Behaviour of low angular momentum relativistic accretion close to the event horizon

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5 May 2014

ABSTRACT
We introduce a novel formalism to investigate the role of the spin angular momentum of astrophysical black holes in influencing the behaviour of low angular momentum general relativistic accretion. We propose a metric independent analysis of axisymmetric general relativistic flow, and consequently formulate the space and time dependent equations describing the general relativistic hydrodynamic accretion flow in the Kerr metric. The associated stationary critical solutions for such flow equations are provided, as well as the stability of the stationary transonic configuration is examined using a novel linear perturbation technique. We examine the properties of infalling material for both the prograde as well as the retrograde accretion as a function of the Kerr parameter at the extreme close proximity of the event horizon. Our formalism can be used to identify a new spectral signature of black hole spin, and has the potential of performing the black hole shadow imaging corresponding to the low angular momentum accretion flow.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – gravitation – dynamical systems

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Astrophysical black holes are the terminal states of the gravitational collapse of massive celestial objects that can be conceived as singularities in space time censored by a mathematically defined 'one way barrier' – the event horizon, and are not amenable to any direct physical observation. As a result, their presence can only be realized through the gravitational influence they exerts on the matter accreting onto those objects. The infalling matter inevitably plunges through the event horizon on a relativistic scale of velocity. Given a set of physically realizable outer boundary conditions, such accretion eventually manifests transonic properties to obey the aforementioned inner boundary conditions (Liang & Thomson 1980). Subsonic at a large distance, accretion thus reaches the event horizon supersonically.

The hypothesis that most (if not all) of the supermassive black holes and the stellar mas black holes powering the active galactic nuclei and the galactic micro quasars respectively possesses non zero value of spin angular momentum has gained widespread currency in recent times (Miller, Reynolds, Fabian, Miniutti & Gallo 2009; Kato, Miyoshi, Takahashi, Negoro & Matsumoto 2010; Ziolkowski 2010; Tchekhovskoy,Narayan & McKinney 2010; Daly 2011; Buliga, Globina, Gnedin, Natsslhivili, Pitrovich, & Shakht 2011; Reynolds et al. 2011; McClintock et al. 2011; Martinez-Sansigre & Rawlings 2011; Dauser, Wilms, Reynolds & Brenneman 2010; Nixon, Cossing, King & Pringle 2011; Tchekhovskoy & McKinney 2012). The black hole spin plays a deterministic role in influencing various characteristic features of the the dynamical and spectral features of accretion and related phenomena in the characteristic black hole metric – the energy extraction from a Kerr black hole through the Blandford Znajek Mechanism (Garofalo 2009a; Das & Czerny 2012a; Tchekhovskoy & McKinney 2012), the spin dependence of the black hole shadow imaging (Falcke, Melia & Agol 2000; Takahashi 2004; Huang, Cai, Shen & Yuan 2007; Hioki & Maeda 2009; Zakharov, Paolis, Ingrosso & Nucita 2012; Straub, Vincent, Abramowicz, Gourgoulhon, & Paumard 2012), various evolutionary properties of the normal and the active galaxies (Garofalo 2009b; Garofalo, Evans, & Samburna 2010; McNamara, Rohanizadegan & Nulsen 2011), and the QPO phenomena associated with the extra-galactic sources (Czerny, Lachowicz, Dovciak, Karas, Pechacek, & Das 2010; Das & Czerny 2011), to mention a few.

The aforementioned issues necessitate investigation of how the black hole spin angular momentum, i.e., the Kerr parameter $a$, influences the dynamical and the radiative behaviour of the general relativistic transonic accretion at the close vicinity of the event horizon of a rotating black hole. The central objective of our work presented in this paper is to investigate what properties of the accretion flow are the principal attributes of the black hole spin at the extreme close proximity of the event horizon of a rotating black hole.

To accomplish such task, we conduct a detailed and multi-step investigation of the transonic properties of general relativistic axisymmetric hydrodynamic inviscid accretion of low angular momentum as realized on the equatorial plane of the Kerr metric using the Boyer Linsquist (Boyer & Lindquist 1967) co-ordinate. We begin with a general prescription where we consider a four dimensional stationary axisymmetric manifold with two commuting Killing vector fields and subsequently construct the general relativistic Euler and the continuity equation from the appropriate energy momentum tensor. Quite interestingly, we have been able to demonstrate, using certain symmetry arguments, that for the three dimensional submanifold the fluid equations are separable using analytical scheme and the corresponding flow velocity components can completely be determined once the equation of state is specified. We thus formulate a general framework for studying the equations for fluid flow in rotating black hole spacetime.

The transonic flow properties in phase portrait, however, can not be determined completely analytically for certain issues which will be elaborated in the subsequent sections. For such purpose, we demonstrate the equivalence between the set of differential equations describing the space gradient of the stationary flow velocity and that of the acoustic perturbation (stationary barotropic sound speed), and the first order autonomous dynamical systems. The emergence of the multi-transonic behaviour manifests through the critical point analysis. Such multi-transonic accretion solution, as we will see in the subsequent sections, may contain stationary shock, properties of which are obtained by the explicit solution of the general relativistic Rankine-Hugoniot conditions. The properties of the post shock flow are then studied as a function of the black hole spin – the Kerr parameter $a$. The post shock flow solution are then followed up to a sufficiently close proximity of the event horizon to demonstrate how the terminal values of the shocked accretion variables are influenced by the black hole spin angular momentum, and the consequences of such dependence are discussed in detail.

The entire procedure to the study the spin dependence of the behaviour of accreting matter close to the event horizon as described above is based on the stationary solutions of the differential equations describing the accretion phenomena. Along with the understanding of the transonic behaviour of the stationary flow solutions, it is equally important (rather necessary) to ensure the stability of such stationary configurations. Criteria for such sustained stability can be examined by studying the time evolution of a linear acoustic like perturbation (applied around the stationary configuration) in the full time dependent flow equations. The existence of the stable stationary transonic solution is associated with the non-divergent linear perturbation of the aforementioned kind. In this work, we develop a novel linear perturbation scheme applicable to the axisymmetric potential flow as realized on the equatorial plane of the Kerr metric. We perturb the velocity potential corresponding to the advective velocity of the low angular momentum accretion considered in our work and demonstrate that such perturbation does not diverge in any significant astrophysically relevant time scale. We thus formally establish the consistency of the formalism, which, for the first time in the literature, has been introduced in the present work to study the black hole spin dependence of the terminal behaviour of accreting matter at the extreme close vicinity of the event horizon of a Kerr black hole.
2 MULTI-TRANSONICITY IN BLACK HOLE ACCRETION: RETROSPECTIVE AND CONTEMPORARY ASPECTS

For accretion onto astrophysical black holes, the transonicity is characterised by a transition from the subsonic state \( (M < 1) \), where \( M \) is the Mach number of the flow) to the supersonic state \( (M > 1) \), or vice versa. For the present work, the Mach number \( M \) is considered to be the local radial Mach number for stationary transonic accretion solution. For our purpose, we define \( M \) to be the ratio of the local advective velocity \( u \) (defined in the subsequent section) and the local speed of the propagation of the acoustic perturbation (local barotropic sound speed) \( c_s \) as defined in the subsequent section. Such transition may be a regular one through the sonic point and is associated with the transition of \( M < 1 \rightarrow M > 1 \) type or may be a discontinuous one through a stationary shock and is associated with the \( M > 1 \rightarrow M < 1 \) type transition. As a consequence of the fact that the non linear equations describing the steady, inviscid stationary axisymmetric flow can be tailored to form a first order autonomous dynamical system, the physical transonic accretion solution for the stationary axisymmetric flow can formally be realized as critical solution on the phase portrait spanned by \( M \) and the radial distance measured from the event horizon – see, e.g., Chaudhury, Ray & Das (2006); Goswami, Khan, Ray & Das (2007) and references therein.

For low angular momentum sub-Keplerian accretion, such transonic features may be exhibited more than once on the phase portrait of the stationary solutions. Such multi-transonicity as well as the resulting shock formation phenomena for axisymmetric accretion under the influence of various post Newtonian potentials, mainly; under the influence of the Paczyński and Wiita (Paczyński & Wiita 1980) pseudo-Schwarzschild potential \(^1\), has been widely studied in the literature, see e.g., Liang & Thomson (1980); Abramowicz & Zurek (1981); Muchotrzeb & Paczyński (1982); Muchotrzeb (1983); Muchotrzeb-Czerny (1986); Chakrabarti (1989); Abramowicz & Kato (1989); Abramowicz & Chakrabarti (1990); Das (2002); Das et al. (2003); Fukue (2004) and references therein.

Since a regular stationary accretion solution can not encounter more than one transonic point, multi-transonicity implies a particular flow configuration with three critical points where two transonic solutions through two different saddle type critical points are connected by a discontinuous stationary shock transition. The inner boundary condition imposed by teh event horizon indicates that such a combined multi-transonic shocked solution originates from a large distance as a subsonic flow and encounters the outermost saddle type sonic point to become supersonic for the fist time. Subjected to the appropriate initial boundary conditions, such supersonic flow makes a \( M > 1 \rightarrow M < 1 \) type discontinuous transition through a stationary shock and the shock induced subsonic flow becomes supersonic again at the innermost saddle type sonic point.

One expects that a shock formation in black-hole accretion discs might be a general phenomenon because shock waves in rotating astrophysical flows potentially provide an important and efficient mechanism for conversion of a significant amount of the gravitational energy into radiation by randomizing the directed infall motion of the accreting fluid. Hence, the shocks play an important role in governing the overall dynamical and radiative processes taking place in astrophysical fluids accreting onto black holes. The hot and dense post shock flow is considered to be a powerful diagnostic tool in understanding various astrophysical phenomena like the spectral properties of the galactic black hole candidates ((Chakrabarti & Titarchuk 1995)) and that of the supermassive black hole at our Galactic centre ((Moscibrodzka, Das & Czerny 2006)), the formation and dynamics of accretion powered galactic and extra-galactic outflows ((Das & Chakrabarti 1999; Chattopadhyay & Das 2007; Das & Chattopadhyay 2007)), shock induced nucleosynthesis in the black hole accretion disc and the metalicity of the intergalactic matter ((Mukhopadhyay 1999) and references therein), and the origin of the quasi-periodic oscillation in galactic and extra-galactic sources (Spongholz & Molteni 1994; Das, Rao & Vadawale 2003; Okuda et al. 2004, 2007; Czerny, Lachowicz, Dovciak, Karas, Pechacek, & Das 2010; Das & Czerny 2011) and references therein), to mention a few.

The idea of shock formation in black hole accretion flow has however contested by some authors (see, e.g., Narayan, Mahadevan & Quataert (1998) and references therein for a review). Nevertheless, the issue of not finding shocks in such works perhaps lies in the fact that only one sonic point close to the black hole may usually be explored using the framework of the shock free advection dominated accretion flow solutions.

At this point we would like to emphasize that the concept of low angular momentum (allowing the multi-transonicity and the shock formation) is not a theoretical abstraction. Rather for realistic astrophysical systems, such sub-Keplerian weakly rotating flows are exhibited in various physical situations, such as detached binary systems fed by accretion from OB stellar winds ((Illarionov & Sunyaev 1975; Liang & Nolan 1984)), semi-detached low-mass non-magnetic binaries ((Bisikalo et al. 1998)), and super-massive black holes fed by accretion from slowly rotating central stellar clusters ((Illarionov 1988; Ho 1999) and references therein). Even for a standard Keplerian accretion disc, turbulence may produce such low angular momentum flow (see, e.g., Igumenshchev & Abramowicz 1999), and references therein).

Multi-transonicity in black hole accretion has been addressed using the general relativistic framework as well. The legacy of the pioneering contributions by Bardeen, Press & Teukolsky (1972) and Novikov & Thorne (1973) to study the general relativistic axisymmetric black hole accretion in the Kerr metric followed two different avenues, quite often in a non overlapping fashion. One school of thought essentially studied the transonic accretion without paying much attention to the appearance of the multi-transonicity and the formation of shock, but rather putting emphasis on other crucial behaviours of the flow, see, e.g., Lemos & Letelier (1994); Riffert & Herold (1995); Abramowicz, Chen, Granath, & Lasota (1996); Pariev (1996); Peitz & Appl (1997); Gamnie & Popham (1998); Popham & Gamnie (1998); Takahashi (2007); Sadowski (2009) and references therein.

The appearance of the multiple critical points as general relativistic flow onto a spinning black hole was observed and consequently

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\(^1\) This one is believed to be the most effective black hole potential introduced in the literature till date, see, e.g., Artemova et al. (1996); Das (2002) and references therein, for further detail.
the formation of the standing shock has been conjectured in the alternative set of (sometime contesting the aforementioned category of work dealing with accretion flow without the appearance of shock transition) literature, with the main motivation to explain the spectral state by incorporating the physics of the post shock accretion flow, as already mentioned in the previous sections. The profound work by Fukue (1987) is credited as the first ever consistent attempt to provide the complete formalism for the shock formation in a multiply critical accretion flow in the Kerr metric, although it is worth mentioning that even before Fukue (1987), multiplicity in the critical points for the general relativistic axisymmetric flow was addressed (Lu 1985, 1986) without mentioning the issue of the shock formation. By revisiting the concept of the Keplerian circular motion for rotating fluids in general relativity Lu, Yu & Young (1995) intuitively explained certain issues related to the shock formation for multi-transonic accretion onto a Kerr black hole.

The full general relativistic formalism introduced by Lu (1985, 1986) and Fukue (1987) was followed by Chakrabarti (1996a,b) where a non relativistic calculation for the shock formation for accretion (in the hydrostatic equilibrium in the vertical direction – the flow model we consider in our present work) and other issues were incorporated within the relativistic framework and some of the results valid for the isothermal flow had directly been applied to study the polytropic flow without adequate justification. Some of such discrepancies has recently been modified by Mondal (2010), Chakrabarti (1990); Lu et al. (1997a,b); Lu & Yuan (1998); Lu & Gu (2004) used the general relativistic shock condition to study the multi-transonic flow for the conical model 2. Peitz & Appl (1997) studied the general relativistic accretion for multi-transonic flow but the shock issues were not dealt in detail. While all the above works dealt with the polytropic accretion, shock transition in general relativistic isothermal flow was discussed in Kafatos & Yang (1994); Yang & Kafatos (1995); Yuan, Dong, & Lu (1996). Shocked accretion for MHD flows in Kerr geometry has also been studied by various authors (Takahashi, Rilett, Fukumura, & Tsuruta 2002; Takahashi, Goto, Fukumura, Rilett & Tsuruta 2006; Fukumura, Takahashi, & Tsuruta 2007).

