Weak-Chaos Ratchet Accelerator

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Classical Hamiltonian systems with a mixed phase space and some asymmetry may exhibit chaotic ratchet effects. The most significant such effect is a directed momentum current or acceleration. In known model systems, this effect may arise only for sufficiently strong chaos. In this paper, a Hamiltonian ratchet accelerator is introduced, featuring a momentum current for arbitrarily weak chaos. The system is a realistic generalized kicked rotor and is exactly solvable to some extent, leading to analytical expressions for the momentum current. While this current arises also for relatively strong chaos, the maximal current is shown to occur, at least in one case, precisely in a limit of arbitrarily weak chaos.

I. INTRODUCTION

Classical and quantum Hamiltonian ratchets have attracted a considerable theoretical interest during the last decade. Also, several kinds of quantum ratchets have been experimentally realized using atom-optics methods with cold atoms or Bose-Einstein condensates. The classical Hamiltonian ratchet effect is a directed current in the chaotic region generated by an unbiased force (having zero mean in space and/or time) and due to some spatial and/or temporal asymmetry. This is analogous to the ordinary ratchet time-periodic Hamiltonians by an unbiased force (having zero mean in space and/or time) and due to some spatial and/or temporal asymmetry. This is analogous to the ordinary ratchet.

A well studied class of systems are those described by time-periodic Hamiltonians $H(x, p, t)$ for which both the force $F = -\partial H/\partial x$ and the velocity $v = \partial H/\partial p$ are periodic in $x$ and $F$ has zero mean over $(x, t)$. The classical ratchet current is usually defined as the average of $v$ over $(x, p, t)$, where $(x, p)$ is restricted to the chaotic region, see, e.g., Refs. 1, 2. It is assumed that $v$ is bounded, e.g., by Kolmogorov-Arnol’d-Moser (KAM) tori. Then, necessary conditions for the ratchet current to be nonzero are the breaking of some symmetry and a mixed phase space featuring “transporting” stability islands which propagate in the $x$ direction. In the presence of bounding KAM tori, one can get, in principle, a well-defined ratchet current also in near-integrable regimes, corresponding to relatively weak and local chaos.

A different and much more significant Hamiltonian ratchet effect was discovered in work (rather than the usual position velocity $v$) of the global chaotic region (see more details in Sec. II). Quantum analogs of the classical ratchet acceleration were found in several systems, either for special, “quantum-resonance” values of a scaled Planck constant $\hbar$ or for generic values of $\hbar$. Quantum-resonance ratchet accelerators have been experimentally realized in recent works.

In this paper, we show for the first time that the phenomenon of ratchet acceleration is not limited to strong-chaos regimes. We introduce a realistic Hamiltonian system exhibiting this phenomenon most significantly in near-integrable regimes, corresponding now to arbitrarily weak but global chaos. The system is a generalized kicked rotor whose force function has zero mean and is characterized by two nonintegrability parameters $b_1$ and $b_2$. A global chaotic region in the $p$ direction arises also for arbitrarily small values of these parameters, i.e., the KAM scenario is not satisfied. As $b_1, b_2 \to 0$, this “non-KAM” system tends to the well-known elliptic sawtooth map, which has been used as a paradigmatic model of “pseudochaos” dynamical complexity with zero Lyapunov exponent in studies of both classical and quantum systems. We show that accelerator-mode islands exist for arbitrarily small $b_1$ and $b_2$. Then, when the system is asymmetric ($b_1 \neq b_2$), a ratchet acceleration $A$ may arise for arbitrarily weak chaos. In one particular case, we derive analytical expressions for $A$ as a function of $b_1$ and $b_2$. Paths of maximal $A$ in the $(b_1, b_2)$ parameter space are determined. We then show that, in sharp contrast with the systems considered in work, $A$ is most significant for relatively small Lyapunov exponent and that its maximal value is attained precisely in a limit $b_1, b_2 \to 0$ of arbitrarily weak chaos.

This paper is organized as follows. In Sec. II, we give a short background on Hamiltonian ratchet accelerators. In Sec. III, we introduce our general model system and describe its basic properties. In particular, in Sec. III C we derive the existence conditions for the main accelerator-mode islands of the system. In Sec. IV,
analytical expressions for the ratchet acceleration $A$ in one case are obtained for all values of the parameters. In Sec. V, we show that the maximal value of $A$ is attained in a limit of arbitrarily weak chaos. Conclusions are presented in Sec. VI. Detailed derivations of several analytical results are given in the Appendices.

II. BACKGROUND ON HAMILTONIAN RATCHET ACCELERATORS

The concept of Hamiltonian ratchet accelerator was introduced in Ref. [6] by adaptation of a formalism developed in Refs. [2, 3]. We give here a self-contained summary of these works, leading to the main result [6] below, a sum rule for the ratchet acceleration $A$. We shall focus on realistic models, the generalized kicked-rotor systems with scaled Hamiltonian $H = p^2/2 + KV(x, t) \sum_{s=-\infty}^{\infty} \delta(t - s)$, where $K$ is the nonintegrability parameter and the potential $V(x, t)$ is periodic in $x$, $V(x + 1, t) = V(x, t)$. Particular cases of these systems where considered in Ref. [6]. The map for $H$ from $t = s - 0$ to $t = s + 1 - 0$ is given by

\[ M : \ p_{s+1} = p_s + Kf_s(x_s), \ x_{s+1} = x_s + p_{s+1} \mod(1), \]  

where the force function $f_s(x) = -dV(x, t = s)/dx$. Because of the periodicity of $V(x, t)$ in $x$, $f_s(x)$ satisfies the ratchet (zero-flux) condition: $\langle f_s(x) \rangle = \int_0^1 f_s(x) dx = 0$.

