Estimation of Small $s$-$t$ Reliabilities in Acyclic Networks

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Abstract

In the classical $s$-$t$ network reliability problem a fixed network $G$ is given including two designated vertices $s$ and $t$ (called terminals). The edges are subject to independent random failure, and the task is to compute the probability that $s$ and $t$ are connected in the resulting network, which is known to be $\#P$-complete. In this paper we are interested in approximating the $s$-$t$ reliability in case of a directed acyclic original network $G$. We introduce and analyze a specialized version of a Monte-Carlo algorithm given by Karp and Luby. For the case of uniform edge failure probabilities, we give a worst-case bound on the number of samples that have to be drawn to obtain an $\epsilon - \delta$ approximation, being sharper than the original upper bound. We also derive a variance reduction of the estimator which reduces the expected number of iterations to perform to achieve the desired accuracy when applied in conjunction with different stopping rules. Initial computational results on two types of random networks (directed acyclic Delaunay graphs and a slightly modified version of a classical random graph) with up to one million vertices are presented. These results show the advantage of the introduced Monte-Carlo approach compared to direct simulation when small reliabilities have to be estimated and demonstrate its applicability on large-scale instances.

1 Introduction

In the classical $s$-$t$ network reliability problem, a fixed graph $G$ with two special vertices $s$ and $t$ is given whose edges fail (disappear) independently of each other with some given probability. The task is to determine the probability that $s$ and $t$ are still connected in the resulting network after edge failures. A famous related problem is the all-terminal reliability problem, where the goal is to determine the probability that all vertices are still connected to each other after edge failures (for further information on various reliability problems see [1]). Both problems are known to be computationally hard ($\#P$-complete) even for very
restricted classes of graphs $G$ \cite{13, 16}. In particular, the $s$-$t$ reliability problem remains hard in the case when $G$ is a directed acyclic planar graph of maximal degree three \cite{12}. Therefore we are interested in finding approximations.

Randomized algorithms respecting some relative error bound with high probability have shown to be interesting approaches for different reliability problems \cite{9, 7, 8}. As we are interested in relative error bounds, there is a significant difference between the estimation of the failure probability of the network and estimating the probability that the network is intact after edge failures. This results from the fact that estimating small probabilities by sampling is in general much more difficult than estimating large probabilities. Moreover, the techniques used for the estimation of reliability values often differ significantly from those used for the estimation of the failure probability. Most of the literature concentrates on estimating the failure probability because in typical applications, as communication or electrical networks, we have a highly reliable network and are interested in accurately estimating the probability of rare failures.

In this paper, we concentrate on estimating the probability that the network is intact, i.e., the probability that there is a path from $s$ to $t$ after edge failures. This is motivated by certain models of spreading processes on networks, such as disease spreading, which can be mapped onto reliability problems \cite{4, 14, 17, 10, 11}. In this context, a path from $s$ to $t$ represents the spread of a disease from $s$ to $t$, and we are interested in estimating the probability of the rare event that the disease spreads over a long distance from $s$ to $t$.

The basis of our approach is a method presented by Karp and Luby \cite{9} for estimating the failure probability in an $s$-$t$ reliability problem when the graph $G$ is planar, which can be easily adapted for estimating the probability of connectedness of $s$ and $t$ on an arbitrary graph $G$. To be efficiently applicable, however, the underlying graph $G$ needs to fulfill some additional properties, such as very low intactness probabilities of the edges and low vertex degrees. Furthermore, a computationally expensive preprocessing phase is required, which computes different quantities needed to efficiently sample from the proposed sample space. This makes the method not suited for application on large networks. We show in this paper that in the case of a directed acyclic network $G$, some of these problems can be circumvented, and the resulting algorithm can be applied to large-scale instances. The simplifications are due to the fact that generating an $s$-$t$ path uniformly at random in an acyclic graph can be performed in linear time (see Section 5). Except for very restricted classes of initial networks such as series-parallel networks (see \cite{15} for further information), no practically useful methods for estimating small $s$-$t$ reliabilities on large networks are known. Considering directed acyclic graphs can thus be seen as a natural next step.

The paper is organized as follows. We begin with some preliminaries in Section 2. In Section 3 the direct Monte-Carlo approach is described and its efficiency is analyzed. Section 4 then presents our algorithm for the estimation of small reliabilities in directed acyclic graphs up to some technical details that are explained in Section 5. In Section 6 we discuss how many samples have to be drawn in our algorithm to obtain a good estimate of the reliability with high probability. Section 7 contains computational results on two types of random
directed acyclic networks comparing the direct Monte-Carlo approach with our algorithm and demonstrating the applicability of our algorithm on large-scale instances.

2 Preliminaries

Let $G = (V, E, p)$ be a directed acyclic network, where
- $V$ is the set of vertices (with $|V| = n$),
- $E$ is the set of edges (with $|E| = m$), and
- $p : E \rightarrow [0, 1]$ is a function associating a failure probability with every edge of the network.

We call a network $G$ satisfying these properties an acyclic reliability network. Furthermore, we fix two special vertices $s, t \in V$. Every edge $e \in E$ fails with probability $p(e)$ independently of the others. Let $G'$ be the resulting graph over the intact edges after the realization of the edge failures. We say that $G'$ is intact if it contains a path from $s$ to $t$, otherwise we say that $G'$ is in a failed state. Finally, the $s$-t reliability $REL_{s,t}(G)$ of the network $G$ is defined as the probability of $G'$ being intact.

Let $q : E \rightarrow [0, 1]$ be the function which associates with every edge its probability of being intact, i.e., $q(e) = 1 - p(e) \forall e \in E$. It is sometimes easier to look at our reliability model in a different way where edges appear rather than disappear. In this model we would begin with an empty network and flip a biased coin for all potential edges $e \in E$ to determine whether they appear.

