A TORELLI THEOREM FOR HIGHER-DIMENSIONAL FUNCTION FIELDS

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Abstract. We prove a Torelli-like theorem for higher-dimensional function fields, from the point of view of “almost-abelian” anabelian geometry.

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1. Introduction

The classical Torelli Theorem, in its cohomological form, can be stated as follows:

Theorem. Let \( X \) be a smooth compact complex curve. Then the isomorphism type of \( X \) is determined by the singular cohomology group \( H^1(X, \mathbb{Z}) \), endowed with its canonical polarized Hodge structure.

In this paper, we develop and prove a higher-dimensional birational variant of this theorem. As expected, one must include not only \( H^1 \) (with its mixed Hodge structure) in this higher-dimensional context, but also some additional non-abelian data. It turns out that the “two-step nilpotent” information, encoded as the kernel of the cup-product, provides sufficient non-abelian information in this setting. Also, as discussed below, our result works even with rational coefficients, in contrast to the classical Torelli theorem mentioned above. Finally, in addition to a result in the Hodge-theoretic context, which is directly analogous to the classical Torelli theorem, we also prove a Galois-equivariant analogue of our main result in the context of \( \ell \)-adic cohomology.

1.1. Main result (Hodge context). Let \( k \) be an algebraically closed field of characteristic 0, and let \( \sigma : k \hookrightarrow \mathbb{C} \) be a complex embedding. Let \( \Lambda \) be a subring of \( \mathbb{Q} \). For \( k \)-varieties \( X \), consider \( X^{an} := X(\mathbb{C}) \) (computed via \( \sigma \)) endowed with the complex topology, as well as the the Betti cohomology of \( X 

\[ H^i(X, \Lambda) := H^i_{\text{Sing}}(X^{an}, \Lambda). \]
Following Deligne [Del71, Del74], one can endow $H^i(X, \Lambda)$ with a canonical mixed Hodge structure (over $\Lambda$). We will denote this mixed Hodge structure by $H^i(X, \Lambda)$, whereas $H^i(X, \Lambda)$ will denote the underlying plain $\Lambda$-module.

Following the usual conventions, we write $\Lambda(i)$ for the unique mixed Hodge structure (over $\Lambda$) whose underlying $\Lambda$-module is $\Lambda$, and which is of Hodge type $(-i, -i)$. We then write $H^i(X, \Lambda(j)) := H^i(X, \Lambda) \otimes \Lambda(j)$. To keep the notation consistent, we write $H^i(X, \Lambda(j))$ for the underlying $\Lambda$-module of $H^i(X, \Lambda(j))$. However, we consider $H^i(X, \Lambda(j))$ only as an abstract $\Lambda$-module. In particular, the $j$ in the notation will also be used to keep track of the (cyclotomic) Tate twists in $\ell$-adic cohomology.

Now let $K$ be a function field over $k$, and let $X$ be a model of $K/k$ – i.e. $X$ is an integral $k$-variety whose function field is $k$. We define

$$H^i(K|k, \Lambda(j)) := \lim_{U \to k} H^i(U, \Lambda(j)), \quad H^i(K|k, \Lambda(j)) := \lim_{U \to k} H^i(U, \Lambda(j))$$

where $U$ varies over the non-empty open $k$-subvarieties of $X$. We consider $H^i(K|k, \Lambda(j))$ as an (infinite-rank) mixed Hodge structure whose underlying $\Lambda$-module is $H^i(K|k, \Lambda(j))$. It is easy to see that this construction doesn’t depend on the original choice of model $X$ of $K/k$.

Finally, note that the cup-product in singular cohomology yields a well-defined cup-product on

$$H^*(K|k, \Lambda(*)) := \bigoplus_{i \geq 0} H^i(K|k, \Lambda(i)),$$

making it into a graded-commutative ring. The cup-product in this ring will play an important role throughout the paper. In fact, in the statement of the main theorem we will consider the kernel of the cup-product, denoted

$$\mathcal{R}(K|k, \Lambda) := \ker(\cup : H^1(K|k, \Lambda(1)) \otimes_{\Lambda} H^1(K|k, \Lambda(1)) \to H^2(K|k, \Lambda(2))).$$

With this notation and terminology, our main result (in the Hodge context) reads as follows.

**Theorem A** (See Theorem 7.1). Let $\Lambda$ be a subring of $\mathbb{Q}$. Let $k$ be an algebraically closed field of characteristic 0, and let $\sigma : k \to \mathbb{C}$ be a complex embedding. Let $K$ be a function field over $k$ such that $\text{tr. deg}(K|k) \geq 2$. Then the isomorphy type of $K|k$ (as fields) is determined by the following data:

- The mixed Hodge structure $H^1(K|k, \Lambda(1))$, with underlying $\Lambda$-module $H^1(K|k, \Lambda(1))$.
- The $\Lambda$-submodule $\mathcal{R}(K|k, \Lambda)$ of $H^1(K|k, \Lambda(1)) \otimes_{\Lambda} H^1(K|k, \Lambda(1))$.

1.2. **Main result** ($\ell$-adic context). Let $k_0$ be a field of characteristic 0 with algebraic closure $k$, and let $\sigma : k \to \mathbb{C}$ be a complex embedding. For a $k_0$-variety $X_0$, we write $X := X_0 \otimes_{k_0} k$ for the base-change of $X_0$ to $k$.

Fix a prime $\ell$, a subring $\Lambda$ of $\mathbb{Q}$, and put $\Lambda_{\ell} := \mathbb{Z}_{\ell} \otimes_{\mathbb{Z}} \Lambda$. Even though $\Lambda_{\ell}$ can only ever be either $\mathbb{Z}_{\ell}$ or $\mathbb{Q}_{\ell}$, we use the notation $\Lambda_{\ell}$ for the sake of consistency. For a $k_0$-variety $X_0$, we consider the $\ell$-adic cohomology of $X$ (with coefficients in $\Lambda_{\ell}$), defined and denoted as:

$$H^i_{\ell}(X, \Lambda_{\ell}(j)) := \left( \lim_{i} H^i_{\ell}(X, \mathbb{Z}/\ell^n(j)) \right) \otimes_{\mathbb{Z}_{\ell}} \Lambda_{\ell}.$$

Note that $H^i_{\ell}(X, \Lambda_{\ell}(j))$ is a $\Lambda_{\ell}$-module endowed with a canonical continuous action of $\text{Gal}_{k_0}$. In other words, we may consider $H^i_{\ell}(X, \Lambda_{\ell}(j))$ as a module over the completed group-algebra $\Lambda_{\ell}[[\text{Gal}_{k_0}]]$.

Now let $K_0$ be a regular function field over $k_0$, and let $K := K_0 \cdot k$ denote its base-change to $k$. Given a model $X_0$ of $K_0|k_0$, i.e. a geometrically-integral $k_0$-variety whose function field is $K_0$, we define

$$H^i_{\ell}(K|k, \Lambda_{\ell}(j)) := \lim_{U_0} H^i_{\ell}(U, \Lambda_{\ell}(j))$$

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where $U_0$ varies over the non-empty open $k_0$-subvarieties of $X_0$. As before, it is easy to see that $H_1^i(K|k, \Lambda_\ell(j))$, considered as a $\Lambda_\ell[[\operatorname{Gal}_{k_0}]]$-module, doesn’t depend on the original choice of model $X_0$ of $K_0|k_0$. What’s more, as a $\Lambda_\ell$-module, $H_1^i(K|k, \Lambda_\ell(j))$ is also independent from the choice of field $k_0$ with algebraic closure $k$ and the regular function field $K_0|k_0$ whose base-change is $K$.

Finally, one has Artin’s comparison isomorphism between $\ell$-adic and singular cohomology (see [AGV71] Expose XI), which, for smooth $X$, is a functorial isomorphism of $\Lambda_\ell$-modules

$$\mathcal{G}_\ell: H^1(X, \Lambda(j)) \otimes \Lambda \Lambda_\ell \cong H^1(X, \Lambda_\ell(j)) \cong H^1(X, \Lambda_\ell(j)),$$

Here singular cohomology is computed with respect to the embedding $\sigma : k \hookrightarrow \mathbb{C}$. Letting $X_0$ be a model of $K_0|k_0$ as above, we note that as $U_0$ varies over the smooth open $k_0$-subvarieties of $X_0$, the base-change $U = U_0 \otimes_{k_0} k$ varies over a cofinal system of open neighborhoods of the generic point of $X = X_0 \otimes_{k_0} k$. As $X$ is a model of $K|k$, we thereby obtain a canonical comparison isomorphism of $\Lambda_\ell$-modules:

$$\mathcal{G}_\ell: H^1(K|k, \Lambda(1)) \otimes \Lambda \Lambda_\ell \cong H^1(K|k, \Lambda_\ell(1)).$$

With the above notation and terminology, we may now state the $\ell$-adic variant of our main result.

**Theorem B (See Theorem 8.1).** Let $\Lambda$ be a subring of $\mathbb{Q}$, and let $\ell$ be a prime. Let $k_0$ be a finitely-generated field of characteristic 0 with algebraic closure $k$, and let $\sigma : k \hookrightarrow \mathbb{C}$ be a complex embedding. Let $K_0$ be a regular function field over $k_0$ such that $\operatorname{tr.deg}(K_0|k_0) \geq 2$. Then the isomorphy type of $K_0|k_0$ (as fields) is determined by the following data:

- The profinite group $\operatorname{Gal}_{k_0}$ and the $\Lambda_\ell[[\operatorname{Gal}_{k_0}]]$-module $H^1_1(K|k, \Lambda_\ell(1))$.
- The $\Lambda_\ell$-module $H^1(K|k, \Lambda(1))$, endowed with Artin’s comparison isomorphism

$$\mathcal{G}_\ell: H^1(K|k, \Lambda(1)) \otimes \Lambda \Lambda_\ell \cong H^1(K|k, \Lambda_\ell(1)).$$

- The $\Lambda_\ell$-submodule $\mathcal{R}(K|k, \Lambda)$ of $H^1(K|k, \Lambda(1)) \otimes \Lambda H^1(K|k, \Lambda(1))$.

**1.3. A comment about the proofs.** Theorems [A] and [B] are perhaps not too surprising, especially to the reader who is familiar both with results concerning 1-motives and their Hodge resp. $\ell$-adic realizations, and with certain recent results from birational anabelian geometry. Indeed, the main results essentially follow by combining the following:

1. The comparison of a 1-motive with its Hodge realization, due to Deligne [Del74], resp. its $\ell$-adic realization, due to Faltings [Fal83] (in the case of abelian varieties) and Jannsen [Jan95] (in general).
2. The construction of the Picard 1-motive of a smooth variety, due essentially to Serre [Ser58], and/or the work of Barbieri-Viale, Srinivas [BVS01].
3. Methods for reconstructing function fields over algebraically closed fields in birational anabelian geometry, due to Bogomolov [Bog91], Bogomolov-Tschinkel [BT08], [BT09] and Pop [Pop02], [Pop12b], [Pop12a].

In addition to the above points, there are a few hurdles that one must overcome, specifically in the case where $\Lambda = \mathbb{Q}$, where the known “global” anabelian techniques (e.g. from Pop [Pop12b], [Pop12a] and/or Bogomolov-Tschinkel [BT08], [BT09]) break down, as one can no longer distinguish between the “divisible” and “non-divisible” (see also Remark 6.2). We overcome these difficulties by relying on arguments surrounding the connection between algebraic dependence and the cup-product, which are in some sense analogous to the ideas from [Top16b] and [Top15b].

Nevertheless, as we see it, the primary novelty of this work comes from the fact that it applies anabelian techniques in a purely motivic setting.

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2. Betti Cohomology

Throughout the paper, we will work with a coefficient ring \(\Lambda\), which will always be an integral domain of characteristic 0. At certain situations, we will need to restrict to the case where \(\Lambda\) is a subring of \(\mathbb{Q}\), although we will make this restriction explicit when it is needed. For a field \(k_0\), by a \(k_0\)-variety, we mean a separated scheme of finite type over \(k_0\).

Throughout the paper, \(k\) will denote an algebraically closed field of characteristic 0 which is endowed with a complex embedding \(\sigma : k \hookrightarrow \mathbb{C}\). Given a \(k\)-variety \(X\), we write \(X^{\text{an}} := X(\mathbb{C})\) for the set of complex points (via \(\sigma\)) endowed with its natural complex topology. We define Betti Cohomology (with respect to \(\sigma\)) in the usual way as the singular cohomology of \(X^{\text{an}}:\)

\[
H^i(X, \Lambda(j)) := H^i_{\text{Sing}}(X^{\text{an}}, \Lambda).
\]

As suggested by the notation, we include the \(j\) in the coefficients in order to keep track of Tate twists, which will play an important role later on in the Hodge and \(\ell\)-adic contexts. Finally, it is important to note that the mere construction of \(H^i(X, \Lambda(j))\) depends on the choice of complex embedding \(\sigma\). We will always exclude \(\sigma\) for the notation, while making sure that it is understood from context.

2.1. Models. Throughout the paper, we will work with a function field \(K\) over \(k\). By a model of \(K|k\), we mean an integral \(k\)-variety whose function field is \(K\). Given such a model \(X\) of \(K|k\), we define

\[
H^i(K|k, \Lambda(j)) := \lim_{U \rightarrow} H^i(U, \Lambda(j))
\]

where \(U\) varies over the non-empty open \(k\)-subvarieties of \(X\). At certain times, we may tacitly restrict the \(U\) that appear in the colimit to any cofinal system of open neighborhoods of the generic point of \(X\). As any two models of \(K|k\) agree on some non-empty open \(k\)-subvariety, it follows that this definition doesn’t depend on the original choice of model \(X\).

The cup product in singular cohomology yields a natural cup-product:

\[
\cup : H^i(K|k, \Lambda(j)) \otimes_{\Lambda} H^r(K|k, \Lambda(s)) \rightarrow H^{i+r}(K|k, \Lambda(j+s)).
\]

This makes \(H^*(K|k, \Lambda(\ast)) := \bigoplus_{i \geq 0} H^i(K|k, \Lambda(i))\) into a graded-commutative \(\Lambda\)-algebra, where \(\Lambda\) is identified with \(H^0(K|k, \Lambda(0))\) in the obvious way.

2.2. Injectivity. Let \(X\) be a smooth model of \(K|k\), and let \(U\) be a non-empty open \(k\)-subvariety of \(X\). It is well known that the inclusion \(U \hookrightarrow X\) induces a morphism

\[
H^1(X, \Lambda(1)) \rightarrow H^1(U, \Lambda(1))
\]

in cohomology which is injective. In particular, the structure map \(H^1(X, \Lambda(1)) \rightarrow H^1(K|k, \Lambda(1))\) is injective as well. In other words, \(H^1(K|k, \Lambda(1))\) can be considered as an inductive union of all \(H^1(U, \Lambda(1))\), as \(U\) varies over the smooth models of \(K|k\). We will henceforth identify \(H^1(X, \Lambda(1))\) with its image in \(H^1(K|k, \Lambda(1))\) for any smooth model \(X\) of \(K|k\).

2.3. Functoriality. Let \(\iota : L \hookrightarrow K\) be a \(k\)-embedding of function fields over \(k\). By a model of \(\iota\), we mean a dominant morphism \(f : X \rightarrow Y\), where \(X\) is a model of \(K|k\), \(Y\) is a model of \(L|k\), and the induced map

\[
f^* : L = k(Y) \hookrightarrow k(X) = K
\]

agrees with the original embedding \(\iota\). Given such a model \(f : X \rightarrow Y\), we obtain a canonical map

\[
\iota_* : H^i(L|k, \Lambda(j)) = \lim_{U'} H^i(U, \Lambda(j)) \xrightarrow{f^*} \lim_{U} H^i(f^{-1}(U), \Lambda(j)) \rightarrow H^i(K|k, \Lambda(j)),
\]
where $U$ varies over the non-empty open $k$-subvarieties of $Y$. It is easy to see that $\iota_*$ doesn’t depend on the choice of model $f : X \to Y$, and that this construction makes the assignment $K \mapsto H^1(K|k, \Lambda(j))$ covariantly functorial with respect to $k$-embeddings of function fields over $k$.

2.4. Kummer theory. Recall that one has a canonical isomorphism $\Lambda \cong H^1(\mathbb{G}_m, \Lambda(1))$, corresponding to the canonical holomorphic orientation of $\mathbb{G}_m^\text{an} = \mathbb{C}^\times$. We will henceforth identify $\Lambda$ with $H^1(\mathbb{G}_m, \Lambda(1))$ and simply write $\Lambda = H^1(\mathbb{G}_m, \Lambda(1))$.

Remark 2.1. The identification $\Lambda = H^1(\mathbb{G}_m, \Lambda(1))$ is compatible with both the Hodge structure and the Galois action on $\ell$-adic cohomology. More precisely, assume that $\Lambda$ is a subring of $\mathbb{Q}$. Then one has $H^1(\mathbb{G}_m, \Lambda(1)) = \Lambda$, where $\Lambda = \Lambda(0)$ is the pure Hodge structure of Hodge type $(0,0)$. In the $\ell$-adic context, if $k_0$ is a field whose algebraic closure is $k$, then one has $H^1_{\ell}(\mathbb{G}_m, k_0(1)) = \Lambda$ on which $\text{Gal}_{k_0}$ acts trivially.

Let $X$ be a model of $K|k$. Let $U$ be a non-empty open $k$-subvariety of $X$, and let $f \in \mathcal{O}^\times(U)$ be given. Then $f$ corresponds to a morphism $f : U \to \mathbb{G}_m$ of $k$-varieties. We define $\mathcal{R}_U(f) \in H^1(U, \Lambda(1))$ to be the image of $1 \in \Lambda$ under the canonical map

$$\Lambda = H^1(\mathbb{G}_m, \Lambda(1)) \xrightarrow{f^*} H^1(U, \Lambda(1)).$$

We similarly write $\mathcal{R}_K(f)$ for the image of $f \in \mathcal{O}^\times(U) \subset K^\times$ in $H^1(K|k, \Lambda(1))$.

It is a straightforward consequence of the Künneth formula that the corresponding maps

$$\mathcal{R}_U : \mathcal{O}^\times(U) \to H^1(U, \Lambda(1)), \quad \mathcal{R}_K : K^\times \to H^1(K|k, \Lambda(1))$$

are both homomorphism of abelian groups. Furthermore, it is a straightforward consequence of the definition that $k^\times$ is contained in the kernel of $\mathcal{R}_U$ and $\mathcal{R}_K$.

We will write $\mathcal{X}_\Lambda(K|k) := (K^\times/k^\times) \otimes_{\mathbb{Z}} \Lambda$. For $t \in K^\times$, we will write $t^\circ$ for the image of $t$ under the canonical map $K^\times \to \mathcal{X}_\Lambda(K|k)$. We will always write (the $\Lambda$-module) $\mathcal{X}_\Lambda(K|k)$ additively. On the other hand, $K^\times/k^\times$ will be written multiplicatively, even though it maps in to $\mathcal{X}_\Lambda(K|k)$ in the obvious way.

Note that the assignment $K \mapsto \mathcal{X}_\Lambda(K|k)$ is covariantly functorial with respect to $k$-embeddings of function fields over $k$. For a $k$-embedding $\iota : L \hookrightarrow K$ of function fields over $k$, we will write $\iota_* : \mathcal{X}_\Lambda(L|k) \to \mathcal{X}_\Lambda(K|k)$ for the corresponding morphism of $\Lambda$-modules. Finally, note that the homomorphism $\mathcal{R}_K$ mentioned above induces a canonical $\Lambda$-linear morphism

$$\mathcal{R}_K^\Lambda : \mathcal{X}_\Lambda(K|k) \to H^1(K|k, \Lambda(1)),$$

which is uniquely defined by the rule $\mathcal{R}_K^\Lambda(t^\circ) = \mathcal{R}_K(t)$ for $t \in K^\times$. It is easy to see from the construction that $\mathcal{R}_K^\Lambda : \mathcal{X}_\Lambda(-|k) \to H^1(-|k, \Lambda(1))$ is a natural transformation of covariant functors.

2.5. Milnor K-theory. Recall that the Milnor $K$-Ring of $K$ is the graded-commutative ring which is denoted and defined as

$$K^M_*(K) := \frac{T_*(K^\times)}{\langle x \otimes (1-x) : x \in K \setminus \{0,1\} \rangle},$$

where $T_*$ denotes the (graded) tensor algebra of (the abelian group) $K^\times$. Following the usual conventions, we will write $\{f_1, \ldots, f_r\} \in K^M_*(K)$ for the product of $f_1, \ldots, f_r \in K^\times = K^M_1(K)$ in $K^M_*(K)$. It is an easy consequence of the fact that

$$H^2(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \Lambda(2)) = 0$$

that one has $\mathcal{R}_K(t) \cup \mathcal{R}_K(1-t) = 0$ in $H^2(K|k, \Lambda(2))$ for all $t \in K \setminus \{0,1\}$. In particular, we see that $\mathcal{R}_K$ extends to a well-defined morphism of graded-commutative rings

$$\mathcal{R}_K^* : K^M_*(K) \to H^*(K|k, \Lambda(*)).$$
The $r$-th component of this map, denoted by $\mathcal{R}_K^r : K^M_r(K) \to H^r(K|k, \Lambda(r))$, is defined by the rule $\mathcal{R}_K^r(f_1, \ldots, f_r) = \mathcal{R}_K(f_1) \cup \cdots \cup \mathcal{R}_K(f_r)$ for $f_1, \ldots, f_r \in K^X$.

3. Birational Thom-Gysin Theory

In this section, we discuss the birational manifestation of the classical Thom-Gysin Theory in codimension 1. The calculations in this section, in some sense, go back to Grothendieck \cite{Gro66} in the de Rham context, and to Bloch-Ogus \cite{BO74} in the context of the Gersten conjecture. It is important to note that many of the constructions in this section work with an arbitrary cohomology theory, and we refer the reader to the extensive work of Dégol \cite{Dég08, Dég12} for these details. We have decided to focus on Betti cohomology in this discussion to prevent straying too far from the focus of this paper.

We will need to discuss Betti cohomology with supports in this section, so we briefly introduce the relevant notation. Let $X$ be a $k$-variety and let $Z$ be a closed $k$-subvariety of $X$. Put $U := X \setminus Z$.

We write

$$H^i_Z(X, \Lambda(j)) := H^i_{\text{Sing}}(X^\text{an}, U^\text{an}, \Lambda(j))$$

for the relative singular cohomology of the pair $(X^\text{an}, U^\text{an})$. Recall that the long exact sequence of the pair reads as follows:

$$\cdots \to H^i(X, \Lambda(j)) \to H^i(U, \Lambda(j)) \to H^{i+1}_Z(X, \Lambda(j)) \to H^{i+1}(X, \Lambda(j)) \to \cdots$$

3.1. Thom-Gysin theory in codimension one. We begin by recalling some well-known constructions and facts surrounding the classical Thom-Gysin theory in codimension 1. Let $X$ be a smooth $k$-variety and let $Z$ be closed $k$-subvariety of $X$ which is smooth, and pure of codimension 1 in $X$. Let $N_X(Z)$ denote the normal bundle of $Z$ in $X$.

Let us briefly recall the classical theory of deformation to the normal cone. This theory considers $\tilde{Z}$, the blowup of $X \times \mathbb{A}^1$ along $Z \times \{0\}$, where $\mathbb{A}^1$ denotes the affine line with parameter $t$. One has a canonical morphism $\tilde{Z} \to Z \to \mathbb{A}^1$. The fibre of this morphism above $t \neq 0$ is $X$, while the fibre above $t = 0$ is a union

$$\tilde{Z}_{t=0} = \mathbb{P}(N_X(Z) \oplus 1) \cup X,$$

where $\mathbb{P}(N_X(Z) \oplus 1)$ denotes the projective completion of $N_X(Z)$, as usual. The intersection of $X$ and $\mathbb{P}(N_X(Z) \oplus 1)$ in this union is along $Z$, embedded in the obvious way in $X$ and as the section at infinity in $\mathbb{P}(N_X(Z) \oplus 1)$. In particular, one has $\tilde{Z}_{t=0} \setminus X = N_X(Z)$.

Furthermore, one has a canonical embedding $Z \times \mathbb{A}^1 \hookrightarrow \tilde{Z}$, which is obtained by identifying $Z \times \mathbb{A}^1$ with the blow-up of $Z \times \mathbb{A}^1$ along $Z \times \{0\}$. The fibre of this inclusion $Z \times \mathbb{A}^1 \hookrightarrow \tilde{Z}$ above $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, is precisely the obvious inclusion $Z \times \mathbb{G}_m \hookrightarrow X \times \mathbb{G}_m$, while the fibre of this inclusion above $t = 0$ is the inclusion of $Z$ in $N_X(Z)$ as the zero-section, followed by the natural inclusion

$$N_X(Z) \hookrightarrow \mathbb{P}(N_X(Z) \oplus 1) \hookrightarrow \mathbb{P}(N_X(Z) \oplus 1) \cup X = \tilde{Z}_{t=0}.$$

We put $X := \tilde{Z} \setminus X$, where $X$ is the copy in the fibre above $t = 0$, as described above.

To summarize, this variety $X$ is endowed with a canonical flat surjective map

$$f : X \hookrightarrow \tilde{Z} \twoheadrightarrow X \times \mathbb{A}^1 \to \mathbb{A}^1,$$

which satisfies the following properties:

(1) One has an isomorphism $f^{-1}(\mathbb{G}_m) \cong X \times \mathbb{G}_m$ over $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, and the fibre of $Z \times \mathbb{A}^1 \hookrightarrow X$ over $\mathbb{G}_m$ corresponds to the inclusion $Z \times \mathbb{G}_m \hookrightarrow X \times \mathbb{G}_m$ induced by $Z \hookrightarrow X$.

