Abstract

We show that the spontaneous compactification of the Abelian and non-Abelian two-form gauge field theories from $D = 4 + 1$ to $D = 3 + 1$ leads to the same theories plus the Maxwell and Yang-Mills ones, respectively. The vector potential comes from the zero mode of the fifth component of the tensor gauge field in $D = 5$. Concerning to the non-Abelian case, it is necessary to make a more refined definition of the three-form stress tensor in order to be compatible, after the compactification, with the two-form stress tensor of the Yang-Mills theory.
I. Introduction

A significant number of quantum field theories we have to describe the real world in $D = 4$ are effective theories, in a sense that they result from the absorption of some degrees of freedom of more general theories. For example, the vector particles related to the weak force are massful even though the corresponding gauge theory consider them initially massless. This occurs because spontaneous symmetry breaking together with Higgs mechanism leads to an effective theory where gauge particles become actually massive.

Another interesting aspect of mass generation for gauge fields, as it was initially pointed out by Cremmer and Scherk [1], is by means of a vector-tensor gauge theory [2] where these fields are coupled in a topological way. Let me present some details of this mechanism in order to make some comparison with the work we are going to develop in this paper. The Lagrangian for the vector-tensor gauge theory with topological coupling is given by (we consider the Abelian case first)

$$L = \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m \epsilon_{\mu\nu\rho\lambda} A^\mu \partial^\nu B^\rho$$

where the antisymmetric stress tensors $H_{\mu\nu\rho}$ and $F_{\mu\nu}$ are defined in terms of (antisymmetric) tensor and vector potentials $B_{\mu\nu}$ and $A_\mu$ as

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This theory is invariant under the gauge transformations

$$\delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$
$$\delta A_\mu = \partial_\mu \epsilon$$

If one considers the path integral formalism and integrates out the tensor fields $B_{\mu\nu}$, the resulting effective theory is massive for the vector field [3, 4]. This can be considered as an alternative mechanism of mass generation without Higgs bosons. The non-Abelian version of this theory requires some care, because the reducibility condition is only achieved in the vanishing surface of the Maxwell stress tensor [5], or by using a kind of Stuckelberg field [6, 7].

Another example that reinforces this point of view can be found in the string scenario. It is well-known that consistent string theories can only be formulated in spacetime dimensions higher than four. Consequently, the theories we have to describe the world in $D = 4$ might be effective theories of those ones formulated in, say, $D = 10$ or $D = 11$, and conveniently compactified to $D = 4$. Of course, it is
not an easy task to know what are those original theories in $D = 10$ or $D = 11$. However, it might be an interesting subject to investigate how the theories we have in $D = 4$ can come from extended theories formulated in spacetimes with dimensions higher than four. In this sense we mention Kaluza-Klein [8] that is formulated in $D = 5$ as a pure Einstein theory and gives Einstein and Maxwell theories in $D = 4$ (the gauge symmetry of the Maxwell theory is originated from the fifth spacetime coordinate transformation).

The purpose of the present paper is to follow a similar procedure as the Kaluza-Klein, but starting from a pure two-form gauge field theory in $D = 5$, both Abelian and non-Abelian. We show that, after spontaneous compactification, Maxwell and Yang-Mills theories naturally emerge as the zero mode of infinite Fourier excitations. However, contrarily to Kaluza-Klein, the gauge symmetry does not come from a spacetime coordinate transformation, but from the fifth component of the tensor gauge symmetry. Another interesting aspect of this mechanism is that photon and color fields remain massless when tensor fields are integrated out (only higher excitations become massive). We also show that the topological coupling term of expression (I.1) does not come from the compactification of any term formulated at $D = 5$.

This paper is organized as follows: In Sec. II we develop the compactification of the Abelian case, where the electromagnetic Maxwell theory is obtained. In the first part of In Sec. III we present some details of the non-Abelian formulation for the two-form gauge theory. Our goal in this section is to figure out the action we are going to compactify from $D = 5$ to $D = 4$ in order to obtain the Yang-Mills theory. We show that it is necessary to make an appropriate definition of the three-form stress tensor different of that one we usually find in literature. Sec. IV contains the compactification of the non-Abelian case and we left Sec. V for some concluding remarks. We also include an Appendix to illustrate the Abelian compactification in the language of differential forms.

