Drinfeld realization of the centrally extended $\mathfrak{psl}(2|2)$ Yangian algebra with the manifest coproducts

Takuya Matsumoto†

Abstract

The Lie superalgebra $\mathfrak{psl}(2|2)$ is recognized as a pretty special one in both mathematics and theoretical physics. In this paper, we present the Drinfeld realization of the Yangian algebra associated with the centrally extended Lie superalgebra $\mathfrak{psl}(2|2)$. Furthermore, we show that it possesses the Hopf algebra structures, particularly the coproducts. The idea to prove the existence of the manifest coproducts is the following. Firstly, we shall introduce them to Levendorskii's realization, a system of a finite truncation of the Drinfeld generators. Secondly, we show that Levendorskii's realization is isomorphic to the Drinfeld realization by induction.

† Department of Applied Physics, Faculty of Engineering, University of Fukui, 3-9-1 Bunkyo, Fukui-shi, Fukui 910-8507, Japan
E-mail: takuyama@u-fukui.ac.jp
Contents

1 Introduction and main theorem 1

2 Central extensions of the Lie superalgebra $\mathfrak{psl}(2|2)$ 4

3 Levendorskii’s realization of the Yangian $Y_L(g)$ 6
   3.1 Definition of the Yangian $Y_L(g)$ 7
   3.2 Hopf algebra structures of $Y_L(g)$ 8

4 Drinfeld realization of the Yangian $Y_D(g)$ 23

A Relation to Drinfeld’s first realization 41

1 Introduction and main theorem

It is known that the Lie superalgebra $\mathfrak{psl}(2|2)$ has the three-dimensional universal central extension \[ \mathfrak{g} \], which we denote by $\mathfrak{g}$ in this paper,
\[ \mathfrak{g} := \mathfrak{psl}(2|2) \oplus \mathbb{C}^3, \]
where $\mathbb{C}^3$ is the three-dimensional abelian Lie algebra. As shown in \[ \], except the Lie superalgebra $\mathfrak{psl}(2|2)$, the dimension of the universal central extension of the basic classical Lie superalgebra is one at most. In this sense, this algebra is an outstanding one.

In addition to the mathematical property, this algebra has special interests from physics, which were discovered by N. Beisert \[2, 3\]. He proposed a certain integrable spin-chain model having the symmetry of the Lie superalgebra $\mathfrak{g}$ and succeeded in solving the spectrum problem of the planar AdS/CFT correspondence in the string theory \[2\]. In this model, the dispersion relation of the excitation on the spin chain, so-called magnon, is completely determined by the presentation theory of the Lie superalgebra $\mathfrak{g}$. More precisely, the value of energy is given by the eigenvalue of one of the central elements $\mathbb{C}^3$ and the eigenvalues of the other two are the momenta of the magnon. The dispersion relation is mathematically interpreted as the atypical condition for the Kac module of $\mathfrak{g}$. The finite-dimensional irreducible representations of $\mathfrak{gl}(2|2)$ and $\mathfrak{psl}(2|2)$ are discussed \[4, 5\] and \[6\], respectively, and that of the Lie superalgebra $\mathfrak{g}$ is investigated in detail by \[7\]. Another remarkable property of this model is that the S-matrix describing the scattering of two magnons is completely determined by the symmetry of the Lie algebra $\mathfrak{g}$ up to an overall phase factor \[2\]. This S-matrix satisfies the Yang-Baxter equation. Thus, this model is integrable. Furthermore, as pointed out in \[3\], it is surprising that this S-matrix also appears in a
completely different physical context, the one-dimensional Hubbard model. Beisert’s S-matrix \([2]\) coincides with Shastry’s S-matrix for the one-dimensional Hubbard model \([8]\) up to a particular twist \(1\).

Since it is known that the Hubbard hamiltonian has the infinite-dimensional Yangian symmetry \(Y(sl(2)) \oplus Y(sl(2))\) \([10]\), it is natural to expect a similar symmetry of the S-matrix. Indeed, it turned out to be the Yangian symmetry \(Y(g)\) associated with the centrally extended Lie superalgebra \(g\) \([11]\), which includes the Yangian \(Y(sl(2)) \oplus Y(sl(2))\) as its subalgebra. The superYangian algebra \(Y(g)\) is the main object to consider in this article.

The Yangian algebra \(Y(g)\) in \([11]\) is given by the formulation of the Drinfeld’s original proposal, so-called the Drinfield’s first realization \([12, 13]\) . However, this formulation is not preferable for studying the representation theory. For this purpose, the Drinfel’d realization\(^2\) is more appropriate presentation since all Cartan generators are explicitly included in the defining relations \([14]\). Another important formulation is so-called RTT-realization, which is the formulation based on the Yang-Baxter equations. With this presentation, the coproducts structures are more transparent. The RTT-realization of \(Y(g)\) is proposed in \([15]\) . However, the closure of the algebra is not wholly proved, and consequently, the relations to the other realizations are not very obvious.

Thus it is desirable to consider the Drinfeld realization of the Yangian \(Y(g)\). For this point, there is already a proposal by Spill-Torrielli \([16]\). However, the Hopf algebra structure of \(Y(g)\) is not proved completely. In particular, it is far from evident that the coproducts \(\Delta : Y(g) \rightarrow Y(g) \otimes Y(g)\) are compatible with the defining relations. Regarding the defining relations in \([16]\), they are associated with the Dynkin diagram \(\otimes \otimes \otimes\), where all Chevalley generators are odd. With these generators, it is hard to see the decomposition of the module of \(Y(g)\) into that of \(Y(sl(2)) \oplus Y(sl(2))\). Therefore, in this paper, we will formulate the Drinfeld realization of \(Y(g)\) based on the Dynkin diagram \(O \otimes O\), where we only have the odd simple root for the second node and the other two are even. In particular, the first and third even roots are associated with the Chevalley generators of the two \(Y(sl(2))\) subalgebras respectively.

The centrally extended Lie superalgebra \(g\) is also obtained from the exceptional Lie superalgebra \(D(2, 1; \alpha)\) by a certain limit for the parameter \(\alpha\). The Drinfeld realization of the quantum affine algebra associated with \(D(2, 1; \alpha)\) is proposed by \([17]\) using the notation of the Weyl groupoid. Hence, it would be interesting to see the relation between our Drinfeld realization of the Yangian and that of the quantum affine algebra.

\(^1\)The R-matrix based on the quantum affine algebra associated with \(sl(2|2)\) is also proposed by \([9]\).

\(^2\)This is also referred as to Drinfel’d’s second realization, in contrast to Drinfel’d’s first realization. In Drinfeld’s original article, it is called the new realization \([14]\).
To introduce the coproduct structures to the Drinfeld realization of $\mathcal{Y}(\mathfrak{g})$, we shall start with Levendorskii’s realization \cite{Lev}, which we denote by $\mathcal{Y}_L(\mathfrak{g})$. In this formulation we only need the finite number of the Drinfeld generators; 
\[ h_{i,0}, \ x_{i,0}^\pm, \ h_{i,1}', \ x_{i,1}^\pm \ \text{with} \ i = 1, 2, 3, \ \text{and} \ P_0^\pm, \ P_1^\pm, \]  
(1.2)
where the generators $h_{i,0}, x_{i,0}^\pm$ and $P_0^\pm$ are degree zero, and nothing but the Chevalley generators of the Lie superalgebra $\mathfrak{g}$. The extended central elements are $P_0^\pm$. On the other hand, the generators $h_{i,1}', x_{i,1}^\pm$ and $P_1^\pm$ are degree one Yangian generators in $\mathcal{Y}(\mathfrak{g})$, which have the non-trivial coproduct structures. Therefore, Levendorskii’s realization \cite{Lev} is a truncation of the Drinfeld realization up to degree zero and one generator. With this realization, we can show that the coproducts are indeed compatible with the defining relations. Next, we will prove that the Levendorskii’s realization $\mathcal{Y}_L(\mathfrak{g})$ is isomorphic to the Drinfeld realization $\mathcal{Y}(\mathfrak{g})$, where we have the infinite numbers of generators; 
\[ h_{i,r}, \ x_{i,r}^\pm, \ \text{with} \ i = 1, 2, 3, \ \text{and} \ P_r^\pm, \]  
(1.3)
where the degrees are all non-negative integers $r = 0, 1, 2, \ldots$. The isomorphism is explicitly proved by the induction with respect to the degree $r$. Now we are ready to state our main theorem as below;

**Main Theorem.** The Yangian $\mathcal{Y}_D(\mathfrak{g})$ is generated by $h_{i,r}, x_{i,r}^\pm$ with $i = 1, 2, 3$ and the central elements $P_r^\pm$ with $r = 0, 1, 2, \ldots$, satisfying the following relations:

\[ [h_{i,r}, h_{j,s}] = 0 \]  
(1.4)
\[ [x_{i,r}^+, x_{j,s}^-] = \delta_{ij} h_{i,r+s} \]  
(1.5)
\[ [h_{i,0}, x_{j,r}^\pm] = \pm a_{ij} \xi_{j,r}^\pm \]  
(1.6)
\[ [h_{i,r+1}, x_{j,s}^\pm] - [h_{i,r}, x_{j,s+1}^\pm] = \pm \frac{1}{2} a_{ij} \{ h_{i,r}, x_{j,s}^\pm \} \]  
(1.7)
\[ [x_{i,r+1}^\pm, x_{j,s}^\pm] - [x_{i,r}^\pm, x_{j,s+1}^\pm] = \pm \frac{1}{2} a_{ij} \{ x_{i,r}^\pm, x_{j,s}^\pm \} \]  
(1.8)
\[ [x_{1,r}^\pm, x_{3,s}^\pm] = [x_{2,r}^\pm, x_{2,s}^\pm] = 0 \]  
(1.9)
\[ [x_{j,r}^\pm, [x_{j,s}^\pm, x_{2,t}^\pm]] + [x_{j,s}^\pm, [x_{j,r}^\pm, x_{2,t}^\pm]] = 0 \quad \text{for} \quad j = 1, 3 \]  
(1.10)
\[ [[x_{1,r}^\pm, x_{2,s}^\pm], [x_{3,s}^\pm, x_{2,0}^\pm]] = P_{r+s}^\pm \]  
(1.11)
where the indices of subscript $r, s, t$ runs non-negative integers and $i, j = 1, 2, 3$. The symmetrized Cartan matrix is given by

\[ (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \]  
(1.12)
The generators $x^\pm_{2,r}$ are odd, and the others are even. All commutators \([\ , \]\) (and anti-commutators \(\{\ , \}\)) are generalized as super (anti-)commutators, respectively. The Yangian $Y_D(\mathfrak{g})$ has a Hopf algebra structure with the coproducts $\Delta : Y_D(\mathfrak{g}) \rightarrow Y_D(\mathfrak{g}) \otimes Y_D(\mathfrak{g})$, counits $\epsilon : Y_D(\mathfrak{g}) \rightarrow \mathbb{C}$, and antipodes $S : Y_D(\mathfrak{g}) \rightarrow Y_D(\mathfrak{g})$, which are given in Proposition 3.4.

In particular, the coproducts structure allows us to prove the PBW theorem for the Yangian $Y_D(\mathfrak{g})$ along the idea of [19]. We will report the PBW theorem for $Y_D(\mathfrak{g})$ in the near future.

This paper is organized as follows. In Section 2, we review the centrally extended Lie superalgebra $\mathfrak{g}$. In Section 3, we define the Yangian algebra $Y_L(\mathfrak{g})$ based on Levendorskii’s realization and prove the Hopf algebra structures, in particular, the coproducts. In Section 4, we will construct Drinfeld realization of the Yangian $Y_D(\mathfrak{g})$ from Levendorskii’s realization $Y_L(\mathfrak{g})$, inductively. Consequently, the Hopf algebra structures of $Y_D(\mathfrak{g})$ are induced from $Y_L(\mathfrak{g})$. In Appendix A, we explain how our Yangian $Y_L(\mathfrak{g})$ is relating to the Drinfeld’s first realization.

## 2 Central extensions of the Lie superalgebra $\mathfrak{psl}(2|2)$

We start to define the central extensions of the Lie superalgebra $\mathfrak{psl}(2|2)$.

**Definition 2.1.** The centrally extended Lie superalgebra $\mathfrak{g} = \mathfrak{psl}(2|2) \oplus \mathbb{C}^3$ over $\mathbb{C}$ has the generators $h_{i,0}, x^\pm_{i,0}$ with $i = 1, 2, 3$ and the central elements $P^\pm_0$, and they satisfy the following relations;

\[
[h_{i,0}, h_{j,0}] = 0 \\
[h_{i,0}, x^\pm_{j,0}] = \pm a_{ij} x^\pm_{j,0} \\
[x^+_i, x^-_j] = \delta_{ij} h_{i,0} \\
[x^+_i, x^+_j] = [x^-_i, x^-_j] = 0 \\
[[x^+_i, x^+_j], [x^-_i, x^-_j]] = 0 \quad \text{for} \quad i = 1, 3 \\
[[x^+_i, x^-_j], [x^+_j, x^-_i]] = P^\pm_0.
\]

The $\mathbb{Z}_2$-grading $p : \mathfrak{g} \rightarrow \mathbb{Z}_2$ is defined by setting $p(x^+_{2,0}) = 1$ and $p(\text{others}) = 0$. The symmetrized Cartan matrix $a_{ij}$ is given in (1.12).

**Remark 2.2.** The Lie superalgebra $\mathfrak{g}$ is 17-dimension over $\mathbb{C}$. The abelian Lie subalgebra $\mathbb{C}^3$ is spanned by the two central elements $P^+_0, P^-_0$ and

\[
C_0 := -\frac{1}{2} h_{1,0} - h_{2,0} - \frac{1}{2} h_{3,0}.
\]

Noting that the generator $C_0$ is the central element of the Lie superalgebra $\mathfrak{sl}(2|2)$.
Remark 2.3. Definition 2.1 is associated with the Dynkin diagram $O \otimes O$ rather than $\otimes \otimes \otimes$ in [10], where the nodes $O$ and $\otimes$ denote even and odd roots, respectively. In particular, the sets of generators $\{x^\pm_{1,0}, h_{1,0}\}$ and $\{x^\pm_{3,0}, h_{3,0}\}$ span two $\mathfrak{sl}(2)$ subalgebras of $\mathfrak{g}$.

The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ has the Hopf algebra structures with the coproducts $\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$, counits $\epsilon : U(\mathfrak{g}) \rightarrow \mathbb{C}$ and antipodes $S : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ defined by

$$
\Delta(a) = a \otimes 1 + 1 \otimes a, \quad \epsilon(a) = 0, \quad S(a) = -a \quad (2.20)
$$

for $a \in U(\mathfrak{g})$. Since $U(\mathfrak{g})$ is $\mathbb{Z}_2$-graded algebra, to see the homomorphism of the coproducts, we need to take into account the following rule

$$(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)} ac \otimes bd. \quad (2.21)$$

for $a, b, c, d \in U(\mathfrak{g})$.

For the later arguments, it would be convenient to introduce the matrix presentation of the Lie superalgebra $\mathfrak{g}$ to express the non-simple root generators. The matrix presentation of $\mathfrak{g}$ is a quotient of the centrally extended general linear Lie superalgebra $\mathfrak{gl}(2|2)$. Let $\mathfrak{gl}(2|2) \oplus \mathbb{C}^2$ be a Lie superalgebra generated by the standard basis $E_{ij}$ with $1 \leq i, j \leq 4$ and the central elements $P, K$, satisfying the relations;

$$[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}(-1)^{p(i)+p(j)(p(k)+p(l))} + \bar{\epsilon}_{ik}\epsilon_{jl}P + \epsilon_{ik}\bar{\epsilon}_{jl}K, \quad (2.22)$$

where the square bracket is a super-commutator and the constants $\epsilon_{ij}$ and $\bar{\epsilon}_{ij}$ are zero except for the values

$$\epsilon_{12} = -\epsilon_{21} = 1 \quad \text{and} \quad \bar{\epsilon}_{34} = -\bar{\epsilon}_{43} = 1. \quad (2.23)$$

The $\mathbb{Z}_2$-grading of the indices is defined by $p(1) = p(2) = 0$, $p(3) = p(4) = 1$ and that of the generators are $p(E_{ij}) = p(i) + p(j)$. Consider a quotient of the Lie superalgebra $\mathfrak{gl}(2|2) \oplus \mathbb{C}^2$ by the relation

$$E_{11} + E_{22} - E_{33} - E_{44} = 0, \quad (2.24)$$

and denote this algebra by $\tilde{\mathfrak{g}} = \mathfrak{sl}(2|2) \oplus \mathbb{C}^2$. This notation is consistent with the following direct sum decomposition

$$\mathfrak{gl}(2|2) = \mathfrak{sl}(2|2) \oplus \mathbb{C}(E_{11} + E_{22} - E_{33} - E_{44}). \quad (2.25)$$

---

3Here we have omitted the abelian braiding factor introduced in [2, 3].
Proposition 2.4. The centrally extended Lie super algebra $\mathfrak{g}$ in Definition 2.1 is isomorphic to the Lie superalgebra $\tilde{\mathfrak{g}}$. The isomorphism $\mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ is given by

$$
\begin{align*}
x_{1,0}^+ &\mapsto E_{21} & x_{1,0}^- &\mapsto E_{12} & h_{1,0}^+ &\mapsto E_{22} - E_{11} \\
x_{2,0}^+ &\mapsto -E_{32} & x_{2,0}^- &\mapsto E_{23} & h_{2,0}^- &\mapsto -E_{33} - E_{22} \\
x_{3,0}^+ &\mapsto -E_{43} & x_{3,0}^- &\mapsto E_{34} & h_{3,0}^- &\mapsto -E_{44} + E_{33} \\
P_0^+ &\mapsto P & P_0^- &\mapsto -K
\end{align*}
$$

(2.26)

Proof. It is straightforward to see that the map in (2.26) preserves the defining relations (2.13)–(2.18) by using the relations (2.22). Thus, it is a homomorphism. The surjectivity is also obvious. Precisely, the non-simple roots are given by

$$
[x_{1,0}^+, x_{2,0}^+] \mapsto E_{31} \quad [x_{1,0}^-, x_{2,0}^-] \mapsto E_{13} \\
-[x_{2,0}^+, x_{3,0}^+] \mapsto E_{42} \quad [x_{2,0}^-, x_{3,0}^-] \mapsto E_{24} \\
[[x_{1,0}^+, x_{2,0}^+], x_{3,0}^+] \mapsto E_{41} \quad [[[x_{1,0}^-, x_{2,0}^-], x_{3,0}^-] \mapsto E_{14} .
$$

(2.27)

To prove the injectivity, we need to see that $\dim \mathfrak{g} \leq \dim \tilde{\mathfrak{g}} = 17$. Setting

$$
\Gamma = \{x_{i,0}^\pm, h_{i,0}, P, K\}_{i=1,2,3},
\Gamma' = \{[x_{1,0}^\pm, x_{2,0}^\pm], [x_{2,0}^\pm, x_{3,0}^\pm], [x_{1,0}^\pm, [x_{2,0}^\pm, x_{3,0}^\pm]]\} \cup \Gamma ,
\text{and } V = \text{span}_C \{X | X \in \Gamma'\} \subset \mathfrak{g},
$$

(2.28)

it is sufficient to show $\mathfrak{g} = V$. To confirm this, it is enough to check $[\Gamma, V] \subset V$. Since it is obvious that $[\Gamma, \Gamma] \subset V$, all we need to see are the relations $[\Gamma, Y] \subset V$ for $Y \in \{[x_{1,0}^\pm, x_{2,0}^\pm], [x_{2,0}^\pm, x_{3,0}^\pm], [x_{1,0}^\pm, [x_{2,0}^\pm, x_{3,0}^\pm]]\}$. We can verify this by direct computations using the relations (2.13)–(2.18). This completes the proof.

