SUBQUOTIENTS OF A FINITE PRODUCT AND THEIR SELF-HOMOTOPY EQUIVALENCES

Hiroshi KIHARA and Nobuyuki ODA

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Abstract. Given a set $X = (X_1, X_2, \ldots, X_m)$ of pointed spaces, we introduce a family \( \{X_{(k,l)}\} \) of subquotients of \( X_1 \times X_2 \times \cdots \times X_m \). This family extends the family of subspaces of \( X_1 \times X_2 \times \cdots \times X_m \) introduced by G. J. Porter and contains the product, the fat wedge, the wedge and the smash product. The (co)homology with field coefficients of \( X_{(k,l)} \) is completely determined, which is used to study the group \( \mathcal{E}(X_{(k,l)}) \) of self-homotopy equivalences of \( X_{(k,l)} \). Especially, in the case of \( X_1 = X_2 = \cdots = X_m = X \), we construct a homomorphism \( \Psi_{(k,l)} \) from the semi-direct product of the \( m \)-fold product \( \mathcal{E}(X)^m \) and the symmetric group \( S_m \) to \( \mathcal{E}(X_{(k,l)}) \) and give sufficient conditions for \( \Psi_{(k,l)} \) to be injective. We apply this result to the case where \( X = S^n, \mathbb{C}P^n \), or \( K(A', n) \) with \( A \) a subring of \( \mathbb{Q} \) or a field \( \mathbb{Z}/p \), providing an important subgroup of \( \mathcal{E}(X_{(k,l)}) \).

1. Introduction

The groups of homotopy classes of self-homotopy equivalences of product spaces have been studied by, for example, Heath [5], Pavešić [9, 10] and Sieradski [12]. However, systematic studies seem to be unknown for product spaces. On the other hand, several general results on the group \( \mathcal{E}(X^\sqcup m) \) of homotopy classes of self-homotopy equivalences for the \( m \)-fold smash product of a space \( X \) were established in our previous paper [7]. In the present paper, for given pointed spaces \( X_1, X_2, \ldots, X_m \), we introduce a family of subquotients of the product space \( X_1 \times X_2 \times \cdots \times X_m \) which contains the product, the fat wedge, the wedge and the smash product of \( X_1, X_2, \ldots, X_m \). Then, we establish general results on the groups of homotopy classes of self-homotopy equivalences for these subquotients, especially in the case \( X_1 = X_2 = \cdots = X_m = X \).

We outline our results more precisely: Let \( X = (X_1, X_2, \ldots, X_m) \) be a set of \( m \) spaces with base point. For integers \( k \) and \( l \) with \( m \geq k \geq l \geq 0 \), we define the subquotient \( X_{(k,l)} \) of the product \( \prod_{i=1}^m X_i \) by

\[
X_{(k,l)} = X_{(k)}/X_{(l)},
\]

where the subspace \( X_{(k)} \) of \( \prod_{i=1}^m X_i \) is defined by

\[
X_{(k)} = \{ x = (x_1, x_2, \ldots, x_m) \mid \text{at least} (m-k) \text{coordinates of} \ x \ \text{are at base point} \}.
\]

It should be noted that \( X_{(m,0)} = X_{(m)} = X_1 \times X_2 \times \cdots \times X_m \) and \( X_{(1,0)} = X_{(1)} = X_1 \vee X_2 \vee \cdots \vee X_m \) hold. Moreover, since \( X_{(m-1)} \) is the fat wedge of \( X_1, X_2, \ldots, X_m \), we have

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Remark 4) to several important cases. First, we completely determine the kernel of an analogous to them (see Remark 2).

The homology group $\tilde{H}_s(\mathbf{X}, \mathbb{F})$ is completely determined for a field $\mathbb{F}$, and the cohomology algebra $\tilde{H}^s(\mathbf{X}, \mathbb{F})$ is also determined under the finite type condition on $\tilde{H}_s(X_i; \mathbb{F})$ $(1 \leq i \leq m)$ (Theorem 1). We use this result to study the group $\mathcal{E}(\mathbf{X}, \mathbb{F})$ of homotopy classes of self-homotopy equivalences. For this purpose, we introduce a few canonical homomorphisms as follows (see Section 3 for details).

First, we define the homomorphism $\psi(k,l) : \prod_{i=1}^m \mathcal{E}(X_i) \to \mathcal{E}(\mathbf{X}_{(k,l)})$ for $m \geq k \geq l \geq 0$ to satisfy

$$\psi(k,l)(f_1, f_2, \ldots, f_m)([x_1, x_2, \ldots, x_m]) = [f_1(x_1), f_2(x_2), \ldots, f_m(x_m)],$$

where $[x_1, x_2, \ldots, x_m] \in \mathbf{X}_{(k,l)} = \mathbf{X}_{(k)}/\mathbf{X}_{(l)}$ denotes the class of a point $(x_1, x_2, \ldots, x_m)$ of $\mathbf{X}_{(k)}$. We prove that the homomorphism $\psi(k,0) = \psi(k) : \prod_{i=1}^m \mathcal{E}(X_i) \to \mathcal{E}(\mathbf{X}_{(k)})$ is always a monomorphism for $1 \leq k \leq m$ (Theorem 2). However, $\psi(k,l)$ is not always a monomorphism even for $k > l$, as we see by Example 1.

In the rest of this section, we restrict ourselves to the case of $\mathbf{X} = (X, X, \ldots, X)$, namely $X_1 = X_2 = \cdots = X_m = X$, in which case $\mathbf{X}_{(k,l)}$ is denoted by $X_{(k,l)}^m$. We can then define the homomorphism $\varphi(k,l) : S_m \to \mathcal{E}(X_{(k,l)}^m)$ from the symmetric group $S_m$ on the $m$ letters $\{1, 2, 3, \ldots, m\}$ to $\mathcal{E}(X_{(k,l)}^m)$ to satisfy

$$\varphi(k,l)(\sigma)([x_1, x_2, \ldots, x_m]) = [x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(m)}].$$

Further, we define the homomorphism $\Psi(k,l) : \mathcal{E}(X)^m \rtimes S_m \to \mathcal{E}(X_{(k,l)}^m)$ by assigning to $((f_1, f_2, \ldots, f_m), \sigma)$ the composite

$$X_{(k,l)}^m \xrightarrow{\varphi(k,l)(\sigma)} X_{(k,l)}^m \xrightarrow{\psi(k,l)(f_1, f_2, \ldots, f_m)} X_{(k,l)}^m,$$

unifying the homomorphisms $\varphi(k,l)$ and $\psi(k,l)$ (see Section 3.2, especially Definition 2 for the semi-direct product $G^m \rtimes S_m$). We show that if $X \not\cong \ast$, then $\Psi(k,0) = \Psi(k) : \mathcal{E}(X)^m \rtimes S_m \to \mathcal{E}(X_{(k,l)}^m)$ is a monomorphism for any $1 \leq k \leq m$ (Theorem 3).

In Section 4 we investigate the homomorphism $\Psi(k,l) : \mathcal{E}(X)^m \rtimes S_m \to \mathcal{E}(X_{(k,l)}^m)$ more precisely to find conditions for $\Psi(k,l)$ to be a monomorphism; such conditions can be used to provide a non-trivial subgroup of $\mathcal{E}(X_{(k,l)}^m)$. The main results in Section 4 (Theorems 4 and 5) give such a condition in the case of $(k,l) \neq (m, m-1)$; these results are complementary to the results in [7], which deal with the case of $(k, l) = (m, m-1)$, but are not completely analogous to them (see Remark 2).

In Section 5, we apply Theorem 5 along with the related results (Proposition 2 and Remark 4) to several important cases. First, we completely determine the kernel of $\Psi(k,l) : \mathcal{E}(S^n)^m \rtimes S_m \to \mathcal{E}((S^n)^m_{(k,l)})$ (Proposition 3). Second, we show that

$$(\pm 1)^m \rtimes S_m \cong \mathcal{E}(\mathbb{C}P^n)^m \rtimes S_m \xrightarrow{\Psi(k,l)} \mathcal{E}((\mathbb{C}P^n)^m_{(k,l)})$$

is a monomorphism for $n \geq 2$ and $0 \leq l < k \leq m$ (Proposition 4). Last, we provide several sufficient conditions for

$$\text{GL}_r(A)^m \rtimes S_m \cong \mathcal{E}(K(A', n))^m \rtimes S_m \xrightarrow{\Psi(k,l)} \mathcal{E}(K(A', n)^m_{(k,l)})$$

to be a monomorphism, where $A$ is a subring of $\mathbb{Q}$, or $A = \mathbb{Z}/p$ for a prime number $p$ (Proposition 5).
2. Subquotients of a finite product and their (co)homology

In this section, we introduce the family \( \{X_{(k,l)}\} \) of subquotients of the product \( X_1 \times X_2 \times \cdots \times X_m \) of pointed spaces \( X_1, X_2, \ldots, X_m \) and then calculate their (co)homology with field coefficients.

2.1. Subquotients of a finite product

We consider topological spaces with base point and base point preserving continuous functions in this paper, which we refer to as spaces and maps, respectively. We use the same symbol \( f : X \to Y \) for a map and its homotopy classes for simplicity.

