Can a skier make a circular turn without any active movement?

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Abstract

A skier's motion was analyzed by a simple model consist of point mass \( m \) and a single rod connected to a single ski plate. We studied the conditions for the stable ski turn as functions of the linear velocity and the radius of the turn. The solutions for the stable ski turn in our model do not require any extra skier's movement to complete a stable circular turn. The solution may then give the skier the most comfortable skiing method without any active movement to control the ski. The generalized force supporting the point mass from the ski plate was calculated. We obtained the force from the ground (rebound force) without any geometrical structure of the ski plate. Adding an active movement to the direction of the ski plate, the conditions for the stable ski turn were also analyzed. Our result gives some insight for the skier who wants to develop technique.

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I. INTRODUCTION

The skill of the alpine ski has developed over a long time. Classical alpine ski styles, such as the American style, Austrian style, and French style, have been briefly introduced from the physicist’s point of view[1]. They explained the basic dynamics of rotations in alpine ski from torque equations. Without external torque, a ski can make turns with counter-rotations by unweighting. In the Austrian style, a vigorous unweighting is applied to initiate the turns. On the other hand, the American style uses less counter-rotation and more external torque by edging control. The French style makes strong use of the external torques produced by having the skier’s weight backwards on an edged ski to commence a turn.

The dynamics of ski skating has also been studied with mathematical model of ski skating on a level plane[2]. They obtained the maximized averaged speed for a given power. They showed that the skating technique, where the ski moves at an angle to the direction of motion, is much faster than the classical technique, in which the ski stays in a track parallel to the direction of motion.

As the measurement technology developed, analysis of ski technique was carried out. Digitizing synchronized video sequences, the ski turn techniques of experienced and intermediate skiers were analyzed with measuring the selected kinematic variables for different ski turns[3]. In 2000, the carving ski turn was studied with specially developed joint angle sensors attached to the skier’s leg, and force sensors fitted between the binding plate and the ski[4]. The joint motion using the carving ski is moderate, with no impact resistance. In contrast, in a short turn using a conventional ski, the skier makes a quick rotation of the thigh in the early part of the turn. As a result, the reacting force occurs instantly, and its amplitude of change is larger.

Using analytical techniques including electromyography, kinetic, and kinematic methods and computer simulations, new insights into the skills of both alpine skiing and ski-jumping have been studied [5]. Biomechanical aspects of new techniques are analyzed according to the specific conditions in alpine skiing and the effects of equipment and individual specific abilities on performance. In 2012, kinetic analysis of ski turns in alpine skiing based on the measured ground reaction forces were published[6]. Their method was based on a theoretical analysis of physical forces acting during the ski turn. They defined two elementary standard phases of a ski turn by analyzing the external forces acting on the system during a turn.
The initiation phase is for the preparing to turn and the steering phase is for the actual turning, during which the center of gravity of the skier-ski system moves along a curvilinear trajectory.

Other recent studies on skiing include a machine learning algorithm to detect and label turns in alpine skiing using a single sensor placed on a skier’s knee \cite{7}. The development of an interactive ski-simulation motion recognition system by physics-based analysis has also been studied\cite{8}.

The development of skiing technique from the ancient era has been studied until now. Some works explained how the ski makes turns, and what is the optimum path theoretically. Other works analyzed the skier’s motion by measuring the position, and forces applied to the skier. In this article we try to analyze the skier’s motion with a point mass $m$ on the single rod connected to a single ski plate. This model excludes the skier’s active motion, and finds the particular solution for the point mass $m$ to make stable ski turns. This solution may give the skier the most comfortable skiing method without any active movement to control the ski. We also assumed that the ski plate moves following a circular trajectory on an inclined slope.

The present paper is organized as follows: Section II sets the Lagrangian for a point mass $m$ on a massless rod (length $l_0$) that is connected to a point mass and a single ski plate. The ski plate moves following a circular trajectory on the inclined slope. We assumed that the mass of the ski plate is zero, and we neglect any internal movement of the ski plate. We include two constraints in order to find two generalized forces. The first one keeps the ski plate moving along a circular path on the slope, and the second one keeps the rod attached to the ski plate. For the skier’s point of view, the second generalized force is related to the rebound force, or the force from the ground to the skier.

Section III solves the differential equations numerically with setting the angle $\beta$ as a constant value, and finds the time-dependent function of the angles $\alpha$ and $\theta$. The initial condition of two angles $\alpha$ and $\beta$ is studied for both making a complete circular turn without falling, and making a successive turn. In general, there is no initial condition that the final condition at $\theta = 90^\circ$ is another new initial condition for the next turn. Since the skier can adjust her/his motion by her/his two legs and poles, we checked the angular velocity of $\theta$ at first, in order to make the same initial condition for the next turn in this section. We found solutions for the functions of the angle of the slope, the speed of the ski, and the radius of
the turn.

On the slope, a skier usually actively moves her/his body in order to keep a stable position and control the ski path. Section IV includes the skier’s active motion parallel to the ski plate in our model, while in Section III our model moves passively during the ski turn, in other words, once the initial angles $\alpha_i$ and $\beta_i$ are chosen, the point mass $m$ on the bar moves down hill with the same angle $\beta_i$ along the circular ski trajectory, without any other active movement. In Section IV, the angle $\beta$ is not constant during a ski turn. We set the angle $\beta$ as a time-dependent function, which can duplicate the skier’s motion along the ski plate back and forth. We analyzed the motion of the point mass $m$ that corresponds to the center of the skier’s mass as the angle $\beta$ changes as a function of time.

