Anisotropic Multiverse with Varying $c$, $G$ and Study of Thermodynamics

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Abstract

We assume the anisotropic model of the Universe in the framework of varying speed of light $c$ and varying gravitational constant $G$ theories and study different types of singularities. For the singularity models, we write the scale factors in terms of cosmic time and found some conditions for possible singularities. For future singularities, we assume the forms of varying speed of light and varying gravitational constant. For regularizing big bang singularities, we assume two forms of scale factors: sine model and tangent model. For both the models, we examine the validity of null energy condition and strong energy condition. Start from the first law of thermodynamics, we study the thermodynamic behaviours of $n$ number of Universes (i.e., Multiverse) for (i) varying $c$, (ii) varying $G$ and (iii) both varying $c$ and $G$ models. We found the total entropies for all the cases in the anisotropic Multiverse model. We also found the nature of the Multiverse if total entropy is constant.

1 Introduction

The Multiverse theory states that there are multiple versions of the universe, each of them are slightly different from the other. Each universe is known as parallel Universe and the whole collection of these parallel Universes is known as the Multiverse (for review see [1]). The notion of a cosmic Multiverse has been introduced in the context of eternal inflation [2,3] caused by inflaton field. Thus due to inflationary expansion, the continuously growing number of subuniverses produced which is surrounded by rapidly growing regions of inflating space [4]. In ref. [4], the model of Multiverse has been discussed by taking scalar field model of dark energy. Observational consequences of an interacting Multiverse have been studied in [5]. Marosek et al [10] have analyzed isotropic cyclic Multiverse model and studied several kinds of singularities for varying speed of light $c$ and varying gravitational constant $G$ and regularizing the singularities in cyclic universe. Also they have studied the thermodynamics for doubleverse model and found the nature of the doubleverse if the total entropy of the doubleverse is constant. Also the regularizing cosmological singularities have been studied in [10,11,12] by varying physical constant (say, $c$, $G$) theories. Motivated by their works, here we study the singularities in the anisotropic model of the universe for varying $c$ and $G$. We study the regularizing singularities for varying constants in anisotropic model of the cyclic universe. We discuss the thermodynamic behaviors in anisotropic Multiverse ($n$ number of universes) model and investigate the nature of the Multiverse if total entropy of the Multiverse is constant. The paper is organized as follows: In section II, we study the anisotropic space-time model with varying $c$ and $G$. In section III, we describe some kinds of singularities. In section IV, we discuss regularizing big bang and big rip singularities for cyclic model. In section V, we study the thermodynamics in Multiverse with varying $c$ and $G$. The discussions and some concluding remarks are presented in section VI.

2 Anisotropic Model of the Universe

Marosek et al [10] have studied isotropic model of the universe for varying speed of light and varying gravitational constant [13,14,15]. In this section, we study the anisotropic model of the Universe and we write the Einstein’s field equations with continuity equation in vary-
ing speed of light and varying gravitational constant. Now we consider homogeneous and anisotropic space-time model of the Universe described by the line element in [16, 17, 18, 19]

\[ ds^2 = -c^2(t)dt^2 + a^2(t)dx^2 + b^2(t)d\Omega_k^2 \] (1)

where \( a, b \) are functions of time \( t \) only and

\[ d\Omega_k^2 = \begin{cases} 
  dy^2 + dz^2, & \text{when } k = 0 \\
  d\theta^2 + \sin^2\theta d\phi^2, & \text{when } k = +1 \\
  d\theta^2 + \sinh^2\theta d\phi^2, & \text{when } k = -1
\end{cases} \] (Bianchi I, Kantowski-Sachs model) (Bianchi III model)

Here \( k = 0, +1, -1 \) is the curvature index of the corresponding 2-space, so that the above three types are described by Thorne [20] as flat, closed and open respectively and \( c = c(t) \) is the time varying speed of light.

As per the proposal of varying speed of light [21], the Friedman equations remain valid even when \( \dot{c} \neq 0 \). Due to the same prescription, the Einstein’s field equations in the framework of varying speed of light and varying gravitational constant theories in anisotropic universe are [16, 17, 18]

\[ \frac{\dot{b}^2}{b^2} + 2 \frac{\ddot{b}}{b} + \frac{k c^2(t)}{b^2} = 8\pi G(t) \rho, \]

\[ \frac{\ddot{b}}{b} + 2 \frac{\dot{b}^2}{b^2} + \frac{k c^2(t)}{b^2} = -8\pi G(t) \rho \frac{\dot{c}^2(t)}{c^2(t)}, \]

\[ \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a} \dot{b}}{ab} = \frac{8\pi G(t) \rho}{c^2(t)} \] (3) (4) (5)

where \( \rho \) and \( p \) are respectively the mass density and pressure. Consequently, \( \rho c^2 \) is the energy density. Here \( G = G(t) \) is the time varying gravitational constant.

