Level Crossing Probabilities II: Polygonal Recurrence of Multidimensional Random Walks

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Abstract

In part I we proved for an arbitrary one-dimensional random walk with independent increments that the probability of crossing a level at a given time $n$ is $O(n^{-1/2})$. In higher dimensions we call a random walk ‘polygonally recurrent’ if there is a bounded set, hit by infinitely many of the straight lines between two consecutive sites a.s. The above estimate implies that three-dimensional random walks with independent components are polygonally transient. Similarly a directionally reinforced random walk on $\mathbb{Z}^3$ in the sense of Mauldin, Monticino and v. Weizsäcker [1] is transient. On the other hand we construct an example of a transient but polygonally recurrent random walk with independent components on $\mathbb{Z}^2$.

1 Introduction

This is a continuation of the paper [2] which gave a $O(n^{-1/2})$ bound for the level crossing probabilities of an arbitrary one-dimensional random walk. We want to apply this result to study ‘polygonal’ transience and recurrence in higher dimensions and to directionally reinforced random walks in the sense of [1].

Definition 1 Let $(S_n)$ be a random walk in $\mathbb{Z}^d$ or $\mathbb{R}^d$. We call $(S_n)$ polygonally recurrent (resp. polygonally transient) if there is a bounded set $B$ (resp. there is no bounded set) such that a.s. there are infinitely many $n$ with the straight line between $S_n, S_{n+1}$ hitting $B$. 
A priori polygonal recurrence is a weaker statement than classical recurrence, e.g. in one dimension every symmetric nontrivial random walk oscillates between arbitrarily high negative and positive values and hence is polygonally recurrent even if it is classically transient. In higher dimensions it is less clear whether the two concepts really differ.

In three dimensions every (truly three-dimensional) random walk is transient. If the components are independent then we get as a straightforward consequence of our $O(n^{-1/2})$ estimate

**Theorem 1** A three-dimensional random walk whose three components are independent is polygonally transient.

We want to extend this result to the following situation, referred to as "directionally reinforced random walk" in [1]: Let a particle move around in $\mathbb{Z}^d$ or $\mathbb{R}^d$. Assume that the particle moves with a constant velocity along straight lines which are parallel to the coordinate axes, keeping its direction of motion $\pm e_k, k \in \{1, \ldots, d\}$ for a nonnegative finite random time $T$ with $P(T > 0) > 0$ (in contrast to [1] we do not require this random time to be strictly positive, cf. the first remark after Theorem 4), then changing to a different direction which is chosen by equal chance among the $2^d - 1$ possibilities. This choice and the random time spent to move in this new direction are assumed to be completely independent of the past of the motion process. This process is, of course, in general not a Markov process but, assuming that the first direction is fixed, the successive locations of change into the first direction form a truly $d-$dimensional random walk embedded in our process. In [1] it was conjectured that in dimension 3 this scheme is always transient in the sense that any bounded set is visited only finitely often a.s. and that in dimension 2 the scheme is transient if the embedded random walk is transient. (It is not difficult to see that for $d = 1$ we have always recurrence and for $d > 3$ always transience, [1], th. 3.1 and end of section 3). We prove in section 2 the transience conjecture for $d = 3$ using the above $O(n^{-1/2})$-bound.

However, we give in section 3 a somewhat involved example in 2 dimensions of a directionally reinforced random walk which is recurrent whereas the embedded random walk is transient but polygonally recurrent. Thus the level crossing probabilities can be sufficiently higher than the return probabilities to change a transience statement into recurrence.
2 Transience in three dimensions

Let us first give the simple

Proof of Theorem 1. Let $A_n$ be the event that the straight line between $S_n, S_{n+1}$ hits $[-1,1]^3$. Then by our independence assumption

$$P(A_n) \leq P(A_1^n)P(A_2^n)P(A_3^n)$$

where $A_i^n$ denotes for $i = 1, 2, 3$ the event that the interval with the endpoints $S_i^n, S_{i+1}^n$ meets $[-1,1]$, since $A_n$ implies each of the $A_i^n$. Clearly if $A_i^n$ occurs then either $S_i^n \in [-1,1]$ or the the random walk $(S^n)$ crosses at time $n$ at least one of the levels $-1$ or $1$. Both events have probability $O(n^{-1/2})$ by [3], p.72 (for the $Z$-case) resp. [4], Theorem 1 (for $R$) and [2], Theorem 2. Hence $P(A_1^n) = O(n^{-1/2})$ and $P(A_n) = O(n^{-3/2})$ which implies the result by Borel-Cantelli.

Consider now the model of $d$-dimensional directionally reinforced random walk as it was defined in the introduction. We have, as a consequence of the estimate for the probability of sign changes in symmetric one-dimensional random walks, the transience of three-dimensional directionally reinforced random walks. In [1] this was shown only under the assumption that the waiting time between changes of direction has a finite expectation and only for $d \geq 4$ without this moment condition.

Theorem 2 For any dimension $d \geq 3$ the $d$-dimensional directionally reinforced random walks are always transient in the sense that bounded sets are visited only finitely often a.s.

Proof: 1. Let us first modify the model to make the problem easier. Assume that, when changing the direction of the travelling object, the next direction is not chosen by equal chance from the $2d - 1$ possible values which are different from the previous one, but only from the $2d - 2$ perpendicular directions. We want to prove that a bounded set is visited only finitely often. Fix a coordinate axis and call it 'vertical', and the others 'horizontal'. It is sufficient to show that the cube $[-1,1]^d$ is penetrated or touched only finitely often, coming from vertical direction (up or down). Consider those times when the particle changes from a horizontal to vertical direction, or vice versa. Considering only these times, the particle constantly changes from an independent symmetric random walk in vertical direction ($R^1$) to a horizontal symmetric and independent random walk ($R^{d-1}$). Hitting the cube in the assumed way means that the
The $\mathbb{R}^{d-1}$ random walk is just in the cube $[-1,1]^{d-1}$, whereas the $\mathbb{R}^1$ random walk crosses the levels 1 or $-1$ or is in $[-1,1]$.

We have shown in [2], Theorem 2, that the probability of the second event is $O(n^{-\frac{d}{2}})$. The first event concerns a genuinely $d-1$-dimensional random walk and has a probability $O(n^{-(d-1)/2})$. To see this we apply Theorem 3 of [5] which gives an estimate for the maximum probability of multi-dimensional rectangular domains with respect to the probability law of a sum of independent random vectors. In our case this estimate reads as

$$P(S_n = X_1 + X_2 + ... X_n \in [-1,1]^{d-1})$$

$$\leq C(\lambda) \left( 1 - \sup_{x \in \mathbb{R}^{d-1}} P(X_1 \in D_\lambda + x) \right)^{-\frac{d-1}{2}} n^{-\frac{d-1}{2}},$$

for any $\lambda \leq 1$, where $D_\lambda := \{y = (y_1, y_2, ..., y_{d-1}) \in \mathbb{R}^{d-1} : |y_j| \leq \lambda \text{ for at least one } j \in \{1, 2, ..., d-1\}\}$. This result is shown under a symmetry condition (S) meaning in our case that the law of $X_1 - X_2$ is invariant under any combined reflection of the coordinate axes, i.e. under any application of a diagonal matrix of the form

$$\begin{pmatrix}
\pm1 & 0 & \cdots & 0 \\
0 & \pm1 & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & \pm1
\end{pmatrix}.$$  

This is obviously fulfilled in the case considered here, since not only the symmetrized law of $X_1 - X_2$ but already the law of $X_1$ itself is invariant under re-orientations of any axes.

