A Conformal Field Theory for the Quantum Hall Effect

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Abstract

The QHE is studied in the context of a CFT. An effective field of $N$ “spins” associated with the cyclotron motion of particles is taken as an order parameter from which an effective Hamiltonian may be defined. This effective Hamiltonian describes the COM motion of the $N$ particles (with coupling $\kappa_0$) together with a current-current interaction (of strength $\kappa_1$). Such a system gives rise to a CFT in the large $N$ limit when $\kappa_0 = \kappa_1$. The Laughlin wavefunction is derived from this CFT as an $N'$-point correlation function of winding state vertex operators.
I. Introduction

It has been suggested by Fubini [1] that the QHE might be nicely understood through the use of CFT techniques. In particular, Fubini [1] and Stone [2] have outlined vertex operator techniques which may be used to obtain the Laughlin wavefunction. We show how the vertex operators may be naturally defined in terms of an order parameter arising from an effective Hamiltonian which includes a momentum-momentum interaction between different particles [3]. This order parameter has an interpretation of a field of “bosonic spins” on scales much larger than the magnetic length. Under certain conditions (i.e. \( \lim N \to \infty, \kappa_0 = \kappa_1, \theta_i = \theta_j \) and radial ordering) the \( N \)-particle system seems to exhibit a long-range order which may be associated with the formation of winding states. These winding states may be characterized by its zero-modes representing the center of mass (COM) motion of the system. Such winding states transform as tensors under the group of conformal transformations; the rank of the tensors being characterized by the energy or angular momentum eigenvalues of the zero-modes.

The system is analogous to that of hydrodynamic turbulence in which the assumption of scale invariance leads to a picture of “eddies upon eddies”. These classical pictures are valid for special values of the current-current couplings \((\kappa_0, \kappa_1)\) in the theory. They correspond to points in field space in which there exists a conformal symmetry. As in string theories, these fixed points correspond to different CFT’s which describe ground states of the system (i.e. classical solutions to the field equations).

In section II we define the collective coordinates which serve as our order parameter for the theory. From these, the winding state vertex operators of Fubini’s [1] are derived. The \( N' \) point correlation function of the winding state vertex operators are then calculated in section III. Here it is shown that for special values of the coupling, this \( N' \)-point correlation function gives Laughlin’s wavefunction in the large \( N \)-limit. The anyonic statistics of the “quasiparticle” states created by these vertex operators is then shown. The quantization of the Hall conductance is then speculated to arise from the topological properties of these winding states; associating the winding states with solitons which wind around a compact configuration space characterized by a two-dimensional modular parameter \( \vec{L} \) (as for a torus). A discussion of the relation between the QHE and CFT is then presented.

II. Generalized Collective Coordinates for Winding States

The collective coordinates of [3] may be generalized to include arbitrary coefficients for the various modes of the fields

\[
\varphi_L(\zeta) \equiv \varphi_0 + i\kappa_0 \alpha_0^* \ln \zeta + \kappa_1 \sum_{n=1}^{N} \frac{1}{n} \left( \alpha_n \zeta^n + \alpha_n^* \zeta^{-n} \right)
\] (1)
\[ \varphi_R(\zeta^*) \equiv \varphi_0 R + i \kappa_0 \beta_0^* \ln \zeta^* + \kappa_1 \sum_{n=1}^{N} \frac{1}{n} \left( \beta_n \zeta^{*n} + \beta_n^* \zeta^{-*n} \right). \] (2)

The mode coefficients \( \alpha_0, \alpha_n, \beta_0 \) and \( \beta_n \) may be viewed as annihilation-creation operators of a second quantized theory governed by the \( N \)-particle Hamiltonian

\[ H = \frac{1}{2\mu} \left( \pi_0^2 + \sum_{n=1}^{N} \pi_n^2 \right) \] (3)

\[ \pi_0 \equiv \vec{p}_0 + \frac{e}{c} \vec{A}(N\vec{x}_0) \quad ; \quad \pi_n \equiv \vec{p}_n + \frac{e}{c} \vec{A}(\vec{x}_n) \]

with the modes given by

\[ \alpha_0 \equiv \frac{\ell}{\sqrt{2Nh}} \left( \pi_0^1 + i \pi_0^2 \right) \quad ; \quad \beta_0 \equiv \frac{\ell}{\sqrt{2Nh}} \left( \pi_0^2 + i \pi_0^1 \right) \]

\[ \alpha_0^* \equiv \frac{\ell}{\sqrt{2Nh}} \left( \pi_0^1 - i \pi_0^2 \right) \quad ; \quad \beta_0^* \equiv \frac{\ell}{\sqrt{2Nh}} \left( \pi_0^2 - i \pi_0^1 \right) \]

\[ \alpha_n \equiv \frac{\ell}{\sqrt{2h}} \left( \pi_n^1 + i \pi_n^2 \right) \quad ; \quad \beta_n \equiv \frac{\ell}{\sqrt{2h}} \left( \pi_n^2 + i \pi_n^1 \right) \]

