Quantum Phase Transition of Bosons in a Shaken Optical Lattice

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Recently, lattice shaking technique has been used to couple different Bloch bands resonantly. For 1D case, in which shaking is along only one direction, experimental observation of domain wall formation has been explained by superfluid Ising transition. Inspired by these, we generalize to a 2D case, in which shaking is along two orthogonal directions. Analogy to the 1D case, we find quantum phase transition from normal superfluid(NSF) phase to $D_4$ symmetry breaking superfluid($D_4$SF) phase. And interaction effect is demonstrated to be responsible for modified critical shaking amplitude. Unlike in the 1D case, here the interaction effect is originated not only in inhomogeneous band mixing in momentum space, but also in different shaking types. We construct a low-energy effective field theory to study the quantum criticality of bosons near the tricritical point of NSF, $D_4$SF and Mott insulator(MI) phases. Furthermore, we find Bose liquid with anisotropically algebraic order and propose to change Bose-Einstein condensation(BEC) into non-condensed Bose liquid by tuning shaking amplitude approaching the critical value.

I. INTRODUCTION

More and more interest has been attracted in ultracold atoms trapped in a time-periodically driven optical lattice. There are two cases, off-resonance and resonance. For off-resonance case, it was stated that shaken lattice system can be described by an effective time-independent Hamiltonian with renormalized hopping amplitudes for large shaking frequency [1]. It was also experimentally demonstrated that hopping amplitude can be changed dynamically with maintained phase coherence of condensation by shaking lattice [2]. Lattice shaking technique can be used to tune hopping parameters and even invert the signs in a coherent way, which opens a new way to simulate quantum phase transitions in ultracold atom systems. Coherent control of the superfluid-Mott insulator(MI) phase transition has been realized in a shaken three-dimensional optical lattice [3]. Synthetic gauge flux can be realized in a shaken optical lattice [4–7], and this is equivalent to insetting a $\pi$ flux in each plaquette in a shaken square lattice, which generates staggered-vortex superfluid state [1], or in each triangle in a shaken triangular lattice, which generates various types of frustrated states [5–7]. In addition, interparticle interaction can be tuned from repulsive to attractive in fermionic lattice systems by ac forcing, which allows one to simulate an attractive Hubbard model effectively with temperatures below the superconducting transition temperature [8].

Resonance case starts from experimental observation of domain wall formation for bosons condensed in a 1D shaken optical lattice [9], in which lattice shaking technique hybrids different Bloch bands. Effective Hamiltonian can not be described by renormalized hopping amplitudes or interactions as used in off-resonant cases, which may lead to novel phases. Finite-momentum superfluid phase with spontaneously broken $Z_2$ symmetry called $Z_2$SF phase has been observed [9] and corresponding NSF-$Z_2$SF-MI phase transition has been described by a low-energy effective field theory in the 1D case [10].

Finite-momentum condensate has been realized by spin-orbit(SO) coupling generated by Raman transitions [11, 12], or in a staggered magnetic field [13], or in a shaken optical lattice [4–6, 9]. The condensate with finite momentum has spatially inhomogeneous order parameter, which is a bosonic analog to FFLO phase in superconductors [14]. Inspired by discovery of finite-momentum condensate by resonantly shaking lattice along one direction [9, 10], in this paper, we generalize to a 2D case. Using Floquet theory, we demonstrate theoretically and numerically lattice shaking leads to phase transition from NSF phase to $D_4$SF phase as shaking amplitude increases. We further show inhomogeneous band mixing induced interaction effect modifies the critical shaking amplitude, which is consistent with the 1D case [10]. There is a notable difference between our model in the 2D case with the model in the 1D case [10]. There are various shaking types. For example, lattice can be shaken along one diagonal of the lattice(linear shaking), or elliptically(elliptical shaking), or circularly(circular shaking). Since separability of the system along two primitive vectors, quasi-energy dispersion is independent of shaking types. However, linear shaking preserves time reversal(TR) symmetry, while elliptical or circular shaking breaks TR symmetry. Analogy to orbital Hund’s rule [15], there is the largest interaction energy at fixed momentum for repulsive bosons with linear shaking than other shaking types. Together with inhomogeneous band mixing, we predict the smallest critical shaking amplitude for linear shaking. Then we construct a low-energy effective theory to describe phase transitions. Critical correlation length exponent is calculated by momentum shell renormalization group(RG) method. In the end, we study existence of BEC in a general shaken lattice system. A lot of efforts have been devoted to realising quantum states which are not Bose condensed [16–27]. We find Bose liquid with anisotropically algebraic order in three dimensional lattice with two directions shaken and propose to change BEC into non-condensed Bose liquid via tuning shaking amplitude approaching the critical value.

The paper is organized as follows. In Sec. I, we intro-
duce the model for bosons in a shaken optical lattice. In Sec. [III] we calculate the quasi-energy spectrum and obtain finite momentum superfluid phase. Next, we study the interaction effect on this phase in Sec. [IV] and [V]. In Sec. [VI] we construct a low-energy effective field theory to study the quantum criticality of the phase transition. The existence of BEC in a general shaken lattice system is discussed in Sec. [VII]. Finally, conclusions are presented in Sec. [VIII].

II. MODEL

The system we consider is two counter-propagating laser beams along \( x \) direction and two along \( y \) direction, which forms a square lattice. The lattice is shaken by time-periodically modulating relative phase \( \phi_x(t) \) between laser beams along \( x \) direction and \( \phi_y(t) \) between that along \( y \) direction via acousto-optic modulators (AOMs). The Hamiltonian reads

\[
\hat{H}(t) = \frac{\hat{p}^2}{2m} + V \cos^2(k_x x + \frac{\phi_x(t)}{2}) + V \cos^2(k_y y + \frac{\phi_y(t)}{2}),
\]

where \( k_x \) denotes photon momentum, \( \phi_x(t) = f \cos \omega t, \phi_y(t) = f \cos(\omega t + \phi) \), \( f \) denotes shaking amplitude and \( \phi \) denotes relative phase between \( \phi_x \) and \( \phi_y \). \( \phi = 0 \) or \( \pi \) means preserving TR symmetry, while \( \phi \neq 0 \) and \( \pi \) means breaking TR symmetry. \( \Delta \equiv f/(2k_x) \) denotes the maximum lattice displacement along \( x \) or \( y \) direction. This model is separable along \( x \) and \( y \) direction.

