Quasi Static Evolution of Compact Objects in Modified Gravity

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Abstract

In this paper, the quasi static-approximation on the hydrodynamics of compact objects is proposed in $f(R,T)$ gravity, where $R$ is the scalar curvature and $T$ is the trace of stress-energy tensor, by exploring the axial and reflection symmetric space time stuffed with anisotropic and dissipative matter contents. The set of invariant-velocities is defined to comprehend the concept of quasi static-approximation. As a consequence, the evolution of compact objects is shown by analyzing the corresponding modified field, dynamical and scalar equations in this approximation to evoke all the feasible outcomes. Furthermore, the significance of kinematical quantities, modified heat-fluxes and scalar variables are found through the proposed approximation.

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1 Introduction

Einstein gravity theory is considered as the foundation of cosmology and relativistic astrophysics. The observational consequences such as Λ-cold dark matter turn out to be stable with varied cosmological controversies other than certain deviations like cosmic coincidence and fine-tuning [1, 2]. The surveys of cosmic-microwave background radiations (CMBR), red-shift, Supernovae Type Ia and very large scale-structures are the evidence of accelerated-expansion of our cosmos [3, 4]. All such observations affirmed the existence of anti-gravity entitled as dark energy (DE). The DE is a kind of energy with enough-negative pressure and have repulsive nature of gravity whose existence is taking for guaranteed to illustrate the observed accelerated-expansion of our cosmos [6, 7]. In order to modify Einstein gravity theory, various mathematical-models have been introduced to describe the DE and dark matter. The dark matter does not interact with ordinary matter and is basically invisible to light. This is the reason to call that type of matter as Dark.

The current physical cosmology is dominated by DE era [8], which illustrates the accelerated expanding nature of universe. On behalf of its mysterious nature, the investigation of DE is regarded as the most assertive field of research in cosmology. The modified gravity theories (MGTs) are very good approach to deal with that type of force which is accountable for current accelerated speed up of universe. In last decades, various number of MGTs are presented to comprehend the current cosmic-epochs. The models of MGTs are proposed by modifying the geometric section of Einstein-Hilbert action (for detailed reviews on MGTs and DE, see, for illustration [9–22]). Nojiri and Odintsov [23] presented \( f(R) \) gravity and analyzed this theory is well consistent to understand the accelerated-expansion of our cosmos. The MGTs comprise \( f(R) \) [24, 26], \( f(G) \) [27, 28], \( f(R, T) \) [29, 30] (where \( G, R \) and \( T \) describe Gauss Bonnet-invariant, Ricci-scalar and the trace of stress-energy tensor, respectively) and \( f(R, T, Q) \) [21, 31, 34] theories (here \( Q = R_{\lambda\omega} T^{\lambda\omega} \), including non-minimal coupling related to geometric and matter contents) etc.

Stellar evolution is a process which demonstrates the changes that astrophysical object undergoes in its lifetime. The anisotropic nature of matter contents have great relevance in the formation and evolution of astrophysical objects. The gravitational collapse is the most significant part of the stellar evolution. The investigation of dynamical characteristics of self-gravitating body is a key problem. Therefore, the study of gravitational collapse has gained wide attention in Einstein gravity theory [35, 36] as well as in MGTs [37, 38]. Herrera and collaborators [39, 40] explored Some dynamical structures for cylindrical as well as spherical collapsing fluids after the evaluation of matching conditions. Yousaf and Bhatti [41] specified the instability constraints and the appearance of cavity in relativistic interiors and studied the effects of dark-source terms of modified gravity on the unstable regions. Bhatti and Yousaf [42] also examined the effects of \( f(R) \) models on the dynamical configuration of
collapsing body and studied that the collapsing process slows down because of charge and constituents of $f(R)$ gravity.

Scalar variables (structure-scalars) have notable significance to apprehend the physical aspects of self-gravitating stellar systems. This idea has gained more attention of astrophysicists. In 2009, Herrera and collaborators [43] presented a detailed study on relativistic set of equations for spherical configuration controlled by scalar-variables called as structure scalars. After this, they [44] utilized the same idea for $(1 + 3)$ cylindrical formalism, and figured out four-set of scalar variables, they also related these scalars to the basic physical aspects of anisotropic matter contents. Yousaf [45] has examined the role of modified scalars-variables in the context of $f(G,T)$ gravity and studied the influence of such modified scalars on the evolutions of kinematical quantities. It was also examined that these scalars are beneficial for the review of Penrose Hawking-singularity. In 2016, Yousaf et al. [46,47] studied the influence of extended gravity on the dynamical behavior of radiating star by evaluating modified scalars. Recently, Bhatti et al. [48] have studied the significance of scalar-variables for the evolution of massive stars. They have also calculated some stellar equations in the direction of $f(R)$ gravity.

The research on dynamical study of axial-symmetric anisotropic matter contents exists in large numbers in direction of Einstein gravity theory. However, it is a little bit inspiring to deal with such spacetime in MGTs. Herrera et al. [49] studied the dynamics of axial-symmetric and anisotropic relativistic system by evaluating scalar-variables in static configuration. After this attempt they [50] also generalized the same work in order to demonstrate the evolving axial and reflection symmetric anisotropic stellar objects and revealed nice outcomes corresponding to physical aspects by means of these scalars. Their contributions delivered gravitational-radiations, heat-dissipation and flow of super energy associated with magnetic parts of the Weyl-tensor, heat-flux vector and vorticity, respectively. Bhatti and his collaborators [51] considered axial-symmetric configuration to analyze the stability of compact bodies by imposing perturbation scheme in the direction of $f(R,T)$ gravity. For this purpose they investigated Newtonian as well post-Newtonian realm for particular $f(R,T)$ model. Recently, we [52] have presented the general study on axial and reflection symmetric sources in the onset of $f(R,T)$ gravity. The relativistic equations for the chosen system are calculated, and the generalized transport equation is also presented to discuss the thermodynamics of the system. They concluded that the generalized structure scalars have a significant role in the dynamics of the system.

In this paper, we bring out the effects of extra terms associated to $f(R,T)$ theory of gravity, on the dynamics of evolving fluid in the quasi-static regime, by following the program outlined in [52]. This paper is outlined as follows: We provide formalism of $f(R,T)$ gravity in section 2. The representation of axially-symmetric anisotropic and dissipative source, and related kinematical variables in section 3, we will also discuss $f(R,T)$ scalar variables in this
section. Section 4 covers kinematics of the system, where we would like to discuss specific velocities. These velocity functions have special role to define the quasi static-approximation (QSA). The next section is devoted to the quasi static-regime to evaluate the dynamics of our relativistic self-gravitating system. In last section, we discuss our findings.

2 The \( f(R, T) \) Formalism

We take into account \( f(R, T) \) gravity proposed on the basis of non minimal coupling between system’s geometry and its fluids contents. The scalar-curvature \( R \) in Einstein’s gravity action function is substituted with its generic function of scalar-curvature and trace of stress-energy tensor i.e., \( R, T \). This gravity is the extension of \( f(R) \) gravity, the \( f(R, T) \) gravity includes certain quantum effects and is regarded to be even more effective than \( f(R) \) gravity. It is highlighted that such modification in the Lagrangian may be noticed as the additional degrees of freedom. Therefore, the equation of motion that develops from this type of Lagrangian will be different from Einstein’s gravity. In that scenario, the cosmological constant might be omitted from the equations describing the universe’s acceleratory phase. These types of Lagrangians are extremely important for studying dark matter and dark energy concerns (for review, please see [16,18,53–55]). The generalized action for \( f(R, T) \) gravity is expressed as [29]

\[
S_{f(R,T)} = \frac{1}{2\kappa} \int \sqrt{-g} \left[ f(R, T) + L_m \right] d^4x, \tag{1}
\]

where \( L_m \) is the relative Lagrangian-density of matter contents. The stress energy-tensor is given as

\[
T^{(m)}_{\lambda\omega} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g}L_m)}{\delta g^{\lambda\omega}} ,
\]

applying variation on Eq.(1) with respect to metric tensor \( g_{\lambda\omega} \) and we receive the following set of equations

\[
R_{\lambda\omega} f_R - \frac{1}{2} g_{\lambda\omega} f + (g_{\lambda\omega} \Box - \nabla_\lambda \nabla_\omega) f_R = \kappa T_{\lambda\omega} - f_T (\Theta_{\lambda\omega} + T_{\lambda\omega}), \tag{2}
\]

where \( g \) is the determinant of metric tensor and \( \nabla_\lambda \) is the operator for covariant-derivative, while \( \Box = g^\lambda_\omega \nabla_\lambda \nabla_\omega \) identifies d’Alembert’s operator. Also,

\[
\Theta_{\lambda\omega} = g^{\alpha\beta} \frac{\delta T^\alpha_\beta}{\delta g^\lambda_\omega} = -2T_{\lambda\omega} + g_{\lambda\omega} L_m - 2g_{\alpha\beta} \frac{\partial^2 L_m}{\partial g^\lambda_\omega \partial g^\alpha_\beta} ,
\]

