Distributed Approximate Distance Oracles*

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Abstract

Data structures that allow efficient distance estimation (distance oracles or distance sketches) have been extensively studied, and are particularly well studied in centralized models and classical distributed models such as the CONGEST model. We initiate their study in newer (and arguably more realistic) models of distributed computation such as the Congested Clique model and the Massively Parallel Computation (MPC) model, as well as in related big data models such as streaming and semi-streaming. We provide algorithms for constructing the well-known Thorup-Zwick distance oracle (or distance sketches) in multiple different computational models and discuss the tradeoffs between different complexity measures such as space, time, approximation ratio, and communication complexity.

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1 Introduction

A common task when performing graph analytics is to compute distances between vertices. This has motivated the study of shortest path algorithms in every interesting model of computation, from the traditional centralized model to newer models such as the Congested Clique model \cite{20} and the Massively Parallel Computation (MPC) model \cite{4}. However, in many applications the time it takes to compute the exact distance is unacceptable, and similarly the memory that it would take to store all \( \binom{n}{2} \) distances is also unacceptable. This motivated Thorup and Zwick \cite{22} to define the notion of an approximate distance oracle: a small data structure which can quickly report an approximation of the true distance for any pair of vertices. In other words, by spending some time up front to compute this data structure (known as the preprocessing step) and then storing it (which can be done since the structure is small), any algorithm used in the future can quickly obtain provably accurate distance estimates.

More formally, an approximate distance is said to have stretch \( t \) if, when queried on \( u, v \in V \), it returns a value \( d'(u, v) \) such that \( d(u, v) \leq d'(u, v) \leq t \cdot d(u, v) \) for all \( u, v \in V \), where \( d(u, v) \) denotes the shortest-path distance between \( u \) and \( v \). The important parameters of an approximate distance oracle are the size of the oracle, the stretch, the query time, and the preprocessing time. For any constant \( k \), Thorup and Zwick’s construction has size \( O(kn^{1+1/k}) \), stretch \( (2^k - 1) \), query time \( O(k) \), and preprocessing time \( O(kmn^{1/k}) \).

Since Thorup and Zwick’s seminal work \cite{22}, there has been a large amount of followup work on improving the achievable tradeoffs, and in particular achieving query time of \( O(1) \) with size \( O(n^{1+1/k}) \) \cite{23,7} or giving more refined bounds \cite{18,19}. However, with the notable exception of a very interesting construction due to Mendel and Naor \cite{17}, the vast majority of followup work has essentially been refinements and improvements to the approach pioneered by Thorup and Zwick. Thus understanding the Thorup-Zwick distance oracle is an important first step to understanding the limits of distance oracles, and showing how to construct the Thorup-Zwick oracle in different computational models gives almost state-of-the-art bounds while also developing the basic tools and framework needed to design more sophisticated structures.

This motivated Das Sarma et al. \cite{21} to initiate the study of Thorup-Zwick distance sketches (the natural distributed version of distance oracles) in distributed networks, and in particular in the CONGEST model of distributed computing \cite{20}. In the CONGEST model we are trying to compute distance oracles/sketches for the communication graph itself, which is an important goal in many distributed settings such as overlay networks and peer-to-peer systems.

But for modern graph analytics, usually the communication graph can be abstracted away by assuming that the datacenter storing the graph is sufficiently well-provisioned. This motivated two different but related models of distributed computation: Congested Clique \cite{20} and MPC \cite{4}. Both of them allow any machine to directly communicate with any other machine, but there are important differences: in Congested Clique the number of servers is equal to the number of vertices in the graph, while MPC allows more flexibility; in Congested Clique every node can send an \( O(\log n) \)-bit message to every other node in every round, while in MPC every machine has a total I/O bound for each round; and the memory at each machine is unbounded in Congested Clique but bounded (and usually small) in MPC.

These models are a particularly good fit for distance oracles/sketches, since in these types of “big data” models we naturally want to compress the data while still being able to
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quickly and accurately report an answer to a query. Moreover, in many actual distributed systems it is reasonable to think of one particular machine as being the “interface” to the rest of the system, so if we can make our distance oracle small enough to fit into the memory of a single machine, we will be able to answer approximate distance queries without even a single network access! So we are left with an obvious question: can distance oracles (and the TZ oracle in particular) be constructed in these newer models? If so, how much do we benefit from these models compared to traditional centralized algorithms or the CONGEST model?

1.1 Our Results

In this paper we initiate the study of distance oracles and sketches in three popular computational models for “big data”: Congested Clique, MPC, and streaming. In all of these models we give tradeoffs between different performance measures such as space, time, stretch, and communication, but we show that in all of them we can build the Thorup-Zwick distance oracle (or a close modification) quite efficiently. In particular, unlike the CONGEST model we will not need any dependence on the diameter in the preprocessing time, and unlike traditional centralized algorithms we will take time (or passes in the case of streaming) that is \( o(n) \). Our main focus will be the Congested Clique model; our results in the other models will be modifications and extensions of the techniques that we develop for the Congested Clique. Since we are constructing the Thorup-Zwick oracle in all of these models (possibly after doing some initial computation), we do not describe the query algorithm and will often not describe the space requirement, as they are the same as in [22]. In particular, the query time will be \( O(k) \) and the space usage will be \( \tilde{O}(kn^{1+1/k}) \).

**Congested Clique.** In Section 3 we focus on the Congested Clique model. In this case, we assume that some node in the network is the *coordinator* at which the distance oracle will be stored (i.e., the machine with which users will interact with the distributed system). So at query time, the user can just query the coordinator rather than initiating an expensive distributed computation. We prove a variety of tradeoffs and bounds, which are given in Section 3 and are somewhat technical. The precise statements can be found in Theorems 9 and 12 but for simplicity we state one particularly interesting corollary obtained by some specific parameter settings:

**Theorem 1.** Given a weighted graph \( G = (V, E, w) \), for all \( k \geq 1 \) and \( \epsilon > 0 \) we can construct a distance oracle with stretch \( (1+\epsilon)(2k-1) \), query time \( O(k) \), and space \( O(kn^{1+1/k}) \) in the Congested Clique model. If \( k = O(1) \) then the number of rounds is \( \tilde{O}(n^{1/k}) \), and if \( k = \Omega(\log n) \) then the number of rounds is \( 2^{\tilde{O}(\sqrt{\log n})} \) (where here \( \tilde{O} \) is suppressing poly log log terms).

We also discuss how we can use spanners to reduce the communication complexity of constructing a distance oracle. For fixed space, we give a tradeoff between the stretch and the communication complexity by combining our approach with a distributed spanner construction of [3]. This gives us a distributed distance oracle with stretch \( O(kt) \) for any \( t, k \geq 2 \) and size \( \tilde{O}(n^{1+1/k}) \) with \( \tilde{O}(kn^{1+1/k}) \) total communication. Therefore, as long as \( n^{1/t} = o(m) \), for a distance oracle of fixed size there is a tradeoff between the communication and the stretch (by varying \( t \)). This setting is very natural for many applications, as constructing distance oracles is most interesting with dense graphs.
MPC. In Section 4, we will discuss the MPC model. Since in MPC model servers often have small memory, it makes more sense to focus on distance sketches. After the preprocessing algorithm, for each node \( v \in V \), a distance sketch of size \( O(kn^{1/k} \log n) \) will be stored and mapped to a machine with key \( v \). Given distance sketches of any pair of nodes, the distance can be queried locally by the machines that store their sketches. The general result is stated in Theorem 20, but a simplified corollary that considers constant and logarithmic stretch sketches is as follows:

**Theorem 2.** Given a weighted graph \( G = (V, E, w) \), and \( \rho \leq \gamma \leq 1, \rho \geq 1/k, 0 < \epsilon < 1 \) we can construct a Thorup-Zwick distance oracle with stretch \( (2k-1)(1+\epsilon) \) and size \( O(kn^{1/k}\log n) \) w.h.p. in \( \widetilde{O}(\frac{\sqrt{2n^{1/k}}}{\gamma} \cdot \min((\log n \cdot \rho), \beta)) \) rounds of MPC with \( O(n^\gamma) \) memory per machine, where \( \beta = \min(O(\log^k n), 2^{\tilde{O}(\sqrt{\log n})}) \). In particular, in case \( k = \Theta(1) \), w.h.p. we require \( \tilde{O}(n^{1/k}) \) rounds, and in case \( k = \Theta(\log n) \), w.h.p. we require \( 2^{\tilde{O}(\sqrt{\log n})} \) rounds.

