A COMBINATORIAL PROBLEM IN INFINITE GROUPS

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Abstract. Let $w$ be a word in the free group of rank $n \in \mathbb{N}$ and let $V(w)$ be the variety of groups defined by the law $w = w(x_1, \ldots, x_n) = 1$. Define $V(w^*)$ to be the class of all groups $G$ in which for any infinite subsets $X_1, \ldots, X_n$ there exist $x_i \in X_i$, $1 \leq i \leq n$, such that $w(x_1, \ldots, x_n) = 1$. Clearly, $V(w) \cup F = V(w^*)$; $F$ being the class of finite groups. In this paper, we investigate some words $w$ and some certain classes $P$ of groups for which the equality $(V(w) \cup F) \cap P = P \cap V(w^*)$ holds.

Introduction and results

Let $w$ be a word in the free group of rank $n \in \mathbb{N}$ and let $V(w)$ be the variety of groups defined by the law $w = w(x_1, \ldots, x_n) = 1$. P. Longobardi, M. Maj and A. Rhemtulla in [29] defined $V(w^*)$ to be the class of all groups $G$ in which for any infinite subsets $X_1, \ldots, X_n$ there exist $x_i \in X_i$, $1 \leq i \leq n$, such that $w(x_1, \ldots, x_n) = 1$ and raised the question of whether $V(w) \cup F = V(w^*)$ is true; $F$ being the class of finite groups. There is no example, so far, of an infinite group in $V(w^*) \setminus V(w)$. In fact the origin of this problem is the following observation:

Let $G$ be an infinite group such that in every two infinite subsets of $G$ there exist two commuting elements, then $G$ is abelian. This is an immediate consequence of the answer of B. H. Neumann to a question of P. Erdős; B. H. Neumann proved that an infinite group $G$ is centre-by-finite if and only if every infinite subset of $G$ contains two distinct commuting elements [37]. Since this first paper, problems of a similar nature have been the object of several articles (for example [3], [4], [5], [11], [12], [15], [24], [27], [28]).

As far as we know, the equality $V(w) \cup F = V(w^*)$ is known for the following words: $w = x^m$, $w = [x_1, \ldots, x_n]$ [2], $w = [x, y] = [x, y] = 1$, $w = [x, y, z]$ [1], $w = [x, y, y]$ [12], $w = (xy)^{-3}x^3y^3$ [1], $w = x^{1^{a_1} \cdots n^{a_m}}$ where $\alpha_1, \ldots, \alpha_m$ are non-zero integers [1], $w = (xy)^2(yx)^2$ or $w = [x^m, y]$ where $m \in \{3, 6\} \cup \{2k \mid k \in \mathbb{N}\}$ [1], $w = [x^n, y][x, y]^{-1}$ where $n \in \{\pm 2, 3\}$ [43] and $w = [x^m, y^n]$ or $w = (x^m y^n)^2$ where $m \in \{2k \mid k \in \mathbb{N}\}$ [4].

In [18], P. Puglisi and L. S. Spiezia proved that every infinite locally finite group (or locally soluble group) in $V((x, k)^n)$ is a $k$-Engel group; (recall that $(x, k)^n$ is defined inductively by $(x, 0)^n = x$ and $(x, k)^n = [(x, k-1)^n]^n$, for $k \in \mathbb{N}$). In [14], C. Delizia proved the equality $V(w) \cup F = V(w^*)$ on the classes of hyperabelian, locally soluble and locally finite groups where $w = [x_1, \ldots, x_k, x_1]$ and $k$ is an integer greater than 2. Later G. Endimioni generalized these results by proving that every infinite locally finite or locally soluble group in $V(w^*)$ belongs to the variety $V(w)$, where $w$ is a word in a free group such that finitely generated soluble groups in $V(w)$ are nilpotent (see Theorem 3 of [14]) (recall that the variety $V((x_1, \ldots, x_k, x_1))$ ($k > 2$) is exactly the variety of nilpotent groups of nilpotency class at most $k$ [55] and every finitely generated soluble Engel group is nilpotent [17]).