We are mainly interested in $\epsilon - \delta$ approximations for $REL_{s,t}(G)$, i.e., algorithms returning an estimate of $REL_{s,t}(G)$ accurate to within a relative error of $\epsilon$ with probability at least $1 - \delta$. To determine how many samples we have to draw in a Monte-Carlo algorithm to obtain an $\epsilon - \delta$ approximation we often refer to the Generalized Zero-One Estimator Theorem introduced by Dagum et al. in [2]. The theorem is repeated below.

**Theorem 1** (Generalized Zero-One Estimator Theorem). Let $X$ be a random variable taking values in $[0, 1]$ and let $X_1, X_2, \ldots, X_N$ denote independent random variables distributed identically to $X$. If $\epsilon < 1$ and $N \geq 4(e - 2) \ln(2/\delta) \cdot \max\{\text{Var}[X], \epsilon E[X]\}/(\epsilon E[X])^2$ then

$$P\left[1 - \epsilon \leq \frac{1}{N} \sum_{i=1}^{N} X_i \leq (1 + \epsilon) E[X]\right] > 1 - \delta.$$ 

We typically use the theorem in the following form. If $N \geq 4(e - 2) \ln(2/\delta) \cdot (1/\epsilon^2 E[X])$ then $(\sum_{i=1}^{N} X_i)/N$ is an $\epsilon - \delta$ approximation for $E[X]$ (using the fact that a random variable $X$ taking values in $[0, 1]$ satisfies $E[X] \geq \text{Var}[X]$).

3 A direct Monte-Carlo approach

In this section we consider a simple Monte-Carlo approach and show that it is efficient for sufficiently large values of $REL_{s,t}(G)$, but inefficient for the estima-
tion of small reliabilities. In this approach we simply flip a biased coin for every edge $e \in E$ and observe whether $s$ and $t$ are connected in the resulting graph. Let $X$ be the random variable corresponding to this approach where $X = 1$ if the resulting network is intact and 0 otherwise. The random variable $X$ has thus a Bernoulli distribution with parameter $REL_{s,t}(G)$ and the reliability is estimated without bias by generating $N$ independent realizations of $X$ and returning their empirical mean which we denote by $Y_N$. A central question when using this method is how large $N$ has to be chosen to obtain an $\epsilon - \delta$ approximation. A direct application of Theorem 1 gives the following:

**Theorem 2.** $Y_N$ is an $\epsilon - \delta$ approximation of $REL_{s,t}(G)$ if $N$ satisfies

$$N \geq 4(e - 2) \ln \left( \frac{2}{\delta} \right) \cdot \frac{1}{e^2 REL_{s,t}(G)}.$$

When $REL_{s,t}(G)$ is bounded below by $1/poly(n, m)$, $Y_N$ is an FPRAS ($\epsilon - \delta$ approximation algorithm with running time bounded by a polynomial in $1/\epsilon, \log(1/\delta)$ and the input size). The difficult case is the estimation of small values, i.e., small reliabilities $REL_{s,t}(G)$. This problem motivated the construction of the algorithm to be presented next.

4 Monte-Carlo method for estimating small reliabilities

The backbone of our algorithm is an adaption of the Monte-Carlo method presented in [9]. Our algorithm exploits that, in a directed acyclic network, we can easily (in linear time) calculate the mean number of intact paths from $s$ to $t$ after the edge failures. This value is normally a good estimate for the reliability in highly unreliable networks as in such networks an intact state typically contains only a few paths from $s$ to $t$. Using a Monte-Carlo approach, our algorithm then estimates the ratio between $REL_{s,t}(G)$ and the mean number of intact paths from $s$ to $t$ after edge failures. Multiplying this estimate with the mean number of intact paths from $s$ to $t$ yields finally an estimate for the $s$-$t$ reliability of the network.

Note that the ratio between the mean number of intact paths from $s$ to $t$ after edges failures and the reliability to estimate, i.e., the reciprocal of the value we estimate in our algorithm, is exactly the mean number of intact paths from $s$ to $t$ after edge failures conditioned on the event that the network will be intact after the failure process. One of the main problems for the development of methods for directly estimating this ratio is the difficulty of choosing a sample out of the pool of intact states with probability proportional to its real appearance probability. In fact such a sampling procedure could easily be transformed into an FPRAS for the estimation of $REL_{s,t}(G)$ using techniques presented by Jerrum et al. [6].
4.1 Notation

Let \( H = \{ h \subseteq E \} \) be the set of all possible states of the network \( G \) after the realization of the edge failures, where a state \( h \in H \) represents the collection of intact edges. Furthermore, let \( A \subset H \) be the set of all intact states. For every state \( h \in H \) we denote by \( w(h) \) the probability that \( h \) occurs after the edge failure process, i.e., \( w(h) = \prod_{e \in h} q(e) \prod_{e \in E \setminus h} p(e) \). In particular, the weight of the set \( A \) is the reliability we want to estimate, \( w(A) = \sum_{a \in A} w(a) = \text{REL}\_{s,t}(G) \).

Let \( P \) be the set of all paths in \( G \) from \( s \) to \( t \). A path \( \gamma \) is simply represented by a subset of the edges \( E \). For every state \( h \in H \) we denote by \( \mathcal{P}(h) \) the set of all paths from \( s \) to \( t \) in state \( h \). A state \( h \in H \) is intact if and only if we have \( \mathcal{P}(h) \neq \emptyset \). With every path \( \gamma \in \mathcal{P} \) we associate a weight \( w(\gamma) \) which is the probability that all edges on the path will be intact after the edge failure process, \( w(\gamma) = \prod_{e \in \gamma} q(e) \).

The mean number of intact paths from \( s \) to \( t \) after edge failures can thus be written as \( \sum_{\gamma \in \mathcal{P}} w(\gamma) \). In Section 5 we will see how this quantity can be efficiently calculated in acyclic graphs.

