Long strings condensation and FZZT branes

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Abstract

We propose a matrix model description of extended D-branes in 2D noncritical string
1 Introduction

The singlet sector of large $N$ gauged quantum mechanics have long been given an interpretation as an exact description of $c = 1$ non-critical string theory. The closed string field theory describes the collective motion of the matrix eigenvalues\(^3\)\(^6\)\(^7\)\(^8\)\(^9\)\(^11\). The non-singlet sectors of the same quantum mechanics have long resisted such a detailed interpretation. The recent work \(^1\) interprets the most basic non-singlet sector, the adjoint, as a sector of $c = 1$ string theory in which one macroscopic fundamental string (long string) is stretched in space from the weak-coupling infinity and folded back.

The turning point of this long string (the tip) is pulled back by the string tension and repelled by the tachyon wall, hence the string tip follows a scattering trajectory coming in from weak coupling infinity in the far past and disappearing back in the far future. The adjoint sector of the quantum mechanics is shown to describe a Fermi sea of free eigenvalues with an interacting impurity, and the impurity dynamics is identified with the dynamics of the tip of the folded string.

\(^1\) makes a further identification to obtain a worldsheet CFT description of the adjoint sector: the folded string is seen as the appropriate limit of an open string mode living on an FZZT brane with Neumann conditions in time. (see for example \(^12\)\(^13\)\(^26\)\(^27\) for a definition and review of FZZT branes in noncritical string theories)

While most D-branes in $c < 1$ noncritical strings have been understood nicely in terms of resolvents of the corresponding matrix models \(^28\)\(^21\)\(^30\)\(^31\)\(^32\), in $c = 1$ these FZZT branes with Neumann boundary conditions in time have until now resisted a matrix model interpretation. These D-branes are extended in time and space, and support a massless propagating degree of freedom (the open string tachyon) that can considerably enrich the dynamics of the $c = 1$ model. The open string tachyon modes are kept away from the strong coupling region both by the closed string tachyon wall $\mu e^{-\phi}$ and by an open tachyon wall $\mu_B e^{-\phi_o}$.
In [1] the folded string is related to a very energetic open string attached to an FZZT brane of very large boundary cosmological constant $\mu_B$. The large open tachyon wall $\mu_B e^{-\phi}$ pins the string endpoints far in the weak coupling region at $\log \mu_B$, but the large energy allows the open string bulk to stretch macroscopically towards the strong coupling region. The stretched string limit is reached by keeping $E_{\text{open}} - \log \mu_B$ constant while sending $\mu_B$ to infinity, so to keep finite the energy available to the string tip’s motion. (see Figure 1)

If one keeps $\mu_B$ large but finite, an observer far enough in the weak coupling region to see the FZZT brane will interpret the scattering process as a coherent pulse of open string radiation coming in and condensing into a long stretched string. After a while the stretched string will then decay back to open string radiation.

We observe here an obvious similarity between this scenario and the description of unstable ZZ branes in the usual matrix model: consider a very energetic close string mode travelling in a background with very large bulk cosmological constant $\mu$. If $E_{\text{closed}} - \log \mu$ is finite there will be a large amount of time for which the closed string mode will admit a matrix model description as one (or few) eigenvalues detached from the rest of the Fermi sea, i.e. a decaying ZZ brane with a $\Lambda \cosh X^0$ tachyon deformation turned on the worldvolume. [2, 4, 33, 34]

The limit $\mu \to \infty$, $E_{\text{closed}} - \log \mu$ finite drains the Fermi sea and leaves a finite number
Figure 3: A single impurity in the Fermi sea and a pulse of open string radiation are related by \( \mu_B \to \infty \) \( E_{op} - \log \mu_B = \text{const.} \).

of eigenvalues with a finite distance from the top of the potential. (see Figure 2) It is then possible to recover a finite \( \mu \) by condensing infinitely many of such eigenvalues and producing again a Fermi sea. This is the statement that \( c = 1 \) noncritical string theory can be recovered from the collective coordinates of infinitely many condensed ZZ branes, and that condensation of ZZ branes is equivalent to a renormalization of \( \mu \).