II. Spontaneous compactification - Abelian case

Let us start from the Lagrangian

\[ \mathcal{L} = \frac{1}{12} H_{MNP} H^{MNP} \quad M, N, P = 0, \ldots, 4 \] (II.1)

where we use capital indices to characterize the $D = 5$ spacetime components. The stress tensor $H_{MNP}$ is defined as in the first relation (I.2) and, consequently, the gauge transformation of $B_{MN}$ is similar to the one given by (I.3).

In order to perform the spontaneous compactification to $D = 4$, which is achieved by integrating out the coordinate $x_4$ in a circle of radius $R$, we consider the tensor potential $B_{MN}$ split as
\[ B_{MN} = (B_{\mu\nu}, B_{4\mu}) \quad \mu, \nu = 0, \ldots, 3 \]  

(II.2)

So, we get the action

\[
S = \int d^4x \int_0^R dx^4 \left( \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} H_{4\mu\nu} H^{4\mu\nu} \right)
\]

(II.3)

Developing the stress component \( H_{4\mu\nu} \), we may write

\[
H_{4\mu\nu} = \partial_4 B_{\mu\nu} + \partial_{\nu} B_{4\mu} + \partial_{\mu} B_{4\nu}
\]

\[
= \partial_4 B_{\mu\nu} + \partial_{\mu} B_{4\nu} - \partial_{\nu} B_{4\mu}
\]

\[
= \partial_4 B_{\mu\nu} + F_{\mu\nu}
\]

(II.4)

where we have defined

\[
B_{\mu4} = A_\mu
\]

(II.5)

with the purpose of making future comparisons in the Maxwell theory. However, the quantity \( A_\mu \) given by (II.5) is not the photon field yet. We notice that all fields in the action (II.3) depend on \( x_\mu \) and \( x_4 \) and the gauge transformation of \( A_\mu \) is not the usual gauge transformation of the photon field (from (I.4), we have \( \delta A_\mu = \partial_\mu \xi_4 - \partial_4 \xi_\mu \)).

Using the result given by (II.4), the general form of the action turns to be

\[
S = \int d^4x \int_0^R dx^4 \left( \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} \partial_4 B_{\mu\nu} \partial^4 B^{\mu\nu} \right.
\]

\[
+ \left. \frac{1}{2} F_{\mu\nu} \partial^4 B^{\mu\nu} \right)
\]

(II.6)

The next step is to expand the fields \( B_{\mu\nu} \) and \( A_\mu \), as well as the gauge parameters \( \xi_\mu \) and \( \xi_4 \), in terms of Fourier harmonics

\[
B_{\mu\nu}(x, x_4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{+\infty} B_{(n) \mu\nu}(x) \exp(2i\pi x^4/R),
\]

\[
A_\mu(x, x_4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{+\infty} A_{(n) \mu}(x) \exp(2i\pi x_4/R)
\]

\[
\xi_\mu(x, x_4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{+\infty} \xi_{(n) \mu}(x) \exp(2i\pi x^4/R)
\]

\[
\xi_4(x, x_4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{+\infty} \xi_{(n)}(x) \exp(2i\pi x_4/R)
\]

(II.7)
Since the fields $B_{\mu\nu}$ and $A_\mu$ are real, as well as the gauge parameters, the Fourier modes must satisfy the conditions $B_{(n)\mu\nu}^{\dagger} = B_{(-n)\mu\nu}^{\dagger}$, $A_{(n)\mu}^{\dagger} = A_{(-n)\mu}$, etc., where dagger means complex conjugation.

