Abstract

In this paper we prove that \(\ell\)-group tensor product of archimedean \(f\)-rings is an \(f\)-ring. We will use this result to characterize multiplicative \(\ell\)-bimorphisms between unital \(f\)-rings.

1 Introduction

Since Martinez in [15], constructed the \(\ell\)-group tensor product, several authors studied the tensor product of ordered structures, namely Fremlin in the framework of Riesz spaces (see [11]) and Banach Lattices (see [12]). Buskes And Van Rooij used the Fremlin tensor product to reconstruct the \(\ell\)-group tensor product. (see [8]). In this end, they used the Lattice cover of \(\ell\)-groups (see [10]).

Recently, Azouzi, Ben Amor and Jaber (see [2]) and separately Buskes and Wicksted (see [9]) proved that the Riesz (Fremlin) tensor product of archimedean \(f\)-algebras is an \(f\)-algebra.

In this work we will use the recent works in the framework of \(f\)-algebras to prove that the \(\ell\)-group tensor product of archimedean \(f\)-rings is an \(f\)-ring. We will use this tensor product to generalize a result of Ben Amor and Boulabiar in [1].

We assume that the reader is familiar with the basic concepts of the theory of lattice ordered groups (\(\ell\)-groups) and \(f\)-rings. For unexplained terminology and notations we refer to the books [1], [5] and [16].

2 Tensor product of \(f\)-rings

Theorem 1 If \(G\) is an archimedean \(f\)-ring then the vector lattice cover of \(G\), \(R[G]\), is an \(f\)-algebra.

Proof. According to [10], \(R[G]\) is the \(\ell\)-subspace of \(\overline{G^d}\) generated by \(G\), where \(\overline{G^d}\) is the Dedekind-MacNeille completion of the divisible hull \(G^d\).
$G^d$ is obviously an archimedean $f$-ring, this with the lemma 3 in [14] lead to $G^d$ is an archimedean $f$-ring. Since $R[G]$ is the $\ell$-subspace of the $f$-ring $G^d$ generated by the $f$-ring $G$, $R[G]$ is itself an $f$-ring and then an $f$-algebra (see Theorem 3.3 in [13]).

**Corollary 1.** If $G$ is a unital archimedean $f$-ring with $e_G$ as unit then the vector lattice cover of $G$, $R[G]$, is a unital $f$-algebra with the same unit.

**Proof.** We proved in Theorem 1 that $R[G]$ is an $f$-ring. It remains to prove that $e_G$ is a unit in $R[G]$. The map

$$\pi_e : R[G] \to R[G] \quad g \mapsto g.e_G$$

is an $\ell$-homomorphism which extends the canonical embedding of $G$ in $R[G]$. Then, according to Theorem 2 in [6], $\pi_e$ is the identity and $e_G$ is the unit element of the $f$-algebra $R[G]$.

**Theorem 2.** Let $G$ and $H$ be two $f$-rings, then the archimedean $\ell$-group tensor product $G \otimes H$ is itself an $f$-ring.

**Proof.** In [8], Buskes and Van Rooij stated that the archimedean $\ell$-group tensor product $G \otimes H$ is the $\ell$-subgroup of $R[G \otimes H]$ generated by the algebraic tensor product $G \otimes H$. But according to the same paper (Proposition 8), $R[G \otimes H]$ is group and lattice isomorphic to $R[G] \otimes R[H]$. So we can consider that $G \otimes H$ is the $\ell$-subgroup of $R[G] \otimes R[H]$ generated by the algebraic tensor product $G \otimes H$. Theorem 1 with [2] lead to $R[G] \otimes R[H]$ is a $f$-subring. Using another time Theorem 3.3 in [13], we can conclude that $G \otimes H$ is an $f$-ring which ends the proof.

**Corollary 2.** Let $G$ and $H$ be two unital $f$-rings with unit element $e_G$ and $e_H$ respectively, then the archimedean $\ell$-group tensor product $G \otimes H$ is itself a unital $f$-ring with $e_G \otimes e_H$ as unit element.

