A fast convergence theorem for nearly multiplicative
connections on proper Lie groupoids

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Abstract

Motivated by the study of global geometric properties of differentiable stacks associated with proper Lie groupoids we investigate the existence of multiplicative connections on such groupoids. Under mild technical assumptions (source properness) we show that one can always deform a given connection which is only approximately multiplicative into a genuinely multiplicative connection. The proof of this fact presented here relies on a recursive averaging technique. We regard our results as an initial step towards the construction of an obstruction theory for multiplicative connections on proper Lie groupoids.

Introduction

This paper is supposed to be the first of a series devoted to the study of multiplicative connections on proper Lie groupoids. These connections are relevant for the understanding of the global transversal geometry of proper Lie groupoids (that is roughly speaking the global geometry of their orbit spaces viewed as “generalized effective orbifolds”) in that they make it possible to construct effective actions of such groupoids. Whenever an effective, proper Lie groupoid action can be found, one can generalize the classical result in the theory of effective orbifolds that any effective, proper, étale Lie groupoid is equivalent (in the sense of Morita) to the translation groupoid associated to some compact Lie group action.

Let us first of all outline the contents of the paper. We say that a connection on the source fibration of a Lie groupoid is non-degenerate if the induced pseudo-action of the groupoid on the tangent bundle of its own base (that is, the pseudo-action obtained by composing the horizontal lift of the connection with the tangent groupoid target mapping) is invertible. We say it is effective, if the same pseudo-action is an action (i.e., respects units and composition). We start by pointing out that any proper Lie groupoid admits an averaging operator which is defined on non-degenerate connections and has

*Part of the results contained in this paper were obtained while the author was a guest of the Max Planck Institute for Mathematics in Bonn, Germany. The author acknowledges support from the Portuguese Foundation for Science and Technology (Fundação para a Ciência e a Tecnologia) through grant # SFRH/BPD/81810/2011.
the property that the average of an effective connection is a multiplicative connection, that is, a connection which as a distribution constitutes a subgroupoid of the tangent groupoid of the given groupoid. If a connection is already multiplicative, it is left unchanged by the averaging operator. Even when one is only given some non-degenerate connection which is not effective, one can in principle study the sequence of connection averaging iterates obtained by recursive application of the averaging operator. Provided the initial connection is in a suitable technical sense close enough to being effective (in which case we say—slightly incorrectly—that it is nearly multiplicative), it turns out that the sequence of connection averaging iterates is actually (defined and) convergent to a multiplicative connection of the same class of differentiability as the original connection. This is the fast convergence theorem mentioned in the title. The rigorous statement of our theorem involves an assumption of source properness on the Lie groupoid, which is not a serious limitation from the point of view of the applications we have in mind (see below). Our fast convergence theorem for connections is obtained essentially as a corollary of a similar fast convergence theorem for arbitrary groupoid pseudo-actions. A consistent amount of technical work is required especially in order to prove the claim about the differentiability class of the limiting connection or pseudo-action.

The present study is part of a general plan aimed at understanding the precise obstructions to the existence of multiplicative connections on proper Lie groupoids. Once these obstructions were precisely known, one could possibly use them to manufacture new vector bundles (other than tangent bundles of groupoid bases) on which proper Lie groupoids might act effectively. This would provide one with a widely applicable method for the construction of effective representations of proper Lie groupoids.

A strategy which we regard as promising in this respect is to work stratum after stratum starting from the most singular part of the groupoid (formed by those arrows that connect base points lying on orbits of minimal dimension). To give an idea of what role the above-mentioned fast convergence theorem is going to play here, consider an arbitrary source-proper Lie groupoid. If we fix a connection at random, there will be an invariant open neighborhood of the semi-stable points (by which we mean those base points that lie on zero-dimensional orbits) such that the connection is nearly multiplicative over this open neighborhood. Our theorem then implies that over the same open set there must be a multiplicative connection. With a little more effort, one can use this remark to show that for a general proper Lie groupoid there is always some invariant open neighborhood of the maximally singular points which has the property that the groupoid admits an effective representation over this open set (on some vector bundle which in general need not be a tangent bundle).

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1. Groupoid connections and pseudo-actions

By a differentiable manifold we mean a (non-empty) locally compact \( C^\infty \)-manifold (we use this term in the sense of Lang’s Fundamentals [9]). A general differentiable manifold may not be Hausdorff nor second countable, and may have components modeled on different (finite-dimensional real) vector spaces. We say that a differentiable manifold is smooth whenever it is Hausdorff, second countable and of constant dimension. A differentiable mapping will be a mapping of class \( C^\infty \) between differentiable manifolds, and a smooth mapping will be a mapping of class \( C^\infty \) between smooth manifolds.

By a differentiable groupoid we mean a small groupoid \( \Gamma \Rightarrow X \) in which \( \Gamma \) and \( X \) are differentiable manifolds, the source \( s \) and the target \( t \) are submersive differentiable mappings, and the other structure mappings (namely the composition \( m \), the unit \( u \), and the inverse \( i \)) are differentiable. We call Lie groupoid any differentiable groupoid \( \Gamma \Rightarrow M \) in which \( M \) is smooth and in which \( \Gamma \) is second countable. A homomorphism of differentiable groupoids will be a differentiable functor. A differentiable groupoid \( \Gamma \Rightarrow X \) will be said to be Hausdorff, second countable, of constant dimension whenever the corresponding property holds for both \( \Gamma \) and \( X \), and essentially connected if the only non-empty, open, closed, \( \Gamma \)-invariant subset \( U \subset X \) is \( U = X \) itself.

The notion of Lie groupoid we adopt here differs from that in the standard textbook [10] in two respects. First, we require second countability. Second, we allow the arrow manifold to have components of different dimensions. The latter convention is a very natural one in the context of representation theory for reasons which we do not intend to discuss here. As to second countability, we point out that in most classical textbooks Lie groups are assumed to be second countable. In [10], the authors also postulate that the source fibers of a Lie groupoid ought to be Hausdorff, but this hypothesis is redundant (compare below).

**Fundamental Structure Theorem.** Let \( \Gamma \Rightarrow X \) be an arbitrary differentiable groupoid. Then, the following statements hold.

(a) For each pair of base points \( x, y \in X \), the subset \( \Gamma^x_y = \Gamma(x, y) \subset \Gamma \) is a differentiable submanifold.

(b) For each base point \( x \in X \), the isotropy group \( \Gamma^x_x = \Gamma(x, x) \) with the differentiable structure inherited from \( \Gamma \) according to (a) is a differentiable group.

(c) For each base point \( x \in X \), the composition of arrows restricts to a free, differentiable, right action \( \Gamma^x \times \Gamma^x_x \to \Gamma^x \) of the isotropy group \( \Gamma^x_x \) on the source fiber \( \Gamma^x = \Gamma(x, -) \subset \Gamma \). This action has the property that there exists a (unique) differentiable manifold structure on the quotient set \( \Gamma^x / \Gamma^x_x \) which makes the quotient projection \( \Gamma^x \to \Gamma^x / \Gamma^x_x \) into a submersion.

**Proof.** The statements (a) and (b) can be proved as in [10]. The statement (c) is essentially a consequence of Godement’s Theorem (a proof of which—valid for arbitrary differentiable manifolds—can be found in [14]); details will be left as an exercise. \( \square \)
Let $\Gamma \Rightarrow X$ be an arbitrary differentiable groupoid. For each base point $x \in X$, the corresponding source fiber $\Gamma^x$ is a differentiable submanifold of $\Gamma$. The differentiable group $G_x = \Gamma^x$ acts differentiably and freely from the right on $\Gamma^x$, and there is a unique differentiable structure on the quotient $\Gamma^x/G_x$ which makes the quotient projection $pr_x^F : \Gamma^x \to \Gamma^x/G_x$ submersive. We indicate the resulting differentiable manifold by $O^x/F$. This is injectively immersed into $X$ in a canonical fashion. Namely, there is a unique map $in^x_F : O^x_x \to X$ such that $in^x_F \circ pr_x^F = t^x \mid \Gamma^x$, and this map is necessarily differentiable, injective and immersive. We refer to $(in^x_F : O^x_x \to X)$ as the orbit of $\Gamma$ (shortly, $\Gamma$-orbit) through $x$. We also refer to the set-theoretic image $\text{im}(in^x_F) \subset X$ by means of the notation $\Gamma x$. The differentiable $G_x$-bundle $pr_x^F : \Gamma^x \to O^x_x$ is $G_x$-equivariantly locally trivial. Indeed, any local differentiable section $\gamma$ to $pr_x^F$ will induce an equivariant local trivialization of $pr_x^F : \Gamma^x \to O^x_x$ according to the prescription $(a, g) \mapsto \gamma(a)g$, and every equivariant local trivialization will be of that form for a unique local differentiable section $\gamma$.

If the base manifold $X$ of a differentiable groupoid $\Gamma \Rightarrow X$ is Hausdorff then by the local triviality of $pr_x^F$ so must be every source fiber $\Gamma^x$ of $\Gamma$. (Indeed in that case the orbit $O^x_x$ must be Hausdorff because it is injectively immersed into $X$, whereas $G_x$, being a differentiable group, is always Hausdorff.) One can use local triviality, moreover, to show that if $X$ is of constant dimension then the same must be true of $O^x_x$ and $\Gamma^x$ for every $x$. Thus, in particular, we see that any essentially connected Lie groupoid must be of constant dimension.

Let $\mathbb{K}$ denote a fixed number field, which we take to be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. By a $\mathbb{K}$-linear differentiable vector bundle over a differentiable manifold $X$ we mean a triplet $E = (p_E, +_E, \cdot_E)$ consisting of the following data: i) a surjective submersion of differentiable manifolds $p_E : E \to X$ (the abuse of notation is intentional here); ii) a differentiable mapping $+_E : E \times_X E \to E$ such that $p_E(e +_E e') = p_E(e) = p_E(e')$ for all $(e, e') \in E \times_X E$, called “sum”, and a differentiable mapping $\cdot_E : \mathbb{K} \times E \to E$ such that $p_E(a \cdot_E e) = p_E(e)$ for all $(a, e) \in \mathbb{K} \times E$, called “multiplication by scalars”, which make each fiber $E_x := p^{-1}_E(x)$ into a vector space over $\mathbb{K}$. When $\mathbb{K} = \mathbb{R}$ (resp., $\mathbb{K} = \mathbb{C}$), we will also refer to $E$ as a real (resp., complex) differentiable vector bundle. When no danger of confusion arises, we will gladly omit the subscript ‘$E$’ from ‘$p_E$', ‘$+_E$’ or ‘$\cdot_E$’, and simply write ‘$a e$’ instead of ‘$a \cdot_E e$’.

A morphism $\alpha : E \to F$ between two $\mathbb{K}$-linear differentiable vector bundles $E$ and $F$ over a given differentiable manifold $X$ is a differentiable mapping $\alpha$ of $E$ into $F$ which satisfies the condition $p_F \circ \alpha = p_E$ and which for each point $x \in X$ induces a $\mathbb{K}$-linear map $\alpha_x$ of $E_x$ into $F_x$. With this notion of morphism, the $\mathbb{K}$-linear differentiable vector bundles over $X$ form a category. One can show that any $\mathbb{K}$-linear differentiable vector bundle $E$ over $X$ is locally trivial, of locally finite rank; that is to say, for each point $x \in X$ there exist an open neighborhood $U \ni x$ in $X$ and an isomorphism of $\mathbb{K}$-linear differentiable vector bundles over $U$

$$E \mid U := p^{-1}_E(U) \cong U \times \mathbb{K}.$$ 

We do not digress into a proof of this fact here; the reader may regard the condition of local triviality as being part of the definition, if they prefer to do so. The non-negative integer $r = \dim_{\mathbb{K}} E_x$ will be called the rank of $E$ at $x$, and denoted by $rk_E(x)$. When the
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locally constant function \( \text{rk}_E : X \rightarrow \mathbb{N} \) is overall constant, we say that \( E \) is of constant rank. Local triviality implies that if \( E \) is any \( \mathbb{K} \)-linear differentiable vector bundle over a smooth manifold then \( E \) is a vector bundle of constant rank if and only if its total space is a smooth manifold. We call smooth vector bundle any differentiable vector bundle (over a smooth manifold) which satisfies these equivalent conditions.

Let \( X \) be an arbitrary differentiable manifold, and let \( E \) be an arbitrary \( \mathbb{K} \)-linear differentiable vector bundle over \( X \). For any value of \( k = 0, 1, 2, \ldots, \infty \) and for any differentiable submanifold \( Y \) of \( X \) the notation \( \Gamma^k(Y; E) \) shall indicate the vector space over \( \mathbb{K} \) formed by all the cross-sections of class \( C^k \) of \( E \) over \( Y \) (that is to say all the mappings \( \xi : Y \rightarrow E \) of class \( C^k \) such that \( p_E \circ \xi = in_Y^X \), where \( in_Y^X : Y \hookrightarrow X \) denotes the submanifold inclusion). When \( Z \) is another submanifold of \( X \) which is contained in \( Y \), we let \( \rho^Y_Z : \Gamma^k(Y; E) \rightarrow \Gamma^k(Z; E) \) denote the restriction map \( \xi \mapsto \xi \mid Z \).

By a connection on a differentiable groupoid \( \Gamma \rightrightarrows X \) we mean an (Ehresmann) connection on the source fibration \( s = s^* : \Gamma \rightarrow X \). Explicitly: a connection on \( \Gamma \rightrightarrows X \) is a right splitting \( \eta \) for the following short exact sequence of morphisms of differentiable vector bundles over the arrow manifold \( \Gamma \).

\[
0 \rightarrow T^1\Gamma \xrightarrow{\mathbb{C}} T\Gamma \xrightarrow{\mathbb{C}} s^*TX \xrightarrow{\eta} 0
\]

\( (T^1\Gamma \) denotes the \( s \)-vertical subbundle of the tangent bundle of the manifold \( \Gamma \), defined by setting \( T^1\Gamma = \ker T_s^*s = T^*_s\Gamma^{s\gamma} \), and \( s \) denotes the unique morphism that corresponds to the tangential source mapping \( T s : T\Gamma \rightarrow TX \) by virtue of the universal property of the pullback \( s^*TX \rightarrow TX \); ‘right splitting’ means \( s_* \circ \eta = \text{id}_{s^*TX} \). We will usually identify \( \eta \) with the subbundle \( H = \text{im} \eta \subset T\Gamma \) (‘H’ here is to be read as ‘capital \( \eta \)’) and refer to \( \eta = \eta^H := (s_* \mid H)^{-1} \) as the horizontal lift associated to \( H \). We also let \( \beta^H := \text{id}_T - \eta^H \circ s_* : T\Gamma \rightarrow T^1\Gamma \) denote the vertical projection associated to \( H \).

When the condition \( \eta_1 = T_s1 : T_sX \rightarrow T_1s\Gamma \) is fulfilled for every \( x \in X \) (where \( 1 = u^\Gamma : X \rightrightarrows \Gamma \) denotes the unit bisection of our groupoid) we call \( \eta \) unital. Otherwise stated, \( \eta \) is unital if and only if the following composite morphism of differentiable vector bundles over \( X \)

\[
TX \cong (s \circ 1)^*TX \cong 1^*s^*TX \xrightarrow{1^*\eta} 1^*T\Gamma
\]
equals \( 1_* : TX \rightarrow 1^*T\Gamma \) (the unique morphism corresponding to \( T1 : TX \rightarrow T\Gamma \) by virtue of the pullback universal property). An easy argument using partitions of unity shows that any differentiable groupoid which is Hausdorff and second countable admits unital connections.

We let \( \text{Conn}^k(\Gamma) \) denote the space of all connections on \( \Gamma \rightrightarrows X \) of class \( C^k \) \((k = 0, 1, 2, \ldots, \infty)\), that is the affine subspace of \( \Gamma^k(\Gamma; L(s^*TX, T\Gamma)) \) formed by all those global cross-sections \( \eta \) of class \( C^k \) of the real differentiable vector bundle \( L(s^*TX, T\Gamma) \) (over \( \Gamma \)) that are solutions for the equation \( s_* \circ \eta = \text{id}_{s^*TX} \). We let \( \text{Conn}^k_0(\Gamma) \) denote the subset of \( \text{Conn}^k(\Gamma) \) formed by the unital connections.

By a pseudo-action of class \( C^k \) of \( \Gamma \rightrightarrows X \) on a (real or complex) differentiable vector bundle \( E \) over \( X \) we mean a morphism of class \( C^k \) (of differentiable vector bundles over \( \Gamma \)) from \( s^*E \) into \( t^*E \), in other words a global cross-section of class \( C^k \) of the differentiable vector bundle \( L(s^*E, t^*E) \) (over \( \Gamma \)). To each arrow \( g \in \Gamma \) a pseudo-action
\( \lambda : s^*E \to t^*E \) assigns a linear map \( \lambda_g : E_{sg} \to E_{tg} \) between the fibers of \( E \) corresponding to the source and to the target of \( g \). If \( \lambda_g \) is for each \( g \in \Gamma \) a linear isomorphism of \( E_{sg} \) onto \( E_{tg} \), we say that \( \lambda \) is invertible. If \( \lambda_{1x} = id_{E_x} \) for all \( x \in X \), we call \( \lambda \) unital. If \( \lambda \) is unital and \( \lambda_{s'g} = \lambda_s \circ \lambda_g \) for every composable pair of arrows \( (g', g) \in \Gamma \times \Gamma, \) we call \( \lambda \) an action or representation. We write \( \text{Psa}^k(\Gamma; E) \) for \( \Gamma^k(\Gamma; L(s^*E, t^*E)) \), the space of all pseudo-actions of class \( C^k \) of \( \Gamma \to X \) on \( E \).

For any connection \( H \subset T\Gamma \) on \( \Gamma \to X \) of class \( C^k \), we may compose the horizontal lift \( \eta^H : s^*TX \to T\Gamma \) with the vector-bundle morphism \( t_* : T\Gamma \to t^*TX \) corresponding in the way described above to the target mapping \( t = t^\gamma \) so as to obtain a pseudo-action \( \lambda^H : s^*TX \to t^*TX \) of class \( C^k \) of \( \Gamma \to X \) on \( TX \) which we call the effect of \( H \). By an effective connection we mean one whose effect is a representation.

### 2. Multiplicative connections

If \( X \xrightarrow{f} B \xleftarrow{g} Y \) are differentiable mappings which are transversal then the same must be true of their tangent mappings \( TX \xrightarrow{f^*} TB \xleftarrow{g^*} TY \). The differentiable submanifold \( TX \times_{Tg} TY \subset TX \times TY \) inherits the structure of a differentiable vector bundle over \( X \times_{Tg} Y \subset X \times Y \) from the differentiable vector bundle \( TX \times TY = pr_X^*TX \oplus pr_Y^*TY \) (\( pr_X \) and \( pr_Y \) here denote the two projections \( X \leftarrow X \times Y \to Y \), respectively) because it corresponds under the canonical vector-bundle identification \( TX \times TY = T(X \times Y) \) to the subbundle \( T(X \times_{Tg} Y) \subset T(X \times Y) \). The induced vector-bundle identification

\[
TX \times_{Tg} TY = T(X \times_{Tg} Y)
\]

admits the following intuitive description. For any pair of tangent vectors \( v \in T_xX, w \in T_yY \) such that \((T_xf)(v) = (T_yg)(w) \in T_{f(x)=g(y)}B \) one can by transversality find two differentiable paths \( \alpha : \mathbb{R} \to X \) and \( \beta : \mathbb{R} \to Y \) with \( \alpha(0) = x, \dot{\alpha}(0) = v \) and \( \beta(0) = y, \dot{\beta}(0) = w \) such that \( f(\alpha(t)) = g(\beta(t)) \) for all \( t \in \mathbb{R} \); the path \((f, g) : \mathbb{R} \to X \times_{Tg} Y \) then represents the vector, tangent to \( X \times_{Tg} Y \) at \((x, y)\), corresponding to \((v, w)\) under \((1)\).

Let \( \Gamma \to X \) be an arbitrary differentiable groupoid. Its tangent groupoid \( T\Gamma \to TX \) has source \( s^{T\Gamma} := Ts^\Gamma \), target \( t^{T\Gamma} := Tt^\Gamma \), composition law \( T\Gamma \xrightarrow{\cdot} T\Gamma \) and \( T\Gamma \xrightarrow{\circ} T\Gamma \), unit \( ut^{T\Gamma} := Tu^\Gamma \) and inverse \( \iota^{T\Gamma} := Ti^\Gamma \). The tangent composition law \( m^{T\Gamma} := Tm^\Gamma \circ (1) \) can be given a more explicit definition which makes it evident that the structure just introduced satisfies the algebraic axioms for a groupoid and therefore constitutes a differentiable groupoid. Namely, let \( w' \in T_{s'\Gamma} \) and \( w \in T_{s\Gamma} \) satisfy \((T_{g^*})^\cdot w' = (T_g)^\cdot w \), where \( sg' = tg \). Choose any two \( C^\infty \) paths \( \gamma, \gamma' : \mathbb{R} \to \Gamma \) with \( \gamma(0) = g, \dot{\gamma}(0) = w, \gamma'(0) = g' \) and \( \dot{\gamma}'(0) = w' \) such that \( s\gamma'(\tau) = t\gamma(\tau) \) for all \( \tau \in \mathbb{R} \). Then

\[
w'w = \frac{d}{dt}|_{t=0} \gamma'(\tau)\gamma(\tau) \in T\gamma'(0)g(0)=g'g\Gamma.
\]

Clearly, any homomorphism of differentiable groupoids \( \phi : \Gamma \to \Delta \) induces a tangent homomorphism \( T\phi : T\Gamma \to T\Delta \). We thus obtain a tangent functor \( T(-) \) from the category of differentiable groupoids into itself.

Let \( \Gamma \to X \) be an arbitrary differentiable groupoid. As in the preceding section, we let \( T^1\Gamma \subset T\Gamma \) denote the \( s \)-vertical subbundle. The algebroid bundle of \( \Gamma \to X \) is
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defined to be the differentiable vector bundle over \( X \) given by \( \text{alg} \Gamma := 1^*T^1\Gamma \) (pullback along \( 1 = u^1 : X \hookrightarrow \Gamma \)). Let us put \( g = \text{alg} \Gamma \) for brevity. For every arrow \( g \in \Gamma \) the right translation mapping \( \tau_{r^{-1}} : \Gamma^g \cong \Gamma^g (h \mapsto hg^{-1}) \) is a diffeomorphism which makes \( g \) correspond to the unit \( 1_g \). Taking its differential at \( g \) we obtain an isomorphism of tangent spaces \( T_g\tau_{r^{-1}} : T_g\Gamma^g \cong T_{1_g}\Gamma^g \). The various linear maps \( \omega^f_g := T_g\tau_{r^{-1}} \) (as \( g \) ranges over \( \Gamma \)) fit together into an isomorphism of differentiable vector bundles over \( \Gamma \)

\[
\omega^f : T^1\Gamma \cong \tau^*g
\]

which shall be called the Maurer–Cartan isomorphism (or Maurer–Cartan form) associated to \( \Gamma \).

A connection \( H \subset \Gamma \) on a differentiable groupoid \( \Gamma \rightrightarrows X \) is said to be multiplicative if \( H \Rightarrow TX \) constitutes a subgroupoid (by necessity, over the whole of \( TX \)) of the tangent groupoid \( \Gamma \rightrightarrows TX \). Trivially, multiplicative connections are unital. They are also always effective, as we will see presently. In order to minimize notational clutter in the discussion to follow, we will resort to the simplicial notation for the manifold of tangent spaces \( \Gamma \). The series by setting \( \Gamma_{(0)} := X \) and \( \Gamma_{(1)} := \Gamma \). We let \( X \rightrightarrows \Gamma \) denote the two mappings given respectively by \( (g_1, g_2, \ldots, g_k) \mapsto sg_k \) and \( \mapsto tg_1 \). For completeness, we also set \( \Gamma_{(0)} := s \), \( \Gamma_{(1)} := t \) and \( \Gamma_{(0)} := \text{id}_X \). We will occasionally allow the abridged versions \( \Gamma_k := \Gamma_{(k)} \) etc. of the (official) notations just introduced, but only in such situations when no ambiguity with the notation \( \Gamma_x := \Gamma(-,x) \) for the target fiber at \( x \) is likely to arise.

Recall that for an arbitrary connection \( H \subset \Gamma \) on a given differentiable groupoid \( \Gamma \rightrightarrows X \) we have the associated horizontal lift \( \eta^H : s^*TX \to \Gamma \) and pseudo-action \( \lambda^H : s^*TX \to r^*TX \). Letting \( pr_1, pr_2, m : \Gamma_{(2)} \to \Gamma \) respectively denote the 1st projection, the 2nd projection and the groupoid’s composition law, we form the following three morphisms of differentiable vector bundles over \( \Gamma_{(2)} \).

\[
\begin{align*}
\tau_{s^1}^*s^*TX & \cong pr_2^*s^*TX \xrightarrow{pr_2^*\lambda^H} pr_2^*r^*TX \cong pr_1^*s^*TX \xrightarrow{pr_1^*\eta^H} pr_1^*T\Gamma \\
\tau_{s^2}^*s^*TX & \cong pr_2^*s^*TX \xrightarrow{m^*\eta^H} m^*T\Gamma \\
\tau_{s^3}^*s^*TX & \cong m^*s^*TX \xrightarrow{m^*\eta^H} m^*T\Gamma
\end{align*}
\]

The first two of them can be combined into a single morphism, say, \( \alpha = (pr_1^*\eta^H \circ pr_2^*\lambda^H, pr_2^*\eta^H) : \tau_{s^3}^*TX \to pr_1^*T\Gamma \oplus pr_2^*T\Gamma \), which is easily recognized to factor through
the subbundle \( T \Gamma \times_{T_1} T \Gamma \subset pr_1^* T \Gamma \oplus pr_2^* T \Gamma \) as in the commutative diagram below.

\[
\begin{array}{c}
\xymatrix{
s_{(2)}^* TX \ar[d] \ar[r]^-{\alpha} & pr_1^* T \Gamma \oplus pr_2^* T \Gamma \ar[d] \ar[r] & T(\Gamma \times \Gamma) |_{\Gamma(2)} \ar[d] \ar[r] & m^* m T \Gamma \\
T \Gamma \times_{T_1} T \Gamma \ar[r] & T(\Gamma \times_{T_1} \Gamma) \ar[r]^{m_\ast} & m^* m T \Gamma
}\end{array}
\]

One defines what is known as the *basic curvature* of \( H \) (cf. [1], Subsection 2.4), here denoted by \( R^H \), to be the morphism of differentiable vector bundles over \( \Gamma(2) \) that results from the expression

\[
R^H \overset{\text{def}}{=} m^* \omega^f \circ (m^* \eta^H - m_* \circ (pr_1^* \eta^H \circ pr_2^* \lambda^H, pr_2^* \eta^H)) \in \text{Hom}_{\Gamma(2)}(s_{(2)}^* T X, t_{(2)}^* g).
\]

More explicitly, at any point \((g', g) \in \Gamma(2)\) the basic curvature \( R^H_{g', g} \) is given by the linear map

\[
T_{sg}X \ni v \mapsto R^H_{g', g} v \overset{\text{def}}{=} \omega^f_{g' g} (\eta^H_{g' g} v - (\eta^H_{g' g} \lambda^H_{g' g}) \eta^H_{g' g}) \in \mathfrak{g}(s'_{g' g} g' g),
\]

**Proposition 2.1.** The following properties are equivalent for an arbitrary unital connection \( H \subset T \Gamma \) on a differentiable groupoid \( \Gamma \rightrightarrows X \).

(a) \( H \) is multiplicative.

(b) The identity \( \eta^H_{g' g} v = (\eta^H_{g' g} \lambda^H_{g' g}) \eta^H_{g' g} v \) holds for every composable pair of arrows \((g', g)\) and for all tangent vectors \( v \in T_{sg}X \).

(c) \( R^H = 0 \).

The proof is straightforward. Observe that under the assumption that the linear endomorphism \( \lambda^H_{1 x} \in \text{End}(T_x X) \) is surjective (equivalently, injective, or bijective) the property (b) alone is sufficient for the unitality of \( H \) at any given point \( x \in X \). Indeed, if that property holds then \( \eta^H_{1 x} v = \eta^H_{1 x} v = (\eta^H_{1 x} \lambda^H_{1 x}) \eta^H_{1 x} v \) on the right within \( m^* \omega^f \rightrightarrows TX \) for variable \( v \in T_x X \), we get the identity of linear maps \( T_x 1 \circ \lambda^H_{1 x} = \eta^H_{1 x} \circ \lambda^H_{1 x} \) which, if \( \lambda^H_{1 x} \) is cancellable, yields the desired conclusion. However, it is very easy to construct examples of non-unital connections with the aforesaid property.

