Error-Feedback Output Regulation of Linear Stochastic Systems: a Hybrid Nonlinear Approach

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Abstract: The problem of output regulation for linear stochastic systems is addressed. We first define and solve the ideal problem of output regulation via error feedback. We note that its solution is not implementable in practice because the Brownian motion is not available for measure. Therefore, we define an approximate problem for which we provide a practical solution. The implemented controller is hybrid, in that a continuous-time, deterministic control law is supplemented by a discrete-time, stochastic correction. This correction is performed using an a-posteriori approximation of the variations of the Brownian motion provided by a nonlinear estimator. The resulting hybrid closed-loop system is nonlinear, as the scalars approximating the increments of the Brownian motion depend nonlinelyrly on the states and the inputs. The error between the solution of the approximate problem and the solution of the ideal problem is characterised. We show that the ideal solution is retrieved as the sampling time tends to zero. We illustrate the results by means of a numerical example.

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1. INTRODUCTION

Output regulation is a classical control problem consisting in the design of a controller that attains closed-loop asymptotic stability and regulation. In particular, it is required that the closed-loop system tracks reference signals or rejects disturbances generated by a known exogenous system. In the 1970s-80s extensive studies were performed in the framework of linear deterministic systems, which led to a thorough understanding of its solution, see Davison (1976), Francis and Wonham (1976), Francis (1977). The nonlinear counterpart of the problem was addressed and solved for the first time in Isidori and Byrnes (1990), whilst further developments were achieved in, e.g., Serrani et al. (2001), Huang (2004) and Pavlov et al. (2006). Other case studies, involving wider classes of dynamical systems, have been addressed in, e.g., Ichikawa and Katayama (2006), Marconi and Teel (2013), Carnevale et al. (2016).

In this paper we study the problem of output regulation in the context of stochastic dynamical systems, i.e., systems that are modelled by a set of stochastic differential equations. Stochastic systems are commonly used to model uncertain processes. In fact, it is easy to incorporate model uncertainties in the dynamics of the system by introducing random coefficients and noise. Applications can be found in several fields, from production planning, finance, technology diffusion, research funding to classical control problems like the optimal stopping, see Yong and Zhou (1999), Øksendal (2003).

While control of stochastic systems is an active research area (see e.g. Krstic and Deng (1998), Yong and Zhou (1999)), output regulation remains an open problem and a systematic solution has not yet been achieved.

Output regulation applied to linear stochastic systems has been addressed in the preliminary work by Scarciotti (2018); in that paper the description of the steady-state of a linear stochastic system is provided (see also Scarciotti and Teel (2018) for a more general result). Therein, the full-information case is dealt with and solved in an ideal scenario where, not only the controlled state and the state of the exogenous system are available for feedback, but also the Brownian motion is assumed to be known. The practical impossibility of this approach was then discussed in Mellone and Scarciotti (2019b). Therein, the authors propose a hybrid control law that provides an approximate solution to the full-information problem, thus overcoming the limitation of Scarciotti (2018).

The goal of this paper is twofold. On the one hand, we aim at solving the error-feedback output regulation problem in the ideal case, i.e. assuming that the Brownian motion is available for measure. On the other hand, we also seek a practically sound solution to the problem. We achieve this by proposing an approximate hybrid framework. To this end, we first design a nonlinear estimator that generates a sequence of scalars that asymptotically approximate a posteriori the variations of the Brownian motion. This approximation improves as the sampling period is made smaller. Then, we use this estimate to implement a hybrid dynamic regulator that solves the approximate problem and stabilises the resulting nonlinear closed-loop system. The difference between the ideal and the approximate solution is characterised and shown to tend to zero as the sampling period and/or the amplitude of the exogenous signal tend to zero.

Due to the page limitation, we omit the proofs of the results. These are available in the extended preprint Mellone and Scarciotti (2019a).

The rest of the paper is organised as follows. In Section 2 we report some preliminaries. In Section 3 we formulate the ideal error-feedback output regulation problem; we
solve the problem and point out that its solution cannot be implemented in practice. Therefore, in Section 4 we define a new approximate problem, we show how to construct an asymptotic approximation of the variations of the Brownian motion and we provide an implementable solution that approximates the ideal one. In Section 5 we illustrate the validity of the theory by means of a numerical simulation. Finally, Section 6 contains some concluding remarks.

