On the integrability of the geodesic flow on a Friedmann–Robertson–Walker spacetime

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Abstract

We study the geodesic flow on the cotangent bundle $T^*M$ of a Friedmann–Robertson–Walker (FRW) spacetime $(M, g)$. On this bundle, the Hamilton–Jacobi equation is completely separable and this property allows us to construct four linearly independent integrals in involution, i.e. Poisson commuting amongst themselves and pointwise linearly independent. As a consequence, the geodesic flow on an FRW background is completely integrable in the Liouville sense. For a spatially flat or spatially closed universe, we construct submanifolds that remain invariant under the action of the flow. For a spatially closed universe these submanifolds are topologically $R \times S^1 \times S^1 \times S^1$, while for a spatially flat universe they are topologically $R \times R \times S^1 \times S^1$. However, due to the highly symmetrical nature of the background spacetime, the four integrals in involution also admit regions where they fail to be linearly independent. We identify these regions although we have not been able in a mathematically rigorous fashion to describe the structure of the associated invariant submanifolds. Nevertheless, the phase space trajectories contained in these submanifolds when projected on the base manifold describe radial timelike geodesics or timelike geodesics ‘comoving’ with the cosmological expansion.

Keywords: cotangent bundle, integrable geodesic flows, Hamilton–Jacobi equation, FRW spacetimes

Introduction

In the past, studies of the geodesic flow associated with a Riemannian metric belonged to a field explored exclusively by mathematicians who aimed to analyze the global behavior of geodesics on a given Riemannian manifold (for an overview the reader is referred to: [1–5]). As it turns out, geodesics are affected by the topological and geometrical properties of the background manifold and disentangling these effects constitutes a problem of immense mathematical complexity. A systematic treatment of the geodesic flow begins by passing to an equivalent Hamiltonian system defined on the cotangent bundle of the underlying manifold and in that manner the flow defined by the corresponding Hamiltonian vector field is what mathematicians refer to as the geodesic flow. The ‘phase portrait’ of the resulting dynamical system provides insights on the global behavior of the geodesics on the background manifold and this ‘phase space’ description of the problem has been proven to be a fruitful one. It allows methods of the symplectic geometry to be called upon and successfully addresses thorny issues such as whether the geodesics exhibit a regular or chaotic behavior or whether a given geometry admits closed

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2 For an introduction to this bundle and an introduction to the symplectic geometry see for instance [6–8].

3 It is worth mentioning that an alternative treatment of issues related to the global behavior of geodesics is based on Lie point symmetries, i.e. continuous Lie groups of transformations acting on a suitable space leaving the geodesic equation form invariant (for an introduction see for instance [9–11]). Determining the Lie group of point symmetries of the geodesic equation is a major undertaking and these groups has been obtained only for particular background spacetimes (see for example [12] and references therein). As far as we are aware, no work has been done in obtaining solutions of the geodesic equation from the knowledge of the Lie group of point symmetries. Although the Lie point symmetries approach essentially is formulated on the tangent bundle, it would be of interest to investigate a possible connection of this approach and the Hamiltonian methods advocated in the symplectic approaches.
geodesics, amongst others. This field is an area of intense investigations and for an overview, progress and open problems regarding the geodesic flow the reader is referred to [1–7].

Geodesic flows defined by Lorentzian metrics have become lately a relevant topic for relativists. The seminal work by Carter [13] on the separability of the Hamilton–Jacobi equation on a Kerr background showed that causal geodesics reveal many properties of the family of Kerr spacetimes. For some astrophysical implications of the geodesic flow on a Kerr background see for instance the interesting account of [14]. However, geodesic flows are relevant in other context as well. The description of black hole shadows (see for instance [15]) employs congruences of null geodesics on the underlying black hole background while (cosmological) gravitational lensing deals with congruences of null geodesics on a (perturbed) Friedmann–Robertson–Walker (FRW) cosmology (for an introduction see [16] and further references therein). Moreover, the propagation of the elusive cosmic rays involves the behavior of timelike geodesics either on a cosmological FRW background or on the gravitational field of the Milky way (see for instance [17–19]).

Another setting where geodesic flows is relevant is offered by the kinetic theory of relativistic gases. In this theory, it is postulated that gas particles between collisions move along future directed timelike geodesics of the background metric at least for a gas composed of neutral particles. Therefore, knowledge of the geodesic flow offers insights into the behavior of such gases. For an introduction to the relativistic kinetic theory see [20, 21] and for more recent accounts see [22–26]. Clearly, in these scenarios we are not any longer dealing with the behavior of a single geodesic but rather the focus is on the global behavior of congruences of geodesics and here geodesic flows are becoming relevant.

In this work, we study the geodesic flow defined by the family of FRW spacetimes $(M, g)$. Motivations for such undertaking come from two independent reasons. From the physical viewpoint, future directed null or timelike geodesics are very important in the cosmological context. For instance, within the geometric optics approximation null geodesics are the messengers of information regarding properties of remote cosmological systems while timelike geodesics describe massive particles such as cosmic rays within a cosmological context. The reader is referred to standard textbooks for cosmological applications of such geodesics (see for example [27, 28]). From the mathematical viewpoint, as we shall show in this paper, the geodesic flow on a FRW spacetime is a completely soluble model and thus becomes a theoretical laboratory for analyzing the complex behavior of completely integrable relativistic geodesic flows.

We begin by introducing the cotangent bundle $T^*M$ associated with a background manifold $(M, g)$ and for completeness we provide a brief description of the natural symplectic structure of this bundle and some of its basic properties that would be relevant later on. We introduce the Hamiltonian $H$ whose associated Hamiltonian vector field $L_H$ defines the geodesic flow over $T^*M$. Because the background metric has Lorentzian signature, the projection of the flow onto the base manifold $(M, g)$ describes families of timelike, null or spacelike geodesics. For reasons that will become clear further ahead, we restrict our attention to the timelike component of this flow and our primary focus is to investigate whether this timelike component is completely integrable in the Liouville–Arnold sense. We show that for any Killing vector field $\xi$ admitted by the background metric $g$, there corresponds an integral of motion defined over $T^*M$. Since any FRW metric admits six linearly independent Killing vector fields, we construct six integrals whose Poisson bracket with the Hamiltonian $H$ vanishes over $T^*M$. We find that these six integrals fail to be in involution, i.e. the Poisson brackets amongst themselves fails to be vanishing. However, based on the separability of the Hamilton–Jacobi equation on an FRW background, we construct four new integrals $F_i$, $i = (1, 2, 3, 4)$ that are Poisson commuting amongst themselves and moreover are pointwise linearly independent over regions of $T^*M$ and this establishes that the geodesic flow on an FRW is indeed completely integrable in the Liouville–Arnold sense. Although this conclusion is welcomed, unfortunately by itself it does not yield insights regarding the global behavior of the flow. By studying the structure of the exterior product $dF_1 \wedge dF_2 \wedge dF_3 \wedge dF_4$, we show that for a spatially closed FRW universe, there exist families of $R \times S^3 \times S^3 \times S^3$ submanifolds of $T^*M$ that are invariant under the flow while for a spatially flat FRW universe we find invariant submanifolds that are topologically $R \times R \times S^3 \times S^3$. However that is not the end of the story. We show that the product $dF_1 \wedge dF_2 \wedge dF_3 \wedge dF_4$, vanishes over particular regions of $T^*M$ and over such regions the integrals $F_i$, $i = 1, 2, 3, 4$, become linearly dependent. The nature of the invariant submanifolds over regions where the integrals in involution of a completely integrable system become dependent, is a very subtle problem and has been the subject of thorough mathematical investigations (see for instance [29, 30]). For our part in this paper, we discuss the role of the background Killing fields on the vanishing property of the product $dF_1 \wedge dF_2 \wedge dF_3 \wedge dF_4$ and we argue–although not in very rigorous mathematical manner—that the singular invariant submanifolds associated with these regions are of lower than four dimensions. Moreover the phase space trajectories included in this submanifolds when projected on the base manifold describe either radial geodesics or geodesics comoving with the cosmological expansion.

The structure of the present paper is as follows: in the next section we introduce the cotangent bundle $T^*M$ over a spacetime $(M, g)$ and discuss some of its basic properties. We introduce the family of the Hamiltonian vector fields, the Lie algebra of observables and define the notion of Liouville–Arnold integrability. In section 2, we restrict the spacetime $(M, g)$ to be an FRW spacetime and based on the Killing symmetries of the background metric $g$, we construct and study properties of the integrals of motion associated to the geodesic flow. In section 3, we discuss properties of the Hamilton–Jacobi equation on an
FRW background and this analysis allows to conclude the Liouville–Arnold integrability property of the geodesic flow. In section 4, we introduce and discuss properties of the invariant, by the flow, submanifolds and in the conclusion section, we discuss some applications and open problems.

1. On the symplectic structure of the cotangent bundle

In this section, we introduce the cotangent bundle and some basic tools of the Hamiltonian dynamics. Although the material is standard, it has been included partially to set up notation and partially to introduce some structures that are of crucial importance for the development of this work.

