LATTICE POINTS IN POLYTOPES, BOX SPLINES, AND TODD OPERATORS

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Abstract. Let $X$ be a list of vectors that is totally unimodular. In a previous
document the author proved that every real-valued function on the set of interior
lattice points of the zonotope defined by $X$ can be extended to a function
on the whole zonotope of the form $p(D) BX$ in a unique way, where $p(D)$ is
a differential operator that is contained in the so-called internal $P$-space. In
this paper we construct an explicit solution to this interpolation problem in
terms of Todd operators. As a corollary we obtain a slight generalisation of
the Khovanskii-Pukhlikov formula that relates the volume and the number of
integer points in a smooth lattice polytope.

1. Introduction

Box splines and multivariate splines measure the volume of certain variable poly-
topes. The vector partition function that measures the number of integral points
in polytopes can be seen as a discrete version of these spline functions. Splines and
vector partition functions have recently received a lot of attention by researchers
in various fields including approximation theory, algebra, combinatorics, and rep-
resentation theory. A standard reference from the approximation theory point of
view is the book [10] by de Boor, Höllig, and Riemenschneider. The combinatorial
and algebraic aspects are stressed in the book [11] by De Concini and Procesi.

Khovanski and Pukhlikov proved a remarkable formula that relates the volume
and the number of integer points in a smooth polytope [18]. The connection is
made via Todd operators, i.e. differential operators of type $\frac{\partial}{\partial x}$. The formula
is closely related to the Hirzebruch-Riemann-Roch Theorem for smooth projective
toric varieties (see [7, Chapter 13]). De Concini, Procesi, and Vergne have shown
that the Todd operator is in a certain sense inverse to convolution with the box
spline [13]. This implies the Khovanskii-Pukhlikov formula and more generally the
formula of Brion-Vergne [6].

In this paper we will prove a slight generalisation of the deconvolution formula
by De Concini, Procesi, and Vergne. The operator that we use is obtained from
the Todd operator, but it is simpler, i.e. it a polynomial contained in the so-called
internal $P$-space. Our proof uses deletion-contraction, so in some sense we provide
a matroid-theoretic proof of the Khovanskii-Pukhlikov formula.

Furthermore, we will construct bases for the spaces $P_-(X)$ and $P(X)$ that were
studied by Ardila and Postnikov in connection with power ideals [2] and by Holtz
and Ron within the framework of zonotopal algebra [15]. Up to now, no general
construction for a basis of the space $P_-(X)$ was known.

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Oxford).
Let us introduce our notation. It is similar to the one used in [11]. We fix a 
d-dimensional real vector space $U$ and a lattice $\Lambda \subseteq U$. Let $X = (x_1, \ldots, x_N) \subseteq \Lambda$ be a finite list of vectors that spans $U$. We assume that $X$ is totally unimodular with respect to $\Lambda$, i.e., every basis for $U$ that can be selected from $X$ is also a lattice basis for $\Lambda$. Note that $X$ can be identified with a linear map $X : \mathbb{R}^N \to U$. Let $u \in U$. We define the variable polytopes

$$\Pi_X(u) := \{ w \in \mathbb{R}^N_{\geq 0} : Xw = u \} \quad \text{and} \quad \Pi_X^1(u) := \Pi_X(u) \cap [0; 1]^N. \quad (1)$$

Note that any convex polytope can be written in the form $\Pi_X(u)$ for suitable $X$ and $u$. The dimension of these two polytopes is at most $N - d$. We define the

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vector partition function $T_X(u) := |\Pi_X(u) \cap \mathbb{Z}^N|$, \quad (2)
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the box spline $B_X(u) := \det(XX^T)^{-1/2} \text{vol}_{N-d} \Pi_X^1(u)$, \quad \quad (3)
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and the multivariate spline $T_X(u) := \det(XX^T)^{-1/2} \text{vol}_{N-d} \Pi_X(u)$. \quad (4)

Note that we have to assume that 0 is not contained in the convex hull of $X$ in order for $T_X$ and $T_X$ to be well-defined. Otherwise, $\Pi_X(u)$ is an unbounded polyhedron. The zonotope $Z(X)$ is defined as

$$Z(X) := \left\{ \sum_{i=1}^N \lambda_i x_i : 0 \leq \lambda_i \leq 1 \right\} = X \cdot [0; 1]^N. \quad (5)$$

We denote its set of interior lattice points by $Z_\ast(X) := \text{int}(Z(X)) \cap \Lambda$. The symmetric algebra over $U$ is denoted by $\text{Sym}(U)$. We fix a basis $s_1, \ldots, s_d$ for the lattice $\Lambda$. This makes it possible to identify $\Lambda$ with $\mathbb{Z}^d$, $U$ with $\mathbb{R}^d$, $\text{Sym}(U)$ with the polynomial ring $\mathbb{R}[s_1, \ldots, s_d]$, and $X$ with a $(d \times N)$-matrix. Then $X$ is totally unimodular if and only if every non-singular square submatrix of this matrix has determinant 1 or $-1$. The base-free setup is more convenient when working with quotient vector spaces.

We denote the dual vector space by $V = U^*$ and we fix a basis $t_1, \ldots, t_d$ that is dual to the basis for $U$. An element of $\text{Sym}(U)$ can be seen as a differential operator on $\text{Sym}(V)$, i.e., $\text{Sym}(U) \cong \mathbb{R}[s_1, \ldots, s_d] \cong \mathbb{R}[\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_d}]$. For $f \in \text{Sym}(U)$ and $p \in \text{Sym}(V)$ we write $f(D)p$ to denote the polynomial in $\text{Sym}(V)$ that is obtained when $f$ acts on $p$ as a differential operator. It is known that the box spline is piecewise polynomial and its local pieces are contained in $\text{Sym}(V)$. We will mostly use elements of $\text{Sym}(U)$ as differential operators on its local pieces.

Note that a vector $u \in U$ defines a linear form $u \in \text{Sym}(U)$. For a sublist $Y \subseteq X$, we define $p_Y := \prod_{y \notin Y} y$. For example, if $Y = ((1, 0), (1, 2))$, then $p_Y = s_1^2 + 2s_1s_2$. Furthermore, $p_0 := 1$. Now we define the

central $\mathcal{P}$-space $\mathcal{P}(X) := \text{span}\{p_Y : \text{rk}(X \setminus Y) = \text{rk}(X)\}$ \quad (6)

and the internal $\mathcal{P}$-space $\mathcal{P}_\ast(X) := \bigcap_{x \in X} \mathcal{P}(X \setminus x)$. \quad (7)

The space $\mathcal{P}_\ast(X)$ was introduced in [15] where it was also shown that the dimension of this space is equal to $|Z_\ast(X)|$. The space $\mathcal{P}(X)$ first appeared in approximation theory [11] [9] [13]. These two $\mathcal{P}$-spaces and generalisations were later studied by various authors, including [2] [5] [16] [19] [21] [22].

In [20], the author proved the following theorem, which will be made more explicit in the present paper.

