Abstract. In this paper we presented the semantics of database mappings in the relational DB category based on the power-view monad $T$ and monadic algebras. The objects in this category are the database-instances (a database-instance is a set of n-ary relations, i.e., a set of relational tables as in standard RDBs). The morphisms in DB category are used in order to express the semantics of view-based Global and Local as View (GLAV) mappings between relational databases, for example those used in Data Integration Systems. Such morphisms in this DB category are not functions but have the complex tree structures based on a set of complex query computations between two database-instances. Thus DB category, as a base category for the semantics of databases and mappings between them, is different from the Set category used dominantly for such issues, and needs the full investigation of its properties.

In this paper we presented another contributions for an intensive exploration of properties and semantics of this category, based on the power-view monad $T$ and the Kleisli category for databases. Here we stressed some Universal algebra considerations based on monads and relationships between this DB category and the standard Set category. Finally, we investigated the general algebraic and induction properties for databases in this category, and we defined the initial monadic algebras for database instances.

1 Introduction

The computational significance of monads has been stressed [1,2] in suggestions that they may help in understanding programs "as functions from values to computations". The idea, roughly, is to give a denotational semantics to computations, and it suggests an alternative to the conceptual gap between the intensional (operational) and the extensional (denotational) approach to the semantics of programming languages.

The idea of a monad, based on an endofunctor $T$ for a given category, as a model for computations is that, for each set of values of type $A$, $T A$ is the object of computations of "type $A$".

Let us explain in which way we can use such a denotational semantics, based on monads, in the case of relational databases. It is well known that the relational databases are complex structures, defined by sets of n-ary relations, and the mappings between them are based on sets of view-mappings between the source database $A$ to the target database $B$. We consider the views as an universal property for databases (possible observations of the information contained in some database).
We assume a view of a database $A$ the relation (set of tuples) obtained by a "Select-Project-Join + Union" (SPJRU) query $q(x)$ where $x$ is a list of attributes of this view. We denote by $\mathbf{L}_A$ the set of all such queries over a database $A$, and by $\mathbf{L}_A/\approx$ the quotient term algebra obtained by introducing the equivalence relation $\approx$, such that $q(x) \approx q'(x)$ if both queries result with the same relation (view). Thus, a view can be equivalently considered as a term of this quotient-term algebra $\mathbf{L}_A/\approx$ with carrier set of relations in $A$ and a finite arity of their operators, whose computation returns with a set of tuples of this view. If this query is a finite term of this algebra it is called a "finitary view".

Notice that a finitary view can have an infinite number of tuples also.

Such an instance level database category $DB$ has been introduced first time in Technical report [3], and used also in [4]. General information about categories the reader can find in classic books [5], while more information about this particular database category $DB$, with set of its objects $\text{Ob}_{DB}$ and set of its morphisms $\text{Mor}_{DB}$, are recently presented in [6]. In this paper we will only emphasize some of basic properties of this $DB$ category, in order to render more selfcontained this presentation.

Every object (denoted by $A, B, C, ...$) of this category is a database instance, composed by a set of $n$-ary relations $a_i \in A$, $i = 1, 2, ...$ called also "elements of $A$".

In [3] has been defined the power-view operator $T$, with domain and codomain equal to the set of all database instances, such that for any object (database) $A$, the object $TA$ denotes a database composed by the set of all views of $A$. The object $TA$, for a given database instance $A$, corresponds to the quotient-term algebra $\mathbf{L}_A/\approx$, where carrier is a set of equivalence classes of closed terms of a well defined formulae of a relational algebra, "constructed" by $\Sigma_R$-constructors (relational operators in SPJRU algebra: select, project, join and union) and symbols (attributes of relations) of a database instance $A$, and constants of attribute-domains. More precisely, $TA$ is generated by this quotient-term algebra $\mathbf{L}_A/\approx$, i.e., for a given evaluation of queries in $\mathbf{L}_A$, $\text{Eval}_A : \mathbf{L}_A \rightarrow TA$, which is surjective function, from a factorization theorem, holds that there is a unique bijection $is_A : \mathbf{L}_A/\approx \rightarrow TA$, such that the following diagram commutes

$$
\begin{array}{ccc}
\mathbf{L}_A & \xrightarrow{\text{Eval}_A} & TA \\
\downarrow \text{nat}_\approx & & \\
\mathbf{L}_A/\approx & \xrightarrow{is_A} & TA
\end{array}
$$

where the surjective function $\text{nat}_\approx : \mathbf{L}_A \rightarrow \mathbf{L}_A/\approx$ is a natural representation for the equivalence $\approx$.

For every object $A$ holds that $A \subseteq TA$, and $TA = TTA$, i.e., each (element) view of database instance $TA$ is also an element (view) of a database instance $A$.

Closed object in $DB$ is a database $A$ such that $A = TA$. Notice that also when $A$ is finitary (has a finite number of relations) but with at least one relation with infinite number of tuples, then $TA$ has an infinite number of relations (views of $A$), thus can be an infinitary object. It is obvious that when a domain of constants of a database is finite then both $A$ and $TA$ are finitary objects.

From a behavioral point of view based on observations we can define equivalent (cate-
gorically isomorphic) objects (database instances) as follows: each arrow (morphism) is composed by a number of “queries” (view-maps), and each query may be seen as an observation over some database instance (object of DB). Thus, we can characterize each object in DB (a database instance) by its behavior according to a given set of observations. Thus databases A and B are equivalent (bisimilar) if they have the same set of its observable internal states, i.e. when TA is equal to TB: \( A \approx B \ \text{iff} \ TA = TB. \) This equivalence relation corresponds to the isomorphism of objects in DB category [6]. It is demonstrated that this powerview closure operator can be extended also to arrows of DB category, thus that it is an endofunctor and that defines a monad (see Section 2).

Basic properties of this database category DB as its symmetry (bijective correspondence between arrows and objects, duality [DB is equal to its dual \( DB^{OP} \)) so that each limit is also colimit (ex. product is also coproduct, pullback is also pushout, empty database \( \bot^0 \) is zero objet, that is, both initial and terminal object, etc..), and that it is a 2-category has been demonstrated in [3,6].

Generally, database mappings are not simply programs from values (relations) into computations (views) but an equivalence of computations: because of that each mapping, from any two databases A and B, is symmetric and gives a duality property to the category DB. The denotational semantics of database mappings is given by morphisms of the Kleisli category \( DB_T \) which may be ”internalized” in DB category as ”computations” [7].

The product \( A \times B \) of a databases A and B is equal to their coproduct \( A + B \), and the semantics for them is that we are not able to define a view by using relations of both databases, that is, these two databases have independent DBMS for query evaluation. For example, the creation of exact copy of a database \( A \) in another DB server corresponds to the database \( A + A \).

In the paper [8,9,8] have been considered some relationships of DB and standard Set category, and has been introduced the categorial (functors) semantics for two basic database operations: matching \( \otimes \), and merging \( \oplus \), such that for any two databases A and B, we have that \( A \otimes B = TA \cap TB \) and \( A \oplus B = T(A \cup B) \). In the same work has been defined the algebraic database lattice and has been shown that DB is concrete, small and locally finitely presentable (lfp) category. Moreover, it was shown that DB is also V-category enriched over itself, was developed a metric space and a subobject classifier for this category, and demonstrated that it is a weak monoidal topos.

In this paper we will develop the denotational semantics for database mappings based on power-view endofunctor \( T \), monadic \( T^{-}(co)\)algebras and their computational properties in DB category, and Kleisly category of a monad \( T \) used for categorial semantics of database mappings and database queries.

Plan of this paper is the following: After brief introduction of DB category and its power-view monad \( T \), taken from [3,6,10], in Section 3 consider its behavioral equivalence and category symmetry. In Section 4 we will consider universal algebra theory for databases and monadic coalgebras for database mappings. In Section 5 will be developed the categorial semantics of database mappings, based on Kleisly category of the monad \( T \). Finally in Section 6 are developed the theoretical considerations of (co)algebras and (co)inductions for databases.
2 Monad over DB Category

In this section we will present a short introduction for a DB category, based on work in [3,10]. As default we assume that a domain of every database is arbitrary large set but is finite. It is reasonable assumption for real applications. We define an universal database instance \( T \), as the union of all database instances, i.e., \( T = \{ a_i | a_i \in A, A \in Ob_{DB} \} \). It is a top object of this category. We have that \( T = T \), because every view \( v \in TT \) is a database instance also, thus \( v \in T \); and vice versa, every element \( r \in T \) is also a view of \( T \), thus \( r \in TT \).

Every object (database) \( A \) has also an empty relation \( \bot \). The object (database) composed by only this empty relation is denoted by \( \bot^0 = \{ \bot \} \).

Any empty database (a database with only empty relations) is isomorphic to this bottom object \( \bot^0 \).

Morphisms of this category are all possible mappings between database instances based on views. Elementary view-map for a given database \( A \) is given by a SPCU query \( f_i = q_{A_i} : A \rightarrow TA \). Let us denote by \( \| f_i \| \) the extension of the relation obtained by this query \( q_{A_i} \). Suppose that \( r_{i1}, ..., r_{ik} \in A \) are the relations used for computation of this query, and that the correspondent algebraic term \( \hat{q}_i \) is a function (it is not a T-coalgebra) \( \hat{q}_i : A^k \rightarrow TA \), where \( A^k \) is k-th cartesian product of \( A \). Then, \( \| q_{A_i} \| = \hat{q}_i(r_{i1}, ..., r_{ik}) \).

Differently from this algebra term \( \hat{q}_i \) which is a function, a view-map \( q_{A_i} : A \rightarrow TA \), which is a T-coalgebra, is not a function.

Consequently, an atomic morphism \( f : A \rightarrow B \), from a database \( A \) to database \( B \), is a set of such view-mappings, thus it is not generally a function.

We can introduce two functions, \( \partial_0, \partial_1 : Mor_{DB} \rightarrow \mathcal{P}(T) \) (which are different from standard category functions \( \text{dom}, \text{cod} : Mor_{DB} \rightarrow Ob_{DB} \)), such that for any view-map \( q_{A_i} : A \rightarrow TA \), we have that \( \partial_0(q_{A_i}) = \{ r_{i1}, ..., r_{ik} \} \subseteq A \) is a subset of relations of \( A \) used as arguments by this query \( q_{A_i} \), and \( \partial_1(q_{A_i}) = \{ v \}, v \in TA \) (\( v \) is a resulting view of a query \( q_{A_i} \)). In fact, we have that they are functions \( \partial_0, \partial_1 : Mor_{DB} \rightarrow \mathcal{P}(T) \) (where \( \mathcal{P} \) is a powerset operation), such that for any morphism \( f : A \rightarrow B \) between databases \( A \) and \( B \), which is a set of view-mappings \( q_{A_i} \), such that \( \| q_{A_i} \| \in B \), we have that \( \partial_0(f) \subseteq A \) and \( \partial_1(f) \subseteq TA \cap B \subseteq B \). Thus, we have

\[
\partial_0(f) = \bigcup_{q_{A_i} \in f} \partial_0(q_{A_i}) \subseteq \text{dom}(f) = A, \quad \partial_1(f) = \bigcup_{q_{A_i} \in f} \partial_1(q_{A_i}) \subseteq \text{cod}(f) = B
\]

Based on atomic morphisms (sets of view-mappings) which are complete arrows (c-arrows), we obtain that their composition generates tree-structures, which can be incomplete (p-arrows), in the way that for a composed arrow \( h = g \circ f : A \rightarrow C \), of two atomic arrows \( f : A \rightarrow B \) and \( g : B \rightarrow C \), we can have the situations where \( \partial_0(f) \subsetneq \partial_0(h) \), where the set of relations in \( \partial_0(h) \setminus \partial_0(f) \subsetneq \partial_0(g) \) are denominated "hidden elements".

Definition 1. The following BNF defines the set \( Mor_{DB} \) of all morphisms in DB:

\[
p - \text{arrow} := c - \text{arrow} \mid c - \text{arrow} \circ p - \text{arrow} \quad \text{(for any two c-arrows } f : A \rightarrow B \text{ and } g : B \rightarrow C \text{)}
\]

\[
morphism := p - \text{arrow} \mid c - \text{arrow} \circ p - \text{arrow} \quad \text{(for any p-arrow } f : A \rightarrow B \text{ and c-arrow } g : B \rightarrow C \text{)}
\]
whereby the composition of two arrows, \( f \) (partial) and \( g \) (complete), we obtain the following p-arrow (partial arrow) \( h = g \circ f : A \rightarrow C \)

\[
h = g \circ f = \bigcup_{q_B \in g \& \partial_0(q_B) \cap \partial_1(f) \neq \emptyset} \{q_B\} \circ \bigcup_{q_A \in f \& \partial_1(q_A) = (v) \& v \in \partial_0(q_B)} \{q_A, \text{tree}\}
\]

\[
= \{q_B \circ \{q_A, \text{tree}\} \mid \partial_1(q_A) \subseteq \partial_0(q_B) \} \mid q_B \in g \& \partial_0(q_B) \cap \partial_1(f) \neq \emptyset
\]

\[
= \{q_B, \text{tree} \mid q_B \in g \& \partial_0(q_B) \cap \partial_1(f) \neq \emptyset\}
\]

where \( q_A, \text{tree} \) is the tree of the morphisms \( f \) below \( q_A \).

We define the semantics of mappings by function \( B_T : \text{Mor}_{DB} \rightarrow \text{Ob}_{DB} \), which, given any mapping morphism \( f : A \rightarrow B \) returns with the set of views ("information flux") which are really "transmitted" from the source to the target object.

1. for atomic morphism, \( \text{f} = B_T(f) \triangleq T\{\parallel f_i \parallel \mid f_i \in f\}\).
2. Let \( g : A \rightarrow B \) be a morphism with a flux \( g \), and \( f : B \rightarrow C \) an atomic morphism with flux \( \text{f} \) defined in point 1, then \( \text{f} \circ g = B_T(f \circ g) \triangleq \text{f} \circ g \).

