COTANGENT BUNDLE TO THE GRASSMAN VARIETY

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Abstract. We show that there is an affine Schubert variety in the infinite dimensional partial Flag variety (associated to the two-step parabolic subgroup of the Kac-Moody group $\hat{SL}_n$, corresponding to omitting $\alpha_0, \alpha_d$) which is a natural compactification of the cotangent bundle to the Grassmann variety.

1. Introduction

Let the base field $K$ be the field of complex numbers. Consider a cyclic quiver with $h$ vertices and dimension vector $\underline{d} = (d_1, \cdots, d_h)$:

$$
\begin{array}{ccc}
1 & \rightarrow & 2 \\
& \rightarrow & \cdots \\
& & h-2 \\
& \rightarrow & h-1 \\
& & h
\end{array}
$$

Denote $V_i = K^{d_i}$. Let

$$Z = \text{Hom}(V_1, V_2) \times \cdots \times \text{Hom}(V_h, V_1), \quad GL_{\underline{d}} = \prod_{1 \leq i \leq h} \text{GL}(V_i)$$

We have a natural action of $GL_{\underline{d}}$ on $Z$: for $g = (g_1, \cdots, g_h) \in GL_{\underline{d}}, f = (f_1, \cdots, f_h) \in Z$,

$$g \cdot f = (g_2f_1g_1^{-1}, g_3f_2g_2^{-1}, \cdots, g_1f_hg_h^{-1})$$

Let

$$\mathcal{N} = \{(f_1, \cdots, f_h) \in Z \mid f_h \circ f_{h-1} \circ \cdots \circ f_1 : V_1 \rightarrow V_1 \text{ is nilpotent}\}$$

Note that $f_h \circ f_{h-1} \circ \cdots \circ f_1 : V_1 \rightarrow V_1$ is equivalent to $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_1 \circ f_h \circ \cdots \circ f_i : V_1 \rightarrow V_i$ is nilpotent. Clearly $\mathcal{N}$ is $GL_{\underline{d}}$-stable. Lusztig (cf. [3]) has shown that an orbit closure in $\mathcal{N}$ is canonically isomorphic to an open subset of a Schubert variety in $\hat{SL}_n/Q$, where $n = \sum_{1 \leq i \leq h} d_i$, and $Q$ is the parabolic subgroup of $\hat{SL}_n$ corresponding to omitting $\alpha_0, \alpha_{d_1}, \alpha_{d_1+d_2}, \cdots, \alpha_{d_1+\cdots+d_{h-1}}$

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(\alpha_i, 0 \leq i \leq n - 1 \text{ being the set of simple roots for } \widehat{SL}_n). \text{ Corresponding to } h = 1, \text{ we have that } \mathcal{N} \text{ is in fact the variety of nilpotent elements in } M_{d_1,d_1}, \text{ and thus the above isomorphism identifies } \mathcal{N} \text{ with an open subset of a Schubert variety } X_\mathcal{N} \text{ in } \widehat{SL}_n/\mathcal{P}, \mathcal{P}, \text{ being the maximal parabolic subgroup of } \widehat{SL}_n \text{ corresponding to “omitting” } \alpha_0.

Let now \( h = 2, \) and let \( Z_0 = \{ (f_1, f_2) \in \mathbb{Z} \mid f_2 \circ f_1 = 0, f_1 \circ f_2 = 0 \} \)

Strickland (cf. [4]) has shown that each irreducible component of \( Z_0 \) is the conormal variety to a determinantal variety in \( M_{d_1,d_2,d_2} \). A determinantal variety in \( M_{d_1,d_2,d_2} \) being canonically isomorphic to an open subset in a certain Schubert variety in \( G_{d_1,d_1+d_2} \) (the Grassmannian variety of \( d_2 \)-dimensional subspaces of \( K^{d_1+d_2} \)) (cf.[2]), the above two results of Lusztig and Strickland suggest a connection between conormal varieties to Schubert varieties in the (finite-dimensional) flag variety and the affine Schubert varieties. This is the motivation for this article. We define a canonical embedding of \( T^*G_{d,n} \) (\( G_{d,n} \) being the Grassmann variety of \( d \)-dimensional subspaces of \( K^n \)) inside a \( P \)-stable affine Schubert variety \( X(\kappa_d) \) inside \( \widehat{SL}_n/Q \) (\( Q \) being the two-step parabolic subgroup of \( \widehat{SL}_n \), corresponding to omitting \( \alpha_0, \alpha_d \)), and show that \( X(\kappa_d) \) gives a natural compactification of \( T^*G_{d,n} \) (cf. Theorem 5.2).

The fact that the affine Schubert variety \( X(\kappa_d) \) is a natural compactification of \( T^*G_{d,n} \), suggests similar compactifications for conormal varieties to Schubert varieties in \( G_{d,n} \) (by suitable affine Schubert varieties in \( \widehat{SL}_n/Q, Q \) being as above). The details will appear in a subsequent paper.

The sections are organized as follows. In §2, we fix notation and recall affine Schubert varieties. In §3, we introduce the element \( \kappa_d \) (in \( \widehat{W} \), the affine Weyl group), and prove some properties of \( \kappa_d \). In §4, we prove one crucial result on \( \kappa_d \) needed for proving the embedding of \( T^*G_{d,n} \) inside \( \widehat{SL}_n/Q \). In §5 we present the results for \( T^*G_{d,n} \).

