Strongly Coupled Dark Energy Cosmologies yielding large mass Primordial Black Holes

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Accepted XXXX . Received XXXX; in original form XXXX

ABSTRACT

Large primordial Black Hole (PBH) formation is enhanced if strongly coupled scalar and spinor fields (Φ and ψ) are a stable cosmic component since the primeval radiative expansion (SCDEW models). In particular, we show that PBH formation is easier at a specific time, i.e., when the asymptotic mass $m_H$, acquired by the ψ field at the higgs scale, becomes dominant, so that the typical BH mass $M_{BH}$ depends on $m_H$ value. For instance, if $m_H \sim 100 \text{ eV} \ (1 \text{ keV})$ and the coupling $\beta \sim 8.35 \, (37)$, PBH with $M_{BH} \approx 10^7 - 10^8 M_⊙ \, (\sim 10^3 - 10^4 M_⊙)$ could form. The very mechanism enhancing PBH formation also causes technical difficulties to evaluate the transfer function of SCDEW models at high k. A tentative solution of this problem leaves only minor discrepancies from ΛCDM, also at these scales, gradually vanishing for greater $m_H$ values. We conclude that, for suitable parameter choices, SCDEW models could be the real physics underlying ΛCDM, so overcoming its fine tuning and coincidence problems, with the extra bonus of yielding large BH seeds.

Key words: cosmology: dark matter,dark energy–black holes

1 INTRODUCTION

The possible relevance of a cosmic scalar field Φ, coupled to Dark Matter (DM), was envisaged even before SN1a Hubble diagrams (Riess et al. (1998); Perlmutter et al. (1999)) forced us to conform with the existence of Dark Energy (DE). The option of DE being such self–interacting scalar field Φ was then widely explored. As first outlined by Wetterich (1995), Amendola (2000) and Amendola & Tocchini-Valentini (2002), it could ease some ΛCDM conundrums, also allowing DE to be a substantial cosmic component through the whole matter–dominated era.

Their approach however required a self–interaction potential $V(\Phi)$, including specific parameters. In turn, no initial conditions for Φ needed to be specified, when using tracking potentials (Steinhardt et al. (1999), see also Ratra & Peebles (1988), Wetterich (1995) , Brax & Martin (1999)).

The ΛCDM option however prevailed as, in spite of the extra parameter(s) in $V(\Phi)$ expressions, data fitting did not improve. The option of ΛCDM being inadequate to fit (new) data was then explored just by testing a DE state parameter $w \neq -1$ or being a first degree polynomial $w = w_0 + w_a (1 - a)$ (Linder (2003)).

A Φ–DM interaction was then discussed, in a fully different context, by Bonometto, Sassi & La Vacca (2012). Instead of displaying its main action in “recent” times, they supposed it to modify the radiative era. Uncoupled DM and kinetic–Φ densities would then scale $\propto a^{-3}$ and $a^{-6}$. A suitable energy flow from coupled DM (coDM, a spinor field ψ) to Φ, however allows for both densities to be $\propto a^{-4}$, so keeping them a constant fraction of the radiative cosmic content, and allowing for a primeval Conformally Invariant (CI) expansion. Bonometto, Sassi & La Vacca (2012) then verified that such regime is not only possible, but is a cosmic attractor. Φ could be a scalar field involved in inflation while, at low $z$, it will safely become DE.

Before discussing the successive work on fluctuation evolution, let us then soon outline the main point made by this paper: without ad–hoc assumptions, this kind of cosmologies allows for the formation of large mass Primordial Black Holes (PBH), when the mass $m_H$, acquired by the ψ field at the higgs scale, becomes dominant. PBH average mass is given by the expression

$$M_{BH} \sim 3.32 \times 10^9 M_⊙ \left(\frac{100 \text{ eV}}{m_H}\right)^{3.31} ,$$

(1)

when β is is selected to allow for close values of early wDM (see...
below) and coDM densities. Giant BH as those in high–z QSO’s could then form through matter accretion, without pushing it to the Eddington limit. Moreover, in principle, PBH number density can approach the observed galaxy number, while the late PBH masses depend on individual histories of matter accretion. This however requires BH seeds with masses \( \lesssim 10^6 M_\odot \) (as preferred \( m_{PBH} \) range, values \( \Omega (1 \text{ keV}) \) could be suggested), but immediately rises the question of early cosmic reionization and microlensing. We shall further comment on this point later on, but the bulk of this discussion is postponed to further analysis.

Let us now summarize further previous work on these models:

(i) In Bonometto & Mainini (2014) fluctuation modes were studied and an algorithm yielding linear transfer functions was described. 

(ii) In Bonometto, Mainini & Macciò (2015) the lagrangian approach was deepened, and a late interaction screening, due to the coDM field \( \psi \) acquiring a mass at the higgs scale, was also envisaged. This is the very mechanism exploited here. 

(iii) In Macciò et al. (2015), N–body simulations of this cosmology, named SCDEW (Strongly Coupled Dark Energy plus Warm DM), were discussed. They showed that, when compared with sub–galactic data (MW and M31 satellites, dwarf rotation curves, etc.), SCDEW performed much better than ΛCDM N–body simulations. 

(iv) A series of papers [Bonometto & Mainini (2014); Bonometto, Mainini & Macciò (2015); Bonometto, Mezzetti & Mainini (2017); Bonometto & Mainini (2017a,b)] was then dedicated to analyse the peculiar evolution of coDM fluctuations, through the radiation dominated era: In spite of radiation domination, their amplitude fastly increases. 

The numerical algorithm built by Bonometto & Mainini (2014) allows us to follow its growth during horizon crossing, when it is linear but fully relativistic, as well as later on, until the amplitude allows a linear treatment. If we describe such growth by a power law \( \delta_{\alpha} \propto a^\alpha \), we find an exponential \( \alpha \geq 2 \) around horizon crossing and until \( \delta_{\alpha} \) overcomes its horizon value by 1-1.5 o.m.; then the rate of increase softens and we gradually settle on a regime \( \alpha \approx 1.6 \), as already envisaged by Amendola & Toncini-Valentini (2002). A still faster growth is expected when approaching non–linearity; this will occur a suitable time after horizon crossing, whose precise evaluation is made possible by the fair treatment of horizon crossing. The very non linear growth regime was then explored by using the spherical top–hat approximation. Their findings will be resumed here below.

It is also worth mentioning soon that the very rapid growth of coDM perturbations plays a key role in yielding small scale fluctuations of the warm DM (wDM) component which, in SCDEW cosmologies, is supposed to dominate the present DM density, although the warm particle mass \( m_w \) is small. The relation between wDM and coDM will be furtherly discussed below.

Let us outline that a part of the above findings were again outlined in a recent work by Amendola et al (2018), with no mention of the above literature. Their basic aim was to outline that the early formation of coDM non–linearities could favor PBH formation, in an epoch when neither radiation nor matter fluctuation amplitudes are allowed to grow. Accordingly, their paper was the first to outline a possible relation between SCDEW models and PBH formation. However, they focused on the early reach of a non–linear regime while, according to the analysis in this work, PBH formation is hardly directly enhanced by that. On the contrary, a key role is played by the coDM field acquiring a mass at the higgs scale. This option, not even mentioned by them, seems to bear a key role in favouring PBH formation on a specific mass scale range. Such acquisition, therefore, is not only a “screening mechanism”, whose need was mentioned also by Amendola et al (2018) to allow for a late data fitting, but also the key to open a passage through the virialization wall.

In fact, Bonometto & Mainini (2017a,b) had showed that spherical top–hat fluctuations, during the radiation dominated era, meet virialization conditions at a low density contrast \( \Delta_{vir} \approx 3-4 \) and, even more significantly, such virialization is just a transitory step, as the peculiar dynamics of coupled particles causes a fast dissolution of virializing lumps. Lump dissolution had been independently noticed by Casas et al. (2016), in a set of numerical simulations involving particles with variable mass, although within a different context.

Our point here, however, is that the action of intrinsic forces pushing towards disruption of spherical geometry as well as the very post–virialization dissolution are suppressed when coupling fades, because of higgs screening. This conclusion is obtained by extending the study of spherical top–hat density enhancements to the epoch when the acquisition of a (tiny) mass by \( \psi \), at the higgs scale, causes a rapid fading of the \( \Phi–\psi \) effective coupling \( \beta_{eff} \); this is one of the main technical contributions of this work. In particular, we find a significant peak on the virial density contrast \( \Delta_{vir} \), reaching values \( \sim 500 \) (or even much more, depending on \( m_{PBH} \) value) while \( \beta_{eff} \) fades, to later reconverge to low values. By itself, however, the reach of such larger \( \Delta_{vir} \), does not ease PBH formation. We rather argue that the same physical reasons allowing the reach of an anomalously large \( \Delta_{vir} \), in a narrow and specific scale interval, can also favour the procession of spherical collapse, towards its relativistic regime. In this connection, the question of the actual likelihood of a spherical geometry, as a function of the fluctuation amplitude at the horizon, will be suitably debated.

