The structure of restricted Leibniz algebras

Baoling Guan\textsuperscript{1,2}, Liangyun Chen\textsuperscript{1}

\textsuperscript{1} School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
\textsuperscript{2} College of Sciences, Qiqihar University, Qiqihar 161006, China

Abstract: The paper studies the structure of restricted Leibniz algebras. More specifically speaking, we first give the equivalent definition of restricted Leibniz algebras, which is by far more tractable than that of a restricted Leibniz algebras in [6]. Second, we obtain some properties of $p$-mappings and restrictable Leibniz algebras, and discuss restricted Leibniz algebras with semisimple elements. Finally, Cartan decomposition and the uniqueness of decomposition for restricted Leibniz algebras are determined.

Keywords: Restricted Leibniz algebras; restrictable Leibniz algebras; $p$-mapping; semisimple elements; Tori; Cartan decomposition

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1 Introduction

The concept of a restricted Lie algebra is attributable to N. Jacobson in 1943. It is well known that the Lie algebras associated with algebraic groups over a field of characteristic $p$ are restricted Lie algebras \cite{13}. Now, restricted Lie algebras attract more and more attentions. For example: restricted Lie superalgebras \cite{9}, restricted color Lie algebras \cite{2}, restricted Lie triple system \cite{8} and restricted Leibniz algebras \cite{7} were studied, respectively. As is well known, restricted Lie algebras play predominant roles in the theories of modular Lie algebras \cite{14}. Analogously, the study of restricted Leibniz algebras will play an important role in the classification of the finite-dimensional modular simple Leibniz algebras.

Leibniz algebras were first introduced as nonantisymmetric generalization of Lie algebras in 1979 \cite{10}. In recent years the study of Leibniz algebras over a field of prime characteristic obtained some important results. In \cite{6}, Dzhumadil'daev and Abdykassymova (2001) introduced the notion of restricted Leibniz algebras(left Leibniz algebras).
In [7], the authors mainly proved that there is a functor-$p$-Leib from the category of diassociative algebras to the category of restricted Leibniz algebras (right Leibniz algebras) and constructed its restricted enveloping algebra. As a natural generalization of a restricted Lie algebra, it seems desirable to investigate the possibility of establishing a parallel theory for restricted Leibniz algebras. However, in dealing with a restricted Leibniz algebra, we cannot employ all methods of restricted Lie algebras. This is because the product in Leibniz algebras does not have skew symmetry.

Similar to restricted Lie algebras, the paper gives the structure of restricted Leibniz algebras (left Leibniz algebras). Let us briefly describe the content and setup of the present article. In Sec. 2, the equivalent definition of restricted Leibniz algebras is given, which is by far more tractable than that of a restricted Leibniz algebras in [6]. In Sec. 3, we obtain some properties of $p$-mappings and restrictable Leibniz algebras. In Sec. 4, we study restricted Leibniz algebras whose elements are semisimple. In Sec. 5, Tori and Cartan decomposition of restricted Leibniz algebras are characterized. In Sec. 6, the uniqueness of decomposition for restricted Leibniz algebras is determined.

In the paper, $F$ is a field of prime characteristic. Let $L$ denote a finite-dimensional Leibniz algebra (left Leibniz algebras) over $F$. We write $\mathbb{N}$ for nonnegative integers. For restricted Leibniz algebra, the concepts of homomorphisms and $p$-homomorphisms, derivations, $p$-representations are similar to restricted Lie algebras [13]. Der$L$ is also denoted by the set consisting of all derivations of Leibniz algebra $L$. All other notions and concepts refer to the reference [13].

**Definition 1.1.** [12] A Leibniz algebra over $F$ is an $F$-module $L$ equipped with a bilinear mapping, called bracket,

$[-,-]: L \times L \rightarrow L$

satisfying the Leibniz identity:

$[[x,y],z] = [x,[y,z]] - [y,[x,z]]$

for all $x,y,z \in L$.

**Lemma 1.2.** [13] Let $V$ and $W$ be $F$-vector spaces and $f: V \rightarrow W$ be a $p$-semilinear mapping. Then the following statements hold:

1. $\ker(f)$ is an $F$-subspace of $V$.
2. $f(V)$ is an $F^p$-subspace of $W$. If $F$ is perfect, then $f(V)$ is an $F$-subspace of $W$.
3. $\dim_F V = \dim_F \ker(f) + \dim_{F^p} f(V)$.
4. If $\langle f(V) \rangle = W$ and $\dim_F W = \dim_F V$, then $\ker(f) = 0$.

**Lemma 1.3.** [13] Let $f: V \rightarrow V$ be $p$-semilinear. Then the following statements are equivalent.

1. $\langle f(V) \rangle = V$.
2. For every $v \in V$, there exist $\alpha_1, \cdots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_if^i(v)$. 2
Lemma 1.4. \cite{13} Let $f$ be an endomorphism of a vector space $V$ and let $\chi$ be a polynomial such that $\chi(f) = 0$. Then the following statements hold:

1. If $\chi = q_1 q_2$ and $q_1, q_2$ are relatively prime, then $V$ decomposes into a direct sum of $f$-invariant subspaces $V = U \oplus W$ such that $q_1(f)(U) = 0 = q_2(f)(W)$.

2. $V$ decomposes into a direct sum of $f$-invariant subspaces $V = V_0 \oplus V_1$, for which $f|_{V_0}$ is nilpotent and $f|_{V_1}$ is invertible.

Lemma 1.5. \cite{13} Let $V$ be a vector space over $\mathbb{F}$ and let $x, y$ be elements of $\text{End}_\mathbb{F}(V)$ such that there is $t \in \mathbb{N}\setminus\{0\}$ with $(\text{ad} x)^t(y) = 0$. Suppose that $q \in \mathbb{F}[\chi]$ is a polynomial, then $V_0(q(x))$ is invariant under $y$.

Definition 1.6. \cite{12} Let $H$ be a subspace of Leibniz algebra $L$. $H$ is called a subalgebra of $L$, if $[H, H] \subseteq H$; $H$ is called a left ideal of $L$, if $[L, H] \subseteq H$; $H$ is called a right ideal of $L$, if $[H, L] \subseteq H$.

Definition 1.7. \cite{12} Let $L$ be a Leibniz algebra. The sequence $(L^n)_{n \in \mathbb{N}\setminus\{0\}}$ of Leibniz algebra $L$ given by $L^1 := L$, $L^{n+1} := [L, L^n]$. Then $(L^n)_{n \in \mathbb{N}\setminus\{0\}}$ is the descending central series of $L$. $L$ is called nilpotent, if there is $t \in \mathbb{N}\setminus\{0\}$ such that $L^t = 0$. An abelian Leibniz algebra $L$ is described by the condition $L^2 = 0$.

Definition 1.8. \cite{1} Let $L$ be a Leibniz algebra. The sequence $(L^{[n]})_{n \in \mathbb{N}\setminus\{0\}}$ defined by means of $L^{[1]} := L$, $L^{[n+1]} := [L^n, L^n]$ is called the derived series of $L$. $L$ is called solvable, if there is $t \in \mathbb{N}\setminus\{0\}$ such that $L^{[t]} = 0$.

Theorem 1.9. \cite{3} (Engel’s Theorem) Let $L$ be a Leibniz algebra. Suppose that the left multiplication operator $L_a$ is nilpotent for all $a \in L$. Then $L$ is nilpotent.

Definition 1.10. \cite{4} A bimodule of Leibniz algebra $L$ is a vector space $M$ over $\mathbb{F}$ equipped with two bilinear compositions denoted by $ma$ and $am$, for any $a \in L$ and $m \in M$, satisfy

$$(ma)b = m[a, b] - a(mb),$$

$$(am)b = a(mb) - m[a, b],$$

$$(a, b)m = a(bm) - b(am).$$

In [4], the author denotes by $\text{End}(M)$ the associative algebra of all endomorphisms of the vector space $M$. If $M$ is a bimodule of Leibniz algebra $L$, then each of the mappings $S_a : m \to ma$ and $T_a : m \to am$ is an endomorphism of $M$, and the mappings $S : a \to S_a$, $T : a \to T_a$ of $L$ into $\text{End}(M)$ are linear. Moreover, $L_{[a, b]} = L_a L_b - L_b L_a$ for all $a, b \in L$. Thus the set $\{L_a | a \in L\}$ forms a Lie algebra of linear transformations of $L$.

Definition 1.11. \cite{12} A representation of a Leibniz algebra $L$ on a vector space $M$ is a pair $(S, T)$ of linear maps $S : a \to S_a$, $T : a \to T_a$ of $L$ into $\text{End}(M)$ such that

$$S_a \circ S_b = S_{[a, b]} - T_a \circ S_b,$$

$$S_b \circ T_a = T_a \circ S_b - S_{[a, b]};$$

$$T_{[a, b]} = T_a \circ T_b - T_b \circ T_a$$

for all $a, b \in L$. 

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The reference [12] also pointed out that the vector space $M$ equipped with the compositions $ma = S_a(m)$ and $am = T_a(m)$ is a bimodule of $L$. Clearly, the two concepts of representation and bimodule are equivalent. Let $L$ be a Leibniz algebra. The right multiplication $R_a$ (resp., the left multiplication $L_a$) of $L$ determined by any element $a \in L$ is the endomorphism of $L$ defined by $R_a(x) = [x, a]$ (resp., $L_a(x) = [a, x]$) for all $x \in L$. The pair $(R, L)$ of linear mappings $R : a \rightarrow R_a, L : a \rightarrow L_a$ is a representation of $L$ on $L$ itself. In particular, $L : a \rightarrow L_a$ is called the adjoint representation of $L$.