As in Ref. [6], we shall assume that the kicking parameter $K$ is large enough that all the rotational (“horizontal”) KAM tori are broken. Then, there are no barriers to motion in the $p$ direction, leading to a global and strongly chaotic region. These barriers cannot exist if there are “accelerator modes”, i.e., orbits that are periodic under map (1) in the following sense:

\[ p_{s+m} = p_s + w, \ x_{s+m} = x_s, \]  

where $m$ is the period and $w$, the winding number, is an nonzero integer. Periodic orbits can be defined in the generalized way (2) $(w \neq 0)$ due to the obvious periodicity of the map (1) in $p$ with period 1. If an accelerator mode is linearly stable, each of its points $(x_s, p_s)$ will be usually surrounded by an island $I_s$, an “accelerator-mode island” (AI). Because of (2), $I_{s+m}$ is just $I_s$ translated by $w$ in the $p$ direction. For arbitrary initial conditions $z_0 = (x_0, p_0)$ in phase space, the mean acceleration (momentum current/velocity) in $n$ iterations of (1) is

\[ A_n(z_0) = \frac{p_n - p_0}{n} \]  

and the average of (3) in some region $R$ with area $S_R$ is

\[ \langle A_n \rangle_R = \frac{1}{S_R} \int_R A_n(z_0) dz_0. \]  

In the case that $R$ is an AI $I$ with winding number $w$, it follows from Eqs. (2, 3, 4) that

\[ \lim_{n \to \infty} \langle A_n \rangle_I = \nu = \frac{w}{m}. \]  

Now, because of the periodicity of the map (1) in $p$ with period 1, one can take also $p_{s+1}$ modulo 1 in (1), leading to a map $\tilde{M}$ on the unit torus $\mathbb{T}^2 : 0 \leq x, p < 1$; this is the unit cell of periodicity of map (1). The reduced phase space $\mathbb{T}^2$ can be fully partitioned into the global chaotic region $C$ with area $S_C$ and all the stability islands $\mathbb{T}^j$ with areas $S_j$, where $j$ labels the island: $S_C = \sum_j S_j = 1$. The average acceleration $\langle A \rangle$ of $\mathbb{T}^j$ is $\nu_j = \nu_{j/m}$, where $\nu_j = 0$ for a normal (non-accelerating) island. We then have the following sum rule relating $\nu_j$ to the ratchet acceleration $\langle A \rangle_C = \lim_{n \to \infty} \langle A_n \rangle_C$ of the global chaotic region:

\[ S_C \langle A \rangle + \sum_j S_j \nu_j = 0. \]  

Eq. (6) is easily derived from the obvious relation $\langle A_n \rangle_{\mathbb{T}^2} = S_C \langle A_n \rangle_C + \sum_j S_j \langle A_n \rangle_{\mathbb{T}^j}$ by taking $n \to \infty$ and using $\langle A_n \rangle_{\mathbb{T}^j} = 0$, a result following straightforwardly from the map (1):

\[ \langle A_n \rangle_{\mathbb{T}^2} = \left\langle \sum_{s=1}^{n} \frac{p_s - p_{s-1}}{n} \right\rangle_{\mathbb{T}^2} = K \sum_{s=0}^{n-1} dz_0 f_s(x) \]  

where we used area preservation ($dz_0 = dz_s$), the invariance of $\mathbb{T}^2$ under $\tilde{M}$, and the ratchet condition $\langle f_s(x) \rangle = 0$. An immediate consequence of relation (6) is that $A$ vanishes if the map (1) is invariant under inversion, $(x, p) \to (-x, -p)$, i.e., one has the inversion (anti)symmetry $f_s(-x) = -f_s(x)$. This is because under this symmetry for each AI with mean acceleration $\nu_j \neq 0$ there exists an AI with the same area but with mean acceleration $-\nu_j$. As we shall see in the next sections for a simple case of $f_s(x)$, $A$ is generally nonzero when AIs are present and inversion symmetry is absent.

III. THE GENERAL MODEL SYSTEM AND ITS BASIC PROPERTIES

A. General

The general model system introduced and studied in this paper is the generalized kicked-rotor system described by a simple map (1):

\[ M : \ p_{s+1} = p_s + Kf_s(x_s), \ x_{s+1} = x_s + p_{s+1} \mod(1), \]  

where $0 < K < 4$ and, for $0 \leq x < 1$,

\[ f(x) = \begin{cases} 
    l_1 x & \text{for } 0 \leq x \leq b_1, \\
    c - x & \text{for } b_1 < x < 1 - b_2, \\
    l_2(1 - x) & \text{for } 1 - b_2 \leq x < 1,
\end{cases} \]  

with $f(x+1) = f(x)$. Here $b_1$ and $b_2$ are positive parameters with $b_1 + b_2 < 1$ while $l_1, l_2$, and $c$, also positive, are
fixed by requiring \( f(x) \) to be continuous and to satisfy the ratchet condition \( \int_0^1 f(x) dx = 0 \) (see Appendix A):

\[
l_1 = \frac{(1 - b_1)(1 - b_1 - b_2)}{b_1(2 - b_1 - b_2)}, \quad l_2 = \frac{(1 - b_2)(1 - b_1 - b_2)}{b_2(2 - b_1 - b_2)},
\]

\[
c = \frac{1 - b_2}{2 - b_1 - b_2}.
\]

Kicked systems with a smooth piecewise linear force function such as (9) have been studied either on the phase plane [21] or on a cylindrical phase space [22], corresponding to the very special case of map (8) with \( b_1 = b_2 = 1/4 \). Apparently, however, these systems have not been considered yet in the context of Hamiltonian ratchet transport, i.e., for general values of \( b_1 \) and \( b_2 \) with \( b_1 \neq b_2 \), leading to an asymmetric force function (9). This general system is realistic since it may be experimentally realized using, e.g., optical analogs as proposed in Ref. [26]. As we shall see below, the system generally does not satisfy the KAM scenario assumed in Sec. II, i.e., it is a non-KAM system.

