EMBEDDING RIGHT-ANGLED ARTIN GROUPS INTO GRAPH BRAID GROUPS

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Abstract. We construct an embedding of any right-angled Artin group $G(\Delta)$ defined by a graph $\Delta$ into a graph braid group. The number of strands required for the braid group is equal to the chromatic number of $\Delta$. This construction yields an example of a hyperbolic surface subgroup embedded in a two strand planar graph braid group.

1. Introduction

Let $\Delta$ be a finite simple graph. We examine a group $A = G(\Delta)$, called the right-angled Artin group associated to $\Delta$, defined with the following presentation: for each vertex $a_i$ of $\Delta$, there exists a corresponding generator for $A$, and two generators $a_i$ and $a_j$ commute (for $i \neq j$) if and only if the vertices $a_i$ and $a_j$ are connected by an edge in $\Delta$. In particular, a right-angled Artin group is a group which has a presentation where the only relators are commutators of generators. Right-angled Artin groups are of interest as sources of subgroups with complicated homology and homotopy finiteness properties; see for example [19]. Right-angled Artin groups are well studied; for a more extensive reference, see for example [3], [7].

Given a graph $\Gamma$, the ordered configuration space of $n$ points on $\Gamma$, denoted $C^n\Gamma$, is the open subset of the direct product $\Pi^n\Gamma$ obtained by removing the diagonal, $\text{Diag} := \{(x_1, \ldots, x_n) \in \Pi^n\Gamma | x_i = x_j \text{ for some } i \neq j\}$. The fundamental group of the ordered configuration space $C^n\Gamma$ of $n$ points on some graph $\Gamma$, denoted $\text{PB}_n\Gamma$, is called a pure graph braid group. The unordered configuration space of $n$ points on $\Gamma$, denoted $UC^n\Gamma$, is the quotient of $C^n\Gamma$ by the action of the symmetric group permuting the factors. Thus, $\text{UC}^n\Gamma := \{\{x_1, \ldots, x_n\} | x_i \in \Gamma, x_i = x_j \text{ for some } i \neq j\}$, where we use braces to indicate that the coordinates are unordered. The fundamental group $\pi_1\text{UC}^n\Gamma$ of $n$ strands on $\Gamma$ is called a graph braid group, denoted $B_n\Gamma$.

Graph braid groups are of interest because of their connections with robotics and mechanical engineering. Graph braid groups can, for instance, model the motions of robots moving about a factory floor ([14], [9], [10]), or the motions of microscopic balls of liquid on a nano-scale electronic circuit ([13]). For more information about graph braid groups see for instance [1], [12], [2].

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In [14], Ghrist conjectured that every graph braid group is a right-angled Artin group. In [1], Abrams was able to find counterexamples to this conjecture: namely, $PB_2K_5$ and $PB_2K_{3,3}$ are surface groups. Connolly and Doig [5] proved a partial positive result, that braid groups on linear trees are right-angled Artin, where a tree is linear if there exists an embedded interval containing all of the vertices of degree at least 3. In general, though, it appears that graph braid groups are usually not right-angled Artin groups. Mautner reported via private communication that he has some examples of graph braid groups on planar graphs with nontrivial fundamental group which are not right-angled Artin [17]. More recently, in [11] it is proved that tree braid groups are right-angled Artin if and only if the tree is linear or there are less than 4 strands.

A related line of inquiry asks whether there exist embeddings between right-angled Artin groups and graph braid groups. A positive result in this direction is due to Crisp and Wiest: for any finite graph $\Gamma$ and any $n$, there exists a graph $\Delta$ such that $B_n \Gamma$ embeds in $G(\Delta)$.

The goal of this paper is to prove the opposite direction of Crisp and Wiest’s theorem:

**Theorem 1.1.** For every finite graph $\Delta$ and any coloring $C$ of $\Delta$ with $n$ colors, there exists a graph $\Gamma$ such that the right-angled Artin group $G(\Delta)$ embeds into the graph braid group $B_n \Gamma$.

