A finite axiomatization of positive MV-algebras

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Abstract. Positive MV-algebras are the subreducts of MV-algebras with respect to the signature \{⊕, ⊙, ∨, ∧, 0, 1\}. We provide a finite quasi-equational axiomatization for the class of such algebras.

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1. Introduction

An MV-algebra is an algebraic structure with a binary operation ⊕, a unary operation ¬ and a constant 0 satisfying certain equational axioms. MV-algebras arose in the literature as the algebraic semantics of Lukasiewicz logic and they are categorically equivalent to lattice-ordered abelian groups with strong unit (unital abelian ℓ-groups, for short). Throughout the paper we assume familiarity with MV-algebras and unital abelian lattice-ordered groups; see [4,9] and [3,6] for background information.

Boolean algebras are the algebraic semantics of classical propositional logic and they constitute a subvariety of MV-algebras; indeed, Boolean algebras are the MV-algebras satisfying the additional axiom \(x ⊕ x = x\). It is well known that bounded distributive lattices are precisely the algebras \(B = ⟨B, ∨, ∧, 0, 1⟩\) that are isomorphic to a subreduct of some Boolean algebra. In this paper we investigate certain algebras, called positive MV-algebras, which are to MV-algebras what bounded distributive lattices are to Boolean algebras.

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Positive MV-algebras are defined as the algebras \( A = \langle A, \oplus, \odot, \lor, \land, 0, 1 \rangle \) that are isomorphic to a subreduct of some MV-algebra.

Whereas bounded distributive lattices are a variety axiomatized by finitely many equations, positive MV-algebras are a quasivariety but not a variety. Our main result is an axiomatization of positive MV-algebras via finitely many quasi-equations. The strategy is the following. Mundici’s equivalence between MV-algebras and unital abelian \( \ell \)-groups was generalized by the first author to an equivalence between the categories of MV-monoidal algebras and unital commutative distributive \( \ell \)-monoids [1]. We observe that this equivalence restricts to an equivalence between positive MV-algebras and those unital commutative distributive \( \ell \)-monoids that are cancellative. We derive the axiomatization for positive MV-algebras by adding to the axioms of monoidal MV-algebras a quasi-equation expressing the cancellation property.

Finally, in the appendix, we obtain an analogue of Booleanization for positive MV-algebras. In particular, we prove that the inclusion of a positive MV-algebra into an MV-algebra that is generated by the image of such inclusion is universal.

The paper is organized as follows. In Section 2 we provide basic notions and examples regarding positive MV-algebras. In Section 3 we recall the definition of MV-monoidal algebras, their equivalence with unital commutative distributive \( \ell \)-monoids, and we restrict this equivalence to positive MV-algebras. In Section 4 we obtain our main result: a finite quasi-equational axiomatization of positive MV-algebras. In the appendix we present the analogue of the free Boolean extension in the MV-algebraic setting.

2. Positive MV-algebras

**Definition 2.1.** An **MV-algebra** is an algebra \( \langle M, \oplus, \neg, 0 \rangle \) (arities 2, 1, 0) with the following properties.

1. \( \langle M, \oplus, 0 \rangle \) is a commutative monoid.
2. \( \neg \neg x = x \).
3. \( x \oplus \neg 0 = \neg 0 \).
4. \( \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \).

A prime example is the so-called **standard MV-algebra** \([0, 1]\), with operations \( x \oplus y := \min \{ x + y, 1 \} \), \( \neg x := 1 - x \), \( 0 := 0 \).

It is usual to expand the signature of MV-algebras with the constant 1 and the binary operations \( \odot, \lor, \land \) defined as follows: \( 1 := \neg 0 \), \( x \odot y := \neg(\neg x \odot \neg y) \), \( x \lor y := (x \odot \neg y) \oplus y \), \( x \land y := \neg(\neg x \lor \neg y) \). Let \( \mathcal{M}V \) be the variety of MV-algebras in the signature containing all the above operations. Using this enriched signature we can denote a member of \( \mathcal{M}V \) as

\[ M = \langle M, \oplus, \odot, \lor, \land, \neg, 0, 1 \rangle. \]
Note that all the operations of $M$ except $\neg$ are positive (= order-preserving in each argument). In this paper we focus on the positive subreducts of MV-algebras, which are precisely the algebras

$$A = \langle A, \oplus, \odot, \lor, \land, 0, 1 \rangle,$$

where $A$ is a subreduct of an MV-algebra $M$.

**Definition 2.2.** An algebra of type $\{\oplus, \odot, \lor, \land, 0, 1\}$ is a positive MV-algebra if it is isomorphic to a positive subreduct of some MV-algebra. We let $\mathcal{MV}_+$ denote the class of all positive MV-algebras.

A prime example of a positive MV-algebra is the positive reduct of the standard MV-algebra $[0, 1]$. We call this reduct the standard positive MV-algebra.

**Definition 2.3.** A McNaughton function is a function $g: [0, 1]^n \to [0, 1]$ that is continuous and piecewise linear with integer coefficients, i.e., there exist linear polynomials $p_1, \ldots, p_t: [0, 1]^n \to [0, 1]$ with integer coefficients, such that for any $(x_1, \ldots, x_n) \in [0, 1]^n$ there is $j$, $1 \leq j \leq t$ with $g(x_1, \ldots, x_n) = p_j(x_1, \ldots, x_n)$.

