Some homological aspects of idempotents in idempotented algebras

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ABSTRACT
We study special idempotents (as described by Bushnell and Kutzko) and split idempotents in the context of module and derived categories for idempotented algebras. We then characterize these concepts for path algebras of quivers.

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1. Introduction

An associative algebra $A$ is said to be idempotented if given any element $a \in A$, there exists an idempotent $e \in A$ (i.e. $e^2 = e$) such that $ae = ea = a$, and given any two idempotents $e_1, e_2 \in A$, there exists an idempotent $e \in A$ such that $ee_i = e_i e = e_i$ for $i = 1, 2$. Every unital algebra is idempotented, but non-unital examples do appear in natural contexts.

The notion of a special idempotent was introduced in [6] for Hecke algebras of $p$-adic reductive groups, which form a prominent class of non-unital idempotented algebras (see Section 3.1 for details). These idempotents play a fundamental role in the theory of types and are objects of interest in the representation theory of $p$-adic groups. In [11], analogs of results in [6] for special idempotents were obtained for the Harish-Chandra Schwartz algebra of a $p$-adic reductive group. Given their importance, it is natural to study these idempotents in greater generality and develop their formalism: this is one of the purposes of this note. In Section 3.2, we study special idempotents for arbitrary idempotented algebras: an idempotent $e \in A$ is special if the category $\mathcal{M}_e := \{M \in \mathcal{M}(A) \mid AeM = M\}$ is closed under subquotients, where $\mathcal{M}(A)$ is the category of non-degenerate $A$-modules. In particular, we generalize some results from [6] to module categories over such algebras.

In Section 3.3, we introduce and study the notion of a split idempotent (see Definition 3.10): this notion isolates idempotents which are well behaved from a certain homological perspective. Derived techniques have become increasingly important in the representation theory of $p$-adic groups, especially in the context of modular representations (see [7] for an exposition). Using our
results on special and split idempotents, in Section 3.3, we obtain the derived versions of the
results in Section 3.2.

When \( e \in A \) is split, the category \( \text{M}(A)_e \) appears as a direct summand of the category \( \text{M}(A) \) (see Proposition 3.18). In Section 3.4, we isolate the conditions on a family of special and split idempotents \( \{e_i\} \) that ensure a decomposition of the (derived) category of non-degenerate \( A \)-modules into a product of the categories \( \text{M}(A)_{e_i} \) (see Theorem 3.28 and Theorem 3.31).

As evidenced by the case of Hecke algebras, it can be a difficult problem to classify special and split idempotents for a given idempotent algebra. Path algebras of quivers form a natural class of idempotent algebras. In Section 4 we achieve a complete classification of such idempotents for these algebras. The proofs of the results in this section are elementary, but of a technical combinatorial nature.

**Notation and conventions.** All algebras will be defined over a fixed, unital, commutative base ring \( \mathbb{K} \). An unadorned \( \otimes \) will mean tensoring over \( \mathbb{K} \). The terms “ring”, “module”, and “complex” used without any prefix will always mean over \( \mathbb{K} \). All complexes are indexed cohomologically.

Recall that an associative algebra \( A \) is said to be idempotented if given any element \( a \in A \), there exists an idempotent \( e \in A \) (i.e. \( e^2 = e \)) such that \( ae = ea = a \), and given any two idempotents \( e_1, e_2 \in A \), there exists an idempotent \( e \in A \) such that \( e_i = e_i e = e_i \) for \( i = 1, 2 \). The reader may convince themselves that \( A \) is idempotented if and only if given elements \( a_1, \ldots, a_n \in A \), there exists an idempotent \( e \in A \) such that \( ea_i = a_i e = a_i \) for all \( i \in \{1, \ldots, n\} \). In this note, \( A \) and \( B \) will denote idempotent algebras. We use \( A_{\text{idm}} \) to represent the set of all idempotents in \( A \). A morphism between two idempotented algebras means a morphism between the underlying associative algebras: a morphism of unital algebras viewed as idempotented algebras is not necessarily unital.

A left module \( M \) over \( A \) is called non-degenerate if for any given \( m \in M \), there is some \( e \in A_{\text{idm}} \) such that \( em = m \). We use \( \text{M}(A) \) to represent the category of non-degenerate left \( A \)-modules. The category of all left \( A \)-modules is denoted by \( \overline{\text{M}}(A) : \text{M}(A) \) is a full abelian subcategory of \( \overline{\text{M}}(A) \).

If \( E \) is an abelian category, we denote the category of complexes of objects in \( E \) by \( \mathcal{C}(E) \) and the corresponding derived category by \( \mathcal{D}(E) \). That being said, when \( E \) is \( \text{M}(A) \) or \( \overline{\text{M}}(A) \) for an idempotented algebra \( A \) we will use more specialized notation: the category of complexes of objects of \( \text{M}(A) \) will be denoted by \( \mathcal{C}(A) \), with associated homotopy category \( K(A) \) and derived category \( \mathcal{D}(A) \), while the category of complexes of objects of \( \overline{\text{M}}(A) \) is denoted by \( \overline{\mathcal{C}}(A) \), with associated homotopy category \( \overline{K}(A) \) and derived category \( \overline{\mathcal{D}}(A) \).

Both \( A^{\text{op}} \) and \( A \otimes B^{\text{op}} \) are idempotented algebras. Right modules will be dealt with using the formalism of the opposite ring. Left \( A \) and right \( B \) bimodules are identified with left modules over \( A \otimes B^{\text{op}} \); such a bimodule is non-degenerate on both sides if and only if it is non-degenerate as an \( A \otimes B^{\text{op}} \)-module.

If \( M \in \overline{\mathcal{C}}(A) \), we use \( H^i(M) \) for its \( i \)-th cohomology \( A \)-module, and \( Z^i(M) \) for the \( A \)-module of \( i \)-th cocycles. If \( M \) and \( N \) are objects of \( \overline{\mathcal{M}}(A) \), we use \( \text{Hom}_A(M, N) \) to denote the module of \( A \)-linear maps between them. More generally, if \( M \) and \( N \) are objects of \( \overline{\mathcal{C}}(A) \), we use \( \text{Hom}_A(M, N) \) for the complex of morphisms between them, where

\[
\text{Hom}_A(M, N)^k = \prod_{i \in \mathbb{Z}} \text{Hom}_A(M^i, N^{i+k})
\]

and \( \partial(f) = \partial_N \circ f - (-1)^k f \circ \partial_M \) for \( f \in \text{Hom}_A(M, N)^k \). If \( M \in \overline{\mathcal{C}}(A^{\text{op}}) \) and \( N \in \overline{\mathcal{C}}(A) \), \( M \otimes_A N \) is the complex with \( k \)-th term

\[
(M \otimes_A N)^k = \bigoplus_{i+j=k} M^i \otimes_A N^j
\]
and $\partial^i(m \otimes n) = \partial(m) \otimes n + (-1)^i m \otimes \partial(n)$ where $m \in M^i, n \in N^j$. For morphisms in $\text{C}(A)$ or $\text{K}(A)$, we use $\text{Hom}_{\text{C}(A)}(M, N)$ and $\text{Hom}_{\text{K}(A)}(M, N)$, respectively, with similar notation for morphisms in $\text{C}(A)$ and $\text{K}(A)$. Recall that $\text{Hom}_{\text{C}(A)}(M, N) = Z^0(\text{Hom}_A(M, N))$ and $\text{Hom}_{\text{K}(A)}(M, N) = H^0(\text{Hom}_A(M, N))$.

An object $P \in \text{K}(A)$ is called $K$-projective if $\text{Hom}_A(P, N)$ is acyclic for all acyclic complexes $N$ in $\text{K}(A)$. Similarly, an object $I \in \text{K}(A)$ is called $K$-injective if $\text{Hom}_A(N, I)$ is acyclic for all acyclic complexes $N$ in $\text{K}(A)$. There are similar notions of $K$-projectivity and $K$-injectivity for $\text{K}(A)$, where we test over all acyclic complexes in $\text{K}(A)$. The $K$-projectivity or $K$-injectivity of a complex does depend on whether we are viewing it as an object of $\text{K}(A)$ or $\text{K}(A)$. See Lemma 2.4.

The notation $\text{RHom}_A$ is used for the right derived functor of $\text{Hom}_A$, and $\text{LHom}_A$ is used for the left derived functor of $\text{Hom}_A$. The construction of $\text{LHom}_A$ and $\text{RHom}_A$ is independent of whether we are constructing them in $\text{D}(A)$ or $\text{D}(A)$ (Lemma 2.4).

2. Preliminaries on module and derived categories

Let $A$ be an idempotented algebra. In this section, we collect facts regarding the category $\text{M}(A)$ that will be relevant to this work. Many of these facts appear explicitly in the literature, while others follow as consequence of more general considerations.

2.1. Module categories of idempotented algebras

It is clear that kernels and cokernels of maps between non-degenerate modules remain non-degenerate, and that arbitrary coproducts of non-degenerate modules are non-degenerate. It then follows that the category $\text{M}(A)$ is closed under colimits.

