UNIFIED DESCRIPTION OF QUANTUM AFFINE (SUPER)ALGEBRAS $U_q(A_1^{(1)})$ AND $U_q(C(2)^{(2)})$*

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Abstract

We show that the quantum affine algebra $U_q(A_1^{(1)})$ and the quantum affine superalgebra $U_q(C(2)^{(2)})$ admit unified description. The difference between them consists in the phase factor which is equal to 1 for $U_q(A_1^{(1)})$ and is equal to $-1$ for $U_q(C(2)^{(2)})$. We present such a description for the construction of Cartan-Weyl generators and their commutation relations, as well for the universal R-matrices.

*Talk given by V.N. Tolstoy
1 Introduction

Among variety of all affine Lie (super)algebras (both quantized and non-quantized) the affine (super)algebras of rank 2 play a key role. In the first place, all affine series of the type $A(n|m)^{(1)}$, $B(n|m)^{(1)}$, $C(n)^{(1)}$, $D(n|m)^{(1)}$, $A(2n|2m−1)^{(2)}$, $A(2n−1|2m−1)^{(2)}$, $C(n)^{(2)}$, $D(n|m)^{(2)}$ and $A(2n|2m)^{(4)}$ begin with the affine (super)algebras of rank 2. Secondly, the contragredient Lie (super)algebras of rank 2 are basic structural blocks of any affine (super)algebras of arbitrary rank. This fact permits, for example, the reduction of the proofs of basic theorems for the extremal projector and the universal R-matrix to the proofs of such theorems for the (super)algebras of rank 2 (see Refs. [1], [11] - [13], [5] - [8]). Further, the representation theory of the affine (super)algebras (both quantized and non-quantized) contains some typical elements of the representation theory of the affine (super)algebras of rank 2. Besides, in applications of the affine (super)algebras, first of all the affine (super)algebras of rank 2 are used by virtue of their simplicity.

In this paper along the line of considerations presented in [9] for rank 2 affine superalgebra $U_q(B(0,1)^{(1)})$ we give detailed description of the quantum untwisted affine algebra $U_q(A_1^{(1)})$ ($\simeq U_q(\hat{sl}(2))$) and the quantum twisted affine superalgebra $U_q(C(2)^{(2)})$. Moreover our goal is to show that these quantum (super)algebras are described in unified way. Namely, we present in unified way their defining relations, the construction of the Cartan-Weyl bases, the complete list of all commutation relations of the Cartan-Weyl generators corresponding to all root vectors and finally the unified form of their universal R-matrices. Difference between both considered quantum (super)algebras is only determined by a phase factor which is equal to 1 for $U_q(A_1^{(1)})$ and it is equal to $-1$ for $U_q(C(2)^{(2)})$. This situation is similar to the finite-dimensional case. Namely, in the paper [4] it was shown that all quantum (super)algebras $U_q(g)$, where $g$ are the finite-dimensional contragredient Lie (super)algebras of rank 2, are divided into three classes. Each such class is characterized by the same Dynkin diagram and has the same reduced root system, provided that we neglect the type (colour) of the roots (white, grey or black). Consequently, all the (super)algebras of the same class have unified defining relations, unified construction of the Cartan-Weyl basis and

\footnote{We introduce the prefix "super" in brackets to stress that the Lie (super)algebras include the Lie algebras as well as the Lie superalgebras.}
its properties, as well as unified universal R-matrix. Difference between the 
(super)algebras of the same class is determined by some phase factor which 
takes values $\pm 1$ depending on the colour of the nodes of the Dynkin diagram.

Basic information about the (super)algebras $A^{(1)}_1$ and $C^{(2)}_2$ is presented 
in the tables 1a and 1b (see Refs. [3, 4, 14]). In the table 1a there are listed 
the standard and symmetric Cartan matrices $A$ and $A^{sym}$, the corresponding 
extended symmetric matrices $\bar{A}^{sym}$ and their inverses $(\bar{A}^{sym})^{-1}$, and also the 
sets of odd roots (odd roots), the Dynkin diagrams (diagram), and the dimen-
sions of these (super)algebras (dim). We remind some elementary definitions 
of the colour of the roots:

- All even roots are called white roots. A white root is pictured by the 
  white node $\bigcirc$.

- An odd root $\gamma$ is called a grey root if $2\gamma$ is not a root. Such an odd 
  root is pictured by the grey node $\otimes$.

