Improved First Estimates to the Solution of Kepler’s Equation

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The manuscript provides a novel starting guess for the solution of Kepler’s equation for unknown eccentric anomaly \( E \) given the eccentricity \( e \) and mean anomaly \( M \) of an elliptical orbit.

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I. KEPLER’S EQUATION

A. Mean and Eccentric Anomaly

The track of the orbit for a 2-body potential proportional to the inverse distance of the two bodies leads to solutions which may be ellipses with eccentricity \( 0 \leq e \leq 1 \). The time dependence is described by the parameter \( M \) of the mean anomaly, which is an angle measured from the center of the ellipse, and which is a product of a parameter \( n \) called the mean motion (essentially the square root of the square of the major axis of the ellipse—allows to recover solutions for negative \( M \) only the cases where \( M \geq E \) and \( M = E = 0 \) are considered.

For the manuscript at hand, \( M \) and \( e \) are considered fixed parameters. To compute the circular coordinates of the mean anomaly, which is an angle measured from the center of the ellipse, and which is a product of a parameter \( n \) called the mean motion (essentially the square root of the coupling parameter in the numerator of the 2-body potential divided by the cube of the major semi-axis) and a time elapsed since some reference epoch:

\[
M = n(t - t_0). \tag{1}
\]

\( E = M + e \sin E \)

\( E - M - e \sin E = 0. \tag{3} \)

\( E \) and \( M \) are angles measured in radian in the range \(-\pi \leq E, M \leq \pi \). To simplify the notation, we discuss only the cases where \( M \geq 0 \), because the parity

\[
E(-M) = -E(M), \tag{4}
\]

—equivalent to flipping the entire orbit along the major axis of the ellipse—allows to recover solutions for negative \( M \) as well.

B. The Inverse Problem

The numerical problem considered here is to find the root of the function

\[
f(E) \equiv E - M - e \sin E \tag{5}
\]

in an efficient and numerically stable fashion.

The expansion of \( E \) in a Taylor series of \( e \) can be written as

\[
E \equiv \sum_{i \geq 0} m_i(M)e^i \tag{6}
\]

supported by the table

| \( m_i(M) \) |
|---|
| 0 |
| 1 |
| 2 |
| 3 |
| 4 |
| 5 |
| 6 |
| 7 |
| 8 |

where the function and its derivatives with respect to the unknown \( E \) are

\[
f \equiv E - e \sin E - M; \quad f' \equiv 1 - e \cos E; \quad f'' \equiv e \sin E. \tag{10}
\]

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* http://www.mpia-hd.mpg.de/~mathar
(8) is for example used in the \texttt{iauPlan94IAU} function for planets' ephemerides [7]. Note that, since the evaluation of the trigonometric functions is expensive compared to the fundamental operations [8, 9], the second-order iteration is preferred since \( \sin E \) in \( f'' \) is already calculated in conjunction with \( f \).

### III. INITIAL VALUE PROBLEM

#### A. Standard Initial Guesses

If the initial guess is the second step of (7),

\[
E^{(0)} = M + e \sin M,
\]  

and the iteration (8) is used with \( e > 0.99 \), a known problem is that the iterations may converge to secondary roots of the equation with the wrong sign [10]. This is basically triggered by starting with an underestimate of \( E \) as illustrated in Figure 1.

A well-known remedy is to start with the initial guess

\[
E^{(0)} = \pi
\]  

which is known to converge [11, 12]. The speed of convergence with the two basic Newton methods is illustrated in Figures 2 and 3.

#### B. Eccentricity One

If \( e = 1 \), \( M = E - \sin E \) has the power series

\[
M = \sum_{i \geq 1} (-1)^{i+1} E^{2i+1}/(2i+1)!. 
\]

The associate power series for the cube root

\[
\sqrt[3]{6M} = \bar{M} = E - \frac{1}{60} E^3 + \frac{1}{8400} E^5 + O(E^9),
\]

can be reversed [13, (3.6.25)]

\[
E = \bar{M} + \frac{1}{60} \bar{M}^3 + \frac{1}{1400} \bar{M}^5 + \frac{1}{25200} \bar{M}^7 + \frac{43}{720720000} \bar{M}^9 + \frac{1213}{381180800000} \bar{M}^{11} \\
+ \frac{151439}{127135008000000} \bar{M}^{13} + \frac{43227}{16542537833} \bar{M}^{15} + \cdots
\]

Again this is not converging well to \( \rightarrow \pi \) as \( M \rightarrow \pi \), but since \( E \) is a increasing function of \( e \) at constant \( M \), this approximation is slightly better than (12) as an initial estimator from above. This is illustrated by the lower number of iterations in Fig. 4 compared to Fig. 2.

