Minimal length effect on the broadening of free wave-packets and its physical implications

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Abstract

We study the Generalized Uncertainty Principle (GUP) modified time evolution for the width of wave-packets for a scalar potential. Free particle case is solved exactly where the wave-packet broadening is modified by a coupling between the GUP parameter and higher order moments in the probability distribution in momentum space. Unlike the standard case, here the GUP modified broadening rate not only depends on the initial size (both in position and momentum space) of the wave-packet, but also on the initial probability distribution and momentum of the particle. The new rate of wave-packet broadening is modified by a handful of new terms - such as the skewness and kurtosis coefficients, as well as the (constant) momentum of the particle. Comparisons with the standard Heisenberg uncertainty principle based results show potentially measurable differences in the rates of free wave-packet broadening for physical systems such as the $C_{60}$ and $C_{176}$ molecules, and more so for large organic molecular wave-packets. In doing so we also scan the GUP parameter space by several orders of magnitude inside the best existing upper bound.

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I. INTRODUCTION

One of the key features of Quantum Mechanics (QM) is the fact that it sets, by means of the Heisenberg Uncertainty Principle (HUP), a fundamental limit on the precise and simultaneous knowledge of two canonically conjugate dynamical variables for any quantum system. This, along with other fundamental principles, when put together, ensures the dispersion of free wave-packets through space in a manner that the width of the packet tends to always increase over time. These insights are important to understand classical-quantum correspondence in general. For example, one can easily compute that the wave-packet corresponding to a free electron will disperse in space very rapidly and therefore the likelihood of pointing down a free electron to be present at a specific point in space is negligible. Whereas, for a classical particle the wave-packet does not have a detectable...
dispersion in space over the age of the universe.

The above features are of course very well in agreement with our experiences which, nonetheless, are also verified in certain cases. On the other hand there is a growing consensus that, inspired by certain quantum gravity theories, are advocating about the existence of a fundamental minimal length scale (at the Planck length). Among them, studies in string theory [2–8], Doubly Special Relativity [9–12], black hole physics [13–15], Loop Quantum Gravity (LQG) [16,17], non-commutative quantum geometries [18–20] and more general approaches concerning QM and General Relativity [21–30] manifest this existence of a minimal length by replacing the HUP by a Generalized Uncertainty Principle (GUP) whose exact form, however, often disagrees among various proposals (for a broad overview see [7,13,31–35] and references therein). The GUP based approaches have a motivation to provide a short hand exercise in the search for quantum gravity effects, hypothesized to be realized in the form a minimal length, in low energy physics and, if it is indeed found then ask for an appropriate fundamental theory, from first principles, to explain this effective description of physical reality [36] (which may well be one of the existing theories - LQG or string theory or an entirely new theory).

One main focus of GUP based studies is to calculate the modified spectrum of different observables which can be useful to test the validity of the theory and in case no measurable differences are found it may still give bounds on the GUP parameters. Some of the studies in this line are reported in several works [37], and in fact a number of new experiments have been proposed [38] to measure these GUP contributions.

We, on the other hand, are opening a new avenue in this quest of understanding the fundamental insights that are brought in by the GUP modification (or the minimal length scale) on the wavefunction itself and thereby giving some new information on a distributional level. To do this we consider the wave-packet corresponding to a free particle which can give an account of the bare effect of minimal length scale on the otherwise very well understood situation. There exist preliminary works on the GUP effect on free particle wave-packets [39] but they were not developed enough to highlight the theoretical and experimental impact discussed here. Particularly, in this article, we present a detailed account of the basic setting of wave-packet evolution both within the standard HUP setting (which is well-known) and within the GUP framework (which is a new study). The GUP modification will be shown to imply a non-trivial distributional ramification on the rate and fundamental properties
of broadening of the free wave-packets. We shall also compare two situations, i.e., GUP vs. HUP explicitly to clarify various outcomes, both mathematically and physically. There will be a considerable effort to estimate the time difference between the GUP and HUP broadening of the free wave-packets and the likelihood of experimentally detecting this departure. Interestingly, while doing so, we can also put some bounds on the GUP parameter and that will be an improvement of several orders of magnitude to the existing best bounds that come from studying the spectrum of a number of observables.

This paper is organized in the following manner: in the next section (II) we provide a review of the basic set up to derive the evolution law for the width of the free wave-packet. In section (III) we shall take first step to include the GUP effect by generalizing Ehrenfest’s equations. Section (IV) is used for the derivation of the governing equation for the spreading of free wave-packets in GUP scenario which will be followed by solving it exactly for the case of free particle in section (V) The next section (VI) will be dedicated to physically explain the new results. Moving on, in section (VII) we shall elaborate on the possibility of testing our results within the present technology. Finally, in section (VIII) we conclude.

II. THE MOTION AND SPREADING OF WAVE PACKETS

In this section we review the standard picture of wave packet broadening in quantum mechanics. This is a standard textbook exercise (see for example [1]), however, it is important to review it here for the sake of clarity and completeness of the paper.

A. Ehrenfest’s Theorem: a Classical Analogy

In Quantum Mechanics, the fundamental principle that sets a limit in the precision to which one can simultaneously measure two given physical quantities, is the Heisenberg Uncertainty Principle (HUP)

\[
[q_i, p_j] = i\hbar\delta_{ij}, \quad i, j = 1, 2, \ldots, N,
\]

where \( N \) is the number of spatial dimensions under consideration. This is equivalent to the uncertainty relationship between the position and momentum of a particle satisfying

\[
\Delta x \Delta p \geq \frac{\hbar}{2}.
\]
We also have the identity applied to the time derivative of average (expectation) value of the observable
\[ i\hbar \frac{d}{dt} \langle A \rangle = \langle [A, H] \rangle + i\hbar \frac{\partial A}{\partial t}, \]
where the observable \( A \) is understood as an self-adjoint operator and \( H = H(q_1, \ldots, q_N; p_1, \ldots, p_N) \) is the system’s Hamiltonian. Using this identity on coordinates of position and momentum, we obtain Ehrenfest’s equations
\[ \frac{d}{dt} \langle q_i \rangle = \frac{1}{i\hbar} \langle [q_i, H] \rangle = \left\langle \frac{\partial H}{\partial p_i} \right\rangle, \quad i = 1, 2, \ldots, N \]
(4)
and
\[ \frac{d}{dt} \langle p_j \rangle = \frac{1}{i\hbar} \langle [p_j, H] \rangle = -\left\langle \frac{\partial H}{\partial q_j} \right\rangle, \quad j = 1, 2, \ldots, N. \]
(5)
which are deduced from the Ehrenfest’s theorem. Notice that these equations are formally identical to Hamilton’s equations in classical mechanics, although this formal analogy can only be rigorously made when the conditions
\[ \left\langle \frac{\partial}{\partial p_i} H(q_1, \ldots, q_N; p_1, \ldots, p_N) \right\rangle = \frac{\partial}{\partial p_i} H(\langle q_1 \rangle, \ldots, \langle q_N \rangle; \langle p_1 \rangle, \ldots, \langle p_N \rangle) \]
and
\[ \left\langle \frac{\partial}{\partial q_j} H(q_1, \ldots, q_N; p_1, \ldots, p_N) \right\rangle = \frac{\partial}{\partial q_j} H(\langle q_1 \rangle, \ldots, \langle q_N \rangle; \langle p_1 \rangle, \ldots, \langle p_N \rangle) \]
are fulfilled. The above equations need not hold for an arbitrary potential, however, both of them hold perfectly up to the quadratic potential which then include the cases such as the free particle and the harmonic oscillator.