Notation. The symbol $\mathbb{Z}$ denotes the set of integer numbers, while $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively; by adding the subscript \(\mathbb{R}_0^+\) (\(\mathbb{C}_0^+\)) to any symbol indicating a set of numbers, we denote that subset of numbers with negative (non-negative, zero) real part. If a function $g$ experiences a jump variation at time $t$, the symbols $g(t^-)$ and $g(t^+)$ denote the value of $g$ immediately before and after the jump, respectively. The identity matrix is denoted by $I$. $A^T$ indicates the transpose of $A$. The symbol $\otimes$ denotes the Kronecker product. $(\nabla, A, \mathcal{P})$ is a probability space given by the set $\nabla$, the $\sigma$-algebra $A$ defined on $\nabla$ and the probability measure $\mathcal{P}$ on the measurable space $(\nabla, A)$. $E[X]$ denotes the expected value of the random variable $X : \nabla \to (\nabla, A)$ (Arnold, 1974, Section 1.3). A stochastic process with state space $\mathbb{R}^n$ is a family \(\{x_t, t \in \mathbb{R}\}\) of $\mathbb{R}^n$-valued random variables, i.e. for every fixed $t$, $x_t(\cdot)$ is an $\mathbb{R}^n$-valued random variable and, for every fixed $w$ in $\nabla$, $x_t(w)$ is an $\mathbb{R}^n$-valued function of time (Arnold, 1974, Section 1.8). For ease of notation, we often indicate a stochastic process \(\{x_t, t \in \mathbb{R}\}\) simply with $x_t$ (this is common in the literature, see e.g. Arnold (1974)). The symbol $W_t$ indicates a standard Wiener process, also referred to as Brownian motion. This denotes an auxiliary output for state-estimation purposes (Arnold, 1974, Chapter 3). The stochastic integrals, and thus the solution of stochastic differential equations, are meant in Rö’s sense.

2. PRELIMINARIES

Consider the linear stochastic single-input single-output system

$$
\begin{align*}
&dx_t = (Ax_t + Bu + P\omega)dt + (Fx_t + Gu + R\omega)dW_t, \\
&y^i_t = C_x x_t + Du, \\
&e_t = y^i_t + Q\omega,
\end{align*}
$$

(1)

where $x_t \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the control input, $\omega(t) \in \mathbb{R}^n$ is the exogenous input, $y_t \in \mathbb{R}$ is the controlled output, $e_t \in \mathbb{R}$ is the tracking error, $A \in \mathbb{R}^{n \times n}$, $F \in \mathbb{R}^{n \times 1}$, $B \in \mathbb{R}^{n \times 1}$, $G \in \mathbb{R}^{n \times 1}$, $P \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$, $D \in \mathbb{R}^{1 \times 1}$ and $Q \in \mathbb{R}^{1 \times 1}$. We assume that the initial condition $x_0$ is independent of $W_t$ for all $t \in \mathbb{R}_0^+$. The exogenous signal $\omega$ is assumed to be the state of a so-called exogenous system, which is described by the equations

$$
\dot{\omega} = S\omega, \quad \omega(0) = \omega_0,
$$

(2)

where $S \in \mathbb{R}^{n \times n}$ and the initial condition $\omega_0$ is independent of $W_t$ for all $t \in \mathbb{R}_0^+$. It is assumed that the state $x_t$ is not available for measure, while measurement outputs are given by

$$
y^o_t = C_o x_t, \quad y^b_t = C_b x_t,
$$

with $y^o_t \in \mathbb{R}$, $y^b_t \in \mathbb{R}$, $C_o \in \mathbb{R}^{1 \times n}$, $C_b \in \mathbb{R}^{1 \times n}$ and $C_o$ and $C_b$ are assumed to be linearly independent row vectors. The state $\omega$ of the exogenous system is assumed to be available for measure\(^1\). Suppose that the following assumption, which guarantees the marginal stability of the exogenous system, holds.

Assumption 1. All the eigenvalues of the matrix $S$ are on the imaginary axis and have the same algebraic and geometric multiplicity.

By Assumption 1 and independence of $x_0$ from $W_t - W_0$ for all $t \in \mathbb{R}_0^+$, the initial value problem associated to (1) has a unique (global) solution (Arnold, 1974, Theorem 8.1.5).

Remark 1. The name error-feedback problem would require us to use the tracking error $e_t$ as the measurement output; instead, as shown in Sections 3 and 4, we use the output $y^o_t$ for state-estimation purposes ($y^o_t$ being an auxiliary output necessary for the estimation of the Brownian motion). This is not a loss of generality, as long as we set $y^o_t = e_t - Du - Q\omega = C_x x_t$ and $C = C_o$.

3. IDEAL ERROR FEEDBACK PROBLEM

In this section we formulate the ideal, continuous-time, error-feedback output regulation problem for system (1) and we provide its solution. A detailed discussion of the assumptions and the results is provided in the extended preprint Mellone and Scarciotti (2019a).

