Blow up of solutions for a Parabolic-Elliptic chemotaxis system with gradient dependent chemotactic coefficient

J. Ignacio Tello
Departamento de Matemáticas Fundamentales, Facultad de Ciencias, Universidad Nacional de Educación a Distancia, Madrid, Spain

ABSTRACT
We consider a Parabolic-Elliptic system of PDE’s with a chemotactic term in a $N$-dimensional unit ball describing the behavior of the density of a biological species “$u$” and a chemical stimulus “$v$.” The system includes a nonlinear chemotactic coefficient depending of “$\nabla v$,” i.e. the chemotactic term is given in the form
\[-\text{div}(\chi|\nabla v|^{\frac{p-2}{2}}\nabla v), \quad \text{for} \quad p \in \left(\frac{N}{N-1}, 2\right), \quad N > 2\]
for a positive constant $\chi$ when $v$ satisfies the poisson equation
\[-\Delta v = u - \frac{1}{\Omega} \int_{\Omega} u_0 dx.\]
We study the radially symmetric solutions under the assumption in the initial mass
\[\frac{1}{\Omega} \int_{\Omega} u_0 dx > 6.\]
For $\chi$ large enough, we present conditions in the initial data, such that any regular solution of the problem blows up at finite time.

1. Introduction
Chemotaxis is among the most important processes in Natural Sciences. It is defined as the biological phenomenon of living organism in respond to a chemical stimulus, orientating its movement toward the higher concentration of the chemical or away from it.

Two main magnitudes appear in the process: the concentration of one or several chemical substances and the density of one or several biological species.

From the pioneering works of Keller and Segel in the 70 s to the present, PDE’s systems of chemotaxis have been studied by a large number of authors. The extensive literature in the field shows the relevance of the problem, we refer the reader to Horstmann [11, 12], Bellomo et al. [3], Hillen and Painter [10] and references therein for more details concerning previous results of such systems. The classical system proposed in Keller and Segel [14] and [15] considers a linear dependence of the chemotactic term respect to the gradient of the chemical substance. Denoting by $u$ the living
organism concentration and by \( v \) the chemical stimulus, the original system reads as follows

\[
\begin{align*}
    u_t - \Delta u &= -\text{div}(\chi u \nabla v) + g(u, v), & x \in \Omega, \ t > 0, \\
    \tau v_t - \Delta v &= h(u, v), & x \in \Omega, \ t > 0,
\end{align*}
\]

for some constants \( \chi \in \mathbb{R}, \ \tau \geq 0, \) and known functions \( g \) and \( h. \) To complete the system, Neumann boundary conditions and initial data are given.

Nevertheless, any biological individual presents a natural limitation to the velocity of movement, and therefore, it is not expected that for large values of \( \nabla v, \) the behavior of the individuals will be proportional to the case of small values of \( \nabla v. \) In that sense, a nonlinear term to limit the growth of \( \nabla v \) give us a large range of applications.

Recently, motivated by different biological phenomena, several authors have considered the chemotactic sensitivity coefficient \( \chi \) as a continuous function of \( v, \) for instance, in Bellomo and Winkler [4] and [5] (see also Bellomo et al. [2]), a chemotaxis system is analyzed for a chemotactic term in the form

\[-\text{div} \left( \frac{\chi u}{\left(1 + |\nabla v|^2\right)^{\frac{1}{2}}} \nabla v \right) \]

In [5], the authors prove the existence of blow up for some initial data with nonlinear diffusive term

\[-\text{div} \left( \frac{u}{\sqrt{u^2 + |\nabla u|^2}} \nabla u \right)
\]

where \( v \) satisfies the elliptic problem

\[-\Delta v = u - \frac{1}{|\Omega|} \int_{\Omega} u_0 \text{d}x.\]

Recently, Chiyoda, Mizukami and Yokota [8] study the parabolic-elliptic system for a chemotaxis term

\[-\text{div} \left( \frac{u^q}{\sqrt{1 + |\nabla v|^2}} \nabla v \right), \]

where the diffusive term for “\( u \)” generalizes the previous model in [5],

\[-\text{div} \left( \frac{u^p}{\sqrt{u^2 + |\nabla u|^2}} \nabla u \right).\]

In [8], the authors obtain blow up of solutions for \( p, q \geq 1 \) by using a sub-solution method.

Recently, M. Winkler [26] proves blow up of solutions for a general chemotactic term

\[-\text{div}(\chi uf(|\nabla v|^2) \nabla v)\]

for a regular function \( f \) satisfying
\[ f(\xi) > c(1 + |\nabla v|^2)^{-x}, \ \text{where} \ \frac{N - 2}{2(N - 1)}. \]  

(1.1)

In Bianchi, Painter and Sherratt [6, 7] the authors consider the term

\[-\text{div}\left( \frac{\chi u}{1 + \omega u} \frac{\nabla v}{1 + \eta |\nabla v|} \right),\]

for some positive constants \( \chi, \omega \) and \( \eta \), where a parabolic equation is coupled to an ODE modeling Lymphangiogenesis in wound healing in a one-dimensional spatial domain.

We express a general chemotactic term in the form

\[-\text{div}[u\tilde{\chi}(u, v, |\nabla v|)|\nabla v|],\]

for a prescribed continuous function \( \tilde{\chi} \). Several authors have studied the problem where \( \tilde{\chi} \) depends only of \( u \) or \( v \), such examples can be found for instance in Laurenc¸ot and Wrzosek [16], Stinner, Tello and Winkler [22], Negreanu and Tello [18], Stinner and Winkler [23] and Winkler [25] among others. In the present article we focus our attention in the case where \( \tilde{\chi} \) depends only on \( |\nabla v| \) in the following way

\[-\text{div}[u\tilde{\chi}(u, v, |\nabla v|)|\nabla v|] = -\chi \text{div}[u|\nabla v|^{p-2}\nabla v],\]

for some positive constant \( \chi \) and \( p \in (N/(N - 1), 2) \) for \( N > 2 \). Notice that the chemotactic term presents a singularity when \( |\nabla v| = 0 \). Assumption (1.1) in [26] is equivalent to \( p \in (N/(N - 1), 2) \) in this article (see assumption (1.8)).

The previous nonlinearity has been already studied in Negranu and Tello [19] and in Wang and Li [24]. In [19], the authors consider the system in a bounded domain \( \Omega \subset \mathbb{R}^N \),

\[
\begin{align*}
\{ & u_t - \Delta u = -\text{div}(\chi u|\nabla v|^{p-2}\nabla v), \quad x \in \Omega, \quad t > 0, \\
& -\Delta v = u - M, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]

(1.2)

with homogeneous Neumann boundary conditions and non-negative initial data satisfying

\[
\frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx = M.
\]

(1.3)

Under assumptions

\[
\begin{align*}
\{ & p \in (1, \infty), \quad \text{if} \ N = 1, \\
& p \in (1, \frac{N}{N-1}), \quad \text{if} \ N \geq 2,
\end{align*}
\]

the authors obtain uniform bounds in \( L^\infty(\Omega) \) for any \( t > 0 \). Similar result is obtained if \( v \) satisfies

\[-\Delta v + v = u, \quad x \in \Omega.\]

The steady states of the one-dimensional case are also considered in [19], where infinitely many non-constant solutions appear for \( p \in (1, 2) \) for any \( \chi \) positive and a prescribed positive mass.
The parabolic-parabolic equation is considered in [24] for $v$ satisfying

$$vt - \Delta v = -uv, \quad t > 0, \quad x \in \Omega$$

where $\Omega$ is a $N$-dimensional bounded domain for $N \geq 2$ with Neumann boundary conditions and bounded initial data $u_0$ and $v_0$. The authors obtain global existence of weak solutions for initial data $v_0$ satisfying

$$v_0 \leq \frac{1}{4k_v}, \quad k_v := \sup_{s \geq 0} \left\{ \frac{s}{(1 + s^2) \ln (1 + s)} \right\}$$

when the exponent $p$ satisfies

$$N < \frac{8 - 2(p - 1)}{p - 1}, \quad \text{i.e.} \quad p < \frac{N + 10}{N + 2}.$$  

Notice that the non-linear term “$uv$” in the equation is not equivalent to the linear term $u - v$ in [19].

In this article we study a mathematical prototype of chemotaxis with flux limitation in the $N$-dimensional open unit ball $B_N$ defined

$$B_N := \{ x \in \mathbb{R}^N, |x| < 1 \},$$

that we denote by $B$ if not explicitly stated otherwise. We denote by $\bar{n}$ the outward pointing normal vector on the boundary $\partial B$. The equation for $v$ is restricted to the elliptic case, for simplicity, we assume that $v$ satisfies the Poisson equation and the system studied is the following

$$ut - \Delta u = -\text{div}(\chi u |\nabla v|^p \nabla v), \quad x \in B, \quad t > 0, \quad (1.4)$$

$$-\Delta v = u - M, \quad x \in B, \quad t > 0, \quad (1.5)$$

$$\frac{\partial u}{\partial \bar{n}} = \frac{\partial v}{\partial \bar{n}} = 0, \quad x \in \partial B, \quad t > 0, \quad (1.6)$$

$$u(0, x) = u_0(x), \quad x \in B, \quad (1.7)$$

where $p$ satisfies

$$p \in \left( \frac{N}{N - 1}, 2 \right), \quad \text{for} \quad N > 2$$

and $M$, defined in (1.3), fullfills

$$M > 6,$$  

i.e.

$$\int_B u_0 dx > 6|B| = 6 \frac{\pi^N}{\Gamma\left(\frac{N}{2} + 1\right)},$$

and for the three dimensional case, the previous inequality reads

$$\int_B u_0 dx > 8\pi.$$  

Notice that the previous assumption is also assume in Semba [21], where the author proves blow up for $p = 2$. 
We define the function
\[
\phi(0, \rho) = \begin{cases} \\
\frac{\rho^\gamma}{\rho^\gamma + 1} & 0 < \rho \leq \frac{1}{2}, \\
\frac{2^{1-\gamma}}{2^{-\gamma} + 1} \left( 1 - \rho + (1 + \gamma) \frac{(\rho_2 - \rho) + (\rho - \frac{1}{2})}{(\rho - \frac{1}{2})^2} \right), & \frac{1}{2} < \rho < 1,
\end{cases}
\]
for \(\gamma > 1\) satisfying
\[
\gamma < \min \left\{ 1 + \frac{2 - p}{N(p - 1)}, \frac{\chi N^{N-p}}{2} - 1, 1 + \frac{M - 6}{4}, \frac{N + 1}{N}, \gamma^* \right\},
\]
(1.10)
for \(\gamma^* > 1\) with the following property:
\[
\gamma^* > 1 \text{ such that } q(\gamma) > \frac{1}{5} \text{ for any } \gamma \in (1, \gamma^*),
\]
where
\[
q(\gamma) = \frac{(\gamma + 2)}{4(\gamma + 1)^2} \left[ \frac{3\gamma}{2} + 1 \right].
\]
We study the blow up of solutions under the following assumptions in the initial data \(u_0\)
\[
u_0 \text{ is a radial function,}
\]
(1.11)
\[
u_0 \in C^{1,2}(B), \quad \frac{\partial u_0}{\partial n} = 0, \quad x \in \partial B,
\]
(1.12)
\[
\frac{N \pi^\frac{N}{2} \chi}{\Gamma(\frac{N}{2} + 1)} \int_{|x| \leq \rho^{1/N}} (u_0(x) - M) dx \geq \phi(0, \rho).
\]
(1.13)
\[
\chi > N^p \max \left\{ 4, \frac{3 \cdot 2^6 (4^{\frac{1}{3}} - p)}{M - 6} \left( \frac{4}{3} \right)^{2-p} \right\}
\]
the previous assumption guaranties
\[
\chi_N := \frac{\chi}{N^p} > \max \left\{ 4, \frac{\rho_2^{2-\frac{4}{3}} 2^2 (1 + \gamma)}{(\rho_2 - \frac{1}{2}) (1 - \rho_2) (M - 6)} \left( \frac{4}{3} \right)^{2-p} \right\}
\]
(1.14)
in view of \(\gamma < 2\) and
\[
\rho_2 = \frac{2 + \gamma}{2(1 + \gamma)} > \frac{5}{8}.
\]
The main result of the article is enclosed in the following theorem.

**Theorem 1.1.** Let \(B\) be the \(N\)-dimensional open unit ball in \(\mathbb{R}^N\), then, under assumptions (1.8)–(1.14), there exists a positive number \(T_{bu} < \infty\) such that, there exists at least a solution \(u\) to problem (1.4)–(1.7) such that the function \(\int_{|x| < \rho^N} (u(t,x) - M) dx\) exists in \((0,T_{bu})\) and
\[
\lim_{t \to T_{bu}} \|u\|_{L^\infty(B)} = \infty,
\]
for some \(T_{bu} \leq T_{\max} := \frac{1}{\epsilon}\) with \(\epsilon\) defined as a function of \(p, N, M\) and \(\gamma\) in (4.6).
Notice that, as a consequence of the previous theorem, the classical solutions of the problem, for initial data satisfying (1.11)–(1.13) do not exist (in a classical sense) for $t \geq T_{bu}$.

The article is organized as follows: In Sec. 2, we introduce the mass accumulation function $U$ and we deduce the equation satisfied by $U$. In Sec. 3, the local existence of solutions and continuous regularity is given for the mass accumulation function $U$. Section 4 is devoted to the construction of a subsolution (denoted by $\phi$) and its properties. In Sec. 5, the comparison result is given for the equation obtained in Sec. 2. Uniqueness of solutions is obtained following the steps of the comparison results given in the same section. Finally, in the last section, the end of the proof of the theorem is presented.

The key of the proof of the results is the sub-solution $\phi$, such function have been constructed by modifying the sub-solution presented in Jäger and Luckhaus [13] where the authors prove finite time blow up for the minimal Keller-Segel system, i.e. for $p = 2$.