\[ \alpha_n^* \equiv \frac{\ell}{\sqrt{2h}} \left( \pi_n^1 - i \pi_n^2 \right) \quad ; \quad \beta_n^* \equiv \frac{\ell}{\sqrt{2h}} \left( \pi_n^2 - i \pi_n^1 \right) \]

\[ \pi_0 \equiv \vec{p} - \frac{e}{c} \vec{A}(N\vec{x}_0) \quad ; \quad \pi_n \equiv \vec{p} - \frac{e}{c} \vec{A}(\vec{x}_n) \]

where \( \vec{x}_0, \vec{p}_0 \) are the COM position and momentum and \( \vec{\pi} \) is the bariocentric coordinate. We have defined all but the zero-mode commutation relation in previous papers [4,3].

Fubini [1] has shown that the Laughlin wavefunction may be derived from vertex operators which appear to us to have the properties of winding state vertex operators. We shall show that this idea is true. Toward this end, we define the zero-modes \( \varphi_0_L \) and \( \varphi_0_R \) in such a way that these commutation relations will be proper to describe winding states, i.e.

\[ [\varphi_0_L, \alpha_0^\dagger] \sim -i \quad ; \quad [\varphi_0_R, \beta_0^\dagger] \sim -i. \] (5)

This definition for the commutator will allow us to define winding state vertex operators \( V_\xi \sim e^{i\xi \varphi} \) (as in the usual formulation of CFT) such that these states are eigenstates of the zero-mode operator \( \alpha_0^\dagger \). We thus define the “coordinate” zero mode as

\[ \varphi_0_L \equiv -i \alpha_0 \kappa_0 \quad ; \quad \varphi_0_R \equiv -i \beta_0 \kappa_0 \]

\[ [\varphi_0_L, \alpha_0^\dagger] = -i \kappa_0 \quad ; \quad [\varphi_0_R, \beta_0^\dagger] = -i \kappa_0 \] (6)

It is natural to introduce a factor of \( \kappa_0 \) for the COM zero mode to annihilate \( N \)-particle state.
We must now enlarge our Hilbert space to include both particle and pseudohole modes (such as in a closed-string theory). A generalized collective coordinate may then be defined as

\[ \varphi(\zeta, \zeta^*) \equiv \varphi_L(\zeta) + \varphi_R(\zeta^*). \]  

(7)

The L(eft) and R(ight) subscripts denote the fact that while the L(eft) and R(ight) fields are defined at the same positions \((x^1, x^2)\), they "move" in opposite directions under the influence of an external magnetic field due to the fact that they are charge conjugates of each other.

Winding state vertex operators may then be defined as the normal ordered product (denoted by : : ) operators with different weightings for the particles and holes

\[ V_{\xi_L, \xi_R}(\zeta, \zeta^*) \equiv : e^{i[\xi_L \varphi_L(\zeta) + \xi_R \varphi_R(\zeta^*)]} : . \]

(8)

If we define our Hilbert space in the usual way with respect to annihilation-creation operators

\[
\begin{align*}
\alpha_0 |0\rangle &= 0 = \alpha_n |0\rangle ; & <0|\alpha_0^\dagger = 0 =<0|\alpha_n^\dagger \quad (9) \\
\beta_0 |0\rangle &= 0 = \beta_n |0\rangle ; & <0|\beta_0^\dagger = 0 =<0|\beta_n^\dagger,
\end{align*}
\]

we then see that the zero mode piece of these vertex operators form eigenstates of these annihilation-creation operators (i.e. they form coherent states). To see this, it is convenient to further separate our collective coordinates into \(\ell(eft)\) and \(r(ight)\) - ordered pieces

\[
\begin{align*}
\varphi^\ell_L(\zeta) &\equiv i\kappa_0 \alpha_0^\dagger \ln \zeta + \kappa_1 \sum_{n=1}^{N} \frac{1}{n} \alpha_n^* \zeta^{-n} \\
\varphi^r_L(\zeta) &\equiv \varphi_0 + \kappa_1 \sum_{n=1}^{N} \frac{1}{n} \alpha_n \zeta^{-n} \\
\varphi^\ell_R(\zeta) &\equiv i\kappa_0 \beta_0^\dagger \ln \zeta^* + \kappa_1 \sum_{n=1}^{N} \frac{1}{n} \beta_n^* \zeta^{*-n} \\
\varphi^r_R(\zeta) &\equiv \varphi_0 + \kappa_1 \sum_{n=1}^{N} \frac{1}{n} \beta_n \zeta^{*-n}
\end{align*}
\]

(10)

so that the effect of our vertex operator on the ground state is

\[
<0|V_{\xi_L, \xi_R}(\zeta, \zeta^*) = <0|e^{i[\xi_L \varphi^\ell_L(\zeta) + \xi_R \varphi^\ell_R(\zeta^*)]} e^{i[\xi_L \varphi^r_L(\zeta) + \xi_R \varphi^r_R(\zeta^*)]} \\
= <0|e^{i[\xi_L \varphi^\ell_L(\zeta) + \xi_R \varphi^r_R(\zeta^*)]} \\
\]

