A NONLINEAR LAZAREV–LIEB THEOREM:
L²-ORTHOGONALITY VIA MOTION PLANNING

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Abstract. Lazarev and Lieb showed that finitely many integrable functions from the unit interval to \( \mathbb{C} \) can be simultaneously annihilated in the \( L^2 \) inner product by a smooth function to the unit circle. Here we answer a question of Lazarev and Lieb proving a generalization of their result by lower bounding the equivariant topology of the space of smooth circle-valued functions with a certain \( W^{1,1} \)-norm bound. Our proof uses a relaxed notion of motion planning algorithm that instead of contractibility yields a lower bound for the \( \mathbb{Z}/2 \)-coindex of a space.

1. Introduction

In 1965 Hobby and Rice established the following result:

**Theorem 1.1** (Hobby and Rice [4]). Let \( f_1, \ldots, f_n \in L^1([0,1]; \mathbb{R}) \). Then there exists \( h: [0,1] \to \{ \pm 1 \} \) with at most \( n \) sign changes, such that for all \( j \),

\[
\int_0^1 f_j(x) h(x) \, dx = 0.
\]

If we restrict the \( f_j \) to lie in \( L^2([0,1]; \mathbb{R}) \), we can view this as an orthogonality result in the \( L^2 \) inner product. The Hobby–Rice theorem and its generalizations have found a multitude of applications, ranging from mathematical physics [6] and combinatorics [1] to the geometry of spatial curves [2].

The theorem also holds for \( f_1, \ldots, f_n \in L^1([0,1]; \mathbb{C}) \), provided \( h \) is allowed \( 2n \) sign changes, by splitting the \( f_j \) into real and imaginary parts. Lazarev and Lieb showed that for complex-valued \( f_j \), the function \( h \) can be chosen in \( C^\infty([0,1]; S^1) \), where \( S^1 \) denotes the unit circle in \( \mathbb{C} \):

**Theorem 1.2** (Lazarev and Lieb [5]). Let \( f_1, \ldots, f_n \in L^1([0,1]; \mathbb{C}) \). Then there exists \( h \in C^\infty([0,1]; S^1) \) such that for all \( j \),

\[
\int_0^1 f_j(x) h(x) \, dx = 0.
\]

If \( h \) is obtained by smoothing the function \( h_0 \) guaranteed by Theorem 1.1, then we would expect its \( W^{1,1} \)-norm, given by

\[
\| h \|_{W^{1,1}} = \int_0^1 |h(x)| \, dx + \int_0^1 |h'(x)| \, dx
\]

to be approximately \( 1 + 2\pi n \), since \( |h(x)| = 1 \), and each sign change of \( h_0 \) contributes approximately \( \pi \) to \( \int_0^1 |h'(x)| \, dx \). However, Lazarev and Lieb did not establish any bound on the \( W^{1,1} \)-norm of \( h \) and left this as an open problem; this was accomplished by Rutherfoord [9], who established a bound of \( 1 + 5\pi n \). Here we improve this bound to \( 1 + 2\pi n \); see Corollary 1.4.

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The Hobby–Rice theorem has a simple proof due to Pinkus [8] via the Borsuk–Ulam theorem, which states that any map $f: S^n \rightarrow \mathbb{R}^n$ with $f(-x) = -f(x)$ for all $x \in S^n$ has a zero. Lazarev and Lieb asked whether there is a similar proof of their result and write: “There seems to be no way to adapt the proof of the Hobby–Rice Theorem (which involves a fixed-point argument).” Rutherford [9] offered a simplified proof of Theorem 1.2 based on Brouwer’s fixed point theorem. Here we give a proof using the Borsuk–Ulam theorem directly, which adapts Pinkus’ proof of the Hobby–Rice theorem. The advantage of this approach is that our main result gives a nonlinear extension of the result of Lazarev and Lieb; see Section 4 for the proof:

**Theorem 1.3.** Let $\psi: C^\infty([0,1]; S^1) \rightarrow \mathbb{R}^n$ be continuous with respect to the $L^1$-norm such that $\psi(-h) = -\psi(h)$ for all $h \in C^\infty([0,1]; S^1)$. Then there exists $h \in C^\infty([0,1]; S^1)$ with $\psi(h) = 0$ and $\|h\|_{W^{1,1}} \leq 1 + \pi n$.

This is a non-linear extension of Theorem 1.2 since for given $f_1, \ldots, f_n \in L^1([0,1]; \mathbb{C})$ the map $\psi(h) = (\int_0^1 f_j(x)h(x)dx)_j$ is continuous (see Section 2) and linear, so in particular, $\psi$ satisfies $\psi(-h) = -\psi(h)$. Using the $L^1$-norm is no restriction; as we show in the next section, the $L^p$ norms on $C^\infty([0,1]; S^1)$ for $1 \leq p < \infty$ are all equivalent, so we could replace $L^1$ with any such $L^p$. In fact, the only relevant feature of the $L^1$-norm is that functions $h_1, h_2$ are close in the $L^1$-norm if $h_1, h_2$ are uniformly close outside of a set of small measure. As a consequence, we recover the result of Lazarev and Lieb, with a $W^{1,1}$-norm bound of $1 + 2\pi n$ since $\psi$ takes values in $\mathbb{C}^n \cong \mathbb{R}^{2n}$; see Section 2 for the proof:

**Corollary 1.4.** Let $f_1, \ldots, f_n \in L^1([0,1]; \mathbb{C})$. Then there exists $h \in C^\infty([0,1]; S^1)$ with $\|h\|_{W^{1,1}} \leq 1 + 2\pi n$ such that for all $j$,

$$\int_0^1 f_j(x)h(x)dx = 0.$$  

Given a space $Z$ with a $\mathbb{Z}/2$-action $\sigma: Z \rightarrow Z$, the largest integer $n$ such that the $n$-sphere $S^n$ with the antipodal $\mathbb{Z}/2$-action (i.e. $x \mapsto -x$) admits a continuous map $f: S^n \rightarrow Z$ with $f(-x) = \sigma(f(x))$ for all $x \in S^n$ is called the $\mathbb{Z}/2$-coindex of $Z$, denoted $\text{coind} Z$. We show that the coindex of the space of smooth $S^1$-valued functions in the $L^1$-norm with $W^{1,1}$-norm at most $1 + \pi n$ is between $n$ and $2n - 1$; see Theorem 6.2. Determining the coindex exactly remains an interesting open problem. Our proof proceeds by constructing $\mathbb{Z}/2$-maps from $S^n$, i.e., commuting with the antipodal $\mathbb{Z}/2$-actions, via elementary obstruction theory, that is, inductively dimension by dimension.

We find it illuminating to phrase our proof using the language of motion planning algorithms. A motion planning algorithm (mpa) for a space $Z$ is a continuous choice of connecting path for any two endpoints in $Z$; see Section 3 for details and Farber [3] for an introduction. An mpa for $Z$ exists if and only if $Z$ is contractible. Here we introduce the notion of (full) lifted mpa, which does not imply contractibility but is sufficiently strong to establish lower bounds for the coindex of $Z$; see Theorem 3.5. We refer to Section 3 for details.