4
by choosing relativistic units $c = G = 1$, so for $\kappa = 8\pi$ and energy density ($L_m = \mu$), then the expression of $\Theta_{\lambda\omega}$ becomes

$$\Theta_{\lambda\omega} = -2T_{\lambda\omega} + \mu g_{\lambda\omega}.$$  

From Eq. (2), the field equations in $f(R, T)$ gravity are

$$G_{\lambda\omega} = T_{\lambda\omega}^{\text{eff}} = \frac{1}{f_R} \left[ (1 + f_T) T_{\lambda\omega}^{(m)} + \mu g_{\lambda\omega} + \frac{f}{2} - \frac{R}{2} f_R g_{\lambda\omega} + \nabla_{\lambda} \nabla_{\omega} f_R - g_{\lambda\omega} \Box f_R \right],$$  

where $f \equiv f(R, T)$, $R$ is the Ricci-scalar and $T$ describes the trace of stress energy-tensor and $(f_R = \frac{\partial f}{\partial R}, f_T = \frac{\partial f}{\partial T})$ and $G_{\lambda\omega}$ represents the Einstein-tensor.

3 Axially Symmetric Geometry and Kinematical Quantities

We consider the axial and reflection symmetric spacetime. For this system, the generic form of the Weyl spherical-coordinates is written as

$$ds_+^2 = -A^2(t, r, \theta) dt^2 + B^2(t, r, \theta) (dr^2 + r^2 d\theta^2) + C^2(t, r, \theta) d\phi^2 + 2G(t, r, \theta) d\theta dt,$$  

where the geometric quantities such as $A, B$ are dimensionless and at the same time $C$ and $G$ have dimension of $r$. With the preceding mathematical form, we suppose that our axially symmetric geometry is occupied with anisotropic and dissipative collapsing matter distribution

$$T_{\lambda\omega}^{(m)} = (P + \mu) U_\lambda U_\omega + Pg_{\lambda\omega} + \Pi_{\lambda\omega} + q_\lambda U_\omega + q_\omega U_\lambda,$$  

where $T_{\lambda\omega}^{(m)}$ is describing some usual energy components, and four-velocity $U_\lambda$ is assigned by particular observer. We are dealing the configuration, where the fluid contents are at rest position. In our case, we have chosen the fluid contents to be co-moving, next

$$U^\lambda = \left( \frac{1}{A}, 0, 0, 0 \right), \quad U_\lambda = \left( -A, 0, \frac{G}{A}, 0 \right).$$  

Now, we present the unit space-like vectors in component form as follows

$$K_\lambda = (0, B, 0, 0), \quad L_\lambda = \left( 0, 0, \frac{\sqrt{r^2 A^2 B^2 + G^2}}{A}, 0 \right), \quad S_\lambda = (0, 0, 0, C).$$
holding the relation
\[ U^\lambda U_\lambda = -K_\lambda K^\lambda = -L_\lambda L^\lambda = -S_\lambda S^\lambda = -1, \]
\[ U^\lambda K_\lambda = U^\lambda L_\lambda = U^\lambda S_\lambda = S_\lambda K^\lambda = K^\lambda L_\lambda = S_\lambda L^\lambda = 0. \]  

(8)

(9)

The unitary vectors \( U_\lambda, K_\lambda, L_\lambda, S_\lambda \) generate orthonormal-tetrad \( (e^a_\lambda) \) such as
\[ e^\lambda(0) = U_\lambda, \quad e^\lambda(1) = K_\lambda, \quad e^\lambda(2) = L_\lambda, \quad e^\lambda(3) = S_\lambda, \]
here \( a = 0, 1, 2, 3 \) and the representation of dual-vector tetrad is given by
\[ \eta^{(a)(b)} = g^{\lambda\omega} e^{\lambda}_a e^{\omega}_b, \]
here \( \eta^{(a)(b)} \) shows the Minkowski space-time. The anisotropic tensor is defined with the help of scalar functions as
\[ \Pi_{\lambda\omega} = \frac{1}{3}(2\Pi_I + \Pi_{II})(K_\lambda K^\omega - \frac{h_{\lambda\omega}}{3}) + \frac{1}{3}(\Pi_I + 2\Pi_{II})(L_\lambda L^\omega - \frac{h_{\lambda\omega}}{3}) + \Pi_{KL} K_\lambda L^\omega. \]

where
\[ \Pi_I = (2K^\lambda K^\omega - L^\lambda L^\omega - S^\lambda S^\omega)T_{\lambda\omega}, \quad \Pi_{II} = (2L^\lambda L^\omega - K^\lambda K^\omega - S^\lambda S^\omega)T_{\lambda\omega}, \quad \Pi_{KL} = T_{\lambda\omega} K^\lambda L^\omega. \]

This particular choice of above scalars is helpful to evaluate the relevant equations in more easier and compact form. Now, we introduce the heat-flux vector in form of two scalars \( q_I \) and \( q_{II} \) as follows
\[ q_\lambda = q_I K_\lambda + q_{II} L_\lambda, \]
and it is observed that \( U_\lambda q^\lambda = 0 \), so for in coordinate-components
\[ q_\lambda = \left(0, Bq_I, \frac{\sqrt{r^2 A^2 B^2 + G^2 q_{II}}}{A}, 0\right). \]

(11)

3.1 Kinematical Quantities

In the study of self-gravitating system, the kinematical quantities play significant role. Any celestial object undergoes different phases such as distortion of shape, it can contract or expand. Here, we would also like to express the shear-tensor \( (\sigma_{\lambda\omega}) \), expansion-scalar \( (\Theta) \) and the component of vorticity other than the four-acceleration
\[ a_\lambda = U^\gamma U_{\lambda;\gamma} = a_I K_\lambda + a_{II} L_\lambda, \]

(12)
along with

$$a_I = \frac{A'}{AB}, \quad a_{II} = \frac{A}{\sqrt{r^2 A^2 B^2 + G^2}} \left[ \frac{A_{\theta}}{A} - \frac{G}{A^2} \left( \frac{\dot{A}}{A} - \frac{\dot{G}}{G} \right) \right],$$

(13)

here $\dot{A} = \frac{\partial A}{\partial t}$, $A' = \frac{\partial A}{\partial r}$ and $A_{\theta} = \frac{\partial A}{\partial \theta}$. The expansion scalar is a kinematical quantity which calculates the fractional change of matter volume with respect to time. Whose mathematical form and expression for our relativistic system is

$$\Theta = U^{\lambda}_\lambda = \frac{AB^2 r^2}{A^2 B^2 r^2 + G^2} \left[ \frac{G^2}{A^2 B^2 r^2} \left( \frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{C}}{C} + \frac{\dot{G}}{G} \right) + \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right].$$

(14)

If $\Theta < 0, \Theta > 0$, then it represents contracting and expanding nature of matter contents, respectively. However, $\Theta = 0$ indicates the presence of vacuum cavity inside the matter contents. Moreover, the shear tensor quantifies the distortion in shape such that its volume remains constant. Defined by

$$\sigma_{\lambda\omega} = \sigma^{(a)(b)} e^{(a)}_{\lambda} e^{(b)}_{\omega} = U_{(\lambda; \omega)} + a_{(\lambda} U_{\omega)} - \frac{h_{\lambda\omega}}{3} \Theta.$$

(15)

The shear-tensor $\sigma_{\mu\nu}$ may also be expressed in form of scalar-functions $\sigma_I$ and $\sigma_{II}$ as

$$\sigma_{\lambda\omega} = \frac{1}{3} (2\sigma_I + \sigma_{II})(K_{\lambda K_{\omega}} - \frac{h_{\lambda\omega}}{3}) + \frac{1}{3} (\sigma_I + 2\sigma_{II})(L_{\lambda L_{\omega}} - \frac{h_{\lambda\omega}}{3}),$$

(16)

where

$$2\sigma_I + \sigma_{II} = \frac{3}{A} \left\{ \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right\},$$