We want to follow a similar logic to renormalize the very large \( \mu_B \) relevant for the long string setting back to a finite value, recovering the full open string tachyon dynamics from the collective coordinates of a large number of long strings. Indeed there are sectors of the matrix model labelled by certain higher representations of \( U(N) \) that can be used to describe several stretched strings, i.e. several impurities in the Fermi sea, in such a way that these impurities have Fermi statistics themselves.

By analogy with the closed string case, we propose the following statement: the dynamics of an FZZT brane of finite \( \mu_B \) in \( c = 1 \) noncritical strings is recovered by condensation of an infinite number of long strings, and is encoded in sectors of the matrix model corresponding to representations with a very large number of boxes (impurities). We intend to show that in presence of a chemical potential for the number of impurities a new Fermi sea of impurities arises, whose collective fluctuations encode the open string tachyon living on the FZZT brane. Stretched strings attached to a finite \( \mu_B \) FZZT brane will correspond to single impurities well above the impurity Fermi sea. (see Figure 3)

The presence of a Fermi level for impurities implies that the energy of one impurity above the vacuum state is finite (the extra impurity cannot have an energy lower that the Fermi
The results in [1] cover the case of $0 \mathcal{A}$ and $0 \mathcal{B} \hat{c} = 1$ as well. A similar long string condensation process as the one described here could be used to understand Neumann FZZT branes in those non-critical superstring models.

The detailed organization of the paper can be gleaned from the table of contents. In section two contains the main results of this note: we will review some properties of non-singlet sectors, give our precise matrix model prescription for the FZZT brane and study the degrees of freedom of the system in the asymptotic region. In section three we will compare the modified matrix model action with the OSFT on a large number of decaying ZZ branes and one FZZT brane. Section four contains some observations about the consequences of our proposal in the theory with compact Euclidean time. In section five we will give a proposal for the matrix model description of an arbitrary number of FZZT branes in the $c = 1$ model and attempt to give a rough comparison with the expected target space dynamics.

2 Non-singlet sectors and impurities

We want to propose an exact description of extended D-branes in $c = 1$ noncritical string theory. We will show that the sea of eigenvalues in the sector of Matrix Quantum Mechanics associated with a condensate of long strings can supports two independent collective degrees of freedom, to be identified with the closed string field and the open string field living on a FZZT brane.

Matrix quantum mechanics is described by a very simple $SU(N)$ invariant action [3, 6, 10]

$$S(M) = \int dt \text{Tr} \left[ \dot{M}^2 - V(M) \right]$$

The restriction to the $SU(N)$ invariant sector of the quantum mechanics is typically done by gauging it: $\dot{M} \rightarrow \dot{M} - [A_0, M]$. This gauging has a straightforward interpretation when the matrix model is taken to be the BSFT for a large number of unstable ZZ branes. (see section 3 for more on this.)

The gauging removes the angular degrees of freedom, the eigenvalues of $M$ behave like free fermions in the potential $V(\lambda)$ and form a Fermi sea in the potential well. An appropriate double scaling limit, that makes $V(M)$ into an upside-down harmonic oscillator $V(M) = -M^2$ while keeping the Fermi level at finite distance from the top of the potential yields the $c = 1$ non-critical string theory. The only propagating mode, the massless closed string tachyon, is mapped to fluctuations of the fermi level by an appropriate bosonization map.

Without gauging, the dynamics of the $N^2$ degrees of freedom in $M$ still decomposes in infinitely many independent sectors, as the large $SU(N)$ symmetry commutes with the Hamiltonian. Different sectors are essentially labelled by the representations of $SU(N)$ that appear in powers of the adjoint. More precisely the change of variables

$$M = \Omega \text{diag}\{\lambda_1, \cdots, \lambda_N\} \Omega^{-1}$$

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splits the phase space into eigenvalues and angular degrees of freedom

\[ \frac{R^N \otimes SU(N)/U(1)^{N-1}}{S_N} \]  

(3)

The quotient by \( U(1)^{N-1} \) represents the freedom of acting from the right on \( \Omega \) with elements in the Cartan subalgebra, \( S_N \) the freedom to permute eigenvalues while acting at the same time on \( \Omega \) with the appropriate element of the Weyl group.