(11)
It is then easy to show that this is an eigenstate of $\alpha_0^+$

$$< 0|V_{\xi_L \xi_R}(\zeta, \zeta^*)\alpha_0^+ = < 0|e^{i\xi_L \varphi_L^L(\zeta)} \left[ e^{i\xi_L \varphi_L^R(\zeta)}, \alpha_0^+ \right] V_{\xi_R}(\zeta^*)$$

$$= < 0|e^{i\xi_L \varphi_L^L(\zeta)} \xi_L e^{i\xi_L \varphi_L^R(\zeta)} V_{\xi_R}(\zeta^*) = < 0|V_{\xi_L \xi_R}(\zeta, \zeta^*)\xi_L$$

(12)

where

$$\left[ e^{i\xi_L \varphi_L^L(\zeta)}, \alpha_0^+ \right] = e^{i \xi_L (\varphi_L^L(\zeta) - \varphi_0^L)} \left[ e^{i\xi_L \varphi_0^L}, \alpha_0^+ \right] = \xi_L e^{i\xi_L \varphi_L^L(\zeta)},$$

(13)

and

$$\left[ e^{i\xi_L \varphi_L^R(\zeta)}, \alpha_0^+ \right] = \kappa_0 \xi_L e^{i\xi_L \varphi_0^R}.$$ (14)

Likewise, it is an eigenstate of $\beta_0^+$

$$< 0|V_{\xi_L \xi_R}(\zeta, \zeta^*)\beta_0^+ = < 0|V_{\xi_L \xi_R}(\zeta, \zeta^*)\kappa_0 \xi_R.$$

(15)

### III. Laughlin’s Wavefunction from Winding State Vertex Operators

Laughlin’s $N'$ quasiparticle - quasihole wavefunction may now be derived as a “modular” ordered $N'$-point correlation function of winding state vertex operators (apart from the conformal symmetry breaking exponential)

$$\Psi_{\xi_L \xi_R}(\zeta_1, \ldots, \zeta_{N'}; \zeta_1^*, \ldots, \zeta_{N'}^*) = e^{-\sum_{i=1}^{N'} |\zeta_i|^2} < 0|M \left( \prod_{i=1}^{N'} V_{\xi_L \xi_R}(\zeta_i, \zeta_i^*) \right) |0 >$$

(16)

This is easily seen by contracting the $N'$-point correlation function:

$$< 0|M \left( \prod_{i=1}^{N'} V_{\xi_L \xi_R}(\zeta_i, \zeta_i^*) \right) |0 > = < 0|M \left( \prod_{i<j=1}^{N'} \frac{V_{\xi_L \xi_R}(\zeta_i, \zeta_i^*) V_{\xi_L \xi_R}(\zeta_j, \zeta_j^*)}{V_{\xi_L \xi_R}(\zeta_i^*, \zeta_i) V_{\xi_L \xi_R}(\zeta_j^*, \zeta_j)} \right) |0 >$$

(17)

$$= < 0|M \left( \prod_{i<j=1}^{N'} e^{-\xi_L^2 [\varphi_L(\zeta_i), \varphi_L^*(\zeta_i)]} e^{-\xi_R^2 [\varphi_R(\zeta_i), \varphi_R^*(\zeta_i)]} \right) |0 >$$

where

$$\left[ \varphi_L^*(\zeta_i), \varphi_L(\zeta_j) \right] = \kappa_0^2 \ln |\zeta_j| + \kappa_1^2 \sum_{n=1}^{N} \frac{1}{n} \left( \frac{\zeta_i}{\zeta_j} \right)^n.$$ (18)

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and we have changed to “ordinary” oscillators strictly for computational ease. The
relation between these two different oscillators is given by

\[ \alpha_n \equiv \sqrt{n}a_n \quad ; \quad \alpha_n^* \equiv \sqrt{n}a_n^* \quad ; \quad \alpha_{-n} \equiv \sqrt{n}a_{-n} \]
\[ \beta_n \equiv \sqrt{n}a_n \quad ; \quad \beta_n^* \equiv \sqrt{n}a_n^* \quad ; \quad \beta_{-n} \equiv \sqrt{n}a_{-n} \]  

(19)

Now, in the large \( N \) limit, the series converges to a logarithm if we take \( \theta_i = \theta_j \) and \( r_i < r_j \) for \( \zeta \equiv r e^{i\theta} \). Also taking \( \kappa_0 = \kappa_1 \), we then have

\[ \lim_{N \to \infty} \lim_{\theta_i \to \theta_j, \kappa_1 \to \kappa_0} \left[ \varphi_L^r(\zeta_i), \varphi_L^r(\zeta_j) \right]_{r_i < r_j} = \kappa_0^2 \ln(\zeta_j - \zeta_i) \]
\[ \lim_{N \to \infty} \lim_{\theta_i \to \theta_j, \kappa_1 \to \kappa_0} \left[ \varphi_L^r(\zeta_i^*), \varphi_L^r(\zeta_j^*) \right]_{r_i < r_j} = \kappa_0^2 \ln(\zeta_j^* - \zeta_i^*) . \]  