Section V summarize the main results, and discusses their application.

II. LAGRANGIAN FOR THE SKIER’S SIMPLIFIED MOTION.

Although a skier moves on two plates with active motions, we studied the simplified skier’s motion as a point mass $m$ on the massless rod (length $l_0$), which connected the point mass and the single ski plate. We did not include the motion of the plate, in other words, we assumed the mass of the ski plate is zero and the ski plate moves following a circular trajectory on an inclined slope. Figure I defines the coordinates: the origin of the coordinate is at the center of the circle, and the $x$-axis is parallel to the fall line, and the $z$-axis is normal to the slope. The angle $\theta$ is measured from the $y$-axis and the radius of the circle on the slope is $r_0$. The position of the rod which is connected to the ski plate moving along a circular trajectory can be written as follows:

$$\vec{R}_p = \{r(t) \sin \theta(t), r(t) \cos \theta(t), 0\}$$

(1)

$$\hat{u} = \frac{\vec{R}_p}{|\vec{R}_p|},$$

(2)

$$\hat{t} = \hat{u} \times \vec{z},$$

(3)

where, $\hat{u}$ is the unit vector, and the unit vector $\hat{t}$ is the tangential vector pointing in the direction of motion of the ski plate.

In Fig. 2 the angle $\alpha$ is the angle between the $z$-axis and the rod, and the angle $\beta$ is the angle between the unit vector $-\vec{u}$ and the projection of the rod on the slope. If angle $\beta$ is zero, the angle $\alpha$ represents how much the skier body leans toward the center of the circular
FIG. 1: Ski trajectory is a circle with a radius \( r_0 \), and its center is at the origin of the \( xyz \) coordinate system. The \( x \)-axis is along the fall line, and the \( z \)-axis is normal to the slope. The direction of \( g \) is along the gravitational field. \( \vec{R}_p \) represents the position of the ski center and \( \vec{t} \) is along the ski direction. \( \phi \) is the angle between the slope and the horizontal.

trajectory. For a given angle \( \alpha \), if the angle \( \beta \) increase, the skier’s body leans toward the direction of the ski movement.

The position \((\vec{Q})\) of the point mass \( m \) is;

\[
\vec{Q} = \vec{R}_p - l_0 \sin \alpha(t) \cos \beta(t) \hat{u} + l_0 \sin \alpha(t) \sin \beta(t) \hat{t} + (l_0 \cos \alpha(t) + b_z(t)) \hat{z},
\]

\[
h(t) = \vec{Q}.\hat{z} \cos \phi - \vec{Q}.\hat{z} \sin \phi,
\]

where, \( h(t) \) is the height of the point mass \( m \) relative to the origin of the circular trajectory.

FIG. 2: The human body is simplified as a massless rod and point body. The length of the rod is \( l_0 \) and the mass of the body is \( m \). The unit vector \( \vec{u} \) is parallel to the radius vector and the unit vector \( \vec{t} \) is along the ski direction. \( \alpha(\beta) \) is the angle between the rod and \( \vec{z} \) (\( \vec{u} \)).

In our model, we can calculate the Lagrangian for the point mass \( m \) that moves on the
In order to calculate the generalized force we used two constraints, as follow:

\[ f_r = r(t) - r_0, \tag{7} \]
\[ f_z = b_z(t), \tag{8} \]

where, \( f_r \) can be used to find the generalized force that keeps the ski plate moving along the circular path on the slope. The other constraint \( f_z \) is also used to check the generalized force that keeps the rod attached to the ski plate.

The Lagrangian equations give the following five equations of motion:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} - \lambda_r \frac{\partial f_r}{\partial \dot{\theta}} - \lambda_z \frac{\partial f_z}{\partial \dot{\theta}} = 0, \tag{9}
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\alpha}} \right) - \frac{\partial L}{\partial \alpha} - \lambda_r \frac{\partial f_r}{\partial \dot{\alpha}} - \lambda_z \frac{\partial f_z}{\partial \dot{\alpha}} = 0, \tag{10}
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\beta}} \right) - \frac{\partial L}{\partial \beta} - \lambda_r \frac{\partial f_r}{\partial \dot{\beta}} - \lambda_z \frac{\partial f_z}{\partial \dot{\beta}} = 0, \tag{11}
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial r} \right) - \frac{\partial L}{\partial r} - \lambda_r \frac{\partial f_r}{\partial r} - \lambda_z \frac{\partial f_z}{\partial r} = 0, \tag{12}
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial b_z} \right) - \frac{\partial L}{\partial b_z} - \lambda_r \frac{\partial f_r}{\partial b_z} - \lambda_z \frac{\partial f_z}{\partial b_z} = 0. \tag{13}
\]

From Eqs. 12-13, we obtain \( \lambda_r \) and \( \lambda_z \) as the following, under the constraint condition \( f_r = 0 \) and \( f_z = 0 \).

\[
\lambda_r = l_0 \cos \beta \sin \alpha \dot{\alpha}^2 - mg \sin \phi \sin \theta + l_0 \cos \beta \cos \alpha \dot{\beta}^2 + 2l_0 \cos \alpha \sin \beta \ddot{\theta} (\dot{\beta} - \dot{\theta}) - 2l_0 \cos \alpha \sin \beta \dot{\alpha}^2 \tag{14}
\]
\[
+ l_0 \sin \alpha \sin \beta \ddot{\beta} - l_0 \sin \alpha \sin \beta \dot{\theta}
\]
\[
\lambda_z = mg \cos \phi - l_0 \cos \alpha \dot{\alpha}^2 - l_0 \sin \alpha \dot{\alpha} \tag{15}
\]