Developing the terms of the action (II.6) by using the expansions given by (II.7), we obtain

$$S = \int d^4x \sum_{n=0}^{\infty} \left( \frac{1}{12} H_{(n)\mu\rho\nu} H_{(n)}^{\mu\rho\nu} - \frac{1}{4} F_{(n)\mu\nu} F_{(n)}^{\mu\nu} + \frac{\pi^2 n^2}{R^2} B_{(n)\mu\nu} B_{(n)}^{\mu\nu} \right)$$

which is invariant for the gauge transformations

$$\delta B_{(n)\mu\nu} = \partial_\mu \xi_{(n)\nu} - \partial_\nu \xi_{(n)\mu} \quad \text{(II.9)}$$

$$\delta A_{(n)\mu} = \partial_\mu \xi_{(n)} - \frac{2i\pi n}{R} \xi_{(n)\mu} \quad \text{(II.10)}$$

We then notice that $-\frac{1}{4} F_{(0)\mu\nu} F_{(0)}^{\mu\nu}$ is the Maxwell Lagrangian. In fact, from (II.10) we have that the gauge transformation of $A_{(0)\mu}$ is just $\partial_\mu \xi_{(0)}$. Further, contrarily to the vector-tensor gauge theory in $D = 4$, the photon field $A_{(0)\mu}$ does not acquire mass after integrating over the tensor field $B_{(0)\mu\nu}$ (for $n = 0$, $A_{(n)\mu}$ and $B_{(n)\mu\nu}$ decouple). However, higher excitations are massive.

### III. Non-Abelian formulation of the two-form gauge theory

We start this section by reviewing the main aspects of the non-Abelian two-form gauge field theory. We shall see that there is an arbitrariness in defining the corresponding field strength (what does not happen in the Abelian counterpart). The definition that usually appears in literature [6, 7] is not in agreement, after the compactification, with the Yang-Mills theory. We find this an important point because the coherency between compactification and the correct obtainment of the Yang-Mills theory might be the guidance to a precise definition of the non-Abelian field strength tensor.

From this section on, we opt to work with differential forms because the notation is simpler and it is easier to make comparisons between one and two-form gauge theories. We found convenient does not work with differential forms in the previous sections because this compact notation would hide some details we would like to
emphasize at that opportunity. We display in the Appendix A the compactification of the Abelian case in the language of differential forms.

Let us first consider the one-form case. We then start from the introduction of the one-form connection

\[ \Gamma = A_\mu \, dx^\mu \]
\[ = A_\mu^a \, T^a \, dx^\mu \tag{III.1} \]

that is a Lie algebra valued on the \( SU(N) \) symmetry group \((a = 1, \ldots, N^2 - 1)\), whose generators satisfy

\[ [T^a, T^b] = i f^{abc} T^c \]
\[ \text{Tr} (T^a T^b) = \frac{1}{2} \delta^{ab} \]
\[ (T^a)^{bc} = -i f^{abc} \tag{III.2} \]

Of course, in the definition of a differential \( p \)-form, the spacetime can have any dimension \( D \) with \( p \leq D \). In this brief review, we take the usual spacetime dimension \( D = 4 \) what is implicit in the use of Greek indices.

The connection permit us to define the exterior covariant derivative as

\[ D \omega = d \omega - i \Gamma \wedge \omega + i (-1)^p \omega \wedge \Gamma \equiv d \omega - i [\Gamma, \omega] \tag{III.3} \]

where \( \omega \) is a Lie algebra valued \( p \)-form \((\omega = \omega^a T^a)\) and \( d \) represents the usual exterior derivative.

The curvature two-form is defined to be

\[ F = d \Gamma - i \Gamma \wedge \Gamma \tag{III.4} \]

It is important to observe that \( F \) is not the covariant derivative of \( \Gamma \). At this point resides the arbitrariness in the definition of the three-form strength as we are going to see soon.

The definition of the exterior derivative and the curvature two-form permit us to introduce the Bianchi identities

\[ D D \omega = i [\omega, F] \tag{III.5} \]
\[ DF = 0 \tag{III.6} \]
that are satisfied for any gauge connection \( \Gamma \) and any algebra valued \( p \)-form \( \omega \). A fundamental consequence of (III.5) is that if one defines the gauge variation of the one-form connection like

\[
\delta \Gamma = D \epsilon
\]  

(III.7)

where \( \epsilon \) is an infinitesimal Lie-algebra valued zero-form parameter \( \epsilon = \epsilon^a T^a \), the curvature two-form transforms as

\[
\delta F = d \delta \Gamma - i \delta \Gamma \wedge \Gamma + i \Gamma \wedge \delta \Gamma
\]

\[
= DD \epsilon
\]

\[
= i [\epsilon, F]
\]  

(III.8)

We observe, in the second step above, that \( \delta F \) is the covariant derivative of \( \delta \Gamma \), even though \( F \) does not have this property with respect to \( \Gamma \).

