PROPERTIES OF RAPIDLY ROTATING PNS WITH ANTIKAON
CONDENSATES AT CONSTANT ENTROPY

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Abstract. Neutron stars have super nuclear density. Even though the mass of a
neutron star is well constrained, the uncertainty in its radius gives a lot of parameter
space for the coexistence of different equations of state (EoS). In this work, we derive
the EoS for a neutrino-free proto-neutron star (PNS) that is at finite entropy. The
core includes antikaon condensates as a constituent. We find EoS for a range of
entropies and antikaon optical potentials. In a non-rotating compact star, the mass-
radius sequence relation has been well established through Tolman-Oppenheimer-
Volkoff equation. We study the change of mass and constituents of a static PNS star
for different parameters and thermodynamic conditions. However rotation induces
many changes in the star equilibrium and its structural properties evolve. In this
work, we also study the effect of rapid rotation on the shape of PNS that is at
finite-entropy and the core of which contains antikaon condensates.

1. Introduction

Neutron star (NS) core matter is super dense, where matter density ($n_b$) can reach
upto a few times normal nuclear matter density ($n_0$), which is unlike anything found
on earth. The behaviour of matter upto nuclear densities has been studied and is well
documented by numerous nuclear physics experiments. Lack of data at higher nuclear
matter densities means we do not completely understand how matter behaves at super
nuclear densities. Consequently, there exist huge uncertainties in understanding the
behaviour of super dense matter such as those found in NS cores.

NS are the nature’s laboratories for studying such matter. One indirect way to
understand highly dense matter properties is by knowing the Mass-Radius relation of
a NS. The NS mass is fairly established by many observational studies, with maximum
mass observed till now to be for neutron stars PSRs J1614-2230 and J0348+0432 with
masses $1.928 \pm 0.017$ and $2.01 \pm 0.04 M_{\text{Solar}}$ respectively [1, 2, 3]. Any EoS intending to
describe NS core matter must be able to reach upto this level. This constraint in itself
rules out many of the proposed EoS for NS core matter. This alone is insufficient to
rigidly pinpoint towards the EoS and a knowledge of radius of a NS observationally
is simultaneously required [1, 5]. This is the major hindrance in knowing EoS of such
matter using observational mass-radius studies.

For the matter whose density is just below nuclear density, its major constituents
are protons, neutrons and electrons. The muons begin to appear as soon as the
density reaches the nuclear limit. However we do not know for certain what the
matter constituents are. We also are not sure about the interactions between them

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when the matter density surpasses nuclear limit. Many studies have theorised the appearance of hyperons at higher matter densities\cite{6,7}. It has also been suggested that Bose-Einstein condensates such as those of pions and/or antikaons are favourable to appear at highly dense NS cores\cite{8,9}. The appearance of new particles such as antikaon condensates results in the softening of matter EoS as negatively charged leptons are replaced by the slow-moving massive condensates, that do not contribute to pressure. The overall pressure thus increases less steeply with density thereby resulting in a softer EoS. This has the effect of lowering of maximum mass reached in a NS.

A NS is born in a core-collapse SN explosion, which is believed to be adiabatic, i.e. the entropy per baryon of each mass element remains constant during the collapse except for during the passage of the shock \cite{10}. In this work we are interested in studying such a proto-neutron stars(PNS) that has been born in an adiabatic environment. The temporal evolution of proto-neutron star (PNS) has been thoroughly studied by Pons et. al \cite{11}. We assume an isoentropic profile for the hot star. Following deleptonisation it has a high entropy\cite{11}. However, it has yet to cool down to Fermi temperature in its super-dense core that contains exotic matter such as antikaon condensates.

NS is a rotating entity rather than a stationary one. Till date $\sim 2500$ pulsars have been observed in our galaxy\cite{12}, many of them can have a period of as low as a millisecond, or a frequency that is greater than 500 Hz. The fastest known pulsar, PSR J1748-2446ad, that has been observed to date was discovered \cite{13} in 2004. It has a rotation period of 1.397 ms and frequency of 716 Hz. PNS may rotate with higher frequency than this or the NS may gain rotational velocity by accreting material from a companion of the X-ray binary. The frequency is limited by mass-shedding phenomenon only. At mass shedding limit a rigidly-rotating NS rotates with maximum frequency which is defined as its Kepler frequency.

In general, the equilibrium of a rotating NS depends considerably on rotational effects. The mass-radius (M-R) relation for a static NS has been established theoretically by the well known Tolman-Oppenheimer-Volkoff (TOV) equations which gives the upper bound to the mass of a static NS. The internal structure of a rotating NS changes as compared to a non-rotating star. As the centrifugal force increases with the rotational velocity, a rotating star can support larger mass compared to a static one. Also, the rotating stars tend to have larger radii\cite{14}. This change is not only due to the appearance of centrifugal force but also due to what is known as ‘frame-dragging’ of inertial reference frames\cite{6,15}. The maximum mass reached in a sequence is a function of the constituent composition as well as the temperature profile inside a NS and is said to be a strong indicator of the underlying EoS.

