integrals is sometimes quite large, logarithmic sampling of \( s - 1 \) is favored. Use of the functional form in equation (37) comfortably produces smooth results for \( \Theta_b \) down to even smaller than 0.01, and so use of the alternative representation in equation (39), or a series in \( 1 - \zeta \), is unnecessary for the purposes of this paper.

Numerical determinations of the optical depth formed by the combination of equations (17) and (37) are presented in Figure 2 for different fractional shell thicknesses \( \Delta R/R_0 \), illustrating its strong dependence on \( \Gamma \) and its significant dependence on the source opening angle for smaller \( \Gamma \) only. The inclusion of \( \beta \ll 1 \) cases is intended only to provide a general guide to the behavior of the optical depth (perhaps to an order-of-magnitude accuracy), but this is strictly incorrect since they neglect limb contributions. The quantity actually plotted in Figure 2 is the scaled optical depth

\[
\tau_{\gamma\gamma}(\varepsilon) \frac{\Delta t (\text{ms})}{d^2_{\text{Gpc}}} \left[ \varepsilon(1 + z) \right]^{1-a} = 1.06 \times 10^{13} \frac{c \Delta t}{R_0} \left( x; \beta, \frac{\Delta R}{R_0} \right),
\]

which is dimensionless since \( \mathcal{F} \) is measured in photons per square centimeter per second, where \( \mathcal{R}(\alpha; \beta, \Delta R/R_0) \) is just \( \mathcal{R}(\alpha)/(1 + \beta^{1+a}) \) times the ratio of the two integrals in equations (17) and (37). Here \( \Delta t \) is the observed source variability timescale, typically in the range of \( 10^{-2}-1 \) s, inferred, for example, from time histories such as those exhibited in the BATSE 1B catalog (Fishman et al. 1994), and \( z \) is the cosmological redshift of the source (it is set to zero in Fig. 2). The coefficient of this equation clearly defines the optical depth scale for cosmological bursts and would be \( 8-10 \) orders of magnitude smaller for Galactic halo sources. A “canonical” spectral index of \( \alpha = 2.5 \) (see Table 2 for specific values) is chosen in Figure 2 for simplicity; increasing (reducing) \( \alpha \) just increases (lowers) the slope of the curves in the \( \Gamma \approx 1 \) regimes. Remembering that the energy \( \varepsilon \) is expressed in units of \( m_ec^2 \), it is evident from Figure 2 that the maximum observed energies of EGRET sources (see Table 2) imply that Lorentz factors in the range of approximately 50–500 are required to render these bursts optically thin to pair production.

The curves in Figure 2a clearly delineate three regimes of parameter space in order of increasing bulk Lorentz factor: (i) nonrelativistic flows where the optical depth is independent of the expansion speed, (ii) thin-shell expansions \( (\Delta R/R_0 \ll 1 - \beta \) and \( \Gamma \gg 1 \), which is the portion of phase space sampled by the work of Fenimore et al. (1993), yielding a strong inverse dependence of \( \tau_{\gamma\gamma} \) on \( \Gamma \), and (iii) at the highest Lorentz factors \( \Gamma \), above the break at \( (\Delta R/R_0)/(1 - \beta) \approx 1 \), thick-shell expansions, where the inverse dependence on \( \Gamma \) is slightly stronger [by two powers of \( \Gamma \); compare equation (41), where \( \tau_{\gamma\gamma} \propto (1 - \beta)^{a}/R_0 \), and equation (42), where \( \tau_{\gamma\gamma} \propto (1 - \beta)^{a+1}/\Delta R \). We note that Fenimore et al. (1993) produced an optical depth versus log \( \Gamma \) plot for their infinitely thin shell analysis that exhibited a dramatic reduction of \( \tau_{\gamma\gamma} \) above some critical Lorentz factor. These turnovers were found to be artificial, being caused by a coding error (E. E. Fenimore 1996, private communication). The ratio \( \Delta R/R_0 \) is independent of \( \Gamma \) in Figure 2a, although other choices are quite plausible. For opening angles \( \Theta_b \approx 1/\Gamma \), the ratio \( \chi = (\Delta R/R_0)/(1 - \beta) \) is the only critical parameter delineating the thin-shell and thick-shell causes, and it is intimately related to the portion of the emission region that is causally connected to the test photon. As is evident from equations (41) and (42), the optical depth is independent of the thickness of the shell when this parameter is much less than unity, and inversely proportional to \( \Delta R \) when \( \chi \gg 1 \); the transition between these regimes is quite gradual. The curves exhibit a lack of dependence on the opening angle \( \Theta_b \) when \( \Theta_b \gtrsim \min \{1/\Gamma, (\Delta R/R_0)^{1/2}\} \) (the curves in Fig. 2a are coincident with the curves in Figure 2b).

| GRB       | Detecting Instrument | \( \varepsilon_{\text{MAX}} \) (MeV) | \( \alpha \) | \( f(1\text{ MeV}) \) (cm\(^{-2}\)s\(^{-1}\)MeV\(^{-1}\)) |
|-----------|----------------------|-------------------------------------|-------------|------------------------------------------|
| 910503…  | EGRET                | 170                                 | 2.2 ± 0.2   | 8.71 ± 0.49                             |
| 910601…  | EGRET                | 3.5                                 | 3.7 ± 0.2   | 0.98 ± 0.08                             |
| 910814…  | EGRET                | 60                                  | 2.8 ± 0.2   | 0.50 ± 0.10                             |
| 930131…  | EGRET                | 1000                                | 2.0 ± 0.3   | 1.95 ± 0.26                             |
| 940217…  | EGRET                | 3380\(^a\)                         | 2.5 ± 0.1   | 0.36 ± 0.03                             |
| 950425…  | EGRET                | 120                                 | 1.93 ± 0.04 | 1.62 ± 0.09                             |

\(^{a}\) The EGRET detections are from Schneid et al. 1992 (GRB 910503), Kow et al. 1993 (GRBs 910601 and 910814), Sommers et al. 1994 (GRB 930131), Hurley et al. 1994 (GRB 940217), and Catelli et al. 1996 (GRB 950425). The COMPTEL measurement of GRB 910601 (see Hanlon et al. 1994) is included because it was seen at a higher energy than the EGRET detection of this source, and it is used in the estimates for lower bounds to \( \Gamma \) in Table 3.

\(^{b}\) The maximum energy of detection.

\(^{c}\) The high-energy spectral index.

\(^{d}\) The (measured or extrapolated) source photon flux at 1 MeV for six bursts observed by EGRET.

\(^{e}\) The famous 18 GeV photon from GRB 940217 was omitted because it was not contemporaneous with any emission below 100 MeV.
for $\Theta_B = 90^\circ, 10^\circ$, and $1^\circ$): this is a consequence of causality dominating geometry in the restriction of the integration phase space. When $\Theta_B \lesssim 1/\Gamma$, the optical depth $[\chi(1 - \beta^2)^{3/2}/(\Theta_B R_0)]$ is actually increased by the angular reduction of the available phase space, an effect that is particularly evident for nonrelativistic expansions. As mentioned in §2.3.3, the reason for this increase in optical depth with declining $\Theta_B$ is that the pair production rate is proportional to the angle between the test and interacting photons (which is limited linearly by $\Theta_B$), while the flux scales as the solid angle (i.e., $\Theta_B^2$ when $\Theta_B \ll 1$) of the expansion. Hence, the density of photons in the source that is inferred for a given flux actually increases as when $1/\Gamma$.

For $10^\circ$, and $1^\circ$): this is a consequence of causality dominating geometry in the restriction of the integration phase space. When $\Theta_B \lesssim 1/\Gamma$, the optical depth $[\chi(1 - \beta^2)^{3/2}/(\Theta_B R_0)]$ is actually increased by the angular reduction of the available phase space, an effect that is particularly evident for nonrelativistic expansions. As mentioned in §2.3.3, the reason for this increase in optical depth with declining $\Theta_B$ is that the pair production rate is proportional to the angle between the test and interacting photons (which is limited linearly by $\Theta_B$), while the flux scales as the solid angle (i.e., $\Theta_B^2$ when $\Theta_B \ll 1$) of the expansion. Hence, the density of photons in the source that is inferred for a given flux actually increases as when $1/\Gamma$.

