On the long time asymptotic behaviour of the modified Korteweg de Vries equation with step-like initial data

Tamara Grava† and Alexander Minakov‡

†SISSA, via Bonomea 265, 34136 Trieste, Italy and School of Mathematics, University of Bristol, UK
‡Institut de Recherche en Mathématique et Physique (IRMP), Université Catholique de Louvain (UCL), Louvain-la-Neuve, Belgium

Abstract

We study the long time asymptotic behaviour of the solution \( q(x,t) \), \( x \in \mathbb{R}, t \in \mathbb{R}^+ \), to the modified Korteweg de Vries equation (MKDV) \( q_t + 6q^2q_x + q_{xxx} = 0 \) with step-like initial datum

\[
q(x,0) \to \begin{cases} 
    c_- & \text{for } x \to -\infty \\
    c_+ & \text{for } x \to +\infty,
\end{cases}
\]

with \( c_- > c_+ > 0 \). For the exact shock initial data

\[
q(x,0) = \begin{cases} 
    c_- & \text{for } x < 0 \\
    c_+ & \text{for } x > 0,
\end{cases}
\]

the solution develops an oscillatory regions called dispersive shock wave that connects the two constant regions \( c_+ \) and \( c_- \). We show that the dispersive shock wave is described by a modulated periodic travelling wave solution of the MKDV equation where the modulation parameters evolve according to the Whitham modulation equation. The oscillatory region is expanding within a cone in the \((x,t)\) plane defined as

\[
-6c_-^2 + 12c_-^2 < \frac{1}{7} < 4c_+^2 + 2c_-^2,
\]

with \( t \gg 1 \).

For step like initial data we show that the solution decomposes for long times in three main regions

- a region where solitons and breathers travel with positive velocities on a constant background \( c_+ \);
- an expanding oscillatory region (that can contain solitons and breathers);
- a region of breathers travelling with negative velocities on the constant background \( c_- \).

When the oscillatory region does not contain solitons or breathers, it coincides up to a phase shift with the dispersive shock wave solution obtained for the exact step initial data. The phase shift depends on the solitons, breathers and the radiation of the initial data. This shows that the dispersive shock wave is a coherent structure that interacts in an elastic way with solitons, breathers and radiation.

Subject classification: 35Q15, 35Q51, 35Q53

Keywords: Integrable system, Riemann-Hilbert problem, dispersive shock waves, long time asymptotic analysis.

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1 Introduction

We consider the Cauchy problem for the focusing modified Korteweg–de Vries (MKDV) equation
\[
q_t(x,t) + 6q^2(x,t)q_x(x,t) + q_{xxx}(x,t) = 0, \quad x \in \mathbb{R}, \ t \in \mathbb{R}^+ \tag{1.1}
\]
with step-like initial datum
\[
q(x,0) = q_0(x) \to c_\pm \quad \text{as} \quad x \to \pm \infty, \tag{1.2}
\]
where \( c_\pm \) are some real constants. We are interested in the long-time behavior of the solution.

Due to symmetries \( q \mapsto -q \) and \( x \mapsto -x, t \mapsto -t \) it is enough to consider the case
\[
c_- \geq |c_+|. \tag{1.3}
\]
The focusing MKDV equation is a canonical model for the description of nonlinear long waves when there is polarity symmetry and it has many physical applications. This includes internal waves of a stratified fluid over a bottom step [54].

For the class of initial data considered, the classical mass and momentum have to be replaced by the conserved quantities

\[ H_0 = \int_{-\infty}^{x} (q(\tilde{x}, t) - c_-) d\tilde{x} + \int_{x}^{+\infty} (q(\tilde{x}, t) - c_+) d\tilde{x} + (c_- - c_+) x - 2(c_3^2 - c_3^4) t, \]

\[ H_1 = \int_{-\infty}^{x} (q^2(\tilde{x}, t) - c_-^2) d\tilde{x} + \int_{x}^{+\infty} (q^2(\tilde{x}, t) - c_+^2) d\tilde{x} + (c_-^2 - c_+^2) x - 4(c_4^2 - c_4^4) t. \]

The long time asymptotic behaviour of integrable dispersive equations with initial datum vanishing at infinity was initiated in the mid seventies using inverse scattering in the works of Ablowitz and Segur [2] and Manakov and Zakharov [59]. In the seminal paper [20] Deift and Zhou introduced the steepest descent method for oscillatory Riemann-Hilbert problems to study the long time asymptotic behaviour of the defocusing MKDV equation with initial data vanishing at infinity. Such technique was extensively implemented in the asymptotic analysis of a wide variety of integrable problems and also in the non integrable case, like the long time behaviour of the perturbed defocusing nonlinear Schrödinger equation [10]. An extension of the steepest descent method for oscillatory Riemann-Hilbert problems, called \( \bar{\partial} \) method was introduced in [48] and applied to study long time behaviour of integrable dispersive equation with initial data with low regularity [8], [13], [22] in the strongly nonlinear regime. In particular for the MKDV equation there is a vast body of literature studying existence of solution for initial data with low regularity (see e.g. [46]). Regarding the weakly nonlinear regime the long time asymptotic behaviour with small initial data was obtained without using the integrability property of the MKDV equation in [27], [36] and [37]. In the last years a vast literature of results concerns the long time dynamics of initial boundary value problems of nonlinear dispersive equations. For a review see [9].

The study of the long time behaviour of dispersive equations with shock initial conditions, was initiated in the papers of Gurevich, Pitaevsky [33] and Khruslov [38], where they considered the long time behaviour of the Korteweg de Vries equation (KdV) with a step initial condition. The oscillatory region emerging from the step has been called a ”dispersive shock wave” or ”collisionless shock wave”, and it was identified with the modulated travelling wave solution of the KdV equation of maximal amplitude about twice the size of the step. The modulation of the wave parameters is determined by the self-similar solution of the Whitham modulation equations [57] with step initial conditions. The KdV dispersive shock wave solution has been widely used in applications, such as the resonant flow of a stratified fluid over topography, see [30], [52].

The dispersive shock wave region was surprisingly found in the long time asymptotics for KdV also for decaying initial conditions [2], [18], but in this case the amplitude of the oscillations is decaying in time.

Since then the long time asymptotic behaviour of dispersive equation with step-like initial conditions has been studied for KdV in [25], [38], for the nonlinear Schrödinger equation in [6], [10], [11], and for the focusing MKDV equation in [5], [40], [41], [42], [47], [45]. The main features in the long time behaviour that distinguish step-like initial conditions from decaying initial conditions is the formation of an oscillatory region that connects the different behaviour at \( \pm \infty \) of the solution. Very surprisingly this oscillatory region was also derived in the long time asymptotic behaviour of the focusing nonlinear Schrödinger equation with compact support perturbation of the constant solution, [6].

For the KdV equation it has been shown [25], [26] that for step-like initial condition the long time behaviour of the solution is characterized by a soliton region, a radiation region on a constant background and a dispersive shock wave region that coincides with the Gurevich Pitaevski solution up to shifts and rescalings showing that the dispersive shock wave interact elastically with solitons.
The scattering problem for MKDV with non vanishing initial condition was developed in [40]. The linear spectral problem is a non self-adjoint problem and for a step like initial data \(q_0(x)\) satisfying certain assumption (see below) the Zakharov-Shabat or AKNS operator for \([1.1]\) has a continuous spectrum \(r(k) : \Sigma \to \mathbb{C}\) where \(\Sigma = \mathbb{R} \cup (-ic_-, ic_-)\) and it might have discrete spectrum anywhere in \(\mathbb{C} \setminus \mathbb{R} \cup (ic_+, ic_-)\cup (-ic_-, -ic_+)\), that corresponds to the zeros of \(a(k)\), the inverse of the transmission coefficient. Pure imaginary couples of conjugated eigenvalues correspond to solitons, while quadruplets of complex conjugated eigenvalues correspond to breathers [55]. A breather is a solution that is periodic in the time variable and decay exponentially in the space variable. Unlike the KdV equation the MKDV equation can have higher order solitons and breathers. In this manuscript we consider the generic case when only first order solitons and breathers appear. Nongeneric cases are considered in the Appendix [2]. First order solitons and breathers are the fundamental localised non radiating solutions of the MKDV equation. Since the MKDV equation is not Galilean invariant, solitary wave solutions and breathers on a constant background \(c > 0\) cannot be mapped to solutions on zero background.

Our main result that is contained in Theorem 1.4 below, is to show that the long time asymptotic solution of the MKDV equation with step like initial data of the form \([1.3]\) with \(c_- > c_+ > 0\) decomposes into three main regions:

- a region of solitons and breathers on a constant background \(c_+\) travelling in the positive direction;
- a region of breathers on a constant background \(c_-\) travelling in the negative direction. This region contains also radiation decaying in time;
- a dispersive shock wave region, which connects the two different asymptotic behaviours of the initial data and interact elastically with breathers and solitons. This region is described by a modulated travelling wave solution of MKDV, or by a modulated travelling wave solution interacting with breathers and solitons.

The localised travelling wave solution on a constant background \(c > 0\) is parametrised by two real constants \(\nu\) and \(\kappa_0 > c\), where \(\pm i\kappa_0\) is the discrete spectrum of the Zakharov-Shabat linear operator and takes the form

\[
q_{\text{soliton}}(x, t; c, \kappa_0, x_0) = c - \frac{2\text{sign}(\nu)(\kappa_0^2 - c^2)}{\kappa_0 \cosh \left[ \sqrt{\kappa_0^2 - c^2}(x - (2c^2 + 4\kappa_0^2)t) + x_0 \right] - \text{sign}(\nu)c},
\]  

where the phase shift \(x_0\) depends on the spectral data via the relations \(x_0 = \log \frac{2(\kappa_0^2 - c^2)}{|\nu|\kappa_0} \in \mathbb{R}\). The solution with \(\nu < 0\) is called soliton and corresponds to a positive hump, while the solution with \(\nu > 0\) is called anti-soliton and corresponds to a negative hump. In both cases the speed is \(4\kappa_0^2 + 2c^2\), namely the speed of the soliton increases with the size of the step. The maximum amplitude of the soliton is \(2\kappa_0 - c\) while the minimum amplitude of the anti-soliton is \(-2\kappa_0 - c\).

The breather solution on a constant background \(c\) has been obtained using the bilinear method and Dabroux transformations in [14], and inverse scattering in [5]. In this manuscript we obtain the breather on a constant background as a solution of a Riemann-Hilbert problem that is parametrised by the point spectrum \(\kappa = \kappa_1 + i\kappa_2\) with \(\kappa_{1,2} > 0\) and by the complex parameter \(\nu\). Introducing the complex number \(\chi\) defined as \(\chi = \chi_1 + i\chi_2 = \sqrt{\kappa^2 + c^2}\), with \(\chi_1 > 0\), \(\chi_2 > 0\), the breather solution on a constant background takes the form

\[
q_{\text{breather}}(x, t; c, \kappa, \nu) = c + 2\partial_x \arctan \left[ \frac{|\chi|}{\kappa_0^2} \cos \varphi + \frac{c|\nu|\chi_2^2}{\kappa_0^2\chi_2^2} e^{-2Z\chi_2} \right],
\]  

where

\[
Z = x + 4t(3\chi_1^2 - \chi_2^2 - \frac{3}{2}c^2),
\]

\[
\varphi = 2(Z - 8t|\nu|^2)\chi_1 + \theta_1 - \theta_2.
\]
and phases \( \theta_1 = \arccos \frac{\nu_1}{|\nu|} \) and \( \theta_2 = \arccos \frac{\chi_1}{|\chi|} \). On the line \( Z = 0 \) the breather oscillates with period \( \frac{\pi}{8|\chi|^2\chi_1} \) and the envelope of the oscillations moves with a speed

\[
V = 4\chi_1^2 + 6\nu_1^2 - 12\chi_1^2. \tag{1.6}
\]

We observe that for fixed \( \kappa \) and large values of \( c \) the velocity of the breathers is always negative. The level set in the complex \( k \)-plane of breathers with the same speed is shown in Figure 1. It has been shown in [11] that solitons and breathers on a constant background \( c > 0 \) interact in an elastic way as in the case \( c = 0 \). The breather (1.5) turns into a couple of the soliton and anti-soliton (1.4) on a constant background when we let \( \chi_1 = 0 \) and \( \chi_2 > 0 \).

For \( c = 0 \) we have \( \chi_1 = \chi_1 \) and \( \chi_2 = \chi_2 \) and the solution (1.5) reduces to the standard breather [50]

\[
q_{\text{breather}}(x, t; c = 0, \kappa, \nu) = 2\partial_x \arctan \left[ \frac{\kappa_2 \cos \varphi}{\kappa_1 \cosh \Theta} \right] = -4\kappa_1 \kappa_2 \left( \kappa_2 \sinh \Theta \cos \varphi + \kappa_1 \cosh \Theta \sin \varphi \right),
\]

where

\[
\Theta = 2\kappa_2 \left( x + 4 \left( 3\kappa_1^2 - \kappa_2^2 \right) t \right) + \log \left( \frac{2\kappa_2 \sqrt{\kappa_1^2 + \kappa_2^2}}{\kappa_1 |\nu|} \right),
\]

and

\[
\varphi(x, t) = 2\kappa_1 \left( x + 4 \left( \kappa_1^2 - 3\kappa_2^2 \right) t \right) - \arccos \frac{\nu_1}{|\nu|} - \arccos \frac{\kappa_1}{|\kappa|}.
\]

It has been shown in [15] that the formation of breathers is generic for certain compact support initial conditions.

The periodic travelling wave solution of the MKDV equation takes the form (see Appendix A)

\[
q_{\text{per}}(x, t; \beta_1, \beta_2, \beta_3, x_0) = -\beta_1 - \beta_2 - \beta_3 + \frac{2(\beta_2 + \beta_3)(\beta_1 + \beta_3)}{\beta_2 + \beta_3 - (\beta_2 - \beta_1)\text{cn}^2 \left( \sqrt{\beta_3^2 - \beta_1^2} (x - Vt) + x_0 |m \right)}, \tag{1.7}
\]

where \( \beta_3 > \beta_2 > \beta_1 \), the speed \( V = 2(\beta_1^2 + \beta_2^2 + \beta_3^2) \) and \( x_0 \) is an arbitrary phase. The function \( \text{cn}(z|m) \) is the Jacobi elliptic function of modulus

\[
m = \frac{\beta_3^2 - \beta_1^2}{\beta_3^2 - \beta_2^2}, \tag{1.8}
\]

and period \( 2K(m) \), where \( K(m) \) is the elliptic integral of the first kind. The periodic solution (1.7) has wave number \( k \), frequency \( \omega \) and amplitude \( a \) given by

\[
k = \frac{\pi \sqrt{\beta_3^2 - \beta_1^2}}{K(m)}, \quad \omega = \nu k, \quad a = 2(\beta_2 - \beta_1),
\]

Figure 1: The level set of curves of the breather speed \( V \) in (1.6) in the plane \( (\kappa_1, \kappa_2) \) for \( c = 1 \). In black the level curves with positive velocity, in blue with negative velocity and in red the line with zero velocity. The snapshot of a breather at \( t = 0 \) (red) and at later times in black. In the first figure \( c = 1, \kappa_1 = 1, \kappa_2 = 1.5 \) and \( V < 0 \) and in the second figure \( c = 1, \kappa_1 = 0.5, \kappa_2 = 1.5 \) and \( V > 0 \).
respectively. When $m \to 1$, the travelling wave solution (1.7) converges to the soliton solution (1.4) with $\beta_2 = \beta_3 = \kappa_0$ and $\beta_1 = c$.

Our main result concerns the asymptotic description for large times of the MKDV initial value problem for step-like initial data of the form (1.2). Before stating our result we remark that the description in [40] of the long time behaviour of the solution of MKDV with the shock initial data

$$q_0(x) = \begin{cases} c_+ & \text{as } x > 0 \\ c_- & \text{as } x < 0, \end{cases}$$

(1.9)

with $c_ > c_+ > 0$ is as follows: there are two constant regions $\frac{x}{t} < -6c_+^2 + 12c_+^2 - \delta$ and $\frac{x}{t} > 4c_-^2 + 2c_+^2 + \delta$ for some $\delta > 0$, where the solution $q(x,t) = c_+ + o(1)$ as $t \to \infty$ and the connection region between the two constant solutions is obtained by an oscillatory region described in terms of a genus 2 quasi-periodic solution. In our asymptotic analysis we show that such genus 2 solution can be reduced to the modulated travelling wave solution (1.7) of MKDV, namely

$$q(x,t) = q_{per}(x,t,c_-,d,c_+,x_0) + O(t^{-1}), \quad -6c_+^2 + 12c_+^2 + \delta < \frac{x}{t} < 4c_-^2 + 2c_+^2 - \delta,$$

(1.10)

where $d = d(x,t)$ depends on space and time according to

$$\frac{x}{t} = W_2(c_+,d,c_-).$$

(1.11)

Here

$$W_2(\beta_1, \beta_2, \beta_3) = 2(\beta_1^2 + \beta_2^2 + \beta_3^2) + 4 \frac{(\beta_2^2 - \beta_1^2)(\beta_3^2 - \beta_1^2)}{\beta_2^2 - \beta_3^2 + (\beta_2^2 - \beta_1^2) \frac{E(m)}{K(m)}},$$

(1.12)

with $E(m)$ the complete elliptic integral of the second kind. We have that

$$W_2(c_-,c_+,c_+) = -6c_+^2 + 12c_+^2 < \frac{x}{t} < 4c_-^2 + 2c_+^2 = W_2(c_-,c_-,c_+).$$

The quantity $W_2(\beta_1, \beta_2, \beta_3)$ is the speed of the Whitham modulation equations [23] for $\beta_1, \beta_2$ and $\beta_3$ referring to the Riemann invariants $\beta_2$. The solution (1.10) is the dispersive shock wave, and it was first derived for the Korteweg de Vries equation [33]. The Whitham equations for MKDV are strictly hyperbolic and genuinely nonlinear [43] only for $\beta_2 \neq 0$ which implies that the equation (1.11) can be inverted for $d$ as a function of $\xi$ only when $c_+ > 0$. The case $c_- > -c_+ > 0$ can be studied in a similar way, by getting a slightly different evolution of the wave parameters (see the Appendix [3] for further comments). A comprehensive set of cases arising in the long-time asymptotic solution for the MKDV equation with step initial data for various values of the parameters $c_-$ and $c_+$ has been discussed in [24, 45, 47].

The phase $x_0$ of the travelling wave solution (1.10) is given explicitly in terms of the scattering data associated to the MKDV equation.

We consider general step-like initial data and we subject the initial function to the following condition:

**Assumption 1.1.** The initial data $q_0(x)$ is assumed to be a function of bounded variation $BV_{loc}(\mathbb{R})$ and satisfying the following conditions

$$\int_{-\infty}^{+\infty} |x|^2 |d_0(x)| < \infty$$

(1.13)

and

$$\int_{\mathbb{R}} e^{2\sigma|x|} |q_0(x) - c_\pm|dx < \infty,$$

(1.14)

where $\sigma > \sqrt{c_-^2 - c_+^2} > 0$, and $d_0(x)$ is the corresponding signed measure (distributional derivative of $q_0(x)$).
This class includes the case of exact (discontinuous) step function.

**Theorem 1.2.** Under condition [1.1] the initial value problem of the MKDV equation (1.1) has a classical solution for all \( t > 0 \).

Under assumption [1.1], the inverse of the transmission coefficient \( a(k) \) is analytic for \( k \in \mathbb{C}_+ \setminus [ic, 0] \), and it can be extended continuously up to the boundary, with the exception of the points \( ic_- \), \( ic_+ \), where \( a(k) \) may have at most a fourth root singularity (see Lemma 2.2). The zeros of \( a(k) \), that we assume to be generically simple, form the point spectrum and by analyticity, the number of zeros is finite. We fix the number of zeros in the quarter plane \( \text{Im } k > 0 \) and \( \text{Re } k \geq 0 \) equal to \( N \).

In order to formulate our result we need to number the speeds of the solitons and the breathers corresponding to the discrete spectrum. We recall that a soliton with point spectrum \( \kappa_0 \) on a constant background \( c \) has a speed \( 2c^2 + 4\kappa_0^2 \), while a breather with point spectrum \( \kappa_1 + i\kappa_2 \) on a constant background \( c \) has the speed specified in (1.6). To simplify the exposition, we assume the following.

**Assumption 1.3.** We assume that the speed \( V \) of each soliton and breather is outside the range of the speed of the dispersive shock wave, namely \( V < -6c_2^- + 12c_2^+ \) or \( V > 4c_2^- + 2c_2^+ \). Furthermore we assume that breathers and solitons have distinct velocities.

Under the above assumptions all the solitons and breathers that move in the positive \( x \)-direction have a speed bigger than \( 4c_2^- + 2c_2^+ \). We number the speeds of the corresponding solitons and breathers on a constant background \( c_+ \) are ordered as

\[
V_1 > V_2 > \cdots > V_n > 4c_2^- + 2c_2^+.
\]

We number the velocities of the \( N - n \) breathers moving to the negative \( x \)-direction to the left of the dispersive shock wave and on the constant background \( c_- \) as

\[
V_N < V_{N-1} < \cdots < V_{n+1} < -6c_2^- + 12c_2^+.
\]

To each velocity \( V_j \) we number the corresponding point spectrum as \( \kappa_j \) with \( \text{Im } \kappa_j > 0 \) and \( \text{Re } \kappa_j \geq 0 \), and norming constant \( \nu_j \).

The question we address is how, under assumptions [1.1] and [1.3] in the generic case, the solitons, breathers and the dispersive shock wave interact in the long time, (see Figure 2). Theorem 1.4 characterises these interactions in the general setting and (1.16), (1.17) and (1.19) below show explicitly how these interactions affect the asymptotic phase shifts of individual solitons, breathers and the dispersive shock wave.

![Figure 2](image-url)  
Figure 2: On the left a step-like initial data, on the right its evolution into a breather with negative speed, a dispersive shock wave and a soliton.

Now define

\[
\tilde{T}_j(k) = \prod_{l<j, \text{Re } \kappa_l > 0} \frac{k - \kappa_l}{k - \kappa_l}^{k + \kappa_l - \kappa_l} \cdot \prod_{l<j, \text{Re } \kappa_l = 0} \frac{k - \kappa_l}{k - \kappa_l}^{k + \kappa_l}, \quad j \geq 2, \quad \tilde{T}_1(k) = 1.
\]  
(1.15)
Theorem 1.4. For step-like initial data, satisfying Conditions 1.1, 1.3, the solution of the Cauchy problem for the MKDV equation behaves for large $t$ in the following way:

(a) Soliton and breather region: $\frac{x}{t} > 4c^2_+ + 2c^2_+ + \delta_1$, $\delta_1 > 0$. For $\frac{x}{t}$ such that $|\frac{x}{t} - V_j| > \varepsilon$ for $j = 1, \ldots, n$,

$$q(x, t) = c_+ + O(e^{-Ct}),$$

with some $C > 0$;

for $|\frac{x}{t} - V_j| < \varepsilon$, $j = 1, \ldots, n$,

$$q(x, t) = \begin{cases} 
q_{\text{soliton}}(x, t; c_+, \kappa_j, x_j) + O(e^{-Ct}), & \text{if } \Re \kappa_j = 0, \\
q_{\text{breather}}(x, t; c_+, \kappa_j, \hat{\nu}_j) + O(e^{-Ct}), & \text{if } \Re \kappa_j > 0,
\end{cases} \tag{1.16}$$

where $q_{\text{soliton}}$ and $q_{\text{breather}}$ are defined in (1.4), (1.5) respectively and

$$\hat{\nu}_j = \frac{\nu_j}{T_j^2(\kappa_j)} , \quad x_j = \ln \left( \frac{|\kappa_j|^2 - c^2_+ T^2_j(i|\kappa_j|)}{|\nu_j||\kappa_j|} \right)$$

and

$$T_j(k) = \tilde{T}_j(k) \cdot \exp \left[ \frac{\sqrt{k^2 + c^2_+}}{2\pi i} \int_{ic_+}^{ic_-} \frac{\ln \tilde{T}_j^2(s)}{(s-k)\sqrt{s^2 + c^2_+}} ds \right] ,$$

with $\tilde{T}_j(k)$ as in (1.15).

(b) Dispersive shock wave region: $-6c^2_+ + 12c^2_+ + \delta_1 < \frac{x}{t} < 4c^2_+ + 2c^2_+ - \delta_1$, $\delta_1 > 0$, one has

$$q(x, t) = q_{\text{per}}(x, t; c_-, d, c_+, x_0) + O(t^{-1}) , \tag{1.17}$$

where the travelling wave $q_{\text{per}}$ has been defined in (1.7), the quantity $d = d(x, t)$ depends on $x$ and $t$ as in (1.14) and the phase

$$x_0 = -\frac{K(m)\Delta}{\pi} + K(m)$$

depends on the discrete and continuous spectrum via the relation

$$\Delta(x, t) = -\frac{\sqrt{c^2_--c^2_+}}{K(m)} \left[ \int_{a(x,t)}^{ic_-} \frac{\ln(|a(s)|^2\tilde{T}_n^2(s))} {\sqrt{(s^2 + c^2_+)(s^2 + c^2_-)(s^2 + d^2)}} ds + \int_{ic_+}^{ic_-} \frac{\ln \tilde{T}_n^2(s)} {\sqrt{(s^2 + c^2_+)(s^2 + c^2_-)(s^2 + d^2)}} ds \right] \tag{1.18}$$

with $\tilde{T}_n(s)$ as in (1.15) for $j = n$ and $a(k)$ is the inverse of the transmission coefficient associated to the Zakharov-Shabat spectral problem.

(c) Breathers on a constant background: $\frac{x}{t} < -6c^2_+ + 12c^2_+ - \delta_1$, $\delta_1 > 0$. For $t$ large and such that $|\frac{x}{t} - V_j| > \varepsilon$ for all $V_j$,

$$q(x, t) = c_- + O(t^{-1/2}) ,$$

and for $x/t$ such that $|\frac{x}{t} - V_j| < \varepsilon$ for $j = n+1, \ldots, N$,

$$q(x, t) = q_{\text{breather}}(x, t; c_-, \kappa_j, \hat{\nu}_j) + O(t^{-1/2}) , \tag{1.19}$$

where $q_{\text{breather}}$ is defined in (1.5) and the phase $\nu_j$

$$\hat{\nu}_j = \frac{\nu_j}{T_j^2(\kappa_j, V_j^{-2})}.$$
with \[ T_j(k, \xi) = \tilde{T}_j(k) \exp \left[ \frac{-\sqrt{k^2 + c_-^2}}{2\pi i} \left\{ \int_{i\xi_-}^{id_0(s)} - \int_{i\xi_-}^{ic_-} \ln |a(s)|^2 ds \right\} \right] \]

for \(-\frac{c_-^2}{2} < \xi < -\frac{c_-^2}{2} + c_+^2\) with \(d_0(\xi) = i\sqrt{\xi + \frac{c_-^2}{2}}\), and

\[ T_j(k, \xi) = \tilde{T}_j(k) \exp \left[ \frac{-\sqrt{k^2 + c_-^2}}{2\pi i} \left\{ \int_{i\xi_-}^{0} - \int_{i\xi_-}^{ic_-} \ln |a(s)|^2 ds \right\} \right] \]

\[ + \int_{ic_-}^{id_1(s)} \ln \tilde{T}_j^2(s) ds = \int_{id_1(\xi)}^{ic_-} \ln (1 + \sqrt{s^2 + c_-^2}) ds \]

for \(\xi < -\frac{c_-^2}{2}\) and \(d_1(\xi) = \sqrt{-\xi - \frac{c_-^2}{2}}\) and \(\tilde{T}_j(k)\) as in \((1.15)\).

Our analysis is obtained by formulating the inverse scattering problem for the MKDV equation with step initial data as a Riemann-Hilbert problem and then we implement the long time asymptotic analysis via the Deift-Zhou steepest descent method \[20\]. The case in which solitons or breathers overlap the dispersive shock wave, we do not get an explicit formula for the solution, but we leave the result at the level of the Riemann-Hilbert problem as illustrate in the Appendix E. We illustrate our results with the following examples.

**Example 1.5.** For the exact step \((1.9)\) one has that the reflection coefficient and the inverse of the transmission coefficients are \[20\]

\[ r(k) = \frac{\gamma(k)^2 - 1}{\gamma(k)^2 + 1}, \quad a(k) = \frac{1}{2} \left( \gamma(k) + \frac{1}{\gamma(k)} \right), \quad \gamma(k) = \left( \frac{k - ic_-}{k + ic_-} \right)^{\frac{1}{2}}. \]

The dispersive shock wave is given by the relation \((1.17)\) with phase shift

\[ x_0 = -\frac{c_-^2 - c_+^2}{\pi} \int_{id_0(x, t)}^{ic_-} \ln |a(s)|^2 ds \]

\[ \sqrt{(s^2 + c_-^2)(s^2 + c_+^2)(s^2 + d^2)} + K(m), \]

The evolution for such initial data is illustrated in Figure 3. The leading edge of the oscillatory region has been studied in \[39\], where it has been identified with a train of asymmetric solitons, and it has been shown that the amplitude of the first soliton is approximately described

\[ q(x, t) \simeq q_{\text{soliton}}(x, t; c_+, c_-, \tilde{x}_0) \]

where \(\tilde{x}_0 = \frac{4}{5} \log t + \alpha\) for some constant \(\alpha\). Here \(q_{\text{soliton}}(x, t; c_+, c_-, \tilde{x}_0)\) is the soliton solution \((1.4)\) on the constant background \(c_+\) with spectrum \(c_-\). The highest peak of the first soliton is approximately located at the position \(x_0(t) = 2(c_+^2 + 2c_-^2)t - \frac{2}{5} \log t + \alpha\). The transition region between the dispersive shock wave and asymptotic solitons turned out to be very rich, and has been studied recently in \[37\].

**Example 1.6.** For an initial function in the form of a soliton on a constant background \((1.4)\) on the left, and a constant on the right, i.e.

\[ q_0(x) = \begin{cases} q_{\text{soliton}}(x, 0; c_-, \kappa_0, x_0), & x < 0, \quad \text{where} \quad \kappa_0 > c_- > 0, \\ c_+, & x > 0, \quad \text{where} \quad c_+ \in \mathbb{R}. \end{cases} \]
Figure 3: The initial data in blue for $c_- = 0.8$ and $c_+ = 0.4$. The black line shows the evolution at time $t = 10$.

