Inverse eigenvalue problems for checkerboard Toeplitz matrices

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Abstract.
The inverse eigenvalue problem for real symmetric Toeplitz matrices motivates this investigation. The existence of solutions is known, but the proof, due to H. Landau, is not constructive. Thus a restriction, namely the required eigenvalues are to be equally spaced, is considered here. Two types of structured matrices arise, herein termed, “checkerboard” and “outer-banded”. Examples are presented. Properties of these structured matrices are explored and a full characterization of checkerboard matrices is given. The inverse eigenvalue problem is solved within the class of odd checkerboard matrices. In addition the “symmetric-spectrum” inverse eigenvalue problem is solved within a subclass of Hankel matrices. A regularity conjecture of H. Landau for the Toeplitz inverse eigenvalue problem is discussed and a similar conjecture for checkerboard Toeplitz matrices is given.

1. Introduction
1.1. Motivation
The reconstruction of a matrix to possess specified spectral data is termed an inverse eigenvalue problem. Following the classification of such problems by M. Chu \cite{1} our attention will be directed to the class of “structured inverse eigenvalue problems” (SIEP), where the objective is to construct a matrix with a certain structure which has the prescribed set of eigenvalues. For an extensive survey of such problems, see \cite{2}, \cite{3}. Within the class of structured inverse eigenvalue problems, the Toeplitz inverse eigenvalue problem (TIEP) has received significant attention:

\begin{equation}
\text{(TIEP)} \quad \text{construct a real, symmetric Toeplitz matrix of size } n \times n \text{ with the prescribed eigenvalues: } \lambda_1, \lambda_2, \ldots, \lambda_n.
\end{equation}

Recall that a Toeplitz matrix, $T_n \in \mathbb{R}^{n \times n}$, is one for which the entries are constant along each diagonal:

\[
T_n = \begin{bmatrix}
    a_0 & a_1 & a_2 & \cdots & a_{n-1} \\
    a_1 & a_0 & a_1 & \cdots & a_{n-2} \\
    a_2 & a_1 & a_0 & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    a_{1-n} & a_{2-n} & \cdots & a_0
\end{bmatrix}
\]

When attention is restricted to symmetric matrices, the Toeplitz matrix is determined by its top row. Accordingly, (after relabeling) we denote it: $T_n = T_n(a_1, a_2, \ldots, a_n), \ a_i \in \mathbb{R}, \ i = 1, \ldots, n.$
Given the prescribed set of distinct values,

\[ \lambda_1 < \lambda_2 < \cdots < \lambda_n, \]

the existence of a real symmetric, \( n \times n \) Toeplitz matrix possessing these as its eigenvalues was demonstrated by H. J. Landau [4], within a class of Toeplitz matrices having a certain additional regularity. This result is described in Subsection 5.1. Unfortunately, the proof does not present a construction, and the constructive TIEP remains, in Landau’s words, “challenging because of its analytic intractability” ([4] p. 749).

While the TIEP has been solved up to size \( n \leq 4 \) by P. Delsarte and Y. Genin [5], forewarned by the statement of Landau, we did not expect to make headway on the analytic reconstruction of larger size general Toeplitz matrices. So we decided to impose further structure on the inverse problem. Following a strategy employed in [6] for a different SIEP, the problem we consider here is the TIEP with prescribed, \textit{equally spaced} eigenvalues, listed in decreasing order:

\[ \lambda_i = \lambda_1 - (i - 1)r, \quad i = 1, \ldots, n, \quad r \in \mathbb{R}^+. \] (1.1)

This particular choice of eigenvalues is motivated by the idea that the reciprocal of the minimum separation between eigenvalues can be thought of as a condition number on the sensitivity of the eigenvectors (invariant subspaces) to perturbation (see [7], Theorem 8.1.12). Therefore a solution to the TIEP with equally spaced, distinct (simple) eigenvalues will be important as a class of arbitrary size Toeplitz test matrices having perfect conditioning with respect to the (forward) eigenvalue-eigenvector problem. The availability of such test matrices may be helpful when comparing the efficacy of the various iterative schemes for the TIEP, such as Newton’s method [8], algebraic techniques [9], [10], and isospectral flows [11], [12].

1.2. Problem Statement

\textbf{Proposition 1.1}. If a symmetric Toeplitz matrix \( T(a_1, a_2, \ldots, a_n) \in \mathbb{R}^{n \times n} \) has eigenvalues, \( \{\mu_1, \mu_2, \ldots, \mu_n\} \), then the Toeplitz matrix, \( T^* = \alpha(T - a_1I) = T^*(0, \alpha a_2, \ldots, \alpha a_n) \) has eigenvalues: \( \{\alpha(\mu_1 - a_1), \alpha(\mu_2 - a_1), \ldots, \alpha(\mu_n - a_1)\} \), \( \forall \alpha \in \mathbb{R} \).

\textbf{Proof}: Adding two Toeplitz matrices produces another Toeplitz matrix, as does scaling one by a real constant. Since \( a_1I \) is Toeplitz, therefore \( T^* \) has the required Toeplitz form, and has eigenvalues equal to those of \( T \) shifted (reduced) by \( a_1 \) and then scaled by \( \alpha \). \( \square \)

Consequently, the prescribed set, (1.1) can be shifted for convenience to have zero mean and (equal) spacing of two. Thus we take \( \lambda_1 = n - 1 \) and \( r = 2 \), in (1.1) as the desired eigenvalue set:

\[ \{(n - 1), (n - 3), \ldots, -(n - 3), -(n - 1)\}, \quad n \in \mathbb{Z}^+. \] (1.2)

So our restricted problem is:

\textbf{(rTIEP)} construct a real, symmetric Toeplitz matrix, \( T_n \) of size \( n \times n \) with the prescribed eigenvalues (1.2).