Meanwhile, it was realized that it is instructive to incorporate an expression for the flow thickness for flow in hydrostatic equilibrium in the vertical direction such that the corresponding flow equation will remain non singular on the horizon. Both the thin accretion disc as well as the quasi-spherical flow structure can be accommodated using such a disc height. Abramowicz, Lanza & Pervival (1997) provided such an expression for the general flow structure. The disc height introduced by Abramowicz, Lanza & Pervival (1997) had further been modified to study the multi-transonic flow structure around Kerr black holes in Barai, Das, & Wiita (2004); Goswami, Khan, Ray & Das (2007); Das & Czerny (2012b).

Owing to the strong curvature of space time close to the black hole, accreting fluid is expected to manifest extreme behaviour just before plunging into the event horizon. The spectral signature of this tremendously hot ultra fast matter with its characteristic density and pressure profile is expected to provide the key features of the strong gravity space time to the close proximity of the event horizon. A detail study of the role of the black hole spin angular momentum in influencing the dynamical features of the transonic matter close to the event horizon is thus very important to perform to understand the salient features of the general relativistic black hole space time, and, in turn, to understand the physical properties of the Kerr metric itself. Gammie & Popham (1998) and Popham & Gammie (1998) were the first to make attempt to understand the flow properties close to the black hole by studying the general relativistic optically thin advection dominated accretion flow (ADAF) in the Kerr metric. Later on, Becker & Le (2003) applied the method of post Newtonian asymptotic analysis to investigate the properties of the inner region of ADAF to obtain their results that has been argued to be in agreement with the relativistic flow description. Subsequently, Barai, Das, & Wiita (2004) studied the influence of black hole spin in determining the properties of the accretion variables sufficiently close to the event horizon for multi-transonic flow, although the shock conditions were not taken into account in their work. It has recently been demonstrated that the multi-transonicity can only be realized through the presence of a standing shock since a smooth flow can never make more than one regular sonic transition Das & Czerny (2012b).

Motivated by the aforementioned issues, in this work we would like to study the behaviour of the low angular momentum multi-transonic shocked accretion extremely close to the black hole event horizon. The present work differs from all other previous works on general relativistic accretion, including Gammie & Popham (1998); Popham & Gammie (1998) and Barai, Das, & Wiita (2004) since for the first time in literature, not only a multi-transonic shocked flow has been studied at the close vicinity of the event horizon to understand the role of the black hole spin angular momentum on determining the salient features of such flow, but also a complete description of the linear perturbation analysis has also been provided which ensures the stability of such accretion solutions. In addition, a formal analytical description for the general fluid flow in axisymmetric black hole space time has also been provided in our work.

As an obvious extension of Barai, Das, & Wiita (2004), we introduce the standing shock by solving the relativistic Rankine-Hugoniot conditions, and study the behaviour of the post shock flow to the close vicinity of the horizon. We then compare such results with the hypothetical flow solutions for which the flow would not pass through a shock (and hence would behave like a mono-transonic flow passing through the saddle type outermost sonic point formed at a large distance from the black hole event horizon) for the same set of initial boundary conditions describing the flow. This allows us to understand whether the shock formation phenomena can alter the dynamical and thermodynamic state of matter extremely close to the event horizon and whether such change may show up through the spectral properties of the black hole candidates.

From recent theoretical and observational findings, the relevance of the counter rotating accretion in black hole astrophysics is being increasingly evident (Dauser, Wilms, Reynolds & Brenneman 2010; Nixon, Cossing, King & Pringle 2011; Tchekhovskoy & McKinney 2012). The conical model for the accretion was first introduced in Abramowicz & Zurek (1981) for flow under the influence of the Paczyński & Wiita (1980) black hole potential.
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2012). It is thus instructive to study whether the characteristic features of the terminal values of the accretion variable for the prograde flow differs considerably form that of the retrograde fl ow. To the best of our knowledge, for the first time in literature, our work presents the detailed spin dependence of the terminal behaviour of infalling matter for retrograde accretion onto a Kerr black hole using the complete general relativistic framework.

We, however, do not explicitly consider the viscous transport of the angular momentum and the specific angular momentum of the accretion flow has been taken to be invariant. Reasonably large radial advective velocity for the slowly rotating sub-Keplerian flow implies that the infall time scale is considerably small compared to the viscous time scale for the flow profile considered in this work. Large radial velocities even at larger distances are due to the fact that the angular momentum content of the accreting fluid is relatively low (Beloborodov & Illarionov 1991; Igumenshchev & Beloborodov 1997; Proga & Begelman 2003). The assumption of inviscid flow for the accretion profile under consideration is thus justified from an astrophysical point of view. Such inviscid configuration has also been address by other authors using detial numerical simulation works (Proga & Begelman 2003; Janiuk, Sznajder, Mościbrodzka, & Proga 2009).

3 METRIC INDEPENDENT FORMULATION FOR GENERALIZED FLOW VELOCITY FOR AXISYMMETRIC ACCRETION

As already mentioned, low angular momentum non self-gravitating axisymmetric inviscid accretion will be considered in our work. To begin with, consider a generic (3+1) stationary axisymmetric space-time endowed with two commuting Killing fields, within which the the back reactionless three dimensional general relativistic fluid equations will be looked upon. In such a space-time, the combined Euler and continuity equations take the form

\[ v^\mu \nabla_\mu v^\nu + \frac{c_s^2}{\rho} \nabla_\mu \rho \left( g^{\mu \nu} v^\nu + v^\mu v^\nu \right) = 0, \]  

(1)

where \( v^\mu \) is the time like fibre bundle (a tangent vector field in the present context) defined on the manifold constructed by the family of streamlines. The normalisation condition corresponding to the velocity vector field \( v^\mu \) is taken to be \( v^\mu v_\mu = -1 \). \( c_s \) is the speed of propagation of the acoustic perturbation embedded in the accreting fluid and \( \rho \) is the local rest mass energy density of the fluid. For a single temperature fluid, \( \rho \) can be replaced by the particle number density.

The space-time considered is a stationary axisymmetric manifold of dimension four endowed with two killing vector fields \( \xi^\mu \) and \( \phi^\mu \) so that

\[ \nabla_{(\mu} \xi_{\nu)} = 0 = \nabla_{(\mu} \phi_{\nu)}. \]  

(2)

The locally timelike killing field \( \xi^\mu \) (of norm \( \zeta \)) is the generator of stationarity whereas the locally spacelike killing field \( \phi^\mu \) (of norm \( \varphi \)) with closed spacelike integral curves generate the axisymmetry. It is usually not possible to obtain any orthogonal basis for the space-time of our consideration since \( \xi^\mu \phi^\mu \neq 0 \) for stationary axisymmetric space-time. We would intend to specify an orthogonal basis using which the space time metric can directly be expressed. To accomplish such task we first define

\[ \Upsilon_\mu \:= \xi_\mu - \frac{\xi \phi}{(\phi, \phi)} \phi_\mu \equiv \xi_\mu - \iota \phi_\mu, \]  

(3)

so that generically \( \Upsilon_\mu \phi^\mu = 0 \). Norm of \( \Upsilon^\mu \) can thus be expressed as,

\[ \Upsilon_\mu \Upsilon^\mu = - ( - \zeta^2 + \varphi^2 ) = - \varpi^2, \]  

(4)

where \( \Upsilon_\mu \) is timelike and \( \varpi^2 \) is positive. The metric element can now be expressed in the orthogonal bases as follows

\[ g_{\mu \nu} = - \varpi^{-2} \Upsilon_\mu \Upsilon_\nu + \varphi^{-2} \phi_\mu \phi_\nu + R^{-2} R_\mu R_\nu + \vartheta^{-2} \Theta_\mu \Theta_\nu, \]  

(5)

\( \{ R^\mu, \Theta^\mu \} \) being the spacelike basis vectors orthogonal to \( \{ \Upsilon^\mu, \phi^\mu \} \). For a stationary axisymmetric space-time the hypersurface \( \zeta^2 = 0 \) defines an ergosphere rather than the horizon. \( \zeta^2 \) is negativew inside the ergosphere since \( \xi^\mu \) is spacelike in that region. On the otherhand, a compact \( \varpi^2 = 0 \) hypersurface defines a killing horizon (e.g. a blackhole event horizon in our case). This can be demonstrated by constructing the null geodesic congruence on such a surface.

The aforementioned formalism is valid for a very general kind of stationary axisymmetric space-time, which includes, but certainly not limited to, the space-time defined by the Kerr family of solutions. With reference to eq. (5), the normalisation condition for velocity vector field may be expressed as

\[ v^\mu v^\nu g_{\mu \nu} = - \zeta^{-2} v_0^2 + \varphi^{-2} v_1^2 + R^{-2} v_2^2 + \vartheta^{-2} v_3^2 = -1, \]  

(6)

where \( v_0 = v_\mu \Upsilon^\mu \) etc. are scalars. Contracting the equation (1) with \( \xi^\mu \) we obtain

\[ v^\mu \nabla_\mu (v^\nu \xi_\nu) + \frac{c_s^2}{\rho} \left[ \xi^\nu \nabla_\nu \rho + (\xi_\mu v^\mu) v^\nu \nabla_\nu \rho \right] = 0, \]  

(7)

where \( v^\mu v^\nu \nabla_\mu \xi_\nu = \frac{1}{2} v^\mu v^\nu \nabla_{(\mu} \xi_{\nu)} = 0 \) is ensured by virtue of the killing equation, i.e. eq. (2). Through similar procedure we also obtain

\[ v^\mu \nabla_\mu v_1 + \frac{c_s^2}{\rho} \left[ \phi^\nu \nabla_\nu \rho + v_1 v^\nu \nabla_\nu \rho \right] = 0. \]  

(8)
Note that all the differential terms appearing in Eqs. (7-8) involve partial derivatives only since \( v_\mu \xi^\mu \) and \( v_1 \) are scalars.

Since we consider the stationary, axisymmetric flow in three dimensions, all the directional partial derivatives with respect to \( \xi^\mu \), \( \phi^\mu \) vanish to yield,

\[
\frac{\text{d}v_1}{\text{d}R} + \frac{c_0^2}{\rho} v_1 \frac{\text{d}p}{\text{d}R} = 0,
\]

from eq. (8), \( R \) being a parameter along \( R^\mu \). Integration of eq. (9) provides

\[
v_1 = A \exp \left( - \int \frac{c_0^2}{\rho} \text{d}R \right),
\]

\[A\] being a constant to be evaluated from the initial boundary conditions.

In a similar fashion, eq. (7) provides the expression for \( v_\mu \xi^\mu \), which will formally be same as \( v_1 \) up to an integration constant, since \( \xi^\mu \) is a Killing field. One thus finds,

\[
v_0 = v_\mu \xi^\mu - \rho v_1 \phi^\mu.
\]

Substitution of \( v_0 \) from eq. (11) and \( v_1 \) from eq. (10) into eq. (6) provides the expression of \( v_2 \).

In this section we thus provide a general formalism for evaluating all relevant bulk velocity components of a rotating accretion flow in a most general axisymmetric space-time. \( \{v_0, v_1, v_2\} \) are, however, the general solutions and evaluation of their specific numerical values for a particular flow configuration in a predetermined (by the accreting black hole) metric is a rather involved procedure since the integration constant appearing in the expressions of \( \{v_0, v_1, v_2\} \) can be evaluated only if the appropriate set of initial boundary conditions are provided. It is also required that sound speed as well as the rest mass energy density is to be known a priori to find the specific values of \( \{v_0, v_1, v_2\} \). For our particular purpose, the axisymmetric space-time metric is of Kerr type and \( \{v_1\} \equiv \{v_t, v_r, v_\theta, v_\phi\} \), for which, the initial boundary conditions cannot be evaluated analytically for barotropic equation of state and for a certain geometric configuration of the accreting fluid. \( c_s \) and \( \rho \) are also unknown a priori.

Our aforementioned formalism for finding the general solution of \( \{v_1\} \) is thus useful for the flow configuration with known values of \( \{c_s, \rho\} \) and initial boundary conditions. Axisymmetric low angular momentum accretion onto an astrophysical black body however constitutes a complex gravitational system for which such predetermined informations are not available in general. In subsequent sections, we thus plan to develop a metric specific formalism to understand the spatial velocity profile of the stationary axisymmetric flow.

### 4 SPACE-TIME METRIC AND THE CONSERVATION EQUATIONS

Hereafter, any relevant radial distance will be scaled in units of \( GM_{BH}/c^2 \) and any relevant velocity will be scaled by \( c \), where \( G, M_{BH} \) and \( c \) are universal gravitational constant, mass of the black hole and speed of light in vacuum respectively. \( G = c = M_{BH} = 1 \) is adopted. The allowed domain of \( a \), the Kerr parameter is taken as \(-1 < a < 1\) as usual.

Using Boyer-Lindquist (Boyer & Lindquist 1967) co-ordinates, the corresponding metric element for the Kerr family of solutions in the spherical polar co-ordinate can be expressed as

\[
ds^2 = -\left(1 - \frac{2}{\mu r}\right) dt^2 + \frac{\mu r}{\Delta} dr^2 + \mu r^2 d\theta^2 - \frac{4a \sin^2 \theta}{\mu r} dt d\phi + r^2 \sin^2 \theta \left(1 + \frac{a^2}{r^2} + \frac{2a^2 \sin^2 \theta}{\mu r^3}\right) d\phi^2,
\]

where \( \theta \) is the polar angle, \( \mu = 1 + \frac{a^2}{2} \cos^2 \theta \) and \( \Delta = r^2 - 2r + a^2 \).