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The specific choice of $R_0 = \Gamma c \Delta t$ is made in Figure 2 for simplicity; alternative dependences on $\Gamma$ are possible, corresponding to different interpretations of source timescales (as discussed in §3), and these result in only a slightly different appearance from the curves in Figure 2. The results presented in this subsection are obtained under the assumption that the test photon starts its life at the rear of the expansion; equation (40) provides a simple scheme of substitution for equation (37) that yields optical depths for arbitrary initial positions of the test photon within the expansion. It is evident that imposing a $\tau_{\gamma\gamma} = 1$ condition on EGRET bursts can potentially map over into both thin- and thick-shell regimes, depending on the assumed source distance, measured flux, and maximum photon energy observed. We note that using a $\tau_{\gamma\gamma} = 1$ criterion for source transparency actually leads to conservative lower bounds for $\Gamma$, since significant attenuation is already present when $\tau_{\gamma\gamma} = 1$.

This can be easily seen if the spectrum is attenuated by an exponential factor $\exp (-\tau_{\gamma\gamma})$, a common choice. A central consequence of the assumption of an infinite power-law burst spectrum is that the optical depth is an increasing (and power-law) function of energy, so that spectral attenuation arises only at high energies; departures from this behavior will be discussed briefly at the end of the next section. We remark also that opacity skin effects (which depend on the spatial distribution of photons and therefore are model dependent) can sometimes render the exponential $\exp (-\tau_{\gamma\gamma})$ a poor descriptor of attenuation, with $1/(1 + \tau_{\gamma\gamma})$ perhaps being an improvement for uniformly distributed photons, leading to broken power laws rather than exponential turnovers. A variety of signatures of spectral attenuation are possible, particularly if pair-cascading is involved, and some of these are illustrated in the work of Baring & Harding (1997).

Undoubtedly, the most crucial piece of information to be gleaned from Figure 2 is that causality minimizes the role of the opening angle $\Theta_B$ of the expansion in faster expansions, and specifically that the $\tau_{\gamma\gamma} = 1$ condition will be virtually independent of $\Theta_B$ in the range $90^\circ \geq \Theta_B \gtrsim 1/\Gamma$ for burst sources. This insensitivity to the angular extent of the expansion is the keystone to the elimination of the number problem for cosmological bursts, as is discussed below.

3. BULK LORENTZ FACTORS FOR EGRET SOURCES

The calculations for $\tau_{\gamma\gamma}$ that we have performed can be applied readily to gamma-ray bursts detected by EGRET. The pair production optical depth in equation (37) depends on the free parameters ($\Gamma$, $\Delta R/R_0$, $\Theta_B$, and $d$) and on the observed parameters: source flux $\mathcal{F}$ (defined in eq. [16]), high-energy spectral index $\alpha$ (see eq. [15]), test photon energy $\epsilon_\gamma$ ($= \epsilon_{\text{MAX}}$), and $\Delta t$. The burst variability timescale $\Delta t$ can be used to infer an upper limit on the source size, which we take for the moment to be $R_0 = \Gamma c \Delta t$, based on the apparent size of the expanding shell perpendicular to the light of sight, as seen by a stationary observer (e.g., see Rees 1966). Alternative size estimates, such as those that couple the source variability to dimensions along the line of sight to the source, are possible. In addition, measured minimum variability timescales for bursts range from milliseconds in the BATSE range to supersecond values in the EGRET data. Motivations for choosing either of these $\Delta t$, and also different source size determinations, are discussed below. Both variability timescales are addressed in the results presented here, for the sake of completeness.

The key observable parameters, to be used in the pair production opacity calculations of this paper, are displayed in Table 2 for six of the burst detections by EGRET. Note that there are eleven EGRET bursts in total (Schneid et al. 1996), three of which have insufficient (published) data for the purposes of our analyses; GRB 920622 and GRB 940301 have the required observational parameters in Schneid et al. (1995) but suffer from poor statistics above about 2 MeV. We therefore conservatively opt to study just the most significant six sources of the EGRET population. In Table 2, the fluxes are expressed in observer-friendly units, via $f(1 \text{ MeV})$, which is just the flux, evaluated at 1 MeV, per MeV energy interval (the flux $\mathcal{F}$ is per $m^2$ at 511 keV). These fluxes are obtained via extrapolations down to 1 MeV of the best-fit power laws to the time-integrated super-MeV spectral data, and they are not necessarily the actual fluxes measured at 1 MeV. Note that both EGRET and COMPTEL parameters are listed for GRB 910601 since this burst was relatively soft and actually had a slightly more significant detection by COMPTEL than by the EGRET TASC in the 3–5 MeV range. We remark that the COMPTEL listings for GRB 910601 (based on Hanlon et al. 1994) differ slightly from those quoted by & Harding that were obtained from Winkler et al. (1993). As noted in a footnote to the table, in the computations of this paper, we neglect the highest energy (18 GeV) photon detected for GRB 940217. This conservative step is taken because the statistically limited sample provided by a single photon leads to a large uncertainty in the spectral form at these energies, which is compounded by the lack of contemporaneous spectral information at lower (i.e., sub-MeV) energies (e.g., Hurley et al. 1994); time-resolved spectra of good statistical quality in the super-100 MeV range await future generations of instrumentation. A nice depiction of the relative fluxes and spectra of four of these bursts is given in Hurley (1996).

3.1. Geometries with $\Theta_B \sim 1/\Gamma$

Before presenting the results for the bulk Lorentz factor constraints inferred from our optical depth calculations here, it is instructive to first review the results of previous, more primitive bulk motion determinations. As outlined in §1, the earlier work of Krolik & Pier (1991) and Baring (1993), and subsequent papers, considered “blobs” of radiation-emitting material moving at relativistic speeds more or less along the line of sight to an observer. The angular extent of these blobs, i.e., the width of the angular distribution of photons as measured in the observer’s frame, was assumed to be comparable to $1/\Gamma$,
where $\Gamma$ was the Lorentz factor of the blob ($\Gamma \gg 1$). For such relativistically moving blobs, the minimum bulk Lorentz factors at redshift $z$ are obtained (i.e., for $t_{\gamma \gamma} = 1$) from the result derived by Baring & Harding (1993; corrected in Harding 1994):

$$
\Gamma^{1+2z} \gtrsim \left[ \frac{3(3.83)(1+z)^{2-1}}{3\lambda^{5/3}(4/3 + z)^{27/11}} \frac{d_{pc}^2}{\Delta t_{\text{ms}}} \left( \frac{E_{\text{MAX}}}{1 \text{ MeV}} \right)^{r-1} \right] f(1 \text{ MeV}).
$$

(47)

Here $f(1 \text{ MeV})$ is just the source flux evaluated at 1 MeV, per MeV energy interval, and therefore is proportional to $f$. This formula can be reproduced approximately from equation (41) by setting $R_0 \sim c\Delta t$. Alternatively, the thick-shell approximation in equation (42) gives more or less the same estimate if $\Delta R \sim c\Delta t/\Gamma$. It therefore follows that the “blob calculation” corresponds to the boundary between thin- and thick-shell cases where the line-of-sight and transverse (i.e., variability) timescales are comparable. Furthermore, it also coincides with the boundary of narrow beam expansions, namely, for $\Omega_b \sim 1/\Gamma$, as can be established by setting $1 - \zeta = 1 - \beta$ in equation (43) and choosing intermediate-shell thicknesses.