A less-restricted version of the above structured inverse eigenvalue problem will be termed the \textit{symmetric spectrum} problem:

\textbf{(ssSIEP)} construct a real, symmetric matrix of size \( n \times n \), in a given class of structured matrices, with prescribed eigenvalues symmetric about 0, that is, when \( \lambda \) is an eigenvalue \( -\lambda \) is also an eigenvalue of the same multiplicity.
The eigenvalues are \( \Lambda = \text{diag}(1) \) if \( n = 2 \), since \( T(-a_1, -a_2, \ldots, -a_n) \) has the symmetric spectrum \((1, 2, \ldots, n)\). Additional sign choices were observed. However for all sizes up to 10, we found, with the aid of Maple\(^\text{\textregistered}\)15, at least one non-negative rTIEP solution (i.e., matrices with all entries non-negative).

Further non-uniqueness arises from two different patterns of non-zero entries observed in rTIEP solution matrices. Up to size \( n = 4 \) we verified that these are the only possible patterns. The first is a checkerboard pattern, with zeroes occupying all positions where the sum of the row and column indices is even. So a symmetric Toeplitz checkerboard matrix, \( C_n \), is characterized as
\[
C_{2m} \left( c_1, \ c_2, \ldots, \ c_m \right) := T_{2m} \left( 0, \ c_1, \ 0, \ c_2, \ 0, \ c_3, \ldots, \ 0, \ c_m \right),
\]
when the size \( n = 2m \) is even, and the same (appending a trailing 0 entry in the \( T \) matrix) when of odd size, \( n = 2m + 1 \).

The second observed pattern is termed outer-banded. In this pattern the non-zero elements occupy the diagonal bands furthest from the main diagonal, while zeroes occupy the main diagonal and the first \( \left\lfloor \frac{n-1}{2} \right\rfloor \) off-diagonal bands. So a symmetric Toeplitz outer-banded matrix, \( B_n \), is characterized as
\[
B_{2m} \left( b_1, \ b_2, \ldots, \ b_m \right) := T_{2m} \left( 0, \ 0, \ldots, \ 0, \ b_1, \ b_2, \ldots, \ b_m \right),
\]
when the size \( n = 2m \) is even, and the same (inserting an extra leading 0 entry in the \( T \) matrix) when of odd size, \( n = 2m + 1 \).

2.2. Non-negative Examples by Increasing Size

For each size \( n \), the examples listed below have the eigenvalues arranged in decreasing order, and the columns of the (orthogonal) eigenvector matrices, \( V \in \mathbb{R}^{n \times n} \) arranged consistently. The first component of each eigenvector is chosen to be non-negative. Thus we have the eigenvalue/eigenvector decomposition: \( T_n = VAV^T \) where the diagonal matrix of eigenvalues is: \( \Lambda = \text{diag}(n-1, n-3, \ldots, 3-n, 1-n) \).

For size \( n = 2 \), the checkerboard and outer-banded patterns coincide and are necessary to solve the rTEIP.

\[
B_2 \left( 1 \right) = C_2 \left( 1 \right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

The eigenvalues are \( \Lambda = \text{diag}(1, -1) \), and the unit eigenvectors are \( V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \).

\[
B_3 \left( 2 \right) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad C_3 \left( \sqrt{2} \right) = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}
\]

The eigenvalues are \( \Lambda = \text{diag}(2, 0, -2) \), and the unit eigenvectors are:
Equating the characteristic polynomial of the generic $T_3(0, a, b)$ to the target polynomial with roots $\{2, 0, -2\}$, yields: $\lambda^3 - (2a^2 + b^2)\lambda - 2a^2b = \lambda(\lambda^2 - 4)$, which implies either $a = 0$ (i.e. outer-banded, and $b = \pm2$) or $b = 0$ (i.e. checkerboard, and $a = \pm\sqrt{2}$).

$$B_4(\sqrt{3}, 2) = \begin{bmatrix} 0 & 0 & \sqrt{3} & 2 \\ \sqrt{3} & 0 & 0 & \sqrt{3} \\ 2 & \sqrt{3} & 0 & 0 \end{bmatrix}, \quad C_4\left(\frac{1+\sqrt{7}}{2}, \frac{3-\sqrt{7}}{2}\right) = \begin{bmatrix} 0 & \frac{1+\sqrt{7}}{2} & 0 & \frac{3-\sqrt{7}}{2} \\ \frac{1+\sqrt{7}}{2} & 0 & \frac{1+\sqrt{7}}{2} & 0 \\ 0 & \frac{1+\sqrt{7}}{2} & 0 & \frac{1+\sqrt{7}}{2} \end{bmatrix}$$

The eigenvalues are $\Lambda = \text{diag}(3, 1, -1, -3)$, and the unit eigenvectors are:

$$V_{B_4} = \frac{1}{\sqrt{8}} \begin{bmatrix} \sqrt{3} & 1 & 1 & \sqrt{3} \\ 1 & -\sqrt{3} & -\sqrt{3} & 1 \\ \sqrt{3} & -\sqrt{3} & -\sqrt{3} & -1 \\ -1 & -\sqrt{3} & -\sqrt{3} & \sqrt{3} \end{bmatrix}, \quad V_{C_4} = \frac{1}{\sqrt{24-8\sqrt{7}}} \begin{bmatrix} \sqrt{7} - 2 & 1 & 1 & \sqrt{7} - 2 \\ 1 & \sqrt{7} - 2 & 2 - \sqrt{7} & 1 \\ 2 - \sqrt{7} & 2 - \sqrt{7} & 1 & 1 \\ 1 & 1 & 2 - \sqrt{7} & 2 \end{bmatrix}$$

