ON COINCIDENCE OF DIMENSIONS IN CLOSED ORDERED DIFFERENTIAL FIELDS

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Abstract. Let $\mathcal{K} = (\mathcal{R}, \delta)$ be a closed ordered differential field in the sense of Singer [20], and $C$ its field of constants. In this note, we prove that for sets definable in the pair $(\mathcal{R}, C)$, the $\delta$-dimension from [4] and the large dimension from [10] coincide. As an application, we characterize the $\mathcal{K}$-definable sets that are internal to $C$, as those sets that are definable in $(\mathcal{R}, C)$ and have $\delta$-dimension 0. We further show that having $\delta$-dimension 0 does not generally imply co-analyzability in $C$.

1. Introduction

Pairs of fields have been extensively studied in model theory and arise naturally in various ways. If $\mathcal{K} = (\mathcal{R}, \delta)$ is a differentially closed field of characteristic zero (DCF$_0$), a closed ordered differential field (CODF) or the differential field of transseries $\mathcal{T}$ constructed in [1], and $C = \{a \in R : \delta a = 0\}$ is the field of constants in each case, then the reduct $\mathcal{M} = (\mathcal{R}, C)$ is a pair of algebraically closed fields ([13], [16]), a dense pair of real closed fields ([6], [17]) or a tame pair ([15], [9]), respectively. In all three cases, there is a natural notion of dimension, the differential or $\delta$-dimension for definable sets in $\mathcal{K}$. Let $\mathcal{K}$ be a sufficiently saturated DCF$_0$, CODF or the differential field of transseries $\mathcal{T}$.

Definition 1.1. The $\delta$-dimension of a definable set $X$ is defined as

$$\delta\text{-dim}(X) = \max_{a \in X} \delta\text{-trdeg}_S S\langle a \rangle$$

where $S$ is any differential field (of size smaller than the saturation of $\mathcal{K}$) over which $X$ is defined and $\delta\text{-trdeg}_S S\langle a \rangle$ denotes the differential transcendence degree over $S$ of the differential field generated by $a$.

In [4, Corollary 5.27] it is shown that in the case of CODFs this notion of dimension coincides with the one obtained from $\delta$-cell decomposition.

In the above three cases (DCF$_0$, CODF, and transseries), the following implications hold for a $\mathcal{K}$-definable set $X$:

$X$ is internal to $C \Rightarrow X$ is co-analyzable in $C \Rightarrow X$ has $\delta$-dimension 0.

In the case of transseries, it is shown in [2] that the latter two properties are equivalent, and in [9] that, when restricted to definable sets in the pair $(\mathcal{R}, C)$, all

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three properties are equivalent. In this note, we prove that in the case of CODFs, the latter two properties are different (see example in Section 4), whereas when restricted to the pair \((\mathbb{R}, C)\), all three properties are equivalent. The latter result is a consequence of Theorem 1.2 below, which states that in \((\mathbb{R}, C)\) the \(\delta\)-dimension coincides with the large dimension from [10], which we describe next.

The primary example of a dense pair of real closed fields is that of \(\langle \mathbb{R}, \mathbb{Q}^{rc} \rangle\), where \(\mathbb{R}\) is the real field and \(\mathbb{Q}^{rc}\) the subfield of real algebraic numbers, studied by A. Robinson in his classical paper [17], where the decidability of its theory was proven. A systematic study of dense pairs \(\mathcal{M} = \langle \mathbb{R}, P \rangle\) of o-minimal structures (that is, \(P\) is a dense elementary substructure of \(\mathbb{R}\)) occurred much later in [6] by van den Dries. In [10] (as well as [3] and [11]), the pregeometric dimension localized at \(P\) was studied. Let \(\text{dcl}\) denote the usual definable closure in the o-minimal structure \(\mathcal{R}\), and denote, for \(A \subseteq \mathbb{R}\),

\[\text{scl}(A) = \text{dcl}(A \cup P).\]

We call the scl-dimension of an \(\mathcal{M}\)-definable set \(X\) \textit{large dimension} and denote it by \(\text{ldim}(X)\). In [10], in a much broader setting that includes dense pairs, the large dimension was given a topological description via a structure theorem (Fact 2.7 below), much alike the topological description of the usual dcl-dimension via the cell decomposition theorem in the o-minimal setting.