Remark 2.5. The defining relation (2.22) is almost same with that of $\mathfrak{gl}(2|2)$ except for the following nontrivial relations

$$
\begin{align*}
[E_{13}, E_{24}] &= -[E_{23}, E_{14}] = K ,
[E_{31}, E_{42}] &= -[E_{32}, E_{41}] = P.
\end{align*}
$$

(2.29) (2.30)

Through the isomorphism in (2.26), the above relations correspond to the extended Serre relations in (2.18). See [20] for the relating arguments.

3 Levendorskii’s realization of the Yangian $\mathcal{Y}_L(\mathfrak{g})$

This section defines the Yangian algebra $\mathcal{Y}_L(\mathfrak{g})$ associated with the Lie superalgebra $\mathfrak{g}$ by Levendorskii’s realization. In Subsection 3.1, we define $\mathcal{Y}_L(\mathfrak{g})$ (Definition 3.1). In Subsection 3.2, we prove the hopf algebra structures, especially the coproducts (Proposition 3.4).
3.1 Definition of the Yangian \( Y_L(g) \)

Let us now define the Yangian algebra \( Y_L(g) \) associated with \( g \).

**Definition 3.1.** The Yangian \( Y_L(g) \) associated with the Lie superalgebra \( g \) is generated by \( h_{i,0}, x^\pm_{i,0}, \tilde{h}_{i,1}, x^\pm_{i,1} \) with \( i = 1, 2, 3 \) and the central elements \( P^\pm_0, P^\pm_1 \). They satisfy the relations (2.13)–(2.18) and

\[
\begin{align*}
[\tilde{h}_{i,1}, h_{j,0}] &= 0 \quad (3.1) \\
[\tilde{h}_{i,1}, \tilde{h}_{j,1}] &= 0 \quad (3.2) \\
[\tilde{h}_{i,1}, x^\pm_{j,0}] &= \pm a_{ij} x^\pm_{j,1} \quad (3.3) \\
[x^\pm_{i,1}, x^\pm_{j,0}] &= \delta_{ij} h_{i,1} \quad (3.4) \\
[x^\pm_{i,1}, x^\pm_{j,0}] - [x^\pm_{i,0}, x^\pm_{j,1}] &= \pm \frac{1}{2} a_{ij} \{x^\pm_{i,0}, x^\pm_{j,0}\} \quad (3.5) \\
[x^\pm_{2,1}, x^\pm_{2,0}] &= 0 \quad (3.6) \\
[\tilde{h}_{j,1}, [x^+_i, x^-_i]] &= 0 \quad \text{for } j = 1, 3 \quad (3.7) \\
[h_{1,1}, [x^+_2, x^-_2]] &= 0 \quad (3.8) \\
[[x^+_1, x^+_2, x^-_0], [x^+_3, x^-_2]] &= P^+_1 \quad (3.9)
\end{align*}
\]

where the generator \( h_{i,1} \) in (3.4) is defined by

\[
h_{i,1} = \tilde{h}_{i,1} + \frac{1}{2} (h_{i,0})^2. \quad (3.10)
\]

The \( \mathbb{Z}_2 \)-grading \( p : Y_L(g) \rightarrow \mathbb{Z}_2 \) is defined by setting \( p(x^\pm_{2,0}) = p(x^\pm_{2,1}) = 1 \) and \( p(\text{others}) = 0 \). The symmetrized Cartan matrix \( (a_{ij}) \) is given in (1.12).

**Remark 3.2.** This realization of Yangian, which is associated with a finite-dimensional complex simple Lie algebra, was originally proposed by Levendorskii \[18\]. Its generalization to the Yangians of the Lie superalgebras \( A(m,n) = \mathfrak{sl}(m+1|n+1) \) is given by \[21\]. Compared with the Yangian of the Lie superalgebra \( \mathfrak{sl}(2|2) \), the Serre relations (3.9) are extended.

Here, let us introduce a terminology *degree*. The degree of the generators of \( Y(g) \) is the second index of the subscript. For example, \( \deg(x^+_2,0) = 0 \) and \( \deg(x^+_2,1) = 1 \). The degree of a monomial is the sum of the degree of each generator. The degree of a polynomial is the highest degree of the monomials included in the polynomial. The degree of tensor product of monomials is defined by the sum of the degree of each tensor factor. The degree of a polynomial of tensor products is given by the highest degree of the tensor products included in the polynomial.

Set

\[
Y_L(g)_r := \{ x \in Y_L(g) \mid \deg(x) \leq r \}. \quad (3.11)
\]
Then, the Yangian $Y(g)$ has a natural filtration with respect to the degree,

$$\{0\} = Y_L(g)_{-1} \subset Y_L(g)_0 \subset Y_L(g)_1 \subset Y_L(g)_2 \subset \cdots .$$

(3.12)

**Remark 3.3.** The universal enveloping algebra $U(g)$ in Definition 2 is a subalgebra of $Y_L(g)$ and identified with $Y_L(g)_0$.

### 3.2 Hopf algebra structures of $Y_L(g)$

We show the Hopf algebra structures of $Y_L(g)$ in this subsection.

**Proposition 3.4.** The Yangian $Y_L(g)$ has the Hopf algebra structures with the coproducts $\Delta : Y_L(g) \to Y_L(g) \otimes Y_L(g)$ given by

$$\Delta(x^\pm_{i,0}) = x^\pm_{i,0} \otimes 1 + 1 \otimes x^\pm_{i,0}$$

$$\Delta(h_{i,0}) = h_{i,0} \otimes 1 + 1 \otimes h_{i,0} \quad (i = 1, 2, 3)$$

$$\Delta(P^+_0) = P^+_0 \otimes 1 + 1 \otimes P^+_0$$

$$\Delta(x^\pm_{1,1}) = x^\pm_{1,1} \otimes 1 + 1 \otimes x^\pm_{1,1} + h_{1,0} \otimes x^\pm_{1,0} - \sum_{\mu=3,4} E_{2\mu} \otimes E_{\mu 1}$$

$$\Delta(x^+_{2,1}) = x^+_{2,1} \otimes 1 + 1 \otimes x^+_{2,1} + h_{2,0} \otimes x^+_{2,0} + E_{12} \otimes E_{31} + E_{34} \otimes E_{42} - E_{14} \otimes P^+_0$$

$$\Delta(x^+_{3,1}) = x^+_{3,1} \otimes 1 + 1 \otimes x^+_{3,1} + h_{3,0} \otimes x^+_{3,0} - \sum_{l=1,2} E_{l3} \otimes E_{4l}$$

$$\Delta(\tilde{h}_{1,1}) = \tilde{h}_{1,1} \otimes 1 + 1 \otimes \tilde{h}_{1,1} - 2x^-_{1,0} \otimes x^+_{1,0} + \sum_{\mu=3,4} (E_{1\mu} \otimes E_{\mu 1} - E_{2\mu} \otimes E_{\mu 2})$$

$$\Delta(\tilde{h}_{2,1}) = \tilde{h}_{2,1} \otimes 1 + 1 \otimes \tilde{h}_{2,1} + E_{12} \otimes E_{21} - E_{13} \otimes E_{31} + E_{34} \otimes E_{43} + E_{24} \otimes E_{42} - P^+_0 \otimes P^+_0$$

$$\Delta(\tilde{h}_{3,1}) = \tilde{h}_{3,1} \otimes 1 + 1 \otimes \tilde{h}_{3,1} + 2x^-_{3,0} \otimes x^+_{3,0} + \sum_{l=1,2} (E_{l3} \otimes E_{3l} - E_{l4} \otimes E_{4l})$$

$$\Delta(x_{1,1}) = x^-_{1,1} \otimes 1 + 1 \otimes x^-_{1,1} + x^-_{1,0} \otimes h_{1,0} - \sum_{\mu=3,4} E_{1\mu} \otimes E_{\mu 2}$$

$$\Delta(x^-_{2,1}) = x^-_{2,1} \otimes 1 + 1 \otimes x^-_{2,1} + x^-_{2,0} \otimes h_{2,0} - E_{13} \otimes E_{21} - E_{24} \otimes E_{43} - P^-_0 \otimes E_{41}$$

$$\Delta(x^-_{3,1}) = x^-_{3,1} \otimes 1 + 1 \otimes x^-_{3,1} + x^-_{3,0} \otimes h_{3,0} + \sum_{l=1,2} E_{l4} \otimes E_{3l}$$

$$\Delta(P^+_1) = P^+_1 \otimes 1 + 1 \otimes P^+_1 - 2C_0 \otimes P^+_0$$

$$\Delta(P^-_1) = P^-_1 \otimes 1 + 1 \otimes P^-_1 - 2P^-_0 \otimes C_0 ,$$

(3.13)

the counits $\epsilon : Y_L(g) \to \mathbb{C}$,

$$\epsilon(X) = 0 \quad \text{for} \quad X \in Y_L(g),$$

(3.14)
and the antipodes \( S : Y_L(\mathfrak{g}) \to Y_L(\mathfrak{g}) \),

\[
\begin{align*}
S(x_{1,0}^\pm) &= -x_{1,0}^\pm, & S(h_{i,0}) &= -h_{i,0}, & (i = 1, 2, 3), & S(P_0^\pm) &= -P_0^\pm \\
S(x_{1,1}^\pm) &= -x_{1,1}^\pm + h_{1,0}x_{1,0}^\pm - \sum_{\mu = 3, 4} E_{2\mu}E_{\mu 1} \\
S(x_{2,1}^\pm) &= -x_{2,1}^\pm + h_{2,0}x_{2,0}^\pm + E_{12}E_{31} + E_{34}E_{42} - E_{14}P_0^+ \\
S(x_{3,1}^\pm) &= -x_{3,1}^\pm + h_{3,0}x_{3,0}^\pm - \sum_{l=1,2} E_{l4}E_{4l} \\
S(\tilde{h}_{1,1}) &= -\tilde{h}_{1,1} - 2x_{1,0}^\pm x_{1,0}^\pm + \sum_{\mu = 3, 4} (E_{1\mu}E_{\mu 1} - E_{2\mu}E_{\mu 2}) \\
S(\tilde{h}_{2,1}) &= -\tilde{h}_{2,1} + E_{12}E_{21} - E_{13}E_{31} + E_{34}E_{43} + E_{24}E_{42} - P_0^-P_0^+ \\
S(\tilde{h}_{3,1}) &= -\tilde{h}_{3,1} + 2x_{3,0}^\pm x_{3,0}^\pm + \sum_{l=1,2} (E_{13}E_{3l} - E_{4l}E_{l4}) \\
S(x_{1,1}^-) &= -x_{1,1}^- + x_{1,0}^h_{1,0} - \sum_{\mu = 3, 4} E_{1\mu}E_{\mu 2} \\
S(x_{2,1}^-) &= -x_{2,1}^- + x_{2,0}^- h_{2,0} - E_{13}E_{21} - E_{24}E_{43} - P_0^-E_{41} \\
S(x_{3,1}^-) &= -x_{3,1}^- + x_{3,0}^- h_{3,0} + \sum_{l=1,2} E_{4l}E_{l4} \\
S(P_0^+) &= -P_1^+ - 2C_0P_0^+ \\
S(P_0^-) &= -P_1^- - 2P_0^- C_0 .
\end{align*}
\]

(3.15)

**Remark 3.5.** The coproducts listed in (3.13) are essentially translated from [11], where Drinfeld’s first realization is adopted.

To prove Proposition 3.4, the following lemmas are useful.

**Lemma 3.6.** A map \( \rho : Y_L(\mathfrak{g}) \to Y_L(\mathfrak{g}) \) defined by

\[
\begin{align*}
x_{1,r}^+ &\mapsto x_{3,r}^- & x_{1,r}^- &\mapsto -x_{3,r}^+ & h_{1,r} &\mapsto h_{3,r} \\
x_{2,r}^+ &\mapsto -x_{2,r}^- & x_{2,r}^- &\mapsto -x_{2,r}^+ & h_{2,r} &\mapsto h_{2,r} \\
x_{3,r}^+ &\mapsto -x_{1,r}^- & x_{3,r}^- &\mapsto x_{1,r}^+ & h_{3,r} &\mapsto h_{1,r} \\
P_r^+ &\mapsto -P_r^- & P_r^- &\mapsto -P_r^+
\end{align*}
\]

(3.16)

with \( r = 0, 1 \) is an automorphism of the Yangian \( Y_L(\mathfrak{g}) \). The map \( \rho \) preserve the \( \mathbb{Z}_2 \)-grading

\[
p(\rho(x)) = p(x) \quad \text{for} \quad x \in Y_L(\mathfrak{g}),
\]

(3.17)

and satisfies \( \rho^2 = \text{id} \).
Proof. By direct computations, it is verified that the map $\rho$ preserves all defining relations in (2.13)–(2.18) and (3.1)–(3.9). The properties $\rho^2(x) = x$ and $p(\rho(x)) = p(x)$ for any $x \in Y_L(\mathfrak{g})$ immediately follow from the definition (3.16). \hfill \square

Remark 3.7. It is easy to see that the Lie superalgebra $\mathfrak{g}$ defined in a matrix form (2.22) enjoys an automorphism $E_{ij} \mapsto E_{5-i, 5-j}$ for $i, j = 1, \ldots, 4$; $P \mapsto K$, $K \mapsto P$. (3.18)

The automorphism $\rho$ in (3.16) for the generators with $r = 0$ agrees with this map. Then, $\rho$ is a natural extension of this to Yangian $Y_L(\mathfrak{g})$.

Lemma 3.8. The automorphism $\rho : Y_L(\mathfrak{g}) \to Y_L(\mathfrak{g})$ defined in (3.16) is compatible with the coproducts in (3.13),

$$(\rho \otimes \rho) \circ \tilde{\Delta} = \Delta \circ \rho, \quad (3.19)$$

where we denoted the opposite coproducts by $\tilde{\Delta} = \sigma \circ \Delta$ with the graded permutation $\sigma$ defined by $\sigma(x \otimes y) = (-1)^{p(x)p(y)} y \otimes x$ for $x, y \in Y_L(\mathfrak{g})$.

Proof. Set $G_r = \{ x_{i,r}^\pm, h_{i,r}, P_r^\pm \mid i = 1, 2, 3 \} \subset Y_L(\mathfrak{g})$ with $r = 0, 1$. It is obvious that (3.19) holds for $x \in G_0$ since their coproducts are primitive, $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in G_0$. Then, to prove the lemma, it is sufficient to show

$$(\rho \otimes \rho) \circ \tilde{\Delta}(x) = \Delta \circ \rho(x) \quad \text{for} \quad x \in G_1. \quad (3.20)$$

This is confirmed by direct calculations. For example,

$$(\rho \otimes \rho) \circ \tilde{\Delta}(\bar{h}_{1,1}) = (\rho \otimes \rho) \circ \sigma \left( \bar{h}_{1,1} \otimes 1 + 1 \otimes \bar{h}_{1,1} - 2x_{1,0}^+ \otimes x_{1,0}^- \right.

+ \sum_{\mu=3,4} \left( E_{1\mu} \otimes E_{\mu 1} - E_{2\mu} \otimes E_{\mu 2} \right) \bigg).

= (\rho \otimes \rho) \left( 1 \otimes \bar{h}_{1,1} + \bar{h}_{1,1} \otimes 1 - 2x_{1,0}^+ \otimes x_{1,0}^- \right.

- \sum_{\mu=3,4} \left( E_{1\mu} \otimes E_{\mu 1} - E_{2\mu} \otimes E_{\mu 2} \right) \bigg)

= 1 \otimes \tilde{h}_{3,1} + \tilde{h}_{3,1} \otimes 1 + 2x_{3,0}^- \otimes x_{3,0}^+

- \sum_{\mu=1,2} \left( E_{4\mu} \otimes E_{4\mu} - E_{3\mu} \otimes E_{3\mu} \right)

= \Delta \circ \rho(\bar{h}_{1,1}). \quad (3.21)$$

The computations are similar for the other generators in $G_1$. This completes the proof. \hfill \square
We are ready to prove Proposition 3.4.