Here $x_0 = \log \frac{2(\kappa_0^2 - c_-^2)}{|\nu|\kappa_0} \in \mathbb{R}$ and $\nu \neq 0$. We have that the reflection coefficient $r(k) = b(k)/a(k)$ and the transmission coefficient $1/a(k)$ are given as follows:

$$
\begin{pmatrix}
a(k) \\
b(k)
\end{pmatrix}
= 
\begin{pmatrix}
a_+(k) & -b_+(k) \\
-b_+(k) & a_+(k)
\end{pmatrix}
\begin{pmatrix}
1 + \frac{i\alpha}{k-\kappa_0} + \frac{i\gamma}{k\kappa_0} & \frac{i\beta}{k+i\kappa_0} + \frac{i\delta}{k-i\kappa_0} \\
\frac{i\beta}{k+i\kappa_0} - \frac{i\delta}{k-i\kappa_0} & 1 - \frac{i\alpha}{k-\kappa_0} - \frac{i\gamma}{k\kappa_0}
\end{pmatrix}
\begin{pmatrix}
a_-(k) \\
b_-(k)
\end{pmatrix},
$$

where

$$
a_\pm(k) = \frac{1}{2} \left( \gamma_\pm(k) + \gamma_\pm(k)^{-1} \right), \quad b_\pm(k) = \frac{1}{2} \left( \gamma_\pm(k) - \gamma_\pm(k)^{-1} \right), \quad \gamma_\pm(k) = \sqrt{\frac{k + i\kappa_\pm}{k - i\kappa_\pm}},
$$

and

$$
\beta = \nu \sqrt{\kappa_0^2 - c_-^2} \left\{ -c_- \nu \kappa_0 + 2(\kappa_0^2 - c_-^2)(\kappa_0 + \sqrt{\kappa_0^2 - c_-^2}) \right\} / \left\{ 4(\kappa_0^2 - c_-^2)^2 - 4(\kappa_0^2 - c_-^2)\nu c_+ + \kappa_0^2 \nu^2 \right\},
$$

$$
\delta = \nu \sqrt{\kappa_0^2 - c_-^2} \left\{ c_- \nu \kappa_0 - 2(\kappa_0^2 - c_-^2)(\kappa_0 - \sqrt{\kappa_0^2 - c_-^2}) \right\} / \left\{ 4(\kappa_0^2 - c_-^2)^2 - 4(\kappa_0^2 - c_-^2)\nu c_- + \kappa_0^2 \nu^2 \right\},
$$

$$
\alpha = \delta / c \left( \kappa_0 + \sqrt{\kappa_0^2 - c_-^2} \right), \quad \gamma = -\beta / c \left( \kappa_0 - \sqrt{\kappa_0^2 - c_-^2} \right).
$$

The discrete spectrum of the soliton shifts a bit from the point $i\kappa_0$ to some near point on the imaginary axis. The evolution of $q(x, t)$ in the limit $t \to +\infty$ is characterised by a soliton to the right of the dispersive shock wave. This initial data corresponds to a fast passage of the soliton through the dispersive shock wave.

**Example 1.7.** We consider an initial function in the form of a constant $c > 0$ on the left, and a soliton on the right, i.e., for

$$
q_0(x) =
\begin{cases}
  c > 0, & x < 0, \\
  -\frac{2c\tanh\nu}{\cosh[2\kappa_0(x-x_0)]}, & x > 0,
\end{cases}
$$

where $\nu \in \mathbb{R} \setminus \{0\}$, $c > \kappa_0 > 0$, $x_0 = \frac{1}{2\kappa_0} \ln \left| \frac{c}{\nu} \right|$. This initial data does not satisfy the decay at infinity of Assumption [1.7]. Below we show that the reflection coefficient is singular on the imaginary axis $\{ic, 0\}$. The reflection coefficient is $r(k) = \frac{b(k)}{a(k)}$. 

10
Figure 4: The initial data in blue for $c_- = 0.8$ and $c_+ = 0.4$, $\kappa_0 = 1$ and $x_0 = 10$, and $\nu > 0$. The black line shows the evolution at time $t = 4$ and $t = 9$.

with

$$a(k) = \frac{1}{2} \left[ (\gamma(k) + \gamma(k)^{-1}) \left(1 - \frac{i\alpha}{k + i\kappa_0}\right) - (\gamma(k) - \gamma(k)^{-1}) \frac{i\beta}{k + i\kappa_0} \right],$$

$$b(k) = \frac{1}{2} \left[ (\gamma(k) - \gamma(k)^{-1}) \left(1 + \frac{i\alpha}{k - i\kappa_0}\right) - (\gamma(k) + \gamma(k)^{-1}) \frac{i\beta}{k - i\kappa_0} \right],$$

where

$$\gamma(k) = \sqrt{\frac{k - ic}{k + ic}}, \quad \alpha = \frac{2\nu^2\kappa_0}{4\kappa_0^2 + \nu^2}, \quad \beta = \frac{4\nu\kappa_0^2}{4\kappa_0^2 + \nu^2}.$$

The coefficient $b(k)$ has a pole at $k = i\kappa_0$ and the function $a(k)$ has two zeros $k_1, k_2$ in the half-plane $\text{Im} \ k \geq 0$:

$$k_{1,2} = \pm \beta \sqrt{c^2 + 2c\beta - (\alpha - \kappa_0)^2 + i(\alpha - \kappa_0)(\beta + c)} \frac{c + 2\beta}{c + 2\beta}$$

and they are symmetric w.r.t. imaginary axis: $k_1 = -k_2$. Let us notice that when $x_0 > 0$ is not small, and hence $\nu > 0$ is big, we have $\alpha \sim 2\kappa_0$, $\beta \sim 0$, and hence

$$k_{1,2} \sim \pm 0 + i\kappa_0.$$

Therefore the soliton part of the initial data is a breather from the spectral point of view, with point spectrum very close to the imaginary axis. The velocity of the breather is approximately $4\kappa_0^2 < 4c^2$ while the velocity of the dispersive shock wave is $-6c^2 < \frac{c}{\nu} < 4c^2$. Namely the breather has a positive velocity that is smaller than the velocity of the leading front of the dispersive shock wave and it will remain trapped by the dispersive shock wave. Furthermore, since $b(k)$ has a pole at $k = i\kappa_0$, also the reflection coefficient has a pole at $k = i\kappa_0$, and this pole requires a quite delicate asymptotic analysis that is beyond the scope of the present article. In the physical literature [53], this phenomenon received a name of “soliton trapping” inside the dispersive shock wave (see Figure 5).

This manuscript is organized as follow. We study the main properties of the scattering data and the solvability of the associated Riemann-Hilbert problem in section 2. In section 3 we formulate and solve the Riemann-Hilbert problems associated to the model problems that will be used in asymptotic analysis, namely the Riemann-Hilbert problem for a breather and a soliton on a constant background and then the model problems for the periodic solution that is expressed via hyperelliptic curves. We will then show how to reduce the hyperelliptic solution to a travelling wave solution of MKDV.

In section 4 we start the asymptotic analysis introducing the $g$-function and performing the contour deformation to arrive at our main result, namely the proof of theorem 1.4. Several Appendices are used to prove the most technical results.
2 Preliminaries

The MKDV equation (1.1) admits a Lax pair representation in the form [55],

$$\Phi_x(x, t; k) = (-ik\sigma_3 + Q(x, t))\Phi(x, t; k),$$  \hspace{1cm} (2.1)

$$\Phi_t(x, t; k) = (-4ik^3\sigma_3 + \tilde{Q}(x, t; k))\Phi(x, t; k),$$  \hspace{1cm} (2.2)

where

$$Q(x, t) = \begin{pmatrix} 0 & q(x, t) \\ -q(x, t) & 0 \end{pmatrix}, \quad \tilde{Q}(x, t; k) = 4k^2Q - 2ik(Q^2 + Q_x)\sigma_3 + 2Q^3 - Q_{xx}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \hspace{1cm} (2.3)$$

If we substitute $q(x, t) = c$ (constant) in (2.3), then the Lax pair equations (2.1), (2.2) admit explicit solutions [40]

$$E_0(x, t; k) = e^{-ikx + 4ik^3t}\sigma_3,$$

for $c = 0$ and

$$E_c(x, t; k) = \frac{1}{2} \begin{pmatrix} \gamma(k) + \gamma^{-1}(k) & \gamma(k) - \gamma^{-1}(k) \\ \gamma(k) - \gamma^{-1}(k) & \gamma(k) + \gamma^{-1}(k) \end{pmatrix} e^{-i\chi(k)k + 2it(2k^2 - c^2)c(k)}\sigma_3, \hspace{1cm} (2.4)$$

for $c \neq 0$. For brevity we denote $E_+ \equiv E_{c+}$, $E_- \equiv E_{c-}$. Now we will define the Jost solutions $\Phi_\pm(x; k)$, which satisfy the equation (2.1), and have the (defining) property that

$$\Phi_\pm(x; k) = E_\pm(x, 0; k) + o(1) \quad \text{as} \quad x \to \pm \infty, \quad \text{Im} \ k = 0.$$

The Jost solutions have the following integral representation

$$\Phi_\pm(x; k) = E_\pm(x, 0; k) + \int_{-\infty}^{x} L_\pm(x, y)E_\pm(y, 0; k)dy \hspace{1cm} (2.5)$$

where the kernels $L_\pm(x, y)$ are studied in the Appendix C. The following two Lemmas appeared before in [40], [41], but with much stronger assumptions on the initial data.

**Lemma 2.1.** The Jost solutions (2.5) their columns $\Phi_{-, j}(x; k), \Phi_{+, j}(x; k), j = 1, 2$ and their entries $\Phi_{\pm, ij}$ have the following properties:

1. $\det \Phi_\pm(x; k) = 1$;
2. \( \Phi_{+,1}(x;k) \) is analytic in \( k \in \mathbb{C}_- \setminus [0, -ic_+] \), and continuous up to the boundary \( \Phi_{+,2}(x;k) \) is analytic in \( k \in \mathbb{C}_+ \setminus [0, ic_+] \) and continuous up to the boundary, \( \Phi_{-,1}(x;k) \) is analytic in \( k \in \mathbb{C}_+ \setminus [0, ic_-] \) and continuous up to the boundary, \( \Phi_{-,2}(x;k) \) is analytic in \( k \in \mathbb{C}_- \setminus [0, -ic_-] \) and continuous up to the boundary; here \( \mathbb{C}_\pm = \{ k : \pm \Im k > 0 \} \).

3. symmetry:

\[
\begin{align*}
\Phi_{22}(x;k) &= \Phi_{11}(x;k), & \Phi_{22}(x;-k) &= \Phi_{11}(x;k), \\
\Phi_{12}(x;k) &= -\Phi_{21}(x;k), & \Phi_{12}(x;-k) &= -\Phi_{21}(x;k), \\
\Phi_{ji}(x;-\bar{k}) &= \Phi_{ji}(x;k), & j, l = 1, 2,
\end{align*}
\]

where \( \Phi(x;k) \) denotes \( \Phi_-(x;k) \) or \( \Phi_+(x;k) \).

4. large \( k \) asymptotics:

\[
\begin{align*}
\Phi_{+,1}(x;k)e^{ikx} &= 1 + O \left( \frac{1}{k} \right), \quad k \to \infty, \quad \Im k \leq 0, \\
\Phi_{-,1}(x;k)e^{-ikx} &= 1 + O \left( \frac{1}{k} \right), \quad k \to \infty, \quad \Im k \geq 0;
\end{align*}
\]

5. jump conditions:

\[
\Phi(x;k-0) = \Phi(x;k+0) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in (ic_-, -ic_+),
\]

where \( \Phi(x;k) \) and \( c \) denote \( \Phi_-(x;k) \) and \( c_- \) or \( \Phi_+(x;k) \) and \( c_+ \), respectively, and \( \Phi(x;k \pm 0) \) denote non-tangential limits of matrix \( \Phi(x;k) \) from the left (−) and from the right (+) of the segment \( (ic_-, -ic_+) \), which is oriented from \( ic_+ \) to \( -ic_- \). The positive side is on the left when traversing along the contour.

The proof of the Lemma is very similar to the case considered in [40] and is given in the Appendix C.

Since the matrix-valued functions \( \Phi_-(x;k) \) and \( \Phi_+(x;k) \) are solutions of first order differential equation (2.1), they are linearly dependent, i.e. there exists an independent of \( x, t \) transition matrix \( T(k) \), such that

\[
T(k) = \Phi_+^{-1}(x;k)\Phi_-(x;k), \quad k \in \mathbb{R} \cup (ic_-, -ic_+).
\]

Due to symmetries of the \( x \)-equation (2.1) the transition matrix has the following structure:

\[
T(k) = \begin{pmatrix} a(k) & -b(k) \\ b(k) & a(k) \end{pmatrix}, \quad \text{where } a(k) = \det (\Phi_{-,1}, \Phi_{+,2}), \; b(k) = \det (\Phi_{+,1}, \Phi_{-,1}).
\]

The functions \( a^{-1}(k) \) and \( r(k) := \frac{b(k)}{a(k)} \) are called the (right) transmission and reflection coefficients, respectively.

Lemma 2.2. Under conditions (1.13) and (1.14) the spectral functions \( a(k), b(k), r(k) \) have the following properties:

6. analyticity:

- \( a(k) \) is analytic for \( k \in \mathbb{C}_+ \setminus [ic_-, 0] \), and it can be extended continuously up to the boundary, with the exception of the points \( ic_-, ic_+ \), where \( a(k) \) may have at most a fourth root singularity \((k - ic_\pm)^{-1/4}\); the function \( a(k) \) might have at most a finite number of zeros in \( k \in \mathbb{C}_+ \setminus [ic_-, 0] \); the function \( b(k) \) is analytic for \( k \in \mathbb{C}_- \setminus [0, -ic_-] \), and the function \( r(k) \) is defined in \( k \in \mathbb{R} \cup (ic_+, -ic_-) \) except for the points where \( a(k) = 0 \); under the condition (1.14), the function \( r(k) \) is meromorphic in the \( \delta = \sqrt{a^2 + c_2^2} - c_- \) neighborhood of \( \Sigma = \mathbb{R} \cup [ic_-, -ic_-] \) with poles at zeros of \( a(k) \);
7. asymptotics:

\[ a(k) = 1 + O(k^{-1}) \quad \text{as} \quad k \to \infty, \Im k \geq 0; \]
\[ r(k) = O(k^{-1}); \]

8. symmetries and determinants: in their domains of definition

\[ a(-\overline{k}) = a(k), \quad b(-\overline{k}) = b(k), \quad r(-\overline{k}) = r(k), \quad a(k)a(-\overline{k}) + b(k)b(-\overline{k}) = 1; \]

9. non vanishing:

\[ a(k), b(k) \text{ do not vanish on the segment } (ic_-, ic_+) \cup (-ic_+, -ic_-); \]

10. \[ \frac{(\Phi_{-1})_{-}(x;k)}{a_{-}(k)} - \frac{(\Phi_{-1})_{+}(x;k)}{a_{+}(k)} = f(k)\Phi_{+2}(x;k), \quad k \in (ic_-, ic_+); \]

11. \[ \frac{(\Phi_{-2})_{-}(x;k)}{a_{-}(k)} - \frac{(\Phi_{-2})_{+}(x;k)}{a_{+}(k)} = -\overline{f(k)}\Phi_{+1}(x;k), \quad k \in (-ic_+, -ic_-), \]

where

\[ f(k) := \frac{i}{a_{-}(k)a_{+}(k)} \quad k \in (0, ic_-) \]

and we denote by \((\Phi_{-1})_{\pm} \) the boundary values of the first column of \( \Phi_+ \) on the different sides of the interval \((ic_-, -ic_-)\) and similarly for the other quantities.

13. on \((ic_+, -ic_+)\):

\[ a_{-}(k) = a_{+}(\overline{k}), \quad b_{-}(k) = -b_{+}(\overline{k}), \quad r_{-}(k) = -r_{+}(\overline{k}), \]

on \((ic_+, 0)\):

\[ f(k) = \frac{i}{a_{-}(k)a_{+}(k)} = i(1 - r_{-}r_{+}); \]

14. on \((ic_-, ic_+) \cup (-ic_+, -ic_-):

\[ a_{-}(k) = -i b_{+}(\overline{k}), \quad a_{+}(k) = i b_{-}(\overline{k}), \quad b_{-}(k) = i a_{+}(\overline{k}), \quad b_{+}(k) = -i a_{-}(\overline{k}), \]

on \((ic_-, ic_+)\):

\[ f(k) = \frac{i}{a_{-}(k)a_{+}(k)} = r_{-}(k) - r_{+}(k) = -\frac{1}{a_{+}(k)b_{+}(\overline{k})} = \frac{1}{a_{-}(k)b_{-}(\overline{k})}. \]

**Proof.** Properties 6 follow from the definition of the functions \(a(k), b(k), r(k)\) and the corresponding properties of \( \Phi_{\pm}(x;k) \). Furthermore, the asymptotics 7 follow from Corollary C.2. Symmetries 8 follow from symmetries of \( \Phi_{\pm} \) and the determinantal property follows from the fact that \( \det T \equiv 1 \). Properties 10, 11, 12, 13 follow from property 6 of Lemma 2.1. Finally, property 9 follows after using the relations

\[ a(k)a(-\overline{k}) + b(k)b(-\overline{k}) = 1, \quad a(-\overline{k}) = a(k), \quad b(-\overline{k}) = b(k), \quad a_{-}(k) = -ib_{+}(\overline{k}), \quad (2.8) \]

which imply that \( a_{\pm}(k) \neq 0, b_{\pm}(k) \neq 0 \) on \((ic_-, ic_+) \cup (-ic_+, -ic_-).

Let us denote the first and second columns of the Jost solution \( \Phi_{-} \) and \( \Phi_{+} \) in (2.5) as

\[ \Phi_{-1}(x, t; k) = \begin{pmatrix} \varphi^{-}(x, t; k) \\ \psi^{-}(x, t; k) \end{pmatrix}, \quad \Phi_{+2}(x, t; k) = \begin{pmatrix} \varphi^{+}(x, t; k) \\ \psi^{+}(x, t; k) \end{pmatrix}, \]
Lemma 2.3. Denote the zeros of \(a(k)\) in \(\text{Im } k \geq 0\) and \(\text{Re } k \geq 0\) by \(\kappa_1, \ldots, \kappa_N\). We have \(\kappa_j \notin [\text{ic}_-, \text{ic}_+]\) and if \(\text{Re } \kappa_j \neq 0\), then \(a(-\pi j) = 0\). If the zeros of \(a(k)\) are simple then the residues of \(a^{-1}(k)\) are given by

\[
\text{Res}_{k=\kappa_j} a^{-1}(k) = \frac{i \nu_j}{\mu_j}
\]

where

\[
(2\nu_j)^{-1} := \int_{-\infty}^{+\infty} \varphi^+(x; \kappa_j) \psi^+(x; \kappa_j) dx,
\]

and

\[
\varphi^-(x, t; \kappa_j) = \mu_j \varphi^+(x, t; \kappa_j), \quad \psi^-(x, t; \kappa_j) = \mu_j \psi^+(x, t; \kappa_j).
\]

The lemma is proven in Appendix D.

2.1 Riemann-Hilbert problem in the generic case

The scattering relations \((2.6)\) between the matrix-valued functions \(\Phi_- (x; k)\) and \(\Phi_+ (x; k)\), and the jump conditions 6, 7, 8, can be written as a matrix Riemann–Hilbert problem (RH). Namely, let us notice that the matrix-valued function

\[
M(x, 0; k) = \begin{cases}
\left( \frac{\Phi_{-1}(x; k)}{a(k)} e^{ikx}, \frac{\Phi_{+2}(x; k)e^{-ikx}}{a(k)} \right), & k \in \mathbb{C}_+ \setminus [0, i \text{c}_-], \\
\left( \frac{\Phi_{+1}(x; k)e^{ikx}}{a(k)} , \frac{\Phi_{-2}(x; k)e^{-ikx}}{a(k)} \right), & k \in \mathbb{C}_- \setminus [-i \text{c}_-, 0],
\end{cases}
\]

solves the following Riemann-Hilbert problem (RHP) at the initial value of time \(t = 0\) \((2.10)\).

Let us define the oriented contour \(\Sigma = \mathbb{R} \cup (i \text{c}_-, -i \text{c}_-)\) as in Fig. 6

\[
\begin{pmatrix}
1 & 0 \\
-r(k)e^{i \sigma(k, \xi)} & \frac{1}{1 + |r(k)|^2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
\frac{r(k)e^{i \sigma(k, \xi)}}{1 + |r(k)|^2} & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{1 - f(k)e^{-2i \sigma(k, \xi)}} & 0 \\
\frac{f(k)e^{-2i \sigma(k, \xi)}}{1 + f(k)e^{-2i \sigma(k, \xi)}} & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{|f(k)e^{-2i \sigma(k, \xi)}} & 0 \\
\frac{-f(k)e^{-2i \sigma(k, \xi)}}{1 + f(k)e^{-2i \sigma(k, \xi)}} & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{|f(k)e^{-2i \sigma(k, \xi)}} & 0 \\
\frac{-f(k)e^{-2i \sigma(k, \xi)}}{1 + f(k)e^{-2i \sigma(k, \xi)}} & 1
\end{pmatrix}
\]

Figure 6: The oriented contour \(\Sigma = \mathbb{R} \cup [i \text{c}_-, i \text{c}_-]\) and the corresponding jump matrix \(J(x, t; k)\).

Riemann-Hilbert problem 1. To find a \(2 \times 2\) matrix-valued function \(M(x, t; k)\) with the following properties:

1. \(M(x, t; k)\) is meromorphic in \(\mathbb{C} \setminus \Sigma\), and it has at most fourth root singularities at the points \(\pm \text{ic}_\pm\).

2. The boundary values \(M_\pm (x, t; k)\) on the oriented contour \(\Sigma\) satisfy the jump condition

\[
M_- (x, t; k) = M_+ (x, t; k) J(x, t; k), \quad k \in \Sigma
\]
and

\[
J(x, t; k) = \begin{cases} 
\begin{pmatrix} 1 & -r(k)e^{-2i\theta(k, \xi)} \\ -r(k)e^{2i\theta(k, \xi)} & 1 + |r(k)|^2 \end{pmatrix}, & k \in \mathbb{R} \setminus \{0\}, \\
\begin{pmatrix} 1 & 0 \\ f(k)e^{2i\theta(k, \xi)} & 1 \end{pmatrix}, & k \in (i\epsilon, i\epsilon_+), \\
\begin{pmatrix} ir_-(-k) & ie^{2i\theta(k, \xi)} \\ f(k)e^{-2i\theta(k, \xi)} & -ir_+(k) \end{pmatrix}, & k \in (i\epsilon_+, 0), \\
\begin{pmatrix} i\epsilon_+(-k) & -f(k)e^{-2i\theta(k, \xi)} \\ ie^{2i\theta(k, \xi)} & -i\epsilon_-(-k) \end{pmatrix}, & k \in (0, -i\epsilon_+), \\
\begin{pmatrix} 1 & -f(k)e^{-2i\theta(k, \xi)} \\ 0 & 1 \end{pmatrix}, & k \in (-i\epsilon_+, -i\epsilon_-),
\end{cases}
\]

where \(r(k)\) is the reflection coefficient of the spectral problem of the MKDV, and

\[
f(k) = r_-(k) - r_+(k) = \frac{i}{a_-(-k)a_+(k)}, \quad k \in (i\epsilon_-, i\epsilon_+) \cup (-i\epsilon_+, -i\epsilon_-),
\]

where \(a^{-1}(k)\) is the transmission coefficient.

3. Simple poles: residue conditions: for \(j = 1, \ldots, N\) poles in the upper half-plane:

\[
\text{Res}_{\kappa_j} M(x, t; k) = \lim_{k \to \kappa_j} M(x, t; k) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
\text{Res}_{-\kappa_j} M(x, t; k) = \lim_{k \to -\kappa_j} M(x, t; k) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

poles in the lower half-plane:

\[
\text{Res}_{\kappa_j} M(x, t; k) = \lim_{k \to -\kappa_j} M(x, t; k) \begin{pmatrix} 0 & i\epsilon_j e^{2i\theta(k, \xi)} \\ 0 & 0 \end{pmatrix},
\]

\[
\text{Res}_{-\kappa_j} M(x, t; k) = \lim_{k \to \kappa_j} M(x, t; k) \begin{pmatrix} 0 & i\epsilon_j e^{2i\theta(k, \xi)} \\ 0 & 0 \end{pmatrix}.
\]

4. Asymptotics: \(M(x, t; k) \to I\) as \(k \to \infty\).

The solution \(M(x, t; k) = M(k)\) of the RHP \(\Pi\) automatically satisfies the following symmetries:

\[
M(k) = M(-\overline{k}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M(-k) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M(\overline{k}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The jump matrices in the RH problem \(\Pi\) satisfy the Schwartz symmetry \(J^{-1}(k) = \left( \overline{J^T(k)} \right)\) for \(k \in \Sigma \setminus \mathbb{R}\). Hence, it follows from the vanishing lemma \([30]\) that the solution of the RH problem \(\Pi\) exists. Further, from our assumption on the initial data it follows the analyticity of the reflection coefficient in a small neighbourhood of the contour \(\Sigma\), (see Lemma \([2.2]\)). It follows that we can deform the contour \(\Sigma\) into a new contour such that the corresponding singular integral equation, which is equivalent to the RHP \(\Pi\) admits differentiation with respect to \(x\) and \(t\). Then, in the spirit of the well-known result of Zakharov – Shabat \([33]\), one can prove that the solution of the initial value problem \((1.1), (1.2)\) can be reconstructed by the following formula (see \([32]\), chapter 2 for details):

\[
q(x, t) = \lim_{k \to \infty} (2ik M(x, t; k))_{12} = \lim_{k \to \infty} (2ik M(x, t; k))_{12}.
\]

In order to make the asymptotic analysis of the above RH problem as \(t \to \infty\) it is more convenient to transform the residue conditions to a jump conditions as in \([32]\).

Let \(\kappa_j\) be a pole of \(a(k)\). Let us encircle this pole with a circle \(C_j\) of radius \(\epsilon > 0\). There are several options:
where $\nu$ is standard \[21\], that the RH problem is equivalent to the

Step 1. Solvability of RHP 1.

The proof is the same as in \[50\]. For the convenience of the reader, we will sketch the main
classical solution, which is

\[(1.2)\]

Then the jump matrix

\[M(k) \rightarrow M(k) \left( \begin{array}{cc} 1 \nu e^{2\theta(x,t,k)} & 0 \\ \frac{1}{k^2 - \kappa_j^2} \end{array} \right), \quad k \text{ is inside } C_j,
M(k) \rightarrow M(k) \left( \begin{array}{cc} 0 & 1 \\ \frac{1}{k^2 - \kappa_j^2} \end{array} \right), \quad k \text{ is inside } C_j,
M(k) \rightarrow M(k) \left( \begin{array}{cc} 1 \nu e^{-2\theta(x,t,k)} & 0 \\ \frac{1}{k^2 - \kappa_j^2} \end{array} \right), \quad k \text{ is inside } C_j,
M(k) \rightarrow M(k) \left( \begin{array}{cc} 0 & 1 \\ \frac{1}{k^2 - \kappa_j^2} \end{array} \right), \quad k \text{ is inside } C_j,
\]

where $J(k)$ has been defined in \[2.11\].

2.2 Solvability of RHP and existence of solution for the MKDV

Theorem 1. Let initial data \[1.2\] satisfies Condition \[1.1\]. Then the iep \[1.1\], \[1.2\] has a unique
classical solution, which is $C^\infty(\mathbb{R}_x) \times C^\infty(\mathbb{R}_t \setminus \{0\})$.

Proof. The proof is the same as in \[50\]. For the convenience of the reader, we will sketch the main
steps.

Step 1. Solvability of RHP \[1\]. It is standard \[21\], that the RH problem is equivalent to the
following singular integral equation for the function $\mu = M_+ + I$ :

\[\mu(x, t; k) = K[\mu](x, t; k) + F(x, t; k), \quad (2.15)\]

where

\[K[\mu] = \frac{1}{2\pi i} \int_{\Sigma} \frac{\mu(x, t; s) (I - J(x, t; s))}{(s - k)_+} ds, \quad F(x, t; k) = \frac{1}{2\pi i} \int_{\Sigma} \frac{(I - J(x, t; s))}{(s - k)_+} ds. \]

Further, the contour $\Sigma$ and the jump matrix $J(x, t; k)$ satisfy the Schwartz reflection principle \[50\]:

1. $\text{Re} \kappa > 0$, $\text{Im} \kappa > 0$. In this case $C_j$ is a circle of radius $\epsilon$ and centre $\kappa_j$ and oriented anti-
clockwise with respect to the centre. Let us also define the three other circles $C_j$, $-C_j$, $-\bar{C}_j$
with centre the points $\kappa_j$, $-\kappa_j$, $-\bar{\kappa}_j$, and radius $\epsilon$ and oriented anti-clockwise.

2. $\text{Im} \kappa = 0$, $\text{Re} \kappa > 0$. In this case $C_j$ is a semicircle in the upper half-plane with center at $\kappa_j$ and
radius $\epsilon$ and oriented anti-clockwise. We denote by $\bar{C}_j$ the semicircle in the lower half-plane
centred at $\kappa_j$, with anti-clockwise orientation with respect to the centre. Further, $-C_j$ and $-\bar{C}_j$
are the semicircles around the point $-\kappa_j$, with the same agreement on orientation.

3. $\kappa_j = 0$. This case is impossible for $c_+ = 0$. We define $C_j$, $\bar{C}_j$, $-C_j$, $-\bar{C}_j$ to be quarter-circles,
with the analogous agreement on the orientation as above.

4. $\text{Re} \kappa_j = 0$, $\text{Im} \kappa_j > 0$. We know that in this case $\kappa_j \in [0, \text{i}c_+ \cup (\text{i}c_-, +\text{i}c\infty]$. In this case we
denote $C_j$ a semicircle in the $\text{Re} k \geq 0$ around the point $\kappa_j$, while $-\bar{C}_j$ is another semicircle in
$\text{Re} k \leq 0$ around the point $\kappa_j$. Semicircles $\bar{C}_j$ and $-C_j$ are the corresponding semicircles in the
lower half-plane, with the above agreement on the orientation.

We replace the residue condition with a jump condition having only upper triangular matrices.
For the purpose we redefine the matrix $M(k)$ as

\[J_M(k) = \begin{cases}
\left( \begin{array}{cc}
1 & 0 \\
\nu e^{2\theta(x,t,k)} & 1
\end{array} \right), & k \in C_j, \\
\left( \begin{array}{cc}
0 & 1 \\
\nu e^{2\theta(x,t,k)} & 1
\end{array} \right), & k \in \bar{C}_j, \\
\left( \begin{array}{cc}
1 & 0 \\
\nu e^{-2\theta(x,t,k)} & 1
\end{array} \right), & k \in C_j, \\
\left( \begin{array}{cc}
0 & 1 \\
\nu e^{-2\theta(x,t,k)} & 1
\end{array} \right), & k \in -C_j,
\end{cases}
\]
• the contour $\Sigma$ is symmetric with respect to the real axis $\mathbb{R}$,
• $J(x, t; k)^{-1} = J(x, t; k)^T$ for $k \in \Sigma \setminus \mathbb{R}$,
• $J(x, t; k)$ has a positive definite real part for $k \in \mathbb{R}$.

