String Thermalization in Static Spacetimes

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Abstract

We study the evolution, the transverse spreading and the subsequent thermalization of string states in the Weyl static axisymmetric spacetime. This possesses a singular event horizon on the symmetry axis and a naked singularity along the other directions. The branching diffusion process of string bits approaching the singular black-hole horizon provides the notion of the temperature that is calculated for this process. We find that the solution of the Fokker-Planck equation in the phase space of the transverse variables of the string, can be factored as a product of two thermal distributions, provided that the classical conjugate variables satisfy the uncertainty principle. We comment on the possible physical significance of this result.

I. Introduction

String theory is the most promising candidate for a consistent quantization of gravity and a subsequent unified description of all the fundamental interactions [1-2]. One of the first steps towards a deep understanding of the quantum gravitational phenomena is to study the string evolution and quantization in the presence of a gravitational field [2-3]. This became necessary by the fact that in the spectrum of bosonic string theory gravity appears naturally through the massless spin-2 state having long range couplings that mimic General Relativity.

In the context of Black Hole Physics, strings have been identified with black hole states [4] and also have been used for the resolution of the so called Information Loss
problem. Namely of the apparent inconsistency of the black hole evaporation through the thermal Hawking radiation and the unitary evolution of the quantum states of the infalling matter that produces the gravitational collapse. The notion of Stretched Horizon has been introduced for the distant observer, and it is supposed to absorb and thermalize the quantum states of infalling matter which is in the form of strings [5-7].

Following these arguments, the transverse spreading of a relativistic string which falls towards the Black Hole Horizon has been described as a branching diffusion process [8]. This stochastic process provides the necessary mechanism for thermalization of the quantum state of the string. The resulting temperature calculated agrees in the order of magnitude, with the semiclassical result of Hawking and Bekenstein. Also, other efficient methods have been proposed that use explicit equations of state for the matter in the form of strings, the so called Planckian solid, which prevent the loss of information inside the black hole during the gravitational collapse [9].

All the above efforts have been concentrated on the problem where the singularity is hidden from the asymptotic observer via the Horizon. It would be interesting to discuss the problem of whether thermalization of string states can occur in a more general context where there exists a singular event horizon or a naked singularity in the spacetime manifold. In principle one can get arbitrary close to the singularity in this case. However, if one takes into account the fact that strings are quantum objects, then one can show that there exists thermalization of string states, occurring at finite distance from the singularity. The purpose of this paper is to examine this possibility.

The Weyl Static Axisymmetric Spacetime is the model under consideration that provides the setting and it is a class of Static exact solutions of the Einstein’s field equations [10]. The problem of a line source of length $2\alpha$ and of mass density $\gamma/2$ is known to be described by the $\gamma$ metric, which can be written either in spherical or prolate spheroidal coordinate systems. This family of solutions encompasses the Schwarzschild solution for $\gamma = 1$ and except for that solution, the family possesses singular event horizons [11]. These are in fact directional singularities for $\gamma \geq 2$, and by a proper choice of the coordinate system reveal their nature as extended hypersurfaces [12]. For all the other values of $\gamma$ one in fact has a singularity for all directions $0 \leq \theta \leq \pi$ [12]. From this the $\gamma_A$ metric which is an exact solitonic solution in vacuum is obtained through a limiting procedure [13]. This can also be interpreted as the metric for a counter-rotating disc in General Relativity [14,15].

This paper is organized as follows:
In section II, the general features of Static Axisymmetric solutions to the Einstein’s field equations are reviewed.
In section III, we develop the formalism and calculate the diffusion coefficients for the thermalization process of the string states, in the case where $0 < \gamma < 1$, which is the range of parameters under consideration.
In section IV, The Fokker-Planck equation is solved in the phase space of the transverse variables of the string, that is falling towards the black-hole.
In section V, numerical estimates that connect the above work with real astrophysical systems, is provided.  
In section VI, we give a discussion of the contribution of the quantum aspects of strings, when they are taken into account, for the process of thermalization.  

II. Static Axisymmetric Black-Hole Spacetimes

The curved spacetime manifold that we want to study is given by the Weyl metric \[ [11] \]
\[\begin{align*}
    ds^2 &= -\left(\frac{\xi - 1}{\xi + 1}\right) \gamma dt^2 + \alpha^2 \left(\frac{\xi^2 - 1}{\xi^2 - \eta^2}\right) \gamma^2 \left(\frac{\xi + 1}{\xi - 1}\right) (\xi^2 - \eta^2) \cdot \\
    &\quad \cdot \left[\frac{d\xi^2}{\xi^2 - 1} + \frac{d\eta^2}{1 - \eta^2}\right] + \alpha^2 \left(\frac{\xi + 1}{\xi - 1}\right) (\xi^2 - 1)(1 - \eta^2)d\phi^2,
\end{align*}\]
written in prolate, spheroidal coordinates, \((t, \xi, \eta, \phi)\). The transformation from the Cartesian coordinates, is given by
\[\begin{align*}
    x^2 &= \alpha^2 (\xi^2 - 1)(1 - \eta^2) \cos^2 \phi \tag{2} \\
    y^2 &= \alpha^2 (\xi^2 - 1)(1 - \eta^2) \sin^2 \phi \tag{3} \\
    z &= \alpha \xi \eta, \tag{4}
\end{align*}\]
where \(\alpha, \gamma\) are constants, representing a line source of length \(2\alpha\) and a mass density \(\gamma/2\). The range of the prolate spheroidal coordinates is given by \(-\infty < t < +\infty,\ 1 \leq \xi < +\infty,\ -1 \leq \eta \leq +1,\ 0 \leq \phi < 2\pi.\)

