Stochastic properties and pricing of bitcoin using a GJR-GARCH model with conditional skewness and kurtosis components

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Abstract
Using a flexible statistical framework that accounts for time-varying skewness and leptokurtosis, we examine the stochastic behavior of Bitcoin in comparison to five major currencies. The empirical findings reveal that the distribution of all series is leptokurtic. Once the effect of skewness-kurtosis is considered, the true price of risk is obtained, with implications on policymakers’ and investors’ strategies.

Keywords Conditional skewness and kurtosis · Skewness price of risk · Upside and downside market probabilities · Skewed generalized error distribution

JEL Classification C10 · C58 · G15

1 Introduction
Cryptocurrencies have undergone an extraordinary development, especially if we consider that they are based on a decentralized system (Nakamoto 2008; Yermack 2015). They are used for business and private transactions, as well as speculation and hedging. Unlike traditional currency transactions guaranteed by a third party such as a bank, cryptocurrency transactions are secured by their strong underlying cryptography. Cryptocurrencies are also popular for reasons other than prudent economic behavior. They are often used in illegal activities such as money laundering, tax evasion, terrorist organization financing, and ransom payments (Vigna 2019; Popper 2019; Justin and Demos 2021; Osipovich 2021).

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Moreover, many inexperienced investors engage with cryptocurrencies simply because of a fear of missing out.

More than eight hundred cryptocurrencies have emerged since the early 2000s, including Bitcoin, Cardano, Ethereum and XRP. Bitcoin is, to date, the most popular, with a market capitalization of 80% of the total capitalization of all cryptocurrencies in 2021.¹ Many stylized facts regarding the Bitcoin market have been explored in the literature. Glaser et al. (2014), Baek and Elbeck (2015), Kristoufek (2015), Dyhrberg (2016), and Hafner (2020) have shown that Bitcoin is used for speculation, making it extremely volatile. Similarly, Baur et al. (2018) reported that Bitcoin is widely used as a speculative investment, despite its huge volatility and low correlation with major asset classes. Meanwhile, Brière et al. (2015) indicated that Bitcoin offers significant diversification benefits, and so including the cryptocurrency in a portfolio of financial assets could mean an improvement in that portfolio’s risk-return characteristics.

Because of the above observations, investors have increasingly begun to include cryptocurrencies in their portfolios. Although risk characteristics and higher moments of cryptocurrencies, such as skewness and kurtosis, are important for the construction of a well-diversified portfolio, these have not yet been fully investigated, as ascertained by Osterrieder and Lorenz (2017), Osterrieder et al. (2017), Chu et al. (2017), Phillip et al. (2018), and Takaishi (2018).

The investigation of the stochastic behavior of cryptocurrencies has focused mainly on the first and second conditional moments of the distribution of their returns (Guesmi et al. 2019). This paper expands the literature on the stochastic properties of Bitcoin’s returns and pricing, by enhancing the GJR GARCH model of Glosten et al. (1993) with the time-varying asymmetry and time-varying shape parameters in Theodossiou and Savva (2021). Henceforth, the enhanced model will be referred as the ST-GJR GARCH.²

The time-varying specification of the asymmetry parameter captures systematic shifts of the tails of the distribution of returns to the left or right of its mode. Note that distributional shifts to the left of the mode yield negatively skewed probability curves and vice versa. Other things being equal, risk-averse investors prefer right-skewed to left-skewed portfolios. For example, adding assets that decrease a portfolio’s skewness (thus making them more left-skewed) should result in higher expected returns. Moreover, the time-varying specification of the shape parameter captures systematic expansions or contractions of the outer tails of the distribution of returns. It relates directly to the fourth centered moment or kurtosis, which is a measure of the variance of volatility shocks.

The pricing equations for Bitcoin, derived using risk-neutral arguments, depend on the conditional parameters of the distribution of returns. It is noted that skewness is found to be a key element in models of risk and return, particularly for short- and medium-term investors (Adcock 2014). During periods of excess volatility, investors realize that their expectations will not materialize and switch to less risky assets, thereby introducing increased negative skewness to the price of risk. The inclusion of skewness and kurtosis parameters in risk and return models stems from the fact that the mean and variance are not the only factors driving prices. To gain additional insights, Bitcoin’s stochastic properties are compared to those of major currencies such as the euro, the British pound, the Canadian dollar, the Chinese yuan, and the Japanese yen.

¹ See http://coinmarketcap.com and https://data.bitcoinity.org (accessed April 2021).
² The risk pricing effects in relation to the distributional effects of heavy tails and downside risk are also examined, as in Barndorff-Nielsen et al. (2010).
The conditional parameters of the ST-GJR GARCH model are estimated via the maximum likelihood method using a sample likelihood specification based on the flexible skewed generalized error distribution (SGED). The findings reveal higher-order moment dependencies and important skewness-kurtosis pricing effects. The understanding of the stochastic dynamics and pricing of Bitcoin is important to portfolio managers, forex traders, speculators, policymakers, as well as businesses, since Bitcoin can be accepted as a form of payment (Akhtaruzzaman et al. 2020).

The rest of this paper is organized as follows: Sect. 2 presents the ST-GJR GARCH model, equations for the transition probabilities and risk measures for upside and downside markets, and a risk-neutral pricing equation for Bitcoin. Section 3 discusses the model’s estimation and major empirical findings. The paper ends with a summary and conclusions in Sect. 4.

2 The ST-GJR GARCH model

Volatility clustering and negative asymmetric volatility have been documented in financial, currency, and commodity prices, including Bitcoin (Corbet et al. 2018). Volatility clustering refers to the tendency of large price shocks to set off large price shocks of smaller magnitude but random sign. On the other hand, negative asymmetric volatility refers to the tendency of large drops in the price of financial assets to be followed by larger and more persistent volatility shocks, compared to equivalent increases in the price of financial assets. In general, volatility clustering and asymmetric volatility trigger skewness and excess kurtosis in the unconditional distribution of financial returns (Theodossiou 2015).

2.1 Conditional variance, mode, and mean

Under the GJR GARCH model, the conditional variance of returns in market $i$, based on the information set $I_{t-1}$, is modeled as a function of past innovations, their squared values, and past conditional variances. That is,

$$\sigma^2_{i,t} = \text{var}(r_{i,t} | I_{t-1}) = v_i + (a_{N,i}N_{i,t-1} + a_i)\varepsilon_{i,t-1}^2 + \beta_{i}\sigma^2_{i,t-1}$$

(1)

where $\varepsilon_{i,t} = r_{i,t} - \mu_{i,t}$ is an innovation or an error term. The indicator variable

$$N_{i,t} = \begin{cases} 
1 & \text{for } \varepsilon_{i,t} < 0 \\
0 & \text{for } \varepsilon_{i,t} \geq 0 
\end{cases}$$

takes the value of one for negative values of $\varepsilon_{i,t}$ and zero otherwise.

The coefficient $a_{N,i}$ measures the impact of past negative errors on current volatility. A positive value is indicative of a larger conditional variance during negative markets and vice versa (asymmetric volatility). The coefficients $a_i$ and $\beta_i$ measure the persistence of volatility shocks over time (volatility clustering). Note that for $a_{N,i}=0$, GJR GARCH yields the standard GARCH model of Bollerslev (1986).