For example, letting $C$ be the trivial coloring assigning to each vertex of $\Delta$ a different color, we obtain an embedding of $G(\Delta)$ into $B_n \Gamma$ for some graph $\Gamma$, where $n$ is the number of vertices of $\Delta$. At the other extreme, $n$ can be as small as the chromatic number of $\Delta$, by letting $C$ be a coloring realizing the chromatic number.

In Section 3, we use Theorem 1.1 to obtain an explicit embedding of the right-angled Artin group associated to the cycle of length 6, $C_6$, into a two strand planar graph braid group. By a result of Servatius, Droms, and Servatius [18], the group $G(C_6)$ itself contains an embedded hyperbolic surface subgroup (where a hyperbolic surface group is the fundamental group of a compact hyperbolic surface without boundary). Thus, we prove as a corollary the following theorem:

**Theorem 1.2.** There exists a planar graph braid group which contains a hyperbolic surface subgroup, with only $n = 2$ strands.

There were previously no known examples of hyperbolic surface subgroups in planar graph braid groups.

The proof presented here of Proposition 3.1 was discovered by Dan Farley and is a refinement of the original proof of the author’s. The current construction of the graph $\Gamma$ in Section 2 was inadvertently suggested by Bert Weist and is a slight improvement over the original construction. REU student Go Fujita recognized (also inadvertently) that the number of strands used in Proposition 3.1 corresponds to a coloring of $\Delta$. 
The author would like to thank Dan Farley for numerous helpful discussions on this and related material, as well as his advisor, Ilya Kapovich. The author is also thankful for the (accidental) helpful observations of Bert Wiest and Go Fujita made while trying to understand an earlier proof.

2. Preliminaries

A coloring $C$ of a finite simple graph $\Delta$ with $n$ colors is a function from the vertices $V\Delta$ of $\Delta$ to the finite set $S = \{1, \ldots, n\}$ such that if two vertices $v$ and $w$ in $\Delta$ are neighbors then $C(v) \neq C(w)$. The elements of $S$ are called colors. The color of a vertex is its image under $C$. The chromatic number of $\Delta$ is the smallest number of colors needed to have a coloring of $\Delta$.

An edge loop in a CW-complex is a closed path consisting of a sequence of 1-cells in the space.

**Definition 2.1** (Halo). Let $\Delta$ be a finite simple graph and let $C$ be a coloring of $\Delta$ with $n$ colors $\{1, \ldots, n\}$. A connected graph $\Gamma$ is called a halo of $\Delta$, or a $\Delta$-halo, if for each vertex $a_i$ in $\Delta$ (equivalently, for each generator in $A$) there exists a simple edge loop $\gamma_i$ in $\Gamma$, called an Artin loop of $\Gamma$, such that:

- for each color $c \in \{1, \ldots, n\}$, there exists a vertex $x_c \in \Gamma$ common to all Artin loops of vertices colored $c$ and to no other Artin loops.
- if $a_i$ and $a_j$ are not connected by an edge in $\Delta$, then the Artin loops $\gamma_i$ and $\gamma_j$ intersect in exactly one vertex $v$. If $C(a_i) = C(a_j)$ then $v = x_{C(a_i)}$; otherwise, $v$ is on no other Artin loop.
- if $a_i$ and $a_j$ are connected by an edge in $\Delta$ (i.e. if $a_i$ and $a_j$ commute), then the Artin loops $\gamma_i$ and $\gamma_j$ are disjoint.

We call the point $\{x_1, \ldots, x_n\} \in UC^n\Gamma$ the Artin basepoint of $\Gamma$.

Although there is no clear canonical choice for a halo of an arbitrary $\Delta$, when a particular $\Delta$-halo $\Gamma$ is specified we write $\Gamma = \Gamma(\Delta)$. That $\Delta$-halos exist is clear; see for example Figure 1.

**Convention 2.2.** For any Artin basepoint of $\Gamma = \Gamma(\Delta)$, arbitrarily fix a direction in which to traverse each $\gamma_i$. Let $c$ be the color of the vertex $a_i$. We call the loop in $UD^n\Gamma$ from $\{x_1, \ldots, x_n\}$ corresponding to the strand at $x_c$ making a single traversal of the Artin loop $\gamma_i$ in the chosen direction also by the name of the Artin loop $\gamma_i$.

