PRIMES IN THE DENOMINATORS OF IGUSA CLASS POLYNOMIALS

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1. Introduction

The purpose of this note is to suggest an analogue for genus 2 curves of part of Gross and Zagier’s work on elliptic curves [GZ84]. Experimentally, for genus 2 curves with CM by a quartic CM field $K$, it appears that primes dividing the denominators of the discriminants of the Igusa class polynomials all have the property 1) that they are bounded by $d$, the absolute value of the discriminant of $K$, and 2) that they divide $d - x^2$, for some integer $x$ whose square is less than $d$. A slightly stronger condition is given in Section 3. Such primes are primes of bad reduction for the genus 2 curve and primes of supersingular reduction for the Jacobian of the genus 2 curve.

2. Algorithm to generate Igusa class polynomials of quartic CM fields

The steps of this algorithm are as follows:

1. Choose a CM field $K$ of degree 4 which is either not Galois or Galois with Galois group $\mathbb{Z}/4\mathbb{Z}$. Let $K_0$ be the totally real quadratic subfield. Choose a CM-type, $\Phi$ (a choice of two complex embeddings of $K$, such that one is not the complex conjugate of the other).

2. For each element of the ideal class group of $\mathcal{O}_K$, choose a representative ideal $a_i$, and find an integral basis $\{1, \tau_i\}$ for it over the ring of integers of $K_0$.

3. Each ideal class corresponds to an isomorphism class of an abelian variety via Shimura’s theory [Sh94, p.126]. The principal polarization(s) on the abelian variety and the corresponding $2 \times 2$ period matrix (matrices) $\Omega_i$ are given in [Sp94, Section 4.2 and p.62]. The entries of $\Omega_i$ are given in terms of $\tau_i$, the CM-type $\Phi$, and $\omega$, the generator of the ring of integers of $K_0$.

4. For each period matrix, evaluate the ten even theta constants up to some amount of precision.

5. The three absolute Igusa invariants, $j_1, j_2, j_3$, associated to each period matrix are defined as combinations of the ten even theta constants by formulas given in the appendix.

6. Take the product over all possible period matrices to form the three Igusa class polynomials:

\[ h_1(x) = \prod_{\Omega_i} (x - j_1(\Omega_i)), \]
\[ h_2(x) = \prod_{\Omega_i} (x - j_2(\Omega_i)), \]
\[ h_3(x) = \prod_{\Omega_i} (x - j_3(\Omega_i)). \]

This algorithm was implemented using the software packages Pari and MAPLE in [CL01]. Implementations can be found in the literature in [Sp94], [vW99], [We03].

**Remark 1.** The above algorithm is analogous to computing the Hilbert class polynomial associated to an imaginary quadratic field \( K \). The \( j \)-invariant for each ideal class is evaluated up to some amount of precision, then \( H(x) \) is formed by taking the product over all ideal classes of \( (x - j(\tau_i)) \), where \( \tau_i \) is a particular algebraic integer associated to each ideal class. Then the Hilbert class polynomial, \( H(x) \), has integer coefficients.

The case of a quartic CM field is different because there are three class polynomials, their coefficients are rational numbers, and the amount of precision required to compute them is not known in advance. The next section proposes a constraint on the primes appearing in the denominators which, together with a bound on the power to which each prime appears, would give a bound on the amount of precision required to compute the coefficients of the class polynomials.

### 3. Conjectural Formula for the Primes in the Denominators of the Igusa Class Polynomials

Let \( h_1, h_2, \) and \( h_3 \) be the three Igusa class polynomials with rational coefficients obtained from the algorithm in Section 2. Let \( \{q_i\} \) be the primes appearing in the denominators of the discriminants of all three \( h_i \). Let \( d \) be the absolute value of the discriminant of the number field \( K \) and let \( d_0 \) be the absolute value of the norm of the discriminant of \( K/K_0 \).

Experimentally I have observed that the primes \( q_i \) satisfy the following property:

**Property (1)** Each \( q_i \) divides \( d - x^2 \), for some \( x \), an integer such that \( x^2 \leq d \). In fact in every case so far, each \( q_i \) divides \( d_0 - x^2 \), for some \( x \), an integer such that \( x^2 \leq d_0 \).

**Remark 2.** I have only tested this property on a few handfuls of examples, but Annegret Weng has also tested it on at least that many others. I am currently working on a proof of this property jointly with Annegret Weng, Farshid Hajir, Fernando Rodriguez-Villegas, and Tonghai Yang.

### References

[CL01] Cohn, Henry; Lauter, Kristin. Generating Genus 2 Curves with Complex Multiplication, Internal Technical Report, January 2001.

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[vW99] van Wamelen, Paul. Examples of genus two CM curves defined over the rationals. Math. Comp. 68 (1999), no. 225, 307–320.

[We03] Weng, Annegret. Constructing hyperelliptic curves of genus 2 suitable for cryptography. Math. Comp. 72 (2003), 435-458.
4. Appendix: Igusa invariants

Let \( \{\theta_i\}_{i=1}^{10} \) be the ten even theta constants associated to a given period matrix.