The only thing we have to prove is that $\sup_{x \in \mathbb{R}^{d-1}} P(X_1 \in D_\lambda + x) \neq 1$ for a suitably chosen $\lambda$. Due to construction, the law $P_{X_1}$ of $X_1$ can be written as a convex combination $P_{X_1} = \delta Q_1 + (1-\delta)Q_2$ with $Q_1$ being the law of a random vector $(\varepsilon_1T_1, \varepsilon_2T_2, ..., \varepsilon_{d-1}T_{d-1})$. Here the $\varepsilon_i$ and $T_j$ are completely independent of each other, the $T_j$ are i.i.d. distributed according to the time law of the directionally reinforced random walk, and the $\varepsilon_i$ are i.i.d. coin tossing random variables. This representation reflects the fact that with a positive probability the moving object changes from the vertical motion to the first horizontal direction, then to the second one and so on, and after that it returns to vertical motion. Hence, it is sufficient to prove that $\sup_{x \in \mathbb{R}^{d-1}} Q_1(X_1 \in D_\lambda + x) \neq 1$ for sufficiently small $\lambda$. We have
\[
\sup_{x \in \mathbb{R}^{d-1}} Q_1(X_1 \in D_\lambda + x) = \sup_{x \in \mathbb{R}^{d-1}} (1 - Q_1(|X_{1,j} - x_j| > \lambda, j = 1, 2, \ldots, d-1))
\]

\[
= \sup_{x \in \mathbb{R}^{d-1}} \left( 1 - \prod_{j=1}^{d-1} (1 - Q_1(|X_{1,j} - x_j| \leq \lambda)) \right)^{d-1}
\]

According to our assumption that \( P(T > 0) > 0 \), the law of \( \varepsilon_1 T_1 \) is non-degenerate. Hence the last expression is less than one for sufficiently small \( \lambda \). Thus our first event has probability \( O(n^{-(d-1)/2}) \) and by Borel–Cantelli the modified model is transient for \( d \geq 3 \).

2. Let us turn to the original model. The difference is that during a vertical 'phase' the particle can change from up to down several times. Hence an intermediate visit of \([-1, 1]^d\) does not necessarily imply a level crossing, if we only consider the positions at the beginning and the end of the vertical phase.

But, assuming infinitely many vertical visits to \([-1, 1]^d\) with a positive probability, we may consider the embedded process which, each time the particle visits \([-1, 1]^d\) during a vertical phase, registers whether the following change takes it to a vertical direction or not. This happens completely independently of the past with probabilities \( p = \frac{1}{2d-1} \) and \( q = \frac{2d-2}{2d-1} \), respectively. So the second case would happen infinitely often, too, with the same positive probability. Hence also this model would show infinitely many visits to \([-1, 1]^d\) or crossings of levels \(-1\) or \(1\) of the embedded vertical component during \([-1, 1]^{d-1}\)-visits of the horizontal part. We may again apply Theorem 3 of [5] and Theorem 2 of [2] to disprove the possibility of such a behaviour.

3 A two-dimensional example

Despite the fact that return probabilities and level crossing probabilities admit similar general asymptotic upper estimates nevertheless they can lead to qualitatively different recurrence properties. We construct a transient two-dimensional random walk which is 'polygonally recurrent' in a special way.
Theorem 3 There is a symmetric distribution on the integers such that two independent copies \((S_n), \tilde{S}_n\) of the associated random walk \((S_n)\) satisfy the two conditions

(a) The two-dimensional random walk \((S_n, \tilde{S}_n)\) is transient.

(b) Almost surely the event \(V_n = \{\text{sgn}(S_n) = -\text{sgn}(S_{n+1}), \tilde{S}_n = 0 = \tilde{S}_{n+1}\}\) occurs for infinitely many \(n\).

In particular

\[
\sum_{n=1}^{\infty} P(S_n = 0)^2 < \infty \quad (1)
\]

\[
\sum_{n=1}^{\infty} P(S_n = 0)P(S_nS_{n+1} < 0) = \infty. \quad (2)
\]

Observe that due to (a) the event \(V_n\) in (b) can be replaced by \(\{S_nS_{n+1} < 0, \tilde{S}_n = 0 = \tilde{S}_{n+1}\}\). Moreover, note that in each term of the series (2) both factors are of the order \(O(n^{-1/2})\). Because of (1) the first factor must be actually of slightly smaller order, but the gap must be subtle because of the divergence in (2). This lets us expect a somewhat delicate construction. The construction will also yield a counterexample for a (slightly modified, see the Remark below) conjecture on directionally reinforced random walks in the sense of [1]:

Theorem 4 There is a waiting time distribution on the nonnegative integers such that the associated directionally reinforced random walk \((R_m)\) on \(\mathbb{Z}^2\) is recurrent but at any given lattice point the walk a.s. changes direction only finitely often.

Remark The actual waiting time distribution constructed below gives positive probability to the value 0. One could insist on a strictly positive waiting time in order to be exactly in the framework of [1] but the conceptual arguments based on unimodality below would not be directly applicable.

3.2 The main idea of the proof

Intuitively the construction of Theorem 3 is based on the observation that if the one-dimensional random walk \((S_n)\) has lattice constant 1 and the underlying
symmetric random variable \( X \) has finite variance \( \sigma^2 \) then

\[
P(S_n = 0) \asymp \frac{1}{\sigma \sqrt{n}}
\]  

(3)

and if \( \gamma_\alpha \) is a \((1 - \alpha)\)-quantile of the distribution of \( |X| \) then

\[
P(S_nS_{n+1} < 0) \geq \frac{\alpha}{2} P(|S_n| \leq \gamma_\alpha) \asymp \frac{\alpha \gamma_\alpha}{\sigma \sqrt{n}}
\]  

(4)

So for (1) the variance must be infinite. But we can let the high values of \( X \) occur rarely enough so that with very high probability the behaviour of our random walk up to time \( n \) equals the behaviour of another random walk with variance \( \sigma_n^2 \) and corresponding quantiles \( \gamma_{\alpha_n} \) of the absolute value where these numbers grow in such a balanced manner that if we plug them into (3) and (4) we get (1) and (2). Clearly (1) then implies the transience (a). If the events \( V_n \) would be independent, one would easily infer statement (b) from (2). Since they are not, an extra argument is needed. The key to this step is Lemma 2 (see Appendix) which gives a quantitative version of the fact known e.g. from Markov chains that the frequency of certain events is high with high probability if only its expectation is high enough.

For the estimates in the main body of the proof it is useful to have symmetric unimodal distributions (since the notion of unimodality is used in the literature not completely consistently, see the Appendix for a definition).