The main technical difficulty when moving to low-memory MPC from Congested Clique is showing how to “simulate” a node when running restricted Bellman-Ford. Since the memory of each machine is so small, it takes many machines to store the edges incident on any particular node, so we need to show that vertex-based algorithms such as Bellman-Ford can still be used. Once we develop this tool, we argue that hopsets of Elkin and Neiman [9] can also be constructed in the low memory MPC setting in almost the same number of rounds as the Congested Clique hopset construction. This hopset construction may be of independent interest, in addition to multiple subroutines used in the overall construction (e.g. finding minimum or broadcasting on a range of machines).

**Streaming.** Finally, in Section 5 we provide an algorithm for constructing distance oracles in the multi-pass streaming model. Our general results can be found in Section 5. For the specific settings of constant or logarithmic stretch, we have:

**Corollary 3.** Given a graph \( G = (V, E, w) \), there exists a streaming algorithm that constructs a Thorup-Zwick distance oracle of stretch \( (2k-1)(1-\epsilon) \) of size \( O(kn^{1/k}\log n) \) w.h.p. and expected space \( O(n^{1+1/k} \cdot \log^2 n) \), such that if \( k = \Omega(1) \), w.h.p. we require \( O(\log^k n) \) passes, and if \( k = \Omega(\log n) \), w.h.p. we require \( 2^{\tilde{O}(\sqrt{\log n})} \) passes.

Note that in case of \( k = \Omega(\log n) \) we are in the so-called semi-streaming setting in which the total memory used is \( O(n \cdot \text{polylog } n) \). A closely related concept is a graph spanner, which are extensively studied in the streaming model (e.g. [11], [1]). One could construct a \( k \)-spanner in space \( O(kn^{1+1/k}) \) in a single pass ([11]) and then locally compute a distance oracle on the spanner. While this would only require a single pass, the stretch of the constructed distance oracle would be \( O(k^2) \). Hence, for a fixed space bound, there is a tradeoff between the stretch and number of passes.

### 1.2 Related Work

Das Sarma et al. [21] first proposed an algorithm that constructs Thorup-Zwick distance sketches with stretch \( 2k-1 \) and size \( O(kn^{1/k}\log n) \) in \( O(\Lambda \cdot kn^{1/k}) \) rounds in the CONGEST model, where \( \Lambda \) is the shortest-path diameter. Later on, Lenzen and Patt-Shamir [13] improve the running time by building distance sketches of stretch \( O(k^2) \) and size \( O(n^{1+1/k}) \) in \( O(n^{1+1/k} + D) \) rounds in the Congest model, where \( D \) is the hop-diameter of the graph. More recently, Elkin and Neiman [10] proposed an algorithm that construct distance sketches of size \( O(n^{1/k}\log n) \) that runs in \( O(n^{1/2+1/k} + D) \cdot \min((\log n)^{O(k)}, 2^{\tilde{O}(\sqrt{\log n})}) \) rounds in the
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CONGEST model. They also use the hopset construction of [9] similar to our algorithms, but they only focus on the CONGEST model.

To the best of our knowledge, constructing distance oracle/sketches are not specifically studied for the Congested Clique or MPC model. The closest problems studied in these models are single-source shortest path (SSSP) (e.g. [14], [9]), and all-pairs shortest path (APSP) (e.g. [6]). One of the main tools we use are hopsets. In previous work, hopsets were mainly used for solving approximate SSSP ([14], [9]). We use the hopset construction of [9]. In the congested clique model, our algorithm for constructing distance oracles uses the same number of rounds as the approximate SSSP algorithm of [9]. Similarly, in the streaming model, we use the same number of passes.

We note that there are no bounds for distance computation for weighted graphs in the MPC model that we are aware of. It is known that many PRAM algorithms can be simulated in the MPC model ([15], [13]), and shortest path computation is well studied in the PRAM model (e.g. [8], [9]). However, most of these algorithms use \( \omega(|E|) \) number of processors, in which case the simulations of [15] and [13] do not directly apply in MPC model with total memory of \( O(|E|) \). Hence, these PRAM algorithm do not apply to the MPC setting directly.

2 Preliminaries and Notation

2.1 Models

In the Congested Clique model an input graph of \( G = (V, E) \) is given, and initially each node \( v \in V \) only knows its incident edges. However, the underlying communication graph is an undirected clique, and in each round every node can send a message of \( O(\log n) \) bits to any other node. This model was introduced by [20], and has been studied extensively in recent years. Note that if the communication graph were \( G \) rather than the clique, then this would be exactly the CONGEST model [20], and thus every algorithm in the CONGEST model can be run without change in the Congested Clique. Here we assume that one specific node is the coordinator, at which the distance oracle will be stored.

The second model that we consider is the Massively Parallel Computation model, which we will denote by MPC, which was introduced by [4] to model MapReduce and other realistic distributed settings. In this model the input is arbitrarily distributed over the machines, and machines each have a bounded memory \( S = N^\epsilon, \epsilon \in (0, 1) \), where \( N \) is the total input size. In the variant of MPC that we consider, every machine can communicate with every other machine in the network, but each machine in each round can have total I/O of at most \( S \).

We will also consider the streaming model (more precisely, the insert-only streaming model), which unlike the previous two models is not distributed. Instead, we assume that there is a centralized algorithm, but this algorithm only sees the edges one at a time, in a streaming fashion. Moreover, the algorithm has bounded space (certainly \( o(m) \) and possibly less), so cannot simply store the stream. We will assume that the space available to the algorithm is equal to the size of the distance oracle that we compute: we are trying to compute a distance oracle so that later algorithms can simply query this oracle for distance estimates rather than replaying the stream, so we assume that the space available is equal to this structure size (which we will attempt to make as small as possible).
2.2 Notation

We denote the (weighted) distance between a pair of nodes \( u, v \in V \) by \( d(u, v) \). We define the \( h \) hop-restricted distance between \( u, v \in V \) to be weight of the shortest path between \( u \) and \( v \) that uses at most \( h \) hops and denote this by \( d_h(u, v) \). We will denote the set of neighbors of a node \( v \in V \) by \( N(v) \), and we will use \( \hat{d}(u, v) \) to denote the estimated distance that node \( v \) has computed from node \( u \) during the run of an algorithm. In a weighted graph \( G \), we define the shortest-path diameter of \( G \) to be the maximum over all \( u, v \in V \) of the number of edges in the shortest \( u - v \) path (so if the graph is unweighted this is the same as the diameter, but in weighted settings it can be larger than the unweighted diameter).

