Innovated Higher Criticism for Detecting Sparse Signals in Correlated Noise

Peter Hall¹ and Jiashun Jin²

Abstract

Higher Criticism is a method for detecting signals that are both sparse and weak. Although first proposed in cases where the noise variables are independent, Higher Criticism also has reasonable performance in settings where those variables are correlated. In this paper we show that, by exploiting the nature of the correlation, performance can be improved by using a modified approach which exploits the potential advantages that correlation has to offer. Indeed, it turns out that the case of independent noise is the most difficult of all, from a statistical viewpoint, and that more accurate signal detection (for a given level of signal sparsity and strength) can be obtained when correlation is present. We characterize the advantages of correlation by showing how to incorporate them into the definition of an optimal detection boundary. The boundary has particularly attractive properties when correlation decays at a polynomial rate or the correlation matrix is Toeplitz.

Keywords: Adding noise, Cholesky factorization, empirical process, innovation, multiple hypothesis testing, sparse normal means, spectral density, Toeplitz matrix.

AMS 2000 subject classifications: Primary 62G10, 62M10; secondary 62G32, 62H15.

1 Introduction

Donoho and Jin [14] developed Tukey’s [36] proposal for “Higher Criticism” (HC), showing that a method based on the statistical significance of a large number of statistically significant test results could be used very effectively to detect the presence of very sparsely distributed signals. They demonstrated that HC is capable of optimally detecting the presence of signals that are so weak, and so sparse, that the the signal cannot be consistently estimated. Applications include the problem of signal detection against cosmic microwave background radiation (Cayon et al. [8], Cruz et al. [12], Jin [27, 28, 29], Jin et al. [31]). Related work includes that of Cai et al. [7], Hall et al. [20] and Meinshausen and Rice [32].

The context of Donoho and Jin’s [14] work was that where the noise is white, although a small number of investigations have been made of the case of correlated noise (Hall et al. [20], Hall and Jin [21], Delaigle and Hall [13]). However, that research has focused on the ability of standard HC, applied in the form that is appropriate for independent data, to accommodate the non-independent case. In this paper we address the problem of how to modify HC by developing innovated Higher Criticism (iHC) and showing how to optimize performance for correlated noise.

¹ Department of Mathematics and Statistics, University of Melbourne, Parkville, VIC, 3010, Australia; Department of Statistics, University of California, Davis, CA 95616.
² Department of Statistics, Carnegie Mellon University, Pittsburgh, PA 15213. The research of Jiashun Jin was supported in part by NSF CAREER award DMS-0908613.
Curiously, it turns out that when using the iHC method tuned to give optimal performance, the case of independence is the most difficult of all, statistically speaking. To appreciate why this result is reasonable, note that if the noise is correlated then it does not vary so much from one location to a nearby location, and so is a little easier to identify. In an extreme case, if the noise is perfectly correlated at different locations then it is constant, and in this instance it can be easily removed.

On the other hand, standard HC does not perform well in the case of correlated noise, because it utilizes only the marginal information in the data, without much attention to the correlation structure. Innovated HC is designed to exploit the advantages offered by correlation, and gives good performance across a wide range of settings.

The concept of the “detection boundary” was introduced by Donoho and Jin [14] in the context of white noise. In this paper, we extend it to the correlated case. In brief, the detection boundary describes the relationship between signal sparsity and signal strength that characterizes the boundary between cases where the signal can be detected, and cases where it cannot. In the setting of dependent data, this watershed depends on the correlation structure of the noise as well as on the sparsity and strength of the signal. When correlation decays at a polynomial rate we are able to characterize the detection boundary quite precisely. In particular, we show how to construct concise lower/upper bounds to the detection boundary, based on the diagonal components of the inverse of the correlation matrix, $\Sigma_n$. A special case is where $\Sigma_n$ is Toeplitz; there the upper and the lower bounds to the detection boundary are asymptotically the same. In the Toeplitz case, the iHC is optimal for signal detection but standard HC is not.

The paper is organized as follows. Section 2 introduces the sparse signal model followed by a brief review of the uncorrelated case. Section 3 establishes lower bounds to the detection boundary in correlated settings. Section 4 introduces innovated HC and establishes an upper bound to the detection boundary. Section 5 applies the main results in Sections 3 and 4 to the case where the $\Sigma_n$’s are Toeplitz. In this case, the lower bound coincides with the upper bound and innovated HC is optimal for detection. Section 6 discusses a case where the signals have a more complicated structure. Section 7 investigates a case of strong dependence. Simulations are given in Section 8 and discussion is given in Section 9. Sections 10, 11, and 12 give proofs of theorems, lemmas, and secondary lemmas, correspondingly.

2 Sparse signal model, review of HC in uncorrelated case

Consider an $n$-dimensional Gaussian vector

$$X = \mu + Z \quad \text{where} \quad Z \sim \text{N}(0, \Sigma), \quad (2.1)$$

with the mean vector $\mu$ unknown and the dimension $n$ large. In most parts of the paper, we assume that $\Sigma = \Sigma_n$ is known and has unit diagonal elements (the case where $\Sigma_n$ is unknown is discussed in Section 9). We are interested in testing whether no signal exists (i.e. $\mu = 0$) or there is a sparse and faint signal.

Such a situation may arise in many situations. One example is global testing in linear models. Consider a linear model $Y \sim \text{N}(M\mu, I_n)$, where the matrix $M$ has many rows and columns, and we are interested in testing whether $\mu = 0$. The setting is closely related to Model (2.1), since the least square estimator of $\mu$ is distributed as $\text{N}(\mu, (M'M)^{-1})$. The global testing problem is important in many applications. One is that of testing whether a
clinical outcome is associated with the expression pattern of a pre-specified group of genes (Goeman et al. [17, 18]), where \( M \) is the expression profile of the specified group of genes. Another is expression quantitative Trait Loci (eQTL) analysis, where \( M \) is related to the numbers of common alleles for different genetic markers and individuals (Chen et al. [9]). In both examples, \( M \) is either observable or can be estimated. Also, it is frequently seen that only a small proportion of genes is associated with the clinical outcome, and each gene contributes weakly to the clinical outcome. In such a situation, the signals are both sparse and faint.

Back to Model (2.1). We model the number of nonzero entries of \( \mu \) as

\[
m = n^{1-\beta}, \quad \text{where } \beta \in (1/2, 1).
\] (2.2)

This is a very sparse case for the proportion of signals is much smaller than \( 1/\sqrt{n} \). We suppose that the signals appear at \( m \) different locations—\( \ell_1 < \ell_2 < \ldots < \ell_m \)—that are randomly drawn from \( \{1, 2, \ldots, n\} \) without replacement,

\[
P\{\ell_1 = n_1, \ell_2 = n_2, \ldots, \ell_m = n_m\} = \binom{n}{m}^{-1}, \quad \text{for all } 1 \leq n_1 < n_2 < \ldots n_m \leq n, \quad (2.3)
\]

and that they have a common magnitude of

\[
A_n = \sqrt{2r \log n} \quad \text{where } r \in (0, 1).
\]

We are interested in testing which of the following two hypotheses is true:

\[
H_0 : \mu = 0 \quad \text{vs.} \quad H_1^{(n)} : \mu \text{ is a sparse vector as above.} \quad (2.4)
\]

This testing problem was found to be delicate even in the uncorrelated case where \( \Sigma_n = I_n \). See [14] (also [7, 23, 24, 27, 32]) for details. Below, we briefly review the results in the uncorrelated case.

### 2.1 Detection boundary \((\Sigma_n = I_n)\)

The testing problem is characterized by the curve \( r = \rho^*(\beta) \) in the \( \beta-r \) plane, where

\[
\rho^*(\beta) = \begin{cases} 
\beta - 1/2, & 1/2 < \beta \leq 3/4, \\
(1 - \sqrt{1 - \beta})^2, & 3/4 < \beta < 1,
\end{cases} \quad (2.5)
\]

and we call \( r = \rho^*(\beta) \) the detection boundary. The detection boundary partitions the \( \beta-r \) plane into two sub-regions: the undetectable region below the boundary and the detectable region above the boundary; see Figure 1. In the interior of the undetectable region, the signals are so sparse and so faint that no test is able to successfully separate the alternative hypothesis from the null hypothesis in (2.4): the sum of Type I and Type II errors of any test tends to 1 as \( n \) diverges to \( \infty \). In the interior of the detectable region, it is possible to have a test such that as \( n \) diverges to \( \infty \), the Type I error tends to 0 and the power tends to 1. (In fact, Neyman-Pearson’s Likelihood Ratio Test (LRT) is such a test.) See [14, 23, 27] for example.

The drawback of LRT is that it needs detailed information of the unknown parameters \((\beta, r)\). In practice, we need a test that does not need such information; this is where HC comes in.
2.2 Higher Criticism and its optimal adaptivity ($\Sigma_n = I_n$)

A notion that goes back to Tukey [36], Higher Criticism was first proposed in [14] to tackle the aforementioned testing problem in the uncorrelated case. To apply Higher Criticism, let $p_j = P\{|N(0,1)| \geq |X_j|\}$ be the $p$-value associated with the $j$-th observation unit, and let $p(j)$ be the $j$-th $p$-value after sorting in ascending order. The Higher Criticism statistic is defined as

$$HC^*_n = \max_{\{j: 1/n \leq p(j) \leq 1/2\}} \left\{ \frac{\sqrt{n}(j/n - p(j))}{\sqrt{p(j)(1-p(j))}} \right\}.$$  

There are also other versions of HC; see [14, 15, 16] for example. When $H_0$ is true, $HC^*_n$ equals in distribution to the maximum of the standardized uniform stochastic process [14]. Therefore, by a well-known result for empirical processes [33],

$$\frac{HC^*_n}{\sqrt{2\log\log n}} \to 1 \quad \text{in probability.}$$  

Consider the HC test which rejects the null hypothesis when

$$HC^*_n \geq (1 + a)\sqrt{2\log\log n} \quad \text{where } a > 0 \text{ is a constant.}$$  

It follows from (2.7) that the Type I error tends to 0 as $n$ diverges to $\infty$. For any parameters $(\beta, r)$ that fall in the interior of the detectable region, the Type II error also tends to 0. This is the following theorem, where we set $a = 0.01$ for simplicity of presentation.

**Theorem 2.1** Consider the HC test that rejects $H_0$ when $HC^*_n \geq 1.01\sqrt{2\log\log n}$. For every alternative $H_1^{(n)}$ where the the associated parameters $(r, \beta)$ satisfy $r > \rho^*(\beta)$, the HC test has asymptotically full power for detection:

$$P_{H_1^{(n)}}\{\text{Reject } H_0\} \to 1 \quad \text{as } n \to \infty.$$
That is, the HC test adapts to unknown parameters \((\beta, r)\), and yields asymptotically full power for detection throughout the entire detectable region. We call this the **optimal adaptivity of HC** [14].

Theorem 2.1 is closely related to [14, Theorem 1.2], where a mixture model is used. The mixture model reduces approximately to the current model if we randomly shuffle the coordinates of \(X\). However, despite its appealing technical convenience, it is not clear how to generalize the mixture model from the uncorrelated case to general correlated settings. Theorem 2.1 is a special case of Theorem 4.2.

We now turn to the correlated case. In this case, the exact “detection boundary” may depend on \(\Sigma_n\) in a complicated manner, but it is possible to establish both a tight lower bound and a tight upper bound. We discuss the lower bound first.

## 3 Lower bound to the detectability

To establish the lower bound, a key element is the theory in comparison of experiments (e.g. [34]), where a useful guideline is that adding noise always makes the inference more difficult. Thus, we can alter the model by either adding or subtracting a certain amount of noise, so that the difficulty level (measured by the Hellinger distance, or the \(\chi^2\)-distance, etc., between the null density and the alternative density) of the original problem is sandwiched by those of the two adjusted models. The correlation matrices in the latter have a simpler form and hence are much easier to analyze. Another key element is the recent development of matrix characterizations based on polynomial off-diagonal decay, where it shows that the inverse of a matrix with this property shares the same rate of decay as the original matrix.

### 3.1 Comparison of experiments: adding noise makes inference harder

We begin by comparing two experiments that have the same mean, but where the data from one experiment are more noisy than those from the other. Intuitively, it is more difficult to make inference in the first experiment than in the other. Specifically, consider the two Gaussian models

\[
X = \mu + Z, \quad Z \sim N(0, \Sigma) \quad \text{and} \quad X^* = \mu + Z^*, \quad Z^* \sim N(0, \Sigma^*),
\]

where \(\mu\) is an \(n\)-vector that is generated according to some distribution \(G = G_n\). The second model is more noisy than the first, in the sense that \(\Sigma^* \geq \Sigma\); see the definition below.

**Definition 3.1** Consider two matrices \(A\) and \(B\). We write \(A \geq B\) if \(A - B\) is positive semi-definite.

The second model in (3.1) can be viewed as the result of adding noise to the first. Indeed, defining \(\Delta = \Sigma^* - \Sigma\), taking \(\xi\) to be \(N(0, \Delta)\) (independently of \(Z\)), and noting that \(Z + \xi \sim N(0, \Sigma + \Delta)\), the second model is seen to be equivalent to \(X + \xi = \mu + (Z + \xi)\). Intuitively, adding noise generally makes inference more difficult. This can be stated precisely by comparing distances between distributions, for example Hellinger distances. In detail, if we denote the Hellinger distance between \(X\) and \(Z\) by \(H_n(X, Z; \mu, \Sigma_n)\), and that between \(X^*\) and \(Z^*\) by \(H_n(X^*, Z^*; \mu, \Sigma^*_n)\), then we have the following theorem, which is proved in Section 10.

**Theorem 3.1** Suppose \(\Sigma_n \leq \Sigma^*_n\) in (3.1). Then \(H_n(X, Z; \mu, \Sigma_n) \geq H_n(X^*, Z^*; \mu, \Sigma^*_n)\).
3.2 Inverses of matrices having polynomial off-diagonal decay

Next, we review results concerning matrices with polynomial off-diagonal decay. The main message is that, under mild conditions, if a matrix has polynomial off-diagonal decay, then its inverse as well as its Cholesky factorization (which is unique if we require the diagonal entries to be positive) also have polynomial off-diagonal decay, and with the same rate. This beautiful result was recently obtained by Jaffard [25]; also see [19, 35]. In detail, let \( Z \) be the set of all integers. Write \( \ell^2 \) for the set of summable sequences \( x = \{x_k\}_{k \in Z} \), and let \( A = (A(j, k))_{j, k \in Z} \) be an infinite matrix. Also, let \( |x|_2 \) be the \( \ell^2 \)-vector norm of \( x \), and \( \| A \| \) be the operation norm of \( A \): \( \| A \| = \sup_{\{x: |x|_2 = 1\}} |Ax|_2 \). Fixing positive constants \( \lambda, M, \) and \( c_0 \), we define the class of matrices

\[
\Theta_\infty(\lambda, c_0, M) = \left\{ A = (A(j, k))_{j, k \in Z} : |A(j, k)| \leq \frac{M}{(1 + |j - k|)^{\lambda}}, \| A \| \geq c_0 \right\}.
\]

The following lemma follows directly from [35].

**Lemma 3.1** Fix \( \lambda > 1, c_0 > 0, \) and \( M > 0 \). For any matrices \( A \in \Theta_\infty(\lambda, M) \), there is a constant \( C > 0 \), depending only on \( \lambda, M \), and \( c_0 \), such that \( |A^{-1}(j, k)| \leq C \cdot (1 + |j - k|)^{-\lambda} \).

Now, consider a sequence of matrices of finite but increasingly larger sizes, where the entries have a given rate of polynomial off-diagonal decay and where the operator norm is uniformly bounded from below. Then the same rate of polynomial off-diagonal decay holds for their inverses, as well as for the inverse of their Cholesky factorizations. In detail, writing \( \Theta_n \) for the set of \( n \times n \) correlation matrices, we introduce the set of matrices

\[
\Theta_n^*(\lambda, c_0, M) = \left\{ \Sigma_n \in \Theta_n : |\Sigma_n(j, k)| \leq M(1 + |j - k|)^{-\lambda}, \| \Sigma_n \| \geq c_0 \right\}.
\]

The following corollary follows from Lemma 3.1 and is proved in Section 11.