The cotangent bundle $T^*M$ associated with any smooth $n$-dimensional spacetime $(M, g)$ is defined by

$$T^*M = \{(x, p), x \in M, p \in T^*_xM\} \tag{1}$$

and this $T^*M$ defines the natural projection map

$$\pi : T^*M \to M : (x, p) \mapsto \pi(x, p) = x \tag{2}$$

so that at any $x \in M$, the fiber $\pi^{-1}(x)$ is isomorphic to the cotangent space $T^*_xM$. Moreover the base manifold $(M, g)$ induces upon $T^*M$ an atlas so that $T^*M$ becomes a 2n-dimensional smooth, orientable manifold. This can be seen by noting that any local chart $(U, \phi)$ in the $C^\infty$ atlas of $(M, g)$ defines the map:

$$\Psi : V = \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n : (x, p) \mapsto \Psi(x, p) = (x^1, x^2, ..., x^n, p_1, p_2, ..., p_n) \tag{3}$$

which serves as a local coordinate system over $V = \pi^{-1}(U)$. The family of the charts $\{\Psi_{(\alpha)}\}$ in the $C^\infty$ atlas of $(M, g)$ defines a collection of local charts $\{(V_{\alpha}, \Psi_{(\alpha)})\}$ on $T^*M$ which constitute a $C^\infty$ atlas of $T^*M$. Furthermore, it can be checked that for any two intersecting charts $(V_{\alpha}, \Psi_{(\alpha)})$ and $(V_{\beta}, \Psi_{(\beta)})$ the Jacobian matrix has positive definite determinant and thus the collection $\{(V_{\alpha}, \Psi_{(\alpha)})\}$ defines an oriented atlas over $T^*M$.

Using this atlas, at any $(x, p) \in T^*M$, we construct the tangent space $T_{(x,p)}(T^*M)$, the cotangent space $T^*_{(x,p)}(T^*M)$ and the tensor algebra in the usual manner. The local coordinates $(x^a, p_a)$ generate at any $(x, p)$ the coordinate basis for $T_{(x,p)}(T^*M)$ and for $T^*_{(x,p)}(T^*M)$ described by

$$\left\{ \frac{\partial}{\partial x^1} \bigg|_{(x,p)}, \frac{\partial}{\partial x^2} \bigg|_{(x,p)}, ..., \frac{\partial}{\partial x^n} \bigg|_{(x,p)}, \frac{\partial}{\partial p_1} \bigg|_{(x,p)}, \frac{\partial}{\partial p_2} \bigg|_{(x,p)}, ..., \frac{\partial}{\partial p_n} \bigg|_{(x,p)} \right\}$$

and in terms of these bases, any $Z \in T_{(x,p)}(T^*M)$ can be expanded according to:

$$Z = \sum_{\alpha} Z^\alpha \frac{\partial}{\partial x^\alpha} \bigg|_{(x,p)} + \sum_{p} P_p \frac{\partial}{\partial \phi_p} \bigg|_{(x,p)}$$

with a similar expansion for the elements of $T^*_{(x,p)}(T^*M)$ (for further details of the tensor algebra and properties of the cotangent bundle see for instance [6, 7, 31], also section 2 in [25] introduces the cotangent bundle and discusses applications of this bundle to the description of a relativistic gas.).

One important structure—in fact of crucial importance for the development of this work—is the association of smooth real valued functions on $T^*M$ induced by any smooth contravariant tensor field defined over $(M, g)$. To see this connection, let $A$ be any smooth $(k, 0)$ contravariant tensor field over $(M, g)$, then this $A$ induces the smooth real valued function $\tilde{A}$ via:

$$\tilde{A} : T^*M \to R : (x, p) \mapsto \tilde{A}(x, p) = A(x)(p, p, ..., p), \tag{4}$$

where $A(x)(p, p, ..., p)$ stands for the value of $A$ at $x$ evaluated on $k$ copies of $p \in T^*_xM$. As an example, let $X$ be any smooth vector field on $(M, g)$, then the real valued function $\tilde{X}(x, p)$ induced by this $X$ is described by:

$$\tilde{X}(x, p) = X(x)(p) = \langle p, X(x) \rangle = X^a(x^a)p_a \tag{5}$$

where $<, >$ stands for the natural pairing between elements of $T^*_xM$ and $T_xM$ and in the last equality we evaluated this pairing relative to a set of local coordinates of the background $(M, g)$. The maps defined in (4) and (5) will be frequently employed in the next sections.

However, by far the most important structure that $T^*M$ acquires from the base manifold, is its natural symplectic structure. To define this structure, we note that the map $\pi$ in (2) induces the linear map

$$\pi_{(x,p)} : T_{(x,p)}(T^*M) \to T_{\pi(x)}M : L \mapsto \pi_{(x,p)}(L)$$

where $\pi_{(x,p)}(L) \in T_{\pi(x)}M$ is defined so that for any smooth $f : M \to R$ we have:

$$\pi_{(x,p)}(L)(f) = L(f(x))$$

Since the composition $f \circ \pi : T^*M \to R$ is smooth, it follows that $L(f(x))$ is well defined and thus $\pi_{(x,p)}(L)$ is also well defined. We now define the co-vector $\theta$ over $T^*M$ via

$$\theta_{(x,p)}(L) = p_1[\pi_{(x,p)}(L)], \quad L \in T_{(x,p)}T^*M$$

In this work, all manifolds involved are assumed to be $C^\infty$. Whenever other fields are employed, they are assumed to be smooth enough so that any operation of differentiaton performed upon them is to be well defined. We should mention that even though we begin with a spacetime $(M, g)$, actually for the most part of this section the metric $g$ does not play any role. It becomes important in defining the Hamiltonian $H$ and the Liouville vector field $L_H$ at the very end of this section.

The set $(x^a, p_a) = (x^1, x^2, ..., x^n, p_1, p_2, ..., p_n)$ are the local coordinates assigned by $(V, \Psi)$ to the point $(x, p)$ on $V = \pi^{-1}(U)$. The resulting chart $(V, \Psi)$ is referred as an adapted chart and the associated coordinates are often referred as adapted coordinates.
which is smooth and well defined. In terms of the local coordinates \((x^\mu, p_\mu)\), the form \(\theta\) can be written as:

\[
\theta_{(x,p)} = p_\mu \, dx^\mu|_{(x,p)}.
\]

The exterior derivative of \(\theta\) defines the closed two form \(\Omega = d\theta\) which locally takes the form

\[
\Omega_{(x,p)} = dp_\mu|_{(x,p)} \wedge dx^\mu|_{(x,p)},
\]

and this local representation shows that \(\Omega\) is a non degenerate, closed two-form field on \(T^*M\), i.e. \(\Omega\) serves as a symplectic form over \(T^*M\).

The bundle \(T^*M\) equipped with this symplectic form \(\Omega\), becomes a smooth symplectic manifold taken as the arena of the Hamiltonian dynamics. From this perspective, any smooth function \(H: T^*M \to \mathbb{R}\) can serve as a Hamiltonian and \(\Omega\) determines uniquely the corresponding Hamiltonian vector field \(L_H\) on \(T^*M\) via

\[
dH = -i_\Omega \Omega = \Omega( L_H)
\]

here \(i_\Omega \Omega\) stands for the interior product of \(L_H\) with \(\Omega\). In the sequel, by the term Hamiltonian flow we mean the flow defined by this Hamiltonian vector field \(L_H\).

The symplectic form \(\Omega\) also defines the Poisson bracket \([F, G]\) for any smooth pair \(F, G: T^*M \to \mathbb{R}\) via

\[
[F, G] := dF(L_G) = \Omega(L_F, L_G),
\]

where \(L_F, L_G\) stand for the Hamiltonian vector fields associated to \(F, G\). As a consequence, the space of smooth real valued functions, \(C^\infty(T^*M, \mathbb{R})\) equipped with the bracket \([\ ]\) becomes a real Lie algebra.

From (8), it follows that a smooth \(F: T^*M \to \mathbb{R}\) is an integral for the flow generated by \(H\), if and only if \(F\) and \(H\) are in involution i.e. \([F, H] = 0\). More generally, \(k\) real valued functions \((F_1, F_2, \ldots, F_k)\) are said to be in involution if \([F_i, F_j] = 0\) for all \(i, j = 1, 2, \ldots, k\). Moreover, they are said to be independent (resp. dependent) at \((x, p) \in T^*M\), if their differentials \((dF_1, dF_2, \ldots, dF_k)\) at \((x, p)\) are a set of linearly independent (resp. linearly dependent) forms on \(T^*_x(M)\). This property holds, if and only if the wedge product \((dF_1 \wedge dF_2 \wedge dF_3 \ldots \wedge dF_k)(x,p) \neq 0\) (respectively \((dF_1 \wedge dF_2 \wedge dF_3 \ldots \wedge dF_k)(x,p) = 0\)). Whenever the \(k\) integrals in involution \((F_1, F_2, \ldots, F_k)\) are independent, then any connected component of the set

\[
\Gamma_{a_1, a_2, \ldots, a_k} = \{(x, p) \in T^*M | F_i(x, p) = a_i, i = 1, 2, \ldots, k\}
\]

if non-empty, defines a \(k\)-dimensional smooth submanifold of \(T^*M\). This submanifold remains invariant by the flow generated by the corresponding Hamiltonian vector fields \(L_{F_i}\), \(i = 1, 2, \ldots, k\).

We now state the notion of Liouville–Arnold integrability in Hamiltonian dynamics (for additional discussion on this integrability consult [6, 7]):

**Definition 1.** The flow of a Hamiltonian \(H\) defined on \(T^*M\) (or more generally any flow generated by a Hamiltonian \(H\) over a 2n dimensional symplectic manifold \(\mathcal{M}\)) is said to be integrable, or completely integrable in the Liouville–Arnold sense, if there exist \(n\) independent integrals \((F_1 = H, F_2, \ldots, F_n)\) of the flow which are in involution.

