Higher identities for the ternary commutator

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Abstract

We use computer algebra to study polynomial identities for the trilinear operation $[a, b, c] = abc - acb - bac + bca + cab - cba$ in the free associative algebra. It is known that $[a, b, c]$ satisfies the alternating property in degree 3, no new identities in degree 5, a multilinear identity in degree 7 which alternates in 6 arguments, and no new identities in degree 9. We use the representation theory of the symmetric group to demonstrate the existence of new identities in degree 11. The only irreducible representations of dimension $<400$ with new identities correspond to partitions $2^5, 1$ and $2^4, 1^3$ and have dimensions 132 and 165. We construct an explicit new multilinear identity for partition $2^5, 1$ and we demonstrate the existence of a new non-multilinear identity in which the underlying variables are permutations of $a^2b^2c^2d^2e^2f$.

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1. Introduction

The theory of multioperator algebras ($\Omega$-algebras), by which is meant vector spaces with multilinear operations, was first studied systematically by the school of Kurosh in Moscow; see [5, 29]. In particular, a natural generalization of the Lie bracket to the $n$-ary setting is the alternating $n$-ary sum ($n$-commutator):

$$[a_1, \ldots, a_n] = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1)} \cdots a_{\sigma(n)},$$

where $\epsilon(\sigma)$ is the sign of the permutation $\sigma$. This operation provides unexpected algebraic structures on vector fields [18, 19], and plays an essential role in the construction of universal enveloping algebras of Filippov algebras ($n$-Lie algebras) [20]. For many other applications, especially to theoretical physics, see the survey of $n$-ary analogues of Lie algebras [13].

For $n = 3$, the alternating ternary sum (ternary commutator) has the form

$$[a, b, c] = abc - acb - bac + bca + cab - cba.$$
The first explicit polynomial identity which is satisfied by this operation in the free associative algebra, but which does not follow from the alternating property in degree 3, was found in 1998; see [6]. This identity has degree 7:
\[
\sum_{\sigma \in S_n} \epsilon(\pi) ([[b^\sigma, e^\sigma, d^\sigma], a, e^\sigma], f^\sigma, g^\sigma) + [[a, b^\sigma, e^\sigma], [d^\sigma, e^\sigma, f^\sigma], g^\sigma]) \equiv 0.
\]
Two years later, it was shown that there are no new identities in degree 9; see [7]. Ten years later, the identity in degree 7 was rediscovered [17], and was generalized to all odd \(n\) [12]; the situation is much simpler for even \(n\) [14–16, 22, 28, 30]. For an outline of the historical development of \(n\)-ary generalizations of Lie algebras, and for a clarification of the terminology, see [13, section 1]. For identities relating the ordinary and ternary commutators, see [21]. For the partially alternating ternary sum in an associative dialgebra (Loday algebra), see [10].

In this paper, we use computer algebra to show that further new polynomial identities for the ternary commutator exist in degree 11. We construct an explicit new multilinear identity among the underlying variables are permutations of \(a^2b^2c^de^2f\).

Owing to the large size of the matrices involved in our computations, we used modular arithmetic to save memory. By choosing a suitable modulus, we found it easy to perform ‘rational reconstruction’ of the correct results in characteristic 0 from the results obtained in characteristic \(p\). Underlying all of these computations is the structure theory of the group algebra of the symmetric group \(S_n\), which is semisimple both in characteristic 0 and in characteristic \(p > n\). For further information, see [8, lemma 8] and [9, section 5.5].

2. Preliminaries

For an alternating trilinear operation, every monomial in degree 11 can be written in terms of one of the following eight association types (placements of brackets); we display these types with the identity permutation of the arguments:

\[
\begin{align*}
1: [[[a, b, c], d, e], f, g], h, i, j, k] & \quad 2: [[[a, b, c], [d, e, f], g], h, i, j, k] \quad 3: [[[a, b, c], d, e], [f, g, h], i, j, k] \quad 4: [[[a, b, c], [d, e, f], [g, h, i]], j, k] \\
5: [[[a, b, c], d, e], f, g], [h, i, j], k] & \quad 6: [[[a, b, c], [d, e, f], g], [h, i, j], k] \\
7: [[[a, b, c], d, e], [f, g, h], [i, j, k]] & \quad 8: [[[a, b, c], d, e], [f, g, h], [i, j, k]]
\end{align*}
\]

