The Structure of Minimum Vertex Cuts*

Seth Pettie
University of Michigan
pettie@umich.edu

Longhui Yin
Tsinghua University
ylh17@mails.tsinghua.edu.cn

Abstract

In this paper we continue a long line of work on representing the cut structure of graphs. We classify the types minimum vertex cuts, and the possible relationships between multiple minimum vertex cuts.

As a consequence of these investigations, we exhibit a simple $O(\kappa n)$-space data structure that can quickly answer pairwise $(\kappa + 1)$-connectivity queries in a $\kappa$-connected graph. We also show how to compute the “closest” $\kappa$-cut to every vertex in near linear $\tilde{O}(m + \text{poly}(\kappa)n)$ time.

*This work was supported by NSF grants CCF-1637546 and CCF-1815316.
1 Introduction

One of the strong themes running through graph theory is to understand the cut structure of graphs and to apply these structural theorems to solve algorithmic and data structural problems. Consider the following exemplars of this line of work:

**Gomory-Hu Tree.** Gomory and Hu (1961) [28] proved that any weighted, undirected graph $G = (V, E)$ can be replaced by a weighted, undirected tree $T = (V, E_T)$ such that for every $s, t \in V$, the minimum $s$-$t$ cut partition in $T$ (removing a single edge, partitioning $V$ into two sets) corresponds to a minimum $s$-$t$ cut partition in $G$. These are sometimes called cut-equivalent trees [1].

**Cactus Representations.** Dinitz, Karzanov, and Lomonosov (1976) [12] proved that all the global minimum edge-cuts of any weighted, undirected graph $G = (V, E)$ could be succinctly encoded as an (unweighted) cactus graph. A cactus is a multigraph in which every edge participates in exactly one cycle. It was proved that there exists a cactus $C = (V_C, E_C)$ and an embedding $\phi : V \rightarrow V_C$ such that the minimum edge-cuts in $C$ (2 edges in a common cycle) are in 1-1 correspondence with the minimum edge-cuts of $G$. A corollary of this theorem is that there are at most $\binom{n}{2}$ minimum edge-cuts.

**Picard-Queyrenne Representation.** In a directed $s$-$t$ flow network there can be exponentially many min $s$-$t$ cuts. Picard and Queyrenne (1980) [38] proved that the family $\mathcal{S} = \{S \mid (S, \overline{S}) \text{ is a min } s$-$t$} \}$ corresponds 1-1 with the downward-closed sets of a partial order, and is therefore closed under union and intersection.

**Block Trees, SPQR Trees, and Beyond.** Whitney (1932) [43,44] proved that the cut vertices (articulation points) of an undirected graph $G = (V, E)$ partition $E$ into single edges and 2-edge connected components (blocks). This yields the block tree representation. Di Battista and Tamassia (1989) [4,5] formally defined the SPQR tree, which succinctly encodes all 2-vertex cuts in a biconnected graph, and Kanevsky, Tamassia, Di Battista, and Chen [31] extended this structure to represent 3-vertex cuts in a triconnected graph.\(^1\)

It is natural to ask how, and to what extent, these structures can be extended and generalized. Gusfield and Naor [29] described an analogue of Gomory-Hu trees for vertex connectivity, i.e., a tree that compactly represents the $s$-$t$ vertex connectivity for every $s, t \in V$. It used a result of Schnorr [39] on an analogue of Gomory-Hu trees for “roundtrip” flow-values in directed networks. These claims were refuted by Benczur [6], who illustrated that Schnorr’s and Gusfield and Naor’s proofs were incorrect and could not be rectified. In particular, $s$-$t$ vertex connectivity and directed $s$-$t$ cuts have no tree representation. We take this as a reminder that having published proofs (even incorrect ones) is essential for facilitating self-correction in science.

The inspiration for this paper is an extended abstract of Cohen, Di Battista, Kanevsky, and Tamassia [10] from STOC 1993. Their goal was to find a cactus-analogue for global minimum vertex cuts, or from a different perspective, to extend SPQR trees [5] and [31] from $\kappa \in \{2, 3\}$ vertex cuts to arbitrarily large $\kappa$. As an application of their ideas, they described a data structure for $\kappa$-connected graphs occupying space $O(\kappa^3n)$ that, given $u, v$, decided whether $u, v$ are separated by a $\kappa$-cut or $(\kappa + 1)$-connected. There are no suspect claims in [10]. On the other hand, the paper

\(^1\)Many of the structural insights behind [5,31] were latent in prior work. See, for example. Mac Lane [34] (1937), Tutte [41,42] (1961-6), Hopcroft and Tarjan [30], and Cunningham and Edmonds [11].
Figure 1: (a) A weighted undirected graph; (b) Its Gomory-Hu (cut-equivalent) tree [28]. (c) A weighted undirected graph (unmarked edges have unit weight); (d) the Cactus representation [12] of its minimum edge cuts. (e) A directed s-t flow network; (f) A dag whose downward-closed sets (that include s but not t) correspond to min s-t cuts (Picard-Queyrenne [38]). (g) An abstract representation of a 2-connected graph; (h) The representation of its 3-connected components as an SPQR tree (Di Battista-Tamassia [5]).
is 7 pages and leaves many of its central claims unproven.\textsuperscript{2} We believe that understanding the structure of minimum vertex cuts is a fundamental problem in graph theory, and deserving of a complete, formal treatment.

In this paper we investigate the structure of the set of all minimum vertex cuts and classify the relationships between different minimum vertex cuts. Our work reveals some structural features of minimum $\kappa$-cuts not evident in Cohen, Di Battista, Kanevsky, and Tamassia \cite{10}, and ultimately allows us to develop a simpler data structure to answer pairwise $\kappa$-cut queries in a $\kappa$-connected graph. It occupies (optimal) $O(\kappa n)$ space and can be constructed in randomized $\tilde{O}(m + \text{poly}(\kappa)n)$ time, in contrast to \cite{10}, which occupies $O(\kappa^3n)$ space and is constructed in $\text{exp}(\kappa)n^5$ time.\textsuperscript{3}

1.1 Related Work

Dinitz and Vainshtein \cite{16,17} combined elements of the cactus \cite{12} and Picard-Queyrenne \cite{38} representations, which they called the connectivity carcass. Given an undirected, unweighted $G = (V,E)$ and $S \subseteq V$ of terminals, $\lambda_S$ is the size of the minimum edge-cut that separates $S$. The carcass represents all size-$\lambda_S$ separating cuts in $O(\min\{m, \lambda_SN\})$ space and answers various cut queries in $O(1)$ time.\textsuperscript{4}

Benczur and Goemans \cite{7} generalized the cactus representation \cite{12} in a different direction, by giving a compact representation of all cuts that are within a factor 6/5 of the global minimum edge-cut.

Dinitz and Nutov \cite{13} generalized the cactus representation \cite{12} in another direction, by giving an $O(n)$-space representation of all $\lambda$ and $\lambda + 1$ edge cuts, where $\lambda$ is the edge-connectivity of the undirected, unweighted graph. Unpublished manuscripts \cite{14,15} give detailed treatments of the $\lambda$ odd and $\lambda$ even cases separately.

Georgiadis et al. \cite{21,25–27} investigated various notions of 1- and 2-edge and vertex connectivity in directed graphs, and the compact representation of edge/vertex cuts.

Sparsification. One general way to compactly represent connectivity information is to produce a sparse graph with the same cut structure. Nagamochi and Ibaraki \cite{36} proved that every unweighted, undirected graph $G = (V,E)$ contains a subgraph $H = (V,E_H)$ with $|E_H| < (k+1)n$ such that $H$ is computable in $O(m)$ time and contains exactly the same $k'$-vertex cuts and $k'$-edge cuts as $G$, for all $k' \in \{1, \ldots, k\}$. Benczur and Karger \cite{8} proved that for any capacitated, undirected graph $G = (V,E)$, there is another capacitated graph $H = (V,E_H)$ with $|E_H| = O(\epsilon^{-2}n \log n)$ such that the capacity of every cut in $G$ is preserved in $H$ up to a $(1 \pm \epsilon)$-factor. This bound was later improved to $O(\epsilon^{-2}n)$ by Batson, Spielman, and Srivastava \cite{3}, which is optimal.

In directed graphs, Baswana, Choudhary, and Roditty \cite{2} considered the problem of finding a sparse subgraph that preserves reachability from a single source, even if $d$ vertices are deleted. They proved that $\Theta(2^d n)$ edges are necessary and sufficient for $d \in [1, \log n]$.

$d$-Failure Connectivity. An undirected graph can be compactly represented such that connectivity queries can be answered after the deletion of any $d$ vertices/edges (where $d$ could be much

\textsuperscript{2}The full version of this paper was never written (personal communication with R. Tamassia, 2011, and R. Di Battista, 2016).

\textsuperscript{3}The algorithm enumerates all minimum $\kappa$-cuts, which can be as large as $\Omega(2^n(n/\kappa)^2)$; modern vertex connectivity algorithms \cite{22–24} may reduce the exponent of $n$ in the running time.

\textsuperscript{4}The carcass was introduced in extended abstracts \cite{16,17} and the (simpler) case of odd $\lambda_S$ was analyzed in detail in a journal article \cite{18}. We are not aware of a full treatment of the case when $\lambda_S$ is even.

3
larger than the underlying connectivity of the graph). Improving on [19, 32, 37], Duan and Pettie [20] proved that $d$ vertex failures could be processed in $\tilde{O}(d^2)$ time such that connectivity queries are answered in $O(d)$ time, and $d$ edge failures could be processed in $O(d \log d \log \log n)$ time such that connectivity queries are answered in $O(\log \log n)$ time. The size of the [20] structure is $\tilde{O}(m)$ for vertex failures and $\tilde{O}(n)$ for edge failures. Choudhary [9] gave an optimal $O(n)$-space data structure that could answer directed reachability queries after $d \in \{1, 2\}$ vertex or edge failures.

Labeling Schemes. Benczur’s refutation [6] of [29, 39] shows that all pairwise vertex connectivities cannot be captured in a tree structure, but it does not preclude other representations of this information. Korman [33] proved that the vertices of any undirected $G = (V, E)$ could be assigned $O(k^2 \log n)$-bit labels such that given $(\text{label}(u), \text{label}(v))$, we can determine whether $u$ and $v$ are $(k+1)$-connected or separated by a $k$-vertex cut. That is, $\text{poly}(\pi) \log n$-bit labels suffice to compute $\min\{\kappa(u,v), \pi\}$, where $\kappa(u,v)$ is the pairwise connectivity of $u, v$.

Vertex Connectivity Algorithms. In optimal linear time we can decide whether the connectivity of a graph is $\kappa = 1, \kappa = 2$, or $\kappa \geq 3$ [30, 40]. For larger $\kappa$, the state-of-the-art in vertex connectivity has been improved substantially in the last few years. Forster, Nanongkai, Yang, Saranurak, and Yingchareonthawornchai [22] gave a Monte Carlo algorithm for computing the vertex connectivity $\kappa$ of an undirected graph in $\tilde{O}(m + n\kappa^3)$ time, w.h.p. The best deterministic algorithm, due to Gao, Li, Nanongkai, Peng, Saranurak, and Yingchareonthawornchai [24], computes the connectivity $\kappa < n^{1/8}$ in $O((m + n^{7/4}\kappa^{O(\kappa)})n^{o(1)})$ time or $O((m + n^{19/20}\kappa^{5/2})n^{o(1)})$ time. For $\kappa > n^{1/8}$, Gabow’s algorithm [23] runs in $O(\kappa n^2 + \kappa^2 n \cdot \min\{n^{3/4}, \kappa^{3/2}\})$ time.

1.2 Organization

In Section 2 we review basic definitions and lemmas regarding vertex cuts. Section 3 gives the basic classification theorem for minimum vertex cuts, and lists some useful corollaries. In short, every pair of cuts have laminar, wheel, crossing matching, or small relation. Sections 3.1–3.4 analyze these four categories in more detail. Section 4 exhibits a new $O(\kappa n)$-space data structure that, given two vertices, answers $(\kappa + 1)$-connectivity queries in $O(1)$ time, and produces a separating $\kappa$-cut (if one exists) in $O(\kappa)$ time. We conclude with some remarks and open problems in Section 5.

2 Preliminaries

The input is a simple, connected, undirected graph $G = (V, E)$ with $n = |V|$ and $m = |E|$. The predicate $A \subset B$ is true if $A$ is a strict subset of $B$.

Let the subgraph of $G$ induced by $A$ be denoted $G|_A$. We call $U \subset V$ a cut if the graph $G|_{V \setminus U}$ is disconnected. A side of the cut $U$ is a connected component of $G|_{V \setminus U}$. If $P$ is a side of $U$ and $A \subseteq P$, we say $A$ is within a side of $U$, and let $\text{Side}_U(A) = P$ denote the side containing $A$. A region of a cut $U$ is a side, or the union of several sides of $U$. Denote $\text{Region}_U(A)$ as the region containing the sides of $U$ that intersects with $A$. We say a cut disconnects or separates $A$ and $B$ if they are in distinct sides of $U$. In particular, if $B = V \setminus (A \cup U)$, we say $U$ disconnects or separates $B$ from the rest of the graph.

---

5 The algorithm does not produce a witness, and hence may err with small probability.

6 Note when $A$ is a singleton set $\{u\}$, $\text{Region}_U(A) = \text{Side}_U(A)$. 

4
A path $\pi = v_1v_2\cdots v_l$ is from $A$ to $B$, if $v_1 \in A$ and $v_l \in B$. Two paths $\pi, \pi'$ from $v_1$ to $v_l$ are internally vertex disjoint if they have no common vertices, except for $v_1, v_l$. We say $U$ blocks $\pi$ if $U \cap \{v_2, \ldots, v_{l-1}\} \neq \emptyset$.

A $k$-cut is a cut of size $k$. Define $\kappa(u,v)$ to be the minimum $k$ such that there exists a $k$-cut separating $u$ and $v$, where $\{u,v\} \notin E(G)$. Define $\kappa = \kappa(G)$ to be the minimum of $\kappa(u,v)$ over all pairs $\{u,v\} \in (\binom{V(G)}{2}) \setminus E(G)$. We say $G$ is $k$-connected if $\kappa(G) \geq k$.

In this paper we assume that $\kappa < n/4$ and consider the set of all (minimum) $\kappa$-cuts.

**Remark 1.** There is some flexibility in defining the corner cases. Some authors leave $\kappa(u,v)$ undefined when $\{u,v\} \in E(G)$ or define it to be $n-1$. In [10] a $k$-cut is defined to be a mixed set of edges and vertices whose removal disconnects the graph. Under this definition, when $\{u,v\} \in E(G)$, $\kappa(u,v) = k$ if removing $k-1$ vertices and $\{u,v\}$ disconnects $u$ and $v$. This last definition is compatible with Menger’s theorem, and allows for it to be extended to all pairs of vertices.

**Theorem 1.** (Menger [35]) Let $G = (V,E)$ be an undirected graph and $\{u,v\}$ a pair not in $E$. Let $U \subset V$ be a minimum size cut disconnecting $u$ and $v$ and $\Pi$ be a maximum size set of internally vertex disjoint paths from $u$ to $v$. Then $\kappa(u,v) = |U| = |\Pi|$.

The following categories make sense when applied to non-minimal vertex cuts, but we are only interested in applying them to minimum vertex cuts. Henceforth cut usually means minimum cut.

**Laminar Cuts.** Let $U$ be a cut and $P$ be a side of $U$. If $W$ is a cut and $W \subset U \cup P$, we say $W$ is a laminar cut of $U$ in side $P$.

![Figure 2: A 7-cut $U$ with two sides, and two 7-cuts $W_1, W_2$ that are laminar w.r.t. $U$.](image)

**Small Cuts.** Informally, when a side of a cut tiny we call the cut small. We define three levels of small cuts. Let $U$ be a cut with sides $A_1, A_2, \ldots, A_a$. We say that

1. $U$ is (I, $t$)-small if there exists an index $i^t$ such that $\sum_{i \neq i^t} |A_i| \leq t$. $A_{i^t}$ is called the large side of $U$ and the others the small sides of $U$.