**Proof of Proposition 3.4.** Let \( \mu : \mathcal{Y}_L(g) \otimes \mathcal{Y}_L(g) \to \mathcal{Y}_L(g) \) be the product of \( \mathcal{Y}_L(g) \) defined by \( \mu(x \otimes y) = xy \) for \( x, y \in \mathcal{Y}_L(g) \), and \( u : \mathbb{C} \to \mathcal{Y}_L(g) \) the unit defined by \( u(1) = 1 \in \mathcal{Y}_L(g) \). Then, it is straightforward to check that the antipodes in (3.15) satisfy the antipode relations,

\[
\mu \circ (S \otimes 1) \circ \Delta(x) = \mu \circ (1 \otimes S) \circ \Delta(x) = u \circ \epsilon(x) = 0
\]

for \( x \in \mathcal{Y}_L(g) \).

The non-trivial part is to show that the coproducts given in (3.13) is a homomorphism of Yangian \( \mathcal{Y}_L(g) \). We shall prove this with respect to each degree of the relations.

For the degree zero relations (2.13)–(2.18), since the products of the degree zero generators are primitive,

\[
\Delta(x) = x \otimes 1 + 1 \otimes x \quad \text{for} \quad x \in \mathcal{Y}_L(g)_0,
\]

it is easy to see that they are consistent with the relations

\[
[\Delta(x), \Delta(y)] = \Delta([x, y]) \quad \text{for} \quad x, y \in \mathcal{Y}_L(g)_0.
\]

The non-trivial part is to show that the coproducts given in (3.13) is a homomorphism of Yangian \( \mathcal{Y}_L(g) \). We shall prove this with respect to each degree of the relations.

For the degree zero relations (2.13)–(2.18), since the products of the degree zero generators are primitive,

\[
\Delta(x) = x \otimes 1 + 1 \otimes x \quad \text{for} \quad x \in \mathcal{Y}_L(g)_0,
\]

it is easy to see that they are consistent with the relations

\[
[\Delta(x), \Delta(y)] = \Delta([x, y]) \quad \text{for} \quad x, y \in \mathcal{Y}_L(g)_0.
\]

The degree one relations are (3.1), (3.3), (3.4), (3.5), (3.6), and (3.9). It is not so difficult to see the compatibility of the coproducts with the relations by direct computations.

- **Relation (3.1).**

  \[
  [\Delta(\tilde{h}_{i,1}), \Delta(h_{j,0})] = 0 \quad \text{for} \quad i, j = 1, 2, 3.
  \]

- **Relations (3.3).** After some computations, we can verify

  \[
  [\Delta(\tilde{h}_{i,1}), \Delta(x^+_j,0)] = a_{ij} \Delta(x^+_{j,1})
  \]

  for \( i, j = 1, 2, 3 \). Then, the following relations are obtained from the above by using the automorphism \( \rho \) in (3.16) and Lemma 3.8.

  \[
  [\Delta(\tilde{h}_{i,1}), \Delta(x^-_j,0)] = -a_{ij} \Delta(x^-_{j,1}).
  \]

- **Relations (3.4).** At first, we compute

  \[
  [\Delta(x^+_{1,1}), \Delta(x^-_{1,0})] = \Delta(h_{1,1}),
  [\Delta(x^+_{2,1}), \Delta(x^-_{2,0})] = \Delta(h_{2,1}).
  \]
where $h_{i,1} = \tilde{h}_{i,1} + \frac{1}{2} h_{i,0}^2$. Applying the automorphism $\rho$ in (3.16) for the first relation, we have

$$[\Delta(x^+_{3,1}), \Delta(x^-_{3,0})] = \Delta(h_{3,1}) \, .$$  \hspace{1cm} (3.29)

The remaining task here is to prove

$$[\Delta(x^+_{i,1}), \Delta(x^-_{j,0})] = 0 \quad \text{for} \quad i \neq j \, .$$  \hspace{1cm} (3.30)

Let us assume that $i \neq j$. Acting $\Delta(\tilde{h}_{i,1})$ and $\Delta(\tilde{h}_{j,1})$ on the relations

$$[\Delta(x^+_{i,1}), \Delta(x^-_{j,0})] = 0 \, ,$$  \hspace{1cm} (3.31)

and using (3.26), (3.27), we have

$$a_{ii}[\Delta(x^+_{i,1}), \Delta(x^-_{j,0})] - a_{ij}[\Delta(x^+_{i,0}), \Delta(x^-_{j,1})] = 0$$
$$a_{ji}[\Delta(x^+_{j,1}), \Delta(x^-_{i,0})] - a_{jj}[\Delta(x^+_{j,0}), \Delta(x^-_{i,1})] = 0 \, .$$  \hspace{1cm} (3.32)

Noting that $a_{ii}a_{ij} - a_{ji}a_{jj} \neq 0$ if $i \neq j$, we obtain

$$[\Delta(x^+_{i,1}), \Delta(x^-_{j,0})] = [\Delta(x^+_{i,0}), \Delta(x^-_{j,1})] = 0 \, .$$  \hspace{1cm} (3.33)

This proves (3.30).

• Relations (3.5). Using the automorphism $\rho$ in (3.16), the relations

$$[\Delta(x^+_{i,1}), \Delta(x^+_{j,0})] - [\Delta(x^+_{i,0}), \Delta(x^+_{j,1})] = \frac{1}{2} a_{ij} \{ \Delta(x^+_{i,0}), \Delta(x^+_{j,0}) \}$$  \hspace{1cm} (3.34)

immediately imply that

$$[\Delta(x^-_{i,1}), \Delta(x^-_{j,0})] - [\Delta(x^-_{i,0}), \Delta(x^-_{j,1})] = -\frac{1}{2} a_{ij} \{ \Delta(x^-_{i,0}), \Delta(x^-_{j,0}) \} \, .$$  \hspace{1cm} (3.35)

In addition, the relations (3.34) are symmetric under exchanging $i$ and $j$. Thus, it is sufficient to check the relations with respect to the following six cases:

$$(i, j) = (1, 1), (2, 2), (3, 3), (1, 2), (1, 3), \text{ and } (2, 3) \, .$$  \hspace{1cm} (3.36)

By computing all cases, we prove (3.34).

• Relations (3.6). We can show that

$$[\Delta(x^+_{i,1}), \Delta(x^+_{j,0})] = 0 \, .$$  \hspace{1cm} (3.37)

Applying the automorphism $\rho$ for this, we obtain $[\Delta(x^-_{i,1}), \Delta(x^-_{j,0})] = 0 \, .$
• Relations (3.9). By calculations, we get
\[
[[\Delta(x_{1,1}^+), \Delta(x_{2,0}^+)],[\Delta(x_{3,0}^+), \Delta(x_{2,0}^-)]]
= [[x_{1,1}^+, x_{2,0}^+] + 1 \otimes [x_{1,1}^+, x_{2,0}^+] + E_{32} \otimes E_{21} - E_{11} \otimes E_{31}
- \sum_{\mu=3,4} E_{3\mu} \otimes E_{\mu 1} - E_{24} \otimes P_0^+ , E_{42} \otimes 1 + 1 \otimes E_{42}]
\]
\[
= P_1^+ \otimes 1 + 1 \otimes P_1^+ - E_{32} \otimes E_{41} - E_{11} \otimes P_0^+
+ E_{32} \otimes E_{41} - E_{33} \otimes P_0^+ - (E_{22} + E_{44}) \otimes P_0^+
= P_1^+ \otimes 1 + 1 \otimes P_1^+ - (E_{11} + E_{22} + E_{33} + E_{44}) \otimes P_0^+
= P_1^+ \otimes 1 + 1 \otimes P_1^+ - 2C_0 \otimes P_0^+
= \Delta(P_1^+).
\] (3.38)

Here $C_0$ is defined in (2.19) and we have used the matrix notation
\[
C_0 = \frac{1}{2}(E_{11} + E_{22} + E_{33} + E_{44}),
\] (3.39)
which follows from the isomorphism (2.26). By automorphism $\rho$ in (3.16), we immediately obtain another relation,
\[
[[\Delta(\tilde{h}_{1,1}), \Delta(\tilde{h}_{2,0})],[\Delta(\tilde{h}_{3,0}), \Delta(\tilde{h}_{2,0})]] = \Delta(P_1^-).
\] (3.40)

Thus, the coproducts in (3.13) are compatible with the defining relations of $Y_L(g)$ up to degree one.

Next, let us move to the degree two relations, (3.2).

• Relations (3.2). It is trivial that
\[
[\Delta(\tilde{h}_{i,1}), \Delta(\tilde{h}_{i,1})] = 0 \quad \text{for} \quad i = 1, 2, 3.
\] (3.41)

Due to the automorphism $\rho$, it is sufficient to show that
\[
[\Delta(\tilde{h}_{1,1}), \Delta(\tilde{h}_{2,1})] = 0,
\] (3.42)
\[
[\Delta(\tilde{h}_{1,1}), \Delta(\tilde{h}_{3,1})] = 0.
\] (3.43)

For convenience, we introduce a notation $\Delta^{(k)}(x)$ for $0 \leq k \leq l$, $x \in Y_L(g)_l$ as follows. Let $\Delta(x)$ be a coproduct for an element $x \in Y_L(g)_l$ with degree $l$. Assume that it is symbolically given by
\[
\Delta(x) = \sum_i a_i \otimes b_i \quad \text{with} \quad a_i, b_i \in Y_L(g)_l.
\] (3.44)
For $0 \leq k \leq l$, we set
\[
\Delta^{(k)}(x) = \sum_{i, \deg = k} a_i \otimes b_i \in Y_L(g)_k \otimes Y_L(g)_k, \tag{3.45}
\]
where the summation runs over tensor products of degree $k$, $\deg(a_i) + \deg(b_i) = k$. Then, a coproduct $\Delta(x)$ for $x \in Y_L(g)_l$ is uniquely written as
\[
\Delta(x) = \Delta^{(l)}(x) + \Delta^{(l-1)}(x) + \cdots + \Delta^{(1)}(x) + \Delta^{(0)}(x). \tag{3.46}
\]
For example,
\[
\Delta(\tilde{h}_{1,1}) = \Delta^{(1)}(\tilde{h}_{1,1}) + \Delta^{(0)}(\tilde{h}_{1,1})
\]
where
\[
\Delta^{(1)}(\tilde{h}_{1,1}) = \tilde{h}_{1,1} \otimes 1 + 1 \otimes \tilde{h}_{1,1}
\]
\[
\Delta^{(0)}(\tilde{h}_{1,1}) = -2x^-_{1,0} \otimes x^+_{1,0} + \sum_{\mu=3,4} (E_{1\mu} \otimes E_{\mu 1} - E_{2\mu} \otimes E_{\mu 2}). \tag{3.47}
\]
With this notation, the relation (3.42) is written as
\[
[\Delta(\tilde{h}_{1,1}), \Delta(\tilde{h}_{2,1})] = [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{2,1})]
+ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{2,1})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{2,1})]
+ [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{2,1})]. \tag{3.48}
\]
The degree two part is trivial,
\[
[\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{2,1})] = [\tilde{h}_{1,1}, \tilde{h}_{2,1}] \otimes 1 + 1 \otimes [\tilde{h}_{1,1}, \tilde{h}_{2,1}] = 0. \tag{3.49}
\]
The degree one terms are computed respectively as

\[ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{2,1})] = [\tilde{h}_{1,1} \otimes 1 + 1 \otimes \tilde{h}_{1,1},
\]

\[ E_{12} \otimes E_{21} - E_{13} \otimes E_{31} + E_{34} \otimes E_{43} + E_{24} \otimes E_{42} - P_0^- \otimes P_0^+ \]

\[ = -2(x_{1,1}^- \otimes x_{1,0}^+ - x_{1,0}^- \otimes x_{1,1}^+ ) + \frac{1}{2} [x_{1,1}^+, x_{2,0}^-] \otimes [x_{1,0}^+, x_{3,0}^+] - [x_{1,0}^+, x_{2,0}^-] \otimes [x_{1,1}^+, x_{3,0}^-] + [x_{2,0}^+, x_{3,0}^-] \otimes [x_{1,1}^-, x_{3,0}^+] + [x_{2,0}^+, x_{3,0}^+] \otimes [x_{1,1}^-, x_{3,0}^-] + \frac{1}{2} [x_{1,0}^-, x_{2,0}^-] \otimes [x_{1,0}^+, x_{2,0}^+] + \frac{1}{2} [x_{1,0}^+, x_{2,0}^-] \otimes \{ x_{1,0}^+, x_{2,0}^- \}, \]

\[ [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{2,1})] = [-2x_{1,0}^- \otimes x_{1,0}^+ + \sum_{\mu=3,4} (E_{1\mu} \otimes E_{\mu 1} - E_{2\mu} \otimes E_{\mu 2}), \]

\[ = 2(x_{1,1}^- \otimes x_{1,0}^+ - x_{1,0}^- \otimes x_{1,1}^+ ) - [x_{1,1}^+, x_{2,0}^-] \otimes [x_{1,0}^+, x_{3,0}^+] + [x_{1,0}^+, x_{2,0}^-] \otimes [x_{1,1}^+, x_{3,0}^-] + [x_{2,0}^+, x_{3,0}^-] \otimes [x_{1,1}^-, x_{3,0}^+] + [x_{2,0}^+, x_{3,0}^+] \otimes [x_{1,1}^-, x_{3,0}^-] + \frac{1}{2} \{ E_{13}, E_{34} \} \otimes E_{41} - \frac{1}{2} E_{14} \otimes \{ E_{31}, E_{43} \}
\]

\[ - \frac{1}{2} \{ E_{12}, E_{24} \} \otimes E_{41} + \frac{1}{2} E_{14} \otimes \{ E_{21}, E_{42} \}. \] (3.50)

Combining them, we have

\[ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{2,1})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{2,1})] \]

\[ = -((x_{2,1}^- x_{3,0}^+) - [x_{2,0}^+, x_{3,1}^-]) \otimes [x_{1,0}^+, x_{3,0}^+] + [x_{2,0}^+, x_{3,0}^-] \otimes [x_{2,1}^+, x_{3,0}^+] - [x_{2,0}^+, x_{3,1}^-] ]
\]

\[ + \frac{1}{2} \{ E_{13}, E_{34} \} \otimes E_{41} - \frac{1}{2} E_{14} \otimes \{ E_{31}, E_{43} \}
\]

\[ - \frac{1}{2} \{ E_{12}, E_{24} \} \otimes E_{41} + \frac{1}{2} E_{14} \otimes \{ E_{21}, E_{42} \}
\]

\[ = -\frac{1}{2} \{ E_{23}, E_{34} \} \otimes E_{42} + \frac{1}{2} E_{24} \otimes \{ E_{32}, E_{43} \}
\]

\[ + \frac{1}{2} \{ E_{12}, E_{23} \} \otimes E_{31} - \frac{1}{2} E_{13} \otimes \{ E_{21}, E_{32} \}
\]

\[ + \frac{1}{2} \{ E_{13}, E_{34} \} \otimes E_{41} - \frac{1}{2} E_{14} \otimes \{ E_{31}, E_{43} \}
\]

\[ - \frac{1}{2} \{ E_{12}, E_{24} \} \otimes E_{41} + \frac{1}{2} E_{14} \otimes \{ E_{21}, E_{42} \} \in Y_L(\mathfrak{g})_0 \otimes Y_L(\mathfrak{g})_0, \] (3.51)

where we have used the relations (3.55) in the last equality. Here we observe that the above
The degree two term vanishes trivially. Furthermore, it is equal to the minus sign of
\[ [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{2,1})] = [-2x_{1,0}^- \otimes x_{1,0}^+ + \sum_{\mu=3,4} (E_{1\mu} \otimes E_{\mu 1} - E_{2\mu} \otimes E_{\mu 2})], \]
\[ E_{12} \otimes E_{21} - E_{13} \otimes E_{31} + E_{34} \otimes E_{43} + E_{24} \otimes E_{42} - P_0^- \otimes P_0^+]. \]
Hence, we prove (3.42). Due to the automorphism \( \rho \), this also implies that
\[ [\Delta(\tilde{h}_{3,1}), \Delta(\tilde{h}_{2,1})] = 0. \] (3.52)
Another relation (3.43) is proved similarly. The degree decomposition is

\[ [\Delta(\tilde{h}_{1,1}), \Delta(\tilde{h}_{3,1})] = [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{3,1})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{3,1})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{3,1})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{3,1})]. \] (3.53)

The degree two term vanishes trivially,
\[ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{3,1})] = [\tilde{h}_{1,1}, \tilde{h}_{3,1}] \otimes 1 + 1 \otimes [\tilde{h}_{1,1}, \tilde{h}_{3,1}] = 0. \] (3.54)

The summation of the degree one terms turns out to be
\[ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{3,1})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{3,1})] = -\{E_{12}, E_{23}\} \otimes E_{31} + E_{13} \otimes \{E_{21}, E_{32}\} - \{E_{13}, E_{34}\} \otimes E_{41} + E_{14} \otimes \{E_{31}, E_{43}\} + \{E_{12}, E_{24}\} \otimes E_{41} - E_{14} \otimes \{E_{21}, E_{42}\} + \{E_{23}, E_{34}\} \otimes E_{42} - E_{24} \otimes \{E_{32}, E_{43}\} \in Y_L(\mathfrak{g})_0 \otimes Y_L(\mathfrak{g})_0. \] (3.55)

This is again equal to the opposite sign of
\[ [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{3,1})] = [-2x_{1,0}^- \otimes x_{1,0}^+ + \sum_{\mu=3,4} (E_{1\mu} \otimes E_{\mu 1} - E_{2\mu} \otimes E_{\mu 2})], \]
\[ 2x_{3,0}^- \otimes x_{3,0}^+ + \sum_{l=1,2} (E_{l3} \otimes E_{3l} - E_{l4} \otimes E_{4l})]. \] (3.56)

Thus, we prove (3.43). Therefore, we complete the proof of the compatibility of the coproducts with the degree two relations (3.2).

Finally, we show the compatibility with the degree three relations (3.7) and (3.8), the most challenging part of this proof.