The conditional mode and conditional mean of returns are specified as linear functions of past returns and their contemporaneous conditional standard deviations. That is,

$$m_{i,t} = \text{mode}(r_{i,t} | I_{t-1}) = m_{0,i} + b_i r_{i,t-1} + c_i \sigma_{i,t}$$

(2)

and
where \( m_{0,i} \) is an intercept, \( b_i \) an autoregressive coefficient and \( c_i \) and \( \delta_{i,t} \) are the in-mean and time-varying skewness coefficients, respectively. The sum \( \xi_{i,t} = c_i + \delta_{i,t} \), interpreted as the total price of risk, measures the impact of risk on expected returns. The coefficients \( c_i \) and \( \delta_{i,t} \) are respectively the pure and the skewness prices of risk (Theodossiou and Savva, 2016).

The analytical equation for the skewness price of risk is

\[
\delta_{i,t} = 2G_{1,i,t} \theta_{i,t} \lambda_{i,t}
\]

where

\[
\theta_{i,t} = \frac{1}{\sqrt{\left(3\lambda^2_{i,t} + 1\right)G_{2,i,t} - 4\lambda^2_{i,t} G^2_{1,i,t}}}
\]

and

\[
G_{1,i,t} = k^{\frac{1}{2}}_{i,t} \Gamma\left(\frac{2}{k_{i,t}}\right) \Gamma\left(\frac{1}{k_{i,t}}\right)^{-1}
\]

\[
G_{2,i,t} = k^{\frac{2}{3}}_{i,t} \Gamma\left(\frac{3}{k_{i,t}}\right) \Gamma\left(\frac{1}{k_{i,t}}\right)^{-1}
\]

where \( \lambda_{i,t} \) and \( k_{i,t} \) are respectively the conditional asymmetry and conditional shape parameters of the distribution of returns \( r_{i,t} \) and \( \Gamma(\cdot) \) is the gamma function. These equations are obtained from Theodossiou and Savva (2021).

### 2.1.1 The risk-return relationship

Theoretical models postulate a positive relationship between risk and expected returns, as per Markowitz (1952) and Sharpe (1964). In the intertemporal CAPM of Merton (1973), stochastic factors can influence equilibrium risk premia in financial markets. These factors trigger fluctuations in the risk-return tradeoff and, as such, are a source of skewness and kurtosis when returns are computed over discrete time intervals. Because investors are constantly hedging against such fluctuations, higher moments are likely to be priced. Ignoring the impact of higher moments on the distribution of equilibrium returns has been the source of contradictory and inconclusive empirical findings on the risk and expected return tradeoff of financial assets (Theodossiou and Savva 2016).

Using the analytical framework of Eq. (3), Savva and Theodossiou (2018) investigated the risk and expected returns relationship across a wide range of international financial markets. They concluded that the contradictory findings in the literature regarding risk and return were the result of skewness. That is, negative skewness often switched the total price of risk from positive to negative.
2.2 Conditional asymmetry and shape parameters

Standardized deviations of returns from their conditional mode are used as proxies for market shocks; see also Feunou et al. (2013) for a similar definition. They are computed using the equation

\[ u_{i,t} = \frac{r_{i,t} - m_{i,t}}{\sigma_{i,t}}, \]  

(8)

where \( m_{i,t} \) and \( \sigma_{i,t} \) are respectively the conditional mode, and conditional standard deviation of returns \( r_{i,t} \) in market \( i \). The variable

\[ u_{i,t}^+ = \begin{cases} 0 & \text{for } u_{i,t} \leq 0 \\ u_{i,t} & \text{for } u_{i,t} > 0 \end{cases}, \]

based on positive market shocks is used as a proxy for upside markets. On the other hand,

\[ u_{i,t}^- = \begin{cases} |u_{i,t}| & \text{for } u_{i,t} < 0 \\ 0 & \text{for } u_{i,t} \geq 0 \end{cases}, \]

based on negative market shocks is used as a proxy for downside markets. The conditional asymmetry parameter of the distribution of returns is specified as

\[ \lambda_{i,t} = \text{asym}(r_{i,t}|I_{i-1}) = 1 - \frac{2}{1 + e^{h_{i,t}}}, \]

(9)

where

\[ h_{i,t} = \gamma_{0,i} + \gamma_{N,i} u_{i,t-1}^- + \gamma_{P,i} u_{i,t-1}^+ + \gamma_{h,i} h_{i,t-1} \]

is a time-varying asymmetry index. Note that for \( h_{i,t} = 0, \lambda_{i,t} = 0 \). Moreover, \( h_{i,t} \) has a positive monotonic impact on \( \lambda_{i,t} \) or the skewness price of risk \( \delta_{i,t} \) i.e., \( \partial \lambda_{i,t}/\partial h_{i,t} > 0 \) and \( \partial \delta_{i,t}/\partial h_{i,t} > 0 \). That is,

\[ \begin{cases} h_{i,t} < 0, \lambda_{i,t} < 0, \delta_{i,t} < 0, \text{negatively skewed} \\ h_{i,t} = 0, \lambda_{i,t} = 0, \delta_{i,t} = 0, \text{symmetric} \\ h_{i,t} > 0, \lambda_{i,t} > 0, \delta_{i,t} > 0, \text{positively skewed} \end{cases} \]

The coefficient \( \gamma_{N,i} \) measures the marginal impact of downside market shocks on the asymmetry index \( h_{i,t} \), the asymmetry parameter \( \lambda_{i,t} \), and the skewness price of risk \( \delta_{i,t} \). A positive value indicates that past downside market shocks, on average, have a positive impact on current values of the three parameters and vice versa. Similarly, the coefficient \( \gamma_{P,i} \) measures the marginal impact of past upside market shocks on the three parameters. A positive value implies that past upside market shocks, on average, have a positive impact on the three parameters. The coefficient \( \gamma_{h,i} \) measures the persistence of past market shocks. The intercept \( \gamma_{0,i} \) is an autonomous component.

The time-varying behavior of the conditional shape parameter \( k_{i,t} \) is modeled like in Mazur and Pipień (2018). That is,
where
\[ g_{i,t} = d_{0,i} + d_{N,i}u_{t-1}^- + d_{p,i}u_{t-1}^+ + d_{h,i}g_{i,t-1} \]
is a time-varying shape index and \( k_L \) and \( k_U \) are pre-set lower and upper bounds for the time-varying shape parameter \( k_{i,t} \). Similarly, the relationship between the \( g_{i,t} \) and \( k_{i,t} \) is monotonic and positive. Moreover, for \( g_{i,t} = -\infty \), \( k_{i,t} = k_L \), \( g_{i,t} = 0 \), \( k_{i,t} = (k_L + k_U) / 2 \) and \( g_{i,t} = \infty \), \( k_{i,t} = k_U \). Note that zero values for \( d_{N,i} \), \( d_{P,i} \) and \( d_{h,i} \) are indicative of a time-invariant behavior of \( k_{i,t} \). Positive \( d_{N,i} \) or \( d_{P,i} \) are indicative of a positive marginal impact of downside or upside market shocks on the conditional shape parameter \( k_{i,t} \) and vice versa.

