SUMS OF THREE SQUARES IN $\mathbb{Q}(\sqrt{3})$, AND IN $\mathbb{Q}(\sqrt{17})$

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Abstract. The numbers of representations of totally positive integers as sums of three integer squares in $\mathbb{Q}(\sqrt{3})$ and in $\mathbb{Q}(\sqrt{17})$, are studied by using Shimura lifting map of Hilbert modular forms. We show the following results. In case of $\mathbb{Q}(\sqrt{3})$, a totally positive integer $a + b\sqrt{3}$ is represented as a sum of three integer squares if and only if $b$ is even. In case of $\mathbb{Q}(\sqrt{17})$, a totally positive integer is represented as a sum of three integer squares if and only if it is not in the form $\pi_1^2 \pi_2^2 \mu$ with $\mu \equiv 7 \pmod{\pi_2^2}$ or $\mu \equiv 7 \pmod{\pi_2^2}$ where $\pi_2, \pi_2'$ are prime elements with $2 = \pi_2\pi_2'$. A similar result as Gauss’s three squares theorem in both cases of $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{17})$, and as its application, tables of class numbers of their totally imaginary extensions are given.

1. Introduction

H. Maass [3] showed that all totally positive integers in $\mathbb{Q}(\sqrt{5})$ are expressed as a sum of three integer squares in $\mathbb{Q}(\sqrt{5})$ by giving the explicit formula for the numbers $r_{3,\mathbb{Q}(\sqrt{5})}(\alpha)$ of representations for totally positive integers $\alpha$. The formula involves class numbers of totally imaginary quadratic extensions $\mathbb{Q}(\sqrt{5}, \sqrt{-\alpha})$ of $\mathbb{Q}(\sqrt{5})$. Since the numbers of representations are calculable, by using Maass’s formula H. Cohn [1] gave a table of class numbers of totally imaginary quadratic extensions of $\mathbb{Q}(\sqrt{5})$. In our previous paper [8] we showed, by using Shimura lifting map, that any totally positive integer $a + b\sqrt{2}$ ($a, b \in \mathbb{Z}$) in $\mathbb{Q}(\sqrt{2})$ is expressed as a sum of three integer squares if and only if $b$ is even. Further we gave the formula for the number $r_{3,\mathbb{Q}(\sqrt{2})}(a + b\sqrt{2})$ representing $a + b\sqrt{2}$ as sums of three integer squares which involves the class number of a totally imaginary quadratic extension $\mathbb{Q}(\sqrt{2}, \sqrt{-a - b\sqrt{2}})$ of $\mathbb{Q}(\sqrt{2})$.

In the present paper we are concerned with the fields $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{17})$. By using Shimura lifting map of Hilbert modular forms again, we give the following results. A totally positive integer $a + b\sqrt{3} \in \mathbb{Q}(\sqrt{3})$ is expressed as a sum of three integer squares if and only if $b$ is even. The formula for $r_{3,\mathbb{Q}(\sqrt{3})}(\alpha\nu^2)$ with totally positive square-free $\alpha$ and with products $\nu$ of totally positive prime elements is obtained, which gives as its application, a table of class numbers of totally imaginary quadratic extensions of $\mathbb{Q}(\sqrt{3})$. As for $\mathbb{Q}(\sqrt{17})$, let $\pi_2 := \frac{5 + \sqrt{17}}{2}, \pi'_2 := \frac{5 - \sqrt{17}}{2}$. A totally positive integer is expressed as a sum of three integer squares if and only if it is not in the form $\pi_2^2 \pi_2'^2 \mu$ with nonnegative integers $\epsilon, \epsilon'$ and with $\mu \equiv 7 \pmod{\pi_2^2}$ or $\mu \equiv 7 \pmod{\pi_2^2}$. The formula for $r_{3,\mathbb{Q}(\sqrt{17})}$ is obtained as well as a table of class numbers of totally imaginary quadratic extensions of $\mathbb{Q}(\sqrt{17})$.

To see how the Shimura lifting map works, we illustrate the case of rational integers (see [5, 6, 7]). Let $\mathbb{H}$ be the upper half plane $\{z \in \mathbb{C} \mid \Im z > 0\}$, and let

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e(z) := \exp(2\pi \sqrt{-1}z). For a discriminant $D$ of a quadratic field, $\chi_D$ denotes the Kronecker-Jacobi-Legendre symbol. If $a$ is a square-free natural number, then $a^*$ denotes $a$ or $4a$ according as $a \equiv 1 \pmod{4}$ or not. For $N \in \mathbb{N}$, $(\mathbb{Z}/N)^*$ denotes the group of Dirichlet characters modulo $N$ with $1_N$ as the identity element. Let $M_{k+1/2}(N, \chi)$ be the space of holomorphic modular forms for $\Gamma_0(N)$ of weight $k + 1/2$ with character $\chi \in (\mathbb{Z}/N)^*$ ($4/N$). Then there is a Shimura lifting map $\mathcal{S}_{a^*, \chi}$ of $M_{k+1/2}(N, \chi)$ to $M_{2k}(N/2, \chi^2)$ for each square-free $a \geq 1$ and for $k \geq 1$, such as

$$\mathcal{S}_{a^*, \chi}(f) = C_{f, a} + \sum_{1 \leq n \leq a^*} \sum_{0 \leq d | n} (\chi_n \chi)(d)d^{k-1}c_{an^2/d^2}e(nz)$$

with $f(z) = \sum_{0 \leq n} c_n e(nz)$, where $C_{f, a}$ is a constant for which the left hand side of (1) is a modular form. Let $\theta(z)$ be a theta series, namely $\theta(z) = \sum_{n=-\infty}^{\infty} e(n^2z)$. If $f$ is a product of $\theta(z)$ and an Eisenstein series $E(z)$ of weight $k$, then $C_{f, a}$ is obtained as the constant term of a Hilbert-Eisenstein series over a field $\mathbb{Q}(\sqrt{a})$ where the Hilbert-Eisenstein series is determined only by $E(z)$.

Let $r_3(\mathbb{Q})$ denote the numbers of representations of $n$ as sums of three rational integral squares. The generating function $\sum_{n \leq n} r_3(\mathbb{Q})e(nz)$ of $r_3(\mathbb{Q})$ is $\theta(z)^3 \in M_{3/2}(4, \chi_4)$. Let $n = am^2$ with square-free $a > 0$. Since $\theta(z)^2$ is an Eisenstein series, $\theta(z)^3$ is a product of $\theta(z)$ an the Eisenstein series. The constant term of $\mathcal{S}_{a^*, \chi}(\theta(z)^3)$ is obtained from the constant term of an associated Hilbert-Eisenstein series which involves the class number of $\mathbb{Q}(\sqrt{-a})$. Since $M_{2}(2, 1_2)$ is one dimensional, it is spanned by an Eisenstein series of weight 2, and by comparing the Fourier coefficients of $\mathcal{S}_{a^*, \chi}(\theta(z)^3)$ and those of the Eisenstein series, we obtain the formula for $r_3(\mathbb{Q})$ (12). In particular Gauss’s three squares theorem is derived from the formula, that is $r_3(\mathbb{Q})(a) = 2^23h\mathbb{Q}(\sqrt{-a})$ for square-free $a \equiv 1, 2 \pmod{4}$, and $r_3(\mathbb{Q}) = 2^3h\mathbb{Q}(\sqrt{-a})$ for square-free $a \equiv 3 \pmod{8}$, $h\mathbb{Q}(\sqrt{-a})$ being the class number of $\mathbb{Q}(\sqrt{-a})$. The one of purposes of the present paper is to obtain such a kind of formulas on $\mathbb{Q}(\sqrt{3})$ or on $\mathbb{Q}(\sqrt{17})$ instead of $\mathbb{Q}$.

Our paper [8] gives only a partial result on Shimura lifts of noncuspidal Hilbert modular forms over a totally real field $K$, however it is shown that products of theta series and Hilbert-Eisenstein series over $K$ have Shimura lifts whose constant terms are obtained from Hilbert-Eisenstein series over totally real quadratic extensions of $K$. So there is a way that a similar argument as in the elliptic modular forms can be made for Hilbert modular forms. In the present paper we carry out this for $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{17})$. However while the space $M_{2}(2, 1_2)$ contains no nontrivial cusp forms in the elliptic modular case as well as the cases of $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{2})$, there is a non-trivial cusp form of weight 2 in each case of $\mathbb{Q}(\sqrt{3})$ or $\mathbb{Q}(\sqrt{17})$, and a closer study of Fourier coefficients of Hilbert modular forms is necessary.

2. Notations and some preceding results

Let $K$ be a totally real algebraic number field of degree $g$ over $\mathbb{Q}$. We denote by $O_K$, $d_K$ and $\delta_K$, ring of algebraic integers, the different of $K$ and the discriminant respectively. For $\alpha \in K$, $\alpha^{(1)}, \cdots, \alpha^{(g)}$ denotes the conjugates of $\alpha$ in a fixed order. If $\alpha^{(i)}$ is positive for every $i$, then we call $\alpha$ totally positive, and denote it by $\alpha > 0$. For an integral ideal $\mathfrak{a}$, let $E_{\mathfrak{a}}$ denotes the group of totally positive units congruent to 1 modulo $\mathfrak{a}$, and so $E_{\mathfrak{a}}$ denotes the group of all totally positive units, while $O_K^\times$ denotes the group of all units. We denote by $N$ and $tr$, the norm map and the trace
map of $K$ over $\mathbb{Q}$ respectively, namely $N(\alpha) = \prod_{i=1}^{g} \alpha(i)$ and $\text{tr}(\alpha) = \sum_{i=1}^{g} \alpha(i)$. An integral ideal is called odd or even according as it is coprime to the ideal (2) or not. An integer $\alpha$ in $\mathcal{O}_K$ is called odd or even according as the ideal (\alpha) is odd or even. If $\mathfrak{P}$ is a prime ideal, then $v_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-adic valuation. Let $\mu_K$ denote the Möbius function on $K$, and let $\varphi_K$ denote the Euler function on $K$, that is, $\varphi_K(\mathfrak{N}) = N(\mathfrak{N}) \prod_{\mathfrak{P} | \mathfrak{N}} (1 - N(\mathfrak{P})^{-1})$ for an integral ideal $\mathfrak{N}$. We denote by $C_{\mathfrak{N}}$, the fractional ideal class group modulo $\mathfrak{N}$ in the narrow sense where ideals have denominators coprime to $\mathfrak{N}$, and denote by $C_{\mathfrak{N}}^*$, the group of characters of $C_{\mathfrak{N}}$. An element of $C_{\mathfrak{N}}^*$ is called a (classical) Hecke character. The identity element of $C_{\mathfrak{N}}^*$ is denoted by $1_{\mathfrak{N}}$, for which $1_{\mathfrak{N}}(\mathfrak{N})$ is 1 or 0 according as a numerator of an ideal $\mathfrak{N}$ is coprime to $\mathfrak{N}$ or not. If $\mathfrak{N}$ is principal with a generator $\nu$, then we denote $1_{\mathfrak{N}}$ also by $1_{\nu}$. For $\mathfrak{a} = (a_1, \ldots, a_g) \in \{0, 1\}^g$, we define $\text{sgn}^a(\alpha)$ by setting $\text{sgn}^a(\alpha) := \text{sgn}(\alpha^{(1)})^{a_1} \cdots \text{sgn}(\alpha^{(g)})^{a_g}$ for $\alpha \in K, \neq 0$ where $\text{sgn}^0(0) := 1$. For $\psi \in C_{\mathfrak{N}}^*$, let $e_\psi = (e_1, \ldots, e_n) \in \{0, 1\}^g$ be so that $\psi(\mu) = \text{sgn}^{e_\psi}(\mu)$ for $\mu \equiv 1 \mod \mathfrak{N}, \mu \neq 0$. The character $\psi$ is called even if $e_\psi = (0, \ldots, 0)$, and it is called odd if $e_\psi = (1, \ldots, 1)$.

For a Hecke character $\psi$, we denote by $f_\psi$, the conductor of $\psi$, and put $e_\psi := f_\psi, \prod_{\mathfrak{p} | \mathfrak{a}} \mathfrak{p}^{f_\psi \mathfrak{p}^{(a)}} \mathfrak{P}$. The primitive character associated with $\psi$ is denoted by $\psi$. For an integral ideal $\mathfrak{N}$, we define $\psi_{\mathfrak{N}}$ to be $\psi_{\mathfrak{N}}$. For an integer $\alpha \in K$ not square, we denote by $f_\alpha$ or $d_{K(\alpha)/K}$, the relative discriminant of $K(\sqrt{\alpha})$ over $K$, and denote by $\psi_{\alpha}$ in $C_{\mathfrak{N}}^*$, the character associated with the extension, where $\psi(\mathfrak{N})$ is 1, -1 or 0 according as the prime $\mathfrak{P}$ of $K$ is split at $K(\sqrt{\alpha})$, inert or ramified.