For each natural number $n$, the free $n$-generated MV-algebra is isomorphic to the algebra $F_n$ of McNaughton functions $[0, 1]^n \to [0, 1]$; see [10] or [4, Chapter 3].

**Example 2.4 (Non-decreasing McNaughton functions).** Let us equip $[0, 1]$ with the standard order of reals, and let $\leq$ be the product order of $[0, 1]^n$. Then the algebra $F_n^{\leq}$ of non-decreasing (i.e. order-preserving) McNaughton functions is a positive subreduct of $F_n$.

We remark that McNaughton functions in $F_n^{\leq}$ correspond to positive formulae in Lukasiewicz logic. This observation follows from an easy extension of the original McNaughton theorem; see [5, Theorem 3.5].

**Proposition 2.5.** The class of positive MV-algebras is the quasivariety generated by the standard positive MV-algebra $[0, 1]$.

**Proof.** It is a well-known fact that if $Q$ is a quasivariety of type $\Sigma$ generated by a class $K$, and $\Sigma'$ is a subsignature of $\Sigma$, the class of isomorphic copies of subalgebras of $\Sigma'$-reducts of algebras in $Q$ is the quasivariety generated by the $\Sigma'$-reducts of algebras in $K$. By Di Nola’s representation theorem (see [4, Theorem 9.5.1]), the standard MV-algebra $[0, 1]$ generates the class of MV-algebras as a quasivariety. The desired result follows. \qed

Next, we show that the quasivariety $\mathcal{MV}_+$ is not a variety. To this purpose, we recall the definition of Chang’s MV-algebra.

**Definition 2.6.** Chang’s MV-algebra $C$ is a linearly ordered MV-algebra defined on the set of formal symbols

$$C := \{0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots \} \cup \{\ldots, 1 - 3\varepsilon, 1 - 2\varepsilon, 1 - \varepsilon, 1\}$$
(where we think of $\varepsilon$ as an infinitesimal) and with the following operations:

$$x \oplus y := \begin{cases} 
(m + n)\varepsilon, & \text{if } x = n\varepsilon, y = m\varepsilon, \\
1 - (m - n)\varepsilon, & \text{if } x = 1 - m\varepsilon, y = n\varepsilon \text{ and } 0 < n < m, \\
1 - (n - m)\varepsilon, & \text{if } x = m\varepsilon, y = 1 - n\varepsilon \text{ and } 0 < m < n, \\
1 & \text{otherwise},
\end{cases}$$

and

$$-x := \begin{cases} 
1 - n\varepsilon, & \text{if } x = n\varepsilon, \\
n\varepsilon, & \text{if } x = 1 - n\varepsilon.
\end{cases}$$

**Example 2.7.** Let $C$ be Chang’s MV-algebra. The algebra $C'$ with the universe $\{0, \varepsilon, 2\varepsilon, \ldots, 1\}$ is a positive subreduct of $C$. Let $\theta$ be an equivalence relation on the algebra $C'$ with classes $\{0\}$, $\{\varepsilon, 2\varepsilon, \ldots\}$, and $\{1\}$. Then $\theta$ is a $\mathcal{MV}_+$-congruence on $C'$. The quotient $C'/\theta$ is isomorphic to the three-element algebra $\{\bar{0}, \bar{\varepsilon}, \bar{1}\}$ that satisfies $\bar{\varepsilon} \oplus \bar{\varepsilon} = \bar{\varepsilon}$ and $\bar{\varepsilon} \circ \bar{\varepsilon} = \bar{0}$. The quasi-equation

$$x \oplus x = x \Rightarrow x \odot x = x$$

holds for all MV-algebras, and thus for all positive MV-algebras, as well. However, this quasi-equation does not hold in $C'/\theta$, as witnessed by the element $\bar{\varepsilon}$. Therefore, $C'/\theta$ is not a positive MV-algebra.

**Theorem 2.8** ([5, Theorem 3.5]). For every $n \in \mathbb{N}$, the subalgebra of the power $[0, 1]^{[0,1]^n}$ of the standard positive MV-algebra $[0, 1]$ generated by the projections consists precisely of the non-decreasing (i.e. order-preserving) McNaughton functions from $[0, 1]^n$ to $[0, 1]$.

**Proposition 2.9.** The free $n$-generated positive MV-algebra is isomorphic to the positive subreduct $F_n \leq$ of $F_n$ from Example 2.4.

**Proof.** By proposition 2.5, $\mathcal{MV}_+$ is the quasivariety generated by the standard positive MV-algebra $[0, 1]$; thus, the free $n$-generated positive algebra is (up to isomorphism) the subalgebra of the power $[0, 1]^{[0,1]^n}$ of the standard positive MV-algebra $[0, 1]$ generated by the projections, which, by Theorem 2.8, is $F_n$. \hfill $\Box$

### 3. MV-monoidal algebras

Every positive MV-algebra satisfies the equations (E1–E7) in Definition 3.1 below (see [4]). Since the algebras thus defined are of separate interest [1], we recall their definition.

**Definition 3.1.** An $MV$-monoidal algebra is an algebra $\langle A, \oplus, \odot, \lor, \land, 0, 1 \rangle$ (arities 2, 2, 2, 2, 0, 0) satisfying the following equational axioms.

