Random trees with superexponential branching weights

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Abstract
We study rooted planar random trees with a probability distribution which is proportional to a product of weight factors \(w_n\) associated with the vertices of the tree and depending only on their individual degrees \(n\). We focus on the case when \(w_n\) grows faster than exponentially with \(n\). In this case, the measures on trees of finite size \(N\) converge weakly as \(N\) tends to infinity to a measure which is concentrated on a single tree with one vertex of infinite degree. For explicit weight factors of the form \(w_n = ((n-1)!)^\alpha\) with \(\alpha > 0\), we obtain more refined results about the approach to the infinite volume limit.

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1. Introduction

Random trees have been studied intensively by mathematicians and theoretical physicists in the last few decades. They have direct applications to many branches in science, are essential in many mathematical models used by physicists and are a natural object to study from the point of view of pure mathematics.

The random trees we are concerned with here were originally called simply generated trees by probabilists [10]. Later the same tree ensembles were referred to as random trees with a local action by physicists and viewed as toy models in statistical mechanics and for some aspects of quantum gravity, see e.g. [2].

Simply generated trees with \(N\) vertices can be defined as follows. Let \((w_n)_{n \geq 1}\) be a sequence of non-negative numbers which we will call branching weights. If \(T\) is a tree graph with a vertex set \(V(T)\) having \(N\) elements, we define a probability distribution on the set of
all such trees by the formula
\[ v(T) = Z^{-1} \prod_{v \in V(T)} w_{\sigma(v)}, \]
where \( \sigma(v) \) is the degree of the vertex \( v \) and \( Z \) is a normalization factor called the partition function in physics. One is interested in typical properties of trees with respect to this measure, especially asymptotics for large \( N \) and the existence of a limiting measure as \( N \to \infty \).

Much is known about such trees for ‘nice’ branching weights, which we review briefly below. In this paper, we aim to complement some of these results for weights \( w_n \) which grow faster than exponentially with \( n \). In this case, some of the formalism that has been used to study simply generated trees is no longer applicable, as we will explain below. A physicist would say that the grand partition function is divergent, which normally is a signal of instability in a physical theory. We will indeed see that with superexponential branching weights one vertex gets connected to all the other vertices in the infinite volume limit.

In the following section, we provide a more technical background and summarize our results. The final section contains detailed proofs.

2. Definitions and summary of results

We consider rooted planar trees with a root \( r \) of degree 1. We let \( \Gamma_N \) be the set of trees with \( N \) edges and denote the set of finite and infinite trees by \( \Gamma \). Vertices of infinite order are allowed and for such vertices the links pointing away from the root are ordered as \( \mathbb{N} \), i.e. there is a leftmost edge pointing away from the root. The unique nearest neighbor of the root \( r \) will be denoted by \( s \).

**Remark 2.1.** We include the root \( r \) just for convenience. It is equivalent to omit it and consider \( s \) as the root (now with arbitrary degree), with minor changes in the notation; \( N \) is then the number of vertices in the tree and the degree \( \sigma(v) \) is replaced by \( 1 + \sigma_s(v) \), where \( \sigma_s(v) \) is the outdegree of \( v \). It may be even more convenient to omit \( r \) but keep the pendant edge from \( s \) to \( r \) as an edge with one free endpoint; this point of view is used sometimes in the proofs below.

**Remark 2.2.** We can regard the set \( \Gamma \) as a set of subtrees of the infinite Ulam–Harris tree \( T_\infty \), which is the tree with a vertex set \( V(T_\infty) = \{r\} \cup \bigcup_{k=0}^{\infty} \mathbb{N}^k \), the set of all finite strings of natural numbers (and \( r \)) with \( s = \emptyset \) (the empty string, so \( \mathbb{N}^0 = \{s\} \)) and a vertex \( v = v_1 \cdots v_k \\) having ancestor \( v_1 \cdots v_{k-1} \) when \( k > 0 \). More precisely, \( \Gamma \) can be identified with the set of all rooted subtrees \( T \) of \( T_\infty \) such that if \( v = v_1 \cdots v_k \) is a vertex in \( T \), then so is \( v_1 \cdots v_{k-1} \) for every \( i < v_k \). We call such subtrees of \( T_\infty \) left subtrees and more generally, we say that a tree \( T' \in \Gamma \) is a left subtree of \( T \in \Gamma \) if \( V(T') \subseteq V(T) \).

We endow \( \Gamma \) with a metric \( d \) which is defined as follows. Let \( T \in \Gamma \) and define \( B_R(T) \) as the graph ball of radius \( R \), centered on the root \( r \) in \( T \). The left ball of radius \( R \), \( L_R(T) \), is defined as the maximal left subtree of \( B_R(T) \) with vertices of degree not greater than \( R \). The metric \( d \) is given by
\[ d(T,T') = \inf \left\{ \frac{1}{R+1} \left| L_R(T) = L_R(T') \right|, \quad T, T' \in \Gamma \right. \]  
Convergence in \( \Gamma \), in the metric \( d \), is equivalent to the convergence of the degree \( \sigma(v) \) for every \( v \in V(T_\infty) \) (where we define \( \sigma(v) = 0 \) for \( v \notin T \)), see [7] for details.
To avoid trivialities we assume that the branching weights satisfy $w_1 \neq 0$ and $w_n \neq 0$ for at least some $n > 2$. We define the finite volume partition function

$$Z_N = \sum_{\tau \in \Gamma_N} \prod_{v \in V(\tau) \setminus \{r\}} w_{\sigma(v)}$$

and a probability distribution $\nu_N$ on $\Gamma_N$ by

$$\nu_N(\tau) = Z_N^{-1} \prod_{v \in V(\tau) \setminus \{r\}} w_{\sigma(v)}.$$ 

This probability distribution describes a random tree $T_N$ with $N$ edges.

Let $\rho \geq 0$ be the radius of convergence of the generating function $g(z) = \sum_{n=0}^{\infty} w_{n+1} z^n$.

