Abstract

Proceeding from nonlinear realizations of the most general $N=4$, $d=1$ superconformal symmetry associated with the supergroup $D(2,1;\alpha)$, we construct all known and two new off-shell $N=4$, $d=1$ supermultiplets as properly constrained $N=4$ superfields. We find plenty of nonlinear interrelations between the multiplets constructed and present a few examples of invariant superfield actions for them. The superconformal transformation properties of these multiplets are explicit within our method.
1 Introduction

Supersymmetric quantum mechanics (SQM), being the simplest supersymmetric theory, has a lot of interesting and important applications. In particular, it provides a nice laboratory for the study of many characteristic features of supersymmetric theories in higher dimensions (see e.g. [1] and refs. therein). Superconformally invariant SQM models are of special significance in the AdS$_2$/CFT$_1$ correspondence and black hole moduli spaces [2].

The interest in SQM models with extended supersymmetry is largely due to the fact that extended supersymmetry in one dimension displays a number of surprising peculiarities as compared with its higher-dimensional counterparts. For instance, $N \geq 4$ supersymmetric models feature target space geometries which in general cannot be reproduced from those of higher-dimensional models via dimensional reduction [3]. Another phenomenon (tightly related to the just mentioned one) is the diversity of off-shell multiplets of $N \geq 4$ supersymmetries in $d=1$. Some of them can be directly recovered by dimensional reduction from off-shell multiplets of, say, $N=1$ or $N=2$ supersymmetries in $d=4$, but there exist some off-shell multiplets whose $d=4$ analogs are on shell! A striking example is the $N=4, d=1$ multiplet $(4, 4, 0)$ [4, 5] that comprises four bosonic and four fermionic physical degrees of freedom but contains no auxiliary fields at all. Its analog in $d>1$ is the on-shell hypermultiplet.

In view of this great number of inequivalent off-shell multiplets for $N \geq 4$ supersymmetries in $d=1$ it is desirable to work out a self-consistent method of deriving the appropriate superfields and their irreducibility constraints directly in one dimension, without resorting to the dimensional reduction procedure. To have such a tool and the complete list of off-shell superfields obtained with its help is important both for constructing new SQM models and for establishing precise interrelations between them.

In the present paper we focus on the case of $N=4, d=1$ supersymmetry (with 4 real supercharges) and propose to derive its various irreducible off-shell superfields from different nonlinear realizations of the most general $N=4, d=1$ superconformal group $D(2, 1; \alpha)$ [6]. An advantage of this approach is that it simultaneously specifies the superconformal transformation properties of the superfields, though the latter can equally be used for constructing non-conformal SQM models as well. As the essence of these techniques, any given irreducible $N=4, d=1$ superfield comes out as a Goldstone superfield parametrizing, together with the $N=4, d=1$ superspace coordinates, some supercoset of $D(2, 1; \alpha)$. The method was already employed in our paper [7] where we have re-derived the off-shell multiplet $(3, 4, 1)$ [8-10] from the nonlinear realization of $D(2, 1; \alpha)$ in the coset with an $SL(2, R) \times [SU(2)/U(1)]$ bosonic part (the second $SU(2) \subset D(2, 1; \alpha)$ was placed into the stability subgroup).

Here we consider nonlinear realizations of the same conformal supergroup $D(2, 1; \alpha)$ in its other coset superspaces. In this way we reproduce the $(4, 4, 0)$ multiplet and also derive two new off-shell multiplets which, to the best of our knowledge, have not been used before in $N=4, d=1$ SQM model building. The $(4, 4, 0)$ multiplet is represented by superfields parametrizing a supercoset with the bosonic part being $SL(2, R) \times SU(2)$, where the dilaton and the three parameters of $SU(2)$ are identified with the four physical bosonic fields. One of the new Goldstone multiplets is a $d=1$ analog of the so-called nonlinear multiplet of $N=2, d=4$ supersymmetry [11]. It has the same off-shell contents $(3, 4, 1)$ as the multiplet employed in [7] but it obeys a different constraint and enjoys
different superconformal transformation properties. It corresponds to the specific non-linear realization of $D(2,1;\alpha)$ where the dilatation generator and one of the two $SU(2)$ subgroups are placed into the stability subgroup. One more new multiplet of similar type is obtained by placing into the stability subgroup, along with the dilatation and three $SU(2)$ generators, also the $U(1)$ generator from the second $SU(2) \subset D(2,1;\alpha)$. It has the same field content as a chiral $N=4,d=1$ multiplet, i.e. $(2,4,2)$. Hence, it may be termed the nonlinear chiral supermultiplet. It is exceptional in the sense that no analogs of it are known in $N=2,d=4$ superspace. For the two new multiplets, we construct general off-shell superfield actions and consider in some detail (by passing to components) a few instructive examples for these actions. Also we find a number of surprising interrelations between the superfields in question. Finally, we apply our techniques to a central charge-extended $SU(1,1|2)$ supergroup obtained as a contraction of $D(2,1;\alpha)$ and derive the appropriate analogs of $N=4,d=1$ multiplets considered before.

2 Supergroup $D(2,1;\alpha)$ and its nonlinear realizations

We use the standard definition of the superalgebra $D(2,1;\alpha)$ \cite{7} with the notations of ref. \cite{7}. It contains nine bosonic generators which form a direct sum of $sl(2)$ with generators $P, D, K$ and two $su(2)$ subalgebras with generators $V, V, T, T, T, T$, respectively:

$$i [D, P] = P, i [D, K] = -K, i [P, K] = -2D, \quad i [V_3, V] = -V, \quad i [V_3, V] = V,$$

$$i [V, V] = 2V_3, \quad i [T_3, T] = -T, \quad i [T_3, T] = T, \quad i [T, T] = 2T_3. \quad (2.1)$$

The eight fermionic generators $Q^i, \overline{Q}^i, S^i, \overline{S}^i$ are in the fundamental representations of all bosonic subalgebras (in our notation only one $su(2)$ is manifest, viz. the one with generators $V, V, V_3$):

$$i [D, Q^i] = \frac{1}{2} Q^i, \quad i [D, S^i] = -\frac{1}{2} S^i, \quad i [P, S^i] = -Q^i, \quad i [K, Q^i] = S^i,$$

$$i [V_3, Q^i] = \frac{1}{2} Q^i, \quad i [V_3, Q^2] = -\frac{1}{2} Q^2, \quad i [V, Q^1] = Q^2, \quad i [V, Q^2] = \overline{Q}^1,$$

$$i [V_3, S^i] = \frac{1}{2} S^i, \quad i [V_3, S^2] = -\frac{1}{2} S^2, \quad i [V, S^1] = S^2, \quad i [V, S^2] = \overline{S}^1,$$

$$i [T_3, Q^i] = \frac{1}{2} Q^i, \quad i [T_3, S^i] = \frac{1}{2} S^i, \quad i [T, Q^i] = \overline{Q}^i, \quad i [T, S^i] = \overline{S}^i \quad (2.2)$$

(and c.c.). The splitting of the fermionic generators into the $Q$ and $S$ sets is natural and useful, because $Q^i, \overline{Q}^k$ together with $P$ form $N = 4,d = 1$ super Poincaré subalgebra, while $S^i, \overline{S}^k$ generate superconformal translations:

$$\{ Q^i, \overline{Q}^j \} = -2\delta^i_j P, \quad \{ S^i, \overline{S}^j \} = -2\delta^i_j K. \quad (2.3)$$

The non-trivial dependence of the superalgebra $D(2,1;\alpha)$ on the parameter $\alpha$ manifests itself only in the cross-anticommutators of the Poincaré and conformal supercharges:

$$\{ Q^i, S^j \} = -2(1+\alpha)\epsilon^{ij}T, \quad \{ Q^1, \overline{S}^2 \} = 2\alpha \overline{V}, \quad \{ Q^1, \overline{S}_1 \} = -2D - 2\alpha V_3 + 2(1+\alpha)T_3,$$

$$\{ Q^2, \overline{S}_1 \} = -2\alpha V, \quad \{ Q^2, \overline{S}_2 \} = -2D + 2\alpha V_3 + 2(1+\alpha)T_3. \quad (2.4)$$
The generators $P, D, K$ are chosen hermitian and the remaining ones obey the following conjugation rules:

\[(T)^\dagger = T, \quad (T_3)^\dagger = -T_3, \quad (V)^\dagger = V, \quad (V_3)^\dagger = -V_3, \quad (Q^i)^\dagger = Q_i, \quad (S^i)^\dagger = S_i.\] (2.5)

The parameter $\alpha$ is an arbitrary real number. At $\alpha = 0$ and $\alpha = -1$ one of the $\text{su}(2)$ algebras decouples and the superalgebra $\text{su}(1, 1|2) \oplus \text{su}(2)$ is recovered. The superalgebra $D(2, 1; 1)$ is isomorphic to $\text{osp}(4|2)$.\(^1\) There are some equivalent choices of the parameter $\alpha$ which lead to isomorphic algebras $D(2, 1; \alpha)$ \(^9\).

We will be interested in diverse nonlinear realizations of the superconformal group $D(2, 1; \alpha)$ in its coset superspaces. As the starting point we shall consider the following parametrization of the supercoset

\[g = e^{itP} e^{i\theta_1 Q^1 + i\tilde{\theta} S^1} e^{i\psi S^3 + i\tilde{\psi} \tilde{S}^3} e^{izK} e^{i\mu D} e^{i\varphi V + i\tilde{\varphi} \tilde{V}} e^{i\phi V_3}.\] (2.6)

The coordinates $t, \theta, \tilde{\theta}$ parametrize the $N = 4, d = 1$ superspace. All other supercoset parameters are Goldstone $N = 4$ superfields. The group $\text{SU}(2) \propto (V, \tilde{V}, V_3)$ linearly acts on the doublet indices $i$ of spinor coordinates and Goldstone fermionic superfields, while the bosonic Goldstone superfields $\varphi, \tilde{\varphi}, \phi$ parametrize this $\text{SU}(2)$. Another $\text{SU}(2)$ as a whole is placed in the stability subgroup and acts only on fermionic Goldstone superfields and $\theta$'s, mixing them with their conjugates. With our choice of the $\text{SU}(2)$ coset we are led to assume that $\alpha \neq 0$. We could equivalently choose another $\text{SU}(2)$ to be nonlinearly realized and the first one to belong to the stability subgroup, then the restriction $\alpha \neq -1$ would be imposed.

