A comment on stabilizing reinforcement learning

Pavel Osinenko, Georgiy Malaniya, Grigory Yaremenko, Ilya Osokin

Abstract—This is a short comment on the paper “Asymptotically Stable Adaptive–Optimal Control Algorithm With Saturation Actuators and Relaxed Persistence of Excitation” by Vamvoudakis et al. The question of stability of reinforcement learning (RL) agents remains hard and the said work suggested an on-policy approach with a suitable stability property using a technique from adaptive control—a robustifying term to be added to the action. However, there is an issue with this approach to stabilizing RL, which we will explain in this note. Furthermore, Vamvoudakis et al. seems to have made a fallacious assumption on the Hamiltonian under a generic policy. To provide a positive result, we will not only indicate this mistake, but show critic neural network weight convergence under a stochastic, continuous-time environment, provided certain conditions on the behavior policy hold.

I. SYNOPSIS OF [1]

The design of RL is usually complex compared to the common techniques of nonlinear and adaptive control. Taking this and the page limitation into account, we will try to present the essence of the work [1] as concise as possible. Vamvoudakis et al. started with a continuous-time, deterministic, control-affine environment

\[ \dot{x} = F(x,u) := f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m. \]

They then addressed an infinite-horizon optimal control problem

\[ \min_{\kappa} J^\kappa(x_0) = \int_0^\infty \rho(x,\kappa(x)) \, dt, \]

where \( \rho \) is a stage cost function. The Hamiltonian reads, for a generic smooth function \( h: \mathcal{H}(x,u|\kappa) := \nabla h^T(F(x,u) + \rho(x,u)) \). Denoting the value function as \( V := \min_{\kappa} J^\kappa \) and, accordingly, the optimal policy as \( \kappa^* \), the Hamilton-Jacobi-Bellman equation (HJB) reads:

\[ \mathcal{H}(x,\kappa^*(x)|V) = 0, \quad \forall x. \]

RL in a nutshell amounts to finding approximate solutions \( \hat{V}_0, \kappa_0 \to [1], \) as some \( \theta \), respectively, \( \vartheta \)-weighted neural networks, commonly called critic and, respectively, actor. Let us assume one-hidden-layer topology for both. Then, by an approximation theorem, one can express the value function and the optimal policy for some ideal weights as follows:

\[ V(x) = \theta^T \varphi(x) + \delta(x), \quad \kappa^*(x) = \theta^T \psi(x) + \delta_u(x), \]

where \( \varphi, \psi \) are the activation function, and \( \delta, \delta_u \) are the approximation errors.

To learn the actor and critic weights, one can follow what the HJB prescribes. From now on, let us assume that the action \( u \) is generated by the policy \( \hat{\kappa}_0 \) (this is an on-policy setup). Then, the Hamiltonian under the critic reads:

\[ \mathcal{H}(x,u|\hat{V}_0) = \theta^T \nabla \varphi(x) F(x,u) + \rho(x,u). \]

Let us introduce the Hamiltonian approximation error as \( \delta \hat{\kappa}(x,u) := \mathcal{H}(x,u|\hat{V}_0) - \mathcal{H}(x,u|\hat{V}_0) = \nabla \delta^T F(x,u) + \rho(x,u). \) By a similar token, consider the Hamiltonian temporal difference (TD) as:

\[ e_{\mathcal{H}}(\theta|x,u) := \mathcal{H}(x,u|\hat{V}_0) - \mathcal{H}(x,\kappa^*|V) = \theta^T \nabla \varphi(x) F(x,u) + \rho(x,u). \]

Introducing the weight error \( \theta := \theta - \theta^* \), observe that \( e_{\mathcal{H}}(\theta|x,u) = \theta^T w(x,u) + \mathcal{H}(x,u|V) - \delta \mathcal{H}(x,u). \)

Vamvoudakis et al. suggested to use a data vector \( w(x,u) := \nabla \varphi(x) F(x,u) \), which yields \( e_{\mathcal{H}}(\theta|x,u) = \theta^T w(x,u) + \rho(x,u) \). Following an analogous methodology, one designs the critic weights via stochastic gradient descent as

\[ \hat{\theta} := -\alpha \nabla \theta J_c(\theta|x_{k+1}, u_{k+1}) = -\alpha \sum_{k=1}^M \frac{e_\mathcal{H}(\theta|x_{k+1}, u_{k+1})}{(w_{k+1}^T w_{k+1} + 1)^2}, \]

where we used a shorthand notation \( w_k := w(x(t),u(t)) \). Then, the critic weights are learned via stochastic gradient descent as

\[ \hat{\theta} := -\alpha \nabla \theta J_c(\theta|x_{k+1}, u_{k+1}) = -\alpha \sum_{k=1}^M \frac{e_\mathcal{H}(\theta|x_{k+1}, u_{k+1})}{(w_{k+1}^T w_{k+1} + 1)^2}. \]

