YURI BURMAN

Abstract. This paper is a continuation of [2]. We prove a three-parameter family of identities (Theorem 1.1) involving a version of the Tutte polynomial for directed graphs introduced by Awan and Bernardi [1]. A particular case of this family (Corollary 1.6) is the higher-degree generalization of the matrix-tree theorem proved in [2], which thus receives a new proof, shorter (and less direct) than the original one. The theory has a parallel version for undirected graphs (Theorem 1.2).

1. Definitions and main results

The following theory has two parallel versions — for directed and undirected graphs — so let us introduce notation for both cases.

By \( \Gamma_{n,k} \) we denote the set of directed graphs with \( n \) vertices numbered \( 1, \ldots, n \) and \( k \) edges numbered \( 1, \ldots, k \); in other words, an element \( G \in \Gamma_{n,k} \) is a \( k \)-element sequence \( ([a_1, b_1], \ldots, [a_k, b_k]) \) where \( a_1, \ldots, a_k, b_1, \ldots, b_k \in \{1, \ldots, n\} \). Loops (edges \([a, a]\)) and parallel edges (pairs \([a_i, b_i] = [a_j, b_j]\)) are allowed. Similarly, by \( \Upsilon_{n,k} \) we denote the set of unoriented graphs with \( n \) numbered vertices and \( k \) numbered edges: \( \Upsilon_{n,k} = \{\{a_1, b_1\}, \ldots, \{a_k, b_k\}\} \). The forgetful map \(|\cdot| : \Gamma_{n,k} \to \Upsilon_{n,k}\) relates to every graph \( G \) its undirected version \(|G|\) obtained by dropping the edge orientation: \([a, b] \mapsto \{a, b\}\).

Denote by \( G_{n,k} \) (resp., \( \Upsilon_{n,k} \)) a vector space over \( \mathbb{C} \) spanned by \( \Gamma_{n,k} \) (resp., \( \Upsilon_{n,k} \)). The forgetful map is naturally extended to the linear map \(|\cdot| : G_{n,k} \to \Upsilon_{n,k}\).

1.1. Main theorems. A Bernardi polynomial \( B \) is a map \( B : \Gamma_{n,k} \to \mathbb{Q}[q, y, z] \) defined as

\[
B_G(q, y, z) = \sum_{f: \{1, \ldots, n\} \to \{1, \ldots, q\}} y^{\#f^G_{\geq}} z^{\#f^G_{<}}
\]

where \( f^G_{\geq} \) (resp., \( f^G_{<} \)) is the set of edges \([ab]\) of \( G \) such that \( f(b) \geq f(a) \) (resp., \( f(b) < f(a) \)). See [1] for a detailed analysis of the properties of \( B_G \) (in particular, for the proof of its polynomiality).

Bernardi polynomial is a directed version of the full chromatic polynomial, which is a map \( C : \Upsilon_{n,k} \to \mathbb{Q}[q, y] \) defined as

\[
C_G(q, y) = \sum_{f: \{1, \ldots, n\} \to \{1, \ldots, q\}} y^{\#f^G_{\geq}}
\]

where \( f^G_{\geq} \) is the set of edges \([ab]\) of \( G \) such that \( f(b) \neq f(a) \). The classical Potts polynomial (as defined e.g. in [7]) is related to the full chromatic polynomial by \( Z_G(q, v) = (v + 1)^k C_G(q, 1/(v + 1)) \); see [1] for details.
For any $G \in \Gamma_{n,k}$ (resp., $G \in \Upsilon_{n,k}$) we denote by $\hat{G}$ the graph $G$ with all the loops deleted (and the numbering of the non-loop edges shifted accordingly); we have $\hat{G} \in \Gamma_{n,k-\ell}$ (resp., $\hat{G} \in \Upsilon_{n,k-\ell}$) where $\ell$ is the number of loops in $G$. It follows directly from the definition that $B_G(q,y,z) = B_{\hat{G}}(q,y,z)$; in the undirected case $Z_G(q,v) = (v+1)^\ell Z_{\hat{G}}(q,v)$.