Then the generalized force \( Q_r \) and \( Q_z \) corresponding to the constraint conditions becomes:

\[
Q_r = \lambda_r \frac{\partial f_r}{\partial r} + \lambda_z \frac{\partial f_z}{\partial r},
\]
\[
= \lambda_r, \tag{16}
\]
\[ Q_z = \lambda_r \frac{\partial f_r}{\partial b_z} + \lambda_z \frac{\partial f_z}{\partial b_z}, \]

\[ = \lambda_z \]  

(17)

Under the constraint conditions, the equations of the motion for the three variables \((\theta, \alpha, \beta)\) are as follow:

\[ m_g r_0 \cos \phi - m_g l_0 \cos \beta \cos \phi \sin \phi \sin \alpha - m_g l_0 \sin \phi \sin \alpha \sin \beta \sin \theta \]

\[ + l_0 r_0 \sin \alpha \sin \beta \dot{\alpha}^2 - 2l_0 r_0 \cos \alpha \cos \beta \dot{\alpha} \dot{\beta} + 2l_0^2 \cos \alpha \sin \alpha \dot{\alpha} \dot{\beta} + l_0 r_0 \sin \alpha \sin \beta \dot{\beta} \]

\[ + 2l_0 r_0 \cos \alpha \cos \beta \dot{\alpha} \dot{\theta} - 2l_0^2 \cos \alpha \sin \alpha \dot{\alpha} \dot{\theta} - l_0 r_0 \sin \alpha \sin \beta \ddot{\alpha} \]

\[ - l_0 r_0 \cos \beta \sin \alpha \ddot{\alpha} + l_0^2 \sin \alpha \dot{\beta}^2 - r_0^2 \ddot{\theta} + 2l_0 r_0 \cos \beta \sin \alpha \dot{\theta} - l_0^2 \sin \alpha \dot{\beta}^2 = 0 \]  

(18)

\[ l_0 (m_g \cos \phi \sin \alpha + m_g \cos \alpha \cos \phi \sin \beta - m_g \cos \alpha \cos \beta \sin \phi \sin \theta - l_0 \sin 2\alpha \dot{\beta} \dot{\theta} \]

\[ + l_0 \cos \alpha \sin \alpha \dot{\beta}^2 + \cos \alpha (-r_0 \cos \beta + l_0 \sin \alpha) \dot{\theta}^2 - l_0 \ddot{\alpha} - r_0 \cos \alpha \sin \beta \ddot{\theta} = 0 \]  

(19)

\[ l_0 \sin \alpha (m_g \cos \beta \cos \theta \sin \phi + m_g \sin \phi \sin \beta \sin \theta - 2l_0 \cos \alpha \dot{\alpha} (\dot{\beta} - \dot{\theta}) \]

\[ + r_0 \sin \beta \theta^2 - l_0 \sin \alpha \dot{\beta} - r_0 \cos \beta \ddot{\theta} + l_0 \sin \alpha \ddot{\theta} = 0 \]  

(20)

III. SKIER’S INITIAL SETUP FOR MAKING STABLE AND SUCCESSIVE TURNS.

We found the numerical solutions for the simplified skier’s motion. In our model, the path of the ski is restricted to a circular orbit on an inclined slope, and we assumed the skier as a point mass \(m\) on the top of a rod with length \(l_0\). We also assumed that the ski plate is a point at the bottom of the rod. Solving the differential equations numerically, we set the angle \(\beta\) as a constant value in this section, and found the time-dependent function of the angle \(\alpha\), and \(\theta\).

Figures 3 shows the time traces of the point mass \(m\) on \(Q\) that moves on the circular trajectory on the slope inclined by 20°. At \(\theta = -90°\), the initial angles \(\alpha_i\) and \(\beta_i\) are (33.9°, 8.3°) respectively. The initial angular velocity of \(\dot{\theta}_0 = \frac{120\pi}{180}\). We assumed the radius of the circle \(r_0\) is 2m and the length \(l_0\) between the point mass \(m\) and the ski is 1m. Although these values are not fitted to the real situation, this data set is enough to analyze the ski
movement. We also set the angle $\beta(t)$ as a fixed value as $\beta_i$; if not the motion becomes too complicated and, it is not easy to find stable ski motion on a circular trajectory.

Figure 4 shows new results for the same slope with the same initial conditions, except for the initial angles $\alpha_i, \beta_i$. The initial angles are $(\alpha_i = 29.9^\circ, \beta_i = 26.0^\circ)$. The initial conditions in Fig. 3 give the final angle $\alpha_f$ at $\theta = 90^\circ$ being zero, but the initial conditions in Fig. 4 give $\alpha_f = -29.9^\circ$. For the next turn, the center of the turn is on the left side, so $\alpha = -29.9^\circ$ is the same initial conditions for the angle $\alpha$.

\[ \phi = 20^\circ \]

\[ \alpha = 0 \]

**FIG. 3:** The arrow indicates the point $Q$ and the starting points of arrows follows the circular orbit. The ski moves from $\theta = -90^\circ$ to the angle $\theta = 90^\circ$. At $\theta = 90^\circ$, the angle between the $\hat{z}$ and the arrow $\alpha$ becomes zero. $(\alpha_i = 33.9^\circ, \beta_i = 8.3^\circ)$

