Oriented unicyclic graphs with extremal skew energy

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Abstract
Let $\overrightarrow{G}$ be an oriented graph of order $n$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote all the eigenvalues of the skew-adjacency matrix of $\overrightarrow{G}$. The skew energy $E_s(\overrightarrow{G}) = \sum_{i=1}^n |\lambda_i|$. In this paper, the oriented unicyclic graphs with minimal and maximal skew energy are determined.

Keywords: Unicyclic graph; oriented graph; skew-adjacency matrix, skew energy
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1 Introduction

An important quantum-chemical characteristic of a conjugated molecule is its total $\pi$-electron energy. The energy of a graph has closed links to chemistry. Since the concept of the energy of simple undirected graphs was introduced by Gutman in [4], there have been lots of research papers on this topic. For the extremal energy of unicyclic graph, Hou [8] showed that $S_n^3$ is the graph with minimal energy in all unicyclic graphs; In [6], Huo and Li showed that $P_n^6$ is the graph with maximal energy in all unicyclic graphs. More results on the energy of unicyclic graphs see [5, 7, 9, 11, 14, 15].

There are various generalizations of energy to graph matrices [12], such as Laplacian energy, incidence energy and distance energy. Let $G$ be a simple undirected finite graph of order $n$ and $\overrightarrow{G}$ be an orientation of $G$, which assigns to each edge of $G$ a direction so that $\overrightarrow{G}$ becomes a directed graph. All digraphs in this paper are oriented graphs of some graphs. Let $\overrightarrow{G}$ be an oriented graph of order $n$. The skew adjacency matrix $S(\overrightarrow{G}) = (s_{i,j})$ is a real skew symmetric matrix, where $s_{i,j} = 1$ and $s_{j,i} = -1$ if $i \rightarrow j$ is an arc of $\overrightarrow{G}$, otherwise $s_{i,j} = s_{j,i} = 0$. The skew spectrum $Sp(\overrightarrow{G})$ of $\overrightarrow{G}$ is defined as the spectrum of $S(\overrightarrow{G})$. Note that $Sp(\overrightarrow{G})$ consists of only purely imaginary eigenvalues because $S(\overrightarrow{G})$ is real skew symmetric.

Recently, the skew-energy of an oriented graph $\overrightarrow{G}$ was defined as the energy of matrix $S(\overrightarrow{G})$ in [1], that is,

$$E_s(\overrightarrow{G}) = \sum_{\lambda \in Sp(\overrightarrow{G})} |\lambda|.$$
situation is when vertices represent distinct chemical species and arcs represent the direction in which a particular reaction takes place between the two corresponding species. It is possible that the skew energy has similar applications as energy in chemistry. For a graph $G$, there are any orientations on it, it is also interesting to find what orientation has extremal energy among all orientations of a given graph.

An unicyclic graph is the connected graph with the same number of vertices and edges. In this paper, we are interested in studying the orientations of unicyclic graphs with extremal skew energy. Let $G(n, \ell)$ be the set of all connected unicyclic graphs on $n$ vertices with girth $\ell$. Denote, as usual, the $n$-vertex path and cycle by $P_n$ and $C_n$, respectively. Let $P_n^\ell$ be the unicyclic graph obtained by connecting a vertex of $C_\ell$ with a terminal vertex of $P_{n-\ell}$, $S_n^\ell$ be the graph obtained by connecting $n-\ell$ pendant vertices to a vertex of $C_\ell$ (see Fig. 1).

The rest of the paper is organized as follows. In section 2, a new integral formal for $E_s(\overrightarrow{G})$ is obtained and the oriented unicyclic graph with minimal skew energy is determined. In section 3, the oriented unicyclic graph with maximal skew energy is determined.

## 2 Oriented unicyclic graph with minimal skew energy

Let $G$ be a graph. A linear subgraph $L$ of $G$ is a disjoint union of some edges and some cycles in $G$ ([2]). A $k$-matching $M$ in $G$ is a disjoint union of $k$-edges. If $2k$ is the order of $G$, then a $k$-matching of $G$ is called a perfect matching of $G$. The number of $k$-matching is denoted by $m(G, k)$.