Definition 1. Let \( m \) be a natural number and \( k \) an integer such that \( 0 \leq k \leq m \). Let \( X = (X_1, X_2, \ldots, X_m) \) be a set of \( m \) spaces. Then we define the subspace \( X_{(k)} \) of the product space \( \prod_{i=1}^{m} X_i \) by

\[
X_{(k)} = \{x = (x_1, x_2, \ldots, x_m) \mid \text{at least } (m-k) \text{ coordinates of } x \text{ are at base point}\}.
\]

We have the following inclusion relations among the subspaces \( X_{(k)} \):

\[
X_{(m)} \supset X_{(m-1)} \supset \cdots \supset X_{(k)} \supset \cdots \supset X_{(1)} \supset X_{(0)}.
\]

Note that \( X_{(m)} = X_1 \times X_2 \times \cdots \times X_m \), \( X_{(1)} = X_1 \cup X_2 \cup \cdots \cup X_m \) and \( X_{(0)} = (*, *, \ldots, *) = * \) hold and that \( X_{(m-1)} \) is just the fat wedge of \( X_1, X_2, \ldots, X_m \). The space \( X_{(k)} \) is denoted by \( T_{m-k}(X_1, X_2, \ldots, X_m) \) in [11].

For \( k \geq l \), we define the space \( X_{(k,l)} \) by \( X_{(k,l)} = X_{(k)}/X_{(l)} \). Then, we have the following diagram of natural maps among the subquotients \( X_{(k,l)} \) of \( X_1 \times X_2 \times \cdots \times X_m \):

\[
\begin{array}{cccccccc}
X_{(m,0)} & \leftarrow & X_{(m-1,0)} & \leftarrow & X_{(m-2,0)} & \leftarrow & \cdots & \leftarrow & X_{(1,0)} & \leftarrow & X_{(0,0)} = * \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
X_{(m,1)} & \leftarrow & X_{(m-1,1)} & \leftarrow & X_{(m-2,1)} & \leftarrow & \cdots & \leftarrow & X_{(1,1)} = * \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
X_{(m,2)} & \leftarrow & X_{(m-1,2)} & \leftarrow & X_{(m-2,2)} & \leftarrow & \cdots & & & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & & & \cdots \\
\vdots & & \vdots & & \vdots & & \cdots & & & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & & & \downarrow \\
X_{(m,m-1)} & \leftarrow & X_{(m-1,m-1)} = * \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & & & \downarrow \\
X_{(m,m)} = * \\
\end{array}
\]

The horizontal map \( i = (i_{(k,l)}^{(k+1,l)}) : X_{(k,l)} \to X_{(k+1,l)} \) is a topological embedding and the vertical map \( p = p_{(k,l)}^{(k+1,l)} : X_{(k,l)} \to X_{(k,l+1)} \) is a topological quotient map. Note that \( X_{(k,0)} = X_{(k)} \) and \( X_{(k,k)} = * \) hold for \( 0 \leq k \leq m \) and that \( X_{(m,m-1)} = X_1 \wedge X_2 \wedge \cdots \wedge X_m \) holds.

In this paper, we mainly consider \( X_{(k,l)} \) for \( 0 \leq l < k \leq m \), since, if \( k = l \), then \( X_{(k,l)} = X_{(k,k)} = * \). Further, from Section 3.2 on, we restrict ourselves to the case of \( X_1 = X_2 = \cdots = X_m = X \), namely \( X = (X, X, \ldots, X) \), in which case \( X_{(k)} \) and \( X_{(k,l)} \) are denoted by \( X^m_{(k)} \) and \( X^m_{(k,l)} \), respectively.
2.2. Homology and cohomology of \( X_{(k,l)} \)

In this subsection, we calculate the homology and cohomology of \( X_{(k,l)} \).

For non-negative integers \( p_1, p_2, \ldots, p_m \), we define \( \nu(p_1, p_2, \ldots, p_m) \) by

\[
\nu(p_1, p_2, \ldots, p_m) = \sharp \{ i \mid p_i > 0 \},
\]

where \( \sharp A \) means the number of elements in a set \( A \). Let \( \tilde{H}_*(X; \mathbb{F}) \) and \( \tilde{H}^*(X; \mathbb{F}) \) denote the reduced homology and the reduced cohomology of a space \( X \) with coefficients in \( \mathbb{F} \).

**Theorem 1.** Let \( m, k \) and \( l \) be integers with \( 0 \leq l < k \leq m \) and let \( X = (X_1, X_2, \ldots, X_m) \) be a set of connected CW-complexes. Let \( \mathbb{F} \) be a field and \( \otimes = \otimes_{\mathbb{F}} \).

1. The natural isomorphism of graded \( \mathbb{F} \)-modules

\[
\tilde{H}_*(X_{(k,l)}; \mathbb{F}) \cong \bigoplus_{l < \nu(p_1, \ldots, p_m) \leq k} H_{p_1}(X_1; \mathbb{F}) \otimes H_{p_2}(X_2; \mathbb{F}) \otimes \cdots \otimes H_{p_m}(X_m; \mathbb{F})
\]

exists, where the \( p \)-th component of the right-hand side is the direct sum of the summands \( H_{p_1}(X_1; \mathbb{F}) \otimes H_{p_2}(X_2; \mathbb{F}) \otimes \cdots \otimes H_{p_m}(X_m; \mathbb{F}) \) with \( p_1 + p_2 + \cdots + p_m = p \).

2. Assume that \( \tilde{H}_*(X_i; \mathbb{F}) \) is of finite type for \( 1 \leq i \leq m \). Then, the natural isomorphism of graded \( \mathbb{F} \)-modules

\[
\tilde{H}^*(X_{(k,l)}; \mathbb{F}) \cong \bigoplus_{l < \nu(p_1, \ldots, p_m) \leq k} H^{p_1}(X_1; \mathbb{F}) \otimes H^{p_2}(X_2; \mathbb{F}) \otimes \cdots \otimes H^{p_m}(X_m; \mathbb{F})
\]

exists, where the \( p \)-th component of the right-hand side is the direct sum of the summands \( H^{p_1}(X_1; \mathbb{F}) \otimes H^{p_2}(X_2; \mathbb{F}) \otimes \cdots \otimes H^{p_m}(X_m; \mathbb{F}) \) with \( p_1 + p_2 + \cdots + p_m = p \).

If \( \alpha_1 \otimes \cdots \otimes \alpha_m \in H^{p_1}(X_1; \mathbb{F}) \otimes H^{p_2}(X_2; \mathbb{F}) \otimes \cdots \otimes H^{p_m}(X_m; \mathbb{F}) \) and \( \beta_1 \otimes \cdots \otimes \beta_m \in H^{q_1}(X_1; \mathbb{F}) \otimes H^{q_2}(X_2; \mathbb{F}) \otimes \cdots \otimes H^{q_m}(X_m; \mathbb{F}) \), then the product

\[
(\alpha_1 \otimes \cdots \otimes \alpha_m) \cdot (\beta_1 \otimes \cdots \otimes \beta_m)
\]

coincides with the product in the usual tensor product of graded \( \mathbb{F} \)-algebras

\[
H^*(X_1; \mathbb{F}) \otimes H^*(X_2; \mathbb{F}) \otimes \cdots \otimes H^*(X_m; \mathbb{F})
\]

when \( \nu(p_1 + q_1, \ldots, p_m + q_m) \leq k \) and must be regarded as zero when \( \nu(p_1 + q_1, \ldots, p_m + q_m) > k \).

For the proof, we need the following lemma.

**Lemma 1.** Let \( X = (X_1, X_2, \ldots, X_m) \) be a set of connected CW-complexes and let \( 0 < k \leq m \). Then, there exists a cofiber sequence

\[
X_{(k-1)} \rightarrow X_{(k)} \rightarrow \bigvee_{i_1 < i_2 < \cdots < i_k} X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_k}.
\]

Hence,

\[
X_{(k,k-1)} \cong \bigvee_{i_1 < i_2 < \cdots < i_k} X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_k}
\]

holds.
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Proof. For \( i_1 < i_2 < \cdots < i_k \), the subspace \((X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k})\) of \(X_{(m)}\) is defined to be

\[
\{(x_1, \ldots, x_m) \in X_{(m)} \mid x_i = * \text{ for } i \neq i_1, i_2, \ldots, i_k\},
\]

which is obviously homeomorphic to \(X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k}\). We then have

\[
X_{(k)} = \bigcup_{i_1 < i_2 < \cdots < i_k} (X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k}).
\]

We can also see that if \((i_1, i_2, \ldots, i_k) \neq (i'_1, i'_2, \ldots, i'_k)\), then

\[
(X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k}) \cap (X_{i'_1} \times X_{i'_2} \times \cdots \times X_{i'_k}) \subseteq X_{(k-1)}
\]

holds and that \((X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k}) \cap X_{(k-1)}\) is homeomorphic to the fat wedge of \(X_{i_1}, X_{i_2}, \ldots, X_{i_k}\) in a canonical way. Thus, we have the cofiber sequence

\[
X_{(k-1)} \to X_{(k)} \to \bigvee_{i_1 < i_2 < \cdots < i_k} X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_k}.
\]

\[\square\]

Proof of Theorem 1.

(1) Step 1: the case of \((k, l) = (m, m-1)\). Recall that \(X_{(m,m-1)} = X_1 \wedge X_2 \wedge \cdots \wedge X_m\).

Then, we can easily see from \([7, \text{Lemma 6 and its proof}]\) that the natural isomorphism of graded \(\mathbb{F}\)-modules

\[
\widehat{H}_s(X_{(m,m-1)}; \mathbb{F}) \cong \widehat{H}_s(X_1; \mathbb{F}) \otimes \widehat{H}_s(X_2; \mathbb{F}) \otimes \cdots \otimes \widehat{H}_s(X_m; \mathbb{F})
\]

exists, which shows that the result holds for \((k, l) = (m, m-1)\).