2. $U$ is (II, $t$)-small if there exists $i^t$ such that for every $i \neq i^t$, $|A_i| \leq t$.

3. $U$ is (III, $t$)-small, if there exists $i^t$ such that $|A_{i^t}| \leq t$. In this case $A_{i^t}$ is the small side of $U$.

Note that for any $t$, I-small cuts are II-small, and II-small cuts are III-small. We typically apply this definition with $t = \kappa$, $t = \Theta(\kappa)$, or $t = \lceil \frac{n-\kappa}{2} \rceil$.

---

7These are sometimes called parallel cuts.
Wheel Cuts. Suppose $V$ can be partitioned into a series of disjoint sets $T$, \{${C}_i$, $S_i$\} (1 \leq i \leq w, w \geq 4$, subscripts are taken module $w$), such that the \{${C}_i$\} and \{${S}_i$\} are nonempty ($T$ may be empty), and $C_i \cup T \cup C_{i+2}$ disconnects $S_i \cup C_i \cup S_{i+1}$ from the rest of the graph. We say $(T; C_1, C_2, \ldots, C_w)$ forms a $w$-wheel with sectors $S_1, S_2, \ldots, S_w$. We call $T$ the center of the wheel, \{${C}_i$\} the spokes of the wheel, and $C(i, j) = C_i \cup T \cup C_j$ the cuts of the wheel. Define $D(i, j) = S_i \cup C_{i+1} \cup \cdots \cup C_{j-1} \cup S_{j-1}$.

Recall that we are only interested in wheels whose cuts are minimum $\kappa$-cuts. The cut of the wheels discussed in this paper are all $\kappa$-cuts. It is proved in Lemma 3 that, if $(T; C_1, C_2, \ldots, C_w)$ forms a wheel, then for every $i, j$ such that $j - i \notin \{1, w-1\}$, $C(i, j)$ is a $\kappa$-cut with exactly two sides, namely $D(i, j)$ and $D(j, i)$. Note that a $w$-wheel $(T; C_1, C_2, \ldots, C_w)$ contains $x$-wheels, $x \in [4, w-1]$.

![Figure 3: A 6-wheel of 8-cuts with a center of size $|T| = 2$.](image)

Specifically, for any subset \{${i}_1, {i}_2, \ldots, {i}_x$\} $\subseteq \{1, 2, \ldots, w\}$ with $x \geq 4$, $(T; C_{i_1}, C_{i_2}, \ldots, C_{i_x})$ forms an $x$-wheel called a subwheel of the original. If a wheel is not a subwheel of any other wheel, it is a maximal wheel. If there exists an index $i^*$ such that, $\sum_{i \neq i^*} |S_i| \leq \kappa$, then we say this is a small wheel.\(^8\)

Matching Cuts and Crossing Matching Cuts. Let $U$ be a cut, $A$ be a side of $U$, and $P \subseteq U$ be a subset of the cut. We call a cut $W$ a matching cut of $U$ in side $A$ w.r.t. $P$ if (i) $U \setminus P \subseteq W \subseteq U \cup A$, (ii) $A \setminus W \neq \emptyset$, and (iii) $W$ disconnects $P \cup (V \setminus (U \cup A))$ from $A \setminus W$. The set $\text{Match}_U;A(P) \overset{\text{def}}{=} W \setminus U$ is the neighborhood of $P$ restricted to $A$. Note that a matching cut is a type of laminar cut.

Now suppose $U$ is a cut with exactly two sides $A$ and $B$, and let $P \subseteq U$ be a non-empty subset of $U$. We call $W$ a crossing matching cut of $U$ in side $A$ w.r.t. $P$ if (i) $W \cap B \neq \emptyset$, (ii) $(U \setminus P) \cup (W \cap A)$ is a matching cut of $U$ in side $A$ w.r.t. $P$.

One could view $U$ and a crossing matching cut $W$ as a degenerate 4-wheel, in which one sector $S_1 = \emptyset$ is empty. Such cuts should not be regarded as wheels, as they do not possess key properties of wheels, e.g., that when $U$ and $W$ are (minimum) $\kappa$-cuts, that $|C_1| = \cdots = |C_4| = \frac{\kappa - |T|}{2}$, because $C_1 \cup T \cup C_2$ is not a cut.

Lemmas 1 and 2 are used throughout the paper. Recall here $\kappa = \kappa(G)$ is the vertex connectivity of $G$.

**Lemma 1.** Suppose $U$ is a $\kappa$-cut and $P$ a side of $U$. For every $p \in P$ and $u \in U$, there exists a path from $p$ to $u$ that is not blocked by $V \setminus P$.

\(^8\)For a small wheel, all its cuts $C(i, j)$ are $(\Pi, O(\kappa^2))$-small.
Figure 4: A cut $U$ (drawn vertically) with two sides $A$ and $B$. Dotted lines indicate two crossing matching cuts w.r.t. $P_1$ (bottom 3 vertices of $U$) and $P_2$ (top 2 vertices of $P_1$).

**Proof.** Fix any $v$ in another side of $U$. By Menger’s theorem (Theorem 1) there are $\kappa$ internally vertex disjoint paths from $u$ to $v$, and therefore each must pass through a different vertex of $U$. The prefixes of these paths that are contained in $P \cup U$ are not blocked by $V \setminus P$.

**Lemma 2.** Suppose $U$ and $W$ are two cuts, $P$ is disconnected by $U$ from the rest of the graph $G$ and $Q$ is disconnected by $W$ from the rest of the graph $G$. Then we have the following two rules:

- (Intersection Rule) If $P \cap Q \neq \emptyset$, then $P \cap Q$ is disconnected by $(U \cap Q) \cup (U \cap W) \cup (W \cap P)$ from the rest of the graph $G$;
- (Union Rule) If $V \setminus (U \cup P \cup W \cup Q) \neq \emptyset$, then $P \cup Q$ is disconnected by $(U \setminus Q) \cup (W \setminus P)$ from the rest of the graph $G$.

![Figure 5: (a) Intersection rule; (b) Union rule.](image)

**Proof.** Denote $A = (U \cap Q) \cup (U \cap W) \cup (W \cap P)$, $B = P \cap Q$. First, $A \cup B = (U \cup P) \cap (W \cup Q) \subset V$, so $V \setminus (A \cup B) \neq \emptyset$. Consider a path $v_1v_2 \cdots v_l$ from $B$ to $V \setminus (A \cup B)$. Because $v_1 \in B = P \cap Q$, and $v_1 \notin U \cup P$ or $v_l \notin W \cup Q$, this path must be blocked by $U$ or $W$, so there must exist some $v_j$ in $U \cup W$. Find the smallest $i$ such that $v_i \in U \cup W$, and without loss of generality assume $v_i \in U$. Then the path $v_1v_2 \cdots v_i$ is not blocked by $W$, so $v_i \in Q \cup W$. Since $(Q \cup W) \cap U \subseteq A$ we have $v_i \in A$. It follows that $A$ separates $B$ from the rest of the graph, proving the Intersection Rule.

For the Union Rule, let $R = V \setminus (U \cup P)$, $S = V \setminus (W \cup Q)$. Now that $R \cap S = V \setminus (U \cup P \cup W \cup Q) \neq \emptyset$, by applying the intersection rule above, $R \cap S$ is disconnected by $T = (U \cap S) \cup (U \cap W) \cup (W \cap R) = (U \setminus Q) \cup (W \setminus P)$ from the rest of the graph.
3 The Classification of Minimum Vertex Cuts

The main binary structural theorem for vertex connectivity is, informally, that every two minimum vertex cuts have a relationship that is Laminar, Wheel, Crossing Matching, or Small; cf. [10]. Moreover, any strict subset of this list would be inadequate to capture all possible relationships between two vertex cuts.9

**Theorem 2.** Fix a minimum $\kappa$-cut $U$ with sides $A_1, A_2, \ldots, A_\alpha$, $\alpha \geq 2$, and let $W$ be any other $\kappa$-cut with sides $B_1, B_2, \ldots, B_\beta$, $\beta \geq 2$. Denote $T = U \cap W$, $W_i = W \cap A_i$ and $U_j = U \cap B_j$. Then $W$ may be classified w.r.t. $U$ as follows:

**Laminar type.** $W$ is a laminar cut of $U$, and in particular, there exists indices $i^*$ and $j^*$ such that $B_{j^*}\setminus A_{i^*} = (U\setminus W) \cup (\cup_{i \neq i^*} A_i)$ and $A_{i^*}\setminus B_{j^*} = (W\setminus U) \cup (\cup_{j \neq j^*} B_j)$.

**Wheel type.** $a = b = 2$, and $(T; U_1, W_1, U_2, W_2)$ forms a 4-wheel with sectors $A_1 \cap B_1$, $A_1 \cap B_2$, $A_2 \cap B_2$ and $A_2 \cap B_1$.

**Crossing Matching type.** $a = b = 2$, and w.l.o.g., $A_1 \cap B_1 \neq \emptyset$, $A_2 \cap B_2 \neq \emptyset$, but $A_1 \cap B_2 = \emptyset$. We have $|W_2| = |U_1| > 0$, $|W_1| = |U_2| > 0$, and $W$ is a crossing matching cut of $U$ in side $A_1$ w.r.t. $U_2$. Furthermore, if $A_2 \cap B_1 \neq \emptyset$, then $|U_1| \geq |U_2|$.

**Small type.** $U$ is $(I, \kappa-1)$-small, and the small sides of $U$ are within $W$, or $W$ is $(I, \kappa-1)$-small, and the small sides of $W$ are within $U$.

**Proof.** Suppose there is a single index $i^*$ such that $W_{i^*} \neq \emptyset$ and $W_i = \emptyset$ for all $i \neq i^*$. It follows that $W \subseteq A_{i^*} \cup U$ is a laminar cut of $U$ in side $A_{i^*}$. It remains to prove the other properties of the laminar type. By Lemma 1 there exists paths from any vertex in $A_i$, $i \neq i^*$, to $U\setminus W$ that are not blocked by $W$, so they all lie within one side of $W$; let us denote this side by $B_{j^*}$. Then $(U\setminus W) \cup (\cup_{i \neq i^*} A_i) \subseteq B_{j^*}$, and because $V = U \cup (\cup_{i=1}^\alpha A_i)$, we obtain $B_{j^*}\setminus A_{i^*} = (U\setminus W) \cup (\cup_{i \neq i^*} A_i)$. Now that $U \subseteq W \cup B_{j^*}$ is laminar w.r.t. $W$, so based on the same reasoning we have $A_{i^*}\setminus B_{j^*} = (W\setminus U) \cup (\cup_{j \neq j^*} B_j)$.

We proceed under the assumption that such indices $i^*, j^*$ do not exist, and without loss of generality assume that $W_1, W_2, U_1, U_2 \neq \emptyset$. We now wish to prove that all $U_i, W_i$ are non-empty. Suppose $W_1 \overset{\mathrm{def}}{=} W \cap A_i = \emptyset$ were empty, then $A_i$ would be contained within a side of $W$, say $A_i \subseteq B_j$. By Lemma 2 (intersection rule), whenever $A_i \cap B_j \neq \emptyset$, the set $W_i \cup T \cup U_j$ disconnects $A_i \cap B_j$ from the rest of the graph. It follows that

$$|W_i| + |T| + |U_j| = |T| + |U_j| \geq \kappa = |T| + \sum_{i=1}^b |U_i|,$$

which implies that $U_j$ is the only non-empty $U_i$-set, contradicting $U_1, U_2 \neq \emptyset$. Therefore, $W_i \neq \emptyset$ for all $i$ and similarly, $U_j \neq \emptyset$ for all $j$.

Define $\Omega = \{(i, j) \mid A_i \cap B_j \neq \emptyset\}$ to be the side-pairs whose intersections are non-empty. We consider the following possibilities, which are exhaustive.

1° There exist $(i, j), (i', j') \in \Omega$ such that $i \neq i', j \neq j'$. Then by Lemma 2 (intersection rule)

$$|W_i| + |T| + |U_j| \geq \kappa$$

---

9The existence of Small cuts as a category—an a priori unnatural class—indicates that there may be other ways to capture all minimum vertex cuts through an entirely different classification system.
and \(|W_i'| + |T| + |U_j'| \geq \kappa\).

On the other hand,

\[
|U_j| + |U_j'| + |T| \leq |U| = \kappa \\
\text{and } |W_i| + |W_i'| + |T| \leq |W| = \kappa.
\]

Thus all these inequalities must be equalities, and, adding the fact that all \(W_i, U_j \neq \emptyset\), we conclude that \(a = b = 2\), \(|W_i| = |U_j| = |W_j| = |U_j|\). W.l.o.g. we fix \(i = j = 1\), \(i' = j' = 2\). See Figure 6.

\[\begin{array}{c|c|c}
\text{A}_1 & U & \text{A}_2 \\
\hline
\neq \emptyset & U_1 & \neq \emptyset \\
\hline
W_1 & T & W_2 \\
\hline
\text{B}_1 & & \\
\hline
\text{B}_2 & U_2 & \\
\end{array}\]

Figure 6: A depiction of cuts \(U, W\) in case 1°.

1.1° Suppose \(A_1 \cap B_2 \neq \emptyset\) and \(A_2 \cap B_1 \neq \emptyset\). Then \(|W_i| + |U_j| \geq \kappa - |T|\) for every \(\hat{i}, \hat{j} \in \{1, 2\}\), so we conclude that

\(|U_1| = |U_2| = |W_1| = |W_2| = \frac{\kappa - |T|}{2}\).

Now that \(W_i \cup T \cup U_j\) disconnects \(A_1 \cap B_\hat{j}\) from the rest of the graph, \(U_1 \cup T \cup U_2 = U\) disconnects \((A_1 \cap B_1) \cup W_1 \cup (A_1 \cap B_2) = A_1\) from \((A_2 \cap B_1) \cup W_2 \cup (A_2 \cap B_2) = A_2\), \(W_1 \cup T \cup W_2 = W\) disconnects \((A_1 \cap B_1) \cup U_1 \cup (A_2 \cap B_1) = B_1\) from \((A_1 \cap B_2) \cup U_2 \cup (A_2 \cap B_2) = B_2\). we conclude that \((T; U_1, W_1, U_2, W_2)\) forms a 4-wheel.

Figure 7: A depiction of the cuts \(U, W\) in case 1.2°.

1.2° Suppose \(A_1 \cap B_2 = \emptyset\) (or symmetrically, that \(A_2 \cap B_1 = \emptyset\)). Then \(A_1 = A_1 \cap (B_1 \cup W) = (A_1 \cap B_1) \cup W_1\). By Lemma 2, \(W_1 \cup T \cup U_1\) separates \(A_1 \cap B_1\) from the rest of the graph. Since \(U_2 \subseteq V \setminus ((A_1 \cap B_1) \cup (U_1 \cup T \cup W_1))\), it follows that \(W_1 \cup T \cup U_1\) disconnects \(U_2\) from \(A_1 \cap B_1 = A_1 \setminus (W_1 \cup T \cup U_1)\), i.e., it is a matching cut of \(U\) in side \(A_1\) w.r.t. \(U_2\).
Suppose \( \kappa \) is minimal w.r.t. containment.

If \( A_2 \cap B_1 \neq \emptyset \), by Lemma 2 (intersection rule)
\[
|U_1| + |W_2| + |T| \geq \kappa = |W_1| + |T| + |W_2|,
\]
so \( |U_1| \geq |U_2| \).

2° Suppose there exists a \( j^2 \) such that \( \forall i. \forall j \neq j^2. (i, j) \notin \Omega \), i.e., \( A_i \cap B_j = \emptyset \). This implies that \( \bigcup_{j \neq j^2} B_j \subseteq U \), and because \( U_{j^2} \neq \emptyset \), \( \bigcup_{j \neq j^2} B_j \) is strictly smaller than \( \kappa \). Therefore \( W \) is a (I, \( \kappa - 1 \))-small cut, and all the small sides of \( W \) are within \( U \).