So, that property does not imply unitality in general.

**Corollary 2.2.** Multiplicative connections are effective.

**Proof.** Apply the map \( T_{g' g} t \) to both members of the identity \( \eta^H_{g' g} v = (\eta^H_{g' g} \lambda^H_{g' g}) \eta^H_{g' g} v \). \( \Box \)

A first glance at multiplicative connections through examples

In order to get some feeling for multiplicative connections, we start looking into the simple case of action groupoids, which is already instructive. We are going to provide examples of complete classifications of multiplicative connections for a few of such groupoids. The classification will be achieved by ad hoc, elementary means, mainly by direct computation. We hope to come back to the classification problem with a more systematic treatment at some later point.
Let $G$ be an arbitrary Lie group. The tangent multiplication law $m^G: TG \times TG \to TG$ [when regarded as a vector-bundle morphism $m^G: TG \times TG \to (m^G)_*TG$] fibers over the manifold of all pairs of group elements $g, h \in G$ into linear maps $m^G_{g,h}$ given by

$$T_gG \oplus T_hG = T_{(g,h)}(G \times G) \xrightarrow{T_{(g,h)}m^G} T_{gh}G, \quad (u, v) \mapsto uv.$$  

As before, we let $\tau_g : G \xrightarrow{\sim} G$ denote the right translation by a group element $g \in G$, that is, the mapping of $G$ into itself given by $x \mapsto xg$. We also let $c_g : G \xrightarrow{\sim} G$ denote the conjugation by $g$, that is, the mapping $x \mapsto gxg^{-1}$. We have the following commutative diagram of smooth mappings.

$$
\begin{array}{ccc}
(x, y) & \xrightarrow{m_G} & G \\
\downarrow & & \downarrow \tau^1_g \times (c \circ \tau^1_h) \\
(xg^{-1}, gyh^{-1}g^{-1}) & \xrightarrow{m_G} & G \\
\end{array}
$$

(2)

Recalling our definitions, according to which $T_g \tau^1_g = T_g \tau^1_g : T_gG \xrightarrow{\sim} T_1G$ is the Maurer–Cartan isomorphism $\omega_g = \omega_g^G : T_gG \xrightarrow{\sim} \mathfrak{g} = \text{alg} G (= T_1G)$ at the point $g \in G$, upon differentiation of the diagram (2) at $(g, h) \in G \times G$ we obtain the following identity of Lie-algebra valued linear maps

$$\omega_{gh} \circ T_{(g,h)m^G} = T_{(1,1)}m^G \circ (\omega_g \times [\text{Ad}_G(g) \circ \omega_h])$$

where $\text{Ad}_G : G \to GL(\mathfrak{g})$ denotes the adjoint representation of $G$. Since the linear map $T_{(1,1)}m^G : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ is nothing but the sum operation $(X, Y) \mapsto X + Y$, we conclude that

$$m^G_{g,h} = T_{(g,h)}m^G = \omega^{-1}_{gh} \circ (\omega_g \circ pr_1 + \text{Ad}_G(g) \circ \omega_h \circ pr_1).$$

(3)

Now, suppose we are given some smooth (left) action $a : G \times U \to U$ of our Lie group $G$ on a smooth manifold $U$. Consider the action groupoid $G \bowtie^a U := G \times U \xrightarrow{pr^a_U} U$, which is of course a Lie groupoid. We put $\Gamma = G \bowtie^a U$ ($a$ being understood). Let $H$ be an arbitrary connection on our Lie groupoid $\Gamma \supseteq U$. We define $\chi^H : pr^a_UTU \to pr^a_GTG$ to be the morphism of smooth vector bundles over $\Gamma = G \times U$ given by the composition of $\eta^H$ with $(pr_G)_*$:

$$\chi^H \overset{\text{def}}{=} pr^a_UTU = s^*TU \xrightarrow{\eta^H} TG = T(G \times U) \xrightarrow{(pr_G)_*} pr^a_GTG.$$  

(Of course $pr_G : G \times U \to G$ here denotes the projection onto $G$.) This morphism fibers over $G \times U$ into the linear maps $\chi^H_{g,u} := T_{(g,u)}pr_G \circ \eta^H_{g,u} : T_uU \to T_gG$. We may further compose $\chi^H$ with the Maurer–Cartan form $TG \xrightarrow{\omega^G} \mathfrak{g}_G := G \times \mathfrak{g}$ (= trivial vector bundle over $G$ with fiber $\mathfrak{g}$) to obtain a Lie-algebra valued morphism

$$X^H \overset{\text{def}}{=} pr^a_G\omega^G \circ \chi^H : pr^a_UTU \to pr^a_G\mathfrak{g}_G = \mathfrak{g}_G \times U$$

(4)

whose fiber over each point $(g, u) \in G \times U$ is a linear map $X^H_{g,u} := \omega^G_g \circ \chi^H_{g,u} : T_uU \to \mathfrak{g}$ with values in the Lie algebra of $G$. It is clear that if we start with an arbitrary smooth
vector-bundle morphism \( X : pr_U^*TU \to \delta_{G \times U} \) then there exists exactly one connection \( H \) on \( \Gamma \) if \( U \), for which \( X = X_H \).

We proceed to derive a system of equations for the linear maps \( X^H_{g,u} \) which is to express the condition of multiplicativity for the connection \( H \) in a form particularly suitable for computations. We begin with the identity \( \text{(2.1)} \text{b) \text{, which in the present context reads} } \nabla^H_{gh,u} = (\nabla^H_{g,h} \lambda^H_{h,u}) \nabla^H_{h,u}. \]

Upon applying the linear map \( T_{(g,h,u)}pr_G : T_{(g,h,u)}(G \times U) \to T_{gh}G \) to both sides of this identity,

\[
X^H_{gh,u} = [T_{(g,h,u)}pr_G \circ T_{(g,h,u)}m^G_U](\nabla^H_{g,h} \lambda^H_{h,u} v, \nabla^H_{h,u} v) \\
= [T_{(g,h)}m^G \circ (T_{(g,h,u)}pr_G \times T_{(h,u)}pr_G)](\nabla^H_{g,h} \lambda^H_{h,u} v, \nabla^H_{h,u} v) \\
= (T_{(g,h)}m^G)(\chi^H_{g,h} v, \lambda^H_{h,u} v, \nabla^H_{h,u} v) \\
= (\omega_g^{-1} \omega_h^G(\chi^H_{g,h} v, \lambda^H_{h,u} v) + [Ad_G(g) \circ \omega_h^G](\nabla^H_{h,u} v)) \text{ [by (3)]} \\
= (\omega_g^{-1}(\chi^H_{g,h} v, \lambda^H_{h,u} v + Ad_G(g)\chi^H_{h,u} v))
\]

we see that the condition \( \text{(2.1) b) can be reformulated as a system of cocycle equations imposed on the linear maps \( X^H_{g,u} \). Namely, for all \( g, h \in G \) and for all \( u \in U \),

\[
\text{Ad}_G(g) \circ \chi^H_{h,u} = \chi^H_{gh,u} = \chi^H_{g,h} \lambda^H_{h,u} = 0. \text{ (6a)}
\]

The condition of unitality for \( H \) is expressed by the equations

\[
\chi^H_{1,u} = 0. \text{ (6b)}
\]

We observe that the zero morphism is always trivially a solution for the multiplicativity equations \( \text{(6).} \) Hence the connection \( \Phi \) on \( G \times U \) characterized by the condition \( X^\Phi = 0 \) is always multiplicative. We therefore see that action groupoids always admit multiplicative connections. However, more general Lie groupoids may not admit any; see [15] for a counterexample. As a matter of fact, the existence of a flat (i.e., integrable as a distribution) multiplicative connection is a rather strong requirement; compare [3] Corollary 3.12.

**Example A: Torus bundles**

We now restrict our attention to the case of a trivial \( G \)-action: \( a = pr_U \). In that case, the action groupoid \( \Gamma = G \ltimes U \) is simply a (trivial) bundle of Lie groups over \( U \), with fiber \( G \). For an arbitrary groupoid connection \( H \) on \( \Gamma \), the corresponding pseudo-action \( \lambda^H \) is necessarily trivial: \( \lambda^H_{g,u} = id_{T_U} \) for every \( (g,u) \in G \times U \). In particular, we see that any connection \( H \) on \( \Gamma \) is effective. Because of this, in view of the comments preceding Corollary 2.2, a connection \( H \) on \( \Gamma \) will be multiplicative if, and only if, it satisfies the equations \( \text{(5)} \) or, equivalently, the equations \( \text{(6a)} \), which in the present context take the following shape, respectively.

\[
X^H_{gh,u} = (T_{(g,h)}m^G)(\chi^H_{g,u} v, \chi^H_{h,u} v) \text{ (7)}
\]
A fast convergence theorem for nearly multiplicative connections on proper Lie groupoids

\[ \text{Ad}_G(g) \circ X^H_{\alpha} - X^H_{\alpha g} + X^H_{g,\alpha} = 0 \]

Let \( H \) be an arbitrary connection on the Lie group bundle \( G \ltimes U \xrightarrow{\text{pr}_U} U \). If for any given tangent vector \( v \in T_uU \) we put \( \xi^H_v(g) := X^H_{g,\alpha}v \) for variable \( g \in G \), we obtain a differentiable section \( \xi^H \in \Gamma^{\text{diff}}(G; TG) \) (in other words a \( C^\infty \) vector field \( \xi^H_v \) on \( G \)). If \( H \) is multiplicative then in virtue of the equation (7) the vector field \( \xi^H_v \) must be subject to the following relations for all \( g, h \in G \).

\[ \xi^H_v(gh) = \xi^H_v(g)\xi^H_v(h) \]

In general, we say that a vector field \( \zeta : G \to TG \) on a Lie group \( G \) is multiplicative if the identity \( \zeta(gh) = \zeta(g)\zeta(h) \) holds for all \( g, h \in G \). Rephrasing (7), we may say that our connection \( H \) is multiplicative if, and only if, \( \xi^H_v \) is a multiplicative vector field on \( G \) for each tangent vector \( v \in TU \). The study of multiplicative connections on \( G \ltimes U \xrightarrow{\text{pr}_U} U \) therefore reduces to the study of multiplicative vector fields on \( G \).

For the purpose of analyzing multiplicative vector fields on \( G \) it will be convenient to let \( Z_\zeta : G \to \mathfrak{g} \), for an arbitrary vector field \( \zeta \in \Gamma(G; TG) \), denote the Lie-algebra valued function on \( G \) given by \( Z_\zeta : \zeta \to Z_\zeta(g) := \omega^\zeta_g(\zeta(g)) \). [For instance, for \( \zeta = \xi^H_v \) as in the preceding paragraph, we have \( Z_\zeta^H(g) := Z_\xi^H(g) = \omega^\zeta_g(\xi^H_v) = X^H_v \) ] By the formula (3), a vector field \( \zeta : G \to TG \) will be multiplicative if, and only if, the corresponding function \( Z_\zeta : G \to \mathfrak{g} \) is a 1-cocycle for the Lie group cohomology of \( G \) with coefficients in the adjoint representation \( \text{Ad}_G : G \to GL(\mathfrak{g}) \); that is to say, the condition that \( \zeta \) is multiplicative is expressed by the following formula, in which \( g, h, G \) are arbitrary group elements.

\[ \text{Ad}_G(g)Z_\zeta(h) - Z_\zeta(gh) + Z_\zeta(g) = 0 \]

If our Lie group \( G \) is commutative then \( \text{Ad}_G = \text{id} \) will be the trivial representation and hence a vector field \( \zeta \) on \( G \) will be multiplicative if and only if \( Z_\zeta^\alpha \) is additive: \( Z_\zeta^\alpha(gh) = Z_\zeta^\alpha(g) + Z_\zeta^\alpha(h) \) for all \( g, h \in G \). Now if \( \alpha : \mathbb{R} \to G \) is an arbitrary one-parameter subgroup of \( G \) then the composition \( Z_\zeta \circ \alpha \) is a continuous additive map of \( \mathbb{R} \) into \( \mathfrak{g} \) and hence is an \( \mathbb{R} \)-linear map. If the image of \( \alpha \) is a compact subgroup of \( G \) then the image of this subgroup under \( Z_\zeta \) is a compact linear subspace of \( \mathfrak{g} \) and hence is necessarily equal to \( \{0\} \). We conclude that under the assumption that \( \zeta \) is multiplicative one has \( Z_\zeta(g) = 0 \) for every \( g \in G \) lying on a compact one-parameter subgroup of \( G \).

If the Lie group \( G \) is compact (besides being commutative) then the set of all those group elements that lie on some compact one-parameter subgroup of \( G \) will be dense within the identity component \( G_0 \) of \( G \). Hence the function \( Z_\zeta \) will vanish identically over \( G_0 \) for any multiplicative vector field \( \zeta \) on \( G \). Being additive, \( Z_\zeta \) will then take only finitely many values, one for each connected component of \( G \). But the image of \( Z_\zeta \) is also a \( \mathbb{Z} \)-sublattice of \( \mathfrak{g} \) and therefore can only be zero. We conclude that on a compact commutative Lie group the only multiplicative vector field is the zero vector field.

2.3. For the trivial action \( a = \text{pr}_U : G \times U \to U \) of a compact abelian Lie group \( G \) on a smooth manifold \( U \) the only multiplicative connection on the associated action groupoid \( G \ltimes U \xrightarrow{\text{pr}_U} U \) is the connection \( \Phi \) characterized by the condition \( X^\Phi = 0 \). More generally, any locally trivial bundle of compact abelian Lie groups \( \Gamma \xrightarrow{\text{pr}_U} M \) over a smooth manifold \( M \) admits exactly one multiplicative connection.

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Example B: Homogeneous circle-spaces

At the other extreme, there is the case of a transitive Lie group action. Among the simplest examples there are the transitive actions of the circle group $SO(2)$. Namely, let $SO(2) \times M \to M$ be an arbitrary transitive smooth (left) action of $SO(2)$ on a smooth manifold $M$. One calls this a homogeneous $SO(2)$-space. Of course, $M$ is necessarily connected and compact. The choice of a base point $x_0 \in M$ defines, for brevity $G = SO(2)$ and letting $K = \text{Stab}_G(x_0)$ denote the stabilizer subgroup at $x_0$, a $G$-equivariant diffeomorphism $G/K \overset{\sim}{\to} M$, $gK \mapsto gx_0$ of the (left) $G$-space $G/K$ of right $K$-cosets of $G$ onto $M$. Ruling out the case $K = G$ as plainly uninteresting, we may and will assume that $K$ is zero-dimensional (as a submanifold of $G$) and, therefore, discrete. Since $K$ is also a closed (hence compact) subgroup of $G$, it must actually be finite. In conclusion, our action groupoid $G \ltimes M$ is isomorphic (as a Lie groupoid) to the standard coset-action groupoid $G \ltimes G/K$ for some finite subgroup $K \subset G$ of order, say, $k$. For our purposes, it will not be restrictive to assume that $G \ltimes M$ really is $G \ltimes G/K$.

The action groupoid $G \ltimes G/K$ is isomorphic to the action groupoid $G \ltimes^k G$ associated with the twisted (left) translation action $G \times G \to G$, $(z, u) \mapsto z^k u$. If we identify $SO(2)$ with the group of complex numbers of modulus 1 under complex multiplication, the exponential $\theta \mapsto \exp 2\pi i\theta$ will be a Lie-group homomorphism from the additive group of the real numbers $(\mathbb{R}, +)$ onto $SO(2)$. The same map will be $\exp(2\pi i-)$-equivariant with respect to the action of $(\mathbb{R}, +)$ on itself given by $(\theta, a) \mapsto k\theta + a$ and with respect to the above $k$-twisted self-action of $SO(2)$. It will therefore promote to a Lie-groupoid homomorphism, say $\varepsilon$, between the corresponding action groupoids:

$$\varepsilon : \mathbb{R} \ltimes^k \mathbb{R} \longrightarrow SO(2) \ltimes^k SO(2), \ (\theta, a) \mapsto (\exp 2\pi i\theta, \exp 2\pi ia).$$

Since the tangent bundle of the real line is canonically trivial, one will have a canonical identification between on the one side the vector-bundle morphisms of type (4) for the action groupoid $\mathbb{R} \ltimes^k \mathbb{R}$ (that is to say, the connections on that action groupoid) and on the other side the real functions of class $C^\infty$ on $\mathbb{R}^2$. The connection on $\mathbb{R} \ltimes^k \mathbb{R}$ corresponding in this way to a $C^\infty$ function $X : \mathbb{R}^2 \to \mathbb{R}$ will be related through $\varepsilon$ to some connection on $SO(2) \ltimes^k SO(2)$ precisely when the function $X$ is $\mathbb{Z} \times \mathbb{Z}$-periodic: $X(\theta + l, a + m) = X(\theta, a)$ for all $(l, m) \in \mathbb{Z} \times \mathbb{Z}$.

The canonical triviality of the tangent bundle $T\mathbb{R}$ also allows one to canonically identify the pseudo-actions of the groupoid $\mathbb{R} \ltimes^k \mathbb{R}$ on the vector bundle $T\mathbb{R}$ with the functions on the real plane. The pseudo-action that corresponds to a function $\Lambda : \mathbb{R}^2 \to \mathbb{R}$ will be an action (representation) if, and only if, $\Lambda$ obeys the following constraints:

$$\Lambda(\theta' + \theta, a) = \Lambda(\theta', k\theta + a)\Lambda(\theta, a) \quad (8a)$$

$$\Lambda(0, a) = 1 \quad (8b)$$

The pseudo-action associated to the connection on $\mathbb{R} \ltimes^k \mathbb{R}$ that corresponds to a given function $X : \mathbb{R}^2 \to \mathbb{R}$, in particular, will correspond to the function denoted by $\Lambda^X : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\Lambda^X(\theta, a) = 1 + kX(\theta, a). \quad (9)$$

When rephrased in terms of $X$, the condition of multiplicativity for the connection that corresponds to $X$ takes the following form, the adjoint representation being of course

$$A(\theta' + \theta, a) = A(\theta', k\theta + a)A(\theta, a) \quad (8a)$$

$$A(0, a) = 1 \quad (8b)$$

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When rephrased in terms of $X$, the condition of multiplicativity for the connection that corresponds to $X$ takes the following form, the adjoint representation being of course
now trivial.

\[ X(\theta' + \theta, a) = X(\theta, a) + X(\theta', k\theta + a)[1 + kX(\theta, a)] \] (10a)

\[ X(0, a) = 0 \] (10b)

We notice that these equations are nothing but the equations (8) specialized for the pseudo-action \( \Lambda^k \) given by (2). Thus, we see that an arbitrary connection on \( \mathbb{R} \ltimes^k \mathbb{R} \) and, consequently, on \( SO(2) \ltimes^k SO(2) \) is multiplicative if, and only if, it is effective. We also notice that the connection on \( \mathbb{R} \ltimes^k \mathbb{R} \) or \( SO(2) \ltimes^k SO(2) \) that corresponds to the constant function \( X(\theta, a) = -1/k \), which is a function satisfying (10a) but not (10b), is an example of a connection which is non-unital but nevertheless satisfies the condition 2.1b. The associated pseudo-action vanishes identically.

Given any \( C^\infty \) real function \( X : \mathbb{R}^2 \rightarrow \mathbb{R} \) which satisfies the equations (10) for a fixed value of \( k \in \mathbb{N}_{\geq 1} \), we let \( f(\theta) = X(\theta, 0) \) denote its restriction to the \( x \)-axis. This \( f \) will be a \( C^\infty \) real function on \( \mathbb{R} \) which satisfies the condition \( f(0) = 0 \) and which never takes the value \(-1/k\) [if it did, say if \( f(\theta) = -1/k \) for some \( \theta \), then from (10a) with \( a = 0 \) and \( \theta' = -\theta \) we would deduce a contradiction: \( 0 = f(0) = f(\theta) = -1/k \)]. It completely determines \( X \) via the equation

\[ X(\theta, a) = \frac{f(\theta + a/k) - f(a/k)}{1 + kf(a/k)} \] (11)

[this equation follows at once from (10a) by first setting \( a = 0 \) and then substituting back \( a/k \) for \( \theta \) and \( \theta' \) for \( \theta \) in the resulting expression]. Conversely, given any function \( f : \mathbb{R} \rightarrow \mathbb{R} \) of class \( C^\infty \) such that \( f(\theta) \neq -1/k \) for all \( \theta \) and such that \( f(0) = 0 \), we can define a function \( X : \mathbb{R}^2 \rightarrow \mathbb{R} \) by means of (11). This will be a \( C^\infty \) function satisfying the equations (10), as one can easily check. Moreover, \( X(\theta, 0) \) will be equal to \( f(\theta) \) for all \( \theta \). In conclusion, the equation (11) defines a parameterization of the set of all multiplicative connections on the action groupoid \( \mathbb{R} \ltimes^k \mathbb{R} \) by the set of all those functions \( f \in C^\infty(\mathbb{R}, ]-1/k, \infty[) \) that vanish at zero.

It remains to be seen when exactly a function \( f \in C^\infty(\mathbb{R}, ]-1/k, \infty[) \) with \( f(0) = 0 \) corresponds to a \( \mathbb{Z} \times \mathbb{Z} \)-periodic function under the above parameterization. In one direction, it is obvious that the function \( X \) that corresponds to \( f \) will be \( \mathbb{Z} \times \mathbb{Z} \)-periodic if \( f \) is \( k^{-1}\mathbb{Z} \)-periodic. We claim that this condition is also necessary. Indeed, if \( X \) is \( \mathbb{Z} \times \mathbb{Z} \)-periodic then in particular \( X(\theta, 1) = X(\theta, 0) \) for all \( \theta \in \mathbb{R} \), which amounts to the identity:

\[ f(\theta + 1/k) - f(\theta) = f(1/k)[1 + kf(\theta)]. \]

Now if \( f(1/k) \neq 0 \) then the sequence of real numbers

\[ 0 = f(0), \ f(1/k), \ldots, f((k-1)/k), \ f(k/k) = f(1) \]

must be strictly increasing or decreasing according to the sign of \( f(1/k) \geq 0 \), because \( 1 + kf(\theta) \) is always positive. Hence \( f(1) \geq 0 \), but this is impossible since \( f(1) = X(1, 0) = X(0, 0) = 0 \). Therefore it actually was \( f(1/k) = 0 \) and, consequently, \( f(\theta + 1/k) = f(\theta) \) for all \( \theta \in \mathbb{R} \).
2.4. The set of all multiplicative connections on an arbitrary positive-dimensional homogeneous circle-space $SO(2) \ltimes M$ coincides with the set of all effective connections and is canonically parameterized by the convex subset of $C^\infty(S^1, \mathbb{R})$ consisting of all those functions that vanish at 1 and never take the value $-1/k$, where $k \in \mathbb{N}_{\geq 1}$ is the order of any stabilizer subgroup for the given action of $SO(2)$ on $M$. Thus, any two multiplicative connections on $SO(2) \ltimes M$ can be smoothly deformed into each other through multiplicative connections.

3. The averaging operator

In the present section and in the next three the only differentiable groupoids which we will be considering will be those that are simultaneously Lie and proper. Our goal, in this section, is to show the equivalence of the following two conditions for any such groupoid, say, $\Gamma$. i) $\Gamma$ admits an effective connection. ii) $\Gamma$ admits a multiplicative connection. We will achieve this by means of an averaging technique. As a matter of fact, the averaging technique in question was originally inspired to us by the deformation argument used in [4] to give a new proof of the linearization theorem for proper Lie groupoids [18, 19]; the reader will find many analogies between the subsections 2.3 to 2.5 of [4] and our exposition. Before embarking on a discussion of our averaging method, we will introduce some auxiliary notations and make a couple of general remarks. Let us now once and for all assume that we are given a proper Lie groupoid $\Gamma \Rightarrow M$; this will remain fixed throughout.

Let $\Gamma_\circ$ denote the differentiable submanifold of $\Gamma \times \Gamma$ formed by all the divisible pairs of arrows; $\Gamma_\circ := \Gamma_s \times_s \Gamma = \{(g, h) \in \Gamma \times \Gamma \mid sg = sh\}$. Let $q_\circ : \Gamma_\circ \to \Gamma$ denote the mapping given by $(g, h) \mapsto gh^{-1}$, to be called the quotient or ratio, and let $l_\circ$ and $r_\circ$ denote the left and the right projection from $\Gamma_\circ$ onto $\Gamma$. Furthermore let $s_\circ$ and $t_\circ$ denote the two mappings of $\Gamma_\circ$ onto $M$ given the first one by $s_\circ(g, h) := s(gh^{-1}) = th$ and the second one by $t_\circ(g, h) := t(gh^{-1}) = tg$. We adopt the notation $w_1 \div w_2$ for the quotient $w_1 w_2^{-1}$ of a divisible pair of tangent vectors $(w_1, w_2) \in T\Gamma \times_{\Gamma_\circ} T\Gamma$ within the tangent groupoid $T\Gamma \Rightarrow TM$. (The main reason why we find this notation convenient is that it avoids clutter in the formation of superscripts.) We observe that the operation $(w_1, w_2) \mapsto w_1 \div w_2$ can be expressed as the following composite mapping

$$T\Gamma \times_{\Gamma_\circ} T\Gamma \overset{\mathbf{1}}{\to} T(\Gamma_s \times_s \Gamma) = T(\Gamma_\circ) \overset{Tg_\circ}{\to} T\Gamma$$

(12)

where $\mathbf{1}$ indicates the isomorphism of differentiable vector bundles over $\Gamma_\circ$ that was discussed in a more general context at the beginning of Section 2 as in that section $T\Gamma \times_{\Gamma_\circ} T\Gamma$ is given the vector bundle structure (over $\Gamma_\circ$) that makes it into a subbundle of $t_\circ^* TM \oplus r_\circ^* T\Gamma$.

From Proposition 2.1 we know that for any connection $H$ on $\Gamma \Rightarrow M$ (assuming this is unital, or at least that its associated pseudo-action $A^H$ on $TM$ is unital) the condition

$$\forall (g', g) \in \Gamma_s \times_s \Gamma \quad \eta^H_{g' g} = (\eta^H_g \circ A^H_g) \eta^H_{g'} : T_{s g} M \to T_{g' g} \Gamma$$

(13)

is equivalent to the multiplicativity of $H$. Making $g' = g^{-1}$ in the preceding equation and exploiting the required unitality of $H$ we get:

$$\forall g \in \Gamma \quad (\eta^H_g)^{-1} = \eta^H_{g^{-1}} \circ A^H_g.$$
Combining the above equations and referring to the fact that for each multiplicative connection on $\Gamma \Rightarrow M$ the corresponding pseudo-action is necessarily a functor of $\Gamma$ into $GL(TM)$ we obtain

$$[\forall (g, h) \in \Gamma \times \Gamma, \Gamma] \quad \eta^H_{gh^{-1}} \circ \lambda^H_g = \eta^H_g \div \eta^H_h : T_{sg=sh}M \to T_{gh^{-1}}$$

(14)

where $\eta^H_g \div \eta^H_h$ denotes the linear map given by $T_{sg=sh}M \ni v \mapsto (\eta^H_g v) \div (\eta^H_h v) \in T_{gh^{-1}}$.

**Lemma 3.1.** Under the assumption that $H$ is a unital connection on $\Gamma \Rightarrow M$ the condition (14) expresses the multiplicativity of $H$.

**Proof.** To go back to the equation (13) starting from the equation (14) just set $gh^{-1} = h'$ in the latter and then multiply each one of its members by $\eta^H_h$ on the right. \qed

We shall say that a connection $H$ on $\Gamma \Rightarrow M$ is non-degenerate or invertible if the pseudo-action $\lambda^H$ which $H$ determines on the tangent bundle $TM$ is invertible—that is to say if we have $\lambda^H(g) \in \text{Lis}(T_gM, T_hM)$ for all $g \in \Gamma$. The notion we are introducing here is not entirely new; the non-degeneracy condition for a Lie-groupoid connection is explicitly written down in Definition 2.2 of [15], although in that work it does not really play a distinguished role as it is implied by other conditions (namely, multiplicativity) which are postulated as part of the same definition. For each value of $p = 0, 1, 2, \ldots, \infty$ we let

$$\text{Conn}^p(\Gamma) \subset \text{Conn}^p(\Gamma)$$

denote the subset traced out by all non-degenerate connections within the Fréchet manifold of all connections of class $C^p$ on $\Gamma \Rightarrow M$.