If $C$ be any undirected even cycle of $G$, we say $C$ is evenly oriented relative to the orientation $\overrightarrow{G}$ if it has an even number of edges oriented in clockwise direction. Otherwise $C$ is oddly oriented.

We call a linear subgraph $L$ of $G$ evenly linear if $L$ contains no odd cycle and denote by $\mathcal{E}L_i(G)$ (or $\mathcal{E}L_i$ for short) the set of all evenly linear subgraphs of $G$ with $i$ vertices. For a linear subgraph $L \in \mathcal{E}L_i$, denote by $p_e(L)$ (resp., $p_o(L)$) the number of evenly (resp., oddly) oriented cycles in $L$ relative to $\overrightarrow{G}$. Denote the characteristic polynomial of $S(\overrightarrow{G})$ by

$$P_s(\overrightarrow{G}; x) = \det(xI - S(\overrightarrow{G})) = \sum_{i=0}^{n} b_i x^{n-i}. \quad (1)$$

Then (i) $b_0 = 1$, (ii) $b_2$ is the number of edges of $G$, (iii) all $b_i \geq 0$ and (iv) $b_i = 0$ for all odd $i$ since the determinant of every real skew symmetric matrix is nonnegative.
and is 0 if its order is odd.

\[ b_i(G) = \sum_{L \in \mathcal{E}_1} (-2)^{p_e(L)} 2^{p_o(L)}, \]

where \( p_e(L) \) is the number of evenly oriented cycles of \( L \) and \( p_o(S) \) is the number of oddly oriented cycles of \( L \) relative to \( \overrightarrow{G} \), respectively.

From Lemma 2.1, we can get the following statement.

**Corollary 2.2** Let \( G \) be a unicyclic graph with unique cycle \( C \) of length \( \ell \) and \( \overrightarrow{G} \) be an orientation of \( G \). Then

\[ b_{2k}(\overrightarrow{G}) = \begin{cases} m(G, k), & \text{if } C \text{ is an odd cycle;} \\ m(G, k) - 2m(G - C, k - \frac{\ell}{2}), & \text{if } C \text{ is evenly oriented;} \\ m(G, k) + 2m(G - C, k - \frac{\ell}{2}), & \text{if } C \text{ is oddly oriented.} \end{cases} \]
Lemma 2.3 ([10]) Let $e = uv$ be an edge of $G$ that is on no any even cycle of $G$. Then
\[ P_s(\overrightarrow{G}; x) = P_s(\overrightarrow{G} - e; x) + P_s(\overrightarrow{G} - u - v; x). \]

By Lemma 2.3, we obtain that

Corollary 2.4 Let $e = uv$ be an edge of $G$ that is on no even cycle of $G$. Then
\[ b_{2k}(\overrightarrow{G}) = b_{2k}(\overrightarrow{G} - e) + b_{2k-2}(\overrightarrow{G} - u - v). \] (3)

Furthermore, if $e = uv$ is a pendant edge with the pendant vertex $v$. Then
\[ b_{2k}(\overrightarrow{G}) = b_{2k}(\overrightarrow{G} - v) + b_{2k-2}(\overrightarrow{G} - u - v). \] (4)

For any orientation of a graph which does not contain any even cycle (in particular, a tree, a unicyclic non-bipartite graph) by Theorem 2.9, we have $b_{2k}(\overrightarrow{G}) = m(G, k)$.

For the $k$-matching number of a graph $G$, the following result is well-known.

Lemma 2.5 Let $e = uv$ be an edge of $G$. Then
(i) $m(G, k) = m(G - e, k) + m(G - u - v, k - 1)$.
(ii) If $G$ is a forest, then $m(G, k) \leq m(P_n, k)$, $k \geq 1$.
(iii) If $H$ is a subgraph of $G$, then $m(H, k) \leq m(G, k)$, $k \geq 1$. Moreover, if $H$ is a proper subgraph of $G$, then the inequality is strict.