We need some more general character than a Hecke character. For the purpose it is convenient to use the idelic language. For a prime $\mathfrak{P}$ of $K$, let $K_{\mathfrak{P}}, \mathcal{O}_{\mathfrak{P}}, \mathcal{O}^*_{\mathfrak{P}}$ be the $\mathfrak{P}$-adic completion of $K$, the maximal local ring and the group of its units. For an integral ideal $\mathfrak{N}$ of $K$, let $J(\mathfrak{N})$ denote the group consisting of ideles whose $\mathfrak{P}$-th components are in $\mathcal{O}_{\mathfrak{P}}$ for $\mathfrak{P} | \mathfrak{N}$. Put $U_K := \prod_{\mathfrak{P}} \mathcal{O}_{\mathfrak{P}}^\times \times (\mathbb{R}^\times)^g$ with $\mathbb{R}^\times = \{ x \in \mathbb{R} \mid x > 0 \}$. Let $K^\times(\mathfrak{N}) := K \cap J(\mathfrak{N})$, namely $K^\times(\mathfrak{N})$ is the group of elements in $K^\times$ whose denominators and numerators are both coprime to $\mathfrak{N}$. Let $K^\times_{\mathfrak{N}}$ denote the subgroup of $K^\times(\mathfrak{N})$ consisting of totally positive elements multiplicatively congruent to 1 modulo $\mathfrak{N}$. A homomorphism of the finite idele to $C_{\mathfrak{N}}$ by sending $j = (j_{\mathfrak{P}})$ to an ideal class containing fractional ideal $\prod_{\mathfrak{P}} \mathcal{O}^\times_{\mathfrak{P}} \times (\mathbb{R}^\times)^g$ gives the natural isomorphism between $J(\mathfrak{N})/(K^\times_{\mathfrak{N}}U_K)$ and $C_{\mathfrak{N}}$, and we identify them. Let $U_{\mathfrak{N}}$ be the subgroup of $U_K$ consisting of ideles whose $\mathfrak{P}$-th components are in $1 + \mathcal{O}^\times_{\mathfrak{P}} \mathcal{O}_{\mathfrak{P}}$. The factor group $U_K/U_{\mathfrak{N}}$ is isomorphic to $(\mathcal{O}_K/\mathfrak{N})^\times$. We fix local parameters $\varpi_{\mathfrak{P}}$ of $\mathcal{O}_{\mathfrak{P}}$ for $\mathfrak{P}$, which give the isomorphism of $U_K/U_{\mathfrak{N}}$ onto $(\mathcal{O}_K/\mathfrak{N})^\times$. Let $(\mathcal{O}_K/\mathfrak{N})^\times$ denote the group of characters of $(\mathcal{O}_K/\mathfrak{N})^\times$. For $\phi \in C_{\mathfrak{N}}^*$ and for $\omega \in (\mathcal{O}_K/\mathfrak{N})^\times$ we define

$$
\psi(\xi \cdot \mathfrak{N}) = \omega(\xi) \prod_{\mathfrak{P} | \mathfrak{N}} \varpi_{\mathfrak{P}}^{-\varphi(\mathfrak{N})} \phi(\varpi_{\mathfrak{P}}^{-\varphi(\mathfrak{N})})
$$

for $\xi \in K^\times$ and for an ideal $\mathfrak{N}$ where a denominator of $\xi \mathfrak{N}$ is coprime to $\mathfrak{N}$. We define $\psi(\xi \cdot \mathfrak{N}) = 0$ if a numerator of $\xi \mathfrak{N}$ is not coprime to $\mathfrak{N}$. Obviously there holds an equality $\psi(\xi \cdot \mathfrak{N})\psi(\xi' \cdot \mathfrak{N}') = \psi(\xi' \cdot \xi' \mathfrak{N} \mathfrak{N}')$. This is a character of $J(\mathfrak{N})/(K^\times_{\mathfrak{N}}U_{\mathfrak{N}})$. Since $J(\mathfrak{N})/(K^\times_{\mathfrak{N}}U_{\mathfrak{N}}) \simeq (U_K/U_{\mathfrak{N}})^\times \times J(\mathfrak{N})/(K^\times_{\mathfrak{N}}U_K) \simeq (\mathcal{O}_K/\mathfrak{N})^\times \times C_{\mathfrak{N}}$ as groups, the group $(J(\mathfrak{N})/(K^\times_{\mathfrak{N}}U_{\mathfrak{N}}))^\times$ of characters is isomorphic to $(\mathcal{O}_K/\mathfrak{N})^\times \times C_{\mathfrak{N}}$. We put $\psi(\xi) := \psi(\xi \cdot \mathfrak{N}) = \omega(\xi)\phi(\xi)$ for $\xi \in K^\times(\mathfrak{N})$, and $\psi(\mathfrak{N}) := \psi(1 \cdot \mathfrak{N})$ for
an ideal $\mathfrak{A}$. Also for $\psi \in (O_K/H_\mathfrak{n})^* \times C^-_\mathfrak{n}$, $e_\psi$ is defined as in the case of Hecke characters, however the equality $e_\psi = e_\delta$ holds since $\omega(\xi) = 1$ for $\xi$ with $\xi \equiv 1 \pmod{\mathfrak{n}}$. We call $\psi$ primitive if there is no ideal integral $\mathfrak{M}$ so that $\psi = \psi' \mathfrak{1}_\mathfrak{M}$ with a character $\psi'$ of $J(\mathfrak{M})/(K^*_\mathfrak{M} H_\mathfrak{M})$. In such a case we denote $\mathfrak{M}$ by $f_\psi$, and call it the conductor of $\psi$, and put $e_\psi := f_\psi \prod_{\psi(\mathfrak{p}) = 0, \mathfrak{p} \mid f_\psi} \mathfrak{p}$. For $\psi$ primitive, we define the Gauss sum by

\[
\tau_K(\psi) := \psi(\mu \cdot f_\psi) = \sum_{\xi \in O_K (\text{mod } f_\psi)} \psi(\xi)e(\text{tr}(\mu \xi))
\]

with $\mu \in K, \nu > 0, (\mu f_\psi, f_\psi) = O_K$, where $\tau_K(\psi)$ is determined up the choices of $\mu$. The standard argument shows that $|\tau_K(\psi)| = N(f_\psi)^{1/2}$, and that $\tau_K(\psi)\tau_K(\psi') = \text{sgn}^\psi(-1)\psi(-1)N(f_\psi)$. When $\omega$ is trivial, $\tau_K(\psi)$ of (2) coincides with the Gauss sum of a Hecke character.

Let $\mathfrak{M}, \mathfrak{N}$ be integral ideals. Let $\omega = \omega' \in (O_K/\mathfrak{M})^* \times C^-_\mathfrak{M}, \psi = \psi' \in (O_K/\mathfrak{N})^* \times C^-_\mathfrak{N}$, so that $\psi \omega$ is an even or odd Hecke character in $C^-_\mathfrak{MN}$, namely

\[
e_\psi \omega = (0, \cdots, 0) \text{ or } (1, \cdots, 1), \text{ and } \omega(\xi) = \omega'(\xi) \text{ for } \xi \in K^\times (\mathfrak{M}, \mathfrak{N})).
\]

For a fractional ideal $2\mathfrak{M}$ and for a totally positive $\nu \in K$, we define

\[
\sigma_{k-1,\psi}(\nu; \mathfrak{M}) := \sum_{\nu\mathfrak{M} \subset \mathfrak{A} \subset O_K} \psi(\mathfrak{A})\psi'(\nu \cdot \mathfrak{MN}^{-1})N(\mathfrak{A})^{k-1},
\]

where it is 0 if $\nu \mathfrak{M}$ is not integral. We also define $\sigma_{k-1,\psi}(\nu; \mathfrak{M})$ to be 0 if $\nu$ is not totally positive. By a condition (9), the summation (4) is equal to $\omega'(\nu \prod_{\psi(\mathfrak{p}) = 0, \mathfrak{p} \mid f_\psi} \mathfrak{p})$ $\sum_{\nu \mathfrak{M} \subset \mathfrak{A} \subset O_K} \psi'(\mathfrak{A})\psi(\nu \mathfrak{MN}^{-1})N(\mathfrak{A})^{k-1}$. If $\mathfrak{M} = O_K$, then we denote $\sigma_{k-1,\psi}(\nu; \mathfrak{M})$ simply by $\sigma_{k-1,\psi}(\nu)$, and further if $K$ is of class number 1 and if both of $\psi$ and $\psi'$ are Hecke characters, namely, $\omega$ and $\omega'$ are both trivial, then $\sigma_{k-1,\psi}(\nu)$ is expressed as

\[
\sigma_{k-1,\psi}(\nu) = \sum_{\delta|\nu, \delta \in O_K/O_K^*} \psi(\delta)\psi'(\nu/\delta)N((\delta))^{k-1} \quad (\nu \in O_K, \nu > 0).
\]

We omit $\psi$ from the notation $\sigma_{k-1,\psi}$ if $\mathfrak{M} = O_K$ and $\psi = 1$, and do similar as for $\psi'$.

Let $2^g$ be the product of $g$ copies of the upper half plane $2^g$. For $g, \delta$ in $K$ and for $\delta = (z_1, \cdots, z_g) \in 2^g, N(\gamma + \delta)$ and $\text{tr}(\gamma)$ stand for $\prod_{i=1}^{2g} (\gamma(i)z_i + \delta(i))$ and $\sum_{i=1}^{2g} \gamma(i)z_i$ respectively. For a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(K)$, we put $A_3 = \begin{pmatrix} \alpha(i)z_i + \beta(i) \\ \gamma(i)z_i + \delta(i) \end{pmatrix}$. We define

\[
\Gamma_0(2^g, \mathfrak{M}) := \{ (\alpha, \beta, \gamma, \delta) \in SL_2(K) \mid \alpha, \delta \in O_K, \beta \in 2^g, \gamma \in \mathfrak{M} \}
\]

for a fractional ideal $2^g$ and for an integral ideal $\mathfrak{M}$.

Let $\psi \in (O_K/\mathfrak{M})^* \times C^-_\mathfrak{M}, \psi' \in (O_K/\mathfrak{N})^* \times C^-_\mathfrak{N}$ be as in (4), and let $k$ be a natural number with the same parity as $e_\psi \omega$. We assume that $\psi$ or $\psi'$ is nontrivial when $g = 1$ and $k = 2$. Then we define an Eisenstein series by

\[
G_{k,\psi}(3; \mathfrak{M}, \mathfrak{N}, \epsilon_\psi^{-1}, \mathfrak{M}^{-1}, \epsilon_\psi^{-1}, \mathfrak{N}^{-1}, \epsilon_\psi^{-1}, \mathfrak{M}^{-1}, \mathfrak{N}^{-1}, \epsilon_\psi^{-1}) := C + 2^g \sum_{0 < \nu \in \mathfrak{D}} \sigma_{k-1,\psi}(\nu; \mathfrak{M}^{-1}, \epsilon_\psi^{-1}, \mathfrak{N}^{-1}, \epsilon_\psi^{-1})e(\text{tr}(\nu_3))
where $C$ is a constant term given by (i) $\psi((\mathcal{N}\varepsilon^{-1}_{\psi}\phi)\mathcal{D})\psi(1 - k, \psi\overline{\psi})$ if $k > 1$ and $\mathcal{N} = \mathcal{O}_K$ or if $\mathcal{N} \subset \mathcal{O}_K$ and $\mathcal{N} = \mathcal{O}_K$, (ii) $\psi((\mathcal{N}\varepsilon^{-1}_{\psi}\phi)\mathcal{D})\psi(0, \psi\overline{\psi})$ if $k = 1$, $\mathcal{N} = \mathcal{O}_K$ and $\mathcal{N} \subset \mathcal{O}_K$, (iii) $\psi(\mathcal{D})\psi(0, \psi\overline{\psi}) + \psi(\mathcal{D})\psi(0, \psi\overline{\psi})$ if $k = 1$ and $\mathcal{N} = \mathcal{N}' = \mathcal{O}_K$, and (iv) 0 otherwise. The Eisenstein series $G_{k,\psi}(z; \mathcal{N}\varepsilon^{-1}e\psi^{-1}, \mathcal{D})$ of (5) is a Hilbert modular form for $\Gamma_0(\mathcal{D}^{-1}\mathcal{D}_K^{-1}, \mathcal{N}\mathcal{D}_K)$ of weight $k$ with character $\psi\overline{\psi}$, namely $G_{k,\psi}(z; \mathcal{N}\varepsilon^{-1}e\psi^{-1}, \mathcal{D})$ satisfies

$$G_{k,\psi}(A; \mathcal{N}\varepsilon^{-1}e\psi^{-1}, \mathcal{D}) = (\psi\overline{\psi})(\gamma)N(\gamma + \delta)kG_{k,\psi}(z; \mathcal{N}\varepsilon^{-1}e\psi^{-1}, \mathcal{D})$$

for $A = \begin{pmatrix} \alpha & \beta \\ -\gamma & \delta \end{pmatrix} \in \Gamma_0(\mathcal{D}^{-1}\mathcal{D}_K^{-1}, \mathcal{N}\mathcal{D}_K)$ (6). We denote by $\mathcal{M}_k(\Gamma_0(\mathcal{D}^{-1}\mathcal{D}_K^{-1}, \mathcal{N}\mathcal{D}_K), \psi\overline{\psi})$, the space of Hilbert modular forms of weight $k$ with character $\psi\overline{\psi}$. For $f$ in the space, the value $\kappa(\alpha, \gamma, f) = \kappa(\alpha, \gamma, f)$ of $f(c)$ at a cusp $\alpha/\gamma$ ($\gamma \in \mathcal{D}_K$) is defined by $\kappa(\alpha, \gamma, f) = \lim_{l \to \infty} N(\gamma + l)k f(B)\gamma$ where $B = (\alpha/\gamma) \in SL_2(K)$. The subspace of $\mathcal{M}_k(\Gamma_0(\mathcal{D}^{-1}\mathcal{D}_K^{-1}, \mathcal{N}\mathcal{D}_K), \psi\overline{\psi})$ consisting of cusps forms is denoted by $\mathcal{S}_k(\Gamma_0(\mathcal{D}^{-1}\mathcal{D}_K^{-1}, \mathcal{N}\mathcal{D}_K), \psi\overline{\psi})$, where $\psi\overline{\psi}$ is omitted if $\psi\overline{\psi} = 1$.