The result (III.8) implies that the action

\[
S = -\frac{1}{2} \text{Tr} \int F \wedge \ast F
\]

(III.9)

is gauge invariant, due to the cyclic property of the trace operation. In (III.9), the symbol \( \ast \) represents the Hodge duality operation. So, the integrand is proportional to the oriented volume element in the Minkowski space-time. To be more precise, the duality operation maps the \( p \)-form coordinate basis \( \{1, dx^\mu, dx^\mu \wedge dx^\nu, dx^\mu \wedge dx^\nu \wedge dx^\rho, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma\} \) into the basis \( \{\eta, \eta^\mu, \eta^{\mu\nu}, \eta^{\mu\nu\rho}, \eta^{\mu\nu\rho\sigma}\} \). In these expressions, \( \eta \) is the four-form oriented volume element, \( \eta^\mu \) is a three-form, \( \eta^{\mu\nu} \) is a two-form and so on. They satisfy relations such as

\[
dx^\mu \wedge \eta^\nu = \delta^\mu_\nu \eta, \quad dx^\mu \wedge \eta_{\nu\rho} = 2\delta^\mu_{[\nu} \eta_{\rho]} \]

and

\[
dx^\mu \wedge \eta_{\nu\rho\sigma} = 3\delta^\mu_{[\nu} \eta_{\rho\sigma]}.
\]

As \( F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \), \( \ast F = \frac{1}{2} F_{\mu\nu} \eta^{\mu\nu} \) and consequently \( F \wedge \ast F = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \eta \).

Let us now consider the non-Abelian two-form case. We start by introducing a two-form Lie-algebra valued object \( \Lambda \) in a similar way it was done for the connection \( \Gamma \), i.e.,

\[
\Lambda = \frac{1}{2} B_{\mu\nu}^a T^a dx^\mu \wedge dx^\nu
\]

(III.10)

Even though \( \Gamma \) is not a connection, it looks like natural to assume its gauge transformation as being

\[
\delta_T \Lambda = D \xi
\]

(III.11)
where the subscript $T$ means that the transformation above is just part (related to the tensor sector) of a more general transformation as we are going to see just below. Here, $\xi$ is an infinitesimal Lie-algebra valued one-form gauge parameter.

We see that the gauge transformation (III.11) is a natural extension of (I.3) and (III.7). However, contrarily to the Abelian two-form case, it is not reducible. In fact, if one takes the one-form parameter $\xi$ as the (covariant) derivative of a zero-form parameter, say $\alpha$, we find

$$\delta_T \Lambda = D D \alpha = i \left[ \alpha, F \right]$$  \hspace{1cm} \text{(III.12)}$$

where in the last step we have used the Bianchi identity (III.5). We notice that the reducibility condition is only attained if the curvature $F$ vanishes identically $[5]$.

Since $\Lambda$ is a Lie-algebra valued object, it may couple with the connection $\Gamma$ and, consequently, it can have an additional transformation related to the vector sector. One considers that this additional transformation is given by (see expression (III.8))

$$\delta_V \Lambda = i \left[ \epsilon, \Lambda \right]$$  \hspace{1cm} \text{(III.13)}$$

So, the general gauge transformation for $\Lambda$ is

$$\delta \Lambda = i \left[ \epsilon, \Lambda \right] + D\xi$$  \hspace{1cm} \text{(III.14)}$$

Now, a controversial point is to define the object that will be the extension of $F$. In the Abelian case, this is very simple and direct because since $F$ is just the exterior derivative of $\Gamma$, it is natural to assume that the extension of $F$, that we call $H$, is the exterior derivative of $\Lambda$. However, in the non-Abelian case, $F$ is not the covariant derivative of $\Gamma$. Hence, it is not clear what should be $H$ in this case. What is usually done in literature is to define this stress tensor as the covariant derivative of $\Lambda$, even though $F$ does not have this property with respect to $\Gamma$. Let us then see what happens if this definition is taken, i.e.

