ON THE INFIMUM ATTAINED BY THE REFLECTED FRACTIONAL BROWNIAN MOTION

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Abstract. Let \( \{B_H(t) \colon t \geq 0\} \) be a fractional Brownian motion with Hurst parameter \( H \in (\frac{1}{2}, 1) \). For the storage process \( Q_{BH}(t) = \sup_{-\infty \leq s \leq t} (B_H(t) - B_H(s) - c(t-s)) \) we show that, for any \( T(u) > 0 \) such that \( T(u) = o(u^{2H-1}) \),

\[
P(\inf_{s \in [0, T(u)]} Q_{BH}(s) > u) \sim P(Q_{BH}(0) > u),
\]
as \( u \to \infty \). This finding, known in the literature as the strong Piterbarg property, goes in line with previously observed properties of storage processes with self-similar and infinitely divisible input without Gaussian component.

1. Introduction

The analysis of distributional properties of reflected stochastic processes is continuously motivated both by theory- and applied-oriented open problems in probability theory. In this paper we analyze the asymptotic properties of tail distribution of infimum of an important class of such processes, that naturally appear in models of storage (queueing) systems and, by duality to ruin problems, gained broad interest also in problems arising in finance and insurance risk; see, e.g., [4, 5, 14, 18] or a novel work [10].

Consider a fluid queue with infinite buffer capacity, service rate \( c > 0 \) and the total inflow by time \( t \) modeled by a stochastic process with stationary increments \( X = \{X(t) : t \in \mathbb{R}\} \). Following Reich [20], the stationary storage process that describes the stationary buffer content process, has the following representation

\[
Q_X(t) = \sup_{-\infty \leq s \leq t} (X(t) - X(s) - c(t-s)).
\]

There is a strong motivation for modeling the input process \( X \) by a fractional Brownian motion (fBm) \( B_H = \{B_H(t) : t \in \mathbb{R}\} \) with \( H > 1/2 \), i.e., a centered Gaussian process with stationary increments, continuous sample paths a.s., and variance function \( \sigma_H^2(t) = t^{2H} \). On one hand, such structural properties of fBm as self-similarity and long range dependence, have been statistically confirmed in data analysis of many real traffic processes in modern data-transfer networks. On the other hand, in [13, 22] it was proven that appropriately scaled aggregation of large number of (integrated) On-Off input processes with regularly varying tail distribution of successive On-times, converges to an fBm with \( H > 1/2 \).

The importance of fBm storage processes resulted in a vast interest of analysis of the process \( Q_{BH} \). In particular finding the properties of finite-dimensional (or at least 1-dimensional) distributions of \( Q_{BH} \) has been a long standing goal; see [14, 18]. The stationarity of increments of \( B_H \) implies the stationarity of the process \( Q_{BH} \), so that, for any fixed \( t \), the random variable \( Q_{BH}(t) \) has the same distribution as \( Q_{BH}(0) \). Nevertheless, apart from the Brownian case \( H = \frac{1}{2} \), the exact distribution of \( Q_{BH}(0) \) is not known. Therefore, one usually resorts to the exact asymptotics of \( P(Q_{BH}(0) > u) \), as \( u \to \infty \). These have been found for the full range of parameter \( H \in (0, 1) \) in [11], leading to,

\[
P(Q_{BH}(0) > u) \sim \sqrt{\pi}a^{\frac{1}{2H}}\sup_{t^H}^{\text{sup}}(A u^{1-H}) \frac{1-H}{1-H} \Psi(A u^{1-H}), \quad u \to \infty,
\]
where the constants \( a, b \) and \( A \) can be given explicitly (see Section 4), \( \sup_{t^H}^{\text{sup}} \) is the so-called Pickands constant, and \( \Psi(u) \) denotes the right tail of the standard normal distribution.

Date: April 8, 2014.

2010 Mathematics Subject Classification. Primary: 60F15, 60G70; Secondary: 60G22.

Key words and phrases. Extremes of Gaussian fields, storage processes, fractional Brownian motion.

KK was supported by NWO grant 613.000.701.
Piterbarg \cite{18} considered the supremum of the process $Q_{B_H}$ on the interval $[0, T]$ and found the exact asymptotics of
\[ \mathbb{P}\left( \sup_{t \in [0,T]} Q_{B_H}(t) > u \right) , \quad as \ u \to \infty , \]
for the whole range of the parameter $H$. By comparing them with (1), he observed a remarkable property that, for $H > \frac{1}{2}$, and any positive function $T = T(u)$ such that $T(u) = o(u^{\frac{1}{2H-1}})$,
\begin{equation}
\mathbb{P}\left( \sup_{t \in [0,T]} Q_{B_H}(t) > u \right) \sim \mathbb{P}\left( Q_{B_H}(0) > u \right) , \quad as \ u \to \infty .
\end{equation}
This property is nowadays referred to as the \textit{generalized Piterbarg property}; see \cite{2}. As a corollary from (2) one easily gets that for any fixed $n > 0$ and $t_1, \ldots, t_n \in [0, T]$, with $u \to \infty$,
\[ \mathbb{P}\left( \min_{i=1, \ldots, n} Q_{B_H}(t_i) > u \mid \sup_{t \in [0,T]} Q_{B_H}(t) > u \right) \geq 1 - \frac{1}{n} \left( 1 - \frac{1}{\mathbb{P}\left( \sup_{t \in [0,T]} Q_{B_H}(t) > u \right)} \right) . \]
This leads to the natural question, whether the minimum over finite number of points can be substituted with the infimum functional, which then leads to
\begin{equation}
\mathbb{P}\left( \inf_{t \in [0,T]} Q_{B_H}(t) > u \right) \sim \mathbb{P}\left( \sup_{t \in [0,T]} Q_{B_H}(t) > u \right) , \quad as \ u \to \infty .
\end{equation}
This property shall be referred to as the \textit{strong Piterbarg property}.

The above terminology has been coined by Albin and Samorodnitsky \cite{2}, who, motivated by \cite{18}, considered the case when the input process $X$ belongs to the class of self-similar infinitely divisible stochastic processes with no Gaussian component. They provide general conditions under which (2) and (3) hold with $Q_X$ instead of $Q_{B_H}$. The approach in \cite{2} is based on the assumption that the Lévy measure associated with $X$ has heavy tails, which combined with the absence of a Gaussian component allows for more direct and less delicate methods to be employed. It is the light-tailed nature of the Gaussian distribution that renders the problem of the asymptotics of suprema of Gaussian processes hard. Furthermore, infima of Gaussian processes (apart perhaps from the Brownian case) have not been considered systematically. On the high level, the problem stems from the fact that an infimum is, by definition, an intersection of events. If the number of events grows to infinity, then the intersection is much harder to handle than, for instance, the sum of events (which corresponds to the supremum).

