SEMISTABLE 3-FOLD FLIPS

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1 Introduction. In this short note, I will give a proof of the existence of
semistable 3-fold flips, which does not use the classification of log terminal (i.e.,
quotient) surface singularities. This permits us to avoid a calculation, which is a
sort of logical bottleneck in all the existing approaches to semistable 3-fold flips
([Ka1 pg. 158–159], [Sh pg. 386–389], [Ka3 pg. 483–486]), however different they
may look. The message is that Shokurov’s main reduction step, as refined in [FA,
Ch. 18], can be used profitably in the semistable case also. My main motivation was
to develop an approach to semistable flips that would have some fighting chances
in dimension 4, and the present paper is a first (small) step in that direction.

We always work over an algebraically closed field of characteristic zero.

1.1 Definition. Let $X$ be a normal projective variety, $B \subset X$ a $\mathbb{Q}$-Weil divisor
such that $K_X + B$ is log terminal (this notion is recalled in 2.2). Let $R \subset \overline{NE}(X)
be an extremal ray with $(K + B) \cdot R < 0$, $\varphi_R : X \to U$ the contraction of $R$. $\varphi_R$ is
said to be a flipping contraction if the $\varphi_R$-exceptional locus has codimension $\geq 2$.

A flip of $\varphi_R$ is by definition a variety $X^+$, together with a morphism $\varphi^+ : X^+ \to U$, such that $K^+ + B^+$ is $\mathbb{Q}$-Cartier and $\varphi^+$-ample.

It is easy to see that the flip is unique if it exists. It is not so easy, but true, that
$K^+ + B^+$ is log terminal (see 2.4). The flip conjecture asserts that flips exist and
that there is no infinite sequence of them. An important special class of flips is that
of semistable flips. These are the flips that appear in the minimal model program
for a semistable family. In short, one is given the additional structure of a projective
morphism $f : X \to \Delta$, where $\Delta = \text{Spec} \mathcal{O}$ is the spectrum of a discrete valuation
ring $\mathcal{O}$, with central and generic points $0, \eta \in \Delta$. We denote $X_0, X_\eta$ the fibers
over $0, \eta$. All extremal rays, contractions, flips, are compatible with this structure.
The starting point is a semistable family in the sense of Mumford, but the minimal
model program will soon introduce singularities, which we call semistable terminal
singularities (see 2.5). These are all terminal, but an important point is that not all terminal singularities occur.

1.2 Definition. Let $f : X \to \Delta$ be as above, $\varphi_R : X \to U$ a semistable flipping contraction. We say that $\varphi_R$ is a special semistable flipping contraction if there exists a component $S \subset X_0$ such that $S \cdot R \neq 0$.

In this paper we prove the following:

1.3 Theorem. Semistable 3-fold flips exist if special semistable 3-fold flips exist.

This is satisfactory because special semistable flips are easier to construct. Since I am unable to improve upon existing treatments, I will limit myself to quickly sketching two constructions. Both these constructions are based on the following:

1.4 Lemma. Let $\varphi_R : X \to U$ be a special semistable 3-fold flipping contraction. Then, in a neighborhood of every positive dimensional fiber of $\varphi_R$, there exists a (reduced) surface $B \in | - K_X |$ having Du Val singularities only.

Proof. By assumption, there is a component $S_0$ of $S = X_0$ with $S_0 \cdot R \neq 0$. But $S \sim 0$, so in fact there are components $S_1, S_2$ of $S$ with $S_1 \cdot R < 0$ and $S_2 \cdot R > 0$. By [FA, 19.11], there is $B \in | - K - S | = | - K |$ such that $K + S + B$ is log canonical (in the language of Shokurov and [FA], such a $B$ is called 1-complement). But $S$ is a Cartier divisor, so $K + B$ is canonical, and so is $K_B = K + B |_B$. □

Let $\mathcal{H} = | - K_X |$. A way to formulate 1.4 is to say that $K + \mathcal{H}$ is canonical in a neighborhood of the flipping locus. I now outline the two known methods to construct the flip:

1.5 Use [Ka1, 8.7]: take a double cover $X' \to X$ branched along two general members of $\mathcal{H}$. Then $X'$ has canonical singularities and $K_X', \sim 0$. The flip of $X$ is a $Z/2$ quotient of the flop of $X'$. In [Ka1], canonical flops are reduced to terminal flops with the crepant descent method, generalized in [FA, Ch 6].

1.6 This method is a private communication from S. Mori, related to work of his student T. Hayakawa, and it allows, more generally, to construct canonical $K + \mathcal{H}$-flops (these may be flips, flops or inverse flips). One uses the crepant descent method in the form given in [FA, Ch 6]. The proof is by induction on $e(X)$, the number of $K + \mathcal{H}$-crepant valuations, the starting point is a crepant divisorial contraction $h : X_1 \to X$, then $e(X_1) = e(X) - 1$ and by induction one is reduced to the case $e = 0$, where the flop is a genuine $K$-flop on a 3-fold with terminal singularities of index 1. It seems that one needs some knowledge about terminal singularities in order to construct the blow up $h$.

