Non-Archimedean Whittaker functions as characters: a probabilistic approach to the Shintani-Casselman-Shalika formula

Reda Chhaibi *

Preliminary version

Abstract

For a reductive group $G$ over a non-Archimedean local field (e.g $GL_n(Q_p)$), Jacquet’s Whittaker function is essentially proportional to a character of an irreducible representation of the Langlands dual group $G^\vee(C)$ (a Schur function if $G = GL_n(Q_p)$). We propose a probabilistic approach to this claim, known as the Shintani-Casselman-Shalika formula, when the group $G$ has at least one minuscule cocharacter in the coweight lattice.

Our presentation goes along the following lines. Thanks to a minuscule random walk $W^{(z)}$ on the coweight lattice and a related random walk on the Borel subgroup, we establish a Poisson kernel formula for the non-Archimedean Whittaker function. The expression and its ingredients are similar to the one previously obtained by the author in the Archimedean case. A simple manipulation reduces the problem to evaluating the probability of $W^{(z)}$ never exiting the Weyl chamber. Then, an implementation of the reflection principle forces the appearance of the Weyl character formula and therefore retrieves characters of $G^\vee(C)$.

The construction of the random walk on the Borel subgroup requires some care. It is extracted from a spherical random walk whose increments have a distribution that can be understood as elements from the spherical Hecke algebra.

MSC 2010 subject classifications: 11F70, 11F85, 60B15, 60J45, 60J50

Keywords: Random walks, Reflection principle, Jacquet’s non-Archimedean Whittaker function, Unramified principal series representation, Shintani-Casselman-Shalika formula

*reda.chhaibi@math.uzh.ch
## Contents

1 Introduction .................................................. 3

2 Setting
   2.1 Lie theory .............................................. 4
   2.2 Measures and probability ............................... 7
   2.3 Unramified principal series ............................. 7
   2.4 Jacquet’s Whittaker function (the non-Archimedean case) ........ 8
   2.5 Random walks ............................................ 10

3 Main results ................................................... 11
   3.1 A characterisation of characters ....................... 13
   3.2 Similarities with the Archimedean case .............. 16
   3.3 Relation to crystals ..................................... 16

4 The reflection principle for minuscule walks ............... 16

5 Harmonic properties of Whittaker functions ............... 19
   5.1 Around the Satake isomorphism ....................... 19
   5.2 On Macdonald’s spherical function .................... 21
   5.3 A spherical and a solvable random walk ............. 24
   5.4 Penalisation and proof of theorem 3.2 ............... 27

6 Acknowledgement ................................................ 28
1 Introduction

Following Jacquet [Jac67], the Whittaker function is a function on a reductive group over a local field and is defined as an integral. Its relevance to number theory comes from the fact that it plays an important role in proving functional equations for \(L\)-functions arising from automorphic forms (see for instance Cogdell’s lectures [CKM04]). Here, we are interested in the case of a non-Archimedean field (e.g. the \(p\)-adic field \(\mathbb{Q}_p\)). We recall in the preliminaries how non-Archimedean Whittaker functions appear in the study of representations from the unramified principal series, as well as some of its basic properties.

The non-Archimedean Whittaker function is a function on the group that essentially reduces to a function on the coweight lattice, and vanishes if the coweight is not dominant. A first striking result due to Shintani [Shi76] is that the Whittaker function for \(GL_n(\mathbb{Q}_p)\) is essentially a Schur function for a partition with at most \(n\) rows. One has simply to interpret the dominant coweight as a partition. Later, Casselman and Shalika [CS80] generalised the result to a reductive group \(G\) by proving it is proportional to a character of the Langlands dual \(G^\vee(\mathbb{C})\). Their proof basically boils down to the computation of the integral defining the Whittaker function.

In this paper, we aim at giving a probabilistic understanding of the Shintani-Casselman-Shalika formula, in a similar fashion to the description of the Archimedean Whittaker function in [Chh13] and [Chh14b]. If one is only concerned with the Shintani-Casselman-Shalika formula, most of the results we present are not new and certainly not as general as possible. Nevertheless, the approach is original and has many advantages. The general idea is that the Shintani-Casselman-Shalika formula holds because of the harmonic properties of Whittaker functions, rather than because of their definition as an integral. The companion paper [Chh14a] deals with probabilistic and representation-theoretic aspects of the Archimedean Whittaker functions. Here, we tried using similar notations, although the objects at hand are different. This way, random walks give a unified approach to both cases, while emphasising why Whittaker functions are characters for the Langlands dual.

In the preliminaries (section 2), we set the necessary group theoretic notations and introduce two random walks. Both depend on a parameter \(z\) which can be interpreted as a drift from the point of view of probability and a Langlands parameter from the point of view of representation theory. \(W^{(z)}\) is a random walk on the cocharacter lattice and \(\left(B_t \left(W^{(z)}\right), t \in \mathbb{N}\right)\) is a random walk on the Borel subgroup \(B\), driven by \(W^{(z)}\).

The first main theorem 3.1 can be stated independently, and in fact does not require any knowledge of Whittaker functions. It is a probabilistic characterisation of functions on the group \(G\) that are \(G^\vee\)-characters. The theorem claims that, under the appropriate growth conditions, if we consider \(f_z\)
to be any function that is harmonic for the random walk \((B_t (W(z)) : t \in \mathbb{N})\) and with the same invariance properties as the Whittaker function, then \(f_z\) has to be a \(G'\)-character. The proof is broken down into three steps. There is a Poisson kernel formula for any such \(f_z\) (Proposition 3.4). A simple manipulation akin to tropicalisation relates \(f_z\) to the probability that \(W(z)\) stays in the Weyl chamber (Proposition 3.5). Finally, this latter probability is evaluated by implementing a version of the classical reflection principle and gives a \(G'\)-character (Proposition 3.9). The reflection principle is explained separately in section 4.

The second main theorem 3.2 claims that the actual Whittaker function satisfies the harmonicity hypothesis of theorem 3.1, modulo a renormalisation by the modular character. The other hypotheses are easier to establish. Of course, the combination of two yields the Shintani-Casselman-Shalika formula (theorem 3.3). This second main theorem is the focus of section 5. There, we will see that the random walk \((B_t (W(z)) : t \in \mathbb{N})\) we defined explicitly is not ad-hoc: it is obtained considering only the \(B\) part of a spherical random walk and penalising it by the value of its “diagonal” part. We will require knowledge of the Satake isomorphism (subsection 5.1) and geometric information about double strata from the Cartan and Iwasawa decompositions. It is well-known that this information is encoded in Macdonald’s spherical function, which we briefly review in subsection 5.2.

We hope that this paper will serve as a bridge between usually disjoint fields of mathematics and a stepping stone for future work.

2 Setting

Let \(K\) be a non-Archimedean local field i.e a locally compact field with a discrete valuation denoted \(\text{val} : K \rightarrow \mathbb{Z}\). Write \(O\) for its ring of integers. We make the choice of a uniformizing element \(\varpi \in \{x \in K | \text{val}(x) = 1\}\). The residue field \(O/\varpi O\) is finite with cardinality \(q\). The absolute value \(| \cdot |\) on \(K\) is given by \(|x| = q^{-\text{val}(x)}\) for any \(x \in K\). The two examples to have in mind are the \(p\)-adic field \(\mathbb{Q}_p\) and the field \(\mathbb{F}_q((T))\) of formal power series over the finite field \(\mathbb{F}_q\).

Given an algebraic group or homogenous space \(M\) over \(K\), we will denote by \(\mathcal{F}(M)\) the \(\mathbb{C}\)-vector space of locally constant functions on \(M\), with compact support.

2.1 Lie theory

As usual the multiplicative and additive groups are respectively written \(G_m\) and \(G_a\). Let \(G\) be a split reductive group (scheme), as constructed by Chevalley (see for e.g [Ste68]). We fix a maximal torus \(T \approx (\mathbb{G}_m)^r\), with \(r\) the rank of the group. \(G := G(K)\) is its set of \(K\) points and \(T = T(K) \approx (\mathbb{K}^*)^r\). Let \((X, \Phi, X^\vee, \Phi^\vee)\) be the associated root datum, meaning that the
lattice of algebraic characters of $T$ is $X$ while the algebraic cocharacters are given by $X^\vee$:

$$X = \text{Hom} (T, \mathbb{G}_m) \quad X^\vee = \text{Hom} (\mathbb{G}_m, T)$$

We will use an additive notation in order to denote the group operation in $X$ and $X^\vee$. Hence we will favour the exponential notation where the image of $t \in T$ by a character $\mu$ is $t^\mu$ and the image of $k \in \mathbb{G}_m$ by $\mu^\vee \in X^\vee$ is $k^{\mu^\vee}$.

The canonical pairing between $X$ and $X^\vee$ is denoted $\langle \cdot , \cdot \rangle$ and is obtained by composing a character $\mu$ and a cocharacter $\lambda^\vee$. It yields $\mu \circ \lambda^\vee : k \mapsto k^{\langle \cdot , \cdot \rangle} \in \text{Hom} (\mathbb{G}_m, \mathbb{G}_m) \approx \mathbb{Z}$.

Inside of $\Phi$, we make the choice of a simple root system $\Delta$ within a set of positive roots $\Phi^+$. Also $Q$ is the root lattice spanned by $\Phi$ and $P \subset Q$ is the weight lattice in duality with the coroot lattice $Q^\vee$. Let $W$ be the Weyl group acting on both $X$ and $X^\vee$. When acting on $X$, $W$ is generated as a Coxeter group by the simple reflections $(s_\alpha)_{\alpha \in \Delta}$:

$$\forall x \in X, s_\alpha (x) := x - \langle \alpha^\vee, x \rangle \alpha$$

This action is linearly extended to $X \otimes \mathbb{R}$ and its fundamental domain is the Weyl chamber:

$$C := \{ x \in X \otimes \mathbb{R} | \forall \alpha^\vee \in \Delta, \langle \alpha^\vee, x \rangle \geq 0 \}$$

Its interior is denoted $C^\circ$. The Weyl vector is $\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$. The length function $\ell : W \to \mathbb{N}$ gives the minimal number of reflections need to write an element $w \in W$ as a product of simple reflections. The longest element in $W$ is denoted $w_0$.

