Regge Regime in QCD
and
Asymmetric Lattice Gauge Theory

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Abstract

We study the Regge regime of QCD as a special regime of lattice gauge theory on an asymmetric lattice. This lattice has a spacing $a_0$ in the longitudinal direction and a spacing $a_t$ in the transversal direction. The limit $\frac{a_0}{a_t} \to 0$ corresponds to correlation functions with small longitudinal and large transversal coordinates, i.e. large $s$ and small $t$. On this lattice the longitudinal dynamics is described by the usual two-dimensional chiral field in finite volume and the transversal dynamics is emerged through an effective interaction of boundary terms of the longitudinal dynamics. The effective interaction depends crucially on the spectrum of the two-dimensional chiral field. Massless excitations produce an effective 2-dimensional action which is different from the action recently proposed by H.Verlinde and E.Verlinde. Massive excitations give rise to an effective action located on the contour in the longitudinal plane.

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1 Introduction

In recent years, efforts have been made to obtain an appropriate scheme to study QCD in the Regge regime of large energies $\sqrt{s} \to \infty$ and fixed momentum transfers $q(q^2 = -t)$, $|q| \sim 1\text{Gev}$, i.e. $|q| > \Lambda_{\text{QCD}} \sim 100\text{Mev}$. It was recognized [1] that in this regime neither QCD perturbation theory applies - since $|t|$ is too small -, nor can the usual lattice gauge theory approach give numerical answers directly - since $s$ is too large.

In spite of the fact that strictly speaking studies of QCD in the Regge regime requires a non-perturbative treating it is rather instructive to analyze this region within the usual perturbation expansion. One of the striking results of intensive studies of this region [2, 3] is the Lipatov suggestion that the scattering amplitude can be obtained from a two-dimensional field theory [5].

The conclusion that the scattering amplitudes in the Regge regime are related to two-dimensional field theories [5] came from using the leading logarithmic approximation (LLA) together with an incorporation of unitarity condition, i.e. it is in some sense also non-perturbative. It restores the Froissart boundary: $\sigma_{\text{tot}} < c \ln^2 s$, which is violated in LLA. The Froissart boundary can also be restored in the eikonal approximation. Some years ago, 't Hooft derived the eikonal approximation for the two-particle scattering amplitude arising from their gravitation interaction [3] (see also [4]). Then this problem has been reexamined by H.Verlinde and E.Verlinde [8]. The eikonal approximation for the two-particle scattering amplitude in QED was studied by Jackiw et al [9]. They have shown that high energy eikonal electrodynamics can be given an action formulation where the action is localized on some contour. The picture of diffraction scattering related to eikonal approximation was applied in the Regge regime in QCD by Nachtmann [1]. According this picture the scattering process between two hadrons can be represented as a collection of separate collisions between the individual quarks, and, moreover, $s$-dependence of the amplitudes of quark anti-quark scattering $\Psi_i(p_1) + \Psi_j(p_2) \to \Psi_k(p_3) + \Psi_l(p_4)$ can be treated in the eikonal approximation

$$< \Psi_i(p_1), \Psi_j(p_2)|S|\Psi_k(p_3), \Psi_l(p_4)>_{\text{diff.}} = i \delta_{i_1,i_3} \delta_{i_2,i_4} \bar{u}(p_3) \gamma^\mu u(p_1) \bar{u}(p_4) \gamma^\mu u(p_2) J(q^2)$$

(1)

and $t$-dependence is governed by gluon string operators

$$J(q^2) = - \int d^2z e^{iqz} < \text{tr} [\mathcal{V}_-(\infty, 0, z) - 1] \text{tr} [\mathcal{V}_+(0, \infty, 0) - 1] >_A$$

(2)

$$\mathcal{V}_\pm (y_+, y_-, z) = P \exp \int_{-\infty}^{y_\pm} dy A_\pm (y_+, y_-, z),$$

(3)

Here $\pm \equiv \frac{1}{\sqrt{2}} (0 \pm 1)$ are used for light-cone components, $i_j$ are colours indices, $<>_A$ means an average over gauge fields. Renormalizability of (3) was proved in [10].

As it was noted in [1] $J(q^2)$ should be calculated non-perturbatively. A simple model to calculate $J(q^2)$ within the semi-classical approximation has been proposed by H.Verlinde and E.Verlinde [11]. They formulated a simple model in which the two-dimensional nature of the interaction is manifest. In our previous paper the lattice version of the truncated model [11] was investigated [12].

