Witnessing quantum coherence in the presence of noise

Alex Monras\textsuperscript{1,2}, Agata Che\c{c}ni\k{s}ka\textsuperscript{2} and Artur Ekert\textsuperscript{2,3}

\textsuperscript{1} Física Teòrica: Informació i Fenomens Quàntics, Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain
\textsuperscript{2} Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543 Singapore
\textsuperscript{3} Mathematical Institute, University of Oxford, Woodstock Rd, Oxford OX2 6HD, UK
E-mail: amb.physis@gmail.com

Received 21 January 2014, revised 17 March 2014
Accepted for publication 3 April 2014
Published 19 June 2014

\textit{New Journal of Physics} 16 (2014) 063041
doi:10.1088/1367-2630/16/6/063041

Abstract
We address the problem of assessing the coherent character of physical evolution. We take the quantum Zeno effect (QZE) as a characteristic trait of quantum dynamics, and derive relations among transfer rates as a function of the strength of a measurement. These relations support the intuition that only quantum dynamics is susceptible to the QZE. With the derived bounds on the magnitude of coherent dynamics, we propose an experimentally viable coherence witness. Our results have potential application in assessing the coherence of quantum transport in biological and other complex many-body systems.

Keywords: coherence, open systems, Zeno effect

1. Introduction

In the popular novel ‘Hitchhiker’s Guide to the Galaxy’ by Douglas Adams [1], the author shows that formulating a proper question may turn out to be more difficult than finding the answer (be it 42 or otherwise). Such a situation is all too familiar to physicists, who often need to phrase and refine their questions, in terms of experimental procedures or measurements, to resolve certain issues. Here the issue is the degree of coherence in a quantum system.
What exactly should be measured, and how, to witness a coherent evolution of an open system about which we have a limited knowledge and on which we are allowed to perform a limited set of measurements? The difficulty of formulating a proper question lies in stating, in general terms, what is understood by quantum coherence. Here, we take an instrumental approach. We refer to one of the signatures of quantum behaviour, namely the quantum Zeno effect (QZE), and use it to derive our operational definition of coherence.

Characterization of the dynamics of open systems is of general interest on its own. However, it becomes even more interesting when applied to systems that are known for their complexity. Prominent examples are many-body systems [2, 3], nanostructures [4] and biological complexes [5–9]. Recent debate on the presence of quantum coherence in certain biological complex systems [10–12] is a prime example showing that assessing coherence and its character is far from straightforward.

These issues have been addressed from a variety of perspectives. In addition to the early studies of decoherence in quantum walks [13], a considerable amount of theoretical effort has been devoted to obtaining an understanding of the role of quantum coherence in more general contexts, such as resource theories [14, 15] and thermodynamics [16]. In addition, coherence witnesses of quantum states have been linked to being able to describe the evolution of the system’s populations in terms of stochastic propagators, namely, a necessary condition for the absence of quantum interference [17].

Regarding the assessment of coherence in physical evolution, several approaches have been put forward. Most notably, the methods of quantum process tomography, first developed with the profiling of hypothetical quantum computers in mind, are now being exploited to assess coherence in biological complexes [18]. Other proposals include the Leggett–Garg inequality [19–21], temporal CHSH inequalities [22, 23] and the so-called no-signalling-in-time condition [24]. These proposals rely in one way or another on assuming that classical systems can be subject to measurement without perturbation.

In this article, we draw a quantitative link between quantum coherence and the quantum Zeno effect [25–27]. Our approach draws from the intuition that the quantum Zeno effect has a close relationship with the quadratic build-up of probabilities, a characteristic trait of unitary evolution. In contrast to the case for quantum evolution, it is well understood that classical rate equations are not subject to the QZE even in the presence of measurement back-action. Hence, one may expect that the extent to which a system’s dynamics can be subjected to the quantum Zeno effect is an indicator of the amount of coherence present in the evolution. We provide a rigorous quantitative formulation of these ideas. In addition, we show that quantum coherence of physical evolution can be assessed with a minimal set of state preparations and a single measurement set-up, providing coherence witnesses that are potentially tight. Our approach should be regarded as a proof-of-principle, showcasing the essential features of these ideas, but open to a wide variety of extensions tailored to the particular systems or contexts of interest. To implement the QZE we adopt a continuous decoupling approach. In the latter, a system undergoing unitary dynamics governed by Hamiltonian $H + kH_m$ can be effectively described, in the large $k$ limit, by an effective Hamiltonian $\tilde{H}_Z + kH_m$, where $\tilde{H}_Z = \sum_i P_i H_{Z_i}$ is the effective Zeno Hamiltonian and $P_i$ are the orthogonal projectors on $H_m$’s eigenspaces [26], thus suppressing transitions among $H$’s eigenspaces. Our analysis reproduces this result in the presence of noise, and shows that, under some mild conditions, only the coherent contribution to dynamics is suppressed.
In order to develop these ideas, we make some general assumptions. Most importantly, the dynamics is assumed to be accurately described by some Markovian master equation, and it is assumed that the timescales for which this approximation breaks down are under control. This assumption is necessary to preserve the validity of our model when small times or strong drivings are considered. In addition, the structures of the noise operators of the master equation are assumed to verify certain mild assumptions, which we motivate below. We will discuss in section 7 the validity and limitations of these assumptions. Nevertheless, we note that the Markovianity of the dynamics is instrumental in the quantitative analysis that we develop, but irrelevant for the underlying intuition for our scheme.

The paper is organized as follows. In section 2 we define the problem and outline the main result of our work, stating without proof several facts that will be discussed in greater detail in the following sections. In section 3 we describe in detail our measurement protocol, and discuss the experimentally accessible quantities that will be used to extract our coherence witness. Section 4 derives the effective dynamics within our approximations, by performing an exact adiabatic elimination of population transfer via coherent transitions. This provides a generic closed form for the measured quantities, that will be used in section 5 to extract the witnesses that we are interested in. Section 6 illustrates our procedure with three examples, and discusses the performance of our bounds. Section 7 concludes with some general observations and suggests future lines of work.

2. The general framework and main results

Our analysis is concerned with a generic finite-dimensional quantum system described by a finite-dimensional Hilbert space $\mathcal{H}$. We denote subspaces of $\mathcal{H}$ by script upper-case symbols, e.g. $\Pi$. The most general time-independent evolution in the Markovian approximation is described by the master equation

$$\frac{d}{dt} \rho(t) = \mathcal{L}[\rho(t)],$$

where $\mathcal{L}$ is a time-independent, Lindblad–Kossakowski superoperator $[28, 29]$:

$$\mathcal{L}[\rho] = -i[H, \rho] + \sum_{\mu} \left( W_{\mu} \rho W_{\mu}^\dagger - \frac{1}{2} \left\{ W_{\mu}^\dagger W_{\mu}, \rho \right\} \right).$$

Here $H$ stands for a Hamiltonian and $\{ W_{\mu} \}$ is a set of jump operators representing the noise. As a general rule, we will write linear operators on $\mathcal{H}$ with Greek or upper-case symbols, and superoperators (linear maps thereof) will be written in calligraphic letters (e.g. $\Pi$) with the single exception of $\phi$, which denotes a completely positive superoperator.

Given $\mathcal{L}$ and a set of jump operators $\{ W_{\mu} \}$, Hamiltonian $H$ is uniquely determined. Here we do not assume detailed knowledge of jump operators, but we assume a certain property of a noisy mechanism at work and focus on the question of how to obtain information on the missing $H$. Cases where we cannot uniquely determine the Hamiltonian part of $\mathcal{L}$ will be mentioned later.

We focus our attention on systems for which there exists a set of $n$ orthogonal projections $\{ P_i \}$, $\sum_{i=1}^{n} P_i = 1$, $P_i P_j = P_j \delta_{ij}$, such that
\[ \sum_{r \neq s} \sum_{\mu} P_r \mathcal{W}_\mu P_s \mathcal{W}_\mu = 0, \quad (2) \]

for all \( i \). Intuitively, this condition enforces that there exists a decomposition of the system's Hilbert space,

\[ \mathcal{H} = \bigoplus_{i=1}^{n} \Pi_i, \quad (3) \]

with respect to which the incoherent mechanism does not create coherence, and neither does its adjoint—incoherent states remain incoherent in the Schrödinger picture, and so do observables in the Heisenberg picture. This can be pictorially understood as the action of the completely positive map \( \phi = \sum_{\mu} \mathcal{W}_\mu \mathcal{W}_\mu^\dagger \) and \( \phi^* \), mapping projectors onto incoherent mixtures among different subspaces \( \Pi_i, \ldots, \Pi_n \):

\[ \begin{pmatrix}
0 \\
\vdots \\
\mathbb{P}_i \\
\vdots \\
0
\end{pmatrix} \overset{\phi, \phi^*}{\longrightarrow} \begin{pmatrix}
\ast \\
\vdots \\
\ast \\
\vdots \\
\ast
\end{pmatrix} \quad (4) \]

This is the only property of the noise operators that we assume to be known, and our only assumption on the Markovian dynamics, besides its range of validity being known. In section 6 we will provide natural examples satisfying equation (2), but we can already here discuss some natural examples where this is the case.

Dephasing is the process in which coherent superpositions among different subspaces acquire random relative phases. Averaging over such randomization essentially damps the coherences, characterized by off-diagonal blocks in the density matrix. This type of noise preserves the block structure of the density operator, a property much stronger than that required by equation (2).