2.3 Algorithmic Building Blocks

2.3.1 Thorup-Zwick Distance Oracle

In this section, we briefly describe the centralized construction of the well-known Thorup-Zwick distance oracle \([22]\). Given an undirected weighted graph \( G = (V, E, w) \), and \( k > 1 \) in the preprocessing phase of their algorithm, they first create a hierarchy of subsets \( A_0, A_1, ..., A_k \) by sampling from nodes of \( V \) in the following manner: set \( A_0 = V \), and for \( 1 \leq i \leq k - 1 \), add any node \( v \in A_{i-1} \) to the set \( A_i \) with probability \( n^{-1/k} \). Also, set \( A_k = \infty, \forall u \in V : d(u, A_k) = \infty \). Let \( B_i(u) = \{w \in A_i : d(u, w) < d(u, A_{i+1})\} \) for all \( u \in V, 1 \leq i \leq k \), where \( d(u, A_i) \) is the minimum distance between \( u \) and a node in the set \( A_i \), and set \( B(u) = \cup_{i=1}^{k-1} B_i(u) \). \( B(u) \) is called the bunch of \( u \). We also denote the node that has the minimum distance to \( u \) among all nodes in \( A_i \) by \( p_i(u) \) and call this the \( i \)-center of \( u \), and so \( d(u, A_i) = d(u, p_i(u)) \). The distance oracle consists of \( \{p_i(u)\}_{i=0}^k, B(u) \) for all \( u \in V \) and the corresponding distances between these nodes and \( u \). Thorup and Zwick showed that this data structure has size \( O(kn^{1+1/k} \log n) \) w.h.p. and access to these distances is enough for approximating distances between every pair of vertices in \( O(k) \) time with approximate ratio (a.k.a stretch) \( 2k - 1 \). In all the settings we consider, after preprocessing the distance oracle/sketches, we can locally perform the query algorithm of \([22]\), and thus we do not go into details of the query algorithm.

2.3.2 Hopsets

For parameter \( \epsilon, \beta > 0 \), a graph \( G_H = (V, H, w_H) \) is called a \((\beta, \epsilon)\)-hopset for the graph \( G \), if for every pair \( u, v \in V \) of vertices and graph \( G' = (V, E \cup H, w') \) obtained by adding edges of \( G_H \), we have \( d_G(u, v) \leq d_{G'}(u, v) \leq (1 + \epsilon)d_G(u, v) \) (Here \( d_G(u, v) \) stands for the distance between \( u \) and \( v \) in \( G \)). The parameter \( \beta \) is called the hopbound of the hopset.

Hopsets were originally proposed by Cohen \([5]\), an applied for computing shortest paths in the PRAM model. Throughout this paper we will use the hopset construction algorithm proposed in \([9]\). For our Congested Clique and streaming algorithms we directly use the results in \([9]\), while for making the construction work in the MPC model, we need to use a subroutine that will let us construct the hopsets in low memory settings (see Section 4).

We first give a high level overview of the hopset construction of \([21]\) here. In their algorithm, they consider each distance scale \((2^k, 2^{k+1}], k = 0, 1, 2, ... \) separately. For nodes with a fixed distance scale \( R \in (2^k, 2^{k+1}) \) the algorithm consists of a set of superclustering, and interconnection phases. Initially, the set of clusters is \( \mathcal{P} = \{\{v\}_{v \in V}\} \). Each cluster in \( C \in \mathcal{P} \) has a cluster center which we denote by \( r_C \). The algorithm uses a sequence \( \delta_1, \delta_2, ... \) of distance thresholds and a sequence \( d_{\delta_1}, d_{\delta_2}, ... \) of degree thresholds that determines the sampling probability of clusters. At the \( i \)-th iteration, every cluster \( C \in \mathcal{P} \) is sampled with
probability $1/\deg_i$. Let $S_i$ denote the set of sampled clusters. Now a single shortest-path exploration of depth $\delta_i$ from the set of centers of sampled clusters $R = \{r_C \mid C \in S_i\}$ is performed. Let $C' \in \mathcal{P} \setminus S_i$ be a cluster whose center $r_{C'}$ was reached by the exploration and let $r_C$ be the center in $R$ closest to $r_{C'}$. An edge $(r_C, r_{C'})$ with weight $d_G(r_C, r_{C'})$ is then added to the hopset. A supercluster $\hat{C}$ with center $r_{\hat{C}} = r_C$ is now created that contains all the vertices of $C$ and the clusters $C''$ for which a hopset edge was added. In the next stage of iteration $i$, all clusters within distance $\delta_i/2$ of each other that have not been superclustered at iteration $i$ will be interconnected. In other words, a separate exploration of depth $\frac{\delta_i}{2}$ is performed from each such cluster center $r_C$ and if center of cluster $C''$ is reached, an edge $(r_C, r_{C''})$ with weight $d_G(r_C, r_{C''})$ will be also added to the hopset. The final phase of their algorithm only consists of the interconnection phase.

## 3 Distance Oracles in Congested Clique

In this section, we will explain how the distance oracle can be constructed in the Congested Clique model. We will use an algorithm by Das Sarma et al. [21] that constructs Thorup-Zwick distance oracles with stretch $2k-1$ and size $kn^{1+1/k}$ in $O(k\Lambda n^{1/k})$ rounds in the CONGEST model, where $\Lambda$ is shortest-path diameter. We use an algorithm similar to their algorithm, but we will use hopsets to add edges that will make the diameter smaller, but preserve the distances within a $(1+\epsilon)$ factor. We will show how this distance oracle can be constructed in the Congested Clique model efficiently. In other words, we use hopsets as a tool to obtain faster distributed algorithms. Constructing hopsets in the Congested Clique model can be done more efficiently than in CONGEST model, and this will let us build a distance oracle in time independent of the graph diameter.

### 3.1 Hopsets in Congested Clique

In order to implement the hopset algorithm of Section 2.3.2 in the distributed setting, [9] proposes the following: each superclustering phase is consisted of a single distributed Bellman-Ford exploration of depth $\delta_i$. For an interconnection phase, a separate distributed Bellman-Ford explorations of depth $\delta_i/2$ from cluster centers is performed. However in distributed settings, $O(n)$ hops could be visited for some scales. To overcome this issue, for constructing hopset edges $H_k$ for a distance scale of $(2^k, 2^{k+1}]$, they use the hopsets $\cup_{0 \leq j \leq k-1} H_j$. This will allow each Bellman-Ford exploration to be limited to $O(\beta)$. We use the following theorem proved in [9].

**Theorem 4** ([9]). For any graph $G = (V, E, w)$ with $n$ vertices, and any $2 \leq \kappa \leq (\log n)/4, 1/2 > \rho \geq 1/\kappa, \mu > 0$ and $0 < \epsilon < 1$, there is a distributed algorithm for the Congested Clique model that computes a $(\beta, \epsilon)$-hopset with expected size $O(n^{1+\frac{\kappa}{\mu}} \log n)$ in $O(\frac{n^2}{\rho} \cdot \log^3 n \cdot \beta)$ rounds whp, where $\beta = O(\frac{n}{\rho} \cdot (\log \kappa + \frac{1}{\rho})^{\log \kappa + \frac{1}{\rho}})$. Moreover, the communication complexity of this algorithm is w.h.p. $O(n^{1+\frac{\kappa}{\rho}} \cdot \beta)$.

