Fractional Porous Medium Equations to Non-Darcian Flow by Means of the Bernstein Polynomials

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Abstract. The Bernstein polynomials is used to develop the fractional differential Swartzendruber model in porous media. With the operator matrix of the Bernstein polynomials, the fractional differential Swartzendruber equation is transformed into the products of several dependent matrices which can also be regarded as the system of linear equations after dispersing the variable. By solving the system of linear equations, the numerical solutions are acquired. Some experimental data are provided to show that the method is computationally efficient and accurate.

1. Introduction
Fractional differential equations are generalized from integer order ones, which are obtained by replacing integer order derivatives with fractional ones. In the last few decades fractional calculus and fractional differential equations have found applications in several differential disciplines such as applications of fractional calculus to time-dependent behavior of rocks[1][2] and composites [3].Moreover, a large class of dynamical systems appearing throughout the field of engineering and applied mathematics was described by differential equations of fractional order[4][5].In past decades, extensive efforts have been devoted to modeling approaches of nonlinear relation corresponding between water flux and hydraulic gradient, called non-Darcian flow[6][6].Generally, non-Darcian flow can be described by nonlinear functions of water flux and hydraulic gradient such as exponential and power functions. Hansbo[7][9] proposed a power relationship between water flux and hydraulic gradient for non-Darcian flow in clay media. By analyzing data sets for water flow in clay soils, Swartzendruber[10] proposed an exponential function to validate Darcy’s law, resulting in a nonlinear relation of water flux versus gradient. In[10][12] the Swartzendruber equation as a non-Darcian flow model is generalized to describe the relation between water flux and hydraulic gradient using fractional derivative, resulting in a new model called the fractional derivative Swartzendruber equation flow model, the analytic solution of which is presented. It proves that fractional derivative modeling approach is acceptable for non-Darcian flow in porous media. Motivated by the previous studies, this paper makes an attempt to characterize the non-Darcian flow in low-permeability porous media by using good properties of Bernstein polynomials. By solving fractional differential operators of Bernstein polynomials, convert the fractional derivative Swartzendruber equation into a series of
correlation matrix products. Then numerical solutions can be obtained by discrete variables, and high precision can be achieved at the same time.

2. Definition of the Caputo derivative

**Definition 2.1** Caputo derivative[13]is widely used in physics and mechanics because of its advantages in solving fractional differential equations with initial conditions. For a given function \( f(x) \) Caputo derivative is defined by

\[
D^\gamma f(x) = \frac{1}{\Gamma(1-\gamma)} \int_0^x (x-s)^{-\gamma} \times f'(s)ds, 0 < \gamma < 1
\]

And (1) has the following properties

\[
D^\gamma c = 0, c \text{ is constant}
\]

\[
D^\gamma x^\beta = \begin{cases} 
\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\gamma)} x^{\beta-\gamma}, & \beta = 1,2,3, \ldots
\end{cases}
\]

3. Properties of Bernstein polynomials

3.1. Bernstein-polynomial

**Definition 3.1** The Bernstein polynomials of n-th degree are defined on the interval \([0, 1]\) as [13]

\[
B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, x \in [0,1]
\]

here \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \). By using the binomial expansion of \((1-x)^{n-i}\), it can be written as

\[
B_{i,n}(x) = \sum_{k=0}^{n-i} (-1)^i \binom{n}{i} \binom{n-i}{k} x^i (1-x)^{n-i-k}, x \in [0,1]
\]

here \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \) \( \binom{n-i}{k} = \frac{(n-i)!}{k!(n-i-k)!} \). Also, the Bernstein basis polynomials of degree \( n \) in \([a, b]\) are given by the formula[18]

\[
B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, x \in [a, b]
\]

here \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \).

3.2. Expansion of B-polynomial

**Definition 3.2** The Bernstein basis polynomials of degree \( n \) in \([0, b]\) are given by the formula

\[
B_{i,n}(x) = \binom{n}{i} \frac{x^i (b-x)^{n-i}}{b^n}
\]

By using the binomial expansion of \((b-x)^{n-i}\), we have the formula

\[
B_{i,n}(x) = \left( \begin{array}{c} n \\ i \\ \end{array} \right) \frac{x^i}{b^n} \left( 1-x \right)^{n-i} = \sum_{k=0}^{n-i} (-1)^i \binom{n}{i} \binom{n-i}{k} \left( \begin{array}{c} x \\ b \\ \end{array} \right)^{i+k}
\]

The Bernstein basis polynomials given by (5) can be written in the matrix form[11]

\[
\Phi(x) = \begin{bmatrix} B_{0,n}(x) & B_{1,n}(x) & \cdots & B_{n,n}(x) \end{bmatrix}^T = AT_m(x)
\]

where

\[
A = \begin{bmatrix}
(-1)^0 \binom{n}{0} & (-1)^1 \binom{n}{0} & \cdots & (-1)^i \binom{n}{i} & \cdots & (-1)^0 \binom{n}{n} \\
0 & (-1)^0 \binom{n}{1} & \cdots & (-1)^i \binom{n}{i} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (-1)^0 \binom{n}{n} \\
(\frac{x}{b})^0 & (\frac{x}{b})^1 & \cdots & (\frac{x}{b})^n
\end{bmatrix}
\]

\[
T_n(x) = \left[ \begin{array}{c} \frac{x}{b} \\ \left( \frac{x}{b} \right)^2 & \cdots & \left( \frac{x}{b} \right)^n \end{array} \right]^T
\]

here, matrix A is an upper triangular matrix and \( |A| = \prod_{i=0}^{n} \binom{n}{i} \), so A is an invertible matrix.
4. Function approximation  