**Theorem 1.** Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is totally unimodular and spans $U$. Let $f$ be a real valued function on $Z_\ast(X)$, the set of interior lattice points of the zonotope defined by $X$. 

Then there exists a unique polynomial \( p \in \mathcal{P}_-(X) \subseteq \mathbb{R}[s_1, \ldots, s_d] \), s. t. \( p(D)B_X \) is a continuous function and its restriction to \( \mathcal{Z}_-(X) \) is equal to \( f \).

Here, \( p(D) \) denotes the differential operator obtained from \( p \) by replacing the variable \( s_i \) by \( \frac{\partial}{\partial s_i} \) and \( B_X \) denotes the box spline defined by \( X \).

Let \( z \in U \). As usual, the exponential is defined as \( e^z := \sum_{k \geq 0} \frac{z^k}{k!} \in \mathbb{R}[s_1, \ldots, s_d] \).

We define the \((z\text{-shifted})\) Todd operator

\[
\text{Todd}(X, z) := e^{-z} \prod_{x \in X} \frac{x}{1 - e^{-x}} \in \mathbb{R}[s_1, \ldots, s_d].
\]

The Todd operator can be expressed in terms of the Bernoulli numbers \( B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \ldots \). Recall that they are defined by the equation:

\[
\frac{s}{e^s - 1} = \sum_{k \geq 0} \frac{B_k}{k!} s^k.
\]

One should note that \( e^z \frac{x}{1 - e^{-x}} = \frac{e^{z+x}}{1 - e^{-z}} = \sum_{k \geq 0} \frac{B_k}{k!} (-z)^k \). For \( z \in \mathcal{Z}_-(X) \) we can fix a list \( S \subseteq X \) s. t. \( z = \sum_{x \in S} x \). Let \( T := X \setminus S \). Then we can write the Todd operator as

\[
\text{Todd}(X, z) = \prod_{x \in S} \left( \sum_{k \geq 0} \frac{B_k}{k!} x^k \right) \prod_{x \in T} \left( \sum_{k \geq 0} \frac{B_k}{k!} (-x)^k \right) = 1 + \sum_{x \in T} \frac{x}{2} - \sum_{x \in S} \frac{x}{2} + \ldots
\]

A sublist \( C \subseteq X \) is called a cocircuit if \( \text{rk}(X \setminus C) < \text{rk}(X) \) and \( C \) is inclusion minimal with this property. We consider the cocircuit ideal \( \mathcal{J}(X) := \text{ideal}\{p_C : C \text{ cocircuit}\} \subseteq \text{Sym}(U) \). It is known \[14, 15\] that \( \text{Sym}(U) = \mathcal{P}(X) \oplus \mathcal{J}(X) \). Let

\[
\psi_X : \mathcal{P}(X) \oplus \mathcal{J}(X) \to \mathcal{P}(X)
\]

denote the projection. Note that this is a graded linear map and that \( \psi_X \) maps any homogeneous polynomial to zero whose degree is at least \( N - d + 1 \). This implies that there is a canonical extension \( \psi_X : \mathbb{R}[s_1, \ldots, s_d] \to \mathcal{P}(X) \) given by \( \psi_X(\sum_i (g_i)) := \sum_i \psi_X(g_i) \), where \( g_i \) denotes a homogeneous polynomial of degree \( i \).

Let

\[
f_z = f_z^X := \psi_X(\text{Todd}(X, z)).
\]

**Theorem 2** (Main Theorem). Let \( X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d \) be a list of vectors that is totally unimodular and spans \( U \). Let \( z \) be a lattice point in the interior of the zonotope \( \mathcal{Z}(X) \).

Then \( f_z \in \mathcal{P}_-(X) \), \( \text{Todd}(X, z)(D)B_X \) extends continuously on \( U \), and

\[
f_z(D)B_X|_{\Lambda} = \text{Todd}(X, z)(D)B_X|_{\Lambda} = \delta_z.
\]

Here, \( f_z \) and \( \text{Todd}(X, z) \) act on the box spline \( B_X \) as differential operators.

Dahmen and Micchelli observed that

\[
T_X = B_X *_d T_X := \sum_{\lambda \in \Lambda} B_X(\cdot - \lambda)T_X(\lambda)
\]

(cf. \[11\] Proposition 17.17). Using this result, the following variant of the Khovanskii-Pukhlikov formula \[18\] follows immediately.

**Corollary 3.** Let \( X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d \) be a list of vectors that is totally unimodular and spans \( U \), \( u \in \Lambda \) and \( z \in \mathcal{Z}_-(X) \). Then

\[
|\Pi_X(u - z) \cap \Lambda| = T_X(u - z) = \text{Todd}(X, z)(D)T_X(u) = f_z(D)T_X(u).
\]
Remark 4. The box spline \( B_X \) is piecewise polynomial. Hence each of its local pieces is smooth but the whole function is not smooth where two different regions of polynomiality intersect. De Concini, Procesi, and Vergne [13] proved the following deconvolution formula, where \( B_X \) is replaced by a suitable local piece \( p_\cdot \): Todd(X,0)\((D)p_\cdot\mid_\Lambda = 0_0 \). In Section 2 we will explain the choice of the local piece.

One can deduce from [13] Remark 3.15 that \( \text{Todd}(X, z)(D)B_X \) can be extended continuously if \( z \in Z_-(X) \). It is also not difficult to show that \( \text{Todd}(X, z)(D)B_X = f_z(D)B_X \) (see Lemma 27) and that multiplying the Todd operator by \( e^{-z} \) corresponds to translating \( \text{Todd}(X, z)(D)B_X \) by \( x \). The novelty of the Main Theorem is that the operator \( f_z \) for \( z \in Z_-(X) \) is shorter than the original Todd operator (cf. Example 15), i.e. it is contained in \( P_-(X) \). Furthermore, we provide a new proof for De Concini, Procesi, and Vergne’s deconvolution formula.

We will also prove a slightly different version of the Main Theorem (Theorem 12) that only holds for local pieces of the box spline but where lattice points in the boundary of the zonotope are permitted as well. This theorem implies the following result.

**Corollary 5.** Let \( X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d \) be a list of vectors that is totally unimodular and spans \( U \) and let \( z \) be a lattice point in the zonotope \( Z(X) \). Let \( u \in \Lambda \) and let \( \Omega \subseteq \text{cone}(X) \) be a chamber s.t. \( u \) is contained in its closure. Let \( p_\Omega \) be the polynomial that agrees with \( \Pi_X(u-z) \cap \Lambda = T_X(u-z) = \text{Todd}(X, z)(D)p_\Omega(u) = f_z(D)p_\Omega(u). \) (15)

The original Khovanskii-Pukhlikov formula is the case \( z = 0 \) in Corollary 5. For more information on this formula, see Vergne’s survey article on integral points in polytopes [24]. An explanation of the Khovanskii-Pukhlikov formula that is easy to read is contained in the book by Beck and Robins [4, Chapter 10].