We introduce an equivalence relation over morphisms by, \( f \approx g \iff \text{f = g} \).

Notice that between any two databases \( A \) and \( B \) there is at least an "empty" arrow \( \emptyset : A \rightarrow B \) such that \( \partial_0(\emptyset) = \partial_1(\emptyset) = \emptyset = \{\bot\} = \bot_{\emptyset} \). We have that \( \bot_{\emptyset} \in A \) for any database \( A \) (in \( DB \) all objects are pointed by \( \bot \) databases), so that any arrow \( f : A \rightarrow B \) has a component empty mapping \( \emptyset \) (thus also arrows are pointed by \( \emptyset \)).

Thus we have the following fundamental properties:

**Proposition 1** Any mapping morphism \( f : A \rightarrow B \) is a closed object in \( DB \), i.e., \( f = Tf \), such that \( f \subseteq TA \cap TB \), and

1. each arrow such that \( f = TB \) is an epimorphism \( f : A \rightarrow B \),
2. each arrow such that \( f = TA \) is a monomorphism \( f : A \rightarrow B \),
3. each monic and epic arrow is an isomorphism.

If \( f \) is epic then \( TA \supseteq TB \); if it is monic then \( TA \subseteq TB \). Thus we have an isomorphism of two objects (databases), \( A \simeq B \iff TA = TB \).

We define an ordering \( \preceq \) between databases by \( A \preceq B \iff TA \subseteq TB \).

Thus, for any database \( A \) we have that \( A \simeq TA \), i.e., there is an isomorphic arrow \( \text{is}_A = \{q_{A_v} \mid \partial_0(q_{A_v}) = \partial_1(q_{A_v}) = \{v\} \& v \in A\} : A \rightarrow TA \) and its inverse \( \text{is}_{TA} = \{q_{TA_v} \mid \partial_0(q_{TA_v}) = \partial_1(q_{TA_v}) = \{v\} \& v \in A \subseteq TA\} : TA \rightarrow A \), such that their flux is \( \text{is}_A = \text{is}_{TA} = TA \).

The following duality theorem tells that, for any commutative diagram in \( DB \) there is also the same commutative commutative diagram composed by the equal objects and inverted equivalent arrows: This "bidirectional" mappings property of \( DB \) is a consequence of the fact that the composition of arrows is semantically based on the set-intersection commutativity property for "information fluxes" of its arrows. Thus any limit diagram in \( DB \) has also its "reversed" equivalent colimit diagram with equal objects, any universal property has also its equivalent couniversal property in \( DB \).
Theorem 1 [10] there exists the contravariant functor \( S = (S^0, S^1) : DB \rightarrow DB \) such that

1. \( S^0 \) is the identity function on objects.
2. for any arrow in \( DB \), \( f : A \rightarrow B \) we have \( S^1(f) : B \rightarrow A \), such that \( S^1(f) \triangleq f^{inv} \), where \( f^{inv} \) is (equivalent) reversed morphism of \( f \) (i.e., \( \tilde{f}^{inv} = f \)).
   \[ f^{inv} = is_A^{-1} \circ iTf_{inv} \circ is_B \]
3. The category \( DB \) is equal to its dual category \( DB^{OP} \).

Let us extend the notion of the type operator \( T \) into the notion of the endofunctor in \( DB \) category:

Theorem 2 [10] There exists the endofunctor \( T = (T^0, T^1) : DB \rightarrow DB \), such that

1. for any object \( A \), the object component \( T^0 \) is equal to the type operator \( T \), i.e., \( T^0(A) \triangleq TA \)
2. for any morphism \( f : A \rightarrow B \), the arrow component \( T^1 \) is defined by
   \[ T(f) \triangleq T^1(f) = \bigcup_{\partial_b(qTB_j)=\partial_a(qTA_i)=\{v\} \& v \in \tilde{f}} \{qTB_j : TB \rightarrow TA \} \]
3. Endofunctor \( T \) preserves properties of arrows, i.e., if a morphism \( f \) has a property \( P \) (monic, epic, isomorphic), then also \( T(f) \) has the same property: let \( P_{mono}, P_{epi} \) and \( P_{iso} \) are monomorphic, epimorphic and isomorphic properties respectively, then the following formula is true
   \[ \forall (f \in Mor_{DB})(P_{mono}(f) \equiv P_{mono}(Tf) \& P_{epi}(f) \equiv P_{epi}(Tf) \& P_{iso}(f) \equiv P_{iso}(Tf)) \]

Proof: it can be found in [10]

The endofunctor \( T \) is a right and left adjoint to identity functor \( I_{DB} \), i.e., \( T \simeq I_{DB} \), thus we have for the equivalence adjunction \( < T, I_{DB}, \eta^C, \eta > \) the unit \( \eta^C : T \simeq I_{DB} \) such that for any object \( A \) the arrow \( \eta^C_A \triangleq \eta^C(A) \equiv is_A^1 : TA \rightarrow A \), and the counit \( \eta : I_{DB} \simeq T \) such that for any \( A \) the arrow \( \eta_A \triangleq \eta(A) \equiv is_A : A \rightarrow TA \) are isomorphic arrows in \( DB \) (By duality theorem holds \( \eta^C = \eta^{inv} \)).

The function \( T^1 : (A \rightarrow B) \rightarrow (TA \rightarrow TB) \) is not higher-order function (arrows in \( DB \) are not functions): thus, there is no correspondent monad-comprehension for the monad \( T \), which invalidates the thesis [11] that "monads \( \equiv \) monad-comprehensions". It is only valid that "monad-comprehension \( \Rightarrow \) monads".

We have already seen that the views of some database can be seen as its observable computations: what we need, to obtain an expressive power of computations in the category \( DB \), are categorial computational properties, as known, based on monads:
Proposition 2 The power-view closure 2-endofunctor \( T = (T^0, T^1) : DB \rightarrow DB \) defines the monad \((T, \eta, \mu)\) and the comonad \((T, \eta^C, \mu^C)\) in DB, such that \( \eta : I_{DB} \leq T \) and \( \eta^C : T \simeq I_{DB} \) are natural isomorphisms, while \( \mu : TT \rightarrow T \) and \( \mu^C : T \rightarrow TT \) are equal to the natural identity transformation \( \text{id}_T : T \rightarrow T \) (because \( T = TT \)).

Proof: It is easy to verify that all commutative diagrams of the monad \((\mu_A \circ \mu_{TA} = \mu_A \circ T \mu_A , \mu_A \circ \eta_{TA} = \text{id}_{TA} = \mu_A \circ T \eta_A)\) and the comonad are diagrams composed by identity arrows. Notice that by duality we obtain \( \eta_{TA} = T \eta_A = \mu_A^{^{1_B}} \).

\( \square \)

3 Categorial symmetry and behavioral equivalence

Let us now consider the problem of how to define equivalent (categorically isomorphic) objects (database instances) from a behavioral point of view based on observations: as we see, each arrow (morphism) is composed by a number of "queries" (view-maps), and each query may be seen as an observation over some database instance (object of \( DB \)). Thus, we can characterize each object in \( DB \) (a database instance) by its behavior according to a given set of observations. Indeed, if one object \( A \) is considered as a black-box, the object \( TA \) is only the set of all observations on \( A \). So, given two objects \( A \) and \( B \), we are able to define the relation of equivalence between them based on the notion of the bisimulation relation. If the observations (resulting views of queries) of \( A \) and \( B \) are always equal, independent of their particular internal structure, then they look equivalent to an observer.

In fact, any database can be seen as a system with a number of internal states that can be observed by using query operators (i.e., programs without side-effects). Thus, databases \( A \) and \( B \) are equivalent (bisimilar) if they have the same set of observations, i.e. when \( TA \) is equal to \( TB \):

Definition 2. The relation of (strong) behavioral equivalence \(' \approx '\) between objects (databases) in \( DB \) is defined by

\[ A \approx B \iff TA = TB \]

the equivalence relation for morphisms is given by, \( f \approx g \iff \bar{f} = \bar{g} \).

This relation of behavioral equivalence between objects corresponds to the notion of isomorphism in the category \( DB \) (see Proposition 1).

This introduced equivalence relation for arrows \( \approx \), may be given by an (interpretation) function \( B_T : Mor_{DB} \rightarrow Ob_{DB} \) (see Definition 1), such that \( \approx \) is equal to the kernel of \( B_T \), \( \approx = \ker B_T \), i.e., this is a fundamental concept for categorial symmetry [12]:

Definition 3. CATEGORIAL SYMMETRY:
Let \( C \) be a category with an equivalence relation \( \approx \subseteq Mor_C \times Mor_C \) for its arrows (equivalence relation for objects is the isomorphism \( \simeq \subseteq Ob_C \times Ob_C \)) such that there exists a bijection between equivalence classes of \( \approx \) and \( \simeq \), so that it is
possible to define a skeletal category $|C|$ whose objects are defined by the imagine of a function $B_T : \text{Mor}_C \rightarrow \text{Ob}_C$ with the kernel $\ker B_T = \approx$, and to define an associative composition operator for objects $\ast$, for any fitted pair $g \circ f$ of arrows, by $B_T(g) \ast B_T(f) = B_T(g \circ f)$.

For any arrow in $C$, $f : A \rightarrow B$, the object $B_T(f)$ in $C$, denoted by $\tilde{f}$, is denominated as a conceptualized object.

Remark: This symmetry property allows us to consider all the properties of an arrow (up to the equivalence) as properties of objects and their composition as well. Notice that any two arrows are equal if and only if they are equivalent and have the same source and the target objects.

We have that in symmetric categories holds that $f \cong g$ iff $\tilde{f} \cong \tilde{g}$.

Let us introduce, for a category $C$ and its arrow category $C \downarrow C$, an encapsulation operator $J : \text{Mor}_C \rightarrow \text{Ob}_C \downarrow C$, that is, a one-to-one function such that for any arrow $f : A \rightarrow B$, $J(f) = \langle A, B, f \rangle$ is its correspondent object in $C \downarrow C$, with its inverse $\psi$ such that $\psi(\langle A, B, f \rangle) = f$.

We denote by $F_{st}, S_{nd} : (C \downarrow C) \rightarrow C$ the first and the second comma functorial projections (for any functor $F : C \rightarrow D$ between categories $C$ and $D$, we denote by $F^0$ and $F^1$ its object and arrow component), such that for any arrow $(k_1; k_2) : \langle A, B, f \rangle \rightarrow \langle A', B', g \rangle$ in $C \downarrow C$ (such that $k_2 \circ f = g \circ k_1$ in $C$), we have that $F^0_{st}(\langle A, B, f \rangle) = A, F^1_{st}(k_1; k_2) = k_1$ and $S^0_{nd}(\langle A, B, f \rangle) = B, S^1_{nd}(k_1; k_2) = k_2$.

We denote by $\triangle : C \rightarrow (C \downarrow C)$ the diagonal functor, such that for any object $A$ in a category $C$, $\triangle(A) = \langle A, A, \text{id}_A \rangle$.

An important subset of symmetric categories are Conceptually Closed and Extended symmetric categories, as follows:

Definition 4. Conceptually closed category is a symmetric category $C$ with a functor $T_e = (T^0_e, T^1_e) : (C \downarrow C) \rightarrow C$ such that $T^0_e = B_T \psi$, i.e., $B_T = T^0_e J$, with a natural isomorphism $\varphi : T_e \circ \triangle \cong I_C$, where $I_C$ is an identity functor for $C$.

$C$ is an extended symmetric category if holds also $\tau^{-1} \circ \tau = \psi$, for vertical composition of natural transformations $\tau : F_{st} \rightarrow T_e$ and $\tau^{-1} : T_e \rightarrow S_{nd}$.

Remark: it is easy to verify that in conceptually closed categories, it holds that any arrow $f$ is equivalent to an identity arrow, that is, $f \cong \text{id}_{\tilde{f}}$.

It is easy to verify also that in extended symmetric categories the following holds:

$$\tau = (T^1_e(\tau_1 F^0_{st}; \psi)) \ast (\varphi^{-1} F^0_{st}), \quad \tau^{-1} = (\varphi^{-1} S^0_{nd}) \ast (T^1_e(\psi; \tau_1 S^0_{nd})), $$

where $\tau_1 : I_C \rightarrow I_C$ is an identity natural transformation (for any object $A$ in $C$, $\tau_1(A) = \text{id}_A$).

Example: The Set is an extended symmetric category: given any function $f : A \rightarrow B$, the conceptualized object of this function is the graph of this function (which is a set), $\tilde{f} = B_T(f) = \{(x, f(x)) \mid x \in A\}$.

The equivalence $\cong$ on morphisms (arrows) is defined by: two arrows $f$ and $g$ are equivalent, $f \cong g$, iff they have the same graph.

The composition of objects $\ast$ is defined as associative composition of binary relations (graphs), $B_T(g \circ f) = \{(x, (g \circ f)(x)) \mid x \in A\} = \{(y, g(y)) \mid y \in B\} \cup \{(x, f(x)) \mid x \in A\} = B_T(g) \ast B_T(f)$. 


Remark: For any object $J(f) = \langle A, B, f \rangle$, the component $T^0_e(J(f))$ is defined by:
for any $(x, f(x)) \in T^0_e(J(f))$, $T^0_e(k_1; k_2)(x, f(x)) = (k_1(x), k_2(f(x)))$.
It is easy to verify the compositional property for $T^0_e$, and that $T^0_e(id_A; id_B) = id_{T^0_e(J(f))}$.
For example, $Set$ is also an extended symmetric category, such that for any object $J(f) = \langle A, B, f \rangle > Set \downarrow Set$, we have that $\tau(J(f)) : A \to B_T(f)$ is an epimorphism, such that for any $x \in A$, $\tau(J(f))(x) = (x, f(x))$, while $\tau^{-1}(J(f)) : B_T(f) \to B$ is a monomorphism such that for any $(x, f(x)) \in B_T(f)$, $\tau^{-1}(J(f))(x, f(x)) = f(x)$.
Thus, each arrow in $Set$ is a composition of an epimorphism and a monomorphism.