In Fig. 5 we plot the generalized force $Q_z$ and $Q_R$. When the final angle $\alpha = 0$, the force $Q_z$ remains positive till $\theta_c = 80^\circ$; after that the generalized force becomes negative. In other words, there should be a negative force to keep the circular trajectory motion, which is to say the force should be toward the bottom. From the skier’s view-point, he/she feels an uprising force from the ground. In actual skier’s movement, he/she can absorb this force by bending his/her body, or he/she is going to fall down. We obtained this uprising force without any motion of the ski plate, in other words, there exists a force from the ground when the skier moves to the down hill following a circular trajectory. For the case to get a
FIG. 4: The arrow indicates the point $Q$ and the starting points of arrows follows the circular orbit. The ski moves from $\theta = -90^\circ$ to the angle $\theta = 90^\circ$. At $\theta = 90^\circ$, the angle between $\hat{z}$ and the arrow $\alpha$ is the same as the $-\alpha_0$ at $t = 0$. ($\alpha_i = 29.9^\circ, \beta_i = 26.0^\circ$)

final condition $\delta \alpha = 0$, the generalized force becomes negative just after $\theta_c = 65^\circ$, and the force is greater than for the case $\alpha = 0$ in Fig. 5. Considering the generalized force $Q_R$, the force is negative till the angle $\theta = \theta_c$, but the force becomes positive after the angle $\theta_c$. From the skier’s view-point, to follow the circular trajectory, he/she should generate a force toward the center of the circle by edging of the ski. However, after the angle $\theta_c$, the direction of the force is changed. The skier needs a force outward the center of circle to accomplish the circular motion. In actual skiing, it’s related to the change of the edge during the ski turn.

In order to make continuous ski turns without adding extra force, all variables at $\theta = 90^\circ$ recover the values at $\theta = -90^\circ$, except the $\alpha_f$. Since the center of the circle moves from the right to the left, the $\alpha_f$ should be $-\alpha_i$. In Fig. 6, the solid red line represents the initial angles $\alpha_i$ and $\beta_i$, which gives the final angles $\beta_f = \beta_i$ and $\alpha_f = -\alpha_i$. The angles on the contour line $\delta \alpha = 0$ gives the original initial condition for the angle-$\alpha$ at $\theta = -90^\circ$. As the initial angle $\beta_i$ changes from 0 to 35°, the $\alpha_i$ varies from 33° to 27°, in order to give the final angle $\alpha_f = -\alpha_i$. 
FIG. 5: The solid (dashed) red line represents the generalized force $Q_z$ ($Q_R$) for the case where the angle $\alpha$ becomes zero at $\theta = 90^\circ$. The solid (dashed) black line represents the generalized force $Q_z$ ($Q_R$) for the case where the angle $\alpha$ becomes $-\alpha_0$ at $\theta = 90^\circ$.

Since we fixed the angle $\beta$ during the turn, we check the two variables $\alpha, \theta$ to recover the initial conditions. Considering the angle $\theta$, we already fix the time at which $\theta(\tau) = -90^\circ$, $\tau$ can be an initial time, and the angle may be set as $\theta = -90^\circ$ at the next turn. Therefore the initial angles ($\alpha_i, \beta_i$) related to the contour $\delta\alpha = 0$ give almost all initial values for the second turn.

On the other hand, if we check the angular velocity of two variables $\alpha$ and $\theta$, the two angular velocities are different from the initial angular velocities. The solid black line in Fig. 6 represents the initial angles that satisfy the condition that $\dot{\theta}(\tau)$ equals $\dot{\theta}_0$, the initial angular velocity of $\theta$. Fortunately, the contours $\delta\alpha = 0$ meets $\dot{\theta} = \dot{\theta}_0$ lines. However, at that initial conditions, the final condition that makes the setup perfect for the next turn is not satisfied. The solid blue line in Fig. 6 represents the initial angles which gives the condition $\dot{\alpha}(\tau) = 0$. Unfortunately, there are no initial angles that satisfy $\delta\alpha = 0, \dot{\theta} = \dot{\theta}_0, \dot{\alpha}(\tau) = 0$ simultaneously. This means that without any extra force, there is no successive ski turn in our model. However, in actual ski movement, the skier uses two ski plates, so it is very easy to control the angle $\alpha$, and the poles make it even easier to control the angular movement of $\alpha$.

From now on, we carefully find the critical condition $\dot{\theta}(\tau) = \dot{\theta}_0$ in order to make successive ski turns. For the $\alpha$ angle, we consider two cases, such as $\alpha(\tau) = 0$ and $\alpha(\tau) = -\alpha_i$. Although, $\alpha(\tau) = -\alpha_i$ is the correct one, usually at that condition, the amplitude of the
\( \dot{\alpha}(\tau) \) is very big and the generalized force \( Q_z \) has very high negative values. From the skier’s view-point, the skier needs large body movement, including pole action, to follow a circular trajectory. We think that the condition for \( \alpha(\tau) = 0 \) needs milder movement of the skier to keep the circular trajectory. Therefore, in our calculation, we consider two cases \( \delta \alpha(\tau) = -\alpha_i \) and \( \alpha(\tau) = 0 \) with \( \dot{\theta}(\tau) = \dot{\theta}_0 \) in order to make successive ski turns. We also plotted the contours line that represent the initial angles that gives the generalized force at \( \theta = 90^\circ \) that becomes \( Q_z = 0 \) in Fig. 6.

