THE J-FLOW AND STABILITY

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Abstract. We study the J-flow from the point of view of an algebra-geometric stability condition. In terms of this we give a lower bound for the natural associated energy functional, and we show that the blowup behavior found by Fang-Lai [8] is reflected by the optimal destabilizer. Finally we prove a general existence result on complex tori.

1. Introduction

The J-flow was introduced by Donaldson [4] from the point of view of moment maps, as well as Chen [2] in his study of the Mabuchi energy. To state the equation, let \((M, \omega)\) be a compact Kähler manifold of dimension \(n\), and let \(\alpha\) be a second Kähler metric on \(M\) which is unrelated to \(\omega\). The J-flow is the parabolic equation

\[
\frac{\partial}{\partial t} \omega(t) = -\sqrt{-1} \partial \bar{\partial} \Lambda_{\omega(t)} \alpha \\
\omega(0) = \omega,
\]

where \(\Lambda\) denotes the trace. The stationary solutions of this flow are metrics \(\omega\) such that

\[
\Lambda_{\omega} \alpha = c,
\]

where \(c\) is a constant, which can be calculated from the Kähler classes of \(\omega\) and \(\alpha\) using the equation

\[
\int_M \alpha \wedge \frac{\omega^{n-1}}{(n-1)!} = c \int_M \frac{\omega^n}{n!}.
\]

It was shown by Song-Weinkove [15] (see also Weinkove [22, 23]) that when a solution to Equation (2) exists, then the J-flow converges to this solution. In [15] the following necessary and sufficient condition was given for the existence of a solution:

There exists a metric \(\omega' \in [\omega]\) such that

\[
c\omega'^{n-1} - (n-1)\omega'^{n-2} \wedge \alpha > 0,
\]

where the positivity means positivity of \((n-1, n-1)\)-forms.

Unfortunately this condition is hard to check in concrete examples, and it is not even clear whether the condition depends on the choice of \(\alpha\) in its Kähler class. In this paper we propose a new numerical condition, which we conjecture is equivalent to existence of a solution.

**Conjecture 1.** A solution to Equation (2) exists if and only if for all \(p\)-dimensional subvarieties \(V \subset M\), where \(p = 1, 2, \ldots, n-1\), we have

\[
\int_V c\omega^p - p\omega^{p-1} \wedge \alpha > 0.
\]
It is straightforward to show that this is indeed a necessary condition. On the other hand one can also naturally arrive at this condition from the point of view of an algebro-geometric stability condition, analogous to K-stability, introduced by Tian [21]. In fact, using that the J-flow arises from a moment map, we can develop a theory parallel to that of constant scalar curvature Kähler metrics and K-stability as in Donaldson [5, 6] for instance.

We will now focus on the situation when $\alpha$ and $\omega$ are algebraic, in the sense that they represent the first Chern classes of ample line bundles. Suppose that $\omega \in c_1(L)$, and let $M \hookrightarrow \mathbb{P}^N$ be an embedding using sections of $L^k$ for some large $k$. A test-configuration $\chi$ for $(M, L)$ is obtained by choosing a $\mathbb{C}^*$-action on $\mathbb{P}^N$, and we will define an associated invariant $F_\alpha(\chi)$, analogous to the Donaldson-Futaki invariant. Our first result, in Section 2, is a lower bound for a natural energy functional in terms of this invariant, analogous to Donaldson’s lower bound for the Calabi functional [19].

**Theorem 2.** We have

$$\inf_{\omega \in c_1(L)} \| \Lambda_\omega \alpha - c \|_{L^2} \geq \sup_{\chi \text{ test-config}} \frac{F_\alpha(\chi)}{\| \chi \|},$$

where the $L^2$-norm is computed using $\omega$, and $\| \chi \|$ is a natural norm for test-configurations $\chi$ for $(M, L)$.

A corollary of this result is analogous to Stoppa’s theorem [16] on the K-stability of constant scalar curvature Kähler manifolds.

**Theorem 3.** If Equation (2) has a solution, then $F_\alpha(\chi) > 0$ for all test-configurations $\chi$ for $(M, L)$ satisfying $\| \chi \| > 0$.

In Section 3 we will study deformation to the normal cone, which is a particular type of test-configuration studied extensively by Ross-Thomas [14]. Applying Theorem 3 to deformation to the normal cone of a subvariety $V$ results in the inequality (5). Note, however, that this direction of Conjecture 1 can easily be checked directly. In this section we will also show that Conjecture 1 holds in the two dimensional case.

When Equation (2) has no solution, then it is natural to conjecture that in Theorem 2 equality holds. We will show this in a special case, namely on the blowup of $\mathbb{P}^3$ in one point. The J-flow on this, and other similar, manifolds has been studied carefully by Fang-Lai [8]. Our new contribution can be summarized as follows.

**Theorem 4.** On the blowup $\text{Bl}_p \mathbb{P}^3$ in one point, with $\omega$ and $\alpha$ representing any two Kähler classes, equality holds in Equation (6). In addition the J-flow minimizes the $L^2$-norm of $\Lambda_\omega \alpha$.

This result is analogous to the second author’s work [19] on the Calabi functional on a ruled surface.

In Section 5 we study Conjecture 1 on the complex torus $\mathbb{C}^n/(\mathbb{Z}^n + i\mathbb{Z}^n)$, for $(S^1)^n$-invariant data. It is easy to see that the inequalities (5) are always satisfied, so we expect that a solution always exists. In this situation Equation (2) reduces to an equation for convex functions on $\mathbb{R}^n$. A generalization of this equation can be formulated as follows. Let $a_{ij}(x)$ be a smooth $\mathbb{Z}^n$-periodic symmetric positive definite matrix-valued function. We are trying to find a $\mathbb{Z}^n$-periodic function $u$:
\( \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( f(x) = |x|^2 + u(x) \) is convex, and

\[
\sum_{i,j} a_{ij}(x) f^{ij}(x) = c,
\]

where \( f^{ij} \) is the inverse Hessian of \( f \), and \( c \) is a suitable constant. By using the Legendre transform and the constant rank theorems of Korevaar-Lewis \([11]\) and Bian-Guan \([1]\) we show that this equation always has a solution.

**Theorem 5.** Equation (7) has a smooth convex solution of the form \( f(x) = |x|^2 + u(x) \), where \( u \) is \( \mathbb{Z}^n \)-periodic.

The J-flow can be generalized to the more general inverse \( \sigma_k \)-flows studied by Fang-Lai \([8]\), and the results of this paper, apart from Theorem 5, extend to this case without any difficulties. We will discuss this in Section 6.

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## 2. The Bergman kernel

The goal of this section is to prove Theorem 2. The method of proof is the same as Donaldson’s proof of the analogous lower bound for the Calabi functional \([6]\). The main ingredient is a relevant Bergman kernel expansion.

Let \( \omega \in c_1(L) \) for an ample line bundle \( L \) over \( M \). Choose a Hermitian metric \( h \) on \( L \) such that \( \omega = \frac{1}{2\pi i} F(h) \), where \( F(h) \) is the curvature form of \( h \). On sections of \( L^k \) define the inner product

\[
\langle s, t \rangle_{L^2} = k^n \int_M \langle s, t \rangle_t h^k \alpha \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_M \langle s, t \rangle_t h^k (\Lambda_\omega \alpha) (\frac{k\omega}{n!})^n.
\]

Given an orthonormal basis \( \{s_0, \ldots, s_N\} \) of \( H^0(L^k) \), define the “Bergman kernel”

\[
B_k(x) = \sum_{i=0}^N |s_i(x)|_{h^k}^2.
\]

The following asymptotic expansion is analogous to the Tian-Zelditch-Lu \([20, 26, 12]\) expansion of the usual Bergman kernel associated to the metric \( \omega \).

**Theorem 6.** We have the asymptotic expansion

\[
B_k(x) = \frac{1}{\Lambda_\omega \alpha(x)} + O(k^{-1}),
\]

valid in \( C^l \) for any \( l \).

**Proof.** The expansion follows from Theorem 4.1.1 in Ma-Marinescu \([13]\). Using the notation of \([13]\), we apply the result to \( E \) being the trivial line bundle, with metric \( \Lambda_\omega \alpha \). The endomorphism \( \frac{1}{2\pi i} \hat{R}^L \) is the identity, so in Theorem 4.1.1, equation (4.1.6) we have \( b_0 = \text{Id}_E \). In particular, by equation (4.1.4) in \([13]\) this means that

\[
\sum_{i=0}^N |s_i(x)|_{h^k}^2 (\Lambda_\omega \alpha) = 1 + O(k^{-1}),
\]
from which the required result follows.