Roughly speaking, adding a $(\beta, \epsilon)$-hopset edges will reduce the shortest path diameter to $\beta$ in exchange for a small loss in approximation ratio. In the next section we will explain how we can set the parameters $\rho$ and $\kappa$ depending on the stretch parameter for the distance oracle, to get our desired running time.
3.2 Distributed Distance Oracle Construction

In our algorithm, we repeatedly use the following subroutine: \textit{h-restricted} distributed Bellman-Ford algorithm, which is widely used in previous work on distributed distance estimation (e.g.
see\cite{16}, or \cite{21}). The complete algorithm can be found in Appendix A. The following lemma follows from basic properties of Bellman-Ford algorithm.

\textbf{Lemma 5.} There is a distributed variant of the Bellman-Ford algorithm runs in \(O(h)\) rounds in Congested Clique and for all nodes \(u \in V\), computes \(d_h(s, u)\), the length of the shortest path between \(s\) and \(u\) among the paths that have at most \(h\) edges.

In order to compute the shortest path from \(s\) to all nodes, we will need to set \(h = \Lambda\), the shortest path diameter. But this can be as large as \(\Omega(n)\). Hence we will use a \((\beta, \epsilon)\)-hopset to approximately find the distance in \(O(\beta)\) time only. In other words, by constructing a \((\beta, \epsilon)\)-hopset \(H\), we would know that there is a path of hopbound \(\beta\) with length \((1 + \epsilon)d(u, v)\) among any pair of nodes \(u, v \in V\), and hence Algorithm \cite{2} can approximate the distances \(d(s, v)\) up to a factor of \((1 + \epsilon)\) for all \(v \in V\).

Next, we explain a distributed construction of a Thorup-Zwick distance oracle. We first describe the algorithm that runs in \(O(\Lambda \cdot kn^{1/k})\) time, and then explain how this can be improved. The sampling phase can easily be done in distributed settings. Then for finding \(p_i(v), 1 \leq i \leq k\) for all nodes \(v \in V\), we will do the following: in iteration \(i\), define a virtual source node \(s_i\), and for all nodes in \(u \in A_i\), add an edge between \(u\) and \(s_i\), where \(w(u, s_i) = 0\). Then we will only need to run the Bellman-Ford algorithm once by setting \(s_i\) to be the source, and hence after \(O(k\Lambda)\) time every node \(u \in V\) knows \(p_i(u)\) and \(d(u, A_i)\).

Finally, for all \(1 \leq i \leq k\) we need to compute the distance from \(v \in A_i \setminus A_{i+1}\) to all the nodes \(v\) for which \(w \in B(v)\). Simply running a distributed Bellman-Ford independently from all the sources \(v \in A_i \setminus A_{i+1}\) would be very slow since due to congestion limit on each edge we cannot run all these in parallel at the same time. This is why we need to use a variant of the distributed Bellman-Ford algorithm proposed by \cite{21}. Das Sarma et al. \cite{21} argue that each node \(v\) needs to forward messages in the runs of Bellman-Ford algorithm for a source \(w\) only if \(w \in B(v)\). This means that, roughly speaking, each node \(v\) participates in \(|B(v)| = O(kn^{1/k} \log n)\) runs of Bellman-Ford. If each of these algorithms was to run alone, w.h.p. this would take \(O(\Lambda \cdot kn^{1/k} \log n)\) rounds. However we need to make sure running these for all sources in \(A_i \setminus A_{i+1}\) can be done without violating the congestion bound on each edge. For that, \cite{21} provide a round-robin scheduling to ensure that all of these shortest path algorithms can run in \(O(\Lambda \cdot kn^{1/k} \log n)\). A more detailed version of this algorithm can be found in Appendix A for completeness.

Note that the algorithm in \cite{21} is for the CONGEST model, which we can easily implement in the Congested Clique model. In other words, we are not using the extra power of the Congested Clique model here, rather, we will use this power for constructing hopsets more efficiently. We summarize the result of \cite{21} in the following theorem. Also, algorithm of \cite{21} construct distance labels at each nodes. It is not difficult to see that nodes can send their labels to the coordinator within a constant factor of total number of rounds required to run all the shortest path algorithms.

\textbf{Theorem 6.} \cite{21} Given undirected graph \(G = (V, E, w)\) with shortest path diameter \(\Lambda\), there is an algorithm that runs in \(\tilde{O}(\Lambda \cdot kn^{1/k} \log n)\) rounds of Congested Clique w.h.p. and outputs a Thorup-Zwick distance oracle with stretch \((2k - 1)\) at the coordinator with high probability.

Next, we will utilize the hopset construction of \cite{9} to make the preprocessing algorithm more efficient with respect to time and communication complexity. Let \(G' = (V, E \cup H, w')\)
be the graph obtained by adding a \((\beta, \epsilon)\)-hopset \(H\) to the undirected graph \(G = (V, E, w)\). By running algorithm of Corollary 7 on \(G'\) we will get the following result.

**Corollary 7.** Given a graph \(G = (V, E, w)\) and a \((\beta, \epsilon)\)-hopset \(H\) for \(G\), there is an algorithm that runs in \(\tilde{O}(\beta \cdot kn^{1/k})\) rounds of Congested Clique and outputs a Thorup-Zwick distance oracle with stretch \((2k-1)(1+\epsilon)\) on the graph \(G' = (V, E \cup H, w')\) at the coordinator with high probability.

Next we will analyze the communication complexity of Algorithm 3 and show that w.h.p. \(\tilde{O}(kn^{1/k})\) messages need to be exchanged for running Algorithm 3. Recall that the number of messages exchanged for constructing a \((\beta, \epsilon)\)-hopset is \(\tilde{O}(\beta \cdot n^{1+\rho}/\rho)\). Hence the dominant number of messages exchanged is for running Algorithm 3.

**Lemma 8.** Total number of messages exchanged for running Algorithm 3 on graph \(G' = (V, E \cup H, w')\) is w.h.p. \(O(\beta \cdot m \cdot kn^{1/k} \log n)\).

**Proof.** The algorithm of Corollary 4 has running time \(O(\beta \cdot kn^{1/k} \log n)\) w.h.p. and overall for each edge in the graph \(O(1)\) messages are exchange, therefore the total communication is \(O(\beta \cdot mkn^{1/k} \log n)\).

We now combine the hopset construction and Algorithm 3 together to obtain our main result. We will use Theorem 4 to construct a hopset \(H\) on graph \(G = (V, E, w)\), and then run Algorithm 3 on the obtained graph \(G' = (V, E \cup H, w')\) and get the following:

**Theorem 9.** Given a graph \(G = (V, E, w)\), and \(2 \leq \kappa \leq \log(n)/4, 1/\kappa \leq \rho \leq 1/2, 0 < \epsilon < 1\) we can construct a Thorup-Zwick distance oracle with stretch \((2k-1)(1+\epsilon)\) and size \(O(kn^{1+1/k} \log n)\) w.h.p. in \(O(\tilde{O}(\log n \cdot \beta + n^{1/k} \log n \cdot \beta))\) time, where \(\beta = O(\frac{1}{\epsilon} (\log k + 1/\rho)^{\log \kappa} \frac{1}{\rho})\).

The running time depends both on the parameter \(\rho\) and stretch \(k\). In other words, there is a tradeoff between the stretch and the running time of this algorithm. When stretch \(k\) is smaller, we can choose a larger value for \(\rho\) and the dominant part of the running time would still be the distance oracle construction. On the hand, for larger values of stretch \(k\), since the distance oracle construction algorithm can be performed more efficiently, we need to set a smaller to balance out the running time of constructing a hopset and that of constructing the distance oracle over the new graph. The parameter \(\kappa\) is often \(O(1/\rho)\), and mostly just impacts the constant factor in the exponent of hopbound \(\beta\). Let us consider two special cases of \(k = O(1)\) and \(k = \Theta(\log n)\) to understand these bounds better.

