A REMARK ON HIGHER HOMOTOPY SHEAVES OF DERIVED ARC SPACES

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Abstract. In their work, [4], Gaitsgory and Rozenblyum introduce a derived version of the well-studied arc spaces of classical algebraic geometry. They observe that these derived spaces do not differ from their classical counterparts in the case of smooth schemes. In this note we will see that this is also the case for reduced local complete intersection schemes.

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2. Introduction

Let $X$ be a classical scheme, we will be dealing with the space $X(D)$ of maps from a formal disc $D = \text{spec}(\mathbb{C}[[t]])$ to $X$, we call this the arc space of $X$ and will introduce it and its variants in detail below. This space is obtainable as a tower of simpler truncated arc spaces and this tower is very easy to understand in the case where $X$ is smooth. Each map in the tower is an etale locally trivial affine space bundle. For non-smooth things the situation is substantially subtler, and the study of arcs in this setting has led to some beautiful constructions, perhaps most notably the motivic vanishing cycles of Denef and Loeser, cf. [2]. It is something of a foundational insight of Derived Algebraic Geometry that constructions which are simple for smooth spaces and more complicated for singular ones are often clarified when viewed as derived objects. Gaitsgory and Rozenblyum remark in their work, [4], that the derived versions of arc spaces genuinely can differ from their classical counterparts, this points the way to a potentially interesting avenue of study as it implies that every classical arc space is endowed with a highly canonical
family of quasi-coherent sheaves on it, namely the higher homotopy sheaves of the derived version of the arc space. It had originally been our hope that in the example of a singular hypersurface \( \{ f = 0 \} \) inside a smooth scheme \( U \) we could relate these sheaves to interesting invariants of the singularities of \( f \), e.g. to vanishing cycles cohomology. We will see that this is not the case. In fact we will prove that the derived arc spaces do not differ from their classical counterparts in quite substantial generality. We view this as ultimately disappointing. The proof will proceed by constructing explicit (cofibrant) models for the algebras of functions on the derived spaces in the tower of truncated arcs and using this to define a sequence of degenerations of each of the maps in the tower. Eventually we will prove the following:

**Theorem 2.1.** If \( X \) is a reduced local complete intersection scheme then the inclusion of the classical arcs into derived arcs,

\[
X(\mathcal{D})^{ct} \hookrightarrow X(\mathcal{D}),
\]

is an equivalence.

3. **BASICS**

3.1. **Arcs.** We recall here the basics of the theory of arc spaces. For a more thorough introduction we refer the reader to [2]. \( X \) will be a scheme below. The spaces we define will be defined on the pre-sheaf level and \( A \) will denote an arbitrary test (\( \mathbb{C} \)-) algebra.

**Definition 3.1.**

1. The \( n \)-truncated arc space of \( X \), denoted \( X(\mathcal{D}_n) \), is defined by

\[
X(\mathcal{D}_n)(A) = X(A[t]/t^{n+1}).
\]

2. The formal arc space, denoted \( X(\mathcal{D}_\infty) \) is defined as the pro-limit of the truncated arc spaces, where the limit is induced by the natural maps \( A[t]/t^{n+1} \to A[t]/t^n \).

3. The arc space of \( X \), \( X(\mathcal{D}) \), is defined to have \( A \)-points \( X(A[[t]]) \).

**Remark.** The arc space \( X(\mathcal{D}) \) is endowed with natural maps to all the truncated arc spaces, and thus by definition to the formal arc space. It is a difficult result of Bhatt ([1]) that this map is an isomorphism if \( X \) is assumed to be quasi-compact and quasi-separated. Note that this is obvious in the case that \( X \) is affine.

. We will quickly summarise the representability properties of these pre-sheaves.
Lemma 3.1.  

(1) If \( X \) is a scheme (resp. affine scheme), then the spaces of truncated arcs are schemes (resp. affine schemes).

(2) The same holds true for \( X(D_\infty) \).

(3) If \( X \) is affine then the arc space \( X(D) \) is an affine scheme.

