Generalized Ehrenfest’s Equations and phase transition in Black Holes

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We generalize Ehrenfest’s equations to systems having two work terms, i.e. systems with three degrees of freedom. For black holes with two work terms we obtain nine equations instead of two to be satisfied at the critical point of a second order phase transition. We finally generalize this method to a system with an arbitrary number of degrees of freedom and found there is \( \frac{N(N+1)^2}{2} \) equations to be satisfied at the point of a second order phase transition where \( N \) is number of work terms in the first law of thermodynamics.

I. INTRODUCTION

Black hole thermodynamics has become a very important topic over the past few decades [1]. As a thermodynamic system, black holes undergo a phase transition shown by Davies for the Kerr-Newman black hole [2]. Using the path integral approach, Hawking and Page investigated Schwarzschild-AdS black holes and found the critical temperature and mass at which a phase transition occurs [3]. This subject has ever since been considered by a number of authors [4–18].

It has been shown that the thermal AdS space may collapse to form a black hole at certain temperatures [3]. This is an example of first order phase transition [17] and can also occur for the case of charged and Kerr-Newman black holes in the AdS space [18]. The authors of [17] argued that this kind of transition may also occur in flat and de Sitter spaces. They also showed that for black holes below the critical temperature of Hawking-Page phase transition, or the so called 'supercooled black holes', there is a line of first order phase transition in the phase diagram that terminates at a second order transition point.

The aim of this paper is to consider the critical point at which heat capacity diverges. In the laboratory thermodynamics of the second order phase transition, there is a finite discontinuity in the diagram of specific heat. But as it has been shown by Davies [2] that an infinite discontinuity also exist for the critical point in black hole’s thermodynamics. So, one can argue that it is not a second order phase transition analogous to the ones in ordinary thermodynamics [20].

Discontinuity of specific heat has made it interesting to study the critical phenomena at the divergence point [15, 17, 18]. Specific heat is approximated around the critical point to a power of temperature, and then the critical exponent \( \alpha \) is read of [17]. In [18] it was shown that for the case of charged black holes, the critical exponent does not depend on the dimensions of the space. Specific heat is also important for the study of local stability of black holes [21] in the sense that black hole is locally stable if its heat capacity is positive.

Here, we use the Clapeyron’s and Ehrenfest’s scheme to investigate the critical point. In [18] Clapeyron equation is used to study the coexistence line of first order phase transition. Ehrenfest’s equations must be satisfied at the critical point of a second order phase transition. The critical point is the one in which heat capacity is discontinuous and, as we will see, this is the case for a large class of black holes.

For some years, it was thought that Ehrenfest’s equations are not both satisfied at the divergent point of heat capacity; rather numerical investigation was used to show that black holes undergo a glassy phase transition [22, 23]. Later, examination of the divergence point of other parameters that appear in Ehrenfest’s equations showed that there was no violation in Ehrenfest’s equations, so that the transition is of second order [24]. In this paper, we check Ehrenfest’s equations in the point of phase transition in a large class of black holes. Also, we will obtain a generalized form of Ehrenfest’s equations for the case of black holes that have more than one work term in their associated first law of thermodynamics and check the relations for phase transition in a Kerr-Newman black hole. It is fascinating that all of the nine generalized Ehrenfest’s equations are consistently satisfied at the point of second order phase transition. Finally we generalized Ehrenfest equations to the case of arbitrary number of degrees of freedom.

This paper is organized as follows. In Section II, we review and check the validity of Ehrenfest’s equations for a large number of black holes with two degrees of freedom. In Section III, we generalize Ehrenfest’s equations for systems with three degrees of freedom and obtain nine equations to be satisfied at the point of second order phase transition. Then, we check their validity for Kerr-Newman black hole at the divergence point of specific heat in different ensembles. We generalize our approach to the case of \( N \) work terms in Section IV and conclude the paper in Section V.
II. SECOND ORDER PHASE TRANSITION FOR TWO PARAMETER BLACK HOLES

Phase transition in thermodynamic systems is an important area in physics. Ehrenfest categorized phase transitions by using the free energy as a function of thermodynamic variables. In this way, the order of phase transitions is defined as the smallest order of free energy differentials that have noncontinuous behavior through the transition. For example, the transition of solid-liquid-gas in water is a first order phase transition because entropy and volume (which are the first differentials of Gibbs free energy) are noncontinuous through the transition. To obtain these equations, we need to develop a dynamics. An example of this type of black hole is Kerr-Newman black hole which have both electrical and angular terms.