(17)

$$\sigma_I + 2\sigma_{II} = \frac{3}{r^2 A^2 B^2 + G^2} \left[ r^2 A^2 B^2 \left\{ \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right\} - \frac{G^2}{A} \left( \frac{\dot{A}}{A} + \frac{\dot{C}}{C} - \frac{\dot{G}}{G} \right) \right].$$

(18)

Finally, the vorticity can be represented by $\hat{\omega}_{\lambda}$ (vorticity-vector) or $\Omega^{\omega_{\alpha}}$ as follows

$$\hat{\omega}_{\lambda} = \frac{1}{2} \eta_{\lambda\omega_{\alpha\beta}} U^{(\omega_{\alpha})} U^{\beta} = \frac{1}{2} \eta_{\lambda\omega_{\alpha\beta}} \Omega^{\omega_{\alpha}} U^{\beta},$$

where $\eta_{\lambda\omega_{\alpha\beta}}$ is the Levi-Civita tensor while the vorticity-tensor is

$$\Omega_{\lambda\omega} = U_{[\lambda; \omega]} + a_{[\lambda} U_{\omega]},$$
its single non-zero component is
\[ \Omega_{12} = \frac{G}{2A} \left( \frac{2A'}{A} - \frac{G'}{G} \right). \]
Therefore, it can be written in form of unit-vectors as
\[ \Omega_{\lambda\omega} = (L_\lambda K_\omega - L_\omega K_\lambda) \Omega, \quad \omega_\mu = -S_\mu \Omega, \]
scalar function \( \Omega \) is given as
\[ \Omega = \frac{-G \left( \frac{2A'}{A} - \frac{G'}{G} \right)}{2B \sqrt{r^2 A^2 B^2 + G^2}}. \quad (19) \]
In case of central regular conditions, the above equation shows that \( G = 0 \iff \Omega = 0 \).

3.2 The Weyl-tensor and Structure Scalars

Since the magnetic portion of the Weyl-tensor does not vanish in the axially symmetric setting. Therefore, we would like to introduce the electric \((E_{\lambda\omega})\) as well as magnetic \((H_{\lambda\omega})\) portions of the Weyl-tensor \((C_{\lambda\omega\gamma\alpha})\). Usually, these are defined as \[56\]
\[ E_{\lambda\omega} = C_{\lambda\beta\omega\alpha} U^\beta U^\alpha, \quad H_{\lambda\omega} = \frac{1}{2} \eta_{\lambda\beta\epsilon\rho} C_{\omega\alpha} \epsilon^\rho U^\beta U^\alpha, \quad (20) \]
Thus, for Eq.(14), these might be expressed as
\[ E_{\lambda\omega} = \frac{1}{3} (2\varepsilon_I + \varepsilon_{II}) \left( K_\lambda K_\omega - \frac{h_{\lambda\omega}}{3} \right) + \frac{1}{3} (\varepsilon_I + 2\varepsilon_{II}) \left( L_\lambda L_\omega - \frac{h_{\lambda\omega}}{3} \right) + \varepsilon_{KL} (K_\lambda L_\omega + K_\omega L_\lambda), \]
\[ H_{\lambda\omega} = H_1 (K_\lambda S_\omega + K_\omega S_\lambda) + H_2 (L_\lambda S_\omega + L_\omega S_\lambda), \]
where \( h_{\lambda\omega} = \delta_{\lambda\omega} + U^\lambda U_\omega, \) and \( \varepsilon_I, \varepsilon_{II}, \varepsilon_{KL} \) and \( H_1, H_2 \) are the components of electric and magnetics parts, respectively. In the decomposition of the Riemann-tensor, the Weyl-tensor plays a key role.

Structure scalars play important role to examine the physical aspects of the fluid contents. To calculate these scalars for our problem, let us take into account three-tensors \(X_{\lambda\omega}, Y_{\lambda\omega}\) and \(Z_{\lambda\omega}\) through Riemann-tensor for the evaluation of scalar-variables
\[ X_{\lambda\omega} = \frac{1}{2} \eta_{\lambda\beta} \epsilon^\rho R_{\rho\omega\alpha} U^\alpha U^\beta, \quad Y_{\lambda\omega} = R_{\lambda\beta\omega\alpha} U^\beta U^\alpha, \quad Z_{\lambda\omega} = \frac{1}{2} \epsilon_{\lambda\rho\mu} R_{\alpha\omega} U^\alpha U^\rho, \quad (21) \]
with $\epsilon_{\lambda\rho} = \eta_{\beta\lambda\rho} U^\beta$ and $R^*_{\lambda\beta\alpha} = \frac{1}{2} \eta_{\rho\beta\alpha} R^\rho_{\lambda\alpha}$. The explicit form of these tensors for our problem is calculated as

$$X_{\lambda\omega} = -E_{\lambda\omega} - \frac{\kappa}{f_R} \left( (1 + f_T) \Pi_{\lambda\omega} - \frac{h_{\lambda\omega}}{2} - (f - R f_R) \frac{h_{\lambda\omega}}{6} \right) + \psi_1,$$

with the corresponding four scalar-variables,

$$X_T = \frac{\kappa}{f_R} \left( \mu + \frac{1}{2} (f - R f_R) \right) + \psi^*_1, \quad X_I = -\varepsilon_I - \frac{\kappa}{2 f_R} (1 + f_T) \Pi_I,$$

$$X_{II} = -\varepsilon_{II} - \frac{\kappa}{2 f_R} (1 + f_T) \Pi_{II}, \quad X_{KL} = -\varepsilon_{KL} - \frac{\kappa}{2 f_R} (1 + f_T) \Pi_{KL}. \quad (24)$$

The scalar $X_T$ describes the trace part of $X_{\lambda\omega}$, while the remaining ones are corresponding to unit space-like vectors. Similarly,

$$Y_{\lambda\omega} = E_{\lambda\omega} - \frac{\kappa}{2 f_R} \left[ (1 + f_T) \Pi_{\lambda\omega} + \psi_2 \right] + \frac{\kappa}{3 f_R} \left[ \mu + 3 \mu + 3(f + R f_R) + 2(f - R f_R) \right],$$

with

$$Y_T = \frac{\kappa}{2 f_R} \left[ (\mu + 3 P)(1 + f_T) + 8 \mu f_T + 4(f - R f_R) + \psi_3 \right], \quad Y_I = \varepsilon_I - \frac{\kappa}{2 f_R} (1 + f_T) \Pi_I,$$

$$Y_{II} = \varepsilon_{II} - \frac{\kappa}{2 f_R} (1 + f_T) \Pi_{II}, \quad Y_{KL} = \varepsilon_{KL} - \frac{\kappa}{2 f_R} (1 + f_T) \Pi_{KL}. \quad (27)$$

Finally

$$Z_{\lambda\omega} = H_{\lambda\omega} + \frac{\kappa}{2 f_R} (1 + f_T) q^\gamma \epsilon_{\lambda\omega\gamma} + \frac{\kappa}{2} \psi_4,$$

along with related scalar-variables

$$Z_I = H_I - \frac{\kappa}{2 f_R} (1 + f_T) q_I, \quad Z_{II} = H_I + \frac{\kappa}{2 f_R} (1 + f_T) q_{II}, \quad(29)$$

$$Z_{III} = H_2 - \frac{\kappa}{2 f_R} (1 + f_T) q_I, \quad Z_{IV} = H_2 + \frac{\kappa}{2 f_R} (1 + f_T) q_I.$$

The expressions of $\psi'_i$s are given in Appendix B. The motivation to demonstrate such an analysis and to insight further these structure scalars in the evolution of self-gravitating compact objects arises from their various physical aspects. Dissipation effects in the interior region of stellar objects are defined by generalized structure scalars as presented in Eqs. (29).
and (30), obtained from $Z_{\lambda \omega}$. Consequently, we can say that the incorporation of $Z_{I,II,III,IV}$ has a direct correlation with the magnetic effects of the Weyl-tensor and heat dissipation. Whereas the evolution of expansion and shearing rate for axial and reflection symmetric anisotropic and dissipative fluid is controlled by $Y_T$ and $Y_{I,II}$, respectively (as expressed in Eqs.(62) and (63) in [52]). We argue that, besides Einstein’s gravity structure scalars, the generalized form of such scalars are also significant in describing compact galactic configuration. It is significant to note that super-massive and enormous compact galactic structures in the universe exclusively exist in $f(R,T)$ gravity. The specific choice of these scalar-variables is to evaluate the QSA of basic modified scalar-equations, which are presented in Appendix A. These sets of scalars define various physical aspects for the evolution of self-gravitating celestial bodies.