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2. AFFINE SCHUBERT VARIETIES

Let \( K = \mathbb{C}, F = K((t)), \) the field of Laurent series, \( A^\pm = K[[t^\pm 1]] \). Let \( G \) be a semi-simple algebraic group over \( K, T \) a maximal torus in \( G, B, \) a Borel subgroup, \( B \supset T \) and let \( B^- \) be the Borel subgroup opposite to \( B \). Let \( \mathcal{G} = G(F) \). The natural inclusions

\[ K \hookrightarrow A^\pm \hookrightarrow F \]
induce inclusions
\[ G \hookrightarrow G(A^\pm) \hookrightarrow G \]
The natural projections
\[ A^\pm \rightarrow K, t^\pm 1 \mapsto 0 \]
induce homomorphisms
\[ \pi^\pm : G(A^\pm) \rightarrow G \]
Let
\[ B = (\pi^+)^{-1}(B), B^- = (\pi^-)^{-1}(B^-) \]
Let \( \hat{W} = N(K[t, t^{-1}])/T \), the affine Weyl group of \( G \) (here, \( N \) is the normalizer of \( T \) in \( G \)); \( \hat{W} \) is a Coxeter group on \( \ell + 1 \) generators \( \{s_0, s_1 \cdots s_\ell\} \), where \( \ell \) is the rank of \( G \), and \( \{s_0, s_1 \cdots s_\ell\} \) is the set of reflections with respect to the simple roots, \( \{\alpha_0, \alpha_1 \cdots \alpha_\ell\} \), of \( G \).

**Bruhat decomposition:** We have
\[ G(F) = \bigcup_{w \in \hat{W}} BwB, G(F)/B = \bigcup_{w \in \hat{W}} BwB(mod B) \]
For \( w \in \hat{W} \), let \( X(w) \) be the affine Schubert variety in \( G(F)/B \):
\[ X(w) = \bigcup_{r \leq w} B\tau B (mod B) \]
It is a projective variety of dimension \( \ell(w) \).

2.1. **Affine Flag variety, Affine Grassmannian:** Let \( G = SL(n), \)
\( G = G(F), G_0 = G(A^+) \). Then \( G/B \) is the affine Flag variety, and \( G/G_0 \) is the affine Grassmannian. Further,
\[ G/G_0 = \bigcup_{w \in \hat{W}} c_0 Bw G_0 (mod G_0) \]
where \( W^{G_0} \) is the set of minimal representatives in \( \hat{W} \) of \( \hat{W}/W_{G_0} \).
Denote \( A^+(= k[[t]]) \) by just \( A \). Let
\[ \hat{Gr}(n) = \{A\text{-lattices in } F^n\} \]
Here, by an \( A \)-lattice in \( F^n \), we mean a free \( A \)-submodule of \( F^n \) of rank \( n \). Let \( E \) be the standard lattice, namely, the \( A \)-span of the standard \( F \)-basis \( \{e_1, \cdots, e_n\} \) for \( F^n \). For \( V \in \hat{Gr}(n) \), define
\[ \text{vdim}(V) := dim_K(V/V \cap E) - dim_K(E/V \cap E) \]
One refers to \( \text{vdim}(V) \) as the virtual dimension of \( V \). For \( j \in \mathbb{Z} \) denote
\[ \hat{Gr}_j(n) = \{V \in \hat{Gr}(n) | \text{vdim}(V) = j\} \]
Then \( \hat{Gr}_j(n), j \in \mathbb{Z} \) give the connected components of \( \hat{Gr}(n) \). We have a transitive action of \( GL_n(F) \) on \( \hat{Gr}(n) \) with \( GL_n(A) \) as the stabilizer.
of the standard lattice $E$. Further, let $G_0$ be the subgroup of $GL_n(F)$, defined as,
$$G_0 = \{ g \in GL_n(F) \mid \text{ord}(\det g) = 0 \}$$
(here, for a $f \in F$, say $f = \sum a_i t^i$, order $f$ is the smallest $r$ such that $a_r \neq 0$). Then $G_0$ acts transitively on $\hat{Gr}_0(n)$ with $GL_n(A)$ as the stabilizer of the standard lattice $E$. Also, we have a transitive action of $SL_n(F)$ on $\hat{Gr}_0(n)$ with $SL_n(A)$ as the stabilizer of the standard lattice $E$. Thus we obtain the identifications:

\begin{align*}
GL_n(F)/GL_n(A) & \simeq \hat{Gr}(n) \\
G_0/GL_n(A) & \simeq \hat{Gr}_0(n), SL_n(F)/SL_n(A) \simeq \hat{Gr}_0(n)
\end{align*}

In particular, we obtain

\begin{align*}
G_0/GL_n(A) & \simeq SL_n(F)/SL_n(A)
\end{align*}

2.2. Generators for $\hat{W}$: Following the notation in [1], we shall work with the set of generators for $\hat{W}$ given by \{s_0, s_1, \ldots, s_{n-1}\}, where $s_i, 0 \leq i \leq n-1$ are the reflections with respect to $\alpha_i, 0 \leq i \leq n-1$. Note that $\{\alpha_i, 1 \leq i \leq n-1\}$ is simply the set of simple roots of $SL_n$ (with respect to the Borel subgroup $B$); also note that $\alpha_0 = \delta - \theta$, where $\theta$ is the highest root of the (finite) Type $A_{n-1}$ with simple roots $\{\alpha_1, \ldots, \alpha_{n-1}\}$:

$$\theta = \alpha_1 + \cdots + \alpha_{n-1}$$

We have the following canonical lifts (in $G$) for $s_i, 0 \leq i \leq n-1$; for $1 \leq i \leq n-1$, $s_i$ is the permutation matrix $(a_{rs})$, with $a_{jj} = 1, j \neq i, i+1$, $a_{i,i+1} = 1, a_{i+1,i} = -1$, and all other entries are 0. A lift for $s_0$ is given by

$$
\begin{pmatrix}
0 & 0 & \cdots & t^{-1} \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 \\
-t & 0 & 0 & 0
\end{pmatrix}
$$