We use similar notations for coweights. The symbols $C^\vee$, $\rho^\vee$, $Q^\vee$ and $P^\vee$ are defined in the same fashion. It is well-known that:

$$Q \subset X \subset P \iff Q^\vee \subset X^\vee \subset P^\vee$$

Such a notation is standard, although perhaps confusing because $P^\vee$ is the dual of $Q$ while $Q^\vee$ is the dual of $P$. The isogeny type of $G$ is determined by $X/Q \subset P/Q$. The Langlands dual $G^\vee (\mathbb{C})$ is the connected reductive group over $\mathbb{C}$ whose root datum is $(X^\vee, \Phi^\vee, X, \Phi)$. The highest weight $G^\vee$-module with highest weight $\lambda^\vee \in (X^\vee)^+ \subset P^\vee$ is written $V (\lambda^\vee)$ and its character is denoted $\text{ch} V (\lambda^\vee) \in \mathbb{C} [X^\vee]^W$.

At the level of the group, our choice of positive roots gives a pair of opposite Borel subgroups $(B, B^\pm)$. Each Borel subgroup is the product of the torus $T$ and its unipotent part $B = NT$ and $B^\pm = TU$. We will mainly work in the lower Borel subgroup $B$ and its unipotent subgroup $N$. For every root $\beta \in \Phi$, we fix a group homomorphism $x_\beta : \mathbb{G}_m \to G$ coming from a pinning.
The image of the uniformizing element \( \varpi \in \mathcal{O} \) by the cocharacter \( \mu^\vee \in X^\vee \) is denoted \( \varpi^\mu \in T \). This notation allows to embed \( X^\vee \) as a discrete subgroup of \( T \):

\[
A = \bigsqcup_{\mu^\vee \in X^\vee} \varpi^{-\mu^\vee}
\]

Let \( K = G(\mathcal{O}) \) be the special maximal compact open group of integral points ([Tit79] 3.8.1). It enters in various decompositions that we will often use and that are established thanks to the Bruhat-Tits theory of buildings ([Tit79]). Here are the non-Archimedean analogues of respectively the singular value decomposition and the Gram-Schmidt decomposition, in the context of the general linear group over \( \mathbb{R} \) or \( \mathbb{C} \):

\begin{align}
\text{Cartan decomposition} & \quad G = \bigsqcup_{\lambda^\vee \in (X^\vee)^+} K \varpi^{-\lambda^\vee} K & \quad \text{(2.1)} \\
\text{Iwasawa decomposition} & \quad G = \bigsqcup_{\mu^\vee \in X^\vee} N \varpi^{-\mu^\vee} K = NAK & \quad \text{(2.2)}
\end{align}

The groups \( T(\mathcal{O}), B(\mathcal{O}) \) and \( N(\mathcal{O}) \) correspond to the respective subgroups of \( T, B \) and \( N \) obtained by intersecting with \( K \). Of course, because we are dealing with split groups, these are integral points of the corresponding algebraic groups (e.g \( T(\mathcal{O}) = T(\mathcal{O}) \)).

**Hypothesis 2.1.** We assume that \( G \) has a dominant minuscule cocharacter ([Bou75], VIII, section 7.3), denoted \( \Lambda^\vee \in X^\vee \) and fixed once and for all.

**Remark 2.2.** Let \( \mathfrak{g} \) be the Lie algebra of \( G \). Saying that the group is reductive means that the derived algebra \([\mathfrak{g}, \mathfrak{g}]\) of \( \mathfrak{g} \) is a semisimple algebra. Necessarily, our hypothesis implies that this semisimple algebra is the direct sum of simple Lie algebras of type \( A_n, B_n, C_n, D_n, E_6, \) or \( E_7 \). These are the only ones having a minuscule coweight (and weight) in the Cartan-Killing classification.

This also imposes restrictions on the isogeny type of \( G \), as we demand that the minuscule coweight \( \Lambda^\vee \) gives a cocharacter. In order that coweights of the Lie algebra lift to cocharacters of \( G \), we can assume that \( G \) is of adjoint type \( (X = Q) \) or equivalently that \( G^\vee \) is simply connected \( (X^\vee = P^\vee) \). Such isogeny restrictions could be lifted in a subsequent version. Getting rid of the minuscule hypothesis will however require a much heavier machinery. In short, the semisimple part of \( G \) is a direct product of simple adjoint groups, each factor from the above type.
2.2 Measures and probability

Equality in law, i.e. equality between the probability distributions of two random variables $X$ and $Y$, is denoted $X \overset{\text{L}}{=} Y$. On any compact group $H$, we denote by $U(H)$ a Haar distributed random variable. For example $U \overset{\text{L}}{=} U(O)$ is a Haar distributed random variable on $O$.

For the non-compact groups $G$ and $N$, each Haar measure is normalised so that $K$ and $K \cap N = N(O)$ have unit measure.

Also, a left-invariant random walk on a group $H$ is a Markov chain $(H_t; t \in \mathbb{N})$ such that:

$$\forall t \in \mathbb{N}, H_t := h_1 h_2 \ldots h_t$$

where $(h_s)_{s \in \mathbb{N}}$ are independent and identically distributed random variables, called the increments. A function $f : H \to \mathbb{C}$ is $\alpha$-harmonic (w.r.t that random walk) if

$$\forall h \in H, \forall t \in \mathbb{N}, \mathbb{E}[f(hH_t)] = \alpha^t f(h)$$

If $\alpha = 1$, one simply says that $f$ is harmonic.

2.3 Unramified principal series

Let $\chi : T \to \mathbb{C}^\ast$ be a character of $T$. We assume it is unramified i.e trivial on $T(O) = T \cap K$. Necessarily, it is then entirely determined by a $z \in X \otimes \mathbb{C}$, the Langlands parameter, such that:

$$\forall \mu^\vee \in X^\vee, \chi(\varpi^{-\mu^\vee}) = e^{z(\mu^\vee)} \quad (2.3)$$

After inflating the character $\chi$ to $B$, consider the representation $I(\chi)$ defined as the parabolic induction:

$$I(\chi) := \text{Ind}_B^G \chi = \left\{ f \in \mathcal{F}(G) \mid \forall b \in B, \forall g \in G, f(bg) = \chi(b) \delta_\varpi(b) f(g) \right\}$$

Here $\delta : B \to \mathbb{C}^\ast$ is the modular character. It is trivial on $U$ and $T \cap K$. For $\varpi^\mu \in T/T \cap K$, it is given by:

$$\delta(\varpi^\mu) = \left| \det \left( \text{Ad} \left( \varpi^\mu \right) \right)_{n \in \text{Lie}(N)} \right| = \left| \varpi^{-\langle 2\rho, \mu \rangle} \right| = q^{\langle 2\rho, \mu \rangle} \quad (2.4)$$

Inside the representation $I(\chi)$, there is a unique vector (up to a scalar) $\Phi_\chi$ that is $K$ right-invariant. This is a simple consequence of the $G = NAK$ decomposition:

$$\forall (n, a, k) \in N \times A \times K, \Phi_\chi(nak) = \chi(a) \delta_\varpi(a) \Phi_\chi(Id)$$

We choose the usual normalisation given by $\Phi_\chi(Id) = 1$. Moreover, in the following, we will rather consider the vector $\Phi_{\chi \omega_0}$ inside the representation $I(\chi^\omega)$. Ultimately, the reason of this choice stems from the domain of absolute convergence we want to impose on the Whittaker function.
2.4 Jacquet’s Whittaker function (the non-Archimedean case)

Let $\psi : (K,+) \to (C^*, \times)$ be a character of $K$, trivial on the ring of integers, but non-trivial on $\varpi^{-1}O$. One obtains a character $\varphi_N = \psi \circ \chi_{st} : N \to C^*$ by composing $\psi$ with the standard character of $N$:

$$\chi_{st} = \sum_{\alpha \in \Delta} \chi_\alpha$$

where $\chi_\alpha (x_{-\beta}(t)) = t1_{\{\alpha = \beta\}}$. Jacquet’s Whittaker function associated to the Langlands parameter $z \in X \otimes C$ and the character $\varphi_N$ is by definition ([Jac67]):

$$\forall g \in G, W_z (g) = W_\chi (g) := \int_N \Phi_{\chi_{w_0}(\bar{w}_0ng)} \varphi_N (n)^{-1} dn$$

where $\bar{w}_0$ is any choice of representative for the longest element in the Weyl group. We will invariably use the subscript $z$ or $\chi$ depending on what is more convenient. The integral is not convergent for all $z \in X \otimes C$, but can be analytically extended. That fact is particularly obvious when looking at the Shintani-Casselman-Shalika formula. For now, let us list the following well-known properties:

**Proposition 2.3.**

- (Domain of definition) The integral in equation (2.5) is absolutely convergent for $\Re (z) \in \hat{C}$.
- (Boundedness) $\delta^{-\frac{1}{2}} \chi^{-1} W_\chi$ is bounded.
- (Invariance property)

$$\forall (n, g, k) \in N \times G \times K, \ W_\chi (ngk) = \varphi_N (n) W_\chi (g)$$

hence $W_\chi$ reduces to a function on $A$.
- (Asymptotic behavior) There is a function $c_G (z)$ given explicitly by the Gindikin-Karpelevich formula (theorem 2.5) such that, as $\lambda^\vee \to \infty$ in the dual Weyl chamber:

$$\lim_{\lambda^\vee \to \infty} \left( \delta^{-\frac{1}{2}} \chi^{-1} W_\chi \right) \left( \varpi^{-\lambda^\vee} \right) = c_G (z)$$

**Remark 2.4.** When considering limits $\lambda^\vee \to \infty$ in the dual Weyl chamber, we always mean that $\lambda^\vee$ goes to infinity away from the walls. Formally:

$$\forall \alpha \in \Delta, \ (\alpha, \lambda^\vee) \to \infty$$
Proof. For now, let us suppose that \( \chi \) is chosen so that the integral defining \( W_\chi \) converges absolutely. Equation (2.6) is proven by performing a change of variable in the integral formula:

\[
W_\chi(ngk) = \int_N \Phi_{\chi w_0}(\bar{w}_0 h ng k) \varphi_N(h) \, dh
= \int_N \Phi_{\chi w_0}(\bar{w}_0 h g) \varphi_N(h n^{-1}) \, dh
= \varphi_N(n) W_\chi(g)
\]

Because of the Iwasawa decomposition, we can focus on \( g \in A \). In this case, we start by proving the alternative expression:

\[
\forall a \in A, \ W_\chi(a) = \chi(a) \delta^{-\frac{1}{2}}(a) \int_N \Phi_{\chi w_0}(\bar{w}_0 n a) \varphi_N(ana^{-1}) \, dn \quad (2.8)
\]