(2) The fibre of $Z \times \mathbb{A}^1 \hookrightarrow X$ over $t = 0$ is the inclusion $Z \hookrightarrow N_X(Z)$ of the zero-section of the line bundle $N_X(Z) \to Z$. 

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This construction thereby provides us with two inclusions of closed pairs:

\[
\begin{align*}
Z \times \mathbb{A}^1 & \hookrightarrow \mathcal{X} \\
Z & \hookrightarrow \mathcal{N}_X(Z) \\
Z & \hookrightarrow X
\end{align*}
\]

associated to the fibres over \( t = 0 \) and \( t = 1 \) respectively, and these two inclusions induce corresponding specialization morphisms in cohomology:

\[
\begin{align*}
\text{H}^i_{\mathbb{A}^1}(\mathcal{X}, \Lambda(j)) & \cong \text{H}^i_{Z}(\mathcal{X}, \Lambda(j)) \\
\text{H}^i_{Z}(\mathcal{N}_X(Z), \Lambda(j)) & \cong \text{H}^i_{Z}(X, \Lambda(j))
\end{align*}
\]

which are isomorphisms. We denote by \( \mathcal{E}_{X,Z} : \text{H}^i_{Z}(X, \Lambda(j)) \cong \text{H}^i_{Z}(\mathcal{N}_X(Z), \Lambda(j)) \) the composition of these two isomorphisms, and call \( \mathcal{E}_{X,Z} \) the excision isomorphism associated to \( Z \hookrightarrow X \).

Next, recall that one has a canonical orientation class \( \eta_{X,Z} \in \text{H}^2_{Z}(\mathcal{N}_X(Z), \Lambda(1)) \) associated to the line bundle \( \mathcal{N}_X(Z) \to Z \). The Thom Isomorphism Theorem asserts that the induced map

\[
x \mapsto \eta_{X,Z} \cup x : \text{H}^i(\mathcal{N}_X(Z), \Lambda(j)) \to \text{H}^{i+2}(\mathcal{N}_X(Z), \Lambda(j + 1))
\]

is an isomorphism. Finally, since the fibres of \( \mathcal{N}_X(Z) \to Z \) are all isomorphic to \( \mathbb{A}^1 \), we see that the specialization to the zero-section, \( \text{H}^i(\mathcal{N}_X(Z), \Lambda(j)) \to \text{H}^i(Z, \Lambda(j)) \), is an isomorphism. By composing the various isomorphism described above, we obtain the so-called purity isomorphism in Betti Cohomology:

\[
\mathfrak{P}_{X,Z} : \text{H}^{i+2}(X, \Lambda(j + 1)) \xrightarrow{\cong} \text{H}^i(Z, \Lambda(j)).
\]

Finally, recall that the Residue Morphism associated to \( Z \hookrightarrow X \) is the morphism

\[
\partial_{X,Z} : \text{H}^i(U, \Lambda(j + 1)) \xrightarrow{\delta} \text{H}^{i+1}(X, \Lambda(j + 1)) \xrightarrow{\mathfrak{P}_{X,Z}} \text{H}^{i-1}(Z, \Lambda(j)).
\]

The following calculation seems to be well-known.

**Lemma 3.1.** Let \( X \) be a smooth \( k \)-variety endowed with a morphism \( f : X \to \mathbb{A}^1 \). Let \( Z \) be the fibre of \( f \) above \( 0 \), and assume that \( Z \) is smooth, integral, and of codimension 1 in \( X \). Put \( U := X \setminus Z \), and consider the induced morphism \( f : U \to \mathbb{G}_m \). Let \( \mathfrak{r}_U(f) : = \gamma \in \text{H}^1(U, \Lambda(1)) \) denote the image of \( 1 \in \Lambda \) under the canonical map

\[
\Lambda = \text{H}^1(\mathbb{G}_m, \Lambda(1)) \xrightarrow{\mathfrak{r}} \text{H}^1(U, \Lambda(1)).
\]

Then the following (equivalent) conditions hold:

1. For \( \alpha \in \text{H}^i(X, \Lambda(j)) \), let \( \alpha^u \) denote the image of \( \alpha \) in \( \text{H}^i(U, \Lambda(j)) \) and \( \alpha^s \) the image of \( \alpha \) in \( \text{H}^i(Z, \Lambda(j)) \). Then one has \( \partial_{X,Z}^{u}(\gamma \cup \alpha^u) = \alpha^s \).

2. The orienting class \( \eta_{X,Z} = \text{H}^2_{Z}(\mathcal{N}_X(Z), \Lambda(1)) \) agrees with the image \( \mathcal{E}_{X,Z}(\delta_\gamma) \) of \( \gamma \) under the boundary map \( \delta : \text{H}^1(U, \Lambda(1)) \to \text{H}^2_{Z}(X, \Lambda(1)) \) and the excision isomorphism \( \mathcal{E}_{X,Z} : \text{H}^2_{Z}(X, \Lambda(1)) \cong \text{H}^2_{Z}(\mathcal{N}_X(Z), \Lambda(1)) \).

**Proof.** The fact that these two assertions are equivalent follows from the definition of \( \partial_{X,Z} \). For a purely algebraic proof of assertion (1), which works with any cohomology theory, we refer the reader to [Deg08, Proposition 2.6.5] and the surrounding discussion. \( \square \)
3.2. Divisorial valuations. Recall that a divisorial valuation of the function field $K|k$ is a valuation $v$ of $K$ which satisfies the following properties:

1. The value group $vK$ of $v$ is isomorphic (as an ordered abelian group) to $\mathbb{Z}$. This implies that $v$ is trivial on $k$.
2. The residue field $Kv$ of $v$ is a function field of transcendence degree $\text{tr.deg}(K|k) - 1$ over $k$.

A valuation $v$ is divisorial if and only if $v$ arises from some prime Weil divisor on some normal model of $K|k$. In addition to the notation $vK$ resp. $Kv$ for the value group resp. residue field of $v$, we will write $\mathcal{O}_v$ for the valuation ring, $\mathfrak{m}_v$ for the valuation ideal, $U_v := \mathcal{O}_v^\times$ for the $v$-units, and $U_v^1 := (1 + \mathfrak{m}_v)$ for the principal $v$-units.

Let $X$ be a model of $K|k$. We say that $X$ is a model for $\mathcal{O}_v|k$ provided that the following conditions hold true:

1. The valuation $v$ has a (necessarily unique) center $\xi_{X,v}$ on $X$.
2. The center $\xi_{X,v}$ is a regular codimension 1 point in $X$.

Given such a model $X$ for $\mathcal{O}_v|k$ with $v$-center $\xi_{X,v}$, we will write $X_v := \{\xi_{X,v}\}$ for the closure of $\xi_{X,v}$ in $X$. An open subvariety $U$ of $X$ will be called a $v$-open $k$-subvariety of $X$ provided that $\xi_{X,v} \in U$, or, equivalently, $U \cap X_v$ is dense in $X_v$. Note that any $v$-open $k$-subvariety of $X$ is again a model of $\mathcal{O}_v|k$, and one has $U \cap X_v = U_v$.

Let $X$ be a model of $\mathcal{O}_v|k$. We define

$$H^i(\mathcal{O}_v|k, \Lambda(j)) := \lim_U H^i(U, \Lambda(j)), \quad H^i_v(\mathcal{O}_v|k, \Lambda(j)) := \lim_U H^i_{U_v}(U, \Lambda(j)),$$

where $U$ varies over the $v$-open $k$-subvarieties of $X$. As before, it is easy to see that this definition doesn’t depend on the original choice of model $X$ of $\mathcal{O}_v|k$. And, similarly to before, we may tacitly restrict the $U$ that appear in the colimit to any cofinal system of open neighborhoods of the center $\xi_{X,v}$ of $v$ on $X$.

3.3. Birational Thom-Gysin theory. Let $X$ be a model of $\mathcal{O}_v|k$. For $U$ a $v$-open $k$-subvariety of $X$, we will follow the notation in Lemma 3.1 and denote the maps in cohomology associated to $U \setminus U_v \hookrightarrow U$ resp. $U_v \hookrightarrow U$ as follows:

$$\alpha \mapsto \alpha^u : H^i(U, \Lambda(j)) \rightarrow H^i(U \setminus U_v, \Lambda(j)), \quad \alpha \mapsto \alpha^v : H^i(U, \Lambda(j)) \rightarrow H^i(U_v, \Lambda(j)).$$

Note that as $U$ varies over the $v$-open $k$-subvarieties of $X$, the complement $U \setminus U_v$ varies over the non-empty open $k$-subvarieties of $X \setminus X_v$, while $U_v$ varies over the non-empty open $k$-subvarieties of $X_v$. In particular, by passing to the colimit, we obtain two morphisms associated to $v$ which are denoted similarly:

$$\alpha \mapsto \alpha^u : H^i(\mathcal{O}_v|k, \Lambda(j)) \rightarrow H^i(K|k, \Lambda(j)), \quad \alpha \mapsto \alpha^v : H^i(\mathcal{O}_v|k, \Lambda(j)) \rightarrow H^i(Kv|k, \Lambda(j)).$$

By considering the long exact sequence of the pairs $(U, U \setminus U_v)$, we obtain in the colimit the long exact sequence of the pair $(\mathcal{O}_v, K)$:

$$\cdots \rightarrow H^i(\mathcal{O}_v|k, \Lambda(j)) \xrightarrow{\alpha \mapsto \alpha^u} H^i(K|k, \Lambda(j)) \xrightarrow{\delta} H^{i+1}(\mathcal{O}_v|k, \Lambda(j)) \rightarrow H^{i+1}(\mathcal{O}_v|k, \Lambda(j)) \rightarrow \cdots$$

Similarly, by considering the purity isomorphisms associated to $U_v \hookrightarrow U$, we obtain in the colimit the purity isomorphism associated to $v$:

$$\mathfrak{P}_v : H^{i+2}_v(\mathcal{O}_v|k, \Lambda(j + 1)) \xrightarrow{\cong} H^i(Kv|k, \Lambda(j)).$$

Finally, we consider the residue morphism associated to $U_v \hookrightarrow U$, and we obtain in the colimit the residue morphism associated to $v$:

$$\partial_v : H^i(K|k, \Lambda(j + 1)) \xrightarrow{\delta} H^{i+1}_v(\mathcal{O}_v|k, \Lambda(j + 1)) \xrightarrow{\mathfrak{P}_v} H^{i-1}(Kv|k, \Lambda(j)).$$
Lemma 3.2. Let $v$ be a divisorial valuation of $K|k$, and let $\pi \in K^\times$ be a uniformizer of $v$. Let $\alpha \in H^i(\mathcal{O}_v|k, \Lambda(j))$ be given. Then one has $\partial_v(R_K(\pi) \cup \alpha^0) = \alpha^g$ as elements in $H^i(K^v|k, \Lambda(j))$.

Proof. Since $\pi$ is a uniformizer of $v$, we can find some smooth model $X$ of $\mathcal{O}_v|k$ such that $\pi \in \mathcal{O}(X)$, and, considering $\pi$ as a morphism $\pi : X \to \mathbb{A}^1$, so that the assumptions of Lemma 3.1 are satisfied for $Z := X_v$ and $f = \pi$. The assertion of the lemma follows directly from Lemma 3.1 along with the definition of $\mathcal{R}_K(\pi)$. 

3.4. Tame symbols. In order to put Lemma 3.2 in the right perspective, we recall the existence of a so-called tame symbol in Milnor K-theory associated to a divisorial valuation $v$ of $K|k$. This is a morphism $\partial_v^M : K^M_{r+1}(K) \to K^M(K^v)$ which is uniquely determined by the fact that

$$\partial_v^M\{\pi, u_1, \ldots, u_r\} = \{\bar{u}_1, \ldots, \bar{u}_r\}$$

where $\pi$ is a uniformizer of $v$, $u_1, \ldots, u_r \in U_v$ are $v$-units, and $\bar{u}_i$ denotes the image of $u_i$ in $(K^v)^\times$. With this notation, we obtain the following.

Lemma 3.3. Let $v$ be a divisorial valuation of $K|k$. Then one has a commutative diagram:

$$\begin{array}{ccc}
K^M_{r+1}(K) & \xrightarrow{\partial_v^M} & H^r+1(K|k, \Lambda(r + 1)) \\
\downarrow R_K & & \downarrow \partial_v \\
K^M_r(K^v) & \xrightarrow{\partial_v^M} & H^r(K^v|k, \Lambda(r))
\end{array}$$

Proof. Let $X$ be a model of $\mathcal{O}_v|k$, and let $u \in U_v$ be given. Then for any sufficiently small $v$-open $k$-subvariety of $X$, one has $u \in \mathcal{O}^\times(U)$. Thus $\mathcal{R}_K(u)$ is contained in the image of $H^1(\mathcal{O}_v|k, \Lambda(1)) \to H^1(K^v|k, \Lambda(1))$. The assertion of the lemma now follows directly from Lemma 3.2 along with the characterization of the tame symbol mentioned above. 

4. Algebraic Dependence and Fibrations

In this section, we discuss the connection between the following three concepts:

1. Algebraic (in)dependence in $K|k$.
2. The cup-product in $H^r(K|k, \Lambda(*)$. 
3. Fibrations whose total space is a model of $K|k$.

In this respect, there are two main proposition which we aim to prove in this section. The first shows that algebraic dependence in $K|k$ is controlled via the Kummer map $\mathcal{R}_K$ and the cup-product in $H^r(K|k, \Lambda(*)$. The second provides us with a method to recover submodules of $H^1(K|k, \Lambda(1))$ which arise from relatively algebraically closed subextensions of $K|k$.

4.1. Good models. The following observation will be used several times throughout this section. Let $L$ be a relatively algebraically closed subextension of $K|k$, and let $f : X \to B$ be a model of $L \hookrightarrow K$. Then $f$ has generically geometrically integral fibers. By replacing $X$ and $B$ with non-empty open $k$-subvarieties, we may assume furthermore that $X$ and $B$ are smooth, and that $f : X \to B$ is a smooth surjective morphism. By further replacing $X$ and $B$ with open $k$-subvarieties, we assume that $X \to B$ is a fibration (i.e. that the induced morphism $X^{an} \to B^{an}$ of complex manifolds is topologically a fibre bundle). In this case, we say that $X \to B$ is a good model of $L \hookrightarrow K$.

Such good models are cofinal among the models of $L \hookrightarrow K$. More precisely, let $f : X \to B$ be a good model of $L \hookrightarrow K$. Then for any non-empty open $k$-subvariety $U \subset B$, the induced model $f^{-1}(U) \to U$ is again good. Also, if $V$ is any non-empty open $k$-subvariety of $X$, then there exists a non-empty open $k$-subvariety $W$ of $V$ such that $W \to f(W)$ is good.
4.2. **Cohomological dimension.** Recall that the Andreatti-Frankel Theorem \cite{AF59} combined with the universal coefficient theorem asserts that whenever \(X\) is a smooth affine \(k\)-variety of dimension \(d\), one has \(H^i(X, \Lambda(j)) = 0\) for \(i > d\). As an immediate consequence of this, we deduce the following fact concerning the cohomological dimension of \(K|k\).

**Fact 4.1.** One has \(H^i(K|k, \Lambda(j)) = 0\) for all \(i > \text{tr.deg}(K|k)\).

4.3. **Algebraic dependence and cup products.** We now prove the first main proposition of this section. First, we recall a straightforward construction which will be useful in the proof. Let \(f_1, \ldots, f_r \in K\) be algebraically independent over \(k\). Extend \(f_1, \ldots, f_r\) to a transcendence base \(f_1, \ldots, f_d\) for \(K|k\). Let \(v_0\) be the \(f_1\)-adic valuation of \(k(f_1, \ldots, f_d)\), and let \(v\) be a prolongation of \(v_0\) to \(K\). Then \(v\) is a divisorial valuation of \(K|k\). Furthermore, note that one has \(v(f_1) \neq 0\), and \(v(f_2) = \cdots = v(f_d) = 0\). Letting \(\bar{f}_i\) denote the image of \(f_i\), \(i = 2, \ldots, d\), in the residue field \(Kv\), we see that \(\bar{f}_2, \ldots, \bar{f}_d\) are algebraically independent in \(Kv|k\), since this holds in the residue field of \(v_0\).

**Proposition 4.2.** Let \(f_1, \ldots, f_r \in K^\times\) be given. Then the following are equivalent:

1. The element \(\mathcal{R}_K^r\{f_1, \ldots, f_r\}\) is trivial in \(H^r(K|k, \Lambda(r))\).
2. The element \(\mathcal{R}_K^r\{f_1, \ldots, f_r\}\) is \(\Lambda\)-torsion in \(H^r(K|k, \Lambda(r))\).
3. The elements \(f_1, \ldots, f_r \in K^\times\) are algebraically dependent over \(k\).

**Proof.** The implication (3) \(\Rightarrow\) (1) follows from Fact \ref{fact:independence} and the functoriality of the situation, while the implication (1) \(\Rightarrow\) (2) is trivial. To conclude, assume that \(f_1, \ldots, f_r \in K^\times\) are algebraically independent over \(k\). We will show that \(\mathcal{R}_K^r\{f_1, \ldots, f_r\}\) is non-\(\Lambda\)-torsion in \(H^r(K|k, \Lambda(r))\). We proceed by induction on \(r\), with the base case \(r = 0\) being trivial.

For the inductive case, choose a divisorial valuation \(v\) of \(K|k\) which has the following properties:

1. First, one has \(v(f_1) \neq 0 = v(f_2) = \cdots = v(f_r)\).
2. Second, letting \(\bar{f}_i\), \(i = 2, \ldots, r\), denote the image of \(f_i\) in \(Kv\), the elements \(\bar{f}_2, \ldots, \bar{f}_r \in (Kv)^\times\) are algebraically independent in \(Kv|k\).

Using Lemma \ref{lem:valuation} we may calculate:

\[
\partial_v(\mathcal{R}_K^r\{f_1, \ldots, f_r\}) = v(f_1) \cdot \mathcal{R}_{Kv}^{r-1}\{\bar{f}_2, \ldots, \bar{f}_r\}.
\]

By the inductive hypothesis, the right-hand-side of this equation is non-\(\Lambda\)-torsion as an element of \(H^{r-1}(Kv|k, \Lambda(r-1))\), hence \(\mathcal{R}_K^r\{f_1, \ldots, f_r\}\) is non-\(\Lambda\)-torsion in \(H^r(K|k, \Lambda(r))\).

4.4. **Geometric submodules.** One of the key points in the proof of our main results is the reconstruction of the image of the canonical map

\[H^1(L|k, \Lambda(1)) \to H^1(K|k, \Lambda(1))\]

associated to a relatively algebraically closed subextension \(L\) of \(K|k\). This subsection proves a key results in this direction. First, we show the injectivity of the map on \(H^1\) associated to \(L \hookrightarrow K\).

**Lemma 4.3.** Let \(L\) be a relatively algebraically closed subextension of \(K|k\). Then the canonical map

\[H^1(L|k, \Lambda(1)) \to H^1(K|k, \Lambda(1))\]

is injective.

**Proof.** Let \(\alpha\) be in the kernel of \(H^1(L|k, \Lambda(1)) \to H^1(K|k, \Lambda(1))\). Following the discussion of \ref{lem:embedding}, we may choose a good model \(X \to B\) of \(L \hookrightarrow K\) such that \(\alpha \in H^1(B, \Lambda(1))\). As \(X \to B\) is a fibration, the map \(H^1(B, \Lambda(1)) \to H^1(X, \Lambda(1))\) is injective. Since the map \(H^1(X, \Lambda(1)) \to H^1(K|k, \Lambda(1))\) is injective as well, it follows that \(\alpha = 0\).
Proposition 4.4. Let \( L \) be a relatively algebraically closed subextension of \( K|k \), and let \( \alpha \in H^1(K|k, \Lambda(1)) \) be given. Assume that \( \alpha \) is not contained in the image of the canonical injective map

\[
H^1(L|k, \Lambda(1)) \hookrightarrow H^1(K|k, \Lambda(1)).
\]

Then there exists a smooth model \( B = B_0 \) of \( L|k \), depending on \( \alpha \), such that for all closed points \( b \in B \), and all systems of regular parameters \( f_1, \ldots, f_r \) of \( \mathcal{O}_{B,b} \), the element

\[
\mathcal{R}_K^r \{f_1, \ldots, f_r\} \cup \alpha
\]

is non-\( \Lambda \)-torsion (in particular, non-trivial) in \( H^{r+1}(K|k, \Lambda(r + 1)) \).

Proof. By the discussion in [11], we may choose a good model \( f : X \to B \) of \( L \to K \) such that \( \alpha \in H^1(X, \Lambda(1)) \). We will show that such a \( B \) satisfies the assertion of the proposition.

Let \( b \) be a closed point in \( B \), and let \( f_1, \ldots, f_r \) be a system of regular parameters of \( \mathcal{O}_{B,b} \). By replacing \( B \) with a sufficiently small open neighborhood of \( b \), and \( X \) with the preimage of this open neighborhood under \( f \), we may assume that the following additional conditions hold true:

1. One has \( f_1, \ldots, f_r \in \mathcal{O}(B) \). Let \( W_i \) denote the zero-locus of \( (f_1, \ldots, f_i) \), \( i = 1, \ldots, r \), in \( B \).
2. The closed subvarieties \( W_1, \ldots, W_r \) are smooth in an integral in \( B \).

Put \( W_0 := B \) and \( Z_0 := X \). The two conditions above imply first that \( W_r = \{ b \} \), and that

\[
W_0 \supseteq W_1 \supseteq \cdots \supseteq W_r
\]

is a flag of smooth integral subvarieties of \( B \), with \( W_{i+1} \) having codimension 1 in \( W_i \). Let \( Z_i \) denote the preimage of \( W_i \) in \( X \). Thus, we have \( Z_r = : Z \) is the preimage of \( b \) in \( X \), and that

\[
Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_r
\]

is again a flag of smooth integral subvarieties of \( X \), with \( Z_{i+1} \) having codimension 1 in \( Z_i \). Furthermore, note that for all \( i = 0, \ldots, r - 1 \), the function \( f_{i+1} \) is a regular parameter for the generic point of \( W_{i+1} \) resp. \( Z_{i+1} \) in \( W_i \) resp. \( Z_i \).

Put \( \partial_i := \partial_{Z_{i-1}, Z_i} \) for \( i = 1, \ldots, r \). By applying Lemma [3.1] successively \( r \) times, we deduce that

\[
\partial_r \circ \cdots \circ \partial_1 (\mathcal{R}_{Z_0} \{f_1 \} \cup \cdots \cup \mathcal{R}_{Z_0} \{f_r \} \cup \alpha) = \beta
\]

where \( \beta \) is the image of \( \alpha \) under the specialization morphism \( H^1(X, \Lambda(1)) \to H^1(Z, \Lambda(1)) \).

Since \( X \to B \) is a good model (in particular \( X^{an} \to B^{an} \) is a fibration), we see that this specialization map fits in a canonical exact sequence

\[
0 \to H^1(B, \Lambda(1)) \to H^1(X, \Lambda(1)) \to H^1(Z, \Lambda(1)).
\]

In particular, we find that one has \( \beta \neq 0 \), for otherwise \( \alpha \) would have been in the image of \( H^1(B, \Lambda(1)) \to H^1(X, \Lambda(1)) \), contradicting the assumption of the proposition.

Finally, recall that \( H^1(Z, \Lambda(1)) \) is \( \Lambda \)-torsion-free, while we have identified \( H^1(Z, \Lambda(1)) \) with its image in \( H^1(k(Z)|k, \Lambda(1)) \). For \( i = 1, \ldots, r \), let \( v_i \) denote the divisorial valuation of \( k(Z_{i-1})|k \) associated to the prime Weil divisor \( Z_i \). Then the calculation above shows that

\[
\partial_{v_r} \circ \cdots \circ \partial_{v_1} (\mathcal{R}_K^r \{f_1, \ldots, f_r\} \cup \alpha) = \beta
\]

as elements of \( H^1(k(Z)|k, \Lambda(1)) \), while \( \beta \) is non-torsion in \( H^1(k(Z)|k, \Lambda(1)) \). Hence we deduce that \( \mathcal{R}_K^r \{f_1, \ldots, f_r\} \cup \alpha \) is non-torsion as an element of \( H^{r+1}(K|k, \Lambda(r + 1)) \), as required.

5. PICARD 1-MOTIVES

Let \( k_0 \) be a field whose algebraic closure is \( k \). As in [12] unless otherwise explicitly specified, we will use the subscript 0 to denote objects over \( k_0 \), and drop the subscript to denote their base-change to \( k \). Specifically, if \( X_0 \) is a \( k_0 \)-variety, then we will write \( X := X_0 \otimes_{k_0} k \), and if \( K_0 \) is a regular function field over \( k_0 \), then we will write \( K := K_0 \cdot k \).
5.1. 1-Motives. Recall that a 1-motive over $k_0$ consists of the following data:

1. A semi-abelian variety $G$ over $k_0$.
2. A finitely-generated free abelian group $L$ endowed with a continuous action of $\text{Gal}_{k_0}$.
3. A $\text{Gal}_{k_0}$-equivariant homomorphism $L \to G(k)$.

This data is commonly summarized as a complex $[L \to G]$ of étale group schemes over $\text{Spec } k_0$, where $L$ is placed in degree 0 and $G$ in degree 1. A morphism of 1-motives over $k_0$ is then simply a morphism of complexes of étale group-schemes over $\text{Spec } k_0$. Given two 1-motives $M_1, M_2$ over $k_0$, we write $\text{Hom}_{k_0}(M_1, M_2)$ for the (abelian) group of all morphisms $M_1 \to M_2$, in the above sense. The base-change of a 1-motive $M := [L \to G]$ to any extension $k_1$ of $k_0$ is computed by taking the base-change term-wise in the complex, and is denoted by $M \otimes_{k_0} k_1$.