- Relation (3.7). Set the degree two element by \( h_{1,2} = [x_{1,1}^+, x_{1,1}^-] \). The degree decomposition is

\[ \Delta(h_{1,2}) = [\Delta(x_{1,1}^+), \Delta(x_{1,1}^-)] = \Delta^{(2)}(h_{1,2}) + \Delta^{(1)}(h_{1,2}) + \Delta^{(0)}(h_{1,2}) , \] (3.57)
and each term is explicitly calculated by

\[
\Delta^{(2)}(h_{1,2}) = [x_{1,1}^+, x_{1,1}] \otimes 1 + 1 \otimes [x_{1,1}^+, x_{1,1}]
\]
\[
\Delta^{(1)}(h_{1,2}) = h_{1,1} \otimes h_{1,0} + h_{1,0} \otimes h_{1,1}
- 2(x_{1,1}^- \otimes x_{1,0}^+ + x_{1,0}^- \otimes x_{1,1}^+ \\
+ [h_{1,1}, x_{2,0}^-] \otimes x_{2,0}^+ - x_{2,0}^- \otimes [h_{1,1}, x_{2,0}^+] \\
+ [x_{1,1}^-, x_{2,0}^-] \otimes E_{31} + E_{13} \otimes [x_{1,1}^+, x_{2,0}^+] \\
- [[h_{1,1}, x_{2,0}^-], x_{3,0}^-] \otimes E_{42} - E_{24} \otimes [[h_{1,1}, x_{2,0}^-], x_{3,0}^-] \\
+ [x_{1,1}^- [x_{2,0}^-, x_{3,0}^-] \otimes E_{41} + E_{14} \otimes [x_{1,1}^+, [x_{2,0}^-, x_{3,0}^-]]
\]
\[
\Delta^{(0)}(h_{1,2}) = -\{h_{1,0}, x_{1,0}^+\} \otimes x_{1,0}^- - x_{1,0}^- \otimes \{h_{1,0}, x_{1,0}^+\}
\]
\[+ \frac{1}{2} ((\{h_{1,0}, E_{1\mu}\} + \{x_{1,0}^-, E_{2\mu}\}) \otimes E_{\mu 1} + E_{1\mu} \otimes (\{h_{1,0}, E_{\mu 1}\} + \{x_{1,0}^+, E_{\mu 2}\}) \\
- (\{E_{13}, E_{24}\} + \{E_{23}, E_{14}\}) \otimes P_0^+ - P_0^+ \otimes (\{E_{31}, E_{42}\} + \{E_{32}, E_{41}\}) \] .
\]

(3.58)

Then, the decomposition of the left hand side of (3.7) with \(j = 1\) becomes

\[
[\Delta(\tilde{h}_{1,1}), [\Delta(x_{1,1}^+), \Delta(x_{1,1}^-)]] = [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(2)}(\tilde{h}_{1,2})] \\
+ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{1,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(\tilde{h}_{1,2})] \\
+ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{1,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(\tilde{h}_{1,2})] \\
+ [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(\tilde{h}_{1,2})] .
\]

(3.59)

It is obvious that the degree three term vanishes by using the relation (3.7),

\[
[\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{1,2})] = [\tilde{h}_{1,1}, [x_{1,1}^+, x_{1,1}^-]] \otimes 1 + 1 \otimes [\tilde{h}_{1,1}, [x_{1,1}^+, x_{1,1}^-]] = 0 .
\]

(3.60)
The degree two terms are somehow complicated, but after some calculations,

\[
\left[\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{1,2})\right] + \left[\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{1,2})\right] \\
= \frac{1}{2} \left( -\{h_{1,0}, x_{2,0}^-\} \otimes x_{2,1}^+ + x_{2,1}^- \otimes \{h_{1,0}, x_{2,0}^+\} \right) \\
- \{x_{1,0}^-, x_{2,0}^-\} \otimes [x_{1,0}^+, x_{2,1}^+] - [x_{1,0}^-, x_{2,1}^-] \otimes \{x_{1,0}^+, x_{2,0}^+\} \\
- \{h_{1,0}, [x_{2,0}^-, x_{3,0}^-]\} \otimes [x_{2,1}^+, x_{3,0}^+] + [x_{2,1}^-, x_{3,0}^-] \otimes \{h_{1,0}, [x_{2,0}^+, x_{3,0}^+]\} \\
- \{x_{1,0}^-, [x_{2,0}^-, x_{3,0}^-]\} \otimes \{x_{1,0}^+, x_{2,1}^+, x_{3,0}^+\} - \{x_{1,0}^-, x_{2,1}^-\} \otimes \{x_{1,0}^+, [x_{2,0}^+, x_{3,0}^+]\} \\
- 2(\{h_{1,0}, x_{1,1}^-\} + \{h_{1,1}, x_{1,0}^-\}) \otimes x_{1,0}^+ + 2x_{1,0}^- \otimes (\{h_{1,0}, x_{1,1}^-\} + \{h_{1,1}, x_{1,0}^+\}) \\
+ \frac{1}{2} \left( [x_{1,0}^-, x_{2,1}^-] - \{h_{1,1}, \{h_{1,0}, [x_{1,0}^+, x_{2,0}^-]\}\} + \{h_{1,1}, [x_{1,0}^-, x_{2,0}^-]\} \right) \otimes E_{31} \\
+ \frac{1}{2} E_{13} \otimes (\{x_{1,0}^+, x_{2,1}^+\} - \{h_{1,1}, \{h_{1,0}, [x_{1,0}^+, x_{2,0}^-]\}\} + \{h_{1,1}, [x_{1,0}^-, x_{2,0}^-]\}) \\
+ \frac{1}{2} \left( \{x_{1,0}^+, [x_{2,1}^-, x_{3,0}^-]\} + \{h_{1,0}, [x_{1,1}^-, x_{2,0}^-], x_{3,0}^-\} + \{h_{1,1}, [x_{1,0}^-, x_{2,0}^-], x_{3,0}^-\}\right) \otimes E_{41} \\
+ \frac{1}{2} E_{14} \otimes (\{x_{1,0}^+, [x_{2,1}^+, x_{3,0}^-]\} - \{h_{1,0}, [x_{1,1}^+, x_{2,0}^+], x_{3,0}^+\} - \{h_{1,1}, [x_{1,0}^+, x_{2,0}^+], x_{3,0}^+\}) \\
+ \frac{1}{4} \left( \{h_{1,0}, [x_{1,0}^-, x_{2,0}^-], x_{3,0}^-\} \otimes E_{41} + E_{14} \otimes \{h_{1,0}, [x_{1,0}^-, x_{2,0}^-], x_{3,0}^-\}\right) \\
+ \frac{1}{2} \left( \{-h_{1,1}, x_{2,0}^-\} \otimes x_{2,0}^+ + x_{2,0}^- \otimes \{h_{1,1}, x_{2,0}^+\}\right) \\
+ \frac{1}{2} \left( \{h_{1,1}, E_{24}\} \otimes E_{42} - E_{24} \otimes \{h_{1,1}, E_{42}\}\right), \tag{3.61}
\]

e we observe that these elements are essentially in degree one,

\[
\left[\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{1,2})\right] + \left[\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{1,2})\right] \\
\in Y_L(\mathfrak{g})_1 \otimes Y_L(\mathfrak{g})_0 + Y_L(\mathfrak{g})_0 \otimes Y_L(\mathfrak{g})_1 \tag{3.62}
\]
Furthermore, adding the degree one terms to the above, we have

\[
\begin{align*}
[\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{1,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{1,2})] \\
+ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{1,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{1,2})] \\
= \frac{1}{2} \sum_{\mu=3,4} \left( -\{h_{1,0}, E_{2\mu}\} \otimes E_{\mu 1} + E_{1\mu} \otimes \{h_{1,0}, E_{\mu 2}\}, x_{1,0}^+ \right) \\
+ \frac{1}{2} \sum_{\mu=3,4} \left( \{h_{1,0}, E_{1\mu}\} \otimes \{x_{1,0}^+, E_{\mu 2}\} - \{x_{1,0}^+, E_{2\mu}\} \otimes \{h_{1,0}, E_{\mu 1}\} \right) \\
+ \frac{1}{4} \left( \{\{E_{12}, E_{23}\}, E_{24}\} - \{\{E_{12}, E_{24}\}, E_{23}\} + \{\{h_{1,0}, E_{13}\}, E_{24}\} + \{\{h_{1,0}, E_{13}\}, E_{23}\} \right) \otimes P_0^+ \\
- \frac{1}{4} P_0^- \otimes \left( \{\{E_{21}, E_{32}\}, E_{42}\} - \{\{E_{21}, E_{42}\}, E_{32}\} + \{\{h_{1,0}, E_{31}\}, E_{42}\} + \{\{h_{1,0}, E_{31}\}, E_{32}\} \right) \\
+ \frac{1}{4} \left( \{\{E_{13}, E_{24}\} + \{E_{23}, E_{14}\}\} \otimes \{h_{1,0}, P_0^+\} - \{h_{1,0}, P_0^-\} \otimes \{E_{31}, E_{42}\} + \{E_{32}, E_{41}\} \right) \\
+ \frac{1}{2} \left( \{x_{1,0}^-, P_0^-\} \otimes \{E_{31}, E_{41}\} - \{E_{13}, E_{41}\} \otimes \{x_{1,0}^+, P_0^+\} \right). 
\end{align*}
\]

We see that there is no degree one element and remaining terms are all in degree zero,

\[
\left( [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{1,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{1,2})] \\
+ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{1,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{1,2})] \right) \in Y_L(\mathfrak{g})_0 \otimes Y_L(\mathfrak{g})_0. 
\]

Finally, we notice that the resulting terms in (3.63) exactly coincide with the minus sign of $[\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{1,2})]$. Thus, summing up all, we have

\[
\begin{align*}
[\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{1,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{1,2})] \\
+ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{1,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{1,2})] \\
+ [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{1,2})] \\
= 0. 
\end{align*}
\]

This proves the desired result

\[
[\Delta(\tilde{h}_{1,1}), \Delta(h_{1,2})] = [\Delta(\tilde{h}_{1,1}), [\Delta(x_{1,1}^+), \Delta(x_{1,1}^-)]] = 0. 
\]

- Relation (3.8). We shall repeat similar computations. Set $h_{2,2} = [x_{2,1}^+, x_{2,1}^-]$. The degree decomposition is

\[
\begin{align*}
[\Delta(\tilde{h}_{1,1}), [\Delta(x_{2,1}^+), \Delta(x_{2,1}^-)]] &= [\Delta^{(2)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2})] \\
+ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{2,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{2,2})] \\
+ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{2,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2})] \\
+ [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{2,2})]. 
\end{align*}
\]
Each degree component of

\[ \Delta(h_{2,2}) = [\Delta(x^+_{2,1}), \Delta(x^-_{2,1})] = \Delta^{(2)}(h_{2,2}) + \Delta^{(1)}(h_{2,2}) + \Delta^{(0)}(h_{2,2}), \tag{3.68} \]

is calculated by

\[ \Delta^{(2)}(h_{2,2}) = [x^+_{2,1}, x^-_{2,1}] \otimes 1 + 1 \otimes [x^+_{2,1}, x^-_{2,1}] \]
\[ \Delta^{(1)}(h_{2,2}) = h_{2,1} \otimes h_{2,0} + h_{2,0} \otimes h_{2,1} \]
\[ + x^+_{1,0} \otimes x^+_{1,0} + x^-_{1,0} \otimes x^-_{1,0} \]
\[ - x^+_{3,0} \otimes x^+_{3,0} - x^-_{3,0} \otimes x^-_{3,1} \]
\[ - [x^+_{1,0}, x^-_{2,1}] \otimes E_{31} - E_{13} \otimes [x^+_{1,0}, x^-_{2,1}] \]
\[ + [x^-_{2,1}, x^-_{3,0}] \otimes E_{42} - E_{24} \otimes [x^+_{2,1}, x^+_{3,0}] \]
\[ - [E_{14}, x^+_{2,1}] \otimes P^+_0 - P^-_0 \otimes [E_{41}, x^-_{2,1}] \]
\[ + \frac{1}{2} ([h_{2,0}, x^-_{1,0}] \otimes x^+_{1,0} + x^-_{1,0} \otimes \{h_{2,0}, x^+_{1,0}\}) \]
\[ - \frac{1}{2} ([h_{2,0}, x^-_{3,0}] \otimes x^+_{3,0} + x^-_{3,0} \otimes \{h_{2,0}, x^+_{3,0}\}) \]
\[ \Delta^{(0)}(h_{2,2}) = -\frac{1}{2} \left( \{(h_{2,0}, E_{13}) + \{E_{12}, E_{23}\}\} \otimes E_{31} + E_{13} \otimes \{(h_{2,0}, E_{31}) + \{E_{21}, E_{32}\}\} \right) \]
\[ + \{(E_{12}, E_{24}) - \{E_{34}, E_{13}\}\} \otimes E_{41} + E_{14} \otimes \{(h_{2,0}, E_{42}) - \{E_{43}, E_{31}\}\} \]
\[ - \{(h_{2,0}, E_{24}) + \{E_{34}, E_{23}\}\} \otimes E_{42} - E_{24} \otimes \{(h_{2,0}, E_{42}) + \{E_{43}, E_{32}\}\} \]
\[ + \{h_{2,0}, P^+_0\} \otimes P^+_0 + P^-_0 \otimes \{h_{2,0}, P^+_0\} \right). \tag{3.69} \]

Firstly, it is obvious that the degree three term vanishes by the relation (3.8),

\[ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{2,2})] = [\tilde{h}_{1,1}, [x^+_{2,1}, x^-_{2,1}]] \otimes 1 + 1 \otimes [\tilde{h}_{1,1}, [x^+_{2,1}, x^-_{2,1}]] = 0. \tag{3.70} \]
Though the degree two terms look quite complicated,

\[
[\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{2,2})] = (\{h_{2,1}, x_{-1,0}\} + \{h_{2,0}, x_{-1,1}\}) \otimes x_{1,0} - x_{1,0} \otimes (\{h_{2,1}, x_{1,0}\} + \{h_{2,0}, x_{1,1}\}) + \frac{1}{2} \left( -\{x_{-1,1}, x_{2,0}\} + \{x_{-1,2}, x_{2,1}\} \right) \otimes E_{31} + E_{13} \otimes (\{x_{1,1}, x_{2,0}\} + \{x_{1,0}, x_{2,1}\}) \right)
\]

we observe that the above elements are in degree one,

\[
[\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{2,2})] \in Y_L(\mathfrak{g})_1 \otimes Y_L(\mathfrak{g})_0 + Y_L(\mathfrak{g})_0 \otimes Y_L(\mathfrak{g})_1.
\]

(3.71)
Finally, we observe that the terms in (3.73) exactly coincide with the minus sign of the 

\[
\left[ \Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2}) \right] + \left[ \Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{2,2}) \right] \\
+ \left[ \Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{2,2}) \right] + \left[ \Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2}) \right]
\]

\[
= \frac{1}{4} \left( h_{2,0} \{ x_{1,0}^+, x_{2,0}^- \} \otimes E_{31} + E_{31} \{ x_{1,0}^+, x_{2,0}^- \}\right)
\]

\[
+ \{ h_{2,0}, E_{13} \} \otimes \{ x_{1,0}^+, x_{2,0}^- \} + \{ x_{1,0}^+, x_{2,0}^- \} \otimes \{ h_{2,0}, E_{31} \}
\]

\[
+ \frac{1}{4} \left( -\{ h_{2,0}, \{ x_{2,0}^- , x_{3,0}^- \} \} \otimes E_{42} + E_{24} \{ x_{2,0}^+, x_{3,0}^- \} \right)
\]

\[
+ \frac{1}{4} \left( (\{ h_{2,0}, \{ E_{13}, E_{34} \} \} - \{ h_{2,0}, \{ E_{12}, E_{24} \} \} \right) \otimes E_{41}
\]

\[
+ E_{14} \otimes \left( \{ h_{2,0}, \{ E_{31}, E_{43} \} \} + \{ h_{2,0}, \{ E_{21}, E_{42} \} \right) \right)
\]

\[
+ \frac{1}{2} \left( \{ E_{34}, E_{13} \} \otimes \{ E_{21}, E_{42} \} - \{ E_{12}, E_{24} \} \otimes \{ E_{43}, E_{31} \} \right)
\]

\[
+ \frac{1}{4} \left( (\{ E_{13}, E_{34} \} - \{ E_{12}, E_{24} \} \right) \otimes \{ E_{41}, h_{2,0} \}
\]

\[
+ \{ E_{14}, h_{2,0} \} \otimes \left( \{ E_{31}, E_{43} \} + \{ E_{21}, E_{42} \right) \right)
\]

\[
+ \frac{1}{4} \left( \{ E_{13}, E_{23} \} \otimes \{ E_{34} \} + \{ E_{13}, E_{34} \} \otimes E_{23} \right) \otimes E_{24}
\]

\[
+ \frac{1}{4} \left( \{ E_{24}, E_{12} \} \otimes \{ E_{23} \} + \{ E_{24}, E_{23} \} \otimes E_{12} \right) \otimes E_{24}
\]

\[
- \frac{1}{4} \{ P_{0}^+ \otimes \{ E_{31}, E_{32} \} \} \otimes \{ E_{43} \} - \{ E_{31}, E_{43} \} \otimes E_{32}
\]

\[
- \frac{1}{4} \{ P_{0}^- \otimes \{ E_{42}, E_{21} \} \} \otimes \{ E_{32} \} + \{ E_{42}, E_{32} \} \otimes E_{21}
\]

\[
+ \frac{1}{2} \left( \{ E_{23}, E_{14} \} \otimes \{ E_{31}, E_{42} \} + \{ E_{13}, E_{24} \} \otimes \{ E_{32}, E_{41} \} \right)
\]

\[
+ \frac{1}{2} \left( \{ E_{13}, E_{14} \} \otimes \{ x_{1,0}^+, P_{0}^+ \} + \{ x_{1,0}^+, P_0^- \} \otimes \{ E_{31}, E_{41} \} \right).
\] (3.73)

We see that there is no degree one element and remaining terms are all degree zero,

\[
\left[ \Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2}) \right] + \left[ \Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{2,2}) \right] \\
+ \left[ \Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{2,2}) \right] + \left[ \Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2}) \right] \in Y_{L}(g)_0 \otimes Y_{L}(g)_0.
\] (3.74)

Finally, we observe that the terms in (3.73) exactly coincide with the minus sign of the
degree zero term, \([\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{2,2})]\). Thus, summing up all, we have

\[
\begin{align*}
&[\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(2)}(h_{2,2})] \\
&+ [\Delta^{(1)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{2,2})] + [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(1)}(h_{2,2})] \\
&+ [\Delta^{(0)}(\tilde{h}_{1,1}), \Delta^{(0)}(h_{2,2})] \\
= 0.
\end{align*}
\]  

(3.75)

This proves the desired result

\[
[\Delta(\tilde{h}_{1,1}), \Delta(h_{2,2})] = [\Delta(\tilde{h}_{1,1}), [\Delta(x^+_2), \Delta(x^-_2)]] = 0.
\]  

(3.76)

Therefore, we have proved that the coproducts defined in (3.13) are indeed compatible with the defining relations (3.1)–(3.9) of \(Y_L(g)\).