Given an embedding \( \varphi : M \hookrightarrow \mathbb{P}^{N_k} \) we define a matrix \( M(\varphi) \) with entries

\[
M(\varphi)_{jk} = \int_M \varphi^* \left( \frac{Z^i Z^j}{|Z|^2} \right) \alpha \wedge \frac{\varphi^* \omega_{FS}}{(n-1)!},
\]

where \( Z^i \) are homogeneous coordinates on \( \mathbb{P}^{N_k} \) and \( \omega_{FS} \) is the Fubini-Study metric. Let \( \underline{M}(\varphi) \) denote the trace-free part of \( M(\varphi) \), so that

\[
\underline{M}(\varphi)_{jk} = M(\varphi)_{jk} - \frac{k^{n-1}}{N_k + 1} \int_M \alpha \wedge \frac{\omega^{n-1}}{(n-1)!}.
\]

Similarly to Proposition 1 in \([6]\) we have the following.

**Lemma 7.** There is a sequence of embeddings \( \varphi_k : M \hookrightarrow \mathbb{P}^{N_k} \) using sections of \( L^k \) such that

\[
\|\underline{M}(\varphi_k)\| \leq k^{n/2-1} \|\Lambda_{\omega} \alpha - c\|_{L^2} + O(k^{n/2-2}).
\]

Here \( \|\underline{M}\| = \text{Tr}(\underline{M}^2)^{1/2} \).

**Proof.** We use the sequence of embeddings \( \varphi_k \) defined by orthonormal bases of \( H^0(L^k) \), with respect to the inner product \( \langle \mathbf{8} \rangle \). We have

\[
\varphi_k^* \omega_{FS} = k \omega + \sqrt{-1} \partial \bar{\partial} B_k = k \omega + O(1),
\]

and so

\[
M(\varphi_k)_{ij} = \int_M \varphi_k^* \left( \frac{Z^i Z^j}{|Z|^2} \right) \alpha \wedge \frac{\varphi_k^* \omega_{FS}}{(n-1)!} = \int_M \frac{(s_i, s_j)_{B_k}}{B_k} \alpha \wedge \frac{(k \omega)^{n-1}}{(n-1)!} + O(k^{-2}).
\]

By changing the basis of sections, we can assume that \( M \) is diagonal. We have

\[
M(\varphi_k)_{ii} = k^{n-1} \int_M \frac{|s_i|^2 B_k}{B_k} \alpha \wedge \frac{\omega^{n-1}}{(n-1)!} + O(k^{-2}) \]

\[
= k^{n-1} \int_M |s_i|^2 B_k (\Lambda_{\omega} \alpha) \alpha \wedge \frac{\omega^{n-1}}{(n-1)!} + O(k^{-2}),
\]

and also the dimension of \( H^0(L^k) \), by Riemann-Roch, is

\[
N_k + 1 = k^n \int_M \frac{\omega^n}{n!} + O(k^{n-1}).
\]

It follows that

\[
\sum_{i=0}^{N_k} M(\varphi_k)_{ii} = k^{n-1} \int_M B_k (\Lambda_{\omega} \alpha) \alpha \wedge \frac{\omega^{n-1}}{(n-1)!} + O(k^{n-2}) \]

\[
= k^{n-1} \int_M (\Lambda_{\omega} \alpha) \frac{\omega^n}{n!} + O(k^{n-2}).
\]

The trace free part of \( M \) is therefore

\[
\underline{M}(\varphi_k)_{ii} = k^{n-1} \int_M |s_i|^2 B_k (\Lambda_{\omega} \alpha - c) \alpha \wedge \frac{\omega^{n-1}}{(n-1)!} + O(k^{-2}).
\]
It follows that
\[
\mathcal{M}(\varphi_k)^2 = k^{2n-2} \left( \int_M |s_i|_h^2 (\Lambda_\omega \alpha) (\Lambda_\omega \alpha - c) \frac{\omega^n}{n!} \right)^2 + O(k^{-3})
\]
\[
\leq k^{2n-2} \int_M |s_i|_h^2 (\Lambda_\omega \alpha) (\Lambda_\omega \alpha - c) \frac{\omega^n}{n!} \int_M |s_i|_h^2 \Lambda_\omega \alpha \frac{\omega^n}{n!} + O(k^{-3})
= k^{-2} \int_M |s_i|_h^2 (\Lambda_\omega \alpha) (\Lambda_\omega \alpha - c) \frac{(\omega^n)^2}{n!} + O(k^{-3}),
\]
and so finally summing up over \(i\) and using (11), we have
\[
\|\mathcal{M}(\varphi_k)\|^2 \leq k^{-2} \int_M (\Lambda_\omega \alpha - c) \frac{(\omega^n)^2}{n!} + O(k^{-3}),
\]
from which the result follows. \(\square\)

We can obtain lower bounds for \(\|\mathcal{M}(\varphi_k)\|\) using test-configurations. For this, suppose that \(\lambda : \mathbb{C}^* \rightarrow GL(N_k + 1)\) is a one-parameter subgroup, such that \(\lambda(S^1) \subset U(N_k + 1)\). So \(\lambda(t) = t^4\) for a Hermitian matrix \(A\) with integer eigenvalues. A Hamiltonian function for the corresponding \(S^1\)-action is given by
\[
h_A = A_{jk} Z^j Z^k.
\]
Let \(\varphi'_k = \lambda(t) \circ \varphi_k\), and define the function
\[
f(t) = \text{Tr}(\mathcal{A}(\varphi'_k)) = \text{Tr}(\mathcal{A}(\varphi'_k)),
\]
where \(\mathcal{A}\) is the trace-free part of \(A\). Then
\[
f(t) = \int_M \varphi'_k(h_A) \alpha \wedge (\varphi'_k \omega_{FS})^{n-1} \left( \frac{\omega_{FS}^{n-1}}{(n-1)!} \right) - \frac{\text{Tr}(A) k^{n-1}}{N_k + 1} \int_M \alpha \wedge \frac{\omega_{FS}^{n-1}}{(n-1)!},
\]
and a calculation shows that for real numbers \(t > 0\) we have \(f'(t) \geq 0\):

**Lemma 8.** With the above definition we have \(f'(t) \geq 0\).

**Proof.** We consider the one-parameter group of diffeomorphisms generated by the vector field \(-\text{grad} h_A\) so we are approaching 0 along the positive real axis in \(\mathbb{C}^*\). Then, we have the following derivative at \(s = 0\)
\[
\frac{d}{ds} \bigg|_{s=0} \int_M \varphi'_k(h_A) \alpha \wedge (\varphi'_k \omega_{FS})^{n-1} \left( \frac{\omega_{FS}^{n-1}}{(n-1)!} \right) = - \int_{\varphi_k(M)} |\text{grad} h_A|^2 \varphi_k^* (\alpha) \wedge \frac{\omega_{FS}^{n-1}}{(n-1)!} + \int_{\varphi_k(M)} h_A \varphi_k^* (\alpha) \wedge \frac{\text{grad} h_A \omega_{FS} \wedge \omega_{FS}^{n-2}}{(n-2)!}.
\]
The second term in the r.h.s of (26) can be written as
\[
\int_{\varphi_k(M)} h_A \varphi_k^* (\alpha) \wedge \text{grad} h_A \omega_{FS} \wedge \omega_{FS}^{n-2} = 2 \int_{\varphi_k(M)} |\partial h_A \wedge \partial h_A \wedge \varphi_k^* (\alpha) \wedge \omega_{FS}^{n-2} |
= \frac{2}{n-1} \int_{\varphi_k(M)} |\partial h_A |_M^2 \varphi_k^* (\alpha) \wedge \omega_{FS}^{n-1}
- \frac{2}{n(n-1)} \int_{\varphi_k(M)} |\partial h_A |_M^2 \omega_{FS} \wedge \omega_{FS}^{n-2},
\]
where $|\partial h_A|^2_M = \frac{1}{2} |\text{grad} h_A|^2_M$ is the norm of the tangential part to $\varphi_k(M)$ and $|\partial h_A|^2_{M,\alpha}$ is the norm with respect to $\varphi_{k\ast}(\alpha)$. We obtain

\begin{equation}
\frac{d}{ds} s=0 \int_M \varphi_k^* (h_A) \alpha \wedge \frac{(\varphi_{k\ast}^* \omega_{FS})^{n-1}}{(n-1)!} = - \int_{\varphi_k(M)} |\text{grad} h_A|^2_M \varphi_{k\ast} (\alpha) \wedge \frac{\omega_{FS}^{n-1}}{(n-1)!} \nonumber \end{equation}

- 2 \int_{\varphi_k(M)} |\partial h_A|^2_{M,\alpha} \frac{\omega_{FS}}{n!},

where $|\text{grad} h_A|^2_M$ is the norm of the normal component. Increasing $t$ corresponds to flowing along $\text{grad} h_A$. We deduce that $f'(t) \geq 0$ for real numbers $t > 0$. □

Now it follows that

\begin{equation}
\text{Tr}(A M (\varphi_k)) = f(1) \geq \lim_{t \to 0} f(t),
\end{equation}

and so by the Cauchy-Schwarz inequality

\begin{equation}
\|A\| \|M(\varphi_k)\| \geq \lim_{t \to 0} f(t).
\end{equation}

In particular if $\lim_{t \to 0} f(t) > 0$, then we get a positive lower bound on $\|M(\varphi_k)\|$. We need to compute the limit on the right hand side.

