GOMORY INTEGER PROGRAMS

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ABSTRACT. The set of all group relaxations of an integer program contains certain special members called Gomory relaxations. A family of integer programs with a fixed coefficient matrix and cost vector but varying right hand sides is a Gomory family if every program in the family can be solved by one of its Gomory relaxations. In this paper, we characterize Gomory families. Every TDI system gives a Gomory family, and we construct Gomory families from matrices whose columns form a Hilbert basis for the cone they generate. The existence of Gomory families is related to the Hilbert covering problems that arose from the conjectures of Sebő. Connections to commutative algebra are outlined at the end.

1. Introduction

Given an integer \(d \times n\) matrix \(A\) and a cost vector \(c \in \mathbb{Z}^n\), we consider the family \(IP_{A,c}\) of all feasible integer programs of the form

\[
IP_{A,c}(b) := \min \left\{ c \cdot x : Ax = b, \ x \in \mathbb{N}^n \right\}
\]

as the right hand side vector \(b\) varies. The matrix \(A\) is assumed to have rank \(d\) and \(\text{cone}(A)\), the cone generated by the columns of \(A\), is assumed to be pointed. We also assume that \(\{x \in \mathbb{R}_+^n : Ax = 0\} = \{0\}\) which guarantees that all programs in \(IP_{A,c}\) are bounded.

In [9], Gomory defined the group relaxation of \(IP_{A,c}(b)\):

\[
\min \left\{ \bar{c}_\sigma \cdot x_\sigma : A_\sigma x_\sigma + A_\bar{\sigma} x_{\bar{\sigma}} = b, \ x_{\sigma} \geq 0, (x_\sigma, x_{\bar{\sigma}}) \in \mathbb{Z}^n \right\}
\]

where \(A_\sigma\) is the optimal basis of the linear relaxation of \(IP_{A,c}(b)\) and non-negativity restrictions on the optimal basic variables \(x_\sigma\) have been dropped. The cost vector \(\bar{c}_\sigma\) is the restriction of \((c - c_\sigma A_\sigma^{-1} A)\) to the components indexed by the complement of \(\sigma\). The extended group relaxations of \(IP_{A,c}(b)\) introduced by Wolsey are the \(2^{|\sigma|}\) relaxations obtained by dropping non-negativity restrictions on each subset of the variables in \(x_\sigma\) [24]. The set of \(d\)-dimensional simplicial cones \(\text{cone}(A_\sigma)\), as \(A_\sigma\) varies over the optimal bases of the LP-relaxations of all programs in \(IP_{A,c}\), triangulates \(\text{cone}(A)\). This is called the regular triangulation of \(\text{cone}(A)\) with respect to \(c\) and is denoted as \(\Delta_c\). The collection of all

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sets $\sigma$ indexing the full dimensional cones of $\Delta_c$ along with all their subsets form a simplicial complex on \{1, \ldots, n\} which we also call $\Delta_c$. In this paper we consider the set of all group relaxations of $IP_{A,c}(b)$ obtained by dropping non-negativity restrictions on the variables indexed by each face of $\Delta_c$ (Definition 2.6). This is a larger set of group relaxations for $IP_{A,c}(b)$ than the set of extended group relaxations of Wolsey. We show that these are precisely all the bounded group relaxations of $IP_{A,c}(b)$ (Theorem 2.7). Among these group relaxations, the easiest to solve are those indexed by the maximal faces of $\Delta_c$. We call these the Gomory relaxations of $IP_{A,c}(b)$. The family of integer programs $IP_{A,c}$ is called a Gomory family if all its members can be solved by one of their Gomory relaxations.

Theorem 2.17 characterizes Gomory families. This theorem is a consequence of re-casting algebraic results on toric initial ideals in terms of group relaxations of integer programs [13], [14]. These algebraic results come from Gröbner bases methods in integer programming [19]. No familiarity with these techniques is assumed in this paper. In Section 3 we relate Gomory families to total dual integrality (TDI-ness). Theorem 3.3 shows that $yA \leq c$ is a TDI system if and only if the regular triangulation $\Delta_c$ is unimodular. This leads to Corollary 3.4 that if $yA \leq c$ is TDI then $IP_{A,c}$ is a Gomory family.

In Sections 4 and 5 we exhibit general classes of Gomory families. A matrix $A$ is said to be normal if its columns form a Hilbert basis for $cone(A)$. In Section 4 we introduce $\Delta$-normal matrices which form a proper subset of normal matrices. Theorem 4.7 shows that every $\Delta$-normal matrix $A$ gives rise to a Gomory family $IP_{A,c}$. While we do not know if every normal matrix gives rise to a Gomory family, we show that for small values of $d$ every regular triangulation of $cone(A)$ supports Gomory families (Theorem 5.5). Gomory families induce special covers of $NA$, the semigroup generated by the columns of $A$, which relates their existence to the Hilbert cover questions found in [2], [7] and [17].

Throughout this paper we consider triangulations of $cone(A)$. We wish to point out that a one-dimensional face of any such triangulation must be generated by a column of $A$. All computations in this paper rely on the connections of this material to commutative algebra as described in [13], [14], [19] and [21]. The relevant connections and codes are described briefly in Section 6.
2. Gomory families

In this paper, we fix a matrix $A \in \mathbb{Z}^{d \times n}$ of rank $d$ and a cost vector $c \in \mathbb{Z}^n$ and consider the family $IP_{A,c}$ of all integer programs

$$IP_{A,c}(b) := \min \{ c \cdot x : Ax = b, \ x \in \mathbb{N}^n \}$$

as $b$ varies in the semigroup $NA := \{ Au : u \in \mathbb{N}^n \} \subseteq \mathbb{Z}^d$. This semigroup is contained in the intersection of cone$(A)$ := \{ $Ax : x \in \mathbb{R}^n_{\geq 0}$ \}, and $ZA := \{ Az : z \in \mathbb{Z}^n$ \}, the lattice generated by the columns of $A$. We may assume without loss of generality that $ZA = \mathbb{Z}^d$.

The feasible linear programs from $A$ and $c$ are of the form

$$LP_{A,c}(b) := \min \{ c \cdot x : Ax = b, \ x \geq 0 \}$$

where $b \in$ cone$(A)$. We denote this family as $LP_{A,c}$. For $\sigma \subseteq \{1, \ldots, n\}$, let $A_\sigma$ be the submatrix of $A$ whose set of column indices is $\sigma$.

**Definition 2.1.** For $\sigma \subseteq \{1, \ldots, n\}$, cone$(A_\sigma)$ is a face of the regular subdivision $\Delta_c$ of cone$(A)$ if and only if there exists a vector $y \in \mathbb{R}^d$ such that $y \cdot a_j = c_j$ for all $j \in \sigma$ and $y \cdot a_j < c_j$ for all $j \not\in \sigma$.

**Remark 2.2.** The regular subdivision $\Delta_c$ is gotten by taking the cone in $\mathbb{R}^{d+1}$ generated by the lifted vectors $(a_i, c_i) \in \mathbb{R}^{d+1}$ where $a_i$ is the $i$-th column of $A$ and $c_i$ is the $i$-th component of $c$, and then projecting the lower facets of this lifted cone back onto cone$(A)$. (See [1].)

We assume that $c$ is generic, which means that $\Delta_c$ is a triangulation of cone$(A)$. All cost vectors except those lying on a finite set of hyperplanes of $\mathbb{R}^n$ are generic [1]. Using $\sigma$ to label cone$(A_\sigma)$, the triangulation $\Delta_c$ can be denoted as a set of subsets of $\{1, \ldots, n\}$. This set of sets is closed under inclusion since $\Delta_c$ is a simplicial complex, and hence it is specified completely by its maximal elements. For a vector $x \in \mathbb{R}^n$, let $\text{supp}(x) := \{ i : x_i \neq 0 \}$ denote the support of $x$. The significance of regular triangulations for linear programming is summarized in the following proposition.

**Proposition 2.3.** [20, Lemma 1.4] An optimal solution of $LP_{A,c}(b)$ is any feasible solution $x^*$ such that $\text{supp}(x^*) = \tau$ where $\tau$ is the smallest face of the regular triangulation $\Delta_c$ such that $b \in$ cone$(A_\tau)$.

Proposition 2.3 implies that $\sigma \subseteq \{1, \ldots, n\}$ is a maximal face of $\Delta_c$ if and only if $A_\sigma$ is an optimal basis for all $LP_{A,c}(b)$ with $b$ in cone$(A_\sigma)$. Given a polyhedron $P \subseteq \mathbb{R}^n$ and a face $F$ of $P$, the normal cone of $F$ at $P$ is the cone $NP(F) := \{ \omega \in \mathbb{R}^n : \omega \cdot x' \geq \omega \cdot x, \ \forall x' \in F \ \text{and} \ x \in P \}$. The set of all normal cones of $P$ form the normal fan of $P$ in $\mathbb{R}^n$. 

Proposition 2.4. The regular triangulation $\Delta_c$ of $\text{cone}(A)$ is the normal fan of the polyhedron $P_c := \{ y \in \mathbb{R}^d : yA \leq c \}$.