In principle, we could lift up the second $\text{SU}(2)$ into the coset, with adding the relevant Goldstone superfields. In Sec. 8 we shall briefly discuss this more general situation and argue that it does not add new non-trivial options.

The left-covariant Cartan one-form $\Omega$ with values in the superalgebra $D(2, 1; \alpha)$ is defined by the standard relation

\[g^{-1} d g = \Omega.\] (2.7)

In what follows we shall need the explicit structure of several important one-forms in the expansion of $\Omega$ over the generators,

\[
\omega_D = idu - 2(\psi^1 d\theta_1 + \psi_1 d\tilde{\theta}^1) - 2izd\tilde{t}, \\
\omega_V = \frac{e^{-i\phi}}{1 + \Lambda^2} \left[ id\Lambda + \hat{\omega}_V + \Lambda^2 \hat{\omega}_V - \Lambda \hat{\omega}_{V_3} \right], \\
\omega_{V_3} = d\phi + \frac{1}{1 + \Lambda^2} \left[ i (d\Lambda - \Lambda d\Lambda) + (1 - \Lambda \Lambda) \hat{\omega}_{V_3} - 2(\Lambda \hat{\omega}_V - \Lambda \hat{\omega}_V) \right].
\] (2.8)

Here

\[
\hat{\omega}_V = 2\alpha \left[ \psi_2 d\theta^1 - \psi^1 (d\theta_2 - \psi_2 d\tilde{t}) \right], \quad \hat{\omega}_V = 2\alpha \left[ \psi^2 d\theta_1 - \psi_1 (d\theta^2 - \psi^2 d\tilde{t}) \right], \\
\hat{\omega}_{V_3} = 2\alpha \left[ \psi_1 d\theta^1 - \psi^1 d\theta_1 - \psi_2 d\theta^2 + \psi^2 d\theta_2 + (\psi^1 \psi_1 - \psi^2 \psi_2) d\tilde{t} \right].
\] (2.9)

\(^1\)Sometimes $D(2, 1; \alpha)$ is defined so that these special values of $\alpha$ are excluded \(^9\). In our definition we retain these values in order to be able to consider all inequivalent cases on equal footing.
\[ d\tilde{t} \equiv dt + i \left( \theta_i d\bar{\theta}^i + \bar{\theta}^i d\theta_i \right), \quad (2.10) \]

and
\[ \Lambda = \tan \frac{\sqrt{\varphi^2}}{\varphi}, \quad \Lambda = \tan \frac{\sqrt{\varphi^2}}{\varphi}. \quad (2.11) \]

The semi-covariant (fully covariant only under Poincaré supersymmetry) spinor derivatives are defined by
\[ D^i = \frac{\partial}{\partial \theta_i} + i\bar{\theta}^i \partial_t, \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} + i\theta_i \partial_t, \quad \{ D^i, \bar{D}_j \} = 2i\delta^i_j \partial_t. \quad (2.12) \]

Let us quote the transformation properties of the \( N = 4 \) superspace coordinates and the basic Goldstone superfields under the transformations of supergroup \( D(2, 1; \alpha) \).

The variations of the \( N = 4, d = 1 \) superspace coordinates under the \( N = 4, d = 1 \) Poincaré supergroup are generated by acting on the coset element \( g_0 \) from the left by the element \( g_0 = e^{\varepsilon_i Q^i + e^j \overline{\mathcal{S}}_j} \in D(2, 1; \alpha) \).

The resulting transformations are
\[ \delta t = i \left( \theta \cdot \bar{\varepsilon} - \varepsilon \cdot \bar{\theta} \right), \quad \delta \theta_i = \varepsilon_i, \quad \delta \bar{\theta}^i = \bar{\varepsilon}^i. \quad (2.14) \]

All our superfields are scalars under the latter transformations.

The superconformal transformations are generated by acting on the coset element \( g_1 \) from the left by the element
\[ g_1 = e^{\varepsilon_i S^i + e^j \overline{\mathcal{S}}_j}, \quad (2.15) \]

and explicitly they read
\[
\begin{align*}
\delta t &= -it \left( \varepsilon \cdot \bar{\theta} + \bar{\varepsilon} \cdot \theta \right) + (1 + 2\alpha) \theta \cdot \bar{\theta} \left( \varepsilon \cdot \bar{\varepsilon} + \bar{\varepsilon} \cdot \varepsilon \right), \\
\delta \theta_i &= \varepsilon_i t - 2i\alpha \theta_i (\theta \cdot \bar{\varepsilon}) + 2i(1 + \alpha)\theta_i (\bar{\theta} \cdot \varepsilon) - i(1 + 2\alpha)\varepsilon_i (\theta \cdot \bar{\varepsilon}) , \\
\delta u &= -2i \left( \varepsilon_1 \cdot \bar{\theta} + \varepsilon_2 \cdot \bar{\theta} \right) , \\
\delta \phi &= 2\alpha \left[ \varepsilon_1 \theta_1 - \varepsilon_2 \theta_2 - \varepsilon_1 \bar{\theta}^1 + \varepsilon_2 \bar{\theta}^2 + \left( \varepsilon_1 \theta_1 - \varepsilon_2 \theta_2 \right) \Lambda + \left( \bar{\theta}^1 \theta_1 - \bar{\theta}^2 \theta_2 \right) \Lambda \right], \\
\delta \Lambda &= 2i\alpha \left[ \theta_2 \bar{\varepsilon}^1 - \bar{\theta}^1 \varepsilon_2 + (\bar{\theta}^1 \varepsilon_1 - \theta_1 \varepsilon_2) \Lambda + \left( \bar{\theta}^1 \varepsilon_1 - \theta_1 \varepsilon_2 + \theta_2 \varepsilon_2 - \bar{\theta}^2 \varepsilon_2 \right) \Lambda \right]. \quad (2.16)
\end{align*}
\]

The \( N = 4 \) superspace integration measure \( dt d^4 \theta \) is transformed as
\[ \delta dtd^4 \theta = 2i(\varepsilon \cdot \bar{\theta} + \bar{\varepsilon} \cdot \theta) dt d^4 \theta. \quad (2.17) \]

The covariant derivatives \( D^i, \bar{D}_i \) transform as
\[
\begin{align*}
\delta D^i &= i \left[ (2 + \alpha)(\varepsilon \cdot \bar{\theta}) + \alpha(\theta \cdot \bar{\varepsilon}) \right] D^i - 2i(1 + \alpha)(\bar{\theta} \cdot \varepsilon) \bar{D}^i - 2i\alpha \left[ \varepsilon_i (\bar{\theta}_k) + \theta^k \varepsilon_i \right] D^k, \\
\delta \bar{D}_i &= i \left[ (2 + \alpha)(\varepsilon \cdot \theta) + \alpha(\bar{\theta} \cdot \varepsilon) \right] \bar{D}_i - 2i(1 + \alpha)(\varepsilon \cdot \varepsilon) \bar{D}_i - 2i\alpha \left[ \varepsilon_i (\bar{\theta}_k) + \theta^k \varepsilon_i \right] \bar{D}^k. \quad (2.18)
\end{align*}
\]

\(^2\)When summing over doublet indices we assume them to stay in a natural position; the Grassmann coordinates and their conjugates carry lower case and upper case indices, respectively. We use a shorthand notation \( \psi \cdot \xi = \bar{\psi}^i \xi_i = -\xi \cdot \psi \). The contraction of spinors of equal kind is defined as \( a \cdot b = a^b b^i \), \( \bar{a} \cdot \bar{b} = \bar{a}_a \bar{b}^a \).
From these transformations it follows, in particular, that chiral $N = 4, d = 1$ superfields can be defined covariantly with respect to the superconformal transformations only at $\alpha = -1$, i.e. in the case of the supergroup $SU(1,1|2)$.

Since all other $D(2,1;\alpha)$ transformations appear in the anticommutator of the conformal and Poincaré supersymmetry generators, it is sufficient to require invariance under these two supersymmetries, when constructing invariant actions for the considered system.

For further use, we also give the explicit expressions for the variations of our superspace coordinates and superfields with respect to two $SU(2)$ subgroup. They are generated by the left action of the group element

$$g_2 = e^{i\theta^V + i\bar{a}V} e^{i\theta^T + i\bar{a}T}$$

and read

$$\delta \theta_1 = \bar{b}\theta^2 - \bar{a}\theta_2, \quad \delta \theta_2 = -\bar{b}\theta^1 + a\theta_1,$$

$$\delta \Lambda = a + \bar{a}\Lambda^2, \quad \delta \bar{\Lambda} = \bar{a} + a\bar{\Lambda}^2, \quad \delta \phi = i (a\bar{\Lambda} - \bar{a}\Lambda).$$

### 3 $\mathbf{N=4, d=1 \text{ "hypermultiplet"}}$

The basic idea of our method is to impose the appropriate $D(2,1;\alpha)$ covariant constraints on the Cartan forms (2.7), (2.8), so as to end up with some minimal $N = 4, d = 1$ superfield set carrying an irreducible off-shell multiplet of $N = 4, d = 1$ supersymmetry. Due to the covariance of the constraints, the ultimate Goldstone superfields will support the corresponding nonlinear realization of the superconformal group $D(2,1;\alpha)$.