The number of multilinear monomials in each type can be easily calculated using the alternating property of \([a, b, c]\); the total is
\[
\frac{11!}{6^{12}2^8} \cdot 1 \cdot 6^{12} + \frac{11!}{6^{12}2^8} \cdot 1 \cdot 1 \cdot 6^{12} + \frac{11!}{6^{12}2^8} \cdot 1 \cdot 1 \cdot 1 \cdot 6^{12} + \frac{11!}{6^{12}2^8} \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 6^{12} = 1401400.
\]

Since this number is so large, we cannot process all the monomials at once, so we use the representation theory of the symmetric group \(S_{11}\) to decompose the problem into a sequence of smaller problems, each corresponding to an irreducible representation. (For a detailed discussion of this approach, see [8, section 4] or [9, section 5].)

Using representation theory requires that we enumerate the symmetric groups of the association types (1); each symmetry is a two-term identity expressing the fact that the value of a monomial changes sign after a transposition of two factors. Since the symmetric group \(S_3\) is generated by the transpositions (12) and (23), every symmetry is a consequence of the 43 symmetries corresponding to the monomials \(\pi\) in table 1. In that table, \(\pi\) represents the identity \(i + \pi \equiv 0\), where \(i\) represents the monomial with the identity permutation of the variables in the same association type. These symmetries are the consequences in degree 11 of the alternating properties \([a, b, c] + [b, a, c] \equiv 0\) and \([a, b, c] + [a, c, b] \equiv 0\) in degree 3.

\[
\begin{align*}
1: & \quad [[[a, b, c], d, e], f, g], h, i, j, k] \\
2: & \quad [[[a, b, c], [d, e, f], g], h, i, j, k] \\
3: & \quad [[[a, b, c], d, e], [f, g, h], i, j, k] \\
4: & \quad [[[a, b, c], [d, e, f], [g, h, i]], j, k] \\
5: & \quad [[[a, b, c], d, e], f, g], [h, i, j], k] \\
6: & \quad [[[a, b, c], [d, e, f], g], [h, i, j], k] \\
7: & \quad [[[a, b, c], d, e], [f, g, h], [i, j, k]] \\
8: & \quad [[[a, b, c], d, e], [f, g, h], [i, j, k]]
\end{align*}
\]
We also need to determine the consequences in degree 11 of the known polynomial identity in degree 7; see [6, 12, 17]. We write this identity symbolically as $I(a, b, c, d, e, f, g) \equiv 0$, where

$$I(a, b, c, d, e, f, g) = \sum_{\sigma \in S_9} \epsilon(\sigma) [[[b^\sigma, c^\sigma, d^\sigma], a, e^\sigma], f^\sigma, g^\sigma] + [[[a, b^\sigma, e^\sigma], [d^\sigma, e^\sigma, f^\sigma], g^\sigma]].$$

We collect similar terms in this identity using the alternating property of the ternary commutator, and see that the total number of distinct terms is $\binom{9}{1} + \binom{9}{3} \cdot 2 = 60 + 60 = 120$. From the alternating property of $[a, b, c]$, and the alternating property of $I(a, b, c, d, e, f, g)$ in the arguments $b, \ldots, g$, it follows that every consequence of $I(a, b, c, d, e, f, g) \equiv 0$ in degree 9 is a linear combination of permutations of three identities, the first two obtained by substituting a triple for a variable, and the third obtained by embedding the identity in a triple:

$$I([a, b, c], d, e, f, g) \equiv 0, \quad I([a, b, c], d, e, f, g) \equiv 0, \quad I([a, b, c], d, e, f, g) \equiv 0,$$

Similarly, every consequence of these three identities in degree 11 is a linear combination of permutations of the eight identities in table 2. (It is a coincidence that the number of association types is equal to the number of consequences of $I(a, b, c, d, e, f, g)$.) We call these consequences the ‘liftings’ of $I(a, b, c, d, e, f, g)$ to degree 11. We summarize this discussion in the following lemma.
Table 3. Representations of $S_{11}$ with dimension < 400.