Taking a transformation \( x \rightarrow x - \Delta \cos \omega t, y \rightarrow y - \Delta \cos(\omega t + \phi) \), Hamiltonian in the comoving frame reads

\[
\hat{H}(t, \varphi) = \frac{1}{2m} \hat{p}^2 + V \cos^2(k_x x) + V \cos^2(k_y y) - \frac{A(t) \cdot \hat{p}}{m},
\]

where effective vector potential \( A(t) = m \omega \Delta (\sin \omega t, \sin(\omega t + \varphi)) \). Neutral particals will act as charged particles in a static square lattice and an ac electric field \( \mathbf{E} = -m \omega^2 \Delta (\cos \omega t, \cos(\omega t + \varphi)) \). The effective charge is set to be unity.

The first three static terms in Eq. (2) give a static band structure \( \epsilon_x(k) \) and corresponding Bloch wave function \( \phi_x(k, r) \), which will serve as basis in the following analysis. \( p_x \)- and \( p_y \)-bands are degenerate. In this paper, we consider shaking frequency \( \omega \) is a little blue-detuned from \( p \)-bands, and hence only keep \( s \)-, \( p_x \)- and \( p_y \)-bands. And we numerically verify that our following qualitative results do not change when counting higher bands. In these bases, the tight-binding form of Hamiltonian in the comoving frame is given by

\[
\hat{H}(t, \varphi) = \sum_{k} \left( \begin{array}{c}
\hat{\Psi}^\dagger_{p_x, k} \\
\hat{\Psi}^\dagger_{p_y, k} \\
\hat{\Psi}^\dagger_{s, k}
\end{array} \right) H_k(t, \varphi) \left( \begin{array}{c}
\hat{\Psi}_{p_x, k} \\
\hat{\Psi}_{p_y, k} \\
\hat{\Psi}_{s, k}
\end{array} \right),
\]

where \( \hat{\Psi}_{p_x, k} \) and \( \hat{\Psi}_{s, k} \) denote creation and annihilation operator of a particle with quasi-momentum \( k \) in \( \lambda \)-band, respectively, and \( \pm \) sign denotes orbital phase.

Hamiltonian in momentum space is given by

\[
H_k(t, \varphi) = \begin{pmatrix}
\epsilon_{p_x}(k) & 0 & 0 \\
0 & \epsilon_{p_y}(k) & 0 \\
0 & 0 & \epsilon_s(k)
\end{pmatrix} - \frac{A_x(t)}{m} \begin{pmatrix}
0 & 0 & 2h_p \sin k_x \\
0 & 0 & 2h_s \sin k_y \\
2h_s \sin k_y & 0 & 0
\end{pmatrix} - \frac{A_y(t)}{m} \begin{pmatrix}
0 & 2h_p \sin k_x & 0 \\
0 & 0 & -i\Omega(k_x) \\
2h_p \sin k_x & 0 & 0
\end{pmatrix} - i\Omega(k_{xy}) \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix},
\]

where

\[
\Omega(k_{xy}) = h_{xp} + 2h_{xsp1} \cos k_{x,y},
\]

\[
h_s = \langle w_{s,p_y}(r) | i \hat{p}_x | w_{s,p_y}(r - e_x) \rangle,
\]

\[
h_p = \langle w_{p_x}(r) | i \hat{p}_x | w_{p_x}(r - e_x) \rangle,
\]

\[
h_{sp} = \langle w_{p_x}(r) | i \hat{p}_x | w_s(r) \rangle,
\]

\[
h_{sp1} = \langle w_{p_x}(r) | i \hat{p}_x | w_{s1}(r - e_x) \rangle,
\]

where \( A_x(t) \) denotes \( \gamma \) component of vector potential \( A(t) \), \( e_\gamma \) denotes primitive vector along \( \gamma \) direction, \( \gamma \) denotes \( x \) or \( y \), and \( w_\lambda(r) \) denotes wannier function of \( \lambda \)-band, \( \lambda \) denotes \( p_x \) or \( s \), and \( \{ \cdot \cdot \cdot | \cdot \cdot \cdot \cdot \cdot \} \) denotes integral in coordinate space \( \int dr \cdot \cdot \cdot \). Real coupling amplitudes \( h_s, h_p, h_{sp}, h_{sp1} \) denote shaking induced nearest-neighbour hopping between \( s \)-bands, between \( p \)-bands, between \( s \)- and \( p \)-bands, and onsite coupling between \( s \)- and \( p \)-bands, respectively, as shown in Fig. 1. The first matrix in Eq. (4) represents static band structure. And

![FIG. 1: Shaking induced couplings in real space. Yellow circle denotes \( s \)-orbital, colored shape laid along horizontal direction and that laid along vertical direction denote \( p_x \)- and \( p_y \)-orbital, respectively, and \( \pm \) sign denotes orbital phase.](image-url)
FIG. 2: Band structure with $V = 13E_r, \hbar \omega = 6.6E_r, \varphi = 0$, where $E_r = \hbar^2 k^2_l/(2m)$ is lattice recoil energy. (a) Bands structure before shaking. The half transparent surface denotes the dressed $s$-band with energy lifted by $\hbar \omega$. (b,c) Quasi-energy dispersion of the uppermost band for $f = 0.02$ (b) and $f = 0.08$ (c).

III. FINITE-MOMENTUM PHASE

Diagonalizing Floquet operator, i.e., time revolution operator in a time period $T = 2\pi/\omega$,

$$\hat{U}(T, \varphi) = \hat{T} e^{-i\int_0^T dt \hat{H}(t, \varphi)},$$

one can obtain quasi-energy dispersion, as shown in Fig. 2. There is a certain critical shaking amplitude $f_c^0$ such that for $f < f_c^0$, the uppermost band exhibits a single minimum at zero momentum, and for $f > f_c^0$, the uppermost band exhibits four minima at finite momentums.