2 The equivalent definition of restricted Leibniz algebras

Definition 2.1. [13] Let $L$ be a Lie algebra over $\mathbb{F}$. A mapping $[p] : L \rightarrow L, a \mapsto a^{[p]}$ is called a $p$-mapping, if

1. $L^{[p]} = (L_a)^p, \forall a \in L$.
2. $(aa)^{[p]} = a^p a^{[p]}, \forall a \in L, \alpha \in \mathbb{F}$.
3. $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b),$

where $(L(a \otimes X + b \otimes 1))^{p-1}(a \otimes 1) = \sum_{i=1}^{p-1} is_i(a, b) \otimes X^{i-1}$ in $L \otimes \mathbb{F}[X], \forall a, b \in L$. The pair $(L, [p])$ is referred to as a restricted Lie algebra.

Definition 2.2. Let $L$ be a Leibniz algebra over $\mathbb{F}$. A mapping $[p] : L \rightarrow L, a \mapsto a^{[p]}$ is called a $p$-mapping, if

1. $L^{[p]} = (L_a)^p, \forall a \in L$.
2. $(aa)^{[p]} = a^p a^{[p]}, \forall a \in L, \alpha \in \mathbb{F}$.
3. $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b),$

where $(L(a \otimes X + b \otimes 1))^{p-1}(a \otimes 1) = \sum_{i=1}^{p-1} is_i(a, b) \otimes X^{i-1}$ in $L \otimes \mathbb{F}[X], \forall a, b \in L$. The pair $(L, [p])$ is referred to as a restricted Leibniz algebra.

Clearly, any restricted Lie algebra is a restricted Leibniz algebra. Let $L$ be a Leibniz algebra over $\mathbb{F}$ and $f : L \rightarrow L$ be a mapping. $f$ is called a $p$-semilinear mapping, if $f(\alpha x + y) = \alpha^p f(x) + f(y), \forall x, y \in L, \forall \alpha \in \mathbb{F}$. Let $S$ be a subset of Leibniz algebra $L$. We put $Z_L(S) = \{x \in L | [x, S] = 0\}$ and $C_L(S) = \{x \in L | [S, x] = 0\}$. $Z_L(S)$ and $C_L(S)$ are called the right centralizer of $S$ in $L$ and the left centralizer of $S$ in $L$, respectively. $Z(L) = \{x \in L | [x, L] = 0\}$ is called the right center of $L$; $C(L) = \{x \in L | [L, x] = 0\}$ is called the left center of $L$. Let $V$ be a subspace of $L$. We put $\text{Nor}_L(V) = \{x \in L | [V, x] \subseteq V\}$. $\text{Nor}_L(V)$ is called the left normalizer of $V$ in $L$.

Proposition 2.3. Let $L$ be a Leibniz algebra over $\mathbb{F}$. Then the following states hold:
(1) \( I = \langle [x^{[p]}_i, x^{[p]}_j] | x \in L, i, j \in \mathbb{N} \rangle \) is contained in \( Z(L) \).

(2) If \( Z(L) = 0 \), then \( L \) is a Lie algebra.

Proof. (1) For \( i, j \in \mathbb{N} \), then
\[
[[x^{[p]}_i, x^{[p]}_j], y] = [x^{[p]}_i, [x^{[p]}_j, y]] - [x^{[p]}_j, [x^{[p]}_i, y]] = 0.
\]
We have \( [x^{[p]}_i, x^{[p]}_j] \in Z(L) \). Consequently, \( I = \langle [x^{[p]}_i, x^{[p]}_j] | x \in L, i, j \in \mathbb{N} \rangle \subseteq Z(L) \).

(2) By (1), \( [x, x] \in Z(L) \). If \( Z(L) = 0 \), then \( [x, x] = 0 \). Hence \( L \) is a Lie algebra. \( \square \)

**Definition 2.4.** Let \( (L, [p]) \) be a restricted Leibniz algebra over \( \mathbb{F} \). A subalgebra (ideal or left ideal) \( H \) of \( L \) is called a \( p \)-subalgebra (\( p \)-ideal or \( p \)-left ideal) of \( L \), if \( x^{[p]} \in H, \forall x \in H \).

**Proposition 2.5.** Let \( L \) be a subalgebra of a restricted Leibniz algebra \( (G, [p]) \) and \( [p]_1 : L \to L \) a mapping. Then the following statements are equivalent:

1. \( [p]_1 \) is a \( p \)-mapping on \( L \).
2. There exists a \( p \)-semilinear mapping \( f : L \to Z_G(L) \) such that \( [p]_1 = [p] + f \).

Proof. (1) \( \Rightarrow \) (2). Consider \( f : L \to G \), \( f(x) = x^{[p]_1} - x^{[p]} \). Since \( L_{f(x)}(y) = 0, \forall x, y \in L \), \( f \) actually maps \( L \) into \( Z_G(L) \). For \( x, y \in L, \alpha \in \mathbb{F} \), we obtain
\[
\begin{align*}
f(\alpha x + y) &= \alpha x^{[p]_1} + y^{[p]_1} + \sum_{i=1}^{p-1} s_i(\alpha x, y) - \alpha x^{[p]} - y^{[p]} - \sum_{i=1}^{p-1} s_i(\alpha x, y) \\
&= \alpha p f(x) + f(y),
\end{align*}
\]
which proves that \( f \) is \( p \)-semilinear.

(2) \( \Rightarrow \) (1). We only check the property pertaining to the sum of two elements \( x, y \in L \),
\[
\begin{align*}
(x + y)^{[p]_1} &= (x + y)^{[p]} + f(x + y) \\
&= x^{[p]} + f(x) + y^{[p]} + f(y) + \sum_{i=1}^{p-1} s_i(x, y) \\
&= x^{[p]_1} + y^{[p]_1} + \sum_{i=1}^{p-1} s_i(x, y).
\end{align*}
\]
The proof is complete. \( \square \)
**Corollary 2.6.** The following statements hold.

1. If $Z(L) = 0$, then $L$ admits at most one $p$-mapping.
2. If two $p$-mappings coincide on a basis, then they are equal.
3. If $(L, [p])$ is restricted, there exists a $p$-mapping $[p]'$ of $L$ such that $x[p]' = 0$, $\forall x \in Z(L)$.

**Proof.** (1) We set $G = L$. Then $Z_G(L) = Z(L)$, the only $p$-semilinear mapping occurring in Proposition 2.5 is the zero mapping.

(2) If two $p$-mappings coincide on a basis, their difference vanishes since it is $p$-semilinear.

(3) $[p]|_{Z(L)}$ defines a $p$-mapping on $Z(L)$. Since $Z(L)$ is abelian, it is $p$-semilinear. Extend this to a $p$-semilinear mapping $f : L \to Z(L)$. Then $[p]' := [p] - f$ is a $p$-mapping of $L$, vanishing on $Z(L)$.

In the special case of $G = U(L)^- \supset L$, where $U(L)$ is the universal enveloping algebra of $L$ (see [11]), we obtain

**Theorem 2.7.** Let $(e_j)_{j \in J}$ be a basis of $L$ such that there are $y_j \in L$ with $(L_{e_j})^p = L_{y_j}$.
Then there exists exactly one $p$-mapping $[p] : L \to L$ such that $e_j[p] = y_j, \forall j \in J$.

**Proof.** For $z \in L$, we have $0 = ((L_{e_j})^p - L_{y_j})(z) = [e_j^p - y_j, z]$. Then $e_j^p - y_j \in Z_{U(L)}(L), \forall j \in J$. We define a $p$-semilinear mapping $f : L \to Z_{U(L)}(L)$ by means of

$$f(\sum \alpha_j e_j) := \sum \alpha_j^p(y_j - e_j^p).$$

Consider $V := \{x \in L | x^p + f(x) \in L\}$. The equation

$$(\alpha x + y)^p + f(\alpha x + y) = \alpha^p x^p + y^p + \sum_{i=1}^{p-1} s_i(\alpha x, y) + \alpha^p f(x) + f(y)$$

ensures that $V$ is a subspace of $L$. Since it contains the basis $(e_j)_{j \in J}$, we conclude that $x^p + f(x) \in L, \forall x \in L$. By virtue of Proposition 2.5, $[p] : L \to L, x[p] := x^p + f(x)$ is a $p$-mapping on $L$. In addition, we obtain $e_j[p] = e_j^p + f(e_j) = y_j$, as asserted. The uniqueness of $[p]$ follows from Corollary 2.6.

**Definition 2.8.** A Leibniz algebra $L$ is called restrictable, if $L_L$ is a $p$-subalgebra of $\text{Der}(L)$, that is, $(L_x)^p \in L_L, \forall x \in L$, where $L_L = \{L_x | x \in L\}$, $\text{Der}(L) = \{D \in \text{gl}(L) | D[x, y] = [D(x), y] + [x, D(y)], \forall x, y \in L\}$.

**Theorem 2.9.** $L$ is a restrictable Leibniz algebra if and only if there is a $p$-mapping $[p] : L \to L$ which makes $L$ a restricted Leibniz algebra.
Proof. \( (\Leftarrow) \) By the definition of \( p \)-mapping \([p]\), we have \((L_x)^p = L_{x[p]} \in L_y, \forall x \in L\). Hence \( L \) is restrictable.

\( (\Rightarrow) \) Let \( L \) be restrictable. Then for \( x \in L \), we have \((L_x)^p \in L_L \), that is, there exists \( y \in L \) such that \((L_x)^p = L_y\). Let \((e_j)_{j \in J} \) be a basis of \( L \). Then there exist \( y_j \in L \) such that \((L_{e_j})^p = L_{y_j} (j \in J)\). By Theorem 2.9, then there exists exactly one \( p \)-mapping \([p]: L \to L \) such that \( e_j^p = y_j, \forall j \in J \), which makes \( L \) a restricted Leibniz algebra.

Definition 2.10. \( \bullet \) A Leibniz algebra \( L \) over \( \mathbb{F} \) is called restricted, if for any \( x \in L \), there exists some \( x[p] \in L \) such that \((L_x)^p = L_{x[p]}\).