Now we are ready to present a formal definition for the $DB$ category:

**Theorem 3** The category $DB$ is an extended symmetric category, closed by the functor $T_e = (T^0_e, T^1_e) : (C \downarrow C) \to C$, where $T^0_e = B_T\psi$ is the object component of this functor such that for any arrow $f$ in $DB$, $T^0_e(J(f)) = \widetilde{f}$, while its arrow component $T^1_e$ is defined as follows: for any arrow $(h_1; h_2) : J(f) \to J(g)$ in $DB \downarrow DB$, such that $g \circ h_1 = h_2 \circ f$ in $DB$, holds

$$T^1_e(h_1; h_2) = \bigcup_{\partial_0(q_{\widetilde{f}}) = \partial_1(q_{\widetilde{f}}) = \{v\} \& v \in h_2 \circ f} \{q_{\widetilde{f}}\}$$

The associativity composition operator for objects $\ast$, defined for any fitted pair $g \circ f$ of arrows, is the set intersection operator $\bigcap$.
Thus, $B_T(g) * B_T(f) = \widetilde{g} \cap \widetilde{f} = \widetilde{g \circ f} = B_T(g \circ f)$.

**Proof:** Each object $A$ has its identity (point-to-point) morphism $id_A = \bigcup_{\partial_0(q_A) = \partial_1(q_A) = \{v\} \& v \in A} \{q_A\}$ and holds the associativity $h \circ (g \circ f) = \widetilde{h} \cap (g \circ f)$.
They have the same source and target object, thus $h \circ (g \circ f) = (h \circ g) \circ f$. Thus, $DB$ is a category. It is easy to verify that also $T_e$ is a well defined functor. In fact, for any identity arrow $(id_A; id_B) : J(f) \to J(f)$ it holds that $T^1_e(id_A; id_B) = \bigcup_{\partial_0(q_{\widetilde{f}}) = \partial_1(q_{\widetilde{f}}) = \{v\} \& v \in id \circ f} \{q_{\widetilde{f}}\} = id_{\widetilde{f}}$ is the identity arrow of $\widetilde{f}$. For any two arrows $(h_1; h_2) : J(f) \to J(g)$, $(l_1; l_2) : J(g) \to J(k)$, it holds that $T^1_e(h_1; h_2) \circ T^1_e(l_1; l_2) = T^1_e(h_1; h_2 \circ l_1; l_2) = T^1_e(l_2 \circ g) \cap T^1_e(h_2 \circ f) = \widetilde{l_2} \cap \widetilde{g} \cap \widetilde{h_2} \cap \widetilde{f} = \widetilde{l_2 \circ g \circ h_2} \cap \widetilde{f} = \widetilde{l_2 \circ g} \cap \widetilde{h_2} \cap \widetilde{f} = \widetilde{l_2 \circ g \circ h_2} \cap \widetilde{f} = \widetilde{l_2 \circ g} \cap \widetilde{h_2} \cap \widetilde{f} = \widetilde{l_2 \circ g \circ h_2} \cap \widetilde{f} = \widetilde{l_2 \circ h_2} \circ \widetilde{f} = \widetilde{T_e^1(l_1; l_2) \circ T_e^1(h_1; h_2)} = \widetilde{T_e^1(h_1; h_2 \circ l_1; l_2)} = \widetilde{T_e^1(l_2 \circ g \circ l_1; l_2 \circ h_1)}$, for any identity arrow, it holds that $id_A$, $T^0_e(J(id_A)) = id_{\widetilde{A}} = TA \simeq A$ as well, thus, an isomorphism $\varphi : T_e \circ \star \simeq I_{DB}$ is valid.

**Remark:** It is easy to verify (from $\tau^{-1} \bullet \tau = \psi$) that for any given morphism $f : A \to B$ in $DB$, the arrow $f_{\text{ep}} = \tau(J(f)) : A \to \widetilde{f}$ is an epimorphism, and the arrow $f_{\text{in}} = \tau^{-1}(J(f)) : \widetilde{f} \to B$ is a monomorphism, so that any morphism $f$ in
DB is a composition of an epimorphism and monomorphism \( f = f_{\text{in}} \circ f_{\text{ep}} \), with the intermediate object equal to its "information flux" \( \tilde{f} \), and with \( f \approx f_{\text{in}} \approx f_{\text{ep}} \).

4 Databases: Universal algebra and monads

The notion of a monad is one of the most general mathematical notions. For instance, every algebraic theory, that is, every set of operations satisfying equational laws, can be seen as a monad (which is also a monoid in a category of endofunctors of a given category: the "operation" \( \mu \) being the associative multiplication of this monoid and \( \eta \) its unit). Thus monoid laws of the monad do subsume all possible algebraic laws.

In order to explore universal algebra properties [9,8] for the category \( DB \), where, generally, morphisms are not functions (this fact complicates a definition of mappings from its morphisms into homomorphisms of the category of \( \Sigma_R \)-algebras), we will use an equivalent to \( DB \) "functional" category, denoted by \( DB_{sk} \), such that its arrows can be seen as total functions.

**Proposition 3** Let us denote by \( DB_{sk} \) the full skeletal subcategory of \( DB \), composed by closed objects only.

Such a category is equivalent to the category \( DB \), i.e., there exists an adjunction of a surjective functor \( T_{sk} : DB \to DB_{sk} \) and an inclusion functor \( In_{sk} : DB_{sk} \to DB \), where \( In_{sk}^0 \) and \( In_{sk}^1 \) are two identity functions, such that \( T_{sk} In_{sk} = Id_{DB_{sk}} \) and \( In_{sk} T_{sk} \simeq Id_{DB} \).

There exists the faithful forgetful functor \( F_{sk} : DB_{sk} \to \text{Set} \), and \( F_{DB} = F_{sk} \circ T_{sk} : DB \to \text{Set} \), thus \( DB_{sk} \) and \( DB \) are concrete categories.

**Proof:** It can be found in [8]. The skeletal category \( DB_{sk} \) has closed objects only, so, for any mapping \( f : A \to B \), we obtain the arrow \( f_T = T_{sk}^0(f) : TA \to TB \) can be expressed in a following "total" form such that \( \partial_0(f_T) = T_{sk}^0(A) = TA \),

\[
 f_T \triangleq \bigcup_{\partial_0(q_T(A_i))=\{v\} \land v \in \tilde{f}} \{q_T(A_i)\} \bigcup_{\partial_0(q_T(A_i))=\{v\} \land v \notin \tilde{f} \land \partial_1(q_T(A_i))=l^0} \{q_T(A_i)\}
\]

so that \( f_R = F_{sk}^1(f_T) : TA \to TB \) (the component for objects \( F_{sk}^1 \) is an identity) is a function in \( \text{Set} \), \( f_R = F_{DB}^1(f) \), such that for any \( v \in TA \), \( f_R(v) = v \) if \( v \in \tilde{f} \); \( \perp \) otherwise.

\( \square \)

In a given inductive definition one defines a value of a function (in our example the endofunctor \( T \)) on all (algebraic) constructors (relational operators). What follows is based on the fundamental results of the Universal algebra [13].

Let \( \Sigma_R \) be a finitary signature (in the usual algebraic sense: a collection \( F_{\Sigma} \) of function symbols together with a function \( ar : F_{\Sigma} \to N \) giving the finite arity of each function symbol) for a single-sorted (sort of relations) relational algebra.

We can speak of \( \Sigma_R \)-equations and their satisfaction in a \( \Sigma_R \)-algebra, obtaining the notion of a \( (\Sigma_R, E) \)-algebra theory. In a special case, when \( E \) is empty, we obtain a purely syntax version of Universal algebra, where \( K \) is a category of all \( \Sigma_R \)-algebras, and the quotient-term algebras are simply term algebras.
An algebra for the algebraic theory (type) \((\Sigma_R, E)\) is given by a set \(X\), called the carrier of the algebra, together with interpretations for each of the function symbols in \(\Sigma_R\). A function symbol \(f \in \Sigma_R\) of arity \(k\) must be interpreted by a function \(\hat{f}_X : X^k \rightarrow X\). Given this, a term containing \(n\) distinct variables gives rise to a function \(X^n \rightarrow X\) defined by induction on the structure of the term. An algebra must also satisfy the equations given in \(E\) in the sense that equal terms give rise to identical functions (with obvious adjustments where the equated terms do not contain exactly the same variables). A homomorphism of algebras from an algebra \(X\) to an algebra \(Y\) is given by a function \(g : X \rightarrow Y\) which commutes with operations of the algebra \(g(\hat{f}_X(a_1, \ldots, a_k)) = \hat{f}_Y(g(a_1), \ldots, g(a_k))\). This generates a variety category \(K\) of all relational algebras. Consequently, there is a bifunctor \(E : DB_{sk}^O \times K \rightarrow Set\) (where \(Set\) is the category of sets), such that for any database instance \(A\) in \(DB_{sk}\) there exists the functor \(E(A, _\_ ) : K \rightarrow Set\) with an universal element \((U(A), g)\), where \(g \in E(A, U(A))\), \(g : A \rightarrow U(A)\) is an inclusion function and \(U(A)\) is a free algebra over \(A\) (quotient-term algebra generated by a carrier database instance \(A\)), such that for any function \(f \in E(A, X)\) there is a unique homomorphism \(h\) from the free algebra \(U(A)\) into an algebra \(X\), with \(f = E(A, h) \circ g\).

From the so called “parameter theorem” we obtain that there exists:

- a unique universal functor \(U : DB_{sk} \rightarrow K\) such that for any given database instance \(A\) in \(DB_{sk}\) it returns with the free \(\Sigma_R\)-algebra \(U(A)\) (which is a quotient-term algebra, where a carrier is a set of equivalence classes of closed terms of a well defined formulae of a relational algebra, “constructed” by \(\Sigma_R\)-constructors (relational operators: select, project, join and union SPJRU) and symbols (attributes of relations) of a database instance \(A\), and constants of attribute-domains. An alternative for \(U(A)\) is given by considering \(A\) as a set of variables rather than a set of constants, then we can consider \(U(A)\) as being a set of derived operations of arity \(A\) for this theory. In either case the operations are interpreted syntactically \(\hat{f}([t_1, \ldots, t_k]) = [f(t_1, \ldots, t_k)]\), where, as usual, brackets denote equivalence classes), while, for any “functional” morphism (correspondent to the total function \(F_{sk}(f_T)\) in \(Set\)), \(F_{sk} : DB_{sk} \rightarrow Set\) \(f_T : A \rightarrow B\) in \(DB_{sk}\) we obtain the homomorphism \(f_T = U^1(f_T)\) from the \(\Sigma_R\)-algebra \(U(A)\) into the \(\Sigma_R\)-algebra \(U(B)\), such that for any term \(\rho(a_1, \ldots, a_n) \in U(A)\), \(\rho \in \Sigma_R\), we obtain \(f_T(\rho(a_1, \ldots, a_n)) = f_T(\rho(h(a_1), \ldots, h(a_n)))\), so, \(f_T\) is an identity function for algebraic operators and it is equal to the function \(F_{sk}^1(f_T)\) for constants.
- its adjoint forgetful functor \(F : K \rightarrow DB_{sk}\), such that for any free algebra \(U(A)\) in \(K\) the object \(F \circ U(A)\) in \(DB_{sk}\) is equal to its carrier-set \(A\) (each term \(\rho(a_1, \ldots, a_n) \in U(A)\) is evaluated into a view of this closed object \(A\) in \(DB_{sk}\)) and for each arrow \(T^1(f_T)\) holds that \(F^1U^1(f_T) = f_T\), i.e., we have that \(FU = Id_{DB_{sk}}\) and \(UF = Id_K\).

Consequently, \(U(A)\) is a quotient-term algebra, where carrier is a set of equivalence classes of closed terms of a well defined formulae of a relational algebra, “constructed” by \(\Sigma_R\)-constructors (relational operators in SPJRU algebra: select, project, join and union) and symbols (attributes of relations) of a database instance \(A\), and constants of attribute-domains.
It is immediate from the universal property that the map \( A \mapsto U(A) \) extends to the endofunctor \( F \circ U : DB_{sk} \to DB_{sk} \). This functor carries monad structure \((F \circ U, \eta, \mu)\) with \( F \circ U \) an equivalent version of \( T \) but for this skeletal database category \( DB_{sk} \). The natural transformation \( \eta \) is given by the obvious “inclusion” of \( A \) into \( F \circ U(A) : a \to [a] \) (each view \( a \) in an closed object \( A \) is an equivalence class of all algebra terms which produce this view). Notice that the natural transformation \( \eta \) is the unit of this adjunction of \( U \) and \( F \), and that it corresponds to an inclusion function in \( Set \), \( \iota : A \to U(A) \), given above. The interpretation of \( \mu \) is almost equally simple.

A typical element of \((F \circ U)^2(A)\) is an equivalence class of terms built up from elements of \( F \circ U(A) \), so that instead of \( t(x_1, \ldots, x_k) \), a typical element of \((F \circ U)^2(A)\) is given by the equivalence class of a term \( t([t_1], \ldots, [t_k]) \). The transformation \( \mu \) is defined by map \( [t([t_1], \ldots, [t_k])] \to [t(t_1, \ldots, t_k)] \). This make sense because a substitution of provably equal expressions into the same term results in provably equal terms.

5 Database mappings and monadic coalgebras

We will use monads \([5,14,15]\) for giving denotational semantics to database mappings, and more specifically as a way of modeling computational/collection types \([1,2,16,17]\): to interpret a database mappings (morphisms) in the category \( DB \), we distinguish the object \( A \) (database instance of type \( A \)) from the object \( TA \) of observations (computations of type \( A \) without side-effects), and take as a denotation of (view) mappings the elements of \( TA \) (which are view of (type) \( A \)). In particular, we identify the type \( A \) with the object of values (of type \( A \)) and obtain the object of observations by applying the unary type-constructor \( T \) (power-view operator) to \( A \).