Now, let’s consider a 1-dimensional wave packet \( \Psi(q, t) \) with Hamiltonian
\[ H = \frac{p^2}{2m} + V(q). \]
(6)
In order to study the time evolution of the expectation values \( \langle q \rangle \) and \( \langle p \rangle \) let’s first define their mean-square deviations,
\[ \xi = (\Delta q)^2 = \langle q^2 \rangle - \langle q \rangle^2, \quad \eta = (\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2. \]
(7)
Note that in the classical approximation \( \Psi(q, t) \) represents a particle with position, momentum and energy given by
\[ q_{cl} = \langle q \rangle, \quad p_{cl} = \langle p \rangle \quad \text{and} \quad E_{cl} = \frac{\langle p \rangle^2}{2m} + V(\langle q \rangle). \]
(8)
Now, let us define the quantity which tracks the difference
\[ \varepsilon = \langle H \rangle - E_{cl} = \frac{1}{2m} \eta + \langle V \rangle - V_{cl} \] (9)

where \( V_{cl} = V(\langle q \rangle) \).

For the classical approximation to hold, we require the extension \( \Delta q \) of the wave packet to remain small as compared to the characteristic distances of the problem under consideration, so that we can make the following Taylor expansions around \( \langle q \rangle \):

\[
\begin{align*}
V(q) &= V_{cl} + (q - \langle q \rangle)V'_{cl} + \frac{1}{2}(q - \langle q \rangle)^2 V''_{cl} + \ldots \\
V'(q) &= V'_{cl} + (q - \langle q \rangle)V''_{cl} + \frac{1}{2}(q - \langle q \rangle)^2 V'''_{cl} + \ldots 
\end{align*}
\] (10)

where \( V'_{cl} = \frac{dV}{dq} \bigg|_{q=\langle q \rangle} \). Using this expansion will guarantee the results are entirely general, i.e. valid for any \( V \). Taking the expectation values of (10), we obtain

\[
\langle V \rangle = V_{cl} + \frac{1}{2} \xi V''_{cl} + \ldots \\
\langle V' \rangle = V'_{cl} + \frac{1}{2} \xi V'''_{cl} + \ldots
\] (11)

By (4), (5) and (6), we have

\[
\frac{d}{dt} \langle q \rangle = \frac{\langle p \rangle}{m}, \quad \frac{d}{dt} \langle p \rangle = -\langle V' \rangle. 
\] (12)

Notice that, if we use \( \langle V' \rangle = V'_{cl} \) (11) up to first order, then equations (12) reduce to “classical” equations of motion for the mean values \( \langle q \rangle \) and \( \langle p \rangle \). This result holds if \( V(q) \) varies slowly over a distance \( \sim \sqrt{\xi} \), so that the effect of \( V''' \) and higher derivatives in (11) is negligible. This condition holds trivially for the cases \( V(q) = c q^2 \) (harmonic oscillator) and \( V(q) = 0 \) (free particle), and for every \( V(q) \) of at most order 2 in \( q \). Assuming these conditions hold (i.e. series (11) are rapidly converging), we have (see (9))

\[
\varepsilon \simeq \frac{1}{2m} (\eta + m V''_{cl} \xi) = \text{constant}
\] (13)

B. Deriving the Master Equation

We have described the motion of wave packets, by means of \( \langle q \rangle \) and \( \langle p \rangle \); now, in order to study the spreading of wave packets over time, we want to obtain functions \( \xi(t) \), \( \eta(t) \)
(i.e. *spread* in configuration and momentum space) explicitly. Notice that \( \xi = \langle u \rangle \), where \( u = q^2 - \langle q \rangle^2 \), and \( \langle q \rangle = f(t) \), so applying identity (3) to this operator yields

\[
\frac{d}{dt} \xi = \frac{1}{m} (\langle pq + qp \rangle - 2 \langle p \rangle \langle q \rangle)
\]  

(14)

Analogously, for the operator \( d\xi/dt \), using again (3) and (12) we obtain

\[
\frac{d^2 \xi}{dt^2} = \frac{2\eta}{m^2} - \frac{1}{m} (\langle V'q + qV'' \rangle - 2 \langle q \rangle \langle V' \rangle).
\]  

(15)

By using (10) in (15), we get the approximate equation

\[
\frac{d^2 \xi}{dt^2} \approx \frac{2}{m^2} (\eta - mV'' \xi),
\]  

(16)

and finally, taking (13) into account, we can re-write it as

\[
\frac{d^2 \xi}{dt^2} \approx \frac{4}{m} (\varepsilon - V'' \xi),
\]  

(17)

which we refer here as the *Master equation*. Upon solving it, and knowing the deviations \( \xi_0, \eta_0, \) and \( \dot{\xi}_0 \equiv d\xi_0/dt \) at \( t = t_0 \), we obtain \( \xi(t) \), the spread of the wave function over time in configuration space; \( \eta(t) \) can then be found with (13), using the fact that \( \varepsilon \) is constant.

Two interesting cases arise: the free particle and harmonic oscillator potential, in which the motion of the center of the packet is rigorously identical to that of a classical particle \( \Pi \). In the case of the free particle, \( V = 0 \), and thus from (13) we have \( \eta = 2m\varepsilon = \eta_0 \), that is, \( \eta = (\Delta p)^2 \) remains constant. However, we have rigorously \( d^2 \xi/dt^2 = 2\eta_0/m^2 \) and thus

\[
\xi(t) = \xi_0 + \dot{\xi}_0 t + \frac{\eta_0}{m^2} t^2.
\]  

(18)

This result tells us that the free wave packet *spreads* indefinitely, as is well known, so this sets a limit for the time interval during which the classical-particle analogy holds. If we have \( \xi_0 = 0 \) (e.g., the packet has the *minimum* width at \( t_0 \), so that, \( \xi_0\eta_0 = \frac{1}{2}\hbar^2 \) then (18) is simplified to \( \xi = \xi_0 + \eta_0 t^2/m^2 \) or, equivalently,

\[
\Delta q(t) = \sqrt{\xi(t)} = \left[ (\Delta q_0)^2 + \left( \frac{\Delta p_0 t}{m} \right)^2 \right]^{1/2},
\]  

(19)

where \( \Delta q_0 \) and \( \Delta p_0 \) are the initial uncertainty in position and momentum space corresponding to the minimum wave-packet. This is a truly remarkable equation and fundamental to our physical understanding of quantum theory which explains why we cannot see an electron.
as a localized object and why classical objects are seem to be localized forever. Take for
instance the case of free electron - the second term in (19) increases with time as $t^2$ and
matches the initial width in time $t = \frac{2\pi(\Delta q_0)^2}{\lambda_e}$ (by using the minimum wave-packet uncertainty relation $\Delta q_0 \Delta p_0 = \hbar/2$ and the definition for the Compton wavelength for electron).

Using $\lambda_e = 2.4 \times 10^{-12}$ m and initial width $\Delta q_0 \simeq 10^{-10}$ m we get the time it takes for the second term in (19) to equate the first term is $t \sim 10^{-16}$ s. This is why it is hard to detect electron as a localized object confined to a small space - the wave-packet gets quickly delocalized. On the other hand for most of the classical objects this time is more than the age of the universe.

III. THE GENERALIZED EHRENFEST EQUATIONS

We now begin taking into account the GUP modifications. We have mentioned before that there are various proposals for expressing the GUP and here we shall consider the following form of the commutation relation between the components of position and momentum as recently being used in [35, 40]

$$[q_i, p_j] = i\hbar \left\{ \delta_{ij} - \alpha \left( p\delta_{ij} + \frac{p_ip_j}{p} \right) + \alpha^2 \left( p^2\delta_{ij} + 3p_ip_j \right) \right\},$$

(20)
correct up to the second order in momentum. Here, we have $\alpha = \alpha_0/m_P c$, where $\alpha_0$ is a dimensionless parameter, $m_P$ is the Planck mass and $c$ is the speed of light. The term $m_P c = 6.52485$ kg.m/s is the Planck momentum. It is important to stress that all the machinery that we shall build here is completely general and independent of the particular form one chooses for the commutator, but the results will vary with different choices.