![Contour lines as a function of the initial angles \( \alpha \) and \( \beta \). The solid red line represents the case when the magnitude of the angle \( \alpha \) at \( \theta = 90^\circ \) is the same as the initial angle. The dashed red line represents the case when the magnitude of the angle \( \alpha \) at \( \theta = 90^\circ \) is zero. The solid black line represents the case when the derivative value of the \( \theta \) at \( \theta = 90^\circ \) is the same at the initial value \( \dot{\theta}_0 \). The solid blue line represents the case when the derivative value of the \( \alpha \) at \( \theta = 90^\circ \) is zero. The dashed green line represents the case when the magnitude of the generalized force \( Q_z \) equals zero at \( \theta = 90^\circ \).](image)

**FIG. 6:** Contour lines as a function of the initial angles \( \alpha \) and \( \beta \). The solid red line represents the case when the magnitude of the angle \( \alpha \) at \( \theta = 90^\circ \) is the same as the initial angle. The dashed red line represents the case when the magnitude of the angle \( \alpha \) at \( \theta = 90^\circ \) is zero. The solid black line represents the case when the derivative value of the \( \theta \) at \( \theta = 90^\circ \) is the same at the initial value \( \dot{\theta}_0 \). The solid blue line represents the case when the derivative value of the \( \alpha \) at \( \theta = 90^\circ \) is zero. The dashed green line represents the case when the magnitude of the generalized force \( Q_z \) equals zero at \( \theta = 90^\circ \).

The condition for successive ski turns can be changed as the radius of the circular trajectory. In Fig. 7 we plotted the initial angles \( \alpha_i \) and \( \beta_i \) that give the condition \( \delta \alpha(\tau) = 0 \), or \( \alpha(\tau) = 0 \) at \( \theta = 90^\circ \). At this time each point satisfies the condition \( \dot{\theta}(\tau) = \dot{\theta}_0 \), where \( \dot{\theta}_0 = \frac{120\pi}{180} \). We fixed the slope angle \( \phi = 20^\circ \), and changed the radius \( r_0 \) of the circular trajectory from 1m to 4m. As the radius increase, both the initial angles \( \alpha_i \) and \( \beta_i \) increase. From the skier’s view, it gives counter intuitive result at first glance. Since the skier moves
generally more actively as the radius of the turn becomes smaller, in other words, the skier leans her/his body more forward and more inside of the turn as the radius of the turn becomes smaller. Actually, in our calculation, we fixed not the linear velocity, but the initial angular velocity $\dot{\theta}_0$. In other words, as the radius of the turn becomes larger, the linear velocity also becomes large. Therefore, the results in Fig. 7 shows that as the skier moves fast, the skier leans further forward (bigger $\beta_i$) and more inside of the turn (bigger $\alpha_i$). The interesting thing is that for the condition $\alpha(\tau) = 0$, the initial angle $\beta_i$ is not changed very much, the skier only has to control the initial angle $\alpha_i$ as the speed of the ski becomes large. Considering the $\dot{\alpha}(\tau)$ and the generalized force $Q_z$, the skiers initial condition had better stay between the two lines in Fig. 7.

![FIG. 7: Each point represent the initial angles $\alpha$ and $\beta$ that give the final condition such as $\delta \alpha = 0$ and $\alpha = 0$. The numbers from 1 to 4 on each point indicate the radius ($r_0$) of the circular path of the ski trajectory. The initial angular velocity $\dot{\theta}_0$ is fixed for all $r_0$ values.](image)

Instead of keeping the angular velocity constant, we keep the linear velocity constant, and we find the initial condition to make successive turn as a function of the radius of the circular trajectory. In Fig. 8, we plotted the initial angles $\alpha_i$ and $\beta_i$ that give the condition $\delta \alpha(\tau) = 0$, or $\alpha(\tau) = 0$ at $\theta = 90^\circ$. At this time, each point satisfies the condition $\dot{\theta}(\tau) = \dot{\theta}_0$, where $\dot{\theta}_0$ is not a constant for various radius. We fix the linear velocity $v_0 = \frac{2120\pi}{180}$ and the angular velocity is a function of the radius $r_0$, $\dot{\theta}_0 = \frac{v_0}{r_0}$. The slope angle is fixed as $\phi = 20^\circ$ and the radius($r_0$) of the circular trajectory is changed from 1m to 4m.

As the radius of the circular trajectory decreases, the initial angle $\beta_i$ increases. From the skier’s view-point, it is very natural that the skier should lean forward to make a short
radius turn with the same initial linear velocity. The interesting thing happens with the initial angle \( \alpha_i \) when we get an condition such as \( \delta \alpha(\tau) = 0 \), as the radius \( r_0 \) increases from 1m, the initial angle \( \alpha_i \) increases at first, but when the radius \( r_0 \) is greater than 2m the initial angle \( \alpha_i \) decreases. If we consider the condition such as \( \alpha(\tau) = 0 \), there is a similar trend in the initial conditions.

From the skier’s view-point, the skier should lean forward and inward of the circle as the linear velocity increases. This is the case when the radius changes from 3m to around 2m in Fig. 8. However, the radius is comparable to the length \( l_0 \) (between the ski and point mass), and the motion of the point mass is not so simple. The numerical result shows that the \( \alpha_i \) should decrease as the angular velocity gets higher in some region.

