Complexes, residues and obstructions for log-symplectic manifolds

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Abstract

We consider compact Kählerian manifolds $X$ of even dimension 4 or more, endowed with a log-symplectic structure $\Phi$, a generically nondegenerate closed 2-form with simple poles on a divisor $D$ with local normal crossings. A simple linear inequality involving the iterated Poincaré residues of $\Phi$ at components of the double locus of $D$ ensures that the pair $(X, \Phi)$ has unobstructed deformations and that $D$ deforms locally trivially.

Keywords  Poisson structure · Log-symplectic manifold · Deformation theory · Log complex · Mixed Hodge theory

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Introduction

A log-symplectic manifold is a pair consisting of a complex manifold $X$, usually compact and Kählerian, together with a log-symplectic structure. A log-symplectic structure can be defined either as a generically nondegenerate meromorphic closed 2-form with normal-crossing (anticanonical) polar divisor $D$, or equivalently as a generically nondegenerate holomorphic tangential 2-vector $\Pi$ such that $[\Pi, \Pi] = 0$ with normal-crossing degeneracy divisor $D$. The two structures are related via $\Pi = \Phi^{-1}$. See [3] or [11] or [2] or [12] for basic facts on Poisson and log-symplectic manifolds and [4] (especially the appendix), [5, 7, 8] or [10], and references therein, for deformations.

Understanding log-symplectic manifolds unavoidably involves understanding their deformations. In the very special case of symplectic manifolds, where $D = 0$, the classical theorem of Bogomolov [1] shows that the pair $(X, \Phi)$ has unobstructed deformations. In [13] we obtained a generalization of this result which holds when $\Phi$ satisfied a certain 'very general position' condition with respect to $D$ (the original statement is corrected in the subsequent erratum/corrigendum). Namely, we showed in this case that $(X, \Phi)$ has 'strongly unob-
structured’ deformations, in the sense that it has unobstructed deformations and \( D \) deforms locally trivially.

Further results on unobstructed deformations (in the sense of Hitchin’s generalized geometry \([6]\)) and Torelli theorems in the case where \( D \) has global normal crossings were obtained by Matviichuk, Pym and Schedler \([9]\), based on their notion of holonomicity.

Our purpose here is to prove a more precise strong unobstructedness result compared to \([13]\), nailing down the generality required: we will show in Theorem 6 that strong unobstructedness can fail only when the log-symplectic structure \( \Phi_1 \), more precisely its (iterated Poincaré) residues at codimension-2 strata of the polar divisor \( D \) (which are essentially the (locally constant) coefficients of \( \Phi \) with respect to a suitable basis of the log forms adapted to \( D \)) satisfy certain special linear relations with integer coefficients. Explicitly, at a triple point of \( D \) with branches labeled 1,2,3 and associated residues \( c_{12}, c_{23}, c_{31} \), the condition is

\[
c_{23} + c_{31} \in \mathbb{N}c_{12}.
\]

Essentially, if this never happens over the entire triple locus then \((X, \Phi)\) has strongly unobstructed deformations.

The strategy of the proof as in \([13]\) is to study the inclusion of complexes

\[
(T_X(T\langle - \log D \rangle, [\ldots, \Pi]) \to (T^*_X, [\ldots, \Pi]),
\]

albeit from a more global viewpoint. In fact as in \([13]\) it turns out to be more convenient to transport the situation over to the De Rham side where it becomes an inclusion

\[
(\Omega^*_X(\log D), d) \to (\Omega^*_X(\log^+ D), d)
\]

where the latter ‘log-plus’ complex is a certain complex of meromorphic forms with poles on \( D \). We study a filtration, introduced in \([13]\), interpolating between the two complexes, especially its first two graded pieces. As we show, the first piece is automatically exact, while 0-acyclicity for the second piece leads to the above cocycle condition. See Sect. 3 for details.

We begin the paper with a couple of auxiliary, independent sections. In Sect. 1 we construct a ‘principal parts complex’ associated to an invertible sheaf \( L \) on a smooth variety, extending the principal parts sheaf \( P(L) \) together with the universal derivation \( L \to P(L) \). We show this complex is always exact. In Sect. 2 we show that, for any normal-crossing divisor \( D \subset X \) on any smooth variety, the log complex \( \Omega^*_X(\log D) \)—unlike \( \Omega^*_X \) itself—can be pulled back to a complex of vector bundles on the normalization of \( D \). These complexes play a role in our analysis of the aforementioned inclusion map.

I am grateful to Brent Pym for helpful communications, in particular for communicating Example 8.

1 Principal parts complex

In this section \( X \) denotes an arbitrary \( n \)-dimensional smooth complex variety and \( L \) denotes an invertible sheaf on \( X \).

