Bounds on the Size of Sound Monotone Switching Networks Accepting Permutation Sets of Directed Trees

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Abstract

In this paper, we prove almost tight bounds on the size of sound monotone switching networks accepting permutations sets of directed trees. This roughly corresponds to proving almost tight bounds on the monotone memory efficiency of the directed ST-connectivity problem for the special case in which the input graph is guaranteed to have no path from \textit{s} to \textit{t} or be isomorphic to a specific directed tree.
1 Introduction

One long-standing open problem in computational complexity theory is the minimal space complexity of the ST-connectivity problem. The ST-connectivity problem is formulated as follows: given a directed graph $G$ with starting and ending vertices $s$ and $t$, is there a directed path from $s$ to $t$? This inquiry is not difficult to answer; the challenge is to answer this question with the minimal amount of space necessary. In a celebrated result by Savitch [7], it was shown that one can answer this problem on an $n$-vertex graph with $O((\log n)^2)$ space. Reingold [6] showed that if $G$ is an undirected graph, then only $O(\log n)$ space is required.

One type of computation which gives insight into this problem is monotone computation. This kind of computation operates by making deductions from the existence of edges; it does not make deductions from the absence of certain edges. We analyze the ST-connectivity problem using a structure called a monotone switching network. Defined more precisely in Section 2, a monotone switching network is an undirected graph with labeled edges based on queries that a program may make about the existence of edges in the input graph. We can think of the vertices of this network as representing possible memory states of a program.

Finding bounds on the size of the monotone switching network roughly corresponds to finding bounds on the amount of space needed to compute ST-connectivity in a monotone computation model. Potechin [2] has shown that in the general case, a monotone switching network needs a size of $n^{\Theta(\log n)}$, which corresponds to needing $\Theta((\log n)^2)$ space. However, finding lower bounds on the size of monotone switching networks does not give us lower bounds on the amount of space needed but does tell us the limits of monotone computation. To obtain general lower bounds on the amount of memory needed, one must analyze a broader class of switching networks, the non-monotone switching networks [2].

We determine bounds on the sizes of monotone switching networks for special cases of the ST-connectivity problem. In these special cases, we assume that the input graph is isomorphic to a given graph via permutation of the vertices. For example the results in Theorem 5.3 concern the case where every vertex in the given graph has a unique path from $s$ to itself.

In our main result, Theorem 5.1 bounds are found in the case of a general tree. If we define $m(\sigma(G))$ to be the size of the monotone switching network and $\ell$ to be the length of the path from $s$ to $t$, we found upper and lower bounds $B_1$ and $B_2$, respectively, on $m(\sigma(G))$ such that

$$\log(B_1/B_2) \leq O(\log \log \ell).$$

The previous best bounds satisfied $\log(B_1/B_2) \leq O(\log \ell)$ [2].
1.1 Outline

In Section 2, we formally define monotone switching networks and related terminology. In Section 3, we summarize previous work with monotone switching networks. In Section 4, we present techniques for bounding the sizes of certain classes of monotone switching networks which are crucial in obtaining the results in this paper. Section 5 provides proof of the main result, bounding the size of sound monotone switching networks in the case of general directed trees.

2 Preliminary Definitions

To discuss monotone switching networks and their properties, the following terminology was introduced by Potechin [2], which we also use.

Definition 2.1. Given a set of vertices $V \cup \{s, t\}$, define a monotone switching network for directed connectivity as an undirected graph $G'$ on the set of vertices $V' \cup \{s', t'\}$. Each edge between two vertices of $G'$ is given a label of the form $a \rightarrow b$ where $a, b \in V \cup \{s, t\}$.

Note. For succinctness, we refer to monotone switching networks for directed connectivity as monotone switching networks.

An example of a monotone switching network is depicted in Figure 1.

Definition 2.2. Define the size of a monotone switching network $G'$ as the number of vertices of $G'$.

We wish to analyze how $G'$ relates to various graphs $G$ on the set of vertices $V \cup \{s, t\}$. Definition 2.3 quantifies this.

Definition 2.3. Given a directed graph $G$ on $V \cup \{s, t\}$, called the input graph, say that a monotone switching network $G'$ accepts $G$ if and only if there exists a path from $s'$ to $t'$ in $G'$ such that the label of each edge of the path corresponds to an edge of $G$. For example, the label $a \rightarrow b$ corresponds to the directed edge from $a$ to $b$ in $G$. If $G'$ does not accept $G$, then $G'$ rejects $G$.

For an example, see Figure 1. We analyze monotone switching networks based on which graphs they accept and reject.

Definition 2.4. A monotone switching network $G'$ is complete if it accepts any input graph $G$ for which there is a path from $s$ to $t$. 
Definition 2.5. A monotone switching network $G'$ is **sound** if it rejects any input graph $G$ for which there is no path from $s$ to $t$.

Unless explicitly stated, we assume that all monotone switching networks under consideration are sound, which means that the computations which the monotone switching network simulates involve sound logical reasoning. On the other hand, almost none of the monotone switching networks under consideration are complete. In other words, the monotone switching networks may find the existence of a path from $s$ to $t$ for some input graphs but not others.

Definition 2.6. Given a set $I$ of input graphs on $V \cup \{s, t\}$, where for each graph $G \in I$ there is a path from $s$ to $t$, define $m(I)$ to be the smallest possible size of a sound monotone switching network which accepts all the elements of $I$.

In Sections 3 and 4 we find bounds on the value of $m(I)$ for specific sets of graphs $I$. The sets of graphs we primarily investigate are permutation sets.

Definition 2.7. Let $G$ be a directed graph on the set of vertices $V \cup \{s, t\}$. For any subset $W$ of $V \cup \{s, t\}$, define $\sigma_W(G)$ to be the set of graphs which are all possible permutations of the labels of the vertices $V \cup \{s, t\}$ that fix all vertices in $W$. Let $\sigma(G) = \sigma_{\{s,t\}}(G)$. These sets are called **permutation sets**.
Figure 2: By definition of $\sigma$, we have that $\sigma(G) = \{G, H\}$. Notice that the monotone switching network $G'$ accepts both graphs and is sound. Thus, $m(\sigma(G)) \leq 4$. In fact, $m(\sigma(G)) = 4$.

We bound the value of $m(\sigma(G))$ for various graphs $G$. An example is given in Figure 2. To aid in finding these bounds, the results listed in Section 3 are used.

3 Previous Results

The results discussed in this section, concerning the value of $m(I)$ for various sets of input graphs $I$, were discovered by Potechin [2, 4]. We assume that for each $G \in I$, its vertices are taken from the set of $n$ vertices $V \cup \{s, t\}$. Let $\mathcal{P}$ be the set of directed graphs with $n$ vertices such that there is a path from $s$ to $t$. We then have the following theorem about $\mathcal{P}$.

Theorem 3.1 (Potechin [2]). We have that

$$m(\mathcal{P}) = n^{\Theta(\lg n)},$$

where $\lg n$ stands for $\log_2(n)$.

Note. The bound we get for $m(\mathcal{P})$ uses big $\Theta$ notation in the exponent, instead of as a constant factor as these are the best bounds currently known. These bounds are tight enough for our purposes because they heuristically correspond to an algorithm using $O(\lg(m(\mathcal{P}))) = O((\lg n)^2)$ memory, which is accurate to a constant factor.
Let $\ell$ be a positive integer less than $n$. Consider $P_\ell$, the set of directed graphs such that there is a path from $s$ to $t$ with length $\ell$. Length is defined to be the number of edges along the path. Theorem 3.2 gives a bound for this subset of $P$.

**Theorem 3.2 (Potechin [2])**

\[ m(P_\ell) = n^{\Theta(\log \ell)}. \]

Notice the similarity between Theorem 3.2 and Theorem 3.3.

**Theorem 3.3 (Potechin [5])**. Let $G$ be a graph such that every path from $s$ to $t$ is of length $\ell$, and every vertex besides $s$ and $t$ is on exactly one such path. Then

\[ m(\sigma(G)) = n^{\Theta(\log \ell)}. \]

Figure 3 has an example of $G$. The asymptotic results of Theorem 3.2 and Theorem 3.3 are identical, although $\sigma(G)$ is only a small subset of $P_\ell$. In some sense, much of the work done by the monotone switching network to accept the elements of $P_\ell$ is used to accept the elements of $m(\sigma(G))$. In contrast, Theorem 3.4 shows that some subsets of $P_\ell$ can be accepted by much smaller sound monotone switching networks. First we need the following definition.

