Quantum determinants

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Abstract
We show how to construct central and grouplike quantum determinants for FRT algebras $A(R)$. As an application of the general construction we give a quantum determinant for the $q$-Lorentz group.

1 Introduction

FRT algebras $A(R)$ were introduced in [6] as the associative $\mathbb{C}$-algebras generated by 1 and $t^a_b; a, b = 1, \ldots, n$ with relations

$$R^{ab}_{cd} t^c_e t^d_f = t^b_d t^a_f R^{cd}_{ef},$$

(1)

where $R$ is an invertible solution of the $n$-dimensional matrix quantum Yang-Baxter equation. These algebras are non-commutative generalisations of the algebras of polynomial functions on $n \times n$-matrices and play the rôle of $q$-matrix symmetries for $q$-deformed systems [2]. They are bialgebras with coproduct $\Delta t^a_b = t^a_c \otimes t^c_b$ and counit $\epsilon(t^a_b) = \delta^a_b$, and have so-called fundamental representations $\rho_{\pm}$ in the algebra of $n \times n$-matrices given by

$$\rho_{\pm}(t^a_c) = R^{ab}_{cd}; \rho_{-}(t^a_c) = R^{-1ab}_{de}.$$ 

(2)

The fact that $\rho_{\pm}$ extend as algebra maps has the consequence that $A(R)$ is dual quasitriangular (see [3] for details).

In order to obtain generalisations of unitary matrix groups one often divides by a further ‘metric relation’ of the form

$$t^{ab}_{ae} g^{cd} = g^{cd},$$

(3)
and the quotient $A$ of $A(R)$ by the ideal generated by this relation is then a Hopf algebra with obvious antipode. However, for mathematical reasons, one would like the quotient $A$ also to be dual quasitriangular. This is the case if the fundamental representations $\rho_{\pm}$ descend to a representation of $A$, i.e. if they respect (3). It is easily seen that this requirement is satisfied iff

$$R_{cm}^{ae} R_{df}^{bm} g^{cd} = g^{ab} \delta_{ef}, \quad R^{-1}_{mc} R^{-1}_{fa} R_{bd}^{mb} g^{cd} = g^{ab} \delta_{ef}. \quad (4)$$

These relations place a restriction on the choice of the metric $g$.

Generalisations of special matrix groups (i.e. matrices with determinant one), are obtained by dividing $A$ by a q-deformed determinant relation. Such relations are known for a number of examples, but a general construction seems to be missing so far. In this paper, we provide such a construction and show that the resulting quantum determinants $D$ are central and grouplike, i.e. ‘multiplicative’: $\Delta D = D \otimes D$.

2 Quantum determinants

In order to construct a q-determinant in $A(R)$, we need to find a $C$-valued tensor $\varepsilon_{a_1...a_n}$ which is completely $q$-antisymmetric in the sense that it solves the equation

$$\varepsilon_{a_1...a_k a_{k+1}...a_n} = -\lambda \delta_{ij} \varepsilon_{a_1...i j...a_n} R^{ij}_{a_k a_{k+1}} \quad (5)$$

for any two adjacent indices $a_k$ and $a_{k+1}$. Here $\lambda$ is a suitable normalisation factor. Solutions to these equations can be easily calculated, and in well-behaved cases the space of solutions will be one-dimensional.

There is also a general R-matrix formula for such epsilon tensors [3]. The setting in this paper was that $A(R)$ acted on an algebra of q-forms given by a second R-matrix $R'$, which has to obey certain relations with $R$. This space of forms was assumed to have a unique top form. An epsilon tensor was then constructed from this top form by differentiation and is given in terms of the matrix $R'$. However, this R-matrix formula is not very useful for actually calculating the epsilon tensor. Checking the assumption that there is a unique top form is essentially tantamount to verifying that (5) has a one-dimensional space of solutions. This already gives the epsilon tensor and thus there is no need to use the rather complicated R-matrix formula. Moreover, in our approach there is no need to find the second R-matrix $R'$. 