**Lemma 3.2** Fix \( \lambda > 1, c_0 > 0, \) and \( M > 0 \). For any sequence of matrices \( \Sigma_n, n \geq 1 \), such that \( \Sigma_n \in \Theta_n^*(\lambda, c_0, M) \), let \( U_n \) be the inverse of the Cholesky factorization of \( \Sigma_n \). There is a constant \( C = C(\lambda, c_0, M) > 0 \) such that for any \( n \) and any \( 1 \leq j, k \leq n \),

\[
|\Sigma_n^{-1}(j, k)| \leq C \cdot (1 + |j - k|)^{-\lambda}, \quad |U_n(j, k)| \leq C \cdot (1 + |j - k|)^{-\lambda}.
\]

When \( \lambda = 1 \), the first inequality continues to hold, and the second holds if we adjoin a \( \log n \) factor to the right hand side.

3.3 Lower bound to the detectability

We are now ready for the lower bound. Consider a sequence of matrices \( \Sigma_n \in \Theta_n^*(\lambda, c_0, M) \). Suppose the extreme diagonal entries of \( \Sigma_n^{-1} \) have an upper limit \( 0 < \gamma_0 < \infty \), i.e.

\[
\lim_{n \to \infty} \max_{\{\sqrt{n} \leq k \leq n - \sqrt{n}\}} \{\Sigma_n^{-1}(k, k)\} = \gamma_0.
\]

Recall that the detection boundary in the uncorrelated case is \( r = \rho^*(\beta) \). The following theorem says that if we re-scale and write \( r = \gamma_0^{-1} \cdot \rho^*(\beta) \), then we obtain a lower bound.

**Theorem 3.2** Fix \( \beta \in (1/2, 1) \), \( r \in (0, 1) \), \( \lambda > 1, c_0 > 0, \) and \( M > 0 \). Consider a sequence of correlation matrices \( \Sigma_n \in \Theta_n^*(\lambda, c_0, M) \) that satisfy (3.4). If \( r < \gamma_0^{-1} \rho^*(\beta) \), then the null hypothesis and alternative hypothesis in (2.7) merge asymptotically, and the sum of Type I and Type II errors of any test converges to 1 as \( n \) diverges to \( \infty \).

We now turn to the upper bound. The key is to adapt the Higher Criticism to correlated noise and form a new statistic—innovated Higher Criticism.
4 Innovated Higher Criticism, upper bound to detectability

Originally designed for the independent case, standard HC is not really appropriate for dependent data, for the following reasons. First, HC only summarizes the information that resides in the marginal effects of each coordinate, and neglects the correlation structure of the data. Second, HC remains the same if we randomly shuffle different coordinates of $X$. Such shuffling does not have an effect if $\Sigma = I$, but does otherwise. In this section we build the correlation into the standard Higher Criticism and form a new statistic—innovated Higher Criticism (iHC). We then use iHC to establish an upper bound to detectability. The iHC is intimately connected to the well-known notion of innovation in time series [6] (see (4.1) below), hence the name innovated Higher Criticism.

Below, we begin by discussing the role of correlation in the detection problem.

4.1 Correlation among different coordinates: curse or blessing?

Consider Model (2.1) in the two cases $\Sigma = I$ and $\Sigma \neq I$. Which is the more difficult detection problem?

Here is one way to look at it. Since the mean vectors are the same in the two cases, the problem where the noise vector contains more “uncertainty” is more difficult than the other. In information theory, the total amount of uncertainty is measured by the differential entropy, which in the Gaussian case is proportional to the determinant of the correlation matrix [11]. As the determinant of a correlation matrix is largest when and only when it is the identity matrix, the uncorrelated case contains the largest amount of “uncertainty” and therefore gives the most difficult detection problem. In a sense, the correlation is a “blessing” rather than a “curse”, as one might have expected.

Here is another way to look at it. For any positive definite matrix $\Sigma$, denote the inverse of its Cholesky factorization by $U = U_n(\Sigma)$ (so that $U \Sigma U^* = I$). Model (2.1) is equivalent to $U_n X = U_n \mu + U_n Z$ where $U_n Z \sim N(0,I)$. (4.1)

(In the literature of time series [9], $U_n X$ is intimately connected to the notion of innovation). Compared to the uncorrelated case, i.e.

$X = \mu + Z$ where $Z \sim N(0,I)$.

The noise vectors have the same distribution, but the signals in the former are stronger. In fact, let $\ell_1 < \ell_2 < \ldots < \ell_m$ be the $m$ locations where $\mu$ is nonzero. Recalling that $\mu_j = A_n$ if $j \in \{\ell_1,\ell_2,\ldots,\ell_m\}$, $\mu_j = 0$ otherwise, and that $U_n$ is a lower triangular matrix,

$$(U_n \mu)_{\ell_k} = A_n \sum_{j=1}^{k} U_n(\ell_k, \ell_j) = A_n U_n(\ell_k, \ell_k) + A_n \left\{ \sum_{j=1}^{k-1} U_n(\ell_j, \ell_k) \right\}. (4.2)$$

Two key observations are as follows. First, since $\Sigma$ has unit diagonal entries, every diagonal entry of $U_n$ is greater than or equal to 1, and especially,

$$U_n(\ell_k, \ell_k) \geq 1. (4.3)$$

Second, recall that $m \ll n$, and $\{\ell_1,\ell_2,\ldots,\ell_m\}$ are randomly generated from $\{1,2,\ldots,n\}$, so different $\ell_j$ are far apart from each other. Therefore, under mild decay conditions on $U_n$,

$$U_n(\ell_j, \ell_k) \approx 0, \quad j = 1,2,\ldots,k-1. (4.4)$$

7
Inserting (4.3) and (4.4) into (4.2), we expect that

\[(U_n \mu)_{\ell_k} \gtrless A_n, \quad k = 1, 2, \ldots, m.\]

Therefore, “on average”, \(U_n \mu\) has at least \(m\) entries each of which is at least as large as \(A_n\). This says that, first, the correlated case is easier for detection than the uncorrelated case. Second, applying standard HC to \(U_n X\) yields a larger power than applying it to \(X\) directly.

Next we make the argument more precise. Fix a positive sequence \(\{\delta_n : n \geq 1\}\) that tends to 0 as \(n\) diverges to \(\infty\), and a sequence of integers \(\{b_n : n \geq 1\}\) that satisfy \(1 \leq b_n \leq n\).

Let

\[\tilde{\Theta}_n^*(\delta_n, b_n) = \{\Sigma_n \in \Theta_n, \sum_{j=1}^{k-b_n} |U_n(\Sigma_n)(k,j)| \leq \delta_n, \text{ for all } k \text{ satisfying } b_n + 1 \leq k \leq n\}.\]

Introducing \(\tilde{\Theta}_n^*\) seems a digression from our original plan of focusing on \(\Theta_n^*\) (the set of matrices with polynomial off-diagonal decay), but it is interesting in its own right. In fact, compared to \(\Theta_n^*\), \(\tilde{\Theta}_n^*\) is much broader as it does not impose much of a condition on \(\Sigma_n(j,k)\) for \(|j-k| \leq b_n\). This helps to illustrate how broadly the aforementioned phenomenon holds.

The following theorem is proved in Section 10.

**Theorem 4.1** Fix \(\beta \in (1/2, 1)\) and \(r \in (\rho^*(\beta), 1)\). Let \(b_n = n^{\beta}/3\), and let \(\delta_n\) be a positive sequence that tends to 0 as \(n\) diverges to \(\infty\). Suppose we apply standard Higher Criticism to \(U_n(\Sigma_n)X\) and we reject \(H_0\) if and only if the resulting score exceeds \(1.01 \sqrt{2 \log \log n}\). Then, uniformly in all sequences of \(\Sigma_n\) satisfying \(\Sigma_n \in \tilde{\Theta}_n^*(\delta_n, b_n)\),

\[P_{H_0}\{\text{Reject } H_0\} + P_{H_1(n)}\{\text{Accept } H_0\} \to 0, \quad n \to \infty.\]

Generally, directly applying standard HC to \(X\) does not yield the same result (e.g. [21]).

### 4.2 Innovated Higher Criticism: Higher Criticism based on innovations

We have learned that applying standard HC to \(U_n X\) yields better results than applying it to \(X\) directly. Is this the best we can do? No, there is still space for improvement. In fact, HC applied to \(U_n X\) is a special case of innovated Higher Criticism to be elaborated in this section. Innovated Higher Criticism is even more powerful in detection.

To begin with, we revisit the vector \(U_n \mu\) via an example. Fix \(n = 100\); let \(\Sigma_n\) be a symmetric tri-diagonal matrix with 1 on the main diagonal, 0.4 on two sub-diagonals, and 0 elsewhere; and let \(\mu\) be the vector with 1 at coordinates 27, 50, 71, and 0 elsewhere. Figure[2] compares \(\mu\) and \(U_n \mu\). Especially, the nonzero coordinates of \(U_n \mu\) appear in three visible clusters, each of which corresponds to a different nonzero entry of \(\mu\). Also, at coordinates 27, 50, 71, \(U_n \mu\) approximately equals to 1.2, but \(\mu\) equals 1.

Now we can either simply apply standard HC to \(U_n X\) as before, or we can first linearly transform each cluster of signals to a singleton, and then apply the standard HC. Note that in the second approach, we may have fewer signals, but each of them is much stronger than those in \(U_n X\). Since the HC test is more sensitive to signal strength than to the number of signals, we expect that the second approach yields greater power for detection than the first.
Figure 2: Comparison of $\mu$ (left) and $U_n(\Sigma_n)\mu$ (right). Here $n = 100$ and $\Sigma_n$ is a symmetric tri-diagonal matrix with 1 on the main diagonal, 0.4 on two sub-diagonals, and 0 elsewhere. Also, $\mu$ is 1 at coordinates 27, 50, and 71 and 0 elsewhere. In comparison, the nonzero entries of $U_n(\Sigma_n)\mu$ appear in three visible clusters, each of which corresponds to a nonzero coordinate of $\mu$.

In light of this we propose the following approach. Write $U_n = (u_{kj})_{1 \leq k,j \leq n}$. We pick a bandwidth $1 \leq b_n \leq n$, and construct a matrix $\tilde{U}_n(b_n) = U_n(\Sigma_n, b_n)$ by banding $U_n$: $\tilde{U}(b_n) \equiv (\tilde{u}_{kj})_{1 \leq j,k \leq n}$, $\tilde{u}_{kj} = \begin{cases} u_{kj}, & k - b_n + 1 \leq j \leq k, \\ 0, & \text{otherwise}. \end{cases}$.

We then normalize each column of $\tilde{U}_n(b_n)$ by its own $\ell^2$-norm, and call the resulting matrix $\bar{U}_n(b_n)$. Next, defining $V_n(b_n) = V_n(b_n; \Sigma_n) = \tilde{U}_n'(b_n; \Sigma_n) \cdot U_n(\Sigma_n)$, we transform Model (2.1) into

$$X \mapsto V_n(b_n)X = V_n(b_n)\mu + V_n(b_n)Z.$$ (4.7)

Finally, we apply standard Higher Criticism to $V_n(b_n)X$, and call the resulting statistic innovated Higher Criticism, $iHC_n^*(b_n) = iHC_n^*(b_n; \Sigma_n) = \frac{1}{\sqrt{(2b_n - 1)}} \sup_{j: 1/n \leq p(j) \leq 1/2} \left\{ \sqrt{n} \cdot \frac{j/n - p(j)}{\sqrt{p(j)(1 - p(j))}} \right\}.$ (4.8)

Note that standard HC applied to $U_nX$ is a special case of $iHC_n^*$ with $b_n = 1$.

We briefly comment on the selection of the bandwidth parameter $b_n$. First, for each $k \in \{\ell_1, \ell_2, \ldots, \ell_m\}$, direct calculations show that $(V_n(b_n)\mu)_k \approx A_n \cdot \sqrt{\sum_{j=1}^{b_n} u_{k,k-j+1}^2} \geq A_n$. Second, $V_n(b_n)Z \sim N(0, \tilde{U}_n'(b_n)\tilde{U}_n(b_n))$, where $\tilde{U}_n'(b_n)\tilde{U}_n(b_n)$ is a banded correlation matrix with bandwidth $2b_n - 1$. Therefore, choosing $b_n$ involves a trade-off: a larger $b_n$ usually means stronger signals, but also means stronger correlation among the noise. While it is hard to give a general rule for selecting the best $b_n$, we must mention that in many cases, the choice of $b_n$ is not very critical. For example, when $\Sigma_n$ has polynomial off-diagonal decay, a logarithmically large $b_n$ is usually appropriate.
4.3 Upper bound to detectability

We now establish an upper bound to detectability. The following lemma describes the signal strength in $V_n(b_n) \cdot X$ and is proved in Section 11.

**Lemma 4.1** Fix $c_0 > 0$, $\lambda \geq 1$, and $M > 0$. Consider a sequence of bandwidths $b_n$ that tends to $\infty$. Let $\{\ell_1, \ell_2, \ldots, \ell_m\}$ be the $m$ random locations of signals in $\mu$, arranged in the ascending order. For sufficiently large $n$, there is a constant $C = C(c_0, \lambda, M)$ such that, except for an event with asymptotically vanishing probability,

$$(V_n(b_n)\mu)_k \geq (1 - C b_n^{1/2 - \lambda} + o(1)) \cdot \sqrt{\Sigma_n^{-1}(k,k) \cdot A_n}, \quad \forall k \in \{\ell_1, \ell_2, \ldots, \ell_m\},$$

for all $\Sigma_n \in \Theta_n^*(\lambda, c_0, M)$, where $o(1)$ tends to 0 algebraically fast.

Now, suppose the diagonal entries of $\Sigma_n^{-1}$ has a lower limit as follows,

$$\lim_{n \to \infty} \left( \min_{\{\sqrt{n} \leq k \leq \sqrt{n}\}} \{\Sigma_n^{-1}(k,k)\} \right) = \gamma_0.$$

(4.9)

Recall that the nonzero coordinates of $\mu$ is modeled as $A_n = \sqrt{2\rho \log n}$. So if we let $b_n = \log n$, then a direct result of Lemma 4.1 is that the vector $V_n(b_n) \cdot X$ has at least $m$ nonzero coordinates, each of which is as large as $\sqrt{\rho} A_n = \sqrt{2\rho \cdot n \cdot \log n}$. For the bandwidth, note that a larger $b_n$ cannot improve the signal strength significantly, but may yield a much stronger correlation in $V_n(b_n)Z$. Therefore, a smaller bandwidth is preferred. The choice $b_n = \log n$ is mainly for convenience, and can be modified.

We now turn to the behavior of $iHC^*_n(b_n)$ under the null hypothesis. In the independent case, $iHC^*_n$ reduces to $HC^*_n$ and is approximately equal to $\sqrt{2\log \log n}$. In the current situation, $iHC^*_n$ is comparably larger due to the correlation. However, since the selected bandwidth is relatively small, $iHC^*_n$ remains logarithmically large. This is formally captured by the following lemma.

**Lemma 4.2** Take the bandwidth to be $b_n = \log n$ and suppose $H_0$ is true. Then, except for an algebraically small probability, $iHC^*_n(b_n) \leq C(\log n)^{3/2}$ for some constant $C > 0$, uniformly for all correlation matrices.

Lemma 4.2 is proved in Section 11. The key is to express $iHC^*_n$ as the maximum of $(2b_n - 1)$ standard HC, and apply the well-known Hungarian construction [10]. The following theorem elaborates on the upper bound, and is proved in Section 10.

**Theorem 4.2** Fix $c_0 > 0$, $\lambda > 1$, and $M > 0$, and set $b_n = \log n$. Suppose $\gamma_0 \cdot r > \rho^*(\beta)$. If we reject $H_0$ when $iHC^*_n(b_n; \Sigma_n) \geq (\log n)^2$, then, uniformly in all $\Sigma_n \in \Theta_n^*(\lambda, c_0, M)$,

$$P_{H_0}\{\text{Reject } H_0\} + P_{\hat{H}_1}\{\text{Accept } H_0\} \to 0, \quad \text{as } n \to \infty.$$  

The cut-off value $(\log n)^2$ can be replaced by other logarithmically large terms that tend to $\infty$ faster than $(\log n)^{3/2}$. For finite $n$, this cut-off value may be conservative. In Section 8 (i.e. experiment (a)), we suggest an alternative where we select the cut-off value by simulation.

