Analytical properties of the quark propagator from truncated Dyson-Schwinger equation in complex Euclidean space

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Abstract

In view of the mass spectrum of heavy mesons in vacuum the analytical properties of the solutions of the truncated Dyson-Schwinger equation for the quark propagator within the rainbow approximation are analysed in some detail. In Euclidean space, the quark propagator is not an analytical function possessing, in general, an infinite number of singularities (poles) which hamper to solve the Bethe-Salpeter equation. However, for light mesons (with masses $M_{q\bar{q}} \leq 1$ GeV) all singularities are located outside the region within which the Bethe-Salpeter equation is defined. With an increase of the considered meson masses this region enlarges and already at masses $\geq 1$ GeV, the poles of propagators of $u, d$ and $s$ quarks fall within the integration domain of the Bethe-Salpeter equation. Nevertheless, it is established that for meson masses up to $M_{q\bar{q}} \simeq 3$ GeV only the first, mutually complex conjugated, poles contribute to the solution. We argue that, by knowing the position of the poles and their residues, a reliable parametrisation of the quark propagators can be found and used in numerical procedures of solving the Bethe-Salpeter equation. Our analysis is directly related to the future physics programme at FAIR with respect to open charm degrees of freedom.
I. INTRODUCTION

The description of mesons as quark-antiquark bound states within the framework of the Bethe-Salpeter (BS) equation with momentum dependent quark mass functions, determined by the Dyson-Schwinger (DS) equation, is able to explain successfully many spectroscopic data, such as meson masses [1–5], electromagnetic properties of pseudoscalar mesons and their radial excitations [6–9] and other observables [10–21]. Contrarily to purely phenomenological models, like the quark bag model, the presented formalism maintains important features of QCD, such as dynamical chiral symmetry breaking, dynamical quark dressing, requirements of the renormalization group theory etc., cf. Ref. [22]. The main ingredients here are the full quark-gluon vertex function and the dressed gluon propagator, which are entirely determined by the running coupling and the bare quark mass parameters. In principle, if one were able to solve the Dyson-Schwinger equation, the approach would not depend on any additional parameters. However, due to known technical problems, one restricts oneself to calculations of the few first terms of the perturbative series, usually up to the one-loop approximation. The obtained results, which formally obey all the fundamental requirements of the theory, are then considered as exact ones, however, with effective parameters. This is known as the rainbow-ladder approximation of the DS equation. The merit of the approach is that, once the effective parameters are fixed, the whole spectrum of known mesons is supposed to be described on the same footing, including also excited states.

It should be noted that there exists other approaches based on the same physical ideas but not so sophisticated, e.g. employing simpler interactions, such as a separable interaction for the effective coupling [18]. Such approaches describe also fairly well properties of light mesons, nevertheless, the investigation of heavier mesons and excited states, consisting even of light $u$, $d$ and $s$ quarks, requires implementations of more accurate numerical methods to solve the corresponding equations.

In the present paper we investigate the prerequisites to the interaction kernel of the combined Dyson-Schwinger and Bethe-Salpeter formalisms to describe the meson mass spectrum including heavier mesons and excited states as well. Particular attention is paid to the charm sector which, together with the baryon spectroscopy, is a major topic in the FAIR research programme. Two large collaborations at FAIR [23, 24] plan precision measurements. Note that it becomes now possible to experimentally investigate not only the mass spectrum of the mentioned mesons,
but also different processes of their decay, which are directly connected with fundamental QCD
problems (e.g., $U(1)$ axial anomaly, transition form factors etc.) and with the challenging
problem of changes of meson characteristics at finite temperatures and densities. The latter is
crucial in understanding the di-lepton yields in nucleus-nucleus collisions at, e.g., HADES. All
these circumstances require an adequate theoretical foundation to describe the meson spectrum
and the meson covariant wave functions (i.e. the BS partial amplitudes) needed in calculations
of the relevant Feynman diagrams and observables.

Due to the momentum dependence of the quark mass functions, the BS equation requires an
analytical continuation of the quark propagators in the complex plane of Euclidean momenta
which can be achieved either by corresponding numerical continuations of the solution obtained
along the real axis or by solving directly the DS equation in the complex domain of validity
of the equation itself. With increasing meson mass $M_{qq}$ this region enlarges, and already at
masses $M_{qq} \geq 1$ GeV, singularities of light-quark propagators fall within the domain the
BS equation to be solved. It is found that even so, only the few first poles contribute to the
solution. To analyse the analytical properties of the quark propagators we solve the DS equation
in the rainbow ladder approximation by making use of the hyperspherical harmonics basis to
decompose the propagators and the corresponding potential and solve numerically the resulting
DS equations for the coefficients of such a decomposition. The further analysis of the solution
is based on a combined application of Rouché’s theorem and a graphical representation of the
inverse propagators as vortex fields of the corresponding complex functions. Since the main
goal of our analysis is the use of the quark propagator functions evaluated at such complex
momenta for which the are needed in the BS equation, we focus our attention on this region of
Euclidean space.

Our paper is organized as follows. In Sec. II Subsecs. II A and II B we briefly discuss the
truncated BS and DS equations relevant to describe the mesons as quark-antiquark bound
states. The rainbow approximation for the DS equation kernel is introduced in Subsec. II C,
and the domain of the complex plane of Euclidean momenta, where the solutions are sought, is
specified in Subsec. II D. The explicit form of the DS equation to be solved in the rainbow ladder
approximation with effective infrared kernel is formulated and the corresponding numerical
solution for space-like momentum is discussed in Subsec. II E. Section III is entirely dedicated
to the solution of the truncated DS equation for complex momenta. In Subsec. III A we consider
the analytical continuation of the solution for complex momenta along rays with constant angle.
It is found that in the right hemisphere of complex momenta squared, \( p^2 \), the solutions of DS equation are analytical functions so that all singularities of the quark propagator are to be searched for in the left hemisphere. The subsequent Sections are aimed at investigations of the solutions in the time-like domain of \( p^2 \) determined by the BS equation for mesons with masses \( M_{qq} \lesssim 2 - 3 \text{ GeV} \): In Subsec. III B we present the solutions and the propagator functions for the \( c \) quark, for which it was found that they are analytical functions in the considered region. A reliable parametrization is found, allowing to facilitate the numerical procedure of solving the BS equation for charmed mesons. A thorough investigation of the pole structure of propagators of light quarks is presented in Sec. IV. By combining Cauchy and Rouché’s theorems with an analysis of the force lines of vector fields of propagators as complex functions, the position of first few poles and the corresponding residues the propagator functions are found with a good accuracy. The dependence of pole locations in the complex momentum plane on the bare quark mass is analysed as well. The impact of the ultraviolet term in the interaction kernel is briefly discussed in Sec. V. Summary and conclusions are collected in Sec. VI.

