On the Complexity of Maximizing Social Welfare within Fair Allocations of Indivisible Goods

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We consider the classical fair division problem which studies how to allocate resources fairly and efficiently. We give a complete landscape on the computational complexity and approximability of maximizing the social welfare within (1) envy-free up to any item (EFX) and (2) envy-free up to one item (EF1) allocations of indivisible goods for both normalized and unnormalized valuations.

We show that a partial EFX allocation may have a higher social welfare than a complete EFX allocation, while it is well-known that this is not true for EF1 allocations. Thus, our first group of results focuses on the problem of maximizing social welfare subject to (partial) EFX allocations. For \( n = 2 \) agents, we provide a polynomial time approximation scheme (PTAS) and an NP-hardness result. For a general number of agents \( n > 2 \), we present algorithms that achieve approximation ratios of \( O(n) \) and \( O(\sqrt{n}) \) for unnormalized and normalized valuations, respectively. These results are complemented by the asymptotically tight inapproximability results.

We also study the same constrained optimization problem for EF1. For \( n = 2 \), we show a fully polynomial time approximation scheme (FPTAS) and complement this positive result with an NP-hardness result. For general \( n \), we present polynomial inapproximability ratios for both normalized and unnormalized valuations.

Our results also imply the price of EFX is \( \Theta(\sqrt{n}) \) for normalized valuations, which is unknown in the previous literature.
1 INTRODUCTION

Fair division is a classical topic studied in economics, social science and computer science, and has garnered immense attention due to its wide applications (e.g., school choices [Abdulkadiroğlu et al., 2005], course allocations [Budish and Cantillon, 2012], paper review assignments [Lian et al., 2018], allocating computational resources [Ghodsi et al., 2011], etc.). Generally speaking, fair division studies how to fairly allocate a given set of resources to agents with respect to their heterogeneous preferences. Among all the fair allocations, an optimal allocation is naturally defined as the one with maximum efficiency (a.k.a. social welfare). The problem of finding such an optimal allocation has been well-studied in the cake-cutting literature where resources are (infinitely) divisible [Aumann and Dombb, 2015, Aumann et al., 2013, Bei et al., 2012, Brams et al., 2012, Caragiannis et al., 2012, Cohler et al., 2011]. However, much less attention for this problem has been received for indivisible items. In this paper, we study the computational complexity and approximability of this problem.

For different application scenarios, multiple fairness notions have been defined to cope with agents’ heterogeneous preferences. Among them, envy-freeness (EF) is arguably the most natural fairness criterion, which states that each agent values her own bundle at least as much as the bundle allocated to any other agent. In other words, each agent does not envy any other agent in the allocation. Other common fairness criteria include proportionality (PROP), equitability (EQ), etc. (see, e.g., [Brams and Taylor, 1996, Brandt et al., 2016, Endriss, 2017, Moulin, 2019] for a survey).

However, exact fairness may not be achievable for allocating indivisible items (e.g., when the number of items is less than the number of agents). Corresponding to envy-freeness, Budish [2011] proposed envy-freeness up to one item (EF1), which requires that, for any pair of agents $i$ and $j$, $i$ does not envy $j$ if an item from $j$’s allocated bundle is removed. Caragiannis et al. [2019b] proposed a stricter notion envy-freeness up to any item (EFX), which requires that, for any pair of agents $i$ and $j$, $i$ does not envy $j$ after the removal of any item from $j$’s bundle. Significant attention has been received for EF1/EFX allocations since then, together with other criteria such as PROP1/PROPX and EQ1/EQX which adapt PROP and EQ to the indivisible resource setting in the same “fairness up to one item” or “fairness up to any item” favor (see [Aziz et al., 2020, Conitzer et al., 2017, Freeman et al., 2019, Gourvès et al., 2014, Lipton et al., 2004]).

The above-mentioned relaxed notions of fairness are extensively studied for allocating indivisible items [Amanatidis et al., 2022, Aziz et al., 2022]. The next straightforward step is to optimize the allocation efficiency with a certain fairness notion guaranteed. A natural measurement of efficiency is the social welfare—the sum of all the agents’ utilities for the allocation. The problem of maximizing efficiency within fair allocations can therefore be formulated as a constrained optimization problem. Unfortunately, compared with other domains, this problem is still much less understood in the context of indivisible item allocations. Especially, the study of this constrained optimization problem in the EFX setting is even more rare, as finding an EFX allocation is already challenging. The existence of an EFX allocation is only known for up to three agents [Chaudhury et al., 2020, Plaut and Roughgarden, 2020], and finding “reasonable” partial EFX allocations is also challenging and is only addressed by very recent work [Caragiannis et al., 2019a, Chaudhury et al., 2021].

Some recent work (e.g., [Aziz et al., 2023, Barman et al., 2020, 2019]) studied the computational complexity of finding the optimal social welfare subject to various fairness constraints including EF1 and EFX, or the computational complexity of deciding whether there exists an allocation maximizing social welfare (without subjecting to a fairness constraint) while being fair (see Section 1.2
Table 1. Approximation ratios and inapproximabilities of MSWwithINEFX.

| $n$  | Utility | Positive Results | Negative Results |
|------|---------|------------------|------------------|
| 2    | Normalized | PTAS (Sec 4.2) | NP-hard ([Aziz et al., 2023]) |
|      | Unnormalized | PTAS (Sec 4.2) | NP-hard ([Aziz et al., 2023]) |
|      | Normalized | $\frac{c \cdot \sqrt{n}}{}$ (Thm 5.3) $(1, 1-\varepsilon)$-BC (Thm E.4) | $\frac{1}{\varepsilon} (\sqrt{5n} + 1 - 1)$ (Thm 5.6) |
|      | Unnormalized | $[\text{pseudo}] (2n + 1)$ (Thm 5.2) $(1, 1-\varepsilon)$-BC (Thm E.4) | $\frac{1}{2} (n + 1)$ (Thm 5.5) |
|      | General | $[\text{pseudo}] O(\sqrt{n})$ (Thm 5.3) | $[\text{pseudo}] n^{0.5-\varepsilon}$ (Thm 5.8) |
|      | Unnormalized | $[\text{pseudo}] O(n)$ (Thm 5.2) | $(n^{1-\varepsilon}, 0.5 + \varepsilon)$-BC (Thm E.8) |

★ [pseudo] marks for positive/negative results for pseudo-polynomial time algorithms. ★ $(\alpha, \beta)$-BC: bi-criteria optimization with $\alpha$-approximation on social welfare and $\beta$-approximate EFX (Definition E.1).

Table 2. Approximation ratios and inapproximabilities of MSWwithINEF1.

| $n$  | Utility | Positive Results | Negative Results |
|------|---------|------------------|------------------|
| 2    | Normalized | FPTAS (Sec 4.1) | NP-hard (Thm 4.11) |
|      | Unnormalized | FPTAS (Sec 4.1) | NP-hard (Thm 4.11, [Aziz et al., 2023]) |
|      | Normalized | $\approx 12\sqrt{n}$ [Barman et al., 2020] $(1, 1-\varepsilon)$-BC (Thm E.3) | $n^{\frac{1}{10}}/10$ (Thm 6.2), $\frac{4n}{n^2}$ (Thm 6.1) |
|      | Unnormalized | $(1, 1-\varepsilon)$-BC (Thm E.3) | $\left\lceil \frac{1+\sqrt{4n-3}}{2} \right\rceil$ (Thm 6.3) |
|      | General | $O(\sqrt{n})$ [Barman et al., 2020] | $n^{\frac{1}{3}-\varepsilon}$, $m^{\frac{1}{3}-\varepsilon}$ (Thm 6.4) |
|      | Unnormalized | $n$ (Thm 6.6) | $m^{1-\varepsilon}$ [Barman et al., 2019] $(n^{\frac{1}{3}-\varepsilon}, \varepsilon)$-BC, $(m^{1-\varepsilon}, \varepsilon)$-BC (Thm E.7) |

★ $n$ and $m$ represent the numbers of agents and items respectively. ★ $(\alpha, \beta)$-BC: bi-criteria optimization with $\alpha$-approximation on social welfare and $\beta$-approximate EF1 (Definition E.2).

for details). However, these papers mostly deal with exact optimal social welfare. The design of approximation algorithms and the study of approximability for this constrained optimization problem are mostly absent from the previous literature.

1.1 Our Results

We study the complexity, specifically, the approximability, of the problems of optimizing social welfare subject to the EFX and EF1 constraint respectively. We use MSWwithINEFX and MSWwithINEF1 to denote the two problems. We give a landscape for the approximability of MSWwithINEFX and MSWwithINEF1. Our results are summarized based on different numbers of agents (see Table 1 and Table 2). In the main body of the paper, we mainly discuss the results for MSWwithINEFX (Sect. 4.2, Sect. 5). The results in EF1 are stated in the paper (Sect. 4.1, Sect. 6), with most of the proofs deferred to the appendix (except for the FPTAS result for two agents in Sect. 4.1). The results of bi-criteria optimization are discussed in Appendix E.
Before going to MSWWITHINEFX and MSWWITHINEF1, we study the resource monotonicity for EF1 and EFX allocations in Sect. 3. We show that a partial EFX allocation may have a higher social welfare than any complete EFX allocations, even for the instances where complete EFX allocations exist (Theorem 3.2). That is, the resource monotonicity fails for EFX allocations. Therefore, for MSWWITHINEFX, we should not exclude partial allocations from our consideration. On the other hand, we show that the resource monotonicity holds for EFX allocations with two agents (Theorem 3.4). Some insights (Proposition 3.3) behind this proof will be a crucial component in our algorithm for MSWWITHINEFX with two agents (Sect. 4.2). For EF1, on the other hand, the resource monotonicity holds trivially. Therefore, when dealing with MSWWITHINEF1, we should always consider complete allocations.

Our results imply that the price of EFX is \( \Theta(\sqrt{n}) \) and \( \Theta(\sqrt{n}) \) for general and normalized valuations respectively. The known results for the price of EFX/EF1 are presented in Table 3. The discussion about the price of fairness is deferred to Appendix F.

| Utilities           | \( n = 2 \)                  | general \( n \)                   |
|---------------------|-------------------------------|-----------------------------------|
| EFX                 | normalized [Bei et al., 2021b]| \( \Theta(\sqrt{n}) \) (Thm F.6) |
|                     | unnormalized \( \geq 2 \) (Thm F.4) | \( \Theta(n) \) (Thm F.5)         |
| EFX                 | normalized \([2/\sqrt{3}, 8/7]\) [Bei et al., 2021b] | \( \Theta(\sqrt{n}) \) [Barman et al., 2020, Bei et al., 2021b] |
|                     | unnormalized \(2\) (Thm F.7) | \( n \) (Thm F.7)                 |

Table 3. Know results for the price of fairness

Below, we highlight some interesting features and observations of our results.

(1) For two agents, the prices of EF1 and EFX are bounded away from 1, but we give an FPTAS and a PTAS for MSWWITHINEF1 and MSWWITHINEFX respectively.

(2) For both MSWWITHINEF1 and MSWWITHINEFX with two agents, we have closed the gap between the approximability and the inapproximability ratios.

(3) For MSWWITHINEFX with general numbers of agents, we have asymptotically closed the gap between the approximability and the inapproximability ratios.

(4) For a constant number of agents, the inapproximability ratios grow as the number of agents grows for both MSWWITHINEF1 and MSWWITHINEFX. However, if we slightly relax EF1 and EFX by a factor of \((1 - \epsilon)\), we can compute the optimal solution in polynomial time, as indicated by our results in bi-criteria optimization.

(5) The bi-criteria optimization algorithms fail to extend to the setting with general numbers of agents. Our intractability results indicate polynomial inapproximability factors even if EF1 and EFX are substantially relaxed.

(6) Theorem 4.11 resolves an open problem raised by Aziz et al. [2023]. Theorem 6.1 generalizes and improves the NP-hardness result for three agents by Aziz et al. [2023].

1.2 Related Work

The study of efficient and fair allocation in the context of divisible resources or indivisible resources is extensive [Aumann and Dombb, 2015, Aumann et al., 2013, Barman et al., 2018, Caragiannis et al., 2019b, Murhekar and Garg, 2021, Segal-Halevi and Sziklai, 2019]. Several work considered the problem of maximizing social welfare within fair allocations, which is the focus of this paper. In the setting of divisible goods, the complexity of the problem is well understood [Bei et al., 2012, Brams et al., 2012, Cohler et al., 2011]. Specifically, maximizing social welfare within envy-free/proportional allocations for piecewise-constant valuations can both be solved optimally [Cohler et al., 2011].
However, if each agent is required to receive a connected piece of cake, this problem admits a polynomial inapproximability factor [Bei et al., 2012].

On the other hand, there are some recent works that consider the problem of maximizing social welfare within EF1 allocations for the indivisible goods. Barman et al. [2019] proved that the problem is NP-hard for 2 agents even when the valuations of one agent is the scaled version of the other (Theorem 4), and the problem is NP-hard to approximate to within a factor of $m^{1-\varepsilon}$ for any $\varepsilon > 0$ for general numbers of agents $n$ and items $m$ (Theorem 5). They also presented a $\frac{1}{2}$-approximation algorithm for this problem with the dominant-strategy incentive-compatible (DSIC) property, but this result is under the special setting where all agents’ valuations are scaled versions of a single valuation function (which is named as “single-parameter setting”). Aziz et al. [2023] showed that the problem is still NP-hard for $n \geq 3$ even when the utility function is normalized (we improve it in Theorem 6.1), and left it an open problem for the case when $n = 2$.\footnote{Nevertheless, Aziz et al. [2023] showed that the problem is NP-hard for two agents with unnormalized valuations.} In addition, they proposed a pseudo-polynomial time algorithm when the number of agents is fixed.

Most of the previous work focuses on EF1, and little was known for the relationship between EFX and social welfare. Aziz et al. [2023] showed that the problem of maximizing social welfare subject to the EFX constraint is NP-hard, and the NP-hardness result continues to hold for two agents with normalized valuations. Other than this, to the best of our knowledge, there is no known (in)approximability result on this problem.

Even for EF1, most of the previous work focuses on exact optimal social welfare, not on its approximability (with the two exceptions of the $m^{1-\varepsilon}$ inapproximability and the $O(\sqrt{n})$-approximation algorithm mentioned just now). A complete landscape of the approximability for this constrained optimization problem is still missing before this paper.

Our paper studies the resource monotonicity with respect to the social welfare. The resource monotonicity for other efficiency notions like Pareto-optimality and Nash social welfare has been studied by Caragiannis et al. [2019a] and Chaudhury et al. [2020]. Chaudhury et al. [2020] showed that there exist partial EFX allocations that are not Pareto-dominated by any complete EFX allocations and there exist partial EFX allocations with Nash social welfare higher than any complete EFX allocations.

The price of fairness measures the loss in social welfare if a fairness constraint is imposed. The price of EF1/EFX is defined by the supremum (over all fair division instances) of the ratio of the maximum social welfare over the maximum social welfare within EF1/EFX allocations. The price of EF1 under normalized valuations is $\Theta(\sqrt{n})$: Barman et al. [2020] showed that the price of EF1 is $O(\sqrt{n})$, and Bei et al. [2021b] showed that the price of EF1 is $\Omega(\sqrt{n})$. Bei et al. [2021b] showed that the price of EFX for two agents with normalized valuations is 1.5, while the price of EFX for a general number of agents is still an open problem (we resolve it in Theorem F.6).

## 2 Preliminaries

Let $[k] = \{1, \ldots, k\}$. Denote by $N = [n]$ the set of agents and $M = [m]$ the set of indivisible items. Each agent $i$ has a nonnegative utility function $v_i : \{0, 1\}^M \rightarrow \mathbb{R}_{\geq 0}$. We assume each agent’s utility function is additive: $v_i(S) = \sum_{g \subseteq S} v_i(\{g\})$ for every $i \in N$ and $S \subseteq M$, and we denote $v_i(\{g\})$ by $v_{ig}$ or $v_i(g)$ for notation simplicity. Further, a utility function $v_i$ is said to be normalized if $v_i(M) = \sum_{g \in M} v_{ig} = 1$, i.e., agent $i$ values exactly 1 for the set of indivisible items $M$. We will use the two phrases utility function and valuation interchangeably in this paper.

An allocation of the items is the collection of the $n$ item sets $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ satisfying $A_i \cap A_j = \emptyset$ for any $i, j \in [n]$, where $A_i$ is the bundle of items allocated to agent $i$. An allocation
is complete if \( \bigcup_{i=1}^{n} A_i = M \), i.e., \( \mathcal{A} = (A_1, A_2, \ldots, A_n) \) is a partition of \( M \). We say an allocation is partial if it is not complete. An allocation is envy-free if \( v_i(A_i) \geq v_j(A_j) \) for any two agents \( i \) and \( j \) in \( N \). That is, according to agent \( i \)'s utility function, agent \( i \) does not envy any other agent \( j \)'s allocation. An envy-free allocation may not exist in the problem of allocating indivisible items (e.g., when \( m < n \)). We consider two well-known common relaxations of envy-freeness, envy-freeness up to any item (EFX) and envy-freeness up to one item (EF1), defined below.

**Definition 2.1.** An allocation \( \mathcal{A} = (A_1, A_2, \ldots, A_n) \) is said to satisfy envy-freeness up to any item (EFX), if for any two agents \( i \) and \( j \), \( v_i(A_i) \geq v_j(A_j \setminus \{g\}) \) holds for any \( g \in A_j \).

**Definition 2.2.** An allocation \( \mathcal{A} = (A_1, A_2, \ldots, A_n) \) is said to satisfy envy-freeness up to one item (EF1), if for any two agents \( i \) and \( j \), there exists an item \( g \in A_j \) such that \( v_i(A_i) \geq v_j(A_j \setminus \{g\}) \).

For a verbal description, in an EF1 allocation, after removing some item \( g \) from agent \( j \)'s bundle, agent \( i \) will no longer envy agent \( j \). For EFX, the quantifier some is changed to any. Given an allocation \( (A_1, \ldots, A_n) \), we say that agent \( i \) envies agent \( j \) if \( v_i(A_i) < v_j(A_j \setminus \{g\}) \) for some \( g \in A_j \); when we are talking about EF1, agent \( i \) strongly envies agent \( j \) if \( v_i(A_i) < v_j(A_j \setminus \{g\}) \) for every \( g \in A_j \). By our definition, an allocation is EFX/EF1 if and only if \( i \) does not strongly envy \( j \) for every pair \((i, j)\) of agents.

**Remark 2.3.** Consider an allocation \( \mathcal{A} = (A_1, A_2, \ldots, A_n) \) and two arbitrary agents \( i \) and \( j \). Since we are considering additive valuations, EFX requires that \( v_i(A_i) \geq v_j(A_j \setminus \{g\}) \) where \( g \) is an item in \( A_j \) with minimum \( v_j(\{g\}) \), and EF1 requires that \( v_i(A_i) \geq v_j(A_j \setminus \{g\}) \) where \( g \) is an item in \( A_j \) with maximum \( v_j(\{g\}) \).

It is well-known that a complete EF1 allocation always exists even for general utility functions, and it can be computed in polynomial time [Budish, 2011, Lipton et al., 2004]. However, the existence of a complete EFX allocation is an open problem for \( n \geq 4 \).

Another critical issue is economic efficiency, where we consider social welfare as defined below.

**Definition 2.4.** The social welfare of an allocation \( \mathcal{A} = (A_1, \ldots, A_n) \), denoted by \( SW(\mathcal{A}) \), is the sum of the utilities of all the agents \( SW(\mathcal{A}) = \sum_{i=1}^{n} v_i(A_i) \).

In this paper, we focus on the problem of maximizing social welfare subject to the EFX/EF1 constraint. More formally, we have the following two constrained optimization problems.

**Problem 1 (MSWWITHINEFX, MSWWITHINEF1).** Given a set of indivisible items \( M = [m] \) and a set of agents \( N = [n] \) with their utility functions \( (v_1, \ldots, v_n) \), the problem of

- maximizing social welfare within EFX allocations (MSWWITHINEFX) aims to find an allocation \( \mathcal{A} = (A_1, \ldots, A_n) \) that maximizes social welfare \( SW(\mathcal{A}) \) subject to that \( \mathcal{A} \) is EFX;
- maximizing social welfare within EF1 allocations (MSWWITHINEF1) aims to find an allocation \( \mathcal{A} = (A_1, \ldots, A_n) \) that maximizes social welfare \( SW(\mathcal{A}) \) subject to that \( \mathcal{A} \) is EF1.

