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$N=2$ AND $N=4$ SUBALGEBRAS OF SUPER VERTEX OPERATOR ALGEBRAS

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ABSTRACT. We develop criteria to decide if an $N=2$ or $N=4$ superconformal algebra is a subalgebra of a super vertex operator algebra in general, and of a super lattice theory in particular. We give some specific examples.

1. Introduction

The advent of Mathieu Moonshine [EOT] in recent years has brought renewed interest in $N=4$ superconformal algebras. Mathieu Moonshine concerns an unexpected connection between the largest Mathieu group $M_{24}$ and the complex elliptic genus of a $K3$ surface. The connection between the $K3$ elliptic genus and $N=4$ superconformal algebras has been known for some time in string theory [EOTY, KYY]. The new observation of Mathieu Moonshine [EOT] is that when the $K3$ elliptic genus is decomposed into characters of the $N=4$ central charge $c=6$ superconformal algebra [ET1, ET2], the coefficients are simply related to $M_{24}$ irreducible module dimensions. This conjecture is now supported by a wealth of studies e.g. [EH1, C, GHV, G, EH2]. Recently, Song has also shown [S] that the $c=6$, $N=4$ superconformal algebra is the algebra of global sections of the chiral de Rham complex on a Kummer surface (following earlier results for hyperkähler manifolds [BZHS]), further clarifying the connection between $K3$ surfaces and $N=4$ superconformal algebras. However, despite these developments, an explicit construction of a vertex operator super algebra involving a $c=6$, $N=4$ superconformal subalgebra that ‘explains’ Mathieu Moonshine is still unknown.

All of this means that one can expect $N=4$ superconformal algebras to play a ubiquitous rôle in the further investigation of these subjects, much as the Virasoro algebra does in general CFT. Now the precise definition of the $N=4$ superconformal algebra is awkward, to say the least. For $c=6$, it is usually described as a subtheory of the algebra of 12 free fermions (or 4 free fermions and 4 free bosons) e.g. [ET1, KW]. But in a CFT that contains no free fermions, such as a super lattice theory for an odd lattice with minimum norm 2, one needs some other device to decide if there is an $N=4$ subalgebra present. Such an odd lattice construction is exploited in further work concerning $N=4$ superconformal algebras, odd Niemeier lattices and elliptic genera including the $K3$ elliptic genus [MTY].

The purpose of the present paper is to describe two general recognition theorems which allow one to readily identify the existence of an $N=4$ superconformal algebra as a subalgebra of a suitable Super Vertex Operator Algebra (SVOA) with just a few well-chosen axioms. Some effort is required to obtain efficient characterizations, i.e., without too many assumptions. It would be of interest if a genuine reduction in

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the number of axioms needed can be achieved in our recognition theorems. Our main results are as follows (unexplained notation is clarified below).

**Theorem 1.** Let $U$ be a SVOA of CFT-type. Let $V \subseteq U$ be the subalgebra generated by 4 primary vectors of weight $\frac{3}{2}$ in $U$, so that

$$V = \mathbb{C}1 \oplus V_{\frac{3}{2}} \oplus V_{1} \oplus V_{\frac{3}{2}} \oplus \ldots$$

is a conformally graded subspace of $U$. Assume that the following hold:

(I) The subspace of $V_{\frac{3}{2}}$ spanned by the four generators decomposes as $A \oplus B$, a pair of 2-dimensional vector representations for $\mathfrak{sl}_2$, where

(II) $A(1)B \cong \mathfrak{sl}_2$,

(III) $A(0)A = B(0)B = 0$,

(IV) $\mathfrak{sl}_2 \cap A(0)B \neq 0$.

Then $V$ is an $N=4$ superconformal algebra with central charge $c = 6k$, where $k \in \mathbb{C}$ is the level of the $\mathfrak{sl}_2$ Kac-Moody subalgebra generated by $A(1)B$.

**Theorem 2.** Let $L$ be a positive-definite, odd, integral lattice of minimum norm 2 with $V_L$ the corresponding SVOA. Let $V \subseteq V_L$ be the subalgebra generated by 4 vectors of weight $\frac{3}{2}$, so that

$$V = \mathbb{C}1 \oplus V_{1} \oplus V_{\frac{3}{2}} \oplus \ldots,$$

and assume that the following hold:

(I) The subspace of $V_{\frac{3}{2}}$ spanned by the four generators decomposes as $A \oplus B$, a pair of 2-dimensional vector representations for $\mathfrak{sl}_2$, where

(II) $A(1)B \cong \mathfrak{sl}_2$,

(III) $A(1)A = B(1)B = 0$,

(IV) $\mathfrak{sl}_2$ contains a root of $L$.

Then $V$ is an $N = 4$ superconformal algebra with central charge $c = 6$.

We note that there are many articles in the literature, e.g., [K], [dS], that discuss constructions of $N=4$ vertex algebras from given generating sets. However, Theorems 1 and 2 in this article rather describe the existence of an $N=4$ subalgebra of a given SVOA, so that, for example the central charge of the $N=4$ subalgebra may differ from that of $U$ or $V_L$ respectively. Furthermore, we do not discuss the simplicity of the $N=4$ subalgebra. Our results appear to shed no light on this interesting question. For results in this direction, see [GK].

The paper is organized as follows. After reviewing some background about $N=4$ algebras in Section 2, we give the proof of Theorems 1 and 2 in Sections 3 and 4 respectively. We also briefly describe the more elementary analogous case of the $N=2$ superconformal subalgebra. This is contained in Section 5. In Section 6 we illustrate how the main Theorems may be applied to some known examples. In particular, we give (Proposition 26) a painless new construction of the $N=4$ algebra in a certain rank 6 lattice theory $V_L$ containing no free fermions. This provides an effective way to decide the existence of the $N=4$ algebra in a general super lattice theory by reducing the problem to one about lattice embeddings. We include some appendices in Section 7 containing technical background in SVOA theory that we assume and use throughout the paper.
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2. \( N=4 \) Superconformal Algebras

The abstract generators and relations for the \( N=4 \) superconformal algebra \( V \) of central charge \( c \) are as follows. It is generated by 4 states \( G^\pm, \bar{G}^\pm \) of conformal weight \( \frac{3}{2} \). The nontrivial relations can be expressed as follows (e.g. [ET1, K]):

(a) \( J^0(0)J^\pm = \pm 2J^\pm \), \quad (b) \( J^0(1)J^0 = \frac{c}{3} \), \quad (c) \( J^+(0)J^- = J^0 \),

(d) \( J^+(1)J^- = -\frac{c}{6} \), \quad (e) \( J^0(0)G^\pm = \pm G^\pm \), \quad (f) \( J^0(0)\bar{G}^\pm = \pm \bar{G}^\pm \),

(g) \( J^+(0)\bar{G}^\pm = G^\pm \), \quad (h) \( J^-(0)\bar{G}^\pm = -\bar{G}^\pm \), \quad (i) \( G^+(1)\bar{G}^\pm = 2J^\pm \),

(j) \( G^+(1)\bar{G}^\pm = \pm J^0 \), \quad (k) \( G^+(2)\bar{G}^\pm = \frac{c}{3} \), \quad (l) \( G^+(0)\bar{G}^\pm = TJ^\pm \),

(m) \( G^+(0)\bar{G}^\pm = \omega \pm \frac{1}{2} T J^0 \).