In summary, a lower bound and an upper bound are established as $r = \gamma_0^{-1} \rho^*(\beta)$ and $r = \gamma_0^{-1} \rho^*(\beta)$, respectively, under reasonably weak off-diagonal decay conditions. When $\gamma_0 = \gamma_0$, the gap between the two bounds disappears, and iHC is optimal for detection. Below, we investigate several Toeplitz cases, ranging from weak dependence to strong dependence; for these cases, iHC is optimal in detection.
5 Application in the Toeplitz case

In this section, we discuss the case where $\Sigma_n$ is a (truncated) Toeplitz matrix that is generated by a spectral density $f$ defined over ($-\pi$, $\pi$). In detail, let $a_k = (2\pi)^{-1} \int_{|\theta| < \pi} f(\theta) e^{-ik\theta} d\theta$ be the $k$-th Fourier coefficient of $f$. The $n$-th truncated Toeplitz matrix generated by $f$ is the matrix $\Sigma_n(f)$ of which the $(j,k)$-th element is $a_{j-k}$, for $1 \leq j,k \leq n$.

We assume that $f$ is symmetric and positive, i.e.

$$c_0(f) \equiv \text{essinf}_{-\pi \leq \theta \leq \pi} f(\theta) > 0. \quad (5.1)$$

First, note that $f$ is a density, so $a_0 = 1$ and $\Sigma_n(f)$ has unit diagonal entries. Second, from the symmetry of $f$, it can be seen that $\Sigma_n(f)$ is a real-valued symmetric matrix. Last, it is well-known [5] that the smallest eigenvalue of $\Sigma_n$ is a direct result of [5, Theorem 2.15].

Comparing (5.1) and (5.2) with the definition of $\Theta_n^*$, we conclude that

$$\Sigma_n \in \Theta_n^*(\lambda, c_0(f), M_0(f)). \quad (5.3)$$

In addition, it is known that the inverse of $\Sigma_n(f)$ is typically asymptotically equivalent to the Toeplitz matrix generated by $1/f$. In particular we have the following lemma, which is a direct result of [5, Theorem 2.15].

**Lemma 5.1** Suppose (5.1) and (5.2) hold. For all $\sqrt{n} \leq k \leq n - \sqrt{n}$ and each $1 < \lambda' < \lambda$, $|\Sigma_n^{-1}(f)(k,k) - \Sigma_n(1/f)(k,k)| \leq Cn^{-(\lambda'-1)/2}$.

The diagonal entries of $\Sigma_n(1/f)$ are the well-known Wiener interpolation rates [37]:

$$C(f) = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{1}{f(\theta)} d\theta. \quad (5.4)$$

Therefore, as a direct result of Lemma 5.1, $\max_{\sqrt{n} \leq k \leq n - \sqrt{n}} \left\{ |\Sigma_n^{-1}(f)(k,k) - C(f)| \right\} = o(1)$.

Comparing this with (3.4) and (4.9) we deuce that

$$\tilde{\gamma}_0 = \gamma_0 = C(f). \quad (5.5)$$

Combining (5.3) and (5.5), the following theorem is a direct result of Theorems 3.2 and 4.1 (the proof is omitted).

**Theorem 5.1** Fix $\lambda > 1$, and let $\Sigma_n(f)$ be the Toeplitz matrix generated by a symmetric spectral density $f$ that satisfies (5.1) and (5.2). When $C(f) \cdot r < \rho^*(\beta)$, the null and alternative hypotheses merge asymptotically, and the sum of Type I and Type II errors of any test converges to 1 as $n$ diverges to $\infty$. When $C(f) \cdot r > \rho^*(\beta)$, suppose we apply iHC with bandwidth $b_n = \log n$ and reject the null hypothesis when $\text{iHC}^*_n(b_n, \Sigma_n(f)) \geq (\log n)^2$. Then the Type I error of iHC converges to zero, and its power converges to 1.

The curve $r = C(f)^{-1} \rho^*(\beta)$ partitions the $\beta$-$r$ plane into the undetectable region and the detectable region, similarly to the uncorrelated case. The regions of the current case can be viewed as the corresponding regions in the uncorrelated squeezed vertically by a factor of $1/C(f)$. See Figure 3. (Note that $C(f) \geq 1$, with equality if and only if $f \equiv 1$, which corresponds to the uncorrelated case.)
Figure 3: Phase diagram in the case where $\Sigma_n$ is a Toeplitz matrix generated by a spectral density $f$. Similarly to that in Figure 1, the $\beta-r$ plane is partitioned into three regions—undetectable, detectable, estimable—each of which can be viewed as the corresponding region in Figure 1 squeezed vertically by a factor of $1/C(f)$. In the rectangular region on the top, the largest signals in $V_n(b_n) \cdot X$ (see (4.6)) are large enough to stand out by themselves.

6 Extension: when signals appear in clusters

In the preceding sections (see e.g. [2,3] in Section 2), the $m$ locations of signals were generated randomly from $\{1, 2, \ldots, n\}$. Since $m \ll \sqrt{n}$, the signals appear as singletons with overwhelming probabilities. In this section we investigate an extension where the signals may appear in clusters.

We consider a setting where the signals appear in a total of $m$ clusters, whose locations are randomly generated from $\{1, 2, \ldots, n\}$. Each cluster contains a total of $K$ consecutive signals, whose strengths are $g_0 A_n, g_1 A_n, \ldots, g_{K-1} A_n$, from right to left. Here, $A_n = \sqrt{2r\log n}$ as before, $K \geq 1$ is a fixed integer, and $g_i$ are constants. Approximately, the signal vector can be modeled as follows.

As before, let $\ell_1, \ell_2, \ldots, \ell_m$ be indices that are randomly sampled from $\{1, 2, \ldots, n\}$. Let $\mu = (\mu_1, \ldots, \mu_n)^T$, where $\mu_j = A_n$ if $j \in \{\ell_1, \ell_2, \ldots, \ell_m\}$, and $\mu_j = 0$ otherwise. Let $B = B_n$ denote the “backward shift” matrix, with 0 in every position except that it has 1 in position $(j+1, j)$ for $1 \leq j \leq n-1$. Thus, $B\mu$ differs from $\mu$ in that the components are shifted one position backward, with 0 added at the bottom. We model the signal vector as

$$\nu = g_0 \mu + g_2 B \mu + \ldots g_k B^{K-1} \mu = \left( \sum_{k=0}^{K-1} g_k B^k \right) \mu.$$

Thus, $\nu$ is comprised of $m$ clusters, each of which contains $K$ consecutive signals. Let $g$ be the function $g(\theta) = \sum_{0 \leq k \leq K-1} g_k e^{-ik\theta}$. We note that $\sum_{0 \leq k \leq K-1} g_k B^k$ is the lower triangular Toeplitz matrix generated by $g$. With the same spectral density $f$, we consider an extension of that in Section 5 by considering the following model:

$$X = \Sigma_n(g) \mu + Z \quad \text{where} \quad Z \sim N(0, \Sigma_n(f)),$$

with $f$ denoting the spectral density in Section 5. The model is closely related to that by Arias-Castro et al. [2], with $g_i = 1$, $m = 1$, and $f \equiv 1$. See details therein.
We note that the model can be equivalently viewed as
\[ \tilde{X} = \mu + \tilde{Z} \text{ where } \tilde{Z} \sim \mathcal{N}(0, \tilde{\Sigma}_n) \text{ and } \tilde{\Sigma}_n = \Sigma_n^{-1}(\tilde{g}) \cdot \Sigma_n(f) \cdot \Sigma_n^{-1}(\tilde{g}), \]
with \( \tilde{g} \) denoting the complex conjugate of \( g \). Asymptotically,
\[ \tilde{\Sigma}_n^{-1} \sim \Sigma_n(\tilde{g}) \cdot \Sigma_n^{-1}(f) \cdot \Sigma_n(g) \sim \Sigma_n(|g|^2/f), \]
where the diagonal entries of \( \Sigma_n(|g|^2/f) \) are
\[ C(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|g(\theta)|^2}{f(\theta)} d\theta. \]
If \( \gamma_0 \) and \( \gamma_0 \) are as defined in (3.4) and (4.9), then \( \gamma_0 = \gamma_0 = C(f, g) \), and we expect the detection boundary to be \( r = C(f, g)^{-1} \cdot \rho^*(\beta) \). The is affirmed by the following theorem, which is proved in Section 10.

**Theorem 6.1** Fix \( \lambda > 1 \). Suppose \( g_0 \neq 0 \) and let \( f \) be a symmetric spectral density that satisfies (5.1) and (5.2). When \( C(f, g) \cdot r < \rho^*(\beta) \), the null and alternative hypotheses merge asymptotically, and the sum of Type I and Type II errors of any test converges to 1 as \( n \) diverges to \( \infty \). When \( C(f, g) \cdot r > \rho^*(\beta) \), if we apply iHC to \( \Sigma_n^{-1}(g)X \) with bandwidth \( b_n = \log n \) and reject the null hypothesis when \( iHC_n(b_n, \Sigma_n^{-1}(g)\Sigma_n(f)\Sigma_n^{-1}(g)) \geq (\log n)^2 \), then the Type I error converges to zero, and the power converges to 1.

### 7 The case of strong dependence

So far, we have only discussed weakly dependent cases. In this section, we investigate the case of strong dependence.

Suppose that we observe an \( n \)-variate Gaussian vector \( X = \mu + Z \), where \( \mu \) contains a total of \( m \) signals, of equal strength to be specified, whose locations are randomly drawn from \( \{1, 2, \ldots, n\} \) without replacement, and \( Z \sim \mathcal{N}(0, \Sigma_n) \) where we assume that \( \Sigma_n \) displays slowly decaying correlation:
\[ \Sigma_n(j, k) = \max\{0, 1 - |j - k|^\alpha n^{-\alpha_0}\}, \quad 1 \leq j, k \leq n, \quad (7.1) \]
with \( \alpha > 0 \) and \( 0 < \alpha_0 \leq \alpha \). The range of dependence can be calibrated in terms of \( k_0 = k_0(n; \alpha, \alpha_0) \), denoting the largest integer by \( k < n^{\alpha_0/\alpha} \). Clearly, \( k_0 \approx n^{\alpha_0/\alpha} \). Seemingly, the most interesting range is \( 0 < \alpha_0/\alpha \leq 1 \). The following lemma establishes cases for which \( \Sigma_n \) is a correlation matrix.

**Lemma 7.1** Let \( \Sigma_n \) be as in (7.1). For sufficient large \( n \), a necessary and a sufficient condition for \( \Sigma_n \) to be positive definite are, respectively, \( 0 \leq \alpha \leq 2 \) and \( 0 < \alpha_0 \leq \alpha \leq 1 \).

Lemma 7.1 is proved in Section 12. Model (7.1) has been studied in detail by Hall and Jin [21], who showed that the detectability of standard HC is seriously damaged by strong dependence. However, it remains open as to what is the detection boundary, and how to adapt HC to overcome the strong dependence and obtain optimal detection. This is what we address in the current section.

The key idea is to decompose the correlation matrix as the product of three matrices each of which is relatively easy to handle. To begin with we introduce a spectral density,
\[ f_\alpha(\theta) = 1 - \sum_{k=1}^{\infty} \left[ (k + 1)^\alpha + (k - 1)^\alpha - 2k^\alpha \right] \cos(k\theta). \quad (7.2) \]
Figure 4: Display of $C(f_\alpha, g_0)$. $x$-axis: $\alpha$. $y$-axis: $C(f_\alpha, g_0)$.

Note that the Fourier coefficients of $f_\alpha(\theta)$ satisfy the decay condition in (5.2) with $\lambda = 2 - \alpha$. Also, we have the following lemma, which is derived in Section 12.

**Lemma 7.2** For $0 < \alpha < 1$, we have $\text{essinf}_{-\pi \leq \theta \leq \pi} \{ f_\alpha(\theta) \} > 0$.

Next, let $g_0(\theta) = 1 - e^{-i\theta}$, $a_n = a_n(\alpha_0) = n^{\alpha_0}/2$.

The Toeplitz matrix $\Sigma_n(g_0)$ is a lower triangular matrix with 1’s on the main diagonal, −1’s on the sub-diagonal, and 0’s elsewhere. Additionally, let $D_n$ be the diagonal matrix where on the diagonal the first entry is 1 and the remaining entries are $\sqrt{a_n}$. Let $\tilde{X} = D_n \cdot \Sigma_n(g_0) \cdot X$. Then Model (7.1) can be rewritten equivalently as

$$
\tilde{X} = \tilde{\mu} + \tilde{Z} \quad \text{where} \quad \tilde{\mu} = D_n \cdot \Sigma_n(g_0) \cdot \mu \quad \text{and} \quad \tilde{Z} \sim N(0, \tilde{\Sigma}_n),
$$

with $\tilde{\Sigma}_n = D_n \cdot \Sigma_n(g_0) \cdot \Sigma_n \cdot \Sigma_n(g_0) \cdot D_n$. The key is that $\tilde{\Sigma}_n$ is asymptotically equivalent to the Toeplitz matrix generated by $f_\alpha$. In detail, introduce

$$
\tilde{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & \Sigma_{n-1}(f_\alpha) \end{pmatrix}.
$$

The following lemma is proved in Section 11.

**Lemma 7.3** The spectral norm of $\tilde{\Sigma}_n - \Sigma_n$ tends to 0 as $n$ tends to $\infty$.

Additionally, note that $\tilde{\mu} = \sqrt{a_n} \cdot \Sigma_{n-1}(g) \cdot \mu$ except for the first coordinate. Therefore, we expect Model (7.3) to be approximately equivalent to

$$
\tilde{X} = \sqrt{a_n} \cdot \Sigma_n(g_0) \cdot \mu + \tilde{Z} \quad \text{where} \quad \tilde{Z} \sim N(0, \Sigma_n(f_\alpha)).
$$

This is a special case of the cluster model we considered in Section 6 with $f = f_\alpha$ and $g = g_0$, except that the signal strength has been re-scaled by $\sqrt{a_n}$. Therefore, if we calibrate the nonzero entries in $\mu$ as

$$
a_n^{-1/2} \cdot A_n = a_n^{-1/2} \cdot \sqrt{2r \log n},
$$

(7.4)
then the detection boundary for the model is succinctly characterized by

\[ r = \frac{1}{C(f_\alpha, g_0)} \cdot \rho^*(\beta), \quad C(f_\alpha, g_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{g_0(\theta)}{f_\alpha(\theta)} \right|^2 d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 - \cos(\theta) f_\alpha(\theta) d\theta. \]

See Figure 4 for the display of \( C(f_\alpha, g_0) \). The following theorem is proved in Section 10.

**Theorem 7.1** Let \( 0 < \alpha_0 \leq \alpha < \frac{1}{2}, \beta \in (\frac{1}{2}, 1), \) and \( r \in (0, 1) \). Assume \( X \) is generated according to Model (7.1), with signal strength re-scaled as in (7.4). When \( C(f_\alpha, g_0) \cdot r < \rho^*(\beta) \), the null and alternative hypotheses merge asymptotically, and the sum of Type I and Type II errors of any test converges to 1 as \( n \) diverges to \( \infty \). When \( C(f_\alpha, g_0) \cdot r > \rho^*(\beta) \), if we apply the iHC to \( X \) with bandwidth \( b_n = \log n \) and reject the null when \( iHC_n^*(b_n, \Sigma_n) \geq (\log n)^2 \), then the Type I error converges to zero, and its power converges to 1.

![Figure 5: Sum of Type I and Type II errors as described in experiment (a).](image)

8 Simulation study

We conducted a small-scale empirical study to compare the performance of iHC and standard HC. For iHC, we investigate two choices of bandwidth: \( b_n = 1 \) and \( b_n = \log n \). In this section, we denote standard HC, iHC with \( b_n = 1 \), and iHC with \( b_n = \log n \) by HC, HC-a, and HC-b correspondingly.

