We investigate the existence of time-periodic solutions of the Dirac equation in the Kerr-Newman background metric. To this end, the solutions are expanded in a Fourier series with respect to the time variable $t$ and the Chandrasekhar separation ansatz is applied so that the question of existence of a time-periodic solution is reduced to the solvability of a certain coupled system of ordinary differential equations. First, we prove the already known result that there are no time-periodic solutions in the non-extreme case. Then it is shown that in the extreme case for fixed black hole data there is a sequence of particle masses $(m_N)_{N \in \mathbb{N}}$ for which a time-periodic solution of the Dirac equation does exist. The period of the solution depends only on the data of the black hole described by the Kerr-Newman metric.

I. INTRODUCTION

In this paper we consider the system consisting of a Kerr-Newman black hole and an electron. The Kerr-Newman metric describing the black hole is the most general electrovac solution of Einstein’s field equations. It describes a rotating, electrically charged, massive black hole, see, e.g., [15] and [9]. We are interested in the stability of a system consisting of such a black hole and an electron. To this end, we have to consider the Dirac equation for the electron in the Kerr-Newman metric, see equations (1) and (2)–(6). The Dirac equation is a complicated system of partial differential equations in all four spacetime variables. We call the system consisting of the black hole and the electron stable if there exists a nontrivial time-periodic solution of the Dirac equation which can be interpreted as the wave function of the electron.

Since the black hole is rotating, the background metric is only axisymmetric, whereas in the non-rotating cases (the Schwarzschild and the Reissner-Nordstrøm geometries) the background metric is spherically symmetric. This loss of symmetry leads to a complicated coupling of the angular and the radial coordinates in the Dirac equation. Chandrasekhar [4] showed that, in spite of this complicated coupling, the Dirac equation can be separated into a system of ordinary differential equations, the so-called angular equation (13) and the radial equation (14). These differential equations have realisations as eigenvalue equations in appropriate Hilbert spaces. For the radial equation, $\omega = \nu\omega_0$ plays the role of the eigenvalue parameter. The eigenvalue $\omega$ has the physical interpretation as the energy of the electron. It should be emphasised that, due to the lack of spherical symmetry of the spacetime, the eigenvalues of the radial and the angular equation are intertwined in a highly complex way. Hence to show the stability of the system under consideration, it does not suffice to find eigenvalues of the radial and the angular equation separately, but we need to show that the eigenvalues are compatible with each other.

In section II we present the separation ansatz for time-periodic solutions of the Dirac equation in the Kerr-Newman background metric due to Chandrasekhar with mathematical rigour and we derive the radial and angular equation. In the next section we consider the radial equation. It has been shown by Belgiorno and Martellini [2] that the essential spectrum of the radial operator covers the real axis. Finster et al. [8] show the nonexistence of time-periodic solutions of the Dirac equation in the non-extreme Kerr-Newman geometry. Schmid [12] investigates the Dirac equations in the extreme Kerr-Newman geometry with the help of special functions and shows the existence of bounded states in the extreme Kerr case, which implies the existence theorem of time-periodic solutions. Our aim is to study the difference between the extreme and non-extreme case from the viewpoint of spectral analysis of the radial Dirac operators. We give also the existence theorem of time-periodic solutions in the extreme Kerr-Newman case. In Theorem IV.5 we show that in the non-extreme

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Kerr-Newman metric the Dirac equation has no time-periodic solution that has an interpretation as a particle wave function. The case of the extreme Kerr-Newman metric is investigated in section V. It turns out that in this case there may be an eigenvalue of the radial equation. Schmid [12] proved a sufficient condition for the existence of a time-periodic solution of the Dirac equation. However, it is not easy to verify that for given particle data there are black hole parameters such that this condition can be satisfied. This problem is discussed in section VI. We show that for fixed data of an extreme Kerr-Newman black hole there is a sequence of particle masses such that the system consisting of the quantum particle and the black hole permits time-periodic solutions.

II. SEPARATION OF THE DIRAC EQUATION IN THE KERR-NEWMAN BACKGROUND METRIC

We consider the Dirac equation (see, e.g., Page [11], Chandrasekhar [4])

\[(\mathcal{R} + \mathcal{A})\Psi = 0\]  

for a spin-\(\frac{1}{2}\) particle with the mass \(m \geq 0\) and the charge \(e\) in the Kerr-Newman geometry, where

\[
\mathcal{R} := \begin{pmatrix}
im r & 0 & \sqrt{\Delta} D_+ & 0 \\
0 & -im r & 0 & \sqrt{\Delta} D_- \\
\sqrt{\Delta} D_- & 0 & -im r & 0 \\
0 & \sqrt{\Delta} D_+ & 0 & im r
\end{pmatrix},
\]

\[
\mathcal{A} := \begin{pmatrix}
-am \cos \theta & 0 & 0 & L_+ \\
0 & am \cos \theta & -L_- & 0 \\
0 & L_+ & -am \cos \theta & 0 \\
-L_- & 0 & am \cos \theta & 0
\end{pmatrix},
\]

\[
D_{\pm} := \frac{\partial}{\partial r} \mp \frac{1}{\Delta} \left( (r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \varphi} - ieQr \right),
\]

\[
L_{\pm} := \frac{\partial}{\partial \theta} + \frac{\cot \theta}{2} \mp i \left( a \sin \theta \frac{\partial}{\partial t} + \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \right),
\]

\[
\Delta(r) := r^2 - 2Mr + a^2 + Q^2
\]

and \(\Psi\) is the wave function of the spin-\(\frac{1}{2}\) particle under consideration. If the so-called black hole condition

\[M^2 - a^2 - Q^2 \geq 0\]

holds, then the Kerr-Newman geometry is interpreted as the spacetime geometry generated by a black hole with the mass \(M \geq 0\), the electric charge \(Q\) and the angular momentum \(J\); if \(M \neq 0\), then the so-called Kerr parameter \(a = J/M\) is the angular momentum of the black hole per unit mass. The black hole condition (7) ensures that the function \(\Delta\) can be written as the product

\[\Delta(r) = (r - r_+)(r - r_-)\]

with

\[r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}.\]

The special case \(M^2 - a^2 - Q^2 = 0\), that is the case where \(r_+ = r_- = M\), is referred to as the extreme Kerr-Newman metric.

Let us recall that the Kerr-Newman metric is the most general electrovac solution of Einstein’s field equations [9]. Special cases contained in the Kerr-Newman geometry are the Kerr geometry (if \(Q = 0\)), the Reissner-Nordstrom geometry (if \(a = 0\)) and the Schwarzschild geometry (if \(Q = 0\) and \(a = 0\)).

A solution \(\tilde{\Psi}\) of (1) for \((r, \theta, \varphi, t) \in \Omega := (r_+, \infty) \times (0, \pi) \times (-\pi, \pi) \times (-\infty, \infty)\) such that for every fixed time \(t\) the function \(\tilde{\Psi}(\cdot, \cdot, \cdot, t)\) lies in a suitable \(L^2\)-space \(\mathcal{H}_{r, \theta, \varphi}\) (see (9)) can be interpreted as the wave function of the electron. Hence the existence of such a \(\tilde{\Psi}\) would imply that the system consisting of the black hole and the spin-\(\frac{1}{2}\) particle in its exterior is stable.
Remark II.1 (Dirac equation in flat space time). In the case of flat spacetime, i.e. for $a = 0$, $M = 0$, $Q = 0$, the Dirac equation given in (1) is unitarily equivalent to the familiar Dirac equation

$$\left(-i\frac{\partial}{\partial t} - i \vec{\alpha} \cdot \vec{\nabla} + \beta m\right) \Psi = 0$$

as given, for instance, in Davydov [5]. For the proof we refer to Winklmeier [18].