### B. Phase space and limit cases

The phase space of the map (8) in the basic periodicity torus \( T^2 \) (0 \( \leq \) \( x \), \( p \) \( < \) 1) is illustrated in Fig. 1 for some values of the parameters. We clearly see in all cases a connected chaotic region encircling \( T^2 \) in both the \( x \) and \( p \) directions, implying global chaos and the nonexistence of KAM tori bounding \( p \). An understanding of this numerical observation will be achieved here and in Sec. IIIC. We first consider here the map (8) in the limit of \( b_1, b_2 \to 0 \).

From Eqs. (10) one has \( l_1 b_1, l_2 b_2, c \to 1/2 \) in this limit, so that the function (9) tends to the sawtooth

\[
f(x) = 1/2 - x \quad (0 \leq x < 1), \quad f(x + 1) = f(x),
\]

with discontinuity at \( x = 0 \). The map (8) with (10) and \( 0 \leq K \leq 4 \) is the well-known elliptic sawtooth map (ESM) [21, 22, 24, 25], having the property that its linearization \( DM \) is a constant \( 2 \times 2 \) matrix with eigenvalues \( \lambda_{\pm} \) on the unit circle:

\[
\lambda_{\pm} = \exp(\pm i\alpha), \quad 2 \cos(\alpha) = 2 - K.
\]

This means that orbits of the ESM which do not cross the discontinuity line \( x = 0 \) lie on ellipses with average rotation angle \( \alpha \). In general, however, an orbit will cross the \( x = 0 \) line. Then, the combination of the mod(1) operation in (8) with the local ellipticity of the ESM will usually lead to a complex dynamics with zero Lyapunov exponent, known as “pseudo-chaos” [22]. The phase space generally consists of the pseudochaotic region, associated with all iterates of the discontinuity line (21), and a set of islands. More specifically, one has to distinguish between three main cases of the ESM, illustrated in Fig. 2 for the same values of \( K \) as in Fig. 1: (a) The “integrable” case of integer \( K = 1, 2, 3 \) [corresponding to \( \alpha/(2\pi) = 1/6, 1/4, 1/3 \) in (12)]; in this case, no pseudo-chaos arises and the phase space consists just of a finite number of “separatrix” lines (iterates of the discontinuity line) bounding a finite number of islands, see Fig. 2(a). (b) The case of non-integer \( K \) with rational \( \alpha/(2\pi) \) in (12); in this case, numerical work [21] indicates that one has an infinite set of islands and that the pseudo-chaotic region is a fractal with zero area, see Fig. 2(b) and exact results for the fractal dimension of such regions in other maps with discontinuities [23]. (c) The case of irrational \( \alpha/(2\pi) \); here one typically has again an infinite set of islands but the pseudo-chaotic region appears numerically to cover a finite area [21], see Fig. 2(c). Since the momentum \( p \) assumes all values on the discontinuity line and is thus unbounded, the pseudo-chaos [or the separatrix in case (a)] is global.

For finite and small \( b_1 \) and \( b_2 \), the continuous map (8) may be considered as a perturbed ESM, with the discontinuity line replaced by a vertical strip \( B \) of width \( b_1 + b_2 \) in \( T^2 \) (see also caption of Fig. 2):

\[
B: \quad 0 \leq p \leq 1 \quad \text{and} \quad 0 \leq x \leq b_1 \quad \text{or} \quad 1 - b_2 \leq x < 1.
\]
small values of $b_1$ and $b_2$. This exactly implies global and arbitrarily weak chaos. We shall consider only period-1 AIs, associated with stable accelerator modes satisfying Eq. (2) with $m = 1$ and $w \neq 0$. As we shall see, there appear to be no higher-period AIs at least in the case of $K = 3$ on which we shall focus from next section on. The initial conditions $(x_0, p_0)$ for $m = 1$ stable periodic orbits in Eq. (2) must necessarily lie in the middle interval in Eq. (4), $b_1 < x_0 < 1 - b_2$, since only in this interval the matrix $DM$ exhibits stability eigenvalues [12]. Then, from Eqs. (2), (8), and (9), we get:

$$x_0 = c - \frac{w}{K}, \quad p_0 = 0 \mod(1), \tag{14}$$

$$b_1 < c - \frac{w}{K} < 1 - b_2. \tag{15}$$

For $w = 0$ one has a non-accelerating stable fixed point $(x_0 = c, p_0 = 0)$, the center of a normal (non-accelerating) island. Let us show that $w$ may take only two nonzero values and this only in some interval of $K$:

$$w = \pm 1, \quad 2 \leq K < 4. \tag{16}$$

In fact, from Eqs. (9) and (10) it follows that the maximal value of $|f(x)|$ is $\max(|b_1, b_2|) < 1/2$. Then, since $w = Kf(x_0)$ from Eqs. (2) and (8), we have $|w| \leq [K/2]$, where $[\ ]$ denotes integer part. This implies, for $0 < K < 4$, that $w$ may take the only nonzero values of $\pm 1$ provided $2 \leq K < 4$.