The corresponding co-variant metric components are

\[
g_{tt} = -\left(1 - \frac{2}{\mu r}\right), \quad g_{rr} = \frac{\mu r^2}{\Delta}, \quad g_{\theta\theta} = \mu r^2.
\]

\[
g_{t\phi} = g_{\phi t} = -\frac{2a \sin^2 \theta}{\mu r}, \quad g_{\phi\phi} = r^2 \sin^2 \theta \left(1 + \frac{a^2}{r^2} + \frac{2a^2 \sin^2 \theta}{\mu r^3}\right).
\]
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Corresponding contravariant elements can be expressed as
\[
g^{tt} = -\frac{g_{\phi\phi}}{g_{t\phi} - g_{t\phi}g_{\phi\phi}} = -\left[1 + \frac{2r}{\mu \Delta} \left(1 + \frac{\mu^2}{r^2}\right)\right]
\]
\[
g^{rr} = 1 \frac{g_{\phi\phi}}{g_{t\phi} - g_{t\phi}g_{\phi\phi}} = \frac{\Delta}{\mu r^2}
\]
\[
g^{\theta\theta} = 1 \frac{g_{\phi\phi}}{g_{t\phi} - g_{t\phi}g_{\phi\phi}} = 1 \frac{\mu^2}{r^2}
\]
\[
g^{\phi\phi} = g^{\phi\phi} = \frac{g_{\phi\phi}}{g_{t\phi} - g_{t\phi}g_{\phi\phi}} = \frac{2a}{\mu \Delta r}
\]
\[
g^{\phi\phi} = -\frac{g_{\phi\phi}}{g_{t\phi} - g_{t\phi}g_{\phi\phi}} = -\left(1 - \frac{\mu r}{\Delta \sin^2 \theta}\right)
\]

We thus obtain,
\[
(g_{tt})_{eq} = -\left(1 - \frac{2}{r}\right)
\]
\[
(g_{rr})_{eq} = \frac{r^2}{\Delta}
\]
\[
(g_{\theta\theta})_{eq} = r^2,
\]
\[
(g_{t\phi})_{eq} = (g_{\phi t})_{eq} = -\frac{2a}{r},
\]
\[
(g_{\phi\phi})_{eq} = \frac{A}{r^2},
\]

where \(A = r^4 + r^2a^2 + 2ra^2\). The corresponding contravariant metric elements can thus be evaluated as,
\[
(g^{tt})_{eq} = -\frac{(g_{\phi\phi})_{eq}}{(g_{t\phi})_{eq}^2 - (g_{tt})_{eq}(g_{\phi\phi})_{eq}} = -\frac{A}{\Delta r^2}
\]
\[
(g^{rr})_{eq} = \frac{1}{(g_{rr})_{eq}} = \frac{\Delta}{r^2}
\]
\[
(g^{\theta\theta})_{eq} = \frac{1}{(g_{\theta\theta})_{eq}} = \frac{1}{r^2}
\]
\[
(g^{\phi\phi})_{eq} = \frac{(g_{\phi\phi})_{eq}}{(g_{t\phi})_{eq}^2 - (g_{t\phi})_{eq}(g_{\phi\phi})_{eq}} = \frac{2a}{\Delta r}
\]
\[
(g^{\phi\phi})_{eq} = -\frac{(g_{\phi\phi})_{eq}}{(g_{t\phi})_{eq}^2 - (g_{t\phi})_{eq}(g_{\phi\phi})_{eq}} = \frac{(1 - \frac{2}{r})}{\Delta}
\]

Hereafter, however, we drop all the ‘eq’ subscripts for the sake of brevity. Any \(g_{\mu\nu}\) or \(g^{\mu\nu}\) will thus explicitly imply that the corresponding metric element has been evaluated on the equatorial plane. Using the cylindrical polar co-ordinate the corresponding metric on the equatorial plane can be expressed as,
\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu
\]
\[
= -\frac{r^2}{A} dt^2 + \frac{r^2}{\Delta} dr^2 + \frac{A}{r^2} (d\phi - \omega dt) + dz^2,
\]

where \(z = r \cos \theta, \omega = 2a r / A \) and \(g_{zz} = 1\).

For the metric element expressed using \((r, \theta, \phi), g_{(r, \theta, \phi)} \equiv det(g_{\mu\nu}) = -r^4\), whereas for \(ds^2\) expressed using \((r, \phi, z), g_{(r, \phi, z)} = -r^2\).

Main calculations presented in this work will be based on the line element as expressed in eq. (16).

In this work, the polytropic equation of state of the following form
\[
p = K \rho^\gamma
\]

is considered to describe the flow, where the polytropic index \(\gamma\) (which is equal to the ratio of the specific heats, \(C_p\) and \(C_v\), respectively) of the accreting material is assumed to be constant throughout the fluid. A more realistic flow model would perhaps require the implementation of a variable polytropic index having a functional dependence on the radial distance of the form \(\gamma \equiv \gamma(r)\) (Meliani et. al. 2004; Ryu & Chattopadhyay 2006; Migone & McKinney 2007; Mondal & Basu 2011; Mukhopadhyay & Dutta 2012). However, we have performed our calculations for a reasonable broad spectrum of \(\gamma\) and thus believe that all astrophysically relevant polytropic indices are covered in our analysis.
The proportionality constant $K$ in eq. (17) is related to the specific entropy of the accreting fluid provided no additional entropy generation takes place. Subjected to the condition that the Clapeyron equation of the form

$$p = \frac{k_B \mu m}{\gamma m_p} \rho T,$$

holds (in addition to the eq. (17)); $k_B$, $\mu$, and $m_p \sim m_H$ being the totally measured flow temperature, the mean molecular weight, and the mass of the singly ionised hydrogen atom, respectively, the entropy per particle of an ensemble may be expressed as (Landau & Lifshitz 1981)

$$\sigma = \frac{1}{\gamma - 1} \log K + \frac{\gamma}{\gamma - 1} + \text{constant};$$

where the constant depends on the chemical composition of the accreting matter. $K$ in eq. (17) can now be interpreted as a measure of the specific entropy of the accreting matter.

The specific enthalpy $h$ can thus be formulated as

$$h = p + \epsilon \rho,$$

(19)

where the energy density $\epsilon$ includes the rest mass density and internal energy. Hence

$$\epsilon = \rho + \frac{p}{\gamma - 1}$$

(20)

The adiabatic sound speed $c_s$ is defined as

$$c_s^2 = \left( \frac{\partial p}{\partial \epsilon} \right)_{\text{constant enthalpy}}$$

(21)

The energy-momentum tensor of an ideal fluid can be expressed as

$$T_{\mu\nu} = (\epsilon + p)v^\mu v^\nu + pg_{\mu\nu}.$$ 

Vanishing of the four divergence of the energy momentum tensor provides the general relativistic version of the Euler equation i.e.

$$T_{\mu\nu}^{;\nu} = 0.$$ 

(22)

The continuity equation is obtained from

$$(p v^\mu) = 0.$$ 

(23)

We have defined two killing vectors $\xi^\mu = \delta^\mu_t$ and $\phi^\mu = \delta^\mu_\phi$ corresponding to stationarity and axisymmetry of the flow.

We now contract eq. (22) with $\phi^\mu$ to obtain,

$$\phi_\mu \left[ (\epsilon + p)v^\nu v^\nu \right]_{;\nu} + \phi_\mu p_{\nu}g^{\mu\nu} = 0.$$ 

But $\phi^\nu p_{\nu} = 0$ due to axisymmetry, hence

$$\phi_\mu \left[ (\epsilon + p)v^\nu v^\nu \right]_{;\nu} = 0,$$

which further provides,

$$g_{\mu\phi} \left[ (\epsilon + p)v^\nu v^\nu \right]_{;\nu} = 0.$$ 

(24)

since $\phi^\mu = \delta^\mu_\phi$. Since $g_{\mu\lambda,\nu} = 0$, eq. (24) can be written as

$$\left[ g_{\mu\phi} (\epsilon + p)v^\nu v^\nu \right]_{;\nu} = 0;$$

from where we obtain

$$[\phi_\mu hv^\nu]_{;\nu} = 0.$$ 

(25)

From eq. (25) one thus infers $\phi_\mu hv^\mu = hv_\phi$ and hence $hv_\phi$, the angular momentum per baryon for the axisymmetry flow, is conserved.

It can also be shown (in Fishbone & Moncrief (1976)) that the quantity $v_\phi v^\phi$, which is the angular momentum per unit inertial mass, is conserved for an iso-entropic flow and hence the world lines along which $v_\phi v^\phi$ remains constant is a solution of the general relativistic Euler equation.

In a similar way, we contract eq.(22) with $\xi^\mu$

$$\xi_\mu T_{\mu\nu}^{;\nu} = 0,$$

to obtain that $hv_\phi$ is conserved. Hereafter, $hv_\phi$ will be interpreted as the specific energy of the flow and will be denoted by $E$ (scaled in unit of $m_0 c^2$ using the system of unit adopted in this work).

For polytropic adiabatic accretion, $E$ is a first integral of motion along a streamline, and thus can be identified with the relativistic Bernoulli’s constant (Anderson 1989).

The angular velocity $\Omega$ can be defined in terms of the specific angular momentum $\lambda$, where

$$\lambda = -\frac{v_\phi}{v_\phi},$$

(26)
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as

\[ \Omega = \frac{v^\phi}{v^t} = \frac{g_{t\phi} + g_{\phi\phi} \lambda}{g_{t\phi} + g_{\phi\phi} \lambda} = \frac{r [2a + \lambda (r - 2)]}{A - 2a \lambda r}. \]

The normalisation condition \( v^\nu v_\nu = -1 \) can be expressed as

\[ v^t v_t + v^r v_r + v^\phi v_\phi = -1. \]

In terms of the angular velocity \( \Omega \) and the specific angular momentum \( \lambda \), one thus writes

\[ v^t v_t + v^r v_r + \Omega v^\phi (-\lambda v_t) = -1 \]  

(26)

Since \( v^2 \equiv -\frac{\rho v}{\rho v_t} \), one can rewrite eq. (26) as

\[ v^t v_t (1 - \lambda \Omega - v^2) = -1, \]

as (by converting \( v^r \rightarrow v_t \))

\[ (g^{tt} - \lambda g^{t\phi})(1 - \lambda \Omega - v^2) v_t^2 = -1 \]

and finally, by using the definition of \( \Omega \), since

\[ \left( \frac{g_{\phi\phi} + \lambda g_{\phi t}}{g_{t\phi} - g_{\phi\phi}} \right) (1 - \lambda \Omega - v^2) v_t^2 = 1, \]

and it can be shown that \( u = v/\sqrt{1 - \lambda \Omega} \) (where \( u \) is the advective velocity i.e. the radial velocity of the fluid measured in frame which co-rotates with the fluid such that \( v^t = u/\sqrt{g_{rr}(1 - u^2)} \), see Gammie & Popham (1998)), \( v_t \) can be obtained as

\[ v_t = \sqrt{\frac{g_{t\phi} - g_{\phi\phi}}{(1 - \lambda \Omega)(1 - u^2)(g_{\phi\phi} + \lambda g_{t\phi})}}. \]

(27)

The mass conservation equation (the continuity equation as defined by eq. (23)) provides

\[ \frac{1}{\sqrt{-g}} (\sqrt{-g} g_{\mu\nu})_{,\mu} = 0, \]

(28)

where \( g \equiv \det(g_{\mu\nu}) \). We multiply equation (28) with the co-variant volume element \( \sqrt{-g} \text{d}^4x \) to obtain

\[ (\sqrt{-g} g_{\mu\nu})_{,\mu} \text{d}^4x = 0. \]

(29)

for accretion studied using the spherical polar co-ordinate and

\[ \partial_t (\sqrt{-g} v^\nu) \text{d}r \text{d}\theta \text{d}\phi = 0, \]

(30)

for accretion studied using the cylindrical co-ordinate.

One can integrate eq. (29) for \( \phi = 0 \rightarrow 2\pi \) and \( \theta = -H_0 \rightarrow H_0; \pm H_0 \) being the value of the polar co-ordinates above and below the equatorial plane, respectively, for a local flow half thickness \( H \). \( 2H/r \) is constant for flow thickness in spherical polar co-ordinate for conical wedge shaped flow, (see e.g., Abramowicz & Zurek (1981); Gammie & Popham (1998); Lu, Yu & Young (1995); Lu et al. (1997a,b); Lu & Yuan (1998); Nag et al. (2012) and references therein) and can integrate eq. (30) for \( z = -H_z \rightarrow H_z \) (where \( H_z \) is the local half thickness of the flow) symmetrically over and below the equatorial plane for axisymmetric accretion studied using the cylindrical polar co-ordinate to obtain the conserved mass accretion rate \( \dot{M} \) in the equatorial plane as

\[ \rho v^\nu A(r) = \dot{M}; \]

(31)

where \( A(r) \) is the surface area through which the inward mass flux is estimated. For spherical symmetry, \( A(r) = 4\pi H_\theta r^2 \) (for not so large value of \( \theta \)) and for cylindrical symmetry, \( A(r) = 2\pi H_z r \).

In standard literature of accretion astrophysics, the local flow thickness for an inviscid axisymmetric flow can vary in three different ways, with different degrees of complexity, (see, e.g. Nag et al. (2012)). A constant flow thickness is considered for simplest possible flow configuration where the disc height \( H \) is not a function of the radial distance (Abraham et al. 2006). In its next variant, the axisymmetric accretion can have a conical wedge shaped structure (Abramowicz & Zurek (1981); Lu (1985, 1986); Lu, Yu & Young (1995); Lu et al. (1997a,b); Lu & Yuan (1998); Lu & Gu (2004)) where \( H \) is directly proportional to the radial distance as \( H = A_h r \), and hence the geometric constant \( A_h \) is determined from the measure of the solid angle subtended by the flow. For the hydrostatic equilibrium in the vertical direction, (Matsumoto et al. (1984); Frank et al. (2002); Das & Czerny (2011) and references therein) expression for the local flow thickness can have a rather complex dependence on the radial distance and on the local speed of propagation of the acoustic perturbation embedded inside the accretion flow. In the present work, we consider the accretion flow to be in vertical equilibrium, and assume that the flow has a radius-dependent local thickness with its central plane coinciding with the equatorial plane of the black hole. The equations (Barai,
Das et al. 2004) of motion apply to the equatorial plane of the black hole, whereas the hydrodynamic flow variables are averaged over the half thickness of the disc \( H \). We follow Abramowicz, Lanza & Pervival (1997) to derive the disc height for our flow configuration, and obtain the expression for the local flow thickness to be

\[
H(r) = \sqrt{\frac{2}{\gamma + 1}} r^2 \left[ \frac{(\gamma - 1)c_s^2}{(\gamma - (1 + c_s^2)) \left( \frac{\lambda^2}{\lambda^2} - a^2(v_i^2 - 1) \right)} \right].
\]

Using the expression for \( v_i \) as obtained from its expression in eq. (27) (using the expressions for \( g_{\mu \nu}'s \) at equatorial plane as mentioned on Eqs. (14)) i.e., by writing \( v_i \) as

\[
v_i = \left[ \frac{A^2 r^2 \Delta}{(1 - a^2)} \left( A^2 - 4a \lambda A + \lambda^2 r^2 (4a^2 - r^2 \Delta) \right) \right]^{1/2},
\]

\( H(r) \) can be obtained in terms of the advective velocity \( u \).

