QUANTUM BACKGROUNDS AND QFT

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ABSTRACT. We introduce the concept of a quantum background and a functor QFT. In the case that the QFT moduli space is smooth formal, we construct a flat quantum superconnection on a bundle over QFT which defines algebraic structures relevant to correlation functions in quantum field theory. We go further and identify chain level generalizations of correlation functions which should be present in all quantum field theories.

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1. INTRODUCTION

The work described in this paper grew from an effort to understand quantum field theory mathematically. In particular, we sought to understand the way in which deformations of the action functional of a quantum field theory control the correlation functions of the theory. Our efforts led us to consider what we call quantum backgrounds, or just backgrounds for short. Quantum backgrounds were invented by the first author and introduced during a sequence of lectures [11] in 2004 extending his work on flat families of quantum field theories [13]. We believe they provide the proper algebraic setting in which local deformations of quantum field theories should be discussed. A background $B$ is a four tuple $B = (P, m, N, \varphi)$ where $P$ is a graded noncommutative ring, $m$ is a degree one element of $P$ satisfying $m^2 = 0$, $N$ is a graded left $P$ module, and $\varphi$ is a degree zero element of $N$ satisfying $m\varphi = 0$. Additionally, we require that $P$ and $N$ be free $k[[\hbar]]$ modules for a fixed field $k$, and that $P$ be a $k[[\hbar]]$ algebra with $P/\hbar P$ commutative. The condition
$m\varphi = 0$ is analogous to having specified an initial action functional that satisfies the quantum master equation. In this paper, we develop our ideas about backgrounds alongside a condensed overview of deformation theory; we do this for two reasons.

The first reason is that physical information (such as correlation functions, our generalized correlation functions, and generalized Ward identities) are extracted from the background data with the aid of a set valued functor that is comparable to the classical deformation functor. Essentially, we are interested in the equation

$$mU\varphi = 0$$

where $U$ is an invertible element of $P$ that depends on parameters. We supply two interpretations: a Schrödinger interpretation where $U\varphi$ is regarded as an evolution of $\varphi$, and a Heisenberg interpretation where $U^{-1}mU$ is regarded as an evolution of $m$. From either perspective, the evolution $U$ is not arbitrary and the way it is constrained is intimately tied to the physics of the theory. If the evolution is unobstructed, the constraints provide enough relations to determine the correlation functions up to finite ambiguity. While one might think of $U\varphi$ as a deformation of $\varphi$, or $U^{-1}mU$ as a deformation of $m$, we prefer the interpretation as an evolution since that concept, along with the definition of a quantum background itself, is evocative of a sort of graded quantum mechanics à la Dirac [4], but it is a quantum mechanics in the moduli space of action functionals for quantum field theories having the same fields.

In this paper, we make restrictions on the form of $U$ and the type of parameter space over which $\varphi$ evolves. Recall the Maurer-Cartan equation $d\gamma + \frac{1}{2}[\gamma, \gamma] = 0$ in a differential graded Lie algebra $L$ leads to a covariant set valued functor of parameter rings $\text{Def}_L$ where $\text{Def}_L(A)$ is the set of solutions to the Maurer-Cartan equation with parameters in the ring $A$, modulo a natural equivalence. In the same way, the quantum master equation $mU\varphi = 0$ for a background $B$ leads to a set-value functor of $k[[\hbar]]$ parameter rings $\text{QFT}_B$. Standard deformation theory, enriched to account for $\hbar$, is used to prove (Theorem 3.4) that the functor $\text{QFT}_B$ is (pro, homotopy) representable by a differential algebra $R$ and a versal solution $U$ to Equation (1) with parameters in $R$. Recall that a functor $F$ is representable by a ring $R$ if it is naturally equivalent to the functor $\text{hom}(R, \_)$, in which case the identity homomorphism $\text{Id} : R \to R$ corresponds to an element $U$, which is a versal element in $F(R)$, modulo the obstructions encoded by the differential in $R$. Here, the evolution of $\varphi \rightsquigarrow U\varphi$ resembles a Schrödinger wave function over the space of theories; or $U\varphi$ might be compared to a versal action functional over the space of action functionals.

The main construction in the paper applies when $\text{QFT}_B$ is smooth formal, which in algebraic terms means that the differential in the representing algebra $R$ vanishes; equivalently, there exist no obstructions to constructing a versal action functional (see Definition 3.5). In geometric terms, smooth formal means the initial background data comprises a smooth point in the formal moduli space. In this case, we construct (Section 4) a quantum flat superconnection on a bundle of over the moduli space

$$\nabla : D_\Pi \to D_\Pi \otimes_R \Omega$$
Here, $D_{\Pi}$, $R$, and are the the tangent bundle, ring of functions, and space of differential forms on the moduli space of solutions to Equation (1), all are free $k[[\hbar]]$ module equivalents of the usual notions in formal geometry. We call the map $\nabla$ a quantum superconnection because: (i) $\nabla$ depends on $\hbar$ and satisfies the $\hbar$-connection equation

\begin{equation}
\nabla(fe) = \hbar(df)e + (-1)^{|f|}f\nabla(e)
\end{equation}

for $f \in R$ and a section $e$;

and is a Quillen-type superconnection since the target involves all of $\Omega$ and not only $\Omega^1$. We prove (Theorem 4.5) that $\nabla^2 = 0$; i.e., $\nabla$ is flat. The connection one-form part of $\nabla$, call it $\nabla^1 : D_{\Pi} \to D_{\Pi} \otimes_R \Omega^1$, defines a flat, torsion free connection, which taken together with $h$ in Equation (2), is similar to the pencil of flat connections on the tangent bundle of a moduli space of topological conformal field theories [6]. But $\nabla^1$ will, in general, depend on $h$. There is some evidence for the existence of special coordinates [12, 16] in which the $h$ dependence of $\nabla^1$ reduces it to a pencil of flat connections, equipping the moduli space with the structure of a representation of the operad $H(M_{0,n})$, but this is not proved in the present paper. One can think of the flat quantum superconnection $\nabla$ as furnishing a sort of Frobenius infinity manifold structure to the moduli space associated to a background.

In the physics language, $\nabla^1$ encodes, up to a choice of one point functions, all of the $n$ point correlation functions for $n > 1$ of all the theories in the moduli space. We have the compelling metaphor: a theory of graded quantum mechanics over the moduli space of quantum field theories is tantamount to the correlation functions of the theories in the space. This is accomplished without defining a measure for the path integral. From our point of view, the path integral amounts to nothing more than a choice of the one point functions, essentially a choice of basis of observables. The connection $\nabla^1$ determines the two point functions for every theory in a neighborhood of the moduli space, and the $n$-point functions for $n > 2$ are obtained via products and derivatives in the moduli directions. The rest of the superconnection, $\nabla^k$ for $k > 1$, defines a homotopy algebraic structure which is a chain level generalization of correlation functions which, to our knowledge, is new in physics. We offer an analogy with differential graded algebras: compare the correlation functions derived from $\nabla^1$ to the associative product in the homology of a differential graded algebra and compare our chain-level generalized correlation functions derived from $\nabla^k$ for $k > 1$ to a minimal $A_{\infty}$ structure defined in the homology of a dga that is quasi-isomorphic to the original dga. We suspect that the quantum background is determined up to quasi-isomorphism by the flat quantum superconnection $\nabla$, but we do not pursue this idea in this paper because $\nabla$ is defined here only for smooth formal backgrounds and so it’s difficult to determine the extent to which $\nabla$ can be interpreted as a “minimal model” for the background until obstructed backgrounds are better understood. Also, we acknowledge that our definition of “quasi-isomorphism” for quantum backgrounds may need some refinement—in particular, may need to encompass the concept of special coordinates.

The second reason that we develop $\text{QFT}_B$ alongside $\text{Def}$ is our sense that $\text{QFT}_B$ generalizes $\text{Def}$. The relationship is epitomized in Section 6 where we discuss an example related to differential BV algebras. The deformation theory of quantum field theories (i.e., the functor of $k[[\hbar]]$ algebras $\text{QFT}_B$ for a background $B$) is richer than classical deformation theory (i.e., the functor of $k$-algebras $\text{Def}_L$ for an $L_{\infty}$ algebra $L$). The fact that when $\text{QFT}_B$ is a smooth functor, the moduli space admits the quantum flat connection encoding
correlation functions and homotopy correlation functions is evidence that \( \text{QFT}_B \) is richer since the \( \text{Def}_L \)-moduli space does not naturally carry such an additional structure. The definition of a background is quite general and might be constructed from a wide variety of mathematical data. Our attitude is that the functor \( \text{QFT}_B \) provides a notion of quantum deformation theory that in some examples extends the classical deformation theory in the \( \hbar \) direction. When it is possible, one should attribute a background \( B \) to some initial mathematical data under investigation, rather than an \( L_\infty \) algebra \( L \) defined over \( k \), and study functors of \( k[[\hbar]] \)-algebras. Our expectation is that correlations, at least in special coordinates, arising from \( \text{QFT}_B \) reveal invariants of the initial data that is more subtle than what can be extracted from \( \text{Def}_L \) as a functor of \( k \)-algebras. However, in order to make a precise statement about \( \text{QFT}_B \) generalizing \( \text{Def}_L \), one should have a theory of special coordinates and perhaps generalize even further the kind of parameter rings on which \( \text{QFT}_B \) can be defined. One might think of \( \text{Def}_L \) as the functor that controls deformations of objects over (formal, graded, differential) commutative spaces whereas \( \text{QFT}_B \) controls deformations of objects over something like “quantized spaces.” We will expand on this point of view in another paper.

Acknowledgements. We thank Michael Schlessinger, James Stasheff, and Dennis Sullivan for many helpful and inspiring conversations and, in particular to Dennis Sullivan, for generous and supportive working conditions at CUNY. The authors also thank the referee for excellent suggestions: what clarity there is in the description of the flat quantum superconnection is due to the referee’s recommendations, and while we were not able, in the time allotted, to convert ourselves to the “functors from rings to simplicial sets” point of view, we recognize that it is a very good suggestion we intend to pursue it.

2. Categories, Functors, and Classical Deformation Theory

This section is essentially a technical primer about deformation theory approached via functors of parameter rings. The reader may wish to skip directly to section 3 where our new ideas are introduced.

2.1. Categories. First, we review some of the tools we employ. In order to describe the moduli space of quantum backgrounds, we use functors of parameter rings that are differential \( k[[\hbar]] \)-algebras, in addition to being differential graded local Artin rings.

2.1.1. Parameter rings. We will be working with certain parameter rings, which we now describe. Fix a field \( k \). Although we have interest in the case that \( k \) has characteristic \( p \), let us assume for this paper that the characteristic of \( k \) is zero. A graded Artin local algebra is defined to be a graded commutative associative unital algebra over \( k \) with one maximal ideal satisfying the ascending chain condition. Such algebras are those local algebras whose maximal ideals are nilpotent finite dimensional graded vector spaces. We use \( m_A \) to denote the maximal ideal of an Artin local algebra \( A \). The quotient \( A/m_A \) is a field called the residue field of \( A \). A differential on \( A \) is defined to be a degree one derivation \( d : A \to A \) satisfying \( d^2 = 0 \). We denote the homology of \( A \) by \( H(A) := \ker(d)/\text{Im}(d) \).

Let \( \mathcal{C} \) be the category whose objects are differential graded Artin local algebras with residue field \( k \) such that \([1] \neq [0]\) in \( H(A) \); morphisms are local homomorphisms, i.e.
differential graded algebra maps \( A \to B \) such that \( \text{Im}(m_A) = m_B \) and the induced maps on residue fields is the identity. Let \( \hat{\mathcal{C}} \) denote the category whose objects consist of differential graded complete noetherian local algebras \( A \) for which \( A/(m_A)^j \in \mathcal{C} \) for every \( j \) and whose morphisms are local. Note objects in \( \mathcal{C} \) are colimits of objects in \( \mathcal{C} \): 
\[ A = \varprojlim A/(m_A)^j. \]

Let \( \Lambda \in \text{ob}(\hat{\mathcal{C}}) \), concentrated in degree zero with zero differential, and denote the maximal ideal \( m_\Lambda \) by \( \mu \). We define the category \( \mathcal{C}_\Lambda \) to be the subcategory of \( \mathcal{C} \) consisting of differential graded Artin local \( \Lambda \)-algebras; morphisms in \( \mathcal{C}_\Lambda \) are local differential \( \Lambda \)-algebra homomorphisms. Let \( \mathcal{C}_\Lambda \) be the category of projective limits of \( \mathcal{C}_\Lambda \). We will be concerned primarily with \( \Lambda = k[[b]] \) or \( \Lambda = k \) itself (and \( \mathcal{C}_\Lambda = \mathcal{C} \)).

