Nusselt number correlations for simultaneously developing laminar duct flows of liquids with temperature dependent properties

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Abstract. New correlations, suitable for engineering applications, for the mean Nusselt number in the entrance region of circular tubes and square ducts with uniform heat flux boundary conditions specified at the walls are proposed. These correlations are obtained on the basis of the results of a previous parametric investigation on the effects of temperature dependent viscosity and thermal conductivity in simultaneously developing laminar flows of liquids in straight ducts of constant cross-sections. In these studies, a finite element procedure has been employed for the numerical solution of the parabolized momentum and energy equations. Viscosity and thermal conductivity are assumed to vary with temperature according to an exponential and to a linear relation, respectively, while the other fluid properties are held constant. Axial distributions of the mean Nusselt number, obtained by numerical integration from those of the local Nusselt number, are used as input data in the derivation of the proposed correlations. A superposition method is proved to be applicable in order to estimate the Nusselt number by considering separately the effects of temperature dependent viscosity and thermal conductivity. Therefore, for each of the considered cross-sectional geometries, two distinct correlations are proposed for flows of liquids with temperature dependent viscosity and with temperature dependent thermal conductivity, in addition to that obtained for constant property flows.

1. Introduction
In many laminar duct flows, entrance effects on fluid flow and heat transfer must be taken into account, since the total length of the duct is comparable with that of the entrance region. Moreover, the development of the velocity and temperature fields can be strongly affected by temperature dependence of fluid properties. If the fluid is a liquid, the relative variations of viscosity with temperature are the most relevant, while those of thermal conductivity are, in general, much smaller, and those of density and specific heat capacity are almost negligible [1, 2]. As far as velocity distribution and pressure drop are concerned, the main effects of temperature dependent fluid properties can be retained even if only viscosity is considered to vary with temperature, while the other properties are assumed constant [1, 3]. Instead, if heat transfer characteristics, as the Nusselt number, have to be evaluated, also the temperature dependence of thermal conductivity must be taken into account [2].
In previous articles, we carried out systematic analyses of the effects of temperature dependent viscosity and thermal conductivity on forced convection in simultaneously developing laminar flows of liquids in straight ducts with uniform wall heat flux boundary conditions [4, 5]. In particular, we obtained axial distributions of the local value of the peripherally averaged Nusselt number for ducts of different cross-sections. We assumed that viscosity decreases with increasing temperature according to an exponential relation, while thermal conductivity varies linearly with temperature. Suitable dimensionless parameters, namely, the viscosity and thermal conductivity Pearson numbers, were used to quantitatively express temperature dependence of the corresponding properties. A finite element procedure was employed for the step-by-step solution of the parabolised momentum and energy equations in a domain corresponding to the cross-section of the duct [6]. Finally, a superposition method was proved to be applicable in order to obtain accurate estimates of the local Nusselt number by considering separately the effects of temperature dependent viscosity and thermal conductivity [4, 5].

In this paper, first we use previous numerical results for the local Nusselt number for circular and square cross-sectional geometries [5] to obtain by numerical integration the corresponding axial distributions of the mean, i.e., longitudinally averaged, Nusselt number. Then, on the basis of such distributions, we obtain very accurate correlations, suitable for engineering applications. Actually, taking advantage of the above mentioned superposition method, we propose for each of the considered cross-sectional geometries a correlation valid for constant property flows and two distinct correlations for flows of liquids with temperature dependent viscosity and with temperature dependent thermal conductivity. Finally, we demonstrate that, by properly assembling these correlations, accurate estimates of the mean Nusselt number can be obtained in wide ranges of operative conditions. No correlations are presented here for pressure drop parameters since, as demonstrated in previous works [4, 5], the influence of thermal conductivity variation on pressure drop is almost negligible if compared with that of viscosity variations. Therefore, the correlations for pressure drop parameters already proposed by the authors for liquids with temperature dependent viscosity and constant thermal conductivity [7] can be used for all liquids with temperature dependent properties.

