ON THE SINGULARITIES OF EFFECTIVE LOCUS OF LINE BUNDLES

LEI SONG

Abstract. We prove that every irreducible component of a semi-regular locus of effective line bundles in the Picard scheme of a smooth projective variety has at worst rational singularities. This generalizes Kempf’s result on rational singularities of $W_0^d$ for smooth curves. We also work out an example of such locus for a ruled surface.

1. Introduction

Fix a ground field $k$, which is algebraically closed and of characteristic 0. Let $X$ be a smooth projective curve of genus $g$. For $r, d \geq 0$, the Brill-Noether locus is defined as
$$\text{supp}(W_r^d(X)) = \{ L \in \text{Pic}(X) \mid h^0(L) \geq r + 1, \deg(L) = d \}.$$ There has been extensive research on these loci in literature (cf. [1]). A special kind of Brill-Noether loci is $W_0^d(X)$, which is the image of the Abel-Jacobi map $\varphi : X_d \to \text{Pic}^d(X)$, where $X_d$ denotes the $d$th symmetric product of $X$ and $\text{Pic}^d(X)$ denotes the Picard variety of degree $d$ line bundles on $X$. When $d = g - 1$, $W_0^{g-1}$ is a theta divisor if $\text{Pic}^{g-1}X$ is identified with the Jacobian of the curve.

Kempf [7] proved that $W_0^d(X)$ has only rational singularities, so in particular it is Cohen-Macaulay and normal; and for $1 \leq d \leq g - 1$, the tangent cone over a point $[L] \in W_0^d(X)$ admits a rational resolution from the normal bundle of the fibre $F = \varphi^{-1}([L])$. He also computed the degree of the tangent cone, generalizing Riemann’s formula on multiplicity of theta divisors. Ein [3] studied the normal sheaf $N$ of the fibre $F$. He showed $N$ can be reconstructed from the multiplication map $H^0(O_F(1)) \otimes H^1(N(-1)) \to H^1(N)$, and proved that for a general curve $X$, $N \cong \rho O_F \oplus (H^1(X, L) \otimes \Omega_F(1))$, where $\rho = g - (r + 1)(g + r - d)$ is the Brill-Noether number. Most of his results were built on a locally free resolution of $N^*$.

This paper attempts to extend part of Kempf’s results on $W_0^d$ for curves to higher dimensional varieties using Ein’s approach.

Now let $X$ be a smooth projective variety of arbitrary dimension. Let $\text{Pic}(X)$ and $\text{Div}(X)$ denote the Picard scheme and divisor scheme, which parameterize line bundles and effective divisors on $X$ respectively. We still have the Abel-Jacobi map $\varphi : \text{Div}(X) \to \text{Pic}(X)$, where $\text{Div}(X)$ plays the same role as $X_d$. However, as a closed subscheme of the Hilbert scheme $\text{Hilb}(X)$, $\text{Div}(X)$ may be very singular. Even for $\dim X = 2$, an example due to Severi and Zappa in 1940s shows that $\text{Div}(X)$ is nonreduced. For this reason, we restrict ourselves to those so called semi-regular line bundles, see section 2 for definition, and consider the semi-regular locus $W_{sr}^0(X)$ they form in $\text{Pic}(X)$. We refer to [8] for background of $\text{Pic}(X)$ and $\text{Div}(X)$. We prove

Theorem 1.1. Let $X$ be a smooth projective variety. Then any irreducible component of $W_{sr}^0(X)$ has only rational singularities.

Key words and phrases. semi-regularity, Brill-Noether locus, rational singularities.
When \( X \) is a curve, \( W^0_{\text{eff}} = \bigcup_{d \geq 0} W^0_d \) and \( W^0_d \) is irreducible, and so the theorem recovers Kempf’s result that \( W^0_d \) has rational singularities.

The paper is organized as follows: in section 2, we study the conormal sheaf of fibres of the Abel-Jacobi map, and derive a locally free resolution of the sheaf. We also obtain several interesting consequences of the resolution. With a criterion of rational singularities based on Kovács’s work, we prove our main theorem. In section 3, one example of an irreducible component \( W^0_0 \) of a ruled surface is analyzed in detail. In the appendix we prove a result on varieties swept out by linear spans of divisors of linear systems on an embedded curve.

### 2. Rational singularities of \( W^0_{\text{eff}}(X) \)

#### 2.1. Semi-regular line bundles and their loci.

Let \( X \) be a smooth projective variety. It is well known that \( \text{Pic}(X) \) is separated, locally of finite type, and smooth. As \( \text{Hilb}(X) \) breaks up according to Hilbert polynomials, so does \( \text{Pic}(X) \). Fix an ample line bundle \( O_X(1) \) on \( X \). For each line bundle \( L \) on \( X \), there exists a \( \mathbb{Q} \)-coefficient polynomial \( P_L \) such that \( P_L(n) = \chi(L(n)) \) for \( n \in \mathbb{Z} \). The \( P_L \) is constant over any connected component of \( \text{Pic}(X) \).

The Abel-Jacobi map \( \phi : \text{Div}(X) \to \text{Pic}(X) \), which sends a divisor \( D \) to the associated line bundle \( O_X(D) \), is a projective morphism. For any line bundle \( L \) on \( X \), canonically \( \phi^{-1}([L]) \approx [L] \), where \([L] \) is the corresponding point of \( L \) in \( \text{Pic}(X) \) (cf. \([8]\)).

**Definition 2.1.** An effective Cartier divisor \( D \) on \( X \) is semi-regular if the boundary map

\[
\partial : H^1(O_X(D)) \to H^2(O_X)
\]

is injective. A line bundle \( L \) is semi-regular if \( L \) is effective, and \( D \) is semi-regular for any \( D \in [L] \).

**Remark 2.2.** If \( X \) is a curve, then all effective divisors, line bundles are automatically semi-regular. The reader can check that a necessary condition for \( L \) to be semi-regular is that \( h^1(L) \leq q \) (see Corollary \([13]\)), and sufficient conditions are either \( h^1(L) = 0 \) or \( h^1(O_X(D)) = 0 \) for all \( D \in [L] \). The second one is however rather strong. For instance, when \( X \) is a surface and \( p_g = h^0(\omega_X) > 0 \), \( h^1(O_X(D)) = 0 \) implies that \( \text{supp}(D) \subset \text{Bs}(\omega_X) \).

**Theorem 2.3** (Severi-Kodaira-Spencer). Assume \( \text{char}(k) = 0 \). \( \text{Div}(X) \) is smooth at \([D]\) of the expected dimension

\[
R := h^0(O_X(D)) - h^1(O_X(D)) - 1 + h^1(O_X)
\]

if and only if \( D \) is semi-regular (cf. \([8],[12]\)).