Proof. Cf. \( [2] \), in all cases one simply observes it for the affine line and uses the appropriate compatabilities with (arbitrary) limits and (Zariski) colimits. \( \square \)

Remark. Note that according to the result of Bhatt mentioned above, for \( X \) qcqs it is in fact the case that \( X(D) \) is representable even for \( X \) non-affine.

Example.  

• For arbitrary \( X \), the space \( X(D_0) \) is \( X \) itself.

• For arbitrary \( X \), the space \( X(D_1) \) is the geometric tangent bundle of \( X \), i.e. the total space of the cotangent sheaf.

• The arc space of the affine line is an infinite dimensional affine space, \( \mathbb{A}_\infty = \text{spec}(\mathbb{C}[x_0, x_1, x_2, ...]) \).

3.2. Recollections on Derived Geometry. There are numerous good introductions to derived geometry, let us mention as examples \( [6] \), \( [5] \) and \( [4] \). We choose to use the language of pre-stacks as it is developed in \( [4] \). We find this convenient as it allows us to define the objects of interest to us at the level of their functors of points (valued in \( \infty \)-groupoids.) The following definition is really just the fixing of some notation, the reader familiar with D.A.G. can skip and refer back to it.

Definition 3.2.  

• We denote the \( \infty \)-category of derived algebras \( d\text{Alg}_\mathbb{C} \), its elements are commutative differential graded algebras concentrated in non-positive degree. For non-negative \( i \) we write \( \pi_i A \) for the \( -i^{th} \) cohomology group of a derived algebra \( A \). If these vanish for strictly positive \( i \) then we refer to \( A \) as classical.

• The category of pre-stacks is the \( \infty \)-category of functors from derived algebras to the \( \infty \)-category of spaces, \( \text{Fun}(d\text{Alg}_\mathbb{C}, \text{sSet}) \).

• Given a derived algebra \( A \), we denote by \( \text{Mod}_A \) the stable \( \infty \)-category of modules for \( A \) and by \( \text{Perf}_A \) those which are perfect. Left Kan extension extends both of these notions to an arbitrary pre-stack, \( \mathcal{X} \), we denote the resulting categories \( QC(\mathcal{X}) \) and \( \text{Perf}_\mathcal{X} \).
• If $\mathcal{X}$ is a pre-stack then we define its classical truncation, denoted $\mathcal{X}^{\text{cl}}$, is defined as the right Kan extension (to all of $\text{dAlg}_C$) of the restriction of $\mathcal{X}$ to classical algebras.

• The pre-stack represented by a derived algebra $A$ will be denoted $\text{spec}(A)$ and pre-stacks locally of this form (cf. [5], chapter 7 for a precise definition) will be called derived schemes.

*Remark.* The pre-stacks we deal with throught will all be representable by derived schemes. In the case of a derived scheme locally of the form $\text{spec}(A)$, the classical truncation is locally of the form $\text{spec}(\pi_0 A)$ as one would certainly hope.

There is one additional recollection we require, the proofs below will make use of explicit models for the algebras of functions on derived arc spaces and it is crucial that we be able to work explicitly with them. To this end we make the following remark;

*Remark.* The $\infty$-category of derived algebras is obtainable as the localisation of a model category, which we denote $\text{CDGA}_C$. The elements are non-positively graded commutative differential graded algebras and the weak equivalences are quasi-isomorphisms. Most importantly, maps for which the underlying map of graded algebras is free are all cofibrant, and indeed generate the class of such maps.

## 4. DERIVED ARCS

We are now in a position to mix the objects described in the above sub-sections. Henceforth whenever we mention the arc spaces above we shall use a superscript $^{\text{cl}}$ so as to emphasise that their definition is in terms of classical algebraic geometry. $X$ will be a derived scheme in what follows;

**Definition 4.1.** (1) The pre-stack of $n$-truncated arcs, denoted $X(D_n)$, is defined by $X(D_n)(A) = X(A[t]/t^{n+1})$.