Now, according to Eq. (14) for $x_0$, the values of $w = 1$ and $w = -1$ should correspond, respectively, to a “left” (L) and “right” (R) AI, see Figs. 1 and 2. An explicit existence condition for the left AI ($w = 1$) is derived, after some simple algebra, from the left inequality in (15) using Eq. (10) for $c$:

$$b_2 < F(b_1) \equiv \frac{Kb_1^2 + (1 - 2K)b_1 + K - 2}{K - 1 - Kb_1}, \tag{17}$$

see also note [28]. It is easily verified that the right inequality in (15) is identically satisfied. Similarly, the existence condition for the right AI is $b_1 < F(b_2)$. One thus has three cases (compare with Fig. 3 for $K = 3$):

(a) Both AIs L and R exist (see, e.g., Figs. 1 and 2) if

$$b_2 < F(b_1) \text{ and } b_1 < F(b_2). \tag{18}$$

Clearly, this will be always satisfied for $K > 2$ and sufficiently small $b_1$ and $b_2$ since $F(b_1) \approx (K - 2)/(K - 1)$ for $b_1 \ll 1$ in Eq. (17); thus, both AIs exist in the arbitrarily weak chaos regime. For $K = 3$, this case corresponds to the domain $LR$ in Fig. 3.

(b) Only one AI, say the right one $R$, exists (as, e.g., in Fig. 4) if

$$b_2 \geq F(b_1) \text{ and } b_1 < F(b_2). \tag{19}$$

Similarly, if $b_2 < F(b_1)$ and $b_1 \geq F(b_2)$ only the left AI

This should be contrasted with the perturbed ESM in Ref. [24] for which the discontinuity line is not removed by the perturbation. The linearization $DM$ of (8) is again a constant $2 \times 2$ matrix in each of of the three intervals in Eq. (9). In the middle interval, it is the same matrix as for the ESM, with stability eigenvalues [12]. In the other two intervals, where the strip (13) is located, $DM$ can be easily shown to have real positive eigenvalues $\lambda_+$ with, say, $\lambda_+ > 1$ and $\lambda_- = \lambda_+^{-1} < 1$, i.e., there is local hyperbolicity. One can then expect that already for small $b_1$ and $b_2$ a global chaotic region with positive Lyapunov exponent will emerge from the vertical strip (13) and will replace the global pseudochaos (or separatrix) for $b_1 = b_2 = 0$. This can be clearly seen by comparing Figs. 1 and 2. The nature of the chaotic region will be further discussed in the next sections, where it will be shown numerically that the Lyapunov exponent indeed tends to zero as $b_1, b_2 \to 0$.

C. Accelerator-mode islands (AIs) and their existence conditions

We show here that AIs for the map (8) rigorously exist in broad ranges of the parameters, including arbitrarily

![Diagram](image-url)
It is easy to show from the expression for $F(b_1)$ in Eq. \(17\) (see also note \[28\]) that no period-1 AIs exist if \(b_1 = b_2 \geq 1/4\), for any value of $K$ in the relevant interval of $2 \leq K < 4$. This is consistent with the known fact that bounding KAM tori exist for some $K$ if $b_1 = b_2 = 1/4$ \[27\], which is apparently the only case of map \[8\] studied until now.

**IV. RATCHET ACCELERATION FOR $K = 3$**

In this section, the ratchet acceleration $A$ in the case of $K = 3$ will be calculated analytically in the framework of a plausible assumption (see below), supported by extensive numerical evidence and exact results. To use the sum rule \[9\], we first identify the global chaotic region $C$ in the basic periodicity torus $\mathbb{T}^2$. Let us denote by $C$ the set of all iterates of the vertical strip $B$ in Eq. \[13\] under $M$, i.e., the map \[8\] modulo $\mathbb{T}^2$ (see Sec. II):

$$C = \bigcup_{s = -\infty}^{\infty} M^s B.$$  

\[23\]

Exact results for the set \[24\] are derived in Appendices B-E. Here we note that orbits which never visit $B$ (and thus also $C$) are all stable since they lie entirely within the middle interval in \[9\] where the linearized map $DM$ has stability eigenvalues \[12\]. Thus, the global chaotic region $C$ must be entirely contained within $C$, in agreement with our expectation at the end of Sec. IIIB. Our extensive numerical studies indicate that $C$ is indistinguishable from $C$, compare, e.g., Figs. 1(a) and 4 with Figs. 11 and 12 in Appendix C. In fact, finite-time Lyapunov exponents of orbits starting from initial conditions covering $B$ uniformly were found to be all strictly positive. We shall therefore assume in what follows that $C$ precisely coincides with $C$. The rest of phase space outside $C$ consists of no more than three stability regions [see, e.g., Figs. 1(a) and 4]: The left AI $L$ ($w = 1$), the right AI $R$ ($w = -1$), and a normal island ($w = 0$) lying between $L$ and $R$. Using the sum rule \[9\] with $\nu_j = w_j$ [since $m = 1$ in Eq. \[5\]], we then get a formula for the ratchet acceleration:

$$A = \frac{S_R - S_L}{S_C}.$$  

\[24\]

Exact expressions for the areas $S_L$, $S_R$, and $S_C$ are derived in Appendices D and E using simple geometry; see Eqs. \[49\], \[50\], and \[52\]–\[54\] there. Inserting these expressions in formula \[24\], we obtain after some algebra explicit results for $A$ in different cases:

(a) If both AIs exist, i.e., case \[18\],

$$A = \frac{(b_1 - b_2)[1 - 3(b_1 + b_2)]}{2(2 - b_1 - b_2)(b_1 + b_2)}.$$  

\[25\]
(b) If only one AI, say the right one, exists, i.e., case (19),

\[ A = \frac{(2 - 3b_2 - 3c)^2}{6(b_1 + b_2) - 6(b_1 + b_2)^2 + (3c - 3b_1 - 1)^2}. \]  

(26)

(c) If no AIs exist, i.e., case (21), \( A = 0 \), of course.