**Definition 2.4** (semi-regular locus),

\[
\text{supp}(W^0_{\text{eff}}(X)) = \{ L \in \text{Pic}(X) \mid L \text{ is semi-regular} \}.
\]

**Remark 2.5.** From Theorem \([2,3]\) we see that “semi-regular” is an open condition on the locus of effective line bundles \( W^0_{\text{eff}}(X) \subset \text{Pic}(X) \) and that any connected component of \( W^0_{\text{eff}}(X) \) is irreducible. Thus each component of \( W^0_{\text{eff}}(X) \) is a subvariety of \( \text{Pic}(X) \). It is worth noting that not every irreducible component of \( W^0_{\text{eff}}(X) \) contains some semi-regular line bundle, see remark \([3,5]\) for an example.
Next we explain the idea of proof of Theorem 1.1. Given \([L] \in \Omega\), which is a component of \(W^0_{\phi}\), there exists a unique \(\Delta_0\) among all irreducible components \(\{\Delta_i\}\) of \(\text{Div}(X)\), such that \(\dim \Delta_0 = R\) and \(\phi^{-1}([L]) \subset (\Delta_0)_{\text{reg}} \cup_{\Delta_i \neq 0} \Delta_i\), where \((\Delta_0)_{\text{reg}}\) is the regular locus of \(\Delta_0\). Consider the induced Abel-Jacobi morphism \(\phi : \Delta_0 \to \Omega\). By properness of \(\phi\), there is a smooth neighborhood \(U\) of \(\phi^{-1}([L])\) inside \(\Delta_0\), such that \(\phi(U)\) is open in \(\Omega\) and \(\phi^{-1}\phi(U) = U\). By abuse of notation, we denote this \(U\) by \(\text{Div}(X)\), hence the normal sheaf of the fibre by \(N_{\phi^{-1}([L])/\text{Div}(X)}\) instead of \(N_{\phi^{-1}([L])/U}\), and simply by \(N\) if the fibre is clear in the context. Under the semi-regularity assumption, \(N\) can be calculated from the universal family of divisors associated to \([L]\) by base change theorem. It turns out that the vector bundle \(N^*\) on the projective space \([L]\) has Castelnuovo-Mumford regularity 0, therefore formal function theorem shows that \(R^i\phi_*O_{\Delta_0} = 0\) for \(i > 0\). Finally, though \(\phi\) is not a resolution of singularities of \(\Omega\), as it is not birational in general, Theorem 2.6 guarantees that \(\Omega\) has rational singularities at \([L]\).

### 2.2. A criterion for rational singularities

The theorem below is a characterization of rational singularities. The original assumption is more general than what we state here.

**Theorem 2.6** (Kovács [10]). Let \(f : Y \to X\) be a surjective proper morphism of varieties. Assume that \(Y\) has rational singularities and that \(f_*O_Y \cong O_X\), \(R^if_*O_Y = 0\) for all \(i > 0\). Then \(X\) has rational singularities.

Based on the above, we obtain Theorem 2.9, which is more than the need to prove Theorem 1.1 and maybe applied to other problems.

**Lemma 2.7.** Let \(X\) be a smooth projective of dimension \(d\) with \(H^1(X, O_X) = 0\), and \(L\) a globally generated ample line bundle on \(X\) with the property that \(K_L + (d-1)L\) is non-effective. Then any \(m\)-regular (with respect to \(L\)) coherent sheaf \(\mathcal{F}\) admits a locally free resolution of the form:

\[
\cdots \to V_i \otimes L^{-(m+1)} \cdots \to V_i \otimes L^{-(m+1)} \to V_0 \otimes L^{-m} \to \mathcal{F} \to 0,
\]

where \(V_i\)'s are finite dimensional vector spaces. Consequently if \(\mathcal{F}\) is locally free and \(0\)-regular, then any symmetric power of \(\mathcal{F}\) is \(0\)-regular.

**Proof.** By Kodaira vanishing and the assumption \(H^1(X, O_X) = 1\), the condition that \(K_L \otimes L^{d-1}\) is non-effective implies that \(\text{reg}(O_X) \leq 1\). Then the result follows from [11] Remark 1.8.16.

**Remark 2.8.** Notice that the existence of such \(L\) in Lemma 2.7 imposes a strong restriction on \(X\): \(-K_X\) is big. Examples for Fano varieties are \(\mathbb{P}^d\), quadric hypersurface \(Q \subset \mathbb{P}^{d+1}\), and \(\mathbb{P}(O_{\mathbb{P}^d}^{d-1} \oplus O_{\mathbb{P}^d}(1))\) (with Fano index 1). If \(-K_X\) is also nef, the assumption \(H^1(X, O_X) = 0\) is redundant by Kawamata-Viehweg vanishing.

**Theorem 2.9.** Let \(f : Y \to X\) be a projective morphism from a smooth variety \(Y\) onto a normal variety \(X\). Let \(p \in X\) be a closed point. Suppose the scheme-theoretic fiber \(F\) is a smooth variety of dimension \(d\) with \(H^i(F, O_F) = 0\) for all \(i > 0\), and the conormal sheaf \(N^F_{f/1}\) is \(0\)-regular with respect to a globally generated ample line bundle \(L\) on \(F\), such that \(K_F + (d-1)L\) is non-effective. Then \(X\) has rational singularities in a neighbourhood of \(p\).

**Proof.** Consider the Stein factorization of \(f : Y \to Y' \to X\), where \(f'\) is projective with connected fibers, and \(g\) is a finite morphism. Then \(Z := g^{-1}(p)\) is a reduced closed point, for otherwise \(F = f'^{-1}(Z)\) would be nonreduced. Therefore \(g\) is generically one to one map, hence birational. By shrinking \(X\) properly, we can assume \(g\) is an isomorphism, therefore we assume \(f\) has connected fibres.
Let $\mathcal{I}$ be the ideal sheaf of $F$ in $Y$. By Lemma 2.7, any symmetric power $S^n(N_{F/Y}) \approx \mathcal{I}^n \cap \mathcal{I}^{n+1}$ is 0-regular. In particular, all its higher cohomologies vanish. From the exact sequence

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to O_{\pi+1}/F \to O_n \to 0,$$

we get $H^i(O_{\pi+1}/F) \cong H^i(O_n)$ for all $i, n > 0$. Then we conclude that $H^i(O_n) = 0$ for all $i > 0$. So $R^i f_*(O_Y)$ is 0, $\forall i > 0$ by the formal function theorem (cf. [6] III 11.1). Since the support of the coherent sheaf $\mathcal{O}_F$ is closed, by shrinking $X$, we can assume $R^i f_*(O_Y) = 0$ on $X$ for $i > 0$.

It’s clear that $f_* O_Y = O_X$, since $X$ is normal and fibers are connected. At this point, we apply Theorem 2.6 to conclude the proof.

2.3. Conormal sheaf of the fibre of $\varphi$. In the rest of section 2, $q$ denotes $h^0(X, O_X)$, the irregularity of $X$. $L$ stands for a line bundle on $X$ with $r = \dim [L]$ and $b = h^1(X, L)$. $F$ denotes the fibre $\varphi^{-1}([L]) \approx [L]$, where $\varphi$ is the Abel-Jacobi map $\text{Div}(X) \to \text{Pic}(X)$. Let $m$ be the maximal ideal of $O_{\text{Pic}(X), [L]}$; $m$ is the maximal ideal of $O_X$, and $\mathcal{I}$ the ideal sheaf of $F$ in $\text{Div}(X)$.