4.2 Estimating the ratio between \( \text{REL}_{s,t}(G) \) and the mean number of intact paths from \( s \) to \( t \) after edge failures

To estimate the ratio between the reliability and the mean number of intact paths from \( s \) to \( t \) after edge failures we sample out of the sample space

\[
\Omega = \{ (\gamma, a) \in \mathcal{P} \times A \mid \gamma \subseteq a \},
\]

where we associate the weight \( w(\gamma, a) = w(a) \) with every element \( (\gamma, a) \in \Omega \). This sample space has the following interesting properties. On the one hand, the weight of the sample space is exactly the mean number of intact paths from \( s \) to \( t \) after edge failures, i.e.,

\[
w(\Omega) = \sum_{(\gamma, a) \in \Omega} w(\gamma, a) = \sum_{\gamma \in \mathcal{P}} \sum_{a \in A : \gamma \subseteq a} w(\gamma, a) = \sum_{\gamma \in \mathcal{P}} w(\gamma).
\]

On the other hand, it is easy to sample elements of \( \Omega \) with probability proportional to their weights by the following two-step procedure. In a first step, a path \( \gamma \in \mathcal{P} \) is drawn with probability proportional to its weight \( w(\gamma) \). How this can be done efficiently will be discussed in Section 5. In the second step, all edges not contained in \( \gamma \) are sampled corresponding to their appearance probabilities. The appeared edges of the second step together with the edges in \( \gamma \) form an intact state \( a \in A \). The tuple \( (\gamma, a) \) is finally the sampled state.

We will now introduce influence values for the elements in \( \Omega \) such that the expected influence value of a sample of \( \Omega \) is equal to \( w(A)/w(\Omega) \). The expected influence value can then be estimated by a standard Monte-Carlo algorithm.
The approach taken in [9] was to fix for every intact state \( a \in A \) an arbitrary \( s \)-\( t \) path \( \gamma_a \in \mathcal{P}(a) \). We begin by following this approach and introduce later on another idea to reduce the variance of the estimator. With every sample we associate an influence value which is equal to one if the sample is of the form \((\gamma_a, a)\) and equal to zero otherwise. The influence value of a sample corresponds therefore to the realization of a Bernoulli variable with parameter

\[
\frac{w(\{ (\gamma_a, a) \in \Omega \mid a \in A \})}{w(\Omega)} = \frac{w(A)}{w(\Omega)}
\]

which is precisely the value we would like to estimate. By repeating the sampling procedure \( N \) times and counting the fraction of samples of the form \((\gamma_a, a)\) we therefore obtain an unbiased estimator \( \xi_N \) for \( w(A)/w(\Omega) \) with variance

\[
\text{Var}(\xi_N) = \frac{1}{N} \frac{w(A)}{w(\Omega)} \left( 1 - \frac{w(A)}{w(\Omega)} \right).
\]

Another unbiased estimator \( \psi_N \) for \( w(A)/w(\Omega) \) with smaller variance than \( \xi_N \) can be obtained by associating an influence value \( m(\gamma_a) = 1/|\mathcal{P}(a)| \) with every sample \((\gamma, a)\), and we define \( \psi_N \) to be simply the mean of the influence values of our samples. This approach is particularly interesting in our case as the fact of \( G \) being acyclic allows to compute the value \(|\mathcal{P}(a)|\) in linear time (see Section 5) and thus does not increase the overall worst-case complexity of our algorithm. The reduction of the variance decreases the expected number of iterations to perform for obtaining an \( \epsilon - \delta \) approximation when applying a stopping criterion as presented in Section 6.

Unfortunately, we cannot guarantee some minimal decrease of the variance of the estimator \( \psi_N \) compared to \( \xi_N \) as there are instances of reliability networks where we have an arbitrarily low decrease. On the other hand, it is easy to find instances where the variance reduces by an arbitrarily high factor.

We finally estimate \( REL_{s,t}(G) \) by multiplying \( \psi_N \) by the weight of the sampling space \( w(\Omega) \) which can be calculated efficiently as shown in the next section. Algorithm 1 gives a pseudocode for an implementation of the \( s \)-\( t \) reliability estimation based on \( \psi_N \). Note that the sampling of the edges not on the path performed in Algorithm 1 in lines 1 to 1 can be done more efficiently as in general we do not need to sample all edges for calculating \(|\mathcal{P}(a)|\). In Section 5 we will see how this part of the algorithm can be improved and show how to perform the remaining unspecified parts of our algorithm. More precisely, we will discuss the following operations, where \( \kappa \) represents the time needed for generating a random number uniformly in \([0, 1]\\):

- Sampling in \( O(m\kappa) \) time a path according to line 1 of Algorithm 1
- Determining in \( O(m) \) time the weight \( w(\Omega) \) of the sampling space \( \Omega \) used in line 1 of Algorithm 1
- Sampling edges not being on the initial path \( \gamma \) and calculating \(|\mathcal{P}(a)|\) as needed in lines 1 to 1 of Algorithm 1
Algorithm 1: Estimation of $REL_{s,t}(G)$ based on $\psi_N$

Input: An acyclic reliability network $G = (V,E,p)$ with two special terminals $s,t \in V$ and the number $N$ of iterations to perform

Output: An estimate for $REL_{s,t}(G)$

1. $x = 0$
2. for $i = 1$ to $N$ do
3.     $a = \emptyset$
4.     Draw a random path $\gamma$ out of the set $\mathcal{P}$ with probability proportional to $w(\gamma)$;
5.     $a = a \cup \gamma$
6.     foreach $e \in E \setminus \gamma$ do
7.         Take a sample $x_e$ of a Bernoulli variable with parameter $q(e)$ to determine whether the edge $e$ appears;
8.         if $x_e = 1$ (the edge appears) then
9.             $a = a \cup e$
10.       end
11.     end
12.     Determine $|\mathcal{P}(a)|$;
13.     $x = x + \frac{1}{|\mathcal{P}(a)|}$
14. end
15. $\psi_N = \frac{x}{N}$
16. Determine $w(\Omega)$;
17. return $\psi_N \cdot w(\Omega)$

