Mean Value from Representation of Rational Number as Sum of Two Egyptian Fractions

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Abstract. For given positive integers $n$ and $a$, let $R(n; a)$ denote the number of positive integer solutions $(x, y)$ of the Diophantine equation

$$\frac{a}{n} = \frac{1}{x} + \frac{1}{y}.$$ 

Write

$$S(N; a) = \sum_{n \leq N, (n, a)=1} R(n; a).$$

Recently Jingjing Huang and R. C. Vaughan proved that for $4 \leq N$ and $a \leq 2N$, there is an asymptotic formula

$$S(N; a) = \frac{3}{\pi^2 a} \prod_{p | a} \frac{p-1}{p+1} N (\log^2 N + c_1(a) \log N + c_0(a)) + \Delta(N; a).$$

In this paper, we shall get a more explicit expression with better error term for $c_0(a)$.

1. Introduction

Representation of rational number as sum of unit fractions, or Egyptian fractions, is an interesting topic in number theory. For its history and related problems, one can see R. K. Guy’s book[1].

Recently Jingjing Huang and R. C. Vaughan[2] studied the representation of rational number as sum of two Egyptian fractions. They established two mean value theorems, one of which is

**Proposition 1.** For given positive integers $n$ and $a$, let $R(n; a)$ denote the number of positive integer solutions $(x, y)$ of the Diophantine equation

$$\frac{a}{n} = \frac{1}{x} + \frac{1}{y}.$$
Write
\[ S(N; a) = \sum_{n \leq N, (n, a) = 1} R(n; a). \] (1)

Then for \(4 \leq N\) and \(a \leq 2N\), there is an asymptotic formula
\[ S(N; a) = \frac{3}{\pi^2 a} \prod_{p|a} \frac{p - 1}{p + 1} \cdot N(\log^2 N - c_1(a) \log N + c_0(a)) + \Delta(N; a), \]
where
\[ c_1(a) = 6 \gamma - 4 \frac{\zeta'(2)}{\zeta(2)} - 2 + \sum_{p|a} \frac{6p + 2}{p^2 - 1} \cdot \log p \] (2)
and
\[ c_0(a) = -2 \log^2 a - 4 \log a \sum_{p|a} \frac{\log p}{p - 1} + O\left( \frac{a}{\varphi(a)} \cdot \log a \right), \] (3)
and
\[ \Delta(N; a) \ll N^{\frac{1}{2}} \log^5 N \cdot \frac{a}{\varphi(a)} \prod_{p|a} \left( 1 - \frac{1}{p^2} \right)^{-1}. \] (4)

Here \(p\) denotes prime number, \(\gamma\) is the Euler constant and \(\varphi(a)\) is the Euler totient function.

In this paper, we shall apply results in [3] and [4] to get a more explicit expression with better error term for \(c_0(a)\) in (3). We shall prove

**Theorem.** Let \(S(N; a)\) be defined in (1). Then for \(4 \leq N\) and \(3 \leq a \leq 2N\), we have
\[ S(N; a) = \frac{3}{\pi^2 a} \prod_{p|a} \frac{p - 1}{p + 1} \cdot N(\log^2 N + c_1(a) \log N + c_0(a)) + \Delta(N; a), \]
where \(c_1(a)\) and \(\Delta(N; a)\) are same as in (2) and (4), while
\[ c_0(a) = -2 \log^2 a - 4 \log a \sum_{p|a} \frac{\log p}{p - 1} - 2 \left( \sum_{p|a} \frac{\log p}{p - 1} \right)^2 \]
\[ + \left( \sum_{p|a} \frac{3p + 1}{p^2 - 1} \cdot \log p \right)^2 - \sum_{p|a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p \]
\[ \begin{align*}
&\left(6\gamma - 4\frac{\zeta''(2)}{\zeta(2)} - 2\right) \sum_{p|a} \frac{3p + 1}{p^2 - 1} \cdot \log p \\
&+ \frac{2}{\zeta^2(2)}(2\zeta'(2) + \zeta(2))^2 - \frac{4}{\zeta(2)}(\zeta''(2) + \zeta'(2)) \\
&- 6\gamma \left(\frac{2\zeta'(2)}{\zeta(2)} + 1\right) + 8\gamma^2 - 2\gamma_1 + 2\zeta(2) + O\left(\frac{1}{\phi(a)}\right).
\end{align*} \]

Here \( p \) denotes prime number, \( \phi(a) \) is the Euler totient function, \( \gamma \) is the Euler constant and \( \gamma_1 \) is defined in (6).