The redshift \( z_P \) when \( \beta_{eff} \) fades –and, therefore, the mass scale \( M_P \) of the \( \Delta_{vir} \) peak– are set by the asymptotic mass \( m_{PBH} \). Accordingly, the preferred value of PBH is the mass scale of fluctuations reaching the horizon so earlier to reach the peak at \( z_P \). This sets the \( m_{PBH} \) dependence of PBH mass scale outlined in eq. (1), and already outlines that the choice of \( \beta \) has little impact on that. Such impact is even smaller if we keep to the assumption of early close values for coDM and wDM.

It is however clear that the geometry of most fluctuations is not spherical. A better approximation to treat their initial non–linear stages could be based on the Zel’dovich pancake approach. If, as we argue, dissolution mechanisms display their action soon, when a density contrast \( \sim 3–4 \) is approached, this technique also enables us to provide approximated, but realistic, SCDEW model spectra at large \( k \) values.

Altogether, SCDEW models are characterized by standard radiation, neutrino and baryon components. DE is a quintessential field \( \Phi \). The option of \( \Phi \) having a role in inflation is open, as its energy density could keep non–negligible since then, for its growth through the intermediate eras is just logarithmic. It is also worth outlining soon that, although \( \Phi \) self–interaction is bound to play a key role, no \( V(\Phi) \) potential needs to be specified (but see below). A dual Dark Matter component is then assumed, comprising coDM (coupled with \( \Phi \)) and wDM (uncoupled and light). Most of the roles of coDM could also be covered by a scalar field \( \chi \), an option not deepened here. Rather, we stress the option that coDM and wDM are the coupled and uncoupled components of the same spinor field. This is suggested by the fact that viable models are however characterized by close early densities \( \rho_{PBH} \) and \( \rho_{wDM} \).

Accordingly, we require the masses of coDM and wDM particles to be equal, so that \( m_{wDM} \) is not only the asymptotic mass of coDM, acquired at the higgs scale, but also wDM quanta are supposed to acquire the same mass. Most model features, however, do
not depend on the two spinor fields quanta sharing exactly the same early densities, as ratios in an interval close to unity however yield viable models.

More in detail, most results of this paper are given for a model where \( m_p = 100 \text{eV} \) and \( \beta = 8.35 \), yielding a (primeval) density ratio (wDM/\( \rho_{\text{DM}} \)) \( \approx 0.9 \). The values of other cosmological parameters are: \( \Omega_b = 0.049 \), \( \Omega_{\text{cd}} = 0.6824 \), \( h_0 = 0.671 \), \( T_0 = 2.726 \) (symbols keep their usual meanings). As shown by Macciò et al. (2015), such a low \( m_p \) value eases the problems ACDM N–body simulations exhibit, at scales close or below the galactic scale. However, for the sake of comparison, another model, with \( m_p = 1 \text{keV} \) and \( \beta = 37 \), yielding an equal density ratio, is also considered.

The plan of the paper is as follows. In Section 2 we provide a more detailed reminder on SCDEW models, namely on their back-ground features. In Section 3 we discuss spherical density enhancements, when taking into account both coDM and the other components. A particular emphasis concerns the approach to redshift values when the DM–Φ coupling fades. Section 4 is then devoted to discuss virialization. Section 5 then shows why potential term (\( \Phi \)). Accordingly, \( \Omega_{\text{b}} = \frac{1}{4 \beta^2} \), \( \Omega_{\text{CO}} = \frac{1}{2 \beta^2} \), so that the requirement \( \Omega_{\text{b}} + \Omega_{\text{CO}} \ll 1 \) implies that \( \beta \gg \sqrt{3}/2 \). Values of \( \beta \leq 2.5 \) are however excluded by limits on dark radiation during BBN or when CMB spectra form.

If the metric is (2) and lifting the restriction that \( \Phi \) is purely kinetic, eqs. (4) also read

\[
\Phi_i + \frac{\omega}{a} \Phi_i = \frac{1 + w}{2} C a^2 \rho_{\text{DM},i}, \quad \rho_{\text{DM},i} + \frac{3}{a} \dot{\rho}_{\text{DM},i} = -C \rho_{\text{DM},i},
\]

with \( \Phi_i \equiv \frac{d \Phi}{d t} \) and \( 2 \omega = 1 + 3w - d \ln(1 + w)/d \ln a \). Eqs. (7) yield \( \Phi_i \) and \( \rho_{\text{DM},i} \) evolutions from the \( \Phi \)–field state parameter \( w(a) \), with no need to specify a V(\( \Phi \)) expression.

Bonometto, Mainini & Macciò (2015) showed eqs. (4) or (7) to be consistent with DM being a spinor field \( \phi \), interacting with \( \Phi \) through a generalized Yukawa lagrangian

\[
\mathcal{L}_\phi = -\mu f(\Phi/m) \bar{\psi} \psi
\]

provided that

\[
f = \exp(-\Phi/m)\]

(see also Das et al. (2006)). Here 2 independent mass scales, \( m = m_p/b \) and \( \mu = g m_p \) are introduced. The constant \( b \), however, coincides with the factor \( b \) gauging the DM–Φ interaction strength in eq. (5), so that \( c = 1/m \); on the contrary, \( g \) and an additive constant on \( \Phi \) keep undetermined.

Let us however outline a numerical coincidence: we can assume \( \Phi \equiv m_p \) at the Planck scale, by taking \( g = 2\pi e^{-b} \). This was also done in previous work, finding a data fit when \( b = (4\pi/3)^{1/2} \approx 40 \); this value is also close to the one mostly considered in this paper. It is then noticeable that, with this choice, \( \mu \) is of the order of the electroweak (EW) scale (more precisely, by forgetting the (arbitrary) factor \( 2\pi \), we have: \( \ln(m_{\text{p}}/100\text{GeV}) = 39.34 \), so that \( \beta = 9.61 \). But, of course, it is fully licit to forget such coincidence and explore any other interval. Let us however keep

\[
f = \exp[-C(\Phi - \Phi_i)] \quad \text{with} \quad \Phi_i \equiv m_p,
\]

with \( \mu = 2\pi m_p \) in eq. (8). This choice is quantitatively relevant, when we pass to consider a higgs coupling screening (Bonometto, Mainini & Macciò (2015)); however, close values would not yield significant changes.

Let us then recall that the particle number operator of a spinor field \( n \equiv \bar{\psi} \psi \). Accordingly, the coDM density reads

\[
\rho_{\text{CO}} = \mu f(\Phi) \bar{\psi} \psi
\]

(formally, \( \rho_{\text{CO}} = -\mathcal{L}_\phi \)). It is then worth focusing on the term

\[
\frac{\delta \mathcal{L}_\phi}{\delta \Phi} = \mathcal{L}_\phi \frac{\partial \rho_{\text{CO}}}{\partial \Phi} = -\mu f'(\Phi) \bar{\psi} \psi = -f'(\Phi) f(\Phi) \rho_{\text{CO}} = C \rho_{\text{CO}}
\]

of the Euler–Lagrange equation which, multiplied by a suitable factor, stands at the r.h.s. of the first eq. (7). Incidentally, eqs. (7) can
entiation in eq. (11), we obtain
\[ \frac{\delta \tilde{L}_{\mu}}{\delta \Phi} = -f'(C\Phi) + \frac{\mu_c \rho_{co}}{1 + R \exp[C(\Phi - \Phi_p)]} \rho_{co}. \] (15)

Here \( R = \bar{\mu}/\mu \). Accordingly, the dynamical equations, in the presence of the mass acquired at the higgs scale, keep the form (7), once we replace
\[ C \to C_{eff} = \frac{C}{1 + R \exp[C(\Phi - \Phi_p)]} \] and/or
\[ \beta \to \beta_{eff} = \frac{\beta}{1 + R \exp[C(\Phi - \Phi_p)]}. \] (16)

Then, when the \( \Phi \) increase causes \( \Phi - \Phi_p \) to approach \( -\ln(R)/C \), the denominators in eq. (16) suppress the effective coupling intensity. For the sake of example, by assuming \( \beta = 10 \) (37) and \( \bar{\mu} = 115 \text{eV} \) (1 keV), the dependence of \( \beta_{eff} \) on \( a \) is shown in Figure 1 (Figure 2).

Besides of radiation, baryons, and the coupled \( \Phi \) (DE) and \( \psi \) (coDM) fields, SCDEW models also include a further DM component, that we shall indicate as wDM (warm DM). In the models considered here, wDM is assumed to be closely related to coDM, acquiring the same mass \( (m_{co} = m_w = m_\beta) \) at the higgs scale and exhibiting a close early density (in the cases considered, \( \rho_{co}/\rho_{w} \approx 0.9 \)). Then coDM has a later rise, as the flow of energy from it to \( \Phi \), yielding \( \rho_{co} \propto a^{-4} \) at large \( z \), is (almost) cut off only when \( \beta_{eff} \) has reached its low value domain.

The assumption that coDM and wDM have similar masses and early densities, during the primeval CI expansion, is somehow arbitrary. The only compulsory requirement on masses and \( \beta \) is that they allow for densities \( \rho_{co} \), at \( z = 0 \) (the index \( i \), labels cosmic components) fitting observations. Our requirement however reflects the conjecture that coDM and wDM particles are, somehow, the \( \beta \)-coupled and the neutral state of the same particle. A priori, one could state that SCDEW models require two DM components. However, they are safely viable under the above restrictions, as though we were dealing with a single DM component with 2 charge states.