2. Statement of the problem

The laminar forced convection in the entrance region of straight ducts of constant cross-sections with uniform wall heat flux \( q''_w > 0 \) (fluid heating) is studied. Since the liquid heating is assumed to begin at the duct inlet, uniform values of the axial velocity component \( u \) and of the temperature \( t \) (i.e., \( t = t_e, u = u_e = \pi \) and \( v = w = 0 \)) are specified as the appropriate inlet conditions, being \( \pi \) the average axial velocity and \( v \) and \( w \) the transverse velocity components. The heat flux (Neumann) boundary condition \( q'' = k \partial t/\partial n = q''_w \) is applied at the solid walls, where \( k \) is the thermal conductivity and \( n \) is the outward normal to the boundary.

Since the fluids considered here are liquids, the dynamic viscosity \( \mu \) is assumed to decrease with increasing temperature according to the widely used exponential formula [3]

\[
\mu = \mu_e \exp[-\beta (t - t_e)]
\]  

where \( \mu_e \) is the value of \( \mu \) at \( t_e \) and \( \beta = -(\mathrm{d}\mu/\mathrm{d}t)/\mu = \text{const} \) is positive. The thermal conductivity \( k \) can be assumed both to increase or decrease with increasing temperature, depending on the fluid considered, according to the linear relation

\[
k = k_e [1 + \alpha (t - t_e)]
\]

where \( k_e \) is the value of \( k \) at \( t_e \) and \( \alpha = (\mathrm{d}k/\mathrm{d}t)/k_e = \text{const} \) can be both positive or negative. By means of simple manipulations, equations (1) and (2) can be cast in the following dimensionless forms

\[
\frac{\mu}{\mu_e} = \exp(-P_{n\mu} T)
\]
where $T = (t - t_e) k_e/(q''_w D_h)$ is the dimensionless temperature, $D_h$ is the hydraulic diameter, $Pn_\mu = \beta q''_w D_h/k_e$ is the viscosity Pearson number (representing the ratio of the characteristic process temperature difference $q''_w D_h/k_e$ to the characteristic temperature difference $1/\beta$ that can produce appreciable viscosity variations) and $Pn_\kappa = \alpha q''_w D_h/k_e$ is the thermal conductivity Pearson number (representing the ratio of the characteristic process temperature difference $q''_w D_h/k_e$ to the characteristic temperature difference $1/\alpha$ that can produce appreciable thermal conductivity variations).

It is worth noting that, since the density $\rho$ and the specific heat $c$ are constant, the local Reynolds number $Re = \rho \pi D_h/\mu$, the local Prandtl number $Pr = \mu c/k$ and the local Péclet number $Pe = Re Pr = \rho c \pi D_h/k$ all vary with temperature because of the variations of $\mu$, of the ratio $\mu/k$ and of $k$, respectively. Therefore we have $\mu/\mu_e = Re_e/Re$ and $k/k_e = Pe_e/Pe$, where $Re_e$ and $Pe_e$ are the Reynolds and Péclet numbers evaluated at $t_e$. Moreover, since the viscosity of liquids decreases with increasing temperature ($\beta > 0$), in the case of fluid heating we have $Pn_\mu > 0$ and $Re_e/Re < 1$, while $Pn_\mu = 0$ and $Re_e/Re = 1$ refer to constant viscosity fluids ($\beta = 0$). Instead, since the thermal conductivity can either increase ($\alpha > 0$) or decrease ($\alpha < 0$) with increasing temperature, we have $Pn_\kappa > 0$ and $Pe_e/Pe < 1$ in the first case and $Pn_\kappa < 0$ and $Pe_e/Pe < 1$ in the second one, while $Pn_\kappa = 0$ and $Pe_e/Pe = 1$ refer to constant thermal conductivity fluids ($\alpha = 0$).

As already pointed out, as the fluid temperature rises along the duct, the viscosity decreases while Re increases. Therefore, to ensure laminar flow conditions, the local Reynolds number $Re_b$ evaluated at the bulk temperature is only allowed to reach the maximum value $(Re_b)_{max} = 2,000$, corresponding to the maximum value $x_{max}$ of the axial coordinate $x$, whereupon computations are stopped. Therefore, taking into account equation (3) and the appropriate heat balance for the duct, the following expression for the maximum value $X_{max}$ of the dimensionless axial coordinate $X^* = x/(D_h Pe_e)$ can be obtained [4, 5, 7]

$$X_{max}^* = \frac{1}{4 Pn_\mu} \ln \left[ \frac{(Re_b)_{max}}{Re_e} \right]$$

(5)