3° There exists \( i^2 \) such that \( \forall i \neq i^2. \forall (i, j) \notin \Omega \). Symmetric to case 2°; \( U \) is (I, \( \kappa - 1 \))-small, and all the small sides of \( U \) are within \( W \).

4° \( \Omega = \emptyset \). Then \( \bigcup_{i=1}^{n} A_i \subseteq W \), so \( V = U \cup (\bigcup_{i=1}^{n} A_i) \subseteq U \cup W \), and \( |V| \leq 2\kappa \). This is a possibility, but not one we consider as it contradicts our initial assumption that \( n > 4\kappa \).

\( \square \)

**Corollary 1.** If \( U \) is a \( \kappa \)-cut that is not (I, \( \kappa - 1 \))-small and has at least 3 sides, then all other \( \kappa \)-cuts have a laminar type relation with \( U \), or are themselves (I, \( \kappa - 1 \))-small cuts.

**Corollary 2.** Suppose \( U \) is a \( \kappa \)-cut that is not (I, \( \kappa - 1 \))-small, with exactly two sides \( A \) and \( B \). Suppose \( W \) is a \( \kappa \)-cut with sides \( K \), \( L \) (and possibly others), such that \( W \cap A \neq \emptyset \), \( W \cap B \neq \emptyset \), \( A \subseteq K \cup W \), and \( L \cap U \neq \emptyset \). Then \( W \) only has two sides, and \( W \) is a crossing matching cut of \( U \) in side \( A \) w.r.t. \( L \cap U \).

**Corollary 3.** Define \( \text{Cuts}_{C;D} \) to be the set of all \( \kappa \)-cuts that disconnect disjoint, non-empty vertex sets \( C \) and \( D \). If \( \text{Cuts}_{C;D} \neq \emptyset \), it contains a unique minimal element \( \text{MinCut}_{C;D} \), such that for any cut \( U \in \text{Cuts}_{C;D} \), \( \text{Region}_{\text{MinCut}_{C;D}}(C) \subseteq \text{Region}_{U}(C) \).

**Proof.** This is a corollary of Theorem 2, but also admits a simple, direct proof via the Picard-Queyrenne theorem [38]. Form a flow network \( \overrightarrow{G} \) via the following steps (i) contract \( C \) and \( D \) to vertices \( s \) and \( t \), (ii) replace each vertex \( v \) with a subgraph consisting of vertices \( v_{\text{in}}, v_{\text{out}} \), and a directed edge \( (v_{\text{in}}, v_{\text{out}}) \), (iii) replace each undirected edge \( \{u, v\} \) with directed edges \( (u_{\text{out}}, v_{\text{in}}) \), (v) give edges from (ii) unit capacity and edges from (iii) infinite capacity. If the flow value is \( \kappa \), then the minimum \( (s_{\text{out}}, t_{\text{in}}) \)-cuts are in one-to-one correspondence with the minimum vertex cuts in \( \text{Cuts}_{C;D} \). Since \( (s_{\text{out}}, t_{\text{in}}) \)-cuts in \( \overrightarrow{G} \) are closed under union and intersection [38], there is a unique vertex cut \( \text{MinCut}_{C;D} \in \text{Cuts}_{C;D} \) such that \( \text{Region}_{\text{MinCut}_{C;D}}(C) \) is minimal w.r.t. containment.

\( \square \)

Theorem 2 classifies the pairwise relationship between two minimum \( \kappa \)-cuts. In Sections 3.1–3.4 we further explore the properties of wheel cuts, (crossing) matching cuts, laminar cuts, and small cuts.
3.1 Wheels and Wheel Cuts

Recall that a $w$-wheel $(T; C_1, \ldots, C_w)$ satisfied, by definition, the property that $C_i \cup T \cup C_{i+2}$ formed a $\kappa$-cut, but did not say anything explicitly about $C(i, j) = C_i \cup T \cup C_j$. Lemma 3 proves that these are also cuts, and bounds their number of sides.

**Lemma 3.** Suppose $(T; C_1, C_2, \ldots, C_w)$ forms a $w$-wheel ($w \geq 4$) with sectors $S_1, S_2, \ldots, S_w$. (Subscripts are modulo $w$.) For any $i \neq j$, $C(i, j)$ is a $\kappa$-cut that disconnects $D(i, j)$ from the rest of the graph. Moreover, when $j - i \notin \{1, w - 1\}$, $C(i, j)$ has exactly two sides, which are $D(i, j)$ and $D(j, i)$. Furthermore, $|C_i| = \frac{\kappa-|T|}{2}$.

**Proof.** By definition $C_i \cup T \cup C_{i+2}$ and $C_{i-1} \cup T \cup C_{i+1}$ are two $\kappa$-cuts that, respectively, separate $S_i \cup C_{i+1} \cup S_{i+1}$ and $S_{i-1} \cup C_i \cup S_i$ from the rest of the graph. By Lemma 2 (intersection rule), $C_i \cup T \cup C_{i+1}$ disconnects $S_i$ from the rest of the graph. Thus, whenever $j - i \in \{1, 2\}$, $C(i, j)$ disconnects $D(i, j)$ from the rest of the graph. This is the base case. Assuming the claim is true whenever $j - i \in [1, l - 1]$, we prove it is true up to $l$ as well, $l \leq w - 1$. Fix $i, j$ such that $j - i = l$. Then $C(i, j - 1)$ is a cut that disconnects $D(i, j - 1)$ from the rest of the graph, and $C(i+1, j)$ is a cut that disconnects $D(i+1, j)$ from the rest of the graph. By Lemma 2 (union rule), $(C(i, j - 1) \setminus D(i+1, j)) \cup (C(i+1, j) \setminus D(i, j - 1)) = C_i \cup T \cup C_j$ disconnects $D(i, j - 1) \cup D(i+1, j) = D(i, j)$ from the rest of the graph. This proves the first part.

By Lemma 1, there exist paths from any vertex in $S_i$ to every vertex in $C_r$, and to every vertex in $C_{r+1}$, that is not blocked by $V \setminus S_r$. Thus, when $j - i \notin \{1, w - 1\}$, $C(i, j)$ separates $D(i, j)$ from $D(j, i)$. $D(i, j)$ forms a side since all vertices in $D(i, j)$ have paths to $C_{i+1}$, being distinct from $C_j$, and $D(j, i)$ forms a side since all vertices have paths to $C_{j+1}$, being distinct from $C_i$. (When $j = i + 1$, $D(i, i + 1)$ may be a region consisting of multiple sides.)

Now it is proved that for all $i, j$, $C(i, j)$ is a $\kappa$-cut, so $|C_i| + |C_j| = \kappa - |T|$ for all $i, j$. Therefore all $|C_i|$ are equal to $\frac{\kappa - |T|}{2}$. □

**Remark 2.** Lemma 3 shows that the set $\{C(i, i + 2)\}_{i \in [w]}$ generates all the $(\binom{w}{2})$ wheel cuts. One might think that the sector cuts $\{C(i, i + 1)\}$ would also suffice, but this is incorrect. In Figure 8, $C(i, i + 1)$ is a (minimum) 6-cut for all $i$ separating $S_i$ from the rest of the graph, but this is not a 4-wheel since $C(1, 3)$ and $C(2, 4)$ are not cuts.

![Figure 8: A faux 4-wheel.](image)

**Theorem 3.** Suppose $(T; C_1, C_2, \ldots, C_w)$ forms a $w$-wheel ($w \geq 4$) with sectors $S_1, S_2, \ldots, S_w$. (Subscripts are given by modulo $w$.) Let $X$ be any minimum $\kappa$-cut. Then one of the following is true:

1° $X = C(i, j)$ for some $i \neq j$.  

---

11
\(2^° \) \( X \subseteq C(i, i + 1) \cup S_i \) for some \( i \), i.e., \( X \) is a laminar cut of \( C(i, i + 1) \).

\(3^° \) \( X \) has crossing matching type relation with some \( C(i, i + 1) \) or some \( C(i, i + 2) \).

\(4^° \) \( X \) is a \( (I, \kappa - 1) \)-small cut.

\(5^° \) \( (T; C_1, C_2, \ldots, C_w) \) is a small wheel.

\(6^° \) There exists \( i < j \), such that \( X \subseteq S_i \cup T \cup S_j \), and \( (T; C_1, \ldots, C_i, X \cap S_i, C_{i+1}, \ldots, C_j, X \cap S_j, C_{j+1}, \ldots, C_w) \) forms a \( (w + 2) \)-wheel; or there exists \( i \neq j \), \( X \subseteq S_i \cup T \cup C_j \), and \( (T; C_1, \ldots, C_i, X \cap S_i, C_{i+1}, \ldots, C_w) \) forms a \( (w + 1) \)-wheel. In other words, \( (T; C_1, C_2, \ldots, C_w) \) is a subwheel of some other wheel.

**Proof.** By the definition of wheels and Lemma 1, there exists paths from any vertex in \( S_i \) to \( C_i \), to \( T \), and to \( C_{i+1} \) that are not blocked by \( V \setminus S_i \), and there exist paths from any vertex in \( C_i \) to \( T \) that are not blocked by \( V \setminus (S_{i-1} \cup C_i \cup S_i) \). These facts are used frequently below.

For every \( i \in \{1, 2, \ldots, w\} \), we consider the following cases relating cuts \( X \) and \( C(i, i + 1) \), which are exhaustive according to Theorem 2.

i. \( X = C(i, i + 1) \), this is a trivial case and we are in case \( 1^° \).

ii. \( X \cap S_i = \emptyset \).

iii. \( X \) is a laminar cut of \( C(i, i + 1) \), and \( X \cap S_i \neq \emptyset \), then it must be true that \( X \subseteq C(i, i + 1) \cup S_i \), and we are in case \( 2^° \).

iv. \( X \) is a wheel type cut of \( C(i, i + 1) \), and \( X \cap S_i \neq \emptyset \).

v. \( X \) is a crossing matching type cut of \( C(i, i + 1) \), then we are in case \( 3^° \).

vi. \( X \) and \( C(i, i + 1) \) have a small type relation, such that \( X \) is \( (I, \kappa - 1) \)-small, then we are in case \( 4^° \).

vii. \( X \) and \( C(i, i + 1) \) have a small type relation, such that \( C(i, i + 1) \) is \( (I, \kappa - 1) \)-small, and \( D(i+1, i) \) is the small side, which implies that \( |D(i, i + 1)| < \kappa \), so the wheel is a small wheel, and we are in case \( 5^° \).

viii. \( X \) and \( C(i, i + 1) \) have a small type relation, such that \( C(i, i + 1) \) is \( (I, \kappa - 1) \)-small, and \( S_i = D(i, i + 1) \) is the region of the small sides. Then we have that \( S_i = D(i, i + 1) \subseteq X \).

It can be seen that if for any \( i \), we are in case \( i, iii, v, vi, or vii \), there is nothing left to prove. Otherwise, we may proceed under the assumption that every index \( i \) is in case \( ii, iv, or viii \). We define the index sets \( I_1, I_2, I_3 \) as follows.

\[
I_1 = \{ i \mid \text{case iv applies to } i \},
I_2 = \{ i \mid \text{case viii applies to } i \},
I_3 = \{ i \mid \text{case ii applies to } i \},
\]

and hence \( I_1 \cup I_2 \cup I_3 = [w] = \{1, 2, \ldots, w\} \).

We split the possibilities into the following cases.

I. \( |I_i| \geq 2 \).
II. \(|I_1| \leq 1, |I_2| > 0, |I_3| > 0\).

III. \(|I_1| = 1, |I_2| = 0\).

IV. \(|I_1| \leq 1, |I_3| = 0\).

V. \(|I_1| = |I_2| = 0\).

We will show that I and III lead to case 6°, II leads to case 3°, IV leads to case 5°, and V leads to 1°. Define \(Y_i, Z_i, Q\) as follows

\[
Y_i = X \cap S_i, \\
Z_i = X \cap C_i, \\
Q = X \cap T.
\]

I. Suppose \(i \neq j\) are in \(I_1\). Because \(X\) has a wheel type relation with some other cut (namely \(C(i, i + 1)\) and \(C(j, j + 1)\)), it must have exactly two sides; let them be \(K\) and \(L\). Without loss of generality we assume \(i = 1\). We consider two subcases depending on whether \(j \not\in \{2, w\}\) (I.a) or \(j \in \{2, w\}\) (I.b).

I.a. \(j \not\in \{2, w\}\), i.e., \(j\) is not adjacent to \(i\). We prove the following claims, culminating in Claim 3, which puts us in case 6°.

**Claim 1.** \(X \subseteq S_1 \cup T \cup S_j\).

Because \(X\) has a wheel type relation with \(C(1, 2)\) and \(C(j, j + 1)\), \(|X \cap D(1, 2)| = |X \cap D(2, 1)|\) and \(|X \cap D(j, j + 1)| = |X \cap D(j + 1, j)|\). Thus, in terms of the \(Y_s, Z_s\) sets,

\[
|Y_1| = \sum_{r \in \{1, 2\}} |Z_r| + \sum_{r \neq 1} |Y_r| \quad \text{and} \quad |Y_j| = \sum_{r \not\in \{j, j + 1\}} |Z_r| + \sum_{r \neq j} |Y_r|.
\]

Therefore, \(|Z_r| = 0\) for all \(r\), \(|Y_r| = 0\) for all \(r \not\in \{1, j\}\), and \(|Y_1| = |Y_j| = \frac{\kappa - |Q|}{2}\). This means that \(I_1 = \{1, j\}\), \(I_2 = \emptyset\), \(I_3 = [w] \setminus I_1\), and \(X \subseteq S_1 \cup T \cup S_j\).

**Claim 2.** \(C_2 \cup D(2, j) \cup C_j\) and \(C_{j+1} \cup D(j + 1, 1) \cup C_1\) are in different sides of \(X\), and \(Q = T\).

By Lemma 1, the subgraphs induced by \(C_2 \cup D(2, j) \cup C_j\) and \(C_{j+1} \cup D(j + 1, 1) \cup C_1\) are connected, and therefore each is contained in a side of \(X\). Suppose they are contained in the same side, say \(K\). Any vertices in \(T \setminus X\) must be in \(K\) as well, so \(L \subseteq S_1 \cup S_j\) and w.l.o.g. we assume \(L \cap S_1 \neq \emptyset\). Applying Lemma 2 (intersection rule) to \(X\) and \(C(1, 2)\) we conclude that

\[
(L \cap C(1, 2)) \cup (S_1 \cap X) \cup (X \cap C(1, 2)) = (S_1 \cap X) \cup (X \cap C(1, 2)) = Y_1 \cup Q
\]

is a cut that disconnects \(L \cap S_1\) from the rest of the graph. It follows that \(|Y_1| + |Q| = \frac{\kappa + |Q|}{2} \geq \kappa\), but this contradicts with \(|Q| < \kappa\). Therefore, we conclude that \(C_2 \cup D(2, j) \cup C_j\) and \(C_{j+1} \cup D(j + 1, 1) \cup C_1\) are in different sides of \(X\), and since each of these sides are adjacent to all vertices of \(T\), that \(Q = T\) as well.

**Claim 3.** \((T; C_1, Y_1, C_2, \ldots, C_j, Y_j, C_{j+1}, \ldots, C_w)\) forms a \((w + 2)\)-wheel.

By Claim 2, \(T \subseteq X\), so \(|Y_1| = |Y_j| = \frac{\kappa - |T|}{2}\) and \(C_2 \cup D(2, j) \cup C_j \subseteq K\) and \(C_{j+1} \cup D(j + 1, 1) \cup C_1 \subseteq L\) are in different sides of \(X\). To verify that this is a wheel we simply need to confirm that the four new cuts involving \(Y_i, Y_j\) are in fact
cuts. We illustrate this for \( Y_1 \cup T \cup C_3 \). Applying Lemma 2 (intersection rule) to \( X \) with side \( K \) and \( C(1,3) \) with side \( D(1,3) \), we conclude that \((K \cap S_1) \cup C_2 \cup S_2\) is disconnected by \( Y_1 \cup T \cup C_3 \) from the rest of the graph. The other three new cuts are confirmed similarly, hence \((T; C_1, Y_1, C_2, \ldots, C_j, Y_j, C_{j+1}, \ldots, C_w)\) forms a \((w+2)\)-wheel.