Problem 1. (Ideal Error-Feedback Output Regulation Problem) Consider system (1), driven by the signal generator (2). The ideal error-feedback output regulation problem consists in determining a dynamic regulator of the form

$$
\begin{align*}
&\dot{z}_t = (G^{\omega} z_t + G^{\omega} \omega + G^1 y^o_t)dt + (G^{\omega} z_t + G^{\omega} \omega) dW_t, \\
&u_t = K z_t + \Gamma \omega,
\end{align*}
$$

(3)

with $G^{\omega} \in \mathbb{R}^{n \times n}$, $G^{\omega} \in \mathbb{R}^{n \times n}$, $G^1 \in \mathbb{R}^{n \times 1}$, $G^{\omega} \in \mathbb{R}^{n \times n}$, $G^{\omega} \in \mathbb{R}^{n \times n}$, $K \in \mathbb{R}^{1 \times n}$, $\Gamma_t \in \mathbb{R}^{1 \times 1}$ such that the following conditions hold.

(SI) The closed-loop system obtained by interconnecting system (1) and the regulator (3) with $\omega \equiv 0$ is almost surely asymptotically stable (see (Kozin, 1963, Section 2)).

(RI) The closed-loop system obtained by interconnecting system (1), the signal generator (2) and the regulator (3) satisfies $\lim_{t \to \infty} e_t = 0$ almost surely.

Assumption 2. System (1) with $\omega \equiv 0$ is almost surely asymptotically stabilisable\(^2\).

Assumption 3. System (1) is almost surely asymptotically detectable\(^2\).

The solution of the ideal error-feedback output regulation problem is provided by the following theorem.

Theorem 1. (Mellone and Scarciotti (2019a)) Consider the ideal error-feedback regulator problem. Suppose Assumptions 1, 2, 3 hold. Then, for any almost surely

\(^1\) Traditionally, the state of the exogenous system is assumed to be unavailable for measure in the error-feedback problem; indeed, the measurement output is assumed to be a linear combination of the states of the controlled system and of the exogenous system, which are then both estimated. Since in our framework the exogenous system is assumed to be deterministic, estimating its state is trivial. Therefore, we will assume it to be available for measure to focus the attention of the reader on the stochastic estimation.

\(^2\) See Mellone and Scarciotti (2019a) for a definition.
asymptotically stabilising $K$ and for any $L$ such that $d\eta_t = (A + LC_a)\eta_t dt + F\eta_t dW_t$ is almost surely asymptotically stable, there exist bounded matrices $\Pi_t \in \mathbb{R}^{r \times v}$ and $\Gamma_t \in \mathbb{R}^{r \times r}$ solving the equations

$$d\Pi_t = [(A + BK)\Pi_t - \Pi_t S + P + BT]\Pi_t dt + [(F + GK)\Pi_t + R + GT]\Pi_t dW_t, \quad 0 = (C + DK)\Pi_t + Q + DT,$$

for any consistent initial condition $(\Pi_0, \Gamma_0)$. Moreover, (3) solves Problem 1 for this selection of $\Pi_t$ and $\Gamma_t$.

**Remark 2.** Problem 1 is ideal because it cannot be solved in practice. In fact, as discussed in Mellone and Sciacchitano (2019b), the stochastic differential $dW_t$ should be available for measure to implement the regulator (3), as well as to solve the regulator equations (4) providing the matrix $\Gamma_t$. This, however, never happens in real situations.

### 4. $\varepsilon$-APPROXIMATE ERROR FEEDBACK PROBLEM

In this section we provide a hybrid control architecture that solves a weaker version of the error-feedback problem. We also show that the ideal solution can be retrieved by a limit procedure. In particular, first we define an approximate problem which we aim at solving using a hybrid regulator that employs estimates of the variations of the Brownian motion; then we describe the steady state of the resulting hybrid closed-loop system, we design an estimator that reconstructs the Brownian motion and we provide the solution to the approximate problem. The proofs of the results are provided in the extended preprint Mellone and Sciacchitano (2019a).

Consider the following hybrid system

$$dz_t = (G^1 z_t + G^2 y^a + G^3 \omega)dt \quad \forall t \in [k\varepsilon, (k+1)\varepsilon),$$

$$z^+_k = z_k + G^2 z^+_{k-1} + G^3 \omega(t^+_{k-1}) \quad k \in \mathbb{Z}_{\geq 0},$$

$$u_t = K z_t + \hat{\Gamma}\omega_t \quad \forall t \in \mathbb{R}_{\geq 0},$$

with $\hat{\Gamma} \in \mathbb{R}^{r \times v}$. The parameter $\varepsilon = t_k - t_{k-1}$, for all $t \in \mathbb{R}_{\geq 0}$, is the sampling period. We can now formulate the approximate output regulation problem.

**Problem 2.** ($\varepsilon$-Approximate Error-Feedback Output Regulation Problem). Consider system (1) driven by the signal generator (2). The *$\varepsilon$-approximate error-feedback output regulation problem* consists in determining a regulator of the form (5) such that the following conditions hold.