In this work we study the rotating NS sequences with EoS containing different exotic particles and entropy ranges. We also compare the same with their static configuration and also with the nucleon only matter constitution. Further, we study the relativistic equilibrium configurations of rotating PNS with different EoS and entropies in terms of their enthalpy profiles and how they are affected by the change in rotation frequency upto Keplerian limits. Just after its birth in a core-collapse Supernovae mechanism, the PNS is expected to be differentially rotating, due to lack of enough viscous forces soon-after explosion \cite{16,17}. In the present work, we however consider the PNS after it has deleptonised and the rigid rotation has set in due to viscosity but wherein the temperature has not cooled down. In this paper we stick to
rigid rotation of a PNS about an axisymmetric axis. We also make a rough estimate of the GW amplitude for a PNS with strong magnetic field. The paper is organised the following way. In section 2 we describe our model EoS of compact star, both core and crust. In section 2.3 we discuss the rotation and axisymmetric instabilities. Section 3 contains our results and the related explanation. Finally, in section 4 we conclude with a summary of the results achieved and further research work being done in continuation.

2. Models of Compact stars

2.1. Equation of state of core. We consider nuclear and $K^-$ condensed matter in the dense interior of neutron star and calculate the equation of state in the framework of relativistic mean field (RMF) model with density-dependent coefficients. The nucleons of mass $m_N$, denoted by spinors $\psi_N$ interact through exchange particles $\sigma$, $\omega$, and $\rho$ mesons. The density-dependent RMF model Lagrangian density is given by [18]

$$\mathcal{L}_N = \sum_N \bar{\psi}_N \left( i\gamma^\mu \partial_\mu - m_N + g_{\sigma N}\sigma - g_{\omega N}\gamma^\mu\omega_\mu - g_{\rho N}\gamma^\mu\tau_N \cdot \rho^\mu \right) \psi_N$$

$$+ \frac{1}{2} \left( \partial_\mu \sigma \partial_\nu \sigma - m_\sigma^2 \sigma^2 \right) - \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu$$

$$- \frac{1}{4} \rho_{\mu\nu} \cdot \rho^{\mu\nu} + \frac{1}{2} m_\rho^2 \rho_\mu \cdot \rho^\mu. \quad (1)$$

Here $\omega^{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$ and $\rho^{\mu\nu} = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu$ are the field strength tensors for the vector mesons, and $\tau_N$ is the isospin operator.

The meson couplings are dependent on the Lorentz scalar functionals $\hat{j}_\mu = \bar{\Psi} \gamma_\mu \Psi$ and are determined following the prescription of Typel et. al [19, 20]. They reproduce the bulk properties of nuclear matter at saturation density, nuclear compressibility, symmetry energy and its slope parameter corresponding to the density dependence of symmetry energy [18, 21]. The details of the parameters used in this calculation are found in ref [18]. In the mean field approximation, the nucleon-meson couplings become function of total baryon density $n$ i.e. $< g_{\alpha N}(\hat{n}) >= g_{\alpha N}(<\hat{n}>) = g_{\alpha N}(n)$ [20, 22]. The density-dependence of the meson-baryon couplings [22] gives rise to rearrangement term $\Sigma^{(r)}$ given by

$$\Sigma^{(r)} = \sum_N \left[ -g_{\sigma N}^' \sigma n_N^\ast + g_{\omega N}^' \omega_0 n_N + g_{\rho N}^' \tau_3 N \rho_0 n_N \right]. \quad (2)$$

Here $g_{\alpha N}^' = \frac{\partial g_{\alpha N}}{\partial n_\alpha}$, $\alpha = \sigma$, $\omega$, $\rho$ and $\tau_3 N$ is the isospin projection of $N = n, p$. We consider a static and isotropic star to compute the dense matter equation of state. In mean field approximation the meson fields are replaced by their expectation values and the only non-vanishing meson field components are $\sigma_0$, $\omega_0$ and $\rho_0$ where $m_\rho^2 \rho_0 = \frac{1}{2} g_{\rho N} \left( n_p - n_n \right)$. $n_p$ and $n_n$ are the number densities of proton and neutron respectively.

The number density of nucleon at finite temperature is given by

$$n_N = 2 \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{e^{\beta(E-n_N)} + 1} - \frac{1}{e^{\beta(E+y_N)} + 1} \right). \quad (3)$$
The Dirac equation for the interacting nucleons is given by \[\gamma_\mu (i\partial^\mu - \Sigma_N^{(r)}) - m_N^* \psi_N = 0\]. The effective nucleon mass is defined as \[m_N^* = m_N - g_{\sigma N} \sigma\]. The chemical potential for the nucleon is \[\mu_N = v_N + g_{\omega N} \omega_0 + g_{\rho N} \tau_3 N \rho_{03} + \Sigma_N^{(r)}\]. Next we calculate, the pressure due to nucleons as \[P_N = -\frac{1}{2} m_N^* \omega_0^2 + \frac{1}{2} m_N^* \omega_2^2 + \frac{1}{2} m_N^* \rho_{03}^2 + \Sigma_N^{(r)} \sum_N n_N + 2T \sum_{N=\nu,\bar{\nu}} \int \frac{d^3k}{(2\pi)^3} \left[ \ln(1 + e^{\beta (E - \nu \omega_0)}) + \ln(1 + e^{\beta (E + \nu \omega_0)}) \right]\] (5).