Numerical results of interest for the present study concern the axial distributions of the peripherally averaged local Nusselt number $Nu = \bar{h} D_h/k_e$, where $\bar{h}$ is the peripherally averaged local convective heat transfer coefficient defined as

$$\bar{h} = \frac{q''_w}{\bar{t}_w - \bar{t}_b} = \frac{k_e}{D_h (\bar{T}_w - \bar{T}_b)}$$

(6)

In equation (6), $\bar{t}_w$ and $\bar{t}_b$ are the peripherally averaged wall temperature and the mean bulk temperature, respectively, and $\bar{T}_w$ and $\bar{T}_b$ are their dimensionless forms. According to equation (6), the peripherally averaged local Nusselt number can be expressed as

$$Nu = \frac{1}{\bar{T}_w - \bar{T}_b}$$

(7)

Axial distributions of the longitudinally averaged Nusselt number, defined as [1]

$$\bar{Nu} = \frac{1}{x} \int_0^x Nu \, dx = \frac{1}{X^*} \int_0^{X^*} Nu \, dX^*$$

(8)

can be obtained by means of an appropriate numerical integration rule.
3. Numerical procedure

If the effects of axial diffusion can be neglected (Re<sub>e</sub> > 50 and Pe<sub>e</sub> > 50) and there is no recirculation in the longitudinal direction, steady-state flow and heat transfer in straight ducts of constant cross-sections are governed by the continuity and the parabolized Navier-Stokes and energy equations [8, 9]. These equations are not reported here for lack of space, but they are reported elsewhere together with the boundary conditions specified on the boundaries of the computational domain [5, 7].

A finite element procedure for the analysis of the forced convection of fluids with temperature dependent properties in the entrance region of straight ducts [6, 7, 10] is used to solve the model equations. The adopted procedure is based on a segregated approach which implies the sequential solution of the momentum and energy equations on a two-dimensional domain in the case of three-dimensional geometries and on a one-dimensional domain in axisymmetric problems, corresponding to the cross-section of the duct. A marching method is then used to move forward in the axial direction. The pressure-velocity coupling is dealt with using an improved projection algorithm already employed by one of the authors (C.N.) for the solution of the Navier-Stokes equations in their elliptic form [11]. The procedure has already been validated, with reference to both constant and temperature dependent property fluids, by comparing heat transfer and pressure drop results with existing literature data for simultaneously developing laminar flows in straight ducts [6, 10, 12, 13].

4. Numerical results

In this study, two cross-sectional geometries are considered, namely circular and square, chosen as representative of axisymmetric and three-dimensional duct geometries, respectively. The corresponding computational domains, defined taking into account existing symmetries, are one-dimensional and two-dimensional for the circular and the square cross-sections, respectively. Details of finite element domain and time discretisations are reported elsewhere [4, 5].

Moreover, it has been demonstrated that if the effects of temperature dependent viscosity and thermal conductivity can have comparable effects on the Nusselt number, according to the values of P<sub>n</sub><sub>µ</sub> and P<sub>n</sub><sub>k</sub> considered, i.e., P<sub>n</sub><sub>µ</sub> > 0 and P<sub>n</sub><sub>k</sub> = 0 (under the assumptions of temperature dependent viscosity and constant thermal conductivity, i.e., P<sub>n</sub><sub>µ</sub> > 0 and P<sub>n</sub><sub>k</sub> = 0). Therefore, we can move forward in the axial direction. The pressure-velocity coupling is dealt with using an improved projection algorithm already employed by one of the authors (C.N.) for the solution of the Navier-Stokes equations in their elliptic form [11]. The procedure has already been validated, with reference to both constant and temperature dependent property fluids, by comparing heat transfer and pressure drop results with existing literature data for simultaneously developing laminar flows in straight ducts [6, 10, 12, 13].
and $\text{Nu}_k$ (under the assumptions of temperature dependent thermal conductivity and constant viscosity, i.e., $\text{Pr}_k \neq 0$ and $\text{Pr}_\mu = 0$), a superposition method is applicable in order to obtain approximate values of the Nusselt number $\nu_{\mu k}$, according to the relation

