Local behavior of local times of super Brownian motion

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Abstract

For \( x \in \mathbb{R}^d - \{0\} \), in dimension \( d = 3 \), we study the asymptotic behavior of the local time \( L^x_t \) of super-Brownian motion \( X \) starting from \( \delta_0 \) as \( x \to 0 \). Let \( \psi(x) = ((1/2\pi^2) \log(1/|x|))^{1/2} \) be a normalization, Theorem 1 implies that \((L^x_t - (1/2\pi|x|))^{1/2}/\psi(x) \) converges in distribution to a standard normal distributed random variable as \( x \to 0 \). For dimension \( d = 2 \), Theorem 2 implies that \( L^x_t - (1/\pi) \log(1/|x|) \) is \( L^1 \) bounded as \( x \to 0 \). To do this, we prove a Tanaka formula for the local time which refines a result in Barlow, Evans and Perkins [1].

1 Introduction and main results

1.1 Introduction

Super Brownian Motion arises as a scaling limit of critical branching random walk. Let \( M_F = M_F(\mathbb{R}^d) \) be the space of finite measures on \( \mathbb{R}^d \) equipped with Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^d) \) and \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) be a filtered probability space. The Super-Brownian Motion \( X \) starting at \( \mu \in M_F(\mathbb{R}^d) \) is a continuous \( M_F(\mathbb{R}^d) \)-valued adapted strong Markov process defined on \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) with \( X_0 = \mu \) a.s. which is the unique in law solution of a martingale problem (see (1) below).

For \( 0 \leq t < \infty \), the weighted occupation time process is defined to be

\[ Y_t(A) := \int_0^t X_s(A)ds, \quad A \in \mathcal{B}(\mathbb{R}^d). \]
If $\mu$ is a measure on $\mathbb{R}^d$ and $\psi$ is a real-valued function on $\mathbb{R}^d$, we write $\mu(\psi)$ for $\int_{\mathbb{R}^d} \psi(y) d\mu(y)$.

Local times of superprocesses have been studied by many authors. Sugitani [7] has proved that given the joint continuity of $\mu_q(t)(x) = \int \mu(dy) \int_0^t p_s(x - y)ds$ in $(t,s)$, the local time $L^x_t$ has a jointly continuous version which satisfies that for any $\phi \in C_b(\mathbb{R}^d)$,

$$\int_0^t X_s(\phi)ds = \int_{\mathbb{R}^d} L^x_t \phi(x)dx.$$  

$L^x_t$ is called the local time of $X$ at point $x \in \mathbb{R}^d$ and time $t > 0$ and it also can be defined as

$$L^x_t := \lim_{\epsilon \to 0} \int_0^t X_s(p^x_{t\epsilon}) ds,$$

where $p^x_{t\epsilon}(y) = p_{\epsilon}(y - x)$ is the transition density of Brownian motion. In general, for any fixed $\epsilon > 0$, $L^x_t - L^x_{\epsilon}$ is jointly continuous in $t \geq \epsilon$ and $x \in \mathbb{R}^d$.

However, the condition of continuity of $\mu_q(t)(x)$ fails in $x = 0$ when $\mu = \delta_0$ in $d = 2$ and $d = 3$ (joint continuity still holds for $L^x_t - L^x_{\epsilon}$). Our main result Theorem 1 gives precise information about the local behavior of local times of super-Brownian motion in dimension $d = 3$. Let $x \in \mathbb{R}^d - \{0\}$ and $X$ be a super-Brownian motion initially in $\delta_0$, and $L^x_t$ be the local time of $X$ at time $t$ and point $x$. Theorem 1 tells us that as $x \to 0$ $L^x_t$ blows up like $1/|x|$ and has a variation like $\sqrt{\log 1/|x|}$. We can view this as an analogue to the classical Central Limit Theorem. For $d = 2$, we derive a refined Tanaka formula in Proposition 3 compared to the one in [1] and Theorem 2 tells us that $|L^x_t - \frac{1}{\pi} \log 1/|x||$ is $L^1$ bounded.

### 1.2 Notations and Properties of super-Brownian motion

We denote by $p_t(x) = (2\pi t)^{-d/2}e^{-|x|^2/2t}, t > 0, x \in \mathbb{R}^d$ the transition density of $d$-dimensional Brownian motion $B_t$. Let $P_t$ be the corresponding Markov semigroup, then for any function $\phi$,

$$P_t \phi(x) = \int p_t(y) \phi(x - y)dy.$$
Let $C_0^2(\mathbb{R}^d)$ denotes the set of all twice continuously differentiable functions on $\mathbb{R}^d$ with bounded derivatives of order less than 2. It is known that super-Brownian motion $X$ solves a martingale problem (Perkins [5], II.5): For any $\phi \in C_0^2(\mathbb{R}^d)$,

$$X_t(\phi) = X_0(\phi) + M_t(\phi) + \int_0^t X_s(\frac{\Delta}{2}\phi)ds,$$  

(1)

where $M_t(\phi)$ is an $\mathcal{F}_t$ martingale such that $M_0(\phi) = 0$ and the quadratic variation of $M(\phi)$ is

$$[M(\phi)]_t = \int_0^t X_s(\phi^2)ds.$$

For the first two moments of Super-Brownian motion, Konno and Shiga [4] gives us

$$E_X_0 X_t(\phi) = X_0(P_t\phi),$$

and

$$E_X_0(X_t(\phi)^2) = \left( X_0(P_t\phi) \right)^2 + \int_0^t X_0\left( P_s((P_{t-s}\phi)^2) \right)ds.$$  

We drop the subscript $X_0$ when there is no confusion.

**Notations.** $c_3 = 1/2\pi$, $c_{3.1} = 2c_3^2 = 1/2\pi^2$, $c_2 = 1/\pi$. The weird order here is to emphasize the dimension the constant is for.

### 1.3 Main result

**Theorem 1.** (d=3) Let $\psi(|x|) = (c_{3.1} \log 1/|x|)^{1/2}$, and $X$ be a super-Brownian motion in $\mathbb{R}^3$ with initial value $\delta_0$. Then for each $0 < t \leq \infty$ as $x \to 0$, we have

$$\left( X, \frac{L_t^x - c_3 \frac{1}{|x|}}{\psi(|x|)} \right) \xrightarrow{d} (X, Z)$$

where $Z$ is a random variable with standard normal distribution and independent of $X$. Moreover, convergence in probability fails.

**Theorem 2.** (d=2) Let $X$ be a super-Brownian motion in $\mathbb{R}^2$ with initial value $\delta_0$. Then we have

$$\limsup_{x \to 0} E \left| L_t^x - c_2 \log \frac{1}{|x|} \right| < \infty.$$
2 Proof of Theorem 1

Fix \( x \in \mathbb{R}^3 - \{0\} \), we will use the Tanaka formula for local times of super-Brownian motion (see [1], Theorem 6.1). Let \( \phi_x(y) = c_3/|y - x| \), under the assumption \( X_0(\phi_x) = \delta_0(\phi_x) = c_3/|x| < \infty \), we have \( P_{\delta_0} \) almost surely that

\[
L^x_t = c_3 \frac{1}{|x|} + M_t(\phi_x) - X_t(\phi_x),
\]

where \( M_t(\phi_x) \) is an \( \mathcal{F}_t \) martingale, with \( M_0(\phi_x) = 0 \) and quadratic variation

\[
[M(\phi_x)]_t = \int_0^t X_s(\phi_x^2) ds = \int_0^t \int \frac{c_3^2}{|y - x|^2} X_s(dy) ds.
\]

To prove Theorem 1, we need several propositions which are stated below and proofs of them will be shown in Section 2.2 after finishing the proof of Theorem 1.

Notations. We define \( g_x(y) := \log |y - x| \) for \( x, y \in \mathbb{R}^3 \).

Proposition 1. For \( d = 3 \), we have almost surely that

\[
X_t(g_x) = \delta_0(g_x) + M_t(g_x) + \frac{1}{2} \int_0^t \int \frac{1}{|y - x|^2} X_s(dy) ds.
\]

Proposition 2. For \( d > 1 \), we have

\[
\int_0^t \int \frac{1}{|y - x|} p_s(y) dy ds \leq \frac{2}{d-1} \mathbb{E}|B_t|, \forall x.
\]

2.1 Proof of Theorem 1

Before proceeding to the proof, we state some lemmas which will be used in proving Theorem 1.

Lemma 1. For any \( u, v \in \mathbb{R}^d - \{0\} \), we have

\[
| \log \frac{|u + v|}{|v|} | \leq \sqrt{\frac{|u|}{|v|}} + \sqrt{\frac{|u|}{|u + v|}}.
\]
Proof. Let $f(u) = \sqrt{u} - \log(1 + u)$ for $u \geq 0$. Observe that $f(0) = 0$ and

$$f'(u) = \frac{1}{2\sqrt{u}} - \frac{1}{1 + u} = \frac{(\sqrt{u} - 1)^2}{2\sqrt{u}(1 + u)} \geq 0,$$

therefore $f(u) \geq 0$ and $\log(1 + u) \leq \sqrt{u}$ for all $u \geq 0$.

If $|u + v| \geq |v|$, then

$$\left| \log \frac{|u + v|}{|v|} \right| = \log \frac{|u + v|}{|v|} \leq \log \frac{|u|}{|v|} \leq \sqrt{\frac{|u|}{|v|}} \leq \sqrt{\frac{|u|}{|u + v|}}.$$

If $|u + v| \leq |v|$, then

$$\left| \log \frac{|u + v|}{|v|} \right| = \log \frac{|v|}{|u + v|} \leq \log \frac{|v + u|}{|u + v|} \leq \sqrt{\frac{|u|}{|u + v|}} \leq \sqrt{\frac{|u|}{|v|}} + \sqrt{\frac{|u|}{|u + v|}}.$$

So Lemma 1 follows.

