Magnetoelectric classification of skyrmions

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We develop a general theory to classify magnetic skyrmions and related spin textures in terms of their magnetoelectric multipoles. Since magnetic skyrmions are now established in insulating materials, where the magnetoelectric multipoles govern the linear magnetoelectric response, our classification provides a recipe for manipulating the magnetic properties of skyrmions using applied electric fields. We apply our formalism to skyrmions and anti-skyrmions of different helicities, as well as to magnetic bimerons, which are topologically, but not geometrically, equivalent to skyrmions. We show that the non-zero components of the magnetoelectric multipole and magnetoelectric response tensors are uniquely determined by the topology, helicity and geometry of the spin texture. Therefore, we propose straightforward linear magnetoelectric response measurements as an alternative to Lorentz microscopy for characterizing insulating skyrmionic textures.

The concept of skyrmions, originally invoked by Skyrme to describe the stability of hadrons in particle physics more than half a century ago [1,2], has found fertile ground in condensed matter systems as diverse as liquid crystals [3], Bose-Einstein condensates [4], quantum Hall systems [5], and helimagnets [6-10]. The magnetic skyrmions that form in the latter are metastable, topologically protected, nanometer-sized, swirling spin textures, with potential application as data bits in future high-density data storage devices [11,13].

Magnetic skyrmions have been found in bulk chiral and polar magnets [8-10,14-20], thin film heterostructures [12,13,21], and multilayer nanostructures, [22-28]. Their radially symmetric spin texture is described by a local magnetization vector \( \hat{n} \), which characterizes the radial profile and the twisting angle respectively, while the skyrmion number, \( N_{sk} \), characterizes the topology of the spin texture and manifests in the various exotic topological transport properties [29-31]. For example, the topological invariants \( N_{sk} = \pm 1 \) represent the skyrmion and the anti-skyrmion respectively (see Figs. 1(a) and (b)).

In addition to their well-explored topological order, the lack of both space-inversion \( \mathcal{I} \) and time-reversal \( \tau \) symmetries in skyrmion-like spin textures makes them potential hosts for magnetoelectric (ME) multipoles [32,33], formally defined as \( M_{ij} = \int \overline{r} \overline{r} \mu_{ij}(\overline{r}) d^3r \), with \( \overline{\mu}(\overline{r}) \) being the magnetization density. The three irreducible (IR) components of the \( M_{ij} \) tensor, the ME monopole \( (\alpha) \), toroidal moment \( (\tilde{f}) \), and the ME quadrupole moment \( q_{ij} \), are quintessential to the linear ME response \( \alpha_{ij} \), which is the generation of magnetization (polarization) by an applied electric (magnetic) field. They have also been associated with other exciting properties and phases of matter, including hidden ferrotoroidic order [34,36], current induced Néel vector switching in antiferromagnetic spintronics [37,38], and even with axionic dark matter [39]. The recent observation [14,20] of skyrmions in insulators opens the door to combined electric and magnetic field manipulation of the skyrmions, mediated via these ME multipoles (MEMs).

In spite of this intriguing connection, to our knowledge, the only link between MEMs and skyrmions mentioned to date, is to their toroidization \( \mathcal{T}_z \) (toroidal moment per unit volume, \( \tilde{f}/V \)) [40]. Here we present a complete ME classification of skyrmions and anti-skyrmions with different helicities, \( \gamma \), (see Fig. 1) as well as magnetic bimerons [see Fig. 1(d)] taking all components of the magnetoelectric multipolization (that is the MEM per unit volume) into account. We find that the ME multipolization \( \mathcal{A} \) (ME monopole per unit volume, \( a/V \)) and the quadrupolization \( \mathcal{Q}_{ij} \) (quadrupole moment per unit volume, \( q_{ij}/V \)) can be non-zero, in addition to the toroidization \( \tilde{f} \), with the form of the MEM tensor depending on both the topology and geometry of the spin texture. These distinct MEMs in skyrmions, anti-skyrmions, and bimerons, further, manifest in the corresponding ME polarizability \( \alpha_{ij} \), implying that the skyrmion type can be determined from a straightforward magnetoelectric measurement, and pointing to combined electric- and magnetic-field control of skyrmions.