Here, \( \omega \) is a Virasoro element of central charge \( c \), \( T \) is the translation operator and \( J^\pm, J^0 \) are weight one vectors. Relations (a)–(m) hold in addition to the usual Virasoro relations between \( J^\pm, J^0 \) and \( \omega \). The initial segment of the Fock space of \( V \) is

\[ V = \mathbb{C}1 \oplus V_1 \oplus V_{\frac{3}{2}} \oplus V_2 \oplus \ldots \]

where

\[ V_1 = \langle J^\pm, J^0 \rangle \cong \mathfrak{sl}_2 \]
\[ V_{\frac{3}{2}} = \langle G^\pm \rangle \oplus \langle \bar{G}^\pm \rangle \], a pair of vector representations for \( \mathfrak{sl}_2 \)
\[ V_2 = T \mathfrak{sl}_2 \oplus \mathbb{C} \omega . \]

For arbitrary subsets \( X, Y \subseteq U \) of a SVOA \( U \) and any integer \( n \), we have already used, and will continue to use, the convenient notation \( X(n)Y \) for the linear subspace of \( U \) spanned by all products \( u(n)v \) (\( u \in X, v \in Y \)). For the \( N=4 \) algebra \( V \) we set

\[ A := \langle G^\pm \rangle, \quad B := \langle \bar{G}^\pm \rangle . \]

Then it follows from the relations (a)–(m) that

\[ A(n)A = B(n)B = 0 \quad (n \geq 0), \quad A(0)B = V_2 , \quad A(1)B = \mathfrak{sl}_2, \quad A(2)B = \mathbb{C}c1 . \]

3. Proof of Theorem 1

Let \( \omega^U \) be the Virasoro element of the given SVOA \( U \), with vertex operator

\[ Y(\omega^U, z) = \sum_{n \in \mathbb{Z}} L^U(n) z^{-n-2} . \]
3.1. **Assumptions (I) and (II).** In the interests of keeping track of just which of the axioms (I)–(IV) in the statement of Theorem 1 are needed when, we begin by recording some consequences of axioms (I) and (II) alone. We set

(1) \( \mathfrak{sl}_2 := \langle h, x^\pm \rangle \subseteq V_1 \)

(2) \( A := \langle \tau^+, \tau^- \rangle, \quad B := \langle \tau^+, \tau^- \rangle. \)

Here,

(3) \( h(0)x^\pm = \pm 2x^\pm, \quad x^+(0)x^- = h \)

are standard generators and relations for \( \mathfrak{sl}_2 \), and \( \tau^\pm, \tau^\pm \) are weight vectors for \( h(0) \) with weights \( \pm 1 \), with

(4) \( x^\pm(0)\tau^\pm = \tau^\pm, \quad x^\pm(0)\tau^\mp = \mp \tau^\pm, \)

(5) \( x^\pm(0)\tau^\pm = x^\pm(0)\tau^\mp = 0. \)

All of this is just a choice of notation based on the hypotheses of the Theorem 1. It amounts to the existence of an isomorphism of \( \mathfrak{sl}_2 \)-modules

\[ \varphi : A \rightarrow B, \quad \tau^\pm \mapsto \tau^\pm. \]

Note that some latitude in scaling the generators of \( A \) and \( B \) remains - a fact that we make use of later. In any case, there is a canonical \( \mathfrak{sl}_2 \)-invariant decomposition into trivial and adjoint modules

\[ A \otimes B = \Lambda \oplus \Sigma, \]

where

(6) \( \Lambda = \mathbb{C}(\tau^+ \otimes \tau^- - \tau^- \otimes \tau^+), \)

(7) \( \Sigma = \langle \tau^+ \otimes \tau^+, \tau^- \otimes \tau^-, \tau^+ \otimes \tau^- + \tau^- \otimes \tau^+ \rangle. \)

Of course, there is a similar decomposition of \( A \otimes A = \Lambda^2(A) \oplus S^2(A) \).

Note that \( A(1)B \) and \( A \oplus B \) consist of primary states. This is clear for \( A \oplus B \), while for \( A(1)B \) we note that \( L^U(1) : V_1 \rightarrow \mathbb{C} 1 \) is a morphism of \( \mathfrak{sl}_2 \)-modules. Restriction to \( A(1)B = \mathfrak{sl}_2 \) must therefore be trivial, and the assertion follows. Thus we have

(8) \[ [L^U(m), u(n)] = -nu(m+n), \]

(9) \[ [L^U(m), v(n)] = \left( \frac{1}{2} (m+1) - n \right) v(m+n), \]

for all for \( u \in A(1)B \) and \( v \in A \oplus B \). In particular, (9) implies

**Lemma 3.** \( A(n+1)A = L^U(1)A(n)A \) for all \( n \neq 1 \) and \( A(n+2)A = L^U(2)A(n)A \) for all \( n \). Similar relations hold for \( B(n+1)B, B(n+2)B, A(n+1)B \) and \( A(n+2)B \). \( \square \)

For \( u, v \in A(1)B = \mathfrak{sl}_2 \) we have from commutativity that

(10) \[ [u(m), v(n)] = (u(0)v)(m+n) + m \langle u, v \rangle \delta_{m+n, 0}, \]

with \( \langle u, v \rangle \) given by \( u(1)v = \langle u, v \rangle 1 \). Define \( k \in \mathbb{C} \) by

(11) \[ \langle h, h \rangle = 2k. \]

**Lemma 4.** \( V \) is a module for the Kac-Moody algebra \( \hat{\mathfrak{sl}}_2 \) of level \( k \).
We set also holds for $k \in (13)$ by commutativity. Hence, for $k \neq 0$, (12) also holds for $k = 0$. Hence, since $u(0)v = -v(0)u$ defines a commutator bracket on $sl_2$, (10) implies the desired result. \qed

Lemma 5. Properties (a)–(h) above hold for $c = 6k$.

Proof. We set
\begin{equation}
J^0 := h, \quad J^\pm := x^\pm, \quad G^\pm := \tau^\pm, \quad \text{and} \quad G^\pm := \mp \tau^\pm.
\end{equation}
By (3)–(5), (11)–(12), we have
\begin{align*}
J^0(0)J^\pm &= \pm 2J^\pm, \quad J^0(1)J^0 = 2k1, \\
J^+(0)J^- &= J^0, \quad J^+(1)J^- = k1, \\
J^0(0)G^\pm &= \pm G^\pm, \quad J^0(0)G^\pm = \pm G^\pm, \\
J^+(0)G^\pm &= G^\pm, \quad J^+(0)G^\pm = -G^\pm.
\end{align*}
These are the needed relations. \qed

Remark 6. In addition, we have $J^\pm(0)G^\pm = J^\pm(0)G^\pm = 0$.

Next, we will obtain properties (i) and (j):

Lemma 7. We can choose the normalization for $\tau^\pm, \tau^\pm$ so that
\begin{equation}
h = \tau^+(1)\tau^-, \quad \tau^+(1)\tau^- = \mp 2x^\pm.
\end{equation}
In particular, we have $G^+(1)G^\mp = 2J^\pm$, and $G^+(1)G^\mp = \pm J^0$.

Proof. Thanks to the $sl_2$-morphism $u \otimes v \mapsto u(1)v$ from $A \otimes B$ on to $A(1)B$, we get generators of $A(1)B$ just by replacing $u \otimes v$ by $u(1)v$ in the generators for $\Sigma$ in (7). Thus
\begin{equation}
A(1)B = sl_2 = \langle \tau^+(1)\tau^+, \quad \tau^-(1)\tau^-, \quad \tau^+(1)\tau^- + \tau^-(1)\tau^+ \rangle.
\end{equation}
Similarly, the generator for $\Lambda$ in (6) maps to 0, leading to $\tau^+(1)\tau^- - \tau^-(1)\tau^+ = 0$.

$\tau^+(1)\tau^- + \tau^-(1)\tau^+ = 2\tau^+(1)\tau^+$ is a nonzero semisimple element of $sl_2$. As such, it is a nonzero multiple of $h$. Since we are free to simultaneously scale the $\tau$’s by a nonzero constant, we may choose the scale so that $h = \tau^+(1)\tau^- = \tau^-(1)\tau^+$. Since $x^\pm(0)h = -h(0)x^\pm = \mp 2x^\pm$ we find, using supercommutativity, that
\begin{align*}
-2x^+ &= x^+(0) \left( \tau^+(1)\tau^- \right) \\
&= (x^+(0)\tau^+) \left( 1)\tau^- + \tau^+(1) (x^+(0)\tau^-) \right) \\
&= 0 + \tau^+(1)\tau^+.
\end{align*}
Similarly, $\tau^-(1)\tau^- = 2x^-$. \qed
3.2. Assumption (III). Lemma 3 implies
(16) \[ A(1)A = A(2)A = B(1)B = B(2)B = 0. \]
Thus super commutativity implies