Recall that, given a closed ordered differential field \(\mathcal{K} = \langle \mathbb{R}, \delta \rangle\) is a dense pair of real closed fields (see, for example, [4]). The main result of this note is the following theorem.

**Theorem 1.2.** Let \(\mathcal{K} = \langle \mathbb{R}, \delta \rangle\) be a closed ordered differential field and \(C\) its field of constants. Let \(X\) be a set definable in \((\mathbb{R}, C)\). Then

\[\text{ldim}(X) = \delta\text{-dim}(X).\]

With a bit more work, we can characterize the notion of being internal to \(C\) for \(\mathcal{K}\)-definable sets.

**Corollary 1.3.** Let \(\mathcal{K} = \langle \mathbb{R}, \delta \rangle\) be a closed ordered differential field and \(C\) its field of constants. Let \(X\) be a \(\mathcal{K}\)-definable set. Then

\(X\) is internal to \(C\) \iff \(\delta\text{-dim}(X) = 0\) and \(X\) is definable in \((\mathbb{R}, C)\).

Finally, in Section 4 in contrast to the case of transseries [2, Proposition 6.2], we give an example of a \(\mathcal{K}\)-definable set with \(\delta\)-dimension 0 which is not co-analyzable in \(C\) (and hence also not internal to \(C\)).

## 2. Preliminaries

For the rest of this note, we fix a real closed field \(\mathcal{R} = \langle \mathbb{R}, <, +, \cdot \rangle\) and a derivation \(\delta\) on \(\mathbb{R}\), such that \(\mathcal{K} = \langle \mathbb{R}, \delta \rangle\) is a CODF, as in [20]. Namely, \(\langle \mathbb{R}, \delta \rangle\) is existentially closed among ordered differential fields. We recall that the theory CODF admits quantifier elimination [20] and elimination of imaginaries [18] in the language of ordered differential rings. We let \(C\) be the field of constants. By ‘definable’ we mean definable in the pair \(\mathcal{M} = \langle \mathbb{R}, C \rangle\), by ‘\(\mathcal{R}\)-definable’ or ‘semialgebraic’ we mean definable in \(\mathcal{R}\), and by ‘\(\mathcal{K}\)-definable’ we mean definable in \(\mathcal{K}\). Definability is always meant with parameters.

In this section, we recall some of the basics from CODFs and dense pairs, and prove a preliminary result (Lemma 2.2). Let \(\mathcal{N}\) be any of \(\mathcal{K}, \mathcal{M}, \mathcal{R}\), and \(X \subseteq \mathbb{R}^m\).
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We say that $X$ is $\mathcal{N}$-internal to $C$, if there is an $\mathcal{N}$-definable function $f : R^n \to R^m$, such that $X \subseteq f(C^n)$. In Lemma 2.1 we prove that for $\mathcal{K}$-definable sets, $\mathcal{N}$-internality to $C$ is invariant under varying $\mathcal{N}$ among $\mathcal{K}$, $\mathcal{M}$ and $\mathcal{R}$. We will need the following result on definable functions in a CODF