(20)

To see this, note that

\[ \left[ \varphi_L^r(\zeta_i), \varphi_L^r(\zeta_j) \right] = \kappa_0^2 \ln \zeta_j + \kappa_1^2 \sum_{n=1}^{N} \frac{1}{n} \left( \frac{\zeta_i}{\zeta_j} \right)^n \]
\[ = \kappa_0^2 \ln \zeta_j + \kappa_1^2 \sum_{n=1}^{N} \frac{1}{n} \left( \frac{r_i}{r_j} \right)^n e^{i(\theta_i - \theta_j)} \]  

(21)

such that, if \( \theta_i = \theta_j \), \( r_i < r_j \) and \( N \to \infty \), the sum converges to a logarithm

\[ \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{r_i}{r_j} \right)^n = \ln \left( 1 - \frac{r_i}{r_j} \right) = \ln \left( 1 - \frac{\zeta_i}{\zeta_j} \right) , \]  

(22)

giving

\[ \left[ \varphi_L^r(\zeta_i), \varphi_L^r(\zeta_j) \right] = \kappa_0^2 \ln \zeta_j + \kappa_1^2 \ln \left( 1 - \frac{\zeta_i}{\zeta_j} \right) \]  

(23)

which reduces to (20) if \( \kappa_0 = \kappa_1 \) (we have the similar limit for the anti-analytic piece).

Our \( N' \)-point correlation function then becomes

\[ \lim_{N \to \infty} <0|M \left( \prod_{i=1}^{N'} V_\xi_L \zeta_R(\zeta_i, \zeta_i^*) \right)_{\theta_i = \theta_j}^{\theta_i = \theta_j} |0> = \prod_{i<j=1}^{N'} \left( \zeta_j - \zeta_i \right) - \kappa_0^2 \left( \zeta_j^* - \zeta_i^* \right)^2 \]
\[ = \prod_{i<j=1}^{N'} \left( \zeta_j - \zeta_i \right) - \kappa_0^2 \left( \zeta_j^* - \zeta_i^* \right)^2 \]  

(24)

\[ ^2 \text{We could have performed all calculations in this ordinary oscillator basis as well.} \]
Here we see that the conformal weights of these vertex operators is given by
\[
(h, \tilde{h}) = \left( \kappa_0^2 \xi_L^2, \kappa_0^2 \xi_R^2 \right)
\]
where \(\kappa_0\) plays the role of the slope parameter \(\alpha'\) in a closed string theory.

Thus, Laughlin’s wavefunction may be obtained as a coherent state in a triple limit of an \(N\)-particle Hamiltonian with momentum-momentum coupling between different particles. In this triple limit (i.e. \(N\) large) the large angle \(\theta_i = \theta_j\) and for special values of the couplings \((\kappa_0 = \kappa_1)\), such a state corresponds to a modularly ordered \(N'\)-point correlation function of winding state vertex operators. It then transforms as a tensor of rank \((\kappa_0^2 \xi_L^2, \kappa_0^2 \xi_R^2)\) under the group of two-dimensional conformal transformations whose field-theoretical Hamiltonian may be taken as
\[
H = \hbar \omega \left( \kappa_0^2 \alpha_0 \alpha_0 + \kappa_1^2 \sum_{n=1}^{N'} \alpha_n^\dagger \alpha_n \right).
\]  
(26)
In the case that \(\kappa_0 \neq \kappa_1\), the “equal angle” commutator between fields may be written (in the large \(N\)-limit) as
\[
\left[ \varphi_L^r(\zeta_i), \varphi_L^s(\zeta_j) \right]_{\theta_i=\theta_j} \xrightarrow{N \to \infty} \kappa_0^2 \ln \zeta_j + \kappa_1^2 \frac{1}{n} \left( \frac{\zeta_i}{\zeta_j} \right)^n
\]  
(27)

The \(N'\)-point correlation function then becomes
\[
<0|M \left( \prod_{i=1}^{N'} V_{\xi_L \xi_R}(\zeta_i, \zeta_i^\dagger) \right) |0> = \prod_{i<j=1}^{N'} (\zeta_j - \zeta_i)^{-\kappa_0^2 \xi_L^2 (\zeta_i^\dagger - \zeta_j^\dagger) \xi_R^2 |\zeta_j|^2 (\kappa_0^2 - \kappa_1^2)}
\]
(28)
which we may identify as the wavefunction for a quasiparticle and quasipseudohole fluctuation (or excitation) centered at \(\zeta = 0\).