**Definition 3.1.** A vertex $v$ of a directed graph $G$ is a **lollipop** if $s \to v$ or $v \to t$ is an edge of $G$.

Theorem 3.4 tells us that lollipops hardly increase the asymptotic value of $m(\sigma(G))$. 
Figure 4: Example input graph $G$ for Corollary 3.5 with many lollipops.

**Theorem 3.4** (Potechin [3, 4]). For any $n$, $k$, and $\ell$, there is a sound monotone switching network of size at most

$$n^{O(1)}k^{O(\lg \ell)}$$

which accepts all input graphs $G$ such that $G$ has $n$ vertices, there is a path of length $\ell$ from $s$ to $t$, and at most $k$ vertices are not lollipops.

**Corollary 3.5** (Potechin [3, 4]). Let $G$ be a graph with $n$ vertices for which there is a path of length $\ell$ from $s$ to $t$, and all but $k$ of the vertices are lollipops. Then,

$$m(\sigma(G)) = n^{O(1)}k^{O(\lg \ell)}.$$  

Figure 4 depicts an example of $G$ with a single path from $s$ to $t$ in which all other vertices are lollipops. When $k$ is asymptotically smaller than $n$, the value of $m(\sigma(G))$ is asymptotically much smaller for the graphs described in Corollary 3.5 than the graphs described in Theorem 3.3, which means that it is much easier to find the existence of a path from $s$ to $t$ for the graphs described in Corollary 3.5 when using monotone computation. We use Theorem 3.4 when constructing upper bounds in the proofs of Theorems 5.3 and 5.1.

### 4 Techniques for Bounding $m(\sigma(G))$

In this section, we provide results which aid in the process of determining $m(\sigma(G))$ for arbitrary graphs $G$. 


Proposition 4.1. Consider two graphs $G$ and $H$ such that every edge of $G$ is also an edge of $H$. If a monotone switching network $G'$ accepts $G$, then $G'$ also accepts $H$.

Corollary 4.2. Given a directed graph $G$, consider the directed graph $H$ which results from adding an edge to $G$. Then, $m(\sigma(G)) \geq m(\sigma(H))$.

Proof. Consider a minimal-size sound monotone switching network $G'$ which accepts all elements of $\sigma(G)$. For any element of $\sigma(G)$ there is a corresponding element of $\sigma(H)$ with the same edges. Hence, by Proposition 4.1, $G'$ also accepts the elements of $\sigma(H)$. Therefore, $m(\sigma(G)) \geq m(\sigma(H))$. \qed

From Proposition 4.1 and its corollary, we infer that for graphs with the same number of vertices the ones with more edges typically have smaller monotone switching networks.

Theorem 4.3. Given a directed graph $G$ with an edge $a \rightarrow b$, let $H$ be the graph where this edge is replaced with $s \rightarrow b$ and $\bar{H}$ be the graph where this edge is replaced with $a \rightarrow t$. Then

$$m(\sigma(G)) \geq \max(m(\sigma(H)), m(\sigma(\bar{H}))).$$

Proof. We prove that $m(\sigma(G)) \geq m(\sigma(H))$. The inequality $m(\sigma(G)) \geq m(\sigma(\bar{H}))$ is a symmetric argument by reversing every edge and swapping $s$ and $t$. Consider the minimal-size sound monotone switching network $G'$ which accepts the elements of $\sigma(G)$. Construct a new monotone switching network $H'$ which contains the same vertices and edges as $G'$. For any edge $e$ with a label of the form $v_1 \rightarrow v_2$ in $H'$, add an additional edge, parallel to $e$, with the label $s \rightarrow v_2$. We now prove two properties about $H'$.

Lemma 4.4. Every element of $\sigma(H)$ is accepted by $H'$.

Proof. Let $H_1$ be an element of $\sigma(H)$. Let $G_1$ be a corresponding element of $\sigma(G)$. Because $G_1$ is accepted by $G'$, there exists a path $P'$ from $s'$ to $t'$ using only edges of $G_1$. Since no edges were deleted in the construction of $H'$, this same path $P'$ exists in $H'$.

If $P'$ uses only edges in $H_1$, we are done. If not, $P'$ uses precisely one edge label which is not in $H_1$, the edge $v_1 \rightarrow v_2$ which was replaced by $s \rightarrow v_2$. Consider the path in $H'$ which follows $P'$ but instead of using the edges labeled $v_1 \rightarrow v_2$, it uses the parallel edges labeled $s \rightarrow v_2$. This is clearly an accepting path for $H_1$ in $H'$. Thus, all the elements of $\sigma(H)$ are accepted by $H'$. \qed

Lemma 4.5. $H'$ is sound.
Proof. It is sufficient to prove that modifying a switching network \( G' \) by adding one parallel edge with label \( s \rightarrow v_2 \) to an edge with label \( v_1 \rightarrow v_2 \) must preserve the soundness of \( G' \). Let \( G'_2 \) be the modified switching network and assume for sake of contradiction that \( G'_2 \) is not sound. Then there exists a graph \( G \) with no path from \( s \) to \( t \) which is accepted by \( G'_2 \). This implies there is a path \( P' \) from \( s' \) to \( t' \) in \( H' \) which uses only edge labels in \( G \). However, because \( G' \) is sound, it rejects \( G \) so \( P' \) must go through the one additional edge in \( G'_2 \), the edge labeled \( s \rightarrow v_2 \), and this edge must be in \( G \). But then if we add the edge \( v_1 \rightarrow v_2 \) to \( G \), we obtain a graph \( G_2 \) which is accepted by \( G' \), as we can follow \( P' \) except that we use the original edge labeled \( v_1 \rightarrow v_2 \) rather than the added parallel edge. Thus, \( G_2 \) must have a path from \( s \) to \( t \). But this is impossible, as if we let \( V \) be the set of vertices reachable from \( s \) in \( G \), \( V \) is also the set of vertices reachable from \( s \) in \( G_2 \). To see this, note that \( v_2 \) is reachable from \( s \) in \( G \), so adding the edge \( v_1 \rightarrow v_2 \) cannot possibly allow us to reach any additional vertices from \( s \). This is a clear contradiction, so \( H' \) is sound.

Since there exists a sound monotone switching network \( H' \) of size \( m(\sigma(G)) \) which accepts every element of \( \sigma(H) \), we have that

\[
m(\sigma(G)) \geq m(\sigma(H))
\]

as desired.

Heuristically, Theorem 4.3 implies that when the number of vertices and edges is the same for two graphs, the one with more edges connected from \( s \) or to \( t \) typically has a smaller monotone switching network.

In contrast to Theorem 4.3, Proposition 4.6 demonstrates the case in which there are similar edges but in the reverse direction.

Definition 4.1. An edge is **useless** if it is of the form \( v \rightarrow s \) or \( t \rightarrow v \) for some vertex \( v \).

This definition is motivated by the fact that having an edge of this form gives no information about whether there is a path from \( s \) to \( t \).

Theorem 4.6. Let \( G \) be a graph with useless edges. Let \( H \) be a copy of \( G \) with the useless edges removed. Then \( m(\sigma(G)) = m(\sigma(H)) \).

Proof of Theorem 4.6. This result follows from Lemma 4.7. The case with an edge of the form \( t \rightarrow a \) follows by a symmetrical argument.

Lemma 4.7. Let \( G \) be a graph with the edge \( a \rightarrow s \). Consider \( H \), an identical graph except \( a \rightarrow s \) is removed. Then \( m(\sigma(G)) = m(\sigma(H)) \).
Figure 5: Example showing $G$ (left) and the merge graph $G_{(S,T)}$ (right). The dashed edge from $\bar{t}$ to $\bar{s}$ is useless and can be removed without affecting the value of $m(\sigma(G_{(S,T)}))$.

**Proof.** Since an edge was removed from $G$ to yield $H$, from Theorem 4.2 we have that $m(\sigma(G)) \leq m(\sigma(H))$. Consider a sound monotone switching network $G'$ of minimal size which accepts all the elements of $\sigma(G)$. Replace every edge whose label is of the form $v \rightarrow s$, for some $v$, with an unlabeled edge (an unlabeled edge can be traversed under any condition) to produce a monotone switching network $G'_2$. This monotone switching network $G'_2$ accepts all the elements of $\sigma(H)$. It is sufficient to prove now that $G'_2$ is sound.