The values of Eisenstein series (5) at cusps are computed in (5). We can take $\alpha, \gamma$ so that $\mathcal{B} := (\alpha, \gamma, \mathcal{D}^{-1}\mathcal{D}_K^{-1})$ is coprime to $\mathcal{N}\mathcal{D}_K$. The value $\kappa(\alpha, \gamma, G_{k,\psi}(z; \mathcal{N}\varepsilon^{-1}e\psi^{-1}, \mathcal{D}))$ at the cusp $\alpha/\gamma$ is 0 if $\gamma$ is not an integral ideal $\mathcal{M}_\gamma$, $\mathcal{M}_\gamma'$ with $\mathcal{M}_\gamma, \mathcal{M}_\gamma'$ coprime to $\mathcal{N}\mathcal{D}_K$, $\mathcal{N}\varepsilon^{-1}e\psi^{-1}$ and with $(\gamma, \mathcal{D}^{-1}\mathcal{D}_K^{-1}, \mathcal{N}\mathcal{D}_K^{-1}) = \mathcal{M}_\gamma^{-1}\mathcal{N}\varepsilon^{-1}e\psi^{-1}\mathcal{M}_\gamma^{-1}$. Suppose otherwise, and let $\mathcal{M}_\gamma$ be the largest such ideal. Then the value is given by (7)

$$sgn^{\psi}(\alpha)sgn^{\psi\overline{\psi}}(\gamma)\mu_K((\mathcal{E}_{\psi}\varepsilon^{-1}e\psi^{-1}, \mathcal{M}_\gamma, \mathcal{M}_\gamma'))N(\alpha \cdot \mathcal{B}^{-1}\mathcal{M}_\gamma, \mathcal{M}_\gamma, \mathcal{M}_\gamma')^{-1} \times \psi(\gamma)N(\mathcal{M}_\gamma^{-1}\mathcal{E}_{\psi}\varepsilon^{-1}e\psi^{-1}, \mathcal{M}_\gamma, \mathcal{M}_\gamma')^{-1}\tau_K(\mathcal{M}_\gamma^{-1})^{-1}N(\mathcal{M}_\gamma)^{-1}N(\mathcal{M}_\gamma^{-1})^{-1}$$

$$\times \prod_{\mathcal{P}|\mathcal{M}_\gamma} (1 - N(\mathcal{P}))(1 - (1 - N(\mathcal{P}))^{-1})$$

where if $\gamma = 0$, then the value is non-zero only when $\mathcal{N} = \mathcal{O}_K$ and it is given by replacing $\gamma$ in (7) by $N(\mathcal{N})$, and where if $\alpha = 0$, the value is non-zero only when $f_{\psi} = \mathcal{O}_K$ and it is given by replacing $\alpha$ in (7) by 1.

When $k = 1$, the values of Eisenstein series at cusps may have an additional term. Let $\mathcal{L}_{\gamma} := \gamma \mathcal{D}^{-1}\mathcal{D}_K^{-1}\mathcal{M}_\gamma^{-1}\mathcal{E}_{\psi}\varepsilon^{-1}e\psi^{-1}$ and $\mathcal{L}'_{\gamma} := \gamma \mathcal{D}^{-1}\mathcal{D}_K^{-1}\mathcal{M}_\gamma^{-1}\mathcal{E}_{\psi}\varepsilon^{-1}e\psi^{-1} \cap e\psi^{-1}f_{\psi}$. If there is an integer divisor $\mathcal{R}$ of $\mathcal{E}_{\psi}\varepsilon^{-1}e\psi^{-1}$ so that the numerator of $\mathcal{L}_{\gamma}\mathcal{R}^{-1}$ is coprime to $\mathcal{N}$ and the denominator is coprime to $f_{\psi}\mathcal{R}$, then there is the additional term. Let $\mathcal{R}_\gamma$ be the divisor of $(\mathcal{N}, \mathcal{E}_{\psi}\varepsilon^{-1}e\psi^{-1})$ satisfying $v_{\mathcal{P}}(\mathcal{L}_{\gamma}\mathcal{R}_\gamma^{-1}) = 0$ for any prime divisor $\mathcal{P}$ of $(\mathcal{N}, \mathcal{E}_{\psi}\varepsilon^{-1}e\psi^{-1})$. Then $\kappa(\alpha, \gamma, G_{k,\psi}(z; \mathcal{N}\varepsilon^{-1}e\psi^{-1}, \mathcal{D}))$ has the additional term

$$sgn^{\psi}(\gamma)\mu_K(\mathcal{R}_{\gamma})\psi(-\gamma \cdot \gamma^{-1}(\mathcal{L}_{\gamma}\mathcal{R}_\gamma^{-1} \cap \mathcal{O}_K))N(\mathcal{L}_{\gamma}\mathcal{R}_\gamma^{-1})^{-1}\tau_K(\mathcal{M}_\gamma^{-1})^{-1}N(\mathcal{M}_\gamma)^{-1}$$

(8)
where if $\gamma = 0$, then the value is non-zero only when $\mathfrak{M} = \mathcal{O}_K$ and it is given by replacing $\gamma$ in (5) by $N(\mathfrak{M})$, and where if $\alpha = 0$, the value is non-zero only when $f_{\psi'} = \mathcal{O}_K$ and it is given by replacing $\alpha$ in (5) by 1.

Let $\theta(\overline{z}) = \sum_{\mu \in \mathcal{O}_K} e(\text{tr}(\overline{\mu} z))$ be a theta series. It is a modular form of weight 1/2 for $\Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K)$. Let $j$ be its factor of automorphy, namely, $\theta(\overline{A} \overline{z}) = j(A, \overline{z}) \theta(\overline{z})$ for $A \in \Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K)$. Suppose that 4|$\mathfrak{M}$. Let $\psi_0 \in C^*_{\mathfrak{M}}$ and let $k$ be a natural number with same parity as $\psi_0$. We denote by $\mathbf{M}_{k+1/2}(\Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K), \psi_0)$, the space of modular forms with $\psi_0(\gamma) j(A, \overline{z}) N(\gamma \mathfrak{M} + \delta)^k (A = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K))$ as a factor of automorphy. Assume that the class number of $K$ is 1 in the wide sense. Let $\alpha$ be a totally positive square-free integer in $K$. Then the Shimura lifting map $\mathcal{A}_{\alpha, \psi_0}$ associated with $\alpha$ should be a linear map of $\mathbf{M}_{k+1/2}(\Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K), \psi_0)$ to $\mathbf{M}_{2k}(\Gamma_0(\mathfrak{d}_K^{-1}, 2^{-1}4\mathfrak{d}_K), \psi_0^0)$ satisfying

$$
\mathcal{A}_{\alpha, \psi_0}(f) = C_{f, \alpha} + \sum_{\nu \in \mathcal{O}_K, \nu \not| \epsilon} \sum_{\mu \in \mathcal{O}_K, \mu \not| \epsilon} (\psi_{\alpha} \psi_0)(\delta) N((\delta))^{-k-1} c_{\alpha, \nu} \epsilon(\text{tr}(\nu \mathfrak{M}))
$$

with $f(\overline{z}) = c_0 + \sum_{\nu \in \mathcal{O}_K, \nu \not| \epsilon} \sum_{\mu \in \mathcal{O}_K, \mu \not| \epsilon} \epsilon(\text{tr}(\nu \mathfrak{M})) \in \mathbf{M}_{k+1/2}(\Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K), \psi_0)$ where $C_{f, \alpha}$ is a constant for which the left hand side of (9) is a Hilbert modular form. As for a space of Hilbert cusp forms, Shimura [8] established such lifting map. However the existence of the map for non-cuspidal Hilbert modular forms is not yet proved unconditionally. We show in [8], for example that if $k \geq 2$ and 16|$\mathfrak{M}$, then there is a map $\mathcal{A}_{\alpha, \psi_0}$ of $\mathbf{M}_{k+1/2}(\Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K), \psi_0)$, and further that there is a map of the subspace of $\mathbf{M}_{k+1/2}(\Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K), \psi_0)$ with $4 \mathfrak{M}$ \text{ mod } $\mathfrak{d}_K$ generated by the products of the theta series and Eisenstein series $\mathfrak{E}$ with $\psi_0 = \psi, \psi'$ for even or odd Hecke characters $\psi, \psi'$. The latter assertion can be slightly generalized to $\psi, \psi'$ as in [8] because the key proposition (Proposition 6.5 in [8]) holds for such characters, where the proof is made by word-to-word translation of the original proof.

Let $f(\overline{z}) = c_0 + \sum_{\nu \in \mathcal{O}_K, \nu \not| \epsilon} \sum_{\mu \in \mathcal{O}_K, \mu \not| \epsilon} \epsilon(\text{tr}(\nu \mathfrak{M})) \in \mathbf{M}_{k}(\Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K), \psi_0)$, and let $\mu \in \mathcal{O}_K, \mu \not| \epsilon$ be so that $\mu \epsilon | \mathfrak{M}$ and $\mu^2 | \mathfrak{M}$. Then we define

$$
(U(\mu ; f))(\overline{z}) := c_0 + \sum_{\nu \in \mathcal{O}_K, \nu \not| \epsilon} c_{\mu \nu} \epsilon(\text{tr}(\nu \mathfrak{M}))
$$

which is in $\mathbf{M}_{k}(\Gamma_0(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K), \psi_0)$.

### 3. Hilbert modular forms on $\mathbb{Q}(\sqrt{3})$

Let $K = \mathbb{Q}(\sqrt{3})$. Then $\mathfrak{d}_K = (2\sqrt{3})$, and the class number of $K$ in the wide sense is 1. Let $\epsilon_0 := 2 + \sqrt{3}$ be a fundamental unit, which has a positive norm. Put $\pi_2 := 1 + \sqrt{3}$ and $p_2 = (\pi_2)$. Then the ideal (2) is decomposed as (2) = $p_2^2$. Let $\chi_{-4}^K := \chi_{-4} \circ N \in C^*_{p_2}$. For $\mu \in \mathcal{O}_K$, $\chi_{-4}^K(\mu) = \text{sgn}(N(\mu))$ for $\pi_2 \nmid \mu$, and $\chi_{-4}^K(\mu) = 0$ for $\pi_2 | \mu$. The Hecke character $\chi_{-4}^K$ is odd. The conductor $f_{\chi_{-4}^K}$ of $\chi_{-4}^K$ is $\mathcal{O}_K$, and $\chi_{-4}^K(\mu) = \text{sgn}(N(\mu))$, in particular $\chi_{-4}^K(p_2) = \text{sgn}(N(p_2)) = -1$. We have $\tau_K(\chi_{-4}^K) = \chi_{-4}^K(2\sqrt{3}) = -1$. Let $\rho_2$ be the unique nontrivial character of
Transformation formulas for theta series (see for example [8], Sect.3) give the values of \( \theta \) as cusps of Eisenstein series. Taking values at cusps of Eisenstein series, the square \( \theta^2 \) of the theta series is a Hilbert modular form for \( \Gamma_0(1,4\mathcal{O}_K) \) of weight 1 with character \( \chi^{-1} \). We take as a set \( C_0(4) \) of representatives of cusps of \( \Gamma_0(1,4\mathcal{O}_K) \),

\[
C_0(4) = \left\{ \frac{1}{8\sqrt{3}}, \frac{1}{4\pi_2 \sqrt{3}}, \frac{1}{4\sqrt{3}}, \frac{4\pi_2}{2\pi_2 \sqrt{3}}, \frac{1}{2\sqrt{3}} \right\},
\]

and as a set \( C_0(2) \) of representatives of cusps of \( \Gamma_0(1,2\mathcal{O}_K) \),

\[
C_0(2) = \left\{ \frac{1}{4\sqrt{3}}, \frac{1}{2\pi_2 \sqrt{3}}, \frac{1}{2\sqrt{3}} \right\}.
\]

We express \( \theta^2 \) as a linear combination of Eisenstein series of weight 1. By (7) and (8), and by equations \( L_K(0, \chi^{-1}_4) = 3^{-1} \) and \( L_K(0, \chi^{-1}_4) = 2^{-1}3^{-1} \), the values at cusps of Eisenstein series \( G_{1,\chi^{-1}_4}(3; \mathcal{O}_K, \mathcal{O}_K), G_{1,\chi^{-1}_4}(3; \mathcal{O}_K, \mathcal{O}_K) \), \( G_{1,\chi^{-1}_4}\psi(3; \mathcal{O}_K, \mathcal{O}_K) \) are obtained as in Table 2, where these Eisenstein series are modular forms for \( \Gamma_0(1,4\mathcal{O}_K) \) of weight 1 with character \( \chi^{-1}_4 \).
Corollary 3.3.

The following equality holds;

\[ \theta(\zeta)^2 = 3G_{1,\chi_{24}}(\zeta; \mathcal{O}_K, \mathcal{O}_K) - 2^{-1}G_{1,\chi_{24}}^4(\zeta; \mathcal{O}_K, \mathcal{O}_K) \]

\[ - 2G_{1,\chi_{24}}^2(\zeta; (2), \mathcal{O}_K) + 2^{-1}G_{1,\psi}^\varphi(\zeta; \mathcal{O}_K, \mathcal{O}_K). \]

Proof. Let \( \Gamma(\mathfrak{d}_K^{-1}, \mathcal{O}_K)[2] \) := \( \{ \left( \begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \in \text{SL}_2(K) \mid \alpha \equiv \delta \equiv 1 \pmod{2}, \beta \in 2\mathfrak{d}_K, \gamma \in 2\mathfrak{d}_K \} \), which is called a congruence subgroup of level 2. Then \( \Gamma(\mathfrak{d}_K^{-1}, \mathcal{O}_K)[2] \) acts freely on \( \mathfrak{S}_2 \). The arithmetic genus of a compactified nonsingular model of \( \Gamma(\mathfrak{d}_K^{-1}, \mathcal{O}_K)[2] \) is determined by the volume and by the contribution from cusp singularities (van der Geer [2] Chap. II ~ IV), and it is computed to be 4 + 1/12 (4 + 4 + 4 + 4 + 4 + 4 + 4 + 4) = 6. Hence \( \text{dim} \mathcal{S}_2(\Gamma(\mathfrak{d}_K^{-1}, \mathcal{O}_K)[2]) = 5 \) where \( \mathcal{S}_2(\Gamma(\mathfrak{d}_K^{-1}, \mathcal{O}_K)[2]) \) denotes the space of cusp forms for \( \Gamma(\mathfrak{d}_K^{-1}, \mathcal{O}_K)[2] \) of weight 2. Since \( (\frac{\mathfrak{d}}{\mathfrak{d}})^{-1}\Gamma(\mathfrak{d}_K^{-1}, \mathcal{O}_K)[2] \) is a subgroup of \( \Gamma(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K) \) of index 2, the dimension of the space of cusp forms for \( \Gamma(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K) \) of weight 2 is at most 5.