As we will show soon, Ehrenfest’s equations are necessary for writing Ehrenfest’s equations and we obtain nine equations for the charged black holes in the canonical ensemble. In canonical ensemble, the suitable form of free energy is defined as follows:

\[ G = M - TS \]

Considering (1), we have:

\[ dG = -SdT + \Phi dQ. \]

We can expand the first differentials \( S \) and \( \Phi \) in terms of independent parameters \( Q \) and \( T \):

\[ dS = \left( \frac{\partial S}{\partial T} \right)_Q dT + \left( \frac{\partial S}{\partial Q} \right)_T dQ = \frac{C_Q}{T} dT + \Phi dQ \]

where, \( C_Q \) is the specific heat in constant charge and \( \alpha = \frac{1}{T} \left( \frac{\partial S}{\partial Q} \right)_T \). At the point of a second order phase transition, we should have continuous \( S \) and \( \Phi \) through the transition so that \( S_1 = S_2 \) (where the subscripts 1 and 2 are the phases before and after transition). As a result, \( dS_1 = dS_2 \), and from (4) we will have:

\[ -\left( \frac{\partial Q}{\partial T} \right)_S = \frac{(C_Q)_2 - (C_Q)_1}{\Phi T (\alpha_2 - \alpha_1)} \]

which is the first Ehrenfest’s equation. In the same way, we have for \( \Phi \):

\[ d\Phi = -\Phi dT + \Phi dQ \]

where, \( \kappa = \frac{1}{T} \left( \frac{\partial S}{\partial Q} \right)_T \). From the continuity of \( \Phi \) in phase transition, it follows that \( d\Phi_1 = d\Phi_2 \). So, we find the second Ehrenfest’s equation as follows:

\[ \left( \frac{\partial Q}{\partial T} \right)_\Phi = \frac{\alpha_2 - \alpha_1}{\kappa_2 - \kappa_1}. \]

For the case of rotating black holes, \( Q \) and \( \Phi \) are replaced with angular momentum \( J \), and angular velocity \( \Omega \), respectively.

The order of phase transitions can be identified by checking the validity of Ehrenfest’s equations at the critical point in which heat capacity diverges. Satisfaction of Ehrenfest’s equations has been shown for Reissner-Nordstrom and Kerr black holes in AdS space [24, 25]. We check these equations for Einstein-Gauss-Bonnet (EGB), Einstein-Maxwell-Gauss-Bonnet (EMGB), Einstein-Yang-Mills-Gauss-Bonnet (EY MGB), Kerr, Reissner-Nordstrom (RN), Kerr-AdS, RN-AdS, and Horava-Lifshitz (HL) with \( k = -1 \) black holes in the canonical ensemble.

Now, let us consider the grand canonical ensemble, in which \( J \) or \( Q \) are no longer constant and can be exchanged between the black hole and its environment while their conjugate parameters are fixed. By choosing the suitable form of free energy, which is \( \tilde{G} = M - TS - \Phi Q \), we can find Ehrenfest’s equations for the charged black holes in the grand canonical ensemble:

\[ -\left( \frac{\partial \Phi}{\partial T} \right)_S = \frac{(C_\Phi)_2 - (C_\Phi)_1}{T Q (\alpha_2 - \alpha_1)} \]

\[ -\left( \frac{\partial \Phi}{\partial Q} \right)_T = \frac{\alpha_2 - \alpha_1}{\kappa_2 - \kappa_1} \]

where, \( C_\Phi \) is the specific heat at a constant electric potential, \( \alpha = \frac{1}{T} \left( \frac{\partial S}{\partial Q} \right)_T \) is analogous to volume expansion, and \( \kappa = \frac{1}{T} \left( \frac{\partial S}{\partial Q} \right)_T \) is analogous to isothermal compressibility. It is straightforward to prove that both Ehrenfest equations [8] and [9] are satisfied for EMGB, EY MGB and HL black holes.