$$ \text{Nu}_{\mu k} - \text{Nu}_c \approx (\text{Nu}_\mu - \text{Nu}_c) + (\text{Nu}_k - \text{Nu}_c) $$  \hspace{1cm} (9)

with an accuracy which can be considered satisfactory in most situations. With reference to the ratios $\text{Nu}_{\mu k}/\text{Nu}_c$, $\text{Nu}_\mu/\text{Nu}_c$ and $\text{Nu}_k/\text{Nu}_c$, equation (9) can be recast in the form \cite{4, 5}

$$ \frac{\text{Nu}_{\mu k}}{\text{Nu}_c} \equiv \left(\frac{\text{Nu}_{\mu k}}{\text{Nu}_c}\right)' = \frac{\text{Nu}_\mu}{\text{Nu}_c} + \frac{\text{Nu}_k}{\text{Nu}_c} - 1 $$  \hspace{1cm} (10)

where $(\text{Nu}_{\mu k}/\text{Nu}_c)'$ is the approximate value of $\text{Nu}_{\mu k}/\text{Nu}_c$ given by the superposition method. Therefore, the values of the differences $(\text{Nu}_\mu/\text{Nu}_c) - 1$ and $(\text{Nu}_k/\text{Nu}_c) - 1$ approximately measure the separate effects of temperature dependent viscosity and thermal conductivity, respectively.

In this work, previous numerical results concerning axial distributions of the ratios $\text{Nu}_{\mu k}/\text{Nu}_c$, $\text{Nu}_\mu/\text{Nu}_c$ and $\text{Nu}_k/\text{Nu}_c$ \cite{5} have been used to obtain, by means of a suitable numerical integration rule according to equation (8), the corresponding distributions of the ratios $\overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c$, $\overline{\text{Nu}}_\mu/\overline{\text{Nu}}_c$ and $\overline{\text{Nu}}_k/\overline{\text{Nu}}_c$. Then, it has been verified that the following relation, obtained by combining equations (8) and (9), still holds true with an accuracy which can be considered satisfactory in most situations

$$ \overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c \approx \left(\frac{\text{Nu}_{\mu k}}{\text{Nu}_c}\right)' = \frac{\overline{\text{Nu}}_\mu}{\overline{\text{Nu}}_c} + \frac{\overline{\text{Nu}}_k}{\overline{\text{Nu}}_c} - 1 $$  \hspace{1cm} (11)

In equation (11), $(\overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c)'$ represents the approximate value of $\overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c$ given by the superposition method. Axial distributions of the ratio $(\overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c)'$ for the highest values of $\text{Pr}_\mu$, i.e., $\text{Pr}_\mu = 2$, and $\text{Pr}_\mu = 4$, different values of $\text{Pr}_k$ and $\text{Pr}_c = 20$ are reported in figures 1 and 2 for circular tubes and square ducts, respectively, to allow the comparison with the corresponding distributions of $\overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c$. As can be seen, even for the highest values of $\text{Pr}_\mu$ and the highest/lowest values of $\text{Pr}_k$ the dashed curves, representing axial distributions of the ratio $(\overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c)'$, are very close to the solid ones giving the numerical solutions for $\overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c$.

**Figure 1.** Comparison of axial distributions of the ratios $\overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c$ and $(\overline{\text{Nu}}_{\mu k}/\overline{\text{Nu}}_c)'$ for simultaneously developing laminar flows in circular tubes with $\text{Pr}_c = 20$, different values of $\text{Pr}_k$ and: (a) $\text{Pr}_\mu = 2$ and (b) $\text{Pr}_\mu = 4$. 


Figure 2. Comparison of axial distributions of the ratios $\frac{\text{Nu}_{\mu,k}}{\text{Nu}_c}$ and $\left(\frac{\text{Nu}_{\mu,k}}{\text{Nu}_c}\right)'$ for simultaneously developing laminar flows in square ducts with $\text{Pr}_e = 20$, different values of $\text{Pn}_k$ and: (a) $\text{Pn}_\mu = 2$ and (b) $\text{Pn}_\mu = 4$.