In this paper we derive exact asymptotics of
\begin{equation}
\mathbb{P}\left( \inf_{t \in [0,T]} Q_{B_H}(t) > u \right) , \quad as \ u \to \infty ,
\end{equation}
and prove that the strong Piterbarg property (3) holds for the same range of functions $T(u)$ as in the generalized Piterbarg property (2), i.e., $T(u) = o(u^{\frac{1}{2H-1}})$, $H > \frac{1}{2}$. The idea of the proof is based on finding the exact asymptotics of
\begin{equation}
\mathbb{P}\left( \Phi(X_u) > u \right) , \quad as \ u \to \infty ,
\end{equation}
for a broad class of functionals $\Phi : C(T) \to \mathbb{R}$ acting on the space $C(T)$ of continuous functions on compacts $T \subset \mathbb{R}_+^d$, $d \geq 1$, and a broad class of Gaussian fields $X_u = \{ X_u(t) : t \in \mathbb{R}_+^d \}$. The connection between (4) and (5) can be seen by setting $d = 1$, $\Phi(f) = \inf_{t \in [0,1]} f(t)$ and $X_u(t) = Q_{B_H}(T(u)t)$, although the relation is far from straightforward since $Q_{B_H}$ is not Gaussian.

\textbf{Structure of the paper:} The exact asymptotics of (5) are given in Lemma 1 (see \textit{Section 3}), which is the first contribution of this paper. Interestingly, the asymptotics of (5) involve a new type of constants of the form
\[ \mathcal{H}_{\eta}^\Phi(T) = \mathbb{E} \exp(\Phi(\sqrt{2}\eta(\cdot) - \sigma_{\eta}^2(\cdot))) , \]
where $\eta$ is a Gaussian random field with variance function $\sigma_{\eta}^2$. These new constants extend the notion of the classical Pickands’ constants $\mathcal{H}_{B_H}^{\sup}(S) = \mathbb{E} \exp(\sup_{t \in [0,S]}(\sqrt{2}B_H(t) - t^{2H}))$, $S > 0$, dating back to Pickands \cite{16}. Recall that $\mathcal{H}_{B_H}^{\sup} = \lim_{S \to \infty} \mathcal{H}_{B_H}^{\sup}(0, S) / S$ in (1). In \textit{Theorem 1} (\textit{Section 4}) we give the strong Piterbarg property, which is the second contribution of this paper. More precisely, we show that (3) holds for $H > \frac{3}{2}$ and $T(u) = o(u^{\frac{1}{2H-1}})$, i.e., the same order of functions for which (2) holds. In \textit{Section 5} and \textit{Section 6} we give the proofs of our main results.
2. Notation

Before we begin, let us set the notation that will be used throughout the paper. By \( B_H = \{ B_H(t) : t \in \mathbb{R} \} \) we denote the fBm with Hurst parameter \( H \in (0, 1) \), that is, a Gaussian process with zero mean and covariance function given by

\[
\text{Cov}(B_H(t), B_H(s)) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right).
\]

Let \( \Psi \) be the right tail of the standard normal distribution. Recall that

\[
\Psi(u) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{u^2}{2} \right) \left( 1 + O(u^{-2}) \right), \quad \text{as } u \to \infty.
\]

For any vector \( t \in \mathbb{R}^d \), \( d \geq 1 \), we denote \( t = (t_1, \ldots, t_n) \). By \( \eta = \{ \eta(t) : t \in \mathbb{R}^d \} \) we denote a centered Gaussian field, with almost surely continuous sample paths, \( \eta(0) = 0 \) and variance function \( \sigma^2_H(t) = \text{Var}(\eta(t)) \). Let us introduce the following condition:

**E1:** \( \mathbb{E} (\eta(t_1) - \eta(t_2))^2 \leq G \| t_1 - t_2 \|^\gamma \), for some \( \gamma, G > 0 \) and every \( t_1, t_2 \in \mathbb{R}^d \).

Condition **E1** is a standard regularity requirement; see, e.g., [17]. Now let \( \Phi : C(T) \to \mathbb{R} \) be a functional acting on \( C(T) \), the space of continuous functions on compacts \( T \subset \mathbb{R}^d \), \( d \geq 1 \). Assume that:

**F1:** \( |\Phi(f)| \leq \sup_{t \in T} f(t) \).

**F2:** \( \Phi(af + b) = a\Phi(f) + b \), for every \( a, b > 0 \).

For \( \Phi \) satisfying **F1** we define a constant \( \mathcal{H}_\Phi^\mathbb{T}(T) \) via

\[
\mathcal{H}_\Phi^\mathbb{T}(T) = \text{E} \exp \left( \Phi \left( \sqrt{2}\eta(\cdot) - \sigma^2_H(\cdot) \right) \right).
\]

Note that the dependence on \( T \) is implicit via \( \Phi : C(T) \to \mathbb{R} \). To see that the above constant is well defined, notice that due to **F1**, \( P \left( \Phi \left( \sqrt{2}\eta(\cdot) - \sigma^2_H(\cdot) \right) > u \right) \leq P \left( \sup_{t \in T} \eta(t) > u/\sqrt{2} \right) \). Now since \( \eta \) is continuous, then it has bounded sample paths a.s. and \( \sigma^2_H = \sup_{t \in T} \sigma^2_H(t) < \infty \). Let \( m = \mathbb{E} \sup_{t \in T} \eta(t) \). Borell’s inequality; see, e.g., [1], implies that for \( x > m \), \( P \left( \sup_{t \in T} \eta(t) > x \right) \leq 2 \exp \left( -(x - m)^2/(2\sigma^2_H) \right) \) and, as a consequence, \( \mathcal{H}_\Phi^\mathbb{T}(T) = \int_{-\infty}^\infty e^x P \left( \Phi \left( \sqrt{2}\eta(\cdot) - \sigma^2_H(\cdot) \right) > x \right) \, dx < \infty \).

3. Generalized Pickands’ lemma

In this section we present a lemma that shall play a crucial role in proving the strong Piterbarg property in the remaining part of the paper.

