Some Statistical Properties of Strange Attractors: Engineering View

Mario Mijangos¹, Valeri Kontorovich¹* and Jorge Aguilar-Torrentera²

¹ Electrical Engineering Department, Communications Group, CINVESTAV-IPN. Av. IPN #2508, Col. San Pedro Zacatenco, C.P. 07360, México, D.F. Phone: 52-55-5061-3764; Fax: 52-55-5060-3977.

² Intel Tecnologías de México, Parque Industrial. Tecnológico II, Periférico Sur, 7980, Jalisco, México.

E-mail: valeri@cinvestav.mx

Abstract. In this paper, the statistical characterization of strange attractors is investigated via the so-called ‘model distribution’ approach. It is shown that in order to calculate the first four cumulants, which are necessary to create a model distribution of kurtosis approximation, a systematic method for the calculus of the variance needs to be considered. Correspondently, an analytical method based on the Kolmogorov–Sinai (K–S) entropy for variance approximation is herein proposed. The methodology is of interest for its application in the statistical analysis of chaotic systems that model physical phenomena found in some areas of electrical (communication) engineering.

1. Introduction
The application of chaos in non-linear dissipative dynamic systems, or also termed strange attractors, has recently attracted a keen interest in the research community, particularly in the electrical engineering area. References on this topic are [1-3]. Strange attractors enjoy a mixed deterministic/stochastic nature and lend themselves to a prospective approach for the analysis of many physical phenomena (see p.ex. [4,5]). In this paper, the interest is focused on the development of engineering tools applied to the statistical analysis of chaotic systems. The approach of the so-called ‘degenerated equations’ for stationary cumulants was recently proposed in [6,7] as a tool aimed for the analysis of physical phenomena found in different fields of the electrical engineering. For instance, it has been found that the statistical properties of some strange attractors are similar to those presented in the distribution of error bits in a digital channel. Recently, the modeling of the digital channel in the presence of stationary interference has been carried out through chaotic systems, making clear the potential of this technique. In other application, in the Electromagnetic Compatibility (EMC) field, stationary cumulants allows describing statistically the behavior of chaotic systems that model Radio Frequency Interference (RFI) generated by high-speed digital interconnects, and that in turn, interfere to the wireless systems.

* To whom any correspondence should be addressed.
This paper deals with the statistical description of strange attractors from the engineering point of view and is organized as follows. Section II presents the analytical approximations of PDF’s (probability density function) for some approximations of the components of Chua and Lorenz attractors. The approximations are based on experimental data obtained from RFI measurements obtained in a computing platform. Section III is devoted to the mathematical background of “degenerated cumulant” equations. In Section IV the analytical expressions for the first four cumulants, obtained from the methods developed in Section III, are provided. Section V presents the analytical approximation method based on the Kolmogorov-Sinai entropy for the variance calculus.

2. Distribution functions for Chua and Lorenz attractors and their analytical approximations.

The components of strange attractors, such as those of Chua and Lorenz attractors, involve an approximation of their PDF’s. It was found that for the Chua attractor case, the PDF of the $x$ component well correspond to the bimodal and symmetric distribution function [8]:

$$W(x) = C(p,q)\exp \left( \frac{px^2 - qx^4}{2} \right),$$

where $q > 0$, $C(p,q)$ – normalization constant, and $x \in [-\infty, \infty]$.

The even central moments for the distribution (1) are given by:

$$\mu_{2n} = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{2^n \pi D_{\frac{1}{2}}(-\delta)} (2q)^{n/2}},$$

where $n = 1, 2, \ldots$; $\mu_{2n}$ are central moments; $\delta = \frac{p}{\sqrt{2q}}$; $\Gamma(\ )$ is Gamma Function and $D_{\frac{1}{2}}(x)$ represents a parabolic cylinder function. The two central moments of (1) are:

$$\sigma^2 = \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{2\pi q D_{\frac{1}{2}}(-\delta)}}$$

where $n = 1, 2, \ldots$; $\mu_{2n}$ are central moments; $\delta = \frac{p}{\sqrt{2q}}$; $\Gamma(\ )$ is Gamma Function and $D_{\frac{1}{2}}(x)$ represents a parabolic cylinder function. The two central moments of (1) are:

$$\mu_2 = \frac{\Gamma\left(\frac{5}{2}\right)}{\sqrt{2\pi q D_{\frac{1}{2}}(-\delta)}}$$

Equation (3) is a non-linear algebraic equation for estimation of $p$ and $q$ [14]. Once (3) has been solved and substituting $q$ and $\delta$ into (2), one can get values for the analytical approximation. By doing so, the approximations are: $p \approx 3.5$ and $q \approx 1.5$.