Then theorem 9.3 from [60] (p.984) guarantees the $L_2$ invertibility of the operator $\Id - \mathcal{K}$ ($\Id$ is the identical operator). Therefore, the singular integral equation (2.15) has a unique solution $\mu \in L_2(\Sigma)$ for any fixed $x, t \in \mathbb{R}$, and the solution of the above RH problem can be obtained by the formula

$$M(x, t; k) = \Id + \frac{1}{2\pi i} \int_\Sigma \frac{(I + \mu(x, t; s))(I - J(x, t; s))ds}{s - k}.$$  

**Step 2.** Differentiability of $M(x, t; k)$ with respect to $x, t$. First of all we notice that it is impossible to differentiate the equation (2.15) with respect to $x, t$, since the function $r(k)$, as well as the matrix $I - J(x, t; k)$, vanishes as $k^{-1}$ as $k \to \infty$ along the real axis. To avoid a weak rate of decreasing of the matrix $I - J(x, t; k)$ for large real $k$, we use an equivalent RH problem on such a contour, where the jump matrix $J(x, t; k)$ for large $k$ becomes exponentially close to $I$. We must consider distinctively the case $t > 0$ and $t < 0$.

For the case $t > 0$, let take a positive $K > 0$. Then for $k > K$ and $k < -K$ we use the following factorization of the jump matrix:

$$J(x, t; k) = \begin{pmatrix} 1 & 0 \\ r(k)e^{2it\theta(k, \xi)} & 1 \end{pmatrix},$$

which allows to transform the RH problem [1] into another one, where the parts of the contour $(-\infty, -K), (K, +\infty)$ are split into two lines in the upper and lower complex plane. As a result, the jump $\tilde{J}(x, t; k)$ of the transformed RH problem is exponentially close to $I$ for large $k$ on the transformed contour $\tilde{\Sigma}$. The corresponding singular integral equation is as follows:

$$\tilde{\mu}(x, t; k) = \tilde{\mathcal{K}}[\tilde{\mu}](x, t; k) + \tilde{\mathcal{F}}(x, t; k),$$  

(2.16)

and for the same reasons as for (2.15), it has a unique solution from $L_2(\tilde{\Sigma})$. However, the equation (2.16) has that advantage, that we can differentiate it with respect to $x, t$ as many times as we wish. Indeed, while the function $I - J(x, t; k)$ vanishes as $\frac{1}{k}$ on $\mathbb{R}$, the function $I - \tilde{J}(x, t; k)$ decays exponentially on the infinite parts of the contour $\tilde{\Sigma}$. The singular integral equation, obtained by differentiating from (2.16), has the same form as (2.16) (only right-hand-side of this equation changes). It provides unique solvability of the partial derivatives of $\tilde{\mu}(x, t; k)$ with respect to $x, t$. Hence, the same is true for $M(x, t; k)$.

In the case $t < 0$ one would need to use upper-triangular factorization of the jump matrix on the real axis in order to split the infinite parts of the contour $\Sigma$ on the the real axis into upper and lower half planes.

**Step 3. Zakharov-Shabat scheme.** It is a standard Lax-pair argument (see, for instance, [50]) to show, that the function

$$\tilde{\Phi}(x, t; k) = \tilde{M}(x, t; k)e^{(ikx + 4ik^3t)\sigma_3}$$

satisfies the Ablowitz-Kaup-Newell-Segur system of equations (see e.g. [1])

$$\tilde{\Phi}_x(x, t; k) + ik\sigma_3\tilde{\Phi}(x, t; k) = Q(x, t)\tilde{\Phi}(x, t; k),$$
$$\tilde{\Phi}_t(x, t; k) + 4ik^3\sigma_3\tilde{\Phi}(x, t; k) = \tilde{Q}(x, t)\tilde{\Phi}(x, t; k),$$

where

$$Q(x, t; k) = \begin{pmatrix} 0 & q(x, t) \\ -q(x, t) & 0 \end{pmatrix},$$

$$\tilde{Q}(x, t; k) = 4k^2Q(x, t; k) - 2ik (Q^2(x, t; k) + Q_z(x, t; k)) \sigma_3 + 2Q^3(x, t; k) - Q_{xx}(x, t; k).$$

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with the function \( q(x, t) \) given by
\[
q(x, t) = 2i \lim_{k \to \infty} k [M(x, t; k)]_{21} = -\frac{1}{\pi} \int_{\Sigma} (I + \mu(x, t; k)) (I - J(x, t; k)) \, dk.
\]

3 Model problems

The asymptotic analysis of the Cauchy problem as \( t \to \infty \) consists of several steps:

- the first step is to change the original phase function, which is present in the exponents of the RHP, with an appropriate function \( g(k, \xi) \) that will be determined later;
- the second step consists in performing a chain of "exact" transformations of the RHP;
- the third step consists in approximating the new RHP to some model problem;
- the fourth step consists in solving the model problems.

Before starting our asymptotic analysis, we introduce the Riemann-Hilbert model problems that will come from such analysis. Namely the RHP for the soliton and breather solution on a constant background and the RH problem for the travelling wave solution \([17]\). In particular for the travelling wave solution, we show that it can be obtained from a model problem solvable in terms of elliptic theta functions but also from a model problem that is solvable via hyperelliptic theta functions that are defined on a genus two hyperelliptic Riemann surface with symmetries. The first case turns out when the step like initial data is such that \( c_+ = 0 \) while the second case occurs when \( c_+ > 0 \). We then show that the genus 2 solutions can nevertheless be written in a genus 1 form.

3.1 One-soliton solution on a constant background \( c > 0 \)

Here we derive a one-soliton solution on a constant background. We use the notation \([ic, -ic]\) to denote the interval oriented downward.

**Riemann-Hilbert problem 2.**

1. Find a \( 2 \times 2 \) matrix \( M(k) = M(k; x, t) \) meromorphic for \( k \in \mathbb{C} \setminus [ic, -ic] \) with simple poles at \( k = \pm i \kappa_0, \kappa_0 > c > 0 \), with the following properties

2. the boundary values \( M_{\pm}(k) \) for \( k \in (ic, -ic) \) satisfy the following jump relations
\[
M_-(k) = M_+(k) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in (ic, -ic);
\]  \( (3.1) \)

3. pole conditions:
\[
\text{Res}_{i \kappa_0} M(k) = \lim_{k \to i \kappa_0} M(k) \begin{pmatrix} 0 & 0 \\ i \nu e^{2ig_c(k, x, t)} & 0 \end{pmatrix},
\]
\[
\text{Res}_{-i \kappa_0} M(k) = \lim_{k \to -i \kappa_0} M(k) \begin{pmatrix} 0 & ire^{-2ig_c(k, x, t)} \\ 0 & 0 \end{pmatrix},
\]  \( (3.2) \)

where \( \nu \) is a non zero real constant and
\[
g_c(k; x, t) = (2(2k^2 - c^2)t + x) \sqrt{k^2 + c^2};
\]  \( (3.3) \)

4. asymptotics:
\[
M(k) = I + O \left( \frac{1}{k} \right), \quad \text{as } k \to \infty.
\]  \( (3.4) \)
The solution of the MKDV equation is obtained from the matrix $M(k; x, t)$ by the relation
\begin{equation}
q(x, t) = 2i \lim_{k \to \infty} k(M)_{21} = 2i \lim_{k \to \infty} k(M)_{12}. \tag{3.5}
\end{equation}

**Lemma 3.1.** The solution of the MKDV $q_t + 6q^2 q_x + q_{xxx} = 0$ obtained from the solution of the RH problem is equal to a soliton on a constant background $c$, namely
\begin{equation}
q_{\text{soliton}}(x, t; \ c, \kappa_0, \nu) = c - \frac{2 \text{sgn}(\nu)(\kappa_0^2 - c^2)}{\kappa_0 \cosh 2 \sqrt{\kappa_0^2 - c^2}(x - (2c^2 + 4\kappa_0^2)t) + x_0} - \text{sgn}(\nu)c. \tag{3.6}
\end{equation}

where
\[ x_0 = \ln \frac{2(\kappa_0^2 - c^2)}{|\nu|\kappa_0}. \]

**Proof.** We first obtain the solution of the RH problem \textbf{3.1} and \textbf{3.4} in the form
\[ M_{\text{reg}}(k) = \begin{pmatrix} \psi_2(k) & \psi_1(k) \\ \psi_1(k) & \psi_2(k) \end{pmatrix}, \tag{3.7} \]
where
\[ \psi_2 = \frac{1}{2} \left( \sqrt{\frac{k - ic}{k + ic}} + \sqrt{\frac{k + ic}{k - ic}} \right), \quad \psi_1 = \frac{1}{2} \left( \sqrt{\frac{k - ic}{k + ic}} - \sqrt{\frac{k + ic}{k - ic}} \right). \]

Since $M(k)M_{\text{reg}}(k)^{-1}$ does not have jumps on $\mathbb{C}$ but only poles, the solution $M(k)$ can be found in the form
\[ M(k; x, t) = \left( 1 + \frac{i\alpha(x, t)}{k - \kappa_0} + \frac{i\beta(x, t)}{k + \kappa_0} + \frac{i\gamma(x, t)}{k - \kappa_0} + \frac{i\delta(x, t)}{k + \kappa_0} \right) M_{\text{reg}}(k), \]
where $\alpha, \beta, \gamma, \delta$ are real parameters to be determined. The solution of the MKDV equation is obtained from the matrix $M(k; x, t)$ through the formula
\begin{equation}
q(x, t) = 2i \lim_{k \to \infty} k(M)_{21} = 2i \lim_{k \to \infty} k(M)_{12} = c - 2\beta - 2\delta. \tag{3.8}
\end{equation}

We observe that due to the symmetry of the problem, it is enough to consider the residue condition only at one of the poles $k = \pm i\kappa_0$. For example the condition \textbf{3.2} at $k = i\kappa_0$ gives
\begin{align*}
\alpha \psi_1(i\kappa_0) + \delta \psi_2(i\kappa_0) &= 0 \\
\beta \psi_1(i\kappa_0) - \gamma \psi_2(i\kappa_0) &= 0 \\
\frac{1}{\nu} (\alpha \psi_2(i\kappa_0) + \delta \psi_1(i\kappa_0)) e^{-2ig_c(i\kappa_0)} &= \psi_1(i\kappa_0) + \frac{\gamma}{2\kappa_0} \psi_1(i\kappa_0) + \frac{\beta}{2\kappa_0} \psi_2(i\kappa_0) + i\alpha \psi_1'(i\kappa_0) + i\delta \psi_2'(i\kappa_0) \\
\frac{1}{\nu} (\beta \psi_2(i\kappa_0) - \gamma \psi_1(i\kappa_0)) e^{-2ig_c(i\kappa_0)} &= \psi_2(i\kappa_0) + \frac{\delta}{2\kappa_0} \psi_1(i\kappa_0) - \frac{\alpha}{2\kappa_0} \psi_2(i\kappa_0) + i\beta \psi_1'(i\kappa_0) - i\gamma \psi_2'(i\kappa_0)
\end{align*}
\begin{equation}
\tag{3.9}
\end{equation}
where ' stands for derivative with respect to $k$. Solving the above system of equations we obtain
\begin{equation}
\alpha = -\delta \frac{\psi_2'(i\kappa_0)}{\psi_1(i\kappa_0)}, \quad \gamma = \frac{\beta \psi_1'(i\kappa_0)}{\psi_2'(i\kappa_0)}, \tag{3.10}
\end{equation}
and
\begin{align*}
\beta(x, t) &= \sqrt{\kappa_0^2 - c^2} \nu e^{2ig_c} \left\{ -ce^{-2ig_c} \left\{ \frac{-c\nu \kappa_0 e^{-2ig_c} + 2(\kappa_0^2 - c^2)(\kappa_0 + \sqrt{\kappa_0^2 - c^2})}{(4(\kappa_0^2 - c^2)^2 - 4(\kappa_0^2 - c^2)c\nu e^{2ig_c} + \kappa_0^2 \nu^2 e^{4ig_c})} \right\} \right\}, \\
\delta(x, t) &= \sqrt{\kappa_0^2 - c^2} \nu e^{2ig_c} \left\{ -ce^{-2ig_c} \left\{ \frac{c\nu \kappa_0 e^{2ig_c} - 2(\kappa_0^2 - c^2)(\kappa_0 - \sqrt{\kappa_0^2 - c^2})}{(4(\kappa_0^2 - c^2)^2 - 4(\kappa_0^2 - c^2)c\nu e^{2ig_c} + \kappa_0^2 \nu^2 e^{4ig_c})} \right\} \right\},
\end{align*}
where $g_c = g_c(i\kappa_0)$.

Plugging the above expressions for $\beta$ and $\delta$ into (3.8) we obtain the statement of the lemma. \hfill $\square$
Observe that when \( c = 0 \), the formula (3.6) coincides with the one-soliton solution (1.4).

Let us notice that the denominator in the formula (3.6) is always non zero. Since \( \kappa_0 > c \), in the case \( \nu > 0 \) we have antisoliton, oriented downward, and for \( \nu < 0 \) we have a soliton, oriented upward.

We see that for \( \kappa_0 - c \to +0 \) and \( \nu < 0 \) the amplitude of the soliton tends to 0, while for \( \nu > 0 \) the amplitude of the anti-soliton tends to a constant \( 2\kappa \).

**Degenerate case of antisoliton.**

When \( \nu > 0 \) and we let \( k_0 \to c + 0 \), we obtain the special case of an antisoliton, namely a rational solution.

In this case

\[
q(x, t; c, c, \nu) = c - \frac{4c}{1 + (2c(x - x_0) - 12c^3t)^2}.
\]

**Remark 3.1.** Let us observe that the MKDV admits a RH problem with poles of higher order. This case is non-generic, and we assume that our initial data will develop generically first order poles.

### 3.2 Simple breathers on a constant background.

In this section we consider a breather on a constant background.

**Riemann-Hilbert problem 3.** Find a \( 2 \times 2 \) matrix \( M(x, t; k) \) meromorphic in \( k \in \mathbb{C} \setminus [\text{ic}, \text{ic}] \) with simple poles at \( \kappa \equiv \kappa_1 + i\kappa_2, \pi \equiv \kappa_1 - i\kappa_2, -\kappa \equiv -\kappa_1 - i\kappa_2, -\pi \equiv -\kappa_1 + i\kappa_2 \), and such that

1. \( M_-(k) = M_+(k)J(k) \), \( k \in (\text{ic}, -\text{ic}) \), where

\[
J(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in (\text{ic}, -\text{ic})
\]

2. **pole conditions:**

\[
\text{Res}_\kappa M(k) = \lim_{k \to \kappa} M(k) \begin{pmatrix} 0 & 0 \\ \nu e^{2itg_\kappa(k, x, t)} & 0 \end{pmatrix}, \\
\text{Res}_{-\pi} M(k) = \lim_{k \to -\pi} M(k) \begin{pmatrix} 0 & 0 \\ -\nu e^{2itg_{-\pi}(k, x, t)} & 0 \end{pmatrix}, \\
\text{Res}_\pi M(k) = \lim_{k \to \pi} M(k) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\text{Res}_{-\kappa} M(k) = \lim_{k \to -\kappa} M(k) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( \nu \) is a non zero complex number and \( g_\kappa(k, x, t) \) as in (3.3);

3. **asymptotics:** \( M(k) = I + O\left(\frac{1}{k}\right) \) as \( k \to \infty \).

The solution of MKDV(+) \( q_t + 6q^2 q_x + q_{xxx} = 0 \) is obtained from the solution of this RH problem by one of the following formulas:

\[
q_{breather}(x, t) = 2i \lim_{k \to \infty} k(M)_{21} = 2i \lim_{k \to \infty} k(M)_{12}, \quad (3.11)
\]

or

\[
q_{breather}^2(x, t) = c^2 + 2i\partial_x \left( \lim_{k \to \infty} k(M - I)_{11} \right).
\]

**Theorem 3.2.** The solution (3.11) of the MKDV equation obtained from the solution of the RH problem corresponds to a breather on a constant background \( c \) with discrete spectrum \( \kappa = \kappa_1 + i\kappa_2 \) and complex parameter \( \nu \) and takes the form

\[
q_{breather}(x, t; c, \kappa, \nu) = c + 2\partial_x \arctan \left[ \frac{\chi \cos (2 \text{Re} g_\kappa(x, t) + \theta_1 - \theta_2) + \frac{c|\nu|^2}{2\chi^2 x^2} e^{-2\text{Im} g_\kappa(x, t)} + c \sin (2 \text{Re} g_\kappa(x, t) + \theta_1 - 2\theta_2)}{\frac{1}{|\nu|}|\nu| e^{2\text{Im} g_\kappa(x, t)} + \frac{c^2|\nu|^2}{4\chi^2 x^2} e^{-2\text{Im} g_\kappa(x, t)} + c \sin (2 \text{Re} g_\kappa(x, t) + \theta_1 - 2\theta_2)} \right]
\]
where $\chi = \chi_1 + i \chi_2 = \sqrt{\kappa^2 + c^2}$, with $\chi_1 > 0$, $\chi_2 > 0$ and real

$$\text{Re} \ g_c(x, t) = \chi_1 \left( 4 \left( \chi_1^2 - 3 \chi_2^2 - \frac{3}{2} c^2 \right) t + x \right), \quad \text{Im} \ g_c(x, t) = \chi_2 \left( 4 \left( 3 \chi_1^2 - \chi_2^2 - \frac{3}{2} c^2 \right) t + x \right).$$

and phases $\theta_1 = \text{arccos} \frac{\nu}{|\nu|}$ and $\theta_2 = \text{arccos} \frac{\chi_1}{|\chi|}$. Formula (3.12) coincides with formula (1.5) for the breather provided in the introduction.

**Proof.** The solution of the RHP can be found in the form

$$M(k) = M_{pol}(k)M_{reg}(k),$$

where $M_{reg}(k)$ has been defined in (3.7) and $M_{pol}$ admit an ansatz of the form

$$M_{pol}(x, t; k) = \left( \frac{1 + A(x, t)}{k - \kappa} - \frac{A(x, t)}{k + \kappa} \right) \frac{k - \kappa}{k + \kappa} + \frac{C(x, t)}{k + \kappa} - \frac{C(x, t)}{k - \kappa} \frac{k + \kappa}{k - \kappa} - \frac{B(x, t) - A(x, t)}{k - \kappa} + \frac{B(x, t) - A(x, t)}{k + \kappa}, \quad (3.13)$$

where $A = A(x, t), B = B(x, t), C = C(x, t)$ and $D = D(x, t)$ are unknown functions to be determined.

Writing down the pole conditions at the point $\kappa$, $\text{Re} \ \kappa > 0$, $\text{Im} \ \kappa > 0$, which is enough due to symmetries of the problem, we obtain the following system of equations:

$$
\begin{align*}
A \psi_1 + D \psi_2 &= 0, \\
B \psi_1 - C \psi_2 &= 0, \quad \text{(pole free condition for the 2nd row at $\kappa$)} \\
A \psi_2 + D \psi_1 &= \nu \psi_1^2 \psi_2 \left( 1 - \frac{\kappa}{\kappa + \nu} + \frac{\kappa}{\kappa - \nu} + \psi_2 \left( \frac{\kappa}{\kappa + \nu} + \frac{\kappa}{\kappa - \nu} \right) + \psi_1 \psi_2 \right), \\
B \psi_2 - C \psi_1 &= \nu \psi_1^2 \psi_2 \left( 1 - \frac{\kappa}{\kappa + \nu} + \frac{\kappa}{\kappa - \nu} + \psi_1 \psi_2 \right),
\end{align*}
$$

(3.14)

where we use the compact notation $\psi_2 = \psi_2(\kappa)$ and $\psi_2' = \psi_2'(\kappa)$, where $\psi_2'(\kappa) = \frac{d}{d\kappa} \psi_1(k)\big|_{k=\kappa}$ and

$$\psi_1 = \frac{1}{2} \left( \sqrt{k - \kappa} - \sqrt{k + \kappa} \right), \quad \psi_2 = \frac{1}{2} \left( \sqrt{k - \kappa} + \sqrt{k + \kappa} \right).$$

Let us introduce

$$R(k) = \frac{\psi_1(k)}{\psi_2(k)} = \frac{1}{c}(k - \sqrt{k^2 + c^2}).$$

With the above notation it is straightforward to obtain the solution (3.11) of the MKDV equation in the form

$$g_{\text{breath}}(x, t) = c + 4 \text{Im}(A(x, t)R(\kappa) - B(x, t)).$$

(3.15)

We need to determine the quantities $A(x, t)$ and $B(x, t)$ in (3.15). These constants are obtained by solving the system of equations (3.14) that can be written in the form (we use here the determinantal property $\psi_2^2 - \psi_1^2 = 1$)

$$
\begin{cases}
\left( \frac{e^{-2g_c(\kappa)}}{\nu \psi_2^2(\kappa)} - \mathcal{R}'(\kappa) \right) A + \frac{R(\kappa) - \overline{R(\kappa)}}{\kappa + \nu} A - \frac{R^2(\kappa) + 1}{2k} B + \frac{1 + |R(\kappa)|^2}{\kappa - \nu} B = \mathcal{R}(\kappa), \\
\left( \frac{e^{-2g_c(\kappa)}}{\nu \psi_2^2(\kappa)} - \mathcal{R}'(\kappa) \right) B + \frac{R(\kappa) - \overline{R(\kappa)}}{\kappa + \nu} B + \frac{R^2(\kappa) + 1}{2k} A - \frac{1 + |R(\kappa)|^2}{\kappa - \nu} A = 1.
\end{cases}
$$

(3.16)

The system of equations (3.16) for $A, B$ is clearly a system of four linear equations for the four real variables $A = A_1 + iA_2, B = B_1 + iB_2$. Introducing the variables $Z = A + iB$ and $W = A - iB$ the system of equations (3.16) can be recast in the form

$$
\begin{cases}
\left( \frac{e^{-2g_c(\kappa)}}{\nu \psi_2^2(\kappa)} - \mathcal{R}'(\kappa) + \frac{1 + |R(\kappa)|^2}{2\kappa} \right) Z + \left( \frac{R(\kappa) - \overline{R(\kappa)}}{\kappa + \nu} - \frac{1 + |R(\kappa)|^2}{2\kappa} \right) W = \mathcal{R}(\kappa) + i, \\
\left( \frac{R(\kappa) - \overline{R(\kappa)}}{\kappa + \nu} + \frac{1 + |R(\kappa)|^2}{2\kappa} \right) Z + \left( \frac{e^{-2g_c(\kappa)}}{\nu \psi_2^2(\kappa)} - \mathcal{R}'(\kappa) \right) W + \frac{R^2(\kappa) + 1}{2k} W = \overline{\mathcal{R}(\kappa)} + i.
\end{cases}
$$

(3.17)
Defining the quantities

\begin{align*}
E &= \frac{e^{-2g_\kappa}}{\nu e^{2\kappa}} - \mathcal{R}'(\kappa) \\
F &= \frac{\mathcal{R}^2(\kappa) + 1}{2\kappa} \\
H &= \frac{\mathcal{R}(\kappa) - \overline{\mathcal{R}(\kappa)}}{\kappa + \overline{\kappa}} \\
G &= \frac{1 + |\mathcal{R}(\kappa)|^2}{\kappa - \overline{\kappa}}.
\end{align*}

(3.18)

The solution of (3.17) is obtained as

\[
Z = A + iB = \begin{pmatrix} \mathcal{R} + i & H - iG \\ \overline{\mathcal{R}} + i & \overline{H} + i\overline{G} \end{pmatrix} = \frac{r + is}{f + ih} = \frac{fr + sh}{f^2 + h^2} + i\frac{sf - rh}{f^2 + h^2},
\]

and similarly \(W = A - iB = \frac{fr + sh}{f^2 + h^2} - i\frac{sf - rh}{f^2 + h^2},\) where

\[
\begin{align*}
r &= \mathcal{R}E - \mathcal{R}H - \mathcal{F} - G, \\
s &= \mathcal{E} - H + \mathcal{R}F + \overline{\mathcal{R}}G, \\
f &= |E|^2 - |F|^2 - G^2 + H^2, \\
h &= 2\text{Re} (\mathcal{E}F - \mathcal{R}G).
\end{align*}
\]

(3.19)

Let us notice, that \(f, h\) are real, while \(r, s\) are complex-valued functions.

The solution to MKDV is given by the formula

\[
q_{breather} = c + 4\text{Im}(AR - B) = c + 4f \frac{(R_2r_1 + R_1r_2 - s_2) + h(R_2s_1 + R_1s_2 + r_2)}{f^2 + h^2},
\]

(3.20)

where we denote \(r = r_1 + ir_2\) with \(r_1, r_2, \in \mathbb{R}\) and similarly for the other quantities. To obtain the expression for \(f\) defined in (3.19) we first observe that

\[
H = \frac{\mathcal{R}(\kappa) - \overline{\mathcal{R}(\kappa)}}{\kappa + \overline{\kappa}} = 1 - \frac{\kappa_1 - \chi_1}{c\kappa_1} = 1 - \frac{k_1\kappa_2 - \chi_1\kappa_2}{c\kappa_1\kappa_2} = 1 - \frac{\chi_2 - \kappa_2}{c\chi_2}
\]

(3.21)

where we use the identity \(k_1k_2 = \chi_1\chi_2\), and

\[
G = \frac{1 + |\mathcal{R}(\kappa)|^2}{\kappa - \overline{\kappa}} = \frac{c^2 + (\kappa_1 - \chi_1)^2 + (\kappa_2 - \chi_2)^2}{2i\kappa_2c^2} = \frac{(k_2^2 + \chi_2^2 - k_1\chi_1 + k_2\chi_2)\kappa_1}{2i\kappa_1\kappa_2c^2}
\]

\[
= \frac{\kappa_2\chi_2 + \chi_1\kappa_1 - \kappa_1^2 + \kappa_2^2}{2i\chi_2c^2} = \frac{c^2 - |\kappa - \chi|^2}{4\chi_2c^2}
\]

(3.22)

where we use the identity \(\chi_1^2 + \kappa_2^2 - c^2 = \chi_2^2 + \kappa_1^2\) repeatedly. Plugging in (3.19) the expressions for \(E, F\) from (3.18), and \(G\) and \(H\) from (3.21) and (3.22) respectively we obtain

\[
|f|\psi_2^4 = J |\kappa + \chi|^2 = \frac{|e^{-2g_\kappa}}{4|\chi|^2} - \frac{ic}{2\chi^2} \left( \frac{1}{\nu} \chi + \kappa + |\chi|^2 \right) + \frac{1}{\nu^2} \left( \frac{1}{4} (\kappa - \chi)^2 - c^2 \kappa_2 - \chi_2^2 \right) \frac{|\kappa + \chi|^2}{4|\chi|^2} + \frac{\chi_2^2 (|\chi|^2 - c^2)}{4\chi_2^2} + \frac{\gamma_1 (|\chi|^2 - c^2)}{4\chi_2^2}
\]

(3.23)

where we use the relation \((\kappa + \chi)(\kappa - \chi) = c^2\) and

\[
\text{Re} g_\kappa = \chi_1 \left( 4 \left( \chi_1^2 - 3\chi_2^2 + \frac{3}{2}c^2 \right) t + x \right), \quad \text{Im} g_\kappa = \chi_2 \left( 4 \left( 3\chi_1^2 - \chi_2^2 - \frac{3}{2}c^2 \right) t + x \right).
\]
In a similar way
\[
|\psi_2|^4 = e^{2 \text{Im} g_2(\kappa)} \left( \frac{e^{-2i \text{Re} g_2(\kappa)}}{2\nu \chi} + \frac{e^{2i \text{Re} g_2(\kappa)}}{2\nu \chi} \right) = e^{2 \text{Im} g_2(\kappa)} \left( \frac{\nu}{2\chi^2} + \frac{\nu}{2\chi^2} \right) \text{M}
\]

\[ (3.24) \]

Long and involved algebraic manipulations give the identities
\[
(h_x - 4\chi^2 h)|\psi_2|^4 = 2(\mathcal{R}_2 r_1 + \mathcal{R}_1 r_2 - s_2)|\psi_2|^4
\]

and
\[
(f_x - 4\chi^2 f)|\psi_2|^4 = -2(\mathcal{R}_2 s_1 + \mathcal{R}_1 s_2 + r_2)|\psi_2|^4.
\]

Combining the above two expressions with \[3.20\], \[3.23\] and \[3.24\] we can write the breather solution on a constant background in the form
\[
\eta_{\text{breather}}(x,t) = c + 2\frac{f_x - f x f}{f^2 + h^2} = c + 2\partial_x \arctan \frac{h}{f},
\]

with \( f \) and \( h \) as in \[3.23\] and \[3.24\] which coincides with the statement of the theorem.

\[ \square \]

### 3.3 Model problem for the periodic travelling wave solution: the elliptic case

We consider a model problem that can be solved via elliptic functions. This model problem was first solved in \[17\] in the context of asymptotic analysis for orthogonal polynomials related to Hermitian matrix models. The same problem appeared in the long time asymptotic analysis of the MKDV solution with step initial data when \( c_+ = 0 \) \[40\]. We introduce two real constant parameters \( \bar{c} > \bar{d} > 0 \). The RH problem is as follows.

**Riemann-Hilbert problem 4.** To find a \( 2 \times 2 \) matrix \( M(x,t,;k) \) such that

1. \( M(x,t;k) \) is analytic in \( k \in \mathbb{C} \setminus \{i\bar{c}, -i\bar{c}\} \),

2. \( M_-(k) = M_+(k) J(k), J(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, k \in (i\bar{c}, i\bar{d}) \cup (-i\bar{d}, -i\bar{c}) \),

where \( \Delta \) is a real constant and

\[
U = -\frac{\pi \bar{c}}{K(m)}, \quad V = 2(\bar{c}^2 + \bar{d}^2)U, \quad m = \frac{\bar{d}}{\bar{c}},
\]

where \( K(m) \) is the complete elliptic integral of the first kind;

3. \( M(k) \to I \) as \( k \to \infty \).