Symmetry of a periodically driven system must be considered at the Floquet operator $\hat{U}(T, \varphi)$ level [29]. Since $\hat{U}(T, \varphi)$ is separable along $z$ and $y$ direction, quasi-energies do not depend on the relative phase $\varphi$. For $\varphi = 0$, the original Hamiltonian $\hat{H}(t, \varphi = 0)$ has $D_4$ symmetry, so does the quasi-energy dispersion. So the uppermost band dispersion has $D_4$ symmetry for any $\varphi$.

To describe the system, an effective static Hamiltonian $\hat{H}_{eff}$ is defined as

$$\hat{U}(T, \varphi) \equiv e^{-i\hat{H}_{eff}(\varphi)T}.$$  \hfill (11)

We will analyze rotating-wave-approximation (RWA) Hamiltonian, i.e., the zero order $(1/\omega)^0$ term of effective Hamiltonian [30], which is given by

$$\hat{H}_k^{RWA}(\varphi) = (O^\dagger(t)(\hat{H}_k(t) - i\partial_t)O(t))^{(0)} = \begin{pmatrix} \epsilon_p(k) & 0 & \tilde{\Omega}(k_x) \\ 0 & \epsilon_p(k) & e^{-i\varphi}\tilde{\Omega}(k_y) \\ \tilde{\Omega}(k_x) & e^{i\varphi}\tilde{\Omega}(k_y) & \epsilon_s(k) + \omega \end{pmatrix},$$ \hfill (12)

where

$$\tilde{\Omega}(k_x, y) = -\frac{\omega \Delta}{2}\Omega(k_x, y),$$ \hfill (13)

$$O(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\omega t} \end{pmatrix},$$ \hfill (14)

and superscript $(0)$ denotes static part. Coupling strength $\tilde{\Omega}(k_x, y)$ is proportional to shaking frequency and amplitude. RWA Hamiltonian in Eq. (12) indicates that lattice shaking induced couplings result in level repulsion effect, which is the strongest along $k_x = 0$ and $k_y = 0$ directions. The level repulsion effect combined with $D_4$ symmetry will give rise to four global minima at finite momentums ($\pm k_c, \pm k_c$) instead of one at zero momentum in the uppermost band as coupling strength, i.e., shaking amplitude $f$ increases.

Finite-momentum BEC in a non-separable square lattice by non-resonant shaking has been proposed in [31]. Our model has more orbital physics, which will be shown in Sec. IV.

IV. SPONTANEOUS SYMMETRY BREAKING

Let us consider interacting bosons condensing at the finite-momentum state with minimal kinetic energy in the uppermost band. Interaction reads

$$H_{int}(t) = g \int d\mathbf{r} \hat{\Psi}^\dagger(r, t)\hat{\Psi}(r, t)\hat{\Psi}(r, t)\hat{\Psi}(r, t),$$ \hfill (15)

where $\hat{\Psi}^\dagger(r, t)$, $\hat{\Psi}(r, t)$ are creation and annihilation operator of condensate state, and positive $g$ denotes repulsive interaction strength.

Bosons can either condense at one of the four degenerate finite-momentum states or at the superposition state. Assume the singe particle ground state is a superposition state

$$\psi_{k_i}(r, t) = \sum_{i=1}^4 a_i \psi_{k_i}(r, t),$$ \hfill (16)

where $\psi_{k_i}(r, t)$ denotes one of the four degenerate states, $k_i = (\pm k_c, \pm k_c)$ denotes condensate momentum, and constant $a_i$ satisfies $\sum_{i=1}^4 |a_i|^2 = 1$. One can write $\psi_{k_i}(r, t)$ in the co-moving frame as

$$\psi_{k_i}(r, t) = b_{p_x, k_i} \phi_{p_x, k_i}(r) + b_{p_y, k_i} \phi_{p_y, k_i}(r)$$
where $b_{\lambda,k_c}$ is combination coefficient and dependent of $\lambda$ and $k_c$, and $\lambda$ denotes $p_x, p_y$ or $s$. Time-average mean-filed interaction energy per particle condensing at the superposition state $\psi_{k_c}(r,t)$ in laboratory frame is given by

$$
\epsilon_{\text{int}}(k_c) = \frac{1}{T} \int_0^T dt g \int dr |\psi_{k_c}(r,t)|^4. \tag{18}
$$

By minimizing the interaction energy $\epsilon_{\text{int}}(k_c)$ with respect to $\{a_i\}_{i=1}^4$, one obtains $(a_1, a_2, a_3, a_4) = (\pm 1, 0, 0, 0)$ or $(0, \pm 1, 0, 0)$ or $(0, 0, \pm 1, 0)$ or $(0, 0, 0, \pm 1)$. Since there is only one nonzero $a_i$, we neglect the phase of $a_i$. So bosons only condense at one of the four finite-momentum states, which breaks $D_4$ symmetry spontaneously.

In the process of turning on shaking adiabatically, bosons will remain in the uppermost band. When shaking amplitude across a critical value, phase transition from NSF phase to $D_4$SF phase happens.

V. CRITICAL SHAKING AMPLITUDE

Interaction effect also modifies critical shaking amplitude. By minimizing total energy consisted of kinetic and interaction energy of bosons in the uppermost band with respect to quasi-momentum, one can obtain condensate momentum $(\pm k_c, \pm k_c)$. Here the methods we use to calculate kinetic and interaction energy are the same as the methods used in Sec. 11 and 14 respectively. When $k_c$ turns to be nonvanishing with increasing shaking amplitude, critical shaking amplitude $f_c$ is obtained.

Fig. 3(a,b) show repulsive interaction effect enlarges the critical shaking amplitude $f_c$ in deep lattice limit. This is because inhomogeneous band mixing in momentum space causes a global minimum of interaction energy at zero momentum in deep lattice as shown in Fig. 3(c), which has been illustrated in the 1D case [10]. Instead, a local maximum has also been predicted at zero momentum in shallow lattice limit, which leads to a smaller critical shaking amplitude $f_c < f_c^0$ [10]. We will focus on deep lattice case in this paper.