**Lemma 2.1.** If $f : R^n \to R$ is a $K$-definable function, then there is an $R$-definable function $F : R^{n(d+1)} \to R$ for some $d$ such that

$$f(a) = F(a, \delta a, \ldots, \delta^d a) \quad \text{for all } a \in R^n.$$  

**Proof.** It is well known, see for instance [5, §1.4.2], that for any $A \subseteq R$ the definable closure $\text{dcl}^K(A)$ equals the real closure of the differential field generated by $A$. It then follows, by a standard compactness argument, that there is a partition $X_1, \ldots, X_r$ of $R^n$ into $K$-definable sets and, for some $d$, there are $R$-definable functions $F_i : R^{n(d+1)} \to R$, for $i = 1, \ldots, r$, such that

$$f(a) = F_i(a, \delta a, \ldots, \delta^d a) \quad \text{for all } a \in X_i.$$  

By quantifier elimination and after possibly increasing $d$, for each $i$, there is an $R$-definable set $T_i \subseteq M^{n(d+1)}$ such that

$$X_i = \{ a \in R^n : (a, \delta a, \ldots, \delta^d a) \in T_i \}$$

Now define $F : R^{n(d+1)} \to R$ as

$$F(b) = \begin{cases} 
F_1(b) & \text{if } b \in T_1 \\
F_2(b) & \text{if } b \in T_2 \setminus T_1 \\
\vdots & \vdots \\
F_r(b) & \text{if } b \in T_r \setminus (T_1 \cup \cdots \cup T_{r-1}) \\
0 & \text{otherwise}
\end{cases}$$

It follows that this yields the desired function. \hfill \Box

**Lemma 2.2.** Assume $X \subseteq R^n$ is $K$-definable. If $X$ is $K$-internal to $C$, then $X$ is definable in $(R, C)$ and $\mathcal{R}$-internal to $C$.

**Proof.** Since $X$ is $K$-internal to $C$, there is a $K$-definable function $f : R^n \to R^m$ such that $f(C^n) = X$ for some $n$. By Lemma 2.1 there is an $R$-definable function $F : R^{n(d+1)} \to R$ for some $d$ such that

$$f(a) = F(a, \delta a, \ldots, \delta^d a) \quad \text{for all } a \in R^n.$$  

Hence, $f(a) = F(a, 0, \ldots, 0)$ for all $a \in C^n$. This shows that $X$ is defined by

$$X = \{ y \in R^m : \exists x \in C^n \text{ with } y = F(x, 0, \ldots, 0) \}$$

Thus, $X$ is definable in the pair $(R, C)$ and $\mathcal{R}$-internal to $C$. \hfill \Box

In view of Lemma 2.2 we call a $\mathcal{K}$-definable set $C$-internal if it is $\mathcal{N}$-internal to $C$, for any $\mathcal{N} = \mathcal{K}, \mathcal{M}, \mathcal{R}$.

\footnote{We thank Marcus Tressl for pointing out the argument in the proof.}
2.1. Notions from dense pairs. In this subsection we recall all necessary background from [10]. The key statements are Facts 2.7 and 2.8 below, which provide a structure theorem for all definable sets, and a topological characterization of large dimension, respectively. The following definition is taken essentially from [8].

Definition 2.3. Let $X \subseteq R^n$ be a definable set. We call $X$ large if there is some $m$ and an $R$-definable function $f : R^{nm} \to R$ such that $f(X^m)$ contains an open interval in $R$. We call $X$ small if it is not large.

Fact 2.4. A definable set is small if and only if it is $C$-internal.

Proof. By [10, Corollary 3.11]. □

Definition 2.5 ([10]). A supercone $J \subseteq R^k$, $k \geq 0$, and its shell $sh(J)$ are defined recursively as follows:

1. $R^0 = \{0\}$ is a supercone, and $sh(R^0) = R^0$.
2. A definable set $J \subseteq R^{n+1}$ is a supercone if $\pi(J) \subseteq R^n$ is a supercone and there are $R$-definable continuous $h_1, h_2 : sh(\pi(J)) \to R \cup \{\pm \infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, $J_a$ is contained in $(h_1(a), h_2(a))$ and it is co-small in it. We let $sh(J) = (h_1, h_2)_{sh(\pi(J))}$.