### 2. Relationship between topologies on $C^\infty([0,1]; S^1)$

We now make precise our introductory comments about the topologies on $C^\infty([0,1]; S^1)$ induced by the various $L^p$-norms and the $d_{0,\infty}$ metric.
Proposition 2.1. The $L^p$-norms for $1 \leq p < \infty$ induce equivalent topologies on $C^\infty([0, 1]; S^1)$.

Proof. For $1 \leq p < \infty$, let $Z_p$ be $C^\infty([0, 1]; S^1)$, equipped with the topology induced by the $L^p$-norm. Note that $\|h\|_p < \infty$ for all $h \in C^\infty([0, 1]; S^1)$, so the identity maps $1_{p,q}: Z_p \to Z_q$ are well-defined as functions. It suffices to show that $1_{p,q}$ is continuous for all $p, q \in [1, \infty)$.

It is a standard fact that $1_{p,q}$ is continuous for $p \geq q$ when the domain has finite measure, as is the case here for $[0, 1]$. For $p < q$, we have

$$
\|h_2 - h_1\|_q = \left(\int_0^1 |h_2(x) - h_1(x)|^q \, dx\right)^{1/q} \\
\leq \left(\int_0^1 |h_2(x) - h_1(x)|^p \cdot (\text{diam}(S^1))^{q-p} \, dx\right)^{1/q} \\
\leq (\text{diam}(S^1))^{(q-p)/q} \cdot \|h_2 - h_1\|_p^{p/q}
$$

Since $S^1$ is bounded, $1_{p,q}$ is continuous. Hence the $Z_p$ are all homeomorphic. □

In the introduction, we claimed that “the only relevant feature of the $L^1$-norm is that functions $h_1, h_2$ are close in the $L^1$-norm if $h_1$, $h_2$ are uniformly close outside of a set of small measure.” To give content to this statement, we define a metric $d_{0,\infty}$ on $C^\infty([0, 1]; S^1)$ by

$$
d_{0,\infty}(h_1, h_2) = \inf\{\delta > 0 : |h_2(x) - h_1(x)| < \delta \text{ for all } x \in [0, 1] \setminus S, \text{ for some } S \subseteq [0, 1] \text{ with } \mu(S) < \delta\}.
$$

Proposition 2.2. The function $d_{0,\infty}$ is a metric.

Proof. By the continuity of maps in $C^\infty([0, 1]; S^1)$, we have $d_{0,\infty}(h_1, h_2) = 0$ iff $h_1 = h_2$. For the triangle inequality, suppose:

- $|h_2(x) - h_1(x)| < \delta_1$ for all $x \in [0, 1] \setminus S_1$, where $\mu(S_1) < \delta_1$.
- $|h_3(x) - h_2(x)| < \delta_2$ for all $x \in [0, 1] \setminus S_2$, where $\mu(S_2) < \delta_2$

Then $|h_3(x) - h_1(x)| < \delta_1 + \delta_2$ for all $x \in [0, 1] \setminus (S_1 \cup S_2)$, and $\mu(S_1 \cup S_2) < \delta_1 + \delta_2$. Hence $d_{0,\infty}(h_1, h_3) \leq \delta_1 + \delta_2$. Taking the infimum over $\delta_1, \delta_2$, we obtain $d_{0,\infty}(h_1, h_3) \leq d_{0,\infty}(h_1, h_2) + d_{0,\infty}(h_2, h_3)$. □

Proposition 2.3. The metric $d_{0,\infty}$ and the norm $\|\cdot\|_1$ induce equivalent topologies on $C^\infty([0, 1]; S^1)$.

Proof. Let $Z_{0,\infty}$ be $C^\infty([0, 1]; S^1)$, equipped with the topology induced by $d_{0,\infty}$; it suffices to show that the identity maps between $Z_{0,\infty}, Z_1$ are continuous.

For the identity map $1: Z_{0,\infty} \to Z_1$, suppose $d_{0,\infty}(h_1, h_2) < \delta$, so that there exists $S \subseteq [0, 1]$ with $\mu(S) < \delta$ such that $|h_2(x) - h_1(x)| < \delta$ on $[0, 1] \setminus S$. Then

$$
\int_0^1 |h_2(x) - h_1(x)| \, dx \leq \int_S \text{diam}(S^1) \, dx + \int_{[0, 1] \setminus S} \delta \, dx \leq \delta(\text{diam}(S^1) + 1).
$$

This shows that $1: Z_{0,\infty} \to Z_1$ is continuous.

For the identity map $1: Z_1 \to Z_{0,\infty}$, let $\varepsilon > 0$ and suppose $\|h_2 - h_1\|_1 < \delta$ for $\delta = \varepsilon^2$. If $d_{0,\infty}(h_1, h_2) \geq \varepsilon$, then $|h_2(x) - h_1(x)| \geq \varepsilon$ on a set $S$ with $\mu(S) \geq \varepsilon$, implying $\|h_2 - h_1\|_1 \geq \varepsilon^2$, a contradiction. Hence $d_{0,\infty}(h_1, h_2) < \varepsilon$, and $1: Z_1 \to Z_{0,\infty}$ is continuous. □
Now we expand our view to consider $L^p$ spaces under other measures $\mu$. We show that finite, absolutely continuous measures can only produce coarser topologies than Lebesgue measure:

**Proposition 2.4.** Let $\mu$ be a finite measure on $[0,1]$ that is absolutely continuous with respect to Lebesgue measure. Let $Z_1$ be $C^\infty([0,1];S^1)$, equipped with the topology induced by the $L_1$-norm with respect to Lebesgue measure, and let $Z_{1,\mu}$ be $C^\infty([0,1];S^1)$, equipped with the topology induced by the $L_1$-norm with respect to $\mu$. Then the identity function $1: Z_1 \to Z_{1,\mu}$ is continuous.

**Proof.** By Proposition 2.3, it suffices to show that $1: Z_{0,\infty} \to Z_{1,\mu}$ is continuous. The argument is similar to the argument that $1: Z_{0,\infty} \to Z_1$ is continuous. Using $\lambda$ to denote Lebesgue measure, suppose $d_{0,\infty}(h_1, h_2) < \delta$, so that there exists $S \subseteq [0, 1]$ with $\lambda(S) < \delta$ such that $|h_2(x) - h_1(x)| < \delta$ on $[0, 1] \setminus S$. Then

$$\int_{[0,1]} |h_2(x) - h_1(x)|d\mu \leq \int_S \text{diam}(S^1)d\mu + \int_{[0,1]\setminus S} \delta d\mu \leq \text{diam}(S^1)\mu(S) + \delta\mu([0,1])$$

Note that since $\mu$ is finite, we have $\mu([0,1]) < \infty$. As $\delta \to 0$, we have $\lambda(S) \to 0$, so $\mu(S) \to 0$ by absolute continuity, hence the right side approaches 0. This shows the desired continuity. \qed

The relationships between the topologies on $C^\infty([0,1];S^1)$ can be summarized as follows, where $1 < p_1 < p_2 < \infty$ and $\mu$ is a finite measure on $[0,1]$ which is absolutely continuous with respect to Lebesgue measure:

$$Z_{\infty} \not\rightarrow Z_{p_2} \leftarrow \cong Z_{p_1} \leftarrow \cong Z_1 \leftarrow \cong Z_{0,\infty}$$

$$Z_{p_2,\mu} \leftarrow \cong Z_{p_1,\mu} \leftarrow \cong Z_{1,\mu}$$

Therefore, when establishing the continuity of $\psi$ for the sake of applying Theorem 1.3, we may use any $L^p$ norm on $C^\infty([0,1];S^1)$, with respect to any finite measure $\mu$ on $[0,1]$ which is absolutely continuous with respect to Lebesgue measure. (If we use a measure $\mu$ other than Lebesgue measure, we can precompose $\psi$ with $1: Z_1 \to Z_{1,\mu}$ before applying Theorem 1.3.)