The waiting time distribution in Theorem 4 will be given as a mixture of uniform distributions

\[
\mathcal{L}(T) = \sum_{l=1}^{\infty} p_l R[0, y_l], \sum_{l=1}^{\infty} p_l = 1
\]  

(5)

where \( R[0, y_l] \) denotes the uniform distribution on an integer interval \([0, y_l] \) and the increasing resp. decreasing sequences \((y_l)\) and \((p_l)\) will be constructed recursively below. Observe that \( T \) is a random variable with non-increasing weights, i.e. \( P(T = k) \geq P(T = k + 1) \) for each \( k \in \{0, 1, 2, \ldots\} \).

In the following we will make use of a coupled sequence of random walks with finite variance waiting times which approximates our infinite variance random walk given by the above waiting time distribution. For this end we first define a one-sided sequence \( T = \{T, \kappa, T^{(1)}, T^{(2)}, \ldots\} \) of random variables with the property \( T^{(\kappa)} = T^{(\kappa+1)} = \ldots = T \) for the random index \( \kappa \) and such that for all
\( m > k \) we have

\[
\mathcal{L}(T^{(k)}|\kappa = m) = \mathcal{L}(T^{(k)}) = z_k^{-1} \sum_{l=1}^{k} p_l R[0, y_l], \quad z_k = \sum_{l=1}^{k} p_l.
\]  

(6)

To do this, think of \( T \) as the result of a two-step construction: First the probability distribution \( \{p_l\} \) is used to find a random index \( \kappa \), then \( T \) is realized by choosing a random integer from \([0, y_\kappa]\) according to the uniform distribution on this interval. Given \((T, \kappa)\), choose \( \{T^{(1)}, T^{(2)}, ..., T^{(\kappa-1)}\} \) as a sequence of independent realizations of \( \mathcal{L}(T^{(i)}), i < \kappa \), respectively, and let \( T^{(\kappa)} = T^{(\kappa+1)} = ... = T \).

This leads to a hierarchical structure: The random walk will be a certain mixture of a sequence of random walks on different scales of time and space. The walk at level \( k \) runs on a space scale determined by \( y_k \) and a step frequency determined by \( p_k \). The reader should think of really rapidly increasing scales. In fact the simplest lower estimates for which the construction below (cf. (17)-(20)) can be carried out show that the \( y_k \) grow at least like in a recursion of the form

\[
y_{k+1} = e^{const \cdot y_k}.
\]

The main part of the proof of Theorem 3 lies in the careful choice of the parameters \( p_k \) and \( y_k \) of the waiting time \( T \) given by (5). Once they are determined we choose an iid sequence \((T_i)\) with the same law as \( T \) and consider

i) The directionally reinforced random walk \((R_m)\) on \( \mathbb{Z}^2 \) which moves in unit size steps and starts at the origin horizontally in either the positive or the negative direction. After the waiting time \( T_1 \) it switches with uniform probability to one of the three other directions. After waiting time \( T_2 \) it changes direction again and so on. We want to show that for our particular law of waiting times a.s. \( R_m \) visits the origin infinitely often but it changes direction at the origin only finitely many times. This will prove Theorem 4.

ii) The sequence \((S_n, \tilde{S}_n)\) consisting of the successive locations at which \((R_m)\) changes from vertical movement back to horizontal movement. By the properties of \( R_m \) the increments of \((S_n, \tilde{S}_n)\) are iid. with independent components. In fact the law of \( S_n - S_{n-1} \) is equal to the law of the random variable \( X \) defined in Lemma 4 where \( \epsilon \) determines the sign of the first part of the horizontal movement of \( R_m \) after the visit of \((S_{n-1}, \tilde{S}_{n-1})\) and \( G \) is a geometric random
variable with parameter $2/3$ which determines the number of horizontal flips before the next vertical step at the location $(S_n, \tilde{S}_{n-1})$. Similarly the second component $\tilde{S}_n - \tilde{S}_{n-1}$ of the increment has the same law and it determines the following vertical movement from $(S_n, \tilde{S}_{n-1})$ to $(S_n, \tilde{S}_n)$.

Thus by Lemma 4 $(S_n, \tilde{S}_n)$ is a random walk with independent components which have a symmetric law as required in Theorem 3. Now it is not hard to see that the assertion of Theorem 3 implies Theorem 4. In fact, consider first the conditional probability $z_{r,t}$ that the directionally reinforced random walk considered above never again after time $t$ changes direction at the origin, given that at time $t$ it changes direction at the origin, coming from direction $r$. Obviously by construction this probability does not depend on $r, t$, so we denote it by $z$. At each instance where the walk changes direction at the origin, with probability $z$ it will never do so again, completely independent from the past. So in the case $z > 0$ there will be a.s. only finitely many changes of direction at the origin, while for $z = 0$ there will be a.s. infinitely many such events. Now assume the latter, i.e. $z = 0$. Note that at each change of direction at the origin the following event is independent of the past and has positive probability: The walk changes to a perpendicular direction, makes zero steps in this direction, then changes again to a perpendicular direction. By assumption $z = 0$ this will happen a.s. infinitely often, too. But by construction of the embedded random walk $(S_n, \tilde{S}_n)$ this means that $(S_n, \tilde{S}_n)$ visits the origin infinitely often a.s. Hence the transience part (a) of Theorem 3 implies $z > 0$ which means that there are a.s. only finitely many changes of direction at the origin for the directionally reinforced random walk. The same argument applies to any other lattice point. On the other hand, part (b) of Theorem 3 immediately implies that there are a.s. infinitely many visits of the origin for the directionally reinforced random walk $(R_m)$. Thus it suffices to find a waiting time distribution ensuring the validity of Theorem 3.

### 3.3 The hierarchical construction

In this section the underlying parameters $p_k$ and $y_k$ of the the law of the waiting time (5) are not yet fixed.

We start with an i.i.d. two-dimensional array

$$(T_{i,n}) = ((T_{i,n}, \kappa_{i,n}, T_{i,n}^{(1)}, T_{i,n}^{(2)}, \ldots))$$

of random sequences as constructed above. Let $(\epsilon_n)$ and $(G_n)$ be two iid sequences of cointossing resp. geometric (with parameter $2/3$) random variables
chosen independently from \((T_{i,n})\) and let

\[
X_n = \epsilon_n \sum_{i=1}^{G_n} (-1)^i T_{i,n}, \quad (7)
\]

\[
X^{(k)}_n = \epsilon_n \sum_{i=1}^{G_n} (-1)^i T^{(k)}_{i,n}. \quad (8)
\]

Then we can define the two-dimensional random walks \((S_n, \tilde{S}_n)\) and \((S^{(k)}_n, \tilde{S}^{(k)}_n)\) by

\[
S_n = \sum_{j=1}^{n} X_j, \quad \tilde{S}_n = \sum_{j=1}^{n} \tilde{X}_j,
\]

\[
S^{(k)}_n = \sum_{j=1}^{n} X^{(k)}_j, \quad \tilde{S}^{(k)}_n = \sum_{j=1}^{n} \tilde{X}^{(k)}_j,
\]

where we take \((\tilde{T}_{i,n}, \tilde{T}^{(1)}_{i,n}, \tilde{T}^{(2)}_{i,n}, \ldots, \tilde{T}^{(K)}_{i,n})\), \((\tilde{\epsilon}_n)\), and \((\tilde{G}_n)\) as above independent of \((T_{i,n})\), \((\epsilon_n)\), and \((G_n)\), and then use them to construct \(\tilde{X}_n, \tilde{X}^{(k)}_n, \tilde{S}_n\) and \(\tilde{S}^{(k)}_n\) as we constructed \(X_n, X^{(k)}_n, S_n\) and \(S^{(k)}_n\).