III. GENERALIZED EHRENFEST’S EQUATIONS FOR THREE PARAMETER BLACK HOLES

In this section, we will generalize the Ehrenfest equations to the case of three parameter black holes that have two work terms in their associated first law of thermodynamics. An example of this type of black hole is Kerr-Newman black hole which have both electrical and angular terms in it’s mass formula. After a brief notation of thermodynamics of Kerr-Newman black hole we modify Ehrenfest’s equations and we obtain nine equations which are supposed to be satisfied at the critical point of second order phase transition. The Smarr mass relation for the Kerr-Newman black hole states that [20]:

\[ M = (2S + \frac{1}{8S} (J^2 + \frac{1}{4} Q^2) + \frac{1}{2} Q^2)^{\frac{1}{2}} \].

(10)
The first law of thermodynamics is
\[ dM = TdS + \Omega dJ + \Phi dQ \]  
(11)

Using the first law of thermodynamics one can determine temperature, \( T \), angular velocity, \( \Omega \), and electrical potential, \( \Phi \).

The suitable form of free energy for canonical ensemble is:
\[ \hat{G} = M - TS. \]  
(12)

So, using Equation (11) we will have:
\[ d\hat{G} = -SdT + \Omega dJ + \Phi dQ. \]  
(13)

We see that \( \hat{G} \) is a function of \( T, J, Q \). We can write \( d\hat{G} \) as follows:
\[ d\hat{G} = (\frac{\partial \hat{G}}{\partial T})_{J,Q}dT + (\frac{\partial \hat{G}}{\partial J})_{T,Q}dJ + (\frac{\partial \hat{G}}{\partial Q})_{T,J}dQ. \]  
(14)

Comparing this equation with (13), we obtain
\[ S = -(\frac{\partial \hat{G}}{\partial T})_{J,Q}, \quad \Omega = (\frac{\partial \hat{G}}{\partial J})_{T,Q}, \quad \Phi = (\frac{\partial \hat{G}}{\partial Q})_{T,J}. \]  
(15)

And thereby we obtain Maxwell’s equations:
\[ (\frac{\partial S}{\partial T})_{J,Q} = -\frac{\partial \Omega}{\partial T}, \quad (\frac{\partial S}{\partial J})_{T,Q} = -\frac{\partial \Phi}{\partial J}, \quad (\frac{\partial \Omega}{\partial Q})_{T,J} = (\frac{\partial \Phi}{\partial Q})_{T,Q}. \]  
(16)

Ehrenfest equations can be obtained by expanding the first differentials of free energy \( (S, \Omega \) and \( \Phi \)) in terms of the independent parameters \( (T, J, Q) \) and by fixing the differentials at the transition point. For \( S \), we have:
\[ dS = (\frac{\partial S}{\partial T})_{J,Q}dT + (\frac{\partial S}{\partial J})_{T,Q}dJ + (\frac{\partial S}{\partial Q})_{T,J}dQ = C_{J,Q}dT + \Omega dJ + \Phi \alpha dQ \]  
(17)

where,
\[ \alpha = \frac{1}{\Omega} (\frac{\partial S}{\partial J})_{T,Q} = -\frac{1}{\Omega} (\frac{\partial \Omega}{\partial T})_{J,Q}, \]  
\[ \alpha' = \frac{1}{\Phi} (\frac{\partial S}{\partial Q})_{T,J} = -\frac{1}{\Phi} (\frac{\partial \Phi}{\partial T})_{J,Q}. \]

Since entropy is continuous, we have \( dS_1 = dS_2 \), in which the indices show the states before and after the transition.

\[ (C_{J,Q})_1 + T\Omega \alpha (\frac{\partial \Omega}{\partial T})_{J,Q} + T\Phi \alpha' (\frac{\partial \Phi}{\partial T})_{J,Q} = (C_{J,Q})_2 + T\Omega \alpha_2 (\frac{\partial \Omega}{\partial T})_{J,Q} + T\Phi \alpha'_2 (\frac{\partial \Phi}{\partial T})_{J,Q} \]  
(18)

which can be written in the following form:
\[ -\frac{dJ}{dT}S = (C_{J,Q})_2 - (C_{J,Q})_1 + \frac{\Phi (\alpha'_2 - \alpha'_1)}{\Omega (\alpha_2 - \alpha_1)} dQ \]  
(19)