![Graph showing initial angles](image)

**FIG. 8:** Each point represent the initial angles \( \alpha \) and \( \beta \) that give the final condition such as \( \delta \alpha = 0 \) and \( \alpha = 0 \). The numbers from 1 to 4 on each point indicate the radius \( r_0 \) of the circular path of the ski trajectory. The initial linear velocity of \( r_0 \dot{\theta}_0 \) is fixed for all \( r_0 \) values.

Keeping the radius of the circular trajectory \( r_0 = 2m \), we change the angular velocity \( \dot{\theta}_0 \) from \( \frac{70\pi}{180} \) to \( \frac{160\pi}{180} \) in Fig. 9. Filled black circles represent the case where the initial angles \( \alpha_i \) and \( \beta_i \) give the final condition such as \( \delta \alpha(\tau) = 0 \) for the slope angle \( \phi = 20^\circ \). The condition \( \alpha(\tau) = 0 \) is also shown in Fig. 9 as the filled red squares. As we expect, as the angular velocity \( \dot{\theta}_0 \) increases, the skier should lean more forward and towards the center of the circle. We also find the initial angles for the slope angle \( \phi = 30^\circ \). The filled circle represent the initial angles for the slope \( \phi = 20^\circ \) and the empty circle does so for the slope \( \phi = 30^\circ \), which satisfy the condition \( \delta \alpha(\tau) = 0 \). Considering the slope angle, one might think that the skier should make a larger angle toward the center of the turn, and lean forward with an large
angle. Actually, the graph in Fig. 9 shows what we expect. What we missed at first is the fact that the angles are defined from the z-axis, and the z-axis is normal to the slope. Therefore, as the slope increase, the z-axis already moves far from the $-\vec{g}$ direction by the increasing angle. From the skier’s view-point, he/she leans more severely towards the fall line direction. In other words, the initial angle for the slope $\phi = 20^\circ$ is generally greater than the initial angle for the slope $\phi = 30^\circ$. However, this does not mean that the skier leans more severely to the fall line as the slope angle increases.

![Graph](image)

**FIG. 9:** The numbers from 70 to 160 on each point indicate the initial angular velocity of $\theta$ from $\dot{\theta}_0 = \frac{70\pi}{180}$ to $\dot{\theta}_0 = \frac{160\pi}{180}$. Filled (open) circles represent the case the initial angles $\alpha$ and $\beta$ that give the final condition such as $\delta \alpha = 0$ for the slope angle $\phi = 20^\circ$ ($\phi = 30^\circ$). Filled (open) squares represent the case where the initial angles $\alpha$ and $\beta$ give the final condition $\alpha = 0$ for the slope angle $\phi = 20^\circ$ ($\phi = 30^\circ$).

In this section, we solved the differential equations (Eq. 18–20) numerically, with setting angle $\beta$ as a constant value. The initial condition of two angles $\alpha$ and $\beta$ is studied for both making a complete circular turn without falling, and making a successive turn. In general, there is no initial condition that the final condition at $\theta = 90^\circ$ is another new initial condition for the next turn. Since the skier can adjust his/her motion by his/her two legs and poles, we checked the angular velocity of $\theta$ at first, in order to make the same initial condition for the next turn in this section.
IV. EFFECT OF THE SKIER’S ACTIVE MOVEMENT.

On the slope, a skier usually actively move his/her body in order to maintain stable position and control the ski path. In section III, our model moves passively during the ski turn, in other words, once the initial angles $\alpha_i$ and $\beta_i$ are chosen, the point mass $m$ on the bar moves down hill with the same angle $\beta_i$ along the circular ski trajectory without any other active movement. The equations of motion determine the motion of the point mass $m$ and give the final angle $\alpha_f$ and the angular velocity $\dot{\theta}(\tau)$.

In this section, we add active movement on the angle $\beta$, which relates the skier’s motion along the ski plates. We set

$$\beta(t) = b_0 + b_1 \cos(\omega t + \psi),$$  \hspace{1cm} (21)

where $\omega = \frac{2\pi}{\tau}$, and $\tau$ is the time $\theta(\tau) = 90^\circ$ for each turn. The $\omega$ is roughly twice the frequency of the ski turn. In other words, the skier moves back and forth with amplitude $b_1$ and makes complete motion during the time from $\theta = -90^\circ$ to $\theta = 90^\circ$. With $b_1 = 0$, the skier motion is the same as the result discussed in section III.

In Fig. 10, the contour surfaces represent the $(b_0, b_1, \psi)$ set that gives $\alpha(\tau) = 0$ with $\dot{\theta}(\tau) = \dot{\theta}_0$. With the same slope angle ($\phi = 30^\circ$, we set the initial angle of $\alpha$ from $32.5^\circ$ to $29^\circ$, then the contour surface changes as in Fig. 11. These figures show that the active movement on the angle $\beta$ may give various motion of the ski.

Figures 12 shows the position of the point mass $m$ on the rod ($l_0 = 1m$), which connects the point mass $m$ and the ski plate. $L_c$ is the projected length to the ground toward the center of the circular ski trajectory. $L_f$ is the projected length to the ground towards the forward direction of the ski movement. For the same slope, with the same constant $b_0 = 0.3, b_1 = 0.2$, we plot the trace of the point mass by a solid line and dashed line for the set $(\alpha_0 = 38.5^\circ, \psi = 0.2)$ and $(\alpha_0 = 46.5^\circ, \psi = -1.58)$, respectively. The red circle indicates the point mass at the initial position and the blue circles indicates the point mass when the ski is parallel to the fall line, i.e. $\theta = 0^\circ$.