The notion of a $\Delta$-halo is similar in flavor to the proof of Theorem 10 of [6], though the notions were discovered independently.

Recall that the ordered configuration space $C^n\Gamma$ is defined to be a subset of the direct product $\Pi^n\Gamma$ of $n$ copies of $\Gamma$, where we subtract off the diagonal $Diag := \{(x_1, \ldots, x_n)|x_i = x_j$ for some $i \neq j\}$. The unordered configuration space $UC^n\Gamma$ is then the quotient of this space by the action of $S_n$ permuting the factors. As $\Gamma$ is a graph, it is also a 1-dimensional CW-complex, where the interiors of edges are the open 1-cells. This means
that there is a product CW-complex structure on $\Pi^n\Gamma$, where an open $k$-cell consists of an ordered collection of exactly $k$ edges and $n - k$ vertices of $\Gamma$. Let $Diag'$ be the set of all open cells whose closure intersects $Diag$. Thus

$$Diag' = \{(y_1, \ldots, y_n) \mid y_i \text{ is a cell in } \Gamma, \text{ and } y_i \cap y_j \neq \emptyset \text{ for some } i \neq j\}.$$ 

Let $D^n\Gamma$ be the CW-complex $\Pi^n\Gamma - Diag'$, the discretized configuration space of $n$ strands on $\Gamma$, and let $UD^n\Gamma$ be the quotient of $D^n\Gamma$ given by permuting the factors of $\Pi^n\Gamma$ via the action of $S_n$, the unlabelled discretized configuration space of $n$ strands on $\Gamma$.

For a general graph, call the vertices of valence at least 3 essential. In [1], Abrams proved that if $\Gamma$ is sufficiently subdivided, then $UC^n\Gamma$ is homotopy equivalent to $UD^n\Gamma$, where sufficiently subdivided means every nontrivial path connecting two essential vertices has at least $n - 1$ edges, and every nontrivial edge loop has at least $n + 1$ edges. In particular, this implies that $B_n\Gamma \cong \pi_1UD^n\Gamma$ for $\Gamma$ sufficiently subdivided. It is clear that by subdividing edges of a graph $\Gamma$ we do not change the associated graph braid group, so we always assume that our graph $\Gamma$ is sufficiently subdivided. Note the loops Artin $\gamma_i$ in $UC^n\Gamma$ are in fact simple edge loops in $UD^n\Gamma$ since the Artin basepoint $x = \{x_1, \ldots, x_n\}$ is a 0-cell in $UD^n\Gamma$.

3. Embeddings and Corollaries

In [3], Droms proved that right-angled Artin groups have a rigidity property: two graphs $\Delta$ and $\Delta'$ are graph isomorphic if and only if the groups $G(\Delta)$ and $G(\Delta')$ are group isomorphic. Thus, we may refer to a right-angled Artin group and the graph which defines it interchangeably.

Let $\Delta$ be a simple graph with corresponding right-angled Artin group $A = G(\Delta)$. For any vertex $v \in \Delta$, let $link(v)$ denote the full subgraph of $\Delta$ whose
vertices are exactly those vertices adjacent to \(v\) in \(\Delta\), not including \(v\) itself. Let \(\Delta - v\) denote the subgraph of \(\Delta\) formed by deleting the vertex \(v\) and any edge adjacent to \(v\). The group \(A\) has the structure of an HNN-extension [4] with stable letter \(v\), where the base group is \(G(\Delta - v)\), the associated subgroup is \(G(\text{link}(v))\), and \(v\) conjugates a given element of \(G(\text{link}(v))\) to itself.

Britton’s Lemma [16] tells us that if \(g_0v^{\epsilon_1}g_1v^{\epsilon_2}g_2\ldots v^{\epsilon_k}g_k = 1\), where \(\epsilon_i \in \{\pm 1\}\) and \(g_i \in G(\Delta - v)\), then there must exist some \(i\) such that \(\epsilon_{i+1} = -\epsilon_i\) and \(g_i \in G(\text{link}(v))\). Thus, \(v^{\epsilon_i}g_i v^{\epsilon_{i+1}} = g_i\), and it must be that \(k\) is not minimal. We call \(v\) an edge adjacent to \(v\) a \(v\)-pinch.