The universal Bernardi polynomial is an element of $G_{n,k}[q,y,z]$ defined as

$$B_{n,k}(q,y,z) \overset{\text{def}}{=} \sum_{G \in \Gamma_{n,k}} B_G(q,y,z)G.$$ 

For a polynomial $P \in \mathbb{C}[q,y,z]$ denote by $[P]_k$ the sum of terms containing monomials $q^s y^i z^j$ with $i + j = k$ (and any $s$). The universal truncated Bernardi polynomial is an element of $G_{n,k}[q,y,z]$ defined as

$$\hat{B}_{n,k}(q,y,z) \overset{\text{def}}{=} \sum_{G \in \Gamma_{n,k}} [B_G]_k(q,y,z)G.$$ 

Note that $[B_G]_k = 0$ if (and only if) $G$ contains at least one loop; that is, $\hat{B}_{n,k}$ contains only loopless graphs. $\hat{B}_{n,k}$ is homogeneous of degree $k$ with respect to $y$ and $z$ and is not homogeneous with respect to $q$.

The universal Potts polynomial and the universal truncated Potts polynomial are elements of $\Upsilon_{n,k}[q,v]$ defined, respectively, as

$$Z_{n,k}(q,v) \overset{\text{def}}{=} \sum_{G \in \Gamma_{n,k}} Z_{\hat{G}}(q,v)G,$$

$$\hat{Z}_{n,k}(q,v) \overset{\text{def}}{=} \sum_{G \in \Gamma_{n,k} \text{ has no loops}} Z_G(q,v)G.$$ 

For any $i = 1, \ldots, k$ and $p, q = 1, \ldots, n$ denote by $R_{p,q,i} : \Gamma_{n,k} \to \Gamma_{n,k}$ the map replacing the edge number $i$ of every graph $G \in \Gamma_{n,k}$ by the edge $[p,q]$ carrying the same number $i$. Also denote by $B_i : \mathcal{G}_{n,k} \to \mathcal{G}_{n,k}$ the linear operator acting on the graph $G \in \Gamma_{n,k}$ such that $[a, b]$ is its edge number $i$ as

$$B_i(G) = \begin{cases} 
G, & a \neq b, \\
- \sum_{m \neq a} R_{a,m,i}G, & a = b.
\end{cases}$$

Following [2] call the product $\Delta \overset{\text{def}}{=} B_1 \cdots B_k : \mathcal{G}_{n,k} \to \mathcal{G}_{n,k}$ the Laplace operator. The undirected version of the Laplace operator is defined as follows: if $G \in \Upsilon_{n,k}$ then $\Delta(G) \overset{\text{def}}{=} |\Delta(\Phi)|$ where $\Phi \in \Gamma_{n,k}$ is any directed graph such that $G = |\Phi|$.

The main results of this paper are the following two theorems:

**Theorem 1.1.**

$$\Delta B_{n,k}(q,y,z) = \hat{B}_{n,k}(q,y-1,z-1).$$

and its undirected version

**Theorem 1.2.**

$$\Delta Z_{n,k}(q,v) = (-1)^k \hat{Z}_{n,k}(q,-v).$$

1.2. Corollaries.
1.2.1. Universal chromatic polynomials. Following [1], denote by $\chi_G^\geq$ (a chromatic polynomial of the directed graph $G \in \Gamma_{n,k}$) a polynomial such that for any $q = 1, 2, \ldots$ the value $\chi_G^\geq(q)$ is equal to the number of mappings $f : \{1, \ldots, n\} \to \{1, \ldots, q\}$ such that $f(a) \geq f(b)$ for every edge $[ab] \in G$. Also denote by $\chi_G^>$ (a strict chromatic polynomial of $G$) a polynomial such that for any $q = 1, 2, \ldots$ the value $\chi_G^>(q)$ is equal to the number of mappings $f : \{1, \ldots, n\} \to \{1, \ldots, q\}$ such that $f(a) > f(b)$ for every edge $[ab] \in G$.

