Secrecy Amplification for Distributed Encrypted Sources with Correlated Keys using Affine Encoders

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Abstract

This paper proposed the application of post-encryption-compression (PEC) to strengthen the secrecy in the case of distributed encryption where the encryption keys are correlated to each other. We derive the universal code construction for the compression and the rate region where codes with achievability and secrecy are obtainable.

Our main technique is to use affine encoders which are constructed from certain linear encoders to encode the ciphertexts before sending them to public communication channels. We show that if the rates of linear codes are within a certain rate region: (1) information leakage on the original sources from the encoded ciphertexts without the keys is negligible, while (2) one who has legitimate keys is able to retrieve the original source data with negligible error probability.

Keywords

Distributed encryption, Slepian-Wolf network, secrecy amplification, affine encoders

I. INTRODUCTION

Background

In this paper, we consider the problem of strengthening the security of communication in multi-source single-destination network. Especially, we are interested on practical solutions with minimum modifications which can be applied even on already running systems. More precisely, we consider a network system described as follows: multiple sources $X_1$ and $X_2$ are processed in separated nodes, and then sent through their respective public communication channels to a joint sink node. Now suppose that an already running system has a potential secrecy/privacy problem such that $(X_1, X_2)$ might be leaked to an adversary which is eavesdropping all public communication channel.

A common measure to prevent the leaking of $(X_1, X_2)$ to such eavesdropper is by encrypting each source using one time pad encryption in its respective corresponding node before it is sent to the public channel. For $i = 1, 2$, let $X_i$ be encrypted using key $K_i$ into $C_i = X_i \oplus K_i$. Instead of sending $X_1$ and $X_2$, the system sends the ciphertexts $C_1$ and $C_2$ to public communication channels. Obviously, if $K_1$ and $K_2$ are ideally generated such that each is following uniform distribution and is independent to each other, no problem is left as $H(X_1X_2|C_1C_2) = H(X_1X_2)$ holds automatically. Note that this means that the pair of ciphertexts $(C_1, C_2)$ does not give any additional information.
about \((X_1, X_2)\) and thus no one is able to reveal \((X_1, X_2)\) using \((C_1, C_2)\) with a better success probability than that of randomly guessing \((X_1, X_2)\) based on the distribution of \((X_1, X_2)\) only (without knowing \((C_1, C_2)\)).

**Problem Framework: Secrecy/Privacy Guarantee under Correlated Keys in Distributed Encryption**

However, in real world, there is no guarantee that keys are always ideally generated, and encryption keys in a system might be correlated to each other. It is easy to see that if \(K_1\) and \(K_2\) are correlated to each other, i.e., \(H(K_1|K_2) < H(K_1)\), the following automatically holds:

\[
H(X_1X_2|C_1C_2) < H(X_1X_2).
\]  

(1)

Notice that the inequation (1) means that we are posed with a security challenge. The first reason is that (1) means that we can no longer directly guarantee that the ciphertexts in pair, i.e., \((C_1, C_2)\), do not give additional information about \((X_1, X_2)\) to the eavesdropper. Furthermore, (1) still holds although \(K_1\) and \(K_2\) are generated randomly from uniform distribution over their respective domain such that any single separated ciphertext do not reveal additional information about the corresponding source data, i.e., \(H(X_i|C_i) = H(X_i)\) for \(i = 1, 2\). In other words, we lost the security guarantee for secrecy against eavesdroppers which access all public communication channels when key are correlated to each other.

Here, we restate our research problem into the following question: Is there any method which: (1) strengthens the secrecy such that it can guarantee that under the condition shown by inequation (1) the eavesdropper can not easily extract \((X_1, X_2)\) from \((C_1, C_2)\), and (2) is implementable with small cost even on already running systems?

**Search for Solution**

Since the root of the problem posed by (1) is the correlation between the keys \(K_1\) and \(K_2\), it is natural to think that if we can somehow reduce the effect of correlation between \(K_1\) and \(K_2\) in \((C_1, C_2)\), then we might be able to amplify the secrecy to an extent that we can guarantee a certain level of secrecy close to perfect secrecy.

**Naïve method (Additional New Secret Randomness):** A naïve method to reduce the correlation between \(K_1\) and \(K_2\) is by introducing additional independent randomness to each node. For example, we can put additional independently generated randomness \(R_1\) and \(R_2\) to each node respectively, and use \(K_1 \oplus R_1\) and \(K_2 \oplus R_2\) as the new inputs as keys to the encryption. However, this method has serious drawbacks. Firstly, it requires new private channels to send the randomness and secondly, it requires each node to bear additional security costs that each node has additional private storage to keep the new randomness securely in private manner. We conclude that this naïve method is not feasible to implement in real world, especially for an already running system and for a system with nodes of lightweight devices such as wireless sensors.

Since a simple additional secret randomness technique as above sounds impractical in real world, we need to find approach to reduce the effect of correlation between \(K_1\) and \(K_2\) in \((C_1, C_2)\) without new secret randomness.
Fundamental Idea for Practical Solution: Compression of Keys

Our fundamental idea for solution is based on our intuition that the correlation between compressed keys is smaller than correlation of uncompressed keys. We can explain our intuition as follows. First, recall that the amount of correlation between two random sources $K_1, K_2$ is directly proportional to the mutual information between $K_1$ and $K_2$, i.e., $I(K_1; K_2)$. For simplicity, let $K_1$ and $K_2$ are taking values from the same set of $n$ dimensional vectors $\mathcal{K}^n$. Let $\varphi$ be a mapping from $\mathcal{K}^n$ onto a set of $m$ dimensional vectors $\mathcal{K}^m$, where where $n > m$. One may simply treat $\varphi$ as a kind of compression function. Using the fact that $H(K_2|\varphi(K_1)) \geq H(K_2|K_1)$ and $H(\varphi(K_1)|\varphi(K_2)) \geq H(\varphi(K_1)|K_2)$ hold, it is easy to derive the following inequations.

$$I(K_1; K_2) \geq I(\varphi(K_1), K_2) \geq I(\varphi(K_1); \varphi(K_2)).$$

The above inequations basically says that compressing the keys $(K_1, K_2)$ may reduce the effect of correlation between them. Thus, one immediate approach is to compress the keys directly before inputting them into the encryption process.

Infeasibility of Direct Compression of Keys: However, recall that there are two points of inputs to the encryption in each node, i.e., source and key. Thus, if we want to use the compressed keys as the new keys to the encryption, in order to guarantee secrecy, in general, we also have to compress the messages. Especially in the case of one-time pad encryption, we need to compress the messages to an extent that the lengths are same with the compressed keys. This means we have to perform compression two times for each node. Moreover, in real world, the devices at the nodes might have the keys hardwired into the electronic circuit, and thus modification of the keys before the encryption will require us a special technique to perform a hardware intrusion without bringing down the already running system. Obviously, this kind of modification is risky or impossible in some cases. Therefore, we conclude that direct compression of keys at the point of inputs to encryption is practically infeasible in general.

Hence, we narrow the research question into the following: Can we find a better method for compression such that we do not need to modify inputs of encryption directly and requires less number of compressions than two times for each node?

Proposed Solution: Indirect Compression of Keys through Compression of Ciphertexts using Affine Encoders

The main result in this paper is that we discover a method to perform compression on the keys indirectly by compressing the ciphertexts. We only need to perform the compression only once for each node and thus the implementation cost is only about half of the method which performs compressions on inputs before encryption described above. The core of our discovery is the specific construction of an affine encoder as a good compression function. We prove that the result of compression of a ciphertext from one-time pad encryption using our affine encoder can be seen as one-time pad encryption of a compressed message with a compressed key.

1 Also recall that when $K_1$ and $K_2$ are independent and have no correlation, $I(K_1; K_2) = 0$ holds, while if $K_1$ and $K_2$ are not independent and have some correlation, $I(K_1; K_2) > 0$. 

January 17, 2018 DRAFT
As a illustration, for $i$-th node ($i = 1, 2$), let affine encoder $\varphi_i$ be associated with a linear encoder $\phi_i$ and a vector $a_i$, and let $\varphi_i$ be defined such that $\varphi_i(x) = \phi_i(x) \oplus a_i$. Using $\varphi_i$, we compress the ciphertext of $i$-th node, $C_i = X_i \oplus K_i$, into $\tilde{C}_i = \varphi_i(X_i \oplus K_i) = \phi_i(X_i \oplus K_i) \oplus a_i$. Thanks to the homomorphic property of linear encoder $\phi_i$, we can expand $\phi_i(X_i \oplus K_i)$ into $\varphi_i(X_i) \oplus \phi_i(K_i)$ and we obtain the following equation.

$$\tilde{C}_i = \varphi_i(C_i) = \varphi_i(X_i \oplus K_i) = \phi_i(X_i) \oplus \phi_i(K_i) \oplus a_i = \phi_i(X_i) \oplus \varphi_i(K_i) = \phi_i(X_i) \oplus \tilde{K}_i. \tag{3}$$

Here we set $\tilde{K}_i := \varphi_i(K_i)$, $i = 1, 2$. We can see $\phi_i(X_i)$ as the compressed source and $\tilde{K}_i$ as the compressed key. Hence, an eavesdropper which collects $(\tilde{C}_1, \tilde{C}_2)$ from public communication channels will see $(Y_1, Y_2)$ as results of one-time pad encryption with compressed keys $(\varphi_1(K_1), \varphi_2(K_2))$ which has less correlation compared to the original keys $(K_1, K_2)$.

We borrow the technique of Oohama [1] on generating randomness using Slepian-Wolf coding [2] to make the joint distribution of compressed keys which are hidden within the compressed ciphertexts exponentially close to the uniform distribution that the effect of correlation between the keys becomes negligible. Furthermore, we borrow the result of Csiszár [3] to show that we can obtain good linear encoders and decoders such that in joint sink node the original sources data can be retrieved with exponentially negligible error probability.

We prove that the code construction can be made to depend on only transmission rates using the universal code technique. As far as our knowledge, our result is the first to show explicitly that the preserving of code structure which is the property of affine encoders constructed from linear encoders is essential in order to amplify the secrecy and to preserve the achievability at the same time in the case of distributed encodings/encryption. One can see that our result is in parallel with the existing work of Körner and Marton [4] in the sense that [4] shows that the preserving of code structure by linear encoders is essential in order to prove the optimal transmission rate in the case of two helper network systems.

Practical Feasibility of Proposed Solution

In practice, our approach does not require hardware intrusion to the terminal devices. We can modify the output of the encryption easily by simply connecting the already existing device in each node with an additional external equipment which is capable to receive the ciphertext from the encryption process as inputs, encode them using specified linear codes, and then finally output the encoded ciphertext to the public communication channel. In order to prevent that the leak of pre-encoded original ciphertexts to the eavesdropper in the case of wireless communication, we can apply a simple idea to enclose the existing device and the additional equipment in a Faraday cage so that no electronic signal carrying the pre-encoded ciphertexts leaks outside.

On modification of joint sink node: We remark that our proposed solution which will be described in detail at later sections actually requires the modification of the input and the output of the joint sink node. We argue that despite this requirement, our approach is still feasible and practical. We can consider the joint sink node as a kind of information processing center in real world. And it is quite natural to assume that in such center, the processing tasks are carried by general-purpose machines with high modularity, that the components are easy to be separated, modified, and recombined without disrupting the already running system.
in this paper we consider the secrecy in concrete construction satisfies the universality. Moreover, Johnson et al. [6] only consider secrecy in asymptotic setting, while in this paper we consider the secrecy in concrete setting with concrete exponential upper bound of the success probability of an eavesdropper revealing the sources from compressed ciphertexts in public channel.

II. Preliminaries

In this section, we show the basic notations and related consensus used in this paper. Also, we explain the basic system setting and basic adversarial model we consider in this paper.