**Definition 3.2.** For any non-degenerate connection $H$ on $\Gamma \Rightarrow M$ we put

$$\delta^H(g, h) \overset{\text{def}}{=} (\eta^H_g \div \eta^H_h) \circ (\lambda^H_g)^{-1} \in L(T_{th}M, T_{gh^{-1}})$$

(15a)

for every pair $(g, h) \in \Gamma \times \Gamma$; as above, $\eta^H_g \div \eta^H_h$ indicates the linear map of $T_{sg=sh}M$ into $T_{gh^{-1}}$ given by $v \mapsto (\eta^H_g v) \div (\eta^H_h v)$. We shall refer to the global vector-bundle section defined by (15a) namely to

$$\delta^H \in \Gamma(\Gamma; L(s^*TM, q^*T\Gamma))$$

(15b)

as the **difference cocycle** associated to $H$, since in a certain sense this measures for every divisible pair $(g, h) \in \Gamma \times \Gamma$ the difference—or better, the ratio—between $\eta^H_g$ and $\eta^H_h$ relative to the tangent multiplicative structure of $\Gamma \Rightarrow M$. The sense in which $\delta^H$ should be understood to be a cocycle will be made somehow more precise later. For the time being, we limit ourselves to the vague remark that in general it is reasonable to expect the ratio to be a cocycle with respect to multiplication: $\frac{g}{h} \frac{h}{k} = \frac{g}{k}$.

We observe straightaway that the difference cocycle $\delta^H$ is of the same class of differentiability as the connection $H$ that gives rise to it. Indeed, suppose that we have $H \in \text{Conn}^p(\Gamma)$ for some value of $p = 0, 1, 2, \ldots, \infty$. Recall that this means that we have $\eta^H \in \Gamma^p(\Gamma; L(s^*TM, T\Gamma))$ whence, in particular, $\lambda^H \in \Gamma^p(\Gamma; L(s^*TM, T^*TM))$. It
is evident from \((15a)\) that \(\delta^H\) can be decomposed into the sequence of vector-bundle morphisms (over \(\Gamma\)) reproduced below,

\[
s^\ast TM \cong r^\ast_1 TM \xrightarrow{r^\ast_1 (d^H)^{-1}} r^\ast_2 s^\ast TM (\cong \Gamma^\ast s^\ast TM) \xrightarrow{(l^\ast_2, l^\ast_1, s^\ast, d^H)} l^\ast_2 T \Gamma \oplus l^\ast_1 s^\ast TM \xrightarrow{} r^\ast_2 T \Gamma = T \Gamma T_1 \times_{T_2} T \Gamma \xrightarrow{} q^\ast_1 T \Gamma
\]

in which each constituent morphism is of class \(C^p\), when not \(C^\infty\).

Making now use for the first time of the fact that our Lie groupoid \(\Gamma \Rightarrow M\) is proper, we choose a normalized left Haar system \(\nu = (\mu, \kappa)\) on \(\Gamma \Rightarrow M\). (We know this is always feasible in virtue of the propositions \(B.4\) and \(B.6\).) This choice will not be reconsidered in the sequel. We remind the reader that the left invariance of the system is expressed by the law: \(\int_{h = s} f(hk) d\nu_{sh}(k) = \int_{k = sh} f(k') d\nu_{th}(k')\).

**Definition 3.3.** For any given non-degenerate connection \(H\) on \(\Gamma \Rightarrow M\) and for every arrow \(g \in \Gamma\) we let \(\hat{\eta}^H_g\) denote the linear map

\[
T_{s^g} M \ni \nu \mapsto \hat{\eta}^H_g \nu \overset{\text{def}}{=} \int_{h = s^g} \delta^H(gk, k) \nu d\nu_{sg}(k) \in T_g \Gamma.
\]  

(Notice that this expression makes sense because \(\delta^H(gk, k)\) is a linear map of \(T_{gk^{-1}} M\) into \(T_g \Gamma\) for all \(g \leftarrow k\).) We shall refer to the global vector-bundle section thus defined

\[
\hat{\eta}^H \in \Gamma(\Gamma; L(s^\ast TM, T \Gamma))
\]

as the (multiplicative) average of the connection \(H\) (taken with respect to the chosen Haar system \(\nu\)).

**Lemma 3.4.** The multiplicative average \(\hat{\eta}^H\) of any non-degenerate \(C^p\) connection \(H \in \text{Conn}^p(\Gamma)\) is itself the horizontal lift for a (unique) connection \(\hat{H}\) of class \(C^p\) on \(\Gamma \Rightarrow M\) which is always unital and which we also call the multiplicative average of \(H\).

**Proof.** Let us in the first place address the question of the differentiability of the global section \((16b)\). We know that \(\delta^H\) is a global \(C^p\) section of the (real) differentiable vector bundle \(L(s^\ast TM, q^\ast_1 T \Gamma)\). Pulling it back along the diffeomorphism \(a : \Gamma_1 \times_t \Gamma_2 \overset{\cong}{\rightarrow} \Gamma_3\) given by \((g, k) \mapsto (gk, k)\) we obtain a global \(C^p\) section of the differentiable vector bundle \(a^\ast L(s^\ast TM, q^\ast_1 T \Gamma) \cong L(a^\ast s^\ast TM, a^\ast q^\ast_1 T \Gamma) \cong L(pr^\ast_1 s^\ast TM, pr^\ast_1 T \Gamma) \cong pr^\ast_1 L(s^\ast TM, T \Gamma)\) (where \(pr_1\) denotes the projection on the 1st factor : \(\Gamma_1 \times_t \Gamma \rightarrow \Gamma\)) which, applied to \(\nu\), occurs as the integrand in the expression \((16a)\). The lemma \([B.7]\) about Haar integrals depending on parameters then implies at once that \(\hat{\eta}^H \in \Gamma^p(\Gamma; L(s^\ast TM, T \Gamma))\).

We proceed to check that \(s^\ast \circ \hat{\eta}^H = \text{id}_{s^\ast TM}\).

\[
(s^\ast)_{g} \circ \hat{\eta}^H_g = T_{s^g} s \circ \int_{h = s^g} \delta^H(gk, k) d\nu_{sg}(k) = \int_{h = s^g} T_{s^g} s \circ (\eta^H_{gk} \div \eta^H_k) \circ (\lambda^H_k)^{-1} d\nu_{sg}(k)
\]
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\[
\pi = \pi^\Phi \overset{\text{def}}{=} T\Gamma \frac{(\omega \circ \beta^\Phi, \alpha)}{s^* \Theta} \oplus s^* TM
\]

[Diagram]

\[\pi = \pi^\Phi \overset{\text{def}}{=} (T^\pi T \circ \eta^\Phi \circ s, \Theta) \oplus s^* TM\]

\[
\int_{tk=sg} (T t \circ \eta^H_k) \circ (\lambda^H_k)^{-1} d\nu_{sg}(k)
\]

\[
= \int_{tk=sg} d\nu_{sg}(k) id_{T \pi M} = (id_{s^* TM})_g
\]

To conclude, we must show that \(\hat{\eta}^H_1\) equals \(T_1 \pi\) for all \(x \in M\). We have \(\delta^H(k, k) = (\eta^H_k \div \eta^H_k^H) \circ (\lambda^H_k)^{-1} = (1^T T \circ t^T T \circ \eta^H_k) \circ (\lambda^H_k)^{-1} = T_1 \circ (T t \circ \eta^H_k) \circ (\lambda^H_k)^{-1} = T_1 \pi\) and hence \(\hat{\eta}^H_1 = \int \nu_{sg}(k) T_1 \pi = T_1 \pi\). The proof is finished. □

Just like any other connection on \(\Gamma \Rightarrow M\), \(\hat{\eta}^H\) will determine a pseudo-action of \(\pi\) on \(T \pi M\), hereafter denoted by \(\hat{\lambda}^H\), for which we have the following explicit formula in terms of \(\lambda^H\).

\[
\hat{\lambda}^H_g = T_g T \circ \hat{\eta}^H_g = \int_{tk=sg} T_g T \circ (\eta^H_g \div \eta^H_k) \circ (\lambda^H_k)^{-1} d\nu_{sg}(k)
\]

Multiplicativity equations for the vertical component of a connection

In Section 2 when dealing with multiplicative connections on action groupoids, we exploited the natural splitting of the tangent bundle of an action groupoid into its vertical and its horizontal subbundle to the purpose of rewriting the multiplicativity condition in terms of the vertical component of a connection. This proved to be useful, then, from the point of view of computations. We intend to work out an analogous reformulation of the multiplicativity condition in the more general context of the present section. Even though in the case of an arbitrary proper Lie groupoid there are no global—let alone canonical—trivializations of the groupoid source mapping available, we can at least always find such trivializations at the infinitesimal level. Namely, on our groupoid \(\Gamma \Rightarrow M\) let us randomly fix some connection \(\Phi\), which we agree to call our “background” connection. This choice will give rise to a splitting of the tangent bundle of \(\Gamma\) into a vertical and a horizontal component, relative to the tangent source mapping:

\[\pi = \pi^\Phi \overset{\text{def}}{=} T(\omega \circ \beta^\Phi, \alpha) \oplus s^* TM\]

[z]

\[\pi = \pi^\Phi \overset{\text{def}}{=} \text{image of vertical projection associated to } \Phi\; \omega \text{ denotes the Maurer–Cartan isomorphism between the } s\text{-vertical tangent bundle } T\Gamma \subset T\Gamma \text{ and the pullback of the algebroid bundle } \Theta = 1^* T^\pi \Gamma \text{ along the target mapping}\]

The splitting (18) leads to a corresponding decomposition of the tangent quotient operation (12). Namely, for each divisible pair of arrows \((g, h) \in \Gamma_\pi\) we have two linear maps, denoted by \(\hat{\eta}^\phi_{g,h}, \hat{\phi}^\phi_{g,h}\) and characterized through the commutativity of the diagram
below, which go from the vector space $\mathfrak{g}_g \oplus \mathfrak{g}_h \oplus T_{sg=sh}M$ into respectively the vector space $\mathfrak{g}_g$ and the vector space $T_{sh}M$.

\[
T_g \Gamma_{\mathfrak{g}_g} \times T_{sh} \Gamma_{\mathfrak{g}_h} \xrightarrow{\pi^\Phi \times \pi^\Phi} (\mathfrak{g}_g \oplus T_{sg}M) \oplus (\mathfrak{g}_h \oplus T_{sh}M)
\]

\[
T_{(g,h)}(\Gamma_{\mathfrak{g}_g}) \xrightarrow{T_{(g,h)}s} T_{g=(gh^{-1})} \mathfrak{g}_g \oplus \mathfrak{g}_h \oplus T_{sg=sh}M \xrightarrow{\pi^\Phi} \mathfrak{g}_g \oplus \mathfrak{g}_h \oplus T_{sh}M
\]

It is easily recognized that $\dot{s}_{g,h} = \dot{s}^\Phi_{g,h}$ coincides with the composite linear map

\[
\mathfrak{g}_g \oplus \mathfrak{g}_h \oplus T_{sg=sh}M \xrightarrow{pr} \mathfrak{g}_h \oplus T_{sh}M \xrightarrow{(\pi^\Phi_h)^{-1}} T_h \Gamma \xrightarrow{T_{sh}} T_{sh}M
\]

where of course $pr$ denotes the projection $(X, Y, v) \mapsto (Y, v)$. In particular we see that the expression $\dot{s}^\Phi_{g,h}(X, Y, v)$ is independent of $g$ and $X$, so we shall abbreviate that expression into $\dot{s}^\Phi_h(Y, v)$. Let us introduce a bunch of related abbreviations, of which we will make repeated use.

\[
\begin{align*}
\dot{q}^\Phi_{g,h}(g, h)(X, Y) & := \dot{q}^\Phi_{g,h}(X, Y, 0) \quad \dot{s}^\Phi_h(Y, 0) := \dot{s}^\Phi_{g,h}(Y, 0) & (21a) \\
\dot{q}^\Phi_{g,h}(g, h)v & := \dot{q}^\Phi_{g,h}(0, 0, v) \quad \dot{s}^\Phi_h(0, v) := \dot{s}^\Phi_{g,h}(0, v) & (21b)
\end{align*}
\]

### 3.5

Let $H$ be an arbitrary connection on $\Gamma \Rightarrow M$. Any such $H$ will be entirely encoded in its vertical component relative to the chosen background connection $\Phi$; by definition, this is the vector-bundle morphism $s' \rightarrow \Gamma \rightleftharpoons (\mathfrak{g}_g)$ (over $\Gamma$) denoted by $X^{H/\Phi}$ given at any point $g \in \Gamma$ by

\[
X^H_{g/\Phi} \overset{\text{def}}{=} \omega_g \circ \beta^\Phi_g \circ \eta^H_g : T_{sg}M \rightarrow \mathfrak{g}_g.
\]

When the background connection $\Phi$ is fixed, like in our case, we may of course simply write $X^H$ instead of $X^{H/\Phi}$. By the above definitions, we have $\pi^\Phi_g(\eta^H_g v) = ((\omega_g \circ \beta^\Phi_g)\eta^H_g v, (T_g s)\eta^H_g v) = (X^H g, v)$ for all $v \in T_{sg}M$. Since $\pi^\Phi_g$ is an invertible linear map we see that the condition (14) is satisfied if and only if the identity below holds for all $(g, h) \in \Gamma_+, v \in T_{sg=sh}M$.

\[
(X^H_{g,h})^{-1} (\dot{\lambda}^H_{g,h} v) = \pi^\Phi_{g,h} (\eta^H_{g,h} \dot{\lambda}^H_{g,h} v)
\]

\[
= \pi^\Phi_{g,h} (\eta^H_{g,h} (v + \eta^H_{g,h} v))
\]

\[
= [\pi^\Phi_{g,h} \circ T_{(g,h)} s] (\eta^H_{g,h} v, \eta^H_{g,h} v)
\]

\[
= (\dot{q}^\Phi_{g,h}(X^H g, v), \dot{s}^\Phi_{g,h}(X^H g, v))
\]

After suppressing $v$ and making use of the abbreviations (21), we obtain the following two equations.

\[
X^H_{g,h} \circ \dot{\lambda}^H_{h} = \dot{q}^\Phi_{g,h}(X^H g, X^H h) + \dot{s}^\Phi_{g,h}(g, h) & (22a) \\
\dot{\lambda}^H_h = \dot{s}^\Phi_{g,h}(h) \circ \dot{X}^H_h + \dot{s}^\Phi_{g,h}(h) & (22b)
\]
Notice that the second of these equations is a tautology; indeed, by the above remark to the effect that \( \dot{s}_h(Y, v) = (T_{t, h})\pi^{-1}_h(Y, v) \) [compare (20)], we have:

\[
\lambda_h^H - \dot{s}_h \circ (X_h^H, id) = T_{h, t} \circ \eta_h^H - T_{h, t} \circ \pi^{-1}_h \circ (X_h^H, id) = T_{h, t} \circ (\eta_h^H - \eta_t^H) = 0.
\]

It follows that for any connection \( H \) the condition (14) is equivalent to the following system of equations involving only the vertical component \( X^H \) of \( H \), which one obtains by substituting (22b) into (22a).

\[
[\forall (g, h) \in \Gamma_+ \] \( \dot{q}_t^\Phi(g, h) \circ (X^H_g, X^H_h) = X^H_{gh^{-1}} \circ (\dot{s}_t(h) \circ X^H_h + \dot{s}_s(h)) - \dot{q}_s(g, h) \quad (23)
\]

3.6. In the last expressions (22) (23) the horizontal terms \( \dot{q}_s(\ldots) \) and \( \dot{s}_s(\ldots) \) can be given a slightly more intuitive description, as follows. To begin with, we have \( \dot{s}_h(0, v) = (T_{t, h})(\pi_h^\Phi)^{-1}(0, v) = (T_{t, h})\eta_h^\Phi v \) for all \( v \in T_{sh}M \), whence

\[
\dot{s}_s^\Phi(h) = \lambda_h^\Phi. \quad (24)
\]

Secondly, for all \( v \in T_{sg=sb}M \) we have (at least, when \( \Phi \) is non-degenerate):

\[
\dot{q}_s^\Phi(0, 0, v) = [pr_1 \circ \pi_{gh^{-1}} \circ T_{(g, h)} \eta_{gh^{-1}} ((\pi^\Phi_h)^{-1}(0, v), (\pi^\Phi_h)^{-1}(0, v)) = [\omega_{gh^{-1}} \circ \beta_{gh^{-1}} \circ T_{(g, h)} \eta_{gh^{-1}}] (\eta_h^\Phi v, \eta_g^\Phi v) = [\omega_{gh^{-1}} \circ \beta_{gh^{-1}} \circ \delta^\Phi(g, h)] \lambda_h^\Phi v.
\]

Thus, letting \( \Delta^\Phi(g, h) := \omega_{gh^{-1}} \circ \beta_{gh^{-1}} \circ \delta^\Phi(g, h) : T_{th}M \to \delta_{gh^{-1}} = t_s \) denote the vertical component of the difference cocycle, we obtain

\[
\dot{q}_s^\Phi(g, h) = \Delta^\Phi(g, h) \circ \lambda_h^\Phi. \quad (25)
\]

Substituting the last two expressions into (23) we reach the following conclusion. For any choice of a non-degenerate background connection \( \Phi \) on \( \Gamma \Rightarrow M \), any other connection \( H \) on \( \Gamma \Rightarrow M \) will be multiplicative if and only if it is unital and its vertical component \( \lambda^{H/\Phi}_h \) relative to \( \Phi \) satisfies the following equation for all divisible pairs \( (g, h) \in \Gamma_+ \)

\[
\text{(Multiplicativity Equation)} \quad \begin{align*}
\dot{q}_s^\Phi(g, h) \circ (X^H_{gh^{-1}}, X^H_h) &= X^H_{gh^{-1}} \circ \lambda^H_h - \Delta^\Phi(g, h) \circ \lambda^\Phi_h \\
\end{align*}
\]

where

\[
\lambda^H_h = \lambda^\Phi_h + \dot{s}_s^\Phi(h) \circ X^H_{h/\Phi} \quad (26b)
\]

Cocycle equations for the background connection

Let \( g, h, k \in \Gamma \) satisfy \( sg = sh = sk \). Then

\[
q_s(g, k) = gk^{-1} = gh^{-1}hk^{-1} = gh^{-1}(kh^{-1})^{-1} = q_s(g, h)q_s(h, k).
\]

If for each pair of indices \( i, j \in \{1, 2, 3\} \) with \( i \neq j \) we let \( q_{ij} \) denote the mapping of \( \Gamma_+ \times \Gamma_+ \) \( \Gamma_+ \) into \( \Gamma \) given by \( (g_1, g_2, g_3) \mapsto q_s(g_i, g_j) \), we can rephrase the last identity more succinctly as

\[
q_{13} = q_s \circ (q_{12}, q_{32}).
\]
Differentiating this identity at any point \((g, h, k) \in \Gamma \times \Gamma \times \Gamma\) and taking into account the obvious relations \(T_{(g, h, k)} q_{ij} = T_{(g, h)} q_{ij} \circ pr_{ij}\), where \(pr_{ij}\) denotes the projection of \(T_{\gamma i} T_{\gamma j} \times T_{\gamma j} T_{\gamma k} \Gamma\), onto \(T_{\gamma i} T_{\gamma j} T_{\gamma k} \Gamma\) given by \((w_1, w_2, w_3) \mapsto (w_i, w_j)\), we obtain

\[
T_{(g, k)} q_{ij} \circ pr_{13} = T_{(g, k, h)} q_{ij} \circ (T_{(g, h)} q_{ij} \circ pr_{12}, T_{(k, h)} q_{ij} \circ pr_{32}).
\]

Composing to the left with the invertible linear map \(\pi_{g k} = gh^{-1} (k h)^{-1}\) and to the right with the linear map \((\eta_g^\Phi, \eta_h^\Phi, \eta_k^\Phi) : T_{\gamma g = (h k) M} \rightarrow T_{\gamma} T_{\gamma} T_{\gamma} \Gamma\), and making repeated use of the commutativity of the diagram \((21)\) and of the relation \(\pi^T \circ \eta^\Phi = (0, id_{\gamma T M})\), we obtain for every tangent vector \(v \in T_{\gamma g = (h k) M}\) the following pair of equations.

\[
\begin{align*}
\dot{q}_{g, k}^\phi(0, 0, v) &= \dot{q}_{g h^{-1}, k h^{-1}}^\phi(q_{g, h}^\phi(0, 0, v), q_{k, h}^\phi(0, 0, v), s_h^\phi(0, v)) \\
\dot{s}_h^\phi(0, v) &= s_{k h^{-1}}^\phi(q_{g, h}^\phi(0, 0, v), s_h^\phi(0, v))
\end{align*}
\]

Recalling the abbreviations \((21)\) and the identities \((24)\) \((25)\) (as in \((3, 6)\), we are now assuming that our background connection \(\Phi\) is non-degenerate), we can rewrite these equations as follows.

\[
\begin{align*}
\Delta^\phi(g, k) \lambda_k^\phi h v &= \dot{q}_{g h^{-1}, k h^{-1}}^\phi(q_{g, h}^\phi(g, h) v, q_{k, h}^\phi(k, h) v) \\
&+ \dot{q}_{g h^{-1}, k h^{-1}}^\phi(s_{g, h}^\phi(h) v) \\
&= [\dot{q}_{g h^{-1}, k h^{-1}}^\phi(\Delta^\phi(g, h) \Delta^\phi(k, h)) \lambda_h^\phi v]_h \\
&+ [\Delta^\phi(g h^{-1}, k h^{-1}) \lambda_h^\phi (h)]_h v \\
\lambda_k^\phi h v &= \dot{s}_h^\phi(k h^{-1}) q_{g, h}^\phi(k, h) v + \dot{s}_{g, h}^\phi(k h^{-1}) s_{h}^\phi(h) v \\
&= [\dot{s}_h^\phi(k h^{-1}) \Delta^\phi(k, h)]_h v + \lambda_{k h^{-1}}^\phi(h) v \\
\Delta^\phi(g, h) \lambda_h^\phi - \lambda_{h k^{-1}}^\phi \lambda_h^\phi &= \dot{s}_h^\phi(h') \Delta^\phi(h' h, h) \lambda_h^\phi (h') v
\end{align*}
\]

In synthesis, after suppressing \(v\) from these equations and setting \(k h^{-1} = h'\) in the second of them, we are left with the following tautological expressions, which we call the “Cocycle Equations”.

\[
\begin{align*}
\dot{q}_{g h^{-1}, k h^{-1}}^\phi(\Delta^\phi(g, h) \Delta^\phi(k, h)) \circ \lambda_h^\phi &= \Delta^\phi(g, k) \circ \lambda_k^\phi h v = \Delta^\phi(g h^{-1}, k h^{-1}) \circ \lambda_{k h^{-1}}^\phi (h) v \\
\lambda_{k h^{-1}}^\phi (h) v &= \dot{s}_h^\phi(h') \circ \Delta^\phi(h' h, h) \circ \lambda_h^\phi (h') v
\end{align*}
\]

Proposition 3.7. Assume that \(\Phi\) is an effective (hence non-degenerate) connection on a proper Lie groupoid \(\Gamma \rightrightarrows M\). Then, the multiplicative average \(\hat{\Phi}\) of \(\Phi\) (computed with respect to any normalized left Haar system \(v\) on \(\Gamma \rightrightarrows M\)) is a multiplicative connection.

Proof. We must check the validity of the Multiplicativity Equation \((26a)\) for the (in view of Lemma \((3, 4)\) unital) connection \(H = \hat{\Phi}\), relative to the (non-degenerate) background connection \(\Phi\). It will not be a bad idea to abridge \(\lambda^\phi\) into \(\hat{\lambda}\) and \(\lambda^\phi / \phi\) into \(\tilde{\lambda}\). We will also systematically suppress \(\Phi^\phi\)-superscripts and thus, for instance, simply write \(\Delta(g, h)\) in place of \(\Delta^\phi(g, h)\) or \(\Delta^\phi(g, h)\) in place of \(\lambda^\phi\).

As a side remark, we observe that when \(H = \hat{\Phi}\) the tautology \((26a)\)—which, we stress, is merely a consequence of \(H\) being a connection—can alternatively be deduced from the cocycle equation \((27b)\). Indeed, referring back to the formula \((17)\), we have

\[
\hat{\lambda}_{h'} - \hat{\lambda}_h = \int_{h = h'} \left[ \lambda_{h', h}^{-1} \lambda_{h'} - \lambda_{h'} \right] d \nu_{sh}(h)
\]

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\[ = \int_{\theta h = s h'} \hat{s}_\lambda(h') \circ \Delta(h', h) \, d\nu_{s h'(h)} \quad \text{[by (27b)]} \]
\[ = \hat{s}_\lambda(h') \circ \hat{X}_{h'} \quad \text{.} \]

(28)

We proceed to check the validity of the Multiplicativity Equation for \( \hat{\Phi} \).

\[ \hat{q}_\lambda(g, h) \circ (\hat{X}_g, \hat{X}_h) = \hat{q}_\lambda(g, h) \circ \int_{\lambda_k = x} (\Delta(gk, k), \Delta(hk, k)) \, d\nu_s(k) \]

[By (27b): ]
\[ = \int_{\lambda_k = x} [\Delta(gk, hk) \circ \nu_{k} \circ \nu_{k}^{-1} - \Delta(g, h) \circ \nu_{k}] \, d\nu_s(k) \]
[By (27b): ]
\[ = \int [\Delta(gk, hk) \circ \hat{s}_\lambda(h) \circ \Delta(hk, k) + \lambda_h] \, d\nu_s(k) \]
[By (28): ]
\[ = \int \Delta(gk, hk) \circ \hat{s}_\lambda(h) \circ \Delta(hk, k) \, d\nu_s(k) \]
\[ + \int \Delta(gk, hk) \circ [\hat{\lambda}_h - \hat{s}_\lambda(h) \circ \hat{X}_k] \, d\nu_{s h'(k)} - \Delta(g, h) \circ \lambda_h \]
\[ = \int \Delta(gk, hk) \circ \hat{s}_\lambda(h) \circ [\Delta(hk, k) - \hat{X}_k] \, d\nu_s(k) \]
\[ + \int \Delta(gk, hk) \circ \hat{s}_\lambda(h) \circ [\Delta(hk, k) - \Delta(hk', k')] \, d\nu_s(k) \, d\nu_s(k') \]
\[ + \hat{X}_{s h^{-1}} \circ \hat{\lambda}_h - \Delta(g, h) \circ \lambda_h \]

Thus far we have not used the assumption that \( \Phi \) was effective. Now, if that is the case then by (27b) the double integral term must vanish. \( \square \)

A connection \( \Phi \) on a proper Lie group bundle \( \Gamma \xrightarrow{x=1} M \) is always effective since for any such groupoid the target mapping equals the source mapping and therefore \( \lambda_g^\Phi = id_{T_gM} \) for all \( g \in \Gamma \). Thus the preceding proposition has the following immediate consequence.

**Corollary 3.8.** Any proper Lie group bundle admits multiplicative connections.

### 3.9 (Concluding remarks).

The equations (27) look like a reworking of the equations (5) and (7) which appear in [11] statement of Proposition 2.15 and which underlie the construction of the adjoint representation of a Lie groupoid as a representation up to homotopy. We decided to call these equations “cocycle equations” simply by analogy with the terminology adopted in the above-mentioned reference; in fact, we have not even attempted giving a really convincing justification for our choice of terminology, partly because this issue does not seem so relevant for the purposes of the present paper. It would probably be instructive to look for a more conceptual description of the averaging operator [16] in the framework of Lie groupoid cohomology with “up-to-homotopy” coefficients, hopefully in this way shedding more light on the role of the
non-degeneracy condition appearing in our definitions and of the “longitudinal” obstruction arising from the fact that the effect of a connection may in general fail to be an action; in particular, we believe there should be a way of relating Proposition 3.7 above to the Cohomology Vanishing Theorem of [1]. Evidence in this direction also seems to be provided by the deformation argument used by Weinstein in [18, proof of Theorem 7.1], which may be turned into a proof of Corollary 3.8 above. However, the approach we adopt here seems to us preferable at least from the point of view of a self-contained exposition. In any case, we do not really know how to adapt the arguments of [18] to a situation more general than that considered in the statement of Corollary 3.8.