2. Equation of the mass accumulation for radial symmetric solutions

Let $u$ be the solution to (1.4)–(1.7) and

$$\omega_{N-1} := |S^{N-1}| = \frac{N\pi^N}{\Gamma\left(\frac{N}{2} + 1\right)},$$

the (N-1)-dimensional volume of the sphere $S^{N-1}$ (the surface of the N-dimensional ball) for the well known function $\Gamma$, already defined by Euler in 1729 and given by

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt, \quad \Re e(z) > 0.$$

Let $\tilde{u}(t, r)$ be defined by $\tilde{u}(t, |x|) := u(t, x)$ for a radially symmetric function $u$. For simplicity we drop the tilde and introduce the following change of unknowns

$$U(t, \rho) := \int_{|x| < \rho^{1/N}} (u(t, x) - M)dx = \omega_{N-1} \int_0^{\rho^{1/N}} (u(t, r) - M)r^{N-1}dr, \tag{2.1}$$

for $M$ defined in (1.3). Notice that, thanks to Leibniz’s rule we obtain

$$U_\rho = \frac{\partial}{\partial \rho} \int_{|x| < \rho^{1/N}} (u(t, x) - M)dx$$

$$= \frac{\partial}{\partial \rho} \omega_{N-1} \int_0^{\rho^{1/N}} (u(t, r) - M)r^{N-1}dr$$

$$= \frac{\omega_{N-1}}{N} \left[ u(t, \rho^{1/N}) - M \right] \rho^{\frac{N-1}{N}}$$

$$= \frac{\omega_{N-1}}{N} \left[ u(t, \rho^{1/N}) - M \right],$$

therefore

$$U_{\rho \rho} = \frac{\omega_{N-1}}{N} \frac{\partial}{\partial \rho} u(t, \rho^{1/N}) \tag{2.2}$$

and

$$u(t, \rho^{1/N}) = \frac{N}{\omega_{N-1}} U_\rho + M. \tag{2.3}$$
We have that
\[ -\int_{|x|<\rho^{1/N}} \Delta u \, dx = -\omega_{N-1} \int_0^{\rho^{1/N}} \left[ r^{-N} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial u}{\partial r} \right) \right] r^{N-1} \, dr \]
\[ = -\omega_{N-1} \int_0^{\rho^{1/N}} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial u}{\partial r} \right) \, dr \]
\[ = -\omega_{N-1} \rho^{\frac{N-1}{N}} \frac{\partial u}{\partial \rho^{1/N}} \]
\[ = -\omega_{N-1} N \rho^{\frac{2N-3}{N}} \frac{\partial u}{\partial \rho}. \]

Thanks to (2.2) we have
\[ -\int_{|x|<\rho^{1/N}} \Delta u \, dx = -N^2 \rho^{\frac{2N-3}{N}} U_{\rho \rho}. \quad (2.4) \]

The term
\[ \int_{|x|<\rho^{1/N}} \text{div}(u|\nabla v|^{p-2} \nabla v) \, dx = \omega_{N-1} \int_0^{\rho^{1/N}} \left( r^{N-1} u \left| \frac{\partial v}{\partial r} \right|^{p-2} \frac{\partial v}{\partial r} \right) r^{N-1} \, dr \]
\[ = \omega_{N-1} \int_0^{\rho^{1/N}} \frac{\partial}{\partial r} \left( r^{N-1} u \left| \frac{\partial v}{\partial r} \right|^{p-2} \frac{\partial v}{\partial r} \right) \, dr \]
\[ = \omega_{N-1} \left( \rho^{\frac{N-1}{N}} u \left| \frac{\partial v}{\partial \rho} \right|^{p-2} \frac{\partial v}{\partial \rho} \right) \]
\[ = \omega_{N-1} N^{p-1} \left( \rho^{\frac{N-1}{N}} u(t, \rho^{1/N}) \left| \frac{\partial v}{\partial \rho} \right|^{p-2} \frac{\partial v}{\partial \rho} \right). \]

As before we have that
\[ -\int_{|x|<\rho^{1/N}} \Delta v \, dx = -\omega_{N-1} N \rho^{\frac{2N-3}{N}} \frac{\partial v}{\partial \rho} = U(\rho^{1/N}, t). \quad (2.5) \]
i.e.
\[ \frac{\partial v}{\partial \rho} = -\frac{\rho^{\frac{2N-3}{N}}}{N \omega_{N-1}} U(\rho^{1/N}, t). \quad (2.6) \]

Then
\[ -\int_{|x|<\rho^{1/N}} \text{div}(u|\nabla v|^{p-2} \nabla v) \, dx = -\omega_{N-1} \chi N^{p-1} \left( \rho^{\frac{N-1}{N}} u(t, \rho^{1/N}) \left| \frac{\partial v}{\partial \rho} \right|^{p-2} \frac{\partial v}{\partial \rho} \right) \]
\[ = \omega_{N-1}^{2-p} \int_{|x|<\rho^{1/N}} \left( \frac{N}{\omega_{N-1}} U_{\rho}(t, \rho^{1/N}) + M \right) \left| U(\rho^{1/N}, t) \right|^{p-2} U(\rho^{1/N}, t) \]
\[ = \omega_{N-1}^{2-p} \int_{|x|<\rho^{1/N}} \left( \frac{N}{\omega_{N-1}} U_{\rho}(t, \rho^{1/N}) + M \right) \left| U(t, \rho^{1/N}) \right|^{p-2} U(t, \rho^{1/N}). \]

After integration in (1.4), thanks to (2.2)–(2.6) and the last equation, we get
\[ U_t - N^2 \rho^{2N-3} U_{\rho\rho} = \omega_{N-1}^2 (2-p) \frac{N}{\omega_{N-1}} U_\rho(t, \rho^{1/N}) + M \left| U(t, \rho^{1/N}) \right|^{p-2} U(t, \rho^{1/N}) \]  

(2.7)

with the boundary condition

\[ U(t, 0) = U(t, 1) = 0 \]

and the initial data

\[ U(0, \rho) = \int_{|\rho|<\rho^{1/N}} (u_0 - M) \, dx. \]

We re-escalate the problem in the following way

\[ \tilde{t} = N^2 t, \quad \tilde{U} = \frac{N}{\omega_{N-1}} U \]

to get

\[ \tilde{U}_t - \rho^{2N-3} \tilde{U}_{\rho\rho} = \chi N^{-p} \rho^{(2-p) \frac{N}{2}} (\tilde{U}_\rho + M) \left| \tilde{U} \right|^{p-2} \tilde{U}. \]

For simplicity, we drop the tilde and introduce the constant \( \chi_N \), already defined in (1.14)

\[ \chi_N = \chi N^{-p} \]

to get

\[ U_t - \rho^{2N-3} U_{\rho\rho} = \chi_N \rho^{(2-p) \frac{N}{2}} (U_\rho + M) \left| U \right|^{p-2} U. \]  

(2.8)

The problem is completed with the boundary conditions

\[ U(t, 0) = U(t, 1) = 0 \]  

(2.9)

and the initial data

\[ U(0, \rho) = \int_{|\rho|<\rho^{1/N}} (u_0 - M) \, dx. \]  

(2.10)

3. A priori estimates and local existence of weak solutions

In this section we obtain some a priori estimates of the solution and prove the local existence of weak solutions for Eq. (2.8). First, we introduce the new variables \( s \) and \( W \) defined by

\[ \rho = s^N, \quad W(t, s) = s^{-N} U(t, s^N) \]

then,

\[ \frac{\partial W}{\partial s} = -Ns^{-N-1} U(t, s^N) + Ns^{-1} \frac{\partial U}{\partial \rho}, \]

\[ \frac{\partial^2 W}{\partial s^2} = N(N+1)s^{-N-2} U(t, s^N) - N(N+1)s^{-2} U_\rho + N^2 s^{-2} \frac{\partial^2 U}{\partial \rho^2}, \]

i.e.
\[
\frac{\partial U}{\partial \rho} = \frac{s}{N} \frac{\partial W}{\partial s} + s^{-N} U(t, s^N) = \frac{s}{N} \frac{\partial W}{\partial s} + W,
\]

\[
\frac{U_{\rho\rho}}{N^2} = \frac{s^{-N}}{N^2} \frac{\partial^2 W}{\partial s^2} - \frac{N + 1}{N} s^{-2N} U(t, s^N) + \frac{N + 1}{N} s^{-N} U_{\rho}
\]

\[
= \frac{s^{-N}}{N^2} \frac{\partial^2 W}{\partial s^2} - \frac{N + 1}{N} s^{-N} W + \frac{N + 1}{N} s^{-N} \left[ \frac{s}{N} \frac{\partial W}{\partial s} + W \right]
\]

\[
= \frac{s^{-N}}{N^2} \frac{\partial^2 W}{\partial s^2} + \frac{N + 1}{N^2} s^{-N+1} \frac{\partial W}{\partial s},
\]

\[
\rho^{1-\delta} U_{\rho\rho} = \frac{1}{N^2} \left[ \frac{\partial^2 W}{\partial s^2} + \frac{N + 1}{s} \frac{\partial W}{\partial s} \right].
\]

We replace in (2.8) and multiply by \(s^{-N}\) to obtain

\[
W_t - N^{-2} \left[ W_{ss} + \frac{N + 1}{s} W_s \right] = \chi_N s^\rho - 2 \left( \frac{s}{N} W_s + W + M \right) |W|^{\rho - 2} W
\]  \hspace{1cm} (3.1)

with the corresponding Dirichlet boundary conditions and initial data. Notice that as far as \(u\) is bounded, we have that

\[
|U| \leq c \rho, \quad \text{and} \quad |W| \leq c.
\]

The proof of the existence of solutions is based on the Hardy inequality

\[
\int_0^1 \rho^{-\delta} u^2 \, d\rho \leq c \int_0^1 \rho^{2-\delta} |u_\rho|^2 \, d\rho.
\]

For readers convenience, we introduce the details of the proof in the following lemma.

**Lemma 3.1.** Let \(I = (0, 1)\) and \(\delta \in (0, 1) \cup (1, \infty)\) then, for any function \(u \in H^1_{\rho^{-\delta}}(I)\) such that \(u(1) = 0\) and \(\lim_{\rho \to 0} \rho^{1-\delta} u^2 = 0\), we have

\[
\int_I \rho^{-\delta} u^2 \, d\rho \leq \frac{1}{\epsilon_0(1 - \delta) - \epsilon_0} \int_I \rho^{2-\delta} |u_\rho|^2 \, d\rho
\]  \hspace{1cm} (3.2)

for any \(\epsilon_0 > 0\) such that

\[
\epsilon_0 < |1 - \delta|.
\]

**Proof.** We consider first the case \(\delta < 1\) and take \(\epsilon_0 > 0\) such that

\[
\epsilon_0 < 1 - \delta.
\]

Then,

\[
[\rho^\rho u]_\rho = \epsilon_0 \rho^{\rho - 1} u + \rho^\rho u_\rho,
\]

we take squares in the previous equation and it results

\[
0 \leq \left| [\rho^\rho u]_\rho \right|^2 \leq \epsilon_0^2 \rho^{2\rho - 2} u^2 + \rho^{2\rho} |u_\rho|^2 + 2 \epsilon_0 \rho^{2\rho - 1} uu_\rho.
\]

We now multiply by \(\rho^{-\rho + 2 - \delta}\) and integrate over \(I\),

\[
0 \leq \epsilon_0^2 \int_I \rho^{-\rho} u^2 \, d\rho + \int_I \rho^{2-\delta} |u_\rho|^2 \, d\rho + 2 \epsilon_0 \int_I \rho^{1-\delta} uu_\rho \, d\rho.
\]
Since
\[
2\epsilon_0 \int_I \rho^{1-\delta} uu_\rho d\rho = \epsilon_0 \int_I \rho^{1-\delta} (u^2)_{\rho} d\rho
= \epsilon_0 (1 - \delta) \int_I \rho^{-\delta} u^2 d\rho
\]
then,
\[
0 \leq \epsilon_0 (\epsilon_0 - 1 + \delta) \int_I \rho^{-\delta} u^2 d\rho + \int_I \rho^{2-\delta} |u_\rho|^2 d\rho.
\]
We divide the previous inequality by \(\epsilon_0 (\epsilon_0 - 1 + \delta)\) and the proof of (3.2) ends for \(\delta < 1\).

To prove the case \(\delta > 1\) we consider
\[
\epsilon_0 < \delta - 1
\]
and the equation
\[
[rho^{-\epsilon_0} u]_\rho = -\epsilon_0 \rho^{-\epsilon_0-1} u + \rho^{-\epsilon_0} u_\rho.
\]
We proceed as before to get
\[
0 \leq \epsilon_0^2 \rho^{-2\epsilon_0-2} u^2 + \rho^{-2\epsilon_0} |u_\rho|^2 - 2\epsilon_0 \rho^{-2\epsilon_0-1} uu_\rho.
\]
We multiply by \(\rho^{2\epsilon_0+2-\delta}\) and integrate over \(I\) to obtain, after integration by parts
\[
0 \leq \epsilon_0^2 \int_I \rho^{-\delta} u^2 d\rho + \int_I \rho^{2-\delta} |u_\rho|^2 d\rho + \epsilon_0 (1 - \delta) \int_I \rho^{-\delta} u^2 d\rho.
\]
Since \(\delta > 1\) and \(\epsilon_0 < \delta - 1\), the proof ends after dividing by \(\epsilon_0 (\delta - 1 - \epsilon_0)\).

**Corollary 3.2.** Under assumptions of Lemma 3.1 we have
\[
\int_I \rho^{-\delta} u^2 d\rho \leq \frac{4}{|1 - \delta|^2} \int_I \rho^{2-\delta} u_\rho^2 d\rho.
\]

**Proof.** The proof of Corollary 3.2 is an immediate consequence of Lemma 3.1 for \(\epsilon_0 = \frac{1 - \delta}{2}\). □

**Definition 3.1.** Let \(I = (0, 1)\), \(I_T = (0, T) \times I\), then, for any initial data \(W_0 \in H^2_{\mathcal{N},+1}(I) \cap H^1_{0,\mathcal{N},+1}(I)\), a weak solution of (3.1) in \(I_T\) is a function
\[
W \in L^\infty((0, T) : H^1_{0,\mathcal{N},+1}(I)) \cap H^1((0, T) : L^2_{\mathcal{N},+1}(I)) \cap L^2((0, T) : H^2_{\mathcal{N},+1}(I))
\]
and
\[
W \in C^0([0, T) : L^2_{\mathcal{N},+1}(I))
\]
such that \(W : I_T \to \mathbb{R}\) satisfies
\[
- \int_{I_T} \zeta_s W s^{N+1} ds dt + \int_{I_T} \zeta(T) W(T) s^{N+1} ds + N^{-2} \int_{I_T} \zeta_s W_s s^{N+1} ds dt
= \int_{I_T} \zeta(0) W_0 s^{N+1} ds + \chi_N \int_{I_T} s^{p+1-N} \zeta \left( \frac{s}{N} W_s + W + M \right) |W|^{p-2} W ds dt
\]
for all \( \zeta \in C^1([0,T]:C^2(I)) \).