3. The element $\kappa_d$

Let $P$ be a parabolic subgroup, and $W_P$ the Weyl group of $P$. Let $R_P$ be the set of roots of $P$, and $S_P$ the set of simple roots of $P$, The Schbert varieties in $G/P$ are indexed by $W/W_P$. We gather some well-known facts on $W^P$, the set of minimal representatives in $W$ of the elements of $W/W_P$.

For $wP \in W/W_P$, there exists a (unique) representative $w_{\min} \in W$ with the following properties:
Fact 1: Among all the representatives in $W$ for $wP$, $w_{\min}$ is the unique element of smallest length.

Fact 2: $w_{\min}(\alpha) > 0, \forall \alpha \in S_P$.

Fact 3: $\dim X(w) = l(w_{\min})$ $(X(w)$ is the Schubert variety in $G/P$, corresponding to $w$).

Fact 4: Let $P_\alpha$ be the parabolic subgroup with $\{\alpha\}$ as the associated set of simple roots. Then $X(w)$ is stable for left multiplication by $P_\alpha$ if and only if $s_\alpha w < w (mod W_P)$. More generally, given a parabolic subgroup $Q$, $X(w)$ is stable for left multiplication by $Q$ if and only if $s_\alpha w < w (mod W_P), \forall \alpha \in S_Q$.

Remark 3.1. These results hold for Kac-Moody groups also.

Denote the Weyl group of $SL(n)$ by $W$ (note that $W$ is just the symmetric group $S_n$). Consider the Type $A_{n-1}$ Dynkin diagram with simple roots $\alpha_1, \ldots, \alpha_{n-1}$, namely

(A) \[ \alpha_1 \cdots \alpha_{n-1} \]

Let $W_{P_d}$ be the Weyl group of $P_d$, the maximal parabolic subgroup of $SL(n)$ corresponding to omitting the simple root $\alpha_d$. Note that $P_d$ consists of $\{(a_{ij}) \in SL(n)\}$ such that $a_{ij}, j \leq d < i \leq n$ are 0. Note also that $W_{P_d} = S_d \times S_{n-d}$. We may suppose $d \leq n - d$, since $G/P_d \cong G/P_{n-d}$. Denote the set of minimal representatives of $W/W_{P_d}$ by $W^{P_d}$. Then the Schubert varieties in $G_{d,n}(\cong G/P_d)$ are indexed by $W^{P_d}$.

3.2. The elements $w_1, w_2$. The unique maximal element $w_0^{P_d}$ corresponding to $G_{d,n}$ has the following reduced expression

$w_0^{P_d} = u_1 u_2 \cdots u_d, \quad u_k = s_{n-d+k} s_{n-d+k-2} \cdots s_k$

Let us denote $w_0^{P_d}$ by just $w_1$. Similarly, considering the Type $A_{n-1}$ Dynkin diagram with simple roots

$\alpha_{d-1}, \alpha_{d-2}, \ldots, \alpha_1, \alpha_0, \alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_{d+1}$ (taken in that order), namely,

(B) \[ \alpha_{d-1} \cdots \alpha_1 \alpha_0 \alpha_{n-1} \cdots \alpha_{d+1} \]

the unique maximal element $w_2$ in the set of minimal representatives corresponding to omitting $\alpha_0$ has the following reduced expression

$v_{d-1} v_{d-2} \cdots v_1 v_0, \quad v_k = s_{d+k+1} s_{d+k+2} \cdots s_{n-s_0} s_1 s_2 \cdots s_k, 0 \leq k \leq d - 1$

Note that for $0 \leq k \leq d-2$, we have, $d+k+1 \leq n-1$ (since $d \leq n-d$). For $k = d - 1$ again, $d+k+1 \leq n-1$, if $d < n - d$; if $d = n - d$, then $v_k = s_0 s_1 s_2 \cdots s_{d-1}$, and in this case, $s_{d+k+1} = (s_{2d} = s_n)$ is to be understood as $s_0$. 
In the sequel, we shall refer to the two systems as system A, system B, respectively. We shall index system B as $\alpha_1', \ldots, \alpha_{n-1}'$, and denote the corresponding reflections by $s_1', \ldots, s_{n-1}'$; we have,

$$s_{d-k}' = s_k, 1 \leq k \leq d - 1, \quad s_{n+d-\ell}' = s_\ell, d + 1 \leq \ell \leq n - 1$$

We define $\kappa_d = w_1w_2$. Then using the lifts for $s_i, 0 \leq i \leq n - 1$, as described in \[2.2\] it is easily checked that $\kappa_d$ is the diagonal matrix with three diagonal blocks:

$$\kappa_d = \text{diag}([tI_d], [I_{n-2d}], [t^{-1}I_d])$$

(here, $I_r$ denotes the identity $r \times r$ matrix).