**Proof.** According to corollary 1 $R[G]$ and $R[H]$ are unital $f$-algebras with $e_G$ and $e_H$ as unit element respectively. Theorem 8 in [2] lead to $R[G] \otimes R[H]$ is a unital $f$-algebra with $e_G \otimes e_H$ as unit element. Since $G \otimes H$ is an $f$-subring of $R[G] \otimes R[H]$, the result follows immediately.

**3 An application**

Let $G$, $H$ and $K$ be archimedean $f$-rings. We recall that a biadditive map $b : G \times H \to K$ is said $\ell$-bimorphism if the maps:

$$b_1 : G \to K \quad g \mapsto b(g, b)$$
and

\[ b_2 : \quad H \rightarrow K \]
\[ h \mapsto b(a, h) \]

are lattice homomorphisms for all \( a \in G \) and \( b \in H \).

An \( \ell \)-bimorphism \( b : G \times H \rightarrow K \) is said to be multiplicative if

\[ b(ac, bd) = b(a, b)c(d) \]

for all \( a \) and \( c \) in \( G \) and \( b \) and \( d \) in \( H \).

Boulabiar and Toumi proved in [7] that if \( G \) and \( H \) are archimedean \( \Phi \)-algebras (that is untital \( f \)-algebras) with \( e_G \) and \( e_H \) as units, and \( K \) is semiprime (that is \( K \) has no idempotent element) then the positive bilinear map \( b \) is multiplicative if and only if \( b \) is an \( \ell \)-bimorphism and \( b(e_G, e_H) \) is idempotent.

We will generalize this result in two directions. First we will deal with \( f \)-rings rather than \( f \)-algebras. Finally, we shall prove that the range \( f \)-ring need not be reduced, which is, we believe, an important improvement.

We pointed out that the tensor product we asked about in Theorem 2 is the \( \ell \)-group tensor product that Buskes and Van Rooij studied in [8] and earlier Martinez in [13]. The following universal propriety is still valid. Let \( G \) and \( H \) two archimedean \( f \)-rings. For any archimedean \( f \)-ring \( K \) and every \( \ell \)-bimorphism \( \varphi : G \times H \rightarrow K \) there exists an \( \ell \)-group homomorphism \( \Phi : G \otimes H \rightarrow K \) such that \( \varphi(a, b) = \Phi(a \otimes b) \) for every \( a \) in \( G \) and \( b \) in \( H \).

The next proposition is a generalization of the Theorem 3.2 in [4] and it will play a key role in the generalization of Boulabiar-Toumi’s theorem.

**Proposition 1** Let \( G \) be a unital \( f \)-ring with unit element \( e_G \), \( H \) be an archimedean \( f \)-ring and \( T \) be a positive homomorphism between \( G \) and \( H \). Then \( T \) is a ring homomorphism if and only if \( T \) is an \( \ell \)-homomorphism and \( T(e_G) \) is idempotent.

**Proof.** The "Only if" part is unchanged from Theorem 3.2 in [4]. Only the "if" part needs some details. Since \( T \) is a ring homomorphism then \( T(e_G) \) is an idempotent element and for every \( a \) in \( G \) we have \( T(a) = T(e_G)T(a) \). This means that the range of \( T \) is included in the set \( T(e_G)^{\perp \perp} \) which is an \( f \)-ring with \( T(e_G) \) as unit element (see for example Lemma 3.4 and 3.5 in [3]). Now, Take \( a \) and \( b \) in \( G \) such that \( a \wedge b = 0 \). From

\[ 0 = T(ab) = T(a)T(b), \]

and the fact that \( T(e_G)^{\perp \perp} \) is reduced, we can affirm that \( T(a) \wedge T(b) = 0 \). And we are done. ■

We have now gathered all the ingredients we need to prove the following Theorem.
Theorem 3  Let $G$ and $H$ be archimedean unital $f$-rings with unit element $e_G$ and $e_H$ respectively and $K$ be an archimedean $f$-ring. Let $b : G \times H \rightarrow K$ be a positive biadditive homomorphism. The following conditions are equivalent:

i) $b$ is multiplicative.

ii) $b$ is an $\ell$-bimorphism and $b(e_G, e_H)$ is idempotent.