In general, the results (25) and (26) were found to agree very well with numerical calculations of \( A \), see examples at the end of the next section. This is additional evidence for the validity of the basic assumption above concerning the chaotic region, \( C = C \).

V. MAXIMAL RATCHET ACCELERATION FOR ARBITRARILY WEAK CHAOS

In this section, we show that the maximal ratchet acceleration \( A \) for \( K = 3 \) is attained in a limit \( b_1, b_2 \to 0 \) of arbitrarily weak chaos. In Fig. 5, we plot \( |A| \) as function of \( b_1 \) and \( b_2 \) using formulas (25), (26), and \( A(b_2, b_1) = -A(b_1, b_2) \). The Lyapunov exponent \( \sigma \) of the chaotic region as function of \( b_1 \) where \( |A(b_1, b_2)| \) is maximal at fixed \( b_2 \); this will define a path \( b_1(b_2) \) in the \((b_1, b_2)\) plane [a path \( b_2(b_1) \) can be similarly defined]. We then show that \( |A(b_1, b_2)| \) is maximal on this path in the limit of \( b_1, b_2 \to 0 \). Let us take the partial derivative of the function (25) with respect to \( b_1 \) and require that \( \partial A/\partial b_1 = 0 \). After a tedious but straightforward calculation, we find that the latter equation reduces to a quadratic one with the only positive root:

\[ b_1(b_2) = \frac{2[5b_2(1 - b_2)]^{1/2} - b_2(7 - 6b_2)}{5 - 6b_2}. \]  

(27)

The path (24) corresponds to the lower curve in Fig. 5, with \( b_1 \geq b_2 \). This curve starts at \( b_1 = b_2 = 0 \), with \( b_1 \approx 2\sqrt{b_2/5} \) for \( b_2 \ll 1 \), and terminates at \( b_1 = b_2 = 1/6 \), on the boundary of the LR domain. For \( b_2 < 1/6 \), we find that \( \partial^2 A/\partial b_1^2 < 0 \) at the value (27) of \( b_1 \), which thus corresponds to a local maximum. From Eqs. (25) and (27), the ratchet acceleration on the path (27) is:

\[ A(b_2) = \frac{\left\{ 5 - 6[5b_2(1 - b_2)]^{1/2} \right\}^2}{20 \left\{ 5 - 4b_2 - [5b_2(1 - b_2)]^{1/2} \right\}}. \]  

(28)

In the limit of \( b_2 \to 0 \) \((b_1 \approx 2\sqrt{b_2/5})\), we get from Eq. (28):

\[ \lim_{b_2 \to 0} A(b_2) = 1/4. \]  

(29)

After a simple but lengthy calculation, we find that the function (25) satisfies \( \partial A/\partial b_2 < 0 \) for \( b_2 < 1/6 \). Thus,
$A(b_2)$ decreases monotonically from $1/4$ (at $b_1 = b_2 = 0$) to 0 (at $b_1 = b_2 = 1/6$) on the path (27). Since this path gives the single extremum (a local maximum) of $A(b_1, b_2)$ for $b_1 \geq b_2$ at fixed $b_2$ and since $A(b_1, b_2) = 0$ for $b_1 = b_2$, we conclude that in the lower part ($b_1 \geq b_2$) of the LR domain $A(b_1, b_2) \geq 0$ and $A(b_1, b_2)$ assumes its maximal value of $1/4$ in the limit $b_1, b_2 \to 0$ of arbitrarily weak chaos on the path (27).

Since $A(b_2, b_1) = -A(b_1, b_2)$, in the upper ($b_2 > b_1$) part of the LR domain $A(b_1, b_2) \leq 0$ and $A(b_1, b_2)$ assumes its maximal negative value $-1/4$ in the limit of $b_1, b_2 \to 0$ on a path $b_2(b_1)$ (the upper curve in Fig. 5), defined similarly to $b_1(b_2)$. The difference between the limiting values of $A(b_1, b_2)$ on the two paths reflects the discontinuity of the ESM, i.e., the map (8) in the limit of $b_1, b_2 \to 0$. In general, $|A(b_1, b_2)|$ can assume in this limit all values $< 1/4$ on other, “non-maximal” paths. For example, on the straight-line path $b_1 = b_2/a$, where $a$ is some arbitrary constant, we find from Eq. (25) that

$$\lim_{b_2 \to 0} A(b_2) = \frac{(1-a)}{4(1+a)}. \quad (30)$$

We remark that the path (27) is tangent to the $b_1$ axis at $b_1 = b_2 = 0$, since $b_1 \approx 2\sqrt{b_2/3}$ for $b_2 \ll 1$. Similarly, the second maximal path $b_2(b_1)$ is tangent to the $b_2$ axis in this limit. Thus, as expected, the maximal value of $|A| = 1/4$ is associated with the largest possible “asymmetry”, $b_1/b_2 = \infty$ or $b_2/b_1 = \infty \sim [a = 0$ or $a = \infty$ in Eq. (30)].

Figs. 7 and 8 show plots of $A$ versus the Lyapunov exponent $\sigma$ for small $b_2$ on both the maximal path (27) and the path $b_1 = 3b_2$. We see in both plots an excellent agreement between the values of $A$ calculated numerically and those calculated from formulas (25), (26), and (28).