**Random Sources of Information and Keys:** Let \((X_1, X_2)\) be a pair of random variables from a finite set \(X_1 \times X_2\). Let \(\{(X_{1,t}, X_{2,t})\}_{t=1}^{\infty}\) be a stationary discrete memoryless source (DMS) such that for each \(t = 1, 2, \ldots\), the pair \((X_{1,t}, X_{2,t})\) takes values in finite set \(X_1 \times X_2\) and obeys the same distribution as that of \((X_1, X_2)\) denoted by 

\[ P_{X_1,X_2} = \left\{ P_{X_1,X_2}(x_1,x_2) \right\}_{(x_1,x_2) \in X_1 \times X_2}. \]

The stationary DMS \(\{(X_{1,t}, X_{2,t})\}_{t=1}^{\infty}\) is specified with \(P_{X_1,X_2}\). Also, let \((K_1, K_2)\) be pair of random variables taken from the same finite set \(X_1 \times X_2\) representing the pair of keys used for encryption at two separate terminals, of which the detailed description will be presented later. Similarly, let \(\{(K_{1,t}, K_{2,t})\}_{t=1}^{\infty}\) be a stationary discrete memoryless source such that for each \(t = 1, 2, \ldots\), the pair \((K_{1,t}, X_{K,t})\) takes values in finite set \(X_1 \times X_2\) and obeys the same distribution as that of \((K_1, K_2)\) denoted by 

\[ P_{K_1,K_2} = \left\{ P_{K_1,K_2}(k_1,k_2) \right\}_{(k_1,k_2) \in X_1 \times X_2}. \]

The stationary DMS \(\{(K_{1,t}, K_{2,t})\}_{t=1}^{\infty}\) is specified with \(P_{K_1,K_2}\).

**Random Variables and Sequences:** We write the sequence of random variables with length \(n\) from the information source as follows: \(X_1 := X_{1,1}X_{1,2} \cdots X_{1,n}, \ X_2 := X_{2,1}X_{2,2} \cdots X_{2,n}\). Similarly, the strings with length \(n\) of \(X_1^n\) and \(X_2^n\) are written as \(x_1 := x_{1,1}x_{1,2} \cdots x_{1,n} \in X_1^n\) and \(x_2 := x_{2,1}x_{2,2} \cdots x_{2,n} \in X_2^n\) respectively. For \((x_1, x_2) \in X_1^n \times X_2^n\), \(P_{X_1,X_2}(x_1, x_2)\) stands for the probability of the occurrence of \((x_1, x_2)\). When the information source is memoryless specified with \(P_{X_1,X_2}\), we have the following equation holds: 

\[ P_{X_1,X_2}(x_1, x_2) = \prod_{t=1}^{n} P_{X_1,X_2}(x_{1,t}, x_{2,t}). \]

In this case we write \(P_{X_1,X_2}(x_1, x_2)\) as \(P_{X_1,X_2}(x_1, x_2)\). Similar notations are used for other random variables and sequences.
Consensus and Notations: Without loss of generality, throughout this paper, we assume that \( X_1 \) and \( X_2 \) are finite fields. The notation \( \oplus \) is used to denote the field addition operation, while the notation \( \ominus \) is used to denote the field subtraction operation, i.e., \( a \ominus b = a \oplus (-b) \) for any elements \( a, b \) of a same finite field. All discussions and theorems in this paper still hold although \( X_1 \) and \( X_2 \) are different finite fields. However, for the sake of simplicity, we use the same notation for field addition and subtraction for both \( X_1 \) and \( X_2 \).

A. Basic System Description

First, let the information sources and keys be generated independently by different parties \( S_{\text{gen}} \) and \( K_{\text{gen}} \) respectively. In our setting, we assume the followings.

- The random keys \( K_1 \) and \( K_2 \) are generated by \( K_{\text{gen}} \) from uniform distribution.
- The key \( K_1 \) is correlated to \( K_2 \).
- The sources \( X_1 \) and \( X_2 \) are generated by \( S_{\text{gen}} \) and are correlated to each other.
- The sources are independent to the keys.

Next, let the two correlated random sources \( X_1 \) and \( X_2 \) from \( S_{\text{gen}} \) be sent to two separated nodes \( L_1 \) and \( L_2 \) respectively. And let two random key (sources) \( K_1 \) and \( K_2 \) from \( K_{\text{gen}} \) be also sent separately to \( L_1 \) and \( L_2 \). Further settings of our system are described as follows, as shown in Fig. 1.

1) **Distributed Sources Processing:** At node 1, \( X_1 \) is encrypted with the key \( K_1 \) using encryption scheme \( Enc_1 \), and at node 2, \( X_2 \) is encrypted with the key \( K_2 \) using encryption scheme \( Enc_2 \). The ciphertexts \( C_i, i = 1, 2 \) are defined by

\[
C_i := Enc_i(X_i, K_i) = X_i \oplus K_i.
\]

2) **Transmission:** Next, the ciphertexts \( C_1 \) and \( C_2 \) are sent to a common information processing center \( D \) through two separated public communication channels. Meanwhile, the keys \( K_1 \) and \( K_2 \) are sent to \( D \) through private communication channels.
3) Joint Sink Node Processing: In D, we decrypt the ciphertexts \((C_1, C_2)\) using the keys \((K_1, K_2)\) through the corresponding decryption procedure \(\text{Dec}_i\), \(i = 1, 2\) which is defined by \(\text{Dec}_i(C_i, K_i) = (C_i \oplus K_i)\) for \(i = 1, 2\). It is obvious that for each \(i = 1, 2\), we can correctly reproduce the source outputs \(X_i\) from \(C_i\) and \(K_i\) by the decryption function \(\text{Dec}_i\).

Eavesdropper Adversarial Model (Informal Description)

An eavesdropper adversary \(A\) eavesdrops all public communication channels in the system and output/estimate the original data from information sources.

III. PROPOSED IDEA: AFFINE ENCODERS AS PRIVACY AMPLIFIER

Let \(\phi^{(n)} := (\phi_1^{(n)}, \phi_2^{(n)})\) be a pair of linear mappings \(\phi_1^{(n)} : X_1^n \rightarrow X_1^{m_1}\) and \(\phi_2^{(n)} : X_2^n \rightarrow X_2^{m_2}\). For each \(i = 1, 2\), we define the mapping \(\phi_i^{(n)} : X_i^n \rightarrow X_i^{m_i}\) by

\[
\phi_i^{(n)}(x_i) = x_i A_i \quad \text{for} \ x_i \in X_i^n,
\]

where \(A_i\) is a matrix with \(n\) rows and \(m_i\) columns. For each \(i = 1, 2\), entries of \(A_i\) are from \(X_i\). We fix \(b_i^{m_i} \in X_i^{m_i}\), \(i = 1, 2\). For each \(i = 1, 2\), define the mapping \(\varphi_i^{(n)} : X_i^n \rightarrow X_i^{m_i}\) by

\[
\varphi_i^{(n)}(k_i) := \phi_i^{(n)}(k_i) \oplus b_i^{m_i} = k_i A_i \oplus b_i^{m_i},
\]

for \(k_i \in X_i^n\). (5)

For each \(i = 1, 2\), the mapping \(\varphi_i^{(n)}\) is called the affine mapping induced by the linear mapping \(\phi_i^{(n)}\) and constant vector \(b_i^{m_i} \in X_i^{m_i}\). By the definition (5) of \(\varphi_i^{(n)}\), \(i = 1, 2\), those satisfy the following affine structure:

\[
\varphi_i^{(n)}(x_i \oplus k_i) = (x_i \oplus k_i) A_i \oplus b_i^{m_i} = x_i A_i \oplus (k_i A_i \oplus b_i^{m_i}) = \phi_i^{(n)}(x_i) \oplus \varphi_i^{(n)}(k_i),
\]

for \(x_i, k_i \in X_i^n\). (6)

Set \(\varphi^{(n)} := (\varphi_1^{(n)}, \varphi_2^{(n)})\). Next, let \(\psi^{(n)}\) be the corresponding joint decoder for \(\phi^{(n)}\) such that \(\psi^{(n)} : X_1^{m_1} \times X_2^{m_2} \rightarrow X_1^n \times X_2^n\). Note that \(\psi^{(n)}\) does not have a linear structure in general.

**Description of Proposed procedure**: We describe the procedure of our privacy amplified system as follows.

1) Encoding of Ciphertexts: First, we use \(\varphi_1^{(n)}\) and \(\varphi_2^{(n)}\) to encode the ciphertexts \(C_1^n = X_1^n \oplus K_1^n\) and \(C_2^n = X_2^n \oplus K_2^n\). Let \(\tilde{C}_i^{m_i} = \varphi_i^{(n)}(C_i)\) for \(i = 1, 2\). Then, instead of sending \(C_1\) and \(C_2\), we send \(\tilde{C}_1^{m_1}\) and \(\tilde{C}_2^{m_2}\) to public communication channel. By the affine structure (6) of encoders we have that for each \(i = 1, 2\),

\[
\tilde{C}_i^{m_i} = \varphi_i^{(n)}(X_i \oplus K_i) = \phi_i^{(n)}(X_i) \oplus \varphi_i^{(n)}(K_i)
\]

\[
= \tilde{X}_i^{m_i} \oplus \tilde{K}_i^{m_i},
\]

where \(\tilde{X}_i^{m_i} := \phi_i^{(n)}(X_i), \tilde{K}_i^{m_i} := \varphi_i^{(n)}(K_i)\).

2) Decoding at Joint Sink Node D: First, using the pair of linear encoders \((\varphi_1^{(n)}, \varphi_2^{(n)})\), D encodes the keys \((K_1, K_2)\) which are received through private channel into \((\tilde{K}_1^{m_1}, \tilde{K}_2^{m_2}) = (\varphi_1^{(n)}(K_1), \varphi_2^{(n)}(K_2))\). Receiving
(\(\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2}\)) from public communication channel, D computes \(\tilde{X}_i^{m_i}, i = 1, 2\) in the following way. From (7), we have that for each \(i = 1, 2\), the decoder D can obtain \(\tilde{X}_i^{m_i} = \phi_i^{(n)}(X_i)\) by subtracting \(\tilde{K}_i^{m_i} = \varphi_i^{(n)}(K_i)\) from \(\tilde{C}_i^{m_i}\). Finally, D outputs \((\tilde{X}_1, \tilde{X}_2)\) by applying the joint decoder \(\psi^{(n)}\) to \((\tilde{X}_1^{m_1}, \tilde{X}_2^{m_2})\) as follows:

\[
(\tilde{X}_1, \tilde{X}_2) = (\psi^{(n)}(\tilde{X}_1^{m_1}, \tilde{X}_2^{m_2})) = (\psi^{(n)}(\phi_1^{(n)}(X_1), \phi_2^{(n)}(X_2))). \tag{8}
\]

Our privacy amplified system described above is illustrated in Fig. 2.

**On Reliability**

From (8), it is clear that the decoding error probability \(p_e\) is as follows:

\[
p_e = p_c(\phi_1^{(n)}, \psi^{(n)} | P_{X_1, X_2}^n) := \Pr[\psi^{(n)}(\phi_1^{(n)}(X_1), \phi_2^{(n)}(X_2)) \neq (X_1, X_2)].
\]

**On Security**

An eavesdropper \(A\) tries to estimate \((X_1, X_2) \in \mathcal{X}_1^n \times \mathcal{X}_2^n\) from \((\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2}) = (\phi_1^{(n)}(X_1 \oplus K_1), \phi_2^{(n)}(X_2 \oplus K_2)) \in \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2}\). The information leakage \(\Delta^{(n)}\) on \((X_1, X_2)\) from \((\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2})\) is measured by the mutual information between those two random pairs. This quantity is formally defined by

\[
\Delta^{(n)} = \Delta^{(n)}(\varphi^{(n)} | P_{X_1, X_2}^n, P_{K_1, K_2}^n) := I(\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2}; X_1, X_2).
\]
Reliable and Secure Framework

Definition 1: The quantity $(R_1, R_2, F, G)$ is achievable for the system $Sys$ if there exists a sequence $\{(\varphi^{(n)}, \psi^{(n)})\}_{n \geq 1}$ such that $\forall \delta > 0, \exists n_0 = n_0(\delta) \in \mathbb{N}, \forall n \geq n_0,$
\[
\frac{1}{n} \log |A_{\varphi}^{m_1}| = \frac{m_1}{n} \log |X_i| \leq R_i, \quad i = 1, 2,
\]
\[
p_e(\varphi^{(n)}, \psi^{(n)})|P_{X_1,X_2}^n \leq 2^{-n(F-\delta)},
\]
\[
\Delta^{(n)}(\varphi^{(n)}|P_{X_1,X_2}^n, P_{K_1,K_2}^n) \leq 2^{-n(G-\delta)}.
\]

Definition 2: (Rate Reliability and Security Region) Let $D_{Sys}(P_{X_1,X_2}, P_{K_1,K_2})$ denote the set of all $(R_1, R_2, F, G)$ such that $(R_1, R_2, F, G)$ is achievable. We call $D_{Sys}(P_{X_1,X_2}, P_{K_1,K_2})$ the rate reliability and security region.

