Research Article

Second-Order Differential Equation with Multiple Delays: Oscillation Theorems and Applications

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Abstract

Oscillation Theorems and Applications

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1. Introduction

The differential equation of second order appears in physical applications such as fluid dynamics, electromagnetism, acoustic vibrations, and quantum mechanics. In this paper, necessary and sufficient conditions are established of the solution to second-order half-linear delay differential equations of the form

\[ (c(y)(u'(y))^\alpha)'+\sum_{j=1}^{m} p_j(y)u^{\beta_j}(y) = 0, \]

for \( y \geq y_0 \),

(A1) \( \beta_j \in C([0,\infty), \mathbb{R}) \), \( \beta_j(y) < y \), \( \lim_{y \to \infty} \beta_j(y) = \infty \)

(A2) \( c \in C^1([0,\infty), \mathbb{R}) \), \( p_j \in C([0,\infty), \mathbb{R}) \); \( 0 < c(y) \), \( 0 \leq p_j(y) \) for all \( y \geq 0 \) and \( j = 1, 2, \ldots, m \); \( \sum p_j(y) \) is not identically zero in any interval \([b, \infty)\)

(A3) \( Y(y) = \int_{y_1}^{y} \zeta^{-1/\alpha}(\eta)d\eta \) with \( \lim_{y \to \infty} Y(y) = \infty \)

(A4) The existence of a differentiable function \( \theta_0 \) such that

\[ 0 < \theta_0(y) = \inf \{ \theta_j(y) \}, \quad \theta_0(y) \geq \theta_j(y) > 0 \quad \text{for} \quad y \geq y_0, \quad j = 1, 2, \ldots, m \]

In [2, 3], Baculíková and Džurina have considered

\[ (c(y)(z'(y))^\alpha)' + p(y)z^{\beta}(\theta(y)) = 0, \]

\[ z(y) = u(y) + q(y)u(\tau(y)), \quad y \geq y_0, \]

and obtained oscillation criteria for the solutions of (2) using comparison techniques when \( a = c = 1 \), \( 0 \leq q(y) < \infty \) and \( \lim_{y \to \infty} Y(y) = \infty \). In the same technique, Baculíková and Džurina [4] have studied the oscillatory behavior of the solutions of (2) under the assumptions \( 0 \leq q(y) < \infty \) and \( \lim_{y \to \infty} Y(y) = \infty \). In [7], Tripathy et al. have studied oscillatory and asymptotic behavior of (4) when \( \lim_{y \to \infty} Y(y) = \infty \) and \( \lim_{y \to \infty} Y(y) < \infty \) for different ranges of the neutral coefficient \( q \). In [5], Bohner et al. have studied the oscillatory behavior of solutions of (2) under \( a = c, \lim_{y \to \infty} Y(y) < \infty \) and \( 0 \leq q(y) < 1 \). Grace et al. [6]
have studied the oscillatory behavior of (4) when \( a = c \) and 
\[ \lim_{y \to -\infty} Y(y) < \infty, \lim_{y \to -\infty} Y(y) = \infty \text{ and } 0 \leq q(y) < 1. \]
In [7], Li et al. have studied the oscillatory behavior of the solutions of (2), under the assumptions \( \lim_{y \to -\infty} Y(y) < \infty \)
and \( q(y) \geq 0. \) Karpuz and Santra [8] have studied the oscillatory behavior of
\[ (c(y)(u'(y)q(y)u(\tau(y)))')' + p(y)f(u(\theta(y))) = 0, \quad (3) \]
by considering the assumptions \( \lim_{y \to -\infty} Y(y) < \infty \) and 
\( \lim_{y \to -\infty} Y(y) < \infty \) for different ranges of \( q. \)

For further work on the oscillation of this type of equations, we refer the readers to [6, 9–32]. Note that the majority of works consider only sufficient conditions, and merely a few consider both necessary and sufficient conditions. Hence, the objective of this work is to establish both necessary and sufficient conditions for the oscillation of solutions of (1) without using the comparison and the Riccati techniques. In this paper, we restrict our attention to study (1), which includes the class of functional differential equations of the delay type.