If $G'_2$ were not sound, then a disconnected graph $K$ would exist which $G'_2$ accepts. Thus, an accepting path $P'$ must traverse an unlabeled edge. Let $\bar{K}$ be identical to $K$ except the edge $v \rightarrow s$ is added for all $v$. We have that $\bar{K}$ is accepted by $G'$ by following the path $P'$, except that the unlabeled edges are replaced with edges whose labels are of the form $v \rightarrow s$. As $G'$ is sound, $\bar{K}$ must a path from $s$ to $t$. However, the addition of edges of the form $v \rightarrow s$ to a graph without a path from $s$ to $t$ cannot produce a graph with a path from $s$ to $t$. This is a contradiction; thus, $G'_2$ is sound.

Therefore,

$$m(\sigma(G)) = |V(H)| \geq m(\sigma(H)).$$

Theorem 4.6 is very useful for proving lower bounds of $m(\sigma(G))$, especially when combined with Theorem 4.8. To introduce this theorem, we first define the concept of a merge graph. Figure 5 depicts an example of a merge graph.

**Definition 4.2.** Given a graph $G$, let $S$ be a set of vertices such that $s \in S$ and $t \not\in S$. Also let $T$ be a set of vertices such that $t \in T$ and $s \not\in T$. Consider the graph $G_{(S,T)}$ whose vertex set is identical to $G$ except the vertices of $S$ have been merged into a single vertex $\bar{s}$, and the vertices of $T$ have been merged into a single vertex $\bar{t}$. Any edge between two
elements of $S$ or two elements of $T$ is removed. Any edge with exactly one endpoint in $S$ is replaced with an edge whose corresponding endpoint is $\bar{s}$. Likewise, any edge with exactly one endpoint in $T$ is replaced with an edge whose corresponding endpoint is $\bar{t}$. Any remaining edges remain unchanged. Define $G_{(S,T)}$ to be the $(S,T)$-merge graph of $G$.

**Theorem 4.8.** Given a graph $G$, let $S$ and $T$ be subsets of $G$ defined as in Definition 4.2. Then $m(\sigma(G)) \geq m(\sigma(G_{(S,T)}))$.

**Proof.** Because $\sigma_{S\cup T}(G) \subset \sigma(G)$, we have that $m(\sigma_{S\cup T}(G)) \leq m(\sigma(G))$. Consider the sound monotone switching network $H'\prime$ which accepts all of the elements of $\sigma_{S\cup T}(G)$. Construct a new monotone switching network $\bar{H}'\prime$ by taking every edge label of this monotone switching network which has an endpoint in $S$ or $T$ and replace it with $\bar{s}$ or $\bar{t}$, respectively. This new monotone switching network accepts all elements of $\sigma(G_{(S,T)})$. As any graph without a path from $\bar{s}$ to $\bar{t}$ cannot correspond to a graph with a path from $s$ to $t$, we have that $\bar{H}'\prime$ is sound. Thus,

$$m(\sigma(G_{(S,T)})) \leq m(\sigma_{S\cup T}(G)) \leq m(\sigma(G)).$$

\hfill \Box

Theorem 4.6 and Theorem 4.8 can work together well to obtain bounds for arbitrary graphs, as shown in Section 5.

A natural question after observing Theorem 4.8 is whether merging arbitrary sets of vertices which do not necessarily contain $s$ or $t$ provides the same result. This is not the case: there exists a graph $G$ for which contracting particular sets of vertices results in an increase in the value of $m(\sigma(G))$. For proof, see Appendix A.

## 5 Main Results

We now demonstrate bounds on the value of $m(\sigma(G))$ where $G$ is any tree. The result we prove is as follows.

**Theorem 5.1.** Let $G$ be an arbitrary directed tree with a path from $s$ to $t$. Define $d_i^s$ to be the number of vertices which are accessible from $s$ with $i$ as the maximum distance of its descendants from $s$. Let $d_i^t$ to be the number of vertices which can access $t$ with the maximum distance of its ancestors to $t$ being $i$. Define the sequence $c_1^s, \ldots, c_{\lceil \lg \lg \ell \rceil}^s$ such that

$$c_1^s = \sum_{i=1}^{n} d_i^s$$

and

$$c_k^s = \sum_{i=2^k}^{n} d_i^s$$

where $k \geq 2$.
Define $c_1^t, \ldots, c_{\lceil \log \log \ell \rceil}^t$ similarly. Let $\bar{d}$ be the number of vertices which are not accessible from $s$ or $t$. Let $\ell$ be the length of the path from $s$ to $t$. Then $m(\sigma(G))$ can be bounded by

$$(\ell + \bar{d})^{O(\log \ell)} \prod_{i=1}^{\lceil \log \log \ell \rceil} (c_i^s + c_i^t)^{O(2^i)} \leq m(\sigma(G)) \leq n^{O(\log \log \ell)} (\ell + \bar{d})^{O(\log \ell)} \prod_{i=1}^{\lceil \log \log \ell \rceil} (c_i^s + c_i^t)^{O(2^i)}.$$ 

Note. If we let $B_1$ be the upper bound and $B_2$ be the lower bound, then

$$\log(B_1/B_2) = O(\log \log \ell).$$

This proof is divided into proving the lower and upper bounds.

### 5.1 Lower Bound

First we prove the following lemma.

**Lemma 5.2.** Let $G$ be a graph and $H$ be a tree disconnected from $G$. Let $n_H$ be the number of vertices of $H$. Let $P$ be a path with $\lceil \sqrt{n_H} \rceil$ vertices which is disconnected from $G$. Then $m(\sigma(G \cup H)) \geq m(\sigma(G \cup P))$.

**Remark.** The length of $P$ can be significantly increased. See Appendix B.

**Proof.** Give the vertices of $H$ a depth labeling $d(v)$ in the following way: pick an arbitrary vertex $v$ and let $d(v)$ be 0. We can now define $d$ recursively. For any edge $v_1 \to v_2$, we have $d(v_2) - d(v_1) = 1$. If there is a path from vertex $w_1$ to vertex $w_2$, then $d(w_2) - d(w_1) > 0$. Thus, any two vertices of the same depth do not have a directed path between them. Let $d_{\text{min}}$ be the minimum depth and $d_{\text{max}}$ be the maximum depth. Note that $d_{\text{min}}$ may be negative if there are edges directed toward $v$. We now have two cases to consider.

**Case 1:** $d_{\text{max}} - d_{\text{min}} + 1 < \lceil \sqrt{n_H} \rceil$.

By the pigeonhole principle, there exists a depth $\bar{d}$ which has at least $\lceil \sqrt{n_H} \rceil$ vertices. Call this set of vertices $\bar{D}$. We can merge all of the vertices of depth less than $\bar{d}$ with $t$ and all the vertices of depth greater than $\bar{d}$ with $s$. After removing useless edges, the vertices of $\bar{D}$ are isolated. We then add edges between the vertices of $\bar{D}$ to create a path of length $\lceil \sqrt{n_H} \rceil$, as desired. If there are additional vertices, they can be merged with $s$.

**Case 2:** $d_{\text{max}} - d_{\text{min}} + 1 \geq \lceil \sqrt{n_H} \rceil$.

First consider the case where if we ignore directions, then $H$ consists of only a single undirected path $P$ from a vertex $w_{\text{min}}$ at depth $d_{\text{min}}$ to a vertex $w_{\text{max}}$ at depth $d_{\text{max}}$. Now looking at the edge directions, let $c^+$ be the number of edges along the path which go from a vertex of lesser depth to one of greater depth. Define $c^-$ to be the number of remaining
edges. We have that $c^+ = d_{\text{max}} - d_{\text{min}} + c^-$. Now note that whenever we have vertices $w_1, w_2, w_3$ such that the path $P$ from $w_{\text{min}}$ to $w_{\text{max}}$ contains an edge from $w_1$ to $w_2$ followed by an edge from $w_3$ to $w_2$, we can do the following. We can merge $w_2$ with $s$, remove both edges (as they are now useless), then add an edge from $w_1$ to $w_3$ and increase the depth of $w_3$ and all later vertices on the path by 1. This keeps $c^+$ the same and reduces $c^-$ by 1. In this way, we can eliminate all edges going the opposite direction as $P$, and when we are done, we will have a path of length $c^+ \geq d_{\text{max}} - d_{\text{min}}$, which will have at least $d_{\text{max}} - d_{\text{min}} + 1$ vertices.

For the general case, note that $H$ will contain at least one such path $P$ as a subgraph and then note that we can ignore all of the other other vertices of $H$ by merging them with $s$ or $t$ and then removing useless edges.

Before handling the case of a general directed tree $G$, we find bounds in the case where $G$ is a flow-out tree, whose bounds are the foundation of our proof of the general case.