Another paradigmatic example of a noise source obeying equation (2) is incoherent hopping between nearest neighbours in spin lattices. This will essentially map any localized state onto a probabilistic mixture of localizations among different sites. Therefore, the generator of this dynamics, here represented by superoperator \( \phi \), will follow the pattern expressed by equation (2) or equation (4) with subspaces corresponding to single sites.

Notice that in both of these cases, the noise operators admit a property much stronger than that required by equation (2), namely, that any operator restricted to a diagonal block is mapped onto a block-diagonal operator. In contrast, (2) is only concerned with the action of the map on the corresponding subspace projectors \( \mathbb{P}_i \). As we will see, this condition is sufficiently mild to be applicable to various systems of interest.

Our work aims at providing experimentally accessible measures of coherence obtained from the outcomes of a unique measurement characterized by projectors \( \{ \mathbb{P}_i \} \). By implementing a Hamiltonian \( k H_m = k \sum_i \mathbb{P}_i \) in addition to the system’s dynamics \( \mathcal{L} \), the coherent coupling among subspaces \( \Pi_i \) is suppressed at leading order in \( 1/k \). The actual transition probabilities are mediated, at leading order \( \mathcal{O}(1) \), by the effective incoherent dynamics derived from the dissipative part of \( \mathcal{L} \). At subleading order \( \mathcal{O}(1/k) \), the population transfer among \( H_m \)'s eigenstates is mediated via virtual transitions through superposition states. It will be this
dependence on $k$ that will reveal the contribution of quantum coherence to the observed population transfer rates.

We consider coherence measures of two types. At the level of decomposition (3), we consider the off-diagonal blocks $H_{ij} = PH_{ij}P$ and their 2-norm $\|H_{ij}\|_2 = \left(\text{tr}\left[H_{ij}^* H_{ij}\right]\right)^{1/2}$ as a measure of how strongly subspaces $\Pi_i$ and $\Pi_j$ are coupled. These quantities are readily accessible from our formalism. In addition, our scheme provides lower bounds to the spectral spread of the system’s Hamiltonian $H$,

$$\mathcal{C}(H) \equiv \lambda_{\max}(H) - \lambda_{\min}(H),$$

(5)

where $\{\lambda_i(H)\}$ represent $H$’s eigenvalues.

Consider the probabilities of preparing the system in states $\rho_j = \Pi_j P_{\text{dim}}$ at time 0 and obtaining outcome $P_i$ at time $t$, when the coupling strength is set at $k$. These probabilities are arranged in $[P(k, t)]_{ij}$. As a general rule all matrices of observable magnitudes derived from $P$ will be written in sans-serif caption. With a suitably chosen tensor $W_{ijk}$ whose components depend on the chosen values of $k$, the time $t$ and the eigenvalues $\eta_i$ of $H$, we have that for $i \neq j$,

$$\|H_{ij}\|_2 = \sqrt{\text{dim} \Pi_j \sum_{k \in K} W_{ijk} \left[\hat{P}(k, t)\right]_{ij}},$$

(6)

with $\hat{P}$ denoting the probabilities’ derivatives w.r.t. time, and the sum over $k$ running over a finite set of values $K$. The role of $W$ and the summation over $k$ is a type of inverse linear transform. In our formalism the contributions to a given $i\leftrightarrow j$ transition are mediated through virtual second-order coherent superpositions. All of these contributions are then coherently combined to provide the transition amplitude. However, each coherent transition evolves at a different frequency, always proportional to the energy difference and $k$. The role of such a linear transform is to disentangle all of these contributions by resolving all different frequencies, which can be achieved by sampling sufficiently many values of $k$.

Equation (6) provides a quantifier for the size of each off-diagonal block in the Hamiltonian. In the ideal case of decomposition (3) into one-dimensional subspaces, this procedure determines the norm of each off-diagonal entry in $H$. This information is sufficient, in itself, to bound $\mathcal{C}(H)$ away from zero. However, the $\|H_{ij}\|$ do not account, in principle, for all the information present in $\hat{P}$. In the following we provide a lower bound based on the latter.

A coherence witness for $\mathcal{C}(H)$ is an experimentally accessible quantity $\Omega$ such that

$$\Omega \leq \mathcal{C}(H).$$

(7)

As we will show, the following is a coherence witness for $\mathcal{C}(H)$:

$$\Omega = \left\|\left(\sum_{i \neq j} \sum_{k \in K} W_{ijk} D^{-1/2} \hat{P}(k, t) D^{1/2}\right)\right\|_{\infty},$$

(8)

where $\|X\|_\infty$ denotes the operator bound norm, i.e. $X$’s largest eigenvalue, $D = \text{diag}(\text{dim } \Pi_1, \ldots, \text{dim } \Pi_n)$ and $K$ is a finite set of values of parameter $k$. 

5
In deriving inequality (7), we will show that it is the system’s susceptibility to the Zeno effect, namely, the potential for altering the system’s dynamics by continuous measurement, that reveals the presence of a coherent contribution to the system’s evolution. Our coherence witness $\Omega$ essentially captures this susceptibility through the dependence of the transition rates $(\dot{\mathcal{P}}(k, t))$ on $k$. A nonzero magnitude of $\Omega$ certifies that there is a nonzero gap between the largest and the smallest energy eigenvalues, $\mathcal{C}(H) \neq 0$, thus leading to a nontrivial Hamiltonian. This, in turn implies that the system’s dynamics cannot be expressed in terms of rate equations.

The present work is a proof-of-principle for establishing noise-independent lower bounds on $\mathcal{C}(H)$ and estimates of $\|H_\mu\|$ with minimal assumptions and preparation/measurement set-ups.

3. The measurement scheme

We begin with a generic master equation of the form of equation (1) with time-independent generator $\mathcal{L}$, subject to an additional controllable Hamiltonian

$$H_m = \sum_{i=1}^{n} \eta_i P_i,$$

which we call the measurement Hamiltonian. We will use an overall parameter $k$ to denote the intensity of this, and thus regard $H_m$ as dimensionless.

We use Hamiltonian $H_m$ to induce a continuous coherent driving, with $H_m$ having no more degeneracy than that imposed by the ranks of the projectors $P_i (\eta_i = \eta_j \Leftrightarrow i = j)$. The effective result of this driving is an induced quantum Zeno effect on the coherent part of the system’s dynamics, described by the effective Zeno Hamiltonian $H_Z = \sum_i P_i H_m P_i$. The dissipative part of the dynamics, characterized by $W_\mu$, will potentially be affected by the driving, but not suppressed.

With our set-up, the Zeno subspaces are $\{\Pi_i\}$, and we make no assumptions on their dimensions other than them being known, $d_i \equiv \text{tr} P_i$. It is not necessary for us to assume any specific values of $\eta_i$; however, from the perspective of the technique used in the course of this work, our preferred choice of the measurement design is to make the differences $\big(\eta_i - \eta_j\big)$ unique.

The dynamics of the system is given by the equation

$$\frac{d}{dt} \rho(t) = \mathcal{L} [\rho(t)] - ik \left[ H_m, \rho(t) \right].$$

We assume that the frequencies related to $kH_m$ are not higher than nor comparable to the frequencies related to the processes underlying decoherence in the dynamics given by jump operators $\{W_\mu\}$. In other words, the magnitude of $kH_m$ does not conflict with the Markovian approximation underlying equation (1). In the following, when we refer to the large $k$ limit, and denote it by $k \to \infty$, one must bear in mind that this limit is constrained within the validity of the Markovian approximation underlying equation (1). With this consideration in mind, we can
safely assume that the Lindblad representation of the evolution is valid throughout the entire measurement and the system’s dynamics remains Markovian [30].

We consider the system initialized in the maximally mixed state in one of the measurement subspaces, namely, $\rho(0) = \rho_i \equiv P_i/d_i$. Next, we introduce the projection superoperator, a centralizer of $H_{m}$, defined as

$$\mathcal{P}[\rho] \equiv \sum_i P_i \rho P_i,$$  \hspace{1cm} (11)

along with its complementary projector

$$Q[\rho] \equiv (\mathcal{I} - \mathcal{P})[\rho] = \sum_{k \neq l} P_k \rho P_l,$$  \hspace{1cm} (12)

where $\mathcal{I}$ is the identity superoperator. With the above mentioned choice of the initial state, $\rho(0) = \mathcal{P}[\rho(0)]$ is satisfied.

We choose the measurement protocol to be the following:

1. **Preparation:** The system is prepared at time 0 in one of the measurement subspaces: $\rho(0) = \rho_j$.
2. **Evolution:** Let the system evolve for the appropriately chosen small time $t$, with continuous driving $kH_{m}$ (with strength $k$).
3. **Measurement:** At time $t$ a conclusive projective measurement $\{P_i\}$ is performed.
4. **Estimation:** Repetition of this process with different initial preparations yields the probabilities of finding outcome $i$ at time $t$, when the system was prepared at time 0 in state $j$, and the evolution is continuously driven with $H_{m}$ at strength $k$.