4 The Kinematics

As an area of study, the kinematics is frequently referred to describe the the geometry of motion and is sometimes considered as subdivision of mathematics. A kinematical problem undertakes by illustrating the geometry of systems and pointing out the initial conditions of some known values of position of the systems. In order to examine the large scale structure of cosmos, the kinematics is deployed in astrophysics to exhibit the motion of celestial objects such as stars, galaxies and collection of such objects. The discussion of this section figures on the kinematical variables distinguishing the motion of the medium unveiled in [57]. We may defined the set of invariant-velocities from the expression, containing space-like triad $(\epsilon_{(i)}^\lambda; i = 1, 2, 3)$ given as ( for detailed study, please see [56]),

$$\left(\frac{D_T(\delta l)}{\delta l}\right)_{(i,j)} = \epsilon_{(i)}^{\lambda} e_{(j)}^{\omega} \left(\sigma_{\lambda \omega} + \frac{h_{\lambda \omega}}{3} \Theta + \Omega_{\lambda \omega}\right). \quad (31)$$

From Eq.(31), we get

$$U_{(1)} = K^\lambda K^\omega \left(\sigma_{\lambda \omega} + \Omega_{\lambda \omega} + \frac{h_{\lambda \omega}}{3} \Theta\right), \quad U_{(2)} = L^\lambda L^\omega \left(\sigma_{\lambda \omega} + \Omega_{\lambda \omega} + \frac{h_{\lambda \omega}}{3} \Theta\right), \quad (32)$$

$$U_{(3)} = S^\lambda S^\omega \left(\sigma_{\lambda \omega} + \Omega_{\lambda \omega} + \frac{h_{\lambda \omega}}{3} \Theta\right), \quad U_{(1,2)} = K^\lambda L^\omega \left(\sigma_{\lambda \omega} + \Omega_{\lambda \omega} + \frac{h_{\lambda \omega}}{3} \Theta\right), \quad (33)$$

$$U_{(1,3)} = K^\lambda S^\omega \left(\sigma_{\lambda \omega} + \Omega_{\lambda \omega} + \frac{h_{\lambda \omega}}{3} \Theta\right). \quad (34)$$

Using Eqs.(14), (16), (19), we have

$$U_{(1)} = \frac{1}{3}(\Theta + \sigma_I), \quad U_{(2)} = \frac{1}{3}(\Theta + \sigma_{II}), \quad U_{(1,3)} = 0, \quad U_{(1,2)} = -\Omega. \quad (35)$$
\[
U_{(3)} = \frac{1}{3} (\Theta - \sigma_I - \sigma_{II}), 
\]
satisfying the relation
\[
U_{(1)} + U_{(2)} + U_{(3)} = \Theta. 
\]

It is observed that the proper time-variation of \( \delta l \) is defined by these specific quantities. The geometrical as well a physical demonstration of such specific quantities is controlled by kinematical variables along with the unit space-like vectors as presented in Eqs. (32)-(34).

5 The Quasi-static Regime

In order to investigate self-gravitating celestial bodies, we may take into consideration three feasible evolutionary regimes, namely: static evolution, quasi-static evolution and dynamic evolution. In static configuration, a coordinate structure can always be selected in a way that all geometric as well as physical quantities are free from time-like coordinate. In this case, time-like hyper-surface (also orthogonal) killing vector is revealed by spacetime. Afterwards, system undergoes complete dynamic phase, where it is regarded to be out of equilibrium condition (either dynamic or thermal). In between the two phases mentioned above, there is quasi static-evolution. The system evolves slowly at every moment, so for it may be regarded in state of equilibrium in such an evolution.

Consequently, we can say that system faces changes slowly on a time-scale, this is very long as to that typical time in which the system responds to small perturbed configuration of hydro-static state of equilibrium. Thus, we can say that in this phase our system is convenient to hydro-static state of equilibrium, and such system may be evaluated in sequence of equilibrium-models. Now, we would like to describe the conditions of the QSA inform of kinematical quantities, specific-velocities and \( f(R, T) \) corrections discussed in above sections. These conditions are entailed due to the fact that hydro-static time of system under consideration must be much larger than any characteristic time-scale of that system. Therefore

- All quantities having order \( O(\epsilon^2) \) and higher will be neglected, where \( \epsilon << 1 \).
- The specific-velocities such as \( U_{(1), (2), (3)} \) and \( U_{(1,2)} \) defined in Eqs. (32)-(34) are smaller quantities, therefore have order \( O(\epsilon) \).
- It follows from Eqs. (35), (36) that the scalars \( \Omega, \Theta, \sigma_{I, II} \) have order \( O(\epsilon) \), also Eqs. (14), (17)-(19) indicate that \( G, A, B \) and \( C \) are of \( O(\epsilon) \).
• It is also observed from Eqs. (14), (17), (18) that $\tilde{\sigma} \equiv \sigma_I = \sigma_{II}$, having the same order i.e, $O(\epsilon)$ and

$$\Theta - 2\tilde{\sigma} = \frac{3}{A} \left( \frac{C}{\dot{C}} \right), \quad \tilde{\sigma} + \Theta = \frac{3}{A} \left( \frac{\dot{B}}{B} \right)$$

(38)

• The dark sources terms such that $f_R \equiv \tilde{f}_R, f_T \equiv \tilde{f}_T, q^I_{\text{eff}} \equiv \tilde{q}^I_{\text{eff}}, q^I_{\text{eff}} \equiv \tilde{q}^I_{\text{eff}}, \mu^I_{\text{eff}} \equiv \tilde{\mu}^I_{\text{eff}}$ and other effective fluid components in the QSA accordingly.

Moreover, it is also suppose that the relaxation-time in the evolution of modified transport-equation must be neglected. In fact, relaxation-time is the time required by the system to come back instinctively in its steady state, after it has been abruptly took away from it. However, it deduces from the nature of the QSA that all the processes evolve on larger time-scale than the time taken for transient-phenomena, inferring that we are expecting the modified heat-fluxes to characterize a constant heat flow along with the effect of $f(R, T)$ corrections. Therefore, the relaxation-time $\tau$ is neglected in both components of modified transport-equation (Eqs.(49) and (50) in [52]) for our relativistic-system then the results are obtained as

$$\tilde{q}^I_{\text{eff}} = -\frac{\kappa}{B} (BTa_I + T'),$$

(39)

$$\tilde{q}^I_{\text{eff}} = \frac{\kappa}{A} \left( A T a_I + C \dot{T} + A^2 T \theta - r_{AB} \tilde{f}_R \chi_{\approx} \right).$$

(40)

Since $\dot{T}$ has order $O(\epsilon)$, and using thermal-equilibrium conditions [58]. Therefore, from above equations, we receive the following expressions in the quasi static-regime

$$(T A)' = \frac{1}{r B f_R} \chi_1^{\approx}, \quad (T A)' = \frac{1}{r A B f_R} \chi_2^{\approx},$$

(41)

where “approx” is used to illustrate the QSA on corresponding quantities. One can be easily evaluated the quasi-static approximated values of these quantities (presented in Appendix in [52], by using the above defined QSA. Moreover, the scalar components of Eq.(13) are turned out to be of order $O(\epsilon)$ after imposing such an approximation, in order that

$$a_I = \frac{1}{B} \left( \frac{A'}{A} \right), \quad a_{II} = \frac{1}{r B} \left( \frac{A_{\theta}}{A} \right).$$

(42)
5.1 QSA on Modified Field Equations:

Here, we would like to evaluate the MFEs in the QSA. By implementing the proposed conditions, as defined earlier in this section, we acquire the MFEs (Eqs.(14)-(20) in [52]) as follows

\[
G_{00} = \frac{\kappa A^2}{f_R} \left[ \mu - \frac{1}{2}(\ddot{f} - \ddot{R}f_R) \right] + \frac{\kappa}{f_R} \left[ \frac{1}{r^2B^2} \ddot{f}_{R,\theta\theta} + \ddot{f}_R \left\{ \frac{A^2}{B^2 C} + \frac{A^2 C'}{B^2 C} \right\} \right] \\
+ \ddot{f}_{R,\theta} \frac{A^2 C \theta}{r^2B^2 C},
\]

\( (43) \)

\[
G_{01} = \frac{\kappa}{f_R} \left[ -AB(1 + \ddot{f}_T)q_I \right],
\]

\( (44) \)

\[
G_{02} = -\frac{ABr\kappa}{f_R} \left[ \frac{\mu G}{ABr} + (1 + \ddot{f}_T)q_{II} \right] + \frac{\kappa G}{2f_R} (\ddot{f} - \ddot{R}f_R) - \frac{\kappa G}{2r^2B^2 f_R} [ \ddot{f}_{R,\theta\theta} \\
+ C_{\theta}(f_{R,\theta}) ],
\]