It follows from the standard scheme introduced by Zakharov, Shabat \[58\], that
\[
q(x,t) = 2i \lim_{k \to \infty} k M_{21}(x,t; k) = 2i \lim_{k \to \infty} k M_{12}(x,t; k)
\]

satisfies MKDV equation \[1.1\]. The explicit formula for \( M(k) \) from \[40\] \[17\] is constructed as follows. We introduce the normalized holomorphic differential
\[
\omega = -\frac{\bar{c}}{4iK(m)} \frac{dz}{\sqrt{(z^2 + \bar{d}^2)(z^2 + \bar{c}^2)}}, \quad 2 \int_{-\bar{d}}^{i\bar{d}} \omega = 1,
\]

(3.27)
with $K(m)$ as in (3.25). For our purpose we fix the function $\sqrt{(z^2 + \tilde{d}^2)(z^2 + \tilde{c}^2)}$ by condition that it is analytic off the intervals $(i\tilde{c}, i\tilde{d}) \cup (-i\tilde{d}, -i\tilde{c})$ and positive at $z = 0$. The intervals $(i\tilde{c}, i\tilde{d})$ and $(-i\tilde{d}, -i\tilde{c})$ are oriented downwards. We define the $\beta$-cycle the close clock-wise loop encircling $(i\tilde{c}, i\tilde{d})$ and the $\alpha$-cycle the path starting on the cut $(ic, id)$ on the left, going to the cut $(-id, -ic)$ on the left and passing to the second sheet and reaching the cut $(ic, id)$ from the right on the second sheet. We define the quantity

$$\tau = \int_\beta \omega.$$  

It follows that

$$\tau = \frac{i}{2} \frac{K'(m)}{K(m)}, \quad m = \frac{\tilde{d}}{\tilde{c}}$$

where $K'(m) = K(\sqrt{1 - m^2})$. Using the relations [12, 165.05, 162.01], the quantity $\tau$ can also be written in the form

$$\tau = \frac{1}{K(\tilde{m})} K(\tilde{m}) = (1 + m)K(m), \quad \tilde{m}^2 = \frac{4m}{(1 + m)^2}. \quad (3.28)$$

Let us introduce the Jacobi theta-function with modulus $\tau$

$$\theta(\zeta) \equiv \theta(\zeta, \tau) = \sum_{n=-\infty}^{\infty} \exp \left\{ \pi i m n^2 + 2\pi i n \zeta \right\}.$$  

It is an even function of $\zeta$ and it has the following periodicity properties

$$\theta(\zeta + 1) = \theta(\zeta), \quad \theta(\zeta + \tau) = e^{-\pi i \tau - 2\pi i c} \theta(\zeta).$$

Next we introduce the Abel map with base point $i\tilde{c}$

$$A(k) := \int_{i\tilde{c}}^{k} \omega,$$  

(3.29)

where $\omega$ is the holomorphic differential (3.27) and we observe that $A(i\infty) = \frac{1}{4}$. Let us also introduce the quantity $\gamma(k) = \sqrt{\frac{k-i\tilde{c}}{k-id}} \sqrt{\frac{k+id}{k+i\tilde{c}}}$. Then the solution for the Riemann-Hilbert problem [4] is given by [10]

$$M(k) = \frac{\theta(0)}{2\theta(\Omega)} \left( \frac{\theta(A(k) - \Omega - \frac{1}{4})}{\theta(A(k) + \frac{1}{4})} \right) \frac{\theta(A(k) + \frac{1}{4})}{\theta(A(k) - \frac{1}{4})} \left( \frac{\theta(A(k) - \Omega - \frac{1}{4})}{\theta(-A(k) - \frac{1}{4} - \Omega)} \right) \frac{\theta(-A(k) - \frac{1}{4} - \Omega)}{\theta(-A(k) + \frac{1}{4})}, \quad (3.30)$$

where

$$\Omega = \frac{xU + tV + \Delta}{2\pi}.$$  

Using the expression for $M(k)$ above, it follows from (3.26) that the MKDV solution $q(x,t)$ is given by

$$q(x,t) = (\tilde{c} - \tilde{d}) \frac{\theta(\Omega + \frac{1}{2}; \tau) \theta(0; \tau)}{\theta(\Omega; \tau) \theta(\frac{1}{2}; \tau)}.$$  

(3.31)

We recall the relation between the Jacobi $\theta$-function and the elliptic function $\text{dn}$, [14], namely

$$\text{dn}(2K(\tilde{m})z|\tilde{m}) = \frac{\theta(\frac{1}{2}; \tau)}{\theta(0; \tau)} \frac{\theta(z; \tau)}{\theta(z + \frac{1}{2}; \tau)}$$  

(3.32)

with $\tilde{m}$ as in (3.28).
Using the above identities the solution \( 3.31 \) can be written in the form

\[
q(x, t) = (\tilde{c} + \tilde{d}) \text{dn} \left( 2K(\tilde{m})(\Omega + \frac{1}{2})|\tilde{m} \right).
\]  

According to [12], formulas (162.01), p.38, (165.05), p 41,

\[
dn(u(1 + m)|\tilde{m}) = \frac{1 - m \text{sn}^2(u|m)}{1 + m \text{sn}^4(u|m)}, \quad K(\tilde{m}) = K(m)(1 + m).
\]  

Hence, the expression \( 3.33 \) can be rewritten as

\[
q(x, t) = (\tilde{c} + \tilde{d}) \frac{\frac{1}{2} \text{dn} (2K(m)\Omega + K(m)|m)}{\frac{1}{2} \text{dn} (2K(m)\Omega + K(m)|m)}
\]

\[
= -\tilde{c} - \tilde{d} + \frac{2\tilde{c} + \tilde{d}}{\tilde{c} + \tilde{d} - \tilde{d} \text{cn} \left( \tilde{c}(x - 2(\tilde{c}^2 + \tilde{d}^2)t - \frac{\Delta}{\pi} K(m) + K(m)|m \right)}
\]

where to obtain the second expression we use the relation \( \text{cn}^2(u|m) + \text{sn}^2(u|m) = 1 \) and the explicit expressions of \( U \) and \( V \) as in \( 3.25 \). The second expression in \( 3.35 \) coincides with the travelling wave solution \( 1.7 \) identifying \( \beta_1 = \tilde{c}, \beta_2 = \tilde{d} \) and \( \beta_1 = 0 \) and \( x_0 = -\frac{\Delta}{\pi} K(m) + K(m) \) (see also Appendix \( A \)).

### 3.4 Model problem for the periodic travelling wave solution: the hyperelliptic case

The model problem we are considering below is obtained from the longtime asymptotic analysis of the MKDV RH problem in the oscillatory region when the step \( c_+ > 0 \). It can be solved using hyperelliptic theta-functions. The goal of this section is to show that such model problem still gives the periodic travelling wave solution \( 1.7 \) identifying \( \beta_1 = \tilde{c}, \beta_2 = \tilde{d} \) and \( \beta_1 = 0 \) and \( x_0 = -\frac{\Delta}{\pi} K(m) + K(m) \) (see also Appendix \( A \)).

**Riemann-Hilbert problem 5.** Find a \( 2 \times 2 \) matrix-valued function \( W(k) \) analytic in \( \mathbb{C} \backslash [\text{i}c_+ - \text{i}c_-] \) such that

1. \( W_-(k) = W_+(k)J_W(k), \quad k \in (\text{i}c_-, -\text{i}c_-) \) with

\[
J_W(x, t; k) = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \quad k \in (\text{i}c_-, -\text{i}c_+) \cup (-\text{i}d, -\text{i}c_-) \cup (\text{i}d, -\text{i}c_+)
\]

\[
J_W(x, t; k) = \begin{pmatrix} \text{e}^{i(xU + tV)} + i\Delta(\xi) & 0 \\ 0 & \text{e}^{-i(xU + tV) - i\Delta(\xi)} \end{pmatrix}, \quad k \in (\text{i}d, \text{i}c_+) \cup (-\text{i}c_+, -\text{i}d);
\]

where \( c_- > d > c_+ \), and

\[
U = -\frac{\pi \sqrt{c_-^2 - c_+^2}}{K(m)} \quad V = -2(c_-^2 + c_+^2 + d^2)U,
\]

with \( m^2 = \frac{d^2 - c_+^2}{c_-^2 - c_+^2} \) and \( \Delta \) real constant.

2. \( W(k) = I + O \left( \frac{1}{k} \right) \) as \( k \to \infty \);

3. \( W(k) \) has at most fourth root singularities at the points \( \pm \text{i}c_-, \pm \text{i}c_+ \) and \( \pm \text{i}d \).
Then the quantity
\[ q(x,t) = \lim_{k \to \infty} 2ikW(k;x,t)_{12} = \lim_{k \to \infty} 2ikW(k;x,t)_{21} \]  
(3.38)
is a solution of the MKDV equation.

**Theorem 3.3.** The solution of the MKDV equation (3.38) obtained from the RH problem 5 is the travelling wave solution (1.7), namely
\[ q_{\text{per}}(x,t,c_+,d,c_-,x_0) = -c_- - d - c_+ + \frac{(c_- + d)(c_+ + c_-)}{d + c_- - (d - c_+)} \bigg( \frac{c_-^2}{c_+^2} + c_+^2(x - Vt) + x_0|m \bigg), \]  
(3.39)with \( V = 2(c_-^2 + c_+^2 + d^2) \) and the phase \( x_0 \) takes the form
\[ x_0 = -\frac{K(m)\Delta}{\pi} + K(m). \]  
(3.40)Here \( \text{cn}(u|m) \) is the Jacobi elliptic function of modulus \( m^2 = \frac{d^2 - c_+^2}{c_-^2 - c_+^2} \) and \( K(m) \) is the complete elliptic integral of the first kind of modulus \( m \).

The Riemann-Hilbert problem 5 has been considered in [49] where it was solved in terms of hyperelliptic theta-function defined on the Jacobi variety of the surface \( \Gamma := \{ (k,y) \in \mathbb{C}^2 | y^2 = (k^2 + d^2)(k^2 + c_+^2)(k^2 + c_-^2) \} \). Such a surface has two automorphisms \( \tau_1 : (y,k) \to (y,-k) \) and \( \tau_2 : (y,k) \to (-y,k) \). Therefore the curve \( \Gamma \) covers two elliptic curves \( \Gamma_\pm := \Gamma/\tau_1 \) and \( \Gamma_\mp = \Gamma/(\tau_1\tau_2) \). The corresponding genus 2 theta-function can be factorized as a product of Jacobi theta-function. However, pursuing this strategy, we did not see a simple way to arrive to the travelling wave solution (3.39). For this reason, we change strategy and we formulate an auxiliary RH-problem that produces the desired solution and we connect such problem to our RH problem 5.

### 3.4.1 Auxiliary Riemann-Hilbert problem

We consider the two real numbers \( \tilde{c} > \tilde{d} > 0 \) with \( \tilde{c}^2 = c_-^2 - c_+^2 \) and \( \tilde{d}^2 = d^2 - c_+^2 \) and construct the following RH problem for a \( 2 \times 2 \) matrix \( M_{el} = M_{el}(\lambda) \):

1. \( M_{el}(\lambda) \) is analytic in \( \lambda \in \mathbb{C} \setminus [i\tilde{c}, -i\tilde{c}] \),
2. \( M_{el,-}(\lambda) = M_{el,+}(\lambda)J_{el}(\lambda) \), where
\[ J_{el}(\lambda) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \lambda \in (i\tilde{c}, i\tilde{d}) \cup (-i\tilde{d}, -i\tilde{c}), \]

\[ J_{el}(\lambda) = e^{i(\lambda U + \pi V + i\Delta - \Delta_4)\sigma_3}, \quad \lambda \in (i\tilde{d}, -i\tilde{d}), \]

where \( \Delta_4 \) is a constant to be determined and \( U, V \) and \( \Delta \) as in the Riemann-Hilbert problem 5.
3. \( M_{el}(\lambda) = I + O \left( \frac{1}{\lambda} \right) \) as \( \lambda \to \infty \).

The explicit formula for \( M_{el}(\lambda) \) can be obtained from the solution of the RH problem 4. The solution of this RH problem possesses the symmetry
\[ M_{el}(\lambda \pm i\tilde{c}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M_{el}(-\lambda \pm i\tilde{c}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]
for \( \lambda \notin [i\tilde{c}, -i\tilde{c}] \). To proceed further, we make a transformation of the complex plane to reduce the Riemann-Hilbert problem for \( M_{el}(\lambda) \) to the one in RHP 5. We introduce a change of variable \( \lambda \to k \) defined as
\[ \lambda = \sqrt{k^2 + c_+^2}, \quad \text{and denote} \quad c_- = \sqrt{\tilde{c}^2 + c_+^2}, \quad d = \sqrt{\tilde{d}^2 + c_+^2}. \]
The function $\lambda = \lambda(k)$ is analytic for $k \in \mathbb{C}\setminus [ic_+, -ic_+]$. Next we introduce the matrix

$$\Lambda(k) = \frac{1}{2} \begin{pmatrix} a(k) + \frac{1}{a(k)} & -i(a(k) - \frac{1}{a(k)}) \\ i(a(k) - \frac{1}{a(k)}) & a(k) + \frac{1}{a(k)} \end{pmatrix} M_d(\lambda(k)),$$

where

$$a(k) = \sqrt{\frac{k^2 + c_+^2}{k^2}}.$$

Then the matrix $\Lambda(k)$ is analytic for $k \in \mathbb{C}\setminus [ic_-, -ic_-]$ and satisfies the following conditions:

$$\Lambda_-(k) = \Lambda_+(k) J_\Lambda(k), \quad k \in [ic_-, -ic_-]$$

with

$$J_\Lambda(k) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & k \in (ic_+, 0), \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & k \in (0, -ic_+), \\ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & k \in (ic_-, id) \cup (-id, -ic_-), \\ e^{i(xU + iv + \Delta_4)\sigma_3}, & k \in (id, ic_+) \cup (-ic_+, -id). \end{cases}$$

This is not exactly the RH problem [3] in [3.36]. We need to do some extra work. For the purpose we introduce a scalar function $F = F(k)$ analytic in $k \in \mathbb{C}\setminus [ic_-, -ic_-]$, which satisfies the following conditions:

$$F_+(k) F_-(k) = 1, \quad k \in (ic_-, id) \cup (-id, -ic_-),$$

$$F_+(k) F_-(k) = 1, \quad k \in (ic_+, 0), \quad F_+(k) F_-(k) = -i, \quad k \in (0, -ic_+),$$

$$F_+(k) F_-(k) = e^{\Delta_4}, \quad k \in (id, ic_+) \cup (-ic_+, -id).$$

The quantity $\Delta_4$ is independent from $k$ and it has to be chosen in such a way that $F(k)$ is bounded at $k \to \infty$. The function $F(k)$ can be represented in the following way:

$$F(k) = \exp \left\{ \frac{R(k)}{2\pi i} \int_{ic_+}^{0} \frac{2\pi i}{s-k} R(s) \ ds - \int_{-ic_+}^{ic_+} \frac{2\pi i}{s-k} R(s) \ ds + \int_{id}^{ic_+} \frac{\Delta_4}{s-k} R(s) \ ds + \int_{-ic_+}^{id} \frac{\Delta_4}{s-k} R(s) \ ds \right\},$$

where $R(k) = \sqrt{(k^2 + c_+^2)(k^2 + c_-^2)(k^2 + d^2)}$. The function $R(k)$ is analytic off the intervals $[ic_-, id] \cup [ic_+, -ic_+] \cup [-id, -ic_-]$ and positive and real at $k = +0$. The function $F(k)$ is bounded at infinity provided that

$$\Delta_4 = -\frac{\pi i c_+}{2} \int_{id}^{ic_+} \frac{d\omega}{R(s)} = -2\pi i \int_{0}^{c_+} \frac{d\omega}{R(s)}, \quad (3.41)$$

where the one-form $\omega$ has been defined in [3.27]. Hence

$$F(k) = 1 + \mathcal{O}(k^{-1}) \quad \text{as} \quad k \to \infty.$$ 

Denote $\Lambda^{(1)}(k) = \Lambda(k) F^{-\sigma_3}(k)$. The jump conditions for the matrix $\Lambda^{(1)}$ are as follows: $\Lambda^{(1)}(k) = \Lambda^{(1)}_+(k) J^{(1)}(k)$ with

$$J^{(1)}(k) = \begin{cases} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & k \in (ic_-, id) \cup (ic_+, -ic_+) \cup (-id, -ic_-), \\ e^{i(xU + iv + \Delta_4)\sigma_3}, & k \in (id, ic_+) \cup (-ic_+, -id). \end{cases}$$

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It is not exactly the solution of the hyperelliptic model problem \(3.36\) since it has poles at the point \(k = 0\). This is because the function \(F(k)\) is vanishing as \(\sqrt{k}\) as \(k \to 0\) with \(\Re k > 0\), and is growing as \(\sqrt{k}\) as \(k \to 0\) with \(\Re k < 0\). Hence, the second column of \(\Lambda^{(1)}\) has a pole of the first order when \(k \to 0\) with \(\Re k < 0\), and the first column of \(\Lambda^{(1)}\) has a pole when \(k \to 0\) with \(\Re k > 0\).

Direct analysis of the behavior of \(\Lambda^{(1)}(k)\) at \(k \to 0\) shows, that the matrix function

\[
\Lambda^{(2)}(k) := \begin{pmatrix} 1 + \frac{\alpha}{k} & \frac{i \alpha}{1 - \frac{\alpha}{k}} \\ \frac{i \alpha}{1 - \frac{\alpha}{k}} & 1 - \frac{\alpha}{k} \end{pmatrix} \Lambda^{(1)}(k)
\]

(3.42)
does not have pole at \(k = 0\) provided that

\[
\alpha = -\frac{c_+ M_{el,11}(c_+) - i M_{el,21}(c_+)}{2 M_{el,11}(c_+) + i M_{el,21}(c_+)},
\]

(3.43)
where \(M_{el}(k)\) is as in \(3.36\) by taking care of replacing \(\Delta\) by \(\Delta + i \Delta_4\). We arrive to the following lemma.

**Lemma 3.2.** The \(2 \times 2\) matrix \(\Lambda^{(2)}(k)\) defined in \(3.42\) with \(\alpha\) as in \(3.43\) is the unique solution to the hyperelliptic model problem in \(3.36\).

Further, the solution of the MKDV equation is given by the following formula:

\[
q_{hel}(x,t) = \lim_{k \to \infty} 2i k \Lambda^{(2)}(k) = 2i \lim_{k \to \infty} k M_{el,21}(k) - 2\alpha = 2i \lim_{k \to \infty} k M_{el,12}(k) - 2\alpha,
\]

so that, plugging in the explicit expression of \(M_{el}(k)\) we obtain

\[
q_{hel}(x,t) = (\tilde{c} - \tilde{d}) \frac{\theta(\frac{1}{2} + \Omega - \frac{2\Delta_4}{\pi i}; \tau)}{\theta(\frac{1}{2} + \frac{\pi}{2}; \tau)} \theta(0; \tau) + c_+ M_{el,11}(c_+) - i M_{el,21}(c_+) \frac{M_{el,11}(c_+) + i M_{el,21}(c_+)}{M_{el,11}(c_+) - i M_{el,21}(c_+)},
\]

(3.44)
with \(\Omega = \frac{xU + tV + \Delta}{2\pi}\) with \(U, V\) and \(\Delta\) as in the int RH problem \(3.36\). Summarizing we have obtained the solution of the hyperelliptic RH problem \(3.36\) and therefore of the MKDV equation in terms of elliptic functions. We need to do some extra work to show that the expression \(3.44\) coincides with the travelling wave solution of the MKDV equation.

**Proof of Theorem 3.3.** In order to prove Theorem \(3.3\) we need to show that the quantity \(q_{hel}(x,t)\) defined in \(3.44\) is equal to modulated travelling wave solution of the MKDV equation defined in \(1.5\), namely we have to prove the relation

\[
q_{hel}(x,t) = -c_- - d - c_+ + 2 \frac{(c_- + d)(c_+ + c_-)}{d + c_- - (d - c_+ \alpha)^2 (\sqrt{c_+^2 - c_-^2} (x - \bar{U} t) + x_0 |m|),
\]

(3.45)
where the phase \(x_0\) takes the form

\[
x_0 = \frac{K(m)\Delta}{\pi} + K(m), \quad m^2 = \frac{d^2 - c_+^2}{c_-^2 - c_+^2}.
\]

(3.46)
For the purpose we need a series of identities among elliptic functions. We first consider the term \(\alpha\) in \(3.43\). We observe from the relation \(3.27\) and \(3.41\) that

\[
A(c_+) = \int_{i\infty}^{c_+} \omega = -\frac{\tau}{2} + \frac{1}{4} - \frac{\Delta_4}{2 \pi i}
\]

so that the quantity \(\alpha\) in \(3.43\) takes the form

\[
\alpha = \frac{c_+}{2} \frac{(\gamma(c_+) + \gamma^{-1}(c_+))}{(\gamma(c_+) + \gamma^{-1}(c_+))} \frac{\theta(\frac{\pi}{2} - \Omega; \tau)}{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)} \frac{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)}{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)} + \frac{i (\gamma(c_+) - \gamma^{-1}(c_+))}{(\gamma(c_+) + \gamma^{-1}(c_+))} \frac{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)}{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)} \frac{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)}{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)} \frac{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)}{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)} \frac{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)}{\theta(\frac{\pi}{2} + \frac{\pi}{2}; \tau)},
\]

(3.47)
In order to simplify the above expression we use the following identities [40]

\[
\frac{\theta(A(k) - \frac{1}{4}; \tau)}{\theta(A(k) + \frac{1}{4}; \tau)} = \sqrt{\frac{c + d}{c - d}} \cdot \gamma(k) + \gamma^{-1}(k)
\]

where \(\gamma(k) = \sqrt{\frac{k-\Omega}{k+\Omega}} \sqrt{\frac{k-\mu}{k+\mu}}\) and the following periodicity property of elliptic function

\[
dn(u + iK'|\tilde{m}) = -\frac{\cncn(u|\tilde{m})}{\snm(u|\tilde{m})}, \quad K' = K(\sqrt{1 - \tilde{m}^2}).
\]

Using the above three identities and [3.32] we arrive to the following form for \(q_{hel}(x,t)\) in (3.44):

\[
q_{hel}(x,t) = (\tilde{d} + \tilde{c})dn(2K(\tilde{m}))(\frac{1}{2} + \Omega - \frac{\Delta_4}{2\pi i})|\tilde{m}| + 
\]

\[
c + \frac{i\sqrt{1 - m \snm(2K(\tilde{m}))\Omega + K(\tilde{m})|\tilde{m}|}}{1 + m \cnn(2K(\tilde{m}))\Omega + K(\tilde{m})|\tilde{m}|} - \frac{\sqrt{c_+ - d}}{c_+ + d}.
\]

Next we use the addition formula for Jacobi elliptic function \(dn\) [12 123.01, p.23]

\[
dn(u + v|\tilde{m}) = \frac{dn(u|\tilde{m})dn(v|\tilde{m}) - \tilde{m}^2 \snm(u|\tilde{m})\snm(u|\tilde{m})\snm(v|\tilde{m})\snm(v|\tilde{m})}{1 - \tilde{m}^2 \snm(u|\tilde{m})\snm(v|\tilde{m})},
\]

(3.49)

and the following relations [12 162.01, 165.05]

\[
\begin{align*}
\snm(u|\tilde{m}) &= (1 + m)\frac{\snm(u|\tilde{m})}{1 + m \snm^2(u|\tilde{m})}, \\
\cnn(u|\tilde{m}) &= \frac{\cnn(u|\tilde{m})}{1 + m \snm^2(u|\tilde{m})}, \\
\dnm(u|\tilde{m}) &= \frac{\dnm(u|\tilde{m})}{1 - m \snm^2(u|\tilde{m})}.
\end{align*}
\]

(3.50)

We also obtain the following relations

\[
\frac{1}{\pi} \snm(iK|\tilde{m})\Delta_4|\tilde{m}| = \frac{i}{(c_+ + c - d + \tilde{d})\sqrt{cd}} (c_+ + c), \quad \cnn\frac{iK|\tilde{m})\Delta_4|\tilde{m}|}{\pi} = \frac{\sqrt{c_+ + c}}{\sqrt{c_+ - c - d + d} (c_+ + c - d + d)}
\]

\[
dnm(iK|\tilde{m})\Delta_4|\tilde{m}| = \frac{c_+ + c - d - \tilde{d}}{c_+ - c - d + d} = \frac{c_+ + c - d}{c_+ - c - d + d},
\]

(3.51)

and the further identity

\[
\frac{2(c_+ + c)(c_+ + d + d)}{(c_+ + c - d + d)(c_+ + c + d - d)(d + d)} = 1.
\]

Substituting the above relations in (3.48) and defining \(\tilde{\Omega} = 2K(m)\Omega + K(m)\), we obtain

\[
q_{hel}(x,t) = \frac{(d + c_+)(1 - m^2\snm(\tilde{\Omega}|m)) + c_+^2(c_+ + d)^2\snm(\tilde{\Omega}|m) - c_+ \cnn(\tilde{\Omega}|m)dnm(\tilde{\Omega}|m)}{c_+^2 - c_+ + c - c_+ - d + 2(c_+ + c_+)\cnn(\tilde{\Omega}|m)}
\]

\[
= -c_+ + (d + c_+) - d + \frac{2(c_+ + d)}{(c_+ + d)(c_+ + c_+) - (d - c_+)\cnn(\tilde{\Omega}|m)}
\]

\[
= -c_+ - c_+ - d + 2\frac{(c_+ + d)(c_+ + c_+)}{c_+ - d - (d - c_+)\cnn(\tilde{\Omega}|m)}
\]

(3.52)
where
\[ x_0 = -\frac{K(m)\Delta}{\pi} + K(m). \]
which concludes the proof of Theorem 3.3. \qed

4 Large time asymptotic

We study the long time asymptotic of the Riemann-Hilbert problem by applying Deift-Zhou steepest descend method for oscillatory RH problems. The high oscillatory terms of the matrix entries of \( J_M(k) \) defined in (2.14) comes from the exponential factors \( e^{\pm i\theta(k)} \). Since the stationary point of \( \theta(k) = xk + 4k^3t \) is \( k = \sqrt{-x/12t} \), we introduce a new independent variable
\[ \xi = \frac{x}{12t}, \]
and the function \( \hat{\theta}(k,\xi) \) with \( t\hat{\theta}(k,\xi) = \theta(x,t,k) \).

To perform the asymptotic analysis of the Riemann Hilbert problem our first step is the introduction of an unknown scalar function \( g = g(k,\xi) \), which will be specified later on, such that
\[ g(k,\xi) = \hat{\theta}(k,\xi) + O\left(\frac{1}{|k|}\right), \quad |k| \to \infty. \]

Then we define the first transformation of the RHP
\[ Y(x,t;k) = M(x,t;k)e^{i(tg(k,\xi) - \theta(x,t;k))\sigma_3}, \]
so that
\[ Y_-(x,t;k) = Y_+(x,t;k)J_Y(x,t;k), \quad k \in \Sigma, \]
where
\[ J_Y(x,t;k) = e^{-i(tg_+(k,\xi) - \theta(x,t;k))\sigma_3}J_M(x,t;k)e^{i(tg_-(k,\xi) - \theta(x,t;k))\sigma_3} \]
with \( J_M \) as in (2.14). It is shown in [10] that the function \( g(k,\xi) \) is always analytic on the complex plane \( \mathbb{C}\setminus[i\epsilon_-, -i\epsilon_-] \). Furthermore, the asymptotic analysis in [11] produces three \( g \) functions that will be verified a-posteriori. Namely

- a dispersive shock wave region;
- a soliton and breather region;
- a breather region.

We start by performing the asymptotic analysis of the dispersive shock wave region.

4.1 Proof of theorem 1.4 part (b): dispersive shock wave region with \( -\frac{c^2}{2} + c^2_+ < \frac{x}{12t} < \frac{c^2}{3} + \frac{c^2}{6} \)

In this region we will verify a-posteriori that the \( g \)-function is analytic in \( \mathbb{C}\setminus[i\epsilon_-,-i\epsilon_-] \) and takes the form
\[ g(k,\xi) = 12 \int_{ic_-}^{ic_+} s \left( s^2 + \xi + \frac{c^2 + c^2_+ - d^2}{2}\right) \sqrt{s^2 + d^2(\xi)} ds, \]
for \( -\frac{c^2}{2} + c^2_+ < \xi < \frac{c^2}{3} + \frac{c^2}{6} \), where the quantity \( d = d(\xi) \) is determined by
\[ \int_{id}^{ic_+} \frac{s \left( s^2 + \xi + \frac{c^2 + c^2_+ - d^2}{2}\right) \sqrt{s^2 + d^2} ds}{\sqrt{(s^2 + c^2_+)(s^2 + c^2_+)}} = 0. \]
The solvability of this equation for \( d = d(\xi) \) was established in [41]. Below we give a different derivation of (4.2) and we show the solvability of \( d = d(\xi) \) using the hyperbolic nature of the Whitham modulation equations.

\[
\begin{align*}
\text{Im } g(k) < 0 & \quad \text{id} \\
\text{Im } g(k) > 0 & \quad \text{ic}_- \\
\text{Im } g(k) < 0 & \quad \text{ic}_+ \\
\text{Im } g(k) > 0 & \quad \text{id}
\end{align*}
\]

Figure 7: (a) Distribution of signs of \( \text{Im } g = 0 \) for \( c_2^+ - \frac{c_2^2}{2} < \xi < \frac{c_2^2}{2} + \frac{c_2^2}{4} \). (b) Contour deformation of the Riemann-Hilbert problem for \( X \).

A plot of the signs of the \( \text{Im } g(k, \xi) \) is shown in Figure 7 which describes the regions of the \( k \)-plane where the quantity \( e^{i g(k, \xi)} \) is exponentially small.

Our first step in the asymptotic analysis is to take care of the discrete spectrum. We introduce the function \( T(k, \xi) \) defined as

\[
T(k, \xi) = \tilde{T}(k) H(k), \quad \tilde{T}(k) = \prod_{\kappa_j \text{ Im } g(\kappa_j) < 0} \left( \frac{k - \kappa_j}{k - \kappa_j - \kappa_j^+} \right) \prod_{\text{Re } \kappa_j > 0, \text{Im } \kappa_j > 0} \left( \frac{k - \kappa_j}{k - \kappa_j} \right) \prod_{\text{Re } \kappa_j = 0} \left( \frac{k - \kappa_j}{k - \kappa_j} \right)
\]

where \( H(k) \) is analytic in \( \mathbb{C} \setminus [ic_-, \text{ic}_+] \) and \( H(k) = 1 + O(k^{-1}) \) as \( |k| \to \infty \). Further properties of \( T(k) \) and \( H(k) \) will be determined later.

Then we define the first transformation

\[
\tilde{Y}(k, \xi) = Y(k, \xi) T^{-\sigma_3}(k, \xi) \times
\]
\[
\begin{pmatrix}
1 & -e^{-2i\gamma(k,\xi)} \\
\frac{\gamma_j}{(k-\kappa_j)} & 0
\end{pmatrix}, \quad |k - \kappa_j| < \epsilon, \quad \text{Im}\ g(\kappa_j) < -2\delta
\]
\[
\begin{pmatrix}
1 & e^{-2i\gamma(k,\xi)} \\
\frac{\gamma_j}{(k+\kappa_j)} & 0
\end{pmatrix}, \quad |k + \kappa_j| < \epsilon, \quad \text{Im}\ g(-\kappa_j) < -2\delta
\]
\[
\begin{pmatrix}
e^{-2itg(k,\xi)} & 0 \\
\frac{\gamma_j}{(k-\kappa_j)} & 1
\end{pmatrix}, \quad |k - \kappa_j| < \epsilon, \quad \text{Im}\ g(\kappa_j) > 2\delta,
\]
\[
\begin{pmatrix}
e^{-2itg(k,\xi)} & 0 \\
\frac{\gamma_j}{(k+\kappa_j)} & 1
\end{pmatrix}, \quad |k + \kappa_j| < \epsilon, \quad \text{Im}\ g(-\kappa_j) > 2\delta,
\]
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad \text{elsewhere.}
\]

where \(\forall \delta > 0\) we assume \(\epsilon\) sufficiently small so that \(\text{Im}\ g(\kappa_j) < -2\delta\) and similarly for the other cases. In this way we are taking control of the exponentially big terms in the jump matrix relative to some points of the discrete spectrum. If some of the points \(\kappa_j\) lie exactly on the line \(\text{Im}\ g(k) = 0\), then a different treatment is needed because the corresponding soliton or breather will lie on the dispersive shock wave region. A formulation of the corresponding model problems associated to these cases is given in the Appendix E.