This procedure yields the generalized transition probabilities $p_{i\rightarrow j}(k, t)$, which can be conveniently arranged in a matrix $[\mathcal{P}(k, t)]_{ij} = p_{i\rightarrow j}(k, t)$. By measuring $\mathcal{P}(k, t)$ at various times and coupling strength values, $k$, one obtains sufficient information about the dynamics to be able to place lower bounds on the amount of coherence and decoherence present in the dynamics. We regard the rates $[\dot{\mathcal{P}}(k, t)]_{ij} = \frac{d}{dt}p_{i\rightarrow j}(k, t)$ as the time derivative of the transition probabilities

$$\dot{\mathcal{P}}_{ij}^{(k)} \equiv \left[\dot{\mathcal{P}}(k, t)\right]_{ij} \equiv \frac{d}{dt}p_{ij}(k, t).$$  \hspace{1cm} (13)

Our set of experimental data will consist of rates of transition between various measurement subspaces $(i, j)$ for a set of measurement strengths $K = \{k_1, \ldots, k_N\}$, measured at appropriately chosen small time $t$.

The next requirement in our analysis is to establish an analytical correspondence between transition rates and specific properties of the Lindblad superoperator that we are interested in.
4. The effective dynamics

Now we obtain the dynamics of the centralized density operator $\mathcal{P}[\rho(t)]$ at suitably chosen small times $t$, in terms of the initial state $\rho(0) = \mathcal{P}[\rho]$ and the driving strength $k$. It is convenient for our purposes to write $\mathcal{L}$ as a combination of two terms:

$$\frac{d}{dt}\rho = -i \text{ad}_H[\rho] + \mathcal{L}_\phi[\rho],$$  \hspace{1cm} (14)

where we have defined

$$\text{ad}_H[\rho] = [H, \rho],$$  \hspace{1cm} (15a)

$$\mathcal{L}_\phi[\rho] = \phi[\rho] - \frac{1}{2} \left\{ \phi^*(1), \rho \right\}.$$  \hspace{1cm} (15b)

Here, $\text{ad}_H$ is the adjoint action well known in the theory of Lie algebras [31], and $\phi$ is a completely positive map,

$$\phi(\rho) = \sum_\mu W_\mu \rho W_\mu^\dagger.$$  \hspace{1cm} (16)

In (15b), $\phi^* = \sum_\mu W_\mu^\dagger \cdot W_\mu$ denotes the adjoint of $\phi$ with respect to the Hilbert–Schmidt inner product, i.e. $\text{tr}[X^\dagger \phi(\rho)] = \text{tr}[(\phi^*(X))^\dagger Y]$.

We start with the generalized Liouville equation [32] including $kH_m$, where $k$ controls the strength of the driving mechanism:

$$\frac{d}{dt}\rho(t) = \mathcal{L}[\rho(t)] - ik \text{ad}_{H_{ikm}}[\rho(t)].$$  \hspace{1cm} (17)

We are interested in the dynamics of the system for small times $t \ll 1/\|\mathcal{L}\|$ as compared with the typical timescales of the Lindblad generator. The details of the derivation are contained in appendix A. At next-to-leading order in $t$ we have

$$\frac{d}{dt}\mathcal{P}[\rho(t)] = \mathcal{P}\mathcal{L}\mathcal{P} + \int_0^t ds \mathcal{U}(s-t)\mathcal{L}\mathcal{P} + \mathcal{O}(t^2\|\mathcal{L}\|^2)\Big[\rho(0)\Big],$$  \hspace{1cm} (18)

where $\mathcal{U}(t)[x] = e^{ik H_m x} e^{-ik H_m}$. One can readily see that first term $\mathcal{P}\mathcal{L}\mathcal{P} = -i \text{ad}_{H_{ikm}}\mathcal{P} + \mathcal{P}\mathcal{L}\phi\mathcal{P}$ gives rise to the effective Zeno Hamiltonian $H_\text{Z} = \mathcal{P}[H]$, together with the effective dissipative dynamics among subspaces $\{\Pi_i\}$. The second term contains the adiabatically eliminated population transfer due to coherences, which occur only at next-to-leading order in the strong driving $kH_m$. This expression is amenable to exact integration, thus yielding

$$\frac{d}{dt}\mathcal{P}[\rho(t)] = \left(\mathcal{D}_0(t) + \mathcal{D}_1(t) + \mathcal{L} \times \mathcal{O}(t^2\|\mathcal{L}\|^2)\right)[\rho(0)],$$  \hspace{1cm} (19a)
This equation is valid for small times defined by \( t \ll 1/\| \mathcal{L} \| \) and for all values of \( k \). We neglect the term \( \mathcal{P} \mathcal{L} \times O(t^2 \| \mathcal{L} \|^2) \) as it is subleading w.r.t. the other terms. The detailed derivation of equation (19) can be found in appendix A.

Note that when we take \( k \to \infty \), the master equation reduces to

\[
\frac{d}{dt} \mathcal{P} = \mathcal{D}_0(t) \left[ \rho(0) \right] = \mathcal{P} \mathcal{L} \mathcal{P} \left( I + t \mathcal{P} \mathcal{L} \mathcal{P} \right) [\rho(0)].
\] (20)

The above can be understood as an effective (Zeno) dynamics characterized by \( \mathcal{L}_Z = \mathcal{P} \mathcal{L} \mathcal{P} \), which, acting on centralized states \( \rho(0) = \mathcal{P} [\rho(0)] \), can be expressed as

\[
\mathcal{L}_Z = -i \text{ad}_{\mathcal{H}_a} + \mathcal{L}_{\phi_{\text{eff}}}
\] (21)

where \( \mathcal{H}_z = \mathcal{P} [H] \) is the Zeno Hamiltonian, \( \phi_{\text{eff}} = \mathcal{P} \phi \mathcal{P} \) describes the effective decoherence process and the order-\( t \) term in equation (20) corresponds to the first term in the expansion \( \rho(t) = \exp \left( t \mathcal{L}_Z \right) \rho(0) \).

Our main interest lies in the operator \( \mathcal{D}_1(t) \) which (a) depends on \( k \mathcal{H}_a \), (b) couples subspaces defined by \( \mathcal{P} \) and \( Q \) and (c) imprints phases onto the \((i,j)\) blocks in \( Q \). Notice that the adjoint action \( \text{ad}_{\mathcal{H}_a} \) has support on the subspace defined by \( \mathcal{Q} \); thus the expression \( Q \text{ad}^{-1}_{\mathcal{H}_a} Q \) is well-defined, and can be written as

\[
Q \text{ad}^{-1}_{\mathcal{H}_a} Q [X] = \sum_{i \neq j} \frac{1}{\eta_i - \eta_j} P_i X P_j,
\] (22)

with immediate generalization to the expression encountered in equation (19b).

Superoperators \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \) capture the essential physics revealed by the measurements, and the population transfer rates (equation (13)) are given by

\[
\left[ \mathcal{P} (k, t) \right]_{ij} = \text{tr} \left[ P_i (\mathcal{D}_0(t) + \mathcal{D}_1(t)) [\rho_j] \right].
\] (23)

To discuss the consequences of both \( \mathcal{D}_0(t) \) and \( \mathcal{D}_1(t) \) we need to treat them on an equal footing. Therefore we need to guarantee that the second-order term in \( \mathcal{D}_0(t) \) can be compared with \( \mathcal{D}_1(t) \), due to the magnitudes of \( t \mathcal{L} \) and \( k \). We find that \( kt \sim 1 \) is a suitable regime of \( k \) to work with.

5. Bounds on coherence

We are now in a position where we can derive bounds for coherence measures of \( \text{ad}_{\mathcal{H}_a} \), and, thus, come to the main results of the present work. As has been discussed in the introduction, in the strong driving regime (Zeno regime, \( k \to \infty \)) all coherent population transfer between the Zeno
subspaces is suppressed, and the remaining dynamics between them can ultimately be attributed to the incoherent processes of the system. We illustrate this with examples in section 6. This does not mean that the incoherent dynamics is unaffected by the measurement. As shown in equation (21), the map $\phi$ describing the incoherent process is also modified, but it remains relevant as long as population transfer is considered.

The main observation which will be recurring in the following is that the rate matrix, $\dot{P}$, can be regarded as a minor of the matrix representation of the superoperator $\mathcal{D}_\theta + \mathcal{D}_1$ in a suitably chosen basis of $L(\mathcal{H})$, $\{P_i, ..., P_n, T_1, ..., T_{g-n}\}$, where the $T$ complete the basis defined by $\{P_i\}$. Since this basis is not orthonormal, we introduce the orthonormal operators $\tilde{P}_i = P_i/\sqrt{d_i}$, such that

$$\text{tr}\left[\tilde{P}_i\tilde{P}_j\right] = \frac{\text{tr}\left[PP_j\right]}{(d_d)^{1/2}} = \delta_{ij}. \quad (24)$$

In addition, we arrange dimensions $d_i$ in the matrix $D = \text{diag}(d_1, ..., d_n)$, so that we can define the normalized rate matrix $R$:

$$R^{(k)}(t) = D^{-1/2}\hat{P}(k, t)D^{1/2}. \quad (25)$$

The normalized rates $R_{ij}^{(k)}(t)$ can be written as

$$R_{ij}^{(k)}(t) = \text{tr}\left[\tilde{P}_i\left(\mathcal{D}_\theta(t) + \mathcal{D}_1(t)\right)[\tilde{P}_j]\right]. \quad (26)$$

Recalling that $Q(\cdot) = \sum_r P_r(\cdot)P_s$ and $H_mP_r = \eta_rP_r$ we can write

$$R_{ij}^{(k)} = \text{tr}\left[\tilde{P}_i\left(\mathcal{D}_\theta(t) + \sum_{r\neq s} \frac{1-e^{i\delta(\eta_r-\eta_s)}}{ik(\eta_r-\eta_s)} P\mathcal{L}Q_{rs}\mathcal{L}P\right)[\tilde{P}_j]\right], \quad (27)$$

where $Q_{rs}[\cdot] = P_r\cdot P_s$ projects onto a specific off-diagonal block corresponding to the pair $(r, s)$:

$$Q_{rs}[\rho] = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & P_rP_s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}. \quad (28)$$

With the right choice of $\{\eta_r\}$, the differences $\eta_r - \eta_s$ are unique, and the off-diagonal blocks $Q_{rs}$ gain unique frequencies. Next, we recover the matrix representation of $\mathcal{P}\mathcal{L}Q_{rs}\mathcal{L}\mathcal{P}$ by means of a suitable linear transform.