**Lemma 3.3.** Let \( W \) the solution to (3.1), then, there exists \( T_1 > 0 \) such that
\[
\left| \int_{I_T} W^2 s ds \right| + \int_{I_T} |W_s|^2 ds dt \leq C(T) < \infty, \quad \text{for any } T < T_1. \tag{3.4}
\]
Moreover, we have that there exists \( T_2 > 0 \) such that
\[
\int_{I_T} W_s^2 s^{N+1} ds dt + \int_{I_T} |W_s|^2 s^{N+1} ds \leq C(T) < \infty, \tag{3.5}
\]
and for any \( \epsilon \in \left( 0, \frac{1}{pN} \right) \) we have
\[
\int_0^T \| s^N W \|_{L^\infty(I)}^2 dt \leq C(T), \tag{3.6}
\]
and
\[
\int_{I_T} \left| W_s + \frac{N+1}{s} W_s s^{N+1} ds dt \leq C(T), \tag{3.7}
\]
for any \( T < T_2 \).

**Proof.** We multiply (3.1) by \( sW \) and integrate by parts to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{I_T} W^2 s ds + N^{-2} \int_{I_T} |W_s|^2 ds
= \frac{\chi_N}{N} \int_{I_T} s^p |W|^p W_s ds + \chi_N \int_{I_T} s^{p-1} |W|^p W ds + \chi_N M \int_{I_T} s^{p-1} |W|^p ds.
\]
Since
\[
\int_{I_T} s^p |W|^p W_s ds = \frac{1}{p+1} \int_{I_T} s^p (|W|^p W)_s ds = -\frac{p}{p+1} \int_{I_T} s^{p-1} |W|^p W ds
\]
and
\[
\int_{I_T} s^{p-1} |W|^p ds \leq \int_{I_T} s^{p-1} |W|^{p+1} ds + 1
\]
we have
\[
\frac{1}{2} \frac{d}{dt} \int_{I_T} W^2 s ds + N^{-2} \int_{I_T} |W_s|^2 ds dt \leq c \int_{I_T} s^{p-1} |W|^{p+1} ds + \chi_N M. \tag{3.8}
\]
Let \( \epsilon \) be a positive number such that
\[ \epsilon \leq \frac{1}{pN} < \frac{p - 1}{p}, \]

then

\[ s'W = \int_0^s [\tau'W]_\tau d\tau, \]

therefore

\[
|s'W| \leq \int_I s'|W_s| ds + \epsilon \int_I s'^{-1}|W| ds
\]

\[
\leq \left[ \int_I s^{2\epsilon - 1} ds \int_I |W_s|^2 ds \right]^{\frac{1}{2}} + \left[ \int_I s'^{-1} ds \int_I s^{-1}|W|^2 ds \right]^{\frac{1}{2}}
\]

which implies, in view of Lemma 3.1

\[ |s'W| \leq c \left[ \int_I s|W_s|^2 ds \right]^{\frac{1}{2}} \quad (3.9) \]

and the term

\[
\int_I s^{p - 1}|W|^{p + 1} ds \leq ||s'|W||^p_{L^\infty(I)} \int_I s^{p - 1 - p\epsilon}|W| ds
\]

\[
\leq ||s'|W||^p_{L^\infty(I)} \left[ \int_I s^{2(p - 1 - p\epsilon) - 1} ds \int_I |W|^2 ds \right]^{\frac{1}{2}} \quad (3.10)
\]

as a consequence of the election of \( \epsilon \), the inequality \( 2(p - 1 - p\epsilon) > 0 \) is satisfied and we have

\[ \int_I s^{2(p - 1 - p\epsilon) - 1} ds \leq c < \infty \]

and (3.8) becomes, thanks to (3.9) and (3.10)

\[
\frac{1}{2} \frac{d}{dt} \int_I W^2 s ds + N^{-2} \int_I |W_s|^2 ds \leq c \left[ \int_I s|W_s|^2 ds \right]^{\frac{1}{2}} \left[ \int_I s|W|^2 ds \right]^{\frac{1}{2}} + \chi_{N,M}.
\]

Thanks to Young’s Inequality

\[
\frac{1}{2} \frac{d}{dt} \int_I W^2 s ds + N^{-2} \int_I |W_s|^2 ds \leq \frac{N^{-2}}{2} \int_I |W_s|^2 ds + c \left[ \int_I W^2 ds \right]^{\frac{1}{2p}} + \chi_{N,M}.
\]

which implies, in view of \( \frac{1}{2p} > \frac{p + 1}{2} \) that

\[
\frac{d}{dt} \frac{1}{2} \int_I W^2 s ds + \frac{N^{-2}}{2} \int_I |W_s|^2 ds \leq c \left[ \int_I |W|^2 ds \right]^{\frac{1}{2p}} + c.
\]

After integration, and thanks to Gronwall’s Lemma, we get that there exists \( T_1 > 0 \) such that

\[ \frac{1}{2} \int_I W^2 s ds + \frac{N^{-2}}{2} \int_I |W_s|^2 ds \leq c(T), \quad \text{for any } T < T_1 \]

and we prove (3.4).
To obtain (3.5) we multiply (3.1) by $s^{N+1} W_t$ and integrate by parts to obtain

$$I_W := \int_I W_t^2 s^{N+1} ds + \frac{N-2}{2} \frac{d}{dt} \int_I |W_t|^2 s^{N+1} ds$$

$$= \frac{\chi_N}{N} \int_I s^{p+N} |W|^{p-2} WW_t W_t ds + \chi_N \int_I s^{p-1+N} |W|^p W_t ds$$

$$+ \chi_N M \int_I s^{p-1+N} |W|^{p-2} WW_t ds,$$

thanks to Young inequality, the previous integrals are bounded as follows

$$\frac{\chi_N}{N} \int_I s^{p+N} |W|^{p-2} WW_t W_t ds \leq \frac{1}{4} \int_I |W_t|^2 s^{N+1} ds$$

$$+ \frac{\chi_N^2}{N^2} |s^{\frac{2p-N-2}{p-2}} W|_{L^\infty(I)} \int_I |W_s|^2 s^{N+1} ds$$

$$\frac{\chi_N}{N} \int_I s^{p+N-1} |W|^p W_t ds \leq \frac{1}{4} \int_I |W_t|^2 s^{N+1} ds + \frac{\chi_N^2}{N^2} \int_I s^{2p+N-4} |W|^2 ds$$

$$\leq \frac{1}{4} \int_I |W_t|^2 s^{N+1} ds + \frac{\chi_N^2}{N^2} |s^{\frac{2p-N-4}{p-2}} W|_{L^\infty(I)} \int_I s^2 ds$$

$$\frac{\chi_N}{N} \int_I s^{p-1+N} |W_t| ds \leq \frac{1}{4} \int_I |W_t|^2 s^{N+1} ds + \frac{\chi_N^2}{N^2} s^{2p+N-3} |W|_{L^\infty(I)}$$

$$\leq \frac{1}{4} \int_I |W_t|^2 s^{N+1} ds + \frac{\chi_N^2}{N^2} \left( \int_I s^2 ds + 1 \right),$$

where the last inequality is a consequence of $s^{2p+N-4} \leq 1$ Notice that $N \geq 3$ and therefore

$$2p + N - 4 > 0.$$

Now, we apply (3.9) and (3.4) and it results

$$I_W \leq \frac{3}{4} \int_I |W_t|^2 s^{N+1} ds + d(t) \left[ \int_I |W_s|^2 s^{N+1} ds + 1 \right] + c$$

where

$$d(t) := c \left[ \int_I \left| \frac{\partial W}{\partial s} \right|^2 sds + 1 \right]^{p-1} \in L^{\frac{3}{2p}}(0, T_1).$$

We apply Gronwall’s lemma to the previous inequality to prove that for any $T < T_1$ there exists $c(T)$ such that

$$\int_{I_T} W_t^2 s^{N+1} ds dt + \int_I |W_s|^2 s^{N+1} ds \leq c(T) < \infty,$$

which proves (3.5).

(3.6) is a consequence of (3.9) and (3.5).
Finally, to obtain (3.7) we first multiply by $s^{N+1}[W_{ss} + \frac{N+1}{s} W_s]$ the following equation

$$-N^{-2} \left[ W_{ss} + \frac{N+1}{s} W_s \right] = -W_t + \zeta_N s^{p-2} \left( \frac{s}{N} W_s + W + M \right) |W|^{p-2} W,$$

and integrate over $I$. Now, we proceed as before, and apply Young’s inequality to the right-hand-side terms, after integration over $(0, T)$ and thanks to (3.5) and (3.6) the proof of the lemma ends as a consequence of the following inequality

$$\frac{1}{2} \int_I |W_{ss}|^2 s^{N+1} ds \leq \int_I \left| W_{ss} + \frac{N+1}{s} W_s \right|^2 s^{N+1} dsdt + 2(N+1)^2 \int_I |W_s|^2 s^{N-1} dsdt$$

Lemma 3.4. Let $N \geq 3$, and $W_0 \in H^2_{(N+1)}(I) \cap H^1_{0,s^{N+1}}(I)$, then, there exists $T_{bu} > 0$ and at least a weak solution $W$ to (3.1) in $(0, T_{bu})$.

Proof. We first consider $T > 0$, such that $T < \min \{ T_1, T_2 \}$ and $k > 0$ large satisfying

$$k > 1 + C(T) \quad t \leq T,$$

for $C(T)$ defined in Lemma 3.3. We define the subset

$$Q := \left\{ W \in L^2((0, T) : L^2_{s^{N+1}}(I)), \quad \int_I |W_s|^2 s^{N+1} ds < k \right\}.$$

For a given $W^{n-1} \in Q$, we consider the problem

$$\begin{align*}
W_t^n - N^{-2} \left[ W_{ss}^n + \frac{N+1}{s} W_s^n \right] &= \zeta_N s^{p-2} \left( \frac{s}{N} W_s^n + W^n + M \right) |W^{n-1}|^{p-2} W^{n-1} \\
W^n(t, 0) &= W^n(t, 1) = 0 \\
W^n(0, s) &= W_0(s).
\end{align*}$$

(3.11)

In analogous fashion to Definition 3.1, we define the notion of weak solution to (3.11) i.e. $W^n$ is a weak solution to (3.11), if for any $W^{n-1} \in Q$, $W^n$ satisfies

$$\int_I \zeta_s W^n s^{N+1} dsdt + \int_I \zeta(T) W^n(T) s^{N+1} ds + N^{-2} \int_I \zeta_s W^n s^{N+1} dsdt$$

$$= \int_I \zeta(0) W^n s^{N+1} ds + \zeta_N \int_I s^{p+N-1} \zeta \left( \frac{s}{N} W_s^n + W^n + M \right) |W^{n-1}|^{p-2} W^{n-1} dsdt$$

(3.12)

for all $\zeta \in C^1([0, T] : C^2_s(I))$. We construct a functional $J(W^{n-1}) = W^n$ where $W^n$ is the solution to (3.11). We obtain, in the same way as in Lemma 3.3, the following estimates

$$\int_I |W_t^n|^2 s^{N+1} dsdt + \int_I |W_s^n|^2 s^{N+1} dsdt \leq c(T) < \infty,$$

(3.13)

$$\int_I \left| W_{ss}^n + \frac{N-1}{s} W_s^n \right|^2 s^{N+1} dsdt \leq C(T),$$

(3.14)
(3.15) implies that
\[
(s^{N+1}W^n)_k = s^{N+1}W^n, \quad (s^{N+1}U)_k = s^{N+1}U^n.
\]

Let \(H^2_{rad}(B_{N+2})\) be the Sobolev functional space of radially symmetric functions in \(L^2(B_{N+2})\) defined over the \(N+2\)-dimensional unit ball \(B_{N+2}\), with derivatives in \(L^2(B_{N+2})\) up to order two. Since \(H^2_{rad}(B_{N+2}) \equiv H^2_{sN+1}(I)\)

(see [9] Theorem 2.3) and \(H^2(B_{N+2}) \hookrightarrow H^1(B_{N+2})\) is a compact embedding we have that \(H^2_{sN+1}(I) \hookrightarrow H^1_{sN+1}(I)\) and \(H^1_{sN+1}(I) \hookrightarrow L^2_{sN+1}(I)\) are also compact. Now, Aubin-Lions Theorem and Schauder fixed point Theorem, provides the existence of a fixed point \(W^*\), which is a weak solution of (3.12) for \(T\) small enough. It is possible to extend the solution as far as \(W\) satisfies (3.3), i.e. there exists a \(T_{bu}\) such that there exists a weak solution to (3.1) in \((0, T_{bu}) \times I\).

Now, we introduce the notion of weak solution to (2.8)–(2.10).

**Definition 3.2.** Let \(I = (0, 1)\) and \(U_0 \in H^2 \left( \frac{1}{sN+1} (I) \cap H^2_{0} (I) \cap L^2_{0,sN+1} (I) \right).\) Then, a weak solution of (2.8)–(2.10) in \((0, T) \times I\) is a function \(U \in L^2((0, T) : H^1_0 (I)) \cap H^1((0, T) : L^2 (\rho^{1+sN+1} (I))) \cap L^2((0, T) : L^2 (\rho^{1+sN+1} (I)))\) such that \(U : [0, T] \times I \to \mathbb{R}\) satisfies

\[
-\int_{I_T} \zeta_t U \rho^{\frac{1}{sN+1}} d\rho dt + \int_I \zeta(T) U(T) \rho^{\frac{1}{sN+1}} d\rho + \int_{I_T} \zeta U_{\rho} d\rho dt = \int_I \zeta(0) U_0 \rho^{\frac{1}{sN+1}} d\rho \int_{I_T} \rho^{\frac{1}{sN+1}} [\rho^\frac{1}{sN} + M] [U] d\rho d\rho dt
\]

for all \(\zeta \in C^1([0, T] : C^2(I))\).

**Lemma 3.5.** Let \(N \geq 3\), and \(U_0 \in H^2 \left( \frac{1}{sN+1} (I) \cap H^2_{0} (I) \cap L^2_{0,sN+1} (I) \right),\) and \(U\) a weak solution to (2.8)–(2.10), then, for any \(T \in (0, T_{bu})\) we have

\[
\int_I U^2 \rho^{\frac{1}{sN+1}} d\rho \leq C(T), \quad \int_I |U_t|^2 \rho^{\frac{1}{sN+1}} d\rho dt + \int_I |U_{\rho}|^2 d\rho + \int_I U^2 \rho^{-\frac{1}{sN+1}} d\rho \leq C(T),
\]

\[
\int_I \rho^{-\frac{1}{sN+1}} |U_{\rho}|^2 d\rho dt \leq C(T), \quad |U(t, \rho)| \leq c \rho^{\frac{1}{sN+1}} C(T),
\]

moreover \(U \in C((0, T) : C^0_{sN+1}(I))\).