- **(SA)** The closed-loop system obtained by interconnecting system (1) and the regulator (5) with $\omega \equiv 0$ is almost surely asymptotically stable.

- **(RA)** The closed-loop system obtained by interconnecting system (1), the signal generator (2) and the regulator (5) yields a steady-state response of the tracking error $e^{ss}(\omega, \varepsilon)$, with $\varepsilon \in \mathbb{R}_{\geq 0}$, which is bounded and such that

$$\lim_{\|\omega\| \to 0} e^{ss}(\omega_0, \varepsilon) = 0 \quad \forall \varepsilon \in \mathbb{R}_{\geq 0},$$

$$\lim_{\varepsilon \to 0} e^{ss}(\omega_0, \varepsilon) = 0 \quad \forall \omega_0 \in \mathbb{R}^v,$$

almost surely.

Note that if the regulator (5) solves Problem 2, then it approximates the solution of Problem 1 with a degree of accuracy that depends on $\varepsilon$. In particular, condition *(RA)* states that, even though the steady-state tracking error is allowed to be almost surely non-zero, the ideal case is recovered by reducing the sampling period $\varepsilon$ and/or by reducing the amplitude of the exogenous signal.

Let the regulator matrices be

$$G^1 = A + BK + LC_a, \quad G^2 = P + BT,$$

$$G^3 = (F + GK)\Delta \hat{\omega}(k), \quad G^2 = (R + G\Gamma\hat{\Gamma}^{-1})\Delta \hat{\omega}(k),$$

where $\Delta \hat{\omega}(k)$ is an approximation of the variation of the Brownian motion $\Delta \hat{\omega}(k) = \hat{\omega}(t_k) - \hat{\omega}(t_{k-1})$, which is constructed in Section 4.2. Note that with these selections the variable $z_t$ is an estimation of the state $x_t$. We also define $\Delta z(k) = x_{t_k} - x_{t_{k-1}}$ and $\Delta y(k) = y_{t_k} - y_{t_{k-1}}$.

**Remark 3.** Since the exogenous system is deterministic and continuous, then $\omega(t^+_k) = \omega(t_k)$ and $\omega(t^+_{k-1}) = e^{-S\varepsilon}\omega(t_k)$ for all $k \in \mathbb{Z}_{\geq 0}$.

Define the variable $z^+_k = z^+_{k-1}$ for all $t \in [t_k, t_{k+1})$ and let $\hat{x}_t = [x^T_t, z^T_t, z'^{T T}_t]^T$. Then the closed-loop system obtained connecting systems (1), (2) and (5) with the selections (6) has the dynamics

$$d\hat{x}_t = (A^e \hat{x}_t + P^e \omega)dt + (F^e z_t + P^e \hat{\Gamma}\omega) dW_t,$$

$$\hat{x}^+_t = \hat{x}_t + (F^e \hat{x}_t + P^e \hat{\Gamma}\omega(t_k)) \Delta \hat{\omega}(k),$$

with

$$A^e = [A + BK \quad 0 \quad 0], \quad P^e = [P + BT],$$

$$F^e = \begin{bmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R^e = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}, \quad D^e = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

$$\tilde{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GK \end{bmatrix}, \quad \tilde{P}^e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G\Gamma \tilde{\Gamma}^{-1} \end{bmatrix} e^{-S\varepsilon}.$$

Note that the auxiliary variable $z^+_k$ holds the value of $z_t$ between sampling times and allows us to write the hybrid dynamics without explicitly introducing a delay.

### 4.1 Steady State of the Closed-Loop System

In this section we characterise the steady-state response of the closed-loop hybrid system (7).

**Lemma 1.** (Mellone and Sciacchitano (2019a)) Assume that there exist matrices $K$ and $L$ such that condition *(SA)* is satisfied. Then the steady-state response of the state $\hat{x}_t$ is $\hat{x}^{ss} = \hat{X}^{ss} \omega(t)$ almost surely, where the matrix $\hat{X}^{ss} \in \mathbb{R}^{3n \times v}$ is the steady-state solution of the hybrid equations

$$d\hat{x}_t = (A^e \hat{x}_t - \hat{x}_t S + \hat{P}^e) dt + (\hat{F}^e \hat{x}_t + \hat{P}^e) dW_t,$$

$$\hat{X}^{ss}_k = \hat{X}^d \hat{X}^{ss}_t + (\hat{F}^d \hat{X}^{ss}_t + \hat{P}^d) \Delta \hat{\omega}(k).$$

The following corollary characterises the steady-state responses of $x_t$ and $z_t$. This result will be used in the following to derive the solution of Problem 2.