Up to this point, the background spacetime metric \(g\) has not played any role. However when it exists, it defines a natural Hamiltonian function \(H\) via

\[
H: T^*M \to \mathbb{R}; (x, p) \mapsto H(x, p) = \frac{1}{2} \bar{g}(x)(p, p)
\]

\[
= \frac{1}{2} g^{\mu \nu}(x) p_\mu p_\nu,
\]

where \(g^{\mu \nu}(x)\) are the contravariant components of \(g\) relative to the local coordinates \((x^1, x^2, \ldots, x^n)\) of \((M, g)\). For this \(H\), it follows from (7) that the Hamiltonian vector field \(L_H\) in the local canonical coordinates \((x^\mu, p_\mu)\) takes the form

\[
L_H = g^{\mu \nu} p_\nu \frac{\partial}{\partial x^\mu} - \frac{1}{2} \frac{\partial g^{\alpha \beta}}{\partial x^\mu} p_\beta \frac{\partial}{\partial p_\mu}
\]

and it is easily verified that the projections of the integral curves of this \(L_H\) on the base manifold describe geodesics on the spacetime \((M, g)\). Due to this property, the flow generated by the Liouville vector field \(L_H\) (or equivalently by the \(H\) in (9)) is referred to as a geodesic flow. Since \(dH(L_H) = 0\) and the Lie derivative of \(H\) along the flow of \(L_H\) satisfies \(\mathcal{L}_{L_H} H = 0\), the geodesic flow defined by any smooth Lorentzian metric can be timelike, null or spacelike, depending on whether the integral curves of the Liouville vector field \(L_H\) lie on the hypersurfaces defined respectively by \(H(x^\mu, p_\mu) = -m^2 < 0\), \(H(x^\mu, p_\mu) = 0\) or \(H(x^\mu, p_\mu) = m^2 > 0\).

The main purpose of the present paper is to discuss properties of the timelike component of the geodesic flow for the case where \((M, g)\) corresponds to a spatially homogeneous and spatially isotropic spacetimes, i.e. \((M, g)\) belongs to the family of FRW spacetimes. In the next section, we set up the geodesic flow on this family of spacetimes.

### 2. Constructing the integrals of motion

In this section, we introduce the family of FRW spacetimes and for reasons that will become apparent further below, we describe this family by employing two coordinate gauges: the spherical \((t, r, \theta, \phi)\) and the Cartesian\(^8\) \((t, x, y, z)\) so that \(g\) takes the form

\[
g = -dr^2 + a^2(t)f^2(r) \left( \frac{dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)}{dx^2 + dy^2 + dz^2} \right),
\]

\(^8\) The vector field \(L_{a_t}\) is also referred as the Liouville vector field.

\(^9\) The reason for employing two coordinate gauges is due to the fact that some of the equations in the main text appear to become singular when expressed in spherical coordinates but are perfectly regular when expressed in the Cartesian gauge. Furthermore, some computations become much shorter when performed in the Cartesian gauge.
where
\[
f(r) = \left(1 + \frac{k r^2}{4}\right)^{-1}, \quad r^2 = x^2 + y^2 + z^2, \quad k \in \{-1, 0, 1\}
\]
(12)

Here \(a(t)\) is the scale factor while \(k\) takes the discrete value of \(-1, 0, 1\) depending upon the curvature of the spatial \(t = \text{const}\) space like hypersurfaces. The choice \((k = -1)\) corresponds to negative curvature, \((k = 0)\) to zero curvature while \((k = 1)\) corresponds to positive curvature. The range of the coordinate \(t\) will be taken in the interval \((0, b)\) with \(b > 0\) (the case where \(b \to \infty\) is not excluded). For the spherical chart, \(r\) takes its values in \((0, \infty)\) while \((\theta, \phi)\) take their values in the familiar range while for the Cartesian chart, \((x, y, z)\) take their values over \((-\infty, \infty)\).

It should be mentioned that even though these coordinate gauges become pathological as \(r \to 0\) or \(r \to \infty\), these pathologies do not generate serious problems as long as we restrict our attention to the spatially flat, i.e. \(k = 0\) or the case of closed \(k = 1\) universe. However for the case of \(k = -1\), the conformal factor of the spatial metric becomes singular at \(r^2 = 4\) and this singularity requires special treatment. Because of these technicalities, our analysis covers the \(k = 0\) and \(k = 1\) cases (even though the techniques are extendable to the \(k = -1\) case).

We denote by \(T^*M\) the eight dimensional cotangent bundle associate with this family of spacetimes, and choose as the Hamiltonian \(H\) the function defined in (9). For the spherical gauge this \(H\) takes the form
\[
H(x, p) = \frac{1}{2} \left[ -(p_x)^2 + \frac{1}{a^2 f^2} \left( (p_y)^2 + \frac{(p_y)^2}{r^2} + \frac{(p_z)^2}{r^2 \sin^2 \theta} \right) \right]
\]
(13)
while for the Cartesian gauge reduces to:
\[
H(x, p) = \frac{1}{2} \left[ -(p_x)^2 + \frac{1}{a^2 f^2} \left( (p_y)^2 + (p_y)^2 + (p_z)^2 \right) \right]
\]
(14)
The local spherical or cartesian coordinates \((x^\mu, p_\mu)\) over \(T^*M\) are defined having in mind that for an arbitrary co-vector \(p \in T^*_x(M)\) the following expansions holds:
\[
p = p_0 dx + p_\mu dx^\mu + p_0 d\theta + p_\mu d\phi = p_\mu dx^\mu + p_\lambda dy + p_\zeta dz.
\]
We now describe a few properties of the flow defined by \(H\) via the proposition:

**Proposition 1.** Let \((M, g)\) a manifold, \(\xi\) a Killing field of \(g\) and let on \(T^*M\) the Hamiltonian \(H(x, p) = \frac{1}{2} \hat{g}(x)(p, p)\), then:

(a) the real valued function
\[
\hat{\xi} : T^*M \to R (x, p) \to \hat{\xi}(x, p) = p(\xi) = \xi_\mu p_\mu
\]
is an integral of motion in the sense \([H, \hat{\xi}] = 0\).

(b) the Hamiltonian vector field \(\hat{L}_\xi\) associated with \(\hat{\xi}\) is described by:
\[
\hat{L}_\xi = \xi^\mu \frac{\partial}{\partial x^\mu} - \frac{\partial \xi^\alpha}{\partial \varphi^\alpha} p_\nu \frac{\partial}{\partial p_\nu}.
\]
(16)

(c) If \(\xi_i\) and \(\xi_j\) are two linearly independent Killing vector fields of \(g\) and \(\hat{\xi}_i\) and \(\hat{\xi}_j\) the corresponding functions as in (15), then their Poisson bracket satisfies:
\[
\{\hat{\xi}_i, \hat{\xi}_j\} = \Omega(L_{\hat{\xi}_i}, L_{\hat{\xi}_j}) = [\xi_i, \xi_j]_g p_\theta,
\]
(17)

where \([\xi_i, \xi_j]\) is the commutator between \(\xi_i\) and \(\xi_j\).

**Proof.** To prove (a) we note from the definition of the Poisson bracket in (8), we obtain
\[
\{\hat{\xi}, H\} = \hat{d}\hat{\xi}(L_H) = L_{\hat{H}}(\xi)
\]
\[
= g^{\mu\nu} p_\mu \frac{\partial}{\partial x^\nu} - \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\rho} p_\rho p_\nu \frac{\partial}{\partial p_\rho} (\xi^\rho p_\nu).
\]
\[
= \left[ -\frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\rho} + g_{\rho\sigma} \frac{\partial \xi^\nu}{\partial x^\rho} \right] p^\mu p^\nu = 0,
\]
(18)
where in the third equality we used the local representation of the Hamiltonian vector field \(L_H\) defined in (10), and the local representation of \(\hat{\xi}\) defined in (15) and the last equality follows from the fact that \(\xi\) satisfies the Killing equation.

To prove (b), we return to (7), replace \(H\) for \(\hat{\xi}\), use local canonical coordinates \((x^\mu, p_\mu)\) to express \(L_{\hat{\xi}}\) in the form
\[
L_{\hat{\xi}} = L^{\nu}_{\hat{\xi}} \frac{\partial}{\partial x^\nu} + \hat{H}_{\mu} \frac{\partial}{\partial p_\mu}
\]
and compare:
\[
\xi^\mu \frac{\partial p_\mu}{\partial \xi^\mu} + \frac{\partial \xi^\mu}{\partial x^\mu} dx^\mu = dx^\nu (L_{\hat{\xi}}) dp_\nu - dp^\nu (L_{\hat{\xi}}) dx^\nu.
\]
(19)
This comparison leads immediately to (16).

To prove (c), we appeal to part (b) combined with the local representation of \(\Omega\) in (6):
\[
\{\hat{\xi}_i, \hat{\xi}_j\} = \Omega(L_{\hat{\xi}_i}, L_{\hat{\xi}_j}) = dp_\nu (L_{\hat{\xi}_i}) dx^\nu (L_{\hat{\xi}_j}) - dp_\nu (L_{\hat{\xi}_j}) dx^\nu (L_{\hat{\xi}_i}) = [\xi_i, \xi_j]_g p_\theta.
\]
\[\square\]

The properties of the geodesic flow described by proposition (1) will be very useful further ahead, here we only mention that this proposition is general in the sense that holds irrespectively whether the background metric \(g\) has Lorentzian, Riemannian or Semi-Riemannian signature, it requires however a special Hamiltonian, namely the one defined in equation (9).