Let us elaborate on this in some detail. It was the desire to keep $N = 4, d = 1$ Poincaré supersymmetry unbroken that has led us to associate Grassmann coordinates $\theta_i, \bar{\theta}^i$ with the Poincaré supercharges in (2.6) and fermionic Goldstone superfields $\psi_i, \bar{\psi}^i$ with the remaining four supercharges which generate conformal supersymmetry. The minimal number of physical fermions in an irreducible $N = 4, d = 1$ supermultiplet is four and it nicely matches with the number of fermionic Goldstone superfields in (2.6) the first components of which can so be naturally identified with the fermionic fields of the ultimate Goldstone supermultiplet. On the other hand, we can vary the number of bosonic Goldstone superfields in (2.6): by putting some of them equal to zero we can enlarge the stability subgroup by the corresponding generators and so switch to another coset with a smaller set of parameters. Thus, for different choices of the stability subalgebra the coset (2.6) will contain different numbers of the bosonic superfields, but always the same number of fermionic superfields $\psi_i, \bar{\psi}^i$. Yet, the corresponding sets of bosonic and fermionic Goldstone superfields contain too many field components, and it is natural to impose on them the appropriate covariant constraints in order to reduce the number of components as much as possible. For preserving off-shell $N = 4$ supersymmetry these constraints must be purely kinematical, i.e. not imply any dynamical restriction like equations of motion.

Some of the constraints just mentioned should express the Goldstone fermionic superfields in terms of spinor derivatives of the bosonic ones. On the other hand, as soon as the first components of the fermionic superfields $\psi_i, \bar{\psi}^k$ are required to be the only physical fermions, we are led to impose much stronger condition that all spinor derivatives of all
bosonic superfields are properly expressed in terms of $\psi_i, \bar{\psi}^i$. Remarkably, these latter conditions will prove to be just the irreducibility constraints picking up irreducible $N = 4$ supermultiplets. In this and next Sections we shall demonstrate how this procedure works for various cosets which correspond to placing some of the original coset bosonic generators $D, V, \nabla, V_3$ into the stability subalgebra. Note that in the cases when the dilatation generator $D$ is among the coset generators (this is true for the options considered here and in the next Section) we should also keep the conformal boosts generator $K$ in the coset on equal footing with the translation generator $P$. This requirement is necessary for the Cartan form $\omega_D$ in (2.8) to be separately covariant under the left action of $D(2, 1; \alpha)$. Actually, this is the standard way of doing with nonlinear realizations of the $d = 1$ conformal group $SL(2, R) \sim SO(1, 2)$ [12].

We shall start with the most general case when the coset [26] contains all four bosonic superfields $u, \varphi, \bar{\varphi}, \phi$. Looking at the structure of the Cartan 1-forms (2.8), it is easy to find that the covariant constraints which express all spinor covariant derivatives of bosonic superfields in terms of the Goldstone fermions amount to setting equal to zero the spinor projections of these 1-forms (these conditions are particular case of inverse Higgs effect [13]). Thus, in the case at hand we impose the following constraints

$$\omega_D = \omega_V \mid = \bar{\omega}_V = \omega_{V_3} = 0\,,$$

(3.1)

where $\mid$ means restriction to spinor projections. These constraints are manifestly covariant under the whole supergroup $D(2, 1; \alpha)$. They allow one to express the Goldstone spinor superfields as the spinor derivatives of the residual bosonic Goldstone superfields $u, \Lambda, \bar{\Lambda}, \phi$ and imply some irreducibility constraints for the latter:

$$D^1 \Lambda = -2i\alpha \Lambda \left(\bar{\psi}^1 + \Lambda \bar{\psi}^2\right), \quad D^1 \bar{\Lambda} = -2i\alpha \left(\bar{\psi}^2 - \bar{\Lambda} \bar{\psi}^1\right), \quad D^1 \phi = -2\alpha \left(\bar{\psi}^1 + \Lambda \bar{\psi}^2\right),$$

$$D^2 \Lambda = 2i\alpha \left(\bar{\psi}^1 + \Lambda \bar{\psi}^2\right), \quad D^2 \bar{\Lambda} = -2i\alpha \left(\bar{\psi}^2 - \bar{\Lambda} \bar{\psi}^1\right), \quad D^2 \phi = 2\alpha \left(\bar{\psi}^2 - \bar{\Lambda} \bar{\psi}^1\right),$$

$$D^1 u = 2i\bar{\psi}^1, \quad D^2 u = 2i\bar{\psi}^2, \quad \dot{u} = 2z$$

(3.2)

(and c.c.). The irreducibility conditions in this and other cases which we shall consider further arise due to the property that the Goldstone fermionic superfields are simultaneously expressed by (3.2) in terms of spinor derivatives of different bosonic superfields. Then, eliminating these spinor superfields, we end up with the relations between the spinor derivatives of bosonic Goldstone superfields. To make these constraints most feasible, it is advantageous to pass to the new variables

$$q^1 = \frac{e^{\frac{i}{2}(\alpha + i\phi)}}{\sqrt{1 + \Lambda \bar{\Lambda}}} \Lambda, \quad q^2 = -\frac{e^{\frac{i}{2}(\alpha - i\phi)}}{\sqrt{1 + \Lambda \bar{\Lambda}}} \bar{\Lambda}, \quad \bar{q}_1 = \frac{e^{\frac{i}{2}(\alpha - i\phi)}}{\sqrt{1 + \Lambda \bar{\Lambda}}} \bar{\Lambda}, \quad \bar{q}_2 = -\frac{e^{\frac{i}{2}(\alpha + i\phi)}}{\sqrt{1 + \Lambda \bar{\Lambda}}} \Lambda.$$  

(3.3)

In terms of these variables the irreducibility constraints acquire the manifestly $SU(2)$ covariant form

$$D^{(i\bar{q}^j)} = 0, \quad \overline{D}^{(i\bar{q}^j)} = 0.$$  

(3.4)

This $N = 4, d = 1$ multiplet was already considered, in the component and $N = 1$ superfield approaches, in [4, 5, 14, 15] and also was recently studied in $N = 4, d = 1$ harmonic superspace [16]. Off shell it contains $4$ bosonic and $4$ fermionic fields and no auxiliary fields. In this respect it resembles the $N = 2, d = 4$ hypermultiplet. But in
contrast to the $d = 4$ case the constraints define an off-shell multiplet in $d = 1$. In fact, all other known $N = 2, d = 4$ supermultiplets also have $N = 4, d = 1$ descendants. Their defining constraints follow from the $N = 2, d = 4$ ones by discarding space-time spinor indices of the covariant derivatives. In the forthcoming Sections we shall reproduce all these $N = 4, d = 1$ supermultiplets within our nonlinear realizations framework. Note that the $q^i$ supermultiplet can be considered as a fundamental one since all its components have the interpretation as Goldstone fields: 4 fermions are Goldstino for the nonlinearly realized conformal $N = 4$ supersymmetry, while 4 bosons are Goldstone fields for the nonlinearly realized dilatations and $SU(2)$ transformations. All other irreducible multiplets derived in next Sections contain auxiliary fields which admit no immediate interpretation as Goldstone fields.

Let us note that in the present approach it is easy to define the transformation properties of $q^i$ under the group $D(2, 1; \alpha)$ because we know the transformation properties of $N = 4, d = 1$ superspace coordinates and all original Goldstone superfields in (2.6), while the constraints are covariant by construction. In particular, the transformations of conformal supersymmetry read

$$\delta q^i = 2i\alpha \left( \bar{\theta}^i \epsilon_j - \theta^i \bar{\epsilon}_j \right) q^j.$$

Superconformally invariant superfield action of the sigma model type can be easily found to be

$$S_q^{(\alpha)} = \int dt d^4 \theta \left( q \cdot \bar{q} \right)^{\frac{1}{2} \alpha}.$$  (3.6)

The invariance of (3.6) follows from the transformation properties of the integration measure and $q \cdot \bar{q}$:

$$\delta (q \cdot \bar{q}) = -2i\alpha \left( \epsilon \cdot \bar{\theta} + \bar{\epsilon} \cdot \theta \right) (q \cdot \bar{q}).$$  (3.7)

The case with $\alpha = -1$ should be considered independently, because as a consequence of we have

$$D^j D_i \left( \frac{1}{q \cdot \bar{q}} \right) = \bar{D}_j \bar{D}_i \left( \frac{1}{q \cdot \bar{q}} \right) = [D^i, \bar{D}_i] \left( \frac{1}{q \cdot \bar{q}} \right) = 0$$  (3.8)

and therefore the action vanishes in this special case. Hence, we instead consider the action

$$S_q^{(\alpha = -1)} = - \int dt d^4 \theta \ln \left( \frac{q \cdot \bar{q}}{q \cdot \bar{q}} \right),$$  (3.9)

which is invariant up to a total derivative in the integrand. One can check that $\frac{1}{1+\alpha} S_q^{(\alpha)}$ is regular for any $\alpha$ and coincides with $S_q^{(\alpha = -1)}$ for $\alpha = -1$. This situation is similar to the case of $N = 4, d = 1$ supermultiplet considered in [7] (see next Section).

A more detailed discussion of possible actions for $q^i$ multiplet can be found in [16]. In particular, there exists a superpotential-type off-shell invariant which, however, does not give rise in components to any scalar potential for the physical bosons. Instead, it produces a Wess-Zumino type term of first order in time derivative. It can be interpreted as a coupling to a four-dimensional background abelian gauge field. The superpotential just mentioned admits a concise manifestly supersymmetric superfield formulation as an integral over an analytic subspace of $N = 4, d = 1$ harmonic superspace [16].
4 N=4, d=1 “tensor” multiplet

This multiplet has been derived from a nonlinear realization of $D(2, 1; \alpha)$ and considered in detail in [7]. Here we shortly recall the basic points of the construction of [7] as a particular case of the general method described in the beginning of the previous Section. The relevant formulas will be needed for establishing relationships of this $N = 4, d = 1$ “tensor” multiplet with other ones.