**Lemma 8.** \([v(m), w(n)] = 0 \text{ for all } v, w \in A \text{ for all } v, w \in B.\]

**Lemma 9.** \(u(1)v = 0 \text{ for all } u \in \mathfrak{sl}_2 \text{ and } v \in A \oplus B.\)

**Proof.** The vector space \(\mathfrak{sl}_2(1)A\) is spanned by the 6 vectors of the form \(u(1)\tau^\pm\) for \(u = x^+\) or \(h\). Recall that \(\tau^+(1)\tau^+ = -2x^+\) from (15). Hence, by super associativity
\[
x^+(1)\tau^+ = -\frac{1}{2} \sum_{i \geq 0} (-1)^i \binom{1}{i} (\tau^+(1-i)\tau^+(1+i) - \tau^+(2-i)\tau^+(i)) \tau^+
\]
using (16), \(\tau^+(2)\tau^+ \in \mathbb{C}1\) and that \(\tau^+(1)\tau^+ = -\tau^+(1)\tau^+ = 2x^+\) by super skew symmetry. But, by super skew symmetry again we directly have \(x^+(1)\tau^+ = +\tau^+(1)x^+\). Thus \(x^+(1)\tau^+ = 0\). Furthermore, \((x^-(0))^k x^+(1)\tau^+ = 0\) for \(k = 1, 2, 3\) results in the identities
\[
h(1)\tau^+ - x^+(1)\tau^- = x^-(1)\tau^+ + h(1)\tau^- = x^-(1)\tau^- = 0.
\]
We can then repeat a similar argument based on super associativity and skewsymmetry using \(h = \tau^-(1)\tau^+\) to show that \(h(1)\tau^\pm = 0\). Thus \(\mathfrak{sl}_2(1)A = 0\). By an identical argument we also find \(\mathfrak{sl}_2(1)B = 0.\) \(\square\)

We next establish property (k):

**Lemma 10.** We have
\[
\tau^+(1)\tau^+ = \pm 2k1.
\]
Consequently, for \(c = 6k\) we have
\[
G^+(2)G^+ = 0, \text{ and } G^+(2)G^+ = \frac{c}{3}1.
\]

**Proof.** The image of \(\Sigma \to A(2)B = \mathbb{C}1\) is \(\mathfrak{sl}_2\)-invariant, hence is 0. Then
\[
\tau^+(2)\tau^+ = \tau^+(2)\tau^- + \tau^-(2)\tau^+ = 0.
\]
The invariant \(\tau^+(2)\tau^- - \tau^-(2)\tau^+\) is computed as follows: \(h(1)\tau^\pm = h(1)\tau^\pm = 0\) (Lemma 9) together with commutativity imply
(17) \([h(1), \tau^\pm(n)] = \pm \tau^\pm(n+1), \quad [h(1), \tau^\pm(n)] = \pm \tau^\pm(n+1).\]
Using (15) we thus find
\[
2k1 = h(1)h = h(1)\tau^+(1)\tau^+ = \pm \tau^+(2)\tau^+.
\]
\(\square\)

**Lemma 11.** Let
\[
\sigma := \frac{1}{2}(\tau^+(0)\tau^- - \tau^-(0)\tau^+).
\]
Then \(\sigma(0)u = Tu, \sigma(1)u = u\) and \(\sigma(2)u = 0\) for all \(u \in \mathfrak{sl}_2\). In particular, \(\sigma\) is a nonzero \(\mathfrak{sl}_2\)-invariant.
Proof. \( \sigma \) is an \( \mathfrak{sl}_2 \)-invariant in \( A(0)B \subseteq V_2 \) because it generates the image of \( \Lambda \) under the \( \mathfrak{sl}_2 \)-morphism defined by \( A \otimes B \rightarrow A(0)B \).

\( \sigma(2) \mathfrak{sl}_2 \in \mathbb{C}1 \) implies \( \sigma(2) \mathfrak{sl}_2 = 0 \) since \( \mathbb{C}1 \) is a trivial \( \mathfrak{sl}_2 \) representation. By skewsymmetry, \( \sigma(1)u = u(1)\sigma \) for all \( u \in \mathfrak{sl}_2 \). Hence, using (17) and Lemma 9, we find

\[
\sigma(1)h = \frac{1}{2} h(1) \left( \tau^+(0)\overline{\tau} - \tau^-(0)\overline{\tau} \right)
= \frac{1}{2} \left( \tau^+(1)\overline{\tau} + \tau^-(1)\overline{\tau} \right) = h,
\]

from (15). This proves, in particular, that \( \sigma \neq 0 \). Since \( \sigma \) is \( \mathfrak{sl}_2 \)-invariant and \( h \) generates \( A(1)B \) as an \( \mathfrak{sl}_2 \)-module, it follows that \( \sigma(1)u = u \) for all \( u \in \mathfrak{sl}_2 \). Furthermore, from skew-symmetry we find that

\[
\sigma(0)u = -u(0)\sigma + T(u(1)\sigma)
= 0 + T(\sigma(1)u) = Tu,
\]

using the \( \mathfrak{sl}_2 \)-invariance of \( \sigma \). \( \square \)

**Lemma 12.** \( \sigma(1)v = \frac{3}{2}v \) and \( \sigma(2)v = 0 \) for all \( v \in A \oplus B \).

**Proof.** Using superassociativity we find

\[
\sigma(1) = \frac{1}{2} [\tau^+(0), \tau^-(1)] - \frac{1}{2} [\tau^-(0), \tau^+(1)].
\]

Hence

\[
\sigma(1)\tau^+ = \frac{1}{2} \tau^+(0)\tau^-(1)\tau^+ - \frac{1}{2} \tau^-(0)\tau^+(1)\tau^+
= -\frac{1}{2} \tau^+(0)h - \frac{1}{2} \tau^-(0)(2x^+)
= \frac{1}{2} \tau^+ + x^+ = \frac{3}{2} \tau^+,
\]

using \( \tau^\pm(0)\tau^\pm = 0 \) (by assumption (III), (15) and super skew-symmetry). Using superassociativity, we similarly find

\[
\sigma(2) = \frac{1}{2} [\tau^+(0), \tau^-(2)] - \frac{1}{2} [\tau^-(0), \tau^+(2)].
\]

Hence

\[
\sigma(2)\tau^+ = \frac{1}{2} \tau^+(0)\tau^-(2)\tau^+ - \frac{1}{2} \tau^-(0)\tau^+(2)\tau^+ = 0.
\]

Similar results follow for \( \tau^- \) by \( \mathfrak{sl}_2 \)-symmetry and for \( \tau^\pm \). \( \square \)

3.3. Assumption (IV). We next obtain properties (l) and (m).

**Lemma 13.** We have

\[
\tau^\pm(0)\overline{\tau} = T x^\pm, \quad \tau^+(0)\overline{\tau} + \tau^-(0)\overline{\tau} = Th.
\]

Consequently,

\[
G^\pm(0)\overline{G}^\mp = T J^\pm, \quad \text{and} \quad G^\pm(0)\overline{G}^\mp = \sigma \pm \frac{1}{2} T J^0.
\]
Lemma 16. For all $A$, $\sigma(0) = [\tau^+(0), \tau^-(0)]$.

Thus we find using super skew-symmetry that

$$\sigma(0)\tau^+ = \tau^+(0)\tau^-(0)\tau^+ + 0$$
$$= \tau^+(0) \left( \tau^+(0)\tau^- - T(\tau^+(1)\tau^-) \right)$$
$$= 0 - \tau^+(0)Th$$
$$= (Th)(0)\tau^+ - T((Th)(1)\tau^+) + 0$$
$$= 0 + T\tau^+,$$

using $(Th)(0) = 0$ and $(Th)(1) = -h(0)$. A similar argument applies to the remaining elements of $A \oplus B$. \hfill \Box

Lemma 17. $\sigma$ is a Virasoro vector for central charge $c = 6k$. \hfill \Box
Proof. We have to check the relations
\[\sigma(0)\sigma = T\sigma, \quad \sigma(1)\sigma = 2\sigma, \quad \sigma(2)\sigma = 0, \quad \sigma(3)\sigma = 3k1.\]
Using (21) we find
\[\sigma(1)\sigma = \frac{1}{2} \left( \frac{1}{2} \tau^+(0)\tau^- + \tau^+(0) \frac{3}{2} \tau^- - \frac{1}{2} \tau^-(0)\tau^+ - \tau^-(0) \frac{3}{2} \tau^+ \right) = 2\sigma,\]
\[\sigma(2)\sigma = \frac{1}{2} \left( \frac{1}{2} \tau^+(0)\tau^- - \tau^+(0) \frac{3}{2} \tau^- \right) = 0,\]
\[\sigma(3)\sigma = \frac{1}{2} \left( \frac{3}{2} \tau^+(0)\tau^- - \frac{3}{2} \tau^-(0)\tau^+ \right) = 3k1.\]
Lastly, by skew symmetry
\[\sigma(0)\sigma = -\sigma(0)\sigma + T(\sigma(1)\sigma) = -\sigma(0)\sigma + 2T\sigma,\]
so that \(\sigma(0)\sigma = T\sigma.\)

Thus Theorem 1 holds since all the defining relations for the \(N=4, c=6k\) superconformal algebra are satisfied.