Proof. The polyhedron $P_c$ is the feasible region of $\text{max} \{ y \cdot b : yA \leq c, y \in \mathbb{R}^d \}$, the dual program to $LP_{A,c}(b)$. The normal fan of $P_c$ is supported on $\text{cone}(A)$, i.e. the union of the normal cones of $P_c$ is $\text{cone}(A)$, since this is the polar cone of the recession cone $\{ y \in \mathbb{R}^d : yA \leq 0 \}$ of $P_c$. Suppose $b$ is any vector in the interior of a maximal face $\text{cone}(A_\tau)$ of $\Delta_c$. Then by Proposition 2.3, $LP_{A,c}(b)$ has an optimal solution $x^\ast$ with support $\sigma$. The optimal solution $y$ to the dual of $LP_{A,c}(b)$ satisfies $y \cdot a_j = c_j$ for all $j \in \sigma$ and $y \cdot a_j \leq c_j$ otherwise, by complementary slackness. Since $\sigma$ is a maximal face of $\Delta_c$, in fact, $y \cdot a_j < c_j$ for all $j \notin \sigma$. This shows that $y$ is unique, and $\text{cone}(A_\tau)$ is contained in the normal cone of $P_c$ at the vertex $y$. If $b$ lies in the interior of another maximal face $\text{cone}(A_\tau)$ then $y'$, the dual optimal solution to $LP_{A,c}(b)$ satisfies $y' \cdot A_\tau = c_\tau$ and $y' \cdot A_\tau < c_\tau$ where $\tau \neq \sigma$. Hence $y'$ is distinct from $y$ and each maximal cone in $\Delta_c$ lies in a distinct maximal cone in the normal fan of $P_c$. Since $\Delta_c$ and the normal fan of $P_c$ have the same support, they must therefore coincide. 

Corollary 2.5. The polyhedron $P_c$ is simple if and only if the regular subdivision $\Delta_c$ is a triangulation of $\text{cone}(A)$. \hfill \Box

Regular subdivisions were introduced in [8] and have since been studied from various points of view. They play a central role in the algebraic study of integer programming ([19], [20]). We use them here to define group relaxations of $IP_{A,c}(b)$.

A subset $\tau$ of $\{1, \ldots, n\}$ partitions $x = (x_1, \ldots, x_n)$ as $x_\tau$ and $x_\bar{\tau}$ where $x_\tau$ consists of the variables indexed by $\tau$, and $x_\bar{\tau}$ the variables indexed by the complementary set $\bar{\tau}$. Similarly, the matrix $A$ is partitioned as $A = [A_\tau, A_\bar{\tau}]$ and the cost vector as $c = (c_\tau, c_\bar{\tau})$. If $\sigma$ is a maximal face of $\Delta_c$ then $A_\sigma$ is nonsingular and $Ax = b$ can be written as $x_\sigma = A_\sigma^{-1}(b - A_\sigma x_\sigma)$. Then $c \cdot x = c_\sigma(A_\sigma^{-1}(b - A_\sigma x_\sigma)) + c_\bar{\sigma}x_\bar{\sigma} = c_\sigma A_\sigma^{-1}b + (c_\sigma - c_\sigma A_\sigma^{-1}A_\sigma)x_\bar{\sigma}$. Let $\tilde{c}_\sigma := c_\sigma - c_\sigma A_\sigma^{-1}A_\sigma$ and for any face $\tau$ of $\sigma$, let $\tilde{c}_\tau$ be the extension of $\tilde{c}_\sigma$ to a vector in $\mathbb{R}^{|\tau|}$ by adding zeros.

Definition 2.6. The group relaxation of the integer program $IP_{A,c}(b)$ with respect to the face $\tau$ of $\Delta_c$, denoted as $G^\tau(b)$, is the program

$$\min \{ \tilde{c}_\tau \cdot x_\tau : A_\tau x_\tau + A_\bar{\tau} x_\bar{\tau} = b, x_\bar{\tau} \geq 0, (x_\tau, x_\bar{\tau}) \in \mathbb{Z}^n \}.$$ 

The group relaxation $G^\tau(b)$ solves $IP_{A,c}(b)$ if its optimal solution is non-negative. These relaxations contain among them the usual group relaxations of $IP_{A,c}(b)$ found in the literature. The program $G^\tau(b)$
where $A_\sigma$ is an optimal basis of the linear relaxation $LP_{A,c}(b)$ is precisely Gomory’s group relaxation of $IP_{A,c}(b)$ \[9\]. The set of relaxations $G^\tau(b)$ as $\tau$ varies among the subsets of this $\sigma$ are the extended group relaxations of $IP_{A,c}(b)$ defined by Wolsey \[24\]. Since $\emptyset \in \Delta_c$, $G^\emptyset(b) = IP_{A,c}(b)$ is a group relaxation of $IP_{A,c}(b)$ by Definition \[24\], and hence $IP_{A,c}(b)$ will certainly be solved by one of its extended group relaxations. However, it is easy to construct examples where a group relaxation of $IP_{A,c}(b)$ solves $IP_{A,c}(b)$, but $G^\tau(b)$ is neither Gomory’s group relaxation of $IP_{A,c}(b)$ nor one of its nontrivial extended Wolsey relaxations (see Example \[24\]). Theorem \[24\] will show that the relaxations in Definition \[24\] are precisely all the bounded group relaxations of $IP_{A,c}(b)$. Hence this definition considers all the group relaxations of each integer program in $IP_{A,c}$ that can possibly solve the program.

For our purposes it is convenient to reformulate $G^\tau(b)$ as follows. Let $B \in \mathbb{Z}^{n\times (n-d)}$ be any matrix such that the columns of $B$ generate the $(n-d)$-dimensional lattice $L = \{ x \in \mathbb{Z}^n : Ax = 0 \} \subset \mathbb{Z}^n$ and let $u$ be a feasible solution of $IP_{A,c}(b)$. Then

\[
IP_{A,c}(b) = \min \{ c \cdot x : Ax = b, \ x \in \mathbb{N}^n \} = \min \{ c \cdot x : x \equiv u \ (\text{mod } L), \ x \geq 0 \}
\]

The last problem is equivalent to $\min \{ c \cdot (u - Bz) : Bz \leq u, \ z \in \mathbb{Z}^{n-d} \}$ and hence, $IP_{A,c}(b)$ is equivalent to the problem

\[
min \ \{ (-cB)z : Bz \leq u, \ z \in \mathbb{Z}^{n-d} \}.
\]

There is a bijection between the set of feasible solutions of \[11\] and the set of feasible solutions of $IP_{A,c}(b)$ via the isomorphism $z \mapsto u - Bz$. In particular, $0 \in \mathbb{R}^{n-d}$ is feasible for \[11\] since it is the pre-image of $u$.

Let $\pi_\tau$ be the projection map from $\mathbb{R}^n \to \mathbb{R}^{|\tau|}$ that kills all coordinates indexed by $\tau$. If $B^\tau$ denotes the $|\tau| \times (n-d)$ submatrix of $B$ obtained by deleting the rows indexed by $\tau$, then we denote by $L_\tau$ the lattice $\pi_\tau(L) = \{ B^\tau z : z \in \mathbb{Z}^{n-d} \}$. It can be deduced from \[22\] that the group relaxation $G^\tau(b)$ is equivalent to the lattice program

\[
\min \ \{ \tilde{c}_\tau \cdot x_\tau : x_\tau \equiv \pi_\tau(u) \ (\text{mod } L_\tau), \ x_\tau \geq 0 \}
\]

which can be reformulated as above to be

\[
\min \ \{ (-\tilde{c}_\tau B^\tau)z : B^\tau z \leq \pi_\tau(u), \ z \in \mathbb{Z}^{n-d} \}.
\]

Since $\tilde{c}_\tau = \pi_\tau(c - c_\sigma A_\sigma^{-1}A)$ for any maximal face $\sigma$ of $\Delta_c$ containing $\tau$, and the support of $c - c_\sigma A_\sigma^{-1}A$ is contained in $\tau$, we get that $\tilde{c}_\tau B^\tau = (c - c_\sigma A_\sigma^{-1}A)B = cB$ since $AB = 0$. Hence $G^\tau(b)$ is equivalent to

\[
\min \ \{ (-cB)z : B^\tau z \leq \pi_\tau(u), \ z \in \mathbb{Z}^{n-d} \}.
\]
The feasible solutions to (1) are the lattice points in the rational polyhedron \( P_u := \{ z \in \mathbb{R}^{n-d} : Bz \leq u \} \) and those to (2) are the lattice points in the relaxation \( P^r_u := \{ z \in \mathbb{R}^{n-d} : B^\tau z \leq \pi^\tau(u) \} \) of \( P_u \) obtained by deleting the inequalities indexed by \( \tau \). In theory, one could define group relaxations of \( IP_{A,c}(b) \) with respect to any \( \tau \subseteq \{1, \ldots, n\} \). The following result justifies Definition 2.6.

**Theorem 2.7.** The group relaxation \( G^\tau(b) \) of \( IP_{A,c}(b) \) has a finite optimal solution if and only if \( \tau \subseteq \{1, \ldots, n\} \) is a face of \( \Delta_c \).

**Proof.** Since all data are integral it suffices to prove that the linear relaxation \( \min \{(-cB)z : z \in P^r_u\} \) is bounded if and only if \( \tau \in \Delta_c \).