\( (45) \)

\[
G_{12} = \frac{\kappa}{f_R} \left[ (1 + \ddot{f}_T)(B^2r\Pi_{KL}) + \frac{BG}{A}q_I + \ddot{f}_{R,\theta} - \frac{B_\theta}{B} \ddot{f}_R - \frac{(Br)^\prime}{Br} \right],
\]

\( (46) \)

\[
G_{11} = \frac{\kappa B^2}{f_R} \left[ (1 + \ddot{f}_T)(P + \Pi_{III}) + \mu \ddot{f}_T + \frac{1}{2}(\ddot{f} - \ddot{R}f_R) + \frac{1}{r^2B^2} (\ddot{f}_{R,\theta\theta} + \frac{A_\theta}{A} - \frac{B_\theta}{B} \\
+ \frac{C_{\theta}}{C} f_{R,\theta} ) \right],
\]

\( (47) \)

\[
G_{22} = \frac{\kappa r^2B^2}{f_R} \left[ (1 + \ddot{f}_T)(P + \Pi_{III}) + \mu \ddot{f}_T + \frac{1}{2}(\ddot{f} - \ddot{R}f_R) - \frac{1}{r^2B^2} (\ddot{f}_{R,\theta\theta} + (\frac{A_\theta}{A} - \frac{B_\theta}{B} \\
+ \frac{C_{\theta}}{C} f_{R,\theta} ) \right],
\]

\( (48) \)

\[
G_{33} = \frac{\kappa C^2}{f_R} \left[ (1 + \ddot{f}_T) \left( P - \frac{1}{3}(\Pi_I + \Pi_{III}) \right) + \mu \ddot{f}_T + \frac{1}{2}(\ddot{f} - \ddot{R}f_R) - \frac{1}{r^2B^2} (\ddot{f}_{R,\theta\theta} \\
+ \frac{A_\theta}{A} f_{R,\theta} ) \right],
\]

\( (49) \)

Here \( f_{R,\theta} = \frac{\partial f_R}{\partial \theta} \) and \( B_\theta = \frac{\partial B}{\partial \theta} \).

5.2 QSA on Hydro-dynamics:

The dynamical equations describe the change in the parameters of the physical system with respect to time. These equations are related to the study of motion of celestial objects
which is supported by stress-energy tensor. As a result of gravitational collapse, static celestial objects become un-stable. To deal such situation, the gravitational field equations are helpful to provide the dynamical equations. Therefore, we want to execute the evolution of dynamical equations in the quasi static constraints. Thus, the quasi static-configuration of dynamical equation (Eq.(54) in [52]) is given as

$$\frac{1}{f_R}\left[(1 + \tilde{f}_R) \left\{ \mu A + \Theta(\mu + P) + \frac{1}{9}(\Pi_I(2\sigma_I + \sigma_{II}) + \Pi_{II}(\sigma_I + 2\sigma_{II})) + \frac{q_I}{B} + \frac{1}{Br}(q_\theta \\
+ G A^2 q_{II}) + 2(q_I a_I + q_{II} a_{II}) + \frac{q_I}{B}(C' + (Br)' + \frac{q_{II}}{Br}(B_{\theta} + C_{\theta}) + \frac{1}{9} (\Pi_I(2\sigma_I + \sigma_{II}) + \Pi_{II}(\sigma_I + 2\sigma_{II})) \right\} \right] = \frac{1}{f_R} \chi^\approx_{54}. \quad (50)$$

From the modified Euler-lagrange equation (Eq.(55) in [52]), the following two equations are attained in the QSA

$$\frac{1}{f_R} \left[(1 + \tilde{f}_R) \left\{ \frac{1}{B} \left(P + \frac{\Pi_I}{3}\right)' + \frac{1}{Br} \left(\Pi_{KL,\theta} + G A^2 \Pi_{KL}\right) + (\mu + P + \frac{\Pi_I}{3})a_I + a_{II} \Pi_{KL} \\
+ \frac{\Pi_I}{3B} \left(2C' + (Br)'\right) + \frac{\Pi_{II}}{3B} \left(\frac{C'}{C} - \frac{(Br)'}{Br}\right) + \frac{\Pi_{KL}}{Br} \left(\frac{2B_{\theta} + C_{\theta}}{B} + \frac{1}{9} (\Pi_I(2\sigma_I + \sigma_{II}) + \Pi_{II}(\sigma_I + 2\sigma_{II})) \right) \cdot \tilde{q}_I \right\} \right] \quad (51)$$

and

$$\frac{1}{f_R} \left[(1 + \tilde{f}_R) \left\{ \frac{1}{Br} \left(P + \frac{\Pi_I}{3}\right,\theta + G A^2 \left(P + \frac{\Pi_{II}}{3}\right) \right) + \frac{\Pi_{KL}}{B} + (\mu + P + \frac{\Pi_{II}}{3})a_{II} + a_I \Pi_{KL} \\
+ \frac{\Pi_I}{3Br} \left(-\frac{B_{\theta}}{B} + \frac{C_{\theta}}{C}\right) + \frac{\Pi_{II}}{3Br} \left(\frac{B_{\theta}}{B} + 2C_{\theta}'\right) + \frac{\Pi_{KL}}{Br} \left(\frac{C'}{C} + \frac{(Br)'}{Br}\right) \cdot \tilde{q}_I \right\} \right] \quad (52)$$

### 5.3 QSA on Modified Scalar Equations:

In our regime, it is followed from Eq.(19) that the time derivative of vorticity-scalar i.e, \( \dot{\Omega} \) have order \( O(\dot{\epsilon}^2) \). The evolution of generalized Ricci-Identities that are Eqs.(66) and (67) in [52], in quasi static constraints yields respectively,

$$\frac{2}{3B} \Theta' - \frac{\Omega}{Br} \left(\frac{2A_{\theta}}{A} + \frac{C_{\theta}}{C}\right) - \frac{\Omega_{\theta}}{Br} - \frac{\dot{\sigma}'}{3B} - \frac{\ddot{a} C'}{BC} = \kappa \tilde{q}_I^{\text{eff}}, \quad (53)$$
\[ \frac{2}{3Br} \Theta_{,\theta} + \frac{\Omega'}{B} - \frac{\Omega}{B} \left( \frac{2A'}{A} + \frac{C'}{C} \right) - \bar{\sigma}_{,\theta} \frac{C_{,\theta}}{3Br} - \bar{\sigma} C_{,\theta} = \kappa \tilde{q}_{II} \text{eff}. \]

It depicts that dissipative fluxes have also order \( O(\epsilon) \). So far, we summarize all the outcomes deduced from the QSA as

- Order of \( \dot{\Omega}, \dot{G} \) is \( O(\epsilon^2) \).

- \( \bar{\sigma}, \Omega, \Theta, G, \dot{A}, \dot{B}, \dot{C}, \dot{a}_I, \dot{a}_{II} \) are of order \( O(\epsilon) \).

- \( \dot{q}_I, \dot{q}_{II}, \tilde{q}_I \text{eff}, \tilde{q}_{II} \text{eff}, \tilde{f}_R, \tilde{f}_T \) all are of order \( O(\epsilon) \).

Since the hydro-static equilibrium state can be detained at any time, therefore the corresponding equations containing \( \sigma_{32}, \sigma_{33} \) components of \( \sigma_{\lambda\omega} \) hold \[50\]. It is obtain from above mentioned equations, respectively

\[ \dot{\Pi}_{KL} \approx O(\epsilon); \quad \dot{q}_I \approx O(\epsilon^2); \quad \ddot{C} \approx O(\epsilon^2); \quad \ddot{B} \approx O(\epsilon^2), \quad (55) \]

\[ \dot{\Pi}_{II} \approx O(\epsilon); \quad \dot{P} \approx O(\epsilon); \quad \dot{q}_{II} \approx O(\epsilon^2). \quad (56) \]

It has been imposed the fact that \( P, \Pi_I, \Pi_{II} \) include terms with \( \ddot{C} \) and \( \ddot{B} \) other than the terms involving some spatial-coordinate derivatives of corresponding line-element. Now, it is followed immediately from Eqs.\((38)\)

\[ \dot{\Theta} \approx O(\epsilon^2); \quad \dot{\bar{\sigma}} \approx O(\epsilon^2). \quad (57) \]