Let \( f(\zeta) \) be a Hilbert modular form given as the left hand side minus the right hand side of the equation \( \Theta(\zeta) \). Then it is a cusp form from Table 1 and Table 2. It is easy to check that the Fourier coefficients of \( f \) are 0 at least for \( \nu = 1, 2, 2 \pm \sqrt{3} \). We show that \( f = 0 \). Suppose that \( f \neq 0 \). Then \( f(\zeta)G_{1,\chi_{24}}(\zeta; \mathcal{O}_K, \mathcal{O}_K), f(\zeta)G_{1,\chi_{24}}^4(\zeta; (2), \mathcal{O}_K), f(\zeta)G_{1,\psi}^\varphi(\zeta; \mathcal{O}_K, \mathcal{O}_K), f(\zeta)^2 \) are linearly independent. The first four Fourier coefficients of these cusp forms vanish. Then these cusp forms together with \( G_{1,\chi_{24}}^4(\zeta; \mathcal{O}_K, \mathcal{O}_K)G_{1,\psi}^\varphi(\zeta; \mathcal{O}_K, \mathcal{O}_K) \) and \( \Xi(\zeta) \) are linearly independent by the Fourier expansions of \( 10 \) and \( 11 \). This contradicts to \( \text{dim} \mathcal{S}_2(\Gamma(\mathfrak{d}_K^{-1}, 4\mathfrak{d}_K)) \leq 5 \). Hence \( f = 0 \).

Comparing the Fourier coefficients of the both sides of \( \Theta(\zeta) \), we obtain the following:

Corollary 3.2. Let \( r_{2,K}(\nu) \) denote the number representing \( \nu \) as sums of two integer squares in \( K \). Let \( \mathfrak{S}_{0,\psi}^{\varphi} \) be as in \( \Theta(\zeta) \). Then

\[ r_{2,K}(\nu) = 2\{6\sigma_{0,\chi_{24}}(\nu) + (\nu^2 - 5)\sigma_{0,\chi_{24}}^2(\nu) - 4\sigma_{0,\chi_{24}}^4(\nu/2) \} \quad (\nu \in \mathcal{O}_K, \nu \neq 0). \]

In other words, for \( \nu = a + b\sqrt{3} > 0 \), we have (i) \( r_{2,K}(\nu) = 0 \) if \( v_{p_2}(\nu) = 0 \) and \( b \) is odd, or if \( v_{p_2}(\nu) \) is odd, and (ii) \( r_{2,K}(\nu) = 4\sigma_{0,\chi_{24}}(\nu) \) if \( v_{p_2}(\nu) = 0 \) and \( b \) is even, or if \( v_{p_2}(\nu) = 2 \), and (iii) \( r_{2,K}(\nu) = 12\sigma_{0,\chi_{24}}(\nu) \) if \( 2\sigma_{p_2}(\nu) \geq 4 \).

The following corollary is for later use, which is rather technical.

Corollary 3.3. (i) Assume that \( \nu \) is odd, and it is represented as a sum of two integer squares. Then \( x_1, x_2 \in \mathcal{O}_K \) satisfying \( x_1^2 + x_2^2 = \nu^2 \), are both even.

(ii) Assume that \( \nu > 0, v_{p_2}(\nu) \geq 4 \), and \( \nu \) is represented as a sum of two integer squares. Then there are \( x_1, x_2 \in \mathcal{O}_2 \) satisfying \( x_1^2 + x_2^2 = \nu \).
Proof. (i) Since \( \nu \) is represented as a sum of two squares, \( \rho_2(\nu) = 1 \). Then \( r_{2,K}(\pi^2_2 \nu) = 4\sigma_0,_{K}^\nu(\pi^2_2 \nu) = 4\sigma_0,_{K}^\nu(\nu) = r_{2,K}(\nu) \). Hence all the solutions of \( x^2_1 + x^2_2 = \pi^2_2 \nu \) is obtained by \( x_1 = \pi_2 x'_1, x_2 = \pi_2 x'_2 \) with \( x'^2_1 + x'^2_2 = \nu \), which shows the assertion.

(ii) Since \( \nu \) is represented as a sum of two squares, \( v_{p_2}(\nu) \) is even. Then \( r_{2,K}(\pi^2 v_{p_2}(\nu)) > 0 \). For solutions \( x'_1, x'_2 \) of \( x'^2_1 + x'^2_2 = \pi^2 v_{p_2}(\nu) \), \( x_1 = \pi_2 v_{p_2}(\nu)/2 - 1 \) and \( x_2 = \pi_2 v_{p_2}(\nu)/2 + 1 \) satisfy \( x'^2_1 + x'^2_2 = \nu \).

We derive from Corollary 3.3, the following result on sums of two integer squares in \( K \). Let \( p_l > 0 \) denote an odd prime element of \( \mathcal{O}_K \) with positive norm, and let \( q_j > 0 \) denote an odd prime element of \( \mathcal{O}_K \) with negative norm. We may assume that the coefficient of \( \sqrt{3} \) in \( p_l \) is even by multiplying by the fundamental unit \( \epsilon_0 \) if necessary. Then a totally positive integer \( \nu \) in \( \mathcal{O}_K \) has a prime factorization as \( \nu = \epsilon_0^l \pi^2_1 \nu_1 \cdots \pi^2_s \nu_s q_1^1 \cdots q_t^t \) with \( e + \sum_{1 \leq j \leq t} n_j \equiv 0 \pmod{2} \). Then the necessary and sufficient condition that \( \nu \) is expressed as a sum of two integers in \( K \), is that (I) \( e = 0 \) and \( k \equiv n_1 \equiv \cdots \equiv n_t \equiv 0 \pmod{2} \), or that (II) \( 2e > 0 \) and \( n_1 \equiv \cdots \equiv n_t \equiv 0 \pmod{2} \). The number of representations is \( 4 \prod_i (1 + m_i) \) for \( e = 0, 2 \), and it is \( 12 \prod_i (1 + m_i) \) for \( e \geq 4, e \equiv 0 \pmod{2} \).

4. Quadratic extensions of \( \mathbb{Q}(\sqrt{3}) \)

Let \( \alpha \in \mathcal{O}_K \) be square-free and not necessarily totally positive, and let \( F = K(\sqrt{\alpha}) \). We denote by \( \psi, f_\alpha(= d_{F/K}) \) and \( \mathcal{O}_F \), the character associated the extension, the conductor of the extension and the relative different of \( F \) over \( K \) respectively. The norm map of \( F/K \) and the norm map of \( F/Q \) are denoted by \( N_{F/K} \) and \( N_F \) respectively. Let \( \chi_F := \chi_1 \circ N_F = \chi_1^3 \circ N_{F/K} \). We classify quadratic extensions by the congruence condition on \( \alpha \) as in Table 3, where we understand that \( \alpha \equiv 1 \pmod{4p_2} \) or \( \alpha \equiv 3 \pmod{4p_2} \) in Case (A) and so on.

**Table 3.** Quadratic extensions \( F = K(\sqrt{3}, \sqrt{\alpha}) \) of \( \mathbb{Q}(\sqrt{3}) \).

| \( \alpha \equiv \mod{4p_2} \) | \( v_{p_2}(f_\alpha) \) | \( \mathcal{O}_F/K \) | \( p_2 \) at \( F \) |
|------------------|-----------------|------------------|-----------|
| (A) \( 1, 3 \) | 0 | \((\sqrt{\alpha})\) | split |
| (B) \( 5, 7 \) | 0 | \((\sqrt{\alpha})\) | iner |
| (C) \( 1 + 2\sqrt{3}, 3 + 2\sqrt{3} \mod{4} \) | 2 | \(\pi_2 \sqrt{\alpha}\) | rami |
| (C) \( \sqrt{3} \mod{2} \) | 4 | \(2\sqrt{\alpha}\) | rami |
| (D) \( 1 + \sqrt{3} \mod{2} \) | 5 | \(2\sqrt{\alpha}\) | rami |

Let \( \mathfrak{P}_2 \subset \mathcal{O}_F \) be the ideal with \( \mathfrak{P}^2_2 = p_2 \mathcal{O}_F \) in Cases (C), (C2) and (D). Then \( \chi_{F,4}^* \in C_{\mathcal{O}_F}^* \) in Cases (A) and (B), and \( \chi_{F,4}^* \in C_{\mathfrak{P}^2_2}^* \) in Cases (C), (C2) and (D). In either case, the conductor \( f_{\chi_{F,4}^*} \) is \( \mathcal{O}_F \). There holds

\[
\chi_{F,4}^{K}\psi_\alpha = \begin{cases}
\psi_\alpha^1 \mathfrak{P}_2 & \text{((A), (B))}, \\
\psi_{-\alpha} & \text{((C1), (C2), (D))}.
\end{cases}
\]

Let \( \chi_{F,4}^* \) be the primitive character associated with \( \chi_{F,4}^* \). Then \( \chi_{F,4}^* \in C_{\mathcal{O}_F}^* \), and \( \tau_F(\chi_{F,4}^*) = 1 \). In Cases (C), (C2) and (D), the value of \( \chi_{F,4}^* \) at \( \mathfrak{P}_2 \) is \(-1\). There holds

\[
L_F(s, \chi_{F,4}^*) = L_K(s, \chi_{F,4}^{K})L_K(s, \psi_\alpha)(1 - \psi_{-\alpha}(\mathfrak{P}_2)2^{-s}),
\]
(13) \[ L_F(s, \chi_E^F) = L_K(s, \chi_K^F) L_K(s, \psi_{-\alpha}), \]

and hence \( L_F(0, \chi_E^F) \) equals \( 2^{-3} L_K(0, \psi_{-\alpha}) \), and \( L_F(0, \chi_E^F) \) equals \( 2 \times 3^{-1} \times L_K(0, \psi_{-\alpha}) \) in Case (A), 0 in Case (B), or \( 3^{-1} L_K(0, \psi_{-\alpha}) \) in Case (C1), (C2) or (D). Let \( \psi, \psi' \) be as in the preceding section. Then \( f_0 \circ N_{F/K} = f_{\psi' \circ N_{F/K}} = (2) \) and \( \tau_F(\psi \circ N_{F/K}) = \tau_F(\psi' \circ N_{F/K}) = 4 \) in Cases (A) and (B), and \( f_0 \circ N_{F/K} = f_{\psi' \circ N_{F/K}} = \psi_2 \) and \( \tau_F(\psi \circ N_{F/K}) = \tau_F(\psi' \circ N_{F/K}) = 2 \) in Cases (C1), (C2) and (D).

Let \( \alpha \) be a totally positive square-free integer of \( K \). By (12), we have

\[
\mathcal{J}_{\alpha, \chi_K^4}(\theta(3)^{3}) = 3 \mathcal{J}_{\alpha, \chi_K^4}(\theta(3)G_{1, \chi_K^4}(\overline{3}; O_K, O_K)) - 2^{-5} \mathcal{J}_{\alpha, \chi_K^4}(\theta(3)G_{1, \chi_K^4}(\overline{3}; O_K, O_K))
\]

which is a Hilbert modular form of weight 2 for \( \Gamma_0(\mathcal{D}^{-1}, 2\mathcal{D}) \).

 Shimura lifts of products of the theta series and Eisenstein series are explicitly constructed in [8] Sect. 6, Sect. 7, which are essentially the restricts to the diagonal, of Hilbert-Eisenstein series on the field \( F \). Let \( \iota: \mathbb{H}^2 \to \mathbb{H}^2 \) be the diagonal map associated with the inclusion of \( K \) into \( F \). For a Hilbert modular form \( f(\overline{3}) (\overline{3} \in \mathbb{H}^2) \) on \( F \) of weight \( k \), \( f(\iota(\overline{3})) (\overline{3} \in \mathbb{H}^2) \) is a Hilbert modular form on \( K \) of weight \( 2k \). We put

\[
\lambda_{\alpha, \chi_K^4}(\overline{3}; (\alpha), O_K) := \begin{cases} \lambda_{\alpha, \chi_K^4}(\iota(\overline{3}), O_F, \overline{\mathcal{D}}_{F/K}^{-1}) & \text{(A), (B)}, \\ \lambda_{1, \chi_K^4}(\iota(\overline{3}), \mathcal{O}_F, \overline{\mathcal{D}}_{F/K}^{-1}) & \text{(C1), (C2), (D)}, \end{cases}
\]

\[
\lambda_{\alpha, \chi_K^4}(\overline{3}; (\alpha), (2)) := \begin{cases} \lambda_{\alpha, \chi_K^4}(\iota(\overline{3}), \mathcal{O}_F, \overline{\mathcal{D}}_{F/K}^{-1}) & \text{(A), (B), (C1), (C2), (D)}, \\ \lambda_{1, \chi_K^4}(\iota(\overline{3}), \mathcal{O}_F, \overline{\mathcal{D}}_{F/K}^{-1}) & \text{(C1), (C2), (D)}, \end{cases}
\]

\[
\lambda_{\psi, \psi}(\overline{3}; (\alpha), O_K) := \begin{cases} \lambda_{\psi, \psi}(\iota(\overline{3}), O_F, \overline{\mathcal{D}}_{F/K}^{-1}) & \text{(A), (B), (C1), (C2), (D)}, \\ \lambda_{1, \chi_K^4}(\iota(\overline{3}), \mathcal{O}_F, \overline{\mathcal{D}}_{F/K}^{-1}) & \text{(C1), (C2), (D)}, \end{cases}
\]

In [8], \( \lambda_{\alpha, \chi_K^4}(\overline{3}; (\alpha), O_K) \), \( \lambda_{\alpha, \chi_K^4}(\overline{3}; (\alpha), O_K) \), \ldots, are denoted by \( \lambda_{\alpha, \chi_K^4}(\iota(\overline{3}), O_K, O_K, O_K) \), \( \lambda_{1, \chi_K^4}(\iota(\overline{3}), O_K, O_K, O_K) \), \ldots. We drop the last two ideals from the notation \( \lambda_{\alpha, \chi_K^4}(\iota(\overline{3}), O_K, O_K, O_K) \) and so on because they are always \( O_K \) in the present paper.