Definition 3 (Reliable and Secure Rate Region): We define the reliable and secure rate region $R_{Sys}(P_{X_1,X_2}, P_{K_1,K_2})$ for the system $Sys$ by
\[
R_{Sys}(P_{X_1,X_2}, P_{K_1,K_2}) := \{(R_1, R_2) : (R_1, R_2, F, G) \in D_{Sys}(P_{X_1,X_2}, P_{K_1,K_2}) \text{ for some } F, G > 0\}.
\]
We call $R_{Sys}(P_{X_1,X_2}, P_{K_1,K_2})$ the reliable and secure rate region.

In this paper we derive good inner bounds of $D_{Sys}(P_{X_1,X_2}, P_{K_1,K_2})$ and $R_{Sys}(P_{X_1,X_2}, P_{K_1,K_2}).$

IV. MAIN RESULTS

In this section we state our main results. To describe our results we define several functions and sets. Let $\overline{X}_1$ and $\overline{X}_2$ be arbitrary random variables over $X_1$ and $X_2$ respectively and $P_{\overline{X}_1,\overline{X}_2}$ is their joint distribution. Let $P(X_1 \times X_2)$ denote the set of all probability distributions on $X_1 \times X_2.$ Similar notations are adopted for other random variables. For $R \geq 0$ and $P_{X_1,X_2} \in P(X_1 \times X_2),$ we define the following three functions:

\[
F_1(R|P_{X_1,X_2}) := \min_{P_{\overline{X}_1,\overline{X}_2} \in P(X_1 \times X_2)} \left\{ [R - H(\overline{X}_1|\overline{X}_2)]^+ + D(P_{\overline{X}_1,\overline{X}_2}||P_{X_1,X_2}) \right\},
\]
\[
F_2(R|P_{X_1,X_2}) := \min_{P_{\overline{X}_1,\overline{X}_2} \in P(X_1 \times X_2)} \left\{ [R - H(\overline{X}_2|\overline{X}_1)]^+ + D(P_{\overline{X}_1,\overline{X}_2}||P_{X_1,X_2}) \right\},
\]
\[
F_3(R|P_{X_1,X_2}) := \min_{P_{\overline{X}_1,\overline{X}_2} \in P(X_1 \times X_2)} \left\{ [R - H(\overline{X}_1\overline{X}_2)]^+ + D(P_{\overline{X}_1,\overline{X}_2}||P_{X_1,X_2}) \right\},
\]
where $[a]^+ := \max\{0, a\}.$ Furthermore, define
\[
F(R_1, R_2|P_{X_1,X_2}) := \min_{i=1,2,3} F_i(R_i|P_{X_1,X_2}),
\]
where $R_3 := R_1 + R_2.$ For random variable $Z$ with distributions $P_Z$ on finite set $Z$ and any $R > 0,$ we define the following function:
\[
G(R|P_Z) := \min_{P_Z \in P(Z)} \{ [H(Z) - R]^+ + D(P_Z||P_Z) \}.
\]
For given $P_{K_1K_2} \in \mathcal{P}(X_1 \times X_2)$, we define

$$G(R_1,R_2|P_{K_1K_2}) := \min\{G(R_1|P_{K_1}), G(R_2|P_{K_2}), G(R_3|P_{K_1K_2})\}.$$ 

Let us define the following two regions of $(R_1,R_2)$:

$$\mathcal{R}_{sw}(P_{X_1X_2}) := \{(R_1,R_2) : R_1 > H(X_1|X_2), R_2 > H(X_2|X_1), R_1 + R_2 > H(X_1X_2)\},$$

$$\mathcal{R}_{key}(P_{K_1K_2}) := \{(R_1,R_2) : R_1 < H(K_1), R_2 < H(K_2), R_1 + R_2 < H(K_1K_2)\}.$$ 

Then we have the following property:

**Property 1:**

a) $F(R_1,R_2,P_{X_1X_2}) > 0$ if and only if $(R_1,R_2) \in \mathcal{R}_{sw}(P_{X_1X_2})$.

b) $G(R_1,R_2,P_{K_1K_2}) > 0$ if and only if $(R_1,R_2) \in \mathcal{R}_{key}(P_{X_1X_2})$.

Our main result is as follows.

**Theorem 1:** For any $R_1, R_2 > 0$, there exists a sequence of mappings $\{(\varphi^{(n)}(\cdot), \psi^{(n)}(\cdot))\}_{n=1}^{\infty}$ such that for any $(P_{X_1X_2}, P_{K_1K_2})$ with $(R_1,R_2) \in \mathcal{R}_{sw}(P_{X_1X_2}) \cap \mathcal{R}_{key}(P_{K_1K_2})$, we have

$$\frac{1}{n} \log |X_i^{m_i}| = \frac{m_i}{n} \log |X_i| \leq R_i, i = 1, 2,$$

$$p_e(\varphi^{(n)}(\cdot), \psi^{(n)}|P_{X_1X_2}^n) \leq 2^{-n[F(R_1,R_2,P_{X_1X_2}) - \delta_{1,n}]},$$

$$\Delta^{(n)}(\varphi^{(n)}|P_{X_1X_2}^n, P_{K_1K_2}^n) \leq 2^{-n[G(R_1,R_2,P_{K_1K_2}) - \delta_{2,n}]}.$$ 

where $\delta_{i,n}, i = 1, 2$ are defined by

$$\delta_{1,n} := \frac{1}{n} \log \left[24(n + 1)^3|X_1||X_2|\right],$$

$$\delta_{2,n} := \frac{1}{n} \log \left[6(\log e)(\log(|X_1||X_2|))\right] \times n(n + 1)^3|X_1||X_2|.$$ 

Note that for $i = 1, 2$, $\delta_{i,n} \to 0$ as $n \to \infty$.

The functions $F(R_1,R_2,P_{X_1X_2})$ and $G(R_1,R_2,P_{K_1K_2})$ take positive values if and only if $(R_1,R_2) \in \mathcal{R}_{sw}(P_{X_1X_2}) \cap \mathcal{R}_{key}(P_{K_1K_2})$. Thus, by Theorem [1] under $(R_1,R_2) \in \mathcal{R}_{sw}(P_{X_1X_2}) \cap \mathcal{R}_{key}(P_{K_1K_2})$, we have the followings:

- On the reliability, $p_e(\varphi^{(n)}(\cdot), \psi^{(n)}|P_{X_1X_2}^n)$ goes to zero exponentially as $n$ tends to infinity, and its exponent is lower bounded by the function $F(R_1,R_2,P_{X_1X_2})$.

- On the security, $\Delta^{(n)}(\varphi^{(n)}|P_{X_1X_2}^n, P_{K_1K_2}^n)$ goes to zero exponentially as $n$ tends to infinity, and its exponent is lower bounded by the function $G(R_1,R_2,P_{K_1K_2})$. 

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The code that attains the exponent functions $F(R_1, R_2 | P_{X_1 X_2})$ and $G(R_1, R_2 | P_{K_1 K_2})$ is the universal code that depends only on $(R_1, R_2)$ not on the value of the distribution $P_{X_1 X_2}$ and $P_{K_1 K_2}$.

Define

$$\mathcal{R}_{\text{Sys}}^{(\text{in})}(P_{X_1 X_2}, P_{K_1 K_2}) := \mathcal{R}_{\text{sw}}(P_{X_1 X_2}) \cap \mathcal{R}_{\text{key}}(P_{K_1 K_2}),$$

$$\mathcal{D}_{\text{Sys}}^{(\text{in})}(P_{X_1 X_2}, P_{K_1 K_2}) := \{ (R_1, R_2, F(R_1, R_2 | P_{X_1 X_2}), G(R_1, R_2 | P_{K_1 K_2}) ) : (R_1, R_2) \in \mathcal{R}_{\text{sw}}(P_{X_1 X_2}) \cap \mathcal{R}_{\text{key}}(P_{K_1 K_2}) \}.$$

From Theorem 1, we immediately obtain the following corollary.

**Corollary 1:**

$$\mathcal{R}_{\text{Sys}}^{(\text{in})}(P_{X_1 X_2}, P_{K_1 K_2}) \subseteq \mathcal{R}_{\text{sys}}(P_{X_1 X_2}, P_{K_1 K_2}),$$

$$\mathcal{D}_{\text{Sys}}^{(\text{in})}(P_{X_1 X_2}, P_{K_1 K_2}) \subseteq \mathcal{D}_{\text{sys}}(P_{X_1 X_2}, P_{K_1 K_2}).$$

A typical shape of $\mathcal{R}_{\text{sw}}(P_{X_1 X_2}) \cap \mathcal{R}_{\text{key}}(P_{K_1 K_2})$ is shown in Fig. 3.

### V. Security Criterion Based on the Correct Probability of Decoding

**On Security**

An eavesdropper $\mathcal{A}$ who tries to estimate $(X_1^n, X_2^n) \in \mathcal{X}_1^n \times \mathcal{X}_2^n$ from
\[(\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2}) = (\varphi_1^{(n)}(X_1A_1 \oplus K_1), \varphi_2^{(n)}(X_2A_2 \oplus K_2)) \in \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2}\]
is always associated with its estimator function \(\psi_A\) defined by
\[
\psi_A : \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2} \rightarrow \mathcal{X}_1^n \times \mathcal{X}_2^n.
\]

For given \((P^n_{X_1, X_2}, P^n_{K_1, K_2})\), let \(p_c(\varphi^{(n)}, \psi_A)\) denote the success probability of \(A\) correctly estimating \((X_1, X_2)\) from \((Y_1^{m_1}, Y_2^{m_2})\) using its estimation function \(\psi_A\) with respect to the pair of linear encoders \(\varphi^{(n)} = (\varphi_1^{(n)}, \varphi_2^{(n)})\), under \((P^n_{X_1, X_2}, P^n_{K_1, K_2})\).

**Definition 4:** The quantity \((R_1, R_2, F, G)\) is achievable for the system \(\text{Sys}\) if there exists a sequence \(\{(\varphi^{(n)}, \psi^{(n)})\}_{n \geq 1}\) such that \(\forall \delta > 0, \exists n_0 = n_0(\delta) \in \mathbb{N}_0, \forall n \geq n_0,
\[
\frac{1}{n} \log |\mathcal{X}_i| = \frac{m_i}{n} \log |\mathcal{X}_i| \leq R_i, \quad i = 1, 2, 
\]
and for any eavesdropper \(A\) with \(\psi_A\):
\[
p_c(\varphi^{(n)}, \psi_A) \leq 2^{-n(G-\delta)}.
\]

**Definition 5 (Rate Reliability and Security Region):** We define the rate reliability and security region \(\overline{D}_{\text{Sys}}(P_{X_1, X_2}, P_{K_1, K_2})\) for the system \(\text{Sys}\) by
\[
\overline{D}_{\text{Sys}}(P_{X_1, X_2}, P_{K_1, K_2}) := \{(R_1, R_2, F, G) \mid (R_1, R_2, F, G) \text{ is achievable for Sys}\}.
\]

**Definition 6 (Reliable and Secure Rate Region):** We define the reliable and secure rate region \(\overline{R}_{\text{Sys}}(P_{X_1, X_2}, P_{K_1, K_2})\) for the system \(\text{Sys}\) by
\[
\overline{R}_{\text{Sys}}(P_{X_1, X_2}, P_{K_1, K_2}) := \left\{(R_1, R_2) \mid (R_1, R_2, F, G) \in \overline{D}_{\text{Sys}}(P_{X_1, X_2}, P_{K_1, K_2}) \text{ for some } F, G > 0 \right\}.
\]

Our aim is to derive good inner bounds of \(\overline{D}_{\text{Sys}}(P_{X_1, X_2}, P_{K_1, K_2})\) and \(\overline{R}_{\text{Sys}}(P_{X_1, X_2}, P_{K_1, K_2})\). To describe our result we define a quantity related to a correct probability of source estimation.

**Definition 7 (Source Uniformity):** Let us define the following quantity:
\[
P_{X_1, X_2}^{\ast} := \max_{(x_1, x_2) \in \mathcal{X}_1^n \times \mathcal{X}_2^n} P_{X_1, X_2}^{n}(x_1, x_2).
\]
Let \(P_{\text{max}} := \max_{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2} P_{X_1, X_2}(x_1, x_2)\). Then, by simple computation we have
\[
P_{X_1, X_2}^{\ast} = P_{\text{max}} = 2^{-n \log \frac{1}{P_{\text{max}}}}.
\]

We state the following lemma which is implied directly by the results of Oohama [7].