The contribution to the $R$ matrix away from the Zeno regime ($k \to \infty$) has been shown to be $R^{(k)} = R^{(Z)} + R^{(1)}$, where $R^{(Z)}$, $R^{(1)}$ are given by

$$R_{ij}^{(Z)} = \text{tr}\left[\tilde{P}_i\mathcal{L}(I - t\mathcal{P}\mathcal{L})[\tilde{P}_j]\right] \quad (29a)$$
In order to recover the details of the superoperator \( \text{ad}_{\mathcal{H}} \), it is convenient to work in the superoperator eigenbasis of \( \text{ad}_{\mathcal{H}} \). Operators of the form \( P \mathcal{O} P \) span the eigenspace of \( \text{ad}_{\mathcal{H}} \) with eigenvalue \( \omega_{rs} = \eta_r - \eta_s \). Namely, eigenvectors consist of operators with nonzero entries lying in a unique off-diagonal block, labelled by row–column indices \((r, s)\). We label pairs of indices with Greek letters, so if \((r, s) = \mu\) we write \( \omega_{\mu} \) for the frequencies in \( \mathcal{D}_1 \) and \( Q_{\rho} [\rho] = P \rho P \). Notice that \( \ker(\text{ad}_{\mathcal{H}}) = \ker Q \), so the expression \( Q \left( 1 - \mathcal{U}^* (t) \right) \text{ad}^{-1}_{\mathcal{H}} Q \) appearing in equation (29a) is well-defined even though \( \text{ad}_{\mathcal{H}} \) is singular. Defining
\[
M_{\mu} = \frac{e^{-i k \omega_{\mu}} - 1}{i k \omega_{\mu}}
\] (30)
we can write \( R^{(1)} \) as
\[
R^{(1)}_{ij} (k) = -\sum_{\mu} M_{\mu} \text{tr} \left[ \tilde{P} P \mathcal{L} Q \mathcal{P} \left[ \tilde{P} \right] \right].
\] (31)
The matrix \( M \) is indexed by the different values of \( k \in K \) and the frequencies \( \mu = \{1, \ldots, n(n-1)\} \). Therefore \( M \) is not necessarily a square matrix, and the number of rows depends on our choice of how many values of \( k \) are sampled. Let us assume that we chose to sample one more value of \( k \) than there are frequencies, i.e. \( |K| = n(n-1) + 1 \); thus \( k \in K = \{ k_0, k_1, \ldots, k_{n(n-1)} \} \).

Let us introduce the pseudoinverse of \( M \), \( W \), such that
\[
\sum_{k \in K} W_{\mu k} M_{kv} = \delta_{\mu v},
\] (32a)
\[
\sum_{k \in K} W_{\mu k} = 0.
\] (32b)
This can always be satisfied when \( |K| = n(n-1) + 1 \) by taking the singular value decomposition of \( M \):
\[
M = U \begin{pmatrix} 0 \\ S \end{pmatrix} V
\] (33)
where we have highlighted the lower \( n(n-1) \times n(n-1) \) block corresponding to the \( S = \text{diag} \left( s_1, \ldots, s_{n(n-1)} \right) \) singular values of \( M \). Let \( u_{kk} \) be the matrix elements of \( U \). Then the choice
\[
W = V^\dagger \left( x \left| S^{-1} \right\rangle \right) U^\dagger,
\] (34)
where $x$ is a vector defined as
\[ x_{\mu} = -\sum_{k=1}^{n} \sum_{n=1}^{n} u_{\mu k}^* u_{n k}, \quad \mu = 1, \ldots, n(n-1), \] (35)
is a solution to equations (32). A few words of caution are in order. In choosing the energy levels of $H_m$ and values of $k$, one must ensure that $M$ has $n(n-1)$ nonzero singular values. Otherwise equation (35) is ill-defined. Fortunately, the freedom in choosing these values is large enough for making sure that this case is always avoidable.

Having at hand the pseudoinverse $W$, one can use it to obtain
\[ T_{\mu} = \sum_{k} W_{\mu k} R^{(k)} \] (36)
\[ = \sum_{k} W_{\mu k} \left[ R^{(z)} + R^{(i)}(k) \right]. \] (37)
The first sum vanishes due to equation (32b), leading to
\[ \left[ T_{\mu} \right]_{ij} = -\text{tr} \left[ \tilde{P} \mathcal{P} \mathcal{Q}_{\mu} \mathcal{L} \mathcal{P} \left[ \tilde{P} \right] \right]. \] (38)
The $T_{\mu}$ matrix may be called the Zeno susceptibility as it determines how the system’s frequency $\omega_{\mu}$ responds under the Zeno measurement, or continuous driving. Finally, by taking into account assumptions discussed at the beginning of this work (2), one can show that (see appendix B.1)
\[ Q \mathcal{L}_{\delta} \mathcal{P} \left[ P \right] = 0 \] (39a)
\[ Q \mathcal{L}^{\delta} \mathcal{P} \left[ P \right] = 0, \] (39b)
so $T_{\mu}$ can be written as
\[ \left[ T_{\mu} \right]_{ij} = \text{tr} \left[ \tilde{P} \text{ad}_{H} Q_{\mu} \text{ad}_{H} \left[ \tilde{P} \right] \right]. \] (40)
$T_{\mu}$ is the matrix representation of $\text{ad}_{H} Q_{\mu} \text{ad}_{H}$ within the subspace of $L(\mathcal{H})$ spanned by $\{P\}$. These matrices contain the essential information in which we are interested. The motivation of assumption (2)—or equivalently (39)—is now clear. It ensures that incoherent dynamics does not couple Zeno subspaces in the adiabatically eliminated virtual—second-order—transitions.

5.1. Coherent coupling between zeno subspaces

The magnitude of $\mathcal{E}(H)$ characterizes the fastest timescales at which the system can coherently evolve and will concern us later. However, as will be seen in section 6, the measurement specified by $\{P\}$ often represents a physically meaningful decomposition of the system’s Hilbert space. It is therefore of interest to quantify the coupling among subspaces $\Pi$, induced by the system’s Hamiltonian $H$. Consider a pair of subspaces $\Pi_i, \Pi_j$, and the corresponding block in the Hamiltonian, given by the operator $H_{ij} = \mathcal{P} H \mathcal{P}_j$. The norm on the latter immediately quantifies the strength of the coupling between $\Pi_i$ and $\Pi_j$. In particular, the Hilbert–Schmidt norm, $\|X\|_2 = \sqrt{\text{tr}[XX^\dagger]}$, is directly related to the Zeno susceptibility $T_{\mu}$. Indeed, an easy calculation shows that, for $i \neq j$,
\[ \left( \frac{d}{d\lambda} \right)^{1/2} \left[ T_{ij} \right] = \| H_{ij} \|_2 \] (41)

which shows that the 2-norm of \( H_{ij} \) is readily available from our measurement scheme. Equation (41) combined with equations (36) and (25) gives the claimed result, equation (6).

The singular value decomposition of \( H_{ij} \) suggests that there are bases in \( \Pi_i, \Pi_j \) such that (supposing that \( \dim(\mathcal{P}_i) > \dim(\mathcal{P}_j) \))

\[
H_{ij} = U \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} V^+ \] (42)

with \( U, V \) unitaries in \( \Pi_i, \Pi_j \) respectively. Hence the singular values of \( h_{ij} \) are the largest coupling strengths that two orthogonal sets of vectors in \( \Pi_i, \Pi_j \) can have. Thus, \( \| H_{ij} \|_2 = \sqrt{\sum_{s \neq s'} \bar{s}^2} \) measures the strength of the coherent coupling between subspaces \( \Pi_i \) and \( \Pi_j \).

### 5.2. A universal measure of coherence

The magnitudes \( \| H_{ij} \|_2 \) characterize the strengths of coupling among Zeno subspaces \( \Pi_i, \Pi_j \), but one may be interested in obtaining estimates of the overall total strength of the Hamiltonian \( H \). This is of course not always possible due to couplings occurring within any given Zeno subspace, which are not accessible to our measurement scheme. However, the dependence of the rates \( R \) on \( k \) can provide nontrivial lower bounds to \( \mathcal{C}(H) \).

