MULTIPLE SOLUTIONS OF FRACTIONAL KIRCHHOFF EQUATIONS INVOLVING A CRITICAL NONLINEARITY

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Abstract. In this paper, we are concerned with the following fractional Kirchhoff equation

\[
\begin{cases}
(a + b \int_{\mathbb{R}^N} |(-\Delta)^s u|^2) (-\Delta)^s u = \lambda u + \mu |u|^{q-2} u + |u|^{2^*_s-2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( N > 2s \), a, b, \( \lambda, \mu > 0 \), s \( \in (0, 1) \) and \( \Omega \) is a bounded open domain with continuous boundary. Here \((-\Delta)^s\) is the fractional Laplacian operator. For \( 2 < q \leq \min\{4, 2^*_s\} \), we prove that if b is small or \( \mu \) is large, the problem above admits multiple solutions by virtue of a linking theorem due to G. Cerami, D. Fortunato and M. Struwe [7, Theorem 2.5].

1. Introduction and main results.

1.1. Overview. In the present paper, we investigate the multiplicity of solutions to the following critical fractional Kirchhoff equation

\[
\begin{cases}
(a + b \int_{\mathbb{R}^N} |(-\Delta)^s u|^2) (-\Delta)^s u = \lambda u + \mu |u|^{q-2} u + |u|^{2^*_s-2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( N > 2s \), a, b, \( \lambda, \mu > 0 \), s \( \in (0, 1) \) and \( \Omega \) is a bounded open domain with continuous boundary. Here \( 2^*_s = 2N/(N - 2s) \) is the fractional Sobolev critical exponent and the operator \((-\Delta)^s\) is the fractional Laplacian defined as \( \mathcal{F}^{-1}(\xi |^{2s} \mathcal{F}(u)) \), where \( \mathcal{F} \) denotes the Fourier transform on \( \mathbb{R}^N \). If \( s = 1, a = 1, b = \mu = 0 \), then problem (1) reduces to

\[
\begin{cases}
-\Delta u = \lambda u + |u|^{2^*_s-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

which arises in a geometric context in the well-known Yamabe problem. Here \( 2^* = 2N/(N - 2) \) is the critical Sobolev exponent. In the literature, problems with critical growth have received a considerable attention. Problem (2) is the so-called Brezis-Nirenberg problem. In the celebrated paper [6], H. Brezis and L. Nirenberg obtained the existence and nonexistence results of positive solutions to

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problem (2) depending on the parameter $\lambda$. Since then, the Brezis-Nirenberg problem has attracted many mathematicians’ interest about the sign-changing solutions, multiple solutions. In [7], by use of a variant of a linking theorem [5, Theorem 2.4], G. Cerami, D. Fortunato and M. Struwe showed a multiplicity result of nontrivial solutions to problem (2). In this aspect, we also would like to cite [8, 26] and the reference therein.

If $s = 1, b \neq 0$, problem (1) becomes the following Kirchhoff equation

$$\begin{aligned}
- \left( a + b \int_\Omega |\nabla u|^2 \right) \Delta u &= \lambda u + \mu |u|^{q-2}u + |u|^{2^* - 2}u & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}$$

which is related to the stationary analogue of the time-dependent Kirchhoff equation

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \left( a + b \int_\Omega |\nabla u|^2 \right) \Delta u &= f(t, x, u) & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega.
\end{aligned}$$

Equation (K) was proposed by G. Kirchhoff [17] in 1883 as a generalization the classic D’Alembert’s wave equation for free vibrations of elastic strings. In the last decade, for different nonlinearities $f(x, u)$, Kirchhoff problems have been studied by many mathematicians. In [22], by the concentration-compactness principle, D. Naimen investigated the existence, nonexistence and uniqueness of positive solutions to the Brezis-Nirenberg problem with a Kirchhoff type perturbation as follows

$$\begin{aligned}
- \left( 1 + b \int_\mathbb{R}^3 |\nabla u|^2 \right) \Delta u &= \lambda u + u^5 & \text{in } \Omega \subset \mathbb{R}^3, \\
u &= 0 & \text{on } \partial \Omega.
\end{aligned}$$

The author gave an extension of the result in [6]. In particular, in contrast to the Brezis-Nirenberg problem (2), several effects of the nonlocal term were observed. Meanwhile, in [21] the author investigated the existence of positive solutions to the following Kirchhoff problem

$$\begin{aligned}
- \left( a + b \int_\Omega |\nabla u|^2 \right) \Delta u &= \lambda |u|^{q-1}u + u^5 & \text{in } \Omega \subset \mathbb{R}^3, \\
u &= 0 & \text{on } \partial \Omega.
\end{aligned}$$

When $q \in (2, 4]$ and $q \in (4, 6)$, the existence results were obtained respectively by variational methods and a cut-off technique. As for the multiplicity results for Kirchhoff problems in the critical case, an analogue result of [7] was obtained in [19]. Precisely, L. Yang, Z. Liu and Z. Ouyang [19] considered problem (3) with $q = 4$ when $\lambda$ is close to some eigenvalue $\lambda_i$ of $-\Delta$ under homogeneous Dirichlet boundary condition and lies in $\lambda_i$’s left neighborhood and show a multiplicity result depending on the multiplicity of $\lambda_i$. For more details about Kirchhoff problems, we refer the readers to [9, 14, 15, 24, 33] and the reference therein.