I.b. \( j \in \{2, w\} \). W.l.o.g. we assume \( j = 2 \). Once again we prove the following claims, culminating in Claim 6 which puts us in case 6°.

**Claim 4.** \( X \subseteq S_1 \cup S_2 \cup T \cup C_2 \).

Because \( X \) has wheel type relation with \( C(1,2) \) and \( C(2,3) \), we have that

\[
|Y_1| = \sum_{r \in \{1,2\}} |Z_r| + \sum_{r \neq 1} |Y_r| \quad \text{and} \quad |Y_2| = \sum_{r \in \{2,3\}} |Z_r| + \sum_{r \neq 2} |Y_r|
\]

Therefore, \( |Z_r| = 0 \) for all \( r \neq 2 \), \( |Y_r| = 0 \) for all \( r \notin \{1,2\} \), and \( |Y_1| = |Y_2| = \kappa - |Q| - \frac{|Z_2|}{2} \). This means \( I_1 = \{1,2\}, I_2 = \emptyset, I_3 = [w]\setminus I_1, \) and \( X \subseteq Y_1 \cup Y_2 \cup T \cup Z_2 \).

**Claim 5.** \( Q = T \) and \( Z_2 = \emptyset \).

By Lemma 1, \( C_3 \cup D(3, w) \cup C_1 \) is within a side of \( X \), say \( K \). Moreover, if \( T \setminus X \neq \emptyset \), then \( T \setminus X \subseteq K \). Because \( X \) has a wheel relation with \( C(1,2) \),

\[
|L \cap C(1,2)| = |K \cap C(1,2)| \geq |C_1|.
\]

Thus, \( X \cap C(1,2) = Q \cup Z_2 \) and \( |X \cap C(1,2)| \leq \kappa - 2|C_1| = |T| \). Since \( T \setminus X \subseteq K \), it follows that

\[
|L \cap C(1,2)| = |L \cap C_2| \leq |C_2| \leq |C_1| + |T \setminus X| = |K \cap C(1,2)|,
\]

implying \( |T \setminus X| = 0 \) and therefore that \( Q = T \) and \( Z_2 = \emptyset \).

**Claim 6.** \((T; C_1, Y_1, C_2, Y_2, C_3, \ldots, C_w)\) forms a \((w+2)\)-wheel.

By Claims 4 and 5 we know \( |Y_1| = |Y_2| = \frac{\kappa - |T|}{2} \). There are three new wheel cuts involving \( Y_1, Y_2 \) that need to be confirmed, namely \( X = Y_1 \cup T \cup Y_2 \), which separates \( C_3 \cup D(3, w) \cup C_1 \) from \( C_2 \), as well as \( C_w \cup T \cup Y_1 \) and \( Y_2 \cup T \cup C_4 \). The latter two are established by applying Lemma 2, as in Claim 3. We conclude that \((T; C_1, Y_1, C_2, Y_2, C_3, \ldots, C_w)\) forms a \((w+2)\)-wheel.

II. At most one index is in \( I_1 \), so there must be indices in \( I_2 \) and \( I_3 \) that are adjacent in the circular order. Without loss of generality let them be \( 1 \in I_2, 2 \in I_3 \), i.e., \( C(1,2) \) is small, \( S_1 \subseteq X \), and \( S_2 \cap X = \emptyset \). By Lemma 1 there exist paths from any vertex in \( S_2 \) to \( C_2 \setminus X \), \( C_3 \setminus X \), and \( T \setminus X \) that are not blocked by \( X \), so they are all on the same side of \( X \), call it side \( K \). Refer to Figure 9 in Claims 7 and 8.

**Claim 7.** There exists another side \( L \) of \( X \), such that \( L \cap C_1 \neq \emptyset \).

Note \( C(1,3) \) separates \( D(1,3) \) from the rest of the graph, and \( X \) separates \( K \) from the rest of the graph. If the claim were not true, then \( C_1 \subseteq X \cup K \), so \( G \setminus (X \cup K \cup C(1,3) \cup D(1,3)) \subseteq G \setminus (X \cup K) \neq \emptyset \). We can now apply Lemma 2 (union rule) to \( C(1,3) \) and \( X \), and deduce that \( K \cup D(1,3) \) is disconnected from the rest of the graph by

\[
(C(1,3) \setminus K) \cup (X \setminus D(1,3)) \subseteq X \setminus S_1.
\]

But \( |X \setminus S_1| < |X| = \kappa \), a contradiction. So there exists a side \( L \) such that \( L \cap C_1 \neq \emptyset \).
III. W.l.o.g. we assume $I_1 = \{1\}$ and therefore $I_3 = \{2, 3, \ldots, w\}$. With $Y_x, Z_x$ defined as in case I, we have

$$X = Y_1 \cup Q \cup (\cup_{i=1}^w Z_i).$$

Because $X$ has a wheel type relation with $C(1, 2)$, $X$ has exactly two sides, say $K$ and $L$, and moreover, the intersections of $X$ with the two sides of $C(1, 2)$ have equal size, i.e.,

$$|Y_1| = \sum_{r \notin \{1, 2\}} |Z_r| = \frac{\kappa - |Q| - |Z_1| - |Z_2|}{2}.$$

Claim 8. $X$ is a crossing matching cut of $C(1, 3)$ in side $D(1, 3)$ w.r.t. $L \cap C_1$, or a crossing matching cut of $C(w, 2)$ in side $D(w, 2)$ w.r.t. $K \cap C_2$.

If $X \cap D(3, 1) \neq \emptyset$ then Corollary 2 implies that $X$ is a crossing matching cut of $C(1, 3)$ in side $D(1, 3)$ w.r.t. $L \cap C_1$.

Otherwise, if $X \cap D(3, 1) = \emptyset$, then because there are paths from every vertex in $D(3, 1)$ to $S_w$ within itself, and paths from $S_0$ to $C_1 \cap L$ not blocked by $X$, $D(3, 1) \subseteq L$. Now that $S_3 \subseteq L$ and $S_2 \subseteq K$, it follows that $C_3 \cup T \subseteq X$. Based on the same reason of Claim 7, we have $K \cap C_2 \neq \emptyset$. Now for cut $C(w, 2)$, we know that $X \cap D(w, 2) \supseteq S_1 \neq \emptyset$, $X \cap D(2, w) \supseteq C_3 \neq \emptyset$, $D(w, 2) = S_w \cup C_1 \cup S_1 \subseteq X \cup L$, and $K \cap C(w, 2) = K \cap C_2 \neq \emptyset$, by Corollary 2, $X$ is a crossing matching cut of $C(w, 2)$ w.r.t. $K \cap C_2$.

Claim 9. $Q = T$.

If $T \setminus X \neq \emptyset$, by Lemma 1 there exists paths from every vertex in $S_r, r \neq 1$, to $T \setminus X$, to $C_r \setminus X$, and to $C_{r+1} \setminus X$, that are not blocked by $X$. Thus, $((D(2, 1) \cup T) \setminus X)$ is within a side of $X$, say side $K$. It follows that the other side $L$ satisfies $L \subseteq S_1$. Applying Lemma 2 (intersection rule) to $C(1, 2)$ and $X$, we find that $L \cap S_1 = L$ is disconnected from the rest of the graph by

$$(L \cap C(1, 2)) \cup (X \cap C(1, 2)) \cup (S_1 \cap X) = Y_1 \cup Q \cup Z_1 \cup Z_2.$$

That means

$$|Y_1| + |Q| + |Z_1| + |Z_2| \geq \kappa,$$

which implies

$$\kappa = |X| = |Y_1| + |Q| + \sum_{i=1}^w |Z_i| \geq |Y_1| + |Q| + |Z_1| + |Z_2| \geq \kappa,$$

and therefore $Z_r = \emptyset$ for all $r \notin \{1, 2\}$, so $X \subseteq (C_1 \cup T \cup C_2 \cup S_1)$, contradicting the fact that $X, C(1, 2)$ have wheel type. Thus $Q = T$.  

15
Claim 10 There exists \( j \not\in \{1,2\} \) such that \( Z_j = C_j \).

If there does not exist such a \( j \), then it follows that \( C_r \setminus X \) is nonempty for every \( r \not\in \{1,2\} \). Thus, there are paths from any vertex in \( S_r \) to \( C_r \setminus X \) and to \( C_{r+1} \setminus X \) that are not blocked by \( X \), so \( D(2,1) \setminus X \) is still contained in a side of \( X \), say \( K \). If \( C_1 \setminus X \) or \( C_2 \setminus X \) is nonempty, then they are also in \( K \) as well, since there exist paths from \( C_1 \setminus X \) to \( S_w \), and from \( C_2 \setminus X \) to \( S_2 \). Then we have \( L \subseteq S_1 \). Based on exactly the same reasoning in Claim 9 we obtain the contradiction that \( X \subseteq (C_1 \cup T \cup C_2 \cup S_1) \). This proves the claim.

Claim 11 \( (T; C_1, Y_1, C_2, \ldots, C_w) \) forms a wheel.

Since \( Q = T \), \( C_j = Z_j \), and
\[
\sum_{r \neq \{1,2\}} |Z_r| = \frac{\kappa - |Q| - |Z_1| - |Z_2|}{2} \leq \frac{\kappa - |Q|}{2} = \frac{\kappa - |T|}{2} = |C_j| = |Z_j|,
\]
it follows that \( |Z_i| = 0 \) for \( i \neq j \). By Lemma 2 (intersection rule) it is easy to verify that \( (T; C_1, Y_1, C_2, \ldots, C_w) \) forms a \((w+1)\)-wheel, as is done in Claims 3 and 6, and this puts us in case 6°.

IV. Recall that \( I_1 \cup I_2 \cup I_3 = \{1,2,\ldots,w\} \). In this case, except for at most one \( i^* \in I_1 \), all other \( i \) are in \( I_2 \) and therefore \( S_i \subseteq X \), then \( \sum_{i \neq i^*} S_i \leq |X| = \kappa \), so this is a small wheel and we are in case 5°.

V. Again recall that \( I_1 \cup I_2 \cup I_3 = \{1,2,\ldots,w\} \). So in this case \( I_3 = \{1,2,\ldots,w\} \). Therefore, \( X \subseteq T \cup (\cup_{i=1}^w C_i) \). Note that the graph induced by \((\cup_{i=1}^w S_i) \cup \{t\}\) is connected for any \( t \in T \), and that every vertex in some \( C_i \) is adjacent to \( S_{i-1} \) and \( S_i \). This implies that \( T \subseteq X \), and that there exists indices \( i \neq j \) such that \( C_i \cup C_j \subseteq X \); otherwise \( X \) is not even a cut. We have deduced that \( X = C(i,j) \) for some \( i,j \), putting us in case 1°.

\[ \square \]

3.2 Matching Cuts and Crossing Matching Cuts

Define \( N(P) \) to be the neighborhood of \( P \subseteq V \) and \( N_A(P) \overset{\text{def}}{=} N(P) \cap A \).

Theorem 4. Let \( U \) be an arbitrary \( \kappa \)-cut and \( A \) a side of \( U \).

1° If there exists a matching \( \kappa \)-cut of \( U \) in side \( A \) w.r.t. \( P \), then it is \( W = (U \setminus P) \cup N_A(P) \), and \( |P| = |N_A(P)| < |A| \). In particular, \( \text{Match}_{U;A}(P) = N_A(P) \).

2° When there is such a matching cut \( W \), \( G \) contains a matching between \( P \) and \( \text{Match}_{U;A}(P) = N_A(P) \).

3° Suppose \( \text{Match}_{U;A}(P) \) and \( \text{Match}_{U;A}(Q) \) exist. If \( P \cap Q \neq \emptyset \), then \( \text{Match}_{U;A}(P \cap Q) \) exists, and
\[
\text{Match}_{U;A}(P \cap Q) = \text{Match}_{U;A}(P) \cap \text{Match}_{U;A}(Q).
\]
If \( |A| > |P \cup Q| \), then \( \text{Match}_{U;A}(P \cup Q) \) exists, and
\[
\text{Match}_{U;A}(P \cup Q) = \text{Match}_{U;A}(P) \cup \text{Match}_{U;A}(Q).
\]
Proof. Part 1°. By definition, $W$ is a matching cut in side $A$ w.r.t. $P$ if (i) $W \cap U = U \cap P$, (ii) $W \subseteq U \cup A$, and (iii) $W$ separates every vertex in $A \setminus W$ from $P$. It must be that $N_A(P) \subseteq W$, for otherwise $W$ would not satisfy (iii). It also follows from (iii) that $N_A(P) \setminus A \neq \emptyset$. Since $W' = (U \setminus P) \cup N_A(P)$ is a cut (separating $A \setminus N_A(P)$ from $P \cup V \setminus (U \cup A)$), it follows that $W' \subseteq W$, but since $W$ is a (minimum) $\kappa$-cut, then $W = W'$ and hence $|P| = |N_A(P)|$. Thus, by definition \text{Match}_{U,A}(P) = N_A(P) = W \setminus U$ is just the neighborhood function $N_A(P)$ whenever such a matching cut $W$ exists.

Part 2°. Define $H$ to be the bipartite subgraph of $G$ between $P$ and $N_A(P) = \text{Match}_{U,A}(P)$. By Hall’s theorem, if $H$ does not contain a matching then there exists a strict subset $P' \subset P$ such that $|P'| > |N_A(P')|$, but if this were the case, $(U \setminus P') \cup N_A(P')$ would be a cut with cardinality strictly smaller than $\kappa$, a contradiction.

Figure 10: Left: a cut $U$ with matching cuts in side $A$. Right: the derived flow network for $P, Q \subset U$.

Part 3°. Redefine $H$ to be the bipartite subgraph of $G$ between $P \cup Q$ and $N_A(P \cup Q)$. We construct a directed flow network $\vec{H}$ on $\{s, t\} \cup V(H)$ as follows; see Figure 10. All edges from $H$ appear in $\vec{H}$, oriented from $P \cup Q$ to $N_A(P \cup Q)$, each having infinite capacity. The edge set also includes edges from $s$ to $P \cup Q$ and $N_A(P \cup Q)$ to $t$, each with unit capacity. Clearly integer flows in $\vec{H}$ correspond to matchings in $H$. By the Kőnig-Egerváry theorem, the size of the minimum vertex cover in $H$ is equal to the size of the maximum matching in $H$. Following the same argument in 2°, they must both be of size $|P \cup Q|$ for otherwise $U$ would not be a minimum vertex cut. In fact, minimum size vertex covers of $H$ are in 1-1 correspondence with minimum capacity $s$-$t$ cuts in $\vec{H}$. The correspondence is as follows. If $(S, T)$ is a minimum capacity $s$-$t$ cut then there can be no (infinite capacity) edge from $S \cap (P \cup Q)$ to $T \cap N_A(P \cup Q)$, so

\[ C = ((P \cup Q) \cap T) \cup (N_A(P \cup Q) \cap S) \]

is a vertex cover in $H$ and

\[ |C| = \text{cap}(S, T). \]

Picard and Queyрене [38] observed that minimum $s$-$t$ cuts are closed under union and intersection, in the sense that if $(S, T), (S', T')$ are min $s$-$t$ cuts, then so are $(S \cup S', T \cap T')$ and $(S \cap S', T \cup T')$. By assumption $(P \setminus Q) \cup N_A(Q)$ and $(Q \setminus P) \cup N_A(P)$ are minimum vertex covers\footnote{This follows from the fact that $(U \setminus Q) \cup N_A(Q)$ and $(U \setminus P) \cup N_A(P)$ are assumed to be matching cuts of $U$ in $A$ w.r.t. $Q$ and $P$, respectively, with $\text{Match}_{U,A}(P) = N_A(P)$ and $\text{Match}_{U,A}(Q) = N_A(Q)$.} of $H$, with cardinality $|P \cup Q|$. Translated to the flow network $\vec{H}$, this implies that
are minimum s-t cuts, which implies that their union and intersection are also minimum s-t cuts:

\[
\begin{align*}
&\{s\} \cup P \cup N_A(P), \ (P \cup Q) \setminus N_A(P \cup Q) \\
&\text{and } \{s\} \cup P \cup N_A(P), \ (Q \setminus P) \cup N_A(P) \\
&\text{and } \{s\} \cup N_A(P), \ (P \setminus Q) \cup N_A(P \cup Q) \\
&\text{and } \{t\}.
\end{align*}
\]

Translated back to \(H\), these correspond to vertex covers

\[
N_A(P \cup Q)
\]

and \((P \cup Q) \setminus (P \cap Q) \cup (N_A(P) \cap N_A(Q))\).