The Hilbert space \( L^2 \left( SU(N)/U(1)^{N-1} \right) \) admits a natural complete orthogonal base given by Fourier theory on coset spaces, on which the Hamiltonian block diagonal. We review in appendix A the details of the decomposition, the well known result is that

\[ \mathcal{H} = \bigoplus_{\mathcal{R}} \mathcal{H}_R^{d_R} \quad \mathcal{H}_R = \frac{L^2(R^N) \otimes \bar{\mathcal{R}}}{S_N} \]  

(4)

Here \( \mathcal{R} \) are \( SU(N) \) representations, \( \bar{\mathcal{R}} \) is the subspace of zero weight vectors in \( \mathcal{R} \), \( d_R \) the dimension of the representation and \( S_N \) acts by simultaneous permutation of the eigenvalues and of the corresponding Cartan generators.

The Hamiltonian in any given sector is

\[ \sum_i \left( -\frac{\partial^2}{\partial \lambda_i^2} - \lambda_i^2 \right) + \sum_{i,j} \frac{1 - \hat{P}_{ij}}{(\lambda_i - \lambda_j)^2} \]  

(5)

\( \hat{P}_{ij} \) acts on \( \bar{\mathcal{R}} \) as a basic transposition in the Weyl group \( S_N \), permuting the Cartan generators \( i \) and \( j \).

Reasoning along the lines of [1], when \( \mathcal{R} \) is the adjoint representation (more properly adjoint plus singlet, i.e. fundamental times anti-fundamental), \( \bar{\mathcal{R}} \) is the span of \( N \) diagonal vectors \( |n\rangle \otimes |\bar{n}\rangle \), \( n = 1 \cdots N \). The Hamiltonian describes the coupling of the \( N \) eigenvalues to a discrete degree of freedom \( n \), that acts as a label whose effect is to distinguish one of the eigenvalues from the others. The remaining \( N-1 \) eigenvalues do not interact with each other, but only with the labelled \( n \)-th eigenvalue. The interaction is the sum of a repulsive term and an exchange term, that allows the label ("impurity") to hop to a nearby eigenvalue.

In the double scaling limit one is left with the coupled system of the Fermi sea plus impurity, and the impurity is identified with the tip of the stretched string. [1] studies the scattering phase shifts for the impurity, in the natural approximation that ignores the back-reaction of the impurity on the second-quantized Fermi sea.

We propose that the sectors relevant to stretched string condensation are the ones labelled by the representation \( \mathcal{R}_k = (N)_{\wedge k} \otimes (\bar{N})_{\wedge k} \), where \( (N) \) is the fundamental, \( (\bar{N}) \) the anti-fundamental of \( SU(N) \). The zero weight subspace \( \bar{\mathcal{R}}_k \) is span by \( (N)_k \) vectors labelled by a multi-index \( I = \{i_1, \cdots, i_k\}, i_1 < i_2 < \cdots < i_k \).

These discrete degrees of freedom are equivalent to an assignment of \( k \) indistinguishable labels marking \( k \) of the eigenvalues. These \( k \) eigenvalues interact as before with the remaining \( N-k \) but do not interact between themselves. Following the analogy, after double scaling
limit it is natural to identify these $k$ impurities in the Fermi sea with $k$ stretched strings, that we claim are attached to a single FZZT brane with very large $\mu_B$.

If there were no interaction terms in the Hamiltonian the system would be equivalent to two sets of independent fermions in original potential. It would be natural to send both $N - k$ and $k$ to infinity in order to describe a condensation of the long strings. The chemical potential of the impurity Fermi sea would be naturally identified with a finite $\mu_B$ for the FZZT brane.