By using Eqs.\((35)\), then the Eq.\((53)\) turned out to be

\[ 2U' = \frac{\Omega}{r} \left[ \ln(\Omega CA^2) \right]_{,\theta} + \bar{\sigma} \left[ \ln(\bar{\sigma} C) \right]' + \kappa B \tilde{q}_I \text{eff}, \quad (58) \]

where \( U \equiv U_1 \equiv U_2 \), after integration we attain

\[ U = U_\Sigma - \frac{1}{2} \int_r^{r_{\Sigma}} \left\{ \frac{\Omega}{r} \left[ \ln(\Omega CA^2) \right]_{,\theta} + \bar{\sigma} \left[ \ln(\bar{\sigma} C) \right]' + \kappa B \tilde{q}_I \text{eff} \right\} dr, \quad (59) \]

it can also be expressed as

\[ U_3 = U_{(3)\Sigma} - \frac{1}{2} \int_r^{r_{\Sigma}} \left\{ \frac{\Omega}{r} \left[ \ln(\Omega CA^2) \right]_{,\theta} + \bar{\sigma} \left[ \ln(\bar{\sigma} C) \right]' + \kappa B \tilde{q}_I \text{eff} \right\} dr, \quad (60) \]

here, the equation \( r = r_{\Sigma} \) described the surface boundary of the source and we have also used the fact that \( U_3 = U - \bar{\sigma} \). In similar fashion, Eq.\((54)\) may be written as follows

\[ 2U_{,\theta} = -\Omega r \left[ \ln(\frac{CA^2}{\Omega}) \right]' + \bar{\sigma} \left[ \ln(\bar{\sigma} C) \right]_{,\theta} + \kappa Br \tilde{q}_{II} \text{eff}, \quad (61) \]
generating

\[ U = U_\Sigma - \frac{1}{2} \int_{\theta}^{\theta_\Sigma} \left\{ -\Omega [\ln(\frac{C A^2}{\Omega})]' + \tilde{\sigma} \left[ \ln(\tilde{\sigma} C) \right]_{,\theta} + \kappa B r \tilde{q}_I^{\text{eff}} \right\} d\theta, \]  

(62)

or

\[ U^{(3)} = U^{(3)}_\Sigma - \frac{1}{2} \int_{\theta}^{\theta_\Sigma} \left\{ -\Omega [\ln(\frac{C A^2}{\Omega})]' + \tilde{\sigma} \left[ \ln(\frac{C}{\sigma}) \right]_{,\theta} + \kappa B r \tilde{q}_I^{\text{eff}} \right\} d\theta, \]  

(63)

in this case, the boundary surface is given by the equation \( \theta = \theta_\Sigma \). Now, we will focus on the physical description of the Eqs. (59), (60), (62), (63). Let us investigate the order of the magnetic part of the Weyl-tensor. From Eqs. (68) and (69) in [52], and the final outcomes in that case are same as presented in [56].

\[ H_1 = -a_I \Omega - \frac{1}{2 Br} \left( \frac{\tilde{\sigma} C_{,\theta}}{C} + \tilde{\sigma}_{,\theta} \right) + \frac{1}{2B} \left( \frac{\Omega C'}{C} - \Omega' \right), \]  

(64)

\[ H_2 = -a_{II} \Omega + \frac{1}{2 Br} \left( \frac{\Omega C_{,\theta}}{C} - \Omega_{,\theta} \right) + \frac{1}{2B} \left( \tilde{\sigma} C' + \tilde{\sigma}' \right), \]  

(65)

inferring that the \( H_1 \) and \( H_2 \) are of order \( O(\epsilon) \). It is worth observing that \( G = 0 = \Omega \) i.e, in vorticity free case, it implies from Eqs. (64), (65) that

\[ H_1 = -\frac{(\tilde{\sigma} C_{,\theta})}{2 r BC}, \quad H_2 = -\frac{\tilde{\sigma} C'}{2 BC}. \]  

(66)

Then from Eqs. (53), (54), using Eqs. (38) and with the assumption \( \Omega = 0 \), yield respectively

\[ 2 \left( \dot{B}_{AB} \right)' - \frac{(\tilde{\sigma} C)'}{C} = \kappa B \tilde{q}_I^{\text{eff}}, \]  

(67)

\[ 2 \left( \frac{\dot{B}}{AB} \right)_{,\theta} - \frac{(\tilde{\sigma} C)_{,\theta}}{C} = \kappa B r \tilde{q}_{II}^{\text{eff}}. \]  

(68)

Now, combining both above equations with Eq. (66), we attain the following relations, respectively

\[ H_1 + \frac{1}{rB} \left( \frac{\dot{B}}{AB} \right)_{,\theta} = \frac{\kappa q_{II}}{2}, \]  

(69)
Thus from Eq.(66) it is followed that disappearance of shear \((\Omega = 0)\) is the necessary as well as sufficient condition for the matter to be purely-electric in the QSA. The expressions of modified heat-fluxes and the other fluid contents are given in Appendix B. However, the modified heat-fluxes have also played an effective role in that case as well. The quasi static-configuration of modified scalar equations is explicitly written in Appendix A.

6 Conclusion

In current research, the compact objects are the most significant class of astrophysical objects. These objects are extremely dense i.e, smaller in size and higher in mass. The study of such objects have gained the main attention of astrophysicists. In this paper, we have investigated the quasi static-evolution of axially and reflection symmetric fluids using the framework constructed in [52]. We have proposed axial and reflection symmetric system, stuffed with anisotropic as well as dissipative fluid contents. So for we have chosen the most generic representation of stress energy-tensor as given in Eq.(5). We developed MFEs to study the motion of relativistic source. To analyze the basic aspects of fluid contents, we discussed the kinematical quantities including \(a_\lambda\) (four-acceleration), \(\Theta\) (expansion-scalar), \(\Omega\) (vorticity-scalar) and \(\sigma_{\lambda\omega}\) (shear-tensor). Moreover, we have discussed the magnetic and electric parts of the Weyl-tensor. Five scalars \(H_1, H_2\) and \(\varepsilon_I, \varepsilon_{II}, \varepsilon_{KL}\) illuminating the magnetic and electric-parts, respectively, are also established.

For the evolution of compact objects, three feasible regimes may be considered, namely: static, quasi-static and dynamic regimes. The QSA is the sensible approach to discuss the hydro-dynamics of self-gravitating compact objects. In this evolutionary phase, the system faces changes sufficiently slow, so for it can be regarded in equilibrium state. For this scenario

- Firstly, the set of invariant-velocities are defined for the comprehension of kinematics as well as for the concept of the QSA. These scalar functions basically hold the relation

\[
U_{(1)} + U_{(2)} + U_{(3)} = \Theta.
\]

It can be seen from Eqs.(32)-(34) that geometrical and physical demonstration of such specific-velocities is governed by kinematical-variables with the unit space-like vectors \(K^\lambda, L^\lambda\) and \(S^\lambda\). Herrera et al. [56] used this approach for the study of axial and reflection symmetric self-gravitating anisotropic and dissipative source in Einstein gravity theory.
• The set of seven MFEs for our (1 + 3) formalism are calculated by using Eq.(3), then quasi static constraints defined in Sec. 4 are imposed to evaluate the proposed approximation. The quasi static-configuration of these MFEs are presented in Eqs.(43)-(49). The continuity as well as generalized Euler-equation are also evaluated in this configuration. Two equations Eqs.(51) and (52) are obtained from generalized Euler-equation in the QSA, containing the extra-curvature terms due to the effects of \( f(R, T) \) gravity as compared to the results presented in Einstein’s gravity theory \([56]\).

• The modified heat-fluxes are also executed to examine the thermodynamic aspects of self-gravitating evolving fluid through the proposed approximation. The significant role played by such kinematical quantities as well as modified heat-fluxes is clearly revealed through the Eqs.(59), (60), (62) and (63). The above mentioned constituents lead to the pattern of various structures. It is to be noted that in the scenario of \( f(R, T) \) gravity, the modified heat-fluxes and extra curvature terms are emerged as shown in Eqs.(39), (40), (41), however in Einstein’s gravity theory \([56]\), the Eqs.(39) and (40) were governed by usual heat-flux components while the result of Eq. (39) was depicted as

\[(TA)' = 0, \quad (TA)_g = 0\]

• Most important, the magnetic-part of the Weyl-tensor is not vanished in the QSA. However, it is followed immediately from Eqs.(64) and (65) that temporal derivatives of \( H_1 \) and \( H_2 \) is at least of order \( O(\epsilon^2) \) i.e, \( H_2 \approx \dot{H}_1 \approx O(\epsilon^2) \), consequently not to be worth considering and are neglected in the QSA accordingly. This suggests that if magnetic-part of the Weyl-tensor disappears at any moment, afterwards the same situation will appear at any time. The transformation of energy is carried by the modified heat-fluxes, these effective constituents express the corrections of \( f(R, T) \) gravity.