We remark here that the choice of variability timescale for use in the estimation of minimum source bulk Lorentz factors is subjective. The smallest $\Delta t$ observed in the BATSE data (see Fishman et al. 1994) is of the order of milliseconds, and it is quite conceivable that intrinsic source variability occurs on even shorter timescales. Such small $\Delta t$ were adopted in the work of Baring (1993), Baring & Harding (1993, 1995, 1996), and Harding (1994) and lead to light-crossing time size determinations of $c\Delta t \sim 3 \times 10^7$ cm. However, variability in the hard gamma-ray band, i.e., for COMPTEL and EGRET data, can only be inferred conclusively from more severely photon-limited samples on the order of 0.1–1 s timescales. Hence, a conservative approach, adopted, for example, by Ryan et al. (1994); for GRB 930131) and Winkler et al. (1995; for GRB 940217), uses these longer $\Delta t$ values in obtaining pair production constraints on bulk motion in bursts. Note that the relative timescales in the different GRB energy bands may not be related at all to intrinsic source properties but may merely reflect current instrumental limitations in the time domain. While experimentalists might prefer the conservative variability values, theorists are often motivated to attribute the shortest timescales to regions emitting the highest energies of radiation. This is frequently justifiable in astrophysics, since a whole host of cosmic objects are powered from a central region and thereby generate their most energetic photons closer to the center; such photons generally would be expected to couple to shorter timescales. If a central “powerhouse” is indeed responsible for gamma-ray burst emission (as suggested by Fishman, D’Amore, & Nayakshin 1997), then submillisecond timescales might accurately reflect the source conditions appropriate to EGRET photons. In contrast, if bursts approximate more closely the class of fireball models that generate the emission we see by impact on the interstellar medium (e.g., Rees & Mészáros 1992; Mészáros & Rees 1993), perhaps through diffusive acceleration of particles at shocks, then the highest energy photons are produced by particles diffusing on the largest scales of the system, and therefore they might be expected to have longer variability timescales than photons in the BATSE energy range. To accommodate a variety of perspectives, this paper considers $\Delta t$ values of 1 ms and 1 s.

The solutions of equation (47) for the minimum bulk Lorentz factor $\Gamma_{\text{MIN}}$ for the burst parameters of Table 2 are listed in Table 3, for the two different variability timescales and for four different GRB source distances that are typical of Galactic disk, Galactic halo, and nearby and distant cosmological populations (although the redshift $z = 0$ is chosen for simplicity). The immediately obvious conclusion is that, except for a disk origin of bursts, relativistic bulk motion is generally inferred for the EGRET sources, because of the detection of energetic photons. The Lorentz factors obtained for cosmological distance scales far exceed those inferred for extragalactic jets in active galaxies. Conversely, Table 3 indicates that isotropic emission cannot be supported in GRBs unless they are quite local, i.e., well within the Galactic disk, a conclusion that differs from

| GRB | $\Delta t$ | 1 kpc | 100 kpc | 100 Mpc | 1 Gpc | Maximum Distance for Isotropic Emission |
|-----|-------|--------|----------|---------|-------|----------------------------------------|
| 910503 (E) ...... | 1 ms | 2.9 | 16 | 199 | 460 | 49 pc |
| 910601 (C) ...... | 1 ms | 1.0 | 4.5 | 56 | 130 | 1.6 kpc |
| 910601 (C) ...... | 1 s | 1.0 | 4.5 | 56 | 130 | 1.6 kpc |
| 910814 (E) ...... | 1 ms | 3.1 | 12 | 101 | 203 | 24 pc |
| 930131 (E) ...... | 1 ms | 2.8 | 17 | 276 | 694 | 79 pc |
| 940217 (E) ...... | 1 ms | 4.2 | 19 | 195 | 419 | 13 pc |
| 950425 (E) ...... | 1 ms | 1.6 | 11 | 188 | 486 | 297 pc |
| 910503 (E) ...... | 1 s | 1.0 | 4.5 | 56 | 130 | 1.6 kpc |
| 910601 (C) ...... | 1 s | 1.0 | 4.5 | 56 | 130 | 1.6 kpc |
| 910814 (E) ...... | 1 s | 1.1 | 4.4 | 35 | 71 | 0.76 kpc |
| 930131 (E) ...... | 1 s | 1.0 | 4.4 | 69 | 174 | 2.5 kpc |
| 940217 (E) ...... | 1 s | 1.3 | 6.2 | 62 | 133 | 0.43 kpc |
| 950425 (E) ...... | 1 s | 1.0 | 2.6 | 45 | 117 | 9.4 kpc |

Note.—The estimated minimum bulk Lorentz factor $\Gamma_{\text{MIN}}$, from eq. (47), and the maximum source timescale $\Delta t$ for GRBs that permits isotropic emission, for the six GRBs detected by EGRET, assuming that two-photon pair production does not influence the spectrum below the maximum detected energy $E_{\text{MAX}}$. The observed spectral index $z$ below $E_{\text{MAX}}$ as well as the source power-law flux $f(1 \text{ MeV})$ at 1 MeV are given in Table 2 ($E = \text{EGRET data}$ and $C = \text{COMPTEL data}$); the EGRET power-law fits were extrapolated down to 1 MeV where necessary to define the normalization measure $f(1 \text{ MeV})$. At 1 kpc, $\Gamma_{\text{MIN}} = 1$ entries represent $\Gamma_{\text{MIN}} < 1$ solutions to (i.e., the breakdown of) eq. (47). Estimates for each burst are given on the two timescales $\Delta t$ typical of source variability and subpulse duration, as seen by BATSE. Note that source redshifts are set to $z = 0$ in order to avoid a choice of cosmology.
Schmidt’s (1978) early work principally because of the positive EGRET detections in the CGRO era. These estimates for the minimum $\Gamma$ in bursts are a good first guide to constraints on bulk motion in their emission regions; the refinements of the burst geometries addressed in this paper only modify the estimates in Table 3 by factors of at most a few, as will become evident shortly. We remark that previous versions of these estimates (e.g., Baring 1993, 1995; Baring & Harding 1993; Harding 1994) have sometimes used slightly different observational parameters. Note that a number of entries in the 1 kpc column have $\Gamma_{\text{MIN}} = 1$. These actually represent unphysical $\Gamma < 1$ solutions to equation (47) that are obtained only because the assumption $\Gamma > 1$ is used to derive equation (47). Hence, $\Gamma_{\text{MIN}} = 1$ entries denote regimes where this “blob” constraint breaks down and refinements are needed. It must be emphasized that equation (47) and the work of this paper implicitly assume that the GRB spectrum extends above 511 keV in the rest frame of the emission region, so that the phase space above the pair creation threshold is nonzero.

It is appropriate to remark on a caveat to these results. Given emission observed out to $\epsilon_{\text{MAX}}$, the maximum photon energy in the source rest frame is of the order of $\epsilon_{\text{MAX}}/\Gamma$, which must exceed unity in order to be above the pair threshold. Hence, the possibility that intrinsic cutoffs could be present in the GRB spectrum anywhere above $\epsilon_{\text{MAX}}$ implies automatically that $\epsilon_{\text{MAX}}$ provides a potential lower bound to $\Gamma$. In particular, when the estimate for $\Gamma_{\text{MIN}}$ obtained from equation (47) exceeds $\epsilon_{\text{MAX}}$, it is quite possible that pair creation never occurs at all, since opacity for photons of energy $\epsilon_{\text{MAX}}$ occurs through interactions with photons at even higher energies, for which there is no observational evidence. Cases with values of $\Gamma_{\text{MIN}}/\epsilon_{\text{MAX}}$ greater than unity generally arise only for small $\Delta t$ and at cosmological distances (see Table 3; GRB 910601 is a perfect example). In such instances, these cutoff considerations become quite relevant, and it becomes necessary to take $\epsilon_{\text{MAX}}$ as the estimate for the minimum Lorentz factor $\Gamma$. Since values of $\epsilon_{\text{MAX}}$ between 1 and 10 MeV in the collection of EGRET bursts are a marker of fainter or steep spectrum bursts, i.e., probably reflecting the observational limitations of EGRET, it is quite realistic to develop bulk motion estimates based on equation (47), in the belief that many (if not most) bursts emit at energies much higher than 100 MeV.