(2) The formal arc space, denoted $X(D_{\infty})$ is defined as the pro-limit of the truncated arc spaces, where the limit is induced by the natural maps $A[t]/t^{n+1} \to A[t]/t^n$.

(3) The arc space of $X$, $X(D)$, is defined to have $A$-points $X(A[[t]])$.

This is of course a carbon copy of the definition in the classical case. We have, as was first observed by Gaitsgory and Rozenblyum, an analogue of the representability results above.
Lemma 4.1.  
(1) If $X$ is a derived scheme (resp. derived affine scheme), then the spaces of truncated arcs are derived schemes (resp. derived affine schemes).

(2) The same holds true for $X(D_{\infty})$.

(3) If $X$ is derived affine then the arc space $X(D)$ is an affine scheme.

Proof. Cf. [GR], it is not fundamentally different from the proof in the classical case. \hfill $\Box$

Remark. We will be particularly interested in the case where $X$ is taken to be a classical scheme. In the case where $X$ os further assumed to be quasi-compact and quasi-separated then the result of Bhatt mentioned above implies once again that the space of (derived) arcs, $X(D)$ is representable by a derived scheme.

We have the following lemma due to [GR];

Lemma 4.2. Let $X$ be a smooth classical scheme, then the spaces $X(D_n)$ and $X(D)$ are classical.

Proof. We reproduce the proof of [4], although the arguments we present below for our main result give an independent proof. It suffices to prove the result for $X$ affine and for the spaces $X(D_n)$ for all $n$. We assume $X$ is given as the zeroes of a smooth map $f : \mathbb{A}^p \to \mathbb{A}^q$. Formation of truncated arc spaces commutes with the formation of limits so $X(D_n)$ is obtained as the zeroes of the induced map $f(D_n)$. The infinitessimal lifting criterion for smoothness implies this map is also smooth. It is finitely presented and thus in fact flat, according to a standard piece of commutative algebra. It follows that there are no tors and the classical fibre product computing $X(D_n)^{cl}$ also computes the derived space $X(D_n)$. \hfill $\Box$

Remark. Gaitsgory and Rozenblyum note further that for non-smooth spaces this equivalence need not hold, and point out that if $X$ is taken to be $\text{spec}(\mathbb{C}[z]/z^2)$ then the derived arc space of $X$ is a non-trivial derived thickening of its classical loop space. They observe further that even for singular spaces the derived thickening can be trivial, for example for an ordinary double-point ($xy = z^2$). In fact they generalise this to the case of the nilpotent cone $\mathfrak{N}$ inside a classical Lie algebra $\mathfrak{g}$, the ordinary double point being the special case of $\mathfrak{sl}_2$. According to a theorem of Kostant, the nilpotent cone is a reduced complete intersection, we take this as our starting point.
5. THE MAIN RESULTS

5.1. Explicit Cofibrant Models. Let $X = \text{spec}(A)$ be an affine derived scheme and assume $A$ is given as a cofibrant element of $\text{CDGA}_C$. We may assume that it is of the form

$$\mathbb{C}\left[x^\lambda \mid \partial_A(x^\lambda) = f_\lambda(x^{\mu})\right],$$

where the $\lambda \in \Lambda$ form an indexing set, $\partial_A$ denotes the differential and $x^\mu$ denotes a multi-variable. We wish to give a description of the algebra of functions on $X(D)$ in these terms, in particular we want to produce a cofibrant model for functions on $X(D)$. We will refer to this algebra as $A(D)$, so that we will have $\text{spec}(A(D)) = X(D)$.

What follows is as much a construction as a definition, we will show below that the algebras constructed in the following definition are indeed models for the algebras on the relevant arc spaces.

**Definition 5.1.**

- $A(D)$ will be freely generated by elements $x_i^\lambda$ for $\lambda \in \Lambda$ and for $i \geq 0$. The differential $\partial_{A(D)}$ is best described via a generating function and to this end we introduce the formal sum, $x^\lambda(t) = \sum x_i^\lambda t^i$. We now define $\partial_{A(D)}$ via the generating function $\partial_{A(D)}(x^\lambda(t)) = f_\lambda(x^\mu(t))$.

