A sharp estimate for the bottom of the spectrum of the Laplacian on Kähler manifolds

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Abstract

On a complete noncompact Kähler manifold we prove that the bottom of the spectrum for the Laplacian is bounded from above by $m^2$ if the Ricci curvature is bounded from below by $-2(m + 1)$. Then we show that if this upper bound is achieved then the manifold has at most two ends. These results improve previous results on this subject proved by P. Li and J. Wang in [L-W3] and [L-W] under assumptions on the bisectional curvature.

1 Introduction

Let $N^n$ be a complete noncompact Riemannian manifold of dimension $n$ and assume that the Ricci curvature has the lower bound $\text{Ric} \geq -(n - 1)$. As a consequence of Cheng’s theorem ([C]) we know that if $\lambda_1 (N)$ denotes the bottom of the spectrum of the Laplacian on $N$ then $\lambda_1 (N) \leq \frac{(n-1)^2}{4}$. This is a sharp upper bound for $\lambda_1 (N)$, the hyperbolic space form $\mathbb{H}^n$ is an example where equality is achieved. Recall that the proof of Cheng’s theorem is relying on the Laplacian comparison theorem for Riemannian manifolds, that is to say on an upper bound of the Laplacian of the distance function on $N$. An interesting question is to study all manifolds satisfying the equality case in Cheng’s theorem, i.e. those manifolds for which $\text{Ric} \geq -(n - 1)$ and $\lambda_1 (N) = \frac{(n-1)^2}{4}$. In [L-W2] P. Li and J. Wang proved that if equality holds in Cheng’s upper bound and $n \geq 3$ then either the manifold has one end or the manifold has two ends in which case $N$ must either be

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(1) a warped product $N = \mathbb{R} \times P$ with $P$ compact and metric given by $ds_N^2 = dt^2 + \exp(2t)ds_P^2$, or

(2) if $n = 3$ a warped product $N = \mathbb{R} \times P$ with $P$ compact and metric given by $ds_N^2 = dt^2 + \cosh^2(t)ds_P^2$.

In [L-W3] P. Li and J. Wang have proved that Cheng’s theorem has an analogue in the Kähler setting. Throughout this paper $M^m$ is a complete noncompact Kähler manifold of complex dimension $m$.

If $ds^2 = h_{\alpha\overline{\beta}}dz^\alpha d\overline{z}^\beta$ denotes the Kähler metric on $M$, then $Re(ds^2)$ defines a Riemannian metric on $M$. Suppose $\{e_1, e_2, ..., e_{2m}\}$ with $e_{2k} = Je_{2k-1}$ for any $k \in \{1, ..., m\}$ is an orthonormal frame with respect to this Riemannian metric, then $\{v_1, ..., v_m\}$ is a unitary frame of $T_x^{1,0}M$, where

$$v_k = \frac{1}{2}(e_{2k-1} - \sqrt{-1}e_{2k}).$$

Recall that the bisectional curvature $BK_M$ of $M$ is defined by

$$R_{\alpha\overline{\alpha}\beta\overline{\beta}} = <R_{\alpha_{\overline{\alpha}}\beta_{\overline{\beta}}} v_\beta, v_{\overline{\beta}}>$$

and we say that $BK_M \geq -1$ on $M$ if for any $\alpha$ and $\beta$

$$R_{\alpha\overline{\alpha}\beta\overline{\beta}} \geq -(1 + \delta_{\alpha\overline{\beta}}).$$

Note that for the space form $\mathbb{C}\mathbb{H}^m$ we have $BK_{\mathbb{C}\mathbb{H}^m} = -1$.

**Theorem 1** (P. Li and J. Wang) If $M^m$ is a complete noncompact Kähler manifold of complex dimension $m$ with $BK_M \geq -1$ then

$$\lambda_1(M) \leq m^2 = \lambda_1(\mathbb{C}\mathbb{H}^m).$$

Li-Wang proved this theorem in the spirit of Cheng’s proof, they first obtained a Laplacian comparison theorem for manifolds with $BK_M \geq -1$ (Theorem 1.6 in [L-W3]) and then the sharp estimate for $\lambda_1(M)$ follows. We would like to point out that the bisectional curvature assumption is essential in their proof of the Laplacian comparison theorem.

An interesting question that one can ask is if the sharp estimate for $\lambda_1(M)$ from Theorem 1 remains true under a Ricci curvature bound from below. This question is motivated in part by the following situation in the compact Kähler case, where we have a version of Lichnerowicz’s theorem. Namely, if for a compact Kähler manifold $N^m$ the Ricci curvature has the
lower bound \( \text{Ric}_N \geq 2(m + 1) \), then the first eigenvalue of the Laplacian has a sharp lower bound, \( \lambda_1 (N) \geq 4(m + 1) \). We are grateful to Lei Ni for pointing out this result to us, for a simple proof of it see [U].

In this paper, our first goal is to show that indeed there is a sharp estimate for \( \lambda_1 (M) \) under only Ricci curvature lower bound. Our proof is based on the variational principle for \( \lambda_1 (M) \) and integration by parts. In fact, our argument can be localized on each end of the manifold.

**Theorem 2** Let \( M^m \) be a complete noncompact Kähler manifold of complex dimension \( m \) such that the Ricci curvature is bounded from below by

\[
\text{Ric} \geq -2(m + 1).
\]

Then if \( E \) is an end of \( M \) and \( \lambda_1 (E) \) is the infimum of the Dirichlet spectrum of the Laplacian on \( E \) then

\[
\lambda_1 (E) \leq m^2.
\]

In particular, we have the sharp estimate

\[
\lambda_1 (M) \leq m^2.
\]

Note that the condition on the Ricci curvature in Theorem 2 means

\[
\text{Ric}(e_k, e_j) \geq -2(m + 1) \delta_{kj}
\]

for any \( k, j \in \{1, \ldots, 2m\} \), which is equivalent to

\[
\text{Ric}_{\alpha \overline{\beta}} \geq -(m + 1) \delta_{\alpha \overline{\beta}},
\]

for the unitary frame \( \{v_1, v_2, \ldots, v_m\} \).

The same as in the Riemannian setting, it is interesting to study the Kähler manifolds for which equality is achieved in Theorem 2. Let us recall that for bisectional curvature lower bound Li-Wang have proved in [L-W] that such manifolds need to have at most two ends.

**Theorem 3** (P. Li and J. Wang) If \( M^m \) is a complete noncompact Kähler manifold with \( \lambda_1 (M) = m^2 \) and \( BK_M \geq -1 \) then \( M \) has at most two ends.

Their proof relies on a study of the Buseman function \( \beta \) on \( M \), so the Laplacian comparison theorem plays again an important role. An intersting
fact about their proof is that it gives an unified approach of the question in the Riemannian and Kählerian case.

We now want to make some comments on the case when $M$ has exactly two ends. In this case, the proof of Li-Wang provides some structure information of the manifold. Namely, not only that for any $t$ the level set $\beta = t$ is diffeomorphic to the level set $\beta = t_0$ for some $t_0$ fixed, but also the metric on $\beta = t$ is determined by the metric on $\beta = t_0$.

Our second goal in this paper is to obtain the same conclusion on the number of ends if equality is achieved in Theorem 2. This will be done by a careful study of the estimates in Theorem 2.

If $M$ is assumed to have exactly two ends, using our approach we will be able to deduce the same structure information of the manifold as discussed above, for the level sets of a function defined by the Li-Tam theory.

Remark. After this paper was written the author was informed by Peter Li that the analysis of the two ends case can be deepened and in fact if $M$ has bounded curvature, then it is isometrically covered by $CH^m$. Examples are also known, with both bounded and unbounded curvature, see [L-W]. We expect this result can be recovered with our way, and in fact this will become apparent towards the end of the proof of Theorem 4.

**Theorem 4** Let $M^m$ be a complete noncompact Kähler manifold of complex dimension $m$ such that the Ricci curvature is bounded from below by

$$\text{Ric} \geq -2 (m + 1).$$

If $\lambda_1 (M) = m^2$ then $M$ has at most two ends.