**Lemma 1:** Fix positive \(\nu\) arbitrary. In the proposed system, for any pair of encoder \(\varphi^{(n)} = (\varphi_1^{(n)}, \varphi_2^{(n)})\), for any eavesdropper \(A\) with estimator function \(\psi_A\), the following holds.
\[
p_c(\varphi^{(n)}, \psi_A) \leq 2^\nu \cdot P_{X_1, X_2}^{\ast} + \frac{1}{\nu} I(\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2}; X_1, X_2)
\]
\[
= 2^\nu \cdot 2^{-n \log \frac{1}{P_{\text{max}}}} + \frac{1}{\nu} \Delta^{(n)}(\varphi^{(n)}) |P_{X_1, X_2}^{n}, P_{K_1, K_2}^{n}|.
\]
Since the proof of Lemma [1] is found in Oohama [7], we omit the detail of the proof. Choosing \( \nu = 1 \) in (18), we have
\[
p_c(\phi(n), \psi_A | P_{X_1,X_2}, P_{K_1,K_2}) \leq 2 \cdot 2^{-n \log \frac{1}{r_{\max}}} + \Delta(n)(\phi(n) | P_{X_1,X_2}, P_{K_1,K_2}).
\]
(19)
From [19] and Theorem [1] we have the following result.

**Theorem 2:** For any \( R_1, R_2 > 0 \), there exists a sequence of mappings \( \{((\phi(n), \psi(n))\}_{n=1}^{\infty} \) such that for any \( (P_{X_1}, P_{K_1}, P_{K_2}) \) with \( (R_1, R_2) \in R_{sw}(P_{X_1,X_2}) \cap R_{key}(P_{K_1,K_2}) \), we have
\[
\frac{1}{n} \log |X_i^n| = \frac{m_i}{n} \log |X| \leq R_i, i = 1, 2,
\]
\[
p_c(\phi(n), \psi(n) | P_{X_1,X_2}) \leq 2^{-n[F(R_1,R_2|P_{X_1,X_2}) - \delta_{1,n}]}.
\]
(20)
and for any eavesdropper \( A \) with \( \psi_A \):
\[
p_c(\phi(n), \psi_A | P_{X_1,X_2}, P_{K_1,K_2}) \leq 2 \cdot 2^{-n \log \frac{1}{r_{\max}}} + 2^{-n[G(R_1,R_2|P_{K_1,K_2}) - \delta_{2,n}]},
\]
(21)
where \( \delta_{i,n}, i = 1, 2 \) are the same quantities as those in Theorem [1]. Note that for \( i = 1, 2, \delta_{i,n} \to 0 \) as \( n \to \infty \).

The functions \( F(R_1, R_2|P_{X_1,X_2}) \) and \( G(R_1, R_2|P_{K_1,K_2}) \) take positive values if and only if \( (R_1, R_2) \in R_{sw}(P_{X_1,X_2}) \cap R_{key}(P_{K_1,K_2}) \). Thus, by Theorem [1] under \( (R_1, R_2) \in R_{sw}(P_{X_1,X_2}) \cap R_{key}(P_{K_1,K_2}) \), we have the followings:

- On the achievability, \( p_c(\phi(n), \psi(n) | P_{X_1,X_2}) \) goes to zero exponentially as \( n \) tends to infinity, and its exponent is lower bounded by the function \( F(R_1, R_2|P_{K_1,K_2}) \).
- On the security, for any \( \psi_A \), \( p_c(\phi(n), \psi_A | P_{X_1,X_2}, P_{K_1,K_2}) \) goes to zero exponentially as \( n \) tends to infinity, and its exponent is lower bounded by the function \( G^*(R_1, R_2|P_{K_1,K_2}) \), where
\[
\begin{align*}
G^*(R_1, R_2|P_{K_1,K_2}) &= \min \left\{ \log \frac{1}{r_{\max}}, G(R_1, R_2|P_{K_1,K_2}) \right\}.
\end{align*}
\]
- The code that attains the exponent functions \( F(R_1, R_2|P_{X_1,X_2}) \) and \( G^*(R_1, R_2|P_{K_1,K_2}) \) is the universal code that depends only on \( (R_1, R_2) \) not on the value of the distribution \( P_{X_1,X_2} \) and \( P_{K_1,K_2} \).

Define
\[
\bar{D}_{\text{Sys}}^{(\text{in})}(P_{X_1,X_2}, P_{K_1,K_2}) := \{ (R_1, R_2, F(R_1, R_2|P_{X_1,X_2}), G^*(R_1, R_2|P_{K_1,K_2})) : (R_1, R_2) \in R_{sw}(P_{X_1,X_2}) \cap R_{key}(P_{K_1,K_2}) \}.
\]
(22)
From Theorem [1] we immediately obtain the following corollary.

**Corollary 2:**
\[
\bar{R}_{\text{Sys}}^{(\text{in})}(P_{X_1,X_2}, P_{K_1,K_2}) \subseteq \bar{R}_{\text{Sys}}(P_{X_1,X_2}, P_{K_1,K_2}),
\]
\[
\bar{D}_{\text{Sys}}^{(\text{in})}(P_{X_1,X_2}, P_{K_1,K_2}) \subseteq \bar{D}_{\text{Sys}}(P_{X_1,X_2}, P_{K_1,K_2}).
\]

VI. PROOF OF THE MAIN RESULT

In this section we prove Theorems [1] and [2]. To prove this theorem we use the method of types developed by Csiszár and Körner [8]. In the first subsection we prepare basic results on the types. Those results are basic tools for our analysis of several quantities related to error provability of decoding or security. In the second subsection
we evaluate upper bounds of \( p_c(\varphi^{(n)},\psi^{(n)}|P^n_{X_1,X_2}) \) and \( p_c(\varphi^{(n)},\psi_A|P^n_{X_1,X_2},P^n_{K_1,K_2}) \). We derive the upper bound \( p_c(\varphi^{(n)},\psi^{(n)}|P^n_{X_1,X_2}) \) which holds for any \( (\varphi^{(n)},\psi^{(n)}) \). This result is stated in Lemma \([4]\). Furthermore we derive the upper bound of \( p_c(\varphi^{(n)},\psi_A|P^n_{X_1,X_2},P^n_{K_1,K_2}) \) which holds for any \( \varphi^{(n)} \) and any adversary \( A \) with \( \psi_A \). This result is stated in Lemma \([7]\). In the third subsection we develop random coding argument to prove an important key lemma (Lemma \([11]\) stating an existence of good universal code \( (\varphi^{(n)},\psi^{(n)}) \)). In the fourth subsection we prove Theorem \([1]\) using Lemma \([4]\), Lemma \([7]\), and Lemma \([11]\).

A. Types of Sequences and Their Properties

In this subsection we prepare basic results on the types. Those results are basic tools for our analysis of several bounds related to error provability of decoding or security.

**Definition 8:** For any \( n \)-sequence \( x_1 = x_1,1x_1,2 \cdots x_1,n \in \mathcal{X}_1^n \), \( n(x_1|x_1) \) denotes the number of \( t \) such that \( x_1,t = x_1 \). The relative frequency \( \{n(x_1|x_1)/n\}_{x_1 \in \mathcal{X}_1} \) of the components of \( x_1 \) is called the type of \( x_1 \) denoted by \( P_{x_1} \). The set that consists of all the types on \( \mathcal{X}_1 \) is denoted by \( \mathcal{P}_n(\mathcal{X}_1) \). Let \( \mathcal{X}_1 \) denote an arbitrary random variable whose distribution \( P_{\mathcal{X}_1} \) belongs to \( \mathcal{P}_n(\mathcal{X}_1) \). For \( P_{\mathcal{X}_1} \in \mathcal{P}_n(\mathcal{X}_1) \), set

\[
T^n_{\mathcal{X}_1} := \{ x_1 : P_{x_1} = P_{\mathcal{X}_1} \}.
\]

Similarly for any two \( n \)-sequences \( x_1 = x_1,1x_1,2 \cdots x_1,n \in \mathcal{X}_1^n \), \( n(x_1,x_2|x_1,2) \) denotes the number of \( t \) such that \( (x_1,t, x_2,t) = (x_1, x_2) \). The relative frequency \( \{n(x_1,x_2|x_1,2)/n\}_{(x_1,x_2) \in \mathcal{X}_1 \times \mathcal{X}_2} \) of the components of \( (x_1,x_2) \) is called the joint type of \( (x_1,x_2) \) denoted by \( P_{x_1,x_2} \). Furthermore, the set of all the joint types on \( \mathcal{X}_1 \times \mathcal{X}_2 \) is denoted by \( \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2) \). Let \( (\mathcal{X}_1,\mathcal{X}_2) \) denote an arbitrary random pair whose distribution \( P_{\mathcal{X}_1,\mathcal{X}_2} \) belongs to \( \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2) \). For \( P_{\mathcal{X}_1,\mathcal{X}_2} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2) \),

\[
T^n_{\mathcal{X}_1,\mathcal{X}_2} := \{ (x_1,x_2) : P_{x_1,x_2} = P_{\mathcal{X}_1,\mathcal{X}_2} \}.
\]

Furthermore, for \( P_{\mathcal{X}_1} \in \mathcal{P}_n(\mathcal{X}_1) \) and \( x_1 \in T^n_{\mathcal{X}_1} \), set

\[
T^n_{\mathcal{X}_1}(x_1) := \{ x_2 : P_{x_1,x_2} = P_{\mathcal{X}_1,\mathcal{X}_2} \}.
\]

For set of types and joint types the following lemma holds. For the detail of the proof see Csiszár and Körner \([8]\).

**Lemma 2:**

a) \( |\mathcal{P}_n(\mathcal{X}_1)| \leq (n+1)^{||\mathcal{X}_1||} |\mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2)| \leq (n+1)^{||\mathcal{X}_1||||\mathcal{X}_2||} \).

b) For \( P_{\mathcal{X}_1} \in \mathcal{P}_n(\mathcal{X}_1) \) and \( P_{\mathcal{X}_1,\mathcal{X}_2} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2) \),

\[
(n+1)^{-||\mathcal{X}_1||} 2^{nH(\mathcal{X}_1)} \leq |T^n_{\mathcal{X}_1}| \leq 2^{nH(\mathcal{X}_1)},
\]

\[
(n+1)^{-||\mathcal{X}_1||||\mathcal{X}_2||} 2^{nH(\mathcal{X}_1,\mathcal{X}_2)} \leq |T^n_{\mathcal{X}_1,\mathcal{X}_2}| \leq 2^{nH(\mathcal{X}_1,\mathcal{X}_2)}.
\]

c) For any \( x_1 \in T^n_{\mathcal{X}_1} \), we have

\[
|T^n_{\mathcal{X}_1}(x_1)| = \frac{|T^n_{\mathcal{X}_1,\mathcal{X}_2}(x_1)|}{|T^n_{\mathcal{X}_1}|}.
\]

d) For \( x_1 \in T^n_{\mathcal{X}_1} \) and \( (x_1,x_2) \in T^n_{\mathcal{X}_1,\mathcal{X}_2} \),

\[
P^n_{\mathcal{X}_1}(x_1) = 2^{-n[H(\mathcal{X}_1)+D(P_{x_1||P_{\mathcal{X}_1}})]},
\]

\[
P^n_{\mathcal{X}_1,\mathcal{X}_2}(x_1,x_2) = 2^{-n[H(\mathcal{X}_1,\mathcal{X}_2)+D(P_{x_1,\mathcal{X}_2||P_{\mathcal{X}_1,\mathcal{X}_2}})]}.
\]
By Lemma 2 (parts b) and d), we immediately obtain the following lemma:

**Lemma 3:** For $P_{X_1 X_2} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)$,

$$P_{X_1 X_2}^n(T_{X_1 X_2}^n) \leq 2^{-nD(P_{X_1 X_2}||P_{X_1 X_2}).}$$

**B. Upper Bounds of $p_e(\phi(n), \psi(n)|P_{n}^n_{X_1 X_2}$ and $p_e(\phi(n), \psi_A|P_{n}^n_{X_1 X_2}, P_{n}^n_{K_1 K_2})$**

In this subsection we evaluate upper bounds of $p_e(\phi(n), \psi(n)|P_{n}^n_{X_1 X_2}$ and $p_e(\phi(n), \psi_A|P_{n}^n_{X_1 X_2}, P_{n}^n_{K_1 K_2})$. For $p_e(\phi(n), \psi(n)|P_{n}^n_{X_1 X_2}$, we derive an upper bound which can be characterized with a quantity depending on $(\phi(n), \psi(n))$ and joint type $P_{X_1, X_2}$ of sequences $(s_1^n, s_2^n) \in \mathcal{X}_1^n \times \mathcal{X}_2^n$. We first evaluate $p_e(\phi(n), \psi(n)|P_{n}^n_{X_1 X_2})$. For $(x_1, x_2) \in \mathcal{X}_1^n \times \mathcal{X}_2^n$ and $P_{X_1 X_2} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)$ we define the following functions.

$$\Xi_{x_1, x_2}(\phi(n), \psi(n)) := \begin{cases} 1 & \text{if } \psi(n)(\phi_1(n)(x_1), \phi_2(n)(x_2)) \neq (x_1, x_2), \\ 0 & \text{otherwise,} \end{cases}$$

$$\Xi_{X_1, X_2}(\phi(n), \psi(n)) := 1 \frac{1}{|T_{X_1 X_2}^n|} \sum_{(x_1, x_2) \in T_{X_1 X_2}^n} \Xi_{x_1, x_2}(\phi(n), \psi(n)).$$

Then we have the following lemma.