For MSWWITHINEFX, we do not restrict ourselves to complete allocations. This is for the following two reasons: firstly, the existence of a complete EFX allocation is still an open problem for \( n \geq 4 \); secondly, as will be shown in Sect. 3, there may exist a partial EFX allocation that has a higher social welfare than any complete EFX allocations. For MSWWITHINEF1, on the other hand, it is well-known that, for any partial allocation, there is a complete allocation with a weakly higher social welfare. Therefore, we can focus exclusively on complete allocations for MSWWITHINEF1 without loss of generality.
3 ON RESOURCE MONOTONICITY

In this section, we study the resource monotonicity for EF1 and EFX allocations: whether we can always find a complete EF1/EFX allocation that has a weakly higher social welfare than a given partial EF1/EFX allocation. We will show that the resource monotonicity holds for EF1 allocations, but not for EFX allocations. Therefore, a partial allocation may provide a better solution for MSWWITHINEFX, while this is not the case for MSWWITHINEF1.

Nevertheless, the resource monotonicity holds for EFX allocations with two agents. We will use this observation in our PTAS algorithm for MSWWITHINEFX with two agents.

3.1 Resource Monotonicity for General Number of Agents

Given a partial EF1 allocation, we can apply the well-known envy-cycle procedure by Lipton et al. [2004] to obtain a complete allocation, and each agent’s received value is non-decreasing throughout the procedure. This immediately implies the resource monotonicity of EF1 allocations described in the theorem below.

**Theorem 3.1.** For any partial EF1 allocation \( A \), we can compute a complete EF1 allocation \( A' \) in polynomial time such that \( SW(A') \geq SW(A) \).

However, the resource monotonicity fails for EFX allocations, even for the more restrictive normalized valuations. The proof of Theorem 3.2 is in Appendix A.

**Theorem 3.2.** There exists an instance \((N, M, (v_1, \ldots, v_n))\) with normalized \( v_1, \ldots, v_n \) where
- complete EFX allocations exist, and
- there is a partial EFX allocation \( A \) such that \( SW(A) > SW(A') \) for any complete EFX allocation \( A' \).

3.2 Resource Monotonicity for EFX Allocations with Two Agents

We first show the following proposition which is the crucial part for proving the resource monotonicity for EFX allocations with two agents. The following proposition is also one of the key components in our PTAS algorithm in Sect. 4.2.

**Proposition 3.3.** Consider two agents with utility functions \( v_1 \) and \( v_2 \). Let \( A, B \subseteq M \) and \( g \in M \) satisfy \( A \cap B = \emptyset \) and \( g \notin (A \cup B) \). If agent 1 envies agent 2 in the allocation \((A, B \cup \{g\})\) and agent 2 envies agent 1 in the allocation \((A \cup \{g\}, B)\), then there exists an allocation \((A', B')\) with \( A' \cup B' = A \cup B \cup \{g\} \) such that
- \((A', B')\) is EFX, and
- \( v_1(A') \geq v_1(A) \) and \( v_2(B') \geq v_2(B) \).

In addition, given \( A, B \) and \( g \), the allocation \((A', B')\) can be computed in polynomial time.

**Proof.** We first show that a bi-partition \((X, Y)\) of \( A \cup B \cup \{g\} \) satisfying the following two properties exists, and it can be computed in polynomial time.

1. \((X, Y)\) is an EFX allocation if both agents’ utility functions are identically \( v_2 \).
2. \( \min\{v_2(X), v_2(Y)\} \geq v_2(B) \).

If such a bi-partition \((X, Y)\) exists, we can obtain an allocation \((A', B')\) that satisfies the two conditions in the proposition. The allocation is defined as follows: let agent 1 pick one of \( X \) or \( Y \) with a higher value, and let agent 2 pick the other bundle. It is clear that the new allocation is EFX (agent 1 does not envy agent 2 since she picks first, and agent 2 does not strongly envy agent 1 due to (1) above). Besides, both agents receive weakly higher values than they would have received in the allocation \((A, B)\). This is obvious for agent 2 due to (2) above. For agent 1, since agent 1 envies
agent 2 in the allocation \((A, B \cup \{g\}), v_1(A) < \frac{1}{2}v_1(A \cup B \cup \{g\})\). Now, by receiving a bundle in \(X, Y\) with a higher value, agent 1 receives at least \(\frac{1}{2}v_1(A \cup B \cup \{g\})\).

It then remains to show such a bi-partition \((X, Y)\) exists and can be computed in polynomial time. We describe a simple algorithm to compute such a bi-partition. Starting with \(X = A \cup \{g\}\) and \(Y = B\). Perform the following until \((X, Y)\) satisfies (1) above: if \(v_2(X) > v_2(Y)\), pick an item with minimum value in \(X\) and move it to \(Y\); if \(v_2(X) < v_2(Y)\), pick an item with minimum value in \(Y\) and move it to \(X\).

We first show that (2) holds when the algorithm terminates. At the beginning of the algorithm, we have \(\min\{v_2(X), v_2(Y)\} = v_2(B)\) since agent 2 envies agent 1 in the allocation \((X = A \cup \{g\}, Y = B)\). Moreover, \(\min\{v_2(X), v_2(Y)\}\) is non-decreasing throughout the algorithm. To see this, we will show the equivalent statement that \(\Delta = |v_2(X) - v_2(Y)|\) is non-increasing (notice that \(\min\{v_2(X), v_2(Y)\} = \frac{v_2(X \cup Y) - \Delta}{2}\) and \(v_2(X \cup Y)\) is a constant). After moving an item \(g\) from one bundle to the other, if the direction of the inequality between \(v_2(X)\) and \(v_2(Y)\) is unchanged, \(\Delta\) clearly does not increase. If the direction of the inequality changes, we have \(\Delta > v_2(g)\) before moving \(g\) (or otherwise (1) is already satisfied and the algorithm should have stopped before moving \(g\)) and \(\Delta < v_2(g)\) after the move, in which case \(\Delta\) decreases.

Secondly, the algorithm terminates in at most \(O(m^2)\) iterations. For each time the direction of the inequality between \(v_2(X)\) and \(v_2(Y)\) changed, the last item \(g\) moved must satisfy \(\Delta < v_2(\{g\})\). Moreover, \(g\) will no longer be moved in any later iterations. Otherwise, if \(g\) is moved in a future iteration, it must be that \(\Delta > v_2(\{g\})\) before this move, which contradicts to \(\Delta < v_2(\{g\})\). Therefore, each change in the inequality direction identifies an item that will never be moved in the future. Thus, the direction of the inequality can be changed at most \(m\) times. The total number of moves is at most \(O(m^2)\) as at most \(m\) items can be moved between two changes of the inequality direction.

The resource monotonicity for EFX allocations with two agents follows straightforwardly from Proposition 3.3.

**Theorem 3.4.** Consider two agents with utility functions \(v_1\) and \(v_2\). For any partial EFX allocation \(A\), we can compute a complete EFX allocation \(A^*\) in polynomial time such that \(SW(A^*) \geq SW(A)\).

**Proof.** Given a partial EFX allocation \((A, B)\) and an unallocated item \(g\), we can compute in polynomial time an EFX allocation \((A', B')\) with \(A' \cup B' = A \cup B \cup \{g\}\) such that \(v_1(A') \geq v_1(A)\) and \(v_2(B') \geq v_2(B)\). To see this, if \((A \cup \{g\}, B)\) is EFX or \((A, B \cup \{g\})\) is EFX, we can directly update the allocation. Otherwise, we can apply Proposition 3.3 to update the allocation. This implies the theorem as the social welfare is clearly non-decreasing and the update from \((A, B)\) to \((A', B')\) can be done for at most \(m\) times.

## 4 Maximizing Social Welfare With Two Agents

This section focuses on MSWWITHINEF1 and MSWWITHINEFX with two agents. We begin by presenting a fully polynomial-time approximation scheme (FPTAS) for MSWWITHINEF1 in Sect. 4.1. Based on the techniques used in Sect. 4.1 and Proposition 3.3 in addition, we present a polynomial-time approximation scheme (PTAS) for MSWWITHINEFX in Sect. 4.2. Finally, in Sect. 4.3, we complement these positive results by proving that both MSWWITHINEF1 and MSWWITHINEFX are NP-hard. Notice that, for both MSWWITHINEF1 and MSWWITHINEFX, our FPTAS and PTAS are applicable to unnormalized valuations, and our NP-hardness results hold even for normalized valuations. In addition, the NP-hardness result for MSWWITHINEF1 with normalized valuations resolves the open problem raised by Aziz et al. [2023].
**ALGORITHM 1:** An FPTAS for MSWwithNEF1

**Input:** two utility functions \( v_1 \) and \( v_2 \), and the parameter \( \varepsilon > 0 \)

**Output:** an EF1 allocation

1. Let \( O_1 \triangleq \{ p \in [m] \mid v_{1p} \geq v_{2p} \} \) and \( O_2 \triangleq \{ q \in [m] \mid v_{1q} < v_{2q} \} \);
2. Let \( \Pi \leftarrow \emptyset \) be the set of all the candidate allocations;
3. for each item \( g \in O_1 \) do
   - for each item \( o \in O_1 \setminus \{ g \} \) do
     - \( v(o) \leftarrow v_{1o} - v_{2o} \) and \( w(o) \leftarrow v_{2o} \);
     - Run the classical FPTAS with parameter \( \varepsilon \) for Knapsack with item set \( [m] \setminus (O_2 \cup \{ g \}) \), value function \( v \), weight function \( w \) and capacity constraint \( \frac{v_2([m]) - v_2}{2} \), and let \( A_1' \) be the output;
     - Let \( A_2' = [m] \setminus (A_1' \cup \{ g \}) \);
     - \( (A_1, A_2) \leftarrow \text{LOCALSEARCHEF1}(A_1' \cup \{ g \}, A_2') \); // See Algorithm 2
     - \( \Pi \leftarrow \Pi \cup \{(A_1, A_2)\} \);
4. return the allocation with the largest social welfare in \( \Pi \)

### 4.1 A Fully Polynomial-Time Approximation Scheme for MSWwithNEF1

This part presents our FPTAS for two agents. We will first handle the easy case where the exact optimal social welfare can be easily computed, without the need for an FPTAS. Next, we will exclusively handle the hard case in the remaining part of this section, where the FPTAS is presented.

**Easy case.** Consider the allocation \((O_1, O_2)\) that maximizes social welfare without the EF1 constraint, where \( O_1 \triangleq \{ p \in [m] \mid v_{1p} \geq v_{2p} \} \) and \( O_2 \triangleq \{ q \in [m] \mid v_{1q} < v_{2q} \} \). If the two agents do not strongly envy each other, the allocation is already EF1, and we have solved MSWwithNEF1.

**Hard case.** After handling the easy case, we will assume \((O_1, O_2)\) is not EF1. Without loss of generality, we make the following assumption in the remaining part of this section.

**Assumption 4.1.** In the allocation \((O_1, O_2)\), agent 2 strongly envies agent 1.

**ALGORITHM 2:** The local search subroutine for MSWwithNEF1

1. **Function** \( \text{LOCALSEARCHEF1}(A_1, A_2) \):
   - while agent 1 strongly envies agent 2 do
     - Find an arbitrary item \( g \in A_2 \setminus O_2 \) and \( A_2 \leftarrow A_2 \setminus \{ g \} \);
     - if agent 2 envies agent 1 under the partial allocation \((A_1, A_2)\) then
       - \( (A_1, A_2) \leftarrow (A_2 \cup \{ g \}, A_1) \);
     - else
       - \( (A_1, A_2) \leftarrow (A_1 \cup \{ g \}, A_2) \);
   - return \((A_1, A_2)\)

The FPTAS is shown in Algorithm 1 and works as follows. The bundle \( O_2 \) is fixed to be given to agent 2 and an item \( g \in O_1 \) is fixed to be given to agent 1. This item \( g \) will be the item in Definition 2.2 whose removal ensures agent 2 does not envy agent 1, and we will enumerate all possibilities of \( g \in O_1 \). Next, we decide the allocation of the remaining items. To ensure agent 2 does not envy agent 1 after removing \( g \) from agent 1’s bundle, the allocation \((A_1, A_2)\) must satisfy \( v_2(A_1 \setminus \{ g \}) \leq v_2(A_2) \), which is equivalent to \( v_2(A_1 \setminus \{ g \}) \leq \frac{1}{2} v_2([m] \setminus \{ g \}) \). Therefore, the problem can be viewed as a classical Knapsack problem, where the capacity of the knapsack is
After this step, we have an allocation with almost optimal social welfare while ensuring agent 2 welfare words, among all the allocations that maximize the social welfare subject to will break after execution of Line 5. The full proof of this lemma is in Appendix B.

Theorem 4.2. Algorithm 1 is an FPTAS for MSWITHINEF1 with two agents.

Before proving Theorem 4.2, we will define some additional notations. Let $C_1$ denote the constraint that agent 1 does not strongly envy agent 2 and $C_2$ denote the constraint that agent 2 does not strongly envy agent 1. Let $OPT$ be the optimal social welfare under the EF1 constraint and $OPT(C_2)$ be the optimal social welfare subject to only $C_2$. Let $ALG$ be the social welfare of the allocation $(A_1, A_2)$ output by Algorithm 1.

First of all, Assumption 4.1 immediately implies the following proposition. (Notice that, if agent 1 envies agent 2, by Assumption 4.1, the allocation $(O_2, O_1)$ has a higher social welfare, which contradicts to that the allocation $(O_1, O_2)$ maximizes social welfare.)

Proposition 4.3. In the allocation $(O_1, O_2)$, agent 1 does not envy agent 2.

Naturally, we have $OPT \leq OPT(C_2)$, as the additional constraints $C_1$ in $OPT$ can only possibly reduce the optimal social welfare. The lemma below shows that the equality is achieved. In other words, among all the allocations that maximize the social welfare subject to $C_2$, there exists one that satisfies $C_1$ as well (and is thus EF1).

Lemma 4.4. $OPT = OPT(C_2)$.

Proof. Suppose for the sake of contradiction that every allocation satisfying $C_2$ with social welfare $OPT(C_2)$ does not satisfy $C_1$. Let $(A_1, A_2)$ be such an allocation among those that agent 1 envies agent 2 the least, i.e., with $v_1(A_2) - v_1(A_1)$ minimized. Firstly, agent 2’s bundle $A_2$ contains at least one item $g$ in $O_1$, since otherwise agent 1 will not envy agent 2 if $O_1 \subseteq A_1$ (by Proposition 4.3).

We consider the partial allocation $(A_1, A_2 \setminus \{g\})$. Agent 1 will still envy agent 2 under this partial allocation, for otherwise, (s)he will not strongly envy agent 2 in $(A_1, A_2)$. We then discuss whether agent 2 envies agent 1 or not in the partial allocation $(A_1, A_2 \setminus \{g\})$.

Case 1: agent 2 envies agent 1. If we exchange the two bundles, the social welfare will increase and both of them will not envy each other. We further give item $g$ to agent 1 and consider the allocation $(A_2, A_1)$. It is easy to see $(A_2, A_1)$ has a higher social welfare (exchanging $A_1$ and $A_2 \setminus \{g\}$ increases the social welfare, and the reallocation of item $g$ from agent 2 to agent 1 weakly increases the social welfare by our definition of the set $O_1$ where $g$ belongs to) and constraint $C_2$ is still satisfied (agent 2 does not envy agent 1 after the exchange, so she does not strongly envy agent 1 if $g$ is additionally given to 1). This violates the assumption that $OPT(C_2)$ maximizes social welfare.

Case 2: agent 2 does not envy agent 1. Then, the allocation $(A_1 \cup \{g\}, A_2 \setminus \{g\})$ has a weakly higher social welfare (since $g \in O_1$) while $C_2$ still satisfies. It violates our assumption that $(A_1, A_2)$ minimizes the amount of envy, as we are able to reallocate item $g$ from agent 2 to agent 1 while still keeping social welfare being optimal subject to $C_2$.

The following lemma shows the correctness of the local search algorithm (Algorithm 2).

Lemma 4.5. Algorithm 2 outputs an EF1 allocation $(A_{1}^{c}, A_{2}^{c})$ that has a weakly higher social welfare than that of the input $(A_{1}^{l}, A_{2}^{l})$, and it terminates after at most $m$ while-loop iterations.

Proof sketch. The analysis in the proof of Lemma 4.4 can be adapted to prove the increment in social welfare. The algorithm terminates after $m$ while-loop iterations because the while-loop will break after execution of Line 5. The full proof of this lemma is in Appendix B.
The following lemma proves the approximation guarantee for Algorithm 1.

**Lemma 4.6.** \( \text{ALG} \geq (1 - \varepsilon) \text{OPT} \).

**Proof.** For the correct guess of \( g \) at Line 3, the Knapsack part will compute an allocation with social welfare at least \( (1 - \varepsilon) \cdot \text{OPT}(C_2) = \text{OPT} \) (Lemma 4.4). The lemma then follows from Lemma 4.5. The full proof of this lemma is in in Appendix B. \( \square \)

With the approximation guarantee proved in Lemma 4.6, Theorem 4.2 holds straightforwardly as it is easy to verify that the algorithm’s running time is polynomial in \( m \) and \( 1/\varepsilon \). The only non-trivial part of the time complexity analysis is the running time for the local search, which is analyzed in lemma 4.5.

### 4.2 A Polynomial-Time Approximation Scheme for MSWwithinEFX

Our polynomial-time approximation scheme for MSWwithinEFX is built upon the algorithm in Sect. 4.1. However, additional techniques are needed because the local search subroutine in Algorithm 2 cannot preserve the EFX property. For an allocation \((A_1, A_2)\), it is possible that i) agent 1 strongly envies agent 2 and ii) moving any item from \( A_2 \) to \( A_1 \) makes agent 2 strongly envy agent 1. Notice that the operation at Line 5 of Algorithm 2, while preserving the EF1 property, cannot preserve the EFX property. This is where Proposition 3.3 comes into play.

The next issue is that, although Proposition 3.3 guarantees that we can find an EFX allocation \((A_1', A_2')\) with a weakly higher social welfare than the allocation \((A_1, A_2 \setminus \{g\})\), the social welfare may be reduced by up to \( v_2(g) \) if we update the allocation from \((A_1, A_2)\) to \((A_1', A_2')\). This is again different from the case with MSWwithinEF1 where the social welfare is non-decreasing throughout the local search subroutine. To ensure that we do not lose too much in the social welfare by applying Proposition 3.3, we need to make sure \( v_2(g) \) is small compared to the optimal social welfare. To accomplish this, we first identify all the “large items” for each of which at least one of the agents has a value higher than \( \varepsilon \cdot \text{SW}(\mathcal{A}') \). There can only be a constant number of large items if the PTAS parameter \( \varepsilon \) is a constant. We can just enumerate all possible allocations of these large items before applying Algorithm 1. This initial enumeration step makes our algorithm a PTAS instead of an FPTAS.

The algorithm is described in Algorithm 3. Firstly, \( \Gamma = \frac{1}{2} \cdot \max\{v_1([m]), v_2([m])\} \) at Line 2 is a lower bound to the optimal social welfare (Proposition 4.8). Line 3 and Line 4 define the large and the small items based on the PTAS parameter \( \varepsilon \). Then, we enumerate all possible allocations \((L_1, L_2)\) of the large items as a start-up (Line 5). The set of the small items \( S \) is then partitioned to \((O_1, O_2)\) in a similar way as we did for the EF1 case. Lines 7-12 handle two trivial cases: the two agents envy each other in the allocation \((L_1 \cup O_1, L_2 \cup O_2)\) and the allocation \((L_1 \cup O_1, L_2 \cup O_2)\) is already EFX. After these, the only possible case is agent \( i \) strongly envies agent \( j \) and agent \( j \) does not envy agent \( i \), for \((i, j)\) being \((1, 2)\) or \((2, 1)\). We assume agent 2 strongly envies agent 1 and agent 1 does not envy agent 2 without loss of generality (Line 13).

At the next step, we need to enumerate the item \( g \) whose removal ensures that agent 2 does not envy agent 1. This is trickier than the EF1 case in two aspects. Firstly, after \( g \) is chosen, all the items with values less than \( v_2(g) \) (based on agent 2’s valuation) must then be allocated to agent 2 (see Remark 2.3). Secondly, we need to consider the possibility that \( g \in L_1 \). Moreover, there should not be any item \( h \in L_1 \) with \( v_2(h) < v_2(g) \) since \( L_1 \) is fixed in agent 1’s bundle at this moment (again, see Remark 2.3). Lines 14-17 handle this step.