1.1 Principal parts

The Grothendieck principal parts sheaf \( P(L) \) (see EGA) is a rank-\((n+1)\) bundle on \( X \) defined as

\[
P(L) = p_1\ast(p^2_2L \otimes (\mathcal{O}_{X \times X}/T^2_\Lambda))
\]
where $\Delta \subset X \times X$ is the diagonal and $p_1, p_2 : X \times X \to X$ are the projections. We have a short exact sequence
\[ 0 \to \Omega_X^1 \otimes L \to P(L) \to L \to 0 \]
whose corresponding extension class in $\text{Ext}^1(L, \Omega_X^1 \otimes L) = H^1(X, \Omega_X^1)$ coincides with $c_1(L)$. The sheaf
\[ P_0(L) = P(L) \otimes L^{-1}, \]
which likewise has extension class $c_1(L)$, is called the normalized principal parts sheaf. The map $P(L) \to L$ admits a splitting $d_L : L \to P(L)$ that is a derivation, i.e.,
\[ d_L(fu) = fd_Lu + df \otimes u. \]
In fact, $d_L$ the universal derivation on $L$. Moreover, $P(L)$ is generated over $O_X$ by the image of $d_L$. Likewise, $P_0(L)$ is generated by elements of the form $d\log(u) := d_Lu \otimes u^{-1}$ where $u$ is a local generator of $L$.

### 1.2 Complex

It is well known that $P(L^{m+1}) \simeq P(L) \otimes L^m$, $m \geq 0$ which in particular yields a derivation $L^{n+1} \to P(L) \otimes L^n$, $n \geq 0$. In fact, this map extends to a complex that we denote by $P_{n+1}(L)$ or just $P^*(L)$ and call the $(n+1)$st principal parts complex of $L$:
\[ P^*(L) : L^{n+1} \to P(L)L^n \to \wedge^2 P(L)L^{n-1} \to \ldots \wedge^{n+1} P(L) = \Omega_X^n \otimes L^{n+1}. \]  
(1)
The differential is given, in terms of local $O_X$-generators $u_1, \ldots, u_k, v_1, \ldots, v_\ell$ of $L$, by
\[ d(u_1 \ldots u_k d_L(v_1)) \wedge \ldots d_L(v_\ell) = \sum u_1 \ldots \hat{u}_i \ldots u_k d_L(u_i) \wedge d_L(v_1) \wedge \ldots \wedge d_L(v_\ell) \]
and extending using additivity and the derivation property. There are also similar shorter complexes
\[ L^m \to P(L)L^{m-1} \to \ldots \to \wedge^m P(L). \]
Note the exact sequences
\[ 0 \to \Omega_X^m L^m \to \wedge^m P(L) \to \Omega_X^m L^m \to 0. \]
These sequences splits locally and also split globally whenever $L$ is a flat line bundle. In such cases, we get a short exact sequence
\[ 0 \to \Omega_X^* L^{n+1}[-1] \to P^*(L) \to \Omega_X^* L^{n+1} \to 0 \]  
(2)
The principal parts complex $P^*(L)$ may be tensored with $L^{j-n-1}$, for any $j > 0$, yielding the $j$-th principal parts complex:
\[ P^*_j(L) : L^j \to P_0(L)L^j \to \wedge^2 P_0(L)L^j \to \ldots \wedge^{n+1} P_0(L)L^j \]  
(3)
The differential is defined by setting
\[ d(d\log(u_1) \wedge \ldots \wedge d\log(u_i)v^j) = j d\log(u_1) \wedge \ldots \wedge d\log(u_i) d\log(v)^j \]
where $u_1, \ldots, u_i, v$ are local generators for $L$, and extending by additivity and the derivation property. Thus, $P^*(L) = P^*_{n+1}(L)$.

An important property of principal parts complexes is the following:
Proposition 1 For any local system \( S \), the complexes \( P^\bullet_j(L) \otimes S \) are null-homotopic and exact for all \( j > 0 \).

Proof The assertion being local, we may assume \( L \) is trivial and \( S = \mathbb{C} \), so the \( i \)-th term of \( P^\bullet_j(L) \otimes S \) is just \( \Omega^{i-1}_X \oplus \Omega^i_X \) and the differential is \( \begin{pmatrix} d & \text{id} \\ 0 & d \end{pmatrix} \). Then, a homotopy is given by \( \begin{pmatrix} 0 & \text{id} \\ 0 & 0 \end{pmatrix} \). Thus, \( P^\bullet_j(L) \) is null-homotopic, hence exact. \( \square \)

1.3 Log version

The above constructions have an obvious extension to the log situation. Thus, let \( D \) be a divisor with normal crossings on \( X \). We define \( P(L)\langle \log D \rangle \) as the image of \( P(L) \) under the inclusion \( \Omega_X^1 \to \Omega_X^1 \langle \log D \rangle \), and likewise for \( P_0(L)\langle \log D \rangle \). Then, as above we get complexes

\[
P^\bullet_j(L)\langle \log D \rangle: L^j \to P_0(L)\langle \log D \rangle L^j \to \ldots \to \wedge^{n+1} P_0(L)\langle \log D \rangle L^j.
\]

(4)

1.4 Foliated version

Let \( F \subset \Omega_X^1(\log D) \) be an integrable subbundle of rank \( m \). Then, \( F \) gives rise to a foliated De Rham complex \( \wedge^\bullet(\Omega_X^1(\log D)/F) \), as well as a foliated principal parts sheaf \( P^\bullet_F(L)\langle \log D \rangle = P^1(L)\langle \log D \rangle/F \otimes L \). Putting these together, we obtain the foliated principal parts complexes (where \( P_{0,F}(L)\langle \log D \rangle := P_0(L)\langle \log D \rangle/F \)):

\[
P^\bullet_{j,F}(L)\langle \log D \rangle: L^j \to P_{0,F}(L)\langle \log D \rangle L^j \to \ldots \to \wedge^{n-m+1} P_{0,F}(L)\langle \log D \rangle
\]

(5)

Note that the proof of Proposition 1 made no use of the acyclicity of the De Rham complex. Hence, the same proof applies verbatim to yield

Proposition 2 For any local system \( S \), the complexes \( P^\bullet_{j,F}(L)\langle \log D \rangle \otimes S \) are null-homotopic and exact for all \( j > 0 \).