We would like to stress that one cannot use the standard continuous perturbative gauge theory in the Regge regime. Note also that the standard Wilson lattice calculations such as strong coupling expansion or Monte-Carlo calculations in the Regge
region also are not applicable since $s$ is too large. These two at first sight incompatible circumstances can be made compatible if we start from an asymmetric lattice.

In this paper we propose to study the Regge regime as a special regime of lattice gauge theory on the asymmetric lattice. This lattice has a spacing $a_0$ in the longitudinal direction and a spacing $a_t$ in the transversal direction. Putting $\frac{a_0}{a_t} \to 0$ we get a possibility to study correlation functions with small longitudinal and large transversal coordinates, i.e. large longitudinal and small transversal components of momentum in the c.m. frame. We will see that the longitudinal dynamics on this lattice is described by the usual two-dimensional chiral field on the corresponding two-dimensional lattice. To remove the lattice in the two-dimensional chiral model we have to perform a renormalization. It is known that due to dimensional transmutation this renormalization produces a mass. For example, for the non-linear $O(N)$ $\sigma$-model in the large $N$ limit one gets $N$ massive free fields with mass $m = M \epsilon^{-4\pi/\gamma}$ \cite{13, 14}.

If this two-dimensional chiral model contains massless excitations than the effective interaction is reduced to an interaction of three discrete point (two degrees of freedom after taking into account gauge invariance) and we get a reduction of the initial 4-dimensional model to a two-dimensional model containing two fields \cite{12}. This degrees of freedom counting disagrees with the H.Verlinde and E.Verlinde \cite{11} counting of fields and as a result we get an effective action which is different from the H.Verlinde and E.Verlinde (VV) effective action. If the two-dimensional chiral model contains massive excitations, as it can be expected from the dimensional transmutation, then an effective interaction of boundary terms cannot be reduced to an interaction of finite number of degrees of freedom since there are extra terms in the effective action. These extra terms are located on some contour in the longitudinal plane and a two dimensional reduction is rather problematic. However under some special assumptions this reduction is still possible.

So, our strategy will be the following. We will start from asymmetric lattice gauge theory which we will study in the limit $\frac{a_0}{a_t} \to 0$. For the resulting theory being the chiral field model we use the non-perturbative knowledge of the behaviour of the theory in the continuous limit $a_0 \to 0$ after suitable renormalization. Then we will perform the Wick rotation and will approximate boundary effects in the finite volume $D=2$ chiral field model by boundary effects of $D=2$ Klein-Gordon theory. We also incorporate the picture of diffraction scattering for amplitudes of quark anti-quark scattering \cite{1}. The calculation of the correlation functions of string operators for two longitudinal Wilson lines \cite{2} on our asymmetric lattice will be reduced to examination of boundary effects in finite volume two-dimensional Klein-Gordon theory. We will conclude with the presentation of the effective action for the transversal theory and we will compare it with the VV effective action. As it has been mentioned above our effective action is different from the VV effective action. This difference comes from the fact that for solutions of the wave equation there are connections between values of the field on characteristics. Only values of the field on two characteristics can be treated as independent variables.

In conclusion we discuss a connection between our asymmetric lattice gauge theory and Bardeen, Pearson and Rabinovici \cite{16} and Klebanov and Susskind \cite{15} light-cone lattice gauge theory.

\footnote{This degrees of freedom counting agrees with our previous estimations performed within an approximation scheme which takes into account one-dimensional excitations \cite{12}.}
Asymmetric Lattice Gauge Theory

Since we intend to study the theory in the region of small longitudinal coordinates and large transversal coordinates and we want to take into account fluctuations in different direction with an approximately equal precision it is relevant to assume that the lattice spacing in the longitudinal and transversal directions are different so that there are equal number of points in the different directions (see fig 1). If the number of points in different directions is the same, i.e. \( n_0 = n_1 = ... n_{D-1} \) then we get a theory in an asymmetric space-time volume \( L_0 L_1...L_{d-1} \), \( L_i \) is a typical size in the i-direction, with lattice spacing \( a_0, a_1, ... a_{D-1} \), such that \( L_0/L_1 = a_0/a_1, ..., L_0/L_D = a_0/a_D, \). A general form of the lattice action on an asymmetric lattice with lattice spacing \( a_0, a_1, ..., a_{D-1} \) in 0,1,...D-1-directions has the form

\[
S = \frac{1}{4g^2} a_0 a_1 ... a_{D-1} \sum_{x,\mu,\nu} \frac{1}{(a_\mu a_\nu)^2} \text{tr} \left( (\square_{\mu,\nu}) - 1 \right),
\]

Here \( x \) are points of the 4-dimensional lattice, \( \square_{\mu,\nu} \) is a single plaquette attached to the links \( (x, x + \mu) \) and \( (x, x + \nu) \), \( (\square_{\mu,\nu}) = U_{x,\mu} U_{x+\mu,\nu} U_{x,\nu} U_{x+\nu,\mu} U_{x,\mu} \) and link variables \( U_{x,\mu} \) are associated with the link between the lattice sites \( x \) and \( x + \hat{\mu} \). \( U_{x,\mu} \) belongs to a representation of the gauge group \( SU(N) \).