**Lemma 4:** In the proposed system, for any pair of linear encoders $\phi(n) = (\phi_1(n), \phi_2(n))$ and for any joint decoder $\psi(n)$, we have

$$p_e(\phi(n), \psi(n)|P_{n}^n_{X_1 X_2}) \leq \sum_{P_{X_1 X_2} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2)} \Xi_{X_1, X_2}(\phi(n), \psi(n))2^{-nD(P_{X_1 X_2}||P_{X_1 X_2}).}$$

**Proof:** We have the following chain of inequalities:

$$p_e(\phi(n), \psi(n)|P_{n}^n_{X_1 X_2}) = \sum_{P_{X_1 X_2} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2)} \sum_{(x_1, x_2) \in T_{X_1 X_2}^n} \Xi_{x_1, x_2}(\phi(n), \psi(n))P_{X_1 X_2}(x_1, x_2)$$

$$= \sum_{P_{X_1 X_2} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2)} \frac{1}{|T_{X_1 X_2}^n|} \sum_{(x_1, x_2) \in T_{X_1 X_2}^n} \Xi_{x_1, x_2}(\phi(n), \psi(n))|T_{X_1 X_2}^n|P_{X_1 X_2}(x_1, x_2)$$

$$= \sum_{P_{X_1 X_2} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2)} \Xi_{X_1, X_2}(\phi(n), \psi(n))P_{X_1 X_2}(T_{X_1 X_2}^n)$$

$$= \sum_{P_{X_1 X_2} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2)} \Xi_{X_1, X_2}(\phi(n), \psi(n))2^{-nD(P_{X_1 X_2}||P_{X_1 X_2}).}$$

Step (a) follows from the definition of $\Xi_{x_1, x_2}(\phi(n), \psi(n))$. Step (b) follows from that the probabilities $P_{X_1 X_2}(x_1, x_2)$ for $(x_1, x_2) \in T_{X_1 X_2}^n$ take an identical value. Step (c) follows from the definition of $\Xi_{X_1, X_2}(\phi(n), \psi(n))$. Step (d) follows from Lemma 3.

We next discuss upper bounds of

$$\Delta^{(n)}(\phi(n)|P_{X_1 X_2}^n, P_{K_1 K_2}^n) = I(C_1^{m_1}, \tilde{C}_2^{m_2}; X_1, X_2).$$

On an upper bound of $I(C_1^{m_1}, \tilde{C}_2^{m_2}; X_1, X_2)$, we have the following lemma.
Lemma 5:

\[ I(\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2}; X_1, X_2) \leq D(P_{\tilde{K}_1^{m_1} \tilde{K}_2^{m_2}} \| P_{U_1^{m_1} U_2^{m_2}}), \]  

where \( P_{U_1^{m_1} U_2^{m_2}} \) represents the uniform distribution over \( \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2} \).

Proof: We have the following chain of inequalities:

\[
I(\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2}; X_1, X_2) = H(\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2}) - H(\tilde{C}_1^{m_1}, \tilde{C}_2^{m_2}|X_1, X_2) \\
\leq H(\tilde{C}_1^{m_1} \tilde{C}_2^{m_2}) - H(\tilde{K}_1^{m_1} \tilde{K}_1^{m_1}|X_1, X_2) \\
\leq \log(|\mathcal{X}_1^{m_1}|) - H(\tilde{K}_1^{m_1} \tilde{K}_2^{m_2}) = D(P_{K_1^{m_1} K_2^{m_2}} \| P_{U_1^{m_1} U_2^{m_2}}).
\]

Step (a) follows from \( \tilde{C}_i^{m_i} = \tilde{K}_i^{m_i} \oplus \tilde{X}_i^{m_i} \) and \( \tilde{X}_i^{m_i} = \phi_i^{(n)}(X_i) \) for \( i = 1, 2 \). Step (b) follows from \( (\tilde{K}_1^{m_1}, \tilde{K}_2^{m_2}) \perp (X_1, X_2) \).

To evaluate \( D(P_{K_1^{m_1} K_2^{m_2}} \| P_{U_1^{m_1} U_2^{m_2}}) \), we define the following quantities:

\[
\Omega_{k_1, k_2; \pi(n)}(\tilde{k}_1^{m_1}, \tilde{k}_2^{m_2}) := \begin{cases} 1, & \text{if } (\phi_1^{(n)}(k_1), \phi_2^{(n)}(k_2)) = (\tilde{k}_1^{m_1}, \tilde{k}_2^{m_2}), \\ 0, & \text{otherwise.} \end{cases}
\]

\[
\Omega(\tilde{k}_1^{m_1}, \tilde{k}_2^{m_2}) := \sum_{(k_1, k_2) \in T_{\mathcal{X}_1 \mathcal{X}_2}^m} \Omega_{k_1, k_2; \pi(n)}(k_1, k_2)^2.
\]

From the above definition, we can regard \( \Omega(\tilde{k}_1^{m_1}, \tilde{k}_2^{m_2}) \) as a probability distribution on \( \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2} \). We denote this probability distribution by \( \Omega(\pi_1, \pi_2; \pi(n)) \). By the definition of \( \Omega(\pi_1, \pi_2; \pi(n)) \), for \( \pi_1, \pi_2 \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2) \), we have the following:

\[
P_{\tilde{K}_1^{m_1} \tilde{K}_2^{m_2}}(\tilde{k}_1^{m_1}, \tilde{k}_2^{m_2}) = \sum_{\pi_1, \pi_2 \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)} \Omega(\tilde{k}_1^{m_1}, \tilde{k}_2^{m_2}) P_{\pi_1 \pi_2}^n(T_{\pi_1 \pi_2}^m)
\]

for \( (\tilde{k}_1^{m_1}, \tilde{k}_2^{m_2}) \in \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2} \).

Furthermore, we define

\[
\Delta(\tilde{K}_1 \tilde{K}_2; \pi^{(n)}):= \sum_{(\tilde{k}_1^{m_1}, \tilde{k}_2^{m_2}) \in \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2}} |X_1|^{m_1} |X_2|^{m_2} \left( \Omega(\tilde{k}_1^{m_1}, \tilde{k}_2^{m_2}) - \frac{1}{|X_1|^{m_1} |X_2|^{m_2}} \right)^2.
\]

Then we have the following lemma.

Lemma 6:

\[
D(P_{\tilde{K}_1^{m_1} \tilde{K}_2^{m_2}} \| P_{U_1^{m_1} U_2^{m_2}}) = D(P_{\phi^{(n)}(K_1) \phi^{(n)}(K_2)} \| P_{U_1^{m_1} U_2^{m_2}}) \\
\leq (\log e) \log(|X_1|) |X_2| n \cdot \sum_{\pi_1, \pi_2 \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)} \Delta(\tilde{K}_1 \tilde{K}_2; \pi^{(n)}))^{-n} D(P_{\pi_1 \pi_2} || P_{\pi_1 \pi_2}),
\]

where \( \Delta(\tilde{K}_1 \tilde{K}_2; \pi^{(n)}) := \min\{1, \Delta(\tilde{K}_1 \tilde{K}_2; \pi^{(n)})\} \).
Proof: By (25) and the convexity of divergence we have
\[ D(P_{\tau_1^{m_1}, \tau_2^{m_2}} \| P_1^{m_1} U_2^{m_2}) \leq \sum_{(\tau_1^{m_1}, \tau_2^{m_2}) \in \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2}} \sum_{\pi_1, \pi_2 \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} P_{\tau_1^{m_1}, \tau_2^{m_2}}^n(T_{\pi_1, \pi_2}^n) \]
\[ \times \Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \log \left( \frac{|X_1^{m_1}| |X_2^{m_2}| \Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2})}{K_1^{m_1}, K_2^{m_2}} \right) \]
\[ = \sum_{\pi_1, \pi_2 \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2)} P_{\pi_1, \pi_2}^n(T_{\pi_1, \pi_2}^n) D(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \| P_1^{m_1} U_2^{m_2}) \]
\[ \leq (a) \sum_{\pi_1, \pi_2 \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2)} D(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \| P_1^{m_1} U_2^{m_2}) 2^{-nD(P_{\pi_1, \pi_2}^n \| P_{\pi_1, \pi_2}^n)} . \quad (27) \]
Step (a) follows from Lemma 5. Hence, it suffices to derive an upper bound of \( D(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \| P_1^{m_1} U_2^{m_2}) \). Since
\[ D(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \| P_1^{m_1} U_2^{m_2}) = m_1 \log |X_1| + m_2 \log |X_2| - H(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2})) \leq \log(|X_1| |X_2|) n, \]
\( D(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \| P_1^{m_1} U_2^{m_2}) \) has the obvious upper bound \( \log(|X_1| |X_2|) n. \) Note that this quantity is larger than \( n. \)

We next derive another upper bound of \( D(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \| P_1^{m_1} U_2^{m_2}) \). Using that the inequality
\[ u \log(u/v) \leq (\log e) \left\{ u - v + (u - v)^2/v \right\} \]
holds for any positive number \( u, v, \) we obtain
\[ D(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \| P_1^{m_1} U_2^{m_2}) \leq (\log e) \sum_{(\tau_1^{m_1}, \tau_2^{m_2}) \in \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2}} |X_1^{m_1}| |X_2^{m_2}| \]
\[ \times \left( \Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) - \frac{1}{|X_1^{m_1}| |X_2^{m_2}|} \right)^2 = (\log e) \Delta_{K_1, K_2}(\phi(n)) . \quad (28) \]
From (28) and the upper bound \( \log(|X_1| |X_2|) n \) of \( D(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \| P_1^{m_1} U_2^{m_2}) \) larger than \( n, \) we have
\[ D(\Omega_{\tau_1^{m_1}, \tau_2^{m_2}}(\tau_1^{m_1}, \tau_2^{m_2}) \| P_1^{m_1} U_2^{m_2}) \leq (\log e) \left[ \log(|X_1| |X_2|) n \min \{ 1, \Delta_{K_1, K_2}(\phi(n)) \} \right] \]
\[ = (\log e) \log(|X_1| |X_2|) n \Delta_{K_1, K_2}(\phi(n)) . \quad (29) \]
Combining (27) and (29), we have the bound (26) of Lemma 6.

Combining Lemma 5 and Lemma 6, we have the following lemma.

Lemma 7: In the proposed system, for any pair of encoder \( \phi(n) = (\phi_1^{(n)}, \phi_2^{(n)}) \), for any eavesdropper \( \mathcal{A} \) with estimator function \( \psi_A \), we have
\[ \Delta^{(n)}(\phi(n)|P_{X_1 X_2}, P_{K_1 K_2}) \]
\[ \leq (\log e) \left[ \log(|X_1| |X_2|) n \min \{ 1, \Delta_{K_1, K_2}(\phi(n)) \} \right] 2^{-nD(P_{\pi_1 \pi_2}^n \| P_{\pi_1 \pi_2}^n)} . \quad (30) \]

The bound (23) in Lemma 4 implies that upper bounds of \( \Xi_{X_1 X_2}(\phi(n), \psi(n)) \) for \( P_{X_1 X_2} \) lead to derivations of good error bounds on \( p_e(\phi(n), \psi(n)) |P_{X_1 X_2} \). Furthermore, the bound (30) in Lemma 7 implies that good upper bounds of \( \Delta_{K_1, K_2}(\phi(n)) \) for \( P_{K_1 K_2} \) lead to derivations of good secure upper bounds on \( p_e(\phi(n), \psi_A)|P_{X_1 X_2}, P_{K_1 K_2} \). In the next subsection we discuss an existence of universal code
\[ (\phi(n), \psi(n)) = (\phi_1^{(n)}, \phi_2^{(n)}, \psi(n)) = (\phi(n) \oplus a_1^{m_1}, \phi_2^{(n)} \oplus a_2^{m_2}, \psi(n)) \]
such that the quantities \( \Xi_{X_1 X_2}(\phi(n), \psi(n)) \) for \( P_{X_1 X_2} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2) \) and \( \Delta_{K_1, K_2}(\phi(n)) \) for \( P_{K_1 K_2} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2) \) attain the bound of Theorem 7.
C. Random Coding Arguments

We construct a pair of affine encoders $\varphi^{(n)} = (\varphi_{1}^{(n)}, \varphi_{2}^{(n)})$ using the random coding method. For the joint decoder $\psi^{(n)}$, we propose the minimum entropy decoder used in Csiszár [3] and Oohama and Han [9].