Lemma 2. For any $t > 0$, we have

$$\limsup_{x \to 0} E\left( \left( \int \frac{1}{|y - x|} X_t(dy) \right)^2 \right) < \infty.$$

Proof.

$$E\left( \left( \int \frac{1}{|y - x|} X_t(dy) \right)^2 \right) = \left[ \int p_t(y) \frac{1}{|y - x|} dy \right]^2 + \int_0^t ds \int p_s(y) dy \left( \int p_{t-s}(y - z) \frac{1}{|z - x|} dz \right)^2.$$

For the first term,

$$\int p_t(y) \frac{1}{|y - x|} dy \leq 1 + \int_{|y - x| < 1} \left( \frac{1}{\sqrt{2\pi t}} \right)^3 e^{-\frac{|y|^2}{2t}} \frac{1}{|y - x|} dy \leq 1 + \left( \frac{1}{\sqrt{2\pi t}} \right)^3 \int_{\mathbb{R}^3} \frac{1}{|y - x|} \mathbf{1}_{|y - x| < 1} dy = 1 + \left( \frac{1}{\sqrt{2\pi t}} \right)^3 4\pi \int_0^1 r^2 dr \frac{1}{r} < \infty.$$
For the second term, we use Cauchy Schwarz to get
\[
\left( \int p_{t-s}(y-z) \frac{1}{|z-x|} dz \right)^2 \\
\leq \int p_{t-s}(y-z) dz \cdot \int p_{t-s}(y-z) \frac{1}{|z-x|^2} dz \\
= \int p_{t-s}(y-z) \frac{1}{|z-x|^2} dz,
\]
and by Chapman-Kolmogorov
\[
\int_0^t ds \int p_s(y) dy \left( \int p_{t-s}(y-z) \frac{1}{|z-x|} dz \right)^2 \\
\leq \int_0^t ds \int p_s(y) dy \int p_{t-s}(y-z) \frac{1}{|z-x|^2} dz \\
= \int_0^t ds \int \frac{1}{|z-x|^2} dz \int p_s(y)p_{t-s}(y-z) dy \\
= \int_0^t ds \int \frac{1}{|z-x|^2} dz \cdot p_t(z) = t \int \frac{1}{|z-x|^2} p_t(z) dz.
\]
Using the same trick in the first term, we get
\[
\int \frac{1}{|z-x|^2} p_t(z) dz \leq 1 + \left( \frac{1}{\sqrt{2\pi t}} \right)^3 4\pi < \infty.
\]
Therefore we get
\[
\limsup_{x \to 0} E \left[ \left( \int \frac{1}{|y-x|} X_t(dy) \right)^2 \right] < \infty.
\]

Lemma 3. For any $t > 0$,

(i) $\limsup_{x \to 0} E \left( X_t^2(g_x) \right) < \infty$

and

(ii) $\limsup_{x \to 0} E \left( M_t^2(g_x) \right) < \infty$. 

\[ \square \]
Proof. (i) For $|y - x| < 1$, we bound $|g_x(y)| = \log 1/|y - x|$ by $1/|y - x|$, so

$$
\limsup_{x \to 0} E \left[ \left( \int_{|y - x| < 1} \log |y - x| X_t(dy) \right)^2 \right] 
\leq \limsup_{x \to 0} E \left[ \left( \int \frac{1}{|y - x|} X_t(dy) \right)^2 \right] < \infty
$$

according to Lemma 2.

For $|y - x| \geq 1$, we bound $|g_x(y)| = \log |y - x|$ by $|y - x|$, so

$$
E \left[ \left( \int_{|y - x| \geq 1} \log |y - x| X_t(dy) \right)^2 \right] \leq E \left[ \left( \int |y - x| X_t(dy) \right)^2 \right] 
= \left( \int p_t(y)|y - x| dy \right)^2 + \int_0^t ds \int p_s(z)dz \left( \int |y - x| p_{t-s}(z - y)dy \right)^2.
$$

It is clear that the first term is finite for any $x$ and for the second term,

$$
\int_0^t ds \int p_s(z)dz \left( \int p_{t-s}(z - y)|y - x|dy \right)^2 
\leq \int_0^t ds \int p_s(z)dz \int p_{t-s}(z - y)|y - x|^2 dy
\leq \int_0^t ds \int p_t(y)|y - x|^2 dy < \infty.
$$

So

$$
\limsup_{x \to 0} E \left[ \left( X_t(g_{x}) \right)^2 \right] 
= \limsup_{x \to 0} E \left[ \left( \int_{|y - x| < 1} \log |y - x| X_t(dy) + \int_{|y - x| \geq 1} \log |y - x| X_t(dy) \right)^2 \right] 
\leq 2 \limsup_{x \to 0} E \left[ \left( \int_{|y - x| < 1} \log |y - x| X_t(dy) \right)^2 \right] + 2 \limsup_{x \to 0} E \left[ \left( \int_{|y - x| \geq 1} \log |y - x| X_t(dy) \right)^2 \right] < \infty.
$$

(ii) Since $M_t(g_x)$ is a martingale with quadratic variation $[M(g_x)]_t = \int_0^t X_s(g_x^2)ds$, 

we get
\[
E\left(M_t^2(g_x)\right) = E \int_0^t X_s(g_x^2) \, ds = \int_0^t ds \int p_s(y) \left( \log |y - x| \right)^2 dy
\]
\[
\leq \int_0^t ds \int p_s(y) \frac{1}{|y - x|} 1_{|y - x| < 1} dy
+ \int_0^t ds \int p_s(y) |y - x| 1_{|y - x| \geq 1} dy
\]
\[
\leq \int_0^t ds \int p_s(y) \frac{1}{|y - x|} dy
+ \int_0^t ds \int p_s(y) |y - x| dy. \quad (\star)
\]

We use the fact that \( \log u \leq \log(1 + u) \leq \sqrt{u} \) for \( u \geq 1 \) by Lemma 1.

By Proposition 2 in \( d = 3 \), we get
\[
\int_0^t ds \int p_s(y) \frac{1}{|y - x|} dy \leq E \left| B_t \right| < \infty.
\]

As it is obvious that the latter term in (\( \star \)) above is finite, we get
\[
\limsup_{x \to 0} E\left(M_t^2(g_x)\right) < \infty.
\]

\[\square\]

2.1.1 Convergence in distribution

Observe that combining (3) and (4), we obtain
\[
[M(\phi_x)]_t = 2c_3^2 \left(X_t(g_x) - \delta_0(g_x) - M_t(g_x)\right).
\]

Note that \( \delta_0(g_x) = \log |x| = -\log 1/|x| \), so
\[
E\left(\left([M(\phi_x)]_t - c_{3.1} \log \frac{1}{|x|}\right)^2\right) = c_{3.1}^2 E\left(\left(X_t(g_x) - M_t(g_x)\right)^2\right),
\]
\[5\]
where \( c_{3.1} = 2c_3^2 \).

\[
E\left(\frac{[M(\phi_x)]_t - c_{3.1} \log \frac{1}{|x|}}{c_{3.1} \log \frac{1}{|x|}}\right)^2 = \frac{c_{3.1}^2}{(c_{3.1} \log \frac{1}{|x|})^2} E\left(\left(X_t(g_x) - M_t(g_x)\right)^2\right)
\]
\[
\leq \frac{2}{(\log \frac{1}{|x|})^2} \left[E\left(X_t^2(g_x)\right) + E\left(M_t^2(g_x)\right)\right] \to 0 \text{ as } x \to 0.
\]
by Lemma 3. Hence we have shown that
\[ \frac{|M(\phi_x)|_t}{c_{3.1} \log \frac{1}{|x|}} \xrightarrow{L^2} 1 \text{ as } x \to 0. \]  
(6)

Since \( \frac{|M(\phi_x)|_t}{c_{3.1} \log \frac{1}{|x|}} \) is the quadratic variation of martingale \( \frac{M_t(\phi_x)}{\sqrt{c_{3.1} \log \frac{1}{|x|}}} \), using the Dubins-Schwarz theorem (see [6], Theorem V.1.6), we can find some Brownian motion \( B^x(t) \) in dimension 1 depending on \( x \) such that
\[ \frac{M_t(\phi_x)}{\sqrt{c_{3.1} \log \frac{1}{|x|}}} = B^x \left( \frac{|M(\phi_x)|_t}{c_{3.1} \log \frac{1}{|x|}} \right). \]

For any sequence \( \{x_n\} \) that goes to 0, (6) implies that
\[ \tau_n := \frac{|M(\phi_{x_n})|_t}{c_{3.1} \log \frac{1}{|x_n|}} \xrightarrow{P} 1 \text{ as } n \to \infty, \]
and we claim that
\[ B^x_{\tau_n} = B^x_n \left( \frac{|M(\phi_{x_n})|_t}{c_{3.1} \log \frac{1}{|x_n|}} \right) \xrightarrow{d} Z, \]
where \( Z \sim N(0, 1) \) in dimension 1.