**Remark 1.** When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all \( y \) large enough.

### 2. Necessary and Sufficient Conditions

**Lemma 1.** Let (A1)–(A3) hold and that \( u \) is an eventually positive solution of (1). Then, there exist \( y_1 \geq y_0 \) and \( d > 0 \) such that

\[ 0 < u(y) \leq dY(y), \quad (4) \]

\[ Y(y) \left[ \int_j^y \sum_{j=1}^m p_j(\xi)u^c(\theta_j(\xi))d\xi \right]^{1/a} \leq u(y), \quad (5) \]

for \( y \geq y_1. \)

**Proof.** Let \( u \) be eventually positive solution of (1). Then, by (A1), there exists a \( y^* \) such that \( u(y) > 0 \) and \( u(\theta_j(y)) > 0 \)
for all \( y \geq y^* \) and \( j = 1, 2, \ldots, m. \) From (1), it follows that

\[ (c(y)(u'(y))^a)' = -\sum_{j=1}^m p_j(y)u^c(\theta_j(y)) \leq 0. \quad (6) \]

Therefore, \( c(y)(u'(y))^a \) is nonincreasing for \( y \geq y^*. \)

Next, we show the \( c(y)(u'(y))^a \) is positive. By contradiction, assume that \( c(y)(u'(y))^a \leq 0 \) at a certain time \( y \geq y'. \)

Using that \( \sum p_j(y) \) is not identically zero on any interval \([b, \infty)\)
and by (6), there exist \( y_1 \geq y^* \) such that

\[ c(y)(u'(y))^a \leq c(y_1)(u'(y_1))^a < 0, \quad \text{for all } y \geq y_1. \quad (7) \]

Recall that \( a \) is the quotient of two positive odd integers. Then,

\[ u'(y) \leq \frac{c(y_1)}{c(y)} u'(y_1), \quad \text{for } y \geq y_1. \quad (8) \]

Integrating from \( y_1 \) to \( y, \) we have

\[ u(y) \leq u(y_1) + (c(y_1))^{1/a} u'(y_1)Y(y). \quad (9) \]

By (A3), the right-hand side approaches \( -\infty \) then, \( \lim_{y \to -\infty} u(y) = -\infty. \) This is a contradiction to the fact that \( u(y) > 0. \) Therefore, \( c(y)(u'(y))^a > 0 \) for all \( y \geq y^*. \) From \( c(y)(u'(y))^a \) being nonincreasing, we have

\[ u'(y) \leq \frac{c(y_1)}{c(y)} u'(y_1), \quad \text{for } y \geq y_1. \quad (10) \]

integrating this inequality from \( y_1 \) to \( y, \) and using that \( u \) is continuous,

\[ u(y) \leq u(y_1) + (c(y_1))^{1/a} u'(y_1)Y(y). \quad (11) \]

Since \( \lim_{y \to -\infty} Y(y) = \infty, \) there exists a positive constant \( d \) such that (4) holds.

Since \( c(y)(u'(y))^a \) is positive and nonincreasing, \( \lim_{y \to -\infty} c(y)(u'(y))^a \) exists and is nonnegative. Integrating (1) from \( y \) to \( b, \) we have

\[ c(b)(u'(b))^a - c(y)(u'(y))^a + \int_y^b \sum_{j=1}^m p_j(\eta)u^c(\theta_j(\eta))d\eta = 0. \quad (12) \]

Letting limit as \( b \longrightarrow \infty, \) we obtain

\[ c(y)(u'(y))^a \geq \int_y^{\infty} \sum_{j=1}^m p_j(\eta)u^c(\theta_j(\eta))d\eta. \quad (13) \]

Then,

\[ u'(y) \geq \left[ \frac{1}{c(y)} \int_y^{\infty} \sum_{j=1}^m p_j(\eta)u^c(\theta_j(\eta))d\eta \right]^{1/a}. \quad (14) \]