Let $P_d$ be as above. Let $P_d'$ be the maximal parabolic subgroup of system B, corresponding to omitting $\alpha_d' (= \alpha_0)$.

We now prove three properties of $\kappa_d$. In the discussion below, we shall have the following notation:

- $R$ (respectively, $R^+$) is the root system (respectively, the system of positive roots) of $G$
- $R_{P_d}$ (respectively, $R^+_{P_d}$) is the root system (respectively, the system of positive roots) of $P_d$
- $W_{P_d}$ is the Weyl group of $P_d$
- $W_{P_d}'$ is the set of minimal representatives of $W/W_{P_d}$
- $\hat{W}Q$ is the set of minimal representatives of $\hat{W}/W_Q$, $Q$ being the two-step parabolic subgroup of $G$, corresponding to omitting $\alpha_0, \alpha_d$.

We will also need the description of the set of positive real roots of $G$ (cf. \[1\]): the set of positive real roots is given by

$$\{q\delta + \beta, q > 0, \beta \in R\} \cup \{R^+\},$$

where, recall that $\delta = \alpha_0 + \theta, \theta = (\alpha_1 + \cdots + \alpha_n)$ is the highest root (in $R^+$).

We have the braid relations

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, 1 \leq i \leq n - 2,$$

$$s_0s_1s_0 = s_1s_0s_1, \quad s_0s_{n-1}s_0 = s_{n-1}s_0s_{n-1}$$

and the commuting relations:

$$s_is_j = s_js_i, 1 \leq i, j \leq n - 1, |i - j| > 1, \quad s_0s_i = s_is_0, 2 \leq i \leq n - 2$$

3.3. I. A reduced expression for $\kappa_d$: In this subsection, we shall prove that the expression $\kappa_d = w_1w_2$, with the reduced expressions for $w_1, w_2$ as in \[3.2\] is reduced. We prove the following more general Lemma.

**Lemma 3.4.** Let $y_1, y_2$ be in $W_{P_d}, W_{P_d}'$ respectively, and let $y_1 = s_{i_1}' \cdots s_{i_t}', y_2 = s_{j_1}' \cdots s_{j_t}'$ be reduced expressions for $y_1, y_2$. Then $s_{i_1}' \cdots s_{i_t}'s_{j_1}' \cdots s_{j_t}'$ is a reduced expression for $y_1y_2$. 

Proof. First observe that any reduced expression for $y_1$ (respectively, $y_2$) is a right-end segment of the reduced expression for $w_1$ (respectively, $w_2$) as described in \[3.2\] and hence
\[ (*) \]
$s_{i_t} = s_d, \ s'_{j_t} = s_0$
(see the descriptions of the reduced expressions for $w_1, w_2$ as given in \[3.2\]). Let us denote the word $s_{i_1} \cdots s_{i_t} s'_{j_1} \cdots s'_{j_t}$ as just $r_1 \cdots r_{\ell + t}$, where $r_j, 1 \leq j \leq \ell + t$ equals the reflection $s_{\beta_j}, \beta_j$ being either an $\alpha_a$ or an $\alpha'_b$; let us denote $\ell + t$ by $m$. We shall show that $r_1 \cdots r_{j-1}(\beta_j) > 0$, i.e., a positive root, for any left-end sub word $r_1 \cdots r_{j-1}, 2 \leq j \leq m$, from which the required result will follow. This is clear for $2 \leq j \leq \ell$, since $r_1 \cdots r_{\ell}(= y_2)$ is reduced. Let us then consider $r_1 \cdots r_{j-1}(\beta_j), j \geq \ell + 1$.

We now divide the proof into the following two cases:

**Case 1:** Let $j = \ell + 1$. Then $\beta_j = \alpha_k$, for some $k, 0 \leq k \leq n-1, k \neq d$.
If $k > 0$, then $\alpha_k \in S_P$ (the system of simple roots of $P_d$), and hence $r_1 \cdots r_{j-1}(\beta_j) = y_1(\alpha_k) > 0$, since $y_1 \in W_P$, and the result follows. If $k = 0$, then $r_1 \cdots r_{j-1}(\beta_j) = y_1(\alpha_0)$ is clearly a positive root, since $y_1$ being in $W_P$, the reduced expression for $y_1$ does not involve $s_0$, and the result follows.

**Case 2:** Let $j \geq \ell + 2$. Now $r_{\ell+1} \cdots r_{j-1}(\beta_j) > 0$, a positive (real) root (since $r_{\ell+1} \cdots r_j$ being a left segment of the reduced expression $r_{\ell+1} \cdots r_m$ is reduced); hence, $r_{\ell+1} \cdots r_{j-1}(\beta_j)$ is of the form $q\delta + \gamma, q \geq 0, \gamma \in R$. We now divide the proof into the following two subcases:

**Subcase 2(a):** Let $q > 0$. Then, $r_{\ell+1} \cdots r_{j-1}(\beta_j) = r_{\ell+1} \cdots r_{\ell}(q\delta + \gamma)(= y_1(q\delta + \gamma))$ is of the form $q\delta + \epsilon$, where $\epsilon \in R$ (since $y_1$ does not involve $s_0$), and the result follows.

**Subcase 2(b):** Let $q = 0$. Then $\gamma = r_{\ell+1} \cdots r_{j-1}(\beta_j)$ is a positive root; further, it is in $R_\epsilon^d$. Hence $r_{\ell} \cdots r_{j-1}(\beta_j) = r_{\ell} \cdots r_{\ell}(\gamma)(= y_1(\gamma)) > 0$ (since, $y_1 \in W_P$), and the result follows.