In fact for any bounded uniformly continuous function \( h(x), \forall \epsilon > 0, \exists \delta > 0 \) such that \( |h(x) - h(y)| < \epsilon \) holds for any \( x, y \in \mathbb{R} \) with \( |x - y| < \delta \). So
\[ E|h(B^x_{\tau_n}) - h(B^x_1)| \leq \epsilon + 2\|h\|_{\infty} \cdot P(|B^x_{\tau_n} - B^x_1| > \delta), \]
and for any \( \gamma > 0 \), we have
\begin{align*}
P(|B^x_{\tau_n} - B^x_1| > \delta) &\leq P(|B^x_{\tau_n} - B^x_1| > \delta, |\tau_n - 1| < \gamma) + P(|\tau_n - 1| > \gamma) \\
&\leq P( \sup_{|s-1| \leq \gamma} |B^x_s - B^x_1| > \delta) + P(|\tau_n - 1| > \gamma) \\
&= P( \sup_{|s-1| \leq \gamma} |B_s - B_1| > \delta) + P(|\tau_n - 1| > \gamma) \\
&< \epsilon + P(|\tau_n - 1| > \gamma), \text{ if we pick } \gamma \text{ small enough.}
\end{align*}

Since \( \tau_n \) converge in probability to 1, for \( n \) large enough, we have \( P(|\tau_n - 1| > \gamma) < \epsilon \) and so
\[ E|h(B^x_{\tau_n}) - h(B^x_1)| \leq \epsilon + 2\|h\|_{\infty} \cdot 2\epsilon. \]

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and hence
\[
{M_t(\phi_{x_n}) \over \sqrt{c_{3.1} \log {1 \over |x_n|}}} = B_{x_n} \overset{d}{\to} Z,
\]
where \(Z \sim N(0, 1)\). Recall that \(\phi_{x_n}(y) = c_3/|y - x_n|\) and by Lemma 2
\[
\lim_{n \to \infty} E\left[\left( {X_t(\phi_{x_n}) \over c_{3.1} \log {1 \over |x_n|}} \right)^2 \right] = 0,
\]

hence
\[
{X_t(\phi_{x_n}) \over \sqrt{c_{3.1} \log {1 \over |x_n|}}} \overset{p}{\to} 0.
\]
Combining (7) and (8), by Theorem 25.4 in Billingsley [2], we have
\[
{L_{x_n} - {1 \over |x_n|} \over \sqrt{c_{3.1} \log {1 \over |x_n|}}} = {M_t(\phi_{x_n}) \over \sqrt{c_{3.1} \log {1 \over |x_n|}}} - {X_t(\phi_{x_n}) \over \sqrt{c_{3.1} \log {1 \over |x_n|}}} \overset{d}{\to} Z.
\]
So any sequence that approaches 0 converges in distribution to \(Z\) as above, which implies that
\[
{L^x_t - {1 \over |x|} \over \sqrt{c_{3.1} \log {1 \over |x|}}} \overset{d}{\to} Z \text{ as } x \to 0.
\]

For \(t = \infty\), let \(\rho\) be the life time of super Brownian motion \(X\), then \(L^x_\infty = L^x_\rho\). Chp II.5 in Perkins [5] tells us that \(\rho < \infty\) a.s.. Sugitani [7] gives us
\[
L^x_t - L^x_\epsilon \text{ is continuous in } x \text{ for any } 0 < \epsilon < t,
\]
with the initial condition being \(\delta_0\).

Fix \(\epsilon\) small, we define \(L^x_\rho - L^x_\epsilon = 0\) if \(\rho < \epsilon\). As \(x \to 0\), we get
\[
{L^x_\rho - L^x_\epsilon \over (c_{3.1} \log {1 \over |x|})^{1/2}} \overset{0 \text{ a.s.}}{\to} 0,
\]
and by Theorem 25.4 in Billingsley [2] again we get
\[
{L^x_\rho - {c_3 \over |x|} \over (c_{3.1} \log {1 \over |x|})^{1/2}} = {L^x_\rho - L^x_\epsilon \over (c_{3.1} \log {1 \over |x|})^{1/2}} + {L^x_\epsilon - {c_3 \over |x|} \over (c_{3.1} \log {1 \over |x|})^{1/2}} \overset{d}{\to} Z.
\]
2.1.2 Remaining Part of Theorem 1

(i) Fix $0 < t \leq \infty$, let $Z_{x_n}^{x_n}$ denotes $(L_t^{x_n} - c_3/|x_n|)/(c_{3,1} \log 1/|x_n|)^{1/2}$. By tightness of each component in $(X, Z_{x_n}^{x_n})$, we clearly have tightness of $(X, Z_{x_n}^{x_n})$ as $x_n \to 0$, so it suffices to show all weak limit points coincide. Assume $(X, Z_{x_n}^{x_n})$ converges weakly to $(X, Z)$ for some sequence $x_n \to 0$. Let $(X, Z)$ be defined on $(\Omega, \mathcal{F}_t, \tilde{P})$ where $X$ is super-Brownian motion and $Z$ is standard normal under $\tilde{P}$.

For any $0 < t_1 < t_2 < \cdots < t_m$, let $\phi_0 : \mathbb{R} \to \mathbb{R}$ and $\psi_i : M_F \to \mathbb{R}$, $1 \leq i \leq m$ be bounded continuous, we have

$$\lim_{n \to \infty} E\left[\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m}) \phi_0(Z_{x_n}^{x_n})\right] = \tilde{E}\left[\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m}) \phi_0(Z)\right]$$

since we assume that $(X, Z_{x_n}^{x_n})$ converge weakly to $(X, Z)$.

Pick $\epsilon > 0$ such that $\epsilon < t_1$ and $\epsilon < t$, by Sugitani [7],

$$L_t^x - L_\epsilon^x$$

is continuous in $x$ for any $0 < \epsilon < t$

with the initial condition being $\delta_0$, when $n \to \infty$ we get

$$Z_{x_n}^{x_n} - Z_{x_n}^\epsilon = \frac{L_t^{x_n} - L_\epsilon^{x_n}}{(c_{3,1} \log 1/|x_n|)^{1/2}} \to 0 \text{ a.s.}$$

and hence

$$(0, Z_{x_n}^{x_n} - Z_{x_n}^\epsilon) \to (0, 0) \text{ a.s.}$$

By Theorem 25.4 in Billingsley [2] again

$$(X, Z_{x_n}^\epsilon) = (X, Z_{x_n}^{x_n}) - (0, Z_{x_n}^{x_n} - Z_{x_n}^\epsilon)$$

converge weakly to $(X, Z)$.

Therefore since $Z_{x_n}^{x_n} \in \mathcal{F}_t^X$,

$$I = \tilde{E}\left[\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m}) \cdot \phi_0(Z)\right]$$

$$= \lim_{n \to \infty} E\left[\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m}) \cdot \phi_0(Z_{x_n}^{x_n})\right]$$

$$= \lim_{n \to \infty} E\left[E\left(\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m})|\mathcal{F}_t^X\right) \cdot \phi_0(Z_{x_n}^{x_n})\right]$$

$$= \lim_{n \to \infty} E\left[E_{X_i}\left(\prod_{i=1}^m \psi_i(X_{t_i - \epsilon})\right) \cdot \phi_0(Z_{x_n}^{x_n})\right]$$
Define
\[ F_\epsilon(\mu) = E_\mu \left( \prod_{i=1}^{m} \psi_i(X_{t_i}) \right) \]
for \( \mu \in M_F \) and we prove by induction that \( F_\epsilon \in C_b(M_F) \). For \( m = 1 \) we have
\[ F_\epsilon(\mu) = E_\mu \left( \psi_1(X_{t_1}) \right) = P_{t_1-\epsilon} \psi_1(\mu). \]

By Theorem II.5.1 in Perkins [5], if \( P_t F(\mu) = E_\mu F(X_t) \), then \( P_t : C_b(M_F) \to C_b(M_F) \) so \( F_\epsilon = P_{t_1-\epsilon} \psi_1 \in C_b(M_F) \) since \( \psi_1 \in C_b(M_F) \). Suppose it holds for \( m - 1 \), then
\[
F_\epsilon(\mu) = E_\mu \left( \prod_{i=1}^{m} \psi_i(X_{t_i}) \right)
\[
= E_\mu \left[ \prod_{i=1}^{m-2} \psi_i(X_{t_i}) \cdot E_\mu \left( \psi_{m-1}(X_{t_{m-1}}) \psi_m(X_{t_m}) | F_{t_{m-1}} \right) \right]
\[
= E_\mu \left[ \prod_{i=1}^{m-2} \psi_i(X_{t_i}) \cdot \psi_{m-1}(X_{t_{m-1}}) P_{t_{m-1}} \psi_m(X_{t_m}) \right]
\[
= E_\mu \left[ \prod_{i=1}^{m-2} \psi_i(X_{t_i}) \cdot \tilde{\psi}_{m-1}(X_{t_{m-1}}) \right]
\]
where \( \tilde{\psi}_{m-1} \) defined to be \( \psi_{m-1} P_{t_{m-1}} \psi_m \) is in \( C_b(M_F) \). It is reduced to the case \( m - 1 \) where we already have \( F_\epsilon \in C_b(M_F) \), so it holds for case \( m \).