Now, we focus our attention on the fractional problems. In what follows, we assume $s \in (0, 1)$. The fractional Laplacian $(-\Delta)^s$ also may be defined as

$$(-\Delta)^s u(x) = c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy, \; x \in \mathbb{R}^N,$$

where $u \in \mathcal{L}$, $c_{N,s}$ is a normalization factor and P.V. stands for the principal value. Here $\mathcal{L}$ denotes the Schwartz space of rapidly decaying $C^\infty$ functions in $\mathbb{R}^N$. Obviously, in contrast to the Laplacian $\Delta$, $(-\Delta)^s$ is nonlocal. In the last decades, a deep interest has been shown for non-local operators, due to their applications in a wide range of contexts, such as the phase transition [32], Markov processes [2] and fractional quantum mechanics [18], minimal surfaces and so on. For the further background on the fractional operator, we refer to [10].
When \( a = 1, b = 0 \) and \( \mu = 0 \), problem (1) reduces to the non-local fractional counterpart of the Brezis-Nirenberg problem (2)

\[
\begin{cases}
(-\Delta)^s u = \lambda u + |u|^{2^*_s - 2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]  

By virtue of the mountain pass theorem, R. Servadei and E. Valdinoci \cite{31} investigated problem (5) with \( N \geq 4s \) and showed that there exists a non-trivial solution provided \( \lambda \in (0, \lambda_{1,s}) \), where \( \lambda_{1,s} \) is the first eigenvalue of the non-local operator \((-\Delta)^s\) with homogeneous Dirichlet boundary datum. Subsequently, in \cite{29} the authors completed the investigation carried on in \cite{31}. Precisely, the authors considered problem (5) in low dimension \( 2s < N < 4s \). In \cite{31}, the following general nonlocal problem also was considered

\[
\begin{cases}
-L_K u = \lambda u + |u|^{2^*_s - 2} u + f(x, u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]  

where \( L_K \) is the non-local operator defined as follows:

\[
L_K(u)(x) = \int_{\mathbb{R}^N} (u(x + y) + u(x - y) - 2u(x))K(y)\,dy, \ x \in \mathbb{R}^N.
\]

A model for \( K \) is given by \( K(x) = |x|^{-(N+2s)} \) and in this case, up to a normalization factor, \( L_K \) is nothing but the fractional Laplacian \((-\Delta)^s\) defined above. In \cite{30}, by use of a linking theorem and under some suitable condition on \( f \), the authors obtained an existence result of nontrivial solutions to problem (6) for any \( \lambda > 0 \). In \cite{11}, based on the variational and topological methods, bifurcation and multiplicity results were obtained for problem (6) with \( f(x, u) = 0 \).

Recently, in a bounded regular domain \( \Omega \subset \mathbb{R}^N \), A. Fiscella and E. Valdinoci \cite{13} proposed the following fractional stationary Kirchhoff equation

\[
\begin{cases}
M \left( \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \right) (-\Delta)^s u = \lambda f(x, u) + |u|^{2^*_s - 2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]  

which models the nonlocal aspects of the tension arising from nonlocal measurements of the fractional length of the string. Here \( M(t) \) is an increasing and continuous function and a typical model is given by \( M(t) = a + tb \). In \cite{3}, G. Autuoria, A. Fiscella and P. Pucci dealt with problem (7) involving a general nonlocal integro-differential operator as follows

\[
\begin{cases}
-M(\|u\|^2)L_K u = \lambda f(x, u) + |u|^{2^*_s - 2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]  

and obtained nontrivial solutions for \( \lambda \) large enough. Moreover, as \( \lambda \to \infty \), the asymptotic behavior of solutions was investigated. Related with problem (8), we also would like to cite \cite{25}, in which the authors considered a stationary Kirchhoff eigenvalue problem and established an existence, multiplicity and nonexistence result of nontrivial solutions depending on a parameter \( \lambda \). As for the existence of ground states to the fractional Kirchhoff equations, we refer to \cite{20, 34} and the reference therein.

1.2. Main result. Motivated by the works above, in this paper, we consider the following fractional Kirchhoff equation

\[
\begin{cases}
(1 + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2) (-\Delta)^s u = \lambda u + \mu |u|^{q-2} u + |u|^{2^*_s - 2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]
The main goal of this paper is to focus attention on the multiplicity of solutions to problem (9). To present our main result, we choose any fixed $t_0 \in (0, s/N)$, $s_0 \in (0, 1)$. Let $T = \min \{T_1, T_2\}$, where

$$T_1 = \left( \frac{1-s_0}{4b} \right)^{1/2}, \quad T_2 = \left( \frac{s-t_0N}{bN} \right)^{1/4} \left( s_0 S_s \right)^{N/8s}$$

and $\lambda^* = \min \{\lambda_1^*, \lambda_2^*\}$, where

$$\lambda_1^* = \left( \frac{N t_0}{s |\Omega|} \right)^{\frac{2}{q}} S_s, \quad \lambda_2^* = \left[ \frac{(q-2)NT^2}{4qs|\Omega|} \right]^{\frac{2}{q}}.$$

$\lambda_k (k = 1, 2, \cdots)$ denote the eigenvalues of the fractional Laplacian $(-\Delta)_s^*$ with homogeneous Dirichlet boundary datum (see Section 2).