The algorithm for generating data included the following four steps: (1). Fix \( n, \beta, \) and \( r \), let \( m = n^{1-\beta} \) and \( A_n = \sqrt{2r \log n} \). (2). Given a correlation matrix \( \Sigma_n \), generate a Gaussian vector \( Z \sim N(0, \Sigma_n) \). (3). Randomly draw \( m \) integers \( \ell_1 < \ell_2 < \ldots < \ell_m \) from \( \{1, 2, \ldots, n\} \) without replacement, and let \( \mu \) be the \( n \)-vector such that \( \mu_j = A_n \) if \( j \in \{\ell_1, \ell_2, \ldots, \ell_m\} \) and 0 otherwise. (4). Let \( X = \mu + Z \). Using data generated in this manner we explored three parameter settings, (a)–(c), which we now describe.
In experiment (a), we took $\Sigma_n(\rho)$ as the tri-diagonal Toeplitz matrix generated by $f(\theta) = 1 + 2\rho \cos(\theta)$, $|\rho| < 1/2$. The corresponding detection boundary was $r = \rho^*({\beta})/C(f)$ with $C(f) = (2\pi)^{-1} \int_{-\pi}^{\pi} \frac{1}{1-2\rho \cos(\theta)} d\theta$. Consider all $\rho$ that range from $-0.45$ to $0.45$ with an increment of $0.05$, and four pairs of parameters $({\beta}, r) = (0.5, 0.2), (0.5, 0.25), (0.55, 0.2), (0.55, 0.25)$. (Note that the corresponding parameters $(m, A_n)$ are $(32, 1.66), (32, 2.63), (22, 1.66), (22, 2.63)$). For each triple $({\beta}, r, \rho)$, we generated data according to (1)–(4), applied HC, HC-a, and HC-b to both $Z$ and $X$, and repeated the whole process independently 100 times. As a result, for each triple $({\beta}, r, \rho)$ and each procedure, we got 100 HC scores that corresponded to the null hypothesis, and 100 HC scores that corresponded to the alternative hypothesis.

We report the results in two different ways. First, we report the minimum sum of Type I and Type II errors (i.e. the minimum of the sum across all possible cut-off values); see Figure 5. Second, we pick the upper 10% percentile of the 100 HC scores corresponding to the null hypothesis as a threshold (for later references, we call this threshold the empirical threshold), and calculate the empirical power of the test (i.e. the fraction of HC scores corresponding to the alternative hypothesis that exceeds the threshold). The empirical thresholds are displayed in Table 1 (to save space, only part of the thresholds are reported), and the power is displayed in Figure 6. Recall that in Theorem 4.2 we recommend $(\log n)^2$ as a cut-off point in the asymptotic setting. For moderately large $n$, this cut-off point is conservative, and we recommend the empirical threshold instead.

The results suggest that (1). iHC-b outperforms iHC-a, and iHC-a outperforms HC. (2). As $|\rho|$ increases (note that a larger $|\rho|$ means a stronger correlation), the detection problem is increasingly easier, and the advantage of iHC is increasingly prominent. (3). Under the null hypothesis, the HC-b scores are usually smaller than those of HC and HC-a. This is mainly due to the normalization term $\sqrt{2n_{\rho_1}-1}$ in the definition of iHC (see (4.8)).

In experiment (b), we took $\Sigma_n$ to be the Toeplitz matrix generated by $f(\theta) = 1 + \frac{1}{2} \cos(\theta) + 2\rho \cos(2\theta)$, where $\rho$ ranged from $-0.2$ to $0.45$ with an increment of $0.05$. (the
Table 1: Display of empirical thresholds in experiment (a) for different $\rho$.

| $\rho$  | -0.45 | -0.35 | -0.25 | -0.15 | -0.05 | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 |
|---------|-------|-------|-------|-------|-------|------|------|------|------|------|
| HC      | 3.078 | 2.681 | 2.849 | 2.843 | 2.577 | 2.613| 2.968| 2.659| 3.078| 3.072|
| HC-a    | 2.637 | 2.889 | 2.759 | 2.806 | 2.689 | 2.657| 3.083| 2.788| 2.679| 2.670|
| HC-b    | 0.973 | 0.810 | 0.771 | 0.805 | 0.716 | 0.752| 0.817| 0.764| 0.819| 0.938|

In experiment (c), we investigated the behavior of HC-a/HC-b/HC for larger $n$. We took $(\beta,r) = (0.5,0.25)$, $n = 500 \times (1,2,3,4,5)$, and $\Sigma_n$ as the tri-diagonal matrix in experiment (a) with $\rho = 0.4$. The sum of Type I and Type II errors is reported in Table 2. The results suggest that the performance of HC-a/HC-b/HC improve when $n$ gets larger. (Investigation of the case where $n$ was much larger than 2500 needed much greater computer memory, and so we omitted it.)

Table 2: Display of the sum of Type I and Type II errors in experiment (c) for different $n$.

| $n$   | 500  | 1000 | 1500 | 2000 | 2500 |
|-------|------|------|------|------|------|
| HC    | 0.130| 0.150| 0.090| 0.115| 0.085|
| HC-a  | 0.040| 0.030| 0.015| 0.025| 0.015|
| HC-b  | 0.025| 0.010| 0.005| 0.005| 0.0  |
9 Discussion

We have extended standard HC to innovated HC by building in the correlation structure. The extreme diagonal entries of $\Sigma_n^{-1}$ play a key role in the testing problem. If the extreme value has finite upper and lower limits, $\bar{\gamma}_0$ and $\underline{\gamma}_0$, then in the $\beta$-$r$ plane, the detection boundary is bounded by the curves $r = \gamma_0^{-1} \cdot \rho^*(\beta)$ from above and $r = \bar{\gamma}_0^{-1} \cdot \rho^*(\beta)$ from below. When the correlation matrix is Toeplitz, the upper and lower limits merge and equal the Wiener interpolation rate $C(f)$. The detection boundary is therefore $r = (C(f))^{-1} \cdot \rho^*(\beta)$. The detection boundary partitions the $\beta$-$r$ plane into a detectable region and an undetectable region. Innovated HC has asymptotically full power for detection whenever $(\beta, r)$ falls into the interior of the detectable region, and is therefore optimally adaptive.

9.1 Connection to recent literature

The work complements that of Donoho and Jin [14] and Hall and Jin [21]. The focus of [14] is standard HC and its performance in the uncorrelated case. The focus of [21] is how strong dependence may harm the effectiveness of standard HC; what could be a remedy was however not explored. The innovated HC proposed in the current paper is optimal for both the model in [14] and that in [21].

The work is related to that of Jager and Wellner [26], where the authors proposed a family of goodness-of-fit statistics for detecting sparse normal mixtures. The work is also related to that of Meinshausen and Rice [32] and of Cai, Jin and Low [7], where the authors focused on how to estimate $\epsilon_n$—the proportion of non-null effects.

Recently, HC was also found to be useful for feature selection in high dimensional classification. See Donoho and Jin [15, 16] and Hall et al. [20]. The work concerned the situation where there are relatively few samples containing a very large number of features, out of which only a small fraction is useful, and each useful feature contributes weakly to the classification problem. In a related setting, Delaigle and Hall [13] investigated HC for classification when the data is nonGaussian or dependent.

9.2 Future work

The work is also intimately connected to recent literature on estimating covariance matrices. While the study is focused more on situations where the correlation matrices can be estimated using other approaches (e.g. [9, 17, 18]), it can be generalized to cases where the correlation matrix is unknown but can be estimated from data. In particular, it is noteworthy that it was shown in Bickel and Levina [4] that when the correlation matrix has polynomial off-diagonal decay, the matrix and its inverse can be estimated accurately in terms of the spectral norm. In such situations we expect the proposed approach to perform well once we combine it with that in [4].

Another interesting direction is to explore cases where the correlation matrix does not have polynomial off-diagonal decay, but is sparse in an unspecified pattern. This is a more challenging situation as relatively little is known about the inverse of the correlation matrix.

Our study also opens opportunities for improving other recent procedures. Take the aforementioned work on classification [15, 16, 20] for example. The approach derived in this paper suggests ways of incorporating correlation structure into feature selection, and therefore raises hopes for better classifiers. For reasons of space, we leave explorations along these directions to future study.


10 Proofs of main results

In this section we prove all theorems in preceding sections, except Theorems 2.1 and 5.1. These two theorems are the direct result of Theorems 3.2 and 4.2, so we omit the proofs. For simplicity, we drop the subscript $n$ whenever there is no confusion.

10.1 Proof of Theorem 3.1

Rewrite the second model in (3.1) as $X + \xi = \mu + \xi + Z$, where independently, $Z \sim N(0, \Sigma)$, $\xi \sim N(0, \Delta)$, $\mu \sim G$ for $\Delta = \Sigma_n - \Sigma$ and some distribution $G$. It suffices to show the monotonicity in the Hellinger affinity. Denote the density function of $N(0, \Sigma)$ by $f(x) = f(x_1, x_2, \ldots, x_n)$, and write $dx_1dx_2\ldots dx_n$ as $dx$ for short. Then the Hellinger affinity corresponding to the second model in (3.1) is

$$h(\Sigma, \Delta, G) \equiv \int \sqrt{(E \Delta E_G f(x - \mu - \xi))(E \Delta f(x - \xi))} dx.$$  

By Hölder’s inequality and Fubini’s theorem, $h(\Sigma, \Delta, G)$ is not less than

$$\int [E \Delta \sqrt{E_G f(x - \mu - \xi)} f(x - \xi)] dx = E \Delta \left[ \int \sqrt{E_G f(x - \mu - \xi)} f(x - \xi) dx \right].$$

Note that $\int \sqrt{E_G f(x - \mu - \xi)} f(x - \xi) dx \equiv \int \sqrt{E_G f(x - \mu)} f(x) dx$ for any fixed $\xi$. It follows that

$$h(\Sigma, \Delta, G) \geq E \Delta \left[ \int \sqrt{E_G f(x - \mu - \xi)} f(x - \xi) dx \right] = \int \sqrt{E_G f(x - \mu)} f(x) dx,$$

where the last term is the Hellinger affinity corresponds to the first model of (3.1). Combining these results gives the claim. \hfill \Box

10.2 Proof of Theorem 3.2

It is sufficient to show that the Hellinger distance between the joint density of $X$ and $Z$ converges to 0 as $n$ diverges to $\infty$. By the assumption $\tilde{\gamma}_0 r < \rho^*(\beta)$, we can choose a sufficiently small constant $\delta = \delta(r, \beta, \gamma_0)$ such that $\gamma_0 (1 - \delta)^{-2} r < \rho^*(\beta)$. Let $\tilde{\mu} = \mu / \sqrt{1 - \delta}$, let $U$ be the inverse of the Cholesky factorization of $\Sigma$, and let $\tilde{U}$ be the banded version of $U$:

$$\tilde{U}(i, j) = \begin{cases} U(i, j), & |i - j| \leq \log^2(n), \\ 0, & \text{otherwise}. \end{cases}$$

Model (2.1) can be equivalently written as

$$X = \tilde{\mu} + Z \quad \text{where} \quad Z \sim N(0, (1 - \delta)^{-1} \cdot \Sigma). \quad (10.1)$$

The key to the proof is to compare Model (10.1) with the following model:

$$X = \tilde{\mu} + Z \quad \text{where} \quad Z \sim N(0, (\tilde{U}'\tilde{U})^{-1}). \quad (10.2)$$

In fact, by Theorem 3.1 to establish the claim it suffices to prove that (i) $\tilde{U}'\tilde{U} \leq (1 - \delta)^{-1} \Sigma$ for sufficiently large $n$, and (ii) the Hellinger distance between the joint density of $X$ and that of $Z$ associated with Model (10.2) tends to 0 as $n$ diverges to $\infty$. \hfill \Box
Lemma 10.1 Fix \( n \) coordinates are no less than \( 3(\log n) \). In view of the definitions of \( W \) and \( \Theta_n^0(\lambda, c_0, M) \), the \( \ell^1 \)-norm of \( W \) is no greater than \( (\log n)^{-2(\lambda - 1)} \). Therefore, \( \| \tilde{U}^T \tilde{U} - U^T U \| \leq C \| W \| \leq C(\log n)^{-2(\lambda - 1)} \). This, and the fact that all eigenvalues of \( \tilde{U}^T \tilde{U} \) are bounded from below by a positive constant, imply the claim.

We now consider the second claim. Model (10.2) can be equivalently written as

\[
\nu \sim \nu_d \text{ defined, we note that } \nu \text{ is a sparse vector where with probability converging to 1, the inter-distances of nonzero entries of equal strength (1 - \( \delta \))}.
\]

\( \text{Proof of Theorem 4.1} \)

Put \( Y = U_n(\Sigma_n)X, \nu = U_n(\Sigma_n)\mu, \) and \( Z = U_n(\Sigma_n)z \). Model (4.1) reduces to

\[
Y = \nu + Z, \quad Z \sim N(0, I_n).
\]

(10.3)

Recalling that \( HC_n^* \sqrt{2 \log \log n} \rightarrow 1 \) in probability under \( H_0 \), it follows that \( P_{H_0}\{\text{Reject } H_0\} \) tends to 0 as \( n \) diverges to \( \infty \), and it suffices to show \( P_{H_1(\nu)}\{\text{Accept } H_0\} \rightarrow 0 \).

The key to the proof is to compare Model (10.3) with

\[
Y^* = \nu^* + Z \quad \text{where} \quad Z \sim N(0, I_n),
\]

(10.4)

with \( \nu^* \) having \( m \) nonzero entries of equal strength \( (1 - \delta_n)A_n \) whose locations are randomly drawn from \( \{1, 2, \ldots, n\} \) without replacement. By (10.2)–(10.3) and the way \( \tilde{\Theta}_n^0(\delta_n, b_n) \) is defined, we note that \( \nu_j \geq (1 - \delta_n)A_n \) for all \( j \in \{\ell_1, \ell_2, \ldots, \ell_m\} \). Therefore, signals in \( \nu \) is both denser and stronger than those in \( \nu^* \).

(10.5)

Intuitively, standard HC applied to Model (10.3) is no “less” than that applied to Model (10.4). We now establish this point. Let \( F_0(t) \) be the survival function of the central \( \chi_1^2 \) distribution \( \chi_1^2(0) \), and let \( \tilde{F}_n(t) \) and \( \tilde{F}_n^* \) be the empirical survival function of \( \{Y_k^2\}_{k=1}^n \) and \( \{Y_k^2\}_{k=1}^n \), respectively. Using arguments similar to those of Donoho and Jin [14], it can be shown that standard HC applied to Models (10.3) and (10.4), denoted by \( HC_n^{(1)} \) and \( HC_n^{(2)} \) for short, can be rewritten as

\[
HC_n^{(1)} = \sup_{\{t: 1/n \leq F_0(t) \leq 1/2\}} \left\{ \frac{\sqrt{n}(\tilde{F}_n(t) - F_0(t))}{\sqrt{F_0(t)F_0(t)}} \right\},
\]

(10.6)

\[
HC_n^{(2)} = \sup_{\{t: 1/n \leq F_0(t) \leq 1/2\}} \left\{ \frac{\sqrt{n}(\tilde{F}_n^*(t) - F_0(t))}{\sqrt{F_0(t)F_0(t)}} \right\},
\]

(10.7)
respectively. The key fact is now that the family of non-central \( \chi^2 \)-distribution \( \{ \chi^2(\delta), \delta \geq 0 \} \) is a monotone likelihood ratio family (MLR), i.e. for any fixed \( x \) and \( \delta_2 \geq \delta_1 \geq 0 \), \( P\{\chi^2(\delta_2) \geq x\} \geq P\{\chi^2(\delta_1) \geq x\} \). Consequently, it follows from (10.5) and mathematical induction that for any \( x \) and \( t \), \( P\{\bar{F}_n(t) \geq x\} \geq P\{\bar{F}_n(t) \geq x\} \). Therefore, for any fixed \( x > 0 \),

\[
P\{HC_n^{(1)} < x\} \leq P\{HC_n^{(2)} < x\}. \tag{10.6}
\]

Finally, by an argument similar to that of Donoho and Jin [14, Section 5.1], the second term in (10.6) with \( x = 1.01\sqrt{2\log\log n} \) tends to 0 as \( n \) diverges to \( \infty \). This implies the claim. \( \square \)

### 10.4 Proof of Theorem 4.2

In view of Lemma 4.2, it suffices to show that \( P_{H_0}\{\text{Accept } H_0\} \rightarrow 0 \). Put \( \bar{U} = \bar{U}(b_n) \), \( V = V_n(b_n) \), \( Y = VX \), \( \nu = V\mu \), \( \bar{Z} = VZ \). Model (4.7) reduces to

\[
Y = \nu + \bar{Z} \quad \text{where} \quad \bar{Z} \sim N(0, \bar{U}\bar{U}'). \tag{10.7}
\]

Let \( \bar{F}_n(t) \) and \( \bar{F}_0(t) \) be the empirical survival function of \( \{Y^2_k\}_{k=1}^n \) and the survival function of \( \chi^2(0) \), respectively. Let \( q = q(\beta, r) = \min\{((\beta + \gamma_0r)^2/(4\gamma_0r), 4\gamma_0r^2) \} \) and set \( t_n^* = \sqrt{2q\log n} \). Since \( \gamma_0r < \rho^*(\beta) \), then it can be shown that \( 0 < q < 1 \) and \( n^{-1} \leq \bar{F}_0(t_n^*) \leq 1/2 \) for sufficiently large \( n \). Using an argument similar to that in the proof of Theorem 4.1

\[
iHC_n^* = \sup_{\{s: 1/n \leq \bar{F}_0(s) \leq 1/2\}} \frac{\sqrt{n}(\bar{F}_n(s) - \bar{F}_0(s))}{\sqrt{2bn - 1}F_0(s)(1 - F_0(s))} \geq \frac{\sqrt{n}(\bar{F}_n(t_n^*) - \bar{F}_0(t_n^*))}{\sqrt{2bn - 1}F_0(t_n^*)(1 - F_0(t_n^*))},
\]

and it follows that

\[
P\{iHC_n^* \leq \log^{3/2}(n)\} \leq P\left\{ \frac{\sqrt{n}(\bar{F}_n(t_n^*) - \bar{F}_0(t_n^*))}{\sqrt{2bn - 1}F_0(t_n^*)(1 - F_0(t_n^*))} \leq \log^{3/2}(n) \right\}. \tag{10.8}
\]

It remains to show that the right hand side of (10.8) is algebraically small. The proof needs detailed calculation which we summarize in the lemma below, the proof of which is given in Section 11.