Given an effective line bundle $L$, $[L] \approx L(H^0(X, L^*))$. Let $Y = X \times [L]$ and $p, q$ be the two projections. By Künneth formula, $\Gamma(Y, p^* L \otimes q^* O(1)) \approx H^0(X, L) \otimes H^0(X, L^*)$. Fix a basis of the vector space $H^0(X, L)$, say $x_0, \ldots, x_r$. The section $s := \sum x_i \otimes x_i^*$ defines a relative Cartier divisor of $Y$ over $[L]$ via

$$0 \to O_Y \to p^* L \otimes q^* O(1) \to O_{\mathcal{D}}(Y) \to 0.$$ (2.2)

Denote the two induced projections from $\mathcal{D}$ to $X$ and $[L]$ also by $p$ and $q$. The divisor $\mathcal{D}$ is actually an incidence correspondence in the sense: for any $x \in X$, $p^{-1}(x)$ parameterizes the effective divisors passing through $x$; for any $[D] \in [L]$, $q^{-1}([D])$ is precisely the divisor $D$.

By the universal property of $\text{Div}(X)$, there is a unique morphism $j : [L] \to \text{Div}(X)$, such that $\mathcal{D} = j^* \mathcal{U}$, where $\mathcal{U}$ is the universal divisor over $\text{Div}(X)$, see the Cartesian diagrams below. In fact $j$ is a closed immersion.

$$\begin{array}{ccc}
D & \to & \mathcal{D} \\
\downarrow & & \downarrow \\
X & \to & [L] \\
\downarrow & & \downarrow \\
[D] & \to & \text{Div}(X)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{D} & \to & \mathcal{U} \\
\downarrow & & \downarrow \\
[\mathcal{D}] & \to & \text{Div}(X)
\end{array}$$

Suppose $L$ is semi-regular and $D \in [L]$, by Grauert’s theorem and the property of $\pi$,

$$\pi_* O_{\mathcal{U}}(\mathcal{U}) \otimes \kappa([D]) \approx H^0(X, N_{D/X}) \approx T_{\text{Div}(X)} \otimes \kappa([D]),$$

where $\kappa([D])$ is the residue field of $[D]$, and $T_{\text{Div}(X)}$ is the tangent sheaf of $\text{Div}(X)$. So by base change theorem (cf. [12] Lecture 7),

$$j^* \pi_* O_{\mathcal{U}}(\mathcal{U}) \approx q_* j^* O_{\mathcal{U}}(\mathcal{U}).$$

Notice that $j^* O_{\mathcal{U}}(\mathcal{U}) \approx O_{\mathcal{D}}(\mathcal{D})$, therefore

$$q_* O_{\mathcal{D}}(\mathcal{D}) \approx T_{\text{Div}(X)}[L].$$ (2.3)
The key theorem below is a generalization of [3] Theorem 1.1.

**Theorem 2.10.** With notation as above, let $L \in W^0_\mathbb{Z}$, and let $\mathcal{N}^*$ the conormal sheaf of $F$ in $\text{Div}(X)$. Then there is the exact sequence

\begin{equation}
0 \to H^1(L)^* \otimes O_F(-1) \to H^1(O_X)^* \otimes O_F \to \mathcal{N}^* \to 0.
\end{equation}

**Proof.** Note $O_Y(\mathcal{Z}) \simeq p^*L \otimes q^*O_F(1)$. Applying $q_*$ to (2.4), we get the exact sequence

$$0 \to O_F \to H^0(L) \otimes O_F(1) \to T_{\text{Div}(X)}|_F \to H^1(O_X) \otimes O_F \to H^1(L) \otimes O_F(1) \to 0.$$ 

The third term comes from (2.3). The surjectivity of the last map is for the reason as follows. For any point $[D] \in F$, we have the commutative diagram

$$
\begin{array}{ccc}
R^1q_*O_Y \otimes \kappa([D]) & \rightarrow & H^1(O_X) \\
\downarrow & & \downarrow \\
R^1q_*O_Y(\mathcal{Z}) \otimes \kappa([D]) & \rightarrow & H^1(O_X(D))
\end{array}
$$

The two horizontal maps are isomorphisms because of Grauert’s theorem. Since $\partial : H^1(O_D(D)) \to H^2(O_X)$ is injective by the semi-regularity assumption, the right vertical map is surjective, so is the left one.

Since the cokernel of $O_F \to H^0(L) \otimes O_F(1)$ is $T_F$, we get the short sequence

$$0 \to N \to H^1(O_X) \otimes O_F \to H^1(L) \otimes O_F(1) \to 0.$$ 

Dualizing it, we get

$$0 \to H^1(L)^* \otimes O_F(-1) \to H^1(O_X)^* \otimes O_F \to \mathcal{N}^* \to 0 \quad \square$$

**Corollary 2.11.** $\mathcal{N}^*$ has Castelnuovo-Mumford regularity 0. \(\square\)

2.4. **Proof of theorem** (1.1)

**Lemma 2.12.** The natural map $\bigoplus_{n \geq 0} \tilde{m}^n/\tilde{m}^{n+1} \to \bigoplus_{n \geq 0} H^0(\mathcal{Z}^n/\mathcal{Z}^{n+1})$ is a surjective graded $k$-algebra morphism. Furthermore, if $R \leq q$, then it is an isomorphism.

**Proof.** We do induction on $n \geq 1$. Consider the commutative diagram of $k$-vector spaces:

$$
\begin{array}{ccc}
m/m^2 & \rightarrow & \tilde{m}/\tilde{m}^2 \\
\downarrow & & \downarrow \\
H^1(O_X)^* & \rightarrow & H^0(\mathcal{N}^*_{F/\text{Div}(X)})
\end{array}
$$

Clearly, the top horizontal map is surjective. The left vertical map is an isomorphism by [8] Theorem 5.11. The bottom horizontal one is surjective by (2.4). Therefore $\tilde{m}/\tilde{m}^2 \rightarrow H^0(\mathcal{N}^*_{F/\text{Div}(X)})$ is surjective. For $n > 1$, consider the commutative diagram:

$$
\begin{array}{ccc}
S^n(\tilde{m}/\tilde{m}^2) & \rightarrow & \tilde{m}^n/\tilde{m}^{n+1} \\
\downarrow & & \downarrow \\
S^nH^0(\mathcal{Z}/\mathcal{Z}^2) & \rightarrow & H^0(\mathcal{Z}^n/\mathcal{Z}^{n+1})
\end{array}
$$

The bottom horizontal map is surjective, because $\mathcal{Z}/\mathcal{Z}^2 = \mathcal{N}^*_{F/\text{Div}(X)}$ is 0-regular. It follows that the right vertical map is surjective.
A proof for isomorphism when \( R \leq q \) can be found in \(^3\) Proposition 3.1 (c) and Theorem 3.2.

**Proof.** (of Theorem 1.1)

By Theorem 2.9 and Corollary 2.11 to finish the proof, it remains to show that any irreducible component \( \Omega \) of \( W_{sr}^0 \) is normal. Let \( Y = \varphi^{-1}(\Omega) \subset \text{Div}(X) \). Consider the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \tilde{m}^p/\tilde{m}^{p+1} & \rightarrow & O_{\Omega(L)}/\tilde{m}^{p+1} & \rightarrow & O_{\Omega(L)}/\tilde{m}^p & \rightarrow & 0 \\
& & \downarrow{\alpha} & & \downarrow{\beta_{p+1}} & & \downarrow{\beta_p} & & \downarrow{\psi} \\
0 & \rightarrow & H^0(\mathcal{O}/\mathcal{O}^{p+1}) & \rightarrow & H^0(O_{\Omega(L)}) & \rightarrow & H^0(O_{\Omega(L)}) & \rightarrow & 0
\end{array}
\]

By Lemma 2.12 \( \alpha_{p} \)'s are surjective. By Snake lemma and induction on \( n \), we get \( \beta_n \) is surjective for all \( n \geq 1 \). It follows that \( O_{\Omega(L)}^\wedge = \lim O_{\Omega(L)}/\tilde{m}^p \rightarrow \lim H^0(O_{\Omega(L)}) = (\varphi, O_Y)_{\text{form}} \) by formal function theorem. Thus the canonical morphism \( O_{\Omega(L)}^\wedge \rightarrow (\varphi, O_Y)_{\text{form}} \) is an isomorphism. Since the completion is a fully faithful functor, we get that \( \varphi, O_Y \approx O_{\Omega} \). Since \( Y \) is smooth and all fibres are connected, \( \Omega \) is normal. \( \square \)

### 2.5. Some consequences of Theorem 2.10

The corollary below is a generalization of Clifford theorem to higher dimensional varieties. It would be interesting to study when the equalities can be achieved.