Rescaling $w_n \mapsto \alpha \beta^n w_n$ with $\alpha, \beta > 0$ does not affect the distributions $\nu_N$, and it is well known and easy to see that if $\rho > 0$, we can by rescaling assume that $(w_n)$ is a probability distribution, i.e. $\sum_{n=0}^{\infty} w_n = 1$. In that case, the random tree $T_N$ with the distribution $\nu_N$ is a Galton–Watson tree with the offspring distribution $(w_n + 1)_{n=0}^{\infty}$, conditioned to have size $N$. If further $\lim_{z \to \rho} z g'(z)/g(z) \geq 1$, then the distributions $\nu_N$ converge to the distribution of a random tree that is infinite, with all vertex degrees finite and exactly one infinite path, see further [1, 3, 4, 8]. The limiting measure describes an infinite critical Galton–Watson tree conditioned on non-extinction. On the other hand, in the subcritical case when $m = \lim_{z \to \rho} z g'(z)/g(z) < 1$, then (at least under some technical conditions) the limit distribution still exists but now describes a random tree with exactly one vertex of infinite degree; the length of the path from $r$ to this vertex has a geometric distribution with mean $1/(1 - m)$; the rest of the tree can be described by a subcritical Galton–Watson process, see [7] for details.

In this paper, we are interested in the case when the radius of convergence $\rho = 0$. Note that then there is no Galton–Watson interpretation. In section 3 we prove that in this case, under certain conditions on the weights, the measures $\nu_N$ also converge weakly as $N \to \infty$ (in the topology generated by $d$). The result can be seen as a natural limiting case of the result in [7] as $m \to 0$; the resulting limit tree is in this case non-random and is simply an infinite star.

**Theorem 2.3.** If the branching weights satisfy

$$\frac{w_{n+1}}{w_n} \to \infty \quad (n \to \infty),$$

then the measures $\nu_N$, viewed as the probability measures on $\Gamma$, converge weakly to the probability measure that is concentrated on the single tree which has $\sigma(s) = \infty$ and all other vertices of degree 1.

Furthermore, we obtain stronger convergence results for certain explicit choices of weights. In the language of statistical mechanics, these results give an explicit description of the finite size effects.

**Theorem 2.4.** For the branching weights $w_2 = \lambda$ and $w_n = (n - 1)!$, $n \neq 2$, the partition function satisfies

$$\frac{Z_N}{\lambda^N (N-1)!} \to 1$$

and we have

$$N - \sigma(s) \to \text{Pois}(\lambda).$$


as \( N \to \infty \). Moreover, the tree \( T_N \) consists of \( r, s, \) and \( \sigma(s) - 1 \) branches attached to \( s \); with probability tending to 1, \( N - \sigma(s) \) of these branches have size 2 and all others have size 1 (i.e. they contain a single leaf only).

Note that in the limit \( N \to \infty \), the branches of size 2 disappear to infinity, so we do not see them in the limit given by theorem 2.3.

**Theorem 2.5.** Let the branching weights be \( w_n = ((n - 1)!)^\alpha \), where \( 0 < \alpha < 1 \). Then the partition function satisfies

\[
Z_N = ((N - 1)!)^\alpha \exp(O(N^{1-\alpha})) = \exp(\alpha N \log(N) - \alpha N + O(N^{1-\alpha})).
\]

Furthermore, with probability tending to 1, the random tree \( T_N \) has the following properties, with \( K = \lfloor 1/\alpha \rfloor \).

(i) \( \sigma(s) = N - O(N^{1-\alpha}) \).

(ii) All vertices except \( s \) have degrees \( \leq K + 1 \).

(iii) All subtrees attached to \( s \) have sizes \( \leq K + 1 \).

Moreover, let \( X_{i,N} \) be the number of vertices of degree \( i \) in \( T_N \) and let

\[
n_i = i!^\alpha N^{1-i\alpha}.
\]

(iv) If \( 1 \leq i < 1/\alpha \), then \( n_i \to \infty \) as \( N \to \infty \) and

\[
\frac{X_{i+1,N}}{n_i} \xrightarrow{p} 1.
\]

If \( i = 1/\alpha = K \) (which occurs only when \( 1/\alpha \) is an integer), then \( n_K = K!^\alpha \) is constant and

\[
X_{K+1,N} \xrightarrow{d} \text{Pois}(n_K).
\]

With these branching weights, the asymptotic distributions of the numbers \( X_{i,N} \) of vertices of different degrees are Gaussian, except in the Poisson case when (2.11) applies.

**Theorem 2.6.** Let \( w_n = ((n - 1)!)^\alpha \) with \( 0 < \alpha < 1 \) as in theorem 2.5. Then there exist numbers \( n_i^* = n_i^*(N) = (1 + o(1))n_i \), \( 1 \leq i < 1/\alpha \), with \( n_i \) given by (2.9), such that, as \( N \to \infty \),

\[
\frac{X_{i+1,N} - n_i^*}{\sqrt{n_i}} \xrightarrow{d} \mathcal{N}(0, 1), \quad 1 \leq i < 1/\alpha,
\]

\[
X_{i+1,N} \xrightarrow{d} \text{Pois}(n_i), \quad i = K = 1/\alpha.
\]

Moreover, these hold jointly for all \( i \leq K \), with independent limits. More precisely, for each \( i < 1/\alpha \),

\[
n_i^* = n_i(1 - (1 - i\alpha)N^{-\alpha} + O(N^{-2\alpha}) + O(1).
\]

In particular, when \( \alpha \) is not too small, we have the explicit limits

\[
\frac{X_{2,N} - N^{1-\alpha}}{N^{(1-\alpha)/2}} \xrightarrow{d} \mathcal{N}(0, 1), \quad 1 > \alpha > \frac{1}{5},
\]

\[
\frac{X_{2,N} - (N^{1-\alpha} - (1 - \alpha)N^{1-2\alpha})}{N^{(1-\alpha)/2}} \xrightarrow{d} \mathcal{N}(0, 1), \quad 1 > \alpha > \frac{1}{5}.
\]
\[ X_{3,N} - 2^\alpha N^{1-2\alpha} \frac{d}{N^{(1-2\alpha)/2}} \xrightarrow{d} \mathcal{N}(0, 2^\alpha), \quad \frac{1}{2} > \alpha > \frac{1}{3}, \quad (2.17) \]

\[ X_{3,N} - (2^\alpha N^{1-2\alpha} (1 - 2\alpha) 2^\alpha N^{1-3\alpha}) \frac{d}{N^{(1-2\alpha)/2}} \xrightarrow{d} \mathcal{N}(0, 2^\alpha), \quad \frac{1}{2} > \alpha > \frac{1}{6}, \quad (2.18) \]

\[ X_{4,N} - 6^\alpha N^{1-3\alpha} \frac{d}{N^{(1-3\alpha)/2}} \xrightarrow{d} \mathcal{N}(0, 6^\alpha), \quad \frac{1}{3} > \alpha > \frac{1}{5}, \quad (2.19) \]

\[ X_{4,N} - (6^\alpha N^{1-3\alpha} (1 - 3\alpha) 6^\alpha N^{1-4\alpha}) \frac{d}{N^{(1-3\alpha)/2}} \xrightarrow{d} \mathcal{N}(0, 6^\alpha), \quad \frac{1}{3} > \alpha > \frac{1}{7}. \quad (2.20) \]

For smaller \(\alpha\), it is possible to obtain further terms in the expansion of \(n^*_i\), and thus explicit forms of the asymptotic mean of \(X_{i+1,N}\). However, this approach seems to become more and more difficult as \(\alpha\) becomes smaller.