**Random Construction of Affine Encoders:** We first choose $m_{i}, i = 1, 2$ such that

$$m_{i} := \left\lfloor \frac{nR_{i}}{\log |X_{i}|} \right\rfloor,$$

where $\lfloor a \rfloor$ stands for the integer part of $a$. It is obvious that for $i = 1, 2$, we have

$$R_{i} - \frac{1}{n} \leq \frac{m_{i}}{n} \log |X_{i}| \leq R_{i}.$$  

By the definition (4) of $\phi_{i}^{(n)}, i = 1, 2$, we have that for each $i = 1, 2$ and for $x_{i} \in X_{i}^{n}$,

$$\phi_{i}^{(n)}(x_{i}) = x_{i}A_{i},$$

where $A_{i}$ is a matrix with $n$ rows and $m_{i}$ columns. By the definition (5) of $\varphi_{i}^{(n)}, i = 1, 2$, we have that for each $i = 1, 2$ and for $k_{i} \in X_{i}^{n}$,

$$\varphi_{i}^{(n)}(k_{i}) = k_{i}A_{i} + b_{i}^{m_{i}},$$

where $b_{i}^{m_{i}}$ is a vector with $m_{i}$ columns. For each $i = 1, 2$, entries of $A_{i}$ and $b_{i}^{m_{i}}$ are from the field of $X_{i}$. Those entries are selected at random, independently of each other and with uniform distribution. Randomly constructed linear encoders $\phi_{i}^{(n)}, i = 1, 2$ and affine encoders and $\varphi_{i}^{(n)}, i = 1, 2$ have three properties shown in the following lemma.

**Lemma 8 (Properties of Linear/Affine Encoders):**

a) For each $i = 1, 2$, and for any $x_{i}, v_{i} \in X_{i}^{n}$ with $x_{i} \neq v_{i}$, we have

$$\Pr[\phi_{i}^{(n)}(x_{i}) = \phi_{i}^{(n)}(v_{i})] = \Pr[(y_{i} \oplus w_{i})A_{i} = 0^{m_{i}}] = |X_{i}|^{-m_{i}}. \tag{31}$$

b) For each $i = 1, 2$, for any $s_{i} \in X_{i}^{n}$, and for any $\tilde{s}_{i}^{m_{i}} \in X_{i}^{m_{i}}$, we have

$$\Pr[\varphi_{i}^{(n)}(s_{i}) = \tilde{s}_{i}^{m_{i}}] = \Pr[s_{i}A_{i} \oplus b_{i}^{m_{i}} = \tilde{s}_{i}^{m_{i}}] = |X_{i}|^{-m_{i}}. \tag{32}$$

c) For each $i = 1, 2$, for any $s_{i}, t_{i} \in X_{i}^{n}$ with $s_{i} \neq t_{i}$, and for any $\tilde{s}_{i}^{m_{i}} \in X_{i}^{m_{i}}$, we have

$$\Pr[\varphi_{i}^{(n)}(s_{i}) = \varphi_{i}^{(n)}(t_{i}) = \tilde{s}_{i}^{m_{i}}] = \Pr[s_{i}A_{i} \oplus b_{i}^{m_{i}} = t_{i}A_{i} \oplus b_{i}^{m_{i}} = \tilde{s}_{i}^{m_{i}}] = |X_{i}|^{-2m_{i}}. \tag{33}$$

Proof of this lemma is given in Appendix A. We next define the joint decoder function $\psi^{(n)} : X_{1}^{m_{1}} \times X_{2}^{m_{2}} \rightarrow X_{1}^{n} \times X_{2}^{n}$. To this end we define the following quantities.

**Definition 9:** For $(x_{1}, x_{2}) \in X_{1}^{n} \times X_{2}^{n}$, we denote the conditional entropy and entropy calculated from the joint type $P_{x_{1}, x_{2}}$ by $H(x_{1} | x_{2})$ and $H(x_{1}, x_{2})$, respectively. In other words, for a joint type $P_{X_{1}, X_{2}} \in P_{n}(X_{1} \times X_{2})$ such that $P_{X_{1}, X_{2}} = P_{x_{1}, x_{2}}$, we define $H(x_{1} | x_{2}) = H(X_{1} | X_{2})$ and $H(x_{1}, x_{2}) = H(X_{1}, X_{2})$. 

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Minimum Entropy Decoder: For $\phi_i(n) (x_i^n) = \bar{y}_i^{m_i}, i = 1, 2$, we define the joint decoder function $\psi(n) : \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2} \to \mathcal{X}_1^n \times \mathcal{X}_2^n$ as follows:

$$
\psi(n) (\bar{y}_1^{m_1}, \bar{y}_2^{m_2}) :=
\begin{cases}
(\bar{z}_1, \bar{z}_2) & \text{if } \phi_1(n) (\bar{z}_1) = \bar{z}_1^{m_1}, \phi_2(n) (\bar{z}_2) = \bar{z}_2^{m_2}, \\
& \text{and } H(\bar{z}_1, \bar{z}_2) < H(\bar{z}_1, \bar{z}_2) \\
\text{for all } (\bar{z}_1, \bar{z}_2) & \text{that satisfy } \phi_1(n) (\bar{z}_1) = \bar{z}_1^{m_1}, \phi_2(n) (\bar{z}_2) = \bar{z}_2^{m_2}, \\
& \text{and } (\bar{z}_1, \bar{z}_2) \neq (\bar{z}_1, \bar{z}_2), \\
\text{arbitrary} & \text{if there is no such } (\bar{z}_1, \bar{z}_2) \in \mathcal{X}_1^n \times \mathcal{X}_2^n.
\end{cases}
$$

Error Probability Bound: In the following arguments we let expectations and variances based on the randomness of the encoder functions be denoted by $\mathbb{E} [\cdot]$ and $\text{Var} [\cdot]$, respectively. Define

$$
\Psi_{\mathcal{X}_1, \mathcal{X}_2} (R_1, R_2) := 2^{-n [R_1 - H(\mathcal{X}_1 | \mathcal{X}_2)] +} + 2^{-n [R_2 - H(\mathcal{X}_2 | \mathcal{X}_1)] +} + 2^{-n [R_1 + R_2 - H(\mathcal{X}_1, \mathcal{X}_2)]}.
$$

Then we have the following lemma.

**Lemma 9:** For any $n$ and for any $P_{\mathcal{X}_1, \mathcal{X}_2} \in \mathcal{P}_n (\mathcal{X}_1 \times \mathcal{X}_2)$,

$$
\mathbb{E} \left[ \Xi_{\mathcal{X}_1, \mathcal{X}_2} (\phi(n), \psi(n)) \right] \leq 4(n + 1)^{|\mathcal{X}_1||\mathcal{X}_2|} \Psi_{\mathcal{X}_1, \mathcal{X}_2} (R_1, R_2).
$$

Proof of this lemma is given in Appendix B.

**Estimation of Approximation Error:** Define

$$
\Theta_{\mathcal{X}_1, \mathcal{X}_2} (R_1, R_2) := 2^{-n [H(\mathcal{X}_1) - R_1]} + 2^{-n [H(\mathcal{X}_2) - R_2]} + 2^{-n [H(\mathcal{X}_1, \mathcal{X}_2) - (R_1 + R_2)]}.
$$

Then we have the following lemma.

**Lemma 10:** For any $\bar{y}_i^{m_i} \in \mathcal{X}_i^{m_i}, i = 1, 2$

$$
\mathbb{E} \left[ \Omega_{\mathcal{X}_1, \mathcal{X}_2} (\phi(n), \psi(n)) (\bar{y}_1^{m_1}, \bar{y}_2^{m_2}) \right] = \frac{1}{|\mathcal{X}_1|^{m_1} |\mathcal{X}_2|^{m_2}}, \tag{34}
$$

$$
\text{Var} \left[ \Omega_{\mathcal{X}_1, \mathcal{X}_2} (\phi(n), \psi(n)) (\bar{y}_1^{m_1}, \bar{y}_2^{m_2}) \right] \leq \frac{(n + 1)^{|\mathcal{X}_1||\mathcal{X}_2|}}{|\mathcal{X}_1| |\mathcal{X}_2|^2} \cdot \Theta_{\mathcal{X}_1, \mathcal{X}_2} (R_1, R_2). \tag{35}
$$

Proof of this lemma is given in Appendix C.

**Corollary 3:** We have the following corollary from Lemma 10.

**Existence of Good Universal Code ($\phi(n), \psi(n)$):** From Lemma 9 and Corollary 3 we have the following lemma stating an existence of good universal code ($\phi(n), \psi(n)$).

**Lemma 11:** There exists at least one deterministic code ($\phi(n), \psi(n)$) satisfying $(m_i/n) \log |\mathcal{X}_i| \leq R_i, i = 1, 2$, such that for any $P_{\mathcal{X}_1, \mathcal{X}_2}, P_{\mathcal{X}_1, \mathcal{X}_2} \in \mathcal{P}_n (\mathcal{X}_1 \times \mathcal{X}_2)$,

$$
\Xi_{\mathcal{X}_1, \mathcal{X}_2} (\phi(n), \psi(n)) \leq 8(n + 1)^{2|\mathcal{X}_1||\mathcal{X}_2|} \Psi_{\mathcal{X}_1, \mathcal{X}_2} (R_1, R_2),
$$

$$
\Delta_{\mathcal{X}_1, \mathcal{X}_2} (\phi(n)) \leq 2(n + 1)^{2|\mathcal{X}_1||\mathcal{X}_2|} \Theta_{\mathcal{X}_1, \mathcal{X}_2} (R_1, R_2).
$$
Proof: We have the following chain of inequalities:

\[
\mathbb{E} \left[ \sum_{P_{X_1, X_2} \in \mathcal{P}_n(X_1 \times X_2)} \left( 4(n+1)^{\|X_1\|,X_2}(\Psi_{X_1,X_2}(R_1,R_2)) \right)^{-1} \Xi_{X_1,X_2}(\phi(n),\psi(n)) \right] \\
+ \sum_{P_{X_1, X_2} \in \mathcal{P}_n(X_1 \times X_2)} \left( (n+1)^{\|X_1\|,X_2}(\Theta_{K_1,K_2}(R_1,R_2)) \right)^{-1} \Delta_{K_1,K_2}(\varphi(n))
\]

\[
= \sum_{P_{X_1, X_2} \in \mathcal{P}_n(X_1 \times X_2)} \left( 4(n+1)^{\|X_1\|,X_2}(\Psi_{X_1,X_2}(R_1,R_2)) \right)^{-1} \mathbb{E} \left[ \Xi_{X_1,X_2}(\phi(n),\psi(n)) \right] \\
+ \sum_{P_{X_1, X_2} \in \mathcal{P}_n(X_1 \times X_2)} \left( (n+1)^{\|X_1\|,X_2}(\Theta_{K_1,K_2}(R_1,R_2)) \right)^{-1} \mathbb{E} \left[ \Delta_{K_1,K_2}(\varphi(n)) \right]
\]

\[
\leq (a) \sum_{P_{X_1, X_2} \in \mathcal{P}_n(X_1 \times X_2)} 1 + \sum_{P_{X_1, X_2} \in \mathcal{P}_n(X_1 \times X_2)} 1 \leq 2|\mathcal{P}_n(X_1 \times X_2)| \leq 2(n+1)^{\|X_1\|,X_2}.
\]

Step (a) follows from Lemma [9] and Corollary [3]. Step (b) follows from Lemma [2] part a). Hence there exists at least one deterministic code \((\varphi(n),\psi(n))\) such that

\[
\sum_{P_{X_1, X_2} \in \mathcal{P}_n(X_1 \times X_2)} \left( 4(n+1)^{\|X_1\|,X_2}(\Psi_{X_1,X_2}(R_1,R_2)) \right)^{-1} \Xi_{X_1,X_2}(\phi(n),\psi(n)) \\
+ \sum_{P_{X_1, X_2} \in \mathcal{P}_n(X_1 \times X_2)} \left( (n+1)^{\|X_1\|,X_2}(\Theta_{K_1,K_2}(R_1,R_2)) \right)^{-1} \Delta_{K_1,K_2}(\varphi(n)) \leq 2(n+1)^{\|X_1\|,X_2},
\]

from which we have that

\[
\left( 4(n+1)^{\|X_1\|,X_2}(\Psi_{X_1,X_2}(R_1,R_2)) \right)^{-1} \Xi_{X_1,X_2}(\phi(n),\psi(n)) \leq 2(n+1)^{\|X_1\|,X_2},
\]

\[
\left( (n+1)^{\|X_1\|,X_2}(\Theta_{K_1,K_2}(R_1,R_2)) \right)^{-1} \Delta_{K_1,K_2}(\varphi(n)) \leq 2(n+1)^{\|X_1\|,X_2}
\]

for any \(P_{X_1,X_2}, P_{K_1,K_2} \in \mathcal{P}_n(X_1 \times X_2).\) Thus we have

\[
\Xi_{X_1,X_2}(\phi(n),\psi(n)) \leq 8(n+1)^{2\|X_1\|,X_2}(\Psi_{X_1,X_2}(R_1,R_2)),
\]

\[
\Delta_{K_1,K_2}(\varphi(n)) \leq 2(n+1)^{2\|X_1\|,X_2}(\Theta_{K_1,K_2}(R_1,R_2))
\]

for any \(P_{X_1,X_2}, P_{K_1,K_2} \in \mathcal{P}_n(X_1 \times X_2).\) \(\blacksquare\)

D. Proof of Theorem 7

In this subsection we prove Theorem 7 using Lemma 4, Lemma 7, and Lemma 11.