A. Time periodic solutions

In this paper we consider time-periodic solutions $\hat{\Psi}$, that is, solutions such that

$$\hat{\Psi}(r, \theta, \varphi, t) = \hat{\Psi}(r, \theta, \varphi, t + \frac{2\pi}{\omega_0}) \quad ((r, \theta, \varphi, t) \in \hat{\Omega})$$

for some $\omega_0 > 0$. In this case, the solutions can be expanded in a Fourier series

$$\hat{\Psi}(r, \theta, \varphi, t) = \sum_{\nu \in \mathbb{Z}} \exp(-i\nu\omega_0 t) \Psi^{\nu}(r, \theta, \varphi).$$

For physical reasons, each wave function $\Psi^{\nu}$ must be an element of the Hilbert space

$$\mathcal{H}_{r, \theta, \varphi} := L^2 \left((r_0, \infty) \times (0, \pi) \times (-\pi, \pi); \frac{r^2 + a^2}{\Delta(r)} \sin \theta \ dr \ d\theta \ d\varphi\right).$$

with the inner product according to Finster et al. [8]

$$(\Psi, \Phi) = \int_{r_0}^{\infty} \int_0^\pi \int_{-\pi}^{\pi} (\Psi(r, \theta, \varphi), \Phi(r, \theta, \varphi))_{C^4} r^2 + a^2 \Delta(r) \sin \theta \ dr \ d\theta \ d\varphi. \quad (10)$$

Here $\langle \cdot, \cdot \rangle_{C^4}$ is the usual scalar product on $C^4$. In order to separate off also the $\varphi$-dependence of the solution $\Psi^{\nu}$ we use the complete system $\{\exp(i\kappa \varphi) : \kappa \in \mathbb{Z} + \frac{1}{2}\}$ in $L^2((-\pi, \pi); d\varphi)$ to employ the ansatz

$$\Psi^{\nu}(r, \theta, \varphi) = \sum_{\kappa \in \mathbb{Z} + \frac{1}{2}} \exp(-i\kappa \varphi) \Psi^{\nu, \kappa}(r, \theta).$$

Thus, if

$$\hat{\Psi}(r, \theta, \varphi, t) = \sum_{\nu \in \mathbb{Z}} \sum_{\kappa \in \mathbb{Z} + \frac{1}{2}} \exp(-i\omega_0 t) \exp(-i\kappa \varphi) \Psi^{\nu, \kappa}(r, \theta)$$

satisfies (1), then each $\Psi^{\nu, \kappa}$ is a solution of

$$(\mathcal{R}_{\nu, \kappa} + \mathcal{A}_{\nu, \kappa}) \Psi^{\nu, \kappa} = 0 \quad \text{on} \quad (r_0, \infty) \times (0, \pi), \quad (11)$$

where $\mathcal{R}_{\nu, \kappa}$ and $\mathcal{A}_{\nu, \kappa}$ are obtained from (2) and (3) by replacing $\mathcal{D}_\pm$ and $\mathcal{L}_\pm$ by

$$\mathcal{D}_{\pm, \nu, \kappa} := \frac{\partial}{\partial r} \pm \frac{i}{\Delta} \left[\omega_0 \nu (r^2 + a^2) + \kappa a + e Q r\right],$$

$$\mathcal{L}_{\pm, \nu, \kappa} := \frac{\partial}{\partial \theta} + \cot \theta \frac{\partial}{\partial \theta} \pm \left[\omega_0 \nu \sin \theta + \frac{\kappa}{\sin \theta}\right],$$

respectively (see the proof of Theorem IV.5).

B. Separation of the radial and the angular coordinate

To study the equation (11) we consider the formal differential expression

$$\mathfrak{A}_{\nu, \kappa} := \begin{pmatrix} -am \cos \theta & \mathcal{L}_{-, \nu, \kappa} \\ -\mathcal{L}_{+, \nu, \kappa} & am \cos \theta \end{pmatrix}$$
for \( \theta \in (0, \pi) \). For any half integer \( \kappa \) the differential expression \( D_{\nu, \kappa} \) has a unique self-adjoint realisation \( A_{\nu, \kappa} \) in the Hilbert space \( L^2((0, \pi); \sin \theta d\theta)^2 \) which has purely discrete spectrum \( \sigma(A_{\nu, \kappa}) = \{ \lambda_{\nu, \kappa, n} : n \in \mathbb{Z} \setminus \{0\} \} \subseteq \mathbb{R} \) where each eigenvalue is simple (see, e.g., Batic et al. [1], Winklmeier [18]). Hence there is a complete set of orthonormal eigenfunctions of \( A_{\nu, \kappa} \):

\[
g^{\nu, \kappa, n} := \begin{pmatrix} g_1^{\nu, \kappa, n} \\ g_2^{\nu, \kappa, n} \end{pmatrix} \in L^2((0, \pi); \sin \theta d\theta)^2 \quad (n \in \mathbb{Z} \setminus \{0\})
\]

with eigenvalues \( \lambda_{\nu, \kappa, n} \). The family (12) allows us to make the ansatz

\[
\Psi^{\nu, \kappa} = \sum_{n \in \mathbb{Z} \setminus \{0\}} \Psi^{\nu, \kappa, n}(r, \theta)
\]

with

\[
\Psi^{\nu, \kappa, n}(r, \theta) = \begin{pmatrix} X_+^{\nu, \kappa, n}(r) g_2^{\nu, \kappa, n}(\theta) \\ X_-^{\nu, \kappa, n}(r) g_1^{\nu, \kappa, n}(\theta) \\ X_+^{\nu, \kappa, n}(r) g_1^{\nu, \kappa, n}(\theta) \\ X_-^{\nu, \kappa, n}(r) g_2^{\nu, \kappa, n}(\theta) \end{pmatrix} \quad (r \in (r_+, \infty), \ \theta \in (0, \pi))
\]

which leads to a separation of the angular coordinate \( \theta \) and the radial coordinate \( r \) (cf. Chandrasekhar [4]): the angular function \( g^{\nu, \kappa, n} \) satisfies the angular equation

\[
A_{\nu, \kappa} g^{\nu, \kappa, n} = \lambda_{\nu, \kappa, n} g^{\nu, \kappa, n}
\]

with the integrability condition \( g^{\nu, \kappa, n} \in L^2((0, \pi); \sin \theta d\theta) \) and the radial function \( X^{\nu, \kappa, n} = (X_+^{\nu, \kappa, n}, X_-^{\nu, \kappa, n}) \) satisfies the radial equation

\[
\begin{pmatrix} -imr - \lambda_{\nu, \kappa, n} & \sqrt{\Delta} D_{-\nu, \kappa} \\ \sqrt{\Delta} D_{+\nu, \kappa} & imr - \lambda_{\nu, \kappa, n} \end{pmatrix} \begin{pmatrix} X_+^{\nu, \kappa, n}(r) \\ X_-^{\nu, \kappa, n}(r) \end{pmatrix} = 0,
\]

with the integrability condition arising from the inner product on \( \mathcal{H}_{r, \theta, \varphi} \)

\[
\int_{r_+}^{\infty} (|X_+^{\nu, \kappa, n}(r)|^2 + |X_-^{\nu, \kappa, n}(r)|^2) \frac{r^2 + a^2}{\Delta(r)} dr < \infty.
\]

\[\text{C. The radial equation}\]

Let

\[
W := \frac{1}{\sqrt{2}} \begin{pmatrix} i & i \\ -1 & -1 \end{pmatrix}, \quad \tilde{V} := \begin{pmatrix} 0 & -1/\sqrt{\Delta} \\ 1/\sqrt{\Delta} & 0 \end{pmatrix}, \quad \tilde{f}^{\nu, \kappa, n}(r) := \begin{pmatrix} f_1^{\nu, \kappa, n}(r) \\ f_2^{\nu, \kappa, n}(r) \end{pmatrix} := W \begin{pmatrix} X_+^{\nu, \kappa, n}(r) \\ X_-^{\nu, \kappa, n}(r) \end{pmatrix}.
\]