**Corollary 10.** Given a graph \(G = (V, E, w)\), and \(2 < \kappa, 0 < \epsilon < 1\) we can construct a Thorup-Zwick distance oracle with stretch \((2k-1)(1+\epsilon)\) in the Congested Clique model, s.t.,
- In case \(k = O(1)\), w.h.p. we require \(\tilde{O}(n^{1/k})\) rounds.
- In case \(k = \Theta(\log n)\), w.h.p. we require \(2^{\tilde{O}(\sqrt{\log n})}\) rounds.

**Proof.** For stretch \(k = O(1)\) we use Theorem 4 and set \(\rho = 1/\kappa\), and \(\kappa = k\) to get \(\beta = O(1)\) and total running time \(\tilde{O}(n^{1/k})\). In case \(k = \Theta(\log n)\), we will set \(1/\kappa = \rho = \sqrt{\frac{\log \log n}{\log n}}\).

### 3.3 Communication Reduction with Spanners

In this section, we will describe how spanners can be used as a tool for reducing communication in exchange for an extra factor in the stretch. Spanners are the graph-theoretic counterpart of distance oracles, i.e., they attempt to preserve distances by constructing a
subgraph, and do not necessarily allow fast queries. More formally, A t-spanner of a graph $G$ is a subgraph $H$ such that $d_G(u, v) \leq d_H(u, v) \leq d_G(u, v) + t$ for all $u, v \in V$. We will use the spanner construction of Theorem 9 which computes spanners efficiently in the more restricted CONGEST model.

**Theorem 11 (3).** For any weighted graph, a $(2t - 1)$-spanner of expected size $O(tn^{1+1/t})$ can be computed in the CONGEST model in $O(t^2)$ rounds and $O(tm)$ communication complexity.

This construction allows us to turn the input graph for algorithms of Section 3.2 into a sparser graph. By doing so we will lose a factor of $t$ in the approximation ratio but we only need to run Algorithm 3 on a graph with $O(n^{1+1/t})$ edges. Hence, we first run the Algorithm of Theorem 11 to get a spanner $G_t$, and then run the algorithm of Theorem 9 on $G_t$. Then the from Lemma 8 we will get the following result.

**Theorem 12.** Given a graph $G = (V, E, w), t, k > 1, we can construct a Thorup-Zwick distance oracle of size $O(kn^{1+1/k} \log n)$ with stretch $t \cdot (2k - 1)(1 + \epsilon) = O(kt)$ w.h.p. with total communication of $\tilde{O}(k^{1+1/k} \beta + tm)$, where $\beta$ and the running time are the same as in Theorem 9.

This implies that there is a direct tradeoff between the approximation ratio and the amount of communication when size of the distance oracle is fixed. In other words, when $n^{1/t} = o(m)$ the amount of communication required for distance oracles of stretch $O(kt)$ is smaller than the amount required for building distance oracles of stretch $O(k)$, where the size is in both cases $O(kn^{1+1/k} \log n)$.

## 4 Distance Sketches in Massively Parallel Computation Model

In this section we will focus on the Massively Parallel computation model [4], also known as the MPC model. Since in these settings we are often dealing with very large scale graphs, for graph problems with $n$ nodes we are mainly interested in the setting that memory per machine is $S = O(n^\gamma)$ where $0 < \gamma \leq 1$. Let $P$ denote the number of machines. Then we assume $S \cdot P = \Theta(m)$ where $m$ is the number of edges in the input graph. We also make the common assumption (e.g. [4]) that machines have unique IDs that other machines can use for direct communication. Let MPC($n^\gamma$) denote the MPC model with $O(n^\gamma)$ memory per machine. Note that in this setting we do not have enough space to store a distance oracle, and thus we will build distance sketches of small size stored at a machine assigned to each node. For simplicity we assume that we can store the sketches in a single machine. Namely, we require $O(n^{1/k})$ memory per machine for stretch $O(k)$ distance sketches.

First, we note that it is known from [5] that MPC($n$) is equivalent to the Congested Clique model. Therefore, all the results of Section 3 hold when memory per machine is $\Omega(n)$.

For the rest of this section we will mostly focus on the case where the memory is $n^\gamma$ where $\gamma < 1$. The main component of our algorithm are the following subroutines that will allow us to simulate one round of Bellman-Ford algorithm in MPC($n^\gamma$), $\gamma < 1$:

**Sorting** [13] Given a set of $N$ comparable items, the goal is to have the items sorted on the output machines, i.e. the output machine with smaller ID holds smaller items.

**Indexing** [2] Suppose we have sets $S_1, S_2, ..., S_k$ of $N$ items stored in the system. The goal is to compute a mapping $f$ such that $vi \in [k], x \in S_i$, $x$ is the $f(S_i, x)$-th element of $S_i$. After running this algorithm the tuple $(x, f(S_i, x))$ is stored in the machine that stores $x$. 

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Find Minimum \((x, y)\) Finds the minimum of \(N\) values stored over a contiguous set of machines given ID \(x\) of the first machine and ID \(y\) of the last machine.

Broadcast \((b, x, y)\) Broadcasts a message \(b\) to a contiguous group of machines given ID \(x\) of the first machine and ID \(y\) of the last machine.

The sorting and indexing subroutines can be performed in \(O(1/\gamma\text{rounds})\) of MPC\((n^\gamma)\) \([2, 13]\). We briefly explain how we can solve the Find Minimum and Broadcast problems also in \(O(1/\gamma)\) rounds of MPC\((n^\gamma)\).

\begin{theorem}
Given \(n\) items over a contiguous range of machines \(x\) to \(y\), subroutines Find Minimum and Broadcast can be implemented in \(O(1/\gamma)\) rounds of MPC\((n^\gamma)\).
\end{theorem}

\textbf{Proof.}
We will first define a rooted aggregation tree \(T\) with branching factor \(n^\gamma\) where the machines \(M_x, ..., M_y\) are placed at the leaves (Here \(M_x\) denote the machine with ID \(x\)). Note that we don’t need to store this tree explicitly, and just need each node (which is a machine) to know its parent. Consider a level \(\ell\) of the tree (leaves have \(\ell = 0\)). For each node in level \(\ell - 1\) that has the input \(x_i\) (as its smallest input), we set as its parent \(M_{p(i)}\) where \(p(i) = x_i + \lceil \frac{x_i}{n^\gamma} \rceil\). Thus each machine can compute its parent by the index of its input, and a label for \(\ell\). Similarly, each machine can compute the indices of its children. The algorithm Find Minimum proceeds as follows: at each round \(\ell\), each machine first computes minimum over its local input, and then send the outcome to the parent machine. Finally the minimum will be computed and stored at the root machine.