When approaching our epoch, we expect a \( \Phi \) transition from kinetic to potential. Rather than dealing with hardly testable \( V(\Phi) \) expressions, we model the \( \psi \) transition from +1 to -1, by requiring
\[ w(a) = \frac{1 - A}{1 + A} \quad \text{with} \quad A = \left( \frac{a_{k\psi}}{a_k} \right)^{1/4}. \] (17)

As expected, results are scarcely dependent on the exponent \( \epsilon \), whose arbitrariness somehow mimics the arbitrariness in the potential choice: \( a_{k\psi} = (1 + z_{k\psi})^{-1} \) is fixed so to obtain the observational amount of today’s DE.

In Figure 2 the scale dependence of the densities is plotted. Here, DE evolution is drawn by taking \( \epsilon = 2.9 \). Taking different \( \epsilon \) values cannot modify either \( \rho_{co} \) or \( \rho_{w} \) at high \( z \), as the former value is assumed while, at high \( z \), \( \Omega_{cb} \equiv 1/4\beta^2 \). Therefore, only the detailed scale dependence, close to \( a_{k\psi} \), is slightly modified. As a change of \( \epsilon \) mimics a change of \( V(\Phi) \) expression, this is a further indication of the serious difficulty that will be however found to detect \( V(\Phi) \) from observational data.

### 2.2 Linear fluctuation evolution

Linear fluctuations in SCDEW models were first discussed in Bonometto & Mainini (2014). In a synchronous gauge, the metric
shall then consider also field perturbations, by assuming gravity perturbations being described by the 3–tensor $h^{ij}$. In this work, evolution of densities (Figure 2). Because we set $\alpha$, replacing $1 + 3 \rho$ with $1 + 3 \rho_{\text{co}}$, which trace is $h_{ij}$. Besides of density perturbations for all components, one shall then consider also field perturbations, by assuming

$$\phi = \Phi + \frac{b}{m_p} \varphi$$

to be the sum of the background field $\Phi$ considered in the previous subsection and a perturbation described by $\varphi$.

The whole discussion has many technical aspects that are deepened in Bonometto & Mainini (2014). The main critical issue, when trying to use equations provided in the literature, arises because we set $w(a)$, instead of the potential. This is obtainable by replacing

$$2V'' = \frac{A}{1 + A} \left[ \frac{\dot{a}}{a} + \frac{\epsilon}{1 + \epsilon} \right] (e - 6) \frac{\dot{a}^2}{a^2} + 2C \frac{\dot{\rho}_{\text{co}}}{\Phi}$$

$$+ \left[ \frac{\dot{a}}{a^2} \Phi + \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) \right] \epsilon_\delta + 2C \frac{\dot{\rho}_{\text{co}}}{\Phi}$$

with $A$ and $\epsilon$ defined as in eq. (17). Here $\rho_{\text{co}}$ is the background density of coDM.

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**3 NON LINEARITIES IN THE EARLY UNIVERSE**

When entering the horizon, in the early Universe, coDM density fluctuations $\delta_{\text{co\,hor}} > 0$ exhibit an average amplitude $\bar{\delta}_{\text{co\,hor}} \sim 10^{-5}$, close to the top likelihood value for positive fluctuations, (supposedly) with a Gaussian distribution. Fluctuations with a greater horizon amplitude $F \times \bar{\delta}_{\text{co\,hor}}$ will be also considered below.

As shown in Figures 3 and 4, obtained from the linear program by Bonometto & Mainini (2014), in a synchronous gauge $\delta_{\rho}$ undergoes an uninterrupted growth, with a greater rate around horizon crossing. Then, when attaining the non–relativistic regime, $\delta_{\rho} \propto a^\alpha$ with $\alpha \approx 1.6$. It is so in spite of coDM being $\sim 1\%$ (or less) of the total density, in the radiation dominated epoch. This behavior is further illustrated in Figure 4, where we extend the plot of fluctuation evolution, so to approach $z = 0$, for all cosmic components. All that can be straightforwardly understood, on the basis of the newtonian limit of coDM dynamics, as discussed by Macciò et al. (2004) and Baldi et al. (2010), and resumed here below.

Such early growth is critical to allow for SCDEW spectra approaching $\Lambda$CDM, up to $k \sim 1$. At greater $k$'s, however, its effects appear excessive as, even for $\delta_{\text{co\,hor}} \approx \delta_{\text{co\,hor}}^*$ values $\delta_{\rho} \sim 0.1$ are attained earlier than “today”. When this occurs, the linear program yields unphysical outputs. One of the aims of this work is to show how to deal with such non–linearities.

The peculiar behavior of $\delta_{\rho}$, in the non–relativistic regime, can be understood if taking into account that coupling effects are then equivalent to: (i) An increase of the effective gravitational push acting between coDM particles, for the density fraction exceeding the horizon (at high $z$). The dotted line has a steepness $\alpha = 1.6$.

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**Figure 3.** Fluctuation evolution in the cosmic components at their entry in the horizon (at high $z$). The dotted line has a steepness $\alpha = 1.6$. 

**Figure 2.** Evolution of densities ($\rho_i$) in the 2 cosmologies discussed in detail in this work.
The self–gravitational push due to $\delta$, is then proportional to
\[
G^\prime \delta \rho \approx G \delta \rho / F \times \left(1 + 4G/3\right) = G \delta \rho / F \left(2/3 + 1/2F^2\right),
\]
(23) as though concerning the whole critical density $\rho_c$, although with an amplitude reduced by a factor (slightly exceeding) 2/3. We must add to that the extra push due to particle mass decline.

Such fast increase eventually leads $\delta$ into a non–linear regime. As a first step, to gain an insight on non–linear evolution, we can assume $\delta$ to be the amplitude of a spherical top–hat density enhancements. Real fluctuations approaching sphericity are surely rare, at least for $F \sim 1$, although becoming more likely as $F$ increases (see below). There are however significant conclusions we can draw from spherical dynamics, that we discuss in the next Section.

There, we prescind from the actual evolution, in principle, are obtainable in analogy to eq. (25), from the actual values of $\delta$, for wDM, and $\delta$, for baryons. Quite in general, $b$ values obtained in this way exhibit a variable sign with zero average, however being $\ll \delta$. Accordingly, we shall simply assume $b = 0$.

The possible impact of other components on top–hat dynamics is however strengthened, if we assume all of them to be safely non–relativistic. In this way we can compare the growth obtainable if neglecting other component gravity, vs. an over–modified growth, with changes exceeding those possibly due to the gravity of fluctuations in other physical components. On the contrary, the simultaneous growth of fluctuations in the artificially non–relativistic component is overestimated and looses much of its significance.

More in detail: when setting the initial conditions for $c$ evolution, a sphere of background materials with (comoving) radius $b = c$ overlapping the top–hat, starts to be affected. Initial conditions for $b$ evolution, in principle, are obtainable in analogy to eq. (25), from the actual values of $\delta$, for wDM, and $\delta$, for baryons.

In a previous paper, we dealt with $c$ evolution, by assuming $\Delta$, not to cause other component inhomogeneities. This may be a reasonable assumption during the very early expansion, but needs to be tested when we approach $\beta F$ fading and matter–radiation equality $\epsilon_\odot$.

As some of the key issues of this work arise from the analysis of such period, we need to deepen the question of other component involvement, and this point is one of the main technical contributions of this work.

A critical issue, however, is that a top–hat configuration, involving relativistic components, would be rapidly smoothed by particle velocities. If the warm DM mass $m_w \sim 100$ eV, derelativization occurs around $\Delta_w$ and, even afterwards (or for greater $m_w$), particle motions keep non negligible. Henceforth, a test on the effects of/other components would be intricate, if we strictly keep to the model.

The possible influence of other objects on top–hat dynamics is however strengthened, if we assume all of them to be safely non–relativistic. In this way we can compare the growth obtainable if neglecting other component gravity, vs. an over–modified growth, with changes exceeding those possibly due to the gravity of fluctuations in other physical components. On the contrary, the simultaneous growth of fluctuations in the artificially non–relativistic component is overestimated and looses much of its significance.

Figure 4. Fluctuation evolution in the cosmic components from their entry in the horizon until $\zeta = 100$, for model and scale indicated in the frame. Model parameters are selected so to cause an early coupled–DM non–linearity on such scale. Colors as in previous Figure.

Figure 5. $\delta$, evolution across the horizon and until the reach of non–linearity (for average amplitude fluctuations). At the bottom right, the $k$ values considered are listed.
At most times, it will be $b_n < b_{n-1}$, and interpolation is needed to gauge the action of a fraction of the $n$–th shell on the coDM sphere radius $c$.

This technical problem is strictly analogous to the one faced by Mainini (2005) and Mainini & Bonometto (2006), and is debated in Appendix A. Here, let us just outline that the actual variables used in dynamical equations are

$$x = c/\tilde{c}, \quad y_n = b_n/\tilde{c},$$

while the independent time variable will also be normalized at the initial time, by setting $\tau = \tilde{\tau}/\tilde{c}$.

In the early CI expansion, results are independent from $\tilde{\tau}$. Here, however, we extend the treatment to low $z$ values, our main results concerning times when $\beta_{ff}$ fades. Although dynamical equations are then formally $\tilde{\tau}$ independent, several coefficients enclosed in them exhibit a specific dependence on $\tilde{\tau}$.