| No | $d_\lambda$ | $\lambda$ | Sym | Sym+Lif | All | New |
|----|-------------|-----------|-----|---------|-----|-----|
| 1  | 1           | 11        | 8   | 8       | 8   | 0   |
| 2  | 10          | 10, 1     | 80  | 80      | 80  | 0   |
| 3  | 44          | 9, 2      | 352 | 352     | 352 | 0   |
| 4  | 45          | 9, 1$^2$  | 360 | 360     | 360 | 0   |
| 5  | 110         | 8, 3      | 880 | 880     | 880 | 0   |
| 6  | 231         | 8, 2, 1   | 1848| 1848    | 1848| 0   |
| 7  | 120         | 8, 1$^3$  | 960 | 960     | 960 | 0   |
| 8  | 165         | 7, 4      | 1320| 1320    | 1320| 0   |
| 10 | 385         | 7, 2$^2$  | 3080| 3080    | 3080| 0   |
| 12 | 210         | 7, 1$^4$  | 1680| 1680    | 1680| 0   |
| 13 | 132         | 6, 5      | 1056| 1056    | 1056| 0   |
| 19 | 252         | 6, 1$^5$  | 2016| 2016    | 2016| 0   |
| 20 | 330         | 5$^2$, 1  | 2639| 2639    | 2639| 0   |
| 29 | 210         | 5, 1$^6$  | 1676| 1676    | 1676| 0   |
| 40 | 120         | 4, 1$^7$  | 944 | 948     | 948 | 0   |
| 45 | 385         | 3$^2$, 1$^5$ | 3005| 3020    | 3020| 0   |
| 46 | 330         | 3, 2$^4$  | 2639| 2639    | 2639| 0   |
| 49 | 231         | 3, 2, 1$^6$ | 1764| 1795    | 1795| 0   |
| 50 | 45          | 3, 1$^8$  | 333 | 349     | 349 | 0   |
| 51 | 132         | 2$^5$, 1  | 1006| 1020    | 1021| 1   |
| 52 | 165         | 2$^4$, 1$^3$ | 1242| 1269    | 1270| 1   |
| 53 | 110         | 2$^3$, 1$^5$ | 807 | 842     | 842 | 0   |
| 54 | 44          | 2$^2$, 1$^3$ | 302 | 333     | 333 | 0   |
| 55 | 10          | 2, 1$^4$  | 57  | 76      | 76  | 0   |
| 56 | 1           | 1$^{11}$  | 0   | 7       | 7   | 0   |

Lemma 2.1. Every polynomial identity in degree 11 satisfied by the ternary commutator, which is a consequence of identities of lower degree, is a linear combination of permutations of the identities in tables 1 and 2.

3. New identities in degree 11

Let $\lambda$ be a partition of 11 with $k$ parts; we write

$$\lambda = (\lambda_1, \ldots, \lambda_k), \quad \lambda_1 \geq \cdots \geq \lambda_k \geq 1, \quad \lambda_1 + \cdots + \lambda_k = 11.$$ 

Let $d_\lambda$ be the dimension of the corresponding irreducible representation of $S_{11}$. For any permutation $\pi \in S_{11}$ the $d_\lambda \times d_\lambda$ matrix $R_{\pi}^\lambda$ representing $\pi$ in the natural representation can be computed using the methods of [11]; see also [9, section 5]. Taking linear combinations gives the matrix representing any element of the group algebra $\mathbb{Q}S_{11}$ over the rational field $\mathbb{Q}$. This provides an algorithm for explicit computation of the isomorphism $\phi$ from $\mathbb{Q}S_{11}$ to its Wedderburn decomposition, by which we mean the direct sum over all partitions $\lambda$ of matrix algebras of size $d_\lambda \times d_\lambda$. Any multilinear polynomial $P$ of degree 11 in the ternary commutator can be expressed as a sum of eight elements of $\mathbb{Q}S_{11}$, one term for each association type (1).

For each partition $\lambda$, the projection of $P$ to the corresponding component of the Wedderburn decomposition consists of an ordered list of eight $d_\lambda \times d_\lambda$ matrices. The partitions $\lambda$ for which $d_\lambda < 400$ are given in table 3.