Again equating the characteristic polynomial of the generic $T_4(0, a, b, c)$ to the target polynomial with roots $\{3, 1, -1, -3\}$, yields:

$$\lambda^4 - (3a^2 + 2b^2 + c^2)\lambda^2 - 4ab(a + c)\lambda + (a^2 + b^2 - ca - 2ba)(a^2 + b^2 - ac + 2ba) = \lambda^4 - 10\lambda^2 + 9.$$ 

So, either $a = 0$, $b = \pm\sqrt{3}$, $c = \pm2$, (i.e. four possibilities, all outer-banded) or $b = 0$, $a = \pm\left(\frac{1+\sqrt{7}}{2}\right)$, $c = \pm\left(\frac{3+\sqrt{7}}{2}\right)$, where the plus/minus symbols inside the parentheses for $a$ and $c$ match, as do the plus/minus symbols outside the parentheses (i.e. four possibilities, all checkerboard), or $c = -a$ which implies that $a$ is complex, in contradiction of assumption.

Consequently, one of the two patterns, checkerboard or outer-banded, is necessary to solve the rTIEP for all sizes $n \leq 4$, though this result has not been verified for sizes $n \geq 5$.

$n=5, 6$

In $C_5$ there are two sign choices in the non-negative solution presented: either the top symbol in every $\pm$ and $\mp$, or the bottom. The same sign choice is to be applied to the eigenvector matrix.

$$B_5\left(\sqrt{8}, 2\right) \quad C_5\left(\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2}\right)$$

The eigenvalues are $\Lambda = \text{diag}(4, 2, 0, -2, -4)$, and the unit eigenvectors of $C_5$ are:

$$V_{C_5} = \frac{1}{12} \begin{bmatrix} \pm\sqrt{22} - 1 & 6 & 2 \pm \sqrt{22} & 6 & \pm \sqrt{22} - 1 \\ 6 & 6 & 0 & -6 & -6 \\ 2 \pm \sqrt{22} & 0 & 2 \mp \sqrt{22} & 0 & 2 \pm \sqrt{22} \\ 6 & -6 & 0 & 6 & -6 \\ \pm \sqrt{22} - 1 & -6 & 2 \pm \sqrt{22} & -6 & \pm \sqrt{22} - 1 \end{bmatrix}$$

Some of the possible size $n = 6$ matrices are: $B_{6a}\left(\sqrt{15}, \pm\sqrt{13 - 2(\sqrt{15})^2} + 3\sqrt{15}, 3 - \sqrt{15}\right)$, $B_{6b}\left(\sqrt{15}, \pm\sqrt{17 - 2(\sqrt{15})^2} - \sqrt{15}, -1 - \sqrt{15}\right)$, and $C_6\left(c_1, c_2, 3 - c_1 - c_2\right)$, where $c_2$ is
a root of the 6th order polynomial \( P(x) = 54x^6 - 216x^5 + 3x^4 + 612x^3 - 800x^2 + 1500x - 369, \) and
\[
\begin{align*}
&\text{c}_1 = \frac{200}{2176} - 172\frac{43}{554} c_2 + 2273\frac{3}{2} - 605\frac{1}{3} c_2^2 - 131\frac{1}{2} \frac{1}{c_2} + 14\frac{1}{5} c_2^3, \\
&\text{there are three positive roots of} \ P(x), \text{but only one yields a non-negative checkerboard matrix, which is (to a floating point approximation):}
\end{align*}
\]
\[
C_6 = \begin{pmatrix}
2.636649799 & 0.2788346628 & 0.084555382
\end{pmatrix}
\]
The eigenvalues are \( \Lambda = \text{diag}(5, 3, 1, -1, -3, -5) \) (to nine significant figures).

\( n = 7, \ldots, 10 \)

We have been unable, to date, to find checkerboard solutions, even numerically, for sizes \( n \geq 7 \). However, outer-banded examples include: \( B_{7n} \left( \sqrt{48}, \ \pm \sqrt{28 - 2(\sqrt{48})^2}, \ -\sqrt{48} \right), \)
\[
B_{7n} \left( \sqrt{48}, \ \pm \sqrt{20 - 2(\sqrt{48})^2 + 4\sqrt{48}}, \ 4 - \sqrt{48} \right),
\]
\[
B_8 \left( \sqrt{105}, \ 3.737635629, \ \sqrt{21.01070232 - 2\sqrt{105}}, \ 0.262364371 \right),
\]
\[
B_9 \left( \sqrt{384}, \ 3.696457542, \ 0.5161371538, \ 0.303542458, \right), \text{and}
\]
\[
B_{10} \left( \sqrt{945}, \ 4.605343159, \ 0.9009158716, \ 0.3392412014, \ 0.1628007010 \right).
\]

2.3. Observations for the checkerboard examples

(i) The orthogonal matrix of eigenvectors for the checkerboard solutions to rTIEP are symmetric up to size \( n = 5 \) and seems to remain so numerically for larger sizes.

(ii) Up to size \( n = 4 \), the non-negative checkerboard solution to rTIEP is unique. However, we found two such solutions at size \( n = 5 \), and suspect there are additional ones for size \( n = 6 \).

(iii) Up to size \( n = 6 \) we have found exactly one solution to rTIEP which is regular (definition by Landau, to follow in Section 5.1), all of them being non-negative, checkerboard matrices.

3. Symmetry Properties

Following is a brief review of the definitions and basic results for the various matrix symmetries.