Define the functions \( f \) and \( g \) as follows:
\[
f(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8) = (\prod_{i=1}^{8} \theta_{k_i})^4,
\]
\[
g(k_1, k_2, k_3, k_4, k_6) = (\prod_{i=1}^{6} \theta_{k_i})^4.
\]

Now \( h_4, h_{10}, h_{12}, \) and \( h_{16} \) are values of modular forms of weights 4, 10, 12, and 16:
\[
h_4 = \sum_{i=1}^{10} \theta_{k_i}^4,
\]
\[
h_{10} = \prod_{i=1}^{10} \theta_{k_i}^2,
\]
\[
h_{12} = g(1, 5, 2, 9, 6, 10) + g(1, 2, 9, 6, 8, 3) + g(5, 9, 6, 8, 10, 7) + g(5, 2, 6, 8, 3, 7) +
g(1, 5, 2, 10, 3, 7) + g(1, 9, 8, 10, 3, 7) + g(1, 5, 2, 8, 10, 4) + g(1, 5, 9, 8, 3, 4) + g(5, 9, 6, 10, 3, 4) +
g(2, 6, 8, 10, 3, 4) + g(1, 2, 9, 6, 7, 4) + g(1, 5, 6, 8, 7, 4) + g(2, 9, 8, 10, 7, 4) + g(5, 2, 9, 3, 7, 4) +
g(1, 6, 10, 3, 7, 4),
\]
\[
h_{16} = f(8, 1, 5, 2, 9, 6, 8, 10) + f(5, 1, 5, 2, 9, 6, 8, 3) + f(10, 1, 2, 9, 6, 8, 10, 3) + f(3, 1, 5, 2, 9, 6, 10, 3) +
f(1, 1, 5, 2, 9, 6, 8, 10, 7) + f(2, 5, 2, 9, 6, 8, 10, 7) + f(1, 1, 5, 2, 6, 8, 3, 7) + f(9, 5, 2, 9, 6, 8, 3, 7) +
f(9, 1, 5, 2, 9, 10, 3, 7) + f(6, 1, 5, 2, 6, 10, 3, 7) + f(5, 1, 5, 9, 8, 10, 3, 7) + f(2, 1, 2, 9, 8, 10, 3, 7) +
f(6, 1, 9, 6, 8, 10, 3, 7) + f(8, 1, 5, 2, 8, 10, 3, 7) + f(10, 5, 2, 6, 8, 10, 3, 7) + f(3, 5, 9, 6, 8, 10, 3, 7) +
f(7, 1, 5, 2, 9, 6, 10, 7) + f(7, 1, 2, 9, 6, 8, 3, 7) + f(9, 1, 5, 2, 9, 8, 10, 4) + f(6, 1, 5, 2, 6, 8, 10, 4) +
f(2, 1, 5, 2, 9, 6, 3, 4) + f(6, 1, 5, 9, 6, 8, 3, 4) + f(1, 1, 5, 9, 6, 10, 3, 4) + f(2, 5, 2, 9, 6, 10, 3, 4) +
f(1, 1, 2, 6, 8, 10, 3, 4) + f(5, 5, 2, 6, 8, 10, 3, 4) + f(9, 2, 9, 6, 8, 10, 3, 4) + f(8, 5, 9, 6, 8, 10, 3, 4) +
f(10, 1, 5, 9, 8, 10, 3, 4) + f(3, 1, 5, 2, 8, 10, 3, 4) + f(5, 1, 5, 2, 9, 6, 7, 4) + f(2, 1, 5, 2, 6, 8, 7, 4) +
f(9, 1, 5, 9, 6, 8, 7, 4) + f(8, 1, 2, 9, 6, 8, 7, 4) + f(1, 1, 2, 9, 8, 10, 7, 4) + f(5, 5, 2, 9, 8, 10, 7, 4) +
f(6, 2, 9, 6, 8, 10, 7, 4) + f(10, 1, 2, 9, 6, 10, 7, 4) + f(10, 1, 5, 6, 8, 10, 7, 4) + f(1, 1, 5, 2, 9, 3, 7, 4) +
f(6, 5, 2, 9, 6, 3, 7, 4) + f(8, 5, 2, 9, 8, 3, 7, 4) + f(5, 1, 5, 6, 10, 3, 7, 4) + f(2, 1, 2, 6, 10, 3, 7, 4) +
f(9, 1, 9, 6, 10, 3, 7, 4) + f(8, 1, 6, 8, 10, 3, 7, 4) + f(10, 5, 2, 9, 10, 3, 7, 4) + f(3, 1, 2, 9, 6, 3, 7, 4) +
f(3, 1, 5, 6, 8, 3, 7, 4) + f(3, 2, 9, 8, 10, 3, 7, 4) + f(7, 1, 5, 2, 8, 10, 7, 4) + f(7, 1, 5, 9, 8, 3, 7, 4) +
f(7, 5, 9, 6, 10, 3, 7, 4) + f(7, 2, 6, 8, 10, 3, 7, 4) + f(4, 1, 5, 2, 9, 6, 10, 4) + f(4, 1, 2, 9, 6, 8, 3, 4) +
f(4, 5, 9, 6, 8, 10, 7, 4) + f(4, 5, 2, 6, 8, 3, 7, 4) + f(4, 1, 5, 2, 10, 3, 7, 4) + f(4, 1, 9, 8, 10, 3, 7, 4).
\]

The Igusa invariants are
\[
I_2 = \frac{h_{12}}{h_{10}},
\]
\[
I_4 = h_4,
\]
\[
I_6 = \frac{h_{16}}{h_{10}},
\]
\[
I_{10} = h_{10},
\]

and the absolute Igusa invariants are
\[
j_1 = \frac{I_2^5}{I_{10}}.
\]
\[
j_2 = \frac{I_4 I_2^3}{I_{10}}.
\]
\[
j_3 = \frac{I_6 I_2^5}{I_{10}}.
\]