**Remark 2.7.** The proof of (2.12) also shows the stronger result that the joint distribution of \((X_{i+1,N})_{i=1}^K\) can be approximated by the joint distribution of independent Poisson random variables \(Y_{i,N} \sim \text{Pois}(n^*_i)\) in the sense that the total variation distance tends to 0 as \(N \to \infty\):

\[
\frac{1}{2} \sum_{m_1, \ldots, m_K} |\mathbb{P}(X_{i+1,N} = m_i, \forall i) - \mathbb{P}(Y_{i,N} = m_i, \forall i)| \to 0.
\]

(2.21)

**Remark 2.8.** The estimate (2.8) of the partition function can be improved to

\[
\mathcal{Z}_N = (N - 1)!^\alpha \exp \left( N^{1-\alpha} + \left( 2^\alpha - \frac{1 - \alpha}{2} \right) N^{1-2\alpha} + O(N^{1-3\alpha}) + o(1) \right).
\]

(2.22)

In particular, if \(1 > \alpha > \frac{1}{2}\), then \(\mathcal{Z}_N = (N - 1)!^\alpha \exp(N^{1-\alpha} + o(1))\).

Again it seems possible, but more complicated, to obtain further terms in the exponent.

**Remark 2.9.** It is straightforward to show, using the same methods as in the proof of theorem 2.4, that when \(\alpha > 1\),

\[
\mathcal{Z}_N = (N - 1)!^\alpha (1 + o(1)),
\]

(2.23)

and all the branches which are attached to \(s\) have size 1, with a probability which tends to 1 as \(N \to \infty\). In this case, the leading contribution to the partition function comes only from the Boltzmann factor of the vertex \(s\), i.e. \(w_{\sigma(s)}\). The case \(\alpha = 1\) is a marginal case when larger branches start to appear and their entropy adds a contribution to the partition function which appears in the associated exponential.

### 3. Proofs of theorems

In this section, we state and prove a few lemmas and prove theorems 2.3–2.6. In the following, we will always assume that the branching weights satisfy the condition in equation (2.5). Define

\[
\mathcal{Z}(N, n) = \sum_{d_1 + \cdots + d_N = n} \prod_{i=1}^N w_{d_i+1}.
\]

(3.1)
Lemma 3.2. As \( N \to \infty \), the partition function for an ordered forest of \( m \) trees with a total number of edges \( N \) is
\[
Z_N^{(m)} = \frac{m}{N} Z(N, N - m).
\]  

More generally, the partition function for an ordered forest of \( m \) trees with a total number of edges \( N \) is
\[
Z_N^{(m)} = \frac{m}{N} Z(N, N - m).
\]  

Lemma 3.1. For every \( \epsilon > 0 \), there exists a \( C_\epsilon < \infty \) such that for all \( N \) and \( n \)
\[
Z(N, n) \leq \epsilon Z(N, n + 1) + C_\epsilon.
\]

Proof. Consider a finite sequence \( d_1, \ldots, d_N \) for which \( \sum_i d_i = n \). Let \( i^* \) be the smallest index such that \( d_{i^*} = \max_i d_i \). Define a sequence
\[
d_i^* = \begin{cases} 
    d_i + 1 & \text{if } i = i^*, \\
    d_i & \text{otherwise}.
\end{cases}
\]  

Note that \( d_i^* \) is the unique maximum in \( (d_i^*) \), so \( (d_i) \) can be recovered from \( (d_i^*) \) and the map \( (d_i) \mapsto (d_i^*) \) is injective.

Let \( \epsilon > 0 \) be given. Choose a number \( A_\epsilon \) such that \( w_i/w_{i+1} < \epsilon \) if \( i \geq A_\epsilon \). Then,
\[
\sum_{d_1 + \cdots + d_N = n} \prod_{i=1}^N w_{d_i+1} \leq \epsilon \sum_{d_1 + \cdots + d_N = n+1} \prod_{i=1}^N w_{d_i+1} \leq \epsilon Z(N, n + 1)
\]
and, crudely,
\[
\sum_{d_1 + \cdots + d_N = n} \prod_{i=1}^N w_{d_i+1} \leq \left( \sum_{i=0}^{A_\epsilon} w_{i+1} \right)^N.
\]

Taking \( C_\epsilon = \sum_{i=0}^{A_\epsilon} w_{i+1} \) completes the proof.

Lemma 3.2. As \( N \to \infty \), \( \sigma(s) \xrightarrow{p} \infty \).

Proof. It suffices to show that
\[
v_N(\sigma(s) = k) \to 0
\]
for every fixed \( k \geq 1 \), since \( v_N(\sigma(s) = m) = 1 - \sum_{k=0}^{m-1} v_N(\sigma(s) = k) \). If the vertex \( s \), in a tree with \( N \) edges, has degree \( k + 1 \), then removing \( s \) and \( r \) but leaving all edges from \( s \) to its children as pendant edges, cf remark 2.1, leaves a forest with \( k \) trees and \( N - 1 \) edges. Therefore, using (3.2) and (3.3),
\[
v_N(\sigma(s) = k + 1) = \frac{N}{N-1} k w_{k+1} Z(N-1, N-k-1) / Z(N, N-1).
\]

Let \( \epsilon > 0 \) be given. Use lemma 3.1 \( k \) times to obtain
\[
Z(N-1, N-k-1) \leq \epsilon^k Z(N-1, N-1) + k C_\epsilon^{N-1}
\]
and note that
\[
Z(N, N-1) \geq w_1 Z(N-1, N-1).
\]
Since the branching weights satisfy \((2.5)\), \(Z(N-1,N-1) \geq w_N w_{N-2}^N\) grows super exponentially and in particular \(Z(N-1,N-1) \geq (2C_\varepsilon)^{N-1}\) for \(N\) large enough. Therefore,

\[
v_N(\sigma(s) = k + 1) \leq \frac{N}{N-1} kw_{k+1}(w_{1}^{-1} \epsilon^k + k2^{-N+1})
\]

(3.12) and (3.8) follows since \(\epsilon\) is arbitrary. \(\square\)