We say that a group $G$ is locally graded if and only if every finitely generated non-trivial subgroup of $G$ has a non-trivial finite quotient. We proved in Theorem 4 of [4] that an infinite locally graded group in $V((x, k)^n)$ is a $k$-Engel group. We generalize this result as Theorem A, below. In order to state our first result we need the following definition. Following [29] we say that a group $G$ is restrained if and only if $\langle x \rangle^{(n)} = \langle x^n \mid i \in \mathbb{Z} \rangle$ is finitely generated for all $x, y \in G$. We show by Proposition 1 below, why the following theorem improves the above mentioned results.

**Theorem A.** Let $w$ be a word in a free group such that every finitely generated residually finite group in $V(w)$ is polycyclic-by-finite. Then every infinite finitely generated locally graded restrained group in $V(w^*)$ belongs to the variety $V(w)$.

G. Endimioni proved that every infinite locally nilpotent group in $V(w^*)$ belongs to the variety $V(w)$, where $w$ is a word in a free group (see Theorem 1 of [14]). The following theorem generalizes Theorem 1 of [14].
Theorem B. Let \( w \) be a word in a free group and let \( \mathcal{P} \) be a class of groups which satisfies the following conditions:

1. the class \( \mathcal{P} \) is closed under taking subgroups.
2. every \( \mathcal{P} \)-group is soluble.
3. every infinite finitely generated \( (\mathcal{P} \text{-by-finite}) \)-group in \( \mathcal{V}(w^*) \) belongs to the variety \( \mathcal{V}(w) \).

Then every infinite residually \([(locally \mathcal{P}) \text{-by-finite}] \) group in \( \mathcal{V}(w^*) \) belongs to \( \mathcal{V}(w) \).

For example, the classes of nilpotent groups, polycyclic groups, abelian-by-nilpotent groups and soluble residually finite groups satisfy the assumptions of Theorem B.

Here we also obtain some reductions in investigation of the equality \( \mathcal{V}(w) \cup \mathcal{F} = \mathcal{V}(w^*) \) on certain classes of groups and certain words \( w \). For example

Theorem C. Let \( w \) be a non-trivial word in a free group. Then every non-linear simple locally finite group does not belong to the class \( \mathcal{V}(w^*) \).

In [14], G. Endimioni proved that if \( w \) be a word in a free group such that finitely generated soluble groups in \( \mathcal{V}(w) \) are polycyclic, then every finitely generated soluble group in \( \mathcal{V}(w^*) \) belongs to the variety \( \mathcal{V}(w) \). Before stating our next result, we need a notation (see [16]). Let \( \alpha \) be a non-zero element of some field of characteristic \( p \). Denote the group generated by the matrices \( \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \right\} \) by \( M(\alpha, p) \).

Theorem D. Let \( w \) be a word in a free group such that every infinitely presented \( M(\alpha, p) \not\in \mathcal{V}(w) \) for all \( p \geq 0 \) or \( C_q \text{wr} C_\infty \not\in \mathcal{V}(w) \) for all primes \( q \). Then every infinite locally soluble group in \( \mathcal{V}(w^*) \) belongs to the variety \( \mathcal{V}(w) \).

We note that the group \( M(\alpha, p) \) is finitely presented if and only if

1. \( p \neq 0 \) and \( \alpha \) is algebraic over the prime field, or
2. \( p = 0 \) and at least one of \( \alpha \) or \( \alpha^{-1} \) is an algebraic integer (see Lemma 11 of [16]).

Theorem D generalizes Theorems 2 and 3 of [14], since we note that if \( \mathcal{V} \) is a variety of groups in which every finitely generated soluble group in \( \mathcal{V} \) is polycyclic then \( \mathcal{V} \) contains no infinitely presented \( M(\alpha, p) \) since \( M(\alpha, p) \) is finitely generated metabelian; the subgroup \( C_q^{(C_\infty)} \) of \( C_q \text{wr} C_\infty \) is not finitely generated and, \( C_q \text{wr} C_\infty \) is not polycyclic for any prime \( q \).

Proofs

We start the proof of Theorem A.