Remark 18. If \(U\) is \(C_2\)-cofinite and of strong CFT type then \(k\) is a positive integer by [DM] since \(sl_2 \subseteq U_1\). We also note that the \(N=4\) Virasoro vector \(\sigma\) (of Lemma 11) and \(\omega_U\) (the Virasoro element of \(U\)) can be independent vectors of different central charges. Thus Theorem 1 is not a generating theorem for \(N=4\) algebras (such as in [K] or [dS]) but rather describes the existence of an \(N=4\) subalgebra of a given SVOA.

Finally we note that the automorphism group of the \(N=4\) superconformal algebra contains an involution \(g\) defined by
\[g : A \oplus B \rightarrow B \oplus A\]
\[\left( \tau^\pm, \bar{\tau}^\pm \right) \mapsto \left( \bar{\tau}^\pm, -\tau^\pm \right).\]
This follows by directly verifying that the defining relations (a)–(m) are preserved by \(g\) by use of super skew–symmetry i.e. \(u(1)v = v(1)u\) and \(u(0)v - v(0)u = -T(u(1)v) \in T\mathfrak{sl}_2\) for all \(u \in A\) and \(v \in B\). Furthermore, Assumption (IV) of Theorem 1 therefore has the following reformulation

**Lemma 19.** \(T\mathfrak{sl}_2 \cap A(0)B \neq 0\) if and only if \(A(0)B = B(0)A\). \(\square\)

### 4. Proof of Theorem 2

In this Subsection we assume the hypotheses and notation of Theorem 2, in particular \(V\) is contained in a super lattice VOA \(V_L\) (see Subsection 7.3 for the definition and relevant properties). In particular, \((V_L)_1\) is a reductive Lie algebra and each of its components is a simple Lie algebra of type \(ADE\) and of level 1.
Now by hypothesis (IV) of Theorem 2, our Lie algebra $\mathfrak{sl}_2$ contains a root of $L$. It follows that $\mathfrak{sl}_2$ is contained in one of the components of $V_1$ and therefore it also has level $k=1$. Adopting the notation of the previous Section, it follows that $(h, h)=2$ (cf. Lemma 4). Thus $h$ is a root of $L$ and we have $\mathfrak{sl}_2=\langle h, e^{\pm h} \rangle$.

We will deduce Theorem 2 from Theorem 1. To this end, notice that states of weight $2$ in $V_L$ are primary because $L$ has no vectors of norm $1$. Thus it suffices to take $U:=V_L$ in Theorem 1 and show that hypotheses (I)–(IV) of Theorem 1 hold. Then Theorem 1 shows that $V$ is the $N=4$ superconformal algebra of central charge $6k=6$. Parts (I) and (II) hold by assumption, so we only have to establish (III) and (IV).

We need some additional notation. Let $(\cdot, \cdot):L \times L \to \mathbb{Z}$ be the bilinear form on $L$.

Note that this is not the notation used in the proof of Lemma 4, where $(\cdot, \cdot)$ denoted the invariant bilinear form on $\mathfrak{sl}_2$.

The vectors in $L$ of norm $n$ are denoted by $L_n:=\{ \alpha \in L | (\alpha, \alpha)=n \}$, in particular $L_2$ is the root system of $L$. Fix a multiplicative bicharacter $\varepsilon:L \times L \to \{\pm 1\}$ that defines the central extension $\hat{L}$ occurring in the short exact sequence

$$1 \to \{\pm 1\} \to \hat{L} \to L \to 0.$$ 

See Subsection 7.2 for more details on the $\varepsilon$-formalism, in particular for the justification that

$$(22) \quad \varepsilon(\alpha, \alpha) = \varepsilon(\alpha, -\alpha) = \begin{cases} -1 & \alpha \in L_2 \\ 1 & \alpha \in L_3 \cup L_4 \end{cases}$$

Recall (2) that $\tau^+, \tau^-$ are highest weight vectors in $A$ and $B$ respectively. We have already mentioned that $U_{\frac{3}{2}}$ is spanned by states $e^\beta$ ($\beta \in L_3$). Thus there are nonempty subsets $X, Y \subseteq L_3$ and scalars $c_\alpha, d_\lambda$ such that

$$(23) \quad \tau^+:=\sum_{\alpha \in X} c_\alpha e^\alpha, \quad \tau^-:=\sum_{\lambda \in Y} d_\lambda e^\lambda.$$ 

Because $h(0)\tau^+=\tau^-$ then we have $(h, \alpha)=1$ ($\alpha \in X$), and similarly $(h, \lambda)=1$ ($\lambda \in Y$).

As a result, we have the following useful facts that hold for $\alpha, \beta \in X$. $|(\alpha, \beta)|\leq 3$ by the Schwarz inequality, moreover

$$(\alpha, \beta) = \begin{cases} 3 & \text{iff } \alpha = -\beta \\ 2 & \text{iff } \alpha-\beta = \gamma \text{ (root } \gamma \perp h) \\ -2 & \text{iff } \alpha+\beta = h \\ -1 & \text{iff } \alpha+\beta = h+\gamma \text{ (root } \gamma \perp h) \end{cases}$$

Identical formulas hold in case $\alpha, \beta \in Y$. We use these formulas in later calculations.

The elements $x^{\pm} \in \mathfrak{sl}_2$ may be identified (recall that $\varepsilon(h, h) = -1$ by (22)) as $x^{\pm} = \mp e^{\pm h}$. Because $x^-(0)\tau^+=\tau^-$ we have

$$\tau^- = \sum_{\alpha \in X} c_\alpha e^{-h}(0)e^\alpha = \sum_{\alpha \in X} c_\alpha \varepsilon(h, \alpha)e^{\alpha-h},$$


and similarly

$$\tau^- = \sum_{\lambda \in \mathcal{Y}} d_{\lambda}(h, \lambda)e^{\lambda - h}.$$ 

We now consider the consequences of assumption (III) of Theorem 2.

**Lemma 20.** $A(1)A = 0$ implies

(a) $\sum_{\alpha \in X} c_{\alpha}c_{h+\gamma - \alpha}(\alpha, h+\gamma)=0$ (each root $\gamma \perp h$),

(b) $\sum_{\alpha \in X} c_{\alpha}c_{h-\alpha}(h, \alpha)\alpha=0,$

with a corresponding statement concerning $B(1)B = 0.$

**Proof.** $A(1)A$ contains the element $\tau^+(1)\tau^-$, which is equal to

$$\sum_{\alpha, \beta \in X} c_{\alpha}c_{\beta}(h, \beta)\epsilon(1)\epsilon^{\beta-h} = \sum_{(\alpha, \beta-h)=(-2,-3)} c_{\alpha}c_{\beta}(h, \beta)\epsilon^{\alpha}(1)\epsilon^{\beta-h}$$

$$= \sum_{\alpha+\beta=h} c_{\alpha}c_{\beta}(h, \beta)\alpha + \sum_{(\alpha, \beta)=-1} c_{\alpha}c_{\beta}(h, \beta)\epsilon(\alpha, \beta-h)\epsilon^{\alpha+\beta-h}$$

$$= \sum_{\alpha+\beta=h} c_{\alpha}c_{\beta}(h, \beta)\alpha - \sum_{\gamma \perp h} \sum_{\alpha+\beta=h+\gamma} c_{\alpha}c_{\beta}(h, \gamma+\alpha)\epsilon(\alpha, \beta)\epsilon^{\alpha+\beta-h}$$

$$= \sum_{\alpha+\beta=h} c_{\alpha}c_{\beta}(h, \beta)\alpha - \sum_{\gamma \perp h} \sum_{\alpha+\beta=h+\gamma} \epsilon(h, \gamma) \sum_{\alpha+\beta=h+\gamma} c_{\alpha}c_{\beta}(h, \alpha)\epsilon(\alpha, \gamma)\epsilon^{\alpha+\beta-h}$$

$$= - \sum_{\alpha} c_{\alpha}c_{h-\alpha}(h, \alpha)\alpha + \sum_{\gamma \perp h} \epsilon(h, \gamma) \sum_{\alpha} c_{\alpha}c_{h+\gamma-\alpha}(\alpha, h+\gamma)\epsilon^{\gamma},$$

using $\epsilon(h, \beta)=\epsilon(h, h-\alpha)=-\epsilon(h, \alpha)$. Hence $A(1)A=0$ if and only if