**Proof.** Since \(W^2 = (s^{N}W^n) s^{-2N} ds = \frac{1}{N} [Ns^{N-1} ds]\) we have

\[
\int_I W^2 ds = \frac{1}{N} \int_I (Ws^N)^2 s^{-2N} s^{N-1} ds = \frac{1}{N} \int_I U^2 \rho^{\frac{1}{sN+1}} d\rho
\]
which implies

$$\int_I U^2 \rho^{\frac{1}{\gamma} - 3} d\rho \leq c(T) < \infty.$$  \hfill (3.16)

We also have that, $sds = N^{-1} \rho^{\frac{1}{\gamma} - 1} d\rho$, and therefore

$$\int_I |W_s|^2 sds = \frac{1}{N} \int_I \left| \frac{1}{\rho^{\frac{1}{\gamma}}} \left( U_{t} - \frac{U}{\rho} \right) \right|^2 \rho^{\frac{1}{\gamma} - 1} d\rho$$

$$= N \int_I \left| U_{t} - \frac{U}{\rho} \right|^2 \rho^{-1} d\rho$$

and it results

$$\int_{I_T} \left| U_{t} - \frac{U}{\rho} \right|^2 \rho^{-1} d\rho dt \leq C(T).$$  \hfill (3.17)

In the same fashion, we obtain, thanks to (3.5) and (3.16)

$$\int_I |U_t|^2 \rho^{\frac{1}{\gamma} - 2} d\rho dt + \int_I |U|^2 d\rho + \int_I U^2 \rho^{-2} d\rho \leq C(T)$$  \hfill (3.18)

which implies, in view of the embedding $H^1_0(I) \hookrightarrow L^\infty(I)$,

$$|U| \leq C(T).$$  \hfill (3.19)

As before, from (3.7) we deduce

$$\int_{I_T} \rho^{2-\frac{2}{\gamma}} |U_{\rho\rho}|^2 d\rho dt \leq C(T).$$  \hfill (3.20)

Notice that thanks to Cauchy-Schwarz inequality

$$U(t, \rho) = \int_0^\rho U_r dr \leq \rho^{\frac{1}{2}} \left[ \int_0^\rho |U_r|^2 dr \right]^{\frac{1}{2}} \leq \rho^{\frac{1}{2}} \left[ \int_I |U_r|^2 dr \right]^{\frac{1}{2}}$$

which implies

$$|U(t, \rho)| \leq c \rho^{\frac{1}{2}} C(T),$$  \hfill (3.21)

for any $t \leq T < T_{bu}$. From (3.18) we have that

$$U \in L^\infty(0, T : H^1_0(I)) \cap H^1(0, T : L^2_{\frac{2}{\gamma} - 2}(I)).$$

Thanks to the compact embedding

$$H^1_0(I) \hookrightarrow C^{0, \frac{1}{2}}_0(I)$$

where $C^{0, \frac{1}{2}}_0(I)$ denotes the Hölder continuous functions in $I$ with zero boundary values and Aubin-Lions Lemma, we have that

$$U \in C((0, T) : C^{0, \frac{1}{2}}_0(I)), \quad \text{for any } T < T_{bu},$$  \hfill (3.22)

and the proof ends.
Lemma 3.6. Let $N \geq 3$, and $U_0 \in H^2_x(I) \cap H^1_0(I) \cap L^2_{0,s-1}(I)$, then, there exists $T_{bu} > 0$ and at least a weak solution $U$ to (2.8)–(2.10), such that

$$\limsup_{t \to T_{bu}} ||U||_{L^\infty} + t = \infty.$$  

Proof. Notice that, since $\zeta, U \in L^2 \rho^{N-2} (I)$ and thanks to Cauchy-Swartz and Young inequalities the term

$$\int_I \rho^{-(N-1)} \zeta |U|^{p-2} Ud\rho \leq \frac{1}{2} \int_I |U|^{p-2} \zeta^2 d\rho + \frac{1}{2} \int_I \rho^{(N-1)} \zeta^2 |U|^{2(p-1)} d\rho$$

is bounded. Now we see the boundedness of the term

$$\int_{I_T} \rho^{-(N-1)} \zeta U_\rho |U|^{p-2} Ud\rho dt$$

in the weak formulation (3.2). As before, we first apply Cauchy-Schwartz and Young inequalities to obtain

$$\int_{I_T} \rho^{-(N-1)} \zeta U_\rho |U|^{p-2} Ud\rho \leq \frac{1}{2} \int_{I_T} |U_\rho|^2 d\rho + \frac{1}{2} \int_{I_T} \rho^{-(N-1)} \zeta^2 |U|^{2(p-1)} d\rho$$

$$\leq \frac{1}{2} \int_{I_T} |U_\rho|^2 d\rho + \frac{c}{2} \left[ \int_{I_T} \rho^{-\frac{N-1}{2}} \zeta^2 d\rho \right]^{1/2} \left[ \int_{I_T} \rho^{-\frac{N-1}{2}} U^2 d\rho \right]^{1/2}.$$  

Thanks to (3.21) we may replace the term $U^{2p}$ by $\rho^{-1} U^2$ in the last integral of the previous inequality, and it results

$$\int_{I_T} \rho^{-\frac{N-1}{2}} U^{2p} d\rho \leq C(T) \int_{I_T} \rho^{-p-1+\frac{2p}{N}} U^2 d\rho.$$  

Now, (3.16) implies the boundedness of the last term for $p \in (1, 2)$ and we get

$$\int_{I_T} \rho^{-\frac{N-1}{2}} U^{2p} d\rho \leq C(T), \quad \text{for} \quad t \leq T.$$  

In view of

$$U(t, \rho) = \rho W(t, \rho^\frac{1}{2})$$

we replace into Definition 3.1 to obtain that $U$ is a weak solution of (2.8)–(2.10).  

4. Constructing a subsolution

We introduce the operator $\mathcal{L}$ defined as follows

$$\mathcal{L}(\phi) := \phi_t - \rho^{\frac{N-1}{2}} \phi_{\rho \rho} - \chi_N \rho^{(2-p)/2} (\phi_\rho + M) |\phi|^{p-2} \phi.$$ (4.1)

Let $\rho_1$ and $\rho_2$ the following positive numbers
\[
\rho_1 = \frac{1}{2} \quad (4.2)
\]
and
\[
\rho_2 = \frac{2 + \gamma}{2(1 + \gamma)} \quad (4.3)
\]
Notice that in view of \( \gamma > 1 \), (4.3) guarantees
\[
\rho_2 \in \left( \frac{1}{2}, \frac{3}{4} \right).
\]
We consider the function \( q \), already defined in the introduction,
\[
q(c) := \left( c + 1 \right) \left( c + 2 \right) \left( c + 3 \right) - c^2 + 1 \quad (4.7)
\]
and the positive real numbers \( \gamma^* \), \( \gamma_0 \) and \( \gamma \) as follows
\[
\gamma^* > 1, \text{ such that } q(s) > \frac{1}{3} \text{ for any } s \in (1, \gamma^*),
\]
\[
\gamma_0 := \min \left\{ 1 + \frac{2 - p}{N(p-1)}, \frac{M - 6}{4}, \frac{N + 1}{N}, \gamma^* \right\} \quad (4.4)
\]
and
\[
\gamma \in (1, \gamma_0).
\]
Then, we construct the following function
\[
\phi_1(\rho, t) = \frac{\rho^\gamma}{\rho^\gamma + a(t)}, \quad 0 < \rho < \rho_1 \quad (4.5)
\]
where
\[
a(t) := (1 - \epsilon t)^{-\frac{1}{2}}, \quad a'(t) = -\frac{\epsilon}{1 - \theta} a^\theta(t),
\]
for
\[
\theta := \frac{3 - p}{2} > 2 - p, \quad \text{for } p \in (1, 2)
\]
and \( \epsilon \) satisfying
\[
\epsilon \leq \min \left\{ \frac{Z_N \gamma (1 - \theta)}{2p}, \frac{Z_N \gamma (p - 1)}{2}, (p - 1) \rho_1 \left[ \frac{M - 3}{4} \right] ^{p-2} - \frac{2(1 + \gamma) \rho_2^{2 - \frac{\gamma}{2}}}{(\rho_2 - \rho_1)(1 - \rho_2)} \right\} \quad (4.6)
\]

**Lemma 4.1.** Let \( \phi_1 \) be defined in (4.5) and the differential operator \( L \) in (4.1), then, under assumptions (1.8)–(1.14), we have that
\[
L(\phi_1) \leq 0, \quad 0 < \rho < \rho_1.
\]
where \( L(\phi_1) \) is understood in the sense of distributions.
Proof. We first compute the following derivatives of $\phi_1$

\[
\phi_{1t} = -\frac{\rho^\gamma a'(t)}{[\rho^\gamma + a(t)]^2},
\]

\[
\phi_{1\rho} = \frac{a(t)\rho^{-1}}{[\rho^\gamma + a(t)]^2},
\]

\[
\phi_{1\rho\rho} = \frac{\gamma (\gamma - 1)a^2(t)\rho^{\gamma - 2} - a(t)(\gamma + 1)\rho^{2\gamma - 2}}{[\rho^\gamma + a(t)]^3},
\]

\[-\rho^{2\gamma - 2}\phi_{1\rho\rho} = -\frac{\gamma (\gamma - 1)a^2(t)\rho^{\gamma - 2} - a(t)(\gamma + 1)\rho^{2\gamma - 2}}{[\rho^\gamma + a(t)]^3},
\]

\[
\rho^{(2-p)\frac{N-1}{N}}M|\phi_1|^{p-2}\phi_1 = \frac{M\rho^{(p-1)(\gamma - 1) + \gamma - \frac{1}{\gamma}}}{[\rho^\gamma + a(t)]^{p-1}},
\]

\[
\rho^{(2-p)\frac{N-1}{N}}\phi_{1\rho}|\phi_1|^{p-2}\phi_1 = \frac{a(t)\gamma \rho^{(p-1)(\gamma - 1) + \gamma - \frac{1}{\gamma}}}{[\rho^\gamma + a(t)]^{p-1}}.
\]

Then,

\[
\mathcal{L}(\phi_1) = \phi_{1t} -\rho^{2N-2}\phi_{1\rho\rho} - \chi_N\rho^{(2-p)\frac{N-1}{N}}(\phi_{1\rho} + M)|\phi_1|^{p-2}\phi_1
\]

\[
= -\frac{\rho^\gamma a'(t)}{[\rho^\gamma + a(t)]^2} + \frac{\gamma (\gamma - 1)a^2(t)\rho^{\gamma - 2} - a(t)(\gamma + 1)\rho^{2\gamma - 2}}{[\rho^\gamma + a(t)]^3} - \frac{\rho^\gamma a(t)\gamma \rho^{(p-1)(\gamma - 1) + \gamma - \frac{1}{\gamma}}}{[\rho^\gamma + a(t)]^{p+1}} - \frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma - 1) + \gamma - \frac{1}{\gamma}}}{[\rho^\gamma + a(t)]^{p+1}}
\]

\[
\leq -\frac{\rho^\gamma a'(t)}{[\rho^\gamma + a(t)]^2} + \gamma (\gamma + 1)\frac{a(t)\rho^{2\gamma - 2}}{[\rho^\gamma + a(t)]^3} - \frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma - 1) + \gamma - \frac{1}{\gamma}}}{[\rho^\gamma + a(t)]^{p+1}} - \frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma - 1) + \gamma - \frac{1}{\gamma}}}{[\rho^\gamma + a(t)]^{p+1}}.
\]

We now cancel the positive terms in the right hand side. We first split the first negative term in the following way

\[
-\frac{\gamma \rho^{2\gamma - 2}}{[\rho^\gamma + a(t)]^3} = \frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma - 1) + \gamma - \frac{1}{\gamma}}}{[\rho^\gamma + a(t)]^{p+1}} - \frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma - 1) + \gamma - \frac{1}{\gamma}}}{[\rho^\gamma + a(t)]^{p+1}}.
\]

To cancel the term $-\frac{\rho^\gamma a'}{[\rho^\gamma + a(t)]^2}$ we proceed as follows:

\[
-\frac{\rho^\gamma a'(t)}{[\rho^\gamma + a(t)]^2} = \frac{\epsilon \rho^\gamma a(t)}{(1 - \theta)[\rho^\gamma + a(t)]^2}
\]

and consider three different cases.
Case 1. \( a(t) \geq \beta \).

\[
\frac{\varepsilon \rho^\gamma a^0(t)}{(1 - \theta)[\rho^\gamma + a(t)]^2} \leq \frac{\varepsilon \rho^\gamma [\rho^\gamma + a(t)]^\theta}{(1 - \theta)[\rho^\gamma + a(t)]^2} \leq \frac{\varepsilon \rho^\gamma}{(1 - \theta)[\rho^\gamma + a(t)]^{2-\theta}}
\]

and

\[
-\frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma-1+\frac{1}{N})+\gamma-\frac{1}{2}}}{2} \leq -\frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma-1+\frac{1}{N})+\gamma-\frac{1}{2}}}{[\rho^\gamma + a(t)]^{p+1}} \leq \frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma-1+\frac{1}{N})+\gamma-\frac{1}{2}}}{4[\rho^\gamma + a(t)]^{p+1}}.
\]

In view of

\[
(p - 1)\left(\gamma - 1 + \frac{1}{N}\right) - \frac{1}{N} < 0, \quad \text{i.e. } \gamma < 1 + \frac{2 - p}{N(p - 1)} \tag{4.7}
\]

and for

\[
\varepsilon \leq \frac{\chi_N \gamma (1 - \theta)}{2^p}
\]

provided

\[
2 - \theta \leq p
\]

we have that

\[
\frac{\varepsilon \rho^\gamma a^0(t)}{(1 - \theta)[\rho^\gamma + a(t)]^2} \leq \frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma-1+\frac{1}{N})+\gamma-\frac{1}{2}}}{2[\rho^\gamma + a(t)]^{p+1}}, \quad \text{for } a(t) \geq \beta.
\]

Case 2. \( \rho^{\frac{p-1}{1-\theta}} \leq a(t) \leq \rho^\gamma. \)

Notice that

\[
\frac{p - 1}{1 - \theta} \geq 1
\]

in view of

\[
\theta \geq 2 - p
\]

and therefore \( \rho^{\frac{p-1}{1-\theta}} \leq \rho^\gamma. \) Then

\[
-\frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma-1+\frac{1}{N})+\gamma-\frac{1}{2}}}{2} \leq -\frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma-1+\frac{1}{N})+\gamma-\frac{1}{2}}}{2^{p-1}\rho^\gamma[\rho^\gamma + a(t)]^2} \leq \frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma-1+\frac{1}{N})+\gamma-\frac{1}{2}}}{2^p} \leq \frac{\chi_N a(t)\gamma \rho^{(p-1)(\gamma-1+\frac{1}{N})+\gamma-\frac{1}{2}}}{2^p[\rho^\gamma + a(t)]^2}
\]

for


\[
e < \frac{\chi_N \gamma (1 - \theta)}{2^p}
\]

and thanks to (4.7) we claim
\[
\frac{\epsilon \rho^\gamma a^0(t)}{(1 - \theta)[\rho^\gamma + a(t)]^2} \leq \frac{\chi_N a(t) \gamma \rho^{(p-1)(\gamma - 1 + \frac{p}{2}) + \gamma - \frac{1}{p}}{2 \left[\rho^\gamma + a(t)\right]^{p+1}}, \quad \text{for } \rho^{\left(p-1\right)} \leq a(t) \leq \rho^\gamma.
\]