2
For the following, we assume that there is a unique solution $\varepsilon_{a_1...a_n}$ of (5). In terms of this epsilon tensor we define the quantum determinant $\mathcal{D}$ in $A(R)$ as

$$\mathcal{D} = \nu^{-1} \varepsilon_{a_1...a_n} t_{b_1}^{a_1} \ldots t_{b_n}^{a_n},$$

where the normalisation factor $\nu$ is given by

$$\nu = \varepsilon_{a_1...a_n} \varepsilon_{a_1...a_1},$$

and the epsilon tensor with upper indices is defined as

$$\varepsilon_{a_1...a_n} = \varepsilon_{b_1...b_n} g_{a_1 b_1} \ldots g_{a_n b_n}$$  \hspace{1cm} (6)

**Proposition.** The q-determinant $\mathcal{D}$ is central and grouplike and the generators of $A(R)$ obey the relation

$$\varepsilon_{a_1...a_n} t_{b_1}^{a_1} \ldots t_{b_n}^{a_n} = \mathcal{D} \varepsilon_{b_1...b_n}.$$  \hspace{1cm} (7)

In the quotient $A$ of $A(R)$, one also finds $\mathcal{D}^2 = 1$ for the square of the q-determinant.

**Proof.** By virtue of the relations (1) and the properties of the epsilon tensor, the element $\varepsilon_{a_1...a_n} t_{b_1}^{a_1} \ldots t_{b_n}^{a_n}$ is a solution of (3) and thus by assumption eigenvector of the one-dimensional projector $P_{b_1...b_n}^{a_1...a_n} = \nu^{-1} \varepsilon_{a_1...a_1} \varepsilon_{b_1...b_n}$, i.e.

$$\varepsilon_{a_1...a_n} t_{b_1}^{a_1} \ldots t_{b_n}^{a_n} = \varepsilon_{a_1...a_n} t_{c_1}^{a_1} \ldots t_{c_n}^{a_n} P_{b_1...b_n}^{c_1...c_n} = \mathcal{D} \varepsilon_{b_1...b_n}.$$  \hspace{1cm} (5)

This proves (5) and also implies that $\mathcal{D}$ is grouplike:

$$\Delta \mathcal{D} = \nu^{-1} \varepsilon_{a_1...a_n} t_{c_1}^{a_1} \ldots t_{c_n}^{a_n} \otimes t_{b_1}^{c_1} \ldots t_{b_n}^{c_n} \varepsilon_{b_1...b_1}$$

$$= \nu^{-1} \mathcal{D} \varepsilon_{c_1...c_n} \otimes t_{b_1}^{c_1} \ldots t_{b_n}^{c_n} \varepsilon_{b_1...b_1}$$

$$= \mathcal{D} \otimes \mathcal{D}.$$  \hspace{1cm} (2)

Moreover, (2) implies for the square of the q-determinant on the quotient $A$

$$1 = \nu^{-1} \varepsilon_{b_1...b_n} \varepsilon_{b_1...b_1}$$

$$= \nu^{-1} \varepsilon_{a_1...a_n} g_{a_1 b_1} \ldots g_{a_n b_n} \varepsilon_{b_1...b_1}$$

$$= \nu^{-1} \varepsilon_{c_1...c_n} t_{a_1}^{c_1} \ldots t_{d_1}^{c_1} g_{a_1 b_1} \ldots g_{a_n b_n} t_{d_1}^{c_1} \ldots t_{b_1}^{d_1} \varepsilon_{d_1...d_1}$$

$$= \nu^{-1} \mathcal{D} \varepsilon_{a_1...a_n} g_{a_1 b_1} \ldots g_{a_n b_n} t_{b_1}^{d_1} \ldots t_{b_1}^{d_1} \varepsilon_{d_1...d_1}$$

$$= \mathcal{D}^2,$$

where we used (5) and (2).