$$\sum_{\alpha \in X} c_{\alpha}c_{h+\gamma-\alpha}(\alpha, h+\gamma)=0,$$

for each root $\gamma \perp h$, and

$$\sum_{\alpha} c_{\alpha}c_{h-\alpha}(h, \alpha)\alpha = 0.$$

A similar analysis applies for $B(1)B = 0.$ \hfill $\Box$

**Lemma 21.** We have $A(0)A=B(0)B=0.$

**Proof.** We prove that $A(0)A=0.$ The proof that $B(0)B=0$ is similar. Assume, then, that $A(0)A\neq 0.$ Because $A(1)A=0$, by assumption (III), then $A(0)A$ has dimension $\leq 3$ by super skew-symmetry, indeed because we are assuming that $A(0)A\neq 0$ then the image of $A\otimes A \rightarrow A(0)A$ is the adjoint module for sl$_2$. Now it follows that $0\neq\tau^+(0)\epsilon(0)\epsilon(0)^-\in A(0)A.$
But we also have
\[
\tau^+(0)\tau^+ = \sum_{\alpha, \beta \in X} c_{\alpha} c_{\beta} e^{\alpha}(0) e^{\beta} = \sum_{(\alpha, \beta) = -1} c_{\alpha} c_{\beta} \varepsilon(\alpha, \beta) e^{\alpha+\beta} + \sum_{(\alpha, \beta) = -2} c_{\alpha} c_{\beta} \varepsilon(\alpha, \beta) \alpha(-1) e^{\alpha+\beta}
\]
\[
= \sum_{\gamma} \{ \sum_{\alpha+\beta = h+\gamma} c_{\alpha} c_{\beta} \varepsilon(\alpha, h+\gamma) \} e^{h+\gamma} + \{ \sum_{\alpha+\beta = h} c_{\alpha} c_{\beta} \varepsilon(\alpha, h) \alpha \} (-1) e^{h} = 0,
\]
where we used Lemma [20]. This contradiction completes the proof of the Lemma. □

This establishes assumption (III) of Theorem 1.

We now consider consequences of assumptions (II) and (IV) of Theorem 2.

Lemma 22. \( A(1)B \cong sl_2 \) with \( sl_2 = \langle h, e^\pm h \rangle \) if and only if
\( (a) \quad h = -\sum_{\alpha \in X} c_{\alpha} d_{h-\alpha} \varepsilon(h, \alpha) \alpha, \)
\( (b) \quad \sum_{\gamma \perp h} \varepsilon(h, \gamma) \{ \sum_{\alpha \in X} c_{\alpha} d_{h+\gamma-a} \varepsilon(\alpha, h+\gamma) \} e^{\gamma} = 0, \)
where the \( \gamma \) sum is taken over each root \( \gamma \perp h. \)

Proof. By calculations similar to those of Lemma [20] we have
\[
\tau^+(1)\tau^- = \sum_{\alpha \in X, \lambda \in Y} c_{\alpha} d_{\lambda} \varepsilon(h, \lambda) e^{\alpha}(1) e^{\lambda-h} = \sum_{(\alpha, \lambda-h) = -2, -3} c_{\alpha} d_{\lambda} \varepsilon(h, \lambda) e^{\alpha}(1) e^{\lambda-h}
\]
\[
= -\sum_{\alpha \in X} c_{\alpha} d_{h-\alpha} \varepsilon(h, \alpha) \alpha + \sum_{\gamma \perp h} \varepsilon(h, \gamma) \sum_{\alpha \in X} c_{\alpha} d_{h+\gamma-a} \varepsilon(\alpha, h+\gamma) e^{\gamma}.
\]

But from (15) we have that \( h = \tau^+(1)\tau^- \) iff (a) and (b) hold. Note that taking the inner product of (a) with \( h \) implies
\[
(24) \quad \sum_{\alpha \in X} c_{\alpha} d_{h-\alpha} \varepsilon(h, \alpha) = -2.
\]
This implies
\[
\tau^-(1)\tau^+ = \sum_{\alpha \in X, \lambda \in Y} c_{\alpha} d_{\lambda} \varepsilon(h, \alpha) e^{\alpha-h}(1) e^{\lambda} = \sum_{(\alpha-h, \lambda) = -2, -3} c_{\alpha} d_{\lambda} \varepsilon(h, \alpha) e^{\alpha-h}(1) e^{\lambda}
\]
\[
= \sum_{\alpha \in X} c_{\alpha} d_{h-\alpha} \varepsilon(h, \alpha)(\alpha-h) + \sum_{\gamma \perp h} \varepsilon(h, \gamma) \sum_{\alpha \in X} c_{\alpha} d_{h+\gamma-a} \varepsilon(\alpha, h+\gamma) e^{\gamma}
\]
\[
= -h + 2h + 0 = h,
\]
iff (a) and (b) hold. The remaining relations in (15) follow from \( sl_2 \) symmetry. □

Lemma 23. \( A(1)B \cong sl_2 \) implies \( Tsl_2 \cap A(0)B \neq 0. \)

Proof. \( A(0)B \) contains the element
\[
\tau^+(0)\tau^+ = \sum_{(\alpha, \lambda) = -1, -2} c_{\alpha} d_{\lambda} e^{\alpha}(0) e^{\lambda}
\]
\[
= -\frac{1}{2} \{ \sum_{\alpha \in X} c_{\alpha} d_{h-\alpha} \varepsilon(h, \alpha) \} h(-1)e^h = h(-1)e^h = Te^{-h},
\]
by (24) of Lemma 22. Thus \( Tsl_2 \cap A(0)B \neq 0. \) □
This completes the proof of hypothesis (IV) of Theorem [1] and with it the proof of Theorem [2].

5. \( N=2 \) Superconformal Algebras

The \( N=2 \) SVOA of central charge \( c \) is generated by a pair of states \( \tau^\pm \) of conformal weight \( \frac{3}{2} \) satisfying the non-zero relations e.g. [K]

\[
(i) \quad \tau^\pm(2) \tau^\mp = \frac{c}{3} 1, \quad (ii) \quad \tau^\pm(1) \tau^\mp = \pm h, \quad (iii) \quad \tau^\pm(0) \tau^\mp = \omega \pm \frac{1}{2} T, \\
(iv) \quad h(0) \tau^\pm = \pm \tau^\pm, \quad (v) \quad h(1) h = \frac{c}{3} 1,
\]

together with the standard Virasoro relations between \( \tau^\pm, h \) and the Virasoro vector \( \omega \) of central charge \( c \). Similarly to Theorem [1] we have

**Theorem 24.** Let \( U \) be a SVOA of CFT–type. Let \( V \subseteq U \) be the subalgebra generated by 2 primary vectors \( \tau^\pm \) of weight \( \frac{3}{2} \) in \( U \), so that

\[
V = C 1 \oplus V_1 \oplus V_2 \oplus \ldots
\]

is a conformally graded subspace of \( U \). Assume that:

(I) \( h(0) \tau^\pm = \pm \tau^\pm \) where \( h:=\tau^+(1) \tau^- \),

(II) \( \tau^\pm(0) \tau^\mp = 0 \).

Then \( V \) is an \( N=2 \) SVOA with central charge \( c=6k \) where \( h(1) h = 2k 1 \).

**Proof.** We sketch the proof, which is similar in many respects to that for Theorem [1]. We firstly note that \( u(1) v = -v(1) u \) for all \( u, v \in \langle \tau^\pm \rangle \) by super skew-symmetry. Thus Assumptions (I) and (II) above imply the properties (ii), (iv) and (v) for \( c=6k \). As for Lemma [9] we find \( h(1) \tau^\pm = 0 \) which in turn implies (as in Lemma [10]) that \( \tau^\pm(2) \tau^\mp = 2k 1 \). Thus property (i) holds.

Using super skew-symmetry we have

\[
\tau^+(0) \tau^- = \tau^- (0) \tau^+ - T (\tau^-(1) \tau^+ ) = \tau^- (0) \tau^+ + T h.
\]

Thus, in this case, we define

\[
(25) \quad \sigma := \frac{1}{2} (\tau^+(0) \tau^- + \tau^- (0) \tau^+),
\]

so that \( \tau^\pm(0) \tau^\mp = \sigma \pm \frac{1}{2} T h \). Hence

\[
(26) \quad [\tau^+(m), \tau^-(n)] = \sigma (m + n) + \frac{1}{2} (m - n) h (m + n - 1) + m(m - 1) k \delta_{m+n+1,0}.
\]

It remains to show that \( \sigma \) is a Virasoro vector of central charge \( c=6k \). As in Lemma [11] we find \( \sigma(0) h = T h, \sigma(1) h = h \) and \( \sigma(2) h = 0 \). (26) implies

\[
\sigma(0) = [\tau^+(0), \tau^-(0)] , \quad \sigma(1) = [\tau^+(0), \tau^-(1)] + \frac{1}{2} h(0), \quad \sigma(2) = [\tau^+(0), \tau^-(2)] + h(1),
\]

from which it follows that \( \sigma(0) \tau^+ = T \tau^+, \sigma(1) \tau^+ = \frac{3}{2} \tau^+ \) and \( \sigma(2) \tau^+ = 0 \) (cf. Lemmas [12] and [15]). Similar results follow for \( \sigma(n) \tau^- \) for \( n = 0, 1, 2 \). Hence (cf. Lemma [16])

\[
[\sigma(m), \tau^\pm(n)] = \left( \frac{1}{2} m - n \right) \tau^\pm(m+n-1),
\]

which implies that \( \sigma \) is a Virasoro vector of central charge \( 6k \) (cf. Lemma [17]). \( \Box \)
6. Examples

We provide some constructions of $N=4$ and $N=2$ subalgebras of a lattice SVOA $V_L$ for an odd lattice $L$. These examples illustrate Theorems 1, 2 and 24. Throughout, we let $L_n$ denote the set of lattice vectors in $L$ of norm $n$.