By [8], we have

\[
\mathcal{J}_{\alpha, \chi_K^4}(\theta(3)G_{1, \chi_K^4}(\overline{3}; O_K, O_K)) = 2^2 \lambda_{\alpha, \chi_K^4}(\overline{3}; (\alpha), O_K),
\]

\[
\mathcal{J}_{\alpha, \chi_K^4}(\theta(3)G_{1, \chi_K^4}(\overline{3}; O_K, O_K)) = 2^{-2} \lambda_{\alpha, \chi_K^4}(\overline{3}; (\alpha), O_K),
\]

\[
\mathcal{J}_{\alpha, \chi_K^4}(\theta(3)G_{1, \chi_K^4}(\overline{3}; (2), O_K)) = \lambda_{2, \alpha, \chi_K^4}(\overline{3}; (\alpha), (2))
\]

and

\[
\mathcal{J}_{\alpha, \chi_K^4}(\theta(3)G_{1, \chi_K^4}(\overline{3}; O_K, O_K)) = \lambda_{\psi, \psi}(\overline{3}; (\alpha), O_K).
\]
We can obtain the values at cusps of Eisenstein series on $F$ by (7) and (9), which are all rational multiples of $L_F(0, \chi^2_{E})$, and hence we obtain the values at cusps of $\lambda_{2, \chi^4}(\alpha; O_K)$, $\lambda_{2, \psi}^{(1)}(\alpha; O_K)$, $\lambda_{2, \psi}^{(2)}(\alpha; O_K)$ and $\lambda_{2, \psi}^{(3)}(\alpha; O_K)$ by using Lemma 8.2. Then the equation (14) gives the values at cusps of $\mathcal{S}_{a, \chi^4}(\theta(3)^2)$ as in Table 4.

**Table 4. Values of $L_F(0, \chi^2_{E})^{-1} \mathcal{S}_{a, \chi^4}(\theta(3)^2)$ at cusps in $C_0(2)$.**

| $C_0(2)$ | $\frac{1}{4\sqrt{3}}$ | $\frac{1}{2\pi_2\sqrt{3}}$ | $\frac{1}{2\pi_1\sqrt{3}}$ |
|----------|------------------------|-----------------------------|-----------------------------|
| $\mathcal{S}_{a, \chi^4}(\theta(3)^2)$ | 3 | 3 | 2 $^{-3/2}$ |
| $L_F(0, \chi^2_{E})$ | 0 | 0 | $2^{-32}$ |
| | 2 $^{-3}$ | 2 $^{-13}$ | 2 $^{-13}$ |
| | 2 $^{-3}$ | 2 $^{-13}$ | 2 $^{-13}$ |

5. **Sums of three squares in $Q(\sqrt{3})$**

The Hilbert modular surface of the group $\Gamma_0(\mathcal{O}^{-1}_{K}, 2\mathcal{O}_K)$ is a blown up K3 surface (see van der Geer [2] Chap. VII), dim $\mathcal{S}_2(\Gamma_0(\mathcal{O}^{-1}_{K}, 2\mathcal{O}_K))$ is one dimensional. The cusp form $\Xi(\gamma)$ of (11) is a generator of the space $\mathcal{S}_2(\Gamma_0(\mathcal{O}^{-1}_{K}, 2\mathcal{O}_K))$. By (2) and (8), and by using $\zeta(-1) = 1/6$ and $L(-1, \chi_{12}) = -2$, the the values at cusps of Eisenstein series $G_2(\gamma; O_K, O_K)$, $G_{2,1_{\nu_2}}(\gamma; O_K, O_K)$, $G_{2,1_{\nu_2}}(\gamma; p_2, O_K)$ are obtained as in Table 5. From the table we see that these three Eisenstein series are linearly independent, and hence $M_2(\Gamma_0(\mathcal{O}^{-1}_{K}, 2\mathcal{O}_K)) = \langle G_2(\gamma; O_K, O_K), G_{2,1_{\nu_2}}(\gamma; O_K, O_K), G_{2,1_{\nu_2}}(\gamma; p_2, O_K) \rangle, \Xi(\gamma) \rangle$.

**Table 5. Values at cusps, of Eisenstein series of weight 2.**

| $C_0(2)$ | $\frac{1}{4\sqrt{3}}$ | $\frac{1}{2\pi_2\sqrt{3}}$ | $\frac{1}{2\pi_1\sqrt{3}}$ |
|----------|------------------------|-----------------------------|-----------------------------|
| $G_2(\gamma; O_K, O_K)$ | 2 $^{-13}$ | 2 $^{-13}$ | 2 $^{-13}$ |
| $G_{2,1_{\nu_2}}(\gamma; O_K, O_K)$ | 2 $^{-13}$ | 2 $^{-13}$ | 2 $^{-13}$ |
| $G_{2,1_{\nu_2}}(\gamma; p_2, O_K)$ | 2 $^{-13}$ | 2 $^{-13}$ | 2 $^{-13}$ |

Let $r_{3,K}(\nu)$ denote the number of representations of $\nu$ as sums of three integer squares in $K$. Then $\theta(3)^3$ is a generating function of $r_{3,K}(\nu)$’s, namely $\theta(3)^3 = 1 + \sum_{\nu \neq 0} r_{3,K}(\nu)e(\text{tr}(\nu))$. Let $\alpha$ be a totally positive square-free integer in $K$. The Shimura lift $\mathcal{S}_{a, \chi}(\theta(3)^3)$ is in $M_2(\Gamma_0(\mathcal{O}^{-1}_{K}, 2\mathcal{O}_K))$, and by Table 3 and Table 5 it is equal to $3^2 L_F(0, \chi_{E}^F) \{3G_2(\gamma; O_K, O_K) + G_{2,1_{\nu_2}}(\gamma; O_K, O_K) + c_{\alpha}\Xi(\gamma) \}$ in Case (A), $3^2 L_F(0, \chi_{E}^F) \{G_2(\gamma; O_K, O_K) + G_{2,1_{\nu_2}}(\gamma; O_K, O_K) + c_{\alpha}\Xi(\gamma) \}$ in Case (B), $3^2 L_F(0, \chi_{E}^F) \{G_2(\gamma; O_K, O_K) + G_{2,1_{\nu_2}}(\gamma; O_K, O_K) + c_{\alpha}\Xi(\gamma) \}$ in Case (C), and $2^{-13/2} L_F(0, \chi_{E}^F) \{G_2(\gamma; O_K, O_K) - G_{2,1_{\nu_2}}(\gamma; O_K, O_K) + c_{\alpha}\Xi(\gamma) \}$ in Cases (C2) and (D) where $c_{\alpha}$ is a constant depending only on $\alpha$.

**Lemma 5.1.** In all cases, $c_{\alpha} = 0$.

**Proof.** We note that the Fourier coefficient of $\Xi(\gamma)$ for $\nu = 1$ is 1 and that for $\nu = \varepsilon_0$ is $-1$ by (11). At first we consider Case (C2) or (D). In this case the coefficient of $\sqrt{3}$ in $\alpha$ is odd, and hence $\alpha$ can not be represented as a sum of squares. Comparing the Fourier coefficients of the equality before the lemma, we have $r_{3,K}(\alpha) = c_{\alpha}$ since...
the Fourier coefficient for $\nu = 1$, of $G_2(\mathcal{O}_K, \mathcal{O}_K) - G_{2,1_{p_2}}(\mathcal{O}_K, \mathcal{O}_K)$ vanishes. Then $c_\alpha = 0$.

Next, we consider Case (A). Comparing the Fourier coefficients of the both sides of the equality $\mathcal{S}_{\alpha, \chi_{K_4}}(\theta(\mathcal{O}^3)) = 3^2 L_F(0, \chi_{F_4}) \{3 G_2(\mathcal{O}_K, \mathcal{O}_K) + G_{2,1_{p_2}}(\mathcal{O}_K, \mathcal{O}_K)\} + c_\alpha \mathcal{Z}(\mathcal{O})$ for $\nu = 1$ and for $\nu = \varepsilon_0$, we obtain $r_{3,K}(\alpha) = 2^4 3^2 L_F(0, \chi_{F_4}) + c_\alpha$ and $r_{3,K}(\varepsilon_0^2 \alpha) = 2^4 3^2 L_F(0, \chi_{F_4}) - c_\alpha$. However obviously the equality $r_{3,K}(\alpha) = r_{3,K}(\varepsilon_0^2 \alpha)$ holds, and hence $c_\alpha = 0$. The similar argument shows the assertion also in the rest of cases.

**Corollary 5.2.** The Shimura lift $\mathcal{S}_{\alpha, \chi_{K_4}}(\theta(\mathcal{O}^3))$ is equal to

$$3^2 L_F(0, \chi_{F_4}) \{3 G_2(\mathcal{O}_K, \mathcal{O}_K) + G_{2,1_{p_2}}(\mathcal{O}_K, \mathcal{O}_K)\} \quad \text{(A)},$$

$$3^2 L_F(0, \chi_{F_4}) \{G_2(\mathcal{O}_K, \mathcal{O}_K) + G_{2,1_{p_2}}(\mathcal{O}_K, \mathcal{O}_K)\} \quad \text{(B)},$$

$$3^2 L_F(0, \chi_{F_4}) G_2(\mathcal{O}_K, \mathcal{O}_K) \quad \text{(C)},$$

$$2^{-1} 3^2 L_F(0, \chi_{F_4}) \{G_2(\mathcal{O}_K, \mathcal{O}_K) - G_{2,1_{p_2}}(\mathcal{O}_K, \mathcal{O}_K)\} \quad \text{(D)}.$$

Since $\mathcal{S}_{\alpha, \chi_{K_4}}(\theta(\mathcal{O}^3)) = C + \sum_{\delta | \nu \delta \mathcal{O}_K^\mathcal{K}} \sum_{\sigma_0, \sigma_1 \mathcal{O}_K^\mathcal{K}} (\psi - \alpha \mathcal{O}_K^\mathcal{K}) (\sigma_3, K(\alpha(\nu/\delta)^2)) \times \mathbb{E} (\nu_0^2)$, with a constant $C$, comparing the terms corresponding to $\nu$, of the both sides of equations in Corollary 5.2, we have for $\nu > 0$,

$$\sum_{\sigma_0, \sigma_1 \mathcal{O}_K^\mathcal{K}} (\psi - \alpha \mathcal{O}_K^\mathcal{K}) (\sigma_3, K(\alpha(\nu/\delta)^2)) \times \mathbb{E} (\nu_0^2) = 2^2 3^2 L_F(0, \chi_{F_4}) \times \begin{cases} 3 \sigma_1(\nu) + \sigma_{1,1_{p_2}}(\nu) \quad \text{(A)}, \\ \sigma_1(\nu) + \sigma_{1,1_{p_2}}(\nu) \quad \text{(B)}, \\ \sigma_1(\nu) - \sigma_{1,1_{p_2}}(\nu) \quad \text{(C)}, \\ 2^{-1} \{\sigma_1(\nu) - \sigma_{1,1_{p_2}}(\nu)\} \quad \text{(D)}. \end{cases}$$

By (11), $L_F(0, \chi_{F_4}) = 2^{-1} 3^2 L_F(0, \psi - \alpha)$, and there holds $L_F(0, \psi - \alpha) = \frac{2^2 \mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})}{\varepsilon_0^2 \mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})}$, where $\varepsilon_0^2 \mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})$ and $\mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})$ denote the number of units in $K(\sqrt{-\alpha})$ and the relative class number respectively. Here the relative class number is just the class number of $K(\sqrt{-\alpha})$ since $K$ is of class number 1. If square-free $\alpha > 0$ is not a unit, then $\varepsilon_0^2 \mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})$ is 2 and $\mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})$ is 1. For $\alpha = 1$, $\varepsilon_0^2 \mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})$ is 12 and $\mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})$ is 2, and for $\alpha = \varepsilon_0$, $\varepsilon_0^2 \mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})$ is 2 and $\mathcal{H}_K(\sqrt{-\alpha}) \mathcal{Z}(\mathcal{O})$ is 2. Thus $L_F(0, \chi_{F_4})$ is equal to $2^{-2} 3^2 h_K(\sqrt{-\alpha})$ for $\alpha = 1$, $2^{-1} 3^2 h_K(\sqrt{-\alpha})$ for $\alpha = \varepsilon_0$, and $3^{-1} h_K(\sqrt{-\alpha})$ for $\alpha$ non-unit. If $\nu$ is a product of totally positive prime elements, by using the Möbius inversion formula on $K$, it is shown that for square-free $\alpha \neq \varepsilon_0^2 > 0$, $r_{3,K}(\alpha \nu^2)$ is equal to $2^2 3^2 h_K(\sqrt{-\alpha})$ times the following:

$$\sum_{0 < \delta | \nu \delta \mathcal{O}_K \mathcal{E}_K} (\psi - \alpha \mathcal{O}_K^\mathcal{K}) (\delta) \{3 \sigma_1(\nu/\delta) + \sigma_{1,1_{p_2}}(\nu/\delta)\} \quad \text{(A)},$$

$$\sum_{0 < \delta | \nu \delta \mathcal{O}_K \mathcal{E}_K} (\psi - \alpha \mathcal{O}_K^\mathcal{K}) (\delta) \{\sigma_1(\nu/\delta) + \sigma_{1,1_{p_2}}(\nu/\delta)\} \quad \text{(B)},$$

$$\sum_{0 < \delta | \nu \delta \mathcal{O}_K \mathcal{E}_K} (\psi - \alpha \mathcal{O}_K^\mathcal{K}) (\delta) \sigma_1(\nu/\delta) \quad \text{(C)}. $$
Theorem 5.3. Let $K = \mathbb{Q}(\sqrt{3})$.

(i) A totally positive integer $a + b\sqrt{3}$ $(a, b \in \mathbb{Z})$ is represented as a sum of three squares in $K$ if and only if $b$ is even.