IV. Statistics of the Winding States

To determine the statistics of the winding states, we consider the rotation of one “quasiparticle” around another in the Laughlin wavefunction. In the modularly ordered (i.e. radially ordered) correlation function, we define \(\zeta \equiv r e^{i\theta}\) such that \(r_1 < r_2\) and \(\zeta' \equiv r_2 e^{2\pi i}\). The unrotated two-quasiparticle wavefunction may then be written as
\[
\psi_{\xi_L \xi_R}(\zeta_1, \zeta_2; \zeta_1^\dagger, \zeta_2^\dagger) \sim <0|M (V_{\xi_L \xi_R}(\zeta_1, \zeta_1^\dagger) V_{\xi_L \xi_R}(\zeta_2, \zeta_2^\dagger)) |0> \sim e^{\kappa_0^2 \xi_L^2 \ln(\zeta_2 - \zeta_1)} e^{\kappa_0^2 \xi_R^2 \ln(\zeta_2 - \zeta_1)}
\]  
(29)
where the commutators are “equal angle” commutators taken in the large $N$-limit. On the other hand, we have for the rotated vertex operator $V_{\xi_L\xi_R}(\zeta'_2, \zeta''_2)$ the “equal angle” commutators

\[
\xi^2_L \left[ \varphi^r_L(\zeta_1), \varphi^r_L(\zeta'_2) \right]_{\theta_1=\theta_2} = \xi^2_L \left[ \kappa_0^2 \ln \zeta'_2 + \kappa_1^2 \sum_{n=1}^{N} \frac{1}{n} \left( \frac{\zeta_1}{\zeta'_2} \right)^n \right] \\
= \xi^2_L \left[ \kappa_0^2 (\ln \zeta_2 + 2\pi i) + \kappa_1^2 \sum_{n=1}^{N} \frac{1}{n} \left( \frac{\zeta_1}{\zeta_2} \right)^n e^{-2\pi in} \right] \\
= \xi^2_L \left[ \varphi^r_L(\zeta_1), \varphi^r_L(\zeta_2) \right]_{\theta_1=\theta_2} + 2\pi i \kappa_0^2 \xi^2_L \\
\xi^2_R \left[ \varphi^r_R(\zeta^*_1), \varphi^r_R(\zeta^*_2) \right]_{\theta_1=\theta_2} = \xi^2_R \left[ \varphi^r_R(\zeta^*_1), \varphi^r_R(\zeta^*_2) \right]_{\theta_1=\theta_2} - 2\pi i \kappa_0^2 \xi^2_R
\]

giving rise to the rotated wavefunction

\[
\psi_{\xi_L\xi_R}(\zeta'_1, \zeta'_2; \zeta^*_1, \zeta^*_2) = \psi_{\xi_L\xi_R}(\zeta_1, \zeta_2; \zeta^*_1, \zeta^*_2) e^{2\pi i \kappa_0^2 (\xi^2_L - \xi^2_R)}.
\]

This leads to anyonic statistics for the quasiparticles associated with the coherent state of the winding vertex operator. The wavefunction is not required to be single-valued; however particles described by vertex operators with phases which change by integer amounts, i.e. satisfying the condition

\[
\kappa_0^2 (\xi^2_L - \xi^2_R) \in Z
\]

describe particles of “ordinary” statistics. Such states of course give rise to single-valued wavefunctions. If one requires locality and closure of the operator product algebra, one can further show that these vertex operators must have conformal weights such that $(\xi_L, \xi_R) \in \Gamma$ where $\Gamma$ is a lattice of Lorentzian signature.

V. An Interpretation for the Winding States

As we have just seen in the previous section, the statistics of the quasiparticle created by the winding state vertex operator is given by the difference between the conformal weights of its analytic and anti-analytic factors. This follows from the fact that the rotation operator is given by

\[
L_3 = \hbar (L_0 - \bar{L}_0)
\]

The phase change can be attributed to a shift in the zero-mode contributing to the COM angular momentum.

To see this explicitly, we calculate the effect of a rotation of our order parameter $\varphi(\zeta_2, \zeta^*_2)$ around the vertex operator $V_{\xi_L\xi_R}(\zeta_1, \zeta^*_1)$ in the radially ordered product with $r_1 < r_2$. For $\zeta'_2 = \zeta_2 e^{2\pi i}$, we then have
\begin{align}
< 0|V_{\xi_L}(\zeta_1)\varphi_L(\zeta'_2) &= < 0|V_{\xi_L}(\zeta_1) \left[ \varphi_{0_L} + i\kappa_0\alpha_0^+ \ln \zeta'_2 + \sum_{n=-N}^{N'} \alpha_n \zeta_2^n \right] \tag{34} \\
&= < 0|V_{\xi_L}(\zeta_1) \left[ \varphi_{0_L} - 2\pi\kappa_0\xi_L + i\kappa_0\alpha_0^+ \ln \zeta_2 + \kappa_1 \sum_{n=-N}^{N'} \alpha_n \zeta_2^n e^{2\pi in} \right] \\
&= < 0|V_{\xi_L}(\zeta_1) [\varphi_L(\zeta_2) - 2\pi\kappa_0\xi_L] \\
< 0|V_{\xi_R}(\zeta^*_1)\varphi_R(\zeta'^*_2) &= < 0|V_{\xi_R}(\zeta^*_1) \left[ \varphi_R(\zeta'^*_2) + 2\pi\kappa_0(\xi_R - \xi_L) \right] \tag{35}
\end{align}