---

3 Given the marginal stability of system (2) (see Assumption 1), for small $\|\omega\|$ we also have small $\|\omega(t)\|$ for all $t \in \mathbb{R}_{\geq 0}$.
Remark 4. Let $\Pi^f_t$ and $\Pi^s_t$ solve the hybrid equations
\[ d\Pi^f_t = (A\Pi^f_t + BK\Pi^s_t - \Pi^f_tS + P + B\hat{F})dt + (F\Pi^f_t + GK\Pi^s_t + R + G\hat{F})dW_t, \]
\[ d\Pi^s_t = -(LC_a\Pi^f_t + (A + BK + LC_a)\Pi^s_t - \Pi^s_tS + P + B\hat{F})dt, \]
where the matrix $\Pi^f_t$ solves the auxiliary equations
\[ d\Pi^f_t = -\Pi^f_tSdt, \]
\[ \Pi^f_{t_k} = \Pi^f_{t_k} + [(F + GK)\Pi^f_{t_k} + (R + G\hat{F})_e]e^{-\varepsilon t_k} \Delta \hat{W}_e(k), \]
with $\Delta \hat{W}_e(k)$ being the variations of the Brownian motion.

4.2 Estimation of the Variations of the Brownian Motion

We now show how the sequence of scalars $\{\Delta \hat{W}_e(k)\}$ is constructed. In particular, the scalars asymptotically approximate a posteriori the variations of the Brownian motion with a degree of accuracy that depends on $\varepsilon$. To this end, we introduce some notation. Let $L_1$ be the space of functions that are integrable in Itô’s sense. Then with
\[ \Delta \hat{W}_e(k) \rightarrow dW_t, \]
we mean
\[ \lim_{\varepsilon \rightarrow 0} \sum_k f(t_{k-1}, w)d\hat{W}_e(k) = \int_0^t f(\tau, w)dW_\tau, \quad \forall f \in L_1. \]

For ease of notation, define the scalars
\[ v^f(k) = C_b(Fx_{t_k} + Gu(t_k) + R\omega(t_k)), \quad \forall k \in \mathbb{Z}_{\geq 0}, \]
\[ v^s(k) = C_b(Fz_{t_k} + Gu(t_k) + R\omega(t_k)), \quad \forall k \in \mathbb{Z}_{\geq 0}. \]

Note that $v^e(k)$ is the diffusion term of the output dynamics $d\tilde{y}_t^e$ evaluated at time $t_k$ and $v^s(k)$ is its approximation when the state $x_t$ is replaced by its estimate $z_t$.

Remark 4. Since, for all $k \in \mathbb{Z}_{\geq 0}$, $v^f(k)$ and $v^s(k)$ are random variables, the property $P(v^f(k) = 0) = P(v^s(k) = 0)$ holds for all $k \in \mathbb{Z}_{\geq 0}$ (for non-trivial choices of the initial conditions and/or of the system matrices).

Lemma 2. (Mellone and Scarciotti (2019a)) Consider system (1) and the regulator (5). Assume $\lim_{t \rightarrow \infty} (z_t - x_t) = o(\varepsilon^2)$ almost surely. Let the sequence $\{\Delta \hat{W}_e(k)\}_{k \geq 0}$ be defined as
\[ \Delta \hat{W}_e(k) = v^f(k - 1)^{-1}[\Delta y^h_k - C_b(Az_{t_{k-1}} + Bu(t_{k-1}) + P\omega(t_{k-1}))\varepsilon]. \]
Then $\lim_{k \rightarrow \infty} \Delta \hat{W}_e(k) \rightarrow dW_t$ holds.

Lemma 2 also implies that, for smaller discretisation steps $\varepsilon$, better approximations $\Delta \hat{W}_e(k)$ of $\Delta W_e(k)$ are obtained.

4.3 Hybrid State Estimator

The problem we now need to address is the almost sure convergence of $z_t$ to $x_t$. If $K$ and $L$ are such that condition (SA) holds, then both $x_t$ and $z_t$ converge to zero almost surely when $\omega \equiv 0$. The case $\omega \neq 0$, however, requires more care. Indeed, a non-zero exogenous input forces the state of the system to lie on a non-zero manifold at steady-state. The main challenge when studying stochastic error-feedback output regulation is to estimate non-zero noisy states. We show that, using the regulator (5) and under suitable assumptions, it is possible to obtain an estimate that converges to the actual state as the sampling period $\varepsilon$ tends to zero. To this end, we first define the following condition.

(EC) The closed-loop system obtained by interconnecting system (1), the signal generator (2) and the regulator (5) satisfies $\lim_{t \rightarrow \infty} (z_t - x_t) = o(\varepsilon^2)$ almost surely.

Note that the satisfaction of this condition is necessary for Lemma 2 to hold. We now characterise the discrete-time dynamics of the estimation error and provide a choice of the gain $L$ such that (EC) holds.