II. BETHE-SALPETER AND DYSON-SCHWINGER EQUATIONS

A. Bethe-Salpeter equation

To determine the bound-state mass of a quark-antiquark pair one needs to solve the Bethe-Salpeter equation, which in the ladder approximation (hereafter referred to as truncated Bethe-Salpeter (tBS) equation) and in Euclidean space reads

\[
\Gamma(P, k) = -\frac{4}{3} \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S(k_1) \Gamma(P, k) S(k_2) \gamma_\nu \left[ g^2 D_{\mu\nu}(p-k) \right],
\]

(1)

with \( \Gamma \) being the BS vertex function and \( S^{-1}(k_1) = (i\gamma \cdot k_1 + m_1) \) and \( S^{-1}(k_2) = (i\gamma \cdot k_2 + m_2) \) are the inverse propagators of two quarks, which interact via gluon exchange encoded in \( [g^2 D_{\mu\nu}(p-k)] \). The vertex function \( \Gamma(P, k) \) is a \( 4 \times 4 \) matrix and, therefore, may contain 16 different functions. The general structure of the vertex functions describing bound states of spinor particles has been investigated in detail, for example, in [25, 26, 28]. To release from the matrix structure, the vertex function \( \Gamma \) is expanded into functions which in turn are determined by angular momentum and parity of the corresponding meson known as the spin-angular
harmonics 12, 25, 27:

\[
\Gamma(p_4, p) = \sum_{\alpha} g_{\alpha}(p_4, |p|) T_{\alpha}(p).
\] (2)

With Eq. (2) it can be shown 26, 27 that the integral matrix form of the BS equation 11 can be reduced to a system of ordinary algebraic equations. It should be noted that, if one would consider the meson as a bound state of two quarks with constant masses, i.e. if the meson is treated within the Bethe-Salpeter formalism with effective quark masses, then the scalar part of the product of quark propagators becomes purely real and free of any singularities. In such a case the quark masses appear as effective parameters, which need to be different for different mesons. In a more realistic case, where mesons are described on a common footing as bound states of dynamically dressed quarks, the corresponding "masses" are represented by rather complicate functions of the momenta, and the product of two propagators in the above tBS equation remains complex, even in Euclidean space. That means prior proceeding in solving the tBS equation, one needs to know the analytical properties of quark propagators in the complex Euclidean momentum space.

B. Dyson-Schwinger equation

The coupled equations of the quark propagator \( S \), the gluon propagator \( D_{\mu\nu} \) and the vertex function \( \Gamma_\mu \) are often considered as integral formulation being equivalent to full QCD. While there are attempts to solve this coupled set of DS equations by some numerical procedures, for certain purposes some approximations 3, 20, 22 are appropriate. Being interested in dealing with mesons as quark-antiquark bound states within utilization of the tBS equation, one has to provide the quark propagator which depends on the gluon propagator and vertex as well, which in turn depend on the quark propagator. Leaving a detailed discussion of the variety of approaches in dressing of the gluon propagator and vertex function in DS equations (see e.g. Refs. 29, 30 and references therein quoted) we mention only that in solving the DS equation for the quark propagator one usually employs truncations of the exact interactions and replaces the gluon propagator combined with the vertex function by effective interaction kernels. This leads to the truncated Dyson-Schwinger (tDS) equation for the quark propagator. In concrete calculations the choice of the form of the effective interaction is inspired by results from calculations of Feynman diagrams within pQCD maintaining requirements of symmetry and asymptotic behaviour already implemented, cf. Refs. 3, 21, 22, 30. The results of such
calculations, even in the simplest case of accounting only for one-loop diagrams with proper regularization and renormalization procedures, are rather cumbersome for further use in numerical calculations, e.g. in the framework of BS or Fadeev equations. Consequently, in practice, the wanted exact results are replaced by parametrizations of the vertex and gluon propagator. Often one employs an approximation which corresponds to one-loop calculations of diagrams with the full vertex function $\Gamma_\nu$, substituted by the free one, $\Gamma_\nu(p,k) \to \gamma_\nu$ (we suppress the color structure and account cumulatively for the strong coupling later on). In Euclidean space the quark propagator obeys then the tDS equation

$$S^{-1}(p) = S_0^{-1}(p) + \frac{4}{3} \int \frac{d^4k}{(2\pi)^4} \left[ g^2 D_{\mu\nu}(p-k) \right] \gamma_\mu S(k) \gamma_\nu ,$$

where $S_0^{-1} = i\gamma \cdot p + m_q$ and $m_q$ is the bare current quark mass. To emphasize the replacement of combined gluon propagator and vertex we use, as in Eq. (1), the notation $[g^2 D_{\mu\nu}]$, where an additional power of $g$ from the second undressed vertex is included. For a consistent treatment of dressed quarks and their bound states, the dressed gluon propagator $[g^2 D_{\mu\nu}(p-k)]$ must be the same in the tBS, (1), and the tDS equations, (3).