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## 2 The proofs

To prove Theorem 2 and Theorem 4 we first need the following preparation.

Let $E$ be a nonparabolic end of $M$.

Without loss of generality we will henceforth assume that $\lambda_1 (E) > 0$.

From the theory of Li-Tam ([L-T]) we know that there exists a harmonic function $f$ on $E$ that is obtained with the following procedure. Let $f_R$ be
the harmonic function with Dirichlet boundary conditions: $f_R = 1$ on $\partial E$, $f_R = 0$ on $\partial E_p(R)$, where $E_p(R) = E \cap B_p(R)$.

Then $f_R$ admits a subsequence convergent to $f$, with the properties: $0 < f < 1$ on $E$, $f = 1$ on $\partial E$ and $f$ has finite Dirichlet integral. Moreover, since $\lambda_1(E) > 0$, we know by a theorem of Li-Wang that \([L-W1]\)

$$
\int_{E_p(R+1) \setminus E_p(R)} f^2 \leq c_1 \exp \left( -2 \sqrt{\lambda_1(E) R} \right).
$$

Further on integration on the level sets of $f$ will play a central role in our proofs, and for this let us recall the following important property of $f$ \([L-W4]\). Namely, for $t, a, b < 1$ let

$$
l(t) = \{ x \in E \mid f(x) = t \}
$$

and define the set

$$
L(a, b) = \{ x \in E \mid a < f(x) < b \}.
$$

Then for almost all $t < 1$

$$
\int_{l(t)} |\nabla f| = \text{const} < \infty
$$

and we have:

$$
\int_{L(a, b)} |\nabla f|^2 = (b - a) \int_{l(t_0)} |\nabla f|.
$$

Let us denote

$$
L = L(\frac{1}{2} \delta \varepsilon, 2 \varepsilon),
$$

where $\delta, \varepsilon > 0$ are sufficiently small fixed numbers to be chosen later.

Since we will often integrate by parts on $L$, let us construct a cut-off $\phi$ with compact support in $L$. Define $\phi = \psi \varphi$ with $\psi$ depending on the distance function

$$
\psi = \begin{cases}
1 & \text{on } E_p(R - 1) \\
R - r & \text{on } E_p(R) \setminus E_p(R - 1) \\
0 & \text{on } E \setminus E_p(R)
\end{cases}
$$
and \( \varphi \) defined on the level sets of \( f \)

\[
\varphi = \begin{cases} 
  (\log 2)^{-1}(\log f - \log(\frac{1}{2}\delta\varepsilon)) & \text{on } L(\frac{1}{2}\delta\varepsilon, \delta\varepsilon) \\
  1 & \text{on } L(\delta\varepsilon, \varepsilon) \\
  (\log 2)^{-1}(\log 2\varepsilon - \log f) & \text{on } L(\varepsilon, 2\varepsilon) \\
  0 & \text{otherwise.}
\end{cases}
\]

For convenience, let us assume \( R = \frac{1}{\delta\varepsilon} \). We have the following result:

**Lemma 1** For any \( 0 < a < 2 \) the following inequality holds:

\[
\frac{1}{16} \left(-\frac{1}{m} - \frac{(1-a)^2}{a(2-a)}\right) \frac{1}{(-\log \delta)} \int_{L} |\nabla f|^{4} \phi^{2} \leq \frac{a}{2-a} \int_{l(\rho)} |\nabla f| + \frac{c}{(-\log \delta)^{\frac{3}{2}}},
\]

where \( c \) is a constant not depending on \( \delta \) or \( \varepsilon \).

**Proof of Lemma 1.** Note that the gradient and the Laplacian satisfy:

\[
\nabla f \cdot \nabla g = 2 (f_{\alpha} g_{\overline{\alpha}} + f_{\overline{\alpha}} g_{\alpha}) \\
\Delta f = 4 f_{\alpha_{\overline{\alpha}}},
\]

Let \( u = \log f \), then a simple computation shows that

\[
u_{\alpha_{\overline{\alpha}}} = f^{-1}f_{\alpha_{\overline{\alpha}}} - f^{-2}f_{\alpha}f_{\overline{\alpha}}.
\]

Consider now

\[
\int_{L} f |u_{\alpha_{\overline{\alpha}}}^{2}| \phi^{2}
\]

which we estimate from above and from below to prove our claim.

To begin with,

\[
\int_{L} f |u_{\alpha_{\overline{\alpha}}}^{2}| \phi^{2} = \int_{L} f^{-1} |f_{\alpha_{\overline{\alpha}}}^{2}| \phi^{2} - 2 \int_{L} f^{-2}(f_{\alpha_{\overline{\alpha}}f_{\alpha_{\overline{\alpha}}}f_{\phi_{\beta}}) \phi^{2} + \frac{1}{16} \int_{L} f^{-3} |\nabla f|^{4} \phi^{2}.
\]

The first term is

\[
\int_{L} f^{-1} |f_{\alpha_{\overline{\alpha}}}^{2}| \phi^{2} = \int_{L} f^{-1}(f_{\alpha_{\overline{\alpha}}} \cdot f_{\overline{\alpha}}) \phi^{2} = -\int_{L} f_{\alpha} (f^{-1}f_{\alpha_{\overline{\alpha}}} \phi^{2})_{\overline{\alpha}}
\]

\[
= \int_{L} f^{-2}(f_{\alpha_{\overline{\alpha}}f_{\alpha_{\overline{\alpha}}}f_{\phi_{\beta}}) \phi^{2} - \int_{L} f^{-1}f_{\alpha}f_{\overline{\alpha}}f_{\phi_{\beta}} \phi^{2} - \int_{L} f^{-1}f_{\overline{\alpha}}f_{\alpha} (\phi^{2})_{\overline{\alpha}}
\]

\[
\int_{L} f^{-3} |\nabla f|^{4} \phi^{2}.
\]

where \( c \) is a constant not depending on \( \delta \) or \( \varepsilon \).
and using the Ricci identities and $\Delta f = 0$ we see that $f_{\alpha \beta \bar{\beta}} = 0$. It also shows that the last integral needs to be a real number.

This proves that

$$\int_L f \left| u_{\alpha \bar{\beta}} \right|^2 \phi^2 = -\int_L f^{-2}(f_{\alpha \bar{\beta}} f_{\alpha \beta})\phi^2$$

$$+ \frac{1}{16} \int_L f^{-3} |\nabla f|^4 \phi^2 - \int_L f^{-1} f_{\alpha \beta} (\phi^2)_{\bar{\beta}}. \quad (1)$$

Let us use again integration by parts to see that

$$-\int_L f^{-2}(f_{\alpha \bar{\beta}} f_{\alpha \beta})\phi^2 = \int_L f_{\alpha} (f^{-2} f_{\alpha \beta \bar{\beta}} \phi^2)_{\bar{\beta}} =$$

$$-2 \int_L f^{-3} f_{\alpha} f_{\bar{\beta}} f_{\alpha \beta} \phi^2 + \int_L f^{-2} f_{\alpha \bar{\beta}} f_{\alpha \beta} \phi^2 + \int_L f^{-2} f_{\alpha} f_{\alpha \beta} (\phi^2)_{\bar{\beta}}.$$  

Similarly, one finds

$$-\int_L f^{-2}(f_{\alpha \bar{\beta}} f_{\alpha \beta})\phi^2 = -\int_L f^{-2}(f_{\alpha \beta \bar{\beta}} f_{\alpha \beta})\phi^2 = \int_L f_{\alpha} (f^{-2} f_{\alpha \beta \bar{\beta}} \phi^2)_{\bar{\beta}}$$

$$= -2 \int_L f^{-3} f_{\alpha} f_{\bar{\beta}} f_{\alpha \beta} \phi^2 + \int_L f^{-2} f_{\alpha \bar{\beta}} f_{\alpha \beta} \phi^2 + \int_L f^{-2} f_{\alpha} f_{\alpha \beta} (\phi^2)_{\bar{\beta}}.$$  

Combining the two identities we get

$$-\int_L f^{-2}(f_{\alpha \bar{\beta}} f_{\alpha \beta})\phi^2 = -\frac{1}{8} \int_L f^{-3} |\nabla f|^4 \phi^2 + \int_L f^{-2} \text{Re}(f_{\alpha \beta \bar{\beta}} f_{\alpha \beta})$$

$$+ \frac{1}{4} \int_L f^{-2} |\nabla f|^2 \text{Re}(f_{\alpha \beta \bar{\beta}} f_{\alpha \beta}). \quad (2)$$

Note that the following inequality holds on $E$:

$$|f_{\alpha \beta \bar{\beta}} f_{\alpha \beta}| \leq \frac{1}{4} |f_{\alpha \beta}| |\nabla f|^2 \quad (3)$$

We want to insist on the proof of this inequality because it will matter when we study the manifolds with $\lambda_1 (M) = m^2$. Since the two numbers in
are independent of the unitary frame, let us choose a frame at the fixed point \( x \in E \) such that

\[
e_1 = \frac{1}{|\nabla f|} \nabla f.
\]

Certainly, we need \(|\nabla f| (x) \neq 0\) which we assume without loss of generality because if \(|\nabla f| (x) = 0\) there is nothing to prove.