From [1], an integral formula for skew energy was given:
\[ \mathcal{E}_s(\overrightarrow{G}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n + x \frac{P'_s(\overrightarrow{G}, -x)}{P_s(\overrightarrow{G}, -x)} \right) dx, \] (5)
where $P'_s(\overrightarrow{G}, -x)$ is the derivative of $P_s(\overrightarrow{G}, -x)$.

Theorem 2.6 Let $\overrightarrow{G}$ be an orientation of a graph $G$. Then
\[ \mathcal{E}_s(\overrightarrow{G}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2} \ln(1 + \sum_{k=1}^{\left\lceil \frac{n}{2} \right\rceil} b_{2k} t^{2k}) dt. \]
Let $W$, $G$, and $V$ be an orientation of a graph $G$. Then by Lemma 3.1 in [1], the characteristic polynomial of $S(G)$ is a diagonal matrix whose $(i, i)$-entry is $-1$ when $i \in W$ and $1$ when $i \notin W$.

**Proof.** From equality (5), we have

$$
\mathcal{E}_s(\overrightarrow{G}) = \frac{1}{\pi} \int_{-\infty}^{\infty} (n + x P_s(\overrightarrow{G}, -x)) dx = \frac{1}{\pi} \int_{-\infty}^{\infty} (n - \mu P_s(\mu)) d\mu
$$

$$
= \frac{1}{\pi} \int_{0}^{\infty} (n - \frac{1}{t} P_s(\frac{1}{t}))(-\frac{1}{t^2}) dt + \frac{1}{\pi} \int_{-\infty}^{0} (n - \frac{1}{t} P_s(\frac{1}{t}))(-\frac{1}{t^2}) dt
$$

$$
= \frac{1}{\pi} \int_{-\infty}^{\infty} (n - \frac{1}{t} P_s(\frac{1}{t}))\frac{1}{t^2} dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t} d[ln(t^n P_s(\frac{1}{t}))]
$$

$$
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2} ln(t^n P_s(\frac{1}{t})) dt. = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2} ln(1 + \sum_{k=1}^{n} b_{2k}t^{2k}) dt. \quad \square
$$

**Example 2.7** Let $\overrightarrow{C_4}$ be an orientation of $C_4$ such that all edges have the same direction. Then by Lemma 2.1, the characteristic polynomial of $S(\overrightarrow{C_4})$ is $x^4 + 4x^2$. By Lemma 2.6,

$$
\mathcal{E}_s(\overrightarrow{C_4}, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2} ln(1 + 4t^2) dt
$$

$$
= \frac{8}{\pi} \int_{0}^{\infty} \frac{1}{1 + 4t^2} dt = 4.
$$

From Theorem 2.6, $\mathcal{E}_s(\overrightarrow{G})$ is an increasing function of $b_{2k}(\overrightarrow{G})$, $k = 0, 1, \cdots, \lfloor \frac{n}{2} \rfloor$. Consequently, if $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ are oriented graphs of $G_1$ and $G_2$, respectively, for which

$$
b_{2k}(\overrightarrow{G_1}) \geq b_{2k}(\overrightarrow{G_2}) \quad (6)
$$

for all $\lfloor \frac{n}{2} \rfloor \geq k \geq 0$, then

$$
\mathcal{E}_s(\overrightarrow{G_1}) \geq \mathcal{E}_s(\overrightarrow{G_2}). \quad (7)
$$

Equality in (7) is attained only if (6) is an equality for all $\lfloor \frac{n}{2} \rfloor \geq k \geq 0$. If relations (6) hold for all $k$, then we write $G_1 \succeq G_2$ or $G_2 \preceq G_1$. If $G_1 \succeq G_2$, but not $G_2 \preceq G_1$, then we write $G_1 \succ G_2$.