**Lemma 10.2** Under the condition of Theorem 4.2, the right hand side of (10.8) tends to 0 algebraically fast as \( n \) diverges to \( \infty \).

### 10.5 Proof of Theorem 6.1

Inspection of the proof of Theorems 3.2 and 4.2 reveals that the condition that \( \Sigma_n \) is a correlation matrix and that \( \Sigma_n \in \Theta_n(\lambda, c_0, M) \) in those theorems can be relaxed. In particular, \( \Sigma_n \) need not have equal diagonal entries and the decay condition on \( \Sigma_n \) can be replaced by a weaker condition that concerns the decay of \( U_n \) (the inverse of the Cholesky factorization of \( \Sigma_n \)), specifically

\[
|U_n(i, j)| \leq M(1 + |i - j|^\lambda)^{-1}.
\]

Let \( U_n(f) \) be the inverse of the Cholesky factorization of \( \Sigma_n(f) \), and define \( \bar{U}_n = U_n(f)\Sigma_n(g) \). Since \( \Sigma_n(g) \) is a lower triangular matrix with positive diagonal entries, then it is seen that \( \bar{U}_n \) is the inverse of the Cholesky factorization of \( \Sigma_n \). By Lemma 3.2, \( U_n(f) \)
has polynomial off-diagonal decay with the parameter $\lambda$. It follows that $\tilde{U}_n$ decays at the same rate. Applying Theorems \ref{thm:univariate} and \ref{thm:multivariate}, we see that all that remains to prove is that
\begin{equation}
\max_{\sqrt{n} \leq k \leq n - \sqrt{n}} \{ |\tilde{\Sigma}^{-1}_n(k, k) - C(f, g)| \} \rightarrow 0.
\end{equation}

By \cite[Theorem 2.15]{stein1999}, for any $\sqrt{n} \leq k \leq n - \sqrt{n}$, $k - K \leq j \leq k + K$, and $1 \leq \lambda' < \lambda$,
\begin{equation}
|\Sigma^{-1}_n(f)(k, j) - (\Sigma_n(1/f))(k, j)| = o(n^{-(1-\lambda')/2}).
\end{equation}

Since $\tilde{\Sigma}^{-1}_n = \Sigma_n(g) \cdot \Sigma^{-1}_n(f) \cdot \Sigma_n(g)$, it follows that $\sup_{\sqrt{n} \leq k \leq n - \sqrt{n}} |\tilde{\Sigma}^{-1}_n(k, k) - (\Sigma_n(g) \cdot \Sigma_n(1/f) \cdot \Sigma_n(g))(k, k)| \rightarrow 0$. Moreover, direct calculations show that $(\Sigma_n(g) \cdot \Sigma_n(1/f) \cdot \Sigma_n(g))(k, k) = C(f, g)$, $\sqrt{n} \leq k \leq n - \sqrt{n}$. Combining these results gives (10.9) and concludes the proofs. \hfill \Box

\section{Proof of Theorem 7.1}

Consider the first claim. It suffices to show that the Hellinger distance between $\tilde{X}$ and $\tilde{Z}$ in Model (7.3) tends to 0 as $n$ diverges to $\infty$. Since $C(f, g) \cdot r < \rho^*(\beta)$, there is a small constant $\delta > 0$ such that $(1 - \delta)^{-1} \cdot C(f, g) \cdot r < \rho^*(\beta)$. Using Lemma 7.2, we see that $\Sigma_{n-1}(f, g)$ is a positive matrix the smallest eigenvalue of which is bounded away from 0. It follows from Lemma 7.3 and basic algebra that $\Sigma \geq (1 - \delta)\Sigma_n$ for sufficiently large $n$.

Compare Model (7.3) with
\begin{equation}
X^* = \tilde{\mu} + Z^* \quad \text{where} \quad Z^* \sim N(0, (1 - \delta)\Sigma).
\end{equation}

By the monotonicity of Hellinger distance (Theorem 3.1), it suffices to show that the Hellinger distance between $X^*$ and $Z^*$ tends to 0 as $n$ diverges to $\infty$.

Now, by the definition of $\tilde{\mu}$, $\tilde{\mu} - \sqrt{\alpha_n} \cdot \Sigma_n(g_0) \cdot \mu = (\mu_n, \sqrt{\alpha_n} \cdot \mu_n, 0, \ldots, 0)'$. Since $P\{\mu_n \neq 0\} = o(1)$ then, except for an event with negligible probability, $\tilde{\mu} = \mu$. Therefore, replacing $\tilde{\mu}$ by $\sqrt{\alpha_n} \cdot \Sigma_n(g_0) \cdot \mu$ in Model (10.10) alters the Hellinger distance only negligibly. Note that the first coordinate of $X^*$ is uncorrelated with all other coordinates, and its mean equals 0 with probability converging to 1, so removing it from the model only has a negligible effect on the Hellinger distance. Combining these properties, Model (10.10) reduces to the following with only a negligible difference in the Hellinger distance:
\begin{equation}
X^*(2 : n) = \Sigma_{n-1}(g_0)(\sqrt{\alpha_n} \cdot \mu(2 : n)) + Z^*(2 : n), \quad Z^*(2 : n) \sim N(0, (1 - \delta)\Sigma_{n-1}(f_0)),
\end{equation}
where $X(2 : n)$ denotes the vector $X$ with the first entry removed. Dividing both sides by $\sqrt{1 - \delta}$, this reduces to the following model:
\begin{equation}
\tilde{X}(2 : n) = \Sigma_{n-1}(g_0)(\frac{\sqrt{\alpha_n} \cdot \mu(2 : n)}{\sqrt{1 - \delta}}) + \tilde{Z}(2 : n), \quad \tilde{Z}(2 : n) \sim N(0, \Sigma_{n-1}(f_0)),
\end{equation}
which is in fact Model (6.1) considered in Section 6. It follows from (7.4) that $\sqrt{\alpha_n} \cdot \mu(2 : n)/\sqrt{1 - \delta}$ has $m$ nonzero coordinates each of which equals $\sqrt{2(1 - \delta)^{-1} \cdot r \log n}$. Comparing Model (10.11) with Model (6.1) and recalling that $(1 - \delta)^{-1} \cdot r \cdot C(f, g_0) < \rho^*(\beta)$, the claim follows from Theorem 6.1.

Consider the second claim. Since $C(f, g_0) \cdot r > \rho^*(\beta)$, then there is a small constant $\delta > 0$ such that $(1 - \delta) \cdot r \cdot C(f, g_0) > \rho^*(\beta)$. Let $U_n$ be the inverse of the Cholesky
factorization of $\Sigma_n$, and let $\bar{U}_n(b_n)$ and $V_n(b_n)$ be as defined right below (4.5). Write Model (7.1) equivalently as

$$VX = V\mu + VZ \quad \text{where } VZ \sim N(0, \bar{U}'(b_n)\bar{U}(b_n)).$$

Recall that $\bar{U}'(b_n)\bar{U}(b_n)$ is a banded correlation matrix with bandwidth $2b_n - 1$. Let $\ell_1, \ell_2, \ldots, \ell_m$ be the $m$ locations of nonzero means of $\mu$. By an argument similar to that in the proof of Theorem 4.2 all remains to show is that, except for an event with negligible probability,

$$(V\mu)_k \geq \sqrt{2\rho^* \log n} \quad \text{for some constant } r' > \rho^*(\beta) \text{ and all } k \in \{\ell_1, \ell_2, \ldots, \ell_k\}. \quad (10.12)$$

We now show (10.12). First, by Lemma 4.1 and (7.4), except for an event with negligible probability,

$$(V\mu)_k \geq (1 - \delta)^{1/4} \cdot (a_n \cdot \Sigma_n(k, k))^{-1/2} \cdot A_n, \quad k \in \{\ell_1, \ell_2, \ldots, \ell_m\}.$$ 

Second, by the way $\tilde{\Sigma}_n$ is defined,

$$(a_n \Sigma_n^{-1})(k, k) = (\Sigma_n(g_0) \cdot \tilde{\Sigma}_n^{-1} \cdot \Sigma_n(\bar{g}_0))(k, k), \quad \text{for all } k \geq 2,$$

and by the way $\tilde{\Sigma}_n$ is defined and Lemma 7.3 for sufficiently large $n$,

$$\tilde{\Sigma}_n^{-1} \geq (1 - \delta)^{-1/2} \tilde{\Sigma}_n^{-1}, \quad \text{and so } \Sigma_n(g_0) \tilde{\Sigma}_n^{-1} \Sigma_n(\bar{g}_0) \geq (1 - \delta)^{1/2} \Sigma_n(g) \tilde{\Sigma}_n^{-1} \Sigma_n(\bar{g}).$$

Last, by [5] Theorem 2.15, $|\Sigma_n(g_0) \cdot \tilde{\Sigma}_n^{-1} \cdot \Sigma_n(\bar{g}_0))(k, k) - C(f_\alpha, g_0)| = O(1)$ when $\min\{k, n - k\}$ is sufficiently large. Combining these results gives (10.12) with $r' = (1 - \delta) \cdot r \cdot C(f_\alpha, g_0)$, and the claim follows directly.

11 Appendix

This section contains proofs for all lemmas in preceding sections, except Lemmas 3.1, 5.1, 7.1, and 7.2. Lemma 3.1 is the direct result of [55] and Lemma 5.1 is the direct result of [5] Theorem 2.15, so we omit the proofs. Lemmas 7.1 and 7.2 are proved in Section 12.

11.1 Proof of Lemma 3.2

Consider the first claim. Construct an infinite matrix $\Sigma_\infty$ by arranging the finite matrices along the diagonal, and note that the inverses of $\Sigma_\infty$ is the matrix formed by arranging the inverse of the finite matrices along the diagonal. Since $\Sigma_\infty(i, j) \leq M(1 + |i - j|^\lambda)^{-1}$, then applying Lemma 3.1 gives the claim.

Consider the second claim. It suffices to show that $|U_n(k, j)| \leq C/(1 + |k - j|^\lambda)$ for all $1 \leq j < k \leq n$. Denote the first $k \times k$ main diagonal sub-matrix of $\Sigma_n$ by $\Sigma_k$, the $k$-th row of $\Sigma_k$ by $(\xi_{k-1}^{r}, 1)$, and the $k$-th row of $U_n$ by $u_k$. It follows from direct calculations that

$$u_k = (1 - \xi_{k-1}^{r} \Sigma_{k-1}^{-1} \xi_{k-1}^{r})^{-1/2} \cdot (\xi_{k-1}^{r} \Sigma_{k-1}^{-1}, 1). \quad (11.1)$$

At the same time, by (11.1) and basic algebra,

$$\left(1 - \xi_{k-1}^{r} \Sigma_{k-1}^{-1} \xi_{k-1}^{r}\right)^{-1} \leq u_k u_k = \Sigma_k^{-1}(k, k). \quad (11.2)$$

23
Combining (11.1) and (11.2) gives
\[ |U_n(k,j)| = |u_k(j)| \leq C|\Sigma_{k-1}^{-1}\xi_{k-1})_j|, \quad 1 \leq j \leq k - 1. \] (11.3)

Now, by Lemma 3.1 \[ |\Sigma_{k-1}^{-1}(j,s)| \leq C(1 + |j - s|^{-\lambda})^{-1} \] for all \( 1 \leq i, j \leq k - 1 \). Note that \( |\xi_{k-1}(s)| \leq C(1 + |s - k|^{-\lambda})^{-1}, \) \( 1 \leq s \leq n \), and \( \lambda > 1 \). It follows from basic algebra that
\[ |(\Sigma_{k-1}^{-1}\xi_{k-1})_j| \leq \sum_{s=1}^{n} \frac{C}{(1 + |j - s|^{-\lambda})(1 + |s - k|^{-\lambda})} \leq \frac{C}{1 + |k - j|^{-\lambda}}. \] (11.4)

Inserting (11.4) into (11.3) gives the claim. □

11.2 Proof of Lemma 4.1
Without loss of generality, assume \( \ell_1 < \ell_2 < \ldots < \ell_m \). By Lemma 11.2, except for an event with negligible probability, \( \ell_1 \geq b_n, \ell_m \leq n - b_n \), and the inter-distances of the \( \ell_j \)'s \( \geq C \log n \cdot n^{-\beta - 1} \). For any \( k \in \{\ell_1, \ell_2, \ldots, \ell_m\} \), let \( d_k = (\sum_{j=k}^{k+b_n-1} u_{jk}^2)^{-1/2} \). By the way \( \bar{U}(b_n) \) is defined,
\[ (\bar{U}'(b_n)U\mu)_k = d_k \sum_{s, j=1}^{n} \bar{u}_{ks}u_{sj}\mu_j = d_k \left[ \sum_{s,j=1}^{n} u_{ks}u_{sj}\mu_j - \sum_{s,j=1}^{n} (u_{ks} - \bar{u}_{ks})u_{sj}\mu_j \right]. \] (11.5)

Consider \( d_k \) first. Write
\[ 1/d_k^2 = \sum_{j=k}^{k+b_n-1} u_{jk}^2 = \sum_{j=k}^{n} u_{jk}^2 - \sum_{j=k-b_n}^{n} u_{jk}^2. \]

First, \( U'U = \Sigma^{-1}, \) \( \sum_{j=k}^{n} u_{jk}^2 = (U'U)(k,k) = (\Sigma^{-1})(k,k) \). Second, by the polynomial off-diagonal decay of \( U \) and basic calculus,
\[ \sum_{j=k+b_n}^{n} u_{jk}^2 \leq C \sum_{j=k+b_n}^{n} \frac{1}{1 + |j - k|^{-\lambda}} \leq Cb_n^{1-2\lambda}. \]

Last, note that the quantities \( \Sigma^{-1}(k,k) \) are uniformly bounded away from 0 and \( \infty \). Combining these results gives
\[ |d_k - \sqrt{\Sigma^{-1}(k,k)}| \leq Cb_n^{1-2\lambda}. \] (11.6)

Consider \( \sum_{s,j=1}^{n} u_{ks}u_{sj}\mu_j \) next. Recall that \( \mu_j = A_n \) when \( j \in \{\ell_1, \ell_2, \ldots, \ell_m\} \) and \( \mu_j = 0 \) otherwise. Since \( U'U = \Sigma^{-1}, \)
\[ \sum_{s,j=1}^{n} u_{ks}u_{sj}\mu_j = \sum_{j=1}^{n} (\Sigma^{-1})(k,j)\mu_j = A_n \Sigma^{-1}(k,k) + A_n \sum_{\ell_j \neq k} \Sigma^{-1}(k,\ell_j). \]

Define \( L_n = n^{\beta - 1/2} \). By Lemma 11.2, except for an event with negligible probability, the inter-distance of \( \ell_j \) is no less than \( L_n \). So by the polynomial off-diagonal decay of \( \Sigma^{-1}, \) the second term is algebraically small. Therefore,
\[ \sum_{s,j=1}^{n} u_{ks}u_{sj}\mu_j = A_n[(\Sigma^{-1})(k,k) + o(b_n^{1-\lambda})]. \] (11.7)
Last, we consider $\sum_{s,j=1}^n (u_{ks} - \tilde{u}_{ks})u_{sj}\mu_j$. Direct calculations show that

$$\left|((U - \tilde{U})'U)(k, j)\right| \leq \begin{cases} \frac{C}{1+|k-j|^{1+\delta}}, & \lambda > 1, \\ \frac{C\log n}{1+|k-j|^{\delta}}, & \lambda = 1, \end{cases}$$

so by a similar argument,

$$\left|\sum_{s,j=1}^n (u_{ks} - \tilde{u}_{ks})u_{sj}\mu_j\right| = \left|\sum_{j=1}^n ((U - \tilde{U})'U)(k, j)\mu_j\right| \leq A_n \cdot ((U - \tilde{U})'U)(k, k) + o(1),$$

where $o(1)$ is algebraically small. Moreover, by the Cauchy-Schwartz inequality,

$$((U - \tilde{U})'U)(k, k) \leq \sum_{s=1}^n \left|(u_{ks} - \tilde{u}_{ks})u_{sk}\right| \leq b_n^{1/2-\lambda},$$

and the claim follows. \(\Box\)

### 11.3 Proof of Lemma 4.2

Without loss of generality, suppose $n$ is divisible by $2b_n - 1$, and let $N = N(n, b_n) = n/(2b_n - 1)$. Let $p_i$ be $N$ iid samples from $U(0, 1)$, and $F_N(t)$ be the empirical cdf. The normalized uniform stochastic process is defined as

$$\mathbb{W}_N(t) = \sqrt{N} [F_N(t) - t]/\sqrt{t(1-t)}.$$ 

The following lemma is proved in Section 11.3.1.