While a single-particle Fermi sea description is available exactly in the singlet sector and approximately in the small $k$ sectors, this double scaling limit procedure in $k$ may seem foolhardy, as it is far from obvious that for large $N - k$ and $k$ this highly interacting (but integrable! \cite{13,15}) spin-Calogero model will admit any such description.

On the other hand there is a simple canonical transformation in the original matrix model that makes evident the integrable structure of the problem and shows that this picture of two Fermi surfaces for two kind of (quasi-)particles is indeed appropriate.

Consider $P$, the matrix momentum conjugate to $M$, and make the hermitian combinations $A^\pm = P \pm M$.

The Hamiltonian is naturally written as Tr{$A^+, A^-$}. It is appropriate to make a change of base from wavefunctions of $M$ to wavefunctions of, say, $A^+$. The power of this change of coordinates is the fact that diagonalization of $A^+ = \Omega^+ \text{diag}\{a_i^+\}\Omega^+$ reorganizes each sector subtly, so that the eigenvalues $a_i^+$ become free fermions, with Hamiltonian

$$\sum_i \{a_i^+, -i \frac{\partial}{\partial a_i^-}\}$$

and no two-particle interaction terms!

In the singlet sectors the changes of variable between $M$, $A^+$ and $A^-$ ”commute” with the diagonalization, and translate directly in the corresponding change of variables between $\lambda_i$ and $a_i^\pm$. In a generic sector the matrix change of variables maps each sector of the angular decomposition of the Hilbert space for one of the variables into the corresponding sector for the other variables. The change of base mixes in a complicated way the eigenvalue degrees of freedom with the discrete ones in $\tilde{R}$, so that eigenvalues in one description become quasi-particles in another. See appendix B for more details on this.

The time evolution of $a^\pm$ is rescaling by $e^{\pm i t}$, as the Hamiltonian is the dilatation operator for $a^\pm$. In the singlet sector the variables $\tau^+ = \log a^+$ and $\tau^- = \log a^-$ are appropriate to describe the asymptotic outgoing and ingoing regions, making the relativistic nature of the motion obvious. Fourier transform of eigenfunctions from $a^-$ to $a^+$ is the traditional swift way to get the single fermion phase shifts.

The energy eigenfunctions for the system in a generic sector can be written down directly:

$$\psi_J^+(a_n^+, \epsilon_n) = \sum_{\sigma \in S_N} \epsilon(\sigma)c_{\sigma(J)} \prod_i (a_i^+)^{\epsilon_n - \frac{1}{2}}$$
Scattering requires convolution with an integral kernel $G^K_J(a_m^-, a_n^+)$ that can be computed by evaluating certain Izkinson-Zuber-like $SU(N)$ integrals.

$$\psi_J^-(a_m^-) = \int \prod_k da_k^+ \sum_{K \in \tilde{R}} G^K_J(a_m^-, a_n^+) \psi^K_J(a_n^+)$$ (8)

It would be interesting to pursue this path to evaluate the scattering phases of $[14]$ on the whole energy spectrum. Less ambitiously we just remark that in the sector corresponding to the representations $R_k$ the system is equivalent to two independent sets of $N - k$ and $k$ free fermions. In fact the internal degrees of freedom in this sector are essentially equivalent to the choice of a subset of $k$ out of $N$ indistinguishable eigenvalues. One can trade this choice for a reduction of the permutation symmetry from $S_N$ to $S_{N-k} \otimes S_k$, and describe the system in terms of two different species of fermionic eigenvalues.

To second-quantize the system, consider a direct sum of sectors with different $k$

$$\mathcal{R}_{\text{Fock}} = \sum_{k=0}^N (N)^{\wedge k} \otimes (\tilde{N})^{\wedge k}$$ (9)

The sector $\mathcal{R}_{\text{Fock}}$ of matrix quantum mechanics describes a system of two kind of fermions, or a system of $N$ fermions with an $SU(2)$ isospin $[15][14][11]$. One can then naturally define two interacting Fermi fields $\Psi_i(\lambda)$ that creates/destroys eigenvalues of the two kinds, changing the values of $N$ and $k$.