Theorem 2.11. Definition 2.10 is equivalent to Definition 2.2.

Proof. If \([p]\) satisfies \((L_x)^p = L_{x[p]}, \forall x \in L\). By Definition 2.8, \( L \) is restrictable. By Theorem 2.9, \( L \) satisfies Definition 2.2. Conversely, it is clear. Hence Definition 2.10 is equivalent to Definition 2.2

Remark 2.12. Definition 2.10 is by far more tractable than Definition 2.2, but just for convenient use it, we give the Definition 2.2.

3 Properties of \( p \)-mappings and restrictable Leibniz algebras

One advantage in considering restrictable Leibniz algebras instead of restricted ones rests on the following theorem.

Theorem 3.1. Let \( f: L_1 \to L_2 \) be a surjective homomorphism of Leibniz algebra. If \( L_1 \) is restrictable, so is \( L_2 \).

Proof. Since \( f \) is a surjective mapping, one gets \( L_2 = f(L_1) \). Then

\[
(L_{f(x)})^p(f(y)) = f(x, \cdots [f(x), f(y)] \cdots ) = f(x, \cdots [x, y] \cdots ) = f((L_x)^p(y))
\]

Since \( L_1 \) is restrictable, we have \((L_{f(x)})^p = L_{f(x[p])} \in L_{L_2} \). Hence \( L_2 \) is restrictable.

Definition 3.2. Let \((L, [p])\) be a restricted Leibniz algebra. A derivation \( D \) is called a restricted derivation, if \( D(a[p]) = (L_a)^{p-1}(D(a)) \).

Definition 3.3. Let \( A \) be a Leibniz algebra and \( B \) be a Lie algebra and \( \varphi: A \to \text{Der}(B) \) a homomorphism. On the vector space \( A \oplus B \), define a multiplication by means of

\[
[(a, b), (a', b')] := ([a, a'], \varphi(a)(b') - \varphi(a')(b) + [b, b']).
\]

This algebra, which is denoted by \( A \oplus_{\varphi} B \), is called the semidirect product of \( A \) and \( B \).
Theorem 3.4. Notions such as Definition 3.3 then $A \oplus \varphi B$ is a Leibniz algebra.

Proof. Let $(a, b), (a', b'), (a'', b'') \in A \oplus \varphi B$, $k, k' \in \mathbb{F}$. Then

$$[k(a, b) + k'(a', b'), (a'', b'')]$$
$$= [(ka + k'a', kb + k'b'), (a'', b'')]$$
$$= (k[a, a''] + k'[a', a''], \varphi(ka + k'a')(b'') - \varphi(a'')(kb + k'b') + k[b, b'] + k'[b', b'']).$$

On the other hand, one gets

$$k[(a, b), (a', b')] + k'[(a', b'), (a'', b'')]$$
$$= k((a, a'') - (a''', a'')) - (a', b') + [b, b''] + (a', a'') - (a'', b'') - (a', b') + [b, b'']$$
$$= (k[a, a''] + k'[a', a''], k\varphi(a)(b'') - k\varphi(a')(b') + k[b, b'] + k'[b', b''] + k'\varphi(a')(b'') - k'\varphi(a')(b')$$
$$= (k[a, a''] + k'[a', a''], \varphi(ka + k'a')(b'') - \varphi(a'')(kb + k'b') + k[b, b'] + k'[b', b'']).$$

Hence $[k(a, b) + k'(a', b'), (a'', b'')] = k[(a, b), (a', b')] + k'[(a', b'), (a'', b'')]$.

Note that $\varphi(a, a') = \varphi(a')(a') - \varphi(a'')(a)$. Moreover, we have

$$[[a, b], (a', b')] + [a, [b, b']] + [a', [b', b'']]$$
$$= [[a, a'], (a''', a'')] - [a, b], [(a', b'), (a'', b'')] + [(a', b'), (a'', b'')] + [(a', b'), (a', b')]$$
$$= (k[a, a''] + k'[a', a''], k\varphi(a)(b'') - k\varphi(a')(b') + k[b, b'] + k'[b', b''].$$

As a result, $A \oplus \varphi B$ is a Leibniz algebra. The result follows.

Theorem 3.5. Let $(A, [p])$ be a restricted Leibniz algebra and $(B, [p])$ be a restricted Lie algebra. If $\varphi : A \to \text{Der}(B)$ be restricted homomorphism such that $\varphi(x)$ is restricted for every $x \in A$, then $A \oplus \varphi B$ is restricted.
Proof. Let \( x \in A \). Then \((L_x)^p - L_{x[p]}|_A = 0\) and \((L_x)^p - L_{x[p]}|_B = \varphi(x)^p - \varphi(x[p]) = 0\) holds, hence \((L_x)^p \in L_{A \oplus B}, \forall x \in A\). If \( x \in B \), then \((L_x)^p - L_{x[p]}|_B = 0\) and for \( y \in A \), we obtain
\[
((L_x)^p - L_{x[p]})(y) = -(L_x)^{p-1} \circ \varphi(y)(x) + \varphi(y)(x[p]) = 0,
\]
hence \((L_x)^p \in L_{A \oplus B}, \forall x \in B\). Therefore, \( A \oplus \varphi B \) is restrictable by Theorem 2.7.

Corollary 3.6. Let \( A, B \) be ideals of a Leibniz algebra \( L \) such that \( L = A \oplus B \). Then \( L \) is restrictable if and only if \( A, B \) are restrictable.

Proof. If \( A, B \) are restrictable, by Theorem 3.5 and setting \( \varphi = 0 \), we conclude that \( L \) is restrictable. If \( L \) is restrictable, so are \( A \cong L/B, B \cong L/A \) by Theorem 3.1.

Corollary 3.7. Let \( A, B \) be restrictable ideals of a Leibniz algebra \( L \) such that \( L = A + B \) and \([A, B] = [B, A] = 0\). Then \( L \) is restrictable.

Proof. Define a mapping \( f : A \oplus B \to L, (x, y) \mapsto x + y \). Clearly, \( f \) is a surjective homomorphism. For \((x_1, y_1), (x_2, y_2) \in A \oplus B\), by \([A, B] = [B, A] = 0\), one gets \([x_1, y_2] = [y_1, x_2] = 0\). We have
\[
f([x_1, y_1], [x_2, y_2]) = [x_1 + y_1, x_2 + y_2] = [x_1, x_2] + [y_1, y_2] = [f(x_1, y_1), f(x_2, y_2)].
\]
By Corollary 3.6, we have \( A \oplus B \) is restrictable. By Theorem 3.1, one gets \( L \) is restrictable.

Definition 3.8. Let \( L \) be a Leibniz algebra and \( \psi \) be a symmetric bilinear form on \( L \). \( \psi \) is called associative, if \( \psi([x, z], y) = \psi(x, [z, y]) \).

Definition 3.9. Let \( L \) be a Leibniz algebra and \( \psi \) a symmetric bilinear form on \( L \). Set \( L^\perp = \{ x \in L | \psi(x, y) = 0, \forall y \in L \} \). \( L \) is called nondegenerate, if \( L^\perp = 0 \).

Theorem 3.10. Let \( L \) be a subalgebra of the restricted Leibniz algebra \((G, [p])\). Assume \( \lambda : G \times G \to F \) to be an associative symmetric bilinear form, which is nondegenerate on \( L \times L \). Then \( L \) is restrictable.

Proof. Since \( \lambda \) is nondegenerate on \( L \times L \), every linear form \( f \) on \( L \) is determined by a suitably chosen element \( y \in L \) : \( f(z) = \lambda(y, z), \forall z \in L \). Let \( x \in L \). Then there exists \( y \in L \) such that
\[
\lambda(x[p], z) = \lambda(y, z), \forall z \in L.
\]
This implies that \( 0 = \lambda(x[p] - y, L^{(1)}) = \lambda([x[p] - y, L], L) \) and \([x[p] - y, L] = 0\). Therefore, we have
\[
(L_x|_L)^p = L_{x[p]}|_L = L_y|_L,
\]
proving that \( L \) is restrictable.
Corollary 3.11. Let \((S,T)\) be a finite-dimensional representation of \(L\) such that \(k_T\) is nondegenerate on \(L\times L\), where \(S : L \rightarrow \text{End}(M)\) and \(T : L \rightarrow \text{End}(M)\). Then \(L\) is restrictable.

Proof. The associative form \((x,y) \mapsto \text{tr}(x,y)\) on \(\text{End}(M) \times \text{End}(M)\) is nondegenerate on \(T(L) \times T(L)\). Hence \(T(L)\) is restrictable, since \(T\) is faithful, \(L\) is restrictable. \(\□\)

Proposition 3.12. Let \(L\) be a restrictable Leibniz algebra and \(H\) a subalgebra of \(L\). Then \(H\) is a \(p\)-subalgebra for some mapping \([p]\) on \(L\) if and only if \((L_H|L)^p \subseteq L_H|L\).

Proof. \((\Rightarrow)\) If \(H\) is a \(p\)-subalgebra, then \(x^{[p]} \in H\), \(\forall x \in H\). \(L_x^p = L_x^{[p]} \subseteq L_H|L\).
Hence, \((L_H|L)^p \subseteq L_H|L\).

\((\Leftarrow)\) If \((L_H|L)^p \subseteq L_H|L\), then \(H\) is restrictable. By Theorem 3.9, \(H\) is restricted. Therefore, \(H\) is a \(p\)-subalgebra of \(L\). \(\□\)

Proposition 3.13. Let \(L\), \(L'\) be restrictable Leibniz algebras and \(f : L \rightarrow L'\) a surjective homomorphism. If \(Z(L') = 0\), then \(\ker(f)\) is a \(p\)-ideal for every \(p\)-mapping on \(L\).