We identify a family $J = \{J_g\}_{g \in S}$ with $\bigcup_{g \in S} \{g\} \times J_g$. The following notion of a cone is a simplified one and weaker than that in [10], but strong enough for our purposes.

Definition 2.6 (Cones [10]). A set $C \subseteq R^n$ is a $k$-cone, $k \geq 0$, if there are a definable $S \subseteq C^m$, a definable family $J = \{J_g\}_{g \in S}$ of supercones in $R^k$, and an $R$-definable continuous function $h : V \subseteq R^{m+k} \to R^n$, such that

1. $C = h(J)$, and
2. for every $g \in S$, $h(g, -) : V_g \subseteq R^k \to R^n$ is injective.

Fact 2.7. Every definable set is a finite union of cones.

Proof. By [10, Theorem 5.1 & Remark 4.5(7)]. □

Fact 2.8. Let $X \subseteq R^n$ be definable and non-empty. Then

$$\text{ldim} X = \max\{k \in \mathbb{N} : X \text{ contains a } k\text{-cone}\}.$$

In particular,

$$\text{ldim} X = 0 \iff X \text{ is small}.$$

Proof. By [10, Proposition 6.9 & Lemma 6.11(3)]. □

We finally remark that both the $\delta$-dimension and the large dimension are dimension functions in the sense [7]. This follows from [4, Section 5.3] and [10, Lemma 6.11], respectively. In particular, they satisfy the usual additivity and fiber properties, which we will be using throughout without specific mentioning.

3. Proof of Theorem 1.2

We proceed through a series of lemmas.

Lemma 3.1. $\delta\text{-dim} C = 0$.

Proof. Any point $a$ in $C$ is a zero of the nonzero differential polynomial $\delta x$, and hence the $\delta$-transcendence degree of $a$ is zero. □
Lemma 3.2. Let $X \subseteq R^n$ be definable. If $\text{ldim}X = 0$, then $\delta\dim X = 0$.

Proof. Suppose $\text{ldim}X = 0$. By Facts 2.3 and 2.8, $X$ is $C$-internal; namely, there is a $K$-definable function $f : R^m \to R^n$ such that $X = f(C^m)$. By Lemma 3.1,

$$\delta\dim X = \delta\dim f(C^m) \leq \delta\dim C^m = 0,$$

as required. \qed

The following lemma is a key result on CODFs.

Lemma 3.3. Let $p(x)$ be a nonzero differential polynomial over $R$. The solution set $\{ c \in R : p(c) = 0 \}$ is co-dense in $R$ with respect to the order topology.

Proof. This fact follows from the axioms of CODF (see [20, §2]) and appears to be folklore. We provide the details. Let $(a, b)$ is a nonempty open interval, we prove that there is $c \in (a, b)$ such that $p(c) \neq 0$. Let $f(x) = \delta^m x$, with $m > \text{ord}(p(x))$, $g_0(x) = p(x)$, $g_1(x) = x - a$ and $g_2(x) = b - x$. Let also $F(x_0, \ldots, x_m)$ and $G_i(x_0, \ldots, x_m)$ be the corresponding algebraic polynomials, for $i = 0, 1, 2, 3$; that is, $f(x) = F(x, \delta x, \ldots, \delta^m x)$ and similarly for the $g_i$'s. Then, we can find $\bar{c} = (c_0, \ldots, c_m) \in R^m$ such that $F(\bar{c}) = 0$, $G_0(\bar{c}) \neq 0$, $G_1(\bar{c}) > 0$ and $G_2(\bar{c}) > 0$. By Singer’s axioms of CODF [20] we can find $c \in R$ such that $g_0(c) \neq 0$, $g_1(c) > 0$ and $g_2(c) > 0$. This yields $p(c) \neq 0$ and $c \in (a, b)$, as desired. \qed

Lemma 3.4. Let $X \subseteq R^n$ be definable. If $\delta\dim X = 0$, then $\text{ldim}X = 0$.