This completes the proof of the Lemma. \(\blacksquare\)

### 3.5. II. Minimal representative property for $\kappa_d$:
In this subsection, we shall prove that $\kappa_d$ is in $\tilde{W}^Q$ (where recall that $Q$ is the two-step parabolic subgroup of $G$ corresponding to omitting $\alpha_0, \alpha_d$). We prove the following more general Lemma.

**Lemma 3.6.** Let $y_1, y_2$ be in $W_P, W_P'$ respectively. Then $y_1y_2$ is in $\tilde{W}^Q$.

**Proof.** We shall show that $y_1y_2(\alpha_i) > 0, i \neq 0, d$, from which the required result will follow. Let $y_2(\alpha_i) = \beta$. Then $\beta > 0$, a positive real root (since $y_2 \in W_P'$), say, $\beta = q\delta + \gamma, q \geq 0, \gamma \in R$. We now divide the proof into the following two cases:
Case 1: Let $q > 0$. Then $y_1 y_2(\alpha_i) = y_1(q \delta + \gamma)$. Now $y_1(q \delta + \gamma)$ is of the form $q \delta + \epsilon$, where $\epsilon \in R$ (since $y_1$ does not involve $s_0$). Hence $y_1 y_2(\alpha_i) > 0$.

Case 2: Let $q = 0$. Then $y_1 y_2(\alpha_i) = y_1(\gamma)$. Now $\gamma = y_2(\alpha_i)$ is a positive root; further, it is in $R^{p^+}$ (since $y_2$ does not involve $s_d$). Hence $y_1(\gamma) > 0$ (since $y_1 \in W^{P_d}$), and therefore $y_1 y_2(\alpha_i)(= y_1(\gamma)) > 0$, as required.

This completes the proof of the Lemma. \hfill \Box

Combining Lemmas 3.4, 3.6 we obtain the following

Corollary 3.7. For the Schubert variety $X(\kappa_d)$ in $G/Q$, we have,

$$\dim X(\kappa_d) = 2d(n - d) (= 2 \dim G_{d,n})$$

3.8. III. $G_0$-stability: As in 3.2 we shall index system B as $\alpha'_1, \cdots, \alpha'_{n-1}$, and denote the corresponding reflections by $s'_1, \cdots, s'_{n-1}$; we have,

$$s'_{d-k} = s_k, 1 \leq k \leq d - 1, \quad s'_{n+d-\ell} = s_{\ell}, d + 1 \leq \ell \leq n - 1$$

Let $P_d, P'_d$ be as in 3.3

Lemma 3.9. For system A, we have,

1. $s_k w_1 = w_1 s_{d+k}, 1 \leq k \leq n - d - 1$
2. $s_\ell w_1 = w_1 s_{\ell-(n-d)}, n + 1 - d \leq \ell \leq n - 1$

(here, $w_1$ is as in 3.2).

Proof. As an element of $S_n$, we have, $w_1 = ([n-d+1, n][1, n-d])$ (here, for a pair of integers $i < j$, $[i, j]$ denotes the set $\{i, i+1, \cdots, j\}$; also, the notation $w = (a_1 \cdots, a_n)$ for a permutation is the usual one-line notation, namely, $w(i) = a_i, 1 \leq i \leq n$). Given a permutation $w = (a_1 \cdots, a_n)$, and a pair of integers $i, j, 1 \leq i, j \leq n$ let $i = k, j = \ell$. Then

$$s_{(i,j)}w = w s_{(k,l)}$$

(here, for $1 \leq a, b \leq n, a \neq b$, $s_{(a,b)}$ denotes the transposition (of switching $a$ and $b$)). The assertions 1 and 2 follow from this and the facts that writing $w_1 = (a_1 \cdots, a_n)$, we have,

$$w_1(d+k) = k, 1 \leq k \leq n - d - 1, \quad w_1(\ell - (n-d)) = \ell, n+1-d \leq \ell \leq n - 1$$

As a straight forward consequence, we have similar results for system B:

Corollary 3.10. For system B, we have,

1. $s'_k w_2 = w_2 s'_{d+k}, 1 \leq k \leq n - d - 1$
We have \( s_k w_2 = w_2 s'_{\ell-(n-d)} \), \( n + 1 - d \leq \ell \leq n - 1 \)

(here, \( w_2 \) is as in §3.2).

Using Lemma 3.9 Corollary 3.10 §3.3 and §3.5 we shall now show the \( G_0 \)-stability for the Schubert variety \( X_{\kappa_d} \) in \( \mathcal{G}/\mathcal{Q} \) (for the action on the left by multiplication).

**Lemma 3.11.** For \( 1 \leq k \leq n - 1, k \neq n - d, d, s_k \kappa_d = \kappa_d s_k \).

**Proof.** Recall that \( \kappa_d = w_1 w_2 \). We may suppose that \( d \leq n - d \) (since \( SL(n/P_d \cong SL(n))/P_{n-d} \)). We divide the proof into the following two cases.