**Lemma 9.** Suppose that $\alpha \in \mathfrak{d}_1(K)$ for a very ample line bundle $K$ over $M$, and let $D \subset M$ be a sufficiently general element in the linear series defined by $K$. Let $D_0 = \lim_{t \to 0} \varphi_k(D)$ denote the flat limit, and $|D_0|$ the corresponding algebraic cycle. Then

\begin{equation}
\lim_{t \to 0} \int_M \varphi_k^* (h_A) \alpha \wedge \frac{(\varphi_{k\ast}^* \omega_{FS})^{n-1}}{(n-1)!} = \int_{|D_0|} h_A \frac{\omega_{FS}^{n-1}}{(n-1)!}.
\end{equation}

**Proof.** According to Theorem [19] we can write $\alpha$ as a linear combination of currents of integrations:

\begin{equation}
\alpha = \int_{|K|} [D] d\mu(D),
\end{equation}

where $\mu$ is a smooth signed measure on the linear series $|K|$. It follows that

\begin{equation}
\int_M \varphi_k^* (h_A) \alpha \wedge \frac{(\varphi_{k\ast}^* \omega_{FS})^{n-1}}{(n-1)!} = \int_{|K|} \left( \int_D \varphi_k^* (h_A) \frac{(\varphi_{k\ast}^* \omega_{FS})^{n-1}}{(n-1)!} \right) d\mu(D) \\
= \int_{|K|} \left( \int_{\varphi_k^*(D)} h_A \frac{\omega_{FS}^{n-1}}{(n-1)!} \right) d\mu(D).
\end{equation}

The integrands are uniformly bounded, so Lebesgue’s convergence theorem implies that

\begin{equation}
\lim_{t \to 0} \int_M \varphi_k^* (h_A) \alpha \wedge \frac{(\varphi_{k\ast}^* \omega_{FS})^{n-1}}{(n-1)!} = \int_{|K|} \left( \int_{|D_0|} h_A \frac{\omega_{FS}^{n-1}}{(n-1)!} \right) d\mu(D),
\end{equation}

where we abuse notation somewhat, denoting by $D_0 = \lim_{t \to 0} \varphi_k^*(D)$ the flat limit, which depends on $D$. As in [18], we now use the fact that the limit

\begin{equation}
\int_{|D_0|} h_A \frac{\omega_{FS}^{n-1}}{(n-1)!}.
\end{equation}
is related to the Chow weight of the divisor $D$ under the $\mathbb{C}^*$-action $\lambda$, to see that for all $D$ outside a Zariski closed subset of $|K|$, the value of the integral $(\ref{integral})$ is the same. The result we want follows, since the Zariski closed subset has measure zero with respect to $\mu$. \hfill \Box

Finally, just as in \cite{Donaldson} we can compute the asymptotics of this lower bound on $\|\mathcal{M}(|\varphi_k|)\|$ as $k \to \infty$, when using the same test-configuration embedded into larger and larger projective spaces. This leads to the following definition.

**Definition 10.** A test-configuration $\chi$ for $(M, L)$ consists of an embedding $M \subset \mathbb{P}^{N_k}$ of $M$ using sections of $L^k$, together with a one-parameter subgroup $\chi: \mathbb{C}^* \hookrightarrow GL(N_k + 1)$. Let $M_0 = \lim_{t \to 0} \chi(t) \cdot M$ denote the flat limit, with polarization $L_0^k$ obtained by restricting the $O(1)$ bundle (so $L_0$ is a $\mathbb{Q}$-line bundle). There is an induced dual $\mathbb{C}^*$-action on $(M_0, L_0^m)$ for each sufficiently divisible $m$, and we let $d_m = \dim H^0(M_0, L_0^m)$, and denote by $w_m$ the total weight of the action. Define $a_0$ and $b_0$ using the expansions

\begin{align}
d_m &= a_0m^n + O(m^{n-1}) \\
w_m &= b_0m^{n+1} + O(m^n)
\end{align}

As above, let $D \subset M$ be a sufficiently general element of the linear series defined by $\alpha$ (we can replace $\alpha$ and $\omega$ by multiples if necessary). The test-configuration $\chi$ induces a test-configuration for $D$, and we denote by $a'_0$ and $b'_0$ the corresponding constants. Finally we define

\begin{equation}
F_\alpha(\chi) = b'_0 - \frac{a'_0}{a_0}b_0 = b'_0 - cb_0,
\end{equation}

where $c$ is the average of $A^{-}\alpha$ as above. The norm $\|\chi\|$ is defined exactly as in \cite{Donaldson} using the asymptotics of $\text{Tr}(A^m_\alpha)$, where $A_m$ is the generator of the $\mathbb{C}^*$-action on $H^0(M_0, L_0^m)$.

Note that from Proposition 3 in \cite{Donaldson}, it follows that if $\lambda$ is a one-parameter subgroup of $GL(N_k + 1)$ such that $\lambda(S^1) \subset U(N_k + 1)$ as before, then we have

\begin{equation}
\int_{|D_0|} h_A \frac{\omega_{FS}^{n-1}}{(n-1)!} = -b_0
\end{equation}

in the notation of the above definition, and a similar equality holds relating $b_0$ to the integral of $h_A$ over $|M_0|$ (note that the eigenvalues of the dual action in the definition of $b_0$ are the negatives of the eigenvalues of $\lambda$). It follows that an alternative definition for $F_\alpha(\chi)$ is given by

\begin{equation}
F_\alpha(\chi) = c \int_{|M_0|} h_A \frac{\omega_{FS}^n}{n!} - \int_{|D_0|} h_A \frac{\omega_{FS}^{n-1}}{(n-1)!},
\end{equation}

where $|D_0|$ is the algebraic cycle representing the limit $\lim_{t \to 0} \lambda(t) \cdot D$, for a generic divisor $D \subset M$ in the linear series defined by $\alpha$. Note that this is the same expression that one needs to add to the usual Futaki invariant when dealing with metrics with conical singularities along a divisor; see Equation (30) in Donaldson \cite{Donaldson}.

From the arguments above, in an identical way to the proof in \cite{Donaldson} we obtain the following.
Theorem 11. We have
\[
\inf_{\omega \in c_1(L)} \|A_\omega \alpha - c\|_{L^2} \geq \sup_{\chi \text{ test-config}} -\frac{F_\alpha(\chi)}{\|\chi\|},
\]
where the $L^2$-norm is computed using $\omega$.

An immediate consequence of this result is that if a metric $\omega \in c_1(L)$ exists for which $A_\omega \alpha = c$, then $F_\alpha(\chi) \geq 0$ for all test-configurations $\chi$. We now strengthen this to strict positivity, with a perturbation method similar to Stoppa’s work \cite{10} on K-stability.

Theorem 12. If Equation (2) has a solution, then $F_\alpha(\chi) > 0$ for all test-configurations $\chi$ for $(M, L)$ satisfying $\|\chi\| > 0$.