5.2. The Hodge realization. Let $M = [L \to G]$ be a 1-motive over $k$. We recall the construction of the Hodge realization of $M$ (associated to the complex embedding $\sigma : k \to \mathbb{C}$). The Hodge realization of $M$ will be a torsion-free integral mixed Hodge structure, which we will denote by $H(M)$ (or $H(M, Z)$).

The underlying abelian group of $H(M)$ is constructed as follows. First, consider the exponential exact sequence of $G^{an}$, which reads as follows:

$$0 \to H_1(G^{an}, Z) \to \text{Lie } G^{an} \to G^{an} \to 0.$$ 

Next, note that one has a canonical map $L \to G(k) \subset G^{an}$ which is part of the data associated to $M$. The underlying abelian group of $H(M)$, which we denote by $H(M)$, is the pull-back of $\text{Lie } G^{an} \to G^{an}$ with respect to this morphism $L \to G^{an}$. In other words, $H(M)$ fits in an exact sequence of the form

$$0 \to H_1(G, Z) \to H(M) \to L \to 0.$$

The mixed Hodge structure $H(M)$ is constructed as follows. First, recall that $G$ is an extension

$$1 \to T \to G \to A \to 1$$

of an abelian $k$-variety $A$ by a $k$-torus $T$. The weight filtration on $H(M)$ is defined as:

$$W_{-2}(H(M)) = H_1(T^{an}, Z), \quad W_{-1}(H(M)) = H_1(G^{an}, Z), \quad W_0(H(M)) = H(M).$$

Finally, the only non-trivial term in the Hodge filtration on $H(M) \otimes \mathbb{C}$ is given by

$$F^0(H(M) \otimes_{\mathbb{Z}} \mathbb{C}) = \ker(H(M) \otimes_{\mathbb{Z}} \mathbb{C} \to \text{Lie } G^{an}),$$

where the map $H(M) \otimes_{\mathbb{Z}} \mathbb{C} \to \text{Lie } G^{an}$ is the one induced by the morphism $H(M) \to \text{Lie } G^{an}$ given as part of the construction of $H(M)$. According to DELIGNE [Del74, Lemma 10.1.3.2], this construction defines a mixed Hodge structure $H(M)$ with underlying abelian group $H(M)$, which fits in an extension of mixed Hodge structures of the form

$$0 \to H_1(G^{an}, Z) \to H(M) \to L \to 0.$$

Here, the homology group $H_1(G^{an}, Z)$ is endowed with its canonical mixed Hodge structure of Hodge type $\{(−1, 0), (0, −1), (−1, −1)\}$, while $L$ is considered as a pure Hodge structure of weight 0.

Given any subring $\Lambda$ of $\mathbb{Q}$, we will write $H(M, \Lambda) := H(M) \otimes_{\mathbb{Z}} \Lambda$ for the base-change of the integral mixed Hodge structure $H(M)$ to $\Lambda$. The construction above is functorial, yielding a (covariant) functor $H(−, Z)$ resp. $H(−, \Lambda)$ from the category of 1-motives over $k$ to the category $\text{MHS}_Z$ of integral Mixed Hodge structures resp. $\text{MHS}_\Lambda$ of mixed Hodge structures over $\Lambda$. The following well-known theorem of DELIGNE will play a crucial role in the rest of the paper.
**Theorem 5.1** ([Deligne [Del74], 10.1.3]). Let $\Lambda$ be a subring of $\mathbb{Q}$, and let $M_1, M_2$ be two 1-motives over $\mathbb{C}$. Then the canonical map

$$\text{Hom}_\mathbb{C}(M_1, M_2) \otimes \mathbb{Z} \Lambda \rightarrow \text{Hom}_{\text{MHS}_\Lambda}(H(M_1, \Lambda), H(M_2, \Lambda))$$

is a bijection.

5.3. The $\ell$-adic realization. Let $\ell$ be a prime and let $M = [L \rightarrow G]$ be a 1-motive over a field $k_0$ whose algebraic closure is $k$. We now recall the construction of the $\ell$-adic realization of $M$. This $\ell$-adic realization, which we will denote by $H_\ell(M)$ (or $H_\ell(M, \mathbb{Z}_\ell)$), will be a finitely-generated torsion-free $\mathbb{Z}_\ell$-module endowed with a canonical continuous action of $\text{Gal}_{k_0}$.

The $\mathbb{Z}_\ell[[\text{Gal}_{k_0}]]$-module $H_\ell(M)$ is constructed in analogy with the $\ell$-adic Tate module, as follows. Let $u : L \rightarrow G(k)$ denote the structure morphism associated with $M$. First, we define

$$H_\ell(M, \mathbb{Z}/\ell^n) := \frac{\{(x, g) \in L \times G(k) : u(x) = -\ell^n \cdot g\}}{\{(\ell^n \cdot x, -u(x)) : x \in L\}}.$$  

Note that $H_\ell(M, \mathbb{Z}/\ell^n)$ has a natural action of $\text{Gal}_{k_0}$. We then define

$$H_\ell(M) = H_\ell(M, \mathbb{Z}_\ell) := \lim_{\ell \to \infty} H_\ell(M, \mathbb{Z}/\ell^n)$$

considered as a $\mathbb{Z}_\ell[[\text{Gal}_{k_0}]]$-module.

For a semi-abelian variety $G$, which we may consider as a 1-motive via $G = [0 \rightarrow G]$, we note that one has

$$H_\ell(G, \mathbb{Z}/\ell^n) = G(k)[\ell^n],$$

hence $H_\ell(G, \mathbb{Z}_\ell)$ is indeed the $\ell$-adic Tate-module of $G$. More generally, for a 1-motive of the form $M = [L \rightarrow G]$, the $\mathbb{Z}_\ell[[\text{Gal}_{k_0}]]$-module $H_\ell(M)$ is as an extension of the form

$$0 \rightarrow H_\ell(G, \mathbb{Z}_\ell) \rightarrow H_\ell(M, \mathbb{Z}_\ell) \rightarrow L \otimes \mathbb{Z}_\ell \rightarrow 0.$$  

Given any subring $\Lambda$ of $\mathbb{Q}$, we write $\Lambda_\ell := \Lambda \otimes \mathbb{Z}_\ell$ and $H_\ell(M, \Lambda_\ell) := H_\ell(M, \mathbb{Z}_\ell) \otimes \Lambda_\ell$ for the base-change of $H_\ell(M, \mathbb{Z}_\ell)$ to $\Lambda_\ell$. The construction above is functorial, yielding a (covariant) functor $H_\ell(-, \mathbb{Z}_\ell)$ resp. $H_\ell(-, \Lambda_\ell)$ from the category of 1-motives over $k_0$ to the category of (continuous) $\mathbb{Z}_\ell[[\text{Gal}_{k_0}]]$ resp. $\Lambda_\ell[[\text{Gal}_{k_0}]]$-modules. The following theorem, which is due to Jannsen [Jan95], generalizes the famous theorem due to Faltings [Fal83] concerning morphisms of abelian varieties over finitely-generated fields.

**Theorem 5.2** ([Jannsen [Jan95], Theorem 4.3]). Let $\Lambda$ be a subring of $\mathbb{Q}$. Assume that $k_0$ is a finitely-generated field whose algebraic closure is $k$. Let $M_1, M_2$ be two 1-motives over $k_0$. Then the canonical map

$$\text{Hom}_{k_0}(M_1, M_2) \otimes \mathbb{Z} \Lambda_\ell \rightarrow \text{Hom}_{\Lambda_\ell[[\text{Gal}_{k_0}]]}(H_\ell(M_1, \Lambda_\ell), H_\ell(M_2, \Lambda_\ell))$$

is a bijection.

5.4. Picard 1-motives. Let $X_0$ be a smooth proper geometrically-integral $k_0$-variety, and let $U_0$ be a non-empty open $k_0$-subvariety of $X_0$. Put $Z := X \setminus U$. Consider the group $\text{Div}_0^0(X)$ of algebraically-trivial Weil divisors on $X$, as well as the subgroup $\text{Div}_Z^0(X)$ of algebraically trivial Weil divisors on $X$ which are supported on $Z$. Note that $\text{Div}_Z^0(X)$ is a finitely-generated free abelian group endowed with a canonical continuous action of $\text{Gal}_{k_0}$.

Next, consider $\text{Pic}^0_{X_0}$, the Picard variety of $X_0$. Recall that one has a canonical morphism $\text{Div}_Z^0(X) \rightarrow \text{Pic}^0_{X_0}(k) = \text{Pic}^0_{X_0}(k)$, mapping a Weil divisor to its associated line bundle. We thereby obtain the so-called Picard 1-Motive of $U_0$ (associated to the inclusion $U_0 \rightarrow X_0$), a 1-motive over $k_0$ which is defined and denoted as

$$M_{1,1}(U_0) := [\text{Div}_Z^0(X) \rightarrow \text{Pic}^0_{X_0}]_1.$$
Whenever $V_0 \subset U_0$ is a non-empty open $k_0$-subvariety, we obtain a canonical morphism
\[ M^{1,1}(U_0) \to M^{1,1}(V_0) \]
of 1-motives over $k_0$, which just arises from the inclusion $\text{Div}^0_{X\setminus U}(X) \hookrightarrow \text{Div}^0_{X\setminus V}(X)$. Furthermore, the construction of $M^{1,1}(U_0)$ is clearly compatible with base-change. For instance, one has $M^{1,1}(U_0) \otimes k = M^{1,1}(U)$ as 1-motives over $k$. Here $M^{1,1}(U_0)$ is computed with respect to the inclusion $U_0 \hookrightarrow X_0$ and $M^{1,1}(U)$ is computed with respect to the inclusion $U \hookrightarrow X$.

The following two theorems, due to Barbieri-Viale, Srinivas [BVS01], describe the Hodge and $\ell$-adic realizations of such Picard 1-motives. They will also play a crucial role later on in the proofs of the main results of this paper.

**Theorem 5.3 ([BVS01 Theorem 4.7]).** Let $\Lambda$ be a subring of $\mathbb{Q}$. Let $X$ be a smooth proper integral variety over $k$, and let $U$ be a non-empty open $k$-subvariety of $X$. Consider the Picard 1-motive $M^{1,1}(U)$ of $U$, computed with respect to the inclusion $U \hookrightarrow X$, as defined above. Then one has a canonical isomorphism of mixed Hodge structures $H(U \otimes \Lambda(1)) \cong H^1(U, \Lambda(1))$. Moreover, this isomorphism is functorial with respect to embeddings $V' \hookrightarrow V$ of open $k$-subvarieties of $X$.

**Theorem 5.4 ([BVS01 Theorem 4.10]).** Let $\Lambda$ be a subring of $\mathbb{Q}$. Let $X_0$ be a smooth proper geometrically-integral variety over $k_0$, and let $U_0$ be a non-empty open $k_0$-subvariety of $X_0$. Consider the Picard 1-motive $M^{1,1}(U_0)$ of $U_0$, computed with respect to the inclusion $U_0 \hookrightarrow X_0$, as defined above. Then one has a canonical isomorphism of $\Lambda$-modules $H_1(U_0, \Lambda(1)) \cong H^1_1(U_0, \Lambda(1))$. Moreover, this isomorphism is functorial with respect to embeddings $V_0 \hookrightarrow U_0$ of non-empty open $k_0$-subvarieties of $X_0$.

**Remark 5.5.** To be completely precise, our definition of the Picard 1-motive agrees with the definition from [BVS01] only in the case where the boundary $Z = X \setminus U$ has simple normal crossings. See Remark 4.5 of loc.cit. However, it seems to be well-known that the construction discussed above yields an equivalent result. Below is a sketch of this argument, which uses embedded resolution of singularities.

Let $k_0$ be a field whose algebraic closure is $k$. Let $X_0$ be a smooth proper geometrically-integral $k_0$-variety, and let $U_0$ be a non-empty open $k_0$-subvariety of $X_0$. Following Hironaka [Hir64], there exists a modification $\tilde{X}_0 \to X_0$ obtained by successive blowups at smooth centers concentrated away from $U_0$ (hence $\tilde{X}_0 \to X_0$ is an isomorphism above $U_0$), such that $\tilde{X}_0 \setminus U_0$ has geometrically simple normal crossings. Put $Z := X \setminus U$ and $\tilde{Z} := \tilde{X} \setminus U$. Note that one has a canonical morphism of 1-motives
\[ [\text{Div}^0_Z(X) \to \text{Pic}^0_{X_0}] \to [\text{Div}^0_{\tilde{Z}}(\tilde{X}) \to \text{Pic}^0_{\tilde{X}_0}] \]
We claim that this is an isomorphism. Indeed, it is well-known that the pull-back morphism $\text{Pic}^0_{X_0} \to \text{Pic}^0_{\tilde{X}_0}$ is an isomorphism. On the other hand, the inclusion $\text{Div}^0_{\tilde{Z}}(\tilde{X}) \hookrightarrow \text{Div}^0_{\tilde{Z}}(\tilde{X})$ is also an isomorphism, as $\tilde{Z}$ is the proper transform of $Z$ in the modification $\tilde{X} \to X$.

In fact, the assertion concerning $\text{Div}^0_{\tilde{Z}}(\tilde{X}) \hookrightarrow \text{Div}^0_{\tilde{Z}}(\tilde{X})$ can actually be proven in cohomological terms, using [BVS01 Theorem 4.7] directly, as follows. First, by applying [BVS01 Theorem 4.7] for the Picard 1-motive associated to $U \hookrightarrow \tilde{X}$, we note that we have a surjective morphism
\[ H^1(U, \mathbb{Z}(1)) \to \text{Div}^0_{\tilde{Z}}(\tilde{X}) \]
which is given by the the sum of the residue morphisms associated to the irreducible codimension 1 components of $\tilde{Z}$. However, it is easy to see, using cohomological purity, that this morphism actually factors through the inclusion $\text{Div}^0_{\tilde{Z}}(\tilde{X}) \hookrightarrow \text{Div}^0_{\tilde{Z}}(\tilde{X})$. Indeed, let $W$ denote the closed subvariety of $X$ which consists of all irreducible components of $Z$ whose codimension in $X$ is $\geq 2$,
along with the singular locus of $Z$. Then $W$ has codimension $\geq 2$ in $X$, hence the map

$$H^2_Z(X, Z(1)) \to H^2_{Z \setminus W}(X \setminus W, Z(1))$$

is an isomorphism by purity. The purity isomorphism $\Psi_{X \setminus W, Z \setminus W}$ identifies $H^2_{Z \setminus W}(X \setminus W, Z(1))$ with $H^0(Z \setminus W, Z) = \text{Div}_Z(X)$, the group of Weil divisors on $X$ supported on $Z$. The corresponding map

$$H^1(U, Z(1)) \to H^2_{Z \setminus W}(X \setminus W, Z(1)) \cong \text{Div}_Z(X)$$

is the sum of the residue morphisms associated to the codimension 1 irreducible components of $Z$. This map fits in an exact sequence of the form

$$H^1(U, Z(1)) \delta \to H^2_{Z \setminus W}(X \setminus W, Z(1)) \to \text{Div}_Z(X),$$

where the map $\text{Div}_Z(X) \to H^2(X, Z(1))$ is the usual cycle class map. By Severi’s Theorem of the Base and the equivalence of homological and algebraic equivalence for divisors, we see that the image of $\delta : H^1(U, Z(1)) \to \text{Div}_Z(X)$ is precisely $\text{Div}_Z^0(X)$. From this, it is easy to see that the map $H^1(U, Z(1)) \to \text{Div}_Z^0(X)$, which is the sum of the residue morphisms associated to the codimension 1 irreducible components of $\bar{Z}$, must factor through the inclusion $\text{Div}_Z^0(X) \hookrightarrow \text{Div}_{\bar{Z}}^0(X)$.

A similar argument shows that for smooth $U_0$, the 1-motive $M^{1,1}(U_0)$ is independent from the choice of embedding $U_0 \hookrightarrow X_0$ in to a smooth proper geometrically-integral $k_0$-variety $X_0$. Such an embedding always exists by Nagata [Nag62] and Hironaka [Hir64].

**Remark 5.6.** Concerning Theorem 5.4, it is important to note that [BVS01] constructs the full étale realization of a 1-motive, resulting in a free $\hat{\mathbb{Z}}$-module. The $\ell$-adic realization we have described is just the pro-$\ell$ primary component of the full étale realization discussed in loc.cit.

Also, it is important to note that the canonical isomorphism

$$H_\ell(M^{1,1}(U)) = H_\ell(U, Z(1))$$

from loc.cit. is only stated for algebraically closed base-fields. Our Theorem 5.4 still follows from this. Indeed, if $k_0$ is a field whose algebraic closure is $k$, and $U_0$ is a smooth $k_0$-variety embedded in a smooth proper geometrically-integral $k_0$-variety $X_0$, then it follows directly from the definition that, on the level of $\mathbb{Z}_\ell$-modules, one has

$$H_\ell(M^{1,1}(U)) = H_\ell(M^{1,1}(U_0)).$$

Loc.cit. then proves that one has $H_\ell(M^{1,1}(U)) = H_\ell(U, Z(1))$, while the construction from loc.cit. is visibly compatible with the action of $\text{Gal}_{k_0}$.

### 6. An Anabelian Result

In this section we discuss an anabelian result, to which we will reduce our two main theorems. Throughout this section, we assume that $\Lambda$ is a subring of $\mathbb{Q}$. Recall that we have defined

$$\mathcal{K}_\Lambda(K|k) := (K^\times / k^\times) \otimes_\mathbb{Z} \Lambda.$$ 

Also recall that, for $t \in K^\times$, we write $t^\circ$ for the image of $t$ in $\mathcal{K}_\Lambda(K|k)$. Note that for any $x \in \mathcal{K}_\Lambda(K|k)$, there exists some $t \in K^\times$ such that $t^\circ \in \Lambda \cdot x$. Given two elements $x, y \in \mathcal{K}_\Lambda(K|k)$, and elements $u, v \in K^\times$ such that $u^\circ \in \Lambda \cdot x$, $v^\circ \in \Lambda \cdot y$, we say that $x, y$ are (in)dependent provided that $u, v$ are algebraically (in)dependent over $k$. It is easy to see that this definition doesn’t depend on the choice of $u, v$ as above, and that $x, y$ are dependent if and only if $x, y$ are not independent.

Next, note that for a subextension $M$ of $K|k$, the canonical map

$$\mathcal{K}_\Lambda(M|k) \to \mathcal{K}_\Lambda(K|k)$$

is injective. We will always identify $\mathcal{K}_\Lambda(M|k)$ with its image in $\mathcal{K}_\Lambda(K|k)$ via this inclusion.
For a subset $S \subset K$, we write
\[\text{acl}_K(S) := \overline{k(S)} \cap K\]
for the relative algebraic closure of $k(S)$ in $K$. A submodule $\mathcal{X}$ of $\mathcal{X}_\Lambda(K|k)$ will be called a rational submodule provided that there exists some $t \in K \setminus k$ such that $\text{acl}_K(t) = k(t)$, and such that $\mathcal{X} = \mathcal{X}_\Lambda(k(t)|k)$.

Next, suppose that $L|l$ is a further function field over an algebraically closed field $l$ of characteristic 0, and let
\[\phi : \mathcal{X}_\Lambda(K|k) \cong \mathcal{X}_\Lambda(L|l)\]
be an isomorphism of $\Lambda$-modules. We say that
1. $\phi$ is compatible with acl provided that for all $x, y \in \mathcal{X}_\Lambda(K|k)$, the pair $x, y$ is dependent in $\mathcal{X}_\Lambda(K|k)$ if and only if the pair $\phi(x), \phi(y)$ is dependent in $\mathcal{X}_\Lambda(L|l)$.
2. $\phi$ is compatible with rational submodules provided that $\phi$ induces a bijection on rational submodules of $\mathcal{X}_\Lambda(K|k)$ resp. $\mathcal{X}_\Lambda(L|l)$.

The collection of all isomorphisms $\mathcal{X}_\Lambda(K|k) \cong \mathcal{X}_\Lambda(L|l)$ which are compatible with acl and with rational submodules will be denoted by
\[\text{Isom}^{\text{acl}}(\mathcal{X}_\Lambda(K|k), \mathcal{X}_\Lambda(L|l)).\]

Note that for any $\phi : \mathcal{X}_\Lambda(K|k) \cong \mathcal{X}_\Lambda(L|l)$ which is compatible with acl and with rational submodules, and any $\epsilon \in \Lambda^\times$, the corresponding isomorphism $\epsilon \cdot \phi$ is again compatible with acl and with rational submodules. In particular, we have a canonical action of $\Lambda^\times$ on $\text{Isom}^{\text{acl}}(\mathcal{X}_\Lambda(K|k), \mathcal{X}_\Lambda(L|l))$, and we denote the orbits of this action by
\[\text{Isom}^{\text{acl}}(\mathcal{X}_\Lambda(K|k), \mathcal{X}_\Lambda(L|l))/\Lambda^\times.\]

Finally, note that any isomorphism of fields $K \cong L$ restricts to an isomorphism on the base-fields $k \cong l$, since $k$ resp. $l$ is precisely the set of multiplicatively divisible elements of $K$ resp. $L$. Thus, any such isomorphism $K \cong L$ induces in the canonical way an isomorphism $\mathcal{X}_\Lambda(K|k) \cong \mathcal{X}_\Lambda(L|l)$ which is compatible with acl and with rational submodules. In other words, we obtain a canonical map
\[\text{Isom}(K, L) \to \text{Isom}^{\text{acl}}(\mathcal{X}_\Lambda(K|k), \mathcal{X}_\Lambda(L|l)) \to \text{Isom}^{\text{acl}}(\mathcal{X}_\Lambda(K|k), \mathcal{X}_\Lambda(L|l))/\Lambda^\times,\]
which is the subject of our key anabelian result.

**Theorem 6.1.** Let $\Lambda$ be a subring of $\mathbb{Q}$. Let $k, l$ be algebraically closed fields of characteristic 0 and let $K$ resp. $L$ be function fields over $k$ resp. $l$, such that $\text{tr.deg}(K|k) \geq 2$. Then the canonical map
\[\text{Isom}(K, L) \to \text{Isom}^{\text{acl}}(\mathcal{X}_\Lambda(K|k), \mathcal{X}_\Lambda(L|l))/\Lambda^\times\]
is a bijection.

**Remark 6.2.** Although we have stated Theorem 6.1 as a theorem, one may deduce it using known results from the literature, in certain cases. In the case where $\Lambda = \mathbb{Z}$, Theorem 6.1 follows from the main result of BOGOMOLOV-TSCHINKEL [BT09]. More generally, if $\Lambda$ is a proper subring of $\mathbb{Q}$, then one may deduce Theorem 6.1 by reducing to the main result of POP [Pop12b]. Finally, if $\text{tr.deg}(K|k) \geq 5$, then one may deduce Theorem 6.1 from the work of EVANS-HRUSHOVSKI [EH91] and GISMATULLIN [Gis08], along with some arguments similar to the ones in 6.6 below (see Remark 6.4). Moreover, in all these cases the condition of compatibility with rational submodules can be relaxed.

In this respect, the most interesting case of Theorem 6.1 is where $\Lambda = \mathbb{Q}$, and where one considers function fields of transcendence degree $\geq 2$. In such cases, we do not know of a straightforward way to deduce Theorem 6.1 from known results in the literature. In particular, it is unclear whether the condition of compatibility with rational submodules can be relaxed in this case.
The goal for the rest of this section will be to prove Theorem 6.1. The bulk of the proof is devoted to constructing a (functorial) left inverse of the canonical map

\[ \text{Isom}(K, L) \rightarrow \text{Isom}_{\text{rat}}^{\text{acl}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))_{/\Lambda^\times}. \]

Because of this, for most of this section, we will work primarily with a fixed element \( \phi \) in the set \( \text{Isom}_{\text{rat}}^{\text{acl}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l)) \), and show how to produce an associated element of \( \text{Isom}(K, L) \). We will henceforth assume that \( \text{tr}. \deg(K|k) \geq 2 \).

6.1. Compatibility with the geometric lattice. As an expository tool, we will consider the so-called geometric lattice associated to the function field \( K|k \), which is denoted by \( (K|k) \). As a set, \( (K|k) \) is the collection of relatively algebraically closed subextensions of \( K|k \).

We consider \( (K|k) \) as a graded lattice, as follows. The (complete) lattice structure of \( (K|k) \) is given by the intersection (the infimum) and the relative algebraic closure \( \text{acl} \) (the supremum) in \( K \). The \( * \) in \( (K|k) \) denotes the grading, which is determined by transcendence degree over \( k \). In other words,

\[ (K|k) = \bigcap_{r \geq 0} \mathcal{G}^r(K|k) \]

where \( \mathcal{G}^r(K|k) \) denotes the relatively algebraically closed subextensions of \( K|k \) which are of transcendence degree \( r \) over \( k \). Finally, note that the lattice structure of \( (K|k) \) is strictly compatible with the grading, in the sense that, whenever \( L_1, L_2 \in (K|k) \) are given, the inclusion \( L_1 \subset L_2 \) implies that \( \text{tr}. \deg(L_1|k) \leq \text{tr}. \deg(L_2|k) \). If furthermore \( \text{tr}. \deg(L_1|k) = \text{tr}. \deg(L_2|k) \), then one has \( L_1 = L_2 \).