### 3.2. Generalized Geometries

Generalizing from equation (47) to the expansion geometries considered in this paper, a pair production transparency condition is obtained by setting the optical depth that is obtained from equations (17) and (37) equal to unity, i.e., effectively reading off abscissa values for chosen ordinates from curves like those in Figure 2. Since the expressions for the optical depth and flux from the expansion involve integrals with integrands that depend on the bulk Lorentz factor, the solutions for the $\Gamma_{\text{MIN}}$ roots to $t_\gamma = 1$ must be solved iteratively: we adopt a bisection technique. The minimum bulk Lorentz factors $\Gamma_{\text{MIN}}$ for EGRET burst sources that result from our pair production transparency calculations in equations (17) and (37) are depicted in Figure 3 as functions of the fractional shell thickness $\Delta R/R_0$. This illustration limits shell thicknesses to regimes where $\Delta R/R_0 \leq 1$, since $R_0$ is tied to the time variability via $R_0 = \Gamma \Delta t$. This coupling becomes inappropriate for $\Delta R \gg R_0$ regimes, where $\Delta R \sim c \Delta t$ is a more apt choice. Again, the source parameters from Table 2 are used, and results are presented for large expansion half-angles, $\Theta_\theta = 90^\circ$, variability timescales of $\Delta t = 1$ ms, and source distances of 100 kpc and 1 Gpc that represent Galactic halo and cosmological burst scenarios, respectively. Cosmological redshift modifications, which depend on the choice of cosmology, are neglected for simplicity. The 12 curves in the four panels exhibit similar behavior, with $\Gamma_{\text{MIN}}$ independent of $\Delta R/R_0$ when $\Delta R \ll 1 - \beta$, the so-called thin-shell limit, and $\Gamma_{\text{MIN}}$ declining roughly as $\delta^4$ for $\delta \gg 1 - \beta$ when the filled-sphere regime is realized. These dependences on $\Delta R/R_0$ appear explicitly in the asymptotic forms in equations (41) and (42). Domains where $\Delta R/R_0 > 1$ (i.e., for a true filled sphere) are not depicted since they are unlikely to be encountered in gamma-ray bursts. The values of $\Gamma_{\text{MIN}}$ obtained when $\Delta R/R_0 \sim 1 - \beta$ (i.e., the transition regions) are comparable to those listed in Table 3 for all bursts; this reflects the broad applicability of the constraint in equation (47) that is the hallmark of the so-called “blob” calculation.

The curves in Figure 3 generally concur with the global trends of increasing $\Gamma_{\text{MIN}}$ for higher $\epsilon_{\text{MAX}}$ and/or declining spectral index $\alpha$. Yet the crossover of two curves in the lower left panel exemplifies how the expansion geometry can complicate trends, and generate nonmonotonic behavior in $\epsilon_{\text{MAX}}$ or $\alpha$. For comparison, Figure 4 reproduces the bottom right panel of Figure 3, but for a variability timescale of $\Delta t = 1$ s. The curves resemble those of Figure 3. However, at these cosmological distances, $\Gamma_{\text{MIN}}$ is reduced from the corresponding values in Figure 3 solely because the large $\Delta t$ dilutes the density of internal photons inferred for the source. Again, a crossover of curves arises, indicating that nonmonotonicity of $\Gamma_{\text{MIN}}$ in $\epsilon_{\text{MAX}}$ and $\alpha$ does not belong exclusively to Galactic halo scenarios. Note that the uncertainties in the observational quantities that are listed in Table 2 are as large as around 10%. These lead to uncertainties in the $\Gamma_{\text{MIN}}$ determinations for both these figures of the order of 20%, which are largely masked by the ranges of $\Gamma_{\text{MIN}}$ produced by varying model parameters. Hence, further consideration of experimental uncertainties is omitted from this paper. Note also that the results we have presented have assumed that the test photons in the source start from the rear of the expansion. This maximizes the computed optical depths, implying that it is possible to lower our estimates for somewhat. Permitting test photons to be emitted throughout the source will perhaps lower the mean optical depth by a factor of 2 or so, leading to a reduction of $\Gamma_{\text{MIN}}$ of around 15%. Hence, the detailed consideration of the distribution of test photons is neglected in this paper since this will only have a minor impact on the $\Gamma_{\text{MIN}}$ obtained.

The results presented in Figures 3 and 4 clearly define the behavior of our $\Gamma_{\text{MIN}}$ solutions for large opening angles, but they obviously do not address the entire available phase space for model parameters. Hence, in Figure 5, we depict the variations of $\Gamma_{\text{MIN}}$ with expansion opening angle $\Theta_\theta$ for fixed $\Delta R/R_0$. These curves demonstrate unequivocally how insensitive $\Gamma_{\text{MIN}}$ is to $\Theta_\theta$ when $\Theta_\theta \gtrsim 3 \Delta R/R_0$, i.e., for a range from modestly small $\Theta_\theta$ up to 90°. Note that such a substantial range is realized for the adopted values of $\Delta R/R_0$, chosen to correspond to the transition between thin-shell and thick-shell regimes. This insensitivity, whose important implications for cosmological models are discussed at the end of this subsection, is a principal conclusion of this paper. The values of $\Gamma_{\text{MIN}}$ at the intercept with the right-hand side axis of each panel in Figure 5 are just
Fig. 3.—The minimum bulk Lorentz factor $\Gamma_{\text{MIN}}$ for six EGRET GRBs, as obtained from the pair production condition $\tau_{\gamma\gamma}(\epsilon) = 1$ in eq. (37). Here $\epsilon$ is the maximum energy $\epsilon_{\text{MAX}}$ detected by EGRET; values of $\epsilon_{\text{MAX}}$ and other observational parameters are listed in Table 2 (COMPTEL data are used for GRB 910601, as in Table 3). The top two panels consider the first three EGRET bursts, while the bottom three are for the most significant of more recent events. Results are shown for two different source distances, $d = 100$ kpc (left panels) and $d = 1$ Gpc (right panels), corresponding to Galactic halo and cosmological scenarios, respectively. The expansion opening half-angle is set to be $\Theta_B = 90^\circ$, and the variability timescale is $\Delta t = 1$ ms. Flat portions of the curves define the thin-shell limit $\Delta R/R_0 \ll 1 - \beta$, while the complementary sloping portions correspond to thick-shell expansions.

those obtained by vertically slicing the plots in the bottom two panels of Figure 3 at the appropriate value of $\Delta R/R_0$. For $\Theta_B \ll \Delta R/R_0$, power-law asymptotic behavior of $\Gamma_{\text{MIN}}$ is observed in the figure, with the dependence being easily deduced (for $1 - \beta \ll \Delta R/R_0$) from equation (43): $\Gamma_{\text{MIN}} \propto \Theta_B^{-1}(2\alpha + 3)$. This is a very weak dependence (for $\alpha$ in the range 2–3 typical of EGRET bursts) on $\Theta_B$, generated by the strong variation of the optical depth with $\Gamma$.

It is quite instructive to augment these plots by summarizing the behavior of the optical depth results with phase-space diagrams using well-chosen variables. This can be achieved in an enlightening manner for both observational and theoretical (i.e., model) phase-space parameters via contour plots, i.e., exhibiting curves of constant $\Gamma = \Gamma_{\text{MIN}}$ that satisfy the criterion $\tau_{\gamma\gamma} = 1$ for the pair production optical depth. First of all, we focus on the space of observational parameters given by the maximum energy observed, $\epsilon_{\text{MAX}}$, and the EGRET spectral index $\alpha$, both being listed in Table 2 for EGRET bursts. Fixing the source flux and the source time variability at the “canonical” values of $f(1 \text{ MeV}) = 3 \text{ cm}^{-2} \text{ s}^{-1} \text{ MeV}^{-1}$ and $\Delta t = 1$ ms, respectively, the resulting contour plot is shown in Figure 6. These contours, which represent lower boundaries to regions of opacity (i.e., $\tau_{\gamma\gamma} > 1$), display a number of trends that are hallmarks of the optical depth properties of the relativistically expanding radiation gas.