**VI. CONCLUSIONS**

In this paper, we have introduced a realistic non-KAM system exhibiting, in weak-chaos regimes, the most significant Hamiltonian ratchet effect of directed acceleration. The system, defined by the generalized standard map (8) with (9), may be viewed as a perturbed elliptic sawtooth map (ESM) with a perturbation that removes the ESM discontinuity. Then, the global weak chaos featured by the system may be generally considered as a perturbed global pseudochaos. Our main exact result is that for $K = 3$ the maximal ratchet acceleration $A$ is attained precisely in a limit $b_1, b_2 \to 0$ of arbitrarily weak chaos with infinite asymmetry parameter ($b_1/b_2 = \infty$ or $b_2/b_1 = \infty$). Despite this fact, the limiting system is interestingly the completely symmetric ESM (see phase spaces in Fig. 2). By continuity considerations, one expects that at least for values of $K$ sufficiently close to $K = 3$ one should again observe a significant increase of the absolute value $|A|$ of the acceleration as the chaos strength decreases. We have verified this numerically in parameter regimes where good accuracy could be achieved within the limitations of our available computational resources. An example is shown in Fig. 9.

Our main result that the strongest Hamiltonian ratchet effect can arise in a limit of arbitrarily weak chaos has apparently no analog in ordinary ratchets if chaos is viewed as the deterministic counterpart of random noise. In fact, a sufficiently high level of noise is essential for the func-
tioning of ordinary ratchets or Brownian motors. Actually, it was recently shown that for a Lévy ratchet the current decreases algebraically with the noise level, in clear contrast with our results.

The quantized version of our non-KAM system may be experimentally realized using, e.g., optical analogs as proposed in Ref. [26] and is expected to exhibit in general a rich variety of quantum phenomena, including the quantum signatures of the weak-chaos ratchet acceleration. The study of these phenomena is planned to be the subject of future works.

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APPENDIX A

We derive here Eqs. (10). First, continuity of the function (11) at $x = b_1$ and $x = 1 - b_2$ implies that

$$l_1 b_1 = c - b_1, \quad l_2 b_2 = 1 - b_2 - c.$$  

Then, using Eqs. (19) and (31) in the ratchet condition

$$\int_0^1 f(x)dx = 0,$$

we find that

$$\int_0^1 f(x)dx = c - \frac{b_1}{2} c + \frac{b_2}{2} (1 - c) - \frac{1}{2} = 0,$$

yielding the expression for $c$ in Eqs. (10). After inserting this expression in Eqs. (31), we get the expressions for $l_1$ and $l_2$ in Eqs. (10).

APPENDIX B: THE REGION $C$ FOR $K = 3$

In this Appendix and in the next ones, we derive, for $K = 3$, exact results for the region $C$ in Eq. (23). As mentioned in Sec. IV, several arguments and extensive numerical evidence indicate that $C$ coincides with the chaotic region $C$ for $K = 3$. We show here that one has the simple relation

$$C = C' = B \cup \bar{M} B \cup \bar{M}^2 B.$$  

To show this, we first denote

$$\bar{B}^{(1)} = \bar{M} B - B \cap \bar{M} B,$$  

$$\bar{B}^{(2)} = \bar{M} \bar{B}^{(1)} - B \cap \bar{M} \bar{B}^{(1)}.$$  

We derive below the relation

$$\bar{M} \bar{B}^{(2)} \subseteq B.$$  

Then, from the definition of $C'$ in Eq. (32) and from Eqs. (36 - 39) it follows that

$$\bar{M} C' \subseteq C'.$$  

Eq. (36) and the fact that $\bar{M}$ is area preserving imply that $MC' = C'$ or $M^{-1}C' = C'$. Thus, $C' = M^s B \cup M^{s+1} B \cup M^{s+2} B$ for all integers $s$, which is possible only if $C'$ is equal to $C$ in Eq. (23). Relation (32) is thus proven.

To derive Eq. (35), we start by obtaining an explicit expression for $\bar{B}^{(1)}$ in Eq. (33). For $K = 3$, the iterate of any initial condition $(x_0, p_0)$ under $\bar{M}$ satisfies $p_1 = x_1 - x_0 \text{ mod}(1)$, $x_1 = x_0 + p_0 + 3f(x_0) \text{ mod}(1)$. Clearly, when $p_0$ varies in $[0, 1)$ at fixed $x_0$, $x_1$ varies in the whole interval $[0, 1)$. Then, taking $(x_0, p_0)$ in $B$ and using Eqs. (38) and (39), we get

$$\bar{M} B = \{(x, p) | 0 \leq x < 1, \quad x - b_1 \leq p \leq x + b_2 \} \mod(T^2),$$  

$$\bar{B}^{(1)} = \{(x, p) | b_1 < x < 1 - b_2, \quad x - b_1 \leq p \leq x + b_2 \}.$$  

The region (35) is a strip (parallelogram) of slope 1, shown in Fig. 10(b) and corresponding to the strip $B$ in Fig. 10(a). Next, we determine the region

$$\bar{B}^{(2)} = \bar{M} \bar{B}^{(1)}.$$  

The second iterate $(x_2, p_2)$ of $(x_0, p_0)$ under $\bar{M}$, with $(x_0, p_0) \in B$ and $(x_1, p_1) \in \bar{B}^{(1)}$, is given by

$$p_2 = p_1 - 3(x_1 - c) \mod(1) = -2x_1 - x_0 + 3c \mod(1),$$  

$$x_2 = x_1 + p_2 \mod(1) = -2x_1 + p_1 + 3c \mod(1).$$  

From Eq. (40) and the first Eq. (41), we find that

$$p_2 = 2x_2 + x_0 - 3c \mod(1).$$  

FIG. 9: Circles: Numerical values of $A$ (obtained as described in the caption of Fig. 7) for $K = 2.99$ on the path $b_1 = 0.25$ and $0.005 \leq b_2 \leq 0.14$; this path lies entirely in domain $R$, see Fig. 3 for the nearby value of $K = 3$. The Lyapunov exponent on the path varies in the interval $0.4787 < \sigma < 0.8326$. Solid line: Analytical results from formula (26) for $K = 3$. 