**Lemma 11.1** There is a generic constant $C > 0$ such that for sufficiently large $n$,

$$P\left\{ \sup_{1/n \leq t \leq 1/2} |\mathbb{W}_N(t)| \geq C(\log n)^{3/2} \right\} \leq Cn^{-C}.$$

We now prove Lemma 4.2. Define $Y = \tilde{U}'UX$. Under the null hypothesis, $Y \sim N(0, \tilde{U}'\tilde{U})$ and the coordinates $Y_k$ are block-wise dependent with a bandwidth $\leq 2b_n - 1$. Split the $Y_k$’s into $2b_n - 1$ different subsets $\Omega_j = \{Y_k : k \equiv j \mod (2b_n - 1)\}$, $1 \leq j \leq 2b_n - 1$. Note that the $Y_k$’s in each subset are independent, and that $|\Omega_j| = N$, $1 \leq j \leq 2b_n - 1$.

Let $\bar{F}_n(t)$ and $\bar{F}_0(t)$ be as in the proof of Theorem 4.1 and let

$$\bar{F}_{n,j} = \frac{2b_n - 1}{n} \sum_{k=1}^n 1\{y_{k}^2 \geq t, Y_k \in \Omega_j\}, \quad 1 \leq j \leq 2b_n - 1.$$ 

Note that $\bar{F}_n(t) = \frac{1}{2b_n - 1} \sum_{j=1}^{2b_n - 1} \bar{F}_{n,j}(t)$. By arguments similar to that of Donoho and Jin [14], and basic algebra, it follows that

$$iHC_n^* = \sup_t \left\{ \frac{\sqrt{n}(\bar{F}_n(t) - \bar{F}_0(t))}{\sqrt{(2b_n - 1)\bar{F}_0(t)}} \right\} \leq \sum_{j=1}^{2b_n - 1} \sup_t \left\{ \frac{\sqrt{N}(\bar{F}_{n,j}(t) - \bar{F}_0(t))}{\sqrt{\bar{F}_0(t)\bar{F}_0(t)}} \right\},$$

and so for any $x > 0$,

$$P\{iHC_n^* \geq x\} \leq \sum_{j=1}^{2b_n - 1} P\left\{ \sup_t \left\{ \frac{\sqrt{N}(\bar{F}_{n,j}(t) - \bar{F}_0(t))}{\sqrt{\bar{F}_0(t)\bar{F}_0(t)}} \right\} \geq x \right\}. \quad \text{(25)}$$
Finally, since that $\tilde{F}_{n,j}$’s are the empirical survival functions of $N$ independent samples from $\chi^2_1(0)$, then

$$
\sup_{1/n \leq \tilde{F}_0(t) \leq 1/2} \left\{ \frac{\sqrt{N}(\tilde{F}_{n,j}(t) - \tilde{F}_0(t))}{\sqrt{\tilde{F}_0(t)F_0(t)}} \right\} = \sup_{\{1/n \leq t \leq 1/2\}} \{\bar{W}_N(t)\} \text{ in distribution.}
$$

Therefore,

$$
P\{iHC^*_n \geq x\} \leq (2b_n - 1)P\sup_{\{1/n \leq t \leq 1/2\}} \{\bar{W}_N(t)\} \geq x\}
$$

Taking $x = C(\log n)^{3/2}$, the claim follows from Lemma $11.1$. \hfill \Box

### 11.3.1 Proof of Lemma $11.1$

By the Hungarian construction [10], there is a Brownian bridge $B(t)$ such that

$$
P\left\{ \sup_{\{1/n \leq t \leq 1/2\}} \left| \sqrt{N}(F_N(t) - t) - B(t) \right| \geq \frac{C(\log N + x)}{\sqrt{N}} \right\} \leq Ce^{-Cx},
$$

where $C > 0$ are generic constants. Noting that $1/\sqrt{t(1-t)} \leq \sqrt{n} \leq C\sqrt{N\log N}$ when $1/n \leq t \leq 1/2$, it follows that

$$
P\left\{ \sup_{\{1/n \leq t \leq 1/2\}} \left| \frac{\sqrt{N}(F_N(t) - t) - B(t)}{\sqrt{t(1-t)}} \right| \geq C(\log N)^{1/2}(\log N + x) \right\} \leq Ce^{-Cx}. \quad (11.8)
$$

At the same time, by [33, Page 446],

$$
P\left\{ \sup_{\{1/n \leq t \leq 1/2\}} \left| \frac{B(t)}{\sqrt{t(1-t)}} \right| \geq C(\log N)^{1/2} \right\} \leq C\log N \cdot e^{-Cx}. \quad (11.9)
$$

Combining $(11.8)$, $(11.9)$, taking $x = C \log N$ and using triangle inequality, gives the claim. \hfill \Box

### 11.4 Proof of Lemma $7.3$

By direct calculations and the way $\hat{\Sigma}$ is defined, we have

$$
\hat{\Sigma} = \begin{pmatrix} \Sigma^* & \xi_{n-1} \\ \xi_{n-1} & 1 \end{pmatrix}, \quad (11.10)
$$

where

$$
\xi_{n-1} = \sqrt{2n^{-\alpha}} \times (0, \ldots, n^{\alpha} - k_0(n)_{\alpha}, k_0(n)_{\alpha} - (k_0(n) - 1)_{\alpha}, \ldots, 2^\alpha - 1, 1) \quad (11.11)
$$

and $\Sigma^*$ is a symmetric matrix with unit diagonal entries, and with the following on the $k$-th sub-diagonal:

$$
\begin{align*}
&\begin{cases} 
2k^\alpha - (k + 1)^\alpha - (k - 1)^\alpha, & k \leq k_0(n) - 1, \\
1 + ((k - 1)^\alpha - 2k^\alpha)/n^{\alpha_0} = O(n^{-\alpha_0/\alpha}), & k = k_0(n), \\
-(1 - (k - 1)^\alpha/n^{\alpha_0}) = O(n^{-\alpha_0/\alpha}), & k = k_0(n) + 1, \\
0, & k \geq k_0(n) + 2.
\end{cases}
\end{align*}
$$
Note that $\Sigma_{n-1}(g_0)$ and $\Sigma^*$ share the same $2k_0(n)-1$ sub-diagonals that are closest to the main diagonal (including the main diagonal). Let $H_1$ be the matrix containing all other sub-diagonals of $\Sigma_{n-1}(g_0)$, and let $H_2$ be the matrix which contains the $k_0(n)$-th and the $(k_0(n)+1)$-th diagonals (upper and lower) of $\Sigma^*$. It is seen that

$$\tilde{\Sigma} - \Sigma = \begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} H_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \xi'_{n-1} \\ \xi'_{n-1} & 0 \end{pmatrix} \equiv B_1 + B_2 + B_3.$$

Let $\| \cdot \|_1$ and $\| \cdot \|_2$ denote the $\ell^1$ matrix norm and the $\ell^2$ matrix norm, respectively. First, by direct calculations, since $\alpha < 1/2$, $\| B_1 + B_2 \|_1 \leq C\alpha_0^{(\alpha-1)/\alpha} \leq C\alpha^{-\alpha_0}$.

Since the spectral norm is no greater than the $\ell^1$-matrix norm and the $\ell^2$-matrix norm, the spectral norm of $B_1 + B_2 + B_3$ is no greater than $C\alpha^{-\alpha_0/2}$, and the claim follows.

### 11.5 Proof of Lemma 10.1

Let $a = \sqrt{(1-\delta)/\tilde{\gamma}_0}$, $r' = \tilde{\gamma}_0(1-\delta)^{-1}r$, $U_1 = a\bar{U}$, and $\bar{\mu} = \frac{1}{a}\tilde{\mu}$. Model (10.2) can be equivalently written as

$$X = \bar{U}\bar{\mu} + Z = U_1\bar{\mu} + Z \quad \text{where} \quad Z \sim N(0, I_n). \quad (11.12)$$

Using the argument in the first paragraph of the proof of Theorem 3.2, it is not hard to verify that (I) $\bar{\mu}$ has $m = n^{1-\beta}$ nonzero entries each of which is equal to $\sqrt{2r' \log n}$ with $r' < \rho^*(\beta)$, and whose locations are randomly sampled from $(1,2,\ldots,n)$; (II) $U_1$, where $U_1(k,j) = 0$ if $|k-j| > (\log n)^2$, is a banded lower triangular matrix; and (III) $\lim_{n \to \infty} \max\{\sqrt{n} \leq k \leq n-\sqrt{n}\} \{U_1\} = (1-\delta) < 1$.

From now on, write $\mu = \bar{\mu}$ and $r = r'$ for short. Note that the Hellinger affinity associated with Model (10.2) is $E_0(\sqrt{W_n})$, where $E_0$ denotes the law $Z \sim N(0, I_n)$, and

$$W_n^* = W_n^*(r, \beta; Z_1, Z_2, \ldots, Z_n) = \left(\frac{n}{m}\right)^{-1} \sum_{\ell = (\ell_1, \ell_2, \ldots, \ell_m)} e^{r'U'_{1}Z-\|U_1\mu\|^2/2}. $$

Introduce the set of indices

$$S_n = \{\ell = (\ell_1, \ell_2, \ldots, \ell_m), \quad \min_{1 \leq j \leq m-1} |\ell_{j+1} - \ell_j| \geq 3(\log n)^2, \ell_1 \geq \sqrt{n}, n-\ell_m \geq \sqrt{n}\}. \quad (11.13)$$

The following lemma is proved in Section 11.5.1

**Lemma 11.2** Let $\ell_1 < \ell_2 < \ldots < \ell_n$ be $m$ distinct indices randomly sampled from $(1,2,\ldots,n)$ without replacement. Then for any $1 \leq K \leq n$, (a) $P\{\ell_1 \leq K\} \leq K m/n$, (b) $P\{\ell_m \geq n-K\} \leq K m/n$, and (c) $P\{\min_{1 \leq i \leq m-1} |\ell_{i+1} - \ell_i| \leq K\} \leq K m(m+1)/n$.

As a result, $P\{\ell = (\ell_1, \ell_2, \ldots, \ell_n) \notin S_n\} = O\{(\log n)^2 n^{1-2\beta}\} = o(1)$.

Applying Lemma 11.2, we make only a negligible difference by restricting $\ell$ to $S_n$ and defining

$$W_n = \frac{1}{m} \sum_{\ell = (\ell_1, \ell_2, \ldots, \ell_m) \in S_n} e^{r'U'_{1}Z-\|U_1\mu\|^2/2}. \quad (11.14)$$

27
in which case
\[ E(W_n^{1/2}) = E(W_n^{1/2} + o(1)). \] (11.15)
Define \( Y = U'Z \),
\[ \sigma_j^2 = \text{var}(Y_j) = (U_1^2)(j,j), \quad 1 \leq j \leq n, \] (11.16)
and the event
\[ D_n = \{ Y_j/\sigma_j \leq \sqrt{2\log n}, \ 1 \leq j \leq n \}. \]
By direct calculation, \( P\{D_n\} = o(1) \), and so by Hölder's inequality, \( E(W_n^{1/2} 1_{\{D_n\}}) = E(W_n^{1/2}) + o(1) \). Combining this result and (11.15) we deduce that \( E(W_n^{1/2}) = E(W_n^{1/2} 1_{\{D_n\}}) + o(1) \), and comparing this property with the desired result we see that it is sufficient to show that
\[ E(W_n^{1/2} 1_{\{D_n\}}) = 1 + o(1). \] (11.17)

The key to (11.17) is the following lemma, which is proved in Section 11.5.2.

**Lemma 11.3** Consider Model (11.12), where \( U_1 \) and \( \mu \) satisfy (I)-(III). As \( n \to \infty \), \( E(W_n 1_{\{D_n\}}) = 1 + o(1) \), and \( E(W_n^2 1_{\{D_n\}}) = 1 + o(1) \).

Since
\[ |W_n^{1/2} 1_{\{D_n\}} - 1| \leq \frac{|W_n 1_{\{D_n\}} - 1|}{1 + W_n^{1/2} 1_{\{D_n\}}} \leq |W_n 1_{\{D_n\}} - 1|, \]
then by Hölder’s inequality,
\[ (E|W_n^{1/2} 1_{\{D_n\}} - 1|)^2 \leq |W_n 1_{\{D_n\}} - 1|^2 = E(W_n^2 1_{\{D_n\}}) - 2E(W_n 1_{\{D_n\}}) + 1. \] (11.18)
Combining (11.18) with Lemma 11.3 gives (11.17).

**11.5.1 Proof of Lemma 11.2**

The last claim follows once (a)–(c) are proved. Consider (a)–(b) first. Fixing \( K \geq 1 \),
\[ P\{\ell_1 = K\} = \binom{n-K}{m-1} \binom{n}{m} \leq m/n, \]
so \( P\{\ell_1 \leq K\} \leq Km/n \). Similarly, \( P\{n - \ell_m \leq K\} \leq Km/n \). This gives (a) and (b).

Next we prove (c). Denote the minimum inter-distance of \( \ell_1, \ell_2, \ldots, \ell_m \) by
\[ L(\ell) = L(\ell; m, n) = \min\{1 \leq i < m - 1\} \{\ell_{i+1} - \ell_i\}. \]
Note that
\[ P\{L(\ell) = K\} \leq \sum_{j=1}^{m-1} \sum_{k=1}^{n} P\{\ell_j - \ell_{j+1} = k + K, k \leq j \leq m\}. \]
Writing \( P\{\ell_j = k, \ell_{j+1} = k + K\} = \binom{n}{m}^{-1} \binom{k}{j-1} \binom{n-k}{m-j-1} \), we have:
\[ P\{L(\ell) = K\} \leq \frac{1}{\binom{n}{m}} \sum_{j=1}^{m-1} \sum_{k=1}^{n} \binom{k-1}{j-1} \binom{n-k-1}{m-j-1} = \frac{1}{\binom{n}{m}} \sum_{k=1}^{n} \sum_{j=1}^{k} \binom{k-1}{j-1} \binom{n-k}{m-j-1}, \]
where the last term is no greater than
\[ \frac{1}{\binom{n}{m}} \sum_{k=1}^{n} \binom{n-K-1}{m-2} \leq \frac{n}{\binom{n-2}{m}} \binom{n}{m-2} \leq m^2/n, \]
and the claim follows.
11.5.2 Proof of Lemma [11.3]

Define $T_n = \sqrt{2 \log n}$. We need the following lemma, which is proved in Section 12.