It is worth to mention that the Hamiltonian vector field \(L_{\hat{\xi}}\) in (16) originates in the Killing field \(\xi\) of the background metric \(g\) and thus the one parameter (in general local) group of diffeomorphisms that this \(\xi\) generates leaves \(g\) invariant. In the present context, the one parameter (in general local) group of diffeomorphisms generated by \(L_{\hat{\xi}}\) leaves \(H\) invariant as
expressed by the vanishing Poisson bracket of $\dot{\xi}$ with $H$ since:

$$0 = \{\dot{\xi}, H\} = \partial H(L_{\dot{\xi}}) = L_{\dot{\xi}}(H) = \xi_{L_{\dot{\xi}}} = H.$$

In this work, the vector field $L_{\dot{\xi}}$ over $T^*M$ has been introduced as the unique Hamiltonian vector field defined by the real valued function in (15) which in turn is uniquely determined by the Killing field $\xi$ admitted by the background metric $g$. In the approach of [25], the field $L_{\dot{\xi}}$ has been defined via a different route. They considered a one parameter group of diffeomorphisms (not necessary isometries) acting on $(M, g)$ generated by (a not necessary Killing) vector $\xi$ and subsequently lifted this group to the bundle $T^*M$. Using this lifted group of diffeomorphisms they constructed the infinitesimal generator. Interestingly, the resulting generator coincides with the vector field $L_{\dot{\xi}}$ introduced in this work.

Clearly, the proposition (1) implies that the flow generated by symmetric metrics admit integrals of motion. For the case of the FRW metrics, an integration of the Killing equations $L_{\dot{\xi}}g = 0$, yields the following set of linearly independent Killing vector fields:

$$\xi_{(1)} = \frac{y}{\partial \tau} - \frac{x}{\partial \rho} = -\sin \phi \frac{\partial}{\partial \theta} - \cos \theta \cos \phi \frac{\partial}{\partial \phi},$$

$$\xi_{(2)} = \frac{z}{\partial \tau} - \frac{x}{\partial \rho} = \cos \phi \frac{\partial}{\partial \theta} - \cos \theta \frac{\partial}{\partial \phi},$$

$$\xi_{(3)} = \frac{y}{\partial \tau} - \frac{y}{\partial \rho} = \frac{\partial}{\partial \phi}.$$  

$$\eta_{(1)} = \left(1 + \frac{k}{4} (x^2 - y^2 - z^2) \right) \frac{\partial}{\partial x} + \frac{k}{2} y \frac{\partial}{\partial y} + \frac{k}{2} \frac{\partial}{\partial z} = 1 + \frac{k}{4} \sin \theta \cos \phi \frac{\partial}{\partial \tau} + \frac{1}{r} \left(1 - \frac{4}{r} \right) \frac{\partial}{\partial r} \left(\cos \psi \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \frac{\partial}{\partial \phi}\right).$$

$$\eta_{(2)} = \frac{k}{2} y \frac{\partial}{\partial x} + \left(1 - \frac{k}{4} (x^2 - y^2 + z^2) \right) \frac{\partial}{\partial y} + \frac{k}{2} \frac{\partial}{\partial z} = \frac{1}{r} \left(1 - \frac{4}{r} \right) \frac{\partial}{\partial r} \left(\cos \psi \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi}\right).$$

$$\eta_{(3)} = \frac{k}{2} x \frac{\partial}{\partial x} + \frac{k}{2} y \frac{\partial}{\partial y} + \left(1 - \frac{k}{4} (x^2 + y^2 - z^2) \right) \frac{\partial}{\partial z} = \left(1 - \frac{4}{r} \right) \frac{\partial}{\partial r} \left(\frac{\partial}{\partial \phi}\right).$$

where we expressed those fields relative to both spherical coordinates $(r, \theta, \phi)$ and to Cartesian\cite{11} $(x, y, z)$ coordinates (of course these sets are related via $(x, y, z) = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$). The commutators\cite{12} between these Killing fields read:

$$[\xi_{(1)}, \xi_{(2)}] = -\xi_{(3)}, \quad [\xi_{(1)}, \eta_{(2)}] = -\eta_{(3)}, \quad [\eta_{(1)}, \xi_{(2)}] = -k \xi_{(3)},$$

(20) and cyclic permutations. For later use, we record the following brackets:

$$[\xi_{(1)}, \eta_{(3)}] = \eta_{(2)}, \quad [\xi_{(2)}, \eta_{(1)}] = -\eta_{(3)}, \quad [\xi_{(2)}, \eta_{(3)}] = -\eta_{(1)},$$

$$[\xi_{(3)}, \eta_{(1)}] = -\eta_{(2)}, \quad [\xi_{(3)}, \eta_{(2)}] = \eta_{(1)}.$$  

By appealing to the proposition (1) combined with (5), it follows that the six functions:

$$\hat{\xi}_{(i)}(x, p) = \xi_{(i)}(x) p_i, \quad \hat{\eta}_{(i)}(x, p) = \eta_{(i)}(x) p_i \quad i = 1, 2, 3,$$

(22) are Poisson commuting with the Hamiltonian $H$ in (13) (the equivalent form of $H$ in (14)). However by appealing to parts (b) and (c) of proposition (1) and the algebra of the commutators in (20), it follows that these six integrals fail to commute amongst themselves. In fact we have the following expressions for the Poisson brackets:

$$\{\hat{\xi}_{(i)}, \hat{\xi}_{(j)}\} = -\hat{\xi}_{(k)}, \quad \{\hat{\xi}_{(i)}, \hat{\eta}_{(j)}\} = -\hat{\eta}_{(k)},$$

$$\{\hat{\eta}_{(i)}, \hat{\eta}_{(j)}\} = -k \hat{\xi}_{(k)}.$$  

modulo cyclic permutations. Even though the integrals $\hat{\xi}, \hat{\eta}, i = 1, 2, 3$, in (22) fail to Poisson commute amongst themselves, one expects that a combination of them may yield integrals with the desired properties. However, it is not clear how to combine these integrals in a manner that they yield new Poisson commuting integrals. In order to resolve this issue, in the next section we turn our attention to the analysis of the Hamilton–Jacobi equation. But before we do so, we consider some particular combinations of the integrals in (22) that will be useful in the next section.

At first we consider the azimuthal component $L_\phi$ and the magnitude of the angular momentum $L^2$ defined by

$$L_\phi(x, p) = \hat{\xi}_{(i)}(x) p_i = \xi_{(i)}(x) p_i = xp_y - yp_x = p_\phi,$$

(24)

$$L^2 = \hat{\xi}_{(i)}(x^i, p_\mu) p^\mu j + \hat{\xi}_{(j)}(x^j, p_\mu) p^\mu j + \hat{\xi}_{(k)}(x^k, p_\mu) p^\mu k = \left(\xi_{(i)}(x^i, p_\mu) p^\mu j + \xi_{(j)}(x^j, p_\mu) p^\mu j + \xi_{(k)}(x^k, p_\mu) p^\mu k\right)^2 = \left(\xi_{(i)}(x^i, p_\mu) p^\mu j + \xi_{(j)}(x^j, p_\mu) p^\mu j + \xi_{(k)}(x^k, p_\mu) p^\mu k\right)^2 = \left(\eta_{(i)}(x^i, p_\mu) p^\mu j + \eta_{(j)}(x^j, p_\mu) p^\mu j + \eta_{(k)}(x^k, p_\mu) p^\mu k\right)^2 = \left(\eta_{(i)}(x^i, p_\mu) p^\mu j + \eta_{(j)}(x^j, p_\mu) p^\mu j + \eta_{(k)}(x^k, p_\mu) p^\mu k\right)^2,$$

(25)

On the other hand, using the functions $\hat{\eta}_{(i)}$, we construct:

$$K^2 = \hat{\eta}_{(i)}(x^i, p_\mu) p^\mu j + \hat{\eta}_{(j)}(x^j, p_\mu) p^\mu j + \hat{\eta}_{(k)}(x^k, p_\mu) p^\mu k = \left(\eta_{(i)}(x^i, p_\mu) p^\mu j + \eta_{(j)}(x^j, p_\mu) p^\mu j + \eta_{(k)}(x^k, p_\mu) p^\mu k\right)^2$$

(26)

\footnote{This coincidence is not an accident. Notice that the parts (b) and (c) of the proposition (1) hold irrespectively whether $\xi$ is Killing or not. Therefore, starting from a smooth vector field $\xi$ on $(M, g)$, we first introduce the smooth real valued function $\xi$ defined on $T^*M$ and then using the symplectic form we construct the Hamiltonian vector field $L_{\dot{\xi}}$. This $L_{\dot{\xi}}$ generates (at least locally) a one parameter group of diffeomorphisms acting upon $T^*M$ which in essence is the same set of operations employed in the [25].}

\footnote{We have chosen to express the Killing fields in both spherical and Cartesian components since some of computations using Cartesian gauge appear much shorter.}

\footnote{For a compact representation of the commutation relations (20), as well as for the representation of the six spatial Killing fields see equations (2)–(6) of [32].}
and a straightforward but long algebra, using the Cartesian representation of the generators yields

\[
K^2 = p_x^2 + p_y^2 + p_z^2 + \frac{k^2}{16} [p_x^2 + p_y^2 + p_z^2]
\]

\[
\times [x^2 + y^2 + z^2]^2
\]

\[
+ k \frac{1}{2} [p_x^2 (x^2 - y^2 - z^2) + p_y^2 (y^2 - z^2 - x^2)
\]

\[
+ p_z^2 (z^2 - x^2 - y^2)] + 2k [xp_x p_y + xp_y p_z + yz p_p].
\]

The representation of the right-hand side in terms of the spherical coordinates is long and not very revealing. Interestingly however, the combination \(K^2 + kl^2\) has the following simple form:

\[
K^2 + kl^2 = \frac{1}{f(r)} \left[ p_x^2 + p_y^2 + p_z^2 \right]
\]

\[
= \frac{1}{f(r)^2} \left[ p_x^2 + p_y^2 + \frac{p_z^2}{r^2} \right].
\] (27)

As we shall see in the next section, the right-hand sides of (24), (25) and (27) appear naturally in the Hamilton–Jacobi equation. This remarkable property is consequence of the complete separability of the Hamilton–Jacobi equation on an FRW background expressed in the spherical gauge and this problem is analyzed in the next section.