**Definition 2.1.** We call objects in \( \mathcal{C}_\Lambda \) parameter rings.

We have functors (“free”, “zero action”, and “underlying”) 
\[ \text{fr} : \hat{\mathcal{C}} \to \mathcal{C}_\Lambda, \quad z : \hat{\mathcal{C}} \to \hat{\mathcal{C}}, \quad \text{un} : \hat{\mathcal{C}} \to \hat{\mathcal{C}} \]
defined by \( \text{fr}(A) = A \hat{\otimes}_k \Lambda, \) \( z(A) = A \) where the action of \( \Lambda \) on \( A \) is given by \( \lambda a = 0 \) for \( \lambda \in \mu \) and \( a \in A \), and \( \text{un}(A) = A \), the underlying Artin algebra with the \( \Lambda \) action forgotten. Note that \( \text{un} \circ z = \text{Id} : \hat{\mathcal{C}} \to \hat{\mathcal{C}}. \)

2.1.2. **Tensor and fiber products of objects.** For \( A, B \in \text{ob}(\hat{\mathcal{C}}_\Lambda) \), we define \( A \otimes B := A \hat{\otimes}_\Lambda B \in \text{ob}(\hat{\mathcal{C}}_\Lambda) \) with grading determined by \( \text{deg}(a \otimes b) = \text{deg}(a) + \text{deg}(b) \) and differential \( d \) defined as \( d(a \otimes b) = d(a) \otimes b + (-1)^{\text{deg}(a)} a \otimes d(b) \), for homogeneous \( a \in A, b \in B \).

Let \( A, B, C \in \text{ob}(\mathcal{C}_\Lambda) \) and \( \alpha \in \text{hom}(A, C) \) and \( \beta \in \text{hom}(B, C) \). We define the fiber product \( A \times_C B \in \text{ob}(\mathcal{C}_\Lambda) \) by \( A \times_C B = \{(a, b) : \alpha(a) = \beta(b)\} \). In \( \mathcal{C}_\Lambda \), fibered products do not exist since \( A \times_C B \) may fail to be noetherian.

2.1.3. **Homotopy equivalence of morphisms and small extensions.** Let us now recall the homotopy model for the interval \( I \).

**Definition 2.2.** Define \( I \in \text{ob}(\hat{\mathcal{C}}_\Lambda) \) by \( I = k[[u, v]], \) \( \text{deg}(u) = 0, \text{deg}(v) = 1, \) and \( d(p(u) + q(u)v) = p'(u)v. \)

One cannot evaluate an arbitrary element of \( I \) at particular values of \( u \) and \( v \), but for the (non-Artin) subalgebra \( I' \) consisting of polynomials in \( u \) and \( v \) and one does have evaluation maps \( ev_i : I' \to k \) and \( ev_i : \text{fr}(I') \to \Lambda \), where \( ev_i \) is evaluation at \( (i, 0) \) for \( i = 0, 1 \).

**Definition 2.3.** Two morphisms \( \tau_0, \tau_1 : A \to B \) in \( \hat{\mathcal{C}}_\Lambda \) are called homotopy equivalent, or homotopic, if there exists an Artin subalgebra \( B' \subset B \otimes \text{fr}(I') \) and a morphism \( \tau(u, v) : A \to B' \) in \( \hat{\mathcal{C}}_\Lambda \) satisfying \( \tau(0, 0) := ev_0 \tau(u, v) = \tau_0 \) and \( \tau(1, 0) := ev_1 \tau(u, v) = \tau_1 \). As suggested by the terminology, homotopy equivalence is an equivalence relation. We denote the set of homotopy equivalence classes of morphisms \( A \to B \) by \( \text{hot}(A, B) \).

**Definition 2.4.** For \( \mathfrak{A} = \mathcal{C}_\Lambda, \hat{\mathcal{C}}_\Lambda \), we define the category \( [\mathfrak{A}] \) to be the homotopy category of \( \mathfrak{A} \). That is, the category defined by \( \text{ob}([\mathfrak{A}]) = \text{ob}(\mathfrak{A}) \) and \( \text{mor}(A, B) = \text{hot}(A, B) \).

Let \( \rho : A \to B \) be a surjective morphism in \( \mathcal{C}_\Lambda \).
Definition 2.5. We say $\rho$ is a small extension if $\ker(\rho)$ is a nonzero ideal $J$ such that $m_A J = 0$. We say that $\rho$ is an acyclic small extension if, in addition, $H(J) = 0$.

2.2. Functors of parameter rings. We now recall some basic definitions and properties of set valued functors of parameter rings. For details see [14, 8].

2.2.1. Representability. Any covariant functor $F : C_\Lambda \to \text{Sets}$, the category of sets, can be extended naturally to a functor $\hat{F} : C_\Lambda \to \text{Sets}$ by $\hat{F}(A) = \varprojlim F(A/m_A^j)$. A functor $\hat{F} : C_\Lambda \to \text{Sets}$ satisfying $F(A) = \varprojlim F(A/m_A^j)$ is called continuous. We consider continuous, covariant functors $\hat{F} : C_\Lambda \to \text{Sets}$ satisfying $\hat{F}(k) = F(\Lambda) = \{\text{one point}\}$. Such functors form a category $\text{Fun}_\Lambda$ whose morphisms are natural transformations. In the case that $\Lambda = k$, we will denote $\text{Fun}_\Lambda$ simply by $\text{Fun}$.

Precomposition with $\fr, z$, and $\un$ defined in (3) gives functors (which we denote by the same letters)

\[(4) \quad \fr : \text{Fun}_\Lambda \to \text{Fun}, \quad z : \text{Fun}_\Lambda \to \text{Fun} \text{ and } \un : \text{Fun} \to \text{Fun}_\Lambda.\]

Definition 2.6. We define a couple for $F \in \text{Fun}_\Lambda$ to be a pair $(A, \xi)$ where $A \in C_\Lambda$ and $\xi \in F(A)$.

Definition 2.7. For $R \in C_\Lambda$, we define $h_R : C_\Lambda \to \text{Sets}$ by $h_R(A) = \text{hom}_\Lambda(R, A)$, and define $[h_R] : [C_\Lambda] \to \text{Sets}$ by $([h_R](A)) = \text{hot}_\Lambda(R, A)$.

Definition 2.8. We say that $F \in \text{Fun}_\Lambda$ is represented by $R$ if $F$ is isomorphic to $h_R$ for some $R \in C_\Lambda$. We say that $F \in \text{Fun}_\Lambda$ is homotopy represented by $R$ if $F$ induces a functor $[C_\Lambda] \to \text{Sets}$ that is represented by $[h_R]$.

In the original terminology [14], Schlessinger called representable functors and couples “pro-representable” and “pro-couples”—we drop the prefix. Homotopy representable functors were treated by Manetti in [8] and Schlessinger-Stasheff in [15].

Remark 2.1. If $F$ is represented by $R$, then under the identification $\text{hom}(R, R) \simeq F(R)$, there is an element $\Pi \in F(R)$ corresponding to $\text{Id} \in \text{hom}(R, R)$. The element $\Pi$ is versal in the sense that for all $A \in C_\Lambda$, and for all $\xi \in F(A)$, there exists a unique morphism $\tau : R \to A$ so that $F(\tau)(\Pi) = \xi$. One may say that $F$ is represented by the couple $(R, \Pi)$.

Likewise, if $F$ is homotopy represented by $R$, then under the identification $\text{hot}(R, R) \simeq F(R)$, there is an element $\Pi \in F(R)$ corresponding to $[\text{Id}] \in \text{hot}(R, R)$. The element $\Pi$ is versal in the sense that for all $A \in C_\Lambda$, and for all $\xi \in F(A)$, there exists a morphism $\tau : R \to A$ unique up to homotopy, so that $F(\tau)(\Pi) = \xi$. One may say that $F$ is homotopy represented by the couple $(R, \Pi)$.

2.2.2. Smoothness and quasi-smoothness.

Definition 2.9. We say $F \in \text{Fun}_\Lambda$ is smooth if and only if the map $F(\tau) : F(A) \to F(B)$ is surjective for all small extensions $\tau : A \to B$.

Definition 2.10. We say $F \in \text{Fun}_\Lambda$ is quasi-smooth if and only if the map $F(\tau) : F(A) \to F(B)$ is surjective for all acyclic small extensions $\tau : A \to B$. 
2.2.3. The Schlessinger condition. Let $A, B, C \in \text{ob}(\mathcal{C}_A)$ and $\alpha \in \text{hom}(A,C)$, $\beta \in \text{hom}(B,C)$ and consider the fibered product $A \times_C B$. For any functor $F \in \mathfrak{Fun}_A$, the versal property of fibered products (of sets) gives a map

$$\eta : F(A \times_C B) \to F(A) \times_{F(C)} F(B)$$

(5)

**Definition 2.11.** We say that $F \in \mathfrak{Fun}_A$ satisfies the Schlessinger condition if the map $\eta$ in equation (5) is a bijection.

Readers familiar with Schlessinger’s work may recognize what we call the Schlessinger condition as a version of his condition (H4) (see theorem 2.11 in [14]). This condition has also been called the Mayer-Vietoris property [10] because of the relationship to abstract homotopy theory [3].

2.2.4. Homotopy equivalence. Let $G \in \mathfrak{Fun}_A$ and $A \in \text{ob}(\mathcal{C}_A)$, we define a relation on the set $G(A)$ by setting $\xi_0 \sim \xi_1$ if and only if there exists an Artin subalgebra $A' \subset A \otimes \text{fr}(I)$ and a $\xi(u,v) \in G(A')$ with $\xi(0,0) = \xi_0$ and $\xi(1,0) = \xi_1$. We’re using the shorthand $\xi(i,0)$ for $G(ev_i)(\xi(u,v))$ where $ev_i : A \otimes \text{fr}(I') \to A$ is the evaluation map determined by $(u,v) = (i,0)$ for $i = 0, 1$. We refer the reader to [8] for details.

**Proposition 2.1.** If $G$ is quasi-smooth and satisfies the Schlessinger condition, then $\sim$ is an equivalence relation and the functor $F \in \mathfrak{Fun}_A$ defined by $F(A) = G(A)/\sim$ induces a functor on the category $[\mathcal{C}_A]$.

**Proof.** The nontrivial, but standard, part is proving that the homotopy relation is transitive. (see [8] theorem 2.8).

2.2.5. Tangent space and tangent module. For $i \in \mathbb{Z}$, we define the parameter ring $E_i$ by $E_i := k[\epsilon_i]/\epsilon_i^2$ where $\epsilon_i$ has degree $-i$ and $d(\epsilon_i) = 0$.

**Definition 2.12.** Then, for any $F \in \mathfrak{Fun}_A$, we define the tangent space to $F$ by

$$T_F = \oplus_i T_F^i := \oplus_i F(E_i).$$

If $F$ commutes with certain products of rings with trivial multiplicative structures then $T_F$ naturally has the structure of a graded vector space. Specifically, if

$$F(A \times_k B) \simeq F(A) \times F(B)$$

for all rings $A$ and $B$ with $m_A^2 = m_B^2 = 0$ and $d(A) = d(B) = 0$ (the so-called vector space objects in $\mathcal{C}_A$) then $F(A)$ will be a vector space. For example, if $F$ satisfies the Schlessinger condition, then $T_F$ is a vector space. Also, and importantly, if $F = G/\sim$ as in proposition 2.1 then $T_F$ is a vector space. Furthermore, $T$ defines a functor from the subcategory of $\mathfrak{Fun}_A$ consisting of those functors satisfying (6) to $\mathfrak{Vect}$, the category of graded vector spaces.