The general expansion of \( J \) in terms of \( T, S, \) and \( Q \) is:
\[ dJ = (\frac{\partial J}{\partial T})_{S,Q}dT + (\frac{\partial J}{\partial S})_{T,Q}dS + (\frac{\partial J}{\partial Q})_{T,S}dQ \]

By differentiating \( J \) with respect to \( T \) for a constant \( S \), we have:
\[ (\frac{dJ}{dT})_S = (\frac{\partial J}{\partial T})_{S,Q} + (\frac{\partial J}{\partial Q})_{S,T} \frac{dQ}{dT} \]  
(20)

Using (19) and (20), we find the first two generalized Ehrenfest’s equations for systems with three degrees of freedom:
\[ -\frac{dJ}{dT}S = \frac{(C_{J,Q})_2 - (C_{J,Q})_1}{\Omega(T(\alpha_2 - \alpha_1))} + \frac{\Omega(\alpha_2 - \alpha_1)}{\Phi(\alpha_2 - \alpha_1)} (\frac{dJ}{dT})_S \]  
(23)

It is straightforward to show that:
\[ \frac{dQ}{dT}S = (\frac{\partial Q}{\partial T})_{S,J} + (\frac{\partial Q}{\partial Q})_{S,T} \frac{dJ}{dT} \]  
(24)

Comparing (23) and (24), we find the third and fourth generalized Ehrenfest equations:
\[ -\frac{dQ}{dT}S,J = \frac{(C_{J,Q})_2 - (C_{J,Q})_1}{\Phi(T(\alpha'_2 - \alpha'_1))} + \frac{\Phi(\alpha'_2 - \alpha'_1)}{\Omega(\alpha'_2 - \alpha'_1)} (\frac{dJ}{dT})_S \]  
(25)

Now we write \( \Omega \) as a function of \( T, J \) and \( Q \), so we have:
\[ \Omega = -\Omega_dT + \Omega_dJ + \Phi \chi dQ. \]  
(27)

Where
\[ \kappa = \frac{1}{\Omega} (\frac{\partial \Omega}{\partial J})_{T,Q}, \quad \chi' = \frac{1}{\Phi} (\frac{\partial \Phi}{\partial Q})_{T,J}. \]

We can repeat the above calculations for \( \Omega \). The result is four Ehrenfest equations:
\[ (\frac{\partial \Omega}{\partial T})_{J,Q} = \frac{\alpha_2 - \alpha_1}{\kappa_2 - \kappa_1} \]  
(28)
\[ (\frac{\partial \Omega}{\partial Q})_{J,T} = \frac{\Omega(\kappa_2 - \kappa_1)}{\Phi(T(\alpha_2 - \alpha_1))} \]  
(29)
\[ (\frac{\partial \Omega}{\partial J})_{T,Q} = \frac{\Omega(\alpha_2 - \alpha_1)}{\Phi(\alpha'_2 - \alpha'_1)} \]  
(30)
\[ (\frac{\partial \Omega}{\partial Q})_{J,T} = \frac{\Omega(\kappa_2 - \kappa_1)}{\Phi(\alpha'_2 - \alpha'_1)}. \]  
(31)

For \( \Phi \) as a function of \( T, J \) and \( Q \), we have:
\[ d\Phi = -\Phi \alpha' dT + \Omega \chi dJ + \Phi \kappa' dQ. \]  
(32)
Where
\[ \kappa' = \frac{1}{\Phi} \frac{\partial \Phi}{\partial Q}, \quad \chi = \frac{1}{\Omega} \frac{\partial \Phi}{\partial J}. \]

And for \( \Phi \):
\[ \frac{\partial J}{\partial T}_{\Phi, Q} = \frac{\Phi(\alpha_2' - \alpha_1')}{\Omega(\chi_2' - \chi_1')} \quad (33) \]
\[ \frac{\partial Q}{\partial T}_{\Phi, J} = \frac{\alpha_2' - \alpha_1'}{\kappa_2' - \kappa_1'} \quad (34) \]
\[ \frac{\partial Q}{\partial J}_{\Phi, T} = \frac{\Omega(\chi_2' - \chi_1')}{\Phi(\kappa_2' - \kappa_1')} \quad (35) \]
\[ \frac{\partial \Phi}{\partial J}_{\Phi, T} = \frac{\Omega(\chi_2' - \chi_1')}{\Phi(\kappa_2' - \kappa_1')} \quad (36) \]

All the twelve generalized Ehrenfest’s equations are presented in Table I. It should be noted that some of the equations are the same and that a total number of nine independent equations exist.