thus confirming the validity of equation (11). Moreover, the maximum positive and negative relative errors in the approximation of $\frac{\text{Nu}_{\mu,k}}{\text{Nu}_c}$ by means of $\left(\frac{\text{Nu}_{\mu,k}}{\text{Nu}_c}\right)'$ are $\varepsilon^{+}_{\text{max}} = 0.15\%$ and $\varepsilon^{-}_{\text{max}} = \varepsilon_{\text{max}} = -0.28\%$ for circular ducts and $\varepsilon^{+}_{\text{max}} = 0.42\%$ and $\varepsilon^{-}_{\text{max}} = -1.39\%$ for square ducts in the range $10^{-4} \leq X^* \leq X^*_{\text{max}}$ with $\text{Pr}_e = 5$, 20 and 100 and different values of $\text{Pn}_\mu$ and $\text{Pn}_k$. Therefore, we can conclude that the agreement between computed and approximate results is very good.

5. Correlations for the mean Nusselt number
Taking advantage of equation (11), a correlation for $\text{Nu}_{\mu,k}$ has been obtained by assembling the different correlations for $\text{Nu}_c$, $\frac{\text{Nu}_{\mu,k}}{\text{Nu}_c}$ and $\frac{\text{Nu}_k}{\text{Nu}_c}$ obtained from computed results for flows of liquids with constant property ($\text{Pn}_k = \text{Pn}_\mu = 0$), temperature dependent viscosity ($\text{Pn}_k = 0$ and $\text{Pn}_\mu > 0$) and temperature dependent thermal conductivity ($\text{Pn}_\mu = 0$ and $\text{Pn}_k \neq 0$), respectively.

5.1. Correlations for the mean Nusselt number for flows of liquids with constant properties
The results obtained for circular and square cross-sectional geometries under the assumption of constant property flow ($\text{Pn}_k = \text{Pn}_\mu = 0$) have been used to obtain correlations for the mean Nusselt number $\text{Nu}_c$ as a function of $X^*$ in the form

$$\text{Nu}_c = (\text{Nu}_c)_{fd} + \frac{a}{1 + b\left(X^*\right)^{-n}}$$  \hspace{1cm} (12)

where $(\text{Nu}_c)_{fd}$, i.e., the asymptotic value of $\text{Nu}_c$ for fully developed conditions, the coefficients $a$ and $b$ and the exponent $m$ only depend on the cross-sectional geometry, while the exponent $n$ also depends on the Prandtl number $\text{Pr}_e$ according to the relation

$$n = n_1 \text{Pr}_e^s - n_2 \text{Pr}_e$$  \hspace{1cm} (13)

whose coefficients $n_1$ and $n_2$ and the exponent $s$ only depend on the cross-sectional geometry. The appropriate forms of equations (12) and (13) for circular and square cross-sections, valid
Table 1. Correlations for $\overline{Nu}_c$ corresponding to equations (12) and (13) for circular and square cross-sections.

| Cross-section | Correlations for $\overline{Nu}_c$ |
|---------------|-------------------------------------|
| circular      | $\overline{Nu}_c = 4.3636 + \frac{0.065 (X^*)^{-1.3}}{1 + 0.10 (X^*)^{-n}}$ $n = 0.761 Pr_e^{0.0224} - 0.000109 Pr_e$ |
| square        | $\overline{Nu}_c = 3.0874 + \frac{0.065 (X^*)^{-1.3}}{1 + 0.11 (X^*)^{-n}}$ $n = 0.755 Pr_e^{0.0255} - 0.000136 Pr_e$ |

for $5 \leq Pr_e \leq 100$ and $10^{-4} \leq X^* \leq X_{\text{max}}^*$, are reported in table 1. Since, for $Pr_e = 5, 20$ and 100, the maximum positive and negative relative errors in the approximation of $\overline{Nu}_c$ by means of equation (12) are $\varepsilon_{\text{max}}^+ = 0.80\%$ and $\varepsilon_{\text{max}}^- = -1.29\%$ for circular ducts and $\varepsilon_{\text{max}}^+ = 2.20\%$ and $\varepsilon_{\text{max}}^- = -1.31\%$ for square ducts, we can conclude that the agreement between computed and approximate results is very good.