Then one can see that

\[
f_{e_1} = |\nabla f|, \quad f_{e_2} = 0, \ldots, f_{e_{2m}} = 0
\]

or, in the unitary frame

\[
f_1 = f_\bar{1} = \frac{1}{2} |\nabla f|, \quad f_\alpha = f_{\bar{\alpha}} = 0 \text{ if } \alpha > 1.
\]

This proves the inequality because

\[
|f_{\alpha\beta} f_\alpha f_\beta| = \frac{1}{4} |\nabla f|^2 |f_{11}| \leq \frac{1}{4} |f_{\alpha\beta}| |\nabla f|^2.
\]

Moreover, we learn that equality holds in (3) if and only if

\[
f_{\alpha\beta} = 0 \text{ for } (\alpha, \beta) \neq (1, 1),
\]

with respect to the frame chosen above.

Since the following holds:

\[
Re (f_{\alpha\beta} f_\alpha f_\beta) \leq |f_{\alpha\beta} f_\alpha f_\beta| \leq \frac{1}{4} |f_{\alpha\beta}| |\nabla f|^2,
\]

we get for an arbitrary \( a > 0 \)

\[
2 \int_L f^{-2} Re (f_{\alpha\beta} f_\alpha f_\beta) \phi^2 \leq \int_L 2 (f^{-1/2} |f_{\alpha\beta}| \phi) \left( \frac{1}{4} f^{-3/2} |\nabla f|^2 \phi \right)
\leq a \int_L f^{-1} |f_{\alpha\beta}|^2 \phi^2 + \frac{1}{16a} \int_L f^{-3} |\nabla f|^4 \phi^2.
\] (4)

Moreover, again integrating by parts we have

\[
\int_L f^{-1} |f_{\alpha\beta}|^2 \phi^2 = \int_L f^{-1} f_\alpha f_{\bar{\alpha}} \phi^2 = - \int_L f_\alpha (f^{-1} f_{\bar{\alpha}} \phi^2)_\beta
= \int_L f^{-2} f_{\alpha\beta} f_\alpha \phi^2 - \int_L f^{-1} f_\alpha f_{\bar{\alpha}} \phi^2 - \int_L f^{-1} f_\alpha f_{\bar{\alpha}} \phi^2
\]
and on the other hand

\[
\int_L f^{-1} |f_{\alpha\beta}|^2 \phi^2 = \int_L f^{-1} f_{\alpha\beta} f_{\bar{\alpha}\bar{\beta}} \phi^2 = -\int_L f_{\alpha} (f^{-1} f_{\alpha\beta} \phi^2)_{\bar{\beta}} \\
= \int_L f^{-2} f_{\alpha\beta} f_{\bar{\alpha}\bar{\beta}} \phi^2 - \int_L f_{\alpha} f_{\alpha\beta} \phi^2 - \int_L f^{-1} f_{\alpha} f_{\alpha\beta} (\phi^2)_{\bar{\beta}}
\]

so that combining the two identities we get

\[
\int_L f^{-1} |f_{\alpha\beta}|^2 \phi^2 = \int_L f^{-2} Re(f_{\bar{\alpha}\bar{\beta}} f_{\alpha\beta}) \phi^2 - \int_L f^{-1} f_{\alpha} f_{\alpha\beta} \phi^2 \\
- \int_L f^{-1} Re(f_{\alpha} f_{\alpha\beta} (\phi^2)_{\bar{\beta}}).
\]

Note that the Ricci identities imply

\[
f_{\bar{\alpha}\bar{\beta}} = f_{\bar{\beta}\alpha} = f_{\bar{\beta}\bar{\alpha}} + Ric_{\beta\alpha} f_{\beta}
\]

and therefore we obtain

\[
\int_L f^{-1} |f_{\alpha\beta}|^2 \phi^2 \leq \int_L f^{-2} Re(f_{\bar{\alpha}\bar{\beta}} f_{\alpha\beta}) \phi^2 + \frac{m+1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 \\
- \int_L f^{-1} Re(f_{\alpha} f_{\alpha\beta} (\phi^2)_{\bar{\beta}}).
\]

Plug this inequality into (4) and it follows

\[
(2 - a) \int_L f^{-2} Re(f_{\bar{\alpha}\bar{\beta}} f_{\alpha\beta}) \phi^2 \leq a \frac{m+1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 \\
+ \frac{1}{16a} \int_L f^{-3} |\nabla f|^4 \phi^2 - a \int_L f^{-1} Re(f_{\alpha} f_{\alpha\beta} (\phi^2)_{\bar{\beta}}).
\]

Let us fix henceforth \(0 < a < 2\) to make sure that \(2 - a > 0\). Now we are getting back to (2) and obtain

\[
- \int_L f^{-2}(f_{\alpha\beta} f_{\bar{\alpha}\bar{\beta}}) \phi^2 \leq \left( -\frac{1}{8} + \frac{1}{16a(2-a)} \right) \int_L f^{-3} |\nabla f|^4 \phi^2 \\
+ \frac{a}{2 - a} \frac{m+1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 - \frac{a}{2 - a} \int_L f^{-1} Re(f_{\alpha} f_{\alpha\beta} (\phi^2)_{\bar{\beta}}) \\
+ \frac{1}{4} \int_L f^{-2} |\nabla f|^2 Re(f_{\beta} (\phi^2)_{\bar{\beta}}).
\]

(5)
We have thus proved that
\[
\int_L f |u_{\alpha\overline{\beta}}|^2 \phi^2 \leq \left( -\frac{1}{16} + \frac{1}{16a(2-a)} \right) \int_L f^{-3} |\nabla f|^4 \phi^2 \\
+ \frac{a}{2-a} \frac{m+1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 + \frac{1}{4} \int_L f^{-2} |\nabla f|^2 \text{Re}(f_{\overline{\beta}} (\phi^2)_{\beta}) \\
- \int_L f^{-1} f_{\alpha\overline{\beta}} f_{\alpha} (\phi^2)_{\overline{\beta}} - \frac{a}{2-a} \int_L f^{-1} \text{Re}(f_{\alpha} f_{\alpha\overline{\beta}} (\phi^2)_{\beta}).
\]

(6)

To finish the upper estimate of \( \int_L f |u_{\alpha\overline{\beta}}|^2 \phi^2 \) we need to estimate the terms involving \((\phi^2)_{\beta}\). We will prove that they can be bounded from above by a constant \(-\log \delta)^{1/2}\).