Lemma 3. (Mellone and Scarciotti (2019a)) The forward-Euler discretisation of the estimation error dynamics $\eta_t = x_t - z_t$ is
\[ \eta_t = \eta_{t-1} + [(I - \Phi_{t_{k-1}})A + LC_a]\eta_{t_{k-1}}\varepsilon + (I - \Phi_{t_{k-1}})F\eta_{t_{k-1}} \Delta W_e(k), \]
where $\Phi_{t_{k-1}} = ((F + GK)z_{t_k} + (R + G\hat{F})\omega(t_k))v^e(k)^{-1}C_b$.

Remark 5. Since $C_b(I - \Phi_{t_{k-1}}) = 0$ for all $k \in \mathbb{Z}_{\geq 0}$, the couple $(I - \Phi_{t_{k-1}})A, C_b$ is not observable. This justifies the need for two different measurement outputs and, additionally, the linear independence of the row vectors $C_a$ and $C_b$. Therefore, $y^e_k$ is used to estimate the state of the system, as the couple $(I - \Phi_{t_{k-1}})A, C_a$ is almost surely observable, whereas $y^s_k$ is used to approximate the variations of the Brownian motion $\Delta \hat{W}_e(k)$, according to expression (9).

The forward-Euler discretisation (10) of the dynamics of the estimation error is nonlinear and time-varying. Moreover, the choice of the gain $L$ that stabilises the estimation error dynamics cannot be independent of the choice of the gain $K$. This is consistent with the fact that the design of the observer and of the controller for stochastic systems cannot be separated in practice (Damm, 2004, Section 1.8).

Assumption 4. There exist matrices $K$ and $L(k)$ such that system (10) is almost surely asymptotically stable.

Note that Assumption 4 is the counterpart of Assumption 3 in the discretised hybrid context.

The following result shows that if the time-varying gain $L(t)$ is chosen piecewise constant, then the hybrid estimator produces an estimation error arbitrarily small.

Lemma 4. (Mellone and Scarciotti (2019a)) Consider the closed-loop system obtained interconnecting (1), (2) and (5) with the selections (6) and let Assumptions 1 and 4 hold. Let $L(t) = L(k)$ for all $t \in [t_k, t_{k+1}]$. Then (EC) is satisfied.

Lemma 4 shows that, if Assumption 4 holds, it is possible to find a matrix $L(t)$ such that $\lim_{t \rightarrow \infty} (z_t - x_t) = o(\varepsilon^2)$ almost surely. However, there might be other functions
\( K(t) \) and \( L(t) \) such that the same results hold.

4.4 Solution to the Approximate Problem

Now we analyse the dynamics of the closed-loop system obtained interconnecting systems (1), (2) and the hybrid regulator (5) with the selections (6) and \( \omega \neq 0 \).

Lemma 5. (Mellone and Scarciotti (2019a)) Consider the closed-loop system obtained interconnecting (1), (2) and (5) with the selections (6) and let Assumption 1 hold. Suppose that there exist matrices \( K \) and \( L \) such that conditions \((\text{SA})\) and \((\text{EC})\) are satisfied. Then there exist almost surely bounded matrices \( \hat{\Pi}_i \in \mathbb{R}^{n \times r} \) and \( \hat{\Gamma}_i \in \mathbb{R}^{1 \times r} \) solving

\[
\begin{align*}
&d\hat{\Pi}_t = [(A + BK)\hat{\Pi}_t - \hat{\Pi}_t S + P + BR] dt, \\
&\hat{\Pi}_t (t_{k+1}^T = \hat{\Pi}_t (t_k) + [(F + GK)\hat{\Pi}_t (t_k) + R + G\hat{\Gamma}_t (t_k)] \Delta \hat{W}_t (k), \\
&0 = (C + DK)\hat{\Pi}_t + Q + D\hat{\Gamma}_t,
\end{align*}
\]

where \( \Delta \hat{W}_t (k) \) is given by (9). Moreover, the steady-state response of the tracking error of the closed-loop system is bounded and given by

\[
e_{ss}^* = [\hat{C}\hat{\Pi}^*_t + DK\hat{\Pi}^*_t + Q + D\hat{\Gamma}^*_t] e^{\hat{c}t} \omega_0, \]

almost surely, where \( \hat{\Pi}^*_t \) and \( \hat{\Gamma}^*_t \) solve (8).

Remark 6. The previous lemma states that, if the matrices \( \Pi_t \) and \( \Gamma_t \) solving the ideal regulator equations exist, then, under additional assumptions, two matrices \( \hat{\Pi}_t \) and \( \hat{\Gamma}_t \) solving the hybrid equations (11) exist as well. Moreover, if the control law \( u_t = Kz_t + \hat{\Gamma}_t \omega \) is adopted, the steady-state response of the tracking error is given by (12).