With these results in hand, we can now deduce Corollary 1.4 from Theorem 1.3:

**Proof of Corollary 1.4.** Let $\psi: C^\infty([0,1];S^1) \to \mathbb{C}^n$ be given by component maps

$$\psi_j: h \mapsto \int_0^1 f_j(x)h(x)dx.$$  

We claim $\psi_j$ is continuous. Since $f_j \in L^1([0,1];\mathbb{C})$, $f_j$ induces a finite measure $\mu_f$ which is absolutely continuous with respect to Lebesgue measure, given by

$$\mu_f(S) = \int_0^1 |f_j(x)|dx.$$
By the above, we may view $C^\infty([0,1];S^1)$ as having the topology induced by the $L^1$-norm $\| \cdot \|_1$ with respect to $\mu_f$. Then

$$|\psi_j(h_2) - \psi_j(h_1)| \leq \int_0^1 |f_j(x)| \cdot |h_2(x) - h_1(x)|dx$$

$$\leq \int_{[0,1]} |h_2 - h_1|d\mu_f$$

$$\leq \|h_2 - h_1\|_1.$$ 

Therefore, $\psi_j$ is continuous, so $\psi$ is continuous. Viewing the codomain $\mathbb{C}^n$ of $\psi$ as $\mathbb{R}^{2n}$, we may apply Theorem 1.3 and get $\|h\|_{W^{1,1}} \leq 1 + 2\pi n$. \qed

3. Lifts of motion planning algorithms and the coindex

Our proof of Theorem 1.3 makes use of motion planning algorithms; see Farber [3]. We use $Y,Z$ in the following definitions to match our notation later:

**Definition 3.1.** Let $Z$ be a topological space, and let $PZ$ be the space of continuous paths $\gamma: [0,1] \to Z$, equipped with the compact-open topology. Then a motion planning algorithm (or mpa) is a continuous map $s: Z \times Z \to PZ$, such that $s(z_0, z_1)(0) = z_0$ and $s(z_0, z_1)(1) = z_1$.

For $Z$ a locally compact Hausdorff space, using the compact-open topology for $PZ$ ensures that a function $s: Z \times Z \to PZ$ is continuous if and only if its uncurried form $\tilde{s}: Z \times Z \times [0,1] \to Z$ given by $(z_0, z_1, t) \mapsto s(z_0, z_1)(t)$ is continuous; see Munkres [7, Thm. 46.11]. One basic fact is that an mpa for $Z$ exists if and only if $Z$ is contractible [3].

We weaken the definition above for our purposes:

**Definition 3.2.** Let $Y,Z$ be topological spaces, and let $\phi: Y \to Z$ be continuous. Let $(\preceq)$ be a preorder on $Y$, and let $Y^2_\preceq = \{(y_0, y_1) \in Y^2 : y_0 \preceq y_1\}$, giving $Y^2$ the product topology and $Y^2_\preceq$ the resulting subspace topology.

A lifted motion planning algorithm (or lifted mpa) for $(Y,Z,\phi,\preceq)$ is a family of maps $s_w: Y^2_\preceq \to PY$ for $w \in (0,1]$ with $s_w(y_0, y_1)(0) = y_0$ and $s_w(y_0, y_1)(1) = y_1$, assembling into a continuous map $s: (0,1] \times Y^2_\preceq \to PY$, with the following continuity property:

For all $y \in Y$ and all neighborhoods $V$ of $\phi(y) \in Z$,

there exists a neighborhood $U$ of $\phi(y) \in Z$ and $\delta > 0$ such that:

if $\phi(y_0), \phi(y_1) \in U$, $w < \delta$,

then $\phi(s_w(y_0, y_1)(t)) \in V$ for all $t \in [0,1]$.

**Definition 3.3.** A lifted mpa $s: (0,1] \times Y^2_\preceq \to PY$ for $(Y,Z,\phi,\preceq)$ is full if $y_0 \preceq y_1$ for all $y_0, y_1 \in Y$.

In this case we say $s$ is a full lifted mpa for $(Y,Z,\phi)$, omitting $(\preceq)$.

The continuity property essentially says that if two points $y_1, y_2 \in Y$ have images in $Z$ close to $\phi(y) \in Z$, then $s_w$ carries $(y_0, y_1)$ to a path whose image under $\phi$ is a path that stays close to $\phi(y)$, provided $w$ is small.
Note that an mpa $s: Z \times Z \to PZ$ satisfying $s(z, z) = e_z$ for all $z \in Z$ extends to a full lifted mpa for $(Z, Z, 1_Z)$ by taking $s_w = s$ for all $w$; the continuity property just restates the continuity of $s$ at diagonal points $(z, z) \in Z \times Z$.

This relaxed notion of mpa still provides lower bounds for the (equivariant) topology of $Z$ that are weaker than contractibility. Recall that for a topological space $Z$ with $\mathbb{Z}/2$-action generated by $\sigma: Z \to Z$ the $\mathbb{Z}/2$-coindex of $Z$ denoted by $\text{coind} Z$ is the largest integer $n$ such that there is a $\mathbb{Z}/2$-map $f: S^n \to Z$, that is, a map satisfying $f(\alpha) = \sigma(f(x))$.

**Definition 3.4.** Let $x \in S^k$, and let $x = (x_1, \ldots, x_{k+1})$. We say that $x$ is **positive** if its last nonzero coordinate is positive, and **negative** otherwise.

Our main tool in proving Theorem 1.3 will be the following theorem:

**Theorem 3.5.** Let $Y, Z$ be topological spaces, equip $Y$ with a $\mathbb{Z}$-action generated by $\rho: Y \to Y$, and equip $Z$ with a $\mathbb{Z}/2$-action generated by $\sigma: Z \to Z$. Let $\phi: Y \to Z$ be continuous and equivariant, i.e., $\sigma \circ \phi = \phi \circ \rho$. Let $(\preceq)$ be a preorder on $Y$ and $s: (0, 1] \times Y^2_2 \to PY$ a lifted mpa for $(Y, Z, \phi, \preceq)$ such that:

1. $y \preceq \rho(y)$.
2. $\rho(y_0) \preceq \rho(y_1)$ if and only if $y_0 \preceq y_1$.
3. $y_0 \preceq y_1$ implies $y_0 \preceq s_w(y_0, y_1)(t) \preceq y_1$, for all $w \in (0, 1]$, $t \in [0, 1]$.