In the previous section we derived Ehrenfest’s equations, which give the laws of motion of the expectation values of the coordinates $q_i$ and conjugate momenta $p_i$ of a quantum system. However, these equations must be modified in order to account for the GUP, given in (20), and the identity (3). Since the Hamiltonian is a function of the $q_i$’s and $p_i$’s, we have

$$[q_i, H] = \sum_{j=1}^{N} [q_i, p_j] \left( \frac{\partial H}{\partial p_j} \right)$$
$$= i\hbar \sum_{j=1}^{N} \left\{ \delta_{ij} - \alpha \left( p\delta_{ij} + \frac{p_ip_j}{p} \right) + \alpha^2 \left( p^2\delta_{ij} + 3p_ip_j \right) \right\} \left( \frac{\partial H}{\partial p_j} \right),$$

(21)
where we have used the explicit form of (20). Similarly, for \( p \):

\[
[p_i, H] = -\sum_{j=1}^{N} [q_j, p_j] \left( \frac{\partial H}{\partial q_j} \right)
= i\hbar \sum_{j=1}^{N} \left\{ -\delta_{ij} + \alpha \left( p \delta_{ij} + \frac{p_j p_i}{p} \right) - \alpha^2 \left( p^2 \delta_{ij} + 3p_i p_j \right) \right\} \left( \frac{\partial H}{\partial q_j} \right),
\]

Using (21) and (22) along with (3), we obtain what we call the generalized Ehrenfest’s equations

\[
dt \langle q_i \rangle = \sum_{j=1}^{N} \left\{ \delta_{ij} \left( \frac{\partial H}{\partial p_j} \right) - \alpha \left( \delta_{ij} \left< p \frac{\partial H}{\partial p_j} \right> + \left< p_i p_j \frac{\partial H}{\partial p_j} \right> \right) \right\} + \alpha^2 \left( \delta_{ij} \left< p^2 \frac{\partial H}{\partial p_j} \right> + 3 \left< p_i p_j \frac{\partial H}{\partial p_j} \right> \right).
\]

(23)

\[
dt \langle p_i \rangle = \sum_{j=1}^{N} \left\{ -\delta_{ij} \left( \frac{\partial H}{\partial q_j} \right) + \alpha \left( \delta_{ij} \left< p \frac{\partial H}{\partial q_j} \right> + \left< p_i p_j \frac{\partial H}{\partial q_j} \right> \right) \right\} - \alpha^2 \left( \delta_{ij} \left< p^2 \frac{\partial H}{\partial q_j} \right> + 3 \left< p_i p_j \frac{\partial H}{\partial q_j} \right> \right).
\]

(24)

Comparing (23) and (24) with (4) and (5) one can readily see we have extra terms, i.e., corrections to the original Ehrenfest equations due to GUP effects. Furthermore, putting these equations into the form

\[
dt \langle q_i \rangle = \sum_{j=1}^{N} \left\{ \delta_{ij} \left< \frac{\partial H}{\partial p_j} \right> - \alpha \left( \delta_{ij} \left< p \frac{\partial H}{\partial p_j} \right> + \left< p_i p_j \frac{\partial H}{\partial p_j} \right> \right) + \alpha^2 \left( \delta_{ij} \left< p^2 \frac{\partial H}{\partial p_j} \right> + 3 \left< p_i p_j \frac{\partial H}{\partial p_j} \right> \right) \right\}
\]

(25)

and

\[
dt \langle p_i \rangle = \sum_{j=1}^{N} \left\{ -\delta_{ij} \left< \frac{\partial H}{\partial q_j} \right> + \alpha \left( \delta_{ij} \left< p \frac{\partial H}{\partial q_j} \right> + \left< p_i p_j \frac{\partial H}{\partial q_j} \right> \right) + \alpha \left< p_i p_j \frac{\partial H}{\partial q_j} \right> - \alpha^2 \left< p_i p_j \frac{\partial H}{\partial q_j} \right> \right\}.
\]

(26)

we recognize, in the \( \delta_{ij} \)-term, the pattern \((1 - \alpha p + \alpha^2 p^2)\) that arises in various results of the GUP modified angular momentum algebra [41].
IV. ONE DIMENSIONAL WAVE PACKETS AND GENERALIZED MASTER EQUATION

For a one-dimensional wave packet we have \( p_i = p, \ q_i = q \) and the GUP (20) becomes

\[
[q, p]_{GUP} = 1 - 2\alpha p + 4\alpha^2 p^2 \equiv \gamma
\]  

(27)

With this definition, we can write (23) and (24) as

\[
\frac{d}{dt} \langle q \rangle = \frac{1}{m} \langle \gamma p \rangle \ ; \ \frac{d}{dt} \langle p \rangle = -\langle \gamma V' \rangle
\]

and the commutation relations

\[
[q, H]_{GUP} = i\hbar \gamma \frac{p}{m} \ ; \ [p, H]_{GUP} = -i\hbar \gamma V'.
\]

These results will be useful for our ensuing discussion. Using the above results, along with the identity (3), we obtain the equation

\[
\dot{\xi}_{GUP} = \frac{1}{m} \left( \langle \gamma pq + q\gamma p \rangle - 2q_d \langle \gamma p \rangle \right)
\]

(29)

where \( q_d = \langle q \rangle \). Notice that if \( \alpha = 0 \), that is, \( \gamma = 1 \), we get back the result (14) obtained using the HUP algebra. This can be further continued to obtain the second derivative

\[
\ddot{\xi}_{GUP} = \frac{2}{m^2} \left\{ \langle \gamma p^2 \rangle - \langle \gamma p \rangle^2 - 2\alpha \langle \gamma p^3 \rangle + 4\alpha^2 \langle \gamma p^4 \rangle \right\}
\]

\[
+ \frac{1}{m} \left\{ 2q_d \langle \gamma V' \rangle - \langle \gamma V'q + q\gamma V' \rangle \right. \\
\left. + 2\alpha \left[ \langle \gamma \chi q \rangle + \langle q\gamma \chi \rangle - 2q_d \langle \gamma \chi \rangle \right] \\
- 4\alpha^2 \left[ \langle \gamma \varpi q \rangle + \langle q\gamma \varpi \rangle - 2q_d \langle \gamma \varpi \rangle \right] \right\}.
\]

(30)

We refer this as the generalized master equation, where

\[
\chi = V'p + pV' \ , \ \varpi = V'p^2 + 2pV'p + p^2V' \ , \ \gamma = 1 - 2\alpha p + 4\alpha^2 p^2.
\]

(31)

Notice here that \( V' = V'(q) \) and thus it does not commute with \( p \). Just to check the consistency, if we set \( \alpha = 0 \), (30) becomes

\[
\ddot{\xi} = \frac{2}{m^2} \eta - \frac{1}{m} \left( \langle V'q + qV' \rangle - 2 \langle q \rangle \langle V' \rangle \right) = \ddot{\xi}_{HUP},
\]