First, $\epsilon_{\text{MAX}}$ is an increasing function of $\alpha$. This arises because, for the particular values of $f(1 \text{ MeV})$, $\Delta t$ and $\Gamma$ chosen, solutions with $\epsilon_{\text{MAX}} < \Gamma$ are always realized. For such solutions, the test photons at energy $\epsilon_{\text{MAX}}$ interact with photons near the pair production threshold in the CM frame, i.e., with photons of energy around $\epsilon_{\text{MAX}}/\Gamma^2 (< 1 \text{ MeV})$ in the observer’s frame. Since the optical depth is held constant (i.e., unity), and the flux is pinned at 1 MeV, thereby providing a “pivot point”
in the spectrum, increasing \( \alpha \) then raises the number of interacting photons (with energies below 1 MeV) so that, correspondingly, \( \epsilon_{\text{MAX}} \) must be increased to compensate. This trend solely finds its origin in the realization of an \( \epsilon_{\text{MAX}} < \Gamma \) branch of solutions. As \( \alpha \to \infty \), the energy of the interacting photons must approach the pivot point, i.e., 1 MeV. Hence, the \( \epsilon_{\text{MAX}} \) curves asymptotically approach \( \sim \Gamma^2 \) as \( \alpha \) becomes very large, behavior that is conspicuous in the Galactic halo cases in Figure 6. In the particular examples shown, \( \epsilon_{\text{MAX}} \) drops off rapidly as \( \alpha \) approaches unity, a singularity of these curves (e.g., see eq. [37]). The monotonic increase of \( \epsilon_{\text{MAX}} \) with \( \alpha \) can be inverted to yield a declining tendency if an \( \epsilon_{\text{MAX}} > \Gamma \) solution branch can be encountered, so that interacting photons are always above the pivot point at 1 MeV. This occurs when the product \( f(1 \text{ MeV})d^2/\Delta t \) is small enough, i.e., forcing \( \epsilon_{\text{MAX}} \) higher for given \( \Gamma \) and \( \alpha \) (e.g., see eq. [37]). Then the contours would rise rapidly as \( \alpha \) approached unity, but they would still asymptote to a roughly \( \Gamma^2 \) dependence if \( \alpha \to \infty \). This situation is more likely to arise for \( \Delta t = 1 \text{ s} \) variability timescales. Note that the larger values of \( f(1 \text{ MeV})d^2/\Delta t \) in the cosmological cases in Figure 6 yield stronger dependences of \( \epsilon_{\text{MAX}} \) on \( \alpha \), primarily because these cases generally have larger “lever arms” for the interacting photons (at energy \( \sim \epsilon_{\text{MAX}}/\Gamma^2 \)) around the pivot point at 1 MeV.

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**Fig. 4.** The minimum bulk Lorentz factor \( \Gamma_{\text{MIN}} \) for the most recent three of the six EGRET bursts depicted in Fig. 3, but now for \( \Delta t = 1 \text{ s} \) and \( d = 100 \text{ kpc} \), and \( d = 1 \text{ Gpc} \), to illustrate the effect of lengthening \( \Delta t \). Again, the curves represent \( \epsilon_{\text{MAX}} \) solutions to eq. (37), and the source parameters are taken from Table 2. A comparison with the bottom panels of Fig. 3 reveals that the \( \Gamma_{\text{MIN}} \) are reduced significantly for these much longer variability timescales; however, the dependence on \( \Delta t \) is somewhat weak.

**Fig. 5.** The minimum bulk Lorentz factor \( \Gamma_{\text{MIN}} \) for the most recent three of the six EGRET bursts depicted in Fig. 3, but now as a function of the expansion opening angle \( \Theta_0 \). The left panel is for \( d = 100 \text{ kpc} \) and \( \Delta R/R_0 = 10^{-2} \), and the right panel depicts the cosmological case of \( d = 1 \text{ Gpc} \) and \( \Delta R/R_0 = 10^{-4} \); the choices of \( \Delta R/R_0 \) correspond to transitions between the thin- and thick-shell regimes. Again, the curves represent \( \epsilon_{\text{MAX}} \) solutions to eq. (37), and the source parameters are taken from Table 2. The independence of \( \Gamma_{\text{MIN}} \) on opening angle when \( \Theta_0 \ll \Delta R/R_0 \), and a very weak power-law dependence (whose slope depends on a burst’s spectral index \( \alpha \) via eq. [43]) for \( \Theta_0 \ll \Delta R/R_0 \), are clearly evident.
The dependence of $\epsilon_{\text{MAX}}$ on $\Gamma$ is very closely given by the blob calculation in equation (47), namely, $\epsilon_{\text{MAX}} \propto \Gamma^{-(1 + 2a)(\alpha - 1)}$. For example, doubling $\Gamma$ at $\alpha = 2$ in either the Galactic halo case or the cosmological case yields an increase of around 29.4, which is close to the “blob” estimate of 32. Similarly, doubling $\Gamma$ at $\alpha = 3$ in either the Galactic halo case or the cosmological case yields an increase of around 11.2, which is very close to the “blob” estimate of 11.3. For fixed $\alpha$, these amplification ratios are independent of the source distance $d$, since $d$ just forms part of the proportionality constant for the relationship between $\epsilon_{\text{MAX}}$ and $\Gamma$. Another trend that is apparent in Figure 6 is that $\epsilon_{\text{MAX}}$ declines with decreasing $\delta = \Delta R/R_0$. The variation of $\epsilon_{\text{MAX}}$  with $\delta$ is depicted using the thin solid curves for the Galactic halo scenario, with the $\delta = 0.3(1 - \beta)$ and $\delta = 3(1 - \beta)$ cases visually defining a band around the $\delta = 1 - \beta$ case (the behavior for cosmological source distances is similar). When $\delta$ is reduced, the average density of photons within the source increases, pushing the optical depth up. Hence, to compensate, $\epsilon_{\text{MAX}}$ must also decline with $\delta$, so defining the observed trend; in Figures 3 and 4, this effect forces $\Gamma_{\text{MIN}}$ to increase when $\epsilon_{\text{MAX}}$ is held constant. From those figures, it is evident that when $\delta \leq 1 - \beta$, the thin-shell limit produces insensitivity of the optical depth to $\delta$, a feature that is also apparent in Figure 6, for which the $\delta = 0.3(1 - \beta)$ curves are more proximate to the $\delta = 1 - \beta$ ones than are the $\delta = 3(1 - \beta)$ cases. Finally, we note that the value of $f(1 \text{ MeV})$ for each of the EGRET bursts that are depicted as points in Figure 6 differs from the chosen canonical value. Hence, for those sources with higher $f(1 \text{ MeV})$ (GRB 910503 and GRB 910814), the exhibited curves should be slid down somewhat to visualize the situation (i.e., infer bulk Lorentz factors) appropriate for these bursts. Likewise, for the remaining EGRET bursts, the curves should be moved upward to deduce $\Gamma$ values that are consistent with the results depicted in Figures 3 and 4.