We apply this discussion to the symmetries of the association types in table 1 and the consequences of $I(a, b, c, d, e, f, g)$ in table 2. We first construct a $43d_\lambda \times 8d_\lambda$ matrix consisting...
of $d_i \times d_j$ blocks: for $i = 1, \ldots, 43$ and $j = 1, \ldots, 8$, block $(i, j)$ contains the representation matrix for the terms of symmetry $i$ in association type $j$. Since each symmetry has the form $\iota + \pi$ in one association type, for each $i$ there will be one nonzero block containing the matrix $I + R^\pi_i$. We compute the row canonical form of this matrix; for each $\lambda$, the nonzero rows form a basis for the space of identities in degree 11 which are consequences of the alternating property of the ternary commutator. The rank $s_\lambda$ of this matrix is given in column ‘Sym’ of table 3.

We next construct a $51d_i \times 8d_i$ matrix consisting of $d_i \times d_i$ blocks; the first 43 rows of blocks are the same as in the preceding matrix. For $i = 1, \ldots, 8$ and $j = 1, \ldots, 8$, block $(43+i, j)$ contains the representation matrix for the terms in association type $j$ of the $i$th consequence of $I(a, b, c, d, e, f, g)$. We compute the row canonical form of this matrix; for each $\lambda$, the nonzero rows form a basis for the space of identities in degree 11 which are consequences of all the identities of lower degree. The rank $sl_\lambda$ of this matrix is given in column ‘Sym+Lif’ of table 3. In the next section we use the name $\text{oldmat}(\lambda)$ for the $8d_i \times 8d_i$ matrix in row canonical form containing the nonzero rows.

Finally, we construct a matrix of size $8d_i \times 9d_i$ and use it to find all the identities satisfied by the ternary commutator in degree 11. The first column of $d_i \times d_i$ blocks corresponds to the associative multilinear polynomials, which we identify with the group algebra $Q[S_1]$. The remaining columns correspond to the eight association types (1). For $i = 1, \ldots, 8$ we put the identity matrix in block $(i, i+1)$; in block $(i, 1)$ we put the representation matrix for the expansion of association type $i$ (with the identity permutation of the variables) in the free associative algebra. By the expansion of an association type, we mean the associative polynomial obtained by replacing each occurrence of $[a, b, c]$ by the alternating ternary sum of its arguments; thus each expansion is a sum of $6^7 = 7776$ terms with coefficients $\pm 1$. In the resulting matrix, the $8d_i \times 8d_i$ submatrix obtained by deleting the first column of blocks is the identity matrix; hence the matrix has rank $8d_i$. We compute the row canonical form of this matrix, and delete the rows whose leading 1s occur within the first $d_i$ columns. From the remaining matrix, we delete the first $d_i$ columns, all of whose entries are 0. The result is a matrix of size $a_\lambda \times 8d_i$, with rank $sl_\lambda$ for some $a_\lambda \geq 0$; this number is given in column ‘All’ of table 3. For each $\lambda$, the (nonzero) rows of this matrix provide a basis for the space of all identities in degree 11 satisfied by the ternary commutator. In the next section we call this matrix $\text{allmat}(\lambda)$.

It is clear that $a_\lambda \geq sl_\lambda$ for every $\lambda$: the space of identities which are consequences of identities of lower degree is a subspace of the space of all identities. If $a_\lambda = sl_\lambda$ for some $\lambda$ then there are no new identities for partition $\lambda$. In this case we also need to verify that the two matrices are exactly the same: the first matrix, $\text{oldmat}(\lambda)$ of size $sl_\lambda \times 8d_i$, containing the symmetries of the association types and the consequences of $I(a, b, c, d, e, f, g)$; and the second matrix, $\text{allmat}(\lambda)$ of size $a_\lambda \times 8d_i$, containing all the identities satisfied by the ternary commutator. If $a_\lambda > sl_\lambda$ for some $\lambda$ then there exist new identities in degree 11 for the representation of $S_{11}$ corresponding to $\lambda$. The difference $a_\lambda - sl_\lambda$ is given in column ‘New’ of table 3.