**Definition 5.1.** A flow-out tree $G$ is a tree with a special vertex $r$ (the root) such that there is a path from $r$ to every other vertex of $G$.

**Theorem 5.3.** Let $G$ be a flow-out tree with root $s$ and a path of length $\ell$ from $s$ to $t$. For $i \geq 1$, define $d_i$ to be the number of vertices whose descendants have a maximum distance of $i$ from $s$. Define an additional sequence $c_1, c_2, \cdots, c_{[\lg \lg \ell]}$ with the property that $c_1 = n$ and for all $i \geq 2$,

$$c_i = \sum_{j=2^i}^n d_j.$$  \hspace{1cm} (1)

We then have that

$$\ell^{\Omega(\lg \ell)} \max_{1 \leq i \leq [\lg \lg \ell]} c_i^{\Omega(2^i)} \leq m(\sigma(G))$$  \hspace{1cm} (2)

**Proof.** To prove the bound, we show that for all $i \leq [\lg \lg \ell]$,

$$m(\sigma(G)) \geq \ell^{\Omega(\lg \ell)} c_i^{\Omega(2^i)}.$$  

Consider the set of vertices which are not on the path from $s$ to $t$. If we merge these vertices with $s$ and remove useless edges directed toward $s$, we are left with a single path of length $\ell$. From Theorem 4.8 and Theorem 3.3 we obtain

$$m(\sigma(G)) \geq \ell^{\Omega(\lg \ell)}.$$  

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Thus, it is sufficient to prove that
\[ m(\sigma(G)) \geq c_i^{\Omega(2^i)} \]
and take the geometric mean of these two bounds. If \( i = 1 \), take every edge \( a \to b \) not on the path from \( s \) to \( t \) and change it to \( s \to b \). By Theorems 3.4 and 4.3 we get
\[ m(\sigma(G)) \geq n^{\Omega(1)} \ell^{\Omega(\log \ell)} \geq c_1^{\Omega(2^1)} \]
Now consider \( i \geq 2 \), let \( k = 2^{2^i} \). Let \( S_1 \) be the set of vertices \( v \) such for any vertex \( w \) which is a descendant of \( v \), the distance from \( s \) to \( w \) is less than \( k \). Merge the elements of \( S_1 \) with \( s \). The remaining vertices have descendants which are a distance of at least \( k \) from \( s \). Thus, there are exactly \( c_i \) vertices remaining by definition of \( c_i \).

We split the problem into two cases. Let \( \tilde{d}_k \) be the number of vertices with a depth at most \( \lceil k/2 \rceil \).

Case 1: \( \tilde{d}_k \geq \sqrt{c_i} \).

For each vertex \( v \) at depth \( \lceil k/2 \rceil \) we can choose a path \( P_v \) from that vertex whose length is \( \lceil k/2 \rceil \). For the vertex \( v \) with descendant \( t \), we can choose \( P_v \) so that it is contained in the path from \( v \) to \( t \). For each \( v \), let \( w_v \) be the other endpoint of \( P_v \). Merge all vertices which are not any path \( P_v \) and are not descendants of any \( w_v \) with \( s \). Merge all descendants of each \( w_v \) with \( t \). If a \( w_v \) has no descendants, add an edge from \( w_v \) to \( t \). The number of vertices at exactly depth \( \lceil k/2 \rceil \) is at least \( \tilde{d}_k/\lceil k/2 \rceil \), as the number of vertices at depth \( i + 1 \) is at least the number of vertices at depth \( i \). Each path has \( \lceil k/2 \rceil \) vertices, implying that there are at least \( \tilde{d}_k \) vertices total. Thus, from Theorem 4.8 and Theorem 3.3 we obtain
\[ m(\sigma(G)) \geq (\tilde{d}_k)^{\Omega(\log(k/2))} = (c_i)^{\Omega(2^i)}. \]

Case 2: \( \tilde{d}_k < \sqrt{c_i} \).

Let \( D_{\lceil k/2 \rceil} \) be the set of points at a distance of \( \lceil k/2 \rceil \) from \( s \). Let \( v_{\lceil k/2 \rceil} \) be the element of \( D_{\lceil k/2 \rceil} \) with the largest subtree \( H \). Merge \( v_{\lceil k/2 \rceil} \) with the vertex \( t \). We now have two subcases to consider.

Subcase 1: \( t \) is in \( H \). Merge the path from \( v_{\lceil k/2 \rceil} \) to \( t \) with \( t \). There may now be subtrees disconnected from \( G \).

Subcase 2: \( t \) is not in \( H \). Let \( w \) be the vertex which is at a distance of \( \lceil k/2 \rceil \) away from \( t \). Merge \( w \) with \( s \).

We now treat both subcases identically. Let \( S_1 \) be the set of vertices which are not in \( H \) and are not on a path from \( s \) to \( t \). Merge \( S_1 \) with \( s \). Because \( H \) is the largest subtree, it has at least
\[ \frac{c_i}{d_k} \geq \sqrt{c_i} \]
vertices. Some of these vertices were removed in the subcase where \( t \in H \), but at most \( \ell - k/2 + 1 \) such vertices were removed. Thus, \( G \) has at least \( \sqrt{c_i} - \ell \) vertices.

After removing useless edges from the mergings, we have a collection of disconnected trees. Let \( j \) be the number of such trees and let the sizes of these trees be \( a_1, a_2, \ldots, a_j \).

By Lemma 5.2 we can reduce these trees to paths with

\[
\lceil \sqrt{a_1} \rceil, \lceil \sqrt{a_2} \rceil, \ldots, \lceil \sqrt{a_j} \rceil
\]

vertices, respectively. We can then add edges and perform mergings to obtain a collection of paths from \( s \) to \( t \) of length \( \lceil k/2 \rceil \) containing at least

\[
\frac{\sqrt{c_i} - \ell}{3}
\]

vertices. From Theorem 3.3 we get a lower bound of

\[
m(\sigma(G)) = \left( \frac{\sqrt{c_i} - \ell}{3} \right)^{\Omega(\lg k/2)} = (\sqrt{c_i} - \ell)^{\Omega(\lg k)}.
\]
If $\sqrt{c_i} \geq 2\ell$, then $m(\sigma(G)) = c_i^{\Omega(\lg k)}$. Otherwise, $m(\sigma(G)) = \ell^{\Omega(\lg \ell)}$ is a better lower bound as $\ell \geq k$.

A dual structure of the flow-out tree is the flow-in tree.

**Definition 5.2.** A **flow-in tree** $G$ is a tree with a special vertex $r$ (the **sink**) such that there is a path to $r$ from every other vertex of $G$.

Theorem 5.3 has the following corollary which states an analogous bound for flow-in trees.

**Corollary 5.4.** Let $G$ be a flow-in tree with sink $t$ and a path of length $\ell$ from $s$ to $t$. For $i \geq 1$, define $d_i$ as the number of vertices whose ancestors, including itself, have a maximum distance of $i$ from $s$. Define an additional sequence $c_1, c_2, \ldots, c_{\lceil \lg \ell \rceil}$ with the property that $c_1 = n$ and for all $i \geq 2$, the element $c_i$ satisfies (1). Hence, $m(\sigma(G))$ satisfies the bounds (2).

**Proof.** We reverse the direction of every edge, and swap the labels of $s$ and $t$. The obtained tree satisfies the hypothesis of Theorem 5.3. Thus, the same bounds hold.

**Proof of lower bound.** For this graph $G$, let $H_s$ be the graph induced by the set of vertices $v$ for which a path from $s$ to $v$ exists. Let $H_t$ be the graph induced by the set of vertices $v$ from which a path from $v$ to $t$ exists. In both cases, we do not include the vertices on the path from $s$ to $t$.

The proof of this bound is divided into two parts. The first is to show that $m(\sigma(G)) \geq (\ell + \bar{d})^{\Omega(\lg \ell)}$. The second is to show that $m(\sigma(G)) \geq (c_i^s + c_i^t)^{\Omega(2^i)}$ for all $i$. We then take the geometric mean of these two bounds.