Let us take another example of PDF. For the Lorenz attractor, the “$x$” component follows an approximation in its distribution function in the following way:

$$W(x) \approx W_{\delta}(x) \left[ 1 + \frac{\gamma_3}{3!} H_3(x) + \frac{\gamma_4}{4!} H_4(x) \right],$$

where: $H_n(x) = (-1)^n \exp \left( -\frac{x^2}{2} \right) \frac{d^n}{dx^n} \exp \left( -\frac{x^2}{2} \right)$; $H_n(\ )$ is a Hermitian polynomial of $n$-order,

$W_{\delta}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $W_{\delta}(\ )$ is a Gaussian distribution, $\gamma_3 = \frac{\kappa_3}{\sigma^3}$, $\gamma_3$ is the skewness coefficient, and $\gamma_4 = \frac{\kappa_4}{\sigma^4}$, where $\gamma_4$ is the kurtosis coefficient; $\gamma_3 = 0$, $\gamma_3 \geq -2$, $\gamma_4 = \gamma_3 + 2 \geq 0$, $\mu_3 = \gamma_3$, and $\mu_4 = \gamma_4 + 3(\sigma^2)^2$;

being $\kappa_3, \kappa_4$ the third and fourth cumulants of the PDF, respectively.

It follows from (2) to (4) that, in order to succeed with the aforementioned approximations for the chaotic models, the first four cumulants of the Chua and Lorenz attractors need to be applied.
Henceforth, we will limit ourselves in developing analytical methods for the investigation of \( \kappa_i - \kappa_s \) for the given strange attractor ODE.

3. Brief description of the degenerated cumulant equations method.

From the point of view of system theory, strange attractors are nonlinear dissipative continuous-time dynamic systems, which are defined by the following vector ODE [4].

\[ \mathbf{x} = f(\mathbf{x}(t)), \quad \mathbf{x} \in \mathbb{R}^n, \]  

where \( f(\mathbf{x}) = [f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})]^T \) is a differentiable vector function.

It was shown long ago (see [4,10] for details) that in the context of ergodic theory, equation (5), which forms an explicit basis of our approach, one has to apply the fundamental idea of Kolmogorov. An “external noise” \( \xi(t) \) needs to be considered in (5); i.e., (5) has to be rewritten to take the form:

\[ \mathbf{x} = f(\mathbf{x}(t)) + \epsilon \mathbf{e}(t), \]  

where \( \mathbf{e}(t) \) is a vector of external white noise with the related positive matrix of “intensities” \( \epsilon = [e_{ij}]_{n \times n} \). Mathematical background can be found in [4,10].

Equation (6) is a Stochastic Differential Equation (SDE), with a PDF \( W_s(x, t) \) of its solution \( x(t) \), which is a \( n \)-dimensional Markov process [8]. Nevertheless; we consider henceforth only the one-dimensional PDF \( W_s(x) \) as:

\[ W_s(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \exp \left( \sum_{i=1}^{n} \frac{(ju)^s}{s!} \kappa_i \right) du, \]  

where \( \kappa_i \) is the cumulant of the \( s \)-th order.

If one assume that cumulants for all \( s > m \) are equal to zero, then for the given set of cumulants we can introduce the model distribution \( W_m(x) \) of the \( m \)-th order (see [6-8] for details and references therein), and its characteristic function is defined by:

\[ \theta_m(\mathbf{u}) = \exp \left( \sum_{i=1}^{m} \frac{(ju)^s}{s!} \kappa_i \right) = \theta_m(\mathbf{u}; \kappa_1, \ldots, \kappa_m). \]  

It is clear that \( W_m(x) \) is an inverse Fourier transform of (8). However, as it is pointed out in [6-8], \( W_m(x) \) is not a true PDF and only an approximation of it. For a Gaussian model distribution \( m = 2 \) while for a kurtosis model distribution \( m = 4 \) \( (\gamma_4 \neq 0) \).