A non-trivial field redefinition exists to map these fields into corresponding quasi-particle free Fermi fields $\Psi_i^\pm(a^\pm)$, built along the lines of the finite $N$ case. Ingoing and outgoing vacua can be defined in a straightforward fashion, with appropriate Fermi levels for the two species.

The story then follows closely the description in the singlet sector, with twice as many fields. In terms of $\tau^\pm = \log a^\pm$ these free Fermi fields represent free massless relativistic fermions. The standard bosonization map can convert them to two massless chiral bosonic fields $\Phi_i^\pm(\tau^\pm, t)$.

We propose then to associate the closed and open string tachyons with these two collective fields. We leave the precise identification, that is the choice of appropriate two-by-two leg pole matrices, to future work. These leg-pole coefficients should be fixed by comparison of the simplest $1 \rightarrow 1$ scattering amplitudes in the matrix model against the corresponding bulk two point function on the sphere for the closed string tachyon, the boundary two point function on the disk for the open string tachyon and the mixed bulk to boundary two point function on the disk.

After the leg poles have been fixed appropriately, higher correlation functions may be compared.

A second kind of comparison would involve a careful study of the boundary equivalent of the ground ring (see for example [35][25]) and comparison with the symmetry algebra of this $SU(2)$ spin-Calogero system, that extends the $A_{\infty}$ of the singlet sector.
The presence of an $SU(2)$ symmetry between the two species of eigenvalues is an interesting and unexpected fact from the point of view of the target space physics. As long as $\mu_B^2 \gg \mu$ it is natural to identify excitations of the upper Fermi sea with the closed strings, that can reach as deep as $-\log \mu$ in the strong coupling region, and excitations of the lower Fermi sea with open strings, that reflect back on the open tachyon wall already at $-\log \mu_B$.

It is possible to lower the boundary cosmological constant until the two Fermi levels become comparable, but then as they overcome each other and become well separated again, the upper Fermi sea should again be identified with closed strings. It appears that the moduli space for $\frac{\mu_B^2}{\mu}$ should be an half line, starting from a point of extended $SU(2)$ symmetry. It would be interesting to verify if the special properties of this point are manifest in the worldsheet BCFT description of the model, for large but comparable $\mu$ and $\mu_B$.

A second important issue is to understand the relation between the incoming vacuum defined by two Fermi seas of incoming quasi-particles and the outgoing vacuum defined by the two Fermi seas of outgoing quasi-particles. While it is possible that they are mapped onto each other by the second-quantized version of the matrix Fourier transform, it would be a fairly non-trivial fact. It is more probable that the incoming vacuum for a FZZT brane will map into a non-trivial state with open and closed radiation in the outgoing region. This should be checked against the possibility of pair creation in the background of an FZZT brane.

3 FZZT - ZZ open strings

There is an alternative, suggestive way to describe a second-quantized ensemble of representations $\bigoplus_k (N^\wedge k \otimes \bar{N}^\wedge k)$. Consider an extended gauged quantum mechanics, with action

$$S(M, u, \bar{u}) = \int dt \, \text{Tr} \left[ (D_0 M)^2 + M^2 \right] + u^\dagger D_0 u + \bar{u}^\dagger D_0 \bar{u}$$

We added two fermionic oscillators, $u$ in the fundamental of $U(N)$ and $\bar{u}$ in the anti-fundamental. The gauging reduces the Hilbert space to the global singlet representation, that is

$$\mathcal{H}_s = \bigoplus_k \mathcal{H}^N_{u} \otimes \mathcal{H}^N_{\bar{u}} \otimes \mathcal{H}^N_{M}$$