Since \( u(y_1) > 0, \) integrating the above inequality yields

\[ u(y) \geq \int_{y_1}^{y} \left[ \frac{1}{c(y)} \int_{y_1}^{\infty} \sum_{j=1}^m p_j(\xi)u^c(\theta_j(\xi))d\xi \right]^{1/a} d\eta. \quad (15) \]

Since the integrand is positive, we can increase the lower limit of integration from \( \eta \) to \( y \) and then use the definition of \( Y(y) \) to obtain

\[ u(y) \geq Y(y) \left[ \int_{y_1}^{y} \sum_{j=1}^m p_j(\xi)u^c(\theta_j(\xi))d\xi \right]^{1/a}, \quad (16) \]

which yields (5). \( \square \)

**Theorem 1.** Assume that there exists a constant \( b_1 \) and the quotient of two positive odd integers, such that \( 0 < c_j < b_1 < a. \)
If (A1)–(A3) hold, then each solution of (1) is oscillatory if and only if

\[ \int_0^{\infty} \sum_{j=1}^m p_j(\xi)Y^c(\theta_j(\xi))d\xi = \infty. \quad (17) \]
Proof. On the contrary, we assume that a solution \( u \) is eventually positive. So, Lemma 1 holds, and then there exists \( y_1 \geq y_0 \) such that

\[
u(y) \geq Y(y)w^{1/a}(y) \geq 0, \quad \text{for } y \geq y_1, \quad (18)\]

where

\[
\begin{align*}
\nu(y) &= \int_{y_1}^{y} \sum_{j=1}^{m} \rho_j(\zeta)u^{c}(\theta_j(\zeta))d\zeta.
\end{align*}
\]

Computing the derivative of \( \nu \), we have

\[
\nu'(y) = -\int_{y_1}^{y} w^{1/a}(y)u^{c}(\theta_j(\zeta))d\zeta.
\]

(21)

Thus, \( \nu \) is nonnegative and nonincreasing. Since \( u > 0 \), by (A2), it follows that \( \sum_{j=1}^{m} \rho_j(\zeta)u^{c}(\theta_j(\zeta)) \) cannot be identically zero in any interval \([b, \infty)\); thus, \( \nu' \) cannot be identically zero, and \( \nu \) cannot be constant on any interval \([b, \infty)\). Therefore, \( \nu(y) \to 0 \) for \( y \to \infty \). Computing the derivative,

\[
(\nu^{1/b/a}(y))' = \left(1 - \frac{b}{a}\right)w^{b/a}(y)\nu'(y).
\]

Integrating (21) from \( y_2 \) to \( y \) and using that \( \nu > 0 \), we have

\[
\nu^{1/b/a}(y_2) \geq \left(1 - \frac{b}{a}\right)\int_{y_2}^{y} w^{b/a}(\zeta)\nu'(\zeta)d\zeta
\]

\[
= \left(1 - \frac{b}{a}\right)\int_{y_2}^{y} w^{b/a}(\zeta)\left(\sum_{j=1}^{m} \rho_j(\zeta)u^{c}(\theta_j(\zeta))\right)d\zeta.
\]

(22)

Next, we find a lower bound for the right-hand side of (22), independent of the solution \( u \). By (4) and (19), we have

\[
u^{c}(\theta_j(\zeta)) = \left(\frac{1}{\zeta(\eta)} + \frac{m}{\zeta(\eta)}\sum_{j=1}^{m} \rho_j(\zeta)u^{c}(\theta_j(\zeta))\right)\left(\frac{1}{\zeta(\eta)} + \frac{m}{\zeta(\eta)}\sum_{j=1}^{m} \rho_j(\zeta)u^{c}(\theta_j(\zeta))\right)\left(\frac{1}{\zeta(\eta)} + \frac{m}{\zeta(\eta)}\sum_{j=1}^{m} \rho_j(\zeta)u^{c}(\theta_j(\zeta))\right)\\\int_{y_1}^{y_2} \left[1 + \sum_{j=1}^{m} \rho_j(\zeta)u^{c}(\theta_j(\zeta))\right]d\zeta
\]

\[
rho_j(\zeta)u^{c}(\theta_j(\zeta))d\zeta.
\]

(28)

(Note that when \( u \) is continuous, \( \Omega u \) is also continuous on \([0, \infty)\). If \( u \) is a fixed point of \( \Omega \), i.e., \( \Omega u = u \), then \( u \) is a solution of (1).)