**Corollary 6.** Let \( X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d \) be a list of vectors that is totally unimodular and spans \( U \). Then

\[
\sum_{z \in Z_-(X)} B_X(z)f_z = 1. \quad (16)
\]

This implies formula (13).

The central \( P \)-space and various other generalised \( P \)-spaces have a canonical basis [15] [21]. Up to now, no general construction for a basis of the internal space \( P_-(X) \) was known (cf. [3] [15] [19]). The polynomials \( f_z \) form such a basis.

**Corollary 7.** Let \( X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d \) be a list of vectors that is totally unimodular and spans \( U \). Then \( \{f_z : z \in Z_-(X)\} \) is a basis for \( P_-(X) \).

We also obtain a new basis for the central \( P \)-space. Let \( w \in U \) be a short affine regular vector, i.e. a vector whose Euclidian length is close to zero that is not contained in any hyperplane generated by sublists of \( X \). Let \( Z(X,w) := (Z(X) - w) \cap \Lambda \) (see Figure 3 for an example). It is known that \( \dim P(X) = |Z(X,w)| = \text{vol}(Z(X)) \). (15).

**Corollary 8.** Let \( X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d \) be a list of vectors that is totally unimodular and spans \( U \). Then \( \{f_z : z \in Z(X,w)\} \) is a basis for \( P(X) \).

This corollary will be used to prove the following new characterisation of the internal space \( P_-(X) \).

**Corollary 9.** Let \( X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d \) be a list of vectors that is totally unimodular and spans \( U \). Then

\[
P_-(X) = \{f \in P(X) : f(D)B_X \text{ is a continuous function}\}. \quad (17)
\]
Remark 10. There is a related result due to Dahmen-Micchelli ([8] or [11] Theorem 13.21): for every function \( f \) on \( \mathcal{Z}(X, w) \) there exists a unique function in DM\((X)\) that agrees with \( f \) on \( \mathcal{Z}(X, w) \). Here, DM\((X)\) denotes the so-called discrete Dahmen-Micchelli space. The proof of the deconvolution formula in [13] relies on this result.

Organisation of the article. The remainder of this article is organised as follows: in Section 2 we will first define deletion and contraction. Then we will describe a method to make sense of derivatives of piecewise polynomial functions via limits and state a different version of the Main Theorem.

In Section 3 we will give some examples. In Section 4 we will recall some facts about zonotopal algebra, i.e., about the space \( \mathcal{P}(X) \), the dual space \( \mathcal{D}(X) \) and their connection with splines. These will be needed in the proof of the Main Theorem and its corollaries in Section 5. In the appendix we will give an alternative proof of the Main Theorem in the univariate case that uses residues.

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2. Deletion-contraction, limits, and the extended Main Theorem

In the first subsection, we will first define deletion and contraction and discuss the idea of the deletion-contraction proof of the Main Theorem. In the second subsection, we will describe a method to make sense of derivatives of piecewise polynomial functions via limits and then state a different version of the Main Theorem.

2.1. Deletion and contraction. Let \( x \in X \). We call the list \( X \setminus x \) the deletion of \( x \). The image of \( X \setminus x \) under the canonical projection \( \pi_\alpha : U \to U/\text{span}(x) =: U/x \) is called the contraction of \( x \). It is denoted by \( X/x \).

The projection \( \pi_\alpha \) induces a map \( \text{Sym}(U) \to \text{Sym}(U/x) \) that we will also denote by \( \pi_\alpha \). If we identify \( \text{Sym}(U) \) with the polynomial ring \( \mathbb{R}[s_1, \ldots, s_d] \) and \( x = s_d \), then \( \pi_\alpha \) is the map from \( \mathbb{R}[s_1, \ldots, s_d] \) to \( \mathbb{R}[s_1, \ldots, s_{d-1}] \) that sends \( s_d \) to zero and \( s_1, \ldots, s_{d-1} \) to themselves. The space \( \mathcal{P}(X/x) \) is contained in the symmetric algebra \( \text{Sym}(U/x) \).

Note that since \( X \) is totally unimodular, \( \Lambda/x \subseteq U/x \) is a lattice for every \( x \in X \) and \( X/x \) is totally unimodular with respect to this lattice.

Let \( x \in X \). Using matroid theory terminology, we call \( x \) a loop if \( \text{rk}(X \setminus x) < \text{rk}(X) \).

Recall that we defined \( f_z = \psi_X(\text{Todd}(X, z)) \) for \( z \in \mathcal{Z}_-(X) \). By Theorem 1 there is a unique polynomial \( q_z^X = q_z \in \mathcal{P}_-(X) \) s.t. \( q_z(D)B_X|_{\Lambda} = \delta_z \) for every \( z \in \mathcal{Z}_-(X) \). In order to prove the Main Theorem it is sufficient to show that \( f_z = q_z \).

In fact, \( q_z \) and \( f_z \) behave in the same way under deletion and contraction: they both satisfy the equalities \( xq_z^{X\setminus x} = q_z^X - q_{z+x}^X \) and \( \pi_\alpha(q_z^X) = q_z^{X/x} \). Unfortunately, it is not obvious that \( f_z \in \mathcal{P}_-(X) \).

Therefore, we have to make a detour. Since \( \mathcal{P}_-(X) \) is in general not spanned by polynomials of type \( p_Y \) for some \( Y \subseteq X \) (cf. [3]), it is quite difficult to handle this space. The space \( \mathcal{P}(X) \) on the other hand has a basis which is very convenient for deletion-contraction (cf. Proposition 2). Therefore, we will work with the larger space \( \mathcal{P}(X) \) and do a deletion-contraction proof there. An extended version of the Main Theorem will be stated in the next subsection. This will require some adjustments since for \( f \in \mathcal{P}(X) \), \( f(D)B_X|_{\Lambda} \) might not be well-defined.
2.2. Differentiating piecewise polynomial functions and limits.

Definition 11. Let $H$ by a hyperplane spanned by a sublist $Y \subseteq X$. A shift of such a hyperplane by a vector in the lattice $\Lambda$ is called an affine admissible hyperplane. An alcove is a connected component of the complement of the union of all affine admissible hyperplanes.

A vector $w \in U$ is called affine regular, if it is not contained in any affine admissible hyperplane. We call $w$ short affine regular if its Euclidian length is close to zero.

Note that on the closure of each alcove $c$, $B_X$ agrees with a polynomial $p_c$. For example, the six triangles in Figure 1 are the alcoves where $B_X$ agrees with a non-zero polynomial.