Then we obtain from (14)

\[
0 = \tilde{V} W \begin{pmatrix} -imr - \lambda_{\nu, \kappa, n} & \sqrt{\Delta} D_{-\nu, \kappa} \\ \sqrt{\Delta} D_{+\nu, \kappa} & imr - \lambda_{\nu, \kappa, n} \end{pmatrix} W^{-1} \tilde{f}^{\nu, \kappa, n}(r)
\]

\[
= \begin{pmatrix} \frac{mr}{\sqrt{\Delta}} - \frac{ak + eQr + \omega_0 \nu (r^2 + a^2)}{\Delta} & - \frac{d}{dr} + \frac{\lambda_{\nu, \kappa, n}}{\sqrt{\Delta}} \\ \frac{d}{dr} + \frac{\lambda_{\nu, \kappa, n}}{\sqrt{\Delta}} & - \frac{mr}{\sqrt{\Delta}} - \frac{ak + eQr + \omega_0 \nu (r^2 + a^2)}{\Delta} \end{pmatrix} \tilde{f}^{\nu, \kappa, n}(r).
\]

for \( r \in (r_+, \infty) \) (see Winklmeier [18]).
Remark II.2. If we take into account only terms of first order in $1/r$ for large $r$, then the radial equation becomes
\[
\begin{pmatrix}
 m - \frac{eQ}{r} + \omega_0 \nu & -\frac{d}{dr} + \frac{\lambda_{\nu,\kappa,n}}{r}
\end{pmatrix}
\begin{pmatrix}
 f(r)
\end{pmatrix}
= 0,
\]
which is exactly the radial equation for the relativistic hydrogen atom, see, e.g., Bjorken and Drell [3].

Remark II.3. It is important to note that in the case $a \neq 0$ the eigenvalue $\lambda$ for the angular equation does depend on $\omega_0 \nu$. In the case $a = 0$, the eigenvalues can be calculated explicitly; they are given by
\[
\sigma(A_{\nu,\kappa}) = \{ \lambda_{\nu,\kappa,n} = \text{sign}(n)(|n| - \frac{1}{2} + |n|) : n \in \mathbb{Z} \setminus \{0\} \} \subseteq \mathbb{R}.
\]
In particular, the eigenvalues $\lambda_{\nu,\kappa,n}$ of the radial equation do not depend on the eigenvalues of the radial equation. Hence, in the case $a = 0$, for a solution of the complete problem (14)–(13) first the angular problem $(A_{\nu,\kappa} - \lambda_{\nu,\kappa})g = 0$ is solved for the eigenvalues $\lambda_{\nu,\kappa,n}$, and then the radial equation (14) can be attacked for fixed $\lambda_{\nu,\kappa,n}$.

We introduce a new coordinate $x$ such that
\[
dx = \frac{r^2 + a^2}{\Delta(r)} \quad (r > r_+),
\]
that is,
\[
x(r) = \begin{cases}
r + \frac{r^2 + a^2}{r_- - r} \log (r - r_+) - \frac{r^2 + a^2}{r_+ - r_-} \log (r - r_-) + x_0 & (r_+ \neq r_-),
\end{cases}
\]
\[
+ 2r_+ \log (r - r_+) - \frac{r^2 + a^2}{r - r_+} + x_0 & (r_+ = r_-),
\]
where $x_0$ is a constant of integration that can be set $x_0 = 0$. The correspondence between $r > r_+$ and $x \in (-\infty, \infty)$ is a bijection. With the new coordinate $x$ equation (17) becomes
\[
H_{\nu,\kappa,n} f_{\nu,\kappa,n} := \begin{pmatrix}
\frac{mr\sqrt{\Delta}}{r^2 + a^2} - \frac{ak + eQ}{r^2 + a^2} & d & \frac{\lambda_{\nu,\kappa,n}\sqrt{\Delta}}{r^2 + a^2}
\end{pmatrix}
\begin{pmatrix}
f_{\nu,\kappa,n}
\end{pmatrix}
= \omega_0 \nu f_{\nu,\kappa,n},
\]
where $r$ has to be understood as $r(x)$ and $f_{\nu,\kappa,n}(x) = \tilde{f}_{\nu,\kappa,n}(r(x))$, $f_{j,\nu,\kappa,n}(x) = \tilde{f}_{j,\nu,\kappa,n}(r(x))$ for all $x \in (-\infty, \infty)$ and $j \in \{1, 2\}$. In view of (16) and (19) the integrability condition (15) becomes
\[
\int_{-\infty}^{\infty} \| f_{\nu,\kappa,n}(x) \|^2_{L^2} dx = \int_{-\infty}^{\infty} \left[ \| f_{1,\nu,\kappa,n}(x) \|^2 + \| f_{2,\nu,\kappa,n}(x) \|^2 \right] dx < \infty.
\]
The operator $H_{\nu,\kappa,n}$ is formally symmetric in the Hilbert space $\mathcal{H}_x := L^2((-\infty, \infty); dx)^2$, so it is natural to look for an operator theoretical realisation of $H_{\nu,\kappa,n}$ in the Hilbert space $\mathcal{H}_x$. The purpose of this note is to investigate the spectral properties of the self-adjoint operator $H_{\nu,\kappa,n}$ in $\mathcal{H}_x = L^2((-\infty, \infty); dx)^2$ and to study the non-existence of non-trivial solutions satisfying (11). In section VI we will investigate the existence of so-called energy eigenvalues of $H_{\nu,\kappa,n}$ (cf. Schmid [12]).

Definition II.4. We call $\omega \in \mathbb{R}$ an energy eigenvalue of $H_{\nu,\kappa,n}$ if there are $\lambda_{\nu,\kappa,n}$ such that $\omega$ is an eigenvalue of $H_{\nu,\kappa,n}$ and $\lambda_{\nu,\kappa,n}$ is an eigenvalue of $A_{\nu,\kappa}$, that is, if equations (13) and (14) can be solved simultaneously with functions satisfying the corresponding integrability conditions.
In this and the following sections we consider the eigenvalue equation \((H_{\nu,\kappa,n} - \omega\nu_0)f^{\nu,\kappa,n} = 0\) from (21) on the Hilbert space \(\mathcal{H}_x = L^2((-\infty, \infty); \, dx)^2\).

If there is no ambiguity, we omit the indices \(\nu, n\) and \(\kappa\) in the following for the sake of clarity; for instance, we write simply \(H\) instead of \(H_{\nu,\kappa,n}\), \(\lambda\) instead of \(\lambda_{\nu,\kappa,n}\) and \(\omega\) instead of \(\omega_0\nu\). We decompose the operator \(H\) into the sum

\[
H = H_0 + V
\]

where

\[
H_0 = \left(\begin{array}{cc} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{array}\right), \quad \mathcal{D}(H_0) = C_c^\infty(-\infty, \infty)^2,
\]

\[
V(x) = \begin{pmatrix} A(x) & B(x) \\ B(x) & C(x) \end{pmatrix},
\]

\[
A(x) = \frac{m r(x) \sqrt{\Delta(r(x))}}{r(x)^2 + a^2} \frac{ak + eQr(x)}{r(x)^2 + a^2},
\]

\[
B(x) = \lambda \frac{\sqrt{\Delta(r(x))}}{r(x)^2 + a^2},
\]

\[
C(x) = -\frac{m r(x) \sqrt{\Delta(r(x))}}{r(x)^2 + a^2} \frac{ak + eQr(x)}{r(x)^2 + a^2},
\]

\[
\Delta(x) = (r(x) - r_+)(r(x) - r_-).
\]

The operator \(H_0\) is symmetric and has a unique self-adjoint extension on the space \(L^2((-\infty, \infty); \, dx)^2\), see, e.g., Weidmann [17, Theorem 6.8]. Since \(x \to -\infty\) is equivalent to \(r(x) \to r_+\) and \(x \to \infty\) is equivalent to \(r(x) \to \infty\), we have

\[
\lim_{x \to -\infty} A(x) = -\frac{ak + eQr_+}{r_+^2 + a^2} =: A_0, \quad \quad \lim_{x \to +\infty} A(x) = m,
\]

\[
\lim_{x \to -\infty} C(x) = A_0, \quad \lim_{x \to +\infty} C(x) = -m,
\]

\[
\lim_{x \to +\infty} B(x) = 0
\]

which implies that the functions \(A(\cdot), B(\cdot)\) and \(C(\cdot)\) are bounded. Since we assume that the black hole condition (7) holds, the multiplication operator \(V\) is symmetric. Therefore, \(H = H_0 + V\) has a unique self-adjoint extension which we again denote by \(H\).