APPENDIX B: THE REGION $C$ FOR $K = 3$
FIG. 10: (Color online) The figure shows, for $K = 3$, $b_1 = 0.1$, and $b_2 = 0.05$: (a) the strip $B$, see Eq. (13); (b) the region $B^{(1)}$, see Eq. (33) or (38); (c) the region $B^{(2)}$, see Eq. (39) or (43).

Eq. (12) and the second Eq. (11) imply that the region (39) is the following set of phase-space points:

$$B^{(2)} = \{(x, p)\} : \begin{cases} x = -2x_1 + p_1 + 3c \mod(1), & (x_1, p_1) \in B^{(1)}, \\ p = 2x + x_0 - 3c \mod(1), & -b_2 \leq x_0 \leq b_1. \end{cases} \tag{43}$$

The region (13) is clearly a parallelogram of slope 2 folded into $T^2$, as shown in Fig. 10(c). The region $\bar{B}^{(2)}$ in Eq. (43) is given by Eq. (43) with $x$ restricted to the interval $b_1 < x < 1 - b_2$. Then, using also Eq. (12), we get for $(x_0, p_0) \in B$ and $(x_2, p_2) \in \bar{B}^{(2)}$:

$$\bar{M} \bar{B}^{(2)} = \{(x, p)\} : \begin{cases} p = p_2 - 3(x_2 - c) \mod(1) = x_0 - x_2 \mod(1), \\ x = x_2 + p \mod(1) = x_0 \mod(1). \end{cases} \tag{44}$$

The result (44) and Eq. (13) imply Eq. (35).

APPENDIX C: AIs AND THE SHAPE OF $C$

We study here the shape of the region $C$ for $K = 3$ in several cases. Let us write Eq. (32) as $C = C' = B \cup B^{(1)} \cup B^{(2)}$, i.e., the union of the three sets in Fig. 10. This union is shown in Fig. 11, exhibiting a case in which both the $L$ and $R$ accelerator-mode islands (AIs) exist (see Secs. IIIC and IV). For values of $b_1$ and/or $b_2$ larger than those in Figs. 10 and 11, there may exist only one AI or no AIs (see Fig. 3). A case of $C$ for which only the $R$ AI exists is shown in Fig. 12 and is clearly different from that in Fig. 11. We show below that the existence of AIs and the shape of $C$ in different cases depends on the location of the vertices $(x^{(j)}, p^{(j)})$ ($j = 1, 2, 3, 4$) of the parallelogram (13), shown in Fig.
10(c), relative to the strip $B$. Because of Eq. (39), one has $(x^{(j)}, p^{(j)}) = M(x^{(j)}, \tilde{p}^{(j)})$, where $(\tilde{x}^{(j)}, \tilde{p}^{(j)})$ ($j = 1, 2, 3, 4$) are the vertices of the parallelogram $\mathcal{X}$ in Fig. 10(b). Clearly,

\[
(x^{(1)}, \tilde{p}^{(1)}) = (b_1, 0), \quad (x^{(2)}, \tilde{p}^{(2)}) = (b_1, b), \quad (x^{(3)}, \tilde{p}^{(3)}) = (1 - b_2, 1), \quad (x^{(4)}, \tilde{p}^{(4)}) = (1 - b_2, 1 - b),
\]

where $b = b_1 + b_2$. To derive explicit expressions for $(x^{(j)}, p^{(j)}) = M(x^{(j)}, \tilde{p}^{(j)})$, one has to properly determine the additive integers from the modulo operations in $M$ so that $(x^{(j)}, p^{(j)})$ will lie within the basic torus $T^2$. We find that the values of $x^{(j)}$ are

\[
x^{(1)} = 3c - 2b_1 - 1, \quad x^{(2)} = x^{(4)} = x^{(1)} + b, \quad x^{(3)} = x^{(1)} + 2b,
\]

indeed satisfying $0 < x^{(j)} < 1$ in the relevant cases in which at least one AI exists. In fact, in these cases one has $b < 0.5$ [from Eq. (21) with $K = 3$] and the latter inequality implies by simple algebra that the smallest value of $x^{(j)}$ in Eqs. (16), i.e., $x^{(1)}$, satisfies $x^{(1)} > 0$ while the largest value $x^{(3)}$ satisfies $x^{(3)} < 1$. In addition, it is clear from Figs. 10-12 that the RA Is exist only if $x^{(3)} < 1 - b_2$ (vertex 3 is outside B); it is easy to show that the latter inequality is indeed equivalent to the existence condition $b_1 < F(b_2)$ for the RA I, derived in Sec. IIIIC. Similarly, the LA I exists only if $x^{(1)} < b_1$ (vertex 1 is outside B), which can be easily shown to be equivalent to the existence condition (17). Thus, when both AIs exist, $b_1 < x^{(j)} < 1 - b_2$ ($j = 1, 2, 3, 4$).