- An odd root $\gamma$ is called a dark root if $2\gamma$ is a root. Such an odd root is 
  pictured by the dark node $\bullet$.

We also remind the definition of the reduced system of the positive root 
system $\Delta_+$ for any contragredient (super)algebras of finite growth.

- The system $\Delta_+$ is called the reduced system if it is defined by the 
  following way: $\Delta_+ = \Delta_+ \backslash \{2\gamma \in \Delta_+ | \gamma \text{ is odd}\}$. That is the reduced 
  system $\Delta_+$ is obtained from the total system $\Delta_+$ by removing of all 
  doubled roots $2\gamma$ where $\gamma$ is a dark odd root.

The total and reduced root systems of the (super)algebras $A^{(1)}_1$ and $C^{(2)}_2$ 
are listed in the table 1b. It is convenient to present the total $\Delta = \Delta_+ \cup (-\Delta_+)$ 
and reduced $\Delta = \Delta_+ \cup (-\Delta_+)$ root systems by the pictures: Figs. 1, 2a, 2b. 
Comparing Fig. 1 and Fig. 2b we see that the reduced root systems of $A^{(1)}_1$ 
and $C^{(2)}_2$ coincide if we neglect colour of the roots.
Table 1a

| $g(A, \Upsilon)$ | $A = A_{sym}$ | $\bar{A}_{sym}$ | $(\bar{A}_{sym})^{-1}$ | odd diagram |
|------------------|--------------|-----------------|-----------------------|------------|
| $A_{1}^{(1)}$    | $\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$ | $\delta - \alpha, \alpha$ |
| $C(2)^{(2)}$     | $\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | $\delta - \alpha, \alpha$ |

Table 1b

| $g(A, \Upsilon)$ | $\Delta_{+}$ | $\Delta_{+}$ |
|------------------|-------------|-------------|
| $A_{1}^{(1)}$    | $\{\alpha, n\delta \pm \alpha, n\delta \mid n \in \mathbb{N}\}$ | $\{\alpha, n\delta \pm \alpha, n\delta \mid n \in \mathbb{N}\}$ |
| $C(2)^{(2)}$     | $\{\alpha, 2\alpha, n\delta \pm \alpha, 2n\delta \pm 2\alpha, n\delta \mid n \in \mathbb{N}\}$ | $\{\alpha, n\delta \pm \alpha, n\delta \mid n \in \mathbb{N}\}$ |

Fig. 1. The total and reduced root system ($\Delta = \bar{\Delta}$) of $A_{1}^{(1)}(\simeq \widehat{sl}_2)$.

Fig. 2a. The total root system $\Delta$ of $C(2)^{(2)}$. 

4
2 Defining Relations of $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$

The quantum (q-deformed) affine (super)algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$ are generated by the Chevalley elements $k_{d}^{\pm1} := q^{\pm h_{d}}$, $k_{\alpha}^{\pm1} := q^{\pm h_{\alpha}}$, $k_{\delta - \alpha}^{\pm1} := q^{\pm h_{\delta - \alpha}}$, $e_{\pm \alpha}$, $e_{\pm (\delta - \alpha)}$ with the defining relations:

\begin{align}
&k_{\gamma}^{-1}k_{\gamma}^{-1} = 1, \quad [k_{\gamma}^{\pm1}, k_{\gamma}^{\pm1}] = 0, \quad (2.1) \\
&k_{\gamma}e_{\pm \alpha}k_{\gamma}^{-1} = q^{\pm (\gamma, \alpha)}e_{\pm \alpha}, \quad k_{\gamma}e_{\pm (\delta - \alpha)}k_{\gamma}^{-1} = q^{\pm (\gamma, \delta - \alpha)}e_{\pm (\delta - \alpha)}, \quad (2.2) \\
&[e_{\alpha}, e_{-\delta + \alpha}] = 0, \quad [e_{-\alpha}, e_{\delta - \alpha}] = 0, \quad (2.3) \\
&[e_{\alpha}, e_{-\alpha}] = [h_{\alpha}]_q, \quad [e_{\delta - \alpha}, e_{-\delta + \alpha}] = [h_{\delta - \alpha}]_q, \quad (2.4) \\
&[e_{\pm \alpha}, [e_{\pm \alpha}, e_{\pm (\delta - \alpha)}]_q]_q = 0, \quad (2.5) \\
&[[e_{\pm \alpha}, e_{\pm (\delta - \alpha)}]_q, e_{\pm (\delta - \alpha)}]_q, e_{\pm (\delta - \alpha)}]_q = 0, \quad (2.6)
\end{align}

where $(\gamma = d, \alpha, \delta - \alpha), (d, \alpha) = 0, (d, \delta) = 1$, and $[h_{\beta}]_q := (k_{\beta} - k_{\beta}^{-1})/(q - q^{-1})$.