Lemma 6.3. Assume that \( \phi : \mathcal{K}_\Lambda(K|k) \cong \mathcal{K}_\Lambda(L|l) \) is an isomorphism of \( \Lambda \)-modules which is compatible with acl. Then there exists an isomorphism of geometric lattices \( \phi^\#: \mathcal{G}^*(K|k) \cong \mathcal{G}^*(L|l) \) such that for all \( M \in \mathcal{G}^*(K|k) \), and setting \( N := \phi^\# M \), the dotted arrow in the following diagram can be (uniquely) completed to an isomorphism of \( \Lambda \)-modules:

\[
\begin{array}{ccc}
\mathcal{K}_\Lambda(K|k) & \xrightarrow{\phi} & \mathcal{K}_\Lambda(L|l) \\
\wedge \downarrow & & \downarrow \\
\mathcal{K}_\Lambda(M|k) & \rightarrow & \mathcal{K}_\Lambda(N|l)
\end{array}
\]

Proof. We say that a submodule \( \mathcal{K} \) of \( \mathcal{K}_\Lambda(K|k) \) is dependently-closed provided that \( \mathcal{K} \) contains all \( y \in \mathcal{K}_\Lambda(K|k) \) such that there exists some non-trivial \( x \in \mathcal{K} \) with \( x, y \) dependent in \( \mathcal{K}_\Lambda(K|k) \). Since \( \Lambda \) is a subring of \( \mathbb{Q} \), we see that the submodules of \( \mathcal{K}_\Lambda(K|k) \) of the form \( \mathcal{K}_\Lambda(M|k) \) for \( M \in \mathcal{G}^*(K|k) \) are precisely the \( \Lambda \)-submodules of \( \mathcal{K}_\Lambda(K|k) \) which are dependently-closed. The assertion follows easily from this observation, since \( \phi \) is compatible with acl.

Remark 6.4. Note that Lemma 6.3 yields a canonical map

\[ \text{Isom}_{\text{rat}}^{\text{acl}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))_{/\Lambda^\times} \rightarrow \text{Isom}(\mathcal{G}^*(K|k), \mathcal{G}^*(L|l)), \]

which is easily seen to be functorial with respect to isomorphisms. The canonical map

\[ \text{Isom}(K, L) \rightarrow \text{Isom}(\mathcal{G}^*(K|k), \mathcal{G}^*(L|l)) \]

factors through the above mentioned map. In the case where \( \text{tr}. \deg(K|k) \geq 5 \), one may use the results of Evans-Hrushovski [EH91], [EH95] and Gismatullin [Gis08] to deduce that the map

\[ \text{Isom}(K, L) \rightarrow \text{Isom}(\mathcal{G}^*(K|k), \mathcal{G}^*(L|l)) \]
is a bijection, hence the map mentioned in Theorem 6.1 has a functorial left-inverse. Using arguments similar to the ones mentioned in [6.6 and one can further deduce that the map

\[
\text{Isom}_\text{rat}^{\text{acl}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))/\Lambda \times \rightarrow \text{Isom}(G^*(K|k), G^*(L|l))
\]

is injective (see also the similar arguments in TOPAZ [Top16b], hence proving Theorem 6.1 in the case where \(\text{tr.\,deg}(K|k) \geq 5\).

In contrast to this, the proof which we present below in the case where \(\text{tr.\,deg}(K|k) \geq 2\) is much more technical, as it uses \(\Lambda\)-module structure of \(\mathcal{K}_\Lambda(K|k)\) resp. \(\mathcal{K}_\Lambda(L|l)\) in a more fundamental way, while eventually relying on the so-called Fundamental Theorem of Projective Geometry (cf. [Art88]).

6.2. Compatibility with divisorial valuations. For a divisorial valuation \(v\) of \(K|k\), we will write

\[
\mathcal{U}_v := \text{Image}((U_v/k^\times) \otimes_{\mathbb{Z}} \Lambda \rightarrow \mathcal{K}_\Lambda(K|k)), \quad \mathcal{U}_v^1 := \text{Image}((U_v^1 \cdot k^\times/k^\times) \otimes_{\mathbb{Z}} \Lambda \rightarrow \mathcal{K}_\Lambda(K|k)).
\]

Note that one has \(\mathcal{U}_v^1 \subset \mathcal{U}_v \subset \mathcal{K}_\Lambda(K|k)\), and that the map \(U_v \rightarrow (K^\times)^v\) induces a canonical isomorphism \(\mathcal{U}_v/\mathcal{U}_v^1 \cong \mathcal{K}_\Lambda(K_v|k)\).

We will need to use a variant of the local theory from almost abelian anabelian geometry, in order to recover \(\mathcal{U}_v\) and \(\mathcal{U}_v^1\) for divisorial valuations \(v\) of \(K|k\) from the given data. Such “almost-abelian” local theories are now extensively developed – see [BY10, Pop10, Top15a, Top16a]. However, the precise statement which we need in our context has not appeared in the literature. Because of this, we have given the full details for this local theory in an appendix to this paper. The following fact, which follows directly from Theorem A.1 from the appendix, summarizes the result which we need.

Fact 6.5. Assume that \(\phi : \mathcal{K}_\Lambda(K|k) \cong \mathcal{K}_\Lambda(L|l)\) is an isomorphism of \(\Lambda\)-modules which is compatible with acl. Then for all divisorial valuations \(v\) of \(K|k\), there exists a unique divisorial valuation \(v^\phi\) of \(L|l\) such that

\[
\phi(\mathcal{U}_v) = \mathcal{U}_{v^\phi}, \quad \phi(\mathcal{U}_v^1) = \mathcal{U}_{v^\phi}^1.
\]

6.3. Rational submodules. Given \(t \in K \setminus k\), recall that \(\text{acl}_K(t) = \overline{k(t)} \cap K\) denotes the relative algebraic closure of \(t\) in \(K\), and put

\[\mathcal{K}_t := \mathcal{K}_\Lambda(\text{acl}_K(t)|k)\]

Also recall that we have identified \(\mathcal{K}_t\) with its image in \(\mathcal{K}_\Lambda(K|k)\) via the canonical (injective) map \(\mathcal{K}_t \hookrightarrow \mathcal{K}_\Lambda(K|k)\). An element \(t \in K \setminus k\) is called general in \(K|k\) provided that \(K\) is regular over \(k(t)\). In particular, if \(t\) is general in \(K|k\) then \(\mathcal{K}_t\) is a rational submodule of \(\mathcal{K}_\Lambda(K|k)\). And conversely, any rational submodule \(\mathcal{K}\) of \(\mathcal{K}_\Lambda(K|k)\) is of the form \(\mathcal{K}_t\) for some general element \(t\) of \(K|k\).

Note that if \(t\) is general in \(K|k\), then any element of the form

\[
u := \frac{a \cdot t + b}{c \cdot t + d}, \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \text{GL}_2(k)
\]

is again a general element of \(K|k\) and one has \(\mathcal{K}_t = \mathcal{K}_u\) as rational submodules of \(\mathcal{K}_\Lambda(K|k)\). The following so-called Birational Bertini Theorem shows the abundance of general elements in higher-dimensional function fields.

Fact 6.6 (Birational Bertini, cf. [Lan72, Ch. VIII, pg. 213]). Let \(x, y \in K\) be algebraically independent over \(k\). Then for all but finitely many \(a \in k\), the element \(x + a \cdot y\) is general in \(K|k\).
6.4. **Divisors on one-dimensional subfields.** Let \( t \in K \setminus k \) be a transcendental element, and put \( \mathcal{K} := \mathcal{K}_t \), which is considered as a submodule of \( \mathcal{K}_\Lambda(K|k) \), as always. We will consider the following collection of submodules of \( \mathcal{K} \):

\[
\mathcal{D}_t = \mathcal{D}_{\mathcal{K}} := \{ \mathcal{U}_v \cap \mathcal{K} : \mathcal{K} \not\subset \mathcal{U}_v \}_v
\]

where \( v \) varies over the *divisorial* valuations of \( K|k \). We also write \( \mathcal{D}_t = \mathcal{D}_{acl_K(t)} \) for the collection of all divisorial valuations of \( acl_K(t)|k \). I.e. \( \mathcal{D}_t \) is in bijection with the closed points of the unique projective normal model \( C_t \) of \( acl_K(t)|k \); this bijection maps \( v \in \mathcal{D}_t \) to its unique center on \( C_t \), as usual. The following lemma compares the two sets \( \mathcal{D}_t \) and \( \mathcal{D}_t \).

**Lemma 6.7.** Let \( t \in K \setminus k \) be a transcendental element in \( K|k \). Put \( M := acl_K(t) \) and \( \mathcal{K} := \mathcal{K}_t \). Then the following hold:

1. For all \( \mathcal{U} \in \mathcal{D}_t \), the quotient \( \mathcal{K}/\mathcal{U} \) is isomorphic to \( \Lambda \).
2. One has a canonical bijection \( \mathcal{D}_t \cong \mathcal{D}_t \) defined by \( w \mapsto \mathcal{U}_w \), for \( w \in \mathcal{D}_t \). Here \( \mathcal{U}_w \) is considered as a submodule of \( \mathcal{K}_\Lambda(M|k) = \mathcal{K} \). The inverse \( \mathcal{D}_t \cong \mathcal{D}_t \) is given by sending \( \mathcal{U} = \mathcal{U}_v \cap \mathcal{K} \) to the restriction of \( v \) to \( M \), where \( v \) is a divisorial valuation of \( K|k \) such that \( \mathcal{K} \not\subset \mathcal{U}_v \).

**Proof.** Concerning assertion (1), let \( v \) be a divisorial valuation of \( K|k \) such that \( \mathcal{K} \not\subset \mathcal{U}_v \) and put \( \mathcal{U} = \mathcal{U}_v \cap \mathcal{K} \). Recall that \( \mathcal{K}_\Lambda(K|k)/\mathcal{U}_v \) is isomorphic \( \Lambda \), hence one has a canonical *injective* morphism of \( \Lambda \)-modules:

\[ \mathcal{K}/\mathcal{U} \to \mathcal{K}_\Lambda(K|k)/\mathcal{U}_v \cong \Lambda. \]

The image of this map is non-trivial as the restriction of \( v \) to \( M \) is non-trivial. Since \( \Lambda \) is a subring of \( \mathbb{Q} \) (in particular, it's a PID of characteristic 0), we see that the quotient \( \mathcal{K}/\mathcal{U} \) is isomorphic to \( \Lambda \).

Now we prove assertion (2). First, let \( w \) be a divisorial valuation of \( M|k \). Then there exists a divisorial valuation \( v \) of \( K|k \) whose restriction to \( M \) is \( w \). It is easy to see in this case that one has \( \mathcal{U}_w \subset \mathcal{U}_v \cap \mathcal{K} \), while \( \mathcal{K} \not\subset \mathcal{U}_v \). Since both \( \mathcal{K}/\mathcal{U}_w \) and \( \mathcal{K}/\mathcal{U}_v \cap \mathcal{K} \) are isomorphic to \( \Lambda \), and since \( \Lambda \) is a PID of characteristic 0, it follows that \( \mathcal{U}_w = \mathcal{U}_v \cap \mathcal{K} \in \mathcal{D}_t \).

Similarly, let \( \mathcal{U} \in \mathcal{D}_t \) be given, and let \( v \) be a divisorial valuation of \( K|k \) such that \( \mathcal{K} \not\subset \mathcal{U}_v \) and such that \( \mathcal{U}_v \cap \mathcal{K} = \mathcal{U} \). Consider the restriction \( w \) of \( v \) to \( M \). Then \( w \) is non-trivial on \( M \), hence \( w \) is a divisorial valuation of \( M|k \). Note also that \( \mathcal{U}_w \subset \mathcal{U} \). Since both \( \mathcal{K}/\mathcal{U} \) and \( \mathcal{K}/\mathcal{U}_w \) are isomorphic to \( \Lambda \), we find that \( \mathcal{U} = \mathcal{U}_w \) similarly to before. \( \square \)

6.5. **Rational-like collections.** Assume now that \( t \) is a general element of \( K|k \), so that \( \mathcal{K} = \mathcal{K}_t \) is a rational submodule of \( \mathcal{K}_\Lambda(K|k) \). Recall that \( \mathcal{K}/\mathcal{U} \cong \Lambda \) for every \( \mathcal{U} \in \mathcal{D}_t \) by Lemma 6.7. Consider a collection of such isomorphisms:

\[ \Phi = (\Phi_\mathcal{U} : \mathcal{K}/\mathcal{U} \cong \Lambda)_{\mathcal{U} \in \mathcal{D}_t}. \]

As any element of \( \mathcal{K} \) is contained in all but finitely many of the \( \mathcal{U} \in \mathcal{D}_t \) by Lemma 6.7, we see that this collection induces a canonical map

\[ \text{div}_\Phi : \mathcal{K} \to \bigoplus_{\mathcal{U} \in \mathcal{D}_t} \Lambda \cdot [\mathcal{U}], \]

defined by \( \text{div}_\Phi(x) = \sum_{\mathcal{U} \in \mathcal{D}_t} \Phi_\mathcal{U}(x + \mathcal{U}) \cdot [\mathcal{U}] \). Here \( [\mathcal{U}] \) is merely a placeholder specifying the \( \mathcal{U} \in \mathcal{D}_t \) in the direct sum.

We say that \( \Phi \) is a *rational-like collection* provided that this map \( \text{div}_\Phi \) fits in a short exact sequence of the form

\[ 0 \to \mathcal{K} \overset{\text{div}_\Phi}{\to} \bigoplus_{\mathcal{U} \in \mathcal{D}_t} \Lambda \cdot [\mathcal{U}] \overset{\text{sum}}{\to} \Lambda \to 0. \]
If $\Phi = (\Phi_U)_{U \in \mathcal{D}_t}$ is such a rational-like collection and $\epsilon \in \Lambda^\times$ is given, then we obtain an induced rational like collection $\epsilon \cdot \Phi := (\epsilon \cdot \Phi_U)_{U \in \mathcal{D}_t}$.

By Lemma 6.7, there is a canonical rational-like collection for $\mathcal{K}$, which is constructed from the field structure of $M := acl_K(t) = k(t)$, as follows. For $U \in \mathcal{D}_t$, choose a divisorial valuation $w$ of $M|k$ such that $\mathcal{U}_w = U$. Consider the isomorphism $\Phi_{\mathcal{U}}^\text{can}$ which is the unique one making the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{(M^\times/k^\times) \otimes_{\mathbb{Z}} \Lambda} & \mathcal{K}/\mathcal{U} \\
\downarrow & & \downarrow \\
\mathcal{K}/\mathcal{U} & \xrightarrow{\Phi_{\mathcal{U}}^\text{can}} & \Lambda
\end{array}
$$

We write $\Phi_{\mathcal{U}}^\text{can} := (\Phi_{\mathcal{U}}^\text{can})_{U \in \mathcal{D}_t}$, and call $\Phi_{\mathcal{U}}^\text{can}$ the canonical rational-like collection associated to the rational submodule $\mathcal{K}$. Also, we will simplify the notation by writing $\text{div}_\text{can} := \text{div}_{\Phi_{\mathcal{U}}^\text{can}}$.

In particular, the exact sequence corresponding to the canonical rational-like collection:

$$
0 \to \mathcal{K} \xrightarrow{\text{div}_\text{can}} \bigoplus_{U \in \mathcal{D}_t} \Lambda \cdot [U] \xrightarrow{\text{sum}} \Lambda \to 0
$$

is nothing other than the usual divisor exact sequence

$$
0 \to k(t)^{\times}/k^\times \xrightarrow{\text{div}} \text{Div}(\mathbb{P}^1_k) \xrightarrow{\text{deg}} \mathbb{Z} \to 0,
$$
tensored with $\Lambda$, and obtained by identifying $\mathcal{K}_\Lambda(k(t)|k) = (k(t)^{\times}/k^\times) \otimes_{\mathbb{Z}} \Lambda$ with $\mathcal{K}_t = \mathcal{K}$ via the inclusion $k(t) \hookrightarrow K$, and identifying $\mathcal{D}_t$ with $\mathcal{D}_t$ via Lemma 6.7.

In general, there is no way to reconstruct the canonical rational-like collection associated to $\mathcal{K}_t$ on the nose. Nevertheless, any rational-like collection differs from the canonical one by some (unique) element $\epsilon \in \Lambda^\times$, as the following lemma shows.

**Lemma 6.8.** Let $\mathcal{K}$ be a rational submodule of $\mathcal{K}_t(K|k)$, let $\Phi$ be a rational-like collection for $\mathcal{K}$, and consider the canonical rational-like collection $\Phi_{\mathcal{K}}^\text{can}$ associated to $\mathcal{K}$. Then there exists a (unique) $\epsilon \in \Lambda^\times$ such that $\Phi = \epsilon \cdot \Phi_{\mathcal{K}}^\text{can}$.

**Proof.** For each $U \in \mathcal{D}_t$, we may choose an element $\epsilon_U \in \Lambda^\times$ such that $\Phi_{\mathcal{U}} = \epsilon_U \cdot \Phi_{\mathcal{U}}^\text{can}$. We must show that $\epsilon_U$ doesn’t depend on the choice of $U \in \mathcal{D}_t$. For two different $U, V \in \mathcal{D}_t$, there exists some $x \in \mathcal{K}$ such that

$$
\text{div}_\Phi(x) = [U] - [V].
$$

This implies that $\text{div}_\Phi(x) = \epsilon_U \cdot [U] - \epsilon_V \cdot [V]$. The “exactness” in the definition of a rational-like collection (applied to $\Phi$ particularly) shows that $\epsilon_U - \epsilon_V = 0$, as required. □

### 6.6. Rational synchronization

A key point in the proof of Theorem 6.1 is a so-called synchronization step. The compatibility with rational submodules allows us to carry out this synchronization process, and the following proposition is the key step in this direction. We first introduce some additional notation, which will help us in the course of the proof of this proposition.

Let $t$ be a general element of $K|k$. By Lemma 6.7, the set $\mathcal{D}_t$ is parametrized by $\mathbb{P}^1(k) = k \cup \{\infty\}$. Given $a \in k \cup \{\infty\}$, we write $\mathcal{U}_{t,a}$ for the element of $\mathcal{D}_t$ which corresponds to the point $t = a$ on $\mathbb{P}^1$. To be explicit, the point $a \in k \cup \{\infty\}$ corresponds to a closed point $t = a$ on $\mathbb{P}^1$ (the projective line parameterized by $t$), which in turn corresponds to a unique divisorial valuation $w$ of $k(t)|k$. This divisorial valuation $w$ corresponds to an element of $\mathcal{D}_t$ via Lemma 6.7 and this element of $\mathcal{D}_t$
is denoted by \( \mathcal{U}_{t,a} \). It is important to note that this parameterization of \( \mathcal{D}_t \) depends on the choice of general element \( t \) which generates the field \( k(t)|k \). Nevertheless, with this choice made, we have

\[
\text{div}_{\text{can}}(t - c)^o = [\mathcal{U}_{t,c}] - [\mathcal{U}_{t,\infty}]
\]

for all constants \( c \in k \). On the other hand, if \( \mathcal{U}_1, \mathcal{U}_2 \in \mathcal{D}_t \) are two distinct elements, then there exists a general element \( x \) of \( K|k \) such that \( k(x) = k(t) \), and such that

\[
\text{div}_{\text{can}}(x^o) = [\mathcal{U}_1] - [\mathcal{U}_2].
\]

With this notation and the observations above, we can now state and prove the key following key proposition.

**Proposition 6.9.** Let \( \phi : \mathcal{K}(K|k) \cong \mathcal{K}(L|l) \) be an isomorphism of \( \Lambda \)-modules which is compatible with \( \text{acl} \) and with rational submodules. Let \( x \) be a general element of \( K|k \). Then there exists a general element \( y \) of \( L|l \), a unit \( \epsilon \in \Lambda^\times \), and a set-theoretic bijection \( \eta : k \cong l \), such that \( \eta 0 = 0 \), \( \eta 1 = 1 \), and such that one has

\[
\phi(x - a)^o = \epsilon \cdot (y - \eta a)^o
\]

for all \( a \in k \).

**Proof.** Let \( \kappa := \mathcal{K}_x \), and recall that \( \mathcal{L} := \phi \mathcal{K} \) is a rational submodule of \( \mathcal{K}(L|l) \). By Fact 6.8, we see that \( \phi \) induces a bijection

\[
\mathcal{U} \mapsto \phi \mathcal{U} : \mathcal{D}_{\mathcal{K}} \xrightarrow{\phi} \mathcal{D}_{\mathcal{L}}.
\]

Consider the canonical rational-like collection \( \Phi := \phi_{\text{can}} \) on \( \mathcal{K} \). We may further consider the push-forward \( \phi_* \Phi := \Psi \) of \( \Phi \) to \( \mathcal{L} \). In explicit terms, for \( \mathcal{V} \in \mathcal{D}_{\mathcal{L}} \) such that \( \mathcal{V} = \phi \mathcal{U} \) with \( \mathcal{U} \in \mathcal{D}_{\mathcal{K}} \), the isomorphism \( \Psi_{|\mathcal{V}} : \mathcal{L}|\mathcal{V} \cong \Lambda \) is given by

\[
\mathcal{L}|\mathcal{V} \xrightarrow{\phi^{-1}} \mathcal{K}|\mathcal{U} \xrightarrow{\Phi_{|\mathcal{U}}} \Lambda.
\]

Note that \( \Psi \) is a rational-like collection on \( \mathcal{L} \), hence, by Lemma 6.3, there exists an \( \epsilon \in \Lambda^\times \) such that \( \Psi = \epsilon^{-1} \cdot \phi_{\text{can}} \), while the construction of \( \Psi \) ensures that one has a canonical commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathcal{K} & \xrightarrow{\text{div}_\Psi} & \bigoplus_{\mathcal{U} \in \mathcal{D}_{\mathcal{K}}} \Lambda \cdot [\mathcal{U}] & \xrightarrow{\sum} & \Lambda & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{L} & \xrightarrow{\text{div}_\Psi} & \bigoplus_{\mathcal{V} \in \mathcal{D}_{\mathcal{L}}} \Lambda \cdot [\mathcal{V}] & \xrightarrow{\sum} & \Lambda & \rightarrow & 0
\end{array}
\]

Note that one has \( \text{div}_\Psi(x^o) = [\mathcal{U}_x,0] - [\mathcal{U}_x,\infty] \), hence \( \text{div}_\Psi(\phi x^o) = [\phi \mathcal{U}_x,0] - [\phi \mathcal{U}_x,\infty] \). Since \( \Psi = \epsilon^{-1} \cdot \phi_{\text{can}} \), we see that \( \text{div}_\Psi = \epsilon^{-1} \cdot \text{div}_{\text{can}} \), hence \( \text{div}_{\text{can}}(\epsilon^{-1} \cdot \phi x^o) = [\phi \mathcal{U}_x,0] - [\phi \mathcal{U}_x,\infty] \). On the other hand, there exists a general element \( y \) of \( L|l \) such that \( \mathcal{K}_y = \mathcal{L} \), and such that \( \phi \mathcal{U}_y,0 = \mathcal{U}_y,0 \) and \( \phi \mathcal{U}_y,\infty = \mathcal{U}_y,\infty \). By replacing \( y \) with an element of the form \( c \cdot y \) for some \( c \in l^\times \), we may assume furthermore that \( \phi \mathcal{U}_y,a = \mathcal{U}_y,a \) for all \( a \in k \). Then, for all \( a \in k \), one has

\[
\text{div}_{\text{can}}(\epsilon^{-1} \cdot \phi(x - a)^o) = \text{div}_\Psi(\phi(x - a)^o)
\]

\[
= [\phi \mathcal{U}_x,a] - [\phi \mathcal{U}_x,\infty]
\]

\[
= [\mathcal{U}_y,a] - [\mathcal{U}_y,\infty]
\]

\[
= \text{div}_{\text{can}}((y - \eta a)^o)
\]

The injectivity of \( \text{div}_{\text{can}} \) implies that \( \phi(x - a)^o = \epsilon \cdot (y - \eta a)^o \), hence proving the assertion. \( \square \)
6.7. Multiplicative synchronization. At this point, the proof of Theorem 6.1 uses an adaptation of the arguments in Pop [Pop12b, §6]. This is particularly true for the proofs of Propositions 6.10 and 6.11.

Following Proposition 6.9, we will say that the element \( \phi \in \text{Isom}^{\text{acl}}_{\text{rat}}(\mathcal{H}(K/k), \mathcal{H}(L/l)) \) is synchronized provided that there exists some general element \( x \) of \( K/k \) and some general element \( y \) of \( L/l \), and some bijection \( \eta : k \cong l \) such that \( \eta 0 = 0 \), \( \eta 1 = 1 \) and such that
\[
\phi(x-a) = (y-\eta a)^0
\]
for all \( a \in k \). If we wish to specify \( x \), \( y \) (and \( \eta \)) as above, we will say that \( \phi \) is synchronized by \( x \) and \( y \) (via \( \eta \)). Furthermore, by Proposition 6.9, there always exists some \( \epsilon \in \Lambda^X \) such that \( \epsilon \cdot \phi \) is synchronized.

As \( K \) is a function field over \( k \), the quotient \( K^\times/k^\times \) is a free finitely-generated abelian group. Indeed, for any normal proper model \( X \) of \( K/k \), the group \( K^\times/k^\times \) embeds in the free abelian group \( \text{Div}(X) \) via the divisor map on rational functions. Thus, the canonical map
\[
K^\times/k^\times \to \mathcal{H}(K/k)
\]
is injective. We will identify \( K^\times/k^\times \) with its image in \( \mathcal{H}(K/k) \), and we similarly identify \( L^\times/l^\times \) with its image in \( \mathcal{H}(L/l) \). We now proceed to show that a synchronized \( \phi \) is actually multiplicatively synchronized, in the sense that \( \phi \) restricts to an isomorphism (of abelian groups) \( K^\times/k^\times \cong L^\times/l^\times \).

**Proposition 6.10.** Assume that \( \phi \in \text{Isom}^{\text{acl}}_{\text{rat}}(\mathcal{H}(K/k), \mathcal{H}(L/l)) \) is synchronized. Then one has \( \phi(K^\times/k^\times) = L^\times/l^\times \).

**Proof.** It suffices to prove that \( \phi(K^\times/k^\times) \subset L^\times/l^\times \), since \( \phi^{-1} : \mathcal{H}(L/l) \cong \mathcal{H}(K/k) \) is also synchronized. Put \( M := \phi^{-1}(L^\times/l^\times) \cap (K^\times/k^\times) \) and let \( M^\times \) denote the preimage of \( M \) in \( K^\times \).