Function approximation \( f(x) \) is a square integral function defined on the interval \([0,b]\) which can be expanded by Bernstein polynomials. Usually consider the first \( n+1 \) item,

\[
f(x) \approx \sum_{j=0}^{n} c_j B_{j,n}(x) = c^T \phi(x) \tag{12}
\]

where the coefficient \( c = [c_0, c_1, \ldots, c_n]^T \) and it can be determined by inner product as follows

\[
c = Q^{-1}(f, \phi(x)) \tag{13}
\]

where \( Q \) is \((n+1) \times (n+1)\) order matrix and called the inner product matrix of \( \phi(x) \),

\[
Q = \int_0^b \phi(x) \phi^T(x) \, dx = \int_0^b (A T_n(x))(A T_n(x))^T \, dx = A \left( \int_0^1 T_n(x)T_n(x)^T \, dx \right) A^T = AHAT \tag{14}
\]

where \( H \) is a Hibert matrix:

\[
H = \begin{bmatrix}
1 & \frac{1}{2} & \ldots & \frac{1}{n+1} \\
\frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
n+1 & n+2 & \ldots & 2n+1
\end{bmatrix}
\]

5. Numerical algorithm  

Swartzendruber[10] proposed an exponential relation between water flux and hydraulic gradient to modify Darcy’s law,

\[
\begin{cases}
\frac{dq}{dt} = k \left(1 - \exp\left(-\frac{1}{i}\right)\right), \\
q(0) = 0.
\end{cases}
\tag{15}
\]

where \( I \) is the threshold gradient and actually refers to the intersection of the linear part in plot of the hydraulic gradient and the water flux. H.W.Zhou and S.Yang [15] replace integer derivative with fractional derivative and have the fractional derivative Swartzendruber equation, i.e.,

\[
\frac{d^{\gamma} q}{dt^{\gamma}} = k \left(1 - \exp\left(-\frac{1}{i}\right)\right), 0 \leq \gamma \leq 1. \tag{16}
\]

Application of the Laplace transform and inverse Laplace transform to (16) and get the analytic solution

\[
q = k \frac{i^{\gamma+1}}{\Gamma(\gamma+2)} E_{1,\gamma+2}\left(\frac{-i}{\gamma}\right) \tag{17}
\]

where \( E_{1,\gamma+2}(\cdot) \) refers to Mittag-Leffler function, i.e., \( E_{a,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(ak+\beta)} \).

Then \( q(i) \) is approximated by Bernstein polynomial as shown in equation (17) shown. By using the Caputo definition of variable order differentiation, we can get

\[
D^{\gamma} q(i) = D^{\gamma} c^T \phi(i) = c^T D^{\gamma} \phi(i) = c^T D^{\gamma} A T_n(i) = c^T A D^{\gamma} = c^T A \begin{bmatrix}
\frac{1}{i} \\
\frac{1}{b} \\
\vdots \\
\frac{1}{n}
\end{bmatrix} = c^T A \begin{bmatrix}
\frac{\Gamma(2)}{\Gamma(2-\gamma)} i^{-\gamma} \\
\frac{\Gamma(3)}{\Gamma(3-\gamma)} i^{-\gamma} \\
\vdots \\
\frac{\Gamma(n+1)}{\Gamma(n+1-\gamma)} i^{-\gamma}
\end{bmatrix} = c^T A M A^{-1} \phi(i) \tag{18}
\]

where

\[
M = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & \frac{1}{\Gamma(2)} i^{-\gamma} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{\Gamma(n+1)} i^{-\gamma}
\end{bmatrix}
\]
\[ M = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & \frac{r(2)}{r(\gamma)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{r(n+1)}{r(n+1-\gamma)}
\end{bmatrix} \]

where \( M \) is the \( \gamma \) differential operator matrix of Bernstein polynomial.

Substituting equations (18) into equation (17), we can get

\[ c^T A M A^{-1} \phi(i) = k(1 - \exp(-i/I)). \tag{19} \]

The coefficient \( c \) can be obtained from the discrete variable \( i \), then the approximate solution can be obtained

\[ q(i) = c^T \phi(i) \tag{20} \]

6. Error Analysis

Suppose that \( f(x) \) is continuous derivative function of \( n+1 \) order on the interval \([0, b] \) and \( c^T \phi(x) \) is the best approximation to \( f \) which is in linear space

\[ Y = \text{span}\{B_{0,n}(x), B_{1,n}(x), \ldots, B_{n,n}(x)\}. \]

then the maximum error is estimated as follows:

\[ \|f - c^T \phi(x)\|_2 \leq \sqrt{\frac{2m+2}{N}} \tag{21} \]

Here \( N = \max\{b - x_0, x_0\}; M = \max|f^{(m+1)}(x)|, x \in [0, b] \).