The application of the fundamental representation $\rho_-$ from (2) to equation (5) yields

$$\varepsilon_{a_1...a_n} R_{b_1}^{-1} t_{c_1 b_1} R_{c_2 b_2}^{-1} \ldots R_{c_n b_n}^{-1} = \rho_-(\mathcal{D}) \varepsilon_{b_1...b_n} \delta_m^i,$$
and since $\mathcal{D}^2 = 1$ on the quotient $A$, one immediately finds $\rho_-(\mathcal{D}) = 1$. Together with (3) – here used in the form $R^{-1 eb}_{fd}g^{ad} = R^{-1 e}_{fd}g^{db}$ – this relation also implies

$$R^{ib}_{c_1a_1}R^{c_2b_2}_{c_2a_2} \ldots R^{c_{n-1}b_n}_{m a_n} \varepsilon^{a_1 \ldots a_n} = \varepsilon^{b_1 \ldots b_n} \delta_i^m$$

and finally with (4) that $\mathcal{D}$ is central in $A(R)$:

$$t^i_j \nu \mathcal{D} = t^i_j \varepsilon_{a_1 \ldots a_n} t^{a_1}_{b_1} \ldots t^{a_n}_{b_n} \varepsilon^{b_n \ldots b_1} \varepsilon^{a_1 \ldots a_n} R^{-1 e_{n-1} \varepsilon_{n \ldots n}}_{m a_n} t^{d_1}_{e_1} \ldots t^{d_n}_{e_n} R^{i e_1}_{f_1 b_1} \ldots R^{f_{n-1} \varepsilon_{n \ldots n}}_{m b_n} \varepsilon^{b_n \ldots b_1} = \nu \mathcal{D} t^i_j$$

This proves the proposition. \hfill \textbf{q.e.d.}

This construction reproduces all known quantum determinants, and has the advantage that it works quite generally. As a new example, one can construct a quantum determinant for the quantum Lorentz group $\mathcal{L}_q[3]$. This algebra is given as a quotient of $A(\mathbf{R}_L)$ by a metric relation of the form (3) which satisfies (2). In terms of the standard $SU_q(2)$ R-matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad q \in \mathbb{R},$$

the R-matrix $\mathbf{R}_L$ is given by

$$\mathbf{R}_L^{ABC D} = R^{dk}_{lb} R^{b \prime l}_{ma} R^{a \prime m}_{nc} \mathbf{R}^{nc}_{dk}$$

where we use multi-indices $A = (aa^\prime)$ and $\mathbf{R}$ is defined as $((\overline{R})^{\prime 2})\overline{t}_2$ where $t_2$ denotes transposition in the second tensor component. In an earlier paper 3 we already solved the equations (3) for this special case. The one-dimensional space of solutions has a basis vector $\varepsilon_{ABCD}$ the non-zero entries of which are

$$\begin{align*}
\epsilon_{1234} &= 1 & \epsilon_{1243} &= -q^{-2} & \epsilon_{1324} &= -1 & \epsilon_{1342} &= q^2 & \epsilon_{1414} &= 1 - q^2 \\
\epsilon_{1423} &= 1 & \epsilon_{1432} &= -1 & \epsilon_{1444} &= 1 - q^{-2} & \epsilon_{2134} &= -1 & \epsilon_{2143} &= q^{-2} \\
\epsilon_{2314} &= 1 & \epsilon_{2341} &= -1 & \epsilon_{2413} &= -q^{-2} & \epsilon_{2431} &= q^{-2} & \epsilon_{2434} &= q^{-2} - 1 \\
\epsilon_{3124} &= 1 & \epsilon_{3142} &= -q^2 & \epsilon_{3214} &= -1 & \epsilon_{3241} &= 1 & \epsilon_{3412} &= q^2 \\
\epsilon_{3421} &= -q^2 & \epsilon_{3424} &= 1 - q^2 & \epsilon_{4123} &= -1 & \epsilon_{4132} &= 1 & \epsilon_{4141} &= q^2 - 1 \\
\epsilon_{4144} &= q^{-2} - 1 & \epsilon_{4213} &= 1 & \epsilon_{4231} &= -1 & \epsilon_{4243} &= 1 - q^{-2} & \epsilon_{4312} &= -1 \\
\epsilon_{4321} &= 1 & \epsilon_{4342} &= q^2 - 1 & \epsilon_{4414} &= 1 - q^{-2} & \epsilon_{4441} &= q^{-2} - 1
\end{align*}$$

The normalisation factor $\nu$ is given by

$$\nu = 2q^{-2}(1 + q^2 + q^4)(1 + q^2)^2.$$
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