4. Basic recursive estimates

In the preceding section we saw that the process that to each non-degenerate connection $\Phi$ on a proper Lie groupoid $\Gamma \rightrightarrows M$ assigns the corresponding multiplicative average $\hat{\Phi}$ (taken with respect to some fixed normalized left Haar system on $\Gamma \rightrightarrows M$) results for any order of differentiability $p = 0, 1, 2, \ldots, \infty$ in an averaging operator

$$\text{Conn}^p(\Gamma) \to \text{Conn}_p^0(\Gamma), \ \Phi \mapsto \hat{\Phi}.$$ 

We have shown that this operator carries effective connections into multiplicative connections. It is immediate to see that every multiplicative connection belongs to the fixed-point set of this operator. On the basis of these considerations, for a generic non-degenerate connection $\Phi$ it seems interesting to investigate the sequence of connections $\hat{\Phi}, \hat{\hat{\Phi}}, \ldots$ which one obtains by repeatedly averaging $\Phi$. Of course, there is in the first place the question of whether this sequence is at all defined. Provided it is, one can study its convergence. In view of the integral formula (17) and of Proposition 3.7, it is reasonable to expect the behavior in the limit of the iterated averages of $\Phi$ to depend essentially on the behavior in the limit of their effects. These are known to coincide with the pseudo-actions of $\Gamma \rightrightarrows M$ on $TM$ obtained from the pseudo-action $\lambda^\Phi$ by recursive application of the formula (17). At this point, there is no particular reason for restricting one’s analysis only to pseudo-actions arising from connections.

As in the previous section, we shall be assigned a proper Lie groupoid $\Gamma \rightrightarrows M$ and a normalized left Haar system $\nu = (\mu, \kappa)$ on it. In addition, we shall be given a $K$-linear ($K = \mathbb{R}$ or $\mathbb{C}$) differentiable vector bundle $E$ over the base $M$ of $\Gamma$. These data shall be kept fixed for the rest of the section.

4.1. For any order of differentiability $p = 0, 1, 2, \ldots, \infty$ we shall let

$$\text{Psa}^\infty_p(\Gamma; E) \subset \text{Psa}^p(\Gamma; E)$$

(by analogy with the notations introduced in Section 3) denote the set of all invertible $C^p$ pseudo-actions of $\Gamma$ on $E$. If for any pseudo-action $\lambda$ of $\Gamma$ on $TM$ which is of the form $\lambda = \lambda^\Phi$ for some non-degenerate connection $\Phi$ on $\Gamma$ we set

$$\Delta^\lambda(h'h, h) := \lambda^{\Phi}(h') \circ \Delta^\Phi(h'h, h)$$

in the tautological expression (27b) and then take $h' = gh^{-1}$, we obtain the following identity:

$$\Delta^\lambda(g, h) = \lambda_g \circ \lambda_h^{-1} - \lambda_{gh^{-1}}. \quad (29)$$
Regarding this as the definition of $\Delta^4$ when $\lambda$ is an arbitrary invertible pseudo-action of $\Gamma$ on our vector bundle $E$, we obtain a global vector-bundle section

$$\Delta^4 \in \Gamma(\Gamma_G; L(s^*_p E, t^*_p E)).$$

This section will be of class $C^p$ whenever the given pseudo-action $\lambda$ is of class $C^p$. Motivated by the identity (17) which expresses the effect of the multiplicative average of a connection in terms of the effect of the connection itself, for any invertible $C^p$ pseudo-action $\lambda \in \text{Psap}^*_p(\Gamma; E)$ we set:

$$\hat{\lambda}(g) \overset{\text{def}}{=} \int_{k=sg} \lambda(\lambda k) \circ \lambda(k)^{-1} \, dv_{sg}(k) \quad (30)$$

(for every $g \in \Gamma$). By the same argument as in the proof of Lemma 3.4, it follows from the lemma on Haar integrals depending on parameters (3.7) that $\hat{\lambda}$ belongs to $\text{Psap}_p^*(\Gamma; E)$; otherwise stated $\hat{\lambda}$ is a unital $C^p$ pseudo-action of $\Gamma$ on $E$. We have the following two fundamental equations.

$$\hat{\lambda}(g'g) - \hat{\lambda}(g'') \circ \hat{\lambda}(g) = \int_{k=sg} \Delta^4(g'gk, gk) \circ \Delta^4(gk, k) \, dv_{sg}(k)$$

- $$\int_{h=sg} \Delta^4(g'gh, gh) \circ \Delta^4(gk, k) \, dv_{sg}(h) \, dv_{sg}(k) \quad (31a)$$

$$\hat{\lambda}(g) = \lambda(g) + \int_{k=sg} \Delta^4(gk, k) \, dv_{sg}(k) \quad (31b)$$

[Proof. The second equation is an immediate consequence of our Haar system’s being normalized: $\hat{\lambda}_g = \int \lambda_{gk} \lambda_k^{-1} \, dv_{sg}(k) = \int \lambda_g \, dv_{sg}(k) + \int (\lambda_{gk} \lambda_k^{-1} - \lambda_g) \, dv_{sg}(k) = \lambda_g + \int \Delta^4(gk, k) \, dv_{sg}(k)$. As to the first equation, we use both left invariance and normality of the Haar system:

$$\lambda_{g'} - \lambda_{g'} \lambda_g = \int \lambda_{g'gk} \lambda_k^{-1} \, dv_{sg}(k) - \left( \int \lambda_{g'k'} \lambda_k^{-1} \, dv_{sg}(k') \right) \circ \left( \int \lambda_{gk} \lambda_k^{-1} \, dv_{sg}(k) \right)$$

[setting $k' = gh$: ]

$$= \int \lambda_{g'gk} \lambda_k^{-1} \, dv_{sg}(k) - \int \lambda_{g'gk} \lambda_k^{-1} \lambda_g \, dv_{sg}(k) - \lambda_g \lambda_g$$

- $$\left( \int \lambda_{g'gh} \lambda_{ghh}^{-1} \, dv_{sg}(h) \right) \circ \left( \int \lambda_{ghh} \lambda_{ghh}^{-1} \, dv_{sg}(k) \right)$$

$$= \int (\lambda_{g'gk} \lambda_k^{-1} - \lambda_g) \circ (\lambda_{gk} \lambda_k^{-1} - \lambda_g) \, dv_{sg}(k)$$

- $$\int (\lambda_{g'gh} \lambda_{ghh}^{-1} - \lambda_g) \circ (\lambda_{ghh} \lambda_{ghh}^{-1} - \lambda_g) \, dv_{sg}(h) \, dv_{sg}(k),$$

which is the desired relation.]
4.2. Let us now endow our vector bundle $E$ with some metric $\phi$ of class $C^\infty$ (Riemannian or Hermitian, depending on whether $E$ is real or complex). This is of course always possible because $M$ admits partitions of unity of class $C^\infty$. We shall keep $\phi$ fixed throughout the sequel. For each pair of base points $x, x' \in M$ we obtain a norm $\| \cdot \|_{x,x'}$ on the vector space $L(E_x, E_{x'})$ by setting

$$\|\lambda\|_{x,x'} = \sup_{|v|_x=1} |\lambda v|_{x'}$$

for every linear map $\lambda$ of $E_x$ into $E_{x'}$; here $| \cdot |_x$ denotes the norm on $E_x$ given by $|v|_x = \sqrt{\phi_x(v,v)}$. Observe that the norms $\| \cdot \|_{x,x'}$ satisfy the following inequalities (for all $x, x', x'' \in M$).

$$\|\lambda' \circ \lambda\|_{x,x''} \leq \|\lambda'\|_{x',x''} \cdot \|\lambda\|_{x,x'}$$

(32)

These inequalities say in particular that $\text{End}(E_x)$ is a Banach algebra under the norm $\| \cdot \|_{x,x}$.

**Lemma 4.3.** Let $A$ be a Banach algebra with unit element $e$. For any given real constant $0 \leq c < 1$, and for every element $v \in A$ such that $|v| \leq c$, the element $e - v$ is invertible, and one has:

$$|(e - v)^{-1} - e| \leq c(1 - c)^{-1}.$$  

**Proof.** Since $|v| < 1$, one has that $e - v$ is invertible with inverse given by 

$$(e - v)^{-1} = e + v + v^2 + v^3 + \cdots.$$  

Then $|(e - v)^{-1} - e| \leq |v| + |v|^2 + |v|^3 + \cdots = |v|(1 - |v|)^{-1} \leq c(1 - c)^{-1}$. \qed

4.4. For any continuous pseudo-action $\lambda$ of $\Gamma$ on $E$ we make the following definitions.

$$b(\lambda) = b_\phi(\lambda) \overset{\text{def}}{=} \sup_{g \in \Gamma} \|\lambda(g)\|_{sg,tg}$$

(33a)

$$c(\lambda) = c_\phi(\lambda) \overset{\text{def}}{=} \sup_{(g',g) \in \Gamma \times \Gamma} \|\lambda(g'g) - \lambda(g') \circ \lambda(g)\|_{sg,tg'}$$

(33b)

These quantities may be infinite, of course. However if $b(\lambda) < \infty$ then $c(\lambda) < \infty$ also.

4.5 (Remark). Let $\lambda \in \text{Psa}_u(\Gamma; E)$ be a unital, continuous, pseudo-action of $\Gamma$ on $E$. Suppose $c(\lambda) < 1$. Then $\lambda$ must be invertible. [Proof. The assumptions entail the inequality: $1 > \|id - \lambda^{-1} \circ \lambda\|_{sg,sg}$ (arbitrary $g$). Since $\text{End}(E_{sg})$ equipped with the norm $\| \cdot \|_{sg,sg}$ is a unital Banach algebra, it follows that $\lambda^{-1} \circ \lambda$ must be an invertible element of $\text{End}(E_{sg})$ and therefore that $\lambda_x$ must be a left invertible (hence injective) linear map. Similarly one sees that $\lambda_x$ must be right invertible (hence surjective).]

4.6. One has the two estimates below for any unital, continuous, pseudo-action $\lambda \in \text{Psa}_u(\Gamma; E)$ which satisfies the condition $c(\lambda) < 1$.

$$[\forall g \in \Gamma] \quad \|\lambda(g)^{-1}\|_{sg,sg} \leq \frac{b(\lambda)}{1 - c(\lambda)}$$

(34a)

$$[\forall (g, h) \in \Gamma \times \Gamma] \quad \|\Delta^\lambda(g, h)\|_{bh,tg} \leq c(\lambda) \frac{b(\lambda)}{1 - c(\lambda)}$$

(34b)
As to (34b), this is an immediate consequence of (34a) since when

\[ \| \lambda^{-1} \circ \lambda^{-1} - id \| = \| (id - (id - \lambda^{-1} \circ \lambda^{-1}^{-1})^{-1} - id \| \leq c(1 - c)^{-1}. \]

Using the inequalities (32) we obtain

\[ \| \lambda^{-1} - \lambda^{-1} \| = \| (\lambda^{-1} \circ \lambda^{-1}^{-1} - id) \circ \lambda^{-1} \| \leq c(1 - c)^{-1} \| \lambda^{-1} \| \]

whence, finally,

\[ \| \lambda^{-1} \| \leq \| \lambda^{-1} \| + \| \lambda^{-1} - \lambda^{-1} \| \leq \left( 1 + \frac{c}{1 - c} \right) \| \lambda^{-1} \| = \frac{\| \lambda^{-1} \|}{1 - c}. \]

4.7. For any unital, continuous, pseudo-action \( \lambda \in \text{Psa}^0_0(\Gamma; E) \) which satisfies the condition \( c(\lambda) < 1 \) the corresponding average pseudo-action \( \hat{\lambda} \in \text{Psa}^0_0(\Gamma; E) \) [defined by (30)] respects the following numerical bounds.

\[ [\forall g \in \Gamma] \quad \| \hat{\lambda}(g) \|_{\text{sg,sg}} \leq \frac{b(\lambda)}{1 - c(\lambda)} \quad (35a) \]

\[ [\forall (g', g) \in \Gamma \times \Gamma, g] \quad \| \hat{\lambda}(g'g) - \hat{\lambda}(g') \circ \hat{\lambda}(g) \|_{\text{sg,sg}} \leq 2c(\lambda)^2 \frac{b(\lambda)^2}{[1 - c(\lambda)]^2} \quad (35b) \]

[Proof. Both estimates are an immediate consequence of the preceding inequalities (34) and of the identities (31). Indeed, the Haar system \( \nu \) involved in (31) is normalized, so one can estimate each one of the integrals that appear in (31) simply by the sup norm of its integrand.]

**Lemma 4.8.** Let \( \{b_0, b_1, \ldots, b_l\} \) and \( \{c_0, c_1, \ldots, c_l\} \) be finite sequences of non-negative real numbers, say of length \( l + 1 \geq 2 \). Suppose that for every index \( i = 0, \ldots, l - 1 \) the following implication is true.

\[ c_i < 1 \Rightarrow \begin{cases} b_{i+1} \leq \frac{b_i}{1 - c_i} \quad \text{and} \\ c_{i+1} \leq 2c_i^2 \left[ \frac{b_i}{1 - c_i} \right]^2 \end{cases} \quad (36) \]

Also suppose \( b_0 \geq 1 \) and \( \varepsilon = 6b_0^2c_0 \leq \frac{2}{3} \). Then the inequalities below must hold for every index \( i = 0, 1, \ldots, l \).

\[ c_i \leq \frac{\varepsilon^2}{6b_0^2} \quad (37a) \]

\[ \frac{b_i}{1 - c_i} \leq \sqrt{3}b_0 \quad (37b) \]
This inequality is obviously also true when $t \leq n - 1$ we must have that $c_i < 1$ (because by hypothesis $b_0 \geq 1$ and $\varepsilon < 1$) and therefore that $b_{i+1} \leq b_i/(1 - c_i)$ [by (38)]. Combining recursively all these inequalities as $i$ runs from zero up to $n - 1$ we deduce that $b_n \leq b_0/(1 - c_0) \cdots (1 - c_{n-1})$. Thus:

$$b_n/(1 - c_n) \leq b_0/(1 - c_0) \cdots (1 - c_n). \tag{38}$$

This inequality is obviously also true when $n = 0$. We proceed to investigate the quantity

$$1/\prod_{i=0}^n(1 - c_i) = [\exp \log \prod_{i=0}^n(1 - c_i)]^{-1} = \exp(-\sum_{i=0}^n \log(1 - c_i)).$$

For every real number $x$ such that $|x| < 1$ we have: $-|x| + |\log(1 + x)| \leq |x - \log(1 + x)| = \left|\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots\right| \leq \frac{|x|^2}{2} + \frac{|x|^3}{2} + \frac{|x|^4}{2} + \cdots = \frac{|x|^2}{1 - |x|}$. This last quantity is $\leq |x|^2$ whenever $|x| \leq 1/2$. Hence setting $x = -t$ we see that

$$-\log(1 - t) \leq t + t^2 \quad \forall t \in [0, \frac{1}{2}]. \tag{39}$$

Since $2^i \geq 2i$ for every integer $i \geq 0$ and since by hypothesis $b_0 \geq 1$ and $\varepsilon \leq 2/3 < 1$, it will be true for all $0 \leq i \leq n$ that $c_i \leq \varepsilon^{2^i}/(6b_0^2) \leq \varepsilon^{2i}/6$. In particular, $c_i < 1/2$ for all $0 \leq i \leq n$, whence by (39)

$$\exp\left(\sum_{i=0}^n -\log(1 - c_i)\right) \leq \exp\left(\sum_{i=0}^n c_i + c_i^2\right) \leq \exp\left(\frac{1}{6} \sum_{i=0}^n \varepsilon^{2i}\right) \exp\left(\frac{1}{6^2} \sum_{i=0}^n \varepsilon^{4i}\right) \leq \exp\left(\frac{1}{6} \varepsilon^2\right) \exp\left(\frac{1}{6^2} 1 - \varepsilon^4\right) \leq \exp\left(\frac{1}{6} \varepsilon^2\right) \exp\left(\frac{1}{6^2} 1 - \varepsilon^4\right) \leq \exp(1/2) \leq \sqrt{3}.$$  

Combining the above with (38) we obtain the claimed inequality $b_n/(1 - c_n) \leq \sqrt{3}b_0$.

To finish the proof of the lemma, we proceed to argue that the inequality (37a) holds for every index $i$ between zero and $n$, by induction on $n$. By hypothesis, that is certainly true when $n = 0$. Assume that the inductive hypothesis holds for a certain value of $0 \leq n \leq l - 1$. Then by the above we have $b_n/(1 - c_n) \leq \sqrt{3}b_0$, whence by (36):

$$c_{n+1} \leq 2c_n \left[\frac{b_n}{1 - c_n}\right]^2 \leq 2 \left(\frac{\varepsilon^{2n}}{6b_0^2}\right)^2 \left[\frac{3b_0^2}{6b_0^2}\right] = \frac{\varepsilon^{2n+1}}{6b_0^2}.$$  

\[\square\]

4.9 (Bibliographic notes). Fast convergence of recursive approximation processes is a familiar phenomenon in various contexts of mathematics. It generally applies to the construction of exact solutions of problems which a priori are only known to admit
approximate solutions (Newton’s method for finding zeros of mappings [8, p. 139] is a basic example). In the theory of topological groups, it has been used to show the existence of a homomorphism near any given “almost homomorphism” between two compact Lie groups [6], or the existence, for any compact group, of a representation by bounded Hilbert space operators near any given “approximate representation” by such operators [5]. In the more general context of groupoids, the use of a recursive averaging process was originally proposed by Weinstein [17] as a possible technique for proving a conjecture of his about the local linearizability of proper Lie groupoids around fixed points. Weinstein’s suggestion was eventually put into practice by Zung [19].

5. Fast Convergence Theorem I (pseudo-actions)

Let \( \lambda \in \text{Psa}^0(\Gamma; E) \) be an arbitrary continuous pseudo-action of a proper Lie groupoid \( \Gamma \Rightarrow M \) on (a real or complex) differentiable vector bundle \( E \) over the base \( M \) of \( \Gamma \). For any (non-empty) invariant open subset \( U = \Gamma U \subseteq M \) and for any (Riemannian or Hermitian) metric \( \phi \) of class \( C^\infty \) on \( E|U \) we may apply the definitions (33) to the pseudo-action \( \lambda|U \in \text{Psa}^0(\Gamma|U; E|U) \) which \( \lambda \) induces (upon restriction) on the open subgroupoid \( \Gamma|U := \Gamma^U = \Gamma U \Rightarrow U \), thus obtaining a pair of (possibly infinite) quantities to which we shall refer by means of the notations: \( b_{U,\phi}(\lambda), c_{U,\phi}(\lambda) \).

**Definition 5.1.** We shall say that a unital, continuous, pseudo-action \( \lambda \in \text{Psa}^0(\Gamma; E) \) is nearly multiplicative or a near action if each base point of \( \Gamma \) possesses an invariant open neighborhood \( U = \Gamma U \) with the property that the inequality below holds for some choice of a \( C^\infty \) metric \( \phi \) on \( E|U \).

\[
c_{U,\phi}(\lambda) \leq \frac{1}{2} b_{U,\phi}(\lambda)^{-2}
\]  

(For \( b_{U,\phi}(\lambda) = \infty \) this condition simply reads: \( c_{U,\phi}(\lambda) = 0 \).) We shall say that a unital connection \( \Phi \) on \( \Gamma \) is nearly multiplicative (although it would be more appropriate to call it nearly effective) if the associated pseudo-action \( \lambda^\Phi \) of \( \Gamma \) on the tangent bundle of \( M \) is a near action (in the above sense).

Our first comment is that every nearly multiplicative pseudo-action \( \lambda \in \text{Psa}^0(\Gamma; E) \) is invertible. This is an immediate consequence of the remark [4,5] indeed, locally around each point of \( M \) we must have \( c_{U,\phi}(\lambda) \leq \frac{1}{2} < 1 \) [by the condition (40) since \( b_{U,\phi}(\lambda) \geq 1 \) in virtue of the unitality of \( \lambda \)].

Because of invertibility, it certainly makes sense for any nearly multiplicative pseudo-action \( \lambda \in \text{Psa}^0(\Gamma; E) \) to consider the pseudo-action \( \hat{\lambda} \in \text{Psa}^0(\Gamma; E) \) which one obtains by averaging \( \lambda \) with respect to any given normalized left Haar system on \( \Gamma \) by means of the formula (30). We contend that any pseudo-action obtained in this way must itself be nearly multiplicative. Indeed let \( U \) and \( \phi \) be as in the above definition and in particular satisfy (40). Let us abbreviate \( b_{U,\phi}(\lambda), c_{U,\phi}(\lambda), b_{U,\phi}(\hat{\lambda}) \) and \( c_{U,\phi}(\hat{\lambda}) \) respectively into \( b, c, \hat{b} \) and \( \hat{c} \). By the unitality of \( \lambda \) we have \( b \geq 1 \) and hence \( c \leq \frac{1}{2} < 1 \). By the estimate (35a) we have \( \hat{b} \leq b/(1 - c) \leq \frac{2}{3} b \) and consequently \( \hat{b}^{-2} \geq (\frac{3}{2})^2 b^{-2} \geq \frac{1}{2} b^{-2} \). On the other hand, by the estimate (35b) and by (40) we have \( \hat{c} \leq 2c^2 b^2/(1 - c)^2 \leq 2(1/9 b^2)^2 b^2/(1 - \frac{2}{3})^2 = 2(\frac{3}{2})^2 (\frac{3}{2})^2 b^{-2} \leq \frac{1}{2} b^{-2} \). Thus, \( \hat{c} \leq \frac{1}{2} \hat{b}^{-2} \).
From the above we see that each nearly multiplicative pseudo-action \( \lambda \in \text{Psa}_0^p(\Gamma; E) \) gives rise to a sequence \( \{\hat{\lambda}^{(i)}\}_{i=0}^\infty \) of averaging iterates \( \hat{\lambda}^{(i)} \in \text{Psa}_0^p(\Gamma; E) \) which one constructs recursively by setting \( \hat{\lambda}^{(0)} := \lambda \) and \( \hat{\lambda}^{(i+1)} := (\hat{\lambda}^{(i)})^\wedge \) for every \( i \in \mathbb{N} \). (Such a construction depends of course on the preliminary choice of a normalized left Haar system on \( \Gamma \).) Since from the formula (17) we know that \( \lambda^\wedge \) equals \( (\lambda^\wedge)^\wedge \) for an arbitrary non-degenerate connection \( \Phi \) of class \( C^0 \) on \( \Gamma \), we also see that each nearly multiplicative connection \( \Phi \in \text{Conn}_{\lambda}^0(\Gamma) \) gives rise to a similar sequence \( \hat{\Phi}^{(i)} \in \text{Conn}_{\lambda}^0(\Gamma) \) \((i = 0, 1, 2, \ldots)\).

**Theorem 5.2.** Let \( \Gamma \rightrightarrows M \) be a proper Lie groupoid whose target mapping is proper. Let \( \lambda \in \text{Psa}_0^p(\Gamma; E) \) be a unital pseudo-action of class \( C^p \) \((p = 0, 1, 2, \ldots, \infty)\) of the groupoid on some \( \mathbb{K}\)-linear differentiable vector bundle \( E \) over the groupoid base \( M \). Assume that \( \lambda \) is nearly multiplicative. Then for any choice of a left invariant Haar probability system on \( \Gamma \) (which one can view as a left Haar system normalized by the constant function of value one) the sequence of successive averaging iterates of \( \lambda \) obtained by recursive application of the formula (30)

\[
\hat{\lambda}^{(0)} := \lambda, \quad \hat{\lambda}^{(1)} := \hat{\lambda}, \quad \ldots, \quad \hat{\lambda}^{(i+1)} := (\hat{\lambda}^{(i)})^\wedge, \quad \ldots \in \text{Psa}_0^p(\Gamma; E)
\]

is convergent in the Fréchet space \( \text{Psa}_0^p(\Gamma; E) \) (that is, in the \( C^p \)-topology) to a unique multiplicative pseudo-action (that is, action) \( \hat{\lambda}^{(\infty)} \in \text{Psa}_0^p(\Gamma; E) \) of class \( C^p \).

The proof of this theorem will occupy the rest of the section. It will be substantially all devoted to showing that each point of the base \( M \) of \( \Gamma \) possesses an invariant open neighborhood \( U = \Gamma U \) with the following couple of properties.

(A) The sequence of induced pseudo-actions \( \{\hat{\lambda}^{(i)}|U\}_{i=0}^\infty \) is Cauchy within the Fréchet space \( \text{Psa}_0^p(\Gamma|U; E|U) \).

(B) There exist metrics \( \phi \) on \( E|U \) for which one has \( \lim_{i \to \infty} c_{U,\phi}(\hat{\lambda}^{(i)}) = 0 \).

Once the claim in question is proven, our theorem will be proven as well. Indeed, suppose \( M \) can be covered with open invariant sets \( U \) which satisfy the properties (A) and (B). The corresponding sets \( \Gamma|U \) will form an open cover of \( \Gamma \). Take all those standard trivializing charts \((\varphi, \tau)\) for the vector bundle \( L(s^*E, t^*E) \) over \( \Gamma \) whose domains are contained in some of the open sets of this cover. The associated seminorms \( \rho_{r, s, \varphi, \tau, \varrho} \) [where \( \varrho = \varphi^{-1}(\overline{B}_1(0)) \) and \( 0 \leq r \leq p \)] will still generate the \( C^p \)-topology on the vector space \( \Gamma^p(\Gamma, L(s^*E, t^*E)) = \text{Psa}_0^p(\Gamma; E) \). Since by (A) the sequence \( \{\hat{\lambda}^{(i)}\} \) is Cauchy with respect to every such seminorm, the same sequence will be Cauchy and hence convergent in the Fréchet space \( \text{Psa}_0^p(\Gamma; E) \) to a unique pseudo-action \( \hat{\lambda}^{(\infty)} \in \text{Psa}_0^p(\Gamma; E) \) of class \( C^p \). Next, since \( C^p \)-convergence implies pointwise convergence, for each arrow \( g \in \Gamma \) the sequence of linear maps \( \{\hat{\lambda}^{(i)}(g)\} \) will be convergent in the finite-dimensional vector space \( L(E_{vg}, E_{tg}) \) to \( \hat{\lambda}^{(\infty)}(g) \) (with respect to whatever norm). It follows that \( \hat{\lambda}^{(\infty)}_{1x} = \lim_{i \to \infty} \hat{\lambda}^{(i)}_{1x} = id_{E_x} \) for all \( x \in M \), since every pseudo-action \( \hat{\lambda}^{(0)} \) was unital. Moreover

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1In fact, one could compute pseudo-action averages relative to an arbitrary normalized left Haar system, without affecting the conclusions of the theorem; compare the comments near the end of Appendix B.
for any given composable pair of arrows \( g', g \in \Gamma \) we must be able to find some open set \( U \) as in (A) such that \( g', g \in \Gamma \cap U \) and some metric \( \phi \) such that (B) holds, in which case we have the following inequality (the norms appearing below depend on \( \phi \) as described in [4,2], but here for conciseness’ sake they wear no subscripts)

\[
||\lambda_{g'}^{(\infty)} - \lambda_{g}^{(\infty)}\lambda_{g}^{(\infty)}|| \leq ||\lambda_{g'}^{(\infty)} - \lambda_{g'}^{(0)}|| + ||\lambda_{g}^{(0)} - \lambda_{g}^{(0)}\lambda_{g}^{(0)}|| + ||\lambda_{g}^{(0)}\lambda_{g}^{(0)} - \lambda_{g}^{(0)}\lambda_{g}^{(0)}||(\lambda_{g}^{(0)} - \lambda_{g}^{(0)}\lambda_{g}^{(0)}),
\]

whose right-hand side becomes arbitrarily small when \( i \) grows sufficiently large. It is thus established that \( \lambda^{(\infty)} \) is an action.