5 CRITICAL POINT CONDITIONS

The two first integrals of motion, the conserved specific energy \( E \) of the flow and the mass accretion rate \( \dot{M} \) can thus be expressed as

\[
E = \left[ \frac{(\gamma - 1)}{\gamma - (1 + c_s^2)} \right] \left[ \frac{1}{1 - u^2} \right] \left[ \frac{\lambda^2}{\lambda^2} - a^2 \left( \frac{1}{2} \right) \right] \frac{A^2 r^2 \Delta}{(1 - a^2)} \left( A^2 - 4a \lambda A + \lambda^2 r^2 (4a^2 - r^2 \Delta) \right),
\]

\[
\dot{M} = 4\pi \Delta \frac{1}{2} H(r) \rho \frac{u}{\sqrt{1 - u^2}},
\]

by integrating the stationary part of the energy momentum conservation equation and the continuity equation, respectively. The set of equations (32 – 33) can not directly be solved simultaneously since it contains three unknown variables \( u, c_s \), and \( \rho \), all of which are functions of the radial distance \( r \). We would like to express \( \rho \) in terms of \( u, c_s \) and other related constant quantities. To accomplish such task, we make a transformation \( \tilde{\Xi} = \dot{M} \gamma^{1/2} K^{1/2 - 2} \). Employing the corresponding equation for the sound speed as well as the equation of state, \( \tilde{\Xi} = \dot{M} \gamma^{1/2} K^{1/2 - 2} \) can be expressed as

\[
\tilde{\Xi} = \left( \frac{1}{\gamma} \right) \left( \frac{1}{2} \right) \left( \frac{1}{1 + c_s^2} \right) \left( \frac{1}{1 - u^2} \right) \left( \frac{\gamma - 1}{\gamma - (1 + c_s^2)} \right) \left( \frac{1}{2} \right) \frac{A^2 r^2 \Delta}{(1 - a^2)} \left( A^2 - 4a \lambda A + \lambda^2 r^2 (4a^2 - r^2 \Delta) \right) H(r).
\]

The entropy per particle \( \sigma \) is related to \( K \) and \( \gamma \) as Landau & Lifshitz (1994)

\[
\sigma = \frac{1}{\gamma - 1} \log K + \frac{\gamma}{\gamma - 1} \text{ constant}
\]

where the constant depends on the chemical composition of the accreting material. The above equation implies that \( K \) in some sense is a measure of the specific entropy of the accreting matter. Hence \( \tilde{\Xi} \) may be interpreted as the measure of the total inward entropy flux associated with the accreting material and thus we label \( \tilde{\Xi} \) to be the entropy accretion rate. It is worth mentioning here that the concept of the entropy accretion rate was first introduced in Abramowicz & Zurek (1981); Blaes (1987) for accretion under the influence of the Paczyński & Wiita Paczyński & Wiita (1980) pseudo-Schwarzschild Newtonian like black hole potential. \( \tilde{\Xi} \) is thus conserved for the shock free polytropic accretion and increases discontinuously at the shock (if forms). Since \( \tilde{E} \) and \( \tilde{\Xi} \) remains conserved along a streamline, the spatial derivative (since we are dealing the stationary flow) of \( \tilde{E} \) globally for shocked flow as well as the shock free flow and of \( \tilde{\Xi} \) globally vanishes only for shock free flow. However, even if the shock forms, the spatial derivative of \( \tilde{\Xi} \) vanishes locally, and hence \( d\tilde{\Xi}/dr = 0 \) holds separately for the pre and the post shock flow, where the pre and the post shock flow implies the transonic accretion solution passing through the outermost and the innermost saddle type critical points, respectively. This point will further be clarified in the subsequent sections.

The relationship between the space gradient of sound speed and that of the advective velocity can now be established by differentiating eq. (34)

\[
dc_s/dr = c_s \left( \frac{\gamma - 1 - c_s^2}{1 + \gamma} \right) \left[ \frac{\chi \psi_a}{4} - \frac{2}{r} - \frac{1}{2u} \left( \frac{2 + 3u \psi_a}{1 - u^2} \right) \frac{du}{dr} \right].
\]

Differentiation of eq. (32) provides another relationship between \( dc_s/dr \) and \( du/dr \). We substitute \( dc_s/dr \) as obtained in eq. (36) into that relationship and finally obtain

\[
\frac{du}{dr} = \frac{2c_s^2}{(\gamma + 1) r} \left[ \frac{r - 1}{\Delta} + \frac{2}{r} - \frac{v_i^2 \chi}{4\psi} \right] - \frac{\chi}{2} \left( \frac{2}{r} \right) \frac{1}{1 - u^2} - \frac{2c_s^2}{(\gamma + 1) (1 - u^2) u} \left[ 1 - \frac{u^2 v_i^2}{2\psi} \right],
\]

(37)
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\[
\psi = \lambda^2 v_t^2 - a^2 (v_t - 1) , \quad \psi_a = \left(1 - \frac{a^2}{\psi}\right) , \quad \sigma = 2\lambda^2 v_t - a^2 ,
\]

\[
\chi = \frac{1}{\Delta} \frac{d\Delta}{dr} + \frac{\lambda}{(1 - \Omega \Delta)} \frac{d\Omega}{dr} = \left(\frac{dg_{\phi\phi}}{dr} + \lambda \frac{dg_{\phi\phi}}{dr}\right) ,
\]

Eq. (37) as well as eq. (36) can now be identified with a set of non-linear first order differential equations representing autonomous dynamical systems, and their integral solutions will provide phase trajectories on the radial Mach number \( M \) vs \( r \) plane. The regular critical point condition for these integral solutions is obtained by simultaneously making the numerator and the denominator of eq. (37) vanish. The critical point condition may thus be expressed as

\[
\left. c_s\right|_{(r=r_c)} = \left[\frac{u^2 (\gamma + 1) \psi}{2\psi - u^2 v_t \sigma}\right]^{1/2} \bigg|_{(r=r_c)} , \quad \left. u\right|_{(r=r_c)} = \left[\frac{\chi r}{2r (r - 1) + 4\Delta}\right]^{1/2} ,
\]

Eq. (39) provides the critical point condition but not the location of the critical point(s). It is necessary to solve eq. (32) under the critical point condition for a set of initial boundary conditions as defined by \([\mathcal{E}, \lambda, \gamma, a]\). The value of \( c_s \) and \( u \), as obtained from eq. (39), may be substituted at eq. (32) to obtain a complicated non-polynomial algebraic expression for \( r = r_c, r_c \) being the location of the critical point. A particular set of values of \([\mathcal{E}, \lambda, \gamma, a]\) will then provide the numerical solution for such algebraic expression to obtain the exact value of \( r_c \). It is thus important to find out the astrophysically relevant domain of numerical values corresponding to \( \mathcal{E}, \lambda, \gamma \), and \( a \). An appropriate choice for the four parameter initial boundary conditions is thus (Barai, Das, & Wiita 2004) \([1 < \mathcal{E} < 2, 0 < \lambda \leq 4/3, 0 \leq \gamma \leq 3/3, -1 \leq a \leq 1]\).

Although to be mentioned here that an upper limit for the Kerr parameter has been set to 0.998 in some works, see, e.g., Thorne (1974). We, in our work, however, do not consider any such interaction of accreting material with the black hole itself which might allow the evolution of the mass and the spin of the hole as was considered in Thorne (1974) to arrive at the conclusion about such upper limit for the black hole spin. Hence we theoretically consider the domain for the Kerr parameter to be \([-1 \leq a \leq 1]\).

The aforementioned four parameters may further be classified into three different categories, according to the way they influence the characteristic properties of the stationary transonic solutions. \([\mathcal{E}, \lambda, \gamma]\) characterizes the flow, and not the spacetime since the accretion is assumed to be non-self-gravitating. The Kerr parameter \( a \) exclusively determines the nature of the spacetime and hence can be thought of as some sort of ‘inner boundary condition in qualitative sense since the effect of gravity is determined within the full general relativistic framework only up to several gravitational radii, beyond which it asymptotically follows the Newtonian description, \([\mathcal{E}, \lambda] \subset [\mathcal{E}, \lambda, \gamma]\) determines the dynamical aspects of the flow, whereas \( \gamma \) determines the thermodynamic properties. To follow a holistic approach, one needs to study the variation of the relevant features of the transonic accretion on all of these four parameters.

Having equipped with the values of \([\mathcal{E}, \lambda, \gamma, a]\), one can compute the location of the critical points by solving the algebraic equation as obtained by the substitution of eq. (39) in eq. (32). For convenience, the four dimensional hypersurface spanned by \([\mathcal{E}, \lambda, \gamma, a]\) can be projected onto \(4C_2\) different two dimensional or \(4C_3\) different three dimensional parameter submanifolds to identify the regions of the parameter sub-space for which a multi-transonic (multi-critical solutions with stationary shock) accretion flow can be obtained. The accretion solution, as already mentioned, may be mono-critical or multi-critical with three critical points where two saddle type critical points are separated by a centre type critical point. The nature of a given critical point (whether it is of saddle type or a centre type) can be examined through the outermost critical point condition for a set of initial boundary conditions as defined by \([\mathcal{E}, \lambda, \gamma, a]\).

For multi-critical solution, the criteria for the accretion flow to have three critical points is associated with the value of the entropy conservation rate \( \dot{\Xi} \) (obtained from eq. (34)), will be denoted by \( \dot{\Xi}_o \) henceforth, evaluated for the solution passing through the innermost saddle type critical point is greater than the value of \( \dot{\Xi} \) evaluated for the solution passing through the outermost saddle type critical point, which will be denoted by \( \dot{\Xi}_o \). The reverse situation, i.e., \( \dot{\Xi}_o > \dot{\Xi} \), provides the stationary configuration for which accretion solution connecting the infinity with the event horizon can have one critical point (the innermost one). For such a \( M - r \) phase portrait, the accretion solution through the outermost critical point is a part of the homoclinic orbit which can not be connected with the solution passing through the innermost critical point, and hence the stationary accretion for such configuration is essentially monocritical even if one obtains three formal solution of the critical point determining algebraic expression.

If \([\mathcal{E}, \lambda, \gamma, a]_{\text{mca}} \subset [\mathcal{E}, \lambda, \gamma, a] \) represents the region of the four dimensional parameter space for which one obtains three critical points (‘mca’ stands for ‘multi-critical’), \([\mathcal{E}, \lambda, \gamma, a]_{\text{mca}} \subset [\mathcal{E}, \lambda, \gamma, a]_{\text{mca}} \) denotes the region embedded in \([\mathcal{E}, \lambda, \gamma, a]_{\text{mca}} \) for which stationary accretion
solution can have three critical points - 'mca' being the acronym used for the phrase 'multi-critical accretion'. Hence it is the \([\mathcal{E}, \lambda, \gamma, a]_{mca}\) which we are interested in to identify the shocked multi-transonic flow.

The space gradient for the advective flow velocity at the critical point is computed by solving the following quadratic equation

\[
\alpha \left( \frac{du}{dr} \right)^2 |_{r=r_c} + \beta \left( \frac{du}{dr} \right) |_{r=r_c} + \zeta = 0,
\]

(40)

where the respective coefficients, all evaluated at the critical point \(r_c\), are obtained as

\[
\alpha = \frac{(1 + u^2)}{(1 - u^2)} - \frac{2 \delta_1 \delta_5}{\gamma + 1}, \quad \beta = \frac{2 \delta_1 \delta_6}{\gamma + 1} + \tau_4, \quad \zeta = -\tau_5;
\]

\[
\delta_1 = \frac{c_s^2 (1 - \delta_2)}{u (1 - u^2)}, \quad \delta_2 = \frac{u^2 v_c \sigma}{2 \psi}, \quad \delta_3 = \frac{1}{\alpha} + \frac{2 \lambda^2}{\sigma} - \frac{\psi}{2}, \quad \delta_4 = \frac{2}{u} + \frac{u v_c \delta_1}{1 - u^2},
\]

\[
\delta_5 = \frac{3 u^2 - 1}{u (1 - u^2)} - \frac{\delta_4 - u (\gamma - 1 - c_s^2)}{1 - \delta_2}, \quad \delta_6 = \frac{(\gamma - 1 - c_s^2) \chi}{2 c_s^2} + \frac{\delta_3 \chi v_c}{2 (1 - \delta_2)}.
\]

\[
\tau_1 = \frac{r - 1}{\Delta} + \frac{2 \sigma u \chi}{4 \psi}, \quad \tau_2 = \frac{(4 \lambda^2 v_c - a^2) \psi - v_c \sigma^2}{\psi}, \quad \tau_3 = \frac{\sigma \tau_2 \chi}{4 \psi}, \quad \tau_4 = \frac{1}{\Delta} - \frac{2 (r - 1)^2}{\Delta^2} - \frac{2}{r^2} - \frac{u v_c d \chi}{4 \psi \dr},
\]

\[
\tau_5 = \frac{2}{\gamma + 1} \left[ c_s^2 r_4 - \left( \frac{\gamma - 1 - c_s^2}{\tau_1} + v_c^2 r_3 \right) \frac{\chi}{2} - \frac{1}{2} \frac{d \chi}{dr} \right], \quad \tau_6 = \frac{2 v_c u}{(\gamma + 1) (1 - u^2)} \left[ \frac{\tau_1}{u} (\gamma - 1 - c_s^2) + c_s^2 \tau_3 \right].
\]

(41)

Note, however, that all quantities defined in eq. (41) can finally be reduced to an algebraic expression in \(r_c\) with real coefficients that are functions of \([\mathcal{E}, \lambda, \gamma, a]\). Hence \((du/dr)_{r=r_c}\) is found to be an algebraic expression in \(r_c\) with constant coefficients those are non linear functions of \([\mathcal{E}, \lambda, \gamma, a]\). Once \(r_c\) is known for a set of values of \([\mathcal{E}, \lambda, \gamma, a]\), the critical slope, i.e., the space gradient for \(u\) at \(r_c\) for the advective velocity can be computed as a pure number, which may either be a real (for transonic accretion solution to exist) or an imaginary (no transonic solution may be found) number. The critical advective velocity gradient for accretion solution may be computed as

\[
\left( \frac{du}{dr} \right)_{r=r_c} = -\frac{\beta}{2 \alpha} \pm \sqrt{\frac{\beta^2}{4 \alpha^2} - 4 \alpha \zeta},
\]

(42)

by taking the positive sign. The negative sign corresponds to the outflow/self-wing solution on which we would not like to concentrate in this work. The critical acoustic velocity gradient \((dc_s/dr)_{r=r_c}\) can also be computed by substituting the value of \((du/dr)_{r=r_c}\) in eq. (36) and by evaluating other quantities in eq. (36) at \(r_c\).