Proof of Theorem A. Let \( G \) be an infinite finitely generated locally graded restrained group in \( \mathcal{V}(w^*) \) and let \( R \) be the finite residual of \( G \). Then \( G/R \) is a finitely generated residually finite group in \( \mathcal{V}(w^*) \) and so, by Lemma 1 of [14], it belongs to \( \mathcal{V}(w) \). Thus by hypothesis, \( G/R \) is polycyclic-by-finite. Therefore by repeated use of Lemma 3 of [20], \( R \) is finitely generated. If \( R \) is finite then \( G \) is residually finite and so by Lemma 1 of [14], \( G \) belongs to the variety \( \mathcal{V}(w) \). Now suppose, for a contradiction, that \( R \) is infinite. By hypothesis, \( R \) has a normal proper subgroup of finite index in \( R \), then the finite residual subgroup \( T \) of \( R \) is proper in \( R \). Therefore \( R/T \) is a residually finite group in \( \mathcal{V}(w) \) and so \( G/T \) is polycyclic-by-finite. Thus \( G/T \) is residually finite and \( R \subseteq T \), a contradiction. This completes the proof. \( \square \)

The following proposition generalizes the result of [3].

Proposition 1. Finitely generated residually finite groups in a variety \( \mathcal{V} \) in which every finite group is nilpotent, are nilpotent.

Proof. We first prove that there exists a positive integer \( k \) depending only on the variety \( \mathcal{V} \) such that for all primes \( p \), \( C_p \text{wr} C_{p^k} \not\in \mathcal{V} \). By the Lemma of [13], there exists an integer \( t \) depending only on \( \mathcal{V} \) such that every 2-generated metabelian group in \( \mathcal{V} \) is nilpotent of class at most \( t \). Now suppose that \( C_p \text{wr} C_{p^m} \in \mathcal{V} \) for some prime \( p \) and positive integer \( m \). Since \( C_p \text{wr} C_{p^m} \) is a 2-generated metabelian group then it is nilpotent of class at most \( t \). But the nilpotency class of \( C_p \text{wr} C_{p^m} \) is exactly \( p^m \), by a result of Liebeck (see [25] or Theorem 2.5 in page 76 of [36]) and so \( p^m \leq t \). Now the same argument as
Let all finitely generated residually finite groups are polycyclic-by-finite. Therefore by Proposition 1, every variety in which all finitely generated soluble groups are nilpotent is contained in a variety in which all finitely generated residually finite groups are polycyclic-by-finite.

**Corollary 2.** Let \( w \) be a word in a free group such that finitely generated soluble groups in \( V(w) \) are nilpotent. Then every infinite locally graded restrained group in \( V(w^*) \) belongs to the variety \( V(w) \).

**Proof.** As noticed before, every finitely generated residually finite group in \( V(w) \) is polycyclic-by-finite. Let \( G \) be an infinite locally graded restrained group in \( V(w^*) \) and assume that \( w \) is a word in the free group of rank \( n \in \mathbb{N} \). Let \( x_1, \ldots, x_n \in G \), we must prove that \( w(x_1, \ldots, x_n) = 1 \). Assume that there exists an infinite finitely generated subgroup \( H \) of \( G \) which contains \( x_1, \ldots, x_n \). Then by Theorem A, \( H \in V(w) \). Now, we may assume that every finitely generated subgroup of \( G \) containing \( x_1, \ldots, x_n \) is finite. Thus there exists an infinite locally finite subgroup \( L \) which contains \( x_1, \ldots, x_n \) and so by Theorem 3 of [13], \( L \) belongs to the variety \( V(w) \). This completes the proof. \( \square \)

In the following lemmas we use some notion: we say that a word \( w \neq 1 \) in a free group is a semigroup word if \( w \) is of the form \( uv^{-1} \), where \( u \) and \( v \) are words in a free semigroup and we say, following [11], that a group \( G \) has no free subsemigroups if and only if for every pair \( (a, b) \) of elements of \( G \), the subsemigroup generated by \( a, b \) has a relation of the form

\[
(1) \quad a^{r_1}b^{s_1} \cdots a^{r_j}b^{s_j} = b^{m_1}a^{n_1} \cdots b^{m_k}a^{n_k}
\]

where \( r_i, s_i, m_i, n_i \) are all non-negative and \( r_1 \) and \( m_1 \) are positive integers. If \( (a, b) \) is a pair of elements in \( G \) satisfying a relation of type (1), then we call \( j + k \) the width of the relation and the sum \( r_1 + \cdots + r_j + n_1 + \cdots + n_k \) the exponent of \( a \) (denoted \( \exp(a) \)) in the relation.