In principle, results are more and more reliable when greater $N$ values are considered, so that outputs do not rely on the interpolation inside the $n$–the shell. Our tests however show that results are already stable for $N = 10$, as in Figure 6. Clearly, the deviations of $b_n$ from straight lines are just marginally appreciable.

This Figure, as well as the whole results on spherical top–hat evolution, are obtained for a model with $\beta = 8.35$ and $m_B = 100\text{ eV}$. Results will be later extrapolated to other couplings and asymptotic mass values.

In Figure 7 we then show the evolution of top–hat radii, and its dependence on the “initial redshifts” $z_{\text{in}}$, selected so that the linear growth yields then a coDM fluctuation amplitude $\delta_{f0} = 0.01$. As expected, $R/\bar{R}$, starting from unity, reaches a maximum value, and then re–decreases (for graphical reasons, in the Figures, $\bar{R}_c$, $\bar{R}_f$, etc., are replaced by $R_{\text{in}}, \tau_{\text{in}}$, etc.). The decrease, if we assume a never–violated spherical symmetry, stops only when re–approaching $R = 0$ and $\bar{R}$ yields a velocity approaching the speed of light; then, non–relativistic equations fail to work.

In top of each black curve a dashed red curve is also plotted, for the 4 greater $z_{\text{in}}$ considered. For the last value (4.70) we plot (in blue) $R/10$ instead of $R$. As expected, $R/\bar{R}$, starting from unity, reaches a maximum value, and then re–decreases (for graphical reasons, in the Figures, $\bar{R}_c$, $\bar{R}_f$, etc., are replaced by $R_{\text{in}}, \tau_{\text{in}}$, etc.). The decrease, if we assume a never–violated spherical symmetry, stops only when re–approaching $R = 0$ and $\bar{R}$ yields a velocity approaching the speed of light; then, non–relativistic equations fail to work.

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In top of each black curve a dashed red curve is also plot, stopping when the virialization conditions are attained. The reach of such conditions and the significance of red curves are discussed in the next sub–section.

For the model considered, Figure 7 confirms that, at large $z$, the growth and recollapse process is substantially independent from the initial redshift. A small, initial deviation from such self–similarities takes place when $\log(z_{\text{in}}) \ll 7$, being visible in our plot for 6.70. Deviations then become wider; for $\log(z_{\text{in}}) = 6.13$, e.g., a significant enhancement of $R$ evolution is appreciable.
Further examples, down to $\log(z_{a=0.01}) = 5.70$ and 5.42 then show further abrupt enhancements, with a duration of the whole process finally boosted by a factor $\sim 30$. This is however quite small, in comparison with the case $\log(z_{a=0.01}) = 4.70$ or still smaller initial redshift values. This last example is shown in the Figure by plotting $(R/R_0)/10$. In spite of that, the Figure frame is unsuitable to contain the whole evolution, leading to a top $R/R_0 \approx 1272$ when $\tau/\bar{\tau} \approx 369$, while the recollapse to zero takes place when $\tau/\bar{\tau} \approx 476$, i.e. at $z \sim 50$. For just slightly smaller $z_{a=0.01}$ values, full recollapse is not yet attained at the present time.

The evolution of the density contrast $\Delta_c$ is then strictly related to $R$ evolution. For the sake of example, Figure 8 shows the $\Delta_c$ dependence on $\tau$ for the 4 greatest values of $z_{a=0.01}$ in the previous Figure.

The evolution found depends on $\bar{\tau}$, when $\beta_{eff}$ decrease modifies the coefficient in dynamical equations. A visual confirm comes from Figure 1, where the location of the abrupt $\beta_{eff}$ decrease is shown.

Before concluding this Section we can also report our estimate on the relevance of non-coDM components, in the spherical growth analysis. When assuming them not to be involved in the dynamical process and to keep homogeneous, the growth rate found has a slight decrease. As expected, the discrepancy increases towards lower redshifts, but however keeps within $\sim 3-4\%$. Let us outline, once more, that this is an overestimate. Accordingly, results obtained by considering only coDM fluctuations are however significant, bearing more than a qualitative validity.

### 3.2 Virialization

As shown in Figure 7, an ideal top–hat expands and eventually re-contracts down to a relativistic regime, as indicated by the black curves. Since early treatments of top–hat evolution (Press & Schechter 1974), hereafter PS), it was outlined that minimal deviations from sphericity, scarcely mattering during expansion, become determinant during recontraction, so leading the system to virialization. In the case of coDM, there is a specific argument strengthening this expectation, which will be discussed in Section 6, herebelow.

In order to evaluate when virialization is approached, however need to evaluate the potential energy and the kinetic energy $T_{co}(R)$; we shall do by treating all components but coDM as homogeneous, so that the only relevant kinetic energy contribution reads

$$T_{co}(R) = \frac{3}{10} \frac{M_{co}}{R^2} \left( \frac{dR}{dt} \right)^2. $$

Being then $dR/dt = (1/a)dR/d\tau = \dot{a}/a \dot{c}$.\n
$$2 \times \frac{5}{3} \frac{T_{co}}{M_{co}} \frac{\dot{c}^2}{c^2} = \left( \dot{v} + \frac{1}{u} \right)^2, $$

with $'$ indicating differentiation in respect to $u$.

The potential energy is then made of two terms: (i) coDM self-interaction; (ii) coDM interaction with the background of all the other components. Therefore, in agreement with Mainini (2005); Mainini & Bonometto (2006), we obtain

$$U(R) = -\frac{3}{5} \frac{\left[ (M_{co} + \gamma \Delta M_{co} R^2 \right]}{R} - \frac{4\pi}{5} G \rho_{b,cd} R^2 = -\frac{3}{5} \frac{\Delta M_{co}}{R} - \frac{4\pi}{5} G \rho_{b,cd} R^2. $$

The related mass values

$$M_\rho = \frac{4\pi}{3} \left( \frac{2\pi}{k_F} \right)^3 \rho_{b,cd} \Omega_{\Delta_c} $$

(here $\rho_{b,cd}$ is the present critical density, $\Omega_{\Delta_c}$ is the present coDM 4.2 Density Contrast and Virial Radius

![Figure 9](image_url)  

**F=10^{1/2}**  

**F=1**

**Log(k/Mpc^{-1}h)**

**Log(\Delta_c)**

**Figure 9.** Upper plot: The dependence of virial density contrast on the redshift when $\delta_{co} = 10^{-2}$. Lower plot: Dependence on the scale $k$. and, from here, proceeding as we did for eq. (A2) in the Appendix, we finally obtain:

$$\frac{5}{3} \frac{U_{co}(R)}{M_{co}} \frac{\dot{c}^2}{c^2} = -h_2 \left[ \frac{x^3}{2} + \frac{3}{x} (\Delta_{co} - x^2) \right]. $$

Here $h_2 = (8\pi/3)G \rho_{b,cd} (\bar{t})^2$ deviates from unity when purely radiative expansion is abandoned. In turn, owing to eq. (21), $q = \gamma \Omega_{\Delta_c}/2 \equiv 1/3 + 1/(4\bar{t}^2)$ exhibits just a mild $\beta$ dependence. The virialization condition reads then

$$ax^3 + h_2^{1/2} x^2 - qh_2 (\Delta_{co} - x^2) - h_2 x^2/2 = 0. $$

From the $c$, and $\tau$, values fulfilling this equation, we then derive the virial radius $R_v = c_s a$. This procedure allows us to work out the virial density contrast $\Delta_{vir}$ for coDM fluctuations, as a function of the redshift when: (i) $\delta_{co}$ has a prescribed value, if, at the horizon, (ii) the actual fluctuation on that scale exceeded average by a factor $F$. In Figure 9 such dependence is shown for $F = \sqrt{10}$ and 1. The peak of the density contrast in coDM, $\Delta_c$, occurs at $k \approx 21.6 \text{Mpc}^{-1}$, for $F = \sqrt{10}$ ($\approx 10.4 \text{Mpc}^{-1}$, for $F = 1$).

The related mass values

$$M_F \equiv \frac{4\pi}{3} \left( \frac{2\pi}{k_F} \right)^3 \rho_{b,cd} \Omega_{\Delta_c} $$

(MNRAS 000, 1–15 (2018))
density) are $M_P \approx 3.37 \times 10^9 M_⊙ h^{-1}$ for $F = 1$ and $M_P \approx 2.84 \times 10^9 M_⊙ h^{-1}$ for $F = \sqrt{10}$. According to eq. (32), here we are taking into account only the mass $m_\nu$ for coDM particles. Furthermore, the infall of other components during the spherical growth is also neglected. This could imply an error up to $\sim 5\%$; the most significant matter infall, however, is expected to occur at low $\zeta$’s (see also the Discussion Section).

