HAUSDORFF DIMENSION IN QUASIREGULAR DYNAMICS
WALTER BERGWEILER AND ATHANASIOS TSANTARIS

Abstract. It is shown that the Hausdorff dimension of the fast escaping set of a quasiregular self-map of $\mathbb{R}^3$ can take any value in the interval $[1, 3]$. The Hausdorff dimension of the Julia set of such a map is estimated under some growth condition.

1. Introduction
Quasiregular maps are a natural generalization of holomorphic maps to higher dimensions; see § 2.1 for their definition and basic properties. We will be concerned with quasiregular self-maps of $\mathbb{R}^d$. These are the analogues of entire maps $f : \mathbb{C} \to \mathbb{C}$. A quasiregular map $f : \mathbb{R}^d \to \mathbb{R}^d$ is said to be of polynomial type if $\lim_{|x| \to \infty} |f(x)| = \infty$ and of transcendental type otherwise. Here $|x|$ is the Euclidean norm of a point $x \in \mathbb{R}^d$.

The escaping set $\mathcal{I}(f)$ of a quasiregular map $f : \mathbb{R}^d \to \mathbb{R}^d$ is defined by
$$\mathcal{I}(f) = \{ x \in \mathbb{R}^d : |f^n(x)| \to \infty \}.$$ Here $f^n$ is the $n$-th iterate of $f$. For transcendental entire functions the escaping set was introduced by Eremenko [16] who, in particular, showed that it is always non-empty. This result was extended to quasiregular maps of transcendental type in [8].

The proofs in [8, 16] actually show that not only the escaping set $\mathcal{I}(f)$ is non-empty, but that this in fact holds for a subset $\mathcal{A}(f)$ of $\mathcal{I}(f)$ called the fast escaping set. This set was first considered in [10].

In order to define it, recall that the maximum modulus is given by
$$M(r, f) = \max_{|x|=r} |f(x)|.$$ Let $M^n(r, f)$ denote the $n$-th iterate of $M(r, f)$ with respect to the first variable. We thus have $M^1(r, f) = M(r, f)$ and $M^n(r, f) = M(M^{n-1}(r, f), f)$ for $n \geq 2$. For a quasiregular map $f$ of transcendental type there exists $R_0 \geq 0$ such that $M(R, f) > R$ for $R > R_0$. For $R > R_0$ we then have $M^n(R, f) \to \infty$ as $n \to \infty$. For such $R$ the fast escaping set is defined by
$$\mathcal{A}(f) = \{ x \in \mathbb{R}^d : \text{there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(x)| \geq M^n(R, f) \text{ for all } n \in \mathbb{N} \}.$$ This definition does not depend on $R$ as long as $M^n(R, f) \to \infty$; see [28, Theorem 2.2, (b)] for transcendental entire functions and [7, Proposition 3.1, (i)] for quasiregular maps. These papers also contain some equivalent definitions of $\mathcal{A}(f)$.

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The Julia set $\mathcal{J}(f)$ of a transcendental entire function $f$ is defined as the set of all points where the iterates of $f$ do not form a normal family. Eremenko [16] showed that $\mathcal{J}(f) = \partial \mathcal{I}(f)$. As noted by Rippon and Stallard [27, p. 1125, Remark 1], we also have $\mathcal{J}(f) = \partial \mathcal{A}(f)$.

For quasiregular mappings the set where the iterates are not normal does not have the properties one would normally associate with a Julia set. Therefore, instead of using normality, the Julia set $\mathcal{J}(f)$ was defined in [5, 12] as the set of all $x \in \mathbb{R}^d$ such that $\mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} f^k(U)$ has capacity zero for every neighborhood $U$ of $x$; see § 2.1 for the definition of capacity. As shown in these papers, the set $\mathcal{J}(f)$ defined this way shares many properties with Julia sets of entire functions. However, there are examples [12, Example 7.3] where $\mathcal{J}(f) \neq \partial \mathcal{I}(f)$. On the other hand, we always have $\mathcal{J}(f) \subset \partial \mathcal{I}(f)$; see [12, Theorem 1.3].

By [9, Theorem 1.1] we also have $\mathcal{J}(f) \subset \partial \mathcal{A}(f)$. Moreover, it was shown in [9, Theorem 1.2] that under the additional hypothesis

$$\liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log \log r} = \infty$$

we even have $\mathcal{J}(f) = \partial \mathcal{A}(f)$. Thus for quasiregular maps the fast escaping set is perhaps even more relevant for the dynamics than the escaping set.

There is a large literature concerned with the Hausdorff dimension of the above sets for transcendental entire functions; see [20] and [31] for surveys. We will describe some of these results, denoting by $\dim X$ the Hausdorff dimension of a subset $X$ of $\mathbb{R}^d$.

Baker [2] proved that the Julia set of a transcendental entire function $f$ contains continua. Thus $\dim \mathcal{J}(f) \geq 1$. On the other hand, for all $\rho \in [1, 2]$ there exists a transcendental entire function $f$ satisfying $\dim \mathcal{J}(f) = \rho$. Here the case $\rho \in (1, 2)$ is due to Stallard [30, Theorem 1.1] while the case $\rho = 1$ is due to Bishop [14]. Examples with $\dim \mathcal{J}(f) = 2$ had been known already before. Perhaps the simplest ones are those with $\mathcal{J}(f) = \mathbb{C}$. The first example of such a function $f$ was given by Baker [1].

Rippon and Stallard [27, Theorem 1] showed that every component of $\mathcal{A}(f)$ is unbounded. While a conjecture of Eremenko [16] saying that every component of $\mathcal{I}(f)$ is unbounded was recently disproved [21], the result by Rippon and Stallard implies of course that $\mathcal{I}(f)$ has at least one unbounded component. In particular, $\dim \mathcal{I}(f) \geq \dim \mathcal{A}(f) \geq 1$. On the other hand, it was shown in [25, Corollary 1.2] that for all $\rho \in [1, 2]$ there exists a transcendental entire function $f$ such that $\dim \mathcal{I}(f) \geq \rho$. The main contribution of [25] is the case $\rho = 1$. The case $\rho \in (1, 2)$ is deduced from results in [30] and [11]. Inspection of the proofs in [11] shows that we actually may achieve $\dim \mathcal{A}(f) = \rho$ for given $\rho \in (1, 2)$. For $\rho = 2$ such an example is given by the exponential function; this follows from the arguments in [23, Theorem 1.2]. Further examples are given in [29]. Altogether we see that for all $\rho \in [1, 2]$ there exists a transcendental entire function $f$ such that $\dim \mathcal{A}(f) = \rho$.

It is the main purpose of this paper to extend this result to quasiregular maps of transcendental type. We will restrict to the case that $d = 3$. The reason is that this allows to use a construction of Nicks and Sixsmith [24].

**Theorem 1.1.** For all $\rho \in [1, 3]$ there exists a quasiregular map $f : \mathbb{R}^3 \to \mathbb{R}^3$ of transcendental type such that $\dim \mathcal{A}(f) = \rho$. 
As already mentioned, we have \( \dim I(f) \geq 1 \) for a quasiregular map \( f : \mathbb{R}^d \to \mathbb{R}^d \) of transcendental type. We do not know whether this bound is sharp. It seems plausible that we have \( \dim I(f) \geq d - 1 \) for such maps; see Remark 4.7 below.

Perhaps we also have \( \dim J(f) \geq d - 1 \). However, no positive lower bound is known for \( \dim J(f) \). It was shown in [12, Theorem 1.7] that \( \dim J(f) > 0 \) if \( f \) is locally Lipschitz continuous. We obtain a lower bound under the growth condition (1.1).

**Theorem 1.2.** Let \( f : \mathbb{R}^d \to \mathbb{R}^d \) be a quasiregular map satisfying (1.1). Then \( \dim J(f) \geq 1 \).

The structure of the rest of this paper is as follows: In Sections 2 and 3 we recall some definitions and some preliminaries. In Section 4 we prove Theorem 1.1 and finally in Section 5 we prove Theorem 1.2.

We conclude the introduction with a remark on quasiregular self-maps of \( \mathbb{R}^d \).

**Remark 1.3.** Fletcher and Vellis [18] studied the more general question of when can a Cantor set be the Julia set of a uniformly quasiregular map. It follows from [18, Theorem 1.2] that for any \( \rho \in (0, d - 1) \) there is a uniformly quasiregular map \( f : \mathbb{R}^d \to \mathbb{R}^d \) such that \( \dim J(f) = \rho \).

Moreover, by [18, Corollary 1.4] we can find a uniformly quasiregular map \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) whose Julia set is a Cantor set which is defined as the limit set of an iterated function system of similarities acting on \([0, 1]^3\). By choosing the right similarities we can make this set have any Hausdorff dimension in \((0, 3)\).

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2. Basic results about quasiregular maps and Hausdorff dimension

2.1. Quasiregular maps. We only sketch the definition and basic properties of quasiregular maps. For a detailed treatment we refer to Rickman’s book [26].

Let \( \Omega \subset \mathbb{R}^d \) be a domain and let \( f : \Omega \to \mathbb{R}^d \) be a map. If \( f \) is differentiable at a point \( x \in \Omega \), let \( Df(x) \) be the derivative, \( |Df(x)| = \sup_{|h|=1} |Df(x)(h)| \) its norm and \( J_f(x) = \text{det} J_f(x) \) the Jacobian determinant. We will also use the notation \( \ell(Df(x)) = \inf_{|h|=1} |Df(x)(h)| \). The map \( f \) is called quasiregular if it is continuous, belongs to the Sobolev space \( W^{1}_{d, \text{loc}}(\Omega) \), and there exists \( K \geq 1 \) such that

\[
|Df(x)|^d \leq K J_f(x)
\]

almost everywhere in \( \Omega \). The smallest such \( K \) is called the outer dilatation \( K_O(f) \).

The last condition is equivalent to the existence of a constant \( K' > 1 \) such that

\[
J_f(x) \leq K' \ell(Df(x))^d
\]

almost everywhere in \( \Omega \). The smallest such \( K' \) is called inner dilatation \( K_I(f) \) of \( f \). An injective quasiregular map is called quasiconformal.

For a survey of results concerning the dynamics or quasiregular maps we refer to [4].
In the proof of Theorem 1.2 we also need the concept of the modulus of a path family. We only recall here the relevant definitions, for a more thorough discussion we refer to [26].

Let $\Gamma$ be a family of paths in $\mathbb{R}^d$, with $d \geq 2$. We denote by $\mathcal{F}(\Gamma)$ the set of all Borel functions $\rho: \mathbb{R}^d \to [0, \infty]$ such that $\int_\gamma \rho \, ds \geq 1$ for every locally rectifiable path $\gamma \in \Gamma$. Then the modulus $M(\Gamma)$ of $\Gamma$ is defined as

$$M(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^d} \rho \, dm,$$

where $m$ is the Lebesgue measure. If $E, F, D \subset \mathbb{R}^d$, we will denote by $\Delta(E, F; D)$ the family of paths $\gamma: [0, 1] \to \mathbb{R}^d$ such one of the endpoints $\gamma(0)$ and $\gamma(1)$ belongs to $E$, the other one to $F$, while $\gamma(t) \in D$ for all $t \in (0, 1)$.