The \(\gamma\) metric in the \((t, r, \theta, \phi)\) coordinate system, using the transformation \[\begin{align*}
    \rho^2 &= (r^2 - 2mr) \sin^2 \theta \tag{5} \\
    z &= (r - m) \cos \theta, \tag{6}
\end{align*}\]
reads
\[\begin{align*}
    ds^2 &= -\mathcal{A} dt^2 + \frac{A^{\gamma - 1} - 1}{B^{\gamma - 1}} dr^2 + \frac{A^{\gamma - 1}}{B^{\gamma - 1}} r^2 d\theta^2 + A^{1 - \gamma} r^2 \sin^2 \theta d\phi^2, \tag{7}
\end{align*}\]
where
\[\begin{align*}
    \mathcal{A} &= \left(1 - \frac{2m}{r}\right) \tag{8} \\
    B &= \left(1 - \frac{2m}{r} + \frac{m^2}{r^2} \sin^2 \theta\right). \tag{9}
\end{align*}\]
The null outgoing vector is given by

\[ l^\alpha = \left( -A^{-\gamma}, \frac{B^{(\gamma^2 - 1)/2}}{A^{(\gamma^2 - 1)/2}}, 0, 0 \right) \]  

(10)

\[ l_\alpha = \left( 1, \frac{A^{(\gamma^2 - 2\gamma - 1)/2}}{B^{(\gamma^2 - 1)/2}}, 0, 0 \right) \]  

(11)

\[ l^\alpha l_\alpha = 0. \]  

(12)

Introducing the timelike Killing vector \( \xi^\mu_{(t)} = \delta^\mu_t \), we have \( \xi^\alpha l_\alpha = 1 \).

The surface gravity is defined as \( \kappa = l^\alpha \xi b \nabla_b \xi_\alpha \) [11]. The result is

\[ \kappa = m \frac{1}{r^2} A^{(\gamma^2 - 1)/2} B^{-(\gamma - 1)^2/2}. \]  

(13)

We have the following two cases, when \( r \to 2m \).

For \( \theta \neq 0, \pi \) when \( \gamma \neq 1 \) the surface gravity tends to infinity. For \( \theta = 0, \pi \) we have

\[ \kappa = m \frac{1}{r^2} (1 - \frac{2m}{r})^{\gamma - 1} \]  

(14)

and it tends to zero when \( \gamma > 1 \), while it tends to infinity as \( 0 < \gamma < 1 \).

Therefore, intuitively one expects that in the former class of models it is difficult to have thermalization process while in the second it can occur naturally. This is because in the first case the strong tidal gravitational forces tend to shrink the string, so it is impossible for the string bits to become uncorrelated (via thermalization process) as the relativistic string approaches the singular event horizon.

### III. String Thermalization on the Symmetry Axis.

The \( \gamma \)-metric for \( \theta = 0, \pi \) is given by

\[ ds^2 = -\left( 1 - \frac{2m}{r} \right)^\gamma dt^2 + \left( 1 - \frac{2m}{r} \right)^{-\gamma} dr^2 = -\left( 1 - \frac{2m}{r} \right)^\gamma dudv, \]  

(15)

where

\[ du = dt + \frac{dr}{1 - \frac{2m}{r}} \]  

(16)

\[ dv = dt - \frac{dr}{1 - \frac{2m}{r}}. \]  

(17)

We introduce the new variables \( (s, p) \), by

\[ du = 4m(ds/cos^\gamma(s)) \]  

(18)

\[ dv = 4m(dp/sin^\gamma(p)). \]  

(19)
The coefficients have been chosen in such a way so as to reproduce the well known result [8] in the case of Schwarzschild Spacetime, as will be shown below.

The metric assumes the Rindler-type form

\[ ds^2 = -4Q^2dsdp \] \tag{20} \\
\[ Q^2 = \frac{4m^2 \left( 1 - \frac{2m}{r} \right)^\gamma}{\cos\gamma(s) \sin\gamma(p)}. \] \tag{21} \\

where \( Q|_{r=2m} = \sqrt{A} \) is finite, continuous and non-zero on the horizon [12].