The algorithm Broadcast will similarly use an aggregation tree, but this time it routes the message top-down. First message \(b\) is sent to the first machine \(M_x\), and then starting from \(M_x\) in each round any machine that receives message \(b\) sends this value to all of its children, which can be determined from the machine’s ID and \(y\). In the end all the machines at the leaves have received \(b\). The number of rounds each of these subroutines take are the height of the aggregation tree which is \(O(\log_{n^\gamma} n) = \frac{1}{\gamma}\). \(\square\)

Next we use the subroutines to create the following setting: Given a graph \(G = (V, E)\), the goal is to store all the edges incident to each node \(v\) in a contiguous group of machines, which we denote by \(M(v)\). More precisely, let \(M_1, ..., M_P\), where \(P = O(\frac{m}{n^\gamma})\), be the list of machines ordered by their ID, and let \(v_1, ..., v_n\) be the list of vertices sorted by their ID. \(M(v_i)\) is consisted of \(i\)-th smallest contiguous group of machines, such that \(|M(v_i)| = \lceil \frac{\deg(v_i)}{n^\gamma} \rceil\). Throughout the algorithm, let \(M_{(u,v)}\) denote the machine that stores the edge \((u,v)\). Also, for all \(u \in V\), let \(r_u\) be the first machine in \(M(u)\), and for any edge \((u,v) \in E\) let \(i_u(v)\) be the index of \((u,v)\) (based on the lexicographic order) among all the edges incident to \(v\). For running the restricted Bellman-Ford algorithm, we also need to store the following information at \(M_{(u,v)}\): \(\deg(u), \deg(v), r_u, r_v, i_u, i_v\) (here by storing \(r_u\) we mean ID of \(r_u\)). Details of how these labels can be constructed and then used to implement restricted Bellman-Ford is presented in Appendix \(\text{[3]}\). A high-level summary of these steps can be found in Algorithm \(\text{[1]}\).

\begin{theorem}
Given a graph \(G = (V, E)\) and a source node \(s \in V\) the restricted Bellman-Ford algorithm computes distances \(d^p(s, v)\) for all \(v \in V\) in \(O(\frac{\gamma}{\lambda})\) rounds of MPC\((n^\gamma)\).
\end{theorem}

\textbf{Proof.}
After storing the tuples \(((u,v), i_u, i_v, r_u, r_v, \deg(u), \deg(v))\) at \(M_{(u,v)}\) for each \((u,v) \in E\), the restricted Bellman-Ford proceeds as follows: in each round, for each node \(v\), we first aggregate the current distances based on a tree \(T_v\) to find the minimum distance estimate for \(v\) and send it to \(r_v\). Then \(r_v\) will broadcast the minimum distance found back down to all the machines in \(M(v)\). Then for each \((v,u) \in N(v)\), \(M_{(v,u)}\) will send the updated distance directly to \(M_{(u,v)}\) which is at machine \(r_u + \lceil \frac{\gamma}{n^\gamma} \rceil\). \(\square\)
Theorem 15. For any graph \( G = (V, E) \) with \( n \) vertices, and any \( \rho \leq \gamma \leq 1, 1 \leq \kappa \leq (\log n)/4, 1/2 > \rho \geq 1/\kappa \) and \( 0 < \epsilon < 1 \), there is an in \( \text{MPC}(n^\gamma) \) model that computes a \((\beta, \epsilon)\)-hopset with expected size \( O(n^{1+\frac{\beta}{2}} \log n) \) in \( O(\frac{n^\gamma}{\rho} \cdot \log^2 n \cdot \beta) \) rounds w.h.p., where \( \beta = O((\frac{\log n}{\epsilon \cdot (\log \kappa + 1/\rho)})^{\log \kappa + 1/\rho}) \).

The complete proof can be found in Appendix B. The high-level idea is that the explorations of interconnections that reach each node \( v \) is at most \( O(n^\rho \log n) \) (See Lemma 3.3 of [9]), and this will allow us to run multiple runs of Bellman-Fords for the interconnection phases at the same time.

After constructing a \((\beta, \epsilon)\)-hopset, we store the edges added to each node \( v \) by redistributing among machines \( M(v) \) that simulate \( v \). Let \( G' = (V, E \cup H, w') \) be the graph obtained by adding hopset edges. For constructing distance sketches with stretch \( 2k - 1 \), we run an algorithm similar to the one proposed by [21]. This algorithm takes \( O(\beta \cdot kn^{1/k}) \) rounds of Bellman-Ford, each of which takes \( O(1) \) in \( \text{MPC}(n^\gamma) \). We will get the following result.

Theorem 16. Given a graph \( G = (V, E, w) \), and \( \rho \leq \gamma \leq 1, 1 \leq \kappa \leq (\log n)/4, 1/2 > \rho \geq 1/\kappa, 0 < \epsilon < 1 \) we can construct Thorup-Zwick distance sketches with stretch \((2k - 1)(1 + \epsilon)\) and size \( O(kn^{1/k} \log n) \) w.h.p. in \( O(\frac{\log n}{\rho} \cdot \log^2 n \cdot \beta + n^{1/k} \log n \cdot \beta) \) time where \( \beta = O((\frac{\log n}{\epsilon \rho} \cdot (\log \kappa + 1/\rho))^{\log \kappa + 1/\rho}) \).

After constructing the distance sketches, the sketch for node \( v \) is stored at the first machine in \( M(v) \). We then map each sketch to a machine with key \( v \), so that sketches of each vertex can directly be accessed by every other vertex, and the query algorithm can be performed in a single round of communication. Similarly to previous section, let us look at two special cases of \( k = O(1) \) and \( k = \Theta(\log n) \).

Corollary 17. Given a graph \( G = (V, E, w) \), and \( \frac{1}{k} \leq \gamma \leq 1, 0 < \epsilon < 1 \) we can construct Thorup-Zwick distance sketches with stretch \((2k - 1)(1 + \epsilon)\) in the \( \text{MPC}(n^\gamma) \) model, such that,
If $k = O(1)$, w.h.p. we require $O(n^{1/k} \ polylog n)$ rounds.
If $k = \Omega(\log n)$, w.h.p. we require $2^{O(\sqrt{\log n})}$ rounds.

5 Distance Oracles in the Streaming Model

In this section we will describe how the Thorup-Zwick distance oracles can be constructed in the insert-only streaming model. For graph problems, the stream is a sequence of edges (and their weights), and the goal is to solve the problem in space strictly sublinear in number of edges. For some problems we might need to see multiple passes of the stream. Similar to the distributed settings, we will use the hopset construction of [9]. They show that in streaming settings a $(\beta, \epsilon)$-hopset, where with the following guarantees can be constructed.

\textbf{Theorem 18 ([9]).} For any graph $G = (V, E, w)$ with $n$ vertices, and any $2 \leq \kappa \leq (\log n)/4, 1/2 > \rho \geq 1/\kappa, 1 \leq t \leq \log n$ and $0 < \epsilon < 1/2$, there is a streaming algorithm that computes a $(\beta, \epsilon)$-hopset with expected size $O(n^{1+\frac{1}{\kappa}} \log^2 n)$, where $\beta = O(\frac{1}{\rho} \cdot (\log(\kappa) + 1) / \kappa)$ requiring either of the following resources:
- $O(\beta \log n)$ passes w.h.p. and expected space $O(\frac{n^{1+\rho}}{\rho} \cdot n^{1+1/\kappa} \cdot \log^2 n)$, or
- $O(n^\rho \cdot \beta \cdot \log^2 n)$ passes w.h.p. and expected space $O(n^{1+1/\kappa} \cdot \log^2 n)$.

We will next explain how the distance oracle can be constructed in $O(\beta)$ passes given a hopset with hopbound $\beta$. First, we need a variant of restricted Bellman-Ford for streaming settings. The idea of using Bellman-Ford in streaming settings has been previously used for shortest path computation (e.g. [9, 14]). This algorithm is similar to the distributed variant: on receipt of each edge $(u, v) \in E$ we will check to see if the distance from any of the sources in $S$ should be updated. After $i$ passes of the algorithm, all nodes have the $i$-restricted distance to nodes in $s$. The details can be found in Algorithm 3 in Appendix C. This algorithm uses $O(|S| \cdot nh)$ total space.