Start with
\[
2 \int_L f^{-2} |\nabla f|^2 \text{Re}(f_{\overline{\beta}} (\phi^2)_{\beta}) \leq \frac{1}{2} \int_L f^{-2} |\nabla f|^3 |\nabla \phi^2| \\
\leq \int_L f^{-2} |\nabla f|^3 |\nabla \varphi| + \int_L f^{-2} |\nabla f|^3 |\nabla \psi| \varphi
\]

Now it is easy to see that by the gradient estimate and co-area formula
\[
\int_L f^{-2} |\nabla f|^3 |\nabla \varphi| \leq c_2 \left( \int_{L(\frac{1}{2} \delta \varepsilon, \delta \varepsilon)} f^{-1} |\nabla f|^2 + \int_{L(\varepsilon, 2\varepsilon)} f^{-1} |\nabla f|^2 \right) \\
\leq c_3,
\]
while by the decay rate of \( f^2 \) we get
\[
\int_L f^{-2} |\nabla f|^3 |\nabla \psi| \leq c_4 \frac{1}{\delta \varepsilon} \exp \left( -2 \sqrt{\lambda_1(E)} R \right) \leq c_5,
\]
using that \( R = \frac{1}{\delta \varepsilon} \). Clearly, the constants so far do not depend on the choice of \( \delta \) or \( \varepsilon \).

To estimate the other terms one proceeds similarly. For example,
\[
-2 \int_L f^{-1} \text{Re}(f_{\alpha} f_{\alpha\overline{\beta}} (\phi^2)_{\beta}) \leq \int_L f^{-1} |f_{\alpha\overline{\beta}}| |\nabla f| \phi |\nabla \phi| \\
\leq \left( \int_L f^{-1} |\nabla f|^2 |\nabla \phi|^2 \right)^{1/2} \left( \int_L f^{-1} |f_{\alpha\overline{\beta}}|^2 \phi^2 \right)^{1/2} \\
\leq c_6 \left( \int_L f^{-1} |f_{\alpha\overline{\beta}}|^2 \phi^2 \right)^{1/2},
\]

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However, using an inequality proved above we get

$$
\int_L f^{-1} |f_{\alpha\beta}|^2 \phi^2 \leq \int_L f^{-2} \text{Re}(f_{\alpha\overline{\alpha}} f_{\beta\overline{\beta}}) \phi^2 + \frac{m + 1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 \\
- \int_L f^{-1} \text{Re}(f_{\alpha} f_{\overline{\alpha}} (\phi^2)_{\beta}) \\
\leq \frac{1}{4} \int_L f^{-2} |f_{\alpha\beta}| |\nabla f|^2 \phi^2 + \frac{m + 1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 \\
+ \frac{1}{2} \int_L f^{-1} |f_{\alpha\beta}| |\nabla f| \phi |\nabla \phi| \\
\leq \frac{1}{8} \int_L f^{-1} |f_{\alpha\beta}|^2 \phi^2 + \frac{1}{8} \int_L f^{-3} |\nabla f|^4 \phi^2 \\
+ \frac{m + 1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 \\
+ \frac{1}{4} \int_L f^{-1} |f_{\alpha\beta}|^2 \phi^2 + \frac{1}{4} \int_L f^{-1} |\nabla f|^2 |\nabla \phi|^2,
$$

which shows there exists constants $c_7$ and $c_8$ such that:

$$
\int_L f^{-1} |f_{\alpha\overline{\beta}}|^2 \phi^2 \leq c_7 \int_L f^{-1} |\nabla f|^2 \phi^2 + c_8 \int_L f^{-1} |\nabla f|^2 |\nabla \phi|^2 \\
\leq c_9 (-\log \delta).
$$

We have proved that

$$
\int_L f^{-1} |f_{\alpha\overline{\beta}}| |\nabla f| \phi |\nabla \phi| \leq c_{10} (-\log \delta)^{1/2}.
$$

Let us gather the information we have so far:

$$
\int_L f |u_{\alpha\overline{\beta}}|^2 \phi^2 \leq \left( -\frac{1}{16} + \frac{1}{16a(2-a)} \right) \int_L f^{-3} |\nabla f|^4 \phi^2 \\
+ \frac{a}{2-a} \frac{m + 1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 + c(-\log \delta)^{1/2}.
$$

The estimate from below is straightforward:

$$
|u_{\alpha\overline{\beta}}|^2 \geq \sum_{\alpha} |u_{\alpha\overline{\alpha}}|^2 \geq \frac{1}{m} \left| \sum_{\alpha} u_{\alpha\overline{\alpha}} \right|^2 = \frac{1}{16m} f^{4-4} |\nabla f|^4. \quad (7)
$$
Hence, this shows that

\[
\frac{1}{16} \left( 1 + \frac{1}{m} - \frac{1}{a(2-a)} \right) \int_L f^{-3} |\nabla f|^4 \phi^2 \leq \frac{a}{2-a} \frac{m}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 + c(-\log \delta)^{\frac{1}{2}},
\]

which proves the Lemma. Q.E.D.

In the following Lemma, we will estimate \( \int_L f^{-3} |\nabla f|^4 \phi^2 \) from below. To serve our purpose, we need this estimate to depend on \( \lambda_1 (E) \) and this is done using the variational principle. Recall that \( E \) is a nonparabolic end, \( \lambda_1 (E) > 0 \) and we set \( L = L(\frac{1}{2} \delta \varepsilon, 2 \varepsilon) \) for \( \delta, \varepsilon \) sufficiently small.

**Lemma 2**

\[
\frac{1}{(-\log \delta)} \int_L f^{-3} |\nabla f|^4 \phi^2 \geq 4 \lambda_1 (E) \int_{l(t_0)} |\nabla f| - \frac{c_0}{(-\log \delta)^{\frac{1}{2}}}.
\]

**Proof of Lemma 2.**

By the variational principle for \( \lambda_1 (E) \),

\[
\lambda_1 (E) \int_E f \phi^2 \leq \int_E \left| \nabla \left( \phi f^{\frac{1}{2}} \right) \right|^2,
\]

which means that

\[
\lambda_1 (E) \int_L f \phi^2 \leq \frac{1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 + \int_L f |\nabla \phi|^2 + \int_L \phi |\nabla f | |\nabla \phi |
\leq \frac{1}{4} \int_{L(\delta \varepsilon, \varepsilon)} f^{-1} |\nabla f|^2 + c_{11},
\]

based on estimates similar to what we did in Lemma 1.

This implies that

\[
\frac{1}{(-\log \delta)} \int_L f \phi^2 \leq \frac{1}{4 \lambda_1 (E)} \int_{l(t_0)} |\nabla f| + \frac{c_{11}}{(-\log \delta)}.
\]
Finally, using the Schwarz inequality we get

\[
\int_{l(t_0)} |\nabla f| = \frac{1}{(- \log \delta)} \int_{L(\delta \varepsilon, \delta \varepsilon)} f^{-1} |\nabla f|^2 \\
\leq \frac{1}{(- \log \delta)} \int_{L} f^{-1} |\nabla f|^2 \phi^2 + \frac{1}{(- \log \delta)} \int_{L \cap (E \setminus S \setminus (R-1))} f^{-1} |\nabla f|^2 \\
\leq \left( \frac{1}{(- \log \delta)} \int_{L} f^{-3} |\nabla f|^4 \phi^2 \right)^{\frac{1}{2}} \left( \frac{1}{(- \log \delta)} \int_{L} f \phi^2 \right)^{\frac{1}{2}} \\
+ \frac{c_{12}}{(- \log \delta)} \frac{1}{\delta \varepsilon} \exp \left( -2 \sqrt{\lambda_1(E) R} \right) \\
\leq \left( \frac{1}{(- \log \delta)} \int_{L} f^{-3} |\nabla f|^4 \phi^2 \right)^{\frac{1}{2}} \times \\
\times \left( \frac{1}{4 \lambda_1(E)} \int_{l(t_0)} |\nabla f| + \frac{c_{11}}{(- \log \delta)} \right)^{\frac{1}{2}} + \frac{c_{13}}{(- \log \delta)},
\]

which proves the Lemma. \textbf{Q.E.D.}

Suppose now that \( M \) has a parabolic end \( F \). A theorem of Nakai (\cite{N}, see also \cite{N-R}) states that there exists an exhaustion function \( f \) on \( \overline{F} \) which is harmonic on \( F \) and \( f = 0 \) on \( \partial F \). In this case we consider for \( T, \beta > 0 \) fixed

\[
\phi = \begin{cases} 
(\log 2)^{-1}(\log f - \log(\frac{1}{2}T)) & \text{on } L(\frac{1}{2}T, T) \\
1 & \text{on } L(T, \beta T) \\
(\log 2)^{-1}(\log(2\beta T) - \log f) & \text{on } L(\beta T, 2\beta T) \\
0 & \text{otherwise}, 
\end{cases}
\]

where the level sets are now defined on \( F \). Since \( f \) is proper, there is no need for a cut-off depending on the distance function. Our point now is that Lemma 1 and Lemma 2 hold for this choice of \( \phi \) also, the proofs are identical. Note that if \( \tilde{L} = L(\frac{1}{2}T, 2\beta T) \)

then the following inequalities hold on \( \tilde{L} \):

\[
\frac{1}{16} \left( \frac{1}{m} - \frac{(1 - a)^2}{a(2 - a)} \right) \frac{1}{\log \beta} \int_{\tilde{L}} |\nabla f|^4 \phi^2 \leq \frac{a}{2 - a} \frac{m + 1}{4} \int_{l(t_0)} |\nabla f| \\
+ \frac{\tilde{c}}{(\log \beta)^{\frac{1}{2}}},
\]