3. Separability of the Hamilton–Jacobi equation

The Hamilton–Jacobi equation for the Hamiltonian described by (9) has the form\(^{13}\)

\[
g_{\mu\nu}(x) \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = -m^2, \quad p_\mu = \frac{\partial S}{\partial x^\mu}
\] (28)

and for the case of the spherical gauge defined in (11) reduces to:

\[
- a^2(t) \left( \frac{\partial S}{\partial \theta} \right)^2 + m^2 a^2(t)
\]

\[
+ \frac{1}{f^2(r)} \left[ \left( \frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] = 0.
\] (29)

The ansatz

\[
S(t, r, \theta, \phi) = S_t(t) + S_r(r) + S_\theta(\theta) + S_\phi(\phi)
\] (30)

implies

\[
\left( \frac{\partial S_t(t)}{\partial t} \right)^2 = m^2 + \frac{\lambda^2}{a^2(t)},
\] (31)

\[
\left( \frac{\partial S_r(r)}{\partial r} \right)^2 = \frac{\lambda f^2(r)}{a^2} - \frac{a_0^2}{r^2},
\] (32)

\[
\left( \frac{\partial S_\theta(\theta)}{\partial \theta} \right)^2 = \frac{a_0^2}{\sin^2 \theta}.
\] (33)

where \((\lambda^2, a_0^2, a_0^2)\) are separations constants. The complete separability\(^{14}\) of (29) is welcomed and implies that the function

\[
S(t, r, \theta, \phi, m, \lambda, a_0, a_0) = \int m^2 + \frac{\lambda}{a^2(t)} \, dt'
\]

\[
+ \int \sqrt{\frac{\lambda f^2(r')}{a^2} - \frac{a_0^2}{r'^2}} \, dr'
\]

\[
+ \int \frac{a_0^2}{\sin^2 \theta} \, d\theta' + a_0 \phi
\] (35)

generates canonical transformations that trivialize the Hamiltonian (see for example [6, 8]). Indeed, defining \((Q^\mu, P_\mu)\) on \(T^*M\) via

\[
P_0 = m, \quad P_1 = \lambda, \quad P_2 = a_0, \quad P_3 = a_0,
\]

\[
Q_1 = \frac{\partial S}{\partial m}, \quad Q_2 = \frac{\partial S}{\partial \lambda}, \quad Q_3 = \frac{\partial S}{\partial a_0}, \quad Q_4 = \frac{\partial S}{\partial a_0},
\]

it is easy to see that the set \((Q^\mu, P_\mu)\) defines new canonical coordinates and relative to these new coordinates the Hamiltonian is trivial. Moreover, since by definition \(p_\mu = \frac{\partial S}{\partial \omega_\mu} \Rightarrow p_\mu = m \frac{dx_\mu}{dt} = g_\mu^\nu p_\nu\), we find in an almost labor free manner that the first integrals describing causal geodesics on an FRW background take the form:

\[
\left( \frac{dt}{d\tau} \right)^2 = m^2 + \frac{\lambda^2}{a^2(t)}, \quad \left( \frac{dr}{d\tau} \right)^2 = \frac{1}{a^2 f^2} \left( \lambda^2 - a_0^2 \right),
\]

\[
\left( \frac{d\theta}{d\tau} \right)^2 = \frac{1}{a^2 f^2 r^2} \left[ a_0^2 - \frac{a_0^2}{\sin^2 \theta} \right], \quad \left( \frac{d\phi}{d\tau} \right)^2 = \frac{a_0^2}{a^2 f^2 r^2 \sin^2 \theta},
\] (36)

where we absorbed the mass parameter \(m\) in the redefinition of the proper time \(\tau\).

In this section we do not analyze the above integrals of geodesic motion, nor we explore the implications of the new canonical chart \((Q^\mu, P_\mu)\) over \(T^*M\) although both of these issues are worth of further analysis. The purpose of this section is to settle the issue regarding the number of integrals in involution with themselves and the Hamiltonian \(H\).

In order to see how the separability of (29) settles this problem, we return to (31)–(34) and first we solve the separation constants \((\lambda^2, a_0^2, a_0^2)\) in terms of the canonical momenta \(p_\mu = \frac{\partial S}{\partial \omega_\mu}\). After some manipulations, we get:

\[
-m^2 = -(p_t)^2 + \frac{\lambda^2}{a^2} = -(p_t)^2
\]

\[
+ \frac{1}{a^2 f^2} \left[ \frac{p_t^2}{r^2} + \frac{p_0}{r^2} + \frac{p_0}{r^2} \right].
\] (37)

\(^{13}\) Note that the form of the Hamilton–Jacobi equation in (28) assumes that \(H\) in (9), is normalized according to \(H(x, p) = -\frac{p^2}{2}\).

\(^{14}\) A referee kindly points out to us that the separability of the Hamilton–Jacobi equation (28) on an FRW background has been suggested in [33]. However beyond this suggestion no further analysis has been pursued in [33] on the implications of this separability.
\[ \lambda^2 = \frac{1}{f^2} \left( p_x^2 + \frac{p_y^2}{r^2} + \frac{p_z^2}{r^2 \sin^2 \theta} \right). \]  

(38)

\[ a_3^2 = \frac{\dot{p}_y^2}{\sin^2 \theta}, \quad a_5^2 = \frac{\dot{p}_y^2}{\sin^2 \theta}. \]  

(39)

However, the right-hand sides of (37)–(39) in combination to (24), (25) and (27), suggest to introduce the functions:

\[ F_i: T^*M \to \mathbb{R}: (x, p) \to F_i(x, p), \quad i = 1, 2, 3, 4 \]  

(40)

with:

\[ F_1(x, p) = 2H(x, p), \quad F_2(x, p) = L^2(x, p), \]
\[ F_3(x, p) = L_3(x, p), \quad F_4(x, p) = K^2(x, p) + kL^2(x, p). \]  

(41)

Expressing these functions in terms of the local coordinates \((x^\alpha, p_\alpha)\), it is seen that relations (37)–(39) just describe the level surfaces of \(F_i\), i.e.:

\[ F_1(x, p) = 2H(x, p) = -m^2, \]
\[ F_3(x, p) = K^2(x, p) + kL^2(x, p) = \lambda^2, \]
\[ F_2(x, p) = L^2(x, p) = a_3^2, \quad F_3(x, p) = L_3(x, p) = a_5. \]  

(42)

We now show that the functions \((F_1 = 2H, F_2, F_3, F_4)\) defined in (40) and (41) are a set of Poisson commuting integrals. For that, it is convenient to construct first the associated Hamiltonian vector fields denoted by \(L_{F_i}\). Clearly \(L_{F_1} = L_H\) is just the Louvillian vector field described in (10) while \(L_{F_2} = L_{L_3} = \frac{\partial}{\partial \dot{p}_3}\) On the other hand, a computation based on (7) shows that the Hamiltonian field associated with \(L^2\) has the form:

\[ L_{F_2} = L_{L^2} = \xi_{(1)}^L p_L L_{\xi_{(1)}} + \xi_{(2)}^L p_L L_{\xi_{(2)}} + \xi_{(3)}^L p_L L_{\xi_{(3)}} \]
\[ = a_1 L_{\xi_{(1)}} + a_2 L_{\xi_{(2)}} + a_3 L_{\xi_{(3)}}, \quad a_i = \xi_{(i)}^L p_L, \quad i = 1, 2, 3, \]

where the second representation of \(L_{F_1}\) will be used shortly. A similar computation shows that the Hamiltonian field \(L_{K^2}\) has the form:

\[ L_{F_3} = L_{K^2} = \eta_{(1)}^L p_L L_{\eta_{(1)}} + \eta_{(2)}^L p_L L_{\eta_{(2)}} + \eta_{(3)}^L p_L L_{\eta_{(3)}} \]
\[ = \hat{a}_1 L_{\eta_{(1)}} + \hat{a}_2 L_{\eta_{(2)}} + \hat{a}_3 L_{\eta_{(3)}}, \quad \hat{a}_i = \eta_{(i)}^L p_L, \quad i = 1, 2, 3. \]

Using these Hamiltonian vector fields, we get as an immediate consequence of the part a) of the proposition (1) that \([H, F_i] = 0\) for all \(i \in 1, 2, 3, 4\). On the other hand, by employing the local canonical coordinates \((x^\alpha, p_\alpha)\), a short computation shows that: \([L_3, L^2] = [L_3, K^2] = 0\). We are therefore left to investigate whether the Poisson bracket between \(L^2\) and \(K^2\) is also vanishing. By appealing to the definition of the Poisson bracket in (8) it follows that

\[ [L^2, K^2] = \Omega(L_{L^2}, L_{K^2}) = \hat{a}_1 \hat{a}_2 \xi_{(1)}^L p_L + \hat{a}_1 \hat{a}_3 \xi_{(2)}^L p_L + \hat{a}_2 \hat{a}_3 \xi_{(3)}^L p_L. \]

where the last equality comes from taking into account the commutators between \(\xi\) and \(n\) listed earlier on.

Thus, the functions \((F_1 = 2H, F_2, F_3, F_4)\) in (40) and (41) are Poisson commuting amongst themselves. Using these integrals, we now introduce the sets:

\[ \Gamma_{(i, j, k, a, a)} = \{ (x, p) \in T^*_x M, F_1(x, p) = 2H(x, p) = -m^2, F_2(x, p) = a_3^2, F_4(x, p) = a_5, F_3(x, p) = \lambda^2 \}. \]  

(43)

i.e. these sets are the common level surface of the integrals.