The explicit form of the energy density is given below,

\[\epsilon_N = \frac{1}{2} m_N^* \omega_0^2 + \frac{1}{2} m_N^* \omega_0^2 + \frac{1}{2} m_N^* \rho_{03}^2 + 2 \sum_{N=\nu,\bar{\nu}} \int \frac{d^3k}{(2\pi)^3} E^* \left( \frac{1}{e^{\beta (E - \nu \omega_0)} + 1} + \frac{1}{e^{\beta (E + \nu \omega_0)} + 1} \right)\].

The rearrangement term does not contribute to the energy density explicitly, whereas it occurs in the pressure through baryon chemical potentials. It is the rearrangement term that accounts for the energy-momentum conservation and thermodynamic consistency of the system \[[22]\].

The antikaon \(K^-\) is described by the Lagrangian density \[\mathcal{L}_K = D^\mu \bar{K} D_\mu K - m_K^2 \bar{K} K\], where the covariant derivative is \[D_\mu = \partial_\mu + ig_{\omega K} \omega_\mu + ig_{\rho K} \tau_3 \rho_{03}\] and the effective mass of (anti)kaons is given by \[m_K^* = m_K - g_{\sigma K} \sigma\] and \(m_K\) is the bare kaon mass. The net (anti)kaon number density is given by \(n_K = n_K^c + n_K^\tau\), where \(n_K^c\) gives the condensate density and \(n_K^\tau\) represents the thermal density given by,

\[n_K^c = 2 \left( \omega_{K^-} + g_{\omega K} \omega_0 + \frac{1}{2} g_{\rho K} \rho_{03} \right) \bar{K} K = 2 m_K^* \bar{K} K\] (7).

\[n_K^\tau = \int \frac{d^3p}{(2\pi)^3} \left( \frac{1}{e^{\beta (\omega_{K^-} - p)} - 1} - \frac{1}{e^{\beta (\omega_{K^-} + p)} - 1} \right)\]. (8)

For s-wave (\(p = 0\)) condensation, the in-medium energies of \(K^-\) is given by \[\omega_{K^-} = m_K^* - g_{\omega K} \omega_0 - g_{\rho K} \rho_{03}\]. The requirement of chemical equilibrium fixes the onset condition of antikaon condensations in neutron star matter.

\[\mu_n - \mu_p = \mu_{K^-} = \mu_e\],

where \(\mu'_e\) in this equation are the chemical potentials of neutron, proton, \(K^-\) and electron respectively. The antikaon condensates implicitly change the rearrangement term of Eq. 2 via the values of the meson fields.
The condensate does not contribute to the pressure. The energy density of (anti)kaons is given by

\[ \epsilon_K = m_K^* n_K^* + \left( g_{\omega K} \omega_0 + \frac{1}{2} g_{\rho K} \rho_0 \right) n_K^T + \int \frac{d^3 p}{(2\pi)^3} \left( \frac{\omega_{K^*}}{e^{\beta (\omega_{K^*} - \mu_{K^*})} - 1} + \frac{\omega_{K^-}}{e^{\beta (\omega_{K^-} + \mu_{K^-})} - 1} \right). \]  

(10)

The first term is the contribution due to the condensate and second and third terms are the thermal contributions to the energy density in \( \epsilon_K \).

In addition to nucleons and \( K^- \) mesons, we have leptons in the system. They are treated as non-interacting particles and relevant physical properties for equation of state i.e. number densities, energy densities and pressure are calculated following the similar method used for nucleons from the Lagrangian density \( \mathcal{L}_i = \sum_l \bar{\psi}_l \left( i\gamma_\mu \partial^\mu - m_l \right) \psi_l \). Here \( \psi_l \) (\( l \equiv e, \mu \)) denotes the lepton spinor. The total energy density in the presence of \( K^- \) condensates is therefore, \( \epsilon = \epsilon_n + \epsilon_K + \epsilon_l \).

We generate the equation of state at constant entropy. The entropy of nucleons and leptons is related to energy density and pressure through Gibbs-Duhem relation \( \mathcal{F}_i = \beta (\epsilon_i + P_i - \sum_l \mu_l n_l) \), where i=n, p, e, \( \mu \). The entropy density of (anti)kaons is \( \mathcal{F}_K = \beta (\epsilon_K + P_K - \mu_K n_K) \), where \( n_K = n_K^* + n_K^T \).