**Definition 2.1.** Define the functor

$$\nu_A : \overline{\text{M}}(A) \to \text{M}(A)$$

by

$$\nu_A(M) := \{m \in M \mid \exists e \in \text{idm} \text{ s.t. } em = m\}.$$

Since $A$ is idempotented, $\nu_A(M)$ is an $A$-submodule of $M$ for any $M \in \overline{\text{M}}(A)$. The functor $\nu_A$ is right adjoint to the canonical inclusion of $\text{M}(A)$ in $\overline{\text{M}}(A)$. The reader will easily check that $\nu_A$ is exact.

The content of the following lemma is elementary and very well-known: we repeat it here for the benefit of the reader as it will be used frequently in the sequel.

**Lemma 2.2.** Let $e \in \text{idm}$ and $M \in \overline{\text{M}}(A)$.

(i) The correspondence $f \mapsto f(e)$ establishes an isomorphism of modules

$$\text{Hom}_A(Ae, M) \cong eM.$$

(ii) For $M = Ae$, the isomorphism in (i) induces an isomorphism of rings

$$\text{End}_A(Ae) \cong eAe^{\text{op}}.$$

(iii) The correspondence $ea \otimes m \mapsto eam$ establishes an isomorphism of modules

$$eA \otimes_A M \cong eM.$$

Using Lemma 2.2, the reader may verify that the family $\{Ae \mid e \in \text{idm}\}$ forms a family of compact projective generators for the category $\text{M}(A)$. From this and the remarks made in the
introduction to Section 2.1, it then follows that $M(A)$ is a Grothendieck abelian category with enough projective objects.

We will use the following proposition later in this paper. We leave its proof as a straightforward exercise for the reader.

**Proposition 2.3.** Let $e \in A_{\text{idm}}$. Let $M \in M(eAe)$, and $N \in M(A)$. Then, there are the following natural isomorphisms of modules:

1. $\text{Hom}_{eAe}(eN, M) \cong \text{Hom}_A(N, \nu_A(\text{Hom}_{eAe}(eA, M)))$
2. $\text{Hom}_A(A \otimes_{eAe} M, N) \cong \text{Hom}_{eAe}(M, eN)$

### 2.2. Derived categories of idempotented algebras

Every object in the category of complexes over a Grothendieck abelian category with enough projective objects admits a quasi-isomorphism from a K-projective complex, and a quasi-isomorphism to a K-injective one ([1, Proposition 4.3, Theorem 5.4]). In particular, every complex in $\mathcal{K}(A)$ admits a quasi-isomorphism to a K-injective complex, and a quasi-isomorphism from a K-projective one.

We now provide derived analogs of the results in Section 2.1. It is easy to see that the canonical functors $\mathcal{C}(A) \to \mathcal{C}(A)$ and $\mathcal{K}(A) \to \mathcal{K}(A)$ are fully faithful. The functor $\nu_A$ naturally extends to functors from $\mathcal{C}(A) \to \mathcal{C}(A)$ and $\mathcal{K}(A) \to \mathcal{K}(A)$; in both cases, these functors are right adjoint to the corresponding inclusion functors.

**Lemma 2.4.** The following hold:

1. If $P$ is a K-projective object in $\mathcal{K}(A)$, then it is K-projective in $\mathcal{K}(A)$.
2. If $I$ is a K-injective object in $\mathcal{K}(A)$, then $\nu_A(I)$ is K-injective in $\mathcal{K}(A)$.

**Proof.** We prove (i). Let $P$ be a K-projective complex in $\mathcal{K}(A)$, and suppose $N$ is an acyclic complex in $\mathcal{K}(A)$. Observe that $\nu_A(N)$ is an object in $\mathcal{K}(A)$; since $\nu_A$ is exact, $\nu_A(N)$ is acyclic. Since $\nu_A$ is right adjoint to the inclusion of $\mathcal{K}(A)$ in $\mathcal{K}(A)$ there is an isomorphism of modules $\text{Hom}_{\mathcal{K}(A)}(P, N) \cong \text{Hom}_{\mathcal{K}(A)}(P, \nu_A(N))$. Since $P$ is K-projective in $\mathcal{K}(A)$ and $\nu_A(N)$ is acyclic, $\text{Hom}_{\mathcal{K}(A)}(P, \nu_A(N)) = 0$. The claim follows. The proof of (ii) is similar.

By Lemma 2.4, it follows that the canonical functor $\mathcal{D}(A) \to \mathcal{D}(A)$ is fully faithful as well. Since $\nu_A$ is exact, it admits an extension $\mathcal{D}(A) \to \mathcal{D}(A)$, which we again denote by $\nu_A$. Again by Lemma 2.4, $\nu_A : \mathcal{D}(A) \to \mathcal{D}(A)$ is right adjoint to the inclusion of $\mathcal{D}(A)$ in $\mathcal{D}(A)$.

The following is a derived analog of Proposition 2.3. As before, we leave its proof to the reader.

**Proposition 2.5.** Let $e \in A_{\text{idm}}$. Let $M \in D(eAe)$, and $N \in D(A)$. Then, there are the following natural isomorphisms of modules:

1. $\text{Hom}_{D(eAe)}(eN, M) \cong \text{Hom}_{D(A)}(N, \nu_A(\text{RHom}_{eAe}(eA, M)))$
2. $\text{Hom}_{D(A)}(A \otimes_{eAe} M, N) \cong \text{Hom}_{D(eAe)}(M, eN)$

### 3. Special idempotents, split idempotents, and categorical decompositions

Before moving on to the results in this section, we survey a concrete example coming from the theory of complex representations of $p$-adic groups. This example is purely for context; a reader not familiar with the subject can skip it entirely.
3.1. The case of reductive groups over non-archimedean local fields

Let $F$ be a non-archimedean local field and let $G$ denote the topological group comprising of $F$-rational points of a connected reductive algebraic group defined over it. Consider the collection of all pairs $(M, \sigma)$ where $M$ is an $F$-Levi subgroup of $G$ and $\sigma$ is a supercuspidal representation of $M$. Define an equivalence relation on the collection of such pairs by deeming two such pairs $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ to be equivalent if there exists an element $g \in G$ and unramified quasi-character $\chi$ of $M_2$ such that

$$M_2 = g^{-1}M_1g \quad \text{and} \quad \sigma_2 = \sigma_1^g \otimes \chi,$$

where $\sigma_1^g$ is the representation of $M_2$ given by composition of $\sigma_1$ by the g-conjugation map. The equivalence classes of this relation are known as inertial supports, and the set of all inertial supports for $G$ is denoted by $\mathcal{B}(G)$.

Let $\mathcal{R}(G)$ denote the category of smooth complex representations of $G$. Bernstein provided a direct product decomposition of this abelian category (see [3, 4]) into full subcategories indexed by inertial supports, that is,

$$\mathcal{R}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{R}^s(G).$$

Let $\mathcal{H}(G)$ denote the Hecke algebra of $G$ consisting of all locally constant, compactly supported, complex valued functions on $G$. It is well known that the category of non-degenerate left modules over $\mathcal{H}(G)$ is equivalent to $\mathcal{R}(G)$. For an idempotent $e \in \mathcal{H}(G)$, set $\mathcal{R}_e(G)$ to be the full subcategory of $\mathcal{R}(G)$ whose objects are those $V \in \mathcal{R}(G)$ which satisfy the condition that $V = \mathcal{H}(G)eV$, i.e. $V$ is generated as a $G$-module by its subspace of $e$ fixed vectors. It is shown in [6, Proposition 3.3] that the category $\mathcal{R}_e(G)$ is closed relative to its $G$-subquotients if and only if it is naturally equivalent to the category of left modules over the unital ring $e\mathcal{H}(G)e$. The idempotents $e \in \mathcal{H}(G)$ which satisfy this aforementioned property (i.e. the category $\mathcal{R}_e(G)$ is closed relative to its $G$-subquotients) are known as special idempotents. An example of a special idempotent is the characteristic function of an Iwahori spherical subgroup of $G$ (see [5, Lemma 4.8], [10]; also c.f. [6, Section 9.2]). In this case the algebra $e\mathcal{H}(G)e$ is the well known Iwahori Hecke algebra.

For any finite subset $\mathcal{S}$ of $\mathcal{B}(G)$ define $\mathcal{R}^\mathcal{S}(G) = \prod_{s \in \mathcal{S}} \mathcal{R}^s(G)$. It is shown in [6, Proposition 3.6] that given any special idempotent $e \in \mathcal{H}(G)$, there exists a finite subset $\mathcal{S}$ of $\mathcal{B}(G)$ such that $\mathcal{R}_e(G) = \mathcal{R}^\mathcal{S}(G)$. Conversely, given a finite subset $\mathcal{S}$ of $\mathcal{B}(G)$, there exists a special idempotent $e(\mathcal{S}) \in \mathcal{H}(G)$ such that $\mathcal{R}^\mathcal{S}(G) = \mathcal{R}_{e(\mathcal{S})}(G)$ (see [6, Proposition 3.13]). These results on special idempotents are the basis of the “theory of types” in the representation theory of $p$-adic groups.