#### C. Taylor series at various \( M \)

The derivate of \( E \) with respect to \( M \) is according to (2)

\[
\frac{dE}{dM} = 1 + e \cos E \frac{dE}{dM} \quad (15)
\]

or

\[
\frac{dE}{dM} = \frac{1}{1 - e \cos E} \quad (16)
\]
By repeated differentiation and using the initial value $M = E = \pi$ builds a Taylor expansion of $E$ in powers of $M - \pi$:

$$E = \pi + \frac{1}{1 + e} (M - \pi) + \frac{e}{(1 + e)^4} \frac{(M - \pi)^3}{3!} + \frac{e(9e - 1)}{(1 + e)^7} \frac{(M - \pi)^5}{5!} + \frac{e(1 - 54e + 225e^2)}{(1 + e)^{10}} \frac{(M - \pi)^7}{7!} + \cdots \quad (17)$$

This approximation as the starting value has excellent quality for $M > 1$. With the same method the Taylor expansion with the initial value $E = M = 0$ is constructed:

$$E = \frac{1}{1 - e} M + \frac{e}{(1 - e^2)^2} \frac{M^3}{3!} - \frac{e(1 - 8e - e^2)}{(1 - e^2)^4} \frac{M^5}{5!} + \frac{e(1 - 52e + 170e^2 + 52e^3 - 35e^4)}{(1 - e^2)^6} \frac{M^7}{7!} + \cdots \quad (18)$$

but this is only advantageous if $e < 0.5$. A third variant is to build a Taylor expansion around $M = \pi/2 - e$,

$$E = \frac{\pi}{2} + (M - \frac{\pi}{2} + e) \frac{1}{2} \frac{(M - \frac{\pi}{2} + e)^3}{3!} - \frac{e(1 - 8e - 5e^2)}{(1 + e)^4} \frac{(M - \frac{\pi}{2} + e)^5}{5!} + \frac{e(1 - 52e + 144e^2 + 224e^3 + 61e^4)}{(1 + e)^6} \frac{(M - \frac{\pi}{2} + e)^7}{7!} + \cdots \quad (19)$$

A fourth variant is to build a Taylor expansion around $M = \pi/6 - e/2$, $E = \pi/6$:

$$E = \frac{\pi}{6} + (M - \frac{\pi}{6} + e) \frac{1}{2} \frac{(M - \frac{\pi}{6} + e)^3}{3!} + \frac{e(1 + e)^2 (M - \frac{\pi}{6} + e)^5}{2! (1 - \sqrt{3}e/2)^2} + \cdots \quad (20)$$

The relative merits of these 4 Taylor series are a complicated function of $e$ and $M$. As a guideline

- for simplicity (19) would not be used at all;
- (17) be used where $\frac{\pi}{4}(1 - e) < M \leq \pi$;
- (18) be used where $0 \leq M < \frac{\pi}{4} - \frac{1}{2} e$;
- (20) be used elsewhere.
D. Improved Initial Value, Version 1

A starting value of $E$ is obtained by inserting the approximation

$$\sin E \approx 1 - \frac{4}{\pi^2} (E - \pi/2)^2$$

into the equation. [Similar approximations could be obtained by truncating the Chebyshev series approximation of the $\sin E$ after the second term [14, 15].]

This leads to a quadratic equation for $E$

$$E^{(0)} = M + e \left[ 1 - \frac{4}{\pi^2} (E^{(0)} - \pi/2)^2 \right],$$

which is solved by

$$\bar{e} \equiv \frac{\pi}{4e} - 1;$$

$$E^{(0)} = \frac{\pi}{2} \bar{e} \left[ \text{sgn}(\bar{e}) \sqrt{1 + \frac{M}{ee^2}} - 1 \right].$$

The error of this estimate relative to the accurate solution is shown in Figure 5. It increases where $M/e \to 0$ and $\bar{e} \to 0$. The figure shows that the $E^{(0)}$ has the same benefit as (12) of approximating the solution from above, therefore converging [11], but being more accurate. In consequence, the convergence is faster, as demonstrated in Figure 6 if compared with Figure 3.