Fix a short affine regular vector $w \in U$. Let $u \in U$. Let $c \subseteq U$ be an alcove s.t. $u$ and $u + \varepsilon w$ are contained in its closure for some small $\varepsilon > 0$ and let $p_c$ be the polynomial that agrees with $B_X$ on the closure of $c$. We define $\lim_w B_X(z) := p_c(z)$ and for $f \in \text{Sym}(U)$

$$\lim_w f(D_{pw})B_X(u) := f(D)p_c(u)$$

(pw stands for piecewise). Note that the limit can be dropped if $f(D)B_X$ is continuous at $u$. Otherwise, the limit is important: note for example that $\lim_w B_{(1)}(0)$ is either 1 or 0 depending on whether $w$ is positive or negative. More information on this construction can be found in [13] where it was introduced. We will later see that $f_c(D)B_X(D)$ is continuous if $z \in Z(X) \cap \Lambda$ is in the interior of $Z(X)$ and discontinuous if it is on the boundary.

Recall that $Z(X,w) := (Z(X) - w) \cap \Lambda$.

Theorem 12. Let $X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d$ be a list of vectors that is totally unimodular and spans $U$. Let $w$ be a short affine regular and let $z \in Z(X,w)$. Then

$$\lim_w f_z(D_{pw})B_X|_{\Lambda} = \lim_w \text{Todd}(X,z)(D_{pw})B_X|_{\Lambda} = \delta_z.$$  

3. Examples

Example 13. We consider the three smallest one-dimensional examples. For $X = (1,1)$ we obtain $\text{Todd}((1,1),1) = (1 + B_1 s + \ldots)(1 - B_1 s + \ldots) = 1 + 0s + \ldots$

Hence $f_1^{(1,1)} = 1$. Furthermore,

$$f_1^{(1,1,1)} = 1 + \frac{s}{2}, \quad f_2^{(1,1,1)} = 1 - \frac{s}{2},$$

$$f_1^{(1,1,1)} = 1 + s + \frac{s^2}{3}, \quad f_2^{(1,1,1)} = 1 - \frac{s^2}{6}, \quad \text{and} \quad f_3^{(1,1,1)} = 1 - s + \frac{s^2}{3}. $$
Example 14. Let $X = \{(1,0),(0,1),(1,1)\} \subseteq \mathbb{Z}^2$. Then $\mathcal{P}_- (X) = \mathbb{R}$, $\mathcal{P}(X) = \text{span}\{1,s_1,s_2\}$, $\mathcal{Z}_-(X) = \{(1,1)\}$, and $f_{(1,1)} = 1$. $\Pi_X(u_1,u_2) \cong [0, \min(u_1,u_2)] \subseteq \mathbb{R}$. The multivariate spline and the vector partition function are:

$$T_X(u_1,u_2) = \begin{cases} u_2 & \text{for } 0 \leq u_2 \leq u_1 \\ u_1 & \text{for } 0 \leq u_1 \leq u_2 \end{cases}$$

and $T_X(x,y) = \begin{cases} u_2 + 1 & \text{for } 0 \leq u_2 \leq u_1 \\ u_1 + 1 & \text{for } 0 \leq u_1 \leq u_2 \end{cases}$.

(20)

Corollary 3 correctly predicts that $T_X(u)|_{\mathbb{Z}^2} = T_X(u - (1,1))$. Figure 1 shows the six non-zero local pieces of $B_X$ and the seven polynomials $f_\varepsilon$ attached to the lattice points of the zonotope $Z(X)$.

Example 15. We consider the polygons in Figure 2, which we will denote by $\bigcirc$, $\square$, and $\Delta$. These polytopes are defined by the matrix

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

(21)

where the first three columns correspond to slack variables and the rows of the last two are the normal vectors of the movable facets. The corresponding zonotope has two interior lattice points: $(1,1,1)$ corresponds to $\square$ and $(1,1,2)$ to $\Delta$. The projections of the Todd operators are $f_{\square} = 1 + s_3/2$ and $f_{\Delta} = 1 - s_3/2$, where $s_3$ corresponds to shifting the diagonal face of the pentagon. Elementary calculations show that shifting this face outward by $\varepsilon \in [-1,1]$ increases the volume of the pentagon by $(-1/2)\varepsilon^2$. This implies $\frac{\partial}{\partial s_3} \text{vol}(\text{pentagon}) = 1$.

The volume of the pentagon is 3.5. Corollary 3 correctly predicts that

$$|\square \cap \mathbb{Z}^2| = \text{vol}(\bigcirc) + \frac{1}{2} \frac{\partial}{\partial s_3} \text{vol}(\bigcirc) = 4$$

(22)

and $|\Delta \cap \mathbb{Z}^2| = \text{vol}(\bigcirc) - \frac{1}{2} \frac{\partial}{\partial s_3} \text{vol}(\bigcirc) = 3$.

(23)

The projection to $\mathcal{P}(X)$ of the unshifted Todd operator $\text{Todd}(X,0)$ is a lot more complicated: $f_{\bigcirc} = 1 + s_1 + s_2 + 1/2 s_3 + s_1 s_2 + s_1 s_3 + s_2 s_3 + s_3^2$.

On the other hand, since $\mathcal{Z}_-(X)$ contains only two points, the box spline $B_X$ must assume the value $1/2$ at both points. Then Corollary 13 correctly predicts that the volume of the pentagon is the arithmetic mean of the number of integer points in the square and the triangle.

Remark 16. If $X$ is not totally unimodular, then $\psi_X(\text{Todd}(X,z))$ is in general not contained in $\mathcal{P}_-(X)$. Consider for example $X = (2,1)$ and $\text{Todd}(X,z) = \frac{2z^2}{2z^2 - z^2 - 1 + z^2 z}$. Then

$$\psi_X(f_z) = \psi_X(2(1 + 2B_1 x)(1 - B_1 x)) = 2 - x \notin \mathcal{P}_-(X) = \mathbb{R}.$$

(24)

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1 This means replacing the inequality $a + b \leq 3$ by $a + b \leq 3 + \varepsilon$. 
Example 17. Let $X$ be a reduced oriented incidence matrix of the complete graph on 4 vertices / the set of positive roots of the root system $A_3$ (cf. Figure 3), i.e.

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix}.$$  (25)

$$f_{(1,1,0)} = -\frac{1}{6} t_1 t_2 - \frac{1}{6} t_1 t_3 + \frac{1}{3} t_2 t_3 + \frac{1}{2} t_1 - \frac{1}{2} t_2 - \frac{1}{2} t_3 + 1$$  (26)

$$f_{(1,1,-1)} = -\frac{1}{6} t_1 t_2 + \frac{1}{3} t_1 t_3 - \frac{1}{6} t_2 t_3 + \frac{1}{2} t_1 - \frac{1}{2} t_2 + \frac{1}{2} t_3 + 1$$  (27)

$$f_{(2,0,-1)} = -\frac{1}{6} t_1 t_2 - \frac{1}{6} t_1 t_3 + \frac{1}{3} t_2 t_3 - \frac{1}{2} t_1 + \frac{1}{2} t_2 + \frac{1}{2} t_3 + 1$$  (28)

$$f_{(1,0,0)} = \frac{1}{3} t_1 t_2 - \frac{1}{6} t_1 t_3 - \frac{1}{6} t_2 t_3 + \frac{1}{2} t_1 + \frac{1}{2} t_2 - \frac{1}{2} t_3 + 1$$  (29)

$$f_{(2,1,-1)} = \frac{1}{3} t_1 t_2 - \frac{1}{6} t_1 t_3 - \frac{1}{6} t_2 t_3 - \frac{1}{2} t_1 - \frac{1}{2} t_2 + \frac{1}{2} t_3 + 1$$  (30)

$$f_{(2,0,0)} = -\frac{1}{6} t_1 t_2 + \frac{1}{3} t_1 t_3 - \frac{1}{6} t_2 t_3 - \frac{1}{2} t_1 + \frac{1}{2} t_2 - \frac{1}{2} t_3 + 1$$  (31)