Using the expression of the Hamiltonian TD through the Hamiltonian approximation error, one obtains:

\[ \hat{\theta} := -\alpha \sum_{k=1}^M \frac{w_{k+1}^T w_{k+1}}{(w_{k+1}^T w_{k+1} + 1)^2}. \]

Denote \( E_t := \sum_{k=1}^M \frac{w_{k+1}^T w_{k+1}}{(w_{k+1}^T w_{k+1} + 1)^2} \). Then, one can perform a stability analysis of the differential equation [1] under a suitable persistence of excitation condition, i.e., \( E_t \geq \varepsilon I, \forall t \geq 0 \) with \( I \) being the identity matrix of proper dimension, and norm-boundedness of \( Z_{t_k} \). For the latter condition, it is crucial that the environment state \( x \) be norm-bounded! Regarding persistence of excitation, [1] suggested to update the experience replay until the respective condition is fulfilled. In other words, one does not have to assume a kind of running persistence of excitation on the buffer, where a data vector is pushed into it at each \( t_k \).

Following an analogous methodology, one designs the actor to minimize an actor loss of the form \( J_a(\vartheta|x) = \frac{1}{2} \text{tr} (e_{\vartheta}(\vartheta|x) e_{\vartheta}(\vartheta|x)^T) \), where the actor error is \( e_{\vartheta}(\vartheta|x) := \hat{\kappa}_0(x) - \kappa^*(x) \). The learning of the actor weights \( \hat{\vartheta} \) can also be done by stochastic gradient descent.
II. ISSUE WITH GUARANTEEING STABILITY

The full details of the actor can be found in \( l \), whereas for us, the essential part is the robustifying term that Vamvoudakis et al. used to guarantee closed-loop stability. The ground idea comes from adaptive control [2]. We depict it in a very short form. Let us suppose that \( x \) is scalar and \( f, g \) are unknown, learned via models \( \hat{f}(x) = \theta_1 x + \theta_2 \varphi_1(x) \), \( \hat{g}(x) = \theta_3 \varphi_2(x) \). Let the ideal weights be \( \theta_1^*, \theta_2^* \) and the respective weight errors be \( \theta_1, \theta_2 \). For brevity, we omit the \( x \)-argument from now on.

Let us consider a policy defined by \( \kappa = \frac{1}{\theta}(\bar{K}x - \hat{f}) \), \( K > 0 \). Then, \( \dot{x} = -Kx + \theta_1 \varphi_1 + \theta_2 \varphi_2 \kappa \). Taking weight update rules as \( \theta_1 := \alpha x \varphi_1, \theta_2 := \alpha_g x \varphi_2 \kappa \), and using a Lyapunov function candidate \( L := \frac{1}{2} \rho^2 \theta_1^2 + \frac{1}{2} \theta_1 \theta_2 \theta_2 \theta_2 + \frac{1}{2} \theta_2 \theta_2 \theta_2 \theta_2 \), one can show that \( \dot{L} \leq -Kx^2 \). A projection might be required in the update of \( \theta_2 \) to ensure \( \theta_2 \) be bounded away from zero, which is central to the described technique.

Now, \( l \) followed, roughly, this idea and added to the actor-generated policy \( \kappa \), a robustifying term of the form \( -K \|x\|^2 \frac{t}{A + \|x\|^2} \). The constants \( K, A \) were then chosen to account for various bounds on the quantities involved (the activation function, its gradient, the Hamiltonian error etc.). With this at hand, the value function \( V \) was picked as the Lyapunov function candidate. Let \( L \) denote this candidate. Then, one has:

\[
\dot{L} = \nabla V^\top \left( f - g \hat{\theta}^\top \psi + g(k^* - \delta_u) - gK \|x\|^2 \frac{t}{A + \|x\|^2} \right) = -\rho^* \dot{V} + \nabla V \bar{g} \kappa^* \|x\|^2 \frac{t}{A + \|x\|^2},
\]

where \( \rho^* := \rho(x, k^*(x)) \) and the last displayed identity follows from the HJB \( \nabla V^\top f = -\rho^* - \nabla V g \kappa^* \).

Therefore, using the value function as a Lyapunov function candidate does not seem as a viable option in online RL.

There is another, conceptual, issue with the robustifying term as described above. Namely, its interference with the nominal actor. The gain \( K \) would need to be chosen large enough depending on the various assumed bounds (if the overall routine was correct). Such a setting might turned out quite conservative. But this harms one of the key principles of safe RL – minimal interference – that says that the safety measures (like closed-loop stability in the considered case) should not restrict or damp down the actor’s learning. Thus, even if one followed the methodology of stabilization via adaptive control correctly, one would face the possible trouble of RL effects simply fading behind robustifying terms.

