Toric manifolds over 3-polytopes

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Abstract. In this note we gather and review some facts about existence of toric spaces over 3-dimensional simple polytopes. First, over every combinatorial 3-polytope there exists a quasitoric manifold. Second, there exist combinatorial 3-polytopes, that do not correspond to any smooth projective toric variety. We restate the proof of the second claim which does not refer to complicated algebro-geometrical technique. If follows from these results that any fullerene supports quasitoric manifolds but does not support smooth projective toric varieties.

1. Introduction

For a 3-dimensional simple polytope \( P \) one can construct a 6-dimensional manifold with the action of the compact torus \( T^3 \), whose orbit space is \( P \). The topology of this manifold tells a lot about the combinatorics of the polytope. There exist several constructions of such manifolds arising in different areas of mathematics: toric varieties in algebraic geometry and singularity theory, symplectic toric manifolds in symplectic geometry, quasitoric manifolds in algebraic topology. Each construction requires certain properties from the polytope, and these properties affect the geometrical structure of the resulting manifold. For example, the construction of a quasitoric manifold as an identification space \([5]\) requires only the combinatorial type of a polytope, and the resulting manifold is just a topological manifold. However, if we fix the affine realization of a polytope, the resulting quasitoric manifold attains smooth structure, see \([4]\). The construction of a symplectic toric manifold requires a polytope to be Delzant \([6]\). There also exist certain conditions on the polytope, which imply the existence of almost complex, or algebraical structure on the corresponding manifold.

There are several natural questions. What does the existence of certain geometrical structure on a toric space say about the combinatorics of the polytope? The constructions of toric topology and toric geometry allow to construct a lot of examples of 6-dimensional manifolds. But how large is the set of examples having certain geometrical structure? In this paper we gather some known results.

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There is a well-known correspondence between toric varieties and rational fans. Projective toric varieties correspond to normal fans of convex polytopes and smooth projective toric varieties correspond to unimodular polytopal fans (i.e. normal fans of Delzant polytopes). Quasitoric manifolds are the algebro-topological generalization of smooth projective toric varieties. A smooth compact manifold $M$ of real dimension $2n$ with the action of half-dimensional compact torus $T^n$ is called **quasitoric** if

1. the action is locally standard (i.e. locally modeled by the standard action of $T^n$ on $\mathbb{C}^n$ by coordinate-wise rotations);
2. the orbit space (which is a manifold with corners as follows from pt.1) is isomorphic to some simple polytope $P$.

In this case we say that $M$ is a quasitoric manifold over $P$. Recall that $n$-dimensional convex polytope is called **simple** if each of its vertices lies in exactly $n$ facets (equivalently: each vertex lies in exactly $n$ edges). Among all convex polytopes only simple polytopes are manifolds with corners.

Every smooth projective toric variety $X$ is a quasitoric manifold: we can restrict the action of an algebraic torus $(\mathbb{C}^*)^n$ on $X$ to its compact subtorus; and the orbit space of this action can be identified with the image of the moment map, which is a simple polytope. However there exist many quasitoric manifolds which are not toric varieties. The simplest example is $\mathbb{C}P^2 \# \mathbb{C}P^2$: this is a quasitoric manifold which is not even algebraical. On the other hand there also exist smooth non-projective toric varieties which are not quasitoric \cite{13}.

In this paper we discuss two basic theorems:

**Theorem 1** (\cite{5, 3}). There exists a quasitoric manifold over any 3-dimensional simple polytope.

**Theorem 2** (\cite{7}). If there exists a smooth projective toric variety over a simple 3-dimensional polytope $P$, then $P$ has at least one triangular or quadrangular face.

The recent interest to these results arose in connection with fullerenes. Mathematically, a fullerene is a simple 3-dimensional polytope having only pentagonal and hexagonal faces. Buchstaber \cite{2} suggested to study fullerenes from the perspective of toric topology: presumably, both fields may benefit from such interaction.

Two theorems above show that (1) there exist quasitoric manifolds over fullerenes; (2) there are no smooth projective toric varieties over fullerenes. Therefore, due to their rigid geometrical nature, smooth projective toric varieties are not suited for the study of fullerenes. However, quasitoric manifolds do: this subject will be discussed in the forthcoming paper of Buchstaber, Erokhovets, Masuda, Panov, and Park.