In both methods, the flip is reduced to a flop. Flops are very difficult, their existence rests upon: 1) the very hard implication terminal Gorenstein $\Rightarrow$ isolated
cDV (this is not an issue in the semistable case) and 2) simultaneous resolution of surface Du Val singularities.

It would be very interesting to find a better way to use the 1-complement $B$ of $1.4$.

1.7 Remark. If $\varphi : X \to U$ is a semistable 3-fold flip, and more generally for every 3-fold flip, $K + \mathcal{H}$ is canonical in a neighborhood of the flipping locus. In fact, Mori’s proof of the existence of 3-fold flips consists in establishing the above statement. We are, of course, with our method, unable to see this even for semistable flips.

After some preliminaries, the main result is proven in §4. In §5 I give a short proof of the main point in [Sh]. The material in the beginning of §3 was expanded, beyond what is strictly required for the proof of theorem 1.3, to fit the needs of §5.

1.8 Acknowledgments. The material of this note was conceived in July 1994, during the 3rd Utah Summer School on moduli of surfaces of general type. I thank the organizers for providing a very nice environment. I am also very grateful to J. Kollár and S. Mori for a very useful discussion that solved two major technical problems.

2 Various kinds of singularities. We begin with the following:

2.1 Definition. Let $X$ be a smooth variety, $S \subset X$ be a reduced Cartier divisor. We say that $S$ is a smooth normal crossing divisor if

2.1.1 $X$ is a curve, or:

2.1.2 Every irreducible component $S_0 \subset S$ is smooth and, for each such component, $(S - S_0)|_{S_0}$ is a smooth normal crossing divisor.

We will work with the following definition of log terminal singularities:

2.2 Definition. Let $X$ be a normal variety, $B \subset X$ a $\mathbb{Q}$-Weil divisor. Assume that $B = \sum b_i B_i$ is a formal linear combination of reduced and irreducible codimension 1 subvarieties $B_i \subset X$ with rational coefficients $0 < b_i \leq 1$.

The divisor $K + B$ is log terminal if it is $\mathbb{Q}$-Cartier and there exists a projective morphism $h : Z \to X$, from a nonsingular variety $Z$, satisfying the following conditions:

2.2.1 The exceptional locus $E \subset Z$ of $h$ is a divisor, and $E \cup \text{Supp} h_*^{-1}B$ is a divisor with smooth normal crossings in $Z$.

2.2.2 Let $E = \sum E_i$ with $E_i$ irreducible. Then:

$$K_Z + h_*^{-1}B + E = h^*(K_X + B) + \sum a_i E_i$$

where $a_i > 0$ for all $i$. 

In particular, if \( K + B \) is log terminal, every component \( B_i \subset [B] \) is normal and every intersection \( B_i \cap B_j \) of two distinct such components is irreducible. By [Sz], the notion just given corresponds to divisorial log terminal of [FA, 2.13.3], and is equivalent to weakly Kawamata log terminal of [FA, 2.13.4] (see below 2.4). At the moment 2.2 is the best candidate for the “correct” definition of log terminal singularities.

2.3 Definition. Let \( X \) be a normal variety, \( B \subset X \) a \( \mathbb{Q} \)-Weil divisor as in 2.2. The divisor \( K + B \) is log canonical if it is \( \mathbb{Q} \)-Cartier and for all morphisms \( h : Z \to X \), from a nonsingular variety \( Z \), with exceptional divisors \( E_i \):

\[
K_Z + h_*^{-1}B + E = h^*(K_X + B) + \sum a_i E_i
\]

with all \( a_i \geq 0 \).

It is easy to see that log terminal singularities are preserved under divisorial contractions. They are also preserved by flips, but this is harder to see, due to an intrinsic drawback of definition 2.2: we require the existence of a resolution satisfying some properties, rather than asking a similar property of all resolutions as in 2.3. This difficulty is resolved via the following result due to Szabó [Sz]:

2.4 Resolution lemma. Let \( X \) be an irreducible variety over an algebraically closed field of characteristic 0, \( S \subset X \) a subvariety of pure codimension 1, \( U \subset X \) a smooth open subvariety such that \( S \cap U \subset U \) is a smooth normal crossing divisor. Then, there exists a projective morphism \( h : Z \to X \), from a smooth \( Z \), satisfying the following conditions:

2.4.1 \( h : h^{-1}U \to U \) is an isomorphism,

2.4.2 \( h^{-1}(S \cup (X \setminus U)) \) is a smooth normal crossing divisor in \( Z \). \( \square \)

I now introduce a class of singularities large enough to allow the minimal model program of a projective semistable family. For a smaller class in dimension 3, see §5.

In what follows we fix a projective morphism \( f : X \to \Delta \), where \( \Delta = \text{Spec} \mathcal{O} \) is the spectrum of a discrete valuation ring \( \mathcal{O} \), with central and generic points \( 0, \eta \in \Delta \). We denote \( S = X_0, X_\eta \) the fibers over \( 0, \eta \). All minimal model programs, divisorial contractions, flips, will be tacitly required to be compatible with this structure.