After the enumeration of the item \( g \), we solve the Knapsack problem as we did in the EF1 case (Lines 18-24). Here, it is possible that the capacity constraint for the Knapsack problem is negative,
ALGORITHM 3: A PTAS for MSWithineFX

Input: two utility functions $v_1$ and $v_2$, and the parameter $\varepsilon > 0$

Output: an EFX allocation

1. Let $\Pi \leftarrow \emptyset$ be the set of all the candidate allocations;
2. Let $\Gamma = \frac{\varepsilon}{2} \cdot \max\{v_1(\{m\}), v_2(\{m\})\}$;                  // a lower bound for the optimal social welfare
3. Let $L = \{g \in [m] \mid \exists i \in \{1, 2\} : v_i(\{g\}) \geq \frac{\varepsilon}{2} \cdot \Gamma\}$;  // large items
4. Let $S = [m] \setminus L$;                                      // small items
5. for each allocation $(L_1, L_2)$ of $L$ do
   6. Let $O_1 \leftarrow \{p \in S \mid v_1p \geq v_2p\}$ and $O_2 \leftarrow \{q \in S \mid v_1q < v_2q\}$;
   7. if $(L_2 \cup O_2, L_1 \cup O_1)$ is envy-free then
      8. $\Pi \leftarrow \Pi \cup \{(L_2 \cup O_2, L_1 \cup O_1)\}$;
      9. break;
   10. if $(L_1 \cup O_1, L_2 \cup O_2)$ is EFX then
     11. $\Pi \leftarrow \Pi \cup \{(L_1 \cup O_1, L_2 \cup O_2)\}$;
     12. break;
6. Suppose w.l.o.g. agent 2 strongly envies agent 1 in the allocation $(L_1 \cup O_1, L_2 \cup O_2)$;  
   13. $\ell \leftarrow \arg\min_{t \in L} v_2(t')$;
   14. $G \leftarrow \{g \in L_1 \cup O_1 \mid v_2(g) \leq v_2(\ell)\}$;
5. for each item $g \in G$ do
   16. $H \leftarrow \{h \in O_1 \mid v_2(h) < v_2(g)\}$;
      17. for each item $o \in O_1 \setminus (\{g\} \cup H)$ do
         18. $v(o) \leftarrow v_{1o} - v_{2o}$ and $w(o) \leftarrow v_{2o}$; // values and weights of items in Knapsack
         19. $C \leftarrow \frac{\varepsilon}{2} v_2([m] \setminus \{g\}) - v_2(L_1 \setminus \{g\})$; \quad // capacity for Knapsack
         20. if $C < 0$ then
             21. continue;
         22. Run the classical FPTAS with parameter $\frac{\varepsilon}{2}$ for Knapsack with item set $O_1 \setminus (\{g\} \cup H)$, value
         23. function $v$, weight function $w$ and capacity constraint $C$, and let $S_1$ be the output;
         24. $A_1' \leftarrow L_1 \cup S_1 \cup \{g\}$ and $A_2' \leftarrow [m] \setminus A_1'$;
         25. $(A_1, A_2) \leftarrow \text{LocalSearchEFX}(A_1', A_2', O_1, O_2)$;
         26. $\Pi \leftarrow \Pi \cup \{(A_1, A_2)\}$
6. return the allocation with the largest social welfare in $\Pi$

in which case we just abort the mission. For example, it is possible that agent 1 already receives too much for $L_1$, and the for-loop at Line 16 will do nothing in this case.

Finally, if agent 1 envies agent 2 in the allocation obtained from the Knapsack solution, we perform a local search algorithm as described in Algorithm 4. The algorithm iteratively moves an item from agent 2’s bundle to agent 1’s, and we only move those small items where agent 1 have higher values (i.e., items in $O_1$). This keeps going until agent 1 does not envy agent 2, and the algorithm will terminate at some point since we have assumed agent 1 does not envy agent 2 in the allocation $(L_1 \cup O_1, L_2 \cup O_2)$. If, at some middle stage, agent 2 begins to envy agent 1, the pre-condition of Proposition 3.3 is met, and we can apply Proposition 3.3 to finalize the allocation (notice that, at this point, we no longer fix $L_1$ and $L_2$ in the two agents’ bundles). The social welfare is non-decreasing except for the application of Proposition 3.3. However, the application of Proposition 3.3 only reduces the social welfare by at most $v_2(g)$ for some small item $g$ in $O_1$, which is acceptable.

**Theorem 4.7.** Algorithm 3 is a PTAS for MSWithineFX.
Again, we let $\OPT$ for the value of the optimal solution to MSWWITHNEFX. Let $\ALG$ be the social welfare of the allocation output by Algorithm 3. We first show the following two propositions.

**Proposition 4.8.** $\Gamma \leq \frac{1}{2} \cdot \max\{v_1([m]), v_2([m])\} \leq \OPT$.

**Proof.** Assume $v_1([m]) \leq v_2([m])$ without loss of generality. Let $(X, Y)$ be an EFX allocation in the valuation profile where both agents’ utility functions are $v_i$. Consider the EFX allocation $\mathcal{A}$ where agent 2 gets one of the bundles $X$ and $Y$ with a higher value and agent 1 get the other bundle. Then $SW(\mathcal{A}) \geq \frac{1}{2}v_2([m]) = \frac{1}{2} \cdot \max\{v_1([m]), v_2([m])\}$.

**Proposition 4.9.** For $L$ defined at Line 3 of Algorithm 3, we have $|L| \leq \frac{8}{\epsilon}$.

**Proof.** Let $L^{(1)} = \{g \in [m] | v_1(\{g\}) \geq \frac{\epsilon}{2} \cdot \Gamma\}$ and $L^{(2)} = \{g \in [m] | v_2(\{g\}) \geq \frac{\epsilon}{2} \cdot \Gamma\}$. Then $L = L^{(1)} \cup L^{(2)}$. Suppose for the sake of contradiction that $|L| > \frac{8}{\epsilon}$. There must exist $i \in \{1, 2\}$ with $|L^{(i)}| > \frac{4}{\epsilon}$. Then $v_i([m]) \geq v_i(L^{(i)}) > \frac{4}{\epsilon} \cdot \frac{\epsilon}{2} \cdot \Gamma = 2\Gamma \geq v_i([m])$, which is a contradiction.

The following lemma proves the approximation guarantee for Algorithm 3.

**Lemma 4.10.** $\ALG \geq (1 - \epsilon) \OPT$.

**Proof.** Let $(S_1, S_2)$ be the allocation corresponding to $\OPT$. Let $L_1$ and $L_2$ be the sets of the large items in $S_1$ and $S_2$ respectively. Consider the for-loop iteration at Line 5 where $(L_1, L_2)$ is in consideration. The maximum social welfare is attained at the allocation $(L_1 \cup O_1, L_2 \cup O_2)$. Therefore, if the for-loop is broken at Line 9 or Line 12, we have $\ALG = \OPT$. We assume that agent 2 strongly envies agent 1 in the allocation $(L_1 \cup O_1, L_2 \cup O_2)$ from now on.

Let $\OPT(C_2)$ be the maximum social welfare of the allocation $(S_1', S_2')$ where i) agent 2 does not strongly envy agent 1 and ii) $L_1 \subseteq S_1'$, $L_2 \subseteq S_2'$. We have $\OPT \leq \OPT(C_2)$ since we do not require that agent 1 does not strongly envy agent 2 in regarding $\OPT(C_2)$. It then suffices to show that $\ALG \geq (1 - \epsilon) \OPT(C_2)$.

Since agent 2 does not strongly envy agent 1 in $(S_1', S_2')$, we have $v_2(S_2') \geq v_2(S_1' \setminus \{g\})$ for $g \in S_1'$ with minimum $v_2(\{g\})$ (Remark 2.3). Consider the for-loop iteration at line 16 where $g$ is in consideration. The social welfare of the allocation obtained by the Knapsack solution is at least $(1 - \frac{\epsilon}{2}) \OPT(C_2)$. As we have mentioned before, after the local search step, the social welfare can be decreased by at most $v_2(\{g\}) \leq \frac{\epsilon}{2} \cdot \Gamma$ for some $g \in O_1$, which is at most $\frac{\epsilon}{2} \cdot \OPT \leq \frac{\epsilon}{2} \cdot \OPT(C_2)$ by Proposition 4.8. Therefore, $\ALG \geq (1 - \epsilon) \OPT(C_2)$.

To conclude Theorem 4.7, it remains to show that the algorithm runs in polynomial time if $\epsilon$ is a constant. Proposition 3.3 and the simple observation that Algorithm 4 can be performed for at most $m$ iterations indicate that Algorithm 4 runs in polynomial time. The only non-trivial part
of showing a polynomial running time for Algorithm 3 is analyzing the number of the iterations for the for-loop at Line 5. Proposition 4.9 implies that the number of the large items is a constant. Therefore, the number of the for-loop iterations is a constant.

### 4.3 NP-Hardness for Two Agents with Normalized Valuations

We complement our positive results with the following NP-hardness results. Aziz et al. [2023] proved the following theorem.

\[\square\]

which contradicts to our assumption.

In this section, we present our results for the for-loop at Line 5. Proposition 4.9 implies that the number of the large items is a constant.

**Theorem 4.11.** MSW\((w.sc/i.sc/t.sc/h.sc/i.sc/n.scEF)\) is NP-hard for \(n = 2\) even under normalized valuations.

**Proof.** We show a reduction from the partition problem. Fix a partition instance \(S = \{e_1, \ldots, e_\ell\}\), such that \(\sum_{i=1}^{\ell} e_i = 2x \in \mathbb{R}^+\). Without loss of generality, we can assume \(x = 1\), and we construct an instance as shown in the table below, where \(C \in \mathbb{R}^+\) is a sufficiently large number. Note that though \(\sum_{o \in [m]} v_1(o)\) and \(\sum_{o \in [m]} v_2(o)\) are not normalized, they are both equal to \(2C + 2\) and can be rescaled to 1. Therefore, the normalization assumption is not violated.

| item | \(k (1 \leq k \leq \ell)\) | \(\ell + 1\) | \(\ell + 2\) | \(\ell + 3\) |
|------|-----------------|------|------|------|
| \(v_1\) | \(e_k\) | \(C\) | \(C\) | 0 |
| \(v_2\) | \(e_k/2\) | \((2C + 1)/3\) | \((2C + 1)/3\) | \((2C + 1)/3\) |

If the partition instance is a yes-instance, suppose \(A_1 \subseteq [\ell]\) and \(A_2 \subseteq [\ell]\) correspond to the partition \((S_1, S_2)\) of \(S\) with equal sum. It is not hard to verify that the allocation \(\mathcal{A} = (A_1 \cup \{\ell + 1, \ell + 2\}, A_2 \cup \{\ell + 3\})\) satisfies EF1, and \(SW(\mathcal{A}) = (16C + 11)/6\).

If the partition instance is a no-instance, we show that the maximum social welfare is less than \((16C + 11)/6\). If there exists an allocation \(\mathcal{A}'\) with social welfare of at least \((16C + 11)/6\), then it is easy to see that \(\ell + 1\) and \(\ell + 2\) must be given to agent 1 and \(\ell + 3\) must be given to agent 2. Since \(SW(\mathcal{A}') \geq (16C + 11)/6\), agent 1 should take a bundle \(O_1 \subseteq [\ell]\) with at least half of the value of the first \(\ell\) items. Due to the EF1 constraint, agent 2 should also take a bundle \(O_2 \subseteq [\ell]\) with at least half of the value of the first \(\ell\) items. This would imply the partition instance is a yes-instance, which contradicts to our assumption. \(\square\)

The NP-hardness also holds for MSW\((w.sc/i.sc/t.sc/h.sc/i.sc/n.scEFX)\) with two agents and normalized valuations. Aziz et al. [2023] proved the following theorem.

**Theorem 4.12 (Aziz et al. [2023]).** MSW\((w.sc/i.sc/t.sc/h.sc/i.sc/n.scEFX)\) is NP-hard for \(n = 2\) even under normalized valuations.

## 5 MSW\((w.sc/i.sc/t.sc/h.sc/i.sc/n.scEFX)\) FOR MORE THAN TWO AGENTS

In this section, we present our results for MSW\((w.sc/i.sc/t.sc/h.sc/i.sc/n.scEFX)\) with general numbers of agents. We first go through some terms and notations.

We will use Algorithm 5 as a subroutine multiple times. Given a partial allocation where agent \(i\) gets \(A_i\) and a set of items \(B\) with \(A_i \cap B = \emptyset\), if agent \(i\) envies \(B\), Algorithm 5 computes \(X_i \subseteq B\) and \(k_i \in \mathbb{Z}^+\) such that \(k_i\) is the minimum integer with \(v_i(X_i) > v_i(A_i)\), where \(X_i \subseteq B\) is the set of the \(k_i\) items in \(B\) with the largest values with respect to agent \(i\)’s valuation \(v_i\). If \(v_i(A_i) \geq v_i(B)\), we set \(k_i = \infty\).

We will use the idea of the most envious agent by Chaudhury et al. [2021]. Given a partial allocation \((A_1, \ldots, A_n)\) and a set of items \(B\) such that \(B \cap A_i = \emptyset\) for each \(i = 1, \ldots, n\), the most envious agent to set \(B\) is an agent with minimum \(k_i = |X_i|\).
**ALGORITHM 5**: The replacement subroutine

1. **Function** `Replace(v_i, A_i, B)`:
2. Sort items in `B` in descending order based on `v_i`;
3. Let `B[k]` be the set of the first `k` items in `B`;
4. `X_i ← B[k]` where `k` is the minimum integer `k` with `v_i(B[k]) > v_i(A_i)`;
5. **return** `X_i`

**Proposition 5.1.** Consider a partial EFX allocation `(A_1, ... , A_n)` and a set of items `B` such that `B ∩ A_i = ∅` for each `i = 1, ..., n`. Let `i` be the most envious agent to `B`. Then `(A_1, ..., A_{i-1}, X_i, A_{i+1}, ..., A_n)` is an EFX allocation, where `X_i` is the output of `REPLACE(v_i, A_i, B)`.

**Proof.** Agent `i` receives a strictly higher value by updating the bundle from `A_i` to `X_i`, so she will not strongly envy any other agent since the original allocation `(A_1, ..., A_n)` is EFX. It remains to show that any agent `j ≠ i` will not strongly envy the bundle `X_i`. If `v_j(A_i) ≥ v_j(B)`, `j` will not envy `X_i` as `X_i ⊆ B`. Otherwise, let `k_j = |X_i|` where `X_i` is the output of `REPLACE(v_j, A_i, B)`. Agent `j` will not envy the subset of `B` consisting of the `(k_j - 1)` items with the highest values to her and thus will not envy any subset of `B` with at most `k_j - 1` items. We have `k_j ≥ k_i` by the definition of the most envious agent. Therefore, agent `j` will not envy any subset of `B` with at most `k_i - 1` items. Thus, agent `j` will not envy `X_i` after removing any item from `X_i`. □

**5.1 An O(n)-Approximation Algorithm for Unnormalized Valuations**

Our approximation algorithm is described in Algorithm 6. The algorithm starts by allocating one item to each agent so that the social welfare of the resultant allocation is maximized. Notice that this is just the problem of finding a maximum weight perfect matching, which can be solved in polynomial time by the Hungarian algorithm. Then, the algorithm applies the subroutine `REPLACE` for the most envious agent whenever there is an agent that envies the pool of the unallocated items.

**ALGORITHM 6**: An O(n)-approximation algorithm for MSWWITHINEFX

**Input**: `(v_1, ..., v_n)`

**Output**: an EFX allocation (that is allowed to be partial)

1. **if** `m < n` **then** Add dummy items with value 0 to all agents so that `m = n`;
2. **Initialize** `A = (A_1, ..., A_n)` that maximizes `SW(A)` subject to `|A_1| = ... = |A_n| = 1`;
3. `B ← [m] \ ∪_{i=1}^{m} A_i`;
4. **while** there exists an agent that envies `B` **do**
5.   **Let** `i` be the most envious agent to `B`;
6.   `X_i ← REPLACE(v_i, A_i, B);`  // see Algorithm 5
7.   `B ← B ∪ A_i \ X_i;`
8.   `A_i ← X_i;`
9. **return** `(A_1, ..., A_n)`

**Theorem 5.2.** Algorithm 6 is a pseudo-polynomial time algorithm that outputs an EFX allocation `A` with `(2n + 1) · MSW(A) ≥ ∑_{i=1}^{n} v_i([m])`.

Since `∑_{i=1}^{n} v_i([m])` is a trivial upper bound to MSW (see Definition F.1 for MSW), Theorem 5.2 implies that the price of EFX is at most `(2n + 1)`. In addition, since MSW is a trivial upper bound for the optimal solution to MSWWITHINEFX, Theorem 5.2 implies that Algorithm 6 is a `(2n + 1)-approximation algorithm.`
To prove Theorem 5.2, we first notice that Proposition 5.1 implies \((A_1, \ldots, A_n)\) is EFX. Each iteration of the while-loop strictly increases the social welfare, so the algorithm runs in a pseudo-polynomial time. The only non-trivial part is the proof for the approximation guarantee \((2n + 1)\). The proof of this is similar to Lemma 1 of Barman et al. [2020]. The high-level intuitions are described as follows. Suppose first \(|A_i| \geq 2\) for all \(i\) in the allocation output by Algorithm 6. The EFX property then ensures \(v_i(A_j)\) is at most twice as much as \(v_i(A_i)\). In addition, \(v_i(A_j) \geq v_i(B)\) for otherwise the while-loop of the algorithm should be carried on. Therefore, among the \(n+1\) bundles \(A_1, \ldots, A_n, B\), agent \(i\) gets a bundle with value at least \(1/2n\) fraction of \(v_i([m])\), which implies Algorithm 6 is a \(2n\)-approximation. The case where \(|A_i| \leq 1\) for some agent \(i\) is trickier, and the maximum-matching initialization at Line 2 is aimed to handle this. The full proof of Theorem 5.2 is deferred to Appendix C.

### 5.2 An \(O(\sqrt{n})\)-Approximation Algorithm for Normalized Valuations

We prove the following theorem in this section.

**Theorem 5.3.** For normalized valuations, there exists a pseudo-polynomial time algorithm that always outputs an EFX allocation \(A\) with \(O(\sqrt{n}) \cdot SW(A) \geq MSW\) (see Definition F.1 for MSW).

Theorem 5.3 implies an \(O(\sqrt{n})\)-approximation pseudo-polynomial time algorithm and that the price of EFX is \(O(\sqrt{n})\).

For normalized valuations, we have \(v_i([m]) = 1\) for each agent \(i\). If the maximum social welfare is upper-bounded by \(O(\sqrt{n})\), say, MSW \(\leq 10\sqrt{n}\), then we can just use Algorithm 6. By Theorem 5.2, we have \(SW(A) \geq n/(2n+1)\), so \(SW(A) = \Omega(1)\). The algorithm is already an \(O(\sqrt{n})\)-approximation, and the price of EFX is \(O(\sqrt{n})\). Notice also that MSW can obviously be computed in polynomial time. Therefore, in this section, we will assume MSW > 10\(\sqrt{n}\) from now on.

**Algorithm 7: An \(O(\sqrt{n})\)-approximation algorithm for MSW\textsc{within}EFX with normalized valuations**

**Input:** \((o_1, \ldots, o_n)\)

**Output:** an EFX allocation (that is allowed to be partial)

1. Compute an allocation \((O_1, \ldots, O_n)\) with social welfare MSW;
2. Initialize \((A_1, \ldots, A_n)\) such that \(A_i\) contains one item \(g\) with \(\max_{g \in O_i} v_i(g)\) if \(O_i \neq \emptyset\) and \(A_i = \emptyset\) if \(O_i = \emptyset\);
3. For each \(i = 1, \ldots, n\), set \(B_i \leftarrow O_i \setminus A_i\);
4. **while** there exists \(j\) such that an agent envies \(B_j\) **do**
5. \hspace{1cm} Let \(i\) be the most envious agent to \(B_j\); \hspace{1cm} // \(i\) may or may not be \(j\)
6. \hspace{1cm} \(X_j \leftarrow \text{Replace}(v_i, A_i, B_j);\)
7. \hspace{1cm} Release all items in \(A_i\) such that each \(g \in A_i\) is added to \(B_k\) if \(g \in O_k\);
8. \hspace{1cm} \(A_i \leftarrow X_i;\)
9. **return** \((A_1, \ldots, A_n)\)

Our algorithm is presented in Algorithm 7. The algorithm starts by computing a social welfare maximizing allocation \((O_1, \ldots, O_n)\) where \(O_i\) consists of those items where agent \(i\) values the most (break tie arbitrarily). The allocation \((A_1, \ldots, A_n)\) is initialized such that agent \(i\) takes one most valuable item in \(O_i\), or \(A_i = \emptyset\) if \(O_i = \emptyset\). Each \(B_i\) denotes the pool of the unallocated items in \(O_i\). The algorithm then enters a while-loop. Whenever there is a pool of the unallocated items \(B_j\) that some agent envies, we find the most envious agent \(i\) to \(B_j\) and replace \(A_i\) by some \(X_i\) in \(B_j\).