Then for each integer $n \geq 0$, there exists a $\mathbb{Z}/2$-map $\beta_n: S^n \to Z$. Moreover, for any choice of initial point $y^* \in Y$, the maps $\beta_n$ can be chosen such that $\beta_n$ maps each positive point of $S^n$ to a point in $Z$ of the form $\phi(y)$, with $y^* \preceq y \preceq \rho^n(y^*)$, that is, the subspace of these points $\phi(y)$ and their antipodes $\sigma(\phi(y))$ in $Z$ has coindex at least $n$.

We will apply Theorem 3.5 by taking $Z$ to be $C^\infty([0, 1]; S^1)$ with the topology induced by the $L^1$-norm, and $Y$ to be $C^\infty([0, 1]; \mathbb{R})$ with the $L^1$-norm, restricted to increasing functions. Using lifted mpa’s allows us to reason about paths in $Y$, which are simpler than paths in $Z$. The theorem encapsulates the inductive construction of a function $\alpha_n: S^n \to Y$, from which we produce $\beta_n: S^n \to Z$; the continuity property of a lifted mpa is needed for this construction to work. The last part of the theorem will give us the $W^{1,1}$-norm bound.

**Proof of Theorem 3.5.** We will inductively construct a function $\alpha_n: S^n \to Y$ and then take $\beta_n = \phi \circ \alpha_n$. We will allow $\alpha_n$ to be discontinuous on the equator of $S^n$, but in such a way that $\phi \circ \alpha_n$ is continuous everywhere.

Specifically, let $\alpha_k: S^k \to Y$ be a function, not necessarily continuous. Let $m: S^k \to S^k$ be given by $(x_1, \ldots, x_k, x_{k+1}) \mapsto (x_1, \ldots, x_k, -x_{k+1})$, so that $m$ mirrors points across the plane perpendicular to the last coordinate axis. Then we say that $\alpha_k$ is **good** if

- **(α-1)** For $x$ positive, $y^* \preceq \alpha_k(x) \preceq \rho^k(y^*)$, and $\alpha_k(-x) = \rho(\alpha_k(x))$.
- **(α-2)** For $x$ in the open upper hemisphere, $\alpha_k(x) \preceq \alpha_k(m(x))$.
- **(α-3)** $\alpha_k$ is continuous on the open upper hemisphere.
- **(α-4)** $\phi \circ \alpha_k$ is continuous.
Let $u, l: B^{k+1} \to S^k$ be the projections to the closed upper and lower hemispheres, that is, $u(x)$ is the unique point in the closed upper hemisphere sharing its first $k$ coordinates with $x$, and similarly for $l(x)$ for the lower hemisphere. Then we have the following claim:

**Claim.** If $\alpha_k: S^k \to Y$ is good, then $\alpha_k$ extends to $\tilde{\alpha}_k: B^{k+1} \to Y$, such that:

$(\tilde{\alpha}-1)$ For all $x \in B^{k+1}$, we have $y^* \preceq \tilde{\alpha}_k(x) \preceq \rho^{k+1}(y^*)$.

$(\tilde{\alpha}-2)$ For all $x \in B^{k+1}$, we have $\alpha_k(u(x)) \preceq \tilde{\alpha}_k(x) \preceq \alpha_k(l(x))$.

$(\tilde{\alpha}-3)$ $\tilde{\alpha}_k$ is continuous in the interior of $B^{k+1}$.

$(\tilde{\alpha}-4)$ $\phi \circ \tilde{\alpha}_k$ is continuous.

**Proof of Claim.** Let $E \subset S^k$ be the equator, the set of points neither in the open upper or lower hemisphere. The set $E$ is compact, so the distance $d(x, E)$ for $x \in B^{k+1}$ is well-defined and nonzero for $x \notin E$. Define $\tilde{\alpha}_k: B^{k+1} \to X_{k+1}$ by

$$
\tilde{\alpha}_k(x) = \begin{cases} 
\alpha_k(x) & x \in E \\
 s_w(x)(\alpha_k(u(x)), \alpha_k(l(x)))(t(x)) & x \notin E
\end{cases}
$$

where $w(x) = \min(d(x, E), t(x), 1 - t(x))$

$$
t(x) = \frac{d(u(x), x)}{d(u(x), l(x))}
$$

Note that $l(x) = m(u(x))$, so $(\tilde{\alpha}-2)$ implies $\alpha_k(u(x)) \preceq \tilde{\alpha}_k(x) \preceq \alpha_k(l(x))$, so $s_w(x)(\alpha_k(u(x)), \alpha_k(l(x)))$ is well-defined, and $(3)$ gives $\alpha_k(u(x)) \preceq \tilde{\alpha}_k(x) \preceq \alpha_k(l(x))$, establishing $(\tilde{\alpha}-2)$.

By $(\tilde{\alpha}-1)$, we have $\rho(y^*) \preceq \rho(\alpha_k(x)) \preceq \rho^{k+1}(y^*)$ for $x$ negative, so $y^* \preceq \alpha_k(x) \preceq \rho^{k+1}(y^*)$ for all $x \in S^k$. Along with the inequality above, this implies $y^* \preceq \tilde{\alpha}_k(x) \preceq \rho^{k+1}(y^*)$, establishing $(\tilde{\alpha}-1)$.

The function $\tilde{\alpha}_k$ is continuous for $x \notin E$, since $u(-), l(-), d(-,-), d(-, E)$ are all continuous, $u(x), l(x) \notin E$, and $\alpha_k$ is continuous on the open upper (and hence lower) hemisphere. In particular, $\tilde{\alpha}_k$ is continuous in the interior of $B^{k+1}$, establishing $(\tilde{\alpha}-3)$.

It remains to show $\phi \circ \tilde{\alpha}_k$ is continuous at $x \in E$. Let $V$ be a neighborhood of $\phi(\tilde{\alpha}_k(x)) = \phi(\alpha_k(x)) \in Z$, and obtain $\delta > 0$ and a neighborhood $U$ of $\phi(\alpha_k(x)) \in Z$ as in the lifted mpa definition. Since $u(-), l(-), d(-, E)$ are continuous, there exists a neighborhood $W \subseteq B^{k+1}$ of $x$ such that for all $x' \in W$ we have $d(x', E) < \delta$ and $u(x'), l(x') \in (\phi \circ \alpha_k)^{-1}(U)$, using the continuity of $\phi \circ \alpha_k$ given by $(\tilde{\alpha}-4)$. Then $\phi(\alpha_k(u(x'))), \phi(\alpha_k(l(x'))) \in U$, so the lifted mpa property implies $\phi(\tilde{\alpha}_k(x)) \in V$, which shows $\phi \circ \tilde{\alpha}_k$ is continuous at $x$, establishing $(\tilde{\alpha}-4)$.