**Case 1:** Let \( 1 \leq k \leq n - d - 1 \)

We have (cf. Lemma 3.2 (1)),

\[
s_k \kappa_d = s_k w_1 w_2 = w_1 s_{d+k} w_2.
\]

Now, in view of §3.8 (with \( \ell = d + k \)), we have that \( s_{d+k} = s'_{n+d-(d+k)} = s'_{n-k} \), and hence \( s_{d+k} w_2 = s'_{n-k} w_2 \). Thus we get

\[
(*) \quad s_k \kappa_d = w_1 s'_{n-k} w_2
\]

To compute \( s'_{n-k} w_2 \), we further divide this case into the following two subcases:

**Subcase 1(a):** Let \( k \geq d + 1 \).

This implies \( n - k \leq n - d - 1 \). Hence, in view of Corollary 3.10 we get that \( s'_{n-k} w_2 = w_2 s'_{d+n-k} \). Now, \( d+n-k \geq d+1 \) (since \( k \leq n-1 \)). Hence we obtain (in view of §3.2), \( w_2 s'_{d+n-k} = w_2 s_k \). Thus \( s'_{n-k} w_2 = w_2 s_k \); substituting in (\( * \)), we get the required result.

**Subcase 1(b):** Let \( k \leq d - 1 \).

This implies \( n - k \geq n - d + 1 \). Hence in view of Corollary 3.10 we get that \( s'_{n-k} w_2 = w_2 s'_{n-k-(n-d)} = w_2 s'_{d-k} \). Now the hypothesis that \( k \leq d - 1 \) implies (cf. §3.2) that \( w_2 s'_{d-k} = w_2 s_k \). Thus \( s'_{n-k} w_2 = w_2 s_k \); substituting in (\( * \)), we get the required result.

**Case 2:** Let \( k \geq n + 1 - d \).

We have, by Lemma 3.9 (2), \( s_k w_1 = w_1 s_{k-(n-d)} \). Now, \( k-(n-d) \leq d-1 \), since \( k \leq n-1 \). Hence, we have (cf. §3.2), \( s_{k-(n-d)} = s'_{d-(k-(n-d))} = s'_{n-k} \). Hence \( s_{k-(n-d)} w_2 = s'_{n-k} w_2 = w_2 s'_{n-k+d} \) (cf. Corollary 3.10), note that \( n - k \leq d - 1 \), since \( k \geq n + 1 - d \). Now, \( k \geq d + 1 \), since \( k \geq n + 1 - d \) and \( n - d \geq d \). Hence, by (cf. §3.2), we get that \( s'_{n-k+d} = s_k \), and therefore, \( s_{k-(n-d)} w_2 = w_2 s_k \); substituting in (\( * \)), we get the required result. \( \Box \)

**Proposition 3.12.** The Schubert variety \( X(\kappa_d) \) in \( \mathcal{G}/\mathcal{Q} \), \( \mathcal{Q} \) being the two-step parabolic subgroup of \( \mathcal{G} \), corresponding to omitting \( \alpha_0, \alpha_d \) is stable for multiplication on the left by \( G_0 \).
Proof. We need to show that \( s_k \kappa_d \leq \kappa_d (\text{mod } Q) \), \( 1 \leq k \leq n - 1 \). For \( k = n - d \), this is clear, since in the reduced expression (Proposition 3.4) for \( \kappa_d = w_1 w_2 \), we have that the reduced expression for \( w_1 \) starts with \( s_{n-d} \) (cf. §3.2).

For \( 1 \leq k \leq n - 1, k \neq d, n - d \), we have, by Lemma 3.11, \( s_k \kappa_d = \kappa_d s_k \), and the result follows (since, for \( 1 \leq k \leq n - 1, k \neq d \), \( s_k \in W_Q \)).

Let now \( k = d \). If \( d = n - d \), then as above, we have that the reduced expression for \( w_1 \) starts with \( s_d \) (cf. §3.2) and the result follows. Let then \( d \leq n - d - 1 \). We have, by Lemma 3.9, (1), \( s_d w_1 = w_1 s_d \). Now \( s_{2d} w_2 = s_{2d} v_{d-1} v_{d-2} \cdots v_0 \),

where \( v_k, 0 \leq k \leq d - 1 \) has the reduced expression \( v_k = s_{d+k+1} s_{d+k+2} \cdots s_{n-1-s_0 s_1 s_2 \cdots s_k} \) (cf. §3.2). In particular, \( v_{d-1} \) begins with \( s_{2d} \). Hence, we obtain \( s_d w_1 w_2 = w_1 s_{2d} w_2 \leq w_1 w_2 \), and the result follows.

\[\square\]

4. The main Lemma

In this section, we prove one crucial result involving \( \kappa_d \) which will be used for proving the main result (namely, Theorem 5.2).

Lemma 4.1. Let \( Y = \sum_{1 \leq i \leq d < j \leq n} a_{ij} E_{ij}, E_{ij} \) being the elementary \( n \times n \) matrix with 1 at the \( (i,j) \)-th place and 0’s elsewhere. Let \( \overline{Y} = \text{Id}_{n \times n} + \sum_{1 \leq i \leq n-1} t^{-i} Y^i \) (note that \( Y^n = 0 \)). There exist \( g \in G_0, h \in Q \) such that \( g \kappa_d = \overline{Y} h \) (recall that \( Q \) is the two-step parabolic subgroup of \( \widehat{SL}_n \), corresponding to omitting \( \alpha_0, \alpha_d \)).