Let $\overrightarrow{G}$ be an orientation of a graph $G$. Let $W$ be a subset of $V(G)$ and $\overrightarrow{W} = V(G) \setminus W$. The orientation $\overrightarrow{G}$ of $G$ obtained from $\overrightarrow{G}$ by reversing the direction of all arcs between $\overrightarrow{W}$ and $W$ is said to be obtained from $\overrightarrow{G}$ by switching with respect to $W$. Moreover, two orientations $\overrightarrow{G}$ and $\overrightarrow{G}'$ of a graph $G$ are said to be switching-equivalent if $\overrightarrow{G}'$ can be obtained from $\overrightarrow{G}$ by a sequences of switching. If two orientations $\overrightarrow{G}$ and $\overrightarrow{G}'$ of a graph $G$ are switching-equivalent then their skew-adjacency matrices are similar by a diagonal matrix whose $(i, i)$-entry is $-1$ when $i \in W$ and $1$ when $i \notin W$ by Lemma 3.1 in [1] and hence they have the same skew spectrum. Thus,
Lemma 2.8 Let $\overrightarrow{G}$ and $\overrightarrow{G}'$ be two orientations of a graph $G$. If $\overrightarrow{G}$ and $\overrightarrow{G}'$ are switching-equivalent, then $\mathcal{E}_s(\overrightarrow{G}) = \mathcal{E}_s(\overrightarrow{G}')$.

By Lemma 2.8 and switching-equivalence, there are only two different orientations on a unicyclic graph $G$. All edges on the unique cycle $C$ have the same direction or just one edge on the cycle has the opposite direction to the directions of other edges on the cycle regardless how the edges not on the cycle $C$ are oriented. Denote by $\overrightarrow{G}^-$ ($\overrightarrow{G}^+$, resp.) the orientation of $G$ in first (second, resp.) case above.

Lemma 2.9 For any unicyclic graph $G$, $\overrightarrow{G}^+ \succeq \overrightarrow{G}^-$.

Proof. By Corollary (2.2), if the girth $\ell$ of $G$ is odd, then

$$b_{2k}(\overrightarrow{G}^+) = b_{2k}(\overrightarrow{G}^-) = m(G, k) \text{ for all } 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor.$$  

If the girth $\ell$ of $G$ is even, then

$$b_{2k}(\overrightarrow{G}^+) = m(G, k) + 2m(G - C_\ell, k - \left\lfloor \frac{\ell}{2} \right\rfloor),$$

$$b_{2k}(\overrightarrow{G}^-) = m(G, k) - 2m(G - C_\ell, k - \left\lfloor \frac{\ell}{2} \right\rfloor)$$

for all nonnegative integer $\left\lfloor \frac{n}{2} \right\rfloor \geq k \geq \left\lfloor \frac{\ell}{2} \right\rfloor$, and

$$b_0(\overrightarrow{G}^+) = b_0(\overrightarrow{G}^-) = 1, \quad b_{2k}(\overrightarrow{G}^+) = b_{2k}(\overrightarrow{G}^-) = m(G, k)$$

for all $0 < k < \left\lfloor \frac{\ell}{2} \right\rfloor$. Thus the result follows immediately. \qed

Lemma 2.10 Let $\overrightarrow{G}$ be an orientation of a unicyclic graph $G \in G(n, \ell)$, $G \neq S^\ell_n$. If unique cycle $C_\ell$ in $\overrightarrow{G}$ and $S^\ell_n$ is the same orientation, then $\overrightarrow{G} \succ S^\ell_n$.

Proof. We prove the statement by induction on $n$. Since $G \neq S^\ell_n$, $n > \ell + 2$. For $n = \ell + 2$, $G$ is one of the two graphs in Fig. 2.

By Corollary 2.4,

$$b_{2k}(\overrightarrow{P^\ell_{\ell+2}}) = b_{2k}(\overrightarrow{P^\ell_{\ell+1}}) + b_{2k-2}(\overrightarrow{C_\ell}),$$

$$b_{2k}(\overrightarrow{G_1}) = b_{2k}(\overrightarrow{P^\ell_{\ell+1}}) + b_{2k-2}(\overrightarrow{T}),$$

where $T$ is a graph obtained by connecting one of the vertices in $P_{\ell-1}$ to a pendant vertex.