We say that a word \( w \) in a free group \( F \) generated by \( x_1, \ldots, x_n \), is a commutator word whenever \( w \) belongs to the derived subgroup of \( F \). In the following we study infinite groups in \( V(w^*) \) where \( w \) is not a commutator word. We note that if \( w \) is not a commutator word then there is a positive integer \( e \) depending only on \( w \) such that every group in the variety \( V(w) \) is of exponent dividing \( e \); for let \( G \) be a group in the variety generated by a non-commutator word \( w \), since \( w \) is not a commutator word, for some \( i \) the sum of the exponents of \( x_i \) in \( w \) is non-zero: let this sum be \( r \) and let \( g \in G \). If we replace \( x_i \) by \( g \) and \( x_j \) by 1 when \( j \neq i \), then \( w \) assumes the value \( g^r \). Thus \( g \) has a finite order \( r \) and \( G \) is of finite exponent.

**Lemma 3.** Let \( w \) be a semigroup word in the free group of rank 2. Then every group in \( V(w^*) \) has no free subsemigroups, and there exist positive integers \( M \) and \( N \) depending only on \( w \) such that for all pairs \( (a, b) \) of elements in \( G \) there is a relation of the form (1) whose width and \( \exp(a) \) is at most \( M \) and \( N \), respectively.

**Proof.** Let \( a, b \) be in \( G \). If \( b \) is of finite order \( m \) then \( ab^m = b^m a, \exp(a) = 2 \) and the width is 2. Now, assume that \( b \) is of infinite order and consider the two sets \( X = \{ a^{b^n} \mid n \in \mathbb{N} \} \) and \( Y = \{ b^m \mid m \in \mathbb{N} \} \). If \( X \) is finite then the centre of \( H = \langle a, b \rangle \) is infinite and so by Lemma 3 of [14], \( H \) belongs to the variety \( V(w) \). Therefore \( w(a, b) = w(b, a) = 1 \) and so the pair \( (a, b) \) satisfies a relation of the form (1) whose width and \( \exp(a) \) is at most \( M_1 \) and \( N_1 \), respectively, where \( M_1 \) and \( N_1 \) are positive fixed integers depending only on \( w \). Now we may assume that \( X \) is infinite, then by the property \( V(w^*) \), there exists a relation of the form

\[
(a^{b^{r_1}})^{s_1} \cdots (a^{b^{r_j}})^{s_j} = b^{m_1}(a^{b^{r_1}})^{n_1} \cdots b^{m_k}(a^{b^{r_1}})^{n_k}
\]

where \( r_i, s_i, m_i, n_i \) are non-negative integers and \( r_1, m_1, t \) are positive integers; also the sum \( r_1 + \cdots + r_j + n_1 + \cdots + n_k \) is the same \( N_2 \) and \( j + k = M_1 \). Therefore the pair \( (a, b) \) satisfies a relation of the form (1) whose width is at most \( M := \max\{2, M_1\} \) and \( \exp(a) \) is at most \( N = \max\{2, N_1\} \). \( \square \)

Recall that a group \( G \) is right orderable if there exists a total order relation \( \geq \) on \( G \) such that for all \( a, b, g \) in \( G \), \( a \geq b \) implies \( ag \geq bg \), equivalently, if there exists a subset \( P \) in \( G \) such that \( PP = P, \) \( P \cup P^{-1} = G, \) and \( P \cap P^{-1} = 1 \).
Proposition 4. Let \( w \) be a semigroup word in the free group of rank 2. Then every right orderable group in \( \mathcal{V}(w^*) \), belongs to the variety \( \mathcal{V}(w) \).

Proof. By Theorem 5 of [30] and Lemma 3, \( G \) is locally nilpotent-by-finite. Let \( x_1, \ldots, x_n \in G \). Since \( G \) is right orderable, \( G \) is torsion-free. Thus every finitely generated subgroup of \( G \) is an infinite finitely generated nilpotent-by-finite group and so residually finite. Therefore by Lemma 1 of [14], \( G \) belongs to the variety \( \mathcal{V}(w) \). □

Lemma 5. Let \( w \) be a semigroup word in a free group. Then every group in \( \mathcal{V}(w^*) \) is restrained.