2.5 Definition. Let \( f : X \to \Delta \) be as above. We say that \( f \) (or \( X \), when there is no danger of confusion) has semistable terminal singularities if the following conditions are satisfied:

2.5.1 \( X \) itself has terminal singularities,

2.5.2 \( S = X_0 = f^*(0) \) is reduced and \( K_X + S \) is log terminal.
In particular, a projective semistable family in the sense of Mumford has semistable terminal singularities. Note that every component \( S_i \) of \( S \) is normal and \( K_S + (S - S_i)|_{S_i} \) is log terminal, as a further consequence every intersection \( S_i \cap S_j \) of two distinct such components is irreducible and normal, etc.

An important observation is that \( S \sim 0 \) is linearly equivalent to 0, so \( K_X \sim K_X + S \) and a divisorial contraction (resp. flip) for \( K_X \) is the same thing as a divisorial contraction (resp. flip) for \( K_X + S \). As a consequence, semistable terminal singularities are preserved by (semistable) divisorial contractions and flips.

I will now recall two results from classification theory. The first classifies log terminal 3-fold singularities with “large” boundary divisor, up to analytic equivalence. The result is an easy consequence of inversion of adjunction [FA, 17.6] and [Sz], and is proven in [FA, 16.15]:

2.6 Lemma. Let \( x \in B \subset X \) be a 3-fold germ with \( \emptyset \neq B \) reduced and \( K_X + B \) log terminal. Then \( B \) has at most 3 irreducible components, and:

2.6.1 If \( B \) has three irreducible components, \( x \in B \subset X \) is analytically isomorphic to: \( 0 \in (xyz = 0) \subset \mathbb{A}^3 \)

2.6.2 If \( B = B_1 + B_2 \) has two irreducible components, one of the following happens:

2.6.2.1 \( B_1 \) and \( B_2 \) are both \( \mathbb{Q} \)-Cartier and \( x \in B \subset X \) is analytically isomorphic to: \( 0 \in (xy = 0) \subset \mathbb{A}^4 \)

2.6.2.1 Neither \( B_1 \) nor \( B_2 \) is \( \mathbb{Q} \)-Cartier and \( x \in B \subset X \) is analytically isomorphic to: \( 0 \in (t = 0) \subset (xy + tg(z,t) = 0) \), everything taking place in affine toric 4-space \( \mathbb{A}^4 \) with \( (q_1, q_2, 1, a) \) where \( (q_1, a, m) = (q_1, q_2, m) = 1 \). □

The next result is a classification of semistable terminal singularities, up to analytic equivalence. For a simple proof, see [Ka3, 4.1]:

2.7 Lemma. Let \( X \) be a 3-fold and \( f : X \to \Delta \) have semistable terminal singularities. Let \( t \in \mathcal{O} \) be the uniformizing parameter, \( x \in S = (t = 0) \) be a point, \( r \) the index of \( K_X \) at \( x \). Then \( x \in X \) is analytically equivalent to one of the following:

2.7.1 \( xyz = t \subset A^4 \)

2.7.2 \( (xy = t) \subset A \) where \( A = \frac{1}{r}(a, -a, 1, 0) \) for some \( (a, r) = 1 \).

2.7.3 Two cases:

2.7.3.1 \( r > 1 \) and \((xy = f(z^r, t)) \subset A \) with \( A \) as in 2.7.2. Here \( f(Z, t) = 0 \) is an isolated curve singularity in the \( Z, t \)-plane and \( f(0, t) \neq 0 \). Also \( f(Z, 0) \neq 0 \), otherwise we are in case 2.7.2.

2.7.3.2 \( r = 1 \) and \( x \in X \) is an isolated singularity of the form: \((g(x, y, z) = tf(x, y, z, t)) \subset \mathbb{C}^4 \) where \((g(x, y, z) = 0) \) is a surface Du Val singularity and \( f \) is arbitrary. □

I wish to emphasize that 2.7 is quite easy, unlike the classification of all terminal 3-fold singularities.
3 The main construction.

3.1 Definition. Let $X$ be a normal variety, $Y \subset X$ a closed subvariety. We say that $X$ is $\mathbb{Q}$-factorial (resp. analytically, resp. formally $\mathbb{Q}$-factorial) along $Y$ if every Weil divisor on the Zariski germ (resp analytic germ, resp. formal completion) of $X$ along $Y$ is $\mathbb{Q}$-Cartier. $X$ is (analytically, formally) $\mathbb{Q}$-factorial if it is so along every subvariety $Y \subset X$ (it is clearly enough to check this at all closed points $y \in X$).

If $X$ is $\mathbb{Q}$-factorial along $Y$, $X$ is $\mathbb{Q}$-factorial at every point $y$ of $Y$. Not so (obviously) if $X$ is analytically or formally $\mathbb{Q}$-factorial along $Y$. The following however is easy:

3.2 Lemma. Let $X$ be analytically (resp. formally) $\mathbb{Q}$-factorial along $Y$. If $Y \subset X$ has codimension $\geq 2$, $X$ is analytically (resp. formally) $\mathbb{Q}$-factorial along every point $y$ of $Y$. □

3.3 Definition. Let $\Delta$ be as usual, $X$ a normal variety and $f : X \to \Delta$ a morphism, $Y \subset X$ a subvariety. $X$ is stably (analytically, formally) $\mathbb{Q}$-factorial along $Y$ if, for every base change $\Delta' \to \Delta$, $X'$ is (analytically, formally) $\mathbb{Q}$-factorial along $Y'$, where $X'$ is the normalized pull-back, and $Y' \subset X'$ the inverse image.