In the sequel, a truncated version of this coupled sequence of random walks will be useful. It is obtained by cutting the random variable \(\kappa\) defining the hierarchy level at a given value \(K\). Hence we define a truncated version of the sequence \(T\) by \(\mathcal{L}(T^{[K]}_n) = \mathcal{L}(T^{[K]}_n, \kappa^{[K]}_n, T^{(1)}_n, T^{(2)}_n, \ldots, T^{(K)}_n)\) with \(\mathcal{L}(T^{[K]}_n) = \mathcal{L}(T^{(K)}_n)\), \(\mathcal{L}(T^{[K]}_n) = \mathcal{L}(T^{[K]}_n)\) and \(\mathcal{L}(T^{(K)}_n) = \mathcal{L}(T^{(K)}_n)\). Observe that this truncation shares with the original coupling the property that \(T^{(1)}_n, T^{(2)}_n, \ldots, T^{(\kappa^{[K]}_n)}_n\) are (conditionally with respect to \(\kappa^{[K]}_n\)) independent, while \(T^{(\kappa^{[K]}_n)}_n, T^{(\kappa^{[K]}_n+1)}_n, \ldots, T^{(K)}_n\) coincide.

So the truncated version of the coupled sequence of random walks is obtained by substituting the i.i.d. sequence \((T_{i,n})\) with the sequence \((T^{[K]}_{i,n})\). We define

\[
X^{(k)}_n^{[K]} = \epsilon_n \sum_{i=1}^{G_n} (-1)^i T^{(k)}_{i,n}^{[K]}, \quad (10)
\]

This yields a finite sequence of random walks \(\{S^{(k)}_n^{[K]}, \tilde{S}^{(k)}_n^{[K]}\}_{K=1}^{K}\) with the property

\[
\mathcal{L}(S^{(k)}_n^{[K]}) = \mathcal{L}(S^{(k)}_n), k \leq K. \quad (11)
\]
For the transience proof we prepare now an upper estimate of the return probabilities of $S_n^{(k)}$. For this end we use the following estimate

**Lemma 1** There is a positive constant $A$ such that the return probabilities $s_n^k = P(S_n^{(k)} = 0)$ satisfy the recursive estimate

$$s_n^k \leq s_n^{k-1}(1-p_k)^n + \frac{A}{p_k y_k \sqrt{n}}. \quad (12)$$

**Proof:**

We set the truncation parameter $K = k$ and denote by $Z_n^{(k)}$ the number of pairs $(i, j)$ with $j \leq n$ and $i \leq G_j$ such that in the representation (10) the r.v. $T_i^{(k)}$ actually is of the maximal level $k$, i.e. $\kappa_{i,j}^{(k)} = k$. For each index pair this happens with probability $p_k/z_k$. On the set $\{Z_n^{(k)} = 0\}$ we have $S_n^{(k)|k} = S_n^{(k-1)|k}$ by construction, hence by (11)

$$P(S_n^{(k)} = 0) = P(S_n^{(k)|k} = 0)$$

$$= P(S_n^{(k-1)|k} = 0) P(Z_n^{(k)} = 0 | S_n^{(k-1)|k} = 0)$$

$$+ P(S_n^{(k)|k} = 0, Z_n^{(k)} > 0)$$

$$\leq P(S_n^{(k-1)} = 0) P(\kappa_{1,j}^{(k)} < k \text{ for } j \leq n | S_n^{(k-1)|k} = 0)$$

$$+ P(S_n^{(k)|k} = 0, Z_n^{(k)} > 0).$$

Our whole construction is designed to ensure that the conditional law of $\{S_n^{(k-1)|k} = 0\}$ given $\{\kappa_{i,j}^{(k)} = k\}$ is equal to the unconditional law for any $i$ and $j \leq n$. Therefore, the two events $\{\kappa_{i,j}^{(k)} < k \text{ for } j \leq n\}$ and $\{S_n^{(k-1)|k} = 0\}$ are independent of each other, and the probability of the first event is $(1-p_k/z_k)^n \leq (1-p_k)^n$. Consequently,

$$P(S_n^{(k)} = 0) \leq P(S_n^{(k-1)|k} = 0)(1-p_k)^n + P(S_n^{(k)|k} = 0, Z_n^{(k)} > 0)$$

$$= P(S_n^{(k-1)} = 0)(1-p_k)^n + P(S_n^{(k)|k} = 0, Z_n^{(k)} > 0). \quad (13)$$

In order to estimate the second term we note that conditionally on the knowledge of the set of pairs $(i, j)$ with a contribution of level $k$, and of the signs $\epsilon_j$ the law of the sum $S_n^{(k)|k}$ is of the form as in (32) with $y = y_k$, except for an additional convolution factor (coming from the contribution of terms of level less than $k$) which does not increase the maximum probability. An upper estimate of this maximum probability is conserved under convex combinations and hence the maximum probability of the conditional law of $S_n^{(k)|k}$ given the value of $Z_n^{(k)}$ is at most $\frac{P}{y_k \sqrt{Z_n^{(k)}}}$. Thus
\[ P(S_n^{(k)} = 0, Z_n^{(k)} > 0) = \sum_{m=1}^{\infty} P(S_n^{(k)} = 0|Z_n^{(k)} = m) P(Z_n^{(k)} = m) \tag{14} \]
\[ \leq \sum_{m=1}^{\infty} \frac{D}{m y_k} P(Z_n^{(k)} = m) \]
\[ = \frac{D}{y_k} E \left( \frac{1}{\sqrt{Z_n^{(k)}}} \mathbf{1}_{\{Z_n^{(k)} > 0\}} \right) \cdot \]

Clearly \( E(Z_n^{(k)}) = np_k/z_k E(G) \) and by Wald's identity \( \text{Var}(Z_n^{(k)}) = np_k/z_k (1 - p_k/z_k) E(G) + n(p_k/z_k)^2 \text{Var}(G) \). Thus Chebyshev's inequality gives two positive constants \( b, B \) such that

\[ P(bnp_k/z_k \leq Z_n^{(k)}) \geq 1 - \frac{np_k/z_k (1 - p_k/z_k) E(G) + n(p_k/z_k)^2 \text{Var}(G)}{((E(G) - b np_k/z_k))^2} \]
\[ \geq 1 - \frac{B}{np_k}. \tag{15} \]