We should make a remark that Theorems \cite{12} are based on completely different methods. Theorem \cite{1} essentially relies on the four color theorem: there are no known proofs which do not use this result. Theorem \cite{2} was formulated and proved by Delaunay in \cite{7}, and its proof is based on the work of Reid \cite{11} concerning Mori’s minimality theory for toric varieties. We restate the proof in more combinatorial topological terms, without referring to this.
algebro-geometrical theory, to make the difference between toric and quasitoric cases more transparent.

2. Quasitoric manifolds

Let \( M \) be a quasitoric manifold of dimension \( 2n \). Its orbit space under the action of \( T^n \) is a simple polytope \( P \). Let \( F_1, \ldots, F_m \) be the facets of the polytope \( P \). Any point \( x \) in the interior of a facet \( F_i \) represents an \( n \)-dimensional orbit of the action. The stabilizer of this orbit is a 1-dimensional toric subgroup \( G_i \subset T^n \). We may assume that \( G_i = \exp(\lambda_{i,1}, \ldots, \lambda_{i,n}) \), where \( (\lambda_{i,1}, \ldots, \lambda_{i,n}) \in \mathbb{Z}^n \) is a primitive integral vector determined uniquely up to sign. One-dimensional stabilizer subgroups define the so called characteristic function. Let \( \{m\} = \{1, \ldots, m\} \) be the index set of facets of \( P \). Consider the function \( \lambda: \{m\} \rightarrow \mathbb{Z}^n \), \( \lambda: i \mapsto (\lambda_{i,1}, \ldots, \lambda_{i,n}) \). Actually, the value is determined uniquely up to sign, however we make a choice of this sign arbitrarily (this corresponds to the choice of orientation of each stabilizer \( G_i \)). Since the action of the torus is locally standard, characteristic function satisfies the condition:

(1) if facets \( F_{i_1}, \ldots, F_{i_n} \) intersect in a vertex, then \( \lambda(i_1), \ldots, \lambda(i_n) \) is a basis of \( \mathbb{Z}^n \).

Therefore, with any quasitoric manifold, one can associate a characteristic pair \((P, \lambda)\), where \( P \) is a simple polytope and \( \lambda \) is a characteristic function.

CONSTRUCTION 3. The construction above can be reverted \([5]\). Given any simple polytope \( P \) with facets \( F_1, \ldots, F_m \) and a function \( \lambda: \{m\} \rightarrow \mathbb{Z}^n \) satisfying condition (1), we can construct a quasitoric manifold \( M_{(P, \lambda)} \) as follows. For each \( i \in \{m\} \) let \( G_i = \exp(\lambda(i)) \subset T^n \) be the corresponding circle subgroup. Take any point \( x \in P \); it lies in the interior of some face \( F \in P \). We have \( F = F_{i_1} \cap \cdots \cap F_{i_k} \). Let \( G_x \) denote the toric subgroup \( G_{i_1} \times \cdots \times G_{i_k} \subset T^n \). Consider the identification space

\[
M_{(P, \lambda)} = (P \times T^n)/\sim
\]

where \((x, t) \sim (x', t')\) whenever the points \( x, x' \) coincide and \( t't^{-1} \in G_x \). One can check that \( M_{(P, \lambda)} \) is a topological manifold, and there is a locally standard action of \( T^n \), which rotates the second coordinate and gives the orbit space \( P \). A canonical smooth structure on \( M_{(P, \lambda)} \) was constructed in \([4]\). Thus \( M_{(P, \lambda)} \) is a quasitoric manifold.

To prove Theorem \([1]\) one needs to show that every simple 3-polytope admits a function \( \lambda \), satisfying condition (1). This is done by the four colors theorem.

PROOF OF THEOREM \([1]\) Let \( c: \{F_1, \ldots, F_m\} \rightarrow \{a, b, c, d\} \) be the coloring of facets of \( P \) in four colors such that adjacent facets have distinct colors. Let \( e_1, e_2, e_3 \) be the basis of the lattice \( \mathbb{Z}^3 \). Replace colors by the vectors as follows: \( a \mapsto e_1, b \mapsto e_2, c \mapsto e_3, d \mapsto e_1 + e_2 + e_3 \). This gives a characteristic function, since every three vectors among \((e_1, e_2, e_3, e_1 + e_2 + e_3)\) form a basis of the lattice. \( \Box \)
3. Toric varieties

Let $V \cong \mathbb{R}^n$ be an oriented real vector space with the fixed lattice $\mathbb{Z}^n \cong N \subset V$. Recall that a fan in $\mathbb{R}^n$ is a collection of convex cones with apex at the origin such that the intersection of each two cones of the collection is a face of both and lies in the collection. The fan is called complete if the union of all cones is the whole space $V$. The fan is called rational if all cones are generated by rational vectors. The cone is called simplicial (resp. unimodular) if it is generated by linearly independent vectors of $V$ (resp. part of a basis of the lattice $N$). The fan is called simplicial (resp. unimodular) if all its cones are such. Every unimodular fan is simplicial.