5 Algorithmic details

5.1 Sampling $s$-$t$ paths

We begin by studying how paths can be efficiently sampled according to line 1 of Algorithm 1. The idea is to start at the terminal $s$ and then to construct a path to $t$ by successively adding new edges. The choice of a new edge augmenting the current partial path is done in the following way. With every edge $(v,u) \in E$ we associate a weight $\tilde{w}(v,u)$ which is the sum of the weights of all paths from $v$ to $t$ using edge $(v,u)$, i.e.,

$$\tilde{w}(v,u) = \sum_{\gamma: \text{path from } v \text{ to } t \text{ with } (v,u) \in \gamma} \prod_{e \in \gamma} q(e).$$

(1)

During the path sampling method, after a partial path from $s$ to some vertex $v$ is constructed, we choose an outgoing edge of vertex $v$ with probability proportional to the weights $\tilde{w}$. It is easy to verify that this procedure effectively samples a path $\gamma \in \mathcal{P}$ with probability proportional to $w(\gamma)$ as desired. Furthermore, the edge weights $\tilde{w}$ can be easily computed by the following procedure.
We suppose without loss of generality that every edge lies on at least one path from \( s \) to \( t \) in \( G \). All edges not satisfying this condition can be eliminated in \( O(m) \) time in a preprocessing step. We then determine a topological order of the vertices. By the condition mentioned above it is clear that \( t \) will be the last vertex in the topological order. We go through the vertices in reverse topological order and determine at each step the weights \( \bar{w} \) of the edges entering the current vertex. At the first step we look at every edge \( e \) incident to \( t \) and determine its weight \( \bar{w}(e) \) which is equal to \( q(e) \). The weights of the other edges can then be determined in linear time by using the following recursive formula which follows from the definition of the weights \( \bar{w} \) in (1),

\[
\bar{w}(v, u) = q(v, u) \cdot \sum_{e \in \delta^+(u)} \bar{w}(e),
\]

where for \( v \in V \), \( \delta^+(v) \) (resp. \( \delta^-(v) \)) denotes the set of all edges going out of \( v \) (resp. all edges entering \( v \)).

5.2 Determining \( w(\Omega) \)

Another subproblem in Algorithm 1 is the calculation of \( w(\Omega) \), the weight of the sample space. Having already calculated the weight function \( \bar{w} \), this problem can easily be solved by expressing \( w(\Omega) \) in terms of \( \bar{w} \) as follows.

\[
w(\Omega) = \sum_{\gamma \in \mathcal{P}} w(\gamma) = \sum_{e \in \delta^+(s)} \sum_{\gamma \in \mathcal{P} \atop e \in \gamma} w(\gamma) = \sum_{e \in \delta^+(s)} \bar{w}(e) = \bar{w}(e)
\]

5.3 Sampling edges outside the initial path and calculating \( |\mathcal{P}(a)| \)

As the sampling of the remaining edges in lines 1 to 4 of Algorithm 1 is used only for the calculation of \( |\mathcal{P}(a)| \), we do not have to know all intact edges but only those on a path from \( s \) to \( t \). One way of improving the procedure is to sample only edges that can be reached from \( s \). This can be done by keeping track of the set of nodes \( \mathcal{L} \) that are currently reachable from \( s \). At the beginning, these are the nodes on the initial path \( \gamma \). Then all edges going out of \( \mathcal{L} \) will be sampled, and for every appeared edge we add its endpoint to \( \mathcal{L} \). This procedure is repeated until there are no edges outgoing from \( \mathcal{L} \) that have not already been sampled.

Finally, the determination of \( |\mathcal{P}(a)| \) in line 1 of Algorithm 1 can be easily done by an analogue technique to the one used to calculate \( w(\Omega) \) where this time we work on the subgraph of \( G \) over the edges in \( a \) where we give a weight of one to every edge.
6 Number of samples to draw

In this section we analyze how many samples have to be drawn in our algorithm to obtain an $\epsilon - \delta$ approximation.

6.1 A priori bounds

Using Theorem 1 we can derive the following bound.

**Theorem 3.** If the number of samples $N$ satisfies

\[ N \geq 4(e - 2) \ln \left( \frac{2\delta}{w(A)} \right) \]

then $w(\Omega) \cdot \xi_N$ and $w(\Omega) \cdot \psi_N$ are both $\epsilon - \delta$ approximations of $\text{REL}_{s,t}(G)$.

In Subsection 6.3, we discuss upper bounds for the ratio $w(\Omega)/w(A)$ which allow to apply Theorem 3 in practice and to formulate conditions a network has to satisfy under which Algorithm 1 is an FPRAS for estimating $\text{REL}_{s,t}(G)$.

6.2 Stopping criteria

In practice, the use of a stopping criterion typically allows to reduce the number of samples to take as we can profit from information gained during the execution of the algorithm. Furthermore, we do not suffer from a possibly weak upper bound for $w(A)/w(\Omega)$. In [2], Dagum et al. present two stopping criteria, applicable to our algorithm and ensuring that the result of the algorithm is an $\epsilon - \delta$ approximation of $\text{REL}_{s,t}(G)$. In the computational results, the second algorithm (named Approximation Algorithm AA) is used.

6.3 Bounding the ratio $w(\Omega)/w(A)$

In this subsection, we discuss upper bounds on the ratio $w(\Omega)/w(A)$. Together with Theorem 3 these bounds allow us to bound the number of iterations needed for our algorithm to deliver an $\epsilon - \delta$ approximation. The bounds discussed in this section are correct not only for the case of directed acyclic reliability networks but also for the more general case of arbitrary directed, undirected or even mixed reliability networks.

**Previous results** Karp an Luby [9] gave the following upper bound on the ratio $w(\Omega)/w(A)$ (in a slightly different context).

**Theorem 4.**

\[ \frac{w(\Omega)}{w(A)} \leq \prod_{e \in E} (1 + q(e)) \]
When combining the above theorem with Theorem 3 we obtain that Algorithm 1 is an FPRAS if \( \prod_{e \in E} (1 + q(e)) \) is bounded by a polynomial in the input size of the reliability network \( G \). In the special case of uniform edge failure probabilities, i.e., \( p(e) = \overline{p} = 1 - \overline{q} \ \forall e \in E \), Theorem 4 reduces to

\[
\frac{w(\Omega)}{w(A)} \leq (1 + \overline{q})^m
\]

and implies that if

\[
\overline{q} = O(\log(m)/m)
\]

then Algorithm 1 is an FPRAS for estimating \( REL_{s,t}(G) \).