For excessively volatile assets (highly leptokurtic), \( k_L \) is set to a value of less than one. In this paper, the lower and upper bounds of the shape parameter are set to \( k_L = 0.4 \) and \( k_U = 1.6 \), respectively. This range can accommodate Pearson’s moment coefficient of kurtosis \( KU \) values between 3.5 and 73.2. Such a wide range of values can accommodate the empirical distributions of the series of returns investigated, as well as many other financial assets. For example, for \( k_L = 0.4 \), \( k_U = 1.6 \) and \( \lambda_{i,t} = 0 \), the \( KU \) values associated with the index values of \( g_{i,t} = -\infty \), \( 0 \), \( \infty \) are:

\[
\begin{align*}
g_{i,t} &= -\infty, \quad k_{i,t} = k_L = 0.4 \quad \text{and} \quad KU_{i,t} = 51.9 \\
g_{i,t} &= 0, \quad k_{i,t} = (k_L + k_U) / 2 = 1, \quad \text{and} \quad KU_{i,t} = 6 \\
g_{i,t} &= \infty, \quad k_{i,t} = k_U = 1.6, \quad \text{and} \quad KU_{i,t} = 3.5
\end{align*}
\]

The Pearson’s moment coefficient of kurtosis, being the fourth moment of standardized returns, \( z_{i,t} = (r_{i,t} - \mu_{i,t}) / \sigma_{i,t} \), can be written equivalently as
\[
KU_i = E z_{i,t}^4 = E \left( z_{i,t}^2 \right)^2 - \left( E z_{i,t}^2 \right)^2 + \left( E z_{i,t}^2 \right)^2 = \text{var} \left( z_{i,t}^2 \right) + 1,
\]
thus
\[
\text{var} \left( z_{i,t}^2 \right) = KU_i - 1.
\]

Therefore, \( KU \) measures the dispersion of volatility shocks. Larger \( KU \) values imply greater dispersion. Note that there is an inverse relationship between the shape parameter \( k \) and \( KU \). Lower values of \( k \) correspond to fat tails and peakness around the mode.

### 2.3 Conditional distribution of returns

The returns for each series are modeled using the centered SGED of Theodossiou (2015). That is,

\[ k_{i,t} = k_U + \frac{(k_L - k_U)}{1 + e^{g_{i,t}}} , \]
where $\mu_{i,t}$, $\sigma_{i,t}$, $\delta_{i,t}$, $\theta_{i,t}$, $\lambda_{i,t}$, and $k_{i,t}$ are as defined previously. Because of its flexibility, the SGED has been employed in the literature for the measurement of risk, pricing of options, and modeling of the time-series behavior of, amongst others, returns of stock indices, currencies, as well as oil and precious metals. We chose the SGED instead of the skewed generalized t, often used in empirical work, because it enables the development of pricing equations for Bitcoin. Note that the SGED gives for $k_{i,t}=1$ the skewed Laplace or double exponential distribution, and for $k_{i,t}=2$ the skewed normal distribution used in Feunou et al. (2013) and Roon and Karehnke (2017).

### 2.4 Upside and downside probabilities and risk measures

The conditional probabilities for downside and upside markets are computed using the equations

\begin{equation}
P(r_{i,t} \leq m_{i,t}) = \int_{-\infty}^{m_{i,t}} f_{r}(r_{i,t}) \, dr_{i,t} = \frac{1 - \lambda_{i,t}}{2} = \frac{1}{1 + e^{h_{i,t}}} \quad (12)
\end{equation}

and

\begin{equation}
P(r_{i,t} > m_{i,t}) = \frac{(1 + \lambda_{i,t})}{2} = \frac{1}{1 + e^{-h_{i,t}}} \quad (13)
\end{equation}

For investors with long positions in assets, downside returns are synonymous to downside risk, computed using the equation

\begin{equation}
\text{var}(r_{i,t} \mid r_{i,t} \leq m_{i,t}) = (1 - \lambda_{i,t})^2 \left( G_{2,i,t} - G_{1,i,t}^2 \right) \theta_{i,t}^2 \sigma_{i,t}^2 \quad (14)
\end{equation}

On the other hand, returns above their conditional mode can be viewed as measures of upside uncertainty, computed using the equation

\begin{equation}
\text{var}(r_{i,t} \mid r_{i,t} > m_{i,t}) = (1 + \lambda_{i,t})^2 \left( G_{2,i,t} - G_{1,i,t}^2 \right) \theta_{i,t}^2 \sigma_{i,t}^2 \quad (15)
\end{equation}

### 2.5 Conditional skewness and kurtosis

The equations for the conditional Pearson’s moment coefficients of skewness and kurtosis implied by the SGED are respectively

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4 See Theodossiou and Savva (2021) for the derivations of Eqs. (12) to (17).
\[
SK_{i,t} = \frac{E(r_{i,t} - \mu_{i,t})^3}{\text{var}(r_{i,t})^{3/2}} = \frac{A_{3,i,t} - 3A_{2,i,t}A_{1,i,t} + 2A_{1,i,t}^3}{(A_{2,i,t} - A_{1,i,t}^2)^{3/2}},
\]

and
\[
KU_{i,t} = \frac{E(r_{i,t} - \mu_{i,t})^4}{\text{var}(r_{i,t})^2} = \frac{A_{4,i,t} - 4A_{3,i,t}A_{1,i,t} + 6A_{2,i,t}A_{1,i,t}^2 - 3A_{1,i,t}^4}{(A_{2,i,t} - A_{1,i,t}^2)^2},
\]

where
\[
A_{i,j,t} = \frac{1}{2} \left[ (-1)^j (1 - \lambda_{i,t})^{j+1} + (1 + \lambda_{i,t})^{j+1} \right] k_{i,j}^{j+1} \Gamma \left( \frac{j+1}{k_{i,j}} \right) \Gamma \left( \frac{1}{k_{i,j}} \right) ^{-1},
\]

\(j = 1, 2, 3, 4\) and \(t = 1, 2, \ldots, T\).

### 2.6 Forecasting Prices

The expected price of asset \(i\) at time \(t\) is
\[
E(X_{i,t}) = X_{i,t-1}e^{\mu_{i,t}} = X_{i,t-1} \int_{-\infty}^{\infty} e^{\mu_{i,t} + \sigma_{i,t}z_{i,t}} f(z_{i,t}) dz_{i,t} = X_{i,t-1}e^{\mu_{i,t} + \ln E_{i,t}^c},
\]

where
\[
E_{i,t}^c \equiv \int_{-\infty}^{\infty} e^{\sigma_{i,t}z_{i,t}} f(z_{i,t}) dz_{i,t}
\]

and \(z_{i,t}\) is a standardized return for asset \(i\). The pdf for \(z_{i,t}\) is obtained from that of Eq. (11) by setting \(\mu_{i,t} = 0\) and \(\sigma_{i,t} = 1\).

In the absence of arbitrage opportunities, prices will be martingale processes. Therefore, their expected value at time \(t\) discounted by the period’s required rate of return,
\[
\hat{r}_{i,t} = r_{f,i,t} + \rho_{i,t},
\]

where \(r_{f,i,t}\) is the conditional risk-free rate of interest in the country of asset \(i\) and \(\rho_{i,t}\) the conditional risk premium for asset \(i\) in period \(t-1\) to \(t\), gives
\[
E(X_{i,t})e^{-\hat{r}_{i,t}} = X_{i,t-1} e^{\mu_{i,t} + \ln E_{i,t}^c - r_{f,i,t} - \rho_{i,t}} = X_{i,t-1}
\]

The above equation implies that \(\mu_{i,t} + \ln E_{i,t}^c - r_{f,i,t} - \rho_{i,t} = 0\), thus
\[
\hat{r}_{i,t} = r_{f,i,t} + \rho_{i,t} = \mu_{i,t} + \ln E_{i,t}^c
\]

and
\[
\hat{X}_{i,t} = X_{i,t-1}e^{\hat{r}_{i,t}}.
\]
3 Empirical findings

This section discusses the estimation of the model and presents the empirical findings.