Let $P$ be a convex polytope in the dual space $V^\ast$. With any such polytope one associates the normal fan: for each face $F \subset P$ take the cone spanned by outward normal vectors to the facets of $P$ containing $F$, and take the collection of these cones. Normal fan is complete. Normal fan of a simple polytope is simplicial. Normal fans of polytopes are called polytopal fans. Note that there exist non-polytopal complete fans.

**Definition 4.** A polytope $P$ is called Delzant if its normal fan is unimodular.

It follows that every Delzant polytope is simple. Toric varieties are classified by rational fans. Compact toric varieties correspond to complete fans. Smooth toric varieties correspond to unimodular fans. Projective toric varieties correspond to polytopal fans. Therefore, smooth projective toric varieties correspond to normal fans of Delzant polytopes (i.e. polytopal unimodular fans). Theorem 2 can be restated as follows.

**Proposition 5 ([7]).** Let $P$ be a 3-dimensional Delzant polytope. Then $P$ has at least one triangular or quadrangular face.

Let $P$ be an $n$-dimensional Delzant polytope with facets $F_1, \ldots, F_m$ and let $\lambda(i) \in \mathbb{Z}^n$ be the primitive outward normal vector to $F_i$. The unimodularity property of the normal fan of $P$ implies that the function $\lambda: [m] \to \mathbb{Z}^n$ defined this way satisfies condition (1). Therefore each Delzant polytope determines the characteristic pair $(P, \lambda)$ in a natural way. Smooth projective toric variety corresponding to the normal fan of $P$ is equivariantly diffeomorphic to the quasitoric manifold determined by the pair $(P, \lambda)$, see [5]. Due to this observation smooth projective toric varieties are particular cases of quasitoric manifolds.

Now we introduce some notation to be used in the following. Let $\Delta$ denote a complete unimodular fan in $V \cong \mathbb{R}^n$ and $m$ be the number of rays in $\Delta$. Let $X_\Delta$ be the smooth compact toric variety determined by this fan. The underlying simplicial sphere $K$ of the fan $\Delta$ has $m$ vertices and dimension $n - 1$. Let $\lambda: [m] \to N$ be the characteristic function, that is $\lambda(i) \in N$ is the primitive generator of $i$-th ray of the fan $\Delta$.

**Cohomology.**

**Theorem 6 (Danilov–Jurkiewicz).** $H^\bullet(X_\Delta; \mathbb{Z}) \cong \mathbb{Z}[K]/\Theta$, where

\[
\mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/(v_{i_1} \cdots v_{i_s} \mid \{i_1, \ldots, i_s\} \notin K), \quad \deg v_i = 2
\]

is the Stanley–Reisner ring of the sphere $K$, and ideal $\Theta$ is generated by linear forms $\sum_{i \in [m]} \langle \mu, \lambda(i) \rangle v_i$, for each linear functional $\mu: N \to \mathbb{Z}$.
A similar theorem was proved by Davis and Januszkiewicz for quasitoric manifolds: in this case \( K \) is a simplicial sphere dual to a polytope, and \( \lambda \) is a characteristic function. Similar theorems hold with real coefficients instead of integers.

Let \( \int_{X_\Delta} : H^{2n}(X_\Delta; \mathbb{Z}) \to \mathbb{Z} \) denote the pairing with the fundamental class of \( X_\Delta \). Consider a subset \( I = \{i_1, \ldots, i_n\} \subset [m] \). We have

\[
\int_{X_\Delta} v_{i_1} \cdots v_{i_n} = \begin{cases} 1, & \text{if } I \in K \\ 0, & \text{otherwise.} \end{cases}
\]

In the following we also need the description of tangent Chern classes of \( X_\Delta \).

**Theorem 7 ([8]).** Under the isomorphism of Theorem 6 the \( j \)-th Chern class of the tangent bundle of the manifold \( X_\Delta \) corresponds to the elementary symmetric polynomial in the variables \( v_i \):

\[
c_j(X_\Delta) = \sigma_j(v_1, \ldots, v_m) = \sum_{I \in K, |I| = j} \prod_{i \in I} v_i \in H^{2j}(X_\Delta; \mathbb{Z}).
\]

A completely similar theorem was proved in [5] for a quasitoric manifold after introducing certain stably complex structure on it.