Owing to the large size of many of the irreducible representations of $S_{11}$, and the time required to compute the representation matrices $R^\pi_\lambda$, we were able to complete these computations only for the 25 partitions in table 3, corresponding to the representations with dimensions $< 400$, slightly less than half of the total of 56 representations. We found two representations which have new identities: number 51 (dimension 132, partition $2^7, 1$) and number 52 (dimension 165, partition $2^4, 1^3$). We summarize this discussion in the following theorem.
Theorem 3.1. New identities in degree 11 for the ternary commutator exist for partitions $2^5, 1$ and $2^4, 1^3$, and these are the only partitions with corresponding irreducible representations of dimension $<400$ which have new identities.

4. A new multilinear identity for representation 51

Representation 51 is the smaller of the two representations with new identities in table 3. In this section we obtain an explicit form of a new identity for this representation. (Similar computations could be performed for representation 52.)

From the computations in the previous section we obtain two matrices:

- oldmat of size $1020 \times 1056$: this full rank matrix contains the rows representing the symmetries of the association types and the consequences of $I(a, b, c, d, e, f, g)$ for the representation corresponding to partition $\lambda = 2^5, 1$.
- allmat of size $1021 \times 1056$: this full rank matrix contains the rows representing all the polynomial identities satisfied by the ternary commutator for the representation corresponding to partition $\lambda = 2^5, 1$.

The row space of oldmat is a subspace of the row space of allmat. For a matrix $A$ in row canonical form, we write leading($A$) for the set of column indices for those columns which contain the leading 1 of some row. We have

$$\text{leading(oldmat)} \subset \text{leading(allmat)},$$

$$\text{leading(allmat)} - \text{leading(oldmat)} = \{251\}.$$

The row of allmat which has its leading 1 in column 251 is row 246; this is the row which represents the new identity. This row has 24 nonzero entries, with 16 distinct integer coefficients:

$\begin{align*}
-432, & \quad -60, \quad -36, \quad -34, \quad -24, \quad -9, \quad 9, \quad 18, \quad 24, \quad 36, \quad 54, \quad 72, \quad 96, \quad 108, \quad 144, \quad 216.
\end{align*}$

The columns of allmat correspond to eight blocks of length $d_\lambda = 132$; the blocks correspond to the association types (1) and the columns in each block correspond to the standard tableaus for partition $\lambda = 2^5, 1$ in lexicographical order:

$\begin{align*}
1 & \quad 2, \\
3 & \quad 4, \\
5 & \quad 6, \\
7 & \quad 8, \\
9 & \quad 10.
\end{align*}$

Table 4 gives complete information about the row representing the new identity, where $t$ is the association type, $j$ is the tableau index and $c$ is the coefficient; the standard tableaus are given in flattened form as a sequence of rows.

To convert this data into an explicit identity for the ternary commutator, we use the correspondence between matrix units in the representation matrices and elements of the group algebra [8, remark 2, p 2004]. We summarize this result in the general case. Given a partition $\lambda$ of $n$, let $d_\lambda$ be the dimension of the corresponding irreducible representation of $S_n$. For $1 \leq i, j \leq d_\lambda$ we construct the element of the group algebra $\mathbb{Q}S_n$ corresponding to the matrix unit $E^{ij}_\lambda$ under the isomorphism of $\mathbb{Q}S_n$ with a direct sum of full matrix algebras. Let $T_1, \ldots, T_{d_\lambda}$ be the standard tableaus for $\lambda$ in lexicographical order. For each $i = 1, \ldots, d_\lambda$, let $R_i$ (respectively $C_i$) be the subgroup of $S_n$ which leaves the rows (respectively columns) of $T_i$
fixed as sets. For \( i, j = 1, \ldots, d \), let \( s_{ij} \) be the permutation for which \( s_{ij}T_i = T_j \). We define elements \( D_{ij} \in \mathbb{Q}S_n \) as follows:

\[
D_{ii} = d_{\lambda} n! \sum_{\sigma \in \mathcal{R}_i} \sum_{\tau \in \mathcal{C}_i} \epsilon(\tau) \sigma \tau,
\]

\[
D_{ij} = D_{ii}s_{ij}^{-1}.
\]