Let $\Omega \subset \mathbb{R}^d$ be open and let $C$ be a compact subset of $\Omega$. Then the pair $(\Omega, C)$ is called a condenser and its capacity $\text{cap}(\Omega, C)$ is defined by

$$\text{cap}(\Omega, C) = \inf_u \int_\Omega |\nabla u|^d \, dm,$$

where the infimum is taken over all non-negative functions $u \in C^\infty_0(\Omega)$ satisfying $u(x) \geq 1$ for $x \in C$. If $\text{cap}(\Omega, C) = 0$ for some bounded open set $\Omega$ containing $C$, then, by [26, Lemma III.2.2], we have $\text{cap}(\Omega', C) = 0$ for every bounded open set $\Omega'$ containing $C$. In this case we say that $C$ is of capacity zero, while otherwise $C$ has positive capacity. A closed subset $C$ of $\mathbb{R}^d$ is said to have capacity zero if this is the case for all compact subsets of $C$.

The connection between capacity and the modulus of a path family is given by [26, Proposition II.10.2]

$$\text{cap}(\Omega, C) = M(\Delta(C, \partial \Omega; \Omega)).$$

Zorich [34, p. 400] introduced an important example of a quasiregular self-map of $\mathbb{R}^3$ which can be considered as a 3-dimensional analogue of the exponential function. In fact, the construction is quite flexible and also works in $\mathbb{R}^d$ for all $d \geq 2$; see [22, § 8.1]. We follow [19, § 6.5.4] in the definition of a Zorich map and consider the cube

$$Q := \{x \in \mathbb{R}^{d-1}: \|x\|_\infty \leq 1\} = [-1, 1]^{d-1}$$

and the upper hemisphere

$$U := \{x \in \mathbb{R}^d: \|x\|_2 = 1, x_d \geq 0\}.$$

Let $h: Q \to U$ be a bi-Lipschitz map and define

$$Z: Q \times \mathbb{R} \to \mathbb{R}^d, \quad Z(x_1, \ldots, x_d) = e^{xd}h(x_1, \ldots, x_{d-1}).$$

The map $Z$ is then extended to a map $Z: \mathbb{R}^d \to \mathbb{R}^d$ by repeated reflection at hyperplanes. Many results about the dynamics of exponential functions have been extended to Zorich maps [3, 6, 15, 32, 33]. In particular, we note that it follows from the arguments given in [3, § 4] that for suitable $c \in \mathbb{R}^3$ and $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = Z(x) - c$, we have $\dim \mathcal{A}(f) = 3$. Thus we may restrict in our proof of Theorem 1.1 to the case that $1 \leq \rho < 3$. We will use a modification of the Zorich map introduced by Nicks and Sixsmith [24]; see § 3.1.
2.2. **Hausdorff dimension.** For the definition and a thorough treatment of Hausdorff dimension we refer to the book by Falconer [17]. To obtain a lower bound for the Hausdorff dimension we will use the following result known as the mass distribution principle [17, Proposition 4.9]. Here and in the following we use the notation $B(x, r)$ for the open ball of radius $r$ around a point $x \in \mathbb{R}^d$. Sometimes we will emphasize the dimension by writing $B_d(x, r)$ instead of $B(x, r)$. The closed ball is denoted by $\overline{B}(x, r)$.

**Lemma 2.1.** Let $E$ be a compact subset of $\mathbb{R}^d$. Suppose that there exist a probability measure $\mu$ supported on $E$ and positive constants $c, r_0$ and $\rho$ such that
\[
\mu(B(x, r)) \leq cr^\rho
\]
for each $x \in E$ and each $r \in (0, r_0)$. Then $\dim E \geq \rho$.

To obtain an upper bound for the Hausdorff dimension we will use the following result [3, Lemma 5.2]. It is a simple consequence of a standard covering result [17, Covering Lemma 4.8].

**Lemma 2.2.** Let $E \subset \mathbb{R}^d$ and $\rho > 0$. Suppose that for all $x \in E$ and $\delta > 0$ there exist $r(x) \in (0, 1)$, $d(x) \in (0, \delta)$ and $N(x) \in \mathbb{N}$ satisfying
\[
N(x)d(x)^\rho \leq r(x)\delta
\]
such that $B(x, r(x)) \cap E$ can be covered by $N(x)$ sets of diameter at most $d(x)$. Then $\dim E \leq \rho$.

In [3] it is assumed in addition that $E$ is bounded, but this hypothesis can be omitted since the Hausdorff dimension of an unbounded set is the supremum of the Hausdorff dimensions of its bounded subsets. Note that in [3] it was assumed that $\rho > 1$ but the proof also works for $\rho > 0$.

Let $\text{meas} X$ denote the Lebesgue measure of a measurable set $X$. We will need the concept of the density $\text{dens}(A, B)$ of a set $A$ in a set $B$, where $A, B \subset \mathbb{R}^d$ are measurable with $\text{meas} B > 0$. It is defined by
\[
\text{dens}(A, B) = \frac{\text{meas}(A \cap B)}{\text{meas} B}.
\]

To estimate how the density changes under a quasiconformal map we will use the following result. We omit its simple proof.

**Lemma 2.3.** Let $f: \Omega \to \mathbb{R}^d$ be quasiregular and let $A, B \subset \Omega$ be measurable with $\text{meas} B > 0$. Suppose that $f$ is injective on $B$. Then
\[
\left(\frac{\text{ess inf}_{x \in B} \ell(Df(x))}{\text{ess sup}_{x \in B} |Df(x)|}\right)^d \leq \frac{\text{dens}(f(A), f(B))}{\text{dens}(A, B)} \leq \left(\frac{\text{ess sup}_{x \in B} |Df(x)|}{\text{ess inf}_{x \in B} \ell(Df(x))}\right)^d.
\]

3. **Preliminaries for the proof of Theorem 1.1**

3.1. **Definition of $f$.** Nicks and Sixsmith [24, Section 5] considered a variant of the Zorich map where $h$ maps $[-1,1]^2$ not to the upper hemisphere but to the upper faces of a square based pyramid. Specifically, they worked with
\[
h(x_1, x_2) = (x_1, x_2, 1 - \max\{|x_1|, |x_2|\}).
\]
The map $Z$ is again defined by

$$Z(x) = e^{x_3} h(x_1, x_2) \quad \text{for } |x_1| \leq 1 \text{ and } |x_2| \leq 1,$$

and extended to a map $Z: \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$ by reflections. They used this map $Z$ to construct a quasiregular map $G: \mathbb{R}^3 \to \mathbb{R}^3$ which is equal to the identity in a half-space.

For $t \in \mathbb{R}$ we put $\mathbb{H}_{\geq t} = \{(x_1, x_2, x_3) \in \mathbb{R}^3: x_3 \geq t\}$. The half-spaces $\mathbb{H}_{\leq t}$, $\mathbb{H}_{> t}$ and $\mathbb{H}_{< t}$ are defined analogously. The quasiregular map $G$ constructed by Nicks and Sixsmith [24, Section 6] has the property that there exists $L > 0$ such that

$$G(x) = \begin{cases} x + Z(x) & \text{for } x \in \mathbb{H}_{\geq L}, \\ x & \text{for } x \in \mathbb{H}_{< 0}. \end{cases}$$

The difficult part in the construction is, of course, to define $G$ in the remaining domain $\{x \in \mathbb{R}^3: 0 < x_3 < L\}$. Here we only note that the construction yields that there exists a constant $C$ such that

$$|G(x) - x| \leq C \quad \text{for } x \in \mathbb{H}_{< 0}. \quad \text{(3.1)}$$

We will also consider the maps $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$,

$$\varphi(x_1, x_2, x_3) = (x_1, x_2, x_3 - |x_1| - |x_2|)$$

and $H: \mathbb{R}^3 \to \mathbb{R}^3$,

$$H = Z \circ \varphi.$$

The map $\varphi$ is quasiconformal and hence $H$ is quasiregular.

Considering $H$ instead of $Z$ has the advantage that while $Z$ is bounded in a half-space, $H$ is bounded in a larger domain. In fact, with

$$\Omega = \{x \in \mathbb{R}^3: x_3 > |x_1| + |x_2|\} \quad \text{(3.2)}$$

we have

$$|H(x)| \leq \max_{y \in [-1,1]^2} |h(y)| = \sqrt{2} \quad \text{for } x \in \mathbb{R}^3 \setminus \Omega. \quad \text{(3.3)}$$

For $s = (s_1, s_2) \in \mathbb{Z}^2$ we consider the beam

$$T(s) = [2s_1 - 1, 2s_1 + 1] \times [2s_2 - 1, 2s_2 + 1] \times \mathbb{R}$$

$$= \{x \in \mathbb{R}^3: |x_j - 2s_j| \leq 1 \text{ for } j \in \{1, 2\}\}.$$

By construction, $Z$ is injective in $T(s)$. Since $\varphi$ maps $T(s)$ bijectively onto itself, $H$ is also injective in $T(s)$. The definition of $Z$ also yields that there exists $R \geq L$ such that if $x, y \in T(s) \cap \mathbb{H}_{\geq R}$ and $x \neq y$, then $|Z(x) - Z(y)| > |x - y|$ and hence

$$|G(x) - G(y)| \geq |Z(x) - Z(y)| - |x - y| > 0.$$ 

We deduce that $G$ is injective in $T(s) \cap \mathbb{H}_{\geq R}$.

We also note that if $s = (s_1, s_2) \in \mathbb{Z}^2$ and $s_1 + s_2$ is even, then both $Z$ and $G$ map $T(s) \cap \mathbb{H}_{\geq R}$ into $\mathbb{H}_{> 0}$, provided $R$ is large enough.

Since $\varphi$ leaves $\partial T(s)$ invariant while $Z$ maps $\partial T(s)$ to the plane $\{x \in \mathbb{R}^3: x_3 = 0\}$ we find that

$$G(H(x)) = H(x) \quad \text{for } x \in \partial T(s). \quad \text{(3.4)}$$
We will define \( f \) by putting \( f(x) = G(H(x)) \) in some beams while \( f(x) = H(x) \) in the other beams.

In order to define this partition of the set of beams, we will use the following result.

**Lemma 3.1.** Let \( 0 < \beta < 1 \). Then there exists a subset \( S \) of \( \mathbb{Z}^2 \) with \((0,0) \in S\) such that \( s_1 + s_2 \) is even for \( s = (s_1, s_2) \in S \) and such that the following two conditions hold:

(a) If \( s, s' \in S \) and \( s \neq s' \), then \( |s - s'| \geq |s|^\beta + |s'|^\beta \).

(b) For all \( x \in \mathbb{R}^2 \) there exists \( s \in S \) such that \( |x - s| \leq |s|^\beta + (|x| + 1)^\beta + 1 \).

**Proof.** Let \( \Sigma \) be the set of all \( s = (s_1, s_2) \in \mathbb{Z}^2 \) such that \( s_1 + s_2 \) is even. We put \( s_0 = (0,0) \) and define a sequence \( (s_k) \) recursively as follows. Assuming that \( s_0, s_1, \ldots, s_{k-1} \) have been defined, let \( A_k \) denote the set of all \( s \in \Sigma \) such that \( |s - s_j| \geq |s|^\beta + |s_j|^\beta \) for all \( j \in \{0, 1, \ldots, k - 1\} \). Since \( \beta < 1 \) this clearly holds if \( |s| \) is sufficiently large so that \( A_k \neq \emptyset \). We then choose \( s_k \in A_k \) such that \( |s_k| = \min_{s \in A_k} |s| \). Finally we put \( S = \{s_k : k \geq 0\} \).