The values of the advective velocity \(u\) and the sound speed \(c_s\) evaluated at the critical point indicates that the Mach number at the critical point is not unity, and not even a constant as well. eq. (39) implies that the Mach number at the critical point is a function of the location of the critical point itself, and hence for any \([\mathcal{E}, \lambda, \gamma, a] \in [\mathcal{E}, \lambda, \gamma, a]_{mca}\) one obtains three different Mach numbers corresponding to the three critical points for a multi-critical stationary solution. It is easy to show that the Mach number at the critical point is a function of the expression \(r_c\) for all values of \(r_c\) for a transonic flow, whether it is monocritical or multi-critical. Since a regular sonic point is identified with the radial distance (measured from the event horizon) where the transonic solution makes a continuous \(M < 1 \rightarrow M > 1\) transition, the Mach number at the sonic point must be equal to unity. Hence the critical points and the sonic points are not topologically (as well as numerically) isomorphic. Such distinction between the critical and the sonic point is observed for polytropic accretion in the hydrostatic equilibrium in the vertical direction only and not for the polytropic flow with wedge shaped conical geometry or with constant thickness. This is a manifestation of the fact that the expression for the flow thickness (the disc height) only for accretion in vertical equilibrium is a function of the non constant sound speed and the expression for such disc height is obtained using a set of simplified assumptions, hence the dependence of the flow thickness on \(c_s\) is not exact. For isothermal accretion, critical points and the sonic points are identical even for flow in hydrostatic equilibrium in the vertical direction. Let alone for the conical flow and for flow with constant thickness, since the isothermal sound speed is a constant and hence its space gradient does not contribute in the calculation of the space gradient of the dynamical velocity (from which the critical point conditions are derived). Since we concentrate on the polytropic accretion in the hydrostatic equilibrium in the vertical direction, we need to find out the sonic point by numerically integrating the flow equations. Two out of the three critical points in a multi-transonic accretion are of saddle type and the third one is the centre type. A physically acceptable transonic solution, however, can be constructed only through a saddle type critical point. No centre type critical point allows any transonic flow solution to pass through it, hence every saddle type critical point is accompanied by a sonic point \(r_s\), generally located at a radial distance (measured from the event horizon) smaller than the respective critical point \(r_c\). The criteria \(r_s < r_c\) is always satisfied since \((u/c_s)_{r_s} < 1\) whereas \((u/c_s)_{r_c} = 1\) and for a smooth transonic accretion, the Mach number anti-correlates with the radial distance measured from the event horizon.
integrate eq. (36) - (37) simultaneously to obtain the stationary transonic branch ABB of the outermost saddle type critical point of the sonic point is found by integrating the same radial distance, AB is not the subsonic branch of the flow, neither the segment BCDE represents the supersonic branch. The location and the sonic points. Hence the line segment measured along the X axis and corresponding to BB₁ represents the logarithmic value of the space gradient of the advective velocity of the outermost sonic point and the inner sonic points, respectively. The actual multi-transonic accretion flow connecting infinity with the black hole event horizon and contains the integral solutions constructed through the outer critical point. The homoclinic orbit FUC₁D₁IJ consists of the transonic accretion and its associated self-wind solutions passing through the saddle type inner critical point I (located at a distance 6.999 in units of $GM_{BH}/c^2$ measured from the horizon). The corresponding inner sonic point (not shown in the figure) is located at a distance 5.985 in units of $GM_{BH}/c^2$ measured from the horizon. The centre type middle critical point shown by an asterisk (*) and marked by L located at a distance 17.215 in units of $GM_{BH}/c^2$ measured from the horizon. Among two formally obtained shock transitions CC₁ and DD₁, the shock formed between the outer sonic point and the middle critical point is found to be the stable one. ABB₁(CC₁D₁IJ thus represents the actual multi-transonic accretion flow connecting infinity with the black hole event horizon and contains the integral solutions constructed through the outer and the inner sonic points, respectively.

6 CONSTRUCTION OF A TYPICAL MULTI-TRANSONIC PHASE TRAJECTORY

6.1 The phase portrait

We now demonstrate how to construct the integral multi-transonic solution with a stationary shock. In figure 1 we plot one such flow topology for $[\mathcal{E} = 1.00001, \lambda = 2.6, \gamma = 1.43, a = 0.215]$. The radial Mach number and the radial distance in logarithmic scale and in units of $GM_{BH}/c^2$ measured from the black hole event horizon has been plotted along the abscissa and the ordinate, respectively. ABB₁CDE is the transonic accretion solution constructed through the saddle type outer critical point B (located at a distance 10815.150 in units of $GM_{BH}/c^2$ measured from the horizon) and the corresponding outer sonic point B₁ (located at a distance 8437.850 in units of $GM_{BH}/c^2$ measured from the horizon), FBG is the associated transonic self-wind solution passing through the outer critical point. The homoclinic orbit FUC₁D₁IJ consists of the transonic accretion and its associated self-wind solutions passing through the saddle type inner critical point I (located at a distance 6.999 in units of $GM_{BH}/c^2$ measured from the horizon). The corresponding inner sonic point (not shown in the figure) is located at a distance 5.985 in units of $GM_{BH}/c^2$ measured from the horizon. The centre type middle critical point shown by an asterisk (*) and marked by L located at a distance 17.215 in units of $GM_{BH}/c^2$ measured from the horizon.

Figure 1. Phase topology corresponding to the multi-transonic shocked accretion and its associated wind branches obtained for a prograde flow characterized by $[\mathcal{E} = 1.00001, \lambda = 2.6, \gamma = 2.6, a = 0.215]$. The radial Mach number and the radial distance in logarithmic scale and in units of $GM_{BH}/c^2$ measured from the black hole event horizon has been plotted along the abscissa and the ordinate, respectively. ABB₁CDE is the transonic accretion solution constructed through the saddle type outer critical point B (located at a distance 10815.150 in units of $GM_{BH}/c^2$ measured from the horizon) and the corresponding outer sonic point B₁ (located at a distance 8437.850 in units of $GM_{BH}/c^2$ measured from the horizon), FBG is the associated transonic self-wind solution passing through the outer critical point. The homoclinic orbit FUC₁D₁IJ consists of the transonic accretion and its associated self-wind solutions passing through the saddle type inner critical point I (located at a distance 6.999 in units of $GM_{BH}/c^2$ measured from the horizon). The corresponding inner sonic point (not shown in the figure) is located at a distance 5.985 in units of $GM_{BH}/c^2$ measured from the horizon.
\[ \Delta r_{cwa} \] for the particular values of the \([E, \lambda, \gamma, a]\) used to obtain figure 1. One can study the dependence of \( \Delta r_{cwa} \) on \([E, \lambda, \gamma, a]\) as well for the entire domain of the four parameters \([E, \lambda, \gamma, a]\) to apprehend the effect of the black hole space time as well as the dynamical and the thermodynamic properties of the flow on the distinction between the sonic and the critical points. It is important to note that however small can \( \Delta r_{cwa} \) be made, it never vanishes for any value of \([E, \lambda, \gamma, a]\). This indicates that the non isomorphism of the critical and the sonic properties of the flow are not any artifact of the choice of the initial boundary conditions describing the stationary transonic accretion.

If the value of the \( M \) is also provided along with \([E, \lambda, \gamma, a]\), one can calculate the values of all possible thermodynamic quantities corresponding to the flow, the pressure \( p \), the density \( \rho \) and the ion temperature \( T \) of the accreting fluid, for example, at all radial distances stating from the infinity up to a very close proximity of \( r_{+} = 1 + \sqrt{1 - a^{2}} \). The transcritical solution passing through the outermost critical point \( r_{cwa}^{out} \) seems to be doubly degenerate as is observed from the appearance of the phase topology FBG on the \( M - \log_{10}(r) \) phase plane. FBG is obtained by integrating eq. (36 – 37) using \([u, c, du/dr, dc/d\ln r]_{cwa}^{out} \) but for the values of \((du/dr)_{cwa}^{out} \) corresponding to the negative sign in eq. (42). Such twofold degeneracy is the consequence of the \( \pm u \) degeneracy appearing in the expression for the energy first integral of motion as defined in eq. (32). Such degeneracy has, however, been apparently removed by orienting the phase portrait so that each phase topology represents either the accretion or the wind. The wind branch FBG, obtained by the advective velocity reversal symmetry, is a mathematical counterpart of the accretion flow and is usually termed as the ‘self -wind’. Had it been the situation that instead of starting from the infinity and heading toward the compact object, transcritical solution would generate from the close proximity of the accretor and would fly off from such object, FBG would denote the phase trajectory along which it would escape to infinity. The phrase ‘wind solution’ stems from the fact that the phase portrait corresponding to the solar wind solution due to Parker (1965) was topologically similar with the aforementioned mathematical counterpart of the accretion solution associated with the classical Bondi (1952) flow.

Similar procedure may be used to obtain the transonic stationary accretion and the wind solutions passing through the innermost saddle type critical point \( r_{cwa}^{in} \) which is located at a radial distance \( r = 6.999 \) and is marked by I on the \( M - \log_{10}(r) \) phase plane. The corresponding value of the radial distance for the sonic point \( r_{cwa}^{in} \) comes out to \( r = 9.85 \).

The transcritical accretion solution HJJ constructed through the inner critical point \( r_{cwa}^{in} \) folds back onto itself and joins with the corresponding transcritical self-wind branch HIK. The combined transcritical accretion-wind solution through the \( r_{cwa}^{in} \) thus forms a homoclinic orbit \( K \) on the \( M - \log_{10}(r) \) phase plane. Such a homoclinic phase trajectory encompasses the centre type critical point \( r_{cwa}^{mid} \) flanked between \( r_{cwa}^{out} \) and \( r_{cwa}^{in} \). The point of inflection \( H \) of the homoclinic orbit is actually a ‘irregular’ and ‘singular’ critical point. A tangent drawn through such a point of inflection \( H \) is parallel to the Y axis. The Mach number \( M \) is defined at that point, whereas its space gradient \( dM/dr \) is not. The advective velocity \( u \) is continuous at that point but its space gradient diverges. At the point of inflection of the homoclinic orbit, the denominator of eq. (37) vanishes, allowing the corresponding numerator to assume a non-zero value. It is therefore understood that along with three regular critical points \([r_{cwa}^{out}, r_{cwa}^{mid}, r_{cwa}^{in}]\), multi-critical stationary flow solution always possesses one more critical point which is of singular type. The only exception observed for a very special case where the multi-critical flow consists of two heteroclinic orbits\(^3\) since no homoclinic orbit for such configuration can further be realized. The transcritical heteroclinic orbits on \( M - \log_{10}(r) \) phase plane is characterized by the identical value of the entropy accretion rate \( \bar{\Xi} \) evaluated for the solution passing through the innermost saddle point \( r_{cwa}^{in} \) as well as for the solution constructed through the outermost saddle point \( r_{cwa}^{out} \).

A homoclinic orbit has its existence only in isolation and such trajectory does not qualify as a global transcritical solution since any realistic transcritical solution has to connect with the event horizon to ensure the existence of the corresponding transonic flow. A local transcritical homoclinic integral flow solution can be made physically realizable by joining it with the transcritical non-homoclinic solution constructed through the outermost saddle type critical point \( r_{cwa}^{out} \) through a discontinuous shock transition since for non-dissipative inviscid flow two different transonic solution can not be smoothly connected to each other through any regular transition. In connection to the astrophysical flow. Such statement thus translates to the fact that no regular smooth stationary transonic solution can encounter more than one sonic point, and multi-transonic solution can only be realized when two different smooth transonic solutions can be connected through a stationary shock. The entropy accretion rate \( \bar{\Xi}_{in} \) for the accretion solution HJJ is greater than the entropy accretion rate \( \bar{\Xi}_{out} \) for the accretion solution ABB \( CDE \) subjected to the appropriate perturbative environment – a standing shock for which \( \Delta \bar{\Xi} = \left( \bar{\Xi}_{in} - \bar{\Xi}_{out} \right) \) amount of entropy accretion rate would be increased for the transonic solution passing through the outer sonic point B, would allow the aforementioned solution to make a discontinuous transition onto its subsonic homoclinic counterpart, i.e., the subsonic part of the transonic accretion solution HJJ. The combined multi-transonic shocked accretion solution would thus be consists of a segment (both subsonic and supersonic) of ABB \( CDE \) and a segment (both subsonic and supersonic) of HJJ connected by a discontinuous shock, the location of which is to be determined by solving certain set of algebraic equations.

\(^3\) A homoclinic orbit on a phase portrait is realized as an integral solution that re-connects a saddle type critical point to itself and embarrasses the corresponding centre type critical point. For a detail description of such phase trajectory from a dynamical systems point of view, see, e.g., Jordan & Smith (1999); Chicone (2006); Strogatz (2001). Subjected to the slightest possible perturbation, the heteroclinic loop opens up by forming a homoclinic orbit either through the inner saddle type point or through the outer saddle type point.
6.2 The relativistic shock of Rankine-Hugoniot type

In the present work, the first integrals of motion are the conserved specific energy and the mass accretion rate. For the non-dissipative inviscid accretion considered in our work, the shock produced is of energy preserving Rankine-Hugoniot (Rankine 1870; Hugoniot 1887a, b; Landau & Lifshitz 1987; Salas 2006) type. The corresponding shock thickness has to be negligibly small compared to any characteristic length scale of the flow to allow no dissipation of energy as a consequence of the strong temperature gradient in between the inner and the outer boundaries of the shock, where the terms ‘inner’ and the ‘outer’ are referred with respect to the proximity to the black hole event horizon.