2.2. Equation of state of crust. We use the crust equation of state of Hempel and Schaffner-Bielich, calculated in the extended Nuclear statistical equilibrium model(NSE) [25]. The crust consists of non-uniform matter of light and heavy nuclei along with unbound nucleons at low temperature and densities below nuclear saturation. Interaction among the unbound nucleons are described by considering the same Lagrangian density of Eq. 1 and density dependent formalism [20-25]. As the \( K^- \) condensates appear only at high density and relatively high temperature, the nuclei and exotic matter are never found to coexist. Therefore, we simply use the crust of HS(DD2) EoS [25,26] upto density of 0.088 fm\(^{-3}\). However this EoS table is for supernova for a wide range of temperature, baryon number density and electron fraction. We impose an additional condition of \( \beta \) equilibrium on the chemical potentials \( \mu_n = \mu_p = \mu_e \) for a given temperature and baryon number density, the electron fraction is determined by finding the zero of the function \( f(Y_e) \) at a fixed entropy [27]. We finally add the crust to our constant entropy core EoS.

2.3. Rotation and axisymmetric instabilities. To compute and compare the hydrostatic equilibrium configurations of rotating neutron stars with DD2 EoS as described above, we use nrotstar code in the numerical library LORENE, which implements multi-domain spectral method for calculating accurate models of rotating neutron stars in full general relativity [28].

In this formalism, field equations are derived using 3+1 formulation. This forms a system of four elliptic partial differential equations, which are then solved numerically using the self-consistent-field method. While solving, the spacetime is assumed to be asymptotically flat and axisymmetric. The corresponding metric that obeys these partial differential equations is given by:

\[ g_{\alpha\beta} dx^\alpha dx^\beta = -N^2 dt^2 + A^2 \left( dr^2 + r^2 d\theta^2 \right) + \beta^2 r^2 sin^2 \theta (d\phi - \omega dt)^2. \]  

(11)

The metric tensor is thus fully specified by four functions (metric potentials): \( N(r,\theta) \), \( A(r,\theta) \), \( B(r,\theta) \), \( w(r,\theta) \) in spherical polar coordinates \((t, r, \theta, \phi)\). The accuracy of...
Table 1. The value of baryon density \( n_b \) in \( \text{fm}^{-3} \) when \( K^- \) condensate starts appearing in PNS core. This is listed for different values of \( U_{\bar{K}} \) and also for different thermodynamic states inside the core.

| \( U_{\bar{K}} \) | T=0 MeV | S=1 \( k_B \) | S=2 \( k_B \) | S=3 \( k_B \) | S=4 \( k_B \) | S=5 \( k_B \) |
|---------------|---------|----------------|----------------|----------------|----------------|----------------|
| -60           | 0.613   | 0.687          | 0.868          | 1.129          | 1.396          | -              |
| -80           | 0.558   | 0.622          | 0.777          | 1.006          | 1.245          | -              |
| -100          | 0.507   | 0.559          | 0.690          | 0.889          | 1.097          | 1.312          |
| -120          | 0.459   | 0.502          | 0.608          | 0.777          | 0.957          | 1.126          |
| -140          | 0.416   | 0.449          | 0.535          | 0.674          | 0.828          | 0.969          |
| -150          | 0.395   | 0.425          | 0.449          | 0.626          | 0.765          | 0.896          |

the solutions obtained using nrotstar is checked using the general relativistic virial theorem GRV3 and GRV2 [29], which gives a typical value of \( \sim 10^{-4} \).

Lorene/nrotstar is formulated primarily for cold EoS, or a barotropic EoS [30]. Our EoS are temperature dependent but they are formulated so as to have constant entropy. This results in a homoeotropic flow, thereby making the EoS barotropic. Thus as long as we have EoS that is isoentropic, Lorene formalism can be used to do the calculations.

Using Lorene/nrotstar we compute stable neutron star configurations for different EoS as are described in earlier section. We measure the change in mass radius profile as the neutron star is rotating at different frequencies. We also study quadrupole moment. Finally, we make an estimate of the strength of gravitational wave, that can be emitted from a uniformly rotating neutron star as it is rotating at near Keplerian frequency, and is on the verge of the onset of secular instabilities.

3. Result

We have generated a number of isoentropic equation of state (EoS) profiles and calculated the properties of a reasonably rapidly rotating and deleptonised PNS using DD2 model. We have considered nucleons-only system consisting of proton, neutron, electron and muon and denote it by "np". The matter when consists of antikaon condensates (\( K^- \)) and thermal kaons(\( K^T \)), are denoted by "npK". The value of the antikaon optical potential(\( U_{\bar{K}} \)) at normal nuclear matter density has been calculated by several methods such as in coupled-channel model, chiral analysis of \( K^- \) atomic and scattering data, but they have no definite consensus [22, 31]. However, it is definitely attractive as suggested by all the studies. We choose a wide range for \( U_{\bar{K}} \) from a shallow value of -60 to a deeper one of -150 MeV [22] in this work. The coupling constants for kaons at the saturation density for different values of \( U_{\bar{K}} \) in the DD2 model are listed in Table 2 of Ref [22].

