NON-SINGULAR ADVERSARIAL ROBUSTNESS OF NEURAL NETWORKS

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Abstract
Adversarial robustness has become an emerging challenge for neural network owing to its over-sensitivity to small input perturbations. While being critical, we argue that solving this singular issue alone fails to provide a comprehensive robustness assessment. Even worse, the conclusions drawn from singular robustness may give a false sense of overall model robustness. Specifically, our findings show that adversarially trained models that are robust to input perturbations are still (or even more) vulnerable to weight perturbations when compared to standard models. In this paper, we formalize the notion of non-singular adversarial robustness for neural networks through the lens of joint perturbations to data inputs as well as model weights. To our best knowledge, this study is the first work considering simultaneous input-weight adversarial perturbations. Based on a multi-layer feed-forward neural network model with ReLU activation functions and standard classification loss, we establish error analysis for quantifying the loss sensitivity subject to \( \ell_\infty \)-norm bounded perturbations on data inputs and model weights. Based on the error analysis, we propose novel regularization functions for robust training and demonstrate improved non-singular robustness against joint input-weight adversarial perturbations.

Index Terms— adversarial example, input perturbation, non-singular adversarial robustness, neural network, weight perturbation

1. INTRODUCTION

Despite recent success achieved by machine learning in a variety of tasks such as object recognition, semantic segmentation, speech recognition and so on, classifiers or predictors remain to perform destructively under the presence of manipulated data subject to perturbations that are imperceptible to human, known as adversarial examples [1][2]. Adversarial examples have been the crux of many attack and defense algorithms tending towards a more adversarially robust model. The notion and mathematical framework of attacks and defenses spins off with the development of such algorithms [3][4].

Specifically, adversarial examples are often generated from unperturbed data within a norm-ball of radius \( \epsilon \). Moreover, the robustness of a model is largely defined as the minimum perturbation that the input could make so as to change a network’s correct output [5][6]. In [7], the definition is taken to be modified in order to fit weight (model parameters) perturbation, another type of attack that could cause model to ill-perform. We note that considering input or weight perturbation alone is myopic and incomplete, as it only contributes to singular adversarial robustness assessment. For further reasoning, in Section 4 (Fig. 1), we show that models trained under only input perturbation would still suffer when encountering weight perturbation, and vice versa, which suggests that those two singular robustness results poses the risk of offering limited, or even false, sense of the comprehensive model robustness.

This paper bridges the gap by formalizing non-singular adversarial robustness of neural networks and studying simultaneous input-weight perturbations. We develop a novel margin bound analysis on the classification loss for multi-layer neural networks with ReLU activations. Moreover, based on the analysis, we propose a new loss function towards training robust neural networks against joint input-weight perturbation and validate its effectiveness via empirical experiments. We summarize our contributions as follows.

- We study non-singular robustness of neural network using the worst-case bound on pairwise class margin function against joint perturbations in neural networks (Theorems 1 and 2).
- We propose a theory-driven approach for training non-singular adversarial robust neural networks, including fusing weight perturbation into conventional adversarial training on data inputs [8].
- We validate our findings via empirical comparisons with standard and singular adversarial robust neural networks.

2. RELATED WORKS

Recent findings showed that a well-trained neural network can fail catastrophically when adversarial examples are present. Such adversarial examples can be found by searching within an \( \ell_p \) norm-ball of radius \( \epsilon \) using gradient-based approaches [2][9][10][11][12][13] or simply using prediction outputs [14][15][16]. Several attack and defense methods were proposed afterwards for studying adversarial robustness. The state-of-the-art robust model presented by [8] is composed by a procedure known as adversarial training, where the model weights are updated with the aim of minimizing the worst-case adversarial perturbations, forming a min-max training objective. [17] further proves the convergence of such training process.