Next, we will explain how distance oracles can be constructed in $O(\Lambda)$ passes, where $\Lambda$ is the shortest path diameter. We will then use a hopset of hopbound $\beta$ to reduce the number of passes to $O(\beta)$. This algorithm is again similar to the distributed algorithm. Sets $A_1, \ldots, A_{k-1}$ can easily be sampled in sublinear space. Here again for finding the distances from each set $A_i$ to all nodes, we will add a virtual node $s_i$ and add an edge of weight 0 between $s_i$ and all the nodes in $A_i$. We then run the restricted Bellman-Ford algorithm from each of these sources $s_i$ separately. This phase requires $O(kn\Lambda)$ space and $O(\Lambda)$ passes. In the final phase, we need to find the distance from each node in $s \in A_i \setminus A_{i+1}$ to all nodes in $C(s) = \{v \mid v \in B(s)\}$. We will run a variant of the Bellman-Ford algorithm in which each node $v$ only stores a distance only if $d(s, v) < d(v, A_{i+1})$ or if this condition holds after receiving an update from a neighbor. We will get the following lemma. For completeness the algorithm details and proof of this lemma can be found in Appendix C.

\textbf{Lemma 19.} There is an algorithm that runs in $O(\Lambda)$ passes, and w.h.p. constructs a $2k-1$ stretch Thorup-Zwick distance oracle of size $O(kn^{1+1/k} \log n)$ using $O(kn^{1+1/k} \log n)$ total space.

Similar to the distributed case, given a graph $G = (V, E, w)$, we can use the $(\beta, \epsilon)$-hopset construction of Theorem 18 to obtain a graph $G' = (V, E \cup H, w')$ which has shortest path diameter $O(\beta)$, and the distances in $G$ are preserved up to a factor of $(1 + \epsilon)$. Then by running Algorithm 5 on $G'$, we would require $O(\beta)$ passes to build a distance oracle with stretch $(2k - 1)(1 + \epsilon)$. 

If $k = O(1)$, w.h.p. we require $O(n^{1/k} \ polylog n)$ rounds.
If $k = \Omega(\log n)$, w.h.p. we require $2^{O(\sqrt{\log n})}$ rounds.
We set the parameters in such a way that we have space to store all the hopset edges locally. Thus while running the algorithm of Theorem 20 we also consider the hopset edges to decide when to update the distance. However, for readability of our algorithm, here we assume that the hopset edges are also appearing in the stream. Hence, by first running the hopset construction algorithm of Theorem 18 and then running the algorithm of Theorem 20 we will get the following result:

Theorem 20. Given a graph $G = (V, E, w)$, there exists a streaming algorithm that constructs a Thorup-Zwick distance oracle of stretch $(2k - 1)(1 + \epsilon)$ of size $O(kn^{1/k} \log n)$ w.h.p. using either of the following resources:

- $O(\beta \log n)$ passes w.h.p. and expected space $O\left(\frac{n^{1+\epsilon}}{\rho} + n^{1+1/k} \cdot \log^2 n\right)$,
- $O(n^\rho \cdot \beta \cdot \log^2 n)$ passes w.h.p. and expected space $O(n^{1+1/k} \cdot \log^2 n)$, where $\beta = O\left(\frac{1}{\epsilon} \cdot (\log(k) + 1/\rho) \log n\right)^{\log(k) + \frac{1}{k}}$ and $1/k \leq \rho \leq 1/2$.

In particular, when $k = O(1)$ we will use the first case of Theorem 20 and set $\rho = 1/k$, and when $k = \Omega(\log n)$ we will use the second case and set $\rho = \sqrt{\log \log n \log n}$. We have,

Corollary 21. Given a graph $G = (V, E, w)$, there exists a streaming algorithm that constructs a Thorup-Zwick distance oracle of stretch $(2k - 1)(1 - \epsilon)$ of size $O(kn^{1/k} \log n)$ w.h.p. and expected space $O(n^{1+1/k} \cdot \log^2 n)$ such that:

- If $k = O(1)$, we require $O(\log^2 n)$ passes with high probability.
- If $k = \Omega(\log n)$, we require $2^{O(\sqrt{\log n})} = n^{o(1)}$ passes with high probability.

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1 All the bounds expressed in expectation can be turned into high probability bound with an additional factor of $\log n$ in the number of passes.
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A Algorithms Omitted from the Section 3

First in Algorithm 2 we review the restricted distributed Bellman-Ford algorithm that is used as a building block in our distance oracle construction. Algorithm 2: Distributed Bellman-Ford with hopbound $h$.

**Input**: Undirected weighted graph $G = (V, E, w)$, and source node $s \in V$.

**Output**: $h$-hop restricted distances from the source $s$ to all nodes $u \in V$, $d^h(s, v)$.

1. Set $\forall v \in V$: $\hat{d}(s, v) = \infty$.
2. for Rounds $i = 0$ to $h$
   3. for $\forall v \in V$
      4. if $\exists u \in N(v) : \hat{d}(s, v) > \hat{d}(s, u) + w(u, v)$ then
         5. Set $\hat{d}(s, v) := \min_{u \in N(v)}(\hat{d}(s, u) + w(u, v))$, and send $\hat{d}(s, v)$ to all neighbors.

Next we review the distance oracle algorithm for the Congested Clique model in Algorithm 3. This algorithm was proposed by [21] for constructing distance sketches in the CONGEST model.

Algorithm 3: Preprocessing distributed distance oracle for stretch $2k - 1$ due to [21].

**Input**: Undirected graph $G = (V, E, w)$, and a coordinator node.

**Output**: Approximate distance oracle stored at the coordinator.

1. Set $A_0 = V, A_k = \emptyset$.
2. for every $v \in V$
   3. for $i = 1$ to $k - 1$
      4. If $v \in A_{i-1}$ with probability $n^{-1/k}$ add $v$ to $A_i$.
6. Coordinator runs Algorithm 2 out of set $A_1$.
7. for $\forall v \in V$
   8. Set $p_i(v) = \arg\min_{u \in A_i} d(u, v)$, and $d(v, A_i) = d(p_i(v), v)$.
9. for $w \in A_i \setminus A_{i+1}$
   10. Coordinator runs the algorithm in Theorem 6.

B Algorithms and Proofs Omitted from Section 4

In this appendix section, we will provide the details of running the restricted Bellman-Ford algorithm in the MPC($n^\gamma$) where $\gamma < 1$. We first describe multiple building blocks that will be required for our algorithm, each of which can be implemented in $O(1/\gamma)$ rounds of MPC($n^\gamma$). Recall that for running the restricted Bellman-Ford algorithm, we need to store the following information at $M_{u,v} : \deg(u), \deg(v), r_u, r_v, i_u, i_v$ (here by storing $r_u$ we mean ID of $r_u$). Here we explain how these labels can be computed for all edges in $O(1/\gamma)$ rounds and then we explain how we can form the above setting. Let $N(v)$ be the set of edges incident on node $v$. With out loss of generality, let us assume that both tuples of form $(u, v)$ and $(v, u)$ are present in the system for each edge and we assume $(u, v) \in N(u)$ and $(v, u) \in N(v)$ (note that the graph is still undirected). First, we use the indexing subroutine...
of \cite{2} on the sets \( \{N(v)\}_{v \in V} \) and to store index \( i_u \) at \( M_{(u,v)} \) and index \( i_v \) at \( M_{(v,u)} \). After this step tuples of form \( ((u, v), w(u, v), i_u) \) are stored at \( M_{(u,v)} \).