In what follows, a prime \(\prime\) always denotes differentiation with respect to \(x\).

**Lemma III.1 (Asymptotic behaviour of \(V\) for \(x \to -\infty\)).**

(i) For \(x \to -\infty\) the functions

\[
A(x) - A_0, \quad B(x), \quad C(x) - A_0
\]

decay exponentially in the case \(r_+ \neq r_-\), and they are of order \(O(x^{-1})\) in the case \(r_+ = r_-\). More precisely, in the latter case we have

\[
A(x) - A_0 = \frac{mM - \mu}{-x} + O\left(\frac{1}{x^2}\right) \quad \text{as} \quad x \to -\infty,
\]

\[
A_0 - C(x) = \frac{mM + \mu}{-x} + O\left(\frac{1}{x^2}\right) \quad \text{as} \quad x \to -\infty,
\]
We prove the assertions for the functions \( \alpha \).

(ii) The derivatives \( A', B' \) and \( C' \) are integrable with respect to \( x \) on \((-\infty, 0] \).

(iii) \( A'(x) = O(x^{-2}) \), \( B'(x) = O(x^{-2}) \) and \( C'(x) = O(x^{-2}) \) hold as \( x \to +\infty \).

\textbf{Proof.} We prove the assertions for the functions \( B \) and \( A \) only. The corresponding assertions for \( C \) can be obtained from those for \( A \) by substituting \( m \) by \(-m\). We remark that (20) shows

\[ r(x) - r_+ = O(\exp(ax)) \quad \text{as} \quad x \to -\infty \quad (r_+ \neq r_-), \quad \ldots \quad (23) \]

\[ r(x) - r_+ = O(x^{-1}) \quad \text{as} \quad x \to -\infty \quad (r_+ = r_-) \quad \ldots \quad (24) \]

for a positive constant \( a \). To keep notation simple, let us write \( r \) instead of \( r(x) \) in this proof.

(i) Note that

\[ \frac{1}{r^2 + a^2} = \frac{1}{r_+^2 + a^2} - \frac{(r - r_+)(r + r_+)}{(r_+^2 + a^2)(r^2 + a^2)} = \frac{1}{r_+^2 + a^2} + O(r - r_+). \quad \ldots \quad (25) \]

Hence it follows immediately that

\[ B(x) = \frac{\lambda}{x} + O\left(\frac{1}{x^2}\right) \quad \text{as} \quad x \to -\infty, \]

where

\[ \mu := O\left(\frac{2\alpha M}{M^2 + a^2} - eQ \frac{1}{M^2 + a^2}\right). \]

(ii) A simple calculation gives

\[ \frac{d}{dx} A(x) = \frac{d}{dx} \frac{d}{dr} A(x(r)) = \frac{\Delta(r)}{r^2 + a^2} \frac{d}{dr} A(x(r)). \]
\[ \frac{\sqrt{\Delta(r)}}{r^2 + a^2} \left\{ m \left( \frac{\Delta(r)}{r^2 + a^2} + \frac{r\Delta'(r)}{2(r^2 + a^2)} - \frac{2r^2 \Delta(r)}{(r^2 + a^2)^2} + \frac{2(\alpha \kappa + eQr)r\sqrt{\Delta(r)}}{(r^2 + a^2)^2} - \frac{eQ\sqrt{\Delta(r)}}{r^2 + a^2} \right) \right\} \]

\[ = \begin{cases} O((r - r_+)^{1/2}) = O(\exp[(1/2)\alpha x]), & (r_+ \neq r_-), \\
O((r - r_+)^2) = O(x^{-2}), & (r_+ = r_-), \\
O((r - r_-)^{1/2}) = O(\exp[(1/2)\alpha x]), & (r_+ \neq r_-), \\
O((r - r_-)^2) = O(x^{-2}), & (r_+ = r_-). \end{cases} \]

as \( x \to -\infty \).

\[ \frac{d}{dx} B(x) = \frac{d}{dx} \frac{d}{dr} B(x(r)) = \frac{\Delta(r)}{r^2 + a^2} \frac{d}{dr} B(x(r)) \]

\[ = \lambda \frac{\sqrt{\Delta(r)}}{r^2 + a^2} \left\{ \frac{\Delta'(r)}{2(r^2 + a^2)} - \frac{2r\Delta(r)}{(r^2 + a^2)^2} \right\} \]

\[ = \begin{cases} O((r - r_+)^{1/2}) = O(\exp[(1/2)\alpha x]), & (r_+ \neq r_-), \\
O((r - r_+)^2) = O(x^{-2}), & (r_+ = r_-). \end{cases} \]

as \( x \to -\infty \).

(iii) Since

\[ \frac{dr}{dx} = \frac{\Delta(r)}{r^2 + a^2} \sim 1, \quad x \sim r \quad \text{as} \quad x \to +\infty, \]

and

\[ 2(r^2 + a^2)\Delta(r) + r(r^2 + a^2)\Delta'(r) - 4r^2 \Delta(r) = 2Mr^3 + 2r^2(a^2 - Q^2) - 6a^2Mr + 2a^2(a^2 + Q^2) \sim r^3 \quad \text{as} \quad r \to \infty \]

the assertion follows from (26). The proof of the assertion concerning \( B'(x) \) follows directly from (27). □

IV. ABSOLUTELY CONTINUOUS SPECTRUM

The following proposition has been shown by Belgiorno and Martellini \[2\].

Proposition IV.1. \( \sigma_{ess}(H) = \mathbb{R} \). □

We point out that the proof of the proposition relies on the fact that

\[ \lim_{x \to -\infty} [V(x) - A_0 I_2] = 0, \]

where \( I_2 \) is the 2 \( \times \) 2 unit matrix. Lemma III.1 yields the following theorem.

Theorem IV.2. (i) \( H \) has purely absolutely continuous spectrum in \( \mathbb{R} \setminus \{A_0\} \).

(ii) \( H \) has purely absolutely continuous spectrum in \( (-\infty, -m) \) and \( (m, +\infty) \),

that is, \( H \) is absolutely continuous in the complement of \( [-m, m] \cap \{A_0\} \).

Proof. (i) Lemma III.1 gives (28) and any component of \( V' \) is integrable at \( -\infty \). Therefore we can prove the theorem in view of Weidmann \[16\], Schmidt \[13\] and also Thaller \[14, Theorem 4.18\].

(ii) The proof is the same as in (i) by using Lemma III.1 (iii). □

Remark IV.3. The above theorem has already been proven by Schmid by different means (see Schmid \[12, Corollary 3.4\]); in addition, he has shown that neither \( \omega = m \) nor \( \omega = -m \) is an eigenvalue of \( H \) \([12, Lemma 3.5] \).
Theorem IV.4. If \( r_+ \neq r_- \), then \( A_0 \) is not an eigenvalue of \( H \).