**Case 3.** \(a(t) < \rho^{\left(p-1\right)}\)

Notice that \(a(t) < \rho^{\left(p-1\right)}\) is equivalent to
\[
\frac{a(t)}{\rho^{\left(p-1\right)}} < 1, \quad \text{i.e.} \quad \frac{a^0(t)}{\rho^{\left(p-1\right)}} < 1,
\]
then
\[
-\chi_N \rho^{p(\gamma - 1 + \frac{1}{p} + 1 - \frac{1}{p})}[\rho^\gamma + a(t)]^{p-1} \leq -\chi_N \frac{a^0(t) \rho^{(p-1)(\gamma - 1 + \frac{1}{p}) + 1 - \frac{1}{p}}}{\rho^{\left(p-1\right)} + a(t)]^{p-1}},
\]

since \(\theta = \frac{3-\gamma}{2}\) we have that
\[
-\chi_N \rho^{p(\gamma - 1 + \frac{1}{p} + 1 - \frac{1}{p})}[\rho^\gamma + a(t)]^{p-1} \leq -\chi_N \frac{a^0(t) \rho^{(p-1)(\gamma - 1 + \frac{1}{p}) + 1 - \frac{1}{p}}}{\rho^\gamma + a(t)]^{p-1}},
\]
then
\[
\frac{\epsilon \rho^\gamma a^0(t)}{(1 - \theta)[\rho^\gamma + a(t)]^2} \leq \chi_N \rho^{p(\gamma - 1 + \frac{1}{p} + 1 - \frac{1}{p})}[\rho^\gamma + a(t)]^{p-1}, \quad \text{for } a(t) < \rho^{\left(p-1\right)}.
\]

Cases 1, 2 and 3 imply
\[
\frac{\rho^\gamma a'}{[\rho^\gamma + a(t)]^2} - \frac{\chi_N a(t) \gamma \rho^{(p-1)(\gamma - 1 + \frac{1}{p}) + \gamma - \frac{1}{p}}}{2 \left[\rho^\gamma + a(t)\right]^{p+1}} - \chi_N \rho^{p(\gamma - 1 + \frac{1}{p} + 1 - \frac{1}{p})}[\rho^\gamma + a(t)]^{p-1} \leq 0.
\] (4.8)

The second term is expressed in the following way:
\[
\gamma(\gamma + 1) \frac{a(t) \rho^{2\gamma - \frac{2}{\gamma}}}{[\rho^\gamma + a(t)]^3} = \gamma(\gamma + 1) \frac{a(t) \rho^{2\gamma - \frac{2}{\gamma}}}{[\rho^\gamma + a(t)]^{1+\frac{1}{p}}[\rho^\gamma + a(t)]^{p-2}}
\]
\[
\leq \gamma(\gamma + 1) \frac{a(t) \rho^{2\gamma - \frac{2}{\gamma} + \gamma(p-2)}}{[\rho^\gamma + a(t)]^{1+\frac{1}{p}}}
\]
\[
= \gamma(\gamma + 1) \frac{a(t) \rho^{2\gamma - \frac{2}{\gamma}}}{[\rho^\gamma + a(t)]^{1+\frac{1}{p}}}
\]
and
\[
\frac{\chi_N a(t) \gamma \rho^{p-1}(\gamma+1)^p + \gamma - \frac{1}{\rho}}{[\rho^p + a(t)]^{p+1}} = - \frac{a(t) \gamma (\gamma + 1) \rho^{p+\frac{1}{2}}}{[\rho^p + a(t)]^{p+1}} \frac{\chi_N}{2(\gamma + 1)} \rho^{1+p(-1+\frac{1}{\rho})}
\]

thanks to assumption (1.8)

\[
1 + p \left( -1 + \frac{1}{N} \right) < 1 + \frac{N}{N-1} \left( -1 + \frac{1}{N} \right) = 0.
\]

Then, for

\[
\gamma < \frac{\chi_N}{2} - 1
\]

we have that \( \phi_1 \) satisfies

\[
\mathcal{L}(\phi_1) \leq 0, \quad \text{in } \rho < \rho_1.
\]

We now consider the function \( \phi_2 \) defined in \((\rho_1, 1)\) as follows

\[
\phi_2(\rho, t) := \beta(t) \left( 1 - \rho + \frac{\kappa (\rho_2 - \rho)(\rho - \rho_1)}{(\rho_2 - \rho_1)} \right)
\]

for

\[
\rho_2 := \frac{2 + \gamma}{2(\gamma + 1)}
\]

where

\[
\beta(t) = \phi_1(t, \rho_1)(1 - \rho_1)^{-1} = \frac{2}{2 - \gamma} \frac{2 - \gamma}{2 - \gamma + a(t)} \leq 2
\]

and

\[
\kappa = 1 + \gamma.
\]

Notice that the function

\[
\left( 1 - \rho + \frac{\kappa (\rho_2 - \rho)(\rho - \rho_1)}{(\rho_2 - \rho_1)} \right)
\]

attains its maximum at

\[
\frac{\rho_1 + \rho_2}{2} - \frac{\rho_2 - \rho_1}{2k} = \frac{1}{2k} (\rho_1 (\kappa + 1) + \rho_2 (\kappa - 1)) = \frac{(\gamma + 2)(2\gamma + 1)}{4(\gamma + 1)^2}
\]

and at this point

\[
\phi_2 = \beta(t) \left[ 1 - \frac{(\gamma + 2)(2\gamma + 1)}{4(\gamma + 1)^2} + \frac{1 + \gamma}{2(\gamma + 1)} \left( \frac{\gamma + 2}{2(\gamma + 1)} - \frac{(\gamma + 2)(2\gamma + 1)}{4(\gamma + 1)^2} \right) \left( \frac{(\gamma + 2)(2\gamma + 1)}{4(\gamma + 1)^2} - \frac{1}{2} \right) \right].
\]

After some computations we get

\[
\phi_2 = \beta(t) \left[ 1 - \frac{(\gamma + 2)(2\gamma + 1)}{4(\gamma + 1)^2} + \frac{(\gamma + 2)\gamma}{8(\gamma + 1)^2} \right].
\]

which implies
\[
\phi_2 = \beta(t) \left[ 1 - \frac{(\gamma + 2)}{4(\gamma + 1)^2} \left[ \frac{3\gamma}{2} + 1 \right] \right].
\]

Since
\[
\lim_{\gamma \to 1} \frac{(\gamma + 2)}{4(\gamma + 1)^2} \left[ \frac{3\gamma}{2} + 1 \right] = \frac{15}{32} > \frac{1}{3}
\]
there exists \( \gamma^* > 1 \) such that
\[
\frac{(\gamma + 2)}{4(\gamma + 1)^2} \left[ \frac{3\gamma}{2} + 1 \right] \geq \frac{1}{3} > 0
\]
for any \( \gamma \in [1, \gamma^*] \). Therefore for such \( \gamma \) we have that
\[
\phi_2 < \frac{2\beta(t)}{3}.
\]

**Lemma 4.2.** Let \( \phi_2 \) be defined in (4.9) and the differential operator \( \mathcal{L} \) in (4.1), then, under assumptions (1.8)–(1.13), for \( \chi_N \) large enough, satisfying
\[
\chi_N > \frac{\rho_2^{2-\frac{2}{p}} 2^{5-p} (1 + \gamma)}{(\rho_2 - \frac{1}{2}) (1 - \rho_2) (M - 6)}
\]
we have that
\[
\mathcal{L}(\phi_2) \leq 0, \quad 0 < \rho_1 < \rho < 1.
\]
where \( \mathcal{L}(\phi_2) \) is understood in the sense of distributions.

**Proof.** In order to obtain \( \mathcal{L}(\phi_2) \) we compute the derivative of \( \phi_2 \)
\[
(\phi_2)_t = - \frac{\rho_1^t d(t)}{[\rho_1^t + a(t)]^2} \left( 1 - \rho + \kappa \frac{(\rho_2 - \rho) + (\rho - \rho_1)}{(\rho_2 - \rho_1)} \right) (1 - \rho_1)^{-1}
\]
\[
= - \frac{a'(t)}{[\rho_1^t + a(t)]} \phi_2
\]
\[
= \frac{e a'(t)}{(p - 1)(\rho_1^t + a(t))} \phi_2,
\]
\[
(\phi_2)_\rho = \begin{cases} 
-\beta(t) (1 + \kappa \frac{2p - (\rho_1 + \rho_2)}{(\rho_2 - \rho_1)}), & \rho_1 < \rho < \rho_2, \\
-\beta(t), & \rho_2 < \rho < 1,
\end{cases}
\]
\[
 \geq - (1 + \kappa) \beta(t), \quad \rho_1 < \rho < 1,
\]
\[
(\phi_2)_{\rho \rho} = \begin{cases} 
-\frac{2\beta(t)}{(\rho_1^t - \rho_2^t)}, & \rho_1 < \rho < \rho_2, \\
0, & \rho_2 < \rho < 1,
\end{cases}
\]

\[-\chi_N \rho^{(2-p)\frac{1}{p-1}} (\phi_2 + M) |\phi_2|^{p-1} \leq -\chi_N \rho^{(2-p)\frac{1}{p-1}} (M - (1 + \kappa) \beta(t)) |\phi_2|^{p-1}.
\]

Notice that \( (\phi_2)_\rho \) presents a positive jump at \( \rho = \rho_2 \) for any \( \kappa > 0 \), then, we have
\[ L(\phi_2) \leq \frac{\epsilon a^0(t)\phi_2}{(p - 1)(\rho_1^\gamma + \alpha(t))} + 2\kappa \beta(t) \chi_{[\rho_1^\gamma, \rho_2^\gamma]} \rho^{2-\frac{\gamma}{p}} - \chi_N \rho_2^{\gamma(p-2)\frac{\gamma}{N}} (M - (1 + \kappa)\beta(t)) |\phi_2|^{p-1}. \]  

(4.11)

In view of \( a(t) \leq 1 \) and \( \theta = (3 - p)/2 \) it results
\[ \frac{\epsilon a^0(t)\phi_2}{(p - 1)(\rho_1^\gamma + \alpha(t))} \leq \frac{\epsilon \phi_2}{(p - 1)(\rho_1^\gamma + \alpha(t))^{\frac{p-1}{2}}}. \]

Since
\[ \phi_2 \geq \beta(t)(1 - \rho), \]
the second term is bounded as follows
\[ \frac{2\kappa \beta(t) \chi_{[\rho_1^\gamma, \rho_2^\gamma]} \rho^{2-\frac{\gamma}{p}}}{(\rho_2 - \rho_1)} \leq \frac{2(1 + \gamma) \beta(t) \chi_{[\rho_1^\gamma, \rho_2^\gamma]} \rho^{2-\frac{\gamma}{p}}}{(\rho_2 - \rho_1)} \phi_2 \leq \frac{2(1 + \gamma)}{(\rho_2 - \rho_1)(1 - \rho_2)} \rho^{2-\frac{\gamma}{p}} \phi_2. \]

Then, thanks to the previous computations, for
\[ \frac{M}{2} > (1 + \kappa)\beta, \]
we get
\[ L(\phi_2) \leq \phi_2 \left[ \frac{\epsilon}{(p - 1)(\rho_1^\gamma + \alpha(t))^{\frac{p-1}{2}}} + \frac{2(1 + \gamma)}{(\rho_2 - \rho_1)(1 - \rho_2)} \rho^{2-\frac{\gamma}{p}} - \chi_N (M - (1 + \kappa)\beta(t)) |\phi_2|^{p-2} \right]. \]

Since \( p < 2 \) and
\[ \phi_2 \leq \frac{2}{3} \beta \leq \frac{2}{3(1 - \rho_1)} = \frac{4}{3}, \]
it results
\[ L(\phi_2) \leq \phi_2 \left[ \frac{\epsilon}{(p - 1)\rho_1^\gamma} + \frac{2(1 + \gamma)}{(\rho_2 - \rho_1)(1 - \rho_2)} \rho^{2-\frac{\gamma}{p}} - \chi_N (M - 2(1 + \kappa)) \left( \frac{4}{3} \right)^{p-2} \right]. \]  

(4.12)

provided
\[ M > 2(1 + \kappa) = 2(2 + \gamma). \]

Thanks to (4.4) we have
\[ \gamma \leq 1 + \frac{M - 6}{4} \]

and the term
\[ M > 2(1 + \kappa) = 2(2 + \gamma). \]
\[-\chi_N(M - 2(1 + \kappa)) \left(\frac{4}{3}\right)^{\rho - 2} \leq -\chi_N \left(M - 2 \left(3 + \frac{M - 6}{4}\right)\right) \left(\frac{4}{3}\right)^{\rho - 2} = -\chi_N \left(\frac{M}{2} - 3\right) \left(\frac{4}{3}\right)^{\rho - 2}\]

then, for

\[
\chi_N > \frac{2(1 + \gamma)}{(\rho_2 - \rho_1)(1 - \rho_2)} \left(\frac{4}{3}\right)^{2 - \frac{2}{\rho_2}} \rho_2^{\frac{2}{\rho_2}} \left(M - 3\right)^{-1} = \frac{\rho_2^{\frac{2}{\rho_2}} 2^2(1 + \gamma)}{(\rho_2 - \rho_1)(1 - \rho_2)(M - 6) \left(\frac{4}{3}\right)}^{2 - \rho}
\]

and

\[
\epsilon < (p - 1)\rho_1^{\gamma(p-1)2} \left[\chi_N \left(\frac{M}{2} - 3\right) \left(\frac{4}{3}\right)^{\rho - 2} - \frac{2(1 + \gamma)}{(\rho_2 - \rho_1)(1 - \rho_2) \rho_2^{\frac{2}{\rho_2}}} \right]
\]

we have

\[
\mathcal{L}(\phi_2) \leq 0, \quad \rho_1 < \rho < 1,
\]

and the proof ends.

\section*{5. A comparison lemma and uniqueness of solutions}

In this section we prove a comparison lemma and uniqueness of solutions under suitable assumptions in the initial data. We first present some previous results.