Whenever \(N_A(P \cup Q)\) is a strict subset of \(A\) the first corresponds to a matching \(\kappa\)-cut of \(U\)

\[
(U \setminus (P \cup Q)) \cup N_A(P \cup Q)
\]

with \(\text{Match}_{U;A}(P \cup Q) = N_A(P \cup Q) = \text{Match}_{U;A}(P) \cup \text{Match}_{U;A}(Q)\).

Whenever \(P \cap Q \neq \emptyset\) the second corresponds to a matching \(\kappa\)-cut of \(U\)

\[
(U \setminus (P \cap Q)) \cup (N_A(P) \cap N_A(Q))
\]

with \(\text{Match}_{U;A}(P \cap Q) = N_A(P) \cap N_A(Q) = \text{Match}_{U;A}(P) \cap \text{Match}_{U;A}(Q)\).

Fix a \(\kappa\)-cut \(U\) and a side \(A\) of \(U\). Define \(\Theta = \{P \mid \text{Match}_{U;A}(P)\text{ exists}\}\). According to Part 3° of Theorem 4, \(\Theta\) is closed under union and intersection, and is therefore characterized by its
minimal elements. Define $\Theta^* = \{\cap_{u \in P, P \in \Theta} P \mid u \in U\}$. It can be seen from the definition that
$\cap_{u \in P, P \in \Theta} P$ corresponds to the minimum matching cut for vertex $u$.

In the most extreme case $\Theta$ may have $2^k - 1$ elements (e.g., if the graph induced by $U \cup N_A(U)$ is a matching), which may be prohibitive to store explicitly. From definition we know that $|\Theta^*| \leq \kappa$, so it works as a good compression for $\Theta$. Lemmas 4 and 5 also highlights some ways in which $\Theta^*$ is a sufficient substitute for $\Theta$.

**Lemma 4.** Let $U$ be a $\kappa$-cut and let $\Theta$ be defined w.r.t. the matching cuts of $U$ in a side $A$. Suppose that $P \in \Theta^*$ and $P \subseteq Q \in \Theta$, and that $W$ is a crossing matching cut of $U$ in side $A$ w.r.t. $Q$. Then $(W \setminus \text{Match}_{U;A}(Q)) \cup (Q \setminus P) \cup \text{Match}_{U;A}(P)$ is also a crossing matching cut of $U$ in $A$ w.r.t. $P$.

Moreover, if $Q = P_1 \cup P_2 \cup \cdots \cup P_k$ where each $P_i \in \Theta^*$, then any pair disconnected by $W$ is also disconnected by some $(W \setminus \text{Match}_{U;A}(Q)) \cup (Q \setminus P_i) \cup \text{Match}_{U;A}(P_i)$.

**Proof.** Apply Lemma 2 (intersection rule) to cuts $W$ and $(U \setminus P) \cup \text{Match}_{U;A}(P)$ and we have the first statement, that $(W \setminus \text{Match}_{U;A}(Q)) \cup (Q \setminus P) \cup \text{Match}_{U;A}(P)$ is a cut. For the second statement, suppose $W$ separates vertices $u$ and $v$. If both $u, v \notin U \cup \text{Match}_{U;A}(Q)$, then they are separated by any $(W \setminus \text{Match}_{U;A}(Q)) \cup (Q \setminus P_i) \cup \text{Match}_{U;A}(P_i)$. If one of $u, v$ is in $Q$, say $u \in Q$, then there exists at least one $i$ for which $u \in P_i$. Then $v \in A \setminus W$ and therefore $u, v$ are separated by $(W \setminus \text{Match}_{U;A}(Q)) \cup (Q \setminus P_i) \cup \text{Match}_{U;A}(P_i)$.

**Lemma 5.** Let $U$ be a $\kappa$-cut with two sides $A$ and $B$, and let $\Theta$ be defined w.r.t. its matching cuts in side $A$. For $P \in \Theta^*$, define $U^*(P)$ to be the cut separating $P$ from $A \setminus \text{Match}_{U;A}(P)$ minimizing $|\text{Side}_{U^*(P)}(P)|$.

1° $U^*(P)$ is either a crossing matching cut of $U$ in side $A$ w.r.t. $P$, or else there is no such crossing matching cut and $U^*(P) = (U \setminus P) \cup \text{Match}_{U;A}(P)$ is a matching cut.

2° Suppose $X$ is a crossing matching cut of $U$ in side $A$ w.r.t. $P$. If $u, v$ are separated by $X$, then they are also separated by either $U^*(P)$ or $(X \setminus \text{Match}_{U;A}(P)) \cup P$, which is a laminar cut of $U$.

**Proof.** Part 1°. Let $W$ be a cut that disconnects $P$ from $A \setminus \text{Match}_{U;A}(P)$. Because every vertex in $\text{Match}_{U;A}(P)$ is adjacent to $P$ and adjacent to $A \setminus \text{Match}_{U;A}(P)$, it follows that $\text{Match}_{U;A}(P) \subseteq W$. From Corollary 3 we know the set of all cuts separating $P$ from $A \setminus \text{Match}_{U;A}(P)$ has a unique minimum element. This is the cut $U^*(P)$; let the sides of $U^*(P)$ be $K^*, L^*$ with $P \subseteq K^* \subseteq B \cup P$ and $A \setminus \text{Match}_{U;A}(P) \subseteq L^*$.

If $K^* = B \cup P$, then $U^*(P) = (U \setminus P) \cup \text{Match}_{U;A}(P)$. If $K^* \subset B \cup P$, then $U^*(P) \cap B \neq \emptyset$. Now since $\text{Match}_{U;A}(P) \subseteq U^*(P)$ and $(U^*(P) \cap A) \cup (U \setminus P)$ is a matching cut w.r.t. $P$, it follows that $U^*(P)$ is a crossing matching cut of $U$ in side $A$ w.r.t. $P$.

Part 2°. Fix such a crossing matching cut $X$. It disconnects $P$ from $A \setminus \text{Match}_{U;A}(P)$, and has exactly two sides, $K \supseteq P$ and $L \supseteq A \setminus \text{Match}_{U;A}(P)$. By the minimality of $U^*(P)$ we have $K^* \subseteq K$. Because $X \cap B \neq \emptyset$, $K \subset B \cup P$. Thus, if $U^*(P) = (U \setminus P) \cup \text{Match}_{U;A}(P)$ is the matching cut (not a crossing matching cut), this contradicts the existence of $X$. We conclude that $U^*(P)$ is a crossing matching cut of $U$.

Let $u \in K$ and $v \in L$ be disconnected by $X$. If $u \in B \cap K$ and $v \in L$ then by Lemma 2 (intersection rule) applied to $U$ and $X$, $B \cap K$ is disconnected from the rest of the graph by $(X \cap B) \cup (X \setminus U) \cup (U \cap K) = (X \setminus \text{Match}_{U;A}(P)) \cup P$, which is a laminar cut of $U$. By the minimality of $U^*(P)$, $K^* \subseteq K$ and $U^*(P) \subseteq K \cup X$, hence $L \subseteq L^*$. When $u \in P$ and $v \in L$, they are also disconnected by $U^*(P)$. □
Corollary 4. Fix a cut $U$ with two sides $A$ and $B$, and let $\Theta^*$ be defined w.r.t. the matching cuts in side $A$, and let $U^*(P)$ be defined as in Lemma 5. Define $\mathcal{U} = \{U^*(P) \mid P \in \Theta^*\}$.

Let $X$ be a crossing matching cut of $U$ in side $A$ w.r.t. $Q$. If $u$ and $v$ are separated by $X$, then they are also separated by a member of $\mathcal{U}$ or $(X \backslash \text{Match}_{U:A}(Q)) \cup Q$, a laminar cut of $U$ in side $B$.

Proof. By Lemma 4, w.l.o.g. we may assume that $Q \in \Theta^*$. Since $X$ is a crossing matching cut, there is a crossing matching cut $U^*(Q) \in \mathcal{U}$. The claim then follows from Lemma 5.

3.3 Laminar Cuts

In this section we analyze the structure of laminar cuts. Throughout this section, $U$ refers to a cut that is not $(I, \kappa - 1)$-small, not a wheel cut $C(i, j)$ in some wheel, and has a side $A$ with $|A| > 2\kappa$.

Consider the set of all cuts $W$ that are laminar w.r.t. $U$, contained in $U \cup A$ and not $(I, \kappa - 1)$-small. It follows that $W$ has a side, call it $S(W)$, that contains $U \backslash W$ and all other sides of $U$.

Define $R(W)$ to be the region containing all other sides of $W$ beside $S(W)$. We call $W$ a maximal laminar cut of $U$ if there does not exist another laminar cut $W'$ such that $R(W) \subseteq R(W')$.

Theorem 5. Let $U$ be the reference cut.

1. If there exist matching cuts of $U$ in side $A$, define $\Theta^*$ w.r.t. $U, A$, define $Q = \cup_{P \in \Theta^*} P$, and let $X = (U \backslash Q) \cup \text{Match}_{U:A}(Q)$ be the matching cut in side $A$ having the smallest intersection with $U$. Then every laminar cut $W$ of $U$ in side $A$ is

   (i) a laminar cut of $X$ in region $A \backslash \text{Match}_{U:A}(Q)$, or
   (ii) a matching cut of $U$, or
   (iii) a crossing matching cut of $X$.

2. If there are no matching cuts of $U$ in side $A$, every laminar cut of $U$ in side $A$ is a maximal laminar cut, or a laminar cut of some maximal laminar cut $W_i$ in a side of $R(W_i)$. Moreover, whenever $W_i, W_j$ are distinct maximal laminar cuts, $R(W_i) \cap R(W_j) = \emptyset$.

Proof. Part 1. $X$ has a side $K = Q \cup (V \backslash (U \cup A))$ and a region $L = A \backslash \text{Match}_{U:A}(Q)$. According to the number of sides of $X$, we split into two cases:

   I. If $X$ has strictly more than two sides, let its sides be $K$ and $L_1, L_2, \ldots, L_r$, where $L = \cup_{i=1}^r L_i$.

   Then by Corollary 1, any other cut should have laminar type relation with $X$, or themselves be $(I, \kappa - 1)$-small cuts. But here the cut $W$ in our concern are not $(I, \kappa - 1)$-small. For such a $W$ of $U$ in side $A$, it can only be a laminar cut of $X$ in some side. If it was $L_i$, then we have (i). If it was $K$, then we have that $W \subseteq X \cup K$ and also $W \subseteq U \cup A$, so that $W \subseteq (X \cup K) \cap (U \cup A) = U \cup \text{Match}_{U:A}(Q)$. Then by definition of $\Theta$, $W$ is a matching cut of $U$ and we have (ii).

   II. If $X$ has exactly two sides, then they are $K = Q \cup (V \backslash (U \cup A))$ and $L = A \backslash \text{Match}_{U:A}(Q)$.

   Fix any laminar cut $W$ of $U$ in side $A$. If $W \subseteq A \cup (U \backslash Q)$ then $W$ is a laminar cut of $X$ in side $A \backslash \text{Match}_{U:A}(Q)$, and we are in case (i). If $W \subseteq U \cup \text{Match}_{U:A}(Q)$, then by definition of $\Theta$, $W = (U \backslash P) \cup \text{Match}_{U:A}(P)$ for some $P \in \Theta$ and $W$ is a matching cut of $U$, and we are in case (ii).

   Thus, we can proceed with the assumption that $W \cap Q \neq \emptyset$ and $W \cap (A \backslash \text{Match}_{U:A}(Q)) \neq \emptyset$. Therefore, $W \cap K \neq \emptyset$ and $W \cap L \neq \emptyset$. So $W$ must have wheel type or crossing matching.
Part 2. By assumption $U$ does not have matching cuts in side $A$. Enumerate all of its maximal laminar cuts $W = \{W_1, W_2, \ldots, W_{|W|}\}$. Fix any laminar cut $W$ of $U$ in side $A$. If $W \notin W$, then by definition of maximality there exists some $W_i$ such that $R(W_i) \subseteq R(W)$ and $S(W_i) \subseteq S(W)$. It follows that $W \subseteq R(W_i) \cup W_i$, so $W$ is a laminar cut of $W_i$ in one of the sides of $R(W_i)$. 

It remains to prove that for all $i \neq j$, $R(W_i) \cap R(W_j) = \emptyset$. Suppose the statement were false. Because $S(W_i) \cap S(W_j) \neq \emptyset$, $W_i, W_j$ must have laminar, crossing matching, or wheel type relation. They cannot be laminar, for then $R(W_i) \subseteq R(W_j)$, or $R(W_j) \subseteq R(W_i)$, contradicting the maximality of $W_i, W_j$. Otherwise, $W_i$ and $W_j$ must both have exactly two sides, namely $R(W_i), S(W_i)$ and $R(W_j), S(W_j)$. Apply Corollary 3 to $C = R(W_i) \cap R(W_j)$ and $D = S(W_i) \cap S(W_j)$, noticing that $W_i$ and $W_j$ disconnects $C$ and $D$, we may set $Y = \text{MinCut}_{C,D}$. Because $\text{Region}_{R_i}(C) \subseteq \text{Region}_{W_i}(C) \cap \text{Region}_{W_j}(C) = R(W_i) \cap R(W_j) = C$, $C$ is actually a region of $U$. As long as $R(W_i) \cup R(W_j) \neq A$, $Y$ is a laminar cut of $U$, also contradicting the maximality of $W_i, W_j$. Thus, we proceed under the assumption that $R(W_i) \cup R(W_j) = A$, meaning $Y = U$ is not a laminar cut of $U$.

If $W_i$ and $W_j$ have a crossing matching type relation, at least one of $R(W_i) \cap S(W_j)$ and $R(W_j) \cap S(W_i)$ is empty, suppose it is $R(W_i) \cap S(W_j) = \emptyset$. Then $R(W_i) \subseteq R(W_j) \cup W_j$, but we already have that $R(W_i) \cup R(W_j) = A$, so $R(W_j) = A \setminus W_j$, which means $W_j$ is a matching cut of $U$ in side $A$, contradicting the assumption of Part 2 that $U$ has no matching cuts in side $A$.

The last case is when $W_i$ and $W_j$ have a wheel type relation, i.e., they form a 4-wheel with center $T = W_i \cap W_j$, spokes $W_i \cap R(W_j)$, $W_i \cap S(W_j)$, $W_j \cap R(W_i)$, $W_j \cap S(W_i)$, and sectors $R(W_i) \cap R(W_j)$, $R(W_i) \cap S(W_j)$, $S(W_i) \cap R(W_j)$, $S(W_i) \cap S(W_j)$. Thus, the $\kappa$-cut $(W_i \cap S(W_j)) \cup (W_i \cap W_j) \cup (W_j \cap S(W_i))$ disconnects $S(W_j) \cap S(W_i)$ from $R(W_i) \cup R(W_j) = A$. This means $U = (W_i \cap S(W_j)) \cup (W_i \cap W_j) \cup (W_j \cap S(W_i))$ is also a cut of this wheel. This contradicts the original assumption that our reference cut $U$ is not a wheel cut $C(i', j')$ of some wheel. 