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and 
\[ \frac{1}{\log \beta} \int_L f^{-3} |\nabla f|^4 \phi^2 \geq 4\lambda_1 (F) \int_{l(t_0)} |\nabla f| - \frac{\bar{c}_0}{(\log \beta)^{\frac{1}{2}}}. \]

Now we are ready to prove Theorem 2.

**Proof of Theorem 2.**

Let us first prove the Theorem for a nonparabolic end \( E \).
We know from Lemma 1 and Lemma 2 that
\[ \lambda_1 (E) \frac{1}{4} \left( \frac{1}{m} - \frac{(1-a)^2}{a(2-a)} \right) \int_{l(t_0)} |\nabla f| \leq \frac{a}{2-a} \frac{m+1}{4} \int_{l(t_0)} |\nabla f| + \frac{C}{(-\log \delta)^{\frac{1}{2}}}, \]
inequality that holds for any \( \delta > 0 \) and for any \( 0 < a < 2 \).
Therefore, making \( \delta \to 0 \) we get that for any \( 0 < a < 2 \),
\[ \lambda_1 (E) \leq \frac{a(m+1)}{2-a} \left( \frac{1}{m} - \frac{(1-a)^2}{a(2-a)} \right)^{-1}. \]
Let us choose \( a = \frac{m}{m+1} \), then
\[ \frac{1}{m} - \frac{(1-a)^2}{a(2-a)} = \frac{1}{m} \frac{m+1}{m+2} ; \frac{a(m+1)}{2-a} = \frac{m(m+1)}{m+2}, \]
which shows that
\[ \lambda_1 (E) \leq m^2. \]
This proves Theorem 2 for a nonparabolic end \( E \). The proof for a parabolic end \( F \) is verbatim. \textbf{Q.E.D.}

**Proof of Theorem 4:**
Suppose that \( M \) has more than one end. We know from [L-W] that if \( \lambda_1 (M) > \frac{m+1}{2} \) the manifold has only one nonparabolic end. Hence let us set \( E \) this nonparabolic end and consequently \( F = M \setminus E \) will be a parabolic end. Note that an end of \( M \) is defined with respect to a compact subset of
so that writing \( M = E \cup F \) with \( E \) nonparabolic and \( F \) parabolic we are not losing generality, in fact \( M \) can have many ends with respect to other compact subsets. The construction of Li-Tam implies that there exists a harmonic function \( f : M \to (0, \infty) \) with the following properties:

1. On \( E \) the function has the decay rate
\[
\int_{E_p(R) \setminus E_p(R-1)} f^2 \leq c_1 \exp \left( -2\sqrt{\lambda_1(M)R} \right),
\]
2. On \( F \) the function is proper.
3. We have:
\[
\sup_{x \in F} f(x) = \infty, \quad \inf_{x \in E} f(x) = 0.
\]

Let us point out some facts about the proofs of Lemma 1 and Lemma 2. In the two lemmata, the function \( f \) was defined only on a single end, which was first assumed to be nonparabolic, and then we observed that the proofs still work on a parabolic end. In the framework of Theorem 4, we know that \( f \) is defined on the whole manifold, so now \( L = L(b_0, b_1) = \{x \in M \mid b_0 < f(x) < b_1\} \) makes sense for any \( 0 < b_0 < b_1 \). One can see that the computations proved in Lemma 1 are true for \( L \) and moreover we may replace everywhere \( \phi^2 \) with \( \phi^3 \). With this in mind, let us fix \( b_0 = \delta\varepsilon, \quad b_1 = \beta T \), where \( 0 < \delta\varepsilon < \varepsilon < T < \beta T \) and for convenience choose \( \beta = \frac{1}{\delta^2} \).

Hence, everywhere in this proof
\[
L = L(\delta\varepsilon, \beta T),
\]
and
\[
a = \frac{m}{m+1}.
\]

The proof of this theorem is based on a more detailed study of inequalities in Lemma 1 and Lemma 2. We want to prove that \( \lambda_1(M) = m^2 \) forces all the inequalities to become equalities on \( L(\varepsilon, T) \). Since \( \varepsilon, T \) are arbitrary, it will follow that we need to have equalities everywhere on \( M \).

Choose \( \phi = \varphi \psi \), where
\[
\psi = \begin{cases} 
1 & \text{on } E_p(R-1) \cup F \\
R - r & \text{on } E_p(R) \setminus E_p(R-1) \\
0 & \text{on } E \setminus E_p(R)
\end{cases}
\]
and
\[
\varphi = \begin{cases} 
(-\log \delta)^{-1}(\log f - \log(\delta\varepsilon)) & \text{on } L(\delta\varepsilon, \varepsilon) \\
0 & \text{on } L(0, \delta\varepsilon) \cup (L(\beta T, \infty) \cap F) \\
(\log \beta)^{-1}(\log(\beta T) - \log f) & \text{on } L(T, \beta T) \cap F \\
1 & \text{otherwise}.
\end{cases}
\]
Recall that by (6) and (7) we have
\[
\frac{1}{16}(1 - \frac{a}{2-a}) \int f^{-3} |\nabla f|^4 \phi \leq \frac{a}{2-a} \int f^{-1} |\nabla f|^2 \phi^3 \\
+ \frac{1}{4} \int f^{-2} |\nabla f|^2 \text{Re}(f_\beta (\phi^3)_\beta) - \int f^{-1} f_\beta f_\alpha (\phi^3)_{\overline{\beta}} \\
- \frac{1}{2-a} \int f^{-1} \text{Re}(f_\alpha f_{\alpha\beta} (\phi^3)_{\beta}).
\] (8)

On the other hand, Schwarz inequality implies
\[
\left( \int f^{-1} |\nabla f|^2 \phi^3 \right)^2 \leq \left( \int f^{-3} |\nabla f|^4 \phi \right) \left( \int f \phi^3 \right),
\] (9)

and by the variational principle it follows
\[
\lambda_1 (M) \int f \phi^3 \leq \int |\nabla \left( f^{\frac{1}{2}} \phi^3 \right)|^2 \\
= \frac{1}{4} \int f^{-1} |\nabla f|^2 \phi^3 + \frac{9}{4} \int f \phi |\nabla \phi|^2 + \frac{3}{2} \int \phi^2 \nabla f \cdot \nabla \phi.
\]

Our point now is that a careful study of the two \(\nabla \phi\) terms shows that they converge to zero as \(\beta \to \infty\) (and \(\delta = \beta \to 0\)).