2 Calculus on normal crossing divisors

In this section \( X \) denotes a smooth variety or complex manifold and \( D \) denotes a locally normal-crossing divisor on \( X \). Our aim is to show that the log complex on \( X \), unlike its De Rham analogue, can be pulled back to the normalization of \( D \).

2.1 Branch normal

Let \( f_i: X_i \to X \) be the normalization of the \( i \)-fold locus of \( D \). A point on \( X_i \) consists of a point on \( D \) together with a choice of \( i \) distinct local branches of \( D \) at it. There is a canonical induced normal-crossing divisor \( D_i \) on \( X_i \); at a point where \( x_1, \ldots, x_m \) is an equation for \( D \) and \( x_1, \ldots, x_i \) are the chosen branches, the equation of \( D_i \) is \( x_{i+1} \ldots x_m \). Note the exact sequence

\[
0 \to T_X(-\log D) \to T_X \to f_{1*}N_{f_1} \to 0
\]

(6)
where $N_{f_1}$ is the normal bundle to $f_1$ which fits in an exact sequence

$$0 \to TX_1 \to f_1^*T_X \to N_{f_1} \to 0.$$ 

Locally, $N_{f_1}$ coincides with $x_1^{-1}\mathcal{O}_X/\mathcal{O}_X$ where $x_1$ is a 'branch equation': to be precise, if $K$ denotes the kernel of the natural surjection $f_1^{-1}\mathcal{O}_X \to \mathcal{O}_{X_1}$, then $J = K/K^2 = K \otimes f^{-1}\mathcal{O}_X \mathcal{O}_X$ is an invertible $\mathcal{O}_{X_1}$-module locally generated by $x_1$ and $N_{f_1} = J^{-1}$. Note that

$$N_{f_1} \otimes \mathcal{O}_{X_1}(D_1) = f_1^*(\mathcal{O}_X(D)).$$

### 2.2 Pulling back log complexes

Interestingly, even though the differential on the pullback De Rham complex $f_1^{-1}\Omega^\bullet_X$ does not extend to $f_1^{-1}\Omega^\bullet_X \otimes \mathcal{O}_{X_1}$, the analogous assertion for the log complex does hold: the differential on $f_1^{-1}\Omega^\bullet_X(log D)$ extends to what might be called the restricted log complex:

$$f_1^*\Omega^\bullet_X(log D) = f_1^{-1}\Omega^\bullet_X(log D) \otimes \mathcal{O}_{X_1}.$$ 

This is due to the identity (where $x_1$ denotes a branch equation)

$$dx_1 = x_1 d\log(x_1).$$

Note that the residue map yields an exact sequence

$$0 \to \Omega^1_{X_1}(log D_1) \xrightarrow{\iota} f_1^*\Omega^1_X(log D) \xrightarrow{\text{Res}} \mathcal{O}_{X_1} \to 0. \tag{7}$$

Note that the residue map commutes with exterior derivative. Therefore, this sequence induces a short exact sequence of complexes

$$0 \to \Omega^\bullet_{X_1}(log D_1) \to f_1^*\Omega^\bullet_X(log D) \to \Omega^\bullet_{X_1}(log D_1)[-1] \to 0. \tag{8}$$

Furthermore, a twisted form of the restricted log complex, called the normal log complex, also exists:

$$N_{f_1} \otimes f_1^*\Omega^\bullet_X(log D) : N_{f_1} \to N_{f_1} \otimes f_1^*\Omega^1_X(log D) \to \ldots \tag{9}$$

this is thanks to the identity, where $\omega$ is any log form,

$$d(\omega/x_1) = (d\omega)/x_1 - d\log(x_1) \wedge \omega/x_1.$$ 

Now recall the exact sequence coming from the residue map

$$0 \to \Omega^1_{X_1}(log D_1) \to f_1^*\Omega^1_X(log D) \to \mathcal{O}_{X_1} \to 0$$

In fact, it is easy to check that this exact sequence has extension class $c_1(N_{f_1})$ hence identifies $f_1^*\Omega^1_X(log D)$ with $P_0(N_{f_1})$ so that the normal log complex (9) may be identified with the principal parts complex $P^\bullet(N_{f_1})$:

**Lemma 3** The normal log complex $N_{f_1} \otimes f_1^*\Omega^\bullet_X(log D)$ is isomorphic to $P^\bullet(N_{f_1})$, hence is exact.
Similarly, a pull back log complex \( f_k^*\Omega_X^\bullet(\log D) = f_k^{-1}\Omega_X^\bullet(\log D) \otimes \mathcal{O}_{X_k} \) exists for all \( k \geq 1 \). A similar twisted log complex also exists the determinant of the normal bundle \( N_{f_k} \):

\[
\det N_{f_k} \otimes f_k^*\Omega_X^\bullet(\log D) : \det N_{f_k} \to \det N_{f_k} \otimes \Omega_X^1(\log D) \to \ldots
\]

(10)