**Proof. Step I.**

Consider an arbitrary invariant open subset \( U' = \Gamma U' \) of the base \( M \) of \( \Gamma \) with the property that the vector bundle \( E \mid U' \) admits a metric \( \phi' \) such that the basic inequality \( c_{U', \phi'}(\lambda) \leq \frac{1}{2} b_{U', \phi'}(\lambda)^2 \) is satisfied. By definition of what it means for a pseudo-action \( \lambda \) to be nearly multiplicative, such open sets must cover all of \( M \). Also consider any relatively compact, \( \Gamma \)-invariant, (non-empty) open set \( U \subseteq M \) such that \( \overline{U} \subset U' \). Because of our assumption of source properness on \( \Gamma \Rightarrow M \), such subsets of \( U' \) must cover all of \( U' \). We let \( \phi = \phi' \mid U \) denote the restriction of the metric \( \phi' \) to the vector bundle \( E \mid U \subseteq E \mid U' \).

**Claim.** We contend that any \( U, \phi \) arising in the way just described constitute a pair of data satisfying the aforesaid conditions (A) and (B).

For reasons of length, the proof of this assertion will be broken into several steps. We shall regard the data \( U', \phi', U \) and \( \phi \) as fixed once and for all until the end of the section. Before going any further, we shall put in place some auxiliary notations.

To begin with, let us set \( \Omega' = \Gamma \mid U' \) and \( \Omega = \Gamma \mid U \). Notice that \( \Omega \) is a relatively compact open subset of \( \Gamma \) whose closure \( \overline{\Omega} \) lies within \( \Omega' \). [Proof. By invariance of \( U \) we must have \( \Omega = s^{-1}(U) \). By openness of \( s \), we must have \( s^{-1}(\overline{U}) = \overline{s^{-1}(U)} \). Hence \( \overline{\Omega} = s^{-1}(\overline{U}) \) must be a compact set, e.g., by properness of \( s \) contained within \( s^{-1}(U') = \Omega' \) \]. It follows that \( \overline{\Omega} \subseteq \Omega \subseteq \Omega' \). [\( \overline{\Omega} \subseteq \Omega \subseteq \Omega' \) \]. The non-trivial inclusion \( \supset \) can be seen as follows. Since the projection on the 1st factor, \( pr_1 : \Gamma \times_{\Gamma} \Gamma \rightarrow \Gamma \), is submersive and therefore open, one has \( \overline{\Omega} \subseteq \Omega \subseteq \overline{s^{-1}(\overline{U})} = pr_1^{-1}(\overline{U}) = pr_1^{-1}(\overline{U}) = \overline{\Omega} \subseteq \overline{\Omega} \subseteq \overline{\Omega} \).

Next, let us fix any three standard normed atlases \( A_1 \) for \( L(s^*E, t^*E) \) over \( \overline{\Omega}, \overline{\Omega} \) for \( L(t^*E, s^*E) \) over \( \overline{\Omega} \) and \( \overline{\Omega} \) for \( L(s^*E, t^*E) \) over \( \overline{\Omega} \). (Compare Appendix [A] here as usual \( s_2 \) and \( t_2 \) denote the two mappings of \( \Gamma_2 := \Gamma \times_{\Gamma} \Gamma \) into \( M \) given respectively by \( (g', g) \mapsto sg \) and \( \mapsto t^g' \)). We may choose \( A_1 \) (resp. \( A_2, A_2 \)) so that the domains of the standard normed charts that compose it be contained within \( \Omega' \) (resp. \( \Omega' \), \( \Omega' \), \( \Omega' \)). We may further assume that for every standard normed chart in \( A_1 \) the corresponding continuous vector-bundle norm is given by the restriction over the chart domain of the continuous vector-bundle norm on \( L(s^*E, t^*E) \mid \Omega' \equiv L(s_{\overline{\Omega}}^*E(E \mid U'), t_{\overline{\Omega}}^*E(E \mid U')) \) that is associated in the ordinary way (i.e. as described in the ‘Examples’ paragraph following the proof of Lemma [A,2] in Appendix [A]) to the two pullback metrics: \( s_{\overline{\Omega}}^* \phi' \) or \( s_{\overline{\Omega}}^* \phi' \).
\( t^* E \phi' \) on \( \Omega^* (E \mid U^*) \). Similarly for \( \tilde{A}_1 \) and \( A_2 \). Having made these choices, for all natural numbers \( r \) we set:

\[
\begin{align*}
p_r(-) &= \| \|_{C^r(\Omega^* (E \mid r^* E), A_1)} \\
\tilde{p}_r(-) &= \| \|_{C^r(\Omega^* (r^* E), \tilde{A}_1)} \\
q_r(-) &= \| \|_{C^r(\Omega^* (r^* E), A_2)}
\end{align*}
\]

and then, for any \( C^p \) pseudo-actions \( \zeta \in \Gamma^p(\Gamma; L(s^* E, t^* E)), \eta \in \Gamma^p(\Gamma; L(t^* E, s^* E)) \) and for any natural number \( r \leq p \):

\[
\begin{align*}
b^{(r)}(\zeta) &= p_r(t^* E \zeta) \\
\tilde{b}^{(r)}(\eta) &= \tilde{p}_r(t^* E \eta) \\
c^{(r)}(\zeta) &= q_r(t^* E (m^* \zeta - pr^*_1 \zeta \circ pr^*_2 \zeta))
\end{align*}
\]

where \( m, pr_1 \) and \( pr_2 \) respectively denote the arrow composition law, the first and the second projection \( : \Gamma_2 \to \Gamma \), and where the expression \([\ldots] \) (by a mild abuse of notation) is supposed to indicate the difference between the two composite vector-bundle morphisms below.

\[
\begin{align*}
s^*_2 E &\equiv pr^*_2 s^* E \equiv m^* s^* E \xrightarrow{m^* \zeta} m^* t^* E \equiv pr^*_1 t^* E \equiv t^*_2 E \\
s^*_1 E &\equiv pr^*_2 s^* E \xrightarrow{pr^*_2 \zeta} pr^*_1 t^* E \equiv pr^*_1 s^* E \xrightarrow{pr^*_1 \zeta} pr^*_1 t^* E \equiv t^*_2 E
\end{align*}
\]

Notice that our selection criteria for the standard normed atlases \( A_1 \) and \( A_2 \) ensure that \( b^{(0)}(\zeta) \) and \( c^{(0)}(\zeta) \) coincide with the two quantities \( b_{U, \phi}(\zeta), c_{U, \phi}(\zeta) \) defined earlier.

**Proof. Step II.**

Throughout the present stage of the proof, \( r \) will denote a fixed natural number. Let \( \zeta \) be a variable ranging over \( \Gamma^{r+1}(\Gamma; \text{Lis}(s^* E, t^* E)) \) (= invertible \( C^{r+1} \) pseudo-actions of \( \Gamma \) on \( E \)). By (318) we have

\[
p_{r+1}(\rho^L_{\Gamma^2}) \leq p_{r+1}(\rho^L_{\Gamma^2}) + q_{r+1}(\rho^L_{\Gamma^2}([a^* \Delta^2], d_1 \nu))
\]

where: i) \( \Delta^2 \) is given by (29); ii) \( a \) denotes the diffeomorphism \( \Gamma_2 \xrightarrow{\sim} \Gamma_2 \) given by \((g, k) \mapsto (gk, k)\); iii) \([a^* \Delta^2]\) indicates the \( C^{r+1} \) cross-section of the vector bundle \( pr^*_1 L(s^* E, t^* E) \) (over \( \Gamma_2 \)) that corresponds to the pullback cross-section \( a^* \Delta^2 \in \Gamma^{r+1}(\Gamma_2; a^* L(s^*_2 E, t^*_2 E)) \) under the vector-bundle isomorphism \( a^* L(s^*_2 E, t^*_2 E) \equiv L(a^* s^*_2 E, a^* t^*_2 E) \equiv L(pr^*_1 s^* E, pr^*_1 t^* E) \equiv pr^*_1 L(s^* E, t^* E) \); iv) \( \nu \) is the left invariant Haar probability system on \( \Gamma \) referred to in the statement of the theorem; v) \( d_1 \nu \) denotes the integration functional corresponding to \( \nu \),

\[
d^{r+1}_{s, \Gamma^2} \nu : \Gamma^{r+1}(\Gamma_2; pr^*_1 L(s^* E, t^* E)) \longrightarrow \Gamma^{r+1}(\Gamma; L(s^* E, t^* E))
\]

whose general definition can be found in the statement of Lemma [3.7]. Observe that

\[
\rho^L_{\Omega^* (s^* E, t^* E) \nu} = d^{r+1}_{s, \Gamma^2} \nu \circ \rho_{\Omega^* (s^* E, t^* E)}^{r+1}
\]

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where $d^{r+1}\tau|_{pr_1^*L(s^*E, t^*E)}$ is an integration functional of the type considered in Lemma [B.8].

The last equation is a by-product of the general remark that the functionals (71) and (72) are compatible with each other i.e. that the latter is induced by the former. Similar compatibility equations, which will be used repeatedly below, hold between the two linear maps appearing in Lemma [A.6] and between the two appearing in Lemma [A.7].

In the notations introduced before Lemma A.8, thanks to (43), we get the following equivalent result from Lemma B.8:

\[ \|pr_2\mathcal{L}(d^*\Delta^t\zeta, d_1\nu)\|_{C^{r+1}L(L(s^*E, t^*E))} \leq \|pr_2\mathcal{L}(d^*\Delta^t\zeta)\|_{C^{r+1}L(pr_1^*L(s^*E, t^*E))}. \]

(44)

Since the cross-section $[d^*\Delta^t\zeta]$ of the vector bundle $pr_1^*L(s^*E, t^*E)$ corresponds under the canonical vector-bundle isomorphism $pr_1^*L(s^*E, t^*E) \cong L(pr_1^*t^*E)$ to $L(pr_1^*t^*E, t^*E)$, the cross-section of $L(pr_1^*t^*E)$ given by the composition $[\rho \Delta^t\zeta]$ that corresponds to $[\rho \Delta^t\zeta]$ that corresponds to $[\rho \Delta^t\zeta]$, by the lemmas A.6, A.7 we have that the right-hand member of (44) must be equal to:

\[ \|pr_2\mathcal{L}(m^*\zeta - pr_1^*\zeta \circ pr_2^*\zeta)\|_{C^{r+1}L(pr_1^*L(s^*E, t^*E))} \]

by (64b):

\[ \leq \|pr_2\mathcal{L}(m^*\zeta - pr_1^*\zeta \circ pr_2^*\zeta)\|_{C^{r+1}L(L(s^*E, t^*E))} \]

\[ \times \|pr_2\mathcal{L}(pr_2^*\zeta)\|_{C^{r+1}L(L(s^*E, t^*E))} \]

\[ \times \|pr_2\mathcal{L}(pr_2^*\zeta)\|_{C^{r+1}L(L(s^*E, t^*E))} = \|\|pr_2\mathcal{L}(pr_2^*\zeta)\|_{C^{r+1}L(L(s^*E, t^*E))} \]

by (68):

\[ \leq \|\|pr_2\mathcal{L}(pr_2^*\zeta)\|_{C^{r+1}L(L(s^*E, t^*E))} \]

by (68):

\[ \leq \|\|pr_2\mathcal{L}(pr_2^*\zeta)\|_{C^{r+1}L(L(s^*E, t^*E))} \]

Conclusion. There exists some constant $B_\tau > 0$ such that for all invertible $C^{r+1}$ pseudo-actions $\zeta \in \Gamma^{r+1}(\Gamma; \text{Lie}(s^*E, t^*E))$ the following inequality holds.

\[ b^{(r+1)}(\zeta) \leq b^{(r+1)}(\zeta) + B_\tau \cdot \|\|pr_2\mathcal{L}(pr_2^*\zeta)\|_{C^{r+1}L(L(s^*E, t^*E))} \]

(45)

We proceed to obtain a similar estimate for the quantity $c^{(r+1)}(\zeta)$. We remind the reader of our notation: $\Gamma_3 := \Gamma \times \Gamma \times \Gamma$ (see Section 2). Consider the following differentiable mappings.

\[ pr_{12} : \Gamma_3 \to \Gamma_2 \quad (g', g, k) \mapsto (g', g) \]

\[ pr_{23} : \Gamma_3 \to \Gamma_2 \quad (g', g, k) \mapsto (g, k) \]

\[ m_{23} : \Gamma_3 \to \Gamma_2 \quad (g', g, k) \mapsto (g', gk) \]

By the formula (31a), since in virtue of the left invariance of the Haar system $\nu$ the double integral term in that formula can be rewritten as

\[ \int_{s^*g} \left[ \int_{h^*s^*g} \Delta(g'gh, gh) d\nu_{s^*g}(h) \circ \Delta(gk, k) d\nu_{s^*g}(k) \right] \]
we have the inequality
\[
q_{r+1}(\rho_{\Omega}^{f_2}(m^*\xi - pr_1^*\xi \circ pr_2^*\xi)) \leq q_{r+1}(\rho_{\Omega}^{f_2}(m_{23}^*a^*\Delta^\xi \circ pr_{23}^*a^*\Delta^\xi), d_2\nu) + q_{r+1}(\rho_{\Omega}^{f_2}(pr_1^*|\langle a^*\Delta^\xi \rangle, d_1\nu|) \circ \rho_{\Omega}^{f_2}|pr_2^*|\langle a^*\Delta^\xi \rangle, d_1\nu|)
\]
(46)

where: i*) by abuse of notation, the term \([m_{23}^*a^*\Delta^\xi \circ pr_{23}^*a^*\Delta^\xi]\) denotes the cross-section of the vector bundle \(pr_{12}^*L(s^*_2E, t^*_2E)\) (over \(\Gamma_3\)) that corresponds to the composite vector-bundle morphism
\[
pr_{12}^*t^*_2E \cong m_{23}^*a^*t^*_1E \xleftarrow{\frac{m_{23}^*a^*\Delta^\xi}{m_{23}^*a^*s^*_1E}} m_{23}^*a^*s^*_1E \equiv pr_{23}^*a^*t^*_1E \xleftarrow{\frac{pr_{23}^*a^*\Delta^\xi}{pr_{23}^*a^*s^*_1E}} pr_{23}^*a^*s^*_1E \equiv pr_{12}^*s^*_2E;
\]

ii*) \(d_2\nu\) stands for the integration functional
\[
d_{s^*_2;L(s^*_2E, t^*_2E)}^{r+1} \nu : \Gamma^{r+1}(\Gamma_3; pr_{12}^*L(s^*_2E, t^*_2E)) \rightarrow \Gamma^{r+1}(\Gamma_2; L(s^*_2E, t^*_2E));
\]

iii*) the terms \([pr_1^*|\ldots\rangle\) and \([pr_2^*|\ldots\rangle\) respectively denote the two vector-bundle morphisms below.
\[
t^*_2E \equiv pr_{12}^*t^*_1E \xleftarrow{pr_{12}^*(\ldots)} pr_1^*s^*_1E \equiv (s \circ pr_1)^*E
\]
\[
(t \circ pr_2)^*E \equiv pr_{23}^*t^*_1E \xleftarrow{pr_{23}^*(\ldots)} pr_2^*s^*_1E \equiv s^*_2E
\]

By analogy with the derivation of the above estimate (44), we use the identity
\[
pr_{12}^*(\Omega_2) = pr_{12}^*(\Omega_2) = \Omega_2 \times t_1\Omega = \Omega_{2, s^*_2} \times \Omega \text{ in combination with the compatibility relation } d_{s^*_2;L(s^*_2E, t^*_2E)}^{r+1} \nu \circ \rho_{\Omega}^{f_2|\Omega_{2, s^*_2} \times \Omega} \circ d_{s^*_2;L(s^*_2E, t^*_2E)}^{r+1} \nu \text{ in order to deduce the following estimate, in which for convenience we set } \Omega_3 := \Omega_{2, s^*_2} \times \Omega = \Omega_{s^*_2} \times \Omega \text{, from the continuity of the linear map (72) (Lemma B.8).}
\]
\[
\|\rho_{\Omega}^{f_2}(m_{23}^*a^*\Delta^\xi \circ pr_{23}^*a^*\Delta^\xi), d_2\nu\|_{C^{r+1}} \leq \|\rho_{\Omega}^{f_2}(m_{23}^*a^*\Delta^\xi \circ pr_{23}^*a^*\Delta^\xi)\|_{C^{r+1}}.
\]

By the lemmas \([\Delta, \xi] \Delta, \xi\) and \([\Delta, \xi]\) taking into account the identity of sets \(m_{23}^{-1}(\Omega_2) = \Omega_{2, s^*_2} \times \Omega = pr_{23}^{-1}(\Omega_2)\), we see that the right-hand side of the preceding inequality must be \(\leq\):
\[
\|\rho_{\Omega}^{f_2}(m_{23}^*a^*\Delta^\xi)\|_{C^r} \|\rho_{\Omega}^{f_2}(pr_{23}^*a^*\Delta^\xi)\|_{C^{r+1}} + \|\rho_{\Omega}^{f_2}(m_{23}^*a^*\Delta^\xi)\|_{C^{r+1}} \|\rho_{\Omega}^{f_2}(pr_{23}^*a^*\Delta^\xi)\|_{C^r}
\leq\|\rho_{\Omega}^{f_2}(a^*\Delta^\xi)\|_{C^r} \|\rho_{\Omega}^{f_2}(a^*\Delta^\xi)\|_{C^{r+1}} + \|\rho_{\Omega}^{f_2}(a^*\Delta^\xi)\|_{C^{r+1}} \|\rho_{\Omega}^{f_2}(a^*\Delta^\xi)\|_{C^r}
= \|\rho_{\Omega}^{f_2}(a^*\Delta^\xi)\|_{C^r} \|\rho_{\Omega}^{f_2}(a^*\Delta^\xi)\|_{C^{r+1}}.
\]
The second factor can be estimated as done after (44), whilst for the first one we have
\[
\|\rho_{\Omega}^{f_2}(a^*\Delta^\xi)\|_{C^r} \leq \|\rho_{\Omega}^{f_2}(m^*\xi - pr_1^*\xi \circ pr_2^*\xi)\|_{C^r} \|\rho_{\Omega}^{f_2}(m^*\xi - pr_1^*\xi \circ pr_2^*\xi)\|_{C^{r-1}}.
\]
(47)
To begin with, we know that for every unital pseudo-action $S$, we have $\||a||_{C^r} \leq ||b||_{C^{r+1}}$. Estimation of the second summand in the right-hand side of the inequality goes even quicker.

\[
\|r_{r,k}^2|pr_r^x(\ldots)\circ r_{r,k}^2|pr_r^y(\ldots)\|_{C^{r+1}} \\
\leq \|r_{r,k}^2|pr_r^x(\ldots)\|_{C^r}\|r_{r,k}^2|pr_r^y(\ldots)\|_{C^{r+1}} + \|r_{r,k}^2|pr_r^z(\ldots)\|_{C^{r+1}}\|r_{r,k}^2|pr_r^y(\ldots)\|_{C^r} \\
\leq \|r_{r,k}^2(\ldots)\|_{C^r}\|r_{r,k}^2(\ldots)\|_{C^{r+1}} + \|r_{r,k}^2(\ldots)\|_{C^{r+1}}\|r_{r,k}^2(\ldots)\|_{C^r} \\
= \|r_{r,k}^2(\ldots)\|_{C^r}\|r_{r,k}^2(\ldots)\|_{C^{r+1}}
\]

The second factor $\|r_{r,k}^2(\ldots)\|_{C^{r+1}}$ can be dealt with as before, whereas the first factor $\|r_{r,k}^2(\ldots)\|_{C^r}$ must be $\leq$ than the right-hand side of the inequality.

**Conclusion.** There must be some constant $C_r > 0$ such that the following inequality holds for all invertible $C^{r+1}$ pseudo-actions $\zeta \in \Gamma^{r+1}(\Gamma; \text{Lis}(s^*E, t^*E))$.

\[
c^{(r+1)}(\tilde{\zeta}) \leq C_r \cdot \{((\tilde{b}^{(r)}(\zeta^{-1}))^3b^{(r+1)}(\zeta)c^{(r)}(\zeta)) + (\tilde{b}^{(r)}(\zeta^{-1}))c^{(r)}(\zeta)\}c^{(r)}(\zeta)
\]

**Proof, Step III.**

Let us now go back to our nearly multiplicative $C^p$ pseudo-action $\lambda \in \Gamma^p(\Gamma; \text{Lis}(s^*E, t^*E))$. For all $i$, the $i$th averaging iterate $\tilde{\lambda}^{(i)}$ is itself a nearly multiplicative (hence invertible) pseudo-action of class $C^p$. For each order of differentiability $r \leq p$ we introduce the following quantities.

\[
\tilde{b}_{(i)}^{(r)} = \tilde{b}^{(r)}(\tilde{\lambda}^{(i)}) \quad (49a) \\
\tilde{\tilde{b}}_{(i)}^{(r)} = \tilde{b}^{(r)}((\tilde{\lambda}^{(i)})^{-1}) \quad (49b) \\
c_{(i)}^{(r)} = c^{(r)}(\tilde{\lambda}^{(i)}) \quad (49c)
\]

We set $\varepsilon = 6b_{U,\phi}(\lambda)^2c_{U,\phi}(\lambda) \leq \frac{\varepsilon}{2} < 1$. We want to show that the four statements below hold for every natural number $r \leq p$; in order to accomplish this, we will be arguing by induction over all such $r$.

**S1(r).** The sequence $\{b_{(i)}^{(r)}\}_{i=0}^\infty$ is bounded.

**S2(r).** The sequence $\{\tilde{b}_{(i)}^{(r)}\}_{i=0}^\infty$ is bounded.

**S3(r).** There exists a constant $R_r > 0$ such that $c_{(i+1)}^{(r)} \leq R_r \cdot (c_{(i)}^{(r)})^2$ for all $i \in \mathbb{N}$.

**S4(r).** There exists an index $i_r \in \mathbb{N}$ such that $c_{(i)}^{(r)} \leq e^{2(i-i_r)}$ for all $i \geq i_r$.

We observe right away that for $r = 0$ all of these conditions are certainly satisfied. To begin with, we know that for every unital pseudo-action $\tilde{\zeta} \in \text{Ps}_{a}(\Gamma; E)$ which obeys the basic inequality $c_{U,\phi}(\tilde{\zeta}) \leq \frac{1}{2} b_{U,\phi}(\tilde{\zeta})^{-2}$ one must have $b^{(0)}(\tilde{\zeta}) = b_{U,\phi}(\tilde{\zeta}) \geq 1$ and hence $c^{(0)}(\tilde{\zeta}) = c_{U,\phi}(\tilde{\zeta}) < 1$; then, by the estimate (34a), it must be true that

\[
\tilde{b}^{(0)}(\zeta^{-1}) \leq b^{(0)}(\zeta)/(1 - c^{(0)}(\zeta)).
\]
Secondly, we know that the iterates $\hat{\lambda}(i)$ are unital and satisfy the aforesaid basic inequality. Thirdly, by virtue of the estimates (35) the two sequences of non-negative real numbers $\{b_i(0), c_i(0)\}$ satisfy the hypotheses of Lemma 4.3. Hence the following statements must be true.

\textbf{S1(0).} $b_i(0) \leq b_i(0)/(1-c_i(0)) \leq \sqrt{3}b_0(0)$ [by (37b) since $0 < 1 - c_i(0) \leq 1$].

\textbf{S2(0).} $b_i(0) \leq b_i(0)/(1-c_i(0)) \leq \sqrt{3}b_0(0)$ [by (37b) and (50)].

\textbf{S3(0).} $c_i(0) \leq 2(c_i(0))^2 [b_i(0)/(1-c_i(0))]^2 \leq 6(b_0(0))^2 (c_i(0))^2$ [by (35b) and (37b)].

\textbf{S4(0).} $c_i(0) \leq \epsilon^2 /[6(b_0(0))^2] < \epsilon^2$ [by (37a) since $b_0(0) \geq 1$].

Let us pass to the inductive step. Suppose the statements S1(r)–S4(r) are valid for a certain order of differentiability $r \leq p-1$. By the equations (45) and (48) we know that there must be some positive constants $B_r, C_r$ such that the two inequalities below hold for all $i \in \mathbb{N}$,

\begin{align*}
    b_i^{(r+1)} &\leq b_i^{(r+1)} + B_r \cdot (\tilde{b}_i^{(r)})^2 b_i^{(r)} c_i^{(r)} + \tilde{b}_i^{(r)} c_i^{(r+1)} \\
    c_i^{(r+1)} &\leq C_r \cdot (\tilde{b}_i^{(r)})^2 b_i^{(r)} c_i^{(r)} + (\tilde{b}_i^{(r)})^2 c_i^{(r+1)}
\end{align*}

The inductive assumption S2(r) then entails the existence of some positive constant $L_r$ such that the following two inequalities are satisfied for all $i \in \mathbb{N}$,

\begin{align*}
    b_i^{(r+1)} &\leq b_i^{(r+1)} + L_r \cdot [b_i^{(r+1)} c_i^{(r)} + c_i^{(r+1)}] \quad \text{(51a)} \\
    c_i^{(r+1)} &\leq L_r \cdot [b_i^{(r+1)} c_i^{(r)} + c_i^{(r+1)}] \quad \text{(51b)}
\end{align*}

We need to “solve” this recursive system in order to complete the inductive step.

\textbf{Proof. Step IV.}

\textbf{Lemma 5.3.} Let $\{c_0, c_1, c_2, \ldots\}, \{b_0, b_1, b_2, \ldots\}$ and $\{c_0', c_1', c_2', \ldots\}$ be sequences of non-negative real numbers. Let $L$, $R$ and $\epsilon < 1$ be positive real numbers. Suppose for all $i \in \mathbb{N}$ we have

\begin{align*}
    b_{i+1}' &\leq b_i' + L \cdot (b_i' c_i + c_i') \\
    c_{i+1}' &\leq L \cdot (b_i' c_i + c_i') c_i \\
    c_{i+1}' &\leq R \cdot c_i'^2 
\end{align*}

and $c_i \leq c_i'$. Furthermore suppose there is some index $I$ such that $c_i \leq \epsilon^{2^{i-I}}$ for all $i \geq I$.

Then the following statements must be true. (a) The sequence $\{b_i'\}$ is bounded. (b) There exists some constant $R' > 0$ such that $c_i' \leq R' \cdot (c_i')^2$ for all $i$. (c) There exists some index $I' \geq I$ such that $c_i' \leq \epsilon^{2^{i-I'}}$ for all $i' \geq I'$.

\textbf{Proof.} At the expense of re-indexing our sequences it will be no loss of generality to assume that $I = 0$. Under this assumption, for every $i \in \mathbb{N}$ we will have that $c_i \leq \epsilon^2 \leq \epsilon < 1$ and therefore that $c_i^2 \leq c_i$. Let us put $d_i' = b_i' c_i + c_i'$. Then

\begin{align*}
    d_{i+1}' = b_{i+1}' c_{i+1} + c_{i+1}'
\end{align*}

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\[ \leq Rb_i^2 c_i^2 + RLa_i^2 c_i^2 + c_{i+1} \quad \text{[by (52a) and (52c)]} \]
\[ \leq Rb_i^2 c_i^2 + RLa_i^2 c_i + c_{i+1} \quad \text{[because } c_i^2 \leq c_i \text{]} \]
\[ \leq Rb_i^2 c_i^2 + (R + 1)La_i^2 c_i \quad \text{[by (52b)]} \]
\[ \leq (R + 1)La_i^2 c_i + Rb_i^2 c_i + Rc_i c_i \quad \text{[a fortiori]} \]
\[ = (R + 1)La_i^2 c_i + R \cdot (b_i c_i + c_i^2) c_i \]
\[ = (RL + L + R)a_i^i c_i; \]

hence the positive real constant \( K = RL + L + R \) will be such that for every \( i \in \mathbb{N} \)

\[ a_{i+1}^1 \leq Ka_i^i c_i. \]

It follows that \( a_{i}^i \leq Ka_0^0 c_0, a_2^2 \leq K(Ka_0^0 c_0)c_1, a_3^3 \leq K(K^2a_0^0 c_0 c_1)c_2 \) and in general:

\[ a_i^i \leq (b_0^0 c_0 + c_0^0)K^i \prod_{n=0}^{i-1} c_n. \quad (53) \]

By (52a), since \( 1 + 2 + \cdots + 2^{i-1} = 2^i - 1 \), the inequality (53) implies that

\[ b_{i+1}^i \leq b_i^i + La_i^i \leq b_i^i + L \cdot (b_0^0 c_0 + c_0^0)K^i \epsilon^{1+2+\cdots+2^{i-1}} \]
\[ = b_i^i + Le^{-1} \cdot (b_0^0 c_0 + c_0^0)K^i \epsilon^2 \]

and therefore, by induction:

\[ b_i^i \leq b_0^0 + Le^{-1} \cdot (c_0 b_0^0 + c_0^0) \sum_{n=0}^{i-1} K^n \epsilon^2. \]

The last inequality shows that the sequence \( \{b_i^i\} \) has to be bounded, which was our first claim (a). Using this fact in combination with the hypothesis \( c_i \leq c_i^i \) and with (52b), we immediately also deduce our second claim (b):

\[ c_{i+1}^i \leq L \cdot (b_i^i c_i + c_i^2) c_i \leq L \cdot (b_i^i c_i^i + c_i^i) c_i = L \cdot (b_i^i + 1)(c_i^i)^2. \]

As to our third claim (c), we have

\[ c_{i+1}^i \leq La_i^i c_i \quad \text{[by (52b)]} \]
\[ \leq L \cdot (b_0^0 c_0 + c_0^0)K^i \prod_{n=0}^{i-1} c_n \cdot c_i \quad \text{[by (53)]} \]
\[ = L \cdot (b_0^0 c_0 + c_0^0)K^i \prod_{n=0}^{i} c_n \]
\[ \leq L \cdot (b_0^0 c_0 + c_0^0)K^i \epsilon^{1+2+\cdots+2^i} \quad \text{[because } c_n \leq \epsilon^{2^i}] \]
\[ = [Le^{-1} \cdot (b_0^0 c_0 + c_0^0)K^i \epsilon^2] \epsilon^2. \]

Since \( \lim_{i} K^i \epsilon^2 = 0 \), the bracketed term will be \(< 1 \) when \( i \) gets sufficiently large. \( \square \)
Proof, Step V.