$$b_{2k}(\overrightarrow{S^\ell_{\ell+2}}) = b_{2k}(\overrightarrow{P^\ell_{\ell+1}}) + b_{2k-2}(\overrightarrow{P_{\ell-1}}).$$
By Corollary 2.4, if \( \ell \) is odd or \( \ell \) is even but \( k \leq \frac{\ell}{2} \) then
\[
b_{2k-2}(\overrightarrow{C_\ell}) = m(C_\ell, k - 1);
\]
for \( k = \frac{\ell}{2} + 1 \). If \( C \) is oddly oriented, then
\[
b_\ell(\overrightarrow{C_\ell}) = m(C_\ell, \frac{\ell}{2}) + 2 = 4;
\]
If \( C \) is evenly oriented, then
\[
b_\ell(\overrightarrow{C_\ell}) = m(C_\ell, \frac{\ell}{2}) - 2 = 0, \quad b_\ell(\overrightarrow{P_{\ell-1}}) = 0.
\]
Since both \( T \) and \( P_{\ell-1} \) are trees,
\[
b_{2k-2}(\overrightarrow{T}) = m(T, k - 1),
b_{2k-2}(\overrightarrow{P_{\ell-1}}) = m(P_{\ell-1}, k - 1).
\]
Since \( P_{\ell-1} \) is a proper subgraph of both \( C_\ell \) and \( T \), by Lemma 2.5,
\[
m(\overrightarrow{P_{\ell-1}}, k - 1) < m(T, k - 1), m(P_{\ell-1}, k - 1) < m(C_\ell, k - 1).
\]
The result holds immediately for \( n = \ell + 2 \).
Suppose that \( \overrightarrow{G} \succ \overrightarrow{S_n^{n'}} \) for all \( n' < n \). Since \( \overrightarrow{G} \) is a unicyclic digraph, there is at least a pendant edge \( uv \) with pendant vertex \( v \) in \( \overrightarrow{G} \), by equality (4), we get
\[
b_{2k}(\overrightarrow{G}) = b_{2k}(\overrightarrow{G} - v) + b_{2k-2}(\overrightarrow{G} - u - v),
b_{2k}(\overrightarrow{S_n^{n'}}) = b_{2k}(\overrightarrow{S_{n-1}^{n'}}) + b_{2k-2}(\overrightarrow{P_{\ell-1}}).
\]
By induction assumption, it suffices to prove that \( b_{2k-2}(\overrightarrow{G} - u - v) \geq b_{2k-2}(\overrightarrow{P_{\ell-1}}) \)
for all \( 0 \leq k \leq \left\lfloor \frac{n - \ell}{2} \right\rfloor \). For \( k > \left\lfloor \frac{\ell - 1}{2} \right\rfloor \), we have \( b_{2k-2}(\overrightarrow{G} - u - v) \geq b_{2k-2}(\overrightarrow{P_{\ell-1}}) = 0 \). For \( \left\lfloor \frac{\ell - 1}{2} \right\rfloor \geq k \geq 0 \), \( b_{2k-2}(\overrightarrow{G} - u - v) = m(\overrightarrow{G} - u - v, k - 1) \geq m(\overrightarrow{P_{\ell-1}}, k - 1) = b_{2k-2}(\overrightarrow{P_{\ell-1}}) \)
since \( P_{\ell-1} \) is a subgraph of \( G - u - v \). □

**Lemma 2.11** Let \( n \geq \ell \geq 6 \) or \( n > \ell = 5 \), then \( \overrightarrow{S_n^{\ell-}} \prec \overrightarrow{S_n^{\ell+}} \prec \overrightarrow{S_n^{\ell-}} \prec \overrightarrow{S_n^{\ell+}} \).

**Proof.** By Lemma 2.1, the characteristic polynomial of \( \overrightarrow{S_n^{\ell+}} \) is:
\[
P_s(\overrightarrow{S_n^{\ell+}}, x) = x^{n-4}(x^4 + nx^2 + 2n - 4);
P_s(\overrightarrow{S_n^{\ell-}}, x) = x^{n-4}(x^4 + nx^2 + 2n - 8);
\]
Obviously, $S_n^4 < S_n^4$. So it suffices to prove that $b_4(S_n^1) > 2n - 4$. By equality (2),

$$b_4(S_n^+ \mathrel{\rightarrow} ) = b_4(S_n^- \mathrel{\rightarrow} ) = m(P_{\ell-1}, 2) + (n - \ell)m(P_{\ell-1}, 1) + 2m(P_{\ell-2}, 1) = \frac{(\ell - 3)(\ell - 4)}{2} + (n - \ell)(\ell - 2) + 2(\ell - 3) = \frac{2n\ell - \ell^2 + \ell - 4n}{2}.$$  