This comes from (where \( x_1, \ldots, x_k \) are the branch equations at a given point of \( X_k \)):

\[
d(\omega/x_1 \ldots x_k) = d\omega/x_1 \ldots x_k - d\log(x_1 \ldots x_k)\omega/x_1 \ldots x_k).
\]

2.3 Iterated residue

We have a short exact sequence of vector bundles on \( X_k \):

\[
0 \to \Omega_{X_k}(\log D_k) \to f_k^*\Omega_X(\log D) \to v_k \otimes \mathcal{O}_{X_k} \to 0
\]

(11)

where \( v_k \) is the local system of branches of \( D \) along \( X_k \) and the right map is multiple residue. Taking exterior powers, we get various exact Eagon–Northcott complexes. In particular, we get surjections, called iterated Poincaré residue:

\[
f_k^*\Omega_X^i(\log D) \to \Omega_{X_k}^{i-k}(\log D_k) \otimes \text{det}_C(v_k), i \geq k, \quad (12)
\]

\[
f_k^*\Omega_X^i(\log D) \to \wedge_C^i v_k \otimes \mathcal{O}_{X_k}, i \leq k. \quad (13)
\]

\( \text{det}_C(v_k) \) is a rank-1 local system on \( X_k \) which may be called the 'normal orientation sheaf.' The maps for \( i \geq k \) together yield a surjection

\[
f_k^*\Omega_X^\bullet(\log D) \to \Omega_{X_k}^\bullet(\log D_k)[-k] \otimes \text{det}(v_k). \quad (14)
\]

3 Comparing log and log plus complexes

In this section \( X \) denotes a log-symplectic smooth variety with log-symplectic form \( \Phi_1 \) and corresponding Poisson vector \( \Pi = \Phi^{-1} \), and \( D \) denotes the degeneracy divisor of \( \Pi \) or polar divisor of \( \Phi_1 \). Our aim is to prove Theorem 6 which shows that deformations of \((X, \Phi_1)\) coincide with locally trivial deformations of \((X, \Phi, D)\) and are unobstructed.

3.1 Setting up

We will use \( \Omega_X^+ = \bigoplus_{i>0} \Omega_X^i \) and similarly for the log versions. This is to match with the Lichnerowicz–Poisson complex \( T_X^\bullet \) and \( T_X^\bullet(\log D) \). Thus, interior multiplication by \( \Phi \) induces an isomorphism \( T_X^\bullet(\log D) \to \Omega_X^\bullet(\log D) \). Equivalently, \( \Phi \) itself is a form in \( \Omega_X^2(\log D) \) inducing a nondegenerate pairing on \( T_X(-\log D) \). In terms of local coordinates, at a point of multiplicity \( m \) on \( D \), we have a basis for \( \Omega_X(\log D) \) of the form

\[
\eta_1 = d\log(x_1), \ldots, \eta_m = d\log(x_m) \quad \text{and then} \quad \eta_{m+1} = d\log(x_{m+1}), \ldots
\]

and then

\[
\Phi = \sum b_{ij} \eta_i \land \eta_j.
\]

We have an inclusion of complexes

\[
T_X^\bullet(\log D) \to T_X^\bullet
\]
where, for $X$ compact Kähler, the first complex controls 'locally trivial' deformations of $(X, \Pi)$, i.e., deformations of $(X, \Pi)$ inducing a locally trivial deformation of $D = [\Pi^n]$, and the second complex controls all deformations of $(X, \Pi)$. It is known (see, e.g., [13]) that locally trivial deformations of $(X, \Pi)$ are always unobstructed and have an essentially Hodge-theoretic (hence topological) character, so one is interested in conditions to ensure that the above inclusion induces an isomorphism on deformation spaces; as is well known, the latter would follow if one can show that the cokernel of this inclusion has vanishing $H^1$.

Our approach to this question starts with the above 'multiplication by $\Phi_1$' isomorphism

$$(T_X^\bullet(-\log D), [\cdot , \cdot, \Pi]) \to (\Omega_X^\bullet(\log D), d).$$

This isomorphism extends to an isomorphism to $T_X^\bullet$ with a certain subcomplex of $\Omega_X^\bullet(\log D)$, the meromorphic forms regular off $D$, that we call the log plus complex and denote by $\Omega_X^\bullet(\log^+ D)$.

Our goal then becomes that of comparing the log and log-plus complexes. To this end we introduce a filtration on $\Omega_X^\bullet(\log^+ D)$, essentially the filtration induced by the exact sequence

$$0 \to T_X(-\log D) \to T_X \to f_{1*}N_{f_1} \to 0$$

and its isomorphic copy

$$0 \to \Omega_X(\log D) \to \Omega_X(\log^+ D) \to f_*N_{f_1} \to 0$$

where $f_1 : X_1 \to D \subset X$ is the normalization of $D$ and $N_{f_1}$ is the associated normal bundle ('branch normal bundle'). We will show that the first graded piece is always an exact complex. The second graded piece is much more subtle. We will show that it is locally exact in degree 0 unless the log-symplectic form $\Phi_1$, i.e., the matrix $(b_{ij})$ above satisfies some special relations with integer coefficients.

The computations of this section are all local in character, though the applications are global.