For the first part, we can merge all of the vertices of $H_s$ with $s$ and all of the vertices of $H_t$ with $t$. In the graph $G_{(H_s, H_t)}$, there may be edges not on the path from $s$ to $t$ which are connected to $s$ or $t$. These edges are directed towards $s$ or away from $t$, so they are useless. We can remove these useless edges to obtain a graph consisting of a single path from $s$ to $t$ and a collections of trees which are disconnected from the main path. Let the size of these $k$ trees be $a_1, a_2, \ldots, a_k$. Notice that $a_1 + a_2 + \cdots + a_k = \bar{d}$. From Lemma 5.2, we can reduce these trees to paths with $\lceil \sqrt{a_1} \rceil$, $\lceil \sqrt{a_2} \rceil$, $\ldots$, $\lceil \sqrt{a_k} \rceil$ vertices. We can link these paths into a long path of length at least

$$\sum_{i=1}^{k} \sqrt{a_i} \geq \sqrt{\bar{d}}.$$
We can then merge this long path with the path from $s$ to $t$ to create a collection of disjoint paths of length $\ell$ from $s$ to $t$. The total number of vertices is at least $\sqrt{d/3} + \ell$. From Theorem 3.3 we get a lower bound of

$$m(\sigma(G)) \geq \left(\frac{\sqrt{d}}{3} + \ell\right)^{\Omega(\log \ell)} = (d + \ell)^{\Omega(\log \ell)},$$

as desired.

For the second part, merge the all vertices not in $H_s$ nor on the path from $s$ to $t$ with $t$. We are then left with a flow-out tree. By Theorem 5.3, we get a lower bound of

$$m(\sigma(G)) \geq \Omega(2^i),$$

Using a symmetric argument with a flow-in tree, by Corollary 5.4, we have

$$m(\sigma(G)) \geq \Omega(2^i),$$

Thus,

$$m(\sigma(G)) \geq \Omega(2^i),$$

as desired.

### 5.2 Upper Bound

**Proof of upper bound.** Like in the proof of the lower bound, let $H_s$ be the graph induced by the set of vertices $v$ for which a path from $s$ to $v$ exists. Let $H_t$ be the graph induced by the set of vertices $v$ from which a path from $v$ to $t$ exists. In both cases, we do not include the vertices on the path from $s$ to $t$.

First, take the vertices not in $H_s$, $H_t$, or the path from $s$ to $t$, and remove any edges connected to them. Thus, $G$ is a flow-in tree, a flow-out tree, and a collection of disconnected vertices. We use a construction very similar to that used in the proof of Theorem 5.3.

Let $\bar{c}$ be the number of points which are at a distance of more than $\ell$ from $s$ or to $t$. Construct a new graph $G_1$ where all the edges connected to these $\bar{c}$ points are removed. We now construct graphs $G_2, \ldots, G_{[\log \ell]+1}$ inductively as follows. Given $G_{i-1}$, let $P^s_i$ be the set of vertices which are at a distance of less than $2^{2^i-1}$ from $s$ and do not have any children. Let $P^t_i$ be the set of vertices which are at a distance of less than $2^{2^i-1}$ to $t$ and do not have any ancestors. Also, add to $P^s_i$ any vertices which are on the paths from $s$ to $P_i^t$, and add to $P^t_i$ any vertices which are on paths from $P_i^t$ to $t$. To create $G_i$, remove the edges connecting $P^s_i$ to $G_{i-1}$ and add edges directly from $s$ to $P^s_i$. Similarly, remove edges connected $P^t_i$ to $G_{i-1}$ and add edges directly to $t$. From the definition of $c_i^s + c_i^t$, there are at least $n - c_i^s - c_i^t$ vertices which are directly connected to $s$ or $t$. Refer to Figure 7 for an example of this construction.

In the graph $G_{[\log \ell]+1}$, every vertex is directly connected via a single edge to $s$ or $t$, on the path from $s$ to $t$, or disconnected from the graph. From Theorem 3.7, there is a monotone switching network $G'_{[\log \ell]+1}$ of size $n^{O(1)}(\ell + \bar{c} + \bar{d})^{O(\log \ell)}$ accepting the elements of $\sigma(G_{[\log \ell]+1})$. 

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Next, we construct inductively the sequence of sound monotone switching networks $G'_1, G'_2, \ldots, G'_i$ such that $G'_i$ accepts the elements of $m(\sigma(G_i))$. In the monotone switching network $G'_{i+1}$, consider an edge with a label of the form $s \rightarrow a$. Consider a graph $\bar{G}_{i+1} \in \sigma(G_{i+1})$ which crosses that edge in its accepting path. The corresponding graph $\bar{G}_i$ may not be able to cross that edge because the edge was deleted and became a path of length at most $2^{2^{i+1}}$ in $\bar{G}_i$ from $s$ to $a$. Thus, to construct $G'_i$, we replace that edge with a monotone switching network which checks if there is a path of length at most $2^{2^{i+1}}$ from $s$ to $a$, assuming at most $c_s^i + c_t^i + \bar{d} + \ell$ vertices are not lollipops.

To construct the monotone switching network, we take the monotone switching network of size $n^{O(1)}(c_s^i + c_t^i + \bar{d} + \ell)^{O(2^i)}$ guaranteed by Theorem 3.3 which checks if there is a path of length at most $2^{2^{i+1}}$ from $s$ to $t$ on $V(G \setminus \{a\}) \cup \{s, t\}$, assuming at most $c_s^i + c_t^i + \bar{d} + \ell$ vertices are not lollipops. Now for each edge of the switching network, whose label is of the form $v \rightarrow t$, replace it with two parallel edges, one of the form $v \rightarrow a$ and the other of the form $v \rightarrow t$. It is easily verified this modified monotone switching network does indeed check whether there is a path from $s$ to $a$ or a path from $s$ to $t$, given that at most $c_s^i + c_t^i + \bar{d} + \ell$ vertices are not lollipops. We can deal with edges in $G'_{i+1}$ with labels...
of the form \(a \to t\) in a similar way. The number of edges in each such checker is at most
\[
n^2 \left(n^O(1)(c_i^s + c_i^t + d)O(2^i)\right)^2 = n^O(1)(c_i^s + c_i^t + d + \ell)O(2^i).
\]
Thus, the number of edges of \(G'_i\) is at most
\[
|E(G'_i)| \leq |E(G'_{i+1})|n^O(1)(c_i^s + c_i^t + \bar{d} + \ell)O(2^i).
\]
Thus,
\[
V(G') \leq E(G') \leq E(G'_1) = E(G'_{\lfloor \lg \ell \rfloor + 1}) \prod_{i=1}^{\lfloor \lg \ell \rfloor} n^O(1)(c_i^s + c_i^t + \bar{d} + \ell)O(2^i)
\]
\[
= n^2(n^O(1)(\ell + \bar{c} + \bar{d})O(\lg \ell))^2 n^O(1)(\lg \ell) \prod_{i=1}^{\lfloor \lg \ell \rfloor} (c_i^s + c_i^t + \bar{d} + \ell)O(2^i)
\]
\[
= n^O(\lg \ell)(\ell + \bar{c} + \bar{d} + \ell)O(\lg \ell) \prod_{i=1}^{\lfloor \lg \ell \rfloor} (c_i^s + c_i^t + \bar{d})O(2^i). \quad (3)
\]
From its definition, \(\bar{c} \leq c_i^s + c_i^t\), for all \(i\). Therefore when \(i = \lg \lg \ell\), we have
\[
\bar{c}O(\lg \ell) \leq (c_i^s + c_i^t + \bar{d} + \ell)O(2^i),
\]
and
\[
\prod_{j=1}^{\lfloor \lg \ell \rfloor} dO(2^i) \leq dO(\lg \ell).
\]
Hence, inequality \(3\) is equivalent to
\[
m(\sigma(G)) \leq n^O(\lg \lg \ell)(\ell + \bar{d})O(\lg \ell) \prod_{i=1}^{\lfloor \lg \ell \rfloor} (c_i^s + c_i^t)O(2^i).
\]

6 Conclusion

Sound monotone switching networks provide an insightful way of analyzing monotone computation. Previously, Potechin \[2, 4\] found tight bounds in the case where the inputs
were the permutation sets of very specific kinds of trees and acyclic graphs. From these
earlier results, we proved in Theorem 5.1 nearly tight bounds for all directed trees. These
bounds give us insight into the structure of space-efficient monotone computation. From
Theorem 5.1 the exponent for $c_1^s + c_1^t$ is orders of magnitude smaller than the exponent
for $c_{\lceil \lg \lg \ell \rceil}^s + c_{\lceil \lg \lg \ell \rceil}^t$. We can infer from this that monotone computation is more effective
at analyzing vertices closer to $s$ and $t$ than vertices which are farther. This suggests that
the optimal algorithm for ST-connectivity in a monotone computation model is akin to a
breadth-first search.

Possibilities of future investigation include:

- Generalize the bounds to permutation sets of all acyclic graphs and eventually all
  graphs.