In other words the relevant sectors of the Matrix quantum mechanics appear in the global singlet paired up with the appropriate sectors of the $u, \bar{u}$ system. At first sight there is a certain over-counting, but after diagonalization of $M$ it is easy to check that each desired sector $k$ appears exactly once: the residual $U(1)^N$ invariance forces the $u$ Hilbert space to be the span of $\prod (u^\dagger_i \bar{u}^\dagger_i ) |0\rangle$, where $|0\rangle$ is killed by $u_i, \bar{u}_i$. Each base vector gives corresponds to the choice of a subset of $\{1, \cdots, N\}$. The system is the same as the one we propose for the long string condensation.
This offers an alternative language to define the collective fields for the two Fermi surfaces a la Das-Jevicki\cite{17} roughly as
\begin{equation}
\phi(x, t) = \text{Tr} \delta(x - M(t)) \quad \phi_0(x, t) = u^\dagger \delta(x - M(t)) u
\end{equation}

We propose to identify this gauged quantum mechanics with the BSFT for the degrees of freedom of $N$ unstable ZZ branes and one FZZT brane. The matrix $M$ should represent the open string tachyon on the decaying ZZ branes, and $u, \bar{u}$ the open strings stretched between FZZT and ZZ branes. The quantum mechanics is gauged because of the gauge field living on the unstable ZZ branes. In $c < 1$ noncritical strings a similar idea to obtain a description of FZZT branes by integrating away open string modes between the FZZT and the ZZ branes has been successfully implemented. \cite{18}

An important difference with the $c < 1$ case is that here we do not require any non-trivial interaction between $M$ and $u, \bar{u}$ besides the indirect one mediated by the gauge field. This fact allows the gauge-fixed action to be quadratic and solvable. A possible way to argue for the absence of terms like $u^\dagger F(M) u$ from the action is that the double scaling limit procedure will generically scale away the functional dependence $F(M)$ to leave a constant mass term for $u$, to be reabsorbed in the definition of the chemical potential $\mu_B$. See also \cite{5} for a possible alternative to the double scaling limit.

4 FZZT branes and T-duality

The thermal partition function in a given sector of the matrix quantum mechanics can be computed in a straightforward way and has been given an interpretation in the context of of $c = 1$ in compactified Euclidean time. (see for example \cite{19, 20, 21, 22})

The projection to a given representation can be implemented directly in the trace of the thermal green function as
\begin{equation}
\mathcal{Z}_\mathcal{R} = \int dM \ d\Omega \ G^E_{\beta}(M, \Omega M \Omega^{-1}) \text{Tr}_\mathcal{R}(\Omega)
\end{equation}

Correlators of closed string winding modes in the Euclidean time direction are computed by insertion of appropriate traces $\frac{1}{n} \text{Tr} \Omega^n$ in the singlet partition function. Hence the twisted thermal partition function in a given sector $\mathcal{R}$ corresponds to the computation of a winding-modes correlator. What is the correlator that correspond to our long strings condensation process? We need to sum over all the sectors, weighed by the chemical potential for the stretched strings: $e^{-2\mu_B k} \mathcal{Z}_{N^\wedge k} \otimes \bar{N}^\wedge k$.

The corresponding sum of traces of exterior powers of the fundamental can be traded for the insertion of a determinant.
\begin{equation}
\mathcal{Z}_{\text{FZZT}}(\mu_B) = \int dM \ d\Omega \ G^E_{\beta}(M, \Omega M \Omega^{-1}) \det(1 + e^{-\mu_B \Omega}) \det(1 + e^{-\mu_B \Omega^{-1}})
\end{equation}
This makes the relation between Neumann FTTZ branes and a condensate of long strings evident: the insertion of the determinant can regarded as the insertion of an exponentiated winding loop operator \( \exp(\text{Tr} \log(1 + e^{-\mu B} \Omega) + \text{Tr} \log(1 + e^{-\mu B} \Omega^{-1})) \) extended along the circle direction, and interpreted much along the lines of what has been done in \( c < 1 \) in terms of the winding integrable hierarchy.