6.1. Example 1. Consider the lattice SVOA $V_L$ for $L=\mathbb{Z}^6$ – the well-known rank 12 free fermion construction. Let $L=\mathbb{Z}^6$ be generated by $\gamma_1, \ldots, \gamma_6 \in L_1$ with $(\gamma_i, \gamma_j)=\delta_{ij}$. Then $V_L$ is generated by 12 weight $\frac{1}{2}$ fermion vectors $e^{\pm \gamma_i}$.

We firstly note from Section 7.2 that

$$
\varepsilon(\gamma_i, \gamma_j) = \begin{cases} 
-\varepsilon(\gamma_j, \gamma_i) & \text{for } i \neq j, \\
-1 & \text{for } i = j.
\end{cases}
$$

In addition, for convenience, we choose $\varepsilon(\gamma_1, \gamma_2) = 1$ so that $\varepsilon(\gamma_2, \gamma_1) = -1$.

Define $\mathfrak{sl}_2$ generators $h:=\gamma_1+\gamma_2$, $x^\pm:=\mp e^\pm h \in (V_{2^6})_1$ where $h(1)h=21$ i.e. $k=1$ in (13). Then $(e^{\gamma_1}, e^{-\gamma_2})$ and $(e^{\gamma_2}, e^{-\gamma_1})$ form a pair of $\mathfrak{sl}_2$-representations, where using Subsection 7.3 we find

$$
\begin{align*}
    h(0)e^{\pm \gamma_1} &= \pm e^{\pm \gamma_1}, & h(0)e^{\pm \gamma_2} &= \pm e^{\pm \gamma_2}, \\
x^+(0)e^{\mp \gamma_1} &= \mp e^{\mp \gamma_1}, & x^+(0)e^{\mp \gamma_2} &= \mp e^{\mp \gamma_2}.
\end{align*}
$$

(27)

Define $a, \bar{a}, b, \bar{b} \in (V_{2^6})_1$ by

$$
\begin{align*}
a &= \frac{1}{\sqrt{2}}(\gamma_3+i\gamma_4), & \bar{a} &= \frac{1}{\sqrt{2}}(\gamma_3-i\gamma_4), \\
b &= \frac{1}{\sqrt{2}}(\gamma_5+i\gamma_6), & \bar{b} &= \frac{1}{\sqrt{2}}(\gamma_5-i\gamma_6),
\end{align*}
$$

which satisfy non-zero relations $a(1)\bar{a}=b(1)\bar{b}=1$. Lastly, define $\tau^{\pm}, \pi^{\pm}$ by

$$
\begin{align*}
    \tau^+ &= a(-1)e^{\gamma_1} + b(-1)e^{\gamma_2}, & \tau^- &= a(-1)e^{-\gamma_2} - b(-1)e^{-\gamma_1}, \\
    \pi^+ &= \bar{a}(-1)e^{\gamma_2} - \bar{b}(-1)e^{\gamma_1}, & \pi^- &= -\bar{a}(-1)e^{-\gamma_1} - \bar{b}(-1)e^{-\gamma_2}.
\end{align*}
$$

We now show that the sub-SVOA generated by $\tau^{\pm}, \pi^{\pm}$ is the $N=4$ superconformal algebra for central charge $c=6$ by use of Theorem 1. $\tau^{\pm}, \pi^{\pm}$ are clearly primary vectors of weight $\frac{3}{2}$. It is straightforward to confirm (4) and (5) by using (27). Thus Axiom (I) of Theorem 1 holds.

In order to confirm Axioms (II)–(IV) of Theorem 1 we note, using superassociativity, that for all $u, v \in \{a, b\}$ and $\lambda, \nu \in \{\pm \gamma_1, \pm \gamma_2\}$

$$
\begin{align*}
(u(\nu) &\theta^\lambda)(v(\nu) \theta^\nu) = \langle u, v \rangle e^\lambda(-1)e^\nu, \\
(u(\nu) &\theta^\lambda)(0) \theta^\lambda(0) \theta^\nu = u(-1)\nu(-1)\theta^\lambda(0)\theta^\nu + \langle u, v \rangle \theta^\lambda(-2)\theta^\nu,
\end{align*}
$$

where $u(1)v=\langle u,v \rangle 1$. (28) and Subsection 7.3 imply that

$$
\begin{align*}
    \tau^+(1)=\tau^- &= -\varepsilon(\gamma_1, -\gamma_1)\gamma_1(-1)1 - \varepsilon(\gamma_2, -\gamma_2)\gamma_2(-1)1 = h, \\
    \tau^-(-1)=\pi^+ &= \varepsilon(-\gamma_2, \gamma_2)\gamma_2(-1)1 + \varepsilon(-\gamma_1, \gamma_1)(-1)1 = h.
\end{align*}
$$

Therefore $\tau^+(1)\pi^+ = 2x^+$ and $\tau^-(-1)\pi^- = 2x^-$ using $x^{\pm}(0)h=\mp 2x^{\pm}$ and hence $A(1)B \cong \mathfrak{sl}_2$ i.e. Axiom (II) holds. Axiom (III) follows from

$$
\begin{align*}
\tau^+(0)\tau^- &= -\varepsilon(-\gamma_1, -\gamma_1)a(-1)b(-1)1 + \varepsilon(\gamma_2, -\gamma_2)b(-1)a(-1)1 = 0.
\end{align*}
$$
using (29). By skew-symmetry $\tau^-(0) \tau^+ = \tau^+(0) \tau^- = 0$ and so $A(0)A = 0$ using $sl_2$ symmetry. A similar argument applies showing that $B(0)B = 0$. Lastly
$$\tau^+(0) \bar{\tau}^+ = - (\gamma_1 + \gamma_2)(-1)e^{\gamma_1 + \gamma_2} = - T x^+,$$
so that Axiom (IV) holds. Hence the SVOA generated by $\tau^\pm, \bar{\tau}^\pm$ is an $N = 4$ superconformal algebra for central charge $c = 6k = 6$ by Theorem [1].

To finish, we show that $\sigma = \omega$, the standard $V_L$ Virasoro vector of central charge 6. From (29) we find
$$\sigma = \frac{1}{2} (\tau^+(0) \bar{\tau}^+ - \tau^-(0) \bar{\tau}^-)$$
$$= -\frac{1}{2} \left( \varepsilon(\gamma_1, -\gamma_1) a(-1) \bar{\alpha} + e^{\gamma_1} (-2)e^{-\gamma_1} + \varepsilon(\gamma_2, -\gamma_2) b(-1) \bar{b} + e^{\gamma_2} (-2)e^{-\gamma_2} \right.$$
$$+ \varepsilon(-\gamma_2, \gamma_2) a(-1) \bar{\alpha} + e^{-\gamma_2} (-2)e^{\gamma_2} + \varepsilon(-\gamma_1, \gamma_1) b(-1) \bar{b} + e^{-\gamma_1} (-2)e^{\gamma_1} \left.) \right.$$
$$= a(-1) \bar{\alpha} + b(-1) \bar{b} + \frac{1}{2} (\gamma_1(-1) \gamma_1 + \gamma_2(-1) \gamma_2) = \frac{1}{2} \sum_{i=1}^{6} \gamma_i(-1) \gamma_i = \omega.$$ 

Thus we conclude the well known result e.g. [ET1].

**Proposition 25.** $V_{26}$ contains an $N=4$ superconformal subalgebra with the standard lattice Virasoro vector for central charge $c = 6$.