It follows from the construction that \( S \) satisfies (a). To prove (b), let \( x \in \mathbb{R}^2 \). Then there exists \( s' \in \Sigma \) with \( |x - s'| < 1 \). It follows from the construction of \( S \) that there exists \( s \in S \) with \( |s - s'| \leq |s|^\beta + |s'|^\beta \). In fact, otherwise one would have to choose \( s_k = s' \) at some point of the construction. It now follows that \( |x - s| \leq |x - s'| + |s' - s| < 1 + |s|^\beta + |s'|^\beta \leq 1 + |s|^\beta + (|x| + 1)^\beta \).

As noted at the end of § 2.1, we may restrict in our proof of Theorem 1.1 to the case that \( 1 \leq \rho < 3 \). If \( \rho > 1 \), we put

\[
\beta = \frac{3 - \rho}{2}
\]

and choose \( S \) according to Lemma 3.1. If \( \rho = 1 \), we put \( S = \{(0,0)\} \). We now put

\[
T = \bigcup_{s \in S} T(s).
\]

The map \( f \) is then defined by

\[
f(x) = \begin{cases} 
G(H(x)) & \text{if } x \in T, \\
H(x) & \text{if } x \notin T.
\end{cases}
\]

Note that by (3.4) the map \( f \) is quasiregular.

We will need the following estimates for the number of points of \( S \) in certain disks.

**Lemma 3.2.** Let \( \beta \) and \( S \) be as in Lemma 3.1. There exists \( r_0 > 0 \) and \( \alpha_0 > 0 \) such that if \( r > r_0 \) and \( x \in \mathbb{R}^2 \), then

\[
\text{card}(S \cap B(x, r)) \leq \alpha_0 r^{2 - 2\beta}.
\]

Moreover,

\[
\frac{1}{64} |x|^{-2\beta} r^2 \leq \text{card}(S \cap B(x, r)) \leq 6|x|^{-2\beta} r^2 \quad \text{for } 8|x|^\beta \leq r \leq \frac{1}{2} |x|
\]

and

\[
\text{card}(S \cap B(x, r)) \leq 324 \quad \text{for } r < 8|x|^\beta.
\]
It follows from (3.8) that
\[
\text{card}(S \cap B(x, r)) \leq 324|x|^{-2\beta}r^2 \quad \text{for } |x|^\beta \leq r \leq 8|x|^\beta.
\]
We thus have an estimate of the same type as (3.7) for a larger range of values of \(r\).

**Proof of Lemma 3.2.** We begin by proving (3.7) and (3.8), from which we will deduce (3.6) afterwards. So let \(x\) and \(r\) be such that one the conditions in (3.7) or (3.8) is satisfied. If \(r\) is large, then so is \(|x|\). Choosing \(r_0\) large we can thus achieve that \(8|x|^\beta < |x|/2\) for any \(x\) satisfying one of the conditions in (3.7) or (3.8). Thus we have \(r \leq |x|/2\) in both cases. Hence
\[
\frac{1}{2}|x| \leq |y| \leq \frac{3}{2}|x| \quad \text{for } y \in B(x, r).
\]
By the definition of \(S\) we have
\[
B(s, |s|^\beta) \cap B(s', |s'|^\beta) = \emptyset \quad \text{if } s, s' \in S.
\]
Combining the last two equations we find that
\[
B(s, 2^{-\beta}|x|^\beta) \cap B(s', 2^{-\beta}|x|^\beta) = \emptyset \quad \text{if } s, s' \in S \cap B(x, r).
\]
We also have
\[
\bigcup_{s \in S \cap B(x, r)} B(s, 2^{-\beta}|x|^\beta) \subset B(x, r + 2^{-\beta}|x|^\beta).
\]
If \(8|x|^\beta \leq r\) we have \(r + 2^{-\beta}|x|^\beta \leq r + 2^{-\beta}r/8 \leq 9r/8\). Hence
\[
\bigcup_{s \in S \cap B(x, r)} B(s, 2^{-\beta}|x|^\beta) \subset B\left(x, \frac{9}{8}r\right)
\]
in this case. Taking the measure on both sides and noting that by (3.11) the union on the left hand side is disjoint, we deduce that
\[
\pi 2^{-2\beta}|x|^{2\beta} \text{card}(S \cap B(x, r)) \leq \pi \frac{81}{64} r^2
\]
so that
\[
\text{card}(S \cap B(x, r)) \leq 2^{2\beta} \frac{81}{64} |x|^{-2\beta}r^2.
\]
This proves the right inequality in (3.7).
If \(r < 8|x|^\beta\), then \(r + 2^{-\beta}|x|^\beta \leq 9|x|^\beta\). Then (3.12) yields that
\[
\bigcup_{s \in S \cap B(x, r)} B(s, 2^{-\beta}|x|^\beta) \subset B\left(x, 9|x|^\beta\right).
\]
Taking the measure on both sides now yields that
\[
\text{card}(S \cap B(x, r)) \leq 2^{2\beta} \frac{81}{64}
\]
and hence (3.8).
To prove the left inequality in (3.7), let \(y \in B(x, r/2)\). By the definition of \(S\), there exists \(s = s(y) \in S\) such that \(|y - s| \leq |s|^\beta + (|y| + 1)^\beta + 1\). By (3.10) we have
\[
|y - s| \leq |s|^\beta + \left(\frac{3}{2} |x| + 1\right)^\beta + 1.
\]
First we show that this implies that $|s| \leq 2|x|$ if $r_0$ is sufficiently large. To do so, suppose that $|s| > 2|x|$. Using (3.10) again we find that

$$|y - s| \geq |s| - |y| \geq |s| - \frac{3}{2}|x| > |s| - \frac{3}{4}|s| = \frac{1}{4}|s|.$$  

Together with (3.13) this yields that

$$\frac{1}{4}|s| \leq |s|^\beta + \left(\frac{3}{2}|x| + 1\right)^\beta + 1 < |s|^\beta + \left(\frac{3}{4}|s| + 1\right)^\beta + 1,$$

which is a contradiction if $r_0$ is large, since then $|x|$ and hence $|s|$ are also large. Hence we have $|s| \leq 2|x|$. It thus follows from (3.13) for large $r_0$ that

$$(3.14) \quad |y - s| \leq 2^\beta|x|^\beta + \left(\frac{3}{2}|x| + 1\right)^\beta + 1 < 4|x|^\beta.$$  

Since we are assuming that $8|x|^\beta < r$ this implies that $|y - s| < r/2$. Hence $s \in B(y, r/2) \subset B(x, r)$. Altogether we have thus shown that for all $y \in B(x, r/2)$ there exists $s = s(y) \in S \cap B(x, r)$ satisfying (3.14). Hence

$$B(\frac{x}{2}, \frac{r}{2}) \subset \bigcup_{s \in S \cap B(x, r)} B(s, 4|x|^\beta).$$  

Considering the measure on both sides we obtain

$$\pi \frac{r^2}{4} \leq \pi 16|x|^{2\beta} \text{card}(S \cap B(x, r))$$

and hence the left inequality of (3.7).

It follows from (3.7) and (3.8) that (3.6) holds if $|x| \geq 2r$. To prove (3.6) if $|x| \leq 2r$ we introduce the notation $\text{ann}(r_1, r_2) := \{x: r_1 \leq |x| \leq r_2\}$ for an annulus. It is easy to see that $\text{ann}(2r/3, 4r/3)$ can be covered by 8 disks of radius $r/2$ and center on $\partial B(0, r)$. It thus follows from (3.7) that this annulus contains at most $12r^{2-2\beta}$ points of $S$. Replacing $4r/3$ by $r$ we find that

$$\text{card}\left(S \cap \text{ann}\left(\frac{r}{2}, r\right)\right) \leq 12r^{2-2\beta}$$

if $r \geq r_0$. It follows that

$$\text{card}\left(S \cap \text{ann}\left(\frac{r}{2k+1}, \frac{r}{2k}\right)\right) \leq 12 \frac{r^{2-2\beta}}{2^{(2-2\beta)k}}$$

for $k \in \mathbb{N}$ as long as $r/2^k \geq r_0$. This implies that $\text{card}(S \cap B(0, r)) \leq \alpha_1 r^{2-2\beta}$ with some constant $\alpha_1$ for $r \geq r_0$. For $|x| \leq 2r$ we have $B(x, r) \subset B(0, 3r)$ and thus (3.6) follows with $\alpha_0 = 3^{2-2\beta}\alpha_1$. \hfill \Box

3.2. Characterization of $A(f)$. For $m \in \mathbb{N}$ we put

$$F_m = \begin{cases} H & \text{if } k \text{ is odd}, \\ G & \text{if } k \text{ is even}, \end{cases}$$

and

$$(3.15) \quad h_m = F_m \circ F_{m-1} \circ \cdots \circ F_1,$$
with \( h_0 = \text{id} \). Then
\[
(3.16) \quad h_{2n} = f^n \quad \text{and} \quad h_{2n+1} = H \circ f^n.
\]
It follows from the definition of \( H \) that
\[
|H(x)| \leq |Z(x)| \leq \exp |x| \quad \text{for all } x \in \mathbb{R}^3
\]
while the definition of \( G \) and (3.1) yield that
\[
|G(x)| \leq |Z(x)| + |x| + C \leq \exp |x| + |x| + C \quad \text{for all } x \in \mathbb{R}^3.
\]
For large \( R \) we thus have
\[
M(R, F_j) \leq \exp R + R + C \leq \frac{1}{2} \exp(2R)
\]
for all \( j \) and hence
\[
(3.17) \quad M(R, h_m) \leq \frac{1}{2} \exp^m(2R).
\]
On the other hand,
\[
|H(0,0,R)| = |Z(0,0,R)| = \exp R
\]
for all \( R > 0 \) and
\[
|G(0,0,R)| = |Z(0,0,R) + (0,0,R)| = \exp R + R \geq \exp R
\]
for \( R \geq L \). Thus
\[
(3.18) \quad M(R, h_m) \geq \exp^m(R)
\]
for \( R \geq L \). In particular, it follows from (3.16), (3.17) and (3.18) that
\[
\exp^{2n}(R) \leq M^n(R, f) \leq \frac{1}{2} \exp^{2n}(2R)
\]
for large \( R \). Recalling that the definition of \( \mathcal{A}(f) \) is independent of the choice of \( R \) as long as \( R \geq R_0 \), it follows from the above considerations and the definition of \( f \) that for all \( R > 0 \) we have
\[
(3.19) \quad \mathcal{A}(f) = \{ x \in \mathbb{R}^3 : \text{there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(x)| \geq \exp^{2n}(R) \quad \text{and } f^{n+L}(x) \in T \text{ for all } n \in \mathbb{N} \}.
\]
This easily yields that
\[
(3.20) \quad \mathcal{A}(f) = \{ x \in \mathbb{R}^3 : \text{there exists } L \in \mathbb{N} \text{ such that } |h_{m+L}(x)| \geq \exp^m(R) \quad \text{for all } m \in \mathbb{N} \text{ and } h_{m+L}(x) \in T \text{ if } m + L \text{ is even} \}.
\]

The following result can easily be deduced from (3.20).