The “tensor” multiplet corresponds to the choice $\phi = 0$ in the coset element (2.6), which amounts to transferring $U(1) \subset SU(2)$ into the stability subgroup. Thus in this case one deals with a supercoset involving $SL(2, R) \times [SU(2)/U(1)]$ as the bosonic manifold and, respectively, with three bosonic Goldstone superfields $u, \Lambda, \bar{\Lambda}$. In accord with our general approach, we impose the following set of constraints:

$$\omega_D = \omega_V = 0 \ . \quad (4.1)$$

Let us point out that now one cannot impose any constraints on the Cartan form $\omega_V$ because it gets transforming inhomogeneously under $D(2, 1; \alpha)$. Explicitly, the set of constraints (4.1) reads

$$D^1 \Lambda = -2i\alpha \Lambda (\bar{\psi}^1 + \Lambda \bar{\psi}^2) , \quad D^1 \bar{\Lambda} = -2i\alpha (\bar{\psi}^2 - \bar{\Lambda} \bar{\psi}^1) , \quad D^1 u = 2i\bar{\psi}^1 ,$$

$$D^2 \Lambda = 2i\alpha (\bar{\psi}^1 + \Lambda \bar{\psi}^2) , \quad D^2 \bar{\Lambda} = -2i\alpha \Lambda (\bar{\psi}^2 - \bar{\Lambda} \bar{\psi}^1) , \quad D^2 u = 2i\bar{\psi}^2 , \quad \bar{u} = 2z \quad (4.2)$$

(plus c.c.). After introducing a new $N = 4$ isovector real superfield $V^{ij}$ ($V^{ij} = V^{ji}$ and $V^{ik} = \epsilon_{ii'}\epsilon_{kk'}V^{i'k'}$) via the identification

$$V^{11} = -i\sqrt{2} \epsilon_{au} \Lambda \frac{1}{1 + \Lambda \bar{\Lambda}} , \quad V^{22} = i\sqrt{2} \epsilon_{au} \bar{\Lambda} \frac{1}{1 + \Lambda \bar{\Lambda}} , \quad V^{12} = \frac{i}{\sqrt{2}} \epsilon_{au} \frac{1 - \Lambda \bar{\Lambda}}{1 + \Lambda \bar{\Lambda}} ,$$

$$V^2 \equiv V^{ik} V_{ik} = \epsilon_{2au} \ , \quad (4.3)$$

the irreducibility constraints for the bosonic superfields following from (4.2) can be cast in the manifestly $SU(2)$-symmetric form

$$D^{(i}V^{jk)} = 0 , \quad \bar{D}^{(i}V^{jk)} = 0 \ . \quad (4.4)$$

The constraints (4.4) could be obtained by the direct dimensional reduction from the constraints defining $N = 2, d = 4$ tensor multiplet [17] in which one suppress the $SL(2, C)$ spinor indices of $d = 4$ spinor derivatives, thus keeping only the doublet indices of the $R$-symmetry $SU(2)$ group. This is the reason why we can call it $N = 4, d = 1$ “tensor” multiplet. Of course, in one dimension no any differential (notoph-type) constraints arise on the components of the superfield $V^{ij}$. The constraints (4.4) leave in $V^{ik}$ the following independent superfield projections:

$$V^{ik} , \quad D^i V^{kl} = -\frac{1}{3} (\epsilon^{ik} \chi^l + \epsilon^{il} \chi^k) , \quad \bar{D}^i V^{kl} = \frac{1}{3} (\epsilon^{ik} \bar{\chi}^l + \epsilon^{il} \bar{\chi}^k) , \quad D^i \bar{D}^k V_{ik} , \quad (4.5)$$

An alternative way to obtain $V^{ik}$ via a dimensional reduction from $d = 4$ is to start from $N = 1, d = 4$ vector multiplet [8]. The $SL(2, C)$ group is reduced to $SU(2)$ and the indices $\alpha, \dot{\alpha}$ becomes the doublet $SU(2)$ indices. The superfield $V^{ik}$ then comes out as the spatial component of the $d = 4$ abelian gauge vector connection superfield.
where
\[ \chi^k \equiv D^i V^k_i, \quad \bar{\chi}^k = \bar{\chi}^k = \bar{D}^i V^i_k. \] (4.6)

Thus its off-shell component field content is just \((3, 4, 1)\). The \(N = 4, d = 1\) superfield \(V^{ik}\) subjected to the conditions (4.4) was introduced in [18] and later on rediscovered in [8, 9, 10].

Like in the case of the superfield \(q^i\), the \(D(2, 1; \alpha)\) superconformal transformations of \(V^{ij}\) can be deduced from the identification (4.3). In particular, the conformal supersymmetry acts as
\[ \delta V^{ij} = -2i\alpha \left[ (\epsilon \cdot \bar{\theta} + \bar{\epsilon} \cdot \theta) V^{ij} + (\epsilon^{(i} \bar{\theta}^{j)k} - \bar{\epsilon}^{(i} \theta^{j)k}) V^{ij} + (\epsilon_k \bar{\theta}^{(i} - \bar{\epsilon}_k \theta^{(i}) V^{ij} \right]. \] (4.7)

Invariant superfield actions consist of a superfield kinetic term and a superpotential [7, 8, 16]. The superconformally invariant superfield kinetic term reads
\[ S_V^{(\alpha)} = \int dt d^4 \theta \left( V^2 \right)^{\frac{1}{2\alpha}}. \] (4.8)

As in the “hypermultiplet” case, for \(\alpha = -1\) the action (4.8) vanishes and should be replaced by
\[ S_V^{(\alpha=-1)} = -\frac{1}{2} \int dt d^4 \theta \left( V^2 \right)^{-\frac{1}{2}} \ln V^2. \] (4.9)

The potential term can be written in a manifestly \(N = 4\) supersymmetric form either with the help of prepotential solving eqs. (4.4) [7], or as an integral over the \(N = 4, d = 1\) analytic harmonic superspace [16].

As the last remark of this Section let us note that the explicit expressions of the “tensor” multiplet (4.3) and the “hypermultiplet” (3.3) in terms of initial Goldstone superfields give the hint how to construct the former out of the latter. Indeed, from eqs. (4.3) and (3.3) it follows that \(V^{ij}\) can be represented as the following composite superfield
\[ \tilde{V}^{11} = -i\sqrt{2} q^1 \bar{q}^1, \quad \tilde{V}^{22} = -i\sqrt{2} q^2 \bar{q}^2, \quad \tilde{V}^{12} = -\frac{i}{\sqrt{2}} \left( q^1 \bar{q}^2 + q^2 \bar{q}^1 \right). \] (4.10)

One can check that, in consequence of the “hypermultiplet” constraints (3.4), the composite superfield \(\tilde{V}^{ij}\) automatically obeys (4.4). This is just the relations established in [16].

The expressions (4.10) supply a rather special solution to the “tensor” multiplet constraints. In particular, they express the auxiliary field of \(\tilde{V}^{ij}\) through the time derivative of the physical components of \(q^i\) which contains no any auxiliary field.\(^4\) As a consequence, the superpotential of \(\tilde{V}^{ik}\) is a particular case of the \(q^i\) superpotential which produces no genuine scalar potential for physical bosons and gives rise for them only to a Wess-Zumino type term of the first order in the time derivative.

5 \(N=4, d=1\) “nonlinear” multiplet

In the previous two Sections we considered two cases when the generator of dilatations \(D\) is placed into the coset along with some \(SU(2)\) generators. Alternative possibilities arise

\(^4\)This is a nonlinear version of the phenomenon which is generic for \(d = 1\) supersymmetry and was discovered at the linearized level in [19, 20].
if we place the generator of dilatation $D$ into the stability subgroup (together with the
generator of conformal boosts $K$). We firstly consider the case with all generators of one
$su(2)$ being present in the coset. This case is singled out by setting $u = 0, z = 0$ in the
general coset element $[2.0]$.

The “nonlinear” multiplet we are going to construct parametrizes the coset $SU(2)$
bosonic manifold. To express all spinor derivatives from our set of bosonic superfields
$\varphi, \bar{\varphi}, \phi$ in terms of Goldstone fermions $\psi^i, \bar{\psi}^j$ we should impose the following set of con-
straints:

$$\omega_{V3} = \omega_V = \bar{\omega}_V = 0 .$$ (5.1)

They are covariant because the corresponding Cartan forms are still homogeneously trans-
formed among themselves under the left action of $D(2, 1; \alpha)$. On the other hand, the
Cartan form $\omega_D$ which enters the previous constraints (3.1), (4.1) now belongs to t he
stability subgroup algebra and so transforms inhomogeneously. For this reason it cannot
be used for defining any covariant constraint in the case under consideration.