Let us recall that the original Pickands’ lemma [15, 16] concerns with a stationary Gaussian process \( X \) with zero mean and covariance function \( r(t) \) satisfying \( r(t) = 1 - |t|^{2H} + o(|t|^{2H}) \), as \( t \to 0 \), for some \( H \in (0, 1) \), and \( r(t) < 1 \) for all \( t > 0 \). Its conclusion states that, for any \( S > 0 \),

\[
P \left( \sup_{t \in [0,S]} X_u(t) > u \right) \sim \mathcal{H}_{sup}([0,S]) \Psi(u), \quad \text{as } u \to \infty,
\]

where \( X_u(t) = X(u^{-1/2}t) \). Pickands’ lemma has been generalized in various ways, capturing both nonstationarity of \( X \) and extension to Gaussian fields; see, e.g., Piterbarg [17]. Dębicki [6] presented an extension covering broader local covariance structures, than satisfying \( \text{Cov}(X(s), X(t)) = 1 - |s - t|^\alpha + o(|s - t|^\alpha) \) as \( s - t \to 0 \), for some \( \alpha \in (0, 2] \). Among others, notable extensions have been recently considered in [8].

In the following lemma we present a version of Pickands’ lemma that captures the new constant \( \mathcal{H}_{sup}^T(T) \) introduced in the previous section.

**Lemma 1** (Generalized Pickands’ lemma). For any \( u > 0 \), let \( X_u = \{ X_u(t) : t \in \mathbb{R}^d \} \) be a centered Gaussian field with a constant variance equal to one. Let the correlation function \( r_u(t_1, t_2) = \text{Corr}(X_u(t_1), X_u(t_2)) \) satisfy

\[
\lim_{u \to \infty} \sup_{t_1, t_2 \in T} \left| \frac{f^2(u) (1 - r_u(t_1, t_2))}{\text{Var}(\eta(t_1) - \eta(t_2))} - 1 \right| = 0,
\]
for some compact set \( T \subset \mathbb{R}^d_+ \), some function \( f(u) \to \infty \), as \( u \to \infty \), and \( \eta \) satisfying **E1**. Let \( \Phi : C(T) \to \mathbb{R} \) be a functional satisfying **F1-F2**. Then, for any function \( n(u) \) such that \( n(u) \sim f(u) \),

\[
\mathbb{P}(\Phi(X_u) > n(u)) \sim \mathcal{H}^\Phi_{B,H} (T) \Psi(n(u)), \quad as \ u \to \infty.
\]

**Remark 1.** Conditions similar to assumption (8) have been introduced in, among others, [6, 7, 8, 12] as a standard way of capturing nonstationarity. The shape of **Lemma 1** is tailored to the needs of the next section, where asymptotics of tail distribution of \( \inf \sup \) functionals of Gaussian processes are analyzed. Various further extensions of **Lemma 1** can be thought of along the lines of already existing extensions of the classical Pickands’ lemma, especially in the direction allowing nonconstant variance function of the family \( (X_u) \), as in Piterbarg and Prisyazhnyuk [19] or Hashorva et al. [10].

**Example 1.** Assume that \( X = \{X(t) : t \in \mathbb{R}^d_+ \} \) is a centered Gaussian field with unit variance and correlation function satisfying

\[
r(t_1; t_2) = 1 - \sum_{i=1}^d a_i |t_{1,i} - t_{2,i}|^{2H_i} + o \left( \sum_{i=1}^d |t_{1,i} - t_{2,i}|^{2H_i} \right), \quad as \ \sum_{i=1}^d |t_{1,i} - t_{2,i}| \to 0,
\]

for some \( H_i \in (0, 1), \ a_i > 0, \ i = 1, \ldots, d \). Define a new field \( X_u = \{X_u(t) : t \in \mathbb{R}^d_+ \} \) via \( X_u(t) = X(t_1 u^{-\frac{1}{H_1}}, \ldots, t_d u^{-\frac{1}{H_d}}) \). For any compact set \( T \subset \mathbb{R}^d_+ \), the process \( X_u \) satisfies (8) with \( f(u) = u \) and \( \eta(t) = \sum_{i=1}^d B_{H_i} \left( a_i^{2H_i} t_i \right) \), where \( B_{H_i} \) constitute independent fBm’s with Hurst parameters \( H_i \). Hence the conclusion of **Lemma 1** holds for any functional \( \Phi \) on \( C(T) \) satisfying **F1-F2**. In the following section we shall encounter this example in the setting of Theorem 5 and is called the Piterbarg property.

**4. Strong Piterbarg property**

In this section we present the main result of this paper. Let us first recall the definition of the storage process \( Q_{B,H} \) with service rate \( c > 0 \) and input \( B_H \),

\[
Q_{B,H} (t) = \sup_{-\infty \leq s \leq t} \left( B_H(t) - B_H(s) - c(t-s) \right).
\]

Let us define the following constants: \( a = \frac{1}{2} t_0^{-2H}, \ b = \frac{B}{2A}, \ A = \frac{1}{1-H} t_0^{-H}, \ B = H t_0^{-H-2}, \ t_0 = \frac{H}{c(1-H)} \), see (1). Finally, let

\[
(9) \quad \mathcal{H}_{B,H}^\sup = \lim_{S \to \infty} \frac{\mathcal{H}_{B,H}^\sup ([0, S])}{S}
\]

be the classical Pickands’s constant. Now we are in position to state our main result.

**Theorem 1** (Strong Piterbarg property). For \( H > \frac{1}{2} \) and any \( T(u) > 0 \), such that \( T(u) = o(u^{\frac{1-H}{H}}) \),

\[
\mathbb{P} \left( \inf_{t \in [0, T(u)]} Q_{B,H}(t) > u \right) \sim \sqrt{\pi a} b \cdot \mathcal{H}_{B,H}^\sup (Au^{1-H})^{\frac{1}{\tau_0}} \Psi(Au^{1-H}), \quad as \ u \to \infty.
\]

In particular,

\[
\mathbb{P} \left( \inf_{t \in [0, T(u)]} Q_{B,H}(t) > u \right) \sim \mathbb{P} \left( Q_{B,H}(0) > u \right) \sim \mathbb{P} \left( \sup_{t \in [0, T(u)]} Q_{B,H}(t) > u \right), \quad as \ u \to \infty.
\]

**Remark 2.** The asymptotics of \( \mathbb{P} (Q_{B,H}(0) > u) \) were found in [11, Theorem 1]; cf. (1). The asymptotic equivalence between the tail decay of the supremum functional and the value of \( Q_{B,H} \) at 0 was proven in [18, Theorem 5] and is called the Piterbarg property, as mentioned in the introduction; cf. (2). Note that the formula in Piterbarg [18, Theorem 5] should have \( a^{\frac{1}{\tau_0}} \) as cited here instead of \( a^\frac{1}{\tau_0} \).