**3.4 Small Cuts**

Fix a vertex $u$ and a threshold $t \leq \left\lfloor \frac{n-k}{2} \right\rfloor$. Define $\text{Sm}_t(u)$ to be a cut $U$ minimizing $|\text{Side}_v(u)|$ with $|\text{Side}_v(u)| \leq t$. We first show that $\text{Sm}_t(u)$, if it exists, is unique.

**Theorem 6.** If there exists any $(III, t)$-small cut that is small w.r.t. $u$, then there exists a unique such cut, denoted $\text{Sm}_t(u)$, such that for any other cut $U$, $u \notin U$,

$$\text{Side}_{\text{Sm}_t(u)} \subseteq \text{Side}_v(u).$$

**Proof.** It suffices to show that for any two $(III,t)$-small cuts $U, W$, there exists a cut $X$ (possibly $U$ or $W$) such that

$$\text{Side}_X(u) \subseteq \text{Side}_v(u) \cap \text{Side}_v(u).$$

If $\text{Side}_v(u) \subseteq \text{Side}_v(u)$ or $\text{Side}_v(u) \subseteq \text{Side}_v(u)$, we may set $X = U$ or $X = W$. If $V \setminus (U \cup W \cup \text{Side}_v(u) \cup \text{Side}_v(u)) \neq \emptyset$, we may pick an arbitrary vertex $v$ in this set, and apply Corollary 3 to the singleton sets $C = \{u\}$ and $D = \{v\}$, and we may set $X = \text{MinCut}_{C,D}$.

These two cases above rule out the possibility that $U, W$ have a laminar or wheel type relation, except when, using the notation of Theorem 2, $A_{i^*} = \text{Side}_v(u)$ and $B_{j^*} = \text{Side}_v(u)$. But this would lead to a contradiction that

$$|V| = |A_{i^*}| + |B_{j^*}| - |A_{i^*} \cap B_{j^*}| + |U \cap W|$$
\[
\leq 2 \left\lfloor \frac{n - \kappa}{2} \right\rfloor - 1 + (\kappa - 1) < n.
\]

By Theorem 2 the remaining case is that \( U, W \) have crossing matching type, i.e., \( \text{Side}_U(u) \setminus \text{Side}_W(u) \neq \emptyset, \text{Side}_W(u) \setminus \text{Side}_U(u) \neq \emptyset, \) and \( V = U \cup W \cup \text{Side}_U(u) \cup \text{Side}_W(u). \) By Lemma 2 (intersection rule), \((U \cap \text{Side}_W(u)) \cup (W \cap \text{Side}_U(u)) \cup (U \cap W)\) is a cut. If it has size exactly \( \kappa \) then we can set \( X \) to be this cut. We proceed under the assumption that it is strictly larger than \( \kappa. \) Thus, by inclusion/exclusion,

\[
|V| = |U \cup W \cup \text{Side}_W(u) \cup \text{Side}_U(u)| = |U| + |W| + |\text{Side}_W(u)| + |\text{Side}_U(u)| - |\text{Side}_W(u) \cap \text{Side}_U(u)| - (|U \cap \text{Side}_W(u)| + |W \cap \text{Side}_U(u)| + |U \cap W|)
\]

Since \( u \in \text{Side}_W(u) \cap \text{Side}_U(u), \) this is

\[
\leq 2\kappa + 2t - 1 - (\kappa + 1) \\
\leq 2 \left\lfloor \frac{n - \kappa}{2} \right\rfloor + \kappa - 2 < n,
\]

which contradicts the definition of \( n = |V|. \) We conclude that when \( t \leq \lceil (n - \kappa)/2 \rceil, \) \( \text{Sm}_t(u) \) is unique if it exists. \( \square \)

4 A Data Structure for \((\kappa + 1)\)-Connectivity Queries

In this section we design an efficient data structure that, given \( u,v, \) answers \((\kappa + 1)\)-connectivity queries, i.e., reports that \( \kappa(u,v) = \kappa \) and produces a minimum \( \kappa \)-cut separating \( u,v, \) or reports that \( \kappa(u,v) \geq \kappa + 1. \)

We work with the mixed-cut definition of \( \kappa(u,v) \) (see Remark 1), which is the minimum size set of vertices and edges that need to be removed to disconnect \( u,v, \) or equivalently, the maximum size set of internally vertex-disjoint paths joining \( u \) and \( v. \)

**Theorem 7.** Given a \( \kappa \)-connected graph \( G, \) we can construct in \( \tilde{O}(m + \text{poly}(\kappa)n) \) time a data structure occupying \( O(n^2) \) space that answers the following queries. Given \( u,v \in V(G), \) report whether \( \kappa(u,v) = \kappa \) or \( \kappa(u,v) \geq \kappa + 1 \) in \( O(1) \) time. If \( \kappa(u,v) = \kappa, \) report a \( \kappa \)-cut separating \( u,v \) in \( O(\kappa) \) time.

**Sparsification.** In \( O(m) \) time, the Nagamochi-Ibaraki \([36]\) algorithm produces a subgraph \( G' \) that has arboricity \( \kappa + 1 \) and hence at most \((\kappa + 1)n \) edges, such that \( \kappa_G'(u,v) = \kappa(u,v) \) whenever \( \kappa_G(u,v) \leq \kappa + 1, \) and \( \kappa_G'(u,v) \geq \kappa + 1 \) whenever \( \kappa_G(u,v) \geq \kappa + 1. \) Without loss of generality we may assume \( G \) is the output of the Nagamochi-Ibaraki algorithm.

\[\text{If } \{u,v\} \notin E(G) \text{ and } \kappa(u,v) = \kappa, \text{ then there exists } U \subset V, |U| = \kappa, \text{ such that removing } U \text{ disconnects } u,v. \]

**Remark 1.** If \( \{u,v\} \in E(G) \) then there exists \( U \subset V, |U| = \kappa - 1, \) such that removing \( U \) and \( \{u,v\} \) disconnects \( u,v. \) In this case the single-edge path \( \{u,v\} \) would count for one of the \( \kappa \) internally vertex disjoint paths, the other \( \kappa - 1 \) passing through distinct vertices of \( U. \)
The Data Structure. Throughout this section we fix the threshold \( t = \left\lceil \frac{n-k}{2} \right\rceil \). Define \( \text{Sm}(u) = \text{Sm}_t(u) \) to be the unique minimum \( \kappa \)-cut with \( \text{Side}_{\text{Sm}_t(u)}(u) \leq t \), if any such cut exists, and \( \text{Sm}(u) = \bot \) otherwise. The data structure stores, for each \( u \in V(G) \), \( \text{Sm}(u), |\text{Side}_{\text{Sm}(u)}(u)| \), a \( O(\log n) \)-bit identifier for \( \text{Side}_{\text{Sm}(u)}(u) \), and for each vertex \( v \in N(u) \cap \text{Sm}(u) \), a bit \( b_{u,v} \) indicating whether \( \{|u, v|\} \cup \text{Sm}(u) \setminus \{v\} \) is a mixed cut disconnecting \( u \) and \( v \). Furthermore, when \( |\text{Side}_{\text{Sm}(u)}(u)| \leq k - 1 \), we store \( \text{Side}_{\text{Sm}(u)}(u) \) explicitly. When \( \text{Sm}(u) = \bot \) we will say \( \text{Side}_{\text{Sm}(u)}(u) = G \) and hence \( |\text{Side}_{\text{Sm}(u)}| = n \). The total space is \( O(kn) \).

Connectivity Queries. The query algorithm proceeds to the first applicable case. Note in the following, \( \text{Sm}(u) \) may be \( \bot \), and for all vertices \( v \), we define \( v \notin \bot \).

Case I: \( \text{Sm}(u) = \text{Sm}(v) \) and \( \text{Side}_{\text{Sm}(u)}(u) = \text{Side}_{\text{Sm}(v)}(v) \). Then \( \kappa(u, v) \geq \kappa + 1 \).

Case II: \( u \notin \text{Sm}(v) \) and \( v \notin \text{Sm}(u) \). Then \( \kappa(u, v) = \kappa \). Without loss of generality suppose that \( |\text{Side}_{\text{Sm}(u)}(u)| \leq |\text{Side}_{\text{Sm}(v)}(v)| \). Then \( \text{Sm}(u) \) is a \( \kappa \)-cut separating \( u \) and \( v \).

Case III: \( v \in \text{Sm}(u) \cap N(u) \), or the reverse. The bit \( b_{u,v} \) indicates whether \( \kappa(u, v) \geq \kappa + 1 \) or \( \kappa(u, v) = \kappa \), in which case \( \{|u, v|\} \cup \text{Sm}(u) \setminus \{v\} \) is the \( \kappa \)-cut.

Case IV: \( v \in \text{Sm}(u), u \in \text{Sm}(v) \). Then \( \kappa(u, v) \geq \kappa + 1 \).

Case V: \( v \in \text{Sm}(u), u \notin \text{Sm}(v) \), or the reverse. If \( |\text{Side}_{\text{Sm}(v)}(v)| \leq \kappa - 1 \), directly check whether \( u \in \text{Side}_{\text{Sm}(v)}(v) \). If so then \( \kappa(u, v) \geq \kappa + 1 \); if not then \( \text{Sm}(v) \) disconnects them. Thus \( |\text{Side}_{\text{Sm}(v)}(v)| \geq \kappa \). If \( |\text{Side}_{\text{Sm}(v)}(v)| \leq |\text{Side}_{\text{Sm}(u)}(u)| \) then \( \text{Sm}(v) \) is a \( \kappa \)-cut separating \( u \) and \( v \), and otherwise \( \kappa(u, v) \geq \kappa + 1 \).

Lemmas 6, 7, and Theorem 8 establish the correctness of the query algorithm. Its construction algorithm is described and analyzed in Section 4.1.

Lemma 6. If \( v \in \text{Side}_{\text{Sm}(u)}(u) \), then either \( \text{Sm}(v) = \text{Sm}(u) \) or \( \text{Sm}(v) \) is a laminar cut of \( \text{Sm}(u) \) with \( \text{Side}_{\text{Sm}(v)}(v) \subset \text{Side}_{\text{Sm}(u)}(u) \).

Proof. \( \text{Sm}(u) \) is (III, t)-small w.r.t. \( v \). By Theorem 6, \( \text{Sm}(v) \) exists and \( \text{Side}_{\text{Sm}(v)}(v) \subseteq \text{Side}_{\text{Sm}(u)}(v) \). \( \square \)

Lemma 7. Suppose \( u \) and \( v \) are not \( (\kappa + 1) \)-connected, i.e., \( \kappa(u, v) = \kappa \). If \( \{u, v\} \notin E(G) \), then they are disconnected by \( \text{Sm}(u) \) or \( \text{Sm}(v) \), and if \( \{u, v\} \in E(G) \), then they are disconnected by \( \{(u, v)\} \cup \text{Sm}(u) \setminus \{v\} \) or \( \{(u, v)\} \cup \text{Sm}(v) \setminus \{u\} \).

Proof. First suppose \( \{u, v\} \notin E(G) \) and let \( X \) be any cut separating \( u \) and \( v \). When \( t = \left\lceil \frac{n-k}{2} \right\rceil \) either \( |\text{Side}_{X}(u)| \leq t \) or \( |\text{Side}_{X}(v)| \leq t \). W.l.o.g. suppose it is the former, then \( \text{Sm}(u) \) exists and by Theorem 6, \( \text{Side}_{\text{Sm}(u)}(u) \subseteq \text{Side}_{X}(u) \), so \( \text{Sm}(u) \) also separates \( u \) and \( v \).

If \( \{u, v\} \in E(G) \), suppose \( (\kappa - 1) \) vertices \( W = \{w_1, w_2, \ldots, w_{\kappa - 1}\} \) and \( \{u, v\} \) disconnect \( u \) and \( v \). After removing \( W \) from the graph, \( G \setminus W \) is still connected. By deleting the edge \( \{u, v\} \), the graph breaks into exactly two connected components, say \( A \) and \( B \) with \( u \in A \) and \( v \in B \). Then \( W \cup \{u\} \) forms a \( \kappa \)-cut with \( \text{Side}_{W \cup \{u\}}(v) = B \), and \( W \cup \{v\} \) also forms a \( \kappa \)-cut with \( \text{Side}_{W \cup \{v\}}(u) = A \). Clearly we have 
\[
\begin{align*}
n &= |W| + |A| + |B| = \kappa - 1 + |A| + |B|.
\end{align*}
\]
Thus \( \text{Sm}(u) \) exists, \( \text{Side}_{\text{Sm}(u)}(u) \subseteq \text{Side}_{W \cup \{v\}}(u) \), and \( \text{Sm}(u) \) is either \( W \cup \{v\} \) or a laminar cut of \( W \cup \{v\} \) in side \( A \). Since \( \{u, v\} \in E(G) \), we have \( v \in \text{Sm}(u) \). If we remove \( \{u, v\} \) from \( G \), then any path from \( u \) to \( v \) goes through a vertex in \( W \), but any path from \( u \) to a vertex in \( W \) goes through a vertex in \( \text{Sm}(u) \setminus \{v\} \). Therefore, \( \{\{u, v\}\} \cup \text{Sm}(u) \setminus \{v\} \) is a mixed cut separating \( u, v \) as it blocks all \( u-v \) paths.

\[ |A| \leq \left\lfloor \frac{n - \kappa + 1}{2} \right\rfloor = \left\lfloor \frac{n - \kappa}{2} \right\rfloor = t. \]

**Theorem 8.** The query algorithm correctly answers \((\kappa + 1)\)-connectivity queries.

**Proof.** Suppose the algorithm terminates in Case I. It follows that \( u \notin \text{Sm}(v), v \notin \text{Sm}(u), \) and neither \( \text{Sm}(u) \) nor \( \text{Sm}(v) \) disconnect \( u \) and \( v \). Lemma 7 implies that \( \kappa(u, v) \geq \kappa + 1 \).

In Case II, if \( \text{Sm}(u) \neq \perp \) but \( \text{Sm}(v) = \perp \) then \( \text{Sm}(u) \) is the cut separating \( u, v \) and since

\[ |\text{Side}_{\text{Sm}(u)}(u)| < |\text{Side}_{\text{Sm}(v)}(v)| = n, \]

then the query is answered correctly. If both \( \text{Sm}(u), \text{Sm}(v) \neq \perp \), then by Lemma 6, \( v \notin \text{Sm}(u) \) and once again the query is answered correctly.

In Case III, by Lemma 7, if \( u \) and \( v \) are separated by a \( \kappa \)-cut, they are separated by \( \{\{u, v\}\} \cup \text{Sm}(u) \setminus \{v\} \) (if \( \text{Sm}(u) \neq \perp \)) or \( \{\{u, v\}\} \cup \text{Sm}(v) \setminus \{u\} \) (if \( \text{Sm}(v) \neq \perp \)), and this information is stored in the bit \( b_{u,v}^u, b_{v,u}^v \).

If we get to Case IV then \( \{u, v\} \notin E(G) \) and neither \( \text{Sm}(u) \) nor \( \text{Sm}(v) \) separate \( u, v \), hence by Lemma 7, \( \kappa(u, v) \geq \kappa + 1 \) and the query is answered correctly.

Case V is the most subtle. Because \( v \in \text{Sm}(u) \) and \( \{u, v\} \notin E(G) \), Lemma 7 implies that if \( \kappa(u, v) = \kappa \), then \( u, v \) must be separated by \( \text{Sm}(v) \). If \( \text{Sm}(v) = \perp \) then \( \kappa(u, v) \geq \kappa + 1 \) and the query is answered correctly. If \( |\text{Side}_{\text{Sm}(v)}(v)| \leq \kappa - 1 \) then the query explicitly answers the query correctly by direct lookup. Thus, we proceed under the assumption that \( \text{Sm}(v) \neq \perp \) exists and is not small.

If \( u \in \text{Side}_{\text{Sm}(v)}(v) \) then \( \text{Sm}(v) \) does not disconnect \( u \) and \( v \), and by Lemma 6,

\[ |\text{Side}_{\text{Sm}(v)}(v)| > |\text{Side}_{\text{Sm}(u)}(u)|, \]

so the query is handled correctly in this case.