Integration of Eqs (18), (19) gives (0 < \( \gamma < 1 \))

(Appendix II)

\[ u(s) = \frac{4m}{1 - \gamma} \cos^{1-\gamma}(s) \ _2F_1 \left( 1 - \gamma, 1; \frac{3 - \gamma}{2}; \frac{1}{2}[1 + \sin(s)] \right) \] \tag{22} \\
\[ v(p) = \frac{4m}{1 - \gamma} \sin^{1-\gamma}(p) \ _2F_1 \left( 1 - \gamma, 1; \frac{3 - \gamma}{2}; \frac{1}{2}[1 + \cos(p)] \right). \] \tag{23} \\

Now we introduce Kruskal-type coordinates,

\[ U = 2s\sqrt{A} \] \tag{24} \\
\[ V = 2p\sqrt{A}. \] \tag{25} \\

The Light-Cone Gauge corresponds to the setting \( \tau = U/4m \), that is to the choice \( \tau = -s\sqrt{A}/2m \). It is easy to verify, using Eqs (15), (18), (19), (21) and (24)-(25) that \( ds^2 = -d UdV \) in conformity with [8] and as it should be for Kruskal coordinates.

In Eq (22), the range of the parameters is \(-\pi/2 < s < \pi/2\) and also \(0 < u < u_0\), where \( u_0 \equiv 4mC_0 2^{1-\gamma}/(1 - \gamma) \), (Appendix II).

From Eq (18), for this range of \( s \) the function \( u(s) \) as strictly increasing and continuous is invertible. The variable \( u \) corresponds to the cosmic-time \( t \) and we are interested in the asymptotic regime \( u \to u_0 \) where we shall seek steady state correlation functionals and stationary probability density functionals.

We consider a relativistic string that falls freely, along the symmetry axis \( \theta = 0 \), towards the singular horizon. The wave equation in the free-fall frame is given by

\[ \left( \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X^i(\tau, \sigma) = 0. \] \tag{26} \\

for the transverse coordinates \( X^i(\tau, \sigma) \). In our case these are the spacelike transverse coordinates in the vicinity of the hypersurface \( \theta = 0 \), because as it was proven in [12] with a proper choice of coordinates this is a two dimensional hypersurface for \( \gamma \geq 2 \) while for our case of \( 0 < \gamma < 1 \) it is a pure singularity. Therefore we consider a string which falls towards the singularity, with its transverse coordinates being normal to the symmetry axis \( \theta = 0 \).

This can be written as

\[ \left[ \left( \frac{du}{d\tau} \right)^2 \frac{\partial^2}{\partial u^2} + \left( \frac{d^2 u}{d\tau^2} \right) \frac{\partial}{\partial u} - \left( \frac{A}{4m^2} \right) \frac{\partial^2}{\partial \sigma^2} \right] X^i(u, \sigma) = 0. \] \tag{27} 

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We can express the derivatives of \( u(s) \) with respect to \( s \), in terms of \( u \) and this is done in Appendix II.

We proceed to the decomposition of the field and its conjugate momentum,

\[
X^i(u, \sigma) = x^i(u, \sigma) + x'^i(u, \sigma) \quad (28)
\]

\[
\dot{X}^i(u, \sigma) \equiv \frac{\partial X^i(u, \sigma)}{\partial u} = v^i(u, \sigma) + v'^i(u, \sigma). \quad (29)
\]

into a slowly varying, classical part and a fast varying, quantum part. The quantum part is now expanded as a sum over modes in the free-fall frame, with a frequency cutoff that separates the fast modes from the slow ones, \([8]\)

\[
x'^i(u, \sigma) = \sum_{n=1}^{\infty} W \left( n + \frac{\epsilon}{\tau} \right) \left[ \frac{c_i}{\sqrt{n}} x^+_n + \frac{\bar{c}_i}{\sqrt{n}} x^-_n + H.c. \right] \quad (30)
\]

\[
v'^i(u, \sigma) = \sum_{n=1}^{\infty} W \left( n + \frac{\epsilon}{\tau} \right) \left[ \frac{c_i}{\sqrt{n}} \dot{x}^+_n + \frac{\bar{c}_i}{\sqrt{n}} \dot{x}^-_n + H.c. \right]. \quad (31)
\]

Here \( u = u(\tau) \), \( \epsilon > 0 \) is a constant, \( W \) is a Gaussian distribution function and the convention \( x^\pm_n = \sqrt{\frac{\tau}{2}} e^{-in(\tau+\sigma)} \) is used. The commutation relations for the field operators are \([c^i_n, (c^j_n)^\dagger] = \delta_{mn} \delta^{ij} \) and similar for the tilded operators.

From these relations we obtain

\[
\dot{x}'_j(u, \sigma) = \sum_{n=1}^{\infty} W \left( n + \frac{\epsilon}{\tau} \right) \left[ \frac{c_i}{\sqrt{n}} \dot{x}^+_n + \frac{\bar{c}_i}{\sqrt{n}} \dot{x}^-_n + H.c. \right] - \eta^j(u, \sigma) \quad (32)
\]

\[
\eta^j(u, \sigma) = \left( \frac{du}{d\tau} \right)^{-1} \left( \frac{\epsilon}{\tau^2} \right) \sum_{n=1}^{\infty} W' \left( n + \frac{\epsilon}{\tau} \right) \left[ \frac{c_i}{\sqrt{n}} x^+_n + \frac{\bar{c}_i}{\sqrt{n}} x^-_n + H.c. \right]. \quad (33)
\]