Step 2: the case of \(l = 0\). We show that

\[
\widehat{H}_s(X_{(k)}; \mathbb{F}) \cong \bigoplus_{0 < v(p_1, p_2, \ldots, p_m) \leq k} H_{p_1}(X_1; \mathbb{F}) \otimes H_{p_2}(X_2; \mathbb{F}) \otimes \cdots \otimes H_{p_m}(X_m; \mathbb{F}).
\]

For \(k = m\), the result is immediate from the Kunneth formula. Next, consider the cofiber sequence

\[
X_{(m-1)} \to X_{(m)} \to X_1 \wedge X_2 \wedge \cdots \wedge X_m.
\]

By the result for \(k = m\) and Step 1, we then obtain the result for \(k = m - 1\). Similarly, by making use of the cofiber sequence in Lemma 1, we obtain the result for \(k = m - 2, m - 3, \ldots, 1\).

Step 3: the case of \(l > 0\). Since we have the cofiber sequence

\[
X_{(l)} \to X_{(k)} \to X_{(k,l)},
\]

the results

\[
\widehat{H}_s(X_{(k,l)}; \mathbb{F}) \cong \bigoplus_{l < v(p_1, \ldots, p_m) \leq k} H_{p_1}(X_1; \mathbb{F}) \otimes H_{p_2}(X_2; \mathbb{F}) \otimes \cdots \otimes H_{p_m}(X_m; \mathbb{F})
\]

follow by the result of Step 2.

(2) We have the diagram of natural maps

\[
X_{(m)} \leftarrow X_{(k)} \to X_{(k,l)}.
\]
Consider the induced diagram
\[ \tilde{H}^*(\mathbf{X}_{(m)}; \mathbb{F}) \to \tilde{H}^*(\mathbf{X}_{(k)}; \mathbb{F}) \to \tilde{H}^*(\mathbf{X}_{(k,l)}; \mathbb{F}) \]

of commutative graded \(\mathbb{F}\)-algebras (without unit). Then, we obtain the result from Part 1 (see [4, p. 189 and p. 222]).

We end this section by calculating the connectivity of \(\mathbf{X}_{(k,l)}\).

**Remark 1.** Let \(\mathbf{X} = (X_1, X_2, \ldots, X_m)\) be a set of connected CW-complexes. Suppose that \(X_i\) is \(c_i\)-connected for \(1 \leq i \leq m\) and set
\[
equiv \min \{c_i + c_{i+1} + \cdots + c_{i+k} \mid 1 \leq i \leq i+k \leq m\}.
\]
Then \(\mathbf{X}_{(k,l)}\) is \(c+1\)-connected. (We note that, if \(X\) is \(c\)-connected, then \(X\) is homotopy equivalent to a CW-complex with the following cell structure: \(X \simeq * \cup \bigcup \lambda^{c+1} \cup \text{higher cells}\).)

3. **The homomorphism \(\Psi_{(k,l)}\)**

In this section, we introduce the homomorphism \(\Psi_{(k,l)} : \mathcal{E}(X)^m \times S_m \to \mathcal{E}(\mathbf{X}_{(k,l)})\) to study the group \(\mathcal{E}(\mathbf{X}_{(k,l)})\), and establish an injectivity result for \(\Psi_{(k,0)}\) (Theorem 3).

3.1. **The homomorphism \(\psi_{(k,l)} : \prod_{i=1}^m \mathcal{E}(X_i) \to \mathcal{E}(\mathbf{X}_{(k,l)})\)**

In this subsection, we introduce the homomorphism \(\psi_{(k,l)} : \prod_{i=1}^m \mathcal{E}(X_i) \to \mathcal{E}(\mathbf{X}_{(k,l)})\) and show that \(\psi_{(k)} (= \psi_{(k,0)})\) is a monomorphism for \(1 \leq k \leq m\) (Theorem 2).

We define the homomorphism \(\psi_{(k,l)} : \prod_{i=1}^m \mathcal{E}(X_i) \to \mathcal{E}(\mathbf{X}_{(k,l)})\) to satisfy
\[
\psi_{(k,l)}(f_1, f_2, \ldots, f_m)([x_1, x_2, \ldots, x_m]) = [f_1(x_1), f_2(x_2), \ldots, f_m(x_m)],
\]
where \([x_1, x_2, \ldots, x_m]\) denotes the class of \(\mathbf{X}_{(k,l)} = \mathbf{X}_{(k)} / \mathbf{X}_{(l)}\) containing \((x_1, x_2, \ldots, x_m) \in \mathbf{X}_{(k)}\). Note that \(\psi_{(k)}\) is just the restriction of \(f_1 \times f_2 \times \cdots \times f_m\) to \(\mathbf{X}_{(k)}\).

Let \([m] = \{1, 2, \ldots, m\}\) and let \(I = \{i_1, i_2, \ldots, i_s\}\) be a subset of \([m]\) such that \(1 \leq i_1 < i_2 < \cdots < i_s \leq m\). We write
\[
\mathbf{X}^I = X_{i_1} \times X_{i_2} \times \cdots \times X_{i_s}.
\]
For an integer \(k \geq s\), we have the natural inclusion \(i^I_{(k)} : \mathbf{X}^I \to \mathbf{X}_{(k)}\). Let \(J = \{j_1, j_2, \ldots, j_t\}\) be a subset of \([m]\) such that \(1 \leq j_1 < j_2 < \cdots < j_t \leq m\). Then we have the natural projection \(p^J_{(k)} : \mathbf{X}_{(k)} \to \mathbf{X}^J = X_{j_1} \times X_{j_2} \times \cdots \times X_{j_t}\). The equality \(p^J_{(k)} \circ i^I_{(k)} = 1_{\mathbf{X}^I}\) holds for any subset \(I \subset [m]\).

For \(f = (f_1, f_2, \ldots, f_m) \in \prod_{i=1}^m \mathcal{E}(X_i)\), we have the commutative diagram
\[
\begin{array}{ccc}
\mathbf{X}^I & \xrightarrow{f^I} & \mathbf{X}^J \\
\downarrow{i^I_{(k)}} & & \downarrow{i^J_{(k)}} \\
\mathbf{X}_{(k)} & \xrightarrow{\psi_{(k)}(f_1, f_2, \ldots, f_m)} & \mathbf{X}_{(k)} \\
\downarrow{p^J_{(k)}} & & \downarrow{p^J_{(k)}} \\
\mathbf{X}^J & \xrightarrow{f^J} & \mathbf{X}^J
\end{array}
\]
where $f^I : X^I \to X^I$ is defined by

$$f^I = f_{i_1} \times f_{i_2} \times \cdots \times f_{i_s} : X_{i_1} \times X_{i_2} \times \cdots \times X_{i_s} \to X_{i_1} \times X_{i_2} \times \cdots \times X_{i_s}.$$ 

**THEOREM 2.** The homomorphism $\psi(k) : \prod_{i=1}^m \mathcal{E}(X_i) \to \mathcal{E}(X_{(k)})$ is a monomorphism for $1 \leq k \leq m$.

**Proof.** We show that, if $(f_1, f_2, \ldots, f_m) \in \prod_{i=1}^m \mathcal{E}(X_i)$ satisfies

$$\psi(k)(f_1, f_2, \ldots, f_m) \simeq 1 : X_{(k)} \to X_{(k)},$$

then $f_i \simeq 1$ for any $i$.

Consider commutative diagram (3.1) for $I = J = \{i\}$, namely, $X^I = X_i$ and $f^I = f^I = f_i$. Then the relation $\psi(k)(f_1, f_2, \ldots, f_m) \simeq 1$ implies that

$$f_i = f^I \circ p^I_k \circ i^I_k = p^I_k \circ \psi(k)(f_1, f_2, \ldots, f_m) \circ i^I_k \simeq p^I_k \circ 1 \circ i^I_k = p^I_k \circ i^I_k = 1_{X_i},$$

for any $i$. □

**Example 1.** The homomorphism $\psi(k,l) : \prod_{i=1}^m \mathcal{E}(X_i) \to \mathcal{E}(X_{(k,l)})$ is not always a monomorphism even for $k > l$: Let $m = k = 2, l = 1$ and $X_1 = X_2 = S^n (n \geq 1)$. Then $X_{(2,1)} = S^n \wedge S^n = S^{2n}$ and $\psi(2,1) : \mathcal{E}(S^n) \times \mathcal{E}(S^n) \to \mathcal{E}(S^{2n})$ is not a monomorphism (see Proposition 8 of [7]).

### 3.2. The homomorphism $\Psi_{(k,l)} : \mathcal{E}(X)^m \times S_m \to \mathcal{E}(X_{(k,l)})$

From now on, we restrict ourselves to the case of $X = (X, X, \ldots, X)$, namely $X_1 = X_2 = \cdots = X_m = X$. Recall that $X_{(k,l)}$ is denoted by $X_{(k,l)}$ for such $X$.

In this subsection, we define the homomorphism $\varphi_{(k,l)} : S_m \to \mathcal{E}(X_{(k,l)})$ and introduce the homomorphism $\Psi_{(k,l)} : \mathcal{E}(X)^m \times S_m \to \mathcal{E}(X_{(k,l)})$, unifying the homomorphisms $\psi_{(k,l)} : \mathcal{E}(X)^m \to \mathcal{E}(X_{(k,l)})$ and $\psi_{(k,l)} : S_m \to \mathcal{E}(X_{(k,l)})$. Then we show that, if $X \not\simeq \ast$, then $\Psi_{(k,0)} = \psi(k) : \mathcal{E}(X)^m \times S_m \to \mathcal{E}(X_{(k,l)})$ is necessarily a monomorphism for $1 \leq k \leq m$ (Theorem 3).