Therefore for some positive constant \( A \)

\[ E \left( \frac{1}{\sqrt{Z_n^{(k)}}} \mathbf{1}_{\{Z_n^{(k)} > 0\}} \right) \leq \frac{1}{\sqrt{bnp_k/z_k}} + \frac{B}{np_k} \leq \frac{A}{p_k \sqrt{n}}. \tag{16} \]

Plugging the estimates (14) and (16) into (13) yields the desired result. \( \blacksquare \)

### 3.4 Recursive choice of the parameters

We start with \( y_1 = 1 \) and \( p_2 = \frac{3}{4} \). The quantity \( p_1 \) will be chosen only in the end in order to get a total sum 1, but with condition (19) below it is obvious that \( \sum_{k=2}^{\infty} p_k \leq \frac{1}{2} \) and hence \( p_1 \) is at least 1/2. Let now \( k \in \mathbb{N}, k > 1 \) be given and assume that \( y_l \) for \( 1 \leq l < k \) and \( p_l \) for \( 2 \leq l \leq k \) are already defined. Then choose an integer \( c_k \) with

\[ c_k > \frac{k^8}{p_k^2}. \tag{17} \]

Now choose the two numbers \( y_k, p_{k+1} \) such that

\[ y_k \sqrt{p_k} \geq \max(12c_k, y_{k-1} \sqrt{p_{k-1}}) \tag{18} \]
0 < p_{k+1} \leq \frac{1}{2} p_k \quad (19)

and

\frac{1}{2k^4} \leq \left( \frac{A}{p_k y_k} \right)^2 \log \frac{1}{p_k} \leq \frac{1}{k^4} \quad (20)

Observe that we may guarantee (18) to hold for \( k = 2 \), even though \( p_1 \) is unknown in the beginning. This completes the recursive construction.

3.5 Transience of \((S_n, \tilde{S}_n)\)

We now have to verify that the resulting distribution has the desired properties. Let \((\tilde{S}_n)\) be an independent copy of the random walk \((S_n)\). For the transience we prove the convergence of the series (1). By construction, the indicator function of the event \( S_n = 0 \) is the pointwise limit of the indicator function of \( S_n^{(k)} = 0 \). Hence, for a fixed integer \( N \) we get

\[ \sum_{n=1}^{N} P(S_n = 0, \tilde{S}_n = 0) = \sum_{n=1}^{N} \lim_{k \to \infty} P(S_n^{(k)} = 0, \tilde{S}_n^{(k)} = 0)(1 - p_{k+1})^{2n} \leq \lim_{k \to \infty} \sum_{n=1}^{\infty} (s_n^k)^2(1 - p_{k+1})^{2n} \]

The proof of (a) will be complete if we can verify that for each \( k \)

\[ \sqrt{\sum_{n=1}^{\infty} (s_n^k)^2(1 - p_{k+1})^{2n}} \leq 2 \sum_{j=1}^{k} \frac{1}{j^2} \quad (21) \]

because this will show that the series in (1) is \( \leq \frac{\pi^4}{9} \).

We prove (21) by induction. In the case \( k = 1 \) this follows from \( s_1^1 \leq 1 \) and \( p_2 = \frac{1}{4} \). For the induction step observe

\[ \sum_{n=1}^{\infty} \frac{1}{n}(1 - q)^n = - \log(1 - (1 - q)) = \log(1/q) \quad (22) \]

for \( 0 < q < 1 \). We use (12) and the triangle inequality in the sequence space \( \ell^2 \) and get

\[ \sqrt{\sum_{n=1}^{\infty} (s_n^k)^2(1 - p_{k+1})^{2n}} \leq 2 \sum_{j=1}^{k} \frac{1}{j^2} \quad (21) \]
\[\sqrt{\sum_{n=1}^{\infty} (s_n^k)^2 (1 - p_{k+1})^{2n}} \leq 2 \sum_{j=1}^{k-1} \frac{1}{j^2} + \sqrt{\frac{1}{k^4}}.\]

In the last step we have used the induction hypothesis for the first term and (20) for the second term.

This proves (21) and hence part (a) of Theorem 3.

3.6 Recurrence of the events \(V_n\)

Of course we want to compare our two-dimensional random walk with its approximations. Therefore the sets

\[F_{n,k} = \{\kappa_{i,j} \leq k, 1 \leq j \leq n, 1 \leq i \leq G_j\} \cap \{\bar{\kappa}_{i,j} \leq k, 1 \leq j \leq n, 1 \leq i \leq \bar{G}_j\}\]

are important. We consider the events

\[E_{n,k} = \{\bar{S}_n = 0, |S_n| \leq c_k\} \cap F_{n,k}.\]

We introduce the notation \(p_k^* := \sum_{i \geq k} p_k \geq p_k\). We write \(G\) for the \(n\)-tuple \((G_1, G_2, ..., G_n)\) and \(\bar{G}\) for the \(n\)-tuple \((\bar{G}_1, \bar{G}_2, ..., \bar{G}_n)\). By construction, conditioned on \(G\) the random variable \(S_n^{(k)}\) is independent of \(F_{n,k}\) and on the set \(F_{n,k}\) we have \(S_n^{(k)} = S_n\). Hence

\[P(E_{n,k}) = \mathbb{E}(P(F_{n,k}|G, \bar{G})P(|S_n^{(k)}| \leq c_k|G)P(\bar{S}_n^{(k)} = 0|\bar{G})).\] (23)

Moreover, conditioning on \(G\), Lemma 4 gives that the laws of \(X_1^{(k)}, ..., X_n^{(k)}\) are symmetric unimodal and so is the conditional law of \(S_n^{(k)}\). Also by Lemma 4 we have

\[\text{Var}(S_n^{(k)}|G) = \sum_{1 \leq i \leq n} \text{Var}(X_i^{(k)}|G_i) \leq 4 \sum_{1 \leq i \leq n} G_i \text{Var}(T_{1,1}^{(k)})\]

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Var($S_n^{(k)}|G) = \sum_{1 \leq i \leq n} \text{Var}(X_i^{(k)}|G_i) \geq n\text{Var}(T_{1,1}^{(k)}).

Corresponding relations are valid for $\tilde{S}_n^{(k)}$. From (6) we get $E((T_1^{(k)})^2) \geq \frac{1}{4}p_ky_k^2$ and hence by Lemma 3 $\text{Var}(T_{1,1}^{(k)}) \geq \frac{1}{12}p_ky_k^2$ for large enough $k$. Hence $\text{Var}(S_n^{(k)}|G) \geq np_ky_k^2/12$, and this expression is at least $12c_k^2 \geq 12$ by (18).