The theoretical phase-space contour plot is presented in Figure 7, which exhibits curves of constant $\Gamma$ that satisfy the criterion $\tau_{\text{PP}} = 1$, in the space defined by the opening angle $\Theta_B$ and the fractional shell thickness $\delta = \Delta R/R_0$. We focus on regimes where $\Delta R/R_0 \leq 1$ since, as mentioned above, we have tied $R_0$ to the time variability via $R_0 = \Gamma c \Delta t$, a coupling that becomes inappropriate for $\Delta R \gg R_0$ regimes, where $\Delta R \sim c \Delta t$ is a more apt choice. The large range of $\Theta_B$ is chosen intentionally to present the information relating to the reduction of expansion opening angles that is omitted from earlier considerations, such as in Figures 3 and 4. For all contours, the maximum energy and spectral index were set at $\epsilon_{\text{MAX}} = 100$ MeV and $\alpha = 2.21$, respectively, and the variability timescale $\Delta t$ and the flux $f(1 \text{ MeV})$ at 1 MeV are representative of EGRET-like sources. These parameters were tuned somewhat to obtain a maximum of informational content in the figure, so that other choices of parameters can lead to some variations in contour shape. Remembering the general trend of a reduction in $\tau_{\text{PP}}$ with increasing $\Theta_B$, the contours in Figure 7 clearly represent upper boundaries to regions of opacity (i.e., $\tau_{\text{PP}} > 1$).

A number of prominent features appear in this phase-space plot. Foremost among these are the vertical portions of the $\Gamma = 10$ and $\Gamma = 250$ contours, present when the opening angles are significant. These define regimes where the optical depth is independent of $\Theta_B$, and both thin-shell and thick-shell regimes that are well described by the asymptotic formulae in equations (42) and (43) can be realized. It is precisely this upper region of the $\Theta_B$-$\delta$ diagram that is probed in solutions depicted in Figures 3 and 4. The virtual independence of the optical depth to $\delta$ observed in those solutions for $\Gamma_{\text{MIN}}$ manifests itself in Figure 7 as an extreme sensitivity of the position of any vertical sections of the contours to the choice of $\Gamma$ (or $\epsilon_{\text{MAX}}$ or $\alpha$). This sensitivity therefore produces a low density of contours in the upper left-hand portion of Figure 7, so that contours possessing vertical sections occupy a minority of cases if broad ranges of $\Theta_B$ are considered. Note also that low values of $\Gamma$ are obtained only in the upper right of the figure. However, since values of $\Delta R/R_0$ greater than unity are unrealistic, it becomes clear that for this choice of $\epsilon_{\text{MAX}}$ and $\alpha$, only values of $\Gamma \gtrsim 5$ are attained. This signifies the general property of these calculations that relativistic bulk motions are always inferred unless $\epsilon_{\text{MAX}}$ is not too much greater than 1 MeV.
The narrow beam (i.e., $\Theta_B \ll 0.01$) portion of the parameter space exhibits distinctive power-law dependences, with $\Theta_B$ rising as $\sqrt{\delta}$ when $\delta$ is very small, and declining as $1/\delta$ when $\delta$ exceeds $1 - \beta$. This asymptotic behavior can be deduced with the aid of equations (43) and (44). Table 1 identifies four parameter regimes, three of which are relevant to narrow beam situations, the other being the domain of large $\Theta_B$ discussed in the previous paragraph. Equation (44) is pertinent to the lower left-hand portion of Figure 7, contours of unit optical depth thereby defining the dependence $\Theta_B \propto \delta^{1/2}/\Gamma^{x+1/2}$ (for $R_0 = \Gamma R_c$). In this limit, doubling $\Gamma$ decreases $\Theta_B$ by around a factor of 6.54 for $x = 2.21$, regardless of the assumed distance to the source. The lower right-hand portion of the figure is described by the thick-shell (i.e., $\delta > 1 - \beta$) limit of equation (43), which yields contours with $\Theta_B \propto \delta^{-1}\Gamma^{-(2x+3)}$; doubling $\Gamma$ in this limit reduces $\Theta_B$ by a factor of 343 for $x = 2.21$, behavior that is borne out in Figure 7. The third asymptotic domain is defined by the thin-shell limit of equation (43), yielding contours approaching a limit with $\Theta_B \propto \Gamma^{-(2x+2)}$, independent of $\delta$. This domain is almost attained at the broad peaks of the contours that remain within the narrow beam (i.e., lower) portion of phase space, particularly for the $\Gamma = 40$ and $\Gamma = 1000$ cases in the figure. These three limiting forms can be written explicitly as (for $\Gamma > 1$ and $R_0 = \Gamma R_c$):

$$\Theta_B \propto \begin{cases} \sqrt{\frac{\sigma_1 d^2}{c R_v} h(\alpha)} \frac{3}{2^{x+3}} \Gamma^{(x+1/2)} \epsilon_{\text{MAX}}^{x+1} \delta^{1/2} & \sqrt{1 - \zeta} \ll 1 - \beta \ll \delta, \\ \sqrt{\frac{\sigma_1 d^2}{c R_v} h(\alpha)} \frac{3}{2^{x+3}} \Gamma^{(x+1/2)} \epsilon_{\text{MAX}}^{x+1} \delta^{1} & 1 - \zeta \ll \delta \ll 1 - \beta, \end{cases}$$

$$\Theta_B \propto \begin{cases} \sqrt{\frac{\sigma_1 d^2}{c R_v} h(\alpha)} \frac{3}{2^{x+3}} \Gamma^{(x+1/2)} \epsilon_{\text{MAX}}^{x+1} \delta^{1/2} & 1 - \zeta \ll \delta \ll 1 - \beta. \end{cases}$$

These asymptotic formulae are depicted as thin, light, dotted lines in Figure 7 for the $\Gamma = 40$ Galactic halo case, clearly indicating how the contours closely approach these formulae in the appropriate ranges of $\delta$. The lowest $\Gamma$ examples in Figure 7 for each of the Galactic halo and cosmological scenarios do not realize thin-shell portions of phase space without assuming significant opening angles $\Theta_B$. This feature marks the general property that low-$\Gamma$ curves occupy the upper right corner of the $\Theta_B-\delta$ diagram, a domain where the thin-shell parts of the solutions in Figures 3 and 4 are appropriate. Other trends relevant to the $\Theta_B-\delta$ diagram include a general reduction of $\Theta_B$ with increases in $x$ or decreases in $\epsilon_{\text{MAX}}$.

This concludes the survey of observational and theoretical parameter space. In view of the extensive presentation of results in this section, it is important to highlight the implication of this work that is most salient for gamma-ray bursts. The principal conclusion of our work (expounded in brief in Baring & Harding 1996, with a preliminary version given in Harding & Baring 1994) is that $\Gamma_{\text{MIN}}$ is quite insensitive to the choice of $\Theta_B$ when $\Theta_B \gtrsim 1/\Gamma_{\text{MIN}}$, behavior that can be inferred from Figures 2a and 2b. This result arises because causality restricts the available phase space for pair production interactions more effectively than does the expansion opening angle $\Theta_B$, when $\Theta_B \gtrsim 1/\Gamma_{\text{MIN}}$; it has profound repercussions for gamma-ray burst models. For such source models, the principal advantage (e.g., Krolik & Pier 1991) of restricting $\Theta_B$ to small values like $1/\Gamma$ is a lower (solid-angle-reduced) luminosity at the source for a given observed flux. However, the number of nonrepeating sources must then be a factor $\Theta_B^{-2} \sim \Gamma^2$ higher in order to account for the observed burst rate. In the case of cosmological GRBs, this factor could be as high as $10^6$ for the values of $\Gamma_{\text{MIN}}$ determined here, which is unacceptably large for many models, particularly those that involve neutron star–neutron star or neutron star–black hole mergers (Paczynski 1986; Eichler et al. 1989; Narayan et al. 1991; Mészáros & Rees 1992), failed Type 1b supernovae (Woosley 1993), and rapid spin-down of high-field millisecond pulsars (Usov 1992). This defined the commonly perceived "number problem" for cosmological bursts.
Clearly, in view of the results of our analysis, this problem is a nonissue, since imposition of small opening angles $\Theta_B$ in order to satisfy pair production transparency in EGRET bursts is not necessary. Hence, causality restrictions to the optical depth differ so little between $\Theta_B = 90^\circ$ and $\Theta_B \sim 1/\Gamma$ cases that burst population statistical requirements can be satisfied comfortably without resorting to beamed expansion geometries. Of course, opening up the expansion angle then amplifies the energetics requirements accordingly, so that model development must still meet the needs of acceptable bursting rates and energy budgets. In such considerations, it is evident from the work presented here that pair production constraints will play only a secondary role in determining such model requirements, becoming involved purely through the evaluation of permissible Lorentz factors for bulk motion in gamma-ray bursts.