We begin by defining the homomorphism $\varphi_{(k,l)} : S_m \to \mathcal{E}(X_{(k,l)})$. First, define the standard action of $S_m$ on the product space $X^m$ by

$$\sigma(x_1, x_2, \ldots, x_m) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(m)}).$$

Then, this action of $S_m$ induces that on $X^m_{(k,l)}$, and hence that on $X^m_{(k,l)}$, which defines the homomorphism $\varphi_{(k,l)} : S_m \to \mathcal{E}(X^m_{(k,l)})$ in the obvious manner, that is,

$$\varphi_{(k,l)}(\sigma)(x_1, x_2, \ldots, x_m) = \sigma(x_1, x_2, \ldots, x_m) = [x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(m)}].$$

Now, using $\varphi_{(k,l)}$ and $\psi_{(k,l)}$, we define the function $\Psi_{(k,l)} : \mathcal{E}(X)^m \times S_m \to \mathcal{E}(X_{(k,l)})$ by

$$\Psi_{(k,l)}((f_1, f_2, \ldots, f_m), \sigma) = \psi_{(k,l)}(f_1, f_2, \ldots, f_m) \circ \varphi_{(k,l)}(\sigma) : X^m_{(k,l)} \to X^m_{(k,l)} \to X^m_{(k,l)},$$

Recall the following definition from [7, Definition 1].
Definition 2. Let $G$ be a group. Assume that the symmetric group $S_m$ acts on the product group $G^m$ from the left by

$$\sigma(g_1, g_2, \ldots, g_m) = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \ldots, g_{\sigma^{-1}(m)}).$$

This action is called the ‘standard action of $S_m$ on $G^m$’ in this paper. For elements

$$((h_1, h_2, \ldots, h_m), \tau), \ ((g_1, g_2, \ldots, g_m), \sigma) \in G^m \times S_m,$$

we define a binary operation $\ast$ by

$$((h_1, h_2, \ldots, h_m), \tau) \ast ((g_1, g_2, \ldots, g_m), \sigma) = ((h_1, h_2, \ldots, h_m) \cdot \tau(g_1, g_2, \ldots, g_m), \tau \circ \sigma).$$

The set $G^m \times S_m$ with the operation $\ast$ becomes a group, which is denoted by $G^m \rtimes S_m$ and called the ‘(standard) semi-direct product of $G^m$ by $S_m$’ in this paper.

We can prove the following result by arguments similar to those in the proofs of Lemma 2 and Theorem 3 of [7], and hence we omit the proof.

**Proposition 1.** Let $m \geq k \geq l \geq 0$. Then the function $\Psi_{(k,l)} : E(X)^m \rtimes S_m \to E(X_{(k,l)})$ is a homomorphism of groups.

In the rest of this subsection, we concentrate on the case of $l = 0$. The homomorphisms $\psi_{(k,0)}, \varphi_{(k,0)}$ and $\Psi_{(k,0)}$ are often written as $\psi_{(k)}, \varphi_{(k)}$ and $\Psi_{(k)}$, respectively.

**Theorem 3.** If $X \not\cong *$, then $\Psi_{(k)} : E(X)^m \rtimes S_m \to E(X_{(k)})$ is a monomorphism for $1 \leq k \leq m$.

**Proof.** Suppose that an element $((f_1, f_2, \ldots, f_m), \sigma)$ of $E(X)^m \rtimes S_m$ satisfies

$$\Psi_{(k)}((f_1, f_2, \ldots, f_m), \sigma) = 1.$$

We show that $((f_1, f_2, \ldots, f_m), \sigma) = 1$.

**Step 1: we show that $\sigma = 1$.** Consider the commutative diagram

\[ X \xrightarrow{i_{(k)}^{(l)}} X_{(k)}^m \xrightarrow{\Psi_{(k)}} X_{(k)}^m \xrightarrow{i_{(m)}^{(k)}} X_{(m)}^m \]

where $i_{(k)}^{(l)} = i_{(k)}^{(l)}$ (see Section 3.1) and $i_{(m)}^{(k)} = i_{(m,0)}^{(k)}$ (see Section 2.1). Since $X \not\cong *$, $[A, X] \neq 0$ for some space $A$. Noticing that $[A, X_{(k)}^m] = [A, X]^m$, we see that

$$((\Psi_{(m)}((f_1, f_2, \ldots, f_m), \sigma) \circ i_{(m)}^{(k)} \circ i_{(k)}^{(l)}([A, X])) = (0, 0, \ldots, 0, [A, X], 0, \ldots, 0, 0) \subset [A, X]^m = [A, X]^m.$$

On the other hand, since \( \Psi(k)((f_1, f_2, \ldots, f_m), \sigma) = 1 \), we have

\[
(i^{(k)}(m) \circ \Psi(k)((f_1, f_2, \ldots, f_m), \sigma) \circ i_j^{(k)}_v([A, X])
=\]

\[
=\left(0, 0, \ldots, 0, [A, X], 0, \ldots, 0, 0\right) \subset [A, X]^m = [A, X^m].
\]

From these, we see that \( \sigma = 1 \).

**Step 2:** we show that \((f_1, f_2, \ldots, f_m) = (1, 1, \ldots, 1)\). By Step 1, we have

\[
\Psi(k)((f_1, f_2, \ldots, f_m), \sigma) = \Psi(k)((f_1, f_2, \ldots, f_m), 1) = \psi(k)(f_1, f_2, \ldots, f_m) = 1.
\]

Hence we have \((f_1, f_2, \ldots, f_m) = (1X, 1X, \ldots, 1X)\) by Theorem 2.

**Example 2.** We identify the monomorphism \( \Psi(k) : \mathcal{E}(X)^m \rtimes S_m \to \mathcal{E}(X_{(k)}^m) \) in several cases, showing that \( \Psi(k) \) need not be an epimorphism.

1. Let \( X = K(A, n) \), where \( A \) is a subring of \( \mathbb{Q} \) or \( \mathbb{Z}/p \) for a prime number \( p \). Since

\[
[K(A^m, n), K(A^m, n)] \cong \text{Hom}(A^m, A^m),
\]

we see that \( \mathcal{E}(K(A^m, n)) \cong \text{GL}_m(A) \). Thus we can identify the monomorphism

\[
\Psi_{(m)} : \mathcal{E}(K(A, n))^m \rtimes S_m \to \mathcal{E}(K(A, n)^m)
\]

with the homomorphism

\[
\Psi_A : (A^\times)^m \rtimes S_m \to \text{GL}_m(A)
\]

defined by \( \Psi_A((\alpha_i), \sigma) = \text{diag}(\alpha_i) \cdot (e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(m)}) \), where \( A^\times \) is the group of units of \( A \), \( \text{diag}(\alpha_i) \) is the diagonal matrix whose diagonal components are \( \alpha_1, \alpha_2, \ldots, \alpha_m \), and \( \{e_1, e_2, \ldots, e_m\} \) is the standard basis of \( A^m \). The monomorphism \( \Psi_A \), and hence \( \Psi_{(m)} \), is obviously not an epimorphism.

2. Let \( X = S^n \) \((n \geq 2)\) and note that \( (S^n)_{(1)}^m = S^n \vee S^n \vee \cdots \vee S^n \) \( (m \text{ times}) \). Since the composite

\[
[S^n \vee S^n \vee \cdots \vee S^n, S^n \vee S^n \vee \cdots \vee S^n] 
\]

\[
\cong \left[S^n, S^n \vee S^n \vee \cdots \vee S^n\right] \times \cdots \times \left[S^n, S^n \vee S^n \vee \cdots \vee S^n\right]
\]

\[
\cong M_m(\mathbb{Z})
\]

preserves addition and transforms composition to multiplication, we have an isomorphism \( \mathcal{E}(S^n \vee S^n \vee \cdots \vee S^n) \cong \text{GL}_m(\mathbb{Z}) \). Therefore, we can identify the monomorphism

\[
\Psi_{(1)} : \mathcal{E}(S^n)^m \rtimes S_m \to \mathcal{E}((S^n)_{(1)}^m)
\]

with the homomorphism

\[
\Psi_{\mathbb{Z}} : (\pm 1)^m \rtimes S_m \to \text{GL}_m(\mathbb{Z}),
\]

which is not an epimorphism (see part (1) for the symbol \( \Psi_A \)).
Last, we study the monomorphism $\Psi(2) : \mathcal{E}(S^n)^2 \times S_2 \rightarrow \mathcal{E}(S^n)^2 = \mathcal{E}(S^n \times S^n)$ using Kahn’s result on $\mathcal{E}(S^n \times S^n)$ [6, Section 2.3]. If $n$ is odd, then $\mathcal{E}(S^n \times S^n)$ is of infinite order, and hence $\Psi(2) : \mathcal{E}(S^n)^2 \times S_2 \rightarrow \mathcal{E}(S^n \times S^n)$ is not an epimorphism. Suppose that $n$ is even. Then, $\mathcal{E}(S^n \times S^n)$ is a semi-direct product of $G(n) \oplus G(n)$ by the dihedral group $\Delta_8$ of order eight (see [6, p. 35] for the group $G(n)$). We can easily see that the composite
$$
\mathcal{E}(S^n)^2 \times S_2 \xrightarrow{\Psi(2)} \mathcal{E}(S^n \times S^n) \rightarrow \Delta_8
$$
is an isomorphism. Hence, $\Psi(2)$ gives a splitting of the epimorphism $\mathcal{E}(S^n \times S^n) \rightarrow \Delta_8$. Further, with more effort, we can show that $G(n) \neq 0$, and hence that $\Psi(2)$ is not an epimorphism.