Our main result can read as

**Theorem 1.1.** Assume $2 < q \leq \min \{4, 2^*_s\}$, then we have

1. If $\lambda \in (\lambda_1^* - \lambda^*, \lambda_1^*)$, then problem (9) has a pair of nontrivial solutions for $b > 0$ small and any $\mu > 0$;
2. If $\lambda_k - \lambda^* < \lambda < \lambda_k = \cdots = \lambda_{k+m-1} < \lambda_{k+m}$, then problem (9) has $m$ distinct pairs of solutions for $b > 0$ small and any $\mu > 0$;
3. In particular, if $q = 4$ when $N \leq 4s$, the assertions above are still true for any $b > 0$ and $\mu > 0$ large.

**Remark 1.** As $b \to 0$, $T_1, T_2, \lambda_2^* \to \infty$. So for $b > 0$ small enough, $T = T_2$ and $\lambda^* = \lambda_1^*$.

**Remark 2.** In [19], the authors considered problem (9) with $s = 1$ and $N = 3$. So our result extend the result made in [19] to the nonlocal fractional setting with $N > 2s (0 < s < 1)$. Moreover, the case $q < 4$ also is considered.

1.3. **Main difficulties and ideas.** The main difficulties are two-fold. Firstly, because of the presence of the nonlocal term $\int_{\mathbb{R}^N} |(-\Delta)_s^* u|^2$ and $q \leq 4$, the Ambrosetti-Rabinowitz (AR for short) condition is not satisfied. So, it is not easy to get the boundedness of the Palais-Smale (PS for short) sequence. To overcome this difficulty, we apply a cut-off technique introduced in [16] to get a bounded (PS) sequence for the truncated functional. By a linking theorem introduced by G. Cerami, D. Fortunato and M. Struwe (see [7]), we obtained multiple solutions to the truncated problem. When $b$ is small or $q = 4$, we prove that these solutions are actually the solutions of the original problem. Similar argument also can be found in [1, 35]. Secondly, due to lack of compactness of the embedding $X_0^s(\Omega) \hookrightarrow L^{2_s}(\Omega)$, the (PS) condition does not hold in general. In order to recover the compactness, by use of the concentration compactness principle, we show that, if $\mu$ is large enough, the (PS) condition holds when the energy level is lower than some special level (see Lemma 3.1 in Section 3).

The paper is organized as follows.
In Section 2, we introduce the Sobolev space, functional framework and some preliminary results.
In Section 3, we investigate the existence of solutions to the truncated problem.
In Section 4, we complete the proof of Theorem 1.1.
2. Preliminaries and functional setting. In this section, we outline the variational framework for problem (9) and introduce the Sobolev space \(X^s_0(\Omega)\) (see below). Moreover, some preliminary results for fractional Sobolev space are introduced.

2.1. Fractional Sobolev space \(X^s_0(\Omega)\). In order to study problem (9) by variational methods, it is important to give out an appropriate space. We should note that the boundary condition \(u = 0\) in \(\mathbb{R}^N \setminus \Omega\) (which is different from the classical case of the Laplacian, where it is required \(u = 0\) on \(\partial \Omega\)) is the natural counterpart of the homogeneous Dirichlet boundary data \(u = 0\) on \(\partial \Omega\), due to the nonlocal character of the problem and should be encoded in the weak formulation.

Now, we introduce the space \(X^s_0(\Omega)\). Let \(X^s\) be a linear space of Lebesgue measurable functional from \(\mathbb{R}^N\) to \(\mathbb{R}\) such that the restriction to \(\Omega\) of any \(u\) in \(X^s\) belongs to \(L^2(\Omega)\). The norm can be defined as

\[
\|u\|_{X^s} = \|u\|_{L^2(\Omega)} + \left( \int_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

Here \(Q = \mathbb{R}^N \setminus \mathcal{O}, \mathcal{O} = C\Omega \times C\Omega,\) where \(C\Omega = \mathbb{R}^N \setminus \Omega\). \(X^s_0(\Omega)\) is defined by

\[
X^s_0(\Omega) = \{ u \in X^s : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.
\]

By [28, Lemma 6], the norm in \(X^s_0(\Omega)\) can be taken as

\[
\|u\|_{X^s_0} = \left( \int_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

We should notice that \(X^s_0(\Omega)\) is different from the usual fractional space since the boundary condition is \(u = 0\) in \(\mathbb{R}^N \setminus \Omega\). While, thanks to the continuity of the \(\partial \Omega\), by [12, Theorem 6], \(X^s_0(\Omega)\) can be seen as the closure of \(C^\infty_0(\Omega)\) with respect to the norm in \(H^s(\mathbb{R}^N)\) since for any \(u \in X^s_0(\Omega), u = 0\) a.e. in \(\mathbb{R}^N \setminus \Omega\). Thus, in the following, we consider the norm on \(X^s_0(\Omega)\) as follows

\[
\|u\| = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2}
= \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} u(x) \right|^2 \, dx \right)^{1/2}.
\]

Up to a normalization factor, the second equality holds due to the definition of \((-\Delta)^s\) (see [10, Lemma 7]). What’s more, \((X^s_0(\Omega), \| \cdot \|)\) is a Hilbert space with the scalar product

\[
\langle u, v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy.
\]

The more details about the space \(X^s_0(\Omega)\) can be found in [28, 29].