The last equation is obtained as follows:

\[
W_\chi(a) = \int_N \Phi_{\chi w_0}(\bar{w}_0 n a) \varphi_N(n) \, dn
= \Phi_{\chi w_0}(a^{\nu_0}) \int_N \Phi_{\chi w_0}(\bar{w}_0 a^{-1} n a) \varphi_N(n) \, dn
= \chi(a) \delta^{-\frac{1}{2}}(a) \int_N \Phi_{\chi w_0}(\bar{w}_0 n a) \varphi_N(ana^{-1}) \, dn
= \chi(a) \delta^{\frac{1}{2}}(a) \int_N \Phi_{\chi w_0}(\bar{w}_0 n a) \varphi_N(ana^{-1}) \, dn
\]

Then, as the character \( \varphi_N \) is unitary, we see that for all \( a \in A \):

\[
\left| \delta^{-\frac{1}{2}} \chi^{-1} W_\chi \right|(a) = \left| \int_N \Phi_{\chi w_0}(\bar{w}_0 n a) \varphi_N(ana^{-1}) \, dn \right| \leq \int_N \Phi_{|\chi w_0|(\bar{w}_0 n)dn}
\]

where \( |\chi| \) is the real valued character such that for all \( \mu^{\nu} \in X^{\nu} \), \( |\chi| \left( \varpi^{-\mu^{\nu}} \right) = e^{i(\Re(z),\mu^{\nu})} \). Hence, thanks to the following theorem 2.5, the Whittaker function is indeed well-defined for \( \Re(z) \in \mathcal{C} \). Moreover, in such a case, \( \delta^{-\frac{1}{2}} \chi^{-1} W_\chi \) is bounded on \( A \) and a fortiori on the entire group, because of the invariance property.

Now, when considering \( a = \varpi^{-\lambda^{\nu}} \) in equation (2.8), we have:

\[
\left( \delta^{-\frac{1}{2}} \chi^{-1} W_\chi \right)(\varpi^{-\lambda^{\nu}}) = \int_N \Phi_{\chi w_0}(\bar{w}_0 n) \varphi_N(\varpi^{-\lambda^{\nu}} n \varpi^{\lambda^{\nu}}) \, dn
\]

If \( \lambda^{\nu} \to \infty \) while staying in the Weyl chamber \( C^{\nu} \) and away from the walls, the matrix coefficients of \( \varpi^{-\lambda^{\nu}} n \varpi^{\lambda^{\nu}} \) become \( \mathcal{O} \) valued. Hence, for
every $n \in N$, $\varphi_N \left( \varpi^{-\lambda^\vee} n \varpi^{\lambda^\vee} \right)^{-1}$ is stationary at 1 and using Lebesgue’s dominated convergence theorem:

$$\left( \delta^{-\frac{1}{2}} \chi^{-1} W_{\lambda} \right) \left( \varpi^{-\lambda^\vee} \right) \xrightarrow{\lambda \to \infty} \int_N \Phi_{\chi^{w_0}}(\tilde{w}_0 n) dn$$

This is equation (2.7).

The function $c_G$ is the non-Archimedean analogue of the Harish-Chandra $c$ function and appears as:

**Theorem 2.5** (Langlands - [Lan71] formula (4)). When $\Re(z) \in \hat{C}$:

$$\int_N \Phi_{|\chi^{w_0}|}(\tilde{w}_0 n) dn < \infty$$

and we have the Gindikin-Karpelevich formula for the non-Archimedean places:

$$c_G(z) := \int_N \Phi_{\chi^{w_0}}(\tilde{w}_0 n) dn = \prod_{\beta \in (\Phi^\vee)^+} \left( \frac{1 - q^{-1} e^{-\langle \beta^\vee, z \rangle} \chi V(\Lambda^\vee)(z)}{1 - e^{-\langle \beta^\vee, z \rangle}} \right)$$

### 2.5 Random walks

For the purpose of constructing probability measures, in all the following, we will assume that the Langlands parameter $z$ is real and in the interior of the Weyl chamber. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be our working probability space. We consider a random walk $W(z) = \left( W_t(z); t \in N \right)$ on $X^\vee$, starting at zero, with independent and identically distributed increments. We choose the following distribution on the Weyl group orbit of $\Lambda^\vee$:

$$\forall t \in N, \forall \mu^\vee \in W \Lambda^\vee, \mathbb{P} \left( W_{t+1}(z) - W_t(z) = \mu^\vee \right) = \frac{e^{\langle \mu^\vee, z \rangle}}{\text{ch} V(\Lambda^\vee)(z)} \quad \text{(2.9)}$$

Because $\Lambda^\vee$ is minuscule, the set $W \Lambda^\vee$ gives exactly the weights appearing in the representation $V(\Lambda^\vee)$. Therefore, the distribution of these increments is of representation-theoretic significance. Notice that the random walk $W(z)$ depends on the choice of minuscule coweight. When relevant, we will emphasise this dependence by writing $W(z)(\Lambda^\vee)$. Also, the Langlands parameter determines the drift, and is indicated in superscript notation. Since the Langlands parameter $z \in \hat{C}$, one can deduce that $E \left( W_1(z) - W_0(z) \right)$ is in the interior of the dual Weyl chamber $C^\vee$. By the law of large numbers, almost surely, the random walk $W(z)$ goes to infinity inside $C^\vee$. 

10
Now, we define \( B_t(W^{(z)}; t \in \mathbb{N}) \) as a left invariant random walk on \( B(K) \) with independent increments. More precisely:

\[
B_0(W^{(z)}) = b_0
\]

(2.10)

\[
\forall t \in \mathbb{N}, \quad B_{t+1}(W^{(z)}) = B_t(W^{(z)}) b_{t+1} W_t^{(z)} (W_{t+1}^{(z)} - W_t^{(z)}) b_t
\]

(2.11)

with \((b_t, b'; t \in \mathbb{N})\) all independent and distributed as \( \mathcal{U}(B(O)) \). Notice that the distribution of \( B_t(W^{(z)}) \) is bi-invariant under \( B(O) \).

Modulo \( T(O) = T \cap K \) on the right, we have the NA decomposition:

\[
B_t(W^{(z)}) = N_t(W^{(z)}) A_t(W^{(z)}) \mod T(O)
\]

(2.12)

with

\[
\forall t \in \mathbb{N}, \quad A_t(W^{(z)}) = W_t^{(z)}
\]

(2.13)

and

\[
N_0(W^{(z)}) = c_0
\]

(2.14)

\[
\forall t \in \mathbb{N}, \quad N_t(W^{(z)}) = N_{t-1}(W^{(z)}) W_t^{(z)} W_{t-1}^{(z)} c_t W_t^{(z)}
\]

(2.15)

where \((c_t; t \in \mathbb{N})\) are independent and distributed as \( \mathcal{U}(N(O)) \). These are obtained by multiplying some of the \((b_t, b'; t \in \mathbb{N})\) together, and taking the \( N \) part in the Iwasawa decomposition.

### 3 Main results

For \( z \in C \), let \( b(z) \) be the inverse of the Weyl denominator for the group \( G^\vee(C) \), evaluated at \( z \):

\[
b(z) = \frac{1}{\prod_{\beta \in (\Phi^\vee) +} (1 - e^{-\langle \beta^\vee, z \rangle})}
\]

Theorem 3.1. There exists one and only one locally constant function \( \psi_\chi : B \to \mathbb{C} \) such that:

- (Harmonicity) \( \chi^{-1} \psi_\chi : B \to \mathbb{C} \) is harmonic for the random walk \( (B_t(W^{(z)}); t \in \mathbb{N}) \) i.e:

\[
\forall t \in \mathbb{N}, \forall b \in B, \quad \chi(b^{-1} \psi_\chi(b)) = E \left[ (\chi^{-1} \psi_\chi) \left( b B_t(W^{(z)}) \right) \right]
\]

(3.1)

- (Boundedness) \( \chi^{-1} \psi_\chi \) is bounded.
• (Behavior at infinity) If $\lambda^y \to \infty$ in the interior of the dual Weyl chamber, then:

$$(\chi^{-1}\psi)(\varpi^{-\lambda^y}) \to b(z)$$

• (Invariance property) For all $(n, b, k) \in N \times B \times B(\mathcal{O})$, we have:

$$\psi(\eta_{N}(n)\psi(b)$$

It is given by a $G^y$-character:

$$\psi_{\chi}(\varpi^{-\lambda^y}) = \begin{cases} chV(\lambda^y)(z) & \text{if } \lambda^y \text{ dominant} \\ 0 & \text{otherwise} \end{cases}$$ \hspace{1cm} (3.2)

**Proof.** Uniqueness can be proved separately using the following standard martingale argument. Suppose there are two such functions, and name $g$ their difference. The martingale $(\chi^{-1}g)[B_t(W^z)]$ is bounded and goes to zero in infinite time. Therefore $g[B_t(W^z)]$ is identically zero.

In order to prove that $\psi_{\chi}$ is given by a character, one has to invoke successively propositions 3.4, 3.5 and 3.9. These are the subject of the next subsection.

**Theorem 3.2.** The function on $B$ defined by:

$$b \mapsto \left(\delta^{-\frac{1}{2}}\chi^{-1}\mathcal{W}_{\chi}\right)(b)$$

is harmonic for the random walk $(B_t(W^z) ; t \in \mathbb{N})$.

**Proof.** See subsection 5.4

We obtain as announced by putting together theorems 3.1 and 3.2:

**Theorem 3.3** (The Shintani-Casselman-Shalika formula [CS80], [Shi76]).

$$\mathcal{W}_{z}(\varpi^{-\lambda^y}) = \begin{cases} \delta^{-\frac{1}{2}}(\varpi^{-\lambda^y}) chV(\lambda^y)(z) \prod_{\beta \in \Phi^+} \left(1 - q^{-1}e^{-\langle \beta^\vee, z \rangle}\right) & \text{if } \lambda^y \text{ dominant} \\ 0 & \text{otherwise} \end{cases}$$ \hspace{1cm} (3.3)

**Proof.** Let $\psi_{\chi} : B \to \mathbb{C}$ denote the normalised Whittaker function as opposed to Jacquet’s Whittaker function:

$$\forall b \in B, \psi_{\chi}(b) := \frac{\mathcal{W}_{\chi}(b) \delta^{-\frac{1}{2}}(b)}{\prod_{\beta \in \Phi^+} \left(1 - q^{-1}e^{-\langle \beta^\vee, z \rangle}\right)}$$

This function satisfies the hypotheses of theorem 3.1: the harmonicity property is theorem 3.2 and the other properties come from proposition 2.3.
3.1 A characterisation of characters

Theorem 3.1 can be taken independently from the rest and seen as a characterisation of characters. For the rest of this section, we will see that its proof decomposes nicely into three steps.