2. Preliminaries

**Proposition 2.** For the given integer \( a \geq 3 \), let \( \chi \) be a Dirichlet character \( \mod a \) and \( \chi_0 \) denote the principal character. Then we have

\[ \sum_{\chi(\mod a) \neq \chi_0} |L(1, \chi)|^2 = \zeta(2) \phi(a) \sum_{p|a} \left(1 - \frac{1}{p^2}\right) - \frac{\phi^2(a)}{a^2} \left(\log a + \sum_{p|a} \frac{\log p}{p - 1}\right)^2 \]

\[ + \frac{\phi^2(a)}{a^2} \left(\gamma^2 + 2\gamma_1 - 2\zeta(2)\right) + O\left(\frac{\phi(a)}{a^2}\right), \]

where \( p \) denotes prime number, \( \phi(a) \) is the Euler totient function, \( \gamma \) is the Euler constant and \( \gamma_1 \) is defined in (6).

This is Theorem 1 in [3].

**Proposition 3.** For the given integer \( a \geq 3 \), we have

\[ \sum_{\chi(\mod a) \neq \chi_0} |L(1, \chi)|^2 = \frac{\pi^2}{12} \cdot \frac{\phi^2(a)}{a^2} \left(a \prod_{p|a} \left(1 + \frac{1}{p}\right) - 3\right). \]

One can see Theorem A in page 440 of [4].

**Lemma 1.** In the neighborhood of \( s = 1 \), there is Laurent expansion

\[ \zeta^3(s) = \frac{1}{(s - 1)^3} + \frac{3\gamma}{(s - 1)^2} + \frac{3\gamma^2 - 3\gamma_1}{s - 1} + \cdots, \]

where \( \gamma \) is the Euler constant and \( \gamma_1 \) is defined in (6).

**Proof.** We know

\[ \zeta(s) = \frac{1}{s - 1} + \gamma - \gamma_1(s - 1) + \cdots. \]
Hence,
\[
\zeta^3(s) = \left( \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots \right)^2 \left( \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots \right)
\]
\[
= \left( \frac{1}{(s-1)^2} + \gamma^2 + \frac{2\gamma}{s-1} - 2\gamma_1 + \cdots \right)
\cdot \left( \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots \right)
\]
\[
= \left( \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + (\gamma^2 - 2\gamma_1) + \cdots \right)
\cdot \left( \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots \right)
\]
\[
= \frac{1}{(s-1)^2} + \frac{\gamma}{(s-1)^2} - \frac{\gamma_1}{s-1} + \frac{2\gamma}{(s-1)^2} + \frac{2\gamma^2}{s-1} + \frac{\gamma^2 - 2\gamma_1}{s-1} + \cdots
\]
\[
= \frac{1}{(s-1)^2} + \frac{3\gamma}{(s-1)^2} + \frac{\gamma^2 - 3\gamma_1}{s-1} + \cdots.
\]

**Lemma 2.** If \(a \geq 3\), then
\[
\sum_{p|a} \frac{\log p}{p-1} \ll \log \log a.
\]

**Proof.** It is enough to prove for sufficiently large \(a\). When \(x \geq \log a\), the function \(\frac{\log x}{x}\) decreases monotonously. Thus
\[
\sum_{p|a} \frac{\log p}{p-1} \ll \sum_{p|a} \frac{\log p}{p}
\]
\[
= \sum_{p|a \quad p \leq \log a} \frac{\log p}{p} + \sum_{p|a \quad \log a < p} \frac{\log p}{p}
\]
\[
\leq \sum_{p \leq \log a} \frac{\log p}{p} + \frac{\log \log a}{\log a} \sum_{p|a} 1
\]
\[
\ll \log \log a + \frac{\log \log a}{\log a} \cdot \log a
\]
\[
\ll \log \log a.
\]

**Lemma 3.** If \(a \geq 3\), then
\[
\sum_{p|a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p \ll \log \log^2 a.
\]