Recently in [11, Section 1], some of the above results on special idempotents have been proved for the module category over $S(G)$, where $S(G)$ denotes the Harish-Chandra Schwartz algebra of $G$.

3.2. Special idempotents in module categories

We extend some notions from Section 3.1 to module categories over arbitrary idempotent algebras.

Lemma 3.1. Let $e \in A_{\text{idm}}$. The following two subcategories of $\mathcal{M}(A)$ coincide:

(a) $\{ M \in \mathcal{M}(A) \mid \forall K \leq H \leq M, e(H/K) \neq 0 \}$,
(b) $\{ M \in \mathcal{M}(A) \mid \forall K \leq H \leq M, H/K \text{ simple, } e(H/K) \neq 0 \}$. 


Proof. Suppose $M \in \mathcal{M}(A)$. If every subquotient of $M$ admits an $e$-fixed element, then every irreducible subquotient does so as well. Conversely, suppose every irreducible subquotient of $M$ admits an $e$-fixed element, and let $N$ be a non-zero subquotient of $M$. Taking quotients, we can assume that $N \leq M$.

Pick $n \in N$, and consider the submodule $An$; note that $n \in An$. By Zorn’s lemma $An$ admits an irreducible quotient: by hypothesis, this irreducible quotient must admit an $e$-fixed element. Since $Ae$ is projective, this element lifts to an $e$-fixed element in $An$, and thus in $N$. The result follows.

Definition 3.2. Denote the subcategory of $\mathcal{M}(A)$ under consideration in Lemma 3.1 by $\mathcal{M}_{A e}$.

Definition 3.4. An idempotent $e \in A_{idm}$ is said to be left special if the full subcategory $\{M \in \mathcal{M}(A) \mid M = AeM\}$ is closed under subquotients.

Remark 3.5. It is clear that the full subcategory $\{M \in \mathcal{M}(A) \mid M = AeM\}$ is always closed under quotients. Thus, $e \in A_{idm}$ is left special if and only if $\{M \in \mathcal{M}(A) \mid M = AeM\}$ is closed under subobjects.

Lemma 3.6. Let $e \in A_{idm}$ be left special. Then $\mathcal{M}(A)_{e} = \{M \in \mathcal{M}(A) \mid M = AeM\}$. Moreover, $Ae$ is a finitely generated projective generator for the category $\mathcal{M}(A)_{e}$.

Proof. Since $e$ is left special the category $\{M \in \mathcal{M}(A) \mid M = AeM\}$ is closed under subquotients. Thus, if $M = AeM$, then every subquotient of $M$ is also generated by its $e$-fixed elements; in particular, every non-zero subquotient must admit an $e$-fixed element. Conversely, suppose every non-zero subquotient of $M$ admits an $e$-fixed element. If $M \neq AeM$, the quotient $M/AeM$ must do so as well. By the projectivity of $Ae$, such an element must lift to an $e$-fixed element of $M$ lying outside $AeM$. This is a contradiction.

Using Lemma 2.2, the remaining claim in the lemma is an easy consequence of what we have proved.

Remark 3.7. A full subcategory $D$ of a category $C$ is called localizing if the canonical inclusion functor $D \rightarrow C$ admits a right adjoint. In a Grothendieck abelian category, every full subcategory closed under extensions, quotients, subobjects, and arbitrary coproducts is localizing.

For any $e \in A_{idm}$ the category $\mathcal{M}(A)_{e}$ is localizing: we denote the right adjoint to the inclusion $\mathcal{M}(A)_{e} \rightarrow \mathcal{M}(A)$ by $\Gamma_{e}$. In the case where $e$ is left special, $\Gamma_{e}(M) = AeM$ for all $M \in \mathcal{M}(A)$.

Theorem 3.8 (c.f. Proposition 3.3 of [6]). Let $e \in A_{idm}$ be left special. The functor

$$\text{Hom}_{A}(Ae, -) : \mathcal{M}(A)_{e} \rightarrow \mathcal{M}(eAe)$$

is an equivalence of categories, with quasi-inverse

$$A \otimes_{Ae} (-) : \mathcal{M}(eAe) \rightarrow \mathcal{M}(A)_{e}.$$
Theorem 1.3]) the functor $\text{Hom}_A(Ae, -)$ establishes an equivalence of categories between $M(A)_e$ and $M(\text{End}_A(Ae)^{op})$. By Lemma 2.2, $\text{End}_A(Ae)^{op} \cong eAe$. Denote a left adjoint quasi-inverse to $\text{Hom}_A(Ae, -)$ by $H$.

We can construct the following diagram of functors:

\[
\begin{array}{ccc}
\text{inc} & \text{M}(A) & \text{A} \otimes_{eAe} (-) \\
\text{M}(A)_e & \Gamma_e & \text{Hom}_A(Ae, -) \\
\text{M}(eAe) & H & \text{M}(eAe)
\end{array}
\]

By construction the functor inc is left adjoint to $\Gamma_e$. Lemma 2.2 and Proposition 2.3 imply that as functors between $M(A)$ and $M(eAe)$, the functor $A \otimes_{eAe} (-)$ has $\text{Hom}_A(Ae, -)$ as its right adjoint.

Thus, on each of the three sides of the above diagram, the functor represented by the inner arrow is right adjoint to that represented by the outer arrow. Furthermore, the triangular diagram formed by considering only the inner arrows is commutative.

As adjoints compose, $\text{inc} \circ H \cong A \otimes_{eAe} (-)$. This establishes the theorem.

**Corollary 3.9.** Let $e \in A_{idm}$ be left special. Then, $Ae$ is flat as a right $eAe$-module.

**Proof.** By Theorem 3.8, the functor $A \otimes_{eAe} (-)$ establishes an equivalence of categories between $M(eAe)$ and $M(A)_e$; as such, it is exact. As $M(A)_e$ is an abelian subcategory of $M(A)$, this precisely says that any exact sequence in $M(eAe)$ remains exact after applying the functor $A \otimes_{eAe} (-)$. However, for any $M \in M(eAe)$ there is a natural isomorphism $A \otimes_{eAe} M \cong Ae \otimes_{eAe} M$: thus $Ae$ is flat as a right $eAe$-module.

### 3.3. Split idempotents and the derived category

**Definition 3.10.** An idempotent $e \in A_{idm}$ is said to be **left split** if the inclusion $\text{inc}: M(A)_e \hookrightarrow M(A)$ preserves injective objects.

For example, if $e$ is a special idempotent arising from a finite subset of $\mathfrak{B}(G)$ (see Section 3.1), then $e$ is left split: this follows from Bernstein’s direct product decomposition of the category of smooth complex representations of $G$.

**Lemma 3.11.** An idempotent $e \in A_{idm}$ is left split if and only if for all injective objects $I \in M(A)$, $\Gamma_e(I)$ is injective in $M(A)$.

**Proof.** Suppose first that $e$ is left split, and $I$ is an injective object in $M(A)$. The functor $\Gamma_e$ is right-adjoint to the inclusion $\text{inc}$; as $\text{inc}$ is exact, $\Gamma_e$ sends injective objects in $M(A)$ to injective objects in $M(A)_e$. As $e$ is left split, $\Gamma_e(I)$ is therefore injective in $M(A)$.

Conversely assume that $\Gamma_e$, viewed as a functor from $M(A)$ to itself, preserves injective objects. Suppose $I$ is injective in $M(A)_e$, and let $E$ be its injective hull in $M(A)$. By hypothesis, $\Gamma_e(E)$ is injective in $M(A)$; it is also an object of $M(A)_e$. Since $I \leq \Gamma_e(E)$, $I$ is therefore a summand of $\Gamma_e(E)$. As $\Gamma_e(E)$ is injective in $M(A)$, $I$ is so as well.

**Lemma 3.12.** Let $e \in A_{idm}$ be left special, and suppose $N \in M(A)$. Then, $\text{Hom}_A(M, N) = 0$ for all $M \in M(A)_e$ if and only if $eN = 0$.
Then the objects in $\text{RHom}_A(M, N)$ are also natural inclusion functors.

The conclusion follows. □

**Lemma 3.13.** Let $e \in A_{\text{adm}}$ be left special. Suppose $M \in \text{M}(A)_e$ and $eN = 0$. Then $\text{Hom}_A(N, M) = 0$.

**Proof.** The existence of a non-zero morphism $f : N \to M$ implies the existence of a non-zero submodule $0 \neq f(N) \leq M$. However, by Lemma 3.6 and Lemma 3.1, we know that $f(N)$ must contain a non-zero $e$-fixed element: thus, $ef(N) \neq 0$. This is a contradiction. □

**Proposition 3.14.** Let $e \in A_{\text{adm}}$ be left special and left split. Suppose $M \in \text{M}(A)_e$ and $eN = 0$. Then $\text{RHom}_A(N, M) = 0$.