Since we are interested in the case when \( L_0 = L_1, L_2 = L_3 \) and \( L_1/L_3 \to 0 \) we put in (4) \( a_0 = a_1, a_2 = a_3 \) and \( a_0 = \lambda a_3 \), so we get

\[
S = \frac{1}{4\lambda^2 g^2} \sum_x \sum_{\alpha,\beta} \text{tr} \left( (\square_{\alpha,\beta}) - 1 \right) + \frac{1}{4g^2} \sum_x \sum_{\alpha,i} \text{tr} \left( (\square_{\alpha,i}) - 1 \right)
\]

\[
+ \frac{\lambda^2}{4g^2} \sum_x \sum_{i,j} \text{tr} \left( (\square_{i,j}) - 1 \right).
\]

\( \alpha, \beta \) are unit vectors in the longitudinal direction and \( i, j \) are unit vectors in the transversal direction. We also denote the points of 4-dimensional lattice as \( x = (y, z) \), where \( y \) and \( z \) are the points of two two-dimensional lattices, say, \( y \)-lattice (longitudinal) and \( z \)-lattice (transversal). Performing the \( \lambda \to 0 \) limit in the lattice action (5) we get

\[
S_{\text{tr}}^{l,l} = \frac{1}{4g^2} \sum_x \sum_{\alpha,i} \text{tr} \left( (\square_{\alpha,i}) - 1 \right),
\]

with \( U_{x,\alpha} \) being a subject of the relation

\[
(\square_{\alpha,\beta}) = 1,
\]

i.e. \( U_{x,\alpha} \) is a zero-curvature lattice gauge field

\[
U_{x,\alpha} = V_x V_{x+\alpha}^+.
\]

Substituting (8) in (6) we get

\[
S_{\text{tr}}^{l,l} = \frac{1}{4g^2} \sum_x \sum_{\alpha,i} \text{tr} \left( V_{x+\alpha} V_{x+\alpha,i} V_{x+\alpha+i} V_{x+i} V_{x,i} V_x - 1 \right),
\]

or

\[
S_{\text{tr}}^{l,l} = \frac{1}{4g^2} \sum_x \sum_{\alpha,i} \text{tr} \left( \tilde{U}_{x+\alpha,i} \tilde{U}_{x,i} - 1 \right)
\]
The action (10) is ultra-local in the transverse z-direction and is the lattice chiral field action in the longitudinal y-direction. In the formal continuous limit $a_0 \to 0$ we get

$$S_{c,l} = \frac{1}{4g^2} \int dy^0 dy^1 \sum_{y,\beta} \left( \partial_\alpha \bar{U}_{z,i}(y) \partial_\alpha \bar{U}_{z,i}^+(y) \right)$$

(12)

with summation over the repeating indexes $\alpha = 0, 1$ and $\bar{U}_{z,i}(y) = V_{z+1}^+(y)U_{z,i}^+(y)V_{z}(y)$. To get its pure continuous version one has also to make in (12) and (11) a formal limit $a_t \to 0$ under assumption $U_{x,i} = \exp(a_t A_i)$

$$S_{c,c} = \frac{1}{4g^2} \int dy^0 dy^1 d^2 z \sum_{z,\alpha} \left( \partial_\alpha \tilde{A} \partial_\alpha \tilde{A} \right)$$

(13)

where

$$\tilde{A}_i = V^+ D_i V, \quad D_i = \partial_i + A_i,$$

(14)

Therefore in the formal limit $a_t \to 0$ the action (12) reproduces the Verlinde and Verlinde truncated action. However, as we will see in the next section, non-perturbatively there is an essential difference in the behaviour of two theories (13) and (12). Note that the action (5) was considered in [12] as a discrete version of the Verlinde and Verlinde rescaling action.