If \( \tau \) is a face of \( \Delta_c \) then there exists \( y \in \mathbb{R}^d \) such that \( yA_\tau = c_\tau \) and \( yA_\tau < c_\tau \). Using the fact that \( A_\tau B^\tau + A_\tau B^\tau = 0 \) we see that \( cB = c_\tau B^\tau + c_\tau B^\tau = yA_\tau B^\tau + c_\tau B^\tau = y(-A_\tau B^\tau) + c_\tau B^\tau = (c_\tau - yA_\tau)B^\tau \). This implies that \( cB \) is a positive linear combination of the rows of \( B^\tau \) since \( c_\tau - yA_\tau > 0 \). Hence \( cB \) lies in the polar of \( \{ z \in \mathbb{R}^{n-d} : B^\tau z \leq 0 \} \) which is the recession cone of \( P^r_u \) proving that the linear program \( \min \{(-cB)z : z \in P^r_u\} \) is bounded.

The linear program \( \min \{(-cB)z : z \in P^r_u\} \) is feasible since 0 is a feasible solution. If it is bounded as well then \( \min \{c_\tau x_\tau + c_\tau x_\tau : A_\tau x_\tau + A_\tau x_\tau = b, x_\tau \geq 0 \} \) is feasible and bounded. Hence the dual of the latter program \( \max \{y \cdot b : yA_\tau = c_\tau, yA_\tau \leq c_\tau \} \) is feasible. This shows that a superset of \( \tau \) is a face of \( \Delta_c \) which implies that \( \tau \in \Delta_c \) since \( \Delta_c \) is a triangulation. \( \square \)

The reformulations (1) and (2) imply that \( G^\tau(b) \) solves \( IP_{A,c}(b) \) if and only if both programs have the same optimal solution \( z^* \in \mathbb{Z}^{n-d} \).

If \( G^\tau(b) \) solves \( IP_{A,c}(b) \) then \( G^{\tau'}(b) \) also solves \( IP_{A,c}(b) \) for every \( \tau' \) contained in \( \tau \). We say that \( \tau \in \Delta_c \) is associated to \( IP_{A,c} \) if for some \( b \in NA \), \( G^\tau(b) \) solves \( IP_{A,c}(b) \) but \( G^{\tau'}(b) \) does not for all faces \( \tau' \neq \tau \) of \( \Delta_c \) containing \( \tau \). Several results about the structure of the subposet of faces of \( \Delta_c \) that are associated to \( IP_{A,c} \) can be found in [13]. For instance, the associated sets of \( IP_{A,c} \) occur in saturated chains [13, Theorem 3.1]. For a given \( b \in NA \), the most easily solved relaxations of \( IP_{A,c}(b) \) are those \( G^\sigma(b) \) where \( \sigma \) is a maximal face of \( \Delta_c \). We call these “top-level” relaxations the *Gomory relaxations of \( IP_{A,c}(b) \).*

**Definition 2.8.** The family of integer programs \( IP_{A,c} \) is a *Gomory family* if, for every \( b \in NA \), \( IP_{A,c}(b) \) is solved by a group relaxation \( G^\sigma(b) \) where \( \sigma \) is a maximal face of the regular triangulation \( \Delta_c \).

Our goal in the rest of this section is to characterize Gomory families of integer programs (Theorem 2.17). We will assume from now on that every integer program in \( IP_{A,c} \) has a unique solution which is a stricter
Suppose $v \in u$ as the vector element $u$. The affine semigroup $\mathcal{O}_c$ is known to be a down set or order ideal in $\mathbb{N}^n$, i.e. $u \in \mathcal{O}_c$ and $v \leq u$, $v \in \mathbb{N}^n$ implies that $v \in \mathcal{O}_c$ [23]. For a given $A$, there are only finitely many sets $\mathcal{O}_c$ as $c$ varies. Two generic cost vectors $c$ and $c'$ are equivalent if $\mathcal{O}_c = \mathcal{O}_{c'}$ and all equivalence classes of generic cost vectors are open full dimensional cones in $\mathbb{R}^n$ [20]. Since $c$ is generic, $\mathcal{O}_c$ is in bijection with $\mathbb{N}A$ via the linear map $\phi_A : \mathbb{N}^n \to \mathbb{N}A$ where $u \mapsto Au$. Let $Q_u := \{ z \in \mathbb{R}^{n-d} : Bz \leq u, (-cB)z \leq 0 \}$ and $Q_u^r := \{ z \in \mathbb{R}^{n-d} : B^rz \leq \pi_r(u), (-cB)z \leq 0 \}$.

**Lemma 2.9.** (i) A vector $u$ is in $\mathcal{O}_c$ if and only if $Q_u \cap \mathbb{Z}^{n-d} = \{ 0 \}$. (ii) If $u \in \mathcal{O}_c$, then the group relaxation $G^r(Au)$ solves the integer program $IP_{A,c}(Au)$ if and only if $Q_u^r \cap \mathbb{Z}^{n-d} = \{ 0 \}$.

**Proof.** (i) The lattice point $u$ belongs to $\mathcal{O}_c$ if and only if $u$ is the optimal solution to $IP_{A,c}(Au)$ which is equivalent to $0 \in \mathbb{Z}^{n-d}$ being the optimal solution to the reformulation (1) of $IP_{A,c}(Au)$. Since $c$ is generic, the last statement is equivalent to $Q_u \cap \mathbb{Z}^{n-d} = \{ 0 \}$. The second statement follows from the fact that (2) solves (1) if and only if they have the same optimal solution. \hfill \Box

By Lemma 2.9, it is convenient to use the optimal solution to $IP_{A,c}(b)$ as the vector $u$ in (1) and (2), and we will do so from now on. For an element $u \in \mathcal{O}_c$ and a face $\tau$ of $\Delta_c$, let $S(u, \tau)$ be the affine semigroup $u + \mathbb{N}(e_i : i \in \tau)$ in $\mathbb{N}^n$ where $e_i$ denotes the $i$-th unit vector in $\mathbb{R}^n$.

**Lemma 2.10.** Suppose $u$ lies in $\mathcal{O}_c$. If $G^r(Au)$ solves $IP_{A,c}(Au)$, then $G^r(Av)$ solves $IP_{A,c}(Av)$ for all $v \in S(u, \tau)$.

**Proof.** If $v \in S(u, \tau)$, then $\pi_r(u) = \pi_r(v)$, and this implies that $Q_v^r = Q_u^r$. If $G^r(Au)$ solves $IP_{A,c}(Au)$, then $\{ 0 \} = Q_u^r \cap \mathbb{Z}^{n-d} = Q_v^r \cap \mathbb{Z}^{n-d}$ for all $v \in S(u, \tau)$. This implies the result by Lemma 2.9 (ii). \hfill \Box

**Proposition 2.11.** The affine semigroup $S(u, \tau)$ is contained in $\mathcal{O}_c$ if and only if $G^r(Au)$ solves $IP_{A,c}(Au)$.

**Proof.** Suppose $S(u, \tau) \subseteq \mathcal{O}_c$. Then for all $v \in S(u, \tau)$, $Q_v = \{ z \in \mathbb{R}^{n-d} : B^rz \leq \pi_r(v), B^rz \leq \pi_r(u), (-cB)z \leq 0 \} \cap \mathbb{Z}^{n-d} = \{ 0 \}$. Since $\pi_r(v)$ can be any vector in $\mathbb{N}[r]$, and $Q_v^r$ is bounded by Theorem 2.7. $Q_v^r = \{ z \in \mathbb{R}^{n-d} : B^rz \leq \pi_r(u), (-cB)z \leq 0 \} \cap \mathbb{Z}^{n-d} = \{ 0 \}$. Hence, by Lemma 2.9 (ii), $G^r(Au)$ solves $IP_{A,c}(Au)$.

Conversely, if $G^r(Au)$ solves $IP_{A,c}(Au)$, then $\{ 0 \} = Q_u^r \cap \mathbb{Z}^{n-d} = Q_v^r \cap \mathbb{Z}^{n-d}$ for all $v \in S(u, \tau)$. Since $Q_v^r$ is a relaxation of $Q_v$, $Q_v \cap \mathbb{Z}^{n-d} = \{ 0 \}$ for all $v \in S(u, \tau)$ and hence by Lemma 2.9 (i), $S(u, \tau) \subseteq \mathcal{O}_c$. \hfill \Box
Definition 2.12. For $\tau \in \Delta_c$ and $u \in \mathcal{O}_c$, the pair $(u, \tau)$ is called an admissible pair of $\mathcal{O}_c$ if

(i) $G^\tau(Au)$ solves $IP_{A,c}(Au)$ or equivalently, $S(u, \tau) \subseteq \mathcal{O}_c$, and

(ii) the support of $u$ is contained in $\bar{\tau}$.

An admissible pair $(u, \tau)$ is a standard pair of $\mathcal{O}_c$ if the affine semigroup $S(u, \tau)$ is not properly contained in another affine semigroup $S(v, \tau')$ where $(v, \tau')$ is also an admissible pair of $\mathcal{O}_c$.

Definition 2.13. For $u \in \mathbb{N}^n$ and a face $\tau$ of $\Delta_c$, we say that the polytope $Q_{\bar{\tau}}^u$ is a standard polytope of $IP_{A,c}$ if $Q_{\bar{\tau}}^u \cap \mathbb{Z}^n = \{0\}$ and every relaxation of $Q_{\bar{\tau}}^u$ obtained by removing an inequality in $B^\tau z \leq \pi_\tau(u)$ contains a non-zero lattice point.