The brackets $[,]$ and $[,]_q$ are the super-, and q-super-commutators:

\begin{align}
[e_{\beta}, e_{\beta'}] & = e_{\beta}e_{\beta'} - (-1)^{\vartheta(\beta)\vartheta(\beta')}e_{\beta'}e_{\beta}, \\
[e_{\beta}, e_{\beta'}]_q & = e_{\beta}e_{\beta'} - (-1)^{\vartheta(\beta)\vartheta(\beta')}q^{(\beta, \beta')}e_{\beta'}e_{\beta}. \quad (2.7)
\end{align}

Here the symbol $\vartheta(\cdot)$ means the parity function: $\vartheta(\beta) = 0$ for any even root $\beta$ and $\vartheta(\beta) = 1$ for any odd root $\beta$.

Remark. The left-side sides of the relations (2.3) and (2.6) are invariant with respect to the replacement of $q$ by $q^{-1}$. Indeed, if we remove the q-brackets we see that the left-hand of (2.3) and (2.6) contain the symmetric
functions of $q$ and $q^{-1}$. This property permits to write the q-commutators in (2.5) and (2.6) in the inverse order, i.e.

$$[[e_{\pm(\delta-\alpha)}, e_{\pm\alpha}], e_{\pm\alpha}]_q = 0,$$

$$[e_{\pm(\delta-\alpha)}, [e_{\pm(\delta-\alpha)}, [e_{\pm(\delta-\alpha)}, e_{\pm\alpha}]]_q]_q = 0. \quad (2.8)$$

$$[[[e_{\pm(\delta-\alpha)}, e_{\pm\alpha}], e_{\pm\alpha}], e_{\pm\alpha}]_q = 0,$$

$$[[[e_{\pm(\delta-\alpha)}, e_{\pm\alpha}], [e_{\pm(\delta-\alpha)}, e_{\pm\alpha}]]_q]_q = 0. \quad (2.9)$$

The standard Hopf structure of the quantum (super)algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$ is given by the following formulas for the comultiplication $\Delta_q$ and antipode $S_q$:

$$\Delta_q(k_{\gamma}^{\pm 1}) = k_{\gamma}^{\pm 1} \otimes k_{\gamma}^{\pm 1}, \quad S_q(k_{\gamma}^{\pm 1}) = k_{\gamma}^{\mp 1},$$

$$\Delta_q(e_{\beta}) = e_{\beta} \otimes 1 + k_{\beta}^{-1} \otimes e_{\beta}, \quad S_q(e_{\beta}) = -k_{\beta} e_{\beta}, \quad (2.10)$$

$$\Delta_q(e_{-\beta}) = e_{-\beta} \otimes k_{\beta} + 1 \otimes e_{-\beta}, \quad S_q(e_{-\beta}) = -e_{-\beta} k_{\beta}^{-1},$$

where $\beta = \alpha, \delta - \alpha; \gamma = d, \beta$. It is not hard to verify by direct calculations for the defining relations (2.1)-(2.6) that the quantum affine (super)algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$ have the following simple involutive (anti)automorphisms:

(i) The non-graded antilinear antiinvolution or conjugation "$\ast$":

$$(q^{\pm 1})^\ast = q^{\mp 1}, \quad (k_{\gamma}^{\pm 1})^\ast = k_{\gamma}^{\mp 1},$$

$$e_{\beta}^\ast = e_{-\beta}, \quad e_{-\beta}^\ast = e_{\beta}. \quad (2.11)$$

$$(xy)^\ast = y^\ast x^\ast \text{ for } \forall x, y \in U_q(g)).$$

(ii) The graded antilinear antiinvolution or graded conjugation "$\dagger$":

$$(q^{\pm 1})^\dagger = q^{\mp 1}, \quad (k_{\gamma}^{\pm 1})^\dagger = k_{\gamma}^{\mp 1},$$

$$e_{\beta}^\dagger = (-1)^{\theta(\beta)} e_{-\beta}, \quad e_{-\beta}^\dagger = e_{\beta}. \quad (2.12)$$

$$(xy)^\dagger = (-1)^{\deg x \deg y} y^\dagger x^\dagger \text{ for any homogeneous elements } x, y \in U_q(g)).$$