By the inductive hypotheses S3(r) and S4(r), the preceding lemma—when applied to the recursive system of inequalities (51)—immediately implies the validity of the statements S1(r + 1), S3(r + 1) and S4(r + 1). Furthermore, the estimate (68) yields an inequality of the form

$$\tilde{b}^{(r+1)}((\hat{\lambda}^{(i)})^{-1}) \leq \text{const} \cdot \tilde{b}^{(r)}((\hat{\lambda}^{(i)})^{-1})^2 b^{(r+1)}(\hat{\lambda}^{(i)})$$

where \text{const} indicates a suitable positive constant, so the remaining statement S2(r + 1) follows from the inductive hypothesis S2(r) and from the (already proven) statement S1(r + 1). This concludes the inductive step.

Proof, Step VI.

For any order of differentiability \( r \geq 1 \) (the case \( r = 0 \) will be left to the reader) such that \( \lambda \) is of class \( C^r \), on the basis of the relation (31b) and of the estimates derived in Step II it must be true for all \( i \in \mathbb{N} \) that

$$p_i(r_{\Omega}^{(i+1)} - r_{\Omega}^{(i)}) \leq B_{r-1} \cdot ([\tilde{b}^{(r-1)}]b^{(r)}_i c^{(r-1)}_i + \tilde{b}^{(r-1)}_i c^{(r)}_i)$$

$$\leq B_{r-1} \sup_n ([\tilde{b}^{(r-1)}_n]b^{(r)}_n + \tilde{b}^{(r-1)}_n) \cdot c^{(r)}_i;$$

here we are using the inequality \( c^{(r-1)}_i \leq c^{(r)}_i \), which is a consequence of the obvious inequality \( q_{r-1}(-) \leq q_r(-) \). Since by S4(r) \( c^{(r)}_i \leq e^{2(r-i)} \) for all \( i \geq i_r \), we conclude that the sequence \( \{r_{\Omega}^{(i)}) \) is Cauchy within \( \Gamma_r(\Omega; L(s^r E, r^E)) \) relative to the \( C^r \)-norm topology. Corollary A.5 then implies that the sequence \( \{r_{\Omega}^{(i)}) = r_{\Omega}^{(i)} \hat{\lambda}^{(i)} \mid U \) is Cauchy within \( \Gamma_r(\Omega; L(s^r E, r^E)) \) relative to the \( C^r \)-topology, which is the desired property (A) for the given invariant open set \( U \). The property (B) is an immediate consequence of S4(0). Our theorem is thus proven.

6. Fast Convergence Theorem II (connections)

**Theorem 6.1.** Let \( \Gamma \) be a proper Lie groupoid with proper target mapping. Let \( \Psi \in \text{Conn}^p(\Gamma) \) be a unital connection on \( \Gamma \) of class \( C^p \) (\( p = 0, 1, 2, \ldots, \infty \)), and suppose \( \Psi \) is nearly multiplicative in the sense of Definition 5.1. Then, the sequence of successive averaging iterates of \( \Psi \) taken with respect to any given left invariant Haar probability system on \( \Gamma \) by means of the formula (16a)

$$\hat{\Psi}^{(0)} := \Psi, \hat{\Psi}^{(1)} := \hat{\Psi}, \ldots, \hat{\Psi}^{(i+1)} := (\hat{\Psi}^{(i)}), \ldots \in \text{Conn}^p(\Gamma)$$

is convergent within the affine Fréchet manifold \( \text{Conn}^p(\Gamma) \) (which by definition consists of all the \( C^p \) connections on \( \Gamma \)) to a unique multiplicative \( C^p \) connection \( \hat{\Psi}^{(\omega)} \in \text{Conn}^p(\Gamma) \).\(^2\)

---

\(^2\)See Footnote 11 on page 28.
Proof. (The reader is referred to Section 3 for all the notations which we will be using without commentary. As in the preceding section, we let $M$ denote the base manifold of $\Gamma$.) Let $H$ be an arbitrary non-degenerate connection on $\Gamma$. For every divisible pair of arrows $(g, h) \in \Gamma_\circ$ and for every base tangent vector $v \in T_{gh}M$, the tangent vector $(\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} v \in T_{gh^{-1}} \Gamma$ will be $s$-vertical (i.e. will lie in the $s$-vertical subspace $T_{gh^{-1}}^\perp \Gamma \subset T_{gh^{-1}} \Gamma$) because

\[
(T_{gh^{-1}} s)((\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} v) = (T_{h t} \eta^H_v) - (id_{T_M} \lambda^H_t v = \lambda^H_t v - \lambda^H_v = 0.
\]

Hence, for every pair $(g, h) \in \Gamma_\circ$ we will have a linear map

\[
(\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} \circ \lambda^H_t \in L(T_{sh}M, T_{gh^{-1}}^\perp \Gamma).
\]

Since $H$ is non-degenerate and since (as noticed above) $T_{gh^{-1}} s \circ (\eta^H_v \div \eta^H_h) = T_{h t} \circ \eta^H_v = \lambda^H_t$, we see that the following identity holds

\[
\Delta^H(g, h) = \omega_{gh^{-1}} \circ \beta^H_{gh^{-1}} \circ \delta^H(g, h)
\]

\[
= \omega_{gh^{-1}} \circ (\delta^H(g, h) - \eta^H_{gh^{-1}} \circ T_{gh^{-1}} s \circ \delta^H(g, h))
\]

\[
= \omega_{gh^{-1}} \circ ((\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} \circ \lambda^H_t \circ (\lambda^H_{t^{-1}}))
\]

\[
= \omega_{gh^{-1}} \circ ((\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} \circ \lambda^H_t \circ (\lambda^H_{t^{-1}}))
\]

whence for any other non-degenerate connection $\Phi$ on $\Gamma$ and for every tangent vector $v$ as above we infer the equation:

\[
\Delta^H(g, h) \lambda^H_t v = \omega_{gh^{-1}}((\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} \lambda^H_t v)
\]

\[
= \omega_{gh^{-1}}(\beta^H_{gh^{-1}}((\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} \lambda^H_t v)) \quad \text{[by s-verticality (54)]}
\]

\[
= pr_1(\pi^H_{gh^{-1}}((\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} \lambda^H_t v)) \quad \text{[by definition (18)]}
\]

\[
= q^H_{gh}((\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} \lambda^H_t v) \quad \text{[by the paragraph 3.5]}
\]

\[
= q^H_{gh}(g, h)(\eta^H_v \div \eta^H_h) - \eta^H_{gh^{-1}} \lambda^H_t v + \Delta^H(g, h) \lambda^H_t v \quad \text{[by (21) and (25)]}
\]

Now, under the assumption that $\Phi$ is also non-degenerate, making $H = \Phi$ in the equation (56) and referring back to the notational conventions and to the computations in the proof of Proposition 3.7, we deduce the identity

\[
\Delta^\Phi(g, h) \circ \lambda_h = q^H_{g t}(g, h) \circ (\hat{X}_g, \hat{X}_h) - \hat{X}_{gh^{-1}} \circ \lambda_{gh^{-1}} \circ \lambda^H(g, h) \circ \lambda_h
\]

\[
= \iint \Delta^\Phi(gk, h k) \circ \delta^\Phi(hk, k') \circ [\Delta^\Phi(hk, k) - \Delta^\Phi(hk', k')] dv(k) dv(k')
\]

\[
= \iint \Delta^\Phi(gk, h k) \circ \Delta^\lambda(hk, k) dv(k) - \iint \Delta^\Phi(gk, h k) \circ \Delta^\lambda(hk', k') dv(k) dv(k')
\]

\[
= \iint \Delta^\Phi(gk, h k) \circ \Delta^\lambda(hk, k) dv(k) - \left[ \int \Delta^\Phi(gk, h k) dv(k) \right] \circ \left[ \int \Delta^\lambda(hk, k) dv(k) \right]
\]

(57a)

in which $v$ denotes the left invariant Haar probability system that according to the statement of our theorem is supposed to have been fixed on $\Gamma$. This identity is the analog for connections of (31a), whereas the analog of (31b) is given by

\[
\hat{\eta}_g^\Phi - \hat{\eta}_g^\Phi = \int [\delta^\Phi(gk, k) - \eta^\Phi] dv(k)
\]
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\begin{align*}
= \int [(\eta_{\phi k} \div \eta_{\phi l}) \circ \lambda^{-1}_k - \eta_{\phi l}] \, dv(k) \\
= \omega^{-1}_l \circ \Delta^\phi (gk, k) \, dv(k). \quad (57b)
\end{align*}

Let now \( \Phi \) be a variable ranging over the subset of \( \text{Conn}^p(\Gamma) \) (= non-degenerate \( C^p \) connections on \( \Gamma \)) formed by all those connections \( \Phi \in \text{Conn}^p(\Gamma) \) such that \( \hat{\Phi} \) also belongs to \( \text{Conn}^p(\Gamma) \). It will not be restrictive for our purposes to assume that the order of differentiability \( p \) is a natural number—i.e. is \( < \infty \). Recall that \( \Delta^\phi \) coincides by definition with the following composite morphism of vector bundles over \( \Gamma_\gamma \) (each component of which is at least of class \( C^p \) because so was \( \Phi \)).

\[ s_\gamma TM \overset{\delta^\phi \circ \lambda}{\rightarrow} q_\gamma^* T\Gamma \overset{q_\gamma^! \circ \beta^\phi}{\rightarrow} q_\gamma^* T^1 \Gamma \overset{q_\gamma^! \omega}{\rightarrow} q_\gamma^* t_\gamma g \equiv t_\gamma g \]

Let \( U \in M \) be an arbitrary invariant, relatively compact, open set. As in the previous section, we shall put \( \Omega = \Gamma \setminus U \). We shall also put \( \Omega_\gamma := \Omega_\gamma \times_\gamma \Omega; \) this will be a relatively compact open subset of \( \Gamma_\gamma \) because \( \Omega_\gamma \times_\gamma \Omega = \Omega_\gamma \times_\gamma \Omega \) (by the same argument showing that \( \Omega \times_\gamma \Omega = \Omega \times_\gamma \Omega \)). If we let \( [r_\gamma, \lambda] \) denote the vector-bundle isomorphism

\[ r_\gamma^* s^\gamma TM \overset{r_\gamma^! \lambda}{\rightarrow} r_\gamma^* t_\gamma TM \equiv s_\gamma^* TM \quad (58) \]

[where as before we set \( \lambda = (\lambda^\phi)^\gamma = \lambda^\phi \)], we get the following inequality:

\[ \| \rho_{\gamma, \lambda}^\phi \|_{C^p} = \| \rho_{\gamma, \lambda}^\phi \circ [r_\gamma, \lambda] \|_{C^p} \leq \| \rho_{\gamma, \lambda}^\phi \|_{C^p} \| r_\gamma, \lambda \|^{-1}_{C^p} \]

\[ \leq \| \rho_{\gamma, \lambda}^\phi \|_{C^p} \| r_\gamma, \lambda \|_{C^p} \| \rho_{\gamma, \lambda}^\phi \|_{C^p} \| r_\gamma, \lambda \|^{-1}_{C^p}. \quad (59) \]

The first factor in (59) may be estimated through the equation (57a). Namely, let us rewrite (57a) in “implicit form”; we obtain the identity

\[ \Delta^\phi \circ [r_\gamma, \lambda] = \langle [m_\gamma^* \Delta^\phi \circ pr_{2,3}^* \Delta^1], d_\gamma' \rangle - \langle [m_\gamma^* \Delta^\phi], d_\gamma' \rangle \circ [r_\gamma, \Delta^1 \| d_\gamma, v] \]

where: i) \( m_\gamma^* \) denotes the mapping of \( \Gamma_\gamma \) into \( \Gamma_\gamma \) given by \( (g, h; k) \mapsto (gk, hk) \); ii) \( pr_{2,3}^* \) denotes the mapping of \( \text{Conn}_\gamma \times_\gamma \Gamma \) into \( \Gamma_2 \) given by \( (g, h; k) \mapsto (hk, h) \); iii) \( \lambda \) is short for \( \lambda^\phi \); iv) \( [m_\gamma^* \Delta^\phi \circ pr_{2,3}^* \Delta^1] \) (by abuse of notation) indicates the cross-section of the vector bundle \( pr_{2,3}^* L(r_\gamma^* s^\gamma TM, t_\gamma g, \lambda) \) (over \( \Gamma_\gamma \times_\gamma \Gamma \)) that corresponds to the composite vector-bundle morphism reproduced below where \( pr_{2,3}^* : \Gamma_\gamma \times_\gamma \Gamma \rightarrow \Gamma_\gamma \) denotes the projection \( (g, h; k) \mapsto (g, h) \);

\[ pr_{2,3}^* t_\gamma g \equiv m_\gamma^* t_\gamma g \overset{m_\gamma^* \Delta^\phi}{\rightarrow} m_\gamma^* s_\gamma^* TM \equiv pr_{2,3}^* a^* t_\gamma^* TM \overset{pr_{2,3}^* a^* \Delta^1}{\rightarrow} pr_{2,3}^* a^* s_\gamma^* TM \equiv pr_{2,3}^* r_\gamma^* s^\gamma TM \]

v) \( d_\gamma', v \) is an abbreviation for the integration functional

\[ d_{\gamma, L^1}(r_\gamma^* s^\gamma TM, t_\gamma g) : \Gamma^p(\Gamma_\gamma \times_\gamma \Gamma; pr_{2,3}^* L(r_\gamma^* s^\gamma TM, t_\gamma g)) \rightarrow \Gamma^p(\Gamma_\gamma \times_\gamma \Gamma; L(r_\gamma^* s^\gamma TM, t_\gamma g)); \]

vi) \( [m_\gamma^* \Delta^\phi] \) indicates the cross-section of the vector bundle \( pr_{2,3}^* L(s^\gamma TM, t_\gamma g) \) that corresponds to the vector-bundle morphism

\[ pr_{2,3}^* t_\gamma g \equiv m_\gamma^* t_\gamma g \overset{m_\gamma^* \Delta^\phi}{\rightarrow} m_\gamma^* s_\gamma^* TM \equiv pr_{2,3}^* s_\gamma^* TM; \]

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vii) $d''_v\nu$ stands for the integration functional

$$d''_v: \Gamma^p(\Omega; \mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)) \times \Gamma^p(\Omega; \mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)) \rightarrow \Gamma^p(\Omega; \mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)).$$

viii) $[r_u(\ldots)]$ denotes the composite vector-bundle morphism obtained by replacing $\lambda$ with $\lambda'$ in (35), where $\langle \ldots \rangle = \langle a^\Delta^4, d_1\nu \rangle$ has the same meaning as in Step II of the proof of Theorem 5.2. By making repeated use of estimates of type $\|\tilde{\Theta} \circ \tilde{\Theta}\|_{C^p} \leq \|\tilde{\Theta}\|_{C^p} \|\tilde{\Theta}\|_{C^p}$ we see by computations entirely analogous to those in the proof of Theorem 5.2 that

$$\|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (\Delta_{\nu} \circ [r_u(\ldots)])\|_{C^p} \leq \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (\Delta_{\nu})\|_{C^p} \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4)\|_{C^p} + \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4)\|_{C^p} \|d_1\nu\|_{C^p} + \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4, d_1\nu)\|_{C^p} \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4)\|_{C^p} \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4)\|_{C^p} \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4)\|_{C^p}. \quad (60)$$

Putting (59) and (60) together, we obtain

$$\|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (\Delta_{\nu} \circ [r_u(\ldots)])\|_{C^p} \leq \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (\Delta_{\nu})\|_{C^p} \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4)\|_{C^p} \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4)\|_{C^p} \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4)\|_{C^p} \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4)\|_{C^p}. \quad (61)$$

Next, from the identity (57b) we deduce the following estimate for the $\|\cdot\|_{C^p}$-norm of the difference $\hat{\nu}^\phi - \tilde{\nu}^\phi \in \Gamma^p(\Omega; \mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g))$:

$$\|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (\hat{\nu}^\phi - \tilde{\nu}^\phi)\|_{C^p} = \|\omega^{-1} \circ \rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (a^\Delta^4, d_1\nu)\|_{C^p} \leq \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (\Delta_{\nu})\|_{C^p}, \quad (62)$$

where $d_1\nu$ (‘c’ like ‘connection’) denotes the integration functional

$$d''_v: \Gamma^p(\Omega; \mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)) \times \Gamma^p(\Omega; \mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)) \rightarrow \Gamma^p(\Omega; \mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)),$$

and $[a^\Delta^4]$ has the obvious meaning (analogous to the meaning of $[a^\Delta^4]$).

We are now prepared to investigate our given nearly multiplicative connection $\Psi \in \text{Conn}^p(G)$. Recall that the order of differentiability $p$ is assumed to be finite. From now on until the end of the section we let $\lambda$ stand for the near action $\lambda^\Phi$ associated to $\Psi$. We also let $\hat{\lambda}^i (i = 0, 1, 2, \ldots)$ denote the horizontal lift corresponding to $\hat{\lambda}^i$ (= $i$th averaging iterate of $\lambda^\Phi$). Inductive application of the identity (17) yields that for every index $i$ the $i$th averaging iterate of $\lambda$ (which recall was denoted by $\lambda^i$ in Section 5) coincides with the near action associated to the $i$th averaging iterate of $\Psi$.

Let $U, \phi$ be as in Step I of the proof of Theorem 5.2 (relative to the near action $\lambda = \lambda^\Phi$). We carry over all the notations introduced in the course of the proof of that theorem—for example (41), (42) and (49). In addition, we fix an arbitrary standard normed atlas for $\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)$ over $\mathcal{Q}$, say $\mathcal{A}$, and then for any non-degenerate $C^p$ connection $\Phi \in \text{Conn}^p(G)$ set:

$$d''^p(\Phi) = \|\rho\_{\Omega}^{C^p}_{\mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g)} (\Delta_{\nu})\|_{C^p; \mathcal{L}(s\,\cdot\,TM, t\,\cdot\,g), \mathcal{A}}.$$
and, for every index \( i \in \mathbb{N} \):
\[
d_i^{(p)} = d_i^{(p)}(\hat{\eta}^{(i)}).
\]

From the estimate (61) we immediately deduce that there must be some positive constant \( D_p \) such that for every index \( i \in \mathbb{N} \) (notations as in Step III of the proof of Theorem 5.2)
\[
d_{i+1}^{(p)} \leq D_p d_i^{(p)} c_i^{(p)} E_i^{(p)} I_{i+1}^{(p)}.
\]

Now, by the statement S2(\( p \)) in Step III of the proof of Theorem 5.2 the sequence \( \{\hat{b}_i^{(p)}\} \) has to be bounded, so we conclude that there must be some constant \( K > 0 \) such that the inequality below holds for every index \( i \in \mathbb{N} \).
\[
d_i^{(p)} \leq K d_i^{(p)} c_i^{(p)}
\]

Hence \( d_1^{(p)} \leq K d_0^{(p)} c_0^{(p)}, d_2^{(p)} \leq K(K d_0^{(p)} c_0^{(p)}) c_1^{(p)}, d_3^{(p)} \leq K(K^2 d_0^{(p)} c_0^{(p)}) c_1^{(p)} c_2^{(p)} \) and in general [for \( i = i_p + j > i_p \), where \( i_p \) is as in Statement 54(\( p \)), proof of Theorem 5.2]:
\[
d_i^{(p)} \leq d_0^{(p)} K^i \prod_{n=0}^{i_p-1} c_n^{(p)}
\]

\[
\leq d_0^{(p)} \left( K^i \prod_{n=0}^{i_p-1} c_n^{(p)} \right) \left( K^{i_p+j-1} \prod_{n=i_p}^{i_p+j-1} c_n^{(p)} \right)
\]

\[
= d_0^{(p)} \left( K^i \prod_{n=0}^{i_p-1} c_n^{(p)} \right) \varepsilon^{-1} K^j \varepsilon^{2^j} = \left( \text{const} \cdot K^{i+j} \varepsilon^{-i+j+p} \right).
\]

It follows at once from the last inequality and from the estimate (62) that the sequence \( \{\mu_\xi^{\varepsilon}(\hat{\eta}^{(i)} - \hat{\eta}^{(0)})\} \) must be Cauchy within \( \Gamma^p(\Omega; L(s^*TM, T^1\Gamma)) \) with respect to the \( C^p \)-norm topology. By Corollary 8.5, the sequence \( \{p_\xi^{\varepsilon}(\hat{\eta}^{(i)} - \hat{\eta}^{(0)})\} \) must then be Cauchy within \( \Gamma^p(\Omega; L(s^*TM, T^1\Gamma)) \) relative to the \( C^p \)-topology.

Since \( M \) can be covered with open invariant subsets \( U \) of the kind considered in the preceding paragraph, it follows by the same argument as in the proof of Theorem 5.2 that the sequence \( \{\hat{\eta}^{(i)} - \hat{\eta}^{(0)}\} \) must be Cauchy within \( \Gamma^p(\Gamma; L(s^*TM, T^1\Gamma)) \) (with respect to the \( C^p \)-topology) and hence that the sequence \( \{\hat{\eta}^{(i)}\} \) has to be convergent within the affine Fréchet manifold \( \text{Conn}^p(\Gamma) \) to a unique connection \( \hat{\eta}^{(\text{convergent})} \) of class \( C^p \). We contend that \( \hat{\eta}^{(\text{convergent})} \) has to be multiplicative. Indeed, let \( \hat{\eta}^{(\text{convergent})} \) denote its horizontal lift. Since \( C^p \)-convergence implies pointwise convergence, we have \( \hat{\eta}_1^{(\text{convergent})} = \lim_i \hat{\eta}_{1x}^{(i)} = \lim_i T_{x1} \hat{\eta}_1^{(i)} \) for every base point \( x \in M \) (because every \( \hat{\eta}^{(i)} \) is unital). Hence \( \hat{\eta}^{(\text{convergent})} \) is unital. Notice that for every \( g \in \Gamma \) we have \( T_{xl} \circ \hat{\eta}_g^{(\text{convergent})} = \lim_i T_{xl} \circ \hat{\eta}_g^{(i)} = \lim_i \hat{\lambda}_g^{(i)} = \hat{\lambda}_g^{(\text{convergent})} \). Since for every divisible pair \( (g, h) \in \Gamma \), the ratio operation restricts to a linear (hence continuous) map \( \hat{\tau}_{g,h} : T_g \Gamma \oplus_{\tau_{g,h}} T_h \Gamma \rightarrow T_{gh^{-1}} \Gamma \), on the basis of (55) we compute:
\[
(\hat{\eta}_g^{(\text{convergent})} \div \hat{\eta}_h^{(\text{convergent})}) - \hat{\lambda}_h^{(\text{convergent})} = \lim_i (\hat{\eta}_g^{(i)} \div \hat{\eta}_h^{(i)}) - \hat{\lambda}_h^{(i)} = \omega_{g,h}^{-1} \circ \Delta(g,h) \circ \hat{\lambda}_h^{(i)} = \omega_{g,h}^{-1} \circ \lim_i \Delta(g,h) = \hat{\lambda}_h^{(\text{convergent})}.
\]
Appendix A. Uniform convergence topologies on spaces of sections

Throughout the present appendix, $k$ will denote a natural number. Let $E$ be an arbitrary $\mathbb{K}$-linear differentiable vector bundle over a differentiable manifold $X$. (Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, as always.) We say that a cross-section $\xi : S \to E$ defined over an arbitrary subset $S \subset X$ is of class $C^k$ if for each point $x \in S$ the following is true.

(*) There exists an open neighborhood $B \ni x$ in $X$ over which $\xi|S \cap B$ can be extended to some cross-section of class $C^k$ of $E$.

We let $\Gamma^k(S; E)$ denote the vector space of all cross-sections of class $C^k$ of $E$ over $S$. Of course, when $S = Y$ is a submanifold—in particular, when $S = U$ is an open subset—the present notation is consistent with the notation introduced in Section I. However notice that “class $C^k$” does not agree with “continuous” in general unless $S$ is a locally closed subset. For each subset $T \subset S$ we let $\rho^k_T$ denote the restriction map from $\Gamma^k(S; E)$ into $\Gamma^k(T; E)$. Observe that

$$\rho^k_S : \Gamma^k(S; E) \longrightarrow \Gamma^k(S; E)$$

(where $\overline{S}$ denotes the closure of $S$ in $X$) is always an injective map; it identifies $\Gamma^k(\overline{S}; E)$ with the linear subspace of $\Gamma^k(S; E)$ consisting of all those $\xi$ such that the property (*) holds for every $x \in X$ (not just for every $x \in S$).

A.I. $C^k$-Topology. We assume that the reader is familiar with some of the basic notions of the theory of topological vector spaces such as for instance the notion of “locally convex topology generated by a family of seminorms” or the notion of Fréchet space, which are thoroughly discussed in Chapter II of [13]. Acquaintance with the elementary concepts and examples of the theory of Fréchet manifolds (a good self-contained account of which can be found in Part I of [7]) is advisable albeit not indispensable. Finally for a description of the $C^k$-topology on the space of mappings between two differentiable manifolds compare [11]. Let $\varphi : U \supseteq \varphi U \subset \mathbb{R}^n$ be any $C^\infty$ local coordinate chart for the differentiable manifold $X$. Also let $\tau : E \mid U \supseteq U \times \mathbb{R}^N$ be any $C^\infty$ local trivialization for the differentiable vector bundle $E$ over the domain $U$ of the chart $\varphi$. We can express an arbitrary global cross-section $\xi$ of $E$ locally over $U$ in terms of its components relative to the local trivialization $\tau$.

$$\xi^\tau = (\xi^\tau_1, \ldots, \xi^\tau_N) \overset{\text{def}}{=} pr_2 \circ \tau \circ \xi \mid U \quad \xi^{\tau, \varphi} = (\xi^{\tau, \varphi}_1, \ldots, \xi^{\tau, \varphi}_N) \overset{\text{def}}{=} \xi^\tau \circ \varphi^{-1}$$

Obviously, $\xi \mid U \in \Gamma^k(U; E)$ if, and only if, every component $\xi_i^{\tau, \varphi}$ is a function of class $C^k$ (defined on the open subset $\varphi U \subset \mathbb{R}^n$ and with values in $\mathbb{K}$). For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ of order $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$ and for every function $f : \Omega \to \mathbb{K}$ of class $C^k$ which is defined on some open domain $\Omega \subset \mathbb{R}^n$ we adopt the customary
for every global cross-section now consider an arbitrary compact subset $E$. For every natural number $r \in \mathbb{N}$ and for every global cross-section $\xi \in \Gamma^k(X;E)$ of regularity class $k \geq r$ we put:

$$p_{r}^{\tau,\varphi,K}(\xi) \overset{\text{def}}{=} \max_{0 \leq s \leq r} \max_{\alpha \in \mathbb{N}^n, |\alpha| = s} \max_{x \in K} \sup_{I} \max_{1 \leq N} |\partial^\alpha \xi_I^\tau,\varphi(\varphi(x))|.$$  

Evidently, this equation defines a seminorm $p_{r}^{\tau,\varphi,K}$ on the vector space $\Gamma^k(X;E)$. The topology of $k$-th order local uniform convergence—shortly, $C^k$-topology—on $\Gamma^k(X;E)$ is the locally convex vectorial topology generated by all the seminorms $p_{r}^{\tau,\varphi,K}$ which one obtains by letting $\tau, \varphi, K$ vary over all the possible choices of a local vector bundle trivialization $\tau$, a local coordinate chart $\varphi$ and a compact domain $K$ as above, and by letting $r$ vary over all the natural numbers $\leq k$. Since $E$ is locally trivial and $X$ is locally compact, this topology is necessarily separated (i.e., Hausdorff).