Theorem 2.14. The following are equivalent:

(i) An admissible pair $(u, \tau)$ is a standard pair of $\mathcal{O}_c$.

(ii) The polytope $Q_{\bar{\tau}}^u$ is a standard polytope of $IP_{A,c}$.

(iii) The face $\tau$ of $\Delta_c$ is associated to $IP_{A,c}$.

Proof. The proof of $(i) \iff (ii)$ is the content of Theorem 2.5 in [13]. The equivalence $(i) \iff (iii)$ follows from the definition of a standard pair and Lemma 2.10.

Under the linear map $\phi_A : \mathbb{N}^n \rightarrow NA$ such that $u \mapsto Au$, the affine semigroup $S(u, \tau)$ where $(u, \tau)$ is a standard pair of $\mathcal{O}_c$ maps to the affine semigroup $Au + NA_\tau$ in $NA$. Since every integer program in $IP_{A,c}$ is solved by one of its group relaxations, $\mathcal{O}_c$ is covered by its standard pairs. We call this cover and its image in $NA$ under $\phi_A$, the standard pair decompositions of $\mathcal{O}_c$ and $NA$ respectively. Since standard pairs of $\mathcal{O}_c$ are determined by the standard polytopes of $IP_{A,c}$, the standard pair decomposition of $\mathcal{O}_c$ is unique. The terminology used above has its origins in [21] which introduced the standard pair decomposition of a monomial ideal. The specialization to integer programming appear in [19], §12.D], [14] and [13]. For each $\tau \in \Delta_c$, there are only finitely many standard pairs of $\mathcal{O}_c$ that are indexed by $\tau$. Borrowing terminology from [21], we call the number of standard pairs of the form $(u, \tau)$ the multiplicity of $\tau$ in $\mathcal{O}_c$. The total number of standard pairs is called the arithmetic degree of $\mathcal{O}_c$. By Theorem 2.13, multiplicity of $\tau$ in $\mathcal{O}_c$ is the number of distinct standard polytopes of $IP_{A,c}$ indexed by $\bar{\tau}$ and the arithmetic degree of $\mathcal{O}_c$ is the total number of standard polytopes of $IP_{A,c}$. Lemma 2.15 shows that the maximal faces of $\Delta_c$ play a special role in the standard pair decomposition of $\mathcal{O}_c$. Part (ii) can also be deduced from the work of Gomory [3].
Lemma 2.15. [19, §12.D] (i) \((0, \sigma)\) is a standard pair of \(\mathcal{O}_c\) if and only if \(\sigma\) is a maximal face of \(\Delta_c\).

(ii) If \(\sigma\) is a maximal face of \(\Delta_c\), the multiplicity of \(\sigma\) in \(\mathcal{O}_c\) is the index of the sublattice \(ZA_\sigma\) in \(ZA\).

The index of \(ZA_\sigma\) in \(ZA\) is also the determinant of \(A_\sigma\) divided by the g.c.d of the maximal minors of \(A\). In contrast to Lemma 2.15, if \(\tau\) is a lower dimensional face of \(\Delta_c\), it may not index any standard pair of \(\mathcal{O}_c\). This makes the structure of \(\mathcal{O}_c\) complicated. (See [13] for further results.) We give an example below.

Example 2.16. Consider the following \(A \in \mathbb{Z}^{3 \times 6}\) of rank three:

\[
\begin{pmatrix}
5 & 0 & 0 & 2 & 1 & 0 \\
0 & 5 & 0 & 1 & 4 & 2 \\
0 & 0 & 5 & 2 & 0 & 3
\end{pmatrix}.
\]

The first three columns of \(A\) generate \(cone(A)\) which is simplicial. If \(c = (21, 6, 1, 0, 0, 0)\) then the regular triangulation \(\Delta_c\) is:

\[
\{\{1, 3, 4\}, \{1, 4, 5\}, \{2, 5, 6\}, \{3, 4, 6\}, \{4, 5, 6\}\}.
\]

The set \(\mathcal{O}_c\) has arithmetic degree 70 which means that \(\mathcal{O}_c\) has 70 standard pairs which are listed below. Not all lower dimensional faces of \(\Delta_c\) index standard pairs in this example.
The associated sets of IP are:

| \(\tau\) | standard pairs (\(\cdot, \tau\)) |
|---|---|
| \(\{1,3,4\}\) | (0,·), (e₂,·), (e₆,·), (e₅ + e₆,·), (2e₆,·) |
| \(\{1,4,5\}\) | (0,·), (e₂,·), (e₃,·), (e₆,·), (e₂ + e₃,·), (2e₂,·), (3e₂,·), (2e₂ + e₃,·) |
| \(\{2,5,6\}\) | (0,·), (e₃,·), (2e₃,·) |
| \(\{3,4,6\}\) | (0,·), (e₅,·), (2e₅,·), (3e₅,·) |
| \(\{4,5,6\}\) | (0,·), (e₃,·), (2e₃,·), (3e₃,·), (4e₃,·) |
| \(\{1,4\}\) | (e₂ + e₅ + e₆,·), (2e₃ + 2e₅ + e₆,·), (2e₃ + e₅ + e₆,·), (2e₃ + 4e₅,·) |
| \(\{1,5\}\) | (e₂ + e₆,·), (2e₂ + e₆,·), (3e₂ + e₆,·) |
| \(\{2,5\}\) | (e₃ + e₄,·), (e₄,·), (2e₄,·) |
| \(\{3,4\}\) | (e₂,·), (e₁ + e₂,·), (e₁ + 2e₅,·), (e₁ + 2e₅ + e₆,·), (e₂ + e₅,·) |
| \(\{3,6\}\) | (e₂,·), (e₂ + e₅,·) |
| \(\{4,5\}\) | (e₂ + 2e₃,·), (e₂ + 3e₃,·), (2e₂ + 2e₃,·), (3e₂ + 3e₃,·), (4e₂,·) |
| \(\{5,6\}\) | (e₂ + 3e₃,·) |
| \(\{1\}\) | (e₂ + e₃ + e₆,·), (e₂ + e₃ + e₅ + e₆,·), (e₂ + 2e₆,·), (e₂ + e₃ + 2e₆,·), (2e₂ + 2e₆,·), (e₂ + e₃ + 2e₅ + e₆,·) |
| \(\{3\}\) | (e₁ + e₂ + e₆,·), (e₁ + e₂ + 2e₆,·), (e₁ + 2e₃ + e₆,·), (e₁ + e₂ + 2e₃ + e₅,·), (e₁ + 2e₃ + 3e₅,·), (e₁ + e₂ + 2e₃ + 4e₅,·), (e₁ + 3e₃ + 3e₅,·), (e₁ + 3e₃ + 4e₅,·) |
| \(\{4\}\) | (e₁ + e₂ + 2e₃ + e₅,·), (e₁ + e₂ + 2e₃ + 2e₅,·), (e₁ + e₂ + 2e₃ + 2e₅ + e₆,·), (e₁ + 2e₃ + 2e₅ + e₆,·), (e₁ + 2e₂ + e₃ + e₆,·), (e₁ + 2e₂ + e₃ + e₅ + e₆,·), (e₁ + 2e₂ + e₃ + 2e₅ + e₆,·), (e₁ + 2e₂ + e₃ + 2e₆,·), (e₁ + 3e₂ + 2e₆,·) |

Observe that the integer program \(IP_{A,c}(b)\) where \(b = A(e₁ + e₂ + e₃)\) is solved by \(G^{\tau}(b)\) with \(\tau = \{1,4,5\}\). By Proposition 2.3, Gomory’s relaxation of \(IP_{A,c}(b)\) is indexed by \(\sigma = \{4,5,6\}\) since \(b\) lies in the interior of the face \(cone(A_\sigma)\) of \(\Delta_c\). However, neither this relaxation nor any nontrivial extended relaxation solves \(IP_{A,c}(b)\) since the optimal solution \(e₁ + e₂ + e₃\) is not covered by any standard pair (\(\cdot, \tau\)) where \(\tau\) is a non-empty subset of \(\{4,5,6\}\).

The results stated thus far give characterizations of Gomory families.

**Theorem 2.17.** The following conditions are equivalent:

(i) \(IP_{A,c}\) is a Gomory family.

(ii) The associated sets of \(IP_{A,c}\) are precisely the maximal faces of \(\Delta_c\).

(iii) (\(\cdot, \tau\)) is a standard pair of \(\mathcal{O}_c\) if and only if \(\tau\) is a maximal face of the regular triangulation \(\Delta_c\).

(iv) All standard polytopes of \(IP_{A,c}\) are simplices.
Proof. The proof follows from Definition 2.8, Proposition 2.11 and Theorem 2.14.

If there is a generic cost vector \( c \) such that for a triangulation \( \Delta \) of \( \text{cone}(A) \), \( \Delta = \Delta_c \), then we say that \( \Delta \) supports the order ideal \( O_c \) and the family of integer programs \( \text{IP}_{A,c} \). No regular triangulation of the matrix \( A \) in Example 2.16 supports a Gomory family. Here is a matrix with a Gomory family.