(iii) The Chevalley graded involution $\omega$:

$$\omega(q^{\pm 1}) = q^{\mp 1}, \quad \omega(k_{\gamma}^{\pm 1}) = k_{\gamma}^{\mp 1},$$

$$\omega(e_{\beta}) = -e_{-\beta}, \quad \omega(e_{-\beta}) = -(-1)^{\theta(\beta)} e_{\beta}. \quad (2.13)$$
(iv) The Dynkin involution $\tau$ which is associated with the automorphism of the Dynkin diagrams of the (super)algebras $A_1^{(1)}$ and $C(2)^{(2)}$:

$$
\begin{align*}
\tau(q^{\pm1}) &= q^{\pm1}, & \tau(k_{\beta}^{\pm1}) &= k_{\beta}^{\pm1}, \\
\tau(k_{\gamma}^{\pm1}) &= k_{\gamma}^{\pm1}, & \tau(k_{\delta-\beta}^{\pm1}) &= k_{\delta-\beta}^{\pm1}, \\
\tau(e_{\beta}) &= e_{\delta-\beta}, & \tau(e_{-\beta}) &= e_{-\delta+\beta}.
\end{align*}
$$

(2.14)

Here in (2.11)-(2.14) $\beta = \alpha, \delta - \alpha; \gamma = d, \beta$.

It should be noted that the graded conjugation $"\dagger"$ and the Chevalley graded involution $\omega$ are involutive (anti)automorphism of the fourth order, i.e., for example, $(\omega)^4 = \text{id}$. Note also that the Dynkin involution $\tau$ commutes with all other three involutions, i.e. $\tau(x^*) = (\tau(x))^*, \tau(x^\dagger) = (\tau(x))^\dagger$ and $\omega \tau(x) = \tau \omega(x)$ for any element $x \in U_q(g)$ ($g = A_1^{(1)}, C(2,0)^{(2)}$).

In the next Section we construct the Cartan-Weyl basis and describe its properties in detail.

3 Cartan-Weyl Basis for $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$

A general scheme for construction of a Cartan-Weyl basis for quantized Lie algebras and superalgebras was proposed in Ref. [11]. The scheme was applied in detail at first for quantized finite-dimensional Lie (super)algebras [5] and then to quantized untwisted affine algebras [13]. This procedure is based on a notion of “normal ordering” for the reduced positive root system. For affine Lie (super)algebras this notation was introduced in Ref. [12] (see also Refs. [11, 6, 7]).

In our case the reduced positive system has only two normal orderings:

$$
\begin{align*}
\alpha, \delta + \alpha, 2\delta + \alpha, \ldots, \infty \delta + \alpha, \delta, 2\delta, 3\delta, \ldots, \infty \delta, \infty \delta - \alpha, \ldots, 2\delta - \alpha, \delta - \alpha, \\
\delta - \alpha, 2\delta - \alpha, \ldots, \infty \delta - \alpha, \delta, 2\delta, 3\delta, \ldots, \infty \delta, \infty \delta + \alpha, \ldots, 2\delta + \alpha, \delta + \alpha, \alpha.
\end{align*}
$$

(3.1) \hspace{2cm} (3.2)

The first normal ordering (3.1) corresponds to “clockwise” ordering for positive roots in Fig. 1, 2b if we start from root $\alpha$ to root $\delta - \alpha$. The inverse normal ordering (3.2) corresponds to “anticlockwise” ordering for the positive roots when we move from $\delta - \alpha$ to $\alpha$. In accordance with the normal ordering (3.1) we set
\[ e_\delta := [e_\alpha, e_{\delta - \alpha}]_q, \quad \quad e_{-\delta} := [e_{-\delta + \alpha}, e_{-\alpha}]q^{-1}, \quad (3.3) \]

\[ e_{n\delta + \alpha} := \frac{1}{a} [e_{(n-1)\delta + \alpha}, e_\delta], \quad e_{-n\delta - \alpha} := \frac{1}{a} [e_{-\delta}, e_{-(n-1)\delta - \alpha}], \quad (3.4) \]

\[ e_{(n+1)\delta - \alpha} := \frac{1}{a} [e_\delta, e_{n\delta - \alpha}], \quad e_{-(n+1)\delta + \alpha} := \frac{1}{a} [e_{-\delta + \alpha}, e_\delta], \quad (3.5) \]

\[ e'_n := [e_\alpha, e_{n\delta - \alpha}]_q, \quad e'_{-n} := [e_{-n\delta + \alpha}, e_{-\alpha}]q^{-1}, \quad (3.6) \]

where \( n = 1, 2, \ldots \), and \( a \) is given by the formula:

\[ a := [(\alpha, \alpha)]_q = \frac{q^{(\alpha, \alpha)} - q^{-(\alpha, \alpha)}}{q - q^{-1}}. \quad (3.7) \]