**Remark 3.** The case of Brownian motion, that is \( H = \frac{1}{2} \), has been treated in [9, Theorem 3]. The authors found the exact distribution of the infimum of \( Q_{B,H} \) attained on any interval of the form \([0, S], \ S > 0\),

\[
\mathbb{P} \left( \inf_{t \in [0, S]} Q_{B,H}(t) > u \right) = \mathbb{P} \left( Q_{B,H}(0) > u \right) \left( 2(1 + S) \Psi(\sqrt{S}) - \sqrt{\frac{2S}{\pi}} \exp \left( -\frac{S}{2} \right) \right).
\]
Let us recall that $Q_{B^{1/2}}(0)$ has $\frac{1}{2}$-exponential distribution. On the other hand, [18, Theorem 6], gives (note that the original formula in [18] has a misprint)

$$\mathbb{P}\left(\sup_{t \in [0,S]} Q_{B^{1/2}}(t) > u\right) \sim \mathbb{P}\left(Q_{B^{1/2}}(0) > u\right) 2\sqrt{\mathbb{H}^{2/4}(0,2S)}$$, as $u \to \infty$.

Therefore, we see that the strong Piterbarg property does not hold in the case of $H = \frac{1}{2}$.

**Remark 4.** One can envision that the strong Piterbarg property can be applied to functionals $\Phi : C([0,T]) \to \mathbb{R}$ of $Q_{B^H}$, that can be majorized, up to the same magnitude, by the infimum and supremum functionals. A simple example is the integral functional. **Theorem 1** yields, for every $H > \frac{1}{2}$,

$$\mathbb{P}\left(\int_0^T(u) Q_{B^H}(t) dt > u\right) \sim \mathbb{P}\left(Q_{B^H}(0) > \frac{u}{T(u)}\right),$$

for every function $T(u) > 0$ such that $T(u) = \alpha(u \frac{\sqrt{\mathbb{H}^{2/4}}}{1})$. The problem of the area under the graph of the storage process fed by the Brownian motion, i.e., the case when $H = \frac{1}{2}$, has been considered in [3].

5. **Proof of Lemma 1**

The general idea behind the proof follows the one in Piterbarg [17, Lemma D.2]. For any $u > 0$,

$$\mathbb{P}(\Phi(X_u) > n(u)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left( -\frac{v^2}{2} \right) \mathbb{P}(\Phi(X_u) > n(u) | X_u(0) = v) dv

\sim \Psi(n(u)) \int_{\mathbb{R}} \exp \left( -\frac{w^2}{2n^2(u)} \right) \mathbb{P}(\Phi(X_u) > n(u) | X_u(0) = n(u) - \frac{w}{n(u)}) dw,$

where we have used the change of variable $v = n(u) - \frac{w}{n(u)}$. Let $\zeta_u = \{\zeta_u(t) : t \in T\}$ be a Gaussian field defined via $\zeta_u(t) = n(u)(X_u(t) - n(u)) + w$. Then, using F2, the last integral can be written as

$$\int_{\mathbb{R}} \exp \left( w - \frac{w^2}{2n^2(u)} \right) \mathbb{P}(\Phi(\zeta_u) > w | \zeta_u(0) = 0) dw = \int_{\mathbb{R}} \exp \left( w - \frac{w^2}{2n^2(u)} \right) \mathbb{P}(\Phi(\zeta_u) > w) dw,$$

where $\chi_u = \{X_u(t) : t \in T\}$ is a Gaussian field defined as $\chi_u(t) = \chi(t)|\zeta_u(0) = 0$. For the family of Gaussian distributions that appear inside the integral, for every $t \in T$,

$$\mathbb{E} \chi_u(t) = n(u) \mathbb{E} \left( X_u(t) | X_u(0) = n(u) - \frac{w}{n(u)} \right) - n^2(u) + w

\mathbb{E} \chi_u(0) = \mathbb{E} \chi_u(t) = 0.$$

Furthermore, for any $t_1, t_2 \in T$,

$$\text{Var} (\chi_u(t_1) - \chi_u(t_2)) = n^2(u) \left( \text{Var} \left( (X_u(t_1) - X_u(t_2)) | X_u(0) = u - \frac{w}{u} \right) \right)

= 2n^2(u)(1 - r_u(t_1,t_2)) - n^2(u) (r_u(t_1;0) - r_u(t_2;0))^2.$$

Hence from (8) it follows that, as $u \to \infty$, uniformly on $T$,

$$\mathbb{E} \chi_u(t) \to -\sigma^2(t),$$

$$\text{Var} (\chi_u(t_1) - \chi_u(t_2)) \to 2 \text{Var}(\eta(t_1) - \eta(t_2)).$$

Thus the finite dimensional distributions of $\chi_u$ converge to the finite dimensional distributions of $\tilde{\eta} = \{\sqrt{2n(t)} - \sigma^2(t) : t \in T\}$. Therefore $\chi_u \to \tilde{\eta}$ in $C(T)$, as $u \to \infty$, provided that the family $\chi = \{X_u : u > 0\}$ is tight. For this let $\chi^o_u = \{\chi^o_u(t) : t \in T\}$ be a centered Gaussian field defined by $\chi^o_u(t) = \chi_u(t) - \mathbb{E} \chi_u(t)$. In order to prove tightness of the family $\chi = \{X_u : u > 0\}$ it suffices to show tightness of the centered family $\chi^o = \{\chi^o_u : u > 0\}$. Since $\chi^o_u(0) = 0$ for all $u > 0$, then a straightforward consequence of Straf’s criterion for tightness of Gaussian fields, [21], implies that it suffices to show that for any $\mu, \rho > 0$, there exists $\delta \in (0,1)$ and $u_0 > 0$ such that, for each $t_1 \in T$ and $u > u_0$,

$$\mathbb{P}\left( \sup_{\|t_1 - t_2\| \leq \delta} |\chi^o_u(t_1) - \chi^o_u(t_2)| \geq \mu \right) \leq \rho \delta$$.
where \( \|t\| = \max\{|t_1|, \ldots, |t_d|\} \). Note that, for sufficiently large \( u \),
\[
\mathbb{E}(\chi_u^\nu(t_1) - \chi_u^\nu(t_2))^2 \leq C \mathbb{V} \mathbb{a} \mathbb{r}(\eta(t_1) - \eta(t_2)),
\]
for all \( t_1, t_2 \in T \) and some constant \( C > 0 \). Thus, the assumption \( E1 \) implies,
\[
\sup_{\|t_1 - t_2\| \leq \delta} \mathbb{V} \mathbb{a} \mathbb{r}(\chi_u^\nu(t_1) - \chi_u^\nu(t_2)) \leq CG\delta^\gamma,
\]
which combined with the application of Borell’s inequality gives (13).