Proof of Theorem 7: By Lemma 11, there exists \((\varphi(n),\psi(n))\) satisfying \((m_i/n)\log|X_i| \leq R_i, i = 1,2,\) such that for any \(P_{X_1,X_2}, P_{K_1,K_2} \in \mathcal{P}_n(X_1 \times X_2),\)

\[
\Xi_{X_1,X_2}(\phi(n),\psi(n)) \leq 8(n+1)^{2\|X_1\|,X_2}(\Psi_{X_1,X_2}(R_1,R_2)),
\]

\[
\Delta_{K_1,K_2}(\varphi(n)) \leq 2(n+1)^{\|X_1\|,X_2}(\Theta_{K_1,K_2}(R_1,R_2)).
\]
We first prove (9) in Theorem 1. On an upper bound of $p_e(\phi(n), \psi(n)|P^n_{X_1, X_2})$, we have the following chain of inequalities:

\[
p_e(\phi(n), \psi(n)|P^n_{X_1, X_2}) \leq (8(n+1)^2|X_1||X_2| \sum_{P_{X_1, X_2} \in P_n(X_1 \times X_2)} \Psi_{X_1, X_2}(R_1, R_2) 2^{-nD(P_{X_1, X_2}|P_{X_1 X_2})}
\]

\[
\leq 24(n+1)^2|X_1||X_2||P_n(X_1 \times X_2)|2^{-n[\min_{\text{a, b, c}} F_i(R_i|P_{X_1, X_2})]}
\]

\[
\leq 24(n+1)^2|X_1||X_2|2^{-nF(R_1, R_2|P_{X_1, X_2})} = 2^{-n[F(R_1, R_2|P_{X_1, X_2})-\delta_1, n]}.
\]

Step (a) follows from Lemma 7 and (37). Step (b) follows from that for any $P_{X_1, X_2} \in P_n(X_1 \times X_2)$,

\[
\Psi_{X_1, X_2}(R_1, R_2) 2^{-nD(P_{X_1, X_2}|P_{X_1 X_2})} = \left\{ 2^{-n[R_1-H(X_1|X_2)]^+} + 2^{-n[R_2-H(X_2|X_1)]^+} + 2^{-n[R_1+R_2-H(X_1, X_2)]^+} \right\} 2^{-nD(P_{X_1, X_2}|P_{X_1 X_2})}
\]

\[
\leq 3 \cdot 2^{-n[\min_{\text{a, b, c}} F_i(R_i|P_{X_1, X_2})]}.
\]

Step (c) follows from Lemma 2 part a). We next prove (10) in Theorem 1. On an upper bound of $\Delta(n)(\phi(n)|P^n_{X_1, X_2}, P^n_{K_1, K_2})$ we have the following chain of inequalities:

\[
\Delta(n)(\phi(n)|P^n_{X_1, X_2}, P^n_{K_1, K_2}) \leq 2(\log e)[\log(|X_1||X_2|)](n+1)^2|X_1||X_2| \sum_{P_{X_1, X_2} \in P_n(X_1 \times X_2)} \min \left\{ 1, \Theta_{X_1, X_2}(R_1, R_2) \right\} 2^{-nD(P_{X_1, X_2}|P_{X_1 X_2})}
\]

\[
\leq 6(\log e)[\log(|X_1||X_2|)](n+1)^2|X_1||X_2||P_n(X_1 \times X_2)|2^{-n[\min\{G_1(R_1|P_{K_1}), G_2(R_2|P_{K_2}), G_3(R_3|P_{K_1, K_2})\}]} \leq 6(\log e)[\log(|X_1||X_2|)](n+1)^2|X_1||X_2|2^{-nG(R_1, R_2|P_{K_1, K_2})} = 2^{-n[G(R_1, R_2|P_{K_1, K_2})-\delta_2, n]}.
\]

Step (a) follows from Lemma 7 and (37). Step (b) follows from that for any $P_{X_1, X_2} \in P_n(X_1 \times X_2)$, we have

\[
\min \left\{ 1, \Theta_{X_1, X_2}(R_1, R_2) \right\} 2^{-nD(P_{X_1, X_2}|P_{K_1 K_2})}
\]

\[
= \min \left\{ 1, 2^{-n[H(X_1|K_1)-R_1]} + 2^{-n[H(X_2|K_2)-R_2]} + 2^{-n[H(X_1, X_2)-(R_1+R_2)]} \right\} 2^{-nD(P_{X_1, X_2}|P_{K_1 K_2})}
\]

\[
\leq \left\{ 2^{-n[H(X_1|K_1)-R_1]} + 2^{-n[H(X_2|K_2)-R_2]} + 2^{-n[H(X_1, X_2)-(R_1+R_2)]} \right\} 2^{-nD(P_{X_1, X_2}|P_{K_1 K_2})}
\]

\[
\leq 3 \cdot 2^{-n[\min\{G_1(R_1|P_{K_1}), G_2(R_2|P_{K_2}), G_3(R_3|P_{K_1, K_2})\}]}.
\]

Step (c) follows from Lemma 2 part a).

**APPENDIX**

A. *Proof of Lemma 8*

In this appendix we prove Lemma 8. The suffix $i$ in $X_i$ used to distinguish $X_1$ and $X_2$ in Lemma 8 is not essential for the proof. In the following argument we omit this suffix. Let $X$ be a finite field and let $\Lambda$ be an $n \times n$ invertible matrix, whose entries are from $X$. Let $\phi : X^n \to X^m$ be a linear map with $\phi(x\Lambda) = x\Lambda A$ for $x \in X^n$. Here $A$ is a matrix with $n$ rows and $m$ columns. Let $\varphi : X^n \to X^m$ be an affine map with $\varphi(s) = sA \oplus b^m$ for $s \in X^n$. Here $b^m$ is a vector with $m$ colomus. Entries of $A$ and $b^m$ are from the field of $X$. Those entries
are selected at random, independently of each other and with uniform distribution. In this appendix we prove the following lemma.

**Lemma 12:**

a) For any $x, v \in \mathcal{X}^n$ with $x \neq v$, we have

$$
\Pr[\phi(x) = \phi(v)] = \Pr[(x \oplus v)A = 0^m] = |\mathcal{X}|^{-m}.
$$

(b) For any $s \in \mathcal{X}^n$, and for any $\tilde{s}^m \in \mathcal{X}^m$, we have

$$
\Pr[\phi(s) = \tilde{s}^m] = \Pr[sA \oplus b^m = \tilde{s}^m] = |\mathcal{X}|^{-m}.
$$

(c) For any $s, t \in \mathcal{X}^n$ with $s \neq t$, and for any $\tilde{s}^m \in \mathcal{X}^m$, we have

$$
\Pr[\phi(s) = \phi(t) = \tilde{s}^m] = \Pr[sA \oplus b^m = tA \oplus b^m = \tilde{s}^m] = |\mathcal{X}|^{-2m}.
$$

**Proof:** Let $a_i^m$ be the $l$-th low vector of the matrix $A$. For each $l = 1, 2, \cdots, n$, let $A_i^m \in \mathcal{X}^m$ be a random vector which represents the randomness of the choice of $a_i^m \in \mathcal{X}^m$. Let $B^m \in \mathcal{X}^m$ be a random vector which represent the randomness of the choice of $b^m \in \mathcal{X}^m$. We first prove the part a). Since $\Lambda$ is invertible, we have

$$
x \neq v \Leftrightarrow x_i \neq v_i \text{ for some } i \in \{1, 2, \cdots, n\}.
$$

Without loss of generality we may assume $x_1 \neq v_1$. Under this assumption we have the following:

$$
(x \oplus v)A = 0^m \Leftrightarrow \sum_{l=1}^{n} (x_l \oplus v_l) a_l^m = 0^m \Leftrightarrow a_1^m = \sum_{l=2}^{n} \frac{y_l \oplus x_l}{x_1 \oplus v_1} a_l^m.
$$

(41)

Computing $\Pr[\phi(x) = \phi(v)]$, we have the following chain of equalities:

$$
\Pr[\phi(x) = \phi(v)] = \Pr[(x \oplus v)A = 0^m] \stackrel{(a)}{=} \Pr\left[a_1^m = \sum_{l=2}^{n} \frac{y_l \oplus x_l}{x_1 \oplus v_1} a_l^m \right]
\stackrel{(b)}{=} \sum_{\{a_i^m\}_{l=2}^{n} \in \mathcal{X}^{(n-1)m}} \prod_{l=2}^{n} P_{A_l^m}(a_l^m) P_{A_1^m}\left(\sum_{l=2}^{n} \frac{y_l \oplus x_l}{x_1 \oplus v_1} a_l^m \right) = |\mathcal{X}|^{-m} \sum_{\{a_i^m\}_{l=2}^{n} \in \mathcal{X}^{(n-1)m}} \prod_{l=2}^{n} P_{A_l^m}(a_l^m) = |\mathcal{X}|^{-m}.
$$

Step (a) follows from (41). Step (b) follows from that $n$ random vectors $A_i^m, i = 1, 2, \cdots, n$ are independent. We next prove the part b). We have the following:

$$
sA \oplus b^m = \tilde{s}^m \Leftrightarrow b^m = \tilde{s}^m \oplus \left\{ \sum_{l=1}^{n} s_l a_l^m \right\}.
$$

(42)

Computing $\Pr[sA \oplus b^m = \tilde{s}^m]$, we have the following chain of equalities:

$$
\Pr[sA \oplus b^m = \tilde{s}^m] \stackrel{(a)}{=} \Pr\left[b^m = \tilde{s}^m \oplus \left\{ \sum_{l=1}^{n} s_l a_l^m \right\} \right] \stackrel{(b)}{=} \sum_{\{a_i^m\}_{l=1}^{n} \in \mathcal{X}^m} \prod_{l=1}^{n} P_{A_l^m}(a_l^m) P_{B^m}\left(\tilde{s}^m \oplus \left\{ \sum_{l=1}^{n} s_l a_l^m \right\} \right)
= |\mathcal{X}|^{-m} \sum_{\{a_i^m\}_{l=1}^{n} \in \mathcal{X}^m} \prod_{l=1}^{n} P_{A_l^m}(a_l^m) = |\mathcal{X}|^{-m}.
$$
Step (a) follows from (42). Step (b) follows from that \( n \) random vectors \( A_i^m, i = 1, 2, \cdots, n \) and \( B^m \) are independent. We finally prove the part c). We first observe that \( s \neq t \Leftrightarrow \) is equivalent to \( s_i \neq t_i \) for some \( i \in \{1, 2, \cdots, n\} \).
Without loss of generality, we may assume that \( s_1 \neq t_1 \). Under this assumption we have the following:

\[
\begin{align*}
\mathbf{sA} \oplus b^m &= \mathbf{tA} \oplus b^m = \mathbf{s}^m &\Leftrightarrow (\mathbf{s} \oplus \mathbf{t})A = 0, b^m = \mathbf{s}^m \oplus \left\{ \sum_{l=1}^{n} s_l a_l^m \right\} \\
\Leftrightarrow a_1^m &= \sum_{l=2}^{n} \frac{t_l \oplus s_l}{s_1 \oplus t_1} a_l^m, b^m = \mathbf{s}^m \oplus \left\{ \sum_{l=1}^{n} s_l a_l^m \right\} \\
\Leftrightarrow a_1^m &= \sum_{l=2}^{n} \frac{t_l \oplus s_l}{s_1 \oplus t_1} a_l^m, b^m = \mathbf{s}^m \oplus \left\{ \sum_{l=2}^{n} \frac{t_1 s_l \oplus s_1 t_l}{s_1 \oplus t_1} a_l^m \right\}. & (43)
\end{align*}
\]