We now recall the embedding of Crisp and Wiest of graph braid groups into right-angled Artin groups, given by Theorem 2 of [6]. Let \(\Gamma\) be a sufficiently subdivided graph. Let \(\Delta\) be the graph whose vertex set is the edge set of \(\Gamma\), and where two vertices \(e\) and \(e'\) are connected by an edge if and only if the closures of \(e\) and \(e'\) are disjoint in \(\Gamma\). Let \(A_\Gamma := G(\Delta\Gamma)\) be the associated right-angled Artin group.

Crisp and Wiest prove their Theorem 2 by giving an embedding of \(UD^n\Gamma\) into an Eilenberg-MacLane complex for \(A_\Gamma\) which is a locally CAT(0) cubed complex. We apply their argument on the level of groups. In particular, let \(\gamma\) be an edge loop of length \(k\) in \(UD^n\Gamma\). Each edge of \(\gamma\) exactly corresponds to one strand crossing one edge of \(\Gamma\) while the other strands do not move. Let \(e_1, \ldots, e_k\) be the edges of \(\Gamma\) corresponding to each edge of \(\gamma\) given in order. Let \(w\) be the word in the generators of \(A_\Gamma\) given by \(e_1 \ldots e_k\). Let \(\Phi : B_n\Gamma = \pi_1(\Gamma, \{x_1, \ldots, x_n\}) \rightarrow A_\Gamma\) be defined by mapping the homotopy class of \(\gamma\) in \(\pi_1 UD^n\Gamma\) to the element represented by \(w\) in \(A_\Gamma\). Thus \(\Phi\) is the “forgetful” map, ignoring all of the strands in \(\gamma\) except the strand crossing an edge in \(\Gamma\). The map \(\Phi\) is the Crisp and Wiest embedding.

The following proposition implies Theorem 1.1.

**Proposition 3.1.** Let \(\Delta\) be a finite graph and let \(C\) be a coloring of \(\Delta\) using \(n\) colors \(\{1, \ldots, n\}\). Let \(\Gamma = \Gamma(\Delta)\) be a halo of \(\Delta\) with Artin basepoint \(\{x_1, \ldots, x_n\}\), and let \(a_i\) denote the vertices of \(\Delta\) for \(i = 1, \ldots, n\). Set \(A := G(\Delta)\). The map \(a_i \mapsto \gamma_i^2\) (following Convention [2,2]) induces an injective homomorphism \(\Psi : G(\Delta) \rightarrow B_n\Gamma\).

We will see below that the square in the definition of \(\Psi\) is necessary to obtain injectivity.

**Proof.** If \(a_i\) and \(a_j\) commute then so do \(\Psi(a_i)\) and \(\Psi(a_j)\), as \(\gamma_i\) and \(\gamma_j\) are disjoint (Artin) loops in \(\Gamma\) and the associated loops in \(UD^n\Gamma\) involve distinct strands. Thus the map \(a_i \mapsto \gamma_i^2\) induces a homomorphism \(\Psi : G(\Delta) \rightarrow B_n\Gamma\). It only remains to prove that \(\Psi\) is also injective. We do so by showing that the composition map \(\Phi \circ \Psi\) from right-angled Artin groups to right-angled Artin groups is injective.

Assume that \(\Phi \circ \Psi\) is not injective. Fix a nontrivial element \(\alpha\) in the kernel of \(\Phi \circ \Psi\) and let \(w\) be a geodesic word representing \(\alpha\). Since \(\alpha\) is nontrivial,
w is non-empty, and w contains a letter a (or its inverse) representing a generator in A. Then w has the form

\[ w = w_0a^{\ell_1}w_1a^{\ell_2}w_2 \ldots a^{\ell_k}w_k \]

where \( \ell_i \in \{\pm 1\} \) for \( i = 1, \ldots, k \) and each \( w_i \) is a (possibly empty) word in the generators of A not containing \( a \) or \( a^{-1} \).