Let us also outline that average fluctuations entering the horizon are unlikely to approach a spherical geometry. Sphericity becomes more and more likely as $F$ increases. However, fluctuations with greater $F$ are rarer. Here below the competition between the two effects is further discussed. Here, let us rather outline that if, e.g., we take $F = 10$, the time taken by the fluctuation to reach an amplitude $\sim 0.01$ is shorter; accordingly we are referring to a wider horizon and a greater mass scale. In the early expansion, the mass scale also exhibits a clear dependence on $\beta$, as $\Omega_m = 1/2\beta^2$. Accordingly if, e.g., $\beta$ is doubled, the peak corresponds to a mass smaller by a factor $\sim 0.25$. This effect weakens when $\beta_{eff}$ is significantly below $\beta$. In the Discussion Section we shall further comment on the reason why the model(s) used here were selected and how far the parameter choice can be relaxed.

4 AFTER VIRIALIZATION

In previous Sections, the approach to virialization is treated in a schematic way. As a matter of fact, to settle in virial equilibrium, the top–hat needs that (tiny) deviations from full homogeneity existed since the beginning. This requirement, however, is not different from what Press & Schechter (1974) claimed to occur, after recombination, for the evolution of a top–hat fluctuation in baryons and Dark Matter.

There is however a critical difference between their case and the present context. Bonometto, Mezzetti & Mainini (2017) and Bonometto & Mainini (2017a), in fact, showed that the vanishing of the virial requires that the average momentum $p_{eff}^2 \approx \gamma GN_\nu m_{eff}(\tau_c)/R_c$ ;

$$p_{eff}^2 \approx \gamma GN_\nu m_{eff}(\tau_c)/R_c ;$$

here $N_\nu$ is the total number of coupled–DM particles, yielding a total mass $N_\nu m_{eff}$ within a volume of size $R_c$. Also Press & Schechter (1974) require a similar condition, but here $m_{eff}$ exhibits a time dependence.

In their case, oscillations around virial equilibrium may occur, while the (conserved) average particle momentum ($p_r$) remains the momentum yielding virial equilibrium. On the contrary, here until $\beta_{eff}$ is large, $p_r$ soon exceeds the equilibrium momentum and particles with kinetic energy $p_r^2/2m_{eff}$ are able to evaporate.

If one assumes a maxwellian distribution, it has been shown that the fastest particles, while evaporating sooner, are however unable to produce an average momentum decrease sufficient to recover a temporary virial equilibrium. Accordingly, systems virializing with a density contrast $\sim 28$–30, at high $\zeta$, evaporate within the very crossing time

$$t_{cross} = 2t_h(\Delta/\Delta_h)^{5/2} \sim 0.7 t_c .$$

Here $t_c$ is the time when virialization is achieved. This result however opens another question: If the (conformal) time when the virialization condition is attained is $\tau_c$, when does the growth really stop? If this very stop is expected to take place when the (conformal) time has grown enough to allow a full crossing to occur, i.e. at a time $\sim 1.3 \tau_c$, we shift from $\tau/\tau_c \approx 24$ to $\tau/\tau_c \approx 31.3$. As full collapse, if sphericity is not violated, is expected when $\tau/\tau_c = 28$, does this mean that any spherical fluctuation is doomed to turn into a BH? Small violations from sphericity could however prevent to occur and, in the next Sections, we actually discuss on the a–sphericity needed to stop the collapse. However, if we suppose that a nearly spherical fluctuation indeed virializes at a time $t_c$, we just discovered that, afterwards, the escape momentum decreases too rapidly. Henceforth, at the end of the growth, either a BH is formed or no trapping effect is possible; particle simply flow out from the overdensity within a time $\sim 0.7 t_c$.

As soon as $\beta_{eff}$ weakens, however, the balance between the two options is expected to change, as the main mass component is the mass $\mu$ acquired at the higgs scale, so that coDM particle mass decrease has a stop. For spherical systems reaching the virialization condition with density contrast $\sim 450$–480, close to the maximum, therefore, the dissolution option seems excluded. They surely may loose a part of their mass, but the likelihood that contraction continues towards BH formation becomes huger, also because the very ratio between virialization and full collapse (conformal) times decreases from $\sim 1.17$ to $\sim 1.07$–1.08.

5 STABILITY OF SPHERICITY ASSUMPTION

The critical issue, therefore, seems to be sphericity and its stabilization during the contraction stages. The PS approach, when ordinary cold DM and baryons are involved, clearly forgets all hydrodynamical effects, which could be sufficient to modify purely gravitational predictions during the contraction stages. In particular, they would act on any substructure, even if the physical fluctuation is spherical. Hydrodynamics is not the only reason why a PS collapse is unstable, but its action is surely unavoidable.

A first significant difference, in the coDM case, is that baryons, if involved, keep far behind coDM evolution; when coDM fluctuations overcome their top expansion, baryons or other components are still timidly hinting a linear growth. As we saw in detail, forgetting any other component besides coDM is then a fair approximation.

There is however a peculiar feature of coDM dynamics, which could decisively contribute to sphericity disruption. To this end, let us compare the two components of the force slowing down the growth of a density excess, in respect to cosmic expansion, and then causing recontraction. According to eq. (A1), in Appendix A, the gravitational force reads

$$F_\sigma = -\frac{G \Delta M_{co}}{ac^2} ,$$

with a suitable expression for $\Delta M_{co}$, taking into account the boosted gravity; this force is directed towards the gravity center. In top of that, we have an extra push

$$F_p = c\Phi \dot{c} ,$$

directed as particle velocities. In average, for a spherical density enhancement, both forces are indeed radially directed.

We can however use the solutions of dynamical equations, to compare them, once normalized as in eq. (A3) of Appendix A. This is done in Figs. 10 or 11 if the redshift when $\delta_{co} = 10^{-2}$ is $10^5$ or $10^{3.5}$, i.e. for fluctuation scales reached by the horizon either when CI expansion is still going on or during the onset of the $\beta_{eff}$ decrease due to higgs screening.

These plots start at the time when $\delta_{co} = 10^{-2}$ and end when recontraction approaches the speed of light, so indicating that the non-relativistic regime is over. Quite in general, they make clear
Figure 10. Boosted gravity and extra push, for top–hat fluctuations starting with $\delta_{co} = 10^{-2}$ at $z = 10^9$, when still in the high–$z$ regime.

Figure 11. Boosted gravity and extra push, for top–hat fluctuations starting with $\delta_{co} = 10^{-2}$ at $z = 10^5$, i.e., in the proximity of the peak in Fig. 8. The lower 10% of the upper plot is magnified in the lower plot.

Figure 12. Probability for the ratio $c/a$ (maximum/minimum ellipsoidal radius) to keep below 4 possible fixed limits, for increasing $F$ values. Each estimate includes a 2–$\sigma$ errorbar. The vertical dotted line is for $F = \sqrt{10}$.

that extra push intensity can approach self–gravity. As is expected, Figures 10 and 11 describe fairly different evolutions, also because the very top virial density contrasts are different.

In fact, all through Figure 10, $\beta_{eff}$ keeps close to $\beta$ and the mass acquired by the $\psi$ field at the higgs scale keeps negligible. Enhanced gravity and extra push then keep similar intensities all through the process.

If (possible) sphericity violations are mild, the two forces, directed towards the gravity center and in the velocity direction, act coherently. Possible $a$–sphericities, however, would make these forces discrepant, and this is a fair reason, intrinsic to coDM nature, bursting deviations from sphericity, even though initially small. If we suppose that coDM feels no other effective force apart these ones, we then face a precise quantitative problem: how large deviations from sphericity are “tolerable”?

Figures 11 then describe a growth leading to a top density contrast $\sim 500$. Here, the extra push is significant only at the very beginning; then, it even exceeds self–gravity. Both forces then fade, after yielding a strong initial headway, as $\beta_{eff}$ becomes significantly smaller than $\beta$: gravity, then, is no longer enhanced as the $\psi$ field approaches a constant mass. Clearly, the transition from variable to constant mass occurs while expansion is running. It is however clear that, if sphericity is not disrupted during the initial stages, the mismatch between gravity and extra push directions should not be a possible cause for a later mixing up, leading to virialization.

Assuming that deviations from sphericity are similarly distributed, for the two scales considered, it seems then clear that, in the latter case, contraction towards BH formation is favored. In Appendix B, we debate deviations from sphericity as a function of $F$.

As expected, the fraction of fluctuations with a quasi–spherical shape increases with $F$ (see Figure 12). The likelihood of fluctuations with assigned $F$ is however expected to decrease as $\exp(-F^2/2)$. For instance, fluctuations with $F = \sqrt{10}$ are $e^{-5} \approx 10^{-12}$ times less frequent than fluctuations with $F = 1$. In turn, the fraction of fluctuations with $c/a > 0.99$ ($a > b > c$ are the 3 major axis of the fluctuation) increases by a factor $\sim 10^{17}$. The (somehow unexpected) result of this comparison is that “spher-
ical” fluctuations with m.s.a. amplitude are much more frequent than fluctuations whose amplitude exceeds average.

It should be also outlined that the requirement that $c/a > 0.99$ is only a necessary condition for the fluctuation to approach a spherical geometry. See Appendix B for further details on this point.