This completes the proof of Proposition 3.4. \(\square\)

4 Drinfeld realization of the Yangian \(Y_D(g)\)

We define the Drinfeld realization of the Yangian \(Y_D(g)\) in Definition 4.1 and we show that it is isomorphic to \(Y_L(g)\), Theorem 4.3.

**Definition 4.1.** The Yangian \(Y_D(g)\) is generated by the generators \(h_{i,r}, x^\pm_{i,r}\) with \(i = 1, 2, 3\) and the central elements \(P_r^\pm\) with \(r = 0, 1, 2, \ldots\). They satisfy the following relations,

\[
\begin{align*}
[h_{i,r}, h_{j,s}] &= 0 \\
[x^+_{i,r}, x^-_{j,s}] &= \delta_{ij} h_{i,r+s} \\
[h_{i,0}, x^\pm_{j,r}] &= \pm a_{ij} x^\pm_{j,r} \\
[h_{i,r+1}, x^\pm_{j,s}] - [h_{i,r}, x^\pm_{j,s+1}] &= \pm \frac{1}{2} a_{ij} \{h_{i,r}, x^\pm_{j,s}\} \quad \text{for } i, j \text{ not both 2} \\
[h_{2,r}, x^\pm_{2,s}] &= 0 \\
[x^+_{i,r+1}, x^\pm_{j,s}] - [x^+_{i,r}, x^\pm_{j,s+1}] &= \pm \frac{1}{2} a_{ij} \{x^+_{i,r}, x^\pm_{j,s}\} \quad \text{for } i, j \text{ not both 2} \\
[x^\pm_{2,r}, x^\pm_{2,s}] &= 0 \\
[x^\pm_{j,r}, x^\pm_{j,s}, x^\pm_{2,t}] + [x^\pm_{j,r}, x^\pm_{2,t}, x^\pm_{j,s}] &= 0 \quad \text{for } j = 1, 3 \\
[x^\pm_{1,r}, x^\pm_{2,0}] + [x^\pm_{3,s}, x^\pm_{2,0}] &= P_{r+s}.
\end{align*}
\]

(4.1)  
(4.2)  
(4.3)  
(4.4)  
(4.5)  
(4.6)  
(4.7)  
(4.8)  
(4.9)

The \(\mathbb{Z}_2\)-grading \(p : Y_D(g) \to \mathbb{Z}_2\) is defined by setting \(p(x^\pm_{2,r}) = 1\) for \(r \in \mathbb{Z}_{\geq 0}\) and \(p(\text{other}) = 0\). The symmetrized Cartan matrix \(a_{ij}\) is given in (1.12).

**Remark 4.2.** The Yangian \(Y_D(g)\) includes the subalgebra \(Y(sl(2)) \oplus Y(sl(2))\), which is spanned by the generators \(\{x^\pm_{i,r}, h_{i,r}\}\) with \(i = 1, 3\), and \(r \in \mathbb{Z}_{\geq 0}\).
We introduce the generators in $Y_L(g)$ inductively by, for $r \in \mathbb{Z}_{\geq 0}$,

\[
\begin{align*}
x_{1,r+1}^+ &= \pm \frac{1}{2} [\tilde{h}_{1,1}, x_{1,r}^+] , \\
x_{2,r+1}^+ &= \mp \frac{1}{2} [\tilde{h}_{2,1}, x_{2,r}^+] , \\
x_{3,r+1}^+ &= \mp \frac{1}{2} [\tilde{h}_{3,1}, x_{3,r}^+] , \\
h_{i,r} &= [x_{i,r}^+, x_{i,0}^+] \quad (i = 1, 2, 3) , \\
P_r^\pm &= [x_{1,r}^\pm, x_{2,0}^\pm, x_{3,0}^\pm, x_{2,0}^\pm] .
\end{align*}
\] (4.10)

**Theorem 4.3.** The Yangian $Y_D(g)$ is isomorphic to $Y_L(g)$. The isomorphism $\phi : Y_D(g) \to Y_L(g)$ is given by

\[
\begin{align*}
h_{i,r} &\mapsto h_{i,r} , \\
x_{i,r}^+ &\mapsto x_{i,r}^+ , \\
P_r^\pm &\mapsto P_r^\pm ,
\end{align*}
\] (4.11)

where the image of $\phi$ is defined in (4.10).

**Remark 4.4.** The original idea of Theorem 4.3 is based on Levendorskii’s construction of the Yangian associated with the finite-dimensional complex Lie algebra $\mathfrak{sl}(n)$. Its generalization to the super-Yangian associated with Lie superalgebra $\mathfrak{sl}(m|n)$ is obtained by Stukopin [21]. Compared to the defining relations of the Yangian $Y(\mathfrak{sl}(2|2))$, the only differences are the extended Serre relations (4.9).

Theorem 4.3 allows us to induce the Hopf algebra structure to $Y_D(g)$ from $Y_L(g)$ by the following commutative diagrams,

\[
\begin{array}{ccc}
Y_D(g) & \xrightarrow{\phi} & Y_L(g) \\
\downarrow \Delta_D & & \downarrow \Delta \\
Y_D(g) \otimes Y_D(g) & \xrightarrow{\phi \otimes \phi} & Y_L(g) \otimes Y_L(g)
\end{array}
\]

\[
\begin{array}{ccc}
Y_D(g) & \xrightarrow{\phi} & Y_L(g) \\
\downarrow S_D & & \downarrow S \\
Y_D(g) & \xrightarrow{\phi} & Y_L(g) \\
\downarrow \epsilon_D & & \downarrow \epsilon \\
C & \xrightarrow{\epsilon} & C
\end{array}
\]

Here $\Delta_D, S_D,$ and $\epsilon_D$ are induced coproduct, antipode, and counit in $Y_D(g)$, respectively. In particular, the manifest expressions of the Drinfeld coproduct $\Delta_D$ are obtained from (3.13) by the induction (4.10).

**Corollary 4.5.** The Yangian $Y_D(g)$ in Definition 4.1 has the Hopf algebra structures with the coproducts given by

\[
\Delta_D = (\phi^{-1} \otimes \phi^{-1}) \circ \Delta \circ \phi : Y_D(g) \to Y_D(g) \otimes Y_D(g) ,
\]
the antipodes
\[ S_D = \phi^{-1} \circ S \circ \phi : Y_D(\mathfrak{g}) \to Y_D(\mathfrak{g}), \]
and the counits
\[ \epsilon_D = \epsilon \circ \phi : Y_D(\mathfrak{g}) \to \mathbb{C}. \]

**Proof.** Since the map \( \phi \) is isomorphism by Theorem 4.3, so is \( \phi^{-1} \). The maps \( \Delta, S, \epsilon \) define the Hopf algebra structures by Proposition 3.4. Thus, the composite maps \( \Delta_D, S_D, \epsilon_D \) are homomorphism of the Yangian \( Y_D(\mathfrak{g}) \). Hence, these maps define the Hopf algebra structures of \( Y_D(\mathfrak{g}) \). \( \square \)

**Proof of Theorem 4.3.** First, we prove that the map \( \psi : Y_L(\mathfrak{g}) \to Y_D(\mathfrak{g}) \) defined by
\[
\begin{align*}
    h_{i,0} &\mapsto h_{i,0}, & x_{i,0}^\pm &\mapsto x_{i,0}^\pm, & P_0^\pm &\mapsto P_0^\pm, \\
    \tilde{h}_{i,1} &\mapsto h_{i,1} - \frac{1}{2} h_{i,0}^2, & x_{i,1}^\pm &\mapsto x_{i,1}^\pm, & P_1^\pm &\mapsto P_1^\pm.
\end{align*}
\] (4.12)
is a homomorphism. Second, we show that the map \( \phi : Y_D(\mathfrak{g}) \to Y_L(\mathfrak{g}) \) in (4.11) is a homomorphism. Since \( \phi \circ \psi = \psi \circ \phi = \text{id.} \), we can conclude that \( \phi \) in (4.11) is the isomorphism.

First, it is easy to see that the map \( \psi : Y_L(\mathfrak{g}) \to Y_D(\mathfrak{g}) \) is a homomorphism. Indeed, the relations (3.1)–(3.9) immediately follow from (4.1)–(4.9) as truncation of the degree. In particular, by (4.3) and (4.4), the relation (3.3) is derived as
\[
\begin{align*}
    [\tilde{h}_{i,1}, x_{j,0}] &\mapsto [h_{i,1}, x_{j,0}] - \frac{1}{2} \left\{ h_{i,0}, [h_{i,0}, x_{i,0}^\pm] \right\} \\
    &\quad = [h_{i,0}, x_{j,0}] \pm \frac{1}{2} a_{ij} \left\{ h_{i,0}, x_{i,0}^\pm \right\} \pm \frac{1}{2} a_{ij} \left\{ h_{i,0}, x_{i,0}^\pm \right\} \\
    &\quad = \pm a_{ij} x_{j,1}^\pm. \quad (4.13)
\end{align*}
\]
The relations (3.7) and (3.8) follow from (4.1),
\[
\begin{align*}
    [\tilde{h}_{j,1}, \tilde{h}_{j,2}] &\mapsto [h_{j,1} - \frac{1}{2} h_{j,0}^2, h_{j,2}] = 0 \quad (j = 1, 3), \\
    [\tilde{h}_{1,1}, \tilde{h}_{2,2}] &\mapsto [h_{1,1} - \frac{1}{2} h_{1,0}^2, h_{2,2}] = 0. \quad (4.14)
\end{align*}
\]

Second, we show that the map \( \phi : Y_D(\mathfrak{g}) \to Y_L(\mathfrak{g}) \) in (4.11) is a homomorphism for the relations (4.1)–(4.9), respectively. Hereafter, all calculations are understood in \( Y_L(\mathfrak{g}) \).

**Proof of (4.3).** Recall the definition of \( x_{i,0}^\pm \) in (4.10) and the relation (3.1). Commuting \( \tilde{h}_{j,1} \) \((j = 1, 3)\) by \( r \)-times with
\[
[h_{i,0}, x_{j,0}^\pm] = \pm a_{ij} x_{j,0}^\pm
\]
we have the relation (4.3) with \( j = 1, 3 \). When \( j = 2 \), we can prove the relation (4.3) by using \( \tilde{h}_{1,1} \) instead of \( \tilde{h}_{2,1} \).

To prove (4.6), we prepare Lemma 4.6 and Proposition 4.7.

**Lemma 4.6.** The following relations hold in \( Y(\mathfrak{g}) \), for \( r \in \mathbb{Z}_{\geq 0} \), \( i, j \in \{1, 2, 3\} \),

\[
\begin{align*}
[h_{i,1}, x_{j,r}] &= \pm a_{ij} x_{j,r+1} \quad (4.15) \\
[x_{i,2}^+, x_{j,1}^+] - [x_{i,1}^+, x_{j,1}^+] &= \pm \frac{1}{2} a_{ij} \{x_{i,1}^+, x_{j,0}^+\} \quad (4.16) \\
h_{i,2} &= [x_{i,2}^+, x_{i,0}^-] = [x_{i,1}^+, x_{i,1}^-] = [x_{i,0}^-, x_{i,2}] \quad (4.17) \\
[h_{i,2}, \tilde{h}_{j,1}] &= 0 \quad (4.18) \\
[h_{i,2}, x_{j,r}] - [h_{i,1}, x_{j,r+1}] &= \pm \frac{1}{2} a_{ij} \{h_{i,1}, x_{j,r}\} . \quad (4.19)
\end{align*}
\]

**Proof of Lemma 4.6.** We shall prove these relations in order.

The proof of (4.15) is parallel to that of (4.3). For \( j = 1, 3 \), commuting \( \tilde{h}_{j,1} \) with the relation (3.3) by \( r \)-times, we have (4.15). For \( j = 2 \), we use \( \tilde{h}_{1,1} \), and obtain (4.15).

To prove (4.16), we start with the degree one relations (3.5),

\[
[x_{i,1}^+, x_{i,0}^-] - [x_{i,0}^-, x_{j,1}^+] = \pm \frac{1}{2} a_{ij} \{x_{i,0}^+, x_{j,0}^-\} .
\]

Acting \( \tilde{h}_{k,1} \) on both hand sides and using the relation (4.15), we have

\[
a_{ki} R_{ij}(1, 0) + a_{kj} R_{ij}(0, 1) = 0 , \quad (4.20)
\]

where we have introduced the notations

\[
R_{ij}(r, s) = [x_{i, r+1}^+, x_{j, s}^+] - [x_{i, r}^+, x_{j, s+1}^+] \mp \frac{1}{2} a_{ij} \{x_{i, r}^+, x_{j, s}^+\} , \quad (r, s \geq 0) . \quad (4.21)
\]

When \( i \neq j \), by setting \( k = i \) and \( k = j \), we have

\[
a_{ii} R_{ij}(1, 0) + a_{ij} R_{ij}(0, 1) = 0 , \quad (4.22) \\
a_{jj} R_{ij}(1, 0) + a_{ij} R_{ij}(0, 1) = 0 . \quad (4.23)
\]

Since \( a_{ii} a_{jj} - a_{ij} a_{ji} \neq 0 \) for the Cartan matrix (1.12), we obtain

\[
R_{ij}(1, 0) = R_{ij}(0, 1) = 0 . \quad (4.24)
\]

When \( i = j = 1, 3 \), by setting \( k = i \), we have

\[
R_{ii}(1, 0) + R_{ii}(0, 1) = 0 . \quad (4.25)
\]
Due to the symmetry $R_{ii}(1,0) = R_{ii}(0,1)$, we obtain

$$R_{ii}(1,0) = R_{ii}(0,1) = 0.$$  \hfill (4.26)

When $i = 2$, as $a_{22} = 0$, we have

$$R_{22}(1,0) = [x_2^+, x_2^+] - [x_2^+, x_2^-].$$  \hfill (4.27)

On the other hand, commuting $\tilde{h}_{1,1}$ with $[36]$, we obtain

$$[x_{2,2}^+, x_{2,0}^-] + [x_{2,1}^+, x_{2,1}^-] = 0.$$  \hfill (4.28)

Hence, it holds that

$$[x_{2,2}^+, x_{2,0}^-] = [x_{2,1}^+, x_{2,1}^-] = 0.$$  \hfill (4.29)

This means that, for any $i,j = 1, 2, 3$, we have

$$R_{ij}(1,0) = 0.$$  \hfill (4.30)

Thus, (4.16) is proved.

Now we move to (4.17). Acting $\tilde{h}_{k,1}$ on $[x_{i,0}^+, x_{i,0}^-] = h_{i,0}$, we have

$$a_{ki}([x_{i,1}^+, x_{i,0}^-] - [x_{i,0}^+, x_{i,0}^-]) = 0.$$  \hfill (4.31)

Chose $k$ such as $k \neq 4 - i$. Because of $a_{ik} \neq 0$, we have

$$h_{i,1} = [x_{i,1}^+, x_{i,0}^-] = [x_{i,0}^+, x_{i,1}^-].$$  \hfill (4.32)

Repeating this procedure for the above relation, we obtain

$$0 = [x_{i,2}^+, x_{i,0}^-] - [x_{i,1}^+, x_{i,1}^-],$$

$$0 = [x_{i,1}^+, x_{i,1}^-] - [x_{i,0}^+, x_{i,2}^-].$$

Combining these two relations and taking into account the definition of $h_{i,2}$, we have (4.17).

The left hand side of (4.18) is rewritten as

$$[h_{i,2}, \tilde{h}_{j,1}] = [[x_{i,2}^+, x_{i,0}^-], \tilde{h}_{j,1}] = [[x_{i,1}^+, x_{i,1}^-], \tilde{h}_{j,1}],$$  \hfill (4.33)

where the last equality is due to (4.17). Furthermore, by (4.15), we have

$$[[x_{i,1}^+, x_{i,1}^-], \tilde{h}_{j,1}] = -a_{ji}([x_{i,2}^+, x_{i,1}^-] - [x_{i,1}^+, x_{i,2}^-]).$$  \hfill (4.34)
Here, it is noted that the right hand side vanishes due to the relations (3.7), (3.8) and (4.15). Thus, we prove the relation (4.18).

Finally, we shall prove the relation (4.19). Commuting \( x_i^\pm \) with the relation (4.16) and taking (4.17) into account, we have

\[
[h_i, 2, x_j^\pm] - [h_i, 1, x_j^\pm] \mp \frac{1}{2} a_{ij} \{h_i, 1, x_j^\pm\} = 0.
\]

Since the second line on the left hand side is zero due to

\[
[x_i^\pm, h_j, 0] - [x_i^\pm, h_j, 1] = \mp a_{ij} x_i^\pm + [h_i, 1 - \frac{1}{2} h_i^2, x_i^\pm]
\]

it becomes

\[
[h_i, 2, x_j^\pm] - [h_i, 1, x_j^\pm] = \pm \frac{1}{2} a_{ij} \{h_i, 1, x_j, 0\}.
\]

Then, noting that \( \tilde{h}_j, 1 \) commutes with \( h_i, 2 \) by (4.18) and acting \( \tilde{h}_j, 1 \) on the above relation by \( r \)-times, we obtain the last relation (4.19);

\[
[h_i, 2, x_j^\pm] - [h_i, 1, x_j^\pm] = \pm \frac{1}{2} a_{ij} \{h_i, 1, x_j^\pm\}.
\]

This completes the proof of the lemma.