Definition 3.1. The following belong to \( \mathbb{R}^{n \times n} \). A subscript indicates the matrix size if necessary.

(i) \( A = \text{diag}(1, -1, 1, \ldots, (-1)^{n+1}) \) is the alternating diagonal matrix with diagonal entries \( \pm 1 \), starting with a positive 1.

(ii) \( J = \text{antidiag}(1, 1, 1, \ldots, 1) \) is the anti-identity matrix with 1s on the main anti-diagonal. In other words \( [J]_{i,j} = \delta_{1,n+1-j} \). \( J \) is sometimes called the exchange permutation (see Golub and Van Loan \([7]\) p. 20), because premultiplication by \( J \) exchanges the rows of a matrix, completely reversing their order, i.e. premultiplication by \( J \) turns a matrix upside down.

(iii) \( E \) is the exchange matrix: \( [e_1 \ e_2 \ \cdots \ e_{2(\frac{n}{2})-1} \ e_2 \ e_1 \ \cdots \ e_{2(\frac{n}{2})}]^T \), where \( e_i \) is the \( i \)th standard basis vector of \( \mathbb{R}^n \). Note that if \( n = 2k \), the matrix \( E \) is called a mod-2 perfect shuffle matrix, denoted \( P_{2,k} \) (\([7]\) p. 20).

Note that \( A, J, \) and \( E \) are all self-inverses, that is, \( A^2 = J^2 = E^2 = I \).

Definition 3.2. For \( M \in \mathbb{R}^{n \times n} \) the following symmetries are defined:

(i) \( M \) is symmetric when \( M = M^T \).

(ii) \( M \) is centrosymmetric when \( M = JMJ \).

(iii) \( M \) is persymmetric when \( M = JM^TJ \), or equivalently, \( M^T = JMJ \).

(iv) \( M \) is super-symmetric when it possesses all three of the above symmetries.
Definition 3.3. For $M \in \mathbb{R}^{n \times n}$ the following skew-symmetries are defined:

(i) A matrix $M$ is skew-symmetric or anti-symmetric when $M = -M^T$.

(ii) A matrix $M$ is skew-centrosymmetric or anti-centrosymmetric when $M = -JM_J$.

(iii) A matrix $M$ is skew-persymmetric or anti-persymmetric when $M = -JM^T_J$.

Theorem 3.4. Let $M \in \mathbb{R}^{n \times n}$, be a non-zero matrix.

(i) If $M$ possesses any two of the symmetries, (symmetric, centrosymmetric, or persymmetric), then it also has the third, and hence is super-symmetric.

(ii) If $M$ possesses any two of the skew-symmetries, (skew-symmetric, skew-centrosymmetric, or skew-persymmetric), then it also has the third symmetry type, but not skew.

(iii) If $M$ possesses a symmetry and a skew-symmetry, then it also has the remaining skew-symmetry.

Proof: Consider the three relations, $M \sim M^T \sim JM_J$.

When any two of these are symmetries (i.e. replace $\sim$ with equality), so is the third, proving (i). When any two of these relations are skew-symmetries, the third relation must be a symmetry, proving (ii). Finally, if any one relation is a skew-symmetry, and a second is a symmetry, then the remaining relation is a skew-symmetry, proving (iii).

Recall that Toeplitz matrices are persymmetric by definition. Hence the class of symmetric Toeplitz matrices is a class of super-symmetric matrices, and in particular are centrosymmetric.

Following [13], [14] we now recall some of the basic properties of centrosymmetric matrices.

Definition 3.5. An $n$ dimensional vector $v$ is said to have even parity if $Jv = v$, and is said to have odd parity if $Jv = -v$. If $v$ is an eigenvector with even (odd) parity, the corresponding to an eigenvalue, $\lambda$, is said to be even (odd).

Note that some authors use the term “symmetric” to label vectors with the property $Jv = v$, and “skew-symmetric” for vectors satisfying $Jv = -v$.

Proposition 3.6. The eigenspaces of a centrosymmetric matrix have bases which can be written to have parity.

Proof: Let $M = JM_J$ and let $Mv = \lambda v$. Then $Mv = JM_Jv = \lambda v$, so $MJv = \lambda Jv$, so $Jv$ is also an eigenvector with the same eigenvalue. If $v = \alpha Jv$ then $\alpha = \pm 1$ and so $v$ has parity. Otherwise, $v$ and $Jv$ are linearly independent and we can construct $u = v + Jv$ and $w = v - Jv$. The span of $\{u, w\}$ is the same as the span of $\{v, Jv\}$, and $u$ and $w$ have (opposite) parity.

Cantoni and Butler ([13], Theorem. 2 p. 280) proved the following:

Theorem 3.7. Every real, super-symmetric matrix (in [13] they assumed symmetric and centrosymmetric) of size $n$ has $\left\lceil \frac{n}{2} \right\rceil$ even parity eigenvectors and $\left\lfloor \frac{n}{2} \right\rfloor$ odd parity eigenvectors.

Such a set of eigenvectors can always be arranged alternately, with the first column even, so that the columns of the eigenvector matrix alternate in parity.

Definition 3.8. An $n \times n$ matrix $V$ is said to have alternating column parity when $VA = JV$, and is said to have alternating row parity when $AV = JV$.

Cantoni and Butler go on to assert the converse of Theorem 3.7 as well ([13] thm. 3): every real, $n \times n$ matrix with a set of $n$ orthonormal eigenvectors, all with parity, must be super-symmetric. We present a simple proof of a slightly stronger result, and add their result as a corollary:
**Proposition 3.9.** Let $M$ be diagonalizable, with $M = VAV^{-1}$, where $V$ has alternating column parity. Then $M$ is centrosymmetric.