IV. ISSUE WITH THE GENERIC-ACTION HAMILTONIAN

Now, let us get to another issue with the work \( l \), which is less severe and remediable, compared to the previous one. This issue is two-fold: first, \( l \) claimed exponential convergence of the critic weight error (see equation (25) in the cited work), while it should have been ultimate boundedness instead, taking into account the perturbation term (the second one in equation \( l \) above). Second, the bound on the perturbation term was incorrect. Let us recall \( l \). The perturbation term here reads:

\[
\alpha \sum_{k=1}^{M} \frac{w_{ik}(H(x_k, u_k|V) - \delta H(x_k, u_k))}{(w_{ik} + 1)^2}.
\]

In \( l \), the generic-action Hamiltonian \( \mathcal{H}(x_k, u_k|V) \) was erroneously assumed zero, while it is only zero under the optimal policy (see equation \( l \)). What this implies is that the critic learning quality actually depends on the policy applied to the environment. In the case of \( l \), the overall routine is an on-policy RL, which means that the policy being learned is the one applied. Ideally, one would expect the respective generic-action Hamiltonian approach zero as the learned policy approaches the optimal one. Otherwise, the ultimate critic weight error will depend on the quality of the behavior policy. Notice that application of robustifying terms, in general, worsens this quality.

V. STOCHASTIC CRITIC CONVERGENCE

In this section, we study a similar actor-critic design as before, but for an environment described by a stochastic differential equation \( dX_t = f(X_t, U_t) dt + \sigma(X_t, U_t) dB_t \), where \( \{B_t\}_{t>0} \) is a vector Brownian motion.

The optimal control problem is now in the following format:

\[
\min_{\kappa} J^\kappa(x_0) = E \left[ \int_0^\infty e^{-\gamma t} \rho(X_t, \kappa(X_t)) dt \mid X_0 = x_0 \right],
\]

where \( \rho(X_t, \kappa(X_t)) \) represents the cost function.
where we introduced a discounting factor $\gamma$ to be more general. The Hamiltonian now reads, for a generic smooth function $h$,
\[
\mathcal{H}(x,u,h) = \nabla h(x)^T f(x,u) + \frac{1}{2}\text{tr}\left(\sigma^T (x,u)\nabla^2 h(x)\sigma(x,u)\right) + \rho(x,u) - \gamma h(x),
\]
while the HJB is the same as above. Now, the Hamiltonian under the critic reads:
\[
\mathcal{H}(x,u;\hat{V}_\theta) = 
\theta^T \nabla \phi(x)f(x,u) + \frac{1}{2}\theta^T \eta(x,u) + \rho(x,u) - \gamma \theta^T \phi(x),
\]
where $\eta(x,u)$ is the vector with entries $\text{tr}(\sigma^T (x,u)\nabla^2 \phi_i(x)\sigma(x,u))$, $i \in [N_\theta]$, where $N_\theta$ is the number of the critic’s features and the notation $\phi_i(x)$ means the $i$th feature. The Hamiltonian approximation error in our case amounts to
\[
\delta \mathcal{H}(x,u) = 
\nabla \delta(x)^T f(x,u) + \frac{1}{2}\text{tr}\left(\sigma^T (x,u)\nabla^2 \delta(x)\sigma(x,u)\right) - \gamma \delta(x).
\]
The Hamiltonian TD modifies to $e \mathcal{H}(\theta|x,u) = \theta^T \nabla \phi(x)f(x,u) + \frac{1}{2}\theta^T \eta(x,u) + \rho(x,u) - \gamma \theta^T \phi(x)$. The data vector in turn becomes $w(x,u) := \nabla \phi(x)f(x,u) + \frac{1}{2}\eta(x,u) - \gamma \phi(x)$. Then, $e \mathcal{H}(\theta|x,u) = \theta^T w(x,u) + \mathcal{H}(x,u;V) - \delta \mathcal{H}(x,u)$. Following the same procedure as in the first section of this note, the evolution of the critic weight error in norm square follows the following SDE:
\[
d\|\hat{\theta}_t\|^2 = -\alpha \left(\|\hat{\theta}_t\|^2 + \sum_{k=1}^M \hat{\theta}_k W_{tk}^T (\mathcal{H}(X_{tk},U_{tk}|V) - \delta \mathcal{H}(X_{tk},U_{tk})) / (W_{tk}^T W_{tk+1})\right) dt,
\]
where $\|\cdot\|$ means the weighted Euclidean norm and $\mathcal{E}_t := \sum_{k=1}^M W_{tk}^T W_{tk+1}$.