From equations (3) - (5), we obtain

\[ \frac{\ddot{a}}{a} + 2 \frac{\ddot{b}}{b} = -4\pi G(t) \left( \rho + \frac{3p}{c^2(t)} \right) \] (6)

For varying \( c \) and varying \( G \), the modified continuity equation is obtained as

\[ \dot{\rho} + \left( \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} \right) \left( \rho + \frac{p}{c^2(t)} \right) = -\dot{G} \frac{c \dot{c}}{4\pi G b^2} \] (7)

In the next sections, we study the singularities for varying constants in anisotropic space-time model.

3 Singularities

In this section, we study several singularities in the anisotropic Universe. For varying \( c \) and varying \( G \) models, we find the conditions so that singularities can occur. For two scale factors model (anisotropic), big bang initial singularity [22] occurs at \( t \to 0 \) and at this time, \( a \to 0, b \to 0, \rho \to \infty \) and \( |p| \to \infty \). The future singularities can be classified in the following ways:

- **Type I (Big Rip) [23]**: For \( t \to t_s, a \to \infty, b \to \infty, \rho \to \infty \) and \( |p| \to \infty \).
- **Type II (Sudden future) [24]**: For \( t \to t_s, a \to a_s, b \to b_s, \rho \to \rho_s \) and \( |p| \to \infty \).
- **Type III (Finite scale factors) [25]**: \( t \to t_s, a \to a_s, b \to b_s, \rho \to \infty \) and \( |p| \to \infty \).
- **Type IV (Big separation) [25]**: For \( t \to t_s, a \to a_s, b \to b_s, \rho \to 0 \) and \( |p| \to 0 \).
- **Type V (w-singularity) [26]**: For \( t \to t_s, a \to a_s, b \to b_s, \rho \to 0, \rho \to 0, w = p/\rho \to \infty \).
- **Type VI (Little Rip) [27]**: For \( t \to \infty, a \to \infty, b \to b_s, \rho \to \infty \) and \( |p| \to \infty \).

where \( t_s, a_s, b_s, \) and \( \rho_s \) are constants with \( a_s \neq 0, b_s \neq 0 \). The big bang, big rip and little rip are strong singularities while the other singularities mentioned above are weak singularities. Since \( a(t) \) and \( b(t) \) are unknown functions of \( t \), so to keep anisotropy of the Universe, we must have \( a(t)/b(t) \) constant. So for simplicity of calculation, we may assume, \( b(t) \) is related to the power law form of \( a(t) \) i.e., \( b(t) = b_0 a^\alpha(t) \) [16, 17, 18, 19, 25, 27, 30, 31, 32], where \( b_0, \alpha \) are positive constants. The physical importance of this assumption is that it gives constant ratio of shear and expansion scalar [27]. Here we use the two scale factors which after appropriate choice of parameters admit big-bang, big-rip, sudden future, finite scale factor and w-singularities and read as [10, 11]

\[ a(t) = a_s \left( \frac{t}{t_s} \right)^m \exp \left| 1 - \frac{t}{t_s} \right|^n \] (8)

where \( a_s, t_s, m, n \) are positive constants. So \( b(t) \) can be written as

\[ b(t) = b_0 a_s^\alpha \left( \frac{t}{t_s} \right)^{m\alpha} \exp \left( \alpha \left| 1 - \frac{t}{t_s} \right|^n \right) \] (9)

Since first powers of equations (8) and (9) give the big bang singularity, so the second parts of (8) and (9) provide the future singularity. Now we want to analyze the future singularity for our anisotropic model. For future singularity part, taking the expressions of \( a, b \) from equations (8), (9) and putting their values in equations (3) and (4), we directly obtain the energy density and pressure for the future singularity as [10, 11]