Then, the continuous mapping theorem implies
\[
(14) \lim_{u \to \infty} \int_{\mathbb{R}} \exp \left( -\frac{u^2}{2m^2} \right) \mathbb{P}(\Phi(\chi_u) > w) \, dw = \int_{\mathbb{R}} \exp(w) \mathbb{P}(\Phi(\sqrt{2}\eta(\cdot) - \sigma^2(\cdot)) > w) \, dw = \mathbb{E} \exp \left( \Phi(\sqrt{2}\eta(\cdot) - \sigma^2(\cdot)) \right) = \mathcal{H}^\nu_T(T),
\]
provided we can interchange the limit with the integral in (14). From (8) it follows that \((1 - r_u(t; 0)) \to 0\) uniformly in \( t \in T \), therefore (10)–(11) imply that for any \( \varepsilon > 0 \) and sufficiently large \( u \),
\[
w_u := \sup_{t \in T} \mathbb{E} \chi_u^\nu(t) \leq \varepsilon |w|.
\]
Using (12) combined with Sudakov–Fernique’s inequality yields, for sufficiently large \( u \) and some constant \( C > 0 \),
\[
m_u := \mathbb{E} \sup_{t \in T} \chi_u^\nu(t) \leq C \mathbb{E} \sup_{t \in T} \eta(t) =: m.
\]
Furthermore, (12) combined with \( E1 \) implies, for sufficiently large \( u \),
\[
\sigma_u^2 := \sup_{t \in T} \mathbb{V} \mathbb{a} \mathbb{r}(\chi_u^\nu(t)) \leq C \sup_{t \in T} \sigma^2(t) \leq CG(\text{diam}(T))^{\gamma}.
\]
Now, by \( F1 \), Borell’s inequality yields, for \(|w| (1 - \varepsilon) \geq m \),
\[
\mathbb{P}(\Phi(\chi_u) > w) \leq \mathbb{P}\left( \sup_{t \in T} \chi_u^\nu(t) > w - w_u \right) \leq \mathbb{P}\left( \sup_{t \in T} \chi_u^\nu(t) - m_u > w - \varepsilon |w| - m_u \right) \leq 2 \exp \left( -\frac{(w - \varepsilon |w| - m_u)^2}{2\sigma^2_u} \right) \leq 2 \exp \left( -\frac{(w - \varepsilon |w| - m)^2}{2CG(\text{diam}(T))^{\gamma}} \right).
\]
Hence the interchange of the limit with the integral in (14) follows by the dominated convergence theorem and the limit is finite, that is \( \mathcal{H}^\nu_T(T) < \infty \).

This completes the proof of Lemma 1.

6. Proof of Theorem 1

We divide the proof on a number of steps. Before we proceed, let us make the following observation. The time-reversibility property of fBm implies that (on the process level)
\[
Q_{B_H}(t) \overset{\Delta}{=} \sup_{\sigma \geq t} (B_H(\sigma) - B_H(t) - c(\sigma - t)),
\]
which is the form of \( Q_{B_H} \) that we shall use in this section. The relations of Section 6.1 and Section 6.2 were derived in [18].

6.1. Reduction to a Gaussian field. Using new variables \( \tau = (\sigma - t)/u \) and \( s = t/u \), for any \( T > 0 \),
\[
\mathbb{P}\left( \inf_{t \in [0, T]} Q_{B_H}(t) > u \right) = \mathbb{P}\left( \inf_{t \in [0, T]} \sup_{\sigma \geq t} (B_H(\sigma) - B_H(t) - c(\sigma - t)) > u \right) = \mathbb{P}\left( \forall s \in \left[0, \frac{T}{u}\right] \exists \tau \geq 0 : B_H(u(s + \tau)) - B_H(su) > u + cu\tau \right) = \mathbb{P}\left( \inf_{s \in [0, T/u^{-1}]} \sup_{\tau \geq 0} \frac{B_H(u(s + \tau)) - B_H(su)}{\tau^{H}\text{u}^H\nu(\tau)} > u^{1-H} \right) = \mathbb{P}\left( \inf_{s \in [0, T/u^{-1}]} \sup_{\tau \geq 0} Z_u(s, \tau) > u^{1-H} \right),
\]
where \( \nu(\tau) = \tau^{-H + ct^{-H}} \) and \( Z_u = \{ Z_u(s, \tau) : s, \tau \geq 0 \} \) is a Gaussian field given by
\[
Z_u(s, \tau) = \frac{B_H(u(s + \tau)) - B_H(su)}{\tau^{H}\text{u}^H\nu(\tau)}.
\]
The distribution of $Z_u$ does not depend on $u$, hence we deal with $Z = Z_1$.  Note that $Z(s, \tau)$ is stationary in $s$, but not in $\tau$.

6.2. Correlation structure of $Z$. The variance $\sigma^2_Z(\tau)$ of $Z(s, \tau)$ equals $\nu^{-2}(\tau)$ and has a single maximum point at $\tau_0 = \frac{H}{c(1-H)}$. Taylor expansion shows that, as $\tau \to \tau_0$,

$$\sigma_Z(\tau) = \frac{1}{A} - \frac{B}{2A^2}(\tau - \tau_0)^2 + O((\tau - \tau_0)^3),$$

where

$$A = \frac{1}{1 - H} \left( \frac{H}{c(1-H)} \right)^{-H} = \nu(\tau_0),$$

$$B = H \left( \frac{H}{c(1-H)} \right)^{-H-2} = \nu''(\tau_0).$$

Furthermore, denote $a = \frac{1}{2} \tau_0^{-2H}$ and $b = \frac{B}{2a}$. Note that $\tau_0, A, B, a, b$ are the same constants as in Section 4.