**Proof.** Let $0 \to M \to I^0 \to I^1 \to I^2 \to ...$ be an injective resolution of $M$ in $\text{M}(A)_e$. Since $e$ is left split, $I^0 \to I^1 \to I^2 \to ...$ remains an injective resolution of $M$ in $\text{M}(A)$, and can be used to compute $\text{RHom}_A$. As each $I^i$ is in $\text{M}(A)_e$, Lemma 3.13 implies that $\text{Hom}_A(N, I^i) = 0$ for all $i \in \mathbb{N}$. The conclusion follows. □

**Corollary 3.15.** Let $e \in A_{\text{adm}}$ be left special and left split. Suppose $M \in \text{M}(A)_e$ and $eN = 0$. Then $\text{Ext}_A^1(N, M) = 0$.

**Proof.** This follows from the isomorphism $\text{Ext}_A^1(N, M) \cong H^1(\text{RHom}_A(N, M))$. □

**Corollary 3.16.** Let $e \in A_{\text{adm}}$ be left special and left split, and suppose $M \in \text{M}(A)$. Then, the short exact sequence $0 \to \Gamma_e(M) \to M \to M/\Gamma_e(M) \to 0$ splits. In particular, $M \cong \Gamma_e(M) \oplus M/\Gamma_e(M)$.

**Proof.** The short exact sequence $0 \to \Gamma_e(M) \to M \to M/\Gamma_e(M) \to 0$ represents an element of $\text{Ext}_A^1(M/\Gamma_e(M), \Gamma_e(M))$. By Corollary 3.15, $\text{Ext}_A^1(M/\Gamma_e(M), \Gamma_e(M)) = 0$. Thus, the short exact sequence is split. □

**Definition 3.17.** Let $e \in A_{\text{adm}}$ be left special. Denote the full subcategory $\{M \in \text{M}(A) \mid eM = 0\}$ by $\text{M}(A)_{e}^\perp$.

If $E$ and $F$ are arbitrary categories, we can form the product category $E \times F$. There are canonical projection functors $E \times F \to E$ and $E \times F \to F$. When $E$ and $F$ are additive categories there are also natural inclusion functors $E \to E \times F$ and $F \to E \times F$. For example the inclusion functor $E \to E \times F$ sends $X \in \text{Ob}(E)$ to $(X, 0)$. In the case where $E$ and $F$ are additive, we will denote $E \times F$ by $E \oplus F$; $E \oplus F$ is again additive. When $E$ and $F$ are abelian, $E \oplus F$ is so as well.

**Proposition 3.18.** Let $e \in A_{\text{adm}}$ be left special and left split. Then, there is a canonical isomorphism of categories $\text{M}(A) \xrightarrow{\cong} \text{M}(A)_e \oplus \text{M}(A)_{e}^\perp$.

**Proof.** This follows from Lemma 3.12, Lemma 3.13, and Corollary 3.16. □

**Remark 3.19.** Suppose $e \in A_{\text{adm}}$ is left special. By Lemma 3.12, the objects of $\text{M}(A)_{e}^\perp$ are precisely the objects in $\text{M}(A)_e$ which are annihilated by $e$. When $A$ is unital, this observation has an interpretation in the language of torsion theories: in particular, it implies that the triple $\left(\text{M}(A)_e, \text{M}(A)_{e}^\perp, \left(\text{M}(A)_{e}^\perp\right)^\perp\right)$ is a TTF-triple ([12, Chapter IV.8]). An immediate corollary of Proposition 3.18 is that for left special and left split $e \in A_{\text{adm}}$, $\left(\text{M}(A)_{e}^\perp\right)^\perp = \text{M}(A)_e$. When $A$ is
unital, this tells us that the TTF-triple generated by $e$ is centrally splitting (c. f. [12, Proposition 8.5, Chapter IV]). For unital rings, centrally splitting triples are well behaved. It is an interesting question as to whether the obvious generalization of a centrally splitting TTF-triple to the idempotent context continues to have these nice properties. We would like to thank the referee for bringing some of these considerations to our attention.

**Proposition 3.20.** Let $e \in A_{idm}$ be left special and left split. Then, there is a canonical equivalence of categories

$$D(A) \xrightarrow{\cong} D(M(A)_e) \oplus D(M(A)_e^+)$$

**Proof.** A morphism $f$ in $C(A)$ mapping to $(f', f'')$ in $C(M(A)_e) \oplus C(M(A)_e)^+$ under the canonical isomorphism $C(A) \xrightarrow{\cong} C(M(A)_e) \oplus C(M(A)_e)^+$ introduced in Proposition 3.18 (here, we are using an obvious extension of Proposition 3.18 to the category $C(A)$) is a quasi-isomorphism if and only if $f'$ and $f''$ are quasi-isomorphisms in $C(M(A)_e)$ and $C(M(A)_e^+)$ respectively. The proposition follows as localization behaves well with respect to the formation of categorical products. \hfill $\square$

**Remark 3.21.** Proposition 3.20 provides a derived analog of Bernstein’s decomposition of categories when the algebra $A$ is the Hecke algebra of a connected reductive algebraic group defined over a non-archimedean local field.

**Definition 3.22.** Denote the full triangulated subcategory $\{ M \in D(A) \mid H^i(M) \in M(A)_e, \forall \ i \in \mathbb{Z} \}$ by $D(A)_e$.

For left special and left split $e \in A_{idm}$, the reader may check that the full subcategory of $D(A)$ corresponding to $D(M(A)_e)$ under the equivalence of categories established in Corollary 3.20 is $D(A)_e$.

As $\Gamma_e$ is exact for left special and left split $e$, it extends to a functor $D(A) \to D(A)$; we continue to denote this extension by $\Gamma_e$. Using the exactness of $\Gamma_e$, it is straightforward to verify that $D(A)_e = \{ M \in D(A) \mid \Gamma_e(M) \cong M \}$; the reader may also verify that $D(A)_e$ is the essential image of $\Gamma_e$.

**Theorem 3.23.** Let $e \in A_{idm}$ be left special and left split. There is a commutative diagram of functors:

$$
\begin{array}{ccc}
D(A) & \xrightarrow{\cong} & D(M(A)_e) \\
\Gamma_e & \xrightarrow{\cong} & \text{RHom}_A(Ae, -) \\
D(A)_e & \xrightarrow{\cong} & D(eAe)
\end{array}
$$

The functor

$$\text{RHom}_A(Ae, -) : D(A)_e \to D(eAe)$$

is an equivalence of categories, with quasi-inverse

$$A \otimes_{eAe}^L (-) : D(eAe) \to D(A)_e.$$
Lemma 2.2. The fact that $\text{Hom}_A(\color{red}{-}, \color{red}{-})$ is left adjoint to $\text{RHom}_A(\color{red}{Ae}, \color{red}{-})$ as functors between $D(A)$ and $D(Ae)$. The canonical functor is left adjoint to $\text{RHom}_A(\color{red}{Ae}, \color{red}{-})$ as functors between $D(A)$ and $D(Ae)$.

We are left with considering the functors on the lower edge of the diagram. If $M \in D(A)_{e}$ is such that $\text{RHom}_A(\color{red}{Ae}, \color{red}{M}) = 0$, then $\text{Hom}_A(\color{red}{Ae}, \color{red}{H^i(M)}) = 0$ for all $i \in \mathbb{Z}$. Since $H^i(M) \in M(\color{red}{Ae})$, for all $i \in \mathbb{Z}$, we see that $M = 0$. From this, it follows that $\text{Ae}$ is a generator for the category $D(A)_{e}$, it is also compact. Since the triangulated category $D(A)_{e}$ is algebraic ([9, Lemma 7.5]), by [8, Theorem 3.3] the functor $\text{RHom}_A(\color{red}{Ae}, \color{red}{-})$ establishes an equivalence of categories between $D(A)_{e}$ and $D(\text{RHom}_A(\color{red}{Ae}, \color{red}{-})^{\text{op}})$.

3.4. Decompositions

Lemma 3.24. Let $e, e' \in A_{\text{idm}}$ be left special. Then, $eAe' = 0$ if and only if $M(\color{red}{A})_{e} \cap M(\color{red}{A})_{e'} = 0$.

Proof. Suppose $eAe' = 0$, and let $N \in M(\color{red}{A})_{e} \cap M(\color{red}{A})_{e'}$. Then, by definition, $N = AeAe'N$. Thus $N = 0$. Conversely, suppose $M(\color{red}{A})_{e} \cap M(\color{red}{A})_{e'} = 0$. If $f \in \text{Hom}_A(\color{red}{Ae}, \color{red}{Ae'})$, $f(\color{red}{Ae}) \in M(\color{red}{A})_{e} \cap M(\color{red}{A})_{e'}$. Thus $\text{Hom}_A(\color{red}{Ae}, \color{red}{Ae'}) = 0$. By Lemma 2.2, $eAe' \cong \text{Hom}_A(\color{red}{Ae}, \color{red}{Ae'})$. This completes the argument.