\[ \rho = \frac{1}{8\pi G(t)} \left[ (\alpha^2 + 2\alpha n^2) a_s^{2n} \left| 1 - \frac{t}{t_s} \right|^{2n - 2} + kc^2(t) \frac{a_s^{2n}}{\alpha^2} \exp \left( -\alpha \left| 1 - \frac{t}{t_s} \right|^n \right) \right] \] (10)
We observe that for $0 \leq n \leq 1$, there is a finite scale factor singularity, for $1 \leq n \leq 2$, there is a sudden future singularity and for $n \geq 2$, there is a generalized sudden future singularity. Since future singularity occurs at $t = t_s$, so we may assume the variable $G$ is in the power law form\[10\]\[11\]:

$$G(t) = G_0 \left|1 - \frac{t}{t_s}\right|^{-r}$$

where $r > 0$ and $G_0 > 0$ are constants. For this form of $G(t)$, at $t = t_s$, we have $G \to \infty$. At $t = t_s$ the density and pressure are finite for $r > 2 - n$. So sudden singularity is regularized due to strong gravitational coupling ($G \to \infty$ at $t = t_s$). But for $t \to \infty$, the scale factors $a \to \infty$ and $b \to \infty$ and both the density $\rho \to \infty$ and pressure $|p| \to \infty$ which achieved to a little rip singularity. On the other hand, similar to the choice of $G(t)$, we may choose the varying speed of light $c$ as in the power law form\[10\]\[11\]:

$$c(t) = c_0 \left|1 - \frac{t}{t_s}\right|^\beta$$

where $\beta > 0$ and $c_0 > 0$ are constants. For this form of $c(t)$, at $t = t_s$, we have $c \to 0$. For $\beta > 2 - n$ at $t = t_s$, $a \to \infty$ and $b \to \infty$ and both the density $\rho \to \infty$ and pressure $p \to \infty$ which regularized to a sudden singularity. Also for $t \to \infty$, we obtain $a \to \infty$, $b \to \infty$, $\rho \to \infty$ and $|p| \to \infty$ which achieved at little rip singularity. So due to above choices of the varying $c$ and $G$, sudden singularity can be regularized.

4 Regularizing Strong Singularity with Varying $G$

In this section, we study regularizing strong singularities like big bang and big rip singularities for varying $G$ model in anisotropic cyclic universe. So we assume constant $c$ in order to construct regularized anisotropic universe\[11\]. In the following subsection A, we discuss the sine model to study regularizing big bang singularity while in the subsection B, we discuss the tangent model to study regularizing big bang and big rip singularities. Also we study the validity of the energy conditions for both models.

4.1 Regularizing big bang singularity: sine model

Here for cyclic universe, we assume the ‘sine’ model to describe regularizing big bang singularity. So the scale factors $a(t)$ and $b(t)$ can be chosen in the forms\[10\]:

$$a(t) = a_0 \left|\sin\left(\frac{\pi t}{t_c}\right)\right|$$

and

$$b(t) = b_0 a^\alpha(t) = b_0 a_0^\alpha \left|\sin^\alpha\left(\frac{\pi t}{t_c}\right)\right|$$

where $a_0, b_0, \alpha$ are positive constants. Here $t_c$ is the turning point for the cyclic universe where scale factors $a$ and $b$ are zero. For $t \to 0$ we get $a \to 0$ and $b \to 0$, which follows big bang singularity. For constant $c$, we assume

$$G(t) = \frac{G_0}{ab}$$

where $G_0$ is positive constant. So at the big bang singularity ($a = 0$), $G \to \infty$. But at the turning point ($t = t_c$), we have $a \to 0$, $b \to 0$ and $G \to \infty$. That means at the turning point of the cyclic universe, the curvature singularity (big-bang like singularity) can be regularized due to strong gravitational coupling. Here ‘like’ is used because the density and pressure are regular for this singularity but they are not regular at a big bang. Using Einstein’s field equations, we obtain the density and pressure in the forms:

$$\rho = \frac{a_0^{2\alpha + 1} b_0^2}{8\pi G_0} \sin^{2\alpha + 1} \left(\frac{\pi t}{t_c}\right) \left[(\alpha^2 + 2\alpha)\cot^2 \left(\frac{\pi t}{t_c}\right) + \frac{k c_0^{2\alpha} b_0 c^2}{a_0^{2\alpha} b_0^2} \cot^2 \left(\frac{\pi t}{t_c}\right)\right]$$

and

$$p = \frac{\pi c^2}{12G_0 t_c^2} \left[(1 + 2\alpha) - 3\alpha^2 \cot^2 \left(\frac{\pi t}{t_c}\right)\right] - \frac{k}{24\pi G_0 a_0^{2\alpha} b_0} \cot^2 \left(\frac{\pi t}{t_c}\right)$$