We now study stably analytically $\mathbb{Q}$-factorial semistable terminal 3-fold singularities. These are Kawamata’s moderate singularities [Ka2]:

3.4 Definition. Let $X$ be a 3-fold, $f : X \to \Delta$ a not necessarily projective morphism. Let $t \in \mathcal{O}$ be a parameter. $f$ has moderate singularities if the analytic germ at every point $x \in X$ is isomorphic to one of the following germs:

3.4.1 $(xyz = t) \subset \mathbb{C}^4$
3.4.2 $(xy = t) \subset A$ where $A = \mathbb{C}\left(\frac{1}{r}(a, r - a, 1, 0)\right)$ for some $(a, r) = 1$.
3.4.3 $(xy = z^n + t^n) \subset A$ for some $n$, with $A$ as in 3.4.2.

The following is proven in [Ka2] (and below):

3.5 Lemma. Let $X$ be a 3-fold, and let $f : X \to \Delta$ be a projective morphism with semistable terminal singularities. There is then a base change $\Delta' \to \Delta$ and a small, not necessarily projective morphism, $h : X'' \to X'$, where $X'$ is the pull-back, such that $X''$ has moderate singularities ($h$ can be taken to be projective locally analytically over $X'$).

Proof. I will give a quick sketch of the proof. Start with a singularity of the form:

$$xy = f(z^r, t)$$

in affine toric 4-space $\frac{1}{r}(a, r - a, 1, 0)$. As in 2.7.3.1, $f(Z, 0) \neq 0$, and by the Weierstrass preparation theorem there exists a $\mathbb{Z}/r$-equivariant analytic change of
coordinates such that, in the new coordinates, \( f(Z,t) \) is a Weierstrass polynomial in \( Z \):

\[
f(Z,t) = \sum_{i=0}^{k} Z^i f_i(t)
\]

where \( f_i(t) \) is a convergent power series with \( f_i(0) = 0 \). After a base change \( t = u^d \):

\[
f(Z,u^d) = \prod_{i=1}^{k} Z - \varepsilon_i(u)
\]

Using [Ko, 2.2], it is easy to construct a small projective partial resolution with \( k \) singular points:

\[
xy = z^r - u^{n_i}
\]

in \( A \), where \( n_i = \text{ord} \varepsilon_i \). Taking \( d \) large enough one can do this on all of \( X \) simultaneously, but we may loose projectivity. One should also note that base changing introduces some quotient singularities along the curves which are intersection of the components of the central fiber. These are however easily resolved. \( \square \)

The following is an immediate corollary of the proof just given:

3.6 Corollary. A semistable terminal 3-fold singularity is stably analytically \( \mathbb{Q} \)-factorial if and only if it is moderate. \( \square \)

We now begin our proof of existence of semistable 3-fold flips.

3.7 Definition. Let \( \varphi : X \rightarrow U \) be a semistable 3-fold flip, \( C \subset X \) be the \( \varphi \)-exceptional set. The flip is said to be moderate if \( X \) has moderate singularities at every point of \( C \).

3.8 Lemma. Semistable 3-fold flips exist if moderate semistable 3-fold flips exist.

The proof of 3.8 is based upon the following:

3.9 Lemma. Let \( \varphi : X \rightarrow U \) be a moderate semistable flip. Let \( \varphi^+ : X^+ \rightarrow U \) be the flip of \( \varphi \), \( C^+ \) the \( \varphi^+ \)-exceptional set. Then \( X^+ \) has moderate singularities at every point of \( C^+ \).

Proof. By 3.6 \( X \) has analytically \( \mathbb{Q} \)-factorial singularities. It is well known then that \( X^+ \) has analytically \( \mathbb{Q} \)-factorial singularities along \( C^+ \). Since \( C^+ \subset X^+ \) has codimension \( \geq 2 \), by 3.2 \( X^+ \) has analytically \( \mathbb{Q} \)-factorial singularities. This is true after base change because the base change of the flip is the flip of the base change. So \( X^+ \) has stably analytically \( \mathbb{Q} \)-factorial singularities. Then \( X^+ \) has moderate singularities by 3.6. \( \square \)
Proof of 3.8. This is standard using 3.9, let me give a quick outline. Let $X \to U$ be a flip, $\Delta' \to \Delta$ a base change as in 3.5, $X' \to U'$ the base change $X'' \to X'$ as in 3.5, with $X''$ moderate. Note that $X', U'$ are acted upon by the cyclic group $G$ of the covering $\Delta' \to \Delta$. As a first step we run a minimal model program for $X''$ over $U'$. By 3.9, this consists of a finite number of moderate flips $X''/\arrowshort X''''$. Let $X' +$ be the relative canonical model of $X'''' \to U'$, whose existence is granted by the base point free theorem. Then $X' + \to U'$ is the flip of $X' \to U'$, and $X' +/G \to U'/G = U$ is the flip of $X \to U$. □