Summing equation (40) over all distinct pairs \( \mu = (r, s), r \neq s \), we get

\[
C_{ij} = \sum_{\mu} \left[ T_{\mu} \right]_{ij} = \text{tr} \left[ \hat{P}_r \text{ad}_{H} \text{ad}_{H} \left[ \hat{P}_s \right] \right]. \] (43)

Observing that \( \text{ad}_{H} \left[ \hat{P}_s \right] = \text{ad}_{H} \left[ \hat{P}_j \right] \)—namely, commutators only couple populations to coherences and vice versa, so the presence of \( \hat{P}_j \) allows one to remove the coherence projector \( Q \)—leads to the conclusion that

\[
C_{ij} = \text{tr} \left[ \text{ad}_{H} \left[ \hat{P}_j \right] \text{ad}_{H} \left[ \hat{P}_j \right] \right]. \] (44)

Clearly \( C \) is positive semidefinite; thus it has a well-defined square root. Let us define its operator norm

\[
\Omega = \| \sqrt{C} \|_\infty. \] (45)

Exploiting the monotonicity of the operator norm (see appendix B.2) we show that \( \Omega \) is a lower bound to the induced Hilbert–Schmidt superoperator norm of \( \text{ad}_{H} \):

\[
\Omega \leq \max_{\| X \|_2 = 1} \left\| \text{ad}_{H} [X] \right\|_2 \equiv \| \text{ad}_{H} \|_2. \] (46)
The induced Hilbert–Schmidt norm is \( \| \text{ad}_\eta \|_2 = \mathcal{C}(H) \) (see appendix B.3), and
\[
\Omega \leq |E_{\text{max}} - E_{\text{min}}| = \mathcal{C}(H),
\]  
where \( E_{\text{max}}, E_{\text{min}} \) are the highest and lowest energy eigenvalues respectively. This combined with (25), (36) and (43) yields our main result, equation (8). A nonzero value of \( \Omega \) is a witness that there is a nontrivial Hamiltonian contributing to the dynamics. This, in turn is an indicator that the dynamics of the system cannot be explained solely in terms of classical rate equations.

More precisely, the experimentally accessible \( \Omega \) provides a lower bound on the spectral spread of \( H, \mathcal{C}(H) \). Notice that in obtaining this bound, only generic assumptions on the decoherence are made, and no requirement is imposed on its strength. Naturally, if \( \| \mathcal{L} \| \) is very large, the timescales \( t \) at which the system needs to be measured become small; this may be due to very high decoherence rates. However, given the order of magnitude of the \( \| \mathcal{L} \| \), and a properly chosen small time \( t \), the resulting bounds are independent of the details of the decoherence process and its actual strength.

6. Examples

6.1. A qubit undergoing Rabi oscillations

We consider, as a first example, a simple two-level system undergoing a spontaneous emission-type incoherent process. We use the Pauli basis to write
\[
H = \frac{\Delta}{2} \left( \cos \theta \sigma_z + \sin \theta \sigma_x \right),
\]
\[
\mathcal{L}[\rho] = -i [H, \rho] + \gamma \left( \sigma^+ \rho \sigma^- - \frac{1}{2} \{ \sigma^+ \sigma^-, \rho \} \right),
\]
\[
H_m = \eta_1 \ket{1}\bra{1} + \eta_2 \ket{2}\bra{2},
\]
where \( \sigma^- = |0\rangle\langle 1| \) and \( E, \theta \) and \( \gamma \) are parameters of the model. Our measures of coherence in this model are easily shown to be
\[
\| H_{xz} \|_2 = \frac{1}{2} |\Delta \sin \theta|,
\]
\[
\mathcal{C}(H) = |\Delta|.
\]  
In addition, there is only one nontrivial form of decomposition equation (3) consistent with equation (2), namely: \( P_1 = \ket{1}\bra{1} \) and \( P_2 = \ket{2}\bra{2} \).

The effective Zeno dynamics given by
\[
\mathcal{L}_Z[\rho] = -i [H_z, \rho] + \gamma \left( \sigma^- \rho \sigma^+ - \frac{1}{2} \{ \sigma^+ \sigma^-, \rho \} \right),
\]
is governed by the Zeno Hamiltonian \( H_z = \frac{\Delta}{2} \cos \theta \sigma_z \) and the incoherent part which stays unaffected. The relevant superoperators take the forms
and

\[ f = \frac{1 - \exp \left\{ i k t \left( \eta_1 - \eta_2 \right) \right\}}{i k (\eta_1 - \eta_2)}, \]  

yielding the following results:

\[ C = \frac{1}{2} \Delta^2 \sin^2 \theta \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right), \]  

\[ \Omega = |\Delta| \sin \theta. \]  

Figure 1 shows the relation between \( C(H) \) and the bound \( \Omega \) provided by our scheme. It illustrates in the simplest possible scenario the performance and limitations of our proposal. When the effective Zeno Hamiltonian differs the most from the true undriven Hamiltonian \( (\theta = \pi/2) \), our methods provide the best bound \( \Omega = C(H) = 1 \). In all other cases, the bounds may be loose—going to the extreme case of being trivially zero when the dynamics rests unaffected by the decoupling mechanism \( (\theta = 0, \pi) \).

Most remarkably, this simple example shows that the coherence witness \( \Omega \) is potentially tight.

6.2. An N-site spin chain with a roller coaster energy landscape

As a second example we take an N-site spin chain with nearest neighbour coupling \( J \) and a roller coaster type of energy landscape with a gap \( E \) between the consecutive sites:
The action of the environment is given by incoherent hopping among nearest neighbour sites, described by jump operators $W_{nm}$:

$$
W_{nm} = \begin{cases}
\sqrt{\gamma} e^{\beta E} |m\rangle \langle n| & \text{if } \mod_2 n = 1 \& m = n \pm 1,
\sqrt{\gamma} e^{-\beta E} |m\rangle \langle n| & \text{if } \mod_2 n = 0 \& m = n \pm 1,
0 & \text{otherwise}.
\end{cases}
$$  

(61)

The effective dynamics in the Zeno regime is given by

$$H_Z = \sum_i P_i H P_i.$$  

(62)

Clearly, a more coarse grained decomposition in Zeno subspaces results in more off-diagonal terms from $H$ persistent in $H_Z$, whereas with Zeno subspaces that are less coarse grained (i.e. with more ‘resolution’ in the measurement setting), more off-diagonal terms are eliminated in the Zeno regime. Therefore the performance of $\Omega$ will strongly depend on the resolution of the measurement. It is worthy of mention that a coarse grained Zeno subspace decomposition $\{\Pi_i\}$ leaves $L_\phi$ unaffected if the fine grained one does also, i.e. $L_{\phi_{fin \#}} = L_\phi$, as is the case for the model considered here.

Let us go back now to the discussion of the performance of witness $\Omega$. Figure 2 shows $\mathcal{C}(H)$ itself (dotted curve) and its lower bounds $\Omega_{d_1 \ldots d_n}$, where the subscript denotes the number of sites (dimension) in each subspace of the $\{\Pi_i\}$ decomposition. We have evaluated $\Omega_{d_1 \ldots d_n}$ for a variety of decompositions, and plot their performances at different numbers of sites $N = \sum_{i=1}^n d_i$. It is clear that the best bound is obtained with rank-1 (single-site) projectors.
We can see as well that for increasing number of sites the coarse grained type of measurement, $\Omega_{1,N-2,1}$, approaches $\Omega_{1,N-1}$; this is to be expected. More details on the behaviour of $\Omega_{1,...,N}$ can be seen in figure 3: it shows $\mathcal{E}(H)$ and $\Omega_{1,...,N}$ obtained from a single-site-resolving measurement, as functions of both the number of sites and the coupling $J$. As can be seen, $\Omega_{1,...,N}$ follows the behaviour of $\mathcal{E}(H)$.

6.3. An N-site spin chain with a ladder energy landscape

Here we take the $N$-site spin chain with the nearest neighbour coupling $J$ and a ladder type of energy landscape characterized by the energy step $E$:

$$H = E \sum_n (N - n) |n\rangle \langle n| + J \left( \sum_n |n\rangle \langle n + 1| + |n + 1\rangle \langle n| \right).$$

Like for the previous case, the action of the environment is given by incoherent hopping, equation (61). The effective dynamics in the Zeno regime takes a form analogous to that in the previous subsection, equation (62), but with the energy levels decreasing linearly with $n$. As previously, the incoherent part of the dynamics is unaltered in the Zeno regime: $\mathcal{L}_\phi = \mathcal{L}_\text{eff}$.

Figure 2 (dashed curve) shows results for $\mathcal{E}(H)$ for the ladder model. The larger $N$, the looser the bounds obtained. This is due to the fact that for a fixed value of $J$ and increasing $N$, the coherence measure $\mathcal{E}(H)$ is dominated by the diagonal part of $H$ and is of order $N$. This is a general feature of our method: the more ‘aligned’ with the eigenspaces of $H$ the Zeno subspaces are, the less effective our witness is. This occurs because in these situations the Zeno effect has essentially little coupling to suppress.

Nevertheless, figure 4 illustrates that even for the ladder model, our scheme provides bounds sensitive to the strength $J$ and the measurement can reveal the magnitude of the couplings between the Zeno subspaces.