It is well known that each endofunctor defines algebras and coalgebras (the left and right commutative diagrams)

\[
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
\downarrow h & & \downarrow k \\
A & \xrightarrow{f} & B
\end{array}
\quad \quad \quad
\begin{array}{ccc}
TA & \xrightarrow{Tf_1} & TB \\
\downarrow h_1 & & \downarrow k_1 \\
A & \xrightarrow{f_1} & B
\end{array}
\]

We will use the following well-known definitions in the category theory (the set of all arrows in a category \( \mathcal{M} \) from \( A \) to \( B \) is denoted by \( \mathcal{M}(A,B) \)):

**Definition 5.** The categories \( CT_{alg} \) of \( T \)-algebras, \( CT_{coalg} \) of \( T \)-coalgebras, derived from an endofunctor \( T \), are defined \([13]\) as follows:

1. the objects of \( CT_{alg} \) are pairs \((A,h)\) with \( A \in Ob_{DB} \) and \( h \in DB(TA,A) ; \) the arrows between objects \((A,h)\) and \((B,k)\) are all arrows \( f \in DB(A,B) \) such that \( k \circ Tf = f \circ h : TA \to B \).
2. the objects of \( CT_{coalg} \) are pairs \((A,h)\) with \( A \in Ob_{DB} \) and \( h \in DB(TA,A) ; \) the arrows between objects \((A,h)\) and \((B,k)\) are all arrows \( f \in DB(A,B) \) such that \( Tf \circ h = k \circ f : A \to TB \).
Definition 6. The monadic algebras/coalgebras, derived from a monad \((T, \eta, \mu)\), are defined \([3, 5]\) as follows:

- Each \(T\)-algebra \((A, h : TA \rightarrow A)\), where \(h\) is a "structure map", such that holds \(h \circ \mu_A = h \circ TH\) and \(h \circ \eta_A = \text{id}_A\) is a monadic \(T\)-algebra. The category of all monadic algebras \(T_{\text{alg}}\) is a full subcategory of \(CT_{\text{alg}}\).
- Each \(T\)-coalgebra \((A, k : A \rightarrow TA)\), such that holds \(Tk \circ k = \mu_C^T \circ k\) and \(\eta_A^C \circ k = \text{id}_A\) is a monadic \(T\)-coalgebra. The category of all monadic coalgebras \(T_{\text{coalg}}\) is a full subcategory of \(CT_{\text{coalg}}\).

Each \(T\)-coalgebra \((A, k : A \rightarrow TA)\), where \(k\) is a morphism from \(A\) to \(TA\). It holds \(Tk \circ k = \mu_C^T \circ k\) and \(\eta_A^C \circ k = \text{id}_A\).

Note: The monad \((T, \eta, \mu)\) given by commutative diagrams:

\[
\begin{array}{ccc}
T^3A & \xrightarrow{T\mu_A} & T^2A \\
\downarrow{\mu_A} & & \downarrow{\mu_A} \\
T^2A & \xrightarrow{\mu_A} & TA
\end{array}
\quad \begin{array}{ccc}
TA & \xrightarrow{\eta_T} & T^2A \\
\downarrow{id_A} & & \downarrow{\mu_A} \\
TA & \xrightarrow{\mu_A} & TA
\end{array}
\]

defines the adjunction \(< F^T, G^T, \eta^T, \mu^T >: DB \rightarrow T_{\text{alg}}\) such that \(G^T \circ F^T = T : DB \rightarrow DB\), \(\eta^T = \eta\), \(\epsilon^T = \eta^{inv}\) and \(\mu = G^T \epsilon^T F^T\). The functors \(F^T : DB \rightarrow T_{\text{alg}}\) and \(G^T : T_{\text{alg}} \rightarrow DB\) are defined as follows: for any object (database) \(A\), \(F^T(A) = (A, \eta_A^{inv} : TA \simeq A)\), while \(F^T(A, \eta_A^{inv} : TA \simeq A) = TA\); for arrows \(F^T\) and \(G^T\) are identity functions.

Definition 7. Given a monad \((T, \eta, \mu)\) over a category \(M\), we have \([5]\):

- **Kleisli triple** is a triple \((T, \eta, -^*)\), where for \(f : A \rightarrow TB\) we have \(f^* : TA \rightarrow TB\), such that the following equations hold: \(\eta_A^* = \text{id}_{TA}\), \(f^* \circ \eta_A = f\), \(g^* \circ f^* = (g \circ f)^*\), for \(f : A \rightarrow TB\) and \(g : B \rightarrow TC\).

A Kleisli triple satisfies the **mono requirement** provided \(\eta_A\) is monic for each object \(A\).

- **Kleisli category** \(M_T\) has the same objects as \(M\) category. For any two objects \(A, B\) there is the bijection between arrows \(\theta : M(A, TB) \rightarrow M_T(A, B)\). For any two arrows \(f : A \rightarrow B\), \(g : B \rightarrow C\) in \(M_T\), their composition is defined by \(g \circ f \triangleq \theta(\mu_C \circ T\theta^{-1}(g) \circ \theta^{-1}(f))\).

The mono requirement for monad \((T, \eta, \mu)\) \([2]\) is satisfied because \(\eta_A : A \rightarrow TA\) is a isomorphism \(\eta_A = \text{id}_A\) (we denote its inverse by \(\eta_A^{-1}\)), thus it is also monic. Consequently, the category \(DB\) is a computational model for view-mappings (which are programs) based on observations (i.e., views) with the typed operator \(T\), so that:

- \(TA\) is a type of computations (i.e. observations of the object of values \(A\) (of type \(A\)), which are the views of the database \(A\))
- \(\eta_A\) is the inclusion of values into computations (i.e., inclusion of elements of the database \(A\) into the set of views of the database \(A\)). It is the isomorphism \(\eta_A = \text{id}_A : A \rightarrow TA\)
- $f^*$ is the **equivalent extension** of a database mapping $f : A \rightarrow TB$ “from values to computations” (programs correspond to call-by-value parameter passing) to a mapping “from computations to computations” (programs correspond to call-by-name), such that holds $f^* = Tf = \mu_B \circ f \circ \eta_A^{-1}$, so $f^* \approx f$.

Thus, in $DB$ category, call-by-value ($f : A \rightarrow TB$) and call-by-name ($f^* : TA \rightarrow TB$) paradigms of programs are represented by **equivalent** morphisms, $f \approx f^*$. Notice that in skeletal category $DB_{sk}$ (which is equivalent to $DB$) all morphisms correspond to the call-by-name paradigm, because each arrow is a mapping from computations into computations (which are closed objects).

The basic idea behind the semantic of programs [11] is that a program denotes a morphism from computations into computations (which are closed objects). That is, the Kleisli category is a model for database mappings up to the equivalence $\approx$.

It means that, generally, database mappings are not simply programs from values into computations. In fact, the semantics of a database mapping, between any two objects $A$ and $B$, is equal to tell that for some set of computations (i.e., query-mappings) over $A$ we have the same equivalent (in the sense that these programs produce the same computed value (view)) set of computations (query-mappings) over $B$: it is fundamentally an equivalence of computations. This is a consequence of the fact that each database mapping (which is not a function) from $A$ into $B$ is naturally bidirectional, i.e, it is a morphism $f : A \rightarrow B$ and its equivalent reversed morphism $f^{inv} : B \rightarrow A$ together (explained by the duality property $DB = DB^{op}$ [6]). Let us define this equivalence formally:

**Definition 8.** Each database mapping $h : A \rightarrow B$ is an equivalence of programs (epimorphisms), $h_A \triangleq \tau(J(h)) : A \rightarrow TH$ and $h_B \triangleq \tau^{-1}(J(h))^{inv} : B \rightarrow TH$ ($\tau$ and $\tau^{-1}$ are natural transformations of a categorial symmetry), where $H$ generates a closed object $h$ (i.e., $TH = \hat{h}$) and $h_A \approx h \approx h_B$, such that computations of these two programs (arrows of Kleisli category $DB_T$) are equal, i.e., $\delta_1(h_A) = \delta_1(h_B)$.

We can also give an alternative model for equivalent computational extensions of database mappings in $DB$ category:

**Proposition 4** Denotational semantics of each mapping $f$, between any two database instances $A$ and $B$, is given by the unique equivalent “computation” arrow $f_1 \triangleq \eta_B \circ f$ in $TC_{coalg}$ from the monadic $T$-coalgebra $(A, \eta_A)$ into a cofree monadic $T$-coalgebra.
Note that each view-map (query) \( q_A : A \to TA \) is just equal to its denotational semantics arrow in \( T_{conty} \), \( q_A : (A, \eta_A) \to (TA, \mu_A^C) \).

It is well known that for a Kleisli category there exists an adjunction \( < F_T, G_T, \eta_T, \mu_T > \) such that we obtain the same monad \( (T, \eta, \mu) \), such that \( T = G_T F_T, \mu = G_T \varepsilon_T F_T, \eta = \eta_T \). Let us see now how the Kleisli category \( DB_T \) is "internalized" into the \( DB \) category.

**Proposition 5** The Kleisli category \( DB_T \) of the monad \( (T, \eta, \mu) \) is isomorphic to \( DB \) category, i.e., it may be "internalized" in \( DB \) by the faithful forgetful functor \( K = (K^0, K^1) : DB_T \to DB \), such that \( K^0 \) is an identity function and \( K^1 \equiv \phi \theta^{-1} \), where, for any two objects \( A \) and \( B \),

\[
\theta : DB(A, TB) \simeq DB_T(A, B) \quad \text{is Kleisli and}
\]

\[
\phi : DB(A, TB) \simeq DB(A, B) \quad \text{such that } \phi(\_\_) = \eta_{\text{inv}} \circ \_\_ \text{is DB category bijection respectively.}
\]

We can generalize a "representation" for the base DB category (instead of usual Set category): a "representation" of functor \( K \) is a pair \( < T, \varphi > \), \( \varphi \) is the total object and \( \varphi : DB_T(Y, _) \simeq K \) is a natural isomorphism, where the functor \( DB_T(Y, _) : DB_T \to DB \) defines "internalized" hom-sets in \( DB_T \), i.e., \( DB_0^0(Y, B) \equiv TB^T \).

**Proof:** Let prove that \( \phi \) is really bijection in \( DB \). For any program morphism \( f : A \to TB \) we obtain \( \phi(f) = \eta_{\text{inv}} \circ f : A \to B \) and, vice versa, for any \( g : A \to B \) its inverse \( \phi^{-1}(g) \equiv \eta_{\text{inv}} \circ g \), and, for any \( f : A \to B \) its inner \( \phi(f) = \eta_{\text{inv}} \circ f \), thus, \( \phi \circ \phi^{-1}(g) = \phi \circ (\eta_{\text{inv}} \circ g) = (\eta_{\text{inv}} \circ \eta_{\text{inv}}) \circ g = \eta_{\text{inv}} \circ g \). Also \( \phi^{-1}(\phi(f)) = \phi^{-1}(\eta_{\text{inv}} \circ f) = \eta_{\text{inv}} \circ (\eta_{\text{inv}} \circ f) = (\eta_{\text{inv}} \circ \eta_{\text{inv}}) \circ f = id_B \circ f = f \), i.e., \( \phi^{-1} \) is an identity function, \( \phi \) is an identity function, thus \( \phi \) is a bijection.

Let us demonstrate that \( K \) is a functor: For any identity arrow \( id_T = \theta(\eta_A) : A \to A \) in \( DB_T \) we obtain \( K^1(id_T) = \phi \theta^{-1}(\theta(\eta_A)) = \phi(\eta_A) = \eta_{\text{inv}} \circ \eta_A = id_A \) (because \( \eta_A \) is an isomorphism). For any two arrows \( g_T : B \to C \) and \( f_T : A \to B \) in Kleisli category, we obtain, \( K^1(g_T \circ f_T) = K^1(\theta(\mu_C \circ T \theta^{-1}(g_T) \circ T \theta^{-1}(f_T))) \) (from def. Kleisli category) = \( \phi \theta^{-1}(\theta(\mu_C \circ T g_T \circ f_T)) \) (where \( g \equiv \theta^{-1}(g_T) : B \to TC \) and \( f \equiv \theta^{-1}(f_T) : A \to TB \) = \( \phi \circ \eta_{\text{inv}} \circ f \) (easy to verify in \( DB \) that \( \mu_C \circ T g_T \circ f = g \circ \eta_{\text{inv}} \circ f \) = \( \eta_{\text{inv}} \circ \eta_{\text{inv}} \circ f = \phi(g) \circ \phi(f) = \phi \theta^{-1}(\theta(g)) \circ \phi \theta^{-1}(\theta(f)) = K^1(\theta \theta^{-1}(g_T)) \circ K^1(\theta \theta^{-1}(f_T)) = K^1(g_T) \circ K^1(f_T).

Thus, each arrow \( f_T : A \to B \) in \( DB_T \) is "internalized" in \( DB \) by its representation \( f \equiv K^1(f_T) = \phi \theta^{-1}(f_T) = \eta_{\text{inv}} \circ \theta^{-1}(f_T) : A \to B \), where \( \theta^{-1}(f_T) : A \to TB \) is a program equivalent to the database mapping \( f : A \to B \), i.e., \( \theta^{-1}(f_T) \approx f \).

K is faithful functor, in fact, for any two arrows \( f_T, h_T : A \to B \) in \( DB_T \), \( K^1(f_T) = K^1(h_T) \) implies \( f_T = h_T \):

from \( K^1(f_T) = K^1(h_T) \) we obtain \( \phi \theta^{-1}(f_T) = \phi \theta^{-1}(h_T) \), if we apply a bijection that
\phi^{-1} \eta^{-1}(f_T) = \phi^{-1}(\phi^{-1}(h_T)) \text{, i.e., } \theta^{-1}(f_T) = \theta^{-1}(h_T) \text{, i.e., } f_T = h_T \text{ (} \theta^{-1} \text{ and } \phi^{-1} \text{ are identity functions)}.