### 3.2 Residues and duality

Let $f_i : X_i \to X$ be the normalization of the $i$-fold locus of $D$, $D_i$ the induced normal-crossing divisor on $X_i$. Thus, a point of $X_i$ consists of a point $p$ of $D$ together with a choice of an unordered set $S$ of $i$ branches of $D$ through $p$ and $D_i$ is the union of the branches of $D$ not in $S$. We consider first the codimension-1 situation. As above, we have a residue exact sequence

$$0 \to \Omega^1_{X_1}(\log D_1) \to f_{1*}^*\Omega^1_X(\log D) \stackrel{\text{Res}}{\to} \mathcal{O}_{X_1} \to 0 \quad (15)$$

(the right-hand map given by residue is locally evaluation on $x_1 @ x_1$ where $x_1$ is a local equation for the branch of $D$ through the given point of $X_1$). Note that if $\eta$ comes from a closed form on $X$ near $D$, then $\text{Res}(\eta)$ is a constant.

Dualizing (15), we get

$$0 \to \mathcal{O}_{X_1} \to f_{1*}^*\mathcal{O}_{X}(\log D_1) \to (\text{co-residue}) \to 0, \quad (16)$$

where the left-hand map, the 'co-residue,' is locally multiplication by $x_1 @ x_1$ where $x_1$ is a branch equation). Set

$$v_1 = x_1 @ x_1.$$


Then, $v_1$ is canonical as section of $f_1^*T_X(-\log D)$, independent of the choice of local equation $x_1$. By contrast, $\omega_{x_1}$ as section of $f_1^*T_X$ is canonical only up to a tangential field to $X_1$, and generates $f_1^*T_X$ modulo $T_X(-\log D)$.

Now $f_1^*\Omega_X^1(\log D)$ and $f_1^*T_X(-\log D)$ admit mutually inverse isomorphisms
\[ i_{X_1}: \omega_{x_1} := (\omega_{x_1}(\cdot))_{X_1} = f_1^*(\omega_{\cdot})_{X_1}, i_{X_1}: \omega_{x_1} := (\omega_{\cdot})_{X_1} = f_1^*(\omega_{\cdot}). \]
The composite
\[ j \circ i_{X_1}: \Omega_{X_1} \to T_X(-\log D) \]
has a rank-1 kernel that is the kernel of the Poisson vector on $X_1$ induced by $\omega_{x_1}$, aka the conormal to the symplectic foliation on $X_1$. Now set
\[ \psi_1 = i_{X_1}(\omega_{x_1}(v_1)) = (\omega_{x_1}v_1). \]

Then, $\psi_1$ is locally the form in $\Omega_{X_1}(\log D_1)$ denoted by $x_1\phi_1$ in [13]. Again $\psi_1$ is canonically defined, independent of choices and corresponds to the first column of the $B = (b_{ij})$ matrix for a local coordinate system $x_1, x_2, \ldots$ compatible with the normal-crossing divisor $D$. By contrast, $\phi_1$, which depends on the choice of local equation $x_1$, is canonical up to a log form in $\Omega_{X_1}(\log D_1)$ and generates $\Omega_{X_1}(\log^+ D_1)$ modulo the latter.

In $X_1 \setminus D_1$, $\omega_{x_1}$ is locally of the form $d\log(x_1) \wedge dx_2 + \text{(symplectic)}$, so there $\psi_1 = dx_2$. Note that by skew-symmetry we have
\[ \text{Res} \circ i_{X_1}(\omega_{x_1}) \circ \tilde{R}_1 = 0. \]

Thus, locally $\psi_1 \in \Omega_{X_1}(\log D_1)$. In terms of the matrix $B$ above, $\psi_1 = \sum b_{ij} \log(x_j)$.

Note that $\psi_1$ which corresponds to the Hamiltonian vector field $v_1$, is a closed form. Consequently, $\psi_1$ defines a foliation on $X_1$. Let $Q^\bullet = \psi_1\Omega^\bullet_{X_1}$ be the associated foliated De Rham complex $\psi_1\Omega^\bullet_{X_1}$:
\[ Q^0 = \mathcal{O}_{X_1}\phi_1 \to Q^1 = \psi_1\Omega^1_{X_1} \simeq \Omega^1_{X_1}/\mathcal{O}_{X_1} \psi_1 \to \ldots \to Q^i = \wedge^i Q^1 \to \ldots \]

endowed with the foliated differential.

Note that the residue exact sequence (15) induces the Poincaré residue sequence
\[ 0 \to \Omega^\bullet_{X_1}(\log D_1) \to f_1^*\Omega^\bullet_X(\log D) \to \Omega^\bullet_{X_1}(\log D_1)[-1] \to 0. \]

Again the Poincaré residue of a closed form is closed. Now the exact sequence
\[ 0 \to T_X(-\log D) \to T_X \to f_{1*}N_{f_1} \to 0 \]
yields
\[ 0 \to \Omega_X(\log D) \to \Omega_X(\log^+ D) \to f_{1*}N_{f_1} \to 0. \quad (17) \]
and this sequence induces the $\mathcal{F}_\bullet$ filtration on the log-plus complex $\Omega^\bullet_X(\log^+ D)$.

### 3.3 First graded piece

Now consider first the first graded $G^\bullet = (\mathcal{F}_\bullet/\mathcal{F}_0\bullet)[1]$ which is supported in codimension 1. (the shift is so that $G^\bullet$ starts in degree 0). Then, $G^\bullet$ is a (finite) direct image of a complex of $X_1$ modules:
\[ E_1 : N_{f_1} \to N_{f_1} \otimes Q_1 \to N_{f_1} \otimes Q_1^2 \to \ldots \]
Using Lemma 3, we can easily show:

**Proposition 4** \( \mathcal{E}_1 \) is isomorphic to \( P_{R_1^*}(N_{f_1}) \), hence is null-homotopic and exact, hence \( G_1^* \) is exact.