In both cases, the roller coaster and ladder types of energy landscape, one could consider also dephasing. It can be shown that also for this type of noise, $\mathcal{L}_\phi = \mathcal{L}_\text{eff}$. However, we have checked that in the examples provided, dephasing does not bring anything new to the discussion and therefore we have focused on the incoherent hopping as a main source of decoherence.
7. Discussion

We have shown how a measurement protocol and analysis based on the notion of Zeno susceptibility can be used to witness coherence. Our results constitute a proof-of-principle for using the Zeno effect, implemented by means of continuous driving, as a signature of nonclassicality. That the quantum Zeno Effect is a genuinely quantum phenomenon is rather intuitive. The flipside of this statement is that dynamics susceptible to the QZE are necessarily quantum. However, for noisy systems this susceptibility will only be partial. Hence, the amount by which the system’s dynamics is affected upon continuous driving is an indicator of how much of the said dynamics is due to coherent processes. Here we have shown how to make this statement quantitative, rigorous and operational for systems undergoing Markovian dynamics under certain mild constraints.

The approach outlined here may be extended towards situations in which some of our assumptions fail to hold. This could be done in a variety of ways, and an exhaustive study is well beyond the scope of this work. We outline here a few of them.

7.1. Generalizations

The measurement protocol suggested has advantages and disadvantages. Regarding state preparations, we have proposed a measurement set-up that relies on preparing maximally mixed states within each Zeno subspace. For a variety of reasons, these preparations may not always be achievable. However, the analysis can be carried over to different sets of preparations. Clearly, preparations corresponding to coarse grainings of a given decomposition \( \{ \Pi_i \} \) will necessarily yield looser bounds. This is particularly relevant in situations where only part of the system is accessible. However, a discussion of optimality past this observation is beyond the scope of this work.

Analogously, if part of the system is inaccessible, it may be impractical to implement measurements on it. However, as long as the jump operators are local (in terms of the accessible and inaccessible subsystems), one can always find a sufficiently coarse grained decomposition (3), corresponding to practical measurements and where \( H_m \) acts nontrivially on the accessible
subsystem alone. This can potentially extend our approach to systems with non-Markovian
dynamics, thus overcoming the Markovianity assumption.

Regarding the assumption on the noise structure, equation (2), our measurement protocol
does not rely on any specific choice of dimensions of $\Pi_i$, but it does rely on the facts that the
system’s Hilbert space can be decomposed into Zeno subspaces compatible with the noise (in
accordance with equation (2)) and that this decomposition is known. We have been unable to
determine whether these assumptions can be directly verified from the measurements implied
by our protocol. This leads us to consider the possibility of lifting these conditions altogether.
However, lifting these conditions essentially boils down to making no assumptions whatsoever
on the noise mechanisms at work. As it turns out, this approach is much less innocent than it
may appear at face value. Given a Lindblad superoperator of the form of equation (1b), the
Hamiltonian and jump operators are not uniquely defined. The transformations

$$H \rightarrow H + \frac{1}{2i} \sum_{\mu} \left( \alpha^\dagger c \alpha^c W^\dagger_{\mu} - \alpha^a W^a_{\mu} \right)$$

$$W^\dagger_{\mu} \rightarrow W^\dagger_{\mu} + \alpha^a_{\mu} 1,$$

leave $L$ invariant [33]. This nonunique decomposition, in the absence of further assumptions on
$W^a_{\mu}$, forces one to reconsider the definition of coherence, as $\mathcal{C}(H)$ ceases to be unique. This
compels us to generalize our notion of coherence. We propose here candidates for such
generalizations.

A universal measure of coherence which is independent of the particular representation of
$\mathcal{L}$ could be given by

$$\mathcal{C}(\mathcal{L}) = \inf_{\mu \in \mathbb{C}} \left\{ H + \frac{1}{2i} \sum_{\mu} \left( \alpha^\dagger c \alpha^c W^\dagger_{\mu} - \alpha^a W^a_{\mu} \right) \right\},$$

where it is understood that $H$ and $W^\dagger_{\mu}$ are the Hamiltonian and jump operators of any
representation of $\mathcal{L}$. This quantity becomes zero if and only if the system’s master equation can
be entirely written in terms of jump operators—with no Hamiltonian. In addition, it is invariant
under the addition of a decoherence term—e.g. $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\mathcal{L} + \mathcal{L}^{\phi})$ for arbitrary $\phi$—which
supports the notion of being able to single out the coherent part of $L$ despite the presence
of noise. We leave as an open problem determining whether this quantity can be easily computed
and/or measured.

The lifting of $\mathcal{C}(H)$ to $\mathcal{C}(\mathcal{L})$ can be seen as a relaxation or a tightening of our problem. On
the one hand, it tightens our framework by requiring us to optimize a superoperator norm over
all possible representations of $\mathcal{L}$. On the other hand, it is a relaxation because the only natural
set-up for using equation (65) is that where no assumptions are made about the $W^\dagger_{\mu}$. In particular,
relaxation of assumptions (2) seems a natural framework in which to consider quantities which
depend solely on $\mathcal{L}$ and not one of its particular representations, as in equation (65).

Along with the general notion of coherence comes a natural question of a measure of
decoherence that could be used as a reference. In this work we refer to $\|L\|$, the fastest
timescale of the system’s dynamics, as an upper bound to the fastest decoherence timescales
(assuming that they are known) which dictate the smallness of $t$. However, a systematic
approach is suggested by the so-called Leibnitz defect [33] $\Delta_{\mathcal{L}}$: 
\[ \Delta_\mathcal{L}(A, B) = \mathcal{L}(AB) - \mathcal{L}(A)B - A\mathcal{L}(B), \]  

which is zero if and only if \( \mathcal{L} \) can be written solely in terms of a Hamiltonian. \( \Delta_\mathcal{L} \) determines how much \( \mathcal{L} \) fails to be of the commutator form \(-i[H, \cdot] \). Thus, a suitable norm on \( \Delta_\mathcal{L} \) is a reasonable candidate for providing a measure of the decoherence in \( \mathcal{L} \). These two measures will be discussed in more detail in future work.

Also, it would be instructive to establish a detailed relation between our approach and results presented in [4, 17, 24]. We note at this point that our method focuses primarily on detecting quantumness of the unknown physical process, instead of looking at the quantumness of states. Detailed comparison and analysis is, however, beyond the scope of the present work.

7.2. Implementations

Experiments with continuous driving may be challenging in certain physical systems of interest. However, the QZE can be induced by other means, such as continuous measurement [34] and dynamical decoupling [35]. Since the essence of our analysis relies not on the Zeno limit, but on how this limit is approached (the next-to-leading order contribution in the asymptotic expansions), it is not immediate that the same results will translate to these other set-ups. The ideas, however, should not vary in their essence. We leave the task of understanding how our ideas carry over to these set-ups for future work.

In the case of implementation by means of continuous measurement, one may also track the set of measurement outcomes and not only the outcome of the final projective measurement. Then a parameter estimation could be applied, such as that of [36], where the authors discuss parameter inference from a continuously measured system with an application to quantum optics. The difficulty then lies in designing a statistical parametric model capturing the specifics of our problem, while being general enough to accommodate unknown \( \mathcal{L} \) dynamics. This procedure may allow us to obtain more accurate estimates of \( \Omega \) and even tighter bounds of \( \mathcal{C} \).

Regarding the tightness of inequality (47), one should note that only as much as is measured can be learnt: Hamiltonians that commute with the measurement Hamiltonian will be totally beyond the reach of witness \( \Omega \). In such cases, a different measurement basis and/or preparation will be required. For all other systems, \( \Omega \) will provide a nontrivial lower bound to \( \mathcal{C}(H) \). As shown by the examples, the more ‘noncommuting’ the unknown Hamiltonian is, the tighter inequality (47) will be. The virtue of this method is that it assumes noisy dynamics and, in our scheme, provides bounds that are robust against the strength of the noise.

Building the statistics needed for estimating \( \hat{P} \) is simple, but it may be challenging to work in the regime of small times and appropriate values of \( k \). A deeper study of the time frames for which the reduced system dynamics is Markovian (for large \( k \)) will allow us to safely lift this small time constraint.

Acknowledgments

The authors wish to express gratitude for early discussions with L Heaney and A Marais. Fruitful discussions with S Pascazio and M Kastoryano are also acknowledged. We acknowledge support from the National Research Foundation and Ministry of Education, Singapore. AM also acknowledges financial support from ERDF: the European Regional...
Appendix A. Derivation of the effective dynamics

In the derivation of equation (19) we absorb $k$ in the coupling Hamiltonian $H_m$, since our results will be independent of the strength of $H_m$.

We start by decomposing the density operator $\rho$ into parts which we will call populations $P[\rho]$ and coherences $Q[\rho]$. We define the unitary superoperator $U(t) = \exp i\mathcal{L}t$, which is just $U(t)[x] = e^{iHt}xe^{-iHt}$, and write down the density operator in the interaction picture w.r.t. $H_m$:

$$\rho'(t) = U(t)[\rho(t)] \quad (A.1a)$$

$$\mathcal{L}'(t) = U(t)\mathcal{L}U^a(t), \quad (A.1b)$$

where we skip the symbol for the composition of superoperators and regard it as a product of operators in Hilbert–Schmidt space. The master equation, equation (17), reads, in the interaction picture,

$$\frac{d}{dt}\rho'(t) = \mathcal{L}'(t)[\rho'(t)]. \quad (A.2)$$

where, essentially, the Hamiltonian $H_m$ is cancelled out. Our approach follows the derivation of the master equation without time convolution [32], which uses projection operator techniques [37], in order to derive the propagator from time 0 to time $t$.