3.10 From now on we fix a nonspecial moderate semistable 3-fold flip $\varphi : X \to U$, $C \subset X$ the flipping material. We now begin the basic construction for the proof of 1.3. Let $m$ be very large and $H_0 \in | - mK_X|$ a smooth member. Let $\overline{H} \subset U$ be a Cartier divisor satisfying the following conditions:

3.10.1 $\overline{H}$ contains $\overline{H}_0 = \varphi H_0$,
3.10.2 $K_U + \overline{H}$ is log terminal outside $\varphi C$
4. (the existence of $\overline{H}$ is a consequence of the standard Bertini theorem on the quasi projective variety $U \setminus \varphi C$).

The following is the main result of this section:

3.11 Lemma. Possibly after a base change, there exists a projective morphism $h : Z \to X$ satisfying the following conditions:

3.11.1 $Z$ is smooth and the $h$-exceptional set $E$ is a divisor.
3.11.2 $h : Z \setminus p^{-1}C \to X \setminus C$ is an isomorphism. In particular $E \subset Z_0$ and $h : Z_\eta \to X_\eta$ is an isomorphism.
3.11.3 $Z_0$ is reduced and $Z_0 \cup h_*^{-1}H$ is a smooth normal crossing divisor.

The following moreover is true:

3.11.4 For every birational morphism $g : Y \to U$, $N^1(Y/U)$ is generated by the $g$-exceptional divisors and the components of $g_*^{-1}\overline{H}$

Proof. By 3.10.2 and the resolution lemma 2.4, there is $h : Z \to X$ satisfying 3.11.1–3, with the possible exception that $Z_0$ may be nonreduced. 3.11.4 is also satisfied, because, as we will check momentarily, the conditions of the following lemma 3.12 are met. By 3.10.1 $\overline{H}$ contains $\overline{H}_0 = \varphi H_0$, and $\overline{H}_0$ is a generator of $WD(U)/CD(U)$ (this notation is introduced in 3.12 below) because $X$ is $\mathbb{Q}$-factorial and $H_0$ is a generator of $N^1(X/U)$. The conditions of 3.12 are therefore satisfied, and 3.11.4 holds.

We will achieve all properties after base change and semistable reduction. Let $t$ be a parameter in $\Delta$, $\Delta' \to \Delta$ a base change, $u$ a parameter in $\Delta'$ and $t = u^d$. Denote $X'$, $Z'$ the base change, $h'$, $H' \subset X'$ etc. the corresponding objects after base change. By the semistable reduction theorem, if $d$ is divisible enough, there is a projective resolution $h'' : Z'' \to Z'$ such that $Z''_0$ is reduced and smooth normal...
crossing. We will check that $Z'' \to X'$ satisfies all the required properties. 3.11.4 is still true after base change, since $X$ is stably $\mathbb{Q}$-factorial, in fact this is the reason why we introduced stable $\mathbb{Q}$-factorializations in 3.5 to begin with. So we only need to show that $Z'' \cup h''_{*}^{-1}H'$ is a smooth normal crossing divisor, and in fact it is enough to check this locally at every point. That this is the case is more or less obvious, but we will try to explain it carefully. To this end, we need to recall part of Mumford’s construction of the semistable reduction.

Locally analytically $Z = \mathbb{A}^3$, and

$$Z_0 \cup h^{-1}_*H = (\prod_{i=1}^{k} z_i^{n_i} \prod_{k+1}^{l} z_i = 0)$$

where $z_i$ are coordinates on $\mathbb{A}^3$, $Z_0 = (t = 0)$, $t = \prod_{i=1}^{k} z_i^{n_i}$, and $h^{-1}_*H = (\prod_{k+1}^{l} z_i = 0)$. After the base change $t = u^d$, the fiber product $Z'$ is described as:

$$Z' = A \times \mathbb{A}^{3-k}$$

where $A = (u^d = \prod_{i=1}^{k} z_i^{n_i}) \subset \mathbb{A}^{k+1}$, and, inside $Z'$, $Z'_0 = (u = 0)$ and $h'_*^{-1}H' = (\prod_{k+1}^{l} z_i = 0)$ in coordinates $z_{k+1},...z_3$ for $\mathbb{A}^{3-k}$. The construction of the semistable reduction begins with taking the normalization of $A$:

$$\nu : A'' = \prod_{j=1}^{e} A_j \to A$$

Here $(e) = (d, n_i)$ and each $A_j$ is isomorphic to the simplicial affine toric variety $\frac{\Delta}{\mathbb{C}}(n_1,...n_k)$. Now $Z''$ is obtained by gluing together pieces of the form:

$$B_j \times \mathbb{A}^{3-k} \to A_j \times \mathbb{A}^{3-k}$$

where $B_j \to A_j$ is a toric resolution of $A_j$ with the property that $(u = 0) \subset B_j$ is a smooth normal crossing divisor (the proof of the semistable reduction theorem consists in proving that such $B_j \to A_j$ exist, and showing that a choice exists so that the gluing is possible). Now $h''_*H'$ is described, in $Z''$, by the equation $\prod_{k+1}^{l} z_i = 0$, which proves the statement. □