4. Zonotopal Algebra

In this section we will recall a few facts about the space $\mathcal{P}(X)$ and its dual, the space $\mathcal{D}(X)$ that is spanned by the local pieces of the box spline. The theory around these spaces was named zonotopal algebra by Holtz and Ron in [15]. This theory allows us to explicitly describe the map $\psi_X$. This description will then be used to describe the behaviour of the polynomials $f_z$ under deletion and contraction. This section is an extremely condensed version of [21].

Recall that the list of vectors $X$ is contained in a vector space $U \cong \mathbb{R}^d$ and that we denote the dual space by $V$. We start by defining a pairing between the
symmetric algebras \( \text{Sym}(U) \cong \mathbb{R}[s_1, \ldots, s_d] \) and \( \text{Sym}(V) \cong \mathbb{R}[t_1, \ldots, t_d] \):

\[
\langle \cdot, \cdot \rangle : \mathbb{R}[s_1, \ldots, s_d] \times \mathbb{R}[t_1, \ldots, t_d] \to \mathbb{R}
\]

\[
\langle p, f \rangle := \left( p \left( \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_d} \right) f \right)(0),
\]

i.e. we let \( p \) act on \( f \) as a differential operator and take the degree zero part of the result. Note that this pairing extends to a pairing \( \langle \cdot, \cdot \rangle : \mathbb{R}[[s_1, \ldots, s_d]] \times \mathbb{R}[[t_1, \ldots, t_d]] \to \mathbb{R} \).

We will now explain a construction of certain polynomials that is essentially due to Micchelli [12]. Let \( \mathcal{B} = (b_1, \ldots, b_d) \subseteq \mathbb{Z} \) be a basis. It is known that the box spline \( B_X \) agrees with a polynomial on the closure of each alcove. These local pieces and their partial derivatives span the Dahmen-Micchelli space \( \mathcal{D}(X) \). This space can be described as the kernel of the cocircuit ideal \( \mathcal{J}(X) \), namely

\[
\mathcal{D}(X) = \{ f \in \text{Sym}(V) : \langle p, f \rangle = 0 \text{ for all } p \in \mathcal{J}(X) \}. \tag{33}
\]

We will now explain a construction of certain polynomials that is essentially due to De Concini, Procesi, and Vergne [12]. Let \( Z \subseteq U \) be a finite list of vectors and let \( B = (b_1, \ldots, b_d) \subseteq Z \) be a basis. It is important that the basis is ordered and that this order is the order obtained by restricting the order on \( Z \) to \( B \). For \( i \in \{0, \ldots, d\} \), we define \( S_i = S^B_i := \text{span}\{b_1, \ldots, b_i\} \). Hence

\[
\{0\} = S^B_0 \subseteq S^B_1 \subseteq S^B_2 \subseteq \cdots \subseteq S^B_d = U \cong \mathbb{R}^d \tag{34}
\]

is a flag of subspaces. Let \( u \in S_i \backslash S_{i-1} \). The vector \( u \) can be written as \( u = \sum_{v=1}^{\lambda} \lambda_v b_v \) in a unique way. Note that \( \lambda_i \neq 0 \). If \( \lambda_i > 0 \), we call \( u \) positive and if \( \lambda_i < 0 \), we call \( u \) negative. We partition \( Z \cap (S_i \backslash S_{i-1}) \) as follows:

\[
P_i^B := \{ u \in Z \cap (S_i \backslash S_{i-1}) : u \text{ positive} \} \tag{35}
\]

and \( N_i^B := \{ u \in Z \cap (S_i \backslash S_{i-1}) : u \text{ negative} \} \tag{36} \).

We define

\[
T_i^{B+} := (-1)^{|N_i|} \cdot T_{P_i} \star T_{-N_i} \text{ and } T_i^{B-} := (-1)^{|P_i|} \cdot T_{-P_i} \star T_{N_i}. \tag{37}
\]

Note that \( T_i^{B+} \) is supported in \( \text{cone}(P_i, -N_i) \) and that

\[
T_i^{B+}(-x) = (-1)^{|P_i \cup N_i|} T_i^{B+}(-x). \tag{38}
\]

Now define

\[
R_i^B := T_i^{B+} - T_i^{B-} \quad \text{and} \quad R_i^B = R^B := R_1^B \star \cdots \star R_d^B. \tag{39}
\]

We denote the set of all sublists \( B \subseteq X \) that are bases for \( U \) by \( \mathbb{B}(X) \). Fix a basis \( B \in \mathbb{B}(X) \). A vector \( x \in X \backslash B \) is called externally active if \( x \in \text{span}\{b \in B : b \leq x\} \), i.e. \( x \) is the maximal element of the unique circuit contained in \( B \cup x \). The set of all externally active elements is denoted \( E(B) \).

**Definition 18** (Basis for \( \mathcal{D}(X) \)). Let \( X \subseteq U \cong \mathbb{R}^d \) be a finite list of vectors that spans \( U \). We define

\[
\mathcal{B}(X) := \{ |\text{det}(B)| R^B_{X \setminus E(B)} : B \in \mathbb{B}(X) \}. \tag{40}
\]

**Proposition 19** ([13 17]). Let \( X \subseteq U \cong \mathbb{R}^d \) be a finite list of vectors that spans \( U \). Then the spaces \( \mathcal{P}(X) \) and \( \mathcal{D}(X) \) are dual under the pairing \( \langle \cdot, \cdot \rangle \), i.e.

\[
\mathcal{D}(X) \to \mathcal{P}(X)^*, \quad f \mapsto \langle f, \cdot \rangle \tag{41}
\]

is an isomorphism.
Lemma 23. Let $Q$ be a finite list of vectors that spans $U$. A basis for $P(X)$ is given by

$$B(X) := \{Q_B : B \in B(X)\},$$

where $Q_B := p_X \setminus (B \cup E(B))$.

Theorem 21 (24). Let $X \subseteq U \cong \mathbb{R}^d$ be a finite list of vectors that spans $U$. Then $B(X)$ is a basis for $\mathcal{P}(X)$ and this basis is dual to the basis $B(X)$ for the central $\mathcal{P}$-space $\mathcal{P}(X)$.