We plot equation of state (EoS) i.e. pressure versus energy density in left panel of Fig. [1]. The dashed lines are for nucleons only matter(np), while the solid lines represent matter that also contains \( K^- \) condensates and thermal kaons(npK) at \( U_{\bar{K}} = -100 MeV \). The set of npK EoS is softer compared to np matter. Also, for a given composition and antikaon optical potential (\( U_{\bar{K}} \)), the EoS is softer for a higher entropy state. We also compare the EoS for entropy values of S=1, 4 and S=5\( k_B \) MeV with T=0. The S=1 EoS does not change much from T=0 for np case, but the difference is more evident for npK.
The NS mass profiles are a reflection of its EoS profile. In order to compare the rotating neutron star configuration, we need to know the result of the static star for the same set of parameters. We solve Tolman-Oppenheimer-Volkoff equations for a static star with our sets of EoS of Fig. 1. In the right panel of Fig. 1, we plot the mass-central energy density profile for np and npK ($U_K = -100\text{MeV}$) matter for different entropy values. The mass and number density ($n_b$) corresponding to the maximum mass stars are listed in Table 2 along with other values of $U_K$ and entropy ranges $S=1-5k_B$. The dashed lines are for np matter, whereas the solid lines are for npK matter. As the np EoS for $T=0$ and $S=1k_B$ do not differ much, the mass profiles also look similar. However, the colder stars have a denser core, whereas the higher entropy stars have less dense core. For a given $U_K$, a higher entropy star increases in mass (slightly) and in central density (significantly). We also study the radius of the corresponding maximum mass star and notice that the maximum mass (and the corresponding radius) increases from $2.297M_{\odot}(12.15\text{km})$ to $2.768M_{\odot}(35.846\text{km})$ as we move from a cold star ($T=0$ MeV) to isentropic star of entropy $S=5k_B\text{MeV}$. However, there is large uncertainty over determination of radius, as pressure at finite temperature never really becomes zero.

In Fig. 2, we compare np(dashed lines) and npK(solid lines) EoS for entropy values $S=1$ and $4k_B\text{MeV}$. At the onset of $K^-$ condensates, the slope of the EoS changes. The density at which the condensates appear are listed in Table 1. For $U_K = -100\text{MeV}$ $K^-$ condensates appear at 0.507 $n_b$ for a cold star whereas their appearance is delayed until 1.132$n_b$ for a star at $S=5k_B$. We have considered a range of optical potentials from -60 to -150 MeV for $K^-$ condensates in a nuclear medium. $K^-$ does not appear at well for a higher entropy system, unless the optical potential is deep i.e. $|U_K| \geq 100$ MeV. The EoS is softer for deeper potential well. At higher entropy, the EoS fans out at higher matter densities, as compared to lower entropy where the difference in EoS for different $U_K$ was more apparent at lower densities as compared to higher densities.

Mass versus number density profiles of a PNS obeying np and npK EoS of Fig. 2 are plotted in Fig. 3. The dashed lines are for np matter and solid lines are for npK matter with different $U_K$. As before the EoS behaviour is reflected in mass profiles. The maximum gravitational mass and the corresponding number density values are listed in Table 2. For the same thermodynamic condition, a softer EoS (deeper $U_K$) makes a lower maximum mass star, whereas the number density remains almost the same.

In Fig. 4 we plot the particle fractions, for a shallow $U_K=-60\text{MeV}$. At low density the star contains neutrons, protons, electrons and muons. At higher densities, the threshold condition; $\mu_{K^-} = \mu_{n} - \mu_{p} = \mu_e$ is satisfied and $K^-$ appears. The lepton fraction falls off as soon as the negatively charged condensates populate. The Bose-Einstein condensates do not contribute to the pressure and is energetically favourable than the leptons. We compare particle fractions for $S=1$ and $4k_B\text{MeV}$ in the 2 panels of Fig. 4. The dashed line is for thermal kaons $K_T^0$ while the dot-dashed line represents $K^-$ condensates. Thermal kaons are populated well before the onset of $K^-$ condensates. For higher entropy, $K_T^0$ appears at low density of matter, pushing $K^-$ condensates to a higher density. In Fig. 5 the particle fraction is drawn for a deeper $U_K=-150\text{MeV}$. A similar trend is noticed here as well. However, $K^-$ populates at lower density compared to $U_K = -60\text{MeV}$. The threshold densities for onset of $K^-$ condensation for the range of entropy and optical potential $U_K$ are listed in Table 1. Comparing the left
and right panels of Figs. 4 and 5 we also observe that at the onset of $K^-$ condensates, the proton fraction increases rapidly for $S=1$, which is not so pronounced for $S=4$.