### C. Choosing an interaction kernel

This truncation, known also as the ladder rainbow approximation, has been widely used to study the physics of dynamical chiral symmetry breaking [2], decay constants [3–7] and other observables [8, 10–12] and has been found to provide a good agreement with experimental data. The employed vertex-gluon kernel in the rainbow approximation is also known as the Maris-Tandy (MT) model [2]. It is chosen here in the form [2, 3, 19, 21]

$$g^2(k^2) D_{\mu\nu}(k^2) = \left( \frac{4\pi^2 D k^2}{\omega^2} e^{-k^2/\omega^2} + \frac{8\pi^2 \gamma m F(k^2)}{\ln[\tau + \left(1 + \frac{k^2}{\Lambda_{QCD}^2}\right)^2]} \right) \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right),$$

where the first term originates from the effective infrared (IR) part of the interaction determined by soft non-perturbative effects, while the second one ensures the correct ultraviolet (UV) asymptotic behaviour of the QCD running constant. A detailed investigation of the interplay of these two terms has shown [3, 31] that the IR part is dominant for light $u, d$ and $s$ quarks with a decreasing role for heavier quark masses ($c$ and $b$) for which the UV part may be quite important in forming meson masses $M_{qq} > 3 - 4$ GeV as bound states. In the present paper we focus on mesons with masses $M_{qq} < (2 - 3)$ GeV which consist of at least one light quark and
for which the UV term can be seemingly neglected. However, in spite of its minor contribution to meson masses, formally the UV term guarantees the correct asymptotic behaviour of the kernel and in principle it must be included even in case of light quarks forming mesons.

D. The relevant region for the tDS equations

In a first step in our investigation we disregard the UV term and study the analytical properties of the corresponding propagator functions obtained with the IR term solely (cf. also Refs. [5, 19, 32]). Then we reconcile the results with the UV term in the kernel in Section V.

As mentioned above, we are interested in the analytical structure of the propagator function $S(p)$ inside and in the neighbourhood of the complex momentum region in Euclidean space dictated by the tBS equation. This momentum region is displayed as the dependence of the imaginary part of the quark momentum squared $\text{Im} p^2$ on its real part $\text{Re} p^2$ determined by the tBS equation. In terms of the relative momentum $k$ of the two quarks residing in a meson the corresponding dependence is

$$p^2 = -\frac{M_{q\bar{q}}^2}{4} + k^2 \pm iM_{q\bar{q}}k$$

(5)

determining in the Euclidean complex momentum plane a parabola $\text{Im} p^2 = \pm M_{q\bar{q}} \sqrt{\text{Re} p^2 + \frac{M_{q\bar{q}}^2}{4}}$ with vertex at $\text{Im} p^2 = 0$ at $\text{Re} p^2 = -M_{q\bar{q}}^2/4$ depending on the meson mass $M_{q\bar{q}}$; the symmetry axis is the $\text{Re} p^2$ axis, i.e. the parabola extends to $\text{Re} p^2 \to \infty$.

Two examples are depicted in Fig. 1 for $M_{q\bar{q}} = 0.14$ GeV (left) and $M_{q\bar{q}} = 2$ GeV (right).

Note that regardless of the form of the interaction kernel, the investigation of the analytical structure of the quark propagator is of great importance, if the propagators exhibit singularities within the corresponding parabola, thus hampering the numerical procedure of solving the tBS equation. On the other side, the knowledge of the nature of singularities and their exact location in the complex plane will allow one to develop effective algorithms adequate for numerical calculations. For instance, if one determines exactly the domain of analyticity of the propagator functions, one can take advantage of the fact that any analytical function can always be approximated by rational complex functions [33]; then, one can parametrize the integrand in the tBS equation by simple functions which allow to carry out some integrations analytically. Such parametrisations have been suggested in Ref. [35] for meson masses $M_{q\bar{q}} < 1$ GeV for which the propagator functions have been found to be analytical. Unfortunately, for larger meson masses the propagator functions exhibit singularities within the domain of tBS
FIG. 1: (Color online) The Euclidean space, where numerical solutions of the tBS and tDS equations are sought. The quark inverse propagator part, $\Pi(p) = p^2 A^2(p) + B^2(p)$, where $p^2 = -M_{qq}^2/4 + k^2 \pm iM_{qq} k \cos \chi$, entering the tBS equation is defined within the colored areas of the corresponding parabolas ($-1 \leq \cos \chi \leq 1$). Left (right) panel: the integration domain for $M_{qq} = 140$ MeV ($M_{qq} = 2$ GeV). The integration domain is restricted to $k_{\text{max}} \leq 3$ GeV/c; for larger values of $k$ all the partial Bethe-Salpeter amplitudes are already negligible small (see, e.g. Ref. [26]). For light mesons (left panel, $m_q = 0.005$ GeV) there are no singularities in the colored tBS integration domain. In the right panel we display the first six self-conjugated poles of propagators of light, $u$, $d$ quarks (asterisks) and the first four poles for the $s$ quarks (open stars) which enter the solution of the tBS equation. The pole positions are quoted in Table III. Axes are in units of $(\text{GeV}/c)^2$.

integration and, as a consequence, parametrizations by rational functions are not possible. Nevertheless, even in this case, if the propagator functions have only isolated poles with known locations and residues, then calculations can be significantly simplified by splitting the singular functions into two terms, one analytical in the considered region the other one having a simple pole structure, as discussed below.

E. Propagator functions

Coming back to the tDS equation (3) we mention that it is a four dimensional integral equation in matrix form. Simplifications can be achieved by exploiting specific decompositions
of the quark propagator. Then one decomposes the kernel over a complete set of basis functions, performs analytically some angular integrations and considers a new system of equations relative to such a partial decomposition. To be specific, we recall that the calculation of the renormalized Feynman diagrams leads to a fermion propagator depending on two functions, e.g. the renormalization constant $Z_2$ and the self-energy $\Sigma(p)$. Instead of $Z_2$ and $\Sigma(p)$ one can introduce other two quantities $A(p)$ and $B(p)$ or, alternatively, $\sigma_s(p)$ and $\sigma_v(p)$. The latter ones will be often addressed in the present paper to as the propagator functions. In terms of these functions the dressed quark propagator $S(p)$ reads \[ S^{-1}(p) = i\gamma \cdot pA(p) + B(p) , \quad S(p) = -i\gamma \cdot p\sigma_v(p) + \sigma_s(p) , \] (6) with \[ \sigma_v(p) = \frac{A(p)}{p^2A(p)^2 + B(p)^2} , \quad \sigma_s(p) = \frac{B(p)}{p^2A(p)^2 + B(p)^2} . \] (7)