Explicitly, the constraints (5.1) amount to the following set of equat-
ions

$$D^1 \Lambda = -2i\alpha (\bar{\psi}^1 + \Lambda \bar{\psi}^2), \quad D^1 \bar{\Lambda} = -2i\alpha (\bar{\psi}^2 - \bar{\Lambda} \bar{\psi}^1), \quad D^1 \phi = -2\alpha (\bar{\psi}^1 + \Lambda \bar{\psi}^2),$$

$$D^2 \Lambda = 2i\alpha (\bar{\psi}^1 + \Lambda \bar{\psi}^2), \quad D^2 \bar{\Lambda} = -2i\alpha (\bar{\psi}^2 - \bar{\Lambda} \bar{\psi}^1), \quad D^2 \bar{\phi} = 2\alpha (\bar{\psi}^2 - \bar{\Lambda} \bar{\psi}^1)$$ (5.2)

(and c.c.). They express 12 spinor derivatives of the bosonic superfield $\varphi, \bar{\varphi}, \phi$ in terms
of 4 fermions $\psi^i, \bar{\psi}^j$. Evidently, this implies the existence of additional constraints on th e
bosonic Goldstone superfields. These constraints can be put in a more concise form by
passing to the new 4 by 4 matrix variables $N^{ai}$:

$$N^{11} = \frac{e^{-\frac{i}{2} \phi}}{\sqrt{1 + \Lambda \bar{\Lambda}}}, \quad N^{21} = \frac{e^{\frac{i}{2} \phi}}{\sqrt{1 + \Lambda \bar{\Lambda}}}, \quad N^{12} = -\frac{e^{-\frac{i}{2} \phi}}{\sqrt{1 + \Lambda \bar{\Lambda}}}, \quad N^{22} = \frac{e^{\frac{i}{2} \phi}}{\sqrt{1 + \Lambda \bar{\Lambda}}}.$$ (5.3)

Here, the new doublet index $a$ is associated with some extra global $SU(2)$ which commutes
with $D(2, 1; \alpha)$ (and with the $N = 4, d = 1$ Poincaré superalgebra $\subset D(2, 1; \alpha)$). The
superfields $N^{ai}$ by construction obey the algebraic constraint

$$N^{ai} N_{ai} = 2$$ (5.4)

which ensures the number of independent superfields to be three as it should be. The
additional irreducibility constraints which follow from (5.2) can now be easily read off as

$$N^{ai} D^j N_{aj} = 0, \quad N^{ai} (\bar{D}^j N_{aj}) = 0.$$ (5.5)

Comparing them with those of ref. [11], we recognize $N^{ai}$ as a $d = 1$ analog of the $N =
2, d = 4$ superfield which represents the nonlinear multiplet.

In order to reveal the component field content of $N^{ai}$, it is convenient to pass to another useful representation of the $d = 1$ “nonlinear” multiplet in terms of the following set of superfields:

$$X = e^{-i\phi} \Lambda, \quad \bar{X} = e^{i\phi} \bar{\Lambda}, \quad \phi .$$ (5.6)

The constraints (5.5) are rewritten as

$$D^1 X = 0, \quad \bar{D}^2 X = 0, \quad D^2 (e^{i\phi} X) = -i D^1 \phi, \quad \bar{D}^1 (e^{i\phi} X) = i \bar{D}^2 \phi$$ (5.7)
(those including $\bar{X}$ follow by conjugation). Now it is clear that, due to (5.7), derivatives of each $N = 4$ superfield with respect to, say, $\theta_2, \bar{\theta}^2$ can be expressed as derivatives with respect to $\theta_1, \bar{\theta}^1$ of some other superfield. Therefore, only the $\theta_2 = \bar{\theta}^2 = 0$ components of each superfield are independent $N = 2$ superfields. Moreover, the $N = 2$ superfield $X$ (with $\theta_2 = \bar{\theta}^2 = 0$) is chiral. Thus, from $N = 2$ point of view, the “nonlinear” multiplet contains one general $N = 2$ superfield $\phi$ and one chiral superfield $X$, i.e. formally it has the same off-shell field content (3, 4, 1) as the “tensor” multiplet $V_{ij}$ of Section 4. However, their superconformal properties are radically different, and this is the main distinction between these two multiplets. The transformation properties of $N^{ai}$ under conformal supersymmetry (2.15) may be directly determined using (5.3) and (2.16):

$$
\delta N^{ai} = 2i\alpha \left( \epsilon^{(i}\bar{\theta}^{j)} - \epsilon^{(i}\theta^{j)} \right) N_j^a. \tag{5.8}
$$

This has to be compared with the analogous transformation law of $V_{ij}$, eq. (4.7). Under the Poincaré supersymmetry the superfields $N^{ai}$ transform as scalars. All other $D(2, 1; \alpha)$ transformations appear in the anticommutator of the conformal and the Poincaré supersymmetries. Hence, as in the previously considered cases, it suffices to know explicit realization of these two supersymmetries only. E.g., commuting (5.8) with the Poincaré supersymmetry transformation, one finds that the superfields $N^{ai}$ has the dilatation weight zero. The same is true for physical bosonic fields of this multiplet.

As regards the actions of the $d = 1$ “nonlinear” multiplet, the general off-shell action

$$
S_N = \int dt d^4\theta L(N^{ai}) \tag{5.9}
$$

where $L(N^{ai})$ is an arbitrary real function of $N^{ai}$, is obviously invariant under $N = 4$ Poincaré supersymmetry. Keeping in mind that $N^{ai}$ has zero dilatation weight, while the integration measure in (5.9) has the weight +1, one comes to the conclusion that superconformally invariant action for $N^{ai}$ can be constructed only by coupling $N^{ai}$ to some dilaton-containing superfield.

The simplest example of $SU(2) \times SU(2)$ invariant action reads

$$
S_1 = -\int dt d^4\theta \ln(1 + XX) = -\int dt d^4\theta \ln(1 + \Lambda\bar{\Lambda}). \tag{5.10}
$$

It is invariant under two commuting $SU(2)$ groups which one can define on $N^{ai}$: the R-symmetry $SU(2)$ acting on the indices $i$ and transforming superfields $\Lambda, \bar{\Lambda}$ by the rule (2.20) and the second $SU(2)$ commuting with $N = 4$ supersymmetry (and with $D(2, 1; \alpha)$) and acting on the extra doublet indices $a$

$$
\delta_2 N^{ai} = \gamma^{(ab)} N_b^i. \tag{5.11}
$$

In variables (5.6) this second group acts as follows

$$
\delta_2 X = \gamma^{11} - 2\gamma^{12} X + \gamma^{22}(X)^2, \quad \delta_2 \bar{X} = \gamma^{22} + 2\gamma^{12} \bar{X} + \gamma^{11}(\bar{X})^2,
$$

$$
\delta_2 \phi = -2i\gamma^{12} + i\gamma^{22} X - i\gamma^{11} \bar{X}. \tag{5.12}
$$

Using the constraints (5.7), it is easy to check the invariance of the action under both these $SU(2)$ groups. Indeed,

$$
\delta_1 \ln(1 + \Lambda\bar{\Lambda}) = a\bar{\Lambda} + a\Lambda, \quad \delta_2 \ln(1 + XX) = \gamma^{11} \bar{X} + \gamma^{22} X \tag{5.13}
$$
and these variations vanish under the Berezin integral since, e.g., it follows from (5.7) that
\[ D^1 D^2 \Lambda = 0, \]
while \( X \) obeys the twisted chirality conditions.

If we define the physical components of the superfields \( \phi, X, \bar{X} \) as
\[ \phi|, X|, \bar{X}|, \psi = D^1 \phi|, \bar{\psi} = -\overline{D^1} \phi|, \bar{\xi} = -\overline{D^1} X|, \]
the action (5.10) takes the form (auxiliary fields vanish on shell):
\[
S_1 = \int dt \left[ \dot{\phi}^2 - 2i \dot{\phi} (\dot{X} \bar{X} - X \dot{\bar{X}}) - \frac{2 \dot{X} \bar{X}}{1 + X \bar{X}} \right] - \frac{X^2 \dot{X}^2 + 2 X \dot{X} \dot{\bar{X}} + \dot{\bar{X}}^2}{(1 + X \bar{X})^2} + 2i \dot{\psi} \bar{\psi} + 2i \dot{\xi} \bar{\xi} + X \dot{\xi} \bar{\psi} + \bar{X} \dot{\psi} \bar{\xi} - 2i \xi (i \dot{\phi} + X \dot{\bar{X}})
\]
(5.15)
The bosonic part of (5.15) can be concisely rewritten in terms of \( \phi, \Lambda, \bar{\Lambda} \):
\[
S_{1}^{bos} = \int dt \left[ \left( \dot{\phi} - i \frac{\Lambda \dot{X} - \Lambda \dot{\bar{X}}}{1 + \Lambda \bar{\Lambda}} \right)^2 + \frac{4 \Lambda \dot{X}}{(1 + \Lambda \bar{\Lambda})^2} \right].
\]
(5.16)
The term within the parentheses is just the \( U(1) \) Cartan form of the R-symmetry group \( SU(2) \) in the parametrization by \( (\phi, \Lambda, \bar{\Lambda}) \), while the second term is the \( d = 1 \) pullback of the standard metric on the coset \( SU(2)/U(1) \). These two parts of the action (5.10) are separately invariant under the left shifts of the R-symmetry \( SU(2) \), while their strict ratio is fixed by the second \( SU(2) \) invariance (5.12), which rotates the \( U(1) \) and \( SU(2)/U(1) \) Cartan forms through each other. Thus the action (5.10) actually defines a \( d = 1 \) sigma model on the coset \( SU(2) \times SU(2)/SU(2)_{diag} \) or, in other words, a sigma model of principal chiral field on \( SU(2) \). So the superfield action (5.10) yields \( N = 4 \) superextension of this sigma model. In the general action (5.9) both these \( SU(2) \) symmetries can be broken. Note that the action (5.9) and its particular case (5.10) are trivially invariant under third \( SU(2) \) group which is contained in the original supergroup \( D(2,1;\alpha) \) and from the very beginning was placed into the stability subgroup in our construction. It acts only on the fermionic fields and rotates them through their conjugates.