Remark 22. Theorem 21 yields an explicit formula for the projection map $\psi_X : \mathbb{R}[s_1, \ldots, s_d] \to P(X)$ that we have defined on page 3:

$$f_\varepsilon := \psi_X(Todd(X, z)) = \sum_{B \in \mathbb{B}(X)} \langle Todd(X, z), R_B \rangle Q_B.$$  (43)

5. Proofs

In this section we will prove the Main Theorem and its corollaries. The proof uses a deletion-contraction argument. Deletion-contraction identities for the polynomials $f_\varepsilon$ will be obtained based on the following idea: one can write $\psi_X(f)$ as

$$\psi_X(f) = \sum_{B \in \mathbb{B}(X)} \langle f, R_B \rangle Q_B = \sum_{z \notin B \in \mathbb{B}(X)} \langle f, R_B \rangle Q_B + \sum_{x \in B \in \mathbb{B}(X)} \langle f, R_B \rangle Q_B$$

(44)

(cf. Remark 22). We will see that the first sum on the right-hand side of (44) corresponds to $\mathcal{P}_-(X \setminus x)$ and the second to $\mathcal{P}_-(X/x)$. Note that $\psi_X(f)$ is by definition independent of the order imposed on the list $X$, while each of the two sums depends on this order.

It is an important observation that for $x \in X$ and $z \in \Lambda$

$$x \cdot Todd(X \setminus x, z) = Todd(X, z) - Todd(X, z + x)$$

holds because $\frac{d}{dz} (1 - e^{-z}) = x$.

Lemma 23. Let $x \in X$ and let $f \in \text{Sym}(U)$. Then

$$x \psi_X(f) = \psi_X(xf).$$

Furthermore, for all $z \in \Lambda$, $x f_\varepsilon^{X \setminus x} = f_\varepsilon^{X} - f_\varepsilon^{X + x}$.  (47)

Proof. Note that the statement is trivial for $x = 0$, so from now on we assume that $x$ is not a loop. Since $\psi_X(f)$ is independent of the order imposed on $X$, we may rearrange the list elements s.t. $x$ is minimal.

Let $x \in B \in \mathbb{B}(X)$. Since $x$ is minimal, span($X \setminus E(B)$) = \{x\}. By Lemma 4.5 in [21], this implies that $\langle xf, R_B \rangle = \langle f, D_x R_B \rangle = \langle f, 0 \rangle = 0$.

Let $x \notin B \in \mathbb{B}(X)$. Since $x$ is minimal, this implies that $x \notin E(B)$. Then

$$\langle xf, R_X^{X \setminus (E(B) \cup x)} \rangle = \langle f, D_x R_X^{X \setminus (E(B) \cup x)} \rangle = \langle f, R_X^{X \setminus (E(B) \cup x)} \rangle.$$  (48)

Using the two previous observations we obtain:

$$\psi_X(xf) = \sum_{x \notin B \in \mathbb{B}(X)} \langle xf, R_X^{X \setminus (E(B) \cup x)} \rangle Q_B + \sum_{x \in B \in \mathbb{B}(X)} \langle xf, R_X^{X \setminus (E(B) \cup x)} \rangle Q_B = 0$$

(49)

The second to last equality follows from the fact that $Q_B = p_X \setminus (B \cup E(B)) = x p_X \setminus (B \cup E(B))$ if $x \notin B \cup E(B)$.
We can deduce the second claim using (45):
\[ x f_z^{X|z} = x \psi_X^X(\text{Todd}(X \setminus x, z)) = \psi_X^X(\text{Todd}(X \setminus x, z)) \]
\[ = \psi_X^X(\text{Todd}(X, z) - \text{Todd}(X, z + x)) = f_z^{X|z + x}. \]
\[ \square \]

Recall that \( \pi_x : \text{Sym}(U) \to \text{Sym}(U/x) \) denotes the canonical projection.

**Lemma 24.** Let \( x \in X \) be neither a loop nor a coloop. Let \( z \in \Lambda \) and let \( \bar{z} = \pi_x(z) \in \Lambda/x \). Then
\[ \pi_x(f_z) = f_{\bar{z}}. \]  
\[ (52) \]

**Proof.** Since the maps \( \pi_x \) and \( \psi_X \) are independent of the order imposed on \( X \), we may rearrange the list elements s.t. \( x \) is minimal.

Let \( x \in B \in \mathcal{B}(X) \). Given that \( x \) is minimal, \( R^B = (T_x - T_{-x}) \ast R^{B|z} \) follows. Hence \( R^B \) is constant in direction \( x \) so we can interpret it as a function on \( U/x \) and identify it with \( R^B \). In \( \text{Todd}(X, z) \), the factor \( x/(1 - e^{-x}) \) becomes \( 1 \) if we set \( x \) to 0. Note that \( x \) divides \( Q_B \) if \( x \not\in B \) since \( x \) is minimal. Then
\[ \pi_x(f_z) = \pi_x \left( \sum_{x \not\in B \in \mathcal{B}(X)} (\text{Todd}(X, z), R^B) Q_B \right) = 0, \text{since } x | Q_B. \]

\[ + \pi_x \left( \sum_{x \in B \in \mathcal{B}(X)} (\text{Todd}(X, z), R^B) Q_B \right) = \sum_{B \in \mathcal{B}(X/x)} (\text{Todd}(X/x, \bar{z}), R^B) Q_B = f_{\bar{z}}. \]
\[ \square \]

In [20], the author showed that the function
\[ \gamma_X : \mathcal{P}_-(X) \to \Xi(X) := \{ f : \Lambda \to \mathbb{R} : \text{supp}(f) \subseteq \mathcal{Z}_-(X) \} \]
that maps \( p \) to \( p(D_B X|_A) \) is an isomorphism. We will now extend \( \gamma_X \) to a map that is an isomorphism between \( \mathcal{P}(X) \) and a superspace of \( \Xi(X) \).

In a short affine regular vector \( w \in U \) we define
\[ \gamma^w_X : \mathcal{P}(X) \to \Xi^w(X) := \{ f : \Lambda \to \mathbb{R} : \text{supp}(f) \subseteq \mathcal{Z}(X, w) \} \]
\[ p \mapsto \lim_w p(D_{pw} B_X|_A). \]

**Proposition 25.** Let \( X \subseteq \Lambda \subseteq U \cong \mathbb{R}^d \) be a list of vectors that is totally unimodular and spans \( U \). Let \( x \in X \) be neither a loop nor a coloop.
Then the following diagram of real vector spaces is commutative, the rows are exact and the vertical maps are isomorphisms:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{P}(X \setminus x) & \xrightarrow{x} & \mathcal{P}(X) & \xrightarrow{\pi_x} \mathcal{P}(X/x) & \rightarrow & 0 \\
\downarrow{\gamma_{X \setminus x}} & & \downarrow{\gamma_X} & & \downarrow{\gamma_{X/x}} & & \\
0 & \rightarrow & Z^w(X \setminus x) & \xrightarrow{\nabla_x} & Z^w(X) & \xrightarrow{\Sigma_x} Z^w(X/x) & \rightarrow & 0
\end{array}
\]

where \( \nabla_x(f)(z) := f(z) - f(z - x) \)
and \( \Sigma_x(f)(\bar{z}) := \sum_{x \in \mathbb{Z} \cap \Lambda} f(x) = \sum_{\lambda \in \mathbb{Z}} f(\lambda x + z) \) for some \( z \in \bar{z} \).