The time evolution for the populations $P[\rho'(t)]$ and coherences $Q[\rho'(t)]$ then reads

$$\frac{d}{dt}P[\rho'(t)] = (P\mathcal{L}'(t)P + P\mathcal{L}'Q)[\rho'(t)], \quad (A.3a)$$

$$\frac{d}{dt}Q[\rho'(t)] = Q\mathcal{L}'(t)[\rho'(t)]. \quad (A.3b)$$

Now, let $\Gamma(t, t_0) = \mathbb{I}\exp\int_{t_0}^t ds Q\mathcal{L}'(s)$ be the time ordered exponential (see [32]), e.g. the solution to

$$\frac{d}{dt}\Gamma(t, s) = Q\mathcal{L}'(t)\Gamma(t, s), \quad (A.4)$$

with boundary condition $\Gamma(t, t) = \mathbb{I}$. With this one can solve for the coherences $Q[\rho'(t)]$:

$$Q[\rho'(t)] = \Gamma(t, t_0)Q[\rho'(t)] + \int_{t_0}^t ds \Gamma(t, s)Q\mathcal{L}'(s)P[\rho'(s)], \quad (A.5)$$

which, used in equation (A.3a), yields

$$\frac{d}{dt}P[\rho'(t)] = \mathcal{P}\mathcal{L}'(t)\left( \mathcal{P}[\rho'(t)] + \int_{t_0}^t ds \Gamma(t, s)Q\mathcal{L}'(s)P[\rho'(s)] \right). \quad (A.6)$$
We are interested in writing the time derivative of $\mathcal{P}\rho(t)$ at time $t$ as a function of the preparation $\mathcal{P}\rho(0)$: therefore we can write the time evolution channel (in the interaction picture) and the density operator as

$$\mathcal{E}(t, t_0) = \mathcal{T} \exp \int_{t_0}^{t} ds \mathcal{L}'(s), \quad \rho'(t) = \mathcal{E}(t, s)[\rho'(s)].$$

(A.7)

Then, the evolution of populations is given by

$$\frac{d}{dt}\mathcal{P}\rho'(t) = \mathcal{P}\mathcal{L}(t)[\mathcal{P}\mathcal{E}(t, 0) + \int_{0}^{t} ds \mathcal{L}(t, s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\mathcal{E}(s, 0)]\rho(0)].$$

(A.8)

Writing this in the Schrödinger picture, we obtain

$$\frac{d}{dt}\mathcal{P}\rho'(t) = \mathcal{P}\mathcal{L}\mathcal{P}\mathcal{E}(t, 0)[\rho(0)] + \mathcal{P}\mathcal{L}\int_{0}^{t} ds \mathcal{U}^*(t)\mathcal{L}(t, s)\mathcal{U}(s)\mathcal{Q}\mathcal{L}\mathcal{P}\mathcal{E}(s, 0)[\rho(0)].$$

(A.9)

Several remarks are worth making as regards this expression. The first line can be conveniently expressed as $\mathcal{L}(t)[\rho(0)]$, where $\mathcal{L}(t) = \mathcal{P}\mathcal{L}\mathcal{P}$. As we will see, this represents the dominant part of the dynamics at time $t$. On the other hand, the essential part of the second line is characterized by the memory kernel

$$\mathcal{K}(t) = \int_{0}^{t} ds \mathcal{U}^*(t)\mathcal{L}(t, s)\mathcal{U}(s)\mathcal{Q}\mathcal{L}\mathcal{P}\mathcal{E}(s, 0).$$

(A.10)

This operator characterizes the non-Markovianity of $\mathcal{P}\rho(t)$, as its evolution cannot be uniquely determined without reference to the coherences $\mathcal{Q}\rho$. The operator $\mathcal{K}(t, s)$ provides the accumulated non-Markovianity at time $t$. Numerical evidence shows that for relatively large values of $k \sim \|\text{ad}_{\mathcal{L}}\|$, $\mathcal{K}(t)$ is of order $1/k$ up to some time $T$, after which it becomes an important contribution, suggesting that after time $T$ the system acquires enough memory to make the non-Markovian effects relevant. A detailed study of this phenomenon is beyond the scope of this work.

Nevertheless, it is interesting to note that the contribution of the second line in equation (A.9) characterizes the adiabatically eliminated transitions among Zeno subspaces, mediated by coherences originated at time $s$, and evolving over time until $t$. The overall effect of this is a correction to the leading order dynamics. The relative relevance of this term will dictate whether transitions among subspaces occur due to incoherent first-order processes or through these virtual second-order transitions.

Instead of attempting a full solution of equation (A.9) we find it constructive to consider the evolution at small times $t$ such that $\epsilon = t \|\mathcal{L}\| \ll 1$. In this case, $\|\mathcal{L}\|$ has the dimension $1/t$ and determines the magnitude of the fastest timescales arising in equation (1). With the small parameter introduced, we can write

$$\mathcal{E}(t, 0) = \mathcal{I} + \int_{0}^{t} ds \mathcal{L}(s) + O(\epsilon^2),$$

(A.11)

$$\mathcal{L}(t, s) = \mathcal{I} + \int_{s}^{t} ds' \mathcal{Q}\mathcal{L}(s') + O(\epsilon^2).$$

(A.12)
We use the above expansions to write the equation for the evolution of the populations:

$$\frac{d}{dt}\mathcal{P}[^{(\rho(t)}] = \mathcal{P} \mathcal{L}\left(\mathcal{P} + \int_0^t ds \mathcal{U}(s-t) \mathcal{L} \mathcal{P} + O(e^3)\right)[\rho(0)]. \quad (A.13)$$

Here we note that the action of unitaries (related to the ‘adiabatic elimination term’) can be represented as

$$\int_0^t ds \mathcal{U}(s-t) = \mathcal{P} \mathcal{L} \mathcal{P} \mathcal{L} \mathcal{P} + \mathcal{P} \mathcal{L} Q \frac{I - e^{-it\Delta \phi \mathcal{H}_m}}{i \Delta \phi \mathcal{H}_m} Q \mathcal{L} \mathcal{P}, \quad (A.14)$$

Therefore, we can write the second-order term as

$$\mathcal{P} \mathcal{L} \int_0^t ds \mathcal{U}(s-t) \mathcal{L} \mathcal{P} = \mathcal{P} \mathcal{L} \mathcal{P} \mathcal{L} \mathcal{P} + \mathcal{P} \mathcal{L} Q \frac{I - e^{-it\Delta \phi \mathcal{H}_m}}{i \Delta \phi \mathcal{H}_m} Q \mathcal{L} \mathcal{P}, \quad (A.15)$$

and finally

$$\frac{d}{dt}\mathcal{P}[\rho(t)] = \mathcal{P} \mathcal{L}\left(\mathcal{P} + \frac{I - e^{-it\Delta \phi \mathcal{H}_m}}{i \Delta \phi \mathcal{H}_m} \mathcal{L} \mathcal{P} + O(e^3)\right)[\rho(0)]]. \quad (A.16)$$

One can check that choosing $k \sim \|\Delta \phi \mathcal{H}_m\|$ of the order $1/t$, so that $\Delta \phi \mathcal{H}_m \sim 1$, leads to superoperators $\mathcal{D}_0$ and $\mathcal{D}_1$ of similar magnitudes. This is the desirable regime to work in, which will render the dependence of $\mathcal{D}_1$ on $k$ most visible, despite statistical and experimental errors.

**Appendix B. Proofs of results regarding superoperators**

We provide here proofs of some of the facts stated in the text that may not be obvious to all readers.

**B.1. Implications of the preferred basis**

Introducing the anticommutator superoperator

$$\text{ac}_A[X] = \frac{1}{2}(AX + XA) \quad (B.1)$$

one can check that for self-adjoint $A = A^\dagger$, $\text{ac}_A$ is also self-adjoint:

$$\text{ac}_A^* = \text{ac}_A \quad (B.2)$$

Using this notation, $\mathcal{L}_\phi$ reads

$$\mathcal{L}_\phi = \phi - \text{ac}_{\phi(t)} \quad (B.3)$$

and our assumptions (2) read

$$Q\phi[P] = 0 \quad (B.4a)$$

$$Q\phi^*[P] = 0. \quad (B.4b)$$
From this, we go on to show that
\[ Q \mathcal{L}_\phi [P] = 0 \] (B.5a)
\[ (\mathcal{L}_\phi \mathcal{Q})^* [P_i] = 0. \] (B.5b)

From equation (B.4b) it is readily seen that
\[ Q \operatorname{ac}_{\phi^{(1)}} [P_i] = 0 \] (B.6)
which shows that \( Q \mathcal{L}_\phi [P] = Q \phi [P] - Q \operatorname{ac}_{\phi^{(1)}} [P] = 0 \), equation (B.5a). Using self-adjointness of \( Q \) and \( \operatorname{ac}_{\phi^{(1)}} \), we have that equation (B.5b) reads
\[ Q \mathcal{L}_\phi^* [P] = Q \left( \phi^* - \operatorname{ac}_{\phi^{(1)}} \right) [P] = 0. \] (B.7)

This implies that equation (40), \( \left[ T_{ij} \right] = -\operatorname{tr} \left[ \tilde{P}_i \mathcal{L} Q \mathcal{L}[\tilde{P}_j] \right] \), when written as
\[ \left[ T_{ij} \right] = -\frac{1}{\sqrt{d_i d_j}} \operatorname{tr} \left[ (Q \mathcal{L}^* [P_i])^\dagger Q \mathcal{L}[P_j] \right]. \] (B.8)