**Lemma 3.3.** Let \( x \in \mathcal{A}(f) \) and \( C > 0 \). Then \(|h_m(x)| \geq |h_{m-1}(x)|^C \) for all large \( m \). Moreover, \( h_m(x)_3 \to \infty \) as \( m \to \infty \).

It follows from Lemma 3.3 that given \( C > 0 \) there exists \( M \in \mathbb{N} \) such that
\[
\prod_{j=M}^{m-1} |h_j(x)| \leq \prod_{j=M}^{m-1} |h_m(x)|^{Cj-m} = |h_m(x)|^{-m}.
\]
for \( m > M \), with
\[
\gamma_m = \sum_{j=M}^{m-1} C^{j-m} = \sum_{k=1}^{m-M} C^{k-m} \leq \frac{1}{C-1}.
\]

It follows that if \( x \in \mathcal{A}(f) \) and \( \delta > 0 \), then
\[
(3.21) \quad \prod_{j=1}^{m-1} |h_j(x)| \leq |h_m(x)|^\delta
\]
for all large \( m \). It also follows from Lemma 3.3 that if \( x \in \mathcal{A}(f) \) and \( C > 0 \), then there exists \( \delta > 0 \) such that
\[
(3.22) \quad |h_m(x)| \geq \exp(\delta C^m)
\]
for all large \( m \).

### 3.3. Estimates for the derivatives of \( H \) and \( G \)

The derivative of a branch \( \Lambda \) of the inverse of the “classical” Zorich map \( Z \) introduced in § 2.1 was estimated in [3, inequalities (2.6) and (2.7)]. It is easy to see that the inequalities obtained there also hold for the modified Zorich map \( Z \) introduced by Nicks and Sixsmith. Thus there exist constants \( \alpha_1, \alpha_2 \) and \( M \) such that for a branch \( \Lambda \) of the inverse of \( Z \) we have
\[
\frac{\alpha_1}{|x|} \leq \ell(D\Lambda(x)) \leq |D\Lambda(x)| \leq \frac{\alpha_2}{|x|} \quad \text{if } x \in \mathbb{H}_{ \geq M},
\]
provided that the derivative exist, which is the case almost everywhere. Since
\[
D\Lambda(x) = DZ(\Lambda(x))^{-1}
\]
this yields that
\[
(3.23) \quad \alpha_1 \leq \frac{|Z(x)|}{|DZ(x)|} \leq \frac{|Z(x)|}{\ell(DZ(x))} \leq \alpha_2 \quad \text{if } Z(x) \in \mathbb{H}_{ \geq M}.
\]

We may assume here that \( \alpha_1 \leq 1 \leq \alpha_2 \).

The following lemma shows in particular that similar estimates hold for \( H \) and \( G \). Here \( M \) is the constant from (3.23).

**Lemma 3.4.** Let \( 0 < \sigma < \tau < 1 \). Then there exists \( C_1, R > 0 \) with the following properties:

Let \( F \in \{Z, H, G\} \), let \( x \in \mathbb{R}^3 \) and let \( 0 < r < \sigma |F(x)| \) such that \( F(x) \in \mathbb{H}_{ \geq R} \) and
\[
(3.24) \quad B(F(x), r) \subset \mathbb{H}_{ \geq M}.
\]

Let \( U \) be the component of \( F^{-1}(B(F(x), r)) \) containing \( x \). If \( F = G \), suppose in addition that \( |F(x)| \geq 3|x| \). Then
\[
(3.25) \quad \text{ess sup}_{u \in U} \max\{|F(u)|, |DF(u)|\} \leq C_1 \text{ ess inf}_{u \in U} \min\{|F(u)|, \ell(DF(u))\}
\]
and
\[
(3.26) \quad \text{diam } U \leq 2C_1 \frac{r}{|F(x)|}.
\]

The condition (3.24) is satisfied in particular if \( F(x)_3 \geq \tau |F(x)| \).
Proof. Suppose first that \( F = Z \). For \( u \in U \) we have \( Z(u) \in \mathbb{H}_{\geq M} \) by (3.24) and

\[
(1 - \sigma)|Z(x)| \leq |Z(x)| - r \leq |Z(u)| \leq |Z(x)| + r \leq (1 + \sigma)|Z(x)|.
\]

Together with (3.23) we find that

\[
\max\{|Z(u)|, |DZ(u)|\} \leq \frac{1}{\alpha_1}|Z(u)| \leq \frac{1 + \sigma}{\alpha_1}|Z(x)|
\]

while

\[
\min\{|Z(u)|, \ell(DZ(u))\} \geq \frac{1}{\alpha_2}|Z(u)| \geq \frac{1 - \sigma}{\alpha_2}|Z(x)|
\]

if \( DZ(u) \) exists. This yields (3.25) for \( F = Z \) with \( C_1 = (1 + \sigma)\alpha_2/((1 - \sigma)\alpha_1) \).

It follows from this and the chain rule that (3.25) also holds for \( F = H \), with a larger constant \( C_1 \), since \(|D\varphi(x)|/\ell(D\varphi(x))\) is constant for all \( x \) where \( \varphi \) is differentiable. In fact, we have \(|D\varphi(x)|/\ell(D\varphi(x)) = (2 + \sqrt{3})/(2 - \sqrt{3})\) for all such \( x \).

To deal with the case that \( F = G \) we note that by (3.1) we have \( G(u) = Z(u) + u \) for \( u \in U \), provided \( R \) is sufficiently large. Hence

\[
\frac{1}{2}|DZ(u)| \leq |DZ(u)| - 1 \leq |DG(u)| \leq |DZ(u)| + 1 \leq 2|DZ(u)|
\]

and

\[
\frac{1}{2}\ell(DZ(u)) \leq \ell(DZ(u)) - 1 \leq \ell(DG(u)) \leq \ell(D(u)) + 1 \leq \ell(DZ(u))
\]

for \( u \in U \). By hypothesis, \( |G(x)| \geq 3|x| \). We deduce that \( |Z(x)| = |G(x)| + x \geq 2|x| \)

and hence \( |Z(x)|/2 \leq |G(x)| \leq 3|Z(x)|/2 \). It follows that

\[
\frac{1 - \sigma}{2}|Z(x)| \leq (1 - \sigma)|G(x)| \leq |G(u)| \leq (1 + \sigma)|G(x)| \leq \frac{3}{2}(1 + \sigma)|Z(x)|
\]

for \( u \in U \). We deduce from the last three inequalities that (3.25) also holds for

\( F = G \) with some constant \( C_1 \).

To prove (3.26) let \( v \in \partial U \) with \( |v - x| = \max_{u \in \partial U} |u - x| \). Let \( \Gamma \) be the straight line segment connecting \( F(x) \) and \( F(v) \) and let \( \gamma \) be the curve in the closure of \( U \) such that \( \Gamma = F(\gamma) \). It follows from (3.25) that

\[
r = \text{length}(\Gamma) \geq \text{length}(\gamma) \min_{u \in U} \ell(DF(u)) \geq |v - x| \frac{|F(x)|}{C_1} \geq \frac{\text{diam} U}{2C_1} |F(x)|.
\]

Now (3.26) follows.

To prove that (3.24) is satisfied if \( F(x) \geq \tau|F(x)| \) we note that if this is the case and \( w \in B(F(x), r) \), then \( w \geq F(x) - r \geq (\tau - \sigma)|F(x)| \geq (\tau - \sigma)R \). Thus (3.24) follows for large \( R \). \( \square \)

We apply Lemma 3.4 to the function \( h_m \) defined by (3.15).

**Lemma 3.5.** Let \( 0 < \sigma < \tau < 1 \). Then there exists \( C_1, R > 0 \) with the following properties:

Let \( m \in \mathbb{N} \), let \( x \in \mathbb{R}^3 \) and let \( U \) be the component of \( h_m^{-1}(B(h_m(x), \sigma|h_m(x)|)) \) containing \( x \). Suppose that \( h_m(x) \geq \tau|h_m(x)| \) and that \( |h_{j-1}(x)| \geq 3|h_j(x)| \) and \( h_j(x) \in \mathbb{H}_{\geq R} \) for \( 1 \leq j \leq m \). Then

\[
\text{ess sup} \max_{u \in U} \left\{ \prod_{j=1}^{m} |h_j(u)|, |Dh_m(u)| \right\} \leq C_1 \min_{u \in U} \left\{ \prod_{j=1}^{m} |h_j(u)|, \ell(Dh_m(u)) \right\}.
\]
Proof. The chain rule implies that
\[
\prod_{j=1}^{m} \ell(DF_j(h_{j-1}(u))) \leq \ell(Dh_m(u)) \leq |Dh_m(u)| \leq \prod_{j=1}^{m} |DF_j(h_{j-1}(u))|,
\]
provided the derivatives exist. The conclusion follows if we show that
\[
\text{ess sup}_{w \in h_{j-1}(U)} \max\{|F_j(w)|, |DF_j(w)|\} \leq C_1 \text{ ess inf}_{w \in h_{j-1}(U)} \min\{|F_j(w)|, \ell(DF_j(w))\}
\]
for \(1 \leq j \leq m\). For \(j = m\) this follows from (3.25), with \(F = F_m\) and \(r = \sigma|F_m(x)|\). For \(1 \leq j \leq m - 1\) it would also follow from (3.25) if we show that there exists \(r \in (0, \sigma|h_j(x)|)\) such that
\[
h_j(U) \subset B(h_j(x), r) \subset \mathbb{H}_{2}\]
for \(1 \leq j \leq m\). By (3.26) we have \(h_{m-1}(U) \subset B(h_{m-1}(x), r)\) with \(r = \text{diam} h_{m-1}(U) \leq 2C_1\sigma\). Thus \(B(h_{m-1}(x), r) \subset \mathbb{H}_{2}\) if \(R\) is sufficiently large. Hence (3.28) holds for \(j = m - 1\). Inductively we see that (3.28) and hence (3.27) also hold for \(1 \leq j \leq m - 2\).

4. Proof of Theorem 1.1

4.1. The upper bound for the dimension. We may assume that \(\rho < 3\) since otherwise there is nothing to prove. Fix a (large) number \(R\) and let \(A_R(f)\) be the set of all \(x \in T \cap \mathbb{H}_{2}\) such that \(h_m(x) \in \mathbb{H}_{2}\), \(|h_m(x)| \geq \exp^m(R)\) and \(|h_m(x)| \geq 3|h_{m-1}(x)|\) for all \(m \in \mathbb{N}\), as well as \(h_m(x) \in T\) for all even \(m\). It follows from (3.19) and (3.20) that \(\mathcal{A}(f) = \bigcup_{m=0}^{\infty} f^{-m}(A_R(f))\). Since \(f\) is locally bi-Lipschitz, it suffices to show that \(\text{dim} \ A_R(f) \leq \rho\) for all \(L\).