which gives the overall effect on the vertex operator

\begin{align}
< 0|V_{\xi_L\xi_R}(\zeta_1,\zeta^*_1)\varphi_R(\zeta'_2,\zeta'^*_2) &= < 0|V_{\xi_L\xi_R}(\zeta_1,\zeta^*_1) [\varphi_R(\zeta'_2,\zeta'^*_2) + 2\pi\kappa_0(\xi_R - \xi_L)] \tag{36}
\end{align}

where the prime over the sum ($\sum'$) indicates that omission of the zero mode in the sum. Thus, the effect is to shift the zero modes

\begin{align}
\varphi_{0_L} &\quad \longrightarrow \quad \varphi_{0_L} - 2\pi\kappa_0\xi_L \tag{37} \\
\varphi_{0_R} &\quad \longrightarrow \quad \varphi_{0_R} - 2\pi\kappa_0\xi_R. \tag{38}
\end{align}

From equations (37,38)

\begin{align}
\varphi_{0_L} &= \frac{\kappa_0 \ell}{\sqrt{2Nh}} \left( -i\pi^1_0 + \pi^2_0 \right) \\
\varphi_{0_R} &= \frac{\kappa_0 \ell}{\sqrt{2Nh}} \left( -i\tilde{\pi}^1_0 + \tilde{\pi}^2_0 \right). \tag{39}
\end{align}

The shift in the zero mode (37,38) may be absorbed into $\pi^2_0$ and $\tilde{\pi}^1_0$, giving

\begin{align}
\pi^2_0 &\quad \longrightarrow \quad \pi^2_0 - k_L = p^2_0 + \frac{e}{c} A^2(N\bar{x}_0) - k_L \\
\tilde{\pi}^1_0 &\quad \longrightarrow \quad \tilde{\pi}^1_0 + k_R = p^1_0 - \frac{e}{c} A^1(N\bar{x}_0) + k_R \tag{40}
\end{align}

where $k_L \equiv 2\pi\xi_L\sqrt{2Nh}$ and $k_R \equiv 2\pi\xi_R\sqrt{2Nh}$. Then in arbitrary gauge, the winding state leads to a shift in the COM momentum

\begin{align}
p^2_0 &\quad \longrightarrow \quad p^2_0 - k_L \tag{41} \\
p^1_0 &\quad \longrightarrow \quad p^1_0 + k_R
\end{align}
This leads to a shift in the COM angular momentum

\[ x_0^1p_0^2 - x_0^2p_0^1 \rightarrow x_0^1p_0^2 - x_0^2p_0^1 - \Delta L_3^{(0)} \]  \hspace{1cm} (42)

where \( \Delta L_3^{(0)} \equiv x_0^1k_L + x_0^2k_R \). The winding state vertex operator \( V_{\xi_L,\xi_R}(\zeta, \zeta^*) \) has the effect of creating a quasiparticle fluctuation whose COM energy is shifted from that of the original ground state.

An alternative description can also be made in coordinate space. Rather than absorbing the shift into the momentum, we may absorb it into the vector potential which can be written in arbitrary gauge as

\[ A^i(x) = \frac{B}{2} \left( \epsilon^{ij}x^j + \frac{\partial A}{\partial x^i} \right). \]  \hspace{1cm} (43)

From (41) we then have

\[ A^2(N\vec{x}_0) \rightarrow A^2(N\vec{x}_0) - \frac{c}{e}k_L \]  \hspace{1cm} (44)

\[ A^1(N\vec{x}_0) \rightarrow A^1(N\vec{x}_0) - \frac{c}{e}k_R \]

which, in symmetric gauge (\( \Lambda = 0 \)) gives

\[ x_0^1 \rightarrow x_0^1 + L_0^1 \]  \hspace{1cm} (45)

\[ x_0^2 \rightarrow x_0^2 + L_0^2 \]

where \( L_0^1 \equiv 4\pi\xi_L\ell\sqrt{\frac{2}{N}} \) and \( L_0^2 \equiv -4\pi\xi_R\ell\sqrt{\frac{2}{N}} \). Our overall zero-mode then changes by the amount

\[ \varphi_0 \equiv \varphi_{0L} + \varphi_{0R} \rightarrow \varphi_0 + 2\pi\kappa_0(\xi_R - \xi_L) \]

\[ \rightarrow \vec{x}_0 \rightarrow \vec{x}_0 + \vec{L}_0. \]  \hspace{1cm} (46)