4. Conditions for $\Psi_{(k,l)}$ to be a monomorphism

We showed that, if $X \not
\cong \ast$, then $\Psi_{(k,0)} : \mathcal{E}(X)^m \times S_m \rightarrow \mathcal{E}(X^m_{(k,0)})$ is a monomorphism for $1 \leq k \leq m$ (Theorem 3). However, $\Psi_{(k,l)} : \mathcal{E}(X)^m \times S_m \rightarrow \mathcal{E}(X^m_{(k,l)})$ need not be a monomorphism (Example 1). In this section, we study the homomorphism $\Psi_{(k,l)}$ more precisely to find sufficient conditions for $\Psi_{(k,l)}$ to be a monomorphism.

Recall that we dealt with the case of $(k, l) = (m, m - 1)$ in [7, Section 3]. We first establish the following theorem and corollary, which are complementary to [7, Theorem 5 and Corollary 7].

Let $\tilde{H}_s(X; G)$ denote the reduced homology of a space $X$ with coefficients in an abelian group $G$.

**Theorem 4.** Let $X$ be a connected CW-complex and let $m \geq k > l \geq 0$ with $(k, l) \neq (m, m - 1)$. Then for the following conditions (i), (ii), (iii) and (iv), the implications (i) $\iff$ (ii) $\implies$ (iii) $\implies$ (iv) hold:

(i) $\tilde{H}_s(X; \mathbb{Z}) \neq 0$.

(ii) $\dim \tilde{H}_s(X; \mathbb{F}) \neq 0$ for some field $\mathbb{F}$.

(iii) The kernel of $\Psi_{(k,l)} : \mathcal{E}(X)^m \times S_m \rightarrow \mathcal{E}(X^m_{(k,l)})$ is $(\text{Ker } \psi_{(k,l)}) \times 1$, where 1 is the group of one element (and hence $\mathcal{E}(X^m_{(k,l)})$ contains a subgroup isomorphic to $\psi_{(k,l)}(\mathcal{E}(X)^m) \times S_m$).

(iv) The homomorphism $\varphi_{(k,l)} : S_m \rightarrow \mathcal{E}(X^m_{(k,l)})$ is a monomorphism.

The following corollary is immediate from Theorem 4.

**Corollary 1.** Let $X$ be a connected CW-complex and let $m \geq k > l \geq 0$ with $(k, l) \neq (m, m - 1)$. If $\tilde{H}_s(X; \mathbb{Z}) \neq 0$, then the homomorphism $\varphi_{(k,l)} : S_m \rightarrow \mathcal{E}(X^m_{(k,l)})$ is a monomorphism.

**Remark 2.** Comparing Theorem 4 and Corollary 1 with Theorem 5 and Corollary 7 in [7], we see that the former are not completely analogous to the latter and that the case of $(k, l) = (m, m - 1)$ is a special case.

For the proof of Theorem 4, we need the following lemma.

**Lemma 2.** Let $X$ be an arcwise connected space. Then the following conditions are equivalent:
(i) \( \widetilde{H}_s(X; \mathbb{Z}) \neq 0 \);
(ii) \( \widetilde{H}_s(X; \mathbb{F}) \neq 0 \) for some field \( \mathbb{F} \);
(iii) \( \widetilde{H}_s(X; \mathbb{F}) \neq 0 \) for some prime field \( \mathbb{F} \) (namely \( \mathbb{Q} \) or \( \mathbb{F}_p \) for a prime \( p \)).

**Proof.** The equivalence (ii) \( \iff \) (iii) follows from the fact that every field is a direct sum of copies of its prime field.

To show the equivalence (i) \( \iff \) (iii), we recall the exact sequence
\[
0 \longrightarrow \text{H}_i(X; \mathbb{Z}) \otimes \mathbb{F} \longrightarrow \text{H}_i(X; \mathbb{F}) \longrightarrow \text{Tor}_{\mathbb{Z}}(\text{H}_{i-1}(X; \mathbb{Z}), \mathbb{F}) \longrightarrow 0
\]
(the universal coefficient theorem). From this, we see the implication (iii) \( \implies \) (i). We end the proof by showing the implication (i) \( \implies \) (iii): Assume that \( \widetilde{H}_s(X; \mathbb{Z}) \neq 0 \).

**Case 1:** \( \text{H}_1(X; \mathbb{Z}) \) contains a non-torsion element. In this case, a monomorphism \( \mathbb{Z} \to \text{H}_1(X; \mathbb{Z}) \) is defined by sending the unit 1 to a non-torsion element. Therefore, since \( \cdot \otimes \mathbb{Q} \) is an exact functor, we have a monomorphism \( \mathbb{Q} \to \text{H}_1(X; \mathbb{Z}) \otimes \mathbb{Q} \), which shows that \( \text{H}_1(X; \mathbb{Z}) \neq 0 \).

**Case 2:** \( \text{H}_1(X; \mathbb{Z}) \) is a non-zero torsion group. Note that \( \text{Tor}_{\mathbb{Z}}(M, \mathbb{F}_p) = \{ m \in M \mid pm = 0 \} \). Then we see from the above exact sequence that \( \text{H}_{i+1}(X; \mathbb{F}_p) \neq 0 \) for some prime \( p \).

**Proof of Theorem 4.** The equivalence (i) \( \iff \) (ii) is the result of Lemma 2, and the implication (iii) \( \implies \) (iv) is trivial. Therefore, we have only to prove the implication (ii) \( \implies \) (iii).

Let \( ((f_1, f_2, \ldots, f_m), \sigma) \in \text{E}(X)^m \times S_m \). By Theorem 1(1), the induced homomorphism \( \Psi_{(k,l)}(f_1, f_2, \ldots, f_m, \sigma)_* \) on \( \widetilde{H}_*(X^m_{(k,l)}; \mathbb{F}) \) is identified with the homomorphism \( (f_1* \otimes f_2* \otimes \cdots \otimes f_m*) \circ \sigma_* \) on
\[
\bigoplus_{1 \leq v(p_1, \ldots, p_m) \leq k} \text{H}_{p_1}(X; \mathbb{F}) \otimes \text{H}_{p_2}(X; \mathbb{F}) \otimes \cdots \otimes \text{H}_{p_m}(X; \mathbb{F}),
\]
where \( \sigma_*(\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_m) = \pm \alpha_{\sigma^{-1}(1)} \otimes \alpha_{\sigma^{-1}(2)} \otimes \cdots \otimes \alpha_{\sigma^{-1}(m)} \); though the sign in this formula can be determined using \([4, (2.8)]\) in Ch. VII, it is unnecessary for the proof of the theorem.

Since \( (k, l) \neq (m, m - 1) \), the homology group \( \widetilde{H}_*(X^m_{(k,l)}; \mathbb{F}) \) has a direct summand
\[
\bigoplus_{v(p_1, \ldots, p_m) = h} \text{H}_{p_1}(X; \mathbb{F}) \otimes \text{H}_{p_2}(X; \mathbb{F}) \otimes \cdots \otimes \text{H}_{p_m}(X; \mathbb{F})
\]
for some \( h \) with \( 0 < h < m \). Checking the image of
\[
\text{H}_{p_1}(X; \mathbb{F}) \otimes \text{H}_{p_2}(X; \mathbb{F}) \otimes \cdots \otimes \text{H}_{p_m}(X; \mathbb{F}) \quad (v(p_1, \ldots, p_m) = h)
\]
by the homomorphism \( \Psi_{(k,l)}(f_1, f_2, \ldots, f_m, \sigma)_* \), we have \( \sigma = 1 \), which completes the proof of Theorem 4.

For a connected CW-complex \( X \) and a field \( \mathbb{F} \), define the subgroup \( \text{E}_*(X; \mathbb{F}) \) of \( \text{E}(X) \) by
\[
\text{E}_*(X; \mathbb{F}) = \{ f \in \text{E}(X) \mid f_* = 1 : \widetilde{H}_*(X; \mathbb{F}) \to \widetilde{H}_*(X; \mathbb{F}) \}.
\]
We note that the subgroup \( \text{E}_*(X) \) of \( \text{E}(X) \) defined by
\[
\text{E}_*(X) = \{ f \in \text{E}(X) \mid f_* = 1 : H_*(X) \to H_*(X) \}
\]
has been studied extensively (see [1, Section 1] and [2]).
THEOREM 5. Let $X$ be a connected CW-complex with $\tilde{H}_*(X; \mathbb{Z}) \neq 0$ and $\mathbb{F}$ a field, and let $m \geq k > l \geq 0$ with $(k, l) \neq (m, m - 1)$. Assume that one of the following four conditions holds:

(i) $\mathbb{F} = \mathbb{Z}/2$;
(ii) $H^*(X; \mathbb{F})$ has non-trivial cup product;
(iii) $k - l \geq 2$;
(iv) $l + 1 = k < m$ and $\{\alpha \in \mathbb{F}^x | \alpha^k = 1\} = \{1\}$, where $\mathbb{F}^x$ is the group of units in $\mathbb{F}$.

Then, for the homomorphism $\Psi_{(k,l)} : \mathcal{E}(X)^{m} \ast S_{m} \to \mathcal{E}(X^{m}_{(k,l)})$, the inclusion relation

$$\text{Ker } \Psi_{(k,l)} \cong \text{Ker } \psi_{(k,l)} \subset \mathcal{E}_*(X; \mathbb{F})^{m}$$

holds. In particular, if $\mathcal{E}_*(X; \mathbb{F}) = \{1\}$, then $\Psi_{(k,l)}$ is a monomorphism.

For the proof of Theorem 5, we need two lemmas. The following lemma is a generalization of [7, Proposition 11]. For a commutative ring $A$ with unit and a free $A$-module $E$, $\text{GL}(E)$ denotes the group of automorphisms of $E$.