Corollary 3.25. Let $e, e' \in A_{\text{idm}}$ be left special. Then, $eAe' = 0$ if and only if $e'eAe = 0$.

Definition 3.26. Let $e, e' \in A_{\text{idm}}$ be left special. Then $e$ and $e'$ are said to be strongly orthogonal if $eAe' = e'eAe = 0$.

Lemma 3.27. Suppose $e, e' \in A_{\text{idm}}$ are strongly orthogonal. Let $M \in M(\color{red}{A})_{e}$ and $N \in M(\color{red}{A})_{e'}$. Then, $\text{Hom}_A(M, N) = 0$.

Proof. Suppose $f \in \text{Hom}_A(M, N)$. Then, $f(M) \in M(\color{red}{A})_{e} \cap M(\color{red}{A})_{e'}$, so by Lemma 3.24 we get that $f = 0$.

Theorem 3.28. Let $\mathcal{E} = \{e_i\}_{i \in I}$ be a family of left special idempotents in $A$. Then, the following are equivalent:

(a.) For every $M \in M(\color{red}{A})$, the canonical morphism $\bigoplus_{i \in I} \text{Hom}_A(M, e_i(M)) \to M$ is an isomorphism.

(b.) The canonical functor $\pi_\mathcal{E} : M(\color{red}{A}) \to \prod_{i \in I} M(\color{red}{A})_{e_i}$ induced by the functors $\Gamma_{e_i} : M(\color{red}{A}) \to M(\color{red}{A})_{e_i}$ for $i \in I$ is an equivalence of categories.

(c.) The following two conditions hold:
   (i) The family $\mathcal{E}$ is pairwise strongly orthogonal.
   (ii) The ideal $\sum_{i \in I} Ae_iA = A$.

Proof. We first show that $\pi_\mathcal{E}$ is essentially surjective if and only if $\mathcal{E}$ is pairwise strongly orthogonal.

First assume that $\mathcal{E}$ is pairwise strongly orthogonal, and let $(M_i)_{i \in I}$ be an object of $\prod_{i \in I} M(\color{red}{A})_{e_i}$. Consider $M := \bigoplus_{i \in I} M_i; M \in M(\color{red}{A})$. Since the distinct elements of $\mathcal{E}$ are strongly orthogonal to each other, $\Gamma_{e_i}(M) = M_i$ for any $i \in I$. Thus $\pi_\mathcal{E}(M) = (M_i)_{i \in I}$.

Conversely, suppose $\pi_\mathcal{E}$ is essentially surjective. We wish to show that $\mathcal{E}$ is pairwise strongly orthogonal. Apropos, let $e_i, e_j \in \mathcal{E}$, with $i \neq j$. Suppose $0 \neq N \in M(\color{red}{A})_{e_i} \cap M(\color{red}{A})_{e_j}$. Consider the
object \((N_i)_{i \in I}\) with \(N_i = N\) and \(N_k = 0\) for all \(k \neq i\). Suppose \((N_i)_{i \in I} \cong \pi_\mathcal{E}(M)\) for some \(M \in \mathcal{M}(A)\). Then, \(\Gamma_i(M) = N\), while \(\Gamma_j(M) = 0\). This is a contradiction.

If (a.) holds, then for any \(M \in \mathcal{M}(A)\) and \(e_i, e_j \in \mathcal{E}\) such that \(i \neq j\), \(\Gamma_i(M) \cap \Gamma_j(M) = 0\). Thus \(\mathcal{M}(A)_{e_i} \cap \mathcal{M}(A)_{e_j} = 0\), and \(e_i\) and \(e_j\) are strongly orthogonal. Thus the functor \(\pi_\mathcal{E}\) is essentially surjective; using Lemma 3.27, we see that \(\pi_\mathcal{E}\) is faithful. It is clear that (a.) forces \(\pi_\mathcal{E}\) to be full. Thus (a.) implies (b.).

Now assume (b.) If \(\pi_\mathcal{E} : \mathcal{M}(A) \to \prod_{i \in I} \mathcal{M}(A)_{e_i}\) is an equivalence of categories, it is essentially surjective. Thus (i) holds. If \(\sum_{i \in I} A e_i A \neq A\), then \(L := A / \sum_{i \in I} A e_i A\) is a non-zero object of \(\mathcal{M}(A)\) such that \(e_i L = 0\) for all \(i \in I\). Then \(\pi_\mathcal{E}(L) = 0\), which contradicts the fact that \(\pi_\mathcal{E}\) is faithful. Thus (b.) implies (c.).

Assume (c.), and let \(M \in \mathcal{M}(A)\). Since \(M = A M, M = \sum_{i \in I} \Gamma_i(M)\). Since the elements of \(\mathcal{E}\) are pairwise strongly orthogonal, this sum is direct. Thus we obtain (a.).

**Definition 3.29.** A family \(\{e_i\}_{i \in I}\) of left special idempotents in \(A\) which satisfies the equivalent conditions in Theorem 3.28 is said to be full.

**Corollary 3.30.** Let the family \(\mathcal{E} = \{e_i\}_{i \in I}\) be full. Then, every idempotent \(e_i \in \mathcal{E}\) is left split.

**Theorem 3.31.** Let the family \(\mathcal{E} = \{e_i\}_{i \in I}\) be full. Then, the canonical functor \(\tilde{\pi}_\mathcal{E} : \mathcal{D}(A) \to \prod_{i \in I} \mathcal{D}(A)_{e_i}\), induced by the functors \(\Gamma_i : \mathcal{D}(A) \to \mathcal{D}(A)_{e_i}\) for \(i \in I\) is an equivalence of categories.

**Proof.** We only sketch the proof. It is clear from Theorem 3.28 that \(\tilde{\pi}_\mathcal{E}\) is essentially surjective. We therefore only need to show that \(\tilde{\pi}_\mathcal{E}\) is full and faithful. By Corollary 3.30, each \(e_i \in \mathcal{E}\) is left split. If \(M\) and \(N\) are objects in \(\mathcal{D}(A)\), we can assume that both \(M\) and \(N\) are \(K\)-projective. Then, as summands of \(M\) and \(N\) respectively, \(\Gamma_i(M)\) and \(\Gamma_i(N)\) are \(K\)-projective for every \(e_i \in \mathcal{E}\) (here, we are using an obvious extension of Proposition 3.18 to the category \(K(A)\)); in particular, for every \(e \in \mathcal{E}\) \(\text{Hom}_{\mathcal{D}(A)}(\Gamma_i(M), \Gamma_i(N)) = \text{Hom}_{K(A)}(\Gamma_i(M), \Gamma_i(N))\). The result now follows from Theorem 3.28.

### 3.5. Miscellany

It would be interesting and useful to characterize left special and left split idempotents for a given idempotent algebra. This seems to be a difficult question in general: for instance, consider the case of \(\mathcal{H}(G)\), as described in Section 3.1.

For simple idempotent algebras, however, the situation presents no difficulty.

**Proposition 3.32.** Let \(A\) be a simple algebra. Then, every \(e \in A_{\text{idm}}\) is left special and left split.

**Proof.** Suppose \(e \in A_{\text{idm}}\). Since \(A\) is simple, \(A = AeA\). This implies that, for any \(M \in \mathcal{M}(A), AeM = AM = M\). Thus \(\mathcal{M}(A)_{e} = \mathcal{M}(A)\). In particular, since \(\mathcal{M}(A)_{e}\) is closed under subquotients, \(e\) is left special; since \(\mathcal{M}(A)_{e} = \mathcal{M}(A)\), \(e\) is left split.

**Example 3.33.** Let \(F\) be an arbitrary field and let \(V\) be a vector space of countably infinite dimension over \(F\). Suppose \(\{x_i\}_{i \in \mathbb{N}}\) be a fixed ordered basis of \(V\). Define \(e_n \in \text{End}_F(V)\) to be the idempotent linear transformation with \(e_n(x_i) = x_i\) if \(i \leq n\) and \(e_n(x_i) = 0\) if \(i > n\). Set \(A_n = e_n \text{End}_F(V) e_n\) and \(A = \lim_{\rightarrow} A_n\). The algebras \(A_n \cong M_n(F)\) and \(A\) are simple idempotent algebras. Thus, by Proposition 3.32, every idempotent in these algebras is left special and left split.
The following result provides us with examples of idempotents that are both left special and left split, in another specific situation.

**Proposition 3.34.** Let \( e \in A_{\text{idm}} \) be a central idempotent. Then, \( e \) is left special and left split.

**Proof.** Let \( M \in \mathcal{M}(A) \). Since \( e \) is central, \( AeM = eM \). Therefore the category \( \{M \in \mathcal{M}(A) \mid M = AeM\} \) is precisely the full subcategory of all \( M \in \mathcal{M}(A) \) such that \( M = eM \), and is clearly closed under subquotients. Thus \( e \) is left special. Note that \( \Gamma_e(M) = eM \), for any \( M \in \mathcal{M}(A) \); in particular, \( \Gamma_e \) is exact.