Therefore by the weak convergence of \( (X, Z^{x_n}_\epsilon) \) to \( (X, Z) \), we have
\[
\lim_{n \to \infty} E \left[ F_\epsilon(X_\epsilon) \cdot \phi_0(Z^{x_n}_\epsilon) \right] = \tilde{E} \left[ F_\epsilon(X_\epsilon) \cdot \phi_0(Z) \right]
\]
and hence
\[
I = \lim_{n \to \infty} E \left[ E_{X_\epsilon} \left( \prod_{i=1}^{m} \psi_i(X_{t_i}) \right) \cdot \phi_0(Z^{x_n}_\epsilon) \right]
\[
= \lim_{n \to \infty} E \left[ F_\epsilon(X_\epsilon) \cdot \phi_0(Z^{x_n}_\epsilon) \right] = \tilde{E} \left[ F_\epsilon(X_\epsilon) \cdot \phi_0(Z) \right]
\[
= \tilde{E} \left[ \tilde{E}_{X_\epsilon} \left( \prod_{i=1}^{m} \psi_i(X_{t_i}) \right) \cdot \phi_0(Z) \right]
\[
= \tilde{E} \left[ \tilde{E} \left( \psi_1(X_{t_1}) \cdots \psi_m(X_{t_m}) | \mathcal{F}_\epsilon \right) \cdot \phi_0(Z) \right]
\]
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Let $\epsilon \to 0$, by martingale convergence we have
\[
\tilde{E}\left(\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m}) \mid \tilde{F}_0^X\right) \overset{L^1}{\to} \tilde{E}\left(\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m}) \mid \tilde{F}_{0+}^X\right) = \tilde{E}\left(\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m})\right).
\]

The equality follows from Blumenthal 0-1 law that $\tilde{F}_{0+}^X$ is trivial. Therefore
\[
I = \tilde{E}\left[\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m}) \cdot \phi_0(Z)\right] = \lim_{\epsilon \to 0} \tilde{E}\left[\tilde{E}\left(\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m}) \mid \tilde{F}_\epsilon^X\right) \cdot \phi_0(Z)\right]
\]
\[
= \tilde{E}\left[\tilde{E}\left(\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m})\right) \cdot \phi_0(Z)\right]
\]
\[
= \tilde{E}\left(\psi_1(X_{t_1}) \cdots \psi_m(X_{t_m})\right) \cdot \tilde{E}\phi_0(Z)
\]

The above functionals are a determining class on $C([0, \infty), M_F) \times R$ and so we get weak convergence of $(X, Z^x_t) \to (X, Z)$ where the latter are independent.

(ii) Suppose we find convergence in probability for $0 < t \leq \infty$,
\[
\frac{L^{x_n}_t - \frac{1}{|x_n|}}{(c_{3.1} \log \frac{1}{|x_n|})^{1/2}} \overset{P}{\to} Z
\]
for some random variable $Z$, then it must converge in distribution to $Z$ as well, so $Z$ is a standard normal distributed random variable. By taking a further subsequence we may assume a.s. convergence holds:
\[
\frac{L^{x_n}_t - \frac{1}{|x_n|}}{(c_{3.1} \log \frac{1}{|x_n|})^{1/2}} \overset{a.s.}{\to} Z.
\]

By Sugitani [7],
\[
L^x_t - L^\epsilon_t^x \text{ is continuous in } x \text{ for any } 0 < \epsilon < t
\]
with the initial condition being $\delta_0$, we get
\[
\frac{L^{x_n}_t - L^\epsilon^{x_n}_t}{(c_{3.1} \log \frac{1}{|x_n|})^{1/2}} \to 0 \text{ a.s.}
\]
Therefore
\[
\frac{L_t - L_n}{(c_3 \log \frac{1}{|x_n|})^{1/2}} = \frac{L_t - L_n}{(c_3 \log \frac{1}{|x_n|})^{1/2}} = \frac{L_t - L_n}{(c_3 \log \frac{1}{|x_n|})^{1/2}} \rightarrow Z \text{ a.s.} \quad (9)
\]

Because (9) holds for any \( \epsilon > 0 \), we get
\[
Z \in \bigcap_{t>0} \mathcal{F}_X^t = \mathcal{F}_0^{X+},
\]
and Blumenthal 0-1 law tells us that any event in \( \mathcal{F}_0^{X+} \) is an event of probability 0 or 1, hence \( Z \) is a.s. constant. This contradicts the fact that \( Z \) is standard normal. So we get a contradiction by assuming that \( (L_t - c_3/|x_n|)/(c_3 \log \frac{1}{|x_n|})^{1/2} \) converges in probability. \( \square \)

2.2 Proof of Proposition 1 and 2

2.2.1 Some useful lemmas

**Lemma 4.** For any \( 0 < \alpha < 3 \), there exists a constant \( C = C(\alpha) \) such that for any \( x \neq 0 \) and \( t > 0 \),
\[
\int_{\mathbb{R}^3} p_t(y) \frac{1}{|y-x|^\alpha} dy < C \frac{1}{|x|^\alpha}.
\]

*Proof.* Fix \( \delta = |x|/2 \),
\[
\int_{\mathbb{R}^3} p_t(y) \frac{1}{|y-x|^\alpha} dy \\
\leq \frac{1}{\delta^\alpha} + \int_{|y-x|<\delta} p_t(y) \frac{1}{|y-x|^\alpha} dy.
\]

For \( |y-x| < \delta \), we have \( |y| \geq |x| - |y-x| > |x| - \delta = \delta \), therefore
\[
\int_{|y-x|<\delta} \left( \frac{1}{2\pi t} \right)^{3/2} e^{-\frac{|y|^2}{2t}} \frac{1}{|y-x|^\alpha} dy \\
\leq \int_{|y-x|<\delta} \left( \frac{1}{2\pi t} \right)^{3/2} e^{-\frac{\delta^2}{2t}} \frac{1}{|y-x|^\alpha} dy \\
= \left( \frac{1}{2\pi t} \right)^{3/2} e^{-\frac{\delta^2}{2t}} \frac{1}{\delta^\alpha} \int_0^\delta r e^{-\frac{\delta^2}{2r}} dr \cdot 4\pi \\
= 4\pi M(\delta) \cdot \frac{1}{3-\alpha} \delta^{3-\alpha}
\]
where
\[ M(\delta) := \sup_{t>0} \left( \frac{1}{2\pi t} \right)^{3/2} e^{-\frac{2}{2\pi} \frac{u}{\delta^2}} = \frac{1}{\delta^3} \sup_{u \geq 0} \left( \frac{u}{2\pi} \right)^{3/2} e^{-u/2} := C_0 \frac{1}{\delta^3}. \]

Therefore
\[ \int_{\mathbb{R}^3} p_t(y) \frac{1}{|y-x|^\alpha} dy < \frac{1}{\delta^\alpha} + 4\pi \cdot C_0 \frac{1}{\delta^3} \cdot \frac{1}{3-\alpha} \delta^{3-\alpha} = C(\alpha) \frac{1}{|x|^\alpha}. \]

**Corollary 1.** For any \( 0 < \alpha < 3 \), there exists a constant \( C = C(\alpha) \) such that for any \( x \neq 0 \) and \( t > 0 \),
\[ E \int_0^t \int \frac{1}{|y-x|^\alpha} X_s(dy)ds = \int_0^t ds \int_{\mathbb{R}^3} p_s(y) \frac{1}{|y-x|^\alpha} dy < C \frac{1}{|x|^\alpha} t. \]

**Proof.** It directly follows from Lemma 4. \( \square \)

**Lemma 5.** In \( \mathbb{R}^3 \), for any fixed \( s > 0 \) and \( y \neq x \), we have
\[ \Delta_y p_s g_x(y) = \int p_s(y-z) \frac{1}{|z-x|^2} dz. \]

**Proof.** Idea of this proof is from Evans [3]. For any fixed \( s > 0 \), \( p_s(y) = (2\pi s)^{-3/2} e^{-|y|^2/2s} \in C_0^\infty(\mathbb{R}^3) \), we have
\[ \|Dp_s\|_{L^\infty(\mathbb{R}^3)} < \infty \text{ and } \|\Delta p_s\|_{L^\infty(\mathbb{R}^3)} < \infty. \]

Here \( Du = D_x u = (u_{x_1}, u_{x_2}, u_{x_3}) \) denotes the gradient of \( u \) with respect to \( x = (x_1, x_2, x_3) \).

For any \( \delta \in (0, 1) \),
\[ \Delta_y \int_{\mathbb{R}^3} p_s(y-z)g_x(z)dz = \int_{B(x,\delta)} \Delta_y p_s(y-z)g_x(z)dz + \int_{\mathbb{R}^3-B(x,\delta)} \Delta_y p_s(y-z)g_x(z)dz =: I_\delta + J_\delta. \]
Now
\[
|I_\delta| \leq \|\Delta p_s\|_{L^\infty(\mathbb{R}^3)} \int_{B(x,\delta)} |g_x(z)|dz \leq C\delta^3 |\log \delta| \to 0.
\]

Note that \(\Delta_y p_s(y-z) = \Delta_z p_s(y-z)\). Integration by parts yields
\[
J_\delta = \int_{\mathbb{R}^3 - B(x,\delta)} \Delta_z p_s(y-z) g_x(z) dz
\]
\[
= \int_{\partial B(x,\delta)} g_x(z) \frac{\partial p_s}{\partial \nu}(y-z) dz - \int_{\mathbb{R}^3 - B(x,\delta)} D_z p_s(y-z) D_z g_x(z) dz
\]
\[
=: K_\delta + L_\delta,
\]
\(\nu\) denoting the inward pointing unit normal along \(\partial B(x,\delta)\). So
\[
|K_\delta| \leq \|Dp_s\|_{L^\infty(\mathbb{R}^3)} \int_{\partial B(x,\delta)} |g_x(z)|dz \leq C\delta^2 |\log \delta| \to 0.
\]

We continue by integrating by parts again in the term \(L_\delta\) to find
\[
L_\delta = \int_{\mathbb{R}^3 - B(x,\delta)} p_s(y-z) \Delta_z g_x(z) dz - \int_{\partial B(x,\delta)} p_s(y-z) \frac{\partial g_x}{\partial \nu}(z) dz
\]
\[
=: M_\delta + N_\delta.
\]