We use the claim above to inductively construct $\alpha_k: S^k \to Y$, by extending each $\alpha_k$ to a map $\tilde{\alpha}_k: B^{k+1} \to Y$, using $\tilde{\alpha}_k$ for the upper hemisphere of $\alpha_{k+1}$, and extending to the negative hemisphere via $\alpha_{k+1}(-x) = \rho(\alpha_{k+1}(x))$. Specifically, we have the following claim:

**Claim.** For all $k \geq 0$ there exists $\alpha_k: S^k \to Y$, not necessarily continuous, such that $\alpha_k$ is good.

**Proof of Claim.** We use induction. For the base case, use $\pm 1$ to denote the points of $S^0$; then let $\alpha_0$ map $\pm 1$ to $y^*, \rho(y^*)$, respectively. Then $\alpha_0$ is good.

Given $\alpha_k$ good and $\tilde{\alpha}_k$ obtained through the previous claim, we now construct $\alpha_{k+1}: S^{k+1} \to Y$. Let $\pi: S^{k+1}_{\geq 0} \to B^{k+1}$ be the projection of the closed upper hemisphere onto the first $k+1$ coordinates.
We define maps on the two closed hemispheres as follows:

\[(\alpha_{k+1})_{\geq 0}: S_{\geq 0}^{k+1} \to Y \quad x \mapsto \tilde{\alpha}_k(\pi(x))\]

\[(\alpha_{k+1})_{\leq 0}: S_{\leq 0}^{k+1} \to Y \quad x \mapsto \rho(\tilde{\alpha}_k(\pi(-x)))\]

Finally, we define \(\alpha_{k+1}\) by \(x \mapsto (\alpha_{k+1})_{\geq 0}(x)\) for \(x\) positive and \(x \mapsto (\alpha_{k+1})_{\leq 0}(x)\) for \(x\) negative.

For \(\alpha_{k+1}\), (α-1) holds by construction, due to (α-1). Next, since \(\tilde{\alpha}_k\) is continuous in the interior of \(B^{k+1}\), we have that \((\alpha_{k+1})_{\geq 0}\) is continuous on the open upper hemisphere, hence \(\alpha_{k+1}\) is also, so (α-3) holds also.

Since \(\tilde{\alpha}_k\) satisfies \(\tilde{\alpha}_k(-x) = \rho(\tilde{\alpha}_k(x))\) for positive \(x\) on the boundary sphere \(S^k \subset B^{k+1}\), we have \((\alpha_{k+1})_{\leq 0}(x) = \rho^2((\alpha_{k+1})_{\geq 0}(x))\) for positive \(x\) on the equator \(S^k \subset S^{k+1}\), and \((\alpha_{k+1})_{\leq 0}(x) = (\alpha_{k+1})_{\geq 0}(x)\) for negative \(x\) on the equator. Hence \(\phi \circ (\alpha_{k+1})_{\geq 0}, \beta \circ (\alpha_{k+1})_{\leq 0}\) agree on the equator, since \(\phi \circ \rho^2 = \sigma^2 \circ \phi = \phi\). Moreover, both composites are continuous; for the second, we have

\[\phi \circ (\alpha_{k+1})_{\leq 0} = \phi \circ \rho \circ \tilde{\alpha}_k \circ \pi \circ (-) = \sigma \circ (\phi \circ \tilde{\alpha}_k) \circ \pi \circ (-)\]

and \(\sigma, \phi \circ \tilde{\alpha}_k, \pi, (-)\) are continuous. Hence (α-4) holds.

Before showing (α-2), we show that (α-2) implies

\[\tilde{\alpha}_k(x) \leq \rho(\tilde{\alpha}_k(-x))\]

for all \(x \in B^{k+1}\) not on the equator. For such \(x\), \(u(-x)\) is on the open upper hemisphere and hence is positive. By (α-2), we have

\[\tilde{\alpha}_k(x) \leq \alpha_k(l(x)) = \alpha_k(-u(-x)) = \rho(\alpha_k(u(-x))) \leq \rho(\tilde{\alpha}_k(-x)).\]

This proves the inequality above.

Now we show (α-2). For \(x \in S^{k+1}\) in the open upper hemisphere, we have

\[\alpha_{k+1}(x) = \tilde{\alpha}_k(\pi(x)) \leq \rho(\tilde{\alpha}_k(-\pi(x))) = \rho(\tilde{\alpha}_k(\pi(-x))) = \alpha_{k+1}(m(x))\]

by the inequality above. Hence (α-2) holds.

Taking \(\beta_n = \phi \circ \alpha_n\), Theorem 3.5 follows from the claims above. To see that \(\beta_n\) is a \(\mathbb{Z}/2\)-map, note that for \(x \in S^n\) positive, we have

\[\beta_n(-x) = \phi(\alpha_n(-x)) = \phi(\rho(\alpha_n(x))) = \sigma(\phi(\alpha_n(x))) = \sigma(\beta_n(x))\]

The other conclusions of the theorem are clear.

For a full lifted mpa, the preorder conditions of Theorem 3.5 are trivially satisfied, so we get:

**Corollary 3.6.** Let \(Y, Z\) be topological spaces, equip \(Y\) with a \(\mathbb{Z}\)-action generated by \(\rho: Y \to Y\), and equip \(Z\) with a \(\mathbb{Z}/2\)-action generated by \(\sigma: Z \to Z\). Let \(\phi: Y \to Z\) be continuous and equivariant, i.e., \(\sigma \circ \phi = \phi \circ \rho\). If there is a full lifted mpa for \((Y, Z, \phi)\), then there exists a \(\mathbb{Z}/2\)-map \(\beta_n: S^n \to Z\) for all integers \(n \geq 0\).
4. Constructing a lifted mpa

The goal of this section is to prove our main result, Theorem 1.3, by constructing a lifted mpa satisfying the conditions of Theorem 3.5. As a warm-up, we use Theorem 3.5 to prove the Hobby-Rice theorem, Theorem 1.1:

Proof of Theorem 1.1. The idea is to lift the space of functions with range in \( \{\pm 1\} \) to nondecreasing functions with range in \( \mathbb{Z} \). By describing a continuous map from pairs of such functions to paths between them, we will produce a lifted mpa, which will imply the result by Theorem 3.5.

Let \( Y \) be the space of nondecreasing functions \( g; [0,1] \to \mathbb{Z} \) with finite range, and let \( Z \) be the space of functions \( h; [0,1] \to \{\pm 1\} \). Equip \( Y, Z \) with the \( L^1 \)-norm, and define \( \rho(g) = g + 1, \sigma(h) = -h \), and

\[
\phi(g)(x) = \begin{cases} 
1 & g(x) \text{ even} \\
-1 & g(x) \text{ odd} 
\end{cases}
\]

Let \( g_0 \leq g_1 \) if \( g_0(x) \leq g_1(x) \) for all \( x \in [0,1] \). Finally, for \( g_0 \leq g_1 \) define \( s_w(g_0,g_1) \) to be the path (in \( t \)) of functions following \( g_0 \) on \([0,1-t)\) and \( g_1 \) on \([1-t,1)\):

\[
s_w(g_0,g_1)(t)(x) = \begin{cases} 
g_0(x) & x < 1-t \\
g_1(x) & x \geq 1-t \end{cases}
\]

Note that \( s_w \) is independent of \( w \). The conditions of Theorem 3.5 are straightforward to check, except perhaps the continuity property in the lifted mpa definition, which we check now.