First, we estimate \( (\Omega u)(y) \) from below. By (A3), we have

\[
(\Omega u)(y) \geq \int_{y_1}^{y} \left[\frac{1}{\zeta(\eta)} + \frac{m}{\zeta(\eta)}\sum_{j=1}^{m} \rho_j(\zeta)u^{c}(\theta_j(\zeta))\right]d\zeta.
\]

(29)

Now, we estimate \( (\Omega u)(y) \) from above. For \( u \in S \), we have \( u^{c}(\theta_j(\zeta)) \leq (\kappa^{1/a}Y(\theta_j(\zeta)))^\nu \). Then, by (26),

\[
u^{c}(\theta_j(\zeta)) \leq (\kappa^{1/a}Y(\theta_j(\zeta)))^\nu.
\]

Then, by (26),

\[
u^{c}(\theta_j(\zeta)) \leq (\kappa^{1/a}Y(\theta_j(\zeta)))^\nu.
\]

(27)

Therefore, \( \Omega \) maps \( S \) to \( S \).

Next, we find a fixed point for \( \Omega \) in \( S \). Let us define a sequence of functions in \( S \) by the recurrence relation:
\(v_0(y) = 0\), for \(y \geq y_0\),
\[v_1(y) = (\Omega v_0)(y) = \begin{cases} 0 & \text{if } y < y_1, \\ \kappa^{1\eta} Y(y) & \text{if } y \geq y_1, \end{cases}\]
(31)
\[v_{n+1}(y) = (\Omega v_n)(y), \quad \text{for } n \geq 1, \quad y \geq y_1.\]

Note that, for each fixed \(y\), we have \(v_1(y) \geq v_0(y)\). Using mathematical induction, we can show that \(v_{n+1}(y) \geq v_n(y)\). Therefore, the sequence \(\{v_n\}\) converges pointwise to a function \(v\). Using the Lebesgue Dominated Convergence Theorem, we can show that \(v\) is a fixed point of \(\Omega\) in \(S\). This shows under assumption (26) that there is a nonoscillatory solution that does not converge to zero. This completes the proof. \(\square\)

**Theorem 2.** Assume that there exists a constant \(b_2\) and the quotient of two positive odd integers such that \(0 < a < b_2 < c_j\).
If (A1)–(A4) hold and \(\varsigma(y)\) is nondecreasing, then each solution of (1) is oscillatory if and only if
\[
\int_{y_1}^{\infty} \left[ \frac{1}{\varsigma(\eta)} \int_{y}^{\infty} \sum_{j=1}^{m} p_j(\varsigma) d\varsigma \right]^{1/\alpha} d\eta = \infty.
\]
(32)

**Proof.** On the contrary, we assume that \(u\) is an eventually positive solution that does not converge to zero. Using the same argument as in Lemma 1, there exists \(y_1 \geq y_0\) such that \(u(\theta_0(y)) > 0\) and \(\varsigma(y)(u'(y))^\alpha\) is positive and nonincreasing. Since \(\varsigma(y) > 0\), so \(u(y)\) is increasing for \(y \geq y_1\). Using \(u(y) \geq u(y_1)\), we have
\[
u^i(y) \geq \nu^{i-b_2}(y) u^{b_2}(y) \geq \nu^{i-b_2}(y_1) u^{b_2}(y),
\]
and hence
\[
u^i(\theta_0(y)) \geq \nu^{i-b_2}(y_1) u^{b_2}(\theta_0(y)), \quad \text{for } y \geq y_2.
\]
(34)