3.12 Lemma. Let $U$ be a normal variety, $D_1,...,D_k \subset U$ Weil divisors on $U$ generating:

$$\frac{WD(U)}{CD(U)} = \frac{\{\mathbb{Q}\text{-Weil divisors on } U\}}{\{\mathbb{Q}\text{-Cartier divisors on } U\}}$$
Let $f : Y \to U$ be a birational morphism. Then $N^1(Y/U)$ is generated by the $f_*^{-1}D_i$ and the irreducible $f$-exceptional divisors.

**Proof.** Let $A \in N^1(Y/U)$. A linear combination $\sum \lambda_i D_i + f_* A$ is $\mathbb{Q}$-Cartier on $U$. Then $\sum \lambda_i f_*^{-1}D_i + A - f^*(\sum \lambda_i D_i + A)$ is $f$-exceptional. But in $N^1(Y/U)$, $f^*(\sum \lambda_i D_i + A) = 0$. □

4 Proof of 1.3. We use Shokurov addition and subtraction method with the refinements in [FA 18.12]. We use the notation of §3, with one change: we now denote $H$ the divisor $h^{-1}H$ on $Z$. Let $\varphi \circ h = p : Z \to U$. Before coming to the details, I will explain the broad outline of the proof, in three steps as follows:

A We run a MMP for the divisor $K_Z + H \sim K_Z + H + Z_0$, over $U$. We need to show that flips exist and that each step of the program preserves condition 2.5.1, namely all varieties involved in the program have terminal singularities (all steps clearly preserve 2.5.2). The program terminates at the variety $p' : Z' \to U$ and $K_{Z'} + H'$ is $p'$-nef.

B Using the nef threshold method, we progressively subtract pieces of $H'$ until we reach a variety $p'' : Z'' \to U$ where $K_{Z''}$ is $p''$-nef. As in A, we need to show that flips exist and that each step of the program preserves condition 2.5.1.

C The relative canonical model $\varphi^+ : X^+ \to U$ of $p'' : X'' \to U$, whose existence is granted by the base point free theorem, is the flip of $\varphi$. This is obvious.

We will use the following notation:

4.1 The central fiber

$$Z_0 = p^*U_0 = \sum_{i=0}^I S_i$$

The $S_i$ for $i > 0$ are irreducible components and $S_0 = p_*^{-1}U_0$. The $S_i$ for $i \geq 1$ are precisely the $p$-exceptional components. Note that the image of every $S_i$ lies in $\mathcal{H}$, which follows from 3.11.2 and the fact that $\mathcal{H} \supset \varphi C$.

We now carry out steps A and B in detail:

A The MMP for $K_Z + H \sim K_Z + H + Z_0$ constructs varieties $Z = Z^1 \to Z^2 \to \cdots \to Z^\alpha$, with projections $p^\alpha : Z^\alpha \to U$, divisors $Z_0^\alpha = \sum_{i=0}^{I(\alpha)} S_i^\alpha$, etc.. Let us inductively assume that $Z^\alpha$ has already been constructed, and that property 2.5.1 holds for $Z^\alpha$. Let $\psi^\alpha : Z^\alpha \to V^\alpha$ be an extremal contraction, corresponding to a ray $R^\alpha$ such that $(K_{Z^\alpha} + H^\alpha) \cdot R^\alpha < 0$. I will in the sequel drop the superscript $\alpha$ from the notation.

A.1 If $\psi$ is a divisorial contraction, we need to show that property 2.5.1 holds for $V$. Since $H$ contains no proper divisor, $H \cdot R \geq 0$. The ray is a ray for $K_Z$ so 2.5.1 is clearly preserved.

A.2 If $\psi$ flips, we need to show that the flip $\psi^+ : Z^+ \to V$ exists and that property 2.5.1 holds for $Z^+$. Let $C$ be a connected component of the flipping set.
We distinguish two cases, according to whether there exists an irreducible component $M$ of $H$ with $M \cdot C < 0$ or not:

**A.2.1** There is an irreducible component $M$ of $H$ with $M \cdot C < 0$. As $C$ is contained in $S_i$ for some $i$, by 2.6.2.1 we must have that $M$, $S_i$ are irreducible in a neighborhood of $C$, and $C = M \cdot S_i$. Because $C \subset M$ is contractible, $S_i \cdot C < 0$ and there exists $S_j$ with $S_j \cdot C > 0$. By 2.6 again, and since by assumption $(K + H + Z_0) \cdot C < 0$, $C$ intercepts no component of $H$ other than $M$. Then $K + H$ satisfies condition $(*)$ of [FA, 5.1] in a neighborhood of $C$. Incidentally, note that $\psi : M \to \psi(M)$ is a 2-dimensional semistable extremal contraction (for $K_M \sim K_M + M \cdot Z_0$). The flip exists by [FA, 5.4.1], and condition $(*)$ holds for $Z^+$ by [FA, 5.2]. In particular 2.5.1 holds for $Z^+$.