Beyond input perturbations, [18][19] proposed fault-injection attacks which perturb the model parameters stored in memory by physically flipping the logical bits of the memory storage. [20][21] studied weight perturbations applied on the internal architecture for generalization. [7] showed that by taking weight sensitivity into account, the model could maintain its performance after weight quantization. Furthermore, [22] demonstrated that by taking advantages of mode connectivity of the model’s parameters, one could mitigate or preclude the attacks based on weight perturbations. Given the above results, we note that perturbations applied on input or weight has been discussed explicitly but separately, while joint attack remains ambiguous. Meanwhile, it is worth mentioning that adversarial training subject to weight perturbation is not meaningful since the min-max formulation would all be taking place in model’s parameter space. In this work, we consider directly when input and weight are both perturbed and prove bounds towards training a non-singular adversarial robust neural network against joint perturbations.
3. MAIN RESULTS

In this section we offer an overview of the presentation for our main results as follows. We first define in Section 3.1 the mathematical notation and preliminary used in this paper. In Section 3.2 we introduce the analysis of classification error by first considering a motivating example of a 4-Layer feed-forward neural network, and then further diving into the margin bound of error in the general case. In Section 3.3 we proceed to develop a theory-driven loss function.

3.1. Notations and Preliminary

We start by offering some mathematical notations used in this paper. Let \([L]\) be the set containing all positive integers smaller than \(L\); namely, \([L] := \{1, 2, \ldots, L\}\). We write the indicator function as \(\mathbb{1}(E)\) which outputs 1 when \(E\) occurs and 0 otherwise. As for notations of vectors, we use boldface lowercase letter (e.g., \(x\)) and the \(i\)-th element is marked as \(x_i\). On the other hand, matrices are denoted by boldface uppercase letter, for example \(W\). Given a matrix \(W \in \mathbb{R}^{m \times n}\), we write its \(i\)-th row, \(j\)-th column and \((i, j)\) element as \(W_{i,:}, W_{:,j}\) and \(W_{i,j}\) respectively. The matrix \((\alpha, \beta)\) norm is written as \(\|W\|_{\alpha, \beta}\). In the following sections, we would adopt the notion of vector-induced norm upon mentioning \((\alpha, \beta)\) norm of a given matrix \(W\); namely, we have \(\|W\|_{\alpha, \beta} = \max_{x \in \mathbb{R}^n} \frac{\|Wx\|_\alpha}{\|x\|_\beta}\). We may use the shorthand notation \(\|p\|_p := \|p\|_p,\) .

Furthermore, we use the notion of \(B^p_W(e)\) to express an element-wise \(\ell_\infty\) norm ball for both matrix and vector. Specifically, given a matrix \(W \in \mathbb{R}^{m \times n}\) and vector \(x \in \mathbb{R}^n\), we could define the norm ball as \(B^p_W(e) := \{W \mid \|W_{i,j} - W_{i',j'}\|_\infty \leq e, \forall i \in [m], j \in [n]\}\) and \(B^p_x(e) := \{x \mid \|x_j - x_{j'}\|_\infty \leq e, \forall j \in [n]\}\).

Preliminary In order to formally state our results, we start by defining the notion for feed-forward neural networks and laying introduction to a few related quantities. We study multi-classification problem with number of classes being \(K\) in this paper and consider an input vector \(x \in \mathbb{R}^d\), an \(L\)-layer neural network is defined as

\[ f_W(x) = W^L \rho(W^{L-1} \ldots \rho(W_1 x)) \in \mathbb{R}^K \]

with \(W\) being the set containing all weight matrices (i.e. \(W := \{W_i \mid \forall i \in [L]\}\) while \(\rho()\) stands for non-negative monotone activation functions applied element-wise on a vector and is assumed to be 1-Lipschitz which includes popular functions like ReLU and Sigmoid. We further introduce some quantities related to neural networks. The \(i\)-th component of neural network’s output is written as \([f_W(x)]_i\) and we denote the pairwise margin, the difference between two classes \(i, j\) in output of the neural network, as \(f_W^i(x) := [f_W(x)]_i - [f_W(x)]_j\). Finally, we express the output of \(k\)-th \((k \in [L - 1])\) layer-puttered setting as \(z_W^k := \rho(W^{k+1} \ldots \rho(W_1 x)), W_m \in \mathbb{R}^{m \times n} \forall m \in [k]\) and \(z_W := \rho(W^1 \ldots \rho(W^L x))\) where \(x \in \mathbb{R}^n(e_{x})\) respectively.