Proof. Let us assume that \( U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \) satisfies

\[
\begin{pmatrix}
A & B - \frac{d}{dx} \\
B + \frac{d}{dx} & -C
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A_0 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},
\]

that is,

\[
U' = \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \begin{pmatrix} -B & A - C \\
A - A_0 & B \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]

(29)

As seen in section III, \( A(x) - A_0, B(x) \) and \( C(x) - A_0 \) decay exponentially as \( x \to -\infty \). Therefore, the Levinson theorem (e.g. Eastham [6]) gives that there are two linearly independent solutions \( U^1, U^2 \) of (29) such that

\[
U^1(x) = \begin{cases} 
\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \\
\end{cases} \quad \text{as } x \to -\infty,
\]

\[
U^2(x) = \begin{cases} 
\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \\
\end{cases} \quad \text{as } x \to -\infty.
\]

Hence there are constants \( c_1 \) and \( c_2 \) such that \( U(x) = c_1 U^1(x) + c_2 U^2(x) \), which is square integrable on \((-\infty, 0)\) only if \( c_1 = c_2 = 0 \).

As an important corollary we obtain the following theorem.

Theorem IV.5 (Non-existence of time-periodic solutions for \( r_+ \neq r_- \)). Let \( \hat{\Psi} \in C^1((r_0, \infty) \times (0, \pi) \times [\pi, \pi] \times \mathbb{R}, \mathbb{C}^4) \) be a solution of (1) satisfying the periodicity conditions

\[
\hat{\Psi}(r, \theta, \varphi, t) = \hat{\Psi}(r, \theta, \varphi, t + \frac{2\pi}{\omega_0}),
\]

\[
\hat{\Psi}(r, \theta, \pi, t) = \hat{\Psi}(r, \theta, \pi, t),
\]

(30)

for some \( \omega_0 > 0 \). Furthermore, assume that for all \( (\varphi, t) \in [\pi, \pi] \times \mathbb{R} \) we have \( \hat{\Psi}(\cdot, \cdot, \varphi, t) \in C^0((r_0, \infty) \times (0, \pi); \mathcal{H}_{r, \theta}) \) where

\[
\mathcal{H}_{r, \theta} := \mathcal{L}^2 \left((r_+, \infty) \times (0, \pi); \frac{r_+ + \pi^2}{2(r)} \sin \theta dr d\theta \right)^4
\]

with the norm denoted by \( \| \cdot \|_{\mathcal{H}_{r, \theta}} \) and the inner product denoted by \( \langle \cdot, \cdot \rangle_{\mathcal{H}_{r, \theta}} \).

If \( r_- \neq r_+ \), then \( \hat{\Psi} \equiv 0 \).

Proof. Let \( \hat{\Psi} \) be a time-periodic solution satisfying the conditions of the theorem. Then \( \hat{\Psi}(\cdot, \cdot, \cdot, t) \) is an \( \mathcal{H}_{r, \theta, \varphi} \)-valued strongly continuous function with respect to \( t \) since \( \hat{\Psi}(\cdot, \cdot, \varphi, t) \) is uniformly continuous in \( \mathcal{H}_{r, \theta} \) with respect to \( (\varphi, t) \in [\pi, \pi] \times \mathbb{R} \). Therefore we can expand \( \hat{\Psi}(r, \theta, \varphi, t) \) as the Fourier series with respect to \( t \)

\[
\hat{\Psi}(r, \theta, \varphi, t) = \sum_{\nu \in \mathbb{Z}} \exp(-i\omega_0 \nu t) \Psi^\nu(r, \theta, \varphi)
\]

strongly in \( \mathcal{L}^2([0, 2\pi/\omega_0]; \mathcal{H}_{r, \theta, \varphi}; dt) \), where

\[
\Psi^\nu(r, \theta, \varphi) = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \exp(i\omega_0 \nu t) \hat{\Psi}(r, \theta, \varphi, t) dt,
\]

\[
\sum_{\nu} \| \Psi^\nu \|_{\mathcal{H}_{r, \theta, \varphi}}^2 < \infty.
\]

Moreover, each \( \Psi^\nu(r, \theta, \varphi) \) can be expanded as

\[
\Psi^\nu(r, \theta, \varphi) = \sum_{\kappa \in \mathbb{Z} + \frac{1}{2}} \exp(-i\kappa \varphi) \Psi^{\nu, \kappa}(r, \theta)
\]
strongly in $L^2([-\pi, \pi]; H_{r, \theta}; \, d\phi)$ with
$$
\Psi^{\nu}(\theta, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(i\kappa \varphi) \Psi^{\nu}(r, \theta, \varphi) \, d\varphi, \quad \sum_{\kappa} \|\Psi^{\nu, \kappa}\|_{H_{r, \theta}}^2 < \infty.
$$

For every $\Phi \in C_0^\infty((r_0, \infty) \times (0, \pi))^4$ of the form
$$
\Phi(r, \theta) = \begin{pmatrix}
\rho_-(r) \eta_2(\theta) \\
\rho_+(r) \eta_1(\theta) \\
\rho_-(r) \eta_2(\theta) \\
\rho_+(r) \eta_1(\theta)
\end{pmatrix}
$$
with $\rho_{\pm}(r) \in C_0^\infty((r_+, \infty)), \, \eta_1(\theta), \, \eta_2(\theta) \in C_0^\infty((0, \pi))$ we obtain, by using (30),

$$
0 = \int_0^{2\pi/\omega_0} dt \int_{-\pi}^{\pi} d\varphi \left\langle (\hat{\mathcal{R}} + \hat{A}) \hat{\Psi}, \exp(-i\omega_0 t) \exp(-i\kappa \varphi) \right\rangle_{H_{r, \theta}} = 0
$$

$$
= \int_0^{2\pi/\omega_0} dt \int_{-\pi}^{\pi} d\varphi \left\langle \hat{\Psi}, (\hat{\mathcal{R}} + \hat{A})^* \exp(-i\omega_0 t) \exp(-i\kappa \varphi) \right\rangle_{H_{r, \theta}} = 0
$$

$$
= \int_0^{2\pi/\omega_0} dt \int_{-\pi}^{\pi} d\varphi \left\langle \hat{\Psi}, \exp(-i\omega_0 t) \exp(-i\kappa \varphi)(\mathcal{R}_{\nu, \kappa} + A_{\nu, \kappa})^* \Phi \right\rangle_{H_{r, \theta}} = 0
$$

$$
= \frac{2\pi}{\omega_0} \int_{-\pi}^{\pi} \left\langle \hat{\Psi}, \exp(-i\kappa \varphi)(\mathcal{R}_{\nu, \kappa}^* + A_{\nu, \kappa}^*) \right\rangle_{H_{r, \theta}} dt
$$

$$
= \frac{4\pi^2}{\omega_0} \left\langle \Phi^{\nu, \kappa}, (\mathcal{R}_{\nu, \kappa}^* + A_{\nu, \kappa}^*) \Phi \right\rangle_{H_{r, \theta}}.
$$

(31)