One can augment the tangent space slightly and obtain a natural $A$-module structure. For each $\alpha \in A$, one uses the morphisms

$$\oplus_i \text{fr}(E_i) = \oplus_i A[\epsilon_i]/\epsilon_i^2 \xrightarrow{\alpha \epsilon_i} \oplus_i A[\epsilon_i]/\epsilon_i^2 = \oplus_i \text{fr}(E_i)$$

to get maps $\alpha : \oplus_i F(\text{fr}(E_i)) \to \oplus_i F(\text{fr}(E_i))$. 


Definition 2.13. For any \( F \in \mathfrak{Fun}_\Lambda \), we define the tangent \( \Lambda \) module to \( F \) by

\[
D_F := \bigoplus_i D^i_F \oplus_i \mathfrak{fr}(E_i).
\]

Like the tangent space, \( D \) defines a functor from the subcategory of \( \mathfrak{Fun}_\Lambda \) consisting of those functors satisfying (6) to \( \mathfrak{Mod}_\Lambda \), the category of \( \Lambda \)-modules.

Proposition 2.2. Suppose that \( G \in \mathfrak{Fun}_\Lambda \) is quasi-smooth and satisfies the Schlessinger condition. Let \( F = G/\sim \) as in proposition 2.1 and suppose that \( T_F \) is finite dimensional with basis \( \{\xi_i\} \). Let \( \{t_i\} \) be the dual basis for \( T_F \). Then, there exists a differential \( \delta \) on the algebra \( R := \Lambda[[t_i]] \) with \( \delta(m_R) \subseteq (m_R)^2 \) and an element \( \Xi \in F(R, \delta) \) so that \( F \) is homotopy represented by the couple \((R, \delta), \Xi)\).

Proof. See theorem 4.5 and corollary 4.6 in [8]. \( \square \)

Let us recall one more result from the general theory of functors of parameter rings, which provides a theoretical underpinning for the definition of “quasi-isomorphism” in sections 2.3 and 3.

Proposition 2.3. Suppose \( G, G' \in \mathfrak{Fun}_\Lambda \) are quasi-smooth and satisfy the Schlessinger condition. Let \( F = G/\sim \) and \( F' = G'/\sim \) and suppose \( \mathfrak{n} : F \to F' \) is a natural transformation. Then \( \mathfrak{n} \) is an isomorphism of functors if and only if \( D_{\mathfrak{n}} : D_F \to D_{F'} \) is an isomorphism of graded \( \Lambda \) modules.

2.3. Classical deformation theory. We now recall the basic elements of classical deformation theory. In this subsection we define the category of \( L_\infty \) algebras and define a functor \( \text{Def}_L \) for each \( L_\infty \) algebra \( L \). At the end of the section, we highlight the properties of \( \text{Def}_L \) (as described in the section 2.2 for general functors) as they pertain to \( L \).

Now, we set \( \Lambda = k \).

2.3.1. \( L_\infty \) algebras. Let \( V = \bigoplus_{j \in \mathbb{Z}} V^j \) be a graded vector space over \( k \). As usual, let \( V[n] \) denote the shift \( V[n] = \bigoplus_{j \in \mathbb{Z}} V^{n+j} \), where \( V^{n+j} = V_j \). Let \( S^i V \) denote the \( i \)-th symmetric product of \( V \). Note that \( \wedge : S^i V[1] \otimes S^j V[1] \to S^{i+j} V[1] \) makes

\[
C(V) := \bigoplus_{i=0}^\infty S^i (V[1])
\]

into a differential graded commutative associative algebra. It is also a graded cocommutative coassociative coalgebra. The coproduct \( C(V) \to C(V) \otimes C(V) \) is characterized by the property that it is map of differential graded algebras, and that \( x \mapsto x \otimes 1 + 1 \otimes x \) for each \( x \in V[1] \). Indeed, \( C(V) \) is a construction of the free cocommutative coassociative coalgebra over \( V[1] \). The versal property characterizing this free construction is: for any graded coalgebra \( C \) and any graded linear map \( C \to V[1] \), there exists a unique coalgebra map \( C \to C(V) \) extending the linear map. This gives an isomorphism of graded vector spaces: \( \text{hom}_\text{linear}(C, V) \cong \text{hom}_\text{coalgebra}(C, C(V)) \).

We mention a second identification. Any homomorphism \( \sigma : S^i V[1] \to V \) extends uniquely to a map \( \mathfrak{f} \in \text{Coder}(C(V)) \), and any coderivation \( \sigma \in \text{Coder}(C(V)) \) is determined by its components \( \sigma_i : S^i V[1] \to V \). Thus, \( \text{hom}_\text{linear}(C(V), V) \cong \text{Coder}(C(V)) \).

Definition 2.14. An \( L_\infty \) algebra \( L \) is defined to be a pair \( L = (V, D) \) where \( V \) is a graded vector space, and \( D \in \text{Coder}^1(C(V)) \) is a coderivation of degree 1, satisfying \( D^2 = 0 \). A
morphism \( \sigma : L \to L' \) between two \( L_\infty \) algebras \( L = (V, D) \) and \( L' = (V', D') \) consists of a degree zero coalgebra map \( \sigma : C(V) \to C(V') \) with \( \sigma \circ D = D' \circ \sigma \). Let us denote the category of \( L_\infty \) algebras by \( \mathfrak{L} \).

We will use the notation \( d_i \) for \( D_i : S^i V[1] \to V \), the \( i \)-th component of \( D \).

2.3.2. Maurer-Cartan Functor.

**Definition 2.15.** Let \( L = (V, D) \in \text{ob}(\mathfrak{L}) \). We define a functor \( \text{MC}_L \in \mathfrak{g} \text{un} \) by

\[
\text{MC}_L(A) = \{ \text{degree 0 differential coalgebra maps } m_A^* \to C(V) \}.
\]

Here, \( m_A^* \) is the differential coalgebra \( m_A^* := \text{hom}(m_A, k) \).

The functor \( \text{MC}_L \) satisfies the Schlessinger condition and is quasi-smooth. Therefore, we can form the quotient \( \text{MC}_L \sim \), where \( \sim \) is the natural gauge equivalence defined in Section 2.2.4. Let us make a list of definitions:

**Remark 2.2.** Any coalgebra map \( m_A^* \to C(V) \) is determined by a linear map \( m_A^* \to V \); equivalently, by and element in \( V \otimes m_A \). Then

\[
\gamma \in \text{MC}_L \iff d_1(\gamma) + \frac{1}{2!} d_2(\gamma \wedge \gamma) + \frac{1}{3!} d_3(\gamma \wedge \gamma \wedge \gamma) + \cdots = 0.
\]

where the \( \gamma \) on the right is viewed as an element in \( V \otimes m_A \) and \( d_i : S^i((V \otimes m_A)[1] \to V \otimes m_A \) is extended from the components of the \( L_\infty \) structure on \( C(V) \) using the differential and multiplication in \( A \).

**Definition 2.16.** Let \( L \) be an \( L_\infty \) algebra.

1. Define the quotient \( \text{Def}_L := \text{MC}_L \sim \) where \( \sim \) is the natural gauge equivalence.
2. The assignment \( L \mapsto \text{Def}_L \) defines a functor from \( \text{Def} : \mathfrak{L} \to \mathfrak{g} \text{un} \).
3. We define \( H(L) \) to be the vector space \( H(L) := T_{\text{Def}_L}[1] \) and call it the homology of \( L \). The composition \( H := T \circ \text{Def} \) defines a functor \( H : \mathfrak{L} \to \mathfrak{Vect} \).

**Remark 2.3.** In practice, the classical deformation theory of a mathematical object \([14, 7, 8] \) is controlled by an \( L_\infty \) algebra \( L \). This means that given a mathematical object, for which there exists the concept of a deformation of that object over a parameter ring \( A \) and the concept of equivalence of such deformations, there is a bijection of sets

\[
\left\{ \text{(equivalence classes of) deformations of the object over the ring } A \right\} \leftrightarrow \text{Def}_L(A).
\]

**Proposition 2.4.** Suppose \( L \) is an \( L_\infty \) algebra and \( T_{\text{Def}_L} = H(L)[-1] \) is finite dimensional with basis \( \{ \alpha^1, \ldots, \alpha^k \} \). Let \( \{ s_i \} \) be the dual basis and let \( S := k[[s_1, \ldots, s_k]] \). There exists a differential \( d : S \to S \) with \( d(m_S) \subset (m_S)^2 \) and \( \Gamma \in \text{MC}_L(S) \) such that \( \text{Def}_L \) is homotopy represented by the couple \( ((S, d), \Gamma) \).

**Proof.** Apply proposition 2.2 to \( \text{Def}_L \).

**Definition 2.17.** Let \( L \) be an \( L_\infty \) algebra and assume \( H(L) \) is finite dimensional.

1. We call \( L \) finite.
(2) A morphism \( L \to L' \) is called a quasi-isomorphism if the map \( \text{Def}_L \to \text{Def}_{L'} \) is an isomorphism of functors, in which case \( L \) and \( L' \) are called quasi-isomorphic.

(3) We call \( \Gamma \) in proposition 2.4 a versal solution to the Maurer-Cartan equation.

(4) We call \( L \) smooth formal if \( d = 0 \) for \( d : S \to S \) as in proposition 2.4.

Remark 2.4. In light of the proposition 2.3, one can assemble the parts of the definition above to say that \( L \) and \( L' \) are quasi-isomorphic if and only if there exists a morphism \( \sigma : L \to L' \) inducing an isomorphism \( H(L) \to H(L') \), which together with a computation establishing
\[
T_{\text{Def}_L} \simeq \text{Ker}(d_1)/\text{Im}(d_1),
\]
may be used to give an alternative definition of quasi-isomorphism as a morphism \( L \to L' \) inducing an isomorphism in homology.

3. QUANTUM BACKGROUNDS

Quantum backgrounds were invented to provide an algebro-mathematical formulation of quantum field theory. We begin by defining a quantum background (see Definition 3.2) which one can think of as the input data, leading naturally to a functor \( \text{QFT} \) (see Definition 3.4). From now on, we set \( \Lambda = k[[\hbar]] \).

3.1. The category \( \mathcal{B} \).

Definition 3.1. We say that a graded associative, unital \( \Lambda \)-algebra \( P \) has a classical limit provided \( P \) is free as a \( \Lambda \) module and if \( K = P/\hbar P \) is a graded, commutative, associative, unital \( k \)-algebra.

Note that if \( P \) has a classical limit, then \( P \simeq K \oplus \hbar K \oplus \hbar^2 K \oplus \cdots \) and \( [P, P] \subset \hbar P \).

Definition 3.2. We define a background \( B \) to be a four-tuple \( B = (P, N, m, \varphi) \) where

(1) \( P = \bigoplus_i P^i \) is a graded, associative, unital \( \Lambda \)-algebra with a classical limit,
(2) \( N = \bigoplus_i N^i \) is a graded left \( P \) module, which is free as a \( \Lambda \) module,
(3) \( m \in P^1 \) satisfies \( m^2 = 0 \) (\( m \) is called a structure),
(4) \( \varphi \in N^0 \) satisfies \( m \cdot \varphi = 0 \) (\( \varphi \) is called a vacuum).

A morphism between two backgrounds \( B = (P, N, m, \varphi) \) and \( B' = (P', m', N', \varphi') \) consists of a map \( \sigma : P \to P' \) of graded \( \Lambda \) algebras and a map \( \tau : N \to N' \) of graded \( P \) modules (where \( N' \) becomes a \( P \) module via \( \sigma \)) with \( \tau(\varphi) = \varphi' \) satisfying the compatibility
\[
\tau(ma \varphi) = m' \sigma(a) \varphi' \quad \text{for all} \quad a \in P.
\]

Denote the category of backgrounds by \( \mathcal{B} \).

3.2. Quantum master equation.

Definition 3.3. Let \( B = (P, N, m, \varphi) \) be a background, let \( (A, d) \in \text{ob}(\mathcal{C}_\Lambda) \), where \( \mathcal{C}_\Lambda \) is the category of differential graded Artin local \( \Lambda = k[[\hbar]] \) algebras, and \( \pi \in (P \otimes m_\Lambda)^0 \).