(ii) Let $\alpha$ be a totally positive square-free integer in $K$ which is not a unit. We classify $\alpha$ as in Table 3. Then the class number of $K(\sqrt{-\alpha})$ is given by

$$2^{-1} \sum_{0<\delta | \nu, \delta \in \mathcal{O}_K/\mathcal{E}_K} (\psi_{-\alpha}1_{p_2\mathcal{H}K})(\delta)\{\sigma_1(\nu/\delta) - \sigma_{1,1_{p_2}}(\nu/\delta)\} = (2), (D),$$

where $\sigma_1$ and $\sigma_{1,1_{p_2}}$ being as in (5). Further for $\alpha = 1, \varepsilon_0$ and for a product $\nu$ of totally positive primes,

$$r_{3, K}(\nu^2) = h_{K(\sqrt{-\varepsilon_0})} \sum_{0<\delta | \nu, \delta \in \mathcal{O}_K/\mathcal{E}_K} (\psi_{-\alpha}1_{p_2\mathcal{H}K})(\delta)\{3\sigma_1(\nu/\delta) + \sigma_{1,1_{p_2}}(\nu/\delta)\},$$

$$r_{3, K}(\varepsilon_0\nu^2) = 3h_{K(\sqrt{-\varepsilon_0})} \sum_{0<\delta | \nu, \delta \in \mathcal{O}_K/\mathcal{E}_K} (\psi_{-\alpha_0}1_{p_2\mathcal{H}K})(\delta)\{\sigma_1(\nu/\delta) - \sigma_{1,1_{p_2}}(\nu/\delta)\},$$

where $h_{K(\sqrt{-\varepsilon_0})} = 1$ and $h_{K(\sqrt{-\varepsilon_0})} = 2$.

Proof. (i) The necessity is obvious, and only the sufficiency is to be proved. We can write as $a + b\sqrt{3} = \alpha \nu^2$ for $\alpha$ square-free. If $\alpha$ satisfies (A), (B) or (C), then $r_{3, K}(\alpha) > 0$ by the formulas before Theorem. Then $r_{3, K}(\alpha \nu^2) > 0$ for any $\nu \in \mathcal{O}_K$.

Suppose that $\alpha$ satisfies (C) or (D). If $\nu$ is odd, then $\nu^2 \equiv 1 \pmod{2}$ and $b$ is odd, which contradicts to our assumption. Hence $\nu$ must be in $p_2$, and $\alpha \nu^2$ is a product of $\pi_2^2 \alpha$ and some integer square. To prove $r_{3, K}(\alpha \nu^2) > 0$, it is enough to show $r_{3, K}(\pi_2^2 \alpha) > 0$. By (15) in Cases (D) and (D), there holds

$$r_{3, K}(\pi_2^2 \alpha) = 2^23^2h_{K(\sqrt{-\varepsilon_0})} > 0,$$

and there are $x_1, x_2, x_3 \in \mathcal{O}_K$ satisfying $x_1^2 + x_2^2 + x_3^2 = \pi_2^2 \alpha$. At least one of $x_i$’s is even, say $x_3$. If $v_{p_2}(x_3) = 1$, then putting $x_3 = \pi_2x_3'$, there holds $x_1^2 + x_2^2 = \pi_2^2(\pi_2^2 \alpha - x_2^2)$ with $\rho_2(\pi_2^2 \alpha - x_2^2) = 1$. Then by Corollary 3.3 (i), both of $x_1, x_2$ are even, and hence $r_{3, K}(\pi_2^2 \alpha) > 0$. If $x_3 = 0$ or $v_{p_2}(x_3) \geq 2$, then there are even $x_1, x_2$ by Corollary 3.3 (ii), and hence $r_{3, K}(\pi_2^2 \alpha) > 0$.

The assertion (ii) is obtained from formulas before the theorem and from (16) since $\pi_2^2 = 4z_3^2$.

Remark 5.4. A formula for the left hand side of (15) in case $N(\nu) < 0$ is obtained by using the theta lifts of $\theta_{\alpha^{-1}}(j) = \sum_{\mu \in \mathbb{R}_K} e(\text{tr}(\mu^2 \nu^2))$. We omit the argument because it is not necessary to show Theorem 5.3.

6. Hilbert modular forms on $\mathbb{Q}(\sqrt{17})$

Let $K = \mathbb{Q}(\sqrt{17})$. Then $\mathfrak{o}_K = (\sqrt{17})$, and the class number of $K$ is 1. We put $\omega := (1 + \sqrt{17})/2$ and $\omega' := (1 - \sqrt{17})/2$. Let $\varepsilon_0 := 3 + 2\omega$ be a fundamental unit, which has a negative norm. Put $\pi_2 := 2 + \omega$, $\pi_2' := 2 + \omega'$, $\epsilon_2 := (\pi_2')$. Then the ideal (2) is decomposed as $\langle 2 \rangle = \mathfrak{p}_2\mathfrak{p}_2'$ in $K$. Let $\chi_{14}^{\mathfrak{p}_2} := \chi_{-\delta} \circ N \in C_{14}^*$, whose conductor $f_{14}^{\mathfrak{p}_2}$ is (4). Let $\rho_2$ be the unique nontrivial ideal class character
in \( \mathcal{O}_K^* \), and let \( \rho'_2 \) be the unique nontrivial ideal class character in \( \mathcal{O}_K^* \). Then \( f_{\rho_2} = p_2^2, f_{\rho'_2} = p_2^2 \) and

\[
\chi_{K_4}^{-1} = \rho_2 \rho'_2.
\]

For \( a, b \in \mathbb{Z} \), we have \( \rho_2((a + b\omega)) = \text{sgn}(a + b\omega)\chi_{-4}(a), \rho'_2((a + b\omega)) = \text{sgn}(a + b\omega)\chi_{-4}(a + b) \), and \( e_{\rho_2} = (0, 1), e_{\rho'_2} = (1, 0) \). Further \( \rho_2(p_2) = -1, \rho'_2(p_2) = -1, \tau_K(\chi_{K_4}^{-1}) = -4, \tau_K(\rho_2) = \tau_K(\rho'_2) = 2\sqrt{-1}, L_K(0, \chi_{K_4}^{-1}) = 2 \).

The set \( C_0(4) \) of representatives of cusps of \( \Gamma_0(\mathcal{O}_K^{-1}, 4\mathcal{O}_K) \) is taken as

\[
C_0(4) = \{ \frac{1}{4\sqrt{17}}, \frac{1}{2\pi_2\sqrt{17}}, \frac{1}{2\pi_2\sqrt{17}}, \frac{1}{2\sqrt{17}}, \frac{1}{2\pi_2\sqrt{17}}, \frac{1}{\pi_2\sqrt{17}}, \frac{1}{\pi_2\sqrt{17}} \}
\]

and the set \( C_0(2) \) of representatives of cusps of \( \Gamma_0(\mathcal{O}_K^{-1}, 2\mathcal{O}_K) \) is taken as

\[
C_0(2) = \{ \frac{1}{4\sqrt{17}}, \frac{1}{2\pi_2\sqrt{17}}, \frac{1}{\pi_2\sqrt{17}}, \frac{1}{\pi_2\sqrt{17}} \}.
\]

The square \( \theta(\mathfrak{z})^2 \) of the theta series is a Hilbert modular form for \( \Gamma_0(\mathcal{O}_K^{-1}, 4\mathcal{O}_K) \) of weight 1 with character \( \chi_{K_4}^{-1} \), and it takes the values at cusps as in Table 6.

**Table 6. Values of \( \theta(\mathfrak{z})^2 \) at cusps.**

| \( \theta(\mathfrak{z})^2 \) | 1 | 0 | 0 | 0 | 2 \( \sqrt{-1} \) | 2 \( \sqrt{-1} \) | 1 |
|---|---|---|---|---|---|---|---|
| \( C_0(4) \) | 0 | 0 | 0 | 0 | 0 | 0 | 2 \( \sqrt{-1} \) |
| \( \mathfrak{z} \in \mathbb{Z} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

By \( 7 \) and \( 8 \), and by \( L_K(0, \chi_{K_4}^{-1}) = 2 \), the values at cusps of Eisenstein series \( G_{1,\chi_{K_4}^{-1}}(\mathfrak{z}; \mathcal{O}_K, \mathcal{O}_K), G_{1,\rho_2}(\mathfrak{z}; \mathcal{O}_K, \mathcal{O}_K) \in \mathbb{M}_1(\Gamma_0(\mathcal{O}_K^{-1}, 4\mathcal{O}_K), \chi_{K_4}^{-1}) \) are obtained as in Table 7.

**Table 7. Values at cusps, of Eisenstein series of weight 1.**

| \( \mathfrak{z} \in \mathbb{Z} \) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 \( \sqrt{-1} \) |
|---|---|---|---|---|---|---|---|---|
| \( G_{1,\chi_{K_4}^{-1}}(\mathfrak{z}; \mathcal{O}_K, \mathcal{O}_K) \) | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 2 \( \sqrt{-1} \) |
| \( G_{1,\rho_2}(\mathfrak{z}; \mathcal{O}_K, \mathcal{O}_K) \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The product of these Eisenstein series is a cusp form of weight 2, since it vanishes at all the cusps. Put \( \Phi(\mathfrak{z}) := 2^{-3}G_{1,\chi_{K_4}^{-1}}(\mathfrak{z}; \mathcal{O}_K, \mathcal{O}_K)G_{1,\rho_2}(\mathfrak{z}; \mathcal{O}_K, \mathcal{O}_K) \), and put \( \Xi(\mathfrak{z}) := 3^{-1}U(2)(\Phi(\mathfrak{z})) \) which is in \( \mathbb{S}_2(\Gamma_0(\mathcal{O}_K^{-1}, 2\mathcal{O}_K)) \). The first several Fourier coefficients of \( \Phi(\mathfrak{z}), \Xi(\mathfrak{z}), U(\mathfrak{z}_2)(\Phi(\mathfrak{z})), U(\mathfrak{z}_2')(\Phi(\mathfrak{z})), \Xi(2\mathfrak{z}) \) are as in Table 8 where for example we read the second row as \( \Phi(\mathfrak{z}) = e(\text{tr}(\mathfrak{z})) + 3e(\text{tr}(2\mathfrak{z})) - e(\text{tr}(\mathfrak{z}_2')) + 2e(\text{tr}(3\mathfrak{z})) - e(\text{tr}(3\omega\mathfrak{z}_3)) - e(\text{tr}(3\omega\mathfrak{z}_3 + \mathfrak{z})) + 3e(\text{tr}(4\mathfrak{z})) + \cdots \).

**Table 8. Fourier coefficients of cusp forms of weight 2.**

| \( \mathfrak{z} \) | 1 | 2 | \( \pi'_2 \) | \( \pi_2 \) | 3 + \( \omega' \) | 3 + \( \omega \) | 4 |
|---|---|---|---|---|---|---|---|
| \( \Phi(\mathfrak{z}) \) | 1 | 3 | -1 | -1 | 2 | -1 | -1 |
| \( \Xi(\mathfrak{z}) \) | 1 | 1 | -1 | -1 | 2 | -1 | -1 |
| \( U(\mathfrak{z}_2)(\Phi(\mathfrak{z})) \) | -1 | -3 | 3 | 1 | 2 | 1 | 3 |
| \( U(\mathfrak{z}_2')(\Phi(\mathfrak{z})) \) | -1 | -3 | 1 | 3 | 2 | 3 | 1 |
| \( \Xi(2\mathfrak{z}) \) | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
The group $\Gamma_0(\mathcal{O}_K^{-1}, 4\mathcal{O}_K)$ acts freely on $\mathcal{S}^2$, and hence the arithmetic genus of a nonsingular model of the compactified Hilbert modular surface for $\Gamma_0(\mathcal{O}_K^{-1}, 4\mathcal{O}_K)$ is obtained from the volume of the fundamental domain and from the contributions from cusps (van der Geer [2]). It is computed to be 6, and hence $\dim S_2(\Gamma_0(\mathcal{O}_K^{-1}, 4\mathcal{O}_K)) = 5$. We see that cusp forms in Table 5 are linearly independent only by looking at the first five Fourier coefficients. This shows that $S_2(\Gamma_0(\mathcal{O}_K^{-1}, 4\mathcal{O}_K)) = \langle \Phi(\lambda), \Xi(\lambda), U(\pi_2)(\Phi(\lambda)), U(\pi_2')(\Phi(\lambda)), \Xi(2\lambda) \rangle$, and that a cusp form in $S_2(\Gamma_0(\mathcal{O}_K^{-1}, 4\mathcal{O}_K))$ vanishes identically if its first five Fourier coefficients are all 0.

**Lemma 6.1.** The following equality holds:

$$
\theta(\lambda)^2 = 2^{-1}G_{1,\chi_{\mathbb{A}}}(\lambda; \mathcal{O}_K, \mathcal{O}_K) + 2^{-1}G''_{1,\rho}(\lambda; \mathcal{O}_K, \mathcal{O}_K).
$$

**Proof.** Let $f$ be a Hilbert modular form given as the left hand side minus the right hand side of the equation (17). It is a cusp form from Table 6 and from Table 7. As easily checked, the first four Fourier coefficients of $f$ are all 0, and hence so are the first five Fourier coefficients of $f^2 \in S_2(\Gamma_0(\mathcal{O}_K^{-1}, 4\mathcal{O}_K))$. Then $f^2 = 0$, and $f = 0$. \hfill \Box

**Corollary 6.2.** For $\nu \in \mathcal{O}_K, > 0$, there holds

$$
r_{2,K}(\nu) = 2\sigma_{0,\chi_{\mathbb{A}}}(\nu) + 2\sigma_{0,\rho_2}(\nu).
$$

In considering a prime factorization in $K$, we may assume that prime elements are totally positive since a fundamental unit has a negative norm. Let $p_i \in K$ be a totally positive prime element congruent to 1 (mod 4), $q_j \in K$ be a totally positive prime element congruent to 3 (mod 4), and let $r_k \in K$ be a totally positive prime element congruent to $\sqrt{17}$ or to $2 + \sqrt{17}$ modulo 4. Then a totally positive $\nu$ has a prime factorization $\nu = e^2_0 \pi_{2}^{e_2} p_1^{b_1} \cdots p_r^{b_r} q_1^{c_1} \cdots q_u^{c_u} \nu'$. Then by Corollary 6.2, it is expressed as a sum of two integer squares if and only if $e + e' + \sum b_j \equiv c_1 \equiv \cdots \equiv c_u$ (mod 2). The number of representations is given by $4 \prod (1+a_i) \prod (1+b_j)$.