In the above calculation, the superscript $^*$ denotes the formal adjoint operator. As in (12), for fixed $\nu$ and $\kappa$ let

$$
g^{\nu, \kappa, n}(\theta) = \begin{pmatrix} g_{1, \nu, \kappa, n}^{\nu, \kappa, n}(\theta) \\
g_{2, \nu, \kappa, n}^{\nu, \kappa, n}(\theta)
\end{pmatrix} \quad (n \in \mathbb{Z} \setminus \{0\})
$$

be a complete family of orthonormal eigenfunctions of $A_{\nu, \kappa}$ with eigenvalues $\lambda_{\nu, \kappa, n}$, respectively. Then $\Psi^{\nu, \kappa}(r, \theta)$ can be expanded in terms of $g^{\nu, \kappa, n}$ ($n \in \mathbb{Z} \setminus \{0\}$) as follows

$$
\Psi^{\nu, \kappa}(r, \theta) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \begin{pmatrix} X_{\nu, \kappa, n}^{\nu, \kappa, n}(r) g_{1, \nu, \kappa, n}^{\nu, \kappa, n}(\theta) \\
X_{\nu, \kappa, n}^{\nu, \kappa, n}(r) g_{2, \nu, \kappa, n}^{\nu, \kappa, n}(\theta)
\end{pmatrix},
$$

where the series converges in the strong sense in $L^2((0, \pi); \, \sin \theta \, d\theta)^4$ and $X_{\pm, \kappa, n}^{\nu, \kappa}(r)$ satisfies

$$
\sum_{n \in \mathbb{Z} \setminus \{0\}} \int_{r_+}^{\infty} (|X_{\pm, \kappa, n}^{\nu, \kappa}(r)|^2 + |X_{\pm, \kappa, n}^{\nu, \kappa}(r)|^2) \frac{r^2 + a^2}{\Delta(r)} \, dr < \infty.
$$

Since $A_{\nu, \kappa}$ on $C_0^\infty((0, \pi))^2$ is essentially selfadjoint in $L^2((0, \pi); \, \sin \theta \, d\theta)$, for any $g^{\nu, \kappa, n}(\theta)$ there exists a convergent sequence $(\eta^{\ell})_{\ell \in \mathbb{N}} \subset C_0^\infty((0, \pi))^2$ such that

$$
A_{\nu, \kappa} \eta^{\ell} = A_{\nu, \kappa} \begin{pmatrix} \eta_1^{\ell} \\
\eta_2^{\ell}
\end{pmatrix} \rightarrow A_{\nu, \kappa} g^{\nu, \kappa, n} = \lambda_{\nu, \kappa, n} g^{\nu, \kappa, n}
$$

in $L^2((0, \pi); \, \sin \theta \, d\theta)^2$. Substituting $\Phi$ in (31) by

$$
\Phi^{\ell}(r, \theta) = \begin{pmatrix}
\rho_-(r) \eta_2^{\ell}(\theta) \\
\rho_+(r) \eta_1^{\ell}(\theta) \\
\rho_-(r) \eta_2^{\ell}(\theta) \\
\rho_+(r) \eta_1^{\ell}(\theta)
\end{pmatrix}
$$
and taking the limit $\ell \to \infty$, we have

$$
\left\langle \sum_{m \in \mathbb{Z} \setminus \{0\}} \begin{pmatrix} X_{+,\nu,\kappa,m}^\nu(r) g^\nu_{+,\kappa,m}(\theta) \\ X_{-,\nu,\kappa,m}^\nu(r) g^\nu_{-,\kappa,m}(\theta) \end{pmatrix}, \begin{pmatrix} (R^\nu_{+,\nu,\kappa} - \lambda_{\nu,\kappa,m}) & \rho_+(r) g^\nu_{+,\kappa,m}(\theta) \\ \rho_-(r) g^\nu_{-,\kappa,m}(\theta) & \rho_-(r) g^\nu_{-,\kappa,m}(\theta) \end{pmatrix} \right\rangle_{\mathcal{H}_{r,\theta}} = 0,
$$

which gives

$$
0 = \int_{r_+}^{+\infty} dr \frac{r^2 + a^2}{\Delta(r)} \left\langle \begin{pmatrix} X_{+,\nu,\kappa,n}^\nu(r) \\ X_{-,\nu,\kappa,n}^\nu(r) \end{pmatrix}, \begin{pmatrix} \im r - \lambda_{\nu,\kappa,n} & i \sqrt{\Delta} D^\nu_{+,\nu,\kappa} \ule {\nu,\kappa,n} \end{pmatrix} \end{pmatrix}_{\mathbb{C}^2},
$$

which implies

$$
\begin{pmatrix} -\im r - \lambda_{\nu,\kappa,n} & i \sqrt{\Delta} D^\nu_{-,\nu,\kappa} \\ \sqrt{\Delta} D^\nu_{+,\nu,\kappa} & \im r - \lambda_{\nu,\kappa,n} \end{pmatrix} \begin{pmatrix} X_{+,\nu,\kappa,n}(r) \\ X_{-,\nu,\kappa,n}(r) \end{pmatrix} = 0.
$$

If we set (cf. (16) and (22))

$$
f^\nu_{+,\nu,\kappa,n}(x) = \begin{pmatrix} f^\nu_{+,\nu,\kappa,n}(r(x)) \\ f^\nu_{-,\nu,\kappa,n}(r(x)) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ -1 & -1 \end{pmatrix} \begin{pmatrix} X^\nu_{+,\nu,\kappa,n}(r(x)) \\ X^\nu_{-,\nu,\kappa,n}(r(x)) \end{pmatrix},
$$

we have

$$
H^\nu_{+,\nu,\kappa} f^\nu_{+,\nu,\kappa,n} = \omega_{\nu,\kappa} f^\nu_{+,\nu,\kappa,n}
$$

(see (21)). Then, Theorem IV.2 and Theorem IV.4 give $f^\nu_{+,\nu,\kappa,n}(n) = 0$ for any $n \in \mathbb{Z} \setminus \{0\}$ and $\kappa \in \mathbb{Z} + (1/2)$, which yields $\Psi^\nu_{+,\kappa}(r,\theta) = 0$. The non-existence of time-periodic solutions is shown by Finster et al. [8] by different means.

V. THE CASE $r_+ = r_-$

In the previous section we have seen that there are no eigenvalues of (1) in the case $r_+ \neq r_-$. In this section we discuss whether $A_0$ is an eigenvalue of $H$ in the case $r_+ = r_-$. Recall that in this case the function $\Delta$ has only one zero and that $r_+ = r_- = M$.

**Theorem V.1.** If

$$
\lambda^2 + m^2 M^2 - \mu^2 \leq \frac{1}{4},
$$

then $A_0$ is not an eigenvalue of $H$.

**Proof.** Let us assume that $U = \ell(u_1, u_2) \in \mathcal{H}_2$ satisfies (29), that is,

$$
U' = \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \begin{pmatrix} -B & A_0 - C \\ A - A_0 & B \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
$$

Lemma III.1 yields for $x \to -\infty$

$$
U'(x) = -\frac{1}{x} \begin{pmatrix} \lambda & m M + \mu \\ m M - \mu & -\lambda \end{pmatrix} U(x) + O(x^{-2}) U(x).
$$

If we introduce $s = \log(-x)$, we have

$$
\frac{d}{ds} U(x(s)) = \begin{pmatrix} \lambda & m M + \mu \\ m M - \mu & -\lambda \end{pmatrix} U(x(s)) + O(\exp(-s)) U(x(s)) \quad \text{as} \quad s \to +\infty.
$$

(32)
The matrix
\[
S := \begin{pmatrix} \lambda & m M + \mu \\ m M - \mu & -\lambda \end{pmatrix}
\]
can be diagonalised by a non-singular matrix \( T \) as
\[
T^{-1}ST = \begin{pmatrix} \sqrt{\lambda^2 + m^2 M^2 - \mu^2} & 0 \\ 0 & -\sqrt{\lambda^2 + m^2 M^2 - \mu^2} \end{pmatrix}
\]
if \( \lambda^2 + m^2 M^2 - \mu^2 \neq 0 \). If \( \lambda^2 + m^2 M^2 - \mu^2 = 0 \), its Jordan canonical form is
\[
T^{-1}ST = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