**Lemma 3.3.** For any \(N \geq 1\) and \(n \geq 0\),

\[
\sum_{\ell=0}^{n} \ell w_{\ell+1} Z(N-1,n-\ell) = \frac{n}{N} Z(N,n).
\]

(3.13)

**Proof.**

\[
\sum_{\ell=0}^{n} \ell w_{\ell+1} Z(N-1,n-\ell) = \sum_{d_1+\cdots+d_{\ell-1}+\ell=n} \ell w_{\ell+1} \prod_{i=1}^{N-1} w_{d_i+1}
\]

\[
= \sum_{d_1+\cdots+d_{\ell}=n} d_N \prod_{i=1}^{N} w_{d_i+1}.
\]

(3.14)

By symmetry we can replace \(d_N\) in front of the product by any \(d_j\), \(j = 1, \ldots, N\). Summing over \(j\) then gives the desired result. \(\square\)

**Lemma 3.4.** Assume \(N \geq 1\) and let \(s_1\) be the first child of \(s\). If \(L \geq 1\) and \(k \geq L\), then

\[
v_N(L+1 \leq \sigma(s_1) \leq k+1 | \sigma(s) = k+1) \leq \frac{2}{L}.
\]

(3.15)

**Proof.** If \(\sigma(s) = k + 1 \geq 2\) and \(\sigma(s_1) = \ell + 1 \geq 1\), then removing the vertices \(r, s\) and \(s_1\), again leaving pendant edges, leaves a forest with \(k+\ell-1\) trees and \(N-2\) edges. Therefore (assuming \(N \geq 3\),

\[
v_N(\sigma(s) = k + 1, \sigma(s_1) = \ell + 1) = \frac{N(k+\ell-1)w_{k+1}w_{\ell+1} Z(N-2,N-1-k-\ell)}{N-2} Z(N,N-1).
\]

(3.16)

By (3.9) and (3.16),

\[
v_N(\sigma(s_1) = \ell + 1 | \sigma(s) = k+1) = \frac{(N-1)(k+\ell-1)w_{\ell+1} Z(N-2,N-1-k-\ell)}{(N-2)k} Z(N-1,N-1-k).
\]

(3.17)

By lemma 3.3,

\[
\sum_{\ell \geq \ell} w_{\ell+1} Z(N-2,N-1-k-\ell) \leq \frac{1}{L} \sum_{\ell \geq 0} \ell w_{\ell+1} Z(N-2,N-1-k-\ell)
\]

\[
= \frac{1}{L} \frac{N-1-k}{N-1} Z(N-1,N-1-k).
\]

(3.18)

Hence, (3.17) implies

\[
\sum_{\ell = L}^{k} v_N(\sigma(s_1) = \ell + 1 | \sigma(s) = k+1) \leq \frac{N-1-k}{N-2} \frac{2}{L} \leq \frac{2}{L}.
\]

(3.19) \(\square\)
Lemma 3.5. As $N \to \infty$, $v_N(\sigma(s_1) = 1) \to 1$.

Proof. Fix $L > 1$ and an $\ell$ such that $1 \leq \ell < L$. Note that when $\ell \geq 1$ formula (3.16) is symmetric in $k$ and $\ell$. Therefore,

$$v_N(\sigma(s_1) = \ell + 1) = \sum_{k=1}^{\infty} v_N(\sigma(s) = k + 1, \sigma(s_1) = \ell + 1)$$

$$= v_N(\sigma(s) = \ell + 1, \sigma(s_1) \geq 2) \leq v_N(\sigma(s) = \ell + 1)$$

(3.20)

and thus $v_N(\sigma(s_1) = \ell + 1) \to 0$ as $N \to \infty$ by lemma 3.2. Next, lemma 3.4 implies

$$v_N(L + 1 \leq \sigma(s_1) \leq \sigma(s))$$

$$= \sum_{k} v_N(L + 1 \leq \sigma(s_1) \leq k + 1 | \sigma(s) = k + 1) v_N(\sigma(s) = k + 1) \leq \frac{2}{L}$$

(3.21)

Thus,

$$\limsup_{N \to \infty} v_N(2 \leq \sigma(s_1) \leq \sigma(s)) \leq \limsup_{N \to \infty} \left( \sum_{k=1}^{L-1} v_N(\sigma(s_1) = \ell + 1) + \frac{2}{L} \right) = \frac{2}{L}.$$ (3.22)

Since $L$ is arbitrary, $v_N(2 \leq \sigma(s_1) \leq \sigma(s)) \to 0$ as $N \to \infty$. By the symmetry of (3.16) in $k$ and $\ell$ we also find that

$$v_N(2 \leq \sigma(s) \leq \sigma(s_1)) = v_N(2 \leq \sigma(s_1) \leq \sigma(s)) \to 0$$

(3.23)

as $N \to \infty$. Finally, since $\sigma(s) \geq 2$, we have

$$v_N(\sigma(s_1) \geq 2) \leq v_N(2 \leq \sigma(s) \leq \sigma(s_1)) + v_N(2 \leq \sigma(s_1) \leq \sigma(s)) \to 0$$

(3.24)

as $N \to \infty$. \hfill \Box

Proof of theorem 2.3. Let $R > 0$. By lemma 3.2, $\sigma(s) \to^p \infty$, so $v_N(\sigma(s) \geq R) \to 1$. Given that $\sigma(s) \geq R$, denote the first $R - 1$ children of $s$ by $s_1, \ldots, s_{R-1}$. Then by lemma 3.5 and symmetry $v_N(\sigma(s_i) = 1, \sigma(s) \geq R) \to 1$ for every $i \leq R$ and thus we find that

$$v_N(\sigma(s) \geq R, \sigma(s_1) = \cdots = \sigma(s_{R-1}) = 1) \to 1$$

(3.25)

as $N \to \infty$. Since $R$ is arbitrary, the result follows from the definition of the topology on $R$, cf the comment below (2.1). \hfill \Box