Proof. First note that the map \( Z(X, w) \setminus Z(X \setminus x, w) \rightarrow Z(X/x, w) \) that sends \( z \) to \( \bar{z} \) is a bijection. This is a variant of \([20\text{ Lemma 17}]\) and it can be proved in the same way, using the fact that \( \dim \mathcal{P}(X) = \text{vol}(Z(X)) = |Z(X, w)| \), which was established in \([15]\). See also Figure 4. This implies that the second row is exact. Exactness of the first row is known (e.g. \([2\text{ Proposition 4.4}]\)).

If \( X \) contains only loops and coloops, then \( \mathcal{P}(X) = \mathbb{R} \) and \( B_X \) is the indicator function of the parallelepiped spanned by the coloops. For every short affine regular \( w \), \( Z(X, w) \) contains a unique point \( z_w \) and \( \lim_w B_X(z_w) = 1 \). Hence \( \gamma^w_X \) is an isomorphism in this case. Note that this also holds for \( U = \Lambda = \{0\} \). In this case \( w = 0 \), \( Z(X) = \{0\} = Z(X, w) \), and \( B_X = \lim_w B_X = 1 \).

The rest of the proof is analogous to the proof of \([20\text{ Proposition}]\) and therefore omitted.

For \( z \in Z(X, w) \), let \( q^X_w = q^w := (\gamma^w_X)^{-1}(\delta_z) \in \mathcal{P}(X) \). By construction, this polynomial satisfies \( \lim_w q^w_x(D_{pw})B_X|_{\Lambda} = \delta_z \).

**Lemma 26.** Let \( w \in U \) be a short affine regular vector and let \( z \in Z(X, w) \). Then \( q^w_x = f_x \).

In particular, \( q^w_x = q^{w'}_{x} \) for short affine regular vectors \( w \) and \( w' \) s.t. \( z \in Z(X, w) \cap Z(X, w') \).

Proof. Fix a short affine regular \( w \). We will show by induction that \( q^w_x = f_x \) for all \( z \in Z(X, w) \).

If \( X \) contains only loops and coloops, then \( \mathcal{P}(X) = \mathbb{R} \) and \( B_X \) is the indicator function of the parallelepiped spanned by the coloops. Hence \( f_x = \psi_X(1 + \ldots) = 1 = q^w_x \). Note that this is also holds for \( U = \Lambda = \{0\} \). In this case \( w = 0 \), \( Z(X) = \{0\} = Z(X, w) \), and \( B_X = \lim_w B_X = 1 \).

Now suppose that there is an element \( x \in X \) that is neither a loop nor a coloop and suppose that the statement is already true for \( X/x \) and \( X \setminus x \).

Let \( q^w_x = g_z := f_z - q^w_x \in \mathcal{P}(X) \).

**Step 1.** If \( z' = z + \lambda x \) for some \( \lambda \in \mathbb{Z} \), then \( g_z = g_{z'} \): by Lemma 23, \( x f_z^{X \setminus x} = f_z^X - f_z^{X + x} \). Because of the commutativity of \( \frac{[56]}{16} \), \( \gamma^w_X(x f_z^{X \setminus x}) = \nabla_x(\delta_z) \). Since \( \gamma^w_X \) is an isomorphism this implies \( x q^w_z = q^w_{z'} - q^w_{z + 2x} \). By assumption, \( q^w_X = f_z^{X \setminus x} \). Hence we can deduce that \( f_z^X - f_z^{X + x} = q^w_{z'} - q^w_{z + 2x} \) and the claim follows.

**Step 2.** \( \pi_x(g_z) = 0 \): it follows from Proposition 25 that \( \pi_x(q^w_x) = q^w_x \). Taking into account Lemma 23 this implies \( \pi_x(g_z) = f_z - q^w_x \). This is zero by assumption.

**Step 3.** \( g_z = 0 \): using the commutativity of \( \frac{[56]}{16} \) again and steps 1 and 2, we obtain

\[
|Z(X, w) \cap \text{span}(x + z)| = \gamma^w_X(g_z) = \Sigma_x(\gamma^w_X(g_z)) = \gamma^w_X(\pi_x(g_z)) = 0.
\]

Hence \( \gamma^w_X(g_z) = 0 \) and the claim follows since \( \gamma^w_X \) is an isomorphism.
Lemma 27. Let $f \in \mathbb{R}[[s_1, \ldots, s_d]]$ and let $w$ be a short affine regular vector.

1. Then $\lim_{w} f(D_{pw}) B_X = \lim_{w} \psi_X(f)(D_{pw}) B_X$.

2. If $\psi_X(f)(D) B_X$ is continuous, then $f(D) B_X$ can be extended continuously s.t. $\psi_X(f)(D) B_X = f(D) B_X$.

Proof. Let $u \in U$ and let $c$ be the alcove s.t. $u + \varepsilon w \in c$ for all sufficiently small $\varepsilon > 0$. Let $p_t$ be the polynomial that agrees with $B_X$ on the closure of $c$. By definition of the map $\psi_X$, for all $i$ the degree $i$ part of $f = f - \psi_X(f)$ is contained in $\mathcal{F}(X)$. By definition of $\mathcal{D}(X)$, $p_t \in \mathcal{D}(X)$. Then (33) implies $j(D)p_t = 0$. Hence $\lim_{w} f(D_{pw}) B_X(u) = f(D)p_t(u) = \psi_X(f)(D)p_t(u) = \lim_{w} \psi_X(f)(D_{pw}) B_X(u)$.

If $\psi_X(f)(D) B_X$ is continuous, then

$$\psi_X(f)(D) B_X(u) = \lim_{w} \psi_X(f)(D_{pw}) B_X(u) = \lim_{w} f(D_{pw}) B_X(u)$$

for all short affine regular vectors $w$, so if we define $f(D) B_X(u) := \psi_X(f) B_X(u)$ for all $u \in U$, we obtain a continuous extension of $f(D) B_X$ to $U$. □

Proof of Theorem 2 (Main Theorem) and of Theorem 12. If $w$ is a short affine regular vector and $z \in \mathcal{Z}(X,w)$, then $\lim_{w} f_z(D_{pw}) B_X|_\Lambda = \delta_z$ by Proposition 25 and Lemma 26.