Now, we present an embedding lemma which is useful in the sequel.

**Lemma 2.1.** (see [28, Lemma 8]) The embedding \(X^s_0(\Omega) \hookrightarrow L^r(\Omega)\) is continuous for any \(r \in [1, 2_s]\) and it is compact if \(r \in [1, 2_s)\).

Let \(S_s\) be the best fractional Sobolev constant defined as follows

\[
S_s = \inf_{u \in H^s(\mathbb{R}^N), u \neq 0} \frac{\|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2}{\|u\|_{L^2(\mathbb{R}^N)}^2}.
\]
2.2. Weak solutions. The energy functional of problem (9) is defined as

\[ I(u) = \frac{1}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{\lambda}{2} \|u\|^2_2 - \frac{\mu}{q} \|u\|^q_2 - \frac{1}{2x^n} \|u\|^{2\gamma}_{2\gamma}. \]

Here \(\|u\|_r\) means \(\|u\|_{L^r(\Omega)}\).

**Definition 2.2.** We call \(u \in X_0^\delta(\Omega)\) a weak solution of problem (9) if for any \(\phi \in X_0^\delta(\Omega)\), there holds

\[(1 + b\|u\|^2) \int_{\mathbb{R}^N} (-\Delta)^\frac{1}{2} u (-\Delta)^\frac{1}{2} \phi \, dx = \lambda \int_{\Omega} u \phi \, dx + \mu \int_{\Omega} |u|^{q-2} u \phi \, dx + \int_{\Omega} |u|^{2\gamma-2} u \phi \, dx.\]

Obviously, \(I \in C^1(\mathcal{X}_0^\delta(\Omega), \mathbb{R})\) and the critical points of \(I\) are the weak solutions of problem (9).

2.3. Some lemmas. To find multiple critical points of \(I\), the following linking theorem is needed, which was introduced in [7].

**Lemma 2.3.** (Linking Theorem) Let \(H\) be a real Hilbert space with norm \(\|\cdot\|\) and suppose that \(I \in C^1(H, \mathbb{R})\) is a functional on \(H\) satisfying the following conditions:

1. \(I(u) = I(-u), I(0) = 0\) and there exists a constant \(c^*\) such that the Palais-Smale condition holds in \((0, c^*)\).
2. There exist two closed subspace \(W, V \subset H\) and positive constants \(\rho, \gamma, \delta\) with \(\delta < \gamma < c^*\) such that \(I(u) \leq \gamma\) for all \(u \in W\), \(I(u) \geq \delta\) for all \(u \in V\) with \(\|u\| = \rho\).

Here \(\dim W \geq \text{codim} V\) with \(\text{codim} V < +\infty\). Then, there exist at least \(\dim W - \text{codim} V\) pairs of critical points of \(I\) with critical values belonging to the interval \([\delta, \gamma]\).

In order to recover the compactness, we need the following concentration compactness principle which was essentially proved in [23].

**Lemma 2.4.** (Concentration-Compactness) Let \(\Omega \subset \mathbb{R}^N\) be an open subset and \(\{u_n\}\) be a sequence in \(X_0^\delta(\Omega)\) weakly converging to \(u\) as \(n \to \infty\) such that

\(|(-\Delta)^\frac{1}{2} u_n|^2 \, dx \to \mu\) and \(|u_n|^2 \, dx \to \nu\) in \(\mathcal{M}(\mathbb{R}^N)\).

Then, either \(u_n \to u\) in \(L^2_{\text{loc}}(\mathbb{R}^N)\) or there exist a (at most countable) set of distinct points \(\{x_j\}_{j \in J} \subset \Omega\) and positive numbers \(\{\nu_j\}_{j \in J}\) such that we have

\[\nu = |u|^{2\gamma} \, dx + \sum_j \nu_j \delta_{x_j}.\]

If, in addition, \(\Omega\) is bounded, then there exist a positive measure \(\bar{\mu} \in \mathcal{M}(\mathbb{R}^N)\) with \(\text{supp} \bar{\mu} \subset \Omega\) and positive numbers \(\{\mu_j\}_{j \in J}\) such that

\[\mu = |(-\Delta)^\frac{1}{2} u|^2 + \bar{\mu} + \sum_j \mu_j \delta_{x_j}, \quad \nu_j \leq \left(\frac{\mu_j}{S_\gamma}\right)^{\frac{2\gamma}{\gamma}}.\]

2.4. The eigenvalue problem. To apply Lemma 2.3, we present some results about the following eigenvalues problem,

\[
\begin{align*}
(-\Delta)^s u &= \lambda u & \text{in } \Omega, \\
 u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{align*}
\]

Denote by \(\{\lambda_i\}_{i=1}^\infty\) the eigenvalues of problem (10). It is well-known that \(0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_i \to \infty\). Now, we introduce two characterizations of the
eigenvalues. Denote by $M(\lambda)$ the eigenfunction space of $(-\Delta)^s$ corresponding to $\lambda$ and set
\[ W_k = \bigoplus_{n=1}^{k} M(\lambda_n), V_k = \bigoplus_{n \geq k} M(\lambda_n), \]
then from [27, Proposition 9] and [31, Proposition 2.3], we have
\[ \lambda_k = \min_{u \in V_k \setminus \{0\}} \frac{\|u\|^2}{\|u\|_2^2}, \tag{11} \]
and
\[ \lambda_k = \max_{u \in W_k \setminus \{0\}} \frac{\|u\|^2}{\|u\|_2^2}. \tag{12} \]
Let \( \{e_k\}_{k \in \mathbb{N}} \) be the sequence of the eigenfunctions corresponding to \( \lambda_k \) and choose \( \{e_k\}_{k \in \mathbb{N}} \) normalized in such a way that this sequence provides an orthogonal normal basis of \( L^2(\Omega) \) and \( X_0^s(\Omega) \).