Now Lemma 5 can be applied to get

$$P(S_n^{(k)} = 0|\tilde{G}) \geq \frac{d}{\sqrt{\text{Var}(S_n^{(k)}|G)}}.$$

$$P(|S_n^{(k)}| \leq c_k|G) \geq \frac{dc_k}{\sqrt{\text{Var}(S_n^{(k)}|G)}}.$$

Hence

$$P(|S_n^{(k)}| \leq c_k|G) P(S_n^{(k)} = 0|\tilde{G}) \geq \frac{d^2c_k}{4\text{Var}(T_{1,1}^{(k)}) \sqrt{\left(\sum_{1 \leq i \leq n} G_i\right) \left(\sum_{1 \leq i \leq n} \tilde{G}_i\right)}}.$$

We have $P(F_{n,k}|G,\tilde{G}) = (1 - p_{k+1}^*) \sum_{1 \leq i \leq n} G_i + \tilde{G}_i$, hence from (23) we get

$$P(E_{n,k}) \geq \frac{d^2c_k}{4\text{Var}(T_{1,1}^{(k)})} \left(\mathbb{E}(1 - p_{k+1}^*) \frac{\sum_{1 \leq i \leq n} G_i}{\sqrt{\sum_{1 \leq i \leq n} G_i}}\right)^2.$$

We consider the function $\psi_a(\lambda) := \exp(-a\lambda)/\sqrt{\lambda}, a > 0$. It is easily checked that this function is convex for $\lambda > 0$. Hence by Jensen’s inequality we get

$$P(E_{n,k}) \geq \frac{d^2c_k}{4\text{Var}(T_{1,1}^{(k)})} \frac{1}{nE_G} (1 - p_{k+1}^*)^{2nE_G}.$$

We have the estimate

$$\text{Var}(T_{1,1}^{(k)}) \leq \mathbb{E}((T_{1,1}^{(k)})^2) \leq z_k^{-1} \sum_{l=1}^k p_l y_l^2 \leq z_k^{-1} kp_k y_k^2.$$

where the last inequality follows from (18). So we arrive at
\[ P(E_{n,k}) \geq \frac{d^2c_k}{4z_k^{-1}kp_ky_k^2} \frac{1}{nEG}(1 - p_{k+1}^*)^{2nEG}. \]

Consequently, since \( p_{k+1}^* \leq 2p_{k+1} \) by (19)

\[
\sum_{n=1}^{\infty} P(E_{n,k}) \geq \frac{d^2c_k}{EG4z_k^{-1}kp_ky_k^2} \sum_{n=1}^{\infty} \frac{1}{n}(1 - p_{k+1}^*)^{2nEG} 
\geq \frac{d^2c_k}{EG4z_k^{-1}kp_ky_k^2} \sum_{n=1}^{\infty} \frac{1}{n}(1 - 2p_{k+1})^{2nEG} 
\geq \frac{d^2c_k}{EG4z_k^{-1}kp_ky_k^2} \sum_{n=1}^{\infty} \frac{1}{n}(1 - 4EGp_{k+1})^n 
= \frac{d^2c_k}{EG4z_k^{-1}kp_ky_k^2} \log \frac{1}{4EGp_{k+1}}. 
\]

If \( \Phi_k \) denotes the total number of the events \( E_{n,k} \) which occur we get from the conditions (17) and (20) for large enough \( k \)

\[ \mathbb{E}(\Phi_k) > \frac{k^2}{p_k}. \quad (24) \]

Next we want to apply Lemma 2. Let \( \mathcal{F}_m \) denote the \( \sigma \)-field generated by the random variables \( S_j, S_j^{(k)}, G_j, \kappa_i, j \) and \( S_j, S_j^{(k)}, \tilde{G}_j, \tilde{\kappa}_i, j \) with \( 1 \leq j \leq m, 1 \leq i \).

Set \( G_i^m = (G_1, G_{i+1}, ..., G_m), G_i^n = (\tilde{G}_1, \tilde{G}_{i+1}, ..., \tilde{G}_m) \) By conditional independence on \( E_{m,k} \) we have the relation

\[
P(E_{n,k}|\mathcal{F}_m) = \mathbb{E}(P(F_{n,k}|F_{m,k}, G_{m+1}^n, \tilde{G}_{m+1}^n) 
\cdot P(|S_{n,k}^m| \leq c_k|S_{m+1}^n, G_{m+1}^n) P(|\tilde{S}_{n,k}^m| = 0|\tilde{S}_{m}^n, \tilde{G}_{m+1}^n)).
\]

For any \( n - m \)-tuple of positive integers \( a = (a_1, a_2, ..., a_{n-m}) \) and any integer \( x \) we obtain

\[
P(|S_{n,k}^m| \leq c_k|S_{m}^m = x, G_{m+1}^n = a) = P(|S_{n-m}^m - x| \leq c_k|G_{1}^{n-m} = a) 
\leq P(|S_{n-m}^m| \leq c_k|G_{1}^{n-m} = a)
\]

since the conditional law \( P(S_{n-m}^m \in \cdot |G_{1}^{n-m} = a) \) is unimodal symmetric, and for the same reason we get

\[
P(\tilde{S}_{n,k}^m = 0|\tilde{S}_{m}^m = x, \tilde{G}_{m+1}^n = a) \leq P(\tilde{S}_{n-m}^m = 0|\tilde{G}_{1}^{n-m} = a).
\]
Let \( b = (b_1, b_2, \ldots, b_{n-m}) \) another \( n-m \)-tupel of positive integers. We get on \( E_{m,k} \)

\[
P(F_{n,k} | F_{m,k}, G_{m+1}^n = a, \tilde{G}_{m+1}^n = b) = P(F_{n-m,k} | G_{1}^{n-m} = a, \tilde{G}_1^{n-m} = b).
\]

So

\[
P(E_{n,k} | F_m) \leq \mathbb{E}(P(F_{n-m,k} | G_{1}^{n-m}, \tilde{G}_{1}^{n-m})P(|S_{n-m}^{(k)}| \leq c_k | G_{1}^{n-m})P(|\tilde{S}_{n-m}^{(k)}| = 0 | \tilde{G}_1^{n-m}))
\]

\[= P(E_{n-m,k}) \text{ a.s. on } E_{m,k}. \quad (25)\]

Let \( r_k = \left\lfloor \frac{k}{p_k} \right\rfloor \) and define the stopping times \( \tau_{i,k} \) recursively by \( \tau_{0,k} = 0 \) and

\[
\tau_{i,k}(\omega) = \min\{n \in \mathbb{N} : n > \tau_{i-1,k}(\omega) \text{ and } \omega \in E_{n,k}\}
\]

and \( \tau_{i,k}(\omega) = \infty \) if \( \omega \) lies in less than \( i \) of the sets \( E_{n,k} \). Lemma 2 allows us to conclude from (25) and (24) that with probability at least \( 1 - \frac{1}{k} \) at least \( r_k \) of the sets \( E_{n,k} \) occur, i.e.

\[
P(\tau_{r_k,k} < \infty) \geq 1 - \frac{1}{k}. \quad (26)\]

We consider the events

\[
H_{i,k} = \{\tau_{i,k} < \infty \} \cap \{c_k \leq |X_{\tau_{i,k}+1}|, \text{sgn}(X_{\tau_{i,k}+1}) \neq \text{sgn}(S_{\tau_{i,k}}), \tilde{X}_{\tau_{i,k}+1} = 0\}
\]

and \( H_k = \bigcup_{i=1}^{r_k} H_{i,k} \). Let \( W_n = V_n \cup \{\omega : S_nS_{n+1} = 0, \tilde{S}_n = 0 = \tilde{S}_{n+1}\} \), where \( V_n \) is as in part (b) of Theorem 3. By definition \( H_{i,k} \subset E_{\tau_{i,k},k} \) and hence \( H_{i,k} \) is contained in the set \( W_n \) with \( n = \tau_{i,k} \). So, if \( \omega \in H_k \) then there is some \( n \) with \( \omega \in W_n \) and \( |X_{n+1}| \geq c_k \). Since the sequence \( (c_k) \) is unbounded every point in \( \limsup H_k \) lies in infinitely many sets \( W_n \). We show that

\[
\lim_{k \to \infty} P(H_k) = 1. \quad (27)
\]