In Fig. 6, temperature versus number density is plotted for entropy values $S=1$ and $S=4\, k_B\, \text{MeV}$. We compare np, and npK matter at shallow ($U_K= -60\, \text{MeV}$) and extreme potentials ($U_K= -150\, \text{MeV}$). The dashed line represents np matter whereas solid lines are for npK matter. For a fixed entropy per baryon, the temperature increases with baryon density. Or in other words, the temperature falls off from the core of the star to its surface. The temperature differs with EoS in higher density regions of the star.

For an isoentropic star at $S=4\, k_B\, \text{MeV}$, it can rise up to 150 MeV compared to 50 MeV for a star at $S=1\, k_B$. We notice the kinks in the npK lines, which mark the appearance of $K^-$ condensates. Also, the core temperature is less for an np EoS compared to npK EoS, for lower entropy star. For $S=1k_B$, $K^-$ appears at lower density, also its fraction is higher for npK EoS with $U_K= -150\, \text{MeV}$ compared to that with $U_K= -60\, \text{MeV}$. Thus the temperature rises higher for npK with $U_K= -150\, \text{MeV}$, than np and npK ($U_K= -60\, \text{MeV}$). For npK ($U_K= -60\, \text{MeV}$), leptons get depleted at the onset of thermal kaon (at $n_b= 0.43\, \text{fm}^{-3}$) and the temperature runs lower than that of np, only to cross it up as $K^-$ condensates appear at $0.687\, \text{fm}^{-3}$. For a higher entropy star the nature of temperature curve is quite the opposite, here core temperature of PNS with np EoS is higher than that for npK. The same reasoning applies here. As is evident from 4 and 5, the lepton fraction falls off as soon as the thermal condensates populate the star. However, the threshold density of $K^-$ condensation is pushed off to higher density. In fact for $U_K= -60\, \text{MeV}$ the core does not even contain $K^-$ condensates. The core contains neutron, proton and the thermal kaons only. It may be worth mentioning here that at critical temperature, the antikaons melt down\cite{32}. It was shown in Ref \cite{32} that the critical temperature of antikaon condensation is enhanced as baryon density increases.

Next, we derive rotating PNS configurations using Lorene/nrotstar. This is done to see the effect of antikaon optical potential in the stellar configuration while keeping all other parameters constant. Also, there is ambiguity over how fast a PNS rotates. Most of the pulsars discovered so far, spin at a frequency of 1 Hz. The pulsar PSR J1748-2446ad is the fastest known pulsar till date rotating with a frequency of 716 Hz. We study the deformation of the PNS rapidly rotating with frequency upto the Keplerian limit.

Fig. 7 shows the evolution of mass profiles with rotation rate. We plot the changing behaviour of mass-central number density relation from a static PNS, a rotating PNS with frequency 500 Hz and 700 Hz to one that is rotating at mass-shedding limit. We have used Lorene to obtain these mass sequences for EoS with npK ($U_K= -100\, \text{MeV}$). The results in the two panels are for $S=1$ and $4k_B$ MeV. As we go from zero to the Keplerian frequency, the increase in mass is much larger than for a low entropy star than a high entropy star. In the above figures, the dashed lines represent mass sequence for PNS rotating at Keplerian frequencies.

In Fig. 8 we compare the PNS configuration for two extreme EoS (with $U_K= -60\, \text{MeV}$ and $-150\, \text{MeV}$). The isocontours lines are of constant fluid energy density in the meridional plane, $\phi = 0$. The vertical direction is aligned with the stellar angular momentum. The thick solid line marks the stellar surface. The coordinates $(x, z)$ are defined by $x = r \sin \theta$ and $z = r \cos \theta$, where $\theta$ is the polar angle. The upper slice of the figure refers to PNS with EoS having npK at $U_K= -60\, \text{MeV}$ and bottom slice for $U_K= -150\, \text{MeV}$. Both are at a constant entropy of S=1 and same baryon mass.
As we can see from the Fig. 8, the lower potential makes the PNS bulkier. Thus the shape of a PNS depends on EoS. Fig. 9 has exactly same parameters except that they are now at constant $S = 4k_B$ MeV. We notice that, as the entropy increases, the PNS is slightly deformed.

Next we see the effect of rotation rate on a particular PNS configuration. Top slice of Fig. 10 has PNS with npK ($U_R = -100$, $S = 1$) rotating slowly at 10Hz. Bottom slice rotates slightly faster at 200 Hz. Fig. 11 has the same PNS rotating at 800 Hz (top slice) and at Keplerian frequency at 1030 Hz (bottom slice). The PNS is fairly spherical at low frequency of 10 Hz and also at 200 Hz, getting deformed much visibly at 800 Hz. At Keplerian frequency it becomes elongated in an effort to keep PNS from falling apart. Thus the rigid rotation of a PNS changes not only its shape, but also various global parameters such as gravitational mass, equatorial radius, etc. For a PNS rotating at Keplerian frequency, its maximum mass is increased by about 20%.