3. The truncated problem. Since the (AR) condition does not hold, to get a bounded (PS)-sequence, we use a cut-off technique to consider the associated truncated problem.

3.1. Truncated functional. Let \( \phi \) be a smooth function on \([0, \infty)\) such that \( \phi = 1 \) on \([0, 1)\), \( \phi = 0 \) on \([2, \infty)\) and \( \phi \in [0, 1] \) otherwise. Furthermore, we assume \( \phi' \in [-2, 0] \) on \([0, \infty)\). For \( T > 0 \), define a cut of functional \( \Phi_T(u) \) on \( X_0^s(\Omega) \) such that
\[ \Phi_T(u) := \phi \left( \frac{\|u\|^2}{T^2} \right). \]
Corresponding to \( I \), the truncated functional defined on \( X_0^s(\Omega) \) is
\[ J_T(u) = \frac{1}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 \Phi_T(u) - \frac{\lambda}{2} \|u\|_2^2 - \frac{\mu}{q} \|u\|_q^q - \frac{1}{2^*} \|u\|_{2^*}^{2^*}. \]
Obviously, \( J_T \in C^1(X_0^s(\Omega), \mathbb{R}) \) is well defined and
\[ \langle J_T'(u), v \rangle = \left\{ \frac{1}{2} \langle u, v \rangle + \frac{b}{2T^2} \langle u^4 \Phi'(u) \|u\|^2 \|u\|^2 \|u\|_2^2 \rangle \right\} \int_{\mathbb{R}^N} (-\Delta)^{ \frac{s}{2} } u(-\Delta)^{ \frac{s}{2} } v \]
\[ - \lambda \int_{\Omega} u v - \mu \int_{\Omega} |u|^{q-2} u v - \int_{\Omega} |u|^{2^*-2} u v. \]
Noting that \( \|u\|^4 \Phi'(u) \left( \frac{\|u\|^2}{T^2} \right) \geq -8T^4 \) for any \( u \in X_0^s(\Omega) \), by the definition of \( T \), we have
\[ 1 + \frac{b}{2T^2} \|u\|^4 \Phi'(u) \left( \frac{\|u\|^2}{T^2} \right) \geq s_0, \tag{13} \]
and
\[ -bT^4 + \frac{s}{N} (s_0 s_0)^{N/2s} \geq t_0 (s_0 s_0)^{N/2s}. \tag{14} \]
As we can see that, if \( u \in X_0^s(\Omega) \) and \( \|u\| < T \), \( J_T(u) = I(u) \). So, if \( u \) is a critical point of \( J_T \) with \( \|u\| < T \), \( u \) is also a critical point of \( I \).

3.2. Compactness. As we can see, the (PS) condition with respect to the truncated functional is difficult to verify because of the critical nonlinearity. To overcome this difficult, we use the concentration compactness principle to show that the (PS) condition holds when the energy is lower than some level \( c^* \).
Lemma 3.1. \( J_T \) satisfies the (PS) condition in \((0, c^*)\) with 
\[ c^* = t_0(s_0S_s)^{N/2s}. \]

Proof. **Step 1.** We show that \( \{u_n\} \) is bounded in \( X_0^s(\Omega) \). The proof is standard, but for the reader’s convenience, we present the detail in the following. Let \( \{u_n\} \subset X_0^s(\Omega) \) with 
\[ J_T(u_n) \to c < c^*, \quad J'_T(u_n) \to 0, \quad n \to \infty, \]
then
\[
c + o(1) \geq J_T(u_n) - \frac{1}{2}\langle J'_T(u_n), u_n \rangle \\
= -\frac{b}{4}\|u_n\|^4\Phi_T(u_n) - \frac{b}{4T^2}\|u_n\|^6\phi'\left(\frac{\|u_n\|^2}{T^2}\right) \\
- \mu \left( \frac{1}{q} - \frac{1}{2} \right) \|u_n\|^q + \frac{1}{2} - \frac{1}{2s}\|u_n\|^{2s} \\
\geq -bT^4 + \left( \frac{1}{2} - \frac{1}{2s} \right)\|u_n\|^{2s}, \tag{15}
\]
which implies that \( \{u_n\} \) is bounded in \( L^{2s}(\Omega) \). By Hölder’s inequality, for any \( r \in [2, 2s]\), \( \{u_n\} \) is bounded in \( L^r(\Omega) \). Therefore, there exists a constant \( M > 0 \) such that \( J_T(u_n) \geq \frac{1}{2}\|u_n\|^2 - M \). Together with \( J_T(u_n) \to c \), we deduce that \( \{u_n\} \) is bounded in \( X_0^s(\Omega) \).