Now, let us go ahead with the cumulant analysis of the dynamic systems (5) and (6) when all coefficients, \( e_{ij} \) tend to zero. Those cumulants \( \{\kappa_i\}^s \) can be found from the following “degenerated cumulant equations” (see [6,7] for details):

\[ \begin{align*}
\langle K_i (x) \rangle &= 0 \\
2 \langle \{\xi_i, K_j (x)\} \rangle &= 0 \\
3 \langle \{\xi_i, \xi_j, K_m (x)\} \rangle &= 0
\end{align*} \]  

where \( i, j, \beta = 1, n \).

Essentially, equations (9) represent a set of non-linear algebraic equations. It is always possible to “cut” the set of cumulants by neglecting all cumulants of order above \( n \); i.e., creating the aforementioned model distribution from (7).

Equations (9) have to be solved first for each component of \( x = [x_1, x_2, \ldots, x_n] \) (first line), next for couples of components \( \{x_i, x_j\}_{i,j=1}^n \) (second line), and then for triplets \( \{x_i, x_j, x_\beta\}_{i,j,\beta=1}^n \) (third line), and so
on. The way to do it is to “open” the cumulant brackets as shown in Appendix A2 in [8] (and related examples at \([6,7]\)). To illustrate the procedure described above, we present some results of the application of this method for the Lorenz and Chua attractors.

4. Some analytical results for Chua and Lorenz attractors.

Equation (10) for Chua’s attractor has the following form:

\[
\begin{align*}
\dot{x} &= \beta_1 (y - x) - \alpha h(x) \\
\dot{y} &= \beta_2 (x - y) + \beta_3 z \\
\dot{z} &= -\beta_4 y,
\end{align*}
\]

(10)

where \(\beta_1/\beta_4\) and \(\alpha\) are the parameters of the attractor, \(\mathbf{x} = [x, y, z]^T\), and \(h(x) = SAT(x)\) is a non-inertial nonlinearity, i.e.,

\[
SAT(x) = \begin{cases} 
-L & x < -L \\
|y| < L \\
L & x > L.
\end{cases}
\]

(11)

The first kinetic coefficients at (9) are:

\[
\begin{align*}
K_{1,1}(x) &= \beta_1 (y - x) - \alpha h(x) \\
K_{1,2}(x) &= \beta_2 (x - y) + \beta_3 z \\
K_{1,3}(x) &= -\beta_4 y.
\end{align*}
\]

(12)

Next, (12) is substituted into (9). The corresponding cumulant brackets have to be opened in the order and way mentioned above. Here we will complete the set of expressions for the first four cumulants of the components of Chua’s attractor. The set of relationships are:

\[
\kappa_i^x = \kappa_i^y = \kappa_i^z = 0^*; \quad \kappa_i^{x,y} = \frac{\alpha + \beta_1}{\beta_2 + \alpha}; \quad \kappa_i^{x,z} = \frac{\beta_1}{\beta_2 + \alpha}; \quad \kappa_i^{y,z} = \frac{\beta_3}{\beta_2 + \alpha}; \quad \kappa_i^{x,y,z} = \left( x^3 y \right) - \left( \kappa_i^{x,y} \right)^2 \frac{\alpha + \beta_1}{\beta_2 + \alpha} \approx -\left( \kappa_i^{x,y} \right)^2
\]

with the assumption that \(\left< x^3 y \right> \approx 0^*\), \(\kappa_1^{x,y} \approx -\left( \kappa_1^{x,y} \right)^2 \), i.e., \(\gamma_1 \approx -1^*\), and \(\kappa_1^{x,y} \neq \kappa_1^{x,z} \neq \kappa_1^{y,z}; \quad \kappa_i^{x,y} = \kappa_i^{x,z} = 0\). Here the superscript “*” means, that those features were proved by numerical simulations.