**Proof:** Since $V$ is assumed to have alternating column parity, we have $JV = VA$. Then

$$ JM J = JVAV^{-1}J = JVA(JV)^{-1} = VAA(VA)^{-1} = VAAV^{-1} = VAV^{-1} = M $$

since $A$ and $J$ are diagonal and $A = A^{-1}$. 

**Corollary 3.10.** If, in addition to having alternating column parity in Proposition 3.9, $V$ is also orthogonal, then $M = VAV^{-1}$ is also symmetric (and hence super-symmetric).

**Lemma 3.11.** Suppose $V$ is symmetric. Then $V$ has alternating column parity if and only if it has alternating row parity.

4. Checkerboard Matrices

4.1. Properties of Checkerboard Matrices

Checkerboard matrices arose as observed solutions of the (rTIEP) for small sizes in Section 2, and form an interesting class. We now explore some of their basic properties.

**Definition 4.1.** We say that $C \in \mathbb{R}^{m \times n}$ is an odd checkerboard matrix if its entries $(C)_{ij}$ are non-zero only where $i + j$ is odd, $i = 1, \ldots , m, \quad j = 1, \ldots , n$. Likewise $C$ is an even checkerboard matrix if its non-zero entries occur only where $i + j$ is even, $i = 1, \ldots , m, \quad j = 1, \ldots , n$. Denote an even checkerboard matrix by $C_e$, and an odd checkerboard matrix by $C_o$.

**Proposition 4.2.**

(i) Every matrix $M \in \mathbb{R}^{m \times n}$ can be uniquely decomposed into the sum of an even checkerboard matrix and an odd checkerboard matrix.

(ii) The product of two even checkerboard matrices is an even checkerboard matrix when the product is defined.

(iii) The product of two odd checkerboard matrices is an even checkerboard matrix when the product is defined.

(iv) The product of an even and an odd checkerboard matrix is an odd checkerboard matrix when the product is defined.

**Proof:**

(i) Let $M \in \mathbb{R}^{m \times n}$. Define $M_e$ to be the even checkerboard matrix with entries $(M_e)_{ij} = (M)_{ij}$ when $i + j$ is even, and 0 otherwise. Define $M_o$ to be the odd checkerboard matrix with entries $(M_o)_{ij} = (M)_{ij}$ when $i + j$ is odd and 0 otherwise. Then $M = M_e + M_o$. Now suppose that $M = C_e + C_o$. Then $M_e - C_e = C_o - M_o$. Clearly, the left-hand side is an even checkerboard matrix, and the right-hand side is an odd checkerboard matrix. Since they are equal, they must equal the zero matrix, and so $C_e = M_e$ and $C_o = M_o$.

(ii-iv) Any even checkerboard vector in $\mathbb{R}^n$ is orthogonal to any odd checkerboard vector in $\mathbb{R}^n$. The $i$th row and column of an even checkerboard matrix is even when $i$ is odd, and vice-versa. Similarly the $i$th row and column of an odd checkerboard matrix is odd when $i$ is odd, and vice-versa. For checkerboard matrices, $P$ and $Q$, the product $C = PQ$ has entries given by $(C)_{ij} = P(i,:)Q(:,j)$ (using the colon notation of [7]). Thus if $P$ and $Q$ are both even, then $(C)_{ij} = 0$ when $i + j$ is even. If $P$ and $Q$ are both odd, then $C_{ij} = 0$ when $i + j$ is even. Finally, if $P$ is odd and $Q$ is even (or vice-versa), then $C_{ij} = 0$ for $i + j$ odd.
Proposition 4.3. Let $M$ be an $m \times n$ matrix. Then:

(i) $A_m M A_n = M$ if and only if $M$ is an even checkerboard matrix.
(ii) $A_m M A_n = -M$ if and only if $M$ is an odd checkerboard matrix.
(iii) $M + A_m M A_n$ is an even checkerboard matrix, and $M - A_m M A_n$ is an odd checkerboard matrix.

Proof: Note, for any matrix, $(A_m M A_n)_{ij} = (-1)^{i+j} M_{ij}$.

(i) If $A_m M A_n = M$, then $M_{ij} = -M_{ij}$ when $i + j$ is odd. Therefore $M_{ij} = 0$ when $i + j$ is odd, so $M$ is an even checkerboard matrix. If $M$ is an even checkerboard matrix, then since $M_{ij} \neq 0$ only when $i + j$ is even, we have $A_m M A_n = M$.

(ii) If $A_m M A_n = -M$, then $M_{ij} = -M_{ij}$ when $i + j$ is even. Therefore $M_{ij} = 0$ when $i + j$ is even, so $M$ is an odd checkerboard matrix. If $M$ is an odd checkerboard matrix, then since $M_{ij} \neq 0$ only when $i + j$ is odd, we have $A_m M A_n = -M$.

(iii) Consider $C = M \pm A_m M A_n$. Then $A_m C A_n = A_m M A_n \pm M = \pm C$. Thus $M + A_m M A_n$ is an even checkerboard matrix, and $M - A_m M A_n$ is an odd checkerboard matrix by properties 1 and 2 respectively.

Proposition 4.4. Let $C_o$ be an odd checkerboard matrix. If $\lambda$ is an eigenvalue of $C_o$ with associated eigenvector, $v$, then $A v$ is an eigenvector of $C_o$ with eigenvalue $-\lambda$.

Proof: Since $C_o$ is odd checkerboard, if $C_o v = -A C_o A v = \lambda v$, then $C_o A v = -\lambda A v$.