3.1 Data and Preliminary Statistics

The data, obtained from DATASTREAM, includes daily US dollar prices for Bitcoin (BTC), the euro (EUR), the Japanese yen (JPY), the Canadian dollar (CAN), the British pound (GBP), and the Chinese yuan (RMB). It covers the period January 3, 2012 to September 17, 2021. Continuously compounded daily returns are computed using the equation

\[ r_{i,t} = 100(\ln X_{i,t} - \ln X_{i,t-1}), \]

where \( X_{i,t} \) is the US dollar price of asset \( i \) at time \( t \) and \( i = \text{BTC}, \text{EUR}, \text{JPY}, \text{CAN}, \text{GBP}, \text{RMB} \). To simplify the discussion of the results, henceforth, the subscript \( i \) is dropped from the coefficients and time-varying parameters.

Table 1 reports the mean, standard deviation, minimum, median, maximum, Pearson’s moment coefficients of skewness and kurtosis and the Bera-Jarque test statistic for normality of returns for each of the series. BTC’s mean return is positive and large, reflecting its steep upward price trend over time. The mean returns for the other series are negative and close to zero. The standard deviation of returns for BTC is 5.3697 and is between ten to twenty-five times larger than those of the other series.

The Pearson’s moment coefficients of skewness and kurtosis are respectively computed using the equations \( SK = m_3/m_2^{3/2} \) and \( KU = m_4/m_2^2 \), where \( m_j \) is the \( j^{th} \) sample moment around the mean. The Bera-Jarque test statistic for normality are computed using the equation \( BJ = (T/24)(4SK^2 + KU^2) \).

**denotes statistical significance at the 1% level

| Statistics | BTC  | EUR  | JPY  | CAN  | GBP  | RMB  |
|------------|------|------|------|------|------|------|
| Mean       | 0.3613 | −0.0040 | −0.0141 | −0.0089 | −0.0048 | −0.0010 |
| St. Dev    | 5.3697 | 0.4850 | 0.5409 | 0.4585 | 0.5601 | 0.1963 |
| Min        | −66.3948 | −2.2594 | −3.7102 | −2.3922 | −8.3113 | −1.8097 |
| Max        | 48.4776 | 2.6006 | 3.4061 | 2.8697 | 3.2252 | 1.4330 |
| Median     | 0.2731 | −0.0036 | −0.0137 | 0.0000 | 0.0000 | 0.0000 |
| Skewness, \( SK \) | −1.0292 | 0.0621 | 0.0592 | 0.0605 | −1.2047 | −0.2101 |
| Kurtosis, \( KU \) | 22.7010 | 5.2458 | 7.8137 | 5.1790 | 24.3391 | 13.2278 |
| Bera-Jarque, \( BJ \) | 41,427.4** | 534.1** | 2,448.1** | 502.8** | 48,691.2** | 11,063.5** |
| OBS        | 2534 | 2534 | 2534 | 2534 | 2534 | 2534 |

The above statistics are computed using US dollar daily returns over the period January 3, 2012 to September 17, 2021 for Bitcoin (BTC), euro (EUR), Japanese yen (JPY), Canadian dollars (CAN), British pounds (GBP), and Chinese yuan (RMB). The equations for the skewness and kurtosis statistics are \( SK = m_3/m_2^{3/2} \) and \( KU = m_4/m_2^2 \), where \( m_j \) is the \( j^{th} \) sample moment around the mean. The Bera-Jarque test statistic for normality are used to determine the normality of the data.
a few outliers in the data. A large KU value of 13.2278 is also present in RMB returns. The Bera-Jarque test statistics reject the null hypothesis of normality in all cases, indicating the presence of asymmetry and/or leptokurtosis in the data. It is noted that these results do not fully reflect the true extent of skewness and kurtosis in the data. These will be computed more accurately using the SGED estimated parameters for each return series.

3.2 Estimation of the SGED Parameters

Maximum likelihood estimates (MLE) for the coefficients of the unconditional and conditional mean, variance, asymmetry, and shape parameters of the distribution of returns are obtained via the Berndt et al. (1974) optimization procedure of the sample log-likelihood below:

$$L(\theta) = \sum_{i=1}^{T} \log f_r(\theta | r_t, I_{t-1}) = \sum_{i=1}^{T} L_i(\theta),$$

(22)

where $f_r(\theta | r_t, I_{t-1})$ is the SGED sample likelihood function for the returns of each series, given by Eq. (11), and $\theta$ is a column vector of coefficients for the conditional mean, variance, asymmetry, and shape parameters specified previously. The estimated values for the skewness price of risk, denoted by $\delta_t$, are obtained via the substitution of the MLE estimates for $k_t$ and $\lambda_t$ into Eq. (4). In the case of the unconditional distribution, $\theta$ includes the parameters $\mu$, $\sigma$, $\lambda$, and $k$. Robust standard errors for the MLE estimates denoted by $\tilde{\theta}$ are obtained from the equation

$$\text{var}(\tilde{\theta}) = \left( \sum_{i=1}^{T} \frac{\partial^2 L_i(\tilde{\theta})}{\partial \theta \partial \theta'} \right)^{-1} \sum_{i=1}^{T} \frac{\partial L_i(\tilde{\theta})}{\partial \theta} \frac{\partial L_i(\tilde{\theta})}{\partial \theta'} \left( \sum_{i=1}^{T} \frac{\partial^2 L_i(\tilde{\theta})}{\partial \theta \partial \theta'} \right)^{-1}.$$

(23)

As shown in Bollerslev and Wooldridge (1992) and Engle and Gonzalez-Rivera (1991), the above estimators are more appropriate in the case of misspecified sample likelihood functions.

3.3 Estimation of unconditional parameters

Table 2 presents the MLE estimates of the parameters of the unconditional distributions of the six series. The estimated means and standard deviations are similar to those in Table 1. The asymmetry parameter $\lambda$ of the distribution of returns is positive (right skewed) and statistically significant for BTC and statistically insignificant for the remaining series. The estimated values of the shape parameter $k$ for BTC and RMB are lower than one, indicating excessive amounts of kurtosis. For the remaining series, it ranges from 1.0139 (JPY) to 1.2329 (CAN). These values deviate significantly from $k = 2$, which is the value of the shape parameter for the normal distribution.

The Pearson’s moment coefficients of kurtosis $KU$ for BTC and RMB, computed using Eq. (17), are respectively 10.9342 and 13.0491. For the other series, they range between 4.5992 and 5.8844. Thus, the resulting variances of volatility shocks for BTC and RMB, computed using the equation $\text{var}(z^2) = KU - 1$, are about three times larger than those of EUR, JPY, CAN, and GBP. Note that for the normal distribution, $\text{var}(z^2) = 2$.