We are given \( g \in Y \), and we may assume \( V \) is a basis set, so that \( V \) consists of all \( h \in Z \) with \( \|h - \phi(g)\| < \varepsilon \) for some \( \varepsilon > 0 \). By our choice of \( U \) we may ensure that \( g_0, g_1 \in Y \) have the same parity as \( g \) except on a sets \( S_0, S_1 \) with \( \mu(S_i) < \varepsilon/4 \). Then functions \( g' \) along the path \( s_w(g_0,g_1) \) have the same parity as \( g \) except on \( S_0 \cup S_1 \), where \( \mu(S_0 \cup S_1) < \varepsilon/2 \), which implies \( \|\phi(g') - \phi(g)\| < \varepsilon \).

Hence the conditions of Theorem 3.5 are satisfied, so we obtain a \( \mathbb{Z}/2 \)-map \( \beta_n : S^n \to Z \). Applying the Borsuk–Ulam theorem to \( \psi \circ \beta_n : S^n \to \mathbb{R}^n \), where \( \psi : h \mapsto (\int_0^1 f_j(x)h(x)dx)_j \), we obtain \( x \in S^n \) with \( \psi(\beta_n(x)) = 0 \). Hence also \( \psi(\beta_n(-x)) = 0 \), so we may assume \( x \) is positive. Taking \( y^* = 0 \) in the last part of Theorem 3.5, we may ensure that \( \beta_n \) maps each positive point of \( S^n \) to a point in \( Z \) of the form \( \phi(g) \) with \( 0 \leq g \leq n \), so that \( \phi(g) \) has at most \( n \) sign changes. This completes the proof.

Now we prove our main result, Theorem 1.3:

Proof of Theorem 1.3. Consider the space \( C^\infty([0,1];\mathbb{R}) \) with the \( L^1 \)-norm, and let \( Y \) be the subspace of nondecreasing functions in \( C^\infty([0,1];\mathbb{R}) \), equipped with the action \( \rho : g \mapsto g + \pi \). Let \( Z \) be \( C^\infty([0,1];S^1) \) with the \( L^1 \)-norm, equipped with the action \( \sigma : h \mapsto -h \).

Define \( \phi : Y \to Z \) by \( \phi(g)(x) = e^{ig(x)} \); then \( \phi \) is continuous since \( x \mapsto e^{ix} \) is 1-Lipschitz:

\[
\|\phi(g_2) - \phi(g_1)\|_1 = \int_0^1 |e^{ig_2(x)} - e^{ig_1(x)}|dx \\
\leq \int_0^1 |g_2(x) - g_1(x)|dx \\
\leq \|g_2 - g_1\|_1.
\]
Define \( (\leq) \) on \( Y \) as \( (\leq) \) pointwise. Then properties (1) and (2) of Theorem 3.5 and the commutativity property \( \phi \circ \rho = \sigma \circ \phi \) evidently hold.

It remains to construct the lifted map \( s \). Let \( \tau : \mathbb{R} \to [0, 1] \) be a smooth, nondecreasing function with \( \tau(x) = 0 \) for \( x \leq -1 \), and \( \tau(x) = 1 \) for \( x \geq 1 \). (For example, take an integral of a mollifier.) Then define \( s_w : Y^2 \to PY \) by

\[
s_w(g_0, g_1)(t)(x) = \left( 1 - \tau \left( \frac{x - (1 - t)}{w} \right) \right) g_0(x) + \tau \left( \frac{x - (1 - t)}{w} \right) g_1(x).
\]

Since \( \tau \) is smooth, and since \( x \mapsto (x - (1 - t))/w \) is smooth for \( w \neq 0 \), the function \( s_w(g_0, g_1)(t) : [0, 1] \to \mathbb{R} \) is smooth. Also, \( s_w(g_0, g_1)(t) \) is nondecreasing:

\[
\frac{d}{dx} [s_w(g_0, g_1)(t)(x)] = -\frac{1}{w} \cdot \tau' \left( \frac{x - (1 - t)}{w} \right) \cdot g_0(x) + \left( 1 - \tau \left( \frac{x - (1 - t)}{w} \right) \right) \cdot g_0'(x) + \frac{1}{w} \cdot \tau' \left( \frac{x - (1 - t)}{w} \right) \cdot g_1(x) + \tau \left( \frac{x - (1 - t)}{w} \right) \cdot g_1'(x) \geq \frac{1}{w} \cdot \tau' \left( \frac{x - (1 - t)}{w} \right) \cdot (g_1(x) - g_0(x)) \geq 0.
\]

Therefore, \( s_w(g_0, g_1) \) takes values in \( PY \). Since \( g_0 \leq g_1 \), we have \( g_0 \leq s_w(g_0, g_1)(t) \leq g_1 \), so property (3) of Theorem 3.5 holds.

Next we show \( s_w(g_0, g_1)(t) \) is continuous in \( w, g_0, g_1, t \). First we establish a helpful result. Let \( B \) be the subspace of \( L^\infty([0, 1]; \mathbb{R}) \) consisting of smooth functions, and let \( \tilde{Y} \) be the space \( L^1([0, 1]; \mathbb{R}) \), of which \( Y \) is a subspace; then pointwise multiplication \( (b, g) \mapsto b \cdot g \) defines a continuous map \( B \times \tilde{Y} \to \tilde{Y} \), via the following inequality, using Hölder’s inequality:

\[
\|b_2g_2 - b_1g_1\|_1 \leq \|b_2(g_2 - g_1)\|_1 + \|g_1(b_2 - b_1)\|_1 \leq \|b_2\|_\infty \cdot \|g_2 - g_1\|_1 + \|g_1\|_1 \cdot \|b_2 - b_1\|_\infty.
\]

Since \( (w, g_0, g_1, t) \mapsto g_0 \), \( (w, g_0, g_1, t) \mapsto g_1 \) are continuous maps \( [0, 1] \times Y \times Y \times [0, 1] \to Y \), by the result above it suffices to show that

\[
(w, g_0, g_1, t) \mapsto \left( x \mapsto \tau \left( \frac{x - (1 - t)}{w} \right) \right)
\]

is a continuous map to \( B \); the subtraction from 1 in the first term is handled by virtue of the fact that \( B \) is a normed linear space, so that pointwise addition and scalar multiplication by \(-1\) each define a continuous map.