Proof. Suppose towards a contradiction that $\text{ldim}X > 0$. So $X$ is large, and hence there is a $K$-definable function $f : R^{m_k} \to R$ such that $f(X^k)$ contains an open interval. On the other hand, we have

$$\delta\dim f(X^k) \leq \delta\dim X^k = 0,$$

By compactness, there must exist a nonzero differential polynomial $p(x)$ that vanishes in all of $f(X^k)$. As the latter contains an interval, this contradicts Lemma 3.3. \qed

Lemma 3.5. Let $J \subseteq R^k$ be a supercone. Then $\delta\dim J = k$.

Proof. By induction on $k$. For $k = 1$, $J$ is co-small in an interval $I$. Since $I \setminus J$ is small, by Lemma 3.2 it must have $\delta$-dimension $0$. By Lemma 3.3 $\delta\dim J = 1$. Hence $\delta\dim J = 1$.

For $k > 1$, by inductive hypothesis, $\delta\dim \pi(J) = k - 1$. Since $J = \bigcup_{t \in \pi(J)} \{ t \} \times J_t$, and each $J_t$ is a supercone in $R$, we have $\delta\dim J_t = 1$ and

$$\delta\dim J = \delta\dim \pi(J) + 1 = k,$$

as required. \qed

Lemma 3.6. Let $C = h(J)$ be a $k$-cone. Then $\delta\dim C = k$.

Proof. Let $J = \bigcup_{t \in S} \{ t \} \times J_t$, where $\text{ldim}S = 0$. Since each $h(t, -)$ is injective, and by Lemma 3.3 $\delta\dim J_t = k$, we obtain $\delta\dim h(t, J_t) = k$. Since $h(t, J_t) \subseteq C$, we obtain $\delta\dim C \geq k$. But $C = \bigcup_{t \in S} h(t, J_t)$, and hence

$$\delta\dim C = \delta\dim S + k = k.$$

\qed
Proof of Theorem 1.2. By Fact 2.7, $X$ is a finite union of cones. Since, by Lemma 3.6 for each cone the $\delta$-dimension and large dimension agree, we are done. \hfill \Box

Proof of Corollary 1.3. Left-to-right is by Lemma 2.2. For right-to-left, by Theorem 1.2, $\dim(X) = 0$. By Facts 2.4 and 2.8, $X$ is internal to $C$. \hfill \Box

4. An example of a zero dimensional not co-analyzable set

We conclude with this short section where we exhibit an example of a $K$-definable set with $\delta$-dimension 0 which is not co-analyzable in $C$ (and hence also not internal to $C$). We recall that in the case of transseries no such example exists [2, Proposition 6.2]. We carry on our terminology from previous sections. We work in a universal (sufficiently saturated) model $K = \langle R, \delta \rangle$ of CODF with constants $C$.

Our example comes from a classical construction on Rosenlicht’s extensions. We recall the following fact from [19, Proposition 2].

**Fact 4.1.** Let $(F, \delta)$ be a differential field of characteristic zero, with field of constants $C$. Let $f(x)$ be a rational function over $C$ such that $\frac{1}{f(x)}$ is not of the form

$$\frac{c}{u} \frac{dv}{dx} \quad \text{or} \quad \frac{dv}{dx}$$

for any $u, v \in C(x)$ and $c \in C$ (two examples of such a function $f(x)$ are $\frac{x}{x-1}$ and $x^3 - x^2$). If $a$ is any solution of $\delta(x) = f(x)$ then $C_{F(a)} = C$.

The goal of this section is to prove the following.

**Proposition 4.2.** Let $f(x)$ be as in Fact 4.1, say for instance $f(x) = x^3 - x^2$. Then, the subset $X \subseteq R$ defined by $\delta(x) = f(x)$ is not co-analyzable in $C$.