**Step 2.** By Step 1, there exists \( u \in X_0^s(\Omega) \) such that, up to a subsequence,
\[
\begin{align*}
&u_n \rightharpoonup u \text{ weakly in } X_0^s(\Omega), \\
&u_n \to u \text{ in } L^r(\Omega) \text{ with } r \in [1, 2s], \\
&u_n \to u \text{ a.e. in } \Omega.
\end{align*}
\]

In the following, we prove that \( u_n \to u \) strongly in \( X_0^s(\Omega) \). By Lemma 2.4, we obtain an at most countable set of distinct points \( \{x_i\}_{i \in J}, \{\eta_i\}_{i \in J} \) and \( \{\nu_i\}_{i \in J} \) in \( \mathbb{R}^+ \) and a positive measure \( \tilde{\eta} \) with \( \text{supp } \tilde{\eta} \subset \Omega \) such that
\[
|(-\Delta)^{\frac{s}{2}}u_n|^2 \, dx \to |(-\Delta)^{\frac{s}{2}}u|^2 \, dx + \tilde{\eta} + \sum_{i \in J} \eta_i \delta_{x_i}, \tag{16}
\]
and
\[
|u_n|^2 \, dx \to |u|^2 \, dx + \sum_{i \in J} \nu_i \delta_{x_i}. \tag{17}
\]
Here \( \delta_x \) is the Dirac delta measure concentrated at \( x \in \mathbb{R}^N \) with mass 1. In addition, we have \( \eta_i \geq S_s \nu_i^{2/s} \). To conclude the proof, it suffices to show \( J = \emptyset \). If not, fix any \( k_0 \in J \). For any \( \varepsilon > 0 \), define a smooth function \( \varphi \) such that \( \varphi = 1 \) on \( B(x_{k_0}, \varepsilon) \), \( \varphi = 0 \) on \( B(x_{k_0}, 2\varepsilon) \) and \( \varphi \in [0, 1] \) otherwise.

Now, we estimate the term
\[
\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}u_n (-\Delta)^{\frac{s}{2}}(\varphi u_n).
\]
For any \( v, w \in X_0^s(\Omega) \), we have
\[
(-\Delta)^{\frac{s}{2}}(vw) = c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{v(x)w(x) - v(y)w(y)}{|x-y|^{N+s}} \, dy.
\]
\[ c_{N,s}v(x) \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x) - w(y)}{|x - y|^{N+s}} \, dy + c_{N,s}w(x) \text{P.V.} \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x - y|^{N+s}} \, dy \]

\[ := v(x)(-\Delta) \dot{u} + w(x)(-\Delta) \dot{v} - H(v, w). \]

By [4, Lemma 2.8, Lemma 2.9], we obtain

\[ \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} u_n(-\Delta) \dot{u}_n(-\Delta) \dot{\varphi} = 0 \]

and

\[ \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} (-\Delta) \dot{u}_n H(u_n, \varphi) = 0. \]

Together with (16), we get

\[ \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} (-\Delta) \dot{u}_n(-\Delta) \dot{\varphi u_n} = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} \varphi(-\Delta) \dot{u}_n|\varphi u_n|^2 \geq \eta_{k_0}. \]

Moreover,

\[ \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} \varphi|u_n|^2 \to \nu_{k_0} \]

and for any \( r \in [1, 2^*_s) \),

\[ \lim_{n \to \infty} \int_{\Omega} \varphi|u_n|^r \leq \lim_{n \to \infty} \int_{\Omega} |u_n|^r = 0. \]

By \( J_T'(u_n) \to 0 \) and (13), we have

\[ 0 = \lim_{\varepsilon \to 0} \lim_{n \to \infty} (J_T'(u_n), \varphi u_n) \]

\[ = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left[ \left( 1 + b\|u_n\|^2 \Phi_T(u_n) + \frac{b}{T^2}\|u_n\|^4 \phi\left( \frac{\|u_n\|^2}{T^2} \right) \right) \right. \]

\[ \int_{\mathbb{R}^N} (-\Delta) \dot{u}_n(-\Delta) \dot{\varphi u_n} - \lambda \int_{\Omega} \varphi u_n^2 - \int_{\Omega} \varphi|u_n|^q - \int_{\Omega} \varphi|u_n|^{2^*_s} \]

\[ \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left[ s_0 \int_{\mathbb{R}^N} (-\Delta) \dot{u}_n(-\Delta) \dot{\varphi u_n} - \int_{\Omega} \varphi|u_n|^{2^*_s} \right] \]

\[ \geq s_0 \eta_{k_0} - \nu_{k_0}. \]

It implies that \( \eta_{k_0} \leq \frac{\nu_{k_0}}{s_0} \). Thanks to \( \eta_i \geq \gamma_{k_0} s_i^{2^*_s} \) \( (i \in \mathcal{J}) \), we obtain

\[ \nu_{k_0} \geq (s_0 S_s)^{\frac{\gamma_i}{2^*_s}}. \]

From (15),

\[ c \geq -bT^4 + \frac{s}{N}(s_0 S_s)^{\frac{\gamma_i}{2^*_s}}, \]

which is a contradiction with \( c \in (0, c^*) \) by (14). Therefore, \( \mathcal{J} = \emptyset \), which implies that \( u_n \to u \) strongly in \( L^{2^*_s}(\Omega) \).