$$H = D\Lambda$$  \hspace{1cm} \text{(III.15)}$$

Using (III.7) and (III.14), we obtain that the gauge transformation for $H$ reads

$$H = i \left[ \xi, F \right] + i \left[ \epsilon, H \right]$$  \hspace{1cm} \text{(III.16)}$$

We notice that an action for $H$ similar to (III.9), i.e. $-\frac{1}{2} \text{Tr} \int H \wedge * H$, will be invariant for the second part of (III.16), but not for the first.

This initial problem can be circumvented by redefining the two-form $\Lambda$ as [1, 2]
\[ \tilde{\Lambda} = \Lambda + D\Omega \]  

(III.17)

where the one-form quantity \( \Omega \) plays the role of a Stuckelberg field. Considering that \( \Omega \) has the gauge transformation

\[ \delta \Omega = i [\epsilon, \Omega] - \xi \]  

(III.18)

we obtain that the gauge transformation for \( \tilde{\Lambda} \) reads

\[ \delta \tilde{\Lambda} = i [\epsilon, \tilde{\Lambda}] \]  

(III.19)

Keeping the definition that \( \tilde{H} \) is the covariant derivative of \( \tilde{\Lambda} \), we have

\[ \delta \tilde{H} = i [\epsilon, \tilde{H}] \]  

(III.20)

Now, an action like \(-\frac{1}{2} \text{Tr} \int \tilde{H} \wedge * \tilde{H}\) is gauge invariant.

The problem in defining \( H \) as the covariant derivative of \( \Lambda \) (or \( \tilde{H} \) in terms of \( \tilde{\Lambda} \)) is that the Yang-Mills theory is not obtained after the compactification from \( D = 5 \) to \( D = 4 \). This is so because \( F \) is not attained from \( H \) after the compactification. In fact, in the definition of \( F \) we have the product \( \Gamma \wedge \Gamma \), while in the case of \( H \) (or \( \tilde{H} \)) the corresponding product is \([\Gamma, \Lambda]\) (or \([\Gamma, \tilde{\Lambda}]\)) (there is a factor 2 that spoils the correct obtainment of \( F \)).

This problem can also be circumvented by introducing another Stuckelberg-like field in the definition of the three-form stress tensor. Denoting this quantity by \( \tilde{\tilde{H}} \), and considering the definition

\[ \tilde{\tilde{H}} = d\tilde{\Lambda} - \frac{i}{2} \left[ \Lambda + \Xi, \tilde{\Lambda} \right] \]  

(III.21)

we have that an action like

\[ S = -\frac{1}{2} \text{Tr} \int \tilde{\tilde{H}} \wedge * \tilde{\tilde{H}} \]  

(III.22)

will be gauge invariant if \( \Xi \) transforms as

\[ \delta \Xi = d\epsilon - i [\Xi, \epsilon] \]  

(III.23)

This will be the action we are going to use in the compactification from \( D = 5 \) to \( D = 4 \) in order to obtain the Yang-Mills theory.

It might be opportune to mention that the use of two auxiliary Stuckelberg fields is not new. They have already been introduced with the purpose of restoring the
reducible condition in the non-Abelian sector \[^6\], that is a different purpose of the use we are making here.

### IV. Spontaneous compactification of the Non-Abelian case

For comparison, see a similar development of the Abelian case in the Appendix A. We start from the action

\[
S = -\frac{1}{2} \text{Tr} \int_{M_5} \tilde{\mathcal{H}} \wedge \ast \tilde{\mathcal{H}} \tag{IV.1}
\]

where

\[
\tilde{\mathcal{H}} = d\tilde{\Lambda} - \frac{i}{2} [\Gamma + \Xi, \tilde{\Lambda}] \tag{IV.2}
\]

\[
\tilde{\Lambda} = \Lambda + D\Omega \tag{IV.3}
\]