### 7. Quadratic Extensions of $\mathbb{Q}(\sqrt{17})$

Let $\alpha \in \mathcal{O}_K$ be square-free and not necessarily totally positive, and let $F = K(\sqrt{\alpha})$. We denote by $v_0, f_0$, the character associated the extension, the conductor of the extension. Let $\chi_{\mathbb{Q}} F := \chi_{\mathbb{Q}} \circ \mathcal{N}_F$. We classify quadratic extensions by the congruence conditions on $\alpha$ modulo some powers of $p_2$ and modulo some powers of $p_2'$ as in Table 9.

| $\alpha \equiv$ | $v_{p_2}(f_0)$ | $p_2$ at $F$ | $\alpha' \equiv$ | $v_{p_2'}(f_0)$ | $p_2'$ at $F$ |
|-----------------|---------------|---------------|-----------------|---------------|---------------|
| (A) 1 (mod $p_2^2$) | 0 | split | (A') 1 (mod $p_2^2$) | 0 | split |
| (B) 5 (mod $p_2^2$) | 0 | inert | (B') 5 (mod $p_2^2$) | 0 | inert |
| ($C_A$) 7 (mod $p_2^2$) | 2 | ramify | ($C_A'$) 7 (mod $p_2^2$) | 2 | ramify |
| ($C_B$) 3 (mod $p_2^2$) | 2 | ramify | ($C_B'$) 3 (mod $p_2^2$) | 2 | ramify |
| (D) 2 (mod $p_2^2$) | 3 | ramify | (D') 2 (mod $p_2^2$) | 3 | ramify |

If $\alpha$ satisfies both (A) and (A'), then we say that $\alpha$ is in Case (AA'). It is similar for (AB'), (AC' A'), and so on.
Let $\rho_2, \rho_2'$ be as in the preceding section. Put $\rho_2^F := \rho_2 \circ N_{F/K}, \rho_2'^F := \rho_2' \circ N_{F/K}$. Then since $\chi^K_E = \rho_2 \rho_2'$, we have $\chi^E_4 = \rho_2^F \rho_2'^F$ and $\overline{\chi}_E^4 = \overline{\rho_2^F} \overline{\rho_2'^F}$. The conductors of $\rho_2^F, \rho_2'^F$ and their values of Gauss sums are given as in Table [10] where $\mathfrak{p}_2, \mathfrak{p}_2'$ denote prime ideals in $F$ with $\mathfrak{p}_2^2 = p_2, \mathfrak{p}_2'^2 = p_2'$ respectively.

| Table 10. Conduets and Gauss sums of $\rho_2^F, \rho_2'^F$. |
|----------------------------------------------------------|
| $f_{\rho_2^F}$ | $\mathfrak{p}_2^F(\mathfrak{p})_F$ | $\tau_F(\mathfrak{p}_2^F)$ | $f_{\rho_2'^F}$ | $\mathfrak{p}_2'^F(\mathfrak{p})_F$ | $\tau_F(\mathfrak{p}_2'^F)$ |
|---------------|-----------------|-----------------|---------------|-----------------|-----------------|
| (A)           | $p_2\mathcal{O}_F$ | $-4$            | (A')          | $p_2\mathcal{O}_F$ | $-4$            |
| (B)           | $p_2\mathcal{O}_F$ | $-4$            | (B')          | $p_2\mathcal{O}_F$ | $-4$            |
| (C_A)         | $\mathcal{O}_F$  | $-1$            | (C_A')        | $\mathcal{O}_F$  | $-1$            |
| (C_B)         | $\mathcal{O}_F$  | $-1$            | (C_B')        | $\mathcal{O}_F$  | $1$             |
| (D)           | $p_2\mathcal{O}_F$ | $0$             | (D')          | $p_2\mathcal{O}_F$ | $0$             |

We have $f_{\rho_2^F} = f_{\rho_2^F} f_{\rho_2'^F}$. The equality $\tau_F(\overline{\chi}_E^4) = \tau_F(\overline{\rho_2^F}) \tau_F(\overline{\rho_2'^F})$ holds, since $f_{\rho_2^F}, f_{\rho_2'^F}$ are squares in $\mathcal{O}_F$ and since $\rho_2^F, \rho_2'^F$ are real characters. We have

$$
\chi^K_4 \psi_\alpha = \begin{cases} 
\psi_{-\alpha}^1 \mathbf{1}_2 & (C_A C_A') \backslash (C_A C_B') \backslash (C_B C_A'), \\
\psi_{-\alpha}^1 \mathbf{1}_{p_2} & (C_A X), (C_B X') \text{ with } X' \neq C_A', C_B', \\
\psi_{-\alpha}^1 \mathbf{1}_{p_2'} & (X C_A'), (X C_B') \text{ with } X \neq C_A, C_B, \\
\psi_{-\alpha}^1 & \text{(otherwise)},
\end{cases}
$$

and

$$
L_F(s, \chi^K_4) = L_K(s, \chi^K_4) L_K(s, \psi_{-\alpha})(1 - \psi_{-\alpha}(p_2) 2^{-s})(1 - \psi_{-\alpha}(p_2') 2^{-s}),
$$

(18)

In particular $L_F(0, \chi^K_4) = 0$ in Case (C_A) or (C_A'). The standard argument shows the following lemma, and we omit the proof.

**Lemma 7.1.** (i) If $\nu \equiv 7 \pmod{p_2^2}$ or $\nu \equiv 7 \pmod{p_2'^2}$, then $r_{3, K}(\nu) = 0$.

(ii) The equalities $r_{3, K}(\pi_2^2 \nu) = r_{3, K}(\nu)$ and $r_{3, K}(\pi_2'^2 \nu) = r_{3, K}(\nu)$ hold.

In the following argument we exclude Cases (C_A) and (C_A') because of Lemma 7.1 (i).

Let $\alpha$ be a totally positive square-free integer in $K$. By (12), we have

$$
\mathcal{J}_{\alpha, \chi^K_4}(\theta(\bar{\alpha})) = 2^{1} \mathcal{J}_{\alpha, \chi^K_4}(\theta(\bar{\alpha}) G_1 \chi^K_4(\bar{\alpha}; \mathcal{O}_K, \mathcal{O}_K)) + 2^{1} \mathcal{J}_{\alpha, \chi^K_4}(\theta(\bar{\alpha}) G_1 \chi^K_4(\bar{\alpha}; \mathcal{O}_K, \mathcal{O}_K)),
$$

which is a Hilbert modular forms of weight 2 for $\Gamma_0(\mathfrak{d}_K^{-1}, \mathfrak{d}_K)$. Let $\iota: \mathfrak{h}^2 \rightarrow \mathfrak{h}^4$ be the diagonal map associated with the inclusion of $K$ into $F$. We put

$$
\lambda_{2, \chi^K_4}(\bar{\alpha}; \mathcal{O}_K) = \begin{cases} 
U(2)(G_1 \chi^K_4(\iota(\bar{\alpha}); \mathcal{O}_F, \mathfrak{d}_F^{-1})) & (AA'), (AB'), (BA'), (BB') \\
U(\tau_2)(G_1 \chi^K_4(\iota(\bar{\alpha}); \mathcal{O}_F, \mathfrak{d}_F^{-1})) & (AC_B'), (AD'), (BC_B'), (BD') \\
U(\tau_2')(G_1 \chi^K_4(\iota(\bar{\alpha}); \mathcal{O}_F, \mathfrak{d}_F^{-1})) & (C_B A'), (C_B B'), (DA'), (DB') \\
G_1 \chi^K_4(\iota(\bar{\alpha}); \mathcal{O}_F, \mathfrak{d}_F^{-1}) & \text{(otherwise)}
\end{cases}
$$
and
\[
\begin{align*}
\lambda_{\rho_2}^\varepsilon(\delta; (\alpha), \mathcal{O}_K) := \\
U(2)(G_{\rho_2}^\varepsilon(\delta; (\alpha), \mathcal{O}_F, \mathcal{O}_F^{-1})) \\
U(\pi_2)(G_{\rho_2}^\varepsilon(\delta; (\alpha), \mathcal{O}_F, \mathcal{O}_F^{-1})) \\
U(\pi_2')(G_{\rho_2}^\varepsilon(\delta; (\alpha), \mathcal{O}_F, \mathcal{O}_F^{-1})) \\
G_{1, \rho_2}'(\delta; (\alpha), \mathcal{O}_F, \mathcal{O}_F^{-1})
\end{align*}
\]

By [8], we have \( \mathcal{S}_{\alpha, \lambda^2} \mathcal{F}(\delta; (\alpha), \mathcal{O}_K) = 2^{-\lambda_{2, \rho_2}^\varepsilon(\delta; (\alpha), \mathcal{O}_K)} \), \( \mathcal{S}_{\alpha, \lambda^2} \mathcal{F}(\delta; (\alpha), \mathcal{O}_K) = 2^{-\lambda_{2, \rho_2}^\varepsilon(\delta; (\alpha), \mathcal{O}_K)} \).

By (7) and (8), and by using the fact \( \mathcal{S}_{\alpha, \lambda^2} \mathcal{F}(\delta; (\alpha), \mathcal{O}_K) \) has a fundamental unit with negative norm, the arithmetic genus of a compactified Hilbert modular surface for \( \mathcal{S}_{\alpha, \lambda^2} \mathcal{F}(\delta; (\alpha), \mathcal{O}_K) \) is obtained from the volume of fundamental domain and from contributions from elliptic singularities of \( \mathcal{S}_{\alpha, \lambda^2} \mathcal{F}(\delta; (\alpha), \mathcal{O}_K) \).

8. Sums of three squares in \( \mathbb{Q}(\sqrt{3}) \), and in \( \mathbb{Q}(\sqrt{17}) \)

The group \( \Gamma_0(\mathfrak{o}_K^{-1}, 2\mathfrak{d}_K) \) has 4 elliptic fixed points on \( \Gamma_0(\mathfrak{o}_K^{-1}, 2\mathfrak{d}_K) \) of \( \mathcal{S}_2 \), which are all of order 2. Since \( K \) has a fundamental unit with negative norm, the arithmetic genus of a compactified Hilbert modular surface for \( \Gamma_0(\mathfrak{o}_K^{-1}, 2\mathfrak{d}_K) \) is obtained from the volume of fundamental domain and from contributions from elliptic singularities, and it is computed to be 2, and hence \( \dim \mathcal{S}_2(\Gamma_0(\mathfrak{o}_K^{-1}, 2\mathfrak{d}_K)) = 1 \).

By (7) and (8), and by using the fact \( \zeta_K(-1) = 1/3 \) and \( L(-1, \chi_{17}) = -4 \), the values at cusps, of Eisenstein series \( G_2(z; \mathcal{O}_K, \mathcal{O}_K), G_{2,1,\rho_2}(z; \mathcal{O}_K, \mathcal{O}_K), G_{2,1,\rho_2}(z; \mathcal{O}_K, \mathcal{O}_K), G_{2,1,\rho_2}(z; \mathcal{O}_K, \mathcal{O}_K) \) are given as in Table 13. The table shows that these Eisenstein series are linearly independent, and \( \mathcal{M}(\Gamma_0(\mathfrak{o}_K^{-1}, 2\mathfrak{d}_K)) = \langle G_2(z; \mathcal{O}_K, \mathcal{O}_K), G_{2,1,\rho_2}(z; \mathcal{O}_K, \mathcal{O}_K), G_{2,1,\rho_2}(z; \mathcal{O}_K, \mathcal{O}_K), G_{2,1,\rho_2}(z; \mathcal{O}_K, \mathcal{O}_K), \rangle \).
Let $\alpha$ be a totally positive square-free integer in $K$. The Shimura lift $S_{\alpha,\chi_3}(\theta(3))$ is in $M_2(\Gamma_0(3^2, 29K))$, and by Table 12 and Table 13 it is equal to $2^{-3}3L_F(0, \chi_3^-) \times G_{2,16}(j; \mathcal{O}_K, \mathcal{O}_K) + c_0\Xi(j)$ in Case (E), $2^{-2}3L_F(0, \chi_3^-)G_{2,16}(j; \mathcal{O}_K, \mathcal{O}_K) + c_0\Xi(j)$ in Case (F), and $2^{-1}3L_F(0, \chi_3^-)G_{2,16}(j; \mathcal{O}_K, \mathcal{O}_K) + c_0\Xi(j)$ in Case (G) where $c_0$ is a constant depending only on $\alpha$.

**Lemma 8.1.** In all cases, $c_0$ is 0.

**Proof.** We consider the case (E). We note that the Fourier coefficients of $\Xi(j)$ for $\nu = 1$ and for $\nu = \pi_2$ are 1 and $-1$ respectively by Table 8. Comparing the Fourier coefficients of the equality $S_{\alpha,\chi_3}(\theta(3)) = 2^{-3}3L_F(0, \chi_3^-)G_{2,16}(j; \mathcal{O}_K, \mathcal{O}_K) + c_0\Xi(j)$ for $\nu = 1$ and for $\nu = \pi_2$, we have $r_{3,K}(\alpha) = 2^{-1}3L_F(0, \chi_3^-) + c_0$ and $r_{3,K}(\pi_2) = 2^{-1}3L_F(0, \chi_3^-) - c_0$. Then Lemma 7.1 (ii) leads to $c_0 = 0$. The similar arguments show the assertion also in the rest of cases.

**Corollary 8.2.** The Shimura lift $S_{\alpha,\chi_3}(\theta(3))$ is equal to

$$ S_{\alpha,\chi_3}(\theta(3)) = 2^{-3}3cL_F(0, \chi_3^-)G_{2,16}(j; \mathcal{O}_K, \mathcal{O}_K) $$

where $c = 1$ in Case (E), $c = 2$ in Case (F), and $c = 2^2$ in Case (G).