Proof. Let \( G \) be a group in \( \mathcal{V}(w^*) \) and let \( x, y \) in \( G \). We must prove that \( H = \langle x \rangle^{(n)} \) is finitely generated. We may assume that \( y \) is of infinite order. Suppose that \( w \) is in the free group of rank \( n > 0 \). Consider a partition of the set \( X = \{ xy^{-1}, xy^{-2}, \ldots \} \) in \( n \) infinite subsets \( X_1, X_2, \ldots, X_n \). Then by the property \( \mathcal{V}(w^*) \), there exist exist negative integers \( t_1, \ldots, t_n \) such that
\[
xy^{t_1} \cdots xy^{t_m} = xy^{t_1} \cdots xy^{t_s}
\]
for some functions \( f \) from \( \{1, 2, \ldots, m\} \) to \( \{1, 2, \ldots, n\} \) and \( g \) from \( \{1, 2, \ldots, s\} \) to \( \{1, 2, \ldots, n\} \), where \( m \) and \( s \) depend only on \( w \). Now, arguing as in Lemma 1(ii) of [20], \( H \) is finitely generated. This completes the proof. □

Lemma 6. Let \( w \) be a word in a free group such that \( w \) is not a commutator word. Then every group in \( \mathcal{V}(w^*) \) is torsion. In particular, \( G \) is restrained.

Proof. Let \( G \) be a group. Suppose, for a contradiction, that \( G \) has an element \( a \) of infinite order, then, by Lemma 3 of [14], \( \langle a \rangle \) belongs to the variety \( \mathcal{V}(w) \) and so \( a \) is of finite order, a contradiction. □

We note that Theorem A can be applied for the following words \( w \) in a free group: by Proposition 1 and the result of [7], any word \( w \) such that every finitely generated soluble group in the variety \( \mathcal{V}(w) \) is nilpotent; by Zelmanov’s positive solution to the restricted Burnside problem (see [16] and [17]), any non-commutator word \( w \) and by Theorem A of [20], every semigroup word \( w \). By Theorem A and Lemmas 5 and 6 and the above remarks we have

Corollary 7. Let \( w \) be a non-commutator word or a semigroup word in a free group. Then every infinite finitely generated locally graded group in \( \mathcal{V}(w^*) \) belongs to the variety \( \mathcal{V}(w) \).

Lemma 8. Let \( G \) be an infinite group in \( \mathcal{V}(w^*) \) and \( H \) be a finite subgroup of \( G \). If \( G \) has an infinite normal locally soluble subgroup, then \( H \) belongs to \( \mathcal{V}(w) \).

Proof. Let \( S \) be a normal locally soluble infinite subgroup of \( G \). If \( S \) is Černikov, then \( S \) has an infinite normal characteristic abelian subgroup (see [40] vol. I page 68) so \( G \) has an infinite normal abelian subgroup whence \( G \) belongs to \( \mathcal{V}(w) \) by Lemma 3 of [14]. Therefore, we may assume that \( S \) is not Černikov. By a result of Zaicev (see [43]), there is an infinite abelian subgroup \( B \) of \( S \) such that \( H \) normalizes \( B \). Hence \( B \) is an infinite normal subgroup of the group \( BH \) and so again by Lemma 3 of [14], \( H \) belongs to \( \mathcal{V}(w) \). □

Proof of Theorem B. It suffices to prove that an infinite [(locally \( \mathcal{P} \))-by-finite] group in \( \mathcal{V}(w^*) \) belongs to the variety \( \mathcal{V}(w) \). Let \( H \) be a normal locally \( \mathcal{P} \)-subgroup of \( G \) of finite index. If \( G \) is torsion, then \( G \) is locally finite and \( H \) is a locally soluble finite normal subgroup of \( G \), so by Lemma 8, \( G \in \mathcal{V}(w) \). Therefore we may assume that \( G \) has an element \( a \) of infinite order. Let \( x_1, \ldots, x_n \) be arbitrary elements of \( G \). Then \( K = \langle a, x_1, \ldots, x_n \rangle \) is a finitely generated \( \mathcal{P} \)-by-finite infinite group and so by condition (3), \( K \in \mathcal{V}(w) \). □

Corollary 9. Let \( G \) be an infinite locally finite \( \mathcal{V}(w^*) \) group. If \( G \) satisfies one of the following conditions, then \( G \) belongs to the variety \( \mathcal{V}(w) \).