5.2. Correlations for the mean Nusselt number for flows of liquids with temperature dependent viscosity

The results obtained for circular and square cross-sectional geometries for $Pn_k = 0$ and $Pn_{\mu} > 0$ have been used to obtain correlations for the ratio $\overline{Nu}_\mu/\overline{Nu}_c$ as a function of $X^*$ in the form

$$\frac{\overline{Nu}_\mu}{\overline{Nu}_c} = 1 + \left( \frac{\overline{Nu}_\mu}{\overline{Nu}_c} \right)_{fd} - 1 \left\{ 1 - \exp \left[ -a (X^*)^m - b (X^*)^n \right] \right\}$$  (14)

In equation (14), $\left( \frac{\overline{Nu}_\mu}{\overline{Nu}_c} \right)_{fd}$ is the asymptotic value for fully developed conditions of the ratio $\overline{Nu}_\mu/\overline{Nu}_c$, which can be evaluated by means of the very accurate formula

$$\left( \frac{\overline{Nu}_\mu}{\overline{Nu}_c} \right)_{fd} = 1 + d_1 Pn_\mu + d_2 Pn_\mu^2$$  (15)

whose coefficients $d_1$ and $d_2$ depend on the cross-sectional geometry. Besides, still with reference to equation (14), the coefficient $b$ and the exponent $n$ only depend on the cross-sectional geometry, while the coefficient $a$ and the exponent $m$ also depend on the viscosity Pearson number $Pn_\mu$ and on the Prandtl number $Pr_e$, respectively, and can be expressed as

$$a = a_1 + a_2 Pn_\mu$$  (16)

$$m = m_1 Pr_e^{-s}$$  (17)

where the coefficients $a_1$, $a_2$ and $m_1$ and the exponent $s$ only depend on the cross-sectional geometry. For engineering application, the appropriate forms of equations (14) to (17) for circular and square cross-sections, valid for $5 \leq Pr_e \leq 100$, $1 \leq Pn_\mu \leq 4$ and $10^{-4} \leq X^* \leq X_{\text{max}}^*$, are reported in table 2. For $Pr_e = 5, 20$ and 100, the maximum positive and negative relative errors in the approximation of $\overline{Nu}_\mu/\overline{Nu}_c$ by means of equation (14) are $\varepsilon_{\text{max}}^+ = 0.14\%$ and $\varepsilon_{\text{max}}^- = -0.21\%$ for circular ducts and $\varepsilon_{\text{max}}^+ = 0.31\%$ and $\varepsilon_{\text{max}}^- = -0.28\%$ for square ducts. Therefore, we can conclude that the agreement between computed and approximate results is very good.
Table 2. Correlations for $\frac{Nu_\mu}{Nu_c}$ corresponding to equations (14) to (17) for circular and square cross-sections.

| Cross-section | Correlations for $\frac{Nu_\mu}{Nu_c}$ |
|---------------|----------------------------------------|
| circular      | $\frac{Nu_\mu}{Nu_c} = 1 + \left[ \left( \frac{Nu_\mu}{Nu_c} \right)_{fd} - 1 \right] \left( 1 - \exp \left[ -a \left( X^* \right)^m - 2.0 \left( X^* \right)^{0.6} \right] \right)$ |
|               | $\left( \frac{Nu_\mu}{Nu_c} \right)_{fd} = 1 + 3.39 \cdot 10^{-2} \text{Pn}_\mu - 9.2 \cdot 10^{-4} \text{Pn}_\mu^2$ |
|               | $a = 2.62 + 0.084 \text{Pn}_\mu$ |
|               | $m = 0.48 \text{Pr}^{-0.076}$ |
| square        | $\frac{Nu_\mu}{Nu_c} = 1 + \left[ \left( \frac{Nu_\mu}{Nu_c} \right)_{fd} - 1 \right] \left( 1 - \exp \left[ -a \left( X^* \right)^m + 3.0 \left( X^* \right)^{0.8} \right] \right)$ |
|               | $\left( \frac{Nu_\mu}{Nu_c} \right)_{fd} = 1 + 4.74 \cdot 10^{-2} \text{Pn}_\mu - 1.0 \cdot 10^{-3} \text{Pn}_\mu^2$ |
|               | $a = 6.79 + 0.044 \text{Pn}_\mu$ |
|               | $m = 0.57 \text{Pr}^{-0.050}$ |