Now we study null energy condition and strong energy condition for this model. From the above expressions (17) and (18), we obtain

$$p + \rho c^2 = \frac{\pi c^2 a_0^{\alpha + 1} b_0}{12 G_0 t_c^2} \sin^{\alpha + 1} \left(\frac{\pi t}{t_c}\right) \times$$

$$\left[(1 + 2\alpha) + (4\alpha - \alpha^2) \cot^2 \left(\frac{\pi t}{t_c}\right)\right]$$

$$+ \frac{k c_0^{2\alpha}}{12 a_0^{2\alpha} b_0} \cot^2 \left(\frac{\pi t}{t_c}\right)$$

and

$$3p + \rho c^2 = \frac{\pi c^2 a_0^{\alpha + 1} b_0}{4 G_0 t_c^2} \sin^{\alpha + 1} \left(\frac{\pi t}{t_c}\right) \times$$

$$\left[(1 + 2\alpha) + 2(\alpha - \alpha^2) \cot^2 \left(\frac{\pi t}{t_c}\right)\right]$$

The null energy condition $p + \rho c^2 > 0$ is satisfied for $0 < \alpha \leq 4$ with $k \geq 0$. The strong energy condition
3p + ρc^2 > 0 is satisfied for 0 < α ≤ 1.

At the turning points, we have
\[ a(n_{t_c}) = a_0 , \quad b(n_{t_c}) = b_0 a_0^\alpha , \quad G(n_{t_c}) = \frac{G_0}{b_0 a_0^\alpha + 1} , \]
and
\[ \rho(n_{t_c}) = \frac{kc^2}{8\pi G_0 b_0 a_0^\alpha - 1} \]  \hspace{1cm} (21)

where \( n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \). But at \( t = mt_c \) (\( m = 0, 1, 2, \ldots \)), \( a \to 0, \; b \to 0, \; \rho = \) constant, \( p = \) constant and \( G \to \infty \). So the curvature singularity (big-bang like singularity) is regularized in each cycle due to strong gravitational coupling.

Now the Hubble parameter is defined by
\[ H = \frac{1}{3} \left( \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} \right) = \frac{\pi}{3t_c} (1 + 2\alpha) \cot \left( \frac{\pi t}{t_c} \right) \]  \hspace{1cm} (23)
and its derivative is obtained by
\[ \dot{H} = -\frac{\pi^2}{3t_c^2} (1 + 2\alpha) \cosec^2 \left( \frac{\pi t}{t_c} \right) \]  \hspace{1cm} (24)

We observe that for \( \alpha \leq \frac{1}{2}, (m+1)t_c \), \( H > 0 \) which represents the expansion and for \( (m+1)t_c \leq t \leq (m+2)t_c, \; H < 0 \) which represents the contraction. Also since \( \dot{H} < 0, \) so there is no proper bounce of the universe.

4.2 Regularizing big bang and big rip singularities: tangent model

Here we consider ‘tangent’ model to describe regularizing big bang and big rip singularities. So the forms of the scale factors are \[10\]
\[ a(t) = a_0 \left| \tan \left( \frac{\pi t}{t_s} \right) \right| , \]  \hspace{1cm} (25)
and
\[ b(t) = b_0 a^{\alpha}(t) = b_0 a_0^\alpha \left| \tan^\alpha \left( \frac{\pi t}{t_s} \right) \right| \]  \hspace{1cm} (26)
and gravitational constant as
\[ G(t) = \frac{G_s}{b^3(t)} = \frac{G_s}{b_0^3 a_0^{3\alpha} \left| \tan^{\alpha\beta} \left( \frac{\pi t}{t_s} \right) \right|} \]  \hspace{1cm} (27)

The mass density and pressure are obtained by
\[ \rho = \frac{b_0^3 a_0^{3\alpha}}{8\pi G_s t_s^2} \left| \tan^{\alpha\beta} \left( \frac{\pi t}{t_s} \right) \right| \left[ 4\pi^2 (\alpha^2 + 2\alpha) \cosec^2 \left( \frac{\pi t}{t_s} \right) \right. \]
\[ \left. + \frac{kc^2}{b_0^2 a_0^{2\alpha}} \right| \cot^{2\alpha} \left( \frac{\pi t}{t_s} \right) \right] \]  \hspace{1cm} (28)
and
\[ p = \frac{\pi c^2 b_0^{a\beta}}{6G_s t_s^2} \left| \tan^{\alpha\beta} \left( \frac{\pi t}{t_s} \right) \right| \times \]
\[ \left[ -(5\alpha^2 + 2\alpha + 2) \cosec^2 \left( \frac{2\pi t}{t_s} \right) \right. \]
\[ + 2(1 + 2\alpha) \cosec \left( \frac{2\pi t}{t_s} \right) \cot \left( \frac{\pi t}{t_s} \right) \]
\[ - \frac{kc^2}{4\pi^2 b_0^2 a_0^{2\alpha}} \right| \cot^{2\alpha} \left( \frac{\pi t}{t_s} \right) \right] \]  \hspace{1cm} (29)