(32)

as given in (15).
V. GUP MODIFIED BROADENING OF WAVE-PACKETS: THE FREE PARTICLE

Now we are ready to analyze the effects of minimal length scale on the broadening of the free particle wave-packet. For that we need to derive the spread \( \xi \equiv (\Delta q)^2 \) as a function of time. This can be easily done since we have \( V(q) = 0 \), and thus \( \chi = \varpi = 0 \) from 31, and we find from 30

\[
\ddot{\xi}_{\text{free}} = \frac{2}{m^2} \left\{ \eta_0 - 4\alpha \left[ \langle p^3 \rangle - p_{cl} \langle p^2 \rangle \right] + 4\alpha^2 \left[ 3 \langle p^4 \rangle - \langle p^2 \rangle^2 - 2p_{cl} \langle p^3 \rangle \right] \right\}, \tag{33}
\]

where \( \eta_0 = 2m\varepsilon \) (as found in section II B). Using (3) we see that \( p_{cl}, \langle p^2 \rangle, \) and all higher moments \( \langle p^n \rangle \) are constant in time for a free particle, and thus the solution of (33) is found to be

\[
\xi_{\text{free}}(t) = \xi_0 + \dot{\xi}_0 t + \frac{1}{m^2} \left( \eta_0 - 4\alpha \left[ \langle p^3 \rangle - p_{cl} \langle p^2 \rangle \right] + 4\alpha^2 \left[ 3 \langle p^4 \rangle - \langle p^2 \rangle^2 - 2p_{cl} \langle p^3 \rangle \right] \right) t^2. \tag{34}
\]

If we assume that at \( t = t_0 \) the packet has minimum width then we must have \( \dot{\xi}(t_0) = 0 \), and then an exact expression dictating the spread over time for the free wave packet is given by

\[
\Delta q_{\text{free}}(t) = \sqrt{\xi(t)} = \sqrt{\Delta q_0^2 + \frac{1}{m^2} (\Delta p_0^2 - 4\alpha C_1 + 4\alpha^2 C_2) t^2}, \tag{35}
\]

where \( C_1 = \langle p^3 \rangle - p_{cl} \langle p^2 \rangle \) and \( C_2 = 3 \langle p^4 \rangle - \langle p^2 \rangle^2 - 2p_{cl} \langle p^3 \rangle \).

Notice that these coefficients \( C_1 \) and \( C_2 \) involve higher-order moments, which introduce higher-order moments, which introduce a novel statistical interpretation to our discussion, regarding the shape of the probability distribution for free wave-packets. To understand this meaningfully, we need to introduce Pearson’s skewness coefficient (\( \Gamma_1 \)) which represents the third order moment, as

\[
\Gamma_1 = \frac{\langle (p - \langle p \rangle)^3 \rangle}{\sigma^3} = \frac{1}{\eta^{3/2}} \langle (p - \langle p \rangle)^3 \rangle. \tag{36}
\]

Further, we also have to introduce the fourth order moment given by the kurtosis coefficient \( \Gamma_2 \) as

\[
\Gamma_2 = \frac{\langle (p - \langle p \rangle)^4 \rangle}{\sigma^4} = \frac{1}{\eta^2} \langle (p - \langle p \rangle)^4 \rangle. \tag{37}
\]

The term \( \sigma \equiv \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \eta^{1/2} \) is the the standard deviation of the momentum distribution, which also appears in the discussion without GUP modification. It is important to
recall that both $\Gamma_1$ and $\Gamma_2$ measure the departure of the probability distribution from the normal distribution. While $\Gamma_1$ measures the asymmetry about its mean $\langle p \rangle$, $\Gamma_2$ measures its tailedness. While the skewness can either take positive or negative values, kurtosis is positive definite. For a normal (true Gaussian) distribution $\Gamma_1 = 0$ and $\Gamma_2 = 3$. In their expanded form, these coefficients are

$$\Gamma_1 = \frac{1}{\eta^{3/2}} \left( \langle p^3 \rangle + 2 \langle p \rangle^3 - 3 \langle p \rangle \langle p^2 \rangle \right),$$

and

$$\Gamma_2 = \frac{1}{\eta^2} \left( \langle p^4 \rangle - 4 \langle p \rangle \langle p^3 \rangle + 6 \langle p^2 \rangle^2 - 3 \langle p \rangle^4 \right).$$

Including the definitions (36) and (37), we can write the coefficients $C_1$ and $C_2$ in terms of $\Gamma_1$, $\Gamma_2$ and $p_{cl} = \langle p \rangle$ as

$$C_1 = \eta \left( 2p_{cl} + \Gamma_1 \eta^{1/2} \right), \quad (38)$$

and

$$C_2 = (3\Gamma_2 - 1) \eta^2 + 10p_{cl} \eta \left( \Gamma_1 \eta^{1/2} + p_{cl} \right), \quad (39)$$

which, apart from the standard deviation term, also includes the skewness, kurtosis and momentum of the free particle’s wave-packet (which is constant over time). This is a remarkable result since every new correction coming from the GUP has a distributional interpretation and, therefore, can be explained physically.

Now, before going on to the analysis of the GUP-modified spread of free wave-packets, we need to find an expression for $\eta$ as a function of the initial size of the wave-packet $\xi_0 = (\Delta q_0)^2$. We can readily find this from the minimum uncertainty relation

$$(\Delta q_0 \Delta p_0)_{GUP} = \hbar \left[ \frac{1}{2} + \left( \frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2 \right) \Delta p_0^2 + 4\alpha^2 p_{cl}^2 - 2\alpha \sqrt{\langle p^2 \rangle} \right]. \quad (40)$$

Using (40), and the fact that $\langle p^2 \rangle = \eta_0 + p_{cl}^2$, we find that

$$\frac{2}{\hbar} (\Delta q_0 \sqrt{\eta_0}) - [1 + 4\alpha^2 (\eta_0 + p_{cl}^2)] + \alpha \left[ \frac{\eta_0 + 2p_{cl}^2}{\sqrt{\eta_0 + p_{cl}^2}} \right] = 0 \quad (41)$$

Upon solving (41) for $\eta_0$ we find the expression $\eta_0 = \eta_0 (\Delta q_0, \alpha, p_{cl})$ that we were looking for. Notice that, since both $\Delta q_0$ and $p_{cl}$ are constant parameters that depend on the particle (or molecule) under consideration, and $\alpha$ is the GUP parameter, solving (41) will yield a numerical value for $\eta_0$ that will be different for different systems one is considering. We take advantage of this in the following section.
VI. RESULTS AND PHYSICAL INTERPRETATION

Now, let us elaborate on the results obtained in the last sections.

The standard discussion based on the HUP provides a universal time-evolution law (19) for the wave-packet’s width, irrespective of the initial probability distribution at time $t_0$. The only requirement for (19) is that the wave-packet’s width was minimal at $t_0$. This will apply for a normal distribution (which is quite ideal) and also for all other situations where the initial probability distribution is not normal. For all cases, the evolution law is the same and is given by (19). On the other hand, as evident from our analysis, that is not true if we have to believe a GUP based calculation. The modified time evolution (35) is, indeed, dependent on the type of initial probability distribution. That is to say, for two wave-packets of the same initial width but different form (different value of skewness or kurtosis) the dispersion rate will be different. With that said, the distributions do not need to be skewed or with excess kurtosis in order to exhibit GUP-induced effects (the evolution of normal Gaussian wave-packets is modified as well). Furthermore, this rate is dependent on both the initial momentum and uncertainty in momentum, as opposed to the standard case (18) where it only depends on the initial uncertainty in momentum space.

One may now ask the question: Why do we have to consider different initial probability distributions at all, for a free particle? To answer this question we may think about a stream of particles which were under some sort of applied force fields for some time and then those force fields are switched off at time $t_0$, and from that instant on (or a little while after, depending on the relaxation time) these particles start behaving as free wave-packets. Then the initial configuration of the wave-packet at time $t_0$, when all the force fields are switched off, depends on the details of the interaction between the particles and said force fields, which can of course be arbitrary and, therefore, the initial configuration of the stream of free particles at $t_0$ need not be a normal distribution. In fact, it is likely to have any other distribution including the possibility to have a nonzero skewness and excess kurtosis.