These elements in general do not satisfy the multiplication formulas for matrix units. To obtain the matrix units, let \( A_{\pi}^h \) be the matrix defined by Clifton [11] for the permutation \( \pi \). For the identity permutation \( \iota \), the matrix \( A_{\iota}^h \) is not necessarily the identity matrix, but it is always invertible. Let \( (a_{ij}) \) be the inverse matrix \( (A_{\iota}^h)^{-1} \); then the element of \( \mathbb{Q}S_n \) corresponding to the matrix unit \( E_{ij}^\lambda \) is

\[
E_{ij}^\lambda \leftrightarrow \sum_{k=1}^{d_{\lambda}} a_{jk}D_{ik}.
\]

We then have the required relations \( E_{ij}^\lambda E_{\ell k}^\lambda = \delta_{j\ell}E_{ik}^\lambda \).

We now return to our discussion of the new identity in degree 11 for the ternary commutator. Since we are dealing with a single identity we may assume that \( i = 1 \): any row of the representation matrix can be moved to row 1 by left multiplication by an element of the group algebra. Moreover, we need to consider only those values of \( j \) which appear in table 4:

\[ j = 94, 95, 97, 98, 114, 117, 118, 119, 121, 124, 125, 126, 127, 129, 130, 132. \]

We compute the matrix \( A_{\iota}^h \) and find that it has the form \( I + U \) where \( U \) is a strictly upper triangular matrix with 262 nonzero entries from the set \{±1\}. The inverse matrix \( (A_{\iota}^h)^{-1} \) has the form \( I + V \) where \( V \) is a strictly upper triangular matrix with 424 nonzero entries from the set \{±1, ±2\}. For all except one of the values of \( j \) listed above, the corresponding row of \( (A_{\iota}^h)^{-1} \)
The total number of permutations satisfying these conditions is

\[ 6720 + 1980 + 4010 + 180 + 4010 + 1190 + 2000 + 550 = 20640. \]
The resulting bracketed permutations will be called 'nonassociative monomials'; they form a basis of the homogeneous subspace $N_3$ of the free alternating ternary algebra on six generators with multidegree $\delta$.

At this point, we would like to construct a matrix of size $1247\,400 \times 20\,640$ in which the $(i, j)$ entry is the coefficient of the $i$th associative monomial in the expansion of the $j$th nonassociative monomial; as before, by expansion we mean repeated application of the alternating ternary sum. This matrix represents the 'expansion map' $E_3 : N_3 \to A_3$ with respect to the bases of associative and nonassociative monomials. The polynomial identities satisfied by the ternary commutator are the (nonzero) vectors in the kernel $K_3$ of this linear map. The matrix representing $E_3$ is very sparse, since each expansion contains only 7776 terms; more than 99% of the entries are 0. However, processing a matrix of this size is not practical. We therefore begin by storing the expansions of the nonassociative monomials in a matrix of size $7776 \times 20\,640$; the $(i, j)$ entry contains the $i$th term of the $j$th expansion in the form $\pm k$. The sign $\pm 1$ is the coefficient of the term and the absolute value $k$ is the lexicographical index of the associative monomial.

We now observe that $1247\,400 = 77 \times 16\,200$. We construct a matrix with an upper block of size $20\,640 \times 20\,640$ and a lower block of size $16\,200 \times 20\,640$, and initialize it to zero. We then perform the following iteration for $\ell = 1, \ldots, 77$.

- For each column index $j$, extract the terms of the corresponding expansion whose indices $k$ lie in the range $16\,200(\ell-1) < k \leq 16\,200\ell$.
- Store the corresponding coefficients in the appropriate row of the lower block; index $k$ goes to row $k-16\,200(\ell-1)$.
- After all the columns have been processed, and the lower block has been filled, compute the row canonical form. (The lower block is now zero.)

At the end of this iteration, the nullspace of the matrix contains the coefficient vectors of the polynomial identities satisfied by the ternary commutator. The rank of the matrix is 19\,964, and so the nullity is 676. We compute the canonical basis of the nullspace from the row canonical form by setting the free variables equal to the standard basis vectors in dimension 676 and solving for the leading variables. We summarize this discussion in the following lemma.