Proof. We shall now show that a choice of \( g_{ij}, 1 \leq i, j \leq n \), and \( h_{ij}, 1 \leq i, j \leq n \) exists so that \( h \in Q \), and \( g \kappa_d = \overline{Y} h \). We have, \( \overline{Y}^{-1} = \text{Id}_n - t^{-1} Y \). Set

\[ (*) \quad h = (\text{Id}_n - t^{-1} Y) g \kappa_d \]

We have (by definition of \( \kappa_d \) (§3.2))

\[ \kappa_d = \text{diag}([t \text{Id}_d], [\text{Id}_{n-2d}], [t^{-1} \text{Id}_d]) \]

Note that since we want \( h \) to belong to \( Q \), the condition on \( h \) is that \( h(0) \) should belong to \( P_d \). Hence, \( h_{ij} \)’s should satisfy the following conditions:

**Condition 1:** \( h_{ij}, j \leq d < i \leq n \) should have order \( > 0 \) (as an element of \( K[[t]] \))
Condition 2: The remaining $h_{ij}$'s should have order $\geq 0$. Now using (*), we describe below the columns of $h$:

\[
\begin{pmatrix}
  h_{1j} \\
  \vdots \\
  h_{dj} \\
  h_{d+1j} \\
  \vdots \\
  h_{nj}
\end{pmatrix} = \begin{pmatrix}
  t g_{1j} - \sum_{d+1 \leq i \leq n} a_{1i} g_{ij} \\
  \vdots \\
  t g_{dj} - \sum_{d+1 \leq i \leq n} a_{di} g_{ij} \\
  t g_{d+1j} \\
  \vdots \\
  t g_{nj}
\end{pmatrix}, \quad 1 \leq j \leq d
\]

\[
\begin{pmatrix}
  h_{1j} \\
  \vdots \\
  h_{dj} \\
  h_{d+1j} \\
  \vdots \\
  h_{nj}
\end{pmatrix} = \begin{pmatrix}
  g_{1j} - \sum_{d+1 \leq i \leq n} t^{-1} a_{1i} g_{ij} \\
  \vdots \\
  g_{dj} - \sum_{d+1 \leq i \leq n} t^{-1} a_{di} g_{ij} \\
  g_{d+1j} \\
  \vdots \\
  g_{nj}
\end{pmatrix}, \quad d+1 \leq j \leq n-d
\]

\[
\begin{pmatrix}
  h_{1j} \\
  \vdots \\
  h_{dj} \\
  h_{d+1j} \\
  \vdots \\
  h_{nj}
\end{pmatrix} = \begin{pmatrix}
  t^{-1} g_{1j} - \sum_{d+1 \leq i \leq n} t^{-2} a_{1i} g_{ij} \\
  \vdots \\
  t^{-1} g_{dj} - \sum_{d+1 \leq i \leq n} t^{-2} a_{di} g_{ij} \\
  t^{-1} g_{d+1j} \\
  \vdots \\
  t^{-1} g_{nj}
\end{pmatrix}, \quad n+1-d \leq j \leq n
\]

Condition 1 follows from (I). Also, from (I) we get that Condition 2 holds for $h_{ij}, 1 \leq i, j \leq d$. Thus the entries in the first $d$ columns of $h$ satisfy the required conditions.

From (II), we get that Condition 2 holds for $h_{ij}, d+1 \leq i \leq n, d+1 \leq j \leq n-d$. Regarding the entries $h_{ij}, 1 \leq i \leq d, d+1 \leq j \leq n-d$, we shall choose $g_{ij}, d+1 \leq i \leq n, d+1 \leq j \leq n-d$ so that order $g_{ij} > 0$. Thus with this choice, we have that the entries in the $j$-th column, $d+1 \leq j \leq n-d$, of $h$ satisfy the required conditions.

In view of (III), we shall choose $g_{ij}, d+1 \leq i \leq n, n-d+1 \leq j \leq n$ so that order $g_{ij} = 1$. With this choice, we obtain that $h_{ij}, d+1 \leq i \leq n, n-d+1 \leq j \leq n$ satisfy Condition 2. In order to have $h_{ij}, 1 \leq i \leq d, n-d+1 \leq j \leq n$ satisfy Condition 2, we choose order $g_{ij} = 0, 1 \leq i \leq d, n-d+1 \leq j \leq n$, and impose the following
conditions. We write \( g_{ij} = \sum g^{(r)}_{ij} t^r \). Then the conditions are

\[
g^{(0)}_{ij} - \sum_{d+1 \leq m \leq n} a_{im} g^{(1)}_{mj} = 0, 1 \leq i \leq d, n-d+1 \leq j \leq n
\]

Treating \( g^{(0)}_{ij}, 1 \leq i \leq d, g^{(1)}_{mj}, d+1 \leq m \leq n, n-d+1 \leq j \leq n \), (** is a linear homogeneous system of \( d^2 \) equations in \( nd \) variables, and hence there exist non-trivial solutions (note that \( nd > d^2 \), since \( d \leq n-1 \)). Hence, we can choose \( g_{ij}, 1 \leq i \leq d, g_{mj}, d+1 \leq m \leq n, n-d+1 \leq j \leq n \) so that all of the entries in \( j \)-th column, \( n-d+1 \leq j \leq n \) satisfy Condition 2.