We will do so using Lemma 2.2. So let \(x \in A_R(f)\). It follows from (3.3) that if \(m\) is even, then \(h_m(x)\) is contained in the domain \(\Omega\) defined by (3.2). This implies that
\[
|h_m(x)| \leq 2h_m(x)\]
Choose \(t > 0\) large enough and put
\[
K_m = B(h_m(x), \frac{1}{4} |h_m(x)|)
\]
and let \(U_m(x)\) be the component of \(h_m^{-1}(K_m)\) that contains \(x\). We now define \(r_m(x) = \text{dist}(x, \partial U_m(x))\) so that \(B(x, r_m(x)) \subset U_m(x)\). Let \(y_m \in \partial U_m(x)\) with \(|y_m - x| = r_m(x)\) and let \(\gamma\) be the straight line segment connecting \(x\) and \(y_m\). Then \(h_m(\gamma)\) connects \(h_m(x)\) with \(\partial K_m\). Lemma 3.5 yields that
\[
\frac{1}{4} |h_m(x)| \leq \text{length}(h_m(\gamma)) \leq \text{length}(\gamma) \text{ ess sup}_{y \in U_m(x)} |Dh_m(y)| \leq r_m(x)C^m_1 \prod_{j=1}^{m} |h_j(x)|.
\]
Thus
\[
r_m(x) \geq \frac{C^m_1}{4 \prod_{j=1}^{m-1} |h_j(x)|}.
\]
If \(s = (s_1, s_2) \in S\) and if \(T(s)\) intersects \(K_m\), then
\[
|(h_m(x)_1, h_m(x)_2) - (s_1, s_2)| \leq \frac{1}{4} |h_m(x)| + \sqrt{2} \leq |h_m(x)|,
\]
provided $R$ is chosen sufficiently large. Suppose that $\rho > 1$. Then $0 < \beta < 1$ by (3.5) and it follows from (3.6) and (4.2) that
\begin{equation}
\text{card}\{s \in S: T(s) \cap K_m \neq \emptyset\} \leq \alpha_0 |h_m(x)|^{2-2\beta}.
\end{equation}
But if $\rho = 1$ so that $\beta = 1$, then $S = \{(0,0)\}$ and hence card $S = 1$. Thus (4.3) holds trivially also in this case, assuming that $\alpha_0 \geq 1$.

For each $s$ we can cover $T(s) \cap K_m$ by $M$ cubes of sidelength 2, where
\[
M \leq \frac{1}{2}|h_m(x)| + 1 \leq |h_m(x)|.
\]
It follows from the last two inequalities that $T \cap K_m$ can be covered by $N_m(x)$ cubes of sidelength 2, where
\begin{equation}
N_m(x) \leq |h_m(x)|^{3-2\beta} = |h_m(x)|^{\rho}.
\end{equation}
Let $W$ be the image of such a cube under the inverse $\varphi: K_m \to U_m(x)$ of the map $h_m: U_m(x) \to K_m$. As these cubes have diameter $2\sqrt{3}$, Lemma 3.5 yields that
\[
\text{diam } W \leq d_m(x)
\]
where
\begin{equation}
d_m(x) := 2\sqrt{3} \text{ess sup}_{y \in K_m} |D\varphi(y)| = \frac{2\sqrt{3}}{\text{ess inf}_{z \in U_m(x)} \ell(Dh_m(z))} \leq \frac{2\sqrt{3}C_m^m}{\prod_{j=1}^m |h_j(x)|}.
\end{equation}
To summarize we see that $B(x, r_m(x)) \cap A_R(f)$ can be covered by $N_m(x)$ sets of diameter at most $d_m(x)$, where $r_m(x)$, $N_m(x)$ and $d_m(x)$ satisfy (4.1), (4.4) and (4.5), respectively. In order to apply Lemma 2.2, let $\epsilon > 0$. We deduce that
\[
\frac{N_m(x)d_m(x)^{\rho+\epsilon}}{r_m(x)^3} \leq |h_m(x)|^\rho \left(\frac{2\sqrt{3}C_m^m}{\prod_{j=1}^m |h_j(x)|}\right)^{\rho+\epsilon} \left(\frac{4\prod_{j=1}^{m-1} |h_j(x)|}{C_1^m}\right)^{3-\rho-\epsilon}
\]
\[
= 4^3 \cdot \left(2\sqrt{3}\right)^{\rho+\epsilon} C_1^{3+\rho+\epsilon}|h_m(x)|^{-\epsilon} \left(\prod_{j=1}^{m-1} |h_j(x)|\right)^{3-\rho-\epsilon}.
\]
It follows from (3.21) and (3.22) that the right hand side is less than 1 for large $m$. Lemma 2.2 now yields that $\dim A_R(f) \leq \rho + \epsilon$. Since $\epsilon$ can be taken arbitrarily small, we conclude that $\dim A_R(f) \leq \rho$ and hence that $\dim A(f) \leq \rho$.

4.2. Nested sets. To estimate $\dim A(f)$ from below we will construct a subset of $A(f)$ as follows. For $m \in \mathbb{N}$ we construct a collection $E_m$ of disjoint compact subsets of $\mathbb{R}^3$ with the property that every element of $E_{m+1}$ is contained in a unique element of $E_m$. Conversely, every element of $E_m$ contains at least one element of $E_{m+1}$. Put
\[
E_m = \bigcup_{V \in E_m} V \quad \text{and} \quad E = \bigcap_{m=1}^\infty E_m.
\]
The construction will be made in such a way that $E \subset A(f)$.

McMullen [23] used Lemma 2.1 to give a lower bound for the dimension of a set $E$ constructed this way in terms of $\sup_{V \in E_m} \text{diam } V$ and $\inf_{V \in E_m} \text{dens}(E_{m+1}, V)$.
Adapting some ideas from [11], we will instead work with bounds for \( \text{diam} V \) and \( \text{dens}(E_{m+1}, V) \) which depend not only on \( m \), but also on \( V \).

We will choose the sets \( \mathcal{E}_m \) such that \( h_m \) maps the elements of \( \mathcal{E}_m \) bijectively onto a closed ball of the form \( K(t) = \overline{B}((0, 0, t/2)) \) for some large \( t \), say \( t \geq R \). Various conditions imposed later will require that \( R \) is sufficiently large. We will also consider the box

\[
Q(t) = \left\{ x \in K(t) : \frac{t}{6} \leq x_1 \leq \frac{t}{4}, \frac{t}{6} \leq x_2 \leq \frac{t}{4}, |x_3 - t| \leq \frac{t}{4} \right\}.
\]

Note that \( Q(t) \subset K(t) \). The advantage of considering \( Q(t) \) will be that points in \( Q(t) \) have a definite distance from \( \partial K(t) \) and that two beams \( T(s) \) and \( T(s') \) intersecting \( Q(t) \) are a certain distance apart.

Put \( \eta(r) = \log \log r \) and, for \( 2 \leq k \in \mathbb{N} \) and \( s \in S \),

\[
p(s, k) = (2s_1, 2s_2, k\eta(k)).
\]

Let \( P_H(s, k) \) be the component of \( H^{-1}(K(H(p(s, k)))_3) \) containing \( p(s, k) \). Here \( H(p(s, k))_3 \) denotes the third component of \( H(p(s, k)) \). Note that since \( \varphi(p(s, k)) = (2s_1, 2s_2, k\eta(k) - 2|s_1| - 2|s_2|) \) we have \( H(p(s, k)) = (0, 0, \exp(k\eta(k) - 2|s_1| - 2|s_2|)) \) so that \( H(p(s, k))_3 = \exp(k\eta(k) - 2|s_1| - 2|s_2|) \). Note that \( P_H(s, k) \cap P_H(s', k') = \emptyset \) if \( (s, k) \neq (s', k') \), assuming that \( k \) and \( k' \) are large enough.

For large \( t \) we consider the set \( U_H(t) \) of all \( P_H(s, k) \) which are contained in \( Q(t) \). Note that if \( P_H(s, k) \in U_H(t) \), then, in particular, \( p(s, k) \in Q(t) \). This yields that

\[
2|s_i| \leq t/4 \quad \text{for} \quad i \in \{1, 2\} \quad \text{while} \quad k\eta(k) \geq 3t/4.
\]

We deduce that

\[
\eta(r) = \log \log r \quad \text{and, for} \quad 2 \leq k \in \mathbb{N} \quad \text{and} \quad s \in S,
\]

\[
p(s, k) = (2s_1, 2s_2, k\eta(k)).
\]

Let \( P_H(s, k) \) be the component of \( H^{-1}(K(H(p(s, k)))_3) \) containing \( p(s, k) \). Here \( H(p(s, k))_3 \) denotes the third component of \( H(p(s, k)) \). Note that since \( \varphi(p(s, k)) = (2s_1, 2s_2, k\eta(k) - 2|s_1| - 2|s_2|) \) we have \( H(p(s, k)) = (0, 0, \exp(k\eta(k) - 2|s_1| - 2|s_2|)) \) so that \( H(p(s, k))_3 = \exp(k\eta(k) - 2|s_1| - 2|s_2|) \). Note that \( P_H(s, k) \cap P_H(s', k') = \emptyset \) if \( (s, k) \neq (s', k') \), assuming that \( k \) and \( k' \) are large enough.

For large \( t \) we consider the set \( U_H(t) \) of all \( P_H(s, k) \) which are contained in \( Q(t) \). Note that if \( P_H(s, k) \in U_H(t) \), then, in particular, \( p(s, k) \in Q(t) \). This yields that

\[
2|s_i| \leq t/4 \quad \text{for} \quad i \in \{1, 2\} \quad \text{while} \quad k\eta(k) \geq 3t/4.
\]

We deduce that

\[
\eta(r) = \log \log r \quad \text{and, for} \quad 2 \leq k \in \mathbb{N} \quad \text{and} \quad s \in S,
\]

\[
p(s, k) = (2s_1, 2s_2, k\eta(k)).
\]

That is, \( \eta(r) = \log \log r \) and, for \( 2 \leq k \in \mathbb{N} \) and \( s \in S \),

\[
p(s, k) = (2s_1, 2s_2, k\eta(k)).
\]

Let \( P_H(s, k) \) be the component of \( H^{-1}(K(H(p(s, k)))_3) \) containing \( p(s, k) \). Here \( H(p(s, k))_3 \) denotes the third component of \( H(p(s, k)) \). Note that since \( \varphi(p(s, k)) = (2s_1, 2s_2, k\eta(k) - 2|s_1| - 2|s_2|) \) we have \( H(p(s, k)) = (0, 0, \exp(k\eta(k) - 2|s_1| - 2|s_2|)) \) so that \( H(p(s, k))_3 = \exp(k\eta(k) - 2|s_1| - 2|s_2|) \). Note that \( P_H(s, k) \cap P_H(s', k') = \emptyset \) if \( (s, k) \neq (s', k') \), assuming that \( k \) and \( k' \) are large enough.