Lemma 4.6 allows us to prove the following proposition, which will play an important role later.

**Proposition 4.7.** The operators defined by

\[
B_{ij} = h_{i, 2} - h_{i, 0} \tilde{h}_{i, 1} - \frac{1}{6} h_{i, 0}^3 - \frac{1}{12} a_{ij} h_{i, 0}
\]

satisfy the relations, for \( i, j = 1, 2, 3 \) and \( r \in \mathbb{Z}_{\geq 0} \),

\[
[B_{ij}, x_j^\pm] = \pm a_{ij} x_j^\pm_{j, r + 2}.
\]

**Proof.** Adding the following two relations,

\[
[h_{i, 2}, x_j^\pm] - [h_{i, 1}, x_j^\pm_{j, r + 1}] = \pm \frac{1}{2} a_{ij} \{h_{i, 1}, x_j^\pm_{j, r}\},
\]

\[
[h_{i, 1}, x_j^\pm_{j, r + 1}] - [h_{i, 0}, x_j^\pm_{j, r + 2}] = \pm \frac{1}{2} a_{ij} \{h_{i, 0}, x_j^\pm_{j, r + 1}\},
\]

28
we have
\[ [h_{i,2}, x^\pm_{j,r}] - [h_{i,0}, x^\pm_{j,r+2}] = \pm \frac{1}{2} a_{ij} \{ h_{i,1}, x^\pm_{j,r} \} \pm \frac{1}{2} a_{ij} \{ h_{i,0}, x^\pm_{j,r+1} \}. \] (4.42)

By the definition \( h_{i,1} = \tilde{h}_{i,1} + \frac{1}{2} h_{i,0}^2 \), it becomes
\[ [h_{i,2}, x^\pm_{j,r}] - [h_{i,0}, x^\pm_{j,r+2}] = \pm \frac{1}{2} \{ \tilde{h}_{i,1}, h_{j,0} \}, x^\pm_{j,r} \] \pm \frac{1}{4} a_{ij} \{ h_{i,0}^2, x^\pm_{j,r} \}
\[ \iff [h_{i,2} - \tilde{h}_{i,1} h_{j,0}, x^\pm_{j,r}] = \pm a_{ij} x^\pm_{j,r+2} \pm \frac{1}{4} a_{ij} \{ h_{i,0}^2, x^\pm_{j,r} \}. \] (4.43)

Using the following identity,
\[ [h_{i,0}^3, x^\pm_{j,r}] = \pm \frac{3}{2} a_{ij} \{ h_{i,0}^2, x^\pm_{j,r} \} \mp \frac{1}{2} a_{ij} \{ h_{i,0}, [h_{i,0}, x^\pm_{j,r}] \}, \] (4.44)
we obtain
\[ [h_{i,2} - \tilde{h}_{i,1} h_{j,0}, x^\pm_{j,r}] = \pm a_{ij} x^\pm_{j,r+2} + \frac{1}{6} [h_{i,0}^3, x^\pm_{j,r}] \pm \frac{1}{12} a_{ij} \{ h_{i,0}, [h_{i,0}, x^\pm_{j,r}] \}
\[ \iff [h_{i,2} - \tilde{h}_{i,1} h_{j,0} - \frac{1}{6} h_{i,0}^3, x^\pm_{j,r}] = \pm a_{ij} x^\pm_{j,r+2} + \frac{1}{12} a_{ij}^2 [h_{i,0}, x^\pm_{j,r}] \] \pm \frac{1}{12} a_{ij}^2 [h_{i,0}, x^\pm_{j,r}] \cdot \] (4.45)

Thus, we get the desired relations,
\[ [h_{i,2} - \tilde{h}_{i,1} h_{j,0} - \frac{1}{6} h_{i,0}^3, x^\pm_{j,r}] = [B_{ij}, x^\pm_{j,r}] = \pm a_{ij} x^\pm_{j,r+2}. \] (4.46)

Hence, the proposition is proved.

We shall use the \textit{degree-two raising operator} \( B_{ij} \) in (4.39) in the following proof.

\textbf{Proof of (4.6).} Let us use the notation (4.21) with \( i,j = 1,2,3 \) which are not both 2. Note that \( R_{ij}(r,s) \) has the following symmetry;
\[ R_{ij}(r,s) = R_{ji}(s,r) \quad \text{if} \quad (i,j) \neq (2,2). \] (4.47)

The relation (4.6) is now expressed as \( R_{ij}(r,s) = 0 \). We will prove this by induction with respect to \( K = r + s \).

When \( K = 0 \), the relation \( R_{ij}(0,0) = 0 \) is nothing but (3.33). When \( K = 1 \), we have already proved the relations \( R_{ij}(1,0) = R_{ij}(0,1) = 0 \) in Lemma 4.6.

Next, we suppose that, for a fixed \( K \geq 1 \), it holds that
\[ R_{ij}(l, K - l) = 0 \quad \text{for any} \quad 0 \leq l \leq K, \] (4.48)
and show that the above relation holds for $K + 1$.

As we did for $K = 1$ case, commuting $\tilde{h}_{k,1}$ with (4.48), we have

$$a_{ki}R_{ij}(l + 1, K - l) + a_{kj}R_{ij}(l, K - l + 1) = 0.$$  \hspace{1cm} (4.49)

When $i \neq j$, setting $k = i$ and $j$, the above relation becomes

$$a_{ii}R_{ij}(l + 1, K - l) + a_{ij}R_{ij}(l, K - l + 1) = 0,$$

$$a_{ji}R_{ij}(l + 1, K - l) + a_{jj}R_{ij}(l, K - l + 1) = 0.$$  \hspace{1cm} (4.50)

Noting that $a_{ii}a_{jj} - a_{ij}a_{ji} \neq 0 (i \neq j)$ for the Cartan matrix (1.12), we obtain

$$R_{ij}(l + 1, K - l) = R_{ij}(l, K - l + 1) = 0 \quad \text{for} \quad i \neq j.$$  \hspace{1cm} (4.51)

Thus, we prove that

$$R_{ij}(l, K + 1 - l) = 0 \quad \text{for} \quad i \neq j, \quad 0 \leq l \leq K + 1.$$  \hspace{1cm} (4.52)

When $i = j = 1, 3$ and $K$ is even, we denote $K = 2m$ with $m \geq 1$. In this case, by setting $k = i$, the relation (4.49) becomes

$$R_{ii}(l + 1, 2m - l) + R_{ii}(l, 2m - l + 1) = 0 \quad \text{for} \quad 0 \leq l \leq 2m.$$  \hspace{1cm} (4.53)

In particular, when $l = m$, this relation is

$$R_{ii}(m + 1, m) + R_{ii}(m, m + 1) = 0.$$  \hspace{1cm} (4.54)

Taking into account the symmetry (4.47), we obtain

$$R_{ii}(m + 1, m) = R_{ii}(m, m + 1) = 0.$$  \hspace{1cm} (4.55)

Next, setting $l = m + 1$ in (4.53) and substituting $R_{ii}(m + 1, m) = 0$, we get

$$R_{ii}(m + 2, m - 1) = R_{ii}(m - 1, m + 2) = 0.$$  \hspace{1cm} (4.56)

Repeating this procedure for $l = m + 2, \cdots, 2m$, we find that

$$R_{ii}(l, 2m + 1 - l) = 0 \quad \text{for} \quad 0 \leq l \leq 2m + 1.$$  \hspace{1cm} (4.57)

Finally, when $i = j = 1, 3$ and $K$ is odd, we denote $K = 2m' + 1$ with $m' \geq 0$. In addition to the induction hypothesis (4.48), we also suppose that

$$R_{ii}(l, 2m' - l) = 0 \quad \text{for} \quad 0 \leq l \leq 2m'.$$  \hspace{1cm} (4.58)
Commuting the generator $B_{ii}$ in (4.39) with this relation, we have
\[ R_{ii}(l + 2, 2m' - l) + R_{ii}(l, 2m' - l + 2) = 0. \] (4.59)

In particular, setting $l = m'$ and using the symmetry (4.47), we find that
\[ R_{ii}(m' + 2, m') = R_{ii}(m', m' + 2) = 0. \] (4.60)

On the other hand, commuting $\tilde{h}_{i,1}$ with the induction hypothesis (4.48) it turns out that
\[ R_{ii}(l + 1, 2m' + 1 - l) + R_{ii}(l, 2m' + 2 - l) = 0 \quad \text{for} \quad 0 \leq l \leq 2m' + 1. \] (4.61)

Setting $l = m'$, it reduces to
\[ R_{ii}(m' + 1, m' + 1) + R_{ii}(m', m' + 2) = 0. \] (4.62)

Remembering (4.60), we find
\[ R_{ii}(m' + 1, m' + 1) = 0. \] (4.63)

For (4.61), repeating this procedure for $l = m' + 1, \cdots, 2m' + 1$ in order, we prove that
\[ R_{ii}(l, 2m' + 2 - l) = 0 \quad \text{for} \quad 0 \leq l \leq 2m' + 2. \] (4.64)

Therefore, the induction hypothesis (4.48) holds for $K + 1$. This completes the proof of the relation (4.6).

**Proof of (4.7).** Introduce the notation,
\[ Q(r, s) = [x^{\pm}_{2,r}, x^{\pm}_{2,s}]. \] (4.65)

Since $x^{\pm}_{2,r}$ are odd and the square bracket is the super-commutator, we have the symmetry;
\[ Q(r, s) = Q(s, r). \] (4.66)

The relation (4.7) is now expressed as $Q(r, s) = 0$ for any $r, s \geq 0$. We will prove this by induction with respect to $K = r + s$.

For $K = 0, 1$, due to the relations (2.14) and (3.6), the following relations hold,
\[ Q(0, 0) = 0 \quad \text{and} \quad Q(1, 0) = Q(0, 1) = 0. \] (4.67)

Next, we suppose that for a fixed $K \geq 1$ the following relations hold,
\[ Q(l, K - l) = 0 \quad \text{for} \quad 0 \leq l \leq K. \] (4.68)
Commuting $\tilde{h}_{1,1}$ and noting $[\tilde{h}_{1,1}, x_{2,r}^\pm] = \mp x_{2,r+1}^\pm$, we have
\[
Q(l + 1, K - l) + Q(l, K - l + 1) = 0 \quad \text{for} \quad 0 \leq l \leq K.
\] (4.69)

When $K$ is even, set $K = 2m$ $(m \geq 0)$. Then, the above relations become
\[
Q(l + 1, 2m - l) + Q(l, 2m - l + 1) = 0 \quad \text{for} \quad 0 \leq l \leq 2m.
\] (4.70)

In particular, when $l = m$, we have
\[
Q(m + 1, m) + Q(m, m + 1) = 0.
\] (4.71)

Taking into account the symmetry (4.66), this gives us
\[
Q(m + 1, m) = Q(m, m + 1) = 0.
\] (4.72)

Then, substituting $l = m + 1, m + 2, \cdots, 2m$ for the relation (4.70) in order, we obtain
\[
Q(l, 2m + 1 - l) = 0 \quad \text{for} \quad 0 \leq l \leq 2m + 1.
\] (4.73)

When $K$ is odd, setting $K = 2m' + 1$ $(m' \geq 0)$, the relations (4.69) become
\[
Q(l + 1, 2m' + 1 - l) + Q(l, 2m' + 2 - l) = 0 \quad \text{for} \quad 0 \leq l \leq 2m' + 1.
\] (4.74)

On the other hand, by the induction hypothesis, the following relations hold;
\[
Q(l, 2m' - l) = 0 \quad \text{for} \quad 0 \leq l \leq 2m'.
\] (4.75)

Then, acting $B_{12}$ in (4.39) on these relations and noting $[B_{12}, x_{2,r}^\pm] = \mp x_{2,r+2}^\pm$, we get
\[
Q(l + 2, 2m' - l) + Q(l, 2m' + 2 - l) = 0 \quad \text{for} \quad 0 \leq l \leq 2m'.
\] (4.76)

In particular, setting $l = m'$ in (4.74) and (4.76) respectively, we obtain
\[
Q(m' + 1, n + 1) + Q(m', m' + 2) = 0,
\]
\[
Q(m' + 2, n) + Q(m', m' + 2) = 0.
\] (4.77)

Because of the symmetry (4.66), this gives us
\[
Q(m' + 2, m') = Q(m', m' + 2) = Q(m' + 1, m' + 1) = 0.
\] (4.78)

Then, substituting $l = m' + 1, m' + 2, \cdots, 2m' + 1$ for (4.74) in this order, we obtain
\[
Q(l, 2m' + 2 - l) = 0 \quad \text{for} \quad 0 \leq l \leq 2m' + 2.
\] (4.79)

Thus, for any fixed $K \geq 1$, we have proved
\[
Q(l, K + 1 - l) = 0 \quad \text{for} \quad 0 \leq l \leq K + 1.
\] (4.80)

This completes the induction procedure.

In order to prove the relations (4.1), (4.2), (4.4), and (4.5), we prepare Lemma 4.8.
Lemma 4.8. For a fixed $K \geq 1$, suppose that the following relations hold

\[ [h_{i,r}, h_{j,s}] = 0 \]  \hspace{1cm} (4.81)
\[ [x_{i,r}^+, x_{j,s}^-] = \delta_{ij} h_{i,r+s} \]  \hspace{1cm} (4.82)
\[ [h_{i,r+1}, x_{j,s}^\pm] - [h_{i,r}, x_{j,s+1}^\pm] = \pm \frac{1}{2} a_{ij} \{h_{i,r}, x_{j,s}^\pm\} \]  \hspace{1cm} (4.83)

for $r, s \in \mathbb{Z}_{\geq 0}$ such that $r + s \leq K$. Then, the following relations hold,

\[ [h_{i,l}, h_{j,K+1-l}] = 0 \quad \text{for} \quad 0 \leq l \leq K+1. \]  \hspace{1cm} (4.84)

**Proof of Lemma 4.8**

Set

\[ S_{ij}(r, s) = [h_{i,r+1}, x_{j,s}^\pm] - [h_{i,r}, x_{j,s+1}^\pm] \mp \frac{a_{ij}}{2} \{h_{i,r}, x_{j,s}^\pm\}. \]  \hspace{1cm} (4.85)

By the induction hypothesis, we have for, $1 \leq l \leq K$,

\[ 0 = S_{ij}(l - 1, K - l) \]
\[ = [h_{i,l}, x_{j,K-l}^\pm] - [h_{i,l-1}, x_{j,K-l+1}^\pm] \mp \frac{a_{ij}}{2} \{h_{i,l-1}, x_{j,K-l}\}. \]  \hspace{1cm} (4.86)

Commuting $x_{j,1}^\pm$ with the above relation from right, it becomes

\[ 0 = [[h_{i,l}, x_{j,K-l}^\pm], x_{j,1}^\pm] - [[h_{i,l-1}, x_{j,K-l+1}^\pm], x_{j,1}^\pm] \mp \frac{a_{ij}}{2} \{[h_{i,l-1}, x_{j,K-l}^\pm], x_{j,1}^\pm\}. \]  \hspace{1cm} (4.87)

The first term is computed as

\[ [[h_{i,l}, x_{j,K-l}^\pm], x_{j,1}^\pm] = -[[x_{j,K-l}^\pm, x_{j,1}^\pm], h_{i,l}] - [x_{j,K-l}^\pm, [h_{i,l}, x_{j,1}^\pm]] \]
\[ = \mp [h_{j,K-l+1}, h_{i,l}] - [x_{j,K-l}^\pm, [h_{i,l+1}, x_{j,0}^\pm]] \mp \frac{a_{ij}}{2} \{h_{i,l}, x_{j,0}^\pm\} \]
\[ = \mp [h_{j,K-l+1}, h_{i,l}] - [x_{j,K-l}^\pm, [h_{i,l+1}, x_{j,0}^\pm]] - [h_{i,l+1}, [x_{j,K-l}^\pm, x_{j,0}^\pm]] \pm \frac{a_{ij}}{2} \{h_{i,l}, x_{j,K-l}^\pm\} \]
\[ = \mp [h_{j,K-l+1}, h_{i,l}] - [x_{j,K-l}^\pm, [h_{i,l+1}, x_{j,0}^\pm]] + [h_{i,l+1}, [h_{j,K-l}^\pm, x_{j,0}^\pm]] \pm \frac{a_{ij}}{2} \{h_{i,l}, x_{j,K-l}^\pm\} \]
\[ \pm \left( [h_{i,l}, x_{j,K-l}^\pm], x_{j,0}^\pm \right) - \frac{a_{ij}}{2} \{h_{i,l}, h_{j,K-l}^\pm\} \]
\[ = \pm (h_{i,l}, h_{j,K-l+1}) - (h_{i,l+1}, h_{j,K-l}) + [h_{i,l+1}, x_{j,K-l}^\pm], x_{j,0}^\pm] \]
\[ \pm \frac{a_{ij}}{2} \{[h_{i,l}, x_{j,K-l}^\pm], x_{j,0}^\pm\} - \frac{a_{ij}}{2} \{h_{i,l}, h_{j,K-l}^\pm\}. \]  \hspace{1cm} (4.88)

Since the second term is obtained from the first term by shifting the index $l \to l - 1$, the
sum of the first and second term reads
\[
[h_{i,l}, x_{j,K-l}^\pm, x_{j,l}^\mp] - [h_{i,l-1}, x_{j,K-l+1}^\pm, x_{j,l}^\mp] \\
= \mp([h_{i,l+1}, h_{j,K-l}] - 2[h_{i,l}, h_{j,K-l+1}] + [h_{i,l-1}, h_{j,K-l+2}]) \\
+ [h_{i,l+1}, x_{j,K-l}^{\pm}, x_{j,l}^\mp] \\
\pm \frac{a_{ij}}{2} \{[h_{i,l}, x_{j,K-l}^{\pm}, x_{j,l}^\mp] - [h_{i,l-1}, x_{j,K-l+1}^{\pm}, x_{j,l}^\mp]\} \\
- \frac{a_{ij}}{2} \{h_{i,l}, h_{j,K-l\mp}\} + \frac{a_{ij}}{2} \{h_{i,l-1}, h_{j,K-l+1}\} \\
= \mp([h_{i,l+1}, h_{j,K-l}] - 2[h_{i,l}, h_{j,K-l+1}] + [h_{i,l-1}, h_{j,K-l+2}]) \\
+ \frac{a_{ij}^2}{4} \{[h_{i,l-1}, x_{j,K-l}^{\pm}, x_{j,l}^\mp]\} \\
+ \frac{a_{ij}}{2} \{h_{i,l-1}, h_{j,K-l+1}\} \pm \frac{a_{ij}}{2} \{x_{j,K-l}^{\pm}, [h_{i,l-1}, x_{j,l}^\mp]\}. \tag{4.89}
\]