Proof. Suppose that there is a metric $\omega$ satisfying $A_\omega \alpha = c$, and let $\chi$ be a test-configuration for $(M, L)$. For small $t \geq 0$ close to zero we write $\alpha_t = \alpha - t\omega$, which is still Kähler, and let
\[
c_t = \frac{n \int_M \alpha_t \wedge \omega^{n-1}}{\int_M \omega^n}.
\]
A simple argument using the implicit function theorem (similar to that in the proof of Theorem \cite{17}) implies that for $t$ sufficiently close to zero there is a metric $\omega_t$ satisfying $A_{\omega_t} \alpha_t = c_t$. The invariants $F_{\alpha_t}$ a priori are only defined for rational $t$, but they can be extended to all $t$ by continuity, and we have $F_{\alpha_t}(\chi) \geq 0$ for $t$ sufficiently close to zero. In fact it is clear from the definition as the asymptotics of the limit of $f(t)$ in (25), that $F_\alpha(\chi)$ is linear in $\alpha$, and so
\[
F_{\alpha_t}(\chi) = F_\alpha(\chi) - tF_\omega(\chi).
\]
We claim that $F_\omega(\chi) > 0$. If this is the case, then from $F_{\alpha_t}(\chi) \geq 0$ it follows that $F_\alpha(\chi) > 0$.

For simplicity of notation assume that $\chi$ is given by a $C^*$-action $\lambda$ on $\mathbb{P}^N$, and we have an embedding $M \subset \mathbb{P}^N$ using sections of $H^0(M, L)$. Let $M_0$ denote the flat limit $\lim_{t \to 0} \lambda(t) \cdot M$. The assumption that $\|\chi\| > 0$ implies that the action of $\lambda$ is non-trivial on the reduced part of $M_0$, since the nilpotent structure gives rise to lower order terms in the expansion of $\text{Tr}(A_\lambda^2)$ in the definition of $\|\chi\|$ (see the discussion on p. 1406 of Stoppa \cite{16}). Alternatively, the norm is given by
\[
\|\chi\|^2 = \int_{|M_0|} (h_A - \overline{h_A})^2 \frac{\omega_{FS}^n}{n!},
\]
where $\overline{h_A}$ denotes the average of $h_A$ on $|M_0|$, so the norm being positive implies that $h_A$ is non-constant on $|M_0|$.

From the definitions it follows that to have $F_\omega(\chi) > 0$, we need
\[
\int_{|M_0|} h_A \omega_{FS}^n > \int_{|D_0|} h_A \omega_{FS}^{n-1},
\]
where $|D_0|$ denotes the limiting cycle of a generic divisor $D \subset M$ representing the class $[\omega]$ under the $C^*$-action. In terms of Definition \cite{10} this means $b'_0 > nb_0$.

In order to compute the induced test-configuration on $D$, it is useful to think of test-configurations as filtrations of the homogeneous coordinate ring as in \cite{17} (see
also \[24\]). Let
\begin{equation}
R = \bigoplus_{k \geq 0} H^0(M, L^k) = \bigoplus_{k \geq 0} R_k
\end{equation}
be the homogeneous coordinate ring of \((M, L)\). As in \[17\], the test-configuration \(\chi\) gives rise to a filtration
\begin{equation}
C = F_0 R \subset F_1 R \subset F_2 R \subset \ldots
\end{equation}
where if necessary we multiply \(\chi\) by an action with constant weights, to make all weights positive. Given a section \(s \in R_1\), the divisor \(D = (s = 0)\) has homogeneous coordinate ring with \(k^\text{th}\) graded piece \(R_k/sR_{k-1}\), using the inclusion \(sR_{k-1} \subset R_k\). The filtration of \(R\) induces a filtration on the coordinate ring of \(D\) by letting
\begin{equation}
F_i (R_k/sR_{k-1}) = F_i R_k/F_i(sR_{k-1})
\end{equation}
The weight of the action on \(R_k\) is given by
\begin{equation}
w_k = \sum_i (-i) \dim F_i R_k/F_i^{-1} R_k,
\end{equation}
with a corresponding formula for the weight on \(R_k/sR_{k-1}\):
\begin{equation}
w_k' = \sum_i (-i) \dim F_i (R_k/sR_{k-1})/F_i^{-1} (R_k/sR_{k-1})
= w_k - \sum_i (-i) \dim F_i (sR_{k-1})/F_i^{-1} (sR_{k-1}).
\end{equation}
In order to estimate the last sum, we will consider the central fiber of the test-configuration. The homogeneous coordinate ring of the central fiber \((M_0, L_0)\) can naturally be thought of as the associated graded ring of the filtration:
\begin{equation}
\tilde{R} = \sum_i F_i R/F_i^{-1} R = \sum_{k,i} F_i R_k/F_i^{-1} R_k,
\end{equation}
with the induced \(\mathbb{C}^*\)-action acting on the \(i\text{th}\) piece with weight \(-i\). Let us write
\begin{equation}
s = s_1 + \ldots + s_m
\end{equation}
for the weight decomposition of the section \(s\).
We have
\begin{equation}
H^0(M_0, L_0^k) = \bigoplus_j H^0(M_{0,j}, L_0^k)/E_k,
\end{equation}
where \(M_{0,j}\) are the irreducible components of \(M_0\), and \(E_k\) is a suitable subspace of the direct sum, defined by equality of sections on various intersections. Since these intersections are lower dimensional, \(\dim E_k = O(k^{a-1})\), so to leading order in \(k\), we can treat \(H^0(M_0, L_0^k)\) as if it were equal to the direct sum. This way we can focus on each irreducible component separately. We need to check how multiplication by \(s\) affects the weight of sections, and this will depend on which irreducible components the product does not vanish on. For each irreducible component \(M_{0,j}\), let \(m_j\) be the largest integer such that \(s_{m_j}\) is not a zero divisor when restricted to \(M_{0,j}\) (i.e. the reduced support \(|M_{0,j}|\) is not contained in the zerocet of \(s_{m_j}\)). Then multiplication by \(s\) will decrease the weight of sections which do not vanish on \(M_{0,j}\) by at least \(m_j\), and the total contribution of this is \((-m_j) \dim H^0(M_{0,j}, L_0^{k-1})\). Summing up...
over all irreducible components, some of the sections will be counted more than once, but up to order $k^{n-1}$ we get an upper bound
\[
\sum_i (-i) \dim F_i(sR_{k-1})/F_{i-1}(sR_{k-1}) \leq w_{k-1} + \sum_j (-m_j) \dim H^0(M_0,j_t, L_0^{k-1}) + O(k^{n-1})
\]
\[= w_{k-1} - \sum_j m_j k^n \int_{|M_0,j|} \frac{\omega_{FS}}{n!} + O(k^{n-1}).\]  
(53)

If $s$ is a generic section, then it has non-zero component in each weight space for the $\C^*$-action $\lambda$, so for each $j$ we have that $m_j$ is the largest weight of the induced $\C^*$-action on the reduced support $|M_0,j|$. It follows that
\[
\int_{|M_0,j|} h_A \frac{\omega_{FS}}{n!} \leq m_j \int_{|M_0,j|} \frac{\omega_{FS}}{n!},
\]
and there is at least one $j$ such that $h_A$ is not constant on $|M_0,j|$, so that we have strict inequality in (54). Using this in (53) we get
\[
\sum_i (-i) \dim F_i(sR_{k-1})/F_{i-1}(sR_{k-1}) < w_{k-1} - \sum_j m_j \int_{|M_0,j|} h_A \frac{\omega_{FS}}{n!} + O(k^{n-1}).
\]
(55)

From (49) we then get
\[
w_k' > w_k - k^n \int_{|M_0,j|} h_A \frac{\omega_{FS}}{n!} + O(k^{n-1})
\]
\[= (n + 1)b_0 k^n - b_0 k^n + O(k^{n-1}),
\]  
(56)

so by taking leading order terms we have $b_0' > nb_0$. 

3. Deformation to the normal cone

A special type of test-configuration is given by deformation to the normal cone of a subvariety. This was studied in detail by Ross-Thomas [14] in the context of K-stability. Let $(M, L)$ be a polarized manifold as before, and let $V \subset M$ be a subvariety. The deformation to the normal cone of $V$ is the flat family
\[
\text{Bl}_{V \times \{0\}} M \times \C
\]
over $\C$, obtained by blowing up. For sufficiently small rational $\kappa > 0$ one can define the $\Q$-polarization $L_\kappa = \pi^* L - \kappa E$, where $E$ denotes the exceptional divisor. Let us denote this test-configuration by $\chi_{V,\kappa}$. We can compute the invariant $F_\alpha(\chi_{V,\kappa})$ using calculations from [14], and from this we obtain the following.