6 PANCAKES

Fluctuations approaching a spherical geometry are however an exception. Let us then approach the question of how typical fluctuations in coDM evolve, by considering the coDM density distribution $\rho_{\alpha}(\tau, r) = m(\tau) \times n(\tau, r)$ in the neighborhood of a given point $x = ar$, whereabouts an overdensity is located. Once the principal ellipsoidal axes are determined, we can face the problem in a Zel’dovich approximation, by requiring that

$$ n(r) a^3 \sum i [1 + \alpha_i G(a)] = \text{const.}; \quad (37) $$

here $\alpha_i$ are negative coefficients, the product being extended over the 3 ellipsoidal axes. The linear algorithm tells us that $G \propto a^d$ and we shall keep to the case $d = 2.6$; being $\delta_{\alpha \alpha} = \sum_i \alpha_i \propto a^{-2.6}$, we soon obtain a fair translation from the eulerian to the lagrangian picture.

The turnaround time, when a coefficient $[a + \alpha_i a^{2.6}]$ shifts from increase to decrease, is obtainable by requiring

$$ \frac{d}{d\tau} [a + \alpha_i a^{2.6}] = \dot{a} \times [1 + 2.6 a_i a^{1.6}] = 0, \quad (38) $$

so obtaining

$$ a_i a^{1.6} = -0.385 \quad \text{i.e.} \quad a = \left[ \frac{1}{2.6(-a_i)} \right]^{0.25} \quad (39) $$

and

$$ a + a_i a^{2.6} = a \times 0.615; \quad (40) $$

accordingly, for a density contrast $\Delta_{\alpha \alpha} \simeq 1/0.615 = 1.6$ and a fluctuation amplitude

$$ \delta_{\alpha \alpha}(\tau_{\alpha \alpha}) \simeq 0.6, \quad (41) $$

still almost linear, turnaround occurs. Non–linearity corrections, therefore, can be expected to be small.

In a standard Zel’dovich approach, the successive compression is expected to burst pressure, so causing fragmentation. On the contrary, coDM knots are pressureless and their particles are (almost) noninteracting. Rather, we expect a key role to be played by the mismatch between enhanced gravity and extra push directions, acting towards the ellipsoid center and, in average, along the contraction axis, respectively. Accordingly, kinetic energy is preserved but particle velocities are randomly diverted so that a local virial equilibrium can be easily approached, and the growth stops. After a short while, then, as in the spherical case, evaporation starts and the whole fluctuation energy is dissipated into heat.

Notice that previous evaluations referred to $n(r)$ rather than $\rho_{\alpha \alpha}(r)$. For CDM or baryons the two options would be equivalent. For coDM, on the contrary, until the particle mass $m_{\alpha \alpha}(\tau) \propto \tau^{-1}$, we must refer to (conserved) number densities.

One dimensional virial equilibrium however requires that the average particle momentum is

$$ \langle p^2 \rangle \simeq \gamma GN_{\alpha \alpha} m_{\alpha \alpha}^2 \tau_{\alpha \alpha}^2 / R. \quad (42) $$

Here we can take $R \sim (a - |a_i a^{2.6}|) 2\pi/k$ to be the thickness attained

![Figure 13. Transfer functions of SCDEW (parameters in the box) and ACeM compared at $z = 0$.](image)

by the growing fluctuation, $N_{\alpha \alpha}$ is the number of coDM particles involved in the growth.

In the spherical case, the equilibrium set by eq. (33) is reached when $a \simeq 1.4 a_{\alpha \alpha}$ and $R$ has decreased by a factor $\approx 0.52$ to reach a density contrast $\sim 27–28$. (For the sake of comparison, in the PS case the scale factor increased up to $a \simeq 4 a_{\alpha \alpha}$ and, namely, the virial density contrast is $\Delta \sim 180$.) The density fluctuations of coDM particles, therefore, undergo both a fast growth and a rapid virialization, followed by dissolution.

In the one–dimensional picture, while the $\delta_{\alpha \alpha}$ growth is reasonably approximated by linear estimates, until turning around, the expected law $\delta_{\alpha \alpha}(\tau > \tau_{\alpha \alpha})$ is admittedly hard to predict.

7 TRANSFER FUNCTIONS

Accordingly, it is uneasy to predict the effects of coDM non–linearities on other components. This is a critical issue, for coDM non–linearities leave an imprint on baryons and wDM distributions. As a tentative option, we model the effects of $\delta_{\alpha \alpha}$ on other components, by letting it evolve according to the linear algorithm until $\delta_{\alpha \alpha} = 3$, at a time $\tau_3$, and assuming a later exponential decay, due to post–virial dissolution; e.g.:

$$ \delta_{\alpha \alpha}(\tau > \tau_3) = \delta_{\alpha \alpha}(\tau_3) \exp[-(\tau / \tau_3 - 1)^{\beta}] \quad (43) $$

(here we selected $\alpha \simeq 1.5$; $n$ is then derived from continuity requirements at $\tau_3$). Here, a virial density contrast $\Delta_{\alpha \alpha} \simeq 4 \approx 2.5$ times the turn–around density contrast) is assumed, which could be an underestimate; that the time needed to start dissolution is the time needed to reach $\delta_{\alpha \alpha} = 3$, on the contrary, could be an overestimate. Until more precise results are obtained through ad–hoc simulations, the best we can do, probably, is assuming a mutual compensation of these approximations.

A modified cambfast program, allowing us to predict transfer functions and spectra, when making these (or close) assumptions, has also been produced. A discussion of the (slight) spectral dependence on parameters (one of them is $\alpha$, here set to 1.5) enclosed –or to be added– in the expression (43), will be provided in a forthcoming paper.

For the sake of example, in Figure 13 we compare the transfer
function of a SCDEW model, with $\beta = 8.35$ and $m_\nu = 100$ eV, with $\Lambda$CDM, at $z = 0$. Model parameters ($\Omega_\text{s}, h, n_\nu$, etc.) are the same. For the two cosmologies, coinciding with those assumed in the NI-HAO simulation set (e.g., Wang et al. (2015); Tollet et al. (2016); Dutton et al. (2016)). The coDM spectrum exhibits significant deviations from $\Lambda$CDM. But one should remind that it mostly accounts for $1/2\beta^2 = 0.7\%$ of the critical density. Its contribution is however taken into account in the total spectrum (blue curve), exhibiting deviations $< 10\%$ up to $k \approx 30 h$ Mpc$^{-1}$, i.e. up to $\sim 10^{10} M_\odot h^{-2}$, well inside the non–linear spectrum; high–$z$ data fitting could be a more serious test for this model. Most computations in this paper were however based on this model and, in the discussion Section, we recall why it was selected.

In general, however, SCDEW spectra are increasingly closer to $\Lambda$CDM when greater $\beta$ and $m_\nu$ are selected; e.g., in the Figures 14, the model with $\beta = 37$ and $m_\nu = 1$ keV is considered. At $z = 0$, this model almost overlaps $\Lambda$CDM, with a tiny lack of power at $k \approx 10 h$ Mpc$^{-1}$ and some power in excess for $k > 50 h$ Mpc$^{-1}$; shifts are however within $4-5\%$. At $z = 100$, discrepancies are slightly more significant. The coDM component, in particular, has still some extra power at large $k$’s, while baryon oscillations are visible on the baryon component only.

Let us recall that wDM particle masses, in these models, are 100 or 1000 eV. With such masses, a WDM model transfer function exhibits a cut around $k = 1$ or $10 h$ Mpc$^{-1}$, i.e. $\sim 10^{14}$ or $10^{15} M_\odot h^{-2}$, respectively. The role of coDM in rising high–$k$ spectral components is therefore essential.

This takes us back to the problem of improving the approximation in eq. (43). Let us however outline that only wide changes in it would substantially modify the similitude between $\Lambda$CDM and SCDEW at high $k$. Furthermore, the basic issue is that, when coDM effects on baryons and wDM of coDM are evaluated, two possible traps are to be avoided: (i) Using a standard linear program, letting $\delta_{co}$ grow forever, as thought linear equations held also when $|\delta_{co}|$ approaches and exceeds unity. (ii) Testing the effects on baryons and wDM due to a spherical top–hat (or similar) coDM fluctuation growth (and dissolution). Figure 12 shows in detail how rare are spherical fluctuations, so that the story of their evolution is unrelated to average fluctuation evolution. An important point, when trying to improve the expression (43), is also accounting for the contribution of negative fluctuations.

Let us however finally outline that no similar problem exists when transfer functions are evaluated in most different cosmologies. Early nonlinearities are a specific feature of SCDEW. In turn, it is thanks to them that SCDEW can open new perspectives for early BH formation.

8 DISCUSSION

SCDEW models have the ambition to challenge $\Lambda$CDM as concordance cosmology. More precisely: they aim to show how $\Lambda$CDM features can be recovered, by avoiding fine tuning and coincidence problems.

If $\Lambda$CDM is “just” an excellent effective model, we expect that the underlying physics will carry along new parameters. SCDEW involving extra parameters, therefore, is hardly a point against it. The true points is whether, by fixing them, we recover $\Lambda$CDM or even go beyond it, not only avoiding logical conundrums but also fixing some open quantitative questions.

The $\beta$–coupling of SCDEW models could be seen as a remnant of the interaction between a primeval inflatonic field and other cosmic component. When the higgs scale is attained, however, this residual force almost vanishes, without invoking any ad–hoc mechanism. Moreover, one does not need to assume a specific $V(\Phi)$ expression to account for $\Phi$ self–interaction; it is sufficient to indicate the redshift $z_{fp}$ when $V(\Phi)$ starts to exceed $\Phi$ kinetic energy density. Accordingly, $z_{fp}$ tuning just replaces the tuning of the DE density parameter $\Omega_\Lambda$.