(E1) $\langle A, \lor, \land \rangle$ is a distributive lattice.

(E2) $\langle A, \oplus, 0 \rangle$ and $\langle A, \odot, 1 \rangle$ are commutative monoids.

(E3) The operations $\oplus$ and $\odot$ distribute over both $\lor$ and $\land$.

(E4) $(x \oplus y) \odot ((x \odot y) \oplus z) = (x \odot (y \oplus z)) \oplus (y \odot z)$. 

(E5) \((x \odot y) \oplus ((x \oplus y) \odot z) = (x \oplus (y \odot z)) \odot (y \oplus z)\).
(E6) \((x \odot y) \oplus z = ((x \oplus y) \odot ((x \odot y) \odot z)) \lor z\).
(E7) \((x \oplus y) \odot z = ((x \odot y) \oplus ((x \odot y) \odot z)) \land z\).

The main use of (E2), (E4), (E5) is to prove the associativity of the monoid operation + in the enveloping unital commutative distributive \(\ell\)-monoid of \(A\). The main use of (E3), (E6) (resp. (E3),(E7)) is to prove that + distributes over \(\lor\) (resp. \(\land\)).

We denote with \(MVM\) the category of MV-monoidal algebras with homomorphisms.

**Definition 3.2.** An abelian lattice-ordered group (abelian \(\ell\)-group, for short) is an algebra \(\langle G, +, \lor, \land, 0, - \rangle\) (arities \(2,2,2,0,1\)) such that \(\langle G, +, 0, - \rangle\) is an abelian \(\ell\)-group, \(\langle G, \lor, \land \rangle\) is a lattice, and + distributes over \(\lor\) and \(\land\). An element 1 of an abelian \(\ell\)-group \(G\) is a strong unit (or simply a unit) if 1 \(\geq 0\) and, for every \(g \in G\), there is \(n \in \mathbb{N}\) such that \(g \leq n1\). In this paper, by a unital abelian lattice-ordered group we mean an algebra \(\langle G, +, \lor, \land, 0, 1, - \rangle\) such that \(\langle G, +, \lor, \land, 0, - \rangle\) is an abelian \(\ell\)-group, 1 is a unit, and \(-1\) is the group-inverse of 1.

We will further consider the category of MV-algebras (the category whose objects are MV-algebras and whose morphisms are MV-homomorphisms) and the category of unital abelian \(\ell\)-groups (the category whose objects are unital abelian \(\ell\)-groups and morphisms are group- and lattice-homomorphisms preserving units). These two categories are equivalent by a theorem of Mundici [8]. These results can be lifted to an equivalence between unital commutative distributive lattice-ordered monoids and MV-monoidal algebras; see [1] or [2, Chapter 4] for details. Since our proof of quasi-equational characterization of positive MV-algebras is essentially based on this equivalence, we provide a summary and some consequences of this result.

**Definition 3.3.** A commutative distributive lattice-ordered monoid with strong units (henceforth shortened to unital commutative distributive \(\ell\)-monoid) is an algebra \(\langle M, +, \lor, \land, 0, 1, -1 \rangle\) (arities \(2,2,2,0,0,0\)) with the following properties.

(M1) \(\langle M, \lor, \land \rangle\) is a distributive lattice.
(M2) \(\langle M, +, 0 \rangle\) is a commutative monoid.
(M3) The operation + distributes over \(\lor\) and \(\land\) on both sides.
(M4) \(-1 + 1 = 0\).
(M5) \(-1 \leq 0 \leq 1\).
(M6) For all \(x \in M\), there exists \(n \in \mathbb{N}\) such that
\[
(-1) + \cdots + (-1) \leq x \leq 1 + \cdots + 1.
\]

For \(n \in \mathbb{N}\), we write \(n\) for \(1 + \cdots + 1\) (\(n\) times) and \(-n\) for \((-1) + \cdots + (-1)\) (\(n\) times), and, we write \(x - n\) for \(x + (-n)\). For unital abelian \(\ell\)-groups, the constant \(-1\) is term-definable via the constant 1 and the group inverse operation. Since in the context of unital commutative distributive \(\ell\)-monoids...
we do not have the group-inverse operation, we stress the fact that \(-1\) is an indivisible formal symbol, i.e. a constant symbol in the signature.

We let \(u\ell M\) denote the category of unital commutative distributive \(\ell\)-monoids with homomorphisms. For a unital commutative distributive \(\ell\)-monoid \(M\), we set

\[
\Gamma(M) := \{x \in M \mid 0 \leq x \leq 1\}.
\]

We equip \(\Gamma(M)\) with the operations of MV-monoidal algebras: we define \(\lor, \land, 0\) and \(1\) by the restriction, and we set

\[
x \oplus y := (x + y) \land 1, \quad x \odot y := (x + y - 1) \lor 0.
\]

The algebra \(\langle \Gamma(M), \oplus, \odot, \lor, \land, 0, 1 \rangle\) is an MV-monoidal algebra (see [1, Proposition 3.6] or [2, Theorem 4.29]). Given a morphism of unital commutative distributive \(\ell\)-monoids \(f: M \rightarrow N\), we let \(\Gamma(f)\) denote its restriction \(\Gamma(f): \Gamma(M) \rightarrow \Gamma(N)\). This yields a functor

\[
\Gamma: u\ell M \rightarrow MVM.
\]

**Theorem 3.4** ([1, Theorem 8.21] or [2, Theorem 4.74]). The functor

\[
\Gamma: u\ell M \rightarrow MVM
\]

determines an equivalence of categories.