Let prove that \( K \) is an isomorphism: from the adjunction \( < F_T, G_T, \eta_T, \mu_T > : DB \rightarrow DB_T \), where \( F_T^0 \) is identity, \( F_T^{-1} \equiv \theta^{-1} \), we obtain that \( F_T \circ K = I_{DB_T} \), and \( K \circ F_T = I_{DB} \), thus, the functor \( K \) is an isomorphism of \( DB \) and Kleisli category \( DB_T \).

\( \square \)

Remark: It is easy to verify that a natural isomorphism \( \eta : I_{DB} \rightarrow T \) of the monad \( (T, \eta, \mu) \) is equal to the natural transformation \( \eta : K \rightarrow G_T \). (consider that \( G_T : DB_T \rightarrow DB \) is defined by, \( G_T^0 = T^0 \) and for any \( f_T : A \rightarrow B \) in \( DB_T \), \( G_T^1(f_T) \equiv \mu_B \circ \eta^{-1}(f_T) : TA \rightarrow TB \).

Thus, the functor \( F_T \) has two different adjunctions: the universal adjunction \( < F_T, G_T, \eta_T, \mu_T > \) which gives the same monad \( (T, \eta, \mu) \), and this particular (for \( DB \) category only) isomorphism’s adjunction \( < F_T, K, \eta_T, \mu_T > \) which gives banal identity monad.

We are now ready to define the semantics of queries in \( DB \) category and the categorial definition of \textit{query equivalence}. This is important in the context of the Database integration/exchange and for the theory of \textit{query-rewriting} [19].

When we define a mapping (arrow, morphism) \( f : A \rightarrow B \) between two databases \( A \) and \( B \), implicitly we define the "information flux" \( \tilde{f} \), i.e, the set of views of \( A \) "transmitted" by this mapping into \( B \). Thus, in the context of query-rewriting we consider only queries (i.e., view-maps) which resulting view (observation) belongs to the "information flux" of this mapping. Consequently, given any two queries, \( q_{A_i} : A \rightarrow TA \) and \( q_{B_j} : B \rightarrow TB \), they have to satisfy (w.r.t. query rewriting constraints) the condition \( \partial_1(q_{A_i}) \in \tilde{f} \) (the \( \partial_1(q_{A_i}) \) is just a resulting view of this query) and \( \partial_1(q_{B_j}) \in \tilde{f} \).

So, the well-rewritten query over \( B \), \( q_{B_j} : B \rightarrow TB \), such that it is equivalent to the original query, i.e., \( q_{B_j} \approx q_{A_i} \), must satisfy the condition \( \partial_1(q_{B_j}) = \partial_1(q_{A_i}) \in \tilde{f} \).

Now we can give the denotational semantics for a query-rewriting in a data integration/exchange environment:

**Proposition 6** Each database query is a (non monadic) \( T \)-coalgebra. Any morphism between two \( T \)-coalgebras \( f : (A, q_{A_i}) \rightarrow (B, q_{B_j}) \) defines the semantics for relevant query-rewriting, when \( \partial_1(q_{A_i}) \in \tilde{f} \).

**Proof:** Consider the following commutative diagram, where vertical arrows are \( T \)-coalgebras,

\[
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
\downarrow q_{A_i} & & \downarrow q_{B_j} \\
A & \xrightarrow{f} & B
\end{array}
\]

The morphism between two \( T \)-coalgebras \( f : (A, q_{A_i}) \rightarrow (B, q_{B_j}) \) means that holds the commutativity \( q_{B_j} \circ f = Tf \circ q_{A_i} : A \rightarrow TB \), and from duality property we obtain that \( q_{B_j} = Tf \circ q_{A_i} \circ f^{ inv} \). Consequently, we have that for a given mapping \( f : A \rightarrow B \) between databases \( A \) and \( B \), every query \( q_{A_i} \) such that \( \partial_1(q_{A_i}) \in \tilde{f} \) (i.e.,
versal functor

We have seen (from Universal algebra considerations) that there exists the unique uni-

6.1 Algebras and induction

We have seen that \( q_{B_j} \approx q_{A_i} \).

\[ \widetilde{q}_{A_i} \subseteq \widetilde{f} \]

we can have an equivalent rewritten query \( q_{B_j} \) over a data base \( B \). In fact,
we have that \( q_{B_j} = \widetilde{Tf} \cap \widetilde{q}_{A_i} \cap \widetilde{f}^{-1} \) because of the fact that \( \widetilde{q}_{A_i} \subseteq \widetilde{f} \) and
\[ \widetilde{f}^{-1} = \widetilde{Tf} = f. \]

Thus \( q_{B_j} \approx q_{A_i} \).

\[ \square \]

6 (Co)Algebras and (Co)Induction

Let us consider the following properties for monadic algebras/coalgebras in \( DB \):

**Proposition 7** The following properties for the monad \( (T, \eta, \mu) \) and the comonad
\( (T, \eta^C, \mu^C) \) hold:

- The categories \( CT_{alg} \) and \( CT_{coalg} \) of the endofunctor \( T : DB \to DB \), are
isomorphic \( (CT_{coalg} = CT_{alg}^p) \), complete and cocomplete. The object \( (\bot^0, \text{id}_{\bot^0} : \bot^0 \to \bot^0) \) is an initial T-algebra in \( CT_{alg} \) and a terminal T-coalgebra in
\( CT_{coalg} \).
- For each object \( A \) in \( DB \) category there exist the unique monadic T-algebra \( (A, \eta_A^C : TA \to A) \) and the unique comonadic T-coalgebra \( (A, \eta_A : A \to TA) \),
\[ \eta_A^C = \eta_A^C (i.e., \eta_A^C \approx \eta_A \approx \text{id}_A). \]
- The free monadic T-algebra \( (TA, \mu_A : T^2A \to TA) \) is dual (and equal) to
the cofree monadic T-coalgebra \( (TA, \mu_A^C : TA \to T^2A) \), \[ \mu_A^C = \mu_A^C (i.e., \mu_A^C = \mu_A = \text{id}_{TA}). \]
- The Kleisli triple over the category \( DB \) satisfies the mono requirement.

**Proof:** Let define the functor \( F : T_{alg} \to T_{coalg} \), such that for any T-algebra \( (A, h : TA \to A) \) we obtain the dual T-coalgebra \( F^0(A, h) = (A, h^{-1} : A \to TA) \),
with a component \( F^1 \) for arrows an identity function; and the functor \( F : T_{coalg} \to
T_{alg} \), such that for any T-coalgebra \( (A, k : A \to TA) \) we obtain the dual T-algebra
\( G^0(A, k) = (A, k^{-1} : TA \to A) \), with a component \( G^1 \) for arrows an identity function. Thus holds \( FG = I_{T_{coalg}} \) and \( GF = I_{T_{alg}} \). \( T_{alg} \) and \( T_{coalg} \) are complete
and cocomplete as the base \( DB \) category \( (T_{coalg} = T_{alg}^p) \).

The rest is easy to verify: each monadic T-algebra/coalgebra is an isomorphism. The
free monadic T-algebra and the cofree monadic T-coalgebra are equal because \( TA = T^2A \),
thus, \( \mu_A, \mu_A^C \) are identity arrows (by duality theorem).

\[ \square \]

As we can see, each monadic T-coalgebra is an equivalent reversed arrow in \( DB \) of
some monadic T-algebra, and vice versa: the fundamental duality property of \( DB \)
introduces the equivalence of monadic T-algebras and monadic T-coalgebras, thus the
equivalence of the dichotomy "construction versus observation" or duality between
induction and coinduction principles \[ 20 \]

6.1 Algebras and induction

We have seen (from Universal algebra considerations) that there exists the unique uni-

versal functor \( U : DB_{sk} \to K \) such that for any given database instance \( A \) in \( DB_{sk} \)
returns with the free $\Sigma_R$-algebra $U(A)$.

Its adjoint is the forgetful functor $F : K \to DB_{sk}$, such that for any free algebra $U(A)$ in $K$ the object $F \circ U(A)$ in $DB_{sk}$ is equal to its carrier-set $A$ (each term $\rho(a_1, \ldots, a_n) \in U(A)$ is evaluated into some view of this closed object $A$ in $DB_{sk}$).

It is immediate from the universal property that the map $A \mapsto U(A)$ extends to the endofunctor $F \circ U : DB_{sk} \to DB_{sk}$. This functor carries monad structure: the natural transformation $\eta$ is given by the obvious "inclusion" of $A$ into $F \circ U(A) : a \mapsto [a]$.

**Finitariness:** In a locally finitely presentable (lfp) category every object can be given as the directed (or filtered) colimit of the finitely presentable (fp) objects. Hence, if the action of a monad preserves this particular kind of colimits, its action on any object will be determined by its action on the fp objects; such a monad is called finitary.

Let verify that the power-view closure 2-endofunctor $T : DB \to DB$ is just a composition of functors described above and that it is finitary.

**Proposition 8** The power-view closure 2-endofunctor $T : DB \to DB$ is immediate from the universal property of composed adjunction $< UT_{sk}, In_{sk}F, In_{sk}\eta_U T_{sk} \cdot \eta_{sk}, \varepsilon_U \cdot U \varepsilon_{sk} F > : DB \to K$, i.e., $T = In_{sk}FUT_{sk} \simeq Id_{DB}$. It is finitary.

The category $DB$ is equivalent to the (Eilenberg-Moore) category $T_{alg}$ of all monadic $T$-algebras and is equivalent to the category $T_{coalg}$ of all monadic $T$-coalgebras.

Its equivalent skeletal category $DB_{sk}$ is, instead, isomorphic to $T_{alg}$ and $T_{coalg}$.

**Proof:** For any object $A$ in $DB$ holds $In_{sk}FUT_{sk}(A) = In_{sk}T_{sk}(A) = TA$, and for any morphism $f : A \to B$ in $DB$ holds $In_{sk}FUT_{sk}(f) = In_{sk}K_{sk}(f) = In_{sk}(f_T) = Tf$ (where from Proposition $\Xi$ $f_T = T_{sk}(f)$, and $f_T = T_f = f$).

The adjunction equivalence $< T_{sk}, In_{sk}, \eta_{sk}, \varepsilon_{sk} >$ between $DB$ and $DB_{sk}$ and the adjunction-isomorphism $< U, F, \eta_U, \varepsilon_U > DB_{sk} \simeq K$, give the composed adjunction $< UT_{sk}, In_{sk}F, In_{sk}\eta_U T_{sk} \cdot \eta_{sk}, \varepsilon_U \cdot U \varepsilon_{sk} F > : DB \to K$, which is an equivalence.

We have that $K \simeq DB_{sk}$, and, from universal algebra (Back's theorem) theory, $K \simeq T_{alg}$, thus $DB_{sk} \simeq T_{alg}$. From this fact and the fact that $DB$ is equivalent to $DB_{sk}$ we obtain that $DB$ is equivalent to $T_{alg}$. The property for $T_{coalg}$ holds by duality.

To understand the finitary condition, consider the term algebra $U(A)$ over infinite database (infinite set of relations) $A$. Since every operation $\rho \in \Sigma_R$ can only take finitely many arguments, every term $t \in U(A)$ can only contain finitely many variables from $A$; and hence, instead of building the term algebra over the infinite database $A$, we can also build the term algebras over all finite subsets (of relations) $A_0$ of $A$ and take union of these: $U(A) = \bigcup\{U(A_0) \mid A_0 \subseteq A\}$. This result comes from Universal algebra because the closure operator $T$ is algebraic and $< C, \subseteq >$, where $C$ is a set of all closed objects in $DB$, is an algebraic (complete+compact) lattice.

\[ \square \]

The notion of $T$-algebra subsumes the notion of a $\Sigma_R$-algebra ($\Sigma_R$-algebras can be understood as algebras in which operators (of the signature) are not subject to any law, i.e., with empty set of equations). In particular, the monad $T$ freely generated by a signature $\Sigma_R$ is such that $T_{alg}$ is isomorphic to the category of $\Sigma_R$-algebras. Therefore, the syntax of a programming language can be identified with monad, the syntactical monad $T$ freely generated by the program constructors $\Sigma_R$.

We illustrate the link between a single-sorted (sort is a relation) $\Sigma_R$ algebra signature of
a relational algebra operators and the T-algebras of the endofunctor T. The assumption that the signature $\Sigma_R$ is finite is not essential for the correspondence between models of $\Sigma_R$ and algebras of T. If $\Sigma_R$ is infinite one can define T via an infinite coproduct, commonly written as

$$\Sigma_R(A) = \bigcup_{\sigma \in \Sigma_R} A^{ar(\sigma)}.$$  

which is a more compact way of describing the category of $\Sigma_R$-algebras is by taking this coproduct in $Set$ category (disjoint union) $\bigcup_{\sigma \in \Sigma_R} A^{ar(\sigma)}$, $1 \leq ar(\sigma_i) \leq N$, $i = 1, 2, \ldots, n$, where the set $A^m$ is the m-fold product $A \times A \times \cdots \times A$; that is, the disjoint union of domains of the operations $\sigma \in \Sigma_R$ of this ”select-project-join +union” language (SPJRU language [21]). More formally, for the signature $\Sigma_R$ we define the endofunctor $\Sigma_R : Set \to Set$, such that for any object $B$, $\Sigma_R(B) = \bigcup_{\sigma \in \Sigma_R} B^{ar(\sigma)}$, and any arrow in $Set$ (a function) $f : B \to C$, $\Sigma_R(f) \triangleq \bigcup_{\sigma \in \Sigma_R} f^{ar(\sigma)}$.