• It is noticed that, when fluid contents are considered non dissipative, shear-free and irrotational, then the sign of \( U \) and \( U(3) \) is identical to the sign of \( U_\Sigma \) and \( U_{(3)\Sigma} \), however the influence of \( f(R, T) \) constituents illustrate a steady heat-flow. Even so in the emergence of any of the above factors (Shear, vorticity, modified heat-flux), the system may lead to a position where ever velocity swaps of sign in matter distribution according to its sign on surface boundary, along with the effective components of heat-flux. These effective constituents play a productive role to understand the behavior of such velocity functions for the quasi static-evolution of massive objects. Consequently, it can be happened that outer regions move in a direction opposite to that of inner ones which propagate in mono direction.
The role of generalized scalar-variables is analyzed in the dynamics of self-gravitating compact objects. It is also observed that one of $f(R,T)$ scalar variables, $X_T$ demonstrates the energy density of the matter composition along with the additional curvature $f(R,T)$ constituents, however its irregularity in terms of local isotropy is well-described via the remaining ones, which are $X_I$, $X_{II}$, and $X_{KL}$. Finally, the quasi static-evolution of $f(R,T)$ scalar equations is presented in Appendix A, which depicts the salient physical aspects of scalar-variables along with effective fluid contents.

The considerable number of stars end up spending a lot of their active lifetimes in this state of equilibrium, fusing hydrogen into helium, yet it is the steady transition of elements through the fusion mechanism that allows their setup to change in any significant way. To investigate self-gravitating stellar objects, we may take into consideration three feasible evolutionary regimes, in between the static and dynamic evolution, we may have the quasi-static regime. This is the regime where the system is considered to evolve gradually slow so that it can be assumed to be in state of equilibrium at every moment. This indicates that the system evolves slowly on a time-scale larger than the typical one for that fluid responds to a small perturbative configuration of hydro-static equilibrium. The hydrostatic time scale is the term applied to this type of time scale \cite{59}. Consequently, one can say that the stellar objects are in hydro-static equilibrium in this phase. The quasi-static approach is very effective, for several stages of the life of the celestial bodies \cite{60}, because the hydrostatic time frame is of the order $10^{-4}$ seconds for a neutron-star and 4.5 seconds for a white-dwarf and more important 27 minutes for the sun. As hydro-static equilibrium is the state in which a gaseous objects’s internal pressure exactly balances its gravitational pressure, such as celestial objects. In order to discuss the quasi-static evolution of compact objects, we define the QSA in form of kinematical variables, invariant velocities and $f(R,T)$ modifications, as presented in section 5. It is worth referring that $\gamma = 0$ bring out the outcomes of Einstein gravity theory as presented in \cite{56}.

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Data availability statement

All data generated or analysed during this study are included in this published article (and its supplementary information files).

7 Appendix A

Here, we are interested to discuss the evolution of $f(R, T)$ scalar-equations (70)-(78) in \[52\] in the QSA as follows

$$
\frac{1}{3A} \left( \varepsilon_I + \frac{\kappa}{2f_R} (1 + \tilde{f}_T)(\Pi_I + \mu) \right) + \frac{1}{3}(\Theta \varepsilon_I + \tilde{\sigma} \varepsilon_{II}) - \Omega \left( \varepsilon_{KL} + \frac{\kappa}{2f_R} (1 + \tilde{f}_T)\Pi_{KL} \right) - \frac{1}{Br} \\
\times \left( H_{1,\theta} + H_{1} C'_{\theta} C \right) - \frac{H_2}{B} \left( \frac{C'}{C} - \frac{(Br)'}{Br} \right) = 2a_{II}H_1 - \frac{\kappa}{6}(\Theta + \tilde{\sigma}) \left( \tilde{\mu}^{\text{eff}} + (\bar{P} + \bar{\Pi}_I)^{\text{eff}} \right) \\
- \kappa \alpha I \tilde{q}_{I}^{\text{eff}} - \frac{\kappa}{2B} \left[ \tilde{q}_{I}^{\text{eff}} + \frac{B_{\theta}}{B} q_{II}^{\text{eff}} \right], \\
\frac{1}{A} \left( \varepsilon_{KL} + \frac{\kappa}{2f_R} (1 + \tilde{f}_T)\Pi_{KL} \right) + \frac{\Omega}{6} \left[ \varepsilon_I - \varepsilon_{II} + \frac{\kappa}{2f_R} (1 + \tilde{f}_T)(\Pi_I - \Pi_{II}) \right] - (a_{II}H_2 - a_I H_1) - \left( \varepsilon_{KL} + \frac{\kappa}{2f_R} (1 + \tilde{f}_T)\Pi_{KL} \right) (\tilde{\sigma} - \Theta) - \frac{1}{2B} \left[ H_1 \left( \frac{(Br)'}{Br} - \frac{2C'}{C} \right) - H_1 \right] \\
- \frac{1}{2Br} \left[ H_{2,\theta} - H_2 \left( \frac{B_{\theta}}{B} - \frac{2C_{\theta}}{C} \right) \right] = -\frac{2\kappa}{6} (2\tilde{\sigma} - \Theta)\Pi_{KL}^{\text{eff}} - \frac{\kappa}{2} (a_{II}\tilde{q}_{II}^{\text{eff}} + a_I q_{II}^{\text{eff}}) \\
- \frac{\kappa}{4Bf_R} (1 + \tilde{f}_T) \left( q_{II}^{\text{eff}} - q_{II} \left( \frac{(Br)'}{Br} \right) \right) - \frac{\kappa}{4rBf_R} (1 + \tilde{f}_T) \left( q_{I,\theta} - q_{I,\theta} \frac{B_{\theta}}{B} \right), \\
\frac{1}{3A} \left[ \varepsilon_{II} + \frac{\kappa}{2f_R} (1 + \tilde{f}_T)(\Pi_{II} + \mu) \right] + \frac{1}{3}(\Theta \varepsilon_{II} + \tilde{\sigma} \varepsilon_{II}) + \Omega \left( \varepsilon_{KL} + \frac{\kappa}{2f_R} (1 + \tilde{f}_T)\Pi_{KL} \right) \\
- \frac{1}{3}(\Theta \varepsilon_{II} + \tilde{\sigma} \varepsilon_{II}) + \Omega \left( \varepsilon_{KL} + \frac{\kappa}{2f_R} (1 + \tilde{f}_T)\Pi_{KL} \right).
$$

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\[+ 2H_2a_T + \frac{1}{B}\left(\frac{H_2C'}{C} + H_2\right) + \frac{H_3}{rB}\left(\frac{C_\theta}{C} - B_\theta B\right) = -\frac{\kappa}{6}(\Theta + \bar{\sigma})\left(\tilde{\mu}_{\text{eff}} + (\bar{P} + \frac{\Pi_{II}}{3})\right)\]

\[- \kappa a_T\bar{q}_{I\text{eff}} - \frac{\kappa}{2f_R}(1 + \bar{f}_T)\left[\frac{q_{II}}{Br} + q_I (B'r')\right],\]  

\[
\frac{1}{3A}\left[-\frac{\kappa}{2f_R}(1 + \bar{f}_T)(-\mu + \Pi_I + \Pi_{II}) - (\varepsilon_{II} + \varepsilon_I)\right] + \frac{K}{18f_R}(1 + \bar{f}_T)(\Pi_I + \Pi_{II})(2\bar{\sigma} - \Theta)
\]

\[+ 2(a_IH_2 - a_IH_1) - \frac{1}{3}(\Theta + \bar{\sigma})(\varepsilon_I + \varepsilon_{II}) + \frac{1}{B}\left(H_2\frac{(Br')'}{Br} + H_2'\right) + \frac{1}{rB}\left(H_{1,\theta} + B_\theta B H_1\right)\]

\[- \frac{\kappa}{6}(\bar{\mu}_{\text{eff}} + \bar{P}_{\text{eff}})(\Theta - 2\bar{\sigma}) - \frac{\kappa\bar{q}_{I\text{eff}} C'}{2B C} - \frac{\kappa\bar{q}_{I\text{eff}} C_\theta}{2Br C},\]  

\[\frac{1}{3B}\left(\varepsilon_I + \frac{\kappa}{2f_R}(1 + \bar{f}_T)\Pi_I\right)' + \frac{1}{Br}\left(\frac{\kappa}{2f_R}(1 + \bar{f}_T)\Pi_{KL} + \varepsilon_{KL}\right) + \frac{K}{3B}(\varepsilon_I + \frac{\kappa}{2f_R}(1 + \bar{f}_T)\Pi_I\Pi_{II})\]