Now \(Dg_x(z) = \frac{z-x}{|z-x|^2}(z \neq x)\) and \(\nu = \frac{-(z-x)}{|z-x|} = \frac{-(z-x)}{\delta}\) on \(\partial B(x,\delta)\). Hence
\[
\frac{\partial g_x}{\partial \nu}(z) = \nu \cdot Dg_x(z) = -\frac{1}{\delta} \text{ on } \partial B(x,\delta).
\]
Since \(4\pi\delta^2\) is the surface area of the sphere \(\partial B(x,\delta)\) in \(\mathbb{R}^3\), we have
\[
N_\delta = 4\pi\delta \cdot \frac{1}{4\pi\delta^2} \int_{\partial B(x,\delta)} p_s(y-z) dz \to 0 \cdot p_s(y-x) = 0 \text{ as } \delta \to 0.
\]

By direct calculation, we have \(\Delta_z g_x(z) = \frac{1}{|z-x|^2}\) when \(z \in \mathbb{R}^3 - B(x,\delta)\), therefore
\[
M_\delta = \int_{\mathbb{R}^3 - B(x,\delta)} p_s(y-z) \frac{1}{|x-z|^2} dz.
\]

Lemma 4 gives
\[
\int p_s(y-z) \frac{1}{|x-z|^2} dz < \infty,
\]
by Dominated Convergence Theorem, we have
\[
M_\delta = \int p_s(y-z) \frac{1}{|x-z|^2} 1_{\{|z-x| \geq \delta\}} dz \to \int p_s(y-z) \frac{1}{|x-z|^2} dz
\]
as \(\delta \to 0\). \qed
2.2.2 Proof of Proposition 1

Define \( \eta \in C^\infty(\mathbb{R}^d) \) by
\[
\eta(x) := C \exp \left( \frac{1}{|x|^2 - 1} \right) 1_{\{|x|<1\}},
\]
the constant \( C \) selected such that \( \int_{\mathbb{R}^d} \eta dx = 1 \).

Let \( \chi_n \) be the convolution of \( \eta \) and the indicator function of the ball \( B_n = \{ x : |x| < n \} \), we get
\[
\chi_n(x) = \int_{\mathbb{R}^d} 1_{\{|x-y|<n\}} \eta(y) dy = \int_{B_1} 1_{\{|x-y|<n\}} \eta(y) dy.
\]

It is known that \( \chi_n \) is a \( C^\infty \) function with support in \( B_{n+1} \) and for \( x \in B_{n-1} \), we have \( |x-y| < n \) since \( |x| < n-1 \) and \( |y| < 1 \), so
\[
\chi_n(x) = \int_{B_1} 1_{\{|x-y|<n\}} \eta(y) dy = \int_{B_1} \eta(y) dy = 1.
\]

It’s easy to see that \( \chi_n \) increases to 1 as \( n \) goes to infinity.

Recall that \( g_x(y) = \log |y-x| \) and let \( g_{n,x}(y) = g_x(y) \cdot \chi_n(y-x) \), then
\[
P_\epsilon g_{n,x}(z) = \int_{|y-x|<n-1} p_\epsilon(z-y) \log |y-x| dy
+ \int_{n-1<|y-x|<n+1} p_\epsilon(z-y) \log |y-x| \chi_n(y-x) dy \in C^2_b,
\]
and
\[
\Delta_z P_\epsilon g_{n,x}(z) = \int_{|y-x|<n-1} \Delta_z p_\epsilon(z-y) \log |y-x| dy
+ \int_{n-1<|y-x|<n+1} \Delta_z p_\epsilon(z-y) \log |y-x| \chi_n(y-x) dy.
\]

It is easy to see that \( P_\epsilon g_{n,x}(z) \) and \( \Delta_z P_\epsilon g_{n,x}(z) \) increases to \( P_\epsilon g_x(z) \) and \( \Delta_z P_\epsilon g_x(z) \) respectively.

For \( P_\epsilon g_{n,x} \in C^2_b(\mathbb{R}^3) \), we have following equation hold a.s.,
\[
X_t(P_\epsilon g_{n,x}) = \delta_0(P_\epsilon g_{n,x}) + M_t(P_\epsilon g_{n,x}) + \int_0^t X_s(\frac{\Delta}{2} P_\epsilon g_{n,x}) ds,
\]

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where \( M_t(P_\epsilon g_{n,x}) \) is a martingale with quadratic variation

\[
[M(P_\epsilon g_{n,x})]_t = \int_0^t X_s \left( (P_\epsilon g_{n,x})^2 \right) ds.
\]

As \( n \) goes to infinity, by monotone convergence, we have

\[
X_t(P_\epsilon g_{n,x}) \to X_t(P_\epsilon g_x), \quad \delta_0(P_\epsilon g_{n,x}) \to \delta_0(P_\epsilon g_x),
\]

and

\[
\int_0^t X_s(\Delta P_\epsilon g_{n,x}) ds \to \int_0^t X_s(\Delta P_\epsilon g_x) ds.
\]

Note that

\[
E \int_0^t X_s \left( (P_\epsilon g_x)^2 \right) ds
\]

\[
= \int_0^t ds \int p_s(y) dy \left( \int p_\epsilon(y - z) \log |z - x| dz \right)^2
\]

\[
\leq \int_0^t ds \int p_s(y) dy \int p_\epsilon(y - z) \left( \log |z - x| \right)^2 dz
\]

\[
= \int_0^t ds \int p_{s+\epsilon}(z) \left( \log |z - x| \right)^2 dz
\]

\[
\leq \int_0^{t+\epsilon} ds \int p_s(z) \left( \log |z - x| \right)^2 dz < \infty.
\]

The last is by \( (*) \) in Lemma 3 when calculating \( E(M_t^2(g_x)) \). So we conclude that

\[
E \left[ \left( M_t(P_\epsilon g_{n,x}) - M_t(P_\epsilon g_x) \right)^2 \right] = E \int_0^t X_s \left( (P_\epsilon g_{n,x} - P_\epsilon g_x)^2 \right) ds \to 0
\]

by Dominated Convergence Theorem since

\[
(P_\epsilon g_{n,x} - P_\epsilon g_x)^2 \to 0 \quad \text{and} \quad (P_\epsilon g_{n,x} - P_\epsilon g_x)^2 \leq 4(P_\epsilon g_x)^2.
\]

So the \( L^2 \) convergence of a martingale \( M_t(P_\epsilon g_{n,x}) \) to \( M_t(P_\epsilon g_x) \) follows, which makes \( M_t(P_\epsilon g_x) \) a martingale as well. By taking a subsequence we have the following equation holds a.s.

\[
X_t(P_\epsilon g_x) = \delta_0(P_\epsilon g_x) + M_t(P_\epsilon g_x) + \int_0^t X_s(\Delta P_\epsilon g_x) ds, \quad (10)
\]
where $M_t(P_\epsilon g_x)$ is a martingale with integrable quadratic variation

$$[M(P_\epsilon g_x)]_t = \int_0^t X_s((P_\epsilon g_x)^2)\,ds.$$ 

Let $\epsilon$ goes to 0, we will show in (i)-(iv) the $L^1$ convergence of each term in (10) to the corresponding term in Proposition 1, i.e.

$$X_t(g_x) = \delta_0(g_x) + M_t(g_x) + \frac{1}{2} \int_0^t \int \frac{1}{|y-x|^2} X_s(dy)ds.$$ 

(i) 

First we have

$$\left| \delta_0(P_\epsilon g_x) - \delta_0(g_x) \right| = \left| \int_{\mathbb{R}^2} p_\epsilon(y) \log |y-x|\,dy - \log |x| \right|$$

$$\leq \int_{\mathbb{R}^3} p_\epsilon(y) \left| \log |y-x| - \log |x| \right|\,dy = E\left[ \log |B_\epsilon - x| - \log |x| \right]$$

$$= E\left[ \log \frac{|B_\epsilon - x|}{|x|} \right].$$

As a result,

$$\left| \delta_0(P_\epsilon g_x) - \delta_0(g_x) \right| \leq E\left[ \log \frac{|B_\epsilon - x|}{|x|} \right]$$

$$\leq E\left[ \sqrt{\frac{|B_\epsilon|}{|x|}} \right] + E\left[ \sqrt{\frac{|B_\epsilon|}{|B_\epsilon - x|}} \right] \text{ by Lemma 1}$$

$$\leq \frac{1}{|x|^{\frac{1}{2}}} E|B_\epsilon|^{\frac{1}{2}} + \left( E|B_\epsilon| \right)^{\frac{1}{2}} \cdot \left( E\frac{1}{|B_\epsilon - x|} \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{|x|^{\frac{1}{2}}} E|B_\epsilon|^{\frac{1}{2}} + \left( E|B_\epsilon| \right)^{\frac{1}{2}} \cdot \left( C \frac{1}{|x|} \right)^{\frac{1}{2}} \text{ by Lemma 4}$$

$$\to 0 \text{ as } \epsilon \to 0.$$

(ii)
Let $B_t$ and $B'_t$ be two independent standard Brownian motion in $\mathbb{R}^3$,

\[
E\left[|X_t(P_\epsilon g_x) - X_t(g_x)|\right] \leq E\left[X_t\left(|P_\epsilon g_x - g_x|\right)\right]
\]

\[
= \int p_\epsilon(y)dy \int p_\epsilon(z) \log |z - (y - x)|dz - \log |y - x|
\]

\[
\leq \int p_\epsilon(y)dy \int p_\epsilon(z) \log |z - (y - x)| - \log |y - x|dz
\]

\[
= E\left(\log \frac{|B'_\epsilon - (B_t - x)|}{|B_t - x|}\right)
\]

\[
\leq E\left[\sqrt{\frac{|B'_\epsilon|}{|B_t - x|}}\right] + E\left[\sqrt{\frac{|B'_\epsilon + B_t - x|}{|B_t - x|}}\right].
\]

Since $E\sqrt{|B'_\epsilon|} \to 0$ and by Lemma 4

\[
E\sqrt{\frac{|B'_\epsilon|}{|B_t - x|}} = E\sqrt{|B'_\epsilon|} \cdot E\sqrt{\frac{1}{|B_t - x|}} \leq E\sqrt{|B'_\epsilon|} \cdot C \frac{1}{|x|^{3/2}} \to 0.
\]

For the second term, we use Cauchy Schwarz Inequality,

\[
\left(E\sqrt{\frac{|B'_\epsilon|}{|B'_\epsilon + B_t - x|}}\right)^2 \leq E|B'_\epsilon| \cdot E\frac{1}{|B'_\epsilon + B_t - x|} = E|B'_\epsilon| \cdot E\frac{1}{|B_t + \epsilon - x|}.
\]

So again by Lemma 4

\[
E\sqrt{\frac{|B'_\epsilon|}{|B'_\epsilon + B_t - x|}} \leq \left(E|B'_\epsilon|\right)^{1/2} \cdot \left(C \frac{1}{|x|^{3/2}}\right)^{1/2} \to 0 \text{ as } \epsilon \to 0.
\]

and the $L^1$ convergence of $X_t(P_\epsilon g_x)$ to $X_t(g_x)$ follows.