For large \( t \) we consider the set \( U_H(t) \) of all \( P_H(s, k) \) which are contained in \( Q(t) \). Note that if \( P_H(s, k) \in U_H(t) \), then, in particular, \( p(s, k) \in Q(t) \). This yields that

\[
2|s_i| \leq t/4 \quad \text{for} \quad i \in \{1, 2\} \quad \text{while} \quad k\eta(k) \geq 3t/4.
\]

We deduce that

\[
\eta(r) = \log \log r \quad \text{and, for} \quad 2 \leq k \in \mathbb{N} \quad \text{and} \quad s \in S,
\]

\[
p(s, k) = (2s_1, 2s_2, k\eta(k)).
\]

That is, \( \eta(r) = \log \log r \) and, for \( 2 \leq k \in \mathbb{N} \) and \( s \in S \),

\[
p(s, k) = (2s_1, 2s_2, k\eta(k)).
\]
Suppose now that \( m \in \mathbb{N} \) and that \( \mathcal{E}_m \) has been defined. Let \( V \in \mathcal{E}_m \). Then we have \( h_m(V) = K(t_V) \) for some \( t_V > R \) and \( h_m : V \to K(t_V) \) is bijective. Let \( \varphi : K(t_V) \to V \) be the inverse of this map. We put

\[
\mathcal{E}_{m+1}(V) = \{ \varphi(P) : P \in U_{F_{m+1}}(t_V) \} = \begin{cases} \{ \varphi(P) : P \in U_{G}(t_V) \} & \text{if } m \text{ is odd,} \\
\{ \varphi(P) : P \in U_{H}(t_V) \} & \text{if } m \text{ is even,}
\end{cases}
\]

and

\[
\mathcal{E}_{m+1} = \bigcup_{V \in \mathcal{E}_m} \mathcal{E}_{m+1}(V).
\]

Following [23] we construct a probability measure \( \mu \) supported on \( E \). In order to do so, we define a sequence \((\mu_m)\) of probability measures inductively. Here \( \mu_m \) is supported on \( \mathcal{E}_m \), and the restriction of \( \mu_m \) to an element of \( \mathcal{E}_m \) is a rescaled Lebesgue measure.

First we define \( \mu_0 \) as the Lebesgue measure on \( K_0 \), rescaled so that \( \mu_0(K_0) = 1 \). Thus \( \mu_0(A) = \text{meas}(A \cap K_0)/\text{meas }K_0 \). Let \( m \geq 0 \) and suppose that the measure \( \mu_m \) has been defined. To define the measure \( \mu_{m+1} \) it suffices to specify \( \mu_{m+1}(W) \) for \( W \in \mathcal{E}_{m+1} \). For such \( W \) there exists \( V \in \mathcal{E}_m \) with \( W \subset V \). We put

\[
\mu_{m+1}(W) = \frac{\mu_m(W)}{\text{dens}(E_{m+1}, V)}.
\]

Note that for \( V \in \mathcal{E}_m \) we have

\[
\mu_{m+1}(V) = \sum_{W \in \mathcal{E}_{m+1}(V)} \mu_{m+1}(W) = \frac{1}{\text{dens}(E_{m+1}, V)} \sum_{W \in \mathcal{E}_{m+1}(V)} \mu_m(W)
= \frac{1}{\text{dens}(E_{m+1}, V)} \mu_m(V \cap E_{m+1}) = \mu_m(V).
\]

We conclude that

\[
\mu_n(V) = \mu_m(V) \quad \text{for } n \geq m, \ V \in \mathcal{E}_m.
\]

Let \( \mu \) be a weak limit of the sequence \((\mu_m)\). Then \( \mu \) is a probability measure supported on \( E \) such that

\[
\mu(V) = \mu_m(V) \quad \text{for } V \in \mathcal{E}_m.
\]

4.3. **Estimates for the sets in \( \mathcal{E}_m \).** As already mentioned, we will estimate \( \text{diam } V \) and \( \text{dens}(E_{m+1}, V) \) for \( V \in \mathcal{E}_m \).

**Lemma 4.1.** There exist \( C_2, C_3, R > 0 \) such that if \( F \in \{G, H\} \), \( t \geq R \) and \( P \in U_H(t) \), then

\[(4.9)\quad \text{diam } P \leq C_2\]

and

\[(4.10)\quad \text{meas } P \geq C_3.\]

**Proof.** Let \( P = P_F(s, k) \), with \( s \in S \) and \( k \in \mathbb{N} \). Put \( q = F(p(s, k)) \) and \( a = (0, 0, q) \). Then \( F(P) = B(a, q/2) \).
If $F = H$, then $F(p(s, k)) = a$ and thus $|F(p(s, k))| = q$. If $F = G$, then $F(p(s, k)) = (2s_1, 2s_2, q)$. Using (4.6) we can deduce that

$$q = k\eta(k) + \exp(k\eta(k)) \geq 20(|s_1| + |s_2|)$$

if $R$ and hence $k$ are sufficiently large. Hence

$$\frac{9q}{10} \leq q - 2(|s_1| + 2|s_2|) \leq |F(p(s, k))| \leq q + 2(|s_1| + |s_2|) \leq \frac{11q}{10}$$

as well as

$$|F(p(s, k)) - a| = |(2s_1, 2s_2, 0)| \leq \frac{q}{10}$$

for large $k$. It follows from the last two inequalities that

$$F(P) = B\left(a, \frac{q}{2}\right) \subset B\left(F(p(s, k)), \frac{3q}{5}\right) \subset B\left(F(p(s, k)), \frac{2}{3}|F(p(s, k))|\right).$$

Since $F(p(s, k))_3 = q \geq 10|F(p(s, k))|/11$ by (4.11) we may apply Lemma 3.4 with $\tau = 10/11$ and $\sigma = 2/3$. We deduce from (3.26) that (4.9) follows with $C_2 = 4C_1/3$.

It also follows from (4.12) that

$$\text{dist}(F(p(s, k)), \partial F(P)) \geq \frac{2q}{5}.$$

Let $\gamma$ be the line segment connecting $p(s, k)$ with the closest point on $\partial P$. Thus $\text{length}(\gamma) = \text{dist}(p(s, k), \partial P)$. Recalling that $F$ is injective on $P$ we find that $F \circ \gamma$ is a curve connecting $F(p(s, k))$ with $\partial F(P)$. Together with Lemma 3.5 it follows that

$$\frac{2q}{5} \leq \text{length}(F \circ \gamma) \leq \text{ess sup}_{x \in P} |DF(x)| \cdot \text{length}(\gamma)$$

$$\leq C_1|F(p(s, k))| \text{dist}(p(s, k), \partial P).$$

Since $|H(p(s, k))| = q$ we see that the estimate $|F(p(s, k))| \leq 11q/10$ given in (4.11) is valid not only for $F = G$ but trivially also holds for $F = H$. Thus $\text{dist}(p(s, k), \partial P) \geq 4/(11C_1)$. This yields (4.10) with $C_3 = 4\pi/(3 \cdot (11C_1)^3)$. \hfill $\Box$

Remark 4.2. It follows in particular from (4.12) that $F(p(s, k)) \in K(F(p(s, k))_3)$ if $k$ is sufficiently large. Of course this is clear if $F = H$, but (4.12) shows that it also holds for $F = G$ if $R$ is large enough.

It follows from (4.7), (4.8) and (4.9) that if $x \in P_F(s, k) \in U_F(t)$ for some $t \geq R$ and $F \in \{G, H\}$, then

$$|F(x)| \geq \frac{1}{2} |F(p(s, k))| \geq \exp\left(\frac{1}{5} |p(s, k)|\right) \geq \exp\left(\frac{1}{5} |x| - C_2\right) \geq 6 \exp\left(\frac{1}{6} |x|\right),$$

provided $R$ and hence $k$ are sufficiently large. It follows that

$$|h_m(x)| \geq 6 \exp^m\left(\frac{1}{6} |x|\right)$$

for $x \in E$, if $R$ is chosen sufficiently large. Thus $E \subset \mathcal{A}(f)$ by (3.20).
Lemma 4.3. There exists $C_4, C_5, C_6, R > 0$ such that if $F \in \{G, H\}$ and $t \geq R$, then
\begin{align}
\text{card} U_F(t) &\geq C_4 \frac{t^\rho}{\eta(t)} \\
\text{dens}(U_F(t), K(t)) &\geq C_5 \frac{t^{\rho-3}}{\eta(t)}.
\end{align}

If $x \in Q(t)$, then
\begin{align}
\text{card}\{P \in U_F(t) : P \cap B(x, \tau) \neq \emptyset\} &\leq \begin{cases} C_6 (\tau + 1) & \text{if } \tau \leq t^\beta, \\
C_6 \tau^\rho & \text{if } \tau > t^\beta, \end{cases}
\end{align}

with $\beta$ defined by (3.5). In particular
\begin{align}
\text{card} U_F(t) &\leq C_6 (2t)^\rho.
\end{align}

Moreover, if $P, P' \in U_F(t)$, then
\begin{align}
\text{dist}(P, P') &\geq \frac{1}{4} \frac{\sqrt{2t}}{\eta(t)}.
\end{align}

Proof. Recall that $B_2((x_1, x_2), r)$ denotes the planar disk centred at $(x_1, x_2)$ and of radius $r$. Assuming that $R$ is large, it follows from (4.9) that if
\begin{align}
s \in B_2 \left( \frac{1}{5}(t, t), \frac{t}{31} \right) \subset B_2 \left( \frac{1}{5}(t, t), \frac{t}{30} - C_2 \right)
\end{align}

and
\begin{align}
\frac{3t}{4} + C_2 \leq k \eta(k) \leq \frac{5t}{4} - C_2,
\end{align}

then $P_F(s, k) \subset Q(t)$ and thus $P_F(s, k) \in U_F(t)$. By (3.7), the cardinality of the set of all $s \in S$ satisfying (4.19) is at least
\begin{align}
\frac{1}{64} \left( \frac{\sqrt{2t}}{5} \right)^{-2\beta} \frac{t^2}{31^2} = \frac{2^{-\beta}}{64 \cdot 5^{-2\beta} \cdot 31^2} \cdot t^{2-2\beta},
\end{align}

provided $t$ is sufficiently large.

To estimate the cardinality of the set of all $k$ satisfying (4.20) we claim that (4.20) is satisfied if
\begin{align}
\left( \frac{3t}{4} + C_2 \right) \frac{1}{\eta(t) - 1} \leq k \leq \left( \frac{5t}{4} - C_2 \right) \frac{1}{\eta(t)}.
\end{align}

To see this note that (4.22) implies that $\sqrt{t} \leq k \leq t$ and thus $\eta(t) - 1 \leq \eta(k) \leq \eta(t)$. Using (4.9) we can now deduce (4.20) from (4.22).

The cardinality of the set of all $k \in \mathbb{N}$ which satisfy (4.22) and hence also (4.20) is at least
\begin{align}
\frac{1}{3} \frac{t}{\eta(t)},
\end{align}
provided \( t \) is sufficiently large. Combining the bounds (4.21) and (4.23) we find that

\[
\text{card } \mathcal{U}_F(t) \geq \frac{2^{-\beta} \cdot t^{3-2\beta}}{3 \cdot 64 \cdot 5^{-2\beta} \cdot 31^2 \eta(t)}
\]

for large \( t \). Now (4.14) follows from (3.5).

Combining (4.14) with (4.10) we obtain (4.15) with \( C_3 = C_3 C_4/(6\pi) \).