The correlation function $r(s_1, \tau_1; s_2, \tau_2)$ of $Z$ equals

$$r(s_1, \tau_1; s_2, \tau_2) = EZ(s_1, \tau_1)Z(s_2, \tau_2)\nu(\tau_1)\nu(\tau_2)$$

$$= \frac{|s_1 - s_2 + \tau_1|^{2H} + |s_1 - s_2 - \tau_1|^{2H} - |s_1 - s_2 + \tau_2|^{2H} - |s_1 - s_2 - \tau_2|^{2H}}{2^{1+H}|\tau_1|^{1+H}}$$

$$= 1 - a(1 + o(1)) \left( |s_1 - s_2 + \tau_1|^{2H} + |s_1 - s_2|^{2H} \right)$$

as $s_1 - s_2 \to 0, \tau_1 \to \tau_0, \tau_2 \to \tau_0$.

6.3. Asymptotic properties of $Z$. In this step we will be concerned with the asymptotic properties of

$$\Pr \left( \inf_{s \in [0,T]} \sup_{\tau \geq 0} AZ(s, \tau) > u \right)$$

as $u$ grows to infinity. Note that we normalized $Z$ such that now the variance of $AZ(s, \tau)$ equals one at $\tau = \tau_0$ ($Z$ is stationary in $s$). It follows from [18, Lemma 1] that there exists a constant $C$ such that, for any $T > 0$ and sufficiently large $u$,

$$\Pr \left( \inf_{s \in [0,T]} \sup_{|\tau - \tau_0| \geq \log u/u} AZ(s, \tau) > u \right) \leq CT a^{2H} \exp \left( -\frac{1}{2} u^2 - b \log^2 u \right).$$

If we restrict ourselves to the neighborhood $\{\tau : |\tau - \tau_0| \leq \log u/u\}$ of $\tau_0$, then the following step shows that the probability in (17), with $Z$ restricted to the neighborhood of $\tau_0$, on the logarithmic scale decays as $-u^2$ when $u$ grows large. Therefore, the neighborhood of $\tau_0$ has the largest contribution to the asymptotic behavior of (17). In the following step we present its asymptotic contribution.

6.4. The asymptotics of the main contributor. In this step we show that for any $\lambda > 0$, with $H_{B_H}$ defined in (9),

$$\liminf_{u \to \infty} \frac{\Pr \left( \inf_{s \in [0,\lambda u^{-1/H}]} \sup_{|\tau - \tau_0| \leq \log u/u} AZ(s, \tau) > u \right)}{\sqrt{\lambda u^{1-2H}} b(\lambda, H_{B_H} H_{\eta} \left[ \log u \middle/ u \right])} \geq 1.$$

For the Gaussian field $X(s, \tau) = AZ(s, \tau - s)$, we have

$$\Pr \left( \inf_{s \in [0,\lambda u^{-1/H}]} \sup_{|\tau - \tau_0| \leq \log u/u} AZ(s, \tau) > u \right) \geq \Pr \left( \inf_{s \in [0,\lambda u^{-1/H}]} \sup_{\tau \in I} X(s, \tau) > u \right),$$

for sufficiently large $u$, where $I := [\tau_0 - \frac{\log u}{u}, \tau_0 + \frac{\log u}{u}]$ (we use that $I \subset [\tau_0 + s - \frac{\log u}{u}, \tau_0 + s + \frac{\log u}{u}]$ for sufficiently large $u$). From (16) it follows that the correlation function $r_X$ of $X$ is given by

$$r_X(s_1, \tau_1; s_2, \tau_2) = 1 - a(1 + o(1)) \left( |\tau_1 - \tau_2|^{2H} + |s_1 - s_2|^{2H} \right)$$

as $s_1 - s_2 \to 0, \tau_1 - s_1 \to \tau_0, \tau_2 - s_2 \to \tau_0$. Furthermore, (15) implies that the variance function $\sigma^2_X$ of $X$ satisfies

$$\sigma_X(s, \tau) = 1 - b(\tau - s - \tau_0)^2 + O((\tau - s - \tau_0)^3),$$
as $\tau - s \to \tau_0$.

Let us divide the interval $[\tau_0 - \frac{\log u}{\sqrt{2}u}, \tau_0 + \frac{\log u}{\sqrt{2}u}]$ into intervals of length $\gamma u^{-\frac{1}{2}}$ for some fixed $\gamma > 0$,

$$I_k = [\tau_0 - k\gamma u^{-\frac{1}{2}}, \tau_0 + (k + 1)\gamma u^{-\frac{1}{2}}], \quad k = 0, 1, 2, \ldots,$$

$$I_{-k} = [\tau_0 - (k + 1)\gamma u^{-\frac{1}{2}}, \tau_0 - k\gamma u^{-\frac{1}{2}}], \quad k = 0, 1, 2, \ldots,$$

Notice that,

$$\mathbb{P} \left( \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_k} X(s, \tau) > u \right)$$

$$\geq \mathbb{P} \left( \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \max_{k = -[\gamma^{-1} u^{-\frac{1}{2}}] \ldots [\gamma^{-1} u^{-\frac{1}{2}}]} \sup_{\tau \in I_k} X(s, \tau) > u \right)$$

$$\geq \mathbb{P} \left( \max_{k = -[\gamma^{-1} u^{-\frac{1}{2}}] \ldots [\gamma^{-1} u^{-\frac{1}{2}}]} \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_k} X(s, \tau) > u \right)$$

$$\geq 2 \sum_{k=0}^{[\gamma^{-1} u^{-\frac{1}{2}}]} \mathbb{P} \left( \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_k} X(s, \tau) > u \right)$$

$$- 2 \sum_{0 \leq l < k \leq [\gamma^{-1} u^{-\frac{1}{2}}]} \mathbb{P} \left( \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_k} X(s, \tau) > u, \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_l} X(s, \tau) > u \right)$$

$$- \mathbb{P} \left( \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_{-0}} X(s, \tau) > u, \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_0} X(s, \tau) > u \right).$$

Now, for any $\varepsilon > 0$, any $s \in [0, \lambda u^{-\frac{1}{2}}]$ and all $\tau \in I_{k+}$, for sufficiently large $u$,

$$1 - (b + \varepsilon)(k + 1)^2 \gamma^2 u^{-\frac{1}{2}} \leq \sigma_X(s, \tau) \leq 1 - b(1 - \varepsilon)k^2 \gamma^2 u^{-\frac{1}{2}}.$$