**Corollary 2.13.** Assume \( [L] \in W_{sr}^0 \) and \( h^1(L) > 0 \). Then

(i) \( h^0(L) + h^1(L) \leq q + 1 \).

(ii) If \( X \) is a projective surface, then \( h^0(L) \leq \frac{q(q+1)}{2} \).

**Proof.** By \(^3\) Proposition 2.5, shape of the resolution \(^2\) of \( N^* \) forces \( \text{rank}(N^*) \geq r \). Since \( \dim \text{Div}(X) = R \), \( \text{rank}(N^*) = R - r \). With \( R = h^0(L) - h^1(L) + q - 1 \) (see \(^2\)), we get (i).

If \( X \) is a surface, then \( R \leq \chi(L) + q - 1 \). So \( h^0(L) = r + 1 \leq \frac{q(q+1)}{2} \).

**Corollary 2.14.** Let \( [L] \in W_{sr}^0 \). Assume \( R \leq q \). Then up to a constant, the Hilbert-Samuel function \( \psi \) for \( O_{W_{sr}[L]} \) is

\[
\psi(p) = \begin{cases} 
\sum_{i=0}^{b}(-1)^i(p+i-1)\binom{b}{i} & \text{if } b \leq r; \\
\sum_{i=0}^{b}(-1)^i(p+i-1)\binom{b}{i} + \sum_{i=r+1}^{b}(-1)^{r+i}(p+i-1)\binom{b}{i} \binom{i-1}{r} & \text{if } b > r.
\end{cases}
\]

**Proof.** The exact Eagon-Northcott complex associated to \(^2\) is

\[
0 \rightarrow S^{p-b}H^1(O_X)^* \otimes \bigotimes_{i=0}^{b} H^i(L)^* \otimes O_{\varphi}(-b) \rightarrow \cdots \rightarrow S^{p-1}H^1(O_X)^* \otimes H^1(L)^* \otimes O_{\varphi}(-1) \\
\rightarrow S^pH^1(O_X)^* \otimes O_{\varphi} \rightarrow S^pN^* \rightarrow 0.
\]

So for \( p > 0 \),

\[
\chi(S^pN^*) = \sum_{i=0}^{b}(-1)^i\chi(S^{p+i}H^1(O_X)^* \otimes \bigotimes_{i=0}^{i} H^i(L)^* \otimes O_{\varphi}(-i))
\]

\[
\chi(S^pN^*) = \begin{cases} 
\sum_{i=0}^{b}(-1)^i(p+i-1)\binom{b}{i} & \text{if } b \leq r; \\
\sum_{i=0}^{b}(-1)^i(p+i-1)\binom{b}{i} + \sum_{i=r+1}^{b}(-1)^{r+i}(p+i-1)\binom{b}{i} \binom{i-1}{r} & \text{if } b > r.
\end{cases}
\]

Since

\[
\Delta \psi(p) = \psi(p + 1) - \psi(p)
\]
Proof. proper morphism (cf.\cite{5} Chap. 4).

Combinatorial relations for calculating Corollary 2.15.

\[ \chi(S^P N^*) \quad \text{by (2.12)} \]

\[ S^P N^* \text{ is 0-regular,} \]

we get the conclusion by the proof of \cite{6} I 7.3 (b).

The multiplicity \( \mu(O_{W(L)}) \) is defined as (leading coefficient of \( \psi \) \cdot (deg \psi)!. To avoid combinatorial relations for calculating \( \mu \), we resort to intersection theory.

**Corollary 2.15.** Let \( [L] \in W_{g,r}^0 \). Assume that \( \psi \) is birational. Then \( \mu = (g)_r \).

**Proof.** \( \mu \) coincides with top Segre class of \((L), W_{g,r}^0\), which is invariant under a birational proper morphism (cf.\cite{4} Chap. 4).

\[
\mu = s_0([L], W_{g,r}^0) \\
= s_r(\mathbb{P}^r, \text{Div} X) \\
= (-1)^r s_r(N^*).
\]

Again by (2.11),

\[ s_i(N^*) = (1 - Ht)^b, \]

where \( H \) is the class of a hyperplane section in \( \mathbb{P}^r \), which concludes the proof. \( \square \)

3. Examples: Ruled Surface

Throughout this section, \( C \) denotes a smooth projective curve of genus \( g \geq 2 \), and \( B \) a line bundle on \( C \) of degree \( d_0 \geq 3 \). We shall construct a ruled surface \( X = \mathbb{P}(E) \) over \( C \) by the extension \( 0 \rightarrow O_C \rightarrow E \rightarrow B \rightarrow 0 \). A nontrivial example of \( W_{g,r}^0 \) will be given at the end of this section. The idea of realizing an extension class of \( B \) by \( O_C \) as a point in \( \mathbb{P}(H^0(K_C \otimes B)) \) is borrowed from Bertram \cite{2}.

We start by fixing notations. Let \( \pi : X \rightarrow C \) be a ruled surface. \( \text{Pic}^{(i,j)}(X) \) denotes the connected component of \( \text{Pic}(X) \) consisting of line bundles of the type \( O_C(i) \otimes \pi^*M \), where \( \deg M = j \). \( W_{i,j}^0 \) denotes \( W_{C}^0 \cap \text{Pic}^{(i,j)}(X) \), where “sr” is omitted for simplicity. Let \( A \) be a very ample line bundle and \( D \) an effective divisor on \( C \). We have the exact sequence \( 0 \rightarrow H^0(C, A(-D)) \rightarrow H^0(C, A) \rightarrow Q \rightarrow 0 \). The linear span \( \langle D \rangle := \mathbb{P}(Q) \subseteq \mathbb{P}(H^0(C, A)) \).

\( \langle D \rangle \) has \( H^0(C, A(-D)) \) as the space of defining linear equations.

Since \( \deg(K_C \otimes B) = 2g - 2 + d_0 \geq 2g + 1 \) by assumption, the complete linear system \( |K_C \otimes B| \) induces an embedding

\[ \varphi : C \hookrightarrow \mathbb{P}^N = \mathbb{P}(H^0(K_C \otimes B)), \]

where \( N = g + d_0 - 2 \). By Serre duality,

\[ H^0(C, K_C \otimes B)^* = H^1(C, B^{-1}) = \text{Ext}^1(B, O_C). \]

So a point \( \eta \in \mathbb{P}^N \) determines an extension of \( B \) by \( O_C \)

\[ 0 \rightarrow O_C \rightarrow E \rightarrow B \rightarrow 0, \quad (3.1) \]

uniquely up to isomorphism.