Definition 4.5. Let $v \in \mathbb{R}^n$. Define $v_{\text{even}} \in \mathbb{R}^{|\frac{n}{2}|}$ by $(v_{\text{even}})_i = v_{2i}$, $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, and define $v_{\text{odd}} \in \mathbb{R}^{|\frac{n}{2}|}$ by $(v_{\text{odd}})_i = v_{2i-1}$, $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

Proposition 4.6. If $C_o$ is an odd checkerboard matrix of size $(2m - 1) \times (2m - 1)$, then $C_o$ has an even checkerboard eigenvector corresponding to eigenvalue $\lambda = 0$.

Proof: Let $C_o$ be an odd checkerboard matrix of size $(2m - 1) \times (2m - 1)$, $m \in \mathbb{N}$, and let $v$ be a vector such that $C_o v = 0$. Also, we will choose $v_{\text{even}} = 0$, which makes $v$ an even checkerboard matrix. The odd numbered rows of $C_o$ when multiplied by $v$ clearly give 0, thus we only need to verify that the even numbered rows also give zero. There are $m - 1$ even numbered rows in $C_o$ and $m$ unknown entries in $v$, thus we have an underdetermined homogeneous system of linear equations which is guaranteed to have non-trivial solutions. Hence we have a non-zero even checkerboard eigenvector associated with the eigenvalue $\lambda = 0$.

Corollary 4.7. If $C_o$ is an odd checkerboard matrix of size $(2m - 1) \times (2m - 1)$, then $C_o$ is singular.

Proposition 4.8. Let $C_e$ be a square even checkerboard matrix. If $\lambda$ is an eigenvalue of $C_e$ with associated eigenvector, $v$, then $A v$ is an eigenvector of $C_e$ with eigenvalue $\lambda$.

Proof: Since $C_e$ is even checkerboard, if $C_e v = A C_e A v = \lambda v$, then $C_e A v = \lambda A v$.

Corollary 4.9. Let $C_e$ be a square even checkerboard matrix. Suppose that $\lambda$ is an eigenvalue of $C_e$. Then an associated eigenvector is either an even checkerboard vector, an odd checkerboard vector, or we can construct it to be either.
Proof: Consider eigenvectors $\mathbf{v}$ and $A\mathbf{v}$ associated with eigenvalue $\lambda$. If $\mathbf{v} = A\mathbf{v}$ then $\mathbf{v}$ is an even checkerboard vector (Prop. 4.3, part 1). If $\mathbf{v} = -A\mathbf{v}$ then $\mathbf{v}$ is an odd checkerboard vector (Prop. 4.3, part 2). If $\mathbf{v} \neq A\mathbf{v}$, then $\mathbf{u} = \mathbf{v} + A\mathbf{v}$ is also an eigenvector with the same eigenvalue, and $A\mathbf{u} = \mathbf{u}$. Also, $\mathbf{w} = \mathbf{v} - A\mathbf{v}$ has eigenvalue $\lambda$, and $A\mathbf{w} = -\mathbf{w}$.

Proposition 4.10. If $C_o$ is a square odd checkerboard matrix, then $C_o \sim D_n := \begin{bmatrix} O_{\frac{n}{2}} & U \\ V & O_{\frac{n}{2}} \end{bmatrix}$.

If $C_e$ is a square even checkerboard matrix, then $C_e \sim G_n := \begin{bmatrix} Q_{\frac{n}{2}} & O \\ O & R_{\frac{n}{2}} \end{bmatrix}$, where $U \in \mathbb{R}^{\frac{n}{2} \times \frac{n}{2}}$, $V \in \mathbb{R}^{\frac{n}{2} \times \frac{n}{2}}$, and $Q$ and $R$ are square matrices of the indicated size.

Proof: It is clear that $EC_oE^T = D_n$, and $EC_eE^T = G_n$.

Proposition 4.11. The eigenvalues of a square odd checkerboard matrix $C_o$ are symmetric about the origin.

Proof: Since $C_o \sim AC_oA = -C_o$, the eigenvalues of $C_o$ and $-C_o$ are identical, and hence must be symmetric about the origin.

Proposition 4.12. If $C_e$ is invertible then $C_e^{-1}$ is even checkerboard. If $C_o$ is invertible then $C_o^{-1}$ is odd checkerboard.

Proof: Suppose $C_e$ is invertible. Now by Proposition 4.10 we know that $C_e = E^T \text{diag}(Q, R)E$, so $C_e^{-1} = E^T \text{diag}(Q^{-1}, R^{-1})E$. Thus $C_e^{-1}$ is an even checkerboard matrix. Similarly, suppose $C_o$ is invertible. $C_o$ is similar to a block anti-diagonal matrix, the inverse of which is also block anti-diagonal, so $C_o^{-1}$ will also be an odd checkerboard matrix.

4.2. Main Result

We are now able to fully characterize checkerboard matrices in Theorems 4.13 and 4.17 below.

Theorem 4.13. Let $M$ be an $n \times n$ diagonalizable matrix. Then $M$ is an odd checkerboard matrix if and only if $M = VAV^{-1}$, where $V$ can be written so that $AV = VJ$ and $J\Lambda J = -\Lambda$.