5.3. Correlations for the mean Nusselt number for flows of liquids with temperature dependent thermal conductivity

The results obtained for the considered cross-sectional geometries for $\text{Pn}_\mu = 0$ and $\text{Pn}_k \neq 0$ have been used to obtain correlations for the ratio $\frac{Nu_k}{Nu_c}$ as a function of $X^*$ in the form

$$\frac{Nu_k}{Nu_c} = 1 + a \text{Pn}_k + \left( b_1 \text{Pn}_k + b_2 \text{Pn}_k^2 \right) X^*$$

(18)

where the coefficient $a$ can be expressed as

$$a = a_1 \text{Pr}_e^d \left( 1 - \exp \left[ -a_2 (X^*)^n \right] \right)$$

(19)

In equations (18) and (19) the coefficients $a_1$, $a_2$, $b_1$ and $b_2$ and the exponent $d$ only depend on the cross-sectional geometry and on the sign of $\text{Pn}_k$, while the exponent $n$ also depends on the Prandtl number $\text{Pr}_e$ according to the relation

$$n = n_1 \text{Pr}_e^{-s}$$

(20)

where the coefficient $n_1$ and the exponent $s$ only depend on the cross-sectional geometry. For engineering application, the appropriate forms of equations (18) to (20) for circular and square cross-sections, valid for $5 \leq \text{Pr}_e \leq 100$, $0.0125 \leq |\text{Pn}_k| \leq 0.4$ and $10^{-4} \leq X^* \leq X^*_{\text{max}}$, are reported in table 3. For $\text{Pr}_e = 5$, 20 and 100, the maximum positive and negative relative errors in the approximation of $\frac{Nu_k}{Nu_c}$ by means of equation (18) are $\varepsilon^+_{\text{max}} = 0.53\%$ and $\varepsilon^-_{\text{max}} = -0.21\%$ for circular ducts and $\varepsilon^+_{\text{max}} = 0.30\%$ and $\varepsilon^-_{\text{max}} = -0.17\%$ for square ducts. Therefore, we can conclude that the agreement between computed and approximate results is very good.
5.4. Estimation of the mean Nusselt number for flows of liquids with temperature dependent properties by means of the proposed correlations

The correlations proposed in the previous sections can be used to predict the value of \( \overline{\text{Nu}_{\mu k}} \). In fact, according to equation (11), we can write

\[
\overline{\text{Nu}_{\mu k}} = \overline{\text{Nu}_c} \left( \frac{\overline{\text{Nu}_{\mu k}}}{\overline{\text{Nu}_c}} \right) = \overline{\text{Nu}_c} \left( \frac{\overline{\text{Nu}_{\mu k}}}{\overline{\text{Nu}_c}} + \frac{\overline{\text{Nu}_k}}{\overline{\text{Nu}_c}} - 1 \right)
\]

(21)

where \( \overline{\text{Nu}_c}, \overline{\text{Nu}_\mu/\overline{\text{Nu}_c}}, \) and \( \overline{\text{Nu}_k/\overline{\text{Nu}_c}} \) are given by equations (12), (14) and (18), respectively. The maximum absolute values \( \varepsilon_{\text{max}} \) of the relative error in the approximation of \( \overline{\text{Nu}_{\mu k}} \) by means of equation (21) for \( \text{Pr}_e = 5, 20 \) and 100 and different values of \( P_{n_\mu} \) and \( P_{n_k}/P_{n_\mu} \) are reported in table 4 for circular and square cross-sectional geometries. As can be seen, the agreement between computed and approximate results is very good.

6. Conclusions

New correlations, suitable for engineering applications, for the mean Nusselt number in the entrance region of circular tubes and square ducts with uniform heat flux boundary conditions specified at the walls have been proposed. These correlations have been obtained on the basis of the results of a previous parametric investigation on the effects of temperature dependent viscosity and thermal conductivity in simultaneously developing laminar flows of liquids in straight ducts of constant cross-sections. In these studies, a finite element procedure has