### 3.4 Second graded piece

Next we study \( G_2 \), which is supported on \( X_2 \). We consider a connected, nonempty open subset \( W \subset X_2 \), for example an entire component, over which the ‘normal orientation sheaf’ \( \nu_2 : X_{2,1} \to X_2 \), i.e., the local \( \mathbb{Z}_2 \)-system of branches of \( X_1 \) along \( X_2 \), is trivial (we can take \( W = X_2 \) if, e.g., \( D \) has global normal crossings). Such a subset \( W \) of \( X_2 \) is said to be a **normally split** subset of \( X_2 \), and a **normal splitting** of \( W \) is an ordering of the branches is specified. Obviously \( X_2 \) is covered by such subsets \( W \). Likewise, for a subset \( Z \subset X_k \).

#### 3.4.1 Iterated residue

Over a normally split subset \( W \), we have a diagram

\[
\begin{array}{ccccccccc}
0 & \to & 2\mathcal{O}_W & \xrightarrow{\hat{R}_2} & f_2^*T_X(-\log D)|_W & \to & T_{X_2}(-\log D_2)|_W & \to & 0 \\
& & \downarrow & & \downarrow & & & & \\
0 & \to & \Omega_W(\log D_2) & \xrightarrow{\hat{R}_2} & f_2^*\Omega_X(\log D)|_W & \to & 2\mathcal{O}_W & \to & 0 \\
\end{array}
\]

where \( \hat{R}_2 \) is the map induced by \( \hat{R}_1 \). The composite map \( R_2 \hat{R}_2 : 2\mathcal{O}_W \to 2\mathcal{O}_W \) is just the alternating form induced by \( \Phi \) and has the form \( c_W H_2 \) where \( H_2 \) is the hyperbolic plane \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). In terms of a local frame for \( \Omega_X(\log D) \) containing \( d\log(x_1) \), \( d\log(x_2) \), \( c_W \) is the coefficient of \( d\log(x_1) \wedge d\log(x_2) \) in \( \Phi \). Note \( c_W \) must be constant because \( \Phi \) is closed. In fact, we have

\[
c_W = \text{Res}_1 \text{Res}_2(\Phi)
\]

where \( \text{Res}_i \) denotes the (Poincaré) residues along the branches of \( X_1 \) over \( X_2 \). Set

\[
\text{Res}_W(\Phi) := c_W.
\]

This is essentially what is called the biresidue by Matviichuk et al., see [9]. Thus, when \( c_W \neq 0 \), we have a basis for the log forms

\[
\eta_1 = d\log(x_1), \ldots, \eta_m = d\log(x_m), \eta_{m+1} = dx_{m+1}, \ldots, \eta_{2n} = dx_{2n}
\]

\( m \) = multiplicity of \( D \), \( m \geq 2 \), and then

\[
\Phi = \sum b_{ij} \eta_i \wedge \eta_j
\]

where

\[
b_{12} = -b_{21} = c_W.
\]

If \( W \) may be not be normally orientable (e.g., an entire component of \( X_2 \)), then \( c_W \) is defined only up to sign; if \( c_W = 0 \), we say that \( W \) is nonresidual; otherwise, it is residual.
3.4.2 Nonresidual case

Here we consider the case \( c_W = 0 \).

Note that in that case we may express \( \Phi \) along \( W \) in the form

\[
\Phi = \text{dlog}(x_1)\gamma_3 + \text{dlog}(x_2)\gamma_4 + \gamma_5
\]

where the gammas are closed log forms in the coordinates on \( W \), i.e., \( x_1, \ldots, x_{2n} \). Moreover, \( \gamma_3 \land \gamma_4 \neq 0 \) because \( \Phi' \) is divisible by \( \text{dlog}(x_1)\text{dlog}(x_2) \). Also, unless \( \gamma_3, \gamma_4 \) are both holomorphic (pole-free), there is another component \( W' \) of \( X' = X \) such that \( c_{W'} \neq 0 \) (in particular, \( W \cap D_2 \neq \emptyset \)). Hence, if no such \( W' \) exists, we may by suitably modifying coordinates, assume locally that \( \gamma_3 = dx_3, \gamma_4 = dx_4 \). A similar argument, or induction, applies to \( \gamma_5 \). This means we are essentially in the P-normal case considered in [14]. This we conclude:

**Lemma 5** Unless \( \Pi \) is P-normal, there exists a nonempty residual open subset \( W \) of \( X_2 \).

3.4.3 Residual case: identifying \( G_2 \)

Next we analyze a residual normally oriented open subset \( W \subset X_2 \). As above, we get a composite map of \( R_2 : 2\mathcal{O}_W \rightarrow f_2^*\Omega_X (\log D)|_W \), whose image we denote by \( M_{2W} \). It has a local basis \( (\psi_{11} = x_1\phi_1, \psi_{12} = x_2\phi_2) \) corresponding to the basis \((e_1, e_2)\) of \( 2\mathcal{O}_W \). In terms of the \( B \)-matrix, we have

\[
\psi_{11} = \sum b_{1j} \eta_j = -\sum b_{j1} \eta_j, \quad \psi_{12} = -\sum b_{2j} \eta_j = \sum b_{j2} \eta_j.
\]

As \( \psi_{11}, \psi_{12} \) are closed, \( M_2 \) is integrable. Let \( \tilde{\Omega} \) denote the quotient \( f_2^*\Omega_X (\log D)|_W / M_{2W} \).