It is also instructive to compare the action (12) with the Bardeen, Pearson and Rabinovici lattice action [16] for light-cone gauge theory. They also considered $a_0 \to 0$ limit of the Willson lattice theory but in addition they assumed that the gauge field on the longitudinal links fluctuate around the unit element of the group, i.e. $U_{x,\alpha} = \exp(a_0 A_\alpha)$. Their action has the form

$$S_{c,l} = \frac{1}{g^2} \int dy^0 dy^1 \sum_{x,\alpha} \sum_{\alpha,\beta} \frac{a_t^2}{2} \text{tr} (F_{\alpha,\beta} F^{\alpha,\beta}) +$$

$$+ \sum_{z} \sum_{\alpha,i} \text{tr} (D_\alpha U_{z,i}(y) D^\alpha U_{z,i}^+(y)) + \sum_{i,j} \frac{1}{a_t^2} \text{tr} (U(U_{i,j} - 1)).$$

(15)

Examining the above formulae (12) and (15) shows that the action (12) with the zero-curvature condition (8) can be extracted from (15) in the limit of large transversal lattice spacing $a_t$. 

Figure 1: Asymmetric lattice
The lattice version of (3) for the zero curvature longitudinal gauge fields is simply reduced to the boundary values of the field $V_z(y)$

$$V_+(L, 0, z) = V_z(-L, 0)V_z^+(L, 0), \quad V_-(0, L, z) = V_z(0, -L)V_z^+(0, L).$$

and the expectation value of these string operators is given by

$$<V_+(L, 0, 0)V_-(0, L, z)>= \int V_0(-L, 0)V_0^+(L, 0)V_z(0, -L)V_z^+(0, L) \exp S^{tr}(V, U) dV dU$$

with $S^{tr}(V, U)$ as in (11). In the continuum limit in the longitudinal direction we use $S^{tr}$ given by (12). Our goal consists in calculation the functional integral (17) over gauge fields $U$ to get an effective action describing an interaction of fields $V_z(0, \pm L)$ $V_z(\pm L, 0)$.

### 3 Boundary Effects for D=2 Klein-Gordon Action

#### 3.1 Boundary Effects for D=2 massless case

Let us assume that the excitation spectrum of the theory (12) contains massless particles. This means that in this case we can forget that our fields variables are restricted by the unitarity condition and we can deal with the action for the wave equation

$$S = \int dy^0 dy^1 \text{tr} [\partial_\alpha \tilde{M}(y) \partial_\alpha \tilde{M}^+(y)],$$

where $\tilde{M}_{z,i}(y) = V_z^+(y)M_{z,i}(y)V_z(y)$ and we drop for the moment indexes $z$ and $i$. $\tilde{M}(y)$ in (18) is an arbitrary complex matrix. By the usual Feynman integral prescription one can compute the transition function from one given configuration $M_a$ at the initial time to another configuration $M_b$ at time $y^0$. Since we are interested in special correlation functions related with the values of $M_{z,i}(y^+, y^-)$ at $y^\pm = \pm L$, $y^\mp = 0$ respectively, it is more convenient to consider the transition function in the light-cone variables.

Let us consider for a moment the transition function for one component massless field in the light-cone variables

$$\mathcal{K}(\phi_1(y^+), \phi_2(y^-), \phi_3(y^+), \phi_4(y^-)) = \int \exp\{iS(\phi, L)\} \cdot \prod d\phi(y^+ y^-),$$

$$S(\phi, L) = \int_{-L}^{L} dy^+ \int_{-L}^{L} dy^- \partial_\phi(\phi(y^+ y^-)\partial_\phi(\phi(y^+ y^-),$$

where the following boundary conditions are assumed (see fig.2)

$$\phi(y^+, y^-)|_{y^- = -L} = \phi_1(y^+), \quad \phi(y^+, y^-)|_{y^+ = -L} = \phi_2(y^-),$$

$$\phi(y^+, y^-)|_{y^- = L} = \phi_3(y^+), \quad \phi(y^+, y^-)|_{y^+ = L} = \phi_4(y^-)$$

The boundary of the space-time region in (20) consists of four characteristics. The classical solution of the wave equation can be specified by functions on two characteristics (the Goursat boundary problem), say $\phi_1$ and $\phi_2$, having the same value on a common point $\phi_1(-L) = \phi_2(-L)$

$$\phi_{cl}(y^+, y^-) = \phi_1(y^+) + \phi_2(y^-) - \phi_1(-L)$$
Figure 2: Boundary conditions

Other two functions being values of the solution of the wave equation on others two characteristics can be written in terms of $\phi_1$ and $\phi_2$ as

$$
\phi_3^{cl}(y^+) \equiv \phi_3(y^+, y^-)|_{y^- = L} = \phi_1(y^+) + \phi_2(L) - \phi_2(-L),
$$

$$
\phi_4^{cl}(y^-) \equiv \phi_4(y^+, y^-)|_{y^+ = L} = \phi_1(L) + \phi_2(y^-) - \phi_1(-L). \tag{25}
$$