The phase factor $\psi$ in Eq. 21 makes the point mass turn clockwise or counterclockwise. Each circle in the solid line indicate the trace of the point mass per $10^\circ$ in the angle $\theta$. Following the solid line, the skier leans forward till he/she moves till $\theta = -60^\circ$, then he/she leans relatively backward until he/she reaches $\theta = 90^\circ$. In this set up, the final angular velocity $\dot{\theta}(\tau) = 3.14$ and it takes $0.80s$. On the other hand, if we follow the dashed line, the
FIG. 10: Contour surface indicating the initial values of $b_0$, $b_1$ and $\psi$ that give the angle $\alpha = 0$ at $\theta = 90^\circ$. The initial angle of the $\alpha$ is $\alpha_0 = 32.5^\circ$, with the slope angle $\phi = 30^\circ$.

FIG. 11: Contour surface indicates the initial values of $b_0$, $b_1$ and $\psi$ that give the angle $\alpha = 0$ at $\theta = 90^\circ$. The initial angle of the $\alpha$ is $\alpha_0 = 29^\circ$, with the slope angle $\phi = 30^\circ$. 
point mass moves counter clockwise. With the initial angle $\alpha_i = 38.5$ and the phase $\psi = 0.2$, the skier leans relatively backward from the initial condition till he/she reaches $\theta = -20^\circ$, almost until the ski is parallel to the fall line, then he/she moves forward till he/she reaches $\theta = 50^\circ$, after that he/she slowly leans backward to finish his/her turn. In this case, the final angular velocity $\dot{\theta}(\tau) = 3.78$ and it takes 0.89s. Comparing this case with the clockwise movement, it makes the final angular velocity much greater and it takes longer time. Generally, the skier lean forward at the first part of the turn and eventually he/she moves back at the end of the turn.

![FIG. 12: Traces of the point mass m as the $\theta$ increase by 10°. $L_c$ is the projected length to the ground toward the center of the circular orbit. $L_f$ is the projected length to the ground toward the forward direction of ski movement. The dashed line moves counterclockwise and the solid line moves clockwise. Red circles represents the starting point ($\theta = -90^\circ$).](image)

In Fig. 13, we plot the angular velocity $\dot{\theta}$ as a function of $\theta$ for the case $(\alpha_i = 46.5^\circ, \psi = -1.58)$, and $(\alpha_i = 38.5^\circ, \psi = 0.2)$. For the solid line, the point mass $m$ moves clockwise, the angular velocity $\dot{\theta}$ increases just after the fall line, and it decreases as the point mass reaches the end of the turn ($\theta = 90^\circ$). On the other hand, the dashed line, the point mass $m$ moves counter clockwise, the angular velocity does not increase as much as the clockwise case at the fall line, but the angular velocity does not decrease and it stays as a higher value. This means that, at the end of the turn, the skier still has higher velocity, and it is not easy to control his/her body to start a new turn. The worst thing is that the skier takes more time to complete one turn even though his/her final velocity is higher.

In Fig. 14, we plot the position of the point mass $m$ on the rod ($l_0 = 1m$) that connects the point mass $m$ and the ski plate. The solid black line, the dashed green line, and the solid
FIG. 13: The solid lines represent the angular velocity \( \dot{\theta} \) as a function of \( \theta \) for the condition \((\alpha_i = 46.5^\circ, \psi = -1.58)\). The dashed lines represent the angular velocity \( \dot{\theta} \) as a function of \( \theta \) for the condition \((\alpha_i = 38.5^\circ, \psi = 0.2)\).

green line represent the traces of the point mass for \( \alpha = (47^\circ, 46^\circ, 48^\circ) \), respectively. The red circle indicates the initial position and the blue circles indicate the point mass when the ski is parallel to the fall line, i.e. \( \theta = 0^\circ \). With the small change of the initial angle \( \alpha \) by 1°, the final angle \( \alpha_f \) changes a lot. If we increase the initial angle, then the final position of the point mass stays close to the center of the circular trajectory. In contrast, if we decrease the initial angle \( \alpha_i \), the point mass return back to the upright position i.e. \( \alpha_f \sim 0 \). The changes of generalized force \( Q_z \) in Fig. 15 show the differences caused by the change of initial angle \( \alpha_i \). With the initial angle \( \alpha_i = \alpha_0 \), the generalized force \( Q_z \) becomes 0 at \( \theta = 90^\circ \). The generalized force stay positive at \( \theta = 90^\circ \) with the initial angle \( \alpha_i = \alpha_0 + 1^\circ \). This condition makes the point mass \( m \) keep follow the circular trajectory upward against the slope, then the skier has hard time to initiate another downward turn. With \( \alpha_i = \alpha_0 - 1^\circ \), the generalized force becomes zero before the ski arrives \( \theta = 90^\circ \), and it becomes negative at \( \theta = 90^\circ \). From the skier’s view, he/she obtains a repulsive force from the ground, and this force enables him/her to move upward. He/shw may use this force in order to adjust his/her body to initiate a new ski turn from that position.