For $n > \ell = 5$, $b_4(S_n^\mathrel{\rightarrow} ) = 3n - 10 \geq 2n - 4$; For $n \geq 6$, $b_4(S_n^\mathrel{\rightarrow} ) = 4n - 15 > 2n - 4$; for $n \geq 7$, $b_4(S_n^\mathrel{\rightarrow} ) = 5n - 21 > 2n - 4$; for $\ell \geq 8$, $b_4(S_n^\mathrel{\rightarrow} ) \geq \frac{n\ell + \ell - 4n}{2} \geq 2n + 4 > 2n - 4$. By Lemma 2.9, the proof is completed. □

For $n = \ell = 5$, $b_4(S_n^\mathrel{\rightarrow} ) = \frac{2n\ell - \ell^2 + \ell - 4n}{2} = 5$, $b_4(S_n^\mathrel{\rightarrow} ) = 10 - 4 = 6$, $b_4(S_n^\mathrel{\rightarrow} ) = 10 - 8 = 2$. It gives $S_5^< \mathrel{\rightarrow} < S_5^< \mathrel{\rightarrow} < S_5^< \mathrel{\rightarrow}$.

**Lemma 2.12** Let $\overrightarrow{S_n^3}$ be any orientation of unicyclic graph $S_n^3$. Then $\overrightarrow{S_n^3} < \overrightarrow{S_n^4}$ for $n \geq 6$; $\overrightarrow{S_n^3} = \overrightarrow{S_n^3} < \overrightarrow{S_n^4}$ for $n = 5$; $\overrightarrow{S_n^4} < \overrightarrow{S_n^3} < \overrightarrow{S_n^4}$ for $n = 4$.

**Proof.** By Lemma 2.1, the characteristic polynomial of $\overrightarrow{S_n^3}$ is

$$P_s(\overrightarrow{S_n^3}, x) = x^{n-4}(x^4 + nx^2 + n - 3).$$

Since $n - 3 < 2n - 8$, the result holds by Lemmas 2.11 and 2.9. For $n = 5$, it is easy to get that for any orientation $E_s(\overrightarrow{S_5^3}) = 2\sqrt{5} + 2\sqrt{3}$, $E_s(\overrightarrow{S_5^3}) = 2\sqrt{5} + 2\sqrt{3} < 2(\sqrt{2} + \sqrt{3})$, $E_s(\overrightarrow{S_5^4}) = 2(\sqrt{2} + \sqrt{3})$. Then $\overrightarrow{S_5^3} = \overrightarrow{S_5^3} < \overrightarrow{S_5^4}$. For $n = 4$, $E_s(\overrightarrow{S_4^3}) = 2\sqrt{6}$, $E_s(\overrightarrow{C_4^+}) = 4\sqrt{2}$, $E_s(\overrightarrow{C_4^-}) = 4$. Thus $\overrightarrow{S_5^3} < \overrightarrow{S_5^3} < \overrightarrow{S_5^4}$. □

From Lemmas 2.10, 2.11 and 2.12, we obtain one of the main result of this paper.

**Theorem 2.13** Among all orientations of unicyclic graphs on $n$ vertices, $\overrightarrow{S_n^3}$ has the minimal skew energy and $\overrightarrow{S_n^3} < \overrightarrow{S_n^4}$ has the second minimal skew energy for $n \geq 6$; both $\overrightarrow{S_5^3}$ and $\overrightarrow{S_5^4}$ have the minimal skew energy, $\overrightarrow{S_5^+}$ has the second minimal skew energy for $n = 5$; $\overrightarrow{C_4^-}$ has the minimal skew energy, $\overrightarrow{S_4^+}$ has the second minimal skew energy for $n = 4$. 