It is clear that\(\frac{9}{4} \int f \phi |\nabla \phi|^2 \leq \frac{c_1}{\log \beta}\), while
\[
\int \phi^2 \nabla f \cdot \nabla \phi = \frac{1}{(- \log \delta)} \int_{L(\delta \varepsilon, \varepsilon)} f^{-1} |\nabla f|^2 \phi^2 - \frac{1}{\log \beta} \int_{L(T, \beta T) \cap F} f^{-1} |\nabla f|^2 \phi^2.
\]
The integral on \(F\) is readily found by the co-area formula:
\[
\frac{1}{\log \beta} \int_{L(T, \beta T) \cap F} f^{-1} |\nabla f|^2 \phi^2 = \left( \int |\nabla f| \right) \int_T t^{-1} \left( \frac{\log(\beta T) - \log t}{\log \beta} \right)^2 dt \\
= \frac{1}{3} \int |\nabla f|.
\]

It is clear that the same formula holds on \(E\) if we integrate against \(\phi^2\) and therefore:
\[
\frac{1}{(- \log \delta)} \int_{L(\delta \varepsilon, \varepsilon)} f^{-1} |\nabla f|^2 \phi^2 \leq \frac{1}{(- \log \delta)} \int_{L(\delta \varepsilon, \varepsilon)} \phi^2 |\nabla f|^2 f^{-1} = \frac{1}{3} \int |\nabla f|.
\]
For later use, observe that a converse of the latter inequality also holds:

\[
\frac{1}{(-\log \delta)} \int_{L(\delta\varepsilon,\varepsilon)} f^{-1} |\nabla f|^2 \phi^2 \geq \frac{1}{(-\log \delta)} \int_{L(\delta\varepsilon,\varepsilon)} f^{-1} |\nabla f|^2 \varphi^2
\]

\[
- \frac{1}{(-\log \delta)} \int_{L(\delta\varepsilon,\varepsilon) \cap (E \setminus E_p(R-1))} f^{-1} |\nabla f|^2 \varphi^2
\]

\[
\geq \frac{1}{3} \int_{\ell(t_0)} |\nabla f| - \frac{c_2}{(-\log \delta)}.
\]

In particular, from the above estimates it follows

\[
\int_L \phi^2 \nabla f \cdot \nabla \phi \leq 0.
\]

We have thus proved that

\[
\lambda_1(M) \int_L f \phi^3 \leq \frac{1}{4} \int_L f^{-1} |\nabla f|^2 \phi^3 + \frac{c_1}{\log \beta},
\]

which plugged into (9) yields

\[
\int_L f^{-3} |\nabla f|^4 \phi^3 \geq 4 \lambda_1(M) \frac{\left( \int_L f^{-1} |\nabla f|^2 \phi^3 \right)^2}{\int_L f^{-1} |\nabla f|^2 \phi^3 + \frac{c_1}{\log \beta}}
\]

\[
= 4 \lambda_1(M) \int_L f^{-1} |\nabla f|^2 \phi^3 - \frac{c_4}{\log \beta} \int_L f^{-1} |\nabla f|^2 \phi^3 + \frac{c_3}{\log \beta}
\]

\[
\geq 4 \lambda_1(M) \int_L f^{-1} |\nabla f|^2 \phi^3 - \frac{c_4}{\log \beta}.
\]

Now let’s return to (8) and use this lower bound, it follows that we have

\[
0 \leq \frac{c_5}{\log \beta} + \frac{1}{4} \int_L f^{-2} |\nabla f|^2 \Re(f_{\bar{\beta}}(\phi^3)_{\beta})
\]

\[
- \int_L f^{-1} f_{\bar{\alpha}}f_{\alpha} (\phi^3)_{\bar{\beta}} - \frac{a}{2 - a} \int_L f^{-1} \Re(f_{\alpha}f_{\bar{\alpha}}(\phi^3)_{\beta}). \quad (10)
\]

Claim:
There exists a constant \( c \geq 0 \) such that
\[
\frac{1}{4} \int_L f^{-2} |\nabla f|^2 \text{Re}(f_{\bar{\beta}} (\phi^3)_{\beta}) - \int_L f^{-1} f_{\alpha\beta} f_{\alpha} (\phi^3)_{\bar{\beta}} \\
- \frac{a}{2 - a} \int_L f^{-1} \text{Re}(f_{\alpha} f_{\bar{\alpha}\beta} (\phi^3)_{\beta}) \leq \frac{c}{(\log \beta)^2}.
\]

**Proof of the claim.**

Let us study each of the three terms in the left hand side.

I. We have:

\[
\frac{1}{4} \int_L f^{-2} |\nabla f|^2 \text{Re}(f_{\bar{\beta}} (\phi^3)_{\beta}) = \frac{3}{16} \int_L \phi^2 f^{-2} |\nabla f|^2 \nabla f \cdot \nabla \phi \\
= \frac{3}{16} \frac{1}{(- \log \delta)} \int_{L(\delta \varepsilon, \varepsilon)} f^{-3} |\nabla f|^4 \phi^2 \\
- \frac{3}{16} \frac{1}{\log \beta} \int_{L(T, \beta T) \cap F} f^{-3} |\nabla f|^4 \phi^2.
\]

As we stressed above, the estimates in Lemma 1 and Lemma 2 are true on any end. Certainly, \(\phi\) here is not the same on \(L(\delta \varepsilon, \varepsilon)\) with \(\phi\) from Lemma 1. Nevertheless, the computations are the same. In fact, in this case there is no need to consider a cut-off \(\varphi\) on \(L\left(\frac{1}{2} \delta \varepsilon, \delta \varepsilon\right)\), because \(\phi\) already is zero there. On \(L(\varepsilon, 2 \varepsilon)\) the cut-off \(\varphi\) is the same as in Lemma 1. Therefore, one can use Lemma 1 for \(L(\delta \varepsilon, \varepsilon)\) and Lemma 2 for \(L(T, \beta T) \cap F\) to estimate the above subtraction.

By Lemma 1 we know that

\[
\frac{1}{(- \log \delta)} \int_{L(\delta \varepsilon, \varepsilon)} f^{-3} |\nabla f|^4 \phi^2 \leq 4 \lambda_1(M) \frac{1}{(- \log \delta)} \int_{L(\delta \varepsilon, \varepsilon)} f^{-1} |\nabla f|^2 \phi^2 \\
+ \frac{c_6}{(- \log \delta)^{\frac{3}{2}}} \\
\leq \frac{4}{3} \lambda_1(M) \int_{l(t_0)} |\nabla f| + \frac{c_6}{(- \log \delta)^{\frac{3}{2}}},
\]

while Lemma 2 implies that

\[
\frac{1}{\log \beta} \int_{L(T, \beta T) \cap F} f^{-3} |\nabla f|^4 \phi^2 \geq \frac{4}{3} \lambda_1(M) \int_{l(t_0)} |\nabla f| - \frac{c_7}{(\log \beta)^{\frac{3}{2}}}.
\]
Combining the two estimates, it results
\[ \frac{1}{4} \int_L f^{-2} |\nabla f|^2 \Re(f_\beta (\phi^3)_\beta) \leq \frac{c_8}{(\log \beta)^{\frac{3}{2}}}. \]

II. Start with
\[ - \int_L f^{-1} f_\beta f_\alpha (\phi^3)_{\overline{\beta}} = - \frac{3}{(- \log \delta)} \int_{L(\delta, \varepsilon)} f^{-2} (f_\beta f_\alpha f_\overline{\beta}) \phi^2 \]
\[ + \frac{3}{\log \beta} \int_{L(T, \beta T) \cap F} f^{-2} (f_\beta f_\alpha f_\overline{\beta}) \phi^2. \]