The transition function $\mathcal{K}$ is proportional to $\delta$-functions guaranteeing that $\phi_3$ is equal to $\phi_3^{cl}$ and $\phi_4$ is equal to $\phi_4^{cl}$, i.e.

$$
\mathcal{K}(\phi_1(y^+), \phi_2(y^-), \phi_3(y^+), \phi_4(y^-)) = \prod_{-L \leq y^+ \leq L} \delta(\phi_3(y^+) - \phi_3^{cl}(y^+)) \cdot \prod_{-L \leq y^- \leq L} \delta(\phi_4(y^-) - \phi_4^{cl}(y^-)) \exp\{S(\phi_{cl}, L)\}. \tag{26}
$$

where $S(\phi_{cl}, L)$ is given by the classical action being calculated on the classical solution (23). $S(\phi_{cl}, L)$ has the form

$$
S(\phi_{cl}, L) = (\phi_1(L) - \phi_1(-L)) \cdot (\phi_2(L) - \phi_1(-L)) \tag{27}
$$

The representation (26) can be directly verified from (19) by using the symmetric approximation of the functional integral in the LHS of (19).

It is turn out that the classical action depends only on the values of the field $\phi$ on tree points, namely, on points I, II and III on fig.2. Of course, the action (20) on the classical solution can be written in terms of values of the classical solution on four points, say points 1,2,3 and 4 on fig.2 as it was done in (11),

$$
S(\phi_{cl}, L) = (\phi_{cl}(L, 0) - \phi_{cl}(-L, 0)) \cdot (\phi_{cl}(0, L) - \phi_{cl}(0, -L)). \tag{28}
$$

However for the classical solutions there are two constraints which make the differences $(\phi_{cl}(L, y^-) - \phi_{cl}(-L, y^-))$ and $(\phi_{cl}(y^+, L) - \phi_{cl}(y^+, -L))$ not depending on $y^-$ and $y^+$, respectively. We can put $y^- = -L$ and $y^+ = -L$, making this action depending explicitly on the field $\phi$ on tree points, but not on four points as in (11).

Specifying the values of the field $\bar{M}_{z,i}(y^+ y^-)$ on these tree points as $\bar{M}_{z,i}^A$, $A = I, II, III$ we get an effective action

$$
S_{eff} = \sum_{z,i} \text{tr} (\bar{M}_{z,i}^I - \bar{M}_{z,i}^{II})(\bar{M}_{z,i}^{II} - \bar{M}_{z,i}^{III}) \tag{29}
$$
and therefore

\[ S_{\text{eff}} = \sum_{z,i} \text{tr} \left( V_z^+(I)U_{z,i}(I)V_{z+1}(I) - V_z^+(II)U_{z,i}(II)V_{z+1}(II) \right) \]  

\[ \cdot (V_z^+(III)U_{z,i}(III)V_{z+1}(III) - V_z^+(II)U_{z,i}(II)V_{z+1}(II)) . \]

In contrast to \[11\] the effective action (29) depends only on three two-dimensional chiral fields \( V_z^A, A = I, II, III \).

In the continuous limit \( a_t \to 0 \) the effective action (29) yields

\[ S_{\text{eff}} = \int d^2 z \text{tr} \left( V^+(I, z)D_t^I V_I(z) - V^+(II, z)D_t^II V_{II}(z) \right) \]

\[ \cdot (V^+(III, z)D_t^III V_{III}(z) - V^+(II, z)D_t^II V_{II}(z)) . \]

### 3.2 Boundary Effects for D=2 massive case

Unfortunately we cannot restrict ourselves only by massless excitations. The spectrum of the chiral model is more complicated. If we knew the exact behaviour of the chiral model, we could rewrite the action in terms of linear (unrestricted) variables without approximations. Now we assume that non-perturbative spectrum contains massive excitations. A more rigorous argument for the above statement comes from the knowledge of the behaviour of the non-linear \( O(N) \) \( \sigma \)-model in two dimensions. This model in the leading large \( N \) order can be described by \( N \)-component massive free field with mass \( m = \frac{Me^{-4\pi/g^2}}{2} \). This gives us a reason to consider boundary effects for free massive theory\[1\]

\[ S = \int dy dy^1 \text{tr} \left[ \partial_\alpha \bar{M}(y) \partial_\alpha \bar{M}(y) + m^2 \bar{M}(y) \bar{M}(y) \right] , \]

as an approximation to boundary effects for the model (12).