The jump matrix \(\tilde{J}_\tilde{F}\) (\(\tilde{Y}_- = \tilde{Y}_+J_{\tilde{F}}\)), is given by

\[
J_{\tilde{F}}(k) = \begin{pmatrix}
1 & \frac{T^2(k,\xi)e^{-2itg(k,\xi)}}{\left(\frac{\kappa_j}{k+\kappa_j}\right)^{\frac{1}{2}}} \\
0 & 1
\end{pmatrix}, \quad k \in \pm C_j, \quad \text{Im}\ g(k) < -\delta
\]
\[
\begin{pmatrix}
1 & \frac{\gamma_j}{\kappa_j} \\
\pm 1 & 0
\end{pmatrix}, \quad k \in \pm C_j, \quad \text{Im}\ g(k) > \delta
\]
\[
\begin{pmatrix}
1 & \frac{T^2(k,\xi)}{\left(\frac{\kappa_j}{k+\kappa_j}\right)^{\frac{1}{2}}} \\
0 & 1
\end{pmatrix}, \quad k \in \pm \overline{C}_j, \quad \text{Im}\ g(k) < -\delta,
\]
\[
\begin{pmatrix}
0 & \frac{T^{-2}(k,\xi)}{\left(\frac{\kappa_j}{k+\kappa_j}\right)^{\frac{1}{2}}} \\
1 & 1
\end{pmatrix}, \quad k \in \pm \overline{C}_j \quad \text{Im}\ g(k) > \delta
\]
\[
T^\frac{\sigma_3}{2}(k,\xi)J_{\tilde{F}}(k)T^{-\frac{\sigma_3}{2}}(k,\xi), \quad \text{elsewhere.}
\]

It is clear from the form of the above jumps, that the matrix \(J_{\tilde{F}}\) will be exponentially close to the identity as \(t \to \infty\) on the circles \(\pm C_j\) and \(\pm \overline{C}_j\).

The next step is to take care of the continuous spectrum on the real axis. As a first step, we reduce the jump \(J_{\tilde{F}}(t,\xi,k)\) for \(k \in \mathbb{R}\setminus\{0\}\) to a matrix exponentially close to the identity. For the purpose it is sufficient to factorise the matrix \(J_{\tilde{F}}(t,\xi,k)\) to the form

\[
J_{\tilde{F}}(x,t;k) = \begin{pmatrix}
\frac{r(k)}{T^2(k)} & -\frac{r(k)}{T^2(k)}T^2(k)e^{-2itg(k,\xi)} \\
0 & 1 + |r(k)|^2
\end{pmatrix}, \quad k \in \mathbb{R} \setminus \{0\}
\]
\[
= \begin{pmatrix}
1 & 0 \\
-\frac{r(k)}{T^2(k)} & 1
\end{pmatrix} \begin{pmatrix}
1 & -\frac{r(k)}{T^2(k)}T^2(k)e^{-2itg(k,\xi)} \\
0 & 1
\end{pmatrix}.
\]
Then using Deift-Zhou contour deformation method we introduce the new matrix \( X(k) \)

\[
X(k) = \begin{cases} 
\hat{Y}(k) \begin{pmatrix} 1 & 0 \\ \frac{r(k)}{T^2(k)} e^{2itg(k, \xi)} & 1 \end{pmatrix}, \quad k \in \Omega_1 \\
\hat{Y}(k) \begin{pmatrix} 1 & 0 \\ -r(k)T^2(k)e^{-2itg(k, \xi)} & 1 \end{pmatrix}^{-1}, \quad k \in \Omega_2 \\
\hat{Y}(k), \quad \text{elsewhere}
\end{cases}
\]

where the regions \( \Omega_1 \) and \( \Omega_2 \) are specified in Figure [7]. In this way we can reduce the jump of \( X(k) \) on \( \mathbb{R} \setminus \{0\} \) to identity, while the jumps of \( X(k) \) on the lines \( L_{1,2} \setminus U_{\pm \id} \) are exponentially close to the identity, where \( U_{\pm \id} \) is a small neighbourhood of the point \( \pm \id \). The jumps \( J_X(k) \) for \( k \in L_{1,2} \cap U_{\pm \id} \) give the subleading contribution to the asymptotics, analysis. We still need to determine the functions \( g(k, \xi) \) and \( F(k) \). The remaining jumps of the matrix \( X(k) \) are obtained using the identities 13, 14 from Lemma [7.2] and take the following form

\[
J_X(x, t; k) = \begin{cases} 
T^\sigma_+(k) \begin{pmatrix} e^{-it(g_+-g_-)} & 0 \\ f(k)e^{it(g_+-g_-)} & e^{it(g_+-g_-)} \end{pmatrix} T^\sigma_-(k), \quad k \in (ic_-, \id), \\
T^\sigma_+(k) \begin{pmatrix} e^{-it(g_+-g_-)} & 0 \\ 0 & e^{it(g_+-g_-)} \end{pmatrix} T^\sigma_-(k), \quad k \in (i\id, ic_+) \cup (-ic_+, -i\id), \\
T^\sigma_+(k) \begin{pmatrix} 0 & e^{it(g_+-g_-)} \\ -g(k)e^{-it(g_+-g_-)} & 0 \end{pmatrix} T^\sigma_-(k), \quad k \in (ic_+, -ic_+), \\
T^\sigma_+(k) \begin{pmatrix} e^{-it(g_+-g_-)} & 0 \\ 0 & e^{it(g_+-g_-)} \end{pmatrix} T^\sigma_-(k), \quad k \in (-i\id, ic_-) \cup (-ic_+, ic_-).
\end{cases}
\]

We require that the above matrix \( J_X \) has oscillatory diagonal terms and non oscillatory off-diagonal terms as \( t \to \infty \). Therefore we need to require that

\[
\begin{align*}
g_+(k) + g_-(k) &= 0, \quad k \in [ic_-, \id] \cup [ic_+, -ic_] \cup [-i\id, -ic_-] \\
g_+(k) - g_-(k) &= \Re, \quad k \in [ic_-, -ic_-], \\
g_+(k) - g_-(k) &= -B(\xi), \quad k \in (i\id, ic_+) \cup (-ic_+, -i\id)
\end{align*}
\]

where \( B = B(\xi) \) is independent from \( k \). Furthermore, for reasons that will become clear later, we chose the scalar function \( T(k) \) in such a way that

\[
T_+(k)T_-(k) = 1, \quad k \in [ic_-, -ic_+].
\]

Then the above jump matrices are reduced to the form

\[
J_X(x, t; k) = \begin{cases} 
\begin{pmatrix} T_+(k)e^{it(g_+-g_-)} & 0 \\ T_-(k)e^{it(g_+-g_-)} & T_+(k) \end{pmatrix}, \quad k \in (ic_-, \id), \\
\begin{pmatrix} 0 & T_-(k)e^{it(g_+-g_-)} \\ T_+(k)e^{it(g_+-g_-)} & 0 \end{pmatrix}, \quad k \in (i\id, ic_+) \cup (-ic_+, -i\id) \\
0 & 1, \quad k \in (ic_+, -ic_+), \\
T_+(k)e^{-it(g_+-g_-)} & 0 \\
T_-(k)e^{-it(g_+-g_-)} & T_+(k)
\end{cases}
\]

\[
J_X(x, t; k) = \begin{cases} 
\begin{pmatrix} T_+(k)e^{it(g_+-g_-)} & 0 \\ T_-(k)e^{it(g_+-g_-)} & T_+(k) \end{pmatrix}, \quad k \in (ic_-, \id), \\
\begin{pmatrix} 0 & T_-(k)e^{it(g_+-g_-)} \\ T_+(k)e^{it(g_+-g_-)} & 0 \end{pmatrix}, \quad k \in (i\id, ic_+) \cup (-ic_+, -i\id) \\
0 & 1, \quad k \in (ic_+, -ic_+), \\
T_+(k)e^{-it(g_+-g_-)} & 0 \\
T_-(k)e^{-it(g_+-g_-)} & T_+(k)
\end{cases}
\]

\subsection{4.1.1 Determination of the scalar functions \( T(k) \) and \( g(k) \)}

In this subsection we determine the scalar function \( T(k) \) and we derive the expression for the function \( g(k) \) that satisfies (4.5) and (4.1).
The solution is obtained passing to the logarithm and using Plemelj formula. Let us introduce

the function \( \Delta = \Delta(\xi) \) will be independent from \( k \) and needs to be determined. Using the relations

we finally have the following RH problem for the function \( T(k) : \)

\[
T_+(k)T_-(k) = \frac{1}{|a(k)|^2} \quad k \in (ic_-, id) \\
T_+(k)T_-(k) = |a(k)|^2 \quad k \in \mathbb{C} \setminus (-id, -ic_-) \\
T_+(k) = T_-(k)e^{i\Delta(k)^2}, \quad k \in (id, ic_+ \cup (-ic_+, -id), \\
T_+(k)T_-(k) = 1, \quad k \in (ic_+, -ic_+), \\
T(k) = 1 + O(k^{-1}), \quad \text{as } |k| \to \infty.
\]

The solution is obtained passing to the logarithm and using Plemelj formula. Let us introduce

\[ R(k) = \sqrt{(k^2 + c^2_1)(k^2 + d^2)(k^2 + c^2_3)} \quad (4.8) \]

where \( R(k) \) is analytic in \( \mathbb{C} \setminus [ic_-, id] \cup [ic_+, -ic_+] \cup [-id, -ic_-] \) and real positive for \( k = +0 \), then the expression

\[
T(k) = \tilde{T}(k) \exp \left[ \frac{R(k)}{2\pi i} \left\{ \left( \frac{\int_{ic_+}^{ic_-} - \int_{id}^{ic_+}}{\int_{ic_-}^{ic_+}} - \int_{id}^{ic_+} \right) \frac{\ln |a(s)|^2}{(s-k)R_+(s)} ds + \left( \frac{\int_{id}^{ic_+} - \int_{ic_-}^{ic_+}}{\int_{ic_-}^{ic_+}} \right) \frac{i\Delta ds}{(s-k)R(s)} \right\} \right]
\]

solves the scalar RHP for \( T(k) \). The requirement that \( T(k) = 1 + O(k^{-1}) \) as \( |k| \to \infty \) determines \( \Delta(\xi) \). Indeed we have, using the symmetry of the problem, that

\[
\Delta = \int_{id}^{ic_+} \ln |a(s)|^2 R_+(s) ds + \int_{ic_+}^{ic_-} \ln |a(s)|^2 R_+(s) ds - \int_{ic_+}^{ic_-} \frac{\ln |a(s)|^2 R_+(s) ds}{R(s)}
\]

\[
= \frac{\sqrt{c^2_1 - c^2_3}}{K(m)} \left[ \int_{id}^{ic_-} \ln |a(s)|^2 R_+(s) ds + \int_{ic_-}^{ic_+} \ln |a(s)|^2 R_+(s) ds \right].
\]

The scalar function \( g(k) \) satisfies the conditions \( (4.5) \) and \( (4.11) \). This implies that the function \( g'(k) \) is analytic in \( \mathbb{C} \setminus [ic_-, -ic_-] \) and on the interval \( [ic_-, -ic_-] \) satisfies the conditions

\[
g'_+(k) + g'_-(k) = 0, \quad k \in (ic_-, id) \cup (ic_+, -ic_-) \cup (-id, -ic_-) \\
g'_+(k) - g'_-(k) = 0, \quad k \in (id, ic_+ \cup (-ic_+, -id) \\
g'(k) = \tilde{\theta}(k) + O(k^{-2}) \quad \text{as } |k| \to \infty.
\]

From the above conditions it follows that

\[
g'(k) = 12 P(k) R(k) \quad P(k) = k^5 + k^3(\xi + \frac{1}{2}(d^2 + c^2_1 + c^2_3)) + bk
\]

where \( R(k) \) is defined in \( (4.8) \).

The constant \( b \) is determined by requiring that the integral

\[ g(k) = \int_{ic_-}^{ic_+} g'(s) ds \]
Appendix B.

Let us observe that

\[ B(\xi) = 2 \int_{id}^{ic} g'(k)dk = 12 \int_{id}^{ic} s^2 + s(\xi + \frac{1}{2}(d^2 + c_2^2 + c_1^2) + b \frac{s}{\sqrt{(s + d^2)(s + c_1^2) + (s + c_2^2)}} ds \]

\[ = -12\pi \sqrt{c_2^2 - c_1^2} \frac{K(m)}{K(m)}(\xi - \frac{1}{6}(c_2^2 + c_1^2 + d^2)) \in \mathbb{R}. \]

Let us observe that

\[ tB(\xi) = xU + tV \]

where \( U \) and \( V \) have been defined in (3.37). We still need to determine the quantity \( d \). This is obtained by requiring that \( g'(k)|_{k=\pm id} = 0 \) that implies that the polynomial \( P(k) \) in (4.12) has a zero at \( k = \pm i\kappa_0 \), namely

\[ P(\pm id) = \pm i[d^3 - d^3(\xi + \frac{1}{2}(d^2 + c_2^2 + c_1^2)) + bd] = 0 \]

or

\[ 12\xi = W_2(c_+, d, c_-), \]

where \( W_2(\beta_1, \beta_2, \beta_3) \) has been defined in (1.12).

We observe that \( W_2(\beta_1, \beta_2, \beta_3) \) is the speed of the Whitham modulation equations for MKDV derived in [23]. The relation with the speed \( V_2(r_1, r_2, r_3) \) of the Whitham modulation equations for KdV is as follows:

\[ W_2(\beta_1, \beta_2, \beta_3) = V_2(\beta_1^2, \beta_2^2, \beta_3^2). \]

In particular it was shown in [33] that the Whitham modulation equations for KdV are strictly hyperbolic and satisfy the relation \( \partial_{\beta_2} V_2(r_1, r_2, r_3) > 0 \) for \( r_1 < r_2 < r_3 \) which implies that

\[ \frac{\partial}{\partial \beta_2} W_2(\beta_1, \beta_2, \beta_3) = 2\beta_2 \frac{\partial}{\partial r_2} V_2(\beta_1^2, \beta_2^2, \beta_3^2)|_{r_2 = \beta_2^2} > 0, \quad \beta_2 > 0. \]

The above relation shows that the equation (4.16) is invertible for \( d \) as a function of \( \xi \) only when \( d > 0 \) or equivalently when \( c_+ > 0 \). Further comments about the case \( c_- > -c_+ > 0 \) are given in Appendix B.

Using the properties of the elliptic functions [44], we get that as \( m \to 0 \)

\[ K(m) = \frac{\pi}{2} \left( 1 + \frac{m}{4} + \frac{9}{64} m^2 + O(m^3) \right), \quad E(m) = \frac{\pi}{2} \left( 1 - \frac{m}{4} - \frac{3}{64} m^2 + O(m^3) \right), \]

and as \( m \to 1 \)

\[ E(m) \approx 1 + \frac{1}{2} (1 - \sqrt{m}) \left[ \log \frac{16}{1 - m} - 1 \right], \quad K(m) \approx \frac{1}{2} \log \frac{16}{1 - m}. \]

Using the above expansions we have that
\[ \text{as } m \to 0 \text{ or } d \to c_+ \] \[ W_2(c_+, c_+, c_-) = -6c_+^2 + 12c_+^2 \]

and

\[ \text{as } m \to 1 \text{ or } d \to c_- \] \[ W_2(c_+, c_-, c_-) = 4c_+^2 + 2c_-^2 \]

which implies that

\[ \frac{-c_-^2}{2} + c_+^2 < \xi < \frac{c_-^2}{3} + \frac{c_+^2}{6}. \]

Summarizing, the function \( g(k) \) takes the form

\[
g(k) = 12 \int_{\infty}^{k} s^5 + s^3(\xi \mp \frac{1}{2}(d^2 + c_+^2 + c_+^2)) + bs \sqrt{(s^2 + c_+^2)(s^2 + c_+^2)(s^2 + d^2)} ds\]

\[ b = \frac{1}{3}(c_+^2 c_+^2 + c_+^2 d^2 + c_+^2 d^2) + (\xi - \frac{1}{6}(c_+^2 + c_+^2 + d^2)) \left[ c_+^2 - (c_+^2 - c_+^2) \frac{E(m)}{K(m)} \right]. \tag{4.19} \]

The parameter \( d = d(\xi) \) is determined by (4.16). The above derivation for the function \( g(k) \) is equivalent to the one obtained in [40] and written in (4.2) with \( d = d(\xi) \) as in (4.3). The signature of \( \text{Im} \ g(k) \) is given in Figure 7.

**Remark 4.1.** Signature of \( \text{Im} \ g(k) \) can be constructed from the following considerations. Let us look at a level line \( \text{Im} \ g(k) = \text{const} \). It can be parametrised by \( k = k(s) \), where \( s \in \mathbb{R} \), with \( |k'(s)| = 1 \). The function \( g(k(s)) \) equals

\[ g(k(s)) = \int_{s_0}^{s} g'(k(s))k'(s)ds, \]

and since \( \text{Im} \ g(k(s)) = \text{const} \), hence

\[ k'(s) = \frac{g'(k(s))}{|g'(k(s))|} \]

This immediately gives us the direction of the level line for every point \( k \), and there is exactly one line passing through every such point as long as \( g'(k) \neq 0 \).

Except for those regular points, we have several singular points, where \( g'(k) \) is either 0 or infinite. For such a point \( k_0 \), let us take a ray

\[ k(s) = k_0 + se^{i\varphi}, \]

which emanates from the point \( k_0 \) at an angle \( \varphi \). The condition that on that ray \( \text{Im} \ g(k) \) is constant, i.e.

\[ e^{i\varphi} = \frac{g'(k(s))}{|g'(k(s))|}, \]

gives the values for \( \varphi \). For the point \( k_0 = ic \), there is only one angle, \(-\frac{\pi}{2}\), and for the point \( k_0 = id \), there are three angles, \( \frac{\pi}{2}, -\frac{\pi}{3}, -\frac{2\pi}{3} \). For the point \( k = ip \), there are 4 values for the angle. (Here \( \pm \mu \) are the zeros of the numerator in the integrand in formula (4.14).)

Also, there are 6 rays \( \text{Im} \ g = \text{const} \), converging to the point \( k \to \infty \), along directions \( \pi k/3 \), \( k = 0, 1, 2, 3, 4, 5 \). The ray emanating from \( k = ic \), comes into point id, gluing with one of the three rays emanating from the point id. There is only one way to connect all the rays emanating from the singular points. Furthermore, among all the level lines \( \text{Im} \ g = \text{const} \), we need to take only those, where \( \text{Im} \ g = 0 \). They consist of the real line, segments \( \pm [ic \ldots id] \), and hence all the other level rays emanating from \( \pm id \). This gives us all 6 rays \( \text{Im} \ g = 0 \), coming to \( \infty \), and is plotted in Figure 7.
4.1.2 Opening of the lenses

Finally we apply Deift-Zhou steepest descendent method to get rid of the highly oscillatory terms in \( k \) in the diagonal exponents of the matrix \( J_X \) in \([4.7]\). For the purpose we open lenses in the intervals \((ic_-, id)\) and \((-id, -ic_-)\). We first need to define the analytic continuation of the function \( f(k) \) on a neighbourhood of the interval \([ic_-, ic_+], \) which we define as

\[
\tilde{f}(k) = \frac{-1}{a(k)b(k)} = \frac{1 + r(k)r(k)}{-r(k)}, \quad k \in U_\sigma(\Sigma_-).
\]  

(4.20)

Then it is immediate to verify that

\[
\tilde{f}_+(k) = f(k), \quad \tilde{f}_-(k) = -f(k), \quad k \in [ic_-, ic_+] \cup [-ic_+, -ic_-].
\]

In order to get a factorization of the matrices \( J_X(x, t; k) \in (ic_-, id) \cup (-id, -ic_-) \) we assume that

\[
T_+(k)T_-(k) = -if_+(k) = -if(k), \quad k \in (ic_-, id)
\]

\[
T_+(k)T_-(k) = -\frac{i}{f(k)} k \in \cup(-id, -ic_-).
\]

(4.21)

In this way we can factorize

\[
J_X = \begin{pmatrix}
1 & \frac{T^2 e^{-2ixg}}{f_1 - f} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
i & 0 \\
i & 0
\end{pmatrix}
\begin{pmatrix}
1 & -\frac{T^2 e^{-2ixg}}{f - f_1} \\
0 & 1
\end{pmatrix}, \quad k \in (ic_-, id)
\]  

(4.22)

\[
J_X = \begin{pmatrix}
1 & 0 \\
\frac{2 e^{2ixg}}{f_1 - f} & 1
\end{pmatrix}
\begin{pmatrix}
i & 0 \\
i & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{2 e^{-2ixg}}{f - f_1} & 1
\end{pmatrix}, \quad k \in (-id, -ic_-)
\]  

(4.23)

Then we proceed with contour deformation and we define a new matrix \( W(k) \) as

\[
W(k) = X(k)\begin{cases}
\begin{pmatrix}
1 & \frac{T^2 e^{-2ixg}}{f} \\
0 & 1
\end{pmatrix}, & k \in \Omega_5 \cup \Omega_7 \\
\begin{pmatrix}
1 & 0 \\
\frac{2 e^{2ixg}}{f_1 - f} & 1
\end{pmatrix}, & k \in \Omega_6 \cup \Omega_8 \\
I & \text{elsewhere}
\end{cases}
\]

we obtain that the jumps for \( W(k) \) are

\[
J_W(x, t; k) = \begin{cases}
\begin{pmatrix}
1 & -\frac{T^2 e^{-2ixg}}{f} \\
0 & 1
\end{pmatrix}, & k \in L_5 \cup L_7 \\
\begin{pmatrix}
1 & 0 \\
\frac{2 e^{2ixg}}{f_1 - f} & 1
\end{pmatrix}, & k \in L_6 \cup L_8 \\
\begin{pmatrix}
i & 0 \\
i & 0
\end{pmatrix}, & k \in (ic_-, id) \cup (ic_+,-ic_+) \cup (-id,-ic_-) \\
\begin{pmatrix}
T_+(k) e^{-i\Delta(t)} \\
T_-(k)
\end{pmatrix}, & k \in (id,ic_+) \cup (-ic_+, -id).
\end{cases}
\]

(4.24)

Because of the signature of \( \text{Im} \ g(k) \) given in Figure\,7 we have that the matrix \( J_W \) is exponentially close to the identity on \( L_5 \cup L_7 \setminus \{U_{id} \cup U_{ic_-}\} \) and on \( L_6 \cup L_8 \setminus \{U_{-id} \cup U_{-ic_-}\} \), while on the the contours \( L_5 \cup L_7 \cap \{U_{id} \cup U_{ic_-}\} \) and on \( L_6 \cup L_8 \cap \{U_{-id} \cup U_{-ic_-}\} \), the matrix \( J_W \) is close to the identity but not uniformly. A detailed analysis of the error term arising in this case has been obtained in a similar setting for the MKdV equation with \( c_+ = 0 \) \([5.\text{Theorem 2}]\) and also for the KdV equation in \([31]\) where it was shown that the error is of order \( O(t^{-1}) \).

We arrive to the model problem for the matrix \( P^\infty(k) \).
Riemann-Hilbert problem 6. Find a $2 \times 2$ matrix-valued function $P^\infty(k)$ analytic in $\mathbb{C}\backslash\{ic_-, -ic_-\}$ such that

1. $P^-_\infty(k) = P^+_\infty(k)J_{P^-}(k), \quad k \in (ic_-, -ic_-)$ with

$$J_{P^-}(k) = \begin{cases} 
\begin{pmatrix} 0 & i \\
1 & 0 
\end{pmatrix} & k \in (ic_-, id) \cup (ic_+, -ic_-) \cup (-id, -ic_-) \\
\begin{pmatrix} e^{iB(\xi)+i\Delta(\xi)} & 0 \\
e^{-iB(\xi)-i\Delta(\xi)} & 0 
\end{pmatrix} & k \in (id, ic_+) \cup (-ic_+, -id);
\end{cases}
$$

with $B(\xi)$ and $\Delta(\xi)$ defined in [4.14] and [4.10] respectively;

2. $P^\infty(k) \to I$ as $k \to \infty$;

3. $P^\infty(k)$ has at most fourth root singularities at the points $\pm ic_-, \pm ic_+$ and $\pm id$.

Then the quantity

$$q(x, t) = \lim_{k \to \infty} 2ikP^\infty(k; x, t)_{21} = \lim_{k \to \infty} 2ikP^\infty(k; x, t)_{12},$$

approximate the MKDV solution. The solution of the Riemann-Hilbert problem [6] has been considered in [49] where it was solved in terms of hyperelliptic theta-function defined on the Jacobi variety of the surface $\Gamma := \{(k, y) \in C^2 \mid y^2 = R^2(k)\}$. In theorem 3.3 we showed that the MKDV solution obtained from the RHP [6] corresponds to the travelling wave solution [17]. In particular we obtain

$$q(x, t) = q_{per}(x, t; c_-, d, c_+, x_0) + O(t^{-1}),$$

where $q_{per}(x, t; \beta_1, \beta_2, \beta_3, x_0)$ has been defined in [3.39] and

$$x_0 = -\frac{K(m)\Delta}{\pi} + K(m)$$

with $K(m)$ the complete elliptic integral with modulus $m^2 = \frac{d^2 - c_2^2}{c_2^2 - c_1^2}$ and $\Delta$ defined in [4.10]. The error term in a similar case was computed in [5] Theorem 2] where all the local parametrices are the same. The error term is valid strictly inside the dispersive shock wave region. On the boundary of the dispersive shock wave region a different error term is present (see [5] Section 3) for details). We have thus concluded the proof of Theorem 1.4 part (b).

5 Proof of theorem 1.4 part (a) and (c)

Here we continue the proof of Theorem 1.4 in the soliton-breather region and in the breather region.

5.1 Breather region: Theorem 1.4 part (c)

In the breather region the function $g(k, \xi)$ takes the form [40]

$$g(k, \xi) = 2 \left(2k^2 - c_2^2 + 6\xi\right) \sqrt{k^2 + c_2^2}, \quad \xi < -\frac{c_2^2}{2} + c_1^2$$

namely $g(k)$ is analytic in $\mathbb{C}\backslash\{ic_-, -ic_-\}$ and

$$g_+(k) + g_-(k) = 0, \quad k \in (ic_-, -ic_-), \quad g(k) = \hat{\theta}(k) + O(k^{-1}) \quad \text{as } |k| \to \infty.$$

The above two conditions defined $g(k)$ uniquely. We chose $\sqrt{k^2 + c_2^2}$ to be real on $(ic_-, -ic_-)$ and positive for $k = +0$.

We observe that Im $g(k) = 0$ for $k$ on the segment $[ic_-, -ic_-]$ and on the real line. There is an extra couple of symmetric curves such that Im $g(k) = 0$ (see Figure 5.1). For $\xi < -\frac{c_2^2}{2}$ this couple
of curves crosses the real line at the points $k = \pm d_1 = \pm \sqrt{-\frac{c_2}{2} - \xi}$ (first left constant region).
For $-\frac{c_2}{2} < \xi < -\frac{c_2}{2} + c_1^2$, a couple of symmetric curve crosses the imaginary line on the points $k = \pm id_0 = \pm i\sqrt{\frac{c_2}{2}}$ (second left constant region). The points $\pm \kappa_1$ and $\pm id_0$ are the zeros of the equation

$$g'(k) = 2(6k^2 - 3c_2^2 + 6\xi) \frac{k}{\sqrt{k^2 + c_2^2}} = 0.$$  

![Distribution of signs of $\text{Im} g(k)$ in the breather region.](image)

(a) $-\frac{c_2}{2} < \xi < -\frac{c_2}{2} + c_1^2$ (First left constant region) (b) $\xi < -\frac{c_2}{2}$ (Second left constant region)

The asymptotic analysis in these two cases is slightly different. We start our analysis of the first left constant region.

### 5.1.1 First left constant region $-\frac{c_2}{2} < \frac{\xi}{12\pi} \equiv \xi < -\frac{c_2}{2} + c_1^2$

For simplicity we assume that $a(k) \neq 0$ for $k \in (i c_+ , 0)$. We define the first transformation of the RHP $M(k) \to Y(k)$ defined as

$$Y(k) = M(k)e^{i\gamma(k,\xi) - \theta(x,t;k)\sigma_3}T^{-\sigma_3}(k,\xi),$$

where now $g(k,\xi)$ is as in (5.1) and $T(k) = T(k,\xi)$ is defined for a given $\xi$ as follows:

$$T(k,\xi) = \tilde{T}(k,\xi)H(k,\xi),$$

$$\tilde{T}(k,\xi) = \prod_{\Re \kappa_j > 0, \Im \kappa_j > 0} \frac{k - \kappa_j k + \kappa_j}{k - \kappa_j k + \kappa_j}, \quad \prod_{\Re \kappa_j = 0, \Im \kappa_j > 0} \frac{k - \kappa_j k + \kappa_j}{k - \kappa_j k + \kappa_j}, \quad \prod_{\Re \kappa_j < 0, \Im \kappa_j < -\varepsilon} \frac{k - \kappa_j k + \kappa_j}{k - \kappa_j k + \kappa_j}, \quad \prod_{\Re \kappa_j < 0, \Im \kappa_j < -\varepsilon} \frac{k - \kappa_j k + \kappa_j}{k - \kappa_j k + \kappa_j},$$

and $H(k,\xi)$ is suppose to be analytic in $\mathbb{C} \setminus [ic_-,-ic_-]$ and it will be determined later. Here $\varepsilon > 0$.