6.2. **Example 2.** In this example, we construct an $N = 4$ algebra from lattice theories that do not contain free fermions. We also illustrate how to reduce the problem of constructing $N = 4$ subalgebras of lattice theories to questions about lattices.

Let $\alpha_1, \ldots, \alpha_6$ be an orthogonal basis for $\mathbb{R}^6$ consisting of vectors of norm 3, and let $L$ be the lattice spanned by the $\alpha_i$ together with
$$h := \frac{1}{3}(\alpha_1 + \ldots + \alpha_6) \in L_2.$$ 

Then $L$ is an odd, positive-definite, integral lattice with *theta function*
$$\theta_L(\tau) = 1 + 2q + 24q^{\frac{3}{2}} + \ldots$$

In particular, there are no vectors of norm 1, and $\pm h$ are the only roots.

**Proposition 26.** $V_L$ contains an $N=4$ superconformal subalgebra with standard lattice Virasoro vector of central charge $c = 6$.

**Proof.** Let $X \subseteq L_3$ consist of the 6 vectors $\alpha_i$. In the formalism of Section [4], especially (23), we take $Y$ to consist of the vectors $\{h - \alpha_i\}$, so that the four generating states of $A$ of weight $\frac{3}{2}$ will be chosen to take the form
$$\tau^+ := \sum_{\alpha \in X} c_\alpha e^{\alpha}, \quad \tau^- := \sum_{\alpha \in X} c_\alpha \varepsilon(h, \alpha) e^{\alpha - h},$$
$$\tau^+ := \sum_{\alpha \in X} d_{h - \alpha} e^{h - \alpha}, \quad \tau^- := - \sum_{\alpha \in X} d_{h - \alpha} \varepsilon(h, \alpha) e^{-\alpha}.$$ 

We show that the hypotheses of Theorem [2] hold for certain choices of scalars $c_\alpha, d_{h - \alpha}, (\alpha \in X)$. Conditions (a) and (b) of Lemma [20] and condition (b) of Lemma [22]
automatically hold since \(X \cap Y = 0\) and \(\pm h\) are the only roots in \(L\). We may check by direct calculation that these facts imply the assumptions of these two Lemmas (cf. the proofs of the Lemmas), i.e., \(A(1)A = B(1)B = 0\). Now choose the scalars \(c_\alpha, d_{h - \alpha}\) so that

\[
c_\alpha d_{h - \alpha} \epsilon(h, \alpha) = -\frac{1}{3}
\]

for each \(\alpha \in X\), implying condition (a) of Lemma 22. Hence \(A(1)B \cong sl_2\).

Because \(\pm h\) are the only roots of \(L\), there is a unique simple component of the Lie algebra \((V_\nu)_1\), and it is isomorphic to \(sl_2\). It follows that this component is our Lie algebra \(A(1)B = sl_2\), and in particular \(h \in sl_2\) and \(A(1)B = \langle h, e^{\pm h}\rangle\). It is then straightforward to check that \(A\) and \(B\) are indeed vector representations of \(sl_2\), so that all hypotheses of Theorem 2 are satisfied. This completes the proof that \(\tau^\pm, \tau^\pm\) generate an \(N=4\) subalgebra of \(V_L\) with \(c=6\).

The Virasoro vector in this example is (cf. Lemmas 11, 17)

\[
\sigma = \frac{1}{2} (\tau^+ (0) \tau^- - \tau^- (0) \tau^+) = \frac{1}{2} \sum_{\alpha, \beta \in X} \left\{ c_\alpha d_{h - \beta} \epsilon(h, \beta) e^{\alpha} (0) e^{-\beta} + c_\alpha d_{h - \beta} \epsilon(h, \alpha) e^{\alpha - h} (0) e^{h - \beta} \right\}
\]

\[
= \frac{1}{12} \sum_{\alpha \in X} \left( \alpha(-2) + \alpha(-1)^2 + (\alpha - h)(-2) + (\alpha - h)(-1)^2 \right) 1
\]

\[
= \frac{1}{12} \sum_{\alpha \in X} \left( 2\alpha(-1)^2 + h(-1)^2 - 2\alpha(-1)h(-1) + 2\alpha(-2) - h(-2) \right) 1
\]

\[
= \frac{1}{6} \sum_\alpha \alpha(-1)\alpha.
\]

This is indeed the standard Virasoro element of \(V_L\), and the proof is complete. \qed

We next consider three known examples of \(N=2\) superconformal subalgebras of odd lattice SVOAs, illustrating Theorem 24.

6.3. Example 3. Consider the lattice SVOA \(V_{Z^3}\), the rank 6 free fermion construction. Let \(L = \mathbb{Z}^3\) be generated by \(\gamma_1, \gamma_2, \gamma_3 \in L_1\) with \((\gamma_i, \gamma_j) = \delta_{ij}\). Define \(h, a^\pm \in (V_{Z^3})_1\) by

\[
h = \gamma_1, \quad a^\pm = \frac{1}{\sqrt{2}} (\gamma_2 \pm i \gamma_3),
\]

with \(a^+(1)a^- = 1\). Lastly, define \(\tau^\pm = a^\pm(-1)e^{\pm \gamma_1}\).

Using (29) and (28) we find that Axioms (I) and (II) of Theorem 24 hold. Since \(h(1)h = 1\), the central charge is \(c=3\). Using (25) one finds that

\[
\sigma = a^+(1)a^- + \frac{1}{2} \gamma_1(-1)\gamma_1 = \frac{1}{2} \sum_{i=1}^3 \gamma_i(-1)\gamma_i,
\]

the standard lattice Virasoro vector for \(V_{Z^3}\). This establishes the well-known free fermion construction.
\textbf{Proposition 27.} \(V_{23}\) contains an \(N=2\) superconformal subalgebra with the standard lattice Virasoro vector for central charge \(c=3\).

\[\text{6.4. Example 4.}\]

In this example we show that every odd lattice SVOA for which \(L_{3} \neq \emptyset\) contains an \(N=2\) SVOA with central charge \(c=1\).

\textbf{Proposition 28 (}\cite{K}, Example 5.9c\textbf{).} Let \(\gamma \in L_{3}\) and define \(\tau^{\pm} := \frac{1}{\sqrt{3}} e^{\pm \gamma}\) and \(h := \frac{1}{3} \gamma\). Then \(\tau^{\pm}\) generate an \(N=2\) subalgebra of \(V_{L}\) with Virasoro vector \(\omega = \frac{1}{6} \gamma(-1)\gamma\) and \(c=1\).

\textit{Proof.} We find \(h = \tau^{\pm}(1)\tau^{\mp}\) with \(h(0)\tau^{\pm} = \pm \tau^{\pm}\) and \(\tau^{\pm}(0)\tau^{\mp} = 0\). Thus Axioms (I) and (II) of Theorem \[24\] hold, so \(\tau^{\pm}\) generate an \(N=2\) superconformal algebra. Since \(h(1)h = \frac{1}{3} 1\), the central charge is \(c=1\) with Virasoro vector \(\frac{1}{6} \gamma(-1)\gamma\) from \[25\]. \qed

\[\text{6.5. Example 5.}\]

\textbf{Proposition 29 (}\cite{A}, Proposition 6.1\textbf{).} Let \(\alpha, \beta \in L_{3}\) with \((\alpha, \beta) = 1\), and let \(\lambda, \mu\) be nonzero scalars. Define

\[
\begin{align*}
\tau^{+} & = \frac{1}{2} (\lambda e^{\alpha} + \mu e^{\beta}), \\
\tau^{-} & = \frac{1}{2} (\lambda^{-1} e^{-\alpha} + \mu^{-1} e^{-\beta}), \\
\omega & = \frac{1}{8} \left\{ \alpha(-1)\alpha + \beta(-1)\beta + \frac{2\lambda}{\mu} \varepsilon(\alpha, \beta) e^{\alpha - \beta} + \frac{2\mu}{\lambda} \varepsilon(\beta, \alpha) e^{\beta - \alpha} \right\}.
\end{align*}
\]

Then \(\tau^{\pm}\) generate an \(N=2\) subalgebra of \(V_{L}\) with \(c = \frac{3}{2}\).

\textit{Proof.} Axioms (I) and (II) of Theorem \[24\] are easily seen to hold. \(h(1)h = \frac{1}{2} 1\) implies the central charge \(c = \frac{3}{2}\) and \(\omega\) is as given on applying \[25\]. \qed

\[\text{7. Appendices}\]

\[\text{7.1. Axioms for super VOAs.}\]

The underlying Fock space is a \(\frac{1}{2} \mathbb{Z}\)-graded \(\mathbb{C}\)-linear super vector space

\[V = \bigoplus_{k \in \frac{1}{2} \mathbb{Z}} V_{k},\]

with \textit{parity} operator \(p(u) = 2k \mod 2\) for \(u \in V_{k}\). Each state \(u \in V\) has a vertex operator \(Y(u, z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1}; u(n) \in \text{End}(V)\) is the \(n^{th}\) mode of \(u\).