Computing \( \Pr[\mathbf{sA} \oplus b^m = \mathbf{tA} \oplus b^m = \mathbf{s}^m] \), we have the following chain of equalities:

\[
\begin{align*}
\Pr[\mathbf{sA} \oplus b^m = \mathbf{tA} \oplus b^m = \mathbf{s}^m] &= \sum_{\{a_i^m\}_{i=2}^{n} \in \mathcal{X}^{(n-1)m}} \prod_{i=2}^{n} P_{A_i^m}(a_i^m) \\
&= \prod_{i=2}^{n} P_{A_i^m}(a_i^m) \\
&= |\mathcal{X}|^{-2m} \sum_{\{a_i^m\}_{i=2}^{n} \in \mathcal{X}^{(n-1)m}} \prod_{i=2}^{n} P_{A_i^m}(a_i^m) = |\mathcal{X}|^{-2m}.
\end{align*}
\]

Step (a) follows from (43). Step (b) follows from the independent property on \( A_i^m, i = 1, 2, \cdots, n \) and \( B^m \).

\[\text{B. Proof of Lemma} \ 9\]

For simplicity of notation, we write \( M_i = |\mathcal{X}_i|^m, i = 1, 2 \). We also use those notations in the arguments of other appendixes.

\[\text{Proof of Lemma} \ 9\] For \( x_1 \in \mathcal{X}_1^m, x_2 \in \mathcal{X}_2^m \) we set

\[
\begin{align*}
B(x_1, x_2) &= \left\{ (x_1, x_2) : H(\bar{x}_1, \bar{x}_2) \leq H(x_1, x_2), P_{\bar{x}_1} = P_{\bar{x}_2} = P_{x_2} \right\}, \\
B(x_1 | x_2) &= \left\{ \bar{x}_1 : H(\bar{x}_1, x_2) \leq H(x_1, x_2), P_{\bar{x}_1} = P_{x_1} \right\}, \\
B(x_2 | x_1) &= \left\{ \bar{x}_2 : H(\bar{x}_2, x_1) \leq H(x_2, x_1), P_{\bar{x}_2} = P_{x_2} \right\}.
\end{align*}
\]

Using parts a) and b) of Lemma 2 we have following inequalities:

\[
\begin{align*}
|B(x_1, x_2)| &\leq (n + 1)^{|\mathcal{X}_1||\mathcal{X}_2|} 2^{nH(x_1, x_2)}, & (44) \\
|B(x_1 | x_2)| &\leq (n + 1)^{|\mathcal{X}_1||\mathcal{X}_2|} 2^{nH(x_1 | x_2)}, & (45) \\
|B(x_2 | x_1)| &\leq (n + 1)^{|\mathcal{X}_1||\mathcal{X}_2|} 2^{nH(x_2 | x_1)}, & (46)
\end{align*}
\]
On an upper bound of $\mathbb{E}[^{\xi_{x_1,x_2}(\phi^{(n)}, \psi^{(n)})}]$, we have the following chain of inequalities:

$$\mathbb{E}[^{\xi_{x_1,x_2}(\phi^{(n)}, \psi^{(n)})}]$$

$$\leq \sum_{\bar{x}_1 \in B(x_1|x_2), \bar{x}_1 \neq x_1} \Pr\{\phi_1^{(n)}(\bar{x}_1) = \phi_1^{(n)}(x_1)\} + \sum_{\bar{x}_2 \in B(x_2|x_1), \bar{x}_2 \neq x_2} \Pr\{\phi_2^{(n)}(\bar{x}_2) = \phi_2^{(n)}(x_2)\}$$

$$+ \sum_{(\bar{x}_1, \bar{x}_2) \in B(x_1,x_2), \bar{x}_1 \neq x_1, \bar{x}_2 \neq x_2} \Pr\{\phi_1^{(n)}(\bar{x}_1) = \phi_1^{(n)}(x_1), \phi_2^{(n)}(\bar{x}_2) = \phi_2^{(n)}(x_2)\}$$

$$\leq \sum_{\bar{x}_1 \in B(x_1|x_2)} \frac{1}{M_1} + \sum_{\bar{x}_2 \in B(x_2|x_1)} \frac{1}{M_2} + \sum_{(\bar{x}_1, \bar{x}_2) \in B(x_1,x_2)} \frac{1}{M_1 M_2}$$

$$= \frac{|B(x_1|x_2)|}{M_1} + \frac{|B(x_2|x_1)|}{M_2} + \frac{|B(x_1,x_2)|}{M_1 M_2}$$

$$\leq 4(n + 1)^{X_1||X_2|} \left\{2^{-n|R_1-H(X_1|x_2)|} + 2^{-n|R_2-H(x_2|x_1)|} + 2^{-n|R_1+R_2-H(x_1,x_2)|}\right\}.$$ 

Step (a) follows from Lemma 8 part a) and independent random constructions of linear encoders $\phi^{(n)}_1$ and $\phi^{(n)}_2$. Step (b) follows from (44), (45), (46), and $M_i \geq 2^n R_i - 1, i = 1, 2$. On the other hand we have the obvious bound

$$\mathbb{E}[^{\xi_{x_1,x_2}(\phi^{(n)}, \psi^{(n)})}] \leq 1.$$ Hence we have

$$\mathbb{E}[^{\xi_{x_1,x_2}(\phi^{(n)}, \psi^{(n)})}]$$

$$\leq 4(n + 1)^{X_1||X_2|} \left\{2^{-n|R_1-H(X_1|x_2)|} + 2^{-n|R_2-H(x_2|x_1)|} + 2^{-n|R_1+R_2-H(x_1,x_2)|}\right\}.$$ 

Hence we have

$$\mathbb{E}[^{\xi_{X_1,X_2}(\phi^{(n)}, \psi^{(n)})}] = \mathbb{E}\left[\frac{1}{|T^n_{X_1,X_2}|} \sum_{(x_1,x_2) \in T^n_{X_1,X_2}} ^{\xi_{x_1,x_2}(\phi^{(n)}, \psi^{(n)})}\right]$$

$$= \frac{1}{|T^n_{X_1,X_2}|} \sum_{(x_1,x_2) \in T^n_{X_1,X_2}} \mathbb{E}[^{\xi_{x_1,x_2}(\phi^{(n)}, \psi^{(n)})}]$$

$$\leq 4(n + 1)^{X_1||X_2|} \left\{2^{-n|R_1-H(X_1|x_2)|} + 2^{-n|R_2-H(X_2|x_1)|} + 2^{-n|R_1+R_2-H(X_1,X_2)|}\right\},$$

completing the proof. 

C. Proof of Lemma 10

Proof of Lemma 10 We first compute the expectation of $^{\Omega_{K_1,K_2;\phi^{(n)}}(\bar{k}_1^{m_1}, \bar{k}_2^{m_2})}$. We obtain the following:

$$\mathbb{E}[^{\Omega_{K_1,K_2;\phi^{(n)}}(\bar{k}_1^{m_1}, \bar{k}_2^{m_2})}] = \mathbb{E}\left[\frac{1}{|T^n_{K_1,K_2}|} \sum_{(k_1,k_2) \in T^n_{K_1,K_2}} ^{\Omega_{k_1,k_2;\phi^{(n)}}(\bar{k}_1^{m_1}, \bar{k}_2^{m_2})}\right]$$

$$= \frac{1}{|T^n_{K_1,K_2}|} \sum_{(k_1,k_2) \in T^n_{K_1,K_2}} \mathbb{E}[^{\Omega_{k_1,k_2;\phi^{(n)}}(\bar{k}_1^{m_1}, \bar{k}_2^{m_2})}] = \frac{1}{M_1 M_2} = \frac{1}{M_1 M_2}. $$
Step (a) follows from Lemma 8 part c) and independent random constructions of affine encoders $\varphi_1^{(n)}$ and $\varphi_2^{(n)}$. Thus (34) is proved. Next, we prove (35). We have the following chain of equalities:

$$
|T_{K_1K_2}^n|^{2} \mathbb{E}

\left[
\left(\Omega_{K_1K_2}(\bar{k}_1^{m_1}, \bar{k}_2^{m_2})\right)^2
\right]

= \sum_{(k_1,k_2) \in T_{K_1K_2}^n} \sum_{k_1 \neq \hat{k}_1 \in T_{K_2}^n} \mathbb{E}

\left[
\Omega_{k_1,k_2;\varphi^{(n)}}(\bar{k}_1^{m_1}, \bar{k}_2^{m_2})\Omega_{\hat{k}_1,k_2;\varphi^{(n)}}(\hat{k}_1^{m_1}, \hat{k}_2^{m_2})
\right]

\overset{(a)}{=} \sum_{(k_1,k_2) \in T_{K_1K_2}^n} \frac{1}{M_1M_2}

+ \sum_{k_1 \neq \hat{k}_1 \in T_{K_1}^n} \frac{1}{M_1M_2}

+ \sum_{k_2 \neq \hat{k}_2 \in T_{K_2}^n} \frac{1}{M_1M_2}

+ \sum_{(k_1,k_2) \in T_{K_1K_2}^n} \frac{1}{M_1M_2}

+ \sum_{k_1 \neq \hat{k}_1 \in T_{K_1}^n} \frac{1}{M_1M_2}

+ \sum_{k_2 \neq \hat{k}_2 \in T_{K_2}^n} \frac{1}{M_1M_2}

\leq \frac{|T_{K_1K_2}^n|^{2}}{M_1M_2} + \sum_{k_1 \neq \hat{k}_1 \in T_{K_1}^n} \frac{|T_{K_1}^n|^{2}}{M_1^{2}M_2^{2}} + \sum_{k_2 \neq \hat{k}_2 \in T_{K_2}^n} \frac{|T_{K_2}^n|^{2}}{M_1^{2}M_2^{2}} + \sum_{(k_1,k_2) \in T_{K_1K_2}^n} \frac{|T_{K_1}^n|^{2}}{M_1^{2}M_2^{2}} + \sum_{k_1 \neq \hat{k}_1 \in T_{K_1}^n} \frac{|T_{K_1}^n|^{2}}{M_1^{2}M_2^{2}} + \sum_{k_2 \neq \hat{k}_2 \in T_{K_2}^n} \frac{|T_{K_2}^n|^{2}}{M_1^{2}M_2^{2}}

= \frac{|T_{K_1K_2}^n|^{2}}{M_1^{2}M_2^{2}} \left\{ M_1M_2 + M_1 + M_2 + 1 \right\}.

(47)

Step (a) follows from Lemma 8 part c) and independent random constructions of affine encoders $\varphi_1^{(n)}$ and $\varphi_2^{(n)}$. Step (b) follows from Lemma 2 part b). Hence we have

$$
\text{Var}

\left[
\left(\Omega_{K_1K_2;\varphi^{(n)}}(\bar{k}_1^{m_1}, \bar{k}_2^{m_2})\right)^2
\right]

\leq \frac{1}{M_1^{2}M_2^{2}} \left\{ M_1M_2 + M_1 + M_2 \right\}

\overset{(a)}{=} \frac{1}{M_1^{2}M_2^{2}} \left\{ 2^{nR_{1}+R_{2}} + 2^{nR_{1}} + 2^{nR_{2}} \right\}

\overset{(b)}{=} \frac{(n+1)|X_1||X_2|}{M_1^{2}M_2^{2}} \left\{ 2^{-n[H(K_1K_2)-(R_{1}+R_{2})]} + 2^{-n[H(K_1)-R_{1}]} + 2^{-n[H(K_2)-R_{2}]} \right\}.

$$

Step (a) follows from $M_i \leq 2^{nR_i}$, $i = 1, 2$. Step (b) follows from Lemma 2 part b).

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