**Improved bound in the case of uniform edge failure probability**

We now give a new bound on \( w(\Omega)/w(A) \) for the case of uniform edge failure probabilities, which is sharper than (3), especially in the case when the reliability network \( G \) is not too dense and does not contain long paths from \( s \) to \( t \).

To quantify the sparsity of a graph we introduce the notion of edge-vertex bound. We say that a graph \( G = (V,E) \) has an edge-vertex bound of \( \mu \) if for any subset of the vertices \( U \subseteq V \) we have that there are at most \( \mu |U| \) edges in the subgraph of \( G \) induced by the vertices \( U \). The best edge-vertex bound of a graph can be determined in polynomial time by reduction to a flow problem [3]. Furthermore, \( \Delta_{\text{max}}/2 \), where \( \Delta_{\text{max}} \) is the maximum degree of the vertices in \( G \), can be used as a simple valid edge-vertex bound. Our new bound is given by the following theorem (see Appendix for a proof).

**Theorem 5.** Let \( G = (V,E,\overline{p}) \) be a reliability network with uniform edge failure probability \( \overline{p} = 1 - \overline{q} \), edge-vertex bound \( \mu \) and let \( l_{\text{min}} \) respectively \( l_{\text{max}} \) be the minimal and maximal length of any \( s\text{-}t \) path in \( G \). Then we have

\[
\frac{w(\Omega)}{w(A)} \leq \min \left\{ \left( \frac{2}{2 - \overline{q}} \right)^{m-l_{\text{min}}} , 2^{1 + \frac{\mu}{m} [ (\overline{q} m + \ln(2)) (\overline{q} m + \ln(2) + l_{\text{max}}) ]} \right\}.
\]

The first term of the minimum in our bound is a slight improvement over the bound given by [3]. This can be seen by observing that \( 2/(2 - x) \leq (1 + x) \ \forall x \in [0,1] \). Contrary to bound [3], the first term of our bound may be sharp. Furthermore it is independent of the graph topology and can easily be generalized to non-uniform failure probabilities.

The second term of the minimum in Theorem 5 tries to exploit some structure of the underlying network and is particularly interesting for graphs with a low edge-vertex bound \( \mu \) and without long paths. For example, when working with networks where \( \mu \) is bounded by a constant and \( l_{\text{max}} = O(\sqrt{m \log(m)}) \), Theorem 5 implies that Algorithm 1 is an FPRAS when \( \overline{q} = O(\sqrt{\log(m)/m}) \), which is a much weaker condition than the bound given by [4].
7 Computational results

In order to test our algorithm we used two random generators for creating directed acyclic graphs with low reliability. These generators are introduced in the first part of this section. In a second part, we analyze the running time of our algorithm on networks created by these generators with different sizes and reliabilities. Furthermore, the proposed algorithm is compared to a direct Monte-Carlo simulation.

7.1 Test instances

Delaunay graphs (DEL) Our generator for directed acyclic Delaunay graphs takes two parameters, the number of vertices $n$ and a uniform edge intactness probability $q$. We begin by choosing $n$ points uniformly at random in the unit square and consider the undirected graph given by a Delaunay triangulation of these points. The two terminals $s$ and $t$ are chosen as two vertices with maximal Euclidian distance. We give a linear orientation to the edges corresponding to the vector from $s$ to $t$, i.e., an (undirected) edge $\{v, w\}$ is oriented as $(v, w)$ if the vector from $v$ to $w$ and the one from $s$ to $t$ have a non-negative scalar product, otherwise we take the orientation $(w, v)$. Finally, all edges get uniform intactness probability equal to $q$. One can easily observe that this construction guarantees that every vertex lies on a path from $s$ to $t$.

Topological construction (TC) A second generator we use has three parameters, the number of vertices $n$, a density parameter $\lambda \in [0, 1]$ allowing to control the expected number of edges in the graph and a parameter $\alpha$ influencing the intactness probabilities. We begin with an empty graph over $n$ vertices $V = \{v_1, v_2, \ldots, v_n\}$ where $v_1 = s$ and $v_n = t$. The graph will be constructed such that $(v_1, v_2, \ldots, v_n)$ is a topological order of the vertices. In a first step, all edges of the graph are introduced, then intactness probabilities are assigned. For $i \in \{1, 2, \ldots, n-1\}$ we introduce an edge from $v_i$ to $v_{i+1}$. This ensures that all vertices are on a path from $s$ to $t$. All other possible edges will be present with probability $\lambda$, i.e., for every $i, j \in \{1, 2, \ldots, n\}$ with $i + 2 \leq j$ we add an edge $(v_i, v_j)$ with probability $\lambda$.

Finally, the intactness probability of an edge $(v_i, v_j)$ is a number chosen uniformly at random in the interval $[0, \frac{1}{(j-i)-\alpha}]$. By choosing $\alpha < 1$, edges connecting topological near vertices have in general higher intactness probabilities than edges connecting vertices being far away from each other in the topological order. Therefore, smaller values for $\alpha$ result in less reliable networks. The value of $\alpha$ will typically be chosen in $[0, 1]$.

Parameter choice The parameters were fixed in such a way that networks of different reliabilities were obtained for graphs with $10^3, 10^4, 10^5$ and $10^6$ vertices. We generated DEL instances for every $q \in \{0.01, 0.02, \ldots, 1\}$ for every graph...
size \( n \in \{10^3, 10^4, 10^5, 10^6\} \). For the creation of TC instances the parameter \( \lambda \) was always chosen such that the expected degree of every vertex is equal to ten. This ensures that all graphs generated with the TC generator having the same number of vertices also have about the same number of edges and simplifies the comparison of running times. For every graph size \( n \in \{10^3, 10^4, 10^5, 10^6\} \) instances were generated for \( \alpha \in \{0.01, 0.02, \ldots, 1\} \).