Also in the same manner we have

\[
\dot{v}'_j(u, \sigma) = \sum_{n=1}^{\infty} W \left( n + \frac{\epsilon}{\tau} \right) \left[ \frac{\bar{c}_i}{\sqrt{n}} \ddot{x}^+_n + \frac{c_i}{\sqrt{n}} \ddot{x}^-_n + H.c. \right] - \xi^j(u, \sigma) \quad (34)
\]

\[
\xi^j(u, \sigma) = \left( \frac{du}{d\tau} \right)^{-1} \left( \frac{\epsilon}{\tau^2} \right) \sum_{n=1}^{\infty} W' \left( n + \frac{\epsilon}{\tau} \right) \left[ \frac{\bar{c}_i}{\sqrt{n}} \ddot{x}^+_n + \frac{c_i}{\sqrt{n}} \ddot{x}^-_n + H.c. \right]. \quad (35)
\]

Substitution into the field equations for the \( X^i \) results in to two coupled equations for the long-wavelength fields,

\[
\dot{x}^i = v^i + \eta^i \quad (36)
\]

\[
\dot{v}^i = -\frac{u''}{(u')^2} v^i + \frac{1}{(u')^2} \left( \frac{A}{4m} \right) \frac{\partial^2}{\partial \sigma^2} x^i + \xi^i. \quad (37)
\]

Here \( u'', u' \) are the derivatives of \( u \) with respect to \( s \), expressed in terms of \( u \). Substituting into the second equation we obtain

\[
\dot{v}^i(u, \sigma) = -H_1(u)v^i(u, \sigma) + H_2(u) \frac{\partial^2}{\partial \sigma^2} x^i(u, \sigma) + \xi^i. \quad (38)
\]
The form of the functions $H_1(u), H_2(u)$ is given by direct comparison with Eq (37). The spatial derivative term becomes negligible with respect to the first term on the r.h.s. of Eq (38), as $u \to u_0^-$, that is, when the string approaches the singularity at $r = 2m$. An additional supporting argument for this is given in Appendix III.

Proceeding as in [8] we finally obtain

$$
< \eta^i(1)\eta^j(2) > = \frac{\alpha'}{2} \delta^{ij} \left( \frac{du}{d\tau} \right)^{-1} \delta[u(\tau_1) - u(\tau_2)] 
\cdot \exp \left[ -\frac{\beta^2}{4} \left( \frac{\epsilon \Delta \sigma}{\tau} \right)^2 \cos \left( \frac{\epsilon \Delta \sigma}{\tau} \right) \right].
$$

(39)

This correlation function shows that the string bits undergo Brownian motion since the correlation of two string bits is practically zero outside the correlation length $\Delta \sigma = |\tau|/\beta \epsilon$.

The rest of the correlation functions can be computed in the same way. We have

$$
\dot{x}_n^\pm \equiv \frac{\partial}{\partial u} x_n^\pm = -in \left( \frac{du(\tau)}{d\tau} \right)^{-1} x_n^\pm = -i \left( \frac{\epsilon}{\tau} \right) \left( \frac{du(\tau)}{d\tau} \right)^{-1} x_n^\pm,
$$

(40)

because the main contribution comes from $n \simeq (\epsilon/\tau)$. So

$$
< \xi^i(1)\xi^j(2) > = - \left[ \frac{\epsilon}{\tau} \left( \frac{du(\tau)}{d\tau} \right)^{-1} \right]^2 < \eta^i(1)\eta^j(2) >
$$

(41)

$$
< \xi^i(1)\eta^j(2) > = - \left[ i \frac{\epsilon}{\tau} \left( \frac{du(\tau)}{d\tau} \right)^{-1} \right] < \eta^i(1)\eta^j(2) >.
$$

(42)

Now these correlator functions are negligible in comparison to Eq (39) as the string approaches the singularity, that is as $u \to u_0^-$. This is evident from Eq (83).

**IV. The Diffusion Process and the Fokker-Planck equation**

For the case of Schwarzschild spacetime ($\gamma = 1$) we obtain the same results as in [8], because the term $(\tau \frac{du}{dt})^{-1}$ in Eq (39) reduces to $(1/4M)$. For the present case the coefficient of the correlator is the diffusion coefficient for the process. It is given by

$$
\zeta_1^2 \equiv \left( \frac{\alpha'}{8m} \right) \left( \frac{2^{\gamma + 1}}{\tau} \right) \left( \frac{u}{u_0} \right)^{\gamma/2} \gamma^{1 - \gamma} \left[ 1 - \left( \frac{u}{u_0} \right)^{2/1 - \gamma} \right] \gamma^{\gamma/2}.
$$

(43)

By the fluctuation-dissipation theorem the temperature of the process assigned by an asymptotic observer is inversely proportional to the diffusion coefficient [18]. As $u \to u_0^-$ the temperature grows without limit.
From Eqs (41)-(42) one concludes that the diffusion process in momentum space is negligible and can be omitted with respect to the diffusion process in ordinary space. We will however retain this also and consider that it evolves with a diffusion coefficient $\zeta_2 \ll \zeta_1$. We rewrite the two Langevin equations as

\[
\frac{\partial x^i(u, \sigma)}{\partial u} = v^i + \eta^i \quad (44) \\
\frac{\partial v^i(u, \sigma)}{\partial u} = -H_1(u)v^i + \xi^i. \quad (45)
\]