This can be expressed as \( \left[ T_{ij} \right] = \operatorname{tr} \left[ B^i Q_\mu [A] \right] \), with
\[ A = Q \left( -i \operatorname{ad}_H + \mathcal{L}_\phi \right) [P_j] = -i \operatorname{ad}_H [P_j], \] (B.9a)
\[ B = Q \left( -i \operatorname{ad}_H + \mathcal{L}_\phi \right)^* [P_j] = i \operatorname{ad}_H [P], \] (B.9b)

so \( \left[ T_{ij} \right] \) reduces to
\[ \left[ T_{ij} \right] = \operatorname{tr} \left[ \left( \operatorname{ad}_H [\tilde{P}_j] \right)^\dagger Q_\mu \operatorname{ad}_H [\tilde{P}_j] \right] = \operatorname{tr} \left[ \tilde{P}_i \operatorname{ad}_H Q_\mu \operatorname{ad}_H [\tilde{P}_j] \right]. \] (B.10)

### B.2. Monotonicity of the operator norm

Recall that \( \{ \tilde{P}_i \} \) is an orthonormal basis in a subspace \( V \) of \( \mathcal{L} (\mathcal{H}) \) and \( \{ \tilde{P}_i, T_r \} \) is its extension to \( \mathcal{L} (\mathcal{H}) \). In this sense, \( C_{ij} = \operatorname{tr} \left[ \tilde{P}_i \operatorname{ad}_H^2 [\tilde{P}_j] \right] \) is the matrix representation of superoperator \( \operatorname{ad}_H^2 \) restricted to subspace \( V \). It is positive semidefinite, as can be seen from
\[ x^\top C x = \operatorname{tr} \left[ \operatorname{ad}_H [X]^\dagger \operatorname{ad}_H [X] \right] \geq 0, \] (B.11)
where $X = \sum x_j \tilde{P}_j$. Thus, $\sqrt{C}$ is uniquely defined and its operator norm is given by
\[
\|\sqrt{C}\|_\infty = \sup_{\|x\|=1} \|\sqrt{C}x\| = \sup_{\|x\|=1} x^\top C x = \sup_{\|x\|=1} \left\| \text{ad}_H \left[ \sum_j x_j \tilde{P}_j \right] \right\|_2 ,
\]
where $\|x\| = \sqrt{x^\top x}$ is the standard Euclidean norm and $\|X\|_2$ is the Hilbert–Schmidt norm $\|X\|_2 = \sqrt{\text{tr} \left[ X^\top X \right]}$. For any $x$ such that $\|x\| = 1$, we have that $X = \sum x_j \tilde{P}_j$ is
\[
\|X\|_2 = \left( \sum_j x_j^2 \text{tr} \left[ \tilde{P}_j^\top \tilde{P}_j \right] \right)^{1/2} = 1.
\]
Hence, we can upper bound equation (B.12) by relaxing the maximization to all operators in $L(\mathcal{H})$ normalized w.r.t. the 2-norm, thus obtaining the induced Hilbert–Schmidt superoperator norm:
\[
\|\sqrt{C}\|_\infty \leq \sup_{\|x\|=1} \|\text{ad}_H[X]\|_2 \equiv \|\text{ad}_H\|_2 .
\]

**B.3. Computation of the induced Hilbert–Schmidt norm**

Here we show that $\|\text{ad}_H\|_2 = \lambda_{\max}(H) - \lambda_{\min}(H)$, where $\lambda_i(H)$ are $H$’s eigenvalues and we will drop their dependence on $H$. Notice that eigenvectors of $\text{ad}_H$ are given by $|\psi_\alpha\rangle \langle \psi_\beta|$, where the $|\psi_\alpha\rangle$ constitute the eigenbasis of $H$. Then, any operator $X$ can be expressed in the eigenbasis of $\text{ad}_H$:
\[
X = \sum_{\alpha\beta} x_{\alpha\beta} |\psi_\alpha\rangle \langle \psi_\beta|,
\]
and hence
\[
\|\text{ad}_H(X)\|_2 = \left\| \sum_{\alpha\beta} x_{\alpha\beta} \left( \lambda_\alpha - \lambda_\beta \right) |\psi_\alpha\rangle \langle \psi_\beta| \right\|_2 \\
= \left( \sum_{\alpha\beta} \left| x_{\alpha\beta} \right|^2 \left( \lambda_\alpha - \lambda_\beta \right)^2 \right)^{1/2} \\
\leq \lambda_{\max} - \lambda_{\min}.
\]
Clearly, $\sum_{\alpha\beta} |x_{\alpha\beta}|^2 = 1 \iff \|X\|_2 = 1$, and thus the bound is attainable, with $X = |\psi_{\max}\rangle \langle \psi_{\min}| \in L(\mathcal{H})$, where the $|\psi_{\max}\rangle$, $|\psi_{\min}\rangle$ are eigenvectors corresponding to $\lambda_{\max}$.
\[ \lambda_{\text{min}} \] respectively. Thus,

\[
\|\text{ad}_H\|_2 = \sup_{\|X\| = 1} \|\text{ad}_H[X]\|_2 = \lambda_{\text{max}} - \lambda_{\text{min}}. \tag{B.17}
\]

References

[1] Adams D 1979 The Hitchhiker’s Guide to the Galaxy (New York: Harmony Books)

[2] Barontini G, Labouvie R, Stubenrauch F, Vogler A, Guerrera V and Ott H 2013 Phys. Rev. Lett. 110 035302

[3] Barreiro J T, Müller M, Schindler P, Nigg D, Monz T, Chwalla M, Hennrich M, Roos C F, Zoller P and Blatt R 2011 Nature 470 486

[4] Lambert N, Emary C, Chen Y-N and Nori F 2010 Phys. Rev. Lett. 105 176801

[5] Engel G S, Calhoun T R, Read E L, Ahn T-K, Mančal T, Chwalla M, Hennrich M, Roos C F, Zoller P and Blatt R 2007 Nature 446 782

[6] Olaya-Castro A, Lee C F, Olsen F F and Johnson N F 2008 Phys. Rev. B 78 085115

[7] Ishizaki A and Fleming G R 2009 Proc. Natl. Acad. Sci. 106 17255

[8] Chin A W, Huelga S F and Plenio M B 2012 Phil. Trans. R. Soc. A 370 3638

[9] Chin A W, Prior J, Rosenbach R, Caycedo-Soler F, Huelga S F and Plenio M B 2013 Nat. Physics 9 113

[10] Kassal I, Yuen-Zhou J and Rahimi-Keshari S 2013 J. Phys. Chem. Lett. 4 362

[11] Gauger E M, Rieper E, Morton J J L, Benjamin S C and Vedral V 2011 Phys. Rev. Lett. 106 040503

[12] Cai J and Plenio M B 2013 Phys. Rev. Lett. 111 230503

[13] Bauerngratz T, Cramer M and Plenio M B 2013 arXiv:1311.0275 [quant-ph]

[14] Rodriguez-Rosario C A, Frauenheim T and Aspuru-Guzik A 2013 arXiv:1308.1245 [quant-ph]

[15] Li C-M, Lambert N, Chen Y-N, Chen G-Y and Nori F 2012 Scientific Report 2 885

[16] Yuen-Zhou J, Arias D H, Eisele D M, Steiner C P, Krich J J, Bawendi M, Nelson K A and Aspuru-Guzik A 2014 ACS Nano at press (doi:10.1021/nn406107q)

[17] Leggett A J and Garg A 1985 Physical Review Letters 54 857

[18] Wilde M M, McCracken J M and Mizel A 2010 Proc. R. Soc. 466 1347

[19] Chen G-Y, Chen S-L, Li C-M and Chen Y-N 2013 Scientific Report 3 2514

[20] Brukner Č, Taylor S, Cheung S and Vedral V 2004 arXiv:quant-ph/0402127

[21] Fritz T 2010 New J. Phys. 12 083055

[22] Košťál and Brukner Č 2013 Phys. Rev. A 87 052115

[23] Misra B and Sudarshan E C G 1977 J. Math. Phys. 18 756

[24] Facchi P and Pascacio S 2002 Phys. Rev. Lett. 89 080401

[25] Facchi P and Pascacio S 2008 J. Phys. A: Math. Theor. 41 493001

[26] Kossakowski A 1972 Rep. Math. Phys. 3 247

[27] Lindblad G 1976 Commun. Math. Phys. 48 119

[28] Gardiner C and Zoller P 2004 Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics (Berlin: Springer)

[29] Fulton W and Harris J 1991 Representation Theory: A First Course (New York: Springer)

[30] Breuer H-P and Petruccione F 2002 The Theory of Open Quantum Systems (Oxford: Oxford University Press)

[31] Tarasov V 2008 Quantum Mechanics of Non-Hamiltonian and Dissipative Systems (Amsterdam: Elsevier)
[34] Facchi P and Pascazio S 2001 *Fortschritte der Physik* **49** 941–7
[35] Facchi P, Lidar D A and Pascazio S 2004 *Phys. Rev. A* **69** 032314
[36] Gammelmark S and Mømer K 2013 *Phys. Rev. A* **87** 032115
[37] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1992 *Atom-Photon Interactions: Basic Processes and Applications* (New York: Wiley)