(iii)

Next we deal with $M_t(P_\epsilon g_x) - M_t(g_x)$ and we use its quadratic variation
to compute its second moment.

\[
\left( E|M_t(P \epsilon g_x) - M_t(g_x)| \right)^2 \leq E\left[ \left( M_t(P \epsilon g_x) - M_t(g_x) \right)^2 \right]
\]

\[
= E \int_0^t X_s((P \epsilon g_x - g_x)^2) ds
\]

\[
= \int_0^t ds \int p_s(y) dy \left( \int p_\epsilon(z) \left( \log |z + y - x| - \log |y - x| \right) dz \right)^2
\]

\[
\leq \int_0^t ds \int p_s(y) dy \int p_\epsilon(z) \left( \log |z + y - x| - \log |y - x| \right)^2 dz
\]

\[
= \int_0^t E \left( \left( \log |B'_\epsilon + B_s - x| - \log |B_s - x| \right)^2 \right) ds.
\]

By Lemma 1 we get

\[
\int_0^t E \left( \left( \log \frac{|B'_\epsilon + B_s - x|}{|B_s - x|} \right)^2 \right) ds
\]

\[
\leq \int_0^t E \left( \left( \sqrt{\frac{|B'_\epsilon|}{|B_s - x|}} + \sqrt{\frac{|B'_\epsilon|}{|B'_\epsilon + B_s - x|}} \right)^2 \right) ds
\]

\[
\leq 2 \int_0^t E \left( \frac{|B'_\epsilon|}{|B_s - x|} \right) + E \left( \frac{|B'_\epsilon|}{|B'_\epsilon + B_s - x|} \right) ds
\]

\[
:= 2I.
\]

For the first term in \( I \),

\[
\int_0^t E \frac{|B'_\epsilon|}{|B_s - x|} ds = E|B'_\epsilon| \cdot \int_0^t E \frac{1}{|B_s - x|} ds
\]

\[
\leq C\epsilon^{1/2} \int_0^t ds \int p_s(y) \frac{1}{|y - x|} dy \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \text{ by Corollary 1.}
\]

For the second term in \( I \), note that \( B'_\epsilon + B_s \equiv B_{s+\epsilon} \) as they are independent Brownian motion, so

\[
\int_0^t E \frac{|B'_\epsilon|}{|B'_\epsilon + B_s - x|} ds \leq \int_0^t \left( E(|B'_\epsilon|) \right)^{1/2} \cdot \left( E \frac{1}{|B'_\epsilon + B_s - x|} \right) ds
\]

\[
= C\epsilon^{1/2} \int_0^t \left( E \frac{1}{|B_{s+\epsilon} - x|^2} \right)^{1/2} ds \leq C\epsilon^{1/2} \left( \int_0^t E \frac{1}{|B_{s+\epsilon} - x|^2} ds \right)^{1/2} \cdot \left( \int_0^t 1^2 ds \right)^{1/2}
\]

\[
\leq C\epsilon^{1/2} \cdot t^{1/2} \left( \int_0^{t+1} ds \int p_s(y) \frac{1}{|y - x|^2} dy \right)^{1/2} \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \text{ by Corollary 1.}
The $L^1$ convergence of $M_t(P_t g_x)$ to $M_t(g_x)$ follows.

(iv) For the convergence of the last term in (10), by Lemma 5 we get

$$E\left| \int_0^t X_s(\frac{\Delta}{2} P_t g_x) ds - \frac{1}{2} \int_0^t ds \int \frac{1}{|y-x|^2} X_s(dy) \right|$$

$$= \frac{1}{2} E\left| \int_0^t ds \int X_s(dy) \int p_\epsilon(y-z) \frac{1}{|z-x|^2} dz - \frac{1}{2} \int_0^t ds \int X_s(dy) \frac{1}{|y-x|^2} \right|$$

$$\leq \frac{1}{2} E\int_0^t ds \int X_s(dy) \left| \int p_\epsilon(y-z) \frac{1}{|z-x|^2} dz - \frac{1}{|y-x|^2} \right|$$

$$= \frac{1}{2} \int_0^t ds \int_{\mathbb{R}^3} \left| \int p_\epsilon(y-z) \frac{1}{|z-x|^2} dz - \frac{1}{|y-x|^2} \right| p_\epsilon(y) dy.$$

Claim:

$$\left| \int p_\epsilon(y-z) \frac{1}{|z-x|^2} dz - \frac{1}{|y-x|^2} \right| \to 0 \text{ as } \epsilon \to 0 \text{ for } y \neq x.$$

Proof. For $\xi = y - x \neq 0$,

$$\left| \int p_\epsilon(y-z) \frac{1}{|z-x|^2} dz - \frac{1}{|y-x|^2} \right|$$

$$= \left| \int p_\epsilon(z) \frac{1}{|z-(y-x)|^2} dz - \frac{1}{|y-x|^2} \right|$$

$$\leq \int p_\epsilon(z) \left| \frac{1}{|z-\xi|^2} - \frac{1}{|\xi|^2} \right| dz$$

$$= E\left( \frac{1}{|B_\epsilon - \xi|^2} - \frac{1}{|\xi|^2} \right)$$

$$= E\left( |B_\epsilon - \xi| - |\xi| \cdot \left( \frac{|B_\epsilon - \xi| + |\xi|}{|B_\epsilon - \xi|^2 |\xi|^2} \right) \right)$$

$$\leq E\left( |B_\epsilon| \cdot \left( \frac{|B_\epsilon - \xi| + |\xi|}{|B_\epsilon - \xi|^2 |\xi|^2} \right) \right)$$

$$= E\left( |B_\epsilon| \cdot \frac{1}{|B_\epsilon - \xi|^2 |\xi|} \right) + E\left( |B_\epsilon| \cdot \frac{1}{|B_\epsilon - \xi| |\xi|^2} \right).$$

For the first term, we use Holder’s inequality with $1/p = 1/5$ and $1/q = 4/5$ to get

$$E\left( |B_\epsilon| \cdot \frac{1}{|B_\epsilon - \xi|^2 |\xi|} \right) \leq \frac{1}{|\xi|} \cdot \left( E(|B_\epsilon|^5) \right)^{1/5} \cdot \left( E\left( \frac{1}{|B_\epsilon - \xi|^2 |\xi|^2} \right) \right)^{4/5}.$$
By Lemma 4, we have
\[ E \frac{1}{|B_\epsilon - \xi|^{5/2}} \leq C \cdot |\xi|^{-\frac{5}{2}} < \infty, \]
so
\[ E \left( |B_\epsilon| \cdot \frac{1}{|B_\epsilon - \xi|^2|\xi|} \right) \leq \frac{1}{|\xi|} \left( C |\xi|^{-\frac{5}{2}} \right)^{4/5} \left( E |B_\epsilon|^5 \right)^{1/5} \to 0 \text{ as } \epsilon \to 0. \]
Similarly
\[ E \left( |B_\epsilon| \cdot \frac{1}{|B_\epsilon - \xi||\xi|^2} \right) \to 0. \]

Note that we have just proved that
\[ \left| \int p_\epsilon(y - z) \frac{1}{|z - x|^2} dz - \frac{1}{|y - x|^2} \right| \to 0 \]
almost everywhere \((y \neq x)\) as \(\epsilon \to 0\). Corollary 1 gives us
\[ \int_0^t ds \int_{\mathbb{R}^3} p_s(y) \frac{1}{|y - x|^2} dy < \infty, \]
and by Lemma 4
\[ \left| \int p_\epsilon(y - z) \frac{1}{|z - x|^2} dz - \frac{1}{|y - x|^2} \right| \leq (C + 1) \frac{1}{|y - x|^2} \text{ for all } \epsilon, \]
by Dominated Convergence Theorem,
\[ \int_0^t ds \int_{\mathbb{R}^3} \left| \int p_\epsilon(y - z) \frac{1}{|z - x|^2} dz - \frac{1}{|y - x|^2} \right| p_s(y) dy \to 0, \]
and we proved that
\[ \int_0^t X_s \left( \frac{\Delta}{2} P_\epsilon g_x \right) ds \overset{L^1}{\to} \int_0^t \int \frac{1}{|y - x|^2} X_s(dy) ds. \]
(v) Combining (i)-(iv), we build the \(L^1\) convergence of each term in (10) to the corresponding term in (4), therefore (4) holds a.s. and the proof of Proposition 1 is done. \[\square\]
2.2.3 Proof of Proposition 2

