SLS (Single $\ell_1$ Selection): a new greedy algorithm with an $\ell_1$-norm selection rule

Ramzi Ben Mhenni, Sébastien Bourguignon and Jérôme Idier.

1 École Centrale de Nantes - CNRS, Laboratoire des Sciences du Numérique de Nantes, 1 rue de la Noë, F-44321 Nantes, France.

Abstract—In this paper, we propose a new greedy algorithm for sparse approximation, called SLS for Single $\ell_1$ Selection. SLS essentially consists of a greedy forward strategy, where the selection rule of a new component at each iteration is based on solving a least-squares optimization problem, penalized by the $\ell_1$ norm of the remaining variables. Then, the component with maximum amplitude is selected. Simulation results on difficult sparse deconvolution problems involving a highly correlated dictionary reveal the efficiency of the method, which outperforms popular greedy algorithms and Basis Pursuit Denoising when the solution is sparse.

1 Introduction

We consider the cardinality-constrained least-squares problem:

$$
\min_{x} \frac{1}{2} \|y - Ax\|_2^2 \quad \text{subject to} \quad \|x\|_0 \leq K,
$$

where $y \in \mathbb{R}^N$, $x \in \mathbb{R}^M$, $A$ is a known matrix or dictionary, and $\|x\|_0$ is the $\ell_0$ "norm": $\|x\|_0 := \text{Card}\{x_i \neq 0\}$. Problem (1) is encountered in many sparse approximation problems occurring in inverse problems [1] [2] [3], denoising, compression [4], or subset selection in Statistics [5].

Finding such best $K$-sparse solution is NP hard [6], therefore most works in signal processing and statistics have concentrated on developing computationally efficient, suboptimal, algorithms [7]. Forward greedy algorithms such as Orthogonal Matching Pursuit (OMP) [8] and Orthogonal Least Squares (OLS) [9] iteratively add new components to an initially empty model, then providing a $K$-sparse approximation in no more than $K$ iterations. However, the selection step of a new atom in such methods is highly sensitive to interferences between the dictionary atoms, in particular in the case of highly correlated dictionaries [10]. Convex optimization strategies in which the $\ell_0$ norm in problem (1) is replaced by the $\ell_1$ norm $\|x\|_1 := \sum_n |x_n|$ (which is known as the LASSO in Statistics [11]), is another widespread approach, for which many dedicated algorithms have been proposed in the past twenty years. Optimizing all variables together in a convex approach then brings more robustness toward the aforementioned interferences, but the solution may often contain undesired nonzero components with small amplitudes.

In this paper, we propose an algorithm which gathers advantages of the two classes of methods. It essentially consists of a greedy strategy, where the selection rule at each iteration is based on exploiting $\ell_1$-norm solutions. The number of iterations is then controlled by the sparsity level $K$ of the searched solution, limiting the computational burden. Moreover, the selection of each new atom, based on solving a convex optimization problem, is expected to be more robust to interferences between the different atoms than standard greedy methods.

2 Forward greedy algorithms

Forward greedy methods start from an empty set and iteratively construct a sparse solution. Let $\mathcal{S}$ denote the index set of the variables already selected (the current support of the solution, with $k$ components) and let $\bar{\mathcal{S}}$ index the remaining $K$ variables. In the following, $A_{\mathcal{S}}$ denotes the matrix composed of the columns of $A$ indexed by $\mathcal{S}$. Similarly, $x_{\mathcal{S}}$ is the vector collecting the elements of $x$ indexed by $\mathcal{S}$. The component of $A$ (resp. the elements of $x$) indexed by $\mathcal{S}$. The principle of forward selection algorithms is given in Algorithms 1.

Algorithm 1: Forward Selection greedy algorithm

1. Initialization: $\mathcal{S} = \emptyset$
2. while $|\mathcal{S}| < K$ do
3. Variable selection: $j = \arg\max_j \mathcal{F}(j)$
4. Support update: $\mathcal{S} \leftarrow \mathcal{S} \cup \{j\}$
5. (If needed) update the estimate $x_{\mathcal{S}}$ and the residual $r = y - A_{\mathcal{S}}x_{\mathcal{S}}$
6. end
7. Result: Support $\mathcal{S}$ and solution $x_{\mathcal{S}}$