The Fokker-Planck equation for the probability density $\Phi = \Phi(x^i, v^i; u)$, corresponding to the above set of equations is given by [19]

\[
\frac{\partial \Phi}{\partial u} = F \Phi \\
F \equiv \sum_i \left[ \zeta_1^2 \frac{\partial^2}{\partial (x^i)^2} + \zeta_2^2 \frac{\partial^2}{\partial (v^i)^2} - v^i \frac{\partial}{\partial x^i} + H_1(u) \frac{\partial}{\partial v^i} \right]. \quad (46)
\]

The probability density can be normalized for all "times" if it is normalized once. This is because we have

\[
\frac{d}{du} \int dx^i dv^i \Phi(x^i, v^i; u) = \int dx^i dv^i \frac{\partial}{\partial u} \Phi(x^i, v^i; u) = \int dx^i dv^i F \Phi(x^i, v^i; u) = 0, \quad (47)
\]

which vanishes under proper boundary conditions, because the action of the operator $F$ can be written as a divergence with respect to the space variables. We denote collectively the space variables $(x^i)$ by $(q)$ and $(v^i)$ by $(v)$.

Writing $F = F_1 + H_1(u)F_2$ where it is evident the content of the two terms, we can write formally the solution of the Fokker-Planck equation as

\[
\Phi(q, v; u) = \exp[uF_1 + G(u)F_2]\delta(q)\delta(v), \quad (48)
\]

with

\[
G(u) \equiv \int_0^u du' H_1(u'). \quad (49)
\]

Here, the usual convention, $\delta(q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\rho e^{iq\rho}$, $\delta(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\bar{\rho} e^{iv\bar{\rho}}$ is used. We were able to solve completely the time-dependent Fokker-Planck Eq (46) without any sort of approximation. The solution is given by

\[
\Phi(q, v; u) = \exp \left[-\frac{[q - Q_2(u)v]^2}{Q_1(u)}\right] U(v; u) \quad (50)
\]

\[
U(v; u) = \Psi(u) \exp \left[-\frac{v^2}{4\zeta_2^2Q_2}(1 - Q_2H_1(u)Q_2)\right], \quad (51)
\]
where the the three functions \( Q_1(u), Q_2(u), \Psi(u) \) satisfy

\[
Q_1'(u) = 4\zeta_1^2 + 4\zeta_2^2 Q_2^2 \\
(f + H_1)' + (f^2 - H_1^2) + 2f \left[ \frac{Q_1^2 - 4\zeta_1^2}{Q_1} \right] = 0
\]

\[
f \equiv \left( \frac{1 - Q_2'}{Q_2} \right)
\]

\[
\Psi'(u) = -\frac{2\zeta_1^2}{Q_1} - \frac{1}{2}(f + H_1) - \frac{2\zeta_2^2 Q_2^2}{Q_1} + H_1.
\]

Prime denotes ordinary differentiation with respect to the \( u \). This set of equations cannot be solved in closed form for a generic choice of the function \( H_1(u) \). In Appendix III, the asymptotic form of \( H_1(u) \) is given as \( u \to u_0^- \). For this choice the solution is given by

\[
f = -H_1, \\
Q_1 = 4\zeta_1^2 u, \\
Q_2 = 1 - \exp \left[ -\kappa_0 \left( \frac{1 + \gamma}{2} \right)^{-1} \left( u_0 - u \right)^{1+\gamma} \right]
\]

with the only approximation made up to here, is that as \( u \to u_0^- \), \( Q_2(u) \simeq 0 \) so that the second term on the r.h.s. of Eq (52) can be neglected, as being of the second order, in the first order approximation case. Neglecting, for the same argument, the third term in the r.h.s. of Eq (55), one can integrate it, obtaining

\[
\Psi(u) = \ln \frac{1}{\sqrt{a}} - \frac{2\kappa_0}{(1 + \gamma)} (u_0 - u)^{(1+\gamma)/2}.
\]

From Eq (50), we now see, that we can have a product of two thermal distributions, provided that the cross term in the exponential is constant, or can be neglected. Denoting this term as

\[
T(u) \equiv \frac{2\kappa_0}{(1 + \gamma)} \frac{(u_0 - u)^{(1+\gamma)/2}}{2\zeta_1^2 u} qv
\]

and following the same procedure as in Appendix III, we find that the asymptotic form of Eq (43), as \( u \to u_0^- \) is given by \( \zeta_1^2 \propto (u_0 - u)^{\gamma/2} \). Therefore in Eq (60) the overall dependence is given by \( T(u) \propto (u_0 - u)^{1/2} \). Now we assume that the Uncertainty Principle relation holds for the phase space variables \( qv = \hbar, \ \hbar = \text{constant} \). Then from Eq (60) the cross term vanishes and we have a product of two thermal distribution functions.