Let \( \gamma \) be the Artin loop in \( \Gamma \) corresponding to \( a \). Let \( p \) be the point on \( \gamma \) which is in the Artin basepoint \( \{x_1, \ldots, x_n\} \). Let \( e_1e_2 \ldots e_m \) be the edges of \( \gamma \) traversed in order from \( p \) - i.e. \( e_1e_2 \ldots e_m \) is the image of \( \gamma \) under \( \Phi \).

Let \( g_i \in A_T \) be the image under \( \Phi \circ \Psi \) of the element in \( A \) represented by \( w_i \).

Consider \( A_T \) as an HNN-extension with stable letter \( e_1 \). Britton’s lemma tells us that since \( (\Phi \circ \Psi)(w) = 1 \), there must exist an \( e_1 \)-pinch in the appropriate expansion of

\[ g_0(e_1e_2 \ldots e_m)^{2\ell_1}g_1 \ldots (e_1e_2 \ldots e_m)^{2\ell_k}g_k. \]

For any \( e_1 \)-pinch of the form \( e_1^{-1}g_ie_1 \), replace the pinch with \( g_i \). Note each \( g_i \) may be represented without using any of \( e_1, \ldots, e_m \). Since \( \Psi \) involves a square, not every occurrence of \( e_1 \) may be replaced in this way, so we know that there must still exist \( e_1 \)-pinches in the result. The only possible remaining \( e_1 \)-pinches must have one of the following forms:

1. \( e_1e_2 \ldots e_2^{-1}e_1^{-1} \)
2. \( e_1^{-1}e_1^{-1}g_1e_2 \ldots e_m \).

We consider the first case; the second case is dealt with in a similar way.

If \( e_1e_2 \ldots e_mg_ie_1^{-1} \ldots e_2^{-1}e_1^{-1} \) is an \( e_1 \)-pinch, then \( e_2 \ldots e_2^{-1}e_1^{-1} = g_i^\prime \), where \( g_i^\prime \in G(\text{link}(e_1)) \). Thus, \( e_2 \ldots e_2^{-1}e_1^{-1}(g_i^\prime)^{-1} = 1 \). Since \( e_2, e_m \notin \text{link}(e_1) \), it follows that \( e_2 \ldots e_2^{-1}e_1^{-1} = g_i^\prime \) is an \( e_1 \)-pinch in \( A_{T-e_1} \), by Britton’s Lemma. Iterating, we see that \( e_j \ldots e_2^{-1}e_1^{-1} \) is an \( e_j \)-pinch for each \( j = 1, \ldots, m \).

In particular, we have proven that

\[ e_1 \ldots e_mg_ie_1^{-1} \ldots e_1^{-1} = g_i. \]

Thus, \( (e_1 \ldots e_m)^2g_i(e_1^{-1} \ldots e_1^{-1})^2 = g_i \). But then

\[ a^{\ell_1}w_1a^{\ell_2}w_2 \ldots a^{\ell_i-1}w_iw_{i+1}a^{\ell_i+2}w_{i+2} \ldots a^{\ell_k}w_k \]

is a word representing \( \alpha \) which is shorter than \( w \). As \( w \) was assumed to be a geodesic, this is a contradiction. Thus no such \( \alpha \) exists, and \( \Phi \circ \Psi \) is injective.

\[ \square \]

Note that if \( \Psi \) is defined only by sending \( a_i \) to \( \gamma_{2i} \) instead of \( \gamma_i^2 \), injectivity may not hold. For instance, consider Figure 2. This is a possible graph \( \Gamma \) for the right-angled Artin group \( \langle a, b, c | [a, c]\rangle \). If \( a \mapsto \gamma_a \mapsto e_a, e_{a}e_{a}e_{a}e_{a} \), \( b \mapsto \gamma_b \mapsto e_b, e_b, e_{b}e_{b}, e_{b}e_{b} , \) and \( c \mapsto \gamma_c \mapsto e_c, e_c, e_c, e_c \), we leave it to the reader to verify that \( g = cba^{-1}c^{-1}ba^{-1}b^{-1} \) is not trivial but \( (\Phi \circ \Psi)(g) \) is (it is insightful to visualize exactly what is happening here in terms of braids on \( \Gamma \)).
Corollary 3.2. Let $n_0$ be the chromatic number for $\Delta$. Then there exists an embedding of $G(\Delta)$ into the graph braid group $B_{n_0} \Gamma$ for some $\Delta$-halo $\Gamma$.