3.2. Case Study: Joint input and single-layer perturbation

The sensitivity of neural network in study through the lens of pairwise margin bound, \(f_W^i(x)\), especially when \(i\) and \(j\) corresponds to the top-1 and the second-top prediction of \(x\). We note that the margin bound in the above previous can be utilized as an indicator of robustness given a neural network. For simplicity, we consider a motivating example with 4-Layer neural network and explain the margin bound through the propagation of error under joint perturbation. We write the neural network \(f_W(x)\) as

\[ f_W(x) = W^4 \rho(W^3 \rho(W^2 \rho(W^1 x))) \]

where \(W^i\) is the weight matrix for the \(i\)-th layer and assuming that one could perturb any element in the second weight matrix \(W^2\) by \(e_2\) and any element in the input \(x\) by \(e_1\). Namely, we have \(W^i \in B_{\infty, 2}^e(e_2)\) and \(x \in B_\infty(e_1)\). We define the notion of an error vector \(e\), as the entry-wise error after propagating through the \(i\)-th layer under only weight perturbation. Consider first the scenario of single-layer weight perturbation, where the second layer is perturbed, then for any input \(x\), since no perturbation happened prior to the second layer, we can take the output after the first layer and derive an upper bound for entries in the error vector \(e_2\) as

\[
\begin{align*}
|e_2|_i := |W^4_{i,j} z^4_W - W^4_{i,j} z^4_{W^0}| & \leq \sum_j |W^2_{i,j} - W^2_{i,j}| |z^3_W|_j | \quad (1) \\
& \leq \sum_j e_2 |z^3_W|_j | \quad (2) \\
& = e_2 |z^3_W|_1 | \quad (3)
\end{align*}
\]

We next consider each subsequent error vector by the process of propagation. Since no layer after the considered layer is being perturbed, we simply take the magnitude of each element in subsequent weight layer to calculate entries of error vector. Thus, we have that

\[
\begin{align*}
|e_3|_i = \sum_j |W^3_{i,j}| |e_2|_j & \leq e_2 |z^2_W|_1 \sum_j |W^3_{i,j}|, \quad (4)
\end{align*}
\]

With propagation through layers, we arrive at the final layer and are able to evaluate error induced by perturbations. Recall the pairwise margin bound \(f_W^i(x)\), we could derive an upper bound using relative error between entries. Specifically, for any two classes \(e_1\) and \(e_2\), we have the relative error in \(e_1\) as

\[
\begin{align*}
|e_1|_{e_1} - |e_1|_{e_2} & = \sum_k |W^4_{c_1,k} - W^4_{c_2,k}| |e_3|_k | \quad (5) \\
& \leq e_2 |z^3_W|_1 \sum_k |W^3_{c_1,k} - W^3_{c_2,k}| |e_3|_k | \quad (6) \\
& \leq e_2 |z^3_W|_1 \max_k |W^3_{c_1,k}| \sum_k |W^4_{c_1,k} - W^4_{c_2,k}| |e_3|_k | \quad (7) \\
& = e_2 |z^3_W|_1 |z^3_W|_\infty |z^4_W - W^4_{c_2,k}|_1 | \quad (8)
\end{align*}
\]

From the above example, we could see that the upper bound of relative error would be propagating at the rate of weight matrices’ \(\ell_\infty\) norm. Thus far, we have derived an upper bound for relative error under single-layer weight perturbation. We proceed to include input perturbation on the basis of single-layer weight perturbation. We denote the error vector \(e_1\) as the entry-wise error after propagating through the \(i\)-th layer under joint perturbation. We can write the first error vector \(e_1\) as

\[
\begin{align*}
|e_1'|_i := |W^4_{i,j} x - W^4_{i,j} x| & \leq \sum_j |W^4_{i,j}| |x_j - x_j| | \quad (9) \\
& \leq e_1 \max_j \sum_j |W^4_{i,j}| \quad (10) \\
& = e_1 |W^4|_\infty | \quad (11)
\end{align*}
\]