Using (34) and \(\theta_0(y) \geq \theta_0(y)\), from (13), we have
\[
\varsigma(y)(u'(y))^\alpha \geq \nu^{i-b_2}(y_1) u^{b_2}(\theta_0(y)) \int_{y}^{\infty} \sum_{j=1}^{m} p_j(\varsigma) d\varsigma
\]
(35)
for \(y \geq y_2\). From \(\varsigma(y)(u'(y))^\alpha\) being nonincreasing and \(\theta_0(y) \leq y\), we have
\[
\varsigma(\theta_0(y))(u'(\theta_0(y)))^\alpha \geq \varsigma(y)(u'(y))^\alpha.
\]
(36)

We use this in the left-hand side of (35). Then, dividing by \(\varsigma(\theta_0(y))u^{b_2}(\theta_0(y)) > 0\), raising both sides to the \(1/\alpha\) power, we have
\[
u'(\theta_0(y))^\alpha \int_{y}^{\infty} \sum_{j=1}^{m} p_j(\varsigma) d\varsigma \int_{\eta}^{\infty} \eta^{\alpha} d\eta = \infty,
\]
(37)
for \(y \geq y_2\). Multiplying the left-hand side by \(\theta_0(y)/\theta_0 \geq 1\) and integrating from \(y_2\) to \(y\),
\[
\frac{1}{\theta_0} \int_{y_2}^{y} \nu'(\theta_0(y)) \frac{\theta_0'(y)}{\theta_0(y)} d\eta \leq \nu^{i-b_2}(y_1) \int_{\eta}^{\infty} \eta^{1/\alpha} d\eta.
\]
(38)

On the left-hand side, since \(a < b_2\), integrating, we have
\[
\frac{1}{\alpha(1-b_2/\alpha)} \left( e^{1-b_2/\alpha}(\theta_0(y)) \right)^\alpha \int_{y_2}^{y} \int_{\eta}^{\infty} \sum_{j=1}^{m} p_j(\varsigma) d\varsigma \int_{\eta}^{\infty} \eta^{1/\alpha} d\eta \leq \frac{1}{\alpha(1-b_2/\alpha)} e^{1-b_2/\alpha}(\theta_0(y_2)) < \infty.
\]
(39)

On the right-hand side of (38), we use that \(\varsigma(\theta_0(\eta)) \leq \varsigma(\eta),\) to conclude that (32) implies the right-hand side approaching \(+\infty\), as \(y \to \infty\), which is a contradiction. Hence, the solution \(u\) cannot be eventually positive.

For eventually negative solutions, we use the same change of variables as in Theorem 1 and proceed as above.

To prove the necessity part, we assume that (32) does not hold and obtain an eventually positive solution that does not converge to zero. If (32) does not hold, then for each \(\kappa > 0\) there exists \(y_1 \geq y_0\) such that
\[
\int_{y_1}^{\infty} \left[ \frac{1}{\varsigma(\eta)} \int_{y}^{\infty} \sum_{j=1}^{m} p_j(\varsigma) d\varsigma \right]^{1/\alpha} d\eta < \frac{\kappa(1-c_0/\alpha)}{2}, \quad \forall y \geq y_1.
\]
(40)

We define the set of continuous function
\[
S = \{ u \in C([0,\infty)) : \frac{\kappa}{2} \leq u(y) \leq \kappa, \quad \text{for } y \geq y_1 \}.
\]
(41)

Then, we define the operator:
\[
(\Omega u)(y) = \begin{cases} 0, & \text{if } y \leq y_1, \\ \frac{\kappa}{2} + \int_{y_1}^{\infty} \frac{1}{\varsigma(\eta)} \int_{\eta}^{\infty} \sum_{j=1}^{m} p_j(\varsigma) u^{c_0}(\theta_j(\varsigma)) d\varsigma \int_{\eta}^{\infty} \eta^{1/\alpha} d\eta, & \text{if } y > y_1. \end{cases}
\]
(42)