We are now ready to present the solution of the \( \varepsilon \)-approximate full-information output regulation problem.

Assumption 5. There exists matrices \( K(t) \) and \( L(t) \) such that \((\text{SA})\) and \((\text{EC})\) is satisfied.

Theorem 2. (Mellone and Scarciotti (2019a)) Consider the closed-loop system obtained interconnecting (1), (2) and (5) with the selections (6) and let Assumptions 1 and 5 hold. Then, for any pair \((K, L)\) such that conditions \((\text{SA})\) and \((\text{EC})\) hold, there exists a matrix \( \hat{\Gamma}_t \in \mathbb{R}^{1 \times r} \) such that the regulator (5) solves the \( \varepsilon \)-approximate error-feedback problem.

5. EXAMPLES

In this section we illustrate the validity of the theoretical results presented in the previous sections by means of a numerical simulation.

The matrices of system (1), which we have chosen randomly, are given by

\[
A = \begin{bmatrix}
0.3747 & 0.5530 & 2.1545 \\
-0.4961 & 0.9678 & 0.1574 \\
1.2407 & -1.9325 & 1.2932
\end{bmatrix}, \quad B = \begin{bmatrix}
-0.0758 \\
0.9100 \\
-1.0092
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
0.0651 & -0.5973 & -0.8573 \\
-0.4567 & -1.1094 & 1.2968 \\
-1.3330 & -1.2786 & -0.4758
\end{bmatrix}, \quad F = 0.001A,
\]

\[
G = 0.001B, \quad R = \begin{bmatrix}
-1.5157 & -0.0377 & 0.4783 \\
-1.1529 & -1.0464 & 0.7712 \\
0.6951 & 0.4631 & -0.0893
\end{bmatrix},
\]

\[
C = [7.7757 -0.0130 14.9039], \quad D = -0.0984,
\]

\[
Q = [-0.3294 -0.0886 -0.6945], \quad R = [1.2389 -0.7966 1.1422],
\]

\[
C_a = [0.8002 -0.8237 -2.2946].
\]

The matrix \( S \) is

\[
S = \begin{bmatrix}
0 & 0 & c \\
0 & 0 & -c \\
0 & -c & 0
\end{bmatrix},
\]

with \( c \in \mathbb{R} \), thus having \( \sigma(S) = \{0, jc, -jc\} \) and satisfying Assumption 1. We have randomly set \( c = 0.3742 \). The initial condition \( x_0 = [5.3767 18.3389 -22.5885]^T \) has been chosen randomly, while \( \omega_0 \) has been selected as \( \omega_0 = [10 10 10]^T \).

We have set \( K = [2.0970 -1.4851 4.0878]^T \), while \( L(t) \) has been set piecewise constant as shown in Lemma 4. These choices are such that \((\text{SA})\) and \((\text{EC})\) are satisfied. The discrete-time numerical implementation of the hybrid controller has required an integration method involving two different sampling periods: 1) \( \varepsilon \) is the sampling period at which the compensations for the diffusion term have been performed, in both equations (5) (with the selections (6)) and (11); three simulations with different values of \( \varepsilon \) (5 \( \cdot \) \( 10^{-1} \), 5 \( \cdot \) \( 10^{-5} \) and 5 \( \cdot \) \( 10^{-6} \)) have been carried out to illustrate that the ideal solution is recovered for decreasing \( \varepsilon \); 2) a smaller sampling period (5 \( \cdot \) \( 10^{-7} \)) has been used to simulate the continuous-time dynamics via a forward-Euler scheme.

Fig. 1 shows the time history of the norm of the estimation error \( z_{e1} - z_e \). In particular, we observe that the asymptotic estimation error is closer to zero as the compensation for the diffusion term happens more frequently.

Fig. 2 shows the evolution of the quantity \( \Delta \hat{W}_e - \Delta \hat{W}_e \). As theoretically shown in Lemma 2, we observe in simulation that said quantity is, asymptotically, an infinitesimal of order \( \varepsilon^2 \), as long as \( \lim_{t \to \infty} (z_{e1} - x_1) = o(\varepsilon^2) \); this condition has been shown to hold (see Fig. 1).

Finally, Fig. 3 shows the time history of the tracking error \( e_{t1} \) of the closed-loop system and confirms the results of Theorem 2, i.e. the tracking improves as \( \varepsilon \) tends to zero.

It has been observed that a controller performing a regulation in the mean sense, i.e. implementing a purely deterministic regulator, always produces estimation and tracking errors that are orders of magnitude higher than those obtained with the proposed scheme (we therefore omit the figure), thus proving the advantage of using the hybrid architecture.