The deviation of spherical symmetry due to the anisotropy of the energy-momentum tensor in the presence of strong magnetic field has been reported by several authors. The inclusion of magnetic fields effects in the EoS and the interaction between the magnetic field and matter (the magnetization) do not affect the stellar structure considerably [33, 34]. Without considering the magnetic field effects in our equations of state, we made some order estimation for the deformation in magnetized neutron stars for a star with baryon mass $2M_{solar}$. We found the quadrupole moment to be order of $10^{38} \text{kgm}^2$. In Fig. 12 we plot the quadrupole moment versus gravitational mass for the npK matter with $U_R = -60$ and $-150$ MeV for $S=1k_B$ and $2k_B$. We assume the magnetic and the rotation axes are not aligned and the star is rotating at a lower frequency, such that the deformation is due to strong magnetic field only.

An estimation of the gravitational wave amplitude can be done using the relation $h_0 = \frac{6G\Omega^2D}{c^4Q}$, where $G$ is the gravitational constant, $c$ the speed of light, $D$ the distance of the PNS, $\Omega$ is the rotational velocity of the star [35]. Usually, a highly magnetised ( $10^{12}$ G) PNS spins slowly whereas a low magnetic field ( $10^7$ - $10^9$ G) PNS can spin up to several hundred Hz. It could however be possible that a high magnetic field PNS spins faster at the time of birth. For a star with $M_B = 2M_{solar}$ that rotates at a frequency of $f=500\text{Hz}$ at a distance $D =10 \text{kpc}$ we obtain the gravitational wave amplitude of $h_0 = 10^{-20}$. However, for a star rotating with an average frequency of $1\text{Hz}$, this number comes down to $10^{-24}$. From the sensitivity curve of the present day detectors [36], it can be seen that the possibility of detection of this amplitude is severely limited.

4. Conclusions

In the present paper we studied the set of PNS EoS that contain antikaon condensates in its core. This is done within the framework of relativistic mean-field theoretical model with density dependent couplings. We also compare these EoS with nucleon-only EoS. All of these have been studied for a range of entropies in a deleptonised PNS. The maximum entropy state for a PNS can be $1 - 2k_B$ [11], which may increase for merger of neutron stars. We explored entropy states upto $S=5k_B$. The finite entropy PNS is then compared with NS at zero temperature.

The EoS with exotic matter tends to be softer as compared to nucleon only EoS. Moreover, among the EoS with antikaon condensates at different antikaon potential, the EoS with deeper potential makes antikaon condensates appear at lower densities...
in the core than that for a shallower potential. The EoS also appears to soften as the entropy of a PNS core increases.

The current limit in the observation of maximum mass NS is $2 \, M_{\odot}$. It rules out the EoS where maximum mass of NS that cannot reach this value. The set of EoS we studied fall within the required observational limit.

We next studied the evolution of the mass profiles with various rotational frequency for different EoS. It was found that for a low entropy star, the mass increase for a given EoS was about 20% over a static PNS, whereas for a higher entropy star, the mass increase declines considerably. The percent increase in the maximum mass for a $S = 4k_B$ PNS was only about 7.

We compared the population of different constituent particles of the PNS in the presence of $K^-$ condensates for optical potential as shallow as -60MeV with that of a deep potential of -150MeV. We studied the change in particle fractions with variation of entropy per baryon as well. At higher entropy, $K^-$ appears only at quite high density or does not appear at all for shallow $U_{\bar{K}}$. We have also discussed the variation of temperature from core to the surface of the star for a fixed entropy at extreme values of $U_{\bar{K}}$ and entropy states.

Finally, we studied the effect of rotation on the equilibrium structure of a PNS in the form of isocontours of its fluid energy density. A PNS obeying an exotic EoS with shallower potential tends to be slightly bulkier as compared to PNS with an EoS with deeper antikaon potential. Further, the stellar structure is spherical when it is non-rotating and nearly so when rotating at low frequencies, but starts to deviate from spherical symmetry as its rotational frequency increases. The PNS deforms considerably as its rotational frequency reaches the Keplerian limit. We have also made a crude estimate of the gravitational wave amplitude for a highly magnetised PNS, whose magnetic axis is not aligned with the rotation axis.

Results of differentially rotating configurations of PNS and the limits of GW emission as a result of the instabilities triggered by mass-shedding limit as well as strong magnetic fields will be reported subsequently.