First, we estimate \((\Omega u)(y)\) from below. Let \(u \in M\), and we have \((\Omega u)(y) \geq \kappa/2 + 0\), on \([y_1,\infty)\).
Now, we estimate \((\Omega u)(y)\) from above. Let \(u \in M\). Then, \(u \leq \kappa\), and by (40), we have

\[
(\Omega u)(y) \leq \frac{K}{2} + \kappa^{1+\alpha} \int_{y_0}^{\infty} \left[ \frac{1}{\zeta(\eta)} \int_{\eta}^{y} \sum_{j=1}^{m} p_j(\zeta) \, d\zeta \right]^{1/\alpha} \, d\eta \leq \frac{K}{2} + \kappa = \kappa.
\]  

(43)

Therefore, \(\Omega\) maps \(S\) to \(S\). To find a fixed point for \(\Omega\) in \(S\), we define a sequence of functions by the recurrence relation:

\[
v_{0}(y) = 0, \quad \text{for } y \geq y_0,
\]

\[
v_{1}(y) = (\Omega v_{0})(y) = 1, \quad \text{for } y \geq y_1,
\]

\[
v_{n+1}(y) = (\Omega v_{n})(y), \quad \text{for } n \geq 1, \ y \geq y_1.
\]

Note that, for each fixed \(y\), we have \(v_{1}(y) \geq v_{0}(y)\). Using mathematical induction, we can prove that \(v_{n+1}(y) \geq v_{n}(y)\). Therefore, \(\{v_{n}\}\) converges pointwise to a function \(v\) in \(S\). Then, \(v\) is a fixed point of \(\Omega\) and a positive solution of (1). The proof is completed.

**Example 1.** Consider

\[
(e^{-y}(u'(y))^{1/3})' + \frac{1}{y+1} (u(y-2))^{1/3} + \frac{1}{y+2} (u(y-1))^{1/3} = 0.
\]

(45)

Here, \(a = 11/3\), \(\zeta(y) = e^{-y}\), \(\beta_1(y) = y - 2\), \(\beta_2(y) = y - 1\), \(\eta(y) = \int_{y}^{y_0} e^{y/(y-2)} \, dy = (3/11) (e^{11y/3} - e^{11y_1/3})\), \(c_1 = 1/3\), and \(c_2 = 5/3\). For \(b = 7/3\), we have \(0 < \max\{c_1, c_2\} < b < a\). To check (17), we have

\[
\int_{0}^{\infty} \sum_{j=1}^{m} p_j(s) \zeta(s) \, ds \geq \int_{0}^{\infty} p_1(s) \zeta(s) \, ds \geq \frac{1}{\eta + 1} \left( \frac{3}{11} (e^{11y/(y-2)} - e^{11y_1/(y-2)}) \right)^{1/3} \, ds = \infty.
\]

(46)

So, every condition of Theorem 1 holds true. Therefore, all solutions of (45) is oscillatory.

**Example 2.** Consider

\[
\left( (u'(y))^{1/3} \right)' + y (u(y - 2))^{1/3} + (y + 1) (u(y - 1))^{1/3} = 0.
\]

(47)

Here, \(a = 1/3\), \(\zeta(y) = 1\), \(\beta_1(y) = y - 2\), \(\beta_2(y) = y - 1\), \(c_1 = 7/3\), and \(c_2 = 11/3\). For \(b = 5/3\), we have \(\min\{c_1, c_2\} > b > a\). To check (32), we have

\[
\int_{y_0}^{\infty} \left[ \frac{1}{\zeta(s)} \int_{s}^{\infty} \sum_{j=1}^{m} p_j(\zeta) \, d\zeta \right]^{1/\alpha} \, ds \geq \frac{1}{\zeta(s)} \int_{s}^{\infty} p_1(\zeta) \, d\zeta \right]^{1/\alpha} \, ds \geq \frac{1}{\zeta(s)} \int_{s}^{\infty} \zeta(\zeta) \, d\zeta \right]^{3} \, ds = \infty.
\]

(48)

So, every conditions of Theorem 2 hold true. Thus, all solutions of (47) is oscillatory.

**3. Conclusion**

This work aims to study the oscillatory behavior of second-order neutral nonlinear differential equation. The obtained oscillation theorems complement the well-known oscillation results present in the literature.

**Data Availability**

The data used to support the findings of the study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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