In the next section, we investigate whether these sets are non empty and if non empty whether they define a four dimensional smooth submanifolds of \(T^*M\).

### 4. The structure of the invariant submanifolds

As we have mentioned in section 1, the integrals \((F_1 = 2H, F_2, F_3, F_4)\) are considered to be independent at \((x, p)\) provided \((dF_1 \wedge dF_2 \wedge dF_3 \wedge dF_4)_{(x, p)} \neq 0\). Since \(F_2 = L^2\) and \(F_3 = K^2 + kL^2\), it is sufficient to examine the product \((dF_1 \wedge dF_2 \wedge dF_3 \wedge dF_4)_{(x, p)} = 0\).

From the definition, we find that at any \((x, p)\):

\[ dF_1 = 2dH = 2 \left[ g^m_{\alpha \beta} p_\alpha \frac{\partial g^{\alpha \beta}}{\partial p_\mu} dx^\mu \right]. \]  

(44)

\[ dF_2 = 2dL^2 = 2 \left( \xi^L_{(i, j)} p_L \right) \left[ \xi_{(i, j)}^L p_L + \frac{\partial g^{\alpha \beta}}{\partial x^\mu} p_\alpha p_\beta dx^\mu \right] \]
\[ + 2(\xi_{(i, j)}^L p_L \xi_{(i, j)}^L p_L + \frac{\partial g^{\alpha \beta}}{\partial x^\mu} p_\alpha p_\beta dx^\mu \right]. \]  

(45)

\[ dF_3 = dL_3 = \left[ \xi_{(i, j)}^L p_L + \frac{\partial g^{\alpha \beta}}{\partial x^\mu} p_\alpha p_\beta dx^\mu \right]. \]  

(46)

\[ dK^2 = 2 \left( \eta_{(i, j)}^L p_L \right) \left[ \eta_{(i, j)}^L p_L + \frac{\partial g^{\alpha \beta}}{\partial x^\mu} p_\alpha p_\beta dx^\mu \right] \]
\[ + 2(\eta_{(i, j)}^L p_L \eta_{(i, j)}^L p_L + \frac{\partial g^{\alpha \beta}}{\partial x^\mu} p_\alpha p_\beta dx^\mu \right]. \]  

(47)

Using these formulas, a straightforward evaluation of \((dF_1 \wedge dF_2 \wedge dF_3 \wedge dK^2)\) yields a long expression which is not very helpful to simplify these results. However, it turns out, some of these sets are singular submanifolds in the following sense: they are regions of \(T^*M\) where either one or more of the \(dF_i\) vanish, or two or more of the \(dF_i\) become linearly dependent.

---

15 It is important to mention here that the functions \((F_1 = 2H, F_2, F_3, F_4)\) defined in (40) and (41) are defined over the entire bundle \(T^*M\). Locally defined integrals cannot be used to conclude Liouville–Arnold integrability. They are required to be extended as smooth integrals over the entire bundle. Here for convenience, we have chosen the constants determining the level surfaces of the \(F_i\) to be the same as those appearing in the separated Hamilton–Jacobi equations (37)–(39).
illuminating in identifying regions where \((dF_1 \wedge dF_2 \wedge dF_3 \wedge dK^2)\) is non vanishing. However, based on the structure of (44)—(47), a few comments are helpful in addressing that problem.

Firstly we note that (44) implies that \(dH(x, p) = 0\) at any \((x, p) \in T^\ast(M)\), since \(dx^k\) and \(dp^k\) are linearly independent and the possibility that all \(p_k\) vanish contradicts the normalization condition. As far as the forms \((dF_2, dF_3, dK^2)\) are concerned, we observe that they are determined by the Killing fields \(\xi_0\) and \(\eta_0\) and here we recall that any non trivial Killing field \(\xi\) cannot satisfy: \(\xi(q) = d_x(q) = 0\) at any \(q \in (M, g)\) (for a proof of this, see for instance [34]). This property implies that the terms within the brackets in the right-hand sides of (45)–(47) cannot become individually zero (recall not all \(p_k\) can be zero) which means that \(dF_3\) is non zero over \(T^\ast(M)\). However, the factors \(\xi_0(x, p) = \xi_0^0(x, p)\), while \(dK^2\) vanishes when all \(\hat{\eta}_0(x, p) = \eta_0^0(x, p)\) in (45) and (47), imply that \((dF_2, dK^2)\) can vanish. For instance, \(dF_2\) vs. \(dK^2\) vanishes over regions where all \(\xi_0(x, p) = \xi_0^0(x, p)\), while \(dK^2\) vanishes when all \(\hat{\eta}_0(x, p) = \eta_0^0(x, p)\), vanish. Moreover, \(dF_2 = d^2F_2\) and \(dK^2\) vanish simultaneously at those \((x, p)\) satisfying \(\xi_0(x, p) = \eta_0(x, p) = 0\). Further ahead we discuss the significance of these zeros.

Finally, \((dF_1 \wedge dF_2 \wedge dF_3 \wedge dK^2)\) can vanish whenever on the level surfaces of the integrals exist \((x, p)\) where at least two of the forms \((dH, dF_2, dF_3, dK^2)\) become linearly dependent. Clearly, at such points \((dF_1 \wedge dF_2 \wedge dF_3 \wedge dK^2)\) vanishes and below we show this possibility indeed occurs.

With these general remarks in mind, we now study the sets \(I_{m, \alpha, \alpha} \subset \mathbb{R}\) defined in (58), and at first we consider the case where \(m^2 > 0\), \((\lambda^2, a_0^2) \in (0, \infty)\) and \(a_0 \in \mathbb{R}\). Our first task is to investigate whether \(I_{m^2 > 0, \lambda > 0, a_0 = 0}\) is non empty. One way to check this property, is to solve algebraically the system: \(F_1(x_0, p_0) = 2H(x_0, p_0) = -m^2 > 0\), \(F_2(x_0, p_0) = a_0^2 > 0\), \(F_3(x_0, p_0) = a_0\), \(F_4(x_0, p_0) = \lambda^2 > 0\) for an initial point \((x_0, p_0)\) on the bundle. If this system admits an \((x_0, p_0)\) as a solution, then we propagate \(F_1(x_0, p_0) = 2H(x_0, p_0)\), \(F_2(x_0, p_0)\), \(F_3(x_0, p_0)\), \(F_4(x_0, p_0)\) along the integral curves of the Hamiltonian vector fields \(L_F\). The maximal connected components consisting of all \((x, p)\) lying on these integral curves define the set \(I_{m^2 > 0, \lambda > 0, a_0 = 0}\) which by construction is non empty.

Although this seems to be a reasonable way to proceed, from the practical point of view, it is a difficult algorithm to implement since in general the integrals \(F_i\) are complicated expressions. An alternative way to proceed is to use the local representation of the integrals and study the individual levels on the planes \((t, p_t), (r, p_r), \ldots\), along the same lines as pursued in [25]. Via this approach, at first, we consider the levels of the integral \(F_1(x, p) = 2H(x, p)\) which satisfy \(F_1(x, p) = 2H(x, p) = -m^2 < 0\). In view of (37) these levels consist of the points \((t, p_t)\) satisfying

\[
    p_t^2 = \frac{\lambda^2}{a^2(t)} = m^2. \tag{48}
\]

The graph of this equation on the \((t, p_t)\)-plane depends upon the scale factor \(a(t)\) and the parameter \(\lambda^2\). If \(\lambda^2 > 0\), then the levels consists of two disconnected regular branches described by

\[
    T_b = \{(t, p_t) \in T^\ast M, t \in (0, b), b > 0, \quad p_t = \pm \left[ m^2 + \frac{\lambda^2}{a^2(t)} \right]^{1/2} \}. \tag{49}
\]

For an open universe, assuming a power law behavior, i.e. \(a(t) = t^\gamma, t \in (0, \infty), \gamma > 0\), asymptotically, i.e. as \(t \to \infty\), the \(p_t\) component approaches the rest mass of the particle while at the other extreme, i.e. as \(t \to 0\), the rest mass becomes irrelevant since \(p_t\) is dominated by the particles kinetic energy. For the case where \(a(t)\) describes a closed, recollapsing at a finite time \(t_f\) universe, still the level surfaces consist of a family of open regular curves but in this case \(p_t\) diverges as \(t \to 0\) and also \(t \to t_f\). Notice that in the particular case where \(\lambda^2 = 0\), the levels consist again of two disconnected characterized by \(p_t = \pm m\) as can be easily seen from (49). In all cases, as long as \(m^2 \in (0, \infty)\), the levels of \(F_1(x, p) = 2H(x, p) = -m^2 < 0\) consist of open submanifolds diffeomorphic to the real line.

The levels defined by \(F_2(x, p)\) satisfy \(F_2(x, p) = a_0^2 > 0\) and in the spherical gauge they are consist of \((p_\theta, \theta)\) obeying:

\[
    p_\theta^2 + \frac{a_0^2}{\sin^2 \theta} = a_\phi^2, \quad a_\phi \in \mathbb{R}, \tag{50}
\]

which is an equation similar to (48) except for the crucial sign difference in the second term. Equation (50) is the trademark of spherical symmetry and thus it has been encountered in other contexts, for example in the analysis of the Kepler problem (see for instance [8]), or in the analysis of spherical relativistic systems (see for instance [25]). From these studies, it follows that as long as \(0 < a_\phi^2 < a_\phi^2\), then (50) describes a family of closed curves on the \((\theta, p_\theta)\) plane that shrinks to the point \((\theta = \frac{\pi}{2}, p_\theta = 0)\) occurring when \(a_\phi^2 = a_\phi^2\). Therefore, provided \(0 < a_\phi^2 < a_\phi^2\), the levels of the integral \(F_2(x, p)\) are topologically circles (and here we left out particular values of \(a_\phi^2\) and \(a_\phi^2\) that will be analyzed further bellow).