For a perfect fluid, we formulate the general relativistic Rankine-Hugoniot condition to be

\[
\begin{align*}
[p u T'] &= 0, \\
[[T_{\mu\nu} \eta^\mu \eta^\nu]] &= 0, \\
[[T_{\mu\nu} \eta^\mu \eta^\nu]] &= 0,
\end{align*}
\]

(43)

where \( \Gamma_u = 1/\sqrt{1 - u^2} \) is the Lorentz factor and \( T_{\mu\nu} \) is the corresponding energy momentum tensor. In the above equation, \([[f]]\) denotes the discontinuity of any relevant physical quantity \( f \) across the surface of discontinuity, i.e., \([[f]] = f_2 - f_1\), where \( f_1 \) and \( f_2 \) are the boundary values of the quantity \( f \) on the two sides of such surface.

Simultaneous solution of eq. (43) yields the shock invariant quantity which changes continuously only across the shock surface. We obtain an analytical expression for such shock invariant quantity in terms of various local accretion variables and in terms of various initial boundary conditions describing the flow. During the numerical integration of the flow equations along the transonic solution ABB\( \cup \) CDE, we calculate the shock invariant. Simultaneously we calculate the same invariant while integrating the flow equations along the solution JIH starting from the inner sonic point up to the irregular sonic point on the homoclinic orbit (the point of inflection). We then determine the radial distance \( r_{sh} \) where the numerical values of the shock invariant quantity is evaluated by integrating the two different flow segments as described above becomes identical. For every \([E, \lambda, \gamma, a]\) allowing the formation of a stationary shock, one in general obtains two different values of \( r_{sh} \). Out of the two formal shock locations, the inner one (with reference to the proximity of \( r_{bh} \)) is always located in between the innermost and the middle sonic point, whereas the outer shock location is obtained in between the middle and the outermost sonic point. The shock strength \( M_{-}/M_{+} \) is different for these two shocks. Following the standard stability analysis procedure as provided in Yang & Kafatos (1995), one finds that the outer shock location is stable. Hereafter, we will refer to the stable outer shock location only whenever we use the phrase shock, and all shock related calculations will be performed with respect to that stable shock location only.

ABB\( \cup \) CDE\( \cup \) IJ on the \( M - \log_{10}(r) \) phase plane as shown in figure 1. We need to calculate the values of various accretion variables along this segment of the flow topology with information in mind that a sudden discontinuous transition for all such variables at the shock location has to be accounted for. We define the ‘pre-shock variables’ to be the value of any accretion variable at the shock location evaluated at the point C, and denote all such variables by a subscript ‘-’. Similarly, a ‘post shock variable’ is defined to be the value of the same variable starting from the inner sonic point up to the irregular sonic point on the homoclinic orbit (the point of inflection). We then determine the\( f \) variation profile can actually be demonstrated by the combined segment ABCC\( \cup \) IJ. Since a sudden change of the value of a flow variable is associated with the formation of a stationary shock in our model, a careful study of the radial profile of any accretion variable would provide a conclusive information about the appearance of a stationary shock. The radial variation of certain accretion variables (density, velocity, ion temperature etc.) are required to construct the observed spectra emergent from the accreting black hole, and the presence of shock can thus be inferred by investigating such spectral profile.

6.3 Shock induced discontinuous transition of flow variables

In figure 2, we show the variation of the advective velocity \( u \) scaled in units of \( 10^{10} \) cm/sec, the bulk ion temperature \( T \) scaled in units of \( 10^{10} \) degree Kelvin, flow density \( \rho \) in gm/cc and fluid pressure \( p \) in dyne/cm\(^2\) for a black hole with mass \( M_{BH} = 3.6 \times 10^6 M_{\odot} \) and accretion rate \( \dot{M} = XYZ \). \([E = 1.00001, \lambda = 2.6, \gamma = 1.43, a = 0.215]\) has been used as the four initial boundary conditions to set up the flow. B and I are the outermost and the innermost saddle type critical points, respectively, and CC\( \cup \) IJ indicates the shock transition. ABCE indicates the variation of the respective accretion variables along the transonic solution passing through the outer sonic point. If the shock would not form, the value of the respective variables would change continuously and monotonically, and the space evolution of such variables would be presented by the line segment ABCE. One could integrate the flow equations along the solutions passing through the outer sonic point up to the very close proximity of the event horizon and can obtain the value of the respective variable on ABCE at an extreme close vicinity of the black hole event horizon for a shock free solution. If, however, the shock forms, there will be a sudden change of the value of the respective variable and its \( r \) variation profile can actually be demonstrated by the combined segment ABCC\( \cup \) IJ. Once again, one can integrate the set of differential and the algebraic equations governing the flow up to the very close proximity of \( r_{bh} \) and the corresponding value of the respective variable at a radial distance nearly equal to \( r_{bh} \) can be obtained for a shocked multi-transonic integral solution.
7 DEPENDENCE OF THE SHOCK LOCATION AND THE SHOCK RELATED QUANTITIES ON $[\mathcal{E}, \lambda, \gamma, a]$

In this section we study the dependence of the shock location $r_{sh}$ and various pre and post shock values of the accretion variables on $[\mathcal{E}, \lambda, \gamma, a]$. To study such dependence on any particular parameter of the parameter set $[\mathcal{E}, \lambda, \gamma, a]$, on specific flow energy $\mathcal{E}$ for example, other three parameters $\lambda$, $\gamma$ and $a$ have to kept constant for the entire range of $\mathcal{E}$ for which such dependence is studied. Whereas a wide range of choice for $[\mathcal{E}, \lambda, \gamma, a]$ is available to produce a monotransonic accretion, only a limited non linear region of $[\mathcal{E}, \lambda, \gamma, a]$ allows the existence of a shocked multi-transonic flow. A continuous range of all $[\mathcal{E}, \lambda, \gamma, a]$ can not be used to construct such flow configuration since the Rankine Hugoniot condition is satisfied only for a small range of $[\mathcal{E}, \lambda, \gamma, a]_{\text{inac}} \in [\mathcal{E}, \lambda, \gamma, a]_{\text{mca}}$, where ‘mca’ stands for ‘multi-critical accretion with shock’ and ‘mca’ indicates the ‘multi-critical accretion’ in general. $[\mathcal{E}, \lambda, \gamma, a]_{\text{inac}} \subset [\mathcal{E}, \lambda, \gamma, a]_{\text{mca}}$, thus provides a true stationary multi-transonic accretion. We thus use various ‘patches’ of the region $[\mathcal{E}, \lambda, \gamma, a]_{\text{mca}}$ to study the dependence of the shock related entities on $[\mathcal{E}, \lambda, \gamma, a]$. One, however, needs to understand that such a choice of $[\mathcal{E}, \lambda, \gamma, a]$ will indeed provide the generic profile for the aforementioned dependence. Consider a set of fixed values $[\lambda_1, \gamma_1, a_1]$ to study the dependence of, say, the shock location $r_{sh}$ on the available (for which the shock forms, subjected to the fixed set $[\lambda_1, \gamma_1, a_1]$) range of the specific energy starting from $\mathcal{E}_{\text{min}}$ to $\mathcal{E}_{\text{max}}$. Such ‘$r_{sh} - \mathcal{E}$’ profile can also be explored for any other fixed set, say $[\lambda_2, \gamma_2, a_2]$ for which the Rankine Hugoniot condition gets satisfied, only with the obvious difference that the numerical values corresponding to $\mathcal{E}_{\text{min}}$ and $\mathcal{E}_{\text{max}}$ associated with the flow described by $[\lambda_2, \gamma_2, a_2]$ will be different as compared to the values of $\mathcal{E}_{\text{min}}$ and $\mathcal{E}_{\text{max}}$ corresponding to the initial boundary conditions as defined by $[\lambda_1, \gamma_1, a_1]$. Hence for any set of values $[\lambda, \gamma, a]$ for which the shock forms, the dependence of $r_{sh}$ on $\mathcal{E}$ can be studied. Similarly, the dependence of any shock related entity on any one of the initial boundary conditions $[\mathcal{E}, \lambda, \gamma, a]$ can be studied for a fixed set of values of the rest of the initial boundary conditions for which a stationary multi-transonic accretion configuration can be realized.

We observe that the shock location co-relates with the specific angular momentum $\lambda$ and anti-correlates with the specific energy $\mathcal{E}$ and the polytropic index $\gamma$. Such trends are independent of the black hole spin parameter, and hence remain the same for the maximally rotating Kerr as well as for a non rotating Schwarzschild black hole. The aforementioned dependence does not explicitly provides any information

![Figure 2](image-url). Variation of the advective flow velocity $u$ (upper left panel), rest mass density $\rho$ (upper right panel), flow ion temperature $T$ (lower left panel) and pressure (lower right panel) corresponding to the flow topology presented in the figure 1 and for a black hole with mass $3.6 \times 10^3 M_\odot$. The solid vertical line in each panel represents the shock transition and the labeling alphabets are in one to one correspondence with those used in figure 1, see text for further detail.
about the dependence of the shock related quantities on the nature of the space time metric\textsuperscript{5}. In this work, we are, however, mainly interested to study how the properties of the post shock flow at the close proximity of the event horizon are influenced by the spin parameter of the astrophysical black holes. Since such spin parameter determines the space time metric, our motivation is to study how the properties of

\textsuperscript{5} We are dealing with a non self gravitating accretion hence no back reaction is considered and the metric is determined exclusively by the properties of the black hole itself.
Das et. al. characterized by the nature of the black hole metric. Hence in the subsequent sections we study the dependence of the shock location as well as other shock related properties on the Kerr parameter in somewhat great detail.

7.1 Dependence of the shock location on black hole spin

As already argued, the entire range of the Kerr parameter, for the prograde as well as for the retrograde flow, can not be studied for any single fixed set \([E, \lambda, \gamma]\) since no such fixed set is available for which the Rankine-Hugoniot conditions are satisfied for the entire range of the Kerr parameters \([0 \geq a \geq 1]\), and \([-1 \geq a \geq 0]\), for the prograde as well as for the retrograde flow, respectively. This can easily be shown by plotting the region \([E, \lambda, \gamma, a]_{\text{ic}}\) embedded in the entire four dimensional hypersurface \([E, \lambda, \gamma, a]_{\text{ic}}\) to ensure that no single value of \([E, \lambda, \gamma]\) is available for which even a multi-critical solution, let alone a multi-transonic shocked solution, exists for the entire range of the Kerr parameter. We have chosen three different fixed sets \([E = 1.00001, \lambda = 2.6, \gamma = 1.43]\), \([E = 1.00001, \lambda = 2.17, \gamma = 1.43]\) and \([E = 1.00001, \lambda = 2.01, \gamma = 1.43]\) to cover a significant range of the low to moderately high (0.2 \(\lesssim a \lesssim 0.5\)), high (0.85 \(\lesssim a \lesssim 0.925\)), and very high (0.9655 \(\lesssim a \lesssim 0.99\)) values of the Kerr parameters, respectively, to study the dependence of \(r_{sh}\) on the black hole spin for prograde flow. Such values of \([E, \lambda, \gamma]\) is chosen to maximize the available range of the Kerr parameter (for which the shock forms) for three different ranges mentioned above, and to avail such range by minimally varying the initial configuration. For three different ranges, only the specific angular momentum \(\lambda\), that too by a considerably small amount, has been varied for each set of initial \([E, \lambda, \gamma]\), by keeping the subset \([E, \gamma]\) at its fixed value.

As a representative value, we take \([E = 1.00001, \lambda = 3.3, \gamma = 1.4]\) to cover a reasonably large range of the black hole spin for which the shocked multi-transonic accretion solution can be constructed for the retrograde flow. In figure 3, we plot the shock location \(r_{sh}\) as a function of the Kerr parameter \(a\) for there prograde flow with fixed set of \([E = 1.00001, \gamma = 1.43]\) and for three different values of \(\lambda\) (as shown in the figure) for three different ranges of the Kerr parameters (as mentioned in the figure) for which the multi-transonic shocked solutions can be constructed. We observe that the shock location non-linearly co-relates with the black hole spin for the prograde flow. Hence we infer that for similar initial conditions describing the flow, the shock forms closest to the event horizon for a Schwarzschild type black hole, and furthest to the event horizon for an extremely rotating Kerr black hole. We find the same ‘\(r_{sh} - \alpha\)’ profile for the retrograde flow as well. In figure 4 the shock location is plotted against the black hole spin for the retrograde flow characterized by \([E = 1.00001, \lambda = 3.3, \gamma = 1.4]\).

7.2 Dependence of shock induced flow variables on black hole spin

We would like to study how the characteristic dynamical and the thermodynamic features of the post shock flow are influenced by the black hole spin. One way of looking at this problem is to study the ratio of the pre (post) to the post (pre) shock Mach number for various accretion variables, since such ratio is a marker of how the presence of a stationary shock introduces a sudden change in the value of the flow variables which in turn make a observable difference in the characteristic black hole spectra. For any flow variable \(f, f_\text{post}\) denotes its pre shock value evaluated at the shock location on the transonic solution passing through the outer sonic point and \(f_\text{post}\) denotes its post shock value evaluated at the shock location on the transonic solution constructed through the inner sonic point. For prograde flow characterized by \([E = 1.00001, \lambda = 2.6, \gamma = 1.43]\), in figure 5 we plot the ratio of the pre to the post shock Mach number (\(M_{\text{post}}/M_{\text{pre}}\)), and post to the pre shock flow temperature (\(T_+/T_-\)), density (\(\rho_+/\rho_-\)) and pressure (\(p_+/p_-\)), respectively. (\(M_{\text{post}}/M_{\text{pre}}\)) is termed as the shock strength as mentioned earlier and (\(\rho_+/\rho_-\)) is termed as the shock compression ratio. The shock strength anti-correlates with the shock location. The
closer the shock forms to the event horizon, the higher the gravitational potential energy liberated resulting the formation of a stronger shock. A strong shock also compresses the flow by a considerable amount. As a result (since the shock as well as the flow under consideration is assumed to be energy preserving) the temperature and the pressure of the flow also increases. Hence \([(p_+/p_-), (T_+/T_-), (p_+/p_-)]\) anti-correlates with the shock location and hence with the black hole spin angular momentum (the Kerr parameter \(a\)).