Let \( h_{\epsilon,x}(y) = \sqrt{|y - x|^2 + \epsilon} \), then

\[
\nabla h_{\epsilon,x}(y) = \frac{y - x}{|y - x|^2 + \epsilon}
\]

and

\[
\Delta h_{\epsilon,x}(y) = \frac{(d - 1)|y - x|^2 + d\epsilon}{(|y - x|^2 + \epsilon)^{3/2}}.
\]

By Ito’s Lemma, we have

\[
\sqrt{|B_t - x|^2 + \epsilon} = \sqrt{|x|^2 + \epsilon} + \int_0^t \frac{B_s - x}{\sqrt{|B_s - x|^2 + \epsilon}} \cdot dB_s + \frac{1}{2} \int_0^t \frac{(d - 1)|B_s - x|^2 + d\epsilon}{(|B_s - x|^2 + \epsilon)^{3/2}} ds.
\]

Let \( H_s = \frac{B_s - x}{\sqrt{|B_s - x|^2 + \epsilon}} \), then

\[
M^\epsilon_t := \int_0^t \frac{B_s - x}{\sqrt{|B_s - x|^2 + \epsilon}} \cdot dB_s \in cM_{0,loc}.
\]

Since

\[
E[M^\epsilon]_t = E\int_0^t \frac{|B_s - x|^2}{|B_s - x|^2 + \epsilon} ds \leq E\int_0^t 1 ds = t < \infty,
\]

then \( M^\epsilon \) is a martingale and hence by taking expectation

\[
E\sqrt{|B_t - x|^2 + \epsilon} = \sqrt{|x|^2 + \epsilon} + \frac{1}{2} \int_0^t E\frac{(d - 1)|B_s - x|^2 + d\epsilon}{(|B_s - x|^2 + \epsilon)^{3/2}} ds.
\] \quad (11)

By Fatou’s Lemma,

\[
\frac{1}{2} \int_0^t E\frac{d - 1}{|B_s - x|} ds = \frac{1}{2} \int_0^t E \liminf_{\epsilon \to 0} \frac{(d - 1)|B_s - x|^2 + d\epsilon}{(|B_s - x|^2 + \epsilon)^{3/2}} ds
\]

\[
\leq \frac{1}{2} \int_0^t \liminf_{\epsilon \to 0} E\frac{(d - 1)|B_s - x|^2 + d\epsilon}{(|B_s - x|^2 + \epsilon)^{3/2}} ds
\]

\[
\leq \liminf_{\epsilon \to 0} \frac{1}{2} \int_0^t E\frac{(d - 1)|B_s - x|^2 + d\epsilon}{(|B_s - x|^2 + \epsilon)^{3/2}} ds
\]

\[
= \liminf_{\epsilon \to 0} \left[ E\sqrt{|B_t - x|^2 + \epsilon} - \sqrt{|x|^2 + \epsilon} \right] \text{ by (11)}
\]

\[
= E|B_t - x| - |x| \leq E|B_t|.
\]
The last equality is from
\[\sqrt{|x|^2 + \epsilon} - |x| \leq \sqrt{\epsilon} \rightarrow 0,\]
and
\[0 \leq E\sqrt{|B_t - x|^2 + \epsilon} - E|B_t - x| \leq E\left(|B_t - x| + \sqrt{\epsilon}\right) - E|B_t - x| = \sqrt{\epsilon} \rightarrow 0.\]
So
\[\int_0^t \int \frac{1}{|y - x|} p_s(y)dyds = \int_0^t E\frac{1}{|B_s - x|}ds \leq \frac{2}{d-1} E|B_t| < \infty.\]

\[\square\]

3 Proof of Theorem 2

To prove Theorem 2, we need the Tanaka formula for \(d = 2\), which are stated below and the proof will follow after the proof of Theorem 2.

**Proposition 3.** (Tanaka formula for \(d=2\)) Let \(c_2 = 1/\pi\) and \(g_{\alpha,x}(y) = \log |y - x|\), where \(x \neq 0\). Then we have a.s. that
\[L^{x}_t = c_2 \left[ X_t(g_x) - \delta_0(g_x) - M_t(g_x) \right]. \tag{12} \]

**Remark.** Barlow, Evans and Perkins \([1]\) gives a Tanaka formula for local time of Super-Brownian Motion in \(d = 2\), which is
\[X_t(g_{\alpha,x}) = X_0(g_{\alpha,x}) + M_t(g_{\alpha,x}) + \alpha \int_0^t X_s(g_{\alpha,x})ds - L^{x}_t,\]
for all \(t \geq 0\) a.s. Here \(g_{\alpha,x}(y)\) is defined to be \(\int_0^\infty e^{-\alpha t} p_t(x - y)dt\). We can see that \(g_{\alpha,x}\) is not well defined for \(\alpha = 0\) and our result effectively extends the Tanaka formula in \([1]\) to the \(\alpha = 0\) case.
3.1 Proof of Theorem 2

By (12), note that 
\[ \delta_0(g_x) = \log |x| = -\log 1/|x|, \]

\[ L^*_t - c_2 \log \frac{1}{|x|} = c_2 \left[ X_t(g_x) - M_t(g_x) \right], \]

therefore
\[ E \left| L^*_t - c_2 \log \frac{1}{|x|} \right| \leq c_2 E \left| X_t(g_x) \right| + c_2 E \left| M_t(g_x) \right|. \]

For the first term,
\[ E \left| X_t(g_x) \right| \leq EX_t(|g_x|) = \int p_t(y) |\log |y - x|| dy \]
\[ \leq \int p_t(y) \frac{1}{|y - x|} 1_{\{|y - x| < 1\}} dy + \int p_t(y) |y - x| 1_{\{|y - x| \geq 1\}} dy \]
\[ \leq \frac{1}{2\pi t} \int \frac{1}{|y - x|} 1_{\{|y - x| < 1\}} dy + \int p_t(y) (|y| + |x|) dy \]
\[ = \frac{1}{2\pi t} \cdot 2\pi + |x| + E |B_t| \rightarrow \frac{1}{2\pi t} \cdot 2\pi + E |B_t|. \]

For the second term,
\[ \left( E \left| M_t(g_x) \right| \right)^2 \leq E \left( M^2_t(g_x) \right) = E \int_0^t X_s(g_x^2) ds, \]

and by Lemma 1
\[ E \int_0^t X_s(g_x^2) ds = \int_0^t ds \int p_s(y) (|\log |y - x||)^2 dy \]
\[ \leq \int_0^t ds \int p_s(y) \frac{1}{|y - x|} 1_{\{|y - x| < 1\}} dy + \int_0^t ds \int p_s(y) |y - x| 1_{\{|y - x| \geq 1\}} dy \]
\[ \leq \int_0^t ds \int p_s(y) \frac{1}{|y - x|} dy + \int_0^t ds \int p_s(y) (|y| + |x|) dy. \]

By Proposition 2 in \( d = 2 \),
\[ \int_0^t ds \int p_s(y) \frac{1}{|y - x|} dy \leq 2E |B_t| < \infty, \]
and
\[ \int_0^t ds \int p_s(y)(|y| + |x|)dy = |x|t + \int_0^t E|B_s|ds \rightarrow \int_0^t E|B_s|ds < \infty. \]
Therefore
\[ \limsup_{x \to 0} E \left| L_t^x - c_2 \log \frac{1}{|x|} \right| < \infty. \]

### 3.2 Proof of Proposition 3

#### 3.2.1 Some useful lemmas

**Lemma 6.** In \( \mathbb{R}^2 \), for \( 0 < \alpha < 2 \), there exists a constant \( C = C(\alpha) \) such that for any \( x \neq 0 \) and \( t > 0 \),
\[ \int_{\mathbb{R}^2} p_t(y) \frac{1}{|y - x|^\alpha} dy < C \frac{1}{|x|^\alpha}. \]

**Proof.** The proof follows from the proof of Lemma 4 after some modification. \( \square \)

**Corollary 2.** In \( \mathbb{R}^2 \), for any \( 0 < \alpha < 2 \), there exists a constant \( C = C(\alpha) \) such that for any \( x \neq 0 \) and \( t > 0 \)
\[ E \int_0^t \int \frac{1}{|y - x|^\alpha} X_s(dy) ds = \int_0^t ds \int_{\mathbb{R}^2} p_s(y) \frac{1}{|y - x|^\alpha} dy < C \frac{1}{|x|^\alpha} t. \]

**Proof.** It follows from Lemma 6. \( \square \)

**Lemma 7.** Let \( g_x(y) = \log |y - x| \), where \( x, y \in \mathbb{R}^2 \), then for any \( s > 0 \) and \( y \neq x \), we have
\[ \frac{\Delta y}{2} P_s g_x(y) = \pi p_s(y - x). \]

**Proof.** Idea of this proof is from Evans [3]. For any fixed \( s > 0 \), \( p_s(y) = (2\pi s)^{-1} e^{-|y|^2/2s} \in C_0^\infty(\mathbb{R}^2) \), we have
\[ \| Dp_s \|_{L^\infty(\mathbb{R}^2)} < \infty \text{ and } \| \Delta p_s \|_{L^\infty(\mathbb{R}^2)} < \infty. \]
For any \( \delta \in (0, 1) \),
\[ \Delta y \int_{\mathbb{R}^2} p_s(y - z) g_x(z) dz \]
\[ = \int_{B(x, \delta)} \Delta y p_s(y - z) g_x(z) dz + \int_{\mathbb{R}^2 - B(x, \delta)} \Delta y p_s(y - z) g_x(z) dz \]
\[ =: I_\delta + J_\delta. \]
Now
\[ |J_\delta| \leq \| \Delta p_s \|_{L^\infty(\mathbb{R}^2)} \int_{B(x,\delta)} |g_x(z)|dz \leq C\delta^2 |\log \delta| \to 0. \]

Note that \( \Delta y p_s(y - z) = \Delta z p_s(y - z) \). Integration by parts yields
\[
J_\delta = \int_{\mathbb{R}^2} \int_{\partial B(x,\delta)} \Delta z p_s(y - z)g_x(z)dz
dz = \int_{\partial B(x,\delta)} g_x(z) \frac{\partial p_s}{\partial \nu}(y - z)dz - \int_{\mathbb{R}^2} D_z p_s(y - z)D_z g_x(z)dz
dz =: K_\delta + L_\delta,
\]
\( \nu \) denoting the inward pointing unit normal along \( \partial B(x,\delta) \). So
\[
|K_\delta| \leq \| Dp_s \|_{L^\infty(\mathbb{R}^2)} \int_{\partial B(x,\delta)} |g_x(z)|dz \leq C\delta |\log \delta| \to 0.
\]

We continue by integrating by parts again in the term \( L_\delta \) to find
\[
L_\delta = \int_{\mathbb{R}^2} \int_{\partial B(x,\delta)} p_s(y - z)\Delta z g_x(z)dz - \int_{\partial B(x,\delta)} p_s(y - z) \frac{\partial g_x}{\partial \nu}(z)dz
= - \int_{\partial B(x,\delta)} p_s(y - z) \frac{\partial g_x}{\partial \nu}(z)dz
=: M_\delta.
\]

since \( \Delta z g_x(z) = 0 \) when \( z \) is away from \( x \).