LEMMA 3. Let $A$ be a commutative ring with unit, and $E_1, E_2, \ldots, E_m$ non-zero free $A$-modules (possibly of infinite rank). Define the group homomorphism

$$\tau : \text{GL}(E_1) \times \text{GL}(E_2) \times \cdots \times \text{GL}(E_m) \to \text{GL}(E_1 \otimes E_2 \otimes \cdots \otimes E_m)$$

by $\tau(\phi_1, \phi_2, \ldots, \phi_m) = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_m$. Then, the kernel of $\tau$ is isomorphic to the group

$$\{ (\alpha_1, \alpha_2, \ldots, \alpha_m) \in A^x \times A^x \times \cdots \times A^x | \alpha_1 \cdot \alpha_2 \cdots \alpha_m = 1 \},$$

where $A^x$ denotes the group of units in $A$. In particular, if $A = \mathbb{Z}/2$, then $\tau$ is a monomorphism.

Proof. The proof of [7, Proposition 11] also applies to the case of infinite rank. \hfill \square

We introduce the subgroup $\mathcal{E}(X; \mathbb{F})$ of $\mathcal{E}(X)$ by

$$\mathcal{E}_*(X; \mathbb{F}) = \left\{ f \in \mathcal{E}(X) \mid f_* = \alpha 1 : \tilde{H}_*(X; \mathbb{F}) \to \tilde{H}_*(X; \mathbb{F}) \right\}.$$ 

Note that the inclusion relations

$$\mathcal{E}_*(X; \mathbb{F}) \subset \mathcal{E}(X; \mathbb{F}) \subset \mathcal{E}(X)$$

exist. If $\tilde{H}_*(X; \mathbb{F}) = 0$, then the equalities $\mathcal{E}_*(X; \mathbb{F}) = \mathcal{E}(X; \mathbb{F}) = \mathcal{E}(X)$ hold. If $\tilde{H}_*(X; \mathbb{F}) \neq 0$, the obvious exact sequence

$$1 \to \mathcal{E}_*(X; \mathbb{F}) \to \mathcal{E}(X; \mathbb{F}) \to \alpha \to \mathbb{F}^x$$

exists.

LEMMA 4. Let $X$ be a connected CW-complex and $\mathbb{F}$ a field, and let $m \geq k > l \geq 0$. If $\tilde{H}_*(X; \mathbb{F}) \neq 0$, then the inclusion relation

$$\text{Ker } \psi_{(k,l)} \subset \left\{ (f_1, f_2, \ldots, f_m) \in \mathcal{E}_*(X; \mathbb{F})^m \mid \alpha(f_{i_1}) \alpha(f_{i_2}) \cdots \alpha(f_{i_s}) = 1 \text{ for } l < s \leq k \}
 \text{ and } 1 \leq i_1 < i_2 < \cdots < i_s \leq m$$

exists.
Proof. Recall from Theorem 1(1) that we have
\[
\widetilde{H}_s(X^m_{(k,l)}; \mathbb{F}) \cong \bigoplus_{l \leq v(p_1, \ldots, p_m) \leq k} H_{p_1}(X; \mathbb{F}) \otimes H_{p_2}(X; \mathbb{F}) \otimes \cdots \otimes H_{p_m}(X; \mathbb{F})
\]
and note that, if \((f_1, f_2, \ldots, f_m) \in \text{Ker} \psi_{(k,l)}\), then \(\psi_{(k,l)}(f_1, f_2, \ldots, f_m)_s\) is the identity homomorphism on \(H_s(X^m_{(k,l)}; \mathbb{F})\). Then, the result follows from Lemma 3.

Proof of Theorem 5. By Theorem 4, we have
\[
\text{Ker} \psi_{(k,l)} \cong \text{Ker} \psi_{(k,l)} \subset E(X)^m.
\]
Since \(E_s(X; \mathbb{F}) = E(X)\) for \(X\) with \(\widetilde{H}_s(X; \mathbb{F}) = 0\), we may assume that \(\widetilde{H}_s(X; \mathbb{F}) \neq 0\). Thus, we have only to show that \(\text{Ker} \psi_{(k,l)} \subset E_s(X; \mathbb{F})^m\) by using Lemma 4.

The case where condition (i) is satisfied. Since \(\mathbb{F}^x = \{1\}\), \(\widetilde{E}_s(X; \mathbb{F}) = E_s(X; \mathbb{F})\), and hence \(\text{Ker} \psi_{(k,l)} \subset E_s(X; \mathbb{F})^m\) holds.

The case where condition (ii) is satisfied. Let \(f\) be an element of \(\widetilde{E}_s(X; \mathbb{F})\) and let \(x, y \in \widetilde{H}^s(X; \mathbb{F})\) be elements that satisfy \(x \cup y \neq 0\). Then, the equalities
\[
\alpha(f)(x \cup y) = f^*(x \cup y) = f^*x \cup f^*y = \alpha(f)x \cup \alpha(f)y = \alpha(f)^2 x \cup y
\]
hold, which imply that \(\alpha(f)^2 = \alpha(f)\), and hence that \(\alpha(f) = 1\). Thus, we see that \(\widetilde{E}_s(X; \mathbb{F}) = E_s(X; \mathbb{F})\), and hence \(\text{Ker} \psi_{(k,l)} \subset E_s(X; \mathbb{F})^m\) holds.

The case where condition (iii) is satisfied. Let \((f_1, f_2, \ldots, f_m)\) be an element of \(\text{Ker} \psi_{(k,l)}\). Since \(\widetilde{H}_s(X^m_{(k,l)}; \mathbb{F})\) has the \(\mathbb{F}\)-submodules
\[
\bigoplus_{v(p_1, \ldots, p_m) = k} H_{p_1}(X; \mathbb{F}) \otimes \cdots \otimes H_{p_m}(X; \mathbb{F})
\]
and
\[
\bigoplus_{v(p_1, \ldots, p_m) = k-1} H_{p_1}(X; \mathbb{F}) \otimes \cdots \otimes H_{p_m}(X; \mathbb{F})
\]
as direct summands by Theorem 1(1), we have \(\alpha(f_{i_1}) \alpha(f_{i_2}) \cdots \alpha(f_{i_k}) = 1\) for \(1 \leq i_1 < i_2 < \cdots < i_k \leq m\) and \(\alpha(f_{i_1}) \alpha(f_{i_2}) \cdots \alpha(f_{i_{k-1}}) = 1\) for \(1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq m\). Hence, \(\alpha(f_i) = 1\) for \(1 \leq i \leq m\).

The case where condition (iv) is satisfied. By Theorem 1(1) we see that
\[
\widetilde{H}_s(X^m_{(k,l)}; \mathbb{F}) = \bigoplus_{v(p_1, \ldots, p_m) = k} H_{p_1}(X; \mathbb{F}) \otimes \cdots \otimes H_{p_m}(X; \mathbb{F}).
\]
Since \(\alpha(f_{i_1}) \alpha(f_{i_2}) \cdots \alpha(f_{i_k}) = 1\) for \(1 \leq i_1 < i_2 < \cdots < i_k \leq m\), we have \(\alpha(f_1) = \alpha(f_2) = \cdots = \alpha(f_m)\), which we write as, say, \(\beta\). Then, we have \(\beta^k = 1\), and hence \(\beta = 1\) by condition (iv).

Set
\[
G_k[\mathbb{F}] = \{\alpha \in \mathbb{F}^x \mid \alpha^k = 1\}
\]
for a field \(\mathbb{F}\). By weakening condition (iv) in Theorem 5, we have the following result.
Proposition 2. Let X be a connected CW-complex with \( \tilde{H}_s(X; \mathbb{Z}) \neq 0 \) and \( \mathbb{F} \) a field. Then under condition (iv)', there exists a natural homomorphism
\[
\gamma : \text{Ker } \Psi_{(k,l)} \longrightarrow G_k(\mathbb{F})
\]
with \( \text{Ker } \gamma \subset \mathcal{E}_s(X; \mathbb{F})^m \). In particular, if \( \mathcal{E}_s(X; \mathbb{F}) = \{1\} \), then \( \gamma : \text{Ker } \Psi_{(k,l)} \longrightarrow G_k(\mathbb{F}) \) is a monomorphism.

Proof. It is shown that, under condition (iv)', the inclusion relation
\[
\text{Ker } \Psi_{(k,l)} \cong \text{Ker } \Psi_{(k,l)} \subset \left\{ (f_1, f_2, \ldots, f_m) \in \mathcal{E}_s(X; \mathbb{F})^m \left| \alpha(f_i)^k = 1 \text{ for any } i, \alpha(f_1) = \alpha(f_2) = \cdots = \alpha(f_m) \right. \right\}
\]
exists (see the proof of Theorem 5). Thus, the result is immediate.

In Theorem 5 and Proposition 2, we may assume that \( \mathbb{F} \) is a prime field. Thus, for a prime field \( \mathbb{F} \), we calculate \( G_k(\mathbb{F}) \) and identify the condition that \( G_k(\mathbb{F}) = \{1\} \). Since \( \mathbb{Q}^\times = \mathbb{Q} - \{0\} \) and \( \mathbb{F}_p^\times \cong \mathbb{C}_{p-1} \) (the cyclic group of order \( p - 1 \)), we have
\[
G_k(\mathbb{Q}) = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ \{\pm 1\} & \text{if } k \text{ is even,} \end{cases}
\]
and
\[
G_k(\mathbb{F}_p) \cong \mathbb{C}_{\gcd(k, p - 1)},
\]
where \( \gcd(a, b) \) is the greatest common divisor of \( a \) and \( b \). Hence, the condition that \( \{\alpha \in \mathbb{F}^\times \mid \alpha^k = 1\} = \{1\} \) in (iv) of Theorem 5 is equivalent to the following condition:

- In the case \( \mathbb{F} = \mathbb{Q} \): ‘k is odd’.
- In the case \( \mathbb{F} = \mathbb{F}_p \): ‘k and \( p - 1 \) are relatively prime’.