Analogously for the inverse normal ordering (3.2) we set

\[ \tilde{e}_\delta := [e_{-\delta - \alpha}, e_\alpha]_q, \quad \tilde{e}_{-\delta} := [e_{-\alpha}, e_{-\delta + \alpha}]q^{-1}, \quad (3.8) \]

\[ \tilde{e}_{(n+1)\delta - \alpha} := \frac{1}{a} [e_{n\delta - \alpha}, \tilde{e}_\delta], \quad \tilde{e}_{-(n+1)\delta + \alpha} := \frac{1}{a} [\tilde{e}_{-\delta}, \tilde{e}_{-n\delta + \alpha}], \quad (3.9) \]

\[ \tilde{e}_{n\delta + \alpha} := \frac{1}{a} [\tilde{e}_\delta, \tilde{e}_{(n-1)\delta + \alpha}], \quad \tilde{e}_{-n\delta - \alpha} := \frac{1}{a} [\tilde{e}_{-\delta + \alpha}, \tilde{e}_{-\delta}], \quad (3.10) \]

\[ \tilde{e}'_n := [e_{-\alpha}, \tilde{e}_{(n-1)\delta + \alpha}]_q, \quad \tilde{e}'_{-n} := [e_{-\delta + \alpha}, \tilde{e}_{-(n-1)\delta - \alpha}]q^{-1}, \quad (3.11) \]

where \( n = 1, 2, \ldots \). Thus, we have two systems of the Cartan-Weyl generators: ’direct’ and ’inverse’. Each such system together with the Cartan generators \( k_\alpha^{\pm 1}, k_\delta^{\pm 1} \) \( e_{\pm \alpha} \) and \( e_{\pm (\delta - \alpha)} \) are called the q-analog of the Cartan-Weyl basis (or simply the Cartan-Weyl basis) for the quantum (super)algebras \( U_q(A_1^{(1)}) \) and \( U_q(C(2)^{(2)}) \).

Now we consider some properties of these bases. First of all, the explicit construction of the Cartan-Weyl generators (3.3)-(3.6) (or (3.8)-(3.11)) permits easily to find their properties with respect to the (anti)involutions (2.11)-(2.13). For example, it is evident that

\[ (e_{\pm \gamma})^* = e_{\mp \gamma}, \quad \forall \gamma \in \Delta_+. \quad (3.12) \]

Further, it is easy to see that the ’direct’ and ’inverse’ Cartan-Weyl generators (3.3)-(3.6) and (3.8)-(3.11) have very simple connection with the Dynkin
involution $\tau$:
\[
\begin{align*}
\tau(e_{n\delta+\alpha}) &= e_{(n+1)\delta - \alpha}, & \tau(\bar{e}_{n\delta+\alpha}) &= e_{(n+1)\delta - \alpha} \quad (n \in \mathbb{Z}), \\
\tau(e_{n\delta-\alpha}) &= e_{(n-1)\delta + \alpha}, & \tau(\bar{e}_{n\delta-\alpha}) &= e_{(n-1)\delta + \alpha} \quad (n \in \mathbb{Z}), \\
\tau(e_{n\delta}) &= e_{n\delta}, & \tau(\bar{e}_{n\delta}) &= e_{n\delta} \quad (n \neq 0).
\end{align*}
\]