Then, adding the last term
\[
\pm \frac{a_{ij}}{2} \{[h_{i,l-1}, x_{j,K-l}^{\pm}, x_{j,l}^\mp]\} = - \frac{a_{ij}}{2} \{h_{i,l-1}, h_{j,K-l+1}\} \mp \frac{a_{ij}}{2} \{x_{j,K-l}^{\pm}, [h_{i,l-1}, x_{j,l}^\mp]\} \tag{4.90}
\]
to above, we obtain the relation
\[
0 = [h_{i,l}, x_{j,K-l}^\pm, x_{j,l}^\mp] - [h_{i,l-1}, x_{j,K-l+1}^\pm, x_{j,l}^\mp] \mp \frac{a_{ij}}{2} \{[h_{i,l-1}, x_{j,K-l}^\pm, x_{j,l}^\mp]\} \\
= \mp([h_{i,l+1}, h_{j,K-l}] - 2[h_{i,l}, h_{j,K-l+1}] + [h_{i,l-1}, h_{j,K-l+2}]) \\
+ \frac{a_{ij}^2}{4} \{[h_{i,l-1}, x_{j,K-l}^{\pm}, x_{j,l}^\mp]\} \\
\pm \frac{a_{ij}}{2} \{h_{j,K-l}, [h_{i,l}, x_{j,l}^\mp] - [h_{i,l-1}, x_{j,l}^\mp]\} \\
= \mp([h_{i,l+1}, h_{j,K-l}] - 2[h_{i,l}, h_{j,K-l+1}] + [h_{i,l-1}, h_{j,K-l+2}]) \\
+ \frac{a_{ij}^2}{4} \{(h_{i,l-1}, x_{j,K-l}^\pm, x_{j,l}^\mp) - (-1)^{\delta_{lj}} \{h_{i,l-1}, x_{j,l}^\mp, x_{j,K-l}^\pm\}\}. \tag{4.91}
\]

Noting that the second line of the last equality is further simplified and it actually vanishes,
\[
\{[h_{i,l-1}, x_{j,K-l}^\pm, x_{j,l}^\mp]\} - (-1)^{\delta_{lj}} \{h_{i,l-1}, x_{j,l}^\mp, x_{j,K-l}^\pm\} = [h_{i,l-1}, [x_{j,K-l}^\pm, x_{j,l}^\mp]] \\
= \pm [h_{i,l-1}, h_{j,K-l}] \\
= 0. \tag{4.92}
\]
As a result, we have the following recursion relation
\[
[h_{i,l+1}, h_{j,K-l}] - 2[h_{i,l}, h_{j,K-l+1}] + [h_{i,l-1}, h_{j,K-l+2}] = 0 \quad \text{for} \quad 1 \leq l \leq K.
\]  (4.93)

The recursion relation (4.93) implies that
\[
[h_{i,l+1}, h_{j,K-l}] - 2[h_{i,l}, h_{j,K-l+1}] + [h_{i,l-1}, h_{j,K-l+2}] = 0 \quad \text{for} \quad 1 \leq l \leq K.
\]  (4.94)

Since \([h_{i,0}, h_{j,K+1}] = 0\), we have
\[
[h_{i,l+1}, h_{j,K-l}] = (l+1)[h_{i,1}, h_{j,K}] \quad \text{for} \quad 1 \leq l \leq K.
\]  (4.95)

When \(l = K\), it becomes
\[
[h_{i,K+1}, h_{j,0}] = (K+1)[h_{i,1}, h_{j,K}] \quad \text{for} \quad 1 \leq l \leq K.
\]  (4.96)

This gives us \([h_{i,1}, h_{j,K}] = 0\) since the left hand side is zero. Therefore, we obtain
\[
[h_{i,l}, h_{j,K-l+1}] = 0 \quad \text{for} \quad 0 \leq l \leq K+1.
\]  (4.97)

This completes the proof of lemma.

We are now ready to prove (4.1), (4.2), (4.4) and (4.5).

**Proof of (4.1), (4.2), (4.4) and (4.5).** At first, we show that two relations (4.4) and (4.5) are casted into the following unified form,
\[
[h_{i,r+1}, x_{j,s}^{\pm}] - [h_{i,r}, x_{j,s+1}^{\pm}] = \pm \frac{a_{ij}}{2} \{h_{i,r}, x_{j,s}^{\pm}\} \quad \text{for} \quad i, j \in \{1, 2, 3\}.
\]  (4.98)

In fact, if \((i,j) \neq (2,2)\), then (4.98) is nothing but (4.4). If \((i,j) = (2,2)\), it reduces to
\[
[h_{2,r+1}, x_{2,s}^{\pm}] - [h_{2,r}, x_{2,s+1}^{\pm}] = 0
\]  (4.99)

for any \(r, s \in \mathbb{Z}_+\) since \(a_{22} = 0\). This implies that
\[
[h_{2,r}, x_{2,s}^{\pm}] = [h_{2,0}, x_{2,r+s}^{\pm}].
\]  (4.100)

Hence, it is enough for us to prove the relations (4.1), (4.2), and (4.98). We show these relations simultaneously by induction with respect to a number \(K \in \mathbb{Z}_{\geq 0}\) such as \(r + s \leq K\).

When \(K = 0\), they hold by Definition 2.1 and 3.1. When \(K = 1\), they are derived from Definition 3.1 and 4.19 in Lemma 4.6.
For a fixed \( K \geq 1 \) and any \( r, s \in \mathbb{Z}_{\geq 0} \) such that \( r + s \leq K \), suppose that the relations (4.1), (4.2), and (4.98) hold. Then, we will show that these relations also hold for any \( r, s \in \mathbb{Z}_+ \) satisfying \( r + s \leq K + 1 \).

Firstly, the relation (4.1) holds for any \( r, s \) such that \( r + s \leq K + 1 \) by Lemma 4.8.

Secondly, let us prove that the relation (4.2) holds for \( r + s \leq K + 1 \). Commuting \( \tilde{h}_{k,1} \) with the relations

\[
[x_{i,l}^+, x_{j,K-l}^-] = \delta_{ij} h_{i,K} \quad \text{for} \quad 0 \leq l \leq K, \tag{4.101}
\]

we have

\[
a_{kj}[x_{i,l+1}^+, x_{j,K-l}^-] - a_{kj}[x_{i,l}^+, x_{j,K-l+1}^-] = 0 \tag{4.102}
\]

because of \([\tilde{h}_{k,1}, h_{i,K}] = 0\). When \( i \neq j \), since \( a_{ii}a_{jj} - a_{ij}a_{ji} \neq 0 \), we get

\[
[x_{i,l}^+, x_{j,K-l}^-] = 0 \quad \text{for} \quad 0 \leq l \leq K + 1. \tag{4.103}
\]

When \( i = j \), it is always possible to chose \( k \) such as \( a_{ik} \neq 0 \). For this \( k \), the relation (4.102) implies that

\[
[x_{i,l+1}^+, x_{i,K-l}] = [x_{i,l}^+, x_{i,K-l+1}] \quad \text{for} \quad 0 \leq l \leq K. \tag{4.104}
\]

In particular, setting \( l = K \), we see by definition \( h_{i,K+1} \) in (4.10) that

\[
[x_{i,l}^+, x_{i,K+l-1}^-] = h_{i,K+1} \quad \text{for} \quad 0 \leq l \leq K + 1. \tag{4.105}
\]

Therefore, we have proved that, for \( i, j \in \{1, 2, 3\} \), it holds that

\[
[x_{i,l}^+, x_{j,K-l+1}^-] = \delta_{ij} h_{i,K+1} \quad \text{for} \quad 0 \leq l \leq K + 1. \tag{4.106}
\]

Finally, we are going to show (4.98). By the induction assumption, it holds that

\[
0 = S_{ij}(l, K - l) \quad \text{for} \quad 0 \leq l \leq K - 1
\]

\[
= [h_{i,l+1}, x_{j,K-l}^\pm] - [h_{i,l}, x_{j,K-l+1}^\pm] + \frac{a_{ij}}{2} \{h_{i,l}, x_{j,K-l}^\pm\}. \tag{4.107}
\]

Commuting \( \tilde{h}_{k,1} \) with \( k \) such that \( a_{kj} \neq 0 \) with this relation, we have

\[
0 = S_{ij}(l, K - l + 1) \quad \text{for} \quad 0 \leq l \leq K - 1, \tag{4.108}
\]

because we have already proved that \([\tilde{h}_{i,1}, h_{j,m}] = 0\) for \( 0 \leq m \leq K \). Then, the remaining task is to prove the cases of \( l = K \) and \( K + 1 \),

\[
S_{ij}(K, 1) = [h_{i,K+1}, x_{j,1}^\pm] - [h_{i,K}, x_{j,1}^\pm] + \frac{a_{ij}}{2} \{h_{i,K}, x_{j,1}^\pm\} = 0,
\]

\[
S_{ij}(K + 1, 0) = [h_{i,K+2}, x_{j,0}^\pm] - [h_{i,K+1}, x_{j,0}^\pm] + \frac{a_{ij}}{2} \{h_{i,K+1}, x_{j,0}^\pm\} = 0. \tag{4.109}
\]
To prove these relations, we shall start with
\[ R_{ij}(r, s) = \left[ x_{i,r}^+, x_{j,s}^+ \right] - \left[ x_{i,r}^-, x_{j,s}^- \right] = \frac{a_{ij}}{2} \{ x_{i,r}^+, x_{j,s}^- \} = 0, \quad (4.110) \]
for \( r, s \in \mathbb{Z}_{\geq 0} \). Then, commuting \( x_{i,0}^+ \) with \( R_{ij}(r, s) = 0 \) and using the result (4.106), we have
\[ S_{ij}(r, s) + \delta_{ij}(-1)^{\delta_{ij}} S_{ii}(r, s) = 0 \quad \text{for} \quad r, s \in \mathbb{Z}_{\geq 0}. \quad (4.111) \]
When \( i \neq j \), we obtain
\[ S_{ij}(r, s) = 0 \quad \text{for} \quad r, s \in \mathbb{Z}_{\geq 0}. \quad (4.112) \]
This proves the relations (4.109) by setting \((r, s) = (K, 1)\) and \((K + 1, 0)\), respectively. When \( i = j \), we have
\[ S_{ii}(r, s) + (-1)^{\delta_{ii}} S_{ii}(r, s) = 0 \quad \text{for} \quad r, s \in \mathbb{Z}_{\geq 0}. \quad (4.113) \]
In particular, by setting \((r, s) = (K, 1)\) and \((K + 1, 0)\), the above relations reduce to
\[ S_{ii}(K, 1) + (-1)^{\delta_{ii}} S_{ii}(1, K) = 0, \]
\[ S_{ii}(K + 1, 0) + (-1)^{\delta_{ii}} S_{ii}(0, K + 1) = 0. \quad (4.114) \]
Here, we notice that, by the relations (4.19) and (4.15) in Lemma 4.6, the following quantities vanish
\[ S_{ii}(1, K) = [h_{i,2}, x_{i,K}^+] - [h_{i,1}, x_{i,K+1}^+] = \frac{a_{ii}}{2} \{ h_{i,1}, x_{i,K}^+ \} = 0, \]
\[ S_{ii}(0, K + 1) = [h_{i,1}, x_{i,K+1}^+] - [h_{i,0}, x_{i,K+2}^+] = \frac{a_{ii}}{2} \{ h_{i,0}, x_{i,K+1}^+ \} = 0. \quad (4.115) \]
Thus, the relations (4.109) with \( i = j \) are proved.

Therefore, we have completed the induction procedure and proved (4.1), (4.2) and (4.9), which includes the relations (4.4) and (4.5).

We are now left with proving the Serre relations (4.8) and (4.9).

**Proof of (4.8).** We start to prove the following lemma.

**Lemma 4.9.** The operators defined by
\[ \tilde{h}_{i,2} = h_{i,2} - h_{i,0} \tilde{h}_{i,1} - \frac{1}{6} h_{i,0}^3 \quad (4.116) \]
satisfy the following relations, for \( i, j = 1, 2, 3 \) and \( r \in \mathbb{Z}_{\geq 0} \),
\[ [\tilde{h}_{i,2}, x_{j,r}^+] = \pm a_{ij} x_{j,r+2}^+ \pm \frac{1}{12} a_{ij}^3 x_{j,r}^+. \quad (4.117) \]
Proof. By Proposition 4.7 and the definition of \( \tilde{h}_{i,2} = B_{ij} + \frac{1}{12}a_{ij}^2h_{i,0} \), we immediately obtain
\[
[\tilde{h}_{i,2}, x^\pm_{j,r}] = [B_{ij} + \frac{1}{12}a_{ij}^2h_{i,0}, x^\pm_{j,r}] = \pm x^\pm_{j,r} \mp \frac{1}{12}a_{ij}^3x^\pm_{j,r}.
\] (4.118)

This proves the lemma. \( \square \)

We set, for \( r, s, t \in \mathbb{Z}_{\geq 0} \) and \( j = 1, 3 \),
\[
X(r, s; t) = [x^\pm_{j,r}, [x^\pm_{j,r}, x^\pm_{2,t}]] + [x^\pm_{j,s}, [x^\pm_{j,r}, x^\pm_{2,t}]].
\] (4.119)

We prove (4.8) by induction with respect to the number \( K = r + s + t \). We may assume that \( r \geq s \) without loss of generality.

When \( K = 0 \), it is nothing but the Serre relations of the Lie algebra (2.17),
\[
X(0, 0; 0) = 2[x^\pm_{j,0}, [x^\pm_{j,0}, x^\pm_{2,0}]] = 0, \quad (j = 1, 3).
\] (4.120)

When \( K = 1 \), commuting \( \tilde{h}_{k,1} \) with \( X(0, 0; 0) = 0 \), we get
\[
a_{kj}X(1, 0; 0) + a_{k2}X(0, 0; 1) = 0.
\] (4.121)

Setting \( k = j \) and 2 respectively, we obtain
\[
X(1, 0; 0) = X(0, 0; 1) = 0
\] (4.122)
since \( a_{jj}a_{22} - a_{j2}a_{2j} \neq 0 \) \( (j = 1, 3) \).

Next, we suppose that for a fixed \( K \geq 0 \) it holds that
\[
X(r, s; t) = 0, \quad (4.123)
\]
\[
X(r + 1, s; t) = X(r, s + 1; t) = X(r, s; t + 1) = 0, \quad (4.124)
\]
for \( r, s, t \in \mathbb{Z}_{\geq 0} \) such as \( r \geq s, r + s + t = K \). Then, we show that
\[
X(r + 2, s; t) = X(r + 1, s + 1; t) = X(r, s + 2; t) = 0,
\]
\[
X(r + 1, s; t + 1) = X(r, s + 1; t + 1) = 0,
\]
\[
X(r, s; t + 2) = 0.
\] (4.125)

Commuting \( \tilde{h}_{k,1} \) \( (k = j, 2) \) with three relations in (4.124), we get
\[
X(r + 2, s; t) + X(r + 1, s + 1; t) = 0, \quad X(r + 1, s; t + 1) = 0, \quad (4.126)
\]
\[
X(r + 1, s + 1; t) + X(r, s + 2; t) = 0, \quad X(r, s + 1; t + 1) = 0,
\]
\[
X(r + 1, s; t + 1) + X(r, s + 1; t + 1) = 0, \quad X(r, s; t + 2) = 0.
\]
While, by Lemma 4.9, commuting \( \tilde{h}_{k,2} \) \((k = j, 2)\) with the relation in (4.123), we get
\[
X(r + 2, s; t) + X(r, s + 2; t) = 0, \quad X(r, s; t + 2) = 0.
\] (4.127)
Thus, (4.126) and (4.127) give (4.125). This completes the induction procedure. Hence, we have proved (4.8).

Proof of (4.9). For convenience, we introduce a notation,
\[
Z_{(k,l,m,n)} = [[x_{1,k}^\pm, x_{2,l}^\pm, [x_{3,m}^\pm, x_{2,n}^\pm]].
\] (4.128)
By definition (4.10), \( P_r^\pm = Z_{(r,0,0,0)} \). Then, the relations (4.9) claim the following two relations, for \( K \in \mathbb{Z}_{\geq 0} \),
\[
Z_{(K,0,0,0)} = Z_{(K-l,0,l,0)} \quad \text{for} \quad 0 \leq l \leq K, \quad (4.129)
\]
\[
[Z_{(K,0,0,0)}, J] = 0 \quad \text{for all} \quad J \in Y_L(g). \quad (4.130)
\]
We show (4.129) and (4.130) by induction with respect to \( K \in \mathbb{Z}_{\geq 0} \). When \( K = 0 \), these relations hold by Definition 2.1 and 3.1.

Suppose that for a fixed \( K \in \mathbb{Z}_{\geq 0} \) the relations (4.129) and (4.130) hold.