Proof. Clearly, \(\ker(f) \leq L\). Since \(L\) is restrictable, there exists \(y \in L\) such that \((L_x)^p = L_y\), \(\forall x \in \ker(f)\). For \(z \in L\), we have \((L_x)^p(z) = L_y(z)\), i.e., \([x, \cdots [x, [x, z], \cdots] = [y, z]\). Since \(f\) is a homomorphism mapping, \([f(x), \cdots [f(x), [f(x), f(z)], \cdots] = [f(y), f(z)]\), that is, \((L_f(x))^p(f(z)) = L_{f(y)}(f(z))\). Since \(f\) is a surjective mapping, one gets \(L' = \{f(z)|z \in L\}\), hence \((L_f(x))^p = L_{f(y)}\). By Theorem 3.11, we have \(L'\) is restrictable. Moreover, \(L_{f(x)^{[p]}} = L_{f(y)}\). By \(Z(L') = 0\), one gets \(f(y) = f(x)^{[p]} = 0, y \in \ker(f)\). \((L_x)^p = L_y \in \ker(f),\) that is, \(\ker(f)\) is restrictable. Therefore, \(\ker(f) \trianglelefteq p L\). \(\□\)

Theorem 3.14. Let \((L,[p])\) be a restrictable Leibniz algebra and \(D\) a derivation. Then \(D(x^{[p]}) - (L_x)^{p-1}(D(x)) \in Z(L), \forall x \in L\).

Proof. Let \(D \in \text{Der}(L)\) and \(a, x \in L\). If \(A\) is the transformation \(x \mapsto [a,x]\) and \(B\) is the transformation \(x \mapsto [D(a),x]\), then \(A = L_a, B = L_{D(a)}\). We can prove \((L_A)^k(B) = \sum_{i=0}^{k}(-1)^{k-i}C_{k-i}^i A^i B A^{k-i}\) by induction on \(k\).

Then by the result, we have
\[
(L_A)^{p-1}(B) = \sum_{i=0}^{p-1}(-1)^{p-1-i}C_{p-1}^i A^i B A^{p-1-i}.
\]

Since
\[
C_{p-1}^i = \frac{(p-1)(p-2)\cdots(p-i)}{i \cdot (i-1) \cdots 1} = \frac{(-1)(-2)\cdots(-i)}{i \cdot (i-1) \cdots 1} = (-1)^i,
\]
we have \((-1)^{p-1-i}C_{p-1}^i = (-1)^{p-1} = 1\). So
\[
BA^{p-1} + ABA^{p-2} + \cdots + A^{p-1}B = [A, \cdots [A, B] \cdots].
\]
Then
\[ D[a^p, x] = D[a, \cdots [a, x] \cdots] \]
\[ = [D(a), \cdots [a, x] \cdots] + \cdots + [a, \cdots [a, D(a)] \cdots x]. \]

On the other hand, we have \( D[a^p, x] = [D(a^p), x] + [a^p, D(x)] \) since \( D \) is a derivation. Hence \( [D(a^p), x] = [a, a \cdots [a, D(a)] \cdots x] \) for all \( x \in L \), that is, \( D(a^p) - (L_a)^{p-1}(D(a)) \in Z(L), \forall a \in L. \)

\[ \square \]

**Corollary 3.15.** Let \((L, [p])\) be a restricted Leibniz algebra. If \( Z(L) = 0 \), then every derivation of \( L \) is a restricted derivation.

**Corollary 3.16.** Let \((L, [p])\) be a restricted Lie algebra with trivial center. Then every derivation of \( L \) is a restricted derivation.

Let \( S \subseteq L \) be a subset of a restricted Leibniz algebra \((L, [p])\). The intersection of all \( p \)-subalgebras containing \( S \) will be denoted by \( S_p \). \( S_p \) is a \( p \)-subalgebra generated by \( S \) in \( L \). By definition, \( S_p \) is the smallest \( p \)-subalgebra of \((L, [p])\) containing \( S \).

We propose to give a more explicit characterization of \( S_p \) in some special cases. The image of \( S \) under the iterated application of the \( p \)-mapping \([p]\) will be denoted by \( S[p]^i \), that is, \( S[p]^i := \{x[p] | x \in S\} \).

**Proposition 3.17.** Let \((L, [p])\) be a restricted Leibniz algebra over \( \mathbb{F} \) and \( H \subseteq L \) a left ideal. Suppose that \((e_j)_{j \in J}\) is a basis of \( H \). Then

1. \( H_p = \sum_{i \in \mathbb{N}} \langle H[p]^i \rangle = \sum_{j \in J, i \in \mathbb{N}} \mathbb{F}e_j^{[p]^i} \).
2. \([H_p, L] = [H, L]; (H_p)^n = H^n, (H_p)^{n+1} = H^{n+1}, n \geq 1. \)
3. \( H_p \) is solvable (nilpotent) if and only if \( H \) is solvable (nilpotent).
4. \( H_p \) is a \( p \)-left ideal.

**Proof.**

(1) Put \( G := \sum_{j \in J, i \in \mathbb{N}} \mathbb{F}e_j^{[p]^i} \). Then, clearly, \( H \subseteq G \subseteq \sum_{i \geq 0} \langle H[p]^i \rangle \subseteq H_p \). To prove \( H_p \subseteq G \), we observe that by property (1) of the definition of \( p \)-mapping, \([e_k^{[p]^i}, e_l^{[p]^j}] = L_{e_k^{[p]^i}}(e_l^{[p]^j}) = (L_{e_k})^{p^i}(e_l^{[p]^j}) = (L_{e_k})^{p^i-1}[e_k, e_l^{[p]^j}] \in H^2 \subseteq H \subseteq G \). Hence \( G \) is a subalgebra.

Put \( V = \{x \in G | x[p] \in G\} \). Since \( G \) is a subalgebra, \( (2) \) and \( (3) \) of the definition of \( p \)-mapping prove that \( V \) is a subspace containing the generating set \( \{e_j^{[p]^i} | j \in J, i \geq 0\} \) of \( G \). Hence \( V = G \) and \( G \) is closed under the \( p \)-mapping. Consequently, \( G \) is a \( p \)-subalgebra containing \( H \) and \( H_p \subseteq G \).

(2) Considering (1) and (2) of the definition, we get \([H_p, L] \subseteq [H, L] \).

(3) It follows from (2).

(4) By (2), we have \([H_p, L] = [H, L] \subseteq H \subseteq H_p \), \( H_p \) is a left-ideal of \( L \). Moreover, \( H_p \) is a \( p \)-subalgebra of \( L \). Hence \( H_p \) is a \( p \)-left ideal of \( L \). \( \square \)
4 Restricted Leibniz algebras whose elements are semisimple

Definition 4.1. Let \((L, [p])\) be a restricted Leibniz algebra over \(\mathbb{F}\). An element \(x \in L\) is called semisimple if \(x = \sum_{i=1}^{m} \alpha_i x^{[p]^i}\) and toral if \(x^{[p]} = x\).

Proposition 4.2. Let \((L, [p])\) be a restricted Leibniz algebra over \(\mathbb{F}\). Then the following statements hold:

1. Every toral element is semisimple.
2. If \(x\) is semisimple, then \(T(x)\) is semisimple for every finite-dimensional \(p\)-representation \((S, T)\), where \(S : L \to \text{End}(M), T : L \to \text{End}(M)\).
3. If \(\mathbb{F}\) is perfect and \([p]\) is nonsingular, then every element \(x \in L\) is semisimple.
4. An endomorphism \(\sigma \in \text{End}(L)\) is semisimple if and only if it is semisimple as an element of the restricted Leibniz algebra \((\text{gl}(L), [p])\).

Proof. (1) Clearly.

(2) Let \((S, T)\) be a finite-dimensional \(p\)-representation. Then \(T(x^{[p]}) = T(x)^{p}\) and the semisimplicity of \(x\) ensures the existence of \(\alpha_1, \ldots, \alpha_n \in \mathbb{F}\) such that \(T(x) = \sum_{i=1}^{n} \alpha_i T(x)^{p^i}\).

Let \(m_x\) be the minimum polynomial of \(T(x)\). Then there is \(\lambda \in \mathbb{F}[x]\) such that \(\lambda m_x = \sum_{i=1}^{n} \alpha_i x^{p^i} - x\). Taking the derivative we obtain \(\lambda' m_x + \lambda m'_x = -1\), which means that \(T(x)\) is semisimple.

(3) Let \(x\) be an element of \(L\setminus\{0\}\). \(L\) is finite-dimensional and there is a minimal element \(m \in \mathbb{N}\setminus\{0\}\) such that \(x^{[p]^m} \in \langle\{x, \ldots, x^{[p]^{m-1}}\}\rangle\). The set \(\{x, \ldots, x^{[p]^{m-1}}\}\) is therefore linearly independent. We find \(\alpha_1, \ldots, \alpha_m \in \mathbb{F}\) such that \(x^{[p]^m} = \sum_{i=1}^{m} \alpha_i x^{[p]^{i-1}}\).

The assumption \(\alpha_1 = 0\) forces \(x^{[p]^{m-1}} - \sum_{i=2}^{m} \alpha_i^{1/p} x^{[p]^{i-2}}\) to be a zero of \([p]\), thus \(x^{[p]^{m-1}} \in \langle\{x, \ldots, x^{[p]^{m-2}}\}\rangle\). This contradicts the choice of \(m\). We have \(\alpha_1 \neq 0\). Thus \(x = \alpha_1^{-1} x^{[p]^m} - \sum_{i=2}^{m} \alpha_i \alpha_1^{-1} x^{[p]^{i-1}}\). Hence \(x\) is semisimple.