**Proof.** Assume that \(a\) is sufficiently large. When \(x \geq \log a\), the function
\[ \frac{\log^2 x}{x} \text{ decreases monotonously. Thus} \]
\[
\sum_{p \mid a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p \ll \sum_{p \mid a} \frac{\log^2 p}{p}
\]
\[
= \sum_{p \mid a} \frac{\log^2 p}{p} + \sum_{p \mid a} \frac{\log^2 p}{p}
\]
\[
\leq \log \log a \sum_{p \leq \log a} \frac{\log p}{p} + \log \log^2 a \sum_{p \mid a} 1
\]
\[
\ll \log \log^2 a.
\]

3. The proof of Theorem

According to the discussion in [2], we have

\[ S(N; a) = \frac{1}{\varphi(a)} \sum_{\chi \mod a} \chi(-1) \text{Res}_{s=1} \left( f_{\chi}(s) \frac{N^s}{s} \right) + \Delta(N; a), \quad (7) \]

where

\[ f_{\chi}(s) = \frac{L(s, \chi \chi_0)}{L(2s, \chi \chi_0)} \cdot L(s, \chi) L(s, \bar{\chi}) \quad (8) \]

and

\[ \Delta(N; a) \ll N^{\frac{1}{2}} \log^5 N \cdot \frac{a}{\varphi(a)} \prod_{p | a} \left( 1 - \frac{1}{p^2} \right)^{-1}. \]

Write

\[ S_1(N; a) = \frac{1}{\varphi(a)} \sum_{\chi \mod a} \chi(-1) \text{Res}_{s=1} \left( f_{\chi}(s) \frac{N^s}{s} \right) \quad (9) \]

and

\[ S_2(N; a) = \frac{1}{\varphi(a)} \text{Res}_{s=1} \left( f_{\chi_0}(s) \frac{N^s}{s} \right). \quad (10) \]

The discussion in [2] shows that when \( \chi \neq \chi_0 \),

\[
\text{Res}_{s=1} \left( f_{\chi}(s) \frac{N^s}{s} \right) = \text{Res}_{s=1} \left( \frac{L(s, \chi \chi_0)L(s, \chi)L(s, \bar{\chi})N^s}{L(2s, \chi_0)s} \right)
\]
\[
= \frac{6N}{\pi^2} \prod_{p | a} \frac{p}{p + 1} \cdot |L(1, \chi)|^2.
\]
Hence,
\[ S_1(N; a) = \frac{6N}{\pi^2} \cdot \frac{1}{\varphi(a)} \prod_{p|a} \frac{p}{p+1} \cdot \sum_{\substack{\chi \mod a \atop \chi \neq \chi_0}} \chi(-1)|L(1, \chi)|^2. \]

Since \( \chi(-1) = 1 \) or \(-1\), by Propositions 2 and 3, we get
\[
\sum_{\substack{\chi \mod a \atop \chi \neq \chi_0}} \chi(-1)|L(1, \chi)|^2
\]
\[
= \sum_{\substack{\chi \mod a \atop \chi \neq \chi_0 \atop \chi(-1) = 1}} |L(1, \chi)|^2
- \sum_{\substack{\chi \mod a \atop \chi \neq \chi_0 \atop \chi(-1) = -1}} |L(1, \chi)|^2
\]
\[
= \sum_{\substack{\chi \mod a \atop \chi \neq \chi_0 \atop \chi(-1) = -1}} |L(1, \chi)|^2
- 2 \sum_{\substack{\chi \mod a \atop \chi \neq \chi_0 \atop \chi(-1) = -1}} |L(1, \chi)|^2
\]
\[
= \zeta(2)\varphi(a) \prod_{p|a} \left(1 - \frac{1}{p^2}\right) - \frac{\varphi^2(a)}{a^2} \left(\log a + \sum_{p|a} \frac{\log p}{p-1}\right)^2
\]
\[
+ \frac{\varphi^2(a)}{a^2} \left(\gamma^2 + 2\gamma_1 - 2\zeta(2)\right) + O\left(\frac{\varphi(a)}{a^2}\right)
\]
\[
- \zeta(2) \cdot \frac{\varphi^2(a)}{a^2} \left(a \prod_{p|a} \left(1 + \frac{1}{p}\right) - 3\right)
\]
\[
= \zeta(2)\varphi(a) \prod_{p|a} \left(1 - \frac{1}{p^2}\right) - \frac{\varphi^2(a)}{a^2} \left(\log a + \sum_{p|a} \frac{\log p}{p-1}\right)^2
\]
\[
+ \frac{\varphi^2(a)}{a^2} \left(\gamma^2 + 2\gamma_1 - 2\zeta(2)\right) + O\left(\frac{\varphi(a)}{a^2}\right)
\]
\[
- \zeta(2)\varphi(a) \prod_{p|a} \left(1 - \frac{1}{p^2}\right) + 3\zeta(2) \cdot \frac{\varphi^2(a)}{a^2}
\]
\[
= - \frac{\varphi^2(a)}{a^2} \left(\log a + \sum_{p|a} \frac{\log p}{p-1}\right)^2 + \frac{\varphi^2(a)}{a^2} \left(\gamma^2 + 2\gamma_1 + \zeta(2)\right) + O\left(\frac{\varphi(a)}{a^2}\right).
\]