This completes the proof of the Lemma. \( \Box \)

5. Cotangent bundle

The cotangent bundle \( T^*G/P_d \) is the vector bundle over \( G/P_d \), the fiber at any point \( x \in G/P_d \) being the cotangent space to \( G/P_d \) at \( x \); the dimension of \( T^*G/P_d \) equals \( 2 \dim G/P_d = 2(n-d) \). Also, \( T^*G/P_d \) is the fiber bundle over \( G/P_d \) associated to the principal \( P_d \)-bundle \( G \to G/P_d \) for the Adjoint action of \( P_d \) on \( u(P_d) \) (=Lie(\( U(P_d) \)), Lie algebra of \( U(P_d) \), \( U(P_d) \) being the unipotent radical of \( P_d \)). Thus

\[
T^*G/P_d = G \times_{P_d} u(P_d) = G \times u(P_d)/ \sim
\]

where the equivalence relation \( \sim \) is given by \( (g, Y) \sim (g x, x^{-1}Y x) \), \( g \in G, Y \in u(P_d), x \in P_d \).

5.1. Embedding of \( T^*G/P_d \) inside \( G/Q \). Define \( \phi : G \times_{P_d} u(P_d) \to G/Q \) as

\[
\phi(g, Y) = g(\Id + t^{-1}Y + t^{-2}Y^2 + \cdots)(\mod Q), g \in G, Y \in u(P_d)
\]

Note that the sum on the right hand side is finite (since \( Y \) is nilpotent).

In the sequel, we shall denote

\[
\underline{Y} := \Id + t^{-1}Y + t^{-2}Y^2 + \cdots
\]

We shall now list some facts on the map \( \phi \):

(i) \( \phi \) is well-defined: Let \( g \in G, x \in P_d, Y \in u(P_d) \). We have,

\[
\phi((gx, x^{-1}Y x)) = gx(\Id + t^{-1}x^{-1}Y x + t^{-2}x^{-1}Y^2 x + \cdots)(\mod Q)
\]

\[
= g(x + t^{-1}Y x + t^{-2}Y^2 x + \cdots)(\mod Q)
\]

\[
\equiv g(\Id + t^{-1}Y + t^{-2}Y^2 + \cdots)(\mod Q)
\]

\[
= \phi(g, Y)
\]

(ii) \( \phi \) is injective: Let \( \phi((g_1, Y_1)) = \phi((g_2, Y_2)) \). This implies that \( g_1 Y_1 \equiv g_2 Y_2 \mod Q \), where recall that for \( Y \in u(P_d), \underline{Y} = \Id + t^{-1}Y + \cdots \).
\( t^{-2}Y^2 + \cdots \). Hence, \( g_1Y_1 = g_2Y_2x \), for some \( x \in Q \). Denoting \( h =: g_2^{-1}g_1 \), we have, \( hY_1 = Y_2x \), and therefore,

\[
x = Y_2^{-1}hY_1 = Y_2^{-1}(hY_1h^{-1})h = Y_2^{-1}Y_1'h
\]

where \( Y_1' = hY_1h^{-1} \). Hence

\[
xh^{-1} = Y_2^{-1}Y_1' = (Id - t^{-1}Y_2)(Id + t^{-1}hY_1h^{-1} + t^{-2}hY_2'h^{-1} + \cdots)
\]

Now, left hand side is integral (since, \( x \in Q, h = g_2^{-1}g_1 \in G \)). Hence both sides equal \( Id \). This implies

\[
Y_2 = Y_1', \ x = h
\]

The fact that \( x = h \) together with the facts that \( x \in Q, h \in G \) implies that

\[
(*) \quad h \in Q \cap G(= P_d)
\]

Further, the fact that \( Y_2 = Y_1' \) implies that \( Y_1 = h^{-1}Y_2h \) (note from above that \( Y_1' = hY_1h^{-1} \)). Hence

\[
Id + t^{-1}Y_1 + t^{-2}Y_1^2 + \cdots = Id + t^{-1}h^{-1}Y_2h + t^{-2}h^{-1}Y_2'h + \cdots
\]

From this it follows that

\[
(**) \quad Y_1 = h^{-1}Y_2h
\]

Now (*) \( , (**) \) together with the fact that \( h = g_2^{-1}g_1 \) imply that

\[
(g_1, Y_1) = (g_2h, h^{-1}Y_2h) \sim (g_2, Y_2)
\]

From this injectivity of \( \phi \) follows.

(iii) \( G \)-equivariance: \( \phi \) is \( G \)-equivariant (clearly).

**Theorem 5.2.** The map \( \theta \) identifies \( \overline{T^*G/P_d} \) (the closure being in \( G/Q \)) with the affine Schubert variety \( X(\kappa_d) \).

**Proof.** Let \( (g_0, Y), g_0 \in G, Y \in u(P_d) \). Then \( Y \) is of the form:

\[
Y = \sum_{1 \leq i \leq d < j \leq n} a_{ij}E_{ij}, E_{ij}
\]

Now \( \theta(g_0, Y) = g_0(Id + t^{-1}Y + t^{-2}Y_2 + \cdots)(mod Q) = g_0Y (mod Q) \), where \( Y = Id + t^{-1}Y + t^{-2}Y^2 + \cdots \). Then Lemma 4.1 implies that there exist \( g \in G_0, h \in Q \) such that \( g\kappa_d = Y'h \). Hence \( Y' \) belongs to \( X(\kappa_d) \); hence \( g_0Y' \) is also in \( X(\kappa_d) \) (since \( g_0 \) is clearly in \( G_0 \)). Hence \( T^*G/P_d \subset X(\kappa_d) \), and therefore \( \overline{T^*G/P_d} \subset X(\kappa_d) \). Now by dimension considerations, we obtain that \( \overline{T^*G/P_d} = X(\kappa_d) \) (note (cf. Corollary 3.7), \( \dim X(\kappa_d) = 2d(n - d) \) \( \square \).
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