In order to perform our analysis on the continuum spectrum we assume the condition (5.2) for $g(k)$ and

$$T_+(k)T_-(k) = \begin{cases} \frac{1}{|a(k)|^2}, & k \in (i c_-,id_0), \\ |a(k)|^2, & k \in (-id_0,-ic_-), \\ 1, & k \in (id_0,ic_-). \end{cases}$$
Then the function $T(k)$ can be found taking the logarithm of the above expression and applying Plemelj formula so that we obtain

$$T(k, \xi) = \tilde{T}(k, \xi) \exp \left[ \frac{-i \xi}{2 \pi} \left\{ \int_{i c_-}^{i d_0} - \ln |a(s)|^2 - \ln \left( \tilde{T}_2(s, \xi) \right) ds \right\} \right] \times$$

$$\times \exp \left[ \frac{-i c_-}{2 \pi i} \left\{ \int_{-i d_0}^{i c_-} - \ln |a(s)|^2 - \ln \left( \tilde{T}_2(s, \xi) \right) ds \right\} \right] \times \exp \left[ \frac{-i c_-}{2 \pi i} \left\{ \int_{-i d_0}^{i c_-} - \ln |a(s)|^2 - \ln \left( \tilde{T}_2(s, \xi) \right) ds \right\} \right],$$

which can be written in the compact form

$$T(k, \xi) = \tilde{T}(k, \xi) \exp \left[ \frac{-i \xi}{2 \pi} \left\{ \int_{i c_-}^{i d_0} - \ln |a(s)|^2 - \ln \left( \tilde{T}_2(s, \xi) \right) ds \right\} \right] \times$$

$$\times \int_{-i d_0}^{i c_-} - \ln |a(s)|^2 - \ln \left( \tilde{T}_2(s, \xi) \right) ds \right\} \times \exp \left[ \frac{-i c_-}{2 \pi i} \left\{ \int_{-i d_0}^{i c_-} - \ln |a(s)|^2 - \ln \left( \tilde{T}_2(s, \xi) \right) ds \right\} \right].$$

Let us notice, that $T(k) = 1 + O\left(\frac{1}{k}\right), \quad k \to \infty$.

With our choice of $g(k)$ and $T(k)$ the jump matrix $J_Y(k)$ for the RHP for $Y(k)$, $Y_-(k) = Y_+(k)J_Y(k)$ admits a factorization of the form

$$\text{(5.5)}$$
that there are no poles within the regions $\Omega_j$. Riemann-Hilbert problem 7. Final RHP for the

Here

Now we open the lenses as in figure 9 where we assume that we can deform all the curves $L_j$ so that there are no poles within the regions $\Omega_j$. We define a new matrix $X$ as

With the above transformation the jump matrix on $(id_0, 0)$ transforms to

and similarly on $(0, -id_0)$. We obtain that the matrix $X(k)$ solves the following RHP.

Riemann-Hilbert problem 7. Final RHP for the 2nd constant region $\frac{-c^2}{2} < \xi < \frac{-c^2}{2} + c^2$.

- Find a $2 \times 2$ matrix $X(k)$ meromorphic in $\mathbb{C}\backslash \Sigma$ with $\Sigma = \bigcup_{j=1}^{7} L_j \cup (ic_-, -ic_-)$ (see figure 9) and with poles in the points $\kappa_j$ such that

- Jumps

and

$$J_X(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in (ic_-, -ic_-),$$
As in the other case we define 

\[ J_X(k) = \begin{cases} 
\frac{1}{T(k)^2}e^{2itg(k)} & , k \in L_1, \\
\frac{1}{T(k)} & , k \in L_7, \\
\frac{1}{T(k)^2} & , k \in L_8, \\
0 & , k \in L_6, \\
\end{cases} \]


\[ \begin{pmatrix} 1 & 0 \\
0 & 1 \\
\end{pmatrix}, \]

where the function \( g \) and \( -g \) we observe that if \( \Im g(\kappa) > \varepsilon \) for \( \Im \kappa > 0 \) then the first set of pole conditions becomes empty as \( t \to \infty \) while for \( \Im g(\kappa) < -\varepsilon \), the second set if pole conditions becomes empty as \( t \to \infty \). The only pole conditions that remain are for those values of \( \kappa \) such that \( -\varepsilon < \Im g(\kappa) < \varepsilon \).

5.1.2 Second constant region \( \xi < -\frac{\varepsilon}{T} \)

As in the other case we define 

\[ Y(k) = M(k)e^{i(\xi g(k, \xi - \theta(t,\xi)k))}T^{-\sigma_3}(k, \xi), \]

where the function \( g(k) \) is as in [5.1] while for the function \( T(k) \) defined in [5.3] we require the following conditions that are needed in order to apply the Deift-Zhou steepest descent method to deform the contours:

\[ T_+(k)T_-(k) = \begin{cases} 
-\frac{1}{\Im (\kappa)} & , k \in (ic_-, 0), \\
\frac{1}{\Im (\kappa)} = |a(k)|^2 & , k \in (0, -ic_-), \\
\end{cases} \]

Remark 5.1. Using the symmetry properties of the \( g \) function, namely \( \bar{g(k)} = g(k) \) and \( -g(k) \) we observe that if \( \Im g(\kappa) > \varepsilon \) for \( \Im \kappa > 0 \) then the first set of pole conditions becomes empty as \( t \to \infty \) while for \( \Im g(\kappa) < -\varepsilon \), the second set if pole conditions become empty as \( t \to \infty \). The only pole conditions that remain are for those values of \( \kappa \) such that \( -\varepsilon < \Im g(\kappa) < \varepsilon \).
and

\[
\frac{T_+(k)}{T_-(k)} = 1 + |r(k)|^2, \quad k \in (-d_1, d_1), \quad d_1 = \sqrt{-\xi - c^2 / 2},
\]

(5.10)

with \( T(k) \to 1 \) as \(|k| \to \infty \). Observe that in this case \( T(k) \) is not analytic in \([ic_-, -ic_-]\) and also on the real axis.

This function can be found in the form

\[
T(k, \xi) = \tilde{T}(k, \xi) \exp \left[ \frac{\sqrt{k^2 + c^2}}{2\pi i} \int_{-\infty}^{0} \left( \frac{-\ln |a(s)|^2 - \ln \tilde{T}(s, \xi)^2}{(s - k)\sqrt{s^2 + c^2}} \right) ds \right]
\]

\[
\cdot \exp \left[ \frac{\sqrt{k^2 + c^2}}{2\pi i} \int_{0}^{\infty} \left( \frac{-\ln |a(s)|^2 - \ln \tilde{T}(s, \xi)^2}{(s - k)\sqrt{s^2 + c^2}} \right) ds \right]
\]

(5.11)

with \( \tilde{T}(k, \xi) \) as in formula (5.3).

Figure 10: The opening of the lenses in the second left constant region.

Because of (5.9) and (5.10) the jump matrix \( J_Y \) of the RHP for \( Y(k) \) has a factorization on the real axis of the form

\[
J_Y(k) = \begin{cases} 
\frac{1}{r(k)e^{2itg(k, \xi)}} & 0 \\
\frac{-r(k)e^{2itg(k, \xi)}}{1 + |r(k)|^2} & \frac{1}{1 + |r(k)|^2} \\
1 & 0 \\
\frac{-r(k)e^{-2itg}}{1 + |r(k)|^2} & \frac{1}{1 + |r(k)|^2} \\
\end{cases}, \quad k \in (-\infty, -d_0) \cup (d_0, +\infty),
\]

\[
J_Y(k) = \begin{cases} 
\frac{1}{r(k)e^{2itg(k, \xi)}} & 0 \\
\frac{-r(k)e^{2itg(k, \xi)}}{1 + |r(k)|^2} & \frac{1}{1 + |r(k)|^2} \\
1 & 0 \\
\frac{-r(k)e^{-2itg}}{1 + |r(k)|^2} & \frac{1}{1 + |r(k)|^2} \\
\end{cases}, \quad k \in (-d_0, d_0).
\]

(5.12)
We use the above factorization to open the lenses and define the matrix \( X(k) \) as

\[
X(k) = \begin{cases}
    Y(k) \begin{pmatrix}
        1 & 0 \\
        -r(k)e^{2itg(k,\xi)} & 1
    \end{pmatrix}, & k \in \Omega_1 \\
    Y(k) \begin{pmatrix}
        1 & -r(k)T^2(k)e^{-2itg(k,\xi)} \\
        0 & 1
    \end{pmatrix}^{-1}, & k \in \Omega_2, \\
    Y(k) \begin{pmatrix}
        1 & -r(k)T^2e^{-2itg} \\
        0 & 1
    \end{pmatrix}^{-1}, & k \in \Omega_3, \\
    Y(k) \begin{pmatrix}
        1 & -r(k)e^{2itg} \\
        0 & 1
    \end{pmatrix}^{-1}, & k \in \Omega_4, \\
    Y(k), & \text{elsewhere}.
\end{cases}
\]  

(5.13)

We arrive to the following RHP for the function \( X(k) \).

**Riemann-Hilbert problem 8. Final RHP for the second constant region** \( \xi < -\frac{c^2}{2} \).

- Find a \( 2 \times 2 \) matrix function \( X(k) \) meromorphic in \( \mathbb{C}\setminus\Sigma \) with \( \Sigma = \bigcup_{j=1}^{4} L_j \cup (ic, -ic) \) (see figure 10) and such that
  - \( X_-(k) = X_+(k)J_X(k), \quad k \in \Sigma \) with
    \[
    J_X(k) = \begin{pmatrix}
        0 & i \\
        i & 0
    \end{pmatrix}, \quad k \in (ic, -ic),
    \]
  - \( J_X(k) = \begin{cases}
        \begin{pmatrix}
            1 & 0 \\
            -r(k)e^{2itg(k)} & 1
        \end{pmatrix}, & k \in L_1, \\
        \begin{pmatrix}
            1 & -r(k)T^2(k)e^{-2itg(k)} \\
            0 & 1
        \end{pmatrix}, & k \in L_2, \\
        \begin{pmatrix}
            1 & -r(k)T^2e^{-2itg} \\
            0 & 1
        \end{pmatrix}^{-1}, & k \in L_3, \\
        \begin{pmatrix}
            1 & -r(k)e^{2itg(k)} \\
            0 & 1
        \end{pmatrix}^{-1}, & k \in L_4,
    \end{cases}
    \]
  - and the pole conditions are the same as in the RHP [7]
  - Asymptotics: \( X(k) \to I \) as \( k \to \infty \).

**5.1.3 Model problems for the regions** \( \xi < -\frac{c^2}{4}, \quad -\frac{c^2}{4} < \xi < -\frac{c^2}{2} + e^2 \)

Looking at the RHPs [3,7] we observe that the jump matrix is exponentially close to \( I \) everywhere, except for the jump on \( (ic, -ic) \), and the parts of the curves \( L_j, j = 1, \ldots, 8, \) intersecting either the real line (i.e. the vicinities of the points \( \pm d_1(\xi) \) ) , or the interval \( (ic, -ic) \) (i.e. the vicinities of the points \( \pm d_0(\xi) \) ). A careful analysis of the contribution of the points \( \pm d_1(\xi) \), to the asymptotics for \( q(x, t) \) involves constructing of a local (approximate) solution to the RHP in the vicinities of those points, and is usually called parametrix analysis. The parametrices in the vicinities of the points \( \pm d_1 \) can be constructed in terms of parabolic cylinder function, in the similar fashion as was done in [49, section 5], and they give the contribution of the order \( O(t^{-1/2}) \). The parametrices in the vicinities of the points \( \pm d_0 \) can be constructed in terms of the Airy function and its derivative, in a similar fashion as was done in [3, section 3], and they give the contribution of the order \( O(t^{-1}) \). We thus reduce the problem to analysis of the corresponding model problems. The regular model problem in these regions gives the constant \( c_- \) as the main terms in the asymptotic expansion for \( q(x, t) \). The meromorphic model RHPs give us the breathers or solitons on a constant background. The model problem for this case is

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Riemann-Hilbert problem 9. Model RHP for the 1st and 2nd constant regions \(-\frac{c_-^2}{2} < \xi < -\frac{c_+^2}{2}\) and \(-\frac{c_-^2}{2} < \xi < \xi_*\).

Find a \(2 \times 2\) matrix \(M_{\text{mod}}^\prime(k, \xi)\) meromorphic for \(k \in \mathbb{C}\setminus[i\varepsilon_-, -i\varepsilon_-]\) and such that

- **Jump:**
  \[
  M_{\text{mod}}^-(k) = M_{\text{mod}}^+(k) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad k \in (i\varepsilon_-, -i\varepsilon_-),
  \]

- **Pole:** if there are points \(\kappa_j\) with \(\text{Im} \kappa_j > 0\), \(\text{Re} \kappa_j > 0\), \(-\varepsilon < \text{Im} g(\kappa_j, \xi) < \varepsilon\), we have the pole condition
  \[
  \begin{cases}
  M_1^\prime (k, \xi) - \frac{\nu_j}{k - \kappa_j} e^{2i\nu_j (k, \xi)} M_1^\prime (k, \xi) = O(1), & k \to \kappa_j, \\
  M_2^\prime (k, \xi) + \frac{\nu_j}{k + \kappa_j} e^{2i\nu_j (k, \xi)} M_2^\prime (k, \xi) = O(1), & k \to -\kappa_j, \\
  M_1^\prime (k, \xi) - \frac{\nu_j}{k - \kappa_j} e^{-2i\nu_j (k, \xi)} M_1^\prime (k, \xi) = O(1), & k \to \kappa_j, \\
  M_2^\prime (k, \xi) + \frac{\nu_j}{k + \kappa_j} e^{-2i\nu_j (k, \xi)} M_2^\prime (k, \xi) = O(1), & k \to -\kappa_j.
  \end{cases}
  \]

- \(M^\prime(k) = 1 + O(k^{-1})\) as \(|k| \to \infty\).

We observe that \(\text{Im} g(\kappa_j, \xi) = \text{Im} \left( \sqrt{\kappa_j^2 + c_-^2} \right) (12\xi - V_j)\) where \(V_j = 4 \text{Im} \left( \sqrt{\kappa_j^2 + c_-^2} \right)^2 + 6c_-^2 - 12 \text{Re} \left( \sqrt{\kappa_j^2 + c_-^2} \right)\) is the speed of the breather corresponding to the spectrum \(\kappa_j\) on the constant background \(c_-\).

Therefore the condition \(-\varepsilon < \text{Im} g(\kappa_j, \xi) < \varepsilon\), is equivalent to require that

\[
|V_j - 12\xi| < \varepsilon, \quad \xi = \frac{x}{12t}, \quad \varepsilon = \varepsilon/\text{Im} \left( \sqrt{\kappa_j^2 + c_-^2} \right).
\]

Namely we can see the breather if we observe in the direction of the \((x, t)\) plane such that \(|V_j - \frac{x}{t}| < \varepsilon\).

We number the points of the discrete spectrum \(\kappa_j\), with \(\text{Im} \kappa_j > 0\) and \(\text{Re} \kappa_j \geq 0\) according to their velocities and we assume that there are \(n\) breathers or solitons moving with distinct velocities \(V_j\), \(j = 1, \ldots, n\) to the right of the dispersive shock wave and \(N - n\) breathers moving with distinct velocities \(V_j\), \(j = N - n, \ldots, N\) to the left of the dispersive shock wave on the constant background \(c_+\) so that

\[
V_N < V_{N-1} < \cdots < V_{n+1} = -6c_+^2 + 12c_+^2 \quad \text{and} \quad 4c_-^2 + 2c_+^2 < V_n < V_{n-1} < \cdots < V_1 \tag{5.14}
\]

We further observe that if \(\xi\) is such that \(\text{Im} g(\kappa_j, \xi) = 0\) then \(\xi = \frac{V_j}{12}\) and with the above ordering we have that

\[
\text{Im} g \left( \kappa_j, \frac{V_j}{12} \right) < 0, \quad \text{Im} g \left( \kappa_{j+1}, \frac{V_j}{12} \right) > 0.
\]

With this ordering the function \(T(k, \frac{V_j}{12})\) defined in (5.3) takes the form \((1.15)\). The RHP problem \([9]\) corresponds to the case of a single breather on a constant background that we considered in Section 3.2. This problem can be solved explicitly in terms of elementary functions, and

\[
\lim_{k \to \infty} M_{12}^\prime (k, \xi) = q_{\text{breath}}(x, t; c_+, \kappa_j, V_j),
\]

where

\[
\nu_j = \frac{\nu_j}{T_j^\prime (\kappa_j)}, \tag{5.15}
\]

and the function

\[
T_j (k) := T(k, \frac{V_j}{12}) \tag{5.16}
\]
with \( T(k, \xi) \) as in \((5.5)\) for \(-\frac{c_+^2}{2} < \xi < \frac{c_+^2}{2} + c_+^2\) and \(d_0 = i\sqrt{\xi + \frac{c_+^2}{2}}\), and \( T(k, \xi) \) as in \((5.10)\) for \(\xi > -\frac{c_+^2}{2}\), \(d_1 = \sqrt{-\xi - \frac{c_+^2}{2}}\). Summarizing we have showed that the solution of the MKDV equation in the long time in the domains \(\xi < -\frac{c_+^2}{2}, -\frac{c_+^2}{2} < \xi < -\frac{c_+^2}{2} + c_+^2\) is as follows:

1. for those \(\xi, t\) such that \(|\text{Im} g(\kappa_j, \xi)| > \varepsilon\) for all \(\kappa_j\) we have 
   \[
   q(x, t) = c_- + O(t^{-\frac{1}{2}}),
   \]

2. For those \(\xi, t\) such that there exists \(\kappa_j, \text{Re} \kappa_j > 0, \text{Im} \kappa_j > 0\), with \(|\text{Im} g(\kappa_j, \xi)| < \varepsilon\), we have 
   \[
   q(x, t) = q_{\text{breath}}(x, t; c_-, \kappa_j, \hat{\nu}_j) + O(t^{-\frac{1}{2}})
   \]
   and 
   \[
   \hat{\nu}_j = \frac{\nu_j}{T_j^2(\kappa_j)},
   \]
   and \(T_j^2(\kappa_j)\) given in \((5.10)\).

Thus we have concluded the proof of Theorem \[1.4\] part (c).

### 5.2 Proof of Theorem \[1.4\] part (a): soliton and breather region.

In this case we consider the right constant region \(\xi > \frac{c_+^2}{2} + \frac{c_+^2}{6}\) where soliton and breather are moving in the positive \(x\) direction with a speed greater than the dispersive shock wave. As in the previous case we introduce the matrix function

\[
Y(k) = M(k)e^{i(tg(k, \xi) - \hat{\theta}(x, t, k))\sigma_3}T^{-\sigma_3}(k, \xi),
\]

where now the function \(g(k, \xi)\) takes the form \[40\]

\[
g(k, \xi) = 2 (2k^2 - c_+^2 + 6\xi) \sqrt{k^2 + c_+^2},
\]

namely \(g(k)\) is analytic in \(\mathbb{C} \setminus [ic_+, -ic_+]\) and

\[
g_+(k) + g_-(k) = 0, \quad k \in (ic_+, -ic_+), \quad g(k) = \hat{\theta}(k) + O(k^{-1}) \quad \text{as } |k| \to \infty.
\]

We chose \(\sqrt{k^2 + c_+^2}\) to be real on \((ic_+, -ic_+)\) and positive for \(k = +0\). The sign of \(\text{Im} g(k, \xi)\) are plotted in figure \[11\]. The function \(T(k, \xi)\) as is in \[5.3\] with \(c_-\) replaced by \(c_+\) and it satisfies the condition

\[
T_-(k)T_+(k) = 1, \quad k \in (ic_-, -ic_-)
\]

and can be found explicitly as follows:

\[
T(k, \xi) = T_0^0(k, \xi)^\ast \exp \left[ \frac{\sqrt{k^2 + c_+^2}}{2\pi i} \int_{ic_+}^{-ic_+} \frac{(-\ln \hat{T}_2(s, \xi)) ds}{(s - k) \left( \sqrt{k^2 + c_+^2} \right)} \right],
\]

where \(T_0^0(k, \xi)\) is of the form \[5.3\], where now \(g(k, \xi)\) is defined in \[5.1\]. This choice of the function \(T(k, \xi)\) permits the factorization of the jump matrix \(J_Y\) on the real axis in the form

\[
J_Y(k) = \left( \begin{array}{cc}
1 & 0 \\
-r(k)e^{2ig(k, \xi)} & 1
\end{array} \right) \left( \begin{array}{cc}
1 & -r(k)T_2(k)e^{-2ig(k, \xi)} \\
0 & 1
\end{array} \right), \quad k \in \mathbb{R}.
\]
We use the above factorization to open the lenses and define the matrix $X(k)$ as

$$X(k) = \begin{cases} 
Y(k) \begin{pmatrix} 1 & 0 \\ -r(k)e^{2i\theta(k,\xi)} & 1 \end{pmatrix}, & k \in \Omega_1 \\
Y(k) \begin{pmatrix} 1 & -r(k)T^2(k)e^{-2i\theta(k,\xi)} \\ 0 & 1 \end{pmatrix}^{-1}, & k \in \Omega_2, \\
Y(k), & \text{elsewhere}, 
\end{cases} \tag{5.20}$$

where the regions $\Omega_1$ and $\Omega_2$ are specified in figure 11.

**Riemann-Hilbert problem 10. Final RHP for the soliton/breather region $\xi > \frac{c^2}{2} + \frac{c^2}{6}$.**

Find a $2 \times 2$ matrix $X(k)$ meromorphic in $\mathbb{C} \setminus \Sigma$ where the contour $\Sigma$ is specified below and

$$X_-(k) = X_+(k)J_X(k) \quad k \in \Sigma$$

and

$$J_X(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in (ic_+, -ic_+), \quad J^{(2)} = I, \quad k \in (ic_-, ic_+) \cup (-ic_+, -ic_-), \\
\begin{pmatrix} 1 & 0 \\ -r(k)e^{2i\theta(k,\xi)} & 1 \end{pmatrix}, \quad k \in L_1, \\
\begin{pmatrix} 1 & -r(k)T^2(k,\xi)e^{-2i\theta(k,\xi)} \\ 0 & 1 \end{pmatrix}, \quad k \in L_2, \tag{5.21}$$

with $L_1$ and $L_2$ as in figure 11 and at the points $\kappa_j$ with $\text{Im}\kappa_j > 0$, we keep the pole condition as in the RHP. Finally the normalizing condition at infinity is $X(k) = 1 + O(k^{-1})$ as $|k| \to \infty$.

We observe that the jump matrices approach the identity exponentially fast on $L_1$ and $L_2$ and also for those values of $\xi$ and $\kappa_j$ such that $\text{Im}\kappa_j > 0$ and $\text{Im}\kappa_j > \epsilon$ or $\text{Im}\kappa_j < -\epsilon$. We finally arrive to the following model problem.

**Riemann-Hilbert problem 11. Model RHP for the soliton region $\xi > \frac{c^2}{2} + \frac{c^2}{6}$.**

Find a $2 \times 2$ matrix $M^{\text{mod}}(k,\xi)$ meromorphic for $k \in \mathbb{C} \setminus [ic_+, -ic_+]$ and such that

- **Jump:**

$$J^{(\text{mod})} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in (ic_+, -ic_+).$$
\begin{itemize}
  \item Pole: If there are points \( \kappa_j \) with \( \text{Im} \kappa_j > 0 \), \( -\varepsilon < \text{Im} g(\kappa_j) < \varepsilon \), we have the pole condition
  \[
  \begin{align*}
  M_1^{(\text{mod})} &= \left[ \frac{\nu_j}{k - \kappa_j} \right] \frac{2\tau g(k, \xi)}{\tau_j(k, \xi)} M_2^{(\text{mod})} = \mathcal{O}(1), \quad k \to \kappa_j, \quad \text{Im} \kappa_j > 0, \\
  M_2^{(\text{mod})} &= \left[ -\pi \right] T^2(k, \xi) e^{-2\tau g(k, \xi)} M_1^{(\text{mod})} = \mathcal{O}(1), \quad k \to \pi_j, \quad \text{Im} \pi_j < 0,
  \end{align*}
  \]
  and the symmetric conditions at the points \(-\pi_j, -\kappa_j\) when we are considering a breather.

  This model problem is exactly the model problem, which gives solitons or breathers on the constant background \( c_+ \). Thus,
  \[
  \lim_{k \to \infty} 2ik(M_{12}^{\text{mod}}) = q_{\text{sol}}(x, t; c_+, \kappa_j, x_j), \quad \text{if Re} \kappa_j = 0,
  \]
  \[
  \lim_{k \to \infty} 2ik(M_{12}^{\text{mod}}) = q_{\text{breath}}(x, t; c_+, \kappa_j, \tilde{\nu}_j), \quad \text{if Re} \kappa_j > 0,
  \]
  where we use the numbering of solitons and breathers according to their velocities as done in \(5.14\) so that the phase shift can be written in the form
  \[
  \hat{\nu}_j = \frac{\nu_j}{T_j(\kappa_j)}, \quad x_j = \log \frac{2(k_c^2 - c_+^2)T_j^2(|\kappa_j|)}{|\nu_j|\kappa_j},
  \]
  \[
  T_j(k) := T(k, \frac{V_j}{12}) = \tilde{T}(k, \frac{V_j}{12}) \exp \left[ \sqrt{k^2 + c_+^2} - \frac{c_+^2}{2\pi i} \right]
  \]
  where \( \tilde{T}(k, \xi) \) as in \(5.3\) so that \( \tilde{T}(k, \frac{V_j}{12}) \) coincides with \( \tilde{T}_j(k) \) defined in \(1.15\).

  Since the jump matrices are exponentially close to identity matrix uniformly, the contribution of the poles give the leading order asymptotic expansion, and the whole contours give exponentially small contribution. We can conclude that the solution of the Cauchy problem \(1.1\), \(1.2\) in the domain \( \xi > \frac{c_+^2}{3} + \frac{c_+^2}{6} \) has the following asymptotics as \( t \to \infty \):

  (1) For those \( \xi, t \) such that \( |\text{Im} g(\kappa_j, \xi)| > \varepsilon \) for all \( \kappa_j \) we have
  \[
  q(x, t) = c_+ + O(e^{-ct}),
  \]

  (2) For those \( \xi, t \) such that there exists \( \kappa_j, \text{Im} \kappa_j > 0, |\text{Im} g(\kappa_j, \xi)| < \varepsilon \), we have
  \[
  q(x, t) = \begin{cases} 
  q_{\text{breath}}(x, t; c_-, \kappa_j, \tilde{\nu}_j) + O(e^{-ct}) & \text{if Re} \kappa_j > 0 \\
  q_{\text{sol}}(x, t; c_-, \kappa_j, x_j) + O(e^{-ct}) & \text{if Re} \kappa_j = 0,
  \end{cases}
  \]
  for some \( C > 0 \), which is determined by the initial datum. Here \( \tilde{\nu}_j \) and \( x_j \) are defined in \(5.22\).

  We thus have finished the proof of Theorem \ref{thm1} part (a).

Appendices

A Travelling wave solution of MKDV and Whitham modulation equations

We look for the solution of the MKDV equation

\[
  u_t + 6u^2u_x + u_{xxx} = 0
\]

in the form

\[
  u(x, t) = \eta(\theta), \quad \text{where} \ \theta = kx - \omega t, \quad \mathcal{V} := \frac{\omega}{k}.
\]
Substituting this ansatz into MKDV equation, we get after two integrations

\[ k^2 \eta_0^2 = -\eta^4 + v\eta^2 + B\eta + A, \quad (A.1) \]

where \( A, B \) are constants of integration. We assume that the above polynomials has four real roots \( e_1 < e_2 < e_3 < e_4 \), so that

\[ k^2 \eta_0^2 = -(\eta - e_1)(\eta - e_2)(\eta - e_3)(\eta - e_4), \quad \sum_{j=1}^{4} e_j = 0 \]

\[ \mathcal{V} = -\sum_{i<j} e_i e_j. \quad (A.2) \]

Finite real valued periodic motion can take place when \( \eta \) varies between \([e_1, e_2]\) or \([e_3, e_4]\). We develop the calculations for the latter case, the first case can be derived by the symmetry \( \eta \to -\eta \) and \( e_j \to -e_4-j \). We obtain the integral

\[ k \int_{e_3}^{\eta} \frac{d\tilde{\eta}}{\sqrt{(\eta_1 - \tilde{\eta})(\eta_2 - \tilde{\eta})(\eta_3 - \tilde{\eta})(\eta_4 - \tilde{\eta})}} = \theta. \]

After integration we arrive to the expression

\[ u_{\text{per}}(x, t) = \eta(\theta) = e_1 + \frac{e_3 - e_1}{1 - (e_3 - e_2)(e_4 - e_1)} \frac{e_3 - e_1}{\sqrt{(e_3 - e_1)(e_4 - e_2)}} \frac{(x - vt)}{|m|}. \quad (A.3) \]

where the function \( \text{cn}(z|m) \) is the Jacobi elliptic function of modulus

\[ m = \frac{(e_3 - e_2)(e_4 - e_1)}{(e_4 - e_2)(e_3 - e_1)}. \]

Since \( e_1 + e_2 + e_3 + e_4 = 0 \) we make a change of variables \( \{e_1, e_2, e_3, e_4\} \to \{\beta_1, \beta_2, \beta_3\} \) defined as

\[ e_1 = -\beta_1 - \beta_2 - \beta_3, \quad e_2 = \beta_1 + \beta_2 - \beta_3, \quad e_3 = \beta_1 + \beta_3 - \beta_2, \quad e_4 = \beta_2 + \beta_3 - \beta_1, \]

with \( \beta_3 > \beta_2 > \beta_1 \). The inverse transformation is

\[ \beta_1 = -\frac{e_2 + e_3}{2} = -\frac{e_1 + e_4}{2}, \quad \beta_2 = \frac{e_2 + e_4}{2} = -\frac{e_1 + e_3}{2}, \quad \beta_3 = \frac{e_3 + e_4}{2} = -\frac{e_1 + e_2}{2}. \]

This choice of variables relates the elliptic solution to the band spectrum of the Zakharov-Shabat linear operator \([2.1]\). In this way we obtain the expression

\[ u_{\text{per}}(x, t) = -\beta_1 - \beta_2 - \beta_3 + \frac{2(\beta_2 + \beta_3)(\beta_1 + \beta_3)}{\beta_2 + \beta_3 - (\beta_2 - \beta_1)\text{cn}^2 \left( \sqrt{\beta_2^2 - \beta_1^2}(x - vt) + x_0 \right)} \frac{1}{|m|}, \quad (A.4) \]

where \( \beta_3 > \beta_2 > \beta_1 \), the speed \( \mathcal{V} = 2(\beta_1^2 + \beta_2^2 + \beta_3^2) \) and \( x_0 \) is an arbitrary phase. The elliptic modulus \( m \) transforms to

\[ m = \frac{\beta_2^2 - \beta_1^2}{\beta_3^2 - \beta_1^2}. \quad (A.5) \]

When the periodic motion takes place between \( e_1 \) and \( e_2 \) the quantity \( \beta_j \to \tilde{\beta}_j \) with \( \tilde{\beta}_1 = -\beta_1, \tilde{\beta}_2 = \beta_2 \) and \( \tilde{\beta}_3 = \beta_3 \). The limiting case of the travelling wave solutions are

(a) \( e_4 \to e_2 \) which means \( \beta_1 \to -\beta_2 \) or \( m \to 0 \);

(b) \( e_3 \to e_4 \) which means \( \beta_2 \to \beta_1 \) or \( m \to 0 \);

(c) \( e_2 \to e_3 \) which implies \( \beta_2 \to \beta_3 \) or \( m \to 1 \).
In the case (a) and (b) we have that $\cosh z \to \cos z$ as $m \to 0$ so that parametrizing $\beta_2 = \beta + \delta$ and $\beta_1 = \beta - \delta$ the periodic solution (A.4) in case (b) has an expansion of the form

$$u_{\text{per}}(x,t) = \beta_3 - 2\delta + 4\delta \cos^2\left(\sqrt{\frac{\beta_3^2 - \beta_2^2}{\beta^2}}(x - (4\beta^2 + 2\beta_3)t) + x_0\right) + O(\delta^2).$$

which is the small amplitude limit. In case (a) the limit $\beta_1 \to -\beta_2 = -\beta$ we obtain

$$u_{\text{per}}(x,t) \approx -\beta_3 + \frac{2(\beta_3^2 - \beta^2)}{\beta + \beta_3 - 2\beta \cos^2\left(\sqrt{\frac{\beta_3^2 - \beta^2}{\beta^2}}(x - (4\beta^2 + 2\beta_3)t) + x_0\right)},$$

which is a trigonometric solution. The soliton limit is the case (c) when $m \to 1$ or $\beta_3 = \beta_2 = \beta$, and $\cosh z \to \cosh z$ so that

$$u_{\text{per}}(x,t) \approx \beta_1 + \frac{2\beta(\beta + \beta_1)}{\beta \cosh(2\sqrt{\beta^2 - \beta_1^2}(x - (4\beta^2 + 2\beta_1)t) + x_0) + \beta_1}$$

which coincides with the one-soliton solution (1.4) on the constant background by identifying $\beta$ with the spectral parameter $\kappa_0$ and $\beta_1$ with the constant background $c$.