Take a line bundle \( L \) and twist (3.1) by \( B^{-1} \otimes L \), we get the sequence

\[ 0 \rightarrow B^{-1} \otimes L \rightarrow E \otimes B^{-1} \otimes L \rightarrow L \rightarrow 0, \]

and the induced map \( \delta : H^0(L) \rightarrow H^1(B^{-1} \otimes L) \).
Proposition 3.1.

\[ \ker(\delta) = \{ s \in H^0(L) \mid D = (s)_0, \, \eta \in \langle D \rangle \}. \]

Proof. Given \( s \in H^0(L) \). Let \( D = (s)_0 \). There exists an associated sequence \( 0 \to B(-D) \to B \to B \otimes O_D \to 0 \). By Snake Lemma and (3.1), we can complete it as follows

\[
\begin{array}{ccccccccc}
0 & \to & O_C & \to & F & \to & B(-D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & O_C & \to & E & \to & B \otimes O_D & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
B \otimes O_D & \to & B \otimes O_D & \to & 0 & & 0 & & 0 \\
\end{array}
\]

Note the extension class of the first row is \( \delta(s) \in H^1(B^{-1} \otimes L) \cong \text{Ext}^1(B \otimes L^{-1}, O_C) \). If \( \delta(s) = 0 \), then the first row splits. So there exists a lift \( \tilde{\sigma}_D : B(-D) \to E \) of \( \sigma_D \).

\[
\begin{array}{ccccccccc}
0 & \to & O_C & \to & F & \to & B(-D) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & O_C & \to & E & \to & B \otimes O_D & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
B \otimes O_D & \to & B \otimes O_D & \to & 0 & & 0 & & 0 \\
\end{array}
\]

Tensoring the above diagram by \( K_C \) and taking cohomology, we reach the following diagram

\[
\begin{array}{ccccccccc}
0 & \to & H^0(K_C \otimes B(-D)) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(K_C) & \to & H^0(K_C \otimes B) & \to & H^1(K_C) & \to & \cdots \\
\end{array}
\]

Therefore the oblique map is 0, which implies that \( \eta \in \langle D \rangle \).

Conversely if \( \eta \in \langle D \rangle \), then the composite map \( H^0(K_C \otimes B(-D)) \to H^0(K_C \otimes B) \to H^1(K_C) \to 0 \). By Serre duality,

\[
\text{Ext}^1(K_C \otimes B(-D), K_C) \cong \text{Hom}(H^0(K_C \otimes B(-D)), H^1(K_C)),
\]

However, this suggests a potential misalignment in the recommended action for the student. If the goal is to adhere strictly to the instructions, the recommended action would be to provide a clear statement of the proposition and proof without the use of external tools or references, ensuring the text is self-contained and comprehensible within the given format.
so the first row of the diagram below splits.

\[
\begin{array}{cccccc}
0 & 0 & & & & \\
\downarrow & & & & & \\
0 & \rightarrow & \mathbb{K}_C & \rightarrow & \mathbb{K}_C \otimes F & \rightarrow & \mathbb{K}_C \otimes B(-D) & \rightarrow & 0 \\
\downarrow & & & & & & & & \\
0 & \rightarrow & \mathbb{K}_C & \rightarrow & \mathbb{K}_C \otimes E & \rightarrow & \mathbb{K}_C \otimes B & \rightarrow & 0 \\
\downarrow & & & & & & & & \\
\mathbb{K}_C \otimes B \otimes O_D & \rightarrow & \mathbb{K}_C \otimes B \otimes O_D & & & & & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & & \\
\end{array}
\]

It follows immediately that the first row of diagram (3.2) splits, therefore \( \delta(s) = 0 \). \( \square \)

**Corollary 3.2.** Let \( V_L = \{ s \in H^0(L) \mid \eta \in \langle \langle s \rangle \rangle \} \). Then

\[
h^0(E \otimes B^{-1} \otimes L) = h^0(B^{-1} \otimes L) + \dim V_L.
\]

Let \( X_{|L|} \) be the variety swept out by linear spans of divisors in \(|L|\) (see appendix). The corollary below characterizes the effective locus of \( \text{Pic}^{(1,*)}(X) \).

**Corollary 3.3.** Let \( M \) be a line bundle on \( C \). \( H^0(C, E \otimes M) \neq 0 \) if and only if

1. either \( H^0(M) \neq 0 \),
2. or \( M \cong B^{-1} \otimes L \) for some effective line bundle \( L \), such that \( \eta \in X_{|L|} \).

**Proof.** Write \( M \) as \( B^{-1} \otimes L \) for some line bundle \( L \), and apply Corollary 3.2 to \( L \). \( \square \)

**Proposition 3.4.** Let \( \Sigma = \Gamma + \alpha f \in |O_X(1) \otimes \pi^*(B^{-1} \otimes L)| \), where \( \Gamma \) is the image of a section \( \sigma : C \rightarrow X \). Then \( \sigma \) corresponds to

\[
0 \rightarrow B \otimes L^{-1}(\alpha) \rightarrow E \rightarrow L(-\alpha) \rightarrow 0.
\]

And the obstruction group \( H^1(O_\Sigma(\Sigma)) \cong H^0\left(C, \mathbb{K}_C \otimes B \otimes L^{-2}(\alpha)\right) \).

**Proof.** \( \Gamma \) arises from some one dimensional quotient of \( E \)

\[
0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0.
\]

By [6] V. 2.6, \( \pi^*N = O_X(1) \otimes O_X(-\Gamma) \), which is isomorphic to \( \pi^*\left(B \otimes L^{-1}(\alpha)\right) \). Therefore \( N \cong B \otimes L^{-1}(\alpha) \), if \( \det E \cong B, M \cong L(-\alpha) \).

Assume \( \alpha = \sum_{i=1}^m a_i p_i \), where \( a_i \in \mathbb{N} \) and \( p_i \in C \). Let \( \Sigma_i = \Gamma + \sum_{j \leq i} a_j f \), where \( a_j f \) is \( \pi^{-1}(a_j p_j) \). In this notation \( \Sigma_0 = \Gamma, \Sigma_m = \Sigma \). Consider the exact sequence

\[
0 \rightarrow O_{Z_i} \rightarrow O_{Z_{i-1}} \oplus O_{a_f} \rightarrow O_{Z_i} \rightarrow 0,
\]

where \( Z_i \) is the scheme theoretic intersection of \( \Sigma_{i-1} \) with \( a_i f \). Tensoring it with \( \mathcal{L} := O_X(1) \otimes \pi^*(B^{-1} \otimes L) \cong O_X(\Sigma) \), one obtains

\[
\cdots \rightarrow H^0(\mathcal{L} \otimes_{\mathcal{I}_{a_f}}) \rightarrow H^0(\mathcal{L} \otimes_{\mathcal{I}_f}) \rightarrow H^1(\mathcal{L} \otimes_{\mathcal{I}_f}) \rightarrow H^2(\mathcal{L} \otimes_{\mathcal{I}_f}) \oplus H^1(\mathcal{L} \otimes_{\mathcal{I}_{a_f}}) \rightarrow 0.
\]

On the one hand, denote the ideal sheaf of \( \pi^{-1}(p_i) \cong \mathbb{P}^1 \) by \( \mathscr{I} \). For \( k \geq 1 \), there exists the sequence

\[
0 \rightarrow \mathscr{I}^{k-1}/\mathscr{I}^k \rightarrow O_X/\mathscr{I}^k \rightarrow O_X/\mathscr{I}^{k-1} \rightarrow 0.
\]
By flatness of $\pi$ and smoothness of fibre $f$, $\mathcal{J}^{k-1}/\mathcal{J}^k \simeq \mathcal{O}_p$. One obtains $H^1(\mathcal{L}_{k\ell}) \simeq H^1(\mathcal{L}_{k\ell}) \simeq \cdots \simeq H^1(\mathcal{L}_{\ell}) \simeq H^1(\mathcal{O}_p, \mathcal{O}_p(1)) = 0$. In particular, $H^1(\mathcal{L}_{\ell}) = 0$.