### 3.3. Discussion

The choice of coupling the scale of the expansion to the variability timescale via transverse dimensions is subjective, although it is widely adopted in applications of bulk relativistic motion in astrophysics. Other choices are possible, such as using burst or subpulse durations (e.g., Fenimore et al. 1997) and/or relating these to longitudinal dimensions in the source. It is fitting to outline the reasons for adhering to our preference. Let us suppose that timescales larger than the variability time $\Delta t$, for example, the burst duration $T_B$, are used as the observational diagnostic of the source size. For uniform expansions, as we have assumed, the geometrical appearance of the “look-back” volume forces the time profile to maintain a well-defined shape $[(t/T_B)^{x-2}$ for spectral index $x = 2]$ that mimics the so named FRED (fast rise, exponential decay) profile (Fenimore et al. 1997). This profile necessarily has a width of the order of $T_B$ under these assumptions, so that its smooth, decaying shape is inconsistent with the vast majority of burst time histories. Therefore, temporal consistency can be attained only if the appropriate observational timescale is of the order of the variability time $\Delta t$, so that the burst comprises a multitude of shells, or perhaps if a single shell is “patchy” in the transverse dimension. The latter possibility still produces time profiles that do not match many burst histories, so that one is compelled to adopt the many-shell proposition, perhaps produced by a central engine, as advocated by Fenimore et al. (1997). The timescale that is then appropriate is $\Delta t$, precisely our choice, although the value of this depends on whether BATSE or EGRET variabilities are used (as discussed in § 3.1).

The issue of whether the variability should be tied to transverse or longitudinal source dimensions remains to be addressed. Since $\Delta t$ is always close to the threshold of temporal resolution of any of the CGRO instruments (BATSE, COMPTEL, and EGRET), it is appropriate to assume that measured variability is actually an upper bound to the source variability. If we opt to relate this to the direction transverse to the line of sight to the observer, then the inequality $R_0/\Gamma \lesssim c \Delta t$ follows, and it is customary to take the equality to specify $R_0$. By the same token, if dimensions along the line of sight are preferred, this inequality is replaced by $R_0 T^2 \lesssim c \Delta t$. A consistent description of the expansion can be obtained only when both of these inequalities are satisfied, which obviously occurs when the more constraining $R_0/\Gamma \lesssim c \Delta t$ is adopted. This motivates our choice of coupling the variability to the transverse dimension; tying it to the line-of-sight direction is insufficiently restrictive. Notwithstanding, the difference between these two choices is merely 1 Lorentz factor in the optical depth (compared with around 5 or 6 imposed by the spectrum; see eq. [43]), to which the estimates of $\Gamma_{\text{MIN}}$ are quite insensitive: opting for $R_0/\Gamma^2 \lesssim c \Delta t$ reduces $\Gamma_{\text{MIN}}$ by factors of the order of 2 or less. Other possibilities for choosing the scale of the expansion exist, such as $\Delta R = c \Delta t$, and these are discussed at length in the temporal analysis of Fenimore et al. (1997). Note that fixing $\Delta R = c \Delta t$ with either $R_0 \sim \Gamma c \Delta t$ or $R_0 \sim \Gamma^2 c \Delta t$ yields $\Delta R/R_0 \ll 1$, comfortably in the phase space covered by Figure 7. The essential point that should be emphasized is that these subjective alternatives probe details of the expansion microstructure that are beyond the purpose of this analysis, and are largely peripheral to it, primarily because of the relative insensitivity of the $\Gamma_{\text{MIN}}$ estimates to these choices. The principal conclusions of this paper, including the insensitivity of the optical depth to the expansion opening angle $\Theta_B$, are guaranteed regardless of such variations on our assumptions.

One question that naturally arises when obtaining estimates for the bulk Lorentz factors via pair production constraints is why the values of 100–1000 obtained here for cosmological bursts are of the same order as those obtained from fireball expansions (e.g., Paczynski 1986; Shemi & Piran 1990; Rees & Mészáros 1992) of enormous initial optical depths. This similarity is no coincidence. The bulk motions attained by the adiabatic expansion phase of pure electron-positron (pair) fireballs yield Lorentz factors $\Gamma$ that saturate at some value corresponding more or less to the “freezout” of pair production, i.e., the epoch of free expansion is approximately marked by the onset of pair production transparency. For cosmological bursts, where the luminosities can be of the order of $L \sim 10^{42} - 10^{49}$ ergs s$^{-1}$ and the energy deposition can be larger than $10^{51}$ ergs, the optical depth is roughly $\sigma_T (R_0, c^2)/\Gamma^3$ (e.g., for $E^{-2}$ spectra) and can be $10^{10}/R_0$ or larger for $\Gamma = 1$, where $R_{10}$ is the size of the fireball at the end of the epoch of opacity in units of $10^{10}$ cm. This optical depth can be reduced to unity by relativistic beaming with $\Gamma$ in the range of 100–1000 when $R_{10} \sim 1$. Since the Lorentz factor attained during the fireball “acceleration phase” scales roughly as its radius (e.g., Paczynski 1986; Piran, Shemi, & Narayan 1993), then it follows that values of a freezout radius of $R_{10} \sim 1$ would correspond to $\Gamma \sim 100$–1000 for fireballs initiated in regions of diameter $10^{10} - 10^{18}$ cm. These $10^{10}$ cm scale lengths for the onset of expansion transparency are comparable to those used for $R_0 = \Gamma c \Delta t$ in this paper, thereby explaining the similarity of our estimates for $\Gamma_{\text{MIN}}$ to the Lorentz factors of fireball-initiated relativistic expansions.

The results we have presented focus on the energy range appropriate to EGRET detections of gamma-ray bursts. There are now ongoing programs for searches of bursts at TeV energies, specifically the target of opportunity monitoring of BATSE localization error boxes by the Whipple air Čerenkov experiment, using the rapid response that is facilitated by the BACODINE alert network (Barthelmy et al. 1995). While these efforts have failed to provide any positive TeV detections so far, probably because Whipple's sensitivity threshold still inhibits any possibility of detection for all but the very brightest of bursts (see Connaughton et al. 1995 for a discussion of the current Whipple sensitivity), the prospect of large field-of-view monitoring of the sky by the air Čerenkov water tank detector MILAGRO (e.g., Yodh 1996) in the very near future promotes the extension of our bulk motion estimates to the TeV energy range. Such considerations also anticipate future space missions.
like GLAST, which will span the 10 MeV–200 GeV range. Obviously, increasing $\epsilon_{\text{MAX}}$ to TeV-type energies would tend to push estimates of the bulk Lorentz factor up, in order to suppress pair creation. To explore the implications of TeV-emitting burst sources for estimates of $\Gamma_{\text{MIN}}$, we computed the infinite power-law “blob” calculation solutions to equation (47) and depicted them in Figure 8, as a function of the spectral index $\alpha_h$. It is sufficient to focus on this simplest of cases, noting that the complicating effects of expansion geometry mirror those considered at lower maximum energies.