The resulting SGED probability curves of the returns of the six series, constructed using the estimated values of the parameters from Table 2, along with their respective histograms.
Stochastic properties and pricing of bitcoin using a GJR-GARCH…

1.3

Fig. 1. Interestingly, the SGED appears to provide an excellent fit to the empirical distribution of the data. This is attributed to its flexibility in accommodating asymmetry and fat-peaked tails. The normal probability curves, especially for BTC and RMB, deviate significantly from their respective histograms. The visual superiority of the SGED is statistically confirmed by the log-likelihood ratio test statistics for normality (LR-Normal), also presented in Table 2. These statistics reject the null hypothesis of normality for the six series.

3.4 Tests for higher-order moment dependencies

Table 3 presents the test statistics for the presence of non-linearities, such as asymmetric volatility, volatility clustering, and higher-order moment dependencies often present in the distribution of financial returns. Panel A presents the test statistics for asymmetric volatility and volatility clustering. These statistics are computed using Equations (A1) and (A2) in the Appendix. With the exception of BTC and GBP, asymmetric volatility is absent in the remaining return series. The latter is attributed to the double-sided property of exchange rates (Theodossiou, 1994). As expected, the statistics for past squared shocks are highly significant, confirming the presence of strong volatility clustering.

Panel B presents the test statistics for the impact of past shocks and their squared values on skewness shocks, measured by $z^3$, and, indirectly, by the asymmetry parameter $\lambda$. These are computed using Equations (A3) and (A4) in the Appendix. The results suggest that past shocks have a significant impact on current skewness and thus the asymmetry parameter $\lambda$. The test statistics for past squared shocks are generally insignificant, suggesting a weak relationship with the asymmetry parameter.

Table 2  Estimated SGED parameters: unconditional distribution of returns

| Statistics | BTC    | EUR    | JPY    | CAN    | GBP    | RMB    |
|------------|--------|--------|--------|--------|--------|--------|
| $\mu$      | 0.4063 | −0.0037| −0.0140| −0.0083| −0.0058| −0.0011|
|            | (0.0998)**| (0.0109)| (0.0016)**| (0.0091)| (0.0113)| (0.0024)|
| $\sigma$   | 5.1330 | 0.4834 | 0.5326 | 0.4571 | 0.5442 | 0.2029 |
|            | (0.1678)**| (0.0098)**| (0.0127)**| (0.009)**| (0.0156)**| (0.0074)**|
| $k$        | 0.7050 | 1.1558 | 1.0139 | 1.2329 | 1.1434 | 0.6479 |
|            | (0.0263)**| (0.0439)**| (0.0382)**| (0.0504)**| (0.0782)**| (0.0514)**|
| $\lambda$  | 0.0313 | 0.0060 | −0.0005| −0.0129| −0.0207| −0.0048 |
|            | (0.0153)*| (0.0209)| (0.0034)| (0.0168)| (0.034)| (0.0097)|
| $SK$       | 0.2202 | 0.0208 | −0.0020| −0.0407| −0.0725| −0.0384 |
| $KU$       | 10.9342| 4.9609 | 5.8844 | 4.5992 | 5.0297 | 13.0491|
| LogL       | −7233.6| −1647.8| −1824.4| −1532.3| −1942.8| 1231.9 |
| LR-Normal  | 1241.2 | 226.7  | 426.8  | 173.0  | 367.0  | 1,401.6|
| OBS        | 2534   | 2534   | 2534   | 2534   | 2534   | 2534   |

Estimates are based on the maximum likelihood estimation method. Parentheses include robust standard errors. $SK = E(z^3)$ and $KU = E(z^4)$ are the Pearson’s moment coefficients of skewness and kurtosis computed using Eqs. (17) and (18), respectively. LogL is the sample log-likelihood value. LR-Normal = −2 (LogL – Log-Normal) is the sample log-likelihood ratio test statistic for normality of returns; it follows chi-square with 2 d.f. Its 5% and 1% critical values are respectively 5.99 and 9.21 ** and * indicate significance at the 1% and 5% level, respectively

(non-parametric) and normal probability curves are presented in Fig. 1. Interestingly, the SGED appears to provide an excellent fit to the empirical distribution of the data. This is attributed to its flexibility in accommodating asymmetry and fat-peaked tails. The normal probability curves, especially for BTC and RMB, deviate significantly from their respective histograms. The visual superiority of the SGED is statistically confirmed by the log-likelihood ratio test statistics for normality (LR-Normal), also presented in Table 2. These statistics reject the null hypothesis of normality for the six series.
Panel C presents the test statistics for the impact of past shocks and their squared values on kurtosis shocks, measured by $z^4$, and, indirectly, by the shape parameter of the distribution $k$. These are computed using Equations (A5) and (A6) in the Appendix. Interestingly, almost all statistics are significant, suggesting a strong impact of past shocks on the shape of the distribution.

In summary, the results confirm the presence of strong volatility, skewness, and kurtosis clustering. These preliminary findings motivate the formulations of the mean, variance, asymmetry, and shape parameters as time-varying.

![Fig. 1 Unconditional distributions of US Dollar daily returns](image-url)
### 3.5 The ST-GJR Garch estimation

Table 4 presents the results of the estimated coefficients of the conditional equations for the variance, mean, asymmetry, and shape parameters of the six series, given respectively by Eqs. (1), (3), (9), and (10).

As for the estimated parameters of the conditional variance presented in Panel A of Table 4, $\alpha_N$ is positive and significant for CAN and GBP; positive and insignificant for EUR and JPY; and negative and insignificant for BTC and RMB. Although insignificant, the latter result is in line with previous findings regarding the presence of negative asymmetric volatility for cryptocurrencies (Cheikh et al. 2020). This can often be attributed to uninformed investors who do not want to lose out on the potentially rising price of Bitcoin (Baur and Dimpfl 2018), and consequently, on a profitable investment. This behavior...
Table 4  Estimation of the parameters of the ST-GJR GARCH model

|       | BTC  | EUR  | JPY  | CAN  | GBP  | RMB  |
|-------|------|------|------|------|------|------|
| A. Conditional variance, $\sigma_t^2 = \nu + (a \sigma_{t-1} + a) \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$ |     |      |      |      |      |      |
| $\nu$  | 1.0034 | 0.0005 | 0.0024 | 0.0010 | 0.0013 | 0.0017 |
| $a$    | 0.1691 | 0.0256 | 0.0540 | 0.0200 | 0.0179 | 0.1838 |
| $\sigma_{t-1}$ | (0.0302)** | (0.0075)** | (0.0125)** | (0.0078)** | (0.0079)** | (0.0356)** |
| $\beta$ | 0.8179 | 0.9679 | 0.9377 | 0.9596 | 0.9598 | 0.7712 |

| B. Conditional mode, $m_t = m_0 + b r_{t-1} + c \sigma_t$ |     |      |      |      |      |      |
| $m_0$    | −0.1040 | −0.0374 | −0.0055 | −0.0288 | −0.0365 | 0.0064 |
| $b$    | −0.1318 | 0.0385 | −0.1011 | −0.0703 | −0.0757 | −0.0555 |
| $c$    | 0.0942 | 0.0764 | −0.0392 | 0.0871 | 0.1112 | −0.0655 |