- Improve known bounds. Currently, these bounds are within a factor of $O(\lg \lg \ell)$ in
  the exponent. Can this be improved to a factor of $O(1)$ in the exponent?

- Find algorithms corresponding to these monotone switching networks. The existence
  of a monotone switching network of size $m$ heuristically implies that an algorithm
  with $O(\log m)$ memory use exists, but such an algorithm may not necessarily exist.
  Much work can be devoted to determining whether or not these algorithms exist and
  finding elegant implementations if they indeed exist.

- Extend these results to non-monotone switching networks. These more general struc-
  tures account for all possible classical computations. Obtaining tight bounds in this
  case would solve the open log-space versus nondeterministic log-space problem.

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References

[1] N. Immerman. Nondeterministic space is closed under complementation. *SIAM J. Comput.*, 17(5):935–938, Oct. 1988.

[2] A. Potechin. Bounds on monotone switching networks for directed connectivity. arXiv:0911.0664v5.

[3] A. Potechin. Improved upper and lower bound techniques for monotone switching networks for directed connectivity. Submitted.

[4] A. Potechin. Monotone switching networks for directed connectivity are strictly more powerful than certain-knowledge switching networks. arXiv:1111.2127v1.

[5] A. Potechin. Personal Conversation, July 2012.

[6] O. Reingold. Undirected st-connectivity in log-space. *STOC*, 2005.

[7] W. J. Savitch. Relationship between nondeterministic and deterministic tape classes. *J. CSS*, 4:177–192, 1970.
A Arbitrary Merging May Not Decrease $m(\sigma(G))$

Recall from Theorem 4.8, that merging vertices of $G$ with $s$ or $t$ does not increase the value of $m(\sigma(G))$. We now exhibit a construction which shows how merging arbitrary vertices fails in general.

Let $\ell$ be a positive integer, construct a graph $G$ with $\ell - 1$ special sets $C_1, C_2, \ldots, C_{\ell - 1}$. Let each set contain $\left\lfloor \frac{n - 2}{2(\ell - 1)} \right\rfloor$ vertices. Let there be no edges between any of the vertices in each $C_i$, but let all of the vertices in each $C_i$ where $i < \ell - 1$, be directed to all the vertices in $C_{i+1}$. Additionally, let there be edges from $s$ to all the vertices of $C_1$ and edges from all vertices of $C_{\ell - 1}$ to $t$. Let the other approximately $n/2$ vertices of $G$ not already mentioned be disconnected from the graph and from each other.

Let $H$ be the graph where for each $i \geq 1$, the vertices of $C_i$ are merged to a single vertex $v_i$. Notice that $H$ is merely a path of length with $\ell$ edges from $s$ to $t$ and has $n/2 + \ell - 1$ vertices.

**Theorem A.1.** If $n > 20$ and $n/4 > \ell$ then,

- $m(\sigma(G)) = n^{O(1)}\ell^{O(\ell)}$
- $m(\sigma(H)) = n^{\Omega(\lg \ell)}$

**Proof.** Since $\sigma(H)$ is the set of minimal elements of $P_\ell$, Theorem 3.2 implies that

$$m(\sigma(H)) = \left(\frac{n}{2} + \ell - 1\right)^{\Omega(\lg \ell)} = n^{\Omega(\lg \ell)}.$$

To obtain the bound for $m(\sigma(G))$, we will construct the monotone switching network $G'$ as follows. Let

$$p = \left(\frac{1}{4(\ell - 1)}\right)^{\ell - 1}.$$

Construct $C = n^2/p$ internally-disjoint undirected paths of length $\ell$ between $s'$ and $t'$. Each such path has consecutive edge labels of the form

$$s \rightarrow w_1, w_1 \rightarrow w_2, \ldots, w_{\ell - 1} \rightarrow t.$$

Where each vertex $w_i$ is selected uniformly and at random. For any particular graph $G \in \sigma(G)$, the probability that a particular path of $G'$ accepts $G$ is, if $\ell$ is sufficiently large,

$$\left(\frac{n - 2}{2(\ell - 1)}\right)^{\ell - 1} \geq \left(\frac{1}{4(\ell - 1)}\right)^{\ell - 1} = p.$$
Thus, the probability that $\bar{G}$ is rejected is at most $1 - p$. This implies that the expected number of elements of $\sigma(G)$ rejected by $G'$ is at most

$$|\sigma(G)|(1 - p)^C \leq n!e^{-Cp} = n!e^{-n^2} < 1$$

Thus, by the probabilistic method, there exists a choice of $C = n^2/p$ paths which accept all the elements of $\sigma(G)$. The size of $G'$ in this case is

$$|V(G')| \leq C\ell = n^2\ell/p = n^2\ell(4(\ell - 1))^{(\ell-1)} = n^{O(1)} \ell^{O(\ell)}$$

as desired.

Theorem A.1 shows that in the case $n$ is arbitrarily large, and $\ell$ is a sufficiently large constant, $m(\sigma(G)) < m(\sigma(H))$. This demonstrates that arbitrary merging does not necessarily decrease the value of $m(\sigma(G))$.

### B Improved Bounds on Lemma 5.2

Although the bound of $\lceil \sqrt{V(H)} \rceil$ on the path length given in Lemma 5.2 is strong enough for proof of Theorems 5.3 and 5.4, a stronger bound has been found. This results improves $\lceil \sqrt{V(H)} \rceil$ to approximately $V(H)/(2 \log V(H))$. First, we give a definition of the quantity we are bounding.

**Definition B.1.** Given a graph $H$, define a disconnected-path family $\mathcal{P} = \{P_1, \ldots, P_k\}$ to be a family of disjoint paths of $H$ such that for any two paths $P_i$ and $P_j$, there is no unidirectional path connecting these two paths.

**Definition B.2.** Define the size of a disconnected-path family $\mathcal{P}$ to be the total number of vertices in the family. This quantity is denoted by $V(\mathcal{P})$.

**Definition B.3.** Define the disconnected-path length of a directed tree $H$ to be the maximal-size disconnected-path family of $H$. We denote this quantity by $p(H)$.

We now prove a theorem which connects Definition B.3 to Lemma 5.2.

**Lemma B.1.** Let $G$ be a directed graph and $H$ be a directed tree disconnected from $G$. Let $P$ be a path with $p(H)$ vertices. Then,

$$m(\sigma(G \cup H)) \geq m(\sigma(G \cup P))$$
Proof. In this proof, we construct sets \( S \) and \( T \) such that the vertices of \( S \) are to be merged with \( s \) and the vertices of \( T \) are to be merged with \( t \). Let \( \mathcal{P} = \{P_1, \ldots, P_k\} \) be a disconnected-path family of size \( p(H) \). We let \( v \) be any vertex of \( H \) not part of any path of \( \mathcal{P} \). We define \( v \) to be inbound if there exists a path \( P_i \) such that there is a path from \( v \) to some vertex of \( P_i \). Analogously, we define \( v \) to be outbound if there exists a path \( P_i \) such that there is a path from some vertex of \( P_i \) to \( v \). We now have four cases to consider.

Case 1: \( v \) is neither inbound nor outbound.
This is not possible, as \( \mathcal{P} \cup \{\{v\}\} \) would be a disconnected-path family of size greater than \( p(H) \).

Case 2: \( v \) is both inbound and outbound.
This is also not possible, as then there would exist paths \( P_i \) and \( P_j \) such that there is a path from one vertex of \( P_i \) to some other vertex of \( P_j \) via \( v \). This path contradicts the definition of a disconnected-path family.

Case 3: \( v \) is only inbound.
Let \( v \) be an element of \( T \).

Case 4: \( v \) is only outbound.
Let \( v \) be an element of \( S \).