Our goal is to show that \( M^\times = K^\times \).

Let \( x \) be general in \( K/k \) and \( y \) general in \( L/l \) such that \( \phi \) is synchronized by \( x \), \( y \). We immediately see that \( k(x)^\times \subset M^\times \), since \( k(x)^\times \) is multiplicatively generated by elements of the form \( (x-a) \), \( a \in k \).

More generally, assume that \( u \in M^\times \) is general in \( K/k \). By Proposition 6.9, there exists a bijection \( \eta : k \cong l \), a general element \( w \) of \( L/l \), and an \( \epsilon \in \Lambda^X \), such that \( \eta 0 = 0 \), \( \eta 1 = 1 \) and
\[
\phi(u-a) = \epsilon \cdot (w-\eta a)^0
\]
for all \( a \in k \). Note in particular that \( \phi(u^0) = \epsilon \cdot w^0 \), while \( \phi(u^0) \in L^\times/l^\times \).

We claim that \( \epsilon \in \mathbb{Z} \). Write \( \epsilon = n/m \), with \( n, m \in \mathbb{Z} \) relatively prime, \( n > 0 \). By the above observations, and using the fact that \( l^\times \) is divisible, we see that there exists \( g \in L^\times \) such that \( w^m = g^n \). But \( w \) is general in \( L/l \), so \( g \in l(w) \). It is easy to see from this observation that \( n = 1 \).

To summarize, for all \( a \in k \), one has
\[
\phi(u-a)^0 = m \cdot (w-\eta a)^0 = ((w-\eta a)^m)^0 \in L^\times/l^\times.
\]
From this we again see that \( k(u)^\times \) is contained in \( M^\times \).

Finally, since \( \Lambda \subset \mathbb{Q} \), we note that for all \( t \in K^\times \), there exists some integer \( n > 0 \) such that \( n \cdot \phi(t^0) \in L^\times/l^\times \). In other words, \( t^m \in M^\times \), so that \( K^\times/M^\times \) is torsion.

To summarize, the subset \( M^\times \) is a multiplicative subgroup of \( K^\times \) which satisfies the following properties:

1. The quotient \( K^\times/M^\times \) is torsion.
2. If \( u \in M^\times \) is general in \( K/k \), then \( k(u)^\times \subset M^\times \).
3. The element \( x \) is contained in \( M^\times \), and \( x \) is general in \( K/k \).

We claim that \( M := M^\times \cup \{0\} \) is a subfield of \( K \). As \( M \) is multiplicatively closed, it suffices to prove that, for all \( u \in M \), one has \( u + 1 \in M \). As \( k(x) \subset M \), we may furthermore assume that \( u \in M \setminus k(x) \). In particular, \( x, u \) are algebraically independent over \( k \).
By Fact 6.6 there exist \( b \in k^\times \) and \( c \in k \) such that the following elements are all general in \( K|k \):

\[
A_1 := \frac{u}{b \cdot x + c}, \quad A_2 := \frac{2 \cdot u}{b \cdot x + c + 1}, \quad A_3 := \frac{2 \cdot u + b \cdot x + c + 1}{u + b \cdot x + c}.
\]

It is clear from the above properties that \( A_1, A_2 \in M \). Hence

\[
B_1 := (b \cdot x + c) \cdot (A_1 + 1) = u + b \cdot x + c, \quad B_2 := (b \cdot x + c + 1) \cdot (A_2 + 1) = 2 \cdot u + b \cdot x + c + 1
\]

are also elements of \( M \), so that \( A_3 = B_2/B_1 \) is an element of \( M \) as well. As \( A_3 \) is general in \( K|k \), we see that

\[
(A_3 - 1) \cdot B_1 = u + 1
\]

is indeed an element of \( M \), as contended.

The argument above shows that that \( M \) is a subfield of \( K \), which contains \( k \), while \( K^\times/M^\times \) is also torsion. Since \( K \) is a function field over \( k \) and \( k \) has characteristic 0, it follows that \( K = M \). \( \Box \)

6.8. Collination. As mentioned before, our final goal will be to use the fundamental theorem of projective geometry. If \( \phi \) is synchronized, then, by Proposition 6.10, \( \phi \) induces an isomorphism of abelian groups

\[
\phi : K^\times/k^\times \cong L^\times/l^\times.
\]

On the other hand, note that \( K^\times/k^\times \) is precisely the projectivization of \( K \) as a \( k \)-vector space. For distinct \( x, y \in K^\times/k^\times \), considered as \( k^\times \)-cosets in \( K^\times \), the projective line in \( K^\times/k^\times \) between \( x, y \) is precisely the set

\[
\mathcal{L}(x,y) := \frac{x + y}{k^\times} \cup \{x, y\}.
\]

In order to apply the fundamental theorem of projective geometry, we will need to prove that this isomorphism \( \phi : K^\times/k^\times \cong L^\times/l^\times \) is compatible with such projective lines. The following proposition takes care of this.

Proposition 6.11. Assume that \( \phi \in \text{Isom}^{\text{rat}}(\mathcal{X}(K|k), \mathcal{X}(L|l)) \) is synchronized. Then the induced isomorphism

\[
\phi : K^\times/k^\times \cong L^\times/l^\times
\]

is a collination. In other words, for all distinct \( x, y \in K^\times/k^\times \), the map \( \phi \) induces a bijection

\[
\phi : \mathcal{L}(x,y) \cong \mathcal{L}(\phi(x),\phi(y)).
\]

Proof. For \( x \in K^\times/k^\times, x \neq 1^\circ \), we write

\[
\mathcal{L}(x) := \mathcal{L}(x,1^\circ) = \frac{k^\times + x}{k^\times} \cup \{x, 1^\circ\} \subset K^\times/k^\times.
\]

As \( \phi \) restricts to a multiplicative isomorphism \( K^\times/k^\times \cong L^\times/l^\times \), and one has

\[
y \cdot \mathcal{L}(x) = \frac{x + y}{k^\times} \cup \{x, y\} = \mathcal{L}(x,y),
\]

it is enough to show that \( \phi \mathcal{L}(x) = \mathcal{L}(\phi(x)) \) for all \( x \in K^\times/k^\times, x \neq 1^\circ \).

Let \( x \in K \setminus k \) be given, and let \( y \in L \setminus l \) be such that \( \phi(x) = y \). Assume first that \( \phi \mathcal{L}(x^\circ) = \mathcal{L}(y^\circ) \). Let \( t \) be algebraically independent from \( x \) (over \( k \)), and choose \( u \) such that \( \phi t^\circ = u^\circ \). Choose a divisorial valuation \( v \) of \( K|k \) such that \( v \) is trivial on \( \text{acl}_K(x) \) and on \( \text{acl}_K(t) \), while also such that \( t \) and \( x \) have the same residue in \( (Kv)^\times \) modulo \( k^\times \) – this is always possible to do since \( x \) and \( t \) are algebraically independent. Put \( w = v^\phi \), where \( v^\phi \) is as in Fact 6.5. By the Local Theory (Fact 6.5), we see that \( y \) and \( u \) have the same residue modulo \( l^\times \) in \( (Lw)^\times \), while also that \( w \) is trivial on \( \text{acl}_L(y) \) and on \( \text{acl}_L(u) \) by Lemma 6.3.

Note that both maps

\[
\text{acl}_K(x)/k^\times \rightarrow (Kv)^\times/k^\times \leftarrow \text{acl}_K(t)/k^\times
\]
are injective, and recall that $x, t$ have the same image, say $(\bar{x})^\circ$, in $(Kv)^\times/k^\times$. In particular, both $\mathcal{L}(x^0)$ and $\mathcal{L}(t^0)$ map bijectively onto $\mathcal{L}((\bar{x})^\circ)$, via the two injective maps above. Furthermore, since $\mathcal{W}_v^1 \cap (K^\times/k^\times) = (U_v^1 \cdot k^\times)/k^\times$ and $acl_K(t^0)/k^\times = \mathcal{X}_t \cap (K^\times/k^\times)$, we find that one has:

$$\mathcal{L}(t^0) = \mathcal{X}_t \cap (K^\times/k^\times) \cap (\mathcal{L}(x^0) \cdot (\mathcal{W}_v^1 \cap (K^\times/k^\times))).$$

We similarly have the following equality:

$$\mathcal{L}(u^0) = \mathcal{X}_u \cap (L^\times/l^\times) \cap (\mathcal{L}(y^0) \cdot (\mathcal{W}_u^1 \cap (L^\times/l^\times))).$$

Recall that $\phi : \mathcal{X}_\Lambda(K|k) \cong \mathcal{X}_\Lambda(L|l)$ identifies $\mathcal{X}_t$ with $\mathcal{X}_u$, $(K^\times/k^\times)$ with $(L^\times/l^\times)$, $\mathcal{L}(x^0)$ with $\mathcal{L}(y^0)$, and $\mathcal{W}_v^1$ with $\mathcal{W}_u^1$. It follows that one has $\phi \mathcal{L}(t^0) = \mathcal{L}(u^0)$.

Finally, recall that $\phi$ is synchronized. Hence, there exist some $x$ and $y$ as above such that $\phi \mathcal{L}(x^0) = \mathcal{L}(y^0)$. Therefore, by the argument above, for any $t \in K$ which is algebraically independent from $x$, we have $\phi \mathcal{L}(t^0) = \mathcal{L}(\phi(t^0))$. On the other hand, if $z$ is algebraically dependent to $x$, then it is independent from any element $t$ which is independent from $x$. Since $\phi \mathcal{L}(t^0) = \mathcal{L}(\phi(t^0))$, we again deduce that $\phi \mathcal{L}(z^0) = \mathcal{L}(\phi(z^0))$, as required. \qed

6.9. Concluding the proof. We now conclude the proof of Theorem 6.1. The following proposition essentially takes care of the final part of the argument.

Proposition 6.12. Assume that $\phi \in Isom^{acl}_\text{rat}(\mathcal{X}_\Lambda(K|k), \mathcal{X}_\Lambda(L|l))$ is synchronized. Then there exists a unique isomorphism of fields $\Gamma : K \cong L$ such that $\phi(t^0) = \Gamma(t)^0$ for all $t \in K^\times$.

Proof. Since $\phi$ is synchronized, it induces an isomorphism

$$\phi : K^\times/k^\times \cong L^\times/l^\times,$$

which is a colineation by Proposition 6.11. By the Fundamental Theorem of Projective Geometry (cf. [Art88]), there exists an isomorphism of fields $\gamma : k \cong l$ and an isomorphism $\Gamma : K \cong L$ (of $k$ resp. $l$ vector spaces) which is $\gamma$-linear, such that $\Gamma$ induces $\phi$ in the sense that $\Gamma(t)^0 = \phi(t^0)$ for all $t \in K^\times$. Moreover, $\Gamma$ is unique up-to homotheties obtained by scaling by elements of $k^\times$ resp. $l^\times$. By replacing $\Gamma$ with $(1/\Gamma(1)) \cdot \Gamma$, we may further assume that $\Gamma(1) = 1$. We will show that this particular (additive) isomorphism $\Gamma$ is actually a field isomorphism, i.e. that it is compatible with multiplication. We follow an argument which is similar to [BT08 Theorem 7.3].

First, since $\Gamma(1) = 1$, it follows that $\Gamma : K \cong L$ restricts to $\gamma : k \cong l$ on $k$. In particular, if $x \in K$ and $a \in k$, then one has

$$\Gamma(a \cdot x) = \gamma(a) \cdot \Gamma(x) = \Gamma(a) \cdot \Gamma(x).$$

Let us therefore assume that $x, y \in K \setminus k$. Our goal is to show that $\Gamma(x \cdot y) = \Gamma(x) \cdot \Gamma(y)$. Since $\Gamma$ induces $\phi : K^\times/k^\times \cong L^\times/l^\times$ and since $\phi$ is compatible with multiplication, we see that there exists some $c \in l^\times$ such that

$$\Gamma(x \cdot y) = c \cdot \Gamma(x) \cdot \Gamma(y).$$

Note that $x \cdot y$ and $y$ are $k$-linearly-independent and hence $c^{-1} \cdot \Gamma(x \cdot y) = \Gamma(x) \cdot \Gamma(y)$ and $\Gamma(y)$ are $l$-linearly-independent.

Let us consider $\Gamma(x \cdot y + y)$. On the one hand, we have

$$\Gamma(x \cdot y + y) = \Gamma(x \cdot y) + \Gamma(y) = c \cdot \Gamma(x) \cdot \Gamma(y) + \Gamma(y),$$

and on the other hand, there exists some $d \in l^\times$ such that

$$\Gamma(x \cdot y + y) = \Gamma((x + 1) \cdot y) = d \cdot \Gamma(x + 1) \cdot \Gamma(y)$$

$$= d \cdot (\Gamma(x) + 1) \cdot \Gamma(y)$$

$$= d \cdot \Gamma(x) \cdot \Gamma(y) + d \cdot \Gamma(y)$$

In particular, we see that $c = d = 1$, and hence $\Gamma(x \cdot y) = \Gamma(x) \cdot \Gamma(y)$, as required. \qed
We now conclude the proof of Theorem 6.1. Let $\phi \in \text{Isom}_{\text{rat}}^{\text{cl}}(\mathscr{X}_\Lambda(K|k), \mathscr{X}_\Lambda(L|l))$ be given. By Proposition 6.9 there exists some $\epsilon \in \Lambda^\times$ such that $\psi := \epsilon \cdot \phi$ is synchronized, while by Proposition 6.12 there exists a unique isomorphism $\Gamma : K \cong L$ of fields such that $\psi(t^\circ) = \Gamma \psi(t)^\circ$. If furthermore $\phi$ arises from a given isomorphism $\Gamma : K \cong L$, then $\phi$ is synchronized and it is easy to see that $\Gamma = \Gamma_\phi$.

We have thus constructed a left-inverse of the canonical map
\[
\text{Isom}(K, L) \to \text{Isom}_{\text{rat}}^{\text{cl}}(\mathscr{X}_\Lambda(K|k), \mathscr{X}_\Lambda(L|l))/\Lambda^\times,
\]
and it follows from the construction that this left-inverse is, in fact, functorial with respect to composition of isomorphisms. To conclude the proof, we must prove that this map
\[
\text{Isom}_{\text{rat}}^{\text{cl}}(\mathscr{X}_\Lambda(K|k), \mathscr{X}_\Lambda(L|l))/\Lambda^\times \to \text{Isom}(K, L)
\]
just constructed is injective. In order to do this, by the discussion above, it suffices to assume that $K = L$, and to prove that the group homomorphism
\[
\text{Aut}_{\text{rat}}^{\text{cl}}(\mathscr{X}_\Lambda(K|k))/\Lambda^\times \to \text{Aut}(K)
\]
is injective. So, let us assume that $\phi \in \text{Aut}_{\text{rat}}^{\text{cl}}(\mathscr{X}_\Lambda(K|k))$ is synchronized, and that $\Gamma_\phi$ is the identity automorphism of $K$. Then $\phi(t^\circ) = \Gamma_\phi(t)^\circ = t^\circ$ for all $t \in K^\times$. As $\phi$ is $\Lambda$-linear and $\mathscr{X}_\Lambda(K|k)$ is generated (as a $\Lambda$-module) by $K^\times/k^\times$, it follows that $\phi$ is itself the identity automorphism of $\mathscr{X}_\Lambda(K|k)$. This concludes the proof of Theorem 6.1.

7. A Torelli Theorem

Let $\Lambda$ be a subring of $\mathbb{Q}$, and let $\mathbf{H}_i$, $i = 1, 2$ be two mixed Hodge structures over $\Lambda$ whose underlying $\Lambda$-modules are denoted by $H_i$, $i = 1, 2$. We say that a $\Lambda$-linear morphism $f : H_1 \to H_2$ is compatible with the mixed Hodge structure provided that $f$ underlies a morphism $f : \mathbf{H}_1 \to \mathbf{H}_2$ of mixed Hodge structures.

Now suppose that $k$ is an algebraically closed field endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$, and let $K|k$ be a function field. Recall that we have defined $\mathcal{R}(K|k, \Lambda)$ to be the kernel of the cup-product
\[
x \otimes y \mapsto x \cup y : H^1(K|k, \Lambda(1)) \otimes_{\Lambda} H^1(K|k, \Lambda(1)) \to H^2(K|k, \Lambda(2)).
\]
Suppose that $l$ is another algebraically closed field endowed with a complex embedding $\tau : l \hookrightarrow \mathbb{C}$, and that $L|l$ is another function field. We say that a $\Lambda$-linear isomorphism $\phi : H^1(K|k, \Lambda(1)) \cong H^1(L|l, \Lambda(1))$ is compatible with $\mathcal{R}$ provided that the induced isomorphism
\[
\phi \otimes \tau : H^1(K|k, \Lambda(1)) \otimes_{\Lambda} H^1(K|k, \Lambda(1)) \cong H^1(L|l, \Lambda(1)) \otimes_{\Lambda} H^1(K|k, \Lambda(1)),
\]
restricts to an isomorphism $\mathcal{R}(K|k, \Lambda) \cong \mathcal{R}(L|l, \Lambda)$. We may now state and prove the first main theorem of this paper which can be seen as a higher-dimensional birational variant of the classical Torelli theorem.

**Theorem 7.1.** Let $\Lambda$ be a subring of $\mathbb{Q}$. Let $k$ be an algebraically closed field endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$, and let $K$ be a function field of transcendence degree $\geq 2$ over $k$. Then the isomorphy type of $K|k$ (as fields) is determined by the following data:
\begin{itemize}
  \item The mixed Hodge structure $H^1(K|k, \Lambda(1))$ with underlying $\Lambda$-module $H^1(K|k, \Lambda(1))$.
  \item The submodule $\mathcal{R}(K|k, \Lambda) \subset H^1(K|k, \Lambda(1)) \otimes_{\Lambda} H^1(K|k, \Lambda(1))$.
\end{itemize}
In other words, suppose that $l$ is another algebraically closed field which can be embedded in $\mathbb{C}$, and let $L$ be any function field over $l$. Then there exists an isomorphism $K \cong L$ of fields (which automatically restricts to an isomorphism $k \cong l$) if and only if there exists a complex embedding $\tau : l \hookrightarrow \mathbb{C}$, and an isomorphism of $\Lambda$-modules
\[
\phi : H^1(K|k, \Lambda(1)) \cong H^1(L|l, \Lambda(1))
\]
which is compatible with the mixed Hodge structure and with \( R \). Here \( H^1(U|l, \Lambda(1)) \) and \( H^*(U|l, \Lambda(*)) \) are computed with respect to the complex embedding \( \tau \).

As one might expect, we will prove Theorem 7.1 by reducing the situation to Theorem 6.1. The non-trivial implication will proceed by associating to any isomorphism of \( \Lambda \)-modules
\[
\phi : H^1(K|k, \Lambda(1)) \cong H^1(L|l, \Lambda(1)),
\]
which is compatible with the mixed Hodge structures and with \( R \), an element of the isomorphism set \( \text{Isom}_{\text{rat}}(\mathcal{X}_\Lambda(K|k), \mathcal{X}_\Lambda(L|l)) \) which was previously considered in Theorem 6.1. Theorem 6.1 then implies that \( \text{Isom}(K, L) \) is non-empty. Finally, note that that any isomorphism of fields \( K \cong L \) restricts to an isomorphism \( k \cong l \) since \( k \) resp. \( l \) is the set of multiplicatively divisible elements in \( K \) resp. \( L \). We now provide the necessary details.

7.1. Compatibility with Kummer theory. Since \( \Lambda \) is torsion-free as a \( \mathbb{Z} \)-module, it follows from Proposition 4.2 that the map
\[
\mathcal{R}^\Lambda_K : \mathcal{X}_\Lambda(K|k) \to H^1(K|k, \Lambda(1))
\]
is injective. The following Key Lemma, which is a crucial part of the proof of Theorem 7.1, shows how to recover the image of this map. This lemma, which is certainly already known to the experts, follows more-or-less directly from Deligne’s theorem (Theorem 5.2), and the calculation of the Hodge realization of a Picard 1-motive (Theorem 5.3).

**Key Lemma 7.2.** Let \( x \in H^1(K|k, \Lambda(1)) \) be given and consider the \( \Lambda \)-linear morphism
\[
\gamma_x : \Lambda \to H^1(K|k, \Lambda(1))
\]
given by \( \gamma_x(a) = a \cdot x \). Then \( x \) is contained in the image of the injective map \( \mathcal{R}^\Lambda_K : \mathcal{X}_\Lambda(K|k) \to H^1(K|k, \Lambda(1)) \) if and only if \( \gamma_x \) is compatible with the mixed Hodge structure. Here we identify \( \Lambda \) as the underlying \( \Lambda \)-module of \( \Lambda(0) \), the pure Hodge structure of Hodge type \((0, 0)\).

**Proof.** First suppose that \( t \in K^\times \) is given, and consider the map
\[
\gamma_t := \gamma_{\mathcal{R}_K(t)} : \Lambda \to H^1(K|k, \Lambda(1))
\]
as defined in the statement of the lemma. Choose a smooth model \( U \) of \( K|k \) such that \( t \in O^\times(U) \), and recall that \( t \) is considered as a morphism \( t : U \to \mathbb{G}_m \) of \( k \)-varieties. The map \( \gamma_t \) agrees with the composition
\[
\Lambda = H^1(\mathbb{G}_m, \Lambda(1)) \xrightarrow{\cdot t} H^1(U, \Lambda(1)) \hookrightarrow H^1(K|k, \Lambda(1)).
\]
On the other hand, one has \( \Lambda(0) = H^1(\mathbb{G}_m, \Lambda(1)) \), while the inclusion \( H^1(U, \Lambda(1)) \hookrightarrow H^1(K|k, \Lambda(1)) \) is compatible with the mixed Hodge structures. Hence, \( \gamma_t \) is also compatible with the mixed Hodge structures. On the other hand, any \( y \in \mathcal{X}_\Lambda(K|k) \) has the form
\[
y = a_1 \cdot t_1^0 + \cdots + a_r \cdot t_r^0
\]
for some \( a_i \in \Lambda \) and \( t_i \in K^\times \), and with this choice made, one has
\[
\gamma_{\mathcal{R}_K(y)} = a_1 \cdot \gamma_{t_1} + \cdots + a_r \cdot \gamma_{t_r}.
\]
Hence \( \gamma_{\mathcal{R}_K(y)} \) is compatible with mixed Hodge structures.

Conversely, let \( x \in H^1(K|k, \Lambda(1)) \) be such that \( \gamma_x \) is compatible with mixed Hodge structures. Let \( X \) be a smooth proper model of \( K|k \), and choose a sufficiently small non-empty open \( k \)-subvariety \( U \) of \( X \) such that \( x \in H^1(U, \Lambda(1)) \). Then \( \gamma_x \) factors through a morphism
\[
\gamma_x : \Lambda \to H^1(U, \Lambda(1)) \hookrightarrow H^1(K|k, \Lambda(1)),
\]
and the induced morphism \( \gamma_x : \Lambda \to H^1(U, \Lambda(1)) \) is compatible with the mixed Hodge structures. Consider the Picard 1-motive \( \mathbf{M}^{1,1}(U) \) associated to the inclusion \( U \hookrightarrow X \), as well as the 1-motive
Z := [Z → 0]. Note that $H(Z, \Lambda) = \Lambda(0)$, and hence by Theorems 5.3 and 5.1 we have a canonical bijection:

$$\text{Hom}_C(Z, M^{1,1}(U) \otimes k \mathbb{C}) \otimes \mathbb{Z} \Lambda \to \text{Hom}_{\text{MHS}}(\Lambda(0), H^1(U, \Lambda(1))).$$

The morphism $\gamma_x$ lies in the target of this bijection, hence it corresponds to some element $y \in \text{Hom}_C(Z, M^{1,1}(U) \otimes k \mathbb{C}) \otimes \mathbb{Z} \Lambda$. By using the definition of $M^{1,1}(U)$ and the definition of morphisms of 1-motives, we have:

$$\text{Hom}_C(Z, M^{1,1}(U) \otimes k \mathbb{C}) = \ker(\text{Div}^0_{X \setminus U}(X) \to \text{Pic}_X^0(k) \hookrightarrow \text{Pic}_X^0(\mathbb{C})) = \ker(\text{Div}^0_{X \setminus U}(X) \to \text{Pic}^0(X)) = O^\times(U)/k^\times.$$

From this we may consider $y$ as an element of $(O^\times(U)/k^\times) \otimes \mathbb{Z} \Lambda \subset \mathcal{H}_\Lambda(K|k)$. By tracing through the definitions, it is easy to see that one has $\gamma_x = \gamma_y$ for this particular element $y \in \mathcal{H}_\Lambda(K|k)$. □

**Remark 7.3.** One may phrase Key Lemma 7.2 as the equality:

$$\text{Image}(\mathcal{R}_K^1 : \mathcal{H}_\Lambda(K|k) \to H^1(K|k, \Lambda(1))) = H^1(K|k, \Lambda(1)) \cap F^0(H^1(K|k, \Lambda(1)) \otimes \mathbb{C}).$$

The equivalence of this formulation with the one given in Key Lemma 7.2 is a matter of tracing through Deligne’s construction [De174 §10.3], which we have briefly outlined in §5.2.

Alternatively, over $\mathbb{Q}$, we may phrase Key Lemma 7.2 as the equality:

$$\text{Image}(\mathcal{R}_K^2 : \mathcal{H}_\Lambda(K|k) \to H^1(K|k, \mathbb{Q}(1))) = H^1(K|k, \mathbb{Q}(1))^{G_{\text{MT}}},$$

where $G_{\text{MT}}$ denotes the (absolute) Mumford-Tate group, i.e. the fundamental group associated to the Tannakian category of rational mixed Hodge structures. This formulation is particularly nice because it is directly analogous to the $\ell$-adic analogue which we will state in Key Lemma 8.3 (in fact, it would be equivalent under the Mumford-Tate conjecture).