Proposition 13. For sufficiently small $\kappa$ we have
\[
F_\alpha(\chi_{V,\kappa}) = \frac{\kappa^{n+p+1}}{p!(n-p+1)!} \int_V \omega^p - p\omega^{p-1} \wedge \alpha.
\]

Proof. We need to compute the numbers $a_0, a_0', b_0, b_0'$ in Definition 10. First we have
\[
a_0 = \int_M \frac{\omega^n}{n!}
\]
\[a_0' = \int_D \frac{\omega^{n-1}}{(n-1)!} = \int_M \alpha \wedge \frac{\omega^{n-1}}{(n-1)!},
\]  
(59)
where \( D \) is any element of the linear series defined by \( \alpha \). To compute \( b_0 \), we use formula (4.6) from [14]:

\[
(60) \quad b_0 = \int_0^\kappa a_0(x) \, dx - \kappa a_0.
\]

For this we need to compute \( a_0(x) \), defined by the expansion

\[
(61) \quad \dim H^0(M, L^k \otimes \mathcal{I}_V^k) = a_0(x)k^n + O(k^{n-1}),
\]

where \( \mathcal{I}_V \) denotes the ideal sheaf of \( V \). It follows that for small \( x \) we have

\[
(62) \quad a_0(x) = a_0 - \frac{x^{n-p}}{(n-p)!} \int_V \frac{\omega_p}{p!},
\]

where \( p = \dim V \), and so

\[
(63) \quad b_0 = -\kappa_n^{n-p+1} \frac{(n-p+1)!}{(n-p+1)!} \int_D \frac{\omega_p}{p!}.
\]

To compute \( b'_0 \), note that for a generic \( D \), the subvariety \( V \) has no component contained in \( D \) and so the induced test-configuration for \( D \) is deformation to the normal cone of \( D \cap V \). The above formula can then be used in this case too and we obtain

\[
(64) \quad b'_0 = -\kappa_n^{n-p+1} \frac{(n-p+1)!}{(n-p+1)!} \int_{D \cap V} \frac{\omega_p}{(p-1)!} = -\kappa_n^{n-p+1} \frac{(n-p+1)!}{(n-p+1)!} \int_V \alpha \wedge \frac{\omega_p}{(p-1)!}.
\]

Our result then follows from the definition of \( F_\alpha(\chi_{V,\kappa}) \) in Definition 10. \( \square \)

Together with Theorem 3, this result implies one direction of Conjecture 1. Note that this direction also follows directly by examining the eigenvalues of the metrics \( \omega \) and \( \alpha \) along \( V \). Indeed, denoting by \( \alpha_V \) and \( \omega_V \) the restrictions to \( V \), we have

\[
(65) \quad p\omega_V^{p-1} \wedge \alpha_V = (\Lambda_\omega \alpha_V)\omega_V^p,
\]

along \( V \), and also \( \Lambda_\omega \alpha_V < \Lambda_\omega \alpha = c \). It follows that along \( V \) we have

\[
(66) \quad c\omega_V^p - p\omega_V^{p-1} \wedge \alpha_V > 0,
\]

and integrating we get (5).

We should point out that part of the content of Conjecture 1 is that deformation to the normal cone of subvarieties provides a sufficiently large family of test-configurations to check, to ensure that a solution to Equation 2 exists. We will show this in the case when \( M \) is two-dimensional, however, it may be necessary to refine the conjecture allowing for more general test-configurations in the higher dimensional case.

**Proposition 14.** When \( \dim M = 2 \), then Conjecture 1 holds.

**Proof.** We only need to show that if

\[
(67) \quad \int_V c\omega - \alpha > 0
\]

for all curves in \( M \), then there exists a metric \( \omega \) satisfying \( \Lambda_\omega \alpha = c \). According to Chen [2] it is enough to show that \( [c\omega - \alpha] \) is a Kähler class.
We will argue by contradiction, assuming that $[c\omega - \alpha]$ is not Kähler with an argument similar to one in [4]. For any $t > 0$ let us define $\alpha_t = \alpha + t\omega$, and let

$$c_t = \frac{\int_M \alpha_t \wedge \omega}{\int_M \frac{1}{2} \omega^2}.$$  

Then

$$\frac{d}{dt}(c_t\omega - \alpha_t) = 2\omega - \omega = \omega,$$

so for sufficiently large $t$ the class $[c_t\omega - \alpha_t]$ is Kähler. Let

$$T = \inf \{t : [c_t\omega - \alpha_t] \text{ is Kähler}\}.$$  

Then $[c_T\omega - \alpha_T]$ is not Kähler, but it is nef, and in addition also big, since we can compute that

$$[c_T\omega - \alpha_T]^2 = [\alpha_T]^2 > 0.$$  

From the main result of Demailly-Paun [3] it follows that there is a curve $V \subset M$ such that

$$\int_V c_T\omega - \alpha_T = 0.$$  

But then we must also have

$$\int_V c\omega - \alpha \leq 0,$$

because of (69). This contradicts our assumption (67). □

4. Example - $\mathbf{P}^3$ blown up in one point

In this section we will follow Fang-Lai [8] in studying a concrete example, namely the blowup $M = \text{Bl}_p\mathbf{P}^3$. The discussion will be somewhat similar to the second author’s work [19], on the Calabi flow on a ruled surface. We write $M = \mathbf{P}(\mathcal{O}(-1) \oplus \mathcal{O})$ as a ruled manifold over $\mathbf{P}^2$. Let $h$ be a metric on $\mathcal{O}(-1)$ with curvature $-\frac{2}{\pi} i \omega_{FS}$, and write $s = \log |\cdot|_h$ for the log of the fiberwise norm. Given a suitably convex function $f : \mathbf{R} \to \mathbf{R}$ we can write down a Kähler metric

$$\alpha = \sqrt{-1} \partial \bar{\partial} f(s)$$

on $M$. At a point choose local coordinates $z = (z_1, z_2)$ on $\mathbf{P}^2$ and a fiberwise coordinate $w$ such that $d \log h(z) = 0$. At this point we have

$$\alpha = \sqrt{-1} f'(s) \pi^* \omega_{FS} + f''(s) \frac{\sqrt{-1} dw \wedge d\bar{w}}{2|w|^2}.$$  

Similarly we can write

$$\omega = \sqrt{-1} \partial \bar{\partial} g(s) = \sqrt{-1} g'(s) \pi^* \omega_{FS} + g''(s) \frac{\sqrt{-1} dw \wedge d\bar{w}}{2|w|^2}$$

for a different convex function $g$. It follows that

$$\Lambda_\omega \alpha = 2 \frac{f'(s)}{g'(s)} + \frac{f''(s)}{g''(s)}.$$  

Let us write $E_0$ for the zero section (the exceptional divisor) on $M$ and $E_\infty$ the infinity section. Following [8] we will work in the Kähler classes

\begin{align}
\alpha &\in a[E_\infty] - [E_0] \\
\omega &\in b[E_\infty] - [E_0],
\end{align}

for constants $a, b > 1$. In terms of $f, g$ this means that

\begin{align}
\lim_{s \to -\infty} f'(s) &= 1, \\
\lim_{s \to \infty} f'(s) &= a \\
\lim_{s \to -\infty} g'(s) &= 1, \\
\lim_{s \to \infty} g'(s) &= b.
\end{align}

We introduce the coordinate $\tau = g'(s)$, and define the strictly increasing function $F : [1, b] \to [1, a]$ by letting

\begin{equation}
F(g'(s)) = f'(s)
\end{equation}

for all $s$. We can then compute that in terms of $F$ we have

\begin{equation}
\frac{dF}{d\tau} + 2\frac{F}{\tau},
\end{equation}

and moreover if we think of $\alpha$ as being fixed, then we can recover $\omega$ from knowing $F$. The main result of [8] in this special case is that the J-flow on $M$ displays three different behaviors depending on the values of $a, b$:

1. If $\frac{ab^2 - 1}{b^3 - 1} > \frac{2}{3}$, then the J-flow converges to a smooth solution of $\Lambda_\omega \alpha = c$.

2. If $\frac{ab^2 - 1}{b^3 - 1} = \frac{2}{3}$, then the J-flow converges to a singular solution of $\Lambda_\omega \alpha = c$, which is smooth away from $E_0$, and has a conical singularity along $E_0$.

3. If $\frac{ab^2 - 1}{b^3 - 1} < \frac{2}{3}$, then the J-flow converges to a current, which is a smooth solution of $\Lambda_\omega \alpha = c'$ (with a suitable constant $c'$) away from $E_0$, and is a current of integration along $E_0$.