Besides, Fig. 3(a,b) also show shaking types affect critical shaking amplitude $f_c$ through interaction effect. $k_c - f$ curve for interaction case in Fig. 3(a) and $gn - f_c$ curve in Fig. 3(b) change with relative phase $\varphi$, and are bounded by corresponding curves with $\varphi = 0$ or $\pi$ and that with $\varphi = \pm \pi/2$.

One can write eigenstate of the uppermost band in the co-moving frame as

$$
\psi_k(r,t,\varphi) = c_{p_x,k} \phi_{p_x,k}(r) + e^{-i\varphi} c_{p_y,k} \phi_{p_y,k}(r) + e^{i\varphi} c_{s,k} \phi_{s,k}(r), \tag{19}
$$

where $c_{\lambda,k}$ denotes combination coefficients of eigenstate of $H_k^{RWA}(\varphi = 0)$ in the uppermost band and $\lambda$ denotes $p_x, p_y$ or $s$. Since $H_k^{RWA}(\varphi = 0)$ is real and symmetric, $c_{\lambda,k}$ can be set to be real. Here we use RWA Hamiltonian for simplicity of analysis.

Using Eq. [18], the difference between time-average interaction energy per particle in a system with broken TR symmetry and that in a TR symmetric system is given by

$$
\Delta \epsilon_{\text{int}}(k,\varphi) = \epsilon_{\text{int}}(k,\varphi) - \epsilon_{\text{int}}(k,\varphi = 0) = -4\nu \sin^2 \varphi \frac{c^2}{p_x,k} \frac{c^2}{p_y,k} U_{k}^{p_x,p_y}, \tag{20}
$$

where $\nu$ denotes site occupation number, $U_{k}^{p_x,p_y} = g \int dr |\phi_{p_x,k}(r)\phi_{p_y,k}(r)|^2$, which is a positive constant in deep lattice limit.

Eq. [20] shows two important ingredients. One is $-\sin^2 \varphi$ signifying level of TR symmetry breaking.

Generally, when one Hamiltonian with TR symmetry is unitarily transformed to another Hamiltonian, spectrum rather than TR symmetry is always invariant. Assuming $\hat{H} = \hat{Q} \hat{H} \hat{Q}^*$ with TR symmetric Hamiltonian $\hat{H}$ and unitary operator $\hat{Q}$, one obtains

$$
[\hat{T}, \hat{H}] = (\hat{Q} \hat{H} \hat{Q}^* - \hat{Q}^* \hat{H} \hat{Q}) \hat{T}, \tag{21}
$$

where $\hat{T}$ is time reversal operator and $\hat{Q} = \hat{T} \hat{Q} \hat{T}^{-1}$. If $\hat{Q}$ commutes with $\hat{T}$, then $\hat{Q} = \hat{Q}$ and $\hat{H}$ is TR symmetric. Otherwise TR symmetry is broken generally. In our case, $\hat{H} \simeq H_k^{RWA}(0)$ is TR symmetric,

$$
\hat{Q} = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\varphi} & 0 \\
0 & 0 & 1
\end{pmatrix}, \tag{22}
$$
and commutation relation $[\hat{T}, H_{RW}^A(\varphi)]$ is proportional to sin $\varphi$. So TR symmetry is conserved for $\varphi = 0$ or $\pi$ and broken maximally for $\varphi = \pm \pi/2$. $\Delta k_{\text{int}}(k, \varphi)$ in Eq. (20) at fixed momentum decrease as level of TR symmetry breaking increases, which is similar to orbital Hund’s rule [17].

The other key ingredient of interaction energy difference in Eq. (20) is momentum dependence. This is because of inhomogeneous band mixing in momentum space, which causes that the interaction component $c_{p_{x}, k}^2 c_{p_{y}, k}^2 \hat{U}_{k}^{p_{x}, p_{y}}$ has global maximum at zero momentum, as shown in Fig. (4). Here we can see $\varphi$ appears only in interaction components involving $p_{x}$- and $p_{y}$- orbitals. The reason is that other terms involving $\varphi$, such as $c_{k}^2 c_{p_{x}, k}^2 c_{p_{y}, k} \cos 2k \varphi$, contains factor $\cos 2k \varphi$ because of energy difference between $s$- and $p$- bands, and can be neglected in the sense of time average.

The two ingredients determine curvature of interaction energy at zero momentum is minimum when $\varphi = 0$ or $\pi$ and maximum when $\varphi = \pm \pi/2$. So there is the smallest critical shaking amplitude for $\varphi = 0$ or $\pi$ and the largest one for $\varphi = \pm \pi/2$.

VI. EFFECTIVE FIELD THEORY

In this section, we introduce a low-energy effective action to describe all three phases, i.e., NSF, D$_4$SF and MI phases. Based on this action, we will show phase diagram calculated by mean-field theory and critical exponent calculated by momentum shell RG theory.

In order to construct the effective action, two important factors from microscopic analysis above must be considered. First, kinetic energy has quartic form of $k_x^4 + k_y^4 + a(k_x^2 + k_y^2)$ at small momentum. Second, momentum dependent interaction has quadratic form of $\alpha + \beta(k_x^2 + k_y^2)$ at small momentum. These two factors also agree with $D_4$ symmetry. The low-energy effective action of $d$ dimensional lattice shaken along $x$ and $y$ directions can be written as

$$S(\Phi, \Phi^*) = \int_0^{1/T} d\tau \int d^d r \{ K_1 \Phi^* \partial_\tau \Phi + K_2 |\partial_\tau \Phi|^2 + E(\Phi, \Phi^*) \},$$

where

$$E(\Phi, \Phi^*) = |\partial_\tau^2 \Phi|^2 + |\partial_y^2 \Phi|^2 + a |\nabla \Phi|^2 + T + r |\Phi|^2 + \alpha |\Phi|^4 + \beta |\Phi \nabla \Phi|^2,$$

the norm of $\Phi$ denotes NSF order parameter, and the phase of $\Phi$ denotes $D_4$SF order parameter, $\nabla = (\partial_x, \partial_y)$, $T = 0$ for $d = 2$ and $T = |\partial_x \Phi|^2$ for $d = 3$. The signs of parameters $\alpha$ and $r$ can be inverted by tuning shaking amplitude $f$ and interaction strength $g$, respectively. Parameter $\alpha$ is considered to be positive for repulsive interactions. And parameter $\beta$ can be either positive in deep lattice limit, or negative in shallow lattice limit. We will take $\beta > 0$ for simplicity.