Notice that, throughout the algorithm, we have \(B_i \subseteq O_i\). In addition, each \(A_i\) is a subset to some bundle \(O_j\), i.e., an agent cannot get items from more than one bundle of \(O_1, \ldots, O_n\), although \(A_i\) may or may not be contained in \(O_i\).
To show Theorem 5.3, we note that Proposition 5.1 ensures the output allocation is EFX. Each while-loop iteration strictly improves the social welfare by applying REPLACE. Therefore, the algorithm runs in pseudo-polynomial time. Thus, the only non-trivial part is the approximation guaranteed, which is addressed in the lemma below.

**Lemma 5.4.** Suppose $\text{MSW} > 10\sqrt{n}$. Algorithm 7 outputs an allocation $\mathcal{A}$ with $\frac{20\sqrt{n}+10}{9} \text{SW}(\mathcal{A}) \geq \text{MSW}$.

**Proof.** Recall that each $A_j$ is a subset of some $O_i$. Let $n_i$ be the number of the agents $j$ with $A_j \subseteq O_i$. Let $\mathcal{L} = \{ i \in [n] \mid n_i > \sqrt{n}\}$ and $\mathcal{S} = \{ i \in [n] \mid n_i \leq \sqrt{n}\}$. We have $|\mathcal{L}| < \sqrt{n}$ for otherwise $\sum_{i \in \mathcal{L}} n_i > n$ which contradicts to $\sum_{i=1}^n n_i = n$.

Now, consider arbitrary $i$ and $j$ with $A_j \subseteq O_i$. We show that $v_i(A_i) \geq \frac{1}{\sqrt{n}} v_i(A_j)$. If $|A_j| \geq 2$, the inequality holds trivially by the EFX property (the removed item $q$ satisfies $v_i(\{q\}) \leq v_i(A_j \setminus \{q\})$ by Remark 2.3). If $|A_j| = 1$, we have $v_i(A_i) \geq v_i(\{g^*\}) \geq v_i(A_i)$ for $g^*$ being the item in $O_i$ with the largest value to agent $i$, where the first inequality is due to that $A_i = \{g^*\}$ at the beginning of the algorithm and $v_i(A_i)$ is non-decreasing throughout the algorithm.

Next, for each $i \in \mathcal{S}$, we have $v_i(A_i) \geq \frac{1}{2\sqrt{n}+1} v_i(O_i)$. This is because $O_i$ is the disjoint union of at most $\sqrt{n} + 1$ bundles $\{A_j \mid A_j \subseteq O_i\} \cup \{B_j\}$ (by definition of $\mathcal{S}$), $v_i(A_i) \geq \frac{1}{2} v_i(A_j)$ (just proved), and $v_i(A_i) \geq v_i(B_i)$ (otherwise the while-loop should be carried on).

Finally, $\text{SW}(\mathcal{A}) = \sum_{i \in \mathcal{L}} v_i(A_i) \geq \frac{1}{2\sqrt{n}+1} \sum_{i \in \mathcal{S}} v_i(O_i) = \frac{1}{2\sqrt{n}+1} (\text{MSW} - \sum_{i \in \mathcal{L}} v_i(O_i))$. On the other hand, we have $\sum_{i \in \mathcal{L}} v_i(O_i) \leq \sum_{i \in \mathcal{L}} v_i([m]) = \sum_{i \in \mathcal{L}} 1 = |\mathcal{L}| < \sqrt{n} < \frac{1}{10} \text{MSW}$. Putting together, we have $\text{SW}(\mathcal{A}) > \frac{9}{20\sqrt{n}+10} \cdot \text{MSW}$. \qed

### 5.3 Asymptotically Tight Inapproximability Results

In this section, we present a family of inapproximability results for MSWWITHINEFX. Theorem 5.5 and Theorem 5.6 provide inapproximability results with ratios of orders $n$ and $\sqrt{n}$ for unnormalized and normalized valuations respectively. They hold even for a constant number of agents. However, they do not match our algorithms in the last two sections, as our algorithms run in pseudo-polynomial time. Instead, they are in contrast with our bi-criteria algorithm (Theorem E.4) in Appendix E: if we are allowed to relax EFX by a little bit, we can compute the optimal social welfare in polynomial time; if relaxation is not allowed, we have strong inapproximability results.

Theorem 5.7 and Theorem 5.8 match our algorithms in the last two sections by providing asymptotically tight inapproximability results.

**Theorem 5.5.** For any odd number $n = 2k + 1$ of agents with $k \geq 1$, it is NP-hard to approximate MSWWITHINEFX to a factor smaller than $(k + 1)$.

**Proof.** We present a reduction from the partition problem. Given a partition instance $S = \{e_1, \ldots, e_\ell\}$ such that $\sum_{i=1}^\ell e_i = 2x$, we construct a MSWWITHINEFX instance as follows. For any fixed $n = 2k + 1$, we construct $2k + 1$ agents named $\{s, a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$ and $m = (k + 1) + k\ell$ items named $g_0, g_1, \ldots, g_k, \{h_j\}_{i=1, \ldots, k; j=1, \ldots, \ell}$. The utility functions of the $n$ agents are defined in the table below, where $w$ is a very large number.
Let $s$ be the “super agent”, and the social welfare mostly depends on the value that agent $s$ receives for large $w$.

If the partition instance is a yes-instance, for each $i \in [k]$, the two agents $a_i$ and $b_i$ can get a value of exactly $x$ from the item set $\{h_{i1}, h_{i2}, \ldots, h_{il}\}$. In this case, the super agent $s$ can get the bundle $\{g_0, g_1, \ldots, g_k\}$, which has value $w(k + 1)$, and the social welfare is at least $w(k + 1)$.

If the partition instance is a no-instance, the super agent $s$ can get at most one item from $\{g_0, g_1, \ldots, g_k\}$. Otherwise, there exists $i \in \{1, \ldots, k\}$ such that $g_i$ is allocated to agent $s$. Since the partition instance is a no-instance, no matter how we allocate the remaining items, one of $a_i$ and $b_i$ will receive a value of less than $x$ and so will envy agent $s$. Since the super agent receives at least two items, removing an item $g \neq g_i$ from agent $s$’s bundle does not remove the envy from agent $a_i/b_i$ to agent $s$, which violates the EFX condition. In this case, the social welfare is at most $w + 2kx$.

The theorem concludes by making $w \gg 2kx$. \hfill $\Box$

**Theorem 5.6.** For any number $n = k(2k + 1)$ of agents with $k \geq 1$, it is NP-hard to approximate MSWWITHINEFX to a factor smaller than $\frac{k}{2}$ even when agents’ valuations are normalized.

**Proof (sketch).** The reduction in the proof of Theorem 5.5 fails as the valuation functions are not normalized. We would like to normalize the valuation while ensuring the social welfare mostly depends on the utility of the super agent. This can be done by a duplication trick. We duplicate the instance in the proof of Theorem 5.5 for $k$ times (the items and the agents are both duplicated), and create an item for which all the “normal agents” have a value of almost 1. In the yes-partition instance, the $k$ super agents in the $k$ duplicated instance will receive value 1; in the no-partition instance, each super agent can only get a value of $\frac{1}{k+1}$. The full proof is deferred to Appendix C. \hfill $\Box$

**Theorem 5.7.** For any constant $\epsilon > 0$, a pseudo-polynomial time $n^{1-\epsilon}$-approximation algorithm to MSWWITHINEFX implies P = NP.

**Proof (sketch).** The proof of Theorem 5.5 cannot be used here, as the partition problem can be solved in pseudo-polynomial time. We need to find another NP-complete problem that does not depend on numerical values. We use the independent set problem. Given an independent set instance $(G = (V, E), x)$ (where we are deciding if $G$ contains an independent set of at least $x$ vertices), we construct an “independent set gadget” as follows. The gadget contains $|V| + |E|$ items $v_1, \ldots, v_{|V|}, e_1, \ldots, e_{|E|}$ corresponding to the vertices and the edges and $|E| + 1$ agents $a_0, a_1, \ldots, a_{|E|}$ where the last $|E|$ agents correspond to the edges and the first agent $a_0$ corresponds to the agent $a_i/b_i$ in the proof of Theorem 5.5. Each “edge agent” $a_i$ (with $i = 1, \ldots, |E|$) has a value of 1 on the two items corresponding to the two incident vertices of the edge and the “edge item” $e_j$, and she has a value of 0 on the remaining items. Agent $a_0$ has value 1 on all the $|V|$ “vertex items”. It is then easy to see, to guarantee EFX, if agent $a_0$ receives at least three “vertex items”, the items

| item | $g_0$ | $g_1$ | $g_2$ | $\cdots$ | $g_k$ | $h_{1j}$ ($j \in [\ell]$) | $h_{2j}$ ($j \in [\ell]$) | $\cdots$ | $h_{kj}$ ($j \in [\ell]$) |
|------|------|------|------|----------|------|----------------|----------------|----------|----------------|
| $v_s$ | $w$ | $w$ | $w$ | $\cdots$ | $w$ | $0$ | $0$ | $\cdots$ | $0$ |
| $v_{a_1}$ | $0$ | $x$ | $0$ | $\cdots$ | $0$ | $e_j$ | $0$ | $\cdots$ | $0$ |
| $v_{b_1}$ | $0$ | $x$ | $0$ | $\cdots$ | $0$ | $e_j$ | $0$ | $\cdots$ | $0$ |
| $v_{a_2}$ | $0$ | $0$ | $x$ | $\cdots$ | $0$ | $0$ | $e_j$ | $\cdots$ | $0$ |
| $v_{b_2}$ | $0$ | $0$ | $0$ | $\cdots$ | $x$ | $0$ | $0$ | $\cdots$ | $e_j$ |
| $v_{a_k}$ | $0$ | $0$ | $0$ | $\cdots$ | $x$ | $0$ | $0$ | $\cdots$ | $e_j$ |
| $v_{b_k}$ | $0$ | $0$ | $0$ | $\cdots$ | $x$ | $0$ | $0$ | $\cdots$ | $e_j$ |
received by agent $a_0$ must correspond to vertices that form an independent set. Therefore, agent $a_0$ can receive utility at least $x$ if and only if the independent set instance is a yes-instance.

We construct $n^{1-\varepsilon}$ copies of this gadget, where each gadget contains $n^\varepsilon$ agents. The $i$-th copy replaces the $i$-th “partition gadget” ($\{a_i, b_i\}, \{h_{i1}, h_{i2}, \ldots, h_{i\ell}\}$) in the proof of Theorem 5.5. The remaining part of the proof is the same. A complete proof of this theorem is in Appendix C. □

**Theorem 5.8.** For any constant $\varepsilon > 0$, a pseudo-polynomial time $n^{0.5-\varepsilon}$-approximation algorithm to MSWITHINEFX with normalized valuations implies $P = NP$.

**Proof (sketch).** Similar duplication trick in the proof of Theorem 5.6 can extend Theorem 5.7 to this theorem. A complete proof of this theorem is in Appendix C. □

### 6 MSWITHINEFX1 FOR MORE THAN TWO AGENTS

Due to the space limit, we only state our results for MSWITHINEFX1 in this section, with the proofs deferred to Appendix D.

#### 6.1 Constant Number of Agents

In this section, we present a family of inapproximability results. We will see that the inapproximability factor grows in a polynomial speed as $n$ grows. Compared with Theorem E.3 in Appendix E, we see that the approximability of MSWITHINEFX1 is catastrophically reduced if we insist on exact EF1 instead of allowing $(1 - \varepsilon)$-approximate EF1.

**Theorem 6.1.** MSWITHINEFX1 is NP-hard to approximate to within factor $\frac{4n}{3n+1}$ for any fixed $n > 2$, even under normalized valuations.

It should be mentioned that the factor $\frac{4n}{3n+1}$ above does not apply to $n = 2$. When substituting $n = 2$, the ratio is $8/7$. However, we gave an FPTAS in Section 4.1.

The inapproximability factor is upper-bounded by 4/3 in Theorem 6.1. In the theorem below, we provide a much stronger inapproximability factor for large $n$.

**Theorem 6.2.** For any fixed $n > 10^7$, MSWITHINEFX1 is NP-hard to approximate to factor $\frac{n^{1/7}}{10}$.

For unnormalized valuations, we have the following result which states that MSWITHINEFX1 admits a $\Theta(\sqrt{n})$ inapproximability factor as $n$ grows (while still being a constant).

**Theorem 6.3.** For any fixed constant $n \geq 2$, MSWITHINEFX1 is NP-hard to approximate to within any factor that is smaller than $\left\lfloor 1+\sqrt{\frac{4n-3}{2}} \right\rfloor$.

#### 6.2 General Number of Agents

The following theorem shows that MSWITHINEFX1 is NP-hard to approximate within a factor polynomial in $n$ or $m$, when the number of agents $n$ is given as an input.

**Theorem 6.4.** For any $\varepsilon > 0$, MSWITHINEFX1 is NP-hard to approximate to within a factor of $n^{1-\varepsilon}$, or within a factor of $m^{1-\varepsilon}$, even with normalized valuations.

It needs to mention that the proofs for the above two theorems make use of the $n^{1-\varepsilon}$ inapproximability result of the maximum independent set problem due to Hästad [1996], Khot [2001], Zuckerman [2006].

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2 Hästad [1996] showed the inapproximability for the ratio $n^{1-\varepsilon}$ under the stronger assumption ZPP $\neq$ NP. Khot [2001] described more precisely the inapproximability ratio, but with an even stronger assumption. Zuckerman [2006] derandomized the construction weakening its assumption to P $\neq$ NP.
Theorem 6.5 (Inapproximability of Maximum Independent Set). For any \( \varepsilon > 0 \), it is \( \text{NP}- \) hard to approximate the maximum independent set problem to within a factor of \( n^{1-\varepsilon} \).

Finally, we remark that a variant of the round-robin algorithm can achieve the following result.

Theorem 6.6. There exists an \( n \)-approximation algorithm for MSW within \( \text{EF1} \).

7 Conclusion and Future Work

In this work, we provided a complete landscape on the complexity and approximability of maximizing social welfare subject to the \( \text{EF1}/\text{EFX} \) constraint. Our results also provide asymptotically tight ratios for the price of \( \text{EFX} \), which is missing in the previous literature.

Given that this problem has been well understood for the cake cutting problem and is now well understood for the indivisible item allocation problem, an interesting future direction is to study this problem for the mixed divisible and indivisible goods. The allocation of mixed goods was first considered by Bei et al. [2021a] in which a fairness notion “EFM” that adapts \( \text{EF} \) is proposed. Studying maximizing the efficiency while guaranteeing fairness is thus a compelling future direction.

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A PROOF OF THEOREM 3.2

We first consider the following instance with three agents and seven items, introduced by Chaudhury et al. [2020].

|   | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ |
|---|------|------|------|------|------|------|------|
| $v_1$ | 8 | 2 | 12 | 2 | 0 | 17 | 1 |
| $v_2$ | 5 | 0 | 9 | 4 | 10 | 0 | 3 |
| $v_3$ | 0 | 0 | 0 | 0 | 9 | 10 | 2 |

Chaudhury et al. [2020] proved the following proposition.

**Proposition A.1** ([Chaudhury et al., 2020]). The allocation $(A_1, A_2, A_3)$ with
\[ A_1 = \{g_2, g_3, g_4\}, \quad A_2 = \{g_1, g_5\}, \quad \text{and} \quad A_3 = \{g_6\}, \]
is a partial EFX allocation (with $g_7$ unallocated) such that no complete EFX allocation $(B_1, B_2, B_3)$ satisfies
\[ v_1(B_1) \geq v_1(A_1) = 16, \quad v_2(B_2) \geq v_2(A_2) = 15, \quad \text{and} \quad v_3(B_3) \geq v_3(A_3) = 10. \]

Let $w$ be a very large number. Based on Chaudhury et al.’s instance, we construct one more agent and two more items with the new valuation profile shown below.

|   | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $h_1$ | $h_2$ |
|---|------|------|------|------|------|------|------|------|------|
| $v_1$ | 8 | 2 | 12 | 2 | 0 | 17 | 1 | 16 | 16 |
| $v_2$ | 5 | 0 | 9 | 4 | 10 | 0 | 3 | 15 | 15 |
| $v_3$ | 0 | 0 | 0 | 0 | 9 | 10 | 2 | 10 | 10 |
| $v_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $w$ | $w$ |

If we are allowed partial allocation, with $g_7$ unallocated, an EFX allocation can be
\[ A_1 = \{g_2, g_3, g_4\}, \quad A_2 = \{g_1, g_5\}, \quad A_3 = \{g_6\}, \quad \text{and} \quad A_4 = \{h_1, h_2\}. \] (1)

This allocation has social welfare slightly more than $2w$. We will show that agent 4 can only get one of $h_1$ and $h_2$ in any complete EFX allocation, in which case the social welfare is slightly more than $w$, which is less than $2w$.

Suppose for the sake of contradiction that agent 4 gets both $h_1$ and $h_2$. If $A_4 = \{h_1, h_2\}$, then Proposition A.1 suggests that one of the first three agents will get a value of less than 16, 15, and 10 respectively to keep the EFX property among them. In this case, the said agent will strongly envy agent 4. If \( \{h_1, h_2\} \subseteq A_4 \), then, after removing an item in $A_4 \setminus \{h_1, h_2\}$, $A_4$ is worth at least 32, 30, and 20 for the first three agents respectively. Therefore, there should be an allocation of the first seven items to the first three agents such that the three agents get values 32, 30, and 20 respectively. It is easy to see that this is impossible: $g_3$ and $g_6$ must be in agent 1’s bundle, then agent 3 must get $g_5$ and $g_7$; agent 2 cannot get a value of 30 from $\{g_1, g_2, g_4\}$. Therefore, we have proved Theorem 3.2 with unnormalized valuations.

Finally, we show that our example also works if the valuation is normalized, where the numbers in the first three rows of the table are divided by 74, 61, and 41 respectively, and $w = 0.5$. The partial EFX allocation (1) has social welfare $\frac{16}{74} + \frac{15}{61} + \frac{10}{41} + 1 > 1.70$. The same arguments above show that at most one of $h_1$ and $h_2$ can be allocated to agent 4 in any complete EFX allocation. To find an upper bound to the social welfare of an EFX allocation, suppose $h_2$ is allocated to agent 4 and each item is allocated to an agent with the highest value except that $h_1$ cannot be given to agent 4. The upper bound is
\[ \frac{8}{74} + \frac{4}{74} + \frac{4}{61} + \frac{10}{41} + \frac{3}{61} + \frac{15}{61} + \frac{1}{2} < 1.63. \]
The social welfare is strictly less than that of (1).
B OMITTED PROOFS IN SECT. 4

B.1 Proof of Lemma 4.5

The first paragraph in the proof of Lemma 4.4 shows the existence of item $g$ at Line 3 of Algorithm 2. The analysis of the two cases in the proof of Lemma 4.4 shows that the social welfare weakly increases after each while-loop iteration. It remains to show that the while-loop terminates after at most $m$ iterations.

In Case 1 where agent 2 envies agent 1 after removing $g$, the algorithm terminates immediately after exchanging two agents’ bundles. Although the algorithm may not terminate immediately in Case 2, we observe that the size of $A_1$ is increased by 1 in each iteration corresponding to Case 2. If Case 1 happens after many iterations corresponding to Case 2, we know the algorithm will terminate after one more iteration. The increasing size of $A_1$ ensures that the algorithm terminates after at most $m$ iterations in this scenario. If Case 1 never happens, then the algorithm will terminate with an EF1 allocation: Proposition 4.3 ensures agent 1 will not envy agent 2 at some stage when more and more items in $O_1$ are added to $A_1$. Again, the increasing size of $A_1$ ensures that the algorithm terminates after at most $m$ iterations in this scenario.

B.2 Proof of Lemma 4.6

Let $(S_1, S_2)$ be an allocation corresponding to both $OPT$ and $OPT(C_2)$ (see Lemma 4.4). First, it is easy to see that $O_2 \subseteq S_2$, and so $S_1 \subseteq O_1$. Otherwise, if an item in $O_2$ is allocated to agent 1, reallocating this item to agent 2 strictly increases the social welfare while $C_2$ is still satisfied.