**Proposition 3.4** (Poisson kernel formula). If \( \psi \) satisfies the hypotheses of theorem 3.1 then the following Poisson formula holds:

\[
\psi \left( \varpi^{-\lambda^\vee} \right) = b(z) e^{\langle z, \lambda^\vee \rangle} \mathbb{E} \left[ \varphi_N \left( \varpi^{-\lambda^\vee} N_\infty \left( W^{(z)} \right) \varpi^{\lambda^\vee} \right) \right] \tag{3.4}
\]

**Proof.** Because \( f := \chi^{-1} \psi \) is harmonic for \( (B_t \left( W^{(z)} \right); t \in \mathbb{N}) \), we have that for every \( b = na \in B \):

\[
\forall t \in \mathbb{N}, f(b) = \mathbb{E} \left[ f \left( b B_t \left( W^{(z)} \right) \right) \right]
\]

Using the NA decomposition from equation (2.12):

\[
b B_t \left( W^{(z)} \right) = n a N_t \left( W^{(z)} \right) a^{-1} a A_t \left( W^{(z)} \right) \mod T(O),
\]

and the invariance property, we have:

\[
\forall t \in \mathbb{N}, f(na) = \mathbb{E} \left[ \varphi_N \left( n a N_t \left( W^{(z)} \right) a^{-1} \right) f \left( a A_t \left( W^{(z)} \right) \right) \right]
\]

Since \( W^{(z)} \) goes to infinity while staying in the interior of \( C^\vee \), we have:

\[
f \left( a A_t \left( W^{(z)} \right) \right) = f \left( a \varpi^{-W^{(z)}} \right) \to b(z)
\]

We have the result by passing to the limit under the expectation. Boundedness for \( f \) gives the required domination in order to apply Lebesgue’s dominated convergence theorem.

By interpreting the distribution of \( N_\infty \left( W^{(z)} \right) \) as a Poisson kernel, this probabilistic representation formula is indeed a Poisson formula: the value of the harmonic function \( \chi^{-1} \psi \) prescribed by a measure on the “boundary”. Now, we proceed to rearranging the expression in equation (3.4).

**Proposition 3.5.**

\[
\mathbb{E} \left[ \varphi_N \left( \varpi^{\lambda^\vee} N_\infty \left( W^{(z)} \right) \varpi^{-\lambda^\vee} \right) \right] = \mathbb{P} \left( \lambda^\vee + W^{(z)} \text{ stays in } C^\vee \right)
\]

Basically, we claim that if we kill the random walk \( \lambda^\vee + W^{(z)} \) upon exiting \( C^\vee \), the \( \varpi \)-adic integral obtained from the distribution of \( N_\infty \left( W^{(z)} \right) \) is nothing but a probability of survival. The proof of proposition 3.5 is a consequence of three simple lemmas.
Lemma 3.6 (“Trivial key lemma”). If $U$ and $U'$ are independent and distributed as $U(O)$, then for any two integers $a$ and $b$ in $\mathbb{Z}$:

$$\varpi^a U + \varpi^b U' \overset{\mathcal{L}}{=} \varpi^{\min(a,b)} U$$

The lemma can be reformulated by saying that the Haar measure on $O$ is tropicalisation friendly. Although simple, this fact is crucial because the combinatorics of representation theory using crystals happen in the tropical world, where the operations (min, +, −) replace the algebraic (+, ×,/) (see for example [BFZ96], or paragraph 42.2 in [Lus10]). As an application, one observes the natural appearance of the minimal (signed) distances of the random walk $W(z)$ to each wall of the (dual) Weyl chamber:

$$\left( \inf_{0 \leq s \leq t} \langle \alpha, W_s(z) \rangle \right)_{(\alpha \in \Delta)}$$

These are very reminiscent of the Littelmann path model, which we implicitly use. That matter is further discussed in the next subsection.

Lemma 3.7. For every $t \in \mathbb{N} \cup \{\infty\}$, there are independent Haar distributed random variables $(C^\alpha)_{(\alpha \in \Delta)}$ such that:

$$\forall \alpha \in \Delta, t \in \mathbb{N}, \chi^- \left( N_t \left( W(z) \right) \right) = \varpi^{\inf_{0 \leq s \leq t}(\alpha, W_s(z))} C^\alpha$$

The $(C^\alpha)_{(\alpha \in \Delta)}$ implicitly depend on $t$ and are not necessarily independent for different values of $t$.

Proof. First, let us consider $t < \infty$. Because of equations (2.14, 2.15) and $\chi^-$ being a character, we have that for every fixed simple root $\alpha$:

$$\chi^- \left( N_t \left( W(z) \right) \right) = \sum_{s=0}^{t} \chi^- \left( \varpi^{-W_s(z)} c_s \varpi^{W_s(z)} \right)$$

$$= \sum_{s=0}^{t} \varpi^{(\alpha, W_s(z))} \chi^- \left( c_s \right)$$

Since each $c_s$ is Haar distributed on $N(O)$, we see that the random variables $(\chi^- \left( c_s \right))_{(0 \leq s \leq t, \alpha \in \Delta)}$ are independent and Haar distributed on $O$. By applying the “trivial key lemma” 3.6, we obtain the result.

The case $t = \infty$ is obtained by a limit argument.

Recall that $\psi : (\mathcal{K}, +) \to (\mathbb{C}^*, \times)$ is a character, trivial on $O$ but non trivial on $\varpi^{-1}O$. Here is another simple but important property that bridges integration over non-Archimedean fields and taking indicator functions:
Lemma 3.8 (Averaging lemma). If $U \subseteq \mathcal{U}(O)$, then:

$$\forall x \in \mathcal{K}, \mathbb{E}[\psi(xU)] = \mathbbm{1}_{\{x \in O\}}$$

Proof. The claim is clearly true if $x \in O$, as $\psi$ is trivial on $O$. If $x \notin O$, there is an element $y \in O$ such that $xy = \varpi^{-1}$. Therefore $xU \triangleq xU + \varpi^{-1}$. Hence:

$$\psi(xU) \triangleq \psi(xU) \zeta$$

where $\zeta = \psi(\varpi^{-1})$ is a non-trivial $q$-th root of unity. One concludes by averaging roots of unity:

$$0 = \frac{1}{q} \sum_{k=0}^{q-1} \zeta^k = \frac{1}{q} \left( \sum_{k=0}^{q-1} \zeta^k \right) \mathbb{E}(\psi(xU)) = \frac{1}{q} \sum_{k=0}^{q-1} \mathbb{E}(\zeta^k \psi(xU)) = \mathbb{E}[\psi(xU)]$$

Proof of theorem 3.5. The following sequence of equalities hold.

$$\mathbb{E}\left[\varphi_N(\varpi^{-\lambda^\vee}N_{\infty}(W(z))\varpi^{\lambda^\vee})\right] = \mathbb{E}\left[\prod_{\alpha} \psi \circ \chi_{-\varpi^{-\lambda^\vee}N_{\infty}(W(z))\varpi^{\lambda^\vee}}\right]$$

Lemma 3.7

$$\mathbb{E}\left[\prod_{\alpha} \psi(\varpi^{(\alpha,\lambda^\vee) + \inf_{\alpha \leq \lambda} \{\alpha, W(z)\}} C^\alpha)\right]$$

Lemma 3.8

$$\mathbb{E}\left[\prod_{\alpha} \mathbbm{1}_{\{\inf_{\alpha \leq \lambda} \{\alpha, W(z)\} \geq 0\}}\right] \text{ (Conditionally on } W(z))$$

$$= \mathbb{P}\left(\forall \alpha \in \Delta, \inf_{\alpha \leq \lambda} \{\alpha, W(z)\} \geq 0\right)$$

$$= \mathbb{P}\left(\lambda^\vee + W(z) \text{ stays in } C^\vee\right)$$

In section 4, we will prove a reflection principle that allows to relate the random walk $W(z)$ to the character of the irreducible representation $V(\lambda^\vee)$. This is exactly corollary 7.7 in [LLP12], which is proved using the Littelmann path model, also in the minuscule case. We give an elementary proof using basic tool such as the reflection principle.

Proposition 3.9. If $\lambda^\vee$ is dominant:

$$\text{ch} V(\lambda^\vee)(z) = b(z)e^{(\lambda^\vee, z)} \mathbb{P}\left(\lambda^\vee + W(z) \text{ stays in } C^\vee\right)$$

Proof. Consequence of theorem 4.1 and the Weyl character formula. \qed

We will now discuss the relationship of the forementioned results to crystal combinatorics and the similarities with the Archimedean case.
3.2 Similarities with the Archimedean case

The same Poisson formula as in proposition 3.4 holds in the Archimedean case with a Euclidian Brownian motion instead of the random walk \( W^{(z)} \) and a hypoelliptic Brownian motion on the group instead of the random walk \( (B_t (W^{(z)}); t \in \mathbb{N}) \). See [Chh13]. The discrete setting is richer as it depends on the dominant coweight \( \Lambda^\vee \); while the only canonical random walk in continuous time is Brownian motion.

In the same direction, one can notice that the formula in proposition 3.9 is virtually the same formula as for the normalised Archimedean function in [Chh13]: instead of considering the probability of survival of random walk killed upon exiting the Weyl chamber, one has to consider the probability of survival of a Brownian motion with a killing measure given by the Toda potential.

3.3 Relation to crystals

We used very indirectly crystal combinatorics, or more precisely the Littelmann path model, thanks to which each instance of the random walk \( W^{(z)} \) can be viewed as a crystal element. In fact, proposition 3.9 holds in the context of general Kac-Moody groups as proved in the subsequent paper [LLP13] of Lesigne et al., but one has to observe Littelmann paths continuously in time.

In the Archimedean case, there are geometric crystals lurking behind the scene [Chh14a].

4 The reflection principle for minuscule walks

It is understood that minuscule walks are basically the only reasonable random walks for which the reflection principle applies (see [Bia92] end of §2). The fact that increments lie in \( W \Lambda^\vee \), the Weyl group orbit of \( \Lambda^\vee \), insures that the walk \( W^{(z)} \) is reflectable, i.e. that \( W^{(z)} \) cannot exit the \( C^\vee \) without having occupied a lattice point on a wall. For a classification of reflectable walks on general lattices, see [GM93]. Their result boils down to taking minuscule steps or compatible ones.