**Effective cone.**

The notion of effective cone is one of the essential points in the proof of Theorem 2. This notion is defined in algebraic geometry for arbitrary projective varieties, however we restrict to the smooth case, where it has a clear geometrical meaning. This subsection is needed only for the completeness of the exposition: for toric varieties all necessary notions will be defined in combinatorial-geometrical manner below.

Let \( X \) be an arbitrary smooth Kähler manifold. Each compact complex curve \( C \subset X \) determines a homology class \( [C] \in H_2(X; \mathbb{R}) \), which is called effective. The set of all nonnegative linear combinations of effective classes in \( H_2(X; \mathbb{R}) \) is called the effective cone of the manifold \( X \):

\[
\text{NE}(X) = \left\{ \sum r_i [C_i] \in H_2(X; \mathbb{R}) \mid r_i \geq 0 \right\}.
\]

**Proposition 8.** \( \text{NE}(X) \) is a strictly convex cone in \( H_2(X; \mathbb{R}) \).

**Proof.** We need to prove that all effective classes lie in some open half-space of \( H_2(X; \mathbb{R}) \). Consider the class of Kähler form \( \omega \in H^2(X; \mathbb{R}) \). For each complex curve \( C \) we have

\[
\langle \omega, [C] \rangle = \int_C \omega|_C = \text{Vol}(C) > 0.
\]

This means that all effective classes lie in the half-space

\[
\{ \alpha \in H_2(X; \mathbb{R}) \mid \langle \omega, \alpha \rangle > 0 \},
\]

which implies the statement. \( \square \)

**Proposition 9 ([11]).** Let \( X \) be smooth projective toric variety. Then its effective cone \( \text{NE}(X) \) is polyhedral and is generated by the fundamental classes of torus-invariant 2-spheres (preimages of edges of the polytope under the projection to the orbit space).
The generators of the effective cone are called \emph{extremal cycles}. Note that in general not all edges of the polytope define extremal cycles: some of them may lie in the cone generated by others.

**Effective cone in toric case: combinatorial-geometrical approach.**

Here we introduce all the necessary notions from the previous paragraph in combinatorial manner. Let $X_\Delta$ be the smooth projective variety corresponding to a polytopal fan $\Delta$.

The simplices of $K$ of codimension 1 as well as the corresponding cones of $\Delta$ will be called \emph{the walls}. For each wall $J = \{i_1, \ldots, i_{n-1}\} \in K$ consider the class $v_J = v_{i_1} \cdots v_{i_{n-1}} \in H^{2n-2}(X; \mathbb{R})$. Note that $v_J \neq 0$ by obvious reasons. Consider the cone in $H^{2n-2}(X_\Delta; \mathbb{R})$ generated by the classes $v_J$ for all walls $J \in K$:

$$\text{NE}(X_\Delta) = \left\{ \sum r_Jv_J \in H^{2n-2}(X_\Delta; \mathbb{R}) \mid r_J \geq 0 \right\}$$

**Proposition 10.** For each smooth projective toric variety the effective cone $\text{NE}(X_\Delta)$ is a strictly convex polyhedral cone in $H^{2n-2}(X_\Delta; \mathbb{R})$.

**Proof.** Let $V_\Delta \in \mathbb{R}[c_1, \ldots, c_m]$ be the volume polynomial of the fan $\Delta$. By definition,

$$V_\Delta(c_1, \ldots, c_m) = \frac{1}{n!} \int_{X_\Delta} (c_1v_1 + \ldots + c_mv_m)^n.$$

It is known (see [9]), that the values of this polynomial are the volumes of simple polytopes with the normal fan $\Delta$. More precisely, let $P = \{x \in V^* \mid \langle x, \lambda(i) \rangle \leq \tilde{c}_i \}$ be a simple convex polytope with the normal fan $\Delta$ (since $X_\Delta$ is projective, at least one such polytope exists). The numbers $\tilde{c}_i$ are called the support parameters of $P$. Then we have $\text{Vol}(P) = V_\Delta(\tilde{c}_1, \ldots, \tilde{c}_m)$. To avoid the mess, we denote the formal variables of the volume polynomial by $c_i$, and concrete real numbers substituted in this polynomial are denoted by $\tilde{c}_i$.