Since $(S_1, S_2)$ is EF1, there exists $g \in S_1$ such that $v_2(S_2) \geq v_2(S_1 \setminus \{g\})$. For each $o \in O_1$, let $v(o) = v_{1o} - v_{2o}$ as it is in Line 5 of Algorithm 1. We can write $OPT$ and $OPT(C_2)$ as

$$OPT = OPT(C_2) = \sum_{o \in S_1} v_{1o} + \sum_{o \in S_2} v_{2o} = \sum_{o \in S_1} v(o) + \sum_{o=1}^n v_{2o} = v_2([m]) + v(g) + \sum_{o \in S_1 \setminus \{g\}} v(o).$$

Consider the for-loop iteration at Line 3 where item $g$ is in consideration. Since $v_2(S_2) \geq v_2(S_1 \setminus \{g\})$, we have $\sum_{o \in S_1 \setminus \{g\}} v_{2o} \leq \frac{1}{\epsilon} (v_2([m]) - v_2(g))$, so $S_1 \setminus \{g\} \subseteq O_1$ is a valid solution to the Knapsack problem at Line 6. By the nature of FPTAS, $A_1'$ output at Line 6 must satisfy $\sum_{o \in A_1'} v(o) \geq (1 - \epsilon) \sum_{o \in S_1 \setminus \{g\}} v(o)$. The social welfare for the allocation $(A_1' \cup \{g\}, A_2')$ satisfies

$$SW(A_1' \cup \{g\}, A_2') = \sum_{o \in A_1' \cup \{g\}} v_{1o} + \sum_{o \in A_2'} v_{2o} = \sum_{o \in A_1' \cup \{g\}} v(o) + \sum_{o=1}^n v_{2o}
= v_2([m]) + v(g) + \sum_{o \in A_1'} v(o) \geq v_2([m]) + v(g) + (1 - \epsilon) \sum_{o \in S_1 \setminus \{g\}} v(o) > (1 - \epsilon) OPT.$$ 

Finally, Lemma 4.5 and our choice of the allocation with the largest social welfare (Line 10) ensure that the final allocation of Algorithm 1 has a social welfare that is at least $SW(A_1' \cup \{g\}, A_2')$.

C OMITTED PROOFS IN SECT. 5

C.1 Proof of Theorem 5.2

We have proved that the algorithm runs in pseudo-polynomial time and always outputs EFX allocations in Sect. 5.1. It remains to show the approximation guarantee $(2n+1) \cdot SW(A) \geq \sum_{i=1}^n v_i([m])$. The proof is similar to Lemma 1 in Barman et al. [2020]. The algorithm by Barman et al. [2020] also
We present a reduction from the partition problem. Given a partition instance \( \mathcal{A} = (A_1, \ldots, A_n) \) that maximizes \( SW(\mathcal{A}) \) subject to \( |A_1| = \cdots = |A_n| = 1 \). The following fact is proved in Lemma 1 in Barman et al. [2020].

**Proposition C.1** ([Barman et al., 2020]). Let \( \mathcal{A}' \) be the allocation after Line 2 of Algorithm 6. We have \( SW(\mathcal{A'}) \geq \frac{1}{n} \sum_{i=1}^{n} \sum_{g \in G_i} v_i(\{g\}) \), where \( G_i \) is the set of the \( n \) items with the largest values to agent \( i \).

Let \( \mathcal{A} = (A_1, \ldots, A_n) \) be the output of Algorithm 6. Since REPLACE subroutine always increases an agent’s utility, the above proposition implies
\[
SW(\mathcal{A}) \geq \frac{1}{n} \sum_{i=1}^{n} \sum_{g \in G_i} v_i(\{g\}).
\] (2)

Next, we find a lower bound for each \( v_i(A_i) \). For each \( j \neq i \), by EFX property we have proved (in fact, EF1 suffices here), there exists an item \( q_j \in A_j \) such that \( v_i(A_i) \geq v_i(A_j \setminus \{q_j\}) \). By the stopping condition for the while-loop, we have \( v_i(A_i) \geq v_i(B) \). Therefore, by summing over the \( n + 1 \) bundles \( A_i, \ldots, A_n, B \), we have
\[
(n + 1) \cdot v_i(A_i) \geq \sum_{i=1}^{n} v_i(A_j \setminus \{q_j\}) + v_i(B) = \sum_{j=1}^{n} v_i(A_j) + v_i(B) - \sum_{j \neq i} v_i(\{q_j\}).
\]

Since \( [m] = B \cup \bigcup_{j=1}^{n} A_j \) and \( \sum_{j \neq i} v_i(\{q_j\}) \leq \sum_{g \in G_i} v_i(\{g\}) \), this implies
\[
(n + 1) \cdot v_i(A_i) \geq v_i([m]) - \sum_{g \in G_i} v_i(\{g\}).
\]

Summing over \( i = 1, \ldots, n \), we have
\[
(n + 1) \cdot \sum_{i=1}^{n} v_i(A_i) \geq \sum_{i=1}^{n} v_i([m]) - \sum_{i=1}^{n} \sum_{g \in G_i} v_i(\{g\}).
\]

By (2) and \( SW(\mathcal{A}) = \sum_{i=1}^{n} v_i(A_i) \) we have
\[
(n + 1) \cdot SW(\mathcal{A}) \geq \sum_{i=1}^{n} v_i([m]) - n \cdot SW(\mathcal{A}),
\]
which implies the desired result \( (2n + 1) \cdot SW(\mathcal{A}) \geq \sum_{i=1}^{n} v_i([m]). \)

### C.2 Proof of Theorem 5.6

We present a reduction from the partition problem. Given a partition instance \( S = \{e_1, \ldots, e_t\} \) with \( \sum_{i=1}^{t} e_i = 2x \), we construct an MSWTHINEFX instance as follows. The \( n = k(2k + 1) \) agents are partitioned into \( k \) groups each of which consists of \( 2k + 1 \) agents. Agents are indexed by \( \{s^{(i)}, a_1^{(i)}, a_2^{(i)}, b_1^{(i)}, b_2^{(i)}, \ldots, a_k^{(i)}, b_k^{(i)}\}_{t=1, \ldots, \ell} \) where the superscript denotes the group number. There are \( m = 1 + k(k + 1 + k \ell) \) items which consist of one item named \( f \) and \( k \) groups of \((k + 1 + k \ell)\) items indexed below
\[
\left\{g_0^{(i)}, g_1^{(i)}, \ldots, g_k^{(i)}; \{s^{(i)}\}_{i=1, \ldots, k}; \{a_j^{(i)}\}_{j=1, \ldots, k}; \{b_j^{(i)}\}_{j=1, \ldots, k}\right\}_{t=1, \ldots, \ell}.
\]

For each group \( t = 1, \ldots, k \), the utility functions of the agents in group \( t \) over the items in group \( t \) are defined in the same way as they are in the table in the proof of Theorem 5.5. Agents from one group have a value of 0 for items in another group. To make the valuations normalized, we set \( w = \frac{1}{t+1}, \) and let the value of item \( f \) be \( 1 - 3x \) for each agent in \( \{a_j^{(t)}, b_j^{(t)}\}_{i=1, \ldots, k}; t=1, \ldots, k \) where
\(^3\)Barman et al.’s algorithm is used to compute an EF1 allocation, whereas ours is used for EFX. Both algorithms start with allocating each agent one item.
we set $x$ to be a very small positive number by rescaling the partition instance (the $k$ super agents $s^{(1)}, \ldots, s^{(k)}$ have value 0 on $f$).

If the partition instance is a yes-instance, we describe an EFX allocation with social welfare at least $k$. In each group $t$, the items $g^{(t)}_0, g^{(t)}_1, \ldots, g^{(t)}_k$ are allocated to agent $s^{(t)}$, and the items in $\{h^{(t)}_{ij}\}_{i=1, \ldots, k; j=1, \ldots, r}$ are allocated to the agents in $a^{(t)}_1, b^{(t)}_1, a^{(t)}_2, b^{(t)}_2, \ldots, a^{(t)}_k, b^{(t)}_k$ such that each of them receives a value of exactly $x$. The item $f$ is discarded. It is straightforward to check that the allocation is EFX and has social welfare of at least $k$.

If the partition instance is a no-instance, we will show that the social welfare is at most 2 for sufficiently small $x$. A “lucky” agent $a^{(t)}_i$ or $b^{(t)}_i$ can be allocated the item $f$ with the most value. However, for most pairs of the “normal agents” $(a^{(t)}_i, b^{(t)}_i)$, if $g^{(t)}_j$ is not allocated to one of them and is allocated to $s^{(t)}$ instead, one of the two agents will envy $s^{(t)}$. This keeps agent $s^{(t)}$ from getting a bundle more than the single item $g^{(t)}_j$. Therefore, for the $t$ super agents $s^{(1)}, \ldots, s^{(k)}$, at least $k-1$ of them can receive at most one item from $g^{(0)}_0 \cdot g^{(1)}_1, \ldots, g^{(k)}_k$, and at most one of them is allowed to receive two items from $g^{(0)}_0, g^{(1)}_1, \ldots, g^{(k)}_k$. By including the values of the item $f$ and the items $h^{(t)}_{ij}$, the social welfare is at most

$$(1 - 3x) + 3x \cdot k^2 + ((k - 1) \cdot w + 2w).$$

Since $w = \frac{1}{k+1}$, the social welfare can be arbitrarily closed to 2 by having $x \to 0$.

C.3 Proof of Theorem 5.7

We present a reduction from the independent set problem. Given an independent set instance $(G = (V, E), x)$ with $x \geq 3$ (where we are deciding if $G$ contains an independent set of at least $x$ vertices), we construct an MSWITHINEFX instance as follows. The set of agents consists of a super agent $s$ and $k$ groups of normal agents $\{a_i, a_{i1}, a_{i2}, \ldots, a_{i|E|}\}_{i=1, \ldots, k}$, where $a_{i1}, a_{i2}, \ldots, a_{i|E|}$ in each group $i$ correspond to the $|E|$ edges in $G$. Notice that $n = 1 + k(|E| + 1)$, we can let $n$ be sufficiently large (but also of polynomial size with respect to $G$) such that $n^{1-\epsilon} < k + 1$ (just make the number of the agents in each group small compared with $n$, so that we can have a large number of groups). The set of items consists of $k + 1$ “super items” $g_0, g_1, \ldots, g_k$ and $k$ groups of “normal items” $\{v_{i1}, v_{i2}, \ldots, v_{i|E|}\}_{i=1, \ldots, k}$ such that each group of $|V| + |E|$ items correspond to the $|V|$ vertices and $|E|$ edges in $G$. The super agent has a value of $w$ for each super item, where $w$ is a large number that is still polynomial in $n$ (say, $w = n^{100}$), and she has a value of 0 for remaining items. In each group $i$, agent $a_{i0}$ has value $x$ on the super item $g_i$, value 1 on each of $v_{i1}, \ldots, v_{i|V|}$. For $j = 1, \ldots, |E|$, each agent $a_{ij}$ only has positive values on the normal items $v_{ij}, v_{i|V|}, e_{ij}, \ldots, e_{i|E|}$ in group $i$. In particular, $a_{ij}$ has value 1 on $e_{ij}$ and on the two vertex items $v_{iu}, v_{iu}$, where $u_1$ and $u_2$ are the two endpoints of the $j$-th edge.

If the independent set instance is a yes-instance, we describe an EFX allocation with social welfare at least $(k + 1)w$. The super agent $s$ gets all the super items $g_0, g_1, \ldots, g_k$. In each group $i$, agent $a_{i0}$ gets a set of $x$ items from $\{v_{i1}, \ldots, v_{i|V|}\}$ corresponding to an independent set of size $x$, and agent $a_{ij}$ (for $j = 1, \ldots, |E|$) gets the item $e_{ij}$. The remaining items are discarded. It is straightforward to check that the allocation is EFX and is, in fact, envy-free.

If the independent set instance is a no-instance, we will show that the super agent $s$ can get at most one super item in any EFX allocations. Suppose this is not the case. A super item $g_i$ with $i = 1, \ldots, k$ must be allocated to the super agent $s$, and $s$ is allocated at least one more item. By EFX, agent $a_{i0}$ cannot envy agent $s$, and must receive a value of at least $x$ from $v_{i1}, \ldots, v_{i|V|}$. This means at least $x$ items from $v_{i1}, \ldots, v_{i|V|}$. Since the independent set instance is a no-instance, agent $a_{i0}$ must receive two items $v_{iu}, v_{iu}$ such that $(u_1, u_2)$ is an edge. Let $a_{ij}$ and $e_{ij}$ be the agent and
the item in the $i$-th group corresponding to this edge respectively. Then $a_{ij}$ can receive a value of at most 1 by getting $e_{ij}$, and the value she has on agent $a_{i0}$’s bundle is 2. To maintain EFX, agent $a_{i0}$ must not receive more than the two items $v_{i1}$, $v_{i2}$. This contradicts to our assumption $x \geq 3$. Since we have proved agent $s$ can get at most one super item, the social welfare, in this case, is just slightly more than $w$ (to be exact, it is $n^{100} + O(n)$).

Putting the completeness and the soundness parts together, the inapproximability factor is $(k + 1)$, which is more than $n^{1-\varepsilon}$. In addition, all the values of the items are bounded by $n^{100}$. A pseudo-polynomial time algorithm is no more powerful than a polynomial time algorithm.

C.4 Proof of Theorem 5.8

The proof of Theorem 5.7 can be extended to prove this theorem by applying the duplication trick we have used for extending Theorem 5.5 to Theorem 5.6.

We again consider a reduction from the independent set problem. The agents are partition to $k$ groups. Each group $t$ contains a super agent $s^{(t)}$ and $(1 + k(|E| + 1))$ normal agents

$$\{a_{i0}^{(t)}, a_{i1}^{(t)}, \ldots, a_{i|E|}^{(t)}\}_{i=1}^{\ldots,k}.$$ 

Each group $t$ contains $k + 1$ super items $g_0^{(t)}, g_1^{(t)}, \ldots, g_k^{(t)}$ and $k$ groups of normal items

$$\{e_1^{(t)}, \ldots, e_{|V|-1}^{(t)}, e_{|E|}^{(t)}, \ldots, e_{|E|}^{(t)}\}_{i=1}^{\ldots,k}.$$ 

Within the same group, agents’ valuations on the items are defined in the same way as they are in the proof of Theorem 5.7. An agent has a value 0 for items in different groups. To normalize agents’ valuations, we make sure each agent values the set of all items $\{s^{(t)}\}$ satisfied for the super agents. We construct an item gadget contains slightly less than $n^{0.5-\varepsilon}$. Notice that we have $k$ groups of agents, each group consists of $k$ “independent set gadgets”, and each independent set gadget contains slightly less than $n^{0.5-\varepsilon}$. Notice that we have $k$ groups of agents, each group consists of $k$ “independent set gadgets”, and each independent set gadget contains slightly less than $n^{0.5-\varepsilon}$. Notice that we have $k$ groups of agents, each group consists of $k$ “independent set gadgets”, and each independent set gadget contains slightly less than $n^{0.5-\varepsilon}$. Notice that we have $k$ groups of agents, each group consists of $k$ “independent set gadgets”, and each independent set gadget contains slightly less than $n^{0.5-\varepsilon}$. Notice that we have $k$ groups of agents, each group consists of $k$ “independent set gadgets”, and each independent set gadget contains slightly less than $n^{0.5-\varepsilon}$.

Following similar analysis in Theorem 5.7, the social welfare in the yes-instance is at least $k(k + 1)w$. For the no-instance, if ignoring the terms without $w$, the social welfare is at most $(k + 1)w + (k + 1)w$, where the first $(k + 1)w$ is for the item $f$, and the second $(k + 1)w$ is because at least $k - 1$ super agents can get only one super item and one of them can get at most two super items (for the existence of a lucky normal agent who receives $f$). Therefore, the inapproximability ratio is $k/2$, which is $n^{0.5-\varepsilon}$. Finally, all the values of the items are bounded by $w$, which is a polynomial of $n$, so a pseudo-polynomial time algorithm is no more powerful than a polynomial time algorithm.

D OMITTED PROOFS IN SECT. 6

D.1 Proof of Theorem 6.1

We will show a reduction from the partition problem. When given a partition instance $\{e_1, \ldots, e_t\}$ such that $\sum_{i=1}^t e_i = 2x$, we construct an instance for MSW/WITHINEF1 with $n$ agents and $t + n - 1$ items as follows. For agent 1, (s)he has value $\frac{n - x}{x}$ for the item $\ell + 1$ and $\ell + 2$, and value 0 for the remaining items. For agent 2 to agent $n$, they have value $e_o$ for item $o \in [\ell]$ and value $x$ for each of the remaining items. The values will sum up to $nx + x$ for all agents and it can be normalized to 1, which satisfies the definition of normalized valuations.

If the partition instance is a yes-instance, we assume the partition is $(S_1, S_2)$. Let agent 2 receive items corresponding to $S_1$, agent 3 receive items corresponding to $S_2$, and each of the agents $4, \ldots, n$ receive one of the items from item $\ell + 3$ to item $n + \ell - 1$, and all of them receive items with total value $x$ in this case. Let agent 1 receive item $\ell + 1$ and item $\ell + 2$. The allocation is EF1 because
We will present a reduction from the partition problem. Fix a partition instance follows. The construction is illustrated in Fig. 1 and Table 4 with D.2 Proof of Theorem 6.2

agent 1 will not envy any other agent, and when removing one of the items in agent 1’s bundle, other agents will not envy agent 1 either. The social welfare under this case is 2nx.

If the partition instance is a no-instance, agent 1 can only receive one of item t + 1 and item t + 2. Otherwise, if agent 1 receives both items, all other agents should receive items whose values sum up to at least x, which is impossible in a no partition instance. In this case, the social welfare will be at most \( \frac{3nx + x}{2} \). Hence, the inapproximability factor is \( \frac{4n}{3n + 1} \) for any fixed \( n > 2 \).

D.2 Proof of Theorem 6.2
We will present a reduction from the partition problem. Fix a partition instance \( S = \{e_1, \ldots, e_t\} \) such that \( \sum_{i=1}^{t} e_i = 2x \in \mathbb{R}^+ \), where \( x \) is a very small number (with the partition instance rescaled).

Let \( t = \left\lceil \frac{n^2}{2} \right\rceil \), \( k = \left\lfloor \frac{1}{2} + \frac{\sqrt{4n^2 - 7}}{7} - 3 \right\rfloor \), and \( x = \frac{1 + t(k^2 - k) - 2t}{t(k^2 - k) + 2} \). We construct a fair division instance as follows. The construction is illustrated in Fig. 1 and Table 4 with \( n = 14 \) (so that \( t = 2, k = 3, x = \frac{1 + 10n}{14} \)). Nodes with gray backgrounds represent items, and nodes without backgrounds represent agents.

When \( n \) is general, the items are divided into four categories: clique items, partition items, and pool items, dummy items.

- There are \( t \) groups of clique items (the \( i \)-th group is named \( C_i \)). For each group, consider a clique \( G = (V, E) \) with \( k \) vertices, each vertex in the clique corresponds to a clique item. The items in group \( C_i \) are named \( c^{(1)}_i, \ldots, c^{(t)}_i \). As shown in Fig. 1, when \( n = 14 \), there are \( t = 2 \) clique item groups each of which contains \( k = 3 \) items.
- There are \( t(k^2 - k)/2 \) groups of partition items, which are denoted by \( \mathcal{P}^{(i)}, \{u, v\}, 1 \leq i \leq t, 1 \leq u < v \leq k \). Each group corresponds to an edge \( \mathcal{P}^{(i)}, \{u, v\} \) to the edge \( \{u, v\} \) in the \( i \)-th clique and contains \( t \) items, and those \( t \) items, named as \( \mathcal{P}^{(1)}, \{u, v\}, \ldots, \mathcal{P}_{t}^{(1)}, \{u, v\} \), correspond to the \( t \) numbers in the partition instance. For simplicity, we only display one of the six groups, \( \mathcal{P}^{(1)}, \{1, 3\} \), in Fig. 1, and \( \mathcal{P}^{(1)}, \{1, 3\}, \ldots, \mathcal{P}_{t}^{(1)}, \{1, 3\} \) are the items contained in group \( \mathcal{P}^{(1)}, \{1, 3\} \).
- There are \( t(k^2 - k) \) pool items, denoted by \( q_i, 1 \leq i \leq t(k^2 - k) \). As shown in Fig. 1, there are 12 pool items, and they are denoted by \( q_1, \ldots, q_{12} \).
are also $n - t(k^2 - k + 1)$ dummy items. For the example with $n = 14$, since $14 - 2(3^2 - 3 + 1) = 0$, we do not draw any dummy item in Fig. 1.