The levels of \(F_3(x, p) = a_0\) when expressed in the local spherical coordinates take the form \(p_\phi = a_0\) and as long as \(a_\phi\) is fixed, they consist of \((\phi, p_\phi)\) with \(\phi \in [0, 2\pi]\) and thus define a circle \(S^1\).

Finally, we analyze the levels of the integral \(F_4(x, p)\) that satisfy \(F_4(x, p) = \bar{K}^2(x, p) + KL^2(x, p) = \lambda^2\). In view of (25) and (27) these levels consist of those \((r, p_r)\) obeying

\[
    p_r^2 = \frac{\lambda^2 f^2(r) - a_0^2}{r^2}. \tag{51}
\]
For $\lambda^2 > 0$ and $a_\theta^2 > 0$, positivity of the right-hand side demands:
\[
\frac{\lambda^2}{a_\theta^2} \geq V(r) = \frac{1}{f^2 r^2} = \left(\frac{1}{r} + kr^2\right)^2.
\] (52)
and this inequality is satisfied provided
\[
r_* \leq r \leq r_*, \quad r_* = \frac{2}{k} [\hat{a} \pm (\hat{a}^2 - k^2)^{1/2}], \quad \hat{a} = \frac{\lambda}{a_\theta}.
\] (53)
where $r_*$ are the two roots of the equation $\hat{a} = f^{-1} r^{-1}$ which are real and distinct provided $\hat{a}^2 := \left(\frac{\lambda}{a_\theta}\right)^2 \geq k$. For a spatially flat universe, i.e. $k = 0$, this inequality always holds as a consequence of $\lambda^2 > 0$ and $a_\theta^2 > 0$ while for a universe with compact spatial sections, i.e. $k = 1$, requires $\lambda^2 > a_\theta^2$.

For $k = 1$ and for $\lambda^2 > a_\theta^2$ so that $\hat{a} = f^{-1} r^{-1}$ admits two real distinct roots $r_\pm$, we find that $r(\tau)$ obeys
\[
\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{a^2 f^2} \left[\lambda^2 - a_\theta^2 \frac{\lambda^2}{f^2 r^2} - 1 \frac{\lambda^2}{f^2 r^2}\right],
\] (54)
which shows that the radial motion is restricted on the closed interval $[r_-, r_+]$ having turning points at $r_\pm$. For the particular values $\lambda^2 = a_\theta^2$ the motion becomes circular at $r_{\min} = 2$ which corresponds to the minima of the potential $V(r) = (\hat{a}^2 f^{-1})^2$.

For $k = 0$, the potential $V(r)$ in (52) reduces to $V(r) = r^{-2}$ which implies that for a given $\hat{a} = \lambda/(a_\theta)^1$ the motion takes place in $r_{\min} = \infty$ where $r_{\min}$ is the root of $r^2 = a_\theta^2 \lambda^2$.

So far, from this analysis we conclude that the sets $\Gamma_{\text{int}:r > 0, \lambda^2 > a_\theta^2 > 0, a_\phi}$ are not empty and now examine whether these sets define smooth four-dimensional submanifolds of $T^*M$. To check if this is the case, we first evaluate the forms $dF_i$ on $\Gamma_{\text{int}:r > 0, \lambda^2 > a_\theta^2 > 0, a_\phi}$ and determine whether exist values of the parameters where the forms $dF_i$ are linearly independent. Secondly, we also investigate whether $dF_2$ or $dF_4$ (or both) vanish on $\Gamma_{\text{int}:r > 0, \lambda^2 > a_\theta > 0, a_\phi}$.

Starting from $dH(x, p)$ in (44) we evaluate $dH(x, p)$ on the level sets of the integrals. In view of the special form of $K^2 + kl^2$ shown in (27), we find that $dH(x, p)$ takes the form:
\[
dH(x, p) = \frac{1}{2} \left[-2p_r dp_t - \frac{2}{a^2} \frac{da^2}{dr} \lambda^2 dt + \frac{1}{a^2} (dK^2 + k dl^2)\right]
\] (55)
which shows that $dH$ is always linearly independent from $dK^2$ and $dF_2 = dL^2$ even for the values $\lambda^2 = 0$ or when the forms $dK^2$ and $dL^2$ are vanishing at some $(x, p)$. This conclusion holds provided $m^2 > 0$.

On the other hand, (45) shows that $dF_2 = dL^2$ vanishes at any $(x, p)$ satisfying $\xi_{ij}(x, p) = 0$ for all $i = (1, 2, 3)$, i.e. at the point where the conformal factor in (11) becomes singular. The case of a spatially hyperbolic universe will be discussed elsewhere.

whether $dF_2 = dL^2$ is linearly independent from the rest of the forms when it is evaluated on $\Gamma_{\text{int}:r > 0, \lambda^2 > a_\theta^2 > 0, a_\phi}$. Using the representation of $L^2$ in the spherical gauge, we obtain:
\[
dF_2(x, p) = dL^2(x, p) = 2p_\theta dp_\theta - \frac{2a_\phi^2 \cos \theta}{\sin^3 \theta} d\theta + \frac{2a_\phi}{\sin^2 \theta} dF_3, \quad a_\phi \in R,
\] (56)
which shows that $dF_2$ are $dF_3$ are linearly independent except for the values $\theta = \frac{\pi}{2}, p_\theta = 0$ lying on the levels of $L^2$ and $L_3$ obeying: $a_\phi^2 = a_\theta^2$. For this case $dF_2 = dL^2$ and $dF_3$ fail to be linearly independent and this occurs when the motion is confined on the equatorial plane.

Finally, we consider the form $dF_4$ and since $F_4(x, p) = K^2(x, p) + kL^2(x, p)$ it follows from (47) and (45) that $dF_4$ vanishes at any $(x, p)$ obeying $\xi_{ij}(x, p) = \delta_{ij}(x, p) = 0$, $i \in (1, 2, 3)$. This case will be analyzed further ahead. However, evaluating $dF_4$ on $\Gamma_{\text{int}:r > 0, \lambda^2 > a_\theta^2 > 0, a_\phi}$ in view of (27), we find:
\[
dK^2 = \frac{dL^2}{f^2} dp_r + \left[p_r \frac{kr}{f} - \frac{2}{r^2} \left(1 - \frac{kr^2}{4}\right) a_\theta^2\right] dr + \left(\frac{1}{r^2 p_r} - k\right) dL^2
\] (57)
which shows that $dK^2$ and $dL^2$ are always linearly independent unless the coefficients of $dp_r$ and $dr$ both vanish simultaneously. From (51) it follows that $p_r$ vanishes at those $r$ that satisfy $\frac{\lambda^2}{a_\theta^2} V(r) = (\hat{a}^2 f^{-1})^2$, i.e. at $r_\pm$, while for the $k = 1$ case the coefficient of $dr$ in (57) vanishes at those $r$ that obey: $r_1 = r_2 = r_{\min} = 2$. Therefore, for $k = 1$, the forms $dK^2$ and $dL^2$ fail to be linearly independent along the circular orbits supported by the potential $V(r) = (\hat{a}^2 f^{-1})^2$. For the case of $k = 0$, the forms $dK^2$ and $dL^2$ are always linearly independent as long as $a_\theta^2 > 0$.

From this analysis we conclude that on the family of sets $\Gamma_{\text{int}:r > 0, \lambda^2 > a_\theta^2 > 0, a_\phi}$ always $(dF_1 \wedge dF_2 \wedge dF_3 \wedge dF_4)_{k(x, p)} \neq 0$. This in turn implies that these sets define a family of smooth four-dimensional submanifolds of $T^*M$. For the case $k = 0$ this family is topologically $R \times R \times S^1 \times S^1$ while for the case $k = 1$ and as long as $\lambda^2 > a_\theta^2 > 0$, then $\Gamma_{\text{int}:r > 0, \lambda^2 > a_\theta^2 > 0, a_\phi}$ also defines a family of smooth four-dimensional submanifolds topologically $R \times R \times S^1 \times S^1$.

The conclusion that the invariant submanifolds have topologies $R \times R \times S^1 \times S^1$ (case $k = 0$) and $R \times S^1 \times S^1 \times S^1$ (for $k = 1$), at first sight seems to contradict the celebrated property of integrable Hamiltonian flows: their invariant submanifolds are the Cartesian product of $n$-tori (four-tori for our case). However, this property holds whenever the invariant submanifolds are compact and connected (for a proof of this property see for instance [6], p 272). The motion on the $(t, p_r)$ and $(r, p_t)$ planes for the $k = 0$ case, fails

\[21\text{ We are avoiding to address the case of a } k = -1 \text{ since for this case the potential } \{\text{V(r) = (f^2)}\} \text{ vanishes at } r^2 = 4, \text{ i.e. at the point where the conformal factor in (11) becomes singular. The case of a spatially hyperbolic universe will be discussed elsewhere.}

\[22\text{ Here we ignore the special values of the parameters that correspond to cases } \theta = \frac{\pi}{2}, p_\theta = 0 \text{ and } a_\phi^2 = a_\theta^2 \text{ as well the case of the circular orbits supported by the potential V(r) = (f^2^-2.} \]
to be bounded and this property is reflected in the topologies of the invariant submanifolds. In fact in [14, 25, 26] they have also found \( R \times S^1 \times S^1 \times S^1 \) topologies\(^{23}\) for the invariant submanifolds of relativistic Hamiltonian integrable systems.