Therefore, from our discussion it follows that, while an HUP based calculation is blind to the initial template, a GUP based approach does differentiate between two different initial templates; it shows a memory of the initial probability distribution at any later instant of time. Note that, however, since all of the physical parameters such as the skewness and kurtosis (in momentum space) and average momentum are constant in time for a free
FIG. 1: GUP-modified free wave-packets with normal kurtosis and variable negative skewness for (a) \( C_{60} \) and (b) \( C_{176} \), and variable positive skewness for both (c) \( C_{60} \) and (d) \( C_{176} \) molecules. The spatial coordinate is along the \( X \) axis and the wavefunction is along the \( Y \) axis.

particle, their initial values will be unchanged during the future course of time. Further, \( \eta = \langle p^2 \rangle - \langle p \rangle^2 \) is also constant in time for a free particle, so that the initial uncertainty in momentum space remains unchanged over the course of time.

To start analyzing these GUP-induced effects, let us first consider a skewed probability distribution (with vanishing excess kurtosis) of the initial wave-packet. A template of such a wave-packet can be expressed in terms of the following function

\[
f(\Gamma_1, t) = \exp \left( -\frac{q^2}{2\xi(t)} \right) \left( \text{erf} \left( \frac{q\Gamma_1}{2\sqrt{\xi(t)}} \right) + 1 \right) \left( \pi \xi(t) \right)^{1/4}.
\]  

(42)

It is easy to check that the probability distribution associated to this wave-packet (that is, the square of (42)) is normalized over the configuration space and, therefore, satisfies the probability conservation condition at all times. This function corresponds to a skewed distribution with normal kurtosis \( \Gamma_2 = 3 \); its width \( \xi(t) \) satisfies the equation (34), and for a given instant of time \( t \) the shape of the wave-packet will change for a given value of skewness \( \Gamma_1 \). In Fig. 1 we plot this behavior for both \( C_{60} \) and \( C_{176} \) “buckyball” molecules. We choose “buckyball” type (Buckminsterfullerene) molecules for this analysis because they are one of the, commercially available, bigger-sized molecules that behave as a single wave-packet,
thus they can be used for experimental studies on our proposal. This will be further clarified in
the next section where we discuss a possible test of our results. Plots with positive skewness
have more probability that the particle will be found on the right side than the left side
of the mean value and vice-versa. Here we have assumed characteristic values for several
parameters including the mass and the initial size (taken to be the van der Waals diameter
[46]) of the molecule, and the GUP coupling constant $\alpha$ as order unity. These plots are
therefore more for a qualitative understanding. Accurate quantitative analysis for testing
our result will be carried out in the next section.

Notice that, even though $\Gamma_1$ and $\Gamma_2$ are defined as the skewness and kurtosis coefficients
in momentum space (see (36) and (37)), this does of course introduce skewness and kurtosis
in position space as well, so that the shape of the wave-packet in position space will also be
affected, as shown in the figures. The difference is that the skewness and kurtosis coefficients
in position space will change over time; we can readily see this from the fact that, generally
speaking, $\langle q^n \rangle = \langle q^n \rangle (t)$ for the free particle. With that said, notice that the GUP-modified
spread evolution law for free wave-packets does not depend explicitly on these coefficients
in position space, but rather in momentum space, so we do not need to compute these for
our present analysis.

Now, let us plot the time evolution of this wave-packet, governed by (42), in Fig. 2, for
both $C_{60}$ and $C_{176}$ parameters. Again, the shape and the rate at which it spreads depends
on the value of $\Gamma_1$ appearing in (34) through $C_1$ and $C_2$ in (38) and (39). Clearly, the initial
distribution has an important role to play in the time evolution of the wave-packet, and this
is a new insight coming from the GUP based analysis.

In Fig. 3, we compare the wave-packet evolution with and without the GUP modifica-
tions. The sample distribution is again given by (42) with either $C_{60}$ or $C_{176}$ parameters,
and we consider the normal (Gaussian) part of it by setting $\Gamma_1 = 0$. We find some impor-
tant insights by looking at these plots: first, the minimum uncertainty wave-packet, defined
at the initial time, has a smaller width for the GUP-based calculation than the HUP-based
standard result. It is therefore consistent to say that for a physical quantum system, such
as the one given by these “buckyballs”, the existence of a minimal length scale in the form
of [20] minimizes the uncertainty in the probability distribution in position space for the
same momentum distribution. This may be related, of course with certain differences, with
an expectation that gravity might have a natural tendency to localize the wavefunction, as
FIG. 2: Comparison of GUP time evolution (broadening) between normal and negatively skewed wave-packets for (a) $C_{60}$ and (b) $C_{176}$, and between normal and positively skewed wave-packets for (c) $C_{60}$ and (d) $C_{176}$ molecules. The spatial coordinate is along the $X$ axis and the wavefunction is along the $Y$ axis.

FIG. 3: GUP vs HUP time evolution (broadening) of free wave-packets for (a) $C_{60}$ and (b) $C_{176}$ molecules. The spatial coordinate is along the $X$ axis and the wavefunction is along the $Y$ axis.

first pointed out by Penrose and Díosi [42].

So far our discussion did not include a distribution with an excess kurtosis. In order to study this let us assume the probability density function of the logistic distribution, given by

\[ G(q, \xi(t)) := \frac{\exp \left( -\frac{q}{\xi(t)} \right)}{\xi(t) \left( \exp \left( -\frac{q}{\xi(t)} \right) + 1 \right)^2}. \]  

This function has skewness $\Gamma_1 = 0$ and kurtosis $\Gamma_2 = 4.2$. The wave-packet associated with
this probability distribution is

\[
g(q, \xi(t)) := \frac{\exp \left( -q \xi(t) \right)}{\sqrt{\xi(t)} \left( \exp \left( -q \xi(t) \right) + 1 \right)}.
\] (44)

The rate (34) at which the spreading takes place with GUP modification for (44) includes the kurtosis \( \Gamma_2 \) as opposed to the standard prediction from the HUP, where the rate of expansion of \( \xi(t) \) (18) is independent of the value of kurtosis. In Fig. 4 we plot the GUP time evolution of (44) starting from the minimum width wave-packet for (a) \( C_{60} \) and (b) \( C_{176} \) molecules. When considering any type of initial distribution (be it normal, skewed or with excess kurtosis), if one takes the GUP parameter to be \( \alpha \sim 1 \), the time evolution is practically identical to the HUP based calculation and it is hard to differentiate between the two in the plots. However, given that the allowed parameter space for \( \alpha \) is quite wide [37], for a large value \( \alpha = 10^{16} \), these plots do show a significant difference between the width of the wave-packet with or without GUP corrections (see Fig. 4(c) and 4(d)). This characteristic is just similar as before (Fig. 3). In the next section we shall speak more about the numbers and the likelihood of measuring them.

VII. POSSIBLE TESTS

In this section we study the possibility of experimental verification of the minimal length effect on the dispersion of the free wave-packets. The scheme that we propose here is quite simple - one needs to measure the timescale in which the wave-packet (describing a particle or a system of particles behaving as a single wave-packet) doubles its initial width. In fact, one can choose any final size that is permissible, but our calculation here will be done considering that the wave-packet is doubling its size.