For the massive transition function given by the functional integral

\[ K_m(\phi_1(y^+), \phi_2(y^-), \phi_3(y^+), \phi_4(y^-)) = \int \exp\left\{ S_m(\phi, L) \right\} \prod d\phi(y^+ y^-) \]

with an action

\[ S_m(\phi, L) = \int_{-L}^L dy \int_{-L}^L dy^1 \left[ \partial_\phi (y^+ y^-) \partial_\phi (y^+ y^-) + m^2 \phi^2 (y^+ y^-) \right] \]

and with boundary conditions (21) and (22) also takes place the representation (26).

The value of the massive action on classical solutions can be written in terms of values of the classical field on four points I,II,III and IV in fig.2

\[ S_m(\phi_{cd}, L) = \frac{1}{2} \left[ \phi_{cd}^2(L, L) - \phi_{cd}^2(-L, L) - \phi_{cd}^2(L, -L) + \phi_{cd}^2(-L, -L) \right] \]

\[ \text{But there is an essential difference with the massless case. Instead of the representation of the classical solutions in the form (23) we have} \]

\[ \phi_{m}^{cd}(y^+, y^-) = \phi_1(y^+) + \phi_2(y^-) - \phi_1(-L)R(y^+, y^-; -L, -L)\]

\[ \text{Since we consider a theory in finite space-time it is natural to expect that the mass renormalization will depend on the volume. The question was analyzed by Luscher some years ago \[17\]. It was found that finite volume effects on the mass renormalization are related to forward scattering amplitudes of the infinite volume theory.} \]
\[
\int_{-L}^{y^+} d\xi \phi_1(\xi) \partial_\xi R(y^+, y^-; \xi, -L) - \int_{-L}^{y^-} d\eta \phi_1(\eta) \partial_\eta R(y^+, y^-; -L, \eta),
\]

where the Riemann function \( R \) can be written in terms of the Bessel function \( J_0 \)

\[
R(y^+, y^-; \xi, \eta) = J_0(\sqrt{4m^2(\xi - y^+)(\eta - y^-)}).
\]  

(37)

Note that for \( m = 0 \) one has \( R = 1 \) and the formula (36) reproduces the solution (23). \( \phi_{cl}(L, L) \) depends on the full set of the initial data of the Goursat boundary problem and after substitution of \( \phi_{cl}(L, L) \) in the action (32) we get

\[
S_m(\phi_{cl}, L) = \frac{1}{2} \left\{ -\phi_1^2(L) - \phi_2^2(L) + \phi_1^2(-L) \right\}
\]

(38)

\[+\left[ \phi_1(L) + \phi_2(L) - \phi_1(-L)J_0(4mL) - \int_{-L}^{L} d\xi (\phi_1(\xi) + \phi_2(\xi)) \partial_\xi J_0(\sqrt{8m^2L(L - \xi)} \right]\}

Let us now examine the effective action (38). We see that this effective action is located on the contours \( \Gamma_{I,II} \) and \( \Gamma_{I,III} \) in the \( (y^+, y^-) \)-plane.

If we assume that the mass \( m \) is fixed and \( L \) goes to infinity and \( \phi_1 \) and \( \phi_2 \) vary smooth enough on the interval \((-L, L)\) then we can neglect the terms with the Bessel function and we get

\[
S(\phi_{cl}) = \frac{1}{2} \left\{ (\phi_1(L) \cdot \phi_2(L) + \phi_1^2(-L) \right]\}
\]  

(39)

So, as in the massless case the effective action is located on tree points, but the form of action is not the same. Now the effective action is given by

\[
S_{eff} = \int \! d^2z \, \text{tr} \left( V^+(I, z)D^I_1V(I, z) \cdot V^+(III, z)D^I_{III}V(III, z) \right)
\]

(40)

\[+V^+(II, z)D^I_{II}V(II, z) \cdot V^+(II, z)D^I_{II}V(II, z).\]

If we assume that the normalization point \( M \) in the mass renormalization is related with \( L \) so that \( ML \) is fixed for \( L \to \infty \) then in this case we cannot neglect the terms in the effective action located on the contours \( \Gamma_{I,II} \) and \( \Gamma_{II,III} \).

4 Expectation Value of String Operators

4.1 Abelian case

It is instructive to start consideration of expectation value of string operators (1) from the abelian case. We know that the properties of compact and non-compact QED are different (18). So it is natural to expect that the expectations value of string operators (3) will be different in compact and non-compact QED. Below we will present calculations which take into account only the usual excitations postponing a consideration of topologically stable quantized-vortex configurations for future investigations.