### 7.2. Compatibility with the geometric lattice

The next key step in the proof of Theorem 7.1 is to show the compatibility with the geometric lattice in a way which refines Lemma 6.3.

**Proposition 7.4.** Let $\phi : H^1(K|k, \Lambda(1)) \cong H^1(L|l, \Lambda(1))$ be a $\Lambda$-linear isomorphism which is compatible with the mixed Hodge structures and with $\mathcal{R}$, as in the statement of Theorem 7.1. For $M \in G^+(K|k)$ and $N \in G^+(L|l)$, consider the (incomplete) diagram of $\Lambda$-modules:

\[
\begin{array}{ccc}
H^1(K|k, \Lambda(1)) & \xrightarrow{\phi} & H^1(L|l, \Lambda(1)) \\
\downarrow & & \downarrow \\
H^1(M|k, \Lambda(1)) & \xrightarrow{\mathcal{R}_M} & H^1(N|l, \Lambda(1)) \\
\mathcal{H}_\Lambda(M|k) & \xrightarrow{\mathcal{R}_M} & \mathcal{H}_\Lambda(N|l)
\end{array}
\]

Then there exists an isomorphism $\phi^\sharp : G^+(K|k) \cong G^+(L|l)$ of geometric lattices such that for all $M \in G^+(K|k)$ with image $N := \phi^\sharp M$, the following hold:

1. The lower dotted arrow in the above diagram can be (uniquely) completed to an isomorphism of $\Lambda$-modules.
2. If furthermore $\text{tr.deg}(M|k) = 1$ (and hence $\text{tr.deg}(N|l) = 1$ as well), then the middle dotted arrow can also be completed to an isomorphism of $\Lambda$-modules.
Proof. First, we note that the assertion holds true for $M = K$ and $N = L$ by Key Lemma 7.2. That is, the dotted arrow in the following diagram can be uniquely completed to an isomorphism:

\[
\begin{array}{ccc}
H^1(K|k, \Lambda(1)) & \xrightarrow{\Phi} & H^1(L|l, \Lambda(1)) \\
\downarrow \Psi_L & & \downarrow \Psi_L \\
\mathcal{K}_L(K|k) & \longrightarrow & \mathcal{K}_L(L|l)
\end{array}
\]

We also write $\phi$ for the induced isomorphism $\mathcal{K}_L(K|k) \cong \mathcal{K}_L(L|l)$. By Proposition 4.2, we see that this isomorphism $\mathcal{K}_L(K|k) \cong \mathcal{K}_L(L|l)$ is compatible with acl, hence by Lemma 6.3, we obtain an isomorphism $\phi^* : \mathbb{G}^*(K|k) \cong \mathbb{G}^*(L|l)$ of geometric lattices such that, for all $M \in \mathbb{G}^*(K|k)$, one has $\phi^* \mathcal{K}_L(M|k) = \mathcal{K}_L(\phi^* M|l)$ as submodules of $\mathcal{K}_L(L|l)$. Since the inclusion $\mathcal{K}_L(M|k) \hookrightarrow H^1(M|k, \Lambda(1)) \hookrightarrow H^1(K|k, \Lambda(1))$ factors through $\mathcal{K}_L(M|l) \hookrightarrow \mathcal{K}_L(K|k)$ (and similarly for $N|l$), this proves assertion (1) of the proposition.

As for assertion (2), let us assume that $M \in \mathbb{G}^1(K|k)$ is given. Put $N := \phi^* M$ so, in particular, one has $N \in \mathbb{G}^1(L|l)$. By Fact 4.1 and Proposition 4.4, we see that the image of the canonical injective map $H^1(M|k, \Lambda(1)) \hookrightarrow H^1(K|k, \Lambda(1))$ is precisely the submodule

\[
\{ x \in H^1(K|k, \Lambda(1)) : \forall y \in \mathcal{K}_L(M|k), \, \mathcal{R}_L^\Lambda(y) \cup x = 0 \},
\]

and analogously for $N|l$. As $\phi$ is compatible with $\mathcal{R}$, it follows that $\phi$ restricts to an isomorphism of submodules:

\[
\text{Image}(H^1(M|k, \Lambda(1)) \hookrightarrow H^1(K|k, \Lambda(1))) \cong \text{Image}(H^1(N|l, \Lambda(1)) \hookrightarrow H^1(L|l, \Lambda(1))).
\]

This proves assertion (2) of the proposition. □

7.3. Concluding the proof of Theorem 7.1. If there exists an isomorphism $K \cong L$, then it automatically follows that this isomorphism restricts to an isomorphism $k \cong l$ of base-fields. From this it is easy to deduce the existence of an isomorphism $\phi : H^1(K|k, \Lambda(1)) \cong H^1(L|l, \Lambda(1))$ which is compatible with the mixed Hodge structures and with $\mathcal{R}$.

Conversely, let us assume that such an isomorphism $\phi$ exists. By Theorem 6.1, it suffices to construct an element of $\text{Isom}_{\text{rat}}^\Lambda(\mathcal{K}_L(K|k), \mathcal{K}_L(L|l))$. Let $\phi : \mathcal{K}_L(K|k) \cong \mathcal{K}_L(L|l)$ be the unique isomorphism induced by $\phi$ as described in Proposition 4.2 (taking $K = M$). Applying the same proposition (or Proposition 4.4), we see that this $\phi$ is compatible with acl. Finally, it is an easy consequence of Theorem 5.3 that $M \in \mathbb{G}^1(K|k)$ is rational over $k$ if and only if the canonical map

\[
\mathcal{R}_L^\Lambda : \mathcal{K}_L(M|k) \hookrightarrow H^1(M|k, \Lambda(1))
\]

is an isomorphism, and similarly for $N \in \mathbb{G}^1(L|l)$. Thus, Proposition 7.4. implies that $\phi : \mathcal{K}_L(K|k) \cong \mathcal{K}_L(L|l)$ is also compatible with rational submodules. In other words, $\phi : \mathcal{K}_L(K|k) \cong \mathcal{K}_L(L|l)$ is indeed an element of isomorphism-set $\text{Isom}_{\text{rat}}^\Lambda(\mathcal{K}_L(K|k), \mathcal{K}_L(L|l))$. By Theorem 6.1, the set $\text{Isom}(K, L)$ is non-empty. As discussed above, such an isomorphism $K \cong L$ automatically restricts to an isomorphism $k \cong l$ of base-fields. This concludes the proof of Theorem 7.1.

8. An $\ell$-adic Variant

Let $k_0$ be a field whose algebraic closure $k$ is endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. Let $K_0$ be a regular function field over $k_0$, and recall that $K := K_0 \cdot k$ denotes the base-change of $K_0$ to $k$. Let $\Lambda$ be a subring of $\mathbb{Q}$. Recall that $\mathcal{C}_\ell$ denotes Artin’s $\ell$-adic comparison isomorphism

\[
\mathcal{C}_\ell : H^1(K|k, \Lambda(1)) \otimes_\Lambda \Lambda_\ell \cong H^1_\ell(K|k, \Lambda_\ell(1)),
\]

which is an isomorphism of $\Lambda_\ell$-modules.
Let $L_0$ be a regular function field over another field $l_0$ whose algebraic closure $l$ is endowed with a complex embedding $\tau : l \hookrightarrow \mathbb{C}$, and write $L := L_0 \cdot l$. Let $\phi_\ell : H^1_\Lambda(K[k], \Lambda_\ell(1)) \cong H^1_\ell(L[l], \Lambda_\ell(1))$ be an isomorphism of $\Lambda_\ell$-modules, and let $\phi : H^1(K[k], \Lambda(1)) \cong H^1(L[l], \Lambda(1))$ be an isomorphism of $\Lambda$-modules. We say that the pair $(\phi, \phi_\ell)$ is compatible with $\mathcal{C}_\ell$ provided that the following diagram commutes:

\[
\begin{array}{ccc}
H^1(K[k], \Lambda(1)) & \xrightarrow{\text{canon.}} & H^1(K[k], \Lambda(1)) \otimes_\Lambda \Lambda_\ell \\
\downarrow{\phi} & & \downarrow{\phi_\ell} \\
H^1(L[l], \Lambda(1)) & \xrightarrow{\text{canon.}} & H^1(L[l], \Lambda(1)) \otimes_\Lambda \Lambda_\ell \\
\end{array}
\]

With this terminology, we may now state the $\ell$-adic variant of Theorem 7.1.

**Theorem 8.1.** Let $\Lambda$ be a subring of $\mathbb{Q}$ and let $\ell$ be a prime. Let $k_0$ be a finitely-generated field whose algebraic closure $k$ is endowed with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. Let $K_0$ be a regular function field over $k_0$. Then the isomorphism type of $K_0/k_0$ is determined by the following data:

- The profinite group $\text{Gal}_{k_0}$ and the $\Lambda_\ell[[\text{Gal}_{k_0}]]$-module $H^1_\Lambda(K[k], \Lambda_\ell(1))$.
- The $\Lambda$-module $H^1(K[k], \Lambda(1))$, endowed with Artin’s comparison isomorphism $\mathcal{C}_\ell : H^1(K[k], \Lambda(1)) \otimes_\Lambda \Lambda_\ell \cong H^1_\ell(K[k], \Lambda_\ell(1))$.
- The $\Lambda$-submodule $\mathcal{R}(K[k], \Lambda)$ of $H^1(K[k], \Lambda(1)) \otimes_\Lambda H^1(K[k], \Lambda(1))$.

In other words, let $L_0$ be another regular function field over a finitely-generated field $l_0$ whose algebraic closure $l$ can be embedded in $\mathbb{C}$, and put $L = L_0 \cdot l$. Then there exists an isomorphism $K_0 \cong L_0$ of fields which restricts to an isomorphism $k_0 \cong l_0$, if and only if there exists an isomorphism $\phi_{\text{Gal}} : \text{Gal}_{k_0} \cong \text{Gal}_{l_0}$ of absolute Galois groups, an isomorphism $\phi_\ell : H^1_\Lambda(K[k], \Lambda_\ell(1)) \cong H^1_\ell(L[l], \Lambda_\ell(1))$ of $\Lambda_\ell$-modules, a complex embedding $\tau : l \hookrightarrow \mathbb{C}$, and an isomorphism $\phi : H^1(K[k], \Lambda(1)) \cong H^1(L[l], \Lambda(1))$ of $\Lambda$-modules, such that all of the following compatibility conditions hold true:

- The isomorphism $\phi_\ell$ is equivariant with respect to the action of $\text{Gal}_{k_0}$, where $\text{Gal}_{k_0}$ acts on $H^1_\ell(L[l], \Lambda_\ell(1))$ via $\phi_{\text{Gal}}$.
- The pair $(\phi, \phi_\ell)$ is compatible with $\mathcal{C}_\ell$.
- The isomorphism $\phi$ is compatible with $\mathcal{R}$.

Here $H^r(K[k], \Lambda(*)$ is computed with respect to the embedding $\tau$.

The proof of Theorem 8.1 is almost entirely analogous to the proof of Theorem 7.1. The main distinction is that we end up formulating a functorial analogue of Proposition 7.3 using $\ell$-adic cohomology. We then end up recovering the function field $K[k]$ (just as in the context of Theorem 7.1). However, the functorial nature of the reconstruction endows this “reconstructed” function field $K[k]$ with its additional structure of the Galois action of $\text{Gal}_{k_0}$. This finally yields $K_0/k_0$ by taking $\text{Gal}_{k_0}$-invariants.

**Remark 8.2.** Following Pop [Pop94, Pop00], one knows that a finitely-generated field is (functorially) determined up-to isomorphism from its absolute Galois group. Thus, we could have stated Theorem 8.1 under the further assumption that $k_0 = l_0$, and could have obtained an equivalent result. However, even if $k_0 = l_0$, the resulting isomorphism $K_0 \cong L_0$ of function fields can potentially restrict to a non-identity automorphism of the base-field $k_0 = l_0$. Because of this, we have decided to separate the base-fields $k_0$ and $l_0$ explicitly using the notation. This also leads to a formulation which is more analogous to Theorem 7.1.
8.1. Compatibility with Kummer theory. We start with a brief $\ell$-adic refinement of the Kummer map $\mathcal{R}_K$ which was defined in \[2,3\]. Let $f \in K^\times$ be given. Then we may choose a finite extension $k_1$ of $k_0$, and a smooth model $U_0$ of $K_0|k_0$ such that one has $f \in \mathcal{O}^\times(U_0 \otimes_{k_0} k_1)$. We consider $f$ as a morphism $f : U_0 \otimes_{k_0} k_1 \to \mathbb{G}_{m,k_1}$ of $k_1$-varieties, so that the corresponding morphism

$$\gamma_f : \Lambda_\ell = H^1_\ell(\mathbb{G}_{m,k_\ell}(1)) \to H^1_\ell(U_\ell, \Lambda_\ell(1)) \to H^1_\ell(K|k, \Lambda_\ell(1))$$

is Gal$_{k_1}$-equivariant, where Gal$_{k_1}$ acts trivially on $\Lambda_\ell$. We write $\mathcal{R}_K^\ell(f) = \gamma_f(1)$ for the image of $1 \in \Lambda_\ell$ under this morphism. Similarly to before, $\mathcal{R}_K^\ell(f)$ doesn’t depend on the choice of $k_1$ or of $U_0$ as above, and the K"unneth formula shows that the corresponding map

$$\mathcal{R}_K^\ell : K^\times \to H^1_\ell(K|k, \Lambda_\ell(1))$$

is a homomorphism of abelian groups which is trivial on $K^\times$.

Next, note that Gal$_{k_0}$ acts on $K$ via the identification Gal$_{k_0} = Gal(K|K_0)$. For $f \in \mathcal{O}^\times(U_0 \otimes_{k_0} k_1)$ as above, the map $\gamma_f$ is not necessarily Gal$_{k_0}$-equivariant, but rather one has $\sigma \gamma_f(c) = \gamma_f(\sigma(c))$, as elements of $H^1_\ell(K|k, \Lambda_\ell(1))$, for all $\sigma \in$ Gal$_{k_0}$ and $c \in \Lambda_\ell$. This shows that the Kummer homomorphism $\mathcal{R}_K^\ell : K^\times \to H^1_\ell(K|k, \Lambda_\ell(1))$, as defined above, is Gal$_{k_0}$-equivariant. This morphism $\mathcal{R}_K^\ell$ therefore induces a canonical Gal$_{k_0}$-equivariant homomorphism

$$\mathcal{R}^\ell_A : \mathcal{H}_\Lambda(K|k) \to H^1_\ell(K|k, \Lambda_\ell(1)).$$

Finally, due to the functoriality of Artin’s comparison isomorphism for $\ell$-adic cohomology, we see that $\mathcal{R}_K$ is compatible with $\mathcal{R}_K^\ell$ in the sense that the following diagram is commutative:

$$\begin{array}{ccc}
K^\times & \xrightarrow{\mathcal{R}_K} & H^1(K|k, \Lambda(1)) \\
\downarrow & & \downarrow \text{canon} \\
K^\times & \xrightarrow{\mathcal{R}^\ell_A} & H^1_\ell(K|k, \Lambda_\ell(1))
\end{array}$$

The morphisms $\mathcal{R}^\ell_A : \mathcal{H}_\Lambda(K|k) \to H^1(K|k, \Lambda(1))$ and $\mathcal{R}^\ell_A : \mathcal{H}_\Lambda(K|k) \to H^1_\ell(K|k, \Lambda_\ell(1))$ are compatible in a similar way. In particular, it follows (from Proposition \[4,\] for example) that the map $\mathcal{R}^\ell_A : \mathcal{H}_\Lambda(K|k) \to H^1_\ell(K|k, \Lambda_\ell(1))$, as well as the induced map

$$\mathcal{R}^\ell_A : \mathcal{H}_\Lambda(K|k) \otimes \Lambda_\ell \to H^1_\ell(K|k, \Lambda_\ell(1))$$

are both injective.

Key Lemma 8.3. The image of the canonical injective map of $\Lambda_\ell[[\text{Gal}_{k_0}]]$-modules

$$\mathcal{R}_K^\ell_{\Lambda_\ell} : \mathcal{H}_\Lambda(K|k) \otimes \Lambda_\ell \to H^1_\ell(K|k, \Lambda_\ell(1))$$

is precisely the $\Lambda_\ell[[\text{Gal}_{k_0}]]$-submodule

$$\bigcup_N H^1_\ell(K|k, \Lambda_\ell(1))^N,$$

where $N$ varies over the open subgroups of Gal$_{k_0}$.

Proof. The proof of this is completely analogous to that of Key Lemma \[7\]. First, by the Galois equivariance of $\mathcal{R}_K^\ell_{\Lambda_\ell}$, we see that the image of $\mathcal{R}_K^\ell_{\Lambda_\ell}$ is contained in

$$\bigcup_N H^1_\ell(K|k, \Lambda_\ell(1))^N,$$

since any element of $\mathcal{H}_\Lambda(K|k) \otimes \Lambda_\ell$ is contained in $(K_0 \cdot k_1) \times k_1^\times \otimes Z \Lambda_\ell$ for some finite extension $k_1|k_0$. Such an element is invariant under the action of Gal$_{k_1}$.
For the converse, we let $x$ be contained in the aforementioned union, and choose a finite extension $k_1$ of $k_0$ such that $x$ is invariant under $\text{Gal}_{k_1}$. Thus $x$ defines a canonical $\text{Gal}_{k_1}$-equivariant morphism

$$\gamma_x : \Lambda_\ell \to H^1_\ell(K|k, \Lambda_\ell(1))$$

given by $\gamma_x(c) = c \cdot x$.

We choose a smooth proper model $X_0$ of $K_0|k_0$, and a non-empty open $k_0$-subvariety $U_0$ of $X_0$ such that $x$ is contained in the image of the canonical map $H^1_\ell(U, \Lambda_\ell(1)) \to H^1_\ell(K|k, \Lambda_\ell(1))$. Thus $\gamma_x$ factors through $H^1_\ell(U, \Lambda_\ell(1)) \to H^1_\ell(K|k, \Lambda_\ell(1))$, so we may consider $\gamma_x$ as an element of $\text{Hom}_{\Lambda_\ell[[\text{Gal}_{k_1}]]}(\Lambda_\ell, H^1_\ell(U, \Lambda_\ell(1)))$.

Put $X_1 = X_0 \otimes_{k_0} k_1$ and $U_1 = U_0 \otimes_{k_0} k_1$. Consider the Picard 1-motive $M^{1,1}(U_1)$ of $U_1$, as well as the 1-motive $Z := [Z \to 0]$ (here $Z$ is endowed with the trivial $\text{Gal}_{k_1}$-action). Similarly to before, one has $H^1_\ell(Z, \Lambda_\ell) = \Lambda_\ell$, and, by Theorem 5.4, $H^1_\ell(M^{1,1}(U_1)) = H^1_\ell(U, \Lambda_\ell(1))$ (as $\Lambda_\ell[[\text{Gal}_{k_1}]]$-modules). Finally, by Theorem 5.2 the morphism $\gamma_x$ corresponds to an element of $\text{Hom}_{k_1}(Z, M^{1,1}(U_1)) \otimes \Lambda_\ell$, while one has

$$\text{Hom}_{k_1}(Z, M^{1,1}(U_1)) = \ker(\text{Div}^0_X(U) \to \text{Pic}^0_X(k))^{\text{Gal}_{k_1}}$$

$$= (\mathcal{O}^*(U)/k^*)^{\text{Gal}_{k_1}} \subset (K^*/k^*)^{\text{Gal}_{k_1}}.$$  

Hence $\gamma_x$ corresponds to an element $y$ of $(K^*/k^*)^{\text{Gal}_{k_1}} \otimes \Lambda_\ell \subset \mathcal{X}_\Lambda(K|k) \otimes \Lambda_\ell$, and by tracing through the definitions one finally finds that $\mathcal{X}^{\ell,\Lambda}_{K}(y) = x$.  

\[\square\]

8.2. Compatibility with the geometric lattice. Let $M$ be a subextension of $K|k$. Note that the $k$-embedding $M \hookrightarrow K$ induces a canonical map

$$H^1_\ell(M|k, \Lambda_\ell(1)) \to H^1_\ell(K|k, \Lambda_\ell(1)),$$

which is constructed in an analogous manner to the construction in \[\text{2.3}\]. If $M$ is relatively algebraically closed in $K|k$, then (using Lemma 4.3 and the functoriality of the comparison isomorphism $\mathcal{G}_\ell$, for example) this morphism is injective.

**Proposition 8.4.** Let $\phi_\ell, \phi$ be as in the statement of Theorem 8.7, so that $(\phi, \phi_\ell)$ is compatible with $\mathcal{G}_\ell$ and $\phi$ is compatible with $\mathcal{R}$. For $M \in \mathbb{G}^*(K|k)$ and $N \in \mathbb{G}^*(L|l)$, consider the (incomplete) diagram of $\Lambda_\ell$ resp. $\Lambda$-modules:

\[
\begin{array}{ccc}
H^1_\ell(K|k, \Lambda_\ell(1)) & \xrightarrow{\phi} & H^1_\ell(L|l, \Lambda_\ell(1)) \\
\downarrow & & \downarrow \\
H^1_\ell(M|k, \Lambda_\ell(1)) & \xrightarrow{\mathcal{R}^{\Lambda}_M} & H^1_\ell(N|l, \Lambda_\ell(1)) \\
\mathcal{X}_\Lambda(M|k) & \xrightarrow{\mathcal{R}^{\Lambda}_N} & \mathcal{X}_\Lambda(N|l)
\end{array}
\]

Then there exists an isomorphism $\phi^\sharp : \mathbb{G}^*(K|k) \cong \mathbb{G}^*(L|l)$ of geometric lattices, such that for all $M \in \mathbb{G}^*(K|k)$ with image $N := \phi^\sharp M$, the following hold:

1. The lower dotted arrow in the above diagram can be (uniquely) completed to an isomorphism of $\Lambda$-modules.

2. If furthermore $\text{tr.deg}(M|k) = 1$ (and hence $\text{tr.deg}(N|l) = 1$ as well), then the middle dotted arrow can be completed to an isomorphism of $\Lambda_\ell$-modules.
Proof. Again, we start off by noting that $\phi$ induces (in a unique way) an isomorphism

$$\mathcal{X}_\Lambda(K|k) \otimes_\Lambda \Lambda_\ell \cong \mathcal{X}_\Lambda(L|l) \otimes_\Lambda \Lambda_\ell,$$

by Key Lemma 8.3. By the compatibility of $\mathcal{R}_K^{\ell,\Lambda}$ with $\mathcal{R}_L^{\Lambda}$, we see that the image of $\mathcal{R}_K^{\ell,\Lambda}$ is precisely the intersection of the image of $\mathcal{R}_K^{\ell,\Lambda_\ell}$ in $H^1_\ell(K|k, \Lambda_\ell(1))$ with the image of the map

$$H^1(K|k, \Lambda(1)) \to H^1(K|k, \Lambda(1)) \otimes_\Lambda \Lambda_\ell \xrightarrow{\psi_\ell} H^1_\ell(K|k, \Lambda_\ell(1))$$

obtained from Artin’s $\ell$-adic comparison isomorphism. Thus assertion (1) holds true for $M = K$ and $N = L$.

Using Proposition 8.1, the compatibility of $\mathcal{R}_K^{\ell,\Lambda}$ with $\mathcal{C}_\ell$, and the compatibility of $\phi$ with $\mathcal{R}$, we find that this isomorphism $\phi : \mathcal{X}_\Lambda(K|k) \cong \mathcal{X}_\Lambda(L|l)$ is compatible with $\text{acl}$. Hence assertion (1) follows from Lemma 8.3.

Concerning assertion (2), we may first argue as in Proposition 7.4(2), using the compatibility of $\mathcal{R}$, to deduce that $\phi$ induces an isomorphism

$$\text{Image}(H^1_\ell(M|k, \Lambda(1)) \to H^1(K|k, \Lambda(1)) \cong \text{Image}(H^1_\ell(N|l, \Lambda(1)) \to H^1(L|l, \Lambda(1))).$$

Finally, the comparison isomorphism $\mathcal{C}_\ell$ allows us to identify the image of the canonical injective map $H^1_\ell(M|k, \Lambda_\ell(1)) \to H^1_\ell(K|k, \Lambda_\ell(1))$ with the image of

$$\text{Image}(H^1_\ell(M|k, \Lambda(1)) \to H^1(K|k, \Lambda(1)) \otimes_\Lambda \Lambda_\ell \to H^1_\ell(K|k, \Lambda(1)) \otimes_\Lambda \Lambda_\ell \xrightarrow{\phi_\ell} H^1_\ell(K|k, \Lambda_\ell(1)),$$

and similarly for the image of $H^1_\ell(N|l, \Lambda_\ell(1)) \to H^1_\ell(L|l, \Lambda_\ell(1))$. In other words, $\phi_\ell$ induces an isomorphism

$$\text{Image}(H^1_\ell(M|k, \Lambda_\ell(1)) \to H^1_\ell(K|k, \Lambda_\ell(1))) \cong \text{Image}(H^1_\ell(N|l, \Lambda_\ell(1)) \to H^1_\ell(L|l, \Lambda_\ell(1))).$$

This proves assertion (2). \hfill \Box

8.3. Concluding the proof of Theorem 8.1. If there exists a field isomorphism $K_0 \cong L_0$ which restricts to an isomorphism $k_0 \cong l_0$, then the existence of $\phi_{\text{Gal}}, \phi_\ell, \phi$, as in the statement of Theorem 8.1, so that $(\phi, \phi_\ell)$ is compatible with $\mathcal{C}_\ell$ and $\phi$ is compatible with $\mathcal{R}$, is trivial. Conversely, let us assume that such $\phi_{\text{Gal}}, \phi_\ell, \phi$ exist.