Then, we have an isomorphism

\[
\tilde{\Omega} \rightarrow \Omega_W \langle \log D_2 \rangle
\]

(19)

given explicitly by

\[
\tilde{\omega} \mapsto \omega - \text{Res}_1(\omega)\psi_{12}/c_W - \text{Res}_2(\omega)\psi_{11}/c_W
\]

(because \( \text{Res}_2(\psi_{11}) = \text{Res}_1(\psi_{12}) = c_W \), residues with respect to the two branches of \( D \)). Now set \( N_2 = \det N_{f_2} \), an invertible sheaf on \( X_2 \). Then, \( G_2^* = (\mathcal{E}_2^*/\mathcal{E}_1^*)[2] \) is the direct image of a complex on \( X_2 \):

\[
\mathcal{E}_2^*: N_2 \rightarrow N_2 \otimes \tilde{\Omega} \rightarrow N_2 \otimes \wedge^2 \tilde{\Omega} \rightarrow \ldots
\]

(20)

where a local generator of \( N_2 \) has the form \( 1/x_1x_2 \) and the differential has the form

\[
\tilde{\omega}/x_1x_2 \mapsto d\tilde{\omega}/x_1x_2 \pm (\tilde{\omega}/x_1x_2) \text{dlog}(x_1x_2).
\]

3.4.4 Zeroth differential

Using the identification (19), the zeroth differential has the form

\[
\tilde{d}(g/x_1x_2) = \frac{1}{x_1x_2}(dg + g(\text{dlog}(x_1x_2) - (\psi_{11} + \psi_{12})/c_W)), \quad g \in \mathcal{O}_{X_2}.
\]

(21)

The form \( \psi_2 = -\text{dlog}(x_1x_2) + (\psi_{11} + \psi_{12})/c_W \) has zero residues with respect to \( x_1, x_2 \), hence yields a form in \( \Omega_{X_2} (\log D_2) \). Changing the local equations \( x_1, x_2 \) changes \( \psi \) by adding a holomorphic (pole-free) form on \( X_2 \).
For \( g \) nonzero (21) can be rewritten
\[
\tilde{d}(g/x_1x_2) = \frac{g}{x_2x_2} \left( \log(g) - \psi_2 \right)
\] (22)

When does this operator have a nontrivial kernel? First, if \( g \) is constant, then \( \psi_2 = 0 \) on \( W \) which is impossible if \( W \) meets \( D_2 \). Next, locally at a point \( x \in W \setminus D_2 \cap W \), clearly \( g/x_1x_2 \) holomorphic and nonzero in the kernel exists locally since \( \psi_2 \) is closed and holomorphic so \( \psi_2 = dh \) for a holomorphic function \( h \) and we can take \( g = e^h \). Moreover, nonzero solutions to \( d(g/x_1x_2) = 0 \) differ by a multiplicative constant. The condition that the local solutions patch is clearly that \( \frac{1}{2\pi i} \int \gamma \psi \) be an integer for any loop \( \gamma \) in \( W \setminus D_2 \cap W \). Now \( \psi_2 \) is defined only modulo a holomorphic form on \( X_2 \) while \( H_1(W \setminus D_2 \cap W) \) is generated modulo \( H_1(W) \) by small loops normal to components of \( D_2 \), so the relevant condition is just integrality over such loops \( \gamma \).

At a simple point of \( D_2 \cap W \), the condition that \( g \) exist locally as a holomorphic function with no pole on \( D_2 \) is clearly that for \( \gamma \) as above, oriented positively, the integer \( \frac{1}{2\pi i} \int \gamma \psi_2 \) is nonnegative, so that \( g \) has no pole on \( D_2 \). In other words, that the sum of the first 2 columns of the \( B \) matrix, normalized so that \( b_{12} = -b_{21} = 1 \), should be a nonnegative integer vector. Finally by Hartogs, if \( g \) is holomorphic off the singular locus of \( D_2 \cap W \), it extends holomorphically to \( W \).

### 3.4.5 Special components

Now let \( Z \) be a component of \( D_2 \cap W \) and assume \( W \) and \( Z \) are both normally split so that the branches of \( D \) along \( W \) may be labeled 12, while those along \( Z \) may be labeled 123. Thus, branches of \( X_2 \) over \( Z \) are labeled 12, 23, 31 and the preceding discussion shows that the zeroth differential has nontrivial kernel along \( Z \) only if the iterated residues of \( \Phi \) along these branches, denoted \( c_{21}, c_{23}, c_{31} \), assuming \( c_{12} \neq 0 \), satisfy

\[
c_{23} + c_{31} = kc_{21}, \quad k \in \mathbb{N}.
\] (23)

We call such a component \( Z \) special; then, \( W \) is said to be special if every (normally split) component of \( D_2 \cap W \) is special.