Finally, by \( J_T'(u_n) \to 0 \), we have

\[ (J_T'(u_n), (u_n - u)) = o(1). \]

Together with \( u_n \to u \) strongly in \( L^{2^*_s}(\Omega) \), it is easy to verify that

\[ \left\{ 1 + b\|u_n\|^2 \Phi_T(u_n) + \frac{b}{T^2}\|u_n\|^4 \phi\left( \frac{\|u_n\|^2}{T^2} \right) \right\} \int_{\mathbb{R}^N} (-\Delta) \dot{u}_n(-\Delta) \dot{(u_n - u)} = o(1). \]
Moreover, there exist constants \( \rho > q \). In particular, if \( X \) is a decomposition of the space \( X_0^s(\Omega) \), we get
\[
\int_{\mathbb{R}^N} (-\Delta)^{2\nu} u_n(-\Delta)^{2\nu}(u_n - u) = o(1).
\]
On the other hand, by \( u_n \to u \) weakly in \( X_0^s(\Omega) \), we get
\[
\int_{\mathbb{R}^N} (-\Delta)^{2\nu} u(-\Delta)^{2\nu}(u_n - u) = o(1).
\]
Therefore, we get that \( u_n \to u \) strongly in \( X_0^s(\Omega) \). The proof is completed. \( \square \)

3.3. The linking geometry. To prove our main result by Lemma 2.3, we give a decomposition of the space \( X_0^s(\Omega) \) and then verify the second condition in Lemma 2.3.

Set
\[
W_k = \bigoplus_{n=1}^{k} M(\lambda_n), \quad V_k = \bigoplus_{n \geq k} M(\lambda_n),
\]
where \( M(\lambda_n) \) denotes the eigenfunction space of \((-\Delta)^{2\nu}\) corresponding to \( \lambda_n \).

**Lemma 3.2.** For \( b > 0 \) small enough and \( \lambda < \lambda_k \), there holds that
\[
\max_{u \in W_k} J_T(u) \leq r := \frac{s}{N} (\lambda_k - \lambda) \frac{2^{*}}{2} |\Omega|.
\]
In particular, if \( q = 4 \), for any \( b > 0 \) we have
\[
\max_{u \in W_k} J_T(u) \leq r := \frac{s}{N} (\lambda_k - \lambda) \frac{2^{*}}{2} |\Omega|, \quad \text{provided} \quad \mu > 0 \text{ large}.
\]
Moreover, there exist constants \( \rho > 0, \delta \in (0, r) \) such that \( J_T(u) \geq \delta \) for all \( u \in V_k \) with \( \|u\| = \rho \).

**Proof.** For any fixed \( k \in \mathbb{N} \), it follows from \( W_k \) being a finite dimension space that the norms are equivalent in \( W_k \). So, there exists a constant \( l > 0 \) such that \( \|u\|_q \geq l \|u\|_q^* \) for all \( u \in W_k \).

Now, we claim that for \( b > 0 \) small enough, we have
\[
J_T(u) \leq \frac{\lambda_k - \lambda}{2} \|u\|_2^2 + \left( \frac{b}{4} \|u\|^{4-q} \Phi_T(u) - \frac{\mu l}{q} \right) \|u\|_q^* - \frac{1}{2^*_s} \|u\|_2^*.
\]
By (12), we have
\[
J_T(u) \leq \frac{\lambda_k - \lambda}{2} \|u\|_2^2 + \left( \frac{b}{4} \|u\|^{4-q} \Phi_T(u) - \frac{\mu l}{q} \right) \|u\|_q^* - \frac{1}{2^*_s} \|u\|_2^*.
\]
**Case 1.** \( \|u\|^2 > 2T^2 \). By the definition of \( \Phi_T(u) \), we get that for any \( \mu > 0 \),
\[
\frac{b}{4} \|u\|^{4-q} \Phi_T(u) - \frac{\mu l}{q} = -\frac{\mu l}{q} < 0.
\]
So Claim (18) holds.

**Case 2.** \( \|u\|^2 \leq 2T^2 \). By Remark 1, we have for any \( \mu > 0 \),
\[
\frac{b}{4} \|u\|^{4-q} \Phi_T(u) - \frac{\mu l}{q} \leq O(b^{2\nu}) - \frac{\mu l}{q} \to -\frac{\mu l}{q} < 0 \quad \text{as} \quad b \to 0.
\]
So Claim (18) holds.