The computational results presented in this section have been obtained on workstations equipped with an AMD processor 3200+ and 1GB of RAM.

### 7.2 Results and interpretations

As a first observation, we noticed that the dependence of the running time on \( \epsilon \) and \( \delta \) is essentially proportional to \( \log(\frac{1}{\delta})/\epsilon^2 \) as predicted. We therefore fixed \( \epsilon = 0.1, \delta = 0.001 \) for all results presented in this section.

Figure 1 shows the running time of the proposed algorithm as a function of the estimated reliability. As expected, the running time grows when larger reliabilities have to be estimated, as an intact state contains often several paths from \( s \) to \( t \). TC instances with about the same reliability as DEL instances are much easier to tackle. This comes from the fact that DEL instances are rather locally connected, whereas most edges in TC instances were randomly chosen. Local connectedness has the effect that when having an intact path \( P \) from \( s \) to \( t \), there is a good chance that several small subpaths of \( P \) can be replaced by other intact subpaths with the same start and endpoint. Every subpath which can be replaced in this way raises the number of intact paths from \( s \) to \( t \). Furthermore, as the replaceable subpaths are small it is likely that large groups of them are disjoint, implying that various combinations of these subpath replacements yield new intact paths from \( s \) to \( t \). Figure 1 shows that even instances with \( 10^6 \) vertices could be solved in reasonable time as long as the estimated reliability was not too large.

![Figure 1: Running times of the proposed algorithm in function of the estimated reliabilities for DEL instances (on the left) and TC instances (on the right).](image-url)
For the comparison of the proposed Monte-Carlo algorithm with a direct Monte-Carlo approach, instances with 1000 vertices were used as most of these instances could have been solved in reasonable time by both algorithms. As well as the proposed Monte-Carlo algorithm, the direct Monte-Carlo algorithm was implemented by using the sampling technique explained in 5.3 allowing to reduce the time needed per iteration in most instances. Figure 2 shows the running times of both algorithms on DEL and TC instances with 1000 vertices. As expected, the direct Monte-Carlo approach has an approximately linear dependence on the reciprocal of the estimated reliability. Figure 2 shows the strength of the proposed algorithm when low reliabilities have to be estimated.

As the running time of the direct Monte-Carlo approach depends nearly linear in $1/REL_{s,t} = 1/w(A)$ (see Figure 2) and we expect that the proposed algorithm has a running time approximately proportional to $w(\Omega)/w(A)$ (i.e. the reciprocal of the value to estimate), we expect the ratio between the running time of the proposed algorithm and the direct Monte-Carlo approach is approximately linear in $w(\Omega)$. This is confirmed by Figure 3 showing the ratio of the running time of both algorithms in function of $w(\Omega)$ for DEL and TC instances with 1000 vertices. It is not surprising that both algorithms need about the same running time when $w(\Omega)$ is near to one. This observation allows to perform a simple a priori test for deciding which algorithm is better suited for a particular instance. Given an instance, we first calculate $w(\Omega)$ (in linear time). If $w(\Omega) \ll 1$ it is likely that the proposed algorithm will be faster than the direct Monte-Carlo approach. When $w(\Omega) \gg 1$, the direct Monte-Carlo approach is likely to be the more efficient algorithm.
Figure 3: Comparison of the running time of the proposed algorithm and the direct Monte-Carlo approach in function of $w(\Omega)$ for networks with 1000 vertices.

8 Conclusions

An adapted version of the Monte-Carlo algorithm given by Karp and Luby in [9] was presented and analyzed. The new algorithm is specialized for directed acyclic graphs and is suited for the estimation of small reliabilities. Computational results show the successful application of the proposed algorithm on two types of randomly generated large-scale instances and its advantage compared to the direct Monte-Carlo approach when very small reliabilities have to be estimated. Previous algorithms for accurate estimation of $s$-$t$ reliability were only applicable on either very small instances or on a very restricted class of initial networks. For the case of uniform edge failure probabilities, a worst-case bound on the number of samples to be drawn was given that sharpens a bound presented in [9] and is significantly stronger in the case of relatively sparse graphs without long paths from $s$ to $t$.

One important open question in this domain is if there exists an FPRAS for estimating $s$-$t$ reliability in directed acyclic graphs. It would be interesting to find algorithms allowing to tackle instances efficiently that cannot be solved in reasonable time by our algorithm or the direct Monte-Carlo approach. Another point is the generalization of the upper bound for $w(\Omega)/w(A)$ for the case of non-uniform failure probabilities. Additionally for the case of general (not necessarily acyclic) networks there seem to be no practically efficient algorithms at the moment for the estimation of low reliabilities on large instances.
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Appendix

Proof of Theorem 5

Let \((\gamma, a) \in \Omega\) be a sample drawn according to lines 1-1 of Algorithm 1. We define the influence value \(m(\gamma, a)\) of the sample as in Section 4 as a random variable
\[
m : \Omega \longrightarrow \mathbb{R}, \quad m(\gamma, a) = \frac{1}{|\mathcal{P}(a)|}.
\]

As discussed in Section 4, the influence value of a sample from \(\Omega\) can be used as an unbiased estimator for \(w(A)/w(\Omega)\) as we have
\[
E[m(\gamma, a)] = \frac{w(A)}{w(\Omega)}.
\]

By the above equation, the reciprocal of a lower bound on \(E[m]\) is an upper bound on \(w(\Omega)/w(A)\). We will therefore deduce the bound given in Theorem 5 by deriving a lower bound bound on \(w(A)/w(\Omega)\).

Let \(\Omega_{l,k}\) be the subset of all elements of the sample space \(\Omega\) where the initial chosen path has length \(l\) and the total number of appeared edges is \(l + k\), i.e.,
\[
\Omega_{l,k} = \{(\gamma, a) \in \Omega \mid |\gamma| = l, |a| = l + k\}.
\]

It is clear that all elements \(\omega \in \Omega_{l,k}\) appear with equal probability and can be sampled by the method described in Algorithm 2. This method is introduced...
just for theoretical analysis.