A quasi-inverse for \(\Gamma\) is described in [2, Chapter 4, Sect. 6] and is denoted by \(\Xi\); we mention that this construction—based on the notion of good \(\mathbb{Z}\)-sequence—is suggested by the following result.

**Proposition 3.5** ([2, Theorem 4.72]). Let \(M\) be a unital commutative distributive \(\ell\)-monoid, and let \(x, y \in M\). If, for every \(n \in \mathbb{Z}\), we have

\[
((x - n) \lor 0) \land 1 = ((y - n) \lor 0) \land 1,
\]

then \(x = y\).

Proposition 3.5 is a consequence of the following fact.

**Lemma 3.6** ([2, Proposition 4.68]). For every unital commutative distributive \(\ell\)-monoid \(M\), every \(x \in M\), and all \(n \leq m \in \mathbb{Z}\) such that \(-n \leq x \leq m\), we have

\[
x = n + \sum_{i=n}^{m}((x - n) \lor 0) \land 1.
\]

**Proposition 3.7.** The functors \(\Gamma\) and \(\Xi\) preserve and reflect injectivity of morphisms.

Proof. It is enough to show that \(\Gamma\) preserves and reflects injectivity. Since \(\Gamma\) is defined as the restriction on morphisms, \(\Gamma\) preserves injectivity. Let us prove that \(\Gamma\) reflects injectivity. Let \(f: M \rightarrow N\) be a morphism of unital commutative
distributive \(\ell\)-monoids, and suppose \(\Gamma(f)\) to be injective. Let \(x, y \in M\) and suppose \(f(x) = f(y)\). For every \(n \in \mathbb{Z}\), we have
\[
\Gamma(f)(((x - n) \lor 0) \land 1) = f(((x - n) \lor 0) \land 1)
\]
\[
= ((f(x) - n) \lor 0) \land 1
\]
\[
= f(((y - n) \lor 0) \land 1)
\]
\[
= \Gamma(f)(((y - n) \lor 0) \land 1).
\]

Since \(\Gamma(f)\) is injective, we deduce
\[
((x - n) \lor 0) \land 1 = ((y - n) \lor 0) \land 1.
\]

Thus, \(f\) is injective. Since this holds for every \(n \in \mathbb{Z}\), by Proposition 3.5 we have \(x = y\). This proves that \(\Gamma\) reflects injectivity. \(\square\)

Lemma 3.8. The \(\{+, \lor, \land, 0, 1, -1\}\)-reduct of a unital abelian \(\ell\)-group is a unital commutative distributive \(\ell\)-monoid, and the \(\{\oplus, \odot, \lor, \land, 0\}\)-reduct of an MV-algebra is an MV-monoidal algebra. Moreover,
\begin{enumerate}
  \item the equivalence \(\Gamma: u\ell M \leftrightarrow MVM: \Xi\) restricts to an equivalence between isomorphic copies of reducts of unital abelian \(\ell\)-groups and isomorphic copies of reducts of MV-algebras, and
  \item the equivalence \(\Gamma: u\ell M \leftrightarrow MVM: \Xi\) restricts to an equivalence between isomorphic copies of subreducts of unital abelian \(\ell\)-groups and isomorphic copies of subreducts of MV-algebras.
\end{enumerate}

Proof. Up to ((1)) is shown in [1, Appendix A] (see [2, Chapter 4, Sect. 8] for a more detailed proof). ((2)) follows from ((1)), together with the fact that \(\Gamma\) and \(\Xi\) preserve injectivity of morphisms (Proposition 3.7). \(\square\)

4. Main result
We recall the following folklore result.

**Lemma 4.1.** Let \(M\) be a generating subset of an abelian group \(G\), and suppose that \(M\) is closed under + and 0. For every \(z \in G\), there are \(x, y \in M\) such that \(z = x - y\).

Lemma 4.1 generalizes in the setting of (unital) abelian \(\ell\)-groups, as follows.

**Lemma 4.2.** Let \(M\) be a generating subset of an abelian \(\ell\)-group (resp. unital abelian \(\ell\)-group) \(G\), and suppose that \(M\) is closed under +, \(\lor, \land\) and 0 (resp. +, \(\lor, \land\), 0, 1 and \(-1\)). For every \(z \in G\), there are \(x, y \in M\) such that \(z = x - y\).

Proof. It is enough to prove that the set \(\{x - y \mid x, y \in M\}\) contains \(M\) and is closed under +, \(\lor, \land\), 0 and \((-\) (resp. +, \(\lor, \land\), 0, 1, \(-1\) and \(-\)). These verifications are easy; we only show closure under \(\lor\):
\[
(x - y) \lor (x' - y') = (((x + y') \lor (x' + y)) - (y + y')).
\]
Recall that a monoid $M$ is called cancellative if, for all $x, y, z \in M$, the condition $x + z = y + z$ implies $x = y$.