Example 2.18. Consider the \( 3 \times 6 \) matrix

\[
A = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 2 & 2 \\
0 & 0 & 1 & 2 & 3 & 4
\end{pmatrix}.
\]

In this case, \( \text{cone}(A) \) has 14 distinct regular triangulations and 48 distinct sets \( O_c \) as \( c \) varies among all generic cost vectors. Ten of these triangulations support Gomory families; one for each triangulation. For instance, if \( c = (0, 0, 1, 1, 0, 3) \), then

\[
\Delta_c = \{ \sigma_1 = \{1, 2, 5\}, \sigma_2 = \{1, 4, 5\}, \sigma_3 = \{2, 5, 6\}, \sigma_4 = \{4, 5, 6\} \}
\]

and \( \text{IP}_{A,c} \) is a Gomory family since the standard pairs of \( O_c \) are: \((0, \sigma_1), (e_3, \sigma_1), (e_4, \sigma_1), (0, \sigma_2), (0, \sigma_3), \) and \((0, \sigma_4)\).

3. Total dual integrality and Gomory families

We now relate the notion of total dual integrality \([16, \S 22]\) to Gomory families. Recall that \( \mathbb{Z}A = \mathbb{Z}^d \) by assumption.

Definition 3.1. The system \( yA \leq c \) is totally dual integral (TDI) if \( \text{LP}_{A,c}(b) \) has an integral optimal solution for each \( b \in \text{cone}(A) \cap \mathbb{Z}^d \).

Definition 3.2. The regular triangulation \( \Delta_c \) is unimodular if \( \mathbb{Z}A_\sigma = \mathbb{Z}^d \) for every maximal face \( \sigma \in \Delta_c \).

Theorem 3.3. The system \( yA \leq c \) is TDI if and only if the regular triangulation \( \Delta_c \) is unimodular.

Proof. The regular triangulation \( \Delta_c \) is the normal fan of \( P_c \) by Proposition 2.4, and it is unimodular if and only if \( \mathbb{Z}A_\sigma = \mathbb{Z}^d \) for every maximal face \( \sigma \in \Delta_c \). This is equivalent to saying that every \( b \in \text{cone}(A_\sigma) \cap \mathbb{Z}^d \) lies in \( \text{NA}_\sigma \) for every maximal face \( \sigma \) of \( \Delta_c \). By Lemma 2.3, this happens if and only if \( \text{LP}_{A,c}(b) \) has an integral optimum for all \( b \in \text{cone}(A) \cap \mathbb{Z}^d \).

Corollary 3.4. If \( yA \leq c \) is TDI then \( \text{IP}_{A,c} \) is a Gomory family.
Proof. By Lemma 2.15, \((0, \sigma)\) is a standard pair of \(\mathcal{O}_c\) for every maximal face \(\sigma\) of \(\Delta_c\). Theorem 3.3 implies that \(\text{cone}(A_\sigma)\) is unimodular (i.e., \(ZA_\sigma = \mathbb{Z}^d\)), and therefore \(NA_\sigma = \text{cone}(A_\sigma) \cap \mathbb{Z}^d\) for every maximal face \(\sigma\) of \(\Delta_c\). Hence the semigroups \(NA_\sigma\) arising from the standard pairs \((0, \sigma)\) as \(\sigma\) varies over the maximal faces of \(\Delta_c\) cover \(NA\). Therefore the only standard pairs of \(\mathcal{O}_c\) are \((0, \sigma)\) as \(\sigma\) varies over the maximal faces of \(\Delta_c\). The result then follows from Theorem 2.17 (ii).

When \(yA \leq c\) is TDI, the multiplicity of any maximal face \(\sigma\) of \(\Delta_c\) in \(\mathcal{O}_c\) is one, and all other faces have multiplicity zero. While this is sufficient for \(IP_{A,c}(b)\) to be a Gomory family, it is far from necessary. TDI-ness guarantees that \(LP_{A,c}(b)\) has an integral optimum for every integral \(b\) in \(cone(A)\). In contrast, if \(IP_{A,c}(b)\) is a Gomory family, the linear optima of the programs in \(LP_{A,c}\) may not be integral.

If \(A\) is unimodular (i.e., \(ZA_\sigma = \mathbb{Z}^d\) for every nonsingular maximal submatrix \(A_\sigma\) of \(A\)), then the feasible regions of the linear programs in \(LP_{A,c}\) have integral vertices for each integral \(b \in cone(A) \cap \mathbb{Z}^d\), and \(yA \leq c\) is TDI for all \(c\). Hence if \(A\) is unimodular, then \(IP_{A,c}\) is a Gomory family for all generic cost vectors \(c\). However, just as integrality of the optimal solutions of programs in \(LP_{A,c}\) is not necessary for \(IP_{A,c}\) to be a Gomory family, unimodularity of \(A\) is not necessary for \(IP_{A,c}\) to be a Gomory family for all \(c\).

Example 3.5. Consider the seven by twelve integer matrix

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

of rank seven. The maximal minors of \(A\) have absolute values zero, one and two and hence \(A\) is not unimodular. This matrix has 376 distinct regular triangulations supporting 418 distinct order ideals \(\mathcal{O}_c\). In each case, the standard pairs of \(\mathcal{O}_c\) are indexed by just the maximal simplices of the regular triangulation \(\Delta_c\) that supports it. Hence \(IP_{A,c}\) is a Gomory family for all generic \(c\).

4. \(\Delta\)-NORMAL MATRICES

In Section 3 we saw that unimodularity of \(A\) or more generally, unimodularity of a regular triangulation of \(cone(A)\), gives rise to Gomory families of integer programs. In this section, we identify a larger set of
matrices and cost vectors that give rise to Gomory families. A common property of unimodular matrices and matrices with a unimodular triangulation is that they form a Hilbert basis for cone$(A)$. In other words, $NA$ equals $cone(A) \cap \mathbb{Z}^d$ for such matrices. Borrowing a term from commutative algebra we make the following definition.

**Definition 4.1.** A $d \times n$ integer matrix $A$ is normal if the semigroup $NA$ equals $cone(A) \cap \mathbb{Z}^d$.

We first note that if $A$ is not normal, then $IP_{A,c}$ need not be a Gomory family for any cost vector $c$.

**Example 4.2.** The non-normal matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$ gives rise to 10 distinct order ideals $O_c$ supported on its four regular triangulations $\{\{1,4\}\}, \{\{1,2\}, \{2,4\}\}, \{\{1,3\}, \{3,4\}\}$ and $\{\{1,2\}, \{2,3\}, \{3,4\}\}$. Each $O_c$ has at least one standard pair that is indexed by a lower dimensional face of $\Delta_c$.

The matrix in Example 2.16 is also not normal and has no Gomory families. These examples show that normality of $A$ is necessary for the existence of Gomory families. However, we do not know at this time whether every normal matrix $A$ has some generic cost vector $c$ such that $IP_{A,c}$ is a Gomory family. Our goal is to show that under certain additional conditions, normal matrices do give rise to Gomory families.

**Definition 4.3.** A $d \times n$ integer matrix $A$ is $\Delta$-normal if it has a triangulation $\Delta$ such that for every maximal face $\sigma \in \Delta$, the columns of $A$ in $cone(A_{\sigma})$ form a Hilbert basis for $cone(A_{\sigma})$.

**Remark 4.4.** If $A$ is $\Delta$-normal for some triangulation $\Delta$, then it is normal. To see this note that every lattice point in $cone(A)$ lies in $cone(A_{\sigma})$ for some maximal face $\sigma \in \Delta$. Since $A$ is $\Delta$-normal, this lattice point also lies in the semigroup generated by the columns of $A$ in $cone(A_{\sigma})$ and hence in $NA$.

Observe that $A$ is $\Delta$-normal with respect to all the unimodular triangulations of $cone(A)$. Hence triangulations $\Delta$ with respect to which $A$ is $\Delta$-normal generalize unimodular triangulations of $cone(A)$.

Examples 4.5 and 4.6 show that the set of matrices where $cone(A)$ has a unimodular triangulation is a proper subset of the set of $\Delta$-normal matrices which in turn is a proper subset of the set of normal matrices.

**Example 4.5.** Examples of normal matrices with no unimodular triangulations can be found in [2] and [7]. If $cone(A)$ is simplicial for
such a matrix, \( A \) will be \( \Delta \)-normal with respect to its coarsest (regular) triangulation \( \Delta \) consisting of the single maximal face with support \( \text{cone}(A) \). For instance, consider the following example taken from [7]:

\[
A = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 & \\
0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & \\
0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 
\end{pmatrix}.
\]

Here \( \text{cone}(A) \) has 77 regular triangulations and no unimodular triangulations. Since \( \text{cone}(A) \) is simplicial, \( A \) is \( \Delta \)-normal with respect to its coarsest regular triangulation \( \{1, 2, 3, 8\} \).

**Example 4.6.** There are normal matrices \( A \) that are not \( \Delta \)-normal with respect to any triangulation of \( \text{cone}(A) \). To see such an example, consider the following modification of the matrix in Example 4.5 that appears in [19, Example 13.17]:

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\
0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 
\end{pmatrix}.
\]

This matrix is again normal and each of its nine columns generate an extreme ray of \( \text{cone}(A) \). Hence the only way for this matrix to be \( \Delta \)-normal for some \( \Delta \) would be if \( \Delta \) is a unimodular triangulation of \( \text{cone}(A) \). However, this \( \text{cone}(A) \) has no unimodular triangulations.

**Theorem 4.7.** If \( A \) is \( \Delta \)-normal for some regular triangulation \( \Delta \) then there exists a generic cost vector \( c \in \mathbb{Z}^n \) such that \( \Delta = \Delta_c \) and \( IP_{A,c} \) is a Gomory family.