Thus, also for any object $A$ in $Set$ we have the endofunctor $\Sigma_{RA} : Set \to Set$, such that for any object $B$ in $Set$ holds $\Sigma_{RA}(B) = (\Sigma_R + A)(B) = A + \Sigma_R B \triangleq A + \bigcup_{\sigma \in \Sigma_R} B^{ar(\sigma)}$, and any arrow $f : B \to C$, $\Sigma_{RA}(f) \triangleq id_A + \bigcup_{\sigma \in \Sigma_R} f^{ar(\sigma)}$.

Let $\omega$ be the category of natural numbers with arrows $\leq$: $j \to k$ which correspond to the total order relation $j \leq k$, i.e., $\omega = \{0 \to 1 \to 2 \to \ldots\}$. An endofunctor $H : C \to D$ is $\omega$-cocontinuous if preserves the colimits of functors $J : \omega \to C$, that is when $HColimJ \simeq ColimHJ$ (the categories $C$ and $D$ are thus supposed to have these colimits). Notice that a functor $J : \omega \to C$ is a diagram in $C$ of the form $\{C_0 \to C_1 \to C_2 \to \ldots\}$. For $\omega$-cocontinuous endofunctors the construction of the initial algebra is inductive [22].

We define an iterable endofunctor $H$ of a category $D$ if for every object $X$ of $D$ the endofunctor $H(\_)+X$ has an initial algebra. It is well known that the signature endofunctor $\Sigma_R$ in $Set$ category is $\omega$-cocontinuous and iterable.

The initial algebra for a given set of terms with variables in $A$, $\mathcal{T}A$, of the endofunctor $\Sigma_{RA} = A + \Sigma_R : Set \to Set$ comes with an induction principle, and since it is the coproduct $A + \Sigma_R \mathcal{T}A$, we can rephrase the principle as follows: For every $\Sigma_R$-algebra structure $h : \Sigma_R B \to B$ and every mapping $f : A \to B$ there exists a unique arrow $f\# : \mathcal{T}A \to B$ such that the following diagram in $Set$

$$\begin{array}{ccc}
A & \xrightarrow{inl_A} & \mathcal{T}A \\
\downarrow f\# & & \downarrow \Sigma_Rf\# \\
B & \xleftrightarrow{h} & \Sigma_RB
\end{array}$$

commutes, where $f\# = [f, h \circ \Sigma_Rf\#]$ is the unique inductive extension of $h$ along the mapping $f$.

The arrow $inl_A : A \hookrightarrow \mathcal{T}A$ is an inclusion of variables in $A$ into terms with variables $\mathcal{T}A$. Formally, $r_i \in A$ is an element of a set $A$ of relations, and only after applying $inl_A$ tom it that one obtain a variable. The arrow $inr_A : \Sigma_R\mathcal{T}A \to \mathcal{T}A$ is an injection which permits to construct a new term given any n-ary algebraic operator $\sigma \in \Sigma_R$ and
terms \( t_1, \ldots, t_n \) in \( TA \). Also the right injection is usually left implicitly and one writes simply \( \sigma(t_1, \ldots, t_n) \) for the resulting term.

Notice that \( f_\#: \langle TA, \text{inl}_A, \text{inr}_A \rangle \rightarrow \langle B, [f, h] \rangle \) is the unique arrow from

From Lambek’s theorem, this initial \( A + \Sigma_R \)-algebra (that is, the free \( \Sigma_R \) algebra with

**Inductive principle in the \( DB \) category:**

Let us consider any chain in \( \Sigma_{D} \alpha \), for all \( \alpha \in \omega \). Thus, the endofunctor \( \Sigma_{\omega} \) preserves colimits because it is monotone and \( \Sigma_{\omega}^n = TA \) is its

The endofunctor \( \Sigma_{\omega}^n = TA \) is the initial \( A + \Sigma_R \)-algebra (that is, the free \( \Sigma_R \) algebra with

**Proposition 9** For each object \( A \) in the category \( DB \) the "merging with \( A \)" endofunctor \( \Sigma_A = A \oplus \_ : DB \rightarrow DB \), and the endofunctor \( A + TA : DB \rightarrow DB \) are \( \omega \)-cocontinuous.

**Proof:** Let us consider any chain in \( DB \) (all arrows are monomorphisms, i.e., "\( \preceq \)" in a correspondent chain of the \( \langle Ob_{DB}, \preceq \rangle \) algebraic lattice), is a following diagram

\[
\begin{align*}
\downarrow 0 & \preceq_0 (\Sigma_A \downarrow 0) \preceq_1 (\Sigma_A \downarrow 0) \preceq_2 \ldots \preceq_\omega (\Sigma_A \downarrow 0),
\end{align*}
\]

where \( \downarrow 0 \) is the initial object in \( DB \), with unique monic arrow \( \downarrow = \Sigma_{0} \downarrow 0 \rightarrow \Sigma_{0} \downarrow 0 \) with

\[
\Sigma_{0} \downarrow = TA,
\]

for all \( \alpha \geq 1 \), as representation of a functor (diagram) \( J : \omega \rightarrow DB \).

The colimit \( \text{Colim} J = \Sigma_{\omega} A \) of the base diagram \( D \) given by the functor \( J : \omega \rightarrow DB \),
is equal to $Colim J = (A \oplus \bigoplus \omega \downarrow \top_0 = TA$. Thus $\Sigma_A Colim J = T(A \bigcup Colim J) = T(A \bigcup TA) = TA = Colim \Sigma_A J$ (where $Colim \Sigma_A J$ is a colimit of the diagram $\Sigma_A J$).

The $\omega - \text{cocompleteness}$ amounts to chain-completeness, i.e., to the existence of least upper bound of $\omega - \text{chains}$. Thus $\Sigma_A$ is $\omega - \text{cocontinuous}$ endofunctor: a monotone function which preserves lubs of $\omega - \text{chains}$.

Constant endofunctor $A : DB \rightarrow DB$ is $\omega$-cocontinuous endofunctor, identity endofunctors are $\omega$-cocontinuous, colimit functors (thus coproduct $\oplus$) are $\omega$-cocontinuous (because of the standard "interchange of colimits"). Since $\omega$-cocontinuity is preserved by functor composition $\circ$, then for the second endofunctor $A + T_\omega = (A + Id_\omega) \circ T\omega$ it is enough to show that $T\omega$ is $\omega$-cocontinuous endofunctor. In fact consider the following diagram obtained by iterative application of the endofunctor $T\omega$

$$\downarrow_0 \preceq_0 (T \downarrow_0) \preceq_1 (T^2 \downarrow_0 \preceq_2 \ldots T^\omega \downarrow_0),$$

where $\downarrow_0$ is the initial object in $DB$, and all objects $T^n \downarrow_0 = \downarrow_0 = 1^0$, so that all arrows in this chain are identities. Thus we obtain that $Colim J = (T \omega \downarrow_0 = \downarrow_0 = 1^0$, and holds $TColim J = Colim T\omega J = \downarrow_0$, so that $T$ is $\omega$-cocontinuous endofunctor.

In what follows we will make the translation of inductive principle from $Set$ into $DB$ category, based on the following considerations:

- The object $A$ in $Set$ is considered as set of variables (for relations in a database instance $A$) while in $DB$ this object is considered as set of relations. Analogously, the set of terms with variables in $A$, $TA$, used in $Set$ category, is translated into set $TA$ of all views (which are relations obtained by computation of these terms with variables in $A$).

But it is not a carrier set for the initial $(A + \Sigma_D)$-algebra for $\Sigma_D = T$ (see below), just because generally $TA$ is not isomorphic to $A + \Sigma_D(TA) = A + TA$ (in fact $T(TA) = TA \neq T(A + TA) = TA + TA$).

- Cartesian product $\times : Set \rightarrow Set$ is translated into matching operation (tensor product) $\otimes : DB \rightarrow DB$.

This translation is based on observations that any n-ary algebraic operator $\sigma \in \Sigma_R$, is represented as a function (arrow) $\sigma : TA^n \rightarrow TA$ which use as domain the n-fold cartesian product $TA \times \ldots \times TA$, while such an operator in $DB$ category is represented by view-based mapping $f_\sigma = \{q_\sigma^r \mid \partial_0(q_\sigma^r) = \{r_1, \ldots, r_in\}, \partial_1(q_\sigma^r) = \{\sigma(r_1, \ldots, r_in)\} \text{ for each tuple } (r_1, \ldots, r_in) \in A^n\} : TA \rightarrow TA$.

Thus this algebraic operator $\sigma$ is translated into an arrow from $TA$ to $TA$. In fact if we replace $\times$ by $\otimes$ in n-fold $TA \times \ldots \times TA$, we obtain $TA \otimes \ldots \otimes TA = T(TA) \bigcap \ldots \bigcap T(TA) = T(TA) = TA$.

- Any disjoint union $X + \_ : Set \rightarrow Set$ used for construction of $\Sigma_R$ endofunctor is translated into "merging with $X"$ endofunctor $X \oplus : DB \rightarrow DB$.

From the fact that coproduct $\oplus$ is replaced by merging operator $\oplus$, we obtain that the object $\Sigma_R(X) = \bigsqcup_{\sigma \in \Sigma_R} X^{ar(\sigma)}$ in $Set$ is translated by the object $\Sigma_D(X) = \oplus_{\sigma \in \Sigma_R} (X \otimes \ldots \otimes X) = \oplus_{\sigma \in \Sigma_R} TX = TX$, where the endofunctor $\Sigma_D = T :$
$DB_f \rightarrow DB$ is the translation for the relational-algebra signature endofunctor $\Sigma_R : \text{Set} \rightarrow \text{Set}$.

It is well known [5] that for any monoidal category $\mathcal{A}$ with a monoidal product $\otimes$, any two functors $F_1 : \text{Pop} \rightarrow \mathcal{A}$ and $F_2 : \mathcal{P} \rightarrow \mathcal{A}$ have a tensor functorial product $F_1 \otimes F_2 = \int^{p \in \mathcal{P}} (F_1 p) \otimes (F_2 p)$.

In our case we take for $\mathcal{A}$ the lfp enriched (co)complete category $DB$ with monoidal product correspondent to matching database operation $\otimes$, $F_2 = \Sigma_D : DB_f \rightarrow DB$ and for $F_1$ the hom functor for a given database $A$ (object in $DB$), $DB(\_A) \circ K : DB_f \rightarrow DB$, where $K : DB_f \rightarrow DB_f$ is an inclusion functor. Notice that for any finite database, i.e., object $B \in DB_f$, the $DB_f(K(B), A)$ is a hom-object $A^K(B)$ of enriched database subcategory $DB_f$. In this context we obtain that for any object (also infinite) $A$ in $DB_f$ (that is, in $DB$), we have a tensor product $DB_f(\_A) \circ K \otimes_{\mathcal{P}} \Sigma_D = \int^{B \in DB_f} DB_f(K(B), A) \otimes \Sigma_DB = \int^{B \in DB_f} DB_f(B, A) \otimes \Sigma_DB$.

This tensorial product comes with a dinatural transformation $\beta : S \rightarrow A$, where $S = DB_f(\_A) \otimes \Sigma_DB : DB_f \times DB_f \rightarrow DB$ and $A$ is a constant functor between the same categories of the functor $S$. Thus, for any given object $A$ in $DB$ we have a collection of arrows $\beta_B : DB_f(B, A) \otimes \Sigma_DB \rightarrow A$ (for every object $B \in DB_f$).

In the case of standard case of $\text{Set}$, which is (co)complete lfp with monoidal product $\otimes$ equal to cartesian product $\times$, we have that such arrows are $\beta_B : \text{Set}(B, A) \times \Sigma_R(B) \rightarrow A$, where $B$ is a finite set with cardinality $n = |B|$, so that $\text{Set}(B, A)$ is a set of all tuples of arity $n$ composed by elements of the set $A$, while $\Sigma_R(B)$ here is interpreted as a set of all basic $n$-ary algebra operations. So that $\beta_B$ is a specification for all basic algebra operations with arity $n$, and is a function such that for any $n$-ary operation $\sigma \in \Sigma_R(B)$ and a tuple $< a_1, ..., a_n > \in A^n \simeq \text{Set}(B, A)$, (where $\simeq$ is an isomorphism in $\text{Set}$), returns with result $\beta_B(< a_1, ..., a_n >, \sigma) = \sigma(a_1, ..., a_n) \in A$.

In the non standard case, when instead of base category $\text{Set}$ is used another lfp enriched (co)complete category, as $DB$ category in our case, the interpretation for this tensorial product and dinatural transformation $\beta$ is obviously very different, as we will see in what follows.

From considerations explained previously we obtain that the finitary signature functor $\Sigma_D : DB_f \rightarrow DB$ has a left Kan extension $\text{Lan}_K(\Sigma_D) : DB_f \rightarrow DB$ and left Kan extension $\text{Lan}_{J \circ K}(\Sigma_D) : DB \rightarrow DB$ for inclusion functor $J : DB_f \rightarrow DB$ (this second extension is direct consequence of the first one, because $J$ does not introduce extension for objects, differently from $K$, from the fact that $DB_f$ and $DB$ have the same objects). Thus it is enough to analyze only the first left Kan extension given by the following commutative diagram:

$$
\begin{array}{ccc}
DB_f & \xrightarrow{K} & DB_f \\
\downarrow{J} & & \downarrow{\text{Lan}_K(\Sigma_D)} \\
DB & & DB
\end{array}
$$

That is, we have the functor $\text{Lan}_K : DB^{DB_f} \rightarrow DB^{DB_f}$ is left adjoint to the functor $\_ \circ K : DB^{DB_f} \rightarrow DB^{DB_f}$, so that left Kan extension of $\Sigma_D \in DB^{DB_f}$ along $K$
is given by functor $\text{Lan}_K(\Sigma_D) \in DB_{DA}$, and a natural transformation $\varepsilon : \Sigma_D \to \text{Lan}_K(\Sigma_D) \circ K$ is an universal arrow. That is, for any other functor $S : DB_1 \to DB$ and a natural transformation $\alpha : \Sigma_D \to S \circ K$, there is a unique natural transformation $\beta : \text{Lan}_K(\Sigma_D) \to S$ such that $\alpha = \beta K \circ \varepsilon$ (where $\circ$ is a vertical composition for natural transformations).