**Proposition 4.3.** An algebra $\langle M, +, \lor, \land, 0 \rangle$ is isomorphic to a subreduct of an abelian $\ell$-group if and only if it is a cancellative commutative distributive $\ell$-monoid.

**Proof.** The left-to-right implication follows from the fact that the conditions that define cancellative commutative distributive $\ell$-monoids are quasi-equations that hold in all abelian $\ell$-groups.

For the converse implication, let $M$ be a cancellative commutative distributive $\ell$-monoid. Following a standard procedure (see [12]) motivated by Lemma 4.1, one embeds $M$ into its so-called Grothendieck group ([7]), or algebra of fractions. The elements of $G$ are the equivalence classes of the relation $\sim$ on $M \times M$ defined by

$$(a, b) \sim (c, d) \iff a + d = c + b.$$ 

For $a, b \in M$, we write $[a, b]$ for the equivalence class of $(a, b)$ with respect to $\sim$. The group operations on $G$ are defined as follows:

$[a, b] + [c, d] := [a + c, b + d]$;

$0 := [0, 0]$;

$-[a, b] := [b, a]$.

One then defines the following additional operations on $G$:

$[a, b] \lor [c, d] := [(a + d) \lor (c + b), b + d]$;

$[a, b] \land [c, d] := [(a + d) \land (c + b), b + d]$.

The algebra $G = \langle G, +, \lor, \land, 0, - \rangle$ is an abelian $\ell$-group, and the map $\iota: M \rightarrow G$

\[
x \mapsto [x, 0]
\]

is an injective homomorphism from $\langle M, +, \lor, \land, 0 \rangle$ to $\langle G, +, \lor, \land, 0 \rangle$. This shows that $M$ is isomorphic to a subreduct of $G$. □

**Proposition 4.4.** An algebra $\langle M, +, \lor, \land, 0, 1, -1 \rangle$ is isomorphic to a subreduct of a unital abelian $\ell$-group if and only if it is a cancellative unital commutative distributive $\ell$-monoid.

**Proof.** Any subreduct of a unital abelian $\ell$-group is a cancellative unital commutative distributive $\ell$-monoid: (M6) holds because, for any $n \in \mathbb{N} \setminus \{0\}$, the validity of an equation $-n \leq x \leq n$ is preserved by subalgebras. All the remaining axioms are quasi-equations, so their validity is preserved.

For the converse implication, given a cancellative unital commutative distributive $\ell$-monoid $\langle M, +, \lor, \land, 0 \rangle$, we consider the unital abelian $\ell$-group $\langle G, +, \lor, \land, 0, - \rangle$ built from $\langle M, +, \lor, \land, 0 \rangle$ as in the proof of Proposition 4.3. We show that $[1, 0]$ is a unit of the abelian $\ell$-group $\langle G, +, \lor, \land, 0, - \rangle$. We have $[0, 0] \lor [1, 0] = [1, 0]$, and thus $[0, 0] \leq [1, 0]$. Moreover, given $a, b \in M$, let
Both sides of (4.3) hold for every \( n \in \mathbb{N} \setminus \{0\} \) be large enough so that \( a \leq n \) and \( b \geq -n \). Then \( a - n \leq 0 \leq b + n \), and thus
\[
[a, b] \land [2n, 0] = [a, b] \land [n, -n] = [(a - n) \land (b + n), b - n] = [a - n, b - n] = [a, b].
\]
Therefore, \( [a, b] \leq [2n, 0] = 2n[1, 0] \). This shows that \([1, 0]\) is a unit. Denoting the element \([1, 0]\) with 1 and its group-inverse \([-1, 0]\) with \(-1\), we conclude that the algebra \( \langle G, +, V, \land, 0, 1, -1, -\rangle \) is a unital abelian \( \ell \)-group. The function \( \iota : M \rightarrow G \) from the proof of Proposition 4.4 clearly preserves also the constants 1 and \(-1\).

\[\square\]

**Proposition 4.5.** The following are equivalent for a unital commutative distributive \( \ell \)-monoid \( M \).

1. \( M \) is cancellative.
2. For all \( x, y, z \in \{ w \in M \mid 0 \leq w \leq 1 \} \), if \( x + z = y + z \) then \( x = y \).
3. For all \( x, y, z \in \Gamma(M) \), if \( x \oplus z = y \oplus z \) and \( x \odot z = y \odot z \) then \( x = y \).

**Proof.** The implication (1) \( \Rightarrow \) (2) is trivial, and the implication (2) \( \Rightarrow \) (3) is easy.

Let us suppose (3), and let us prove (2). Let \( x, y, z \in \{ w \in M \mid 0 \leq w \leq 1 \} \) be such that \( x + z = y + z \). We claim that, for every \( n \in \mathbb{Z} \), we have
\[
((x + z - n) \lor 0) \land 1 = ((y + z - n) \lor 0) \land 1. \tag{1}
\]

For \( n = 0 \), resp. \( n = 1 \), (1) boils down to \( x \oplus z = y \oplus z \), resp. \( x \odot z = y \odot z \); both of these conditions are true by hypothesis. Since \( x \), \( y \) and \( z \) are below 1, both sides of (1) equal 0 for \( n \geq 2 \). Since \( x \), \( y \) and \( z \) are above 0, both sides of (1) equal 1 for \( n \leq -1 \). This settles the claim. By Proposition 3.5, we deduce \( x = y \).