**Proof.** Without loss of generality we can assume that the columns of \( A \) in \( \text{cone}(A_\sigma) \) form a minimal Hilbert basis of this cone for any maximal face \( \sigma \) of \( \Delta \). If there were a redundant element, the smaller matrix obtained by removing this column from \( A \) would still be \( \Delta \)-normal.

For a maximal face \( \sigma \in \Delta \), let \( \sigma_{in} \subset \{1, \ldots, n\} \) be the set of indices of all columns of \( A \) lying in \( \text{cone}(A_\sigma) \) that are different from the columns of \( A_\sigma \). Suppose \( a_{i_1}, \ldots, a_{i_k} \) are the columns of \( A \) that generate the one dimensional faces of \( \Delta \), and \( c' \in \mathbb{R}^n \) a cost vector such that \( \Delta = \Delta_{c'} \). We modify \( c' \) to obtain a new cost vector \( c \in \mathbb{R}^n \) such that \( \Delta = \Delta_c \) as follows. For \( j = 1, \ldots, k \), let \( c_j := c'_{i_j} \). If \( j \in \sigma_{in} \) for some maximal face \( \sigma \in \Delta \), then \( a_j = \sum_{i \in \sigma} \lambda_i a_i \), \( 0 \leq \lambda_i < 1 \) and we define \( c_j := \sum_{i \in \sigma} \lambda_i c_i \). Hence, for all \( j \in \sigma_{in} \), \((a_j, c_j) \in \mathbb{R}^{d+1} \) lies
in $C_{\sigma} := \text{cone}((a_i, c_i) : i \in \sigma) = \text{cone}((a_i', c_i') : i \in \sigma)$ which was a
facet of $C = \text{cone}((a_i, c_i') : i = 1, \ldots, n)$. If $y \in \mathbb{R}^d$ is a vector as in
Definition 2.1 showing that $\sigma$ is a maximal face of $\Delta_c$ then $y \cdot a_i = c_i$ for
all $i \in \sigma \cup \sigma_{in}$ and $y \cdot a_j < c_j$ otherwise. Since $\text{cone}(A_{\sigma}) = \text{cone}(A_{\sigma \cup \sigma_{in}})$,
we conclude that $\text{cone}(A_{\sigma})$ is a maximal face of $\Delta_c$.

If $b \in \mathbb{N}A$ lies in $\text{cone}(A_{\sigma})$ for a maximal face $\sigma \in \Delta_c$, then $IP_{A,c}(b)$
has at least one feasible solution $u$ with support in $\sigma \cup \sigma_{in}$ since $A$
is $\Delta$-normal. Further, $(b, c \cdot u) = (A_{\sigma}, c \cdot u)$ lies in $C_{\sigma}$ and all feasible
solutions of $IP_{A,c}(b)$ with support in $\sigma \cup \sigma_{in}$ have the same cost value by
construction. Suppose $v \in \mathbb{N}^n$ is any feasible solution of $IP_{A,c}(b)$ with
support not in $\sigma \cup \sigma_{in}$. Then $c \cdot u < c \cdot v$ since $(a_i, c_i) \in C_{\sigma}$ if and only
if $i \in \sigma \cup \sigma_{in}$ and $C_{\sigma}$ is a facet in the lower envelope of $C$. Hence the
optimal solutions of $IP_{A,c}(b)$ are precisely those feasible solutions with
support in $\sigma \cup \sigma_{in}$. The vector $b$ can be expressed as $b = b' + \sum_{i \in \sigma} z_i a_i$
where $z_i \in \mathbb{N}$ are unique and $b' \in \{\sum_{i \in \sigma} \lambda_i a_i : 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$ is also
unique. The vector $b' = \sum_{j \in \sigma_{in}} r_j a_j$ where $r_j \in \mathbb{N}$. Setting $u_i = z_i$ for
all $i \in \sigma$, $u_j = r_j$ for all $j \in \sigma_{in}$ and $u_k = 0$ otherwise, we obtain all
feasible solutions $u$ of $IP_{A,c}(b)$ with support in $\sigma \cup \sigma_{in}$.

If there is more than one such feasible solution, then $c$ is not generic.
In this case, we can perturb $c$ to a generic cost vector $c'' = c + \epsilon \omega$
by choosing $\epsilon > 0$, $\omega_j = 0$ whenever $j = i_1, \ldots, i_k$ and $\omega_j = 0$
otherwise. Suppose $u_1, \ldots, u_t$ are the optimal solutions of the integer
programs $IP_{A,c''}(b')$ where $b' \in \{\sum_{i \in \sigma} \lambda_i a_i : 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$. (Note
that $t = |\{\sum_{i \in \sigma} \lambda_i a_i : 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d|$ is the index of $\mathbb{Z}A_{\sigma}$ in
$\mathbb{Z}A$.) The support of each such $u_i$ is contained in $\sigma_{in}$. For any $b \in
\text{cone}(A_{\sigma}) \cap \mathbb{Z}^d$, the optimal solution of $IP_{A,c''}(b)$ is hence $u = u_i + z$
for some $i \in \{1, \ldots, t\}$ and $z \in \mathbb{N}^n$ with support in $\sigma$. This shows
that $\mathbb{N}A$ is covered by the affine semigroups $\phi_{A}(S(u_i, \sigma))$ where $\sigma$ is a
maximal face of $\Delta$ and $u_i$ as above for each $\sigma$. By construction, the
corresponding admissible pairs $(u_i, \sigma)$ are all standard for $\mathcal{O}_c$. Since
all data is integral, $c'' \in \mathbb{Q}^n$ and hence can be scaled to lie in $\mathbb{Z}^n$.
Renaming $c''$ as $c$, we conclude that $IP_{A,c}$ is a Gomory family. \hfill \Box

**Corollary 4.8.** Let $A$ be any normal matrix such that $\text{cone}(A)$ is
simplicial, and let $\Delta$ be the coarsest triangulation whose single maximal
face has support $\text{cone}(A)$. Then there exists a cost vector $c \in \mathbb{Z}^n$ such
that $\Delta = \Delta_c$ and $IP_{A,c}$ is a Gomory family.

**Example 4.9.** Consider the normal matrix in Example 2.18. Here
$\text{cone}(A)$ is generated by the first, second and sixth columns of $A$
and hence $A$ is $\Delta$-normal with respect to the regular triangulation
$\{\{1, 2, 6\}\}$. There are 13 distinct sets $\mathcal{O}_c$ supported on $\Delta$. Among the
13 corresponding families of integer programs, only one is a Gomory family. A representative cost vector for this \( IP_{A,c} \) is \( c = (0, 0, 4, 4, 1, 0) \). The standard pair decomposition of \( O_c \) is the one constructed in Theorem 4.7. The affine semigroups \( S(\cdot, c) \) from this decomposition are:

\[
S(0, \sigma), S(e_3, \sigma), S(e_4, \sigma), \text{ and } S(e_5, \sigma).
\]

Note that \( A \) is not \( \Delta \)-normal with respect to the regular triangulation supporting the Gomory family \( IP_{A,c} \) in Example 2.18. The columns of \( A \) in \( cone(A_{\sigma_1}) \) are the columns of \( A_{\sigma_1} \) and \( A_3 \). The vector \( (1, 2, 2) \) is in the minimal Hilbert basis of \( cone(A_{\sigma_1}) \) but is not a column of \( A \). This example shows that a regular triangulation \( \Delta \) of \( cone(A) \) can support a Gomory family even if \( A \) is not \( \Delta \)-normal. The Gomory families in Theorem 4.7 have a very special standard pair decomposition.

5. Hilbert covers and Gomory families

The results in the previous section lead to the following problem.

**Problem 5.1.** If \( A \in \mathbb{Z}^{d \times n} \) is a normal matrix, does there exist a generic cost vector \( c \in \mathbb{Z}^n \) such that \( IP_{A,c} \) is a Gomory family?

We do not know the answer to this question. However, in this section we answer a stronger version of this question for small values of \( d \) and state our observations for general \( d \). We begin with the following result.

**Theorem 5.2.** If \( A \in \mathbb{Z}^{d \times n} \) is a normal matrix and \( d \leq 3 \), then there exists a generic cost vector \( c \in \mathbb{Z}^n \) such that \( IP_{A,c} \) is a Gomory family.

**Proof.** It is known that if \( d \leq 3 \) then \( cone(A) \) has a regular unimodular triangulation \( \Delta_c \). [17]. The result then follows from Corollary 3.4. \( \square \)

Before we proceed, we rephrase Problem 5.1 in terms of covering properties of \( cone(A) \) and \( NA \) along the lines of [2], [3], [4], [7] and [17]. To obtain the same set up as in these papers we assume in this section that \( A \) is normal and the columns of \( A \) form the unique minimal Hilbert basis of \( cone(A) \). Using the terminology in [3], the free Hilbert cover problem asks whether there exists a covering of \( NA \) by semigroups \( NA_{\tau} \) where the columns of \( A_{\tau} \) are linearly independent. The unimodular Hilbert cover problem asks whether \( cone(A) \) can be covered by full dimensional unimodular subcones \( cone(A_{\tau}) \) (i.e., \( ZA_{\tau} = \mathbb{Z}^d \)), while the stronger unimodular Hilbert partition problem asks whether \( cone(A) \) has a unimodular triangulation. (Note that if \( cone(A) \) has a unimodular Hilbert cover or partition using subcones \( cone(A_{\tau}) \), then \( NA \) is covered by the semigroups \( NA_{\tau} \).) All these problems have positive answers if \( d \leq 3 \) since \( cone(A) \) admits a unimodular Hilbert partition in this case [2], [17]. Normal matrices (with \( d = 4 \)) such that \( cone(A) \) has
no unimodular Hilbert partition can be found in [2] and [7]. Examples (with \( d = 6 \)) that admit no free Hilbert cover and hence no unimodular Hilbert cover can be found in [3] and [4].