In the sequel, we suppose that all columns in $A$ have unit norm. For OMP, the selected atom is the most correlated to the residual:

$$
\mathcal{F}_{\text{OMP}}(j) = |a_j^T r|, \quad j \in \mathcal{S},
$$

where $a_j$ is the $j$-th column of $A$. OMP includes an additional orthogonalization step of the solution on its support by:

$$
x_{\mathcal{S}} = \arg\min_{x_{\mathcal{S}}} \|y - A_{\mathcal{S}}x_{\mathcal{S}}\|_2^2 = A_{\mathcal{S}}^+ y,
$$

where $A_{\mathcal{S}}^+ := (A_{\mathcal{S}}^T A_{\mathcal{S}})^{-1} A_{\mathcal{S}}^T$ denotes the pseudo-inverse of $A_{\mathcal{S}}$. For OLS [9], the approximation error is minimized among all possible supports including one new component:

$$
\tilde{j} = \arg\min_{j \notin \mathcal{S}} \|y - A_{\mathcal{S}\cup\{j\}}x_{\mathcal{S}\cup\{j\}}\|_2^2,
$$

which amounts to

$$
\mathcal{F}_{\text{OLS}}(j) = \|A_{\mathcal{S}\cup\{j\}} A_{\mathcal{S}\cup\{j\}}^+ y\|_2^2.
$$

Restricting the selection step to models involving no more than one new component is a major limitation of such greedy algorithms. Let us consider the sparse deconvolution problem, where $A$ is composed of shifted versions of the impulse response of the filter. In the toy example of Figure 1, $x$ is composed of two close spikes, giving strongly overlapping echoes in the data $y$. The score function for the first iteration of both OMP and OLS is $\mathcal{F}(j) = |a_j^T y|$, and is shown in Figure 1 (c). It is maximal for the index located in the middle of the two true indices, thus selecting a wrong atom.
3 Single $\ell_1$ Selection

We propose to select a new atom by considering the following $\ell_1$-norm optimization problem at each iteration:

$$
\min_{x_S \in \mathbb{R}^K, x_{\bar{S}} \in \mathbb{R}^N} \frac{1}{2} \| y - A_S x_S - A_{\bar{S}} x_{\bar{S}} \|_2^2 + \lambda \| x_{\bar{S}} \|_1, \quad (3)
$$

Similarly to OLS, such a criterion allows the re-estimation of the amplitudes of previously selected components $x_S$ (whereas they are fixed in the residual for the selection rule of OMP) within the selection step. It also jointly estimates a sparse vector $x_{\bar{S}}$ for the remaining ones, which is not restricted to a single non-zero component as in OLS. Note that for a given $x_{\bar{S}}$, the solution in $x_S$ reads $x_S = A_S^\dagger (y - A_{\bar{S}} x_{\bar{S}})$, therefore the problem in Eq. (3) can be recast as an optimization problem in $x_{\bar{S}}$ only:

$$
\hat{x}_{\bar{S}} = \arg \min_{x_{\bar{S}} \in \mathbb{R}^N} \frac{1}{2} \| y - \tilde{A}_{\bar{S}} x_{\bar{S}} \|_2^2 + \lambda \| x_{\bar{S}} \|_1 \quad (4)
$$

where $P := I_K - A_S A_S^\dagger$, $\tilde{y} := P y$, and $\tilde{A}_{\bar{S}} := P A_{\bar{S}}$.

Then, the new component is selected as the component in $\hat{x}_{\bar{S}}$ with maximum amplitude value:

$$
F_{SLS}(j) = |\hat{x}_{\bar{S}}|_j, \quad j \in \bar{S}. \quad (5)
$$

In this paper, the solution to (4) is computed by the homotopy algorithm [11][12], which iteratively solves (5) for decreasing values of parameter $\lambda$, starting from $\lambda_{\text{max}} := \max_{i \in \bar{S}} |\tilde{a}_i^T \tilde{y}|$, which is the minimum value of $\lambda$ above which the solution is identically zero. Here, we use a stopping rule that controls the number of nonzero components in $\hat{x}_{\bar{S}}$. More precisely, at a given iteration of the forward greedy procedure we impose that

$$
\| \hat{x}_{\bar{S}} \|_0 = 3(K - \text{Card}(S)),
$$

where $K - \text{Card}(S)$ represents the number of non-zero components that still must be determined. Such an empirical rule limits the computation time of the homotopy method, while ensuring that enough components are present in the model in order to overcome the interference issues explained in Section 2.

The SLS selection rule is illustrated on the toy example of Section 2 in Figure 1. At the first iteration of the SLS algorithm, the scoring function is maximum for a true component—it also shows non-zero, but smaller, value at the index erroneously selected by OMP and OLS, see Figure 1(e). Then, after two iterations, the true support is correctly estimated, as shows Figure 1(f).

\section*{4 Simulation results}

We evaluate the performance of the SLS algorithm, compared to several well-known sparse estimation algorithms: OMP [8], OLS [9], $\ell_1$-norm regularization or BPDN (Basis Pursuit DeNoising), also computed here by the homotopy algorithm [11], SBR [10], Subspace Pursuit [13], accelerated Iterative Hard Thresholding (IHT) [14] and $A^\dagger$OMP [15]. All algorithms are implemented in Matlab and are tuned such that all solutions have the true sparsity level $K$.