Finally, we can write a composite version of the “nonlinear” multiplet by expressing \( N^{ai} \) in terms of the “angular” part of the \( d = 1 \) “hypermultiplet”
\[
N^{ai} = \sqrt{2} \frac{q^{ai}}{|q|},
\]
(5.17)
where \( q^{ai} \equiv (q^i, \bar{q}^i) \) and \(|q| = \sqrt{q^{ai} q_{ai}}\). As a consequence of (5.4), such a composite \( N^{ai} \) automatically obeys the constraints (5.5). However, the representation (5.17) is very restrictive, since it expresses the auxiliary field of “nonlinear” multiplet through the time derivative of physical bosonic fields of the “hypermultiplet” \( q^i \). It is worth noting that this substitution linearizes the sigma model action (5.10) when the auxiliary fields of \( N^{ai} \) are retained. The auxiliary fields term in the action becomes a kinetic term for the radial
part \(|q|\) of \(q^{a_i}\) and it combines with the sigma model Lagrangian for the angular part of \(q^{a_i}\) in such a way that one finally ends up with the free \(SU(2) \times SU(2)\) invariant kinetic term for the \(SO(4)\) vector \(q^{a_i}\).

One can reformulate the “nonlinear” multiplet in terms of analytic harmonic \(N = 4, d = 1\) superfield subjected to certain harmonic constraint \([16]\), in a full analogy with its \(N = 2, d = 4\) prototype \([21]\). Then, besides the sigma model action \((5.9)\), a superpotential term can be set up for this multiplet as an integral over the analytic harmonic \(N = 4, d = 1\) superspace. We plan to perform a more detailed analysis of possible actions for the \(d = 1\) nonlinear multiplet and construction of the corresponding SQM models elsewhere.

6 N=4, d=1 “nonlinear chiral” multiplet

In this Section we shall consider the last possibility for choosing different bosonic submanifolds in the coset \((2.6)\). Namely, we shall keep in the coset only bosonic superfields parametrizing the sphere \(S^2 \sim SU(2)/U(1)\). Hence, we put \(u = 0, z = 0, \phi = 0\) in the coset element \((2.6)\). Now we have only two bosonic superfields \(\varphi, \bar{\varphi}\) (or \(\Lambda, \bar{\Lambda}\) related to the former ones by the equivalence relation \((2.11)\)). We impose the following constraints:

\[\omega_V| = \bar{\omega}_V| = 0.\]  

(6.1)

Explicitly, after elimination of the fermionic Goldstone superfields, these constraints amount to the irreducibility conditions

\[D_1 \Lambda = -\Lambda D_2 \Lambda, \quad \bar{D}_2 \Lambda = \Lambda \bar{D}_1 \Lambda \quad \text{(and c.c.)}.\]  

(6.2)

The superfields \(\Lambda, \bar{\Lambda}\) obeying \((6.2)\) are recognized as a nonlinear modification of standard chirality constraints. The crucial difference from the latter lies in that the constraints \((6.2)\) are covariant with respect to \(D(2,1;\alpha)\) group for any \(\alpha\), while the chirality constraints are covariant only for \(SU(1,1|2) \sim D(2,1;-1)\). The constraints \((6.2)\) leave in \(\Lambda, \bar{\Lambda}\) the following independent superfields projections:

\[\Lambda, \bar{\Lambda}, \psi = -D_1 \bar{\Lambda}, \quad \bar{\psi} = \bar{D}_1 \Lambda,\]

\[\xi = D_2 \Lambda, \quad \bar{\xi} = -\bar{D}_2 \bar{\Lambda}, \quad B = \bar{D}_1 D_2 \Lambda, \quad \bar{B} = \bar{D}_2 D_1 \bar{\Lambda},\]  

(6.3)

which are just the irreducible \((2,4,2)\) field content with \(\Lambda, \bar{\Lambda}\) and \(B, \bar{B}\) being the physical and auxiliary bosonic fields, respectively.

Like in the previous case of “nonlinear” multiplet, the general sigma-model type action of \(\Lambda, \bar{\Lambda}\) possesses only \(N = 4, d = 1\) super Poincaré invariance and is given by

\[S_\Lambda = \int dt d^4\theta \, L (\Lambda, \bar{\Lambda}) ,\]  

(6.4)

where \(L (\Lambda, \bar{\Lambda})\) is an arbitrary real function of \(\Lambda, \bar{\Lambda}\). The more restrictive case corresponds to preserving at least \(U(1)\) symmetry generated by \(V_3\). The relevant action is

\[S_2 = \int dt d^4\theta \, f(\Lambda \bar{\Lambda}) ,\]  

(6.5)
where $f$ is a real function of the product $\Lambda \overline{\Lambda}$. After passing to the components (6.3) and eliminating the auxiliary fields by their equations of motion, action (6.5) takes the following form:

$$S_2 = \int dt \left\{ -4g \frac{\dot{\Lambda} \dot{\overline{\Lambda}}}{1 + \Lambda \overline{\Lambda}} + 2ig \left( \psi \dot{\psi} + \xi \dot{\xi} \right) + 2ig' \left( \dot{\Lambda} \overline{\Lambda} \psi \dot{\overline{\psi}} + \Lambda \overline{\Lambda} \xi \dot{\xi} \right) - 2ig \frac{\xi \dot{\psi} + \dot{\xi} \psi}{1 + \Lambda \overline{\Lambda}} \right\} + \left[ \Lambda \overline{\Lambda} (1 + \Lambda \overline{\Lambda}) \left( g'' - \frac{(g')^2}{g} \right) + (1 + \Lambda \overline{\Lambda}) g' + \frac{g}{1 + \Lambda \overline{\Lambda}} \right] \xi \dot{\psi} \dot{\overline{\psi}} .$$

(6.6)

Here, $g \equiv (1 + \Lambda \overline{\Lambda}) \left[ f'' \Lambda \overline{\Lambda} + f' \right]$ and prime means derivative with respect to $(\Lambda \overline{\Lambda})$.

Let us remind that the superfields $\Lambda, \overline{\Lambda}$ do not contain dilaton (generator $D$ is now in the stability subgroup). Therefore, like in the case of “nonlinear” multiplet, it is impossible to construct superconformally invariant actions of the sigma-model type out of $\Lambda, \overline{\Lambda}$ alone. The superconformally invariant action can likely be constructed only by coupling the “nonlinear chiral” multiplet to some other $N = 4$ supermultiplet containing a dilaton among its field components. On the other hand, it is easy to construct the action which is invariant under the R-symmetry $SU(2)$ transformations (2.20)

$$S_3 = -\int dtd^4 \theta \ln (1 + \Lambda \overline{\Lambda})$$

(6.7)

(the general action (6.4) and (6.7) are trivially invariant under the group $SU(2)$ which belongs to the stability subgroup and acts only on fermions). In components the action (6.7) takes the form

$$S_2 = \int dt \left[ \frac{4\dot{\Lambda} \dot{\overline{\Lambda}}}{(1 + \Lambda \overline{\Lambda})^2} - 2i \dot{\psi} \dot{\overline{\psi}} + \frac{\xi \dot{\xi}}{1 + \Lambda \overline{\Lambda}} + 2i \frac{\dot{\Lambda} \overline{\Lambda} \psi \dot{\overline{\psi}} + \Lambda \overline{\Lambda} \xi \dot{\xi} + \dot{\Lambda} \xi \dot{\psi} + \dot{\Lambda} \psi \dot{\overline{\xi}}}{(1 + \Lambda \overline{\Lambda})^2} \right] .$$

(6.8)

It is drastically simplified after passing to the new fermionic fields

$$\tilde{\psi} = \frac{\psi + \xi \overline{\Lambda}}{1 + \Lambda \overline{\Lambda}}, \quad \tilde{\xi} = \frac{\xi - \Lambda \psi}{1 + \Lambda \overline{\Lambda}} ,$$

(6.9)

in terms of which it reads

$$S_3 = \int dt \left( \frac{4\dot{\Lambda} \dot{\overline{\Lambda}}}{(1 + \Lambda \overline{\Lambda})^2} - 2i \dot{\tilde{\psi}} \dot{\tilde{\overline{\psi}}} - 2i \dot{\tilde{\xi}} \dot{\tilde{\overline{\xi}}} \right) .$$

(6.10)

We thus conclude that the action (6.7) describes a $N = 4$ superextension of the $d = 1$ $SU(2)/U(1)$ nonlinear sigma model.

The “nonlinear chiral” multiplet can be constructed as a composite one in terms of the “hypermultiplet”, “tensor” and even “nonlinear” $d = 1$ multiplets. The corresponding expressions can be easily found using (3.3), (4.3) and (5.3). In particular, in terms of the “hypermultiplet” the superfields $\Lambda, \overline{\Lambda}$ can be expressed as:

$$\Lambda = -\frac{q^1}{q^2}, \quad \overline{\Lambda} = -\frac{\bar{q}^1}{\bar{q}^2} .$$

(6.11)
Of course, these realizations are very special since the nonlinear chiral multiplet contains more auxiliary fields (just 2) compared to other multiplets (just 1 or 0). Therefore, in all these realizations some of the auxiliary fields present in Λ, Λ (or even all in the case of “hypermultiplet”) are expressed via the time derivatives of physical components of q^i, V^{ij} or N^a.