More severe constraints on $V$ expressions might derive from its role in inflation. Mutual constraints between inflationary dynamics and $\Omega_\Lambda$ could arise from that. This is clearly still an open point.

In a SCDEW cosmology, all cosmic components keep significant densities all through cosmic expansion, including inflation and today. Admittedly, there is an exception: baryon density and its being comparable with other densities; an open question shared with any other cosmological scenario.

These arguments could be made in support of SCDEW cosmologies even in earlier analysis. Two new points were added here, both related to early coDM non–linearities: (i) How to treat their effects on other cosmic components, so to approach a prediction on
high-κ SCDEW spectra. (ii) A possible role of SCDEW dynamics in producing large scale Black Holes.

8.1 SCDEW high-κ spectra

The high-κ range corresponds to mass scales below the average galaxy mass scale. Being highly non linear today, the main tool to constrain predictions are high-z observations.

First of all, here we outline that SCDEW rises a technical “problem”, that no other cosmology faces. Then, we provide an approximate analytical solution. Admittedly, it is not yet fully satisfactory. Two specific “traps” are however to be avoided: (a) A standard linear program, in this κ-range, lets δ_c grow forever, even when |δ_c| approaches and exceeds unity: extending linear results on such κ-range is surely wrong and, at first sight, might seem an under–evaluation of its effects. (b) Testing coDM effects on baryons and wDM by using a spherical fluctuation growth (and dissolution) is also wrong: for average amplitude fluctuations, spherical fluctuations are a black swan.

The approximated expression suggested here, which can be better fixed by future analysis, is a conceptual step forward in respect to the options (a) or (b). In particular, it shows that an extension of “linear” program results is not a sort of under–evaluation, it is just plainly wrong.

8.2 Early BH formation

As far as the (ii) point is concerned, let us recall that recent observations at z > 6 (see, e.g., Fan et al. (2006); Willott et al. (2007); Mortlock et al. (2011); Venemans et al. (2015); Wu et al. (2015); Banados et al. (2014, 2015, 2018); Matsuoka et al. (2016); Tang et al. (2017); Chehade et al. (2018)) do reveal that SMBH’s (super–massive BH) already existed when the cosmic age was ∼ 10^7y. Their masses approach or exceed 10^9M_⊙ and various attempts have been made to justify their existence. In the standard approach (see, e.g., Volonteri (2010); Volonteri & Bellovay (2012); Haiman (2013); Latif & Ferrara (2016)) SMBH are tentatively explained, by assuming the existence of BH seeds, with mass ∼ 400 M_⊙, since z ∼ 15. They should then coalesce or be subject to accretion. More in detail, the most popular options are: (i) The so–called DCBH (Direct Collapse Black Hole), i.e., the collapse of a protogalactic gas cloud, metal free. (ii) The core collapse of ultra–massive stars. (iii) The collapse of dense nuclear star clusters.

None of these options is devoid of problems, as well as the very formation of their seeds. Much work is in progress in this field and it is even possible that each one of above options is viable, in suitably different contexts.

Naively, the direct formation of BH’s over scales M_{BH} ∼ 10^8 M_⊙ could be the best product a deus ex machina could provide. The scale M_{BH} is however set by the assumption of a wDM particle mass ∼ 100 eV, therefore neglecting the primeval time dependent mass scale μ exp(CF) ∝ τ^-1. At the time when PBH might have formed, however, the latter mass component was still non–negligible. When enclosed in a BH, coDM particle masses might still evolve in time; but time coordinates inside and outside BH’s are different; altogether, it is unclear how the observed BH mass could evolve in time. This point does not modify the expected o.o.m. of BH masses, but could change their values up to some 10%’s. Another reason why BH mass could be greater is the accretion of baryons and wDM during the late collapse stages. This option was however enclosed in the computation of this paper, and causes a correction never exceeding a few percents.

More severe problems could be caused by accretion in the epoch between recombination and reionization due to Pop III stars. Not so much because of the amount of matter then accreted, but because of the radiation emitted, which risks to cause an early reionization. Clearly, this point requires a more detailed analysis, that we postpone to further work.

Let us however keep to the mass at BH formation, by taking into account just m_H masses. We then considered a sequence of SCDEW models, starting from β = 8.35 and m_H = 100 eV, and simultaneously increasing β and m_H, so to keep close values for ρ_c and ρ_co during the early CL expansion. For m_H = 100 eV, it is ρ_c/ρ_co = 0.9. Another model we consider in some detail is m_H = 1 keV, yielding a similar primeval density ratio.

This assumption is in agreement with our conjecture that wDM and coDM own a strictly related origin being, somehow, the β–charged and neutral states of the same particle. Of course, instead of a ratio 0.9, we could fix any value close to unity. Results are however just marginally modified by any such variation.

We then try to apply eq. (1) to find the dependence of M_F (and, therefore, M_{BH}) on m_H, by taking into account the corresponding values of Ω_{co}. Such values are obtained from the background program, whose results are plotted in Figure 15 (black curve with round circles). In the same Figure we also plot the m_H dependence of k_1 (cyan curve), as well as the resulting dependence on m_H of M_{BH} (black curve). Overimposed onto it a dotted red line yields the expression (1).

Let us now outline that the choice of a model with m_e ∼ 100 eV was not casual. In previous work, it allowed us to ease problems that N–body simulations of ΛCDM exhibited, typically below the galactic mass scale: the abundance of MW and M31 satellites, the flattening of dwarf profiles, and also, possibly, the distribution of concentrations. N–body simulations run by Macciò et al. (2015) confirmed such expected result.

Hydro simulations of ΛCDM, as those run for the NIHAO program (see, e.g., Wang et al. (2015); Tollet et al. (2016); Dutton et al. (2016)), showed that these very problems can find a solution within the ΛCDM paradigm. This however requires a tuning of specific baryon physics parameters, while the concentration distribution still exhibits some problem vs. existing (loose) data. Accord-
ingly, however, the alternative of DM being warm has lost much of its appeal; hydro simulations of SCDEW are still being planned, to test whether a small $m_w$ risks to over–solve previous ΛCDM problems, below the average mass scale, as well as to verify the effects of higher baryon spectra, in respect to DM ones, as predicted by low–$m_w$ SCDEW cosmologies.

Within this context, models with greater $m_w$ values are an open option. In order to obtain BH seeds with a mass $\sim 400 M_\odot$, a value of $m_{th} \sim 2$ keV should be favored. If one prefers not to invoke a later accretion at the Eddington limit, a value $m_{th} \sim 1$ keV might be preferable.

According to PS or similar mass functions, at low mass scale, the ratio between the expected numbers of systems forming with different masses $N(> M_1)/N(> M_2) \sim M_2/M_1$. Let is then evaluate this ratio for $M_1 \approx 400 M_\odot$ and $M_2 \approx 4 \times 10^8 M_\odot$ (a minimal galaxy scale); we find that the number of candidate BH seed, for each galaxy, is obtainable by multiplying the ratio $\sim 10^8$ by the $c/\sigma > 0.99$ likelihood -- $2 \times 10^{-4}$. In this case, we therefore obtain 200 “possible” PBH per galaxy. This kind of estimate allows us to rise $n_{th}$ up to $\sim 1$ keV.

Of course, for such a wider number of expected PBH, the question of them causing early reionization is to be deepened. In this paper we show that their spectra are close to the $\sim 1$ keV , SCDEW predicts PBH in the $10^{-5} - 10^{-3} M_\odot$ range. In principle, their expected abundance is consistent with the number of “seeds” yielding observed BH’s at the center of early and/or late galactic systems.

9 CONCLUSIONS

SCDEW models are likely to be the possible physics underlying the successful ΛCDM paradigm, eliminating all conundrums due to Λ and even allowing for the same field being both DM and inflaton. In this paper we show that their spectra are close to ΛCDM also in the high–$k$ range.

The main point made here is that, in top of that, SCDEW predicts primordial BH with a mass–scale up to $\sim 10^{-5} - 10^{-3} M_\odot$, a value reached if present DM particles have a mass $\sim 100$ eV. The recent success of ΛCDM hydro simulations, in explaining data previously met by SCDEW N–body simulations with such $m_w$, allows for $m_w = m_{th}$ values exceeding 100 eV. For instance, if $m_w \sim 1$–2 keV, SCDEW predicts PBH in the $10^{-5} - 10^{-3} M_\odot$ range. In principle, their expected abundance is consistent with the number of “seeds” yielding observed BH’s at the center of early and/or late galactic systems.

ACKNOWLEDGMENTS

Francesco Haardt is thanked for wide discussions, namely on the question of early cosmic reionization. We also wish to thank Javier Rubio for discussions.

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APPENDIX A: EVOLUTION OF OVERDENSITIES

In order to follow the evolution of spherical top–hat overdensities, we made use of a system of equations obtained by perfecting the approach followed in Bonometto, Mezzetti & Mainini (2017); Bonometto & Mainini (2017a).