The second error vector consists of previous error vector under
weight perturbation and the first error vector. Namely, we can write
\[
[e_2]_i = [e_2]_i + \sum_j |W^2_{i,j}| |[e_1]_j|
\]
and proceed to calculate the relative error in \(e_1\) between two classes \(c_1\) and \(c_2\) as
\[
[e_1]_{c_1} - [e_1]_{c_2} = \sum_k |W^3_{c_1,k} - |W^3_{c_2,k}| |[e_1]_k|
\]
\[
\leq \eta \sum_k |W^3_{c_1,k} - W^3_{c_2,k}| \sum_i |W^3_{i,j}|
\]
\[
\leq \eta \max_k |W^3_{c_1,k} - W^3_{c_2,k}| \sum_i |W^3_{i,j}|
\]
\[
= \eta \sum_i |W^3_{i,j}| |[e_1]_k - |W^3_{i,j}||
\]
where \(\tau_W^i(\xi)\) can be expressed as
\[
\tau_W^i(\xi) = \varepsilon_n \left( \sum_{i \neq j} \left| W^L_{i,j} - W^L_{j,i} \right| \right) + 2d \varepsilon_L \Pi_{m=1}^{L-1} \| W^m \|_\infty + \varepsilon_m d_m\]
while \(\zeta_W^i(x, \xi)\) possesses the following form
\[
\zeta_W^i(x, \xi) = \sum_{i \neq j} \left| W^L_{i,j} - W^L_{j,i} \right| \left( \varepsilon_n \| W^m \|_\infty \right) \left( \varepsilon_{L-1} \| W^{L-1} \|_\infty \right)
\]
with \(W^m_{i,j} = W^m_{i,j} + \varepsilon_m, \forall i, j \) and \(\forall m \in [L] \setminus \{1\}\).

**Proof:** See Appendix A.2

### 3.3. Theory-inspired loss towards non-singular robustness

With our theoretical insights on margin bound, we now propose a new regularization function towards training a non-singular adversarial robust neural network. Specifically, consider the new loss function in the following form:
\[
\ell'(f_W(x), y) = \ell_{cLS}(f_W(x), y) + \alpha \max \{\tau_W^y(\xi)\}
\]
\[
+ \beta \max \{\zeta_W^y(x, \xi)\}
\]
where the first term \(\ell_{cLS}\) corresponds to standard (or adversarial) classification loss, while the second and third term regularizes perturbation sensitivity to input and weight space with nonnegative coefficients \(\alpha\) and \(\beta\), respectively. They are inspired from Theorem 2 and can be interpreted as the maximum error on pairwise margin induced by joint input-weight perturbations. Specifically, each regularizer alone corresponds to singular sensitivity, while their mixture governs non-singular adversarial robustness.

### 4. EXPERIMENTS

#### 4.1. Experiment Setup

We used the MNIST image classification dataset containing 10 handwritten digit categories. We trained neural network models with four dense layers (number of neurons are 128-64-32-10) and the ReLU activation function without the bias term. For comparison, five different training methods using the training loss in (23) are presented in our experiments: (i) Standard Model, (ii) Weight Perturb, (iii) Adversarial Training (AT) [8], (iv) Adversarial Training with additional \(\beta\)-term regularization (AT + \(\beta\)), and (v) Joint Input-Weight Perturb (JWP). To obtain reasonable accuracy on the unperturbed testing data, we have tuned the models with weight and input perturbation levels \(\varepsilon_w\) and \(\varepsilon_{x}\) and regularization coefficients \(\alpha\) and \(\beta\) for each model. For the standard model, we used the cross entropy (CE) for \(\ell_{cLS}\) with \(\alpha = \beta = 0\). For the weight perturb model, we used the CE loss function \(\ell' = (\alpha = \beta = 0.25)\) with \(\varepsilon_{w} = 0.01\) and \(\varepsilon_{x} = 0\). For AT, we followed the same min-max training setting with CE loss
Fig. 1. Comparison of test accuracy contour of neural networks under joint input-weight PGD attack (100 steps) with varying input ($\epsilon_x$) and weight ($\epsilon_w$) perturbation levels. AUC refers to the area under curve scores. Comparing to the standard model (a), singular robust models (b) and (c) have comparable or even worse robustness under their respective untrained perturbation type. Non-singular robust models using our proposed regularization function, including (d), (e) and (f), show significantly better AUC scores.