In the standard QED the expectation value of string operators (3) in the Born approximation, i.e. in the approximation neglecting fermionic loops, is given by

\[
< \mathcal{V}_-(L, 0, z)\mathcal{V}_+(0, L, 0)>_{QED} = \int e^{-\frac{i}{\hbar} \int d^4xF_{\mu\nu}^2 + \int d^4xA_{\mu}J_{\mu} \prod \delta(\partial_{\mu}A_{\mu})dA_{\mu}} \]  

(41)

with

\[
J^\mu = \int_{\Gamma_1} \delta^4(x - x(\sigma))dx^\mu(\sigma) + \int_{\Gamma_2} \delta^4(x - x(\sigma))dx^\mu(\sigma)
\]  

(42)
where the contours $\Gamma_{13}$ and $\Gamma_{24}$ are specified as $x^+(\sigma) = \sigma$, $x^-(\sigma) = 0$, $x^i(\sigma) = z^i$ and $x^+(\sigma) = 0$, $x^-(\sigma) = \sigma$, $x^i(\sigma) = 0$, $-L \leq \sigma \leq L$, respectively. Therefore

$$< \mathcal{V}_-(L, 0, z) \mathcal{V}_+(0, L, 0) >_{QED} = \exp\left\{ \frac{ig^2}{2} \int dk_+ dk_- d^2k_i \frac{\sin k_+ L}{k_+} \cdot \frac{\sin k_- L}{k_-} \cdot \frac{e^{ik_iz^i}}{k_+ k_- - k_i^2} \right\}$$

In the limit $L \to \infty$ two first factors in the integrand produce $\delta(k_+) \delta(k_-)$ and we get

$$< \mathcal{V}_-(\infty, 0, z) \mathcal{V}_+(0, \infty, 0) >_{QED} = \exp\left\{ -\frac{ig^2}{2} \int d^2k_i \frac{e^{ik_iz^i}}{k_i^2} \right\}$$

Thus $J(q^2)$ given by (43) for QED can be represented as

$$J_{QED}(q^2) = -\int d^2z e^{i(qz)}(\exp\left\{ -\frac{ig^2}{2} \int d^2k_i \frac{e^{ik_iz^i}}{k_i^2} \right\} - 1)$$

Note that if we introduce infrared regulating mass $\mu$ then (46) for small $\mu$ reproduces the standard eikonal formula [3, 9]

$$J_{QED}(q^2) = \frac{\Gamma(1 + ig^2/4\pi)}{4\pi\mu^2(4\pi)^{1+ig^2/4\pi}}$$

Let us derive this result starting from the action (12). In this case $V_z(y) = e^{i\phi_z(y)}$ and for small $a_t$ $M_{z,j}(y) = a_t \partial_j \phi(y, z)$. According (29) and we have the following effective action

$$S_{eff}^{QED} = \int d^2z [\partial_t \phi_1(L, z) - \partial_t \phi_1(-L, z)][\partial_t \phi_2(L, z) - \partial_t \phi_1(-L, z)]$$

In terms of the phase factor $\phi$ the string operators $\mathcal{V}_\pm$ have the simple form

$$\mathcal{V}_+(0, L, z) = e^{i(\phi(0, L, z) - \phi(0, -L, z))}, \quad \mathcal{V}_-(0, L, z) = e^{i(\phi(0, -L, z))}$$

For $\phi(y^+, y^-, z)$ being a solution of the wave equation on $\pm$ variables the string operators $\mathcal{V}_\pm(z)$ should be written in terms of the initial date of the Goursat problem

$$\mathcal{V}_+(0, L, z) = e^{i(\phi_2(L, z) - \phi_1(-L, z))}, \quad \mathcal{V}_-(0, L, z) = e^{i(\phi_1(L, z) - \phi_1(-L, z))}$$

Therefore the correlator (43) for QED is

$$< \mathcal{V}_-(L, 0, z) \mathcal{V}_+(0, L, 0) >_{QED} = \int \exp\left\{ i(\phi_2(L, z) - \phi_1(-L, z)) + \phi_1(L, 0) - \right. \phi_1(-L, 0) \left. \right\} + \int d^2z [\partial_t \phi_1(L, z) - \partial_t \phi_1(-L, z)][\partial_t \phi_2(L, z) - \partial_t \phi_1(-L, z)]$$

$$\prod_z d\phi_1(L, z) d\phi_2(L, z) d\phi_1(-L, z).$$

Performing the change of variables $\rho_1(z) = \phi_1(L, z) - \phi_1(-L, z)$, $\rho_2(z) = \phi_2(L, z) - \phi_1(-L, z)$ and eliminating gauge degrees of freedom we left with the functional integral over two fields