**Proposition A.1.** Suppose $X$ is a second countable differentiable manifold. Let $E$ be an arbitrary $\mathbb{K}$-linear differentiable vector bundle over $X$. Then, for each value of $k = 0, 1, 2, \ldots, \infty$, the $C^k$-topology makes $\Gamma^k(X;E)$ into a Fréchet space (= complete, metrizable, locally convex, topological vector space). When $X$ is compact and $k$ is finite, the Fréchet space $\Gamma^k(X;E)$ is actually Banachable (= complete normable).

As we will do for various other statements in this appendix, we will leave the proof as an exercise. (One has to pay some attention here because the intersection between two compact subsets of a non-Hausdorff manifold need not itself be compact. For instance, consider the line with a double origin and therein the two compact intervals $[-1, 1]$ centered each around a different origin.)

A.II. $C^k$-Norms. It is sometimes more convenient to work with a slightly more flexible definition of “generating seminorm for the $C^k$-topology”.

By a continuous norm $p$ on a differentiable vector bundle $E$ over a differentiable manifold $X$ we shall mean the datum of a norm $p_x$ on each vector-bundle fiber $E_x$ depending on $x$ in such a manner that the function on $E$ given by $e \mapsto p_x(e)$ is continuous.

**Lemma A.2.** Let $e_1, \ldots, e_N$ be a local frame for $E$ defined over some open set $U \subset X$. Let $K$ be a compact subset of $U$. Let $p$ be a continuous vector-bundle norm on $E \mid U$. Then, there exists a constant $c > 0$ such that if for any section $\xi \in \Gamma^0(U;E)$ we write $\xi = \sum_I a_I e_I$ where $a_I \in C^0(U)$ then $\max_{x} \|a_I\|_K \leq c \sup_{x \in K} p_x(\xi(x))$.

**Proof.** Put

$$c^{-1} = \inf_{\substack{x \in K \mid \|\cdot\|_{L^\infty}}} \inf_{|\alpha|^I = \cdots |\alpha|^N = 1} p_x(z_1 e_1(x) + \cdots + z_N e_N(x)).$$

Whenever $|a_I(x)| > 0$ for some $x \in K$ and for some $I$, let us say $I = 1$, we put $b_I = a_I(x)/a_1(x)$ for all $I$ and let $p^2 = 1 + |b_2|^2 + \cdots + |b_N|^2 \geq 1$. Then $(1/p)^2 + |b_2|^2 + \cdots + |b_N|^2 = 1$, whence $c^{-1} \leq p(e_1(x) + b_2 e_2(x) + \cdots + b_N e_N(x))$ and therefore $c^{-1} \|a_1(x)\| \leq p_x(\xi(x))$. \hfill $\Box$
Examples. Any (Riemannian or Hermitian, depending on whether $E$ is real or complex) metric $\phi$ on $E$ gives rise to a continuous norm on $E$ given for every $x$ by the rule: $E_x \ni e \mapsto \sqrt{\phi_x(e,e)}$. If $p$ and $q$ are any continuous norms on two vector bundles $E$ and $F$ over the same $X$ respectively then the rule $L(E_x,F_x) \ni \lambda \mapsto \sup_{p,\epsilon=1} q, (\lambda e)$ defines another such norm on $L(E,F)$. [The following remarks should provide enough evidence in support of the last claim. Let $B$ and $F$ be arbitrary vector spaces over $K$. Let $p_0$ and $p_1$ be norms on $B$ such that $bp_0 \leq p_1 \leq Bp_0$ for some constants $b, B > 0$ and let $q_0$ and $q_1$ be norms on $F$ such that $cq_0 \leq q_1 \leq Cq_0$ for some other constants $c, C > 0$. Then, the two norms on $L(B,F)$ given by $\|\| = \sup_{p,\epsilon=1} q, (\lambda e) \ (r = 0, 1)$ obey the inequalities:

$$cB^{-1}||||_0 \leq ||||_1 \leq Cb^{-1}||||_0.$$ 

In particular, if for some small number $\epsilon > 0$ the relations $\min\{b, c\} \geq 1 - \epsilon$ and $\max\{B, C\} \leq 1 + \epsilon$ hold then $\frac{1}{1 + \epsilon} ||||_0 \leq ||||_1 \leq \frac{1}{1 - \epsilon} ||||_0.$

Let $X$ and $E$ be as above. By a standard normed chart for $E$ we shall mean a triplet $(\varphi, \tau, \rho)$ consisting of: i) a local coordinate chart for $X$ of the form $\varphi : U \cong \mathbb{R}^n$; ii) a local vector-bundle trivialization $\tau : E | U \cong U \times \mathbb{K}^n$ for $E$ over the domain of definition of $\varphi$; iii) a continuous vector-bundle norm $\rho : E \downarrow U \to \mathbb{R}_{>0}$ on the restriction of $E$ over $U$. For any local coordinate chart $\varphi$ as in i) and for any real number $a > 0$ we shall let $B^a \subset \mathbb{K}^n$ denote the two subsets of the chart domain $U^\varphi \cong \mathbb{R}^n$ that correspond to the two balls $B^a(0) \subset B^a(0)$ of radius $a$ centered at the origin in $\mathbb{R}^n$ under $\varphi$. We shall also abbreviate $B^a$ into $B^a$ and $K^a$ into $K^a$. Suppose $\varOmega \subset X$ is a relatively compact open subset of $X$, in other words an open subset whose closure $\overline{\varOmega}$ is compact. By a standard normed atlas for $E$ over $\overline{\varOmega}$ we shall mean a finite collection $\mathcal{A} = \{(\varphi_i, \tau_i, \rho_i) | i \in \mathcal{I}\}$ of standard normed charts for $E$ such that $\overline{\varOmega} \subset \bigcup_{i \in \mathcal{I}} B^\varphi_i$.

Let $X$, $E$ and $\varOmega$ be as in the preceding paragraph. Suppose we are given a standard normed atlas $\mathcal{A} = \{(\varphi_i, \tau_i, \rho_i) | i \in \mathcal{I}\}$ for $E$ over $\overline{\varOmega}$. For any index $i \in \mathcal{I}$ we set $U_i \equiv U^\varphi_i$, $K_i \equiv K^\varphi_i$, $B_i \equiv B^\varphi_i$ and write $\tau_i : E | U_i \cong U_i \times \mathbb{K}^n = U_i \times \mathbb{B}_i$. For each point $u \in U_i$, we let $\lambda u$ indicate the norm on $\mathbb{B}_i$ that corresponds to $\rho_{i,u}$ under the linear isomorphism $\varphi_{i,u} : \mathbb{E}_u \cong \mathbb{B}_i$. Suppose we are given some cross-section $\xi \in \Gamma^k(\overline{\varOmega}, E)$. A fortiori, $\xi$ will be defined over $\varOmega_i = \varOmega \cap U_i$ and hence will correspond via $\tau_i$ and $\varphi_i$ to a unique vector-valued function $\xi|_{\varOmega_i} : \varphi_i \varOmega_i \to \mathbb{B}_i$ of class $C^k$. Since $\varphi_i \varOmega_i$ is an open subset of euclidean space $\mathbb{R}^n$, for each multi-index $\alpha \in \mathbb{N}^n$ of order $|\alpha| \leq k$ one can make perfect sense of the partial derivative $\partial^\alpha \xi|_{\varOmega_i}$ as a vector-valued function $\varphi_i \varOmega_i \to \mathbb{B}_i$. Then, one puts:

$$\|\xi\|_{C^k(\overline{\varOmega}, E, \mathcal{A})} \overset{\text{def}}{=} \max_{i \in \mathcal{I}} \max_{\alpha \in \mathbb{N}^n, |\alpha| \leq k} \sup_{u \in B_i \cap \varOmega_i} |\partial^\alpha \xi|_{\varOmega_i} (\varphi_i u)|_{\lambda u}. \quad (63)$$

We refer to the function $\|\|_{C^k(\overline{\varOmega}, E)}$ thus defined on $\Gamma^k(\overline{\varOmega}, E)$ as a standard $C^k$-norm. [Of course, in order for (63) to define a norm one has to make sure that each one of the (finitely many) suprema occurring in the defining expression is $< \infty$. This is the case, because $\xi$ is defined over all of $\varOmega \cap U_i$ (which is a closed subset of the 2nd countable Hausdorff manifold $U_i = \mathbb{R}^n$) and is locally $C^k$-extendible within $U_i$ around each point of that closed subset, so by the existence of partitions of unity it admits a $C^k$ extension to all of $U_i (\supset K_i \supset B_i).] \!$ Observe that our notation $\|\|_{C^k(\overline{\varOmega}, E)}$ is correct in that the
quantity (63) only depends on $\overline{\Omega}$, not on $\Omega$; the only relevant object, here, is a closed, compact subset of $X$ which coincides with the closure of its own interior.

**Proposition A.3.** Let $E$ be an arbitrary $\mathbb{K}$-linear differentiable vector bundle over a differentiable manifold $X$, and let $\Omega \subset X$ be an arbitrary relatively compact open subset of $X$. Then, any two standard $C^k$-norms on $\Gamma^k(\overline{\Omega}; E)$ are equivalent.

This is an exercise; you apply Lemma A.2 together with essentially the same arguments you use to prove Proposition A.1. We refer to the (normable linear) topology induced on $\Gamma^k(\Omega; E)$ by any standard $C^k$-norm as the $C^k$-topology on $\Gamma^k(\overline{\Omega}; E)$; we shall mean a norm which induces the $C^k$-topology (i.e. a norm which is equivalent to some standard $C^k$-norm).

**Corollary A.4.** Let $X$, $E$ and $\Omega$ be as in the previous proposition. The restriction map $\rho_X^\Omega : \Gamma^k(X; E) \to \Gamma^k(\Omega; E)$ is continuous (for the $C^k$-topology on the first space and the $C^k$-topology on the second one).

**Corollary A.5.** Let $X$, $E$ and $\Omega$ be as in Proposition A.3. The restriction map $\rho_\Omega^\Omega : \Gamma^k(\Omega; E) \to \Gamma^k(\overline{\Omega}; E)$ is continuous (for the $C^k$-norm topology on the first space and the $C^k$-topology on the second one).

Observe that when $X$ is a compact manifold these corollaries imply that the $C^k$-topology and the $C^k$-topology on $\Gamma^k(X; E)$ coincide.

**A.III. Two basic continuity lemmas.**

**Lemma A.6.** Let $\omega : E \to F$ be an arbitrary morphism of $\mathbb{K}$-linear differentiable vector bundles over a differentiable manifold $X$. For each value of $k = 0, 1, 2, \ldots, \infty$ the $\mathbb{K}$-linear map

$$\Gamma^k(X; \omega) : \Gamma^k(X; E) \to \Gamma^k(X; F), \xi \mapsto \omega \circ \xi$$

is continuous with respect to the $C^k$-topology on $\Gamma^k(X; E)$ and the $C^k$-topology on $\Gamma^k(X; F)$. Moreover for any relatively compact open subset $\Omega \subset X$ the $\mathbb{K}$-linear map

$$\Gamma^k(\overline{\Omega}; \omega) : \Gamma^k(\overline{\Omega}; E) \to \Gamma^k(\overline{\Omega}; F), \xi \mapsto \omega \circ \xi$$

is continuous with respect to the $C^k$-norm topology on $\Gamma^k(\overline{\Omega}; E)$ and the $C^k$-norm topology on $\Gamma^k(\overline{\Omega}; F)$.

For any differentiable mapping $f : Y \to X$ and for any $\mathbb{K}$-linear differentiable vector bundle $E$ over $X$ the pullback vector bundle $f^*E$ has the fiber product $Y \times_X E$ as its total manifold and the first projection $Y \times_X E \to Y$ as its bundle projection onto $Y$. Clearly, a mapping $Z \to Y \times_X E$ will be of class $C^k$ if and only if its two components $Y \leftarrow Z \to E$ are both $C^k$. Hence, by the universal property of the fiber product, for each $C^k$ cross-section $\xi$ of $E$ there will be a unique $C^k$ cross-section $f^*\xi$ of $f^*E$ such that $pr_E \circ f^*\xi = \xi \circ f$ (where $pr_E$ denotes the second projection $Y \times_X E \to E$).
Lemma A.7. Let \( f : Y \to X \) be an arbitrary differentiable mapping and let \( E \) be an arbitrary \( \mathbb{K} \)-linear differentiable vector bundle over \( X \). For each value of \( k = 0, 1, 2, \ldots, \infty \) the cross-section pullback operation gives rise to a \( \mathbb{K} \)-linear map

\[
\Gamma^k(f; E) : \Gamma^k(X; E) \to \Gamma^k(Y; f^* E), \xi \mapsto f^* \xi
\]

which is continuous with respect to the \( C^k \)-topology on \( \Gamma^k(X; E) \) and the \( C^k \)-topology on \( \Gamma^k(Y; f^* E) \). Moreover for any open, relatively compact subset \( O \) of \( X \) and for any similar subset \( O \) of \( Y \) such that \( f(O) \subset \Omega \) the \( \mathbb{K} \)-linear map

\[
\Gamma^k(f : O \to \Omega; E) : \Gamma^k(\Omega; E) \to \Gamma^k(O; f^* E), \xi \mapsto (f^* \xi)|_O
\]

is continuous with respect to the \( C^k \)-norm topology on \( \Gamma^k(\Omega; E) \) and the \( C^k \)-norm topology on \( \Gamma^k(O; f^* E) \).

A.IV. Norm estimates for composition and inversion. To begin with, we introduce a notational device which will spare us the nuisance of keeping track of irrelevant scaling factors throughout. Let \( S \) be an arbitrary set. We introduce a binary relation \( \preccurlyeq \) on the set \( \text{Func}^\mathbb{R}_0(S) \) of all non-negative real valued functions on \( S \) by defining \( f \preccurlyeq g \) to mean: there exists a constant \( C > 0 \) such that \( f \leq Cg \). Since this binary relation is reflexive and transitive, setting \( f \equiv g \Leftrightarrow f \preccurlyeq g \) & \( g \preccurlyeq f \) will give rise to an equivalence relation on \( \text{Func}^\mathbb{R}_0(S) \), and \( \preccurlyeq \) will descend to a partial order \( \preccurlyeq \) on the set of all \( \equiv \)-equivalence classes of functions in \( \text{Func}^\mathbb{R}_0(S) \). Notice that for all \( f, g, h \in \text{Func}^\mathbb{R}_0(S) \) the relation \( f \equiv g \) implies the two relations \( f + h \equiv g + h \) and \( fh \equiv gh \). Moreover, if \( \lambda : S' \to S \) is any mapping then \( f \equiv g \in \text{Func}^\mathbb{R}_0(S) \) implies \( f \circ \lambda \equiv g \circ \lambda \in \text{Func}^\mathbb{R}_0(S') \). Thus, the operations of sum, product, and pullback along a mapping make sense for \( \equiv \)-equivalence classes of functions. Now, for \( X, E \) and \( \Omega \) as in Proposition A.3 let \( ||| \cdot |||_{C^l\Omega,E} \) (or simply \( ||| \cdot |||_{C^l\Omega} \), or even \( ||| \cdot |||_{C^l} \) when omissions do not lead to confusion) denote the \( \equiv \)-class of any \( C^k \)-norm within \( \text{Func}^\mathbb{R}_0(\Gamma^k(\Omega; E)) \).

Lemma A.8. Let \( E, F \) and \( G \) be \( \mathbb{K} \)-linear differentiable vector bundles over the same differentiable manifold \( X \). Let \( \Omega \) be a relatively compact open subset of \( X \). Then, letting \( \eta \) denote a variable ranging over \( \Gamma(\Omega; L(E, F)) \) and \( \vartheta \) one ranging over \( \Gamma(\Omega; L(F, G)) \), we have:

\[
\begin{align*}
(\text{for } l = 0) & \quad ||| \vartheta \circ \eta |||_{C^0} \leq ||| \vartheta |||_{C^0} ||| \eta |||_{C^0}; && (64a) \\
(\text{for } l = k + 1) & \quad ||| \vartheta \circ \eta |||_{C^{k+1}} \leq ||| \vartheta |||_{C^k} ||| \eta |||_{C^{k+1}} + ||| \vartheta |||_{C^{k+1}} ||| \eta |||_{C^{k+1}}. && (64b)
\end{align*}
\]

\((\ast)\) The correct interpretation of these inequalities should be evident; for example, the first inequality is to be understood as

\[
||| \cdot |||_{C^0\Omega,L(E,G)} \circ \omega \leq (||| \cdot |||_{C^0\Omega,L(F,G)} \circ pr_1) (||| \cdot |||_{C^0\Omega,L(E,F)} \circ pr_2),
\]

where \( pr_1 \) and \( pr_2 \) denote the two projections \( \vartheta, \eta \mapsto \vartheta \) and \( \mapsto \eta \) respectively and where \( \omega \) denotes the composition operation \( \vartheta, \eta \mapsto \vartheta \circ \eta. \)

Proof. First of all some generic considerations. Let \( V \subset \mathbb{R}^n \) be an open subset of euclidean \( n \)-space and let \( \mathbb{E}, \mathbb{F}, \mathbb{G} \) be vector spaces of finite dimension over \( \mathbb{K} \). If \( f :
we immediately deduce the desired norm estimates (64) from (67). Similarly, we have induced standard normed charts: \((\varphi, \tau, p)\) and \((\varphi, \rho, r)\) are standard normed charts respectively for \(E, F\) and \(G\) defined over the same local coordinate patch \(\varphi : U \rightarrow \mathbb{R}^p\). Let us write \(\tau : E \mid U \rightarrow U \times E, \sigma : F \mid U \rightarrow U \times F\) and \(\rho : G \mid U \rightarrow U \times G\). We have an induced local trivialization \([\tau, \sigma]\) for \((L(E, F))\) over \(U\) given by

\[
L(E, F) \mid U \cong L(E \mid U, F \mid U) \cong L(U \times E, U \times F) \cong U \times L(E, F)
\]

and a continuous norm \([p, q]\) on \((L(E, F))\) \(U\) given by

\[
L(E_u, F_u) \ni \lambda \mapsto \sup_{\rho_n(1)} q_n(\lambda e).
\]

Similarly, we have induced standard normed charts: \((\varphi, [\sigma, \rho], [q, r])\) for \((L(F, G))\). Now let \(\eta \in \Gamma([\sigma, \rho]; L(E, F))\) and \(\vartheta \in \Gamma([\sigma, \rho]; L(F, G))\). For every multi-index \(\gamma \in \mathbb{N}^n\) of order \(|\gamma| \leq l\) and for every \(u \in \Omega \cap U\) the identity (65) gives us the inequity

\[
\|\partial^\gamma (\vartheta \circ \eta)\|_{L(E_u, F_u)} = \|\partial^\gamma (\vartheta[\sigma, \rho] \circ \eta[\sigma, \rho]) (\varphi u)\|_u \\
\leq \|\partial^\gamma \vartheta[\sigma, \rho] \circ \eta[\sigma, \rho] (\varphi u)\|_u + \sum_{\gamma = \beta + \alpha, |\beta| > 0} l_{\beta, \alpha} \|\partial^\gamma \eta[\sigma, \rho] \circ \varphi (u)\|_u \|\partial^\gamma \vartheta[\sigma, \rho] \circ \eta[\sigma, \rho] (\varphi u)\|_u.
\]

where each occurrence of \(\|\|_u\) refers to the norm on the appropriate vector space among \(L(E, F), L(E, G)\) and \(L(F, G)\) corresponding to \([p, q]_u\) resp. \([p, r]_u\) resp. \([q, r]_u\) under the linear bijection \([\tau, \sigma]_u\) resp. \([\tau, \rho]_u\) resp. \([\sigma, \rho]_u\).

To conclude, let us choose standard normed atlases \(\mathcal{A} = \{(\varphi_i, \tau_i, p_i) \mid i \in \mathcal{J}\}, \mathcal{B} = \{\psi_j, \sigma_j, q_j\} \mid j \in \mathcal{J}'\}\) and \(\mathcal{C} = \{\chi_k, \rho_k, r_k\} \mid k \in \mathcal{K}\}\) respectively for \(E, F\) and \(G\) over \(\Omega\). It is no loss of generality to assume that \(\mathcal{J} = \mathcal{K}\) and that \(\varphi_i = \psi_i = \chi_i\) for all \(i \in \mathcal{J}\). (To achieve this, we simply select any finite family of standard coordinate patches for \(X\) covering \(\Omega\) with their open unit balls and subordinate to the open cover of \(\Omega\) given by the triple intersections \(U_i \cap V_j \cap W_k \mid i, j, k\) where \(U_i, V_j\) and \(W_k\) denote the domains of \(\varphi, \psi_j\) and \(\chi_k\), respectively.) Over \(\overline{\Omega}\) there will be induced standard normed atlases \([A, B]\) for \((L(E, F))\), \([B, C]\) \(\ldots\) for \((L(F, G))\) and so forth. If we pick the standard \(C^1\) norms associated to these standard normed atlases we immediately deduce the desired norm estimates (64) from (67).
Lemma A.9. Let $E$ and $F$ be $\mathbb{K}$-linear differentiable vector bundles over a given differentiable manifold $X$, and let $\Omega \subset X$ be a relatively compact open set. Then, for $\eta$ a variable ranging over $\Gamma^{k+1}(\Omega; \text{Lis}(E, F))$, we have:

$$
||\eta^{-1}||_{C^{k+1}} \leq (||\eta^{-1}||_{C^k})^2||\eta||_{C^{k+1}}. 
$$  

(68)

Proof. Observe that if $f : V \rightarrow \text{Lis}(\mathbb{E}, \mathbb{F})$ is any mapping of class $C^1$ defined on some open subset $V \subset \mathbb{R}^n$ of euclidean $n$-space with values in the set of invertible linear maps between two finite-dimensional vector spaces $\mathbb{E}$ and $\mathbb{F}$ over $\mathbb{K}$ then for every $j = 1, \ldots, n$ letting $f^{-1}$ (by abuse of notation) denote the $C^1$ mapping of $V$ into $\text{Lis}(\mathbb{F}, \mathbb{E})$ given by $y \mapsto f(y)^{-1}$ we have

$$
\partial_j(f^{-1}) = -f^{-1} \circ \partial_j f \circ f^{-1}. 
$$  

(69)

For any given multi-index $\gamma \in \mathbb{N}^n$ of positive order $|\gamma|$ not greater than the order of differentiability of $f$, by making repeated use of the equation (65) we deduce an identity of the following kind from (69):

$$
\partial^\gamma (f^{-1}) = - \sum_{\gamma = \beta + \sigma, \alpha > 0} \sum_{\beta = \beta_1 + \beta_2} l_{\beta_1, \beta_2} \partial_\beta^\gamma (f^{-1}) \circ \partial^\alpha f \circ \partial_\beta^\gamma (f^{-1}) 
$$  

(70)

(the coefficients $l_{\beta_1, \beta_2}$ being appropriate non-negative integer constants).

Let $(\varphi, \tau, p)$ and $(\varphi, \sigma, q)$ be arbitrary standard normed charts respectively for $E$ and $F$ defined over the same coordinate patch $\varphi : U \subset \mathbb{R}^n$. In the notations of the preceding proof, for any given multi-index $\gamma \in \mathbb{N}^n$ of order $|\gamma| > k + 1$ the equation (70) implies the following inequality, which is supposed to be valid for all $u \in \Omega \cap U$ for any cross-section $\eta \in \Gamma^{k+1}(\Omega; \text{Lis}(E, F))$:

$$
||\partial^\gamma (\eta^{-1})^{(r, r)} \varphi (\varphi u)||_a 
\leq \sum_{\gamma = \beta + \sigma, \alpha > 0} \sum_{\beta = \beta_1 + \beta_2} l_{\beta_1, \beta_2} ||\partial_\beta^\gamma ((\eta^{-1})^{(r, r)} \varphi )^{(\varphi u)}||_a 
\times ||\partial_\beta^\gamma ((\eta^{-1})^{(r, r)} \varphi )^{(\varphi u)}||_a 
\times ||\partial^\alpha \eta^{(r, r)} \varphi (\varphi u)||_a. 
$$  

(71)

(The case $|\gamma| = 0$ must be handled separately):

$$
||(\eta^{-1})^{(r, r)} \varphi (\varphi u)||_a = ||(\eta^{-1})^{(r, r)} \varphi (\varphi u) \circ \eta^{(r, r)} \varphi (\varphi u) \circ \eta^{(r, r)} \varphi (\varphi u)^{-1}||_a 
\leq ||(\eta^{-1})^{(r, r)} \varphi (\varphi u)||_a^2 ||\eta^{(r, r)} \varphi (\varphi u)||_a.
$$  

From this point on, our argument substantially follows the same pattern as in the proof of Lemma A.8.

\[\square\]

Appendix B. Haar integrals depending on parameters

References for part of the material covered in the present appendix include [12], [16] and [2].

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B.I. Integration of densities along the fibers of a submersion. Let $E$ be an arbitrary real differentiable vector bundle over a differentiable manifold $X$. For any real number $s > 0$ the $s$-density bundle on $E$, denoted by $\Delta^s E$ hereafter, is the differentiable real line bundle over $X$ constructed as follows. For each point $x \in X$ let $r(x) = \text{rk}_E(x)$ indicate the rank of $E$ at $x$. Define $\Delta^s E_x$ to be the (real) vector space formed by all the functions $h : \wedge^r(x) E_x \to \mathbb{R}$ such that $h(tw) = |t|^r h(w)$ for every $t \in \mathbb{R}$ and $w \in \wedge^r(x) E_x$. Set $\Delta^s E := \bigsqcup_{x \in X} \Delta^s E_x$ (disjoint union). By definition, a $C^\infty$ local trivialization for $\Delta^s E$ over the domain of definition $U$ of an arbitrary $C^\infty$ local trivializing frame $\xi = (\xi_1, \ldots, \xi_s)$ for $E$ is provided by the mapping

$$\bigcup_{u \in U} \Delta^s E_u \ni (u, h) \mapsto (u, \langle h, \xi_1(u) \wedge \cdots \wedge \xi_s(u) \rangle) \in U \times \mathbb{R}.$$ 

By an $s$-density of class $C^k$ on $E$ we shall mean an arbitrary global section of class $C^k$ of the differentiable line bundle $\Delta^s E$.

Suppose we are given a $C^\infty$ mapping $\phi : E' \to E$ between two real differentiable vector bundles $E'$ and $E$ which sends fibers into fibers, thus inducing a differentiable mapping say $f : X' \to X$ of the base $X'$ of $E'$ into the base $X$ of $E$, and which for each point $x' \in X'$ sets up a linear bijection $\phi_{x'} : E'_{x'} \cong E_{f(x')}$ between the fiber of $E'$ at $x'$ and the corresponding fiber of $E$. Of course we may interpret $\phi$ as a vector-bundle isomorphism between $E'$ and $f^* E$ (the pullback of $E$ along $f$) although the latter point of view is less convenient when coming to notations. For any given $s$-density $\delta \in \Gamma^s(X; \Delta^s E)$ of class $C^k$ on $E$ we have a new $s$-density on $E'$, which we call the inverse image of $\delta$ under $\phi$ and indicate by $\phi^* \delta$, also of class $C^k$, given by

$$(\phi^* \delta)(x') := \delta(f(x')) \circ \wedge^r(x') \phi_{x'}.$$ 

In the special case when $\phi$ is the canonical projection from a pullback $f^* E \to E$ we normally write $f^* \delta$ for the inverse image under $\phi$ of a density $\delta$ and refer to it as the pullback of $\delta$ along $f$.