From these cases, we have constructed the sets \( S \) and \( T \). If we performed the mergings, by definition of inbound and outbound, the only new edges connected to \( s \) are directed toward \( s \) and the only new edges connected to \( t \) are directed from \( t \). Hence, all created edges are useless. Removing these edges, we are left with the paths of \( \mathcal{P} \) disconnected from \( G \). Linking these paths with additional edges, we obtain a path \( P \) with \( p(H) \) vertices. Thus it follows from Theorems 4.6 and 4.8 and Corollary 4.2

\[ m(\sigma(G \cup H)) \geq m(\sigma(G \cup P)). \]

\[ \square \]

Lemma B.2.

\[ p(H) \geq \left\lceil \frac{V(H)}{\lg V(H) + 1} \right\rceil. \]

Proof. Let \( r \) be the root of \( H \). Give the vertices of \( H \) a labeling \( j : V(H) \to \mathbb{Z}^+ \) such that for each vertex \( v \):

- If \( v \) has no children, then \( j(v) = 1 \).
- If \( v \) has children \( w_1, \ldots, w_c \), and \( j(w_i) \) has a unique maximum, then let

\[ j(v) = \max_{1 \leq i \leq c} j(w_i). \]
• If \( v \) has children \( w_1, \ldots, w_c \), but multiple children have the same maximal \( j \), then let

\[
j(v) = 1 + \max_{1 \leq i \leq c} j(w_i).
\]

We will now show that \( j(v) \leq \log V(H) + 1 \) for all \( v \). From the definition of \( j \), we have that if \( j(v) > 1 \), then there exist at least two descendants of \( v \), \( w_1 \) and \( w_2 \) such that

\[
j(w_1) = j(w_2) = j(v) - 1.
\]

By induction, if \( j(v) > k \), then there exist at least \( 2^k \) descendants of \( v \), \( w_1, \ldots, w_{2^k} \) such that \( j(w_i) = j(v) - k \) for all \( i \). We can thus see that

\[
V(H) \geq 2^{j(v)-1}.
\]

Therefore, \( j(v) \leq \log V(H) + 1 \).

Define \( k_i \) to be the number of vertices \( w \) such that \( j(w) = i \). Note that for each \( i \), the vertices such that \( j(w) = i \) form a disconnected-path family. Thus \( p(H) \geq k_i \) for all \( i \). By the pigeonhole principle, there exists an \( i \) such that

\[
p(H) \geq k_i \geq \left\lceil \frac{V(H)}{\log V(H) + 1} \right\rceil,
\]

as desired. \( \square \)

**Corollary B.3.** Let \( G \) be a graph and \( H \) be a flow-out tree disconnected from \( G \). Let \( P \) be a path of length \( \lceil V(H)/(\log V(H) + 1) \rceil \). Then

\[
m(\sigma(G \cup H)) \geq m(\sigma(G \cup P)).
\]

**Corollary B.4.** Let \( H \) be a flow-in tree. Then

\[
p(H) \geq \left\lceil \frac{V(H)}{\log V(H) + 1} \right\rceil.
\]

**Corollary B.5.** Let \( H \) be a disjoint collection of flow-in and flow-out trees. Then

\[
p(H) \geq \left\lceil \frac{V(H)}{\log V(H) + 1} \right\rceil.
\]

**Proof.** This inequality follows from the facts that the function

\[
\left\lceil \frac{x}{\log x + 1} \right\rceil
\]

is subadditive. \( \square \)
It turns out it is easy to find $p(H)$ exactly, when $H$ is a flow-out tree. For each vertex $v$, define $d(v)$ to be the maximum number of nodes on a path from $v$ to some other node. Define another function $b(v)$ with the following properties:

- If there are no edges leading out of $v$, then $b(v) = d(v) = 1$.
- Otherwise, let $w_1, \ldots, w_c$ be the nodes leading out of $v$, then

$$b(v) = \max \left( \sum_{i=1}^{c} b(w_i), d(v) \right).$$

Let $r$ be the vertex from which there is a path to every other vertex. We now prove the following three lemmas concerning the value of $b(r)$.

**Lemma B.6.** $b(r) \geq p(H)$.

*Proof.* We will prove this by induction on $|V(H)|$. If $|V(H)| = 1$, then $b(r) = p(H) = 1$. Assume that $|V(H)| > 1$. We shall show for any disconnected-path family $\mathcal{P}$, that $b(r) \geq V(\mathcal{P})$. Let the children of $r$ be $w_1, \ldots, w_c$. We have two cases to consider.

*Case 1: $r$ is an element of $\mathcal{P}$*

In this case, $\mathcal{P}$ must be a path. Thus, $b(v) \geq d(v) = V(\mathcal{P})$.

*Case 2: $r$ is not an element of $\mathcal{P}$*

Since all the subtrees of $r$ are disconnected from the each other, by the inductive hypothesis, the maximal size of $\mathcal{P}$ is at most

$$\sum_{i=1}^{c} b(w_i) \leq b(v),$$

as desired.

In either case $b(r) \geq V(\mathcal{P})$; therefore $b(r) \geq p(H)$. \qed

**Lemma B.7.** $p(H) \geq b(r)$.

*Proof.* We prove this lemma by constructing a disconnected-path family $\mathcal{P}$ of size $b(r)$. Consider a topological ordering of the vertices of $H$,

$$r = v_1, v_2, \ldots, v_{V(H)}$$

with the property that if $j < i$ then there is no path from $v_i$ to $v_j$. We now scan through the list and determine which elements go in $\mathcal{P}$. This is decided as follows.
Figure 8: Values of $b(v)$ for an example tree. Also noted are the vertices which are elements of $S$ and $T$ from Lemma [B.1].

- If there is a path from any element already in $P$ to $b_i$ or $b(v_i) = \sum_{i=1}^{c} b(w_i)$, then do not add $v_i$ to $P$.
- Otherwise, take the longest path $P_i$ from $v_i$ and add this path to $P$.

See Figure 8 for an example. From the recursion, we know that $b(r)$ vertices are in $P$. For each path of $H$ which is preserved, all the vertices leading out of it and leading into it are not in $P$. Hence, $P$ is a disconnected-path family and $p(H) \geq b(v)$. □

Note. By Lemmas [B.6] and [B.7] we have that $b(r) = p(H)$.

From these Lemmas, $p(H)$ can be computed exactly in linear time using the recursive of $b(r)$.

Remark. Utilizing this recursion to compute $p(H)$, we can construct flow-out trees which show that Lemma [B.2] is asymptotically optimal. See figure 9 for an example.

Now we consider the case that $H$ is a general directed tree.

**Lemma B.8.** Let $H$ be a directed tree. Then,

$$p(H) \geq \left\lfloor \frac{V(H)}{2(\lg V(H) + 1)} \right\rfloor.$$
Figure 9: Example of a flow-out tree $H$ (with $b(v)$ values added) such that a generalized construction of $H$ demonstrates Lemma B.2 is asymptotically optimal.

Proof. Let $r$ be an arbitrary vertex with an indegree of 0. Let $B_0 = \{r\}$. Let $B_1$ be the set of all vertices such that there exists a path from $r$ to any vertex of $B_1$. Let $B_2$ be the set of all vertices not in $B_0 \cup B_1$ such that there exists a unidirectional path from $B_2$ to some vertex of $B_1$. More generally, let $B_k$ be the set of all vertices not in $B_0 \cup B_1 \cup \cdots \cup B_{k-1}$ such that there exists a unidirectional path from each vertex of $B_k$ to $B_{k-1}$. Thus, if $i - j \geq 2$, then the vertices of $B_i$ and $B_j$ are disconnected. Define the two sets

$$B_1 = \bigcup_{i \geq 0} B_{2i+1} \quad \text{and}$$

$$B_2 = \bigcup_{i \geq 0} B_{2i}.$$

It is apparent that $B_1$ is a disjoint collection of flow-out trees and that $B_2$ is a disjoint collection of flow-in trees. From Corollary B.3 we have that

$$p(B_i) \geq \left\lceil \frac{V(B_i)}{\log V(B_i) + 1} \right\rceil.$$
Hence,

\[ p(H) \geq \max(p(B_1), p(B_2)) \geq \left\lceil \frac{V(H)/2}{\log(V(H)/2) + 1} \right\rceil \geq \left\lceil \frac{V(H)}{2 \log(V(H) + 1)} \right\rceil, \]

as desired.

**Theorem B.9.** There exists a linear-time algorithm, given a directed tree \( H \) as input, which computes \( p(H) \).

**Proof.** Let \( r \) be an arbitrary vertex of \( H \). Also, let \( n = V(H) \). Consider an arbitrary ordering

\[ v_1 = r, v_2, \ldots, v_n \]

of the vertices of \( H \) such that for any undirected path from \( r \),

\[ (r, v_{i_1}, w_{i_2}, \ldots, w_{i_k}), \]

the indices \( i_1 \) to \( i_k \) are in increasing numerical order. For example, this ordering could be a level-order traversal of \( H \) rooted at \( r \), where the direction of the edges are ignored. Notice that this implies at most one vertex connected to a vertex \( v_j \) has an index less than \( j \).

Define \( V_1, V_2, \ldots, V_n \) to be subgraph of \( H \) such that \( V_i \) has as vertices the \( v_j \) such that \( j \geq i \) and there exists an undirected path from \( v_j \) to \( v_i \) using only the vertices \( v_k \) such that \( k \geq i \). Notice that \( V_1 = H \).