4.2. Performance Evaluation

For non-singular robustness evaluation, we generalize the projected gradient descent (PGD) attack [8] for input perturbation to joint input-weight perturbation, by simultaneously computing the signed gradient of the CE loss with respect to the data input and the model weight, clipping the perturbation within their respective $\ell_\infty$ ball constraints, and iterate this process for 100 steps with step sizes $\alpha_X = 0.01$ and $\alpha_W = 0.0005$. We describe this joint PGD attack as follows. Given an input $X$ and a trained neural network weight $W$, the perturbed weight $\tilde{W}$ and input $\tilde{X}$ are crafted by iterative gradient ascent using the sign of gradient of the CE loss marked as $\text{sgn}(\nabla_{W,X} \ell_{\text{cls}}(f_{\tilde{W}}(\tilde{X}), y))$. The attack iteration with step sizes $\alpha_W$ of weight and $\alpha_X$ of input is formalized as

$$\tilde{W}^{(0)} = W,$$

$$\tilde{W}^{(t+1)} = \text{Clip}_{W,\epsilon_w} \left\{ \tilde{W}^{(t)} + \alpha_W \text{sgn}(\nabla_{W,X} \ell_{\text{cls}}(f_{\tilde{W}}^{(t)}(\tilde{X}^{(t)}), y)) \right\}$$

Fig. 1 demonstrates the non-singular robustness performance for each model. The standard model (a) is vulnerable to both weight and input perturbations. Singular robust models (b) and (c) are only robust to the seen perturbation type, while they only have comparable or even worse robustness against unseen perturbation type. For example, AT (model (c)) is only trained on input perturbation and is observed to be less robust under weight perturbation compared to the standard model. Similarly, the robustness of weight perturb model (b) to input perturbation is only slightly better than the standard model. The results suggest the insufficiency of singular robustness analysis. Comparing the area under curve (AUC) score of test accuracy, non-singular robust models (bottom row, (d)-(f)) using our proposed loss significantly outperform standard and singular robust models (top row). The AUC of best $\text{AT}+\beta$ model (e) improves that of AT by about 24%, validating the effectiveness of our proposed regularizer. $\text{AT}+\beta$ also attains better AUC than JIWP, suggesting that min-max training is crucial to non-singular robustness.
5. CONCLUSION

In this paper, we analyze the robustness of pairwise class margin for neural networks against joint input-weight perturbations. A theory-inspired regularizer is proposed towards training comprehensive robust neural networks. Empirical results against joint input-weight perturbations show that singular robust models can give a false sense of overall robustness, while our proposal can significantly improve non-singular adversarial robustness and offer thorough evaluation.

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A. PROOF OF THEOREMS

A.1. Theorem 1: Single-Layer Bound

We shall first prove when \( N \neq L \) and follow similar reasoning to prove the case when \( N = L \). Consider the difference between set of pairwise margin \( f_{ij}^W(x) - f_{ij}^{\hat{W}}(x) \), we have

\[
f_{ij}^{\hat{W}}(x) - f_{ij}^W(x) = f_{ij}^{\hat{W}}(x) - f_{ij}^W(x) + f_{ij}^W(x) - f_{ij}^W(x)
\]

\[
(\alpha) \leq \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W} - \rho(W_{j}^{L-1}\hat{z}^{L-2}_{W})) \right\| _{\infty} + \epsilon_{N} \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \Pi_{k=1}^{L-1} \left\| (W^{L-k})^{T} \right\| _{1,\infty} \right.
\]

\[
(\beta) \leq \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W} - \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W})) \right\| _{\infty} + \epsilon_{N} \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \Pi_{k=1}^{L-1} \left\| (W^{L-k})^{T} \right\| _{1,\infty} \right.
\]

\[
(\gamma) \leq \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W} - \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W})) \right\| _{\infty} + \epsilon_{N} \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \Pi_{k=1}^{L-1} \left\| (W^{L-k})^{T} \right\| _{1,\infty} \right.
\]

\[
(\delta) \leq \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W} - \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W})) \right\| _{\infty} + \epsilon_{N} \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \Pi_{k=1}^{L-1} \left\| (W^{L-k})^{T} \right\| _{1,\infty} \right.
\]