1. \( G \) has an infinite locally soluble normal subgroup.
2. \( G \) contains an element with finite centralizer.
3. \( G \) contains an element of prime power order with Černikov centralizer in \( G \).
Proof. Let \( x_1, \ldots, x_n \) be arbitrary elements of \( G \), we must prove that \( w(x_1, \ldots, x_n) = 1 \). Since \( G \) is locally finite, \( H = \langle x_1, \ldots, x_n \rangle \) is finite.

If \( G \) has an infinite locally soluble normal subgroup, then, by Theorem B, \( H \in \mathcal{V}(w) \).

If \( G \) satisfies the conditions (2) or (3) then by Hartley’s results of \([32]\) and \([33]\) \( G \) is (locally soluble)-by-finite and so by part (1), the proof is complete. \( \square \)

Let \( w \) be a word in a free group. Now we state some reductions in investigation of the equality \( \mathcal{V}(w) \cup \mathcal{F} = \mathcal{V}(w^*) \) on the class of locally soluble groups and locally finite groups.

Let \( G \) be an infinite locally soluble group in \( \mathcal{V}(w^*) \). If \( G \) is torsion then by Corollary 9(1), \( G \) belongs to the variety \( \mathcal{V}(w) \). Therefore we may assume that \( G \) has an element \( g \) of infinite order and so in order to prove that \( G \in \mathcal{V}(w) \) it suffices to show that for all \( x_1, \ldots, x_n \), the infinite finitely generated soluble subgroup \( \langle x_1, \ldots, x_n, g \rangle \) belongs to the variety \( \mathcal{V}(w) \). Therefore we have

**Remark 10.** Let \( w \) be a word in a free group. Then the following are equivalent:

1. any infinite locally soluble group in \( \mathcal{V}(w^*) \) belongs to the variety \( \mathcal{V}(w) \).
2. any infinite finitely generated soluble group in \( \mathcal{V}(w^*) \) belongs to \( \mathcal{V}(w) \).

We note that, by Lemma 6, every finitely generated soluble group in \( \mathcal{V}(w^*) \), where \( w \) is not a commutator word, is finite.

Let \( G \) be an infinite locally finite group in \( \mathcal{V}(w^*) \). In order to prove that \( G \in \mathcal{V}(w) \), we must show that \( \langle x_1, \ldots, x_n \rangle \in \mathcal{V}(w) \) for all \( x_1, \ldots, x_n \in G \), therefore we may assume that \( G \) is countable. Fix \( x_1, \ldots, x_n \in G \) and let \( H = \langle x_1, \ldots, x_n \rangle \). If \( C_G(H) \) is infinite, then there is an infinite abelian subgroup \( A \) in \( C_G(H) \), as \( G \) is locally finite (see Theorem 3.43 of \([10]\)), therefore the centre of \( K = \langle A, H \rangle \) is infinite and so by Lemma 3 of \([4]\), \( K \in \mathcal{V}(w) \). Thus we may assume that \( C_G(H) \) is finite. Also, by Lemma 4 of \([4]\) and Corollary 9 we may assume that \( H \) is not supersoluble and the centralizer of any element in \( G \) is finite and the centralizer of every element of prime power order is not Černikov. These conditions on a locally finite group lead us to the following definitions.

We say that a group \( G \) is an \( L \)-group whenever \( G \) is an infinite countable locally finite group and there exists a finite subgroup \( H \) of \( G \) such that

1. \( H \) is not supersoluble and \( C_G(H) \) is finite.
2. \( C_G(x) \) is infinite for all \( x \in G \).
3. \( C_G(g) \) is not Černikov for all elements \( g \in G \) of prime power order.
4. the largest normal locally soluble subgroup of \( G \) is finite.

In this case, we say that \( G \) is an \( L \)-group with respect to \( H \). Also, we say that \( G \) is an \( L^* \)-group with respect to \( H \) whenever every infinite subgroup of \( G \) which contains \( H \), is an \( L \)-group with respect to \( H \).