**B Whitham modulation equations**

The Whitham modulation equations are the slow modulations of the wave parameters $\beta_j = \beta_j(X,T)$ of the travelling wave solution (A.4), where $X = cx$ and $T = ct$, where $\epsilon$ is a small parameter. These equations were first derived by Whitham [57] for the KdV case using the method of averaging over conservation laws. The same equations can be obtained requiring that the travelling wave solution with slow variation of the wave parameters satisfies the KdV equation up to an error of order $O(\epsilon)$. After Whitham’s work the modulations equations have been obtained for a large set of partial differential equations. In particular for the MKDV equation they have been obtained by Driscoll and O’Neil [23] using the original Whitham averaging procedure. The Whitham modulation equations for MKDV wave parameters $\beta_3 > \beta_2 > \beta_1$ take the form

$$\frac{\partial}{\partial T} \beta_j + W_j(\beta_1,\beta_2,\beta_3) \frac{\partial}{\partial X} \beta_j = 0, \quad j = 1, 2, 3,$$

where the speeds $W_j = W_j(\beta_1,\beta_2,\beta_3)$ are

$$W_j(\beta_1,\beta_2,\beta_3) = 2(\beta_1^2 + \beta_2^2 + \beta_3^2) + 4 \prod_{k \neq j}(\beta_j^2 - \beta_k^2) \frac{\beta_j^2 + \alpha}{\beta_1^2 + \alpha}, \quad j = 1, 2, 3,$$  \hspace{1cm} (B.1)

$$\alpha = -\beta_3^2 + (\beta_3^2 - \beta_1^2) \frac{E(m)}{K(m)}, \quad m = \frac{\beta_2^2 - \beta_1^2}{\beta_3^2 - \beta_1}, \quad \text{where} \quad E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \psi} d\psi$$

is the complete elliptic integral of the second kind. At the boundary one has

- $m \to 0$ or $\beta_2 = \pm \beta_1$, then $W_1(\beta_1,\beta_1,\beta_3) = W_2(\beta_1,\beta_1,\beta_3) = -6\beta_3^2 + 12\beta_1^2$ and $W_3(\beta_1,\beta_1,\beta_3) = 6\beta_3^2$;

- $m \to 1$ or $\beta_2 = \pm \beta_3$, then $W_2(\beta_1,\beta_3,\beta_3) = W_3(\beta_1,\beta_3,\beta_3) = 4\beta_3^2 + 2\beta_1^2$ and $W_1(\beta_1,\beta_3,\beta_3) = 6\beta_3^2$.

The solution of Whitham modulation equations is obtained as a boundary value problem, namely for an initial monotone decreasing initial data $q_0(X)$, the solution satisfies the boundary conditions

- when $\beta_1 = \beta_2$, then $\beta_3(X,T) = q(X,T)$ that solves the equation $q_T + 6q^2q_X = 0$ with initial data $q_0(X)$;
• when \( \beta_2 = \beta_3 \) then \( \beta_1(X,T) = q(X,T) \) that solves the equation \( q_T + 6q^2q_X = 0 \) with initial data \( q_0(X) \).

We observe that the Whitham modulation equations for MKDV can be obtained from the Whitham modulation equations for KdV with speeds \( V_j(r_1, r_2, r_3) \) as

\[
W_j(\beta_1, \beta_2, \beta_3) = V_j(\beta_1^2, \beta_2^2, \beta_3^2), \quad j = 1, 2, 3.
\]

It was shown in [43] that for \( r_3 > r_2 > r_1 \) one has

• strict hyperbolicity: \( V_2(r_1, r_2, r_3) > V_2(r_1, r_2, r_3) > V_1(r_1, r_2, r_3) \),

• genuine nonlinearity: \( \frac{\partial}{\partial r_j} V_j(r_1, r_2, r_3) > 0 \).

It follows that the Whitham equations for MKDV are strictly hyperbolic and genuinely nonlinear only when all \( \beta_3 > \beta_2 > \beta_1 > 0 \). Indeed one has

\[
\frac{\partial}{\partial \beta_j} W_j(\beta_1, \beta_2, \beta_3) = 2\beta_j \frac{\partial}{\partial r_j} V_j(r_1, r_2, r_3)|_{r_3=\beta_j^2},
\]

so that when one of the \( \beta_j = 0 \) the equations are not genuinely nonlinear.

In particular, if we consider the self-similar solution \( \beta_j(X,T) = \beta_j(z) \) with \( z = X/T = x/t \) we have that the Whitham equations reduce to the form

\[
(W_j - z) \frac{\partial}{\partial z} \beta_j = 0, \quad j = 1, 2, 3.
\]

The initial data that corresponds to such self-similar solution is the step initial data

\[
q_0(X) = \begin{cases} 
  c_+ & \text{as } X > 0 \\
  c_- & \text{as } X < 0.
\end{cases}
\]

We observe that such initial data is invariant under the scaling \( X \rightarrow \epsilon x \) so that we can also consider the initial data \( q_0(x) \) in the fast variable \( x \). The self-similar Whitham equations (B.3) can be uniquely solved in the form

\[
\beta_3 = c_-, \quad \beta_1 = c_+, \quad \frac{X}{T} = \frac{x}{t} = W_2(c_+, \beta_2, c_-), \quad c_- \leq \beta_2 \leq c_+,
\]

as long as we can invert \( \beta_2 \) as a function of \( x/t \) within the region

\[
W_1(c_+, c_-, c_-) < \frac{x}{t} < W_3(c_+, c_-, c_-).
\]

This happens only when \( W_3(c_+, c_-, c_-) \) is a strictly monotone function of \( \beta_2 \), that is when \( \beta_2 > 0 \), namely when \( c_- > c_+ > 0 \). Thus we have obtained the solution appearing in the long-time asymptotic regime for the Cauchy problem for MKDV considered in this manuscript.

When \( c_- > -c_+ > 0 \) the solution of the MKDV will still develop a dispersive shock wave, but the Whitham equations are not in general strictly hyperbolic and genuinely nonlinear. In this case it can be shown that the solution of (B.3) is obtained as follows. We define \( z^* = W_1(c_+, -c_-, c_-) = W_2(c_+, -c_+, c_-) = -6c_-^2 + 12c_1^2 \). Then for \( \frac{x}{t} \leq \frac{X}{T} < W_3(c_+, c_-, c_-) \) we have that the solution is obtained as

\[
\beta_3 = c_-, \quad \beta_1 = c_+, \quad \frac{X}{T} = \frac{x}{t} = W_2(c_+, \beta_2, c_-),
\]

which is solvable for \( \beta_2 = \beta_2(x/t) \) is view of genuinely nonlinearity, and for \( -6c_-^2 = W_1(0, 0, c_-) \leq \frac{x}{t} \leq \frac{X}{T} = \frac{x}{t} = W_2(c_+, \beta_2, c_-) \) we have

\[
\beta_3 = c_-, \quad \frac{X}{T} = \frac{x}{t} = W_1(\beta_1, \beta_2, c_-), \quad \frac{X}{T} = \frac{x}{t} = W_2(\beta_1, \beta_2, c_-).
\]

The proof of the solvability of the above system of equations for \( \beta_1 = \beta_1(x/t) \) and \( \beta_2 = \beta_2(x/t) \) requires further analysis that has been considered in a similar setting in [28] and also in [21] for the Camassa-Holm equation. A discussion of the various cases arising in the long time asymptotic for the MKDV solution can be found in [24] A review of dispersive shock wave for KdV can be found in [29].
C Proofs of properties of Jost solutions

Here we prove Lemma 2.1. For the inspiration, we used the work [34], but our functional spaces are a bit different. Namely, in order to be able to tackle not only continuous, but also discontinuous initial functions \(q_0(x)\), like \(q_0(x) = c_\pm, \pm x > 0\), we work with functions of bounded variations, \(BV_{loc}(\mathbb{R})\), rather than merely continuous functions. Such functions admit representation

\[ q_0(x) \overset{ac}{=} q_{ac}(x) + \sum_j \alpha_j \chi_{(-\infty, x_j]}(x), \]

where the set \(\{x_j\}\) is at most countable, and \(\sum_j |\alpha_j| =: \alpha < \infty\), and \(\chi_A(x)\) is the indicator function of a set \(A\), equals 1 if \(x \in A\) and 0 otherwise. Furthermore, \(q_{ac}\) is an absolutely continuous function, i.e.

\[ q_{ac}(x) = q_{ac}(x_0) + \int_{x_0}^x q'_{ac}(\tilde{x})d\tilde{x}, \quad q'_{ac}(x) \in L^1_{loc}(\mathbb{R}). \]

It follows that \(q_0(x)\) is locally bounded. Furthermore, the associated distributional derivative is the signed measure \(dq_0(x)\),

\[ dq_0(x) = q'_{ac}(x)dx - \sum_j \alpha_j \delta(x-x_j), \]

where \(\delta(.)\) is the Dirac \(\delta\)-distribution (a.k.a. Dirac \(\delta\)-function).

For the readers who would like to restrict themselves only to smooth (absolutely continuous) \(q_0(x)\), we recommend to think \(\alpha_j \equiv 0\), in which case \(dq_0(x) = q'_0(x)dx\) is the usual derivative. We have the following proposition

\textbf{Proposition C.1.} \(\text{(a) Let } q_0(x) \in BV_{loc}(\mathbb{R}), \text{ and} \)

\[ \int_{-\infty}^0 |x||q_0(x) - c_-|dx + \int_{0}^{+\infty} |x||q_0(x) - c_+|dx < \infty, \quad \text{and } \text{ess sup } |q_0(x)| < \infty \quad (C.1) \]

Then there are exist solutions \(\Phi_{\pm}(x;k)\) of \(x\)-equation (2.1), such that

\[ \Phi_{\pm}(x;k) = E_{\pm}(x;k) + \int_{\pm\infty}^x L_{\pm}(x,y)E_{\pm}(y;k)dy, \quad (C.2) \]

where

\[ L_{\pm}(x,y) = \begin{pmatrix} L^\pm_1(x,y) & L^\pm_2(x,y) \\ -L^\pm_2(x,y) & L^\pm_1(x,y) \end{pmatrix} \]

and \(\mp \int_{-\infty}^x |L^\pm_1(x,y)|dy < \infty\), where \(j = 1, 2\). Moreover, \(L_{\pm}\) admit the following estimates in terms of auxiliary quantities

\[ \hat{\sigma}_{\pm}(x) = \mp \int_{-\infty}^x |q_0(\tilde{x}) - c_{\pm}|d\tilde{x} \geq 0, \quad \hat{\sigma}_{1,\pm}(x) = \mp \int_{-\infty}^x \hat{\sigma}_{\pm}(\tilde{x})d\tilde{x} \geq 0, \quad M_{\pm}(x) = \text{ess sup } |q_0(\tilde{x})|, \]

which exist in view of \((C.1)\). Then

\[ |L_{\pm}(x,y)| \leq C[M_{\pm}(x), \hat{\sigma}_{1,\pm}(x)] \cdot \hat{\sigma}(\frac{x+y}{2}), \quad (C.3) \]

where \(C[M_{\pm}(x), \hat{\sigma}_{1,\pm}(x)]\) is a constant, which depend only on \(M_{\pm}(x), \hat{\sigma}_{1,\pm}(x)\), and whose explicit form is not important for us.
(b) Let in addition to assumptions (a)

\[ \pm \int_{\pm \infty}^{0} |dq_0(x)| < \infty, \quad \pm \int_{\pm \infty}^{0} \sup_{x \in (\pm \infty, x]} |q_0(x)| \, dx < \infty \]  

(C.4)

Then (we suppress \( \pm \) everywhere for the ease of reading)

\[ \Phi(x; k) = E(x; k) + \frac{1}{ik} (q_0(x) - c) \sigma_1 E(x; k) - \frac{1}{ik} \int_{\infty}^{x} \sigma_1 E(y; k) \, dq_0\left(\frac{x + y}{2}\right) - \frac{\hat{L}(x, x) \sigma_3 E(x; k)}{ik} \]

\[ - \frac{1}{ik} \int_{\infty}^{x} \left( q_0\left(\frac{x + y}{2}\right) - c \right) \sigma_1 Q_x E(y; k) \, dy + \frac{1}{ik} \int_{-\infty}^{x} \left( \tilde{L}_y(x, y) \sigma_3 + \tilde{L} \sigma_3 Q_x \right) E(y; k) \, dy, \]

where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), and \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and

\[ \left| \int_{\infty}^{x} \tilde{L}_y(x, y) \, dy \right| < \infty, \quad \left| \int_{\infty}^{x} \tilde{L}(x, y) \, dy \right| < \infty. \]

Moreover, (below \( j = 1 \) or \( j = 2 \))

\[ |\partial_y L_j(x, y)| \leq C[M(x), \sigma_1(x), \sigma_2(x)] \cdot \left( \sigma_1 (\frac{x + y}{2}) + \sup_{x \in (\pm \infty, x]} |q_0(\tilde{x})| \right), \]

where \( C[M(x), \sigma_1(x), \sigma_2(x)] \) is a constant, which depend only on \( M_\pm(x) \), \( \sigma_{1, \pm}(x) \), \( \sigma_2(x) \) and whose explicit form is not important for us, and

\[ \tilde{\sigma}_{2, \pm}(x) := \pm \int_{\pm \infty}^{x} |dq_0(x)| \geq 0. \]

(c) Let in addition to assumptions (a), (b),

\[ \pm \int_{\pm \infty}^{0} e^{C|x|} |q_0(x) - c| dx < \infty, \]  

(C.5)

with some \( C > 0 \). Then

\[ \pm \int_{\pm \infty}^{0} e^{C|x|} |L(x, y)| dy < \infty, \quad \pm \int_{\pm \infty}^{0} e^{C|x|} |L_y(x, y)| dy < \infty. \]

Remark C.1. The second of conditions (a) follows from the first of conditions (b).

Remark C.2. Conditions (b) of the Lemma are satisfied if, for example,

\[ \pm \int_{\pm \infty}^{0} |x| |dq_0(x)| < \infty. \]

Remark C.3. Both conditions (a), (b) of the Lemma are satisfied if, for example,

\[ \pm \int_{\pm \infty}^{0} |x|^2 |dq_0(x)| < \infty, \quad \text{and} \quad q_0(x) = c_- + \int_{-\infty}^{x} dq_0(y) = c_+ - \int_{x}^{+\infty} dq_0(y), \quad \text{i.e.} \quad \int_{-\infty}^{+\infty} dq_0(x) = c_+ - c_. \]
Proof. We start the proof of Proposition C.1. With a bit of abuse of notations, we will suppress the indices \( \pm \). Assume first that \( q_0 \) is an absolutely continuous function. Then one can check directly that

\[
\Phi(x, k) = E(x, k) + \int_{-\infty}^{x} L(x, y) E(y, k) dy
\]

is a solution to (2.1) provided that the kernel

\[
L(x, y) = \begin{pmatrix}
L_1(x, y) & L_2(x, y) \\
-L_2(x, y) & L_1(x, y)
\end{pmatrix}
\]

satisfies the following system of equations:

\[
\begin{aligned}
L_{1, x}(x, y) + L_{1, y}(x, y) &= -(q_0(x) + c)L_2(x, y), \quad y \in (x, \infty), \\
L_{2, x}(x, y) - L_{2, y}(x, y) &= (q_0(x) - c)L_1(x, y), \quad y \in (x, \infty), \\
L_2(x, x) &= \frac{(q_0(x) - c)}{2}, \quad \lim_{y \to \infty} L_2(x, y) = 0.
\end{aligned}
\]

(C.6)

In its turn, solutions of (C.6) can be found as solutions of appropriate integral equations. Namely,

\[
H_1(u, v) \equiv L_1(x, y), \quad H_2(u, v) \equiv L_2(x, y),
\]

then we come to the integral equations

\[
H_1(u, v) = -\int_{-\infty}^{u} (q_0(\tilde{u} + v) + c)H_2(\tilde{u}, v) d\tilde{u}, \quad H_2(u, v) = \frac{(q_0(u) - c)}{2} + \int_{0}^{v} (q_0(u + \tilde{v}) - c)H_1(u, \tilde{v}) d\tilde{v},
\]

(C.7)

or

\[
H_1(u, v) = -\frac{1}{2} \int_{-\infty}^{u} (q_0(\tilde{u} + v) + c)(q_0(\tilde{u}) - c) d\tilde{u} - \int_{-\infty}^{u} (q_0(\tilde{u} + v) + c) \int_{0}^{v} (q_0(\tilde{u} + \tilde{v}) - c)H_1(\tilde{u}, \tilde{v}) d\tilde{v} d\tilde{u},
\]

\[
H_2(u, v) = \frac{(q_0(u) - c)}{2} - \int_{0}^{v} (q_0(u + \tilde{v}) - c) \int_{-\infty}^{u} (q_0(\tilde{u} + \tilde{v}) + c)H_2(\tilde{u}, \tilde{v}) d\tilde{u} d\tilde{v}.
\]

(C.8)

We see that \( H_2(x, y) \) has the same level of regularity as \( q_0 \) does. Hence, system (C.6) is appropriate for absolutely continuous functions \( q_0 \), but should have another meaning would we switch to discontinuous functions \( q_0 \) of bounded variations. One might think that now we will need to operate with functions of bounded variations of two variables. However, in our case functions of bounded variations of one variable suffices. Indeed, subtracting function \( q_0(u + v) \), we obtain a regular function. To this end, denote

\[
\hat{L}_1(x, y) = L_1(x, y), \quad \hat{L}_2(x, y) = L_2(x, y) - \frac{1}{2} (q(\frac{x + y}{2}) - c),
\]

\[
\hat{H}_1(u, v) = H_1(u, v), \quad \hat{H}_2(u, v) = H_2(u, v) - \frac{1}{2} (q(u) - c).
\]

Furthermore, \( \hat{H}_1, \hat{H}_2 \) satisfy the following system of integral equations:

\[
\hat{H}_1(u, v) = -\frac{1}{2} \int_{-\infty}^{u} (q_0(\tilde{u}) - c)(q_0(\tilde{u} + v) + c) d\tilde{u} - \int_{-\infty}^{u} (q_0(\tilde{u} + v) + c)\hat{H}_2(\tilde{u}, v) d\tilde{u},
\]

\[
\hat{H}_2(u, v) = \int_{0}^{v} (q_0(u + \tilde{v}) - c)\hat{H}_1(u, \tilde{v}) d\tilde{v},
\]

(C.9)
Indeed, the induction step goes as follows:

$$
\tilde{H}_1(u, v) = -\frac{1}{2} \int_0^u (q_0(\ddot{u} + v) + c)(q_0(\ddot{u} - c) d\ddot{u} - \int_0^u (q_0(\ddot{u} + v) + c) \tilde{H}_1(\ddot{u}, \ddot{v}) d\ddot{v},
$$

$$
\tilde{H}_2(u, v) = -\frac{1}{2} \int_0^u (q_0(u + \ddot{v}) - c) \int_0^u (q_0(\ddot{u} + v) + c)(q_0(\ddot{u} - c) d\ddot{u} d\ddot{v},
$$

$$
- \int_0^u (q_0(u + \ddot{v}) - c) \int_0^u (q_0(\ddot{u} + v) + c) \tilde{H}_2(\ddot{u}, \ddot{v}) d\ddot{v}.
$$

We see that by subtracting from $L_2$ the ‘irregular part’, which is $q_0$, we gain in regularity of $\tilde{L}_2$. Now $\tilde{L}_1, \tilde{L}_2$ have one more level of regularity than $q_0$, and hence we are able to integrate the integral representation for $\Phi$ by parts, which gives us $k^{-1}$ term in the large $k$ asymptotic expansion.

More precisely, apply the successive approximation method to the first of equations \( \text{(C.10)} \), i.e. represent $\tilde{H}_1 = \sum_{j=0}^{\infty} \tilde{H}_1^{(j)}$, where $H_1^{(0)}$ is the inhomogeneous part of the r.h.s. of the equation, and $H_1^{(j)}$ is the homogeneous part of the r.h.s. of the equation, applied to $H_1^{(j-1)}$. Then we have by induction that

$$
|H_1^{(j)}| \leq \left( \frac{M(u + v)}{2} \right)^{j+1} \Delta(u, v) \cdot \left( \delta_1(u) - \dot{\delta}_1(u) \right)^j,
$$

where

$$
\Delta(u) = \left| \int_0^u (q(u) - c) d\ddot{u}, \quad \delta_1(u) = \left| \int_0^u (q(\ddot{u}) - c) d\ddot{u} = \left| \int_0^u (u - \ddot{u}) q_0(\ddot{u}) d\ddot{u}, \quad M(z) = \text{ess sup}_{\ddot{u} \in (\infty, z)} |q_0(\ddot{u}) + c|.
$$

Indeed, the induction step goes as follows:

$$
|\tilde{H}_1^{(j+1)}(u, v)| \leq \int_0^u (q_0(\ddot{u} + v) + c) \int_0^u (q_0(\ddot{u} + v) - c) \tilde{H}_1^{(j)}(\ddot{u}, \ddot{v}) d\ddot{u} d\ddot{v},
$$

$$
\leq \int_0^u (q_0(\ddot{u} + v) + c) \int_0^u (q_0(\ddot{u} + v) - c) \frac{M(\ddot{u} + \ddot{v})^{j+1}}{2 \cdot j!} \dot{\delta}(\ddot{u}) \cdot (\dot{\delta}_1(\ddot{u}) - \ddot{\delta}_1(\ddot{u}))^j d\ddot{v}.
$$

In the latter we estimate

$$
\dot{\delta}(\ddot{u}) \leq \dot{\delta}(u), \quad \dot{\delta}_1(\ddot{u}) - \ddot{\delta}_1(\ddot{u}) \leq \dot{\delta}_1(\ddot{u} + v) - \ddot{\delta}_1(\ddot{u}), \quad M(\ddot{u} + \ddot{v}) \leq M(u + v), \quad |q_0(\ddot{u} + v) + c| \leq M(u + v), \quad \text{and thus obtain}
$$

$$
|H_1^{(j+1)}(u, v)| \leq \frac{M(\ddot{u} + \ddot{v})^{j+2}}{2 \cdot j!} \dot{\delta}(\ddot{u}) \int_0^u (\dot{\delta}_1(\ddot{u}) - \ddot{\delta}_1(\ddot{u}))^j \int_0^u (q_0(\ddot{u} + v) - c) d\ddot{v}.
$$

Here we use the definition of $\dot{\delta}$ as the integral of $|q - c|$, and further make use of $|\dot{\delta}_1(u)| = \dot{\delta}(u)$, and integrating by parts, obtain the induction step.

Hence,

$$
|\tilde{H}_1(u, v)| \leq K_1(u, v) := \frac{1}{2} M(u + v) \dot{\delta}(u) \exp [M(u + v) \cdot (\dot{\delta}_1(u + v) - \ddot{\delta}_1(u))],
$$

and similarly,

$$
|\tilde{H}_2(u, v)| \leq K_2(u, v) := \frac{1}{2} M(u + v) \cdot \dot{\delta}(u) \cdot (\dot{\delta}(u + v) - \ddot{\delta}(u)) \cdot \exp [M(u + v) \cdot (\dot{\delta}_1(u + v) - \ddot{\delta}_1(u))],
$$

and henceforth,

$$
|\tilde{L}_1(x, y)| \leq C \dot{\delta}(\frac{x + y}{2}), \quad |\tilde{L}_2(x, y)| \leq C \dot{\delta}(\frac{x + y}{2}).
$$
where $\tilde{C}$ is a generic constant, which does not depend on $y$. Since $\hat{\sigma} \in L_1$, we henceforth proved the statement (a) of the Lemma.

Passing to the part (b), it is sufficient to prove that $\int_{-\infty}^{x} \left|L_y(x, y)\right|dy < \infty$, and for this it is sufficient to estimate $\partial_u \tilde{H}_1$, $\partial_v \tilde{H}_1$, $\partial_u \tilde{H}_2$, $\partial_v \tilde{H}_2$. We have

$$
\partial_u \tilde{H}_1(u, v) = -(q_0(u + v) + c) \left\{ \tilde{H}_2(u, v) + \frac{1}{2} (q_0(u) - c) \right\}, \quad \partial_u \tilde{H}_2(u, v) = (q_0(u + v) - c) \tilde{H}_1(u, v),
$$

and hence it is enough to estimate $\partial_v \tilde{H}_1$, $\partial_u \tilde{H}_2$. We have

$$
\partial_v \tilde{H}_1(u, v) = \int_{-\infty}^{u} \left[ \frac{q_0(\tilde{u}) - c}{2} - \tilde{H}_2(\tilde{u}, v) \right] d\tilde{u}q_0(\tilde{u} + v) - \int_{-\infty}^{u} (q_0(\tilde{u} + v) + c) \partial_v \tilde{H}_2(\tilde{u}, v)d\tilde{u},
$$

and to estimate, we split $d q_0(z) = q'_0(z)dz + \sum_j \alpha_j \delta(z - x_j)$. We thus have

$$
|\partial_v \tilde{H}_1(u, v)| \leq \tilde{\sigma}_2(u + v) \left( \frac{1}{2} \text{ess sup}_{u \in (-\infty, u)} |q_0(\tilde{u}) - c| + K_2(u, v) \right) + M(u + v) \tilde{\sigma}(u + v) K_1(u, v),
$$

$$
|\partial_u \tilde{H}_2(u, v)| \leq M(u + v) \left( \tilde{\sigma}(u + v) - \tilde{\sigma}(u) \right) \left( \frac{1}{2} \text{ess sup}_{u \in (-\infty, u)} |q_0(\tilde{u}) - c| + K_2(u, v) \right) + (\tilde{\sigma}_2(u + v) - \tilde{\sigma}_2(u)) K_1(u, v),
$$

Since

$$
\left| \int_{-\infty}^{0} \text{ess sup}_{u \in (-\infty, u)} |q_0(\tilde{u}) - c| du \right| + \left| \int_{-\infty}^{0} \tilde{\sigma}(u) du \right| < \infty,
$$

we obtain the statement of (b). The statement (c) follows immediately from the statement (b) and from the estimate (let us stick to $-\infty$ for certainty)

$$
\int_{-\infty}^{x} \sigma(y)e^{C|y|}dy = \int_{-\infty}^{x} e^{C|y|} \int_{-\infty}^{y} |q_0(z) - c| dz dy = \int_{-\infty}^{x} \int_{-\infty}^{z} |q_0(z) - c| e^{C|y|} dy = \frac{1}{C} \int_{-\infty}^{\infty} |q_0(z) - c| \left( e^{C|z|} - e^{C|z|} \right) dz.
$$

\[\square\]

**Corollary C.2.** (a) Provided that conditions \textbf{[C.1]} are satisfied, the first column $\Phi_{-1}$ is analytic in $\text{Im} \sqrt{k^2 + c_+^2} > 0$, i.e. in $\{ k : \text{Im} k > 0 \} \setminus [i\infty, 0]$, the second column $\Phi_{-2}$ is analytic in $\text{Im} \sqrt{k^2 + c_+^2} < 0$, and the first column of the right Jost solution $\Phi_{+1}$ is analytic in $\text{Im} \sqrt{k^2 + c_+^2} < 0$, and the second column $\Phi_{+2}$ is analytic in $\text{Im} \sqrt{k^2 + c_+^2} > 0$.

(b) Furthermore, define the transition matrix $T(k) := \Phi_+(x; k)^{-1}\Phi(x; k)$, and denote its elements by

$$
T(k) = \begin{pmatrix} a(k) & -b(k) \\ b(k) & a(k) \end{pmatrix}, \quad \text{and furthermore} \quad r(k) = \frac{b(k)}{a(k)}.
$$

Then $a(.)$ is analytic in $k \in \{ k : \text{Im} k > 0 \} \setminus [i\infty, 0]$, continuous up to the boundary with the exception of the points $k \in \{ i\gamma, i\gamma_0, 0 \}$, and has uniform w.r.t. $k \in [0, \pi]$ asymptotics $a(k) = 1 + O(k^{-1})$ as $k \to \infty$. Function $r(k)$ is defined and continuous for $k \in \mathbb{R} \setminus \{0\}$, and has the asymptotics $b(k) = O(k^{-1})$ as $k \to \pm \infty$, $k \in \mathbb{R}$.  

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Furthermore, under (C.12) the kernels in (C.2) are summable:

\[ \int_{-\infty}^{0} |q_0(x) - c_-| e^{2i\sqrt{(c_- + \delta)^2 - c_-^2}} dx + \int_{0}^{+\infty} |q_0(x) - c_+| e^{2i\sqrt{(c_+ + \delta)^2 - c_+^2}} dx < \infty, \quad \text{for some} \quad \delta > 0. \]

Then the Jost solutions \( \Phi_- \), \( \Phi_+ \) are analytic in a \( \delta \)--neighborhood of the contour \( k \in \Sigma = \mathbb{R} \cup [ic_-, -ic_] \).

Further, a\( (k) \) and b\( (k) \) have analytic extension to \( k \in U_\delta(\Sigma) \setminus [ic_-, -ic_-] \), and have the asymptotics \( b(k) = O(k^{-1}) \), as \( k \rightarrow \infty \), \( k \in U_\delta(\Sigma) \).