**A.2.2** $M \cdot C \geq 0$ for all irreducible components $M$ of $H$. In particular the flip is also a flip for $K_Z = K_Z + Z_0$ so $Z^+$ satisfies 2.5.1 if it exists. Let us prove that $Z^+$ exists. By 3.11.4 there is a component $M$ of $H + \sum_{i>0} S_i$ such that $M \cdot C \neq 0$. Now:

$$p^* \overline{H} = H + \sum_{i>0} b_i S_i$$

and all $b_i > 0$, since we made sure that the $p$-image of every $p$-exceptional divisor lies in $\overline{H}$. Since $p^* \overline{H} \cdot C = \overline{H} \cdot p_* C = 0$, there is a component $L$ of $H + \sum_{i>0} S_i$ such that $L \cdot C < 0$. By assumption, $L$ is one of the $S_i$. The flip is a special flip, and it exists by assumption.

**B** Let $p' : Z' \to U$ with $K_{Z'} + H'$ $p'$-nef be the model constructed in A. The nef threshold method is a guided version of the minimal model program. Without attempting a general formulation, I will describe what this is in the present circumstances. We create varieties $Z' = Z'^1 \to Z'^2 \to \cdots \to Z'^\alpha$ such that $K_{Z'^\alpha} + \varepsilon^\alpha H'^\alpha$ is nef. Assume $Z'^\alpha$ has been constructed, I will now give the recipe for $Z'^{\alpha+1}$. $\varepsilon^{\alpha+1}$ is the smallest $\varepsilon \leq \varepsilon^\alpha$ such that $K_{Z'^\alpha} + \varepsilon H'^\alpha$ is still nef, and $Z'^\alpha \to Z'^{\alpha+1}$ the modification associated to one of the (finitely many) extremal rays $R^\alpha$ such that

$$\left(K_{Z'^\alpha} + (\varepsilon^{\alpha+1} - \eta) H'^\alpha\right) \cdot R^\alpha < 0$$

for all $\eta$ very small.

Assuming that the contraction $\psi_{R^\alpha} : Z^\alpha \to V^\alpha$ is flipping, I need to prove that the flip exists. I will in the sequel drop the superscript $\alpha$ from the notation. By construction $H' \cdot R > 0$. But $H' \cdot R = -\sum b_i S'_i \cdot R$, so $S'_i \cdot R < 0$ for some $i$. As $Z'_0 \cdot R = 0$, $S'_j \cdot R > 0$ for some $j$. The flip is special, and it exists by assumption. This concludes the proof of 1.3. □

**4.2 Remark.** As a final remark, I wish to say that, using the $G$-minimal model program and a $G$-invariant version of Mumford semistable reduction, it would have
been possible to prove 1.3 avoiding the material on stable \( \mathbb{Q} \)-factorialization and the classification 2.7 (but not 2.6) altogether. Such an argument would have probably been more complicated than the one given in the text.

5 Strictly semistable 3-fold singularities. In this last section, I will prove some results in [Sh]. I would be interested in knowing if the material here has some sort of higher dimensional generalization.

In this section all varieties, germs, etc. are tacitly assumed to come equipped with a morphism, usually a projective one, to \( \Delta \). All minimal models, divisorial cotractions, etc., are required to be compatible with this structure.

5.1 Definition. A germ \( x \in X \) (resp. a variety \( X \)) admits a semistable resolution if there is a resolution \( f : Z \to X \) such that \( Z_0 \subset Z \) is a reduced smooth normal crossing divisor (in other words, \( Z \) is semistable in the sense of Mumford).

The following is well known:

5.2 Lemma. Let \( x \in X \) be a moderate 3-fold singularity. Then \( x \in X \) admits a projective semistable resolution.

Proof. Let \( x \in X \) be described by the equation:

5.2.1 \((xy = t) \subset A\), or
5.2.2 \((xy = z^r + t^n) \subset A\),

where \( A = \frac{1}{r}(a, r - a, 1, 0) \) for some \((a, r) = 1\). In both cases let \( f : B \to A \) be the weighted blow up with weights \( \frac{1}{r}(a, r - a, 1, r) \), \( X' \subset B \) the proper preimage, and let us also denote \( f : X' \to X \) the restriction to \( X' \). The following are easily checked via a small calculation in explicit coordinates:

In case 5.2.1, \( X' \) has two singular points, given by \((xy = t) \subset \frac{1}{a}([r], [a - r], 1, 0)\) and \((xy = t) \subset \frac{1}{a - r}([r], [a - r], 1, 0)\).

In case 5.2.2, \( X' \) has three singular points, given by \((xy = t) \subset \frac{1}{a}([r], [a - r], 1, 0)\), \((xy = t) \subset \frac{1}{r - a}([r], [a - r], 1, 0)\), and \((xy = z^r + t^{n-1}) \subset A\).

It is then immediate that a repeated application of \( f : X' \to X \) gives the desired resolution. \( \square \)

5.3 Definition. A semistable terminal 3-fold singularity is strictly semistable if its analytic \( \mathbb{Q} \)-factorialization is moderate.