The following generalised reflection principle has been rediscovered on many occasions. In enumerative combinatorics, if the goal is to count the number of non-intersecting paths of a given length, the formula is known as the Lindström-Gessel-Viennot principle (see discussion in the section 2.7 of [Sta12]). When dealing with continuous time Markov processes that are evolving independently, and with the goal of computing a non-intersection probability, the result is even older and dates back to Karlin and McGregor [KM59]. Here, we want to consider discrete time random walks with non-uniform weights and an infinite time horizon. Also, if the ambient lattice is
isomorphic to \( \mathbb{Z}^r \), most of the random walks at hand cannot be treated as \( r \) independent walkers. Therefore, we feel that applying existing theorems to our setting is tedious. Our result is more elegantly obtained by a direct proof, while of course using the same ideas.

**Theorem 4.1** (Reflection principle). We have:

\[
P \left( \lambda^\vee + W(z) \text{ stays in } C^\vee \right) = \sum_{w \in W} (-1)^{\ell(w)} e^{(w(\lambda^\vee + \rho^\vee) - (\lambda^\vee + \rho^\vee), z)}
\]

**Proof.** Recall that because of the law of large numbers, almost surely, \( W(z) \) will eventually enter the Weyl chamber and stay there. By performing a translation by the Weyl covector \( \rho^\vee \), we see that \( \lambda^\vee + W(z) \) stays in the dual Weyl chamber (walls included) if and only if \( \rho^\vee + \lambda^\vee + W(z) \) never hits a wall. Let:

\[\tau := \inf \left\{ t \in \mathbb{N} \mid \rho^\vee + \lambda^\vee + W_t(z) \in \partial C^\vee \right\}\]

be the first time \( \rho^\vee + \lambda^\vee + W_t(z) \) hits a wall. The convention is that \( \tau = \infty \) if the event never occurs.

The sample path space is \( \Pi = (X^\vee)^{\mathbb{N}} \). Consider the functional \( F: \Pi \to \mathbb{R} \):

\[
\forall \pi \in \Pi, \quad F(\pi) := \sum_{w \in W} (-1)^{\ell(w)} 1_{\{\pi(0) \in wC^\vee\}}
\]

As a first ingredient for the reflection principle, we want to define an involutive transform \( T: \Pi \to \Pi \) that reflects the portion of a path before \( \tau \), if \( \tau < \infty \). If \( \tau = \infty \), we set \( T(\pi) = \pi \). Hence:

\[\tau = \infty \Rightarrow F \circ T(\pi) = F(\pi) \quad (4.1)\]

The reflection considered is with respect to the wall \( H_\alpha = \text{Ker } \alpha \) that the random walk hit first. However, if the exit point is on more than one wall, there is an ambiguity in our choice of wall. To that effect, one can choose an arbitrary order on the walls (or equivalently on simple roots). If the path \( \pi \) hits the hyperplane \( H_\alpha \) at the time \( \tau \), \( \alpha \) being the smallest in our arbitrary order, we set:

\[T(\pi)_t := \begin{cases} s_\alpha (\pi(t)) & \text{if } t < \tau \\ \pi(t) & \text{otherwise} \end{cases}\]

The transform is clearly involutive on the sample path space \( \Pi \) and leaves \( \tau \) unchanged. Moreover:

\[\tau < \infty \Rightarrow F \circ T(\pi) = -F(\pi) \quad (4.2)\]

At the moment, the transform \( T \) does not preserve the distribution of \( \rho^\vee + \lambda^\vee + W(z) \). In order to compensate for that issue, we randomise the starting point. This is the second ingredient in the reflection principle.
Consider a random coweight $\nu$ distributed on the Weyl group orbit of $\lambda + \rho$ as follows:

$$
P(\nu = w(\lambda + \rho)) = \frac{e^{\langle w(\lambda + \rho), z \rangle}}{\sum_{w' \in W} e^{\langle w'(\lambda + \rho), z \rangle}} \quad (4.3)
$$

We claim that, upon this randomisation, the distribution of $\nu + W(z)$ is preserved under $T$. In order to see that, it suffices to check that for any finite time horizon $t \in \mathbb{N}$:

$$
\left( T(\nu + W(z))_s; 0 \leq s \leq t \right) \overset{d}{=} \left( \nu + W(z)_s; 0 \leq s \leq t \right) \quad (4.4)
$$

Notice that $T(\nu + W(z))$ and $\nu + W(z)$ coincide after $\tau$, which is a stopping time. By the strong Markov property for $W(z)$, the question is reduced to proving that for every path $\pi \in \Pi$, hitting $\partial C^\vee$ at time $t$ for the first time, we have:

$$
P(\forall 0 \leq s \leq t, T(\nu + W(z))_s = \pi(s)) = P(\forall 0 \leq s \leq t, \nu + W(z)_s = \pi(s))
$$

Or equivalently, if $\pi$ hits $H_\alpha = \text{Ker} \alpha$ first, we have thanks to the random walk’s transition probabilities (equation (2.9)):

$$
P(\nu = s_\alpha(0)) e^{\langle s_\alpha(\pi(t) - \pi(0)), z \rangle} = P(\nu = \pi(0)) e^{\langle \pi(t) - \pi(0), z \rangle}
$$

Because $\pi(t) \in H_\alpha$, $s_\alpha(\pi(t)) = \pi(t)$. Hence, in the end, equation (4.4) is equivalent to:

$$
P(\nu = s_\alpha(0)) e^{\langle -s_\alpha(\pi(0)), z \rangle} = P(\nu = \pi(0)) e^{\langle -\pi(0), z \rangle}
$$

From the previous equation, one sees that our choice (equation (4.3)) for the distribution of $\nu$ was in fact the only possible choice.

We are now ready to finish the proof. On the one hand, because of implication (4.1), and the fact that $\nu + W(z)$ has a chance of never hitting $\partial C^\vee$ only in the case of $\nu = \lambda + \rho$, we have:

$$
E \left( F(\nu + W(z)) 1_{\{\tau = \infty\}} \right) = P(\nu = \lambda + \rho) P(\lambda + \rho + W(z) \text{ hits } \partial C^\vee)
$$

On the other hand, because of implication (4.2) and the fact that $T$ preserves the distribution of $\rho + \Lambda + W(z)$, we also have:

$$
E \left( F(\nu + W(z)) 1_{\{\tau < \infty\}} \right) = 0
$$

Therefore, by putting together the two previous equations:

$$
P(\lambda + W(z) \text{ stays in } C^\vee)$$
\[ P(\lambda^\vee + \rho^\vee + W(z) \text{ hits } \partial C^\vee) \]
\[ = \frac{E(F(\mu^\vee + W(z)) \mathbf{1}_{T=\infty})}{P(\mu^\vee = \lambda^\vee + \rho^\vee)} \]
\[ = \frac{E(F(\mu^\vee + W(z)))}{P(\mu^\vee = \lambda^\vee + \rho^\vee)} \]
\[ = \sum_{w \in W} (-1)^{\ell(w)} P(\mu^\vee \in wC^\vee) \]
\[ = \sum_{w \in W} (-1)^{\ell(w)} e(w(\lambda^\vee + \rho^\vee) - (\lambda^\vee + \rho^\vee), z) \]

5 Harmonic properties of Whittaker functions

We will now review the facts we will need around the Satake isomorphism and spherical functions. For more details, the reader is referred to [Gro98] for a pedagogical overview of the Satake isomorphism and to [Car79] for a more complete survey, including p-adic representation theory. The complete reference for spherical functions is [Mac71].

5.1 Around the Satake isomorphism

Given two locally constant functions \((\varphi_1, \varphi_2) \in \mathcal{F}(G)\), one can define their convolution \(\varphi_1 *_G \varphi_2\) by:

\[ \forall g \in G, \varphi_1 *_G \varphi_2(g) := \int_G \varphi_1(gh^{-1}) \varphi_2(h) \, dh \]

This is possible as soon as one of the two functions is compactly supported. Let \(H(G, K) = \mathcal{F}(K\backslash G/K)\) be the space of (compactly supported) \(K\)-bi-invariant functions on the group. It is also known as the spherical Hecke algebra. Since the convolution of two bi-invariant functions is bi-invariant, \(H(G, K)\) is a convolution algebra.

In the same fashion, let \(H(T, T(O))\) denote the space of functions on \(T\), invariant under \(T(O) = T \cap K\). It is also a convolution \(\mathbb{C}\)-algebra with a much simpler structure. It is spanned by the indicator functions \(\mathbf{1}_{\{w^\mu \in X^\vee\}}\). When we identify \(\mathbf{1}_{\{w^\mu \in T(O)\}}\) with the formal exponential \(e^{w^\mu}\), we have a \(\mathbb{C}\)-algebra isomorphism \(H(T, T(O)) \approx \mathbb{C}[X^\vee]\). For an unramified character \(\chi : T \to \mathbb{C}^*\) and a function \(f \in H(T, T(O))\), we write the convolution product:

\[ f(\chi) := \chi *_T f(\text{Id}) = \int_T f(t^{-1}) \chi(t) \, dt \quad (5.1) \]

19
Notice that, if the character $\chi$ is given by a $z \in X \otimes \mathbb{C}$ such that for all $\mu^\vee \in X^\vee$, $\chi(w^{-\mu^\vee}) = e^{(z, -\mu^\vee)}$, we have $e^{\mu^\vee}(\chi) = e^{(z, -\mu^\vee)}$. Therefore, one can simply think of the notation in equation (5.1) as an evaluation.