Proof of theorem 2.4. First, we establish an upper bound on $Z_N$. Consider equation (3.2) for $Z_N$. For a given sequence $(d_i)$, let $m_i$ denote the number of indices $i$ for which $d_i = j$, where $j = 0, \ldots, N-1$. Instead of summing over $(d_i)$ we sum over $(m_i)$. For a given sequence $(m_i)$, there are $(m_0, \ldots, m_{N-1})$ sequences $(d_i)$ and therefore, since $u_1 = 1$,

$$Z_N = \frac{N!}{(N - 1)!} \sum_{m_0 + m_1 + \cdots + m_{N-1} = N} \prod_{i=0}^{N-1} \frac{u_m^{m_i}}{m_i!}$$

$$= \sum_{m_0 + m_1 + \cdots + m_{N-1} = N-1} \frac{1}{(N - \sum_{j=1}^{N} m_j) !} \prod_{j=1}^{N-1} \frac{u_m^{m_i}}{m_i!}.$$ (3.26)
Denote the maximum vertex degree by $M$ and fix a number $L \geq 2$. By lemma 3.2, it is sufficient to consider the case $M > L$. That contribution to (3.26) can be estimated by shifting $m_{M-1} \rightarrow m_{M-1} + 1$ which yields the upper bound

$$
\sum_{m_1+2m_2+\cdots+(M-1)m_{M-1}=N-M \geq L} (M-1)! \frac{N}{M} \prod_{i=1}^{M-1} m_i!
\leq \sum_{m_1+2m_2+\cdots+(M-1)m_{M-1}=N-M \geq L} \frac{\lambda^{m_1}}{m_1!} \prod_{i=2}^{M-1} \frac{(1/M^{i-1})m_i}{m_i!}
\leq \exp \left( \lambda + \sum_{i=2}^{\infty} \frac{i!}{(i \lor L)^{i-1}} \right),
$$

(3.27)

where $A \lor B$ denotes the maximum of $A$ and $B$. The last expression converges to $e^\lambda$ when $L \to \infty$ by dominated convergence.

Next, we establish a corresponding lower bound on $Z_N$. Consider the contribution to (3.26) from terms for which the only non-zero elements in the sequence $(m_i)$ are $m_0$, $m_1$ and $m_k = 1$, where $k \geq 2$ is arbitrary; thus $m_0 = k$, $m_1 = N - k - 1$ and $m_k = 1$. These terms provide the following lower bound of (3.26):

$$
\sum_{\ell=2}^{N-1} \frac{1}{\ell!} \frac{w_{N-1-k}^{N-k}}{(N-1-k)!} = \sum_{\ell=0}^{N-3} \frac{\lambda^{\ell}}{\ell!} \rightarrow e^\lambda
$$

as $N \to \infty$. This and (3.27) prove (2.6).

To complete the proof, note that the probability that $T_N$ has $\sigma(s) = N-j$ and that exactly $j$ of the $\sigma(s) - 1 = N - j - 1$ branches attached to $s$ have size 2 and all others have size 1 is, assuming $N > 2j$ and using (2.6),

$$
\frac{1}{Z_N} \binom{N-j-1}{j} w_1^{N-j-1} w_2^j w_{N-j} = \frac{1}{Z_N} \binom{N-j-1}{j} \lambda^j (N-j-1)! \rightarrow \frac{\lambda^j}{j!} e^{-\lambda}.
$$

(3.29)

These limits sum to 1 and yield the Pois($\lambda$) distribution in (2.7).

\textbf{Proof of theorem 2.5.} Consider the weights $w_{n+1} = n^m$. Write, again by (3.2),

$$
Z_N = \frac{1}{N} \sum_{d_1+\cdots+d_N=N-1} \prod_{i=1}^{N} d_i^m.
$$

(3.30)

We obtain the lower bound

$$
Z_N \geq (N-1)!^m
$$

(3.31)

by considering only the terms in $Z_N$ with one $d_i = N-1$, and all others 0 (i.e. stars).

Define $Z_N(k, \epsilon)$ as the contribution to $Z_N$ when precisely $k \geq 0$ vertices have degree greater than $\epsilon(N-1)$, where $\epsilon$ is some small positive number. First, consider the case when $k = 0$. Let $(d_i)_{i=1}^{N}$ be a sequence which satisfies $d_i \leq \epsilon(N-1)$ for all $i$. Using the simple relation

$$
N!M! \leq (N+1)! (M+1)!, \quad \text{for} \quad N \geq M-1
$$

(3.32)

we can distribute and add the smallest elements in $(d_i)_{i=1}^{N}$, to the larger ones until each of them reaches $\epsilon(N-1)$. Thus we obtain the upper bound, using Stirling’s formula,

$$
\prod_{i=1}^{N} d_i^m \leq \epsilon^{(N-1)}^m \leq C_{1} N^{2m/\epsilon} (N-1)^{m} e^{\omega N},
$$

(3.33)
where $C_1 > 0$ is a number independent of $N$ (but, as other constants below, it may depend on $\alpha$ and $\epsilon$). Therefore,

$$Z_N(0, \epsilon) \leq C_1 N^{2a/\epsilon} (N - 1)!^a \epsilon^{\alpha N} \left(\frac{2N - 2}{N - 1}\right) \leq C_1 N^{2a/\epsilon} (N - 1)!^a \epsilon^{\alpha N} 2^{2N}$$  \hspace{1cm} (3.34)

which is negligible compared to (3.31) as $N \to \infty$ for $\epsilon$ small enough.

Next, consider the case when two or more of the $d_i$ are larger than $\epsilon(N - 1)$, i.e. when $k \geq 2$ in $Z_N(k, \epsilon)$. Clearly, $k < 1/\epsilon$. Denote the $d_i$ which are greater than $\epsilon(N - 1)$ by $d_i_1, \ldots, d_i_k$ and let $D_j = d_i_j$. The indices $i_j$ can be chosen in $\binom{N}{k}$ ways. We will now lump together all the $D_j$ into a single one, i.e. we define

$$D = D_1 + \cdots + D_k.$$  

For each $D$, there are at most $\binom{D + k - 1}{k - 1}$ choices of $D_1, \ldots, D_k$. Note that, with $D_* = \lceil \epsilon(N - 1) \rceil$, using $D_j \geq D_*$ and Stirling’s formula again,

$$\frac{D_1! \cdots D_k!}{D!} \leq \frac{D_*^k}{(kD_*)!} \leq C_2 N^{k} \left(\frac{1}{k}\right)^{\epsilon N},$$  \hspace{1cm} (3.35)

where $C_2 > 0$ is independent of $N$. Thus, we obtain the upper bound

$$\sum_{2 \leq k \leq 1/\epsilon} Z_N(k, \epsilon) \leq C_2 \sum_{2 \leq k \leq 1/\epsilon} \binom{N}{k} N^a k \left(\frac{1}{k}\right)^{\alpha N} \times \sum_{D = d_1 + \cdots + d_k = N - 1, d_i \leq (N - 1), y_i} \frac{D + k - 1}{D - k} \frac{k - 1}{k - 1} D_*^a \prod_{i=1}^{N-k} d_i!^a\ 

\leq C_3 N^{3/\epsilon} \left(\frac{1}{2}\right)^{2aN} Z_N(1, \epsilon),$$  \hspace{1cm} (3.36)

where $C_3 > 0$ is independent of $N$. This estimate, together with (3.34), shows that the main contribution to $Z_N$ for $N$ large comes from $Z_N(1, \epsilon)$.