We now define the six functions \( a_1, a_2, a_3, a_4, a_5, a_6 \) from the set of vertices of \( H \) to the positive integers. There definitions are as follows.

- \( a_1(v_i) \) is the maximal-size disconnected-path family \( P_i^1 \) of \( V_i \) such that \( v_i \) is not a vertex of a path of \( P_i^1 \) and there is no directed path from \( v_i \) to any element of \( P_i^1 \).

- \( a_2(v_i) \) is the maximal-size disconnected-path family \( P_i^2 \) of \( V_i \) such that \( v_i \) is not a vertex of a path of \( P_i^2 \) and there is no directed path from any element of \( P_i^2 \) to \( v_i \).

- \( a_3(v_i) \) is the maximal-size disconnected-path family \( P_i^3 \) of \( V_i \) such that \( v_i \) is a vertex of a path of \( P_i^3 \) and that this path has no edge directed from \( v_i \).

- \( a_4(v_i) \) is the maximal-size disconnected-path family \( P_i^4 \) of \( V_i \) such that \( v_i \) is a vertex of a path of \( P_i^4 \) and that this path has no edge directed toward \( v_i \).
• $a_5(v_i)$ is the maximal-size disconnected-path family $\mathcal{P}_i^5$ of $V_i$ such that $v_i$ is a vertex of a path of $\mathcal{P}_i^5$.

• $a_6(v_i)$ equals $p(V_i)$

Thus, $a_6(v_1)$ equals $p(H)$.

**Lemma B.10.** These six functions satisfy the following recursion.

\[
\begin{align*}
a_1(v_i) &= \sum_{(v_i, v_j) \in E(H)} a_1(v_j) + \sum_{(v_j, v_i) \in E(H)} a_6(v_j) \\
a_2(v_i) &= \sum_{(v_j, v_i) \in E(H)} a_2(v_j) + \sum_{(v_i, v_j) \in E(H)} a_6(v_j) \\
a_3(v_i) &= 1 + \sum_{(v_i, v_j) \in E(H)} a_1(v_j) + \sum_{(v_j, v_i) \in E(H)} a_2(v_j) \\
&\quad + \max_{i \leq j} \left( 0, \max_{(v_i, v_j) \in E(H)} (a_3(v_j) - a_1(v_j)) \right) \\
a_4(v_i) &= 1 + \sum_{(v_j, v_i) \in E(H)} a_2(v_j) + \sum_{(v_i, v_j) \in E(H)} a_1(v_j) \\
&\quad + \max_{i \leq j} \left( 0, \max_{(v_j, v_i) \in E(H)} (a_4(v_j) - a_2(v_j)) \right) \\
a_5(v_i) &= 1 + \sum_{(v_i, v_j) \in E(H)} a_1(v_j) + \sum_{(v_j, v_i) \in E(H)} a_2(v_j) \\
&\quad + \max_{i \leq j} \left( 0, \max_{(v_i, v_j) \in E(H)} (a_3(v_j) - a_1(v_j)) \right) \\
&\quad + \max_{i \leq j} \left( 0, \max_{(v_j, v_i) \in E(H)} (a_4(v_j) - a_2(v_j)) \right) \\
a_6(v_i) &= \max(a_1(v_i), a_2(v_i), a_5(v_i))
\end{align*}
\]

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Proof. For each vertex \( v_i \), we let \( D_i^+ \) be the set of vertices connected to \( v_i \) by an edge directed away from \( v_i \) and \( D_i^- \) be the set of vertices connected to \( v_i \) by an edge directed toward \( v_i \).

\[ a_1 \]: Every element \( v_j \) of \( D_i^+ \) cannot be in \( P_1^i \), nor can vertices \( v_k \) for which there is a directed path from \( v_j \) to \( v_k \) be in \( P_1^i \). Hence, there can be at most \( a_1(v_j) \) vertices of \( V_j \) in \( P_1^i \). For every vertex \( v_j \) in \( D_i^- \), any maximal-size disconnected-path family of \( V_j \) can be chosen. Thus, there can be at most \( a_6(v_j) \) vertices of \( V_j \) in \( P_1^i \). Thus,

\[
 a_1(v_i) \leq \sum_{(v_i,v_j) \in E(H)} a_1(v_j) + \sum_{i \leq j} a_6(v_j)
\]

Because the union of these maximal-size disconnected-path families is a disconnected-path family of \( V_i \), we have that equality holds.

\[ a_2 \]: This is analogous to the proof for \( a_1 \), except the directions of the edges are reversed.

\[ a_3 \]: There are two cases to consider, depending on whether the path through \( v_i \) in \( P_3^i \) contains any vertices besides \( v_i \). If the path does have additional vertices, let \( v_j^+ \) be the element of \( D_i^+ \) through which the path from \( v_i \) in \( P_3^i \) continues. At most \( a_3(v_j^+) \) vertices of \( P_3^i \) can be from the tree \( V_j^+ \). For any other vertex \( v_k^+ \) in \( D_i^+ \), at most \( a_1(v_k^+) \) vertices of \( P_3^i \) can be from the tree \( V_k^+ \). If the path through \( v_i \) does not encompass any other vertices, then each subtree \( V_k^+ \) in \( D_i^+ \) contributes at most \( a_1(v_k^+) \) vertices to \( P_3^i \). Each vertex of \( v_j^- \) in \( D_i^- \) can contribute up to \( a_2(v_j^-) \) vertices to \( P_3^i \). Because there may be choices for which vertex the path from \( v_i \) passes through, we must take the maximum of all the possibilities

\[
 a_3(v_i) \leq 1 + \sum_{(v_i,v_j) \in E(H)} a_1(v_j) + \sum_{i \leq j} a_2(v_j) + \max \left( 0, \max_{i \leq j} (a_3(v_j) - a_1(v_j)) \right)
\]

As the vertices of these maximal-size disconnected-path families form a disconnected-path family of \( V_i \), equality is attainable.

\[ a_4 \]: This is analogous to the proof for \( a_3 \), except the directions of the edges are reversed.
a₅: If the path through $v_i$ in $P^5_i$ has $v_i$ itself as one of its endpoints, then we can use $\max(a_3(v_i), a_4(v_i))$ as an upper bound. Thus, let us assume the contrary. Let $v_{j+}$ be the element of $D^+_i$ through which the path from $v_i$ through $P^5_i$ continues. At most $a_3(v_{j+})$ vertices of $P^5_i$ can be from the tree $V_{j+}$. For any other vertex $v_{k+}$ in $D^+_i$, at most $a_1(v_{k+})$ vertices of $P^5_i$ can be from the tree $V_{k+}$. Let $v_{j-}$ be the element of $D^-_i$ through which the path from $v_i$ enters $v_i$. At most $a_4(v_{j-})$ vertices of $P^5_i$ can be from the tree $V_{j-}$. For any other vertex $v_{k-}$ in $D^-_i$, at most $a_2(v_{k-})$ vertices of $P^5_i$ can be from the tree $V_{k-}$. Because there may be multiple elements in $D^+_i$ or $D^-_i$, we must take the maximum of all possibilities. These facts can be combined to yield the inequality

$$a_5(v_i) = 1 + \sum_{(v_i, v_j) \in E(H)} a_1(v_j) + \sum_{(v_j, v_i) \in E(H)} a_2(v_j)$$

$$+ \max \left( 0, \max_{i \leq j} (a_3(v_j) - a_1(v_j)) \right)$$

$$+ \max \left( 0, \max_{i \leq j} (a_4(v_j) - a_2(v_j)) \right)$$

As from the previous cases, equality is attainable because the vertices of these maximal-size disconnected-path families form a disconnected-path family of $V_i$.

a₆: In the tree $V_i$, either there is a path through $v_i$ in $P^6_i$, which is accounted for in $a_5$, there is no path through $v_i$ and no descendants of $v_i$ are in $P^6_i$, which is accounted for in $a_1$, or there is no path through $v_i$ and no ancestors of $v_i$ are in $P^6_i$, which is accounted for in $a_2$.

Utilizing this recursion, we can construct a linear-time algorithm for computing $p(H)$. First, it takes constant time to compute the values of the six functions for $v_n$. Then, given the values of these six functions for the vertices $v_j, \ldots, v_n$, we can compute these values for $v_{j-1}$ in $O(d(v_{j-1}))$ time. Thus, we can compute $a_6(v_1)$ in

$$O \left( \sum_{i=1}^{n} d(v_i) \right) = O(|E(H)|) = O(n)$$

time, as desired.