**Proof.** The (a) - part of the corollary follows from the fact that under (C.1) the kernels in (C.2) are summable:

\[ \int_{-\infty}^{x} |L_-(x, y)| dy < \infty, \quad \int_{x}^{+\infty} |L_+(x, y)| dy < \infty. \]

The (c)- statement of the corollary follows directly from the estimates (C.3) and the fact that under (C.12)

\[ \int_{-\infty}^{0} e^{2i\sqrt{(c_- + \delta)^2 - c_-^2}} \sigma_-(y) dy < \infty, \quad \int_{0}^{+\infty} e^{2i\sqrt{(c_+ + \delta)^2 - c_+^2}} \sigma_+(y) dy < \infty. \]

The first of the last estimates gives us that \( \Phi_- \) is analytic in \( 0 < |Im \sqrt{k^2 + c_-^2}| < \sqrt{(c_- + \delta)^2 - c_-^2} \), which includes \( \delta \)--neighborhood of \( \Sigma \), and the second estimate gives us analyticity of \( \Phi_+ \) in \( |Im \sqrt{k^2 + c_+^2}| < \sqrt{(c_+ + \delta)^2 - c_+^2} \), which also includes \( \delta \)--neighborhood of \( \Sigma \).

To prove (a) and (d) parts, denote

\[ \Phi_-(x; k) = \begin{pmatrix} \varphi^-(x; k) & -\psi^-(x; k) \\ \psi^-(x; k) & \phi^-(x; k) \end{pmatrix}, \quad \Phi_+(x; k) = \begin{pmatrix} -\psi^+(x; k) & \varphi^+(x; k) \\ \phi^+(x; k) & \psi^+(x; k) \end{pmatrix}, \]

then (the expressions below are independent of \( x \))

\[ b(k) = e^{-i(x^+(k) + x^-(k))} \left[ a^-(k) + \int_{-\infty}^{x} (L_1^-(x, y)a^-(k) + L_2^-(x, y)b^-(k)) e^{i(x-y)\chi^-(k)} dy \right] \right] \cdot \left[ b^+(k) + \int_{x}^{+\infty} (-L_2^+(x, y)a^+(k) + L_1^+(x, y)b^+(k)) e^{i(x-y)\chi^+(k)} dy \right] - \left[ b^-(k) + \int_{-\infty}^{x} (-L_2^-(x, y)a^-(k) + L_1^-(x, y)b^-(k)) e^{i(x-y)\chi^-(k)} dy \right] \right] + \left[ a^+(k) + \int_{+\infty}^{x} (L_1^+(x, y)a^+(k) + L_2^+(x, y)b^+(k)) e^{i(x-y)\chi^+(k)} dy \right]. \]

(c) Suppose that in addition to (C.1) the following conditions are satisfied: \( c_- \geq c_+ \geq 0 \), and
we find that the transition matrix is the same as for unperturbed Jost solutions $E_{a}$.

Proof. First of all, since $\Phi(-\infty)x\pm\infty$, we have that $\Phi$ is integrable by parts, as in Proposition C.1 (b). The jump condition 6 follows from the symmetries 3 follow from the symmetry of equation (2.1). The multiplier in the r.h.s. of (2.1) is trace-less. The 2nd property follows from the possibility of integral representations from Proposition C.1. The symmetries 3 follow from the symmetry of equation (2.1).

The large $k$ asymptotics also follows from integral representations in Proposition C.1 (b), and, furthermore, all the exponential terms with dependence on $k$ are estimated by $e^{|x|\Im \chi(k)} \leq e^{\tilde{C}|x|}$, and hence do not depend on $k$.

**Proof. of Lemma 2.1.** The 1st property follows by the Liouville's formula from the fact that the left multiplier in the r.h.s. of (2.1) is trace-less. The 2nd property follows from the possibility of integral representation from Proposition C.1. The symmetries 3 follow from the symmetry of equation (2.1).

The large $k$ asymptotics also follows from integral representations in Proposition C.1 (b), and the possibility of integrating them by parts, as in Proposition C.1 (b). The jump condition 6 follows from the following observation: First of all, since $\Phi(k \pm 0)$ are 2 solutions of the first order differential equation (2.1), they are connected by a matrix, which does not depend on $x$ (and $t$). Taking then $x \rightarrow \pm \infty$ limit, we find that the transition matrix is the same as for unperturbed Jost solutions $E(k \pm 0)$.

**Proof. of Lemma 2.2.** The properties 7, 8 follows from Corollary C.2. In particular, the analyticity of $a(k)$ in a domain, which contains the real line, ensures that $a(k)$ does not have accumulating zeros on the real line and segment $[ic_{-}, 0]$. Symmetries 9 follow from the corresponding symmetries in Lemma 2.1, 3, for the Jost solutions, and representation (2.7). The first part of the property 13 follows from the symmetries 9.

The first part of Property 14 is obtained as follows: on $\pm (ic_{-}, ic_{+})$, we have that $\Phi_{+}(k)$ is regular, and $\Phi_{-}(k-0)$ is two-sided functions, so that $\Phi_{-}(k \mp 0) = \Phi_{+}(k)T(k \mp 0)$. Dividing one equation by another, we obtain

$$T(k + 0)^{-1}T(k - 0) = \Phi_{-}(k + 0)^{-1}\Phi_{-}(k - 0) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$ 

Property 10, i.e. non-vanishing of $a(k)$, $b(k)$ on $\pm (ic_{-}, ic_{+})$, follows from the following observation: suppose $a(k^{*} + 0) = 0$, $k^{*} \in (ic_{-}, ic_{+})$, then by the first part of property 13, $a(k^{*} - 0) = 0$, and by the first part of Property 14, also $b(k^{*} + 0) = 0$. We see that $T(k^{*} + 0) = 0$, which contradicts $\det T \equiv 1$.

Properties 11, 12 are the element-wise written relation $\Phi_{-}(k) = \Phi(k)T(k)$. The second parts of Properties 13, 14 is the explicitly written relation $\det T(k \pm 0) = 1$.

**D Zeros of $a(k)$ in $\Im \sqrt{k^{2} + c_{2}^{2}} \geq 0.$**

In this section we prove Lemma 2.3 and we derive the RH problem in the case of higher order poles, which gives higher order solitons or breathers.

**Proof.** In order to prove Lemma 2.3 we recall that the coefficient $a(k)$ is defined as...
\begin{equation}
\Phi_{k}\pm \Phi_{\pm}(x; k) \right),
\end{equation}
with \( \Phi_{k} \) the Jost solutions defined in \( \text{(2.5)} \). For simplicity let us use the following notation
\[
\Phi_{-}(x; k) = \begin{pmatrix} \varphi^{-}(x; k) \\ \psi^{-}(x; k) \end{pmatrix}, \quad \Phi_{+}(x; k) = \begin{pmatrix} \varphi^{+}(x; k) \\ \psi^{+}(x; k) \end{pmatrix}.
\]

Let \( a(k_0) = 0 \) and \( \dot{a}(k_0) \neq 0 \), where \( \dot{a} \) means differentiation with respect to \( k \).

Then \( \{ \varphi^{-} = \mu \varphi^{+}, \psi^{-} = \mu \psi^{+}, \mu \neq 0 \} \).

If \( \{ \varphi^{-} \} \) is a solution of the Lax equation \( \text{(2.1)} \), then \( \{ -\varphi^{-}(k) \overline{\varphi^{+}}(k) \} \) is also a solution of the equation \( \text{(2.1)} \). Due to above mentioned symmetry, all zeros of \( a(k) \) either belong to the imaginary axis or go in pairs, \( k \) and \( -\overline{k} \). According to Lemma \( \text{2.2} \) property 9 the zeros of \( a(k) \) are outside the segment \( (ic_{+}, ic_{-}) \).

In order to prove the relation \( \text{(2.9)} \) we differentiate with respect to \( k \) the relation \( \text{(2.1)} \) obtaining
\[
\dot{a}(k_0) = \det \begin{pmatrix} \dot{\varphi}^{-}(k_0) & \frac{1}{\mu} \dot{\varphi}^{+}(k_0) \\ \dot{\psi}^{-}(k_0) & \frac{1}{\mu} \dot{\psi}^{+}(k_0) \end{pmatrix} + \det \begin{pmatrix} \mu \varphi^{+}(k_0) & \dot{\varphi}^{+}(k_0) \\ \mu \psi^{+}(k_0) & \dot{\psi}^{+}(k_0) \end{pmatrix}.
\]

In order to evaluate the above two determinants we define the quantity
\[
W := \det \begin{pmatrix} \varphi^{-} & \dot{\varphi}^{-} \\ \psi^{-} & \dot{\psi}^{-} \end{pmatrix}.
\]

From the Lax equation \( \text{(2.1)} \) we have
\[
\begin{cases}
\varphi_x^{-} + ik \varphi^{-} = q \psi^{-}, \\
\psi_x^{-} - ik \psi^{-} = -q \varphi^{-},
\end{cases}
\]
\[
\begin{cases}
\dot{\varphi}_x^{-} + ik \dot{\varphi}^{-} + i \varphi^{-} = q \dot{\psi}^{-}, \\
\dot{\psi}_x^{-} - ik \dot{\psi}^{-} - i \psi^{-} = -q \dot{\varphi}^{-}.
\end{cases}
\]

We have from the above equations that
\[
W_x = 2i \varphi^{-} \dot{\psi}^{-}, \quad W \big|_{a}^{b} = 2i \int_{a}^{b} \varphi^{-} \dot{\psi}^{-} \, dx.
\]

Using the above relations we obtain
\[
\dot{a}(k_0) = \frac{-1}{\mu} \int_{-\infty}^{0} 2i \varphi^{-}(x; k_0) \psi^{-}(x; k_0) \, dx - \mu \int_{0}^{+\infty} 2i \varphi^{+}(x; k_0) \psi^{+}(x; k_0) \, dx
\]
\[
= -2i \mu \int_{-\infty}^{+\infty} \varphi^{+}(x; k_0) \psi^{+}(x; k_0) \, dx = -\frac{2i}{\mu} \int_{-\infty}^{+\infty} \varphi^{-}(x; k_0) \psi^{-}(x; k_0) \, dx,
\]
which conclude the proof of Lemma \( \text{2.3} \) \( \blacksquare \)

**Lemma D.1.** Let \( k_0, \text{Im}k_0 > 0, k_0 \notin (ic_{-}, ic_{+}] \) be a zero of \( a(.) \) of the order \( n \geq 1 \), i.e.
\[
a(k_0) = \cdots a^{(n-1)}(k_0) = 0, \quad a^{(n)}(k_0) \neq 0.
\]

Then there exist coefficients \( \mu_0, \cdots, \mu_{n-1} \)

independent on \( x, t \), such that the following equalities hold:
\[
f^{(m)}_{-} = \sum_{k=0}^{m} C_{m}^{k} \mu_{m-k} f^{(k)}_{+}, \quad m = 0, \cdots, n-1, \quad C_{m}^{k} := \frac{m!}{k!(m-k)!}
\]

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where \( f_\pm = (\varphi_\pm, \psi_\pm)^T \), i.e. \[
\begin{aligned}
f_- &= \mu_0 f_+ + \mu_1 f_+,
f^- &= \mu_0 f_+ + 2\mu_1 f_+ + \mu_2 f_+,
\end{aligned}
\]

Now, in the above expression each term \( \det[\cdot] \) finishes the proof.

Proof. Denote 
\[
f = \Phi_{-1}, \quad g = \Phi_{+2}
\]
the first and the second columns of \( \Phi_{-1} \), respectively. \( f, g \) are analytic in the upper half-plane. The spectral coefficient \( a(k) \) is given by the formula
\[
a(k) = \det[f(x, t; k), g(x, t; k)].
\]
The base of induction is the fact that if \( a(k_0) = 0 \), then
\[
\exists \mu_0 \text{ such that } f = \mu_0 g.
\]
Further, suppose that the statement of the lemma is satisfied for \( m = 1, \cdots, m-1 \). By the Lebeque formula,
\[
a^{(m)}(k_0) = \sum_{k=0}^{m-1} \det[f^{(m-k)}(k), g^{(k)}].
\]
In the above formula we split off the term containing \( f^m \), and for all the smaller order derivatives of \( f \) we substitute their expression in terms of the derivatives of \( g \). We get
\[
a^{(m)}(k_0) = \det[f^{(m)}, g] + \sum_{k=0}^{m-1} \sum_{j=0}^{m-k} C_m^k C_{m-k}^j \mu_{m-k-j} \det[g^{(j)}, g^{(k)}]. \tag{D.2}
\]
Now, in the above expression each term \( \det[g^{(j)}, g^{(k)}] \) with \( k \geq 1, k \neq j \), will appear twice: once as
\[
C_m^k C_{m-k}^j \mu_{m-k-j} \det[g^{(j)}, g^{(k)}],
\]
and another time as
\[
C_m^j C_{m-j}^k \mu_{m-j-k} \det[g^{(k)}, g^{(j)}]
\]
Since
\[
C_m^j C_{m-j}^k = C_m^k C_{m-k}^j = \frac{m!}{j!(m-k-j)!},
\]
these two terms will cancel each other. The same is for \( j = k \). Hence, the only terms which remains in (D.2) are those corresponding to \( j = 0 \), i.e.
\[
a^{(m)}(k_0) = \det[f^{(m)}, g] - \sum_{k=1}^{m} C_m^k \mu_{m-k} \det[g^{(k)}, g] = \det[f^{(m)}] - \sum_{k=1}^{m} C_m^k \mu_{m-k} g^{(k)} \tag{D.3}
\]
and hence \( a^{(m)}(k_0) = 0 \) implies the existence of a coefficient \( \mu_m \) such that
\[
f^{(m)} - \sum_{k=1}^{m} C_m^k \mu_{m-k} g^{(k)} = \mu_m g, \quad \text{or} \quad f^{(m)} = \sum_{k=0}^{m} C_m^k \mu_{m-k} g^{(k)}.
\]
This finishes the proof.

Lemma D.2. Let \( k_0, \Im k_0 > 0 \), \( k_0 \notin i\mathbb{C} \) be a zero of \( a(.) \) of the \( n^{th} \) order, \( n \geq 1 \), i.e.
\[
a(k_0) = \ldots = a^{(n-1)}(k_0) = 0, \quad a^{(n)}(k_0) \neq 0;
\]
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let the coefficients \( \mu_0, \ldots, \mu_{n-1} \) (independent of \( x, t \)) be as in Lemma D.1, i.e.

\[
f^{(m)}(k_0) = \sum_{j=0}^{m} C^j_m \mu_j f^{(m-j)}(k_0), \quad m = 0, \ldots, n-1, \quad \text{where } C^j_m := \frac{m!}{j!(m-j)!} \text{ are the binomial coefficients,}
\]

and let \( T_1(x, t), \ldots, T_n(x, t) \) be the coefficients in the Taylor expansion of the function \( \frac{e^{2i\theta(x,t;k_0)} - 2i\theta(x,t;k_0)}{a(k)} \) at the point \( k = k_0 \), i.e.

\[
\frac{e^{2i\theta(x,t;k)}}{a(k)} = \sum_{q=1}^{n} T_q(x,t) \frac{e^{2i\theta(x,t;k_0)}}{(k-k_0)^q} + \mathcal{O}(1), \quad k \to k_0.
\]

Then

\[
\frac{1}{a(k)} f_-(x, t, k) e^{i\theta(x,t;k)} = \left[ \sum_{j=1}^{n} A_j(x, t) \frac{e^{2i\theta(x,t;k_0)}}{(k-k_0)^j} \right] f_+(x, t, k) e^{-i\theta(x,t;k)}
\]

is regular in a vicinity of the point \( k = k_0 \) provided that

\[
A_j(x, t) = \sum_{m=j}^{n} T_m(x, t) \frac{\mu_{m-j}}{(m-j)!} = \sum_{m=0}^{n-j} T_{m+j}(x, t) \frac{\mu_{m+j}}{m!}, \quad j = 1, \ldots, n.
\]

**Remark D.1.** It is remarkable that the coefficients \( A_j \) are determined only by \( \mu_j, \ j = 0, \ldots, n-1, \) and not by \( f_+ \) and its derivatives at \( k = k_0 \).

**Remark D.2.** Definition of \( A_j, j = 1, \ldots, n, \) requires up to the \( 2n - 1^{st} \) derivative of \( a(.) \) at the point \( k = k_0, \) i.e. \( a^{(n)}(k_0), \ldots, a^{(2n-1)}(k_0). \)

**Proof.** Regularity of the expression \[D.6\] is equivalent to the regularity of

\[
\frac{1}{a(k)} f_-(x, t, k) e^{2i\theta(x,t;k_0)} - 2i\theta(x,t;k_0) = \left[ \sum_{j=1}^{n} A_j(x, t) \frac{e^{2i\theta(x,t;k_0)}}{(k-k_0)^j} \right] f_+(x, t, k).
\]

Let’s first treat the left summand in \[D.8\]. We have

\[
f_-= \sum_{m=0}^{n-1} \frac{1}{m!} f^{(m)}(k_0)(k-k_0)^m + \mathcal{O}((k-k_0)^n),
\]

and substituting in the above formula the expressions \[D.4\] of \( f^{(m)} \) in terms of \( f_+^{(j)} \), we obtain

\[
f_-= \sum_{m=0}^{n-1} \sum_{j=0}^{m} \mu_{m-j} f_+^{(j)}(k_0) j!(m-j)! (k-k_0)^m + \mathcal{O}((k-k_0)^n).
\]

Multiplying the above formula by \[D.5\], we obtain

\[
\frac{e^{2i\theta(x,t;k_0)} - 2i\theta(x,t;k_0)}{a(k)} f_-(x, t, k) = \sum_{p=1}^{n-p} \sum_{m=0}^{n-p} T_{m+p}(x, t) \frac{\mu_{m-j} f_+^{(j)}(k_0)}{j!(m-j)!} (k-k_0)^{-p} + \mathcal{O}(1).
\]

The second summand in \[D.8\] has the following decomposition in the Taylor series:

\[
\left[ \sum_{j=1}^{n} A_j(x, t) \frac{e^{2i\theta(x,t;k_0)}}{(k-k_0)^j} \right] f_+(x, t, k) = \sum_{p=1}^{n} \sum_{j=0}^{p} A_j(x, t) \frac{f_+^{(j-p)}(k_0)}{(j-p)!} (k-k_0)^{-p} + \mathcal{O}(1).
\]

Equating terms \( (k-k_0)^{-p}, p = 1, \ldots, n \), in \[D.9\] and \[D.10\], we come to the system of equations for \( A_j \):

\[
\sum_{j=p}^{n} A_j \frac{f_+^{(j-p)}(k_0)}{(j-p)!} = \sum_{m=0}^{n-p} T_{m+p} \sum_{j=0}^{m} \frac{\mu_{m-j} f_+^{(j)}(k_0)}{j!(m-j)!}, \quad p = 1, \ldots, n.
\]

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The system (D.11) allows to determine $A_j$, $j = 1, \ldots, n$, one by one, starting from $A_n$. However, it is not clear at this point that $A_j$ do not depend on $f_+$ and its derivatives. In order to solve the system (D.11), let us rearrange the terms in (D.11):

$$
\sum_{j=0}^{n-p} \frac{A_{j+p} f_+^{(j)}(k_0)}{j!} = \sum_{j=0}^{n-p} \left[ \frac{\mu_{m-j}}{(m-j)!} \right] \frac{T_{m+p} f_+^{(j)}(k_0)}{j!}, \quad p = 1, \ldots, n. \tag{D.12}
$$

The above expression (D.12) is satisfied provided that

$$
A_{j+p} = \sum_{m=j}^{n-p} T_{m+p} \frac{\mu_{m-j}}{(m-j)!}, \quad p = 1, \ldots, n, \quad j = 0, \ldots, n-p,
$$

however, (D.13) is not yet the definition of $A_{j+p}$, since the r.-h.-s. might depend on different choices for $j, p$ with $j + p$. However, reorganizing terms in (D.13), we obtain

$$
A_j(x, t) = \sum_{m=j}^{n} T_{m}(x, t) \frac{\mu_{m-j}}{(m-j)!}, \quad j = 1, \ldots, n.
$$

\[\square\]

\section*{D.1 Handling pole conditions.}

Analysis via the RHP for simple poles first was introduced in [32] for KdV (see also [51] for the CH equation).

Here we explain how to proceed, if our spectral problem has poles of multiple order. First of all, the pole conditions 3 of the RHP [1] will change to the following conditions:

- for $\kappa_j, \Re \kappa_j > 0, \Im \kappa_j > 0$, the function $M(\xi, t; k)$ has the following pole conditions:

$$
M_1(k) - \sum_{l=1}^{n_1} \nu^{(l)}_j \frac{e^{2i\theta(k, \xi)}}{(k - \kappa_j)^l} M_2(k) = \mathcal{O}(1), k \to \kappa_j, \quad M_1(k) - \sum_{l=1}^{n_1} \nu^{(l)}_j \frac{e^{2i\theta(k, \xi)}}{(-1)^l(k + \kappa_j)^l} M_2(k) = \mathcal{O}(1), k \to -\kappa_j,
$$

$$
M_2(k) + \sum_{l=1}^{n_2} \nu^{(l)}_j \frac{e^{-2i\theta(k, \xi)}}{(k - \kappa_j)^l} M_1(k) = \mathcal{O}(1), k \to \kappa_j, \quad M_2(k) + \sum_{l=1}^{n_2} \nu^{(l)}_j \frac{e^{-2i\theta(k, \xi)}}{(-1)^l(k + \kappa_j)^l} M_1(k) = \mathcal{O}(1), k \to -\kappa_j,
$$

where $M_1, M_2$ are the first and the second columns of $M$.

\subsection*{D.1.1 Applications: Solitons and breathers of multiple order}

They are generated by the following meromorphic RHP.

**Riemann-Hilbert problem 12.** Find a meromorphic $2 \times 2$ matrix-valued function $M(x, t; k)$ such that

1. $M$ is meromorphic in $\mathbb{C}$ with poles at $\kappa, \bar{\kappa}, -\kappa, -\bar{\kappa}$, for some $\Re \kappa \geq 0, \Im \kappa > 0$.

2. Pole conditions, upper half-plane:

$$
M_1(k) = \sum_{j=1}^{n} \frac{A_j e^{2i\theta(k)}}{(k - \kappa)^j} M_2(k) = \mathcal{O}(1), \quad k \to \kappa,
$$

$$
M_1(k) = \sum_{j=1}^{n} (-1)^j \frac{(\bar{A})_j e^{2i\theta(-\kappa)}}{(k + \bar{\kappa})^j} M_2(k) = \mathcal{O}(1), \quad k \to -\bar{\kappa},
$$

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lower half-plane:

\[ M_2(k) = \left[ \sum_{j=1}^{n} \frac{-\bar{A}_j e^{-2\theta(\kappa)}}{(k - \kappa)^j} \right] M_1(k) = \mathcal{O}(1), \quad k \to \kappa, \]

\[ M_2(k) = \left[ \sum_{j=1}^{n} \frac{(-1)^j A_j e^{2i\theta(\kappa)}}{(k + \kappa)^j} \right] M_1(k) = \mathcal{O}(1), \quad k \to -\kappa; \]

3. asymptotics: \( M(x, t; k) \to I \) as \( k \to \infty \).

Here \( A_j(x, t) \) are as in (D.5), (D.7), i.e.

- simple pole \( n = 1, \ a(\kappa) = 0, \ \dot{a}(\kappa) \neq 0 \). In this case we have

\[ T_1 = \frac{e^{2i\theta(\kappa)}}{\bar{a}(\kappa)}, \quad A_1 = \mu_0 T_1. \]

Here \( \theta(k) = 4k^3t + xk, \) and the dependence on \( x, t \) comes only from \( \theta(k) \).

- double-pole \( n = 2, \ a(\kappa) = 0, \ \dot{a}(\kappa) = 0, \ \ddot{a}(\kappa) \neq 0 \). In this case we have

\[ T_2 = \frac{2e^{2i\theta(\kappa)}}{\bar{a}(\kappa)}, \quad T_1 = \frac{2i e^{2i\theta(\kappa)} [6\bar{a}(\kappa) (x + 12\kappa^2t) + i\ddot{a}(\kappa)]}{3 (\bar{a}(\kappa))^2}, \quad A_2 = T_2 \mu_0, \quad A_1 = \mu_0 T_1 + \mu_1 T_2, \]

- triple-pole \( n = 3, \ a(\kappa) = 0, \ \dot{a}(\kappa) = 0, \ \ddot{a}(\kappa) = 0, \ \dddot{a}(\kappa) \neq 0 \). In this case we have

\[ T_3 = \frac{6e^{2i\theta(\kappa)}}{\bar{a}(\kappa)}, \quad T_2 = \frac{24i \dddot{a}(\kappa) (x + 12\kappa^2t) - 3a^{(4)}(\kappa) e^{2i\theta(\kappa)}}{2 (\bar{a}(\kappa))^2}, \]

\[ T_1 e^{-2i\theta(\kappa)} = \frac{-12 [(x + 12\kappa^2t)^2 - 12i\kappa t]}{\bar{a}(\kappa)} + \frac{3a^{(4)}(\kappa) (x + 12\kappa^2t)}{(\bar{a}(\kappa))^2} + \frac{15 (a^{(4)}(\kappa))^2 - 12a^{(3)}(\kappa) a^{(5)}(\kappa)}{40 (\bar{a}(\kappa))^3}, \]

\[ A_3 = T_3 \mu_0, \quad A_2 = \mu_0 T_2 + \mu_1 T_3, \quad A_1 = \mu_0 T_1 + \mu_1 T_2 + \frac{\mu_2}{2} T_3. \]

Taking into account symmetry (2.13), the solution can be found in the form

\[ M(k) = \left( 1 + \sum_{j=1}^{n} \frac{\alpha_j(x, t)}{(k - \kappa)^j} + \sum_{j=1}^{n} \frac{(-1)^j \beta_j(x, t)}{(k + \kappa)^j} \right) \left( 1 + \sum_{j=1}^{n} \frac{\alpha_j(x, t)}{(k - \kappa)^j} + \sum_{j=1}^{n} \frac{(-1)^j \beta_j(x, t)}{(k + \kappa)^j} \right), \]

(If \( \kappa = -\kappa \neq 0 \), then we make a straightforward reduction of the above expression) Then we find the solution of the MKDV by the formula

\[ q(x, t) = 2i \left( \beta_1(x, t) - \bar{\beta}_1(x, t) \right) = -4 \text{Im} \beta_1(x, t). \]

If \( \text{Im} \kappa = 0, \ \text{Re} \kappa > 0, \) then we have a multiple soliton, if \( \text{Im} \kappa > 0, \ \text{Re} \kappa > 0, \) then we have a multiple breather. As was pointed out in [56], double-pole soliton can be obtained as a limit of a breather with \( \text{Re} \kappa \to 0 \).
E Simple breather on hyperelliptic background

The considerations of section 4 go without significant changes also in the case when there are breathers of velocity \( V \) so that \(-6c_2^2 + 12c_2^2 < V < 4c_2^2 + 2c_2^4\). In this case the model RHP problems will have pole conditions corresponding to the points \( \kappa, -\bar{\kappa}, \bar{\kappa}, \kappa \), and hence instead of RHPs (4), (5) will have the following model problems:

**Riemann-Hilbert problem 13.** Find a \( 2 \times 2 \) matrix \( M(x, t; k) \) meromorphic in \( k \in \mathbb{C}\setminus[-ic, ic] \) with simple poles at \( \kappa \equiv \kappa_1 + i\kappa_2, \bar{\kappa} \equiv \kappa_1 - i\kappa_2, -\kappa \equiv -\kappa_1 - i\kappa_2, -\bar{\kappa} \equiv -\kappa_1 + i\kappa_2 \), and such that

1. \( M_-(k) = M_+(k)J(k), \quad k \in (ic_-,-ic_-), \) where
   \[
   J(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in (ic_-, id) \cup (id, -id) \cup (-id, -ic_-),
   \]
   \[
   J(k) = e^{(ixU + itV + i\Delta)x}, \quad k \in (id, ic_+) \cup (-ic_+, -id),
   \]

2. pole conditions:
   \[
   \text{Res}_\kappa M(k) = \lim_{k \to \kappa} M(k) \begin{pmatrix} 0 & 0 \\ \nu e^{2ig_h(k, x, t)} & 0 \end{pmatrix},
   \]
   \[
   \text{Res}_{-\bar{\kappa}} M(k) = \lim_{k \to -\bar{\kappa}} M(k) \begin{pmatrix} 0 & 0 \\ -\nu e^{2ig_h(k, x, t)} & 0 \end{pmatrix},
   \]
   \[
   \text{Res}_{-\kappa} M(k) = \lim_{k \to -\kappa} M(k) \begin{pmatrix} 0 & 0 \\ 0 & \nu e^{-2ig_h(k, x, t)} \end{pmatrix},
   \]
   \[
   \text{Res}_{-\bar{\kappa}} M(k) = \lim_{k \to -\bar{\kappa}} M(k) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
   \]
   where \( \nu \) is a non zero complex number and \( g_h(k, x, t) = tg(k, x/(12t)) \) with \( g \) as in (4.2);

3. asymptotics: \( M(k) = I + O(\frac{1}{k}) \) as \( k \to \infty \).

For the function
\[
q(x, t) = 2i \lim_{k \to \infty} M_{21}(x, t; k)
\]
to be a solution of the MKdV equation (1.1), the parameters \( U, V \) are not arbitrary numbers, but they are related to \( g_h \) in the following way:
\[
g_h(k - i0) - g_h(k + i0) = xU + tV, \quad k \in (id, ic_+) \cup (-ic_+, -id).
\]
In other words, \( g_h \) is an Abelian integral on the Riemann surface of the function \((k^2 + c_2^2)(k^2 + d^2)(k^2 + c_2^4))^{1/2}\), as in discussion after Theorem 3.3 on p. 27 with zero \( a- \) periods, and asymptotic behavior \( kx + 4k^2t \) as \( k \to \infty \) on the upper sheet, and \( xU + tV \) is the value of its (two) \( b- \)periods. The solution of the MKdV equation that corresponds to this RHP is obtained from its solution by formula (3.6).

The RHP 13 with \( \nu = 0 \) is merely RHP 3 from section 3.4 and using its solution \( W(k) \), we can construct the solution for RHP 13 with \( \nu \neq 0 \). Namely, the ratio \( M(k)W(k)^{-1} \) does not have any jump across the segment \([ic_-,-ic_-]\), and has simple poles at the points \( \kappa, -\bar{\kappa}, \bar{\kappa}, -\kappa \). Hence, using the symmetries (2.13) we get
\[
M(k)W(k)^{-1} = M_{pol}(x, t; k),
\]
with \( M_{pol}(k) = M_{pol}(x; t; k) \) as in (3.13). Substituting then the ansatz \( M(k) = M_{pol}(k)W(k) \) into the pole condition at \( k = \kappa \) of RHP 13 gives us linear system for the unknown parameters \( A, B, C, D \). In view of Schwartz symmetry of the problem, that linear system always has solution (the determinant is never 0). The solution of the MKdV equation \( q(x, t) = 2i \lim_{k \to \infty} M(x, t; k)_{21} \) equals
\[
q(x, t) = q_W(x, t) - 4\text{Im}(B + D),
\]
65
where \( q_W(x,t) = 2i \lim_{k \to \infty} W(k) \). However, the expression for \( \text{Im} B + \text{Im} D \) that we obtain in this way is huge, and we were not able to find a closed simple form for it.

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