8
3 Oriented unicyclic graph with maximal skew energy

By Lemma 2.9, we only need to consider $\overrightarrow{G}$ for considering of maximum skew energy.

Lemma 3.1 Let $\overrightarrow{G}$ be an orientation of unicyclic graph $G \in G(n, \ell)$ and $G \neq P_{n, \ell}$. Then $\overrightarrow{G}^+ \prec P_{n, \ell}^+$.

Proof. We prove the statement by induction on $n$. For $n = \ell + 2$, there are only two cases for $G \neq P_{\ell}^+$: one is $S_{\ell+2}^l$, the other is the graph $G_1$ in Fig. 2. By the proof in Lemma 2.10, we only need to prove that $b_{2k-2}(\overrightarrow{C_{\ell}}) \geq m(T, k - 1)$. By Lemma 2.5,

$$m(\overrightarrow{C_{\ell}}^+, k - 1) = m(C_{\ell}, k - 1) = m(P_{\ell}, k - 1) + m(P_{\ell-2}, k - 2),$$

$$m(T, k - 1) = m(P_{\ell-1}, k - 1) + m(P_s \bigcup P_t, k - 2) \leq m(P_{\ell-1}, k - 1) + m(P_{\ell-2}, k - 2) < m(P_{\ell}, k - 1) + m(P_{\ell-2}, k - 2) = m(C_{\ell}, k - 1) \leq b_{2k-2}(\overrightarrow{C_{\ell}}^+), \quad \text{(by Corollary 2.4)}$$

where $s + t = l - 2$.

Suppose $\overrightarrow{G}^+ \prec P_{n'}^+$ for all $n' < n$. Then there is a pendant edge, say $uv$, with pendant vertex $v$. By Lemma 2.3,

$$b_{2k}(\overrightarrow{G}^+) = b_{2k}(\overrightarrow{G}^+ - v) + b_{2k-2}(\overrightarrow{G}^+ - v - u);$$

$$b_{2k}(P_{n, \ell}^+) = b_{2k}(P_{n-1, \ell}^+) + b_{2k-2}(P_{n-2, \ell}^+).$$

By induction hypothesis, it suffices to prove that $b_{2k-2}(\overrightarrow{G}^+ - v - u) \leq b_{2k-2}(\overrightarrow{P_{n-2, \ell}}^+)$. If $\overrightarrow{G}^+ - v - u$ contains a cycle, then by induction hypothesis the inequality holds. Suppose that $\overrightarrow{G}^+ - v - u$ is a forest, then by Lemmas 2.5 and 2.9,

$$b_{2k-2}(\overrightarrow{G}^+ - v - u) = m(G - v - u, k - 1) \leq m(P_{n-2}, k - 1) \leq m(P_{n-2, \ell}, k - 1) \leq b_{2k-2}(P_{n-2, \ell}^+). \quad \square$$

Lemma 3.2 For $3 \leq \ell \leq n$, $\ell \neq 4$, $P_{n, \ell}^+ \prec P_{n}^{4+}$. 


Proof. We proceed the proof by induction on \( n \). For \( n = \ell = 5 \), \( b_4(P^+_5) = b_4(C^+_5) = 5 \), \( b_4(P^+_4) = b_4(S^+_5) = 6 \), thus \( b_4(P^+_5) < b_4(P^+_4) \). Suppose that \( P^+_n < P^+_m \) for all \( n' < n \), \( 5 \leq \ell \). By Lemma 2.1, we have

\[
\begin{align*}
b_{2k}(P^+_n) &= b_{2k}(P_{n-1}^+) + b_{2k-2}(P_{n-2}^+), \\
b_{2k}(P^+_4) &= b_{2k}(P_{n-1}^+) + b_{2k-2}(P_{n-2}^+).
\end{align*}
\]

By induction hypothesis, the result follows.

Similarly, we can prove that for \( \ell = 3 \) the inequality holds too. \( \square \)

By Lemmas 3.1 and 3.2, we have

**Theorem 3.3** Among all orientations of unicyclic graph, \( E_s(P^+_n) \) is the unique oriented graph (under switching-equivalent) with maximal skew energy.

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