Comparing these definitions with the definition of the Bernardi polynomial in Section 1.1 one obtains the equalities:

$$\chi_G^\geq(q) = B_G(q, 0, 1),$$

$$\chi_G^>(q) = [B_G]_k(q, 0, 1).$$

Thus, one may call the elements of $\mathcal{G}_{n,k}$

$$\chi_{n,k}^\geq(q) \overset{\text{def}}{=} B_{n,k}(q, 0, 1) = \sum_{G \in \Gamma_{n,k}} \chi_G^\geq(q),$$

and

$$\chi_{n,k}^>(q) \overset{\text{def}}{=} B_{n,k}(q, 0, 1) = \sum_{G \in \Gamma_{n,k}} \chi_G^>(q)$$

universal chromatic polynomials. Substitution of $y = 0$ and $z = 1$ in Theorem 1.1 yields

**Corollary 1.3.** $\Delta \chi_{n,k}^\geq(q) = (-1)^k \chi_{n,k}^>(q)$.

1.2.2. Higher matrix-tree theorems. A graph $G \in \Gamma_{n,k}$ is called acyclic if it contains no oriented cycles (in particular, no loops); $G$ is called totally cyclic (or strongly semiconnected, following the terminology of [2]) if every edge of $G$ is a part of a directed cycle.

It is possible to make further specialization of parameters in Corollary 1.3 due to the following

**Proposition 1.4.** For any $G \in \Gamma_{n,k}$

$$\chi_G^\geq(-1) = \begin{cases} (-1)^{\beta_0(G)}, & \text{if } G \text{ is totally cyclic,} \\ 0, & \text{otherwise}, \end{cases}$$

(1.2)

$$\chi_G^>(-1) = \begin{cases} (-1)^k, & \text{if } G \text{ is acyclic,} \\ 0, & \text{otherwise}. \end{cases}$$

(1.3)

where $\beta_0(G)$ is the 0-th Betti number (i.e. the number of connected components) of the graph $G$.

For proof see [1, Eq. (45) and Definition 5.1]. Note that it follows immediately from the definition that $\chi^\geq\equiv 0$ if (and only if) $G$ contains an oriented cycle (e.g. a loop), and that $\chi^>(q) = q^{\beta_0(G)}$ if $G$ is totally cyclic, so one half of each formula is evident (but not the other half).

Consider now (following [2]) the sum

$$\det_{n,k} \overset{\text{def}}{=} \frac{(-1)^k}{k!} \chi_G^\geq(-1) = \frac{(-1)^k}{k!} \sum_{G \in \Gamma_{n,k}} (-1)^{\beta_0(G)} G$$
where by $G_{n,k} \subset \Gamma_{n,k}$ we denote the set of all totally cyclic graphs. Thus, Corollary 1.3 specializes to

**Corollary 1.5.** $\Delta \operatorname{det}_{n,k} = (-1)^n \sum_{G \in G_{n,k}} G$.

where $G_{n,k} \subset \Gamma_{n,k}$ is the set of all acyclic graphs.