We are left to prove the implication (2) \( \Rightarrow \) (1).

Let us assume (2). Let us start by proving the following claim, where we have added the hypothesis \( 0 \leq z \leq 1 \); we will then weaken this hypothesis to \( 0 \leq z' \), and we will finally prove the general case.

**Claim 4.5.1.** For all \( x, y, z \in M \) with \( 0 \leq z \leq 1 \), if \( x + z = y + z \) then \( x = y \).

**Proof of Claim.** Let \( x, y, z \in M \) with \( 0 \leq z \leq 1 \), and suppose \( x + z = y + z \). Since \( x + z = y + z \), we have, for every \( n \in \mathbb{Z} \),
\[
((-n + x + z) \lor z) \land (1 + z) = ((-n + y + z) \lor z) \land (1 + z). \tag{4.1}
\]

By distributivity of \( + \) over \( \lor \) and \( \land \), Eq. (4.1) can be written as
\[
(((\neg n + x) \lor 0) \land 1) + z = (((\neg n + y) \lor 0) \land 1) + z. \tag{4.2}
\]

From Eq. (4.2), using (2), we deduce
\[
((-n + x) \lor 0) \land 1 = ((-n + y) \lor 0) \land 1. \tag{4.3}
\]

Since Eq. (4.3) holds for every \( n \in \mathbb{Z} \), by Proposition 3.5 we have \( x = y \).

**Claim 4.5.2.** For all \( x, y, z \in M \) with \( z \geq 0 \), if \( x + z = y + z \) then \( x = y \).
Proof of Claim. We prove this inductively on \( n \in \mathbb{N} \setminus \{0\} \) such that \( z \leq n \). The case \( n = 1 \) is Claim 4.5.1. Let \( n \in \mathbb{N} \setminus \{0,1\} \), and let us suppose that the statement holds for \( n - 1 \). Let \( x, y, z \in \mathcal{M} \) with \( 0 \leq z \leq n \), and suppose \( x + z = y + z \). Set \( w := z \land (n - 1) \), and \( v := (z - (n - 1)) \lor 0 \). Then
\[
\begin{align*}
w + v &= (z \land (n - 1)) + ((z - (n - 1)) \lor 0) \\
&= (z \land (n - 1)) + (z \lor (n - 1)) - (n - 1) \\
&= z + (n - 1) - (n - 1) \\
&= z,
\end{align*}
\]
and thus the equality \( x + z = y + z \) can be written as
\[
x + w + v = y + w + v.
\]
By Claim 4.5.1, using the fact that \( 0 \leq v \leq 1 \), we deduce
\[
x + w = y + w.
\]
By inductive hypothesis, using the fact that \( 0 \leq w \leq n - 1 \), we conclude \( x = y \).

Let \( x, y, z \in \mathcal{M} \) and suppose \( x + z = y + z \). The element \( z \) can be written as \( u + n \), where \( n \in \mathbb{Z} \) and \( u \geq 0 \). Then the equality \( x + z = y + z \) can be written as \( x + u + n = y + u + n \), which is equivalent to \( x + u = y + u \). By Claim 4.5.2, we deduce \( x = y \). This proves ((1)): \( \mathcal{M} \) is cancellative.

We arrive at our main result: a finite axiomatization of the class of positive MV-algebras.

**Theorem 4.6.** The positive MV-algebras are precisely the MV-monoidal algebras that satisfy, for all \( x, y, a n d z \),

\[
\text{if } x \oplus z = y \oplus z \text{ and } x \odot z = y \odot z, \text{ then } x = y.
\]

**Proof.** For the left-to-right implication, we note that \([0,1]\) satisfies the equations defining MV-monoidal algebras and the quasi-equation in the statement. Therefore, by Proposition 2.5, the same is true for every positive MV-algebra.

Let us prove the converse implication. Suppose \( A = \langle A, \oplus, \odot, \lor, \land, 0, 1 \rangle \) is an MV-monoidal algebra, and suppose that, for all \( x, y, z \in A \), if \( x \oplus z = y \oplus z \) and \( x \odot z = y \odot z \), then \( x = y \). Using the implication (3) \( \Rightarrow \) (1) in Proposition 4.5, we deduce that \( \Xi(A) \) is cancellative. By Proposition 4.4, \( \Xi(A) \) is isomorphic to a subreduct of a unital abelian \( \ell \)-group. By Lemma 3.8, \( A \) is isomorphic to a subreduct of an MV-algebra, i.e. \( A \) is a positive MV-algebra.

Theorem 4.6 provides a finite quasi-equational axiomatization for the class of positive MV-algebras: this consists of the equational axioms defining MV-monoidal algebras (E1–E7), together with the quasi-equation

\[
\text{if } x \oplus z = y \oplus z \text{ and } x \odot z = y \odot z, \text{ then } x = y,
\]

which can be seen as an appropriate version of the cancellation property.
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Appendix A. Free MV-extensions

By standard results in general algebra, the forgetful functor $U$ from MV-algebras to positive MV-algebras has a left adjoint $F$. For every positive MV-algebra $A$, it is immediate that the component $ι_A: A \to UF(A)$ at $A$ of the unit is injective, and that the image of $ι_A$ generates the MV-algebra $F(A)$. We speak of the pair $(F(A), ι_A)$ (or simply of $F(A)$, leaving $ι_A$ understood) as the free MV-extension of $A$. In this section we prove that, given a positive subreduct $A$ of an MV-algebra $B$ such that $A$ generates $B$, and denoting with $i$ the inclusion of $A$ into $B$, $(B, i)$ is the free MV-extension of $A$; moreover, under the same conditions, for every $x \in B$, there are $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ such that $x = \bigoplus_{j=1}^n (a_j \odot b_j)$.