Let $\hat{\partial}_i = \frac{\partial}{\partial c_i}$ be the differential operators, acting on $\mathbb{R}[c_1, \ldots, c_m]$. Let $D = \mathbb{R}[\hat{\partial}_1, \ldots, \hat{\partial}_m]$ be the commutative algebra of differential operators with constant coefficients, and $\text{Ann}V_\Delta = \{D \in D \mid DV_\Delta = 0\}$ be the annihilating ideal of the polynomial $V_\Delta$. According to [10, 12], we have

$$D/\text{Ann}V_\Delta \simeq H^*(X_\Delta; \mathbb{R}), \quad \hat{\partial}_i \leftrightarrow v_i.$$

Moreover, the integration map $\int_{X_\Delta} : H^{2n}(X_\Delta; \mathbb{R}) \to \mathbb{R}$ coincides with the natural map $(D/\text{Ann}V_\Delta)_n \to \mathbb{R}, \ D \mapsto DV_\Delta$. To prove the statement it is sufficient to show that the classes

$$\{\hat{\partial}_J = \hat{\partial}_{i_1} \cdots \hat{\partial}_{i_{n-1}} \in (D/\text{Ann}V_\Delta)_{n-1} \mid J = \{i_1, \ldots, i_{n-1}\} \in K\}$$

lie in one open half-space.

Let $P$ be a convex polytope with the normal fan $\Delta$ and support parameters $\tilde{c}_i$. Consider the element $\hat{\partial}_c = \sum_{i \in [m]} \tilde{c}_i\hat{\partial}_i \in (D/\text{Ann}V_\Delta)_1$. Recall a simple fact: for each homogeneous polynomial $\Psi \in \mathbb{R}[c_1, \ldots, c_m]$ of degree $k$ there holds $\frac{1}{k!}\hat{\partial}_c^k\Psi = \Psi(\tilde{c}_1, \ldots, \tilde{c}_m)$ (this is a particular case of Euler’s theorem on homogeneous functions).

**Lemma 11.** Let $J \in K$ be a wall. Then $\hat{\partial}_c\hat{\partial}_J V_\Delta > 0$. 

PROOF. Note that $\partial J V_\Delta$ is a linear polynomial in variables $c_i$. Therefore, the number $\partial c_\partial J V_\Delta$ coincides with the value of the polynomial $\partial J V_\Delta$ in the point $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_m)$ by the preceding remark. It is known that the value of the polynomial $\partial J V_\Delta$ in the point $\tilde{c}$ coincides up to positive factor with the length of the edge $F_j \subset P$, dual to the wall $J \in K$ (this was noted by Timorin in [12], and in [1] we proved that the factor is the volume of the parallelepiped spanned by $\lambda(i_1), \ldots, \lambda(i_{n-1})$). Thus $\partial c_\partial J V_\Delta > 0$. □

According to lemma, all classes $\partial J \in D/ \text{Ann} V_\Delta$ lie in the half-space $\{ D | D \partial c V_\Delta > 0 \}$ which implies the statement. □

DEFINITION 12. Let $J = \{ i_1, \ldots, i_{n-1} \} \in K$ be a wall such that $v_J \in H^{2n-2}(X_\Delta; \mathbb{R})$ is a generating element of the effective cone $\text{NE}(X_\Delta)$. Then $J$ is called an extremal simplex and $v_J$ is called an extremal class.

The condition of being extremal can be written as follows. Suppose an extremal class $v_J$ is expressed as a sum $\nu_1 + \nu_2$, where $\nu_1, \nu_2 \in \text{NE}(X_\Delta)$. Then both elements $\nu_1, \nu_2$ lie in the ray generated by $v_J$.

REMARK 13. This definition agrees with the general theory. The vector spaces $H^{2n-2}(X_\Delta; \mathbb{R})$ and $H_2(X_\Delta; \mathbb{R})$ can be identified by Poincare duality, and under this identification the class $v_J = v_{i_1} \cdots v_{i_{n-1}}$ corresponds to the fundamental class of torus-invariant 2-sphere obtained as a transversal intersection of characteristic submanifolds $X_i, \ldots, X_{i_{n-1}}$ (preimages of facets $F_i, \ldots, F_{i_{n-1}}$ under the projection to the orbit space).