HUP based calculation gives a precise estimate for that which we already discussed for the case of electrons in section II. Let us re-do the analysis, now in presence of the GUP modifications. Using (35) we can easily calculate this time to be

\[
t_{\text{double}} = \frac{\sqrt{3m} \Delta q_0}{\sqrt{\Delta p_0^2 - 4\alpha C_1 + 4\alpha^2 C_2}}
\] (45)

where the minimum uncertainty wave-packet now satisfies the relationship (40), and plugging
FIG. 4: GUP modified free wave-packets with kurtosis, for (a) \( C_{60} \) and (b) \( C_{176} \) molecules. Also, comparison between wave-packets predicted by GUP and HUP for a probability distribution with excess kurtosis for (c) \( C_{60} \) and (d) \( C_{176} \). There is a notable difference for large values of the GUP parameter \( \alpha \sim 10^{16} \). The spatial coordinate is along the \( X \) axis and the wavefunction is along the \( Y \) axis.

in the definitions (38) and (39) of \( C_1 \) and \( C_2 \), we get

\[
t_{\text{double}} = \frac{\sqrt{3}m\Delta q_0}{\sqrt{\Delta p_0^2 + 4\eta\left[\alpha^2 ((3\Gamma_2 - 1) \eta + 10p_{cl} (\Gamma_1 \eta^{1/2} + p_{cl})) - \alpha (2p_{cl} + \Gamma_1 \eta^{1/2})\right]}}
\]  

(46)

If we, for the sake of simplicity, consider a gaussian wave-packet, then we can set the skewness and kurtosis coefficients to \( \Gamma_1 = 0 \) and \( \Gamma_2 = 3 \), respectively. With this the above expression gets simplified, giving

\[
t_{\text{double}}(\Gamma_1 = 0, \Gamma_2 = 3) = \frac{\sqrt{3}m\Delta q_0}{\sqrt{\Delta p_0^2 + 8\eta\left(\alpha^2 (4\eta + 5p_{cl}^2) - \alpha p_{cl}\right)}}
\]  

(47)

With expressions (46) and (47) at hand, we can use relation (41) to replace \( \eta = \Delta p_0^2 \) in terms of \( \Delta q_0 \) and other parameters. Therefore, we now have everything we need for doing a numerical calculation.

First, let us go back to the case of the free electron, where the initial wave-packet had a width of \( 10^{-10} \) m. We can use (47) and estimate the magnitude of the GUP modification. A
simple numerical check shows that, for the values $1 \leq \alpha \leq 10^{21}$, the result remains effectively the same with or without GUP ($t_{\text{double}} \approx 10^{-16} \text{ s}$), the difference between HUP and GUP being as small as $10^{-30} \text{ s}$ for $\alpha \leq 10^{10}$, and at most $10^{-19} \text{ s}$ for $\alpha \sim 10^{21}$. However, if we go for higher values like $\alpha = 10^{23}$ we start getting a difference of about $10^{-17} \text{ s}$, which is near to the original HUP value, but still unlikely to be detected given the current existing precision of atomic clocks. Furthermore, if we have to believe an upper bound for $\alpha$ (the stringent upper bound comes from the precision observation of Lamb shift which implies $\alpha \leq 10^{10}$ [43]), we clearly see that the GUP modification does not give a major difference in $t_{\text{double}}$ for the case of free electrons, at least in the initial stage where it is more likely to be detectable by a laboratory based experiment.

In order for these effects to be detectable in a laboratory, we must magnify the GUP modifications somehow. To do this, we must consider probes whose wave-packets have initial sizes bigger than that of an electron and that represent a comparatively larger mass. One obvious way to achieve this is to consider atoms instead of electrons or, even better, use bigger molecules which can behave like a single wave-packet. This brings us to the so called ”bucky-ball” systems and Large Organic Molecules (LOM). “Bucky balls” or Buckminsterfullerene are basically a bunch of carbon atoms behaving as a single quantum wave-packet [44]. We shall consider again $C_{60}$ and $C_{176}$ molecules — which we already considered in various plots. On the other hand, LOMs are probably the most exciting since they are the largest molecules (in terms of the combination of size and mass scale) found so far which behave like a single wave-packet [45]. Below we do an analysis for these three objects where we shall keep $\alpha$ a free parameter from the beginning, and see how the wide range of values for this parameter affects the time difference between the HUP and GUP results for the minimal width wave-packet to double its initial width.

In the case of a $C_{60}$ buckyball molecule, with a mass of $1.19668 \times 10^{-24} \text{ kg (720 u)}$ and an initial width $\Delta q_0$ equal to its van der Waals diameter (7 Å) [46], the HUP prediction for the doubling time is $t_{\text{double}}(C_{60}, \text{HUP}) = 1.92719 \times 10^{-8} \text{ s}$. If we start considering GUP modifications, first with $\alpha = 1$ as the value of the GUP parameter, then we get practically the same value $t_{\text{double}}$; the difference between them being

$$t_{\text{double}}(C_{60}, \alpha = 1) - t_{\text{double}}(C_{60}, \text{HUP}) = -6.61744 \times 10^{-24} \text{ s}$$
However, if we take large values like $\alpha = 10^{10}$, then we find that

$$t_{\text{double}}(C_{60}, \alpha = 10^{10}) - t_{\text{double}}(C_{60}, HUP) = 1.15631 \times 10^{-14} s,$$

and if we go even further, like $\alpha = 10^{16}$, we find $t_{\text{double}}(C_{60}, \alpha = 10^{16}) = 2.96189 \times 10^{-8} s$, and

$$t_{\text{double}}(C_{60}, \alpha = 10^{16}) - t_{\text{double}}(C_{60}, HUP) = 1.0347 \times 10^{-8} s.$$

That is, the difference between both predictions is of the order of the original HUP prediction ($\sim 10^{-8} s$) while taking $\alpha \sim 10^{16}$ as the GUP parameter.

This analysis shows that depending on the wide range of values for $\alpha$, the difference between the HUP and GUP predictions for $t_{\text{double}}$ for $C_{60}$ buckyballs stays in an interval where the lower end is undetectable even with the most precise clocks currently available, but the upper end stays well within the available range of precision.

Furthermore, since we want to amplify the GUP-induced effects (and thus make them easier to detect at laboratory-based experiments), let us now consider a $C_{176}$ buckyball. Using this molecule’s parameters ($m = 3.50706 \times 10^{-24} \text{ kg} (2112 \text{ u})$ and $\Delta q_0 = 1.2 \text{ nm}$ [16]), we find that the HUP prediction for the doubling time is $t_{\text{double}}(C_{176}, HUP) = 1.6598 \times 10^{-7} s$ and, again, taking small values of $\alpha$ (like order unity) yields an effectively undetectable difference between the HUP and GUP predictions. However, if we again set $\alpha = 10^{10}$, we get

$$t_{\text{double}}(C_{176}, \alpha = 10^{10}) - t_{\text{double}}(C_{176}, HUP) = 9.9588 \times 10^{-14} s,$$

which is better by a factor more than 8, and going to higher values like $\alpha = 10^{16}$ yields $t_{\text{double}}(C_{176}, \alpha = 10^{16}) = 2.55094 \times 10^{-7} s$ and

$$t_{\text{double}}(C_{176}, \alpha = 10^{16}) - t_{\text{double}}(C_{176}, HUP) = 8.9114 \times 10^{-8} s.$$

This is again an improvement by almost a factor of 9 over the time difference ($1.0347 \times 10^{-8} s$) that we got for the $C_{60}$ molecule. Therefore, we see that bigger (larger van der Waals diameter) and more massive molecules tend to show stronger deviations from the HUP behavior when considering GUP-modified $t_{\text{double}}$ calculations.

Now let us consider the case of recently discovered LOM wave-packets [45]. Considering a TPPF152 or tetraphenylporphyrin molecule (which consists of 430 atoms and is formally known as $C_{168}H_{94}F_{152}O_{8}N_{4}S_{4}$, with a mass of $5.310 \text{ u} (\sim 8.81746 \times 10^{-24} \text{ kg})$ and an initial
size of 60 Å. Taking $\alpha = 1$, once again, does not bring the time difference in a detectable range. However, if we go to larger values of $\alpha$ like $10^{10}$ we find

$$t_{\text{double}}(TPPF152, \alpha = 10^{10}) - t_{\text{double}}(TPPF152, HUP) = 6.25961 \times 10^{-12} \text{s}.$$ 

which improves the result of the $C_{176}$ molecule by a factor of 63 and this number is 500 times better than for the $C_{60}$ molecule. Further, moving to $\alpha \sim 10^{16}$ the difference becomes

$$t_{\text{double}}(TPPF152, \alpha = 10^{16}) - t_{\text{double}}(TPPF152, HUP) = 5.60129 \times 10^{-6} \text{s}.$$ 

which is again better by a factor of 63 from $C_{176}$ and 560 from $C_{60}$.