When \( yA \leq c \) is TDI, the standard pair decomposition of \( NA \) induced by \( c \) gives a unimodular Hilbert partition of \( cone(A) \) by Theorem 3.3. An important difference between Problem 5.1 and the Hilbert cover problems is that affine semigroups cannot be used in Hilbert covers. Moreover, affine semigroups that are allowed in standard pair decompositions come from integer programming. If there are no restrictions on the affine semigroups that can be used in a cover, \( NA \) can always be covered by full dimensional affine semigroups: for any triangulation \( \Delta \) of \( cone(A) \) with maximal subcones \( cone(A_\sigma) \), the affine semigroups \( b + NA_\sigma \) cover \( NA \) as \( b \) varies in \( \{ \sum_{i \in \sigma} \lambda_i a_i : 0 \leq \lambda_i < 1 \} \cap \mathbb{Z}^d \) and \( \sigma \) varies among the maximal faces of the triangulation. A partition of \( NA \) derived from this idea can be found in [18, Theorem 5.2].

In order to state our main theorem, we recall the notion of supernormality which was introduced in [11].

Definition 5.3. A matrix \( A \in \mathbb{Z}^{d \times n} \) is supernormal if for every submatrix \( A' \) of \( A \), the columns of \( A \) that lie in \( cone(A') \) form a Hilbert basis for \( cone(A') \).

Proposition 5.4. For \( A \in \mathbb{Z}^{d \times n} \), the following are equivalent:

(i) \( A \) is supernormal,
(ii) \( A \) is \( \Delta \)-normal for every regular triangulation \( \Delta \) of \( cone(A) \),
(iii) Every triangulation of \( cone(A) \) in which all columns of \( A \) generate one dimensional faces is unimodular.

Proof. The equivalence of (i) and (iii) was established in [11, Proposition 3.1]. Definition 5.3 shows that (i) \( \Rightarrow \) (ii). Hence we just need to show that (ii) \( \Rightarrow \) (i). Suppose that \( A \) is \( \Delta \)-normal for every regular triangulation of \( cone(A) \). In order to show that \( A \) is supernormal we only need to check submatrices \( A' \) where the dimension of \( cone(A') \) is \( d \). Choose a cost vector \( c \) with \( c_i \gg 0 \) if the \( i \)-th column of \( A \) does not generate an extreme ray of \( cone(A') \), and \( c_i = 0 \) otherwise. This gives a polyhedral subdivision of \( cone(A) \) in which \( cone(A') \) is a maximal face. There are standard procedures that will refine this subdivision to a regular triangulation \( \Delta \) of \( cone(A) \). Let \( T \) be the set of maximal faces \( \sigma \) of \( \Delta \) such that \( cone(A_\sigma) \) lies in \( cone(A') \). Since \( A \) is \( \Delta \)-normal, the columns of \( A \) that lie in \( cone(A_\sigma) \) form a Hilbert basis for \( cone(A_\sigma) \) for each \( \sigma \in T \). However, since their union is the set of columns of \( A \) that lie in \( cone(A') \), this union forms a Hilbert basis for \( cone(A') \). \( \Box \)
It is easy to catalog all $\Delta$-normal and supernormal matrices, of the type considered in this paper, for small values of $d$. We say that the matrix $A$ is graded if its columns span an affine hyperplane in $\mathbb{R}^d$. If $d = 1$, $\text{cone}(A)$ has $n$ triangulations $\{\{i\}\}$ each of which has the unique maximal subcone $\text{cone}(A_i)$ whose support is $\text{cone}(A)$. If we assume that $a_1 \leq a_2 \leq \cdots \leq a_n$, then $A$ is normal if and only if either $a_1 = 1$, or $a_n = -1$. Also, $A$ is normal if and only if it is supernormal. If $d = 2$ and the columns of $A$ are ordered counterclockwise around the origin, then $A$ is normal if and only if $\det(a_i, a_{i+1}) = 1$ for all $i = 1, \ldots, n-1$. Such an $A$ is supernormal since it is $\Delta$-normal for every triangulation $\Delta$ — the Hilbert basis of a maximal subcone of $\Delta$ is precisely the set of columns of $A$ in that subcone. If $d = 3$ then as mentioned before, $\text{cone}(A)$ has a unimodular triangulation with respect to which $A$ is $\Delta$-normal. However, not every such $A$ needs to be supernormal: we saw that the matrix in Example 2.18 is not $\Delta$-normal for the $\Delta$ supporting the Gomory family in that example. If $d = 3$ and $A$ is graded, then without loss of generality we can assume that the columns of $A$ span the hyperplane $x_1 = 1$. If $A$ is normal as well, then its columns are precisely all the lattice points in the convex hull of $A$. Conversely, every graded normal $A$ with $d = 3$ arises this way — its columns are all the lattice points in a polygon in $\mathbb{R}^2$ with integer vertices. In particular, every triangulation of $\text{cone}(A)$ that uses all the columns of $A$ is unimodular. Hence, by Proposition 5.4, $A$ is supernormal, and therefore $\Delta$-normal for any triangulation of $A$.

**Theorem 5.5.** Let $A \in \mathbb{Z}^{d \times n}$ be a normal matrix of rank $d$.

(i) If $d = 1, 2$ or $A$ is graded and $d = 3$, every regular triangulation of $\text{cone}(A)$ supports at least one Gomory family.

(ii) If $d = 2$ and $A$ is graded, every regular triangulation of $\text{cone}(A)$ supports exactly one Gomory family.

(iii) If $d = 3$ and $A$ is not graded, or if $d = 4$ and $A$ is graded, then not all regular triangulations of $\text{cone}(A)$ may support a Gomory family. In particular, $A$ may not be $\Delta$-normal with respect to every regular triangulation.

**Proof.** (i) If $d = 1, 2$ or $A$ is graded and $d = 3$, $A$ is supernormal and hence by Proposition 5.4 and Theorem 4.7, every regular triangulation of $\text{cone}(A)$ supports at least one Gomory family.

(ii) If $d = 2$ and $A$ is graded, then we may assume that

$$A = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 2 & \cdots & n-1
\end{pmatrix}.$$
In this case, \( A \) is supernormal and hence every regular triangulation \( \Delta \) of \( cone(A) \) supports a Gomory family by Theorem \[7\]. Suppose the maximal cones of \( \Delta \), in counter-clockwise order, are \( C_1, \ldots, C_r \). Assume the columns of \( A \) are labeled such that \( C_i = cone(a_{i-1}, a_i) \) for \( i = 1, \ldots, r \), and the columns of \( A \) in the interior of \( C_i \) are labeled in counter-clockwise order as \( b_{i1}, \ldots, b_{ik_i} \). Hence the \( n \) columns of \( A \) from left to right are:

\[
a_0, b_{11}, \ldots, b_{1k_1}, a_1, b_{21}, \ldots, a_{r-1}, b_{r1}, \ldots, b_{rk_r}, a_r.
\]

Indexing the columns of \( A \) by their labels, the maximal faces of \( \Delta \) are \( \sigma_i = \{i - 1, i\} \) for \( i = 1, \ldots, r \). Let \( e_i \) be the unit vector of \( \mathbb{R}^n \) indexed by the true column index of \( a_i \) in \( A \) and \( e_{ij} \) be the unit vector of \( \mathbb{R}^n \) indexed by the true column index of \( b_{ij} \) in \( A \). Since the columns of \( A \) form a minimal Hilbert basis of \( cone(A) \), \( e_i \) is the unique solution to \( IP_{A,c}(a_i) \) for all \( c \) and \( e_{ij} \) is the unique solution to \( IP_{A,c}(b_{ij}) \) for all \( c \). Hence the standard pairs of Theorem \[7\] are \((0, \sigma_i)\) and \((e_{ij}, \sigma_i)\) for \( i = 1, \ldots, r \) and \( j = 1, \ldots, k_i \).

Suppose \( \Delta \) supports a second Gomory family \( IP_{A,\omega} \). Then every standard pair of \( O_w \) is also of the form \((\cdot, \sigma_i)\) for \( \sigma_i \in \Delta \), and \( r \) of them are \((0, \sigma_i)\) for \( i = 1, \ldots, r \). The remaining standard pairs are of the form \((e_{ij}, \sigma_k)\). To see this, consider the semigroups in \( NA \) arising from the standard pairs of \( O_w \). The total number of standard pairs of \( O_c \) and \( O_w \) are the same. Since the columns of \( A \) all lie on \( x_1 = 1 \), no two \( b_{ij}s \) can be covered by a semigroup coming from the same standard pair and none of them are covered by a semigroup \((0, \sigma_i)\). We show that if \((e_{ij}, \sigma_k)\) is a standard pair of \( O_w \) then \( k = i \) and thus \( O_w = O_c \).