**Lemma 5.1.** The kernel $K_3$ of the linear map $E_3 : N_3 \to A_3$ has dimension 676.

The next step is to determine the subspace $L_3 \subset K_3$ consisting of the polynomial identities which are consequences of the known identity $I(a, b, c, d, e, f, g)$ in degree 7. (The consequences of the alternating properties in degree 3 have already been excluded by our choice (2) of nonassociative monomials.) For each consequence of $I(a, b, c, d, e, f, g)$ in table 2, we must determine the corresponding substitutions of the variables $a^2b^2c^2d^2e^2f$, recalling that $[a, b, c]$ alternates in all three arguments and $I(a, b, c, d, e, f, g)$ alternates in $b, c, d, e, f, g$. If $q_1 \cdots q_{11}$ denotes an associative monomial, then the eight consequences require the following conditions, where $<$ denotes lexicographical order:

1: $q_1 < q_{10} < q_{11}, \quad q_8 < q_9, \quad q_2 < q_3 < q_4 < q_5 < q_6 < q_7$
2: $q_1 < q_8 < q_9, \quad q_2 < q_{10} < q_{11}, \quad q_3 < q_4 < q_5 < q_6 < q_7$
3: $q_1 < q_8 < q_9, \quad q_2 < q_3 < q_4 < q_5 < q_6 < q_7, \quad q_{10} < q_{11}$
4: $q_2 < q_{10} < q_{11}, \quad q_8 < q_9, \quad q_3 < q_4 < q_5 < q_6 < q_7$
5: $q_2 < q_8 < q_9, \quad q_3 < q_{10} < q_{11}, \quad q_4 < q_5 < q_6 < q_7, \quad q_2q_8q_9 < q_3q_{10}q_{11}$
6: $q_2 < q_8 < q_9, \quad q_3 < q_4 < q_5 < q_6 < q_7, \quad q_{10} < q_{11}$
redo the computation for partition 25 by the linearized polynomial, giving a total of $32 \times I$, 10 292 terms of the polynomial using the alternating ternary sum; then expand each of the 10 292 terms of the polynomial using the alternating ternary sum; after all terms of the polynomial have been expanded and added to the vector, every component of the vector is 0.

We now use the representation theory of the symmetric group as described in section 4 to redo the computation for partition 25, 1 with dimension $d_5 = 132$. We construct a 52, $d_5 \times 8d_5$
matrix consisting of $d_\lambda \times d_\lambda$ blocks; the first 43 rows of blocks contain the representation matrices for the symmetries of the association types, and the next eight rows of blocks contain the representation matrices for the consequences of $I(a, b, c, d, e, f, g)$. The last row of blocks contains the representation matrices for the linearized form of the new identity. We compute the row canonical form of this matrix and find that its rank is 1021, as required. Furthermore, we verify that the nonzero rows of this matrix coincide exactly with the $1021 \times 1056$ matrix $allmat(\lambda)$ obtained from the expansions of the association types.

6. Conclusion

The interest of the theoretical physics community in ternary structures was renewed by the work of Bagger and Lambert [1–4] and Gustavsson [23–27]. A comprehensive survey on $n$-ary generalizations of Lie algebras and their applications in physics up to 2009 has been given by de Azcárraga and Izquierdo [13]. We hope that the algebraic findings in this paper will turn out to be useful in future physical applications. On the mathematical side, there are many open questions related to these structures. We mention the following.

1. Classification of all identities satisfied by the ternary commutator: in particular, are there further new identities in degrees higher than 11? Can all the identities for the ternary commutator be generated by a finite set, or are there further new identities in ever higher degrees?
2. Generalization of the degree 11 identities for the ternary commutator to the $n$-ary commutator for odd $n$: this is related to the work of Curtright et al [12] generalizing the degree 7 identity to all odd $n$.
3. Investigation of higher identities relating the binary and ternary commutators: this would extend the work of Fairlie and Nuyts [21]. More generally, study identities relating $n$-ary commutators for two or more values of $n$.
4. Explicit construction of the universal associative enveloping algebras for simple Filippov algebras: a first step in this direction has been taken by Elgendy and Bremner [20].
5. Classification of simple finite-dimensional alternating ternary algebras which satisfy all the identities for the ternary commutator: this seems to define a much larger class of ternary algebras than Filippov’s 3-Lie algebras.

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