There is a distinguished \textit{vacuum state} \(1 \in V_{0}\) with vertex operator \(Y(1, z) = \text{Id}_{V}\); and a distinguished \textit{Virasoro state} \(\omega \in V_{2}\) with vertex operator

\[Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2},\]

whose modes satisfy the Virasoro relations with \textit{central charge} \(c\):

\[[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{2} \binom{m+1}{3} \delta_{m+n,0} c \text{Id}_{V}.\]

We distinguish the endomorphism \(T \in \text{End}(V)\) defined by \(T(u) := u(-2)1 = L(-1)u\). The \(\frac{1}{2} \mathbb{Z}\) grading is determined by \(L(0)\) with \(L(0)u = ku\) for \(u \in V_{k}\).
Modes satisfy the following axioms, the third being the super Jacobi identity:

(a) \( u(n)v = 0 \) for all \( n \geq n_0 \),
(b) \( u(-1)1 = u; \; u(n)1 = 0 \) for \( n \geq 0 \),
(c) \( \forall r, s, t \in \mathbb{Z}, \)

\[
\sum_{i \geq 0} \binom{r}{i} (u(t+i)v)(r+s-i)w = \\
\sum_{i \geq 0} (-1)^i \binom{r}{i} \{ u(r+t-i)v(s+i) - (-1)^{t+p(u)p(v)}v(s+t-i)u(r+i) \} w,
\]

for all \( u, v, w \in V \). The special cases \( t = 0, r = 0 \) give respectively super commutativity

\[
u(r)v(s) = (-1)^{p(u)p(v)}v(s)u(r) = \sum_{i \geq 0} \binom{r}{i} (u(i)v)(r+s-i),
\]

and super associativity

\[
(u(t)v)(s) = \sum_{i \geq 0} (-1)^i \binom{r}{i} \{ u(t-i)v(s+i) - (-1)^{t+p(u)p(v)}v(s+t-i)u(i) \}.
\]

Taking \( r = -1, s = 0, w = 1 \) leads to super skew-symmetry

\[
v(t)u = (-1)^{t+p(u)p(v)} \sum_{i \geq 0} (-1)^i T^i u(t+i)v.
\]

7.2. The \( \varepsilon \)-formalism. Fix a finitely generated free abelian group \( L \). We are interested in groups \( \hat{L} \) which are central extensions of \( L \) by \( \mathbb{Z}_2 \). So there is a short exact sequence of groups

\[
1 \to \{ \pm 1 \} \to \hat{L} \to L \to 0,
\]

and \( \hat{L} \) can be identified with \( L \times \{ \pm 1 \} \) as a set, with multiplication

\[
(\alpha, e)(\beta, f) = (\alpha+\beta, \varepsilon(\alpha, \beta)ef) \quad (\alpha, \beta \in L, e, f \in \{ \pm 1 \}),
\]

where

\[
\varepsilon : L \times L \to \{ \pm 1 \}.
\]

We may, and shall, take \( \varepsilon \) to be bimultiplicative, i.e.,

\[
\varepsilon(\alpha+\beta, \gamma) = \varepsilon(\alpha, \gamma) \varepsilon(\beta, \gamma), \quad \varepsilon(\alpha, \beta+\gamma) = \varepsilon(\alpha, \beta) \varepsilon(\alpha, \gamma).
\]

This ensures that \( \varepsilon \in Z^2(L, \{ \pm 1 \}) \) is a 2-cocycle and that multiplication in \( L \) is associative. In particular we note that \( \varepsilon(\alpha, 0) = \varepsilon(0, \alpha) = 1 \) and \( \varepsilon(\alpha, \beta) = \varepsilon(\alpha, \beta) = \varepsilon(-\alpha, \beta) \).

If \( L \) is a positive-definite integral lattice with bilinear form \( (\ , \ ) \), then we may further choose \( \varepsilon \) (([K], P. 155)) so that it satisfies

\[
\varepsilon(\alpha, \beta) \varepsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)+(\alpha, \beta)(\beta, \beta)} \quad \varepsilon(\alpha, \alpha) = (-1)^{(\alpha, \alpha)+(\alpha, \alpha)^2)/2}.
\]

Remark 30. Depending on context, various choices for \( \varepsilon \) are used in the literature, although they give equivalent theories. The one used in [FLM], for example, is different to the one we generally use here.
7.3. Super lattice theories. Let $L$ be a positive-definite integral lattice equipped with a bilinear form $(\ ,\ )$, with $\varepsilon$ as in Subsection 7.2. The twisted group algebra $C^\varepsilon[L]$ has basis $e^\alpha (\alpha \in L)$ and multiplication

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta} \quad (\alpha, \beta \in L).$$

It is $\frac{1}{2}\mathbb{Z}$-graded by $wt(e^\alpha) := \frac{1}{2}(\alpha, \alpha)$.

There are Lie algebras (the first is abelian)

$$\mathfrak{h} := \mathbb{C} \otimes \mathbb{Z} \ L, \quad \widehat{\mathfrak{h}} := \mathfrak{h} \otimes \mathfrak{C}[t, t^{-1}] \otimes \mathbb{C}, \quad \mathfrak{h}^+ = \mathfrak{h} \otimes \mathbb{C}[t], \quad \widehat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}],$$

with brackets $[x \otimes t^m, y \otimes t^n] = (x, y)\delta_{m+n,0}\mathbb{C}$, $[c, \widehat{\mathfrak{h}}] = 0$, and an induced $\widehat{\mathfrak{h}}$-module

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes U(\mathfrak{h} \otimes \mathbb{C}[t] \otimes \mathbb{C}) \mathbb{C} \cong S(\widehat{\mathfrak{h}}^-) \text{ (linearly)},$$

$\mathfrak{h} \otimes \mathbb{C}[t]$ acting trivially on $\mathbb{C}$ and $c$ acting as $1$. Fock space for the lattice theory is

$$V_L = M(1) \otimes C^\varepsilon[L] \cong S(\widehat{\mathfrak{h}}^-) \otimes \mathbb{C}[L] \text{ (linearly)}$$

with the usual tensor product grading. The Virasoro vector is $\frac{1}{2}\sum_i h_i(-1)h_i$, the sum ranging over any orthonormal basis $\{h_i\}$ of $\mathfrak{h}$.

For $\alpha \in \mathfrak{h}$ write $\alpha(n) := \alpha \otimes t^n$, $\alpha(z) := \sum_{n \in \mathbb{Z}} \alpha(n)z^{-n-1}$, $z^\alpha \cdot e^\beta \mapsto z^{(\alpha, \beta)}e^\beta$, and set

$$Y(e^\alpha, z) := \exp \left( \sum_{m=1}^{\infty} \frac{\alpha(-m)z^m}{m} \right) \exp \left( -\sum_{m=1}^{\infty} \frac{\alpha(m)z^{-m}}{m} \right) e^\alpha z^\alpha,$$

and for $v = \alpha_1(-n_1)\ldots\alpha_k(-n_k)\otimes e^\alpha \in V_L \ (n_i \geq 1)$ set

$$Y(v, z) := \left( \frac{1}{(n_1 - 1)!} \left( \frac{d}{dz} \right)^{n_1-1} \alpha_1(z) \right) \ldots \left( \frac{1}{(n_k - 1)!} \left( \frac{d}{dz} \right)^{n_k-1} \alpha_k(z) \right) Y(e^\alpha, z),$$

with the usual normal ordering conventions.

For $\gamma, \rho \in L$ we have

$$e^\gamma(n)e^\rho = \begin{cases} 0 & \text{if } (\gamma, \rho) \geq -n \\ \varepsilon(\gamma, \rho)e^{\gamma+\rho} & \text{if } (\gamma, \rho) = -n-1 \\ \varepsilon(\gamma, \rho)\gamma(-1)e^{\gamma+\rho} & \text{if } (\gamma, \rho) = -n-2 \\ \frac{1}{2}\varepsilon(\gamma, \rho)(\gamma(-2)e^{\gamma+\rho} + \gamma(-1)^2e^{\gamma+\rho}) & \text{if } (\gamma, \rho) = -n-3 \end{cases}$$

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