The following Lemma corresponds, essentially, to the fact that Mundici’s bijection between $G$ and $Ξ(Γ(G))$ preserves +. We refer to [2, Proposition 4.67] for a proof in the setting of unital commutative distributive $\ell$-monoids.

**Lemma A.1.** Let $G$ be a unital abelian $\ell$-group, and let $x, y \in G$. For each $n \in \mathbb{Z}$, set $x_n := ((x - n) \lor 0) \land 1$ and $y_n := ((x - n) \lor 0) \land 1$. We have

$$(x + y) \lor 0 \land 1 = \bigoplus_{n \in \mathbb{Z}} (x_n \odot y_{-n-1}) = \bigodot_{n \in \mathbb{Z}} (x_n \oplus y_{-n}).$$

(The expression $\bigoplus_{n \in \mathbb{Z}} (x_n \odot y_{-n-1})$ makes sense because for all but finitely many $n \in \mathbb{Z}$ we have $x_n \odot y_{-n-1} = 0$. Analogously, the expression $\bigodot_{n \in \mathbb{Z}} (x_n \oplus y_{-n})$ makes sense because for all but finitely many $n \in \mathbb{Z}$ we have $x_n \oplus y_{-n} = 1$.)

**Lemma A.2.** Let $G$ be a unital abelian $\ell$-group. Set

$$M := \{ M \subseteq G \mid M \text{ is closed under } +, \lor, \land, 0, 1, -1 \}$$

and

$$A := \{ A \subseteq Γ(G) \mid A \text{ is closed under } ⊕, ⊖, \lor, \land, 0, 1 \}.$$

(1) The sets $M$ and $A$ are in bijection, as witnessed by the functions

$$M \longleftrightarrow A$$

$$M \mapsto \{ x \in M \mid 0 \leq x \leq 1 \}$$

$$\{ x \in G \mid \forall n \in \mathbb{Z} ((x - n) \lor 0) \land 1 \in A \} \longleftrightarrow A.$$

(2) The bijection in (1) restricts to the subsets

$$\{ M \in M \mid M \text{ is closed under } - \} \text{ and } \{ A \in A \mid A \text{ is closed under } - \}.$$
(3) For every \( M \in \mathcal{M} \), \( M \) generates the unital abelian \( \ell \)-group \( G \) if and only if \( \{ x \in M \mid 0 \leq x \leq 1 \} \) generates the MV-algebra \( \Gamma(G) \).

Proof. The fact that the function \( f : \mathcal{M} \to \mathcal{A} \) is well-defined is immediate. Let us prove that the function \( g : \mathcal{A} \to \mathcal{M} \) is well-defined. Let \( A \in \mathcal{A} \), and set

\[
M := \{ x \in G \mid \forall n \in \mathbb{Z} \ ((x - n) \lor 0) \land 1 \in A \}.
\]

The set \( M \) is closed under + by Lemma A.1. Moreover, it is closed under \( \lor \) because, for all \( x, y \in M \), we have \((x \lor y) \lor 0) \land 1 = ((x \lor 0) \land 1) \lor ((x \lor 0) \land 1)\). Analogously, it is closed under \( \land \). It is not difficult to prove that \( 0, 1, -1 \in M \). Therefore, \( M \in \mathcal{M} \), and thus \( g \) is well-defined.

It is easy to see that the composite \( f \circ g : \mathcal{A} \to \mathcal{A} \) is the identity on \( \mathcal{A} \). To prove that \( g \circ f \) is the identity on \( \mathcal{M} \), let \( M \in \mathcal{M} \). We shall prove that

\[
M = \{ x \in G \mid \forall n \in \mathbb{Z} \ ((x - n) \lor 0) \land 1 \in M \}.
\]

The left-to-right inclusion is immediate. For the converse inclusion, let \( x \in G \) be such that, for every \( n \in \mathbb{Z} \), \((x - n) \lor 0) \land 1 \in M \). Let \( n \in \mathbb{N} \) be such that \(-n \leq x \leq n \). Then (cf. Lemma 3.6)

\[
x = -n + \sum_{i=-n}^{n-1} (((x - i) \lor 0) \land 1),
\]

and thus \( x \in M \). This proves (1).

If \( M \in \mathcal{M} \) is closed under \( \neg \), \( f(M) \) is easily seen to be closed under \( \neg \). If \( A \in \mathcal{A} \) is closed under \( \neg \), \( g(A) \) is closed under \( \neg \) because, for every \( x \in g(A) \) and every \( n \in \mathbb{Z} \), we have

\[
((n - x) \lor 0) \land 1 = 1 - (((x + (n + 1)) \lor 0) \land 1) = \neg(((x + (n + 1)) \lor 0) \land 1).
\]

This proves (2).