Therefore, with $\bar{X}(s, \tau) = X(s, \tau)/\sigma_X(s, \tau)$,

$$\mathbb{P} \left( \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_k} \bar{X}(s, \tau) > u \right) \geq \mathbb{P} \left( \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_{k+}} \bar{X}(s, \tau) > u_{k+} \right),$$

where

$$u_{k+} = \frac{u}{1 - (b + \varepsilon)(k + 1)^2 \gamma^2 u^{-\frac{1}{2}}}.$$

Thus by Example 1, as $u \to \infty$,

$$2 \sum_{k=0}^{[\gamma^{-1} u^{-\frac{1}{2}}]} \mathbb{P} \left( \inf_{s \in [0, \lambda u^{-\frac{1}{2}}]} \sup_{\tau \in I_k} \bar{X}(s, \tau) > u \right)$$

$$\geq 2(1 + o(1)) \sum_{k=0}^{[\gamma^{-1} u^{-\frac{1}{2}}]} \mathcal{H}_{B_{H}}^\text{inf}(0, \lambda u^{-\frac{1}{2}}) \mathcal{H}_{B_{H}}^\text{sup}(0, \lambda u^{-\frac{1}{2}}) \Psi(u_{k+}).$$

Notice that (cf. (6)), as $u \to \infty$,

$$\sum_{k=0}^{[\gamma^{-1} u^{-\frac{1}{2}}]} \Psi(u_{k+}) \sim \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{[\gamma^{-1} u^{-\frac{1}{2}}]} \frac{1}{u_{k+}} e^{-\frac{u_{k+}^2}{2}}.$$
Furthermore, as $u \to \infty$, 
\[
\frac{1}{\sqrt{2\pi}} \sum_{k=0}^{[\gamma^{-1} u^{\frac{1-H}{2}}]} \frac{1}{u_{k+}} e^{-\frac{u^{2}}{2}} = \frac{1}{u\sqrt{2\pi}} \sum_{k=0}^{[\gamma^{-1} u^{\frac{1-H}{2}}]} (1 - (b + \epsilon)(k + 1)^2 \gamma^2 u^{-\frac{H}{2}}) 
\times \exp \left( \frac{u^2}{2(1 - (b + \epsilon)(k + 1)^2 \gamma^2 u^{-\frac{H}{2}})^2} \right)
\]
\[
= \frac{1}{u\sqrt{2\pi}} \sum_{k=0}^{[\gamma^{-1} u^{\frac{1-H}{2}}]} \exp \left( \frac{-u^2}{2(1 - (b + \epsilon)(k + 1)^2 \gamma^2 u^{-\frac{H}{2}})^2} \right) (1 + o(1))
\]
\[
= \frac{1}{u\sqrt{2\pi}} \sum_{k=0}^{[\gamma^{-1} u^{\frac{1-H}{2}}]} \exp \left( \frac{-u^2}{2} \sum_{k=0}^{[\gamma^{-1} u^{\frac{1-H}{2}}]} \exp \left( \frac{-u^2(1 + (b + \epsilon)(k + 1)^2 \gamma^2 u^{-\frac{H}{2}})^2}{(1 - (b + \epsilon)^2(k + 1)^4 \gamma^4 u^{-\frac{H}{2}})^2} \right) (1 + o(1))
\]
\[
= \Psi(u) \sum_{k=0}^{[\gamma^{-1} u^{\frac{1-H}{2}}]} \exp \left( -(b + \epsilon)(k + 1)^2 \gamma^2 u^{-\frac{H}{2}} \right) (1 + o(1))
\]
\[
= \Psi(u) u^{\frac{1}{H} - 1} \sum_{k=0}^{[\gamma^{-1} u^{\frac{1-H}{2}}]} u^{1 - \frac{H}{2}} \exp \left( -(b + \epsilon)\gamma^2 \left( (k + 1)u^{-\frac{1}{2}} \right)^2 \right) (1 + o(1))
\]
\[
= \Psi(u) u^{\frac{1}{H} - 1} \int_0^\infty \exp \left( -(b + \epsilon)\gamma^2 x^2 \right) dx (1 + o(1))
\]
and 
\[
\int_0^\infty \exp \left( -(b + \epsilon)\gamma^2 x^2 \right) dx = \frac{\sqrt{\pi}}{2\gamma \sqrt{b + \epsilon}}
\]
Combining these estimates we obtain 
\[
2 \sum_{k=0}^{[\gamma^{-1} u^{\frac{1-H}{2}}]} \mathbb{P} \left( \inf_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_k} X(s, \tau) > u \right)
\]
\[
\geq 2H_{BH}^{\inf}(\mathbb{[0, 1] u^{\frac{1-H}{2}}}) H_{BH}^{\sup}(\mathbb{[0, 1] u^{\frac{1-H}{2}}}) \Psi(u) u^{\frac{1}{H} - 1} \frac{\sqrt{\pi}}{2\gamma \sqrt{b + \epsilon}} (1 + o(1)),
\]
which, by the fact that $\epsilon, \gamma > 0$ are arbitrary and $\lim_{S \to \infty} \frac{1}{S} \mathcal{H}_{BH}^{\sup}(\mathbb{[0, S]}) = \mathcal{H}_{BH}^{\sup}$, yield 
\[
2 \sum_{k=0}^{[\gamma^{-1} u^{\frac{1-H}{2}}]} \mathbb{P} \left( \inf_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_k} X(s, \tau) > u \right) \geq H_{BH}^{\inf}(\mathbb{[0, 1] u^{\frac{1-H}{2}}}) H_{BH}^{\sup}(\mathbb{[0, 1] u^{\frac{1-H}{2}}}) \Psi(u) u^{\frac{1}{H} - 1} \frac{\sqrt{\pi}}{2\gamma \sqrt{b + \epsilon}} (1 + o(1)).
\]
Finally, note that 
\[
2 \sum_{0 \leq t < k \leq [\gamma^{-1} u^{\frac{1-H}{2}}]} \mathbb{P} \left( \inf_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_k} X(s, \tau) > u, \inf_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_t} X(s, \tau) > u \right)
\]
\[
+ \mathbb{P} \left( \inf_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_{t-a}} X(s, \tau) > u, \inf_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_t} X(s, \tau) > u \right)
\]
\[
\leq 2 \sum_{0 \leq t < k \leq [\gamma^{-1} u^{\frac{1-H}{2}}]} \mathbb{P} \left( \sup_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_k} X(s, \tau) > u, \sup_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_t} X(s, \tau) > u \right)
\]
\[
+ \mathbb{P} \left( \sup_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_{t-a}} X(s, \tau) > u, \sup_{s \in [0, 1] u^{\frac{1-H}{2}}} \sup_{\tau \in I_t} X(s, \tau) > u \right).
\]
It has been shown in [18, end of the proof of Lemma 3], that the last expression is of a smaller order than \( u^{-1} \Psi'(u) \), which completes the proof of this step.