In particular the equation $\Lambda_\omega \alpha = c$ has a smooth solution on $M$, if and only if

\begin{equation}
\frac{ab^2 - 1}{b^3 - 1} > \frac{2}{3}.
\end{equation}

This is consistent with Conjecture [1]. Indeed we have

\begin{equation}
c = \frac{3(a[E_\infty] - [E_0]) \cdot (b[E_\infty] - [E_0])^2}{(b[E_\infty] - [E_0])^3} = \frac{3(ab^2 - 1)}{b^3 - 1},
\end{equation}

and so

\begin{equation}
\int_{E_0} c\omega^2 - 2\omega \wedge \alpha = c - 2 = \frac{3(ab^2 - 1)}{b^3 - 1} - 2.
\end{equation}

The latter quantity is positive precisely when the Inequality [82] holds. In addition the fact that in case (2) and (3) the singularities occur along $E_0$ is reflected by the fact that it is deformation to the normal cone of $E_0$ which is the destabilizing test-configuration in these cases.

**Remark 15.** Donaldson [4] pointed out that the obvious conjecture to make is that the J-flow converges whenever the class $[c\omega - \alpha]$ is Kähler. It is easy to check that on $M$, if we set $a = 5$ and $b = 10$, then the latter class is Kähler, but Inequality [82] does not hold. This means that the obvious conjecture is false.
We will now use Theorem 2 to show that in these cases the J-flow minimizes the $L^2$-norm of $(\Lambda_\omega \alpha - c)$. The only interesting case is (3), since in the other two cases the infimum is zero. As a consequence we will also see that equality holds in Theorem 2. In order to work with algebraic Kähler metrics we need to assume that $a, b$ are rational, but a simple approximation argument extends the results to arbitrary $a, b > 1$.

**Theorem 16.** For any $a, b > 1$, the J-flow minimizes the $L^2$-norm of $\Lambda_\omega \alpha$. In addition equality holds in Equation (6).

**Proof.** We will only focus on case (3). In [8], the J-flow is rewritten in terms of the function $F$, resulting in an evolution equation for a time dependent family of functions $F_t : [1, b] \rightarrow [1, a]$. It is then shown in [8], that as $t \rightarrow \infty$, the functions $F_t$ converge to $F_\infty$ satisfying

$$F_\infty(\tau) = \begin{cases} 1, & 1 \leq \tau \leq \lambda \\ G(\tau), & \lambda \leq \tau \leq b, \end{cases}$$

for a suitable constant $\lambda \in (1, b)$, and $G$ satisfies the ODE

$$\frac{d}{d\tau} \left[ \frac{dG}{d\tau} + 2 \frac{G}{\tau} \right] = 0,$$

with boundary conditions $G(\lambda) = 1$, $G(b) = a$. This function $G$ is strictly increasing.

The average of $\Lambda_\omega \alpha$ is the fixed constant $c$ in Equation 83, which we can also compute from the function $F$. Namely, the volume form of $\omega$ is $\frac{1}{2} \tau^2 d\tau$, so

$$\int_1^b \frac{1}{2} \tau^2 d\tau = \frac{b^3 - 1}{6},$$

and

$$\int_M \Lambda_\omega \alpha \frac{\omega^3}{3!} = \int_1^b \left[ \frac{dF}{d\tau} + 2 \frac{F}{\tau} \right] \frac{1}{2} \tau^2 d\tau$$

$$= \frac{1}{2} \int_1^b \frac{d}{d\tau} \left( \tau^2 F \right) d\tau$$

$$= \frac{ab^2 - 1}{2},$$

so the ratio of the two quantities recovers Equation 83. It is therefore equivalent to minimize $\|\Lambda_\omega \alpha\|_{L^2}$ or $\|\Lambda_\omega \alpha - c\|_{L^2}$. Using the work in [8], along the J-flow $\omega(t)$ we have

$$\lim_{t \rightarrow \infty} \int_M (\Lambda_\omega (t) \alpha - c)^2 \frac{\omega^3}{3!} = \int_1^b \left[ \frac{dF_\infty}{d\tau} + 2 \frac{F_\infty}{\tau} - c \right]^2 \frac{1}{2} \tau^2 d\tau.$$
linear approximations to the convex function

\begin{equation}
    h(\tau) = \frac{dF_\infty}{d\tau} + 2 \frac{F_\infty}{\tau} - c = \begin{cases} 
    2\tau^{-1} - c, & 1 \leq \tau \leq \lambda \\
    2\lambda^{-1} - c, & \lambda \leq \tau \leq b.
    \end{cases}
\end{equation}

We choose a sequence of piecewise linear, rational, convex functions \( h_k \), approximating \( h \). We can assume that each \( h_k \) is constant for \( \tau \) close to \( b \). This means that each \( h_k \) is a deformation to the normal cone of a suitable scheme supported on \( E_0 \). Let us denote this test-configuration by \( \chi_k \). A general element in the linear series corresponding to \( \alpha \) has no component contained in \( E_0 \). Let us write \( F = \pi^*(O(1)) \), so that

\begin{equation}
    a[E_\infty] - [E_0] = [F] + (a - 1)[E_\infty].
\end{equation}

In the limit along the central fiber of \( \chi_k \), a generic element of the linear series corresponding to \( \alpha \) will be the same as the induced test-configuration for a generic element of \( [F] \) plus \((a - 1)\)-times \([E_\infty]\). Using this, we can compute the numbers \( b_{0,k}, b'_{0,k} \) in Definition 10 for the invariant \( F_\alpha(h_k) \), and we get

\begin{equation}
    b_{0,k} = - \int_1^b h_k \frac{1}{2} \tau^2 \, d\tau
\end{equation}

\begin{equation}
    b'_{0,k} = - \int_1^b h_k \tau \, d\tau - \frac{(a - 1)b^2}{2} h_k(b)
\end{equation}

\begin{equation}
    ||\chi_k||^2 = \int_1^b h_k^2 \frac{1}{2} \tau^2 \, d\tau.
\end{equation}

To compute the right hand side of (89) we have

\begin{equation}
\begin{aligned}
    & \int_1^b \left[ \frac{dF_\infty}{d\tau} + 2 \frac{F_\infty}{\tau} - c \right]^2 \frac{1}{2} \tau^2 \, d\tau = \frac{1}{2} \int_1^b h \left[ \tau^2 \frac{dF_\infty}{d\tau} + 2\tau F_\infty - c\tau^2 \right] \, d\tau \\
    & = \frac{1}{2} \int_1^b \frac{d}{d\tau}(\tau^2 F_\infty) \, d\tau - c \int_1^b \frac{1}{2} \tau^2 \, d\tau \\
    & = - \int_1^b \frac{dh}{d\tau} F_\infty \frac{1}{2} \tau^2 \, d\tau + \frac{1}{2} \left[ b^2 ah(b) - h(1) \right] - c \int_1^b \frac{1}{2} \tau^2 \, d\tau \\
    & = \frac{1}{2} \int_1^b h \tau \, d\tau + \frac{(a - 1)b^2}{2} h(b) - c \int_1^b \frac{1}{2} \tau^2 \, d\tau \\
    & = \lim_{k \to \infty} \left[ \int_1^b h_k \tau \, d\tau + \frac{(a - 1)b^2}{2} h_k(b) - c \int_1^b h_k \frac{1}{2} \tau^2 \, d\tau \right] \\
    & = - \lim_{k \to \infty} F_\alpha(\chi_k).
\end{aligned}
\end{equation}
where in the fourth line we used that $h'(\tau) = 0$ whenever $F_\infty(\tau) \neq 1$. From Theorem 2 we obtain for any $\omega$ in our Kähler class the lower bound

$$\|\Lambda_\omega \alpha - c\|_{L^2} \geq - \lim_{k \to \infty} \frac{F_\alpha(\chi_k)}{\|\chi_k\|}$$

$$= \frac{\int_1^b h^2 \frac{1}{2} \tau^2 \, d\tau}{\left( \int_1^b h^2 \frac{1}{2} \tau^2 \, d\tau \right)^{1/2}}$$

$$= \left( \int_1^b \left[ \frac{dF_\infty}{d\tau} + 2 \frac{F_\infty}{\tau} - c \right]^2 \frac{1}{2} \tau^2 \, d\tau \right)^{1/2}$$

$$= \lim_{t \to \infty} \|\Lambda_{\omega(t)} \alpha - c\|_{L^2},$$

where we used Equation 89 in the last line. This establishes that the J-flow minimizes the $L^2$-norm of $\Lambda_\omega \alpha$ as well as the fact that equality holds on Theorem 2 on the manifold $M$.