4. Unimodular geometry of fans

An arbitrary wall $J = \{ i_1, \ldots, i_{n-1} \} \in K$ is contained in exactly two maximal simplices: $I = \{ i_1, \ldots, i_{n-1}, i \}$ and $I' = \{ i_1, \ldots, i_{n-1}, i' \}$. Both sets of vectors 
$$\{ \lambda(i_1), \ldots, \lambda(i_{n-1}), \lambda(i) \}, \quad \{ \lambda(i_1), \ldots, \lambda(i_{n-1}), \lambda(i') \}$$
are the bases of the lattice. Write $\lambda(i')$ in the first basis:
$$\lambda(i') = a_1 \lambda(i_1) + \ldots + a_{n-1} \lambda(i_{n-1}) - \lambda(i).$$
(Unimodularity condition of the set $\lambda(I')$ guarantees that the coefficient at $\lambda(i)$ is $\pm 1$. The fact that cones at $I$ and $I'$ lie on opposite sides of the wall $J$ guarantees that the coefficient at $\lambda(i)$ is exactly $-1$.) In what follows we assume that vertices $i_1, \ldots, i_{n-1}$ are ordered such that $\lambda(i_1), \ldots, \lambda(i_{n-1})$ is a positive basis of the lattice, while, respectively, $\lambda(i_1), \ldots, \lambda(i_{n-1}), \lambda(i')$ is a negative basis.

DEFINITION 14. The number
$$\text{curv}(J) = 2 - a_1 - \ldots - a_{n-1} \in \mathbb{Z}$$
is called the unimodular curvature of the wall $J$.

The underlying simplicial complex $K$ of a fan $\Delta$ may be realized in $V \cong \mathbb{R}^n$ as a star-shaped sphere as follows: let us send the vertex $i$ to the point $\lambda(i) \in V$ and continue the map on each simplex by linearity. We denote the image of this map by $\text{st}(K)$; it is a piecewise linear sphere in $V$ winding around the origin.
We say that \( \text{st}(K) \) is concave (resp. convex, resp. flat) at the wall \( J \), if the affine hyperplane, through the points \( \lambda(i_1), \ldots, \lambda(i_{n-1}), \lambda(i) \) separates \( \lambda(i') \) from the origin (resp. does not separate, resp. contains \( \lambda(i') \)).

**Lemma 15.** A unimodular curvature and parameters \( a_1, \ldots, a_{n-1} \), defined above satisfy the following properties.

1. \( a_s = \det(\lambda(i_1), \ldots, \lambda(i_s-1), \lambda(i'), \lambda(i_{s+1}), \ldots, \lambda(i_{n-1}), \lambda(i)) \).
2. The star-shaped sphere \( \text{st}(K) \) is convex (resp. flat, resp. concave) at a wall \( J \) if and only if \( \text{curv}(J) > 0 \) (resp. \( \text{curv}(J) = 0 \), resp. \( \text{curv}(J) < 0 \)).
3. In complete simplicial fan there exists a wall of positive curvature.
4. \( \int_{X} v_i \rho_j = -a_s \), \( \text{curv}(J) = \int_{X} \sum_{t \in \text{m}} \rho_t \).

**Proof.**

1. Take exterior product of the relation

\[
\lambda(i) + \lambda(i') = \sum_{t=1}^{n-1} a_t \lambda(i_t)
\]

with the exterior form \( \lambda(i_1) \wedge \cdots \lambda(i_s) \wedge \cdots \lambda(i_{n-1}) \wedge \lambda(i) \). The result is the desired relation.

2. The convexity of the star-shaped sphere \( \text{st}(K) \) at a wall \( J \) depends on spatial relationship between the affine line through the points \( \lambda(i), \lambda(i') \) and the codimension \( 2 \) affine subspace through the points \( \lambda(i_1), \ldots, \lambda(i_{n-1}) \), that is on the sign of the determinant

\[
\det(\lambda(i_1) - \lambda(i'), \ldots, \lambda(i_{n-1}) - \lambda(i'), \lambda(i) - \lambda(i')) = \\
= \det(\lambda(i_1), \ldots, \lambda(i_{n-1}), \lambda(i)) - \det(\lambda(i_1), \ldots, \lambda(i_{n-1}), \lambda(i')) - \\
- \sum_{s=1}^{n-1} \det(\lambda(i_1), \ldots, \lambda(i'), \ldots, \lambda(i_{s-1}), \lambda(i_s), \lambda(i_{s+1}), \ldots) = 1 - (-1) - \sum_{s=1}^{n-1} a_s = \text{curv}(J).
\]

3. If the curvature of any wall is non-positive, then the star-shaped sphere \( \text{st}(K) \) could not wind around the origin.