To complete the analysis of invariant submanifolds we now consider the family of sets

\[
\Gamma_{(m^2>0,x^3>0,a^2<j_a=y_j=0)} = \{ (x, p) \in T^s_xM, F_i(x, p) = 2H(x, p) = -m^2, F_j(x, p) = F_k(x, p) = 0, F_l(x, p) = \lambda^2 > 0 \}. \tag{58}
\]

and at first we show that these sets are non empty. For this, we employ the spherical gauge and restrict our attention to points on the bundle coordinatized according to: \((x, p) = (t, r, \theta, \phi, p_t, p_r, 0, 0, 0)\). Clearly for such points \(\xi_i(x, p) = <p, \xi_i(x) > = 0\) for all \(i = 1, 2, 3\) (but note that in general for such points \(\tilde{\eta}_i(x, p) = <p, \tilde{\eta}_i(x) > \neq 0\).

An alternative way to implement \(F_2(x, p) = F_3(x, p) = 0\) is to employ the Cartesian gauge and restrict our attention to the points: \((x, p) = (t, x, y, z, p_t, p_r) = Ax, p_y = Ay, p_z = Az\), where \(A\) is an arbitrary but smooth function of \((x, p)\). Clearly, \(F_2(x, p) = F_3(x, p) = 0\) but also notice that \(dF_2(x, p) = a_F(x, p) = 0\) at any \((x, p)\) lying on the levels \(F_2(x, p) = F_3(x, p) = 0\).

Since here \(m^2 > 0\) and \(\lambda^2 > 0\), the levels of \(F_i(x, p) = 2H(x, p) = -m^2 < 0\) are those described by (48) while the levels of \(K^2(x, p) = \lambda^2 > 0\) are topologically lines satisfying:

\[
p^2_{\theta} - \lambda^2j^2(r) = 0, \lambda^2 > 0.
\]

Moreover, for \(a^2_{\theta} = a_\phi = 0\), (50) implies that \(p_{\phi} = 0\) and thus \(\frac{d\phi}{dr} = 0\), implying \(\theta\) is constant, taken without loss of generality to have the equatorial value.

We now consider the forms \((dH, dF_2, dF_3, dK^2)\) evaluated on \(\Gamma_{(m^2>0,x^3>0,a^2<j_a=y_j=0)}\). Since on these sets \(dF_2 = dF_3 = 0\) and only \((dF_1 \wedge dF_2)_x = p_{\phi} = 0\) is non vanishing we interpret this as implying that the invariant submanifolds \(\Gamma_{(m^2>0,x^3>0,a^2<j_a=y_j=0)}\) are topologically \(R \times \mathbb{R}\), although we do not have a rigorous mathematical proof of this claim. As we have discussed in the introduction, the topology of such singular invariant submanifolds is a subtle problem (see [29, 30]) and at this point more work is needed.

The family of the sets

\[
\Gamma_{(m^2>0,x^3>0,a^2<j_a=y_j=0)} = \{ (x, p) \in T^s_xM, F_i(x, p) = 2H(x, p) = -m^2, F_j(x, p) = F_k(x, p) = 0, a^2_\theta, F_l(x, p) = \lambda^2 = 0 \}. \tag{59}
\]

seems to be empty. This follows by noting that \(F_i(x, p) = \lambda^2 = 0 \implies K^2(x, p) + kl^2(x, p) = 0\) and thus by appealing to (27) we conclude that

\[
p^2_{\theta} = \frac{a^2_\theta}{l^2},
\]

which leads to a contradiction unless \(a^2_\theta = 0\).

Interestingly however, the set \(\Gamma_{(m^2>0,x^3>0,a^2<j_a=y_j=0)}\) is non empty. This can be seen by restrict attention on the points \((x, p)\) coordinatized according to \((x, p) = (t, x, y, z, p_t, p_r, 0, 0, 0)\). Clearly at such points \(\xi_i(x, p) = \tilde{\eta}_i(x, p) = 0\) for all \(i = 1, 2, 3\) and in this case the only non trivial level sets are those described by (48). Evaluating again \((dH, dF_2, dF_3, dK^2)\) on \(\Gamma_{(m^2>0,x^3>0,a^2=j_a=y_j=0)}\) then clearly the only non vanishing differential is \(dF_1 = 2dH\) implying that the invariant submanifolds are one dimensional described by the family of lines shown in (49) in the limit of vanishing \(\lambda\).

5. Summary and discussion

In this work we have analyzed the structure of the timelike component of the geodesic flow for the family of spatially flat and spatially closed FRW spacetimes. Following this, in this section we discuss the benefits of this analysis.

In order to do so, it is instructive at first to discuss the connection between a geodesic flow and the notion of a single geodesic. This connection is best illustrated by recalling the analysis of the Kepler problem in Classical Mechanics (see for example [8]). For a single particle moving in the attractive Newtonian spherical potential, the possible orbits are well known. Due to the symmetries, one can always choose a plane that contains the entire orbit and in that way all details of the orbits are easily determined. However the situation becomes more complex (and also more interesting), when more than one test particles are involved. In that case, one considers Hamiltonian methods and constructs the corresponding Hamiltonian flow and associated invariant subspaces along the lines discussed for instance in section 10 of [8]. Once the structure of these invariant submanifolds are known, they provide information regarding the structure of the trajectory through any chosen point on the phase space.

A similar situation occurs for the case of causal geodesics on an FRW spacetime. The large number of the Killing fields admitted by this background makes the analysis of the behavior of a single causal geodesic a trivial problem. Using these Killing fields, one readsjusts the coordinate gauge so that for any chosen initial condition, the motion takes place either on the \((t, r)\)-plane or along the direction orthogonal to the family of the hypersurfaces which are homogeneous and isotropic (see for instance discussion in [34], p 103 or consult [27, 28]). The first family of timelike geodesics corresponds to the choices \(m^2 > 0\), \(\lambda^2 > 0\), \(a^2_\theta = a_\phi = 0\) while the second family to the choice of constants: \(m^2 > 0\), \(\lambda^2 = a^2_\theta = a_\phi = 0\) (see equation (36)).

But, like for the case of the Kepler problem, the situation becomes more involved if we assign over a hypersurface which is homogeneous and isotropic a distribution of initial positions and of timelike future pointing four momenta (velocities) as for example occurs in problems dealing with relativistic kinetic theory. For this case, one cannot any longer readjust the coordinate gauge so that all particles move on the same \((t, r)\) plane or comoving with the expansion of the Universe. A framework to address these problems is provided by the structure of the timelike component of the geodesic.

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\(^{23}\) Our thanks to Olivier Sarbach for discussing this point with us.
flow carried out in this work. If for instance, the initial distribution has its support in the invariant submanifold described by $I_{(g^2 > 0, F^2 > 0, a_2^2 > 0, a_s)}$, then we have an understanding of the behavior of the phase space trajectories. As we have shown in this work, they will lie entirely within $I_{(g^2 > 0, F^2 > 0, a_2^2 > 0, a_s)}$ and the topology of that submanifold provides information regarding the global behavior of the phase space trajectories. By projecting them on the base $(M, g)$ we obtain information regarding the structure of the corresponding timelike geodesics and thus for a collisionless gas, its future evolution. In conclusion, the structure of the geodesic flow offers insights not only to the behavior of a single geodesic, but rather to a family of such geodesics.

The results of this paper can be used to construct the most general solutions of the collisionless Liouville equation on an FRW background by taking advantage of the generating function constructed by the complete integral of the Hamilton–Jacobi equation on an FRW background. Via this function, we introduce suitable coordinates in the cotangent bundle and proceed along the same lines as in [25, 26] extending also the work carried out in [32].

Although the present work addressed some properties of the timelike component of the geodesic flow mainly for the spatially flat and spatially closed universe, it would be of interest to address the properties of the geodesic flow on a spatially hyperbolic universe. Moreover, it would of interest to address the structure of the null component of the geodesic flow on an FRW background since as we have noticed at various stages of this work, the nullity condition introduces some new challenges that are worth to be considered. We hope to discuss these issues in the future.

Finally, this work shows that the geodesic flow on an FRW background is a soluble model and this is welcomed. It offers the possibility to understand subtle issues regarding the dynamics of completely integrable relativistic Hamiltonians such as the foliation of the relativistic phase space by invariant submanifolds, singularities in these foliations, etc. As we have already mentioned, it would be of interest to understand the structure of the invariant submanifolds over points where the integrals in involution fail to be independent. The results of the paper may offer some clues toward this direction due to the presence of the symmetries, but definitely more work is needed in this direction.

The present work was focused on the geodesic flow on a highly symmetric background spacetime, namely the family of spatially flat homogeneous and isotropic spacetimes. It would be very interesting to investigate the Liouville–Arnold type of integrability for the geodesic flow on less symmetric backgrounds such as the family of Bianchi models. In that regard, the authors24 of [35, 36] introduce a particular family of spacetimes referred as Stäckel spacetimes having the property that the Hamilton–Jacobi equation is separable. In [36], they define a particular family of spatially homogeneous (but non spatially isotropic) Stäckel spacetimes that includes as a special case the family of Bianchi cosmologies and have proven separability of Hamilton–Jacobi for geodesic motion.

24 Our thanks to an anonymous referee for bringing to our attention [35, 36]. It would be interesting to examine whether the resulting integrals are globally well defined and study the structure of the invariant by the flow submanifolds. It is our hope that the present work, combined with that in [36], the analysis in [14] and the studies of the geodesic flow on a Schwarzschild black hole [25, 26] may act as a further stimulus for a systematic study of the fascinating subject of relativistic geodesic flows.

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