Finally, we consider $Z_N(1, \epsilon)$. Using the representation as in (3.26) we have, writing $L = \lceil \epsilon(N - 1) \rceil$ for convenience,

$$\frac{Z_N(1, \epsilon)}{(N - 1)!} = \sum_{D = L+1}^{N-1} \sum_{m_1+2m_2+\cdots+Lm_{L} = N-1-D} \frac{D^*}{D - 1 - \sum_{j=1}^{L} m_j} \prod_{i=1}^{L} \frac{i!^{m_i}}{m_i!},$$  \hspace{1cm} (3.37)

where $D + 1$ denotes the degree of the large vertex and $m_i$ denotes the number of vertices which have degree $i + 1$. Consider one term in this sum and let $D = D + \sum_{i=1}^{K+1} m_i$, adding the outdegrees of all vertices which have degree greater than $K + 1$ to the large vertex, where we recall that $K = \lceil 1/\alpha \rceil$. Then,

$$\tilde{D}! \geq D! \cdot L^{\sum_{i=K+1}^{\infty} m_i} \geq D! \cdot L^{\sum_{i=K+1}^{\infty} m_i}$$  \hspace{1cm} (3.38)

and

$$\left(N - 1 - \sum_{i=1}^{K} m_i\right)! \leq \left(N - 1 - \sum_{i=1}^{L} m_i\right)! \cdot N^{\sum_{i=K+1}^{\infty} m_i}.$$  \hspace{1cm} (3.39)

Thus,

$$\frac{D^*}{(N - 1 - \sum_{i=1}^{L} m_i)!} \leq \frac{\tilde{D}^* N^{\sum_{i=K+1}^{\infty} m_i}}{(N - 1 - \sum_{i=1}^{K} m_i)! \cdot L^{\sum_{i=K+1}^{\infty} m_i}}$$  \hspace{1cm} (3.40)
and
\[
Z_N(1, \epsilon) \leq \sum_{\tilde{D} = L+1}^{N-1} \sum_{m_1 + \cdots + K m_k = N - \tilde{D}} \sum_{m_{k+1}, \ldots, m_k \geq 0} \frac{\tilde{D}^{1u}}{(N - 1 - \sum_{j=1}^{K} m_j)!} \prod_{i=1}^{K} \frac{i^{1i} m_i}{m_i!} \prod_{i=K+1}^{L} \left( \frac{N_i}{L_i^{u}} \right)^{m_i} \frac{1}{m_i!}.
\] (3.41)

We have
\[
\sum_{m_{k+1}, \ldots, m_k = K+1}^{L} \prod_{i=K+1}^{L} \left( \frac{N_i}{L_i^{u}} \right)^{m_i} \frac{1}{m_i!} = \prod_{i=K+1}^{L} \exp \left( \frac{N_i}{L_i^{u}} \right)
\]
\[
= \exp \left( \sum_{i=K+1}^{L} \frac{N_i}{L_i^{u}} \right).
\] (3.42)

Let, using $L = \lceil \epsilon (N - 1) \rceil > \epsilon N/2$ (assuming $N$ large),
\[
a_i = \frac{N_i}{L_i^{u}} \leq 2^{u_i} i^{1i} \alpha^{n^{1u}} N^{1-u_i}.
\] (3.43)

Noting that $a_i / a_i = ((i + 1)/L)^a \leq 1$ for $i < L$, we find
\[
\sum_{i=K+1}^{L} a_i \leq (K + 1) a_K + N a_{a K + 2} = O(N^{1 - (K + 1)u}) + O(N^{2 - (2K + 2)u})
\]
\[
= o(1)
\] (3.44)

and thus from (3.41),
\[
Z_N(1, \epsilon) \leq \frac{N-1}{(N-1)!} \sum_{\tilde{D} = L+1}^{N-1} \sum_{m_1 + \cdots + K m_k = N - \tilde{D}} \frac{\tilde{D}^{1u}}{(N - 1 - \sum_{j=1}^{K} m_j)!} \prod_{i=1}^{K} \frac{i^{1i} m_i}{m_i!}.
\] (3.45)

The sum here is just the sum in (3.37) with $m_i = 0$ for $i > K$, so we have shown that $Z_N(1, \epsilon)$ is dominated by such terms. Recalling (3.34) and (3.36), we see that
\[
\frac{Z_N}{(N-1)!} = (1 + o(1)) \sum_{m_1 + \cdots + K m_k = N - L - 1}^{N} \frac{(N - 1 - \sum_{j=1}^{K} m_j)^{1u}}{(N - 1 - \sum_{j=1}^{K} m_j)!} \prod_{i=1}^{K} \frac{i^{1i} m_i}{m_i!}
\] (3.46)

and that $Z_N$ is dominated by trees having exactly one vertex of degree $> \epsilon (N - 1)$ and all other vertices having degrees $\leq K + 1$.

By lemma 3.2, the contribution from trees with $s(s) \leq K + 1$ is negligible, so it suffices to consider the case when the unique vertex with high degree is $s$, which proves (ii).

To obtain the more precise results in (i) and (iv), fix $i \leq K$, fix $m_i$ for $j \neq i$ and denote the summand in (3.46) by $b(m_i)$. Increasing $m_i$ by 1 decreases $D = N - 1 - \sum_{j=1}^{K} m_j$ by 1 and, assuming still $D > L$ and recalling the definition of $n_i$ in (2.9),
\[
\frac{b(m_i + 1)}{b(m_i)} \leq N^{1-u_i} i^{1u} m_i^{1u} = C_i \frac{N^{1-u_i} i^{1u}}{m_i + 1} \leq C_i \frac{n_i}{m_i + 1}.
\] (3.47)

If $m_i \geq [2C_i n_i]$, this ratio is less than 1/2. In particular,
\[
\sum_{m_i \geq [2C_i n_i]} b(m_i) \leq 2 b(3C_i n_i) \leq 2^{1-2C_i} b([2C_i n_i])
\] (3.48)

If $i < 1/\alpha$, then $n_i \to \infty$ as $N \to \infty$. Summing over all $m_j$, $j \neq i$, we see that the contribution to $Z_N$ from $m_i \geq 3C_i n_i$ is negligible, so we may assume that $m_i < 3C_i n_i$. In the exceptional case $i = 1/\alpha$, we obtain by the same argument that we may assume
$m_i < \log N$, say. In particular, since $n_i = O(N^{1-\iota}) = O(N^{1-\sigma})$, we see that we may assume $\sigma(s) = D + 1 = N - \sum_{j=1}^{K} jm_j = N - O(N^{1-\sigma})$, which proves (i).