We are using the boldface notation to represent the geometrical elements in \(M_5\). Following the same steps of the Abelian case, we isolate the \(dx^4\) component from the quantities above. First, we take \(\Lambda, \Omega, \Gamma, \) and \(\Xi\), and introduce some definitions for the \(dx^4\) component,

\[
\Lambda = \Gamma \wedge dx^4 + \Lambda \\
\Omega = \varphi dx^4 + \Omega \\
\Gamma = \phi dx^4 + \Gamma \\
\Xi = \chi dx^4 + \Xi \tag{IV.4}
\]

where \(\Lambda, \Gamma, \varphi, \Omega, \phi, \chi, \) and \(\Xi\) depend on \((x^\mu, x^4)\). Consequently, we have

\[
D\Omega = \left( -\partial_4 \Omega + i [\phi, \Omega] + D\varphi \right) \wedge dx^4 + D\Omega \\
\tilde{\Lambda} = \left( \Gamma - \partial_4 \Omega + i [\phi, \Omega] + D\varphi \right) \wedge dx^4 + \tilde{\Lambda} \\
d\tilde{\Lambda} = \left( \partial_4 \tilde{\Lambda} + d\Gamma - d\partial_4 \Omega + i [d\phi, \Omega] + i [\phi, d\Omega] + dD\varphi \right) \wedge dx^4 + d\tilde{\Lambda} \tag{IV.5}
\]

The combination of (IV.4) and (IV.3) gives
\[ \tilde{H} = \tilde{H} + F \wedge dx^4 + G \wedge dx^4 \]  
(IV.6)

where \( G \) is a compact notation for

\[ G = \partial_4 \Lambda + d \left( D\varphi - \partial_4 \Omega + i [\phi, \Omega] \right) - \frac{i}{2} [\phi + \chi, \Lambda] \]

\[ - \frac{i}{2} [\Gamma + \Xi, D\varphi - \partial_4 \Omega + i [\phi, \Omega] \]  
(IV.7)

Now, the first Fourier component of \( F \), that appears in (IV.6), can be identified as the Yang-Mills stress tensor. It is important to emphasize that this was actually possible by virtue of the factor \( \frac{1}{2} \) we have introduced in the definition of \( \tilde{H} \).

The Hodge duality \( \ast \tilde{H} \) is given by

\[ \ast \tilde{H} = - \ast \tilde{H} \wedge dx^4 + \ast F + \ast G \]  
(IV.8)

Developing the quantities above in terms of Fourier harmonics and replace them in the action (IV.1), we easily obtain the Yang-Mills theory from the first harmonic component when the coordinate \( x_4 \) is integrated out.

V. Conclusion

In this paper, we have studied the spontaneous compactification of the two-form gauge field theory from \( D = 5 \) to \( D = 4 \). In the Abelian case, this leads to the same theory plus Maxwell one. However, in the non-Abelian case, the Yang-Mills theory is only attained if we make a convenient new definition of the non-Abelian three-form stress tensor.

To conclude, let us say that the topological term which couples vector and tensor gauge fields in \( D = 4 \), given at Eq. (I.1), cannot be generated from compactification. At first sight, we could think that it is originated from a Chern-Simon term in \( D = 5 \) like \( \kappa \epsilon^{MNPQR} \partial_M B_{NP} B_{QR} \). But we can directly verify that this term is zero. We may then conclude that the topological term of Eq. (I.1) has its own origin just in \( D = 4 \). In a physical point of view, there are two explanations for this fact. First, we know that the topological term in \( D = 4 \) is the starting point to generate mass for the vector potential if tensor degrees of freedom are integrated out. On the other hand, if one starts from the pure tensor gauge theory in \( D = 5 \) and integrated out the \( x_4 \) component, the excitations for \( n > 0 \) are already massive, without necessity of any topological coupling terms. Another possibility is that this term may have a quantum origin like the usual Chern-Simon term in \( D = 3 \). This second
possibility is presently under study, and possible results shall be reported elsewhere [11].