The minimum values of the mass density and pressure are given by
\[ \rho_{\text{min}} = \rho(\kappa t_s) = \frac{b_0^{\alpha\beta}}{8\pi G_s t_s^2} \left[ 4\pi^2 (\alpha^2 + 2\alpha) + \frac{kc^2 t_s^2}{b_0^{2\alpha}} \right] \]  \hspace{1cm} (30)
and
\[ p_{\text{min}} = p(\kappa t_s) = \frac{c^2 b_0^{\alpha\beta}}{24\pi G_s t_s^2} \left[ 4\pi^2 (5\alpha^2 + 2\alpha + 2) + \frac{kc^2 t_s^2}{b_0^{2\alpha}} \right] \]  \hspace{1cm} (31)

where \( \kappa = \frac{2m+4}{m} \) (\( m = 0, 1, 2, \ldots \)).

We see that the scale factors \( a \to \infty, \; b \to 0, \; G \to 0 \), density \( \rho = \) constant and pressure \( p = \) constant for \( t = t_s \) with \( n = 1/2, 3/2, 5/2, \ldots \). So the big rip like singularity is regularized. But the scale factors \( a \to 0, \; b \to 0, \; G \to \infty \), density \( \rho = \) constant and pressure \( p = \) constant for \( t = mt_s \) with \( m = 0, 1, 2, \ldots \). So the curvature singularity (big-bang like singularity) is also regularized.

Now we study null energy condition and strong energy condition for this model. Now from equations (28) and (29), we get
\[ p + \rho c^2 = \frac{\pi c^2 b_0^{a\beta}}{3G_s t_s^2} \left| \tan^{\alpha\beta} \left( \frac{\pi t}{t_s} \right) \right| \times \]
\[ \left[ -(\alpha^2 + 1) \cosec^2 \left( \frac{2\pi t}{t_s} \right) \right. \]
\[ + (1 + 2\alpha) \cosec \left( \frac{2\pi t}{t_s} \right) \cot \left( \frac{\pi t}{t_s} \right) \]
\[ + \frac{kc^2 t_s^2}{4\pi^2 b_0^{2\alpha}} \right| \cot^{2\alpha} \left( \frac{\pi t}{t_s} \right) \right] \]  \hspace{1cm} (32)

At \( t = mt_s \), we get: (i) \( p + \rho c^2 > 0 \) i.e., null energy condition is satisfied for \( 0 < \alpha \leq 2 \) and \( k \geq 0 \); (ii) \( p + \rho c^2 < 0 \) i.e., null energy condition is violated for \( \alpha > 2 \) and \( k < 0 \). But at \( t = nt_s \), we get: \( p + \rho c^2 < 0 \) i.e., null energy condition is always violated.

Also from equations (28) and (29), we have
\[ 3p + \rho c^2 = \frac{\pi c^2 b_0^{a\beta}}{G_s t_s^2} \left| \tan^{\alpha\beta} \left( \frac{\pi t}{t_s} \right) \right| \times \]
\[ \left[ -(2\alpha^2 + 1) \cosec^2 \left( \frac{2\pi t}{t_s} \right) \right. \]
\[ + (1 + 2\alpha) \cosec \left( \frac{2\pi t}{t_s} \right) \cot \left( \frac{\pi t}{t_s} \right) \]
\[ + \frac{kc^2 t_s^2}{b_0^{2\alpha}} \right| \cot^{2\alpha} \left( \frac{\pi t}{t_s} \right) \right] \]  \hspace{1cm} (33)
At $t = nt_s$, we get: (i) $3p + \rho c^2 > 0$ i.e., strong energy condition is satisfied for $0 < \alpha \leq 1$; (ii) $3p + \rho c^2 < 0$ i.e., strong energy condition is violated for $\alpha > 1$. But at $t = nt_s$, we get: $3p + \rho c^2 < 0$ i.e., strong energy condition is always violated.