Figure 5. Variation of the shock strength (upper left panel), the shock compression ratio (upper right panel), the ratio of the post to the pre shock temperature (lower left panel) and the ratio of the post to the pre shock pressure (lower right panel) with the Kerr parameter \(a\) (for each panel plotted along the abscissa) for the prograde flow characterized by \([\mathcal{E} = 1.00001, \lambda = 2.6, \gamma = 1.43]\).

In this work, the numerical value of any accretion variable \(V\) evaluated at a very close proximity \(r_\delta = r_+ + \delta (r_+ = 1 + \sqrt{1 - a^2}, and \delta \) being a small number lying within the open interval \(0 < \delta << 1\)) of the event horizon is dubbed as the corresponding “quasi terminal value” of \(V\), and is distinguished by a subscript \(\delta\). The quasi-terminal value \(V_\delta\) is computed by integrating the flow equations (along the stationary transonic branch) from the critical point \(r_c\) down to \(r_\delta\). We take \(\delta = 0.001GM_{BH}/c^2\) and perform our calculation for \(V_\delta\) for a \(3 \times 10^6 M_\odot\) black hole accreting at a rate of \(4.29 \times 10^{-6} M_\odot\) yr\(^{-1}\). Such values of \(M_{BH}\) and \(\dot{M}\) corresponds to our Galactic centre black hole and its environment where low angular momentum inviscid advective accretion model is considered as an appropriate approximation (see, e.g., Moscibrodzka, Das & Czerny (2006) and references therein). Also to be noted that \(M_{BH}\) and \(\dot{M}\) used in this work are two representative values, and any other value for the black hole mass as well as for the accretion rate can be considered for our calculation of \(V_\delta\).

For any generic flow variable \(V\) we calculate the corresponding \(V_\delta\) along two branches, either along the solution ABB\(_1\)CC\(_1\)IJ for a multi-transonic shocked accretion flow, or for a hypothetical shock free solution ABB\(_1\)CDE passing through the outer sonic point only. For the same set of initial boundary condition \([\mathcal{E}, \lambda, \gamma, a]\), the dependence \(V_\delta^{\text{shock}}\) and \(V_\delta^{\text{no shock}}\) on the black hole spin angular momentum as well as on \([\mathcal{E}, \lambda, \gamma]\) can be studied to estimate the impact of the shock formation phenomena in determining the properties of the matter.
extremely close to the black hole. This in turn will allow us to infer the influence of the shock formation on the observable spectra generated by the photon flux emanating out from the region inside the ISCO.

For co-rotating flow, in figure 7 we plot the variation of the quasi-terminal values of the Mach number ($M_\delta$, the top left panel), flow density ($\rho_\delta$ in CGS units, top right corner), bulk ion temperature ($T_\delta$, in units of degree Kelvin, bottom left panel) and pressure ($p_\delta$ in CGS unit, bottom right panel) respectively, for both shocked solution (solid red line) and for the shock free solution (dashed blue line) on the black hole spin for multi-transonic flow characterized by $[\delta = 1.00001, \lambda = 2.6, \gamma = 1.43]$. Similar relational dependence are presented in figure 8 and figure 9 for other ranges of the Kerr parameter for which multi-transonic shocked flow can be described by $[\delta = 1.00001, \lambda = 2.17, \gamma = 1.43]$ and $[\delta = 1.00001, \lambda = 2.01, \gamma = 1.43]$, respectively. For the same set of $[\delta, \lambda, \gamma, a]$, we find that $M_\delta^\text{shock} < M_\delta^\text{no shock}$ and $\rho_\delta^\text{shock} < \rho_\delta^\text{no shock}$, whereas $T_\delta^\text{shock} > T_\delta^\text{no shock}$ and $p_\delta^\text{shock} > p_\delta^\text{no shock}$.

From figure 1, one observes that for any $r < r_{\text{in}}^\gamma$, the value of Mach number evaluated along the transonic solution passing through the outer sonic point is always greater than that evaluated on the transonic solution passing through the inner sonic point. At the shock, the Mach number decreases discontinuously, and although the value of the Mach number shoots up at a very high rate (space gradient of the Mach number, i.e., $dM/dr$, becomes large), the Mach number for the supersonic flow in the post shock region can never exceed the value of the Mach number associated with the supersonic segment of the shock free solution at any $r$ since in that case the post shock supersonic flow would have to intersect the shock free supersonic solution on $'M - log_{10}(r)'$ plane. Such a crossover is not allowed since no two phase topology can intersect on a phase plane. Hence what actually one observes is $(dM/dr)^{\text{shock}} > (dM/dr)^{\text{no shock}}$ but the trend $M_\delta^{\text{shock}} < M_\delta^{\text{no shock}}$ is maintained. At the extreme close proximity of the event horizon ($r_\delta << 0.001$) the post shock supersonic branch asymptotically approaches the shock free supersonic branch, hence $M_\delta^{\text{shock}} \rightarrow M_\delta^{\text{no shock}}$ for such an extremely small value of $r_\delta$. Nevertheless, the criteria $M_\delta^{\text{shock}} - M_\delta^{\text{no shock}} \neq 0$ remains valid for for all values of $r_\delta$, however small $r_\delta$ can be made.

Similar situation is observed for the variation of $\rho_\delta$ with the spin as well. Although the density increases at the shock, close to $r_+$, the flow density corresponding to the shocked flow makes a crossover with the density profile corresponding to the shock free transonic flow, and starts decreasing gradually as has been observed in figure 2. For the flow temperature and pressure, no such crossover takes place and hence the trend $T_\delta^{\text{shock}} > T_\delta^{\text{no shock}}$ and $p_\delta^{\text{shock}} > p_\delta^{\text{no shock}}$ are uniformly maintained, see, e.g., figure 2. Similar to $[M_\delta, \rho_\delta, T_\delta, p_\delta] - a'$ profile is observed for the retrograde flow as well, see, e.g., figure 10 for such variations for the counter rotating accretion.

Figure 6. Variation of the shock strength (upper left panel), the shock compression ration (upper right panel), the ratio of the post to the pre shock temperature (lower left panel) and the ratio of the post to the pre shock pressure (lower right panel) with the Kerr parameter $a$ (for each panel plotted along the abscissa) for the retrograde flow characterized by $[\delta = 1.00001, \lambda = 3.3, \gamma = 1.4]$. 

![Figure 6](image-url)
Accretion close to event horizon

Figure 7. Variation of the quasi-terminal Mach number (upper left panel), quasi-terminal density (upper right panel), quasi-terminal temperature (lower left panel) and the quasi-terminal pressure (lower left panel) with the Kerr parameter $a$ (plotted along the abscissa) for shocked (dashed red line) and for the hypothetical shock free (solid black line) prograde flow characterized by $[\mathcal{E} = 1.00001, \lambda = 2.6, \gamma = 1.43]$.

Figure 8. Variation of the quasi-terminal Mach number (upper left panel), quasi-terminal density (upper right panel), quasi-terminal temperature (lower left panel) and the quasi-terminal pressure (lower left panel) with the Kerr parameter $a$ (plotted along the abscissa) for shocked (dashed red line) and for the hypothetical shock free (solid black line) prograde flow characterized by $[\mathcal{E} = 1.00001, \lambda = 2.17, \gamma = 1.43]$. 
Figure 9. Variation of the quasi-terminal Mach number (upper left panel), quasi-terminal density (upper right panel), quasi-terminal temperature (lower left panel) and the quasi-terminal pressure (lower right panel) with the Kerr parameter $a$ (plotted along the abscissa) for shocked (dashed red line) and for the hypothetical shock free (solid black line) prograde flow characterized by $[\mathcal{E} = 1.00001, \lambda = 2.01, \gamma = 1.43]$.

Figure 10. Variation of the quasi-terminal Mach number (upper left panel), quasi-terminal density (upper right panel), quasi-terminal temperature (lower left panel) and the quasi-terminal pressure (lower right panel) with the Kerr parameter $a$ (plotted along the abscissa) for shocked (dashed red line) and for the hypothetical shock free (solid black line) retrograde flow characterized by $[\mathcal{E} = 1.00001, \lambda = 3.3, \gamma = 1.4]$. 
8.1 Dependence of quasi-terminal values on $[E, \lambda, \gamma]$

We also study the dependence of the quasi-terminal values on the specific energy $E$ (figure 11), specific angular momentum $\lambda$ (figure 12), and the flow adiabatic index $\gamma$ (figure 13). For all such cases the following trend is found:

$$M_{\delta}^{\text{shock}} < M_{\delta}^{\no\text{shock}}, \rho_{\delta}^{\text{shock}} < \rho_{\delta}^{\no\text{shock}}, T_{\delta}^{\text{shock}} > T_{\delta}^{\no\text{shock}}, p_{\delta}^{\text{shock}} > p_{\delta}^{\no\text{shock}}$$

as has been observed for the black hole spin dependence of $[M_{\delta}, \rho_{\delta}, T_{\delta}, p_{\delta}]$.

9 SPIN DEPENDENCE OF QUASI-TERMINAL VALUES FOR THE STATIONARY MONO-TRANSONIC ACCRETION

It is instructive to investigate whether the characteristic feature of the black hole spin dependence of the quasi-terminal values remain invariant for a direct spin flip of the astrophysical black hole. In other words, we would like to understand whether the solutions when $a = 0$ to $[E, \lambda, \gamma, a]$ for which a stationary transonic accretion solution can be constructed for the entire range of the black hole spin, i.e., for $-1 \leq a \leq 1$. A considerably slowly rotating substantially hot and almost purely non relativistic flow allows us to study the black hole spin dependence of the quasi-terminal values for the entire range of the Kerr parameters describing both the prograde and the retrograde accretion. For such flow configurations, we obtain that the critical as well as the sonic points are always of the innermost type saddle one. The location of the critical and the sonic points are found to be at close proximity of $r_{c}$ for the prograde accretion onto a maximally rotating hole and are formed at the maximally allowed distance of approximately $(4 - 4.5) GM_{BH}/c^2$ unit for the retrograde accretion onto maximally rotating hole with the negative value of the spin parameter. The value of $\Delta r_{c}^* = (r_{c} - r_{s})$ thus comes out to be minimum for $a \rightarrow 1$ and maximum for $a \rightarrow -1$. For mono-transonic accretion, in figure 14 we show the dependence of $[M_{\delta}, \rho_{\delta}\epsilon\lambda, T_{\delta}, p_{\delta}]$ on the black hole spin parameter for the entire range of $-1 \leq a \leq 1$. 

![Figure 11. Variation of the quasi-terminal Mach number (upper left panel), quasi-terminal density (upper right panel), quasi-terminal temperature (lower left panel) and the quasi-terminal pressure (lower left panel) with the specific flow energy $E$ (plotted along the abscissa) for shocked (dashed red line) and for the hypothetical shock free (solid black line) prograde flow characterized by $[\lambda = 2.17, \gamma = 1.43, a = 0.881049812]$.](image-url)
Figure 12. Variation of the quasi-terminal Mach number (upper left panel), quasi-terminal density (upper right panel), quasi-terminal temperature (lower left panel) and the quasi-terminal pressure (lower left panel) with the specific flow angular momentum $\lambda$ (plotted along the abscissa) for shocked (dashed red line) and for the hypothetical shock free (solid black line) prograde flow characterized by $[\varepsilon = 1.000004, \gamma = 1.43, a = 0.881049812]$.

Figure 13. Variation of the quasi-terminal Mach number (upper left panel), quasi-terminal density (upper right panel), quasi-terminal temperature (lower left panel) and the quasi-terminal pressure (lower left panel) with the adiabatic index $\gamma$ (plotted along the abscissa) for shocked (dashed red line) and for the hypothetical shock free (solid black line) prograde flow characterized by $[\varepsilon = E = 1.000004, \lambda = 2.28, a = 0.881049812]$.
10 LINEAR STABILITY ANALYSIS OF THE STATIONARY SOLUTION

In this work, our entire analysis of the multitransonic flow at the vicinity of the hole is based on the phase space behaviour of the stationary solutions. It is, however, important to ensure that such stationary configurations are stable as well, at least up to the limit of astrophysically relevant time scales. This can be accomplished by studying the time evolution of a linear acoustic-like perturbation applied around a stationary configuration. We will first demonstrate that the axisymmetric accretion can be considered as potential flow and will thus identify the corresponding velocity potential. Next we perturb such velocity potential and will prove that such perturbation will not diverge to destabilize the original stationary solution of our interest. In the following part we present the analysis for irrotational, entropy conserving flow solutions in general, which necessarily includes the stationary ones.

From Thermodynamics,
\[
\frac{dh}{d\omega} = T\frac{\partial}{\partial \rho} + \frac{dp}{\rho} \tag{45}
\]

For polytropic flow along a specified streamline, we have from eq. (22),
\[
h\nu^\mu (\rho v^\nu)_{,\nu} + \rho v^\nu (h\nu^\mu)_{,\nu} + p_{,\mu}g^{\mu\nu} = 0. \tag{46}
\]

Since due to specific entropy conservation along a streamline, \(dh = \frac{d\omega}{\rho}\), from eq. (23) one obtains,
\[
v^\nu (h\nu^\mu)_{,\nu} + \partial_{\mu} h = 0. \tag{47}
\]

We define \(\Omega_{\alpha\beta} \equiv P_{\mu}^\alpha P_{\beta}^\nu \omega_{\mu\nu}\) to be the vorticity of the flow, where \(\omega_{\mu\nu} \equiv (h\nu^\mu)_{,\nu} - (h\nu^\nu)_{,\mu}\), and \(P_{\alpha}^\beta \equiv \delta_{\alpha}^\beta + v_{\alpha} v^\beta\) is the projection tensor.