The resulting system of equations to be solved is (for details see Ref. [12]; here we display only the formulae for the IR part of the kernel)

\[
\begin{align*}
A(p) &= 1 + 2D \int dk \frac{k^4}{p} \frac{A(k)}{k^2A^2(k) + B^2(k)} e^{-(p-k)^2/\omega^2} \left\{ \frac{p^2 + k^2 + 2\omega^2}{kp} I_2^{(s)}(z) - 2I_1^{(s)}(z) \right\} , \quad (8) \\
B(p) &= m_q + 2D \int dk \frac{k^2n}{p} \frac{B(k)}{k^2A^2(k) + B^2(k)} e^{-(p-k)^2/\omega^2} \left\{ \frac{p^2 + k^2}{kp} I_1^{(s)}(z) - 2I_2^{(s)}(z) \right\} , \quad (9)
\end{align*}
\]

where $z = 2pk/\omega^2$ and $I_n^{(s)}(z)$ are the scaled (as emphasized by the label ”(s)” modified Bessel functions of the first kind defined as $I_n^{(s)}(z) \equiv \exp(-z) I_n(z), I_n^{(s)}(z \to \infty) = \frac{1}{\sqrt{2\pi z}} \left[ 1 - \frac{4n^2 - 1}{8z} \right], I_n^{(s)}(z \to 0) = \left( \frac{z}{2} \right)^n e^{-z} \Gamma(n + 1)$. The resulting system of one-dimensional integral equations Eqs. (8) and (9) can be solved numerically by an iteration method. Independent parameters are $\omega$, $D$ and $m_q$. We find that the iteration procedure converges rather fast and practically does not depend on the choice of the trial start functions for $A(p)$ and $B(p)$.

In our subsequent analysis we employ the effective parameters from Refs. [19, 21], $\omega = 0.5$ GeV and $D = 16$ GeV$^{-2}$. Results are shown in Fig.2 as momentum dependence of the functions $A(p), B(p)$ and $\sigma_{s,v}(p)$ for different bare masses: $m_q = 0.005$ GeV for $u, d, m_q = 0.115$ GeV for $s$, and $m_q = 1$ GeV for $c$ quarks. It is seen that all these quantities are smooth, positively defined functions not containing any singularity along the space-like Euclidean momentum $p^2$. The solution then is generalized to complex values of $p^2$ needed to solve the BS equation for bound states. As mentioned above, such solutions provide a good description of many properties of
FIG. 2: (Color online) The solution of the tDS equations (8) and (9) with the IR part of the interaction kernel (11) only as a function of the space-like ($p^2 > 0$) real Euclidean momentum $p^2$. Solid curves correspond to $u, d$ quarks with bare mass $m_q = 5$ MeV, dashed curves depict results for $s$ quarks with bare mass $m_q = 115$ MeV, and dash-dotted curves are for $c$ quarks with $m_q = 1$ GeV.

light mesons (masses, widths, decay rates etc., cf. [3, 4, 6–8, 10–12]). However, an attempt to apply these solutions for heavier mesons with at least one light quark, e.g. for open charm $D$ mesons, leads to instabilities in the numerical procedure of solving the BS equation$^1$. Obviously, this is due to the fact that the integration domain for heavier mesons becomes larger, cf. right panel of Fig. 1 and the singularities in the propagator functions approach closely or even intrude into the corresponding parabola. Hence, a more detailed investigation of the behaviour of the propagator function in the complex Euclidean plane is required.

$^1$ In Ref. [34] the situation is described by the phrase “the ladder-rainbow model kernel has deficiencies in the heavy quark region that are masked in $QQ$ mesons but are plainly evident in $qQ$ mesons”. If one attributes “deficiencies” with singularities in the propagator functions, it is the light-quark propagator which causes the problems in composites at energy scales $\gtrsim 1$ GeV.
III. SOLUTIONS OF THE tDS EQUATION IN COMPLEX PLANE

As evident from Fig. 1, the parabolic integration domain for solving the BS equation can be conveniently divided into two parts: (i) one (infinite) region where $\text{Re } p^2 > 0$, and (ii) a second one where $\text{Re } p^2 < 0$, which is restricted by the meson mass $M_{q\bar{q}}$, i.e. with the minimum (negative) value $\text{Re } p^2 = -M_{q\bar{q}}^2/4$.

A. Solutions along rays $\phi = \text{const}$

From eqs. (8) and (9) it is explicitly seen that, in the right hemisphere, the integrals in the DS equation converge. That means, an analysis of the behaviour of the solution for $\text{Re } p^2 > 0$ for large $|p^2|$ can be accomplished directly by means of Eqs. (8) and (9), i.e. one can perform a rotation of the real axis by an angle $\phi_p < \pi/4$ (for $p^2$, the rotation angle corresponds to $\phi_{p^2} < \pi/2$ thus covering the full half plane $\text{Re } p^2 > 0$) and solve the tDS equation for momenta $p = |p| \exp(i\phi_p)$ along rays $\phi_p = \text{const}$. In such a way one can obtain solutions of DS equation in the whole right hemisphere, including large values of momenta, $|p^2| \to \infty$. That method turns out to be extremely efficient. The iteration procedure converges rather fast and allows for a detailed analysis of the solutions $A(p)$, $B(p)$, $\sigma_s(p)$ and $\sigma_v(p)$ in a large interval $\text{Re } p^2 > 0$ in the complex plane.