In Fig. 5, we have plotted the difference between the doubling times for various values of $\alpha$ (difference between the GUP based and HUP based calculations). This is a log-log plot where values of $\Delta t_{\text{double}}$ are shown for the parameter space $1 \leq \alpha \leq 10^{19}$. Note that for the larger values of $\alpha \geq 10^{16}$ we get a doubling time difference $O(10^{-8} \text{s})$ for $C_{60}$ molecule which can be easily detected by today’s atomic clocks. This result is even better ($10^{-7} \text{s}$) for $C_{176}$ and in $\mu$s range for TPPF152. On the other side, we can scan the complete parameter space of $\alpha$ (up to order unity), if we can measure a time difference of the order of $10^{-21} \text{s}$ to $10^{-23} \text{s}$, just by considering these molecules. However, if we have to believe that highly precise atomic clocks can differentiate the time measurement by at most $10^{-15} \text{s}$, the use of $C_{60}$ molecules can scan the parameter space $\alpha \geq 10^{9}$, and it is again better for $C_{176}$, for which we can scan $\alpha \geq 10^{8}$. The best of the three however stands for TPPF152 which can scan, on the lower side, down to $\alpha \sim 10^6$. Therefore, we get an improvement by four orders of magnitude on the best existing bound found in [43] to constrain $\alpha$. In addition, if we are lucky and Nature behaves in such a manner, we might be able verify (20) with these molecular wave-packets. If not we can put a new bound and move on to redo the experiments with even bigger and more massive wave-packets. This is a totally new avenue that has not been proposed before. In fact any departure from HUP, irrespective of the manner it differs, will be a pathbreaking discovery since it will anyway challenge the standard quantum mechanical prediction. We expect, perhaps colleagues from the experimental side will find this result interesting.

VIII. CONCLUSIONS

We have introduced a novel approach and, to some extent, established the fact that studying the dispersion of free wave-packets might lead to an indirect evidence for the long
FIG. 5: log - log plots between the GUP parameter $\alpha$ and the doubling time difference $\Delta t_{\text{double}} = t_{\text{double}}(\text{GUP}) - t_{\text{double}}(\text{HUP})$ between the GUP and HUP time evolution for (a) $C_{60}$ molecule (the lower plot), (b) $C_{176}$ molecule (the middle plot) and (c) the large organic molecule TPPF152 (the upper plot). The $X$ axis represents the GUP parameter and the $Y$ axis represents the difference between GUP and HUP doubling time. The shaded region indicates the region of parameter space that can be probed by above molecular wave-packets with an atomic clock of maximum precision of $10^{-15}$ s.

anticipated minimal length scale in Nature. Our result here is based on the possibility that HUP should be replaced by a GUP \cite{20} in presence of the minimal length. Nonetheless, it is very important to stress that our approach is very general and independent of the specific manner in which the commutator bracket has to be modified in \cite{20}. This specific study, based on this choice, has several interesting outcomes which we enlist below.

(i) Use of GUP has brought a rich distributional consequence on the expansion rate of free wave-packets. The rate of dispersion not only depends on the initial uncertainty and standard deviation (in position and momentum) but also on the higher order moments in momentum space (such as skewness and kurtosis). In addition, it also depends on the initial
momentum of the wave-packet.

(ii) We have shown that by measuring the “doubling time”, that is the time in which a free, minimal width wave-packet doubles its size, we may get important clues on the minimal length scale. The difference between the doubling times of HUP and GUP based predictions may well be in the detectable range if we use highly precise atomic clocks.

(iii) This difference in broadening time is more for massive molecular wave-packets in comparison with the wave-packets representing smallest objects like electrons. Large organic molecule (such as TPPF152), “buckyball” (such as $C_{60}$, $C_{176}$) wave-packets may be useful on verifying or falsifying the GUP proposal [20].

(iv) In the absence of detecting any difference for doubling time with an atomic clock of precision level $10^{-15}$ s, with $C_{60}$, we can better the best upper bound on $\alpha \sim 10^{10}$ by one order of magnitude. For $C_{176}$ we can improve by two orders of magnitude, and for TPPF152 by four orders of magnitudes, assuming that the precision level of the best atomic clock is $10^{-15}$ s. This will be even better if we can use atomic clocks more precise than that.

(v) There are two ways to improve the numbers presented here and to reach even closer to testing the GUP theory. One of them is to consider larger and heavier molecular wave-packets and the other is to come up with new atomic clocks which can measure the time difference even beyond a femto-second.

Finally, we want to stress that coming up with an experiment to test our results might not be impossible in near future, especially because of the remarkable progress that has been achieved to test the superposition principle with increasingly massive molecular wave-packets [45]. Perhaps, an experiment in our context will be easier to conduct since the wave-packet does not pass through the double slit, rather, it only needs to be set free until it doubles its size.

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[1] A. Messiah, *Quantum Mechanics: Volume I*, North-Holland Publishing, 1967.

[2] D.J. Gross and P.F. Mende, *The High-Energy Behavior of String Scattering Amplitudes*, Phys. Lett. B 197, 129 (1987)

[3] D.J. Gross and P.F. Mende, *String Theory Beyond the Planck Scale*, Nucl. Phys. B 303, 407 (1988).

[4] D. Amati, M. Ciafaloni and G. Veneziano, *Can Space-Time Be Probed Below the String Size*, Phys. Lett. B 216, 41 (1989).

[5] K. Konishi, G. Paffuti and P. Provero, *Minimum Physical Length and the Generalized Uncertainty Principle in String Theory*, Phys. Lett. B 234, 276 (1990).

[6] F. Scardigli, R. Casadio, “Generalized uncertainty principle, extra dimensions and holography,” Class. Quant. Grav. 20, 3915-3926 (2003) [hep-th/0307174].

[7] L. Garay, *Quantum Gravity and minimal length* Int. J. Mod. Phys. A 10, 145-166 (1995) [arXiv:gr-qc/9403008].

[8] C. Castro, *String Theory, Scale Relativity and the Generalized Uncertainty Principle Found*. Phys. Lett. 10 (1997) 273-293 [arXiv:hep-th/9512044].

[9] W. S. Chung and H. Hassanabadi, *New generalized uncertainty principle from the doubly special relativity* [arXiv:1807.11552 [gr-qc]].

[10] J. L. Cortés and J. Gamboa, *Quantum Uncertainty in Doubly Special Relativity*, Phys. Rev. D 71 (2005) 065015 [arXiv:hep-th/0405285].

[11] J. Magueijo and L. Smolin, *Generalized Lorentz invariance with an invariant energy scale* Phys. Rev. D 67 (2003) 044017 [arXiv:gr-qc/0207085].

[12] G. Amelino-Camelia, *Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale* Int. J. Mod. Phys. D 11 (2002) 35-60 [arXiv:gr-qc/0012051].

[13] M. Maggiore, *A Generalized uncertainty principle in quantum gravity*, Phys. Lett. B 304, 65 (1993) [hep-th/9301067].

[14] B. Carr, J. Mureika and P. Nicolini, *Sub-Planckian black holes and the Generalized Uncertainty Principle* JHEP 1507 (2015) 052, [arXiv:1504.07637v2 [gr-qc]].
[15] F. Scardigli, “Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment,” Physics Letters B, 452 (1999) 39-44 [arXiv:hep-th/9904025].

[16] G. M. Hossain, V. Husain and S. S. Seahra, Background independent quantization and the uncertainty principle Class. Quant. Grav. 27 (2010) 165013 [arXiv:1003.2207 [gr-qc]].