Since \( \tau \) is constant outside of the compact set \([-1, 1]\), \( \tau \) is uniformly continuous, hence it suffices to prove that

\[
(w, g_0, g_1, t) \mapsto \left( x \mapsto \frac{x - (1 - t)}{w} \right)
\]

is a continuous map to \( B \). Note that

\[
\sup_{x \in [0, 1]} \left| \frac{x}{w_2} - \frac{x}{w_1} \right| = \left| \frac{1}{w_2} - \frac{1}{w_1} \right|
\]
Since $w \mapsto 1/w$ is a continuous map $\mathbb{R} \setminus \{0\} \to \mathbb{R}$, the map $(w, g_0, g_1, t) \mapsto (x \mapsto x/w)$ is a continuous map to $B$, as is $(w, g_0, g_1, t) \mapsto (x \mapsto -(1-t)/w)$, so the map above is indeed a continuous map to $B$. Hence $s_w(g_0, g_1)(t)$ is continuous in $w, g_0, g_1, t$.

It remains to show the continuity property for a lifted mpa. Let $g \in Y$, then for $g_0, g_1 \in Y$ we have

$$
\|\phi(s_w(g_0, g_1)(t)) - \phi(g)\|_1
= \int_0^{1-t-w} |\phi(g_0)(x) - \phi(g)(x)| dx + \int_{1-t+w}^1 |\phi(g_1)(x) - \phi(g)(x)| dx
+ \int_{1-t-w}^{1-t+w} |\phi(s_w(g_0, g_1)(t))(x) - \phi(g)(x)| dx
\leq \|\phi(g_0) - \phi(g)\|_1 + \|\phi(g_1) - \phi(g)\|_1 + 4w,
$$

where we use the fact that $S^1$ has diameter 2 in the last step. This inequality implies the continuity property for a lifted mpa.

Therefore, we may apply Theorem 3.5 to obtain a $\mathbb{Z}/2$-map $\beta_n: S^n \to Z$. Then $\psi \circ \beta_n: S^n \to \mathbb{R}^n$ is a $\mathbb{Z}/2$-map, so by the Borsuk–Ulam theorem, we have $\psi(\beta_n(x)) = 0$ for some $x \in S^n$, and we may assume $x$ is positive. Taking $y^* = c_0$ in the last part of Theorem 3.5, we have $\rho^n(y^*) = c_n$, so we may ensure that $h = \beta_n(x)$ is of the form $\phi(g)$ for $g \in Y$, where $g$ is an increasing function with range in $[0, \pi n]$. This gives the desired $W^{1,1}$-norm bound:

$$
\int_0^1 \left| \frac{d}{dx} e^{ig(x)} \right| dx = \int_0^1 |g'(x)| dx = g(1) - g(0) \leq \pi n,
$$

which implies $\|h\|_{W^{1,1}} \leq 1 + \pi n$. \hfill \Box

## 5. Improving the Bound Further

In the introduction we argued that a $W^{1,1}$-norm bound of $1 + 2\pi n$ in Theorem 1.2 might be expected from smoothing the Hobby–Rice theorem. In this section, we show an improved bound for Theorem 1.2 in the case where the $f_j$ are real-valued. The idea is to modify the $S^1$ step of our construction so that some functions in the image of $\alpha_k$ have smaller range within $[0, \pi k]$, and to modify the later steps so that functions $h$ in the image of $\alpha_k$ with large range have $\psi(\phi(h)) \neq 0$.

**Theorem 5.1.** Let $f_1, \ldots, f_n \in L^1([0, 1]; \mathbb{R})$. Then there exists $h \in C^\infty([0, 1]; S^1)$ such that for all $j$,

$$
\int_0^1 f_j(x) h(x) dx = 0.
$$

Moreover, for any $\varepsilon > 0$, $h$ can be chosen such that

$$
\|h\|_{W^{1,1}} < 1 + \pi (2n - 1) + \varepsilon.
$$

**Proof.** Define $Y, Z, \rho, \sigma, \phi, s$ as in the proof of Theorem 1.3, let $y^* = c_0$, and let $(\preceq)$ be $(\leq)$. We will produce $\alpha_n: S^n \to Y$ and $\beta_n: S^n \to Z$ by the inductive construction in the proof of Theorem 3.5, but we modify the first step by defining $\alpha_1: S^1 \to Y$ by $e^{ix} \mapsto c_x$ for $x \in [0, 2\pi)$. This $\alpha_1$ differs from the $\alpha_1$ obtained in the proof of Theorem 3.5, which only gives constant functions at $\pm 1 \in S^1$,
but is still good in the sense introduced in the proof of Theorem 3.5. Using this \( \alpha_1 \) as our base case, we inductively construct \( \alpha_k \) as before with the following additional condition:

For \( \delta > 0 \) (depending on \( k \) and the \( f_j \)), \( \alpha_k \) may be chosen such that for all \( x \):

\[
\Re[e^{i\alpha_k(x)(t)}] = \pi_1(x) \quad \text{for } t \in [0,1] \setminus S, \text{ where } \mu_f(S) < \delta \quad (P_{\alpha_k,\delta})
\]

Here \( \mu_f \) is as in the proof of Corollary 1.4, that is,

\[
\mu_f(S) = \int_0^1 |f_j(x)| dx,
\]

and \( \pi_1 : S^k \to [-1,1] \) is the projection to the first coordinate.

The condition \((P_{\alpha_k,\delta})\) holds for \( k = 1 \) and all \( \delta > 0 \) by our definition of \( \alpha_1 \). To show that the condition carries through the inductive step, it suffices to show that given \( \delta > 0 \), there exists \( \delta'' > 0 \) such that given \( \alpha_k \) such that \((P_{\alpha_k,\delta''})\) holds, we can extend \( \alpha_k \) to \( \tilde{\alpha}_k \) as in the first claim in the proof of Theorem 3.5 such that \((P_{\tilde{\alpha}_k,\delta''})\) holds.

We accomplish this by modifying the definition of \( \tilde{\alpha}_k \) in the first claim in the proof of Theorem 3.5 to impose a universal upper bound on \( w(x) \). Since \( \mu_f \) is absolutely continuous with respect to Lebesgue measure \( \lambda \), for \( \delta''' > 0 \) there exists \( \delta'''' > 0 \) such that \( \lambda(S) \leq 2\delta'''' \) implies \( \mu_f(S) < \delta'''' \).

Then we use \( \delta'''' \) as our upper bound on \( w(x) \):

\[
\tilde{\alpha}_k(x) = \begin{cases} 
\alpha_k(x) & x \in E \\
\min(w(x),\alpha_k(u(x)),\alpha_k(l(x)))(t(x)) & x \notin E
\end{cases}
\]

where \( w(x) = \min(d(x,E),t(x),1-t(x),\delta''''\)\)

\[
t(x) = \frac{d(u(x),x)}{d(u(x),l(x))}
\]

This ensures that functions in the image of \( \tilde{\alpha}_k \) are equal to one of the functions \( \alpha_k(u(x)),\alpha_k(l(x)) \) except on a set \( S \) with \( \mu_f(S) < \delta'''' \). Hence we may take \( \delta' = \delta'''' = \delta/2 \); then \((P_{\tilde{\alpha}_k,\delta})\) holds as desired. This shows that for any \( \delta > 0 \), \( \alpha_k \) may be chosen such that \((P_{\alpha_k,\delta})\) holds.