According to Theorem 1.8.1 and Theorem 1.10.1 in Eastham [6], we have two linearly independent solutions \( U^1(s), U^2(s) \) of (32) such that if \( \lambda^2 + m^2 M^2 - \mu^2 \neq 0 \),
\[
\begin{align*}
U^1(s) &= \{v_1 + o(1)\} \exp (\sqrt{\lambda^2 + m^2 M^2 - \mu^2} s) = \{v_1 + o(1)\}(-x)^{\sqrt{\lambda^2 + m^2 M^2 - \mu^2}}, \\
U^2(s) &= \{v_2 + o(1)\} \exp (-\sqrt{\lambda^2 + m^2 M^2 - \mu^2} s) = \{v_2 + o(1)\}(-x)^{-\sqrt{\lambda^2 + m^2 M^2 - \mu^2}}
\end{align*}
\]
as \( x \to -\infty \) and, if \( \lambda^2 + m^2 M^2 - \mu^2 = 0 \),
\[
\begin{align*}
U^1(s) &= v_1 + o(1), \\
U^2(s) &= sv_1 + v_2 + o(s) = v_1 \log (-x) + v_2 + o(\log (-x))
\end{align*}
\]
as \( x \to -\infty \), where \( T = (v_1, v_2) \).
Therefore a necessary condition for the existence of an \( L^2(-\infty, 0) \)-eigenfunction is
\[
\sqrt{\lambda^2 + m^2 M^2 - \mu^2} \in (1/2, \infty), \quad \text{i.e.} \quad \lambda^2 + m^2 M^2 - \mu^2 > 1/4. \]

**Remark V.2.** The above discussions show that for \( \omega \) to be an eigenvalue of \( H \) the following conditions are necessary:
\[
\begin{align*}
r_+ = r_- = M & \quad \text{(Theorem IV.4)}, \\
\omega &= A_0 = -\frac{ak + e Q r_+}{v_+^2 + a^2} & \quad \text{(Theorem IV.2 (i))}, \quad (33) \\
\omega^2 &< m^2 & \quad \text{(Theorem IV.2 (ii)), Remark IV.3),} \\
\lambda^2 + m^2 M^2 - \mu^2 &> \frac{1}{4} & \quad \text{(Theorem V.1).} \quad (35) \\
\end{align*}
\]

However, the solvability of the above system is not yet sufficient for the existence of an energy eigenvalue. Schmid [12] showed that if in addition either
\[
\beta - \sigma \lambda = 0, \quad \alpha + \eta = 0 \quad (37.0)
\]

or
\[
N + \alpha + \eta = 0 \quad \text{for some positive integer} \ N, \quad (37.N)
\]
holds, then the solvability of the system (34)–(37.N) is sufficient for the existence of an eigenvalue \( \omega \) of \( H \).

In the above formulae, we used
\[
\begin{align*}
\sigma := \text{sign} \omega, & \quad \eta := \sqrt{\lambda^2 + M^2 m^2 - \mu^2}, \quad \mu = 2M \omega + eQ = -\frac{2akM}{M^2 + a^2} - eQ \frac{M^2 - a^2}{M^2 + a^2}, \\
\alpha := \frac{Mm^2 - \omega \mu}{\sqrt{m^2 - \omega^2}}, & \quad \beta := \frac{(M |\omega| - \sigma \mu)m}{\sqrt{m^2 - \omega^2}}.
\end{align*}
\]

Note that the variable \( \eta \) is denoted by \( \kappa \) in [12].
If $\omega$ is an energy eigenvalue of (1), then there must be an $N \in \mathbb{N}_0$ such that $\omega$ satisfies the complicated system of conditions (34)–(36) and (37.N). It is not clear that for given data of the black hole and the particle there are tuples $(\nu \omega_0, \kappa, \lambda_{\nu, \kappa, n})$ that solve the system (34)–(37.N). Schmid [12] has shown that in the so-called Kerr case (i.e. if $Q = 0$) for fixed data of the spin-$\frac{1}{2}$ particle there exist two sequences $(a_{\nu, \kappa, n}^+)_{N \in \mathbb{N}}$ such that for $a = a_{\nu, \kappa, n}^+$ the value $\omega_N = -\frac{2a}{a^2}$ is an energy eigenvalue.

Here, we fix the black hole data $M$, $Q$ and $a$ and vary the mass of the fermion to obtain the existence of energy eigenvalues in the case $r_- = r_+$.

**Theorem VI.1.** Fix $M > 0$, $a$, $Q$, $e \in \mathbb{R}$, $\nu \in \mathbb{Z}$ and $\kappa \in \mathbb{Z} + \frac{1}{2}$. Let $\omega := -\frac{\nu \omega_0 + Q M}{a^2 + M^2}$. Take $\lambda = \lambda_{\nu, \kappa, n}$ for any sufficiently large $|n|$. If $\omega(eQ + M \omega) < 0$ then there is a sequence $(m_N)_{N \in \mathbb{N}} \subseteq (\omega, \infty)$ such that if $m \in \{m_N : N \in \mathbb{N}\}$, then $\omega$ is an energy eigenvalue of $H_{\nu, \kappa, n}$. For $N_0$ large enough, the sequence $(m_N)_{N \geq N_0}$ is monotonously decreasing and converges to $|\omega|$. Before we prove the theorem, let us emphasise that $\lambda = \lambda_{\nu, \kappa, n}$ does depend also on $m$. Therefore, we denote it by $\lambda = \lambda_{\nu, \kappa, n}(m)$. In order to check the condition (36) we prepare the following lemma.

**Lemma VI.2.** Fix $\nu \in \mathbb{Z}$ and $\kappa \in \mathbb{Z} + \frac{1}{2}$. If $|n|$ is sufficiently large, then the inequality (36) holds for any $m > 0$, $M > 0$ and $\mu \in \mathbb{R}$, that is,

$$\lambda_{\nu, \kappa, n}^2 + m^2 M^2 - \mu^2 > \frac{1}{4}.$$  

**Proof.** It follows from standard perturbation theory (applied to the angular operator with $m$ as perturbation parameter, see Winklmeier [18], Kato [10]) that

$$\left| \frac{d}{dm} \lambda_{\nu, \kappa, n}(m) \right| \leq \left\| \begin{pmatrix} -a \cos \theta & 0 \\ 0 & a \cos \theta \end{pmatrix} \right\| = |a|,$$

hence, for $m > |\omega|$, 

$$|\lambda_{\nu, \kappa, n}(|\omega|)| - |a|(m - |\omega|) \leq |\lambda_{\nu, \kappa, n}(m)| \leq |\lambda_{\nu, \kappa, n}(|\omega|)| + |a|(m - |\omega|).$$ (38)

Let $\bar{m} = M^{-1}\sqrt{\mu^2 + 1/4}$. Since the sequence $(\lambda_{\nu, \kappa, n})_n$ is monotonously increasing and unbounded from below and from above, there is an integer $n_0$ such that 

$$|\lambda_{\nu, \kappa, n}(|\omega|)| > |a|(|\bar{m} - |\omega|)| + \sqrt{\mu^2 + 1/4}$$

for all $|n| \geq n_0$. If $m \in (|\omega|, \bar{m})$, we have 

$$|\lambda_{\nu, \kappa, n}(m)| \geq |\lambda_{\nu, \kappa, n}(|\omega|)| - |a|(m - |\omega|) \geq |\lambda_{\nu, \kappa, n}(|\omega|)| - |a|(|\bar{m} - |\omega|)| > \sqrt{\mu^2 + 1/4}$$

which implies 

$$\sqrt{\lambda_{\nu, \kappa, n}(m)^2 + M^2 m^2 - \mu^2} > \sqrt{\mu^2 + 1/4 + m^2 M^2 - \mu^2} > 1/2.$$ 

If $m > \bar{m} = M^{-1}\sqrt{\mu^2 + 1/4}$, then we have 

$$\sqrt{\lambda_{\nu, \kappa, n}(m)^2 + M^2 m^2 - \mu^2} > \sqrt{M^2 \bar{m}^2 - \mu^2} = 1/2.$$

Now we shall prove Theorem VI.1.