(4) If \(\sigma\) is a semisimple element of \((\text{gl}(L), [p])\), then (2) entails the semisimplicity of \(\sigma\). Assume conversely that \(\sigma\) is semisimple. Let \(\overline{\mathbb{F}}\) denote an algebraic closure of \(\mathbb{F}\). Then \(\overline{\sigma} := \sigma \otimes \text{id}_{\overline{\mathbb{F}}}\) is a diagonalizable endomorphism of \(M \otimes_{\mathbb{F}} \overline{\mathbb{F}}\). Consequently, \(\overline{\mathbb{F}}[\overline{\sigma}] \subseteq \text{End}_{\overline{\mathbb{F}}}(M \otimes_{\mathbb{F}} \overline{\mathbb{F}})\) does not contain any nonzero nilpotent elements. On the basis of Lemma 1.2 (4), this implies, as \(\overline{\mathbb{F}}\) is perfect, the surjectivity of \(p : \overline{\mathbb{F}}[\overline{\sigma}] \to \overline{\mathbb{F}}[\overline{\sigma}]\). Hence \(\langle \overline{\mathbb{F}}[\overline{\sigma}^p] \rangle = \overline{\mathbb{F}}[\overline{\sigma}]\), as desired.

Theorem 4.3. Let \((L, [p])\) be a restricted Leibniz algebra over \(\mathbb{F}\). For every \(x \in L\), there exists \(k \in \mathbb{N}\setminus\{0\}\) such that \(x^{[p]^k}\) is semisimple.
Proof. The family \((x^{[p]^k})_{i\geq 0}\) is linearly dependent. Then there exist \(k \geq 0, \alpha_1, \ldots, \alpha_n \in \mathbb{F}\) such that 
\[ x^{[p]^k} = \sum_{i=1}^{n} \alpha_i x^{[p]^{k+i}}. \]
This means that \(x^{[p]^k}\) is semisimple. \(\square\)

Proposition 4.4. Let \((L, [p])\) be a restricted Leibniz algebra over a perfect field \(\mathbb{F}\). Then the following states hold:

1. \([p]\) is injective if and only if \([p]\) is nonsingular.
2. If \([p]\) is nonsingular, then \([p]\) is surjective.

Proof. (1) Let \([p]\) be injective and \(0 \neq x \in L\). If \(x^{[p]} = 0\), then \(x \in \ker[p]\), which implies \(\ker[p] \neq 0\). This is a contradiction. Hence \(x^{[p]} \neq 0\). i.e., \([p]\) is nonsingular. Conversely, \(x^{[p]} = 0, \forall x \in \ker[p]\). Since \([p]\) is nonsingular, then \(x = 0\). Hence \(\ker[p] = 0\), i.e., \([p]\) is injective.

(2) Suppose that \([p]\) is nonsingular. Let \(x\) be an element of \(L\). Using Proposition 4.2 (3), we conclude that \(x\) is semisimple, 
\[ x = \sum_{i=1}^{m} \alpha_i x^{[p]^i} = \left( \sum_{i=1}^{m} \alpha_i^{1/p} x^{[p]^{i-1}} \right)^{[p]} \]
We get \(\sum_{i=1}^{m} \alpha_i^{1/p} x^{[p]^{i-1}} \in L\), since \(\mathbb{F}\) is perfect. Hence \(x\) is an image under \([p]\). \(\square\)

Theorem 4.5. Let \((L, [p])\) be a restricted Leibniz algebra over a perfect field \(\mathbb{F}\). Then the following states are equivalent:

1. Every element of \(L\) is semisimple.
2. \([p]\) has no nontrivial zero.
3. \([p]\) is nonsingular.

Proof. (1) \(\Rightarrow\) (3) Let \(x \in L\setminus\{0\}\). By (1), 
\[ x = \sum_{i=1}^{m} \alpha_i x^{[p]^i}. \]
If \(x^{[p]} = 0\), then \(x = 0\), this is a contradiction, hence \(x^{[p]} \neq 0\). i.e., \([p]\) is nonsingular.

(3) \(\Rightarrow\) (1) By Proposition 4.2 (3), (1) holds.

(2) \(\Leftrightarrow\) (3) Clearly. \(\square\)

Proposition 4.6. Let \(\mathbb{F}\) be perfect and \((L, [p])\) a restricted Leibniz algebra such that \([p]\) is nonsingular. Then for any \(p\)-subalgebra \(H\) of \(L\) the implication 
\[ x^{[p]^r} \in H \Rightarrow x \in H \]
holds.

Proof. Since \([p]\) is nonsingular on \(H\), by Proposition 4.4 \([p]\) is surjective. We therefore find \(y \in H\) such that 
\(x^{[p]^r} = y^{[p]^r}\). Since \([p]\) is injective on \(L\), we conclude that \(x = y \in H\). \(\square\)

Proposition 4.7. Let \(\mathbb{F}\) be perfect and \((L, [p])\) a solvable restricted Leibniz algebra with a nonsingular \([p]\)-mapping. Then \(L\) is abelian.
Proof. Let $n \geq 0$ be the minimal integer with respect to the condition $L^{(n)} \subseteq Z(L)$. If $n > 0$, then $x^{[p]} \in Z(L)$ holds for any $x \in L^{(n-1)}$. Hence by Proposition 4.6, $x \in Z(L)$, contradicting the choice of $n$. Therefore, we obtain $n = 0$ and so $L = L^{(0)} = Z(L)$. □

5 Tori and Cartan decomposition

Definition 5.1. Let $Q$ be a Leibniz algebra and $M$ be a $Q$-module. $M$ is called symmetric, if $[x,m] + [m,x] = 0$, for any $x \in Q, m \in M$.

Accordingly, we have the following definition.

Definition 5.2. A representation $(S,T)$ of a Leibniz algebra $L$ on the vector space $M$ is called symmetric, if $S_a + T_a = 0$ for all $a \in L$.

We have the following Theorems 5.3 and 5.5, whose proofs are analogous to restricted Lie algebra(cf. [13]).

Theorem 5.3. Let $H$ be a nilpotent Leibniz algebra and $(S,T)$ a finite-dimensional symmetric representation, where $S : H \rightarrow \text{End}(M), T : H \rightarrow \text{End}(M)$. Then there exists a finite set $B \subseteq \text{Map}(H,F[\chi])$ such that

(1) $\pi_h$ is irreducible, $\forall \pi \in B, \forall h \in H$.

(2) $M_\pi$ is an $H$-submodule, $\forall \pi \in B$.

(3) $M = \bigoplus_{\pi \in B} M_\pi$.

Definition 5.4. A nilpotent subalgebra $H$ of a Leibniz algebra $L$ is a Cartan subalgebra, if $\text{Nor}_L(H) = H$.

Theorem 5.5. Let $H$ be a Cartan subalgebra of Leibniz algebra $L$ over an algebraically closed field $F$. Then $L$ has the decomposition $L = \bigoplus_{\alpha \in \Phi} L_\alpha$, which is referred to as the root space decomposition of $L$ relative to $H$.

Definition 5.6. Let $(L,[p])$ be a restricted Leibniz algebra over $F$. A subalgebra $T \subseteq L$ is called a torus if

(1) $T$ is an abelian $p$-subalgebra.

(2) $x$ is semisimple, $\forall x \in T$.

Remark 5.7. Suppose that $h \in L$ is a semisimple element which acts nilpotently on an element $x \in L$. That is, there is $n \in \mathbb{N}\{0\}$, $(L_h)^n(x) = 0$. In fact, the semisimplicity of $h$ readily yields that $h = \sum_{i \geq 0} \alpha_i h^{[p]} x^{k+i}$. Choose $k \in \mathbb{N}\{0\}$ such that $p^k \geq n$. Then $\sum_{i \geq 0} \alpha_i h^{[p]} x^{k+i}, x = 0$ and $L_h(x) = 0$.

Theorem 5.8. Let $(L,[p])$ be a restricted Leibniz algebra and $H \subseteq L$ a subalgebra of $L$. If there exists a maximal torus $T \subseteq L$ such that $H = Z_L(T) = C_L(T)$, then $H$ is a Cartan subalgebra of $L$. 14
Proof. Assume that $T$ is a maximal torus. Let $x$ be an element of $H = Z_L(T) = C_L(T)$. There is $k \in \mathbb{N}\setminus\{0\}$ such that $x^{[p]^k}$ is semisimple. Since $x^{[p]^k}$ is contained in the $p$-subalgebra $H, T_1 := T + \mathbb{F}x^{[p]^k}$ is a torus of $L$ containing $T$. In fact, $[T, x^{[p]^k}] = [x^{[p]^k}, T] = 0$, since $H = Z_L(T) = C_L(T)$. Hence $T_1$ is abelian. Let $z = y + \beta x^{[p]^k} \in T_1(y \in T, \beta \in \mathbb{F})$. Then $z^p = y^p + \beta^p x^{[p]^{k+1}} \in T + \mathbb{F}x^{[p]^k}$, since $x^{[p]^k}$ is semisimple and $T$ is a $p$-subalgebra. Hence $T_1$ is a $p$-subalgebra. Let $y \in T$. Consider the $p$-mapping on $V := (\mathbb{F}y + \mathbb{F}x^{[p]^k})_p$. \[ [p] : V \to V \text{ is } p\text{-semilinear, since } T_1 \text{ is abelian. The semisimplicity of } y \text{ and } x^{[p]^k} \text{ show that } y, x^{[p]^k} \in \langle V^p \rangle. \] Hence $\langle V^p \rangle = V$. Then $y + \beta x^{[p]^k}$ is semisimple follows from Lemma 1.3.