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Table 2. Maximum mass reached and the corresponding radii for different entropy values, for the given npk and np EoS. The first five columns are for npk eos at different optical potentials and the last column is for np eos with no kaon contribution.

| $U_K$ | T=0 MeV | $S=1$ $k_B$ | $S=2$ $k_B$ | $S=3$ $k_B$ | $S=4$ $k_B$ | $S=5$ $k_B$ |
|-------|--------|-------------|-------------|-------------|-------------|-------------|
|       | $M_{max}(M_\odot)$ | $n_\rho(\text{fm}^{-3})$ | $M_{max}(M_\odot)$ | $n_\rho(\text{fm}^{-3})$ | $M_{max}(M_\odot)$ | $n_\rho(\text{fm}^{-3})$ | $M_{max}(M_\odot)$ | $n_\rho(\text{fm}^{-3})$ | $M_{max}(M_\odot)$ | $n_\rho(\text{fm}^{-3})$ |
| -60   | 2.372  | 0.8199     | 2.375       | 0.8094      | 2.380       | 0.7781      | 2.404       | 0.6991      | 2.487       | 0.5560      | 2.773       | 0.1893      |
| -80   | 2.339  | 0.8199     | 2.342       | 0.8050      | 2.357       | 0.7751      | 2.387       | 0.6991      | 2.473       | 0.5545      | 2.768       | 0.1789      |
| -100  | 2.297  | 0.8318     | 2.295       | 0.8079      | 2.325       | 0.7722      | 2.364       | 0.6991      | 2.457       | 0.5515      | 2.768       | 0.1789      |
| -120  | 2.242  | 0.8601     | 2.233       | 0.8303      | 2.278       | 0.7766      | 2.335       | 0.7006      | 2.436       | 0.5486      | 2.765       | 0.1745      |
| -140  | 2.176  | 0.9123     | 2.157       | 0.8780      | 2.216       | 0.7960      | 2.295       | 0.7006      | 2.411       | 0.5471      | 2.762       | 0.1670      |
| -150  | 2.142  | 0.9481     | 2.115       | 0.9182      | 2.179       | 0.8124      | 2.270       | 0.7021      | 2.396       | 0.5411      | 2.761       | 0.1655      |
| np    | 2.147  | 0.8497     | 2.419       | 0.8348      | 2.427       | 0.7900      | 2.458       | 0.7066      | 2.539       | 0.5694      | 2.788       | 0.2251      |
Figure 1. a) The equation of state (EoS) with pressure plotted against energy density for np and npK (for $U_K = -100$ MeV only) for a core at zero temperature state of $(T = 0 \text{ MeV})$ and at adiabatic state (entropy $S = 1, 4 \text{ and } 5$ in units of $k_B$). b) The mass sequences against corresponding radii for the EoS of the left panel, for entropy state of $1, 4 \text{ and } 5 \, k_B$ and $T = 0 \text{ MeV}$. The dashed lines are for np matter and solid lines are for npK matter for $U_K = 100$ MeV in both the panels.
Figure 2. The EoS, pressure vs. energy density, plotted for a range of values of $U_\bar{K} = -60$ to $-150$ MeV a) for $S = 1$ entropy state b) for higher entropy of $S = 4$. In both the plots np EoS is also included as dashed line for comparison.
Figure 3. The mass-number density profiles generated with EoS in Figure 2 for a) $S=1$ and b) $S = 4$. 
Figure 4. Fraction of different particles in a beta-equilibrated matter with n, p, e, \( \mu \) and antikaon condensates of \( K^- \) and \( K^T \); for \( U_{\bar{K}} = -60 \) MeV; plotted as a function of the normalised baryon density for an entropy of a) \( S = 1 k_B \) and b) \( S = 4 k_B \).

Figure 5. Particle fractions vs baryon number density as in Fig 4, but for a deeper \( U_{\bar{K}} = -150 \) MeV for a) \( S = 1 k_B \) and b) \( S = 4 k_B \).
Figure 6. Temperature in the PNS is plotted as a function of normalised baryon density $n_b$, for a given thermodynamic state, a) $S = 1k_B$ and b) $S = 4k_B$. 
Figure 7. The evolution of mass with frequency for a PNS with npK ($U_K = -100$ MeV) and at a) $S = 1 k_B$ and b) $S = 4 k_B$. The sequences are plotted for static PNS and PNS rotating with a frequency of 500 Hz, 700 Hz and Keplerian frequency.
Figure 8. Iso-contours representing constant fluid energy density in a PNS. The vertical direction is aligned with the stellar angular momentum. The thicker line represents the stellar surface. Top panel shows static PNS with EoS for $U_K = -60$ MeV, at $S = 1k_B$. Lower panel shows the same but for a deeper $U_K = -150$ MeV. Both panels have PNS with same baryon mass.
Figure 9. Energy density iso-contours of PNS as Fig 8, but with higher entropy state $S = 4k_B$.
Figure 10. Effect of rotation on PNS shape. Energy density iso-contours for a PNS with npK ($U_K = -100\text{MeV}$, $S=1$) rotating at 10 Hz (top panel) and 200 Hz (lower panel).
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**Figure 11.** Fluid energy density iso-contours for PNS ($U_K = -100$ MeV, $S=1$) rotating at 800 Hz (top panel) and at 1030 Hz (lower panel) which is its mass shedding limit or the Keplerian frequency.

**Figure 12.** Quadrupole moment versus gravitational mass (in $M_{\odot}$) for a PNS with npK ($U_K = -60$ MeV and $-150$ MeV, with entropy $S=1k_B$ and $2k_B$.}