**Algorithm 2**: Sampling uniformly from $\Omega_{l,k}$

1. Choose uniformly a path $\gamma \in \mathcal{P}$ with length $l$;
2. $a = \gamma$;
3. $R = E \setminus \gamma$;
4. for $i = 1$ to $k$ do
   5. Choose an edge $e_i \in R$ uniformly at random;
   6. $a = a \cup e_i$;
   7. $R = R \setminus e_i$;
5. end
6. return $(\gamma, a)$

With every edge $e_i$ added in the for-loop of Algorithm 2 we associate a multiplicity $M_{e_i} \in \{1/2, 1\}$ equal to $1/2$ if both endpoints of $e_i$ are saturated by edges in $\gamma \cup \{e_j \mid 1 \leq j \leq i - 1\}$ and equal to 1 otherwise. Intuitively, the multiplicity is a measure for the influence of an added edge in Algorithm 2 on the ratio $w(A)/w(\Omega)$. The following lemma formalizes this intuition.

**Lemma 1.** Let $(\gamma, a) \in \Omega_{l,k}$ be an intact state constructed as described in Algorithm 2. Then we have

$$
\frac{1}{|\mathcal{P}(a)|} \geq \prod_{i=1}^{k} M_{e_i}.
$$

Before proving this lemma, we discuss the link between the lemma and Algorithm 2. In general there are multiple ways to get an intact state $a \in A$ with Algorithm 2. Depending on how the state $a$ was obtained, the multiplicities associated with the edges in $a$ are different. As Lemma 1 is true for every possible way of obtaining state $a$ by Algorithm 2 it is applicable even to intact states that were not constructed through Algorithm 2. One just has to fix a possible way how the state $a$ could have been constructed by Algorithm 2, i.e., an $s$-$t$ path $\gamma \subseteq a$ has to be fixed as well as an order for the edges in $a \setminus \gamma$ specifying in which sequence those edges were chosen in Algorithm 2. The multiplicities can then be calculated with respect to this order, and Lemma 1 can be applied.

**Proof of Lemma 1.** Let $(\gamma, a) \in \Omega_{l,k}$ be an intact state constructed as described in Algorithm 2. We define $F = \{e_1, e_2, \ldots, e_k\} = a \setminus \gamma$ to be the set of all edges added to the initial path $\gamma$ during the construction of $a$. Let $F^{\frac{1}{2}}, F^1$ be the following partitioning of the edges in $a$.

$$
F^{\frac{1}{2}} = \{e \in F \mid M_e = \frac{1}{2}\}
$$

$$
F^1 = \gamma \cup \{e \in F \mid M_e = 1\}
$$

We prove the following statement, which immediately implies Lemma 1.

**Proposition.** For every set $H \subseteq F^{\frac{1}{2}}$ there exists at most one $s$-$t$ path in the state $a$ that contains all edges of $H$ and none of $F^{\frac{1}{2}} \setminus H$. 

Lemma 1 follows from the above proposition by the following observation. The proposition implies that there are at most as many different s-t paths in a as there are subsets of $F^\frac{1}{2}$. We therefore have

$$|\mathcal{P}(a)| \leq 2^{|F^\frac{1}{2}|},$$

which finally implies

$$\frac{1}{|\mathcal{P}(a)|} \geq \left(\frac{1}{2}\right)^{|F^\frac{1}{2}|} = \prod_{i=1}^{k} M_{e_i}.$$

It remains to prove the proposition. Let $H \subseteq F^\frac{1}{2}$ and suppose that we have two different s-t paths $\gamma_1, \gamma_2 \subseteq a$ such that both contain all edges of $H$ and none of $F^\frac{1}{2} \setminus H$. This implies that their symmetric difference $\gamma_1 \Delta \gamma_2 \subseteq a$ contains a cycle consisting only of edges in $F^1$, thus contradicting the fact that $F^1$ does not contain cycles (any cycle in $a$ contains at least one element of $F^\frac{1}{2}$).

With the aid of Lemma 1 we prove the following intermediate result.

**Lemma 2.** Let $l \in \mathbb{N}, k \in \{0\} \cup \mathbb{N}$ with $l + k \leq m$ and $\omega = (\gamma, a)$ be a random sample from the sample space according to Algorithm 1. We have

$$\mathbb{E}\left[\frac{1}{|\mathcal{P}(a)|} \mid (\gamma, a) \in \Omega_{l,k}\right] \geq \max\left\{2^{-k}, 2^{-\#(k+l)}\right\}.$$  \hspace{1cm} (5)

**Proof of Lemma 2.** Let $\omega' = (\gamma', a')$ be a sample corresponding to a result of Algorithm 2 for the given $l$ and $k$. Conditioned on $\Omega_{l,k}$, $\omega$ has therefore the same distribution as $\omega'$. Let $M_{e_1}, M_{e_2}, \ldots, M_{e_k}$ be the random variables corresponding to the multiplicities of the edges in $a' \setminus \gamma'$. By Lemma 1 we have

$$\mathbb{E}\left[\frac{1}{|\mathcal{P}(a)|} \mid (\gamma, a) \in \Omega_{l,k}\right] = \mathbb{E}\left[\frac{1}{|\mathcal{P}(a')|}\right] \geq \mathbb{E}\left[\prod_{i=1}^{k} M_{e_i}\right].$$

Let $i \in \{1, \ldots, k\}$. Observe that independently of which path $\gamma'$ and which edges $e_1, \ldots, e_k$ were chosen, we have that at most $l + 1 + 2(i-1)$ vertices in $G$ are saturated by the edges in $\gamma' \cup \{e_1, \ldots, e_{i-1}\}$ ($l + 1$ vertices are saturated through $\gamma'$ and every additional edge saturates at most two new vertices in $G$). Let $V_i$ be the vertices in $G$ which are saturated by $\gamma' \cup \{e_1, \ldots, e_{i-1}\}$.