Now let $z \in \mathcal{Z}_-(X)$. Then by Theorem 1 there exists $q_z^- \in \mathcal{P}_-(X)$ s.t. $q_z^- (D) B_X|_{\mathcal{Z}_-(X)} = \delta_z$ and $q_z^- (D) B_X$ is continuous. Continuity implies that $q_z^- (D) B_X$ vanishes on the boundary of $\mathcal{Z}(X)$, so $q_z^- (D) B_X|_\Lambda = \delta_z$. Furthermore, for any short affine regular $w$, $\lim_{w} q_z^- (D_{pw}) B_X|_\Lambda = \delta_z$. Since $\gamma_X^w$ is injective, $q_z^- = q_z^w = f_z$ must hold. Hence, $f_z \in \mathcal{P}_-(X)$.

To finish the proof, note that Lemma 27 implies that $\lim_{w} f_z(D_{pw}) B_X|_\Lambda = \lim_{w} \text{Todd}(X,z)(D_{pw}) B_X|_\Lambda$ and if $f_z(D) B_X$ is continuous it is possible to extend $\text{Todd}(X,z)(D) B_X$ continuously s.t. $f_z(D) B_X = \text{Todd}(X,z)(D) B_X$. □

Proof of Corollary 6. Let $p := \sum_{z \in \mathcal{Z}_-(X)} B_X(z) f_z$. By the Main Theorem, $p \in \mathcal{P}_-(X)$ and $p(D) B_X$ and $B_X = 1(D) B_X$ agree on $\mathcal{Z}_-(X)$. By the uniqueness part of Theorem 1 this implies $p = 1$.

Using the previous observation and Corollary 8 we can deduce (13):

$$B_X *_d T_X(u) = \sum_{\lambda \in \Lambda} B_X(u - \lambda) T_X(\lambda) = \sum_{z \in \Lambda} B_X(z) T_X(u - z)$$

$$= \sum_{z \in \Lambda} B_X(z) f_z(D) T_X(u) = T_X(u).$$

□

Proof of Corollary 7. By Theorem 2, $\{f_z : z \in \mathcal{Z}_-(X)\}$ is a linearly independent subset of $\mathcal{P}_-(X)$. This set is actually a basis since $|\mathcal{Z}_-(X)| = \dim \mathcal{P}_-(X)$. This equality follows from Theorem 1. It was also proven in [13]. □

Proof of Corollary 8. By construction, $\{f_z : z \in \mathcal{Z}(X,w)\} \subseteq \mathcal{P}(X)$. Proposition 25 and Lemma 26 imply that the set is actually a basis. □

Proof of Corollary 9. “$\supseteq$” is part of Theorem 4.

“$\subseteq$”: Let $p \in \mathcal{P}(X)$ and suppose that $p(D) B_X$ is continuous. Let $w$ be a short affine regular vector. By Corollary 8 there exist uniquely determined $\lambda_z \in \mathbb{R}$ s.t.

$$p = \sum_{z \in \mathcal{Z}(X,w)} \lambda_z f_z.$$ 

Let $z \in \mathcal{Z}(X,w) \setminus \mathcal{Z}_-(X)$. Since $z - w \notin \mathcal{Z}(X)$, $\lim_{w} p(D_{pw}) B_X(z) = 0$ holds. As we assumed that $p(D) B_X$ is continuous, this implies that

$$0 = \lim_{w} p(D_{pw}) B_X(z) = \sum_{y \in \mathcal{Z}(X,w)} \lambda_y \delta_y(z) = \lambda_z.$$

Hence $p = \sum_{z \in \mathcal{Z}_-(X)} \lambda_z f_z$, which is in $\mathcal{P}_-(X)$ by the Main Theorem. □
Appendix A. The univariate case and residues

In this appendix we will give an alternative proof of (a part of) the Main Theorem in the univariate case. This proof was provided by Michèle Vergne.

A totally unimodular list of vectors $X \subseteq \mathbb{Z}^1 \subseteq \mathbb{R}^1$ contains only entries in $\{-1, 0, 1\}$. Suppose that it contains $a$ times $-1$ and $b$ times $1$. The zonotope $Z(X)$ is then the interval $[-a, b]$. We may assume that $X$ does not contain any zeroes and we choose $N$ s. t. $a + b = N + 1$. Then $\mathcal{P}(X) = \mathbb{R}[s]_{\leq N}$ and $\mathcal{P}(-X) = \mathbb{R}[s]_{\leq N-1}$.

Let $z \in Z_-(X) = \{-a + 1, \ldots, b - 1\}$. Then there exist positive integers $\alpha$ and $\beta$ s. t.

$$\text{Tod}(X, z) := e^{-zs} \left( \frac{s}{e^s - 1} \right)^\alpha \left( \frac{s}{1 - e^{-s}} \right)^\beta = \left( \frac{s}{e^s - 1} \right)^\alpha \left( \frac{s}{1 - e^{-s}} \right)^\beta . \quad (61)$$

In this case, $\psi_X : \mathbb{R}[s] \to \mathbb{R}[s]_{\leq N}$ is the map that forgets all monomials of degree greater or equal $N + 1$.

Lemma 28. Let $X \subseteq \mathbb{Z}^1 \subseteq \mathbb{R}^1$ be a list of vectors that is totally unimodular and let $z$ be an interior lattice point of the zonotope $Z(X)$. Then $f_z = \psi_X(\text{Tod}(X, z)) \in \mathcal{P}_-(X)$.

Proof. Suppose that $X$ contains $N + 1$ non-zero entries. $\text{Tod}(X, z)$ agrees with its Taylor expansion

$$\text{Tod}(X, z) = 1 + c_1 s + \ldots + c_N s^N + \ldots \quad (62)$$

The coefficients $c_i$ depend on $z$ and $X$ and can be expressed in terms of Bernoulli numbers. It is sufficient to show that $c_N = 0$. This can be done by calculating the residue at the origin. Let $\gamma \subseteq \mathbb{C}$ be a circle around the origin. Using the residue theorem and the considerations at the beginning of this section we obtain:

$$c_N = \text{Res}_0 \left( z^{-(N+2)} \text{Tod}(X, z) \right) = \text{Res}_0 \left( \frac{1}{(e^s - 1)^\alpha (1 - e^{-s})^\beta} \right) \quad (63)$$

$$= \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{(1 - e^{-s})^\alpha (e^s - 1)^\beta} \, ds = \frac{1}{2\pi i} \oint_{\exp(\gamma)} \frac{1}{(1 - \sigma)^\alpha (\sigma - 1)^\beta} \, d\sigma \quad (64)$$

$$= \frac{1}{2\pi i} \oint_{\exp(\gamma)} \frac{\sigma^{\alpha-1}}{(\sigma - 1)^{\alpha+\beta}} \, d\sigma = \frac{1}{2\pi i} \oint_{\exp(\gamma)} \sum_{i=0}^{\alpha-1} \left( \frac{\alpha - 1}{i} \right) (\sigma - 1)^{-(\alpha+\beta)+1} \, d\sigma .$$

The last equality can easily be seen by substituting $y = \sigma + 1$, expanding, and resubstituting. Note that $\exp(\gamma)$ is a curve around 1. Since $\alpha$ and $\beta$ are positive the residue of the integrand of the last integral at one is 0. Hence $c_N = 0$. □

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