5 Thermodynamics in Multiverse

In ref. [10], the authors have studied the classical thermodynamics of two universes (doubleverse) and their consideration that the entropy of the two universes is changing in such correlated way that the total entropy is always the same (constant). Motivated by their work, in this section, we will study the thermodynamics in Multiverse ($n$ number of universes) model. For this purpose, we use the first law of thermodynamics and calculate the total entropy of the Multiverse for (i) varying $c$ with constant $G$ model, (ii) varying $G$ with constant $c$ model and (iii) both varying $c$ and varying $G$ model. Then we will show that the total entropy is always the same (constant) provided there are some relations between $n$ number of universes.

5.1 Thermodynamics for varying $c$

First we consider the varying $c$ model with constant $G$ and analyze the thermodynamic nature of $n$ number of universes i.e., Multiverse model. The first law of thermodynamics is given by [10]

$$dE = TdS - pdV$$

where $E, T, S, p, V$ are the internal energy, temperature, entropy, pressure and volume respectively. From Einstein’s mass-energy equation we have

$$E = mc^2 = \rho V c^2$$

where $\rho$ is the mass density and $V = ab^2$. From above two equations, we obtain [10]

$$\dot{\rho} + \frac{\dot{V}}{V} (\rho + \frac{p}{c^2}) + 2\rho \frac{\dot{c}}{c} - \frac{T}{Vc^2} \dot{S} = 0$$

Using continuity equation

$$\dot{\rho} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) (\rho + \frac{p}{c^2}) = \frac{kcc}{4\pi Gb^2}$$

and above equation (36), we have

$$\frac{T}{Vc^2} \dot{S} - 2\rho \frac{\dot{c}}{c} = \frac{kcc}{4\pi Gb^2}$$

Defining [10]

$$\bar{\rho} = \frac{1}{8\pi G} \left(\frac{\dot{b}^2}{b^2} + 2\frac{\dot{a}}{a} \frac{\dot{b}}{b^2} + \frac{2kc^2}{b^2}\right) = \rho + \frac{kcc}{8\pi Gb^2}$$

and

$$\bar{\rho} = p - \frac{kc^2}{24\pi Gb^2}$$

we obtain

$$\dot{S} = 2\rho \frac{V c^2}{T} \frac{\dot{c}}{c}$$

For the equation of state of ideal gas we can take [10]

$$\dot{\rho} \frac{V c^2}{T} = \text{constant} = \frac{N k_B}{\bar{w}}$$

where $N$ is the number of particles, $k_B$ is the Boltzmann constant and $\bar{w} = \frac{2c}{3}$. After integrating the above equation, we get the entropy as in the following form:

$$S(t) = \frac{2Nk_B}{\bar{w}} \log[A_0c(t)]$$

where $A_0$ is constant of integration. For flat ($k = 0$) model, we have $\dot{\rho} = \rho$. Now using the barotropic equation of state $p = wpc^2$ with the equation of state parameter $w = \text{const}$, we have

$$S(t) = \frac{2Nk_B}{\bar{w}} \log[A_0c(t)]$$

The nature of entropy depends on the nature of $c(t)$.

Now we want to study the nature of entropy in the Multiverse model. So the total entropy of the $n$ number of universes (i.e., Multiverse) is given by

$$\dot{S} = \sum_{i=1}^{n} \dot{S}_i(t)$$

where the entropy of $i$-th universe is

$$S_i(t) = \frac{2N_i k_B}{\bar{w}} \log[A_0 c_i(t)], \quad i = 1, 2, ..., n$$

We assume the following form of $c(t)$ [11]:

$$A_0 c_i(t) = e^{\lambda_i \phi_i(t)}, \quad i = 1, 2, ..., n$$

where $\lambda_i$’s are arbitrary constants. So the total entropy of the Multiverse is [10]

$$S = \sum_{i=1}^{n} S_i(t) = \sum_{i=1}^{n} \frac{2k_B N_i \lambda_i}{\bar{w}} \phi_i(t)$$

Now we can study the relation between the natures of the $n$ number of universes if we assume total entropy of the Multiverse is constant. In ref [11], the authors have considered the doubleverse system and they have studied the nature of two universes if the total entropy of two universes is constant. For this purpose, here we may assume in the Multiverse there are even number of universes (i.e., $n$ is even). To get constant total entropy of the Multiverse (i.e., $S = \text{constant}$), we may assume [10]

$$\phi_i(t) = \left\{ \begin{array}{ll}
\sin^2 \left(\frac{\pi t}{\tau_s}\right) & , \quad i = 1, 2, ..., \frac{n}{2} \\
\cos^2 \left(\frac{\pi t}{\tau_s}\right) & , \quad i = \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n
\end{array} \right.$$
constant. For varying gravitational constant $G$ with constant $c$, the continuity equation is

$$
\dot{\rho} + \left( \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} \right) \left( \rho + \frac{p}{c^2} \right) = - \rho \frac{\dot{G}}{G}
$$