**V. Numerical Estimates**

There exist a variety of physically interesting systems that one could refer to, in order to ascertain that the range of parameters under study, is acceptable. Using
geometrical units we set \( c = G = K = 1 \) with \( c = 2.99 \times 10^{10} \) cm/sec=1 and \( G/c^2 = 0.74 \times 10^{-28} \) cm/gr=1 for the conversion factors [20].

For cosmic strings, one can consider two extreme cases. A string that originates in symmetry breaking at a mass scale \( \Lambda \simeq 10^{16} \) GeV (the scale of GUT’s) has a mass per unit length of the order \( 10^{22} \) gr/cm [21]. On the other side strings arising in the electroweak scale \( \Lambda \simeq 1 \) TeV has a mass per unit length of the order \( 10^{-6} \) gr/cm. Such a string with the length of a galaxy (radius \( 10^{22} \) cm) would have a mass of the order of \( 10^{16} \) gr and it would be unobservable by gravitational interactions. For the first case we obtain \( \gamma = 0.74 \times 10^{-6} \ll 1 \) while for the second \( \gamma = 0.74 \times 10^{-34} \ll 1 \).

A more realistic prospect comes from the fact that the mass of the Sun is \( M_\odot = 1.989 \times 10^{33} \) gr= \( 1.477 \times 10^{5} \) cm. In order that we get a value of

\[
\gamma = \frac{1M_\odot}{2\alpha} = \frac{1}{2},
\]

we must have \( \alpha = 1.447 \times 10^{5} \) cm, which compared with the Sun radius \( R_\odot \simeq 7 \times 10^{10} \) cm gives a ratio of the order of \( 10^{-5} \).

On the Galactic scale, the most extreme case corresponds to the masses of central regions [22-23]. Here we have \( M_{BH} \simeq 10^9 M_\odot = 1.477 \times 10^{14} \) cm, whereas the central region extends over the scale of \( R \simeq 0.1 \) pc= \( 3.086 \times 10^{17} \) cm. This yields \( \gamma \simeq 0.3 \times 10^{-3} \). Therefore in all cases, the range of parameters under study is the relevant one, for almost all astrophysically interesting systems.

**VI. Discussion**

The concept of the *Stretched Horizon* has been introduced recently, that thermalizes the quantum states of the infalling matter towards the black hole and eventually reemits them in the form of the thermal Hawking radiation. This prevents the loss of information inside the black hole Horizon during the gravitational collapse so that there exists at this first level of analysis no conflict between the notion of gravitational collapse and the unitary evolution of states in quantum theory. This is achieved by considering that matter is in the form of strings, that undergo a diffusion process in the space of the transverse variables.

In our case the spacetime under consideration contains a singular event horizon for \( \theta = 0 \) and \( \gamma \geq 2 \), while for other values of \( 0 < \gamma < 1 \), a naked singularity along all the directions. Therefore there doesn’t exist a Horizon and no natural distinction between an asymptotic and a free-falling observer. Even more we have chosen to describe the process of *Branching Diffusion* in terms of the coordinate-based observer that in principle, can approach arbitrary close to the singularity. If however we invoke the fact that strings are quantum objects, this provides the mechanism for obtaining thermal spectrum for the diffusion processes of string bits in both the configuration and the momentum space of the transverse variables of the string. Therefore we can associate with it a temperature. This has been achieved for the particular class
of static models that have $0 < \gamma < 1$. Although we considered the case of the symmetry axis where there exists the singularity, by continuity our arguments are valid in the vicinity of $\theta = 0$. Therefore in the case of the naked singularity we also have thermalization process.

Now, although it seems that this temperature increases without limit, there exists a very interesting fact here. It has to do with the fact that practically the string is somehow thermalized at some finite distance away from the singularity, in whatever quantum state it had been initially. The string bits become practically uncorrelated over a spatial distance of the same order of magnitude as the quantum correlation length. In principle then, very little or no information, can get lost on the singular event horizon, or can escape from it.

One can reverse this argument and examine whether this is some form of Cosmic Censorship. This is enhanced by the fact that this is a static spacetime and there exists the invariance under coordinate-time reversal $t \leftrightarrow (-t)$. Also in the same spirit, it would be interesting, if one could address the same problem for membranes or other extended objects of modern string theory. Work along these lines is in progress and will be reported promptly.

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**Appendix I**

We give below, for the sake of completeness, the non-zero components of the Riemann curvature tensor for the $(\gamma)$ metric. They are consistent with the conditions

$$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{\gamma\delta\alpha\beta}$$

$$R_{[\alpha\beta\gamma\delta]} = 0. \quad (62)$$

The expressions are considerably complicated. With the help of the definitions

$$\mathcal{A} = \left(1 - \frac{2m}{r}\right) \quad (63)$$

$$\mathcal{B} = \left(1 - \frac{2m}{r} + \frac{m^2}{r^2 \sin^2 \theta}\right) \quad (64)$$