Assuming $\Phi = |\Phi| e^{ik \cdot r}$, $E$ can be rewritten as

$$E(|\Phi|, k) = (k_x^2 + k_y^4 + a k^2 + r) |\Phi|^2 + (\alpha + \beta k^2) |\Phi|^4,$$

where $k = (k_x, k_y, \cdots)$ and $r = (x, y, \cdots)$ are $d$ dimensional vectors. By minimizing $E$ with respect to $|\Phi|$ and $k$, one obtains three different phases: 1) MI phase with $\Phi = 0$; 2) NSF phase with $\Phi \neq 0$ and $k = 0$; 3) $D_4$SF phase with $\Phi \neq 0$, $k_x \neq 0$, $k_y \neq 0$ and $k^2 = k_x^2 - k_y^2 = 0$. Phase boundaries are also obtained: 1) $r = 0$ and $a > 0$ separating NSF and MI phase; 2) $r = 2a \alpha / \beta < 0$ separating NSF and $D_4$SF phase; 3) $r = a^2 / 2$ and $a < 0$ separating $D_4$SF and MI phase, which is different from the 1D case [10]. There is a mean-field tricritical point $(a, r) = (0, 0)$. In the vicinity of this tricritical point, $a$ and $r$ are proportional to $f - f_c$ and $g - g_c$, respectively. The phase diagram in $f$- and $g$-terms is shown in Fig. (5).

Next we will study critical correlation length exponent by momentum shell RG theory.

At zero temperature and in $d$ dimensional momentum and frequency space, the action in Eq. (23) can be written as

$$S(\Phi, \Phi^*) = \int \frac{d^d k}{(2\pi)^d} \frac{d\omega}{2\pi} \Phi^* (k, \omega) \{ -i K_1 \omega + K_2 \omega^2 + k_x^2 + k_y^2 + a(k_x^2 + k_y^2) + T_k + r \} \Phi(k, \omega) + \int_0^\Lambda k \{ \alpha + \beta (k_2^2 + k_4^2 + k_3^2 + k_4^2) \} \Phi^* (k_1, \omega_1) \Phi^* (k_2, \omega_2) \Phi (k_3, \omega_3) \Phi (k_4, \omega_4),$$

where $T_k = 0$ for $d = 2$, $T_k = k_x^2$ for $d = 3$, $\int_0^\Lambda k \omega = \int \frac{d^d k_1 d\omega_1}{(2\pi)^d+1} \delta(k_1 + k_2 - k_3 - k_4) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$ and $A$ denotes high momentum cut-off.
The one-loop Feynman graphs for renormalizing the parameters in Eq. (26) are shown in Fig. (6). The study of critical exponent is divided into two cases.

Case A. Without particle-hole symmetry, $K_1 \neq 0$, so $K_2$-term becomes irrelevant. In this case, scaling dimensions of the parameters read

$$[k_x] = [k_y] = \frac{1}{2}, [\alpha] = 1, [r] = 2, [\omega] = 2,$$

$$[\alpha] = 3 - d, [\beta] = 2 - d, [\Phi] = \frac{d + 3}{2}. \quad (27)$$

The upper critical dimension is 3, which is 1/2 larger than that in 1D shaken lattice [10] due to extra shaking direction. For $d = 2$, $\alpha$-term is relevant and $\beta$-term is marginal. It is different from irrelevant $\beta$-term in the 1D case [10]. So we need to consider corrections from $\beta$-term. For graphs in Fig. (6) (b,c), we need to expand them in powers of external momentums. The one-loop RG flow equations read

$$\frac{da}{dl} = a, \quad \frac{dr}{dl} = 2r, \quad \frac{d\alpha}{dl} = \alpha - \frac{\alpha^2}{1 + r} I_2(a), \quad \frac{d\beta}{dl} = (\epsilon - 1)\beta - \frac{\alpha\beta}{1 + r} I_2(a) + \frac{1}{(1 + r)^2}\left[\alpha^2J_2(a) - \alpha\beta L_2(a)\right] - \frac{\alpha^2}{(1 + r)^3}M_2(a), \quad (31)$$

where

$$I_2(a) = \int_0^{2\pi} \frac{d\phi}{(2\pi)^2} \frac{-a + \sqrt{a^2 + 4(\cos^4 \phi + \sin^4 \phi)}}{4(\cos^4 \phi + \sin^4 \phi)}, \quad (32)$$

$$J_2(a) = \int_0^{2\pi} \frac{d\phi}{(2\pi)^2} \frac{-a + \sqrt{a^2 + 4(\cos^4 \phi + \sin^4 \phi)}}{2(\cos^4 \phi + \sin^4 \phi)} \left[a + 3\cos^2 \phi - a + \sqrt{a^2 + 4(\cos^4 \phi + \sin^4 \phi)}\right] \cos^2 \phi, \quad (33)$$

$$L_2(a) = \int_0^{2\pi} \frac{d\phi}{(2\pi)^2} \left[-a + \sqrt{a^2 + 4(\cos^4 \phi + \sin^4 \phi)}\right] \cos^2 \phi, \quad (34)$$

$$M_2(a) = \int_0^{2\pi} \frac{d\phi}{(2\pi)^2} \cos^2 \phi, \quad (35)$$

and $\epsilon = 3 - d$. 