Now we apply the Borsuk–Ulam theorem as before. We have the following diagram:

\[
S^{2n} \xrightarrow{\phi \circ \alpha_{2n}} \mathbb{Z} / \mathbb{Z}^2 \xrightarrow{\psi} \mathbb{C}^n
\]

The composition \( \psi \circ \phi \circ \alpha_{2n} \) is a \( \mathbb{Z}/2 \)-map, so the Borsuk–Ulam theorem implies that it has a zero; that is, there exists \( x \in S^{2n} \) such that for all \( j \), we have

\[
\int_0^1 f_j(t)e^{i\alpha_{2n}(x)(t)} dt = 0.
\]

Moreover, we may assume \( x \in S^{2n} \) is positive.
But by the above, we have for the real parts, for all $j$,
\[
\text{Re} \left[ \int_0^1 f_j(t)e^{i\alpha_2n(x(t))}dt \right] = \int_0^1 f_j(t) \cdot \text{Re}[e^{i\alpha_2n(x(t))}]dt = \pi_1(x) \cdot \int_0^1 f_j(t)dt + \int_S f_j(t)(\text{Re}[e^{i\alpha_2n(x(t))}]) - \pi_1(x))dx.
\]
We can bound the last term as follows:
\[
\left| \int_S f_j(t)(\text{Re}[e^{i\alpha_2n(x(t))}]) - \pi_1(x))dx \right| \leq \int_S |\text{Re}[e^{i\alpha_2n(x(t))}]) - \pi_1(x)|d\mu_f \leq 2\mu_f(S).
\]
Now if all $\int_0^1 f_j(t)dt$ are 0, then we may take $h$ to be an arbitrary constant, which gives $\|h\|_{W^{1,1}} = 1$. Hence we may assume that some $\int_0^1 f_j(t)dt$ is nonzero. In this case, we may ensure that for the $x$ with $\int_0^1 f_j(t)dt$ is nonzero, we have
\[
|\text{Re}[e^{i\alpha_2n(x(t))}]) - \pi_1(x)| < \delta
\]
for any constant we like, by taking $\delta$ small in $(P_{\alpha_2n}, \delta)$. In particular, choose $\delta$ sufficiently small so that $|\text{Re}[e^{i\theta}]| < \delta$ implies $|\theta - \pi/2| < \varepsilon'$ for $\theta \in [0, \pi]$.

Now we analyze the ranges of functions $\alpha_k(x) : [0, 1] \rightarrow \mathbb{R}$ with $x$ positive and $|\pi_1(x)| < \delta$, using the fact that functions $\alpha_{k+1}(x)$ are produced as transition functions between two functions $\alpha_k(x'), \alpha_k(x'')$ with $\pi_1(x') = \pi_1(x'') = \pi_1(x)$. For $k = 1$, $\alpha_k(x)$ has range in $[\pi/2 - \varepsilon', \pi/2 + \varepsilon']$, and each increment of $k$ extends the right end of this interval by $\pi$. Hence $\alpha_{2n}(x)$ has range in $[\pi/2 - \varepsilon', \pi/2 + \pi(2n - 1) + \varepsilon']$.

Hence taking $h = \phi(\alpha_{2n}(x))$ gives $\|h\|_{W^{1,1}} \leq 1 + \pi(2n - 1) + 2\varepsilon'$. Choosing $\varepsilon' < \varepsilon/2$ gives the desired result.

\[\square\]

6. A LOWER BOUND

We ask whether $\|h\|_{W^{1,1}} \leq 1 + 2n\pi$ is the best possible bound in Theorem 1.2. We prove a lower bound of $1 + n\pi$ in the case that the $f_j$ are real-valued, which implies the same lower bound in the case that the $f_j$ are complex-valued.

**Theorem 6.1.** There exist $f_1, \ldots, f_n \in L^1([0, 1]; \mathbb{R})$, such that for any $h \in C^1([0, 1]; S^1)$ with
\[
\int_0^1 f_j(x)h(x)dx = 0 \quad j = 1, \ldots, n
\]
we have $\|h\|_{W^{1,1}} > n\pi + 1$.

**Proof.** Consider the case $n = 1$, and take $f_1$ constant and nonzero. Suppose for contradiction that $\|h\|_{W^{1,1}} \leq \pi + 1$, and write $h(x)$ as $e^{i\theta(x)}$ for $g \in C^1([0, 1]; \mathbb{R})$, so that $\int_0^1 |g'(x)|dx \leq \pi$. Since $g$ is continuous, $g$ attains its minimum $m$ and maximum $M$ on $[0, 1]$. By adding a constant to $g$, we may assume $m = 0$; then we have $M \leq \pi$.

Since $f_1$ is constant, we have $\int_0^1 h(x)dx = 0$, so $\int_0^1 \text{Im}(h(x))dx = 0$. But $\text{Im}(h(x))$ is continuous in $x$ and nonnegative, so $\text{Im}(h(x)) = 0$ for all $x$. Hence $h$ is constant at either 1 or $-1$, but this contradicts $\int_0^1 h(x)dx = 0$. Therefore, $\|h\|_{W^{1,1}} > \pi + 1$ for $n = 1$. 

Now allow $n$ arbitrary, and take each $f_j$ to be the indicator function on a disjoint interval $I_j$. If $\|h\|_{W^{1,1}} \leq \pi n + 1$, then $\int_{I_j} |g'(x)|dx \leq \pi$ for some $j$, and we obtain a contradiction as above. Therefore, $\|h\|_{W^{1,1}} > \pi n + 1$. □

This $W^{1,1}$-norm bound establishes an upper bound for the coindex of the space of smooth circle-valued functions with norm at most $1 + \pi n$:

**Theorem 6.2.** For integer $n \geq 1$ let $Y_n$ denote the space of $C^\infty$-functions $f: [0, 1] \to S^1$ with $\|f\|_{W^{1,1}} \leq 1 + \pi n$. Then

$$n \leq \text{coind } Y_n \leq 2n - 1.$$ 

*Proof.* In the proof of Theorem 1.3 we constructed a $\mathbb{Z}/2$-map $\beta_n : S^n \to Y_n$, which shows that $\text{coind } Y_n \geq n$. Let $f_1, \ldots, f_n$ be chosen as in Theorem 6.1. Then the map $\psi: Y_n \to \mathbb{R}^{2n}$ given by $\psi(h) = (\int_0^1 f_j(x)h(x)dx)_j$ has no zero and is a $\mathbb{Z}/2$-map. Thus $\psi$ radially projects to a $\mathbb{Z}/2$-map $Y_n \to S^{2n-1}$. A $\mathbb{Z}/2$-map $S^{2n} \to Y_n$ would compose with $\psi$ to a $\mathbb{Z}/2$-map $S^{2n} \to S^{2n-1}$, contradicting the Borsuk–Ulam theorem. This implies $\text{coind } Y_n \leq 2n - 1$. □

**Problem 6.3.** Determine the homotopy type of $Y_n$.

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