6.5. Derivation of the asymptotics. Recall from Section 6.1 that, for any \( T > 0 \),

\[
P(u) := P \left( \inf_{t \in [0,T]} Q_{B_H}(t) > u \right) \leq P \left( \inf_{s \in [0,T]A} \sup_{\tau \geq 0} AZ(s,\tau) > Au^{1-H} \right).
\]

Theorem 1 is a simple reformulation of the observations of the previous steps in terms of the storage process \( Q_{B_H} \). We have,

\[
\left[ 0, TA^{\frac{1}{1-H}}(Au^{1-H})^{-\frac{1}{1-H}} \right] = \left[ 0, \lambda(u)(Au^{1-H})^{-\frac{1}{1-H}} \right],
\]

where \( \lambda(u) = TA^{\frac{1}{1-H}}u^{-\frac{2}{1-H}} \). Let \( T = T(u) \) be such that \( T(u) = o(u^{\frac{2}{1-H}}) \) as \( u \to \infty \). Then, for any \( \varepsilon > 0 \) and all \( u \) such that \( \lambda(u) \leq \varepsilon \),

\[
\left[ 0, TA^{\frac{1}{1-H}}(Au^{1-H})^{-\frac{1}{1-H}} \right] \subset \left[ 0, \varepsilon(Au^{1-H})^{-\frac{1}{1-H}} \right].
\]

Hence,

\[
P(u) \geq P \left( \inf_{s \in [0,\varepsilon(Au^{1-H})^{-\frac{1}{1-H}}] |\tau - \tau_0| \leq \log(Au^{1-H})/(Au^{1-H})} \sup_{\tau \geq 0} AZ(s,\tau) > Au^{1-H} \right)
\]

and by (18) the last expression is asymptotically bounded below by

\[
\sqrt{\pi}a^{\frac{1}{1-H}}b^{-\frac{1}{2}}H_{B_H}^{\sup}H_{B_H}^{\inf} \left( [0,\varepsilon a^{\frac{1}{1-H}}] \right) (Au^{1-H})^{-\frac{1}{1-H}} \Psi(Au^{1-H}).
\]

Observe that by Fatou’s lemma \( \limsup_{t \to 0} H_{B_H}^{\inf}([0,\varepsilon a^{\frac{1}{1-H}}]) = 1 \), which implies the appropriate lower bound for \( P(u) \). Finally, recall from (1), that

\[
P \left( Q_{B_H}(0) > u \right) \sim \sqrt{\pi}a^{\frac{1}{1-H}}b^{-\frac{1}{2}}H_{B_H}^{\sup} (Au^{1-H})^{\frac{1}{1-H}} \Psi(Au^{1-H}), \text{ as } u \to \infty,
\]

which is the upper bound for \( P(u) \). This completes the proof of Theorem 1.

Acknowledgement: K. Dębicki was partially supported by NCN Grant No 2013/09/B/ST1/01778 (2014-2016) and by the project RARE -318984, a Marie Curie FP7 IRSES Fellowship.

References

[1] R.J. Adler. An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes, volume 12 of Lecture Notes-Monograph Series, IMS, 1990.
[2] J. Albin and G. Samorodnitsky. On overload in a storage model, with a self-similar and infinitely divisible input. Ann. Appl. Probab., 14:820–844, 2004.
[3] M. Arendarczyk, K. Dębicki, and M. Mandjes. On the tail asymptotics of the area swept under the Brownian storage graph. Bernoulli, 2013.
[4] S. Asmussen. Applied Probability and Queues. Springer, 2nd edition, 2003.
[5] S. Asmussen and H. Albrecher. Ruin Probabilities. World Scientific Publishing Co. Inc., 2nd edition, 2010.
[6] K. Dębicki. Ruin probability for Gaussian integrated processes. Stochastic Process. Appl., 98:151–174, 2002.
[7] K. Dębicki and P. Kissowski. Asymptotics of supremum distribution of \((a(t),a(t))-locally stationary Gaussian processes. Stochastic Process. Appl., 118:2022–2037, 2008.
[8] K. Dębicki and K. Tabiś. Extremes of time-average stationary Gaussian processes. Stochastic Process. Appl., 121:2049–2063, 2011.
[9] K. Dębicki, K.M. Kosiński, and M. Mandjes. On the infimum attained by a reflected Lévy process. Queueing Syst., 70:23–35, 2012.
[10] E. Hashorva, L. Ji, and V.I. Piterbarg. On the supremum of \( \gamma \)-reflected processes with fractional Brownian motion as input. Stochastic Process. Appl., 2013, in press.
[11] J. Hüsler and V.I. Piterbarg. Extremes of a certain class of Gaussian processes. Stochastic Process. Appl., 83:257–271, 1999.
[12] J. Hüsler and V.I. Piterbarg. On the ruin probability for physical fractional brownian motion. Stochastic Process. Appl., 113:315–332, 2004.
[13] T. Mikosch, S. Resnick, H. Rootzén, and A. Stegeman. Is network traffic approximated by stable Lévy motion or fractional Brownian motion? Ann. Appl. Probab., 12:23–68, 2002.
[14] I. Norros. A storage model with selfsimilar input. Queueing Syst., 16:387–396, 2004.
[15] J. Pickands, III. Asymptotic properties of the maximum in a stationary Gaussian process. *Trans. Amer. Math. Soc.*, 145:75–86, 1969.

[16] J. Pickands, III. Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.*, 145:51–73, 1969.

[17] V.I. Piterbarg. *Asymptotics Methods in the Theory of Gaussian Processes and Fields*, volume 148 of *Translation of Mathematical Monographs*. AMS, 1996.

[18] V.I. Piterbarg. Large deviations of a storage process with fractional Brownian motion as input. *Extremes*, 4:147–164, 2001.

[19] V.I. Piterbarg and V Prisyazhnyuk. Asymptotic behavior of the probability of a large excursion for a nonstationary gaussian processes. *Theory of Probability and Mathematical Statistics*, pages 121–133, 1978.

[20] E. Reich. On the integrodifferential equation of Takács I. *Ann. Math. Stat.*, 29:563–570, 1958.

[21] M.L. Straf. Weak convergence of stochastic processes with several parameters. In *Proceedings of the Sixth Berkeley Symposium in Mathematical Statistics and Probability*, volume 2, pages 187–221, 1972.

[22] M. S. Taqqu, W. Willinger, and R. Sherman. Proof of a fundamental result in self-similar traffic modeling. *Comp. Comm. Review*, 27:5–23, 1997.

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