In Fig. 16 we plot the position of the point mass \( m \) as functions of \( b_0 \) and \( b_1 \) in Eq. 21. The solid black line can be a reference trace as in Fig. 14 with \((\alpha_0 = 47^\circ, b_0 = \bar{b}_0, b_1 = \bar{b}_1, \psi = -\frac{\pi}{2})\), where \( b_0 = 0.3, b_1 = 0.2 \). The actual movement of the angle \( \beta \) is in Fig. 17. The solid red line represents the trace with \( b_0 = 1.2\bar{b}_0 \), and the dashed red line represents
FIG. 14: Traces of the point mass $m$ as the $\theta$ increase by $10^\circ$. $L_c$ is the projected length to the ground toward the center of the circular orbit. $L_f$ is the projected length to the ground toward the forward direction of ski movement. Red circles represents the starting point ($\theta = -90^\circ$) and the blue circle and the black circle represents the mid point ($\theta = 0$) and the final points ($\theta = 90^\circ$), respectively. $\alpha_0 = 47^\circ$. ($b_0 = 0.3, b_1 = 0.2, \psi = -\frac{\pi}{2}$).

FIG. 15: The solid black line represents the generalized force $Q_z$ with the initial angle $\alpha_0 = 47^\circ$. The solid (dashed) green line represents the generalized force $Q_z$ with the initial angle $\alpha_0 + 1^\circ$ ($\alpha_0 + 1^\circ$).

The trace with $b_0 = 0.8\bar{b}_0$. The two traces show that, if we decrease the constant angle $b_0$ by a little, then the trace of the point mass $m$ becomes easier to adapt to the next turn. From the skier’s view-point, the body can be positioned upright to the ski plate.

The dashed blue line represents the trace with $b_1 = 1.2\bar{b}_1$, and the solid blue line represents the trace with $b_1 = 0.8\bar{b}_1$. The value $b_1$ is related to the modulation amplitude of the $\beta$ movement. If the modulation amplitude is smaller than $b_1$, the solid blue line shows that
the trace of the point mass \( m \) is much closed to the solid red line at \( \theta = 90^\circ \)

![Image of traces of the point mass](image)

**FIG. 16**: Traces of the point mass \( m \) as the \( \theta \) increase by 10\(^\circ\). \( L_c \) is the projected length to the ground toward the center of the circular orbit. \( L_f \) is the projected length to the ground toward the forward direction of ski movement. The solid black line is for the case \( b_0 = \bar{b}_0 \) and \( b_1 = \bar{b}_1 \). The initial angle \( \alpha \) for all the cases is the same as \( \alpha_0 \).

The position of the point mass \( m \) around the final turn \( \theta = 90^\circ \) in Fig. 16 shows two groups. The first one is the dashed blue and dashed red lines, while the other is the solid red and solid blue lines. These two groups can be explained in the Fig. 17 only if we consider the angle \( \beta \) after \( \theta = 60^\circ \). In other words, the important \( \beta \) movement in the ski turn happens after the skis cross the fall line, in other words \( \theta \) is greater than 30\(^\circ\).

![Image of time dependence of angle \( \beta \)](image)

**FIG. 17**: Time dependence of the angle \( \beta \). The solid black line is for the case \( b_0 = \bar{b}_0 \) and \( b_1 = \bar{b}_1 \), where \( \bar{b}_0 = 0.3, \bar{b}_1 = 0.2 \).

In this section, the angle \( \beta \) is not a constant in solving the differential equations (Eq. 18).
numerically. We set the angle $\beta$ as a time-dependent function, which can duplicate the skier’s motion along the ski plate back and forth. We analyzed the motion of the point mass $m$ which corresponds to the center of the skier’s mass as the angle $\beta$ changes as a function of time.

V. CONCLUSION AND DISCUSSION.

We studied the skier’s motion based on a simple model, in which the point mass $m$ on a single rod is connected to a single ski plate. We at first excluded the skier’s active motion and find the particular solution for the point mass $m$ to make stable ski turns. Our model neglected the intrinsic motion of the ski plate; we simply assumed that the single massless ski plate moves along a circular trajectory on an inclined plane slope. For a certain initial angle $\beta_i$, we can find a time-dependent angle $\alpha(t)$, which gives the point mass $m$ a complete half-turn without falling.

The generalized force ($Q_z$) supporting the point mass $m$ from the ski plate was calculated. A skier can control $Q_z$ by up and down movement. However, in our model $Q_z$ is calculated without any skier’s active movement. $Q_z$ can be explained by a rebound force from the ground. This rebound force is not related to any geometrical structure of the ski plate.

Although we can’t find a complete solution for successive turns, a final condition at $\theta = 90^\circ$ gives a new solution for the initial angles making a next half turn without falling. In an actual ski turn, the skier uses two legs and poles to adjust the movement of the angle $\alpha$; we tried to find a solution with some tolerance for the angle $\alpha$. As the degree of the slope increases, the initial angles $\alpha_i$ and $\beta_i$ should increase for a stable ski turn. We studied the conditions for the stable ski turn as functions of the linear velocity and radius of the turn. From the skier’s view-point, the solutions for the stable ski turn do not require any extra movement to complete a stable circular turn. Then the solution may give the skier the most comfortable skiing method without any active movement to control the ski.

In an actual ski turn, the skier may add extra movement along the ski plate ($\vec{t}$-direction). Adding an active movement to the direction of the ski plate, the conditions for the stable ski turn were analyzed. With active fine tuning of the angle $\beta$, the motion of the point mass $m$ was studied in detail. The final angle $\alpha_f$ depends on the angle $\beta$ after passing through the fall line ($\theta = 0$).
In our study, we used simple model for the skier’s movement, so the results are not directly applicable to actual skiing process. However, our result gives some insight into the skier who wants to develop his/her technique.

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