As an immediate consequence of 5.2, we have:

5.4 Corollary. 5.4.1 A strictly semistable terminal analytic germ admits a projective semistable resolution.
5.4.2 If $X$ has strictly semistable terminal singularities, there is a semistable resolution $Z \to X$, which is projective locally analytically in $X$.

By 3.9 strictly semistable terminal singularities are preserved by flips. Our goals in this section are to show that they are also preserved by divisorial contractions, and to establish a converse to 5.4. The first result is due to Shokurov [Sh].

5.5 Theorem. Assume that $X$ is projective over $\Delta$ and has strictly semistable terminal singularities. Let $f : X \to Y$ be an extremal divisorial contraction. Then $Y$ has strictly semistable terminal singularities.

5.6 Theorem. Let $x \in X$ be a semistable terminal analytic germ. The following are equivalent:
   5.6.1 $x \in X$ is strictly semistable,
   5.6.2 $x \in X$ has a projective semistable resolution.

5.6 is an easy consequence of 5.5:

Proof of 5.6 using 5.5. We need to show that 5.6.2 implies 5.6.1. Let $Z \to X$ be a projective semistable resolution. Run a minimal model program for $Z \to X$. By 3.2, this terminates at an analytic $\mathbb{Q}$-factorialization $X' \to X$. By 5.5, $X'$ has strictly semistable terminal singularities, and so does $x \in X$. □

The proof of 5.5 uses the following:

5.7 Lemma. Let $x \in X$ be an analytically $\mathbb{Q}$-factorial rational singularity. If $x \in X$ admits a semistable resolution, it is stably $\mathbb{Q}$-factorial. In particular, if $x \in X$ is semistable terminal, it is moderate.

Proof of 5.5 using 5.7. Let $E \subset X$ be the exceptional divisor. Abusing notation I will in the sequel denote $X$ the analytic germ of $X$ along $E$. Let $y = f(E)$. The point here is that $Y$ is not necessarily analytically $\mathbb{Q}$-factorial along $y$, and $X$ is not necessarily analytically $\mathbb{Q}$-factorial along $E$. Let $X' \to X$ be a projective analytic $\mathbb{Q}$-factorialization along $E$. Run now a minimal model program for $X' \to Y$. After a finite number of flips $X' \dasharrow X''$ I meet a divisorial contraction $X'' \to Y$. Here $Y' \to Y$ is an analytic $\mathbb{Q}$-factorialization. Note that, since flips preserve strictly semistable terminal singularities, $X''$ has strictly semistable singularities, so it admits a semistable resolution $Z \to X''$. Then the composition $Z \to Y'$ is a semistable resolution, and since $Y'$ is analytically $\mathbb{Q}$-factorial, by 5.7 it is stably analytically $\mathbb{Q}$-factorial, hence moderate. This proves 5.5. □

Proof of 5.7. Let $\Delta' \to \Delta$ a base change of degree $d$. Let $X' \to \Delta'$ be the normalization of the fiber product, and $\pi : X' \to X$ the natural map. Fix a semistable resolution $h : Z \to X$ and let $Z'$ be the fiber product. The following are the two crucial observations:
5.7.1 $Z'$ is already normal, and in particular there is a natural map $h' : Z' \to X'$, given by the Stein factorization. More to the point, $Z'$ has toroidal simplicial, hence analytically $\mathbb{Q}$-factorial, singularities.

5.7.2 The cyclic group $G = \mu_d$ acts on $Z'$, $X'$ in such a way that $Z'/G = Z$, $X'/G = X$ and $h' : Z' \to X'$ is $G$-equivariant. Most importantly, $G$ acts trivially on the central fiber $Z'_0$.

Let $D' \subset X'$ be a Weil divisor. Our aim is to show that $D'$ is $\mathbb{Q}$-Cartier. Certainly $\sum_{g \in G} gD' = \pi^* \pi_* D'$ is $\mathbb{Q}$-Cartier, and so is its pull back $h'^* \sum gD' = \sum gh'^{-1} D' + \sum da_i E_i$, where $E_i \subset Z'_0$ are the $h'$-exceptional components. First of all I claim that:

$$h'^{-1} D' + \sum a_i E_i \equiv 0$$

is numerically equivalent to zero relatively to $h'$. Let indeed $C \subset Z'_0$ be a curve such that $h'C$ is a point. Then, using 5.7.1 to intersect with $h'^{-1} D'$:

$$h'^{-1} D' \cdot C = gh'^{-1} D' \cdot gC = gh'^{-1} D' \cdot C$$

by 5.7.2. Then:

$$0 = (\sum gh'^{-1} D' + \sum da_i E_i) \cdot C = d(h'^{-1} D' + \sum a_i E_i) \cdot C$$

which shows the claim. $X$ has rational singularities, and so does $X'$. $R^1 h'_* \mathcal{O}_{Z'} = (0)$, hence a multiple $\nu(h'^{-1} D' + \sum a_i E_i) \sim \mathcal{O}_{Z'}$ is linearly equivalent to zero. Then $\nu D'$ is Cartier. □

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