From the Hölder inequality, it follows from (18) that
\[
J_T(u) \leq \frac{\lambda_k - \lambda}{2} \|\Omega\|^{2\nu}_2 \|u\|_2^2 + \frac{1}{2^*_s} \|u\|_2^* \quad \text{for} \quad u \in W_k.
\]
Moreover, one can get that for \( b > 0 \) small enough,
\[
J_T(u) \leq \frac{s}{N} (\lambda_k - \lambda) \frac{s}{2} |\Omega|, \quad u \in W_k.
\]
In particular, if \( q = 4 \), then
\[
J_T(u) \leq \frac{\lambda_k - \lambda}{2} \|u\|_2^2 + \left( \frac{b}{4} \Phi_T(u) - \frac{\mu l}{4} \right) \|u\|^4 - \frac{1}{2s} \|u\|_2^2.
\]
Let \( \mu > \frac{b}{4} := \mu_1 \), then
\[
J_T(u) \leq \frac{\lambda_k - \lambda}{2} \|u\|_2^2 - \frac{1}{2s} \|u\|_2^2.
\]
Similar as above, for any \( b > 0 \) and \( \mu > \frac{b}{4} := \mu_1 \), we have
\[
J_T(u) \leq \frac{s}{N} (\lambda_k - \lambda) \frac{s}{2} |\Omega|, \quad u \in W_k.
\]
So the first assertion of Lemma 3.2 holds. Let \( u \in V_k \), it follows from (11) and the Sobolev embedding theorem that there exists a constant \( C > 0 \) such that
\[
J_T(u) \geq \frac{1}{2} \left( \frac{\lambda_k - \lambda}{\lambda_k} \right) \|u\|^2 - \frac{\mu C}{q} \|u\|^q - \frac{1}{2s} \|u\|_2^2,
\]
which implies that the second assertion of Lemma 3.2 is true. The proof is completed. \( \square \)

4. Proof of the main result. In this section, we use the linking theorem to prove the existence of multiple solutions to the truncated problem. Then, we show that these solutions are also the solutions to the original problem.

Proof of Theorem 1.1. Let \( W_k, V_k \) be defined as above. Taking any fixed \( t_0 \in (0, s/N) \), \( s_0 \in (0, 1) \), from Lemma 3.1 and Lemma 3.2, we deduce that if \( b > 0 \) small, (2) of Lemma 2.3 holds for \( \mu > 0 \),
\[
\lambda \in (\lambda_k - \lambda^*_1, \lambda_k), \quad r = \frac{s}{N} (\lambda_k - \lambda) \frac{s}{2} |\Omega|,
\]
where \( \lambda^*_1 = \left( \frac{N_0}{s(N)} \right)^\frac{2}{N} s_0 S_k \). In particular, if \( q = 4 \) when \( N \leq 4s \), then with an additional assumption \( \mu > 0 \) large, similar assertion above holds for any \( b > 0 \).

Let \( m = \text{dim} W_k - \text{codim} V_k \), which is the multiplicity of \( \lambda_k \). Then by Lemma 2.3 and 3.1 there exist \( m \) distinct pairs of nontrivial critical points \( \pm u_j, \quad j = 1, \cdots, m \) of \( J_T \) such that
\[
0 < J_T(u_j) \leq r.
\]
By the definition of \( \lambda^* \), for \( \lambda \in (\lambda_k - \lambda^*, \lambda_k) \)
\[
r < \min \left\{ c^*, \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) T^2 \right\}. \quad (19)
\]
Now, it is enough to show that \( \|u_j\| < T \). By \( J_T(u_j) < c^* \) and \( J_T'(u_j) = 0 \), we have
\[
c^* > J_T(u_j) - \frac{1}{2} \langle J_T'(u_j), u_j \rangle \geq -bT^4 + \mu \left( \frac{1}{2} - \frac{1}{q} \right) \|u_j\|^q,
\]
which implies that as \( b \to 0 \),
\[
\|u_j\|_q^q \leq \mu^{-1} \left( \frac{1}{2} - \frac{1}{q} \right)^{-1} (c^* + bT^4) = O(1).
\]
Thus, together with Hölder’s inequality, \( \lambda_k \|u_j\|^2 < \frac{T^2}{4} \) for \( b > 0 \) small enough. On the other hand, one can check that if \( q \in (2, 4) \),

\[
b \left( \frac{4}{q} - 1 \right) T^4 < \frac{T^2}{4} \left( \frac{1}{2} - \frac{1}{q} \right)
\]

holds for \( b \) small and if \( q = 4 \),

\[
b \left( \frac{4}{q} - 1 \right) T^4 < \frac{T^2}{4} \left( \frac{1}{2} - \frac{1}{q} \right)
\]

holds for any \( b > 0 \). Therefore, for \( b \) small if \( q \in (2, 4) \) or any \( b > 0 \) if \( q = 4 \),

\[
J_T(u_j) - \frac{1}{q} \langle J'_T(u_j), u_j \rangle
= \left( \frac{1}{2} - \frac{1}{q} \right) \|u_j\|^2 + b \left( \frac{1}{4} - \frac{1}{q} \right) \|u_j\|^4 \Phi_T(u_j)
- \frac{b}{2qT^2} \|u_j\|^6 \phi' \left( \frac{\|u_j\|^2}{T^2} \right) - \left( \frac{1}{2} - \frac{1}{q} \right) \lambda \|u_j\|^2 - \left( \frac{1}{2} - \frac{1}{q} \right) \|u_j\|^2
> \left( \frac{1}{2} - \frac{1}{q} \right) \|u_j\|^2 - b \left( \frac{4}{q} - 1 \right) T^4 - \left( \frac{1}{2} - \frac{1}{q} \right) \lambda_k \|u_j\|^2
> \left( \frac{1}{2} - \frac{1}{q} \right) \|u_j\|^2 - \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) T^2.
\]

Combining with (19), we get \( \|u_j\| < T \). Thus, \( J_T(u_j) = I(u_j) \) and \( u_j \) is a nontrivial solution of the original problem (9). The proof is completed.

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MULTIPLE SOLUTIONS OF FRACTIONAL KIRCHHOFF EQUATIONS

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