Finally, we would like to point out that = 2, d = 4 analog of “nonlinear chiral” multiplet is unknown. A formal generalization of the constraints (6.12) to the d = 4 case reads

$$D^1_\alpha \Lambda = -\Lambda D^2_\alpha \Lambda, \quad \overline{D}^1_\alpha \Lambda = -\Lambda \overline{D}^2_\alpha \Lambda .$$

After passing to the new covariant derivatives defined as

$$\mathcal{D}^1_\alpha = D^1_\alpha + \Lambda D^2_\alpha, \quad \overline{\mathcal{D}}^1_\alpha = \overline{D}^1_\alpha + \Lambda \overline{D}^2_\alpha$$

the constraints (6.12) take the very simple form

$$\mathcal{D}^1_\alpha \Lambda = 0, \quad \overline{\mathcal{D}}^1_\alpha \Lambda = 0 .$$

Formally, the derivatives (6.13) look like covariant derivatives in the harmonic N = 2, d = 4 superspace [21], with Λ, Λ being “harmonics” in a special gauge. Then the constraints (6.14) could be treated as the Grassmann harmonic analyticity conditions for Λ, Λ. Hence, the constraints (6.12) may be interpreted as the ones describing a special nonlinear realization in harmonics superspace, with harmonics becoming Goldstone N = 2 superfields. Of course, the same interpretation can be done in the projective superspace [22]. In this case the extra complex variable, which is usually introduced to define the set of anticommuting spinor derivatives in the projective superspace, should be considered as an N = 2 superfield obeying nonlinear chirality conditions. The detailed discussion of such a new type of N = 2, d = 4 supermultiplets is out of the scope of the present paper and will be given elsewhere.

7 Special case: su(1,1|2) with central charges

In the previous Sections we considered some multiplets on which the supergroup D(2, 1; α) can be realized. They exist for arbitrary values of parameter α. The special case α = −1, when superalgebra D(2, 1; −1) is isomorphic to the semi-direct sum of su(1, 1|2) and su(2), is also admissible. The main peculiarity of this case is the special form of superconformal invariant actions for the “hypermultiplet” and “tensor” multiplet.

Let us remind that the choice α = 0 is inadmissible with our definition of the coset space (2.6) because the Cartan forms ˆω_V, ˆω_V, ˆω_V^3 (2.9) would be equal to zero in this case. Hence, it would be impossible to relate the covariant derivatives of scalar Goldstone superfields φ, φ, φ to the fermions ψ_i, ˘ψ_i. However, there is still a possibility to reach α = 0, but for the properly contracted D(2, 1; α). Namely, let us redefine the su(2) generators V, ˘V, V_3 as

$$Z \equiv \alpha V, \quad \overline{Z} \equiv \alpha \overline{V}, \quad Z_3 \equiv \alpha V_3 .$$

All the (anti)commutation relations of D(2, 1; α) can be rewritten in terms of Z, Z, Z_3. Setting α = 0 in these relations leaves us with a su(1, 1|2) superalgebra extended by three central charges. Indeed, from (2.1)–(2.4) it immediately follows that for the choice α = 0
the generators $Z, \overline{Z}, Z_3$ commute with everything (including themselves) but they still appear in the cross-anticommutators of the Poincaré and conformal supercharges:

\[
\begin{align*}
\{Q^i, S^j\} &= -2e^{ij}T, \quad \{Q^1, \overline{S}_2\} = 2\overline{Z}, \quad \{Q^1, \overline{S}_1\} = -2D - 2Z_3 + 2T_3, \\
\{Q^2, \overline{S}_1\} &= -2Z, \quad \{Q^2, \overline{S}_2\} = -2D + 2Z_3 + 2T_3.
\end{align*}
\] (7.2)

Now we can define a realization of this extension of $SU(1,1|2)$ by three central charges in the following coset:

\[
g = e^{i\theta_1}e^{\phi_1}e^{\psi_1}e^{\overline{\psi}_1}e^{izK}e^{iuD}e^{i\phi Z + i\overline{\phi}\overline{Z} + \phi Z_3}
\] (7.3)

(as before, the other $su(2)$ generators $T, \overline{T}, T_3$ are placed into the stability subgroup).

Now the Cartan forms \(2.8\) are drastically simplified:

\[
\begin{align*}
\omega_D &= i du - 2 \left( \bar{\psi} i d\theta_1 + \psi_1 d\bar{\theta} \right) - 2izd\bar{t}, \\
\omega_Z &= id\varphi + 2 \left[ \psi_2 d\bar{\theta} - \bar{\psi} \left( d\theta_2 - \psi_2 d\bar{t} \right) \right], \\
\omega_{Z_3} &= d\phi + 2 \left[ \psi_1 d\bar{\theta} - \bar{\psi} \left( d\theta_1 - \psi_2 d\bar{t} \right) + \bar{\psi} \left( d\bar{\theta} - \psi_2 d\theta_2 \right) + (d\psi_1 - \bar{\psi} \psi_2) d\bar{t} \right].
\end{align*}
\] (7.4)

The superconformal transformations are generated by the left shifts of the coset element \(7.3\) by the supergroup element \(2.15\)

\[
g_1 = e^{ie_i S^i + e_\psi \overline{S}_i}.
\]

Explicitly, they are

\[
\begin{align*}
\delta t &= -it \left( \epsilon \cdot \bar{\theta} + \bar{\epsilon} \cdot \theta \right) + \theta \cdot \bar{\theta} \left( \epsilon \cdot \bar{\theta} - \bar{\epsilon} \cdot \theta \right), \\
\delta u &= -2i \left( \epsilon \cdot \bar{\theta} + \bar{\epsilon} \cdot \theta \right), \\
\delta \varphi &= 2 \left[ e_1 \theta_1 - e^2 \theta_2 - e_1 \bar{\theta} \right], \\
\delta \overline{\varphi} &= 2i \left( \theta_2 e_1 - \bar{\theta} e_2 \right).
\end{align*}
\] (7.5)

Now we shall list the $N = 4, d = 1$ supermultiplets which can be obtained by applying our procedure to this central charge-extended $su(1,1|2)$.

7.1 “Hypermultiplet”

As in the Section 3, we place all four bosonic superfields $u, \varphi, \bar{\varphi}, \phi$ into the coset and impose the following constraints

\[
\omega_D = \omega_Z = \omega_{Z_3} = 0,
\] (7.6)

or, explicitly,

\[
\begin{align*}
D^1 \varphi &= 0, \quad D^1 \bar{\varphi} = -2i \bar{\psi}^2, \quad D^1 \phi = -2i \bar{\psi}^1, \quad D^1 u = 2i \bar{\psi}, \\
D^2 \varphi &= 2i \bar{\psi}, \quad D^2 \bar{\varphi} = 0, \quad D^2 \phi = 2 \bar{\psi}^2, \quad D^2 u = 2i \bar{\psi}^2, \quad \bar{u} = 2z
\end{align*}
\] (7.7)

(and c.c.). After passing to the new variables

\[
\begin{align*}
q^1 &= \varphi, \quad q^2 = -\frac{1}{2}(u - i\phi), \quad \bar{q}_1 = \bar{\varphi}, \quad \bar{q}_2 = -\frac{1}{2}(u + i\phi),
\end{align*}
\] (7.8)
the constraints acquire the familiar form
\[ D^{(i} q^{j)} = 0, \quad \overline{D}^{(i} q^{j)} = 0. \]  (7.9)

Surely, no \( SU(2) \) group realized on the doublet indices of \( q^i \) and \( D^i \) is present now, despite the fact that the constraints look \( SU(2) \)-covariant. The simplest \( SU(1,1|2) \) invariant action can be written as
\[ S = \int dt d^4 \theta e^{-(q^2 + \bar{q}^2)(q^2 + \bar{q}^2) \equiv -\int dt d^4 \theta e^{u u}. \]  (7.10)

### 7.2 “Tensor” multiplet

In this case the constraints formally coincide with (4.1), but look much simpler when rewritten in the explicit form:
\[ D^1 \varphi = 0, \quad D^1 \bar{\varphi} = -2i \bar{\psi}^2, \quad D^1 u = 2i \bar{\psi}^1, \]
\[ D^2 \varphi = 2i \bar{\psi}^1, \quad D^2 \bar{\varphi} = 0, \quad D^2 u = 2i \bar{\psi}^2 \]  (7.11)

(and c.c.). Once again, we can introduce a new \( N = 4 \) superfield \( V^{ij} \) such that \( V^{ij} = V^{ji} \) and \( V^i_\ell = \epsilon_\ell^\ell' \epsilon_\kappa^\kappa' V^i_\kappa' \) via the identification
\[ V^{11} = -i \sqrt{2} \varphi, \quad V^{22} = i \sqrt{2} \bar{\varphi}, \quad V^{12} = \frac{i}{\sqrt{2}} u \]  (7.12)

and rewrite the constraints (7.11) as
\[ D^{(i} V^{jk)} = 0, \quad \overline{D}^{(i} V^{jk)} = 0. \]  (7.13)

Thus, we have the same “tensor” multiplet as before but with much simpler expression for it in terms of coset fields. The price we have to pay for this simplicity is the lost of \( su(2) \) invariance.

The invariant action for this case in terms of \( N = 2 \) projections of \( N = 4 \) superfields
\[ \tilde{u} = u|, \quad \lambda = \varphi|, \quad \tilde{\lambda} = \bar{\varphi} |, \]  (7.14)

where \( | \) means restriction to \( \theta^2 = \bar{\theta}^2 = 0 \), has the following form
\[ S = -\frac{1}{2} \int dt d^2 \theta \left[ e^{\tilde{u}} \left( D D \tilde{u} + D \lambda \overline{D} \lambda \right) + \kappa \ln \left( \frac{\tilde{u} + \sqrt{\tilde{u}^2 + 4 \lambda \bar{\lambda}}}{2} \right) \right]. \]  (7.15)

It contains the kinetic and potential terms in analogy with the \( SU(2) \) “tensor” multiplet of the Section 4.