A different kind of FZZT brane object that has been given an interpretation in the matrix model is the FZZT brane instanton with Dirichlet boundary conditions in time. This has been identified \([24, 23]\) with a basic matrix resolvent \( \det(z - M(t_0)) \) placed at a particular time. In euclidean time the various images along the circle would add up to an appropriate momentum loop operator.

\[
\prod_n \det(z - M(2\pi Rn)) = \exp \left[ \sum_n \text{Tr} \log(z - M(2\pi Rn)) \right] = \exp \left( \sum_n c_n T_n + \bar{c}_n T_{-n} \right) \tag{3}
\]

\[
\exp \left[ \sum_n \int dt e^{intE} \text{Tr} \log(z - M(t)) \right] \simeq \exp \left( \sum_n c_n T_n + \bar{c}_n T_{-n} \right) \tag{4}
\]

T-duality acts by exchanging loop operators made of winding and the momentum modes, and appropriately exchanges the FZZT brane with Dirichlet and Neumann conditions in time.

5 \( U(n) \) extension

The comparison with the FTTZ-ZZ BSFT offers an immediate generalization of our proposal that may describe the degrees of freedom living on \( n \) FZZT branes.

\[
S(M, U, \bar{U}) = \int dt \text{Tr} \left[ (D_0 M)^2 + M^2 \right] + \text{Tr} \left[ U^\dagger D_0 U \right] + \text{Tr} \left[ \bar{U}^\dagger D_0 \bar{U} \right] \tag{1}
\]

Here the \( U \)'s are appropriate \( N \) by \( n \) matrices of fermionic oscillators, corresponding to the open degrees of freedom between \( n \) FZZT and \( N \) unstable ZZ branes. Note that we do not gauge the \( U(n) \) symmetry of the model, as the presence of a gauge field on FZZT branes side is questionable. One can construct a vertex operator that roughly corresponds to it, but it behaves much like the discrete states in the closed string sector. This matter is of considerable interest and is related to the issue of generalizing the \( A_\infty \) symmetry algebra of the closed string theory to the open-closed string theory.

After diagonalization of \( M \) the fermionic variables \( U \) produce states of the form \( \prod(U^\dagger_{i,m} \bar{U}^\dagger_{i,m} |0\rangle) \). These may be thought as an assignment of \( n^2 \) kind of labels to the eigenvalues, so one has \( n^2 \) types of impurities that can move around the eigenvalue Fermi sea.

On the other hand we can add \( n \) chemical potentials as masses for the \( U \) fermionic oscillators, and identify them with \( n \) boundary cosmological constants on the \( n \) FZZT branes. It is natural to associate the dynamics of the \( n^2 \) types of impurities with the \( n^2 \) types of long strings attached to \( n \) FZZT branes. The situation is considerably more complicated than in
the case of one FZZT brane, and there are considerable subtleties to be understood in the scaling limit. We leave that to future work.

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A Decomposition in non-singlet sectors

The Hilbert space of square integrable functions on a compact group $L^2(G)$ has a natural complete orthogonal base given by the matrix functions that represent $G$ on its various representations.

$$ f(g) = \sum_{\mathcal{R},I,J} f^{\mathcal{R}I}_{I} D^{I}_{IJ}(g) $$

(1)

It is an important theorem of group theory that these matrix element functions are orthogonal for the Haar invariant measure on $G$. A traditional example would be the base $D^{j}_{m,m'}(\theta, \phi, \psi)$ on $SU(2) = S^3$.

The natural action of $G_L \ast G_R$ on $L^2(G)$ maps to the action on the indices $I,J$

$$ f(g_L g g_R^{-1}) \rightarrow D^{I}_{RR}(g_L) f^{R}_{R} D^{T}_{RJ}(g_R^{-1}) $$

(2)

To get a base for functions on a coset $G/H$ one just needs to restrict the index $J$ to the $H$ invariant subspace. Again the basic example is $SU(2)/U(1) = S^2$, with the spherical harmonics $Y_{lm}(\theta, \phi) = D^{l}_{m,m'=0}(\theta, \phi, \psi)$.