So using equation (36), we get

$$
\dot{S} = - \frac{\rho V c^2}{T} \frac{\dot{G}}{G}
$$

Using the equation of state of ideal gas

$$
\frac{\rho V c^2}{T} = \text{constant} = N k_B
$$

we get

$$
S = N k_B \log \left[ \frac{B_0}{G(t)} \right]
$$

where $B_0$ is an integration constant.

So the entropy of $i$-th universe is

$$
S_i(t) = N k_B \log \left( \frac{B_0}{G_i(t)} \right), \quad i = 1, 2, \ldots, n
$$

So the total entropy of the Multiverse is

$$
S = \sum_{i=1}^{n} S_i = \sum_{i=1}^{n} N k_B \log \left( \frac{B_i}{G_i(t)} \right)
$$

Similar to the previous subsection $A$, we assume that the Multiverse contains even number of universes (i.e., $n$ is even). Now we assume the gravitational constant $G(t)$ in terms of the scale factors as in the following form:

$$
G_i(t) = \begin{cases} 
\frac{G_{0i}}{a_i(t)b_i(t)}, & i = 1, 2, \ldots, \frac{n}{2} \\
G_{0i}a_i(t)b_i(t), & i = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n
\end{cases}
$$

If all the scale factors of all the universes are equal i.e., $a_i(t) = a(t)$ and $b_i(t) = b(t)$, $i = 1, 2, \ldots, n$ then all the universes have same types of evolution. To get the constant total entropy, we need the following conditions:

$$
N_1 = N_2 = \ldots = N_n, \quad \text{with} \quad \frac{B_1}{G_{01}} = \frac{B_2}{G_{02}} = \ldots = \frac{B_n}{G_{0n}}
$$

We observe that the entropies and gravitational constant of $n/2$ number of universes are growing (or diminishing) and other $n/2$ number of universes are diminishing (or growing) but the total entropy of the Multiverse will be constant.

5.3 Thermodynamics for varying $c$ and $G$

Finally, we consider both varying $c$ and varying $G$ models and analyze the thermodynamic nature of $n$ number of universes i.e., Multiverse model. For varying $c$ and varying $G$ together, the continuity equation becomes

$$
\dot{\rho} + \left( \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} \right) \left( \rho + \frac{p}{c^2} \right) = - \rho \frac{\dot{G}}{G} + \frac{k c \dot{c}}{4 \pi G b^2}
$$

Using this equation and equation (36), we get

$$
\frac{T}{V c^2} \dot{S} - 2 \rho \frac{\dot{c}}{c} = - \rho \frac{\dot{G}}{G} + \frac{k c \dot{c}}{4 \pi G b^2}
$$

Now we obtain

$$
\dot{S} = 2 \frac{\rho V c^2 \dot{c}}{T} - \rho V c^2 \frac{\dot{G}}{G} + \frac{k c \dot{c}}{4 \pi G b^2}
$$

where $\dot{\rho}$ is defined in equation (39). Due to ideal gas equation of states, we may take

$$
\frac{\dot{\rho} V c^2}{T} = \text{constant} = \frac{N k_B}{\bar{w}},
$$

and \( \rho V c^2 \frac{\dot{G}}{G} = \text{constant} = \frac{N k_B}{\bar{w}} \)

Using the above relations, integrating equation (61), we get the entropy

$$
S(t) = \frac{2 N k_B}{\bar{w}} \log[c(t)] - N k_B \log[G(t)] + \log[D_0]
$$

where $D_0$ is an integration constant. The entropy of $i$-th Universe is

$$
S_i(t) = \frac{2 N k_B}{\bar{w}} \log[c_i(t)] - N k_B \log[G_i(t)] + \log[D_i], \quad i = 1, 2, \ldots, n
$$

So the total entropy of the Multiverse is obtained as

$$
S = \sum_{i=1}^{n} S_i = \sum_{i=1}^{n} \left[ \frac{2 N k_B}{\bar{w}} \log[c_i(t)] - N k_B \log[G_i(t)] + \log[D_i] \right]
$$