Proof: Let $M$ be a diagonalizable odd checkerboard matrix. From Proposition 4.11, we know that the spectrum of $M$ is symmetric about the origin. Let $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ be skew-persymmetric, that is, $\lambda_k = -\lambda_{n+1-k}$. If $n$ even, let $V$ be the block matrix defined by $[U|AUJ]$ where $U$ is the matrix whose columns are the first $\frac{n}{2}$ eigenvectors of $M$ corresponding to $\lambda_1, \ldots, \lambda_{\frac{n}{2}}$. If $n$ is odd, define $V$ to be $[U|v_{\frac{n+1}{2}}|AUJ]$ where $v_{\frac{n+1}{2}}$ is associated with eigenvalue 0. Note that $v_{\frac{n+1}{2}}$ can be chosen to be an even checkerboard vector, by Prop. 4.6. It is clear that $MV = \Lambda$, and the rows of $V$ have alternating parity, or in other words, $AV = VJ$.

Let $M$ be diagonalizable, that is, $MV = VA$. Let $V$ have rows with alternating parity, starting with even, so $AVJ = V$. And let $\Lambda$ be skew-persymmetric, thus $J\Lambda J = -\Lambda$. Decompose $M = C_e + C_o$, where $C_e$ is an even checkerboard matrix, and $C_o$ is an odd checkerboard matrix. Then

$$MV = C_eAVJ + C_oAVJ = AVJ.$$ 

Since $A^2 = I$ and $J^2 = I$, we can rearrange the above equation to

$$AC_eAV + AC_oAV = VJ\Lambda J = -V\Lambda.$$
Because \((AMA)_{ij} = (-1)^{i+j}M_{ij}\), we have that \(AC_eA = C_e\) and \(AC_oA = -C_o\). Therefore we have two equations

\[
\begin{align*}
C_eV + C_oV &= VA \\
C_eV - C_oV &= -VA
\end{align*}
\]

Thus \(C_eV = 0\), and so \(C_e = 0\), proving that \(M\) is an odd checkerboard matrix.

**Corollary 4.14.** If \(M = VAV^{-1}\) with \(JAJ = -\Lambda\) and \(V = AVJ = JVA\), then \(M\) is centrosymmetric odd checkerboard matrix.

**Proof:** Since \(M\) is diagonalizable and \(V = JVA\), \(M\) is centrosymmetric by Proposition 3.9. It is odd checkerboard by Theorem 4.13

**Corollary 4.15.** Under the conditions in Corollary 4.14, if in addition \(V\) is orthogonal, that is \(V^{-1} = V^T\), then \(M\) is supersymmetric.

Using the main result, we can now constructively solve the general SIEP for odd checkerboard matrices.

**Corollary 4.16.** The structured inverse eigenvalue problem for an odd checkerboard matrix is solvable if and only if the spectrum is symmetric with respect to the origin.

**Proof:** Since square, odd checkerboard matrices must have a symmetric spectrum of eigenvalues by the main result, Theorem 4.13, the necessity of a symmetric spectrum for a solution follows. For the converse, assume a skew-persymmetric diagonal eigenvalue matrix \(\Lambda\) formed with the desired eigenvalues. The Kac matrix is a diagonalizable, odd checkerboard matrix of arbitrary size. An explicit formula for the eigenvectors of the Kac matrix can be found in [15]. Use this formula to construct an eigenvector matrix of the desired size. This matrix, labeled \(V\), can be arranged with alternating row parity by Theorem 4.13. Finally, \(VAV^{-1}\) is the desired odd checkerboard matrix, by Theorem 4.13.

**Theorem 4.17.** Let \(M\) be an \(n \times n\) diagonalizable matrix. Then \(M\) is an even checkerboard matrix if and only if \(M = VAV^{-1}\), where \(V\) is an even checkerboard matrix.

**Proof:** Suppose that \(V\) is an invertible even checkerboard matrix. Then \(V^{-1}\) is also even checkerboard by Proposition 4.12 and we also note that a diagonal matrix is even checkerboard. Thus by Proposition 4.2 we know that \(VAV^{-1}\) is also even checkerboard.

Next suppose that \(M\) is a diagonalizable even checkerboard matrix. By Proposition 4.10, \(M\) is similar to a block diagonal matrix \(D = \text{diag}(Q,R)\). Both matrices \(Q\) and \(R\) are therefore diagonalizable, so that \(Q = ULU^{-1}\) and \(R = WMW^{-1}\), where \(L\) and \(M\) are diagonal matrices and \(U\) and \(W\) are matrices of eigenvectors of \(Q\) and \(R\) respectively. Then \(\text{diag}(Q,R) = \text{diag}(U,W)\text{diag}(L,M)\text{diag}(U^{-1},W^{-1})\). And so,

\[
M = E^TDE = E^T\text{diag}(U,W)E\text{diag}(L,M)E E^T\text{diag}(U^{-1},W^{-1})E.
\]

Define \(V = E^T\text{diag}(U,W)E\) and \(\Lambda = E^T\text{diag}(L,M)E\). Then clearly \(V\) is an even checkerboard matrix, and \(\Lambda\) is a diagonal matrix.
5. Toeplitz matrices revisited
The first non-zero entry in an outer-banded matrix satisfying rTIEP can be established.

**Proposition 5.1.** Outer-banded Toeplitz matrices of even size satisfy $|\det B_{2n}| = b_{2n}^2$.

**Proof:** Let $B_{2m}$ be written with lower-triangular, Toeplitz blocks, $L \in \mathbb{R}^{m \times m}$: $B_{2m} = \begin{bmatrix} O & L^T \\ L & O \end{bmatrix}$, and let $M = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$, where $O, I$, are $m \times m$ zero and identity matrices. Then $MB_{2m} = \begin{bmatrix} L & O \\ O & L^T \end{bmatrix}$, and $\det MB_{2m} = \det L \det L^T$. Since $M$ is a permutation matrix, we have $\det B_{2m} = \pm(\det L)^2$. Since $L$ is lower triangular with constant diagonal $b_1$, the result follows. ■

**Corollary 5.2.** If $B_{2n}(b_1, b_2, \ldots, b_n)$ satisfies rTIEP then $b_1 = \sqrt{1 \cdot 3 \cdots (2n-1)}$. If $B_{2n+1}(b_1, b_2, \ldots, b_n)$ satisfies rTIEP, then $b_1 = \sqrt{2 \cdot 4 \cdots (2n)}$.