To determine the values of $p^{(j)}$, we first notice that the vertices $(x^{(j)}, p^{(j)})$ must touch the boundaries of the region $\mathcal{X}$; this is because the vertices $\mathcal{X}$ in Fig. 10(b) obviously touch the boundaries of the strip $B$ in Fig. 10(a) and $(x^{(j)}, p^{(j)}) = M(x^{(j)}, \tilde{p}^{(j)})$. Then, in the case that both AIs exist, i.e., $b_1 < x^{(3)} < 1 - b_2$ (see above), $(x^{(j)}, p^{(j)})$ touch the boundaries $p = x - b_1$ and $p = x + b_2$ of the parallelogram $\mathcal{X}$ [see Figs. 10(b), 10(c), and 11], so that

\[
(p^{(1,2)}) = (x^{(1,2)} - b_1, \quad p^{(3,4)} = x^{(3,4)} + b_2).
\]

Assume now that only the RA I exists, as in Fig. 12. Then, $x^{(1)} \leq b_1$ (from above), i.e., vertex 1 (the point $d$ in Figs. 11 and 12) lies within the left part of strip $B$, on the boundary of region $\mathcal{X}$ given by $p = x - b_1 \text{mod}(1)$; thus, for $x^{(1)} < b_1$ (as in Fig. 12), $p^{(1)}$ in Eq. (17) must be replaced by $x^{(1)} - b_1 + 1$ while $p^{(j)}$ for $j > 1$ remains unchanged. Similarly, when only the LA I exists, vertex 3 lies within the right part of strip $B$, on the boundary of region $\mathcal{X}$ given by $p = x + b_2 \text{mod}(1)$; for $x^{(3)} > 1 - b_2$, $p^{(3)}$ in Eq. (17) must be replaced by $x^{(3)} + b_2 - 1$.

APPENDIX D: AREAS OF AIs

Consider the LA I in Fig. 11. This is the triangle $d_{efg}$ on the torus $T^2$, composed of two triangles, $def$ and $efg$. The point $d$ is vertex 1 in Fig. 10(c) and the segment $de$ is part of the upper boundary of the region $\mathcal{X}$. This boundary is a line of slope 2 passing through vertex 1:

\[
p - p^{(1)} = 2(x - x^{(1)}).
\]

Then, since $p_c = 0$, we get from Eqs. (16)-(18) that $x_c = (3c - b_1 - 1)/2$. Also, $x = b_1$ and $p_f = 0$. The point $g$, with $x_g = b_1$, lies on the line (48) with $p^{(1)}$ replaced by $p^{(1)} + 1$. Thus, $p_g = 2 + 3b - 3c$. The area of the LA I is therefore

\[
S_L = S_{def} + S_{efg} = \frac{1}{2}(x_e - x_f)[p^{(1)} + (1 - p_g)]
\]

\[
= \frac{1}{2}(3c - 3b_1 - 1)^2.
\]

By symmetry arguments, the area of the RA I is obtained from Eq. (19) by inserting the expression for $c$ from Eqs. (10) and performing the exchange $b_1 \leftrightarrow b_2$. We get

\[
S_R = \frac{1}{2}(2 - 3b_2 - 3c)^2.
\]

APPENDIX E: AREA OF C

The area of $C$ can be calculated starting from the relation $C = B \cup B^{(1)} \cup B^{(2)}$ (see above), where $B^{(1)}$ and $B$ do not overlap. Then, because of Eqs. (39) and (38), also $B^{(2)}$ does not overlap with $B^{(1)}$. However, it may overlap with $B$. The area of $C$ is thus given by

\[
S_C = S_B + S_{B^{(1)}} + S_{B^{(2)}} - S_{B^{(1)} \cap B^{(2)}}.
\]

From Eq. (13), $S_B = b$, where $b = b_1 + b_2$. The region $B^{(1)}$ in Eq. (38) is a parallelogram with basis $b$ (in the $p$ direction) and height $1 - b$ (in the $x$ direction), see also Figs. 11 and 12. Then, $S_{B^{(1)}} = b(1 - b)$. From Eq. (39) and the fact that $M$ is area preserving, it follows that $S_{B^{(2)}} = S_{B^{(1)}}$. Finally, concerning the overlap $B \cap B^{(2)}$, we consider first the case that both AIs exist, see Fig. 11. In this case, $B \cap B^{(2)}$ consists of the green (dark grey) regions in Fig. 11. These are two parallelograms having heights $b_1$ and $b_2$ (in the $x$ direction) and basis $b$, i.e., the width of region $\mathcal{X}$ in the $p$ direction. Thus, $S_{B^{(1)} \cap B^{(2)}} = b^2$. The area (51) is therefore

\[
S_C = 3(b_1 + b_2 - 3)(b_1 + b_2)^2.
\]

Consider now the case that only one AI exists, say the RA I as in Fig. 12. In this case, as explained at the end of Appendix C, the point $d$, i.e., the vertex 1 of region $B^{(2)}$, lies inside the left part of strip $B$, on the boundary of region $\mathcal{X}$. This means that the black triangles $def$ and $efg$ in Fig. 12 are not included in the region $B^{(2)}$ or $B \cap B^{(2)}$ but they are actually part of the region $\mathcal{X}$. Thus, to calculate $S_{B^{(1)} \cap B^{(2)}}$ one must subtract from $b^2$ (the value of $S_{B^{(1)} \cap B^{(2)}}$ in the previous case) the areas of $def$ and $efg$. The area (51) is then obtained by adding $S_{def} + S_{efg}$ to the expression (52). By comparing Fig. 12 with Fig. 11, it is clear that the areas $S_{def}$ and $S_{efg}$ can be calculated precisely as in Appendix D and $S_{def} + S_{efg}$
is given again by formula \((19)\). Therefore, the area \((52)\) increases precisely by an amount equal to the area \((49)\) of the missing \(L\) AI:

\[
S_C = 3(b_1 + b_2) - 3(b_1 + b_2)^2 + (3c - 3b_1 - 1)^2/2. \quad (53)
\]

Similarly, when only the \(L\) AI exists, one must add the area \((50)\) to \((52)\):

\[
S_C = 3(b_1 + b_2) - 3(b_1 + b_2)^2 + (2 - 3b_2 - 3c)^2/2. \quad (54)
\]