We begin by recalling the definition of co-analyzability in $C$ for subsets of $R^n$ (see [12] for further details). A $K$-definable set $X \subseteq R^n$ is said to be co-analyzable in $C$ in 0-steps if $X$ is finite, and in $(r+1)$-steps if there is a $K$-definable set $Y \subseteq C \times R^n$ such that

1. the canonical projection $\pi: C \times R^n \to R^n$ maps $Y$ onto $X$, and
2. for each $c \in C$ the fibre $Y_c = \{a \in R^n : (c, a) \in Y\}$ is co-analyzable in $C$ in $r$-steps.

Co-analyzable means co-analyzable in $r$-steps for some $r \geq 0$.

We will make use of the following lemma.

**Lemma 4.3.** Suppose $Z$ is an infinite $K$-definable subset of $R^n$ with the property that for any $a \in Z$, and any differential field $S$ over which $Z$ is defined, the field $S(a)$ is a differential subfield of $R$ with constants equal to $C_S$. Then, $Z$ is not co-analyzable in $C$.

**Proof.** Let $S$ be a model of CODF (of size smaller than the saturation of $K$) over which $Z$ is defined. Towards a contradiction, assume first that $Z$ is 1-step co-analyzable. Namely, there is a $K$-definable $Y \subseteq C \times R^n$ such that the projection $\pi: C \times R^n \to R^n$ maps $Y$ onto $X$ and the fibre $Y_c$ is finite for all $c \in C$. Then, by elimination of imaginaries, we can find an injective $K$-definable function $f: W \to Z$ with parameters from $S$ with $W$ an infinite subset in $\text{dcl}^K(S, C)$. Now take $a$ in the image of $f$ which is not $S$, then the unique $w \in W$ such that $f(w) = a$ is in
dcl^K(S, a). Since C_{S(a)} = C_S, it follows from \[13\] Corollary 2, Chapter II, §1 that \( S(a) \cap S(C) = S \), and hence \( dcl^K(S, a) \cap dcl^K(S, C) = S \) as \( S \) is real closed. Hence, \( w \in S \). But this implies that \( a \in S \), a contradiction.

We now prove \( Z \) is not co-analyzable in \( r + 1 \)-steps. If it were we would have a \( K \)-definable \( Y \subseteq C \times \mathbb{R}^n \) such that the projection \( \pi : C \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) maps \( Y \) onto \( Z \) and all fibres \( Y_c \) are \( r \)-step co-analyzable in \( C \). Taking any \( c \in C \), such that the fibre \( Y_c \) is infinite we have that, as \( Y_c \) contained in \( Z \), \( Y_c \) satisfies the property that for each \( a \in Y_c \) and differential field \( S \) over which \( Y_c \) is defined, the field \( S(a) \) is a differential subfield of \( R \) with constants equal to \( C_S \). We are now done by induction. \( \square \)

We can now easily conclude the proof of Proposition 4.2.

**Proposition 4.2.** The set \( X \) is infinite; indeed, for any \( S \) we can define a derivation on \( S(x) \) that maps \( x \mapsto f(x) \). Furthermore, by Fact 4.1 \( X \) satisfies the property that for each \( a \in X \) and differential field \( S \) over which \( X \) is defined, the field \( S(a) \) is a differential subfield of \( R \) with constants equal to \( C_S \). The proposition now follows from Lemma 4.3. \( \square \)

**Remark 4.4.** We note that the proof of Proposition 4.2 relies essentially only on the fact that for \( A \subseteq R \) the definable closure \( dcl^K(A) \) equals the real closure of the differential field generated by \( A \), and the fact that the theory CODF eliminates imaginaries.

We conclude with an application on the existence of a proper CODF extension with the same field of constants. We are not aware of any such example in the literature.

**Corollary 4.5.** There is a proper extension \( R \cong S \) of CODFs with the same constants.

**Proof.** This follows from the existence of a non-co-analyzable in \( C \) definable set (given by Proposition 4.2 and [2] Proposition 6.1(iv)]. \( \square \)

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