Hence,
\[
S_1(N; a) = \frac{6N}{\pi^2} \cdot \frac{\varphi(a)}{a^2} \prod_{p|a} \frac{p}{p+1} \cdot \left( -\left(\log a + \sum_{p|a} \frac{\log p}{p-1}\right)^2
\right)
\]
\[
+ \left(\gamma^2 + 2\gamma_1 + \zeta(2)\right) + O\left(\frac{1}{\varphi(a)}\right)
\]

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\[
= \frac{3}{\pi^2 a} \prod_{p|a} \frac{p-1}{p+1} \cdot N \left( -2 \log^2 a - 4 \log a \sum_{p|a} \frac{\log p}{p-1} \right.

\left. - 2 \left( \sum_{p|a} \frac{\log p}{p-1} \right)^2 + 2 \gamma^2 + 4 \gamma_1 + 2 \zeta(2) + O \left( \frac{1}{\varphi(a)} \right) \right).
\]

Now we proceed to calculate
\[
\text{Res}_{s=1} \left( f_{\chi_0}(s) \frac{N^s}{s} \right).
\]

The discussion in [2] yields
\[
f_{\chi_0}(s) = \frac{L^3(s, \chi_0)}{L(2s, \chi_0)} = \frac{\zeta^3(s)}{\zeta(2s)} \prod_{p|a} \frac{(p^s - 1)^2}{p^s(p^s + 1)} = \frac{\zeta^3(s)}{\zeta(2s)} \cdot G(s), \quad (11)
\]

where
\[
G(s) = \prod_{p|a} \frac{(p^s - 1)^2}{p^s(p^s + 1)}. \quad (12)
\]

By Lemma 1,
\[
\text{Res}_{s=1} \left( f_{\chi_0}(s) \frac{N^s}{s} \right) = \frac{N^s}{\zeta(2s)} \cdot G(s).
\]

We shall calculate these residues respectively.

1. It is easy to see
\[
(3\gamma^2 - 3\gamma_1) \text{Res}_{s=1} \frac{1}{s-1} \cdot \frac{N^s}{\zeta(2s)} \cdot G(s) = N \cdot \frac{G(1)}{\zeta(2)} \cdot (3\gamma^2 - 3\gamma_1).
\]
2. We have

\[
3\gamma \text{Res}_{s=1} \frac{1}{(s-1)^2} \cdot \frac{N^s}{\zeta(2s)} \cdot G(s)
\]

\[
= 3\gamma \left( \left. \frac{N^s}{\zeta(2s)} \cdot G(s) \right|_{s=1} \right)
\]

\[
= 3\gamma \left( \left. \left( \frac{N^s}{\zeta(2s)} \right)' \cdot G(s) + \frac{N^s}{\zeta(2s)} \cdot G'(s) \right|_{s=1} \right)
\]

\[
= 3\gamma \left( \left. \frac{N^s \log N \cdot \zeta(2s) - N^s \left( \zeta'(2s) \cdot 2s + \zeta(2s) \right)}{(\zeta(2s))^2} \right|_{s=1} \right) \cdot G(s)
\]