Since the bijection in (1) preserves the inclusion order in both directions, we have (3). □

It is a well-known fact that, if we consider a Boolean algebra \( B \) generated by a bounded distributive sublattice \( D \), any element of \( B \) can be written as a finite join of elements of the form \( x \land \neg y \) (or, equivalently, as a finite meet of elements of the form \( x \lor \neg y \)) for \( x, y \in D \). An analogous result holds for positive MV-algebras and MV-algebras. Whereas the result for Boolean algebras can be derived by standard applications of the distributive and De Morgan’s laws, the corresponding one for MV-algebras is not straightforward. We shall present it in the next theorem.

Theorem A.3. Let \( A \) be a positive subreduct of an MV-algebra \( B \), and suppose that \( A \) generates the MV-algebra \( B \). For every \( x \in B \) there are \( n, m \in \mathbb{N} \) and \( s_1, \ldots, s_n, t_1, \ldots, t_n, u_1, \ldots, u_m, v_1, \ldots, v_m \in A \) such that

\[
x = \bigoplus_{i=1}^{n} s_i \circ \neg t_i = \bigodot_{i=1}^{m} u_i \oplus \neg v_i.
\]
Proof. By Lemma A.2, the set
\[ M := \{ x \in G \mid \forall n \in \mathbb{Z} \ ( (x - n) \lor 0 ) \land 1 \in A \} \]
is closed under \(+, \lor, \land, 0, 1\) and \(-1\) and generates the unital abelian \(\ell\)-group \(G\). Let \(z \in B\). By Lemma 4.2, there are \(x, y \in M\) such that \(z = x - y\). For every \(n \in \mathbb{N}\), we have
\[
((-y - n + 1) \lor 0) \land 1 = 1 - (1 - (y + n - 1) \lor 0) \land 1
\]
\[= 1 - (1 + ((y + n - 1) \lor 0) \land 1)
\]
\[= 1 - ((y + n) \land 1) \lor 0
\]
\[= 1 - (y + n) \lor 0 \land 1
\]
\[= \neg((y + n) \lor 0) \land 1).\]

Therefore, by Lemma A.1, we have
\[
z = \bigoplus_{n \in \mathbb{Z}} (((x - n) \lor 0) \land 1) \odot ((-y - n + 1) \lor 0) \land 1)
\]
\[= \bigoplus_{n \in \mathbb{Z}} (((x - n) \lor 0) \land 1) \odot \neg((y + n) \lor 0) \land 1).\]

Since \(x, y \in M\), \((x - n) \lor 0) \land 1, (y + n) \lor 0) \land 1 \in A\). This proves the first equality in the statement. The second one is analogous. \(\square\)

Theorem A.4. Let \(M\) be a generating subset of a unital abelian \(\ell\)-group \(G\), and suppose that \(M\) is closed under \(+, \lor, \land, 0, 1\) and \(-1\). For every unital Abelian \(\ell\)-group \(H\) and every function \(f : M \to H\) that preserves \(+, \lor, \land, 0, 1\) and \(-1\), there exists a unique morphism \(g : G \to H\) of unital abelian \(\ell\)-groups that extends \(f\).

Proof. This follows from Lemma 4.2. For every \(z \in G\) we define the morphism \(g\) as \(g(z) = f(x) - f(y)\) where \(x\) and \(y\) are elements of \(M\) such that \(z = x - y\). The fact that \(g\) is a well-defined morphism follows from the fact that \(x - y = u - v\) is equivalent to \(x + v = u + y\). \(\square\)

The following theorem generalizes an analogous result for bounded distributive lattices, namely that the inclusion of a bounded distributive lattice into a Boolean algebra that is generated by the image of such inclusion is universal, i.e. it is a so-called Booleanization, or free Boolean extension: see [11, Theorem 4.1] for a version of this result for (not necessarily bounded) distributive lattices.

Theorem A.5. Let \(A\) be a positive MV-algebra, \(B\) an MV-algebra, and \(i : A \hookrightarrow B\) an injective function that preserves \(\oplus, \odot, \lor, \land, 0\) and \(1\) and such that its image generates the MV-algebra \(B\). Then \((B, i)\) is the free MV-extension of \(A\).

Proof. Without loss of generality, we may suppose \(A \subseteq B\) and \(i\) to be the inclusion of \(A\) into \(B\). Let \(C\) be an MV-algebra, and let \(f : A \to C\) be a function that preserves \(\oplus, \odot, \lor, \land, 0\) and \(1\). We shall prove that there exists a unique MV-homomorphism \(g : B \to C\) that extends \(f\). Uniqueness follows from the fact that \(A\) generates the MV-algebra \(B\). Let us prove existence. By the
equivalence in Theorem 3.4, we obtain morphisms of unital commutative distributive $\ell$-monoids $\Xi(i) : \Xi(A) \rightarrow \Xi(B)$ and $\Xi(f) : \Xi(A) \rightarrow \Xi(C)$. By Proposition 3.7, $\Xi(i)$ is injective, and we may suppose $\Xi(i)$ to be an inclusion. By Theorem A.4, there exists a unique morphism $g' : \Xi(B) \rightarrow \Xi(C)$ of unital abelian $\ell$-groups that extends $\Xi(f)$. Then, $g := \Gamma(g')$ is an MV-homomorphism that extends $f$. □

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