5. Example - Complex Tori

In this section we will study the J-flow, or rather its critical equation, on a complex torus $M = \mathbb{C}^n / (\mathbb{Z}^n + i\mathbb{Z}^n)$. It is easy to generalize to quotients by other lattices, so for simplicity of notation we will focus on this specific case. The equation can then be reduced to a special case of the following, in which $a_{jk}$ is the Hessian of a function. It turns out that the stability condition in this case is vacuous, which can also be seen from the fact that “constant” metrics in any two Kähler classes always provide solutions of the J-equation.

Let $a_{jk}(x)$ be a smooth, symmetric positive definite matrix valued function on $\mathbb{R}^n$, which is $\mathbb{Z}^n$-periodic. In addition let $B = (b_{jk})$ be a symmetric positive definite matrix. In terms of the J-equation, $a_{jk}(x)$ is determined by the metric $\alpha$, and $B$ determines the Kähler class of $\omega$.

**Theorem 17.** There exists a smooth convex function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = \frac{1}{2} x^T B x + u(x),$$

with $u(x)$ being $\mathbb{Z}^n$-periodic, that satisfies the equation

$$\sum_{j,k} a_{jk}(x) f^{jk}(x) = c.$$  

Here $f^{jk}$ is the inverse of the Hessian of $f$, and $c$ is a constant. The solution $f$ is unique up to addition of a constant.

**Proof.** We argue using the continuity method, connecting $a_{jk}(x)$ to a constant matrix. When $a_{jk}(x)$ is constant, then a solution is given by $u(x) = 0$. To prove openness is standard using the implicit function theorem. The linearization of the operator in (96) at a solution $f$ is given by

$$L : u \mapsto \sum_{j,k,p,q} a_{pq}(x) f^{jp}(x) f^{kq}(x) \frac{\partial^2 u(x)}{\partial x^j \partial x^k},$$

defined on periodic functions $u$. This can be thought of as an elliptic operator on the torus. Deforming it into the Laplacian for the flat metric, we see that it has
index zero, and moreover the strong maximum principle implies that any element of the kernel is constant. So the image of \( L \) has codimension 1. Moreover examining the maximum and minimum point of \( u \) we see that non-zero constants are not in the image. It follows that

\[
C^{k+2,\alpha} \times \mathbb{R} \to C^{k,\alpha}
\]

\[
(u, c) \mapsto L(u) - c
\]

is surjective. This is sufficient for openness.

The uniqueness also follows from the strong maximum principle. Namely, suppose that \( f \) and \( h \) satisfy

\[
\sum_{j,k} a_{jk}(x)f^{jk}(x) = c_f, \quad \sum_{j,k} a_{jk}(x)h^{jk}(x) = c_h.
\]

Then, writing \( f_t = h + t(f - h) \) for \( t \in [0,1] \),

\[
c_f - c_h = \sum_{j,k} a_{jk}(x)(f^{jk}(x) - h^{jk}(x))
\]

\[
= \sum_{j,k} a_{jk}(x) \int_0^1 \frac{d}{dt} f_t^{jk}(x) \, dt
\]

\[
= \sum_{j,k} a_{jk}(x) \int_0^1 \sum_{p,q} f_t^{jp}(x)(f - h)_{pq}(x)f_t^{qk}(x) \, dt
\]

\[
= \sum_{p,q} \left( \int_0^1 \sum_{j,k} a_{jk}(x)f_t^{jk}(x) \, dt \right) \frac{\partial^2 (f - h)}{\partial x^p \partial x^q}.
\]

The coefficients in the brackets define a positive definite symmetric matrix. Examining the maximum and minimum point of the periodic function \( f - h \) we find that \( c_f - c_h = 0 \), and then the strong maximum principle implies that \( f - h \) is a constant.

It remains to prove a priori estimates for the solution. First of all it is easy to obtain a priori lower and upper bounds \( 0 < \xi < c < \bar{c} \), in terms of the smallest and largest eigenvalues of \( a_{jk}(x) \) and \( B \) by examining the maximum and minimum points of \( u \). Consider now the Legendre transform \( g : \mathbb{R}^n \to \mathbb{R} \) of \( f \), defined by

\[
f(x) + g(y) = x \cdot y,
\]

where \( y = \nabla f(x) \). It is standard that then \( x = \nabla g(y) \), and

\[
\left( \frac{\partial^2 f}{\partial x^j \partial x^k} \right)^{-1}(x) = \left( \frac{\partial^2 g}{\partial y^j \partial y^k} \right)(y).
\]

It follows that \( g \) satisfies the equation

\[
\sum_{j,k} a_{jk}(\nabla g(y)) \frac{\partial^2 g}{\partial y^j \partial y^k} = c.
\]

In addition \( g \) is convex, so this equation implies a uniform upper bound on the Hessian of \( g \) (in terms of the lowest eigenvalue of \( a_{jk}(x) \)). In particular we have a uniform \( C^\alpha \) bound on \( \nabla g \), and so Equation (102) implies uniform \( C^{2,\alpha} \) bounds on \( g \). Bootstrapping, we obtain uniform \( C^{k,\alpha} \) bounds on \( g \) for all \( k \). The only question
that remains is to find a positive lower bound on the Hessian of \( g \), since that will then imply the required estimates on the Legendre transform \( f \).

We argue by contradiction. Suppose that there is a sequence of solutions \( f_i \), with Legendre transforms \( g_i \), such that the \( g_i \) do not have a uniform lower bound on their Hessians. The \( C^{k,\alpha} \) bounds imply that we can find a subsequence converging in \( C^\infty \) to a convex function \( g_\infty \), solving an equation of the form

\[
\sum_{j,k} a_{jk}^\infty (\nabla g_\infty(y)) \frac{\partial^2 g_\infty}{\partial y^j \partial y^k} = c_\infty > 0,
\]

but with the Hessian of \( g_\infty \) having a zero eigenvalue at some point. The constant rank result Corollary 1.3 in Bian-Guan [11] (see also Korevaar-Lewis [11] for the two-dimensional case) implies that the Hessian of \( g_\infty \) is degenerate everywhere, and in particular there is a line \( L \subset \mathbb{R}^n \), along which \( g_\infty \) is linear. This contradicts the fact that each \( g_k \) (and so also \( g_\infty \)) is of the form

\[
g_k(y) = \frac{1}{2} y^T B^{-1} y + v(y),
\]

where \( v \) is periodic, with period \( B \cdot \mathbb{Z}^n \). This implies that the solution \( g \) of (103) has a uniform lower bound on its Hessian (depending on bounds on \( a_{jk}(x) \) and \( B \)). □

6. The inverse \( \sigma_k \)-flow

The discussion in the previous sections can be extended to a class of more general equations introduced in Fang-Lai-Ma [9], called the inverse \( \sigma_k \)-flow. We are interested in the elliptic version of the equation, which for \( k = 1, \ldots, n \) can be written as

\[
\binom{n}{k} \alpha^k \wedge \omega^{n-k} = c \omega^n,
\]

where as before \( \alpha \) is a fixed Kähler metric on the \( n \)-dimensional Kähler manifold \( M \), and we are trying to solve for \( \omega \) in a fixed Kähler class. In analogy with the result of Song-Weinkove [15], it is shown in [9] that a necessary and sufficient condition for Equation (106) to have a solution is the following:

There is a metric \( \omega' \in [\omega] \) such that

\[
c \omega'^{n-1} - \binom{n-1}{k} \omega'^{n-k-1} \wedge \alpha^k > 0,
\]

in the sense of positivity of \((n-1,n-1)\)-forms.

In analogy with Conjecture 1 one can make the following conjecture.

**Conjecture 18.** A solution of Equation (106) exists if and only if for all subvarieties \( V \subset M \) of dimension \( p \), for \( p = k, k+1, \ldots, n-1 \), we have

\[
\int_V c \omega^p - \binom{p}{k} \omega^{p-k} \wedge \alpha^k > 0.
\]

Most of the results of the paper have suitable parallels in this case, and we will briefly describe the modifications that need to be made.

For the Bergman kernel expansion in Section 2 we must use the inner product

\[
\langle s, t \rangle_{L^2} = k^n \int_M \langle s, t \rangle_h \frac{\alpha^k \wedge \omega^{n-k}}{k!(n-k)!}.
\]
To obtain the analogous result to Theorem 2 we define $F_\alpha(\chi)$ as in Definition 10, but instead of letting $D \subset M$ be a sufficiently general element of the linear series defined by $\alpha$, we must take $D_1 \cap \ldots \cap D_k \subset M$, for $k$ sufficiently general elements. The coefficient $b'_0$ is defined using the induced test-configuration for this intersection.