On the other hand, $H^0(\mathcal{L}_{\ell}) \to H^0(\mathcal{L}_{\ell})$ is surjective, since its cokernel $H^1(\mathcal{O}_f, \mathcal{L}(-\Gamma)|_{\mathcal{L}_{\ell}}) = 0$ by a similar argument as above.

So $H^1(\mathcal{L}_{\ell}) \simeq H^1(\mathcal{L}_{\ell}^\times)$. Inductively, one gets

$$H^1(\mathcal{O}_\Sigma(\Sigma)) \simeq H^1(\mathcal{L}_{\Sigma}^\times) \simeq H^1(\mathcal{L}_{\Sigma}) \simeq H^1(C, \pi^*\mathcal{O}_\Sigma(\Sigma)) \simeq H^1(C, B^{-1} \otimes L^{-2}(\alpha)) \simeq H^0(C, K_C \otimes B \otimes L^{-2}(\alpha))^*$

**Remark 3.5.** The obstruction group indicates that if $2 \deg L < g + d_0 - 1$, then $\mathcal{O}_\Sigma(1) \otimes \pi^*(B^{-1} \otimes L)$ is not semi-regular. For instance, take $g = 3$, $d_0 = 3$ and $\eta \in S^3C$. In this case $\mathcal{W}_{1,1}^0 \neq 0$, but none of its point is semi-regular. This produces many examples of effective line bundle locus on $\text{Pic}(X)$ which doesn’t contain any semi-regular line bundle.

**Proposition 3.6.** Assume $H^0(C, B^{-1} \otimes L) = 0$ and $\eta \in X_{\ell}$. Then there exists a reducible divisor $\Sigma \in |\mathcal{O}_\Sigma(1) \otimes \pi^*(B^{-1} \otimes L)|$ if and only if there exists a line bundle $L' \subsetneq L$, such that $\eta \in X'_{\ell}$.

**Proof.** Let $\Sigma = \Gamma + \alpha f \in |\mathcal{O}_\Sigma(1) \otimes \pi^*(B^{-1} \otimes L)|$ for some effective $\alpha$ with $\deg \alpha \geq 1$. Then $\mathcal{O}_\Sigma(1) \simeq \mathcal{O}_\Sigma(1) \otimes \pi^*(B^{-1} \otimes L(-\alpha))$. Notice $H^0(C, B^{-1} \otimes L(-\alpha)) = 0$, so by Corollary 3.3 $X'_{\ell} \ni \eta$.

Conversely let $L' \simeq L(-\alpha)$ for some effective $\alpha$ with $\deg \alpha \geq 1$ and $X'_{\ell} \ni \eta$, then $H^0\left(\mathcal{O}_\Sigma(1) \otimes \pi^*(B^{-1} \otimes L')\right) \neq 0$. Let $\Sigma \in |\mathcal{O}_\Sigma(1) \otimes \pi^*(B^{-1} \otimes L)|$, then the reducible divisor $\Sigma + \alpha f \in |\mathcal{O}_\Sigma(1) \otimes \pi^*(B^{-1} \otimes L)|$.

For $i \geq 0$, define the set

$$X^{i+1}_{\ell} = \bigcup_{i \leq D \leq 1 \text{ deg } D = \deg L - i} \langle D \rangle,$$

namely the set swept out by linear spans of all degree $(\deg L - i)$ sub-divisors of $|L|$. The inclusion $X^{i+1}_{\ell} \subset X^i_{\ell}$ is clear.

**Corollary 3.7.** With the above notation, assume $H^0(C, B^{-1} \otimes L) = 0$ and $\eta \in X'_{\ell} \setminus X^1_{\ell}$. Then $\mathcal{O}_\Sigma(1) \otimes \pi^*(B^{-1} \otimes L)$ is semi-regular if and only if $H^0(C, K_C \otimes B \otimes L^2) = 0$.

We give an example of $\mathcal{W}_{1,2}^0(X)$ for a ruled surface $X$, which is a 3 dimensional singular variety and essentially different from Brill-Noether loci of curves.

**Example 3.8.** Let $C$ be a general curve of genus $g = 5$ (hence all Brill-Noether loci $W_{\delta}(C)$ involved below have the expected dimensions $\rho = g - (r + 1)(g - d + r)$). For a $B$ with $d_0 = 3, |K_C \otimes B|$ induces an embedding $\phi : C \hookrightarrow \mathbb{P}(H^0(K_C \otimes B)) \simeq \mathbb{P}^d$.

**Claim A:** There exist $B \in \text{Pic}^3(C)$, $L \in \text{Pic}^5(C)$ on $C$ and $\eta \in \mathbb{P}(H^0(K_C \otimes B))$ satisfying the following conditions:

A.1 $H^0(B^{-1} \otimes L) = 0$,
A.2 $H^0(L) = 2$,
A.3 $L$ is base point free,
A.4 \( h^0(K_C \otimes B \otimes L^{-2}(p)) = 0 \) for all \( p \in C \).

Let \( \Lambda = \cap_{D \in [L]} \langle D \rangle \).

A.5 \( \eta \in \Lambda \setminus X^2_\mu \). The triple \((C, B, \eta)\) chosen determines a ruled surface \( \pi : X = \mathbb{P}(E) \to C \). Then \( \mathcal{L} := O_X(1) \otimes \pi^* \left( B^{-1} \otimes L \right) \in W^0_{1,2}(X) \).

Remark 3.9. Condition A.1 implies that for any \( D \in [L], \langle D \rangle \cong \mathbb{P}^4 \). The use of conditions will be self-evident after the proof of the claim.

Proof. A.1, A.5 and Corollary 3.2 imply that \( h^0(X, \mathcal{L}) = 2 \). Any divisor \( \Sigma \in [\mathcal{L}] \) can be written as \( \Gamma + \alpha f \), where \( \Gamma \) is the image of a section \( \sigma \), \( \alpha \) is an effective divisor on \( C \), and \( f \) is the fibre class. One has

\[ O_X(\Gamma) = O_X(1) \otimes \pi^* \left( B^{-1} \otimes L(-\alpha) \right) \]

Since \( h^0(B^{-1} \otimes L(-\alpha)) = 0 \), \( \eta \) is contained in \( X_{[L(-\alpha)]} \) by Proposition 3.3. Due to A.5, \( \deg \alpha \leq 1 \). Therefore A.4 and Proposition 3.4 imply that \( \mathcal{L} \) is semi-regular.