Proof. Now consider the Taylor expansion of \( f(x) \), we have

\[ f_1(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \cdots + f^{(m)}(x_0)\frac{(x-x_0)^m}{m!} \tag{22} \]

According to the mean value theorem, we can get

\[ |f(x) - f_1(x)| \leq \left| f^{(m+1)}(\xi) \right| \frac{(x-x_0)^{m+1}}{(m+1)!}, \exists \xi \in [0, b] \tag{23} \]

Since \( c^T \phi(x) \) is the best approximation to \( f \), so we have

\[ \|f - c^T \phi(x)\|_2^2 \leq \|f - f_1\|_2^2 = \int_0^b (f(x) - f_1(x))^2 dx \\ = \int_0^b \left( f^{(m+1)}(\xi) \right) \frac{(x-x_0)^{m+1}}{(m+1)!} dx \leq M^2 \frac{N^{2m+2}}{(m+1)!} \tag{24} \]

Here \( N = \max\{b - x_0, x_0\}; M = \max|f^{(m+1)}(x)|, x \in [0, b] \). By taking the square root of the above equation, the maximum error estimate can be obtained.

7. Illustrative Example

H.W.Zhou [15] indicated that the fractional derivative flow model (16) is better agreement with the experimental data [17] than the Swartzendruber equation in (15) with lower mean squared errors. Here we consider solving the fractional derivative Swartzendruber equation by Bernstein polynomials, which was performed numerical computations by a computer program written.

First, take \( n = 2 \) (the highest degree of Bernstein's formula is 2) and combine with the fractional derivative flow model and the data[17], the numerical solution of (16) with different rock block percentages 20%-70% was

\[ q_{12} = (0.04023i^2 + 2.151i - 42.17) \times 10^{-8}, \]
\[ q_{22} = (0.2982i^2 + 14.28i - 263.5) \times 10^{-8}, \]
\[ q_{32} = (0.1214i^2 + 13.17i - 208.3) \times 10^{-8}, \]
\[ q_{42} = (0.2951i^2 + 20.2i - 218.8) \times 10^{-8}, \]
\[ q_{32} = (0.362i^2 + 25.37i - 181.6) \times 10^{-8}. \]

Then take \( n = 3 \) (the highest degree of Bernstein's formula is 3) and the numerical solution of (16) with different rock block percentages 20%-70% was

\[ q_{31} = (-9.371 \times 10^{-5} i^3 + 0.06403i^2 + 0.3764i - 5.721) \times 10^{-8}, \]
\[ q_{32} = (-7.574 \times 10^{-4} i^3 + 0.4712i^2 + 2.418i - 34.52) \times 10^{-8}, \]
\[ q_{33} = (4.929 \times 10^{-1} i^3 - 0.234i^2 + 5.447i - 59.22) \times 10^{-8}, \]
\[ q_{43} = (-1.841 \times 10^{-3} i^3 + 0.5763i^2 + 7.328i - 49.87) \times 10^{-8}, \]
\[ q_{53} = (-3.283 \times 10^{-3} i^3 + 0.7273i^2 + 0.132i - 0.6327) \times 10^{-8}. \]

It can be seen from Table 1 that the numerical solution and exact solution of fractional derivative flow model are highly consistent. From the analysis of numerical examples, the low-order Bernstein polynomial can achieve very high accuracy, indicating that this method is very effective in solving the numerical solutions of variable-order fractional differential equations. The results demonstrate that the proposed numerical solution is also in better agreement with the experimental data.

| SRM specimens | \( n=2 \) | \( n=3 \) | Fractional derivative flow model |
|---------------|---------|---------|----------------------------------|
|                | R²      | MSE     | R²      | MSE     | R²      | MSE     |
| SRM20-1        | 0.9988  | 1.3508e-15 | 0.9999  | 0.2606e-17 | 0.9817 | 0.2242 |
| SRM30-1        | 0.9963  | 0.2794e-13 | 0.9999  | 0.881e-17 | 0.9989 | 0.0531 |
| SRM40-1        | 0.9976  | 1.1824e-14 | 0.9999  | 1.0352e-17 | 0.9908 | 0.0095 |
| SRM60-1        | 0.9979  | 1.0508e-14 | 0.9999  | 0.4224e-16 | 0.9819 | 0.0307 |
| SRM70-1        | 0.9987  | 0.7778e-14 | 0.9999  | 1.3815e-16 | 0.9807 | 0.0193 |

8. Conclusion
The object of this paper is to develop fractional order equations for describing non-Darcian flow behavior between water flux and hydraulic gradient with a class of Bernstein polynomials. By solving fractional differential operator moment of Bernstein polynomial, convert original equation into a series of correlation matrix products, then get a numerical solution by discrete variables. This method provides an effective tool for solving the fractional derivative Swartzendruber equation. Numerical solution provided to show that the method is computationally efficient and accurate.

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