By Lemma 3.11, \( e \) is left split precisely when \( el \) is injective for any injective object \( I \) in \( \mathcal{M}(A) \). This is indeed the case here: if \( f : M \rightarrow N \) is an injective morphism in the category \( \mathcal{M}(A) \) and \( g : M \rightarrow el \) is any morphism in \( \mathcal{M}(A) \), there is a morphism \( g' : N \rightarrow I \) such that \( ig = g'f \), where \( i \) is the canonical inclusion \( el \rightarrow I \). By post-composing \( g' \) with the morphism \( m : I \rightarrow el \) given by multiplication by \( e \), we arrive at a morphism \( mg' : N \rightarrow el \); by the idempotence of \( e \), \( mg'f = mg = g \). Thus \( el \) is injective.

**Remark 3.35.** The converse of Proposition 3.34 is not true. For example, if \( V \) is a two dimensional vector space then any non-central idempotent in \( \text{End}(V) \) is left split and left special by Proposition 3.32.

It is an interesting question to ask whether there are contexts under which the converse of Proposition 3.34 continues to hold. We have been unable to resolve this question in full generality for now, but would like to direct the reader’s attention toward Theorem 4.11, where we answer it in the affirmative for the case of path algebras of quivers. We would like to thank the referee for drawing our attention to this problem.

4. Path algebras

We now study left special and left split idempotents in path algebras. We begin by recalling some preliminaries.

Recall that a quiver is a quadruple \( Q := (V, E, s, t) \), with sets \( V \) and \( E \) and functions \( s, t : E \rightarrow V \). The elements of \( V \) and \( E \) are, respectively, called the vertices and edges of \( Q \), and if \( a \) is an edge, we respectively refer to \( s(a) \) and \( t(a) \) as the source and target of \( a \). Let \( E_V \) be the set of symbols \( \{e_v \mid v \in V\} \); there is a canonical bijection between \( E_V \) and \( V \). A path in \( Q \) is either a finite sequence of edges \( a_1, \ldots, a_n \) such that \( t(a_i) = s(a_{i+1}) \) (we will denote such a path by the concatenation \( a_n \cdots a_1 \)), or an element of \( E_V \). Let \( P \) denote the set of all paths in \( Q \). The elements of \( P \setminus E_V \) are called non-trivial paths, while those in \( E_V \) are called trivial. The path \( e_v \) should be thought of as the path of length 0 at the vertex \( v \).

There are obvious extensions of \( s \) and \( t \) to functions \( s, t : P \rightarrow V \).

**Definition 4.1.** Let \( Q \) be a quiver. The path algebra of \( Q \), denoted by \( \mathbb{K}Q \), is the free module on the set \( P \), with multiplication defined as follows and then extended bilinearly:

1. If \( p \) and \( q \) are paths with \( t(q) \neq s(p) \), then \( pq := 0 \).
2. If \( p \) and \( q \) are non-trivial paths with \( t(q) = s(p) \), then \( pq \) is defined as the concatenation of \( p \) and \( q \).
3. If \( p \) is a path, then \( e_{t(p)}p = p e_{s(p)} := p \).

For example, if we take \( Q \) to be the quiver with a single vertex and \( n \) distinct paths that start and end at that vertex, then \( \mathbb{K}Q \) is the free associative \( \mathbb{K} \)-algebra in \( n \) non-commuting variables.
**Definition 4.2.** Let $Q$ be a quiver. Every element $x \in \mathbb{K}Q$ can be uniquely decomposed as $x = x_0 + x_+$, where $x_0$ is a $\mathbb{K}$-linear sum of paths of length 0, and $x_+$ is a $\mathbb{K}$-linear sum of paths of strictly positive length. We call the decomposition $x = x_0 + x_+$ the *canonical decomposition* of the element $x$. An element $x \in \mathbb{K}Q$ is said to be *homogeneous* if $x = x_0$, or, equivalently, if $x_+ = 0$.

**Remark 4.3.** For every finite subset $S \subseteq V$ the homogeneous element $e_S := \sum_{v \in S} e_v$ is an idempotent in the algebra $\mathbb{K}Q$. It follows that $\mathbb{K}Q$ is always idempotent. It is unital precisely when the vertex set $V$ is finite, in which case $e_V$ is the unit.

**Definition 4.4.** Let $Q$ be a quiver, with vertex set $V$. A finite subset $S \subseteq V$ is said to be *left closed* if whenever $v \in S$ and $a$ is an edge with $s(a) = v$, then $t(a) \in S$. Similarly, a finite subset $S \subseteq V$ is said to be *right closed* if whenever $v \in S$ and $a$ is an edge with $t(a) = v$, then $s(a) = v$.

**Remark 4.5.** Note that a finite set $S \subseteq V$ is left and right closed precisely when it is a union of connected components of the underlying directed graph $(V, E)$ of $Q$.

**Definition 4.6.** Let $S$ be a set, and let $L$ be a subset of $\mathbb{K}$ which contains 0. Let $f$ be a map $f : S \rightarrow L$. The *support* of $f$, denoted by $\text{supp}(f)$, is the complement of $f^{-1}([0])$ in $S$. A map $f : S \rightarrow L$ is said to have *finite support* if $\text{supp}(f)$ is finite.

**Lemma 4.7.** Let $Q$ be a quiver, with vertex set $V$. Let $f : V \rightarrow \mathbb{K}_{\text{idm}}$ be a map with finite support such that $f(t(p))f(s(p)) = f(s(p))$ for all paths $p$ in $P$. The element $e = \sum_{v \in \text{supp}(f)} f(v)e_v$ is a homogeneous left special idempotent in $\mathbb{K}Q$. Moreover, $e\mathbb{K}Qe = \mathbb{K}Qe$.

**Proof.** Let $f : V \rightarrow \mathbb{K}_{\text{idm}}$ be a map with finite support such that $f(t(p))f(s(p)) = f(s(p))$ for all paths $p$ in $P$. Denoting $\lambda_v := f(v)$ and $S := \text{supp}(f)$, define $e = \sum_{v \in S} \lambda_v e_v$. It is straightforward to check that $e$ is a homogeneous idempotent.

We first show that $e\mathbb{K}Qe = \mathbb{K}Qe$. Let $p \in P$, and consider the element $pe$. It is easy to see that $pe = \lambda_{t(p)}p$ (in particular, if $s(p) \notin \text{supp}(f)$, then $pe = 0$). We will show that $epe = pe$. In order to do so, note that $epe = \lambda_{s(p)}\lambda_{t(p)}p = \lambda_{s(p)}\lambda_{t(p)}p$. Since $\lambda_{t(p)}\lambda_{s(p)} = \lambda_{s(p)}$ for all paths $p \in P$, $\lambda_{t(p)}\lambda_{s(p)}p = \lambda_{s(p)}p$; thus, $epe = pe$. It follows that $e\mathbb{K}Qe = \mathbb{K}Qe$.

Choose $M \in \text{M}(\mathbb{K}Q)$ such that $\mathbb{K}QeM = M$. Since $epe = pe$ for every path $p \in P$, $M = \mathbb{K}QeM = eM$, and every element of $M$ is fixed by $e$. Thus $\mathbb{K}QeM = M$ if and only if every element of $M$ is fixed by $e$. As this condition is obviously inherited by submodules, $e$ is left special. \qed

**Theorem 4.8.** Let $Q$ be a quiver, with vertex set $V$ and edge set $E$. Let $e \in \mathbb{K}Q$, with canonical decomposition $e = e_0 + e_+$. Then, the following are equivalent:

(a.) The element $e$ is a left special idempotent
(b.) The element $e$ is a left special idempotent, and $e\mathbb{K}Qe = \mathbb{K}Qe$.
(c.) The element $e_0$ is a homogeneous left special idempotent such that $e_0e_+ = e_+$ and $e_+e_0 = 0$.
(d.) The element $e_0$ is a homogeneous left special idempotent such that $e_0e_+ = e_+$, $e_+e_0 = 0$, and $e_0\mathbb{K}Qe_0 = \mathbb{K}Qe_0$.
(e.) The following exist:
   (i) A map $f : V \rightarrow \mathbb{K}_{\text{idm}}$ with finite support such that $f(t(p))f(s(p)) = f(s(p))$ for all paths $p$ in $P$,
   (ii) A map $g : P \rightarrow \mathbb{K}$ with finite support such that $f(t(p))g(p) = g(p)$ and $f(s(p))g(p) = 0$ for all paths $p$ in $P$,
   such that $e = \sum_{v \in \text{Supp}(f)} f(v)e_v + \sum_{q \in \text{Supp}(g)} g(q)q$. 


Moreover, when the above equivalent statements hold, we have that $e\mathbb{K}Q = e_0\mathbb{K}Q$ and $M(\mathbb{K}Q)_e = M(\mathbb{K}Q)_{e_0}$.