Next, we look for \((B, L, \eta)\) with the constraints. First we define two morphisms:

\[
\begin{align*}
m_B : \text{Pic}^d(C) & \to \text{Pic}^{d+3}(C) & \text{by } M & \mapsto M \otimes B, \\
g : \text{Pic}^d(C) & \to \text{Pic}^{3d}(C) & \text{by } M & \mapsto M^\otimes 2.
\end{align*}
\]

A.1, A.2 are to say that \( L \in W^1_3 \setminus m_B(W^0_2) \). A.3 is to say \( L \not\in \text{Im}(\alpha) \), where \( \alpha : W^1_4 \times C \to \text{Pic}^3(C) \), sending \((M, p) \mapsto M(p) \). Notice \( \dim W^1_4 = 3 \), \( \dim m_B(W^0_2) = 2 \) and \( \dim \text{Im}(\alpha) = 2 \).

By Riemann-Roch and Serre duality, A.4 is translated to the condition: \( h^0(B^{-1} \otimes L^2(-p)) = 2 \) for any \( p \in C \), which turns out to be equivalent to

- \( h^0(B^{-1} \otimes L^2) = 3 \), and
- \( B^{-1} \otimes L^2 \) is base point free.

So \( m_B : \gamma(L) \not\subseteq W^1_7 \cup \text{Im}(\beta) \), where \( \beta : W^2_6 \times C \to \text{Pic}^7(C) \).

Notice \( \dim \text{Im}(\beta) = \dim W^2_7(C) + 1 = 3 \) and \( \dim W^2_7 = 1 \).

To attain A.1-4 simultaneously, it suffices to choose \( B \) with the property that

\[
W^2_6 \not\subseteq \gamma^{-1} \left( m_B \left( W^3_7 \cup \text{Im}(\beta) \right) \right) \cup m_B \left( W^0_2 \right).
\]

Regard \( \text{Pic}^5(C) \) as a homogeneous space with the group \( J = \text{Pic}^3(C) \) acting in the obvious way. Then Kleiman’s transversality (9, Theorem 2) implies that for generic \( B \in \text{Pic}^3(C) \), the intersection of \( \gamma^{-1} \left( m_B \left( W^3_7 \cup \text{Im}(\beta) \right) \right) \cup m_B \left( W^0_2 \right) \) with \( W^2_6 \) has the expected dimension 0. So \( (3.3) \) holds for generic choice of \( B \), and consequently \( L \) has 3 dimensional freedom to choose. The diagram below indicates the relations among the varieties and morphisms above.

\[
\begin{CD}
W^3_7 @> \beta >> \text{Pic}^7(C) @> m_B >> \text{Pic}^{10}(C) \quad & \gamma \quad \text{Pic}^5(C) @> \alpha >> W^1_4 \times C \\
W^2_6 \times C @> \beta >> \text{Pic}^7(C) @> m_B >> \text{Pic}^{10}(C) \quad & \gamma \quad \text{Pic}^5(C) @> \alpha >> W^1_4 \times C
\end{CD}
\]

For A.5, we first show that \( \Lambda \cong \mathbb{P}^2 \). In fact, by Proposition 4.1, \( \Lambda = \langle D_1 \rangle \cap \langle D_2 \rangle \) for any distinct \( D_1, D_2 \in [L] \). Observe \( H^0(\mathcal{F}_{(D_1)\cap(D_2)}/\mathcal{F}(1)) \) is given by \( \text{Im}(H^0(K_C \otimes B(-D_1)) \oplus \text{Im}(H^0(K_C \otimes B(-D_2))) \).
$H^0(K_C \otimes B(-D_2)) \to H^0(K_C \otimes B))$. By base point freeness of $L$ (A.3), one has the short exact sequence

$$0 \to O_C(-D_1 - D_2) \to O_C(-D_1) \otimes O_C(-D_2) \to O_C \to 0,$$

and hence

$$h^0(J_{(D_1) \cap (D_2)}/\mathbb{P}^1(1)) = \dim \text{Im}(H^0(K_C \otimes B(-D_1)) \otimes H^0(K_C \otimes B(-D_2)) \to H^0(K_C \otimes B))$$

$$= 2h^0(K_C \otimes B \otimes L^{-1}) - h^0(K_C \otimes B \otimes L^{-2})$$

$$= 4,$$

which shows $\langle D_1 \rangle \cap \langle D_2 \rangle \cong \mathbb{P}^2$.

To prove $\Lambda \setminus \mathcal{X}_D \neq \emptyset$, it suffices to show for a general degree 3 divisor $Z$ with $h^0(L(-Z)) > 0$, $\langle Z \rangle \cap \Lambda = \text{point}$. We fix $D_0 \in |L|$, and assume $Z + p + q = D \in |L|$, for some points $p, q \in C$ and assume $D \neq D_0$. $\langle Z \rangle \cap \Lambda = \text{point if and only if}$ the image of the diagonal map

$$H^0(K_C \otimes B(-D_0)) \to H^0(K_C \otimes B)$$

$$H^0(K_C \otimes B(-D_0)|_Z) \to H^0(K_C \otimes B|_Z)$$

is a 2 dimensional as a vector space.

From the exact sequence

$$0 \to H^0(K_C \otimes B(-D_0 - Z)) \to H^0(K_C \otimes B(-D_0)) \to H^0(K_C \otimes B(-D_0)|_Z),$$

This happens when $h^0(K_C \otimes B(-D_0 - Z)) = h^0 \left( K_C \otimes B \otimes L^{-2}(p + q) \right) = 0$.

By A.4 and its reformulation, $B^{-1} \otimes L^2$ is base point free and $h^0 \left( K_C \otimes B \otimes L^{-2}(p) \right) = 0$.

Therefore $h^0 \left( K_C \otimes B \otimes L^{-2}(p + q) \right) = 0$ if and only if $|B^{-1} \otimes L^2|$ separates $p$ and $q$.

Let $C'$ be the image of the map $C \xrightarrow{|B^{-1} \otimes L^2|} \mathbb{P}^3$. Obviously $C'$ is not $\mathbb{P}^1$. Since $\deg \left( B^{-1} \otimes L^2 \right) = 7$ is a prime number, the induced map $C \to C'$ cannot be a finite morphism of degree $\geq 2$ and hence $C \to C'$ is birational. The number of pairs $(p, q)$ that $B^{-1} \otimes L^2$ cannot separate is finite. \hfill \Box

**Claim B:** Fix $(C, B, \eta)$ as before. For general $q_1 + \cdots + q_5 \in C_5$, the following conditions hold

- B.1 $h^0(B^{-1}(q_1 + \cdots + q_5)) = 0$,
- B.2 $h^0(q_1 + \cdots + q_5) = 1$,
- B.3 $h^0(K_C \otimes B(-2q_1 - \cdots - 2q_5)) = 0$.

Moreover, for at least 3 dimensional of $[q_i]_{i=1}^5$,

- B.4 $\langle q_1, \cdots, q_5 \rangle \not\supset \eta$.

holds.

**Proof.** First pick two distinct points $q_1, q_2$ such that $h^0(K_C \otimes B(-2q_1 - 2q_2)) = h^0(K_C \otimes B) - 4$. We then proceed by induction on the number of points. Suppose $q_1, \cdots, q_i$ for $2 \leq i \leq 4$ have been picked, such that:

- $h^0(B^{-1}(q_1 + \cdots + q_i)) = 0$,
- $h^0(q_1 + \cdots + q_i) = 1$,
- $h^0(K_C \otimes B(-2q_1 - \cdots - 2q_i)) = \max \{ h^0(K_C \otimes B) - 2i, 0 \}$. 