**Proof of Theorem VI.1.** By definition, $\omega$ satisfies condition (34). As seen in Lemma VI.2, there is an $n_0 \in \mathbb{N}$ such that condition (36) is satisfied for all $|n| \geq n_0$. From now on, let us assume that condition (36) holds. Next we consider the conditions (37.0) and (37.N). To this end we compute

$$\alpha + \eta = \frac{M m^2 - \omega \mu}{\sqrt{m^2 - \omega^2}} + \sqrt{\lambda_{\nu, \kappa, n}^2 + M^2 m^2 - \mu^2}.$$
If \( \omega(eQ + M\omega) \leq 0 \), then \( \alpha + \eta > \frac{1}{2} \) and condition (37.N) cannot be satisfied for any \( N \in \mathbb{N}_0 \). Assume now that \( \omega(eQ + M\omega) > 0 \). Then the function
\[
A : (|\omega|, \infty) \longrightarrow \mathbb{R}, \\
m \mapsto -\frac{\omega(eQ + M\omega)}{\sqrt{m^2 - \omega^2}} + M\sqrt{m^2 - \omega^2} + \sqrt{\lambda_{\nu,\kappa,n}(m)^2 + M^2m^2 - \mu^2}
\]
is continuous, satisfies \( \lim_{m \searrow |\omega|} A(m) = -\infty \), \( \lim_{m \to \infty} A(m) = \infty \) in view of (38). Hence for every \( N \in \mathbb{N} \) there is at least one \( m_N \in (|\omega|, \infty) \) such that \( A(m_N) = -N \) and therefore satisfies condition (37.N). Since the function \( A \) is monotonously increasing in an interval \((|\omega|, |\omega| + \delta)\) for a sufficiently small \( \delta > 0 \), it follows that for \( N \) large enough there is only one \( m_N \) satisfying \( A(m_N) = -N \) and that the sequence \((m_N)_N\) is decreasing. \( \square \)

**Remark VI.3.** The proof shows that for fixed \( m \) only a finite number of \( \lambda_{\nu,\kappa,n} \) is allowed in order to satisfy condition (37.N). The closer \( m \) is to \( |\omega| \), the more (and the larger) values for \( \lambda_{\nu,\kappa,n} \) are allowed.

**Remark VI.4.** The condition \( \omega(eQ + M\omega) > 0 \) is satisfied if the ratio \( eQ/\kappa \) is sufficiently small since
\[
\omega(eQ + M\omega) = -\frac{a}{(a^2 + M^2)^2}(\kappa a + eQM)(eQa - \kappa M)
\]
\[
= \frac{a}{(a^2 + M^2)^2}[aM\kappa^2 + eQ(M^2 - a^2)\kappa - aMe^2Q^2]
\]
\[
= \frac{a^2M}{(a^2 + M^2)^2}\left(\kappa - \frac{eQa}{2M} + \frac{eQM}{2a}\right)^2 - \frac{e^2Q^2}{4}\left(\frac{a}{M} - \frac{M}{a}\right)^2 + 4\]
\]
\[
= \frac{a^2\kappa^2M}{4(a^2 + M^2)^2}\left(\frac{2eQ}{\kappa} + \left(\frac{a}{M} - \frac{M}{a}\right)^2 - 4 + \left(\frac{a}{M} - \frac{M}{a}\right)^2\right).
\]

**Remark VI.5.** Let \( \omega \) be an energy eigenvalue, \( m \in \{m_N : N \in \mathbb{N}\} \) (see Theorem VI.1), and \( f^{\nu,\kappa,n} \) the eigenfunctions of \( H_{\nu,\kappa,n} \). If we set
\[
\begin{pmatrix}
X_{+)^{\nu,\kappa,n}(r) \\
X_{-)^{\nu,\kappa,n}(r)
\end{pmatrix}
= \frac{1}{\sqrt{2}} \begin{pmatrix}
i & -1 \\
-1 & i
\end{pmatrix}
\begin{pmatrix}f_{1}^{\nu,\kappa,n}(x) \\
f_{2}^{\nu,\kappa,n}(x)
\end{pmatrix}
\]
then
\[
\tilde{\Psi}(r, \theta, \phi, t) = \exp(-\imath \omega t) \exp(-\imath \kappa \varphi)
\begin{pmatrix}X_{+)^{\nu,\kappa,n}(r) g_{1}^{\nu,\kappa,n}(\theta) \\
X_{-)^{\nu,\kappa,n}(r) g_{1}^{\nu,\kappa,n}(\theta)
\end{pmatrix}
\]
is a time-periodic solution of (1).

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[1] D. Batic, H. Schmid, and M. Winklmeier. On the eigenvalues of the Chandrasekhar-Page angular equation. *J. Math. Phys.*, 46(1):012504–35, 2005. arXiv:gr-qc/0512112.

[2] F. Belgiorno and M. Martellini. Quantum properties of the electron field in Kerr-Newman black hole manifolds. *Phys. Lett. B*, 453(1-2):17–22, 1999. arXiv:gr-qc/9811060.

[3] J. D. Bjorken and S. D. Drell. *Relativistic quantum mechanics*. McGraw-Hill Book Co., New York, 1964.

[4] S. Chandrasekhar. *The mathematical theory of black holes*, volume 69 of *International Series of Monographs on Physics*. The Clarendon Press Oxford University Press, New York, 1983. Oxford Science Publications.

[5] A. S. Davydov. *Quantum mechanics*. Pergamon Press, Oxford, 1976. Translated from the second Russian edition, edited and with additions by D. ter Haar, International Series in Natural Philosophy, Vol. 1, Pergamon International Library of Science, Technology, Engineering and Social Studies.

[6] M. S. P. Eastham. *The asymptotic solution of linear differential systems*, volume 4 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1989. Applications of the Levinson theorem, Oxford Science Publications.

[7] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau. Erratum: “Nonexistence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry”. *Comm. Pure Appl. Math.*, 53(9):1201, 2000.

[8] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau. Nonexistence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry. *Comm. Pure Appl. Math.*, 53(7):902–929, 2000. arXiv:gr-qc/9905047.

[9] V. P. Frolov and I. D. Novikov. *Black hole physics*, volume 96 of *Fundamental Theories of Physics*. Kluwer Academic Publishers Group, Dordrecht, 1998. Basic concepts and new developments, Chapter 4 and Section 9.9 written jointly with N. Andersson.

[10] T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin Heidelberg New York, second edition, 1980.

[11] D. N. Page. Dirac equation around a charged, rotating black hole. *Phys. Rev.*, D14:1509–1510, 1976.

[12] H. Schmid. Bound state solutions of the Dirac equation in the extreme Kerr geometry. *Math. Nachr.*, 274/275:117–129, 2004. arXiv:math-ph/0207039.

[13] K. M. Schmidt. Absolutely continuous spectrum of Dirac systems with potentials infinite at infinity. *Math. Proc. Cambridge Philos. Soc.*, 122(2):377–384, 1997.

[14] B. Thaller. *The Dirac Equation*. Texts and Monographs in Physics. Springer-Verlag, Berlin Heidelberg New York, 1992.

[15] R. M. Wald. *General relativity*. University of Chicago Press, Chicago, IL, 1984.

[16] J. Weidmann. Absolut stetiges Spektrum bei Sturm-Liouville-Operatoren und Dirac-Systemen. *Math. Z.*, 180(3):423–427, 1982.

[17] J. Weidmann. *Spectral theory of ordinary differential operators*, volume 1258 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo, 1987.

[18] M. Winklmeier. *The angular part of the Dirac equation in the Kerr-Newman metric: Estimates for the eigenvalues*. PhD thesis, Universität Bremen, 2006.