**Proof.** Suppose (d) holds, and let $e = e_0 + e_+$ be the canonical decomposition of $e$. Since $e_0e_+ = e_+$ and $e_+e_0 = 0$, $e_0 = e$ and $ee_0 = e_0$. Consider $\mathbb{K}Q e$. Since $e = e_0 e$, $e_0 = ee_0$, and $e_0\mathbb{K}Q e_0 = \mathbb{K}Q e_0$, $\mathbb{K}Q e = \mathbb{K}Q e_0 e = e_0 \mathbb{K}Q e_0 e = e_0 \mathbb{K}Q e_0 e = e\mathbb{K}Q e$. Thus $e\mathbb{K}Q e = \mathbb{K}Q e$, and $epe = pe$ for every path $p \in P$.

Choose $M \in M(\mathbb{K}Q)$ such that $\mathbb{K}Q e M = M$. Since $epe = pe$ for every path $p \in P$, $M = \mathbb{K}Q e M = eM$, and every element of $M$ is fixed by $e$. Thus $\mathbb{K}Q e M = M$ if and only if every element of $M$ is fixed by $e$. As this condition is obviously inherited by submodules, $e$ is left special. Thus (d) implies (b).

It is obvious that (b) implies (a). We next prove that (a) implies (e). Assume that (a) holds: let $e \in \mathbb{K}Q_{\text{idm}}$, and suppose $e$ is left special. Let $e = e_0 + e_+$ be the canonical decomposition of $e$. Suppose $e_+$ is non-zero. Then there is a finite subset $S \subseteq V$, $\{\lambda_v\}_{v \in S}$, $p_1, \ldots, p_n$, and $\kappa_{11}, \ldots, \kappa_{n}$ such that $e_0 = \sum_{v \in S} \lambda_v e_v$ and $e_+ = \sum_{i=1}^{n} \kappa_i p_i$, where the $\lambda_v$ and $\kappa_i$ are non-zero elements of $\mathbb{K}$, and the $p_i$ are distinct non-trivial paths in $Q$. A straightforward multiplication shows that each $\lambda_v$ is idempotent.

Suppose $S$ is not left closed. Then, there is an edge of $Q$, say $a$, such that $s(a')$ belongs to $S$ but $t(a')$ does not. Define $M = \mathbb{K}_{\lambda_s(a')}$, and consider $M \times M$. We give $M \times M$ the structure of a non-degenerate $\mathbb{K}Q$-module as follows. Let $e_{s(a')} \otimes e_{t(a')}$ act via the map $(x, y) \mapsto (x, 0)$, $e_{s(a')} \otimes e_{t(a')} \otimes e_{s(a')} \otimes e_{t(a')}$ act via the map $(x, y) \mapsto (0, x)$. Define the action of all other edges and trivial paths to be 0. It can be checked that this extends naturally to give $M \times M$ the structure of a $\mathbb{K}Q$-module: call this module $N$. Since $e_{s(a')} + e_{t(a')}$ fixes $N$, $N$ is non-degenerate. Now $e_{s(a')} \otimes e_{t(a')} \in N$ is fixed by $\lambda_{s(a')}$, while $\lambda_{t(a')}$ is the $\lambda$-module $M \times M$; it follows that $\mathbb{K}Q e N = N$. However, it is easily checked that $L := 0 \times M$ is a $\mathbb{K}Q$-submodule of $N$, and that $\mathbb{K}Q e L = 0$. This is a contradiction. Thus $S$ is left closed.

A similar argument gives the condition on idempotents. Since $S$ is left closed, it is enough to show that for any edge $a$ with $t(a), s(a) \in S$, $\lambda_{s(a)} \subseteq \lambda_{t(a)}$: we can assume that $s(a) \neq t(a)$. Define $M' = \mathbb{K}_{\lambda_{s(a)}}$, and construct a $\mathbb{K}_Q$-module $N'$ as we did above for $M$ and $N$, but with the edge $a$ playing the role that $a'$ played in the previous construction. As before, $\mathbb{K}QeN = N'$, and $L' := 0 \times M'$ is a $\mathbb{K}Q$-submodule of $N'$. Since $e$ is assumed left special, $L' = \mathbb{K}Q e L'$; by construction $\mathbb{K}Q e L' = \lambda_{s(a)} e_{t(a)} L'$. In particular, $L'$ is fixed by $\lambda_{t(a)}$. Since $M' = \mathbb{K}_{\lambda_{s(a)}}$, this implies that $\lambda_{s(a)}$ is fixed by $\lambda_{t(a)}$, or equivalently that $\lambda_{s(a)} \subseteq \lambda_{t(a)}$.

Let $P''$ denote those paths in $P' := \{p_i \mid i = 1, \ldots, n\}$ which cannot be written as the concatenation of shorter non-trivial paths in $P'$; note that every path in $P' \setminus P''$ is a concatenation of shorter paths in $P'$. Then, $e = \sum_{v \in S} \lambda_v e_v + \sum_{p_i \in P'} \kappa_i p_i + \sum_{p_j \in P''} \kappa_j p_j$. Let $p_i \in P''$. If $s(p_i) \in S$, then since $S$ is left closed $t(p_i) \in S$. Since $e$ is idempotent, by squaring and comparing both sides of the preceding equation we see that $\kappa_i p_i = \lambda_{s(p_i)} e_{t(p_i)} p_i + \lambda_{s(p_i)} \kappa_i p_i e_{t(p_i)}$. Therefore $\kappa_i = (\lambda_{s(p_i)} + \lambda_{t(p_i)}) \kappa_i$. However, by what we have shown earlier in this proof, in this situation $\lambda_{s(p_i)} \subseteq \lambda_{t(p_i)}$: thus, $\lambda_{s(p_i)} \kappa_i = \kappa_i$, and $\lambda_{s(p_i)} \kappa_i = 0$. If $s(p_i) \notin S$, a similar, but easier, comparison shows that $\lambda_{s(p_i)} \kappa_i = \kappa_i$. Finally, suppose that $p_i \in P' \setminus P''$. If $s(p_i) \in S$, then we can write $\kappa_i p_i = \lambda_{s(p_i)} e_{t(p_i)} p_i + \lambda_{s(p_i)} \kappa_i p_i e_{t(p_i)} + \sum_{j,k} \kappa_j \kappa_k p_j p_k$, where the $p_j$ and $p_k$ are shorter concatenatable non-trivial paths such that $p_j p_k = p_i$. By induction, $\lambda_{s(p_i)} \kappa_i = \kappa_k$, while $\lambda_{s(p_i)} \kappa_j = 0$. Thus $\kappa_j \kappa_k = 0$. It follows that $\kappa_i p_i = \lambda_{s(p_i)} \kappa_i e_{t(p_i)} p_i + \lambda_{s(p_i)} \kappa_i p_i e_{t(p_i)}$, and we can now argue as before. The case where $s(p_i) \notin S$ is similar.

Define a map $f : V \to \mathbb{K}_{\text{idm}}$ as follows: $f(v) = \lambda_v$ if $v \in S$, while $f(v) = 0$ if $v \notin S$. Similarly, define a map $g : P \to \mathbb{K}$ as follows: $g(p) = \kappa_i$ if $p = p_i$ for some $i \in \{1, \ldots, n\}$, while $g(p) = 0$ if
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With these choices of maps for \( f \) and \( g \), the statement in (e) holds. This takes care of the case when \( e_+ \) is non-zero. The proof when \( e_+ = 0 \) is similar, in fact much simpler, and is thus omitted (here \( g \) is the constant map zero). Thus (a) implies (e).

Let (e) hold. The idempotent \( e \) admits a canonical decomposition \( e = e_0 + e_+ \), with \( e_0 = \sum_{v \in \text{Supp}(f)} f(v)e_v \) and \( e_+ = \sum_{q \in \text{Supp}(g)} g(q)q \). By Lemma 4.7, \( e_0 \) is a homogeneous left special idempotent. A straightforward calculation shows that \( e_0e_+ = e_+ \), while \( e_+e_0 = 0 \). Thus (d) holds.

We have shown the equivalence of (a), (b), (d), and (e). It is clear that (d) implies (c). Conversely, if (c) holds, then \( e_0 \) is assumed to be a left special idempotent. Using the equivalence of (a) and (b), it follows that \( e_0 | \mathbb{K}Qe_0 = \mathbb{K}Qe_0 \). Thus (d) holds.

This completes the equivalence of the five statements in the theorem. That \( e \mathbb{K}Q = e \mathbb{K}Q \) is a straightforward consequence of the identities \( ee_0 = e_0 \) and \( e_0e = e \). The equality \( M(\mathbb{K}Q)_e = M(\mathbb{K}Q)_{e_0} \) follows.

The proof of Theorem 4.8 provides us with a useful observation which we record as our next result.

**Proposition 4.9.** Let \( Q \) be a quiver, with vertex set \( V \). Let \( e \in \mathbb{K}Q \) be a left special idempotent, with canonical decomposition \( e = e_0 + e_+ \). If \( e_0 = \sum_{v \in S} \lambda_v e_v \), with \( \lambda_v \neq 0 \) for all \( v \in S \), then the set \( S \) is left closed.