\[
+ \left. \frac{N^s}{\zeta(2s)} \cdot G'(s) \right|_{s=1}
\]

\[
= 3\gamma G(1) \left( \frac{N \log N \cdot \zeta(2) - N \left( 2\zeta'(2) + \zeta(2) \right)}{\zeta^2(2)} + \frac{N \cdot G'(1)}{\zeta(2) \cdot G(1)} \right)
\]

\[
= N \cdot \frac{G(1)}{\zeta(2)} \cdot 3\gamma \left( \frac{\zeta(2) \log N - 2\zeta'(2) + \zeta(2)}{\zeta(2)} + \frac{G'(1)}{G(1)} \right)
\]

\[
= N \cdot \frac{G(1)}{\zeta(2)} \cdot \left( 3\gamma \log N - 3\gamma \left( 2\zeta'(2) + 1 \right) + 3\gamma \cdot \frac{G'(1)}{G(1)} \right).
\]

The derivative of \( \log G(s) \) is

\[
\frac{G'(s)}{G(s)} = \sum_{p|a} \left( 2 \log(p^s - 1) - s \log p - \log(p^s + 1) \right)'
\]

\[
= \sum_{p|a} \left( 2 \frac{2}{p^s - 1} \cdot p^s \log p - \log p - \frac{1}{p^s + 1} \cdot p^s \log p \right)
\]

\[
= \sum_{p|a} \left( 2 \frac{2}{p^s - 1} + \frac{1}{p^s + 1} \right) \log p.
\]

Hence,

\[
\frac{G'(1)}{G(1)} = \sum_{p|a} \left( 2 \frac{2}{p - 1} + \frac{1}{p + 1} \right) \log p = \sum_{p|a} \frac{3p + 1}{p^2 - 1} \cdot \log p,
\]

which is a formula in page 1652 of [2].

3. We have

\[
\text{Res}_{s=1} \frac{1}{(s-1)^3} \cdot \frac{N^s}{\zeta(2s)} \cdot G(s)
\]

\[
= \left. \frac{1}{2!} \left( \frac{N^s}{\zeta(2s)} \cdot G(s) \right)' \right|_{s=1}
\]
We shall calculate these expressions respectively.

a) Since

\[
\frac{N^s}{\zeta(2s)s} = \frac{N^s \log N \cdot \zeta(2s)s - N^s(\zeta'(2s) \cdot 2s + \zeta(2s))}{\zeta(2s)s^2} \]

We shall calculate these expressions respectively.

\[
\frac{N^s}{\zeta(2s)s}'' = \frac{N^s \log N \cdot \zeta(2s)s - N^s(\zeta'(2s) \cdot 2s + \zeta(2s))}{\zeta(2s)s^2}
\]

\[
= \frac{1}{2} \cdot \left( \frac{N^s}{\zeta(2s)s}'' \cdot G(s) + 2 \left( \frac{N^s}{\zeta(2s)s}' \cdot G'(s) + \frac{N^s}{\zeta(2s)s} \cdot G''(s) \right) \right)_{s=1}
\]

\[
= \frac{1}{2} \cdot \left( \frac{N^s}{\zeta(2s)s}'' \cdot G(s) \right)_{s=1} + \frac{1}{2} \cdot \frac{N^s}{\zeta(2s)s} \cdot G''(s)_{s=1}
\]

We shall calculate these expressions respectively.
b) We have

\[
\left. \left( \frac{N^s}{\zeta(2s)} \right) \right|_{s=1} ^{'} \cdot G'(s) = N^s \log N \cdot \zeta(2s) - N^s (\zeta'(2s) \cdot 2s + \zeta(2s)) \cdot G'(s) \] 
\[
= \frac{N \log N \cdot \zeta(2) - N (2\zeta'(2) + \zeta(2))}{\zeta^2(2)} \cdot G'(1)
\]
\[
= N \cdot \frac{G(1)}{\zeta(2)} \cdot \left( \log N - 2 \frac{\zeta'(2)}{\zeta(2)} - 1 \right) \cdot \frac{G'(1)}{G(1)}.
\]

c) Differentiating the equality in (13)