Clearly, $T \subseteq T_1$. The maximality of $T$ then shows that $x^{[p]^k} \in T$. Consequently, $(L_x)^p(H) = 0$, proving that $L_x|_H$ is nilpotent. By Engel’s theorem(cf. [3, Theorem 1.1]), $H$ is nilpotent. Let $x$ be an element of $\text{Nor}_L(H)$. Then $(L_h)^2(x) = 0$ for every $h \in T$. Since $h \in T$ is semisimple, by remark 5.7.1 we obtain $L_h(x) = 0, \forall h \in T$, hence $x \in C_L(T) = H$. As a result, $H$ is a Cartan subalgebra of $L$. \[ \Box \]

**Proposition 5.9.** Let $(L, [p])$ be a restricted Leibniz algebra. If $H$ is nilpotent, then $T := \{ h \in Z(H) | h \text{ semisimple} \}$ is a maximal torus of $H$.

**Proof.** Let $x, y$ be two elements of $T$. Then $x \in Z(H), y \in Z(H)$. We have $[y, x] = [x, y] = 0$. Hence $T$ is abelian. Since $Z(H)$ is a $p$-subalgebra, $x^{[p]} \in Z(H)$. Since $x$ is semisimple, so is $x^{[p]}$. Hence $x^{[p]} \in T$. $T$ is a $p$-subalgebra. Hence, $T$ is a torus of $H$.

Let $T'$ be a torus of $H$ and $T \subseteq T'$. Let $x \in T'$. Then $x \in H$. Since $H$ is nilpotent, there exists $n \in \mathbb{N}\setminus\{0\}$ such that $(L_x)^n = 0, (L_x)^n(h) = 0, \forall h \in H$. Since $x$ is semisimple, by remark 5.7.1 one gets $L_x(h) = 0, x \in Z(H), x \in T$. Then $T' = T$. $T$ is a maximal torus. \[ \Box \]

**Corollary 5.10.** Let $(L, [p])$ be a restricted Leibniz algebra over an algebraically closed field $\mathbb{F}$. Consider the root space decomposition $L = \bigoplus_{\alpha \in \Phi} L_\alpha$ with respect to a Cartan subalgebra $H$. Then the following states hold:

1. If $h \in H$ is semisimple, then $L_h|_{L_\alpha} = \alpha(h)\text{id}_{L_\alpha}$; $\alpha(h) \in \text{GF}(p)$ for all toral $h \in H$, where $\text{GF}(p)$ is a finite field.
2. $\alpha(x^{[p]}) = 0, \forall x \in L_\alpha$.

**Proof.** (1) $L_x|_{L_\alpha}$ is semisimple and consequently diagonalizable. Since $\alpha(h)$ is the only eigenvalue of $L_h|_{L_\alpha}$, we obtain $L_h|_{L_\alpha} = \alpha(h)\text{id}_{L_\alpha}$. Suppose that $h$ is toral. Then $\alpha(h)\text{id}_{L_\alpha} = \alpha(h^{[p]})\text{id}_{L_\alpha} = L_h^{[p]}|_{L_\alpha} = (L_h)^p|_{L_\alpha} = \alpha(h)^p\text{id}_{L_\alpha}$. This proves that $\alpha(h) = \alpha(h)^p$ and $\alpha(h) \in \text{GF}(p)$.

(2) Let $x$ be a nonzero element of $L_\alpha$. Then $[x^{[p]}, x] = 0$ and $0$ is an eigenvalue of $L_x^{[p]}|_{L_\alpha}$. As $\alpha(x^{[p]})$ is the only eigenvalue of $L_x^{[p]}|_{L_\alpha}$, we obtain $\alpha(x^{[p]}) = 0$. \[ \Box \]

**Lemma 5.11.** Let $T$ be a torus of the restricted Leibniz algebra $(L, [p])$.

1. Any $T$-invariant subspace $W \subseteq L$ (i.e., $[T, W] \subseteq W$) decomposes $W = C_W(T) + [T, W]$.
2. If $I \triangleleft L$ is a $p$-ideal such that $L/I$ is a torus, then there exists a torus $T' \supseteq T$ such that $L = T' + I$. 

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Proof. (1) The adjoint representation gives $W$ the structure of a $T$-module. According to Theorem 5.3, we may write $W = \bigoplus_{\pi \in B} W_{\pi}$. Let $\pi_0$ be the function with $\pi_0h = X, \forall h \in T$. Then $W_{\pi_0} \subseteq C_W(T)$ and $[T, W_{\pi}] = W_{\pi}, \forall \pi \neq \pi_0$. Hence $W = C_W(T) + [T, W].$

(2) Let $T' \supseteq T$ be a maximal torus. According to (1), we write $L = C_L(T') + [T', L]$. Since $[T', L] \subseteq [L, L] \subseteq I$, it will suffice to show that $C_L(T') \subseteq T' + I$. Let $x \in C_L(T')$. By virtue of Theorem 4.3, there is $r$ such that $x^{[p]^r}$ is semisimple. As $x + I$ is a semisimple element of $L/I$, we find $n \geq r$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $x - \sum_{i=r}^{n} \alpha_i x^{[p]^i} \in I$.

Since $\sum_{i=r}^{n} \alpha_i x^{[p]^i}$ is a semisimple element of $C_L(T')$ and $T'$ is a maximal torus, we obtain $\sum_{i=r}^{n} \alpha_i x^{[p]^i} \in T'$. This concludes our proof. \qed

**Theorem 5.12.** Let $(L_1, [p]_1), (L_2, [p]_2)$ be restricted Leibniz algebras and $\varphi : L_1 \to L_2$ a surjective $p$-homomorphism.

1. If $T_1$ is a maximal torus of $L_1$, then $\varphi(T_1)$ is a maximal torus of $L_2$.

2. If $T_2$ is a maximal torus of $L_2$ and $T_1$ is a maximal torus of $\varphi^{-1}(T_2)$, then $T_1$ is a maximal torus of $L_1$.

**Proof.** (1) Clearly, $\varphi(T_1)$ is a torus of $L_2$. Suppose that $T' \supseteq \varphi(T_1)$ is a maximal torus of $L_2$. Then $\varphi^{-1}(T')/\ker(\varphi)$ is a torus and by Lemma 5.11 (2) we may write $\varphi^{-1}(T') = T_1 + \ker(\varphi)$. Hence $T' = \varphi(\varphi^{-1}(T')) = \varphi(T_1)$. This shows that $\varphi(T_1)$ is a maximal torus of $L_2$.

(2) It follows from (1) that $\varphi(T_1) = T_2$. Let $T' \supseteq T_1$ be a maximal torus of $L_1$. Then $T_2 \subseteq \varphi(T')$ and the maximality of $T_2$ yields $\varphi(T') = T_2$. Thus $T' \subseteq \varphi^{-1}(T_2)$ and $T' = T_1$, because of the maximality of $T_1$.

\qed

### 6 The uniqueness of decomposition

Similar to Definition 2.1 of the reference \[5\], we give the following definition.

**Definition 6.1.** Let $\varphi$ be an endomorphism of a restricted Leibniz algebra $(L, [p])$. $\varphi$ is called an $L$-endomorphism of $L$, if $\varphi L_x = L_x \varphi$ and $\varphi R_x = R_x \varphi$ for any $x \in L$. An $L$-endomorphism of $L$ $\varphi$ is called an $L$-$p$-endomorphism of $L$, if $\varphi(x^{[p]}) = \varphi(x)^{[p]}, \forall x \in L$. An $L$-endomorphism $(L$-$p$-endomorphism) of $L$ $\varphi$ is called an $L$-automorphism $(L$-$p$-automorphism) of $L$, if $\varphi$ is bijection.

**Example 6.2.** Let $(L, [p])$ be a restricted Leibniz algebra over $\mathbb{F}$ with decomposition $L = A \oplus B$ and $\pi$ be the projection into $A$ with respect to this decomposition, where $A$ and $B$ are $p$-ideals of $L$. Then $\pi$ is an $L$-$p$-endomorphism of $L$.

**Lemma 6.3.** Let $(L, [p])$ be a restricted Leibniz algebra over $\mathbb{F}$. Then

1. If $A$ is a subset of $L$, then $Z_L(A)$ is a $p$-subalgebra of $L$.  

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(2) If $B$ is an ideal of $L$, then $Z_L(B)$ is a $p$-ideal of $L$. In particular, $Z(L)$ is a $p$-ideal of $L$.

Proof. (1) For any $x, y \in Z_L(A), z \in A$, we have $[[x, y], z] = [x, [y, z]] - [y, [x, z]] = 0$, $[x, y] \in Z_L(A)$. Similarly, $[y, x] \in Z_L(A)$. Hence $Z_L(A)$ is a subalgebra of $L$. Since $x \in Z_L(A)$, one gets $x^p, z = (L_x)^p x, z = 0$, $x^p \in Z_L(A)$. As a result, $Z_L(B)$ is a $p$-subalgebra of $L$.

(2) For any $x \in Z_L(B), y \in L, z \in B$, since $B$ is an ideal of $L$, $[y, z] \in B$, we have $[[x, y], z] = [x, [y, z]] - [y, [x, z]] = 0$, $[x, y] \in Z_L(B)$. Similarly, $[y, x] \in Z_L(B)$. Hence $Z_L(B)$ is an ideal of $L$. By (1), $Z_L(B)$ is a $p$-subalgebra of $L$. Therefore, $Z_L(B)$ is a $p$-ideal of $L$. □

Lemma 6.4. Let $(L, [p])$ be a restricted Leibniz algebra over $\mathbb{F}$. Then the following statements hold:

(1) If $f$ and $g$ are $L$-endomorphisms of $L$, then so are $f + g$ and $fg$.

(2) If $f$ and $g$ are $L$-$p$-endomorphisms of $L$, then so is $fg$.

(3) If $f$ is an $L$-$p$-automorphism of $L$, then so is $f^{-1}$.

Proof. (1) Since $f$ and $g$ are $L$-endomorphisms of $L$, $fL_x = L_x f$ and $gL_x = L_x g$ for any $x \in L$. Then $(f + g)L_x = fL_x + gL_x = L_x f + L_x g = L_x (f + g), (fg)L_x = f(gL_x) = f(L_x g) = (fL_x)g = (L_x f)g = L_x (fg)$. Similarly, $(f + g)R_x = R_x (f + g), (fg)R_x = R_x (fg)$. So $f + g$ and $fg$ are $L$-endomorphisms of $L$.