The Satake transform $S : H(G, K) \to H(T, T(O))$ is defined by an integral transform similar to those studied by Harish-Chandra, in the real case. For $\mathcal{H} \in H(G, K)$, $S(\mathcal{H})$ is given by:

$$\forall t \in T, S(\mathcal{H})(t) := \delta(t)^{\frac{1}{2}} \int_N dn \mathcal{H}(tn) = \delta(t)^{-\frac{1}{2}} \int_N dn \mathcal{H}(nt) \quad (5.2)$$

In fact, the Satake transform takes its values in $\mathbb{C}[X^\vee]^W$, the space of functions on $X^\vee$ invariant under $W$. An even more involved statement is:

**Theorem 5.1** (Satake [? - see also for e.g [Gro98] Proposition 3.6 ). *The Satake transform is an algebra isomorphism between the spherical Hecke algebra $H(G, K)$ and $\mathbb{C}[X^\vee]^W$.***

The importance of the Satake transform for us stems from:

**Proposition 5.2.** Both the spherical vector $\Phi_\chi$ and the Whittaker function $W_\chi$ are convolution eigenfunctions by elements in $\mathcal{H} \in H(G, K)$:

$$\Phi_\chi *_G \mathcal{H} = S(\mathcal{H})(\chi) \Phi_\chi \quad (5.3)$$

$$W_\chi *_G \mathcal{H} = S(\mathcal{H})(\chi) W_\chi \quad (5.4)$$

**Proof.** Here, we will use the Iwasawa decomposition $G = KAN$ (instead of $NAK$) along with fact that the Haar measure decomposes as (see e.g [Car79] §4.1):

$$dg = \delta(a) \, dk \, da \, dn$$

We only need to prove both identities on $NA$ since the convolution of two right $K$ invariant functions is right $K$ invariant. Moreover, terms on both side of an equality have the same behavior. In the first identity, they are left $N$ invariant. In the second identity, they have the twisting property of equation (2.6) under the left action of $N$. This reduces the reasoning to proving that for $a \in A$:

$$\Phi_\chi *_G \mathcal{H}(a) = \int_G \Phi_\chi(ah^{-1})\mathcal{H}(h)dh$$

$$= \int_A db \, \delta(b) \int_N dn \Phi_\chi(a(nb)^{-1}) \mathcal{H}(bn)$$

$$= \int_A db \, \delta(b) \, \Phi_\chi(ab^{-1}) \int_N dn \mathcal{H}(bn)$$

$$= \Phi_\chi(a) \int_A db \, \delta(b) \Phi_\chi(b^{-1}) \int_N dn \mathcal{H}(bn)$$

$$= \Phi_\chi(a) \int_A db \, \chi^{-1}(b) \delta(b)^{\frac{1}{2}} \int_N dn \mathcal{H}(bn)$$

20
\[
\Phi \chi(a) \int_A db \chi(b)^{-1} S(H)(b) = S(H)(\chi) \Phi \chi(a)
\]

And:

\[
W_\chi \ast_G H(a) = \int_G W_\chi(ah^{-1})H(h)dh
\]

Eq \((2.5)\)

\[
= \int_N dn \int_G dh \Phi_{\chi^w_0}(\bar{w}_0nah^{-1})H(h)\varphi_N(n)^{-1}
\]

\[
= \int_N dn \Phi_{\chi^w_0} \ast_G H(na)\varphi_N(n)^{-1}
\]

Eq \((5.3)\)

\[
= S(H)(\chi^w_0) \int_N dn \Phi_{\chi^w_0}(\bar{w}_0na)\varphi_N(n)^{-1}
\]

\[
= S(H)(\chi) W_\chi(a)
\]

Interchanging integrals in the previous computations is allowed because the function \(H\) is compactly supported and the integral defining \(W_\chi\) is absolutely convergent.

For a probabilist, being a convolution eigenfunction is very close to being harmonic for a certain random walk on \(G\). To that endeavour, the eigenvalue would need to be one and the spherical Hecke algebra element must give rise to a probability measure. Therefore, the previous theorem is at the heart of our approach, once restated in a probabilistic fashion.

In order to carry out our program, we will need the explicit computation of \(S(H)\) for certain \(H\).

### 5.2 On Macdonald’s spherical function

Because of the Cartan decomposition (Eq. \((2.1)\)), the indicator functions \(\mathbf{1}_{\{K\varpi_\lambda K\}}\) of double \(K\)-cosets form a basis of \(H(G, K)\). Therefore, the symmetric functions \(S(\lambda^\vee) := S(\mathbf{1}_{\{K\varpi_\lambda K\}})\) form a basis of \(\mathbb{C}[X^\vee]^W\). They coincide in fact with the Macdonald spherical functions (\[Mac71\] Proposition 3.3.1). We record in the next proposition two well-known facts. The first one is that the \(S(\lambda^\vee)\) encodes the cardinalities of \(G/K\) cosets when intersecting strata from the Iwasawa and Cartan decompositions. The second one is that \(S(\lambda^\vee)\) is proportional to a character when \(\lambda^\vee\) is minuscule.

If \(A \subset G\) is a compact left \(K\)-invariant subset, then \(A\) is a finite union of \(G/K\) cosets. The number of such cosets is denoted:

\[
\text{Card}_{G/K}(A) = \int_A dg
\]
Proposition 5.3 (Expressions for the spherical functions). For every $\lambda^\vee$ dominant:

$$S(\lambda^\vee) = \sum_{\mu^\vee \in X^\vee} \text{Card}_{G/K} \left( N_{\varpi^{-\mu^\vee} K} \cap K \varpi^{-\lambda^\vee} K \right) q^{-(\mu^\vee, \rho)} e^{\mu^\vee}$$ (5.5)

and if $\lambda^\vee$ is minuscule:

$$S(\lambda^\vee) = q^{(\rho, \lambda^\vee)} \text{ch} V(\lambda^\vee)$$ (5.6)

Proof. We will use the fact that, in the Iwasawa decomposition $G = NAK$, the Haar measure decomposes as (see again e.g. [Car79] §4.1):

$$dg = \delta^{-1}(a) \, dk \, da \, dn$$

With the identification of the indicator function $1_{\{ \varpi^\mu T(O) \}}$ with the formal exponential $e^{\mu^\vee}$ for $\mu^\vee \in X^\vee$, we have:

$$S(\lambda^\vee) = \sum_{\mu^\vee \in X^\vee} S \left( 1_{\{ K \varpi^{\lambda^\vee} K \}} \left( \varpi^{\mu^\vee} \right) e^{\mu^\vee} \right)$$

Eq (5.2)

$$= \sum_{\mu^\vee \in X^\vee} \delta^{\frac{1}{2}} \left( \varpi^{\mu^\vee} \right) e^{\mu^\vee} \int_N \text{dn} 1_{\{ K \varpi^{\lambda^\vee} K \}} \left( \varpi^{\mu^\vee} n \right)$$

$$= \sum_{\mu^\vee \in X^\vee} \delta^{\frac{1}{2}} \left( \varpi^{\mu^\vee} \right) e^{\mu^\vee} \int_K \text{dk} \int_N \text{dn} 1_{\{ K \varpi^{\lambda^\vee} K \}} \left( \varpi^{\mu^\vee} n \right)$$

Iwasawa

$$= \sum_{\mu^\vee \in X^\vee} q^{-(\rho, \mu^\vee)} e^{\mu^\vee} \int_{N_{\varpi^{-\mu^\vee} K} \cap K_{\varpi^{-\lambda^\vee} K}} dg$$

$$= \sum_{\mu^\vee \in X^\vee} q^{-(\rho, \mu^\vee)} \text{Card}_{G/K} \left( N_{\varpi^{-\mu^\vee} K} \cap K \varpi^{-\lambda^\vee} K \right) e^{\mu^\vee}$$

The second relation can be found as equation (3.13) in [Gro98]. A formal argument is sketched as follows. We already know thanks to the Satake isomorphism that $S(\lambda^\vee)$ is $W$-invariant. As a consequence, by expressing the spherical function in the basis of $G^\vee$-characters, there are coefficients $a_{\lambda^\vee, \mu^\vee}$ such that:

$$S(\lambda^\vee) = \sum_{\mu^\vee \in (X^\vee)^+} a_{\lambda^\vee, \mu^\vee} \text{ch} V(\mu^\vee)$$

Let us write $\mu^\vee \leq \lambda^\vee$ for $\lambda^\vee - \mu^\vee \in (Q^\vee)^+$. Because $N_{\varpi^{-\mu^\vee} K} \cap K \varpi^{-\lambda^\vee} K = \emptyset$ unless $\mu^\vee \leq \lambda^\vee$ and $N_{\varpi^{-\mu \lambda^\vee} K} \cap K \varpi^{-\lambda^\vee} K$ is only one coset, we have in fact:

$$S(\lambda^\vee) = q^{(\rho, \lambda^\vee)} \text{ch} V(\lambda^\vee) + \sum_{\mu^\vee \leq \lambda^\vee} a_{\lambda^\vee, \mu^\vee} \text{ch} V(\mu^\vee)$$

If $\lambda^\vee$ is minuscule, only the first term remains.
In the case of $\lambda^\vee$ not minuscule, $S(\lambda^\vee)$ becomes a linear combination of characters $\text{ch} V(\mu^\vee)$ for $\mu^\vee \leq \Lambda^\vee$. Such a change of basis involves Kazhdan-Lusztig polynomials and goes beyond the scope of the present work. Before diving into the probabilistic part of this section, we need to state the following.

**Corollary 5.4.** If $\lambda^\vee$ is dominant and $\mu^\vee \in W\lambda^\vee$, then:

$$N\varpi^{-\mu^\vee}K \cap K\varpi^{-\lambda^\vee}K = N(O)\varpi^{-\mu^\vee}K$$

**Proof.** The fact that the right-hand side is included in the left-hand side is immediate. In order to prove the reverse inclusion, it suffices that both sets have the same cardinalities as $G/K$ cosets. On the one hand, thanks to the $W$-invariance in equation (5.5) and the fact that $N\varpi^{-\mu^\vee}K \cap K\varpi^{-\lambda^\vee}K$ is only one coset we have that:

$$\text{Card}_{G/K}(N\varpi^{-\mu^\vee}K \cap K\varpi^{-\lambda^\vee}K) = q^{-\langle w_0\lambda^\vee - \mu^\vee, \rho \rangle}$$

On the other hand, if $n \in N(O)$ then ([Ste68] lemmas 17 and 18), for any fixed order on the positive roots, there are unique parameters $(o_\beta)_{\beta \in \Phi^+}$ in $O$ such that

$$n = \prod_{\beta \in \Phi^+} x_{-\beta}(o_\beta).$$

By choosing an order such that:

$$n = \prod_{\beta \in \Phi^+, \langle \beta, \mu^\vee \rangle \geq 0} x_{-\beta}(o_\beta) \prod_{\beta \in \Phi^+, \langle \beta, \mu^\vee \rangle < 0} x_{-\beta}(o_\beta),$$

we obtain:

$$n\varpi^{-\mu^\vee}K = \varpi^{-\mu^\vee} \varpi^{\mu^\vee} n\varpi^{-\mu^\vee}K$$

$$= \varpi^{-\mu^\vee} \prod_{\beta \in \Phi^+, \langle \beta, \mu^\vee \rangle \geq 0} x_{-\beta}(\varpi^{-\langle \beta, \mu^\vee \rangle o_\beta}) \prod_{\beta \in \Phi^+, \langle \beta, \mu^\vee \rangle < 0} x_{-\beta}(\varpi^{-\langle \beta, \mu^\vee \rangle o_\beta}) K$$

$$= \varpi^{-\mu^\vee} \prod_{\beta \in \Phi^+, \langle \beta, \mu^\vee \rangle \geq 0} x_{-\beta}(\varpi^{-\langle \beta, \mu^\vee \rangle o_\beta}) K$$