We call the equation
\[
\left( e^{\pi/\hbar} m e^{-\pi/\hbar} - \hbar e^{\pi/\hbar} d \left( e^{-\pi/\hbar} \right) \right) \varphi = 0
\]
the quantum master equation, and denote the set of solutions by $\text{QM}_B(A)$:

$$\text{QM}_B(A) = \left\{ \pi \in (P \otimes m_A)^0 \text{ such that } \left( e^{\pi/h} me^{-\pi/h} - h e^{\pi/h} d \left( e^{-\pi/h} \right) \right) \varphi = 0 \right\}.$$  

Notice that $m_k = 0$, so $\text{QM}_B(k) = \{0\}$, and any morphism $\tau : A \to B \in \text{hom}(\mathcal{C}_A)$ gives rise to a map from $\text{QM}_B(A) \to \text{QM}_B(B)$ given by $\pi \mapsto (1 \otimes \tau)\pi$. Thus, we have defined a functor $\text{QM}_B \in \mathfrak{Fun}$.  

Definition 8 relies on the fact that both terms $e^{\Pi/h} me^{-\Pi/h}$ and $h e^{\Pi/h} d (e^{-\Pi/h})$ are well defined elements of $P \otimes A$, a fact that we clarify in two remarks. An example follows in a third remark.

**Remark 3.1.** First, we establish the meaning of $\frac{[\eta, \zeta]}{h}$ in $P \otimes A$. The ring $P$ by assumption is free as a $\Lambda$ module, but $A$, owing to its nilpotency, cannot be and so $\hbar$ is a zero divisor in $P \otimes A$. However, in $P$, one has $[P, P] \subseteq hP$, so one has an operator

$$\frac{[\cdot]}{h} : P \times P \to P$$

$$\alpha, \beta \mapsto \gamma$$

where $\gamma \in P$ is defined by expressing $[\alpha, \beta] = h \gamma$. Then, for $\eta = \alpha \otimes a$ and $\zeta = \beta \otimes b$ in $P \otimes A$, one defines

$$\frac{[\eta, \zeta]}{h} = \frac{[\alpha \otimes a, \beta \otimes b]}{h} = \gamma \otimes ab \in P \otimes A.$$  

Then, $\frac{1}{h}[\eta_1, \cdots [\eta_{j-1}, \eta_j] \cdots] \subseteq P \otimes A$ is understood in the same way.

**Remark 3.2.** Expanding $e^{\pi/h} me^{-\pi/h}$ and $h e^{\pi/h} d (e^{-\pi/h})$ in terms of repeated commutators:

$$e^{\pi/h} me^{-\pi/h} = \exp \left( \frac{[\cdot, \pi]}{h} \right) (m) = m + \frac{[m, \pi]}{h} + \frac{[m, \pi], \pi}{2h^2} + \cdots$$

displays each term as an element of $P \otimes A$, and the sum terminates since $m_A$ is nilpotent. The same holds for the $d$ term once expanded as

$$h e^{\pi/h} d (e^{-\pi/h}) = f \left( \frac{[\cdot, \pi]}{h} \right) (d \pi),$$

where $f(x) = \frac{e^x - 1}{x} = \sum_{j \geq 0} \frac{x^j}{(j+1)!}$.  

**Remark 3.3.** In certain cases, the quantum background $B$ arises from a differential BV algebra $(V, d, \Delta, \cdot)$ with Lie bracket $(, )$. In these cases, the quantum master equation defined for the background as defined above reduces to the usual quantum master equation seen in the physics literature:

$$d \pi + h \Delta \pi + \frac{1}{2} (\pi, \pi) = 0$$

for $\pi \in V \otimes m_A$. This example is illustrated in Section 6.

**Theorem 3.1.** The functor $\text{QM}_B$ is quasismooth and satisfies the Schlessinger condition.

**Proof.** Since $\text{QM}_B$ is given as a solution set of an equation, $\text{QM}_B$ satisfies the Schlessinger condition. To show that it is quasi-smooth, let $(A, d) \in \text{ob}(\mathcal{C}_A)$ and let $C_A =$
\((P \otimes A)\varphi\) denote the cyclic submodule of \(N \otimes A\) generated by the vacuum \(\varphi (= \varphi \otimes 1)\).

Note that \(C_A\) is a complex with differential \(D : C_A^i \to C_A^{i+1}\) defined by
\[
D((\pi \otimes a)\varphi) = (m\pi \otimes a)\varphi + (-1)^{|a|}h(\pi \otimes d(a))\varphi.
\]

Let \(\tau : A \to A'\) be an acyclic small extension with kernel \(I\) and let \(a' \in \text{QM}_B(A')\).
Since \(\tau\) is surjective, we can choose an \(a \in P \otimes m_A\) so that \(1 \otimes \tau(a) = a'\). Let
\[
i = e^{a/h}me^{-a/h} - he^{a/h}d(e^{-a/h}) = (e^{-a/h}) \in P_A.
\]

Notice that \((1 \otimes \tau)(i\varphi) = 0\). So, \(i\varphi\) is in the kernel of \(1 \otimes \tau : C_A \to C_{A'}\).
Since the functor \(\otimes\) is right-exact, \(i\varphi \in C_I\), and we can write \(i\varphi = i'\varphi\) for some \(i' \in I\). Now, since \(a \in m_A\), we have \(ai' = 0\) and \(e^{-a/h}i' = i'\). Now, we have
\[
i' \varphi = e^{-a/h}i' \varphi = e^{-a/h}i\varphi = e^{-a/h}(e^{a/h}me^{-a/h} - he^{a/h}d(e^{-a/h}))\varphi = (me^{-a/h} - hd(e^{-a/h}))\varphi.
\]

Now, consider
\[
D(i' \varphi) = (m^2e^{-a/h} - hmd(e^{-a/h}) + dme^{-a/h} - hd^2(e^{-a/h}))\varphi = 0.
\]

Since \(I\) is acyclic, \(C_I\) is acyclic, which implies that there exists a \(j\varphi \in C_I^0\) with \(Dj\varphi = i'\varphi\). Define \(a' = a - hj \in (P \otimes m_A)^0\). The claim is that \(a' \in \text{QM}_B(A)\). To show this, we compute. First note that \(aj = 0\) and \(j^2 = 0\) imply that \(e^{a/h} = e^{a/h} + j\) and \(e^{-a/h} = e^{-a/h} - j\). So
\[
\left(e^{a/h}me^{-a/h} - he^{a/h}d(e^{-a/h})\right)\varphi = \left((e^{a/h} - j)m(e^{-a/h} + j) - h(e^{a/h} - j)d(e^{-a/h} + j)\right)\varphi
= i\varphi - jm\varphi - mj\varphi - jm\varphi + hjde^{-a/h}\varphi - he^{a/h}dj\varphi = (i' - Dj)\varphi = 0.
\]

As a corollary, we make the following definition:

**Definition 3.4.** Let \(B\) be a background.

1. We define \(\text{QFT}_B := \text{QM}_B/\sim\) where \(\sim\) is the natural gauge equivalence.
2. The assignment \(B \mapsto \text{QFT}_B\) defines a functor from \(\text{QFT} : \mathcal{B} \to \text{Vect}\).
3. We define \(H(B) := T\text{QFT}_B\) and call it the homology of \(B\). The composition \(H := T \circ \text{QFT}\) defines a functor \(H : \mathcal{B} \to \text{Vect}\). In addition, we define the Dirac module of \(B\) to be the \(\Lambda\)-module \(D(B) := D\text{QFT}_B\).

The composition \(D := D \circ \text{QFT}\) defines a functor \(D : \mathcal{B} \to \text{Mod}\Lambda\).

**Proposition 3.2.** Let \(B\) be a background. There are natural isomorphisms
\[
D(B) \simeq \left\{ a \in P : \left[\frac{m}{h}a\right] \varphi = 0 \right\} / \sim
\]
and
\[
H(B) \simeq \left\{ a \in P \otimes \Lambda k : \left[\frac{m}{h}a\right] \varphi = 0 \right\} / \sim
\]
where \( a_1 \sim a_0 \) if and only if \( (a_1 - a_0)\varphi = \frac{[m,a]}{\hbar} \varphi \) for some \( b \in P \) (or \( P \otimes \Lambda k \)).

**Proof.** We set up the natural isomorphism (10) in homogeneous pieces by

\[ \xi \leftrightarrow a. \]

For \( \xi \in (P \otimes \Lambda m_{E_i})^0 \), with \( \xi = a\epsilon_i \), where \( a \in P^{-i} \otimes \Lambda k \) and \( \epsilon_i \in m_{E_i} \), one writes

\[ e^{\alpha_i/\hbar} me^{-\alpha_i/\hbar} \varphi = \left( m - \frac{1}{\hbar}[m,a]\epsilon_i + \frac{1}{2\hbar^2}[m,a][m,a]\epsilon_i^2 + \cdots \right) \varphi = -\frac{1}{\hbar}[m,a]\epsilon_i \varphi \]

and sees that \( \xi \in QM_B(E_i) \) if and only if \( \frac{[m,a]}{\hbar} \varphi = 0 \).

Now we check equivalence. First suppose \( \frac{[m,a]}{\hbar} \varphi = 0 \) and \( \frac{[m,a]}{\hbar} \varphi = 0 \) and \( (a_1 - a_0)\varphi = \frac{[m,b]}{\hbar} \varphi \) for some \( b \in P \). Then, define \( \xi(u,v) = (a_1 u + a_0 (1 - u) + bv)\epsilon_i \).

One sees that \( \xi(0,0) = \xi_0 = a_0\epsilon_i \) and \( \xi(1,0) = \xi_1 = a_1\epsilon_i \). To see that \( \xi(u,v) \in QM_B(E_i \otimes \mathfrak{fr}(k[u,v]/(u^2,uv,v^2))) \), compute:

\[ e^{\xi(u,v)/\hbar} me^{-\xi(u,v)/\hbar} - h e^{\xi(u,v)/\hbar} \delta \left( e^{-\xi(u,v)/\hbar} \right) \varphi = 0. \]

Conversely, let

\[ \xi(u,v) = (a(u) + b(u)v)\epsilon_i \in P^{-i} \otimes I' \]

for \( a(u), b(u) \in P^{-i} \otimes \Lambda k[u] \) for some \( I' \subseteq k[u,v] \). Then \( \xi(u,v) \in QM_B(E_i \otimes \mathfrak{fr}(I')) \) implies that \( \frac{[m,a(u)]}{\hbar} \varphi = 0 \) and \( a'(u)\varphi = \frac{[m,b(u)]}{\hbar} \varphi \). Define \( b = \int_0^1 b(u)du \). Then, for \( \xi(0,0) = a_0\epsilon_i \) and \( \xi(1,0) = a_1\epsilon_i \), \( a_0, a_1 \in P^{-i} \otimes \Lambda k \), we have \( \frac{[m,a_0]}{\hbar} \varphi = 0 \) and \( \frac{[m,a_1]}{\hbar} \varphi = 0 \) and \( (a_1 - a_0)\varphi = \frac{[m,b]}{\hbar} \varphi \).

The computation for \( D(B) \) is the same, except that the correspondence (9) \( \xi \leftrightarrow [a] \) is setup for \( \xi \in (P \otimes \Lambda m_{\mathfrak{fr}(E_i)})^0 \) \( P^{-i} \otimes k m_{E_i} \), with \( \xi = a\epsilon_i \) with \( a \in P^{-i} \) and \( \epsilon_i \in m_{E_i} \).

**Corollary 3.3.** Let \( C_\Lambda = P \varphi \) and \( C_k = (P \otimes \Lambda k) \varphi \) be the cyclic submodules of \( N \) and \( N \otimes k \) generated by the vacuum. Then multiplication on the left by \( m \) defines differentials \( m : C_\Lambda^i \to C_\Lambda^{i+1} \) and \( m : C_k^i \to C_k^{i+1} \). Then the Dirac space and tangent space are isomorphic to the homology of these complexes \( D(B) \simeq H(C_\Lambda, m) \) and \( H(B) \simeq H(C_k, m) \).

**Theorem 3.4.** Suppose that \( B \) is a background and \( T_{QFT_B} = H(B) \) is finite dimensional. with basis \( \{ \xi^1, \ldots, \xi^\ell \} \). Let \( \{ t_1, \ldots, t_r \} \) be the dual basis and let \( R = \Lambda[[t_1, \ldots, t_r]] \). Then there exists a differential \( \delta : R \to R \) with \( \delta(m_R) \subset (m_R)^2 \) and \( \Pi \in QM_B(R) \) such that \( QFT_B \) is homotopy represented by the couple \( (R, \delta, \Pi) \).