Equation (5) for the Lorenz attractor acquires the form:

\[
\begin{align*}
\dot{x} &= \sigma (y - x) \\
\dot{y} &= R x - y - zx \\
\dot{z} &= xy - Bz,
\end{align*}
\]

(13)

where \(\sigma, B\) and \(R\) are the parameters of the attractor, \(\mathbf{x} = [x, y, z]^T\).

Thus, the first kinetic coefficients for (9) are:

\[
\begin{align*}
K_{1,1}(x) &= \sigma (y - x) \\
K_{1,2}(x) &= R x - y - zx \\
K_{1,3}(x) &= xy - Bz.
\end{align*}
\]

(14)

By repeating the same procedure for (14), one gets: \(\kappa_1^x = \kappa_1^y = \kappa_1^z = 0^*\), \(\kappa_1^{x,y} = \kappa_1^{x,z} = 0^*\), \(\kappa_1^{y,z} = \beta_2 \kappa_1^{x,z} + \beta_3 \kappa_1^{y,z} = 0\), i.e. \(\gamma_1^y = 0^*\); \(\kappa_1^{x,y} + \kappa_1^{x,z} < \frac{BR^2}{4} \), \(\kappa_1^{y,z} > \frac{BR^2}{2}\);

\[
\begin{align*}
\kappa_1^{x,y,z} &= \frac{R - 1}{B} \kappa_1^{x,y} - \frac{\beta_3}{B} \kappa_1^{x,z} \\
\kappa_1^{x,y} &= \kappa_1^{x,z} = \kappa_1^{y,z} = 0^*; \quad \kappa_1^{x,y,z} = \beta_2 \kappa_1^{x,z} + \beta_3 \kappa_1^{y,z} = 0, \text{ i.e. } \gamma_1^y = 0^*; \kappa_1^{x,y} + \kappa_1^{x,z} < \frac{BR^2}{4} \neq \frac{BR^2}{2}.
\end{align*}
\]

(14)

\[
\kappa_1^{x,y,z} = \frac{2 \beta_3 (\beta_2 - \kappa_1^{x,y} - \kappa_1^{x,z})}{B} \neq 0^*, \text{ i.e. } \gamma_1^y \neq 0; \quad \kappa_1^{x,y} = \kappa_1^{x,z} = 0; \quad \kappa_1^{y,z} \neq 0\), and \(\kappa_1^{x,y,z} = R \kappa_1^{x,z} - \kappa_1^{x,y} - 2 \beta_3 \kappa_1^{x,z}\), i.e., \(\gamma_1^y, \gamma_1^z \neq 0^*\).
It is important to notice from the set of relationships the dependence for the components of both Chua and Lorenz on $\kappa_2$ in most of the cumulants considered there. An approximated method to estimate the variance is considered in the next section.

5. Approximated solution for variance.

The solution for $\kappa_2$ is based on the Kolmogorov–Sinai (K–S) entropy [11, page 122] with the Lyapunov exponents for a linear matrix defined by its eigenvalues [11, pages 542–543]. In the proposed methodology the entropy is obtained from the PDF approximation for a specific component of the strange attractor and the result is compared with the K–S entropy computed through the parameters of the attractor. From those results, we can arrive at an algebraic expression for variance with dependence on the parameters of the attractor. Our method can be summarized as follows.

1. The non–linear system describing the chaotic behavior of the attractor has to be linearized, for example, statistically.
2. From the stochastically linearized system we form a coefficient matrix from which its eigenvalues are determined.
3. Once the eigenvalues of the linearized system are obtained, the entropy is estimated by [11]:

$$\log |\lambda_{\text{max}}| < H_{K-S} < \sum_{j=1}^{n} \log |\lambda_j|,$$

where, $\lambda_j$ is the $j$-th eigenvalue of the linearized matrix and $\log(\cdot)$ can be referenced either to the natural logarithm or base 10 logarithm.
4. The entropy is calculated directly from the analytical approximation of the PDF of the strange attractor by:

$$H = -\int_{-\infty}^{\infty} W(x) \log W(x) \, dx,$$

where $W(x)$ represents the analytical approximation of the PDF for the strange attractor component.
5. Then, an algebraic equation depending on the variance based on step 3 and 4 can be created.
6. Solving equation in Step 5, we can now arrive at a solution for the variance.