For the remaining terms, we may now use $D = N - o(N)$ to improve (3.47) to

$$b(m_i + 1) \leq \frac{(C_d n_i)^m_i}{m_i!}.$$  (3.50)

Using this for each $i \leq K$, we see that the general summand in (3.46) is at most $\prod_{i=1}^{K} \frac{(C_d n_i)^m_i}{m_i!}$, and thus (3.46) yields

$$\frac{Z_N}{(N-1)!} \leq (1 + o(1)) \frac{(N-1)!^a}{(N-1)!} \sum_{m_1, \ldots, m_k} \prod_{i=1}^{K} \frac{(C_d n_i)^m_i}{m_i!}$$

which proves (2.8).

Finally, we show (iii). If $\tau$ is a tree in $\Gamma_N$ such that all vertices except $s$ have degrees $\leq K + 1$, but some branch attached to $s$ has more than $K + 1$ vertices, pick the first such branch and find, by breadth-first search, say, a subtree $t_0$ of that branch with exactly $K + 2$ vertices. Rearrange the edges inside $t_0$ so that $t_0$ is replaced by a star with its center adjacent to $s$; this produces a vertex of degree $K + 2$. Let $\tau' \in \Gamma_N$ be the result of making this change inside $\tau$. We have changed the degree of (at most) $K + 2$ vertices, and since all old and new degrees are at most $2K + 1$, the weights of $\tau$ and $\tau'$ differ by at most a constant factor. Furthermore, $\tau'$ has exactly one vertex of degree $K + 2$, and thus only a bounded number of trees $\tau$ can produce the same $\tau'$. Consequently,

$$\mathbb{P}(T_s \text{ has a branch of size } > K + 1) \leq C_N \mathbb{P}(T_s \text{ has a vertex } \neq s \text{ of degree } > K + 1),$$

and this probability tends to 0 by (ii).

**Proof of Theorem 2.6.** Recall that $Z_N$ is given by (3.46), and that the significant terms have $m_i = (1 + o(1)) n_i = O(N^{1-\iota})$, except when $i = 1/\alpha$.

Let us first note that if $1/\alpha$ is an integer and $i = K = 1/\alpha$, then, see the proof of theorem 2.5, (3.49) implies that $X_{K+1,N} = m_K$ has an asymptotic Poisson distribution $\text{Pois}(n_K)$, which further is asymptotically independent of $X_{i,N}, i \leq K$; furthermore, $\sum_{m_K} b(m_K) = \exp(n_K + o(1)) b(0)$. In the following, we thus assume $m_K = 0$ and sum only over $m_i, i < K$, when $i = K = 1/\alpha$; we omit the trivial modifications below in this case.
Define, for a fixed $\eta \in (0, 1)$, $V = \prod_{i=1}^{n}[1 - (1 - \eta)n_i, (1 + \eta)n_i]$. In the following, we consider only $(m_i)_i^K \in V$; recall that it suffices to sum over such $(m_i)$ in (3.46). For more compact notation, write

$$A = \sum_{i=1}^{K} m_i$$ and $$B = \sum_{i=1}^{K} im_i.$$ (3.54)

Note that $A$ and $B$ are $O(N^{1-\alpha})$. Use Stirling’s approximation on the first factor in the sum in (3.46) to obtain

$$\frac{(N - 1 - B)^{\eta}}{(N - 1 - A)!} = \sqrt{(2\pi (N - 1 - B))^{\eta}} \left(\frac{N - 1 - B}{e}\right)^{\alpha(N-1-B)}$$

$$\times \left(\frac{N - 1 - A}{e}\right)^{N-1-A} (1 + O(N^{-1}))$$

$$= \sqrt{(2\pi (N - 1))^{\eta}} \left(\frac{N - 1}{e}\right)^{(\alpha - 1)(N - 1)} (N - 1)^{A-B}$$

$$\times \exp \left\{ \alpha B - A + \alpha (N - 1 - B) \log \left(1 - \frac{B}{N - 1}\right) \right\} (1 + O(N^{-\alpha}))$$

$$= (N - 1)^{\eta - 1} (N - 1)^{A-B} \exp \left\{ \sum_{j=2}^{K} \frac{\alpha B^j - A^j}{j(j-1)(N-1)^{j-1}} \right\} (1 + o(1))$$

$$= (N - 1)^{\eta - 1} N^{A-B} \exp \left\{ \sum_{j=2}^{K} \frac{\alpha B^j - A^j}{j(j-1))^{j-1}} \right\} (1 + o(1)), (3.55)$$

where in the last step we expanded the logarithms and kept only powers of $A$ and $B$ which contribute for large $N$, and then approximated $N - 1$ by $N$. Hence, (3.46) yields, using Stirling’s formula again,

$$\frac{Z_N}{(N - 1)!^\alpha} = (1 + o(1)) \sum_{(m_i)_i^K \in V} \exp \left\{ \sum_{j=2}^{K} \frac{\alpha B^j - A^j}{j(j-1)(N-1)^{j-1}} \right\} \prod_{i=1}^{K} \frac{N^{m_i}_i - \alpha m_i}{m_i!}$$

$$= \sum_{(m_i)_i^K \in V} \exp(f(m_1, \ldots, m_K) + o(1)).$$

where

$$f(m_1, \ldots, m_K) = \sum_{j=1}^{K} \left( (1 - \alpha)i)m_i \log N + \alpha m_i \log i! - m_i \log m_i + m_i - \frac{1}{2} \log(2\pi m_i) \right)$$

$$+ \sum_{j=2}^{K} \frac{\alpha B^j - A^j}{j(j-1)(N-1)^{j-1}}.$$ (3.56)
Regard $f$ as a function of real variables $m_1, \ldots, m_K$. Then, for $m_1, \ldots, m_K \in V$, which entails $A, B = O(N^{1-\alpha})$,

$$\frac{\partial f}{\partial m_i} = (1 - \alpha i) \log N + \alpha \log(i!) - \log m_i - \frac{1}{2m_i} + \sum_{j=2}^{K} \frac{aiB^{j-1} - A^{j-1}}{(j-1)!N^{j-1}}.$$

$$= \log n_i - \log m_i - \frac{1}{2m_i} + \frac{aiB - A}{N} + O(N^{-2\alpha})$$

$$= \log n_i - \log m_i - \frac{1}{2m_i} - \frac{(1 - i\alpha)m_1}{N} + O(N^{-2\alpha})$$

$$= \log n_i - \log m_i + o(1) \quad (3.57)$$

and, similarly,

$$\frac{\partial^2 f}{\partial m_i \partial m_j} = -\frac{\delta_{ij}}{m_i} + O\left(\frac{\delta_{ij}}{m_i m_j}\right) + O\left(\frac{1}{N}\right). \quad (3.58)$$