Similarly to before, it is an easy consequence of Theorem 8.4 that, for a field $M$ of transcendence degree 1 over $k$, the map

$$\mathcal{R}_M^{\ell,\Lambda_\ell} : \mathcal{X}_\Lambda(M|k) \otimes_\Lambda \Lambda_\ell \to H^1_\ell(M|k, \Lambda_\ell(1))$$

is an isomorphism (of $\Lambda_\ell$-modules) if and only if $M$ is rational over $k$.

Motivated by Proposition 8.4, we consider the set

$$\text{Isom}_{\mathcal{R}_K^{\ell,\Lambda}, \mathcal{C}_\ell, \text{acl}}(H^1_\ell(K|k, \Lambda_\ell(1)), H^1_\ell(L|l, \Lambda_\ell(1)))$$

of isomorphisms $\psi_\ell : H^1_\ell(K|k, \Lambda_\ell(1)) \cong H^1_\ell(L|l, \Lambda_\ell(1))$ of $\Lambda_\ell$-modules which satisfy the following two conditions:

(1) First, the dotted arrow in the diagram

$$\begin{array}{ccc}
H^1_\ell(K|k, \Lambda_\ell(1)) & \xrightarrow{\psi_\ell} & H^1_\ell(L|l, \Lambda_\ell(1)) \\
\mathcal{R}_K^{\ell,\Lambda} & \nearrow & \mathcal{R}_L^{\ell,\Lambda} \\
\mathcal{X}_\Lambda(K|k) & \rightarrow & \mathcal{X}_\Lambda(L|l)
\end{array}$$

can be (uniquely) completed to an isomorphism of $\Lambda$-modules which is compatible with $\text{acl}$.32
(2) Second, there exists a bijection \( \phi^\ell : G^1(K|k) \cong G^1(L|l) \) such that, for \( M \in G^1(K|k) \) and \( N := \phi^\ell M \), the dotted arrows in the diagram

\[
\begin{array}{ccc}
H^1_k(K|k, \Lambda_\ell(1)) & \xrightarrow{\psi_\ell} & H^1_k(L|l, \Lambda_\ell(1)) \\
\downarrow & & \downarrow \\
H^1_k(M|k, \Lambda_\ell(1)) & \xrightarrow{r^\ell_\Lambda} & H^1_k(N|l, \Lambda_\ell(1)) \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{K}_\Lambda(M|k) & \xrightarrow{r^\ell_\Lambda} & \mathcal{K}_\Lambda(N|l) \\
\end{array}
\]

can be (uniquely) completed to an isomorphism of \( \Lambda_\ell \) resp. \( \Lambda \)-modules.

By Proposition [5.4] this set \( \text{Isom}_{G^1, \Lambda, \mathcal{G}^1, \text{ac}^\ell}(H^1_\ell(K|k, \Lambda_\ell(1)), H^1_\ell(L|l, \Lambda_\ell(1))) \) is non-empty. Note we have not specified the isomorphisms \( \psi_\ell \) in this set to be compatible with the Galois action. However, we do note that the observation above concerning rationality yields a canonical map

\[
\text{Isom}_{G^1, \Lambda, \mathcal{G}^1, \text{ac}^\ell}(H^1_\ell(K|k, \Lambda_\ell(1)), H^1_\ell(L|l, \Lambda_\ell(1))) \rightarrow \text{Isom}_{\text{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))/_\Lambda^* 
\]

which is easily seen to be compatible with compositions of isomorphisms on either side.

Also, note that one has a canonical map

\[
\text{Isom}(K, L) \rightarrow \text{Isom}_{G^1, \Lambda, \mathcal{G}^1, \text{ac}^\ell}(H^1_\ell(K|k, \Lambda_\ell(1)), H^1_\ell(L|l, \Lambda_\ell(1))) 
\]

which is again compatible with compositions on either side. Furthermore, it is easy to see from the above constructions of these maps that one has a commutative diagram

\[
\begin{array}{ccc}
\text{Isom}(K, L) & \xrightarrow{\text{Isom}_{G^1, \Lambda, \mathcal{G}^1, \text{ac}^\ell}} & \text{Isom}_{G^1, \Lambda, \mathcal{G}^1, \text{ac}^\ell}(H^1_\ell(K|k, \Lambda_\ell(1)), H^1_\ell(L|l, \Lambda_\ell(1))) \\
\downarrow & & \downarrow \\
\text{Isom}_{\text{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))/_\Lambda^* 
\end{array}
\]

where the diagonal map is the canonical one described around §6 Theorem 6.1 states that this diagonal map is a bijection. In particular, the fields \( K \) and \( L \) are isomorphic.

Finally, let us note that one has a canonical action of \( \text{Gal}_{k_0} \) on these isomorphism sets in the commutative triangle above, given in the usual way by

\[
(\sigma \cdot \psi)(x) = \phi_{\text{Gal}}(\sigma) \cdot \psi(\sigma^{-1} \cdot x)
\]

for \( \psi \) an element of one of these three isomorphism sets and \( \sigma \in \text{Gal}_{k_0} \). By tracing through the constructions, especially the Galois-equivariance of \( R^\ell_\Lambda \) (cf. [8.1]), it is easy to see that these maps are equivariant with respect to these actions of \( \text{Gal}_{k_0} \). The invariants under this action are precisely the isomorphisms which are \( \text{Gal}_{k_0} \)-equivariant, with respect to the natural action of \( \text{Gal}_{k_0} \) on the corresponding objects.

Our original isomorphism \( \phi^\ell \) was such a \( \text{Gal}_{k_0} \)-equivariant isomorphism, hence we obtain a corresponding element of

\[
\text{Isom}(K, L)
\]

which is is \( \text{Gal}_{k_0} \)-invariant. In other words, there exists a \( \text{Gal}_{k_0} \)-equivariant isomorphism \( K \cong L \) of fields, where \( \text{Gal}_{k_0} \) acts on \( L \) via \( \phi_{\text{Gal}} \). Taking invariants of \( K \) resp. \( L \) resp \( k \) resp. \( l \) with respect to this action of \( \text{Gal}_{k_0} \), we find that this isomorphism \( K \cong L \) restricts to an isomorphism \( K_0 \cong L_0 \). Since \( K \cong L \) also restricts to an isomorphism \( k \cong l \), it follows that \( K_0 \cong L_0 \) restricts to an isomorphism \( k_0 \cong l_0 \). This concludes the proof of Theorem [8.1].
The local theory in “almost-abelian” anabelian geometry has been extensively developed by Bogomolov [Bog91], Bogomolov-Tscheinkel [BT02], Pop [Pop10], and Topaz [Top15a, Top16a]. Despite the fact that such local theories are by now more-or-less completely understood, the precise formulation which is needed in the above paper hasn’t appeared in the literature, since previous results have mostly focused on the “classical” anabelian point of view of “recovering” decomposition and inertia groups in Galois groups (of function fields, in this case). The goal of this appendix is therefore to give an essentially self-contained account of the local theory, which is required in the main body of the present paper. The arguments we give in this appendix are, in many respects, merely a distillation of the ideas developed in the references mentioned above.

Using the notation introduced in the body of the paper, the main result in the local theory reads as follows.

Theorem A.1. Let $K|k$ and $L|l$ be two function fields over algebraically closed fields, and let $\Lambda$ be a subring of $\mathbb{Q}$. Assume that $\deg(K|k) \geq 2$. Let

$$\phi : \mathcal{K}_\Lambda(K|k) \xrightarrow{\sim} \mathcal{K}_\Lambda(L|l)$$

be an isomorphism of $\Lambda$-modules which is compatible with $\text{acl}$, and let $v$ be a divisorial valuation of $K|k$. Then there exists a unique divisorial valuation $v^\phi$ of $L|l$ such that one has $\phi(\mathcal{U}_v) = \mathcal{U}_{v^\phi}$ and $\phi(\mathcal{U}_v^1) = \mathcal{U}_{v^\phi}^1$.

A.1. Notation. We will work with a fixed function field $K$ over an algebraically closed field $k$. For most of the appendix, we will work with the $\mathbb{Q}$-vector space

$$\mathcal{G}(K|k) := \text{Hom}(K^\times/k^\times, \mathbb{Q}).$$

We consider elements of $\mathcal{G}(K|k)$ as homomorphisms $f : K^\times \to \mathbb{Q}$ of abelian groups which are trivial on $k^\times$. We may also use the notation

$$\mathcal{G}(M|F) := \text{Hom}(M^\times/F^\times, \mathbb{Q})$$

for an arbitrary field extension $M|F$, although extensions which are not function fields will only occur, in our context, as residue fields of valuations of a function field as above.

For a valuation $v$ of $K$ (which may or may not be of geometric origin), we define

$$\mathcal{I}_v := \text{Hom}(K^\times/(U_v \cdot k^\times), \mathbb{Q}), \quad \mathcal{D}_v := \text{Hom}(K^\times/(U_v^1 \cdot k^\times), \mathbb{Q})$$

considered as subspaces of $\mathcal{G}(K|k)$. Note in particular that one has $\mathcal{I}_v \subset \mathcal{D}_v$. The following is immediate from the definitions along with the fact that $\mathbb{Q}$ is an injective object in the category of abelian groups.

Fact A.2. The inclusion

$$(Kv)^\times/(kv)^\times = (U_v \cdot k^\times)/(U_v^1 \cdot k^\times) \hookrightarrow K^\times/(U_v^1 \cdot k^\times)$$

induces a canonical surjective map $f \mapsto f_v : \mathcal{D}_v \to \mathcal{G}(Kv|kv)$ whose kernel is $\mathcal{I}_v$.

In a nutshell, our goal in this appendix is to give a recipe to reconstruct $\mathcal{I}_v$ and $\mathcal{D}_v$ for divisorial valuations $v$ of $K|k$. To conclude Theorem [A.1] we will note that

$$\mathcal{G}(K|k) = \text{Hom}_\Lambda(\mathcal{K}_\Lambda(K|k), \mathbb{Q})$$

hence one has a canonical pairing

$$\mathcal{K}_\Lambda(K|k) \times \mathcal{G}(K|k) \rightarrow \mathbb{Q}.$$ 

We then observe that $\mathcal{U}_v$ resp. $\mathcal{U}_v^1$ agree with the orthogonal of $\mathcal{I}_v$ resp. $\mathcal{D}_v$ with respect to this pairing, for any divisorial valuation $v$ of $K|k$. 

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A.2. Abhyankar’s inequality. Let $v$ be a valuation of $K$. Recall that Abhyankar’s Inequality,

$$\text{rank}_Q(vK/vk) + \text{tr}. \deg(Kv/kv) \leq \text{tr}. \deg(K/k),$$

relates the transcendence degree of the residue extension $Kv/kv$, the rational-rank of $vK/vk$, and the transcendence degree of $K/k$. We say that $v$ has no transcendence defect provided that this inequality is an equality. If this is the case, then it is well-known that $vK/vk$ and $Kv/kv$ are finitely generated (as a group resp. field extension), hence $vK/vk \cong \mathbb{Z}^r$ for some $r \geq 0$, and $Kv$ is a function field over the (algebraically closed) field $kv$. Defectless valuations will play a crucial role in the discussion below.

A.3. acl-pairs. Let $f, g \in \mathcal{G}(K/k)$ be given. We say that $(f, g)$ is an acl-pair provided that for all $x, y \in K^\times$ which are algebraically dependent over $k$, one has $f(x) \cdot g(y) = f(y) \cdot g(x)$. A subset $S \subset \mathcal{G}(K/k)$ will be called an acl-set provided that any pair of elements of $S$ is an acl-pair. Note that $S$ is an acl-set if and only if its span $(S)_Q$ in $\mathcal{G}(K/k)$ is an acl-set.

Lemma A.3. Let $v$ be a valuation of $K$, and let $f, g \in \mathcal{D}_v$ be given. Assume that $(f_v, g_v)$ forms a acl-pair in $\mathcal{G}(Kv/kv)$. Then $(f, g)$ forms an acl-pair in $\mathcal{G}(K/k)$.

Proof. Let $x, y \in K^\times$ be algebraically dependent over $k$. Our goal is to show that one has

$$f(x) \cdot g(y) = f(y) \cdot g(x).$$

If one has $x, y \in U_v \cdot k^\times$, then we are done since $f = f_v$ and $g = g_v$ on $U_v \cdot k^\times$. On the other hand, assume, for example, that $x$ is not contained in $U_v \cdot k^\times$. Then the restriction of $v$ to acl$_K(x) := M$ is non-trivial, while $y \in M$. We let $w$ denote the restriction of $v$ to $M$.

Since $x \notin U_v \cdot k^\times$, we find that $vx \notin vk$, hence $wM/wk$ has rational-rank $\geq 1$. Since $\text{tr}. \deg(M/k) = 1$, it follows that $w$ is without transcendence defect, that $wM/wk \cong \mathbb{Z}$, and that $Mw = kw = kv$. Thus, one has $U_w \cdot k^\times = U^1_w \cdot k^\times \subset U^1_v \cdot k^\times$.

If $vy \in wk$ and hence $wy \in wk$, it follows that $f(y) = g(y) = 0$, so that $f(x) \cdot g(y) = f(y) \cdot g(x)$ trivially. On the other hand, if $wy \notin wk$, then $x$ and $y$ have $\mathbb{Q}$-linearly-dependent images in

$$(K^\times/U_w \cdot k^\times) \otimes \mathbb{Z} \mathbb{Q} = (K^\times/U^1_w \cdot k^\times) \otimes \mathbb{Z} \mathbb{Q},$$

since $wM/wk$ has rational rank $1$ and $U_w \cdot k^\times = U^1_w \cdot k^\times$. As $U^1_w \subset U^1_v$, it follows that the images of $x, y$ in

$$(K^\times/U^1_v \cdot k^\times) \otimes \mathbb{Z} \mathbb{Q}$$

are again $\mathbb{Q}$-linearly-dependent. Since $f, g \in \mathcal{D}_v$, it follows that one has $f(x) \cdot g(y) = f(y) \cdot g(x)$ in this case as well. \qed

A.4. Rigid elements in fields. The theory of Rigid Elements refers to a collection of classical results which were introduced in the course of studying the Witt ring of quadratic forms of fields. There is an aspect of this theory which shows how to reconstruct valuation rings in fields given certain bounds for their units and principal units. This part of the theory of rigid elements will be crucial for our considerations in this appendix. The results needed for our considerations come from the work of Arason-Elman-Jacob [AEJ87], and we summarize the necessary main results from loc.cit. in the following theorem.

Theorem A.4 ([AEJ87] Theorem 2.16). Let $T$ be a subgroup of $K^\times$ such that $\pm 1 \in T$, and let $H$ denote the subgroup of $K^\times$ which is generated by $T$ and all $x \in K^\times \backslash T$ such that $1 + x \notin T \cup x \cdot T$. Then there exists a subgroup $\bar{H} \subset K^\times$ and a valuation $v$ of $K$ such that the following conditions hold

1. One has $H \subset \bar{H}$ and $[\bar{H} : H] \leq 2$.
2. One has $U^1_v \subset T$ and $U_v \subset \bar{H}$. 

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A.5. Valuative elements. Let $S$ be a subset of $\mathcal{G}(K|k)$. We say that $S$ is valuative provided that there exists some valuation $v$ of $K$ such that $S \subset \mathcal{I}_v$. An element $f \in \mathcal{G}(K|k)$ will be called valuative provided that the set $\{f\}$ is valuative.

Lemma A.5. Let $S$ be a valuative subset of $\mathcal{G}(K|k)$. Then there exists a unique coarsest valuation $v$ of $K$ such that $S \subset \mathcal{I}_v$. This valuation $v$ depends only on $S$ and $K$, and, if $w$ is any valuation such that $S \subset \mathcal{I}_w$, then $v$ is a coarsening of $w$.

Proof. Let $w$ be any valuation such that $S \subset \mathcal{I}_w$. Put

$$H = \{x \in K^\times : \forall f \in S, \ f(x) = 0\},$$

and note that one has $U_w \subset H$. Let $v$ be the coarsest coarsening of $v$ such that $U_v \subset H$. In other words, $v$ is the coarsening of $w$ associated to the maximal convex subgroup of $w(H)$. It is straightforward to see, using the ultrametric inequality, that

$$U_v = \{t \in H : \forall x \in K^\times \setminus H, \ 1 + x \in (t + x) \cdot H\}.$$

Hence $v$ doesn’t depend on the original choice of valuation $w$, while $S \subset \text{Hom}(K^\times/H, \mathbb{Q}) \subset \mathcal{I}_v$. □

By Lemma A.5, we can associate a valuation $v$ of $K$ to any valuative subset $S$ of $\mathcal{G}(K|k)$. This valuation $v$ has the property that $S \subset \mathcal{I}_v$, and that $v$ is the coarsest valuation as such. We call this valuation $v$ the valuation associated to $S$. If $f$ is a valuative element, then the valuation associated to $f$ will just be the valuation associated to $\{f\}$.

Lemma A.6. Let $f$ be a valuative element with associated valuation $v$, and let $g \in \mathcal{G}(K|k)$ be an element such that $(f,g)$ is an acl-pair. Then one has $g \in D_v$.

Proof. Let $x \in K^\times$ be such that $v(x) > 0$. Note that one has $g(1 + x) \cdot f(x) = g(x) \cdot f(1 + x)$ while $f(1 + x) = 0$. If $f(x) \neq 0$, we therefore deduce that $g(1 + x) = 0$.

On the other hand, if $f(x) = 0$, then the defining property of $v$ ensures that there exists some $y \in K^\times$ such that $0 < v(y) < v(x)$ and $f(y) \neq 0$. Arguing as above, we have $g(1 + y) = 0$. On the other hand, note that $v(y + x(1 + y)) = v(y)$ hence we also have

$$g(1 + x) + g(1 + y) = g(1 + (y + x(1 + y))) = 0.$$

In other words, $g(1 + x) = 0$, as required. □

Lemma A.7. Let $f_1, f_2$ be two valuative elements of $\mathcal{G}(K|k)$. Then $(f_1, f_2)$ form an acl-pair if and only if there exists a valuation $w$ of $K$ such that $f_1, f_2 \in \mathcal{I}_w$.

Proof. If $f_1, f_2 \in \mathcal{I}_v$ for some valuation $v$, then $(f_1, f_2)$ is an acl-pair by Lemma A.5. Conversely, assume that $(f_1, f_2)$ is an acl-pair. We let $v_1, v_2$ be the valuations associated to $f_1, f_2$ respectively. By Lemma A.6, one has $f_1 \in \mathcal{I}_{v_1}$ and $f_2 \in \mathcal{D}_{v_2}$, while also $f_2 \in \mathcal{I}_{v_2}$ and $f_1 \in \mathcal{D}_{v_2}$. In any case, one has $f_1, f_2 \in \mathcal{D}_{v_1} \cap \mathcal{D}_{v_2}$, so that both $f_1$ and $f_2$ are trivial on $U_{v_1}$ and on $U_{v_2}$. If $v_1, v_2$ are comparable, then we are done since $\mathcal{I}_{v_1}$ and $\mathcal{I}_{v_2}$ are comparable in this case. Otherwise, let $w$ be the finest common coarsening of $v_1, v_2$, and note that one has $U_{v_1} \cdot U_{v_2} = U_w$ by the approximation theorem for independent valuations. Hence $f_1, f_2$ are both trivial on $U_w$, which means that $f_1, f_2 \in \mathcal{I}_w$. □

Lemma A.8. Let $v_1, v_2$ be two valuations of $K$. Assume that there exists an element $g \in \mathcal{D}_{v_1} \cap \mathcal{D}_{v_2}$ which is non-valuative. Then $v_1$ and $v_2$ are comparable.

Proof. Suppose not, and let $w$ be the finest common coarsening of $v_1, v_2$. Then one has $U_{v_1} \cdot U_{v_2} = U_w$ by the approximation theorem for independent valuations, while $g \in \mathcal{D}_{v_1} \cap \mathcal{D}_{v_2}$ implies that $g$ is trivial on $U_{v_1} \cdot U_{v_2}$, hence $g$ is trivial on $U_w$. This means that $g \in \mathcal{I}_w$, hence contradicting the assumption of the lemma. □
A.6. **The Main Theorem of acl-pairs.** The following theorem is the technical core of the results in this section. The proof of this theorem uses an adaptation of ideas due to Bogomolov [Bog91].

**Theorem A.9.** Let $f, g \in \mathcal{G}(K|k)$ be given. Then $(f, g)$ is an acl-pair if and only if there exists a valuation $v$ of $K$ such that $f, g \in \mathcal{D}_v$, and such that $af + bg \in \mathcal{I}_v$ for some $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$.

**Proof.** If a valuation $v$ exists as in the statement of the theorem, then $f, g$ form an acl-pair by Lemma A.3.

We now prove the converse, by using the *Theory of Rigid Elements* as summarized in Theorem A.4. Consider the map

$$\Psi : K^\times/k^\times \to \mathbb{Q}^2 = K^2(Q).$$

As $(f, g)$ forms an acl-pair, we find that $\Psi$ maps projective lines to affine lines. In other words, if $x, y \in K$ are $k$-linearly-independent, then for all $(a, b) \in k^2 \setminus \{(0, 0)\}$, the point $\Psi(ax + by)$ lies on the affine line between $\Psi(x)$ and $\Psi(y)$. The primary goal of the proof is to prove the following.

**Key Claim.** Let $x, y \in K^\times$ be such that $\Psi(x)$ and $\Psi(y)$ are $\mathbb{Q}$-linearly-independent in $\mathbb{Q}^2$. Then one has $\Psi(1 + x) \in \{\Psi(1), \Psi(x)\}$ or $\Psi(1 + y) \in \{\Psi(1), \Psi(y)\}$.

Before we prove the key claim, let us show how to deduce the theorem from this. First, note that we may assume that $f, g$ are $\mathbb{Q}$-independent in $\mathcal{G}(K|k)$, for otherwise the assertion of the theorem is trivial. Put $T := \ker(f) \cap \ker(g)$, and let $H$ be the subgroup of $K^\times$ which is generated by $T$ and all $x \in K^\times \setminus T$ such that $1 + x \notin T \cup x \cdot T$. By Theorem A.4 there exists a valuation $v$ and a subgroup $\widetilde{H} \subset K^\times$ containing $H$ such that $[\widetilde{H} : H] \leq 2$, such that $U_1 \subset T$ and such that $U_v \subset \widetilde{H}$.

Put $\mathcal{D} := \operatorname{Hom}(K^\times/T, \mathbb{Q})$ and $\mathcal{I} := \operatorname{Hom}(K^\times/H, \mathbb{Q}) = \operatorname{Hom}(K^\times/\widetilde{H}, \mathbb{Q})$. In particular, we have $\mathcal{D} \subset \mathcal{D}_v$ and $\mathcal{I} \subset \mathcal{I}_v$, while $\mathcal{I} \subset \mathcal{D}$ and $f, g \in \mathcal{D}$. Finally, the Key Claim ensures that $\mathcal{D}/\mathcal{I}$ is at most 1-dimensional, and the assertion of the theorem follows from this.

The rest of the proof will be devoted to proving the Key Claim. Assume, for a contradiction, that the Key Claim is false, and let $x, y \in K^\times$ be witnesses of this, so that $\Psi(x), \Psi(y)$ are $\mathbb{Q}$-linearly-independent, $\Psi(1 + x) \neq \Psi(1), \Psi(x)$, and $\Psi(1 + y) \neq \Psi(1), \Psi(y)$.

As noted above, the condition that $f, g$ form an acl-pair implies that $\Psi(1 + x) = A \cdot \Psi(x)$ and $\Psi(1 + y) = B \cdot \Psi(1 + y)$ for some $A, B \in \mathbb{Q} \setminus \{0, 1\}$. As an initial reduction step, we may assume that $1 - A > 0$ by replacing $x$ with $x^{-1}$, if needed. Indeed, we see that one has

$$\Psi(1 + x^{-1}) = \Psi\left(\frac{1 + x}{x}\right) = (A - 1) \cdot \Psi(x) = (1 - A) \cdot \Psi(x^{-1}).$$

Hence, replacing $x$ by $x^{-1}$ has the effect of replacing $A$ by $1 - A$, while at least one of $1 - A$ or $A$ must be positive. We will therefore assume henceforth that $1 - A$ is positive.

As a second reduction step, we compose $\Psi$ with a $\mathbb{Q}$-linear automorphism of $\mathbb{Q}^2$ to obtain $\Phi : K^\times/k^\times \to \mathbb{Q}^2$ which satisfies:

$$\Phi(1) = (0, 0), \quad \Phi(1 + x) = (1, 0), \quad \Phi(y) = (0, 1).$$

Hence $\Phi(1 + x) = (A, 0)$ and $\Phi(1 + y) = (0, B)$ with $A, B \in \mathbb{Q} \setminus \{0, 1\}$ as above.

Finally, we embed $\mathbb{Q}^2 = K^2(Q)$ in $\mathbb{P}^2(Q)$ in the usual way via $(a, b) \mapsto (1 : a : b)$. We will write $(a, b) := (1 : a : b)$ and $(a : b) := (0 : a : b)$ to simplify the notation. For two distinct points $p, q \in \mathbb{P}^2(Q)$, we write $L(p, q)$ for the projective line between $p, q$.