In Fig. 3 we present, as an example, the solutions $A(p)$ and $B(p)$ in the complex plane along rays $\phi_p = \text{const}$ for two adjacent values of $\phi_p$. The complex solutions are smooth and smoothly changing under variations of $\phi_p$ as long as $\phi_p < \pi/2$, however, not anymore positively definite so that, in principle, some combinations of $A(p)$ and $B(p)$, in particular the inverse propagator part $\Pi(p^2) = p^2 A^2(p) + B^2(p)$, can vanish at certain values of $p^2$. This would imply the existence of pole-like singularities for the propagator functions $\sigma_{s,v}(p^2)$. To check this we compute integrals of $\Pi(p^2)$ along wedge-shaped closed contours formed by two rays with $\phi_{p(1)}$ and $\phi_{p(2)}$ and an enclosing curve provided by a section of a circle with a radius $|p_0|$. In what follows, such integrals along closed contours are referred to as the Cauchy integrals. A non-zero value of the Cauchy integrals will signal the occurrence of singularities within the chosen contour. We calculate the Cauchy integrals for many choices of integration contours and find them vanishing, i.e. the functions $A$ and $B$ are analytical. Moreover, we find also that the propagator functions $\sigma_{s,v}(p^2)$ are analytical as well. Then, evidently all singularities of the solutions $A$, $B$ and the propagator functions $\sigma_{s,v}$, if any, are to be searched for in the left
FIG. 3: (Color online) The solution of the tDS equations (8) and (9) in the complex Euclidean space $\Re p^2 > 0$ along rays $\phi_p = \text{const}$ for two values of $\phi_p$ (black solid curves: $\phi_p = 32^\circ$, red dashed curves: $\phi_p = 34^\circ$) for $u, d$ quarks with $m_q = 5\,\text{MeV}$.

hemisphere, i.e. at $\Re p^2 < 0$. It should be pointed out that an analogous rotation of axes to this region, where $\phi_{p^2} > \pi/2$, is impossible since, as seen from Eqs. (8) and (9), at arguments larger than $\pi/2$, negative values of $\Re p^2 < 0$ lead to divergent integrals as $|p^2| \to \infty$. Hence, simple analytical continuation of the tDS equation along rays $\phi_p = \text{const}$ in the whole complex plane is impossible.

B. Solution of the tDS equation at $\Re p^2 < 0$

In our calculations we are interested in a restricted domain in the left hemisphere for which $|p^2|$ is relatively small, $\Re p^2 > -M_{q\bar{q}}^2/4$ where the tDS integrals converge. That means, at such values of $|p^2|$ the tDS equation along the positive real axis can still be used to find the solution in the complex plane. It is worth mentioning that, in principle, one does not need to solve Eqs. (8) and (9) for each point inside the parabola (5). It suffices to know whether the solutions $A(p^2)$ and $B(p^2)$ are analytical in the corresponding domain, and if so, one can find
the solutions $A(p^2)$ and $B(p^2)$ only once along the contour defined by the parabola and, due to
the Cauchy’s theorem,

$$A(z) = \frac{1}{2\pi i} \oint A(\xi) \frac{d\xi}{\xi - z}, \quad (10)$$

to determine $A(p^2)$ and $B(p^2)$ in any other desired point, see Ref. [20]. Consequently, we
proceed with an analysis of the domain of analyticity of $A(p^2)$, $B(p^2)$ and $\sigma_{s,v}(p^2)$ in the region
$-M_{q\bar{q}}^2/4 < \text{Re} \, p^2 < 0$.

We begin with an analysis of properties of the propagator of the $c$ quark for which, as we
found in our previous calculations [12], the procedure of solving the BS for $M_{q\bar{q}} < 2$ GeV is
rather stable. This is a hint that the solutions $A(p^2)$ and $B(p^2)$ and the propagator functions
$\sigma_{s,v}(p^2)$ could be analytical functions everywhere in the integration domain. Indeed, we solve the
tDS equation for $A(p^2)$ and $B(p^2)$ in a large interval of Re $p^2 < 0$, compute the corresponding
Cauchy integrals and find them to be always zero. Moreover, we find that $\sigma_s(p^2)$ and $\sigma_v(p^2)$ are
also analytical functions within the corresponding parabola. This means that for such analytical functions one can find convenient parametrizations in terms of rational functions, cf. [33], to be
used in solving the tBS equation. Remind that, as seen from the tDS equation, the solutions
$A(p^2)$ and $B(p^2)$ are self-conjugated functions, i.e. the propagator functions $\sigma_{s,v}(p^2)$ must be real
along the real axis of $p^2$. This restricts the possible form of parametrizations by the condition
$F(p^2, \{\alpha_i\}) = F(p^2, \{\alpha_i^*\})$, where $\{\alpha_i\}$ is the set of free parameters, to be found by fitting the solution along the real axis, see e.g. Ref. [35]. In Fig. 4 we exhibit the behaviour of $\sigma_{s,v}(p^2)$ along the real axis which are used in further parametrizations. The propagator functions are
smooth and obey a simple form which inspires the following choice for the parametrization

$$\sigma_{s,v}(p^2) = \sum_i \frac{\alpha_i(s,v)}{p^2 + \beta_i^2(s,v)} + \sum_i \frac{\alpha_i^*(s,v)}{p^2 + \beta_i^{*2}(s,v)}, \quad (11)$$

where the complex parameters $\alpha_i$ and $\beta_i$ can be easily obtained by fitting the corresponding
solution along the real axis of $p^2$. We use the Levenberg-Marquardt algorithm for fitting. We
find that for each function in Eq. (11) two terms, i.e. eight parameters for each function, are
quite sufficient to obtain a good approximation of the solution. In Table II we present the sets of
parameters $\alpha_{1,2}(s,v)$ and $\beta_{1,2}(s,v)$ obtained for $\sigma_s$ and $\sigma_v$, respectively, from a fit in the interval
$-1.5 \text{(GeV/c)}^2 < p^2 < 10 \text{(GeV/c)}^2$. The quality of the fit is demonstrated in Fig. 4 where
the dashed lines represent the obtained approximation to the corresponding exact propagator
functions (solid curves). In the interval $-1.5 \text{(GeV/c)}^2 < p^2 < 10 \text{(GeV/c)}^2$, the achieved fit
The functions $\sigma_s$ and $\sigma_v$ for real values of $p^2$. The solid curves are for the numerically determined solutions, while dashed curves depict the approximation (11) with $i = 1, 2$. For $m_q = 1$ GeV.

is excellent. Then we randomly calculate the propagator functions at complex values of $p^2$ by solving numerically (8) and (9) and compare with parametrizations (11). The agreement is impressively good. (It should be noted, however that, since the parametrized functions are of a rather simple shape, the set of obtained parameters $\{\alpha_i\}$ and $\{\beta_i\}$ is far from being unique, i.e. one can achieve a similar quality of the fit with many other choices of $\{\alpha_i\}$ and $\{\beta_i\}$. The only restriction is that the ”mass” parameters $\{\beta_i\}$ must not provide singularities, neither along the real axis, nor in the complex plane inside the parabola, see also Ref. [36].) In essence, Eq. (11) with parameters in Table II provide a convenient parametrization of $\sigma_{s,v}(p^2)$ which is easily applicable in the tBS equation for $M_{q\bar{q}} < 3$ GeV.