The agents are also divided into three categories: super agents, normal agents, and dummy agents,

- There are $t$ super agents named $s_1, \ldots, s_t$ corresponding to the $t$ cliques.
- There are also $t$ groups of normal agents corresponding to the $t$ cliques, and each group contains $k^2 - k$ normal agents. For the $i$-th group, each edge $(u, v)$ in the clique corresponds to two normal agents $a_{(u, v)}^{(i)}$ and $a_{(v, u)}^{(i)}$. For the example with $n = 14$, since there are 6 edges for the two cliques in Fig. 1, they correspond to 12 normal agents $a_{(u, v)}^{(1)}$ and $a_{(u, v)}^{(2)}$, $1 \leq u < v \leq 3$.
- There are also $n - t(k^2 - k + 1)$ dummy agents. For the example with $n = 14$, since $14 - 2(3^2 - 3 + 1) = 0$, we do not draw any dummy agent in Fig. 1.

When $n = 14$, the valuations are shown in detail in Table 4. When $n$ is general, the valuations are defined as follows.

- For each super agent $s_i$, (s)he has value $\frac{1}{k}$ for each item in $C_i$ and value 0 for each other items. In the example of $n = 14$, as shown in Table 4, super agent $s_1$ only has a positive value to items in $C_1$.
- For each $j = 1, \ldots, t$ and each of the two normal agents $a_{(u, v)}^{(j)}$ and $a_{(v, u)}^{(j)}$ in group $j$, they both have value $x$ to the clique items $c_u^{(j)}$ and $c_v^{(j)}$. Moreover, they both have value $\epsilon$ to the partition item $p_{w}^{(j), (u, v)}$ for each $w = 1, \ldots, t$ and value 0 to the partition items in other groups. Other than these items, for each normal agent, (s)he has value $x - \epsilon$ to each pool item. As shown in Table 1, those normal agents have the same value $x - \epsilon$ to those pool items $q_1, \ldots, q_{12}$.
- For each dummy agent, (s)he values $\frac{1}{n - t(k^2 - k + 1)}$ to all dummy items (if existing).

If the partition instance is a yes-instance, consider the following allocation. For each $j = 1, \ldots, t$, we allocate super agent $s_j$ all the $k$ clique items in group $C_j$. For each normal agent, we allocate her one pool item with value $x - \epsilon$ and a set of partition items with total value $\epsilon$ (the partition instance is a yes-instance). Each dummy agent receives exactly one dummy item. Hence, each normal agent receives value $x$ in total. It is not hard to verify that this allocation is EF1. In particular, a normal agent will not strongly envy the super agent in the same group because, according to her utility function, the super agent receives two items with value $x$ and all the other items have value 0. In this allocation, the social welfare is given by

$$SW(\mathcal{A}) \geq \frac{t \times 1 + t(k^2 - k) \times (x - \epsilon) + t(k^2 - k) \times \epsilon}{\text{clique items}} + \frac{t(k^2 - k)x}{\text{pool items}} + \frac{\epsilon}{\text{partition items}}.$$

If the partition instance is a no-instance, we will show an upper bound on the social welfare. For each $j = 1, \ldots, k$, we first argue that super agent $s_j$ can take at most $\sqrt{2tk} + 1$ items in $C_j$. Suppose $s_j$ takes $s$ items in $C_j$. Since each item corresponds to a unique vertex and there are $\frac{s^2 - s}{2}$ edges in the clique induced by the $s$ vertices, then there are $s^2 - s$ normal agents in group $j$ who can not receive any value from $C_j$. Since the partition instance is a no-instance, half of these normal agents can only receive less than $\epsilon$ value from the partition items. To ensure they do not strongly envy $s_j$, each of them should take at least two pool items. Since at least $\frac{s^2 - s}{2}$ agents need to take at least two pool items and the number of the pool items equals to the number of the normal agents, at least $\frac{s^2 - s}{2}$ normal agents can not take any pool item. To ensure they do not strongly envy other agents, they need to take at least one clique item. Hence, $\frac{s^2 - s}{2} \leq tk$, which implies $(s - 1)^2 \leq 2tk$, which further implies $s \leq \sqrt{2tk} + 1$ as we claimed at the beginning of this paragraph.
The upper bound of the social welfare is given by
\[
\mathcal{SW}(A) < ts \cdot (1/k) + tk \cdot x + t(k^2 - k) \cdot (x - \varepsilon) + t(k^2 - k) \cdot \varepsilon + \frac{1}{n - t(k^2 - k + 1)}\]
\[
= \frac{ts}{k} + tkx + tx(k^2 - k) + 1
\]
\[
\leq \frac{ts}{k} + 1 + 1 + 1 \quad (x \text{ is defined as } \frac{1 + (t(k - k)/2)}{t(k - k) + 2}, \text{ and } \varepsilon \text{ is small})
\]
\[
= \frac{ts}{k} + 3
\]

For the last inequality, it is straightforward to verify \(tkx \leq 1\) and \(tx(k^2 - k) \leq 1\) by definitions of \(t, k, \) and \(x\).

Finally, the inapproximability factor is no less than
\[
\frac{t + t(k^2 - k)x}{\frac{ts}{k} + 3} \geq \frac{tk + 3}{k} \geq \frac{1}{\sqrt{2t(k + 1)} + \frac{3}{t}} \geq \frac{1}{2\sqrt{n} \frac{1}{k} + \frac{3}{n^2}}
\]

We can notice that:
\[
\frac{t}{k} = \frac{t}{\frac{1}{2} + \frac{1}{2} \frac{4\sqrt{n} - 3}{4\sqrt{n} - 3}} \leq \frac{t}{\frac{1}{2} + \frac{1}{2} \sqrt{n}} \leq \frac{t}{\frac{1}{2} \sqrt{n}} = \frac{2t \frac{1}{2}}{\sqrt{n}} \leq \frac{2(2n^2)^{\frac{1}{2}}}{\sqrt{n}} = \frac{4\sqrt{2}}{n^{\frac{1}{2}}}
\]
where the middle inequality can be obtained by straightforward computations and the fact that \(\sqrt{n}/t \geq 1\).

Therefore, the inapproximability is at least:
\[
\frac{1}{2\sqrt{n} \frac{1}{k} + \frac{3}{n^2}} \geq \frac{1}{2\sqrt{\frac{1}{2} \sqrt{4\sqrt{2}n - 3} \frac{1}{n^2} + \frac{3}{n^2}} \geq \frac{1}{\frac{7}{n^2} + \frac{3}{n^2}} = \frac{n^4}{10}
\]

so the theorem concludes.

**D.3 Proof of Theorem 6.3**

We will present a reduction from the partition problem. When given a partition instance \(S = \{e_1, \ldots, e_r\}\) such that \(\sum_{i=1}^{r} e_i = 2\varepsilon \in \mathbb{R}^+\) (the partition instance is scaled such that \(\varepsilon\) is sufficiently small), we construct an MSWWITHINEF1 instance with \(n\) agents as follows.

The items are divided into two categories: **clique items** and **partition items**. Consider a clique with \(k = \left\lfloor \frac{1 + \sqrt{4n - 3}}{2} \right\rfloor\) vertices. Each vertex in the clique corresponds to a clique item. For partition items, the partition instance is copied for \(k(k - 1)/2\) times corresponding to those \(k(k - 1)/2\) edges. For each edge \((u, v)\), construct a set of items \(p^{(u,v)} = \{p^{(u,v)}_1, \ldots, p^{(u,v)}_i\}\) that corresponds to the partition instance.

The agents are divided into three categories: one **super agent**, \(k(k - 1)\) **normal agents**, and \(n - k(k - 1) - 1\) **dummy agents**. For each edge \((u, v)\) in the clique, it corresponds to two normal agents \(a^{(u,v)}_1\) and \(a^{(u,v)}_2\).
Table 4. The valuations of the construction when \( n = 14 \).

| \( a_{(1,2)} \) | \( x \) | \( x \) | \( 0 \) | \( 0 \) | \( 0 \) | \( e_i \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x - \varepsilon \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( a_{(2,1)} \) | \( x \) | \( x \) | \( 0 \) | \( 0 \) | \( 0 \) | \( e_i \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x - \varepsilon \) |
| \( a_{(1,3)} \) | \( x \) | \( 0 \) | \( x \) | \( 0 \) | \( 0 \) | \( e_i \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x - \varepsilon \) |
| \( a_{(3,1)} \) | \( x \) | \( 0 \) | \( x \) | \( 0 \) | \( 0 \) | \( e_i \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x - \varepsilon \) |
| \( a_{(1,2)} \) | \( 0 \) | \( x \) | \( x \) | \( 0 \) | \( 0 \) | \( e_i \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x - \varepsilon \) |
| \( a_{(2,3)} \) | \( 0 \) | \( 0 \) | \( x \) | \( x \) | \( 0 \) | \( 0 \) | \( e_i \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x - \varepsilon \) |
| \( a_{(1,3)} \) | \( 0 \) | \( 0 \) | \( x \) | \( x \) | \( 0 \) | \( 0 \) | \( e_i \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x - \varepsilon \) |
| \( a_{(3,2)} \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x \) | \( x \) | \( 0 \) | \( 0 \) | \( e_i \) | \( 0 \) | \( 0 \) | \( x - \varepsilon \) |
| \( a_{(2,3)} \) | \( 0 \) | \( 0 \) | \( 0 \) | \( x \) | \( x \) | \( 0 \) | \( 0 \) | \( e_i \) | \( 0 \) | \( 0 \) | \( x - \varepsilon \) |

Note that \( p_i^{1,(1,\alpha)} \) and \( p_i^{1,(\alpha,\alpha)} \) are regarded as the same item.

Since \( n - \ell(k^2 - k + 1) - 2(3^2 - 3 + 1) = 0 \), there exists no dummy agents and dummy items in this example.
The valuations are defined as follows: for the super agent, she has value 1 for each clique item and 0 for each partition item. For each dummy agent, she has value 0 for all items. For each normal agent \(a_{(u,v)}\), \(i \in \{1,2\}\), she has value \(\epsilon\) to the clique items \(u\) and \(v\), and value \(e_i\) to the partition item \(p_{i}^{(u,v)}\) for each \(i \in \{1,\ldots,t\}\). (S)he has value 0 on each of the remaining items.

If the partition instance is a yes-instance, it is possible to allocate the partition items to the normal agents such that each normal agent receives value exactly \(\epsilon\). We will then allocate all the clique items to the super agent and let each dummy agent receive the empty set. It is easy to see that the allocation is EF1. In particular, a normal agent will not strongly envy the super agent because, according to her utility function, the super agent receives two items with value \(\epsilon\) and all the other items have value 0. In this case, the social welfare is at least \(k\) by only accounting for the super agent’s utility.

If the partition instance is a no-instance, the super agent can receive only one clique item. Otherwise, if the super agent receives both item \(u\) and item \(v\), then one of the two normal agents in \(a_{(u,v)}\), \((a_{(u,v)})_2\) will strongly envy the super agent: both agents think the super agent receives two items with value \(\epsilon\), yet, at least one of them can only get a value less than \(\epsilon\) due to that the partition instance is a no-instance. In this case, the social welfare is at most \(1 + 2\epsilon \cdot (k + k(k - 1)/2)\): the super agent receives at most one clique item with value 1, the contribution to the social welfare is at most \(\epsilon\) for each clique item that is not given to the super agent, and the contribution to the social welfare is at most \(2\epsilon\) for each set of items \(p^{(u,v)} = \{p_{1}^{(u,v)}, \ldots, p_{t}^{(u,v)}\}\). Notice that this social welfare can be made arbitrarily close to 1 by making \(\epsilon\) sufficiently small.

Hence, the inapproximability factor is \(\frac{1}{k}\), and the theorem concludes by noticing \(k = \left\lfloor \frac{1+\sqrt{4n-3}}{2} \right\rfloor\).

### D.4 Proof of Theorem 6.4

We will present a reduction from the maximum independent set problem. We begin by describing the construction, which will be used for proving both theorems.

The construction. Fix a maximum independent set instance \(G = (V,E)\), and let \(k = |V|\) and \(\ell = |E|\). We will construct a fair division instance with \(n = k\ell + k\) agents and \(m = 2k^2\) items. We will assume \(\ell > 2k\) (notice that we can add multi-edges without changing the nature of the independent set problem).

The agents are partitioned into \(k\) groups. Agents are named by \(\{a_{1}^{(j)}, \ldots, a_{\ell}^{(j)}\}_{j=1,\ldots,k}\). Group \(j\) consists of \(\ell + 1\) agents \(a_{1}^{(j)}, \ldots, a_{\ell}^{(j)}, s^{(j)}\). Each edge \(e_i \in E\) in the maximum independent set instance corresponds to \(k\) agents \(a_{1}^{(j)}, \ldots, a_{\ell}^{(j)}\). In particular, for each group \(j\), the \(\ell\) agents \(a_{1}^{(j)}, \ldots, a_{\ell}^{(j)}\) represent the \(\ell\) edges in \(G\), and there is a special agent \(s^{(j)}\). We will use \(A^{(j)}\) to denote the set of agents in group \(j\).

The items are partitioned into \(k + 1\) groups defined as follows. For each \(j = 1,\ldots,k\), group \(j\) contains \(k\) items \(b_{1}^{(j)}, \ldots, b_{k}^{(j)}\). Each vertex \(u_i \in V\) in the maximum independent set instance corresponds to \(k\) items \(b_{1}^{(j)}, \ldots, b_{k}^{(j)}\). The \((k+1)\)-th group contains \(k^2\) items \(c_{1}, \ldots, c_{k^2}\). We will use \(B^{(j)}\) to denote the set of items in group \(j\) for each \(j = 1,\ldots,k\) and \(C\) to denote the set of items in group \(k + 1\).

The valuations are defined as follows. For each special agent \(s^{(j)}\), (s)he has value \(1/k\) for each item in \(B^{(j)}\), and value 0 for each of the remaining items. For each agent \(a_{i}^{(j)}\), (s)he has value \(\tau/2\) for each of the two items \(b_{i_1}^{(j)}\) and \(b_{i_2}^{(j)}\) representing the two vertices \(u_{i_1}\) and \(u_{i_2}\) of the edge \(e_i\), (s)he has value \((1-\tau)/k^2\) for each item in \(C\), and (s)he has value 0 for each of the remaining items, where we set \(\tau = 2/k^2\) so that item \(b_{i_1}^{(j)}\) and \(b_{i_2}^{(j)}\) have a slightly higher value than each item in \(C\).
Proof sketch. Notice that, for each $j = 1, \ldots, k$, the agent group $A^{(j)} \setminus \{s^{(j)}\}$ and the item group $B^{(j)}$ resembles the graph $G = (V, E)$ in that each edge is represented by exactly one agent in $A^{(j)} \setminus \{s^{(j)}\}$ and each vertex is represented by exactly one item in $B^{(j)}$. The $k$ groups can be viewed as $k$ copies of the maximum independent set instance.

For each group $j$, the items in $B^{(j)}$ have much higher values to agent $s^{(j)}$ than to any other agents. We need to maximize the number of the items in $B^{(j)}$ that are allocated to $s^{(j)}$.

On the other hand, the number of the “non-special” agents, $k\ell$, is more than the number of items $2k^2$. This implies some non-special agents will receive no item at all. Since all the non-special agents have a common valuation on items in $C$, to guarantee EF1, no agent can receive more than one item in $C$.

It is then easy to see that, in each group $j$, the items allocated to agent $s^{(j)}$ must correspond to an independent set in $G$. For otherwise, if $s^{(j)}$ receives both items $b_i^{(j)}$ and $b_i^{(j)}$ for an edge $e_i = (u_i, u_i)$, agent $a_i^{(j)}$ will have to receive at least two items in $C$ to guarantee EF1 (recall that, for agent $a_i^{(j)}$, the value of $b_i^{(j)}$ or $b_i^{(j)}$ is higher than the value of any item in $C$), and we have seen that this is infeasible.

Since we would like to maximize the number of items in $B^{(j)}$ that are allocated to agent $s^{(j)}$, our problem now naturally becomes the problem of maximizing the size of the independent set in $G$. Notice that the $k$ groups simulate the $k$ identical copies of the maximum independent set instance.

Since $\ell = O(k^2)$, we have $n = O(k^2)$ and $m = \Theta(k^2)$. Theorem 6.4 holds due to Theorem 6.5.

Formal proof. For the completeness part, we prove the following proposition.

PROPOSITION D.1. If $G$ has an independent set of size $t$, then there exists an allocation with social welfare at least $t$.

Proof. Let $I \subseteq V$ be an independent set of $G$ with $|I| = t$. Consider the following allocation. For each $j = 1, \ldots, k$, allocate $\{b_i^{(j)} \mid u_i \in I\}$ to agent $s^{(j)}$. Allocate remaining items arbitrarily subject to that each remaining agent in $\{A^{(j)} \setminus \{s^{(j)}\} \mid j = 1, \ldots, k\}$ receive at most one item.

It is straightforward to see that the allocation is EF1. Since each agent in $\{A^{(j)} \setminus \{s^{(j)}\} \mid j = 1, \ldots, k\}$ receives at most one item, no agent will strongly envy any of them. It remains to show that no one will envy $s^{(j)}$ for each $j$. Firstly, those special agents $\{s^{(j)}\}$ will not envy each other. This is because each $s^{(j)}$ receives items from group $B^{(j)}$ only, and (s)he only has positive values for items in $B^{(j)}$. Secondly, each non-special agent $a_i^{(j)}$ will not envy each special agent $s^{(j)}$. If $j \neq j'$, agent $a_i^{(j)}$ has 0 value for every item in $B^{(j)}$, and thus will not envy $s^{(j)}$. If $j = j'$, the only two items in $B^{(j)}$ that agent $a_i^{(j)}$ has non-zero values are $b_i^{(j)}$ and $b_i^{(j)}$, where $e_i = (u_i, u_i)$ is the edge corresponding to $a_i^{(j)}$. Since $s^{(j)}$ receives items that correspond to an independent set, $s^{(j)}$ receives at most one of $b_i^{(j)}$ and $b_i^{(j)}$. Thus, agent $a_i^{(j)}$ does not envy $s^{(j)}$.

Finally, we find a lower bound on the social welfare by only considering special agents. Each $s^{(j)}$ receives $t$ items, and each of them has value $1/k$. Since there are $k$ special agents, the social welfare is at least $k \cdot t \cdot \frac{1}{k} = t$.

For the soundness part, we prove the following proposition.

PROPOSITION D.2. If $G$ does not have an independent set of size larger than $t$, then the social welfare of any allocation is at most $t + 2$.

Proof. Consider an arbitrary allocation $A$. The crucial observation is that $A_{B^{(j)} \cap B^{(j)}}$ must correspond to an independent set. To see this, suppose for an edge $e_i = (u_i, u_i)$ some agent $s^{(j)}$ receives both items $b_i^{(j)}$ and $b_i^{(j)}$. Recall that agent $a_i^{(j)}$ has value $\tau/2 = 1/k^2$ on each of both items,
We present a variant of the round-robin algorithm. Like the round-robin algorithm, our algorithm shows that our algorithm is a \( \frac{1}{\sqrt{2}} \)-approximation algorithm for MSW(with)EF1, which is more than the total number of items \( 2k^2 \). Thus, EF1 cannot be guaranteed if \( A_{(i)} \cap B^{(j)} \) does not correspond to an independent set.

With this observation and the assumption that the maximum independent set has size no more than \( t \), each special agent \( s^{(j)} \) receives a bundle with at most \( t/k \). Thus, the overall utility for all the special agents is at most \( t \). For the non-special agent, each item is worth at most \( \frac{t}{2} = 1/k^2 \). Even if all the \( 2k^2 \) items are allocated to the non-special agents, the overall utility for all the non-special agents is bounded by \( 2k^2 \cdot \frac{1}{k^2} = 2 \). Therefore, the social welfare of \( \mathcal{A} \) is at most \( t + 2 \). \( \square \)

To conclude the proof, Theorem 6.5 and the two propositions above imply it is NP-hard to approximate \( SW(\mathcal{A}) \) to within a factor of \( k^{1-\epsilon} \) for any \( \epsilon > 0 \). Since \( m = 2k^2 \), the inapproximability factor can be written as \( \frac{1}{(\sqrt{2})} \frac{m^{\frac{1}{2} - \epsilon}}{k^\epsilon} \), which is more than \( m^{\frac{1}{2} - \epsilon} \) by choosing \( \epsilon \) appropriately. This concludes Theorem 6.4 for the part with \( m \). Since \( t = O(k^2) \), we have \( n = O(k^3) \). We can see Theorem 6.4 for the part with \( n \) holds by rewriting \( k^{1-\epsilon} \) in a similar way.