Since one has $A, B \in \mathbb{Q} \setminus \{0, 1\}$ by assumption, there is a unique $\mathbb{Q}$-projective-linear automorphism $\Sigma$ of $\mathbb{P}^2(Q)$ which satisfies the following properties:

$$\Sigma(0, 0) = (1, 0), \quad \Sigma(1, 0) = (0, 1), \quad \Sigma(0, 1) = (0, 1), \quad \Sigma(A, 0) = (1 : 0), \quad \Sigma(0, B) = (0 : 1).$$

A straightforward calculation shows that this automorphism $\Sigma$ sends the line at infinity (i.e. the projective line between $(1 : 0)$ and $(0 : 1)$) to the projective line between $(1 - A, 0)$ and $(0, 1 - B)$.
We write $\Delta := \Sigma \circ \Phi$. To summarize, we have constructed a map

$$\Delta : K^\times / k^\times \to \mathbb{P}^2(\mathbb{Q})$$

which satisfies the following properties:

1. First, the map $\Delta$ is compatible with projective lines. That is, whenever $u, t \in K^\times$ are such that $\Delta(t) \neq \Delta(u)$ (hence $u, t$ are $k$-linearly-independent in $K$), and $(a, b) \in k^2$ is non-zero, then one has $\Delta(at + bu) \in L(\Delta(t), \Delta(u))$.

2. Second, one has $\Delta(1) = (0, 0), \Delta(x) = (1, 0), \Delta(y) = (0, 1), \Delta(1 + x) = (1 : 0)$ and $\Delta(1 + y) = (0 : 1)$.

3. Third, the image of $\Delta$ does not contain any points in the projective line between $(1 - A, 0)$ and $(0, 1 - B)$.

We will obtain our contradiction from the three properties above by showing that properties (1) and (2) imply that $(1 - A, 0)$ is contained in the image of $\Delta$. In fact, we will show that $(r, 0)$ is contained in the image of $\Delta$ for all $r \in \mathbb{Q}, r > 0$.

In the steps below, we will calculate $\Delta(t)$ for various $t \in K^\times$ by exhibiting $t$ as a sum (or difference) in $K$ in two different ways. For example, the fact that $1 + x + y = (1 + x) + y = (1 + y) + x$ implies that $\Delta(1 + x + y) = (1, 1)$. Indeed, the equation above and property (1) implies that $\Delta(1 + x + y)$ lies on the line $L(\Delta(1 + x), \Delta(y))$ and on the line $L(\Delta(1 + y), \Delta(x))$. Using property (2), we find that the intersection of these two lines contains a unique point $(1, 1)$, hence one has $\Delta(1 + x + y) = (1, 1)$. This is the starting point of our calculations.

**Step 1.** One has $\Delta(1 + x + y) = (1, 1)$.

In the subsequent steps, we use essentially the same argument by exhibiting elements $t$ of $K$ as a sum (or difference) in two different ways, which allows us to calculate $\Delta(t)$. To simplify the exposition, we will give explicitly these decompositions of $t$ as a sum/difference in two ways, leaving to the reader the straightforward calculation of the intersections of the corresponding projective lines.

**Step 2.** One has $\Delta(2 + x + y) = (1 : 1)$.

**Proof.** This follows from Step 1 and the fact that one has

$$2 + x + y = (1 + x) + (1 + y) = 1 + (1 + x + y).$$

**Step 3.** For all integers $n \geq 1$, one has $\Delta((2 - n) + x + y) = (1/n, 1/n)$ and $\Delta((1 - n) + x) = (1/n, 0)$.

**Proof.** The base-case $n = 1$ is Property (2) and Step 1 above. For the inductive case, we first calculate $\Delta((2 - (n + 1)) + x + y)$ using the fact that

$$(2 - (n + 1)) + x + y = ((1 - n) + x) + y = ((2 - n) + x + y) - 1$$

combined with the inductive hypothesis and Property (2). This shows that

$$\Delta((2 - (n + 1)) + x + y) = (1/(n + 1), 1/(n + 1)).$$

Finally, we calculate $\Delta((1 - (n + 1)) + x)$ using the fact that

$$(1 - (n + 1)) + x = ((2 - (n + 1)) + x + y) - (1 + y) = ((1 - n) + x) - 1$$

which completes the proof.
combined with the inductive hypothesis, the calculation above, and Property (2). This shows that
\[ \Delta((1 - (n + 1)) + x) = (1/(n + 1), 0), \]
as required. □

Step 4. For all integers \( n, m \geq 1 \), one has
\[ \Delta((1 + m - n) + mx + y) = (m/n, 1/n), \]
\[ \Delta((m - n) + mx) = (m/n, 0). \]

Proof. The proof proceeds by induction on \( m \), with base-case \( m = 1 \) being covered by Step 3. For the inductive case, we first use the equation
\[ (1 + ((m + 1) - n) + (m + 1)x + y) = ((1 + m - n) + mx + y) + (1 + x) \]
\[ = ((m - n) + mx) + (2 + x + y) \]
along with Step 2 and Property (2) to deduce that
\[ \Delta((1 + (m + 1) - n) + (m + 1)x + y) = ((m + 1)/n, 1/n). \]
We then use the equation
\[ ((m + 1) - n) + (m + 1)x = ((m - n) + mx) + (1 + x) \]
\[ = ((1 + (m + 1) - n) + (m + 1)x + y) - (1 + y). \]
along with Property (2), the inductive step, and the above calculation, to deduce that
\[ \Delta(((m + 1) - n) + (m + 1)x) = ((m + 1)/n, 0), \]
as required. □

By Step 4 we deduce that the image of \( \Delta \) contains \((r, 0)\) for all \( r \in \mathbb{Q} \times \mathbb{R}, r > 0\). In particular, the image of \( \Delta \) contains \((1 - A, 0)\), which contradicts Property (3) (recall that we have arranged for \(1 - A > 0\)), hence concluding the proof of the Key Claim. As discussed right after the statement of the Key Claim, this also concludes the proof of Theorem A.9. □

The following proposition refines Theorem A.9.

Proposition A.10. Let \( \mathcal{H} \) be a subspace of \( \mathcal{G}(K|k) \). Then \( \mathcal{H} \) is an acl-subspace if and only if there exists a valuation \( v \) of \( K \) such that \( \mathcal{H} \subset \mathcal{D}_v \) and such that \( \mathcal{H} \cap \mathcal{I}_v \) has codimension \( \leq 1 \) in \( \mathcal{H} \).

Proof. If a valuation \( v \) exists as in the statement of the proposition, then \( \mathcal{H} \) is an acl-subspace by Lemma A.3. Conversely, we let \( \mathcal{I} \) denote the set of all valuative elements of \( \mathcal{H} \). For any \( f \in \mathcal{I} \), let \( v_f \) denote the valuation associated to \( f \). By Lemma A.7 we see that the collection \( (v_f)_{f \in \mathcal{I}} \) of valuations is pairwise comparable. Indeed, for any \( f, g \in \mathcal{I} \), that lemma implies that there exists a valuation \( w \) such that \( f, g \in \mathcal{I}_w \), hence \( v_f \) and \( v_g \) are both coarsenings of \( w \) by Lemma A.3, which implies that \( v_f \) and \( v_g \) are comparable.

Consider the valuation-theoretic supremum \( v \) of the \( (v_f)_{f \in \mathcal{I}} \). In terms of valuations rings,
\[ \mathcal{O}_v = \bigcap_{f \in \mathcal{I}} \mathcal{O}_{v_f}, \]
and it is straightforward to see that \( \mathcal{O}_v \) is a valuation ring of \( K \), while \( v_f \) are all coarsenings of \( v \). Thus, we have \( f \in \mathcal{I}_{v_f} \subset \mathcal{I}_v \) for all \( f \in \mathcal{I} \), hence \( \mathcal{I} \subset \mathcal{I}_v \). This implies, in particular, that \( \mathcal{I} \) is a subspace of \( \mathcal{D} \), while \( \mathcal{I} \) has codimension \( \leq 1 \) in \( \mathcal{D} \) by Theorem A.9.

If \( \mathcal{I} = \mathcal{D} \), then we are done. Otherwise, let \( g \in \mathcal{D} \setminus \mathcal{I} \) be given, and note that \( g \in \mathcal{D}_{v_f} \) for all \( f \in \mathcal{I} \), by Lemma A.6. In particular, \( g \) is trivial on \( U^1_{v_f} \) for all such \( f \), hence \( g \) is trivial on \( \bigcup_{f \in \mathcal{I}} U^1_{v_f} \),
This implies that \( g \in \mathcal{D}_v \), as required. \( \square \)

**A.7. Quasi-divisorial valuations.** A valuation \( v \) of \( K \) will be called a quasi-divisorial valuation of \( K|k \) provided that one of the following (equivalent) conditions holds true:

1. \( v \) is without transcendence defect in \( K|k \) and \( vK/vk \cong \mathbb{Z} \).
2. \( v \) is without transcendence defect in \( K|k \) and \( \text{tr.deg}(Kv/kv) = \text{tr.deg}(K|k) - 1 \).
3. One has \( \text{tr.deg}(Kv/kv) = \text{tr.deg}(K|k) - 1 \) and \( vK/vk \cong \mathbb{Z} \).

Note that a quasi-divisorial valuation \( v \) of \( K|k \) is divisorial (in the sense of Proposition [A.13]) if and only if \( v \) is trivial on \( k \). In this subsection we show how to recover \( \mathcal{D}_v \) and \( \mathcal{I}_v \) for quasi-divisorial valuations of \( K|k \) (see Theorem A.13).

**Lemma A.11.** Put \( d := \text{tr.deg}(K|k) \). Then any acl-subspace \( \mathcal{H} \) of \( \mathcal{G}(K|k) \) has dimension \( \leq d \).

**Proof.** If \( \mathcal{H} \) is an acl-subspace, then by Proposition A.10 there exists a valuation \( v \) such that \( \mathcal{H} \subset \mathcal{D}_v \) and such that \( \mathcal{I}_v \cap \mathcal{H} \) has codimension \( \leq 1 \) in \( \mathcal{H} \). Assume for a contradiction that \( \text{dim} \mathcal{H} > d \). If \( \mathcal{H} \subset \mathcal{I}_v \), then \( \text{rank}_Q(vK/vk) = \text{dim} \mathcal{I}_v > d \), contradicting Abhyankar’s inequality. If \( \mathcal{H} \not\subset \mathcal{I}_v \), then \( \mathcal{H} \cap \mathcal{I}_v \) has dimension \( d \), while \( \mathcal{D}_v \neq \mathcal{I}_v \). In particular, \( \mathcal{G}(Kv/kv) \neq 0 \), so that \( \text{tr.deg}(Kv/kv) > 0 \), while \( \text{dim}_Q(vK/vk) \geq d \); this contradicts Abhyankar’s inequality yet again. \( \square \)

**Proposition A.12.** Put \( d := \text{tr.deg}(K|k) \), and let \( \mathcal{D} \) be an acl-subspace of \( \mathcal{G}(K|k) \) of dimension \( d \). Let \( \mathcal{I} \) be the subset of \( \mathcal{H} \) consisting of all valuative elements of \( \mathcal{D} \). Then the following hold:

1. \( \mathcal{I} \) is a subspace of \( \mathcal{D} \) of codimension \( \leq 1 \), and \( \mathcal{I} \) is valuative.
2. Letting \( v \) denote the valuation associated to \( \mathcal{I} \), one has \( \mathcal{I}_v = \mathcal{I} \) and \( \mathcal{D} \subset \mathcal{D}_v \).
3. The valuation \( v \) is without transcendence defect in \( K|k \).

**Proof.** For each \( f \in \mathcal{I} \), let \( v_f \) denote the valuation associated to \( f \). By Lemma A.7 we see that the valuations in the collection \( \langle v_f \rangle_{f \in \mathcal{I}} \) are pairwise-comparable. We may therefore consider the valuation-theoretic supremum of \( \langle v_f \rangle_{f \in \mathcal{I}} \), which is the valuation \( v \) whose valuation ring is defined by

\[
\mathcal{O}_v = \bigcap_{f \in \mathcal{I}} \mathcal{O}_{v_f}.
\]

It is straightforward to see that this subring of \( K \) is indeed a valuation ring, and furthermore that one has

\[
U_v = \bigcap_{f \in \mathcal{I}} U_{v_f}, \quad U_v^1 = \bigcup_{f \in \mathcal{I}} U_{v_f}^1.
\]

Note also that \( v_f \) is a coarsening of \( v \) for all \( f \in \mathcal{I} \). In particular, one has \( f \in \mathcal{I}_v \subset \mathcal{I}_v \) for all \( f \in \mathcal{I} \) so that \( \mathcal{I} \subset \mathcal{I}_v \). In particular, we see that \( \mathcal{I} = \mathcal{D} \cap \mathcal{I}_v \), so that \( \mathcal{I} \) is a subspace of \( \mathcal{D} \). The fact that \( \mathcal{I} \) has codimension \( \leq 1 \) in \( \mathcal{D} \) follows from Theorem A.9.

Concerning assertions (2) and (3), note first that \( v \) is the valuation of \( K \) associated to the valuative subspace \( \mathcal{I} \), since, by definition, \( v \) is the coarsest valuation which contains every element of \( \mathcal{I} \).

We now show that \( \mathcal{I} = \mathcal{I}_v \), and we have two cases to consider. First, if \( \mathcal{I} = \mathcal{D} \), then \( \mathcal{I} \) has dimension \( d \), so that \( \text{rank}_Q(vK/vk) = d \). In particular, \( v \) is without transcendence defect, and \( K/v = kv \) since \( kv \) is algebraically closed. This means that \( \mathcal{G}(Kv/kv) = 0 \) hence \( \mathcal{D}_v = \mathcal{I}_v \), while \( \mathcal{I}_v \) has dimension \( d \). As \( \mathcal{I} \subset \mathcal{I}_v \), it follows that \( \mathcal{I} = \mathcal{I}_v \) and that \( \mathcal{D} = \mathcal{I} = \mathcal{I}_v = \mathcal{D}_v \).

Second, assume that \( \mathcal{I} \neq \mathcal{D} \), and let \( g \in \mathcal{D} \setminus \mathcal{I} \) be given. By Lemma A.6 we see that \( g \in \mathcal{D}_v \) for all \( f \in \mathcal{I} \). In particular, \( g \) is trivial on \( U_{v_f}^1 \) for all \( f \in \mathcal{I} \), which implies that \( g \) is trivial on \( U_v^1 \), hence \( g \in \mathcal{D}_v \). Therefore, one has \( \mathcal{D} \subset \mathcal{D}_v \). By our definition of \( \mathcal{I} \), we see that \( g \) is not valuative, hence \( \mathcal{I} = \mathcal{I}_v \cap \mathcal{D} \), and the image of \( g \) in \( \mathcal{G}(Kv/kv) \) is non-trivial. In particular, \( \mathcal{G}(Kv/kv) \neq 0 \) so that \( \text{tr.deg}(Kv/kv) \geq 1 \). On the other hand, \( \text{dim} \mathcal{I} = d - 1 \) so that \( \text{rank}_Q(vK/vk) \geq d - 1 \). This implies that \( v \) is without transcendence defect in \( K|k \), and that \( \text{rank}_Q(vK/vk) = d - 1 \). Hence
$\mathcal{I}_v$ has dimension $d - 1$, which is exactly the dimension of the subspace $\mathcal{I} \subset \mathcal{I}_v$. It follows that $\mathcal{I} = \mathcal{I}_v$. □

The criterion in the following theorem is inspired by Bogomolov-Tschinkel [BT08, Proposition 8.4]. It is important to mention, however, that loc.cit. works over the algebraic closure of a finite field, where quasi-divisorial and divisorial valuations happen to coincide.

**Theorem A.13.** Put $d := \text{tr.deg}(K|k)$, and let $\mathcal{I} \subset \mathcal{G}(K|k)$ be a $1$-dimensional subspace. Assume that $d \geq 2$. Then the following are equivalent:

1. There exists a quasi-divisorial valuation $v$ of $K|k$ such that $\mathcal{I} = \mathcal{I}_v$.
2. There exist two $d$-dimensional acl-subspaces $\mathcal{D}_1, \mathcal{D}_2$ of $\mathcal{G}(K|k)$ such that $\mathcal{I}_v = \mathcal{D}_1 \cap \mathcal{D}_2$.

Moreover, if these equivalent conditions hold true and $f \in \mathcal{I} = \mathcal{I}_v$ is a non-trivial element, then one has $g \in \mathcal{D}_v$ if and only if $(f, g)$ is an acl-pair.

**Proof.** Assume (1). Then $Kv$ is a function field of transcendence degree $d - 1$ over $kv$. Choose $w_1$ and $w_2$ two independent valuations of $Kv|kv$ each of which correspond to the composition of $d - 1$ divisorial valuations. In other words, $w_i$ is a valuation without transcendence defect, which is trivial on $kv$, and whose value group is isomorphic to $\mathbb{Z}^{d-1}$ with the lexicographic ordering.

Put $v_i = w_i \circ v$ and $\mathcal{D}_i := \mathcal{I}_{v_i}$ for $i = 1, 2$. Then $\mathcal{D}_1, \mathcal{D}_2$ satisfy condition (2). Indeed, $\mathcal{D}_i$ is $d$-dimensional by construction, and it is an acl-space by Lemma [A.8]. The fact that $\mathcal{I}_v = \mathcal{D}_1 \cap \mathcal{D}_2$ follows from the fact that $v$ is the finest common coarsening of $v_1, v_2$ (since $w_1, w_2$ are independent).

Conversely, assume (2). Let $\mathcal{I}_f$ be the collection of valuative elements in $\mathcal{D}_f$. Then by Proposition [A.12] there exist defectless valuations $v_1, v_2$ of $K$ such that $\mathcal{I}_i = \mathcal{I}_{v_i}$, $\mathcal{D}_i \subseteq \mathcal{D}_{v_i}$, and $\mathcal{I}_i$ has codimension $\leq 1$ in $\mathcal{D}_i$. Let $f$ be a non-trivial element of $\mathcal{D}_1 \cap \mathcal{D}_2$.

We claim that $f$ is valuative. If not, then $v_1, v_2$ are comparable by Lemma [A.8]. Let us assume, for example, that $v_1$ is coarser than $v_2$. Hence, $\mathcal{I}_1 \subset \mathcal{I}_2$, while $\mathcal{I}_1$ is non-trivial. Also, one has $\mathcal{I}_1 \subset I_1 \cap I_2 \subset D_1 \cap D_2$. Since $\mathcal{D}_1 \cap \mathcal{D}_2$ is 1-dimensional, we see that $\mathcal{D}_1 \cap \mathcal{D}_2 = \mathcal{I}_1$ hence $f \in \mathcal{I}_1$ and $f$ is valuative.

Let $v$ be the valuation of $K$ associated to $f$. By Lemma [A.8], we see that $v$ is a coarsening of $v_1$ and of $v_2$, hence $v$ is without transcendence defect. It remains to show that rank$_{\mathbb{Q}}(vK/vk) = 1$. As $f \in \mathcal{I}_v$ is non-trivial, it follows that rank$_{\mathbb{Q}}(vK/vk) \geq 1$. If rank$_{\mathbb{Q}}(vK/vk) > 1$, then dim$_{\mathbb{Q}} \mathcal{I}_v > 1$, while $\mathcal{I}_v \subset \mathcal{I}_1 \cap \mathcal{I}_2 \subset \mathcal{D}_1 \cap \mathcal{D}_2$. Again, this contradicts the fact that dim$_{\mathbb{Q}} \mathcal{D}_1 \cap \mathcal{D}_2 = 1$.

Finally, we have deduced that rank$_{\mathbb{Q}}(vK/vk) = 1$, so that dim$_{\mathbb{Q}} \mathcal{I}_v = 1$. Hence, $f$ is a generator of $\mathcal{I}_v$, which means that $\mathcal{D}_1 \cap \mathcal{D}_2 = \mathcal{I}_v$, as required.

Concerning the final assertion about $\mathcal{D}_v$, we note that any element of $\mathcal{D}_v$ forms an acl-pair with $f$ by Lemma [A.3] while any $g \in \mathcal{G}(K|k)$ which forms an acl-pair with $f$ must be an element of $\mathcal{D}_v$ by Lemma [A.6]. □

**A.8. Divisorial valuations.** In this subsection, we give a criterion for deciding whether a quasi-divisorial valuation is actually divisorial. For $t \in K \setminus k$, consider the canonical $k$-embedding acl$_K(t) \hookrightarrow K$ of function fields over $k$, as well as the induced (surjective) map

$$\mathcal{G}(K|k) \to \mathcal{G}($$ acl$_K(t)|k),$$

whose kernel we denote by $\mathcal{N}_t$. We will write $\mathcal{G}_t$ for the corresponding abstract quotient of $\mathcal{G}(K|k)$:

$$\mathcal{G}(K|k) \to \mathcal{G}(K|k)/\mathcal{N}_t =: \mathcal{G}_t.$$

**Proposition A.14.** Let $v$ be a quasi-divisorial valuation of $K|k$. Then the following are equivalent:

1. The valuation $v$ is divisorial.
2. There exists an element $t \in K \setminus k$ such that $\mathcal{D}_v$ maps surjectively onto $\mathcal{G}_t$.
3. There exists an element $t \in K \setminus k$ such that the image of $\mathcal{D}_v$ in $\mathcal{G}_t$ has finite codimension in $\mathcal{G}_t$. 41
Proof. Suppose that \( v \) is divisorial, and choose \( t \in U_v \setminus k^\times \cdot U_t^1 \). Then \( v \) is trivial on \( \text{acl}_K(t) \). Let \( \bar{t} \) denote the image of \( t \) in \( K^v \). Then one has a canonical \( k \)-embedding of fields \( \text{acl}_K(t) \to \text{acl}_{K^v}(\bar{t}) \).

We have a commutative diagram:

\[
\begin{array}{ccc}
G(K^v|k) & \xleftarrow{D_v} & G(K|k) \\
\downarrow & & \downarrow \\
G(\text{acl}_{K^v}(\bar{t})|k) & \xrightarrow{G_{\bar{t}}} & G_{\bar{t}} = G(\text{acl}_K(t)|k)
\end{array}
\]

The surjectivity of \( D_v \to G(K^v|k) \) comes from Fact \([A, 2]\). The map \( G(K^v|k) \to G(\text{acl}_{K^v}(\bar{t})|k) \) is induced by the inclusion \( \text{acl}_{K^v}(\bar{t}) \hookrightarrow K^v \), hence it is surjective as well, so that \( D_v \to G(\text{acl}_{K^v}(\bar{t})|k) \) is surjective. The surjectivity of \( G(K|k) \to G(\text{acl}_{K^v}(\bar{t})|k) \) and \( G(\text{acl}_K(t)|k) \to G_{\bar{t}} = G(\text{acl}_K(t)|k) \) are similar. It follows that the map \( D_v \to G_{\bar{t}} \) is surjective. This shows the implication \((1) \Rightarrow (2)\), while the implication \((2) \Rightarrow (3)\) is trivial.

Finally, assume that \( v \) is non-trivial on \( k \), and that \( t \in K \setminus k \) is given. We may assume that \( v(t) > 0 \). Indeed, if \( v(t) = 0 \) then we replace \( t \) by \( a \cdot t \) with \( a \in k^\times \) such that \( v(a) > 0 \), while if \( v(t) < 0 \), we just replace \( t \) with \( t^{-1} \).

With this in mind, we have \( 1 + a \cdot t \in U_v^1 \) for all \( a \in k^\times \) such that \( v(a) > 0 \). There are infinitely many such \( a \), all of which are \( \mathbb{Q} \)-linearly-independent in \( (\text{acl}_K(t)^\times / k^\times) \otimes_{\mathbb{Z}} \mathbb{Q} \). It easily follows from this that \( D_v \to G_{\bar{t}} \) has a cokernel of infinite rank.

A.9. Concluding the Proof of Theorem \([A.1]\) We now show how to conclude the proof of Theorem \([A.1]\) using Theorem \([A.13]\) and Proposition \([A.14]\). To see this, let us first note that one has \( \text{Hom}_{\Lambda}(\mathcal{A}(K|k), \mathcal{A}(L|l)) = G(K|k) \) canonically. Thus one has a canonical non-degenerate pairing

\[
\mathcal{A}(K|k) \times G(K|k) \to \mathcal{A}(L|l).
\]

Also, for a divisorial valuation \( v \) of \( K|k \), the orthogonal of \( \mathcal{I}_v \) resp. \( D_v \) in \( (K^\times / k^\times) \otimes_{\mathbb{Z}} \Lambda \) with respect to this pairing is precisely \( \mathcal{Z}_v \) resp. \( \mathcal{Z}_v^1 \). With this observation, the assertion of Theorem \([A.1]\) follows easily from Theorem \([A.13]\) and Proposition \([A.14]\).

To make this more precise, note that \( \phi : \mathcal{A}(K|k) \cong \mathcal{A}(L|l) \) induces an isomorphism

\[
\phi^* : G(L|l) \to G(K|k)
\]

which is adjoint to \( \phi \) with respect to the pairings mentioned above. As \( \phi \) is compatible with acl, one has \( \text{tr. deg}(K|k) = \text{tr. deg}(L|l) \), and the adjoint \( \phi^* \) is compatible with acl-pairs. By Theorem \([A.13]\) for all divisorial valuations \( v \) of \( K|k \), there exists a unique \textit{quasi-divisorial valuation} \( v^\phi \) of \( L|l \) such that \( \phi^*(D_v) = D_v \) and \( \phi^*(\mathcal{I}_v) = \mathcal{I}_v \). We must show that \( v^\phi \) is, in fact, a divisorial valuation.

Arguing similarly to Lemma \([B.3]\), we see that since \( \phi \) is compatible with acl, there exists a bijection \( \phi^\sharp : G^1(K|k) \cong G^1(L|l) \) such that

\[
\ker(G(K|k) \to G(M|k)) = \phi^*(\ker(G(L|l) \to G(\phi^\sharp M|l)))).
\]

In particular, we see that \( v^\phi \) satisfies the equivalent conditions of Proposition \([A.12]\) (since \( v \) does). Hence \( v^\phi \) is divisorial. We conclude by taking the orthogonal in \( \mathcal{A}(K|k) \) resp. \( \mathcal{A}(L|l) \) with respect to the pairings mentioned above, as previously discussed.

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