\[
G'(s) = G(s) \sum_{p|a} \left( \frac{2}{p^s - 1} + \frac{1}{p^s + 1} \right) \log p,
\]

we get

\[
G''(s) = G'(s) \sum_{p|a} \left( \frac{2}{p^s - 1} + \frac{1}{p^s + 1} \right) \log p
\]
\[
+ G(s) \sum_{p|a} \left( \frac{-2ps \log p}{(p^s - 1)^2} - \frac{ps \log p}{(p^s + 1)^2} \right) \log p
\]
\[
= G(s) \left( \frac{G'(s)}{G(s)} \sum_{p|a} \left( \frac{2}{p^s - 1} + \frac{1}{p^s + 1} \right) \log p
\]
\[
- \sum_{p|a} \left( \frac{2}{(p^s - 1)^2} + \frac{1}{(p^s + 1)^2} \right) p^s \log^2 p \right).
\]

Therefore

\[
G''(1) = G(1) \left( \frac{G'(1)}{G(1)} \sum_{p|a} \left( \frac{2}{p - 1} + \frac{1}{p + 1} \right) \log p
\]
\[
- \sum_{p|a} \left( \frac{2}{(p - 1)^2} + \frac{1}{(p + 1)^2} \right) p \log^2 p \right)
\]
\[
= G(1) \left( \left( \frac{G'(1)}{G(1)} \right)^2 - \sum_{p|a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p \right),
\]

which is a formula in page 1652 of [2].
We have
\[
\frac{1}{2} \cdot \frac{N^s}{\zeta(2s)} \cdot G''(s) \bigg|_{s=1} = N \cdot \frac{G'(1)}{\zeta(2)} \cdot \left( \frac{1}{2} \left( \frac{G'(1)}{G(1)} \right)^2 - \frac{1}{2} \sum_{p|a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p \right).
\]
Combining results in cases a), b) and c), we get
\[
\text{Res}_{s=1} \frac{1}{(s-1)^3} \cdot \frac{N^s}{\zeta(2s)} \cdot G(s)
= \left( \frac{G'(1)}{G(1)} \right)^2 \cdot \left( \frac{1}{2} \log^2 N - \frac{1}{\zeta(2)} (2\zeta'(2) + \zeta(2)) \log N + \frac{1}{\zeta^2(2)} (2\zeta'(2) + \zeta(2))^2 - \frac{2}{\zeta(2)} (\zeta''(2) + \zeta'(2)) \right)
+ \left( \frac{G'(1)}{G(1)} \right) \cdot \log N - \left( \frac{2\zeta'(2)}{\zeta(2)} + 1 \right) \cdot \frac{G'(1)}{G(1)}
+ \left( \frac{1}{2} \left( \frac{G'(1)}{G(1)} \right)^2 - \frac{1}{2} \sum_{p|a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p \right)
\]
\[
= N \cdot \frac{G'(1)}{\zeta(2)} \cdot \left( \frac{1}{2} \log^2 N - \frac{1}{\zeta(2)} (2\zeta'(2) + \zeta(2)) \log N + \frac{G'(1)}{G(1)} \cdot \log N + \frac{1}{2} \left( \frac{G'(1)}{G(1)} \right)^2 \right)
- \frac{1}{2} \sum_{p|a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p - \left( 2\zeta'(2) + 1 \right) \cdot \frac{G'(1)}{G(1)}
+ \frac{1}{\zeta^2(2)} (2\zeta'(2) + \zeta(2))^2 - \frac{2}{\zeta(2)} (\zeta''(2) + \zeta'(2)) \right).
\]
Now we can get from cases 1, 2 and 3 that
\[
\text{Res}_{s=1} \left( f_{\chi_0}(s) \frac{N^s}{s} \right)
= N \cdot \frac{G(1)}{\zeta(2)} \cdot \left( \frac{1}{2} \log^2 N - \frac{1}{\zeta(2)} (2\zeta'(2) + \zeta(2)) \log N + \frac{G'(1)}{G(1)} \cdot \log N + \frac{1}{2} \left( \frac{G'(1)}{G(1)} \right)^2 \right)
\]
\[- \frac{1}{2} \sum_{p|a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p - \left(2 \frac{\zeta'(2)}{\zeta(2)} + 1 \right) \cdot \frac{G'(1)}{G(1)} \]
\[+ \frac{1}{\zeta^2(2)} (2\zeta'(2) + \zeta(2))^2 - \frac{2}{\zeta(2)} (\zeta''(2) + \zeta'(2)) \]
\[+ N \cdot \frac{G(1)}{\zeta(2)} \left(3\gamma \log N - 3\gamma \left(2 \frac{\zeta'(2)}{\zeta(2)} + 1 \right) + 3\gamma \cdot \frac{G'(1)}{G(1)} \right) \]
\[+ N \cdot \frac{G(1)}{\zeta(2)} \cdot (3\gamma^2 - 3\gamma_1) \]
\[= N \cdot \frac{G(1)}{\zeta(2)} \left(\frac{1}{2} \log^2 N + 3\gamma \log N \right) \]
\[\frac{1}{\zeta(2)} (2\zeta'(2) + \zeta(2)) \log N + \frac{G'(1)}{G(1)} \cdot \log N \]
\[+ \frac{1}{2} \left( \frac{G'(1)}{G(1)} \right)^2 - \frac{1}{2} \sum_{p|a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p \]
\[+ \frac{3\gamma - 2 \frac{\zeta'(2)}{\zeta(2)} - 1}{\zeta(2)} \cdot \frac{G'(1)}{G(1)} + \frac{1}{\zeta^2(2)} (2\zeta'(2) + \zeta(2))^2 \]
\[- \frac{2}{\zeta(2)} (\zeta''(2) + \zeta'(2)) - 3\gamma \left(2 \frac{\zeta'(2)}{\zeta(2)} + 1 \right) + 3\gamma^2 - 3\gamma_1 \].