By these discussions we have

**Remark 11.** Let \( w \) be a word in a free group. Then the following are equivalent:

1. an infinite locally finite group in \( \mathcal{V}(w^*) \), belongs to the variety \( \mathcal{V}(w) \).
2. an infinite \( L^* \)-group in \( \mathcal{V}(w^*) \), belongs to the variety \( \mathcal{V}(w) \).

We use Remark 11 for the study of an infinite locally finite group \( G \) in \( \mathcal{V}(w^*) \) where \( w \) is not a commutator word in the free group of rank \( n > 0 \), and obtain another condition on such groups \( G \). We prove that \( G \) is of finite exponent dividing \( e \), where \( e \) is a positive integer depending only on \( w \) such that every group in the variety \( \mathcal{V}(w) \) is of exponent dividing \( e \). For, let \( a \) be an element of \( G \), then \( C_G(a) \) is infinite and by Theorem 3.43 of \([10]\) there exists an infinite abelian subgroup \( A \in C_G(a) \). By Lemma 3 of \([4]\), \( A \in \mathcal{V}(w) \). Consider infinite subsets \( X_1 = \ldots = X_n = aA \). Therefore, by the property \( \mathcal{V}(w^*) \), there exist \( a_1, \ldots, a_n \in A \) such that \( w(a_1, \ldots, a_n) = 1 \). Thus \( w(a, \ldots, a)w(a_1, \ldots, a_n) = 1 \). But \( w(a_1, \ldots, a_n) = 1 \) and so \( w(a, \ldots, a) = 1 \) and \( a^e = 1 \). Therefore we have:

**Remark 12.** Let \( w \) be a non-commutator word in a free group and \( e \) be a positive integer depending only on \( w \) such that every group in the variety \( \mathcal{V}(w) \) is of exponent dividing \( e \). Then the following are equivalent:

1. any infinite locally finite group in \( \mathcal{V}(w^*) \) belongs to the variety \( \mathcal{V}(w) \).
2. any infinite \( L^* \)-group of exponent dividing \( e \) belongs to the variety \( \mathcal{V}(w) \).
A natural question which arises is the following: Is there an infinite $L^*$-group of finite exponent? We only know that such a group is not simple. For by a result of L. G. Kovács [22], any infinite, simple, locally finite group $G$ involves infinitely many non-isomorphic non-abelian finite simple groups; hence, if $G$ satisfies non-trivial laws, then according to a result of G. A. Jones (see Theorem of [18]), the variety generated by infinitely many finite simple groups is the variety of all groups. But the variety generated by $G$ is a proper variety, a contradiction.

Now we study infinite simple locally finite groups in $\mathcal{V}(w^*)$, where $w$ is a non-trivial word in a free group. As we have seen earlier, there is no infinite simple locally finite group which satisfies a non-trivial identity. Call a simple locally finite group an $S$-group. The $S$-groups fall into two classes with widely different properties—the linear groups and the non-linear groups. Every linear $S$-group is a group of Lie type over an infinite locally finite field (see [34]).

Proof of Theorem C. Suppose, for a contradiction, there exists a non-linear $S$-group $G$ in $\mathcal{V}(w^*)$. By a result of Hartley [32], there exists a section $C/D$ of $G$ such that $C/D$ is a direct product of finite alternating groups of unbounded orders. Thus $C/D$ is an infinite residually finite group in $\mathcal{V}(w^*)$ and so $C/D$ belongs to the variety $\mathcal{V}(w)$. Since $C/D$ is a direct product of finite alternating groups of unbounded orders, the variety $\mathcal{V}(w)$ contains infinitely many non-isomorphic finite alternating groups. Therefore, by Theorem of [18], $\mathcal{V}(w)$ is the variety of all groups and so $w$ is the trivial word, a contradiction. This completes the proof.

P. S. Kim in [19] studied $\mathcal{V}(w_2^*)$ on the class of locally soluble groups, where $w_2 = [[x_1, x_2], [x_3, x_4]]$. For this word the variety $\mathcal{V}(w_2)$ is the variety of metabelian groups. It is proved in [19], that every infinite locally soluble group in $\mathcal{V}(w_2^*)$ is metabelian and also it is proved that any infinite group belonging to $\mathcal{V}(w_2^*)$ is metabelian if and only if there is no infinite simple group in $\mathcal{V}(w_2)$. We study $\mathcal{V}(w^*)$ on the class of locally finite groups, where $w$ is a soluble word that is $w = w_d$ for some $d \in \mathbb{N}$ where $w_0 = x$, $w_i = [w_{i-1}, w_{i-1}]$ and $w_{i-1}$ is the word on $2^{i-1}$ distinct letters which has been defined inductively, for all $i \in \mathbb{N}$.