**Proof.** Apply proposition (2.2) to \( QFT_B \).

**Definition 3.5.** Let \( B \) be a background and assume \( H(B) \) is finite dimensional.

1. We call \( B \) finite.
2. A morphism of backgrounds \( B \to B' \) is called a quasi-isomorphism if \( QFT_B \to QFT_{B'} \) is an isomorphism of functors, in which case we say \( B \) and \( B' \) are quasi-isomorphic.
4. Flat Quantum Superconnection

One is able to supply derived algebro-geometric interpretations of structures associated to smooth formal backgrounds. Some of these structures have been unearthed in specific examples, and our aim in this section to explain this structure from a unifying perspective afforded by backgrounds. For example, our main construction is a flat superconnection \(\nabla\) on a bundle over the functor \(\text{QFT}_B\). The connection one-form part of \(\nabla\), which we call \(\nabla^1\), defines a flat connection which coincides with the connection defined in [11] in the setting of quantum periods. Throughout this section we assume that \(B\) is a smooth formal background.

4.1. Conceptual description. First, to define a bundle over a functor \(F\) of parameter rings, one assigns an \(A\) module \(M\) to each point \(\pi \in F(A)\), functorial with respect to ring maps \(A \to A'\). In the case of \(\text{QM}_B\) for a background \(B\) and a point \(\pi \in \text{QM}_B(A)\) where \(A\) is a parameter ring with zero differential, we can define a new background \(B_\pi\) as

\[
B_\pi := (B \otimes A, m^\pi, N \otimes A, \phi \otimes 1)
\]

where \(m^\pi := e^{\pi/\hbar}m e^{-\pi/\hbar}\). To see that \(B_\pi\) defines a background, note that \(m^\pi\) squares to zero (trivially) and that \(m^\pi(\phi \otimes 1) = 0\) by the quantum master equation. Let

\[
D_\pi := D(B_\pi) = \{ x \in \pi \otimes A : \frac{[m^\pi, x]}{\hbar} \phi = 0 \} / \sim
\]

where \(C_A = (P \otimes A)\phi\) is the cyclic submodule of \(N \otimes A\) generated by the vacuum. The Dirac space \(D_\pi\) of the background \(B_\pi\) is an \(A\) module. If \(f : A \to A'\) is a map of parameter rings, we obtain a map of background \(B_\pi \to B_{F(f)(\pi)}\) and hence a map of Dirac spaces. Furthermore, if \(\pi \sim \pi'\), the backgrounds \(B_\pi\) and \(B_{\pi'}\) are quasi-isomorphic, hence have isomorphic Dirac spaces. Thus, the construction of \(D_\pi\) from \(\pi \in \text{QM}_B(A)\) defines a bundle over the functor \(\text{QFT}_B\) when \(\text{QFT}_B\) is smooth formal. Call this bundle the versal Dirac bundle and denote it by \(\tilde{D}\). We haven’t defined a background \(B_\pi\) for \(\pi \in QM_B(A)\) when \(A\) is a parameter ring with nonzero differential, but it’s straightforward (but not necessary for smooth backgrounds) to modify the definition of \(D_\pi\) for such a point \(\pi\). Set

\[
D_\pi := H(C_A, m + d)
\]

where \(d\) is the differential in \(A\). The versal Dirac bundle resonates with how one thinks of the quantum master equation: a solution of the quantum master equation over a ring \(A\) with no differential defines a new background \(B_\pi\) defined over \(A\). By taking the Dirac space of \(B_\pi\) one obtains an \(A\) module which is the fiber of a bundle over the moduli space.

Next, we describe the construction of a flat connection on the versal Dirac bundle \(\tilde{D}\) of a smooth formal background. According to Grothendieck, a flat connection on a bundle over a functor is given by a collection of \(A\) module isomorphisms \(D_0 \otimes A \to D_{\pi}\) for each point \(\pi\) over \(A\), identifying the fiber over \(\pi\) with the fiber over 0. It will be convenient to use a representing couple \((R, \Pi)\) where we take \(R\) to have zero differential. Using the representing couple, we replace \(\tilde{D}\) with the \(R\) module \(D_{\Pi}\) and obtain a flat connection by
defining an isomorphism \( D_0 \otimes R \to D_{\Pi} \). This suffices to define a flat connection on the versal Dirac bundle since any other point \( \pi \) over \( A \) is given by a morphism of parameter rings \( R \to A \) inducing \( D_{\Pi} \to D_{\pi} \) and induces an isomorphism \( D_0 \otimes A \to D_{\pi} \).

In terms of a representing couple \((R, \Pi)\), we have the moduli space \( \mathcal{M} = \text{spec}(R) \) and \( R = \Lambda[[H^*]] = k[[h, H^*]] \) where \( H \) is the homology of the background and \( H^* \) is the linear dual \( H^* = \text{hom}_k(H, k) \). Since \( H = \text{hom}_A(R, k[t]/t^2) \) is isomorphic to the derivations of \( R \), the vector space \( H \) is naturally isomorphic to the tangent space of the formal space \( \mathcal{M} \) at its base point. Let \( D_0 = D(B) \) denote the Dirac space of the original background, which is the fiber of the Dirac bundle at the base point. We will prove that \( H \otimes_k R \simeq D_{\Pi} \), which implies that \( D_0 \simeq H[[h]] \) and \( D_0 \otimes R \simeq H \otimes_k R \simeq D_{\Pi} \), which defines a flat connection on \( \widetilde{D} \). From another point of view, bundles over \( M \) correspond to \( R \) modules. In particular, sections of the versal Dirac bundle \( \widetilde{D} \) are elements in the \( R \) module \( D_{\Pi} \), which we will also call the versal Dirac bundle. The connection that we define, \( \nabla^1 \), defines a map

\[
\nabla^1 : D_0 \times D_{\Pi} \to D_{\Pi}
\]

\[
(X, \mathcal{Y}) \mapsto \nabla^1_X \mathcal{Y}
\]

which is \( R \) linear in the first coordinate \( \nabla^1_{fX} \mathcal{Y} = f \nabla^1_X \mathcal{Y} \) and satisfies the \( h \)-connection equation in the second

\[
\nabla^1_X (f \mathcal{Y}) = hX(f) \mathcal{Y} + (-1)^{|f|} f \nabla^1_X \mathcal{Y}
\]

for \( f \in R, X \in D, \mathcal{Y} \in D_{\Pi} \).

We prove that \( \nabla^1 \) is flat:

\[
\nabla^1_X \nabla^1_1 Z - \nabla^1_Y \nabla^1_1 Z - \nabla^1_{[X,Y]} Z = 0.
\]

Equivalently, we can use \( \Omega := R[[H^*[1]]] = k[[h, H^*, H^*[1]]] \), the module of Kahler differentials on \( R \), to write the connection using one-forms:

\[
\nabla^1 : D_{\Pi} \to D_{\Pi} \otimes_R \Omega^1.
\]

In fact, what we define is a “chain level” flat connection on the bundle of cyclic modules \( C_R = (P \otimes_A R)\varphi \)

\[
\nabla^{\text{chain}} : C_R \to C_R \otimes_R \Omega^1
\]

which descends to \( \nabla^1 \) on the Dirac space. However, a deeper analysis of \( \nabla^{\text{chain}} \) on \( C_R \) reveals that when \( \nabla^{\text{chain}} \) is compressed from the chain level to the Dirac space, it manifests as a superconnection

\[
\nabla : D_{\Pi} \to D_{\Pi} \otimes_R \Omega
\]

which can be decomposed as \( \nabla = \nabla^1 + \nabla^2 + \nabla^3 + \cdots \) with each \( \nabla^i : D_{\Pi} \to D_{\Pi} \otimes \Omega^i \).

In a basis of \( D_{\Pi} \), \( \nabla^i = d_{dR} + A^i \) where \( d_{dR} \) is the de Rham differential in \( \Omega \) and \( A^i \) is a matrix of \( i \)-forms. We prove that the quantum superconnection \( \nabla \) is flat. The relationship between \( \nabla^1 \) to \( \nabla \) is analogous to a familiar one concerning differential graded algebras.

Given a dga \((A, d, \cdot)\), the product descends to an associative structure in the homology \( H(A, d) \). However, a more refined structure which always exists is a minimal \( A_\infty \) structure in \( H(A, d) \) that is quasi-isomorphic to the dga \((A, d, \cdot)\). The first part of such a minimal \( A_\infty \) structure on \( H(A, d) \) coincides the the associative structure in \( H(A, d) \), but the \( A_\infty \) structure has more information, it determines the dga \((A, d, \cdot)\) up to homotopy equivalence.
The connection $\nabla^1$ is defined canonically from a representing couple $(R, \Pi)$ but our construction of the superconnection depends on an $R$ module decomposition $C_R \cong D' \oplus E \oplus F$ where $D' \cong D_{\Pi}$ are representatives for the Dirac bundle, and $m^{\Pi} : F \to E$ is an isomorphism. Such a Hodge decomposition of a chain complex is often used in order to obtain a minimal $A_\infty$ structure in the homology of a dga.

4.2. Heisenberg versus Schrödinger. As mentioned above, if $\pi \in \mathcal{QM}_B(A)$ and the differential in $A$ is zero, then we can define a new background with parameters in $A$. The way it was described above was according to the Heisenberg representation. Explicitly,

$$B_{Heis}^\pi := (P \otimes A, m^{\pi}, N \otimes A, \varphi \otimes 1)$$

becomes a background: $(m^{\Pi})^2 = 0$ (trivially) and the quantum master equation says that $m^{\Pi} \varphi = 0$ (nontrivially). We think of $B_{Heis}^\pi$ as an evolution of $B$ in which the structure $m$ that evolves:

$$m \rightsquigarrow m^{\pi} = U^{-1} m U$$

where $U = e^{-\pi/\hbar}$. The structure $m^{\pi}$ depends on a parameter $t$ in a parameter ring $R$. For $t = 0$, one has the initial structure $m^{\pi}(0) = m$ (since $\pi \in P \otimes m_R$), and we imagine as $t$ varies, $m^{\pi}(t)$ varies over the space $\text{spec}(A)$ of structures path connected to the initial one.

There is also a Schrödinger interpretation of the evolution $B$ to the background

$$B_{Schr}^\pi := (P \otimes A, m, N((h)) \otimes A, e^{-\pi/\hbar} \varphi).$$

In $\tilde{B}_{Schr}^\pi$, it is the vacuum $\varphi$, rather than the structure $m$, that evolves:

$$\varphi \rightsquigarrow U \varphi.$$ 

We permit the module $N$ and the evolving vacuum $U \varphi = e^{-\pi/\hbar} \varphi$ to have contain negative powers in $\hbar$, but only finitely many. An expansion in terms of $A$ coordinates $t_i$

$$e^{-\pi/\hbar} \varphi = \sum_{j=0}^{\infty} (-1)^j \frac{\pi^j}{h^j j!} \varphi = \sum_{j \geq 0, i_1, \ldots, i_n \geq 0} t_{i_1} \ldots t_{i_n} \hbar^j p_{j_1 \ldots j_n} \varphi,$$

for $p_{j_1 \ldots j_n} \in P$ shows that $e^{-\pi/\hbar} \varphi$ is a formal Laurent series for each fixed total degree of the $t_i$’s. Now we prove the equivalence of the Heisenberg and Schrödinger perspectives

**Proposition 4.1.** The backgrounds $B_{Heis}^\pi$ and $B_{Schr}^\pi$ are quasi-isomorphic.

**Proof.** The morphism of quantum backgrounds

$$(\sigma, \tau) : B_{Heis}^\pi \to B_{Schr}^\pi$$

defined by

$$\sigma(\alpha) = U \alpha U^{-1} \quad \text{for } \alpha \in P \otimes A \quad \text{and} \quad \tau(\psi) = U \psi \quad \text{for } \psi \in N \otimes A.$$

The Heisenberg Dirac module is computed by putting the parameters in the chain complex and twisting the differential:

$$D(B_{Heis}^\pi) = H(C_{Heis}^\pi, m^{\pi}) \quad \text{where} \quad C_{Heis}^\pi = (P \otimes A) \varphi.$$