Then we sort the tuples based on edge IDs lexicographically. This will result in the setting described above in which edges incident to each node \( u \) are stored in a contiguous group of machines \( M(u) \). Now in order to compute \( \text{deg}(u) \), machines will check whether they are the last machine in \( M(u) \) either by scanning their local memory or communicating with the next machine. Then the last machine in \( M(u) \) sets \( \text{deg}(u) \) to the maximum index \( i_u \) it holds. This machine can also compute \( r_u \), ID of the first machine in \( M(u) \), and then broadcasts \( \text{deg}(u) \) and \( r_u \) to all machines in \( M(u) \). At the end of these computations, each tuple \( ((u, v), w(u, v), i_u) \) will be replaced by the tuple \( ((u, v), w(u, v), r_u, i_u, \text{deg}(u)) \).

Next, we sort these tuples again but this time based on the ID of the smallest endpoint. In other words, for each edge \( (u, v) \in E \), both tuples \( ((u, v), w(u, v), i_u, \text{deg}(u)) \) and \( ((v, u), w(v, u), i_v, \text{deg}(v)) \) will be at the same machine. Now we can easily merge these two tuples to create tuples of form \( ((u, v), w(u, v), i_u, i_v, \text{deg}(u), \text{deg}(v)) \). Finally, we use the sorting subroutine again to redistribute the edges into the initial setting of having contiguous group of machines \( M(u) \) for all \( u \in V \).

Note that size of the input for the sorting an indexing subroutines in Algorithm \( \text{1} \) are \( N = |E| \). The running time of each of these subroutines will still be \( O(1/\gamma) \). Therefore Algorithm \( \text{1} \) runs in \( O(\frac{1}{\gamma}) \) rounds of MPC(\( n^{\gamma} \)) for all \( 0 < \gamma \leq 1 \). Also, note that after each aggregation step the machines remove the extra information.

Next, we show how the hopsets of \( \text{3} \) can be constructed in MPC(\( n^{\gamma} \)).

**Proof of Theorem \( \text{15} \)**

This result follows from an argument similar to result of \[9\] for Congested Clique. As explained in Section \[3\], the distributed implementation of this algorithm just performs multiple restricted Bellman-Ford algorithms in each phase. Each round of a single Bellman-Ford algorithm can be simulated in \( O(\frac{1}{\gamma}) \) rounds of MPC(\( n^{\gamma} \)) by running the algorithm of Theorem \( \text{14} \) on each node, where the inputs are distance estimates of the neighbors. This means that each superclustering phase can be performed in \( O(\frac{1}{\gamma}) \) rounds. But at each interconnection phase multiple separate Bellman-Fords will run from each clustered center remaining. Thus we need to argue that these runs of Bellman-Ford will not violate the memory (and IO memory) limit of each machine. To show this we use Lemma 3.2 and Lemma 3.3 in \[3\].

Lemma 3.3 implies that for each vertex \( v \in V \), w.h.p. the number of explorations of interconnection phase that visit \( v \) is at most \( O(\text{deg}_v \cdot \log n) \) (or \( n^{\rho} \) in the final phase). In other words, each node only forwards messages to at most \( O(\text{deg}_v \cdot \log n) \) in each depth \( \delta_i/2 \) Bellman-Ford explorations performed for an interconnection phase. Moreover, the parameters of their construction is set so that \( \text{deg}_v = O(n^{\rho}) \) throughout the algorithm. We impose that \( \frac{1}{\gamma} \leq \rho \leq \gamma \). Hence, each node \( v \in V \) need to store and forward distance estimates corresponding to at most \( O(n^{\rho} \cdot \log n) \) sources for \( O(\frac{1}{\rho}) \) iterations for each distance scale. This will then be repeated \( O(\beta) \) times to cover all distance scales. Overall, all of the Bellman-Ford explorations can be implemented in \( O(\frac{1}{\gamma} \cdot n^{\rho} \cdot \log n) \).

**C Algorithms and Proofs Omitted from Section \( \text{5} \)**

In this section we present details of the algorithm for constructing distance oracles in the streaming model. The restricted Bellman-Ford in streaming is presented in Algorithm \( \text{4} \). Note that unlike centralized Bellman-Ford nodes do not store and initial distance estimate
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(due to space limitation in the streaming model).

Algorithm 4: Restricted Bellman-Ford in the Streaming Model

Input: Undirected weighted graph $G = (V, E, w)$, and source node $s \in V$.

Output: $h$-hop restricted distances from the source $s$ to all nodes $u \in V$, $\hat{d}^h(s, v)$.

1. for each $O(h)$ passes do
   2. for $(u, v) \in E$ do
      3. if $\hat{d}(s, v) = \emptyset$ or $\hat{d}(s, u) + w(v, u) < \hat{d}(s, v)$ then
         4. $\hat{d}(s, v) = \hat{d}(s, u) + w(v, u)$
      5. if $\hat{d}(s, u) = \emptyset$ or $\hat{d}(s, v) + w(v, u) < \hat{d}(s, u)$ then
         6. $\hat{d}(s, u) = \hat{d}(s, v) + w(v, u)$

Using the restricted Bellman-Ford algorithm, we can construct a Thorup-Zwick distance oracle of stretch $(2k - 1)(1 + \epsilon)$ in $O(\beta)$ passes. The details of this algorithm is presented in Algorithm 5.

Proof of Lemma 19

It is clear that described algorithm takes $O(\Lambda)$ passes, and correctly updates all the distances required for building a Thorup-Zwick distance oracle. We also show that the space required is the same as the distance oracle size. This follows from the fact that for each node $v$, we are only storing distances to the nodes that are in $v$’s bunch $B(v)$, and we know from [22] that w.h.p. $|B(v)| = O(kn^{1/k} \log n)$. Thus the total space is w.h.p. $O(kn^{1+1/k} \log n)$.

Algorithm 5: Preprocessing distance oracle of stretch $2k - 1$ in the streaming model.

Input: Undirected graph $G = (V, E, w)$ of shortest path diameter $\Lambda$.

Output: Approximate distance oracle stored at the coordinator.

1. Set $A_0 = V, A_k = \emptyset$.
2. for $i = 1$ to $k - 1$ do
   3. if $v \in A_{i-1}$ with probability $n^{-1/k}$ add $v$ to $A_i$.
4. Run Algorithm on parallel out of each set $A_i, 1 \leq i \leq k$, to find $p_i(v) = \arg\min_{u \in A} d(u, v)$, and set $d(v, A_i) := d(p_i(v), v)$.

5. for each $O(\Lambda)$ passes do
   6. for $(u, v) \in E$ do
      7. for $i = k - 1$ down to 1 do
         8. for $s \in A_i \setminus A_{i+1}$ do
            9. if $\hat{d}(s, v) < d(v, A_{i+1})$ or $\hat{d}(s, u) + w(u, v) < \hat{d}(v, A_{i+1})$ then
               10. if $\hat{d}(s, u) + w(v, u) < \hat{d}(s, v)$ then
                  11. $\hat{d}(v, s) = \hat{d}(s, u) + w(v, u)$
               12. if $\hat{d}(s, u) < d(u, A_{i+1})$ or $\hat{d}(s, v) + w(u, v) < d(u, A_{i+1})$ then
                  13. if $\hat{d}(s, u) + w(v, u) < \hat{d}(s, u)$ then
                     14. $\hat{d}(s, u) = \hat{d}(s, v) + w(v, u)$