If we require the wavefunction to be single-valued, we must have

\[ \kappa_0(\xi_R - \xi_L) = M \quad ; \quad \text{for } M \in \mathbb{Z} \]  \hspace{1cm} (47)

or, in terms of the real coordinate zero-modes (i.e. COM position)

\[ \vec{x}_0 \sim \vec{x}_0 + \vec{L}_0 \]  \hspace{1cm} (48)

suggesting that the winding mode be associated with the cyclotron motion of a quasiparticle. The equivalence relation defined by (48) suggests that \( \vec{x}_0 \in \hat{\mathcal{M}} \) where \( \hat{\mathcal{M}} \) is
some compact manifold characterized by the independent “winding (modular) parameters” composing $\vec{L}_0$. For a two-quasiparticle wavefunction there will only be two such independent parameters; however for an $N'$-quasiparticle wavefunction

$$\psi_{\xi_{L_1}, \xi_{L_N'}, \xi_{R_1}, \xi_{R_{N'}}}(\zeta_1, \ldots, \zeta_{N'}; \zeta_1^*, \ldots, \zeta_{N'}^*) \sim <0|M \left( \prod_{i=1}^{N'} V_{\xi_{L_i}, \xi_{R_i}}(\zeta_i, \zeta_i^*) \right)|0>$$

the number of parameters will obviously be related to the number of independent phases associated with the rotations of vertex operators around each other. The dimension of $\vec{L}_0$ thus increases with $N'$. For $N' = 1$, $\mathcal{N}$ will be diffeomorphic to $S^1 \times S^1 \sim T^2$. The field $\varphi(\zeta, \zeta^*) = \varphi_L(\zeta) + \varphi_R(\zeta^*)$ can thus be recognized as fields coordinatizing a group manifold $\mathcal{M}$ where

$$g_L(\xi_L) \equiv e^{i\xi_L \varphi_L} \in G_L; \quad g_R(\xi_R) \equiv e^{i\xi_R \varphi_R} \in G_R$$

$$g(\xi_L, \xi_R) \equiv g_L(\xi_L)g_R(\xi_R); \quad g(\xi_L, \xi_R) \in G \sim G_L \times G_R$$

may be recognized as our vertex operators which are elements of the group $G$. This has obvious generalizations to include other interactions (such as spin-spin interactions) in the context of the nonlinear sigma model (NLSM) and Kac-Moody algebras via the Wess-Zumino-Witten theory.

### VI. The QHE as a CFT

In the previous sections we have shown how the Laughlin wavefunction could be derived by defining an order parameter $\varphi(\zeta, \zeta^*)$ from which a winding state vertex operator could be defined. Under the special conditions of

- the large $N$-limit
- modular (or radial) ordering
- equal-angle commutators
- and special values for the couplings $\kappa_1 = \kappa_0$

we found that one can obtain the Laughlin $N'$-quasiparticle wavefunction as an $N'$-point correlation function of winding state vertex operators (apart from an exponential factor which can be attributed to a boundary term). This leads to the picture of the quantum

\[3\text{We expect the conductivity tensor to be associated with the symmetric and antisymmetric couplings of the NLSM. The idea is to relate a metric and torsion tensor on the group manifold to the longitudinal and Hall components of the conductivity tensor; the torsion being related to the Hall conductivity.}\]
Hall effect as a manifestation of some critical phenomena expressible in the language of CFT.

To better understand the relations of the QHE with CFT, we may view our order parameter $\varphi(\zeta, \zeta^*)$ as that of a field of spins (created by $\alpha^\dagger$) and “anti-spins” (created by $\beta^\dagger$). This picture is valid in the low-energy limit - that is at energies $E << \hbar \omega$ in which case the cyclotron motion of the electrons is may be taken as time-independent (the oscillators $\alpha^\dagger$ create energy eigenstates and are thus associated with stationary states). At higher energies, the spins degrees of the electrons must then be taken into account. In a very strong magnetic field, however, the electron spins are all aligned with the external magnetic field (i.e. all electrons are polarized)\(^4\) We expect different effective field theories to be applicable at different energies. From the perspective of the NLSM, the couplings\(^5\) of the effective field theory become renormalized at lower and lower energies according to the renormalization group (RG) equations. Fixed points of these RG equations correspond to critical couplings associated with those of a CFT. These points are viewed as local minima of the effective potential. As a local minimum of an effective potential, the field configurations correspond to those which minimize the action, i.e. they correspond to field equations for a classical field theory. In connection with integrable systems, these classical field theories may be associated with “particle-like” configurations due to a coherent superposition of the many particle system creating a quasiparticle state (i.e. bound state configuration). Here we have seen an example in which the winding state vertex operators create such coherent states which are eigenstates of the COM complex momentum $\alpha_0$. The integrability of the system is due to the large (infinite) number of conserved quantities which commute with the Hamiltonian at the critical point. The long range order associated with the conformal invariance may thus be related to the $N'$-quasiparticle coherent state formed by the $N'$-point correlation function of winding state vertex operators (in the triple limit previously described) which gives rise to the Laughlin wavefunction describing this $N'$-quasiparticle state. It may also be viewed as that long range interaction responsible for the current-current coupling which has led us to this formulation of the QHE in terms of a CFT \(\square\).