From (1) and (7) we have:
\[ \frac{1}{\log \beta} \int_{L(T, \beta T) \cap F} f^{-2} (f_\beta f_\alpha f_\overline{\beta}) \phi^2 \leq \left( \frac{1}{16} - \frac{1}{16m} \right) \frac{1}{\log \beta} \times \]
\[ \times \int_{L(T, \beta T) \cap F} f^{-3} |\nabla f|^4 \phi^2 + \frac{c_9}{(\log \beta)^{\frac{3}{2}}} \]
\[ \leq \left( \frac{1}{16} - \frac{1}{16m} \right) \frac{4}{3} \lambda_1(M) \int_{l(t_0)} \rvert \nabla f \rvert + \frac{c_{10}}{(\log \beta)^{\frac{3}{2}}}, \]

while from (5) we know that
\[ - \frac{1}{(- \log \delta)} \int_{L(\delta, \varepsilon)} f^{-2} (f_\beta f_\alpha f_\overline{\beta}) \phi^2 \leq \left( - \frac{1}{8} + \frac{1}{16a(2-a)} \right) \frac{1}{(- \log \delta)} \times \]
\[ \times \int_{L(\delta, \varepsilon)} f^{-3} |\nabla f|^4 \phi^2 \]
\[ + \frac{a}{2-a} \frac{m+1}{4} \frac{1}{(- \log \delta)} \int_{L(\delta, \varepsilon)} f^{-1} |\nabla f|^2 \phi^2 + \frac{c_{11}}{(- \log \delta)^{\frac{3}{2}}} \]
\[ \leq \left( \left( - \frac{1}{8} + \frac{1}{16a(2-a)} \right) \frac{4}{3} \lambda_1(M) + \frac{a}{2-a} \frac{m+1}{3} \right) \int_{l(t_0)} |\nabla f| \]
\[ + \frac{c_{12}}{(- \log \delta)^{\frac{3}{2}}}, \]

using the estimates in I. It is easy to see that the coefficients of \( \int_{l(t_0)} |\nabla f| \)
cancel out (this comes as no surprise) and therefore
\[ - \int_L f^{-1} f_\beta f_\alpha (\phi^3)_{\overline{\beta}} \leq \frac{c_{13}}{(\log \beta)^{\frac{3}{2}}} \].
Note also that in a similar fashion it can be proved that
\[ \int_L f^{-1} f_{\pi\beta} f_{\alpha} (\phi^3)_{\beta} \leq \frac{c_{14}}{(\log \beta)^{\frac{1}{2}}} . \]

III. Finally, by (2) one has:

\[ - \int_L f^{-1} \text{Re}(f_{\alpha} f_{\bar{\alpha}} (\phi^3)_{\bar{\beta}}) = -\frac{3}{(- \log \delta)} \int_{L(\delta,\varepsilon)} f^{-2} \text{Re}(f_{\pi\beta} f_{\alpha} f_{\bar{\beta}}) \phi^2 \]
\[ + \frac{3}{\log \beta} \int_{L(T,\beta T) \cap F} f^{-2} \text{Re}(f_{\pi\beta} f_{\alpha} f_{\bar{\beta}}) \phi^2 \]
\leq -\frac{3}{8(- \log \delta)} \int_{L(\delta,\varepsilon)} f^{-3} |\nabla f|^4 \phi^2 + \frac{3}{(- \log \delta)} \int_{L(\delta,\varepsilon)} f^{-2}(f_{\alpha\bar{\alpha}} f_{\bar{\alpha}} f_{\bar{\beta}}) \phi^2 \]
\[ + \frac{3}{8 \log \beta} \int_{L(T,\beta T) \cap F} f^{-3} |\nabla f|^4 \phi^2 + \frac{3}{\log \beta} \int_{L(T,\beta T) \cap F} f^{-2}(f_{\alpha\bar{\alpha}} f_{\bar{\alpha}} f_{\bar{\beta}}) \phi^2 \]
\[ + \frac{c_{15}}{(\log \beta)^{\frac{1}{2}}} . \]

By I and II it can be proved that
\[ \int_L f^{-1} \text{Re}(f_{\alpha} f_{\bar{\alpha}} (\phi^3)_{\bar{\beta}}) \leq \frac{c_{16}}{(\log \beta)^{\frac{1}{2}}} . \]

This proves the claim. Q.E.D.

Let us use this result in (10), then we infer that
\[ 0 \leq \frac{c_5}{\log \beta} + \frac{1}{4} \int_L f^{-2} |\nabla f|^2 \text{Re}(f_{\beta} (\phi^3)_{\beta}) \]
\[ - \int_L f^{-1} f_{\pi\beta} f_{\alpha} (\phi^3)_{\beta} - \frac{a}{2 - a} \int_L f^{-1} \text{Re}(f_{\alpha} f_{\bar{\alpha}} (\phi^3)_{\bar{\beta}}) \leq \frac{C}{(\log \beta)^{\frac{1}{2}}} . \] (11)

Since \( \beta \) (and \( \delta = \frac{1}{\beta} \)) is arbitrary it follows that for \( \varepsilon \) and \( T \) fixed the above inequality becomes equality by letting \( \beta \to \infty \).

From (11) we are able to draw the conclusion that the following formulas need to hold on \( M \):

\[ \text{Ric}_{11} = -(m + 1) \]
\[ |\nabla f| = 2 \sqrt{\lambda_1(M)} f \]
\[ u_{\alpha\beta} = -m \delta_{\alpha\beta} \]
\[ u_{\alpha\beta} = m \delta_{\lambda\alpha} \delta_{\lambda\beta} \]

(12)
with respect to the frame
\[ v_\alpha = \frac{1}{2} (e_{2\alpha-1} - \sqrt{-1}Je_{2\alpha-1}), \]
\[ e_1 = \frac{1}{|\nabla f|} \nabla f, \ J e_{2k-1} = e_{2k}. \]

Note that in view of (12) this frame is globally defined on \( M \).

Let us prove that indeed we have these relations on \( M \).

Suppose that there exists a point \( x_0 \in M \) and a positive \( \eta_0 \) such that:
\[ \text{Ric}_{11}(x_0) \geq -(m + 1) + \eta_0. \]

Let us choose \( \varepsilon \) and \( T \) such that \( x_0 \in L(\varepsilon, T) \). Recall that \( L = L(\delta \varepsilon, \beta T) \), for arbitrary \( \beta \) and for \( \delta = \frac{1}{\beta} \).

Then one can see that there exists \( \eta_1 > 0 \) such that
\[ -\int_L f^{-1} f_\alpha f_{\bar{\alpha} \beta} \phi^3 \leq \frac{m+1}{4} \int_L f^{-1} |\nabla f|^2 \phi^3 - \eta_1. \]

It is easy to check that now (11) will become
\[ 0 < \eta_1 \leq \frac{c_5}{\log \beta} + \frac{1}{4} \int_L f^{-2} |\nabla f|^2 \text{Re}(f_{\bar{\beta}} (\phi^3)_{\beta}) - \int_L f^{-1} f_{\bar{\alpha} \beta} f_\alpha (\phi^3)_{\beta} - \frac{a}{2-a} \int_L f^{-1} \text{Re}(f_\alpha f_{\bar{\alpha} \beta} (\phi^3)_{\beta}) \leq \frac{C}{(\log \beta)^{\frac{1}{2}}}, \]
which gives a contradiction if we let \( \beta \to \infty \).

Next, let us focus on the Schwarz inequality (9). Suppose by absurd that there exists no constant \( a \neq 0 \) such that
\[ |\nabla f|(x) = af(x) \text{ for any } x \in U, \]
where \( U \subset L(\varepsilon, T) \) is a fixed open set. It is clear that if
\[ h = f^{-\frac{1}{2}} |\nabla f|^2 \phi^\frac{1}{2}, \ g = f^{\frac{1}{2}} \phi^\frac{1}{2}, \]
then there exists no \( a \in \mathbb{R} \) such that \( g = ah \) on \( U \), which implies that
\[ \eta_0 := \min_{a \in \mathbb{R}} \int_U (g - ah)^2 > 0. \]
This shows that
\[
\eta_0 \leq a^2 \int_U h^2 - 2a \int_U gh + \int_U g^2,
\]
\[
0 \leq a^2 \int_{L \setminus U} h^2 - 2a \int_{L \setminus U} gh + \int_{L \setminus U} g^2,
\]
for any \( a \in \mathbb{R} \). As a consequence, the following inequality is true for any \( a \in \mathbb{R} \):
\[
0 \leq a^2 \int_L h^2 - 2a \int_L gh + \left( \int_L g^2 - \eta_0 \right).
\]
It follows that
\[
\left( \int_L gh \right)^2 \leq \left( \int_L h^2 \right) \left( \int_L g^2 - \eta_0 \right).
\]
Similarly, one can see that there exists an \( \eta_1 > 0 \) such that
\[
\left( \int_L gh \right)^2 \leq \left( \int_L g^2 \right) \left( \int_L h^2 - \eta_1 \right).
\]
Adding these two inequalities and using the arithmetic mean inequality we get that there exists \( \eta_2 > 0 \) with the property
\[
\left( \int_L gh + \eta_2 \right)^2 \leq \int_L g^2 \int_L h^2.
\]
We have thus proved that there exists a constant \( \eta_2 > 0 \) depending on \( U \) but not on \( \beta \) (and \( \delta \)) such that
\[
\left( \int_L f^{-1} |\nabla f|^2 \phi^3 + \eta_2 \right)^2 \leq \left( \int_L f^{-3} |\nabla f|^4 \phi^3 \right) \left( \int_L f \phi^3 \right),
\]
inequality that will be used instead of (9) in the argument that followed. Consequently,
\[
\int_L f^{-3} |\nabla f|^4 \phi^3 \geq 4\lambda_1(M) \frac{\left( \int_L f^{-1} |\nabla f|^2 \phi^3 + \eta_2 \right)^2}{\int_L f^{-1} |\nabla f|^2 \phi^3 + \frac{c_3}{\log \beta}}
\]
\[
\geq 4\lambda_1(M) \int_L f^{-1} |\nabla f|^2 \phi^3 + 8\lambda_1(M) \eta_2 - c_1 \frac{1}{\log \beta}.
\]
However, using the same reasoning as for Ricci one can see that this yields a contradiction.