To prove (4.16) let \( x \in Q(t) \). Arguments similar to the ones used to obtain (4.19) and (4.20) show that if \( P_F(s, k) \in \mathcal{U}_F(t) \) with \( P_F(s, k) \cap B(x, \tau) \neq \emptyset \), then

\[
\text{card } \mathcal{U}_F(t) \geq \frac{2^{-\beta} \cdot t^{3-2\beta}}{3 \cdot 64 \cdot 5^{-2\beta} \cdot 31^2 \eta(t)}
\]

\[
\text{and } |k\eta(k) - x_3| \leq \tau + C_2.
\]

By the definition of \( Q(t) \) we have

\[
\frac{t}{6} \leq |(x_1, x_2)| \leq |x| \leq 2t.
\]

Suppose first that \( t^\beta \leq \tau \leq t/24 \). Then, provided \( R \) is large enough, \( \tau + C_2 \leq 2\tau \) and hence \( B_2((x_1, x_2), \tau + C_2) \subset B_2((x_1, x_2), 2\tau) \). Since \( 2\tau \leq t/12 \leq 2|\langle x_1, x_2 \rangle| \) we can use (3.7) and (3.9) to bound the cardinality of \( S \cap B_2((x_1, x_2), 2\tau) \). This is bigger than the cardinality \( N_1 \) of the set of all \( s \in S \) satisfying (4.24). Thus

\[
N_1 \leq 324 |\langle x_1, x_2 \rangle|^{-2\beta} (2\tau)^2 \leq 324 \cdot 6^{2\beta} \cdot 4t^{-2\beta} \tau^2.
\]

Suppose next that \( \tau \geq t/24 \). Then \( N_1 \leq o_0 (2\tau)^{2-2\beta} \leq o_0 2^{2-2\beta} 2^{2\beta} t^{-2\beta} \tau^2 \) by (3.6). Suppose finally that \( \tau \leq t^\beta \). Then \( \tau + C_2 \leq t^\beta + C_2 \leq 6^\beta |x|^\beta + C_2 \leq 8|x|^{\beta} \) for large \( R \) by (4.25). Now (3.8) yields that \( N_1 \leq 324 \). Altogether it follows that there exists \( C_6 > 0 \) such that

\[
4N_1 \leq \begin{cases} C_6 & \text{if } \tau \leq t^\beta, \\ C_6 t^{-2\beta} \tau^2 & \text{if } t^\beta \leq \tau. \end{cases}
\]

For \( s \in S \) we now consider the cardinality \( N_2(s) \) of the set of all \( k \) such that \( P_F(s, k) \) intersects \( B(x, \tau) \). Since

\[
(k + 1)\eta(k + 1) - k\eta(k) \geq \eta(k) \geq \eta(t) - 1 \geq \frac{1}{2} \eta(t)
\]

if \( P_F(s, k) \in \mathcal{U}_F(t) \) we deduce from (4.9) that

\[
N_2(s) \leq N_2 := \frac{4(\tau + C_2)}{\eta(t)} + 1.
\]

Combining the above estimates we obtain

\[
\text{card } \{ P \in \mathcal{U}_F(t) : P \cap B(x, \tau) \neq \emptyset \} \leq N_1 N_2.
\]

Suppose that \( \tau \leq t^\beta \). Noting that (4.28) yields that \( N_2 \leq \tau + 2 \) we obtain (4.16) from (4.29) and (4.26).
Suppose next that $\tau > t^3$. Since $Q(t) \subset K(t) \subset B(x, 2t)$ we may assume that $\tau \leq 2t$. By (4.28) we have $N_2 \leq \tau$ for large $t$. Together with (4.29) and (4.26) this yields that
\[
\text{card}\{P \in \mathcal{U}_c(t) : P \cap B(x, \tau) \neq \emptyset\} \leq \frac{1}{4} C_6 t^{-23/2 \tau^3} = \frac{1}{4} C_6 \left(\frac{\tau}{t}\right)^{23/2} \tau^3 \leq C_6 t^\rho.
\]
So we obtain (4.16) also in this case. Since, as already mentioned, $Q(t) \subset B(x, 2t)$, this also yields (4.17).

Finally, (4.18) follows from (4.9) and (4.27), provided $R$ is large enough. \hfill \Box

Let $x \in E$. Then $x \in E_m$ for all $m \in \mathbb{N}$ and thus there exists $V_m(x) \in \mathcal{E}_m$ such that $x \in V_m(x)$. Recall that by the definition of $\mathcal{E}_m$ we have
\[
(4.30) \quad h_m(V_m(x)) = K(t_m),
\]
for some $t_m > R$ satisfying $t_m \to \infty$ as $m \to \infty$. It is easy to see that $t_m/2 \leq |h_m(x)| \leq 2t_m$.

**Lemma 4.4.** There exists $C_7 > 0$ such that
\[
dens(E_{m+1}, V_m(x)) \geq C_7 \frac{|h_m(x)|^{p-3}}{\eta(|h_m(x)|)}.
\]

*Proof.* With $h_m(V_m(x)) = K(t_m)$ it follows from Lemma 2.3, Lemma 3.4 and (4.15) that
\[
dens(E_{m+1}, V_m(x)) \geq C_1^{-3m} \dens(U_{F_{m+1}}(t_m), K(t_m)) \geq C_1^{-3m} C_5 \frac{t_m^{p-3}}{\eta(t_m)}.
\]
The conclusion now follows since $t_m \leq 2|h_m(x)|$ and $\eta(2t_m) \leq 2\eta(t_m)$. \hfill \Box

**Lemma 4.5.** There exists $C_8 > 0$ such that if $W \in \mathcal{E}_{m+1}(V_m(x))$, then
\[
(4.31) \quad C_8^{-m} \leq \text{diam } W \cdot \prod_{j=1}^m |h_j(x)| \leq C_8^m.
\]

In particular, it follows from Lemma 4.5 that
\[
(4.32) \quad \frac{C_8^{-m}}{\prod_{j=1}^m |h_j(x)|} \leq \text{diam } V_m(x) \leq \frac{C_8^m}{\prod_{j=1}^m |h_j(x)|}.
\]

*Proof of Lemma 4.5.* We have $h_m(V_m(x)) = K(t_m)$. Let $\varphi : K(t_m) \to V_m(x)$ be the inverse function of $h_m : V_m(x) \to K(t_m)$. Each $W \in \mathcal{E}_{m+1}(V_m(x))$ is of the form $W = \varphi(P)$ for some $P \in \mathcal{U}_{F_{m+1}}(t_m)$. It follows that
\[
\text{diam } W \leq \text{ess sup}_{y \in K(t_m)} |D\varphi(y)| \cdot \text{diam } P = \frac{\text{diam } P}{\text{ess inf}_{w \in V_m(x)} \ell(Dh_m(w))}.
\]

Lemma 3.5 and (4.9) now yield that
\[
\text{diam } W \leq \frac{C_2 C_4^m}{\prod_{j=1}^m |h_j(x)|},
\]
from which the right inequality in (4.31) follows for a suitable $C_8 > 0$. 

\[
\text{diam } W \leq \frac{C_2 C_4^m}{\prod_{j=1}^m |h_j(x)|},
\]
from which the right inequality in (4.31) follows for a suitable $C_8 > 0$. 

To prove the left inequality we proceed analogously, considering the measure of \( W \) instead of its diameter. We have
\[
\text{meas } W \geq \underset{y \in K(t_m)}{\text{ess inf}} \ell(D\varphi(y))^3. \quad \text{meas } P = \frac{\text{meas } P}{\underset{w \in V_m(x)}{\text{ess sup}} |Dh_m(w)|^3}.
\]

Lemma 3.5 and (4.10) now imply that
\[
\text{meas } W \geq C_3 C_1^{3m} \prod_{j=1}^m |h_j(x)|^3.
\]

Adjusting the value of \( C_8 \) if necessary we obtain the left inequality in (4.31). \( \square \)

Lemma 4.6. Let \( W, W' \in \mathcal{E}_{m+1}(V_m(x)) \). Then
\[
\text{dist}(W, W') \geq \text{diam } W.
\]

Proof. Let \( P = h_m(W) \) and \( P' = h_m(W') \). With \( h_m(V_m(x)) = K(t_m) \) we then have \( P, P' \in \mathcal{U}_{F_{m+1}}(t_m) \). Let \( \gamma \) be a straight line segment connecting \( W \) and \( W' \) such that \( \text{length}(\gamma) = \text{dist}(W, W') \). First we assume that \( \gamma \subset V_m(x) \). It then follows from Lemma 4.5 and the arguments in its proof that
\[
\text{dist}(P, P') \leq \text{length}(h_m(\gamma)) \leq \text{dist}(W, W') \text{ ess sup}_{w \in V_m(x)} |Dh_m(w)| \leq \text{dist}(W, W') C_1 C_8 \text{diam } W.
\]

Using (4.18) we deduce that
\[
\frac{\text{dist}(W, W')}{\text{diam } W} \geq C_1^{-m} C_8^{-m} \frac{1}{2} \eta \left( \frac{1}{2} h_m(x) \right).
\]

It now follows from (4.13) that if \( R \) is chosen sufficiently large, then the right hand side of (4.34) is at least 1 for all \( m \).

Suppose now that \( \gamma \not\subset V_m(x) \). Then \( \gamma \) contains a subsegment \( \gamma' \) which connects \( W \) with \( \partial V_m(x) \). Thus \( h_m(\gamma') \) connects \( P \) with \( \partial K(t_m) \). Since \( P \subset Q(t_m) \), we have \( \text{length}(h_m(\gamma')) \geq (2 - \sqrt{3}) t_m / 4 \). Estimating the length of \( h_m(\gamma') \) as in (4.33) we obtain
\[
\frac{\text{dist}(W, W')}{\text{diam } W} \geq C_1^{-m} C_8^{-m} \frac{(2 - \sqrt{3}) t_m}{4}
\]

instead of (4.34). Again the conclusion follows if \( R \) is chosen sufficiently large. \( \square \)

4.4. Proof of the lower bound. We may assume that \( \rho > 1 \). In order to apply Lemma 2.1, let \( x \in E \) and \( r > 0 \). For \( m \in \mathbb{N} \) we put
\[
d_m(x) = \text{diam } V_m(x) \quad \text{and} \quad \Delta_m(x) = \text{dens}(\mathcal{E}_{m+1}, V_m(x)).
\]

We choose \( m \in \mathbb{N} \) such that
\[
d_{m+1}(x) \leq r < d_m(x).
\]

It follows from Lemma 4.6 that
\[
\mu(B(x, r)) = \mu(B(x, r) \cap V_m(x)).
\]
Let
\[ \mathcal{Y}_m = \{ W \in \mathcal{E}_{m+1}(V_m(x)) : W \cap B(x, r) \neq \emptyset \}. \]

Then
\[ \mu(B(x, r)) \leq \sum_{W \in \mathcal{Y}_m} \mu(W) = \sum_{W \in \mathcal{Y}_m} \mu_{m+1}(W) = \sum_{W \in \mathcal{Y}_m} \frac{1}{\prod_{j=0}^{m} \Delta_j(x) \meas K_0} \meas W. \]

Using Lemma 4.4 and Lemma 4.5 we find that
\[
\mu(B(x, r)) \leq \frac{1}{\meas K_0^m} \card \mathcal{Y}_m \cdot \prod_{j=1}^{m} C_7^{-j} \frac{\eta(|h_j(x)|)}{|h_j(x)|^{r-3}} \cdot \left( 2 \frac{C_8^m}{\prod_{j=1}^{m} |h_j(x)|} \right)^3
\]
\[
= \frac{8C_7^{-m(m+1)/2}C_8^m}{\meas K_0} \card \mathcal{Y}_m \cdot \prod_{j=1}^{m} \frac{\eta(|h_j(x)|)}{|h_j(x)|^r}. \tag{4.35}
\]

To estimate \( \card \mathcal{Y}_m \) we note that
\[ h_m(B(x, r)) \subset B \left( h_m(x), r \esssup_{W \in B(x, r)} |Dh_m(w)| \right). \]