**Lemma 11.4** Consider a bivariate zero mean normal variable $(X, Y)'$ that satisfies $\text{Var}(X) = \sigma_X^2$, $\text{Var}(Y) = \sigma_Y^2$, and $\text{corr}(X, Y) = \rho$, where $c_0 \leq \sigma_1, \sigma_2 \leq 1$ for some constant $c_0 \in (0, 1)$. Then there is a constant $C > 0$ such that for sufficiently large $n$,

$$E[\exp(A_nX - \sigma_X^2A_n^2/2) \cdot 1\{Y/\sigma_Y > T_n\}] \leq C \cdot n^{-(1-e^{-\sqrt{r}})^2} \leq Cn^{-(1-\sqrt{r})^2},$$

$$E[\exp(A_n(X + Y) - \frac{\sigma_X^2 + \sigma_Y^2}{2}A_n^2) \cdot 1\{X/\sigma_X, Y/\sigma_Y \leq T_n\}] \leq Cn^{-d(r)},$$

where $d(r) = \min\{2r, 1 - 2(1 - \sqrt{r})^2\}.$

We also need the following definition.

**Definition 11.1** We say that two indices $j$ and $k$ are near to each other if $|j - k| \leq (\log n)^2$.

We now proceed to show Lemma [11.3]. Consider the first claim. Note that for any $\ell = (\ell_1, \ell_2, \ldots, \ell_m) \in S_n$, the minimum inter-distance of $\ell_i$ is no less than $3(\log n)^2$. In view of the definition of $Y_j$ and $\sigma_j$ (see [11.16]), we have

$$\|U_1\mu\ell\|^2 = A_n^2 \sum_{i=1}^m (U_i'U_1)(\ell_i, \ell_i) = A_n^2 \sum_{i=1}^m \sigma_{\ell_i}^2.$$ 

Consequently, we can rewrite $W_n$ as

$$W_n = \frac{1}{\binom{m}{n}} \sum_{\ell = (\ell_1, \ell_2, \ldots, \ell_m) \in S_n} \exp \left( A_n \sum_{i=1}^m Y_{\ell_i} - \frac{A_n^2}{2} \sum_{i=1}^m \sigma_{\ell_i}^2 \right). \quad (11.19)$$

Note that

$$1\{D_n^\ell\} \leq \sum_{j=1}^n 1\{Y_j/\sigma_j > T_n\}. \quad (11.20)$$

Combining (11.19) and (11.20) gives

$$E[W_n \cdot 1\{D_n^\ell\}] \leq \frac{1}{\binom{m}{n}} \sum_{\ell = (\ell_1, \ldots, \ell_m) \in S_n} \sum_{k=1}^n E \left[ \exp \left( A_n \sum_{j=1}^m Y_{\ell_j} - \frac{A_n^2}{2} \sum_{j=1}^m \sigma_{\ell_j}^2 \right) \cdot 1\{Y_k/\sigma_k > T_n\} \right]. \quad (11.21)$$

Now, for each $1 \leq k \leq n$, when $k$ is near one $\ell_j$, say $\ell_{j_0}, Y_k$ must be independent of all other $Y_{\ell_j}$ with $j \neq j_0$. It follows that

$$E \left[ \exp \left( A_n \sum_{j=1}^m Y_{\ell_j} - \frac{A_n^2}{2} \sum_{j=1}^m \sigma_{\ell_j}^2 \right) \cdot 1\{Y_k/\sigma_k > T_n\} \right] = E[\exp(A_nY_{\ell_{j_0}} - \sigma_{j_0}^2A_n^2/2) \cdot 1\{Y_k/\sigma_k > T_n\}].$$

By Lemma 11.4 the right hand side is no greater than $Cn^{-(1-\sqrt{r})^2}$. Therefore,

$$E \left[ \exp \left( A_n \sum_{j=1}^m Y_{\ell_j} - \frac{A_n^2}{2} \sum_{j=1}^m \sigma_{\ell_j}^2 \right) \cdot 1\{Y_k/\sigma_k > T_n\} \right] \leq Cn^{-(1-\sqrt{r})^2}. \quad (11.22)$$
Moreover, for each fixed \( \ell = (\ell_1, \ldots, \ell_m) \in S_n \), there are at most \( 2m(\log n)^2 \) different indices \( k \) that can be near some of the \( \ell_j \)'s; and when they are, they can be near only one such \( \ell_j \). Combining these results gives

\[
E[W_m \cdot 1_{\{D_m\}}] \leq \frac{1}{m} \sum_{\ell=(\ell_1, \ldots, \ell_m) \in S_n} C(\log n)^2 mn^{-(1-\sqrt{\beta})^2} \leq C(\log n)^2 n(1-\beta)^{-(1-\sqrt{\beta})^2}.
\]

By the definition of \( \rho^*(\beta) \) and the assumption of the lemma, \( r < \rho^*(\beta) \leq (1 - \sqrt{1-\beta})^2 \), and so the first claim follows directly from (11.23).

We now consider the second claim. Fix \( 0 \leq N \leq m \), and let \( \tilde{S}_N(\ell) \) denote the set of all \( k = (k_1, k_2, \ldots, k_m) \in S_n \) such that there are exactly \( N \) \( k_j \)'s that are near to one \( \ell_i \). (Clearly, any \( k_j \) can be near to at most one \( \ell_i \).) The two sets of indices \( \ell \) and \( (k_1, k_2, \ldots, k_m) \) form exactly \( N \) pairs, each contains one candidate from the first set and one candidate from the second. These pairs are not near to each other, and not near to any remaining indices outside the pairs. Using (11.19), we write

\[
E[W_n^2 \cdot 1_{\{D_n\}}] = \left( \frac{n}{m} \right)^2 \sum_{\ell=(\ell_1, \ell_2, \ldots, \ell_m) \in S_n} \sum_{N=0}^{m} \sum_{k=(k_1, k_2, \ldots, k_m) \in \tilde{S}_N(\ell)} \exp \left( A_n \sum_{j=1}^{m} (Y_{\ell_j} + Y_{k_j}) - \frac{A_n^2}{2} \sum_{j=1}^{m} (\sigma_{\ell_j}^2 + \sigma_{k_j}^2) \right) \cdot 1_{\{D_n\}}.
\]

For any fixed \( \ell \) and \( k \in \tilde{S}_N(\ell) \), by symmetry, and without loss of generality, we suppose the \( N \) pairs are \( (\ell_1, k_1), (\ell_2, k_2), \ldots, (\ell_N, k_N) \). By independence of the pairs with other indices, and also by independence among the pairs,

\[
E \left[ \exp \left( A_n \sum_{j=1}^{m} (Y_{\ell_j} + Y_{k_j}) - \frac{A_n^2}{2} \sum_{j=1}^{m} (\sigma_{\ell_j}^2 + \sigma_{k_j}^2) \right) \cdot 1_{\{D_n\}} \right]
\]

\[
\leq E \left[ \exp \left( A_n \sum_{j=1}^{m} (Y_{\ell_j} + Y_{k_j}) - \frac{A_n^2}{2} \sum_{j=1}^{m} (\sigma_{\ell_j}^2 + \sigma_{k_j}^2) \right) \cdot 1_{\{Y_{\ell_j}/\sigma_{\ell_j} \leq T_n, Y_{k_j}/\sigma_{k_j} \leq T_n, \text{ for all } 1 \leq j \leq N\}} \right]
\]

\[
\leq E \left[ \exp \left( A_n \left\{ \sum_{j=1}^{N} (Y_{\ell_j} + Y_{k_j}) - \frac{A_n^2}{2} \sum_{j=1}^{N} (\sigma_{\ell_j}^2 + \sigma_{k_j}^2) \right\} \right) \cdot 1_{\{Y_{\ell_j}/\sigma_{\ell_j} \leq T_n, Y_{k_j}/\sigma_{k_j} \leq T_n, \text{ for all } 1 \leq j \leq N\}} \right]
\]

\[
= \prod_{j=1}^{N} \left[ E \left[ \exp \left( A_n (Y_{\ell_j} + Y_{k_j}) - \frac{A_n^2}{2} (\sigma_{\ell_j}^2 + \sigma_{k_j}^2) \right) \cdot 1_{\{Y_{\ell_j}/\sigma_{\ell_j} \leq T_n, Y_{k_j}/\sigma_{k_j} \leq T_n\}} \right] \right].
\]

Here, in the first inequality, we have used the fact that

\[
1_{\{D_n\}} \leq 1_{\{Y_{\ell_j}/\sigma_{\ell_j} \leq T_n, Y_{k_j}/\sigma_{k_j} \leq T_n, \text{ for all } 1 \leq j \leq N\}};
\]

in the second inequality, we have utilized the independence and the fact that

\[
E[\exp(A_n Y_j - \sigma_{\ell_j}^2 A_n^2/2)] = 1, \quad \text{for all } j = 1, \ldots, n;
\]

and in the third equality, we have used again the independence. Moreover, in view of the way \( U_1 \) is defined and Lemma 3.2, there is a constant \( c_0 \in (0, 1) \) such that \( \sigma_j \in [c_0, 1] \). Using Lemma 11.1 for sufficiently large \( n \) and each \( 1 \leq j \leq N \),

\[
E \left[ \exp \left( A_n (Y_{\ell_j} + Y_{k_j}) - \frac{A_n^2}{2} (\sigma_{\ell_j}^2 + \sigma_{k_j}^2) \right) \cdot 1_{\{Y_{\ell_j}/\sigma_{\ell_j} \leq T_n, Y_{k_j}/\sigma_{k_j} \leq T_n\}} \right] \leq Cn^{d(r)},
\]

\[30\]
with $d(r)$ being as in Lemma [11.4] Combining (11.25) and (11.26) gives

$$E[W_n^2 \cdot 1_{\{D_n\}}] \leq \binom{n}{m}^{-2} \sum_{\ell=(\ell_1, \ldots, \ell_m)} \sum_{N=0}^{m} (Cn^d(r))^N |\tilde{S}_N(\ell)|. \quad (11.27)$$

where $|\tilde{S}_N(\ell)|$ denote the cardinality of $\tilde{S}_N(\ell)$. By elementary combinatorics,

$$|\tilde{S}_N(\ell)| \leq \binom{m}{N} (2\log^2 n)^N \binom{n-N}{m-N} \leq (2\log^2 n)^N \binom{m}{N} \binom{n}{m-N}. \quad (11.28)$$

Direct calculations show that

$$\binom{m}{n} \frac{1}{\binom{n-m}{n-m-N}} = \frac{1}{N!} \left( \frac{m!}{(m-N)!} \right)^2 \frac{(n-m)!}{(n-m+N)!} \leq \frac{1}{N!} \left( \frac{m^2}{n} \right)^N. \quad (11.29)$$

Substituting (11.28)-(11.29) into (11.27) and recalling that $m = n^{1-\beta}$, we deduce that

$$E[W_n^2 \cdot 1_{\{D_n\}}] \leq \binom{n}{m}^{-1} \sum_{\ell=(\ell_1, \ell_2, \ldots, \ell_m)} \sum_{N=0}^{m} \frac{1}{N!} \left( \frac{m^2}{n} \right)^N \left( C\log^2 n n^d(r) \right)^N, \quad (11.30)$$

where the last term $\leq \sum_{N=0}^{\infty} \frac{1}{N!} \left( C\log^2 n n^{1+d(r)-2\beta} \right)^N$. By the assumption of the lemma,

$$r < \rho^*(\beta) = \begin{cases} \beta - 1/2, & 1/2 < \beta \leq 3/4, \\ (1 - \sqrt{1-\beta})^2, & 3/4 \leq \beta < 1, \end{cases}$$

thus it can be seen that $1 + d(r) - 2\beta < 0$ for all fixed $\beta$ and $r \in (0, \rho^*(\beta))$. Combining this with (11.30) gives the second claim. \qed

### 11.6 Proof of Lemma [10.2]

The key observation is that, there is a sequence of positive numbers $\delta_n$ that tends to 0 as $n$ diverges to $\infty$ such that $\nu_k \geq (1 - \delta_n)A_n$ for all $k \in \{\ell_1, \ell_2, \ldots, \ell_m\}$, so it is natural to compare Model [10.7] with the following model:

$$Y^* = \nu^* + Z, \quad Z \sim N(0, I_n), \quad (11.31)$$

where $\nu^*$ has $m$ nonzero entries of equal strength $(1 - \delta_n)A_n$ whose locations are randomly drawn from $\{1, 2, \ldots, n\}$ without replacement.

For short, write $t = t^*_n$ and

$$H_n(t) = \frac{\sqrt{n}(\tilde{F}_n(t) - F_0(t))}{\sqrt{(2b_n - 1)F_0(t)(1 - F_0(t))}}. \quad (11.32)$$

Let $\tilde{F}_n^*(t)$ be the empirical survival function of $\{Y_k^*\}_{k=1}^n$, and let $\bar{F}(t) = E[\tilde{F}_n(t)]$ and $\bar{F}^*(t) = E[\tilde{F}_n^*(t)]$. Recall that the family of non-central $\chi^2$-distributions has monotone likelihood ratio. Then $\bar{F}(t) \geq \bar{F}^*(t) \geq \bar{F}_0(t)$. Now, first, since the $Y_k$’s are block-wise dependent with a block size $\leq 2b_n - 1$, it follows by direct calculations that

$$\text{Var}(H_n(t)) \leq C \bar{F}(t)/\bar{F}_0(t). \quad (11.33)$$

Second, by $\bar{F}(t) \geq \bar{F}_n^*(t)$,
\[ E[H_n(t)] = \frac{\sqrt{n}(\tilde{F}(t) - \tilde{F}_0(t))}{\sqrt{(2b_n - 1)\tilde{F}_0(t) - 1 - F_0(t))}} \geq \frac{\sqrt{n}(\tilde{F}^*(t) - \tilde{F}_0(t))}{\sqrt{(2b_n - 1)\tilde{F}_0(t)}}, \quad (11.32) \]

where the right hand side diverges to \( \infty \) algebraically fast, by an argument similar to that in [14]. Combine Chebyshev’s inequality, the identity \( b_n = \log n \), and the calculations of the mean and variance of \( H_n(t) \),

\[ P\{H_n(t) \leq (\log n)^2\} \leq C(\log n) \frac{\tilde{F}(t)}{n(\tilde{F}(t) - F_0(t))^2}. \quad (11.33) \]

It remains to show that the last term in (11.33) is algebraically small. We discuss separately the cases \( \tilde{F}(t)/\tilde{F}_0(t) \geq 2 \) and \( \tilde{F}(t)/\tilde{F}_0(t) < 2 \). For the first case,

\[ \frac{\tilde{F}(t)}{n(\tilde{F}(t) - F_0(t))^2} \leq \frac{C}{nF_0(t)} \leq \frac{C}{nF_0(t)^2}, \]

which is algebraically small since \( t = \sqrt{2q \log n} \) and \( 0 < q < 1 \). For the second case,

\[ \frac{\tilde{F}(t)}{n(F(t) - F_0(t))^2} \leq \frac{CF_0(t)}{n(F(t) - F_0(t))^2} \leq \frac{CF_0(t)}{n(F^*(t) - F_0(t))^2}, \quad (11.34) \]

which is seen to be algebraically small by comparing it to the right hand side of (11.32). This concludes the claim. \( \square \)

12 Complementary technical details

12.1 Proof of Lemma [7.1]

Consider the first claim. Suppose such an autoregression structure exists for \( \alpha \geq \alpha_0 > 0 \). Let

\[ Y_k = \sqrt{\alpha} \cdot (X_{k+1} - X_k)/d, \quad a_n = n^{\alpha_0}/2, \quad k = 1, 2, \ldots, n - 1. \]

Clearly, \( \text{Var}(Y_k) = 1 \). At the same time, direct calculations show that the correlation between \( Y_1 \) and \( Y_{j+1} \) equals to \((j + 1)^\alpha + (j - 1)^\alpha - 2j^\alpha)/2 \) for all \( 1 \leq j \leq n - 2 \), which is no larger than 1. Taking \( j = 2 \) yields \((3^\alpha + 1 - 2 \cdot 2^\alpha)/2 \leq 1 \), and hence \( \alpha \leq 2 \).

Consider the second claim. For any \( k \geq 1 \), define the partial sum \( S_k(t) = 1 + 2 \sum_{j=1}^{k}(1 - \frac{j^\alpha}{n^{\alpha_0}})^+ \cos(kt) \). By a well-known result in trigonometric, to show the positive-definiteness of \( \Sigma_n \), it suffices to show that

\[ S_{k_0 + 1}(t) \geq 0 \text{ for all } t \in [-\pi, \pi] \quad \text{and} \quad S_{k_0 + 1}(t) > 0 \text{ except for a measure 0 set.} \quad (12.1) \]

Here, \( k_0 = k_0(n; \alpha, \alpha_0) \) is the largest integer \( k \) such that \( k^\alpha \leq n^{\alpha_0} \).