\[
(\epsilon) \leq \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W} - \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W})) \right\| _{\infty} + \epsilon_{N} \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \Pi_{k=1}^{L-1} \left\| (W^{L-k})^{T} \right\| _{1,\infty} \right.
\]

where inequality (a) results from applying Hölder inequality, and inequality (b) comes from the contractive property (1-Lipschitz) of activation function \( \rho(\cdot) \). Inequality (c) and (d) come from triangle inequality applied element-wise on vector \( W_{i}^{L-1}(\hat{z}^{L-2}_{W} - \hat{z}^{L-2}_{W}) \) combined with iteration while inequality (e) comes from the constraint of \( \epsilon_{N} \) and \( \epsilon_{x} \).

With analogous analysis, we proof the event when \( N = L \) as following

\[
f_{ij}^{\hat{W}}(x) - f_{ij}^{W}(x) = f_{ij}^{\hat{W}}(x) - f_{ij}^{W}(x) + f_{ij}^{W}(x) - f_{ij}^{W}(x)
\]

\[
(\alpha') \leq \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W} - \rho(W_{j}^{L-1}\hat{z}^{L-2}_{W})) \right\| _{\infty} + 2\epsilon_{L} \left\| (W^{L-1})^{T} \right\| _{1,\infty}
\]

\[
(\beta') \leq \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W} - \rho(W_{j}^{L-1}\hat{z}^{L-2}_{W})) \right\| _{\infty} + 2\epsilon_{L} \left\| (W^{L-1})^{T} \right\| _{1,\infty}
\]

\[
(\gamma') \leq \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W} - \rho(W_{j}^{L-1}\hat{z}^{L-2}_{W})) \right\| _{\infty} + 2\epsilon_{L} \left\| (W^{L-1})^{T} \right\| _{1,\infty}
\]

\[
(\delta') \leq \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\| \rho(W_{i}^{L-1}\hat{z}^{L-2}_{W} - \rho(W_{j}^{L-1}\hat{z}^{L-2}_{W})) \right\| _{\infty} + 2\epsilon_{L} \left\| (W^{L-1})^{T} \right\| _{1,\infty}
\]

where inequality (i) comes from problem definition (within element-wise \( \ell_{\infty} \) norm ball) and since the activation function \( \rho(\cdot) \) is non-negative, we could transform the inner product to its \( \ell_{1} \) norm. Additionally, inequality (ii) can be easily derived from iterating through the weight matrices and applying the setting of input perturbation \( \epsilon_{x} \).

A.2. Theorem 2: Multi-Layer Scenario

In the following proof for Theorem 2 we apply similar steps in Appendix A.1 introduce one lemma in order to help with the proof of Theorem 2 and consider the difference between set of pairwise margin under natural and weight perturbation setting. Firstly, we have the following lemma for weight perturbation.

Lemma 1 (Perturbation of Pure Weight) Let \( f_{W}(x) = W^{L}(\ldots \rho(W^{1}x)\ldots) \) denote an \( L \)-layer (natural) neural network and let \( f_{\hat{W}}(x) = \hat{W}^{L}(\ldots \hat{W}^{1}x)\ldots \) with \( \hat{W}^{k} \in \mathbb{B}_{w,1}(\epsilon_{k}), \forall k \in [L] \), denote its perturbed version. For any set of pairwise margin \( f_{ij}^{W}(x) \) and \( f_{ij}^{\hat{W}}(x) \), we have

\[
f_{ij}^{\hat{W}}(x) \leq f_{ij}^{W}(x) + \left\| W_{i}^{L} - W_{j}^{L} \right\| _{1} \left\{ \epsilon_{1} \left\| x \right\| _{1} \Pi_{k=1}^{L-2} \left\| (W^{L-k})^{T} \right\| _{1,\infty} \right.
\]

\[
+ \sum_{k=1}^{L-3} (\Pi_{m=k+2}^{L-1} \left\| W_{m}^{\infty} \right\| _{1}) \epsilon_{k+1} \left\| h^{k+2} \right\| _{1} + \epsilon_{L-1} \left\| h^{L-2} \right\| _{1} \left. \right\} + 2\epsilon_{L} \left\| h^{L-1} \right\| _{1}
\]