If \( u \notin \text{Side}_{\text{Sm}(v)}(v) \) then \( \text{Sm}(v) \) separates \( u \) and \( v \), so we must argue that

\[ |\text{Side}_{\text{Sm}(v)}(v)| \leq |\text{Side}_{\text{Sm}(u)}(u)| \]

for the query algorithm to work correctly. It cannot be that \( \text{Sm}(v) \) and \( \text{Sm}(u) \) have a laminar relation, so by Theorem 2 they must have a crossing matching, wheel, or small type relation. If they have the small-type relation then the small sides of \( \text{Sm}(u) \) are contained in \( \text{Sm}(v) \) (contradicting \( u \notin \text{Sm}(v) \)) or the small sides of \( \text{Sm}(v) \) are contained in \( \text{Sm}(u) \), but we have already ruled out this case. Thus, the remaining cases to consider are wheel and crossing matching type.

Suppose \( \text{Sm}(u), \text{Sm}(v) \) form a 4-wheel \((T; C_1, C_2, C_3, C_4)\). Then \( u \notin \text{Sm}(v) \) appears in a sector of the wheel, say \( S_1 \). Then \( C(1, 2) \) is a cut violating the minimality of \( \text{Sm}(u) = C(1, 3) \).

Suppose \( \text{Sm}(u), \text{Sm}(v) \) have a crossing matching type relation. Let \( A_1 = \text{Side}_{\text{Sm}(u)}(u) \) and \( A_2 \) be the other side of \( \text{Sm}(u) \), and \( B_1 = \text{Side}_{\text{Sm}(v)}(v) \) and \( B_2 \) be the other side of \( \text{Sm}(v) \). Then \( u \in A_1 \cap B_2 \), and it must be that the diagonal quadrant \( A_2 \cap B_1 = \emptyset \). Suppose otherwise, i.e., \( A_2 \cap B_1 = \emptyset \), and let \( X = (\text{Sm}(u) \cap B_2) \cup (\text{Sm}(v) \cap A_1) \cup (\text{Sm}(u) \cap \text{Sm}(v)) \). Then by Corollary 3 \( X \) is a \( \kappa \)-cut with \( \text{Side}_X(u) = A_1 \cap B_2 \), contradicting the minimality of \( \text{Sm}(u) \). Thus, \( \text{Sm}(v) \) is a crossing matching cut of \( \text{Sm}(u) \) in side \( A_2 \) w.r.t. some \( Q \subseteq \text{Sm}(u) \cap B_1 \) with \( v \in Q \). By Theorem 2 and \( u \in A_1 \cap B_2 = \emptyset \), we have

\[ |A_1 \cap \text{Sm}(v)| = |B_2 \cap \text{Sm}(u)| \geq |A_2 \cap \text{Sm}(v)| = |B_1 \cap \text{Sm}(u)| = |Q|. \]
Thus,
\[ |\text{Side}_{\text{Sm}(u)}(u)| = |A_1| > |(A_1 \cap B_1) \cup Q| = |\text{Side}_{\text{Sm}(v)}(v)|, \]
establishing the correctness in the crossing matching case. (The strictness of the inequality is because \(A_1 \cap B_2 \neq \emptyset\).)

### 4.1 Construction of the Data Structure

We assume Nagamochi-Ibaraki sparsification \[36\] has already been applied, so \(G\) has arboricity \(\kappa+1\) and \(O(\kappa n)\) edges. We use the recent Forster et al. \[22\] algorithm for computing the connectivity \(\kappa = \kappa(G)\) in \(\tilde{O}(\text{poly}(\kappa)n)\) time and searching for \(\kappa\)-cuts.

**Theorem 9** (Consequence of Forster, Nanongkai, Yang, Saranurak, and Yingchareonthawornchai \[22\]). Given parameter \(s \leq \left\lceil \frac{n-\kappa}{2} \right\rceil\) and a vertex \(x\), there exists an algorithm that runs in time \(O(s\kappa^3)\) with the following guarantee. If the cut \(\text{Sm}_s(x)\) exists, it is reported with probability \(\geq \frac{3}{4}\), otherwise the algorithm returns \(\perp\). If the cut \(\text{Sm}_s(x)\) does not exist, the algorithm always returns \(\perp\).

**Corollary 5.** Given \(x \in V(G)\) and an integer \(s \leq \left\lceil \frac{n-\kappa}{2} \right\rceil\), we can, with high probability \(1 - 1/\text{poly}(n)\), compute \(\text{Sm}_s(x)\) in \(\tilde{O}(|\text{Side}_{\text{Sm}_s(x)}(x)| \cdot \kappa^3)\) time, or determine that \(\text{Sm}_s(x)\) does not exist in \(\tilde{O}(s\kappa^3)\) time.

**Proof.** High probability bounds can be accomplished by repeating the algorithm of Theorem 9 \(O(\log n)\) times. Use a doubling search to find \(\text{Sm}_{s_0}(x)\) for each \(s_0 = 2^i, i \leq \lfloor \log s \rfloor\). When \(s_0 \geq |\text{Side}_{\text{Sm}_s(x)}(x)|\) the procedure will succeed w.h.p.

We call the procedure of Corollary 5 FindSmall\((x, s)\).

**Definition 1.** Let \(U\) be a cut, \(A\) a side of \(U\). We use the notation \(\overline{A} = G \setminus (U \cup A)\) to be the region of all other sides of \(U\). Define \(G(U, \overline{A})\) to be the graph induced by \(U \cup \overline{A}\), supplemented with a \(\kappa\)-clique on \(U\). If \(W\) is a cut in \(G(U, \overline{A})\), define \(\text{Side}_{W}^{G(U, \overline{A})}(x)\) to be \(\text{Side}_{W}(x)\) in the graph \(G(U, \overline{A})\).

**Lemma 8.** Let \(U\) be a \(\kappa\)-cut, \(A\) be a side of \(U\), and \(W\) be a set of \(\kappa\) vertices in \(G(U, \overline{A})\). Then \(W\) is a \(\kappa\)-cut in \(G(U, \overline{A})\) if and only if \(W\) is a laminar cut of \(U\) in one of the sides of \(\overline{A}\). Moreover, when \(W\) is such a cut, for any vertex \(u \in U\setminus W\),

\[ \text{Side}_{W}(u) = \text{Side}_{W}^{G(U, \overline{A})}(u) \cup A. \]

**Proof.** A laminar cut \(W\) of \(U\) in one of the sides of \(\overline{A}\) does not disconnect vertices of \(U\), so it is a cut in \(G(U, \overline{A})\). When \(W\) is a cut in \(G(U, \overline{A})\), all vertices of \(U\setminus W\) should be in one side \(B\) of \(W\). Let the other sides of \(W\) form a region \(C\). Then in \(G\), by Lemma 1, there exists paths from vertices in \(A\) to \(U\setminus W\) that are not blocked by \(W\), so vertices in \(A\) and \(B\) together form a side \(D\) of \(W\). But any path from \(C\) to \(A\) should pass through some vertex in \(U\), while any path from \(C\) to \(B\) (involving \(U\setminus W\)) should pass through \(W\), so \(W\) is a cut in \(G\). This proves the first statement.

Note \(B\) here is exactly \(\text{Side}_{W}^{G(U, \overline{A})}(u)\), and \(D\) is exactly \(\text{Side}_{W}(u)\), and we have that \(D = B \cup A\). This proves the second statement.

Lemma 9 shows how, beginning with a cut \(X\) where \(\text{Side}_{X}(u)\) is small, can find another cut \(Y\) (if one exists) where \(\text{Side}_{Y}(u)\) is about \(M_i\), in \(\tilde{O}(M_4\kappa^3)\) time. The difficulty is that there could be an unbounded number of cuts “between” \(X\) and \(Y\) that would prevent the algorithm of Theorem 9 from finding \(Y\) directly.
Lemma 9. Fix any integer $M \leq t/2$, vertex $u$, and cut $X$ with $A = \text{Side}_X(u)$, $|A| \leq 2M$. There exists an algorithm $\text{Expand}(u, A, M)$ that runs in time $\tilde{O}(M \kappa^4)$ and, w.h.p., returns a cut $Y$ satisfying the following properties.

- $\text{Side}_X(u) \subseteq \text{Side}_Y(u)$.
- $|\text{Side}_Y(u)| \leq 2M$.
- If there exists a cut $Z$ that is $(\text{III}, M)$-small w.r.t. $u$, then $|\text{Side}_Y(u)| \geq |\text{Side}_Z(u)|$.

Proof. This algorithm uses Corollary 5 and Lemma 8.

1. Initially $Y \leftarrow X$. While $|\text{Side}_Y(u)| < M$,
   - (a) For each vertex $v \in Y$, in parallel,
     i. In the graph $G(Y, \overline{\text{Side}_Y(u)})$, run $\text{FindSmall}(v, M)$.
   - (b) The moment any call to $\text{FindSmall}$ halts in step (i) with a cut $W$, stop all such calls and set $Y \leftarrow W$. If all $|Y|$ calls to $\text{FindSmall}$ run to completion without finding a cut, halt and return $Y$.

Throughout the algorithm, $Y$ is always a cut such that $\text{Side}_X(u) \subseteq \text{Side}_Y(u)$ and $|\text{Side}_Y(u)| \leq 2M$. Furthermore, $|\text{Side}_Y(u)|$ is strictly increasing, so the algorithm terminates. By Corollary 5 the running time of (a) is $\tilde{O}(\Delta \kappa^4)$ where $\Delta = |\text{Side}_W(u)| - |\text{Side}_Y(u)|$ if a cut $W$ is found, and $\Delta = M$ otherwise. (The extra factor $\kappa$ is due to the parallel search in step (a).) The sum of the $\Delta$s telescopes to $O(M)$, so the overall running time is $\tilde{O}(M \kappa^4)$.

Suppose the cut $Z$ exists. If it were not the case that $|\text{Side}_Y(u)| \geq |\text{Side}_Z(u)|$, then in the last iteration, $|\text{Side}_Y(u)| < |\text{Side}_Z(u)| \leq M$. We argue below that there is another cut to find, and therefore that the probability the algorithm terminates prematurely is $1/\text{poly}(n)$, by Corollary 5.

Note that $|\text{Side}_Y(u)| + |\text{Side}_Z(u)| \leq 2M \leq t$, so $C \overset{\text{def}}{=} G \setminus (\text{Side}_Y(u) \cup \text{Side}_Z(u) \cup Y \cup Z) \neq \emptyset$. Apply Corollary 3 to sets $C$ and $D = \{u\}$, we may set $U = \text{MinCut}_{C,D}$. Because $\text{Region}_U(C) \subseteq \text{Region}_Y(C) \cap \text{Region}_Z(C) = C$, $C$ is actually a region of $U$, so $U$ is a laminar cut of $Y$, $\text{Side}_U(u) \subseteq \text{Side}_Y(u) \cup \text{Side}_Z(u)$. Since $|\text{Side}_Y(u)| < |\text{Side}_Z(u)|$, $U \neq Y$. Then for any vertex $v \in Y \setminus U$,

$$0 < |\text{Side}^Y_G(v, \overline{\text{Side}_Y(u)})| \leq |\text{Side}_U(u)| - |\text{Side}_Y(u)| \leq |\text{Side}_Z(u)| \leq M.$$ 

Therefore, $U$ or some other cut should have been found in step (a) in the last iteration.

Theorem 10 (Consequence of Picard and Queyremolle [38]). Let $H = (V, E, \text{cap})$ be a capacitated s-t flow network and $f$ be a maximum flow. In $O(|E|)$ time we can compute a directed acyclic graph $H' = (V', E')$ with $|E'| \leq |E|$, and an embedding $\phi : V \rightarrow V' \cup \{\perp\}$, such that the downward closed sets of $H'$ are in 1-1 correspondence with the minimum s-t cuts of $H$. I.e., if $V'' \subseteq V'$ is a vertex set with no arc of $E'$ leaving $V''$, then $(\phi^{-1}(V''), \overline{\phi^{-1}(V'')})$ is a min s-t cut in $H$, and all min s-t cuts in $H$ can be expressed in this way. Here $\phi(v) = \perp$ if $v$ appears on the “t” side of every min s-t cut.

We use Corollary 6 to find $\text{Sm}_U(u)$ for potentially many vertices $u$ in bulk.

Corollary 6. Fix two disjoint, non-empty vertex sets $C$ and $D$. In $O(\kappa^2(n - |C| - |D|))$ time, we can output a cut $S(v)$ for every $v \in V \setminus (C \cup D)$, such that if $\text{Sm}(v)$ exists and $C \subseteq \text{Side}_{\text{Sm}(v)}(v)$, then $S(v) = \text{Sm}(v)$. 

26
Proof. Form a flow network $\tilde{G}$ as in the proof of Corollary 3, shrinking $C$ and $D$ to vertices $s$ and $t$, splitting each $v$ into $v_{in}$, $v_{out}$, etc. The minimum $s_{out}$-$t_{in}$ cuts in $\tilde{G}$ are in 1-1 correspondence with the minimum vertex cuts separating $C$, $D$ in $G$. We are only interested in these cuts if they have size $\kappa$, so in this case a maximum flow $f$ can be computed in $O(\kappa^2(n - |C| - |D|))$ time. (Recall that the graph is assumed to have arboricity $\kappa + 1$, so every induced subgraph of $\tilde{G}$ has density $O(\kappa)$.)

Given $\tilde{G}$, $f$, the Picard-Queyrenne representation $\tilde{G}'$, $\phi$ can be computed in linear time. For each $v \in G$, let $S(v)$ be the cut corresponding to the smallest downward-closed set containing $\phi(v)$ in $\tilde{G}'$. Thus, the set $\{S(v)\}_{v \in V \setminus (C \cup D)}$ can be enumerated by a bottom-up traversal of $\tilde{G}'$ in $O(|E(\tilde{G}'|)) = O(\kappa(n - |C| - |D|))$ time. If $C \subseteq \text{Side}_{\text{Sm}(v)}(v)$ then clearly $S(v) = \text{Sm}(v)$.

We are now ready to present the entire construction algorithm.

Preamble. The algorithm maintains some $\kappa$-cut $T(u)$ for each $u$, which is initially $\perp$, and stores $|\text{Side}_{T(u)}(u)|$. If the algorithm makes no errors, $T(u) = \text{Sm}_0(u) = \text{Sm}(u)$ at the end of the computation. The procedure $\text{Update}(u, U)$ updates $T(u) \leftarrow U$ if $U$ is a better cut, i.e., $|\text{Side}_U(u)| \leq \min\{|\text{Side}_{T(u)}(u)| - 1, t\}$, and does nothing otherwise. $\text{Update}(A, U)$ is short for $\text{Update}(u, U)$ for all $u \in A$.

Step 1: very small cuts. For each $u \in V$, let $U \leftarrow \text{FindSmall}(u, 100\kappa)$ and then $\text{Update}(u, U)$. This takes $O(n\kappa^4)$ time.

Step 2: unbalanced cuts. For each index $i$ such that $100\kappa < 2^i \leq t$, let $\alpha = 2^i$, and pick a uniform sample $V_i \subseteq V$ of size $(n \log n)/\alpha$.\footnote{The Forster et al. [22] algorithm samples vertices proportional to their degree. Note that after the Nagamochi-Ibaraki [36] sparsification, the minimum degree is at least $\kappa$ and the density of every induced subgraph is at most $\kappa + 1$, so it is equally effective to do vertex sampling.} For each $u \in V_i$, compute $\text{Sm}_\alpha(u) \leftarrow \text{FindSmall}(u, \alpha)$. If $\text{Sm}_\alpha(u) = \text{Sm}(u) \neq \perp$, we first do an $\text{Update}(\text{Side}_{\text{Sm}(u)}(u), \text{Sm}(u))$, then compute $Y \leftarrow \text{Expand}(u, \text{Sm}(u), \alpha)$. For each $v \in Y$ we compute $W_v \leftarrow \text{FindSmall}(w, \alpha)$ and then $\text{Update}(v, W_v)$. We then run the algorithm of Corollary 6 with $C = \text{Side}_{\text{Sm}(u)}(u)$ and $D = V \setminus (Y \cup \text{Side}_Y(u))$, which returns a set of cuts $\{S(v)\}_{v \in V \setminus (C \cup D)}$. For each such $v \in \text{Side}_Y(u) \setminus \text{Side}_{\text{Sm}(u)}(u)$, do an $\text{Update}(v, S(v))$. For each index $i$, the running time is $O(|V_i| \cdot \kappa^4) = O(n\kappa^4)$, which is $O(n\kappa^4)$ overall.