\textbf{Lemma 5.1.} Let $p$ be a positive constant satisfying $p \in (1, 2)$ (i.e. $p - 1 \in (0, 1)$) and $x$ and $y$ be positive numbers such that 

\[
x \geq 0, \quad y > 0, \quad x < ky,
\]

for some $k \in (0, 1)$. Then,

\[
y^{p-1} - x^{p-1} = (p - 1)\xi^{p-2}(y - x),
\]

where $\xi$ satisfies

\[
\xi \geq k_0 y, \quad \text{for } k_0 := \left((1 - p)(1 - k)\right)^{-\frac{1}{p-1}}. \tag{5.1}
\]

\textbf{Proof.} The proof is a direct application of Mean Value Theorem, where $\xi$ satisfies

\[
\xi^{2-p} = \frac{(p - 1)(y - x)}{y^{p-1} - x^{p-1}} \geq \frac{(p - 1)(1 - k)y}{y^{p-1}} = (1 - p)(1 - k)y^{2-p},
\]

i.e.,

\[
\xi \geq ((1 - p)(1 - k))^{\frac{1}{p-1}}y
\]

which ends the proof. \hfill \Box

\textbf{Lemma 5.2.} Let $\epsilon_1$, $\delta$ and $\kappa$ any strictly positive numbers, such that

\[
0 < \epsilon_1 \leq \kappa.
\]
Let \( I = (0, 1) \), and \( u \in L^2_{\rho} (I) \), then,

\[
\int_I \rho^\varepsilon_1 u^2 \, d\rho \leq \rho_0^\varepsilon_1 \rho^\varepsilon_1 u^2 \, d\rho + \rho_0^{\varepsilon_1 - \kappa} \int_I \rho^{\kappa - \delta} u^2 \, d\rho \tag{5.2}
\]

for any \( \rho_0 \in (0, 1) \).

**Proof.** Since

\[
\int_I \rho^\varepsilon_1 u^2 \, d\rho = \int_0^{\rho_0} \rho^\varepsilon_1 u^2 \, d\rho + \int_{\rho_0}^1 \rho^\varepsilon_1 u^2 \, d\rho,
\]

\[
\int_0^{\rho_0} \rho^\varepsilon_1 u^2 \, d\rho \leq \rho_0^\varepsilon_1 \int_0^{\rho_0} \rho^{\varepsilon_1 - \delta} u^2 \, d\rho,
\]

\[
\int_{\rho_0}^1 \rho^\varepsilon_1 u^2 \, d\rho \leq \rho_0^{\varepsilon_1 - \kappa} \int_{\rho_0}^1 \rho^{\kappa - \delta} u^2 \, d\rho
\]

hold, we obtain (5.2) in view of

\[
\int_0^{\rho_0} \rho^{\varepsilon_1 - \delta} u^2 \, d\rho \leq \int_I \rho^{\varepsilon_1 - \delta} u^2 \, d\rho
\]

and

\[
\int_{\rho_0}^1 \rho^{\kappa - \delta} u^2 \, d\rho \leq \int_I \rho^{\kappa - \delta} u^2 \, d\rho
\]

for any \( \rho_0 \in (0, 1) \).

\[\Box\]

**Lemma 5.3.** Let \( p > 1 \) and \( U \) a solution to (2.8) with the initial data \( U_0 \) satisfying

\[ U_0 \geq \phi(0, x). \]

Then, under assumptions of Theorem (1.1), the solution \( U \) satisfies,

\[ U(t, \rho) \geq \phi(t, \rho), \quad \rho \in (0, 1), \quad t < T_{bu} := \frac{1}{\epsilon} \]

for \( \epsilon \) defined in (4.6).

**Proof.** We proceed by contradiction and assume that there exists \( t_0 \in (0, T_{bu}) \) and \( \rho_3 \in (0, 1) \) such that

\[ U(t_0, \rho_3) < \phi(t_0, \rho_3). \]

Then, due to the continuity of \( U \) we have that

\[
\int_{I_\rho} \rho^{-\delta} (\phi(t, \rho) - U(t, \rho))^2 \, d\rho \, dt > 0. \tag{5.3}
\]

We denote by \( w \) the difference between \( \phi \) and \( U \), i.e.

\[ w = \phi - U \]

which satisfies the equation
\[ w_t - \rho^{\frac{2N-2}{N}} w_{\rho \rho} \leq \int \frac{1}{p} \left( (\phi^p)_\rho - (U^p)_\rho \right) + M \int (|\phi|^{p-2} \phi - |U|^{p-2} U) \]

in the sense of distributions. We now multiply by \( \rho^{\frac{2}{N} - \delta} w_+ \) for some \( \delta \in (1, \gamma) \) then, after integration over \( I_T := I \times (0, T) \) for \( I = (0, 1) \) we get

\[
\frac{1}{2} \int_I \rho^{2-\delta} w_{\rho \rho} w_+ d\rho d\tau \leq \int_{I_T} \rho^{2-\delta} w_{\rho \rho} w_+ d\rho d\tau \\
+ \int_{I_T} \rho^{(2-p)\frac{N+1}{N-1} + \frac{2}{N} - \delta} \left( (\phi^p)_\rho - (U^p)_\rho \right) w_+ d\rho d\tau \\
+ M \int_{I_T} \rho^{(2-p)\frac{N+1}{N-1} + \frac{2}{N} - \delta} (|\phi|^{p-2} \phi - |U|^{p-2} U) w_+ d\rho d\tau.
\]

For simplicity we label the integrals in the previous equation in the following way

\[
I_1 := \int_{I_T} \rho^{2-\delta} w_{\rho \rho} w_+ d\rho d\tau, \\
I_2 := \int_{I_T} \rho^{(2-p)\frac{N+1}{N-1} + \frac{2}{N} - \delta} \left( (\phi^p)_\rho - (U^p)_\rho \right) w_+ d\rho d\tau, \\
I_3 := M \int_{I_T} \rho^{(2-p)\frac{N+1}{N-1} + \frac{2}{N} - \delta} (|\phi|^{p-2} \phi - |U|^{p-2} U) w_+ d\rho d\tau.
\]

Then, (5.5) is expressed as follows

\[
\frac{1}{2} \int_I \rho^{2-\delta} w_{\rho \rho} w_+ d\rho d\tau = I_1 + I_2 + I_3.
\]

Notice that, since

\[
\lim_{\rho \to 0^+} \phi_\rho \phi \rho^{2-\delta} = \lim_{\rho \to 1^-} \phi_\rho \phi \rho^{2-\delta} = 0,
\]

and

\[
\lim_{\rho \to 0^+} \phi^2 \rho^{1-\delta} = \lim_{\rho \to 1^-} \phi^2 \rho^{1-\delta} = 0,
\]

we have

\[
I_1 = \int_{I_T} \rho^{2-\delta} w_{\rho \rho} w_+ d\rho d\tau \\
= -\int_{I_T} \rho^{2-\delta} w_{\rho \rho}^2 d\rho d\tau - (2-\delta) \int_{I_T} \rho^{1-\delta} (w_+) w_+ d\rho d\tau \\
= -\int_{I_T} \rho^{2-\delta} |w_{\rho \rho}|^2 d\rho d\tau - \frac{2-\delta}{2} \int_{I_T} \rho^{1-\delta} (w_+)^2 d\rho d\tau \\
= -\int_{I_T} \rho^{2-\delta} |w_{\rho \rho}|^2 d\rho d\tau + \frac{(2-\delta)(1-\delta)}{2} \int_{I_T} \rho^{-\delta} (w_+)^2 d\rho d\tau
\]

Therefore
\[ I_1 = - \int_{I_T} \rho^{2-\delta}|w_+|^2 \, d\rho dt + \left( \frac{2 - \delta}{2} \right) \left( 1 - \delta \right) \int_{I_T} \rho^{-\delta}(w_+)^2 \, d\rho dt. \] (5.7)

For simplicity in the notation, we label the previous integrals in the following way

\[ I_{1a} = - \int_{I_T} \rho^{2-\delta}|w_+|^2 \, d\rho dt \] (5.8)

and

\[ I_{1b} = \frac{(2 - \delta)(1 - \delta)}{2} \int_{I_T} \rho^{-\delta}w_+^2 \, d\rho dt. \] (5.9)

Now we integrate by parts in \( I_2 \),

\[
I_2 = \frac{Z_N}{2p} \int_{I_T} \rho^{(2-p) \frac{N-1}{N} + \frac{\gamma}{2} - \delta}(\phi^p - U^p)w_+ d\rho dt
- \frac{Z_N}{p} \int_{I_T} \rho^{(2-p) \frac{N-1}{N} + \frac{\gamma}{2} - \delta}(\phi^p - U^p)_+(w_+)_\rho d\rho dt
- (2 - p) \frac{N - 1}{N} + 2 \frac{N - \delta}{\rho} \frac{Z_N}{p} \int_{I_T} \rho^{(2-p) \frac{N-1}{N} + \frac{\gamma}{2} - \delta - 1}(\phi^p - U^p)_+ w_+ d\rho dt,
\]

where the boundary terms
\[ \rho^{(2-p) \frac{N-1}{N} + \frac{\gamma}{2} - \delta}(\phi^p - U^p)_+ w_+, \quad \text{at } \rho = 0, 1 \]
are equal to 0 as a consequence of \(|\phi| < \epsilon \rho^\gamma\) and \(\delta < \gamma\). We label the previous integrals in the following way

\[
I_{2a} = - \frac{Z_N}{p} \int_{I_T} \rho^{(2-p) \frac{N-1}{N} + \frac{\gamma}{2} - \delta}(\phi^p - U^p)_+(w_+)_\rho d\rho dt
\]

\[
I_{2b} := - \left( 2 - p \right) \frac{N - 1}{N} + 2 \frac{N - \delta}{\rho} \frac{Z_N}{p} \int_{I_T} \rho^{(2-p) \frac{N-1}{N} + \frac{\gamma}{2} - \delta - 1}(\phi^p - U^p)_+ w_+ d\rho dt
\]

to write

\[ I_2 = I_{2a} + I_{2b}. \]

Notice that, thanks to Young inequality

\[
I_{2a} := \frac{Z_N}{p} \int_{I_T} \rho^{(2-p) \frac{N-1}{N} + \frac{\gamma}{2} - \delta}(\phi^p - U^p)_+(w_+)_\rho d\rho dt
\]

\[
= \frac{Z_N}{p} \int_{I_T} \rho^{2-\delta-\frac{\gamma}{2}}(\phi^p - U^p)_+(w_+)_\rho d\rho dt
\]

\[
\leq \frac{1}{2} \int_{I_T} \rho^{2-\delta}|(w_+)_\rho|^2 d\rho dt + \int_{I_T} \rho^{2-\delta-2\frac{\gamma}{2}}(\phi^p - U^p)_+^2 d\rho dt
\]

\[
\leq \frac{1}{2} \int_{I_T} \rho^{2-\delta}|(w_+)_\rho|^2 d\rho dt + \int_{I_T} \rho^{2-\delta-2\frac{\gamma}{2}}\phi^{2p-2}w_+^2 d\rho dt.
\]

Notice that the last inequality is obtained by Mean Value theorem since \(U \leq \phi\). In view of
\[
\phi^{2p-2} \leq c \frac{\rho^{2p-2}}{a^{2(p-1)}(t)}
\]

we have that
\[
\int_I \rho^{2-\delta-2pN\frac{1}{N}} \phi^{2p-2} w_+^2 \, dp \, dt \leq c \int_I a^{2(p-1)}(t) \rho^{2-\delta-2pN\frac{1}{N}} w_+^2 \, dp \, dt
\]
\[
= c \int_I a^{2(p-1)}(t) \rho^{-\delta + \frac{2p}{N}} w_+^2 \, dp \, dt.
\]

Then, we have
\[
I_{2a} \leq \frac{1}{2} \int_I \rho^{2-\delta} |(w_+)|^2 \, dp \, dt + c \int_I a^{2(p-1)}(t) \rho^{-\delta + \frac{2p}{N}} w_+^2 \, dp \, dt. \quad (5.10)
\]

We proceed with \(I_{2b}\) in the following way
\[
I_{2b} = \left( \delta - (2-p) \frac{N-1}{N} - \frac{2}{N} \right) \frac{\int a}{\rho} \int_I \rho^{(2-p)N\frac{1}{N} + \frac{2}{N} - \delta - 1} (\phi^p - U^p)_+ w_+ \, dp \, dt
\]
\[
\leq c \int_I \rho^{(2-p)N\frac{1}{N} + \frac{2}{N} - \delta - 1} \phi^{p-1} w_+^2 \, dp \, dt
\]
\[
\leq c \int_I a^{-p}(t) \rho^{(2-p)N\frac{1}{N} + \frac{2}{N} - \delta - 1 + \gamma(p-1)} w_+^2 \, dp \, dt
\]

since
\[
(2-p) \frac{N-1}{N} + \frac{2}{N} - \delta - 1 + \gamma(p-1) = (p-1) \left( \frac{N-1}{N} \right) + \frac{1}{N} - \delta \geq -\delta + \frac{1}{N}
\]

it results
\[
I_{2b} \leq \int_I c a^{-p}(t) \rho^{-\delta + \frac{2p}{N}} w_+^2 \, dp \, dt.
\]

Then, thanks to Lemma 5.2, we have that
\[
I_{2b} \leq \int_I \rho^\frac{1}{p} a^{-p}(t) \rho^{-\delta} w_+^2 \, dp \, dt + \int_I c a^{-p}(t) \rho^\frac{1}{p} \rho^{-2\delta} w_+^2 \, dp \, dt
\]

where
\[
\rho^\frac{1}{p} \leq \frac{(2-\delta)(\delta-1)a^{(p-1)}(t)}{8c}, \quad \text{for } t \leq T.
\]

Then,
\[
I_2 \leq \frac{1}{2} \int_I \rho^{2-\delta} (w_+)_p^2 \, dp \, dt + \int_0^T c a^{2N(1-p)}(t) \int_I \rho^{\frac{2p}{N}} \rho^{-2\delta} w_+^2 \, dp \, dt
\]
\[
+ \frac{(2-\delta)(\delta-1)}{8} \int_I \rho^{-\delta} w_+^2 \, dp \, dt. \quad (5.11)
\]

We consider now
\[ I_3 = M\mathcal{L}\int_{I_T} \rho^{(2-p)\frac{N-1}{N} + \frac{1}{p} - \delta} (|\phi|^{p-2}\phi - |U|^{p-2}U) w_+ d\rho dt. \]

Thanks to Lemma 5.1,

\[ \int_{I_T} \rho^{(2-p)\frac{N-1}{N} + \frac{1}{p} - \delta} (|\phi|^{p-1} - |U|^{p-1}) w_+ d\rho dt \leq c \int_{I_T} \rho^{(2-p)\frac{N-1}{N} + \frac{1}{p} - \delta} |\phi|^{p-2} w_+^2 d\rho dt \]

and

\[ \phi^{p-2} \leq c\rho^{\gamma(p-2)}(1 - \rho)^{p-2} \]

we have

\[ I_3 \leq c \int_{I_T} \rho^{(2-p)\frac{N-1}{N} + \frac{1}{p} - \gamma(p-2)} (1 - \rho)^{p-2} w_+^2 d\rho dt. \]

Since

\[ \gamma \leq \frac{N+1}{N}, \quad \text{and} \quad p < 2. \]

we have that

\[ (2 - p) \left( \frac{N - 1}{N} - \gamma \right) + \frac{2}{N} - \delta \geq \frac{p - 1}{N} - \delta, \quad (5.12) \]

and therefore

\[ I_3 \leq c \int_{I_T} \rho^{\frac{p-1}{N} - \delta} (1 - \rho)^{p-2} w_+^2 d\rho dt. \]

Notice that the previous integral presents two possible singularities, at \( \rho = 0 \) and at \( \rho = 1 \). To treat them, we first consider the positive number \( \rho_0' < \frac{1}{2} \) small enough such that

\[ c_1 (1 - \rho_0')^\delta \left( \frac{\rho_0'}{p-1} \right) \leq \frac{1}{4} \]

and we split the previous integral into three parts in the following way

\[ I_3 \leq c \int_0^T \int_0^{\rho_0'} \rho^{\frac{p-1}{N} - \delta} (1 - \rho)^{p-2} w_+^2 d\rho dt \]

+ \( c \int_0^T \int_{\rho_0'}^1 \rho^{\frac{p-1}{N} - \delta} (1 - \rho)^{p-2} w_+^2 d\rho dt \)

+ \( c \int_0^T \int_{1 - \rho_0'}^1 \rho^{\frac{p-1}{N} - \delta} (1 - \rho)^{p-2} w_+^2 d\rho dt \).