| C. Asymmetry parameter, $\gamma_i = 1 - 2/\left(1 + \exp\left(h_i\right)\right)$, where $h_i = \gamma_0 + \gamma_N \sigma_{t-1} + \gamma_P u_{t-1} + \gamma_h h_{t-1}$ |     |      |      |      |      |      |
| $\gamma_0$ | −0.0218 | 0.0118 | 0.0547 | −0.0248 | −0.0530 | 0.0287 |
| $\gamma_N$ | −0.0864 | 0.0203 | −0.2159 | −0.1617 | −0.1334 | −0.1133 |
| $\gamma_P$ | 0.2031 | −0.0362 | 0.1093 | 0.0674 | 0.0920 | 0.0222 |
| $\gamma_h$ | 0.2458 | 0.8686 | 0.0041 | −0.1762 | −0.1775 | −0.1854 |

| D. Shape parameter, $k_t = k_U + (k_L - k_U)\left(1 + \exp\left(g_i\right)\right)$, where $g_i = d_0 + d_N u_{t-1} + d_P u_{t-1} + d_s g_{t-1}$ |     |      |      |      |      |      |
| $d_0$    | −0.1319 | 2.3770 | 0.1951 | 0.5282 | 2.0704 | −0.7066 |
| $d_N$    | 0.1204 | 0.3681 | −0.0432 | 1.3978 | −0.3722 | −0.2911 |
| $d_P$    | 0.2258 | 0.5061 | 0.1417 | −0.5138 | −1.1448 | −0.5741 |
| $d_s$    | 0.9663 | −0.7727 | 0.7108 | 0.6310 | 0.2322 | −0.2594 |

| E. Sample means of time-varying parameters | BTC  | EUR  | JPY  | CAN  | GBP  | RMB  |
|-------|------|------|------|------|------|------|
| $\bar{\gamma}$ | 0.0136 | −0.0036 | 0.0094 | −0.0279 | −0.0314 | −0.0005 |
| $\bar{\delta}$ | 0.0190 | −0.0054 | 0.0146 | −0.0431 | −0.0483 | −0.0003 |
| $\bar{\epsilon}$ | 0.0022** | (0.0007)** | (0.0024)** | (0.0018)** | (0.0017)** | (0.0010) |
| $\bar{k}$ | 0.1113 | 0.0709 | −0.0246 | 0.0441 | 0.0628 | −0.0658 |
| $\bar{s}$ | 0.9017 | 1.3746 | 1.2285 | 1.4460 | 1.4535 | 0.7787 |
is commonly known as ‘fear of missing out’, or FOMO. Furthermore, the findings of the asymmetric volatility on foreign exchange rates reflect the two-sided effect of these assets (Theodossiou, 1994). The coefficient $a$ is positive and significant for all series. The parameter $\beta$ is positive and significant for all cases, indicating that volatility is persistent over time. Moreover, their sum is close to 0.98, indicating the presence of strong volatility clustering.

Regarding the conditional mode equation and specifically the risk-return tradeoff (Panel B), the pure price of risk $c$ is positive and statistically significant for BTC and GBP; negative and statistically significant for RMB; and insignificant for the remaining series. With the exception of EUR, the estimated values of the autoregressive coefficient $b$ are negative and statistically significant, indicating that, to some extent, returns are predictable by their own lag values.

Panel C reports the estimated coefficients of the conditional asymmetry index $h_t$. The estimated values for $\gamma_N$ are negative and statistically significant for all series except EUR. These findings imply that past downside market shocks have a negative impact on the asymmetry index $h_t$ and asymmetry parameter $\lambda_t$. The estimated values for $\gamma_P$ are positive and statistically significant for BTC only, suggesting that past upside market shocks have a positive impact on the asymmetry index and parameter. Furthermore, this coefficient is greater (in absolute values) for all cases compared to the downside asymmetry coefficient, except in the case of EUR. The persistence of past upside and downside market shocks ($\gamma_h$) is significant for EUR only, while the estimated parameter $\gamma_0$ is insignificant in all cases.

Panel D reports the estimated coefficients of the shape index $g_t$. The constant $d_0$ is negative and significant for BTC and RMB; positive and significant for EUR; and positive but insignificant for the remaining series. The coefficient $d_N$ is positive and significant only for BTC and negative and significant for RMB, while $d_P$ is positive and significant only for BTC, indicating that past downside and upside market shocks impact
the shape of the distribution of returns. The parameter $d_g$ is statistically significant for BTC, EUR, JPY, and CAN.

Furthermore, Panel E presents the simple arithmetic means of the conditional asymmetry and shape parameters with their standard errors. The means of the conditional asymmetry parameters, denoted by $\tilde{\alpha}$, are positive for BTC and JPY and negative for EUR, CAN, GBP, and RMB. Except for RMB, these means are statistically significant at the 1% level. The fact that these means are close to zero indicate that the time-varying distributions of returns revert over time to a symmetric shape. The average estimated coefficients of the skewness price of risk $\tilde{\gamma}$ are positive and significant for BTC and JPY, and negative and significant for EUR, CAN, and GBP. Overall, these findings highlight the importance of skewness for the risk-return relationship, as outlined by León et al. (2005), Theodossiou and Savva (2016), and Savva and Theodossiou (2018). If ignored, the impact of the risk on mean returns could be underestimated or overestimated. The simple arithmetic means of the total price of risk, denoted by $\tilde{\xi}$, is positive and significant for BTC and GBP, and negative and significant for RMB. For the remaining series, these are statistically insignificant.

The arithmetic means of the shape parameters of the six series of returns, denoted by $\tilde{k}$, range between 0.7787 (RMB) and 1.4460 (CAN), suggesting that the distributions of returns are characterized by excessive volatility. RMB and BTC appear to be the most volatile of the six assets under consideration. The simple arithmetic means of the time-varying Pearson’s moment coefficient of kurtosis, denoted by $\tilde{KU}$, indicate significant deviations of the empirical distributions of returns from the normal distribution, which equal to three.

Table 5 presents various quantile values for the conditional means, conditional standard deviations, conditional asymmetry parameters, and conditional shape parameters of the six series. The conditional means of BTC range between $-35.242\%$ and $22.41\%$. The conditional means of the five currencies fall between $-5.5\%$ and $0.643\%$. The median daily returns are $0.271\%$ for BTC, $-0.006$ for EUR, $-0.011$ for JPY, $-0.007$ for CAN, $-0.005$ for GBP, and $-0.002$ for RMB. Clearly, the return series for BTC reflect its extraordinary price growth over the sample period.

The conditional standard deviations (volatility) of daily returns range between $2.40\%$ and $20.473\%$ for BTC, and between $0.085\%$ and $1.22\%$ for the other five currencies. The median volatility values are $4.117\%$ for BTC, $0.445\%$ for EUR, $0.463\%$ for JPY, $0.427\%$ for CAN, $0.492\%$ for GBP, and $0.145\%$ for RMB. BTC prices are relatively more volatile (about ten times more). In this respect, BTC prices are extremely risky for investors, but offer higher compensation.

The conditional asymmetry parameters range between $-0.329$ and $0.567$ for BTC, $-0.147$ and $0.048$ for EUR, $-0.524$ and $0.338$ for JPY, $-0.412$ and $0.157$ for CAN, $-0.317$ and $0.208$ for GBP, and $-0.566$ and $0.131$ for RMB. Finally, the conditional shape parameter of all series ranges between 0.401 and 1.600. These results imply leptokurtic shapes for their empirical distributions. The median values of the shape parameters are $0.885$ for BTC, $1.381$ for EUR, $1.223$ for JPY, $1.484$ for CAN, $1.475$ for GBP, and $0.801$ for RMB.