**D.5 Proof of Theorem 6.6**

We present a variant of the round-robin algorithm. Like the round-robin algorithm, our algorithm consists of \( \lceil m/n \rceil \) iterations. In each iteration except for the last iteration, exactly \( n \) items are allocated such that each agent receives exactly one item. In each iteration, we first find a tuple \((i,g)\) with maximum \( v_i(g)\) such that agent \( i \) has not been allocated an item yet in the current iteration and item \( g \) has not been allocated, and we allocate item \( g \) to agent \( i \). We do this \( n \) times until each agent receives exactly one item in this iteration.

Firstly, the allocation output by our algorithm is EF1, for the same reason that the standard round-robin algorithm is EF1. The crucial observation here is that, for every agent, the item (s)he receives at a particular iteration has a (weakly) larger value than the value of any item that is allocated in the later iterations. Therefore, for any pair of agents \( i \) and \( j \), in agent \( i \)'s valuation, the item allocated to agent \( i \) at the \( t \)-th iteration has a (weakly) larger value than the item allocated to agent \( j \) at the \((t+1)\)-th iteration. Thus, if removing the item allocated to agent \( j \) in the first iteration from agent \( j \)'s bundle, agent \( i \) will not envy agent \( j \).

We will then show that this is a \( n \)-approximation algorithm for MSW(with)EF1. Let \( I_t \) be the set of items allocated in the \( t \)-th iterations. For each item \( g \), let \( u^*_g = \max_{i=1,...,n} v_i(g) \). It is obvious that

\[
\sum_{g=1}^{m} u^*_g = \sum_{i=1}^{[m/n]} \sum_{g \in I_t} u^*_g
\]

is an upper bound to the optimal social welfare. On the other hand, let \( o_t \in I_t \) be the first item allocated in the \( t \)-th iteration, and let \( a_t \) be the agent who receives \( o_t \). By the nature of our algorithm, we have \( v_{a_t}(o_t) = u^*_o \geq u^*_g \) for any \( g \in I_t \). By only accounting for the items \( o_1,\ldots,o_{[m/n]} \), the social welfare for the allocation \( \mathcal{A} \) output by our algorithm satisfies

\[
SW(\mathcal{A}) \geq \sum_{t=1}^{[m/n]} u^*_o \geq \sum_{t=1}^{[m/n]} \left( \frac{1}{n} \sum_{g \in I_t} u^*_g \right) = \frac{1}{n} \sum_{g=1}^{m} u^*_g,
\]

which shows that our algorithm is a \( n \)-approximation algorithm for MSW(with)EF1.
E BI-CRITERIA OPTIMIZATION

In this section, we also consider the bi-criteria optimization version of MSW/ WITHINEFX and MSW/ WITHINEF1, where the EFX and EF1 constraints are relaxed in the following natural way.

Definition E.1. Given a real number $\beta \in (0, 1]$, an allocation is $\beta$-approximately EFX, if for any two agents $i$ and $j$, $v_i(A_i) \geq \beta \cdot v_i(A_j \setminus \{g\})$ holds for any item $g \in A_j$.

Definition E.2. Given a real number $\beta \in (0, 1]$, an allocation is $\beta$-approximately EF1, if for any two agents $i$ and $j$, there exists an item $g \in A_j$ such that $v_i(A_i) \geq \beta \cdot v_i(A_j \setminus \{g\})$.

For $\alpha \geq 1$ and $\beta \in (0, 1]$, an algorithm is an $(\alpha, \beta)$-bi-criteria approximation algorithm for MSW/ WITHINEFX if it always outputs an allocation $\mathcal{A}$ such that

- $\alpha \cdot SW(\mathcal{A}) \geq SW(\mathcal{A}^*)$, and
- $\mathcal{A}$ is $\beta$-approximately EFX,

where $\mathcal{A}^*$ is the optimal solution to MSW/ WITHINEFX, i.e., $\mathcal{A}^*$ is an EFX allocation with the highest social welfare.

An $(\alpha, \beta)$-bi-criteria approximation algorithm for MSW/ WITHINEF1 is defined similarly.

The remaining part of this section is organized as follows. In Sect. E.1, we consider constant numbers of agents, and we state our results of $(1, 1 - \epsilon)$-bi-criteria optimization for both MSW/ WITHINEF1 and MSW/ WITHINEFX. These results are proved in Sect. E.2 and Sect. E.3. Finally, when the number of the agents is not a constant, we show in Sect. E.4 that bi-criteria optimization fails assuming P $\neq$ NP.

E.1 Bi-Criteria Optimization for Constant Number of Agents

For a constant number of agents, we will describe a $(1, 1 - \epsilon)$-bi-criteria algorithm for both MSW/ WITHINEF1 and MSW/ WITHINEFX. Our algorithm’s running time is polynomial in terms of $m$ and $1/\epsilon$ (notice that $n$ is a constant).

Theorem E.3. Fix an arbitrary value of algorithm parameter $\epsilon > 0$. There exists a $(1, 1 - \epsilon)$-bi-criteria approximation algorithm for MSW/ WITHINEF1. In addition, the algorithm’s running time is polynomial in terms of $m$ and $1/\epsilon$ when the number of agents $n$ is a constant.

Theorem E.4. Fix an arbitrary value of algorithm parameter $\epsilon > 0$. There exists a $(1, 1 - \epsilon)$-bi-criteria approximation algorithm for MSW/ WITHINEFX. In addition, the algorithm’s running time is polynomial in terms of $m$ and $1/\epsilon$ when the number of agents $n$ is a constant.

Aziz et al. [2023] provide a pseudo-polynomial time algorithm for MSW/ WITHINEF1 with a constant number of agents. One may expect that standard rounding techniques can achieve the above-mentioned bi-criteria optimization. However, straightforward rounding techniques fail. It is possible that an agent’s valuations to all items (except for a few items allocated to someone else; recall that EF1/EFX allows envy for up to one item) are extremely small so a significant loss of precision occurs after rounding. An EF1/EFX allocation in the rounded instance may be very far from being EF1/EFX in the original instance.

For example, consider an agent who has a very large value on one item $g$ and different small values on the remaining items. The rounding may round the values of the items in $[m] \setminus \{g\}$ to 0. The EF1/EFX allocation may decide to allocate $g$ to another agent. In this case, any allocation after rounding is EF1 (and it is EFX if $g$ is the only item in another agent’s bundle), but the allocation may be far from approximately EF1/EFX before rounding.

To achieve bi-criteria optimization, we will use a careful individualized rounding together with some extra enumeration techniques.
The bi-criteria optimization algorithms for MSWWITHNEF1 and MSWWITHNEFX are mostly the same.

E.2 Proof of Theorem E.3

Before we prove Theorem E.3, we first describe our algorithms. We will first show our high-level designs, and then give the complete algorithms (shown in Algorithm 8 and Algorithm 9). The high-level ideas of our algorithm are here.

The algorithm proceeds in three steps. In the first step, we fix some items in each agent’s bundle. In particular, for each pair of agents $i$ and $j$, we fix an item $g_{ij}$ in agent $i$’s bundle $A_i$. This item will be the item $g$ in Definition 2.2 such that, after removing it from $A_i$, agent $j$ will not envy agent $i$. The first step will enumerate all possible sets for $\{g_{ij}\}$. In the second step, we will apply an individualized rounding technique so that each agent’s valuations to all the remaining items that are not fixed in the first step can only take values from a set of numbers whose cardinality is polynomial in $m$ and $1/\epsilon$. In the third step, we will use a dynamic programming method to solve our problem. It is crucial that the first two steps cannot be swapped. We will see the reason later.

Step 1: items fixing. For each agent $i$, we fix $X_i = \{g_{i1}, g_{i2}, \ldots, g_{i(i-1)}, g_{i(i+1)}, \ldots, g_{in}\}$ with $X_i \subseteq A_i$. For each $g_{ij}$, it is expected that agent $j$ does not envy agent $i$ after removing $g_{ij}$. Notice that it is possible that $g_{ij} = g_{jk}$. We enumerate all possible sets for $\{X_1, \ldots, X_n\}$. For each fixed $\{X_1, \ldots, X_n\}$, we proceed to Step 2 and 3 compute the remaining part of the allocation. Notice that the total number of the sets is at most $m^{n(n-1)}$ (each $g_{ij}$ can be one of the $m$ items), and this is a polynomial of $m$ given that $n$ is a constant.

Step 2: individualized rounding. We first fix an adjustable precision parameter $K$ that is a polynomial of $m$ and $1/\epsilon$. Let $Y_i = \{g_{i1}, \ldots, g_{i(i-1)}, g_{i(i+1)}, \ldots, g_{in}\}$. Notice that, for agent $i$, each item in $Y_i$ has been fixed to another agent. Let $V_i = \nu_i([m] \setminus Y_i)$ be agent $i$’s total value for all the items that are not in $Y_i$. Let $\tau_i = V_i/K$. We will round down the valuations $\nu_i$ of agent $i$ to the items in $[m] \setminus Y_i$ such that the rounded valuations $\overline{\nu}_i$ can only take values from $\{0, \tau_i, 2\tau_i, \ldots, K\tau_i\}$. Notice that valuations of items in $Y_i$ are not rounded, and they can be significantly larger than $K\tau_i$.

In the next step, we will solve one problem similar to MSWWITHNEF1 on the instance after the rounding process above, where we aim to find an allocation that maximizes social welfare in the original valuation subject to the EF1 constraint in the rounded valuation. To show that we can obtain an optimal social welfare with a nearly EF1 allocation, we need to achieve that 1) an EF1 allocation in the original instance is still EF1 in the rounded instance (so that no “high-quality” allocation with high social welfare is ruled out after rounding) and 2) an EF1 allocation in the rounded instance is approximate EF1 in the original instance. Notice that the EF1 here is in terms of the items $g_{ij}$ fixed in Step 1.

To achieve 1), we add $n$ dummy items $d_1, \ldots, d_n$ such that agent $i$ has value $m\tau_i$ on item $d_i$ and value 0 on the remaining $n-1$ items. Each item $d_i$ is allocated to agent $i$. If the allocation satisfies that $\nu_i(A_i) \geq \nu_i(A_j \setminus \{g_{ij}\})$ in the original instance, we must also have $\overline{\nu}_i(A_i \cup \{d_i\}) \geq \overline{\nu}_i(A_j \cup \{d_i\} \setminus \{g_{ij}\})$ in the rounded instance. This is because $\nu_i(A_i)$ can only be reduced by at most $m\tau_i$ after rounding. We can set $K$ to be significantly larger than $m$ such that the value of each dummy item is negligible.

To show 2), a crucial observation is that, in an allocation $\mathcal{A} = (A_1, \ldots, A_n)$ (with the $n$ dummy items added) that is EF1 in the rounded instance, we must have $\overline{\nu}_i(A_i) \geq \frac{1}{n}V_i$. Notice that agent $i$ should not envy any other agent after removing the corresponding item in $Y_i$ from the other agent’s bundle. Therefore, agent $i$ should receive at least the average value of $[m] \cup \{d_1, \ldots, d_n\} \setminus Y_i$. Since the value of $[m] \cup \{d_1, \ldots, d_n\} \setminus Y_i$ is at least $V_i$ (notice that the value of $[m] \setminus Y_i$ is at least $V_i - m\tau_i$ after rounding), we have $\overline{\nu}_i(A_i) \geq \frac{1}{n}V_i$. Suppose, when considering $\mathcal{A}$ in the original instance (with
dummy items removed), agent $i$ envies agent $j$ even after removing $g_{ji}$ from agent $j$’s bundle. We aim to show that the amount of envy is small compared with agent $i$’s value on $j$’s bundle (i.e., the amount of envy is an $\epsilon$ fraction of $i$’s value on $j$’s bundle). Firstly, since the allocation is EF1 in the rounded instance, the amount of envy in the original instance is at most $2m\tau_i$ ($m\tau_i$ for the dummy item, and at most $m\tau_j$ for the loss of the precision in the rounding). We can make this amount considerably smaller than $\frac{1}{n}V_i$ by setting $K$ large enough. Secondly, agent $i$’s value on $j$’s bundle (with $g_{ji}$ removed) in the original instance should be at least $\frac{1}{n}V_i - 2m\tau_i \approx \frac{1}{n}V_i$ (in order to make $i$ possibly envy $j$). The amount of envy, which is at most $2m\tau_i$, is indeed very small.

We now remark that Step 1 and Step 2 cannot be swapped. If $Y_i$ is participated in the rounding, it is possible that items in $Y_i$ have much larger values to agent $i$ compared with the remaining items such that the remaining items have value 0 after rounding. In this case, $V_i = v_i([m] \setminus Y_i)$ may even be rounded to 0, and the amount of envy $m\tau_i$ can be large compared with agent $i$’s value on $j$’s bundle.

**Step 3: dynamic programming.** The remaining part of the algorithm is a standard dynamic program. Let $H\{\{u_{ij}\}_{i=1,\ldots,n}j=1,\ldots,n\}$ be a Boolean function which takes $n^2$ values $\{u_{ij}\}$ as inputs and outputs TRUE if and only if there exists an allocation $\mathcal{A} = (A_1,\ldots,A_n)$ such that $\overline{v}_j(A_j \setminus \{g_{ji}\}) = u_{ij}$. After the rounding and adding dummy items in Step 2, we can assume each $u_{ij}$ only takes values from $\{0, \tau_i, 2\tau_i, \ldots, (K+m)\tau_i\}$. In addition, the total number of possible inputs to $H[\cdot]$ is $(K+m+1)^{n^2}$, which is a polynomial in $m$ and $1/\epsilon$ (as $K$ is a polynomial in $m$ and $1/\epsilon$, and $n$ is a constant). We can use a standard dynamic program to evaluate $H[\cdot]$ for all inputs. We can check each of the corresponding allocations to see if it satisfies EF1, and find out the one with maximum social welfare. In addition, we can get the exact optimal social welfare instead of the $(1 - \epsilon)$-approximation by comparing the social welfare of the stored allocations in terms of their actual values (instead of the rounded values).

**Algorithm 8:** Bi-criteria optimization of MSWITHINEF1

**Input:** utility functions $v_1,\ldots,v_n$, item set $M = [m]$, and the parameter $\epsilon > 0$

**Output:** an $(1 - \epsilon)$-approximate EF1 allocation.

1. Set $K \leftarrow \lceil \frac{3nm}{\epsilon} \rceil$;
2. Initialize $\Pi \leftarrow \emptyset$;
3. /* $\Pi$ stores candidate allocations */
4. for each $X = \{X_i = \{g_{i1,\ldots,gi(i-1),gi(i+1),\ldots,gi(n)\} \mid i = 1,\ldots,n\}$
5.     /* elements in $X_i$ may be repeated, but $X_i \cap X_j = \emptyset$ for any $i, j$ */
6.     for each $i = 1,\ldots,n$
7.         Set $Y_i \leftarrow \{g_{i1,\ldots,gi(i-1),gi(i+1),\ldots,gni}\}$
8.         Set $V_i \leftarrow v_i([m] \setminus Y_i)$
9.         Set $\tau_i \leftarrow V_i / K$;
10.        for each $o \in [m] \setminus Y_i$, set $\overline{v}_i(o) \leftarrow \max_{k: k\tau_i \leq v_i(o)} k\tau_i$;
11.        for each $o \in Y_i$, set $\overline{v}_i(o) \leftarrow 0$;
12.        Add a new item $d_i$ to $M$ such that $\overline{v}_i(d_i) = m\tau_i$ and $\overline{v}_j(d_i) = 0$ for each $j \neq i$ and $v_j(d_i) = 0$ for each $j \in [n]$;
13.        $X_i \leftarrow X_i \cup d_i$;
14.        $\mathcal{A} \leftarrow \text{DYNAMICPROGRAM}(M, \overline{v}_1,\ldots,\overline{v}_n, v_1,\ldots,v_n, \tau_1,\ldots,\tau_n, X, K)$; // see Algorithm 9
15. Include $\mathcal{A}$ in $\Pi$;

**Return** the allocation in $\Pi$ with the largest social welfare with respect to $\{v_1,\ldots,v_n\}$.
Algorithm 9: The dynamic programming subroutine for MSWwithinEF1

Function DynamicProgram(\(M = [m + n], \overline{v}_1, \ldots, \overline{v}_n, v_1, \ldots, v_n, \tau_1, \ldots, \tau_n, X, K\)):

/* \(X = \{X_i = \{g_{i1}, \ldots, g_{i(i-1)}, g_{i(i+1)}, \ldots, g_{im}, d_i\} | i = 1, \ldots, n\} \) */
/* for each \(i\), dummy item \(d_i\) is the \((m+1)\)-th item */
/* for each \(i\), \(\overline{v}(S)/\tau_i \in \{0, 1, \ldots, K + m\}\) holds for any \(S \subseteq M\) */

for each \(\chi \in \{0, 1, \ldots, K + m\}^{n^2}\) and each \(t = 0, 1, \ldots, m + n\) do

- Initialize \(H[\chi, t] \leftarrow \text{NIL}\);
- \(H[0^{n^2}, 0] \leftarrow 0\);

for each \(t = 0, \ldots, m + n - 1\) do

- for each \(\chi\) in the dictionary ascending order such that \(H[\chi, t] \neq \text{NIL}\) do

  - if item \(t + 1\) belongs to \(X\) then

    - Suppose \(t \in X_t\);
    - Update(\(\chi, t, t'\));

  - else

    - for each \(i = 1, \ldots, n\) do

      - Update(\(\chi, t, i\));

  - for each \(\chi \in \{0, 1, \ldots, K + m\}^{n^2}\) do

    - Set \(H[\chi, m + n] \leftarrow \text{NIL}\) if the allocation stored in \(H[\chi, m + n]\) is not envy-free w.r.t. \(\overline{v}_1, \ldots, \overline{v}_n\);

    - if \(H[\chi, m + n] \neq \text{NIL}\) for some \(\chi\) then

      - return the allocation in \(\{H[\chi, m + n] | H[\chi, m + n] \neq \text{NIL}, \chi \in \{0, 1, \ldots, K + m\}^{n^2}\}\) with the largest social welfare w.r.t. \(v_1, \ldots, v_n\), i.e. the one with the largest value \(H[\chi, m + n]\)

    - else

      - return NIL

Function Update(\(\chi, t, i\)):

- for each \(j\), set \(\chi'_j = \chi_j + \overline{v}_j(t + 1)/\tau_j\);
- for each \((i', j)\) with \(i' \neq i\), set \(\chi'_{ji} = \chi_{ji}\);
- if \(H[\chi', t + 1] = \text{NIL}\) or \(H[\chi', t + 1] < H[\chi, t] + v_i(t + 1)\) then

- \(H[\chi', t + 1] \leftarrow H[\chi, t] + v_i(t + 1)\);

E.2.1 Formal Proof of Theorem E.3. In Algorithm 8, we set the precision parameter \(K\) as \(\left\lceil \frac{3mn}{c} \right\rceil\) which is a polynomial of \(m\) and \(1/c\). We enumerate all possible removing item set \(X = \{X_i = \{g_{i1}, \ldots, g_{i(i-1)}, g_{i(i+1)}, \ldots, g_{im}, d_i\} | i = 1, \ldots, n\}\) in Line 3, where each \(g_{ij}\) is expected that agent \(j\) will not envy agent \(i\) after removing \(g_{ij}\) in agent \(i\)'s bundle (Step 1: items fixing). Lines 6-8 perform the individualized rounding step and set the rounded valuation \(\overline{v}_i\) such that the valuations can only take values from \(\{0, \tau_i, 2\tau_i, \ldots, K\tau_i\}\). For convenience, for each agent \(i\), Line 9 sets the valuation of all items in \(Y_i\) as 0 so that we can turn the original EF1 condition with regard to \(Y_i\) to the envy-freeness condition.

Lines 10-11 add the dummy items \(\{d_1, \ldots, d_n\}\) to reach the goal that if the allocation satisfies that \(v_i(A_i) \geq v_i(A_i \setminus \{g_{ij}\})\) in the original instance, we must also have \(\overline{v}_i(A_i \cup \{d_i\}) \geq \overline{v}_i(A_i \cup \{d_i\})\), and update the fixed item set \(X_i\) for each agent \(i\). Finally, Lines 12-14 call Algorithm 9 to find the optimal envy-free allocation after fixing set \(X\) and return the one with the largest social welfare with respect to original valuation among all possible sets \(X\).
For Algorithm 9, it uses the dynamic program to calculate the optimal envy-free allocation after fixing the assigned item set $X = \{X_i = \{g_{i1}, \ldots, g_{i(i-1)}, g_{i(i+1)}, \ldots, g_{im}, d_i\} | i = 1, \ldots, n\}$, where all items in $X_i$ should be assigned to agent $i$ (Step 3: dynamic programming). Since we add the dummy items, for any subset $S \subseteq M$, we have $\overline{\pi}(S) / \tau_i \in \{0, 1, \ldots, K + m\}$ for each $i$. In this algorithm, we use the state $H[\chi, t]$ to store the maximum social welfare value with regard to the original valuation (and record the corresponding allocation simultaneously), where $\chi_{ij}$ represents the value of agent $j$’s bundle with regard to $\pi_i$ when allocating only the first $t$ items.