For deformation to the normal cone of a subvariety $V \subset M$, the discussion is entirely analogous to that in Section 3. Note that if $\dim V < k$, then $V$ will be disjoint from a generic intersection $D_1 \cap \ldots \cap D_k$. This is the reason why we only consider $V$ with $\dim V \geq k$ in Conjecture 18. When $k = n$, then there is no condition at all, and correspondingly in this case Equation (106) is just a prescribed volume form equation for $\omega$, which by Yau’s solution of the Calabi conjecture [25] always has a solution.

Analogous calculations can also be made to those in Section 4 following the work of Fang-Lai [8]. The extension of Theorem 17, however, is more subtle, since one does not automatically obtain $C^2$-bounds on the Legendre transform of the solution. We leave a further study of this equation to future work.

7. Appendix: smooth forms as averages of currents

The goal of this section is to prove the following.

**Theorem 19.** Suppose that $\alpha \in c_1(K)$, where $K$ is a very ample line bundle over $M$. Let $|K|$ denote the projective space of sections of $K$. There is a smooth signed measure $\mu$ on $|K|$ such that

\begin{equation}
\alpha = \int_{|K|} [D] \, d\mu(D),
\end{equation}

where $[D]$ denotes the current of integration along a divisor $D \in |K|$. The equality can be interpreted in the weak sense, i.e. for any smooth $(n-1,n-1)$-form $\beta$ we have

\begin{equation}
\int_M \alpha \wedge \beta = \int_{|K|} \left( \int_D \beta \right) \, d\mu(D).
\end{equation}

This result may be well-known to experts, but we have not found this statement in the literature.

**Proof.** We first prove the statement in the special case when $M = \mathbb{P}^n$ and $K = \mathcal{O}(1)$. Let us denote by $\mathbb{P}^{n*}$ the dual projective space, and let $\mu_{FS}$ be the Fubini-Study volume form. Let $F$ be a smooth function on $\mathbb{P}^{n*}$ with integral 1, and define

\begin{equation}
\alpha_F = \int_{\mathbb{P}^{n*}} [D] F(D) \, d\mu_{FS}(D).
\end{equation}

Then $\alpha_F \in c_1(\mathcal{O}(1))$, and we can compute $\alpha_F$ if we know its trace $\Lambda_{\omega_{FS}} \alpha_F$ relative to the Fubini-Study metric. Indeed if

\begin{equation}
\alpha_F = \omega_{FS} + \sqrt{-1} \partial \bar{\partial} \varphi,
\end{equation}

then

\begin{equation}
\Lambda_{\omega_{FS}} \alpha_F = n + \Delta_{\omega_{FS}} \varphi,
\end{equation}

so $\varphi$ can be recovered from the trace by solving the Poisson equation. We claim that at a point $p \in \mathbb{P}^n$ we have

\begin{equation}
\Lambda_{\omega_{FS}} \alpha_F(p) = \int_{H_p} F(D) \, d\mu_{FS}(D),
\end{equation}

where $H_p$ is the hyperplane at $p$. This is an application of the Poisson equation to the Fubini-Study volume form.
where \( H_p \subset \mathbb{P}^n \) denotes the hyperplanes passing through \( p \), and \( \mu_{FS} \) is the natural Fubini-Study measure on \( H_p \). This equality follows from symmetry considerations, and the fact that \( \alpha_F \) at the point \( p \) depends only on the values of \( F \) on \( H_p \).

In order to show that any \( \alpha \in c_1(O(1)) \) is of the form \( \alpha_F \), we simply need to show that any smooth function \( G \) on \( \mathbb{P}^n \) can be written in the form

\[
G(p) = \int_{H_p} F(D) \, d\mu_{FS}(D),
\]

for some smooth \( F \) on \( \mathbb{P}^n \). This transformation from \( F \) to \( G \) is a generalized Radon transform, and Theorem 4.1 in Helgason [10] says that it is a one-to-one mapping. Our result therefore follows for the special case when \( M = \mathbb{P}^n \).

In the general case, use sections of \( K \) to embed \( \iota : M \rightarrow \mathbb{P}^N \), such that \( K = \iota^*O(1) \). We have

\[
\alpha = \iota^*\omega_{FS} + \sqrt{-1} \partial \bar{\partial} \varphi,
\]

for some smooth \( \varphi \), so if \( \tilde{\varphi} : \mathbb{P}^N \rightarrow \mathbb{R} \) denotes a smooth function such that \( \iota^*\tilde{\varphi} = \varphi \), then we have

\[
\alpha = \iota^*\left(\omega_{FS} + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}\right) = \iota^*\tilde{\alpha},
\]

for a suitable \( \tilde{\alpha} \in c_1(O(1)) \). By the special case of our result, there is a smooth function \( F : \mathbb{P}^N \rightarrow \mathbb{R} \) such that

\[
\tilde{\alpha} = \int_{\mathbb{P}^N} [D]F(D) \, d\mu_{FS}(D).
\]

Restricting to \( \iota(M) \subset \mathbb{P}^N \), the result follows.

\[\square\]

References

[1] B Bian and P. Guan, A microscopic convexity principle for nonlinear partial differential equations, Invent. Math. 177 (2009), 307–335.
[2] X. X. Chen, A new parabolic flow in Kähler manifolds, Comm. Anal. Geom. 12 (2004), no. 4, 837–852.
[3] J.-P. Demailly and M. Paun, Numerical characterization of the Kähler cone of a compact Kähler manifold, Ann. of Math. (2) 159 (2004), no. 3, 1247–1274.
[4] S. K. Donaldson, Moment maps and diffeomorphisms, Asian J. Math. 3 (1999), no. 1, 1–16.
[5] ———, Scalar curvature and stability of toric varieties, J. Differential Geom. 62 (2002), 289–349.
[6] ———, Lower bounds on the Calabi functional, J. Differential Geom. 70 (2005), no. 3, 453–472.
[7] ———, Kähler metrics with cone singularities along a divisor, Essays in mathematics and its applications, Springer, 2012, pp. 49–79.
[8] H. Fang and M. Lai, Convergence of general inverse \( \sigma_k \)-flow on Kähler manifolds with Calabi ansatz, arXiv:1203.5253.
[9] H. Fang, M. Lai, and X. Ma, On a class of fully nonlinear flows in Kähler geometry, J. Reine Angew. Math. 653 (2011), 189–220.
[10] S. Helgason, A duality in integral geometry; some generalizations of the Radon transform, Bull. Amer. Math. Soc. 70 (1964), 435–446.
[11] N. Korevaar and J. L. Lewis, Convex solutions of certain elliptic equations have constant rank Hessians, Arch. Rational Mech. Anal. 97 (1987), no. 1, 19–32.
[12] Z. Lu, On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch, Amer. J. Math. 122 (1998), no. 2, 235–273.
[13] X. Ma and G. Marinescu, Holomorphic Morse inequalities and Bergman kernels, Progress in Mathematics, vol. 254, Birkhäuser Verlag, Basel, 2007.
[14] J. Ross and R. P. Thomas, An obstruction to the existence of constant scalar curvature Kähler metrics, J. Differential Geom. 72 (2006), 429–466.
[15] J. Song and B. Weinkove, On the convergence and singularities of the J-flow with applications to the Mabuchi energy, Comm. Pure Appl. Math. 61 (2008), no. 2, 210–229.
[16] J. Stoppa, K-stability of constant scalar curvature Kähler manifolds, Adv. Math. 221 (2009), no. 4, 1397–1408.
[17] G. Székelyhidi, Filtrations and test-configurations, arXiv:1111.4986.
[18] , A remark on conical Kähler-Einstein metrics, arXiv:1211.2725.
[19] , The Calabi functional on a ruled surface, Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 5, 837–856.
[20] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, J. Differential Geom. 32 (1990), no. 1, 99–130.
[21] , Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 137 (1997), 1–37.
[22] B. Weinkove, Convergence of the J-flow on Kähler surfaces, Comm. Anal. Geom. 12 (2004), no. 4, 949–965.
[23] , On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy, J. Differential Geom. 73 (2006), no. 2, 351–358.
[24] D. Witt Nyström, Test configurations and Okounkov bodies, Compos. Math. 148 (2012), no. 6, 1736–1756.
[25] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I., Comm. Pure Appl. Math. 31 (1978), 339–411.
[26] S. Zelditch, Szegő kernel and a theorem of Tian, Int. Math. Res. Notices 6 (1998), 317–331.

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