Table 6 reports the paired contemporaneous correlations of standardized returns of the six series. These correlations range between 0.0079 (BTC vs EUR) and 0.5483 (EUR vs GBP). Interestingly, the paired correlations between BTC and the five currencies are the lowest. The largest correlation is between BTC and CAN (0.047). These results imply that BTC can be used for hedging purposes by investors and businesses alike (Briere et al. 2015).

Table 7 presents the downside and upside mean probabilities computed using Eqs. (12) and (13). In the case of EUR, CAN, GBP, and RMB, the upside probability is lower than
Stochastic properties and pricing of bitcoin using a GJR-GARCH…

Table 5  Quantiles of the Conditional Mean, Standard Deviation, Shape, and Asymmetry Parameters

|       | Mean     | Min     | 0.5%    | 1%      | 5%      | 50%     | 95%     | 99%     | 99.5%   | Max    |
|-------|----------|---------|---------|---------|---------|---------|---------|---------|---------|--------|
| BTC   | μ_t      | 0.378   | -35.242 | -12.099 | -0.162  | 0.028   | 0.271   | 1.153   | 2.296   | 4.692  | 22.414 |
|       | σ_t      | 4.678   | 2.400   | 2.410   | 2.471   | 2.628   | 4.117   | 8.480   | 12.263  | 19.77  | 20.473 |
|       | λ_t      | 0.014   | -0.329  | -0.253  | -0.145  | -0.084  | -0.004  | 0.166   | 0.277   | 0.538  | 0.567  |
|       | k_t      | 0.902   | 0.513   | 0.519   | 0.555   | 0.605   | 0.885   | 1.263   | 1.411   | 1.505  | 1.510  |
| EUR   | μ_t      | -0.005  | -0.125  | -0.108  | -0.058  | -0.037  | -0.006  | 0.029   | 0.049   | 0.070  | 0.192  |
|       | σ_t      | 0.467   | 0.249   | 0.251   | 0.259   | 0.279   | 0.445   | 0.719   | 0.804   | 0.848  | 0.857  |
|       | λ_t      | -0.004  | -0.147  | -0.120  | -0.065  | -0.042  | -0.002  | 0.029   | 0.038   | 0.043  | 0.048  |
|       | k_t      | 1.375   | 0.712   | 0.967   | 1.133   | 1.231   | 1.381   | 1.493   | 1.529   | 1.568  | 1.593  |
| JPY   | μ_t      | -0.017  | -1.023  | -0.306  | -0.11   | -0.055  | -0.011  | 0.001   | 0.008   | 0.079  | 0.643  |
|       | σ_t      | 0.511   | 0.274   | 0.281   | 0.302   | 0.317   | 0.463   | 0.860   | 0.975   | 1.115  | 1.171  |
|       | λ_t      | 0.009   | -0.524  | -0.412  | -0.243  | -0.140  | -0.028  | 0.117   | 0.177   | 0.259  | 0.338  |
|       | k_t      | 1.229   | 1.108   | 1.119   | 1.160   | 1.176   | 1.223   | 1.300   | 1.335   | 1.387  | 1.442  |
| CAN   | μ_t      | -0.009  | -0.215  | -0.169  | -0.068  | -0.039  | -0.007  | 0.012   | 0.027   | 0.066  | 0.595  |
|       | σ_t      | 0.446   | 0.238   | 0.243   | 0.259   | 0.282   | 0.427   | 0.661   | 0.724   | 0.824  | 0.841  |
|       | λ_t      | -0.028  | -0.412  | -0.336  | -0.207  | -0.145  | -0.012  | 0.049   | 0.076   | 0.129  | 0.157  |
|       | k_t      | 1.446   | 0.699   | 0.867   | 1.028   | 1.175   | 1.484   | 1.592   | 1.598   | 1.600  | 1.600  |
| GBP   | μ_t      | -0.006  | -5.500  | -0.125  | -0.041  | -0.023  | -0.005  | 0.018   | 0.036   | 0.071  | 0.423  |
|       | σ_t      | 0.520   | 0.264   | 0.269   | 0.282   | 0.346   | 0.492   | 0.777   | 1.094   | 1.205  | 1.217  |
|       | λ_t      | -0.031  | -0.317  | -0.276  | -0.189  | -0.136  | -0.024  | 0.049   | 0.093   | 0.157  | 0.208  |
|       | k_t      | 1.435   | 0.433   | 0.479   | 0.934   | 1.224   | 1.475   | 1.515   | 1.519   | 1.522  | 1.523  |
| RMB   | μ_t      | -0.005  | -0.826  | -0.370  | -0.043  | -0.021  | -0.002  | 0.002   | 0.003   | 0.170  | 0.453  |
|       | σ_t      | 0.1660  | 0.085   | 0.085   | 0.085   | 0.091   | 0.145   | 0.325   | 0.423   | 0.488  | 0.517  |
|       | λ_t      | -0.000  | -0.566  | -0.299  | -0.171  | -0.075  | 0.012   | 0.037   | 0.056   | 0.106  | 0.131  |
|       | k_t      | 0.779   | 0.401   | 0.405   | 0.495   | 0.608   | 0.801   | 0.869   | 0.921   | 1.163  | 1.253  |

The quantile values are based on the series of conditional parameters obtained from the MLE estimation of the ST-GJR GARCH model. μ_t is the conditional mean, σ_t the conditional standard deviation, λ_t the conditional asymmetry parameter, and k_t the conditional shape parameter.

Table 6  Correlation matrix of standardized residuals

|       | BTC      | EUR      | JPY      | CAN      | GBP      | RMB      |
|-------|----------|----------|----------|----------|----------|----------|
| EUR   | 0.0282   | 1        |          |          |          |          |
| JPY   | 0.0079   | 0.3392   | 1        |          |          |          |
| CAN   | 0.0467   | 0.4099   | 0.1121   | 1        |          |          |
| GBP   | 0.0150   | 0.5483   | 0.2115   | 0.4270   | 1        |          |
| RMB   | -0.0218  | 0.1896   | 0.0594   | 0.2072   | 0.2042   | 1        |

The correlations are based on standardized residuals, computed using \( z_t = \frac{r_t - \mu_t}{\sigma_t} \), where \( \mu_t \) and \( \sigma_t \) are respectively the conditional mean and conditional standard deviations of returns.
the downside, indicating that there is a higher probability of a negative, rather than a positive, shock to occur (i.e., a negative probability distribution).

Overall, the main findings suggest that the exchange rates are well captured by GARCH models because of the significant first-order positive serial correlation and heteroscedasticity. The SGED estimated parameter, $k$, indicates that the distribution of all series is leptokurtic (especially for Bitcoin; see also Takaishi, 2018). Moreover, the results suggest that ignoring skewness and kurtosis effects from the risk and return relationship may lead to misleading inferences.

### 3.6 Robustness tests

To test the robustness of the results across time, the model is re-estimated using pre-COVID-19 data from the period January 1, 2012 to January 29, 2020 (1836 observations). The results are similar to a large extent with the findings of the full sample. They suggest that all series are leptokurtic with significant asymmetry and kurtosis parameters, indicating that the use of a flexible statistical framework that accounts for time-varying skewness and leptokurtosis is necessary, irrespective of the timespan of the data sample. A full set of results and analysis is available from the authors upon request.