- Restricting to those variables $x_i^\lambda$ with $i$ at most $n$ defines an algebra we will denote $A(D_n)$.

- Setting $x_i^\lambda$ to be of degree $i$ defines a grading on $A(D)$ which we refer to as the grading by conformal weight. It is in fact induced by the rotation action of $\mathbb{G}_m$ on $D$.

We now have the following simple lemma, hinted at above:

**Lemma 5.1.** With notation as above, the algebra $A(D)$ is a model for the algebra of functions on the arc space $X(D)$.

**Proof.** If $A \rightarrow B$ is a cofibration in $\text{CDGA}_C$ then it is easily seen that so too is the induced map $A(D) \rightarrow B(D)$. A cofibrant algebra is a (possibly transfinitely) iterated coproduct of symmetric algebras of derived vector spaces, for which the result is clear.

The result now follows since formation of arc spaces commutes with arbitrary homotopy
limits, and the assignment, \( A \mapsto A(D) \) preserves homotopy colimits, as it preserves classical colimits and cofibrations.

\[ \square \]

**Example.** Set \( X = \text{spec}(\mathbb{C}[z]/z^2) \). A cofibrant model for \( \mathcal{O}(X) \) can be taken to be \( A = \mathbb{C}[x, \zeta \mid \partial(\zeta) = x^2] \). Then \( A(D) \) has
\[
\partial(\zeta_0) = x_0^2, \quad \partial(\zeta_1) = 2x_0x_1, \quad \partial(\zeta_2) = 2x_0x_2 + x_1^2, \quad \&c.
\]

Let us note that the class \( \eta = 2x_1\zeta_0 - x_0\zeta_1 \) is a non-zero element of \( \pi_1A(D) \), and so this genuinely differs from its classical counterpart, as mentioned above. If we now write \( X_n = \text{spec}(\mathbb{C}[z]/z^n) \) we can consider \( \mathcal{O}(X_n(D)) \). This comes with two gradings, one from conformal weight and one from the \( \mathbb{G}_m \)-action on the space \( X_n \). We can compute the bi-graded Euler characteristic of \( \mathcal{O}(X_n(D)) \) as follows, where we write \( q \) for the conformal weight variable and \( z \) for the internal weight one. Then we have
\[
\chi(\mathcal{O}(X_n(D))) = (-z^n; q)_\infty(z; q)_\infty^{-1}.
\]
Here we have written \( (z; q)_\infty = \prod(1 - q^i z) \) as is standard in \( q \)-series literature. We remark that this is a hugely more pleasant answer than one would get by computing the bi-graded dimension of the algebra of functions on the space of classical arcs into \( X_n \).

### 5.2. Weak Smoothness

We require a simple definition before stating our criterion for classicality of derived arc spaces. Before we state it we remind the reader that for \( X \) a scheme there is an object \( \mathbb{L}_X \in QC(X) \) called the cotangent complex. Its 0th homotopy sheaf is the cotangent sheaf \( \Omega^1_X \) and in the case of a smooth scheme \( X \) the two things agree. Inspired by this we define:

**Definition 5.2.** We say a scheme \( X \) is weakly smooth if its cotangent complex has no higher homotopy groups, i.e. if there is an isomorphism \( \mathbb{L}_X = \Omega^1_X[0] \) inside \( QC(X) \).

We may then state the main result of this sub-section:

**Theorem 5.2.** If \( X \) is a classical scheme, then the derived scheme of arcs, \( X(D) \), is classical iff \( X \) is weakly smooth.

**Proof.** We may assume \( X = \text{spec}(A) \) is affine, once again we will assume given a cofibrant model for \( A \) of the form
\[
\mathbb{C}[x^\lambda \mid \partial_A(x^\lambda) = f_A(x^\mu)],
\]
and write \( A(D) \) for the associated cofibrant model for \( \mathcal{O}(X(D)) \).