\[= \frac{2\kappa}{6f_R}(1 + \bar{f}_T)\mu',\]

\[\frac{1}{3Br}\left(\varepsilon_{II} + \frac{\kappa}{2f_R}(1 + \bar{f}_T)\Pi_{II}\right)' + \frac{1}{Br}\left(\frac{\kappa}{2f_R}(1 + \bar{f}_T)\Pi_{KL} + \varepsilon_{KL}\right) + \frac{K}{3Br}(\varepsilon_I + \frac{\kappa}{2f_R}(1 + \bar{f}_T)\Pi_I\Pi_{II})\]

\[\times (1 + \bar{f}_T)\Pi_I\left(\frac{C_\theta}{C} - B_\theta B\right) + \frac{1}{3Br}\left(\varepsilon_{II} + \frac{\kappa}{2f_R}(1 + \bar{f}_T)\Pi_{II}\right)\left(\frac{2C_\theta}{C} + B_\theta B\right) + \frac{1}{B}\]

\[- \frac{\kappa}{2f_R}(1 + \bar{f}_T)\Pi_{KL} + \varepsilon_{KL}\left(\frac{2C'}{C} + \frac{(Br')'}{Br}\right) = \frac{2\kappa}{6Bf_R}(1 + \bar{f}_T)\mu_{\text{II}},\]

\[- \frac{1}{B}\left[H_1\left(\frac{(Br')'}{Br} + \frac{2C'}{C}\right) + H_1'\right] - \frac{1}{rB}\left[H_{2,\theta} + H_2\left(\frac{B_\theta B}{C} + \frac{2C_\theta}{C}\right)\right] + \frac{\kappa}{2Bf_R}(1 + \bar{f}_T)\]

\[\times \left(q_{II}\frac{(Br')'}{Br} + q_{II}'\right) + \Omega\left[\kappa(\tilde{\mu}_{\text{eff}} + \bar{P}_{\text{eff}}) - (\varepsilon_I + \varepsilon_{II}) + \frac{\kappa}{6f_R}(1 + \bar{f}_T)\Pi_I + \Pi_{II}\right] - \frac{\kappa}{2rBf_R}(1 + \bar{f}_T)\Pi_I\]

\[= \frac{1}{B}\left(\frac{\kappa}{6f_R}(1 + \bar{f}_T)\Pi_{KL}\right) + \frac{1}{rB}\left(\frac{\kappa}{3f_R}(1 + \bar{f}_T)\Pi_{KL}\right) - \frac{\varepsilon_I}{3rB}\left(\frac{C_\theta}{C} + \frac{2A_\theta}{A}\right) - \frac{\varepsilon_{II}}{3rB}\]

\[\times \left(\frac{2C_\theta}{C} + \frac{A_\theta}{A}\right) - \frac{\kappa}{Bf_R}(1 + \bar{f}_T)\Pi_{KL}\frac{B'r'}{Br} - \frac{\varepsilon_{KL}}{B}\left(\frac{C'}{C} + \frac{A'}{A}\right) + \frac{\kappa}{6Bf_R}(1 + \bar{f}_T)\Pi_I\]
\[ -\Pi_{II} \frac{B_{\theta}}{B} + \frac{1}{A} \dot{H}_1 = -\frac{2\kappa}{6Bf_R} (1 + \tilde{f}_T)\mu_{\theta}, \quad (78) \]

\[ -\frac{1}{B} \left( \frac{\kappa}{6f_R} (1 + \tilde{f}_T)\Pi_{II} - (\varepsilon_I - \varepsilon_{II}) \right) + \frac{\kappa}{2rf'_R} (1 + \tilde{f}_T) \left( \Pi_{KL} \frac{2B_{\theta}}{B} + \Pi_{KL,\theta} \right) + \frac{1}{A} \dot{H}_2 \]

\[ + \frac{2C' - A'}{3B} \left( \frac{2C'}{C} + \frac{A'}{A} \right) + \frac{2\varepsilon_I}{3B} \left( \frac{C'}{C} + \frac{2A'}{A} \right) + \frac{\kappa}{6Bf_R} (1 + \tilde{f}_T)(\Pi_I - \Pi_{II}) \left( \frac{Br'}{Br} + \varepsilon_{KL} \left( \frac{C_{\theta}}{C} \right) \right) \]

\[ - \frac{A_{\theta}}{A} = \frac{\kappa}{6Bf_R} (1 + \tilde{f}_T)\mu', \quad (79) \]

where \( \tilde{q}_{II}^{\text{eff}} \) shows the quasi static-evolution of effective component of heat-flux \( (q_{II}^{\text{eff}}) \). So, one can easily computed the quasi static-evolution of effective components of relativistic-fluid by using the conditions of the QSA as defined in above section.

## 8 Appendix B

The expression of effective fluid contents for our relativistic system are

\[ \mu^{\text{eff}} = \frac{1}{f_R} \left[ \bar{\mu} - \frac{1}{2}(f - Rf_R) + \chi_0 \right], \quad q_{II}^{\text{eff}} = \frac{1}{f_R} \left[ q_I(1 + \tilde{f}_T) - \frac{1}{AB}\chi_1 \right], \]

\[ \Pi_{KL}^{\text{eff}} = \frac{1}{f_R} (1 + \tilde{f}_T)\Pi_{KL} + \frac{1}{f_R\sqrt{r^2A^2B^2 + G^2}} \left[ G(\chi_0 + \chi_2) \right], \]

\[ (P + \frac{\Pi_{II}}{3})^{\text{eff}} = \frac{1}{f_R} (1 + \tilde{f}_T)(P + \frac{\Pi_{II}}{3}) + \frac{1}{f_R} \left[ \bar{\mu} f_T + \frac{1}{2}(f - Rf_R) + \frac{1}{B^2}\chi_4 \right], \]

\[ (P + \frac{\Pi_{II}}{3})^{\text{eff}} = \frac{1}{f_R} (1 + \tilde{f}_T)(P + \frac{\Pi_{II}}{3}) + \frac{1}{f_R\sqrt{r^2A^2B^2 + G^2}} \left\{ \frac{1}{2}(f - Rf_R)(G - rB^2) - G\chi_0 \right\} \]

\[ 2\left\{ \sqrt{r^2A^2B^2 + G^2}q_{II}f_T + G(\chi_0 + \frac{\chi_2}{r}) \right\} + rB^2(\bar{\mu} f_T + \frac{\chi_5}{C^2}) \right\}. \]

The values of \( \chi_i \)'s appearing in the expression of effective fluid contents and in Eqs.\( 41 \) are presented in Appendix in \[52\].

The extra terms \( \psi_i \)'s appearing in Eqs.\( 22 \), \( 25 \), \( 26 \) and \( 28 \) are

\[ \psi_1 = \frac{\kappa}{8f_R} \epsilon_{\lambda\nu} \left[ \left( \nabla^\mu \nabla_{\nu} f_R \right) \epsilon_{\mu\pi\gamma} - \left( \nabla^\mu \nabla_{\nu} f_R \right) \epsilon_{\mu\omega\pi} - \left( \nabla^\alpha \nabla_{\nu} f_R \right) \epsilon_{\alpha\omega\gamma} + \left( \nabla^\alpha \nabla_{\nu} f_R \right) \epsilon_{\alpha\omega\pi} \right], \]

\[ \psi_2 = \nabla_\lambda \nabla_\omega f_R - \frac{3}{2} U_\lambda U_\omega (f - Rf_R) - \nabla_\lambda \nabla_\gamma f_R U_\gamma U_\omega + 2U_\lambda U_\omega \nabla f_R - \left( \nabla^\lambda \nabla_\omega f_R \right) U_\lambda U_\omega + g_{\lambda\omega} \]
\times (\nabla^\alpha \nabla_\gamma f_R) U_\alpha U_\gamma, \\
\psi_3 = \nabla^\lambda \nabla_\lambda f_R + \frac{3}{2} (f - R f_R) - (\nabla^\lambda \nabla_\gamma f_R) U_\alpha U_\lambda - 2 \Box f_R - (\nabla^\alpha \nabla_\lambda f_R) U^\lambda U_\alpha + 4 (\nabla^\alpha \nabla_\gamma f_R) U_\alpha U_\gamma, \\
\psi_4 = (\nabla^\alpha \nabla_\gamma f_R) U_\gamma \epsilon_{\lambda \alpha \omega}.

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