Corollary 1.5 admits a refinement. A totally cyclic graph may have isolated vertices (the ones not incident to any edge). Let $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$ be a set of vertices. We call a diagonal $I$-minor the element

$$\det^I_{n,k} \defeq \frac{(-1)^k}{k!} \sum_{G \in G_{n,k}} (-1)^{\beta_0(G)} G \in G_{n,k}$$

where $G^I_{n,k}$ is the set of all totally cyclic graphs $G \in \Gamma_{n,k}$ such that the vertices $i_1, \ldots, i_s$, and only they, are isolated. Similarly, denote by $G^I_{n,k} \subset \Gamma_{n,k}$ the set of all acyclic graphs such that $i_1, \ldots, i_s$, and only they, are sinks (vertices without edges starting at them); so Corollary 1.5 now looks like

$$\Delta \sum_{I \subset \{1, \ldots, n\}} \operatorname{det}^I_{n,k} = \frac{(-1)^n}{k!} \sum_{I \subset \{1, \ldots, n\}} \sum_{G \in G^I_{n,k}} G.$$  

It follows from the definition of the Laplace operator that if $G \in G^I_{n,k}$ then $\Delta G = \sum_H x_H H$ where $x_H \in \mathbb{Z}$ and all $H$ have $i_1, \ldots, i_s$, and only them, as sinks. Since $G^I_{n,k}$ with different $I$ do not intersect, and the same is true for $G^I_{n,k}$, there is

**Corollary 1.6 (of Corollary 1.5).** For every $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$ one has $\Delta \operatorname{det}^I_{n,k} = \frac{(-1)^n}{k!} \sum_{G \in G^I_{n,k}} G$.

This is the abstract-matrix tree theorem [2, Theorem 1.7] which, in turn, is a higher-degree generalization of the celebrated matrix-tree theorem (first discovered by G. Kirchhoff in 1847 [3] and extended to the directed graphs by W. Tutte [6]).

**Acknowledgements.** The research was funded by the Russian Academic Excellence Project '5-100' and by the grant No. 15-01-0031 "Hurwitz numbers and graph isomorphism" of the Scientific Fund of the Higher School of Economics.

2. **Proofs**

A graph $H \in \Gamma_{n,m}$ is called a subgraph of $G \in \Gamma_{n,k}$ (notation $H \subseteq G$) if it can be obtained from $G$ by deletion of several edges. (When one deletes the edge number $s$ from the graph, the numbers of the remaining edges are preserved if they are less than $s$ and are lowered by 1 if they are greater than $s$.)

For convenience denote by $e(G)$ the number of edges of the graph $G$ (so $e(G) = k$ if $G \in \Gamma_{n,k}$). Proof of Theorem 1.1 involves the following well-known lemma:

**Lemma 2.1 (Möbius inversion formula, [4]).** Let $f : \bigcup \Gamma_{n,k} \to \mathbb{C}$ be a function on the set of graphs with $n$ vertices, and let the function $h$ on the same set be defined by the equality $h(G) = \sum_{H \subseteq G} f(H)$ for every $G \in \Gamma_{n,k}$. Then one has $(-1)^{e(G)} f(G) = \sum_{H \supseteq G} (-1)^{e(H)} h(H)$.

**Proof of Theorem 1.1.** Let $\Delta B_{n,k}(q, y, z) = \sum_{G \in \Gamma_{n,k}} x_G G$; by the definition of the Laplace operator $x_G \neq 0$ only if $G$ contains no loops. For a graph $H \in \Gamma_{n,k}$ the element $\Delta H \in G_{n,k}$ contains a term $y_{G,H} G$ with $y_{G,H} \neq 0$ if and only if $\Phi \defeq H$. 


(the graph $H$ with all the loops deleted) is a subgraph of $G$. For any subgraph $\Phi \subseteq G$ of a loopless graph $G$ there exists exactly one $H \overset{\text{def}}{=} L(\Phi)$ such that $\hat{\Phi} = \hat{H}$: every edge $[ab]$ present in $G$ but missing in $\Phi$ is replaced by the loop $[aa]$ in $H$.

Eventually, the coefficient $y_{G,H}$ in this case is

$$y_{G,H} = (-1)^{\#\text{of loops in } H} B_H(q,y,z) = (-1)^{k-e(\Phi)} B_{\Phi}(q,y,z).$$

where $\Phi = \hat{H}$.