Proof.

i) $\rightarrow$ ii) Since $b$ is multiplicative then so are the two positive homomorphisms $b_1$ and $b_2$, where

$$b_1 : \ G \rightarrow K \quad g \mapsto b(g, e_H)$$

and

$$b_2 : \ H \rightarrow K \quad h \mapsto b(e_G, h).$$

Proposition 1 yields to $b_1$ and $b_1$ are $\ell$-homomorphisms. Which means that $b$ is an $\ell$-bimorphism. $b(e_G, e_H)$ is idempotent follows immediately.

ii) $\rightarrow$ i) Let $\Phi : G \otimes H \rightarrow K$ be the $\ell$-homomorphism such that

$$b(a, b) = \Phi(a \otimes b)$$

for every $a$ in $G$ and $b$ in $H$. Corollary 2 yields to $G \otimes H$ is a unital archimedean $f$-ring with $e_G \otimes e_H$ as a unit element. This, with Theorem 3.2 in [4] show that $\Phi$ is multiplicative. The result follows immediately since

$$b(ac, bd) = \Phi(ac \otimes bd) = \Phi(a \otimes b)\Phi(c \otimes d) = b(a, b)b(c, d)$$

for every $a$ and $c$ in $G$ and every $b$ and $d$ in $H$. ■

References

[1] M. Anderson and T. Feil. A first course in abstract algebra. CRC Press, Boca Raton, FL, third edition, 2015. Rings, groups, and fields.

[2] Y. Azouzi, M. A. Ben Amor, and J. Jaber. The tensor product of $f$-algebras. Quaest. Math., to appear.

[3] Youssef Azouzi and Mohamed Amine Ben Amor. On von Neumann regular elements in $f$-rings. Algebra Universalis, 78(1):119–124, 2017.

[4] M. A. Ben Amor and K. Boulabiar. Almost $f$-maps and almost $f$-rings. Algebra Universalis, 69(1):93–99, 2013.

[5] A Bigard, K. Keimel, and S. Wolfenstein. Groupes et anneaux réticulés. Lecture Notes in Mathematics, Vol. 608. Springer-Verlag, Berlin-New York, 1977.
[6] R. D. Bleier. Minimal vector lattice covers. *Bull. Austral. Math. Soc.*, 5:331–335, 1971.

[7] K. Boulabiar and M. A. Toumi. Lattice bimorphisms on $f$-algebras. *Algebra Universalis*, 48(1):103–116, 2002.

[8] G. J. H. M. Buskes and A. C. M. van Rooij. The Archimedean $l$-group tensor product. *Order*, 10(1):93–102, 1993.

[9] G. J. H. M. Buskes and A. W. Wickstead. Tensor products of $f$-algebras. *Mediterr. J. Math.*, 14(2):Art. 63, 10, 2017.

[10] P. F. Conrad. Minimal vector lattice covers. *Bull. Austral. Math. Soc.*, 4:35–39, 1971.

[11] D. H. Fremlin. Tensor products of Archimedean vector lattices. *Amer. J. Math.*, 94:777–798, 1972.

[12] D. H. Fremlin. Tensor products of Banach lattices. *Math. Ann.*, 211:87–106, 1974.

[13] M. Henriksen and J. R. Isbell. Lattice-ordered rings and function rings. *Pacific J. Math.*, 12:533–565, 1962.

[14] D. G. Johnson. The completion of an archimedean $f$-ring. *J. London Math. Soc.*, 40:493–496, 1965.

[15] J. Martinez. Tensor products of partially ordered groups. *Pacific J. Math.*, 41:771–789, 1972.

[16] S. A. Steinberg. *Lattice-ordered rings and modules*. Springer, New York, 2010.