The main outcomes of the classification are given in...
Table I. ME classification of skyrmions, anti-skyrmions and bimerons. Only independent multipolization components are listed.

| Properties          | Skyrmions | Anti-skyrmions | Bimerons |
|---------------------|-----------|----------------|----------|
| ME multipolization  | $\mathcal{M}_{ij}/V$, i.e., the MEM moment per unit volume $V$, describes the first order asymmetry in the magnetization density $\vec{m}(\vec{r})$ that couples to derivatives of the magnetic field [33, 34]. Hereafter, we only consider the spin part of the magnetic moment. As stated earlier, the $\mathcal{M}_{ij}$ tensor has three IR components [33]: (a) the scalar ME monopole $a = \frac{1}{2} \mathcal{M}_{ii} = \int \vec{r} \cdot \vec{m}(\vec{r}) d^3r$, (b) the ME toroidal moment $t_i = \frac{1}{2} \varepsilon_{ijk} \mathcal{M}_{jk} = \frac{1}{2} \int \vec{r} \times \vec{m}(\vec{r}) d^3r$, and (c) the symmetric traceless five component quadrupole moment $q_{ij} = \frac{1}{2} \mathcal{M}_{ij} + \mathcal{M}_{ji} - \frac{1}{2} \delta_{ij} \mathcal{M}_{kk} = \frac{1}{2} \int (r_i \vec{r}_j + r_j \vec{r}_i - \frac{1}{2} \delta_{ij} r^2 \vec{m} \cdot \vec{m}) d^3r$. The ME monopole and the $q_{x2-y2}$ and $q_{z2}$ quadrupole moment components form the diagonal of the $\mathcal{M}_{ij}$ tensor, while the symmetric and the anti-symmetric parts of the off-diagonal elements are represented by the $q_{xy}$ and $q_{yz}$ quadrupole moments and $\vec{r}$ respectively.

We now point out some interesting features of the $\mathcal{M}_{ij}$ tensor based on the reduced lattice dimension from $3D \rightarrow 2D$, and the spin geometry. First, a 2D in-plane lattice directly implies vanishing $\mathcal{M}_{zi} = \int z \mu_i d^2r$ components, and the corresponding multipolization (which in this case is multipole moment per unit area $S$, $\mathcal{M}_{ij} = \mathcal{M}_{ij}/S$). Here $i = x, y$, and $z$ are the Cartesian components of $\vec{m}$. Moreover, the radially symmetric $\mu_z$ spin component at each lattice site of a skyrmion crystal forces the $\mathcal{M}_{iz} = \int r_i \mu_z d^2r$ components to vanish. The resulting $\mathcal{M}_{ij}$ tensor can therefore be written as a 2 × 2 matrix of non-zero components [see Fig. 2(a)]. In contrast, the absence of $\mu_z$ radial symmetry in a bimeron texture means that the $\mathcal{M}_{ij}$ tensor does not reduce to a 2 × 2 matrix. The different dimensionality of their $\mathcal{M}_{ij}$ tensors emphasizes that although topologically equivalent, skyrmions and bimerons have distinct ME responses originating from geometrical differences. Secondly, in a 2D lattice, the ME monopole $a$ and the quadrupole $q_{ij}$ are equal and opposite to each other: $a = \frac{1}{3} \int d^2r (x \mu_x + y \mu_y + z \mu_z) = \frac{1}{3} \int d^2r (x \mu_x + y \mu_y + z \mu_z - \frac{1}{3} (x \mu_x + y \mu_y)) = -\frac{1}{3} \int d^2r (x \mu_x + y \mu_y) = -a$. This means that in a 2D lattice $q_{z2}$ ($Q_{z2}$) is the same as an anti-$\alpha$ (anti-A) and vice versa (see Figs. 2(b) and (c)).

**Results and discussion.** We begin by considering a tight-binding model for an electron in a square lattice of spin texture $\hat{n}_i$ [15].

$$H = t \sum_{\langle i,j \rangle} c_{i}^\dagger c_{j} - J_{H} \sum_{i} \hat{n}_{i} \cdot (c_{i}^\dagger \sigma c_{i}). \quad (1)$$

Here $t$ and $J_{H}$ are the nearest-neighbor hopping and Hund’s coupling respectively. For a skyrmionic texture, $\hat{n}_i \equiv (\sin \theta_i(r_i) \cos \phi_i(r_i), \sin \theta_i(r_i) \sin \phi_i(r_i), \cos \theta_i(r_i))$ with $\theta_i = \pi (1 - r_i/\lambda)$ and $\phi_i = \phi_{m} + \gamma$. Here, the vorticity $m = \pm 1$ corresponds to a skyrmionic and an anti-skyrmionic state respectively while the helicity $\gamma$ can take different values; e.g., $\gamma = 0$ and $\pi/2$ correspond to Bloch and Néel skyrmions respectively [see Figs. 1(a) and (c)].

The bimeron configuration, also known as in-plane magnetized version of a skyrmion [see Fig. 1(d)], is obtained from the skyrmion texture via spin rotation by $\pi/2$ around the $y$ axis, $(\hat{n}_x, \hat{n}_y, \hat{n}_z) \rightarrow (\hat{n}_z, \hat{n}_y, -\hat{n}_x)$, losing thereby the radial symmetry of the spin-$z$ component of a skyrmion while keeping the topology intact [16, 17].