\[
= f_{ij}^{W}(x) + \zeta_{ij}^{W}(\epsilon_{x}, \epsilon_{1})
\]

where \( h^{k+2} = \rho(W^{k+1}x) \) with

\[
W_{i}^{m} = W_{i}^{m} + \epsilon_{m}, \forall i, j and \forall m \in [L] \setminus \{1\}
\]

\[
W_{i}^{1} = W_{i}^{1} + \text{sgn}(x_{i}) \epsilon_{1}, \forall i
\]
Proof:

\[
f^{ij}_{\hat{\mathbf{W}}} (\mathbf{x}) - f^{ij}_{\mathbf{W}} (\mathbf{x}) = \{ \hat{W}^L_{i,:} - W^L_{i,:} \} \mathbf{h}^{L-1} - \{ W^L_{i,:} - \hat{W}^L_{i,:} \} \mathbf{h}^{L-1} \tag{32}
\]

\[
\leq \left\| W^L_{i,:} - \hat{W}^L_{j,:} \right\|_1 \left\| \rho(\hat{W}^{L-1} \mathbf{h}^{L-2}) - \rho(W^{L-1} \mathbf{h}^{L-2}) \right\|_{\infty} + 2\epsilon_L 1^T \mathbf{h}^{L-1} \tag{33}
\]

\[
\leq \left\| W^L_{i,:} - \hat{W}^L_{j,:} \right\|_1 \left\{ \left\| W^{L-1} (\mathbf{h}^{L-2} - \mathbf{h}^{L-2}) \right\|_{\infty} + \left\| (W^{L-1} - \hat{W}^{L-1})(\mathbf{h}^{L-2}) \right\|_{\infty} \right\} + 2\epsilon_L \left\| \mathbf{h}^{L-1} \right\|_1 \tag{34}
\]

\[
\leq \epsilon_L \left\{ \left\| (W^{L-1})^T \right\|_{1,\infty} + \sum_{j=1}^{L-3} \left( \Pi_{k=j+2}^{L-1} \left\| (W^k)^T \right\|_{1,\infty} \right) \epsilon_{j+1} \right\} + 2\epsilon_L \left\| \mathbf{h}^{L-1} \right\|_1 \tag{35}
\]

In the above proof for lemma, inequality (a) comes from the problem definition and (b) stems from the contractive property of \( \rho(\cdot) \) combined with triangle inequality. One could achieve (c) through triangle inequality. By induction and maximization of the \( \ell_1 \) norm of perturbed output under weight perturbation \( \hat{\mathbf{z}}^L \), we could attain inequality (d) and (e).

Thus for any set of pairwise margin \( f^{ij}_{\hat{\mathbf{W}}} (\tilde{\mathbf{x}}) \) and \( f^{ij}_{\mathbf{W}} (\tilde{\mathbf{x}}) \), we have

\[
f^{ij}_{\hat{\mathbf{W}}} (\tilde{\mathbf{x}}) - f^{ij}_{\mathbf{W}} (\tilde{\mathbf{x}}) \leq \left\| W^L_{i,:} - \hat{W}^L_{j,:} \right\|_1 \left\{ \Pi_{k=1}^{L-1} \left\| W^k \right\|_{\infty} \right\} \epsilon_X + \zeta^{ij}_{\hat{\mathbf{W}}} (\mathbf{x}, \xi) \tag{38}
\]

\[
\leq \epsilon_X \left\{ \left\| W^L_{i,:} - \hat{W}^L_{j,:} \right\|_1 + 2d_L \epsilon_L \right\} \Pi_{m=1}^{L-1} \left\| W^m \right\|_{\infty} + d_m \epsilon_m + \zeta^{ij}_{\hat{\mathbf{W}}} (\mathbf{x}, \xi) \tag{39}
\]

Inspecting the above proof, inequality (a) results from separating and applying Lemma[1] and Hölder Inequality. In the other hand, by iterating through the perturbed matrix one could derive inequality (b). Lastly, by applying the constraints on perturbation radius \( \epsilon_m \) for all layer \( m \) and \( \epsilon_X \) for the input, we would arrive at the results.