we have

$$R_{\phi\theta\theta \theta} = \frac{1}{2} \gamma \left(\frac{A_\gamma}{\mathcal{A}}\right) \left(\frac{\mathcal{B}}{\mathcal{A}}\right)^{\gamma^2 - 1} \mathcal{A}^\gamma \left[\mathcal{A} - \frac{1}{2} (\gamma - 1) r A_x \right] r \sin^2 \theta \quad (66)$$
\begin{align*}
R_{\theta \theta \theta} &= \frac{1}{2} \gamma \left( \frac{A_r}{A} \right) A^{\gamma} \left[ \frac{1}{2} \gamma (\gamma - 1) r^2 A_r - \frac{1}{2} \gamma (\gamma^2 - 1) r^2 A \left( \frac{B_r}{B} \right) - r A \right] \quad (67) \\
R_{r \theta \theta} &= \frac{1}{4} \gamma (\gamma - 1) \left( \frac{A_r}{A} \right) \left( \frac{B_r}{B} \right) A^{\gamma} \quad (68) \\
R_{r r r} &= \frac{1}{2} A^{\gamma} \left[ \gamma \left( \frac{A_r}{A} \right) - \frac{1}{4} \gamma (\gamma^2 - 1) \left( \frac{A_r}{A} \right) \left( \frac{B_r}{B} \right) - \frac{1}{4} \gamma (\gamma - 1)^2 \left( \frac{A_r}{A} \right)^2 \right] \quad (69) \\
R_{\theta \phi \phi} &= \frac{r^2}{A^{\gamma - 1}} \left[ - \frac{1}{2} \gamma (\gamma - 1) \left( \frac{B_\theta}{B} \right) \sin \theta \cos \theta + (1 + A) \sin^2 \theta + \frac{1}{4} \gamma (\gamma - 1)^2 \left( \frac{A_r}{A} \right) r^2 A_r \sin^2 \theta - \frac{1}{4} \gamma (\gamma - 1)(\gamma^2 - 1) \left( \frac{B_r}{B} \right) r^2 A_r \sin^2 \theta - \frac{1}{2} \gamma (\gamma - 1) r A_r \sin^2 \theta + \frac{1}{2} (\gamma^2 - 1) \left( \frac{B_r}{B} \right) r A \sin^2 \theta \right] \quad (70) \\
R_{\phi \theta \phi} &= \frac{1}{A^{\gamma}} \left[ \frac{1}{2} \gamma (\gamma - 1) r^2 A_r \sin \theta \cos \theta - \frac{1}{2} \gamma (\gamma - 1) \left( \frac{B_r}{B} \right) r^2 A \sin \theta \cos \theta - 2 r A \sin \theta \cos \theta + \frac{1}{4} \gamma (\gamma - 1)(\gamma^2 - 1) \left( \frac{B_r}{B} \right) r^2 A_r \sin^2 \theta - \frac{1}{2} \gamma (\gamma - 1) \left( \frac{B_r}{B} \right) r A \sin^2 \theta \right] \quad (71) \\
R_{\theta \theta r} &= \frac{1}{2} A^{\gamma} \left( \frac{A}{B} \right)^{\gamma - 1} \left[ \gamma (1 - \gamma) r^2 A_{rr} + (\gamma^2 - 1) \left( \frac{A}{B} \right) r^2 B_{rr} + (\gamma^2 - 1) \left( \frac{B_\theta}{B} \right) - r A_r (\gamma^2 - \gamma + 1) + \frac{1}{2} (\gamma^2 - 1) \left( \frac{B_r}{B} \right) r^2 A_r + \gamma (\gamma - 1) \left( \frac{A_r}{A} \right) r^2 A_r - (\gamma^2 - 1) \left( \frac{B_r}{B} \right)^2 r^2 A + (\gamma - 1) \left( \frac{A}{B} \right) r B_r - (\gamma^2 - 1) \left( \frac{B_r}{B} \right)^2 \right] \quad (72) \\
R_{\phi \theta r} &= \frac{1}{2} A^{\gamma} \left[ (\gamma - 1) r^2 A_{rr} + (\gamma^2 - 1) \left( \frac{B_\theta}{B} \right) \cot \theta - 2 r A_r - \frac{1}{2} \gamma (\gamma - 1) \left( \frac{A_r}{A} \right) r^2 A_r - (\gamma^2 - 1) \left( \frac{B_r}{B} \right) r A + (\gamma^2 + \gamma - 1) \left( \frac{A_r}{A} \right) r A + \frac{1}{2} (\gamma - 1)(\gamma^2 - 1) \left( \frac{B_r}{B} \right) r^2 A_r \right] \sin^2 \theta. \quad (73)
\end{align*}
Appendix II

We consider Eq (18) and the case $0 < \gamma < 1$. Setting $u(s) = -4m \cos^{1-\gamma}(s)f(s)$ one finds that $f(s)$ satisfies

$$
\cos(s)f'(s) + (\gamma - 1)\sin(s)f(s) + 1 = 0.
$$

(74)

Setting $h = \sin(s)$ and subsequently $h = (2t - 1)$ ($0 < t < 1$) one finally gets

$$
2t(1-t) \frac{df}{dt} + (1 - \gamma)(1 - 2t)f(t) + 1 = 0.
$$

(75)

It can be shown that the following is a solution to this equation [17],

$$
f(t) = -\frac{1}{1-\gamma} 2F_1(1 - \gamma, 1; \frac{3 - \gamma}{2}; t),
$$

(76)

by expanding in a power series $f(t)$ and substituting into Eq (75). Using MATHEMATICA, Eq (18) is integrated as

$$
u(s) = \frac{4m \cos^{1-\gamma}(s)}{1 - \gamma \sqrt{\sin^2(s)}} 2F_1(1 - \gamma, \frac{1}{2}; \frac{3 - \gamma}{2}; \cos^2(s)) \sin(s).
$$

(77)

These are exactly equivalent due to a property of hypergeometric functions (see [17], p.561, 15.3.30). In the same way one integrates Eq (19).