In the adiabatic limit ($J_{H} \gg t$), the low lying bands of the tight-binding model, Eq. 1, can be approximated as [15]

$$H = \sum_{\langle i,j \rangle} t_{ij}^{eff} \hat{d}_{i} \hat{d}_{j}, \quad (2)$$

where $\hat{d}_{i}, \hat{d}_{j}^\dagger$ are spin-less operators, and $t_{ij}^{eff} =}$
Energy/t

FIG. 2. (a) Schematics showing the evolution of the $\mathcal{M}_{ij}$ tensor as the structural dimension reduces from 3D to 2D, and finally for the specific radially symmetric skyrmion spin texture. Representative spin magnetic moment arrangements (shown in arrows) for a ME monopole $a$ and the quadrupole moment $q_{2z}$ in the (b) $x$-$y$ plane and (c) along $z$, showing that $a$ and $q_{2z}$ in (b) are exactly equal and opposite in the absence of a local $z$ coordinate.

$t \cos(\theta_{ij}/2)e^{ia_{ij}}$ is an effective hopping that depends on the twisting angle difference ($\phi_i - \phi_j$). We first analyze the band structures of 2D periodic crystals of skyrmions, bimerons, and antiskyrmions, computed from the tight-binding model, Eq. (1). The low-lying bands for $J_H/t = 10$, shown in Fig. 2(a), are well described by the adiabatic limit Hamiltonian of Eq. (2) and are identical for the three topological spin textures for a given set of parameters. This is because the effective hopping $t_{ij}^{\text{eff}}$ depends only on the difference in $\phi$, and therefore, remains the same for any constant rotation of the spins. Since Néel and Bloch skyrmions differ only in helicity $\gamma$, and bimerons by a $\pi/2$ rotation around $y$-axis, they therefore have the same $t_{ij}^{\text{eff}}$. For anti-skyrmions, $t_{ij}^{\text{eff}}$ has the opposite phase leading to opposite topological order while keeping the band energies unaltered.

While the bandstructure is insensitive to rotations of the spins, they manifest in the corresponding spin multipolization $\mathcal{M}_{ij}$, which can be computed as the Brillouin zone (BZ) integration over all the occupied states $n$, of $O_{ij}(\vec{k})$, [11][19]:

$$\mathcal{M}_{ij} = -g \mu_B \int_{\text{occ}} \frac{d^2k}{(2\pi)^2} \sum_n O_{ij}^n(\vec{k})$$

where

$$O_{ij}^n(\vec{k}) = \sum_{m \neq n} (\varepsilon_n + \varepsilon_m - 2\varepsilon_k) \text{Im} \left[ \frac{\langle n|v_i|m\rangle\langle m|s_j|n\rangle}{(\varepsilon_n - \varepsilon_m)^2} \right]$$

Here $v_i$ and $s_j$ are the velocity and the Pauli spin operators respectively.

We compute the $\mathcal{M}_{ij}$ tensor for $\gamma$ values ranging from $-\pi$ to $\pi$, assuming that only the lowest band in Fig. 3(a) is occupied. We start with the case of the skyrmion crystal, and discuss pure Bloch- and Néel- type skyrmions first, before analyzing intermediate $\gamma$ values. Our calculations show that the Bloch skyrmions have only non-zero off-diagonal elements, $\mathcal{M}_{xy} = -\mathcal{M}_{yx}$, indicating the presence of only the toroidization $T_z$, consistent with Ref. 10. We note that a pure $T_z$ is unusual [33][34][11][33]; while recently we demonstrated a pure toroidal moment in the reciprocal space of PbTiO$_3$ [50], to the best of our knowledge this is the first prediction of a pure toroidal moment in a real-space spin texture. In complete contrast, Néel skyrmions have only non-zero diagonal elements, $\mathcal{M}_{xx} = \mathcal{M}_{yy}$. Since the ME monopolization $A$, and quadrupolarization $Q_{x^2-y^2}$ and $Q_{z^2}$ contribute to the diagonal elements as

$$\mathcal{M}_{xx} = A + \frac{1}{2}(Q_{x^2-y^2} - Q_{z^2})$$
$$\mathcal{M}_{yy} = A - \frac{1}{2}Q_{x^2-y^2} - \frac{1}{2}Q_{z^2}$$
$$\mathcal{M}_{zz} = A + Q_{z^2},$$

this implies the presence of a ME monopolization $A = -Q_{z^2}$ in a Néel skyrmion, consistent with our previous discussion that $a = -q_{2z}$ in a 2D system.