**Corollary 4.10.** Let \( Q \) be a quiver, with vertex set \( V \). Let \( S \) be a finite subset of \( V \). Then, the idempotent \( e_S \) is left special if and only if \( S \) is left closed.

**Theorem 4.11.** Let \( Q \) be a quiver, with vertex set \( V \). Let \( e \in \mathbb{K}Q \). Then, the following statements are equivalent:

(a) The element \( e \) is a left split and left special idempotent.
(b) There are finite connected components \( S_1, \ldots, S_m \) of \( V \) and non-zero elements \( \lambda_1, \ldots, \lambda_m \) of \( \mathbb{K}_{\text{idm}} \) such that \( e = \sum_{i=1}^m \lambda_i e_{S_i} \).
(c) The element \( e \) is a central idempotent in \( \mathbb{K}Q \).

In particular, an idempotent \( e \in \mathbb{K}Q \) is both left special and left split if and only if it is central.

**Proof.** Suppose first that (b) holds. Each \( e_{S_i} \) is central, for each \( i \in \{1, \ldots, n\} \): if \( p \) is a path in \( Q \), then \( e_{S_i}p = p = pe_{S_i} \) if \( s(p) \) (or equivalently, \( t(p) \)) lies in \( S_i \), while \( ep = pe = 0 \) if \( t(p) \) and \( s(p) \) do not lie in \( S_i \). This implies that \( e \) is central. Since \( \lambda_i \in \mathbb{K}_{\text{idm}} \) for all \( i \) and \( \lambda_i \not\in S_j \) for \( i \neq j \), \( e \) is also an idempotent. Thus (c) holds. That (c) implies (a) is a direct consequence of Proposition 3.34.

Now suppose that (a) holds, i.e. the idempotent \( e \) is left split and left special. Let \( e = e_0 + e_+ \) be its canonical decomposition and suppose that \( e_0 = \sum_{v \in S} \lambda_v e_v \) with \( \lambda_v \neq 0 \) for each \( v \in S \). Let \( e_+ = \sum_{i=1}^n k_i p_i \). By Proposition 4.9, the set \( S \) is left closed. Suppose \( S \) is not right closed. Then there is an edge \( a \) such that \( t(a) \) lies in \( S \) while \( s(a) \) does not. By Corollary 3.16, the algebra \( \mathbb{K}Q \) admits a decomposition \( \mathbb{K}Q = \mathbb{K}Qe\mathbb{K}Q \oplus \mathbb{K}A' \), where \( \mathbb{K}Qe\mathbb{K}Q \) is pointwise fixed by \( e \) (using Theorem 4.8), while \( A' \) is some \( \mathbb{K}Q \)-submodule of \( \mathbb{K}Q \) which is annihilated by \( e \). By Theorem 4.8, we also know that \( e_0e = e \) and \( e_0c_0 = e_0c_0 \). Note that \( e_0A' = e_0c_0A' \subset eA' = \{0\} \), so \( e_0A' = 0 \).

Consider the idempotent \( e_{(a)} \). This idempotent admits a decomposition \( e_{(a)}(a) = x + y \), where \( x \in \mathbb{K}Qe\mathbb{K}Q \) and \( y \in A' \) : note that \( e_0y = 0 \), while \( e_0x = x \) (this last claim follows from the fact that \( e_0e = e \)). Thus \( x = e_0x = e_0e_{(a)} = 0 \), since \( s(a) \not\in S \). This implies that \( e_{(a)} \), and therefore \( \lambda_{(a)}e_{(a)} \), lie in \( A' \). This is a contradiction, as \( a \lambda_{(a)}e_{(a)} = \lambda_{(a)}e_{(a)}d = \lambda_{(a)}e_{(a)}d \in \mathbb{K}Qe\mathbb{K}Q \). Thus \( S \) is right closed. Since \( S \) is both left and right closed, it is a union of connected components of \( V \); it is clear that these components are finite, as \( S \) is.
We are left to show the condition on idempotents. It is enough to show that for any edge \(a\) with \(t(a), s(a) \in S\), \(\lambda_{t(a)} = \lambda_{s(a)}\). By Theorem 4.8, we know that \(K\lambda_{s(a)} \subseteq K\lambda_{t(a)}\). Consider the element \((\lambda_{t(a)} - \lambda_{s(a)})e_{s(a)}\). Since \(e\) is left split, there exist \(x \in KQeKQ\) and \(y \in A'\) such that \((\lambda_{t(a)} - \lambda_{s(a)})e_{s(a)} = x + y\). Then \(e_0(\lambda_{t(a)} - \lambda_{s(a)})e_{s(a)} = e_0x + e_0y\). We have \(e_0(\lambda_{t(a)} - \lambda_{s(a)})e_{s(a)} = (\lambda_{t(a)} - \lambda_{s(a)})\lambda_{s(a)}e_{s(a)} = 0\), since \(\lambda_{s(a)}\) fixes \(\lambda_{s(a)}\). Thus \(e_0x = e_0y = 0\). Arguing as above we get that, \(e_0x = x\). Thus \(x = 0\), and \((\lambda_{t(a)} - \lambda_{s(a)})e_{s(a)} \in A'\). This implies that \(\lambda_{t(a)}a(\lambda_{t(a)} - \lambda_{s(a)})e_{s(a)} = 0\).

Thus \(\lambda_{t(a)} = \lambda_{t(a)}\lambda_{s(a)}\). Since we know that \(\lambda_{s(a)} = \lambda_{t(a)}\lambda_{s(a)}\), we get that \(\lambda_{s(a)} = \lambda_{t(a)}\). Using Theorem 4.8, this is enough to show \(e_+ = \sum_{i=1}^n \kappa_i p_i = 0\) if \(i \in \{1, \ldots, n\}\), then \(\kappa_i = \lambda_{t(p_i)}\kappa_i = \lambda_{s(p_i)}\kappa_i = 0\). Thus (b.) holds.

Corollary 4.12. Let \(Q\) be a quiver, with vertex set \(V\). Let \(S\) be a left closed subset of \(V\). Then, \(e_S\) is left split if and only if \(S\) is right closed.

Example 4.13. Suppose \(Q\) is the quiver \(v_1 \xrightarrow{p} v_2\) and let \(S = \{v_2\}\). By Corollaries 4.10 and 4.12, it follows that the idempotent \(e_S\) is left special but not left split.

Proposition 4.14. Suppose \(K\) has no idempotents apart from 0 and 1. Let \(Q\) be a quiver, with vertex set \(V\). A family of left special idempotents \(E = \{e_i\}_{i \in I}\) is full if and only if there is a family of finite left closed subsets of \(V\), \(\mathcal{S} := \{S_i\}_{i \in I}\), with \(S_i \cap S_j = \emptyset\) for \(i \neq j\) and \(V = \bigcup_{i \in I} S_i\), such that \(e_i = e_{S_i}\) for all \(i \in I\).

Proof. First, consider a family \(\mathcal{S} := \{S_i\}_{i \in I}\) of finite left closed subsets of \(V\), with \(S_i \cap S_j = \emptyset\) for \(i \neq j\) and \(V = \bigcup_{i \in I} S_i\). By Corollary 4.10, each \(e_{S_i}\) is left special. Let \(S_i\) and \(S_j\) belong to \(\mathcal{S}\). \(e_{S_j}KQe_{S_i}\) is the module generated by all paths in \(Q\) which start at a vertex in \(S_i\) and end at a vertex in \(S_j\). Since \(S_i\) is left closed, such a path must have its target lie in \(S_i \cap S_j\). We conclude that \(e_{S_i}KQe_{S_j} = 0\). Thus, for \(i \neq j\) the idempotents \(e_{S_i}\) and \(e_{S_j}\) are strongly orthogonal. \(KQe_{S_i}KQ\) is the module generated by all paths which pass through a vertex in \(S_i\). As \(S_i\) is left closed, \(KQe_{S_i}KQ = e_{S_i}KQ\). Since the target of every path in \(KQ\) lies in some \(S_i \in \mathcal{S}\), it follows that \(\{e_{S_i}\}_{i \in I}\) is full.

For the converse, suppose \(E = \{e_i\}_{i \in I}\) is a full family of left special idempotents. By Corollary 3.30, every idempotent in \(E\) is left split. By Theorems 4.8 and 4.11, for every \(e_i \in E\) there is a finite right and left closed subset \(S_i\) of \(V\) such that \(e_i = e_{S_i}\). Since the elements of \(E\) are pairwise strongly orthogonal, for \(i \neq j\), we get that \(S_i \cap S_j = \emptyset\). It is also straightforward to see that \(V = \bigcup_{i \in I} S_i\).

Remark 4.15. Similar arguments provide a variant of Proposition 4.14 without the additional condition on idempotents in \(K\), which we omit here for the sake of brevity.

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