Since here both $c_i(t)$ and $G_i(t)$ are varying with $t$, to get constant total entropy of the $n$ numbers of universes (where $n$ is considered as even) we can choose $c_i(t)$ from equation (47) where $\phi_i(t)$ is satisfying the equation (49) with the condition (50) and $G_i(t)$ is satisfying the equation (57) with the condition (58). In this case, the speed of light and gravitational constant of $n/2$ number of universes are growing/diminishing and their values of other $n/2$ number of universes are diminishing/growing. So for both varying speed of light and varying gravitational constant, we may conclude that total entropy of the Multiverse is constant.
6 Discussions and Concluding Remarks

We have assumed the anisotropic model of the Universe in the framework of varying speed of light $c$ and varying gravitational constant $G$ theories. We have mentioned different types of weak and strong singularities for the anisotropic model of the Universe. To study the strong and weak singularity models, we have written the scale factors $a(t)$ and $b(t)$ in terms of cosmic time and found some conditions for possible weak and strong singularities. For future singularity, the density and the pressure have been obtained. We have observed that there is a finite scale factor singularity if $0 \leq n \leq 1$. But there is a sudden future singularity if $1 \leq n \leq 2$. On the other hand, there is a generalized sudden future singularity if $1 \leq n \leq 2$. Also for the choice of $G(t)$, at $t = t_s$ (the future singularity time), we have obtained $G \to \infty$. At $t = t_s$ the density and pressure are finite for $r > 2 - n$. So sudden singularity is regularized due to strong gravitational coupling ($G \to \infty$ at $t = t_s$). But for $t \to \infty$, the scale factors $a \to \infty$ and $b \to \infty$ and both the density $\rho \to \infty$ and pressure $p \to \infty$ which achieved to a little rip singularity. For the choice of $c(t)$, at $t = t_s$, we have obtained $c \to 0$. But for $\beta > 2 - n$ at $t = t_s$, $a \to \infty$ and $b \to \infty$, $\rho \to \infty$ and $p \to \infty$ which regularized to a sudden singularity. Also for $t \to \infty$, we have obtained $a \to \infty$, $b \to \infty$, $\rho \to \infty$ and $|p| \to \infty$ which achieved at little rip singularity.

For strong singularity with varying $G$, we have assumed two forms of scale factors: sine model and tangent model. For sine model, at the big bang singularity (at $t = 0$), we have $G \to \infty$. Also at the turning point $t = mt_c$ ($m = 0, 1, 2, ...$), we have obtained $a \to 0$, $b \to 0$, $\rho = constant$, $p = constant$ and $G \to \infty$. That means at the turning point of the cyclic universe, the big bang like singularity is regularized due to strong gravitational coupling. But for tangent model, the scale factors $a \to \infty$, $b \to \infty$, $G \to 0$, $\rho = constant$ and $p = constant$ for $t = nt_a$ with $n = 1/2, 3/2, 5/2, ...$. So the big rip like singularity is regularized. But the scale factors $a \to 0$, $b \to 0$, $G \to \infty$, density $\rho = constant$ and pressure $p = constant$ for $t = mt_a$ with $m = 0, 1, 2, ...$. So the big bang like singularity is also regularized. For both the models, we have examined the validity of null energy condition and strong energy condition. For sine model, null energy condition is satisfied for $0 < \alpha \leq 4$ with $k \geq 0$ and strong energy condition is satisfied for for $0 < \alpha \leq 1$. But for tangent model, null energy condition is satisfied for $0 < \alpha \leq 2$ and $k \geq 0$ and strong energy condition is satisfied for $0 < \alpha \leq 1$.

Finally, we have studied the thermodynamic nature in Multiverse ($n$ number of universes) model. Using the first law of thermodynamics, we have obtained the total entropy of the Multiverse for (i) varying $c$ with constant $G$ model, (ii) varying $G$ with constant $c$ model and (iii) both varying $c$ and varying $G$ model. Then we have shown that the total entropy is always the same (constant) provided there are some relations between $n$ number of universes, where $n$ is considered as even number. We have assumed the speed of light and gravitational constant of $n/2$ number of universes are growing/diminishing and their values of other $n/2$ number of universes are diminishing/growing. So for both varying speed of light and varying gravitational constant, we have concluded that total entropy of the Multiverse is constant if the entropies of $n/2$ number of universes are growing/diminishing and entropies of other $n/2$ number of universes are diminishing/growing.

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