$V$ is compact and $f$ is continuous, so $f$ attains its maximum in $V$ at some point $n^* = (n_1^*, \ldots, n_K^*) \in V$. By (3.57), for large $N$, $\frac{\partial f}{\partial m_i} > 0$ when $m_i = (1 - \eta)n_i$ and $\frac{\partial^2 f}{\partial m_i^2} < 0$ when $m_i = (1 + \eta)n_i$, so the maximum is not attained on the boundary of $V$, i.e. $|n_i^* - n_i| < \eta n_i$. Consequently, by (3.57),

$$0 = \frac{\partial f}{\partial m_i}(n^*) = \log n_i - \log n_i^* + o(1) \quad (3.59)$$

and thus $n_i^* = (1 + o(1))n_i$. A Taylor expansion of $f$ at $n^*$ yields, using (3.59) and (3.58), for $m = (m_1, \ldots, m_K) \in V$,

$$f(m) = f(n^*) - \frac{1}{2} \sum_{i=1}^{K} \left(\frac{(m_i - n_i^*)^2}{n_i^*} + O\left(\frac{|m_i - n_i^*|^3}{n_i^*}\right)\right) + O\left(\frac{|m - n^*|^2}{N}\right). \quad (3.60)$$

Choosing $\eta$ small enough, this implies first (for large $N$)

$$f(m) \leq f(n^*) - \frac{1}{2} \sum_{i=1}^{K} \frac{(m_i - n_i^*)^2}{n_i^*}, \quad (3.61)$$

which implies that it suffices to consider terms in (3.56) with, say, $|m_i - n_i^*| < \eta_1^{1/2} \log N$; let $V_1 \subset V$ be the set of such $m$. For such terms, (3.60) yields

$$f(m) = f(n^*) - \frac{1}{2} \sum_{i=1}^{K} \frac{(m_i - n_i^*)^2}{n_i^*} + o(1), \quad (3.62)$$

and thus by (3.56), letting $\beta = f(n^*)$ be the maximum value of $f$ on $V$,

$$\frac{Z_{\beta}}{(N - 1)!\alpha} = (1 + o(1)) \sum_{(m_i) \in V_1} \exp\left(\beta - \frac{1}{2} \sum_{i=1}^{K} \frac{(m_i - n_i^*)^2}{n_i^*} + o(1)\right). \quad (3.63)$$

Since each term here corresponds to the case $X_{i+1,N} = m_i, i = 1, \ldots, K$, and $n_i^* = (1 + o(1))n_i$, (2.12) follows. Furthermore, (3.63) also yields the Poisson approximation result in remark 2.7, since the Poisson probabilities $\mathbb{P}(Y_{i,N} = m_i, \forall i)$ can easily be approximated by the same Gaussian as in (3.63); we omit the details.

In order to obtain more precise estimates of $n_i^*$, we go back to (3.57) and refine (3.59) to

$$0 = \frac{\partial f}{\partial m_i}(n^*) = \log n_i - \log n_i^* - \frac{(1 - i\alpha)n_i^*}{N} + O(N^{-2\alpha} + N^{\alpha - 1}) \quad (3.64)$$
which yields
\[ \log \frac{n_i^*}{n_i} = -\frac{(1 - i\alpha)n_i^*}{N} + O(N^{-2\alpha} + N^{i\alpha - 1}) \] (3.65)
and thus, recalling \( n_i^*/N = O(N^{-\alpha}) \),
\[ \frac{n_i^*}{n_i} = 1 - \frac{(1 - i\alpha)n_i^*}{N} + O(N^{-2\alpha} + N^{i\alpha - 1}). \] (3.66)
Taking \( i = 1 \) we find \( n_1^*/n_1 = 1 + O(N^{-\alpha}) \), and thus \( n_1^* - n_1 = O(N^{1 - 2\alpha}) \), so (3.66) yields
\[ \frac{n_i^*}{n_i} = 1 - \frac{(1 - i\alpha)n_i^*}{N} + O(N^{-2\alpha} + N^{i\alpha - 1}), \] (3.67)
establishing (2.14).

We obtain (2.15)–(2.20) from (2.12) and (2.14), by checking that in each case the omitted terms in the numerator are of smaller order than the denominator.

Finally, to evaluate the partition function, we approximate the sum in (3.63) by a Gaussian integral and obtain
\[
\frac{Z_N}{(N-1)!^\alpha} = (1 + o(1)) \exp^0 \prod_{i=1}^K \sqrt{2\pi n_i^2} = \exp^{\beta + o(1)} \prod_{i=1}^K \sqrt{2\pi n_i}, \] (3.68)
We have \( \beta = f(n^*) \). Furthermore, (3.67) shows \( n_i^* - n_i = O(n_iN^{-\alpha}) = O(N^{1 - 2\alpha}) \), and it follows from (3.60) that, with \( n = (n_1, \ldots, n_K) \),
\[ f(n) = f(n^*) + O(N^{1 - 3\alpha}) = \beta + O(N^{1 - 3\alpha}), \] (3.69)
so it remains to evaluate \( f(n) \). For \( m = n \), the final sum in (3.56) is
\[
\frac{\alpha B^2 - A^2}{2N} + O(N^{1 - 3\alpha}) = \frac{(\alpha - 1)n_i^2}{2N} + O(N^{1 - 3\alpha}), \] (3.70)
and thus, after some cancellations,
\[ f(n) = \sum_{i=1}^K \left( n_i - \frac{1}{2} \log(2\pi n_i) \right) - \frac{(1 - \alpha)n_i^2}{2N} + O(N^{1 - 3\alpha}). \] (3.71)
Hence, (3.68) yields, with (3.69) and (3.71) and recalling \( n_1 = N^{1 - \alpha} \),
\[ \frac{Z_N}{(N-1)!^\alpha} = \exp \left( \sum_{i=1}^K n_i - \frac{1 - \alpha}{2} N^{1 - 2\alpha} + O(N^{1 - 3\alpha}) + o(1) \right). \] (3.72)
We substitute \( n_1 \) and \( n_2 \) from (2.9) and drop \( n_i \) for \( i \geq 3 \), which yields (2.22).

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