Lines 2-4 initialize the values $H[\chi, t]$ for all possible $\chi$ and $t$. Lines 5-12 transfer the state, where Lines 7-9 is for the case where item $t + 1$ has been assigned to agent $i$ before and Lines 10-12 is for the case where item $t + 1$ has not been assigned and we need to enumerate the allocated agent. Next, Lines 13-18 return the envy-free allocation with the largest social welfare with regard to the original valuation. Lines 19-23 describe the detailed state transfer if we assign item $t + 1$ to agent $i$ from the state $H[\chi, t]$.

After describing the algorithm formally, we now prove Theorem E.3. The proof contains two parts:

- an EF1 allocation in the original instance can be transferred to an EF allocation in the modified instance (after rounding and adding dummy items).
- an EF allocation in the modified instance can be transferred to an $(1 - \varepsilon)$-approximate EF1 allocation in the original instance.

We will use the following two lemmas to show these two points.

**Lemma E.5.** For an EF1 allocation $\mathcal{A} = \{A_1, \ldots, A_n\}$ with regard to $\{v_1, \ldots, v_n\}$ in the original instance, the corresponding allocation $\mathcal{A}' = \{A'_1, \ldots, A'_n\}$ where $A'_i = A_i \cup \{d_i\}$ for each $i$, is an EF allocation with regard to $\{\overline{\pi}_1, \ldots, \overline{\pi}_n\}$ after fixing some set $X = \{X_i = \{g_{i1}, \ldots, g_{i(i-1)}, g_{i(i+1)}, \ldots, g_{im}, d_i\} | i = 1, \ldots, n\}$ in the modified instance.

**Proof.** For an EF1 allocation $A = \{A_1, \ldots, A_n\}$ with regard to $\{v_1, \ldots, v_n\}$, for each $(i, j)$ where $i \neq j$, there exists one $g_{ji} \in A_j$ such that $v_i(A_i) \geq v_i(A_i \setminus \{g_{ji}\})$ by the definition of EF1. We use all these $g_{ji}$ to get the set $X$. We want to show $\overline{\pi}_i(A_i \cup \{d_i\}) \geq \overline{\pi}_i(A_i \cup \{d_i\})$ for each $(i, j)$ where $i \neq j$.

We consider each $(i, j)$ where $i \neq j$. We have $\overline{\pi}_i(A_i \cup \{d_i\}) = \overline{\pi}_i(A_i) + m \tau_i \geq v_i(A_i) - m \tau_i + m \tau_i = v_i(A_i)$ because $A_i$ contains at most $m$ items and the value of each is rounded down by at most $\tau_i$. On the other hand, since $g_{ji} \in Y_i$, we have $\overline{\pi}_i(g_{ji}) = 0$ by Line 9 of Algorithm 8. By EF1, $v_i(A_i)$ is at least $v_i(A_i \setminus \{g_{ji}\}) \geq \overline{\pi}_i(A_j \cup \{d_j\}) = \overline{\pi}_i(A_j)$. Therefore, $\overline{\pi}_i(A_i \cup \{d_i\}) \geq \overline{\pi}_i(A_j \cup \{d_j\})$. □

**Lemma E.6.** For an EF allocation $\mathcal{A}' = \{A'_1, \ldots, A'_n\}$ where $A'_i = A_i \cup \{d_i\}$ for each $i$ with regard to $\{\overline{\pi}_1, \ldots, \overline{\pi}_n\}$ after fixing the set $X = \{X_i = \{g_{i1}, \ldots, g_{i(i-1)}, g_{i(i+1)}, \ldots, g_{im}, d_i\} | i = 1, \ldots, n\}$ in the modified instance, the corresponding allocation $\mathcal{A} = \{A_1, \ldots, A_n\}$ is an $(1 - \varepsilon)$-approximate EF1 allocation with regard to $\{v_1, \ldots, v_n\}$ in the original instance.

**Proof.** By definition, for each pair $(i, j)$ where $i \neq j$, we have $\overline{\pi}_i(A_i \cup \{d_i\}) \geq \overline{\pi}_j(A_j \cup \{d_j\})$, then it suffices to show $v_i(A_i) \geq (1 - \varepsilon)v_i(A_i \setminus \{g_{ji}\})$.

We consider each pair $(i, j)$ where $i \neq j$ such that agent $i$ envies agent $j$ even after removing item $g_{ji}$ from $j$’s bundle: $v_i(A_i) < v_i(A_i \setminus \{g_{ji}\})$ (otherwise, if no such pair exist, the allocation is...
EF1 and we are done. We have

\[
    v_i(A_I) \geq \overline{\ell}_i(A_I) \quad \text{ (values of items are rounded down)}
\]

\[
    = \overline{\ell}_i(A_I \cup \{d_i\}) - m\tau_i \\
    \geq \overline{\ell}_i(A_I \cup \{d_j\}) - m\tau_i \\
    = \overline{\ell}_i(A_j) - m\tau_i \\
    = \overline{\ell}_i(A_j \setminus \{g_{ji}\}) - m\tau_i \\
    \geq v_1(A_j \setminus \{g_{ji}\}) - 2m\tau_i,
\]

where the last inequality is because \(A_j \setminus \{g_{ji}\}\) contains at most \(m\) items and the value of each is rounded down by at most \(\tau_i\). Thus, if we can show \(2m\tau_i \leq \epsilon v_i(A_j \setminus \{g_{ji}\})\), then we have \(v_i(A_I) \geq (1 - \epsilon) v_i(A_J \setminus \{g_{ji}\})\), which finishes the proof.

So the remaining is to show \(2m\tau_i \leq \epsilon v_i(A_j \setminus \{g_{ji}\})\). Since we have \(\overline{\ell}_i(A_I) + m\tau_i = \overline{\ell}_i(A_I \cup \{d_i\}) \geq \overline{\ell}_i(A_{I'} \cup \{d_j\}) = \overline{\ell}_i(A_{I'} \setminus \{g_{ji}\})\) for each \(i' \neq i\), and we also have \(\overline{\ell}_i(A_I) + m\tau_i \geq \overline{\ell}_i(A) + m\tau_i\), we sum all these \(n\) terms up and we have

\[
    \overline{\ell}_i(A_I) + m\tau_i \geq \frac{1}{n} \left( \overline{\ell}_i(A_I) + m\tau_i + \sum_{i' \neq i} \overline{\ell}_i(A_{I'} \setminus \{g_{ji}\}) \right) = \frac{1}{n} \left( \overline{\ell}_i([m] \setminus Y_i) + m\tau_i \right) \geq \frac{1}{n} \overline{\ell}_i([m] \setminus Y_i)
\]

where the last inequality is due to that the value of each item is rounded down by at most \(\tau_i\) and there are at most \(m\) items. Recalling that we have set \(V_i = v_i([m] \setminus Y_i)\), this means \(\overline{\ell}_i(A_I) \geq \frac{V_i}{n} - m\tau_i\).

Next,

\[
    \epsilon v_i(A_j \setminus \{g_{ji}\}) > \epsilon v_i(A_i) \quad \text{ (we have assumed } v_i(A_i) < v_j(A_j \setminus \{g_{ji}\}) \text{ at the beginning)}
\]

\[
    \geq \epsilon \overline{\ell}_i(A_i)
\]

\[
    \geq \epsilon \left( \frac{V_i}{n} - m\tau_i \right) \quad \text{ (we have just proved this)}
\]

\[
    \geq \frac{V_i}{n} - m\tau_i.
\]

To show \(2m\tau_i \leq \epsilon v_i(A_j \setminus \{g_{ji}\})\), it remains to show \(\epsilon \frac{V_i}{n} - m\tau_i \geq 2m\tau_i\), that is \(\tau_i \leq \frac{\epsilon}{3mn} V_i\). Since \(K = \left\lceil \frac{3mn}{\epsilon} \right\rceil \geq \frac{3mn}{\epsilon}\), we have \(\tau_i = V_i/K \leq \frac{\epsilon}{3mn} V_i\), and the lemma concludes.

**Proof of Theorem E.3.** Assume the allocation \(\mathcal{A}' = (A_1, \ldots, A_n)\) is the EF1 allocation with the largest social welfare, from Lemma E.5, it can be transferred to one EF allocation \(\mathcal{A}' = \{A_1', \ldots, A_n'\}\) where \(A_i' = A_i \cup \{d_i\}\) for each \(i\) after fixing some set \(X\). Then, since we choose the allocation with the largest social welfare with regard to original valuation for all possible sets \(X\), the allocation output by Algorithm 8 must have at least the same social welfare with regard to original valuation after removing the dummy items. Furthermore, because of Lemma E.6, this allocation is an \((1 - \epsilon)\)-approximate EF1 allocation with regard to the original valuation. To sum up, the output allocation \(\mathcal{A}\) should be an \((1 - \epsilon)\)-approximate EF1 and \(SW(\mathcal{A}) \geq SW(\mathcal{A}')\). □

**E.3 Proof of Theorem E.4.**

We can follow the same ideas as in the proof of Theorem E.3. Thus, we only discuss the difference in the algorithms. Our algorithm is shown in Algorithm 10 and Algorithm 11.

The main difference is that the \(g_{ij}\) we enumerate now should ensure such \(g_{ij}\) is the smallest item in agent \(i\)'s bundle from agent \(j\)'s perspective. To ensure this, at Step 3 in Algorithm 10, we only need the feasible choices of the set \(X\) such that \(v_i(g_{ji}) \leq v_i(g_{jk})\), for all \(i, j, k\). This means \(g_{ji}\) is the smallest item among all items which have been allocated to agent \(j\), from agent \(i\)'s perspective.
The second difference is at Step 12 in Algorithm 11. Here, when we allocate the items not in the set \( X \) to some agent \( i \), we also need to ensure that the allocated item cannot have a smaller value than any \( g_{ij} \) from agent \( j \)’s perspective.

**ALGORITHM 10**: Bi-criteria optimization of MSWWITHINEFX

**Input**: utility functions \( v_1, \ldots, v_n \), item set \( M = [m] \), and the parameter \( \epsilon > 0 \)

**Output**: an \((1 - \epsilon)\)-approximate EFX allocation.

1. Set \( K \leftarrow \lceil \frac{m}{2m/n} \rceil \);
2. Initialize \( \Pi \leftarrow \emptyset \);
3. for each feasible \( X = \{ X_i = \{ g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n \} \mid i = 1, \ldots, n \} \) do
   4. Set \( Y_i \leftarrow \{ g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n \} \);
   5. Set \( V_i \leftarrow v_i(\lfloor m \rfloor \setminus Y_i) \);
   6. Set \( \tau_i \leftarrow V_i / K \);
   7. for each \( o \in [m] \setminus Y_i \), set \( \overline{v}_i(o) \leftarrow \max_k k \tau_i \leq v_i(o) k \tau_i \);
   8. for each \( o \in Y_i \), set \( \overline{v}_i(o) \leftarrow 0 \);
   9. Add a new item \( d_i \) to \( M \) such that \( \overline{v}_i(d_i) = m \tau_i \) and \( \overline{v}_j(d_i) = 0 \) for each \( j \neq i \) and \( v_j(d_i) = 0 \) for each \( j \in [n] \);
   10. \( X_i \leftarrow X_i \cup d_i \);
11. \( \mathcal{A} \leftarrow \text{DYNAMICPROGRAMEFX}(M, \overline{v}_1, \ldots, \overline{v}_n, v_1, \ldots, v_n, \tau_1, \ldots, \tau_n, X, K) \); // see Algorithm 11
12. Include \( \mathcal{A} \) in \( \Pi \);
13. return the allocation in \( \Pi \) with the largest social welfare with respect to \( \{v_1, \ldots, v_n\} \)

**E.4 Intractability Results for Bi-Criteria Optimization with General \( n \)**

In the previous section, for a constant number of agents, we show that MSWWITHINEFX becomes tractable if we slightly relax EF1 and EFX. In this section, we show that the problem becomes largely intractable for a general number of agents even if EF1 and EFX are relaxed substantially.

**Theorem E.7.** Fix any small \( \epsilon > 0 \). If there exists a \((n^{0.5-\epsilon}, \epsilon)\)-bi-criteria approximation algorithm for MSWWITHINEFX, then \( P = \text{NP} \). If there exists a \((m^{1-\epsilon}, \epsilon)\)-bi-criteria approximation algorithm for MSWWITHINEFX, then \( P = \text{NP} \).

**Proof.** The proof is similar to the proof in Appendix A.2 in the paper [Barman et al., 2019]. We present a reduction from the maximum independent set problem.

Given a maximum independent set instance \( G = (V, E) \), we construct a fair division instance with \( n \) items and \( m + 1 \) agents, where \( n = |V| \) and \( m = |E| \). Those \( n \) items correspond to the \( n \) vertices in \( G \). Those \( m + 1 \) agents consist of a super agent and \( m \) edge agents that correspond to the \( m \) edges in \( G \).

Fix a small number \( \delta > 0 \). The super agent has value 1 for each item. For the edge agent representing the edge \((u, v)\), \( (s)\)he has value \( \delta \) on the two items representing \( u \) and \( v \), and \( (s)\)he has value 0 on the remaining items.

To guarantee \( \epsilon \)-approximate EF1, it is required that the super agent cannot take any pair of items representing vertices \( u \) and \( v \) for an edge \((u, v)\). For otherwise, the agent representing the
Algorithm 11: The dynamic programming subroutine for MSWwithinEFX

```plaintext
Function DynamicProgramEFX(M = [m+n], \( \overline{\tau}_1, \ldots, \overline{\tau}_n, v_1, \ldots, v_n, r_1, \ldots, r_n, X, K \)):

// X = \{X_i = \{g_i, \ldots, g_{i+1}\}, \ldots, g_m, d_i\} | i = 1, \ldots, n
// for each i, dummy item \( d_i \) is the (m+i)-th item
for each \( \chi \in \{0, 1, \ldots, K+m\}^{n^2} \) and each \( t = 0, 1, \ldots, m+n \) do
  Initialize \( H[\chi, t] \leftarrow \text{NIL} \);
  \( H[0,0] \leftarrow 0 \);
for each \( t = 0, \ldots, m+n-1 \) do
  for each \( \chi \) in the dictionary ascending order such that \( H[\chi, t] \neq \text{NIL} \) do
    if item \( t+1 \) belongs to \( X \) then
      Suppose \( t \in X_{\chi} \);
      Update(\( \chi, t, t' \));
    else
      for each \( i = 1, \ldots, n \) do
        if \( v_j(g_{ji}) \leq v_j(t+1) \) for all \( j \neq i \) then
          Update(\( \chi, t, i \));
  for each \( \chi \in \{0, 1, \ldots, K+m\}^{n^2} \) do
    Set \( H[\chi, m+n] \leftarrow \text{NIL} \) if the allocation stored in \( H[\chi, m+n] \) is not envy-free w.r.t. \( \overline{\tau}_1, \ldots, \overline{\tau}_n \);
  if \( H[\chi, m+n] \neq \text{NIL} \) for some \( \chi \) then
    return the allocation in \( \{H[\chi, m+n] | H[\chi, m+n] \neq \text{NIL}, \chi \in \{0, 1, \ldots, K+m\}^{n^2} \} \) with the largest social welfare w.r.t. \( v_1, \ldots, v_n \), i.e. the one with the largest value \( H[\chi, m+n] \);
else
    return NIL

Function Update(\( \chi, t, i \)):
for each \( j \), set \( \chi'_{ji} \leftarrow \chi_{ji} + \overline{\tau}_j(t+1)/\tau_j \);
for each \( (i', j) \) with \( i' \neq i \), set \( \chi'_{ji'} \leftarrow \chi_{ji} \);
if \( H[\chi', t+1] = \text{NIL} \) or \( H[\chi', t+1] < H[\chi, t] + v_i(t+1) \) then
  \( H[\chi', t+1] \leftarrow H[\chi, t] + v_i(t+1) \);
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edge \((u, v)\) will receive a value of 0, and \( \varepsilon \)-approximate EF1 fails to hold for any \( \varepsilon > 0 \). As a result, the set of items taken by the super agent must correspond to an independent set in \( G \).

By making \( \delta \) sufficiently small, the social welfare almost exclusively depends on the super agent’s utility. Theorem E.7 follows easily from Theorem 6.5 and the fact \( m = O(n^2) \).

Theorem E.8. Fix any small \( \varepsilon > 0 \). If there exists a \((n^{3-\varepsilon}, 0.5 + \varepsilon)\)-bi-criteria approximation algorithm for MSWwithinEFX, then \( P = NP \).

Proof. The same reduction in the proof of Theorem 5.7 can be used here. The analysis of the yes-instance is exactly the same. The analysis of the no-instance is almost the same except for the following differences. Firstly, if agent \( a_0 \) gets both \( u_i \) and \( v_i \) for an edge \( j = (u_i, u_j) \), then the fact agent \( a_0 \) cannot get more than these two items (that we have proved in Theorem 5.7) still follows from \((0.5+\varepsilon)\)-EFX requirement. This is where the parameter \( 0.5+\varepsilon \) in the theorem statement comes from. Secondly, we will use the inapproximability result of the independent set problem described
in Theorem 6.5. We need that, for a yes-instance, $G$ has an independent set of size $x$, and, for a no-instance, $G$ has an independent set of size less than $0.5x$. We need a factor 2 inapproximability result of the independent set problem, while Theorem 6.5 says that the inapproximability ratio is more than 2. □

F PRICES OF EFX AND EF1

The price of fairness measures the loss in the social welfare when a fairness constraint is imposed.

**Definition F.1.** Given a valuation profile $(v_1, \ldots, v_n)$, let $MSW(v_1, \ldots, v_n) = \sum_{i=1}^{m} \max_{j \in [n]} v_i(\{j\})$ be the maximum social welfare among all allocations (without any fairness constraint). We simply write $MSW$ when the valuation profile is clear from the context.

**Definition F.2.** The price of EFX is defined by $\sup (v_1, \ldots, v_n) MSW(v_1, \ldots, v_n) / SW(A^*)$, where $A^*$ is an EFX allocation (which is allowed to be partial) with maximum social welfare.

**Definition F.3.** The price of EF1 is defined by $\sup (v_1, \ldots, v_n) MSW(v_1, \ldots, v_n) / SW(A^*)$, where $A^*$ is an EF1 allocation with maximum social welfare.

Under normalized valuations, it is known that the price of EF1 is between $2/\sqrt{3}$ and $8/7$ for $n = 2$ [Bei et al., 2021b] and is $\Theta(\sqrt{n})$ for general $n$ [Barman et al., 2020, Bei et al., 2021b]. For the price of EFX under normalized valuations, it is 1.5 for two agents [Bei et al., 2021b] and is unknown for general $n$.

Under unnormalized valuations, the prices of both EF1 and EFX are large: $n$ is a trivial lower bound.

**Theorem F.4.** For unnormalized valuations, the prices of both EF1 and EFX are at least $n$.

**Proof.** For an arbitrary $n$, consider $n$ agents and $n$ items where agent 1 has value 1 on all the items and each of agents 2, \ldots, $n$ has value $\frac{\varepsilon}{n-1}$ on all the items. $MSW = n$ for the allocation where agent 1 gets all the items. On the other hand, an EF1/EFX allocation must allocate each agent one item, which has social welfare $1 + \varepsilon$. □

Our results in Sect. 5 also make the price of EFX clear.

**Theorem F.5.** The price of EFX is $\Theta(n)$.

**Proof.** Theorem 5.2 implies the price of EFX is $O(n)$, as $\sum_{i=1}^{m} v_i([m])$ is a trivial upper bound to $MSW$. Theorem F.4 then implies this theorem. □

**Theorem F.6.** The price of EFX is $\Theta(\sqrt{n})$ for normalized valuations.

**Proof.** Theorem 5.3 implies the price of EFX is $O(\sqrt{n})$. Since EFX is a stronger notion than EF1, the lower bound $\Omega(\sqrt{n})$ for the price of EF1 by Bei et al. [2021b] can be directly applied here. □

Theorem 6.6 and Theorem F.4 immediately imply the following theorem.

**Theorem F.7.** The price of EF1 is exactly $n$. 