![Figure 5](image1.png)

**FIG. 5.** Mismatch $E^{(0)} - E$ of the initial estimate (24).

![Figure 6](image2.png)

**FIG. 6.** The number of iterations needed for a relative accuracy of $10^{-12}$ in $E$ starting from (24) iterating with (9).
E. Improved Initial Value, Version 2

If (3) is expressed as

$$E - M = e \sin(M + E - M),$$  \hspace{1cm} (25)
both sides may be expanded in a Taylor series of $E - M$,

$$E - M \approx e \sin(M) + (E - M) e \cos M - \frac{(E - M)^2}{2} e \sin M + \ldots$$  \hspace{1cm} (26)

Keeping this series up to $O(E - M)$ yields the estimate

$$E^{(0)} = M + \frac{e \sin(M)}{1 - e \cos M}.$$  \hspace{1cm} (27)

This is basically the estimate of the second step of the fixed point iteration (7) with an enhancement factor of the second term if $e$ or $\cos M$ are large. As pointed out earlier [1], this is also obtained applying the Newton method to the estimator $E^{(0)} = M$.

If the series is kept up to $O((E - M)^2)$, the associated quadratic equation proposes

$$\frac{e}{2} \sin M (E^{(0)} - M)^2 + (1 - e \cos M)(E^{(0)} - M) - e \sin M = 0.$$  \hspace{1cm} (28)

This quadratic equation is solved by

$$E^{(0)} - M = \frac{1 - e \cos M}{e \sin M} \left[ \sqrt{1 + \frac{2e^2 \sin^2 M}{(1 - e \cos M)^2}} - 1 \right].$$  \hspace{1cm} (29)

Figures 7 and 8 show in comparison with Figure 5 that these approximations derived from the Taylor series of $E - M$ are not better than the one from the quadratic estimate of $\sin E$.

Expansion of (26) up to third order in $E - M$ yields a cubic equation for $E^{(0)} - M$, which is even closer to the exact solution as demonstrated in Figure 9. See
Mikkola’s paper for a similar approach [16–18].

F. Adiabatic switching on

One principle in perturbative quantum mechanics switches on the fermionic interaction by increasing the coupling constant from zero (no interaction) up to the value attained by the real-world system. Adopting this concept here means that the eccentricity is started at $e = 0$ at a known solution $E = M$, and the $E$ is tracked until the actual value of $e$ is reached. Let overdots denote derivatives with respect to $e$ at constant $M$, e.g. $\dot{e} = 1$, $\dot{M} = 0$. The derivative of $E$ is

$$\dot{E} = \sin E + e \cos E \dot{E};$$

(30)

To avoid numerically expensive evaluations of the trigonometric functions the auxiliary angle $\phi \equiv \cos E$ is introduced with derivative

$$\dot{\phi} = - \sin E \dot{E}.$$  

(32)

Insertion into the previous equation means $\phi$ satisfies the first order differential equation

$$\dot{\phi} = - \frac{1 - \phi^2}{1 - e \phi}. $$

(33)

The application here is to solve this with a single-step “explicit” classic Runke-Kutta integration with the initial value $\phi(e = 0) = \cos M$ up to the actual $e$ [19, 20]. These estimates $E$ are surprisingly close to the actual solutions, as illustrated in Fig. 10.

Fig. 11 demonstrates that 3 steps of the first-order Newton method suffice to obtain 12 digits accuracy in $E$.

This approach is close in spirit to the Taylor series (6), but not suffering from the singularity at $e \to 1$.

IV. SUMMARY

A initial value (24) combined with Halley’s equation (9) leads to fast and stable convergence for the inverse problem of Kepler’s equation for elliptic orbits. That initial value is simpler but generally worse than Markley’s estimator [18].

Appendix A: C++ Implementation

A reference implementation is reprinted in the anc directory which implements (9) and (24). If compiled with the -DTEST preprocessor symbol, g++ -O2 -DTEST

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig10.png}
\caption{Mismatch $E^{(0)} - E$ of the initial estimate of a single-step Runge-Kutta estimator for (33).}
\end{figure}

-o solveKepler solveKepler.cc, a test program is obtained which can be called with two command line arguments, solveKepler e M, to investigate the convergence with this and other approaches. For a fair comparison of the starting guesses $E^{(0)}$, the routine uses (9) for all starting guesses. It prints for all implemented starting guesses the values of $e$, $M$, the index $i$ of the iteration loop in the Newton method, the estimate $E^{(i)}$ reached so far, and the error $E^{(i)} - E$ relative to the true solution.

The program switches over to solving

$$M = e \sinh E - E \quad (A1)$$

for hyperbolic orbits if the command line parameter $e$ is larger than 1, and starts with the estimate $E^{(0)} = \arcsinh(M/e)$ in that case.

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FIG. 11. The number of iterations needed for a relative accuracy of $10^{-12}$ in $E$ starting from the RK4 solution of (33) iterating with (8).

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