The non-trivial fixed point lies at $(r^*, a^*, \alpha^*, \beta^*) = (0, 0, \frac{J_2(0) - M_2(0)}{L_2(0)} \epsilon^2)$. Defining new variables $\delta r = r - r^*, \delta a = a - a^*, \delta \alpha = \alpha - \alpha^*, \delta \beta = \beta - \beta^*$, the linearized flow equations are given by

$$\frac{d}{dl} \begin{pmatrix} \delta r \\ \delta a \\ \delta \alpha \\ \delta \beta \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 \\ 0 & 2J_2(0) - M_2(0) \epsilon & -1 - \frac{L_2(0)}{J_2(0)} \epsilon & 0 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta a \\ \delta \alpha \\ \delta \beta \end{pmatrix}. \quad (36)$$
Eigenvalues of the matrix in Eq. (36) are 2, 1, $-\epsilon$, $-1 - \epsilon L_2(0)/I_2(0)$. Then the scaling dimension of parameter $r$ at the non-trivial fixed point is $y_r = 2$. The correlation length exponent of superfluid transition can be calculated as $\nu = 1/y_r = 1/2$. It is the same as the mean-field value in usual Bose gas [32, 33] and the value in the 1D case with $K_1 \neq 0$ [10] because of no one-loop corrections on $r$ from interaction, as shown in the flow diagram in Fig. 7(a).

Case B. With particle-hole symmetry, $K_1 = 0$. In this case, scaling dimensions of the parameters read

\[ [k_x] = [k_y] = \frac{1}{2}, [k_z] = 1, [a] = 1, [r] = 2, [\omega] = 1, \]

\[ [\alpha] = 4 - d, [\beta] = 3 - d, [\Phi] = -\frac{d+2}{2}. \]  

(37)

\[
\frac{d\beta}{dt} = (\epsilon - 1)\beta - \frac{1}{2(1+r)^2} \left[ \alpha \beta I_3(a) + \frac{\beta^2}{2} J_3(a) \right] + \frac{1}{(1+r)^2} \left\{ \frac{9}{4} \alpha^2 [a I_3(a) + 3 J_3(a)] + \alpha \beta M_3(a) + \beta^2 N_3(a) \right\}
\]  

+(1+r)^2 \left[ \alpha^2 P_3(a) + \alpha \beta R_3(a) + \beta^2 S_3(a) \right], \]  

(41)

where

\[
I_3(a) = \frac{1}{(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} \sin^2 \theta + \cos^2 \theta \left[ -(a \sin^2 \theta + \cos^2 \theta) + \sqrt{(a \sin^2 \theta + \cos^2 \theta)^2 + 4 \sin^4 \theta (\sin^4 \phi + \cos^4 \phi)} \right] \frac{3}{2}
\]

\[
J_3(a) = \frac{1}{(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} \sin^2 \theta + \cos^2 \theta \left[ -(a \sin^2 \theta + \cos^2 \theta) + \sqrt{(a \sin^2 \theta + \cos^2 \theta)^2 + 4 \sin^4 \theta (\sin^4 \phi + \cos^4 \phi)} \right] \frac{2}{2}
\]

\[
L_3(a) = \frac{1}{8(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} \sin^2 \theta + \cos^2 \theta \left[ -(a \sin^2 \theta + \cos^2 \theta) + \sqrt{(a \sin^2 \theta + \cos^2 \theta)^2 + 4 \sin^4 \theta (\sin^4 \phi + \cos^4 \phi)} \right] \frac{5}{2}
\]

\[
M_3(a) = \frac{3}{2(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} \sin^2 \theta \left[ a(1 + \cos^2 \phi) + \cos^2 \phi (7 + \cos 2\phi) \right] \frac{3}{2}
\]

\[
N_3(a) = \frac{3}{16(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} \sin^2 \theta \left[ a(3 + 2 \cos 2\phi) + \cos^2 \phi (5 + 2 \cos 2\phi) \right] \frac{2}{2}
\]

The upper critical dimension is 4. For $d = 3$, the one-loop RG equations read

\[
\frac{da}{dt} = a + \frac{\beta}{2 \sqrt{1+r}} I_3(a), \]  

(38)

\[
\frac{dr}{dt} = 2r + \frac{1}{\sqrt{1+r}} \left[ 2 \alpha I_3(a) + \frac{\beta}{2} J_3(a) \right], \]  

(39)

\[
\frac{d\alpha}{dt} = \epsilon \alpha - \frac{1}{(1+r)^2} \left[ \frac{5}{2} \beta^2 I_3(a) + \alpha \beta J_3(a) + \beta^2 L_3(a) \right], \]  

(40)
\[ P_3(a) = \frac{5}{(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{\frac{\sin^2 \theta}{4} + \cos^2 \theta} \left[ -\left( a \sin^2 \theta + \cos^2 \theta \right) + \sqrt{\left( a \sin^2 \theta + \cos^2 \theta \right)^2 + 4 \sin^4 \theta \left( a^4 + \cos^4 \theta \right)} \right] \]

\[ \sin^3 \theta \cos^2 \phi \left[ a + \cos^2 \phi \right] \frac{-(a \sin^2 \theta + \cos^2 \theta) + \sqrt{(a \sin^2 \theta + \cos^2 \theta)^2 + 4 \sin^4 \theta (a^4 + \cos^4 \phi)}}{\sin^2 \theta (a^4 + \cos^4 \phi)} \right] \], \quad (47) \]

\[ R_3(a) = \frac{5}{(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{\frac{\sin^2 \theta}{4} + \cos^2 \theta} \left[ -\left( a \sin^2 \theta + \cos^2 \theta \right) + \sqrt{\left( a \sin^2 \theta + \cos^2 \theta \right)^2 + 4 \sin^4 \theta \left( a^4 + \cos^4 \phi \right)} \right] \]

\[ \sin^5 \theta \cos^2 \phi \left[ a + \cos^2 \phi \right] \frac{-(a \sin^2 \theta + \cos^2 \theta) + \sqrt{(a \sin^2 \theta + \cos^2 \theta)^2 + 4 \sin^4 \theta (a^4 + \cos^4 \phi)}}{\sin^2 \theta (a^4 + \cos^4 \phi)} \right]^2 \], \quad (48) \]

\[ S_3(a) = \frac{5}{8(2\pi)^3} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{\frac{\sin^2 \theta}{4} + \cos^2 \theta} \left[ -\left( a \sin^2 \theta + \cos^2 \theta \right) + \sqrt{\left( a \sin^2 \theta + \cos^2 \theta \right)^2 + 4 \sin^4 \theta \left( a^4 + \cos^4 \phi \right)} \right] \]