A different situation occurs when one tries to find such parametrizations for the propagator functions of light quarks. We find that, in spite of zero Cauchy integrals for $A(p^2)$ and $B(p^2)$, the propagator functions $\sigma_{s,v}(p^2)$ provide non-zero Cauchy integrals for $M_{q\bar{q}} > 1$ GeV. Besides, the values of the Cauchy integrals for $\sigma_{s,v}(p^2)$, i.e. their residues, are found to depend on the chosen contour inside or near the parabola. This is a clear indication that $\sigma_{s,v}(p^2)$ have poles in this region and, moreover, the number of poles is greater than one. As noted above, for numerical calculations it is extremely important to find, with a good accuracy, the position of the poles and the corresponding residues for $\sigma_s(p^2)$ and $\sigma_v(p^2)$. In such a case, if the complex valued functions $\sigma_{s,v}(p^2)$ have only isolated poles $p_{0i}^2$ within a certain domain and are analytical
TABLE I: The parameters $\alpha_i(s,v)$ and $\beta_i(s,v)$ for the effective parametrizations, Eq. (11). For $c$ quark of mass $m_q = 1$ GeV.

| j  | $\alpha_1(j)$ [GeV] | $\beta_1(j)$ [GeV] | $\alpha_2(j)$ [GeV] | $\beta_2(j)$ [GeV] |
|----|----------------------|---------------------|---------------------|---------------------|
|    | (Re , Im )           | (Re , Im )          | (Re , Im )          | (Re , Im )          |
| s  | (1.409 , 0.9802)     | (1.8627 , 0.602)    | (-0.909 , -0.245)   | (1.875 , 0.886)     |
| v  | (0.08624 , 0.598)    | (1.773 , 0.7179)    | (0.4145 , -0.267)   | (2.112 , 0.5177)    |

on its closing contour $\gamma$, they can be represented as

$$\sigma_{s,v}(p^2) = \tilde{\sigma}_{s,v}(p^2) + \sum_i \frac{\text{res}[\sigma_{s,v}(p^0_i)]}{p^2 - p^2_0},$$

(12)

where $\tilde{\sigma}_{s,v}(p^2)$ are analytical functions within the considered domain and, consequently, can be computed as

$$\tilde{\sigma}_{s,v}(p^2) = \frac{1}{2\pi i} \oint_{\gamma} \tilde{\sigma}_{s,v}(\xi) d\xi = \frac{1}{2\pi i} \oint_{\gamma} \frac{\sigma_{s,v}(\xi)}{\xi - p^2} d\xi.$$ (13)

Then, Eqs. (12) and (13) imply that, in solving the BS equation, the first term in (12) is free of singularities and does not require modifications of the numerical procedure; the second term also does not cause numerical troubles since integrations can be performed analytically. A similar strategy in solving numerically the tBS equation in presence of poles has been proposed in Ref. [5].

IV. THE POLE STRUCTURE OF SOLUTIONS OF THE tDS EQUATION

In order to be able to use the representation (12) on needs to know the positions of poles and the corresponding residues of the propagator functions. To this end we solve the decomposed tDS equations [8] and [9] for $A(p^2)$ and $B(p^2)$ and investigate the above mentioned scalar part $\Pi(p^2) = p^2 A^2(p^2) + B^2(p^2)$ of the quark propagators $S(p^2)$ together with the functions $\sigma_{s,v}(p^2)$ by computing Cauchy integrals along closed contours. If the corresponding Cauchy integral for $\Pi(p^2)$ is zero and for $\sigma_{s,v}(p^2)$ finite, this immediately implies that $\Pi(p^2)$ has zeros and $\sigma_{s,v}(p^2)$ have poles inside the considered region.
A. Searching for singularities

To find the appropriate contour enclosing zeros of $\Pi(p^2)$ we proceed as follows. The complex function $\Pi(p^2)$ is presented as a vector field in the complex momentum plane $\text{Re}p–\text{Im}p$ in a certain region of $p^2$. A vector field has a smooth distribution of its force lines if it does not contain null vectors (see e.g Ref. [37]). Contrarily, in the vicinity of null vectors (for the vector field these are known as singular vectors) the force lines exhibit a vortex-type behavior. This can essentially facilitate our analysis since a contour around the vortices of $\Pi(p^2)$ definitely contains zeros.

In Fig. 5 we present a few selected regions of the $\text{Re}p–\text{Im}p$ plane, where the vector field of $\Pi(p^2)$ has been found to exhibit vortex-type structures. Clearly, the zeros and singularities must be searched for in the vicinity of these vortices. The force line method, as other methods such as contour plots or three dimensional visualizations, are useful for surveys.

B. Allocation of singularities

The further strategy of finding the positions of singularities more accurately is as follows:

(i) Choose a vortex, enclose it with a contour and compute the Cauchy integrals of $A(p^2)$, $B(p^2)$ and $\Pi(p^2)$. Vanishing integrals imply that these functions are analytical within the chosen contour.

(ii) Compute the Rouché’s integral\(^2\) of the function $\Pi(p^2)$. Since in the previous item we found $\Pi(p^2)$ to be analytical, such an integral, according to the Rouché’s theorem, gives exactly the number of its zeros inside the contour.

(iii) Compute the Cauchy integral of the propagator functions $\sigma_{s,v}(p^2)$ which, if the Rouché’s integral is found to be an integer positive number, clearly must be different from zero. Moreover, the non-zero Cauchy integrals provide us with the residues needed in Eq. (12). To find the desired singularities we compute the following integrals

\(^2\) Rouché’s integral of an analytical complex function $f(z)$ on a closed contour $\gamma$ is defined as $\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz$. 