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Appendix A

In this Appendix we consider the spontaneous compactification of the Abelian case in the language of differential forms. First we introduce the quantity

$$\Lambda = \frac{1}{2} B_{MN} dx^M \wedge dx^N$$  \hspace{1cm} (A.1)

Let us rewrite it by isolating the $dx^4$ component (that shall be integrated out in a circle)

$$\Lambda = B_{\mu 4}(x, x_4) dx^\mu \wedge dx^4 + \frac{1}{2} B_{\mu \nu}(x, x_4) dx^\mu \wedge dx^\nu$$

$$= A_\mu(x, x_4) dx^\mu \wedge dx^4 + \frac{1}{2} B_{\mu \nu}(x, x_4) dx^\mu \wedge dx^\nu$$

$$= \Gamma(x, x_4) \wedge dx^4 + \Lambda(x, x_4)$$  \hspace{1cm} (A.2)

where $\Gamma$ and $\Lambda$ are differential forms in $\mathcal{M}_4$, but they do not correspond to one and two-forms of the Maxwell and tensor gauge theories, respectively, because they still depend on the coordinate $x_4$. Using the expression (A.2), we calculate $H$ by means of the following relation

$$H = d\Lambda$$

$$= (dx^4 \partial_4 + dx^\mu \partial_\mu) \wedge (\Gamma \wedge dx^4 + \Lambda)$$

$$= dx^4 \wedge \partial_4 \Lambda + dx^\mu \wedge \partial_\mu \Gamma \wedge dx^4 + dx^\mu \wedge \partial_\mu \Lambda$$

$$= \partial_4 \Lambda \wedge dx^4 + F \wedge dx^4 + H$$  \hspace{1cm} (A.3)

To construct the action, we need the dual $^*H$, that directly obtained by
\[ \begin{align*}
^*H &= (^*(d\Lambda)) \\
&= \frac{1}{4} \epsilon^{MNPQ} \partial_M B_{NP} \, dx_Q \wedge dx_R \\
&= \partial_4^* \Lambda + ^*F - ^*H \wedge dx^4 
\end{align*} \] (A.4)

where \(^*\Lambda\), \(^*F\), and \(^*H\), even though depend on \(x_4\), are Hodge dualities in \(M_4\). Using the expressions for \(H\) and \(^*H\) given by the expressions above, we have the action

\[ S = \frac{1}{2} \int_{M_5} H \wedge ^* H \]

\[ = \frac{1}{2} \int_{M_5} \left( \partial_4 \Lambda \wedge \partial_4 ^* \Lambda + 2 F \wedge \partial_4 ^* \Lambda + F \wedge ^* F - H \wedge ^* H \right) \wedge dx^4 \] (A.5)

The next step is to integrate the coordinate \(x^4\) over a circle of radius \(R\). We then consider the following expansion of the forms \(\Lambda\), \(F\), and \(H\), as well as their Hodge dualities, in terms of Fourier harmonics

\[ \Lambda(x, x_4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{n=+\infty} \Lambda_{(n)} \exp \left( 2in\pi \frac{x^4}{R} \right) \]

\[ ^*\Lambda(x, x_4) = \frac{1}{\sqrt{R}} \sum_{n=-\infty}^{n=+\infty} ^*\Lambda_{(n)} \exp \left( 2in\pi \frac{x^4}{R} \right) \]

etc. (A.6)

Introducing these expansions into the expression (A.5) and integrating out the coordinate \(x^4\) on a circle of radius \(R\), we obtain

\[ S = \sum_{n=-\infty}^{n=+\infty} \int_{M_4} \left[ \frac{1}{2} H_{(n)} \wedge ^* H_{(-n)} + \frac{1}{2} F_{(n)} \wedge ^* F_{(-n)} \right. \]

\[ \left. + \frac{1}{2} \left( \frac{2n\pi}{R} \right)^2 \Lambda_{(n)} \wedge ^* \Lambda_{(-n)} - \frac{2in\pi}{R} F_{(n)} \wedge ^* \Lambda_{(-n)} \right] \] (A.7)

Since \(\Lambda\) and \(^*\Lambda\) are real quantities, we have that \(\Lambda_{(-n)} = \Lambda_{(n)}^\dagger\) and \(^*\Lambda_{(-n)} = ^*\Lambda_{(n)}^\dagger\).
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