6. CONCLUSION

In this paper we dealt with the error-feedback output regulation problem for linear stochastic systems. We have first defined and solved the ideal problem. However, the ideal regulator cannot be implemented in practice because the stochastic differential \( dW_t \) is not available for measure. By defining a new approximate problem and introducing a way to estimate \textit{a posteriori} the variations of the Brownian motion, we have shown that a practically sound solution can approximate the ideal one with an arbitrary degree of accuracy.

The results obtained in this paper are a further step towards a systematic understanding of output regulation for stochastic systems. The dynamics of the estimation
Fig. 1. Time history of \(\|x_{t} - z_{t}\|\) for different values of \(\varepsilon\): \(\varepsilon = 5 \cdot 10^{-4}\) (blue line), \(\varepsilon = 5 \cdot 10^{-5}\) (orange line), \(\varepsilon = 5 \cdot 10^{-6}\) (yellow line).

Fig. 2. Large: time history of \(\Delta W_{\varepsilon} - \Delta \hat{W}_{\varepsilon}\) for different values of \(\varepsilon\): \(\varepsilon = 5 \cdot 10^{-4}\) (blue line), \(\varepsilon = 5 \cdot 10^{-5}\) (orange line), \(\varepsilon = 5 \cdot 10^{-6}\) (yellow line). Insert: zoomed in detail.

Fig. 3. Time history of the tracking error \(e_{t}\) for different values of \(\varepsilon\): \(\varepsilon = 5 \cdot 10^{-4}\) (blue line), \(\varepsilon = 5 \cdot 10^{-5}\) (orange line), \(\varepsilon = 5 \cdot 10^{-6}\) (yellow line).

error produced by the hybrid controller requires further analysis; in particular, the properties of the matrix \(\Phi_{\varepsilon}\), as well as the research of general criteria for the choice of the estimation gain \(L\) are open problems that will be the subjects of forthcoming studies.

REFERENCES

Arnold, L. (1974). *Stochastic Differential Equations*. A Wiley-Interscience publication. Wiley.

Carnevale, D., Galeani, S., Menini, L., and Sassano, M. (2016). Hybrid output regulation for linear systems with periodic jumps: Solvability conditions, structural implications and semi-classical solutions. *IEEE Transactions on Automatic Control*, 61(9), 2416–2431.

Damm, T. (2004). *Rational Matrix Equations in Stochastic Control*. Number 297 in Lecture Notes in Control and Information Sciences. Springer.

Davison, E. (1976). The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Transactions on Automatic Control*, 21(1), 25–34.

Francis, B. (1977). The linear multivariable regulator problem. *SIAM Journal on Control and Optimization*, 15(3), 486–505.

Francis, B. and Wonham, W. (1976). The internal model principle of control theory. *Automatica*, 12(5), 457 – 465.

Huang, J. (2004). *Nonlinear Output Regulation: Theory and Applications*. International series in pure and applied mathematics. Philadelphia, PA: SIAM Advances in Design and Control.

Ichikawa, A. and Katayama, H. (2006). Output regulation of time-varying systems. *Systems & Control Letters*, 55(12), 999–1005.

Isidori, A. and Byrnes, C.I. (1990). Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control*, 35(2), 131–140.

Kozin, F. (1963). On almost sure stability of linear systems with random coefficients. *Journal of Mathematics and Physics*, 42(1-4), 59–67.

Krstic, M. and Deng, H. (1998). *Stabilization of Nonlinear Uncertain Systems*. Communications and Control Engineering. Springer London.

Marconi, L. and Teel, A.R. (2013). Internal model principle for linear systems with periodic state jumps. *IEEE Transactions on Automatic Control*, 58(11), 2788–2802.

Mellone, A. and Scarciotti, G. (2019a). Error-Feedback Output Regulation of Linear Stochastic Systems: a Hybrid Approach. In *2019 European Control Conference (ECC)*, 287–292.

Öksendal, B. (2003). *Stochastic Differential Equations (Sixth Edition)*. Springer-Verlag.

Pavlov, A., van de Wouw, N., and Nijmeijer, H. (2006). *Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach*. Systems & Control: Foundations & Applications. Birkhäuser Boston.

Scarciotti, G. (2018). Output Regulation of Linear Stochastic Systems: the Full-Information Case. In *2018 European Control Conference (ECC)*, 1920–1925.

Scarciotti, G. and Teel, A.R. (2018). On moment matching for stochastic systems. In: *IEEE Transactions on Automatic Control*. Under review.

Serrani, A., Isidori, A., and Marconi, L. (2001). Semi-global nonlinear output regulation with adaptive internal model. *IEEE Transactions on Automatic Control*, 46(8), 1178–1194.

Yong, J. and Zhou, X.Y. (1999). *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Stochastic Modelling and Applied Probability. Springer New York.