Remark 3. (1) Let X be a connected CW-complex with \( \tilde{H}_s(X; \mathbb{Z}) \) of finite type. Let \( \mathcal{H}_p(X; \mathbb{Z}) \) denote the free part of \( H_p(X; \mathbb{Z}) \), where the free part of a finitely generated \( \mathbb{Z} \)-module \( M \) is defined to be the quotient module of \( M \) by the torsion submodule \( \text{tor}(M) \). Then, we can see that
\[
\mathcal{H}_p(X_{(k,l)}; \mathbb{Z}) \cong \bigoplus_{l < v(p_1, \ldots, p_m) \leq k} \mathcal{H}_{p_1}(X; \mathbb{Z}) \otimes \mathcal{H}_{p_2}(X; \mathbb{Z}) \otimes \cdots \otimes \mathcal{H}_{p_m}(X; \mathbb{Z})
\]
holds for \( p > 0 \) (see the proof of Theorem 1(1)). We define the subgroup \( \mathcal{E}_s(X; \mathbb{Z}) \) of \( \mathcal{E}(X) \) by
\[
\mathcal{E}_s(X; \mathbb{Z}) = \left\{ f \in \mathcal{E}(X) \left| f_s = \alpha 1 : \bigoplus_{p > 0} \mathcal{H}_p(X; \mathbb{Z}) \longrightarrow \bigoplus_{p > 0} \mathcal{H}_p(X; \mathbb{Z}) \text{ for some } \alpha \in \mathbb{Z}^\times = \{\pm 1\} \right. \right\}.
\]
Let \( m \geq k > l \geq 0 \) and suppose that \( \tilde{H}_s(X; \mathbb{Z}) \) has a free direct summand. Then, the inclusion relation
\[
\text{Ker } \Psi_{(k,l)} \subset \left\{ (f_1, f_2, \ldots, f_m) \in \mathcal{E}_s(X; \mathbb{Z})^m \mid \alpha(f_{i_1})\alpha(f_{i_2}) \cdots \alpha(f_{i_s}) = 1 \text{ for } l < s \leq k \right\}
\]
and \( 1 \leq i_1 < i_2 < \cdots < i_s \leq m \).
holds, where \( \alpha : \widetilde{E}_a(X; \mathbb{Z}) \to \{ \pm 1 \} \) is the obvious homomorphism (cf. Lemma 4). Hence, the inclusion relation

\[
\text{Ker }\psi_{(k, l)} \subset \left\{ (f_1, f_2, \ldots, f_m) \in \widetilde{E}_a(X; \mathbb{F})^m \right\} \\
\alpha(f_i) = \pm 1 \text{ for any } i, \\
\alpha(f_{i_1})\alpha(f_{i_2}) \cdots \alpha(f_{i_s}) = 1 \text{ for } 1 < s \leq k \\
\text{and } 1 < i_1 < i_2 < \cdots < i_s \leq m
\]

also holds for any prime field \( \mathbb{F} \). Therefore, in the case where \( \widetilde{H}_a(X; \mathbb{Z}) \) is a graded module of finite type with a free direct summand, the inclusion relation

\[
\text{Ker }\Psi_{(k, l)} \cong \text{Ker }\psi_{(k, l)} \subset \mathcal{E}_a(X; \mathbb{F})^m
\]

holds under condition (ii)

(2) Let \( X \) be a connected CW-complex with \( \widetilde{H}_a(X; \mathbb{Z}) \neq 0 \) and let \( m \geq k > l \geq 0 \) with \( (k, l) \neq (m, m - 1) \). For the homomorphism \( \Psi_{(k, l)} : \mathcal{E}(X)^m \times S_m \to \mathcal{E}(X_{(k, l)})^m \), we showed that the inclusion relation \( \text{Ker }\Psi_{(k, l)} \cong \text{Ker }\psi_{(k, l)} \subset \mathcal{E}_a(X; \mathbb{F})^m \) holds for any field \( \mathbb{F} \) that satisfies one of conditions (i)–(iv) of Theorem 5. Therefore, \( \Psi_{(k, l)} \cong \psi_{(k, l)} \subset \bigcap \mathcal{E}_a(X; \mathbb{F})^m \) holds, where \( \mathbb{F} \) ranges over all such fields. In particular, if \( k - l \geq 2 \), then

\[
\text{Ker }\Psi_{(k, l)} \cong \text{Ker }\psi_{(k, l)} \subset \bigcap_{\mathbb{F}-\text{prime field}} \mathcal{E}_a(X; \mathbb{F})^m.
\]

Remark 4. In the case of \( (k, l) = (m, m - 1) \), we can use [7, Theorem 5] instead of Theorem 4 to obtain the following results analogous to Theorem 5 and Proposition 2 (the essential part of Remark 3 also applies to the case of \( (k, l) = (m, m - 1) \)).

Let \( X \) be a connected CW-complex such that \( \widetilde{H}_a(X; \mathbb{Z}) \) is a graded module of finite type which is isomorphic to neither 0 nor \( \mathbb{Z}[n] \) for any \( n \geq 1 \) (or more generally \( \dim \widetilde{H}_a(X; K) \geq 2 \) for some field \( K \) ) (see [7, Theorem 5]). Let \( \mathbb{F} \) be a field. Then, the following two results hold.

1. Suppose that one of the following conditions holds:
   (i) \( \mathbb{F} = \mathbb{Z}/2 \);
   (ii) \( H^*(X; \mathbb{F}) \) has non-trivial cup product.

   Then, the inclusion relation

   \[
   \text{Ker }\Psi_{(m, m - 1)} \cong \text{Ker }\psi_{(m, m - 1)} \subset \mathcal{E}_a(X; \mathbb{F})^m
   \]

   holds. In particular, if \( \mathcal{E}_a(X; \mathbb{F}) = \{ 1 \} \), then \( \Psi_{(m, m - 1)} \) is a monomorphism.
   (This result is implicitly used in the proof of [7, Theorem 10].)

2. There exists a natural homomorphism

   \[
   \gamma : \text{Ker }\Psi_{(m, m - 1)} \to \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_m) \in (\mathbb{F}^*)^m \mid \alpha_1\alpha_2 \cdots \alpha_m = 1 \right\}
   \]

   with \( \text{Ker }\gamma \subset \mathcal{E}_a(X; \mathbb{F})^m \). In particular, if \( \mathcal{E}_a(X; \mathbb{F}) = \{ 1 \} \), then \( \gamma : \text{Ker }\Psi_{(m, m - 1)} \to \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_m) \in (\mathbb{F}^*)^m \mid \alpha_1\alpha_2 \cdots \alpha_m = 1 \right\} \) is a monomorphism.

Remark 5. The essence of the proofs of Theorem 5, Proposition 2, and Remark 4 is to investigate the kernel of \( \psi_{(k, l)} : \mathcal{E}(X)^m \to \mathcal{E}(X_{(k, l)})^m \), and hence the arguments in the proofs also apply to the study of the kernel of \( \psi_{(k, l)} : \prod_{i=1}^{m} \mathcal{E}(X_i) \to \mathcal{E}(X_{(k, l)}) \).
5. Applications of Theorem 5

In this section, we apply Theorem 5 along with Proposition 2 and Remark 4 to the case where $X = S^n$, $\mathbb{C}P^n$ or $K(A', n)$ with $A$ a subring of $\mathbb{Q}$ or a field $\mathbb{Z}/p$ (Propositions 3–5); we also deal with the co-Moore space $C(A', n)$ using an argument similar to that in the proof of Proposition 5.

**Proposition 3.** Let $n \geq 1$. The kernel of the homomorphism $\Psi_{(k, l)} : \mathcal{E}(S^n)^m \rtimes S_m \to \mathcal{E}((S^n)^m_{(k, l)})$ is given by the following:

$$\ker \Psi_{(k, l)} = \begin{cases} 1 & \text{for } k - l \geq 2, \\ 1 & \text{for } l + 1 = k < m \text{ and } k \text{ odd}, \\ \{\pm 1\} & \text{for } l + 1 = k < m \text{ and } k \text{ even}, \\ \ker \Psi' & \text{for } (k, l) = (m, m - 1), \end{cases}$$

where $\Psi' : \{\pm 1\}^m \rtimes S_m \to \{\pm 1\}$ is defined by $\Psi'((\epsilon_1, \epsilon_2, \ldots, \epsilon_m, \sigma)) = \epsilon_1\epsilon_2 \cdots \epsilon_m \cdot (\text{sgn } \sigma)^n$.

**Proof.** Note that $H_*(\mathbb{Q}) : \mathcal{E}(\mathbb{Q}) \cong \{\pm 1\} \to \text{Aut}(H_n(\mathbb{Q}))$ is a monomorphism, and hence that $E_*(\mathbb{Q}) = \{1\}$. Observe from Lemma 1 that

$$(S^n)^m_{(k, k - 1)} \cong \bigvee_{i_1 < i_2 < \cdots < i_k} S^n \wedge \cdots \wedge S^n = \bigvee_{i_1 < i_2 < \cdots < i_k} S^{kn}.$$ 

Then, by Theorem 5 and Proposition 2, we have the result for $(k, l) \neq (m, m - 1)$. See [7, Proposition 8] for the case of $(k, l) = (m, m - 1)$. \hfill \Box

Next we consider the case where $X = \mathbb{C}P^n$. The following is a generalization of [7, Proposition 9].