The equations (26) and (28) convert to the usual Ehrenfest’s equations in the limit of Reissner-Nordstrom black hole. Also, the equations (21) and (28) go to the usual Ehrenfest’s equations in the limit of the two parameter Kerr black hole.

Explicit expressions of singular parameters appearing in the nine generalized Ehrenfest’s equations are given in Table III. A is a common factor in the denominator of all the singular parameters. This factor is zero at the critical point. Canceling out this factor is the key to the satisfaction of Ehrenfest’s equations. Now we explain how Equation (24) is satisfied. Consider the right hand side of Equation (24). Using the equations for \( C_{LQ} \) and \( \alpha \) from Table III we have:

\[ \frac{(C_{LQ})_2 - (C_{LQ})_1}{\Omega T(\alpha_2 - \alpha_1)} = \frac{2S(-4J^2 + Q^4 + 64S^2)(4J^2 + (Q^2 + 8S)^2)}{8\sqrt{8J^2S^3 + 2S^3(Q^2 + 8S)^2}(4J^2 + (Q^2 + 8S)(Q^2 + 24S))} \times \frac{1}{\alpha_2'} - \frac{1}{\alpha_1'} \]
\[ = \frac{2\sqrt{2S}(4J^2S + Q^4 + 16Q^2S^2 + 64S^3)\dot{\chi}}{J(4J^2 + Q^4 + 32Q^2S + 192S^2)} = \frac{-\partial J}{\partial T}_{S, Q} \quad (37) \]

So the first generalized Ehrenfest’s equation is satisfied. In the same way, we can show that other equations in Table II are also satisfied.

We can use several ensembles to study the thermodynamics of the black hole. In the grand canonical ensemble, the angular velocity \( \Omega \) and the electrical potential \( \Phi \) are constants. We also have two mixed ensembles, one is defined by taking \( J \) and \( \Phi \) to be constant and the other is defined by demanding that \( Q \) and \( \Omega \) be constant. We called them \( J \)-fixed and \( Q \)-fixed ensemble, respectively. To check the existence of second order phase transition in the mixed ensembles we should find generalized Ehrenfest equations in these ensembles by a similar calculations to the previous one for canonical ensemble. But there is a simple method to obtain these relations by comparing energy equations of both ensembles. The suitable form of free energy for \( Q \)-fixed ensemble is:

\[ \tilde{G} = M - TS - J\Omega \quad (38) \]

Using the first law of thermodynamics, \( dM = Tds + \Omega dJ + \Phi dQ \) we have:

\[ d\tilde{G} = -SdT - Jd\Omega + \Phi dQ \quad (39) \]

Comparing this relation and (13) we see that by replacing \( \{ J \to -\Omega \text{ and } \Omega \to J \} \) or \( \{ J \to \Omega \text{ and } \Omega \to -J \} \) we can obtain differential form of free energy of \( Q \)-fixed ensemble from the canonical ensemble. We have already obtained the Ehrenfest equations from free energy (table II). So by the above replacements it is possible to obtain a new set of Ehrenfest equations in \( Q \)-fixed ensemble. In a similar way by replacing \( \{ Q \to \Phi \) and \( \Phi \to Q \) or \( \{ Q \to \Phi \text{ and } \Phi \to -Q \} \) we can obtain another set of generalized Ehrenfest equations in \( J \)-fixed ensemble. By imposing both of the following replacements (\( \{ J \to -\Omega \text{ and } \Omega \to J \} \), \( \{ J \to \Phi \text{ and } \Phi \to Q \} \)) or (\( \{ J \to \Omega \text{ and } \Omega \to -J \} \), \( \{ Q \to \Phi \text{ and } \Phi \to -Q \} \)) on canonical ensemble we can find other relations for the grand canonical ensemble. Also by replacing \( \{ Q \leftrightarrow J \) and \( \Phi \leftrightarrow \Omega \) mixed ensembles convert to each other. So Table II can be used to find Ehrenfest equations of all ensembles. We can easily demonstrate that the generalized Ehrenfest’s equations in the \( Q \)-fixed and \( J \)-fixed mixed ensembles are satisfied. But as shown in [27], the specific heat for constant \( \Omega \) and \( \Phi \) does not diverge and, thus, so the possibility of a second order transition in the grand canonical ensemble is ruled out.