FIG. 5: (Color online) The force lines of the inverse propagator part $\Pi(p^2)$ for $u$, $d$ quarks. The self-conjugated nature of the singularities located within the "turbulent" regions is evident.

$$\oint_{\gamma} \left[ \xi^2 A^2(\xi) + B^2(\xi) \right] d\xi^2 = 0,$$

(14)

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{\xi^2 A^2(\xi) + B^2(\xi) \xi^2}{\xi^2 A^2(\xi) + B^2(\xi)} d\xi^2 = N_z,$$

(15)

$$\frac{1}{2\pi i} \oint_{\gamma} \sigma_{s(v)}(\xi^2) d\xi^2 = \sum_i \text{res}[\sigma_{s(v)}(\xi_i^2)]$$

(16)

(an effective algorithm for numerical evaluations of Cauchy-like integrals can be found in, e.g. Ref. [38]). Then we squeeze the contour around the chosen vortex while holding the conditions
until the desired accuracy of the determination of the pole position is attained. In such a way we find the first few poles of $\sigma_{s,v}(p^2)$ together with their residues relevant for (12). Note that a good numerical test of the performed calculations is the following procedure. Enclose a few vortices by a larger contour and ensure that Rouché’s integral is exactly equal to the number of vortices and the Cauchy integrals of $\sigma_{s,v}(p^2)$ are exactly the sum of individual residues found before for each isolated vortex. The technique of computing Cauchy integrals to determine the analyticity of the propagator functions has been exploited also in Ref. [39] within a simplified kernel with the running coupling replaced by a constant and the IR term taken as $\sim 1/q^4$.

| pole position | 1            | 2            | 3            | 4            |
|---------------|--------------|--------------|--------------|--------------|
| $u, d$ quarks | (-0.2588, ± 0.1961) | (-0.2418, ± 2.597) | (-1.0415,± 2.8535) | (-0.738,0.0) |
| $s$ quarks   | (-0.436, ± 0.513) | (-0.51, ± 3.35) | (-1.45, ± 3.82) | (-3.25,0.0) |

Results of our calculations are presented in Table II and Figs. 1 and 6. From Table II and Figs. 1 and 6 it is seen that for $M_{q\bar{q}} < 1$ GeV all singularities in the light quark propagator are located outside the parabola. This allows to establish easily reliable algorithms for solving numerically the tBS equation [12] for such a case. For larger meson masses, e.g. $M_{q\bar{q}} \sim 2$ GeV, the singularities are either within the tBS domain [5], or in the close vicinity (see Fig. 1, right panel) and, as a result, the numerical procedure adequate for low masses requires a proper modification. We can take advantage of the fact that, for meson masses $0.14 rm GeV < M_{q\bar{q}} \lesssim 2$ GeV, the number of singularities is not too large and one can combine Eqs. (11) and (12) in such a manner that the regular part $\tilde{\sigma}_{s,v}(p^2)$ can be used as above without modifying the algorithm. The pole part, however, for heavier quarks allows to carry out the
FIG. 6: Positions of few first poles in the upper hemisphere of the complex $p^2$ plane, labeled in correspondence to the Table II for $u$, $d$ (asterisks) and $s$ (open stars) quarks. The relevant sections of the parabola (5) corresponding to the meson bound-state mass $M_{q\bar{q}}$ are presented for $M_{q\bar{q}} = 1, 1.5, 2, 2.5$ and 3 GeV, from right to left. To emphasize the dependence of the pole positions on the bare quark mass $m_q$, the ”lowest” poles for the fiducial values $m_q = 0.5$ GeV (open circle), $m_q = 0.9$ GeV (triangle) and $m_q = 1$ GeV (open square) are displayed as well. As an illustration of the behaviour on the real axis, the fourth pole for $m_q = 0.5$ GeV is presented (circle). The tendency is that, with increasing bare quark mass, the corresponding pole is shifted towards larger values of $\text{Im } p^2$ and $|\text{Re } p^2|$. The area to the left after the axis break is the ”unrevealed terrain”, where further singularities could be located.

angular integration analytically by employing parametrizations (11). Note that the propagator functions for the $s$ quark ($m_q = 115 \text{ MeV}$) possess only a self conjugated pole in the vicinity of the considered parabola, cf. Fig. 1. It is located at $p^2 = (-0.436 \pm 0.5131i) \text{ (GeV/c)^2}$ with residues $\text{res}[\sigma_s] = (9.05 \times 10^{-3} \pm 0.491i) \text{ GeV}$ and $\text{res}[\sigma_v] = 0.261 \mp 0.538i$ for $\sigma_s$ and $\sigma_v$, respectively. The second pole, located at $p^2 = (-0.507 \pm 3.35i) \text{ rm (GeV/c)^2}$ (with the respective residues $\text{res}[\sigma_s] = (5.5 \times 10^{-2} \pm 0.10i) \text{ GeV}$ and $\text{res}[\sigma_v] = 1.34, 10^{-2} \mp 6.12, 10^{-2}i$), is located already too far from the corresponding parabola for strange mesons and, consequently is irrelevant in numerical calculations.
With these calculations our analysis of the pole structure is completed. Let us remind
the prepositions: (i) tDS equation Eq. (3), restricted to the momentum range relevant for mesons
as $q\bar{q}$ bound states, (ii) tBS equations and $M_{q\bar{q}} < 3$ GeV, (iii) combined vertex-gluon kernel (1)
with IR part only.