\[ \sin^7 \theta \cos^2 \phi \left[ a + \cos^2 \phi \right] \frac{-(a \sin^2 \theta + \cos^2 \theta) + \sqrt{(a \sin^2 \theta + \cos^2 \theta)^2 + 4 \sin^4 \theta (a^4 + \cos^4 \phi)}}{\sin^2 \theta (a^4 + \cos^4 \phi)} \right]^2 \], \quad (49) \]

and \( \epsilon = 4 - d = 1 \). The non-trivial fixed point lies at

\[ (r^*, a^*, \alpha^*, \beta^*) = \left( - \frac{2\epsilon}{5}, \frac{27J_3(0) + 2P_3(0)}{25I_3(0)} \epsilon^2, \frac{2\epsilon}{5I_3(0)}, \frac{2(27J_3(0) + 2P_3(0))}{25I_3(0)} \epsilon^2 \right). \quad (50) \]

Defining \( \delta r = r - r^*, \delta a = a - a^*, \delta \alpha = \alpha - \alpha^* \) and \( \delta \beta = \beta - \beta^* \), the linearized equations are

\[ \begin{eqnarray*}
\frac{d}{dt} \begin{pmatrix}
\delta r \\
\delta a \\
\delta \alpha \\
\delta \beta
\end{pmatrix} = \begin{pmatrix}
2 - \frac{2\epsilon}{5} & 4J_3(0) & 2I_3(0)(1 + \frac{1}{5}\epsilon) & J_3(0) \\
0 & 1 & 0 & 0 \\
0 & 0 & -\epsilon & 0 \\
0 & 0 & [27J_3(0) + 2P_3(0)] \frac{2}{5I_3(0)} \epsilon - 1 + \frac{2}{5} \frac{27J_3(0) + 2P_3(0)}{I_3(0)} \epsilon
\end{pmatrix}
\begin{pmatrix}
\delta r \\
\delta a \\
\delta \alpha \\
\delta \beta
\end{pmatrix}
\end{eqnarray*} \quad (51) \]

The eigenvalues of the matrix in Eq. (51) are \( 2 - 2\epsilon/5, 1, -\epsilon \) and \( -1 + 4\epsilon/5 + 2\epsilon (M_3(0) + I_3(0))/5I_3(0) \). Here it is the important difference from Case A that \( r \)-term gets corrections from interaction, as shown in flow diagram in Fig. 7(b). The correlation length exponent of superfluid transition is \( \nu = 1/(2 - 2\epsilon/5) = 1/2 + \epsilon/10 = 3/5 \). It is different from bosons with quartic dispersion in only one direction with particle-hole symmetry due to different upper critical dimensions. While it is the same as conventional bosons with quadratic dispersion with particle-hole symmetry in three dimension, belonging to the \( O(2) \) rotor model class, up to \( \epsilon \) order \( \delta \).

Systems with parameters \( r \) and \( \alpha \) that lie in the right region of the critical surface in Fig. 7 will eventually flow towards MI phase, while system with \( r \) and \( \alpha \) in the left region of the critical surface will flow towards NSF.
phase for initial $a > a^*$ or $D_4$SF phase for initial $a < a^*$.

**VII. THE FATE OF BEC IN GENERAL SHAKEN LATTICE SYSTEMS**

In sections above, we suppose BEC exists in the system. However, it is known low-energy density of states(DoS) can be increased by SO coupling or lattice shaking. Fluctuations may destroy long-range order(LRO). In this section we will first study existence of LRO in the 2D cases, and then in general cases.

Let us first assume bosons condensing in NSF phase and write $\Phi = \sqrt{\rho_0 + \delta \rho} e^{i \theta}$, where $\rho_0 = -r/2\alpha$ denotes superfluid density, $\delta \rho$ denotes density fluctuation and $\theta$ denotes phase fluctuation. Then we substitute the $\Phi$ field into Eq.(23) and expand the action in Eq.(23) to quadratic order in $\delta \rho$ and $\theta$. By integrating out $\delta \rho$ field, the low-energy effective action for $\theta$ field is given by

\[ S_{eff}(\theta) = \rho_0 \int d^d r \left\{ \tilde{K}(\partial_\theta \theta)^2 + (\nabla^2 \theta)^2 + \tilde{a}(\nabla \theta)^2 + T \right\}, \]

(52)

where $\tilde{K} = K_0^2 + K_2$, $\tilde{a} = a + \beta \rho_0$.

In Gaussian approximation, the correlation function can be written as

\[ \langle \Phi^*(r) \Phi(0) \rangle = \rho_0 e^{-\frac{1}{2}(\theta(r) - \theta(0))^2}. \]

(53)

At finite temperature, Eq.(53) can be written as

\[ \langle \Phi^*(r) \Phi(0) \rangle = \rho_0 \exp\left\{ -T \int \frac{d^d k}{(2\pi)^d} \left( 1 - e^{ikr} \right) \left( k_x^4 + k_y^4 + \tilde{a}(k_z^2 + k_y^2) + T_k \right)^{-1} \right\}, \]

(54)

where $T$ denotes temperature.

For $d = 2$, in the large separation limit, the integral in Eq.(54) can be approximated by

\[ \int_0^{2\pi} d\theta \int_0^\Lambda \frac{dk}{(2\pi)^2} \frac{1}{k^3(\cos^4 \theta + \sin^4 \theta) + \tilde{a} k}. \]

(55)

As $k \to 0$, the integrand in Eq.(55) behaves as $1/k$ for $\tilde{a} > 0$ and $1/k^3$ for $\tilde{a} = 0$. So the integral is divergent, the correlation function in Eq.(54) approaches zero at large separation and LRO is absent for $\tilde{a} \geq 0$ at any finite temperature.