In the Schrodinger picture, we twist the complex and leave the differential unchanged:

$$D(B_{Schr}^\pi) = H(C_{Schr}^\pi, m) \quad \text{where} \quad C_{Schr}^\pi = (P \otimes A) e^{-\pi/\hbar} \varphi.$$
The map $D(B^H_{\text{Heis}}) \rightarrow D(B^\text{Schr}_\pi)$ induced by $(\alpha, \tau)$ is an isomorphism

$$U^{-1}mUx\varphi = 0 \iff mUxU^{-1}U\varphi = 0 \iff m\sigma(x)\tau(\varphi) = 0$$

and

$$U^{-1}mUx\varphi = y\varphi \iff mUxU^{-1}U\varphi = UyU^{-1}U\varphi \iff m\sigma(x)\tau(\varphi) = \sigma(y)\tau(\varphi).$$

$\square$

4.3. The isomorphism $H \otimes_k R \rightarrow D_{\Pi}$. We now switch to using the Schrödinger representation. Since we use this picture for the remainder, we do not use the superscript $\text{Schr}$. Set $U := e^{-H/\hbar}$. Then the primary chain complex consists of the $R$ module

$$C_{\Pi} = P \otimes RU\varphi$$

with differential given by multiplication by $m$. The closed elements correspond to elements $\mathcal{X} \in P \otimes R$ satisfying $m\mathcal{X}U\varphi = 0$ and the exact elements correspond to elements $\mathcal{Y} \in P \otimes R$ satisfying $\mathcal{Y}U\varphi = m\mathcal{X}U\varphi$. Since $H$ is the $k$-vector space of $k$-linear derivations of $R$, $H$ also acts on $P \otimes R$ as $P$ linear derivations. However, a $P$ linear derivation of $R$ maps $C_{\Pi} \rightarrow D_{\Pi}$ since differentiating $U$ may introduce one negative power of $\hbar$. However $\hbar H$ acts on $C_{\Pi}$ by $P$ linear derivations.

**Theorem 4.2.** If $B$ is a smooth formal background represented by the couple $(R, \Pi)$, then the map $\Phi^0 : H \rightarrow C_{\Pi}$ defined by $\Phi^0(x) = h_xU\varphi$ induces an isomorphism $H \otimes_k R \simeq D_{\Pi}$.

**Proof.** Since $\Pi$ is a solution to the master equation, $0 = mU\varphi$. Apply the derivation $hx$ to get $hxmU\varphi$. Since $m \in P$ and $x$ is a $P$ linear derivation of $R$, $m$ and $x$ commute and we have $0 = hxmU\varphi$. Now, $hx(U)\varphi = hx(U)U^{-1}U\varphi$ and since $hx(U)U^{-1} \in P \otimes R$ contains no negative powers of $\hbar$, $\Phi^0 = hx(U)\varphi$ is a closed element in $C_{\Pi}$. Taking the homology class of $\Phi^0$ defines a map $H \rightarrow D_{\Pi}$.

To see that the map, when the coefficients are extended to $R$ becomes an isomorphism $H \otimes R \rightarrow D_{\Pi}$, first observe that it is injective. If $\Phi^0(s) = hxU\varphi$ satisfies $hxU\varphi = m\mathcal{Y}U\varphi$ for some $\mathcal{Y}U\varphi \in C_{\Pi}$, then reducing this equation modulo $\hbar$ and $H^*$ implies that $x$ satisfies $x\varphi = my\varphi$ for $y = \mathcal{Y} \mod \hbar$, $H^*$.

To see that $\Phi^0$ is surjective $D_{\Pi}$, note that if $m\mathcal{X}U\varphi = 0$ then reducing this equation modulo $H^*$ yields $mX_1\varphi = 0$ for $X_1 = \mathcal{X} \mod \hbar H^* \in (P \otimes R)/H^* \simeq P$. So $X_1$ defines a class in $H \otimes R/H^*$ and $\Phi^0(X_1) = \mathcal{X} \mod H^*$. Then, $\mathcal{X} = \Phi^0(X_1) \mod (H^*)^2$ is an $m$ closed element of $(P \otimes R)/(H^*)^2$ hence defines an element $X_2 \in H \otimes R/(H^*)^2$. Then $\mathcal{X} = \Phi^0(X_1 + X_2) \mod (H^*)^3$, and so on. This builds inductively an element in $X = X_1 + X_2 + X_3 \in H \otimes R$ with $\Phi^0(X) = \mathcal{X} \varphi$. $\square$

By reducing the isomorphism $H \otimes_k R \simeq D_{\Pi}$ modulo $H^*$, we see that there is no $\hbar$ torsion in $D$.

**Corollary 4.3.** If $B$ is a smooth background then $D(B) \simeq H(B)[[\hbar]]$.

**Remark 4.1.** In coordinates, say $\{x_i\}$ is a basis for $H$ with dual basis $\{t_i\}$, we have $x_i$ acts by $\hbar \frac{\partial}{\partial t_i}$ on $R = k[[h, t_i]]$ and $P[[h, t_i]]$. A versal solution to the quantum master equation
\( \Pi \) has the form
\[
\Pi = \sum \left( x_i t_i + \hbar x_i^{(1)} t_i + h^2 x_i^{(2)} t_i + O(h^3) \right) + \left( x_j t_i + \hbar x_j^{(1)} t_i + h^2 x_j^{(2)} t_i + O(h^3) \right) + \cdots
\]
with \( x_i, \ldots, x_n \in P \). The first terms \( \{ x_i \} \) are the \( k \)-basis for \( H \) and the linear in \( t \) terms \( \{ X_i := x_i + \hbar x_i^{(1)} + h^2 x_i^{(2)} + h^3 \cdots \} \) are a \( k[[\hbar]] \)-basis for \( D \) and \( \{ X_i := h^k d_{\pi} x_i \} \) is an \( R = k[[\hbar]] \)-basis for \( D_{\Pi} \). In the case that \( \Pi \) commutes with its derivatives in \( P \), \( X_i = \frac{\partial}{\partial x_i} \). The isomorphisms \( H \otimes \Lambda \to D \) and \( H \otimes R \to D_{\Pi} \) are given by \( x_i \mapsto X_i \) and \( x_i \mapsto X_i, \)

**Remark 4.2.** Recent developments [12, 16] suggest that the converse to Corollary 4.3 is true and the concept of special coordinates in string theory is related to the way in which an isomorphism \( D(B) \simeq H(B)[[\hbar]] \) can be extended to produce a “special” versal solution to the quantum master equation.

So, as described in the conceptual description of the connection, Theorem 4.2 implies that \( D_0 \otimes R \simeq H \otimes_{k} R \simeq D_{\Pi} \) and therefore defines a flat connection on \( D_{\Pi} \). Now we translate from the Grothendieck formal geometry picture to the covariant derivative picture of a flat connection. The translation amounts to observing that \( D_0 = H[[\hbar]] \) is the \( \Lambda = k[[\hbar]] \) module of \( \Lambda \) linear derivations of \( R = k[[\hbar, H^*]] \), hence act as \( P \) linear derivations of \( C_{\Lambda} \) and \( D_{\Lambda} \).

**Definition 4.1.** Define \( \nabla^{ch} : D_0 \times C_{\Pi} \to C_{\Pi} \) by \( (X, Y) \mapsto \nabla^{ch}_X(Y) = hX(Y)U^{-1} \).

First note that \( hX(YU\varphi) = hX(YU)U^{-1}U\varphi \) and that \( hX(YU)U^{-1} \) has no negative powers of \( \hbar \). This proves that \( \nabla^{ch} : C_{\Pi} \to C_{\Pi} \). Now, since \( m \in P \) and \( X \) acts as a \( P \) linear derivation, \( \nabla^{ch}_X(mY) = X(mY) = mX(Y) \) so \( \nabla^{ch} \) is a chain map, hence defines a map
\[
\nabla^1 : D_0 \times D_{\Pi} \to D_{\Pi}
\]

**4.4. Superconnection on the versal Dirac module.** First, refashion \( \nabla^{ch} \) in terms of differential forms. Let \( \Omega^k = R[[H^*]] \otimes_R S^k(H^*[1]) \) be the module of \( k \) forms on spec \( R \) and \( \Omega := \Pi_0 \Omega^k \) be the module of Kahler differential forms. Let \( d_{dR} : \Omega \to \Omega \) denote the de Rham differential; i.e., the shift functor \( H \to H[1] \) extended to a \( \Lambda \)-linear derivation of \( \Omega \). Therefore \( hd_{dR} : C_{\Pi} \otimes_R \Omega \to C_{\Pi} \otimes_R \Omega \) is a square zero \( P \) linear derivation. Since \( m \in P, [m, hd_{dR}] = 0 \). Therefore, \( d_{dR} \) descends to give a map
\[
\nabla^1 : D_{\Pi} \to D_{\Pi} \otimes_R \Omega^1
\]
This is precisely the same as the connection defined before, expressed in terms of a connection one forms.

However, one can do better. We now explain how to transfer \( d_{dR} \) to a superconnection \( \nabla : D_{\Pi} \to D_{\Pi} \otimes_R \Omega \). In order to do so, we use one more piece of data. Consider \( \Phi^0 : H \to C_{\Pi} \) as in Theorem 4.2 defined by \( \Phi^0(x) = \hbar x(U) \). Since \( \Phi^0(H) \) is closed in \( C_{\Pi} \) and \( R\Phi^0(H) \simeq D_{\Pi} \), we may choose an acyclic complement \( E \) in \( C_{\Pi} \) so that
\[
C_{\Pi} = \Phi^0(H) \oplus E.
\]
Now, the equation
\[(16) \quad m\Phi^0 = 0\]
implies \(m\hbar dR\Phi^0 = 0\). Therefore,
\[(17) \quad \hbar dR\Phi^0 = A^1\Phi^0 + m\Phi^1\]
where \(A^1 : D_\Pi \to D_\Pi \Omega^1\) and \(\Phi^1 : H \to E \otimes \Omega^1\). Since \(d^2_{\alpha R} = 0\), applying \(\hbar dR\) to Equation (17) yields
\[(\hbar dR A^1)\Phi^0 - A^1\hbar dR\Phi^0 - m\hbar dR\Phi^1 = 0.\]
Using Equation (17) again get
\[0 = (\hbar dR A^1)\Phi^0 - A^1(A^1\Phi^0 + m\Phi^1) - m\hbar dR\Phi^1\]
This one equation splits into the two equations
\[(18) \quad \hbar dR A^1 - A^1 A^1 = 0\]
\[(19) \quad m(\hbar dR\Phi^1 - A^1\Phi^1) = 0\]
Now, the second equation defines a cycle in \(C_\Pi \otimes R\Omega\) and so implies that \(\hbar dR\Phi^1 - A^1\Phi^1 = A^2\Phi^0 + m\Phi^2\) for unique \(A^2 : D_\Pi \to D_\Pi \otimes R\Omega^2\) and \(\Phi^2 : H \to E \otimes R\Omega^2\). Iterating this procedure proves the following theorem:

**Theorem 4.4.** For every versal solution \(\Pi\) to the quantum master equation, and every choice of acyclic complement \(E\) of \(\Phi^0(H)\) in \(C_\Pi\), there exist unique maps \(A^n : D_\Pi \to D_\Pi \otimes R\Omega^n\) for \(n \geq 1\) and \(\Phi^n : H \to E \otimes R\Omega^n\) for \(n \geq 0\) satisfying
\[(20) \quad \hbar dR A^n = \sum_{j=1}^{n} (-1)^{n-j} A^{n+1-j} A^j,\]
\[(21) \quad \hbar dR \Phi^n_i = \sum_{j=0}^{n} A^{n+1-j} \Phi^j + m\Phi^{n+1}.\]

**Definition 4.2.** Define \(\nabla : D_\Pi \to D_\Pi \otimes R\Omega\) by
\[\nabla := \hbar dR + A^1 + A^2 + \cdots\]

**Theorem 4.5.** \(\nabla\) is flat.

**Proof.** Notice, that \(\nabla\) satisfies
\[\nabla(f \mathcal{Y}) = \hbar dR(f) \wedge \mathcal{Y} + (-1)^{|f|} f \nabla(\mathcal{Y})\]
for any \(f \in R\) and \(\mathcal{Y} \in D_\Pi\). Equation (20) implies \(\nabla^2 = 0\), which is the flatness condition. \(\Box\)

**Remark 4.3.** As discussed in the next section, this small connection carries all of the information about the correlation functions (except for a choice of one-point functions). However, \(\nabla\) carries much more information, what might be thought of as homotopy correlation functions. For example, in the \(B\)-model, the homotopy correlation functions should give a chain level generalization of the Frobenius manifold structure already discovered \([1, 2]\).
5. Path integral, correlation functions, and generalizations

In this section, we relate the quantum superconnection to the correlation functions of the quantum field theory defined by the quantum background $B$. We assume $B$ is finite, but not necessarily smooth formal.