**Theorem 1:** Consider the critic learning via (9) under some behavior policy $\mu$. Let the following conditions hold: (a) $f(\cdot,\mu(\cdot))$ and $\sigma(\cdot,\mu(\cdot))$ are of linear growth and $\mu(0) = 0$, $f(0,0) = 0$, $\sigma(0,0) = 0$; (b) the critic topology is chosen s.t. the value function approximation error $\delta$ is of quadratic growth and $\delta(0) = 0$; (c) $\rho$ and $V$ are of quadratic growth; (d) the behavior policy is persistently exciting, i.e., $\mathcal{E}_t \geq \epsilon I$ a.s.; (e) $\exists \bar{X} > 0 \forall t \geq 0 \mathbb{E}\left[\|X_t\|^4\right] \leq \bar{X}^2$. Then, the critic error weights satisfy:
\[
\mathbb{E}\left[\|\hat{\theta}_t\|^2\right] \leq e^{-\alpha t}\|\hat{\theta}(0)\|^2 + D \sup_{\tau \leq t} \mathbb{E}\left[\|\hat{\theta}_\tau\|^2\right],
\]
for some constant $D$.

**Proof:** Consider (9). Denote $Z_{tk} := \mathcal{H}(X_{tk},U_{tk}|V) - \delta \mathcal{H}(X_{tk},U_{tk})$. Now, introduce a stopping time $T_R := \inf_{t > 0} \{\|X_t\| > R\}$ and integrate (9) from 0 to $T_R := t \wedge T_R$, where $t \wedge T_R := \min\{t, T_R\}$ and take expectation on both sides of (9):
\[
\mathbb{E}\left[\int_0^{T_R} \|\hat{\theta}_\tau\|^2 d\tau\right] = 
\mathbb{E}\left[\int_0^{T_R} -\alpha \|\hat{\theta}_\tau\|^2 + \sum_{k=1}^M \hat{\theta}_k W_{tk} Z_{tk} \right] d\tau \leq 
\mathbb{E}\left[\int_0^{T_R} e^{\alpha t} \|\hat{\theta}_\tau\|^2 + \sup_{\tau \leq t} \mathbb{E}\left[\|\hat{\theta}_\tau\|^2\right]\right].
\]
where the last inequality follows from (d) the fact that $\mathbb{E}\left[\|Z_t\|^2\right] \leq 1$ always. Now, due to (e), $T_R \rightarrow t$ a.s. as $R \rightarrow \infty$, and so using the dominated convergence on the right of (11) and Fatou's lemma on the left, deduce:
\[
\mathbb{E}\left[\|\hat{\theta}_t\|^2\right] \leq e^{-\alpha t}\|\hat{\theta}(0)\|^2 + \mathbb{E}\left[\sup_{\tau \leq t} \mathbb{E}\left[\|\hat{\theta}_\tau\|^2\right]\right].
\]
From (a), (b), (c) $\exists C > 0$ s.t. $\mathbb{E}\left[\mathcal{H}(X_{tk},U_{tk}|V) - \delta \mathcal{H}(X_{tk},U_{tk})\right] \leq C \|x\|^2$. From (e), $\forall t \geq 0 \mathbb{E}\left[\|Z_t\|^2\right] \leq C^2 \bar{X}^2$. Then, using the Fubini’s lemma and the Cauchy-Schwartz inequality, one obtains from (12):
\[
\mathbb{E}\left[\|\hat{\theta}_t\|^2\right] \leq e^{-\alpha t}\|\hat{\theta}(0)\|^2 + \frac{MC\bar{X}}{2} \sup_{\tau \leq t} \mathbb{E}\left[\|\hat{\theta}_\tau\|^2\right].
\]
where $\frac{MC\bar{X}}{2}$ is the desired constant $D$ from the theorem statement.

From (13) one can deduce a suitable ultimate boundedness property for the mean-square critic weight error. A remark should be made: the above stated assumptions are only necessary when we are dealing with an SDE driven by Brownian motion. If the driving noise were bounded, the assumptions could be discarded.

VI. CONCLUSION

In this note, we discussed an attempt at guaranteeing closed-loop stability under online, on-policy RL using a technique from adaptive control as per [1]. To show stability, the value function was used as a Lyapunov function candidate. The discussed issues with the cited work indicate that using the value function for this role is troublesome. It seems to only work when the actor learned the policy close enough to the optimal one. S on the other hand, robustifying terms added to the policy may fade the effects of learning thus harming minimal interference. We conclude this note by saying that stabilizing, and, generally speaking, safe RL remains an open and hard problem of control engineering and machine learning.

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