In Algorithm 2 the edge $e_i$ is chosen uniformly at random from the remaining edges $E \setminus (\gamma' \cup \{e_1, \ldots, e_k\})$. As $G$ is sparse with edge-vertex bound $\mu$ we have that at most $\mu |V_i| \leq \mu (l + 1 + 2(i-1))$ edges have both endpoints in $V_i$. Furthermore, $i - 1$ of these edges were already chosen. We therefore have the following
stochastic inequality (which is true for any realization of $\gamma', M_{e_1}, \ldots, M_{e_{i-1}}$):

$$P\left(M_{e_i} = \frac{1}{2} \mid \gamma', M_{e_1}, \ldots, M_{e_{i-1}}\right) \leq \min\left\{1, \frac{\mu|V_i| - (i - 1)}{m - (i - 1)}\right\}$$

$$\leq \min\left\{1, \frac{\mu|V_i|}{m}\right\}$$

$$\leq \min\left\{1, \frac{\mu(2i + l - 1)}{m}\right\}.$$

The stochastic inequality above allows to give a simple bound on the following conditional expectation:

$$E[M_{e_i} \mid \gamma', M_{e_1}, \ldots, M_{e_{i-1}}] = \frac{1}{2} P\left(M_{e_i} = \frac{1}{2} \mid \gamma', M_{e_1}, \ldots, M_{e_{i-1}}\right) +$$

$$P\left(M_{e_i} = 1 \mid \gamma', M_{e_1}, \ldots, M_{e_{i-1}}\right)$$

$$= 1 - \frac{1}{2} P\left(M_{e_i} = \frac{1}{2} \mid \gamma', M_{e_1}, \ldots, M_{e_{i-1}}\right)$$

$$\geq \max\left\{\frac{1}{2}, 1 - \frac{\mu(2i + l - 1)}{2m}\right\}.$$

Applying the above inequality, the expectation in (5) can be developed as

$$E\left[\prod_{i=1}^{k} M_{e_i}\right] \geq \prod_{i=1}^{k} \max\left\{\frac{1}{2}, 1 - \frac{\mu(2i + l - 1)}{2m}\right\}$$

$$\geq \prod_{i=1}^{k} \max\left\{\frac{1}{2}, \left(\frac{1}{4}\right)^{\frac{\mu(2i + l - 1)}{m}}\right\}.$$

The last inequality comes from the fact that $(1/4)^x \leq 1 - x$ for $x \in [0, 1/2]$ and $(1/4)^x \leq 1/2$ for $x \geq 1/2$. Developing further, we finally get the result of Lemma 2

$$\prod_{i=1}^{k} \max\left\{\frac{1}{2}, \left(\frac{1}{4}\right)^{\frac{\mu(2i + l - 1)}{m}}\right\} \geq \max\left\{2^{-k}, 2^{-\sum_{i=1}^{k} \frac{\mu(2i + l - 1)}{m}}\right\}$$

$$\geq \max\left\{2^{-k}, 2^{-k(l+1)}\right\}.$$

Beginning with the result of Lemma 2 we now prove Theorem 5 by first weakening and then eliminating the conditioning on $\Omega_{l,k}$. Let $\Omega_l$ be the set of all elements of the sample space $\Omega$ where the initial chosen path has length $l$, i.e.,

$$\Omega_l = \{(\gamma, a) \mid |\gamma| = l\}.$$
Let $K$ be the random variable corresponding to the number of edges that appeared additionally to the ones of the initial path, when drawing an element out of $\Omega_l$. Note that $K$ is binomially distributed as $K \sim \text{Bin}(m - l, \eta)$.

Using Lemma 2 we get

$$
E \left[ \frac{1}{|P(a)|} \mid \gamma, a \in \Omega_l \right] \geq E \left[ \max \left\{ 2^{-K}, 2^{-\eta K(K+1)} \right\} \right]
\geq \max \left\{ E [2^{-K}], E [2^{-\eta K(K+1)}] \right\}
= \max \left\{ \left( 1 - \frac{\eta}{2} \right)^{m-l}, E [2^{-\eta K(K+1)}] \right\}.
$$

(6)

By replacing $l$ by $l_{\text{min}}$ in the first term of the above maximum we get the first part of the inequality in Theorem 5, i.e.,

$$
\frac{w(\Omega)}{w(A)} = \left( E \left[ \frac{1}{|P(a)|} \right] \right)^{-1} \leq \left( 1 - \frac{\eta}{2} \right)^{- (m-l_{\text{min}})} = \left( \frac{2}{2-\eta} \right)^{m-l_{\text{min}}},
$$

The remaining part of Theorem 5 will be shown by developing the second term of (6) further. We will use the inequality

$$
E[2^{-\eta K(K+1)}] \geq 2^{-\eta \alpha (a+l)} P[K \leq \alpha],
$$

which is true for any $\alpha \in \mathbb{R}$ and a result shown by Hamza [5] stating that the distance between the median and the mean of a binomial random variable is at most $\ln(2)$. Therefore, when choosing $\alpha = \eta m + \ln(2)$, we have $P[K \leq \alpha] \geq 1/2$ and get the following result:

$$
E[2^{-\eta K(K+1)}] \geq 2^{-1 - \frac{\eta}{2} (\eta (m-l) + \ln(2)) (\eta (m-l) + \ln(2)+l)}
\geq 2^{-1 - \frac{\eta}{2} (\eta m + \ln(2)) (\eta m + \ln(2)+l)}
\geq 2^{-1 - \frac{\eta}{2} (\eta m + \ln(2)) (\eta m + \ln(2)+l_{\text{max}})}.
$$

This implies the second part of the inequality in Theorem 5 as

$$
\frac{w(\Omega)}{w(A)} = \left( E \left[ \frac{1}{|P(a)|} \right] \right)^{-1} \leq 2^{1 + \frac{\eta}{2} (\eta m + \ln(2)) (\eta m + \ln(2)+l_{\text{max}})},
$$

which completes the proof of Theorem 5.