From the well known theorem for left Kan extension, when we have a tensorial product $\int [B \in DB_1] DB_1(K(B), A) \otimes \Sigma DB$ for every $A \in DB_1$, i.e., $A \in DB$, then the function for objects of the functor $\text{Lan}_K(\Sigma_D)$ is defined by (here $B \preceq_{\omega} A$ means that $B \in DB_1$ & $B \preceq A$):

$\text{Lan}_K(\Sigma_D)(A) = \text{def} \int [B \in DB_1] DB_1(K(B), A) \otimes \Sigma DB$

$= \int [B \preceq_{\omega} A] DB_1(B, A) \otimes \Sigma DB$ (from $K(D) = D$)

$= \int [B \preceq_{\omega} A] DB_1(B, A) \otimes \Sigma DB + \int [B \in DB_1 \& B \preceq_{\omega} A] DB_1(B, A) \otimes \Sigma DB$

$= \int [B \preceq_{\omega} A] DB_1(B, A) \otimes \Sigma DB + \top^0$ (hom object $DB_1(B, A)$ is an empty database $\top^0$ (zero object in $DB$) if there is no (monic) arrow from $B$ to $A$)

$\simeq \int [B \preceq_{\omega} A] DB_1(B, A) \otimes \Sigma DB$ (*)

$= \int [B \preceq_{\omega} A] \text{in}_B \otimes \Sigma DB$, where $\text{in}_B : B \hookrightarrow A$ is unique monic arrow into $A$

$= \int [B \preceq_{\omega} A] TB \otimes \Sigma DB$

$= \int [B \preceq_{\omega} A] TB \otimes TB$ (for finite $B$, $\Sigma DB = T(B)$)

$= \int [B \preceq_{\omega} A] TB = \bigsqcup_{B \preceq_{\omega} A} TB$

$= \{ T(B) \mid B \preceq_{\omega} A \}$ (lub of compact elements of directed set $\{ B \mid B \preceq_{\omega} A \}$)

$= T(A)$ (from the fact that the poset $DB_1$ is a complete algebraic lattice $[\mathbb{B}]$ $(DB_1, \preceq)$ with meet and join operators $\otimes$ and $\oplus$ respectively, and with compact elements $TB$ for each finite database $B$).

Consequently, we obtain that $\text{Lan}_K(\Sigma_D)$, the extension of $\Sigma_D$ to all (also infinite non-closed) objects in $DB$, is equal (up to isomorphism) to endofunctor $T$. That is, formally we obtain:

**Corollary 1** The following strong connection between the relational-algebra signature endofunctor $\Sigma_D$ translated into the database category $DB$ and the closure endofunctor $T$ hold:

$T = \Sigma_D$.

**Remark:** Let us consider now which kind of interpretation can be given to the tensor product (see (*) above): $\int [B \in DB_1] DB_1(B, A) \otimes \Sigma DB \simeq \int [B \preceq_{\omega} A] DB_1(B, A) \otimes \Sigma DB$ and its $B$-components (for $B \preceq_{\omega} A$, this is $B \subseteq TB \subseteq TA$), $DB_1(B, A) \otimes \Sigma DB$ in the enriched lfp database category $DB$:

The second component $\Sigma_D$ cannot be set of signature operators, just because an object in $DB$ can not be set of functions, and it is not interesting in this interpretation: in fact it can be omitted from $B$-component, because it is equal to $TB$ which is the lub of the first component, i.e., hom object $DB_1(B, A) = DB_1(B, TA) = \text{in}_B$, for inclusion arrow $\text{in}_B : B \hookrightarrow TA$.

But for the case when $B \subseteq TA$ we have for $f = \text{in}_B \bigsqcup f_B$, where $f_B = \{ f_{\sigma} : B \to TA \mid \partial_B(f_{\sigma}) = B \text{ and } \partial_B(f_{\sigma}) = \{ \sigma(B) \} \}$ for each permutation $B$ of relations in $B$ and each operation $\sigma \in \Sigma$ with $\text{ar}(\sigma) = |B|$, with $f_B \subseteq TB$,
that $DB_1(B, TA) = \tilde{f} = T(\tilde{in}_B \cup \tilde{f}_B) = \tilde{in}_B \oplus \tilde{f}_B = TB \oplus \tilde{f}_B$.

We can enlarge the source object for $f_B$ to object $TA$ (because $B \subseteq \omega TA$), in order to obtain an equivalent mapping $f_B : TA \to TA$ and to obtain a representation of tensor products by signature operations and signature view-based mappings $f_B : TA \to TA$, differently from mappings (i.e. functions) $\sigma : TA^{ar(\sigma)} \to TA$ in the standard case when we use $Set$ category, as the following interpretation:

\[
\sum B \in DB_1(B, A) \otimes \Sigma_D B
\]

\[
= \sum B \subseteq TA \oplus \tilde{f}_B
\]

As we can see the translation of the relational-algebra signature $\Sigma_R$ is given by the power-view endofunctor $\Sigma_D = T : DB \to DB$, as informally presented in introduction.

Consequently, the endofunctor $(A + \Sigma_D) : DB \to DB$, from Proposition 9 is the $\omega$-cocontinuous endofunctor $(A + \Sigma_D) : DB \to DB$, with a chain

\[
\bot^0 \preceq_0 \cdots \preceq_2 \cdots \preceq_1 ((A + \Sigma_D)^2 \bot^0) \preceq_2 \cdots \preceq_1 ((A + \Sigma_D)^2 \bot^0) \preceq_2 \cdots \preceq_1 ((A + \Sigma_D)^2 \bot^0) \preceq_2 \cdots \preceq_1 ((A + \Sigma_D)^2 \bot^0),
\]

where $(A + \Sigma_D) \bot^0 = \bot^0 + A, (A + \Sigma_D)^2 \bot^0 = (A + \Sigma_D)(\bot^0 + A) = \bot^0 + A + TA$, and we obtain that the colimit of this diagram in $DB$ is $(A + \Sigma_D)^\omega = \bot^0 + A + \bigcup \Sigma_D TA$.

From the fact that for coproduct (and initial object $\bot^0$) holds that $\bot^0 + B \simeq B$ for any $B$, then we can take as the colimit $(A + \Sigma_D)^\omega = A + \bigcup \Sigma_D TA$. This colimit is a least fixpoint of the monotone operator $(A + \Sigma_D)$ in a complete lattice of databases in $DB$ (Knaster-Tarski theorem).

Notice that the coproduct of two databases $A$ and $B$ in $DB$ category corresponds to completely disjoint databases, in the way that it is not possible to use relations from these two databases in the same query: because of that we have that $T(A + B) = TA + TB$, that is the set of all views of a coproduct $A + B$ is a disjoint union of views of $A$ and views of $B$.

In fact we have that $(A + \Sigma_D)((A + \Sigma_D)^\omega) = A + T(A + \bigcup \Sigma_D TA) = A + TA + T\bigcup \Sigma_D TA = A + TA + \bigcup \Sigma_D TA = A + TA + \bigcup \Sigma_D TA = A + \bigcup \Sigma_D TA = (A + \Sigma_D)^\omega$.

We can denote this identity arrow in $DB$ category, which is the initial $(A + \Sigma_D)$-algebra, by $[inl_A, inr_A] : (A + \Sigma_D)(A + \bigcup \Sigma_D TA) \to (A + \bigcup \Sigma_D TA)$.

Consequently, the variable injection $inl_A : A \hookrightarrow TA$ in $Set$ is translated into a monomorphism $inl_A : A \hookrightarrow (A + \bigcup \Sigma_D TA)$ in $DB$ category, with information flux $\tilde{inl}_A = TA$. The right inclusion $inr_A : \Sigma_R TA \hookrightarrow TA$ in $Set$ is translated into an isomorphism (which is a monomorphism also) $inr_A : \Sigma_D(A + \bigcup \Sigma_D TA) \simeq (A + \bigcup \Sigma_D TA)$ in $DB$ category, based on the fact that $\Sigma_D(A + \bigcup \Sigma_D TA) = T(A + \bigcup \Sigma_D TA) = TA + T\bigcup \Sigma_D TA = TA + \bigcup \Sigma_D TA \simeq A + \bigcup \Sigma_D TA$.

So that $\tilde{inr}_A = TA + \bigcup \Sigma_D TA$ with $TA \subseteq \tilde{inr}_A$.

Moreover, by this translation, any $\Sigma_R$ algebra $h : \Sigma_R B \to B$ in $Set$ is translated into an isomorphism $h_D : \Sigma_D B \to B$ with $h_D = TB$.

Consequently, the initial algebra for a given database $A$, with a set of view in $TA$, of the $\omega$-cocontinuous endofunctor $(A + \Sigma_D) : DB \to DB$ comes with an induction principle, which we can rephrase the principle as follows: For every $\Sigma_D$-algebra structure
$h_D : \Sigma_D B \to B$ (which must be an isomorphism) and every mapping $f : A \to B$ there exists a unique arrow $f_\# : TA \to B$ such that the following diagram in $DB$

$$
\begin{array}{cccc}
A & \xleftarrow{\text{inl}_A} & A + \coprod_{\omega} TA & \xrightarrow{\text{inr}_A} \Sigma_D(A + \coprod_{\omega} TA) \\
\downarrow{f_\#} & & \downarrow{\Sigma_D f_\#} & \\
B & & \xleftarrow{h_D} \Sigma_D B
\end{array}
$$

commutes, where $\Sigma_D = T$ and $f_\# = [f, h_D \circ \Sigma_D f_\#]$ is the unique inductive extension of $h_D$ along the mapping $f$.

It is easy to verify that it holds. From the fact that $f = f_\# \circ \text{inl}_A$ we have that $\widehat{f} = \widehat{f_\#} \cap \text{inl}_A = \widehat{f_\#} \cap TA = \widehat{f_\#} \subseteq \text{inr}_A$. So there is the unique arrow $f_\#$ that satisfies this condition for a given arrow $f$. From the fact that $\Sigma_D = T$, we obtain that $\Sigma_D f_\# = Tf_\# = T\widehat{f_\#} = \widehat{f}$ because the information fluxes are closed object w.r.t. power-view operator $T$. Consequently, we have that $h_D \cap \Sigma_D f_\# = TB \cap \Sigma_D f_\# = \Sigma_D f_\# = f_\# \cap TA = f_\# \cap \text{inr}_A$, so that holds the commutativity $h_D \circ \Sigma_D f_\# = f_\# \circ \text{inr}_A$.

The diagram above can be equivalently represented by the following unique morphism between initial $(A + \Sigma_D)$-algebra and any other $(A + \Sigma_D)$-algebra:

$$
\begin{array}{cccc}
A + \coprod_{\omega} TA & \xrightarrow{[\text{inl}_A, \text{inr}_A]} & (A + \Sigma_D)(A + \coprod_{\omega} TA) \\
\downarrow{f_\#} & & \downarrow{(A + \Sigma_D)f_\#} & \\
B & & \xrightarrow{[f, h_D]} (A + \Sigma_D)B
\end{array}
$$

Thus we obtain the following Corollary:

**Corollary 2** For each object $A$ in $DB$ category there is the initial $\Sigma_A$-algebra, $\langle A + \coprod_{\omega} TA, [\text{inl}_A, \text{inr}_A] \rangle$, where $\text{inl}_A : A \leftarrow (A + \coprod_{\omega} TA)$ is a monomorphism, while $\text{inr}_A : \Sigma_D(A + \coprod_{\omega} TA) \leftarrow (A + \coprod_{\omega} TA)$ is an isomorphism.

This inductive principle can be used to show that the closure operator $T$ inductively extends to the endofunctor $T : DB \to DB$. Indeed, to define its action $Tf$ on arrow $f : A \to B$, take the inductive extension of $\text{inr}_B : \Sigma_D(B + \coprod_{\omega} TB) \to (B + \coprod_{\omega} TB)$ (of the $(B + \Sigma_D) : DB \to DB$ endofunctor with initial $(B + \Sigma_R)$-algebra structure $[\text{inl}_B, \text{inr}_B] : (B + \Sigma_D)(B + \coprod_{\omega} TB) \to (B + \coprod_{\omega} TB)$) along the composite $\text{inl}_B \circ f$, i.e., $(\text{inl}_B \circ f)_\# = [\text{inl}_B \circ f, \text{inr}_B \circ \Sigma_D(f + \coprod_{\omega} T\hat{f})]$. The
Let us define now a relational algebra signature, and let $x, y, z$ be attribute variables. Coalgebras are suitable mathematical formalizations of reactive systems and their behavior, like to our case when we are considering databases from the query-answering, observation based approach.

Example: Let a database $A$ contain two relations, $r_P$ (of a predicate $P$ with 4 attributes), and $r_Q$ (of the predicate $Q$ with 5 attributes), such that 4-th attribute of $r_P$ and 3-th attribute of $r_Q$ are of the same domain. Let define the mapping at logical level from $A$ to $B$, which contains the relation $r_R$ (of a predicate $R$ with two attributes) by the conjunctive query $R(x, y) \leftarrow P(a, x, z) \land Q(b, y, z)$, (which by completion is an equivalence, $R(x, y) \leftrightarrow P(a, x, z) \land Q(b, y, z)$) where ’a’, ’b’, are constants of a domain and $x, y, z$ are attribute variables.

Let us define now a relational algebra signature, $\Sigma$, with sorts correspondent to tuples.

6.2 A coalgebraic view: corecursion and infinite trees

Coalgebras are suitable mathematical formalizations of reactive systems and their behavior, like to our case when we are considering databases from the query-answering, that is, view-based approach.

This subsection presents an application of corecursion, that is, construction method using final coalgebras [25]. In order to better understand the rest lets give an example for a coalgebraic point of view of a database mappings.