(2) Since $f$ and $g$ are $L$-endomorphisms of $L$, by (1), $fg$ is an $L$-endomorphism of $L$. Clearly, $fg(x^p) = (fg(x))^p$. As a result, $fg$ is an $L$-$p$-endomorphism of $L$.

(3) Since $f$ is an $L$-automorphism of $L$, there is an automorphism $f^{-1}$ such that $f \cdot f^{-1} = f^{-1} \cdot f = id_L$ and $fL_x = L_x f$ for any $x \in L$. As $(f \cdot f^{-1})L_x = L_x (f \cdot f^{-1}), f(f^{-1}L_x) = L_x (f \cdot f^{-1})$ and $f^{-1}fL_x = f^{-1}L_x (f \cdot f^{-1})$, i.e., $f^{-1}L_x = f^{-1}L_x (f \cdot f^{-1}) = f^{-1}L_x f^{-1} = L_x f^{-1}$. Similarly, $f^{-1}R_x = R_x f^{-1}$. So $f^{-1}$ is an $L$-automorphism of $L$. $x^p = (f^{-1}(x))^p = f(((f^{-1}(x))^p), \forall x \in L, f^{-1}(x^p) = (f^{-1}(x))^p$. Hence $f^{-1}$ is an $L$-$p$-automorphism of $L$. □

Lemma 6.5. Let $(L, [p])$ be a restricted Leibniz algebra over $\mathbb{F}$. If $\varphi$ is an $L$-$p$-endomorphism of $L$, then there exists $k \in \mathbb{N}\setminus\{0\}$ satisfying

(1) $L$ has a decomposition of $p$-ideals $L = \ker\varphi^k \oplus \Im\varphi^k$.

(2) If $L$ can not be decomposed into the direct sum of $p$-ideals of $L$, then $\varphi^k = 0$ or $\varphi \in \Aut_p L$, where $\Aut_p L$ is the group of $p$-automorphisms of $L$.

Proof. (1) Let $f(\lambda) = \lambda^k g(\lambda)$ be the minimal polynomial of $\varphi$, where $\lambda$ and $g(\lambda)$ are coprime. Then there are polynomials $u(\lambda)$ and $v(\lambda)$ satisfying $u(\lambda)g(\lambda) + v(\lambda)\lambda^k = 1$. So we have $y = u(\varphi)g(\varphi)(y) + v(\varphi)\varphi^k(y)$ for all $y \in L$. Since $\varphi^k(u(\varphi)g(\varphi)(y)) = (\varphi^k g(\varphi))u(\varphi)(y) = 0, u(\varphi)g(\varphi)(y) \in \ker\varphi^k$, and $v(\varphi)\varphi^k(y) = \varphi^k(v(\varphi)(y)) \in \Im\varphi^k$. Thus $L = \ker\varphi^k + \Im\varphi^k$. If $y \in \ker\varphi^k \cap \Im\varphi^k$, then $\varphi^k(y) = 0$ and $y = \varphi^k(z)$ for some $z \in L$. So

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\[
y = u(\varphi)g(\varphi)\varphi^k(z) + v(\varphi)\varphi^k(y) = u(\varphi)f(\varphi)(z) + v(\varphi)(\varphi^k(y)) = 0, \text{ i.e., } \ker \varphi^k \cap \Im \varphi^k = \{0\}. \text{ Thus } L = \ker \varphi^k \oplus \Im \varphi^k \text{ as a vector space.}
\]

Since \( \varphi \) is an \( L \)-\( p \)-endomorphism of \( L \), \( \varphi^k \) is an \( L \)-\( p \)-endomorphism of \( L \) by Lemma 6.4 (2). Then \( \varphi^k[x, L] = [\varphi^k(x), \varphi^k(L)] = 0, \varphi^k[L, x] = [\varphi^k(L), \varphi^k(x)] = 0 \) for any \( x \in \ker \varphi^k \), i.e., \( \ker \varphi^k \) is an ideal of \( L \). \( \varphi^k(x)[y^p] = \varphi^k-1(\varphi(x)[y^p]) = (\varphi^k(x))[y^p] = 0, \forall x \in \ker \varphi^k \). Hence \( \ker \varphi^k \) is a \( p \)-ideal of \( L \). Let \( x = x_1 + \varphi^k(x_2) \in L \), where \( x_1 \in \ker \varphi^k, x_2 \in L \). Suppose \( a = \varphi^k(y) \) for some \( y \in L \). Since \( \varphi^k \) is an \( L \)-\( p \)-endomorphism of \( L \), then \( [x, a] = [x, \varphi^k(y)] = \varphi^k(x_2, y) \in \Im \varphi^k \). Similarly, \( [a, x] = \Im \varphi^k \). Therefore, \( \Im \varphi^k \) is an ideal of \( L \). Let \( x \in \Im \varphi^k \). Then there exists \( y \in L \) such that \( x = \varphi^k(y), x[y^p] = (\varphi^k(y))[y^p] = \varphi^k(y)[y^p], x[y^p] \in \Im \varphi^k \). Consequently, \( \Im \varphi^k \) is a \( p \)-ideal of \( L \).

(2) If \( L \) cannot be decomposed into the direct sum of \( p \)-ideals, then we can know that \( \ker \varphi^k = L \) or \( \Im \varphi^k = L \) by (1). This means that \( \varphi^k = 0 \) or \( \varphi^k \in \Aut_p L \). So \( \varphi^k = 0 \) or \( \varphi \in \Aut_p L \). \( \square \)

**Lemma 6.6.** Let \((L, [p])\) be a restricted Leibniz algebra over \( \mathbb{F} \). Let \( \varphi_i (1 \leq i \leq n) \), \( \sum_{i=1}^{n} \varphi_i (1 \leq j \leq n) \) be \( L \)-\( p \)-endomorphisms of \( L \) and \( \varphi_1 + \varphi_2 + \cdots + \varphi_n = \id_L \). If \( L \) cannot be decomposed into the direct sum of \( p \)-ideals, then there exists \( i (1 \leq i \leq n) \) satisfying \( \varphi_i \in \Aut_p L \).

**Proof.** We prove this result by induction on \( n \). The result is obviously true for \( n = 1 \). For \( n = 2 \), since \( \varphi_1 + \varphi_2 = \id_L \), \( \varphi_1(\varphi_1 + \varphi_2) = (\varphi_1 + \varphi_2)\varphi_1 \) and \( \varphi_1\varphi_2 = \varphi_2\varphi_1 \). Now, we suppose \( \varphi_1, \varphi_2 \notin \Aut_p L \). By virtue of Lemma 6.3 (2), there is \( k_i (i = 1, 2) \) satisfying \( \varphi_i^k = 0 \). Put \( k > k_1 + k_2 \), then \( \id_L = (\varphi_1 + \varphi_2)^k = \sum_{j=0}^{k} C_k^j \varphi_1^{k-j} \varphi_2^j = 0 \). It is a contradiction. From it we can get \( \varphi_1 \in \Aut_p L \) or \( \varphi_2 \in \Aut_p L \).

Suppose \( n - 1 \) holds and \( \psi := \sum_{i=1}^{n-1} \varphi_i \), then \( \psi + \varphi_n = \id_L \). From the discussion in the case of \( n = 2 \), we get \( \psi \in \Aut_p L \) or \( \varphi_n \in \Aut_p L \). If \( \varphi_n \in \Aut_p L \), then the conclusion is true. If \( \psi \in \Aut_p L \), then \( \psi^{-1}, \varphi_1\psi^{-1}, \cdots, \varphi_{n-1}\psi^{-1} \) are \( L \)-\( p \)-endomorphisms of \( L \) by means of Lemma 6.4 and \( \sum_{i=1}^{n-1} \varphi_i\psi^{-1} = \psi\psi^{-1} = \id_L \). By the inductive assumption, there exists \( i \) such that \( \varphi_i\psi^{-1} \in \Aut_p L \). Hence \( \varphi_i \in \Aut_p L \). \( \square \)

**Lemma 6.7.** Let \((L, [p])\) be a restricted Leibniz algebra over \( \mathbb{F} \). If \( L \) has a decomposition of \( p \)-ideals \( L = A \oplus B \), then the following statements hold:

1. \( Z(L) \) has a decomposition of \( p \)-ideals \( Z(L) = Z(A) \oplus Z(B) \).
2. If \( Z(L) = 0 \), then \( Z_L(A) = B \) and \( Z_L(B) = A \).

**Proof.** (1) According to Lemma 6.3, \( Z(A) \) and \( Z(B) \) are \( p \)-ideals of \( L \). Since \( Z(A) \cap Z(B) = \{0\} \), we have \( Z(A) \oplus Z(B) \subseteq Z(L) \). Now, suppose \( x \in Z(L) \) and \( x = x_1 + x_2 \), where \( x_1 \in A, x_2 \in B \). Then \([x_1, A] = [x - x_2, A] = 0 \). Hence \( x_1 \in Z(A) \). Similarly, \( x_2 \in Z(B) \). Hence \( Z(L) = Z(A) \oplus Z(B) \).

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(2) $B \subseteq Z_L(A)$ is obviously true. It is sufficient to show that $Z_L(A) \subseteq B$. Since $L = A \oplus B$, for any element $x$ of $Z_L(A)$, we have $x = x_1 + x_2$, where $x_1 \in A$, $x_2 \in B$. It follows that

$$0 = [x, a] = [x_1 + x_2, a] = [x_1, a] + [x_2, a] = [x_1, a]$$

for all $a \in A$. Thus $x_1 \in Z(A) = 0$. Hence $x = x_1 + x_2 = x_2 \in B$ and $Z_L(A) \subseteq B$. Consequently, $Z_L(A) = B$. Similarly, we can get that $Z_L(B) = A$. \qed

**Lemma 6.8.** Let $(L, [p])$ be a restricted Leibniz algebra over $\mathbb{F}$ such that $L = A \oplus B$. If $A$ and $B$ are ideals of $L$ and $C$ is a subalgebra of $L$ such that $A \subseteq C$, then $C = A \oplus (C \cap B)$ and $C \triangleleft L$ if and only if $(B \cap C) \triangleleft B$.