What about the normally split hypothesis? Suppose first \( W \) is contained in a connected open set \( W' \) which is not normally split. Then, as \( c_{12} \) is locally constant in \( W' \), it follows that \( c_{12} = 0 \), i.e., \( W \) is not residual. Now suppose \( Z \) is contained in \( Z' \) open connected and not normally split. Then, monodromy acts on the branches of \( X_2 \) along \( Z' \) cyclically and consequently the \( c_{ij} \) above are all equal. Then, (23) holds automatically with \( k = 2 \), so \( Z \) is special.

### 3.4.6 Conclusion

What we have so far proven is the following: if \( W \) is a normally oriented residual open subset of \( X_2 \), then the stalk of the zeroth cohomology \( \mathcal{H}^0(G_{2}^*) \) vanishes somewhere on \( W \) unless

(i) \( W \cap D_2 = \emptyset \), or

(ii) \( W \) is special.

Note that if the stalk of \( \mathcal{H}^0(G_{2}^*) \) vanishes somewhere in \( W \), then because \( G_{2}^* \) is coherent and torsion-free, it follows that \( H^0(G_{2}^*)|_W = 0 \), hence a similar vanishing holds for the entire component of \( X_2 \) containing \( W \). Now recall that, minding the index shift, if \( H^0(G_{2}^*) = 0 \),
then the cokernel of the inclusion \( \Omega^+_X(\log D) \rightarrow \Omega^+_X(\log + D) \) has vanishing \( \mathbb{H}^1 \) (and \( \mathbb{H}^0 \)). On the other hand, it is well known (see, e.g., [13]) that \( \Omega^+_X(\log D) \simeq T^*_X(− \log D) \) controls deformations of \((X, \Phi)\) or \((X, \Pi)\) where \(D\) deforms locally trivially, and those deformations are unobstructed thanks to Hodge theory.

Summarizing this discussion, we conclude:

**Theorem 6** Let \((X, \Phi)\) be a log-symplectic manifold with polar divisor \(D\). With notations as above, let

\[
\Omega^+_X(\log D) = \bigoplus_{i > 0} \Omega^+_X(\log D), \quad \Omega^+_X(\log + D) = \bigoplus_{i > 0} \Omega^+_X(\log + D).
\]

Then, the inclusions

\[
\Omega^+_X(\log D) \rightarrow \Omega^+_X(\log + D),
\]

\[
T^*_X(− \log D) \rightarrow T^*_X
\]

induce isomorphisms on \(\mathcal{H}^2\) and injections on \(\mathcal{H}^3\); hence, isomorphisms on \(\mathbb{H}^1\) and injections on \(\mathbb{H}^2\), unless either

(i) \(X_2\) has a nonresidual component; or

(ii) \(X_2\) has a special component.

As noted above, any component of \(X_2\) that is disjoint from \(D_2\), i.e., contains no triple points of \(D\), is automatically nonresidual.

**Corollary 7** Notations as above, if \(X\) is compact and Kählerian and conditions (i), (ii) both fail, then the pair \((X, \Phi)\) has unobstructed deformations and the polar divisor of \(\Phi\) deforms locally trivially.

In the case where \(D\) has global normal crossings, i.e., is a union of smooth divisors, this result also follows from results in [9], which also states a partial converse: when \(T^*_X(− \log D) \rightarrow T^*_X\) is not a quasi-isomorphism, \((X, \Phi)\) has obstructed deformations and admits deformations where \(D\) either smooths or deforms locally trivially.

**Example 8** (Due to M. Matviichuk, B. Pym, T. Schedler, see [9], communicated by B. Pym)

Consider the matrix

\[
B = (b_{ij}) = \begin{pmatrix}
0 & 1 & 2 & 4 \\
-1 & 0 & 3 & 5 \\
-2 & -3 & 0 & 6 \\
-4 & -5 & 1 & 0
\end{pmatrix}
\]

and the corresponding log-symplectic form on \(\mathbb{C}^4\), \(\Phi = \sum_{i < j} b_{ij} \cdot \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}\) and corresponding Poisson structure \(\Pi = \Phi^{-1}\), both of which extend to \(\mathbb{P}^4\) with Pfaffian divisor \(D = (z_0 z_1 z_2 z_3 z_4)\), \(z_0\) = hyperplane at infinity. Then \(\Pi\) admits the 1st order Poisson deformation with bivector \(z_3 z_4 \circ \frac{dz_1}{z_1} @ z_2\) which in fact extends to a Poisson deformation of \((\mathbb{P}^4, \Pi)\) over the affine line \(\mathbb{C}\), and the Pfaffian divisor deforms as \((z_3 z_4 z_0 (z_1 z_2 - t z_3 z_4))\), hence non locally-trivially. Correspondingly, the log-plus form \(z_3 z_4 \phi_1 \phi_2\) is closed (and not exact). That \(d(z_3 z_4 \phi_1 \phi_2) = 0\) corresponds to the integral column relation

\[
k_1 - k_2 + (e_1 + e_2) - (e_3 + e_4) = 0
\]

where the \(k_i\) and \(e_j\) are the columns of the \(B\) matrix and the identity, respectively, showing that \((z_1 z_2 z_3)\) and \((z_1 z_2 z_4)\) are residual triples of type II and (12), i.e., \((x_1) \cap (x_2)\) is a special component of \(X_2\).
Remark 9 As we saw above, the presence of monodromy on the branches of $D$ is related to nonresidual or special components. This suggests that log-symplectic manifolds with irreducible polar divisor may often be obstructed. However, we do not have specific examples.

Data availability statement There is no data set associated with this paper.

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