| Cross-section | \( P_{n_k} \) | Correlations for \( \overline{\text{Nu}_{k}/\overline{\text{Nu}_c}} \) |
|---------------|---------------|--------------------------------------------------|
| circular      | \( > 0 \)     | \( \overline{\text{Nu}_k} = \overline{\text{Nu}_c} \left( 1 + a \overline{\text{Nu}_k} + \left( 1.97 \overline{\text{Nu}_k} - 0.238 \overline{\text{Nu}_k}^2 \right) X^* \right) \)
|               |               | \( a = 0.032 \text{Pr}^{0.031} \) \( \{ 1 - \exp \left[ -28 \left( X^* \right)^n \right] \} \)
|               |               | \( n = 0.60 \text{Pr}^{-0.024} \) \( \text{Pr}_e \) |
| circular      | \( < 0 \)     | \( \overline{\text{Nu}_k} = \overline{\text{Nu}_c} \left( 1 + a \overline{\text{Nu}_k} + \left( 1.90 \overline{\text{Nu}_k} - 0.399 \overline{\text{Nu}_k}^2 \right) X^* \right) \)
|               |               | \( a = 0.033 \text{Pr}^{0.035} \) \( \{ 1 - \exp \left[ -28 \left( X^* \right)^n \right] \} \)
|               |               | \( n = 0.60 \text{Pr}^{-0.024} \) \( \text{Pr}_e \) |
| square        | \( > 0 \)     | \( \overline{\text{Nu}_k} = \overline{\text{Nu}_c} \left( 1 + a \overline{\text{Nu}_k} + \left( 1.98 \overline{\text{Nu}_k} - 0.295 \overline{\text{Nu}_k}^2 \right) X^* \right) \)
|               |               | \( a = 0.052 \text{Pr}^{0.024} \) \( \{ 1 - \exp \left[ -18 \left( X^* \right)^n \right] \} \)
|               |               | \( n = 0.60 \text{Pr}^{-0.037} \) \( \text{Pr}_e \) |
| square        | \( < 0 \)     | \( \overline{\text{Nu}_k} = \overline{\text{Nu}_c} \left( 1 + a \overline{\text{Nu}_k} + \left( 1.92 \overline{\text{Nu}_k} - 0.716 \overline{\text{Nu}_k}^2 \right) X^* \right) \)
|               |               | \( a = 0.053 \text{Pr}^{0.034} \) \( \{ 1 - \exp \left[ -18 \left( X^* \right)^n \right] \} \)
|               |               | \( n = 0.60 \text{Pr}^{-0.037} \) \( \text{Pr}_e \) |
Table 4. Maximum absolute values $|\varepsilon|_{\text{max}}$ (%) of the relative error in the approximation of $\text{Nu}_{\mu k}$ by means of equation (21) for simultaneously developing flows in circular and square ducts with different viscosity and thermal conductivity Pearson numbers.

| Cross-section | $Pn_\mu$ | \hline | Pn$_{\mu}$/Pn$_k$ |
|---------------|----------|----------------|------------------|
|               | -10      | -20           | -40             | -80          | 80         | 40         | 20         | 10         |
| circular      | 1.26     | 1.26          | 1.26            | 1.26         | 1.25       | 1.25       | 1.25       | 1.25       |
|               | 1.25     | 1.23          | 1.23            | 1.22         | 1.22       | 1.21       | 1.21       | 1.21       |
|               | 1.29     | 1.23          | 1.20            | 1.19         | 1.17       | 1.16       | 1.15       | 1.11       |
| square        | 1.67     | 1.67          | 1.67            | 1.67         | 1.67       | 1.66       | 1.66       | 1.66       |
|               | 1.59     | 1.60          | 1.61            | 1.61         | 1.61       | 1.61       | 1.61       | 1.62       |
|               | 1.45     | 1.40          | 1.43            | 1.44         | 1.46       | 1.47       | 1.57       | 1.76       |

been employed for the numerical solution of the parabolized momentum and energy equations. Viscosity and thermal conductivity have been assumed to vary with temperature according to an exponential and to a linear relation, respectively, while the other fluid properties are held constant. The temperature dependences of viscosity and thermal conductivity have been quantitatively expressed by the corresponding Pearson numbers. Axial distributions of the mean Nusselt number, obtained by numerical integration from those of the local Nusselt number, have been used as input data in the derivation of the proposed correlations. A superposition method has been proved to be applicable in order to estimate the Nusselt number by considering separately the effects of temperature dependent viscosity and thermal conductivity. Therefore, for each of the considered cross-sectional geometries, two distinct correlations have been proposed for flows of liquids with temperature dependent viscosity and with temperature dependent thermal conductivity, in addition to that obtained for constant property flows.

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