In this figure, the cosmological cases exhibited the expected trend of a dramatic increase in $\Gamma_{\text{MIN}}$ for flatter spectra, a consequence of the enhanced supply of interacting photons at $\sim \Gamma^2/\epsilon_{\text{MAX}}$ (generally well above 511 keV) for lower values of $\alpha_h$. For typical EGRET source spectral indices, in the range 2–3, $\Gamma_{\text{MIN}}$ is indeed an increasing function of $\epsilon_{\text{MAX}}$. However, for $\alpha_h < 1$, this behavior is reversed, with $\Gamma_{\text{MIN}}$ declining with $\epsilon_{\text{MAX}}$, because the optical depth in equation (47) is then a decreasing function of $\epsilon_{\text{MAX}}$. Note that in the figure, the Galactic halo case ($d = 100$ kpc) displays a comparative insensitivity of $\Gamma_{\text{MIN}}$ to $\alpha_h$. This insensitivity arises because the energies ($\sim \Gamma^2/\epsilon_{\text{MAX}}$) of the photons interacting with those at $\epsilon_{\text{MAX}}$ are generally relatively near the pivot energy of 511 keV, where the source flux is pinned. In Figure 8, the source flux $f$ at 511 keV is typical of BATSE burst detections; for this flux, the MILAGRO experiment will be sensitive to bursts with $\alpha_h \leq 2.6$. The cosmological redshift was taken to be $z = 0$ for simplicity.

Fig. 8.—Solutions to eq. (47) for the minimum bulk Lorentz factor $\Gamma_{\text{MIN}}$ that guarantees source transparency up to energy $\epsilon_{\text{MAX}}$ for two different source distances $d$ as labeled. Values of $\epsilon_{\text{MAX}}$ are chosen to probe beyond the EGRET energy range up to the domain of air Čerenkov detection techniques. Again, infinite power-law source spectra are assumed. The source flux $f$ at 511 keV is typical of BATSE burst detections; for this flux, the MILAGRO experiment will be sensitive to bursts with $\alpha_h \leq 2.6$. The cosmological redshift was taken to be $z = 0$ for simplicity.
4. CONCLUSION

In this paper, we have presented our extension of pair production transparency calculations in relativistically expanding gamma-ray burst sources to quite general geometries, including shells of finite thickness and arbitrary opening angle. This work includes an extensive analytic reduction of the optical depth from a quintuple integral to a single integral in the special, but quite broadly applicable, case of observing photons only along the axis of the expansion. Such a reduction is extremely expedient for opacity and transparency considerations, providing the reliable and numerically amenable analytic expressions in equations (17) and (37) that completely describe the pair production optical depth. We determine that, for the EGRET sources to be optically thin up to the maximum energies observed, i.e., display no spectral attenuation, the minimum bulk Lorentz factor $\Gamma_{\text{MIN}}$ is only moderately dependent on the shell thickness and virtually independent of its opening solid angle if $\Theta_p \gtrsim 1/\Gamma_{\text{MIN}}$. This insensitivity to $\Theta_p$, which is a consequence of the strong impact that causality has on the available interaction phase space, relieves the commonly perceived number problem for nonrepeating sources at cosmological distances: it is not necessary to invoke small $\Theta_p$ to effect photon escape. This negation of the number problem for a wide range of expansion geometries is the principal conclusion of this paper and an important result for specific cosmological burst models. The values of $\Gamma$ obtained, typically of the order of 10–30 for halo bursts and $\gtrsim 100$ for sources of cosmological origin, depend only moderately on the choice of GRB timescale used to determine the expansion size. Our new limits on required expansion velocity for given source geometries will significantly aid the placing of realistic constraints on gamma-ray burst source models.

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APPENDIX

PROPERTIES OF THE HYPERGEOMETRIC FUNCTION $g_s(z)$

The hypergeometric function appears in the integrand of the expression for the pair production optical depth (eq. [37]):

$$
\mathcal{G}_s(z) \equiv \int_0^1 dq \frac{(1 - q)^{2a}}{(1 - zq)^{1 + z}} = \frac{1}{1 + 2a} F\left(a + 1, 1; 2a + 2; z\right).
$$

This can be represented by the alternative hypergeometric form

$$
\mathcal{G}_s(z) = \left[1 + \frac{1}{2a + 2} \frac{1}{z} \right] F\left[\alpha + 1, 1; 2\alpha + 2; \frac{z}{(z - 1)}\right],
$$

using the transformation formula 9.131.1 of Gradshteyn & Ryzhik (1980).

The argument of $\mathcal{G}_s(z)$ is either $\lambda$ or $\sigma \lambda$, which can be derived from equation (36). It is clear that for $s \geq 1$, $\lambda$ attains values within a range $-\infty < \lambda < \lambda_{\text{max}}$, with $\lambda_{\text{max}} = \beta/(1 + \beta)$ (note that $d\lambda/ds > 0$). Clearly, the condition $\zeta < 1$ renders $\sigma$ less than unity, so that $\sigma \lambda$ is also bounded by the range $[-\infty, \beta/(1 + \beta)]$. It follows that there are two natural ways to evaluate $\mathcal{G}_s(z)$ for the purposes of this paper, both using the series expansion (9.100 of Gradshteyn & Ryzhik 1980):

$$
F(a + 1, 1; 2a + 2; z) = 1 + \frac{\alpha + 1}{2\alpha + 2} z + \frac{(\alpha + 1)(a + 2)}{(2\alpha + 2)(2\alpha + 3)} z^2 + \cdots.
$$

When $|z| < 1$, this series can be used directly with convergence as rapidly as the geometric series $\sum_n (z/2)^n$. When $-\infty < z < -1$, the alternative form for $\mathcal{G}_s(z)$ in equation (A2) can be used, where $1/2 < z/(z - 1) < 1$; the series in equation (A3) with the substitution $z \rightarrow z/(z - 1)$ then also converges like the geometric series $\sum_n [z/(2(z - 1))]^n$, i.e., with the same rapidity. With this scheme, the computation of $\mathcal{G}_s(z)$ to high accuracy is quick.

Note that as $s \rightarrow 1$, $\lambda \rightarrow -\infty$. Using the representation in equation (A2) and also equation (37), it is clear that in this limit $(1 + 2a)\mathcal{G}_s(\lambda)$ approaches $F(a + 1, 1; 2a + 2; 1)/(1 - \lambda)$ and therefore becomes approximately proportional to $s^{-1}$. It follows that the integrand in equation (37) is finite as $s \rightarrow 1$.

Also of use in this paper, specifically in the determination of the optical depth in the limit of filled spherical expansions, is the integral identity

$$
\int_0^1 dz \; z \mathcal{G}_s(z) = \frac{1}{a} - \frac{2}{a - 1} [\psi(2a) - \psi(a) - 1], \quad \psi(x) = \frac{d}{dx} \ln \Gamma(x),
$$

for use in $\beta \approx 1$ situations. This can be established using the integral representation of $\mathcal{G}_s$ in equation (A1), then reversing the order of integration and performing the $z$-integration analytically. An integration by parts then enables the use of identity 3.231.5 of Gradshteyn & Ryzhik (1980), and the result ensues. The finite rational series for $\psi(x + n) - \psi(x)$ was also used in manipulating equation (A4), and for integer $a$, it can be used to obtain rational values for these integrals. For $\beta \ll 1$ cases of
initially filled spherical expansions, the integral
\[
\int_0^1 \frac{dz}{2 - z} \, g(z) = \int_0^{\pi/2} d\theta \, \cos^{2\alpha} \theta \approx \left\{ \frac{1}{1 + 2\alpha} \left[ \frac{3}{2(\pi + 1)} \right]^{9/8} \right\} \tag{A5}
\]
is needed. The identity is established by using the transformation 9.134.1 in Gradshteyn & Ryzhik (1980) for the hypergeometric function in equation (A1), changing to an integration variable of \( z(2 - z)^{-1} \), using the integral representation for general hypergeometric functions in 9.111 of Gradshteyn & Ryzhik (1980), and then reversing the order of integration. The integral, when multiplied by 1 + 2\alpha, is only weakly dependent on \( \alpha \), and the simple approximation obtained in equation (A5) is accurate to better than 1% for 0 < \( \alpha < \frac{1}{2} \) and better than 0.1% for \( \frac{1}{2} < \alpha < 10 \).

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