By (13), $L_F(0, \chi_3^-) = 2L_K(0, \psi_\alpha)$. Let $h_{K(\sqrt{-\alpha})}, w_{K(\sqrt{-\alpha})}, Q_{K(\sqrt{-\alpha})}/K$ be as in Sect. 5. Then $w_{K(\sqrt{-\alpha})}$ is 6 if $\sqrt{-3} \in K(\sqrt{-\alpha})$, 4 if $\sqrt{-1} \in K(\sqrt{-\alpha})$, and 2 if otherwise. The Hasse unit index $Q_{K(\sqrt{-\alpha})}/K$ is always 1. By $L_K(0, \psi_\alpha) = w_{K(\sqrt{-\alpha})}^{-2}h_{K(\sqrt{-\alpha})}^{-1}h_{K(\sqrt{-\alpha})}$, $L_F(0, \chi_3^-)$ is expressed in terms of $h_{K(\sqrt{-\alpha})}$. Applying the Möbius inversion formula on $K$ to the equality between $\nu$-th terms of both sides of (20), we have for $\alpha$ with $K(\sqrt{-\alpha}) \neq K(\sqrt{-1}), K(\sqrt{-3})$,

$$ r_{3,K}(\alpha \nu^2) = 2 \cdot 3c h_{K(\sqrt{-\alpha})} \sum_{\delta < \nu, \delta \in \mathcal{O}_K/\mathcal{E}_K} (\psi_\alpha 1_{2\mu K})(\delta) \sigma_{1,1_2}(\nu/\delta) \quad (\nu \in \mathcal{O}_K, > 0) $$

where $c = 1$ in Case (E), $c = 2$ in Case (F), and $c = 2^2$ in Case (G) and where $\sigma_{1,1_2}$ is as in (5). When $\alpha = 1$ or $\alpha = 3$, we have

$$ r_{3,K}(\nu^2) = 3h_{K(\sqrt{-3})} \sum_{\delta > 0, \delta \in \mathcal{E}_K} (\psi_{-1} 1_{2\mu K})(\delta) \sigma_{1,1_2}(\nu/\delta) \quad (\nu \in \mathcal{O}_K, > 0), $$

$$ r_{3,K}(3\nu^2) = 3h_{K(\sqrt{-3})} \sum_{\delta > 0, \delta \in \mathcal{E}_K} (\psi_{-1} 1_{2\mu K})(\delta) \sigma_{1,1_2}(\nu/\delta) \quad (\nu \in \mathcal{O}_K, > 0), $$

where $h_{K(\sqrt{-3})} = 2, h_{K(\sqrt{-3})} = 1$. For any $\nu \in \mathcal{O}_K$, $\neq 0$, one of $\pm \nu, \pm \varepsilon_0 \nu$ is totally positive and hence the formula for $r_{3,K}(\alpha \nu^2)$ for any $\nu$ is obtained from the above formulas since $r_{3,K}(\alpha \nu^2) = r_{3,K}(\varepsilon_0^2 \alpha \nu^2)$.

We have shown the following:
Theorem 8.3. Let \( K = \mathbb{Q}(\sqrt{17}) \). Let \( \pi_2 = (5 + \sqrt{17})/2, \pi'_2 = (5 - \sqrt{17})/2, \) and \( p_2 = (\pi_2), p'_2 = (\pi'_2) \).

(i) A totally positive integer in \( K \) is represented as a sum of three integer squares in \( K \) if and only if it is not in the form \( \pi_2^{2x} \pi'_2^{2y} \mu \) with nonnegative rational integers \( e, e' \) and with \( \mu \equiv 7 \pmod{p_2^3} \) or \( \mu \equiv 7 \pmod{p'_2^3} \).

(ii) Let \( \alpha \) be a totally positive square-free integer in \( K \) which is congruent to 7 neither modulo \( p_2^3 \) nor modulo \( p'_2^3 \). Further we assume that \( K(\sqrt{-\alpha}) \neq K(\sqrt{-1}), K(\sqrt{-3}) \).

We classify \( \alpha \) as in Table 11. Then the class number of the field \( K(\sqrt{-\alpha}) \) is given by

\[ 2^{-1}3^{-1}r_{3,K}(\alpha) \text{ in Case (E)}, \quad 2^{-2}3^{-1}r_{3,K}(\alpha) \text{ in Case (F)}, \quad \text{and } 2^{-3}3^{-1}r_{3,K}(\alpha) \text{ in Case (G)}. \]

9. Tables of class numbers

A tabulation for class numbers of totally imaginary quadratic extensions \( F = K(\sqrt{-\alpha}) \) of \( K \) is made for 220 selected values of \( \alpha \) in each case of \( K = \mathbb{Q}(\sqrt{3}) \) and \( K = \mathbb{Q}(\sqrt{17}) \).

In case \( K = \mathbb{Q}(\sqrt{3}) \), square-free totally positive integers \( \alpha = a + b\sqrt{3} \) with \( b \geq 0 \) are arranged in lexicographical order, where we omit \( \alpha \) with \( b < 0 \) since rings of integers in \( F \) for \( \alpha = a + b\sqrt{3} \) and for \( \alpha = a - b\sqrt{3} \) are isomorphic to each other and they have the same class number. We omit \( a + b\sqrt{3} \) from the table if there is \( n \in \mathbb{Z} \) so that \( a + b\sqrt{3} = \varepsilon_0^n(a' \pm b'\sqrt{3}) \) with \( a' < a \). By Theorem 5.3, a class number \( h_F \) is obtained from \( r_{3,K}(\alpha) \) or \( r_{3,K}(4\alpha) \), and \( r_{3,K}(a + b\sqrt{3}) \) is obtained by counting integral solutions of

\[ x_1^2 + 3y_1^2 + x_2^2 + 3y_2^2 + x_3^2 + 3y_3^2 = a, \]
\[ 2x_1y_1 + 2x_2y_2 + 2x_3y_3 = b \]

in terms of ordinary integral arithmetic.

In case \( K = \mathbb{Q}(\sqrt{17}) \), square-free totally positive integers \( \alpha = a + b\omega \) (\( \omega = (1 + \sqrt{17})/2 \) with \( b \geq 0 \) and with \( \alpha \not\equiv 7 \pmod{p_2^3} \), \( \alpha \not\equiv 7 \pmod{p'_2^3} \), are arranged in lexicographical order. A tabulation is made applying essentially the same principle as in \( \mathbb{Q}(\sqrt{3}) \), where we note that rings of integers in \( F \) for \( \alpha = a + b\omega \) and for \( \alpha = a + b - b\omega \) are isomorphic to each other. By Theorem 8.3 a class number \( h_F \) is obtained from \( r_{3,K}(\alpha) \), and \( r_{3,K}(a + b\omega) \) is obtained by counting integral solutions of

\[ x_1^2 + 4y_1^2 + x_2^2 + 4y_2^2 + x_3^2 + 4y_3^2 = a, \]
\[ 2x_1y_1 + y_1^2 + 2x_2y_2 + y_2^2 + 2x_3y_3 + y_3^2 = b. \]
Table 14. Table of class numbers of $F = \mathbb{Q}(\sqrt{3}, \sqrt{-\alpha})$

\[ \alpha = a + b\sqrt{3}, \quad d_{F/\mathbb{Q}(\sqrt{3})} = (m\alpha), \quad h_F = \text{the class number of } F. \]

|   |   |   | $h_F$ |   |   |   |   |   |   |
|---|---|---|------|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 16 | 7 | 4 | 6 | 23 | 4 |
| 2 | 1 | 4 | 2 | 17 | 0 | 1 | 4 | 23 | 5 |
| 3 | 1 | 4 | 2 | 17 | 1 | 4 | 16 | 23 | 6 |
| 4 | 1 | 4 | 2 | 17 | 2 | 2 | 10 | 23 | 7 |
| 5 | 0 | 1 | 2 | 17 | 3 | 4 | 22 | 23 | 8 |
| 5 | 1 | 4 | 6 | 17 | 4 | 1 | 8 | 23 | 9 |
| 5 | 2 | 2 | 2 | 17 | 5 | 4 | 14 | 23 | 10 |
| 6 | 1 | 4 | 4 | 17 | 6 | 2 | 6 | 23 | 11 |
| 7 | 0 | 1 | 2 | 17 | 7 | 4 | 12 | 24 | 1 |
| 7 | 1 | 4 | 4 | 17 | 8 | 1 | 2 | 24 | 5 |
| 7 | 2 | 2 | 2 | 18 | 1 | 4 | 16 | 24 | 7 |
| 7 | 3 | 4 | 2 | 18 | 5 | 4 | 12 | 24 | 11 |
| 8 | 1 | 4 | 10 | 18 | 7 | 4 | 8 | 25 | 1 |
| 8 | 3 | 4 | 6 | 19 | 0 | 1 | 2 | 25 | 2 |
| 9 | 1 | 4 | 4 | 19 | 1 | 4 | 10 | 25 | 3 |
| 9 | 2 | 4 | 6 | 19 | 2 | 2 | 6 | 25 | 4 |
| 9 | 4 | 1 | 2 | 19 | 3 | 4 | 8 | 25 | 5 |
| 10 | 1 | 4 | 4 | 19 | 4 | 1 | 6 | 25 | 6 |
| 10 | 3 | 4 | 4 | 19 | 5 | 4 | 8 | 25 | 7 |
| 10 | 5 | 4 | 4 | 19 | 6 | 2 | 4 | 25 | 8 |
| 11 | 0 | 1 | 2 | 19 | 7 | 4 | 10 | 25 | 9 |
| 11 | 1 | 4 | 10 | 19 | 9 | 4 | 6 | 25 | 10 |
| 11 | 2 | 2 | 6 | 20 | 1 | 4 | 14 | 25 | 11 |
| 11 | 3 | 4 | 12 | 20 | 3 | 4 | 18 | 25 | 12 |
| 11 | 4 | 1 | 4 | 20 | 5 | 4 | 20 | 26 | 1 |
| 11 | 5 | 4 | 8 | 20 | 7 | 4 | 20 | 26 | 3 |
| 12 | 1 | 4 | 12 | 20 | 9 | 4 | 18 | 26 | 5 |
| 12 | 5 | 4 | 4 | 21 | 1 | 4 | 16 | 26 | 9 |
| 13 | 0 | 1 | 4 | 21 | 2 | 2 | 8 | 26 | 11 |
| 13 | 1 | 4 | 10 | 21 | 4 | 1 | 2 | 26 | 13 |
| 13 | 2 | 2 | 2 | 21 | 5 | 4 | 16 | 27 | 2 |
| 13 | 3 | 4 | 8 | 21 | 7 | 4 | 12 | 27 | 4 |
| 13 | 5 | 4 | 4 | 21 | 8 | 1 | 6 | 27 | 5 |
| 13 | 6 | 2 | 2 | 21 | 10 | 2 | 4 | 27 | 7 |
| 14 | 1 | 4 | 16 | 22 | 1 | 4 | 12 | 27 | 8 |
| 14 | 7 | 4 | 8 | 22 | 3 | 4 | 12 | 27 | 10 |
| 15 | 1 | 4 | 8 | 22 | 5 | 4 | 12 | 27 | 11 |
| 15 | 2 | 2 | 4 | 22 | 7 | 4 | 8 | 27 | 13 |
| 15 | 4 | 1 | 2 | 22 | 9 | 4 | 8 | 28 | 1 |
| 15 | 5 | 4 | 8 | 22 | 11 | 4 | 8 | 28 | 3 |
| 15 | 7 | 4 | 8 | 23 | 0 | 1 | 12 | 28 | 5 |
| 16 | 1 | 4 | 8 | 23 | 1 | 4 | 32 | 28 | 7 |
| 16 | 3 | 4 | 6 | 23 | 2 | 2 | 12 | 28 | 9 |
| 16 | 5 | 4 | 10 | 23 | 3 | 4 | 22 | 28 | 11 |
Table 15. Table of class numbers of $F = \mathbb{Q}(\sqrt{17}, \sqrt{-\alpha})$.

$\alpha = a + b\omega$ ($\omega = \frac{1 + \sqrt{17}}{2}$), $d_{F/\mathbb{Q}(\sqrt{17})} = (\pi_1 \pi_2' \alpha)$, $h_F$ is the class number of $F$.

| $a$ | $b$ | $e$ | $e'$ | $h_F$ |
|-----|-----|-----|------|-------|
| 1   | 0   | 2   | 2    | 12    |
| 2   | 0   | 2   | 2    | 4     |
| 3   | 0   | 0   | 0    | 2     |
| 5   | 0   | 2   | 4    | 8     |
| 5   | 1   | 2   | 4    | 6     |
| 6   | 0   | 2   | 4    | 1     |
| 6   | 1   | 2   | 0    | 2     |
| 6   | 3   | 2   | 4    | 9     |
| 7   | 3   | 0   | 2    | 20    |
| 9   | 1   | 2   | 8    | 12    |
| 9   | 2   | 0   | 2    | 24    |
| 9   | 4   | 2   | 4    | 24    |
| 9   | 5   | 2   | 4    | 28    |
| 10  | 0   | 2   | 12   | 16    |
| 10  | 3   | 2   | 8    | 24    |
| 10  | 4   | 2   | 8    | 28    |
| 10  | 5   | 0   | 2    | 18    |
| 11  | 0   | 0   | 1    | 7     |
| 11  | 2   | 0   | 2    | 16    |
| 11  | 6   | 0   | 2    | 24    |
| 13  | 0   | 2   | 16   | 16    |
| 13  | 1   | 2   | 8    | 32    |
| 13  | 4   | 2   | 12   | 16    |
| 13  | 5   | 2   | 12   | 32    |
| 13  | 6   | 2   | 4    | 24    |
| 13  | 8   | 2   | 4    | 28    |
| 14  | 0   | 2   | 16   | 32    |
| 14  | 1   | 2   | 0    | 2     |
| 14  | 3   | 2   | 12   | 24    |
| 14  | 4   | 2   | 8    | 4     |
| 14  | 7   | 2   | 8    | 8     |
| 14  | 8   | 2   | 8    | 16    |
| 15  | 3   | 0   | 2    | 4     |
| 15  | 7   | 0   | 2    | 20    |
| 17  | 1   | 2   | 16   | 22    |
| 17  | 2   | 0   | 6    | 24    |
| 17  | 4   | 2   | 12   | 24    |
| 17  | 5   | 2   | 12   | 24    |
| 17  | 9   | 2   | 8    | 16    |
| 17  | 10  | 0   | 2    | 24    |
| 18  | 3   | 2   | 16   | 22    |
| 18  | 4   | 2   | 12   | 28    |
| 18  | 5   | 2   | 4    | 24    |
| 18  | 7   | 2   | 16   | 22    |

$\frac{\sqrt{17}}{2}$
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