The first example is for the Chua’s attractor. From Equation (10) a linearized system is obtained as mentioned in Step 1. From Step 2, the obtained coefficient matrix is:

$$\begin{bmatrix}
-\left(\beta_1 + a\right) & \beta_1 & 0 \\
\beta_2 & -\beta_2 & \beta_4 \\
0 & -\beta_3 & 0
\end{bmatrix},$$

with eigenvalues: $\lambda_1 \approx -11$, $\lambda_2 \approx -0.09 + 3.63i$, $\lambda_3 \approx -0.09 - 3.63i$. From here, the numeric value for K–S entropy result to be: $H_{K-S} = 2.16$.

From the algebraic equation for the variance, using the Step 4, the entropy has the equation:

$$H = q \left(\kappa_2\right)^2 - p\kappa_2 - \log(C),$$

where $H$ is the entropy of (1); $q$, $p$ and $C$ are parameters of (1) and $\kappa_2$ is the variance of the process. From (18) one can obtain the value of the variance, which equals to 2.24. From simulation results we obtain that $\sigma^2 = 2.02$. The difference between the results for the variance is around 11%.

Using the same methodology for Lorenz attractor, the linearized coefficient matrix for the system (13) is:

1 Being $H_{K-S}$ the Kolmogorov–Sinai entropy.
\[
\begin{bmatrix}
-\sigma & \sigma & 0 \\
R & -1 & 0 \\
0 & 0 & -B \\
\end{bmatrix}
\]

with eigenvalues of the matrix (19) with: \(\sigma = 10, B = 8/3\) and \(R = 28\): \(\lambda_1 \approx 11.82, \lambda_2 \approx -22.82, \lambda_3 \approx -2.66\).

The Kolmogorov–Sinai entropy for the Gaussian approximation of (4) is:
\[
H = \frac{1}{2} \log_{10} 2\pi e \sigma^2.
\]

By doing the corresponding numerical substitutions, the variance \(\sigma^2\) results to be equal to 29.34. Now, comparing this value with the variance obtained by the Gaussian approximation \((\sigma^2 \approx 36)\) an error in the approximation equal to 18.48%. Surely not always this methodology can lead to a completely analytical solution. Rather, it could render some numerical results.

### 6. Conclusions.

This paper presents an engineering statistical description of chaotic systems. For these chaotic models we can find an analytical approximation for their PDF’s that can be used as a statistical description of their behavior in terms of their cumulants.

The solution for variance for strange attractors presented in Section V results to be appropriate as a prediction method for this parameter. Moreover, the error in the calculation is considered permissible in the estimation of the variance.

### References

[1] Hasler M, Mazzini G, Ogorzalek M, Rovatti R and Setti G 2002 *Proc. of the IEEE* 90

[2] Yu J, Pan W Z and Zhang R B 2006 *Int. J. Nonlinear Sci.* 7 365

[3] Liu L, Su Y C and Liu CX 2007 *Int. J. Nonlinear Sci.* 7 187

[4] Eckman J P and Ruelle D July 1985 *Rev. Mod. Phys.* 57 (3) part 1 617

[5] Lasota A and Mackey M C 1984 *Chaos, Fractals and Noise* (New York Springer-Verlag)

[6] Kontorovich V A 2007 *Math. Method. Appl. Sci.* 30 1705

[7] Kontorovich V A 2005 *Proc. Of the Intern. Conf. on Numerical Analysis and Applied Math ICNAAM-2005* (Greece Rhodes) pp.320-323

[8] Primak S, Kontorovich V and Lyandres V 2004 *Stochastic Methods and their Applications to Communications. Stochastic Differential Equations Approach* (John Wiley & Sons, Ltd).

[9] Mijangos M and Kontorovich V 2006 *XVI Int. Conference on Electromagnetic Disturbances EMD 2006* (Lituania Kaunas) pp.27-29

[10] Rabinovich M and Trubetskov D 1984 *Introduction to Oscillation Theory* (Nauka, Moscow, in Russian)

[11] Alwyn Scott 2005 *Encyclopedia of Nonlinear Science* (New York: Routledge)