As done there, let c be the comoving top–hat radius of a coDM overdensity. As explained in Section 3, however we try to take into account other components as well. To this aim, we consider a set of N concentric shells of wDM (w) and baryons (b) with comoving radii $b_n$ ($n = 1, ..., N$) such that $b_n(\tau) = c(\tau)$ at the initial (conformal) time $\tau$ (here below, all “tilded” quantities will refer to the initial time $\tau$).

In strict analogy with Mainini (2005) and Bonometto & Mainini (2017b), the evolution equations for the radii read:

\[ \dot{c} = -(1/a) (\dot{a} - C\Phi) \dot{c} - G \frac{\Delta M_{bw}}{a c^3} \]

\[ \dot{b}_n = - \frac{a}{\dot{a}} b_n - G \frac{\Delta M_{bw}}{ab_n} \]  

(A1)
where:
\[ \Delta M_{\text{w}} = \gamma \Delta M_{\text{w}}(<c) + \Delta M_{\text{w}}(<c) + \Delta M_{\text{f}}(<c) \]
\[ \Delta M_{\text{b}} = \Delta M_{\text{b}}(<b) + \Delta M_{\text{w}}(<b) + \Delta M_{\text{f}}(<b) \]
and
\[ \Delta M_r(<r) = M_r(<r) - (M_r(<r)) = \frac{4\pi}{3} \rho_s \Omega_{\Delta} (\Delta - 1) a^3 r^3 \]  
(A2)
is the mass excess of the \( i \) component within a sphere of comoving radius \( r \), \( M_r(<r) \) and \( (M_r(<r)) \) being the actual mass and the average mass respectively.

Let then \( x = c/\bar{c}, y_b = b_0/\bar{c} \) and \( \tau = \tau/\bar{\tau} \); the above equations can then be recast in the more convenient form, suitable to numerical integration:
\[ x'' = -h_0 x' - \left[ y h_{1,\text{w}} (\tilde{\Delta}_{\text{w}} - x^3) + h_{1,\text{b}} (\tilde{\Delta}_{\text{b}} - x^3) \right] \frac{1}{a^3} \]
\[ y'' = \frac{d' a}{a} \left[ y h_{1,\text{w}} (\tilde{\Delta}_{\text{w}} - y^3) + h_{1,\text{b}} (\tilde{\Delta}_{\text{b}} - y^3) \right] \frac{1}{a^3} \]  
(A3)
here \( \cdot' \) indicates differentiation with respect to \( u \).

The coefficients \( h_0 = \frac{d' a}{a} - C \Phi' \), \( h_{1,\text{w}} = \frac{1}{2} \Omega_{H_2}, h_{1,\text{b}} = \frac{8 \pi}{3} \bar{\rho}_s a^2 \tau^2 \),

while the total density contrast for wDM and baryons:
\[ \Delta_{\text{w}} = \frac{\Omega_{\Delta} + \Omega_{\Delta_{\text{b}}}}{\Omega_{\Delta_{\text{b}}}} \Omega_{\Delta_{\text{w}}} \], \[ \Delta_{\text{b}} = \frac{3}{\Omega_{\Delta_{\text{w}}} + \Omega_{\Delta_{\text{b}}}} \Omega_{\Delta_{\text{w}}} \frac{h_{1,\text{w}}}{h_{1,\text{b}}} \]

which initial value can be obtained in analogy to (24) for coDM.

It is worth mentioning that the dependence on \( \Phi \) of the results is carried by the time dependence of the dynamical coefficients \( h_0, h_{1,\text{w}} \) and \( h_{1,\text{b}} \). In the early CI expansion they keep constant and results do not show any dependence on \( \tau \); this regime is however abandoned as soon as the higgs screening becomes efficient.

**APPENDIX B: SPHERICITY LIKELIHOOD**

The scope of this Appendix is determining the expected \( a \)-sphericity of fluctuation entering the horizon, as a function of the factor \( F \) by which the fluctuation amplitude exceeds average.

The average \( a \)-sphericity is expected to decrease with \( F \). However, the fluctuations characterized by a given \( F \) exhibit an \( a \)-sphericity distribution. If we fix suitable \( a \)-sphericity thresholds \( x \), our aim is finding which fraction of fluctuations lay inside \( x \), as a function of \( F \).

To do so, we follow a pattern close to Peacock & Heavens (1985), adding the extra ingredients needed to reply the above question. Another important work, in this field, is Bardeen et al. (1986).

In the proximity of a maximum, the density fluctuation \( \delta(r) \) can be approximated, to the second order, as
\[ \delta(x_1, x_2, x_3) = \delta_0 + \frac{1}{2} (8\delta_0^3 - 3\delta_0 x_2^2 + 3\delta_0 x_3^2 + \delta_0^2 x_1^2) \]  
(B1)
Here \( \delta_0 \) is the density contrast at maximum, while the second derivatives \( \delta''_0 = \partial^2 \delta / \partial x_i^2 \) (\( i = 1, 2, 3 \)), in the directions of the principal axes of the fluctuation, are evaluated at its maximum (such \( \delta''_0 \) must be negative to have a confined fluctuation). The three semi-axes \( a_i \) of this ellipsoidal density distribution read then
\[ a_i^2 = -\frac{\delta_0}{\delta''_0} \]  
(B2)
If \( \delta \) is fully unsmoothed, when extending our sight far from the maximum, the shape of the fluctuation shall hardly be given by eq. (B1). Being interested in the fluctuation symmetry on the horizon scale \( R_H \), therefore, we need to assume \( \delta \) to be suitably smoothed, so hiding detailed features on scales \( \lesssim R_H \).

Accordingly, our result concerns the basic symmetry of the whole fluctuation and any information concerning its profile is lost. Admittedly, they can be important in defining the expected evolution of fluctuations, after entering the horizon (see, e.g., Germani & Musco (2003)).

Furthermore, if we suppose to proceed through a gradual increase of the smoothing ratio, to finally reach a scale \( \sim R_H \), we must expect that the directions of the principal axes can (gradually) rotate in space. This means that, even when we expect \( \delta(r) \) \( a \)-sphericity not to exceed a suitable limit, for the fluctuation taken as a whole, some \( a \)-sphericity may exist inside the structure, with compensations among different scales \( \lesssim R_H \).

As we aim to correlate sphericity to virialization, we have to bear in mind that, at any \( a \)-sphericity level, inner structures may frustrate the effects of an apparent overall sphericity. In spite of that, it is clear that the basic feature to discriminate between virialization or total collapse is the overall sphericity level.

Let us then outline that, for a gaussian noise, \( \delta_u \) and \( \delta''_u \) (\( i = 1, 2, 3 \)) follow a multivariate normal distribution, whose covariance matrix elements are given by Peacock & Heavens (1985), and depend on
\[ \sigma_n^2(R) = \frac{1}{2\pi} \int_0^{\infty} k^{2(n+1)} P(k) W^2(k R) dk \]  
(B3)
with \( n = 1, 2, 3 \) (let us outline that these are not vector components). Here \( R \) is the smoothing scale (close to the Hubble radius \( R_H \)), \( P(k) \) is the spectrum of fluctuations, \( k \) being the wave number, while \( W(kR) \) is the window function used.

For the sake of simplicity, let us then assume that fluctuation entering the horizon have a spectral index \( n_i = 1 \). This forces us to adopt a gaussian window, to avoid \( \sigma_n^m \) possible divergences, in spite of dynamical eluations being based on a top–hat profile. According to the definition (B3) it shall be
\[ \sigma_n \propto R^n \]  
(B4)
so that the density field exhibits an \( R \)-independent standard deviation \( \sigma_0 \).

To generate a column vector \( \mathbf{u} \) whose elements are \( \delta_{\Delta,\Delta_{\text{b}},\Delta_{\text{w}},\Delta_{\text{w}}}, \delta_{\Delta,\Delta_{\text{b}},\Delta_{\text{w}},\Delta_{\text{w}}} \), we use Cholesky decomposition of the covariance matrix \( [\mathbf{V}] \) to get \( \mathbf{u} = [\mathbf{A}] \cdot \mathbf{v} \) (the mean values are equal to zero); here \( \mathbf{v} \) is a column vector whose elements are 4 normal deviates with 0 mean and standard deviation 1; \( [\mathbf{A}] \) is a lower triangular matrix such that the product with its transposed returns \( [\mathbf{V}] \). When one or more values of \( \delta''_0 \) are negative, the vector \( \mathbf{u} \) is rejected.

Owing to eq. (B4), the semi-axes \( a_i \) turn out to be \( \propto R \), so that \( a_i/R_H \) (\( i = 1, 2, 3 \)) do not depend on redshift. Let us then dub a (c) the maximum (minimum) \( a_i \) semi-axes, \( b \) being the intermediate one, and focus on the ratios \( c/a \) and \( b/a \).

We then estimate the fraction of perturbations, for a given value of \( F = (1 + \delta_u)/\sigma_0 \), with \( c/a \) exceeding 4 possible thresholds (0.9, 0.95, 0.98, 0.99). To do so, we fix \( F \) and generate a large number (\( \sim 10^5 \)) of random replicas for the ratio \( c/a \).

The results are shown in Figure 12, in the text. Let us however outline that, also for small values of \( F \), a non negligible fraction of systems appear to be “almost” spherical (i.e. \( c/a > 0.99 \)).

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