V. IMPACT OF THE UV TERM

An unpleasant fact is to be noted here. As seen from the Table II the fourth pole is located
exactly on the real axis, hence formally violating confinement. Presumably this is due to the
simplified interaction we used so far, i.e. ignoring the UV term in Eq. (4). We reiterate that, in
spite of the minor contribution of the UV term for meson masses with light quarks according
to Refs. [12, 19, 31], it can formally improve the behaviour of the solution on the real axis.
Indeed, an analysis of the solution of the tDS equation along the real $p^2$ axis with the UV
term taken into account shows that, at least in the considered domain, the pole singularities
of the propagator functions disappear. However, instead it turns out that with both, IR and
UV, terms the new solutions $A(p)$ and $B(p)$ are not anymore analytical even for $\text{Re } p^2 > 0$, as
evidenced by logarithmic branch point singularities mentioned in [40]. In the present paper we
do not discuss the physical meaning and mathematical implications of such singularities in the
complex plane. Instead, we try to reconcile the absence of singularities on the real axis with
analyticity of $A(p)$ and $B(p)$ in the complex plane, i.e. to keep the logarithmic term only either
on or in a close vicinity of the real $p^2$ axis making it vanishing at larger $\text{Im } p^2$. This can be
accomplished by multiplying the UV term by a damping function which is equal to unity on
the real axis and vanishes elsewhere. Such a procedure has been proposed, e.g. in Ref. [30],
to manipulate the IR and UV structure of the ghosts in dressing the gluon propagator. In such a
manner the solution along the real axis preserves the correct ultraviolet behaviour. Then, the
new solution is used to compute the tDS integrals in (8) and (9) in the complex plane with
the IR term only. Such a modification of the phenomenological truncation of the DS equation
looks somehow artificial, however, it maintains the correct UV behaviour of the solution on the
real axis and provides, as before, analytical solutions $A(p)$ and $B(p)$ in the complex plane and
does not affect the solution of the tBS equation for $M_{q\bar{q}} \lesssim 2$ GeV. Certainly, the solution and
the propagator functions will slightly differ from the ones previously obtained with the IR term
only. Repeating all the above analysis for the new solution one finds the new positions and
residues for the modified kernel. The result is listed in Table III. Note that including the UV term in (4) with additional parameters $\tau, \gamma_m$ and $\lambda_{QCD}$ requires slightly modified parameters $D$ and $\omega$, cf. Ref. [3]. Comparing with Table II with the IR term solely it is seen that the positions

| Table III: The pole structure of the propagator functions for $u, d$ quarks ($m_q = 5$ MeV). The effective parameters for the kernel (4) are from Refs. [3, 19, 21]: $\omega = 0.4$ GeV, $D = 36.45$ GeV$^{-2}$, $\tau = e^2 - 1$, $\Lambda_{QCD} = 0.234$ GeV, $\gamma_m = 0.48$. Notation as in Table II. For completeness, the corresponding poles and residues for the $s$ quark are also displayed (bottom part). |
|---|---|---|
| $u, d$ quarks | 1 | 2 | 3 |
| pole position | (-0.215, \(\pm\) 0.335) | (-0.12, \(\pm\) 2.21) | (-1.091, \(\pm\) 0.49) |
| res[$\sigma_s$] | (-0.017, \(\pm\) 0.325) | (0.035, \(\pm\) 0.09) | (-0.021, \(\pm\) 0.0017) |
| res[$\sigma_v$] | (0.224, \(\pm\) 0.464) | (0.02, \(\pm\) 0.057) | (-0.002, \(\pm\) 0.003) |
| $s$ quark | 1 | 2 | 3 |
| pole position | (-0.21, \(\pm\) 0.41) | (-0.14, \(\pm\) 2.17) | (-0.743, \(\pm\) 2.53) |
| res[$\sigma_s$] | (-0.001, \(\pm\) 0.30) | (0.039, \(\pm\) 0.08) | (-0.039, \(\pm\) 0.06) |
| res[$\sigma_v$] | (0.23, \(\pm\) 0.38) | (0.02, \(\pm\) 0.058) | (0.004, \(\pm\) 0.044) |

of the poles and the residues of the propagator functions are a bit different. However, since the new solution of the DS equation, $\sigma_{s,v}(p^2)$, is correlated with its analytical part and pole structure, cf. Eq. (12), this circumstance does not affect the results of solving the BS equation, implying for instance the robustness of the numerical results reported in [12].

VI. SUMMARY

We analyse analytical properties of the solution of the truncated Dyson-Schwinger (tDS) equation for the quark propagator in the Euclidean complex momentum domain which is determined by the truncated Bethe-Salpeter equation for $q\bar{q}$ bound states with light quarks. It is found that, within the ladder rainbow truncation with only the infrared term in the combined effective vertex-gluon kernel, the solutions $A(p^2)$, $B(p^2)$ and the propagator functions $\sigma_{s,v}(p^2)$ for $c$ quarks are analytical functions in the whole considered domain for $M_{q\bar{q}} < 3$ GeV, while for $u, d$ and $s$ quarks they are analytical only in the right hemisphere $\text{Re} \, p^2 > 0$. At negative $\text{Re} \, p^2$, the functions $A(p^2)$ and $B(p^2)$ are still analytical, however, the propagator functions $\sigma_{s,v}(p^2)$ contain pole singularities. The exact position of the poles and the corresponding residues of the
propagator functions can be found by applying Rouché’s theorem and computing the Cauchy integrals. Prior to that, in order to localize the approximate region with singularities, we analyse the vector fields of the inverse propagators, the vortices of which indicate the positions of null vectors.

The position of the first few poles and the corresponding residues are found with good accuracy relevant to be used in further calculations of the Bethe-Salpeter (BS) equation. It is also found that, with only the effective infrared term in the parametrization of the combined vertex-gluon kernel, the propagator functions exhibit poles on the real axis, formally violating the confinement. These singularities can be removed by taking into account an ultraviolet term. This term is known to have a minor contribution for not too heavy meson masses and, at the same time, to provide additional singularities of logarithmic branch point types in the solution of the tDS equation in the complex plane. These singularities hamper a simple analysis of the pole structure of the solution. Nevertheless, since the ultraviolet term is needed mainly to guarantee the correct asymptotic behaviour along the real axis, we suggest to take it into account only when solving the tDS equation for real $p^2$ and then to use such solutions to compute the tDS integrals in the complex plane with IR term only. In such a way one can assure the absence of poles on the real axis and analytical functions determining the quark propagator in the considered complex domain.

The performed analysis is aimed at elaborating adequate numerical algorithms to solve the BS equation in presence of singularities and to investigate the properties of mesons, such as the open charm $D$ mesons, related directly to physical programmes envisaged, e.g. at FAIR. Furthermore, the knowledge of the analytical structure of the quark propagators is important for designing appropriate phenomenological kernels since it is related to such fundamental characteristics of QCD as confinement and dynamical chiral symmetry breaking phenomena encoded in the chiral condensate being the trace of the quark propagator.

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