Denote by \( U^{(k)} \) a random variable with law \( R[0, y_k] \). By definitions (5) and (29) we have

\[
P(c_k \leq |X|) \geq P(G = 1)p_k P(c_k \leq U^{(k)})
\]

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\[ p_k \geq \frac{2}{3} p_k (1 - \frac{c_k}{y_k}) > \frac{1}{2} p_k \]

for all sufficiently large \( k \) since \( \frac{c_k}{y_k} \to 0 \) because of (18). Let \( \delta = P(X = 0) \). Then the complements \( H_{i,k}^c \) of our sets satisfy

\[
P(\{\tau_{i,k} < \infty\} \cap H_{i,k}^c \mid F_{\tau_{i,k}}) \leq 1 - P(c_k \leq |X|, \text{sgn}(X) > 0, \bar{X} = 0) \leq 1 - \frac{\delta}{4}p_k
\]

and an induction argument shows that

\[
P \left( \{\tau_{r_k,k} < \infty\} \cap \bigcap_{j=1}^{r_k} H_{j,k}^c \right) \leq (1 - p_k \frac{\delta}{4})^{r_k}.
\]

The right-hand side in this inequality becomes arbitrarily small for large \( k \) by the choice of \( r_k \). Hence by (26)

\[
\lim_{k \to \infty} P(H_k) = \lim_{k \to \infty} P(\tau_{r_k,k} < \infty) = 1.
\]

We have shown that almost surely the event \( W_n \) occurs for infinitely many \( n \). By the transience of our random walk, the event \( \{\omega : S_n S_{n+1} = 0, \bar{S}_n = 0 = \bar{S}_{n+1}\} \) a.s. cannot occur infinitely often. So we conclude that a.s. for infinitely many \( n \) the event \( V_n \) occurs. This completes the proof.

4 Appendix: Some tools

**Lemma 2 (A counting variable estimate)** Let \((\mathcal{F}_n)_{0 \leq n \leq N}\) be a (finite or infinite) filtration. Let \((E_n)\) be an adapted sequence of events such that \( E_0 = \Omega \) and for \( m < n \)

\[
P(E_n \mid F_m) \leq P(E_{n-m}) \text{ a.s. on } E_m.
\]

Let \( \Phi \) be the number of events which occur (including \( E_0 \)). Then for each \( r = 0, 1, 2, \ldots \)

\[
P(\Phi > r) \geq 1 - \frac{r}{\mathbb{E}(\Phi)}.
\]  

(28)
Proof: Clearly it suffices to consider the case of finite $N$. We call an index $n$ a success time if $E_n$ occurs. Let $\tau_r(\omega)$ be the $r$-th success time $\geq 1$ and let $\tau_r = N + 1$ if $\Phi \leq r$. Moreover let $\Phi_m$ be the number of success times $\geq m$. Then the inequality in our assumption implies for each $m \leq N$

$$\mathbb{E}(\Phi_m|\mathcal{F}_m) = \sum_{n=m}^{N} P(E_n|\mathcal{F}_m) \leq \sum_{n=m}^{N} P(E_{n-m}) \leq \mathbb{E}(\Phi)$$

(a.s. on $E_m$) and hence also $\mathbb{E}(\Phi_r|\mathcal{F}_r) \leq \mathbb{E}(\Phi)$ on the set $\{\Phi > r\} = \{\tau_r \leq N\} \in \mathcal{F}_r$. Then

$$\mathbb{E}(\Phi) \leq rP(\Phi \leq r) + \mathbb{E}\left(\mathbf{1}_{\{\Phi > r\}}(\Phi_r + r)\right) \leq r + P(\Phi > r)\mathbb{E}(\Phi).$$

Dividing by $\mathbb{E}(\Phi)$ yields the result. 

Lemma 3 Let $T$ be a random variable on $\{0, 1, 2, \ldots\}$ with non-increasing weights. Then the estimate

$$(\mathbb{E}T)^2 \leq \frac{3}{4} \mathbb{E}T^2$$

is valid. It implies $(\mathbb{E}T)^2 \leq 3\text{Var}(T)$.

Proof: It is easy to see that a random variable on $\{0, 1, 2, \ldots\}$ has non-increasing weights iff it can be represented as a mixture of uniform distributions $R[0,y]$ as in (5). So we get by Jensen’s inequality

$$(\mathbb{E}T)^2 = \left(\sum_{l=1}^{\infty} p_l y_l/2\right)^2 \leq \sum_{l=1}^{\infty} p_l (y_l/2)^2 = \frac{3}{4} \sum_{l=1}^{\infty} p_l y_l^2 / 3 \leq \frac{3}{4} \sum_{l=1}^{\infty} p_l y_l (2y_l + 1) / 6 = \frac{3}{4} \mathbb{E}T^2.$$ 

Lemma 4 Let $T_i, i \in \mathbb{N}$ be an identically distributed sequence of random variables on $\{0, 1, 2, \ldots\}$ with non-increasing weights. Let $\epsilon \in \pm 1$ be a cointossing random variable and let $G$ be a random variable with values in $\mathbb{N}$. If all these r.v.’s are independent of each other then the law of

$$X = \epsilon \sum_{i=1}^{G} (-1)^i T_i$$

(29)
is symmetric unimodal with
\[
\Var(T_1) \leq \Var(X) \leq 4\E(G)\Var(T_1),
\]
where the last estimate is interpreted trivially if $T_1$ has infinite variance.

**Proof:** Denote by $\tau$ the law of the $T_i$ and by $\mu_k$ the law of $X$ in the case where $G$ takes the constant value $k$. The convolution of $\tau$ with its reflected image on $-\mathbb{N}$ is symmetric and easily seen to be unimodal. Moreover it is known that the convolution of two symmetric unimodal laws is again symmetric unimodal. (Decompose both laws as mixtures of uniform distributions on suitable centered intervals.) This implies the assertion if $G$ is an even constant, i.e. $\mu_{2m}$ is symmetric unimodal. Now assume that $G = 2m + 1$ is odd. The conditional law of $\epsilon \sum_{i=1}^n T_{2i} - T_{2i-1}$ given $\epsilon$ is by symmetry equal to $\mu_{2m}$ and hence independent of $\epsilon$. Thus $\mu_{2m+1}$ is the convolution of $\mu_{2m}$ with the law of $\epsilon T_{2m+1}$. The latter is also symmetric unimodal and hence $\mu_{2m+1}$ is unimodal symmetric as well. Since this property is also stable under mixtures the result follows also for nonconstant $G$. Set $Y = \epsilon^{-1}X$. We have

\[
\Var(\epsilon Y) = \E(Y^2) = \Var(Y) + (\E Y)^2 \\
= \E G \cdot \Var(T_1) + P(G \text{ odd})(\E T_1)^2 \leq 4\E(G)\Var(T_1)
\]

where the variance of the sum $Y = \epsilon^{-1}X$ is computed by Wald’s identity and Lemma 3 was used. This relation implies both the lower and upper estimate of the variance.