Step 3: balanced cuts. Sample $O(\log n)$ pairs $(x, y) \in V^2$. For each such pair, compute $U \leftarrow \text{FindSmall}(x, t)$. If $U \neq \perp$, apply the algorithm of Corollary 6 to $C = \text{Side}_{\text{Sm}(x)}(x)$ and $D = \{y\}$, which returns a set $\{S(v)\}$. Then do an $\text{Update}(v, S(v))$ for every $v \in V \setminus (C \cup D)$. By Corollary 5 this takes $O(\kappa^3n)$ time.

Step 4: adjacent vertices. At this point it should be the case that $T(u) = \text{Sm}(u)$ for all $u$. For each $v \in T(u) \cap N(u)$ compute and set the bit $b_{u,v}$. (This information can be extracted from the calls to $\text{FindSmall}$ and the algorithm of Corollary 6 in the same time bounds.)

Lemma 10 is critical to proving the correctness of the algorithm’s search strategy.

Lemma 10. Suppose $\text{Sm}(u)$ and $\text{Sm}(v)$ exists, $u \in \text{Side}_{\text{Sm}(v)}(v)$, and suppose there is a cut $W$ such that

$$\kappa \leq |\text{Side}_{\text{Sm}(v)}(v)| < |\text{Side}_W(u)| \leq t.$$
Then \( v \in W \cup \text{Side}_W(u) \).

Proof. Because \( u \in \text{Side}_{\text{Sm}(v)}(v) \), by Lemma 6, \( \text{Sm}(u) \neq \bot \) is a laminar cut of \( \text{Sm}(v) \) and \( \text{Side}_{\text{Sm}(u)}(u) \subseteq \text{Side}_{\text{Sm}(v)}(u) \).

Consider the relationship between \( W \) and \( U = \text{Sm}(v) \). We use the same notation \( U_*, W_*, A_*, B_* \) and \( T \) from Theorem 2. Neither \( U \) nor \( W \) is \((1, \kappa - 1)\)-small, so they may have laminar, wheel, or crossing matching type relation.

If they have laminar type relation, then \( u \in \text{Side}_W(u) \cap \text{Side}_U(u) \), so either \( \text{Side}_W(u) = A_* \) or \( \text{Side}_U(u) = B_* \), but they cannot be both true as otherwise

\[
|V| = n > 2 \left[ \frac{n - \kappa}{2} \right] + \kappa - 2 \geq |A_*| + |B_*| - |A_* \cap B_*| + |T| = |V|.
\]

If \( \text{Side}_W(u) \) is \( A_* \) then \( v \in \text{Side}_W(u) \). If \( \text{Side}_U(u) \) is \( B_* \) then \( |\text{Side}_W(u)| < |\text{Side}_{\text{Sm}(v)}(u)| \), which contradicts the definition of \( W \).

If they have wheel type relation, then \( v \) cannot lie in a sector of the 4-wheel formed by \( W \) and \( U \), as this would violate the minimality of \( \text{Sm}(v) \). Therefore \( v \in W \).

If they have crossing matching type relation, suppose \( A_1 \cap B_1 \neq \emptyset, A_2 \cap B_2 \neq \emptyset \) and \( A_1 \cap B_2 = \emptyset \). Then \( v \in W \) (and the proof is done) or \( v \in A_2 \cap B_1 \), because for \( v \in A_1 \cap B_1 \), \( W_1 \cup T \cup U_1 \) is a cut with side smaller than \( \text{Sm}(v) \), and for \( v \in A_2 \cap B_2 \), \( U_2 \cup T \cup W_2 \) is a cut with a side smaller than \( \text{Sm}(v) \).

If \( v \in A_2 \cap B_1 \), then by Theorem 2, \(|W_2| \geq |U_2|\). Also, \( A_2 \) is \( \text{Side}_{\text{Sm}(v)}(v) \) so \( u \in A_2 \). Thus

\[
|\text{Side}_W(u)| - |\text{Side}_U(v)| = |B_2| - |A_2| = |U_2| - |W_2| - |A_2 \cap B_1| < 0,
\]

which contradicts the condition that \( |\text{Side}_W(u)| > |\text{Side}_{\text{Sm}(v)}(v)| \). \( \square \)

Theorem 11. The construction algorithm correctly computes \( \{\text{Sm}(v)\}_{v \in V} \) and runs in time \( \tilde{O}(n\kappa^4) \).

Proof. If \( |\text{Side}_{\text{Sm}(v)}(v)| \leq 100\kappa \), then \( T(v) = \text{Sm}(v) \) after Step 1, with high probability.

Suppose that \( |\text{Side}_{\text{Sm}(v)}(v)| \in [2^j, 2^{j+1}] \) and \( 2^{j+1} \leq t \). Then with high probability, at least one vertex \( x \in V_j \) is sampled in Step 2 such that \( x \in \text{Side}_{\text{Sm}(v)}(v) \). Step 2 (Expand) computes a cut \( Y \) such that \( |\text{Side}_Y(x)| \geq |\text{Side}_{\text{Sm}(v)}(v)| \), so by Lemma 10, either \( v \in \text{Side}_Y(x) \) or \( v \in Y \). In the former case \( \text{Sm}(v) \) is computed using the Corollary 6 algorithm. In the latter case \( \text{Sm}(v) \) is computed directly using \( \text{FindSmall} \).

Finally, if \( \text{Sm}(v) \) is balanced, say \( |\text{Side}_{\text{Sm}(v)}(v)| \geq t/4 \), then w.h.p. we would pick a pair \( (x, y) \) in Step 3 such that \( x \in \text{Side}_{\text{Sm}(v)}(v) \) and \( y \in V \setminus (\text{Sm}(v) \cup \text{Side}_{\text{Sm}(v)}(v)) \). If this holds the algorithm of Corollary 6 correctly computes \( \text{Sm}(v) \). \( \square \)

5 Conclusion

This paper was directly inspired by the extended abstract of Cohen, Di Battista, Kanevsky, and Tamassia [10]. Our goal was to substantiate the main claims of this paper, and to simplify and improve the data structure that answers \((\kappa + 1)\)-connectivity queries.

We believe that our structural theorems can, ultimately, be used to develop even more versatile vertex-cut data structures. For example, is it possible to answer the following more general queries in \( O(\kappa) \) time using \( O(\kappa n) \) space?

**Is-it-a-cut?** \((u_1, \ldots, u_\kappa)\): Return true iff \( \{u_1, \ldots, u_\kappa\} \) forms a \( \kappa \)-cut.
**Part-of-a-cut?**\((u_1, \ldots, u_{\kappa-g})\): Return true iff the input can be extended to a \(\kappa\)-cut \(\{u_1, \ldots, u_{\kappa-g}\} \cup \{u_{\kappa-g+1}, \ldots, u_{\kappa}\}\).

We assumed throughout the paper that \(\kappa\) was not too large, specifically \(\kappa < n/4\). When \(n < 2\kappa\), all cuts are \((1,\kappa)\)-small by our classification, and the classification theorem (Theorem 2) says very little about the structure of such cuts. Understanding the structure of minimum vertex cuts when \(\kappa\) is large, relative to \(n\), is an interesting open problem.

**References**

[1] A. Abboud, R. Krauthgamer, and O. Trabelsi. Cut-equivalent trees are optimal for min-cut queries. In *Proceedings of the 61st IEEE Annual Symposium on Foundations of Computer Science (FOCS)*, pages 105–118, 2020.

[2] S. Baswana, K. Choudhary, and L. Roditty. Fault tolerant subgraph for single source reachability: generic and optimal. In *Proceedings of the 48th Annual ACM Symposium on Theory of Computing (STOC)*, pages 509–518, 2016.

[3] J. D. Batson, D. A. Spielman, and N. Srivastava. Twice-Ramanujan sparsifiers. *SIAM J. Comput.*, 41(6):1704–1721, 2012.

[4] G. D. Battista and R. Tamassia. Incremental planarity testing (extended abstract). In *Proceedings of the 30th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 436–441, 1989.

[5] G. D. Battista and R. Tamassia. On-line maintenance of triconnected components with SPQR-trees. *Algorithmica*, 15:302–318, 1996.

[6] A. A. Benczúr. Counterexamples for directed and node capacitated cut-trees. *SIAM J. Comput.*, 24(3):505–510, 1995.

[7] A. A. Benczúr and M. X. Goemans. Deformable polygon representation and near-mincuts. In M. Grötschel and G. O. H. Katona, editors, *Building Bridges: Between Mathematics and Computer Science*, volume 19 of Bolyai Society Mathematical Studies, pages 103–135. 2008.

[8] A. A. Benczúr and D. R. Karger. Randomized approximation schemes for cuts and flows in capacitated graphs. *SIAM J. Comput.*, 44(2):290–319, 2015.

[9] K. Choudhary. An optimal dual fault tolerant reachability oracle. In *Proceedings 43rd Int’l Colloq. on Automata, Languages, and Programming (ICALP)*, 2016.

[10] R. F. Cohen, G. Di Battista, A. Kanevsky, and R. Tamassia. Reinventing the wheel: an optimal data structure for connectivity queries (extended abstract). In *Proceedings of the 25th Annual ACM Symposium on Theory of Computing (STOC)*, pages 194–200, 1993.

[11] W. H. Cunningham and J. Edmonds. A combinatorial decomposition theory. *Canadian J. Math.*, 32(3):734–765, 1980.

[12] E. A. Dinic, A. V. Karzanov, and M. V. Lomonosov. On the structure of the system of minimum edge cuts in a graph. *Studies in Discrete Optimization*, pages 290–306, 1976. (in Russian).
[13] Y. Dinitz and Z. Nutov. A 2-level cactus model for the system of minimum and minimum+1 edge-cuts in a graph and its incremental maintenance. In *Proceedings 27th ACM Symposium on Theory of Computing (STOC)*, pages 509–518, 1995.

[14] Y. Dinitz and Z. Nutov. A 2-level cactus tree model for the system of minimum and minimum+1 edge cuts of a graph and its incremental maintenance. Part I: the odd case. Unpublished manuscript, 1999.

[15] Y. Dinitz and Z. Nutov. A 2-level cactus tree model for the system of minimum and minimum+1 edge cuts of a graph and its incremental maintenance. Part II: the even case. Unpublished manuscript, 1999.

[16] Y. Dinitz and A. Vainshtein. The connectivity carcass of a vertex subset in a graph and its incremental maintenance. In *Proceedings of the 26th Annual ACM Symposium on Theory of Computing (STOC)*, pages 716–725, 1994.

[17] Y. Dinitz and A. Vainshtein. Locally orientable graphs, cell structures, and a new algorithm for the incremental maintenance of connectivity carcasses. In *Proceedings of the 6th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 302–311, 1995.

[18] Y. Dinitz and A. Vainshtein. The general structure of edge-connectivity of a vertex subset in a graph and its incremental maintenance. odd case. *SIAM J. Comput.*, 30(3):753–808, 2000.

[19] R. Duan and S. Pettie. Connectivity oracles for failure prone graphs. In *Proceedings 42nd ACM Symposium on Theory of Computing*, pages 465–474, 2010.

[20] R. Duan and S. Pettie. Connectivity oracles in graphs subject to vertex failures. *SIAM J. Comput.*, 49(6):1363–1396, 2020.

[21] D. Firmani, L. Georgiadis, G. F. Italiano, L. Laura, and F. Santaroni. Strong articulation points and strong bridges in large scale graphs. *Algorithmica*, 74(3):1123–1147, 2016.

[22] S. Forster, D. Nanongkai, L. Yang, T. Saranurak, and S. Yingchareonthawornchai. Computing and testing small connectivity in near-linear time and queries via fast local cut algorithms. In *Proceedings of the 31st ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2046–2065. SIAM, 2020.

[23] H. N. Gabow. Using expander graphs to find vertex connectivity. *J. ACM*, 53(5):800–844, 2006.

[24] Y. Gao, J. Li, D. Nanongkai, R. Peng, T. Saranurak, and S. Yingchareonthawornchai. Deterministic graph cuts in subquadratic time: Sparse, balanced, and k-vertex. *CoRR*, abs/1910.07950, 2019.

[25] L. Georgiadis, G. F. Italiano, L. Laura, and N. Parotsidis. 2-edge connectivity in directed graphs. *ACM Trans. Algorithms*, 13(1):9:1–9:24, 2016.

[26] L. Georgiadis, G. F. Italiano, L. Laura, and N. Parotsidis. 2-vertex connectivity in directed graphs. *Inf. Comput.*, 261:248–264, 2018.

[27] L. Georgiadis, G. F. Italiano, and N. Parotsidis. Strong connectivity in directed graphs under failures, with applications. *SIAM J. Comput.*, 49(5):865–926, 2020.
[28] R. E. Gomory and T. C. Hu. Multi-terminal network flows. Journal of the Society for Industrial and Applied Mathematics, 9, 1961.

[29] D. Gusfield and D. Naor. Efficient algorithms for generalized cut trees. In Proceedings First ACM-SIAM Symposium on Discrete Algorithms, pages 422–433, 1990.

[30] J. E. Hopcroft and R. E. Tarjan. Dividing a graph into triconnected components. SIAM J. Comput., 2(3):135–158, 1973.

[31] A. Kanevsky, R. Tamassia, G. D. Battista, and J. Chen. On-line maintenance of the four-connected components of a graph. In Proceedings 32nd IEEE Symposium on Foundations of Computer Science (FOCS), pages 793–801, 1991.

[32] B. M. Kapron, V. King, and B. Mountjoy. Dynamic graph connectivity in polylogarithmic worst case time. In Proceedings of the 24th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1131–1142, 2013.

[33] A. Korman. Labeling schemes for vertex connectivity. ACM Trans. on Algorithms, 6(2), 2010.

[34] S. Mac Lane. A structural characterization of planar combinatorial graphs. Duke Math. J., 3(3):460–472, 1937.

[35] K. Menger. Zur allgemeinen kurventheorie. Fundamenta Mathematicae, 10:96–115, 1927.

[36] H. Nagamochi and T. Ibaraki. A linear-time algorithm for finding a sparse k-connected spanning subgraph of a k-connected graph. Algorithmica, 7(5&6):583–596, 1992.

[37] M. Pătraşcu and M. Thorup. Planning for fast connectivity updates. In Proceedings 48th IEEE Symposium on Foundations of Computer Science (FOCS), pages 263–271, 2007.

[38] J.-C. Picard and M. Queyranne. On the structure of all minimum cuts in a network and applications. In Combinatorial Optimization II, volume 13 of Mathematical Programming Studies, pages 8–16. 1980.

[39] C.-P. Schnorr. Bottlenecks and edge connectivity in unsymmetrical networks. SIAM J. Comput., 8(2):265–274, 1979.

[40] R. E. Tarjan. Depth-first search and linear graph algorithms. SIAM J. Comput., 1(2):146–160, 1972.

[41] W. T. Tutte. A theory of 3-connected graphs. Nederl. Akad. Wetensch. Proc. Ser. A 64 = Indag. Math., 23:441–455, 1961.

[42] W. T. Tutte. Connectivity in Graphs. University of Toronto Press, 1966.

[43] H. Whitney. Congruent graphs and the connectivity of graphs. American J. Mathematics, 54(1):150–168, 1932.

[44] H. Whitney. Non-separable and planar graphs. Trans. Amer. Math. Soc., 34(2):339–362, 1932.