In Figs. 3(b)-(e), we show the $k$-space distributions of $O_{ij}(\vec{k})$ for Bloch and Néel skyrmions. It is interesting to point out the multipolization symmetries, $O_{ij}^{\text{Bloch}}(\vec{k}) = O_{ij}^{\text{Néel}}(\vec{k})$, which follow from the differences in $\gamma$: $\mu_{ij}^{\text{Bloch}} = \sin \theta \sin(\pi/2 + \phi) = \mu_{ij}^{\text{Néel}}$. Furthermore, since the velocity $v_{ij}$ does not depend on $\gamma$, it is easy to see from Eq. (3) that $O_{xy}^{\text{Bloch}} = O_{xy}^{\text{Néel}}$. Similarly, it can be shown that $\mu_{ij}^{\text{Bloch}} = -\mu_{ij}^{\text{Néel}}$, leading to $O_{ij}^{\text{Bloch}}(\vec{k}) = -O_{ij}^{\text{Néel}}(\vec{k})$. Note that, although the explicit value of $O_{ij}$ depends on the Fermi energy $\varepsilon_F$ [see Eq. (3)], the relations discussed here remain intact because they are determined by the symmetries of the spin texture.

Finally, for twisted skyrmions described by intermediate $\gamma$ values, we find that both diagonal and off-diagonal
The spin multipolarization, discussed above, is directly related to the spin ME polarizability, \( \alpha_{ij} = -\epsilon \partial \mathbf{M}_{ij} / \partial \mathbf{e}_\alpha \), which can, therefore, be computed as

\[
\alpha_{ij} = e g \mu_B \int_{\text{occ}} \frac{d^3k}{(2\pi)^2} \sum_n D_{ij}^n(k),
\]

where

\[
D_{ij}^n(k) = -2 \text{Im} \sum_{m \neq n} \left[ \frac{(n|v_1|m)(m|s_1|n)}{(\epsilon_n - \epsilon_m)^2} \right].
\]

Here the integration is over the occupied part of the BZ. We compute the response \( \alpha_{ij} \) for the skyrmion as a function of \( \gamma \) and show our results in Fig. 4(d). Similar to the \( \mathbf{M}_{ij} \) tensor, we find that \( \alpha_{ij} \) has a \( 2 \times 2 \) matrix form, with \( \alpha_{xx} = \alpha_{yy} \) varying as \( \cos \gamma \), and \( \alpha_{xy} = -\alpha_{yx} \) as \( \sin \gamma \). Substituting this expression for \( \mu \), we obtain \( \mathbf{M}_{ij} \propto \sin \gamma \). A similar sine/cosine dependence of \( \alpha_{ij} \) is shown in Fig. 4(c). A similar periodic dependence is also evident in the corresponding polarizability \( \alpha_{ij} \) shown in Fig. 4(e). A similar periodic dependence is also evident in the corresponding polarizability \( \alpha_{ij} \), shown in Fig. 4(f).

To summarize, we have introduced a ME classification of skyrmions and applied it to Bloch and Néel skyrmions \[52\], as well as anti-skyrmions, and bimerons. The formalism is general and can be extended to other spin textures. Our work opens the door for future works examining the implications of MEMs in both metallic and insulating skyrmion-like textures. In particular, dependence of the multipolarization on the helicity demonstrated here, may have useful implications in the context of recent efforts to control the helicity of skyrmions \[53, 54\].
on formal definition is not followed. For example, as pointed out by Khomskii [56, 57], the elementary excitation in spin ice that is referred to as a magnetic monopole carries an electric dipole (and so breaks space-inversion symmetry) in addition to a magnetic charge.

We hope that our work stimulates experimental efforts in ME manipulation of skyrmions and related topological spin textures, opening up new avenues in designing unique skyrmionic (skyrmion-based spintronic) devices with high energy efficiency.

ACKNOWLEDGEMENTS

NAS and SB were supported by the ERC under the EU’s Horizon 2020 Research and Innovation Programme grant No 810451 and by the ETH Zurich. Computational resources were provided by ETH Zurich’s Euler cluster, and the Swiss National Supercomputing Centre, project ID eth3.
The phase factor $a_{ij} = \tan^{-1}\left[\frac{-\sin(\phi_i - \phi_j)}{\cos(\theta_i/2) \cos(\theta_j/2) + \cos(\phi_i - \phi_j)}\right]$ in the effective hopping $t_{ij}^{\text{eff}}$ is analogous to a moving electron in a slowly varying magnetic field, a pedagogical description of which can be found in Ref. [58] and, therefore, points to the emergent magnetic field of a skyrmion [30, 59]. Note that $\theta_{ij}$ in the expression of effective hopping $t_{ij}^{\text{eff}}$ is the angle between the magnetization vectors with $\cos(\theta_{ij}) = \hat{n}_i \cdot \hat{n}_j = \cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos(\phi_i - \phi_j)$.

Note that under both time reversal and space inversion, the integrand $O_n^{ij}(\vec{k}) \rightarrow -O_n^{ij}(\vec{-k})$, implying that a non-zero value of the integral in Eq. (3) requires both symmetries to be broken.