5.1. Path integral.

**Definition 5.1.** Let $B = (P, m, N, \phi)$ be a background. We define a (chain level) path integral pairing for $B$ to be a $k[[h]]$ module map $\int : N \to k[[h]]$, which we denote by $\psi \mapsto \int \psi$, satisfying

- (P Module axiom) $\int (\alpha \beta) \psi = \int \alpha (\beta \psi)$ for all $\alpha, \beta \in P$, and $\psi \in N$,
- (Stokes axiom) $\int m \psi = 0$ for every $\psi \in N$.

We call the condition $\int m \psi = 0$ Stokes axiom because, if we use $\int \psi \alpha$ to denote $\int \alpha \psi$, the condition $\int \psi m = 0$ implies

$$\int \partial \psi \alpha = \int D \alpha,$$

where $\partial$ and $D$ are differentials in $P$ and $N$ defined by

$$\partial(\psi) = m \psi$$

and $D\alpha = m \alpha$.

Given a finite background $B$ and a ring $(R, \delta)$ which represents $\text{QFT}_B$, one can extend the module $N$ to $C_\Lambda$ which is a left $P$ module with the action of $m$ extended to be that of $m \otimes 1$. A path integral pairing $\int$ for $B$ extends

$$C_\Lambda \to R((h))$$

and extends with module compatibility property and the Stokes property.

**Definition 5.2.** If $\Pi$ is a versal solution to the quantum master equation, then $e^{-\Pi/h} \phi \in C_\Lambda \otimes R$ and we define the partition function $Z \in R((h))$ by

$$Z = \int e^{-\Pi/h} \phi.$$

5.2. Generalized Ward identities. For a couple $((R, \delta), \Pi)$ representing $\text{QFT}_B$, we have the master equation

$$me^{-\Pi/h} \phi - h\delta(e^{-\Pi/h})\phi = 0.$$

Integrating (and Stokes axiom) gives

$$\int h\delta(e^{-\Pi/h})\phi = h\delta \int e^{-\Pi/h} \phi = 0.$$

The integral is linear over $R$, giving the following differential equation for the partition function:

(22) $h\delta Z = 0.$
This non-trivial equation captures the most general symmetries of the integral, including, in some example, the Ward identities. In the case that the background is smooth, the Ward identities vanish. The conclusion is

*The obstructions to deforming the background manifest themselves as symmetries of the partition function.*

5.3. **Correlation functions.** Now suppose that $B$ is smooth formal. Choose $R$ so that $\delta = 0$. The tangent space $H = H(B)$ plays the role of the classical observables and the Dirac space $D = D(B)$ plays the role of physical observables. In the smooth case, there is an isomorphism $H(\{h\}) \rightarrow D$, hence we may think of the two classes of observables interchangeably. We think of the Dirac bundle $D_{\Pi}$ as consisting of the observables for all theories in a neighborhood of $B$ in the moduli space.

**Definition 5.3.** For each $n = 0, 1, 2, \ldots$, we define multilinear maps, called $n$-point correlation functions,

$$\langle \cdots \rangle : H^\otimes n \rightarrow R((\hbar))$$

by

$$Z_{i_1} \otimes \cdots \otimes Z_{i_n} \mapsto \langle Z_{i_1}, \ldots, Z_{i_n} \rangle := \int \hbar^n Z_{i_n} \cdots Z_{i_1} e^{-\Pi/\hbar} \varphi.$$

Here, the technique of computing correlation functions by differentiating a family of action functionals here becomes a definition since each observable $Z_i \in H$ is a derivation of the representing ring $R$. The $n$ point correlation functions are completely determined by the one point correlation functions and the small connection $\nabla^1$; i.e., the one-form part of $\nabla$. To see this note that $\langle Z_i \rangle = \langle \Phi^0(Z_i) \rangle$. Applying $h_d^{dR}$ gives a one-form $h_d^{dR}(Z_i)$ which when evaluated on the tangent vector $Z_j$ gives

$$\langle h_d^{dR}(\Phi^0(Z_i))(Z_j) \rangle = \langle Z_j, Z_i \rangle.$$

The equations in Theorem 4.4 imply

$$\langle Z_j, Z_i \rangle = \langle A^1 \Phi^0(Z_i) + m \Phi^1(Z_i) \rangle = A^1 \langle \Phi^0(Z_i) \rangle.$$

Next, one finds that

$$\langle Z_k, Z_j, Z_i \rangle = h_d^{dR} A^1(Z_k) \langle \Phi^0(Z_i) \rangle + A^1 A^1 \langle \Phi^0(Z_i) \rangle.$$ 

By recursively differentiating and eliminating the exact terms, one can compute all correlation functions in terms of the $A^1$, and their derivatives.

5.4. **Homotopy correlation functions.** A chain level path integral pairing gives rise to a linear functional on the versal Dirac bundle and this linear functional on $D_{\Pi}$ is sufficient to determine the correlation functions. So, one may propose to define a path integral, or a “cohomological path integral” to contrast it with a chain level path integral, as any linear functional $F \in \text{hom}(D_{\Pi}, R)$. This avenue will lead to the definition of the one-point correlation functions as $\langle [Z_i] \rangle = F([Z_i])$ and the definition of the $n$-point correlation functions by equations already given. However, there is information, beyond correlation functions, that a chain level path integral pairing can detect that is invisible to any cohomological path integral.
To explain, we first make the correlation function into a function of the moduli by \( \int \Phi^0 \). Then, the \( n \) point correlation functions are obtained as derivatives of the one point correlation functions and are inductively determined by integrating Equation (21) for \( n = 0 \):

\[
\mathcal{H} d_R \int \Phi^0 = A^1 \int \Phi^0.
\]

Thus, one might as well summarize all of the \( n \) point correlation functions into the single fundamental correlation function

\[
\int \Phi^0
\]

with the understanding that the correlation function satisfies Equation (23).

Note that the information contained in the boundary term \( m \Phi^1 \) from equation (21) is lost after integration, but may be retained as a correlation one-form

\[
\int \Phi^1.
\]

Think of the correlation one-form as a homotopy 1-point correlation function, from which many homotopy \( n \)-point functions can be derived by differentiation. These homotopy correlation functions that may be derived from \( \int \Phi^1 \) are summarized inductively by integrating equation (21):

\[
\mathcal{H} d_R \int \Phi^1 = A^1 \int \Phi^1 + A^2 \int \Phi^0.
\]

This discussion suggests an efficient way to handle these correlation and homotopy correlations by defining the correlation \( k \) form by \( \int \Phi^k \) which satisfies \( d_R \int \Phi^k = A^1 \int \Phi^k + A^2 \int \Phi^{k-1} + \cdots + A^k \int \Phi^0 \). Better yet:

**Definition 5.4.** We define the primary correlation form to be the function

\[
\int \Phi : H \to \Omega
\]

where \( \Phi = \Phi^0 + \Phi^1 + \Phi^2 + \cdots \in C_{\Omega} \otimes_R \Omega \)

**Theorem 5.1.** The primary correlation form satisfies \( \nabla \int \Phi = 0 \).

6. APPLICATION—DBV ALGEBRAS

There are many situations which give rise to quantum backgrounds. In each, the resulting quantum flat superconnection provides a penetrating tool for investigating the situation. Here, we give one example which we feel may elucidate the many definitions in this paper. We begin with a differential BV algebra and construct a quantum background from it. Then, we indicate how the various features of the background correspond to the dBV algebra. In particular, the quantum superconnection derived from the QFT functor deeply probes the homotopy theory of the BV algebra. Related ideas appear in [13].

Let \((V, d, \Delta, \wedge)\) be a differential BV algebra. This means that \(V\) is a graded vector space and \(d, \Delta\) are commuting differentials on \(V\) with the properties

- \((V, d, \wedge)\) is a differential graded commutative, associative, algebra, and
\[ (V[-1], d, (, ) ) \] is a differential graded Lie algebra, where the bracket is defined by 
\[ (v, w) := (-1)^{|v|} \Delta(v \wedge w) - (-1)^{|v|} \Delta(v) \wedge w - v \wedge \Delta(w) \] for homogeneous vectors \( v \) and \( w \).

We use parentheses \((, )\) for the Lie bracket, reserving square brackets \([, ]\) always for the graded commutator. For convenience, we assume that the dBV algebra is fairly simple; assume \( V = SU \) for a finite dimensional graded vector space \( U \) for which \( \wedge \) is the associative symmetric product in \( SU \). This assumption allows us to operate easily in coordinates. Let \( \{ q_1, \ldots, q_n \} \) be a homogeneous basis for \( U \). Then elements of \( V \) are polynomials in the variables \( \{ q_i \} \), the wedge product is the ordinary graded commutative product of polynomials, and we may abbreviate \( q_i q_j \) by \( q_i q_j \). The operator \( d \), as a derivation of the product, is a first order differential operator, and \( \Delta \), as a BV operator, is a second order differential operator. Such operators have expressions in the basis \( \{ q_i \} \) as

\[
d = \sum_{i=1}^{n} a_i(q) \frac{\partial}{\partial q_i}
\]

and

\[
\Delta = \sum_{i,j=1}^{n} b_{ij}(q) \frac{\partial^2}{\partial q_j \partial q_i}
\]

for each \( i \) and \( b_{ij}(q) \in V \) and \( b_{ij}(q) \in V \) with \( b_{ij}(q) = -(-1)^{|q_i| |q_j|} b_{ji}(q) \) for each \( i \) and \( j \).

From such a dBV algebra, we now define a background \( B_{V,d,\Delta,\wedge} = (P, m, N, \varphi) \).

6.1. The ring \( P \). We define the ring \( P \) as \( P = W(U) \), a graded Weyl algebra on the vector space \( U \), defined to be

\[
W(U) := T(U \oplus U^*)[[\hbar]]/J
\]

where \( J \) is the left ideal generated by

\[
[q, q'], \quad [p, p'], \quad \text{and} \quad [q, p] = \hbar p(q).
\]

for any \( q, q' \in U \), and \( p, p' \in U^* \), and \([, , ]\) is the graded commutator. The product in \( P \) is induced by the tensor algebra. As a \( k[[\hbar]] \) module, \( P \approx (S(U \oplus U^*))[[\hbar]] \) and as a \( k \)-algebra \( P/\hbar P \approx S(U \oplus U^*) \). In coordinates, say \( \{ q_i \} \) is a basis for \( U \) and \( \{ p'_i \} \) is the dual basis of \( U^* \), each elements of \( P \), which is an equivalence class in \( T(U \oplus U^*)[[\hbar]] \) is represented uniquely by a polynomial in the \( \{ q_i \} \) and \( \{ p'_i \} \) in normal ordering with “all the \( p \)'s on the right.” Multiplication is carried mechanically out by concatenating the two polynomials and using the commutation relation \( p'_i q_j = (-1)^{|q_j|} p'_i q_j p' - \hbar \delta_j^i \) repeatedly until obtaining a polynomial with the \( p \)'s on the right, thereby obtaining the unique representative for the product in \( P \).