If \( B(x, r) \subset V_m(x) \) or, equivalently, \( h_m(B(x, r)) \subset K(t_m) \), with \( t_m \) defined by (4.30), then, by Lemma 3.5,
\[
r \esssup_{W \in B(x, r)} |Dh_m(w)| \leq \tau_m := rC_1^m \prod_{j=1}^{m} |h_j(x)|. \tag{4.36}
\]

Since \( h_m(x) \in Q(t_m), \) this is the case in particular if \( \tau_m \leq (2 - \sqrt{3})t_m/4. \) It follows that
\[ \card \mathcal{Y}_m \leq \card \{ P \in \mathcal{U}_H(t_m) : P \cap B(h_m(x), \tau_m) \neq \emptyset \} \quad \text{if} \quad \tau_m \leq \frac{(2 - \sqrt{3})t_m}{4}. \]

Let \( 0 < \varepsilon < r - 1. \) If \( \tau_m \leq 1, \) we deduce from (4.16) that \( \card \mathcal{Y}_m \leq 2C_8. \) Since \( |h_j(x)| \to \infty \) we have that \( \eta(|h_j(x)|) \leq |h_j(x)|^{\varepsilon/2}, \) for large \( j. \) Together with (4.35), (4.32) and (3.22) we find that
\[ \mu(B(x, r)) \leq \frac{16C_6C_7^{-m(m+1)/2}C_8^m}{\meas K_0} \prod_{j=1}^{m} \frac{\eta(|h_j(x)|)}{|h_j(x)|^r} \]
\[ \leq C_8^{-m(\rho-\varepsilon)} \prod_{j=1}^{m} \frac{1}{|h_j(x)|^{\rho-\varepsilon}} \leq d_{m+1}(x)^{\rho-\varepsilon} \leq r^{\rho-\varepsilon} \]
for large \( m. \)

If \( 1 < \tau_m \leq t_m^2, \) then (4.16) yields that \( \card \mathcal{Y}_m \leq 2C_6 \tau_m. \) Together with (4.35), (4.36), (4.32) and (3.22) we deduce that
\[ \mu(B(x, r)) \leq \frac{16C_6C_7^{-m(m+1)/2}C_8^mC_1^m}{\meas K_0} \prod_{j=1}^{m} \frac{\eta(|h_j(x)|)}{|h_j(x)|^{r-1}} \]
\[ \leq C_8^{-m(\rho-1-\varepsilon)} r \prod_{j=1}^{m} \frac{1}{|h_j(x)|^{\rho-1-\varepsilon}} \leq r d_{m+1}(x)^{\rho-1-\varepsilon} \leq r^{\rho-\varepsilon} \]
for large $m$.

Suppose next that $t_m^2 < \tau_m \leq (2 - \sqrt{3})t_m/4$. It then follows from (4.16) that $\card\mathcal{Y}_m \leq C_6^\rho \tau_m$. Together with (4.35), (4.36), (4.32) and (3.22) this yields that

$$
\mu(B(x, r)) \leq \frac{8C_6C_m^{(m+1)/2}C_8^{3m}C_{11}^{m_1}}{\operatorname{meas} K_0} r^\rho \prod_{j=1}^m \eta(|h_j(x)|)
$$

(4.37)

for large $m$.

Suppose finally that $\tau_m > (2 - \sqrt{3})t_m/4 \geq t_m/15$. It follows from (4.17) that $\card\mathcal{Y}_m \leq \card\mathcal{U}_H(t_m) \leq C_6(2t_m)^\rho \leq C_630^\rho r_0^\rho$. As in (4.37) we find that $\mu(B(x, r)) \leq r^{\rho - \epsilon}$ for large $m$.

Thus in all cases we have $\mu(B(x, r)) \leq r^{\rho - \epsilon}$ if $m$ is sufficiently large, and hence if $r$ is sufficiently small. Lemma 2.1 yields that $\dim E \geq \rho - \epsilon$. Since $E \subset \mathcal{A}(f)$ and since $\epsilon > 0$ can be chosen arbitrarily small, we conclude that $\dim \mathcal{A}(f) \geq \rho$. This completes the proof of the lower bound for $\dim \mathcal{A}(f)$ and hence the proof of Theorem 1.1. \qed

Remark 4.7. The function $f$ considered by Rempe and Stallard [25] that satisfies $\dim \mathcal{I}(f) = 1$ behaves like $\exp\exp(z)$ in a half-strip and is bounded outside this half-strip. This suggests that in order to construct an example of a quasiregular map for which $\dim E$ behaves like $\mathcal{I}(Z(x))$ in a half-beam and which is bounded outside this half-beam. We have been unable to construct such a function.

However, the methods of [6, § 4] indicate that such a function would have an invariant Cantor set of dimension greater than $d - 1$. Arguments similar to those in [13] now suggest that $\dim \mathcal{I}(f) \geq d - 1$ for such a function $f$. Perhaps we have $\dim \mathcal{I}(f) \geq d - 1$ and $\dim \mathcal{J}(f) \geq d - 1$ for all quasiregular maps $f : \mathbb{R}^d \to \mathbb{R}^d$ of transcendental type.

5. Proof of Theorem 1.2

We are going to need the following lemma which is implicit in the proof of [9, Theorem 1.2]. We include a sketch of the proof here for convenience.

Lemma 5.1. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a quasiregular map and $x_1 \in \mathcal{A}(f) \setminus \mathcal{J}(f)$. Let $r > 0$ be such that $B(x_1, 4r) \cap \mathcal{J}(f) = \emptyset$ and $x_2 \in B(x_1, r)$. Then there is a constant $c > 0$, depending only on $d$ and $r$, such that for all large enough $k \in \mathbb{N}$ we have

$$
\log \frac{\log |f^k(x_2)|}{\log |f^k(x_1)|} \leq c \left( K_1(f)K_0(f) \right)^{k/(d-1)}.
$$

Proof. Since $B(x_1, 4r) \cap \mathcal{J}(f) = \emptyset$, the iterates $f^k$ of $f$ omit a set of positive capacity in $B(x_1, r)$. Let $R > 0$ be such that $B(0, R) \cap \mathcal{J}(f) \neq \emptyset$. For all large enough $k$ we will have that $R_1 := |f^k(x_1)| \geq R$. Put $R_2 := |f^k(x_2)|$. We may assume that $R_2 > R_1$ since otherwise there is nothing to prove.

For any such $k$ now consider the sets $X_1 := \{ x \in B(x_1, 2r) : |f^k(x)| \leq R_1 \}$ and $X_2 := \{ x \in B(x_1, 2r) : |f^k(x)| \geq R_2 \}$. Denote by $Y_j$ the component of $X_j$ that contains $x_j$, for $j = 1, 2$. The maximum principle now implies that $Y_2$ connects $x_2$ to $\partial B(x_1, 2r)$. Moreover, $Y_1$ connects $x_1$ to $\partial B(x_1, 2r)$. Indeed, if this was not the
Since $B(0, R) \cap J(f) \neq \emptyset$ by complete invariance of the Julia set, this yields that $B(x_1, 2r) \cap J(f) \neq \emptyset$. This is a contradiction since $B(x_1, 4r) \cap J(f) = \emptyset$.

Let $\Gamma = \Delta(X_1, X_2; B(x_1, 2r))$ and $\Gamma_1 = \Delta(Y_1, Y_2; B(x_1, 2r))$. We now argue as in the proof of [9, Theorem 1.2] in order to estimate the modulus $M(\Gamma)$ of the path family $\Gamma$. We find that

$$c_1 \leq M(\Gamma_1) \leq M(\Gamma) \leq c_2 K_O(f^k) K_I(f^k) \left( \frac{\log |f(x_2)|}{\log |f(x_1)|} \right)^{1-d},$$

where the positive constants $c_1$ and $c_2$ depend only on $r$ and $d$. After rearranging and using the fact that $K_I(f^k) \leq K_I(f)^k$ and $K_O(f^k) \leq K_O(f)^k$ this gives the desired inequality. 

**Proof of Theorem 1.2.** Towards a contradiction assume that there is a function $f$ satisfying (1.1) with $\dim J(f) < 1$. Then $J(f)$ is totally disconnected and $J(f) = \partial A(f)$ by [9, Theorem 1.2]. Hence, the complement of $J(f)$ comprises of one unbounded component $U$ which is contained in $A(f)$.

Choose $\alpha > (K_I K_O)^{1/(d-1)}$. Then $\alpha > 1$. Let also $x_1 \in U$ and, using (1.1), choose $R > |x_1|$ so large that

$$(5.1) \quad M(r, f) > r \quad \text{and} \quad \log M(r, f) > (\log r)^\alpha \quad \text{for all } r \geq R.$$ 

Since $x_1 \in U \subset A(f)$ there exists $L \in \mathbb{N}$ such that

$$(5.2) \quad |f^{k+L}(x_1)| \geq M^{k+1}(R, f)$$

for all $k \in \mathbb{N}$. Since $|x_1| < R$ and hence $|f^k(x_1)| \leq M^k(R, f)$ this implies that

$$(5.3) \quad |f^{k+L}(x_1)| \geq M(M^k(R, f), f) \geq M(|f^k(x_1)|, f)$$

for all $k \in \mathbb{N}$.

Choose an arc $\gamma$ in $U$ connecting $x_1$ and $f^L(x_1)$. Cover this arc by $m$ balls $B(x_i, r_i)$ so that $x_m = f^L(x_1)$ and such that $x_i \in \gamma$ and $B(x_i, 4r_i) \cap J(f) = \emptyset$ for $i = 1, \ldots, m$. Applying now Lemma 5.1 $2m - 2$ times we obtain

$$\log \frac{\log |f^{k+L}(x_1)|}{\log |f^k(x_1)|} \leq C(K_I(f) K_O(f))^{k/(d-1)}$$

for all large enough $k$, where $C$ is a constant that depends on $m, d$ and the radii $r_i$.

Combining this with (5.3) yields that

$$\log \frac{M(|f^k(x_1)|, f)}{\log |f^k(x_1)|} \leq C(K_I(f) K_O(f))^{k/(d-1)}.$$ 

Since $|f^k(x_1)| \to \infty$ as $k \to \infty$ we may apply the second inequality in (5.1) for $r = |f^k(x_1)|$ if $k$ is sufficiently large. It thus follows from the last equation that

$$(5.4) \quad (\alpha - 1) \log \log |f^k(x_1)| \leq C(K_I(f) K_O(f))^{k/(d-1)}$$

for large $k$. 

It follows from (5.1) that
\[ \log M^k(R, f) \geq (\log R)^{\alpha^k} \]
for all \( k \in \mathbb{N} \). Together with (5.2) we thus have
\[ \log \log |f^k(x_1)| \geq \log \log M^{k+1-L}(R, f) \geq \alpha^{k+1-L} \log R \]
for large \( k \). Since \( \alpha > (K_I K_O)^{1/(d-1)} \) this contradicts (5.4) for large \( k \). \( \square \)

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Mathematisches Seminar, Christian–Albrechts–Universität zu Kiel, Heinrich–Hecht–Platz 6, 24098 Kiel, Germany
Email address: bergweiler@math.uni-kiel.de

Department of Mathematics and Statistics, University of Helsinki, Pietari Kalmin katu 5, 00560 Helsinki, Finland
Email address: athanasios.tsantaris@helsinki.fi