For $d = 3$ and $\tilde{a} > 0$, in the large separation limit, the integral in Eq.(54) can be approximated by

\[ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k_x^4 + k_y^4 + \tilde{a}(k_z^2 + k_y^2) + k_z^2}. \]

(56)

After integrating over $k_z$, the integral above is given by

\[ \int_0^{2\pi} d\theta \int_0^\Lambda \frac{dk}{8\pi^2} \frac{1}{\sqrt{k^2(\cos^4 \theta + \sin^4 \theta) + \tilde{a}}}. \]

(57)

As $k \to 0$, the integrand in Eq.(57) behaves like 1. So the integral is finite and LRO exists.

For $d = 3$ and $\tilde{a} = 0$, the integral in the exponent in Eq.(54) reads

\[ \int \frac{d^3 k}{(2\pi)^3} \frac{1 - e^{ikr}}{k_x^4 + k_y^4 + k_z^2}. \]

(58)

The critical point $\tilde{a} = 0$ means critical shaking amplitude $f_c$ in Sec. V, i.e., phase boundary between NSF and $D_4$SF phase in mean field level in Sec. VI. At large separation, the correlation function is given by

\[ \langle \Phi^*(r) \Phi(0) \rangle \sim |x|^{-2\eta T}, \]

(59)

\[ \langle \Phi^*(z) \Phi(0) \rangle \sim |z|^{-\eta T}. \]

(60)

where $\eta = \Gamma(5/4)/\pi^{5/2}$ and $\Gamma(z)$ denotes Euler gamma function. So there exists BEC only at zero temperature and non-condensed Bose liquid at finite temperature. This algebraically ordered Bose liquid is anisotropic. Since fluctuation is enhanced by shaking, the correlation function decays faster along shaking directions.

At zero temperature, Eq.(54) can be written as

\[ \langle \Phi^*(r) \Phi(0) \rangle = \rho_0 \exp\left\{ -\int \frac{d^d k}{(2\pi)^d} \left( 1 - e^{ikr} \right) \left( \tilde{K} \omega^2 + k_x^4 + k_y^4 + \tilde{a}(k_z^2 + k_y^2) + T_k \right)^{-1} \right\}. \]

(61)

The zero temperature results are equivalent to adding a unshaken direction to corresponding finite temperature case. Correlation function in Eq.(61) remains finite at large separation for $\tilde{a} > 0$ and $d = 2$ and 3. At the critical point $\tilde{a} = 0$ and in the large separation limit, vanishing correlation function is found for $d = 2$, which is consistent with results in SO coupled BEC with similar dispersion [26, 27], while finite correlation function is found for $d = 3$.

From calculations above, we know phase fluctuation destroys LRO at any finite temperature for $d = 2$ and $\tilde{a} > 0$. The effect is even stronger at the critical point due to pure quartic dispersion along shaking directions. For $d = 2$ and $\tilde{a} = 0$, LRO does not exist even at zero temperature [26, 27]. For a system with $d = 3$, there exists LRO at any temperature when $\tilde{a} > 0$ and quasi-LRO at finite temperature when $\tilde{a} = 0$. Therefore BEC can be changed into a non-condensed Bose liquid by tuning the shaking amplitude $f$ approaching the critical value $f_c$.

Raman induced SO coupling and lattice shaking have generated quartic dispersion [27]. And higher order terms may be generated in the future. Next we will study the possibility of changing BEC into non-condensed Bose liquid in a system with a general dispersion.

Existences of LRO is obtained by checking if the correlation function in Eq.(54) is finite in large separation limit. And the results at the critical point are shown in Table I and II, where $N$ represents having no LRO.
TABLE I: Existence of LRO for $\tilde{\alpha} = 0$ at finite temperature.

| $\mathbb{T}_k$ | $k_x^n + \mathbb{T}_k$ | $k_x^n + k_y^n + \mathbb{T}_k$ | $k_x^n + k_y^m + k_z^l$ | $k_x^n + k_y^m + k_z^l$ |
|---------------|----------------|-----------------|----------------|----------------|
| (n $\geq$ 4)   | (m $\geq$ n $\geq$ 4) | (l $\geq$ m $\geq$ n $\geq$ 4) | (l $\geq$ m $\geq$ n $\geq$ 4) | (l $\geq$ m $\geq$ n $\geq$ 4) |
| d=2           | N               | N               | N              | N              |
| d=3           | O               | O               | N              | N              |

TABLE II: Existence of LRO for $\tilde{\alpha} = 0$ at zero temperature.

| $\mathbb{T}_k$ | $k_x^n + \mathbb{T}_k$ | $k_x^n + k_y^n + \mathbb{T}_k$ | $k_x^n + k_y^m + k_z^l$ | $k_x^n + k_y^m + k_z^l$ |
|---------------|----------------|-----------------|----------------|----------------|
| (n $\geq$ 4)   | (m $\geq$ n $\geq$ 4) | (l $\geq$ m $\geq$ n $\geq$ 4) | (l $\geq$ m $\geq$ n $\geq$ 4) | (l $\geq$ m $\geq$ n $\geq$ 4) |
| d=2           | O               | O               | O              | N              |
| d=3           | O               | O               | O               | N              |

and O represents having LRO. For $\tilde{\alpha} > 0$, at finite temperature, LRO exists only in systems with $d = 3$. So systems with dispersion $k_x^n + k_y^m + k_z^l + \alpha(k_x^n + k_y^m + k_z^l)$ or $k_x^n + k_y^m + k_z^l + \alpha(k_x^n + k_y^m + k_z^l)$ with $l \geq m \geq n \geq 4$ can be used to change BEC into non-condensed Bose liquid by tuning shaking amplitude approaching the critical value $f_c$.

VIII. CONCLUSIONS

In conclusion, we have investigated quantum phase transition of bosons in a shaken lattice by using Floquet theory and low-energy effective field theory. We have found spontaneous $D_4$ symmetry breaking in $D_4$-SF phase and calculated critical shaking amplitude $f_c$ of NSF-$D_4$-SF transition. We further demonstrated shaking types together with inhomogeneous band mixing induced interaction effect modifies critical shaking amplitude. We identified a quantum tricritical point of NSF, $D_4$-SF and MI phases and studied quantum criticality nearby the tricritical point. Moreover, we found anisotropically algebraic LRO and proposed to turn BEC into non-condensed Bose liquid by tuning shaking amplitude approaching the critical value $f_c$.

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