IV. HIGHER DIMENSIONAL GENERALIZATION OF EHRENFEST’S EQUATIONS

We can generalize the method used in the previous section to the case of \( N \) work terms in the first law of thermodynamics. We assume that we have a free energy with \( N \) work terms as below:

\[ dG = -SdT + \sum_{i=1}^{N} \Omega_i dJ_i \quad (40) \]

To obtain Ehrenfest’s equations, we should first write the differentials of the free energy that are \( S \) and \( \Omega_i \) in terms of independent parameters (i.e., \( T \) and \( J_i \)). So, we have
TABLE I. Generalized Ehrenfest’s equations in the canonical ensemble. There are twelve equations which nine of them are independent.

| $S$ fixed | $\Omega$ fixed | $\Phi$ fixed |
|-----------|---------------|--------------|
| $\frac{\partial J}{\partial T} S,J = \frac{(C-J,Q) - (C,J,Q)}{T \Omega} (2g_2-g_1)$ | $\frac{\partial J}{\partial T} \alpha, Q = \frac{g_2}{g_1}$ | $\frac{\partial J}{\partial T} \Phi, S = \frac{g_2 - g_1}{\Omega}$ |
| $-\frac{\partial J}{\partial T} S,T = \frac{\Phi}{T \Omega} (g_1 - g_2)$ | $-\frac{\partial J}{\partial T} \alpha, T = \frac{g_2 - g_1}{\Omega}$ | $-\frac{\partial J}{\partial T} \Phi, T = \frac{g_2 - g_1}{\Omega}$ |
| $-\frac{\partial J}{\partial T} S,J = \frac{(C,J,Q) - (C,J,Q)}{T \Omega} (2g_2-g_1)$ | $\frac{\partial J}{\partial T} \alpha,J = \frac{Q - g_2}{\Omega}$ | $\frac{\partial J}{\partial T} \Phi, J = \frac{g_2 - g_1}{\Omega}$ |
| $\alpha = \frac{1}{\Omega} \left( \frac{\partial S}{\partial Q} \right)_{T,Q}$ | $\kappa = \frac{1}{\Omega} \left( \frac{\partial \Omega}{\partial J} \right)_{T,Q}$ | $\chi = \frac{1}{\Omega} \left( \frac{\partial \Phi}{\partial J} \right)_{T,Q}$ |
| $\alpha' = \frac{1}{\Omega} \left( \frac{\partial S}{\partial Q} \right)_{T,J}$ | $\kappa' = \frac{1}{\Omega} \left( \frac{\partial \Omega}{\partial J} \right)_{T,J}$ | $\chi' = \frac{1}{\Omega} \left( \frac{\partial \Phi}{\partial J} \right)_{T,J}$ |

The singular parameters appearing in the Generalized Ehrenfest’s equations for the canonical ensemble are:

- $C_{i,j} = 28(-4j^2 + Q^2 + 64S^2)(4j^2 + (Q^2 + 8S)^2)$
- $\alpha = 8\sqrt{7j^2S^2 + 28S(Q^2 + 8S)^2 + (Q^2 + 24S)^2}$
- $\alpha' = 8\sqrt{7j^2S^2 + 28S(Q^2 + 8S)^2 + (Q^2 + 24S)^2}$
- $\kappa = \frac{-4j^2 - 16Q^2 + (Q^2 + 8S)^2}{(Q^2 + 8S)^2}$
- $\kappa' = \frac{-4j^2 - 16Q^2 + (Q^2 + 8S)^2}{(Q^2 + 8S)^2}$
- $\chi = \frac{-8Q(Q^2 + 4j^2 + (Q^2 + 8S)^2)}{(Q^2 + 8S)^2}$
- $\chi' = \frac{-8Q(Q^2 + 4j^2 + (Q^2 + 8S)^2)}{(Q^2 + 8S)^2}$

$A = 48j^4 + (3Q^2 - 8S)(Q^2 + 8S)^3 + 8j^2(3Q^4 + 32Q^2S + 192S^2) + 216j^2(3Q^2S + 192S^2)$.