We use now two well known properties of the hypergeometric function to obtain the limit $t \to 1_-$.

$$
2F_1(a, b; c; t) = (1 - t)^{c-a-b} 2F_1(c-a, c-b; c; t)
$$

$$
2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} (c-a-b > 0, \ b > 0).
$$

(78)

So we obtain

$$
2F_1(1-\gamma, 1; \frac{3 - \gamma}{2}; t) = (1 - t)^{-(1-\gamma)/2} 2F_1(-\frac{1 + \gamma}{2}, \frac{1 - \gamma}{2}; \frac{3 - \gamma}{2}; t \approx 1) =
$$

$$
= (1 - t)^{-(1-\gamma)/2} \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{1-\gamma}{2})}{\Gamma(1-\gamma)}
$$

$$
\equiv C_0(1-t)^{-(1-\gamma)/2}.
$$

(79)

For $t=1$, which corresponds to $s = \pi/2$, we obtain that

$$
u_0 \equiv u(s = \pi/2) = \frac{4mC_0}{1 - \gamma} 2^{1-\gamma}.
$$

(80)

Therefore in the vicinity of $t \simeq 1$, we have

$$
u(s) = \frac{4m}{1 - \gamma} C_0 (4t)^{(1-\gamma)/2},
$$

(81)
so we can invert Eq (22) to obtain

\[ \sin(s) = \left[ 2 \left( \frac{u}{u_0} \right)^{2/(1-\gamma)} - 1 \right] \]  

and

\[ \frac{du}{ds} = \frac{4m}{2^\gamma} \left( \frac{u}{u_0} \right)^{-\gamma/(1-\gamma)} \left[ 1 - \left( \frac{u}{u_0} \right)^{2/(1-\gamma)} \right]^{-\gamma/2} \]

\[ \frac{d^2u}{ds^2} = \frac{4m\gamma}{2^{\gamma+1}} \left( \frac{u}{u_0} \right)^{-(\gamma+1)/(1-\gamma)} \left[ 1 - \left( \frac{u}{u_0} \right)^{2/(1-\gamma)} \right]^{-(\gamma+1)/2} \left[ 2 \left( \frac{u}{u_0} \right)^{2/(1-\gamma)} - 1 \right] \]  

**Appendix III**

From Appendix II we obtain

\[ \lim_{u \to u_0} \frac{H_2(u)}{H_1(u)} = 0 \]  

and the spatial derivative term is negligible. Alternatively, the vector \( \Xi^i \equiv \partial x^i / \partial \sigma \) is the separation vector of two points on the string, obeying the geodesic deviation equation

\[ \frac{D^2 \Xi^i}{d\tau^2} + R^i_{jkl} u^j \Xi^k u^l = 0. \]

For \( i = \phi \), we have from Appendix I, that the second term on the l.h.s. of Eq (66) vanishes. Then it is solved as \( \Xi^\phi(\tau, \sigma) = \Xi_0(\sigma)\tau + \Xi_1(\sigma) \). So the spatial derivative term is proportional to the proper time \( \tau \) which remains finite on the singular horizon.

We proceed now to obtain the asymptotic form of \( H_1(u) \), as \( u \to u_0^- \). Combining Eqs (37), (38) and (83) we have

\[ H_1(u) = \left( \frac{\gamma}{2m} \right) \left( \frac{1}{2\gamma} \right) \left[ 1 - \left( \frac{u}{u_0} \right)^{2/(1-\gamma)} \right]^{-(1-\gamma)/2}. \]

Denoting,

\[ 0 < \epsilon \equiv \left( \frac{u_0 - u}{u_0} \right) \ll 1, \]

we have that

\[ \left[ 1 - \left( \frac{u}{u_0} \right)^{2/(1-\gamma)} \right]^{-(1-\gamma)/2} = \left[ 1 - (1 - \epsilon)^{2/(1-\gamma)} \right]^{-(1-\gamma)/2} \sim \left( \frac{1 - \gamma}{2\epsilon} \right)^{(1-\gamma)/2}, \]

so that

\[ H_1(u) = \frac{\kappa_0}{(u_0 - u)^{(1-\gamma)/2}} \]

\[ \kappa_0 = \left( \frac{\gamma}{2m} \right) \left( \frac{1}{2\gamma} \right) \left( \frac{(1 - \gamma)u_0}{2} \right)^{(1-\gamma)/2}. \]
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