### 4 Summary and conclusions

This paper expands the literature on the investigation of and comparison between the stochastic properties of equilibrium returns of Bitcoin and five major currencies. The investigation extends beyond the conditional mean and conditional variance of the distribution of returns using a stochastic dynamic framework that is based on the flexible SGED.

Our empirical findings suggest that the higher-order dependencies in the series under study affect their risk-return tradeoff. Once the effect of skewness-kurtosis is considered, the true price of risk is obtained.

Furthermore, the dynamic behavior and pricing characteristics of Bitcoin and other currencies can be used in hedging and risk management by speculators and portfolio managers, as well as by regulators and policymakers. Specifically, the estimated equations for computing the conditional mean, variance, asymmetry, and kurtosis parameters provide a way to compute the required rate of return for the purpose of forecasting future prices.

More broadly, these findings have serious implications for theoretical and empirical research on asset pricing, as well as for practitioners interested in asset pricing in global markets.

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**Table 7** Means of downside and upside probabilities

|                | BTC    | EUR    | JPY    | CAN    | GBP    | RMB    |
|----------------|--------|--------|--------|--------|--------|--------|
| Downside Probability | 0.4932 | 0.5018 | 0.4953 | 0.5139 | 0.5157 | 0.5002 |
| Upside Probability   | 0.5068 | 0.4982 | 0.5047 | 0.4861 | 0.4843 | 0.4998 |
| Difference            | 0.0140 | −0.0036| 0.0094 | −0.0278| −0.0314| −0.0005|

Means of downside and upside probabilities, computed using Eqs. (12) and (13)
Appendix

Appendix – Test statistics for higher-order moment dependencies

Let \( z_t = (r_t - \mu) / \sigma \), for \( t = 1, 2, \ldots, T \), be a series of standardized returns. Under the null hypothesis that \( z_t \)'s are normal and identically and independently distributed (i.i.d.) across time, the statistics below, scaled to a zero mean, can be used to test for the presence of asymmetric volatility, volatility clustering, and other higher-order moment dependencies in the data. That is,

\[
nd_1 = \frac{1}{T} \sum_{t=1}^{T} z_{t-j}^2 \quad \text{and} \quad \text{var}(nd_1) = \frac{1}{T} Ez_t^4 = \frac{3}{T},
\]

(A1)

\[
nd_2 = \frac{1}{T} \sum_{t=1}^{T} \left( z_{t-j}^2 - 1 \right) \quad \text{and} \quad \text{var}(nd_2) = \frac{1}{T} \left( Ez_t^4 - 1 \right) = \frac{1}{T} \left( 12 - \frac{4}{T} \right),
\]

(A2)

\[
nd_3 = \frac{1}{T} \sum_{t=1}^{T} z_{t-j}^3 \quad \text{and} \quad \text{var}(nd_3) = \frac{1}{T} Ez_t^6 = \frac{15}{T},
\]

(A3)

\[
nd_4 = \frac{1}{T} \sum_{t=1}^{T} z_{t-j}^4 \quad \text{and} \quad \text{var}(nd_4) = \frac{1}{T} Ez_t^6 = \frac{45}{T},
\]

(A4)

\[
nd_5 = \frac{1}{T} \sum_{t=1}^{T} z_{t-j}^5 \quad \text{and} \quad \text{var}(nd_5) = \frac{1}{T} Ez_t^8 = \frac{105}{T},
\]

(A5)

\[
nd_6 = \frac{1}{T} \sum_{t=1}^{T} \left( z_{t-j}^6 - 3 \right) \quad \text{and} \quad \text{var}(nd_6) = \frac{1}{T} \left( Ez_t^8 - 9 \right) = \frac{1}{T} \left( 378 - \frac{72}{T} \right),
\]

(A6)

where \( j = 1, 2, \ldots \) and \( T \) is the sample size.

Note that test statistic \( nt(nd_i) = nd_i / \sqrt{\text{var}(nd_i)} \), for \( i = 1, 2, \ldots, 6 \), is asymptotically normal.

Proof

For any normal i.i.d. standardized return, the \( s \)th moment is given by.

\[
Ez_t^s = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z_t^s e^{-\frac{z_t^2}{2}} dz_t = \frac{1}{\sqrt{2\pi}} \left[ (-1)^s + 1 \right] \int_{0}^{\infty} z_t^s e^{-\frac{z_t^2}{2}} dz_t.
\]

For \( w_t = z_t^2 / 2, z_t = 2^{\frac{1}{2}} w_t^{\frac{1}{2}} \) and \( dz_t = 2^{-\frac{1}{2}} w_t^{-\frac{1}{2}} dw_t \),
For \( s = 2, 4, 6 \) and \( 8 \) (note that \( \Gamma(1/2) = \sqrt{\pi} \)),

\[
Ez^2_t = \frac{1}{\sqrt{\pi}} 2\Gamma\left(\frac{3}{2}\right) = \frac{1}{\sqrt{\pi}} 2 \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = 1,
\]

\[
Ez^4_t = \frac{1}{\sqrt{\pi}} 2^2 \Gamma\left(\frac{5}{2}\right) = \frac{1}{\sqrt{\pi}} 2^2 \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = 3,
\]

\[
Ez^6_t = \frac{1}{\sqrt{\pi}} 2^3 \Gamma\left(\frac{7}{2}\right) = \frac{1}{\sqrt{\pi}} 2^3 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = 15,
\]

and

\[
Ez^8_t = \frac{1}{\sqrt{\pi}} 2^4 \Gamma\left(\frac{9}{2}\right) = \frac{1}{\sqrt{\pi}} 2^4 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = 105.
\]

All odd moments are equal to zero, i.e., \( Ez^s = 0 \) for \( s = 1, 3, 5, \ldots \)

The statistics given by Equations (A1) to (A6) can be written concisely as

\[
nd_i = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{z^q - z^r}{l} \right)
\]

and

\[
\text{var}(nd_i) = \frac{1}{T^2} E \left( \sum_{t=1}^{T} \frac{z^q - z^r}{l} \right)^2 - l^2,
\]

where \( l = Ez^q_i - Ez^r_i = Ez^q_{i-j} - Ez^r_{i-j}, i = 1, 2, \ldots, 6, q = 1, 2, \) and \( r = 2, 3, 4. \) Note that for \( q = 2, r = 2, Ez^2_{i-j} = 1 \) and \( q = 2, r = 4, Ez^2_{i-j} = 3. \) For other values of \( q \) and \( r, Ez^q_{i-j} = 0. \)

The first term of \( \text{var}(nd_i) \) is

\[
E \left( \sum_{t=1}^{T} \frac{z^q - z^r}{l} \right)^2 = \sum_{t=1}^{T} Ez^2_{i-j} + 2 \sum_{t=1}^{T} Ez^q_{i-j} Ez^{q+r}_{i-j} + \sum_{t=1}^{T} \sum_{t=1}^{T} Ez^q_{i-j} Ez^q_{i-j} Ez^q_{i-j} Ez^r_{i-j} = T Ez^2_{i-j} + 2(T - j) Ez^q_{i-j} Ez^{q+r}_{i-j} + (T^2 - T - 2(T - j)) Ez^q_{i-j} Ez^q_{i-j} Ez^q_{i-j}.
\]

Substitution of the values \( j \) and \( Ez^q_i \) into the above equation gives the variances of the six statistics.
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