By [1, Eq. (21)] one has $\sum_{\Phi \subseteq G} [B_{\Phi}]_{e(\Phi)}(q,y - 1, z - 1) = B_G(q,y,z)$. Applying the Moebius inversion formula (Proposition 2.1) to this identity one obtains

$$x_G = \sum_{\Phi \subseteq G} y_{G,L(\Phi)} = (-1)^k \sum_{\Phi \subseteq G} (-1)^e(\Phi) B_{\Phi}(q,y,z) = [B_G]_{k}(q,y - 1, z - 1).$$

Proof of Theorem 1.3 is similar to that of Theorem 1.1: again, if $\Delta Z_{n,k}(q,v) = \sum_G x_G$ then $G$ entering the sum have no loops. A contribution $y_{G,H}$ of a graph $H$ that into $x_G$ is nonzero if and only if $\hat{\Phi} = \hat{H}$ is a subgraph of $G$. For a subgraph $\Phi \subseteq G$ having $e(\Phi)$ edges there are $2^{k-e(\Phi)}$ graphs $H$ such that $\Phi = \hat{H}$: every edge $[ab]$ present in $G$ but missing in $\Phi$ may correspond either to a loop $[aa]$ or to a loop $[bb]$ in $H$: recall that $a \neq b$ because $G$ is loopless.

The contribution $y_{G,H}$ of all such graphs $H$ into $x_G$ is the same and is equal to $(1)^{k-e(\Phi)} Z_{\Phi}(q,v)$. Now by [3] one has $Z_G(q,v) = \sum_{H \subseteq G} q^{\beta(H)} v^{e(H)}$, and therefore

$$x_G = \sum_{\Phi \subseteq G} 2^{k-e(\Phi)}(-1)^{k-e(\Phi)} Z_{\Phi}(q,v) = (-2)^k \sum_{\Phi \subseteq G} \left(-\frac{1}{2}\right)^{e(\Phi)} Z_{\Phi}(q,v)$$

$$= (-2)^k \sum_{\Phi \subseteq G} \left(-\frac{1}{2}\right)^{e(\Phi)} q^{\beta(\Phi)} v^{e(\Phi)} = (-2)^k \sum_{\Phi \subseteq G} q^{\beta(\Phi)} v^{e(\Phi)} \sum_{\Phi \subseteq \Psi} \left(-\frac{1}{2}\right)^{e(\Phi)}$$

$$= (-2)^k \sum_{\Psi \subseteq G} q^{\beta(\Psi)} v^{e(\Psi)} \left(-\frac{1}{2}\right)^{e(\Psi)} (1 - \frac{1}{2})^{k-e(\Phi)} = (-1)^k Z_G(q,v).$$

References

[1] J. Awan, O. Bernardi, Tutte polynomials for directed graphs, arXiv:1610.01839v2.
[2] Yu. Burman, Abstract matrix-tree theorem, arXiv:1612.03873v3.
[3] G. Kirchhoff, "Uber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem., 72 (1847), S. 497–508.
[4] G.-C. Rota, On the foundations of combinatorial theory I: Theory of Mobius functions, Z. Wahrsch. Verw. Gebiete, 2 (1964) pp. 340–368.
[5] A. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids, in: Surveys in Combinatorics 2005, Cambridge University Press, Jul 21, 2005 — Mathematics — 258 pages.
[6] W.T. Tutte, The dissection of equilateral triangles into equilateral triangles, Math. Proc. Cambridge Phil. Soc., 44 (1948) no. 4, pp. 463–482.
[7] D.J.A. Welsh and C. Merino, The Potts model and the Tutte polynomial, J. of Math. Physics, 41 (2000), no. 3, pp. 1127–1152.
National Research University Higher School of Economics, 119048, 6 Usacheva str., Moscow, Russia, and Independent University of Moscow, 119002, 11 B. Vlassievsky per., Moscow, Russia

E-mail address: burman@mccme.ru