Now we denote the previous integrals by \( I_{3a} \), \( I_{3b} \) and \( I_{3c} \) respectively, i.e.
\[ I_{3a} := c \int_0^T \int_0^1 \rho^{p-1} (1 - \rho)^{\delta-2} w_+^2 \, d\rho \, dt, \]
\[ I_{3b} := c \int_0^T \int_{1/2}^{1-\rho_0} \rho^{p-1} (1 - \rho)^{\delta-2} w_+^2 \, d\rho \, dt, \]
\[ I_{3c} := c \int_0^T \int_{1-\rho_0}^{1} \rho^{p-1} (1 - \rho)^{\delta-2} w_+^2 \, d\rho \, dt \]

to write
\[ I_3 = I_{3a} + I_{3b} + I_{3c}. \]

We consider first \( I_{3a} \), which satisfies
\[ I_{3a} \leq c \int_0^T \int_0^1 \rho^{\delta-\delta} w_+^2 \, d\rho \, dt \]
\[ \leq c \int_{I_T} \rho^{\delta-\delta} w_+^2 \, d\rho \, dt. \]

We now apply Lemma 5.2, for \( \epsilon_1 = \frac{p-1}{N} \) and \( \kappa = \frac{2}{N} \), and \( \rho \) such that \( \rho_0^{\frac{p-1}{N}} < \frac{(2-\delta)(\delta-1)}{8} \) to obtain
\[ I_{3a} \leq \frac{(2 - \delta)(\delta - 1)}{8} \int_{I_T} \rho^{\delta-\delta} w_+^2 \, d\rho \, dt + c \int_{I_T} \rho^{\frac{\delta-\delta}{p}} w_+^2 \, d\rho \, dt. \hspace{1cm} (5.13) \]

In similar way we have that
\[ I_{3b} \leq c \int_0^T \int_{1/2}^{1-\rho_0} \rho^{\delta-\delta} w_+^2 \, d\rho \, dt \leq 2^{\frac{1}{p}} c \int_{I_T} \rho^{\frac{\delta-\delta}{p}} w_+^2 \, d\rho \, dt. \hspace{1cm} (5.14) \]

Now, we consider \( I_{3c} \), then
\[ I_{3c} \leq c \int_0^T \int_{1-\rho_0}^{1} \rho^{\delta-\delta} (1 - \rho)^{\delta-2} w_+^2 \, d\rho \, dt \]
\[ \leq (1 - \rho_0')^{\delta-\delta} c \int_0^T \int_{1-\rho_0'}^{1} (1 - \rho)^{\delta-2} w_+^2 \, d\rho \, dt \]
\[ \leq c_1 \int_0^T \int_{1-\rho_0'}^{1} (1 - \rho)^{\delta-2} ||w_+||_{L^\infty(1-\rho_0', 1)}^2 \, d\rho \, dt. \]

After integration in the term
\[ \int_{1-\rho_0'}^{1} (1 - \rho)^{\delta-2} \, d\rho \]
we have
\[ I_{3c} \leq c_1 (\rho_0')^{\frac{p-1}{p-1}} \int_0^T ||w_+||_{L^\infty(1-\rho_0', 1)}^2 \, dt. \]

Thanks to the embedding \( L^\infty(I) \subset H^1(I) \) for any bounded one-dimensional interval \( I \), we have, in view of \( w=0 \) at \( \rho=1 \), that
\[\|w_+\|_{L^\infty(1,\rho_0)}^2 \leq \rho_0' \int_{1-\rho_0}^1 |(w_+)_\rho|^2 d\rho.\]

Then
\[
I_{3c} \leq c_1 \left( \frac{\rho_0'}{P - 1} \right)^p \int_0^T \int_{1-\rho_0}^1 |(w_+)_\rho|^2 d\rho dt \\
\leq c_1 (1 - \rho_0')^{\delta - 2} \left( \frac{\rho_0'}{P - 1} \right)^p \int_{I_T} \rho^{2 - \delta} |(w_+)_\rho|^2 d\rho dt
\]
i.e.
\[
I_{3c} \leq c_1 (1 - \rho_0')^{\delta - 2} \left( \frac{\rho_0'}{P - 1} \right)^p \int_{I_T} \rho^{2 - \delta} |(w_+)_\rho|^2 d\rho dt
\]
we take \(\rho_0'\) small enough such that
\[
c_1 (1 - \rho_0')^{\delta - 2} \left( \frac{\rho_0'}{P - 1} \right)^p < \frac{1}{4}
\]
which implies,
\[
I_{3c} \leq \frac{1}{4} \int_{I_T} \rho^{2 - \delta} |(w_+)_\rho|^2 d\rho dt.
\] (5.15)

thanks to (5.13), (5.14) and (5.15)
\[
I_3 \leq \frac{(2 - \delta)(\delta - 1)}{8} \int_{I_T} \rho^{-\delta} w_+^2 d\rho dt + c \int_{I_T} \rho^{-\delta} w_+^2 d\rho dt + \frac{1}{4} \int_{I_T} \rho^{2 - \delta} |(w_+)_\rho|^2 d\rho dt
\] (5.16)

We replace (5.7), (5.11) and (5.16) into (5.5) to obtain
\[
\frac{1}{2} \int_{I_T} \rho^{\delta - \delta} w_+^2 d\rho \left|_{t} \right. + \frac{1}{4} \int_{I_T} \rho^{2 - \delta} |(w_+)_\rho|^2 d\rho dt \leq \int_{I_T} c(a(t)) \rho^{\delta - \delta} w_+^2 d\rho dt
\]

Notice that, the terms
\[
\frac{(\delta - 2)(\delta - 1)}{8} \int_{I_T} \rho^{-\delta} w_+^2 d\rho dt
\]
in \(I_2\) and \(I_3\) are canceled with the term in \(I_1\)
\[
- \frac{(2 - \delta)(\delta - 1)}{2} \int_{I_T} \rho^{-\delta} w_+^2 d\rho dt
\]
thanks to (5.4). Now, Gronwall’s lemma provides us
\[
\int_{I_T} \rho^{\delta - \delta} w_+^2 d\rho dt = 0
\]
which contradicts (5.3) and the proof ends for \(t < T_{bu}\). \(\square\)

**Lemma 5.4.** Let \(p\) and \(M\) be positive numbers satisfying (1.8) and (1.9) respectively. Then, the problem (2.8) has at most one solution under assumptions
\[
U(0, \rho) \geq \phi(0, \rho),
\] (5.17)
\begin{equation}
U \leq A\rho^{\frac{N-1}{N}},
\end{equation}
for a positive constant \( A \) large enough.

\textbf{Proof.} We proceed by contradiction, and assume that there exists two different solutions of (2.8) with the same initial data \( U_1 \) and \( U_2 \). We define the function
\[ w = U_1 - U_2 \]
which satisfies, in view of positivity of \( U_1 \) and \( U_2 \)
\begin{equation}
w_t - \rho^{\frac{2N-2}{N+1}}w_{\rho\rho} = \frac{\lambda N}{P} \rho^{(2-p)\frac{N-1}{N+1}}\left((U_1^p)' - (U_2^p)'\right) + \frac{M\lambda N}{P} \rho^{(2-p)\frac{N-1}{N+1}}\left(U_1^{p-1} - U_2^{p-1}\right).
\end{equation}
Notice that, since \( p \geq \frac{N}{N+1} \) we have that
\[(2-p) \frac{N-1}{N} + \frac{2}{N} < 1.\]
We multiply by \( \rho^{\frac{2}{N+1} - \delta}w_+ \) for some \( \delta > 0 \) satisfying
\begin{equation}
\delta \in (1, \gamma)
\end{equation}
for \( \gamma < \gamma_0 \) and \( \gamma_0 \) defined in (4.4). Then, we have, after integration over \( I_T \)
\begin{equation}
\frac{1}{2} \int_{I_T} w_+^2 \rho^{\frac{2}{N+1} - \delta} d\rho \leq \int_{I_T} \rho^{2-\delta} w_{\rho\rho} w_+ d\rho dt
\end{equation}
\begin{equation}
+ \frac{\lambda N}{P} \int_{I_T} \rho^{(2-p)\frac{N-1}{N+1} + \frac{2}{N} - \delta} \left((U_1^p)' - (U_2^p)'\right) w_+ d\rho dt
\end{equation}
\begin{equation}
+ M\lambda N \int_{I_T} \rho^{(2-p)\frac{N-1}{N+1} + \frac{2}{N} - \delta} \left(U_1^{p-1} - U_2^{p-1}\right) w_+ d\rho dt.
\end{equation}

We label the integrals in the previous equation using the same notation as in the previous lemma,
\begin{align*}
I_1 &:= \int_{I_T} \rho^{2-\delta} w_{\rho\rho} w_+ d\rho dt, \\
I_2 &:= \frac{\lambda N}{P} \int_{I_T} \rho^{(2-p)\frac{N-1}{N+1} + \frac{2}{N} - \delta} \left((U_1^p)' - (U_2^p)'\right) w_+ d\rho dt, \\
I_3 &:= M\lambda N \int_{I_T} \rho^{(2-p)\frac{N-1}{N+1} + \frac{2}{N} - \delta} \left(U_1^{p-1} - U_2^{p-1}\right) w_+ d\rho dt.
\end{align*}
We proceed as in \textbf{Lemma 5.3} to obtain, thanks to
\[ \lim_{\rho \to 0} \rho^{2-\delta} w_+ (w_+)' = \lim_{\rho \to 0} \rho^{2-\delta} w_+ (w_+)' = 0.\]
which is obtained as a consequence of
\[ |U| \leq A\rho^{\frac{N-1}{N}}, \quad \delta < 1 + \frac{1}{N}.\]
therefore the terms in the boundary are null in the following integrals
\begin{equation}
I_1 = -\int_{I_T} \rho^{2-\delta} |w_+|^2 d\rho + \frac{(2-\delta)(1-\delta)}{2} \int_{I_T} \rho^{-\delta} (w_+)^2 d\rho dt.
\end{equation}
The term \( I_2 \) is treated in the following way. First, we apply Mean Value Theorem to the term \( U_1^p - U_2^p \) to obtain, thanks to assumption (5.18), that
\[
(U_1^p - U_2^p)_+ \leq (p-1)A^{p-1}\frac{N-1/N}{p}(U_1 - U_2)_+,
\]
then
\[
I_{2a} := \frac{Z_N}{p} \int_I p^{(2-p)\frac{N-1}{N}+\frac{\xi}{p}-\delta} (U_1^p - U_2^p)_+ (w_+)_p \, dp \, dt
\leq c \int_I p^{(2-p)\frac{N-1}{N}+\frac{\xi}{p}-\delta}\left|A^p\frac{N-1}{N}w_+\right| (w_+)_p \, dp \, dt
\]
which implies
\[
I_{2a} \leq c p A^{p-1} \int_I p^{(2-p)\frac{N-1}{N}+\frac{\xi}{p}-\delta+(p-1)\frac{N-1}{N}w_+} (w_+)_p \, dp \, dt
\leq c_0 \int_I p^{2-\delta} (w_+)_p \, dp \, dt + \frac{c p A^{2p-2}}{4\epsilon_0} \int_I p^{-\delta+\frac{\xi}{p}w_+^2} \, dp \, dt.
\]
i.e.
\[
I_{2a} \leq c_0 \int_I p^{2-\delta} (w_+)_p \, dp \, dt + \frac{c p A^{2p-2}}{4\epsilon_0} \int_I p^{-\delta+\frac{\xi}{p}w_+^2} \, dp \, dt.
\]
\( I_{2b} \) has the following expression
\[
I_{2b} = -\left(2 - p\right)\frac{N - 1}{N} + \frac{2}{N} - \delta \frac{Z_N}{p} \int_I p^{(2-p)\frac{N-1}{N}+\frac{\xi}{p}-\delta} (U_1^p - U_2^p)_+ w_+ \, dp \, dt.
\]
Thanks to Mean Value theorem we get
\[
I_{2b} = c \int_I p^{(2-p)\frac{N-1}{N}+\frac{\xi}{p}-\delta} \left|\xi\right| \, dp \, dt
\]
for some \( \xi < A^p \frac{N-1}{N} \) with \( A \) large enough (see assumption (5.18)). Then,
\[
I_{2b} \leq c A^{p-1} \int_I p^{(2-p)\frac{N-1}{N}+\frac{\xi}{p}-\delta+(p-1)\frac{N-1}{N}w_+^2} \, dp \, dt
\leq c A^{p-1} \int_I p^{\frac{\xi}{p}-\delta} \, dp \, dt
\]
and we get
\[
I_2 \leq \frac{1}{2} \int_I p^{2-\delta} (w_+)_p \, dp \, dt + \left(\frac{c p A^{2p-2}}{2} + c A^{p-1}\right) \int_I p^{-\delta+\frac{\xi}{p}w_+^2} \, dp. \quad (5.23)
\]
Thanks to Mean Value Theorem, \( I_3 \) satisfies
\[
I_3 = M \int_{I_T} \left( (2-p) \frac{N+1}{N+\delta} \frac{1}{s^{N+\delta}} \left( U_1^{p-1} - U_2^{p-1} \right) \right) w_+ \rho dt
\]
\[
\leq (p-1) M \int_{I_T} \left( (2-p) \frac{N+1}{N+\delta} \frac{1}{s^{N+\delta}} |\phi|^{p-2} w_+^2 \right) \rho dt
\]
\[
\leq (p-1) M \int_{I_T} \left( (2-p) \frac{N+1}{N+\delta} \frac{1}{s^{N+\delta}} + (p-2)(1-\rho)^{p-2} \right) w_+^2 \rho dt.
\]

We now proceed as in Lemma 5.3 and it results
\[
I_3 \leq c \rho_0^2 \int_{I_T} \rho^{-\delta} w_+^2 d\rho + c \int_{I_T} \rho^{\frac{\delta}{N+\delta}} w_+^2 d\rho dt, \quad (5.24)
\]
for \( \rho_0 \) such that \( \rho_0^2 \leq \frac{(2-\delta)(\delta-1)}{8c} \). Then, we replace (5.22), (5.23) and (5.24) into (5.19) to obtain
\[
\left| \int_{I_T} \rho^{\frac{\delta}{N+\delta}} w_+^2 d\rho \right| \leq c \int_{I_T} \rho^{\frac{\delta}{N+\delta}} w_+^2 d\rho dt,
\]
and Gronwall Lemma ends the proof.