We follow the convention to call a symmetric random $\mathbb{R}$-valued variable $T$ unimodal if it is absolutely continuous with respect to Lebesgue measure and the density can be chosen non-increasing on $\mathbb{R}$. Analogously, if $T$ is a symmetric random variable with values on $\mathbb{Z}$ it is called unimodal if it has non-increasing weights on $\{0, 1, 2, \ldots\}$.

**Lemma 5** There is a positive constant $d$ such that for every symmetric unimodal distribution $\mu$ with finite variance $\sigma^2 > 0$ and every $c > 0$ with $c \leq \sigma$ one has

\[
\mu(\{x : |x| < c\}) \geq \frac{cd}{\sigma}.
\]

**Proof:** 1. First consider the case where $\mu$ is carried by $\mathbb{R}$. Introduce a scaling parameter $\lambda \geq 1$ and observe that the assumption on $\mu$ implies that for any $c > 0$ we have $\mu(\{x : |x| < c\}) \geq \lambda^{-1} \mu(\{x : |x| < \lambda c\})$. (Substitute $x$
by \( \lambda x \) in the integral over the density function of \( \mu \). Hence we have with \( \lambda' := \lambda c/\sigma \geq c/\sigma \) chosen arbitrarily

\[
\mu(\{x : |x| < c\}) \geq \lambda^{-1}(1 - \mu(\{x : |x| \geq \lambda c\})) \\
\geq \lambda^{-1}(1 - \frac{\sigma^2}{\lambda^2 c^2}) = \frac{c}{\sigma}(\lambda')^{-1}(1 - (\lambda')^{-2})
\]

by Chebychev’s inequality. Now we choose \( d := \max_{\lambda' > 1}((\lambda')^{-1}(1 - (\lambda')^{-2})) \).

This proves the assertion in the case that \( \mu \) is carried by \( \mathbb{R} \).

2. Now consider the case that \( \mu \) is carried by \( \mathbb{Z} \). For \( \sigma^2 < 3/4 \) the it follows from Chebychev’s inequality that \( d \) can be chosen to be 1/4. So we may assume \( 3/4 \leq \sigma^2 \) and even \( 3/4 \leq c^2 \leq \sigma^2 \). Let \( \nu \) be the uniform distribution on \([-\frac{1}{2}, \frac{1}{2}]\).

It is easily checked that \( \hat{\mu} := \mu * \nu \) is symmetric unimodal on \( \mathbb{R} \) with variance \( \sigma^2 + \frac{1}{12} \) and \( \mu(\{j\}) = \hat{\mu}([j - \frac{1}{2}, j + \frac{1}{2}]) \). We have for \( 3/4 \leq c^2 \leq \sigma^2 \)

\[
\mu(\{x : |x| < c\}) \geq \hat{\mu}(\{x : |x| < c - \frac{1}{2}\}) \\
\geq \frac{(c - \frac{1}{2})d}{\sqrt{\sigma^2 + \frac{1}{12}}} \geq \frac{cd'}{\sigma}
\]

for a suitable \( d' > 0 \).

---

**Lemma 6** Let \( \{\mu_i\}_{i=1}^m \) be a set of equidistributions on some integer intervals of equal length \( y \geq 1 \), i.e. \( \mu_i = \mathcal{R}[a_i, a_i + y] \). Then we have

\[
\max_{x \in \mathbb{Z}}(\mu_1 * \mu_2 * \ldots * \mu_m)(\{x\}) \leq \frac{D}{\sqrt{my}}.
\]

where \( D \) is some absolute constant.

**Proof:** The result can be obtained by standard estimates for concentration functions involving characteristic functions. The expression on the left-hand side of (32) is the concentration function \( Q(\mu_1 * \mu_2 * \ldots * \mu_m; 1/2) \) of the \( m \)-fold convolution, evaluated for an interval length \( \lambda = 1/2 \). We make use of an estimate given in [6], which is essentially due to Esseen [5]:

\[
Q(\mu_1 * \mu_2 * \ldots * \mu_m; \lambda) \leq \lambda \left(2\tau \left(\frac{\sin(\tau/2)}{\tau/2}\right)^2\right)^{-1} \int_{-2\tau/\lambda}^{2\tau/\lambda} |\varphi_{\mu_1 * \ldots * \mu_m}(t)| dt \quad (33)
\]
being valid for arbitrary $\lambda > 0$ and $0 < \tau < 2\pi$. In order to simplify the considerations, we substitute each $\mu_i$ by the same $\hat{\mu} := \mu_i((\cdot) + a_i + y/2)$ being symmetric with respect to the origin. The resulting shift does not change the concentration function of the convolution, but the characteristic functions become real-valued. The explicit expression for the characteristic function $\varphi_{\hat{\mu}}$ is given by

$$Q(\mu_1 * \mu_2 * \ldots * \mu_m; \frac{1}{2}) \leq C \frac{1}{y} \int_0^1 \left| \frac{\sin \frac{1}{2}t(y + 1)}{(y + 1) \sin \frac{1}{2}t} \right|^m dt.$$ 

Now we make use of the fact that the members in the Taylor expansion of the sine function are alternating and non-increasing in the interval considered. We may continue the estimate as follows

\[
\begin{align*}
&\leq C \frac{1}{y} \int_0^1 \left( \frac{\sin \frac{1}{2}t}{(y + 1) \sin \frac{1}{2}t} \right)^m dt \\
&\leq C \frac{1}{y} \int_0^1 \left( \frac{\frac{1}{2}t - \frac{1}{12}t^3 + \frac{1}{240}t^5}{\frac{1}{2}t - \frac{1}{12(y+1)^2}t^3} \right)^m dt \\
&\leq C \frac{1}{y} \int_0^1 \left( \frac{1 - \frac{1}{6}t^2 + \frac{1}{120}t^4}{1 - \frac{1}{6(y+1)^2}t^2} \right)^m dt \\
&\leq C \frac{1}{y} \int_0^1 \left( \frac{1 - \frac{1}{5}t^2}{1 - \frac{1}{6(y+1)^2}t^2} \right)^m dt \\
&\leq C \frac{1}{y} \int_0^1 \left( \frac{1 - \frac{1}{5}t^2}{1 - \frac{1}{10}t^2} \right)^m dt \\
&\leq C \frac{1}{y} \int_0^1 \left( 1 - \frac{1}{10}t^2 \right)^m dt \\
&\leq C \frac{1}{y} \int_0^1 e^{-\frac{1}{10}t^2} dt \\
&\leq C \frac{1}{y} \int_0^\infty e^{-\frac{1}{10}t^2} dt \\
&= \frac{D}{\sqrt{my}},
\end{align*}
\]
for some absolute constant $D$. ■

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