The Weyl algebra \( P = W(U) \) may be familiar as the “canonical quantization” of the symplectic vector space \( U \oplus U^* \). The ring \( S(U \oplus U^*) \) can be thought of as the ring of (polynomial) functions on a symplectic space, with the Poisson bracket defined by \( \{ q_i, p_j' \} := \delta_i^j \). Then \( P \), isomorphic as a \( k[[\hbar]] \) module to \( S(U \oplus U^*)[[\hbar]] \), is a one-parameter deformation of \( S(U \oplus U^*) \) over \( k[[\hbar]] \), with the infinitesimal deformation being the Poisson bracket. The product we defined in \( P \) might be called a “star product.”
While it is easy (and tedious) to perform calculations in $P$ using explicit polynomials in $p$ and $q$, one can understand the elements in $P$ and the product in $P$ in a coordinate free way. Observe that

$$P/\hbar P \simeq S(U \oplus U^*) \subseteq V \otimes V^* = \text{hom}_k(V, V),$$

and hence, as a $\Lambda$ module,

$$W(U) \subseteq \text{hom}_k(V, V)[[\hbar]]. \tag{27}$$

So, without choosing a basis for $U$, one can regard elements of $P$ as power series in $\hbar$ with coefficients that are the finite $k$-linear operators on $V$; i.e. finite sums of operators $S^iU \to S^kU$. The identification in equation (27) is as free $\Lambda = k[[\hbar]]$ modules; to acquaint oneself with the multiplication in $P$ from this point of view, we expand

$$\alpha\beta = \alpha \wedge \beta + (\hbar \text{ terms}) + (\hbar^2 \text{ terms}) + \cdots + \hbar^j(\alpha \circ \beta)$$

for $\alpha, \beta \in \text{hom}_k(V, V)$. We interpret some of the $\hbar$ terms on the right for a couple of illustrative cases:

**Example 6.1.** Suppose $\alpha : S^i U \to S^k U$ and let $\beta : S^i U \to S^j U$. Then,

$$\alpha\beta = \alpha \wedge \beta + (\hbar \text{ terms}) + (\hbar^2 \text{ terms}) + \cdots + \hbar^j(\alpha \circ \beta) \tag{28}$$

where $\alpha \wedge \beta : S^{i+j} U \to S^{i+k} U$ is the graded commutative wedge product of linear transformations and $\alpha \circ \beta : S^i U \to S^k U$ is composition in $\text{hom}(V, V)$. Hence, $\ast$ extends the commutative associative product on $\text{hom}(V, V)$ toward the noncommutative composition of homomorphisms. The $\hbar^r$ terms for $r = 2, 3, \ldots, j - 1$ can be understood in terms of some multiple-gluing operations using a topological model. In this model $\hbar$ is related to genus or Euler characteristic $[5]$.

**Example 6.2.** Let $\alpha : S^i U \to U$ and let $\beta : S^i U \to U$. Then,

$$[\alpha, \beta] = \hbar [\overline{\alpha}, \overline{\beta}]_G \tag{29}$$

where the bracket on the left is the graded commutator in $P$ and the bracket $[\cdot, \cdot]_G$ on the right is understood by way of the Gerstenhaber bracket in the Coder$(V)$, the Hochschild complex of the algebra $(V[-1], d, \wedge)$. That is, to compute $[\alpha, \beta]$ in $P$, one lifts $\alpha$ and $\beta$ to coderivations $\overline{\alpha}, \overline{\beta} : V \to V$, takes the graded commutator of those two coderivations to obtain a coderivation $[\overline{\alpha}, \overline{\beta}]_G : V \to V$, with the final result of $[\alpha, \beta]$ being $\hbar$ times the component $S^{i+j-1}U \to U$ that determines the coderivation $[\overline{\alpha}, \overline{\beta}]_G$.

6.2. **The structure $m$.** We define $m \in P^1$

$$m = \sum_{i=1}^n a_i(q)p^i + \sum_{i,j=1}^n b_{ij}(q)p^ip^j$$

where $a_i$ and $b_{ij}$ are as in equations (25) and (26). The fact that $(V, d, \Delta, \wedge)$ is a BV algebra implies that as an element of $P$, $m^2 = 0$. 

6.3. **The module** $N$. We have $U^* \subset P$ and the left ideal $PU^*$. The quotient $P/PU^*$ is naturally a $P$ module and we define $N = P/PU^*$. Use $[\alpha]$ to denote the image of $\alpha \in P$ in $P/PU^*$. There is an isomorphism of $k[[\hbar]]$ modules $V[[\hbar]] \simeq N$. In coordinates, this isomorphism is simple: an element of $V[[\hbar]]$ is a polynomial in $q$ which represents a class of polynomials in $P$, and also a class in $P/PU^*$ since $V$ has no $p$'s. To illustrate, a typical element of $P$ might be $\alpha = 2\hbar q_1 q_2 + q_1 p^2 p^3 + h^2 p^3$ and typical element of $N$ might be $\gamma = [2\hbar q_1 - q_1 q_3]$. Then (say $q_1$ and $q_3$ are even and $q_2$ is odd),

$$\alpha \cdot \gamma = [(2\hbar q_1 q_2 + q_1 p^2 p^3 + h^2 p^3)(2\hbar q_1 - q_1 q_3)]$$

$$= |(4\hbar^2(q_1)^2 q_2 - 2h(q_1)^2 q_2 q_3 + 2h(q_1) p^2 p^3 - h(q_1)^2 p^2 + h^2 q_1 q_3 p^3 - h^3 q_1)|$$

$$= |(4\hbar^2(q_1)^2 q_2 - 2h(q_1)^2 q_2 q_3 - h^3 q_1)|.$$

6.4. **The vacuum** $\varphi$. The vacuum $\varphi$ is defined to be $|1\rangle$. Note that $m \varphi = m|1\rangle = |m\rangle = |0\rangle$, as required.

6.5. **Relating the background and the dBV algebra.** We begin with a summary of the results. It will be convenient to state the summary in terms of several algebras built from $(V, d, \Delta, \wedge)$:

$$C := (V, d, \wedge)$$ is a commutative dga over $k$

$$L := (V[-1], d, (, , ))$$ is a dgLa over $k$

$$L^h := (V[-1][[\hbar]], d + h\Delta, (, , ))$$ is a dgLa over $k[[\hbar]]$

$$H := \text{Ker}(d : V \to V)/\text{Im}(d : V \to V)$$ is a vector space over $k$

$$H^h := \text{Ker}(d - h\Delta : V[[\hbar]] \to V[[\hbar]])/\text{Im}(d - h\Delta : V[[\hbar]] \to V[[\hbar]])$$ is a $k[[\hbar]]$ module.

Note that $H \simeq H(C) \simeq H(L)[1]$. By the minimal model theory for $L_\infty$ algebras, there is an $L_\infty$ structure on $H[-1]$, unique up to quasi-isomorphism, that is quasi-isomorphic to $L$. Note also, $H^h \simeq H(L^h)[1]$, so there is an $L_\infty$ structure (over $k[[\hbar]]$) on $H^h[-1]$ quasi-isomorphic to $L^h$.

On the quantum background side, we denote $B_{V, d, \Delta, \wedge}$ simply by $B$, and we have

$$(R = k[[H^*, \hbar]], \delta) := \text{a ring representing QFT}_B$$

$$\Pi \in \text{QMB}_B(R, \delta) := \text{versal solution to quantum master equation}$$

$$(S = k[[H^*]], \delta_0) := (R \otimes_A k, \delta \otimes_A k)$$ is the “classical” $\hbar = 0$ part of $R, \delta$

$$\Pi_0 \in \text{QMB}_B(S, \delta_0) := \text{the image of \Pi in } S$$ is the “classical” part of $\Pi$.

The most immediate relationships between the background $B$ and the dBV algebra $(V, d, \Delta, \wedge)$ are summarized in Figure 1.

In the case that $B$ is smooth formal, we also have $L$ and $L^h$ smooth formal and the moduli space for QFT$_B$ and Def$_L$ are identified. On the BV side, we may define the following algebras, which can be thought of as families of algebras fibered over the moduli...
$H(B) =$ the tangent space to $\text{QFT}_B$

$D(B) =$ the Dirac space of $B$  

$\text{The ring } S$  

$\text{The ring } R$  

$(S, \delta_0)$  

$\Pi_0$  

$(R, \delta)$  

$\Pi$  

\begin{align*}
H & \text{ is the tangent space to QFT}_B \\
H^h & \text{ is the Dirac space of } B \\
k[[H^*]] & \text{ is the dual of an } L_\infty \text{ minimal model of } L \\
k[[\hbar, H^*]] & \text{ is a minimal model map } H \to L \\
\text{the dual of an } L_\infty \text{ minimal model of } L^h & \text{ is a minimal model map } H^h \to L^h
\end{align*}

**Figure 1.** Comparing $B$ to $(V, d, \Delta, \wedge)$

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The ring $S[k[[H^*]]]$ is the dual of an $L_\infty$ minimal model of $L$. The ring $S$ is a commutative associative algebra. Furthermore, by the minimal model theory, $H_t$ has a $C_\infty$ structure, quasi-isomorphic to $C_t$.

On the quantum background side, we have the quantum superconnection

\[
\nabla = d + A^1 + A^2 + \cdots
\]

\[
\nabla^1 = d + A^1
\]

\[
\nabla_0 = d + A^1_0 + A^2_0 + \cdots
\]

\[
\nabla^1_0 = d + A^1_0
\]

We conjecture that there are special coordinates for any smooth formal background. What that means in the present example of a dBV algebra is that there is a particular solution $\Pi$ to the quantum master equation for which the associated flat quantum connection $\nabla$ has no $\hbar$ dependence. This is proved in [12] in a certain semiclassical case. When $\nabla^1$ does not depend on $\hbar$, the flatness equation expressed in terms of the connection one form $A$

\[
hd_{dR} A + A^2 = 0
\]

decouples into two equations $d_{dR} A = 0$ and $A^2 = 0$. The equation $A^2 = 0$ together with the torsion-freeness of $\nabla^1$ implies that $A$ defines a family of commutative, associative algebra structures on $H$ parametrized by $H$. The condition that $dA = 0$ gives additional constraints on the way these algebra structures vary with their parameters, implying that $H$ has the structure of a Frobenius manifold without a metric [9]. We conjecture that the
superconnection in special coordinates affords $H$ with the structure of a minimal algebra over an appropriate resolution of the Frobenius manifold structure.

6.6. More general Weyl-type backgrounds. We illustrate so many details about a background arising from a dBV algebra because the situation may be familiar, but we emphasize that it is only an example. Notice that $m \in P$ obtained from $d$ and $\Delta$ in a dBV algebra is quite special: it is quadratic in $p$ and has no dependence on $\hbar$. One may more generally consider a background $B = (P, m, N, \varphi)$ where $P = W(U)$ is the Weyl algebra on a graded vector space $U$, $m$ is any element, at least linear in $p$ (in order to annihilate the vacuum), satisfying $m^2 = 0$, $N = SU[[\hbar]]$, and $\varphi = |1\rangle$. There seem to be many interesting examples [5]. Furthermore, for a fixed $m$, one may consider different modules, possibly highlighting different aspects of the same structure.

7. PROSPECTUS

In conclusion, let us make some very brief remarks about future directions, to which we are now turning.

In this paper, we constructed a flat quantum superconnection over the moduli space if the background is smooth formal. Our attention is now focused on the non smooth formal case. In the general case, we intend to construct a quantum connection $\nabla$ which interacts with the differential $\delta$ from proposition 3.4. This seems to be necessary to develop a good minimal model theory for quantum backgrounds.

We imagine the ideas in this paper can be applied in two different ways. The first application is rather direct: apply the framework described here to mathematical situations that fit. One noteworthy example arises in symplectic field theory. One might summarize the output of symplectic field theory as an element of square zero in a particular noncommutative ring. In other words, the output of symplectic field theory is a rather good match to the input data of a background. It would be interesting to construct the QFT moduli space and quantum connection $\nabla$ in this example and interpret these structures in terms of symplectic topology.

A second application is a kind of quantized deformation theory. By this we mean start with a classical mathematical structure and produce a quantum background $B$ whose classical limit is the $L_\infty$ algebra $L$ controlling the given structure’s classical deformations. By “classical limit” we mean that $B$ and $L$ share a relationship much the same as the relationship between $B$ and $L$ exemplified in Section 6. Then, the QFT$_B$ moduli space enriches the classical moduli space in the $\hbar$ direction. One advantage is that the superconnection $\nabla$ when expressed in special coordinates, which are invisible from the classical deformation theory, is expected to encode invariants of the original structure.

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