The following relations:

$$dS = \left( \frac{\partial S}{\partial T} \right)_{(J_i)} dT + \sum_{j=1}^{N} \left( \frac{\partial S}{\partial J_j} \right)_{(J_{j-1})} dJ_j$$

$$d\Omega_i = \left( \frac{\partial \Omega_i}{\partial T} \right)_{(J_i)} dT + \sum_{j=1}^{N} \left( \frac{\partial \Omega_i}{\partial J_j} \right)_{(J_{j-1})} dJ_j$$

(41)

We have $N+1$ equations similar to (41), differentiating any of which with respect to $T$ leads to Ehrenfest’s equations. We can choose any one of $N$ work terms in Equations (41) to write the following relations:

$$\left( \frac{dJ_i}{dT} \right)_{(\Omega_k)} = \alpha_i + \sum_{j \neq i} \beta_{ij} \left( \frac{dJ_j}{dT} \right)_{(\Omega_k)}$$

(42)

where, $\alpha_i$ and $\beta_{ij}$ are some constants. Each set of $N+1$ equations of (41) generates $N$ equations similar to (42), each of which leads to $N$ Ehrenfest’s equations. So far we have $N^2(N+1)$ equations some of which are the same. Sentences like $\left( \frac{\partial J_i}{\partial J_j} \right)_{T,(\Omega_k)}$ and $\left( \frac{\partial J_i}{\partial J_j} \right)_{T,S}$ lead to the same Ehrenfest’s equations under exchange of $i \leftrightarrow j$. We have $N^2(N-1)$ terms like $\left( \frac{\partial J_i}{\partial J_k} \right)_{T,(\Omega_k)}$ ($i,k = 1, \ldots, N, i \neq j$) and $N(N-1)$ terms like $\left( \frac{\partial J_i}{\partial J_j} \right)_{T,S}$. So, the final number of Ehrenfest’s equations is given by:

$$N^2(N+1) - \frac{1}{2} \left( N^2(N-1) + N(N+1) \right) = N(N+1)^2$$

(43)

It is clear that we have two equations for $N = 1$, nine equations for $N = 2$, etc.

Also, we can exploit the same argument to determine the number of generalized Ehrenfest’s equations for Myers-Perry black holes [28] that are the generalized form of Kerr-AdS black holes in higher dimensions, with rotation in more than one plane. In higher dimensions, the number of rotating planes will be as follows:

$$N \leq \left[ \frac{d-1}{2} \right]$$
where, $d$ is the number of space-time dimensions and brackets give the integer part. For example, we have two rotating planes for a five dimensional space-time and three for a seven dimensional one. We have nine generalized Ehrenfest’s equations for a five dimensional Myers-Perry (MP) black hole and 24 equations for a seven dimensional MP black hole.

V. CONCLUSIONS

In this paper, we considered the thermodynamic phase transition in black holes strongly focusing on Ehrenfest’s equations. For a large class of black holes which have just one work term in their first law of thermodynamics, we showed that at a second order phase transition in the canonical and grand canonical ensemble the Ehrenfest equations are satisfied.

We also generalized Ehrenfest’s equation to the case of systems with three degrees of freedom. We obtained nine Ehrenfest’s equations that are satisfied at the critical points. As an example, we considered the Kerr-Newman black hole in the canonical ensemble. We showed that the generalized Ehrenfest’s equations are satisfied at the critical point in which heat capacity diverges. So, the infinite discontinuity of specific heat in black hole thermodynamics causes no problem for a second order phase transition to occur. In the case of the Kerr-Newman black hole, we found two mixed ensembles that have their own generalized Ehrenfest’s equations. Finally, we generalized this approach to the case of $N$ work terms in the first law of thermodynamic which can be used to check the order of transitions in black holes that have more than two work terms such as Myers-Perry black hole in higher dimensions.

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