Corollary 13. Let $w$ be a soluble word and let $G$ be an infinite locally finite $\mathcal{V}(w^*)$-group. Then the following are equivalent:

1. $G \in \mathcal{V}(w)$.
2. $G$ has no infinite linear simple locally finite section.

Proof. Suppose that (1) is true. Then $G$ is soluble and (2) is clear. Now suppose that (2) is true and $w = w_d$ for some positive integer $d$. Suppose, for a contradiction, that $G \notin \mathcal{V}(w)$. Thus $G$ is not soluble of derived length at most $d$. Suppose, if possible, that $K = G^{(d+1)}$ is finite. Then $H = G^{(d)}$ is an FC-group and so $H$ is soluble by applying suitably Lemma 1 of [4]. Thus $G$ is a torsion soluble group and so by Theorem B, $G$ is soluble of derived length at most $d$, a contradiction. Hence $K$ is infinite and so $G/K$ is a soluble group of derived length $d$. Therefore $G^{(d)} = G^{(d+1)}$ that is $H = H'$, which implies that $H$ is a perfect group. Suppose that $H$ has an infinite proper normal subgroup $N$, then $H/N$ is soluble of derived length at most $d$, this implies $H = H' = N$ since $H$ is perfect, a contradiction. Let $N$ be a finite normal subgroup of $H$, then $C_H(N)$ has finite index in $H$. Since $H$ has no infinite normal proper subgroups, $C_H(N) = H$. Hence the centre $Z$ of $H$ is the unique maximal normal subgroup of $H$ so that $S = H/Z$ is simple. By Theorem C, $S$ is an infinite linear simple locally finite group, which is a contradiction.

Now, we start proving Theorem D, for this we need the following lemma:

Lemma 14. Every infinite locally soluble group of finite rank in $\mathcal{V}(w^*)$ belongs to the variety $\mathcal{V}(w)$.

Proof. Let $G$ be an infinite locally soluble group of finite rank in $\mathcal{V}(w^*)$. By Remark 10, we may assume that $G$ is finitely generated. Therefore $G$ is a minimax group, and so by Theorem 10.33 of [40], the finite residual $R$ of $G$ is the direct product of finitely many quasicyclic subgroups of $G$, thus $G$ is residually finite or $G$ has an infinite normal abelian subgroup, then, by Lemma 1 or Lemma 3 of [14] respectively, the proof is complete.

Proof of Theorem D. Let $G$ be an infinite locally soluble group in $\mathcal{V}(w^*)$. By Remark 10, we may assume that $G$ is a finitely generated infinite soluble group. Firstly, suppose that $w$ is a word such that $C_p \text{ wr } C_\infty \notin \mathcal{V}(w)$ for all primes $p$. We prove that $G$ is a minimax group and so $G$ is of finite rank, then
Lemma 14 completes the proof.

By a deep result of Krogholler (see [23]), which asserts that every finitely generated soluble group having no sections of type $C_p \wr C_\infty$ is minimax, it suffices to show that if $C_p \wr C_\infty \in V(w)$ then $C_p \wr C_\infty \in V(w)$. But $C_p \wr C_\infty$ has an infinite normal abelian subgroup, therefore by Lemma 3 of [19], $C_p \wr C_\infty \in V(w)$, which is a contradiction.

Now, suppose that $w$ is a word such that every infinitely presented $M(\alpha, p) \notin V(w)$. If $G$ is not semipoly-cyclic group (see [14]), then there exists a subgroup $H$ of a quotient group of $G$ which is isomorphic to an infinitely presented $M(\alpha, p)$. But $M(\alpha, p)$ is an infinite residually finite group in $V(w^*)$ and so $M(\alpha, p) \in V(w)$, a contradiction. Therefore $G$ is semi-polycyclic and so is of finite rank (see [14]). Thus, by Lemma 14, $G \in V(w)$. This completes the proof. □
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