Firstly, we consider the relation (4.129). Commuting \( \tilde{h}_{2,1} \) with (4.129), it yields for \( 0 \leq l \leq K \)
\[
0 = [\tilde{h}_{2,1}, Z_{(K-l,0,l,0)}]
= \mp (Z_{(K-l+1,0,l,0)} - Z_{(K-l,0,l+1,0)}). \quad (4.131)
\]
Hence, this proves that the relation (4.129) holds for \( K + 1 \),
\[
Z_{(K+1,0,0,0)} = Z_{(K+1-l,0,l,0)} \quad \text{for} \quad 0 \leq l \leq K + 1. \quad (4.132)
\]
Next, we show the relation (4.130) for \( K + 1 \). For this purpose, it is sufficient to prove
\[
[Z_{(K,0,0,0)}, J] = 0 \quad \text{for any} \quad J \in \{x_{j,0}^\pm, x_{1,1}^\pm \mid j = 1, 2, 3\}, \quad (4.133)
\]
because the other generators in \( Y_L(g) \) are generated by the above generators \( x_{j,0}^\pm \) with \( j = 1, 2, 3 \), and \( x_{1,1}^+ \). The calculations are straightforward with the help of the following two lemmas.

Lemma 4.10. For \( r, s \in \mathbb{Z}_{\geq 0} \), it holds that
\[
[x_{1,r}^\pm, x_{3,s}^\pm] = 0. \quad (4.134)
\]
Proof. Commuting $\tilde{h}_{1,1}$ by $r$ times and $\tilde{h}_{3,1}$ by $s$ times respectively with
\[ [x_{1,0}^\pm, x_{3,0}^\pm] = 0 , \]
we obtain the desired relations. \hfill $\square$

**Lemma 4.11.** For an odd generator $F \in Y_L(\mathfrak{g})$, if $[F, F] = 0$, then it holds that
\[ [F, [F, x]] = 0 \quad \text{for any} \quad x \in Y_L(\mathfrak{g}). \] (4.135)

*Proof.* By the super-Jacobi identity, we have
\[ [F, [F, x]] = [[F, F], x] - [F, [F, x]] . \] (4.136)

Since $[F, F] = 0$, it gives us $[F, [F, x]] = 0$. \hfill $\square$

The commutativity with the Lie algebraic generators can be shown as follows.

With the generators $x_{1,0}^\pm$, it is calculated as
\[
[Z_{(K+1,0,0,0)}, x_{1,0}^\pm] = [Z_{(0,0,K+1,0)}, x_{1,0}^\pm] = [[[x_{1,0}^\pm, x_{2,0}^\pm], [x_{3,K+1}^\pm, x_{2,0}^\pm]], x_{1,0}^\pm] = 0 . \] (4.137)

Here, the first equality is due to (4.132). The third equality is obtained from the Serre relation $[x_{1,0}^\pm, [x_{1,0}^\pm, x_{2,0}^\pm]] = 0$ and Lemma 4.10. The last equality is owing to Lemma 4.11 because of $\deg([x_{1,0}^\pm, x_{2,0}^\pm]) = 1$ and $[[x_{1,0}^\pm, x_{2,0}^\pm], [x_{1,0}^\pm, x_{2,0}^\pm]] = 0$.

With the generators $x_{1,0}^\mp$, the commutation relation is computed as
\[
[Z_{(K+1,0,0,0)}, x_{1,0}^\mp] = [Z_{(0,0,K+1,0)}, x_{1,0}^\mp] = [[[x_{1,0}^\pm, x_{2,0}^\pm], [x_{3,K+1}^\pm, x_{2,0}^\pm]], x_{1,0}^\mp] = \pm[[h_{1,0}^\pm, x_{2,0}^\pm], [x_{3,K+1}^\pm, x_{2,0}^\pm]] = -[x_{2,0}^\pm, [x_{3,K+1}^\pm, x_{2,0}^\pm]] = 0 , \] (4.138)

where the last equality is due to Lemma 4.11. The commutativity of $P_{K+1}^\pm = Z_{(K+1,0,0,0)}$ with the other Lie algebraic generators $x_{2,0}^\pm, x_{3,0}^\pm$ are proved by the similar computations.
Finally, let us confirm the commutativity with the degree one generator $x_{1,1}^\pm$,

$$
\begin{align*}
[Z_{(K+1,0,0,0)}, x_{1,1}^\pm] &= [Z_{(1,0,K,0)}, x_{1,1}^\pm] \\
&= [[x_{1,1}^\pm, x_{2,0}^\pm], [x_{3,K}^\pm, x_{2,0}^\pm], x_{1,1}^\pm] \\
&= [[x_{1,1}^\pm, x_{2,0}^\pm], [x_{3,K}^\pm, x_{2,0}^\pm, x_{1,1}^\pm]] \\
&= 0.
\end{align*}
$$

Here, the first equality is due to (4.132) with $l = K$. The third equality is obtained from the Serre relation $[x_{1,1}^\pm, [x_{1,1}^\pm, x_{2,0}^\pm]] = 0$ and Lemma 4.10. The last equality is owing to Lemma 4.11 because of $\deg([x_{1,1}^\pm, x_{2,0}^\pm]) = 1$ and

$$
[[x_{1,1}^\pm, x_{2,0}^\pm], [x_{1,1}^\pm, x_{2,0}^\pm]] = -[[x_{1,1}^\pm, [x_{1,1}^\pm, x_{2,0}^\pm]], x_{2,0}^\pm] + [x_{1,1}^\pm, [x_{2,0}^\pm, [x_{1,1}^\pm, x_{2,0}^\pm]]] = 0. \tag{4.140}
$$

Hence, we have proved that

$$
[Z_{(K+1,0,0,0)}, J] = 0 \quad \text{for any} \quad J \in Y_L(g). \tag{4.141}
$$

This completes the induction procedure, and the relation (4.9) is proved.

Now, we have shown that the map $\phi : Y_D(g) \to Y_L(g)$ defined in (4.11) is a homomorphism. Therefore, Theorem 4.3 is proved.

\section*{Acknowledgment}

We would like to thank the hospitality of the School of Mathematics and Statistics, the University of Sydney. Most of this work has been done during his stay at the university. We thank Prof. Hiroyuki Yamane for the variable discussions. We also appreciate our colleagues, Prof. Yoshiyuki Koga, and Prof. Yuji Satoh, for their communication with the related subjects. We also thank Prof. Alessandro Torrielli for his correspondence. Finally, we are very grateful to Prof. Alex Molev for his variable comments and encouragement for this project. This work was supported by JSPS KAKENHI Grant Number JP19K03421.

\section{Relation to Drinfeld’s first realization}

In this Appendix, we propose the relation of our Yangian $Y_L(g)$ to the Drinfeld first realization \cite{Drinfeld1, Drinfeld2}. For this purpose, we introduce the Lie algebraic generators

$$
x_i^\pm = x_i^{\pm,0}, \quad h_i = h_{i,0}, \quad P^\pm = P_0^\pm \quad \text{with} \quad i = 1, 2, 3, \tag{A.1}
$$

Here, the first equality is due to (4.132) with $l = K$. The third equality is obtained from the Serre relation $[x_{1,1}^\pm, [x_{1,1}^\pm, x_{2,0}^\pm]] = 0$ and Lemma 4.10. The last equality is owing to Lemma 4.11 because of $\deg([x_{1,1}^\pm, x_{2,0}^\pm]) = 1$ and

$$
[[x_{1,1}^\pm, x_{2,0}^\pm], [x_{1,1}^\pm, x_{2,0}^\pm]] = -[[x_{1,1}^\pm, [x_{1,1}^\pm, x_{2,0}^\pm]], x_{2,0}^\pm] + [x_{1,1}^\pm, [x_{2,0}^\pm, [x_{1,1}^\pm, x_{2,0}^\pm]]] = 0. \tag{4.140}
$$

Hence, we have proved that

$$
[Z_{(K+1,0,0,0)}, J] = 0 \quad \text{for any} \quad J \in Y_L(g). \tag{4.141}
$$

This completes the induction procedure, and the relation (4.9) is proved.

Now, we have shown that the map $\phi : Y_D(g) \to Y_L(g)$ defined in (4.11) is a homomorphism. Therefore, Theorem 4.3 is proved. \hfill \Box

\section{A Relation to Drinfeld’s first realization}

In this Appendix, we propose the relation of our Yangian $Y_L(g)$ to the Drinfeld first realization \cite{Drinfeld1, Drinfeld2}. For this purpose, we introduce the Lie algebraic generators

$$
x_i^\pm = x_i^{\pm,0}, \quad h_i = h_{i,0}, \quad P^\pm = P_0^\pm \quad \text{with} \quad i = 1, 2, 3, \tag{A.1}
$$

This completes the induction procedure, and the relation (4.9) is proved.

\section*{Acknowledgment}

We would like to thank the hospitality of the School of Mathematics and Statistics, the University of Sydney. Most of this work has been done during his stay at the university. We thank Prof. Hiroyuki Yamane for the variable discussions. We also appreciate our colleagues, Prof. Yoshiyuki Koga, and Prof. Yuji Satoh, for their communication with the related subjects. We also thank Prof. Alessandro Torrielli for his correspondence. Finally, we are very grateful to Prof. Alex Molev for his variable comments and encouragement for this project. This work was supported by JSPS KAKENHI Grant Number JP19K03421.

\section{A Relation to Drinfeld’s first realization}

In this Appendix, we propose the relation of our Yangian $Y_L(g)$ to the Drinfeld first realization \cite{Drinfeld1, Drinfeld2}. For this purpose, we introduce the Lie algebraic generators

$$
x_i^\pm = x_i^{\pm,0}, \quad h_i = h_{i,0}, \quad P^\pm = P_0^\pm \quad \text{with} \quad i = 1, 2, 3, \tag{A.1}
$$

This completes the induction procedure, and the relation (4.9) is proved.

\section*{Acknowledgment}

We would like to thank the hospitality of the School of Mathematics and Statistics, the University of Sydney. Most of this work has been done during his stay at the university. We thank Prof. Hiroyuki Yamane for the variable discussions. We also appreciate our colleagues, Prof. Yoshiyuki Koga, and Prof. Yuji Satoh, for their communication with the related subjects. We also thank Prof. Alessandro Torrielli for his correspondence. Finally, we are very grateful to Prof. Alex Molev for his variable comments and encouragement for this project. This work was supported by JSPS KAKENHI Grant Number JP19K03421.
and the following hatted degree one generators,

\[
\begin{align*}
\hat{x}_1^+ &= x_{1,1}^+ - \frac{1}{2}(h_{1,0}x_{1,0}^+ - \sum_{\mu=3,4} E_{2\mu}E_{\mu1}) \\
\hat{x}_2^+ &= x_{2,1}^+ - \frac{1}{2}(h_{2,0}x_{2,0}^+ + E_{12}E_{31} + E_{34}E_{42} - E_{14}P_0^+) \\
\hat{x}_3^+ &= x_{3,1}^+ - \frac{1}{2}(h_{3,0}x_{3,0}^+ - \sum_{l=1,2} E_{l3}E_{4l}) \\
\hat{h}_1 &= \tilde{h}_{1,1} + x_{1,0}^+ x_{1,0} - \frac{1}{2} \sum_{\mu=3,4} (E_{1\mu}E_{\mu1} - E_{2\mu}E_{\mu2}) \\
\hat{h}_2 &= \tilde{h}_{2,1} - \frac{1}{2}(E_{12}E_{21} - E_{13}E_{31} + E_{34}E_{43} + E_{24}E_{42} - P_0^- P_0^+) \\
\hat{h}_3 &= \tilde{h}_{3,1} - x_{3,0}^+ x_{3,0} - \frac{1}{2} \sum_{l=1,2} (E_{l3}E_{3l} - E_{l4}E_{4l}) \\
\hat{x}_1^- &= x_{1,1}^- - \frac{1}{2}(x_{1,0}^- h_{1,0} - \sum_{\mu=3,4} E_{1\mu}E_{\mu2}) \\
\hat{x}_2^- &= x_{2,1}^- - \frac{1}{2}(x_{2,0}^- h_{2,0} - E_{13}E_{21} - E_{24}E_{43} - P_0^- E_{41}) \\
\hat{x}_3^- &= x_{3,1}^- - \frac{1}{2}(x_{3,0}^- h_{3,0} + \sum_{l=1,2} E_{l4}E_{3l}) \\
\hat{P}^+ &= P_1^+ + C_0 P_0^+ \\
\hat{P}^- &= P_1^- + P_0^- C_0. 
\end{align*}
\]  

(A.2)

Here, the generators $E_{ij}$ of the matrix algebra in (2.22) via the identification (2.26).

We expect that the generators in (A.1) and (A.2) satisfy the defining relations of Drinfeld’s first realization. Though we have not checked all commutation relations, the following proposition supports our proposal.

**Proposition A.1.** The generators in (A.2) satisfy the relations,

\[
\begin{align*}
[\hat{h}_i, x_j^+] &= [h_i, \hat{x}_j^+] = \pm a_{ij} \hat{x}_j^+ \\
[\hat{x}_i^+, x_j^-] &= [x_i^+, \hat{x}_j^-] = \delta_{ij} \hat{h}_i \\
[\hat{x}_i^+, x_j^+] &= [x_i^+, \hat{x}_j^+] \\
[[\hat{x}_1^+, x_2^+], [x_3^+, x_2^+]] &= \hat{P}^\pm.
\end{align*}
\]  

(A.3) (A.4) (A.5) (A.6)

**Proof.** Using the relations (2.13)–(2.18), (2.22) and (3.1)–(3.9), we can verify these relations by direct computations. \qed
Remark A.2. It is noted that the antipodes for the generators listed in (A.2) do not contain the Lie algebraic generators (A.1), i.e.
\[ S(\hat{J}) = -\hat{J} \quad \text{for} \quad \hat{J} \in \{ \hat{x}^\pm, \hat{h}, \hat{P}^\pm \}. \] (A.7)

This is consistent with the fact that the Lie superalgebra \( g \) has a vanishing Killing form.

References

[1] K. Iohara and Y. Koga, *Central extensions of Lie superalgebras*, Comment. Math. Helv. 76 (2001), 110–154.

[2] N. Beisert, *The \( \mathfrak{su}(2|2) \) dynamic S-matrix*, Adv. Theor. Math. Phys. 12 (2008), 945–979.

[3] N. Beisert, *The analytic Bethe ansatz for a chain with centrally extended \( \mathfrak{su}(2|2) \) symmetry*, J. Stat. Mech. Theory Exp. 2007, no. 1, P01017, 63 pp.

[4] A. H. Kamupingene, Nguyen Anh Ky and Tch. D. Palev, *Finite-dimensional representations of the Lie superalgebra \( \mathfrak{gl}(2/2) \) in a \( \mathfrak{gl}(2) \oplus \mathfrak{gl}(2) \) basis. I. Typical representations*, J. Math. Phys. 30 (1989), 553–570.

[5] Tch. D. Palev and N. I. Stoilova, *Finite-dimensional representations of the Lie superalgebra \( \mathfrak{gl}(2/2) \) in a \( \mathfrak{gl}(2) \oplus \mathfrak{gl}(2) \) basis. II. Nontypical representations*, J. Math. Phys. 31 (1990), 953–988.

[6] G. Götz, Th. Quella and V. Schomerus, *Tensor products of \( \mathfrak{psl}(2|2) \) representations*, [arXiv:hep-th/0506072](https://arxiv.org/abs/hep-th/0506072).

[7] T. Matsumoto and A. Molev, “Representations of centrally extended Lie superalgebra \( \mathfrak{psl}(2|2) \),” J. Math. Phys. 55 (2014) 091704 [arXiv:1405.3420 [math.RT]].

[8] B. S. Shastry, “Exact Integrability of the One-Dimensional Hubbard Model,” Phys. Rev. Lett. 56 (1986) 2453.

[9] H. Yamane, *A central extension of \( U_q \mathfrak{sl}(2|2)^{(1)} \) and R-matrices with a new parameter*, J. Math. Phys. 44 (2003), 5450–5455.

[10] D. B. Uglov and V. E. Korepin, *The Yangian symmetry of the Hubbard model*, Phys. Lett. A 190 (1994), 238–242.

[11] N. Beisert, *The S-matrix of AdS/CFT and Yangian symmetry*, in Proceedings of the Solvay workshop “Bethe Ansatz: 75 Years Later”, 2006, paper 002.
[12] V. G. Drinfeld, “Hopf algebras and the quantum Yang-Baxter equation,” Sov. Math. Dokl. 32 (1985) 254 [Dokl. Akad. Nauk Ser. Fiz. 283 (1985) 1060].

[13] V. G. Drinfeld, “Quantum groups,” J. Sov. Math. 41 (1988) 898 [Zap. Nauchn. Semin. 155 (1986) 18].

[14] V. G. Drinfeld, “A New realization of Yangians and quantized affine algebras,” Sov. Math. Dokl. 36 (1988) 212.

[15] N. Beisert and M. de Leeuw, “The RTT realization for the deformed \( \mathfrak{gl}(2|2) \) Yangian,” J. Phys. A 47 (2014) 305201 [arXiv:1401.7691 [math-ph]].

[16] F. Spill and A. Torrielli, “On Drinfeld’s second realization of the AdS/CFT \( \mathfrak{su}(2|2) \) Yangian,” J. Geom. Phys. 59 (2009) 489 [arXiv:0803.3194 [hep-th]].

[17] I. Heckenberger, F. Spill, A. Torrielli and H. Yamane, “Drinfeld second realization of the quantum affine superalgebras of \( D^{(1)}(2, 1; x) \) via the Weyl groupoid,” RIMS Kokyuroku Bessatsu B 8 (2008), 171 [arXiv:0705.1071 [math.QA]].

[18] S. Z. Levendorskii, “On generators and defining relations of Yangians,” Journal of Geometry and Physics, Volume 12, Issue 1, 1993, Pages 1-11, ISSN 0393-0440.

[19] S. Z. Levendorskii, “On PBW bases for Yangians,” Letters in Mathematical Physics, January 1993, Volume 27, Issue 1, pp 37-42.

[20] V. K. Dobrev, “Note on Centrally Extended \( \mathfrak{su}(2|2) \) and Serre Relations,” Fortsch. Phys. 57 (2009) 542 [arXiv:0903.0511 [hep-th]].

[21] V. Stukopin, “Yangians of Lie Superalgebras of Type \( A(m, n) \),” Functional Analysis and Its Applications, Vol.28, No.3, 1994; “On representations of Yangian of Lie superalgebra \( A(n, n) \) type,” J. Phys. Conf. Ser. 411 (2013) 012027.