If \( q_{i+1} \) is chosen by avoiding \( C \cap (q_1, \ldots, q_i) \) and any inflection points of \( |K_C(-q_1-\cdots-q_i)| \) and \( |K_C \otimes B(-2q_1-\cdots-2q_i)| \), which are finite. Then conditions B.1 – B.3 hold for \( i+1 \) as well. We leave B.4 to the interested reader.

Take \( q_1, \ldots, q_5 \) satisfying B.1 – B.4, then \( \mathcal{L}' := O_X(1) \otimes \pi^*B^{-1}(q_1+\cdots+q_5) \in W^0_{1,2}(X) \). \( \mathcal{L}' \) can specialize to the component \( \Omega \) of \( W^0_{1,2}(X) \), which contains \( \mathcal{L} \). Since \( h^0(\mathcal{L}') = 1 \), the induced Abel-Jacobi map \( \varphi : \text{Div} X \to \Omega \) is a birational morphism, and hence

\[
\dim \Omega = R = \chi(\mathcal{L}) + 1 + q(X) = 3.
\]

For \( \mathcal{L}, b = r + 1 - \chi(\mathcal{L}) = r + 2 \) (no \( H^2 \) involved), then the multiplicity \( \mu \) of \( \Omega \) at \( [\mathcal{L}] \) equals \( (r+2) = 3 \) by Corollary 2.15. Our theorem asserts that \( \Omega \) has at worst rational singularities.

Remark 3.10. Friedman and Morgan conjectured in [4] that for a “general” ruled surface \( X \), each component of \( \text{Div}(X) \) is smooth and has the expected dimension \( R \). If the conjecture is true, then for such general \( X \), each component of \( W^0_{1,2}(X) \) has rational singularities, except at points \( [\mathcal{L}] \) where \( \varphi^{-1}([\mathcal{L}]) \) is not contained in one component of \( \text{Div}(X) \).

4. Appendix

In the section, we review the construction of \( X_{\mathcal{L}_i} \), the variety swept out by all linear spans of divisors in \( |\mathcal{L}| \) on a curve \( C \) with respect to an embedding \( C \subset \mathbb{P}^N \).

Assume \( A \) is a very ample line bundle on the curve \( C \) with genus \( g \geq 1 \). Let \( V \) denote \( H^0(C, A) \). Given an effective line bundle \( \mathcal{L} \) of degree \( d \) with \( \dim |\mathcal{L}| = r \). Denote the two projections from \( C \times |\mathcal{L}| \) by \( p \) and \( q \). There exists the sequence

\[
0 \to p^*L^{-1} \otimes q^*O_{\mathcal{L}_i}(-1) \to O_{C \times \mathcal{L}_i} \to O_{\mathcal{D}} \to 0,
\]

where \( \mathcal{D} \) is the universal divisor, see (2.2).

Applying \( q_*(p \ast A \otimes \_ \_ \_ \) to above, we get the exact sequence

\[
0 \to H^0(C, A \otimes L^{-1}) \otimes O_{\mathcal{L}_i}(-1) \to V \otimes O_{\mathcal{L}_i} \to q_*(p^*A \otimes O_D) \to H^1(C, A \otimes L^{-1}) \otimes O_{\mathcal{L}_i}(-1)
\]

\[
\to H^1(C, A) \otimes O_{\mathcal{L}_i} \to 0,
\]

where each term is locally free. Consequently

\[
0 \to H^0(C, A \otimes L^{-1}) \otimes O_{\mathcal{L}_i}(-1) \to V \otimes O_{\mathcal{L}_i} \to Q \to 0,
\]

where \( Q \) is locally free. (4.1) yields the diagram

\[
\begin{array}{ccc}
\mathbb{P}(Q) & \xrightarrow{\phi} & \mathbb{P}(V) \\
\downarrow{\pi} & & \\
|\mathcal{L}| & & \\
\end{array}
\]

\( X_{\mathcal{L}_i} \) is defined as the scheme theoretic image of \( \phi : \mathbb{P}(Q) \to \mathbb{P}(V) \). Geometrically, \( X_{\mathcal{L}_i} \) is the union of all linear spans of divisors \( D \in |\mathcal{L}| \).

**Proposition 4.1.** Notations as above. Assume \( d_0 := \deg A - (2g - 2) \geq 3 \), then any fibre of \( \phi : \mathbb{P}(Q) \to X_{\mathcal{L}_i} \) over a closed point is a projective space. \( \phi \) is birational if and only if \( r \leq h^0(A \otimes L^{-1}) \). Furthermore, if \( r < h^0(A \otimes L^{-1}) \), then \( X_{\mathcal{L}_i} \) is Cohen-Macaulay.
Proof. We write \( A = K_C \otimes B \) with \( \deg B = d_0 \geq 3 \). Any \( p \in X_{|L|} \subset \mathbb{P}(V) \) determines an extension

\[
0 \to O_C \to E \to B \to 0.
\]

By Proposition 4.1, \( \pi \circ \phi^{-1}(p) \approx \mathbb{P}(\ker(H^0(L) \to H^1(B^{-1} \otimes L))) \). From the construction of \( \mathbb{P}(Q) \), we have \( \phi^{-1}(p) \approx \pi \circ \phi^{-1}(p) \) is a projective space.

In our case, \( \phi \) is birational if and only if it is generically finite. Since \( \phi \) is induced by the tautological line bundle \( O_{\mathbb{P}(Q)}(1) \), \( \phi \) is generically finite if and only if \( (c_1(O_{\mathbb{P}(Q)}(1)))^r \mathbb{P}(Q) = s_1(Q^r) > 0 \).

Let \( H \) be a hyperplane section on \( |L| \). Let \( s_r(\omega) \) and \( c_r(\omega) \) denote the Segre and Chern polynomials. Then by (4.1),

\[
s_r(Q^r) = c_r\left(H^0(C,A \otimes L^{-1})^\vee \otimes O_{|L|}(1)\right) = (1 + tH)^{r\left(C \otimes A L^{-1}\right)}.\]

It follows that \( s_r(Q^r) = \left(H^0(A \otimes L^{-1})\right)^r \). So \( s_r(Q^r) > 0 \) if and only if \( r < h^0\left(A \otimes L^{-1}\right) \).

When \( r < h^0\left(A \otimes L^{-1}\right) \), consider the map of vector bundles

\[
\xi : H^0\left(A \otimes L^{-1}\right) \otimes O_{\mathbb{P}(V)}(-1) \to H^0(L)^r \otimes O_{\mathbb{P}(V)}.
\]

\( X_{|L|} = \{ x \in \mathbb{P}(V) | \text{rank}(\xi_x) \leq r \} \), which gives \( X_{|L|} \) a determinantal variety structure.

\[
\dim X_{|L|} = \dim \mathbb{P}(Q) = r + \text{rank}(Q) - 1 = r + h^0(C,A) - h^0(A \otimes L^{-1}) - 1 = \dim \mathbb{P}(V) - \left(h^0(A \otimes L^{-1}) - r\right)\left(h^0(L) - r\right),
\]

which is the expected dimension, therefore \( X_{|L|} \) is Cohen-Macaulay (cf. [11] pp 84).

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