Let us first assume that \( X(D) \) is classical. As mentioned above this means that \( A(D) \)
has no higher homotopy groups. \( A(D) \) is graded by conformal weight \( q \) and this of course descends to the homotopy groups. There is a sub-complex \( V \) of \( A(D) \) consisting of elements of conformal weight 1. This is simply the cotangent complex \( \mathbb{L}_A \), and thus we have deduced weak smoothness of \( A \).

We now focus on the converse. We will show that each space \( X(D) \) is classical, ie that for all \( n \), and \( i > 0 \), we have a vanishing \( \pi_i A(D) = 0 \). Below we will introduce an increasing filtration, \( \mathcal{F}_n^< \), counting weight in the top conformal weight variables \( x_\lambda^i \). Examining the generating function description for the differential \( \partial A(D) \) we see that we have

\[
\partial A(D)(x_\lambda^i) = \sum_{\mu} (\partial_{\mu} f_\lambda)x_\mu^i + \mathcal{O}(< n),
\]

where \( f_\lambda \) is meant to be understood as a polynomial in the weight 0 variables and \( \mathcal{O}(< n) \) denotes a sum of monomials containing no conformal weight \( n \) variables. Now we can define \( \mathcal{F}_n^< \) by letting \( \mathcal{F}_n^< A(D) \) be spanned by monomials of weight at most \( i \) in the conformal weight \( n \) generators. The formula for \( \partial A(D) \) above shows that this respects the differential. Further, it immediately implies that we have

\[
Gr_{\mathcal{F}_n} A(D) \cong sym\ A(D)(\mathbb{L}_A \otimes A(D)).
\]

We have a convergent \( E_1 \) spectral sequence:

\[
\pi_\ast sym\ A(D)(\mathbb{L}_A \otimes A(D)) = \pi_\ast (Gr_{\mathcal{F}_n} A(D)) \implies Gr(\pi_\ast A(D)).
\]

Weak smoothness now allows us to prove by induction on \( n \) that all the algebras \( A(D) \) are classical.

**Remark.** Geometrically (according to the Rees construction) we are constructing a derived \( \mathbb{A}^1 \)-family, \( \mathcal{X} \rightarrow \mathbb{A}^1 \), with generic fibres \( \mathcal{X}_\eta = X(D(\eta)) \) and central fibre

\[
Tot_{X(D)}(X(D) \otimes X \mathbb{L}_X).
\]

### 5.3. Classicality for lci schemes.

We now prove that the derived arc spaces of a reduced locally complete intersection inside a smooth scheme (henceforth an lci scheme) are classical. This is standard commutative algebra given the above characterisation in terms of weak smoothness.

**Lemma 5.3.** If \( X \) is an lci scheme, then it is weakly smooth.
Proof. After an etale localisation we can assume that $X = \text{spec}(R)$ is a complete intersection inside an affine space $\mathbb{A}^d$, with ideal sheaf $I$ being cut out by equations $(f_1, \ldots, f_c)$. Being a complete intersection means that

$$\mathbb{C}[x_1, \ldots, x_d, \zeta_1, \ldots, \zeta_c \mid \partial(\zeta_i) = f_i] \to R$$

is a cofibrant resolution. From this we see that the cotangent complex, $\mathbb{L}_R$ is computed as

$$\text{Jac}(f_1, \ldots, f_c) : R^\oplus c \to R^\oplus d.$$ 

This identifies with the map

$$I/I^2 \to \Omega^1(\mathbb{A}^d) \otimes_{\mathbb{A}^d} X$$

coming from the conormal sequence. As explained in Ex 17.2 of Eisenbud’s book [3], this conormal sequence is exact on the left in the case of a reduced complete intersection and so we deduce that $\pi_1 \mathbb{L}_R = 0$. Noting that $\pi_{>1}$ manifestly vanishes we have proven weak smoothness of $X$. $\square$

Finally, we deduce the main theorem, which we restate here:

**Theorem 5.4.** If $X$ is a reduced local complete intersection scheme then the inclusion of the classical arcs into derived arcs,

$$X(D)^{cl} \hookrightarrow X(D),$$

is an equivalence.

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