A CFT may be characterized by a stress-energy tensor built out of the order parameters of the theory. In particular, the Sugawara construction utilizes the conserved currents such that the stress-energy tensor is quadratic in these currents. Such is the case with our construction here. We may define the stress-energy tensor as

\[
T(\zeta, \zeta^*) \equiv T_L(\zeta) + T_R(\zeta^*)
\]

\[
T_L(\zeta) \equiv \frac{\partial \varphi_L(\zeta)}{\partial \zeta} \frac{\partial \varphi_L(\zeta)}{\partial \zeta^*} ; \quad T_R(\zeta^*) \equiv \frac{\partial \varphi_R(\zeta^*)}{\partial \zeta^*} \frac{\partial \varphi_R(\zeta^*)}{\partial \zeta}.
\]

\(^4\)The spin-flip energy is much larger than that of the gap energy of the Landau levels. We expect this energy to be of order $e^2$ (like the fine structure constant for spin interactions in an atom) as opposed to the order $e$ interaction of the cyclotron motion ($\omega = \frac{eB}{\mu c}$).

\(^5\)In this case we may take our couplings to be associated with the components of the conductivity tensor $\sigma_{ij}(\varphi(\zeta, \zeta^*))$. 

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Here we may recognize the conserved current as that associated with our conjugate momentum \[ j_L(\zeta) \equiv \frac{\partial \varphi_L(\zeta)}{\partial \zeta} = \pi(\zeta) \quad ; \quad j_R(\zeta^*) \equiv \frac{\partial \varphi_R(\zeta^*)}{\partial \zeta^*} = \tilde{\pi}(\zeta^*). \] (52)

The generators of the conformal transformations under which the theory is invariant may be obtained from the moments of the stress-energy tensor. The resulting Virasoro algebra consists of two commuting pieces associated with the analytic and anti-analytic terms (i.e. L(ef)t and R(ight) pieces associated with the particles and pseudoholes, respectively). Expanding our stress-energy tensors, we have

\[
T_L(\zeta) = \frac{1}{\zeta^2} \left[ -\kappa_0^2 \alpha_0^\dagger \alpha_0 + i \kappa_0 \kappa_1 \alpha_0^\dagger \sum_{n=1}^{N} \left( \alpha_0 \zeta^n - \alpha^*_n \zeta^{-n} \right) \\
+ \kappa_1^2 \sum_{n,m=1}^{N} \left( \alpha_n \alpha_m \zeta^{n+m} + \alpha^*_n \alpha^*_m \zeta^{n+m} - \alpha^*_n \alpha_m \zeta^{m-n} \right) \right] \\
\equiv - \sum_{n=-N}^{N} \frac{L_n}{\zeta^{n+2}} \]  

(53)
giving the generator of dilations (in the large \( N \) limit with \( \kappa_0 = \kappa_1 \))

\[
L_0 = \kappa_0^2 \alpha_0^\dagger \alpha_0 + \kappa_1^2 \sum_{n=1}^{N} \left( \alpha_n^* \alpha_n^* + \alpha_n \alpha_n^* \right). \]  

(54)
Likewise for our “psuedoholes”

\[
\tilde{L}_0 \equiv \kappa_0^2 \beta_0^\dagger \beta_0 + \kappa_1^2 \sum_{n=1}^{N} \left( \beta_n^* \beta_n^* + \beta_n \beta_n^* \right). \]  

(55)
Normal ordering rather than symmeterizing the dilation operator gives the usual dilation operator. The effective Hamiltonian is then given by

\[
H(\kappa_0, \kappa_1) = H_L(\kappa_0, \kappa_1) + H_R(\kappa_0, \kappa_1) = \hbar \omega (L_0 + \tilde{L}_0) \\
H_L(\kappa_0, \kappa_1) \equiv \hbar \omega L_0(\kappa_0, \kappa_1) ; \quad H_R(\kappa_0, \kappa_1) \equiv \hbar \omega \tilde{L}_0(\kappa_0, \kappa_1). \]  

(56)
To emphasize the fact that this Hamiltonian corresponds to that of a CFT only for special values of the couplings (and in the large \( N \) limit), we have denoted the coupling dependence explicitly.
VII. Conclusion

Our results suggest that the Laughlin wavefunction describes a state of the system corresponding to a critical point in a phase transition such that it is describable by a CFT. In our example model (which excludes electron spins), the current-current coupling seems to be the source of the long-range order of the system. Cast in the more general context of a 2+1 dimensional NLSM, we expect the effective field theory to be related to a chiral symmetry breaking which gives rise to the QHE.

Many connections between the QHE, CFT and strings still remain to be uncovered. Among the more interesting connections of interest is the connection between the Landau levels and Regge trajectories. This requires a better understanding of such things as the physical state conditions and its relation to the measurements of the conductivity tensor.

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