5.1. Landau’s Conjecture
In a seminal 1994 work [4], H.J. Landau demonstrated that the TIEP is solvable within a certain subclass of the symmetric Toeplitz matrices, for any prescribed set of distinct eigenvalues. We now describe this subclass of Toeplitz matrices ([4], p. 756), and make an observation based on our examples from Section 2.

**Definition 5.3.** A real symmetric Toeplitz matrix, $T_n(a_1, a_2, \ldots, a_n)$ is regular, provided it, and every leading principal submatrix, $T_n(a_1, a_2, \ldots, a_k)$, $1 \leq k \leq n$, have the properties: i) distinct eigenvalues, and ii) when ordered by size, the eigenvalues alternate parity, with the largest one even. Let $T_n$ denote the set of regular matrices $T_n(0, a_2, \ldots, a_n)$.

Landau ends his paper by conjecturing that the solution to TIEP for any set of distinct eigenvalues is unique within $T_n$. Examining the examples of Section 2, we note that i) outer-banded matrices of size $n \geq 3$ are not regular, (since the principal two-by-two submatrix is the zero matrix), ii) we found exactly one regular checkerboard solution of rTIEP of each size $n \leq 6$, iii) we showed that the regular solutions of rTIEP up to size 4 are unique.

Based on these observations we make the following conjecture.

**Conjecture 5.4.** The rTIEP has a unique, regular, non-negative, odd checkerboard solution for all sizes.

6. A Hankel Matrix SIEP
In [16], the authors gave a closed form construction for a real symmetric Toeplitz matrix of even size $T_{2n}$ with arbitrary given eigenvalues, $(\lambda_i)^n_1$, each of double multiplicity. Further, each double eigenvalue was associated with one even parity eigenvector, and one odd. We observe that this matrix can be used to solve the ssSIEP within the class of persymmetric Hankel matrices.

Recall that a Hankel matrix has constant entries along each antidiagonal, so that a Hankel matrix, $H_n$, can be defined by exchanging the rows (or columns) in a Toeplitz matrix. If the Toeplitz matrix is symmetric, the Hankel matrix will be persymmetric, i.e. $H_n = J T_n = T_n J$, and both are supersymmetric.

**Proposition 6.1.** Let $M$ be centrosymmetric, and let $\lambda_e$ and $\lambda_o$ be even and odd parity eigenvalues of $M$ respectively. Then $\lambda_e$ and $-\lambda_o$ are even and odd parity eigenvalues of both of the exchanged centrosymmetric matrices, $JM$ and $MJ$, for the same eigenvectors.

**Proof:** By Proposition 3.9, all eigenvectors of $M$ have parity. Suppose $(v_e, \lambda_e)$ and $(v_o, \lambda_o)$ are eigenvector-eigenvalue pairs for $M$ of even and odd parity, respectively. Then $Mv = M(JJ)v_e = \lambda_e v_e$, which implies $MJv_e = \lambda_e v_e$, and $(M(JJ))v_o = \lambda_o v_o$ which implies $-MJv_o = \lambda_o v_o$, 11
so that $MJ$ has the desired property. On the other hand, $JMv_e = \lambda_e Jv_e = \lambda_e v_e$, and $JMv_o = \lambda_o Jv_o = -\lambda_o v_e$. So $JM$ also has the desired property.

**Theorem 6.2.** If $T_{2n}$ is constructed as per [16] p. 290, with given eigenvalues, $(\lambda_i)_{i=1}^n$, each of double multiplicity, then $H_{2n} = JT_{2n}$ is a supersymmetric Hankel matrix with symmetric spectrum, $((\lambda_i)_{i=1}^n \cup (-\lambda_i)_{i=1}^n)$.

**Proof:** The construction ([16] p. 290) guarantees that each double eigenvalue, $\lambda_i$ of the symmetric, Toeplitz matrix $T_{2n}$ (hence supersymmetric) is associated with one even, and one odd parity eigenvector, $i = 1, \ldots, n$. Since $T_{2n}$ is centrosymmetric, Proposition 6.1 applies, to the effect that the spectrum of the Hankel matrix, $H_{2n} = JT_{2n}$ consists of an even parity set of eigenvalues, $(\lambda_i)_{i=1}^n$, and an odd parity set, $(-\lambda_i)_{i=1}^n$. Clearly, $H_{2n}$ is supersymmetric.

7. Conclusion
Small-sized examples of symmetric Toeplitz matrices with equally spaced eigenvalues were sought, with the goal of finding patterns for the entries of the general size, restricted inverse problem. Two solution forms were found: “(odd) checkerboard”, and “outer banded”. Diagonalizable odd checkerboard matrices were characterized by alternating row parity in their (orthogonal) eigenvector matrix, and a spectrum symmetric about zero. The solution to the inverse problem for this class of matrices was then solved by utilizing a known orthogonal matrix which can be arranged to have alternating row parity.

We conjecture that the restricted Toeplitz inverse eigenvalue problem possesses both odd checkerboard and outer banded solutions of all sizes, $n \in \mathbb{N}$, and that all solutions ($n > 2$) must be one of these two forms. We also suspect that a non-negative odd checkerboard matrix will be the unique solution belonging to Landau’s class of “regular” symmetric Toeplitz matrices.

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