4. We express the class \( v_i \) through the classes \( v_j, j \notin J \), by using linear relations in the cohomology ring. Consider the linear functional \( \mu \) on the space \( V \) such that \( \langle \mu, \lambda(i) \rangle = 0 \) and

\[
\langle \mu, \lambda(i_t) \rangle = \begin{cases} 
0, & t \neq s, \\
1, & t = s.
\end{cases}
\]

Applying \( \mu \) to relation \( \text{(3)} \), we get \( \langle \mu, \lambda(i') \rangle = a_s \). It follows that there is a linear relation \( v_i + a_s v_{i'} = \sum_{j \notin \{i_1, \ldots, i_{n-1}, i, i'\}} C_j v_j \) in the cohomology ring. Let us multiply this relation by \( v_J \). Since \( J \) forms a simplex only with vertices \( i, i' \), Stanley–Reisner relations imply

\[
\int_{X} v_J v_i = \int_{X} -a_s v_J v_{i'} = -a_s.
\]
As before, let

\[ I \]

Consider two cases:

The formula for the curvature easily follows:

\[
\int_{X_\Delta} (v_J \cdot \sum_{i \in [m]} v_i) = \int_{X_\Delta} v_J v_i + \int_{X_\Delta} v_J v_i' + \sum_{s=1}^{n-1} \int_{X_\Delta} v_J v_i_s = 2 - \sum_{s=1}^{n-1} a_s = \text{curv}(J).
\]

There is an interesting corollary (which also gives an alternative proof of point (3) of the previous Lemma in dimension \( n = 3 \)).

**Proposition 16 (Unimodular Gauss–Bonnet theorem).** Let \( \Delta \) be a unimodular simplicial fan of dimension 3. Then the sum of curvatures of all its walls equals 24.

**Proof.** It follows from the previous lemma, that

\[
\sum_{J \in K, |J|=2} \text{curv}(J) = \int_{X_\Delta} (\sum_{J \in K, |J|=2} v_J) (\sum_{i \in [m]} v_i) = \int_{X_\Delta} c_2(X_\Delta) c_1(X_\Delta) = c_{1,2}(X_\Delta).
\]

It is known that for stably complex manifolds of real dimension 6 the Chern number \( c_{1,2}(X_\Delta) \) coincides with 24 Td(\( X_\Delta \)). Todd genus of a smooth compact toric variety equals 1, and the statement follows.

Now we prove Theorem 2. Let \( \Delta \) be the normal fan of a Delzant polytope \( P \). The walls of this fan are simply the edges of the 2-dimensional triangulated sphere \( K \).

**Proof of Theorem 2.** According to Lemma 15(3), there exists a wall \( \tilde{J} \in K \) of positive curvature. On the other hand, Lemma 15(4) implies that the curvature of the wall \( \tilde{J} \) coincides with the value of the linear functional \( H^4(X_\Delta; \mathbb{R}) \rightarrow \mathbb{R}, \ u \mapsto \int_{X_\Delta} (u \cdot c_1(X_\Delta)) \) on the effective class \( v_J \). Since a linear functional takes positive value on some element of the effective cone, this functional should take positive value on some generator of this cone. Therefore, there exists an extremal wall \( J = \{i_1, i_2\} \in K \) having positive curvature.

Let \( a_1, a_2 \) be the parameters of the wall \( J \), defined earlier. Since \( \text{curv}(J) = 2 - a_1 - a_2 > 0 \) and the numbers \( a_1, a_2 \) are integral, we have either \( a_1 \leq 0 \), or \( a_2 \leq 0 \). Assume \( a_1 \leq 0 \). Consider two cases:

1. \( a_1 < 0 \). Let us prove that in this case \( i_2 \) is contained in exactly three maximal cones.

As before, let \( I = \{i, i_1, i_2\} \), \( I' = \{i', i_1, i_2\} \) be the maximal simplices containing the wall \( J \). Suppose that apart from the vertices \( i_1, i, i' \) the vertex \( i_2 \) is connected to the vertices \( k_1, \ldots, k_p, p \geq 1 \) (we assume that the neighbors of the vertex \( i_2 \) are cyclically ordered as \( i_1, i', k_1, \ldots, k_p, i \), see Fig.1).