Here, we write $\mu^\vee = w\lambda^\vee$. As the parameters $\varpi^{-\langle \beta, \mu^\vee \rangle o_\beta} \mod O$ uniquely determine the coset $n\varpi^{-\mu^\vee}K$, we see that

$$\text{Card}_{G/K}(N\varpi^{-\mu^\vee}K \cap K\varpi^{-\lambda^\vee}K) = q^a,$$

23
with \(a\) being the integer

\[
a = \sum_{\beta \in \Phi^+} \langle \beta, \mu^\vee \rangle = \sum_{\beta \in w^{-1} \Phi^+} \langle \beta, \lambda^\vee \rangle = \sum_{\beta \in w^{-1} w_0 \Phi^-} \langle \beta, \lambda^\vee \rangle
\]

Since \(\langle \beta, \lambda^\vee \rangle\) is either zero or negative when \(\beta\) is a negative root, we can restrict the previous summation index to \(\beta \in \Phi^+\) and discard the condition \(\langle \beta, \lambda^\vee \rangle \geq 0\). Hence a summation over the inversion set \(\text{Inv}(w_0^{-1} w \Phi^-) = \Phi^+ \cap w^{-1} w_0 \Phi^-\). Using the fact that (consequence of corollary 1.3.22 in [Kum02])

\[
\forall v \in W, \sum_{\beta \in \text{Inv}(v)} \beta = \rho - v \rho,
\]

we obtain:

\[
a = \left( \sum_{\beta \in \Phi^+ \cap w^{-1} w_0 \Phi^-} \beta, \lambda^\vee \right) = \langle \rho - w^{-1} w_0 \rho, \lambda^\vee \rangle = \langle \rho, \lambda^\vee - w_0 \mu^\vee \rangle
\]

5.3 A spherical and a solvable random walk

The elements of the spherical Hecke algebra can be seen as bi-invariant probability measures. This allows the construction of random walks on the group, for which the Whittaker functions should be harmonic. More precisely, we will work on a slightly different level, on the solvable group \(B\). Because of the Iwasawa (NAK) decomposition, one identifies \(G/K\) with \(B(K) / B(\mathcal{O})\). Therefore harmonic properties for functions in \(\mathcal{F}(G/K)\) have counterparts for functions in \(\mathcal{F}(B)\). For every discrete time Markov process \(X\), we will denote by \(F^X = (F^X_t)_{t \in \mathbb{Z}}\) the natural filtration generated by \(X\).

Let \(\lambda^\vee\) be a dominant cocharacter. Define \((G_t; t \geq 0)\) as the left invariant random walk on \(G\) with i.i.d. increments distributed as \(U(K) \varpi^{-\lambda^\vee} U(K)\):

\[
G_t := g_1 g_2 \ldots g_t \quad \text{and} \quad \forall s \in \mathbb{N}, g_s \in U(K) \varpi^{-\lambda^\vee} U(K)
\]

The previous product has to be understood as a product of independent random variables. We will refer to this walk as the spherical random walk as its increments are \(K\) bi-invariant. The following proposition summarises its properties.

**Proposition 5.5.** The law of the random variable \(U(K) \varpi^{-\lambda^\vee} U(K)\) is given as follows. For every \(f \in \mathcal{F}(G)\):

\[
\mathbb{E} \left[ f \left( U(K) \varpi^{-\lambda^\vee} U(K) \right) \right] = \frac{1}{\text{Card}_{G/K} (K \varpi^{-\lambda^\vee} K)} \int_G f(g) \mathbb{1}_{\{g \in K \varpi^{-\lambda^\vee} K\}} dg
\]
and the fact that the Whittaker function is a convolution eigenfunction can be restated as an \(\alpha\)-harmonicity property:

\[
E \left( W_\chi(G_{t+1}) \mid \mathcal{F}_t^G, G_t = g \right) = \frac{S(\lambda^\vee)(\chi)}{\operatorname{Card}_{G/K}(K\varpi^{-\lambda^\vee}K)} W_\chi(g)
\]

**Proof.** Clearly the law of our random variable is absolutely continuous with respect to the Haar measure on \(G\): because \(K\) is a compact open subgroup, the Haar measure on \(K\) is nothing but \(\mathbf{1}_{\{g \in K\}} dg\). Thus, the expectation \(E \left[ f \left( \mathcal{U}(K) \varpi^{-\lambda^\vee} \mathcal{U}(K) \right) \right] \) is easily written as an integral against \(dg\). The Radon-Nikodym derivative has to be a \(K\) bi-invariant function, and hence proportional to \(\mathbf{1}_{\{g \in K\varpi^{-\lambda^\vee}K\}}\). The proportionality constant is given by the volume of \(K\varpi^{-\lambda^\vee}K\) or equivalently the number of right \(K\) cosets it contains. Hence a proof of the first statement.

Another proof consists in checking the formula by doing the computation backwards. Using the left and right invariance of the Haar measure, we obtain:

\[
\int_G f(g) \mathbf{1}_{\{g \in K\varpi^{-\lambda^\vee}K\}} dg = \int_K dk_1 \int_K dk_2 \int_G f(k_1 g k_2) \mathbf{1}_{\{g \in K\varpi^{-\lambda^\vee}K\}} dg
\]

Fubini

\[
= \int_G dk \mathbf{1}_{\{g \in K\varpi^{-\lambda^\vee}K\}} \int_K dk_1 \int_K dk_2 f(k_1 g k_2)
\]

\[
= \operatorname{Card}_{G/K} \left( K\varpi^{-\lambda^\vee}K \right) \int_K dk_1 \int_K dk_2 f(k_1 \varpi^{-\lambda^\vee}k_2)
\]

\[
= \operatorname{Card}_{G/K} \left( K\varpi^{-\lambda^\vee}K \right) E \left( f \left( \mathcal{U}(K) \varpi^{-\lambda^\vee} \mathcal{U}(K) \right) \right)
\]

For the second statement, \(\mathcal{F}_t^G\) being the natural filtration of the random walk \((G_t; t \in \mathbb{Z})\). We have:

\[
\mathbb{E} \left[ W_\chi(G_{t+1}) \mid \mathcal{F}_t^G, G_t = g \right] = \mathbb{E} \left[ W_\chi(gg_{t+1}) \right]
\]

\[
= \frac{1}{\operatorname{Card}_{G/K}(K\varpi^{-\lambda^\vee}K)} \int_G W_\chi(gh) \mathbf{1}_{\{h \in K\varpi^{-\lambda^\vee}K\}} dh
\]

\(G\) unimodular

\[
= \frac{1}{\operatorname{Card}_{G/K}(K\varpi^{-\lambda^\vee}K)} \int_G W_\chi(gh^{-1}) \mathbf{1}_{\{h \in K\varpi^{\lambda^\vee}K\}} dh
\]

\[
= \frac{1}{\operatorname{Card}_{G/K}(K\varpi^{-\lambda^\vee}K)} \left( W_\chi \ast_G \mathbf{1}_{\{K\varpi^{\lambda^\vee}K\}} \right)(g)
\]

Eq (5.4)

\[
= \frac{S(\lambda^\vee)(\chi)}{\operatorname{Card}_{G/K}(K\varpi^{-\lambda^\vee}K)} W_\chi(g)
\]

25
Now, let us consider a random walk \((B_t; t \in \mathbb{Z})\) on the solvable group \(B\) which will inherit the \(\alpha\)-harmonicity property of \((G_t; t \in \mathbb{Z})\) given in proposition 5.5. When considering the Iwasawa decomposition of the random walk \((G_t; t \in \mathbb{Z})\), we write:

\[ G_t := B_t K_t, \]

but since \(B_t\) is defined only modulo \(B(\mathcal{O})\), a choice has to be made. Nevertheless, this choice happens only in a realisation of our random walk. In term of distribution, it is natural to consider increments for \(B_t\) that are right \(B(\mathcal{O})\) invariant. Therefore we choose \((B_t; t \in \mathbb{Z})\) to be a left-invariant random with independent increments

\[ B_t := b_1 b_2 \ldots b_t \]

with the distribution of each \(b_s\) being right \(B(\mathcal{O})\) invariant. Hence we are left with the task of specifying only the distribution of \(b_s \cdot B(\mathcal{O})\) on \(B/B(\mathcal{O}) \approx G/K\). This is given by \(b_s K = g_s K\). Therefore, the increments \((b_t; t \in \mathbb{N})\) are \(B(\mathcal{O})\) bi-invariant.

We easily see that \(W_\chi\) inherits the \(\alpha\)-harmonicity property with respect to \((B_t; t \in \mathbb{Z})\) from the corresponding property with respect to \((G_t; t \in \mathbb{Z})\):

\[
E (W_\chi (B_{t+1}) \mid F^B_t, B_t = b) = \frac{S (\lambda^\vee) (\chi)}{\text{Card}_{G/K} (K \varpi^{-\lambda^\vee} K)} W_\chi (b) \quad (5.7)
\]

Indeed, by the tower rule of conditional expectation:

\[
E (W_\chi (B_{t+1}) \mid F^B_t, B_t = b) = E (E (W_\chi (G_{t+1}) \mid F^G_t) \mid F^B_t, B_t = b) = \frac{S (\lambda^\vee) (\chi)}{\text{Card}_{G/K} (K \varpi^{-\lambda^\vee} K)} W_\chi (b) \quad (5.7)
\]

The law of the increments \((B_t; t \in \mathbb{Z})\) is only explicit in the case of a spherical random walk with a choice of dominant cocharacter \(\lambda^\vee\) that is minuscule. As of this moment, we will go back to using the letter \(\Lambda^\vee\), referring to the minuscule cocharacter we fixed at the beginning.