Now \( Dg_x(z) = \frac{z-x}{|z-x|^3} (z \neq x) \) and \( \nu = \frac{-(z-x)}{|z-x|} = -\frac{z-x}{\delta} \) on \( \partial B(x,\delta) \). Hence
\( \frac{\partial g_x}{\partial \nu}(z) = \nu \cdot Dg_x(z) = -\frac{1}{\delta} \) on \( \partial B(x,\delta) \). Since \( 2\pi \delta \) is the surface area of the sphere \( \partial B(x,\delta) \) in \( \mathbb{R}^2 \), we have
\[
M_\delta = 2\pi \cdot \frac{1}{2\pi \delta} \int_{\partial B(x,\delta)} p_s(y - z)dz \to 2\pi p_s(y - x) \text{ as } \delta \to 0.
\]

Therefore we proved
\[
\frac{\Delta y}{2} P_s g_x(y) = \pi p_s(y - x) = \pi p_s^x(y).
\]

3.2.2 Proof of Proposition 3

Using the same argument in proving Proposition 1 in Section 2.2.2, by a smooth cutoff $\chi_n$ of log and let $n$ goes to infinity, we have following equation hold a.s.,

$$X_t(P_\epsilon g_x) = \delta_0(P_\epsilon g_x) + M_t(P_\epsilon g_x) + \int_0^t X_s\left(\frac{\Delta}{2} P_\epsilon g_x\right)ds,$$

(13)

where $M_t(P_\epsilon g_x)$ is a martingale with quadratic variation being

$$[M(P_\epsilon g_x)]_t = \int_0^t X_s((P_\epsilon g_x)^2)ds,$$

and $M_t^2(P_\epsilon g_x) - [M(P_\epsilon g_x)]_t$ is also a martingale.

Let $\epsilon$ goes to 0, we will show the a.s. convergence of each term in (13) to the corresponding term in Proposition 3, which is equivalent to

$$X_t(g_x) = \delta_0(g_x) + M_t(g_x) + \pi L^x_t.$$

(14)

By Lemma 7, we have

$$\int_0^t X_s(\frac{\Delta}{2} P_\epsilon g_x)ds = \pi \int_0^t X_s(p^x_\epsilon)ds \xrightarrow{a.s.} \pi L^x_t as \epsilon \to 0.$$

Then in (i)-(iii) we will build the $L^1$ convergence of the rest three terms in (13) to the corresponding term in (14) and we can take a subsequence along which all four terms converge a.s. and therefore (14) holds a.s..

(i)

Let $B_t$ and $B'_t$ be two independent standard Brownian motion in $\mathbb{R}^2$,

$$E\left|X_t(P_\epsilon g_x) - X_t(g_x)\right| \leq E\left[X_t\left(|P_\epsilon g_x - g_x|\right)\right]$$

$$= \int p_t(y)dy \int p_\epsilon(z)\log|z - (y - x)|dz - \log|y - x|$$

$$\leq \int p_t(y)dy \int p_\epsilon(z)\log|z - (y - x)| - \log|y - x|dz$$

$$= E\left[\left|\log\frac{|B'_t - (B_t - x)|}{|B_t - x|}\right|\right]$$

$$\leq E\left\sqrt{\frac{|B'_t|}{|B_t - x|}} + E\left\sqrt{\frac{|B'_t|}{|B'_t + B_t - x|}}\right|\right]$$

by Lemma 1.
Since $E\sqrt{|B|} \to 0$ and $E\sqrt{\frac{1}{|B_t - x|}} < \infty$ by Lemma 6,

$$E\sqrt{\frac{|B'_{\epsilon}|}{|B_{\epsilon} + B_t - x|}} = E\sqrt{|B_{\epsilon}'|} \cdot E\sqrt{\frac{1}{|B_t - x|}} \to 0.$$ 

For the second term, we use Cauchy Schwarz Inequality,

$$\left( E\sqrt{\frac{|B'_{\epsilon}|}{|B_{\epsilon}' + B_t - x|}} \right)^2 \leq E|B_{\epsilon}'| \cdot E\frac{1}{|B_{\epsilon}' + B_t - x|} = E|B_{\epsilon}'| \cdot E\frac{1}{|B_{t+\epsilon} - x|}.$$ 

So by Lemma 6

$$E\sqrt{\frac{|B'_{\epsilon}|}{|B_{\epsilon}' + B_t - x|}} \leq \left( E|B_{\epsilon}'| \right)^{1/2} \cdot \left( E\frac{1}{|B_{t+\epsilon} - x|} \right)^{1/2} \to 0 \text{ as } \epsilon \to 0.$$ 

and the $L^1$ convergence of $X_t(P_{\epsilon}g_x)$ to $X_t(g_x)$ follows.

(ii) Similarly we have

$$\left| \delta_0(P_{\epsilon}g_x) - \delta_0(g_x) \right| = \int_{\mathbb{R}^2} p_{\epsilon}(y) \log |y - x| dy - \log |x|$$

$$\leq \int_{\mathbb{R}^2} p_{\epsilon}(y) \left( |\log |y - x| - \log |x| \right) dy = E \log |B_{\epsilon} - x| - \log |x|$$

$$\leq E\sqrt{\frac{|B_{\epsilon}|}{|x|}} + E\sqrt{\frac{|B_{\epsilon}|}{|B_{\epsilon} - x|}} \to 0.$$
(iii) For the convergence of the martingale term $M_t(P_\epsilon g_x)$ to $M_t(g_x)$,

$$\left( E|M_t(P_\epsilon g_x) - M_t(g_x)| \right)^2 \leq E \left[ \left( M_t(P_\epsilon g_x) - M_t(g_x) \right)^2 \right]$$

$$= E \int_0^t X_s \left( (P_\epsilon g_x - g_x)^2 \right) ds$$

$$= \int_0^t ds \int p_s(y) dy \left( \int p_\epsilon(z) (\log |z - (y - x)| - \log |y - x|) dz \right)^2$$

$$\leq \int_0^t ds \int p_s(y) dy \int p_\epsilon(z) \left( \log |z - (y - x)| - \log |y - x| \right)^2 dz$$

$$= \int_0^t E \left[ \left( \log \frac{|B'_\epsilon - (B_s - x)|}{|B_s - x|} \right)^2 \right] ds$$

$$\leq \int_0^t E \left[ \left( \sqrt{\frac{|B'_\epsilon|}{|B_s - x|}} + \sqrt{\frac{|B'_\epsilon|}{|B'_\epsilon - (B_s - x)|}} \right)^2 \right] ds \text{ (by Lemma 1)}$$

$$\leq 2 \int_0^t E \left( \frac{|B'_\epsilon|}{|B_s - x|} + \frac{|B'_\epsilon|}{|B'_\epsilon - (B_s - x)|} \right) ds$$

$$:= 2J.$$

For the first term in $J$, by Corollary 2,

$$\int_0^t E \frac{|B'_\epsilon|}{|B_s - x|} ds = E|B'_\epsilon| \cdot \int_0^t E \frac{1}{|B_s - x|} ds \to 0.$$

and for the second term in $J$, using Holder’s inequality with $1/p = 1/3$ and $1/q = 2/3$ twice, we get

$$\int_0^t E \frac{|B'_\epsilon|}{|B'_\epsilon + B_s - x|} ds \leq \int_0^t \left( E(|B'_\epsilon|^3) \right)^{1/3} \cdot \left( E \frac{1}{|B'_\epsilon + B_s - x|^{3/2}} \right)^{2/3} ds$$

$$= \left( E(|B'_\epsilon|^3) \right)^{1/3} \cdot \int_0^t \left( E \frac{1}{|B_{s+\epsilon} - x|^{3/2}} \right)^{2/3} ds$$

$$\leq \left( E(|B'_\epsilon|^3) \right)^{1/3} \cdot \left( \int_0^t E \frac{1}{|B_{s+\epsilon} - x|^{3/2}} ds \right)^{2/3} \cdot \left( \int_0^1 t^3 ds \right)^{1/3}$$

$$\leq \left( E(|B'_\epsilon|^3) \right)^{1/3} \cdot \left( \int_0^{t+1} E \frac{1}{|B_{s} - x|^{3/2}} ds \right)^{2/3} \cdot t^{1/3} \to 0$$

as $\epsilon \to 0$ by Corollary 2.

This completes the proof of Proposition 3. \qed

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