If \( r = 1 \), the standard pairs of \( O_w \) are \((0, \sigma_1), (e_{11}, \sigma_1), \ldots, (e_{1k_1}, \sigma_1)\) as in Theorem \[7\]. If \( r > 1 \), consider the last cone \( C_r = cone(a_{r-1}, a_r) \). If \( a_{r-1} \) is the second to last column of \( A \), then \( C_r \) is unimodular and the semigroup from \((0, \sigma_r)\) covers \( C_r \cap \mathbb{Z}^2 \). The subcomplex comprised of \( C_1, \ldots, C_{r-1} \) is a regular triangulation \( \Delta' \) of \( cone(A') \) where \( A' \) is obtained by dropping the last column of \( A \). Since \( A' \) is a normal graded matrix with \( d = 2 \) and \( \Delta' \) has less than \( r \) maximal cones, the standard pairs supported on \( \Delta' \) are as in Theorem \[7\] by induction. If \( a_{r-1} \) is not the second to last column of \( A \) then \( b_{rk_r} \), the second to last column of \( A \) is in the Hilbert basis of \( C_r \) but is not a generator of \( C_r \). So \( O_w \) has a standard pair of the form \((e_{rk_r}, \sigma_i)\). If \( \sigma_i \neq \sigma_r \), then the lattice point \( b_{rk_r} + a_r \) cannot be covered by the semigroup from this or any other standard pair of \( O_w \). Hence \( \sigma_i = \sigma_r \). By a similar argument, the remaining standard pairs indexed by \( \sigma_r \) are \((e_{r(k_r-1)}, \sigma_r), \ldots, (e_{r1}, \sigma_r)\) along with \((0, \sigma_r)\). These are precisely the standard pairs of \( O_c \) indexed by \( \sigma_r \). Again we are reduced to considering the subcomplex comprised
of $C_1, \ldots, C_{r-1}$ and by induction, the remaining standard pairs of $O_w$ are as in Theorem 4.7.

(iii) The $3 \times 6$ normal matrix $A$ of Example 2.18 has 10 distinct Gomory families supported on 10 out of the 14 regular triangulations of $cone(A)$. Furthermore, the normal matrix

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 1 & 1 & 0 \\
0 & 0 & 4 & 3 & 2 & 1
\end{pmatrix}
\]

has 11 distinct Gomory families supported on 11 out of the 19 regular triangulations of $cone(A)$.

6. Computations and Commutative Algebra

As mentioned in the introduction, the computations in this paper were done using the connections of this material to commutative algebra (see [19], [20] and [23]). In this section we give a brief description of our methods.

Our general feeling is that Problem 5.1 has a negative answer. To check whether a matrix $A$ has a Gomory family, we first need to compute all the distinct sets of optimal solutions $O_c$ (to the programs in $IP_{A,c}$) that arise as $c$ varies among the generic cost vectors with respect to $A$. As mentioned in Section 2, there are only finitely many such sets for a fixed $A$. To check whether $IP_{A,c}$ is a Gomory family, we need to compute the standard pair decomposition of $O_c$: $IP_{A,c}$ is a Gomory family if and only if all the standard pairs of $O_c$ are indexed by maximal faces of $\Delta_c$.

A monomial $x^u$ in the polynomial ring $S := \mathbb{Q}[x_1, \ldots, x_n]$ is the product $x^u = x_1^{u_1} \cdots x_n^{u_n}$ where $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$. An ideal $M$ in $S$ is called a monomial ideal if it is generated by monomials. Since every ideal in $S$ is finitely generated, $M = \langle x^{u_1}, \ldots, x^{u_t} \rangle$ for a set of minimal generators $x^{v_1}, \ldots, x^{v_t}$. The toric ideal of $A$, denoted as $I_A$ is the binomial ideal in $S$ defined as:

\[
\langle x^u - x^v : u, v \in \mathbb{N}^n \text{ and } Au = Av \rangle.
\]

The cost of a monomial $x^u$ with respect to a cost vector $c \in \mathbb{R}^n$ is the dot product $c \cdot u$ and the initial term of a polynomial $f = \sum \lambda_u x^u \in S$ is the sum of all terms in $f$ of highest cost. For any ideal $I \subset S$, the initial ideal of $I$ with respect to $c$, denoted as $in_c(I)$, is the ideal generated by all the initial terms of all polynomials in $I$. These concepts come from the theory of Gröbner bases for polynomial ideals [3]. The toric
ideal \( I_A \) provides the algebraic link between integer programming and Gröbner basis theory. For an introduction to this connection see [19].

**Proposition 6.1.** (i) A cost vector \( c \in \mathbb{R}^n \) is generic for \( A \) if and only if the initial ideal \( \text{in}_c(I_A) \) is a monomial ideal.

(ii) For a generic \( c \), the vector \( u \) belongs to \( \mathcal{O}_c \) if and only if \( x^u \) is not in the initial ideal \( \text{in}_c(I_A) \).

There are only finitely many distinct initial ideals for a polynomial ideal [19], and hence there are only finitely many distinct sets \( \mathcal{O}_c \) as \( c \) varies among the generic cost vectors. We say that two cost vectors \( c \) and \( c' \) in \( \mathbb{R}^n \) are equivalent if \( \text{in}_c(I_A) = \text{in}_{c'}(I_A) \).

**Theorem 6.2.** [20, Theorem 3.10] (i) Each equivalence class of generic cost vectors form an open full dimensional polyhedral cone in \( \mathbb{R}^n \). The closure of this cone is called the Gröbner cone of \( c \) where \( c \) is any vector in the interior of the cone.

(ii) The collection of all Gröbner cones of \( A \) form a complete polyhedral fan in \( \mathbb{R}^n \) called the Gröbner fan of \( A \).

(iii) The Gröbner fan of \( A \) is the normal fan of an \((n-d)\)-dimensional polytope called the state polytope of \( A \). □

By the above results, \( \mathcal{O}_c \) can be computed implicitly by computing the monomial ideal \( \text{in}_c(I_A) \). This can be done using a computer algebra package like Macaulay 2 [10]. In order to find all initial ideals of \( I_A \), we use the software package TiGERS [14] for enumerating the vertices of the state polytope of \( A \). At each vertex, TiGERS returns the initial ideal induced by a vector in the interior of the normal cone at that vertex. The standard pair decomposition of the set of monomials outside a monomial ideal described in terms of its minimal generators can be calculated using Macaulay 2. See the chapter Monomial Ideals in [12].

To obtain a normal matrix of the type discussed in this paper, it suffices to start with an arbitrary set of vectors \( a_1, \ldots, a_p \in \mathbb{Z}^d \) such that \( \text{cone}(a_1, \ldots, a_p) \) is pointed and full dimensional and then to compute the Hilbert basis of \( \text{cone}(a_1, \ldots, a_p) \). The elements in the Hilbert basis form the columns of a normal matrix. We used the package Normaliz by Bruns and Koch [3] to compute Hilbert bases.

The regular triangulation \( \Delta_c \) of \( \text{cone}(A) \) is a pure \( d \)-dimensional complex of cones and hence every maximal face \( \sigma \in \Delta_c \) has cardinality \( d \). Hence we get the following algorithm.

**Algorithm 6.3.** How to check if \( A \in \mathbb{Z}^{d \times n} \) gives a Gomory family.

(i) Compute all the initial ideals of the toric ideal \( I_A \) using TiGERS.

(ii) For each initial ideal \( \text{in}_c(I_A) \):

(a) Use Macaulay 2 to find its standard pairs.
(b) $IP_{A,c}$ is a Gomory family if and only if every set $\tau$ in a standard pair $(\cdot, \tau)$ has cardinality $d$.

A monomial $x^u$ is square-free if $u \in \{0,1\}^n$ and a monomial ideal $M$ is square-free if all its minimal generators are square-free. The radical of a monomial ideal $M$ in $S$ is the ideal $\sqrt{M} := \langle f : f^r \in M \text{ for some } r \in \mathbb{N} \rangle$. The radical $\sqrt{M}$ is a square-free monomial ideal. The Stanley Reisner ideal of the regular triangulation $\Delta_c$ is the square-free monomial ideal

$$\langle \Pi_{i \in \tau} x_i : \tau \text{ is a minimal non-face of } \Delta_c \rangle.$$ 

**Theorem 6.4.** [19, Theorem 8.3] The Stanley-Reisner ideal of the regular triangulation $\Delta_c$ is the radical of $\text{in}_c(I_A)$. \hfill \Box

If two distinct initial ideals $\text{in}_c(I_A)$ and $\text{in}_{c'}(I_A)$ have the same radical, then $\Delta_c = \Delta_{c'}$ and we say that $\Delta_c$ supports these initial ideals. Several initial ideals of $I_A$ may have the same radical. Using the above theorem, the initial ideals of $I_A$ output by TiGERS can be grouped according to their radicals or equivalently, the regular triangulations supporting them. This, in combination with Algorithm 6.3, allows us to check whether a regular triangulation supports a Gomory family.

Recall from Theorem 3.3 that $yA \leq c$ is TDI if and only if $\Delta_c$ is unimodular. Using the following result, we obtain an algebraic check for TDI-ness of $yA \leq c$.

**Theorem 6.5.** [19, Corollary 8.9] The regular triangulation $\Delta_c$ is unimodular if and only if the initial ideal $\text{in}_c(I_A)$ is square-free. \hfill \Box

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