**Proposition 5.6.** If the increments of \((G_t; t \in \mathbb{Z}), (g_1, g_2, \ldots)\) are distributed as \(\mathcal{U} (K) \varpi^{-\Lambda^\vee} \mathcal{U} (K)\), with \(\Lambda^\vee\) minuscule, then the increments of \((B_t; t \in \mathbb{Z})\) are also independent and identically distributed with:

\[
B_{t+1}^{-1} B_{t+1} = \mathcal{U} (B(\mathcal{O})) \varpi^{-\mu^\vee} \mathcal{U} (B(\mathcal{O})) \quad (5.8)
\]

with the above product being a product of independent random variables and \(\mu^\vee\) random following

\[
\forall \mu^\vee \in W \Lambda^\vee, P (\mu^\vee = \mu^\vee) = \frac{q^{(\Lambda^\vee - \omega_0 \mu^\vee, \rho)}}{\text{Card}_{G/K} (K \varpi^{-\lambda^\vee} K)} \quad (5.9)
\]
Proof. Let \( b \in B \) be a random increment of \((B_t; t \in \mathbb{Z})\). Because \( \Lambda^\vee \) is minuscule, \( N \omega^{-\mu^\vee} K \cap K \omega^{-\Lambda^\vee} K = \emptyset \) unless \( \mu^\vee \in W \Lambda^\vee \). Hence, there exists a random \( \mu^\vee \in W \Lambda^\vee \) so that \( b \in N \omega^{-\mu^\vee} K \) in the Iwasawa decomposition.

From corollary 5.4, we even have \( b \in N (\mathcal{O}) \omega^{-\mu^\vee} K \). At this moment, the support of the distribution of \( b \) is well identified.

Now, write \( b = b_1 \omega^{-\mu^\vee} b_2 \) with \( b_1, b_2 \in B (\mathcal{O}) \) - not a unique expression. As the distribution of \( b \) is bi-invariant under \( B (\mathcal{O}) \), we see that \( b_1 \) and \( b_2 \) can be picked Haar distributed on \( B (\mathcal{O}) \). This gives equation (5.8).

We are only left with the task of identifying the law of \( \mu^\vee \). As \( bK = gK \) with \( g \overset{\mathcal{L}}{=} \mathcal{U} (K) \omega^{-\Lambda^\vee} \mathcal{U} (K) \), \( bK \) is a uniformly chosen right \( K \) coset in \( K \omega^{-\Lambda^\vee} K \). Therefore, for a fixed \( \mu^\vee \), \( \mathbb{P} (\mu^\vee = \mu^\vee) \) is proportional to the number of such cosets that are of the form \( N \omega^{-\mu^\vee} K \). This number, \( q (\Lambda^\vee - v_0 \mu^\vee, \rho) \), is given in the proof of corollary 5.4, hence the formula in equation (5.9).

\[ \square \]

5.4 Penalisation and proof of theorem 3.2

We will now penalize the random walk \((B_t; t \in \mathbb{Z})\) by its diagonal part in order to obtain \((B_t (W (z)); t \in \mathbb{Z})\). The following lemma tells us how the distribution of their increments are related.

Lemma 5.7. For all \( f \in \mathcal{F} (B) \), we have that:

\[
\mathbb{E} \left[ f \left( B_t^{-1} B_{t+1} \right) \right] = \frac{q (\Lambda^\vee, \rho) \varrho V (\Lambda^\vee) (z)}{\text{Card}_{G/K} (K \omega^{-\Lambda^\vee} K)} \mathbb{E} \left[ \left( \chi^{-1} \delta^{-\frac{1}{2}} f \right) \left( B_t^{-1} (W (z)) B_{t+1} (W (z)) \right) \right]
\]

Proof. We have:

\[
\mathbb{E} \left[ f \left( B_t^{-1} B_{t+1} \right) \right] = \sum_{\mu^\vee \in W \Lambda^\vee} \frac{q (\Lambda^\vee - v_0 \mu^\vee, \rho)}{\text{Card}_{G/K} (K \omega^{-\Lambda^\vee} K)} \mathbb{E} \left[ f \left( \mathcal{U} (B (\mathcal{O})) \omega^{-\mu^\vee} \mathcal{U} (B (\mathcal{O})) \right) \right]
\]

\[
\implies \frac{q (\Lambda^\vee, \rho) \varrho V (\Lambda^\vee) (z)}{\text{Card}_{G/K} (K \omega^{-\Lambda^\vee} K)} \mathbb{E} \left[ \left( \chi^{-1} \delta^{-\frac{1}{2}} f \right) \left( \mathcal{U} (B (\mathcal{O})) \omega^{-\mu^\vee} \mathcal{U} (B (\mathcal{O})) \right) \right]
\]

Now, the \( \alpha \)-harmonicity in equation (5.7) will translate to a strict harmonicity property, which was the basis of our Poisson kernel formula. For more convenient notations, let us define:

\[
\forall b \in B, f_z (b) := \delta \left( b \right)^{-\frac{1}{2}} \chi \left( b \right)^{-1} W_z (b)
\]

27
Then we have:

\[ E \left( f_z \left( B_{t+1} \left( W(z) \right) \right) \mid F_t^B, B_t = b \right) = E \left( f_z \left( b B_t^{-1} \left( W(z) \right) B_{t+1} \left( W(z) \right) \right) \right) \]

\[ = E \left[ \left( \chi^{-1} \delta_{b} z W_z \right) \left[ b B_t^{-1} \left( W(z) \right) B_{t+1} \left( W(z) \right) \right] \right] \]

**Lemma 5.7**

\[ \frac{\text{Card}_{G/K} \left( K \varpi^{-\Lambda \vee} K \right)}{q^{(\Lambda \vee, \phi)} \text{ch} V (\Lambda \vee) (z)} \delta (b)^{-\frac{1}{2}} \chi (b)^{-1} E \left[ W_z \left( b B_t^{-1} B_{t+1} \right) \right] = \]

\[ \frac{\text{Card}_{G/K} \left( K \varpi^{-\Lambda \vee} K \right)}{q^{(\Lambda \vee, \phi)} \text{ch} V (\Lambda \vee) (z)} \delta (b)^{-\frac{1}{2}} \chi (b)^{-1} \frac{S (\Lambda \vee) (\chi)}{\text{Card}_{G/K} \left( K \varpi^{-\Lambda \vee} K \right)} W_z (b) \]

\[ = \frac{S (\Lambda \vee) (\chi)}{q^{(\Lambda \vee, \phi)} \text{ch} V (\Lambda \vee) (z)} f_z (b) \]

**Eq (5.6)**

\[ = f_z (b) \]

This concludes the proof of theorem 3.2.

### 6 Acknowledgement

The author is grateful to Dan Bump, Paul-Olivier Dehaye and Simon Pepin Lehalleur for guidance and many helpful discussions. Also, I would like to mention Gautham Chinta and Philippe Bougerol for their encouragement in pursuing this probabilistic approach.

### References

[BFZ96] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Parametrizations of canonical bases and totally positive matrices. *Adv. Math.*, 122(1):49–149, 1996.

[Bia92] Philippe Biane. Minuscule weights and random walks on lattices. In *Quantum probability & related topics, QP-PQ*, VII, pages 51–65. World Sci. Publ., River Edge, NJ, 1992.

[Bou75] N. Bourbaki. *Éléments de mathématique*. Hermann, Paris, 1975. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées, Actualités Scientifiques et Industrielles, No. 1364.

[Car79] P. Cartier. Representations of \( p \)-adic groups: a survey. In *Automorphic forms, representations and L-functions (Proc. Sympos.
Reda Chhaibi. Littelmann path model for geometric crystals, Whittaker functions on Lie groups and Brownian motion. PhD thesis in Université Paris VI, pages 1–226, 2013, arXiv:1302.0902.

Reda Chhaibi. Archimedean Whittaker functions as characters of geometric crystals. In preparation, 2014.

Reda Chhaibi. The geometric Robinson-Schensted correspondence, the Whittaker process and canonical measures on geometric crystals. In preparation, 2014.

James W. Cogdell, Henry H. Kim, and M. Ram Murty. Lectures on automorphic $L$-functions, volume 20 of Fields Institute Monographs. American Mathematical Society, Providence, RI, 2004.

W. Casselman and J. Shalika. The unramified principal series of $p$-adic groups. II. The Whittaker function. Compositio Math., 41(2):207–231, 1980.

David J. Grabiner and Peter Magyar. Random walks in Weyl chambers and the decomposition of tensor powers. J. Algebraic Combin., 2(3):239–260, 1993.

Benedict H. Gross. On the Satake isomorphism. In Galois representations in arithmetic algebraic geometry (Durham, 1996), volume 254 of London Math. Soc. Lecture Note Ser., pages 223–237. Cambridge Univ. Press, Cambridge, 1998.

Hervé Jacquet. Fonctions de Whittaker associées aux groupes de Chevalley. Bull. Soc. Math. France, 95:243–309, 1967.

Samuel Karlin and James McGregor. Coincidence probabilities. Pacific Journal of Mathematics, 9(4):1141–1164, 1959.

Shrawan Kumar. Kac-Moody groups, their flag varieties and representation theory, volume 204 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2002.

Robert P. Langlands. Euler products. Yale University Press, New Haven, Conn.-London, 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967, Yale Mathematical Monographs, 1.
[LLP12] Cédric Lecouvey, Emmanuel Lesigne, and Marc Peigné. Random walks in Weyl chambers and crystals. Proc. Lond. Math. Soc. (3), 104(2):323–358, 2012.

[LLP13] Cédric Lecouvey, Emmanuel Lesigne, and Marc Peigné. Conditioned random walks from Kac-Moody root systems. Transactions of the AMS (accepted), pages 1–30, 2013, arXiv:1306.3082.

[Lus10] George Lusztig. Introduction to quantum groups. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

[Mac71] I. G. Macdonald. Spherical functions on a group of p-adic type. Ramanujan Institute, Centre for Advanced Study in Mathematics, University of Madras, Madras, 1971. Publications of the Ramanujan Institute, No. 2.

[Shi76] Takuro Shintani. On an explicit formula for class-1 “Whittaker functions” on $GL_n$ over $P$-adic fields. Proc. Japan Acad., 52(4):180–182, 1976.

[Sta12] Richard P. Stanley. Enumerative combinatorics. Volume 1, volume 49 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, second edition, 2012.

[Ste68] Robert Steinberg. Lectures on Chevalley groups. Yale University, New Haven, Conn., 1968. Notes prepared by John Faulkner and Robert Wilson.

[Tit79] J. Tits. Reductive groups over local fields. In Automorphic forms, representations and $L$-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1. Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.