Since \(\Omega_{\alpha\beta} = 0\) for irrotational flow, we obtain
\[
\omega_{\alpha\beta} + v_{\beta} v^\nu \omega_{\alpha\nu} + v_{\alpha} v^\mu \omega_{\mu\beta} + v_{\alpha} v^\mu v^\nu v_{\beta} \omega_{\mu\nu} = 0. \tag{48}
\]

The 2nd, 3rd and the 4th terms in the above expression vanish owing to the relation \(v^\mu v_{\mu} = -1\) and by virtue of eq. (47). We thus obtain \(\omega_{\alpha\beta} = 0\) which implies
\[
(h\nu^\alpha)_{,\beta} - (h\nu^\beta)_{,\alpha} = 0. \tag{49}
\]

eq. (49) indicates that there exists a 4-scalar \(\Psi\) such that,
\[
h\nu^\alpha = -\partial_{\alpha} \Psi. \tag{50}
\]
Ψ is thus the velocity potential of the flow we consider in this work.

Solutions (subjected to different initial boundary conditions) describe the vorticity free polytropic flow. Evidently the flow is entirely determined by the solutions corresponding to the velocity potential Ψ. To analyse the stability of such solutions we may introduce small perturbations on the solutions which satisfy the eq. (50) and investigate whether such perturbations may eventually grow to mask the original solutions. If such perturbation does not diverge, the stationary solutions are proved to be stable within our framework.

Applying small perturbation on the background values of the flow variables as, \( h \rightarrow h + \delta h, \rho \rightarrow \rho + \delta \rho, \nu_\mu \rightarrow \nu_\mu + \delta \nu_\mu \), we obtain,

\[
h \nu_\mu + \delta h \nu_\mu + h \delta \nu_\mu = \partial_\mu \Psi + \partial_\mu \delta \Psi.
\]

Perturbation of the eq. (23) provides,

\[
(\delta \rho \nu\nu)_{,\mu} + (\rho \delta \nu\nu)_{,\mu} = 0.
\]

Using the relation, \( \nu\nu \delta \nu_\mu = 0 \), and denoting \( \delta \Psi \) by \( \tilde{f} \), we obtain,

\[
\partial_\mu \left( -\frac{\rho}{h} g^{\mu\nu} \left[ g^{\mu\nu} - \left\{ 1 - \frac{1}{c_s^2} \right\} \nu\nu \nu\nu \right] \partial_\nu \tilde{f} \right) = 0.
\]  
(51)

For axisymmetric flow configuration as considered in our work,

\[
\nu\nu = (\nu^t, \nu^r, 0, \nu^\phi)
\]

and from radial propagation of perturbation we obtain,

\[
\partial_r \tilde{f} \equiv (\partial_t \tilde{f}, \partial_r \tilde{f}, 0, 0)
\]

Substituting these conditions into the eq. (51), it reduces to,

\[
\partial_\mu (f^{\mu\nu} \partial_\nu \tilde{f}) = 0;
\]  
(52)

where \( \mu \) and \( \nu \) run for 0 and 1 and

\[
f^{\mu\nu} = \sqrt{-g} \frac{\rho}{h} \left[ g^{\mu\nu} - \left\{ 1 - \frac{1}{c_s^2} \right\} \nu\nu \nu\nu \right].
\]

eq. (52) is the equation which determines the time evolution of the first order linearly perturbed velocity potential (see Moncrief (1980); Bilić (1999)).

Substitution of the trial acoustic wave solution of the form \( \tilde{f} = \tilde{f}^\omega \exp(-i\omega t) \) into the equation (52) yields,

\[
(-\omega^2) f^{tt} \tilde{f}^\omega + (-i\omega) \left[ f^{tr} \tilde{f}^r + f^{rt} \tilde{f}^r + f^{rr} \tilde{f}^r + f^{tt} \tilde{f}^t \right] + \left[ f^{tr} f^{r\omega} + f^{rr} f^{r\omega} \right] = 0.
\]

The space dependent part \( \tilde{f}^\omega \) is expressed in terms of the trial power series of the following form,

\[
\tilde{f}^\omega(r) = \exp \left[ \sum_{n=1}^{\infty} \frac{k_n(r)}{\omega^n} \right],
\]

and it is examined whether the solution is bounded within the finite limit at the outer boundary as \( r \rightarrow \infty \). Collecting the coefficients of the same power of \( \omega (\omega \gg 1) \) together we obtain,

\[
\text{for } \omega^2 \text{ containing terms, } - f^{tt} - 2i f^{tr} \frac{dk_{-1}}{dr} + f^{rr} \left( \frac{dk_{-1}}{dr} \right)^2 = 0,
\]  
(53)

\[
\text{for } \omega^1 \text{ containing terms, } - 2i f^{tr} \frac{dk_0}{dr} - \frac{df^{tr}}{dr} \frac{dk_{-1}}{dr} + f^{rr} \left[ 2 \frac{dk_{-1}}{dr} \frac{dk_0}{dr} + \frac{d^2 k_{-1}}{dr^2} \right] = 0,
\]  
(54)

\[
\text{for } \omega^0 \text{ containing terms, } - 2i f^{tr} \frac{dk_1}{dr} + \frac{df^{tr}}{dr} \frac{dk_0}{dr} + f^{rr} \left[ 2 \frac{dk_{-1}}{dr} \frac{dk_1}{dr} + \left( \frac{dk_0}{dr} \right)^2 + \frac{d^2 k_0}{dr^2} \right] = 0.
\]  
(55)

The leading order coefficients turn out to be (from eq. (53)),

\[
k_{-1} = \int f^{tr} \pm \sqrt{(f^{tr})^2 - f^{rr} f^{tt}} \frac{dr}{f^{rr}}
\]  
(56)

and substituting back into the eq. (54) we obtain,

\[
\frac{dk_0}{dr} = \frac{df^{tr} \frac{dk_{-1}}{dr} + f^{rr} \frac{d^2 k_{-1}}{dr^2} - i df^{tr}}{2i f^{tr} - 2f^{rr} \frac{dk_{-1}}{dr}} = \frac{\frac{d}{dr} \left( i f^{tr} \frac{f^{tr} \pm \sqrt{(f^{tr})^2 - f^{rr} f^{tt}}}{f^{rr}} - i f^{tr} \right)}{2i f^{tr} - 2f^{rr} \frac{dk_{-1}}{dr}}
\]

which, upon further simplification becomes

\[
\frac{dk_0}{dr} = -\frac{1}{2} \left( \pm \sqrt{(f^{tr})^2 - f^{rr} f^{tt}} \right)
\]  
(57)
Hence we find,

\[ k_0 = -\frac{1}{2} \ln \left( \sqrt{(f^{tt})^2 - f^{rr} f^{tt}} \right) \]  

(58)

From eq. (55) we obtain

\[ \frac{dk_1}{dr} = \frac{\frac{df^{rr}}{dr} \frac{dk_0}{dr} + f^{rr} \left( \frac{dk_0}{dr} \right)^2 + f^{rr} \frac{d^2 k_0}{dr^2}}{2 \left( f^{tt} - f^{rr} \frac{dk_0}{dr} \right)} \]  

(59)

From eq. (52) using expressions of contravariant metric elements as defined on the equitorial plane in Eqs. (15), one obtains,

\[ f^{tt} = \sqrt{-g} \frac{A}{\rho \Delta} \left( 1 - \frac{1}{c_s^2} \right) (\gamma_L^2 - 1) \]  

(60a)

\[ f^{rr} = \sqrt{-g} \frac{\Delta}{\rho r^2} \left( 1 - \frac{1}{c_s^2} \right) \left( u^2 + 1 \right) \]  

(60b)

\[ f^{rt} = \sqrt{-g} \frac{\gamma L \sqrt{A}}{r^2} \left( 1 - \frac{1}{c_s^2} \right) \frac{u}{1 - u^2} \]  

(60c)

It is easy to see that in the asymptotic limit \( r \to \infty, \Delta \sim r^2 \) and \( A \sim r^4 \). At that limit \( \rho \) tends to a constant ambient value, denoted by \( \rho_\infty \) and subsequently \( h \) tends to its ambient value \( h_\infty \), as also \( c_s \) tends to some ambient value \( c_s\infty \). The Lorentz factor \( \gamma_L \) tends to unity for accretion at the outer boundary condition.

To find out the asymptotic behaviour of \( u \), we make use of the eq. (31). It turns out that

\[ \frac{u}{\sqrt{1 - u^2}} \sim \frac{1}{\rho \mathcal{A}} \]

where \( \mathcal{A} \) is \( 4\pi H_z r \) for the flow considered within the framework of the cylindrical symmetry. It is to be noted that \( H_z \) is constant for the accretion with constant flow thickness. For axisymmetric accretion in vertical equilibrium,

\[ H_z = \sqrt{\frac{2}{\gamma + 1}} \frac{1}{r^2} \left( \frac{(\gamma - 1)c_s^2}{\gamma - (1 + c_s^2) \Delta_{0} \mathcal{A}} \right)^{\frac{\Delta}{\rho \mathcal{A}}} \sim r^2. \]

On the other hand for conical model described within the framework of spherical symmetry for \( -H_\theta \leq \theta \leq H_\theta \), the area \( \mathcal{A} \) is \( 4\pi H_\theta r^2 \), where \( H_\theta \) is constant for a disc. Hence for all the flow configurations at asymptotic limit \( r \to \infty, \mathcal{A} \sim r^\alpha \) where \( \alpha \geq 1 \). Thus,

\[ \frac{u}{\sqrt{1 - u^2}} \sim \frac{1}{r^\alpha}, \]

at this asymptotic limit.

Now it becomes apparent from equations (60) that,

\[ f^{tt} \sim \sqrt{-g} \]  

(61a)

\[ f^{rr} \sim \sqrt{-g} \]  

(61b)

\[ f^{rt} \sim \frac{\sqrt{-g}}{r^3} \]  

(61c)

Hence substituting these expressions (61) into eqs. (56), (58) and (59) one readily obtains that \( k_{-1} \sim r, k_0 \sim \ln r \) and \( k_1 \sim 1/r \) at the asymptotic limit. Hence for the first three terms in the trial power series solution for the space dependent part of the perturbation it is evident that \( \omega/|K_{-1}| \gg |k_0| \gg |k_1|/\omega \) for the high frequency regime at the asymptotic solution. This indicates that \( \omega^\alpha |k_\alpha| \gg \omega^{\alpha+1} |k_{\alpha+1}| \) or in other word the power series converges even at the outer boundary of the flow ruling out any divergence of any possible perturbing component of \( \Psi \) from the solution of eq. (50) for a particular set of boundary conditions.

In this work, we study how the stationary accretion solutions in the Kerr metric behaves close to the black hole horizon. The dependence of such behaviour on the black hole spin is also studied. By employing suitable stability analysis above we ensure that such stationary solutions (which constitute a subcategory of the potential flow, in general) are stable, and hence any spectral profile which might be constructed out of those solutions are reliable, at least upto an astrophysically relevant time scale.

### 11 Discussion

Majority of the works on the role of the black hole spin in influencing the accretion dynamics are focused on high angular momentum disk-like flows, with the central role payed by the Innermost Stable Circular Orbit (ISCO). However, SgrA* and M87 are the two most preferred candidates for direct imaging of the flow close to the black hole horizon since the angular size of the black hole horizon is by far the largest in these two sources due to the interplay between the black hole mass and the distance to the source is measured by the observer. The exact value of the angular momentum of the inflowing material for the aforementioned two sources is difficult to estimate, and the current evaluations indicate values ranging from moderate (Cuadra et. al. 2008) to considerably low (Moscibrodzka, Das & Czerny 2006).
The present constraints inferred on the unresolved components in these two sources in the mm bands are already very impressive (Doeleman et al. 2008, 2012). In the future VLBI Event Horizon Telescope will bring us still much closer to the central compact object for these sources (Doeleman 2010). In order to understand the salient features of these images as well as the properties of the corresponding broadband radiation spectra we need to predict the emissivity distribution and construct the expected black hole silhouettes for various models using the ray tracing techniques as described in (Luminet 1979; Fukue & Yokoyama 1988; Karas et. al. 1992; Falcke, Melia & Agol 2000; Takahashi 2004; Huang, Cai, Shen & Yuan 2007).

Sizable amount of work in this direction has been performed for high angular momentum flows and for ion tori (e.g. Vincent et al. (2012)). The results given in our present paper paper form a starting point for complementary study for the case of low angular momentum accretion. In our next work (Huang and Das, in preparation), we plan to perform the black hole shadow imaging corresponding to the low angular momentum axisymmetric accretion as considered in this work.

In our present work we have studied the radial ion temperature profile as a function of the Kerr parameter, as well as the dependence of the corresponding $T_\delta$ profile on the black hole spin. Our ongoing calculation concentrates on the calculation of the electron temperature from such ion temperature. Knowledge of such electron temperature, along with the density and the velocity profile as calculated in our present work, will then provide us the complete knowledge of the emitted polarized radiation in the millimeter and sub-millimeter band. We also plan to study the influence of shock formation on the polarized emission of SgrA* using our flow model.

Findings as illustrated in Fig. 14 indicate a consistent asymmetry between the prograde and the retrograde flow as far as the quasi-terminal values are concerned. The constructed shadow image is also expected to manifest the asymmetry. We thus expect to propose a novel method to differentiate the co-rotating and counter-rotating flow once we construct the corresponding spectra and the associated shadow images out of the accretion variables as calculated using our flow model. The observations of Sgr A* and M87 by Doeleman et al. (2008, 2012) seem to rule out counter-rotating flow since the images are in both cases much smaller than ISCO. However, the argument applies only withing the frame of disk-like accretion. Low angular momentum flow is much less influenced by the position of the ISCO and the emissivity is more concentrated towards the center as perceived in a spherically symmetric flow. Therefore, the construction of the predicted images in the case of low angular momentum flows requires urgent attention as we believe.

ACKNOWLEDGMENTS

SH and IM would like to acknowledge the kind hospitality provided by HRI, Allahabad, India, under a visiting student research programme. The visits of PB, SN and TN at HRI was partially supported by astrophysics project under the XIth plan at HRI. The work of TKD has been partially supported by a research grant provided by S. N. Bose National Centre for Basic Sciences, Kolkata, India, under a guest scientist (long term sabbatical visiting professor) research programme, as well as is partially funded by the astrophysics project under the XI th plan at HRI. PB acknowledges support from the ERC Starting Grant “cosmoIGM”.

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