**Lemma 5.5.** Let \( p \) and \( M \) be positive numbers satisfying (1.8) and (1.9) respectively. Then, the solution \( U \) to (2.8)–(2.10), is a classical solution in \((0, T)\) (for \( T < T_{bu} \)) in the sense
\[
U \in C_{loc}^{1,2}(I_T) \cap C^\alpha(I_T), \quad \text{for } \alpha = 1 - \frac{N+3}{q}
\]
and \( q \frac{N+2}{2-p} \), where \( U_0 \) is defined in (2.10) for \( u_0 \) satisfying (1.11), (1.12) and (1.13).

**Proof.** As in Sec. 3 we introduce the function \( W \),
\[
W(t, s) = s^{-N} U(t, s^N)
\]
which satisfies (3.1), and \( \tilde{W} : (0, T) \times B_{N+2} \to \mathbb{R} \), defined by \( \tilde{W}(t, x) = W(t, |x|) \) which satisfies
\[
\tilde{W}_t - N^{-2} \Delta_{N+2} \tilde{W} + b(t, x) \nabla \tilde{W} = f
\]
where
\[
b := \chi_N |x|^{p-2} \frac{x}{N} \tilde{W}^{p-2} \tilde{W}, \quad f := \chi_N |x|^{p-2} (\tilde{W} + M) |\tilde{W}|^{p-2} \tilde{W}.
\]
We notice that \( b \) is bounded and continuous and in view of \( U \leq c \rho \), \( f \) satisfies
\[
|f| \leq c |x|^{p-2}
\]
therefore
\[
f \in L^\infty(0, T : L^q(B_{N+2}))
\]
for any \( q < \frac{N+2}{2-p} \). Therefore we have that
\[
\tilde{W} \in W^{1,q}(0, T : L^q(B_{N+2})) \cap L^q(0, T : W^{2,q}(B_{N+2}))
\]
see for instance Quittner-Souplet [20], Theorem 48.1 p. 438. Since
\[
\frac{N + 2}{2 - p} > \frac{N + 2}{2 - \frac{N}{N-1}} = \frac{(N + 2)(N - 1)}{N - 2} > N + 3,
\]
and thanks to the Sobolev embedding (see for instance Theorem 5.4, Adams [1], p. 97)
we have that
\[
\tilde{W} \in C^{0,\alpha}((0, T) \times B_{N+2}))
\]
for \(\alpha \leq 1 - \frac{N+3}{q}\). It implies that
\[
U \in C^{0,\alpha}(I_T)
\]
for any \(T < T_{bu}\). Then, for any \(\epsilon > 0\) we have that
\[
g(t) := U(t, \epsilon) \in C^{0,\alpha}(0, T_{bu})
\]
and \(U\) satisfies
\[
\begin{cases}
U_t - \rho^{\frac{2N-2}{N-1}} U_{pp} + \tilde{b}U_p = \tilde{f}, & \epsilon < \rho < 1, \quad 0 < t < T_{bu}, \\
U(t, \epsilon) = g(t), & U(1, t) = 0, \\
U(0, \rho) = U_0(\rho), & \epsilon < \rho < 1.
\end{cases}
\]
for
\[
\tilde{b} := \chi_{N\rho^{(2-p)\frac{N-1}{N+1}}} |U|^{p-2} U
\]
\[
\tilde{f} := \chi_{N\rho^{(2-p)\frac{N-1}{N+1}}} M |U|^{p-2} U.
\]
Since \(\tilde{b}, \tilde{f} \in C^{0,\beta}((0, T) \times (\epsilon, 1))\) for \(\beta = (p - 1)\alpha\), we have, thanks to Theorem 5.6, Lieberman [17], p. 90, that
\[
U \in C^{1,2}_{t,x}((0, T) \times (\epsilon, 1)),
\]
which implies, in view of (5.26), the wished regularity.

\begin{proof}[6. Blow up of solutions]

We consider the function
\[
\phi(\rho, t) = \begin{cases}
\phi_1(t, \rho), & 0 < \rho \leq \rho_1, \\
\phi_2(t, \rho), & \rho_1 < \rho < 1,
\end{cases}
\]
for \(\phi_1\) and \(\phi_2\) defined in (4.5) and (4.9) respectively. Notice that, thanks to Lemmas 4.1 and 4.2 we have that
\[
\mathcal{L}(\phi) \leq 0, \quad \text{for } \rho \neq \rho_i, \quad (i = 1, 2).
\]
Since
\[
\lim_{\rho \to \rho_i} \phi_{1,\rho}(t, \rho) = \frac{\gamma a \rho_i^{\gamma - 1}}{(\rho_i^{\gamma} + a)^2} \leq \phi_1(t, \rho_i) \frac{\gamma}{\rho_i}
\]
and

\[
\phi_{2\rho}(t, \rho) = \begin{cases} 
-\beta \left( 1 + \kappa \frac{2\rho - (\rho_2 + \rho_1)}{(\rho_2 - \rho_1)} \right), & \rho_1 < \rho < \rho_2 \\
-\beta, & \rho_2 < \rho < 1
\end{cases}
\]

we have that

\[
\lim_{\rho \to \rho_1^+} \phi_{2\rho}(t, \rho) = (\kappa - 1)\beta(t) = \frac{(\kappa - 1)}{1 - \rho_1}\phi_1(t, \rho_1).
\]

In view of

\[
\frac{\kappa - 1}{1 - \rho_1} = \frac{\gamma}{\rho_1}
\]

i.e.

\[
\kappa = 1 + \gamma \left( \frac{1 - \rho_1}{\rho_1} \right) = 1 + \gamma
\]

we have that \( \phi_\rho \) is a continuous function at \( \rho = \rho_1 \). In the same way we compute \( \phi_{2\rho} \) at \( \rho_2 \) to get

\[
\lim_{\rho \to \rho_2^-} \phi_{2\rho}(t, \rho) = -\beta(1 + \kappa) < -\beta = \lim_{\rho \to \rho_2^-} \phi_{2\rho}(t, \rho)
\]

which implies that the first derivative of \( \phi_2 \) respect to \( \rho \) presents a positive jump at \( \rho = \rho_2 \) and therefore

\[
\mathcal{L}(\phi) \leq 0, \quad \text{for } \rho \in (0, 1).
\]

**Lemmas 5.3 and 5.4** provide that for any regular solution \( U \) to (2.8) with initial data

\[
U(0, \rho) \geq \phi(0, \rho)
\]

satisfies

\[
U(t, \rho) \geq \phi(t, \rho), \quad \text{for } t < T_{\text{max}}. \tag{6.1}
\]

Notice that for \( \rho = |a(t)|^{\frac{1}{\gamma}} \) we have

\[
\lim_{t \to T_{\text{max}}} \phi \left( t, |a(t)|^{\frac{1}{\gamma}} \right) = \frac{1}{2}
\]

and

\[
\lim_{t \to T_{\text{max}}} a(t) = 0.
\]

Following standard arguments (see for instance [13]) and thanks to Lemma 5.3 we end the proof.

**Acknowledgment**

The author wants to thank to the anonymous reviewers and also to professor Michael Winkler, for their helpful comments and suggestions.
**Funding**

The author was supported by Ministerio de Ciencia e Innovación, Spain, under grant number MTM2017-83391-P.

**ORCID**

J. Ignacio Tello [http://orcid.org/0000-0003-2671-7803](http://orcid.org/0000-0003-2671-7803)

**References**

[1] Adams, R. A. (1975). Sobolev Spaces. New York: Academic Press.

[2] Bellomo, N., Bellouquid, A., Nieto, J., Soler, J. (2010). Multiscale biological tissue models and flux-limited chemotaxis from binary mixtures of multicellular growing systems. *Math. Models Methods Appl. Sci*. 20(07):1179–1693. DOI: 10.1142/S0218202510004568.

[3] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M. (2015). Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues. *Math. Models Methods Appl. Sci*. 25(09):1663–1763. DOI: 10.1142/S021820251550044X.

[4] Bellomo, N., Winkler, M. (2017). A degenerate chemotaxis system with flux limitation: Maximally extended solutions and absence of gradient blow-up. *Commun. Partial. Differ. Equ*. 42(3):436–473. DOI: 10.1080/03605302.2016.1277237.

[5] Bellomo, N., Winkler, M. (2017). Finite-time blow-up in a degenerate chemotaxis system with flux limitation. *Trans. Amer. Math. Soc. Ser. B*. 4(2):31–67. DOI: 10.1090/btran/17.

[6] Bianchi, A., Painter, K. J., Sherratt, J. A. (2015). A mathematical model for lymphangiogenesis in normal and diabetic wounds. *J. Theor. Biol*. 383:61–86. DOI: 10.1016/j.jtbi.2015.07.023.

[7] Bianchi, A., Painter, K. J., Sherratt, J. A. (2016). Spatio-temporal models of lymphangiogenesis in wound healing. *Bull. Math. Biol*. 78(9):1904–1941. DOI: 10.1007/s11538-016-0205-x.

[8] Chiyoda, Y., Mizukami, M., Yokota, T. (2020). Finite-time blow-up in a quasilinear degenerate chemotaxis system with flux limitation. *Acta Appl. Math*. 167(1):231–259. DOI: 10.1007/s10440-019-00275-z.

[9] de Figueiredo, D. G., Moreira dos Santos, E., Miyagaki, O. H. (2011). Sobolev spaces of symmetric functions and applications. *J. Funct. Anal*. 261(12):3735–3770. DOI: 10.1016/j.jfa.2011.08.016.

[10] Hillen, T., Painter, K. J. (2009). A user’s guide to PDE models for chemotaxis. *J. Math. Biol*. 58(1–2):183–217. DOI: 10.1007/s00285-008-0201-3.

[11] Horstmann, D. (2003). From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. *Jahresber. Dtsch. Math. Ver*. 105(3):103–165.

[12] Horstmann, D. (2011). Generalizing the Keller–Segel model: Lyapunov functionals, steady state analysis, and blow-up results for multi-species chemotaxis models in the presence of attraction and repulsion between competitive interacting species. *J. Nonlinear Sci*. 21(2):231–270. DOI: 10.1007/s00332-010-9082-x.

[13] Jäger, W., Luckhaus, S. (1992). On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc*. 329(2):819–824. DOI: 10.2307/2153966.

[14] Keller, E. F., Segel, L. A. (1970). Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol*. 26(3):399–415. DOI: 10.1016/0022-5193(70)90092-5.

[15] Keller, E. F., Segel, L. A. (1971). A model for chemotaxis. *J. Theoret. Biol*. 30(2):225–234. DOI: 10.1016/0022-5193(71)90050-6.

[16] Laurençot, P., Wrzosek, D. (2005). A chemotaxis model with threshold density and degenerate diffusion. In: Brezis, H., Chipot, M., Escher, J. eds. *Nonlinear Elliptic and Parabolic Problems. Progress in Nonlinear Differential Equations and Their Applications*, Vol. 64. Basel: Birkhäuser.
[17] Lieberman, G. M. (1996). *Second Order Parabolic Differential Equations*. Singapore: World Scientific Publishing.

[18] Negreanu, M., Tello, J. I. (2013). On a parabolic-elliptic chemotactic system with non-constant chemotactic sensitivity. *Nonlinear Analysis: Theory, Methods and Applications*. 80:1–13. DOI: 10.1016/j.na.2012.12.004.

[19] Negreanu, M., Tello, J. I. (2018). On a parabolic-elliptic system with gradient dependent chemotactic coefficient. *J. Differ. Equ.* 265(3):733–751. DOI: 10.1016/j.jde.2018.01.040.

[20] Quittner, P., Souplet, P. (2007). Superlinear parabolic problems: Blow-up. In: *Global Existence and Steady States*, Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel.

[21] Semba, T. (2005). Blowup behavior of radial solutions to Jäger-Luckhaus system in high dimensional domains. *Funkcialaj Ekvacioj*. 48(2):247–271.

[22] Stinner, C., Tello, J. I., Winkler, M. (2012). Mathematical analysis of a model of chemotaxis arising from morphogenesis. *Math. Meth. Appl. Sci*. 35(4):445–465. DOI: 10.1002/mma.1573.

[23] Stinner, C., Winkler, M. (2011). Global weak solutions in a chemotaxis system with large singular sensitivity. *Nonlinear Anal. Real World Appl*. 12(6):3727–3740. DOI: 10.1016/j.nonrwa.2011.07.006.

[24] Wang, H., Li, Y. (2019). On a parabolic-parabolic system with gradient dependent chemotactic coefficient and consumption. *J. Math. Phys*. 60(1):011502. DOI: 10.1063/1.5040958.

[25] Winkler, M. (2019). Global classical solvability and generic infinite-time blow-up in quasilinear Keller-Segel systems with bounded sensitivities. *J. Differ. Equ*. 266(12):8034–8066. DOI: 10.1016/j.jde.2018.12.019.

[26] Winkler, M. (To appear). A critical blow-up exponent for flux limitation in a Keller-Segel system. *Indiana Univ. Math. J*. https://arxiv.org/abs/2010.01553.