If $\gamma$ is a Riemannian metric of class $C^\infty$ on a real differentiable vector bundle $E$, there exists on $E$ a unique 1-density $\delta$ of class $C^\infty$ with the property that $\delta(x)(e_1 \wedge \cdots \wedge e_s) = 1$ for every base point $x$ of $E$ and for every $\gamma$-orthonormal vector-space basis $\{e_1, \ldots, e_s\} \subset E_x$. We refer to such $\delta$ as the volume density associated to the metric $\gamma$ and adopt for it the notation $\text{Vol}_\gamma$. Note that for $\phi : E' \to E$ as in the preceding paragraph we have $\phi^* \text{Vol}_\gamma = \text{Vol}_{\phi^* \gamma}$ (where $\phi^* \gamma$ denotes the inverse image metric). In general, we call volume density on $E$ any 1-density $\delta$ of class $C^\infty$ on $E$ having the property that $\delta(x)w > 0$ for all $x$ and for all non-zero $w \in \wedge^r(x) E_x$. In the sequel we will only deal with volume densities. It will be convenient to simply write $\Delta E$ in place of $\Delta^1 E$ and to abbreviate $\Delta TX$ into $\Delta X$ for any differentiable manifold $X$. We shall also speak about volume densities “on $X$” really meaning “on the tangent bundle of $X$”.

Let $\phi : Z \to Y$ be a surjective submersion between two Hausdorff manifolds. Let $T^{\phi} Z$ denote the $\phi$-vertical tangent bundle of $Z$, that is, the differentiable subbundle of $TZ$ given (as $z$ varies over $Z$) by $T^\phi_z Z := \ker(T_{\phi^* Z}, T^\phi_z Y)$. By a volume density along the fibers of $\phi$ or a $\phi$-vertical volume density we mean a volume density on the $\phi$-vertical tangent bundle of $Z$. We say that a subset $S$ of $Z$ is $\phi$-properly located if the restriction $\phi|S : S \to Y$ is a proper mapping (the inverse images of compact sets
Let $Y$ and $Z$ be Hausdorff differentiable manifolds, and let $\phi : Z \to Y$ be a surjective and submersive differentiable mapping. We say that a function $f$ supported in $Z$ is \textit{$\phi$-properly supported} whenever its support $\text{supp}_{Z} f$ is $\phi$-properly located. Of course, when $\phi$ is fixed throughout a discussion we may abbreviate $T\uparrow z Z$ into $T\uparrow z Z$ and refer to this simply as the vertical tangent bundle of $Z$; similarly for the remaining terminology just introduced.

There is a pairing, which we shall call \textit{integration along the fiber}, between the continuous functions on $Z$ with properly located support and the vertical volume densities on $Z$. Namely, let $f \in C(Z, \mathbb{C})$ be any such function and let $\delta \in \Gamma^\infty(Z; \Delta T\uparrow z Z)$ be any such density. For each point $y$ in $Y$ we have a canonical isomorphism of differentiable line bundles over the fiber $\phi^{-1}(y)$

$$
(\Delta T\uparrow z Z) \mid \phi^{-1}(y) \cong \Delta \phi^{-1}(y)
$$

induced at any $z \in \phi^{-1}(y)$ by the canonical identification of vector spaces $T_z \phi^{-1}(y) = T_z^\uparrow z Z$. Thus, we may regard the restriction $\delta \mid \phi^{-1}(y)$ as an ordinary volume density on $\phi^{-1}(y)$, which we indicate shortly by $\delta_y$. Then, letting $\mu_y$ stand for the positive Radon measure on $\phi^{-1}(y)$ attached to $\delta_y$ in the ordinary way (i.e. through the localization theorem for positive functionals and the change of variables formula as in [8, p. 451]), we obtain a function on $Y$, denoted by $\int f \delta$, given by

$$
Y \ni y \mapsto \int_{\phi^{-1}(y)} f \mid \phi^{-1}(y) \, d\mu_y \in \mathbb{C}
$$

where $f \mid \phi^{-1}(y) \in C_c(\phi^{-1}(y), \mathbb{C})$ indicates the restriction of $f$ to $\phi^{-1}(y) \subset Z$.

**Lemma B.1.** Let $Y$ and $Z$ be Hausdorff differentiable manifolds, and let $\phi : Z \to Y$ be a surjective and submersive differentiable mapping. Let $\delta \in \Gamma^\infty(Z; \Delta T\uparrow z Z)$ be an arbitrary volume density along the fibers of $\phi$. For every natural number $k$, the following statements are true.

(a) If $f$ is any properly supported $C^k$ function on $Z$, the function $\int f \delta$ is also of class $C^k$.

(b) For each properly located subset $S \subset Z$, the operation of integration along the fiber, $f \mapsto \int f \delta$, gives rise to a $C^k$-continuous linear map $C^k_Z(Z, \mathbb{C}) \to C^k(Y, \mathbb{C})$; here $C^k_Z(Z, \mathbb{C}) \subset C^k(Z, \mathbb{C})$ denotes the closed subspace formed by all those functions $f$ such that $\text{supp}_Z f \subset S$.

**Proof.** To begin with, we prove the lemma in the special case when $Y = U$ is an open subset of $\mathbb{R}^n$, $\phi$ is the projection from a product $\mathbb{R}^n \times U = Z$ onto the 2nd factor $U \subset \mathbb{R}^n$ and $S$ is a subset of $Z$ of the form $K \times U$ where $K$ is a compact set in $\mathbb{R}^m$. We let $x = (x_1, \ldots, x_m)$ denote the coordinates in $\mathbb{R}^m$ and $y = (y_1, \ldots, y_n)$ those in $\mathbb{R}^n$. In the situation just described our density $\delta$ can be written as

$$
\delta(x, y) = r(x, y) \, dx = r(x, y) \, dx_1 \cdots dx_m
$$

where $r$ is some positive function of class $C^\infty$. By definition, for all $y \in U$ we have $(\int f \delta)(y) = \int_{\mathbb{R}^m} f(x, y) r(x, y) \, dx$ (integration with respect to Lebesgue measure), $f \in C^k(\mathbb{R}^m \times U, \mathbb{C})$ being an arbitrary properly supported function of class $C^k$. Differentiation
under the integral sign immediately shows that \( \int f \delta \) belongs to \( C^k(U, \mathbb{C}) \). By the same token, whenever \( f \in C^k_s(\mathbb{R}^m \times U, \mathbb{C}) \) it follows for any compact set \( L \subset U \) and multi-index \( \alpha \in \mathbb{N}^n \) of order \( |\alpha| \leq k \) that

\[
\sup_{y \in L} |\partial^\alpha \bigl( \int f \delta \bigr)(y)| = \sup_{y \in L} \int_{\mathbb{R}^m} D^\alpha(f r)(x, y) \, dx \\
\leq \text{meas}(K) \cdot \sup_{(x,y) \in K \times L} |\partial^0(\alpha)(f r)(x, y)| \\
\leq \text{meas}(K) \cdot p_k^{K \times L}(f r).
\]

Thus, since the linear map \( f \mapsto f r \) of \( C^k_s(\mathbb{R}^m \times U, \mathbb{C}) \) into itself is \( C^k \)-continuous by Lemma A.6, we are also done with the continuity statement (b) in the special case considered before since the map in question will be a compact subset of \( Z \) because \( S \) is properly located. We may cover \( \phi^{-1}(A) \cap S \) with a finite family of local trivializing charts for \( \phi \) of the form

\[
 Z \supset \text{open } W_i \rightleftharpoons \mathbb{R}_m \times \mathbb{R}_n \\
 V \supset \text{open } V_i \rightleftharpoons \mathbb{R}_n
\]

Let us fix some \( C^\infty \) partition of unity \( \{ g_j \} \) over \( W = \bigcup_{i \in I} W_i \) subordinated to \( \{ W_i \} \) with compact supports \( \text{supp } g_j \subset W_{i(j)} \). We may find a finite set \( J \) of \( j \)-indices so that \( \sum_{j \in J} g_j = 1 \) in a neighborhood of \( \phi^{-1}(A) \cap S \). For each \( i \in I \) let us put \( g_i = \sum_{j \in J; j(i) = i} g_j \in C^\infty_c(W_i) \). Of course we will have \( \sum_{i \in I} g_i = 1 \) in a neighborhood of \( \phi^{-1}(A) \cap S \). Let us identify \( g_i \) notationally with its extension by zero to all of \( Z \). We contend that for each \( i \in I \) the correspondence \( f \mapsto \int g_i f \delta \) gives rise to a \( C^k \)-continuous linear map of \( C^k(Z, \mathbb{C}) \) into \( C^k(Y, \mathbb{C}) \). This is a straightforward consequence of the two lemmas A.6 and A.7 and of the special case considered before since the map in question can be decomposed into

\[
 C^k(Z, \mathbb{C}) \xrightarrow{\text{supp } g_i} C^k_{\text{supp } g_i}(W_i, \mathbb{C}) \xrightarrow{\int f \delta} C^k_{\phi(\text{supp } g_i)}(V_i, \mathbb{C}) \xrightarrow{j_0} C^k(Y, \mathbb{C})
\]

where \( j_0 \) indicates extension by zero. Now, since for every function \( f \in C^k_s(Z, \mathbb{C}) \) we have \( \int f \delta = \int \sum_i g_i f \delta \) near \( A \), from the inequality

\[
p^k_\phi (\int f \delta) = p^k_\phi A \left( \int \sum_i g_i f \delta \right) \leq \sum_i p^k_\phi (\int g_i f \delta)
\]

and from the \( C^k \)-continuity of the maps \( f \mapsto \int g_i f \delta \) we conclude that the map \( f \mapsto \int f \delta \) must be continuous on \( C^k_s(Z, \mathbb{C}) \) with respect to the given seminorm \( p^k_\phi \) on \( C^k(Y, \mathbb{C}) \).

\[ \square \]

**Lemma B.2.** Let \( \phi : Z \to Y \) and \( \delta \) be as in the statement of Lemma B.7 and assume that \( \phi \) is a proper mapping. Let \( \Omega \subset Y \) be a relatively compact, open, subset. Then, the operation of integration along the fiber gives rise to a continuous linear map \( \Gamma^k(\phi^{-1}(\Omega); \mathcal{C}_Z) \to \Gamma^k(\Omega; \mathcal{C}_Y) \) (both spaces of sections being given the \( C^k \)-norm topology).

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The proof is entirely analogous to that of the preceding lemma. Notice that since 
\( \phi \) is an open mapping we must have \( \phi^{-1}(\overline{D}) = \phi^{-1}(\overline{D}) \). Since \( Z \) is Hausdorff and \( \phi \) is proper, any given function \( f \in \Gamma^k(\phi^{-1}(\overline{D}); \mathbb{C}_Z) \) will admit some local \( C^k \) extension around each fiber \( \phi^{-1}(y) \) \((y \in \overline{D})\), and the domain of definition of any such extension will contain a whole open tube of the form \( \phi^{-1}(V) \) (\( V \) open \( y \)).

**B.II. Haar systems on differentiable groupoids.**

**Definition B.3.** Let \( \Gamma \rightrightarrows X \) be a Hausdorff differentiable groupoid. A **left Haar system** on \( \Gamma \rightrightarrows X \) is a family \( \mu = \{\mu_x\}_{x \in X} \) whose members \( \mu_x \) are positive Radon measures on the target fibers \( \Gamma_x = \tau^{-1}(x) \) which satisfies the following three conditions.

(a) **(Differentiability.)** There exists some volume density along the target fibers, call it \( \delta \), such that for each base point \( x \in X \) the measure \( \mu_x \) coincides with the positive Radon measure on \( \Gamma_x \) associated to the volume density \( \delta_x := \delta \rvert_{\Gamma^1(x)} \).

(b) **(Left invariance.)** For each arrow \( g \in \Gamma \) one has
\[
\mu_{tg}(gA) = \mu_{tg}(A)
\]
for all Borel subsets \( A \) of the target fiber \( \Gamma_{tg} \), where \( gA = \{gh \mid h \in A\} \).

(c) **(Definiteness.)** \( \mu_x(U) > 0 \) for every non-empty open subset \( U \subset \Gamma_x \).

**Comments.** Haar systems on differentiable groups are the same thing as Haar measures, as defined e.g. in [8, p. 351]. The condition (c) is of course redundant, since it is implied by (a). There is an alternative formulation of the left invariance condition (b) which is often more useful in practice, namely let \( L_gf \) denote, for an arbitrary function \( f \) on \( \Gamma_{tg} \), the left translate of \( f \) by \( g \), that is the function on \( \Gamma_{tg} \) given by \( (L_gf)(h) = f(g^{-1}h) \):

(b') For any arrow \( g \in \Gamma \) the left translate \( L_gf \) of an arbitrary function \( f \in \mathcal{L}^1(\mu_{tg}, \mathbb{C}) \) is an element of \( \mathcal{L}^1(\mu_{tg}, \mathbb{C}) \) and one has
\[
\int_{\Gamma_{tg}} L_gf \, d\mu_{tg} = \int_{\Gamma_{tg}} f \, d\mu_{tg}.
\]

**Proposition B.4.** Let \( \Gamma \rightrightarrows X \) be an arbitrary Hausdorff differentiable groupoid based on a second countable manifold \( X \). Then \( \Gamma \rightrightarrows X \) admits a left Haar system.

**Proof.** It will be more natural for us to construct a right Haar system (in the obvious sense). Any right Haar system can be turned into a left Haar system by means of a groupoid inversion mapping. We leave the straightforward details to the reader.

As in Section 2 we let \( \mathfrak{g} = T^\perp \Gamma \) denote the algebroid bundle of \( \Gamma \). Since \( X \) is by assumption a second countable Hausdorff manifold, it admits \( C^\infty \) partitions of unity. As a consequence, the vector bundle \( \mathfrak{g} \) (over \( X \)) will admit some \( C^\infty \) Riemannian metric say \( \gamma \). Letting \( \omega : T^\dagger \Gamma \rightrightarrows T^\dagger \mathfrak{g} \) denote the Maurer–Cartan isomorphism associated to \( \Gamma \), the inverse image \( \omega^\dagger \text{Vol}_{r_{\mathfrak{g}}} \) will be a volume density along the source fibers whose corresponding system of positive Radon measures will constitute a right Haar system on \( \Gamma \rightrightarrows X \).
B.III. Normalizing functions.

Definition B.5. Let $\Gamma \rightrightarrows X$ be a Hausdorff differentiable groupoid and let $\mu = \{\mu_x\}$ be a left Haar system on $\Gamma \rightrightarrows X$. A normalizing (or cut-off) function for $\mu$ is a non-negative function $\kappa : X \to [0, +\infty)$ of class $C^\infty$ with the following two properties.

(a) The composite function $\kappa \circ s^f : \Gamma \to [0, +\infty)$ is $t^f$-properly supported.

(b) $\int \kappa \circ s^f \, d\mu = 1$ (= constant function of value 1 on $X$).

We call the pair $\nu = (\mu, \kappa)$ a normalizing left Haar system on $\Gamma$.

Let $\nu = (\mu, \kappa)$ be a normalizing left Haar system on a given Hausdorff differentiable groupoid $\Gamma \rightrightarrows X$. For each base point $x \in X$ the function $(\kappa \circ s)_x = (\kappa \circ s^f) \big| \Gamma_x$ belongs to $C_c(\Gamma_x, \mathbb{C}) \subseteq \mathcal{L}^1(\mu_x, \mathbb{C})$. Let us consider the finite positive Radon measure $\nu_x$ on $\Gamma_x$ that by Riesz’ theorem corresponds to the bounded positive functional on $C_c(\Gamma_x, \mathbb{C})$ given by $g \mapsto \int g(\kappa \circ s)_x \, d\mu_x$. For every Borel measurable function $f$ on $\Gamma_x$ we have $f \in \mathcal{L}^1(\nu_x, \mathbb{C}) \iff f(\kappa \circ s)_x \in \mathcal{L}^1(\mu_x, \mathbb{C})$ and in that case $\int f \, d\nu_x = \int f(\kappa \circ s)_x \, d\mu_x$. It is then clear that $C(\Gamma_x, \mathbb{C}) \subset \mathcal{L}^1(\nu_x, \mathbb{C})$. By the same token we see that the system of measures on the target fibers $\{\nu_x\}$ is left invariant. By abuse of language and of notation we shall also refer to $\{\nu_x\}$ as a normalizing left Haar system and write $\nu = \{\nu_x\}$.

Example. Suppose you are given a Hausdorff differentiable groupoid $\Gamma \rightrightarrows X$ whose target mapping $t^f : \Gamma \to X$ is proper (the inverse image of each compact set is compact; this will be the case, e.g., whenever $\Gamma$ is a compact manifold). Let $\mu = \{\mu_x\}$ be a left Haar system on $\Gamma \rightrightarrows X$. Then by setting $\kappa(x) = 1 \int 1 \, d\mu_x$ for all $x \in X$ you get a normalizing function $\kappa$ for $\mu$. Notice that $\kappa \circ s^f = \kappa \circ t^f$ in consequence of the left invariance of $\mu$. The normalized system $\nu = \{\nu_x\}$ will be itself a left Haar system and, moreover, a probability system in the sense that $\nu_x(\Gamma_x) = 1$ for all $x$.

Recall that a differentiable groupoid $\Gamma \rightrightarrows X$ is said to be proper whenever it is Hausdorff and its combined source–target mapping $(s, t) : \Gamma \to X \times X$, $g \mapsto (sg, tg)$ is proper.

Proposition B.6. Let $\Gamma \rightrightarrows X$ be a proper differentiable groupoid. Assume that its base manifold $X$ is second countable. Then, every Haar system on $\Gamma \rightrightarrows X$ admits normalizing functions.

Proof. Let $X/\Gamma$ denote the orbit space of $\Gamma$. As a set, this is the quotient of $X$ by the equivalence relation $x \equiv y \Leftrightarrow \exists g \in \Gamma (sg = x \; \& \; tg = y)$. Its topology is the finest making the quotient projection $\pi : X \to X/\Gamma$ continuous. It is evident that $\pi$ is an open mapping and hence that $X/\Gamma$ is a locally compact space. The properness of $\Gamma$ implies that $X/\Gamma$ is Hausdorff. The second countability of $X$ implies that $X/\Gamma$ is second countable.

Fix any sequence of compact sets $A_0 = \emptyset, A_1, A_2, \ldots$ in $X/\Gamma$ such that $A_i \subset \text{Int}(A_{i+1})$ for all $i$ and such that $X/\Gamma = \bigcup_{i=0}^\infty A_i$. (Compare [8] proof of Theorem 11 of Ch. XII, p. 341.) For each $i \geq 1$ set $U_i = \pi^{-1}(\text{Int}(A_{i+1}) \setminus A_{i-1})$. Since the manifold $X$ is second countable and Hausdorff, there will exist over it a $C^\infty$ partition of unity $\{V_j, \psi_j\}$ with compact supports subordinated to the open cover $\{U_i\}$. Put $W_j = \{x \in X | \psi_j(x) > 0\}$. For each non-negative integer $i$ it must be possible to find a finite set say $J(i)$ of $j$-indices...
so that the corresponding open sets \( \pi(W_j) \) already cover the compact set \( A_{i+1} \setminus \text{Int}(A_i) \) and each of them intersects it non-vacuously. Set \( J = \bigcup_{j=0}^{\infty} J(i) \). Clearly, \( \{\pi(V_j)\}_{j \in J} \) will be a locally finite open cover of \( X/\Gamma \). Hence, each point \( x \in X \) will possess some invariant open neighborhood intersecting only finitely many among the open sets \( V_j \) with \( j \in J \). If we set \( \chi = \sum_{j \in J} \psi_j \) we will have that \( \chi \circ s^f \) is \( t^f \)-properly supported and non-zero along every \( t^f \)-fiber. Now, let \( \mu = \{\mu_i\} \) be an arbitrary left Haar system on \( \Gamma \rightrightarrows X \). We must have \( \int \chi \circ s^f \ d\mu > 0 \) (everywhere as a function on \( X \)). It is straightforward to check that

\[
\kappa = \left( 1 \int \chi \circ s^f \ d\mu \right) \chi
\]

must then be a normalizing function for \( \mu \). \( \square \)

**B.IV. Dependence on global parameters under the integral sign.**

**Lemma B.7.** Let \( \Gamma \rightrightarrows X \) be a differentiable groupoid which is Hausdorff and second countable and on which a normalized left Haar system \( \nu = (\mu, \kappa) \) is assigned. Let \( f : P \to X \) be an arbitrary differentiable mapping from some “space of parameters” \( P \) into the base manifold \( X \) of the groupoid. Let \( E \) be an arbitrary \( \mathbb{K} \)-linear differentiable vector bundle over \( P \). Then, letting \( pr_p \) denote the projection on the first factor \( P \times \Gamma \), \( \Gamma \to P, (y, h) \mapsto y \) from the fiber product \( P \times \Gamma \), \( \Gamma = \{(y, h) \in P \times \Gamma \mid f(y) = \theta\} \), for any given order of differentiability \( k = 0, 1, 2, \ldots, \infty \) each global \( C^k \) cross-section \( \vartheta \in \Gamma^k(P \times \Gamma; pr_p^*E) \) of the pullback vector bundle \( pr_p^*E \) can be turned into a global cross-section of class \( C^k \) of \( E \), denoted by \( \int \vartheta \ d\nu \), obtained by integrating \( \vartheta \) along the target fibers:

\[
P \ni y \mapsto \left( \int \vartheta \ d\nu \right)(y) \overset{\text{def}}{=} \int_{h = f(y)} \vartheta(y, h) \ d\nu_{f(y)}(h) \in E_y \quad (71a)
\]

(the integrand being a vector-valued continuous function on the target fiber \( \Gamma_{f(y)} \) with values in the finite-dimensional vector space \( E_y \)). The resulting integration functional, which we denote by

\[
d^\vartheta_{f,E} : \Gamma^k(P \times \Gamma; pr_p^*E) \longrightarrow \Gamma^k(P, E), \ \vartheta \mapsto \langle \vartheta, d^\vartheta_{f,E} \rangle := \int \vartheta \ d\nu, \quad (71b)
\]

is a \( C^k \)-continuous linear map.

**Proof.** Let \( e_1, \ldots, e_N \) be any local trivializing frame of \( C^\infty \) sections for the differentiable vector bundle \( E \) defined over the domain \( V \) of some local coordinate chart \( \psi : V \overset{\sim}{\to} \psi V = \mathbb{R}^p \) for the differentiable manifold \( P \) and let \( \tau : E \mid V \overset{\sim}{\to} V \times \mathbb{K}^N \) denote the corresponding vector-bundle trivialization. The pullback sections \( pr^*e_1, \ldots, pr^*e_N \) (where \( pr = pr_p \)) form a local trivializing frame of class \( C^\infty \) for \( pr^*E \) over \( V' = pr^{-1}(V) = V \times \Gamma \). We let \( pr^*\tau \) denote the corresponding vector-bundle trivialization of \( pr^*E \) over \( V' \). For an arbitrary cross-section \( \vartheta \in \Gamma(P \times \Gamma; pr^*E) \) we have

\[
\vartheta \mid V' = \sum_{i=1}^N \vartheta_i pr^*e_i,
\]

53
where the functions $\vartheta^{pr\tau}_I : V' \to \mathbb{K}$ are uniquely determined. Evidently, $\vartheta \mid V'$ is of class $C^k$ if, and only if, $\vartheta^{pr\tau}_I \in C^k(V', \mathbb{K})$ for all $I = 1, \ldots, N$. By definition of what we mean by “integration of a vector-valued function”, we have for all $y \in V$

\[
\left( \int \vartheta \, dv \right)(y) = \sum_{I=1}^{N} \left[ \int_{t_{h=f(y)}} \vartheta^{pr\tau}_I(y, h) \, dv_{f(y)}(h) \right] e_I(y)
\]

and, therefore,

\[
\left( \int \vartheta \, dv \right)_I = \int_{t_{h=f(-)}} \vartheta^{pr\tau}_I(-, h) \, dv_{f(-)}(h) = \int \vartheta^{pr\tau}_I \, dv \quad [I = 1, \ldots, N].
\]

Henceforth, for every compact set $K \subset V$ and for every integer $r \in \mathbb{N}_g$:

\[
p^{r,K}_\vartheta \left( \int \vartheta \, dv \right) = \max_{I=1,\ldots,N} p^{r,K}_I \left( \left( \int \vartheta \, dv \right)_I \right) = \max_{I=1,\ldots,N} p^{r,K}_I \left( \int \vartheta^{pr\tau}_I \, dv \right).
\]

We are thus reduced to the case when $E = \mathbb{C}_p$ is a trivial line bundle over $P$ (“trivial coefficients”). We may—and will—assume that we are dealing with complex valued functions, rather than with sections of an arbitrary vector bundle. (Of course, the statement for real valued functions will follow from that for complex valued functions.)

Let $\delta \in \Gamma^\omega(\Gamma; \Delta T^1 \Gamma)$ be any volume density along the target fibers whose associated system of positive Radon measures on the target fibers is $\mu$. Observe that the projection $pr_p : P \times \Gamma \to P$ is a surjective submersion because it is the pullback of a surjective submersion. Also notice that the tangent mapping $T(pr_p) : T(P \times \Gamma) \to T\Gamma$ of the other projection $pr_\Gamma : P \times \Gamma \to \Gamma$ induces a vector-bundle isomorphism $T |_{pr_p} (P \times \Gamma) \cong pr_\Gamma^* (T\Gamma)$ so that the pullback $pr_\Gamma^* \delta$ can be seen as a volume density along the fibers of $pr_p$. Now by construction the integration functional (71b) (in the case of trivial coefficients $E = \mathbb{C}_p$) is given by the composition of the following two linear maps

\[
C^k(P \times \Gamma, \mathbb{C}) \xrightarrow{(\omega^{sf \circ pr_p})^{-}} C^k(\supp(\omega^{sf \circ pr_p}) (P \times \Gamma, \mathbb{C}) \xrightarrow{\int (-) pr_\Gamma^* \delta} C^k(P, \mathbb{C})
\]

which are already known to be continuous (by Lemmas [A.6] and [B.1] respectively). \(\square\)

The following three statements are equivalent for any proper differentiable groupoid $\Gamma \Rightarrow X$:

i) Each point in the base $X$ admits a relatively compact $\Gamma$-invariant open neighborhood.

ii) $\Gamma K$ is compact for every compact set $K \subset X$.

iii) The source (equivalently, the target) mapping of $\Gamma$ is proper.

[The equivalence of the statements ii) and iii) is an immediate consequence of the two identities $\Gamma R = t(s^{-1}(R))$ and $s^{-1}(R) = (s, t)^{-1}(R \times \Gamma R)$, which hold for every subset $R \subset X$. The statements i) and ii) are equivalent in virtue of the equality $\overline{FU} = \overline{FU}$, which is valid for any relatively compact (open) subset $U$ of $X$ in consequence of properness.] A proper differentiable groupoid satisfying these conditions is commonly called source proper in the literature.
Lemma B.8. Let $\Gamma \Rightarrow X$ be a proper differentiable groupoid whose target mapping is proper. Let $f : P \to X$ and $E$ be as in the statement of Lemma B.7, and let $\Omega \subset P$ be an arbitrary relatively compact open subset. Then for any left invariant Haar probability system $\nu = \{ \nu_x \}$ on $\Gamma$ and for any given order of differentiability $k = 0, 1, 2, \ldots, \infty$ the operation of integration along the target fibers (71a) gives rise to an integration functional
\[
\int_{\Omega}^k f \nu \in \Gamma^k(\Omega) \times \Gamma^k \times \Gamma^{\nu} \times \Gamma^{\nu}, \theta \mapsto \int \theta \ d\nu
\] (72)
which is a continuous linear map for the $C^k$-norm topology on $\Gamma^k(\Omega) \times \Gamma^k \times \Gamma^{\nu} \times \Gamma^{\nu}$ and the $C^k$-norm topology on $\Gamma^k(\Omega)$.

The proof is based on Lemma B.2 in exactly the same way as the proof of Lemma B.7 is based on Lemma B.1. Notice that the projection $pr_P : P \times \Gamma \to P$ must now be a proper mapping because so was by hypothesis the target mapping of $\Gamma$. Also notice that $\Omega f \times_\Gamma \Gamma = pr_P^{-1}(\Omega) = pr_P^{-1}(\Omega)$ because as observed before $pr_P$ is an open mapping. Finally be aware that in the statement of the lemma we have taken $\nu$ to be a left invariant Haar probability system for concreteness but of course we might as well have taken it to be an arbitrary normalized left Haar system, with no loss of consistency.

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