We now show (12.1). By that in [38] Page 183, if we let \( a_0 = 2 \), and \( a_j = 2(1 - \frac{j^\alpha}{n^{\alpha_0}})^+ \), \( 1 \leq j \leq n - 1 \), then \( S_{k_0 + 1}(t) = \sum_{j=0}^{k_0 - 1}(j + 1)\Delta^2 a_j K_j(t) + (k_0 + 1)K_{k_0}(t)\Delta a_{k_0} + D_n(t)a_{k_0 + 1} \).

Here, \( \Delta a_j = a_j - a_{j+1} \), \( \Delta^2 a_j = a_j + a_{j+2} - 2a_{j+1} \), and \( D_j(t) \) and \( K_j(t) \) are the Dirichlet’s kernel and the Fejér’s kernel, respectively,

\[ D_j(t) = \frac{\sin((j + \frac{1}{2})t)}{2 \sin(\frac{j}{2})}, \quad K_j(t) = \frac{2}{j+1} \left( \frac{\sin(j+1)t}{2 \sin(\frac{j}{2})} \right)^2, \quad j = 0, 1, \ldots. \quad (12.2) \]
In view of the definition of $k_0$, $a_{k_0+1} = (1 - (k_0+1)\alpha/n^0)^+ = 0$. Also, by the monotonicity of \{a_j\}, $\Delta a_{k_0} = a_{k_0} - a_{k_0+1} \geq 0$. Therefore, $S_{k_0+1}(t) \geq \sum_{j=0}^{k_0-1}(j+1)\Delta^2 a_j K_j(t)$.

We claim that the sequence \{\alpha a_1, \ldots, a_{n-1}\} is convex. In detail, since $\alpha \leq 1$, the sequence \{\alpha j\} is concave. As a result, the sequence \{(1 - \frac{\alpha}{j}\sigma)\} is convex, and so is the sequenced \{(1 - \frac{\alpha}{j}\sigma)^+\}. In view of the definition of $a_j$, the claim follows directly. The convexity of $a_j$ implies that $\Delta^2 a_j \geq 0$, $0 \leq j \leq n - 2$. Therefore, $S_{k_0+1}(t) \geq 0$. This proves the first part of (12.1).

We now prove the second part of (12.1). We discuss separately for two cases $\alpha < 1$ and $\alpha = 1$. In the first case, $\Delta a_0 = \frac{\alpha}{j}\sigma (2 - 2\alpha) > 0$ and $K_0(t) = \frac{1}{2}$. As a result, $S_{k_0+1}(t) \geq 2 - 2\alpha^0 > 0$, and the claim follows. In the second case, $\Delta a_j = \frac{1}{n^0}(2j - j - (j + 2)) = 0$, and $\Delta a_{k_0-1} = (1 - \frac{k_0-1}{n^0}) - 2(1 - \frac{k_0}{n^0}) = \frac{1}{n^0}(k_0 + 1) - 1 > 0$. Therefore, $S_{k_0+1}(t) \geq (k_0 + 1)(\frac{k_0+1}{n^0} - 1)K_0(t)$. Clearly, $S_{k_0+1}(t)$ can only assume 0 when $(\frac{k_0+1}{2})t$ is a multiple of $\pi$. Since the set of such $t$ has measure 0, the claim follows directly.

12.2 Proof of Lemma 7.2

Let $a_0 = 2$, and $a_k = 2k\alpha - (k+1)\alpha - (k-1)\alpha$, $1 \leq k \leq n - 1$. Clearly, $a_k > 0$ for all $k$, so $f_\alpha(0; \alpha) > 0$. Furthermore, when $\theta \neq 0$, by [38, Equation 1.7, Page 183],

$$f_\alpha(\theta) = \sum_{\nu=0}^{\infty} (\nu + 1)[a_{\nu+2} + a_\nu - 2a_{\nu+1}] a_\nu K_\nu(\theta),$$

(12.3)

where $K_\nu(\theta)$ is the Fejér’s kernel as in (12.2). By the positiveness of the Fejér’s kernel, all remains to show is that $a_{k+1} + a_{k-1} - 2a_k > 0$, for all $k \geq 2$.

Define $h(x) = (1 + 2x)^\alpha + (1 - 2x)^\alpha - 4(1 + x)^\alpha - 4(1 - x)^\alpha + 6$, $0 \leq x \leq 1/2$. By direct calculations, for all $k \geq 2$,

$$a_{k+1} + a_{k-1} - 2a_k = -k^\alpha[(1 + \frac{2}{k})^\alpha + (1 - \frac{2}{k})^\alpha - 4(1 + \frac{1}{k})^\alpha - 4(1 - \frac{1}{k})^\alpha + 6] = -k^\alpha h\left(\frac{1}{k}\right).$$

(12.4)

Also, by basic calculus, $h''(x) = 4\alpha(\alpha - 1)[(1+2x)^{\alpha-2} + (1-2x)^{\alpha-2} - (1+x)^{\alpha-2} - (1-x)^{\alpha-2}]$.

Since $0 < \alpha < 1$, $x^{\alpha-2}$ is a convex function. It follows that $h''(x) < 0$ for all $x \in (0, 1/2)$, and $h(x)$ is a strictly concave function. At the same time, note that $h(0) = h'(0) = 0$, so $h(x) < 0$ for $x \in (0, 1/2]$. Combining this with (12.4) gives the claim.

12.3 Proof of Lemma 11.4

Denote the density, cdf and survival function of $N(0, 1)$ by $\phi$, $\Phi$, and $\bar{\Phi}$. For the first claim, define $W = X/\sigma_1$ and $V = Y/\sigma_2$ if $\rho \geq 0$ and $V = -Y/\sigma_2$ otherwise. The proofs for two cases $\rho \geq 0$ and $\rho < 0$ are similar, so we only show the first one. In this case, it suffices to show

$$E[\exp(\sigma_1 A_n W - \sigma_1^2 A_n^2/2) \cdot 1_{\{V > T_n\}}] \leq C \cdot n^{-(1 - \rho^2/2)^2}.$$

Write $W = (W - \rho V) + \rho V$, and note that $(1 - \rho^2)^2 + \rho^2 \leq 1$. It is seen that

$$\sigma_1 A_n W - \sigma_1^2 A_n^2/2 \leq [\sigma_1 A_n (W - \rho V) - \sigma_1^2 (1 - \rho^2) A_n^2/2] + [\sigma_1 A_n \rho V - \sigma_1^2 \rho^2 A_n^2/2].$$

(12.5)

Since $W$ and $V$ have unit variance and a correlation $\rho$, then $(W - \rho V)$ is independent of $V$ and is distributed as $N(0, (1- \rho^2)^2)$. Therefore, $E[\exp(\sigma_1 A_n (W - \rho V) - \sigma_1^2 (1 - \rho^2) A_n^2/2)] = 1$. Combining this with (12.5) gives

$$E[\exp(\sigma_1 A_n W - \sigma_1^2 A_n^2/2) \cdot 1_{\{V > T_n\}}] = E[\exp(\sigma_1 \rho A_n V - \sigma_1^2 \rho^2 A_n^2/2) \cdot 1_{\{V > T_n\}}].$$
Now, by direct calculations,
\[ E\left[ \exp\left(A_n V - A_n^2/2\right) \cdot 1\{V > T_n\} \right] = \int_{T_n}^{\infty} \phi(x - \sigma_1 \rho A_n) dx = \Phi(T_n - \sigma_1 \rho A_n). \]

Since \( \Phi(x) \leq C \phi(x) \) if \( x > 0 \), \( \Phi(T_n - \sigma_1 \rho A_n) \leq C \phi(T_n - \sigma_1 \rho A_n) = C n^{-\left(1-\frac{\rho}{\sqrt{2}}\right)^2} \). Combining these results gives the claim.

We now show the second claim. By Hölder’s inequality, it suffices to show that
\[ E[\exp(2A_n X - \sigma_1^2 A_n^2) \cdot 1\{X \leq \sigma_1 T_n\}] \leq C n^{-d(r)}. \]

Recalling that \( W = X/\sigma_1 \), we have
\[ E[\exp(2A_n X - \sigma_1^2 A_n^2) \cdot 1\{X \leq \sigma_1 T_n\}] = E[\exp(2\sigma_1 A_n W - \sigma_1^2 A_n^2) \cdot 1\{W \leq T_n\}]. \]

By direct calculations,
\[ E[\exp(2\sigma_1 A_n W - \sigma_1^2 A_n^2) \cdot 1\{W \leq T_n\}] = e^{\sigma_1^2 A_n^2} \int_{-\infty}^{T_n} \phi(x - 2\sigma_1 A_n) dx = e^{\sigma_1^2 A_n^2} \Phi(T_n - 2\sigma_1 A_n). \]

Since \( \Phi(x) \leq C \phi(x) \) for all \( x < 0 \) and \( \Phi(x) \leq 1 \) for all \( x \geq 0 \),
\[ e^{\sigma_1^2 A_n^2} \Phi(T_n - 2\sigma_1 A_n) \leq \begin{cases} C e^{\sigma_1^2 A_n^2} = C n^{2\sigma_1^2 r}, & \sigma_1^2 r \leq 1/4, \\ e^{\sigma_1^2 A_n^2} \Phi(T_n - 2\sigma_1 A_n) = C n^{1-2(1-\sigma_1 \sqrt{2})^2}, & \sigma_1^2 r > 1/4; \end{cases} \]

in view of the definition of \( d(r) \), \( e^{\sigma_1^2 A_n^2} \Phi(T_n - 2\sigma_1 A_n) \leq C n^{d(\sigma_1^2 r)}. \) Since that \( \sigma_1 \leq 1 \) and that \( d(r) \) is a monotonically increasing function, we have \( d(\sigma_1^2 r) \leq d(r) \). Combining these results gives the claim. \( \square \)

Acknowledgement: JJ would like to thank Christopher Genovese and Larry Wasserman for extensive discussion. He would also like to thank Peter Bickel, Emmanuel Candés, David Donoho, Karlheinz Gröchenig, Michael Leinert, Joel Tropp, Aad van der Vaart, and Zepu Zhang for encouragement and pointers.

References

[1] Abramovich, F., Benjamini, Y., Donoho, D. and Johnstone, I. (2000). Adapting to unknown sparsity by controlling the false discovery rate. *Ann. Statist.* 34 584–653.

[2] Arias-Castro, E., Donoho, D. and Huo, X. (2005). Near-optimal detection of geometric objects by fast Multiscale methods. *IEEE Trans. on Info. Theory* 51(7) 2402–2425.

[3] Benjamini, Y. and Hochberg, Y. (1995). Controlling the false discovery rate: a practical and powerful approach to multiple testing. *J. Roy. Statist. Soc. Ser. B.* 57 289–300.

[4] Bickel, P. and Levina, E. (2007). Regularized estimation of large covariance matrices. *Ann. Statist.* 36(1) 199–227.

[5] Böttcher, A. and Silbarmann, B. (1998). *Introduction to Large Truncated Toeplitz Matrices*. Springer.
[6] Brockwell, P. and Davis, R. (1991). *Time Series and Methods*, 2nd edition. Springer.

[7] Cai, T., Jin, J. and Low, M. (2007). Estimation and confidence sets for sparse normal mixtures. *Ann. Statist.* **35** 2421-2449.

[8] Cayon, L., Jin, J. and Treaster, A. (2005). Higher Criticism statistic: detecting and identifying non-Gaussianity in the WMAP first year data. *Mon. Not. Roy. Astron. Soc.* **362** 826–832.

[9] Chen, L., Tong, T. and Zhao, H. (2005). Considering dependence among genes and markers for false discovery control in eQTL mapping. *Bioinformatics* **24**(18) 2015–2022.

[10] Csörgö, M., Csörgö, S., Horvath, L. and Mason, D. (1986). Weighted empirical and quantile processes. *Ann. Prob.* **14**(1) 31–85.

[11] Cover, T. M. and Thomas, J. A. (2006). *Elementary Information Theory*. Wiely & Sons.

[12] Cruz, M., Cayon, L., Martínez-González, E., Vieva, P. and Jin, J. (2007). The non-Gaussian cold spot in the 3-year WMAP data. *Astrophys. J.* **655** 11–20.

[13] Delaigle, A. and Hall, J. (2008). Higher Criticism in the context of unknown distribution, non-independence and classification. *Platinum Jubilee Proceedings of the Indian Statistical Institute*. To appear.

[14] Donoho, D. and Jin, J. (2004). Higher Criticism for detecting sparse heterogeneous mixtures. *Ann. Statist.* **32** 962–994.

[15] Donoho, D. and Jin, J. (2008a). Higher Criticism thresholding: optimal feature selection when useful features are rare and weak. *Proc. Nat. Acad. Sci.* **105**(39) 14790–14795.

[16] Donoho, D. and Jin, J. (2008b). Higher Criticism thresholding: optimal phase diagram. *arxiv.org/abs/0812.2263*.

[17] Goeman, J., van de Geer, S., de Kort, F. and van Houwelingen, H. (2004). A global test for groups of genes: testing association with a clinical outcome. *Bioinformatics* **20**(1) 93–99.

[18] Goeman, J., van de Geer, S. and van Houwelingen, H. (2006). Testing against a high dimensional alternative. *J. R. Statist. Soc. Ser. B.* **68**(3) 477–493.

[19] Gröchenig, K. and Leinert, M. (2006). Symmetry and inverse-closedness of matrix algebra and functional calculus for infinite matrices. *Trans. Amer. Math. Soc.* **358** 2695-2711.

[20] Hall, P., Pittelkow, Y. and Ghosh, M. (2008). Theoretical measures of relative performance of classifiers for high dimensional data with small sample sizes. *J. Roy. Statist. Soc. Ser. B.* **70** 159–173.

[21] Hall, P. and Jin, J. (2008). Properties of Higher Criticism under strong dependence. *Ann. Statist.* **36** 381–402.
[22] Horn, R.A. and Johnson, C.R. (2006). *Matrix Analysis*. Cambridge University Press.

[23] Ingster, Y.I. (1997). Some problems of hypothesis testing leading to infinitely divisible distribution. *Math. Methods Statist.* 6 47–69.

[24] Ingster, Y.I. (1999). Minimax detection of a signal for $l^p_n$-balls. *Math. Methods Statist.* 7 401–428.

[25] Jaffard, S. (1990). Propriétés des matrices “bien localisées” près de leur diagonale et quelques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 7 461–476.

[26] Jager, L. and Wellner, J. (2007). Goodness-of-fit tests via phi-divergences. *Ann. Statist.* 35 2018–2053.

[27] Jin, J. (2004). Detecting a target in very noisy data from multiple looks. In A *Festschrift to Honor Herman Rubin* (Dasgupta eds.), IMS Lecture Notes Monogr. Ser. 45 255–286. Inst. Math. Statist., Beachwood, OH.

[28] Jin, J. (2006). Higher Criticism statistic: theory and applications in non-Gaussian detection. In *Statistical Problems in Particle Physics, Astrophysics And Cosmology* (L. Lyons and M.K. Ünel. eds.). Imperial College Press, London.

[29] Jin, J. (2007). Proportion of nonzero normal means: universal oracle equivalences and uniformly consistent estimators. *J. Roy. Statist. Soc. Ser. B* 70(3) 461–493.

[30] Jin, J. and Cai, T. (2007). Estimating the null and the proportion of non-null effects in large-scale multiple comparisons. *J. Amer. Statist. Assoc.* 102 496–506.

[31] Jin, J., Starck, J.-L., Donoho, D., Aghanim, N. and Forni, O. (2005). Cosmological non-Gaussian signature detection: comparing performance of different statistical tests. *EURASIP J. Appl. Sig. Proc.* 15 2470–2485.

[32] Meinshausen, M. and Rice, J. (2006). Estimating the proportion of false null hypotheses among a large number of independently tested hypotheses. *Ann. Statist.* 34 373–393.

[33] Shroack, G. and Wellner, J. (1986). *Empirical Processes with Applications to Statistics*. John Wiley & Sons.

[34] Strasser, H. (1998). Differentiability of statistical experiments. *Statist. Decisions* 16 113–130.

[35] Sun, Q. (2005). Wiener’s lemma for infinite matrices with polynomial off-diagonal decay. *Comptes Rendus Mathematique* 340 567–570.

[36] Tukey, J.W. (1989). Higher Criticism for individual significances in several tables or parts of tables. *Internal working paper*, Princeton University.

[37] Wiener, N. (1949). *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*. Wiley, New York.

[38] Zygmund, A. (1959). *Trigonometric Series*, 2nd edition. Cambridge University Press.