$SU(N)$ invariance of the action implies that the Hamiltonian commutes with the left $SU(N)$ action on $\Omega$, hence indices $\mathcal{R}$ and $I$ label independent dynamical sectors. The Hilbert space is then decomposed to

$$ \mathcal{H} = \bigoplus_{\mathcal{R}} \mathcal{H}_{\mathcal{R}} \quad \mathcal{H}_{\mathcal{R}} = \frac{L^2(R_N) \otimes \widetilde{\mathcal{R}}}{S_N} $$

(3)

where $\widetilde{\mathcal{R}}$ is the subspace of zero weight vectors in $\mathcal{R}$, and $S_N$ acts by permutation of the eigenvalues combined with an action by the corresponding Weyl group element.

The wavefunction in a given sector is simply

$$ \Psi(M) = \sum_{J \in \mathcal{R}} \frac{\psi_{J}(\lambda_n)}{\prod_{i>j} (\lambda_i - \lambda_j)} D^{I}_{RJ}(\Omega) $$

(4)
The Vandermonde factor comes as usual from the Jacobian of the diagonalization.

**B Matrix fourier transform in non-singlet sectors**

In the paper we will need to understand the effect of changes of variables from $M$ to $A^\pm = P \pm M$, where $P$ is the momentum conjugate to $M$. These are all roughly of the form of a Matrix Fourier transform.

\[
\Psi(M) = \int dA^\pm e^{Tr[i(A^\pm)^2 + iMA^\pm + iM^2]}\Psi(\pm) \tag{1}
\]

\[
\Psi_-(A^-) = \int dA^+ e^{Tr[iA^+A^-]}\Psi_+(A^+) \tag{2}
\]

We want to show that these transformations map a specific angular sector of the Hilbert space for one set of wavefunctions, say $\Psi_+(A^+)$, to the corresponding sector for the other set, say $\Psi_-(A^-)$.

Starting from the result of the previous appendix

\[
\Psi_+(A^+) = \sum_{J \in \tilde{R}} \psi_J(a_+^+) \prod_{i>j} (a_i^+ - a_j^+) D_{\tilde{R},J}(\Omega^+) \tag{3}
\]

Inserting this into the integral, and diagonalize the integration variables to get

\[
\Psi_-(A^-) = \int \prod_k da_+^k d\Omega^+ e^{Tr[i\Omega^+ diag\{a_+^+\} \Omega^+ A^-]} \sum_{J \in \tilde{R}} \psi_J(a_+^+) \prod_{i>j} (a_i^+ - a_j^+) D_{\tilde{R},J}(\Omega^+) \tag{4}
\]

Diagonalizing $A^-$ and changing variables in the integral $\Omega^+ = \Omega^- \Omega$ gives

\[
\Psi_-(\Omega^- diag\{a^-_m\} \Omega^-) = \int \prod_k da_+^k d\Omega e^{Tr[i\Omega diag\{a_+^+\} \Omega diag\{a^-_m\}]} \sum_{J, K \in \tilde{R}} \psi_J(a_+^+) \prod_{i>j} (a_i^+ - a_j^+) D_{\tilde{R},K}(\Omega^-) D_{\tilde{R},J}(\Omega) \tag{5}
\]

Hence $H_{\tilde{R},J}^K$ is mapped to $H_{\tilde{R},I}^{-}$ by the transformation

\[
\psi_J^-(a^-_m) = \int \prod_k da_+^k \sum_{K \in \tilde{R}} G_{J}^{K}(a^-_m, a_+^+) \psi_K(a_+^+) \tag{6}
\]

where

\[
G_{J}^{K}(a^-_m, a_+^+) = \int d\Omega e^{Tr[i\Omega diag\{a_+^+\} \Omega diag\{a^-_m\}]} \prod_{i>j} (a_i^+ - a_j^+) (a_i^- - a_j^-) D_{\tilde{R},J}^{K}(\Omega) \tag{7}
\]

This integral Kernel can be computed with character expansion in a manner similar to the Izkinson-Zuber integral, but that goes beyond the scope of this note.
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