According to Lemma 15(1), \( a_1 = \det(\lambda(i'), \lambda(i_2), \lambda(i)) < 0 \). This means, that the sum of dihedral angles of the cones \( C(I) = \text{cone}(\lambda(i_1), \lambda(i_2), \lambda(i)) \) and \( C(I') = \text{cone}(\lambda(i_1), \lambda(i_2), \lambda(i')) \) at the edge \( \mathbb{R}_{\geq 0} \lambda(i_2) \) exceeds one straight angle (see left part of Fig.1). There exists a 2-plane \( \Pi \) which contains the ray \( \mathbb{R}_{\geq 0} \lambda(i_2) \) and separates \( \lambda(i_1) \) from the vectors (4)

\[
\lambda(i), \lambda(i'), \lambda(k_1), \ldots, \lambda(k_p).
\]

Let \( \mu \) be the linear functional on \( \mathbb{R}^3 \), annihilating the plane \( \Pi \) and taking value 1 on the vector \( \lambda(i_1) \). By construction, \( \mu \) takes negative values on all vectors from the list (4). In
Figure 1. The vicinity of the ray \( \mathbb{R}_{>0} \lambda(i_2) \) in the first and the second cases. The ray \( \mathbb{R}_{>0} \lambda(i_2) \) points to the reader.

in the cohomology ring we have a linear relation

\[
v_{i_1} = \sum_{t \in \{i, i', k_1, \ldots, k_p\}} C_t v_t + \sum_{t \not\in \{i_1, i_2, i', k_1, \ldots, k_p\}} D_t v_t,
\]

in which all coefficients \( C_t \) are positive. Multiplying this relation by \( v_{i_2} \), we get

\[
v_J = v_{i_1} v_{i_2} = \sum_{t \in \{i, i', k_1, \ldots, k_p\}} C_t v_t v_{i_2}
\]

(the part of expression, having coefficients \( D_t \) vanishes by Stanley–Reisner relations).

Therefore, the class \( v_J \) is expressed as a positive linear combination of the classes \( v_t v_{i_2}, t \in \{i, i', k_1, \ldots, k_p\} \). Since \( v_J \) was chosen to be extremal, each of the classes \( v_t v_{i_2} \) is proportional to the class \( v_{i_2} \). Since all these classes are nonzero, they are all proportional to each other. This leads to contradiction. Indeed, according to relation (2) we have \( (v_t' v_{i_2}) v_{k_1} \neq 0 \) since \( \{i_1, i_2, k_1\} \notin K \), but \( (v_{i_1} v_{i_2}) v_{k_1} = 0 \) since \( \{i_1, i_2, k_1\} \in K \).

(2) \( a_1 = 0 \). We prove that in this case the vertex \( i_2 \) is contained in four maximal cones. The proof is similar to the previous case. Assume the contrary: let the vertex \( i_2 \) have the neighbors \( i, i_1, i', k_1, \ldots, k_p, p \geq 2 \), written in the cyclic order.

According to Lemma [15(1)], the condition \( a_1 = 0 \) implies that the vectors \( \lambda(i_2), \lambda(i), \lambda(i') \) belong to a single 2-plane, say \( \Pi \). Let \( \mu \) be the linear functional annihilating \( \Pi \) and taking value 1 on the vector \( \lambda(i_1) \). Consequently, \( \mu \) takes strictly negative values on the vectors \( \lambda(k_1), \ldots, \lambda(k_p) \). By the same arguments as before we deduce that the class \( v_J = v_{i_1} v_{i_2} \) is written as a positive linear combination of the classes \( v_t v_{i_2}, t \in \{k_1, \ldots, k_p\} \). The extremality of the wall \( J \) implies that all these classes (there are at least two of them by assumption) are proportional to the class \( v_J \). Again, this leads to contradiction: \( v_{k_1} v_{i_2} v_1 = 0 \) since \( \{k_1, i_2, 1\} \notin K \), but \( v_{i_1} v_{i_2} v_1 \neq 0 \) since \( \{i_1, i_2, 1\} \in K \).

We proved that there are no more than four maximal cones containing \( \lambda(i_2) \). There can’t be three maximal cones by obvious geometrical reasons: the vectors \( \lambda(i_2), \lambda(i), \lambda(i') \) belong to a 2-plane and therefore cannot form a maximal cone.

It was shown that in the sphere \( K \) there exists a vertex having either 3 or 4 neighbors. This means that in the dual polytope \( P \) there exists either a triangular or quadrangular face. \( \square \)
Remark 17. The existence of a strictly convex effective cone, and as a corollary, extremal classes, is the fact, which marks out projective smooth toric varieties among all quasitoric manifolds. For general quasitoric manifolds we may still define the cohomology classes \( v_J \in H^{2n-2}(X; \mathbb{R}) \) corresponding to the walls, however their nonnegative linear combinations may span the whole space \( H^{2n-2}(X; \mathbb{R}) \) rather than a strictly convex cone. This is why it is impossible to find “extremal” classes with nice properties.

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