Summing up, we have proved that there exists a constant $a > 0$ such that $|\nabla f| = af$ on $M$. Using Lemma 1 and Lemma 2 one can see that $a = 2\sqrt{\lambda_1(M)}$.

The proofs for the remaining two formulas use the same ideas. Note that in (7) we need to have equality everywhere on $M$, therefore there exists a function $\mu$ on $M$ such that

\[ u_{\alpha\beta} = \mu \delta_{\alpha\beta}. \]

However, taking the trace and using that $f$ is harmonic one can show that $\mu = -m$.

Finally, we pointed out that if equality holds in (3) then

\[ f_{\alpha\beta} = 0 \text{ for } (\alpha, \beta) \neq (1,1), \]

and on the other hand equality holds in (4) if and only if

\[
\begin{align*}
|f_{\alpha\beta}| &= \frac{1}{a} \frac{|\nabla f|^2}{4f} = m(m+1)f, \\
Re(f_{11}) &= |f_{11}|.
\end{align*}
\]

This means that

\[ f_{11} = m(m+1)f, \]

or in terms of $u$ one has

\[ u_{\alpha\beta} = m\delta_{1\alpha}\delta_{1\beta} \]

as claimed.

Now we are ready to complete the proof of Theorem 4. Let us compute the real Hessian of

\[ B := \frac{1}{2m}u. \]

We have:

\[
\begin{align*}
B_{e_1e_1} &= B_{11} + B_{1\bar{1}} + 2B_{1\bar{1}} = 1 - 1 = 0, \\
B_{e_2e_2} &= -(B_{11} + B_{1\bar{1}} - 2B_{1\bar{1}}) = -2, \\
B_{e_{2k-1}e_{2k-1}} &= B_{kk} + B_{kk} + 2B_{kk} = -1, \\
B_{e_{2k}e_{2k}} &= -B_{kk} - B_{kk} + 2B_{kk} = -1, \\
B_{e_ke_j} &= 0 \text{ if } k \neq j.
\end{align*}
\]
for $k \in \{2, \ldots, m\}$. Also, notice that $|\nabla B| = 1$ on $M$.

Since all the computations from now on will be done in the real frame
$\{e_1, \ldots, e_{2m}\}$ with $Je_{2k-1} = e_{2k}$ and $e_1 = \frac{1}{|\nabla f|} \nabla f$, for convenience we will drop the $e_k$ index and use only $k$ in the formulas for the real Hessian and the curvature.

Also, let us make the convention that Roman letters $i, j, k$ run from 1 to $2m$ and Greek letters $\alpha, \beta, \gamma$ run from 3 to $2m$.

We have proved that there exists a smooth function $B$ on $M$ with real Hessian

$$
(B_{ij}) = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & -2 & 0 & 0 & \ldots & 0 \\
0 & 0 & -1 & 0 & \ldots & 0 \\
0 & 0 & 0 & -1 & \ldots & 0 \\
0 & & & & \ddots & \ddots & \ddots & \ddots \\
0 & & & & & 0 & \ldots & \ldots & -1
\end{pmatrix}
$$

and with unit gradient, $|\nabla B| = 1$ on $M$.

Note that our function $B$ satisfies the same properties as the Buseman function $\beta$ in [L-W]. Using the Hessian of $\beta$ Li-Wang proved that the manifold has at most two ends and in the two ends case they inferred some information on the structure of $M$. We will give an outline of their argument below.

Denote the level set of $B$ by

$$
N_t = \{x \in M \mid B(x) = t\}.
$$

Since $|\nabla B| = 1$, $M$ is diffeomorphic to $\mathbb{R} \times N_0$ and $e_1 = \nabla B$ is the unit normal to $N_t$ for any $t$. If $N_0$ is noncompact, then $M$ will have one end, which contradicts our assumption that $M$ has more than one end.

Consequently, $N_0$ is compact, and this implies that $M$ has two ends. For the remainder of this proof $M$ has two ends, and we want to find the metric of $N_t$ depending on the metric of $N_0$.

Knowing $B_{ij}$ is equivalent to knowing the second fundamental form of $N_t$, which implies that if

$$
\nabla e_i = \omega_{ik} e_k,
$$

then one can find

$$
\omega_{11}(e_j) = \begin{cases}
0 & \text{for } i \neq j \\
2 & \text{for } i = j = 2 \\
1 & \text{for } 3 \leq i = j \leq 2m.
\end{cases}
$$
Also, using the Kähler property we know that
\[ \omega_{1k}Je_k = J\nabla e_1 = \nabla Je_1 = \nabla e_2 = \omega_{2k}e_k, \]
which implies
\[ \omega_{\alpha 2}(e_j) = \begin{cases} 
0 & \text{for } j = 1 \text{ or } j = 2 \\
-1 & \text{for } \alpha = 2p + 1, \ j = 2p + 2 \\
1 & \text{for } \alpha = 2p + 2, \ j = 2p + 1.
\end{cases} \]

It is clear that the flow \( \phi_t : M \to M \) generated by \( e_1 \) is a geodesic flow. Since
\[ \nabla_{e_1}e_2 = \nabla_{e_1}Je_1 = J\nabla_{e_1}e_1 = 0 \]
we can conclude that \( e_2 \) is parallel along the geodesic \( \tau \) defined by \( e_1 \). We will consider the rest of the frame so that it is also parallel along this geodesic.

The next step is to prove that
\[ V_2(t) = e^{-2t}e_2, \]
\[ V_\alpha(t) = e^{-t}e_\alpha \]
are the Jacobi fields along the geodesic \( \tau \) with initial conditions
\[ V_2(0) = e_2, \ V'_2(0) = -2e_2 \]
\[ V_\alpha(0) = e_\alpha, \ V'_\alpha(0) = -e_\alpha. \]

This is true because the information on \( \omega_{i1} \) and \( \omega_{\alpha 2} \) allows to find sufficient values for the curvature tensor. Using the second structural equations one can show that
\[ R_{1212} = -4, \ R_{121\alpha} = 0, \]
\[ R_{1\alpha 1\beta} = -\delta_{\alpha\beta}, \] (13)
and this indeed shows that \( V_k(t) \) are Jacobi fields for \( k \in \{2, ..., 2m\} \).

However, \( d\phi_t(e_k) \) for \( k \geq 2 \) are also Jacobi fields with the same initial conditions as \( V_k(t) \), so they must coincide.

The conclusion is that the metrics on \( N_t \) viewed as one parameter of metrics on \( N_0 \) are
\[ ds_t^2 = e^{-4t}\omega_2^2(0) + e^{-2t}\left(\omega_3^2(0) + ... + \omega_{2m}^2(0)\right), \]
where \( \{\omega_1, ..., \omega_{2m}\} \) is the dual frame of \( \{e_1, ..., e_{2m}\} \). Q.E.D.
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