Since
\[
\frac{1}{\varphi(a)} \cdot \frac{G(1)}{\zeta(2)} = \frac{1}{\varphi(a)} \prod_{p|a} \frac{p}{p-1} \cdot \frac{6}{\pi^2} \prod_{p|a} \frac{(p-1)^2}{p(p+1)}
\]
+ \frac{2}{\zeta^2(2)}(2\zeta'(2) + \zeta(2))^2 - \frac{4}{\zeta(2)}(\zeta''(2) + \zeta'(2))
- 6\gamma\left(\frac{2\zeta'(2)}{\zeta(2)} + 1\right) + 6\gamma^2 - 6\gamma_1).

Therefore

S_1(N; a) + S_2(N; a)
= \frac{3}{\pi^2 a} \prod_{p \mid a} \frac{p - 1}{p + 1} \cdot N \left(-2 \log^2 a - 4 \log a \sum_{p \mid a} \frac{\log p}{p - 1}
- 2 \left(\sum_{p \mid a} \frac{\log p}{p - 1}\right)^2 + 2\gamma^2
+ 6 \gamma \left(\frac{2\zeta'(2)}{\zeta(2)} + 1\right) + 6\gamma^2 - 6\gamma_1\right)
+ \frac{3}{\pi^2 a} \prod_{p \mid a} \frac{p - 1}{p + 1} \cdot N \left(\log^2 N + \left(6\gamma - \frac{4\zeta'(2)}{\zeta(2)} - 2\right) \log N
+ \left(\sum_{p \mid a} \frac{6p + 2}{p^2 - 1} \cdot \log p\right) \log N
+ \left(\sum_{p \mid a} \frac{3p + 1}{p^2 - 1} \cdot \log p\right)^2
- \sum_{p \mid a} \frac{3p^3 + 2p^2 + 3p}{(p^2 - 1)^2} \cdot \log^2 p
+ \left(6\gamma - \frac{4\zeta'(2)}{\zeta(2)} - 2\right) \sum_{p \mid a} \frac{3p + 1}{p^2 - 1} \cdot \log p
+ \frac{2}{\zeta^2(2)}(2\zeta'(2) + \zeta(2))^2 - \frac{4}{\zeta(2)}(\zeta''(2) + \zeta'(2))\right)
where $c_1(a)$ and $c_0(a)$ are defined in (2) and (5). By Lemmas 2 and 3, we can see that terms of $c_0(a)$ in (5) are arranged in decreasing order of $a$.

So far the proof of Theorem is finished.

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