**Proof.** Let $C_0$ be a coloring of $\Delta$ using exactly $n_0$ colors. Let $\Gamma$ be a $\Delta$-halo for $C_0$. The result then follows from Theorem 3.1. \hfill $\square$

**Example 3.3.** As an application of Corollary 3.2, consider the right-angled Artin group $A$ for which $\Delta = C_6$, the cycle of length 6. Thus,

$$A = \langle a_1, a_2, a_3, a_4, a_5, a_6 | [a_i, a_{i+1}] (i = 1, \ldots, 5), [a_6, a_1] \rangle.$$ 

Figure 3 shows $C_6$ with a coloring using 2 colors and a possible graph $\Gamma(C_6)$. Theorem 1.1 tells us that we may embed $G(C_6)$ into $B_n \Gamma(C_6)$ where $n = 2!$ Note that $\Gamma(C_6)$ is planar.
Recall that a hyperbolic surface group is the fundamental group of a compact hyperbolic surface without boundary. Servatius et al. [18] (see also [15]) have shown that a right-angled Artin group whose graph has an achordal cycle of length at least 5 contains an embedded hyperbolic surface subgroup. Since $A$ embeds into $B_2\Gamma(C_6)$, it follows that there is a hyperbolic surface subgroup embedded in $B_2\Gamma(C_6)$.

Example 3.3 proves Theorem 1.2 via the following corollary:

**Corollary 3.4.** There exists a planar graph braid group - namely $B_2\Gamma(C_6)$ - which contains a hyperbolic surface subgroup, with only $n = 2$ strands.

It is known that hyperbolic surface groups can be subgroups of graph braid groups. In fact, in [1], Abrams proved that

**Proposition 3.5.** The pure graph braid groups $PB_2K_{3,3}$ and $PB_2K_5$ are the fundamental groups of closed orientable hyperbolic manifolds of genus 4 and 6, respectively.

This proposition, combined with embedding results in [1], yields the existence of hyperbolic surface subgroups in almost every nonplanar graph braid group (the possible exceptions being $B_nK_5$ and $B_nK_{3,3}$ for $n \geq 3$). No planar examples were previously known.

Of course, Corollary 3.3 relied on the observation that the graph $\Gamma(C_6)$ in the above example was planar. It is not a priori true in general that a planar graph $\Delta$ gives rise to a planar $\Delta$-halo. However, the following corollary does hold. Let $\Delta^{op}$ be the graph with vertex set equal to $\Delta$ and an edge connecting two vertices if and only if $\Delta$ does not have that edge.

**Corollary 3.6.** Let $A = G(\Delta)$ be a right-angled Artin group with associated graph $\Delta$ such that $\Delta^{op}$ has a finite-sheeted planar covering. Then $A$ embeds in some planar graph braid group $B_n\Gamma$ for $n \leq |V(\Delta)|$.

**Proof.** In Proposition 19 of [6], Crisp and Wiest showed that if $\Delta^{op}$ has a finite-sheeted planar covering $\tilde{\Delta}^{op}$, then if $\tilde{\Delta}$ is the opposite graph of $\tilde{\Delta}^{op}$, there exists an injective homomorphism $j : G(\Delta) \to G(\tilde{\Delta})$. As $\tilde{\Delta}$ is a cover of $\Delta$, $\tilde{\Delta}$ may be colored with $n \leq |V(\Delta)|$ colors. We leave it to the reader to verify that a planar opposite graph gives rise to a planar halo. 

We end with a question. As mentioned, the notion of a $\Delta$-halo is similar in flavor to the proof of Theorem 10 of [6]. Although Crisp and Wiest deal with pure surface braid groups instead of regular surface braid groups, is it possible to apply the ideas presented here to reduce the number of strands needed for their embedding?

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