$$< \mathcal{V}_-(L, 0, z) \mathcal{V}_+(0, L, 0) >_{QED} = \int \exp\left\{ i(\rho_1(z) - \rho_2(z)) + \int d^2z \partial_t \rho_1(z) \partial_t \rho_2(z) \right\} \prod_z d\rho_1(z) d\rho_2(z)$$

that gives the same answer as the previous calculation (43).
4.2 Nonabelian case

In the non-abelian case the string operators (16) also should be written in terms of independent variables (the initial data of the Goursat problem). In the case when we restrict ourselves by the consideration only massless excitations we have to take into account the following relation (see fig.2)

$$\tilde{M}(4) - \tilde{M}(2) = \tilde{M}(III) - \tilde{M}(II)$$  \hspace{1cm} (52)

So, $V(4)$ should be find as a solution of the difference equation

$$V_{z+i}^+(4)U_{z,i}(4)V_{z+i}(4) = V_{z+i}^+(2)U_{z,i}(2)V_{z+i}(2) \hspace{1cm} (53)$$

$$+ V_{z+i}^+(III)U_{z,i}(III)V_{z+i}(III) - V_{z+i}^+(II)U_{z,i}(II)V_{z+i}(II)$$

In the continuum case this equation has the form

$$D^4_i V(4, z) = V(4, z)(V^+(2, z)D_i^2 V(2, z) + \hspace{1cm} (54)$$

$$V^+(III, z)D_{III}^IV(III, z) - V^+(II, z)D_{II}^IV(II, z))$$

A solution of this equation can be represented as

$$V(4, z) = V(4, 0)P \exp \int_0^z (A_i(4, z') + V^+(2, z')D_i^2 V(2, z') \hspace{1cm} (55)$$

$$+ V^+(III, z')D_{III}^IV(III, z') - V^+(II, z')D_{II}^IV(II, z'))dz'$$

The necessity of taking into account the constraint (52) makes the calculation of the quark-antiquark scattering amplitude more complicated as compare with calculations performed in [11] and we postpone this calculation for future publications. Note here only that motivated by the previous consideration of the abelian case we can assume that in a rough approximation

$$V^+(4, z)V(2, z) = V^+(III, z)V(II, z) + \text{corrections} \hspace{1cm} (56)$$

In this approximation we can reproduce only one type of diagrams describing the quark-antiquark scattering amplitude.
5 Concluding Remarks.

Our paper was stimulated by the Verlinde and Verlinde paper [11] but there are some essential differences between our works.

- We start from the asymmetric lattice action, because the standard QCD perturbation theory cannot be applied in the Regge regime since $|t|$ is too small.

- We have obtained that the longitudinal dynamics of the truncated asymmetric lattice theory is described by the usual two-dimensional chiral field. Its continuous version depends crucially on the spectrum of the usual two-dimensional chiral field model in finite volume. In the framework of the perturbation theory there are only massless excitations and the corresponding continuum version is described by the free massless field. E.Verlinde and H.Verlinde started from this massless field that is correct only perturbatively.

- Transversal dynamics arising from boundary effects for free massless field and chiral field are different. These two theories coincide only in the leading order of the standard perturbation theory. Therefore two approaches give the same results only within the standard perturbation theory. The lattice version permits to take into account non-perturbative effects.

- Even if we start from free massless field as E.Verlinde and H.Verlinde did we have to take into account constrains on the boundary classical fields in the Feynman light-cone transition functions. This reduces the number of independent degrees of freedom and makes more complicated the calculation of string correlation functions in the non-abelian case.

The central observation of the paper [11] is that the leading Regge regime contribution comes from those gauge field configurations that are flat in the longitudinal directions. Consideration of asymmetric lattice gauge theory in the limit $a_0/a_t \to 0$ confirms this statement.

Our limiting theory for the continuous space-time in the longitudinal directions (12) does not reproduce the light-cone lattice action [16] since we do not assume that the longitudinal gauge field fluctuate around the unit element of the gauge group. As it has been mentioned in Sect.2 there is a connection between the truncated lattice action (12) and the light-cone transversal lattice QCD action considered previously by Bardeen, Pearson and Rabinovici [16]. Namely, (12) is the large $a_t$ version of the light-cone transversal lattice QCD [17]. The study of this theory for large $a_t$ corresponds to the regime of small, but non-zero $|t|$. Note that just this model due to the recent work of Klebanov and Susskind [15] has a rather interesting and mysterious string interpretation. Perhaps this is not so surprising since fluctuations of some special excitations of the two-dimensional chiral model, namely one-dimensional excitations, make this model in some sense equivalent to QCD$_2$ which has string interpretation according to recent Gross and Taylor investigations [18].

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