### 7.3 “Nonlinear” and “nonlinear chiral” multiplets

The Cartan forms (7.14) for the case we are considering do not contain any nonlinearities. Hence, we could expect that “nonlinear” multiplets will actually become linear. This is indeed so.
For these “nonlinear” multiplets we obtain the following constraints

\[
D^1 \phi = 0, \quad D^1 \bar{\psi} = -2i \bar{\psi}^2, \quad D^1 \bar{\phi} = -2 \bar{\psi}^1, \\
D^2 \phi = 2i \bar{\psi}^1, \quad D^2 \bar{\phi} = 0, \quad D^2 \bar{\phi} = 2 \bar{\psi}^2 
\]  
(7.16)

(and c.c.). They can be rewritten just as (7.13). Therefore, “nonlinear” multiplet for the contracted superalgebra coincides with the “tensor” one. But their transformation properties are still radically different, because the “tensor” multiplet contains the dilaton \(u\) while in the “nonlinear” case we deal only with Goldstone fields for the central charge generators.

A similar linearization takes place for the nonlinear chiral multiplet. Indeed, the constraints now read

\[
D^1 \phi = 0, \quad D^2 \phi = 0 
\]  
(7.17)

and define a sort of twisted chiral \(d = 1\) multiplet. Any Lagrangian function of these superfields will respect manifest off-shell \(N = 4, d = 1\) supersymmetry. In these cases, as distinct from the options considered in the previous Sections, one cannot construct out of these superfields alone not only superconformally invariant, but also \(SU(2)\) invariant actions (though still persists the “trivial” \(SU(2)\) realized on fermions only).

### 7.4 “Old” multiplets

Here we mention two additional possibilities which exist for the central charge extended \(su(1,1|2)\) superalgebra. Actually, one of them exists as well in the general \(D(2,1;\alpha)\) case.

First of all, we could consider the case when the set of bosonic Goldstone superfields in the coset (7.3) includes only dilaton \(u\) and one extra Goldstone superfield \(\phi\) associated with the central charge \(Z_3\). The set of constraints in this case

\[
D^1 \phi = -2 \bar{\psi}^1, \quad D^1 u = 2i \bar{\psi}^1, \quad D^2 \phi = 2 \bar{\psi}^2, \quad D^2 u = 2i \bar{\psi}^2 
\]  
(7.18)

defines the ordinary \(N = 4\) chiral superfield \(u + i \phi\). Let us remind that the chirality conditions are compatible only with the \(su(1,1|2)\) superalgebra. In more detail such a supermultiplet and the corresponding actions have been considered in [23].

The last possibility corresponds to retaining the single dilaton \(u\) in the bosonic part of the coset (7.3). In this case no any constraints appear since four Goldstone fermions are expressed through four spinor derivatives of \(u\). As was shown in [23], one should impose some additional irreducibility constraints on dilaton \(u\)

\[
D^i D_i e^{-\alpha u} = \overline{D}_i D_i e^{-\alpha u} = [D^i, \overline{D}_i] e^{-\alpha u} = 0 
\]  
(7.19)

in order to pick up in \(u\) the minimal off-shell field content \((1,4,3)\). Once again, the detailed discussion of this case can be found in [23].

In fact, the same multiplet can be derived from the nonlinear realization of \(D(2,1;\alpha)\) in its coset (2.6), with all bosonic superfields except the dilaton being equal to zero. This clearly corresponds to placing both internal \(SU(2)\) groups into the stability subgroup. The only constraint is

\[
\omega_D = 0, 
\]  
(7.20)
and it serves to trade the Goldstone fermions for the spinor derivatives of \( u \) without implying any additional constraints for \( u \). One can further constrain \( u \) by (7.19) and check the covariance of this set of constraints under the whole \( D(2, 1; \alpha) \). This example shows that our method is directly applicable for deriving only those irreducible \( N=4, d=1 \) superfields which are defined by constraints of the first order in spinor derivatives. On the other hand, we could re-derive the multiplet \( u \) from our \( V^{ik} \) discussed in [7] and Section 4. Indeed, one can construct the composite superfield

\[
e^{-\alpha \tilde{u}} = \frac{1}{\sqrt{V^2}},
\]

which obeys just the constraints (7.19) as a consequence of (4.1). The relation (7.21) is of the same type as the previously explored substitutions (4.10) or (5.17) and expresses two out of the three auxiliary fields of \( \tilde{u} \) via physical bosonic fields of \( V^{ik} \) and time derivatives thereof. The \( D(2, 1; \alpha) \) invariant action for the superfield \( u \) at \( \alpha \neq -1 \) and \( \alpha = -1 \) can be then obtained by the substitution \( \sqrt{V^2} \to e^{\alpha u} \) in (4.8) and (4.9)

\[
S_u^{(\alpha)} = \int dt d^4 \theta e^u, \quad S_u^{(\alpha=-1)} = \int dt d^4 \theta e^u.
\]

Note that in [23] only the action for the \( SU(1, 1|2) \) case (i.e. with \( \alpha = -1 \)) was considered.

### 8 Summary and concluding remarks

In this paper we showed that a lot of off-shell \( N=4, d=1 \) supermultiplets with irreducibility constraints of first order in spinor derivatives can be self-consistently derived from non-linear realizations of the most general \( N=4, d=1 \) superconformal group \( D(2, 1; \alpha) \) in its appropriate coset superspaces. Multiplets with irreducibility constraints of higher order in spinor derivatives can be derived from these basic ones via a chain of proper substitutions. In this way we derived most of the previously known multiplets as well as two new multiplets which have not yet been exploited in constructing \( N=4 \) supersymmetric SQM models.

Our results are summarized in the Table below.

| multiplet        | content     | R symmetry coset     | dilaton | \( \alpha \) | superfield |
|------------------|-------------|----------------------|---------|-------------|------------|
| “old tensor”     | (1,4,3)     | –                    | yes     | any         | \( u \)    |
| chiral           | (2,4,2)     | central charge       | yes     | 0, -1       | \( \phi, \bar{\phi} \) |
| nonlinear chiral | (2,4,2)     | \( su(2)/u(1) \)    | no      | any         | \( \Lambda, \bar{\Lambda} \) |
| tensor           | (3,4,1)     | \( su(2)/u(1) \)    | yes     | any         | \( V^{ij} \) |
| nonlinear        | (3,4,1)     | \( su(2) \)         | no      | any         | \( N^{ia} \) |
| hypermultiplet   | (4,4,0)     | \( su(2) \)         | yes     | any         | \( q^{ia} \) |

Using the same method, we also derived \( N=4 \) multiplets associated with nonlinear realizations of the supergroup \( SU(1, 1|2) \) modified by three central charges. Our method automatically yields the superconformal transformation properties of the final irreducible superfields, which is helpful in constructing their superconformally invariant actions. However, these superfields can be used equally well for constructing actions which in general exhibit only Poincaré supersymmetry.
As an interesting project for further study we mention the explicit construction of SQM models associated with the new $N=4, d=1$ supermultiplets found here. Another promising task includes generalizing these techniques to $N>4, d=1$ supersymmetries, classifying with its help the corresponding supermultiplets and constructing examples of the SQM models associated with these supermultiplets. At present not too much is known about SQM models with such a large amount of supersymmetry. As a first step towards this goal one may attack the case of $N=8, d=1$ supersymmetry [24].

Finally, let us briefly argue why there is no need to also consider the situation where generators from both $SU(2)$ groups $(V, \bar{V}, V_3)$ and $(T, \bar{T}, T_3)$ are present in the coset (2.6). For generic $\alpha$ there are not too many possibilities to place generators from both $SU(2)$ groups into the coset. Indeed, we have only four fermionic superfields in the game. Therefore, the maximal number of bosonic superfields must be less then or equal to four. This leaves only three possibilities for the bosonic coset:

a) $SU(2) \times SU(2)/SU(2)_{\text{diag}}$;
b) $SU(2) \times SU(2)/SU(2)_{\text{diag}}$ plus dilaton $u$;
c) $[SU(2)/U(1)] \times [SU(2)/U(1)]$.

Let us dwell on the first case. At the linearized level the corresponding constraints read

$$iD^1 \phi = 0, \quad iD^2 \phi + \alpha \bar{\psi}_1 = 0, \quad i\bar{D}_1 \phi - \alpha \psi_2 + (1 + \alpha) \bar{\psi}_1 = 0, \quad i\bar{D}_2 \phi + (1 + \alpha) \bar{\psi}_2 = 0,$$

$$D^1 \phi_3 + (1 + 2\alpha) \bar{\psi}_1 = 0, \quad D^2 \phi_3 + \bar{\psi}_2 = 0, \quad \bar{D}_1 \phi_3 - \psi_1 = 0, \quad \bar{D}_2 \phi_3 - \psi_2 = 0,$$

where the $N=4$ superfields $\phi, \bar{\phi}, \phi_3$ parametrize the coset $SU(2) \times SU(2)/SU(2)_{\text{diag}}$. It is easy to observe that these constraints together with their complex conjugates imply

$$\dot{\phi} = \dot{\bar{\phi}} = \dot{\phi}_3 = 0.$$

Apparently, these constraints are too strong, and the corresponding case should be excluded from our consideration. The same is true for the other two possibilities mentioned above.

We close by noting that one can construct “mirror” $N=4, d=1$ multiplets by switching between the two internal $SU(2)$ groups: place the one acting on indices $i, k$ into the stability subgroup while lifting the other one up into the coset. Clearly, the resulting superfield constraints will look the same as before, modulo a different splitting of the $N=4, d=1$ superspace coordinates and spinor derivatives into the $SU(2)$ doublets. New interesting possibilities may arise from actions which include both types of multiplets simultaneously.

**Acknowledgments**

This work was partially supported by INTAS grant No 00-00254, RFBR-DFG grant No 02-02-04002, grant DFG No 436 RUS 113/669, RFBR grant No 03-02-17440 and a grant of the Heisenberg-Landau programme. E.I. and S.K. thank the Institute for Theoretical Physics of the University of Hannover for the warm hospitality extended to them during the course of this work.
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