**Proposition 4.** Let $n \geq 2$. Then the homomorphism $\Psi_{(k, l)} : \mathcal{E}(\mathbb{C}P^n)^m \rtimes S_m \to \mathcal{E}((\mathbb{C}P^n)^m_{(k, l)})$ is a monomorphism for $0 \leq l < k \leq m$. In particular, $\mathcal{E}((\mathbb{C}P^n)^m_{(k, l)})$ contains a subgroup which is isomorphic to $\{\pm 1\}^m \rtimes S_m$.

**Proof.** Since the homomorphism $H_*(\mathbb{Q}) : \mathcal{E}(\mathbb{C}P^n) \cong \{\pm 1\} \to \text{Aut}(\mathcal{H}_*(\mathbb{C}P^n \mathbb{Q}))$ is a monomorphism, $E_*(\mathbb{C}P^n \mathbb{Q}) = \{1\}$, Further, since $H^*(\mathbb{C}P^n \mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1})$ with deg $x = 2$, we obtain the result by Theorem 5 and Remark 4(1). \hfill \Box

**Proposition 5.** Let $A$ be a subring of $\mathbb{Q}$, or $A = \mathbb{Z}/p$ for a prime number $p$. Let $A'$ be the free $A$-module of rank $r$ and $K(A', n)$ the Eilenberg–MacLane complex for $r, n \geq 1$. Assume that one of the following conditions is satisfied:

(i) $A = \mathbb{Z}/p$;
(ii) $r \geq 2$ or $n$ is even;
(iii) \( k - l \geq 2; \)
(iv) \( l + 1 = k < m \) and \( k \) is odd.

Then \( \Psi_{(k,l)} : \mathcal{E}(K(A^r, n))^m \rtimes S_m \to \mathcal{E}(K(A^r, n))_{(k,l)}^m \) is a monomorphism. In particular, \( \mathcal{E}(K(A^r, n))_{(k,l)}^m \) contains a subgroup isomorphic to \( \text{GL}_r(A)^m \rtimes S_m. \)

Proof. Let \( \mathbb{F} \) be the field of fractions of \( A \) (that is, if \( A \subseteq \mathbb{Q} \), then \( \mathbb{F} = \mathbb{Q} \); if \( A = \mathbb{Z}/p \), then \( \mathbb{F} = \mathbb{Z}/p \)) and let \( \ell_0 : A \to \mathbb{F} \) be the canonical inclusion. Then the homomorphism \( H_n(\mathbb{F}; \mathbb{F}) : \mathcal{E}(K(A^r, n)) \to \text{Aut}(H_n(K(A^r, n); \mathbb{F})) \) is identified with the composition of the following monomorphisms (see the proof of [7, Theorem 10]) and hence we have \( \mathcal{E}_a(K(A^r, n); \mathbb{F}) = \{1\} \).

\[
\mathcal{E}(K(A^r, n)) \xrightarrow{H_n(\mathbb{F}; \mathbb{F})} \text{Aut}(H_n(K(A^r, n); \mathbb{F})) \cong \text{GL}_r(A)
\]

\[
\xrightarrow{\ell_0} \text{Aut}(H_n(K(A^r, n); \mathbb{F})) \cong \text{GL}_r(\mathbb{F}).
\]

The case of \( (k, l) \neq (m, m - 1) \). By Theorem 5, we have only to show that each of conditions (i) to (iv) implies one of the conditions (i) to (iv) in Theorem 5. Recall the structure of the cohomology algebra \( H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p) \) from Cartan [3]. Then, we see that condition (i) implies condition (ii) in Theorem 5. Therefore, we may assume that \( A \) is a subring of \( \mathbb{Q} \). Then, conditions (ii), (iii) and (iv) imply conditions (ii), (iii) and (iv) in Theorem 5, respectively.

The case of \( (k, l) = (m, m - 1) \). By using Remark 4(1) instead of Theorem 5, we can prove the result by an argument similar to that in the case of \( (k, l) \neq (m, m - 1) \).

An argument similar to that in the proof of Proposition 5 applies to the case where \( X \) is the co-Moore space \( C(A^r, n) \) for \( A = \mathbb{Z} \) or \( \mathbb{Z}/p \) with \( p \) odd prime (for the co-Moore spaces, see [8, Section 1.1], in which the co-Moore space \( C(G, n) \) is denoted by \( P^n(G) \)).

Since \( C(\mathbb{Z}^r, n) \) is the \( r \)-fold wedge \( (S^n)^{\times r} \) of \( S^n \) and \( [C(\mathbb{Z}^r, n), C(\mathbb{Z}^r, n)] \cong \text{Hom}(\mathbb{Z}^r, \mathbb{Z}^r) \) for \( n \geq 2 \) (see Example 2(2)), we assume that \( r \geq 2 \) and \( n \geq 2 \) in the case of \( A = \mathbb{Z} \) (see Proposition 3). Since \( [C((\mathbb{Z}/p)^r, n), C((\mathbb{Z}/p)^r, n)] \cong \text{Hom}((\mathbb{Z}/p)^r, (\mathbb{Z}/p)^r) \) holds for \( n \geq 4 \) [8, Proposition 1.4.2], we assume that \( n \geq 4 \) in the case of \( A = \mathbb{Z}/p \).

COROLLARY 2. (1) Let \( r \geq 2 \) and \( n \geq 2 \). Then, the kernel of

\[
\text{GL}_r(\mathbb{Z})^m \rtimes S_m \cong \mathcal{E}(C(\mathbb{Z}^r, n))^m \rtimes S_m \xrightarrow{\Psi_{(k,l)}} \mathcal{E}(C(\mathbb{Z}^r, n))_{(k,l)}^m
\]

is given by the following:

\[
\text{Ker } \Psi_{(k,l)} = \begin{cases} 
1 & \text{for } k - l \geq 2, \\
1 & \text{for } l + 1 = k < m \text{ and } k \text{ odd}, \\
\{\pm 1\} & \text{for } l + 1 = k < m \text{ and } k \text{ even}, \\
\{(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \{\pm 1\}^m \mid \alpha_1, \alpha_2 \cdots \alpha_m = 1\} & \text{for } l + 1 = k = m.
\end{cases}
\]

(2) Let \( p \) be an odd prime number and let \( n \geq 4 \). Then, the kernel of

\[
\text{GL}_r(\mathbb{Z}/p)^m \rtimes S_m \cong \mathcal{E}(C((\mathbb{Z}/p)^r, n))^m \rtimes S_m \xrightarrow{\Psi_{(k,l)}} \mathcal{E}(C((\mathbb{Z}/p)^r, n))_{(k,l)}^m
\]
satisfies the following:

\[
\text{Ker } \Psi_{(k,l)}(k,l) = \begin{cases} 
1 & \text{for } k - l \geq 2, \\
\subset C_{\gcd(k, p-1)} & \text{for } l + 1 = k < m, \\
\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subset (\mathbb{F}_{p}^\times)^m \{\alpha_1\alpha_2 \cdots \alpha_m = 1\} & \text{for } l + 1 = k = m.
\end{cases}
\]

Proof. (1) Since \(C(\mathbb{Z}^r, n) \cong (S^n)^{\vee r}\), we have

\[
[C(\mathbb{Z}^r, n), C(\mathbb{Z}^r, n)] \cong \text{Hom}(\mathbb{Z}^r, \mathbb{Z}^r),
\]

and hence

\[
\mathcal{E}(C(\mathbb{Z}^r, n)) \cong \text{GL}_r(\mathbb{Z}).
\]

(see Example 2(2)). Then, the result follows from Theorem 5, Proposition 2 and Remark 4 (see the proofs of Propositions 3 and 5).

(2) By the assumption, we have

\[
[C(\mathbb{Z}/p^r, n), C(\mathbb{Z}/p^r, n)] \cong \text{Hom}(\mathbb{Z}/p^r, \mathbb{Z}/p^r),
\]

[8, Proposition 1.4.2], and hence

\[
\mathcal{E}(C(\mathbb{Z}/p^r, n)) \cong \text{GL}_r(\mathbb{Z}/p).
\]

Then, the result follows from Theorem 5, Proposition 2 and Remark 4 (see the proof of Proposition 5).

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satisfies the following:

\[
\begin{align*}
\ker \Psi_1(k, l) &= 1 \\
&\quad \text{for } k - l \geq 2,
\end{align*}
\]

\[
\subset \gcd(k, p_1 - 1)
\]

\[
\subset \{(\alpha_1, \alpha_2, \ldots, \alpha_m) \in (F \times p)^m \mid \alpha_1 \alpha_2 \cdots \alpha_m = 1\}
\]

\[
\text{for } l + 1 = k = m.
\]

Proof.

(1) Since \(C(Z_r, n) \sim= S^{r} \lor r\), we have \([C(Z_r, n), C(Z_r, n)] \sim= \text{Hom}(Z_r, Z_r)\), and hence \(E(C(Z_r, n)) \sim= \text{GL}_r(Z_r)\) (see Example 2(2)). Then, the result follows from Theorem 5, Proposition 2 and Remark 4 (see the proofs of Propositions 3 and 5).

(2) By the assumption, we have \([C((Z/p)^r, n), C((Z/p)^r, n)] \sim= \text{Hom}((Z/p)^r, (Z/p)^r)\), and hence \(E(C((Z/p)^r, n)) \sim= \text{GL}_r(Z/p)\). Then, the result follows from Theorem 5, Proposition 2 and Remark 4 (see the proof of Proposition 5).

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