Algorithms are tested on difficult sparse deconvolution problems, with an up-sampled convolution model in order to achieve high-resolution spike locations [16]. Problems are underdetermined with $M = 1,000$ and $N = 350$. Columns of $A$ are then highly correlated, with mutual coherence $\max_{i \neq j} |a_i^T a_j| = 0.81$. White Gaussian noise $\epsilon$ is then added with $\text{SNR}_{\text{dB}} = 10 \log \| A \epsilon \|_2^2 = 20 \text{ dB}$. Results are averaged over 50 random realizations of the sparse sequence and of noise. A typical signal is shown in Figure 2.

Figure 3 shows the average quadratic error (left), the exact recovery rate (the fraction of solutions which have the correct support) and the average computing time for all algorithms as a function of $K$. SLS clearly achieves the best performance in terms of solution quality, and has a lower computation time than OLS and SBR up to $K = 30$ and slightly higher for $K = 40$. Note that other fast algorithms (OMP, BP, IHT and SP) are always much faster than SLS—but always give worse solutions.

![Figure 1: a) Toy example where standard greedy selection fails. (a) Contribution of two atoms and (b) corresponding noise-free data. (c) Scoring function for OMP and OLS at their first iteration and (d) OLS solution after two iterations. (e) Scoring function at the first iteration of the proposed SLS algorithm and (f) SLS solution after two iterations. Circles (resp. stars) locate the true (resp. estimated) spikes.](image)

![Figure 2: Typical sparse deconvolution problem: sequence $x$ (.) and data $y$ (•).](image)

![Figure 3: Quadratic error (top left), exact recovery rate (top right) and computation time as a function of $K$ for SLS and other sparse approximation methods.](image)
References

[1] H. Taylor, S. Banks, and F. McCoy, “Deconvolution with the $l_1$ norm,” *Geophysics*, vol. 44, no. 1, pp. 39–52, Jan. 1979.

[2] J. Mendel, *Optimal seismic deconvolution: An estimation-based approach*, Monograph Series. Academic Press, 1983.

[3] C. A. Zala, “High-resolution inversion of ultrasonic traces,” *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, vol. 39, no. 4, pp. 458–463, July 1992.

[4] Y. Eldar and G. Kutyniok, *Compressed Sensing: Theory and Applications*, Cambridge University Press, 2012.

[5] A. Miller, *Subset selection in regression*, Chapman and Hall/CRC, 2002.

[6] B. K. Natarajan, “Sparse approximate solutions to linear systems,” *SIAM Journal on Computing*, vol. 24, no. 2, pp. 227–234, 1995.

[7] J. A. Tropp and S. J. Wright, “Computational methods for sparse solution of linear inverse problems,” *Proceedings of the IEEE*, vol. 98, no. 6, pp. 948–958, June 2010.

[8] Y. Pati, R. Rezaiifar, and P. S. Krishnaprasad, “Orthogonal matching pursuit: recursive function approximation with applications to wavelet decomposition,” in *Asilomar Conference on Signals, Systems and Computers*, 1993, pp. 40–44 vol.1.

[9] S. Chen, S. Billings, and W. Luo, “Orthogonal least squares methods and their application to non-linear system identification,” *International Journal of Control*, vol. 50, no. 5, pp. 1873–1896, 1989.

[10] C. Soussen, J. Idier, D. Brie, and J. Duan, “From Bernoulli Gaussian Deconvolution to Sparse Signal Restoration,” *IEEE Transactions on Signal Processing*, vol. 59, no. 10, pp. 4572–4584, 2011.

[11] R. Tibshirani, “Regression shrinkage and selection via the lasso,” *Journal of the Royal Statistical Society, Series B*, vol. 58, pp. 267–288, 1996.

[12] D. L. Donoho and Y. Tsaig, “Fast Solution of $l_1$ Norm Minimization Problems When the Solution May Be Sparse,” *IEEE Transactions on Information Theory*, vol. 54, no. 11, pp. 4789–4812, Nov. 2008.

[13] D. Needell and J. Tropp, “Cosamp: Iterative signal recovery from incomplete and inaccurate samples,” *Applied and Computational Harmonic Analysis*, vol. 26, no. 3, pp. 301 – 321, 2009.

[14] T. Blumensath and M. E. Davies, “Iterative Thresholding for Sparse Approximations,” *Journal of Fourier Analysis and Applications*, vol. 14, no. 5, pp. 629–654, Dec. 2008.

[15] N. B. Karahanoglu and H. Erdogan, “$A^*$ orthogonal matching pursuit: Best-first search for compressed sensing signal recovery,” *Digital Signal Processing*, vol. 22, no. 4, pp. 555 – 568, 2012.

[16] E. Carcreff, S. Bourguignon, J. Idier, and L. Simon, “Resolution enhancement of ultrasonic signals by up-sampled sparse deconvolution,” in *IEEE International Conference on Acoustic, Speech and Signal Processing*, Vancouver, Canada, May 2013.