**Proof.** Since $B$ is an ideal and $C$ is a subalgebra of $L$, $[C \cap B, C] \subseteq [B, C] \subseteq B$ and $[C \cap B, C] \subseteq [C, C] \subseteq C$. Then $[C \cap B, C] \subseteq C \cap B$, i.e., $C \cap B$ is an ideal of $C$. So there is an isomorphism such that $(B + C)/B \cong C/C \cap B$. On the other hand, $(C + B)/B \cong A$. Hence $A \cong C/C \cap B$ and $C = A \oplus (C \cap B)$. The second statement is clear. \qed

**Theorem 6.9.** Suppose that a restricted Leibniz algebra $(L, [p])$ over $\mathbb{F}$ has decompositions of $p$-ideals

$$L = \oplus_{i=1}^{s} M_i = \oplus_{i=1}^{s} M_i, \quad (1)$$

$$L = \oplus_{i=1}^{t} N_i = \oplus_{i=1}^{t} N_i, \quad (2)$$

where $M_1, \ldots, M_s$ and $N_1, \ldots, N_t$ can not be decomposed into the direct sum of $p$-ideals. If $Z(L) = 0$, then $s = t$ and $M_i = N_i$, $i = 1, 2, \ldots, s$ after changing the orders.

**Proof.** We prove this theorem by induction on $n$. If $s = 1$, then $L$ can not be decomposed into the direct sum of $p$-ideals. So $t = 1$ and $M_1 = N_1 = L$.

Now put $s > 1$, naturally $t > 1$, too. Let $\pi$ be the projection of $L$ to $M_1$ with respect to the decomposition (1), $\sigma$ the imbedding of $M_1$ to $L$, $\rho_i$ the projection of $L$ to $N_i$ with respect to the decomposition (2) and $\tau_i$ the imbedding of $N_i$ to $L$. Then $\pi, \rho_1, \ldots, \rho_t$ and $\sum_{i=1}^{k} \rho_i(1 \leq k \leq t)$ are $L$p-endomorphisms of $L$ and $\rho_1 + \rho_2 + \cdots + \rho_t = \text{id}_L$.

Letting $\pi_i^* = \pi \sigma = \pi |_{M_i}, \rho_i^* = \rho_i \sigma = \rho_i |_{M_1}$ for any $i = 1, 2, \ldots, t$, then $\pi_i^* \rho_i^*$ is the $M_1$p-endomorphism of $M_1$.

Defined $\sum_{i=1}^{j} \tau_i \rho_i : L \to L$ by $(\sum_{i=1}^{j} \tau_i \rho_i)(x) = \sum_{i=1}^{j} \tau_i \rho_i(x)$ for all $x \in L, 1 \leq j \leq t$. We verify that the mapping is an $L$p-endomorphism of $L$. In fact, for $x, y \in L$, we write $x = \sum_{i=1}^{t} x_i, y = \sum_{i=1}^{t} y_i$, where $x_i, y_i \in B_i(1 \leq i \leq t)$. So

$$\sum_{i=1}^{j} \tau_i \rho_i(x^{[p]}) = \sum_{i=1}^{j} \tau_i (\rho_i(x)^{[p]}) = \sum_{i=1}^{j} \tau_i(x_i^{[p]}) = \sum_{i=1}^{j} x_i^{[p]}.$$
On the other hand, from $L = N_1 \oplus N_2 \oplus \cdots \oplus N_t$, we can obtain that $[N_i, N_j] = 0$ for $1 \leq i, j \leq t$ with $i \neq j$. Hence we may imply that

$$(\sum_{i=1}^{j} \tau_i \rho_i(x))[p] = (\sum_{i=1}^{j} x_i)[p] = \sum_{i=1}^{j} x_i^p$$

and $\sum_{i=1}^{j} \tau_i \rho_i(x^p) = (\sum_{i=1}^{j} \tau_i \rho_i(x))[p]$. Next we show that $\sum_{i=1}^{j} \tau_i \rho_i$ is an endomorphism of $L$. By virtue of $[N_i, N_j] = 0$, we have

$$\sum_{i=1}^{j} \tau_i \rho_i[x, y] = \sum_{i=1}^{j} \tau_i[\rho_i(x), \rho_i(y)] = \sum_{i=1}^{j} [x_i, y_i] = [\sum_{i=1}^{j} x_i, \sum_{i=1}^{j} y_i] = [\sum_{i=1}^{j} \tau_i \rho_i(x), \sum_{i=1}^{j} \tau_i \rho_i(y)].$$

Finally, using similar method we may verify that

$$(\sum_{i=1}^{j} \tau_i \rho_i)L_x = L_x(\sum_{i=1}^{j} \tau_i \rho_i) \quad \text{and} \quad (\sum_{i=1}^{j} \tau_i \rho_i)R_x = R_x(\sum_{i=1}^{j} \tau_i \rho_i).$$

Thus $\sum_{i=1}^{j} \tau_i \rho_i$ is an $L$-$p$-endomorphism of $L$. Furthermore, $\pi(\sum_{i=1}^{j} \tau_i \rho_i)\sigma = \sum_{i=1}^{j} \pi_i^* \rho_i^*$ is an $M_1$-endomorphism of $M_1$. For each $h \in M_1$, we have $h = \pi(h) = \pi(\sum_{i=1}^{l} \rho_i(h)) = \sum_{i=1}^{l} \pi_i^* \rho_i^*(h)$, then $\sum_{i=1}^{l} \pi_i^* \rho_i^* = \text{id}_{M_1}$. So there exists an index $i$ satisfying $\pi_i^* \rho_i^* \in \text{Aut}_p M_1$ by virtue of Lemma 6.6. If needed, after changing the order of $N_1, N_2, \cdots, N_t$, we can get $i = 1$. $\pi_1^* \rho_1^* \in \text{Aut}_p M_1$. Thus $\rho_1^*$ is a bijection. Let $M = M_2 \oplus M_3 \oplus \cdots \oplus M_s$, $N = N_2 \oplus N_3 \oplus \cdots \oplus N_t$. By Lemma 6.7, we have $Z(M) = Z(N) = \{0\}$ and $M = Z_L(M_1), M_1 = Z_L(M), \ker \rho_1 = N = Z_L(N_1), N_1 = Z_L(N)$. Hence $\{0\} = \ker \rho_1^* = M_1 \cap \ker \rho_1 = M_1 \cap N$. So we have $M_1 \subseteq Z_L(N) = N_1, N_1 = M_1 \oplus (N_1 \cap M)$ by Lemma 6.8. But $N_1$ can not be decomposed into the direct sum of $p$-ideals, then $N_1 = M_1$. By inductive assumption we obtain the desired result.

**Corollary 6.10.** Let $(L, [p])$ be a restricted Lie algebra over $\mathbb{F}$ with trivial center. If $L$ has a decomposition of $p$-ideals $L = A_1 \oplus A_2 \oplus \cdots \oplus A_s$, then the decomposition is unique after changing the orders, where $A_1, \cdots, A_s$ can not be decomposed into the direct sum of $p$-ideals.

**References**

[1] Albeverio, S., Ayupov, Sh. A. and Omirov, B. A., On nilpotent and simple Leibniz algebras. *Comm. Algebra*, 2005, 33: 159-172.

[2] Bahturin, Y., Mikhalev, A., Petrogradski, V. M. and Zaicev, M., Infinite-dimensional Lie superalgebras, Walter de Gruyter, Berlin, New York, 1992.
[3] Barnes, D. W., On Engel’s Theorem for Leibniz algebras. *Comm. Algebra*, 2012, 40: 1388-1389.

[4] Barnes, D. W., Some Theorems on Leibniz algebras. *Comm. Algebra*, 2011, 39: 2463-2472.

[5] Chen, L. Y. and Zhang, Y. Z., Meng, D. J., On the decomposition of restricted Lie superalgebras. *Acta Math. Sci. (Chinese)*, 2007, 27(A)(4): 577-583. (in chinese)

[6] Dzhumadil’daev, A. S., Abdykassymova, S. A., Leibniz algebras in characteristic $p$. *C. R. Acad. Sci. Paris Série I*, 2001, 332: 1047-1052.

[7] Dokas, I., Loday, J. L., On restricted Leibniz algebras. *Comm. Algebra*, 2006, 34: 4467-4478.

[8] Hodge, T. L., Lie triple system, restricted Lie triple system and algebraic groups. *J. Algebra*, 2001, 244: 533-580.

[9] Farnsteiner, R., Note on Frobenius extensions and restricted Lie superalgebras. *J. Pure Appl. Algebra*, 1996, 108: 241-256.

[10] Korshunov, M. K., Associative and Leibniz algebras of distributions. *Izv. Vyssh. Uchebn. Zaved. Mat.* 1979, no. 5: 74-77. (in Russian)

[11] Loday, J.-L., Pirashvili, T., Universal enveloping algebras of Leibniz algebras and (co)homology. *Math. Ann.*, 1993, 296: 139-158.

[12] Patsourakos, A., On nilpotent properties of Leibniz algebras. *Comm. Algebra*, 2007, 35: 3828–3834.

[13] Strade, H., Farnsteiner, R., Modular Lie algebras and their representations. New York: Dekker. 1988.

[14] Strade, H., The classification of the simple modular Lie algebras. *Ann. Math.*, 1991, 133: 577-604.