[17] A. Ashtekar, S. Fairhurst and J. L. Willis, Quantum gravity, shadow states, and quantum mechanics Class. Quant. Grav. 20 (2003) 1031-1062 [arXiv:gr-qc/0207106].

[18] S. Capozziello, G. Lambiase and G. Scarpetta, Generalized Uncertainty Principle from Quantum Geometry Int. J. Theor. Phys. 39 (2000) 15-22 [arXiv:gr-qc/9910017].

[19] H. S. Snyder, Quantized Space-Time Phys. Rev. 71 1 (1947).

[20] S. Doplicher, K. Fredenhagen, J. E. Roberts, The quantum structure of spacetime at the Planck scale and quantum fields Commun. Math. Phys. 172 (1995) 187-220 [arXiv:hep-th/0303037].

[21] P. Pedram, A Higher Order GUP with Minimal Length Uncertainty and Maximal Momentum Phys. Lett. B 714 (2012) 317 [arXiv:1110.2999 [hep-th]].

[22] M. Maggiore, The algebraic structure of the generalized uncertainty principle Phys. Lett. B 319 (1993) 83-86 [arXiv:hep-th/9309034].

[23] M. Maggiore, Quantum Groups, Gravity, and the Generalized Uncertainty Principle Phys. Rev. D 49, 5182 (1994) [arXiv:hep-th/9305163].

[24] A. Kempf, G. Mangano and R. B. Mann, Hilbert Space Representation of the Minimal Length Uncertainty Relation Phys. Rev. D 52 1108-1118, (1995) [arXiv:hep-th/9412167].

[25] R. J. Adler and D. I. Santiago, On Gravity and the Uncertainty Principle Mod. Phys. Lett. A 14 (1999) 1371 [arXiv:gr-qc/9904026].

[26] K. Nozari and A. Etemadi, “Minimal length, maximal momentum and Hilbert space representation of quantum mechanics,” Phys. Rev. D 85, 104029 (2012) [arXiv:1205.0158 [hep-th]].

[27] M. Faizal, A. F. Ali, A. Nassar, Generalized Uncertainty Principle as a Consequence of the Effective Field Theory Phys. Lett. B 765 (2017) 238-243 [arXiv:1701.00341 [hep-th]].

[28] V. E. Kuzmichev and V. V. Kuzmichev, Generalized uncertainty principle in quantum cosmology for the maximally symmetric space [arXiv:1711.10278 [gr-qc]].

[29] A. A. Deriglazov and A. M. Pupasov-Maksimov, “Relativistic corrections to the algebra of position variables and spin-orbital interaction,” Phys. Lett. B 761, 207 (2016) [arXiv:1609.00043 [quant-ph]].
[30] M. Bishop, E. Aiken and D. Singleton, Modified commutation relationships from the Berry-Keating program, Phys. Rev. D 99, no. 2, 026012 (2019), [arXiv:1810.03976 [physics.gen-ph]].

[31] S. Hossenfelder, Minimal length scale scenarios for quantum gravity, Living Rev. Rel. 16, 2 (2013) 1203.6191 [gr-qc].

[32] A. N. Tawfik and A. M. Diab, Generalized Uncertainty Principle: Approaches and Applications, Int. J. Mod. Phys. D 23, no. 12, 1430025 (2014) [arXiv:1410.0206 [gr-qc]].

[33] B. N. Tiwari, On Generalized Uncertainty Principle [arXiv:0801.3402].

[34] M. Sprenger, P. Nicolini and M. Bleicher, Physics on Smallest Scales - An Introduction to Minimal Length Phenomenology Eur. J. Phys. 33 (2012) 853-862 [arXiv:1202.1500 [physics.ed-ph]].

[35] P. Bosso, Generalized Uncertainty Principle and Quantum Gravity Phenomenology [arXiv:1709.04947v1].

[36] S. Ghosh and P. Roy, “Stringy’ Coherent States Inspired By Generalized Uncertainty Principle,” Phys. Lett. B 711, 423 (2012); S. Ghosh, “Quantum Gravity Effects in Geodesic Motion and Predictions of Equivalence Principle Violation,” Class. Quant. Grav. 31, 025025 (2014);

[37] S. Das and E. C. Vagenas, Universality of Quantum Gravity Corrections Phys. Rev. Lett. 101 221301 (2008) [arXiv:0810.5333 [hep-th]]; A. F. Ali, S. Das and E. C. Vagenas, Discreteness of Space from the Generalized Uncertainty Principle Phys. Lett. B 678 497-499 (2009) [arXiv:0906.5396 [hep-th]]; D. Gao and M. Zhan, Constraining the generalized uncertainty principle with cold atoms Phys. Rev. A 94, 013607 (2016) [arXiv:1607.04353 [gr-qc]]; Z. Feng, S. Yang, H. Li and X. Zu, Constraining the generalized uncertainty principle with the gravitational wave event GW150914 Phys. Lett. B 768, 81-85 (2017) [arXiv:1610.08549 [hep-ph]]; F. Scardigli, R. Casadio, “Gravitational tests of the Generalized Uncertainty Principle,” Eur. Phys. J. C 75 (2015) 9, 425 [arXiv:1407.0113 [hep-th]]; F. Scardigli, G. Lambiase, E. Vagenas, “GUP parameter from quantum corrections to the Newtonian potential”, Phys. Lett. B 767 (2017) 242 [arXiv:1611.01469 [hep-th]]; G. Lambiase, F. Scardigli “Lorentz violation and generalized uncertainty principle”, Phys. Rev. D97 (2018) no.7, 075003 [arXiv:1709.00637 [hep-th]].

[38] P. Bosso, S. Das and R. B. Mann, Potential tests of the Generalized Uncertainty Principle in the advanced LIGO experiment [arXiv:1804.03620v2]; I. Pikovski, M. R. Vanner,
M. Aspelmeyer, M. Kim and C. Brukner, *Probing Planck-scale physics with quantum optics* Nat. Phys. 8, 393-397 (2012) [arXiv:1111.1979 [quant-ph]]; M. Bawaj, C. Biancofiore, M. Bonaldi, F. Bonfigli, et. al. *Probing deformed commutators with macroscopic harmonic oscillators* Nat. Comm. 6, 7503 (2015) [arXiv:1411.6410 [gr-qc]].

[39] K. Nozari and S. H. Mehdipour, “Wave packets propagation in quantum gravity,” Gen. Rel. Grav. 37, (1995); K. Nozari, “Some aspects of planck scale quantum optics,” Phys. Lett. B 629, 41 (2005).

[40] A. F. Ali, S. Das and E. C. Vagenas, *The Generalized Uncertainty Principle and Quantum Gravity Phenomenology*, [arXiv:1001.2642 [hep-th]].

[41] P. Bosso and S. Das, *Generalized Uncertainty Principle and Angular Momentum*, arXiv:1709.04947

[42] R. Penrose, *On Gravity’s role in Quantum State Reduction*, Gen. Relativ. and Gravit., 28, 581 (1996).

[43] A. F. Ali, S. Das and E. C. Vagenas, *A proposal for testing Quantum Gravity in the lab* Phys. Rev. D 84 044013 (2011) [arXiv:1107.3164 [hep-th]].

[44] Markus Arndt et. al., *Wave-particle duality of C_{60} molecules*, Nature, vol. 401, pp. 680 - 682 (1999).

[45] S. Gerlich, S. Eibenberger, M. Tomandl, S. Nimmrichter, et. al. *Quantum interference of large organic molecules* Nat. Commun. 2, (2011) 263.

[46] A. Goel, J. B. Howard and J. B. V. Sande, *Size analysis of single fullerene molecules by electron microscopy*, Carbon 42 1907-1915 (2004).