Example: Let a database $A$ contain two relations, $r_P$ (of a predicate $P$ with 4 attributes), and $r_Q$ (of the predicate $Q$ with 5 attributes), such that 4-th attribute of $r_P$ and 3-th attribute of $r_Q$ are of the same domain. Let define the mapping at logical level from $A$ to $B$, which contains the relation $r_R$ (of a predicate $R$ with two attributes) by the conjunctive query $R(x, y) \leftarrow P(a, x, z) \land Q(b, y, z)$, (which by completion is an equivalence, $R(x, y) \leftrightarrow P(a, x, z) \land Q(b, y, z)$) where ’a’, ’b’, are constants of a domain and $x, y, z$ are attribute variables.

Let us define now a relational algebra signature, $\Sigma$, with sorts correspondent to tuples.

\[
\begin{align*}
A & \xleftarrow{\text{inl}_A} A + \coprod_{\omega} TA \xleftarrow{\text{inr}_A} \Sigma_D(A + \coprod_{\omega} TA) \\
B & \xleftarrow{\text{inl}_B} B + \coprod_{\omega} TB \xleftarrow{\text{inr}_B} \Sigma_D(B + \coprod_{\omega} TB)
\end{align*}
\]

commutes, thus $(f + \coprod_{\omega} Tf) \circ \text{inl}_A = \text{inl}_B \circ f$, so, $f + \coprod_{\omega} Tf \cap \text{inl}_A = \tilde{Tf} \cap TA = \tilde{Tf} = \text{inl}_B \cap \tilde{f} = TB \cap \tilde{f} = \tilde{f}$, i.e., $\tilde{Tf} = \tilde{f}$ as originally defined for the endofunctor $T$ ([6]). That $T$ is an endofunctor is easy to verify from the left commutative diagram where the objects $A + \coprod_{\omega} TA$ can be represented as results of the composed endofunctor $E = (I_{DB} + \coprod_{\omega} \circ T) : DB \rightarrow DB$, where $I_{DB}$ is the identity endofunctor for $DB$, while the endofunctor $\coprod_{\omega} : DB \rightarrow DB$ is a $\omega$ coproduct.

It is easy to verify that $\text{inl}_A = \eta_A : A \hookrightarrow EA$, where $EA = A + \coprod_{\omega} TA$, is obtained from the natural transformation $\eta : I_{DB} \longrightarrow E$. Another example is the definition of the operation $\mu_A : E^2 A \longrightarrow EA$ inductively extending $\text{inr}_A : \Sigma_D EA \longrightarrow EA$ along the identity $id_{EA}$ of the object $EA$ (consider the first diagram, substituting $A$ and $B$ with the object $EA = A + \coprod_{\omega} TA$, $f$ with $id_{EA}$ and $h_D$ with $\text{inr}_A$). Inductively derived $\eta_A$ (which is a monomorphism), $\mu_A$ (which is an identity, i.e., $\mu_A = id_{EA}$, because we have that $E^2 = E$) and the endofunctor $E$, define the monad $(E, \eta, \mu)$, i.e., this monad is inductively extended in a natural way from the signature endofunctor $\Sigma_D = T : DB \longrightarrow DB$.

Thus, the monad $(E, \eta, \mu)$, where $E = I_{DB} + \coprod_{\omega} \circ T = I_{DB} + T \circ \coprod_{\omega}$ is an inductive algebraic extension of the "coalgebraic" observation based power-view monad $(T, \eta, \mu)$.
of variables which represent the views and (also infinite) set of unary and binary basic operations:

\[ \Sigma = Op_1 + Op_2, \]

where

\[ Op_1 = \{ \pi(S) \mid S \in \mathcal{P}(N) \text{ for some finite } N \} \cup \{ \text{Where}(C) \mid C \text{ is any selection condition on attributes} \} \]

\[ Op_2 = \{ \text{Join}(v=\_w) \mid v, w \text{ are relation's attributes} \} \cup \{ \text{Union} \} \]

Now, the mapping \( f : A \to B \) given by the logic implication above my be equivalently expressed by the following system of guarded equations:

\[
\begin{align*}
&< x, y > \approx \pi(2,5)(< v_1, x, z, w_1, y, z_1 >) \\
&< v_1, x, z, w_1, y, z_1 > \approx \text{Join}(v=\_w)(< v_1, x, z >, < w_1, y, z_1 >) \\
&< v_1, x, z > \approx \text{Where}(v=\_w')(< v, x, z >) \\
&< w_1, x, z_1 > \approx \text{Where}(w=\_w')(< w, x, z_1 >) \\
&< v, x, z > \approx \pi(2,3,4)(< v_1, x, z, w_1 >) \\
&< w, x, z_1 > \approx \pi(1,2,3)(< w, x, z_1, z_2, z_3 >) \\
&< x_1, v, x, z > \approx r_P \\
&< w, x, z_1, z_2, z_3 > \approx r_Q
\end{align*}
\]

such that the relation \( r_P \) is the solution of this system for the tuple-variable \( < x, y > \).

The polynomial endofunctor of \( \text{Set}, H_\Sigma : \text{Set} \to \text{Set} \), derived by this signature \( \Sigma \), for any given set of tuple variables \( X \) (in example above \( < x, y >, < v_1, x, z >, < w_1, x, z_1 >, < v, x, z >, < w, x, z_1 >, < x_1, v, x, z >, < w, x, z_1, z_2, z_3 >, < v_1, x, z, w_1, y, z_1 > \in X \)), is of the form

\[ H_\Sigma(X) = \bigsqcup_{n<\omega} Op_n \times X^n = \bigsqcup_{n \in \{1,2\}} Op_n \times X^n \]

It is easy to verify that right parts of equations (except two last equations) belong to \( H_\Sigma(X) \). The right parts of the last two equations belong to "parameters" database \( A \), i.e., \( r_P, r_Q \in A \).

Thus, the system of guarded equations above, which define a mapping from a database \( A \) to a database \( B \), my be expressed by the function \( f_c : X \to H_\Sigma(X) + A \) (for example, \( f_c(< v_1, x, z, w_1, y, z_1 >) = \text{Join}(v=\_w)(< v_1, x, z >, < w_1, y, z_1 >) \)), which is just a coalgebra of the polynomial \( \text{Set} \) endofunctor \( H_\Sigma(\_ + A) : \text{Set} \to \text{Set} \) with the signature \( \Sigma_X = \Sigma \cup X \) (the tuple-variables in \( X \) are seen as operations of arity 0).

It is known \(^{[25]}\) that such polynomial endofunctors of \( \text{Set} \) have a **final coalgebra** which is the algebra of all finite and infinite \( \Sigma_X \)-labelled trees, i.e., the set of all views \( T_\infty(A) \), so that \( T_\infty(A) = H_\Sigma(T_\infty(A)) + A \), i.e., \( T_\infty(A) \) is the maximal fixpoint of the endofunctor \( H_\Sigma(\_ + A) \).

So we obtained that for any database \( A \), its complete power-view object \( T_\infty(A) \) corresponds to the final coalgebra of the iterable endofunctor \( H_\Sigma(\_ + A) \), in the way that the guarded system of equations defined by a database mapping \( f \) has the **unique** solution \( s : X \to T_\infty(A) \), which is the \( H_\Sigma(\_ + A) \)-coalgebra homomorphism from the coalgebra \( (X, f_c) \) into the final coalgebra \( (T_\infty(A), \simeq) \), as given by the following...
commutative diagram in $\text{Set}$:

$$
\begin{array}{ccc}
X & \xrightarrow{s} & T_\infty(A) \\
\downarrow{f_\epsilon} & & \downarrow{\simeq} \\
H_\Sigma(X) + A & \xrightarrow{H_\Sigma(s) + A} & H_\Sigma(T_\infty(A)) + A
\end{array}
$$

It means, for example, that $s(<x, y>) \in T_\infty(A)$ is the unique solution of the conjunctive formula $P(a, x, z) \land Q(b, y, z)$ which is given in the body of the mapping query from a database $A$ into a database $B$, and which is part of the minimal Herbrand model for the logic theory expressed by this database mapping.

Let us now consider coalgebra properties in $\text{DB}$ category. We define an *iteratable* endofunctor $H$ of a category $D$ if for every object $X$ of $D$ the endofunctor $H(X) + X$ has a final algebra. We are going to show that the signature endofunctor $\Sigma_R$ is iteratable.

**Proposition 10** Every endofunctor $\Sigma_{RA} = (\Sigma_R + A) : \text{DB} \rightarrow \text{DB}$ has the final $\Sigma_{RA}$-coalgebra, $< T_\infty A, \langle p_l, p_r \rangle : T_\infty A \rightarrow \Sigma_R T_\infty A + A >$, where $p_l : T_\infty A \rightarrow A$ and $p_r : T_\infty A \rightarrow \Sigma_R T_\infty A$ are the unique product epimorphisms of the (co)product $TA \simeq \Sigma_R TA + A$ obtained as a maximal fixpoint of this endofunctor. Thus for any database $A$ its power-view object $T_\infty A$, that is the set of all views of $A$ obtained by finite and infinite tree terms of the SPJRU relational algebra, is a final $\Sigma_{RA}$-algebra.

The final coalgebra $< T_\infty A, \langle p_l, p_r \rangle >$ (where $\langle p_l, p_r \rangle$ is an isomorphism) of the endofunctor $\Sigma_{RA} = \Sigma_R + A : \text{DB} \rightarrow \text{DB}$ comes with a coinduction principle, and since it is the (co)product $\Sigma_R TA + A$, we can rephrase the principle as follows: For every $\Sigma_R$-coalgebra structure $h : B \rightarrow \Sigma_R B$ (which is an isomorphism) and every mapping $f : B \rightarrow A$ there exists a unique arrow $f^\# : B \rightarrow T_\infty A$ such that the diagram

$$
\begin{array}{ccc}
A & \xleftarrow{p_l} & T_\infty A \\
\downarrow{f^\#} & & \uparrow{\Sigma_R f^\#} \\
B & \xrightarrow{h} & \Sigma_R B
\end{array}
$$

commutes in $\text{DB}$, where $f^\# = < f, \Sigma_R f^\# \circ h >$ is the unique *coinductive extension of $h$ along the mapping*.

Note that $f^\# : < B, < h, f > > \rightarrow < T_\infty A, \langle p_l, p_r \rangle >$ is the unique arrow to the final $\Sigma_{RA}$-coalgebra from the coalgebra of the map $< h, f > : B \rightarrow \Sigma_R B + A$:

$$
\begin{array}{ccc}
B & \xrightarrow{f^\#} & T_\infty(A) \\
\downarrow{< h, f >} & & \downarrow{\simeq} \\
\Sigma_R(B) + A & \xrightarrow{\Sigma_R(f^\#) + A} & \Sigma_R(T_\infty(A)) + A
\end{array}
$$
This coinductive principle can be used to show that the closure operator $T_\infty$ coinductively extends to the endofunctor $T_\infty : DB \to DB$. Indeed, to define its action $T_\infty f$ on arrow $f : B \to A$, take the inductive extension of $p_r : T_\infty B \to \Sigma_R T_\infty B$ (of the $\Sigma_R : DB \to DB$ endofunctor with the final $\Sigma_R B$-coalgebra structure $< p_1, p_r > : T_\infty B \to \Sigma_R T_\infty B + B$) along the composite $f \circ p_1$, i.e., $T_\infty f \triangleq (f \circ p_1)^\# = < f \circ p_1, \Sigma_R T_\infty f \circ p_r >$. (Note that $T_\infty f$ can be seen as a homomorphism from the $\Sigma_R$-coalgebra $< T_\infty B, p_r >$ to the $\Sigma_R$-coalgebra $< T_\infty A, p_r >$.)

Thus, final coalgebras of the functors $\Sigma_R$ form a monad $(T_\infty, \eta, \mu)$, called the completely iterative monad generated by signature $\Sigma_R$.

7 Conclusions

In previous work we defined a category $DB$ where objects are databases and morphisms between them are extensional GLAV mappings between databases. We defined equivalent (categorically isomorphic) objects (database instances) from the behavioral point of view based on observations: each arrow (morphism) is composed by a number of "queries" (view-maps), and each query may be seen as an observation over some database instance (object of $DB$). Thus, we characterized each object in $DB$ (a database instance) by its behavior according to a given set of observations. In this way two databases $A$ and $B$ are equivalent (bisimilar) if they have the same set of its observable internal states, i.e. when $TA$ is equal to $TB$. It has been shown that such a $DB$ category is equal to its dual, it is symmetric in the way that the semantics of each morphism is an closed object (database) and viceversa each database can be represented by its identity morphism, so that $DB$ is a 2-category.

In [8,8] has been introduced the categorial (functors) semantics for two basic database operations: matching and merging (and data federation), and has been defined the algebraic database lattice. In the same paper has been shown that $DB$ is concrete, small and locally finitely presentable (lfp) category, and that $DB$ is also monoidal symmetric $V$-category enriched over itself. Based on these results the authors developed a metric space and a subobject classifier for $DB$ category, and they have shown that it is a weak monoidal topos.

In this paper we presented some other contributions for this intensive exploration of properties and semantics of $DB$ category. Here we considered some Universal algebra considerations and relationships of $DB$ category and standard $Set$ category. We defined a categorial coalgebraic semantics for GLAV database mappings based on monads, and of general (co)algebraic and (co)induction properties for databases.
It was shown that a categorial semantics of database mappings can be given by the Kleisly category of the power-view monad $T$, that is, was show that Kleisly category is a model for database mappings up to the equivalence $\approx$ of morphisms in $DB$ category. It was demonstrated that Kleisly category is isomorphic to the $DB$ category, and that call-by-values and call-by-name paradigms of programs (database mappings) are represented by equivalent morphisms. Moreover, it was shown that each database query (which is a program) is a monadic $T$-coalgebra, and that any morphism between two $T$-coalgebras defines the semantics for the relevant query-rewriting.

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