Proposition 3.1 The root vectors (3.13)-(3.17) satisfy the following permutation relations:
\[
\begin{align*}
k_d e_{n\delta \pm \alpha} k_d^{-1} &= q^{n(d,\delta)} e_{n\delta \pm \alpha}, & k_d' e_{n\delta} k_d^{-1} &= q^{n(d,\delta)} e_{n\delta}, \\
k_\gamma e_{n\delta \pm \alpha} k_\gamma &= q^{+\gamma} e_{n\delta \pm \alpha}, & k_\gamma' e_{n\delta} k_\gamma^{-1} &= e_{n\delta}
\end{align*}
\]
for any $n \in \mathbb{Z}$ and any $\gamma \in \Delta_+$, and also
\[
\begin{align*}
[e_{n\delta+\alpha}, e_{-n\delta-\alpha}] &= (-1)^{n(\delta(\alpha))} q^{n(d,\delta)} e_{n\delta \pm \alpha}, & (n \geq 0), \\
[e_{n\delta-\alpha}, e_{-n\delta+\alpha}] &= (-1)^{(n-1)\delta(\alpha)} q^{n(d,\delta)} e_{n\delta \pm \alpha}, & (n > 0), \\
[e_{n\delta+\alpha}, e_{(n+2m-1)\delta + \alpha}] &= (q_\alpha - 1) \sum_{l=1}^{m-1} q_\alpha^{-l} e_{(n+l)\delta + \alpha} e_{(n+2m-1-l)\delta + \alpha}, \\
[e_{n\delta+\alpha}, e_{(n+2m)\delta + \alpha}] &= (q_\alpha - 1)^{m-1} q_\alpha^{2} e_{(n+m)\delta + \alpha} \\
&\quad + (q_\alpha - 1) \sum_{l=1}^{m-1} q_\alpha^{-l} e_{(n+l)\delta + \alpha} e_{(n+2m-l)\delta + \alpha}
\end{align*}
\]
for any integers $n \geq 0$, $m > 0$;
\[
\begin{align*}
[e_{(n+2m-1)\delta - \alpha}, e_{n\delta - \alpha}] &= -(q_\alpha^{2} - 1) \sum_{l=1}^{m-1} q_\alpha^{-l} e_{(n+2m-1-l)\delta - \alpha} e_{(n+l)\delta - \alpha}, \\
[e_{(n+2m)\delta - \alpha}, e_{n\delta - \alpha}] &= -(q_\alpha - 1) q_\alpha^{-m+1} e_{(n+m)\delta - \alpha} \\
&\quad - (q_\alpha^{2} - 1) \sum_{l=1}^{m-1} q_\alpha^{-l} e_{(n+l)\delta - \alpha} e_{(n+2m-l)\delta - \alpha}
\end{align*}
\]
for any integers $n, m > 0$;

$$[e_{-n\delta + \alpha}, e_{(n+2m-1)\delta + \alpha}] = -(-1)^{(n+1)\theta(\alpha)}(q_\alpha^2 - 1)$$

$$\times \sum_{l=n}^{n+m-1} q_\alpha^{-l} k_{n\delta - \alpha} e_{(l-n)\delta - \alpha} e_{(n+2m-1-l)\delta + \alpha}$$

$$+ (q_\alpha^2 - 1) \sum_{l=1}^{n-1} (-1)^{(l+1)\theta(\alpha)} q_\alpha^{-l} k_{l\delta} e_{(-l+1)\delta + \alpha} e_{(n+2m-1-l)\delta + \alpha}$$

(3.21)

$$[e_{-n\delta + \alpha}, e_{(n+2m)\delta + \alpha}] = -(-1)^{(n+1)\theta(\alpha)}(q_\alpha^2 - 1)$$

$$\times \sum_{l=n}^{n+m-1} q_\alpha^{-l} k_{n\delta - \alpha} e_{(l-n)\delta - \alpha} e_{(n+2m-l)\delta + \alpha}$$

$$+ (q_\alpha^2 - 1) \sum_{l=1}^{n-1} (-1)^{(l+1)\theta(\alpha)} q_\alpha^{-l} k_{l\delta} e_{(-l+1)\delta + \alpha} e_{(n+2m-l)\delta + \alpha}$$

$$- (-1)^{(n+1)\theta(\alpha)}(q_\alpha - 1) q_\alpha^{-m+n+1} k_{m\delta - \alpha} e_{m\delta + \alpha}$$

(3.22)

for any integers $n, m > 0$;

$$[e_{(n+2m-1)\delta - \alpha}, e_{-n\delta - \alpha}] = (-1)^{(n+1)\theta(\alpha)}(q_\alpha^2 - 1)$$

$$\times \sum_{l=n+1}^{n+m-1} q_\alpha^{-l} e_{(n+2m-1-l)\delta + \alpha} e_{(l-n)\delta - \alpha} k_{n\delta + \alpha}^{-1}$$

$$- (q_\alpha^2 - 1) \sum_{l=1}^{n-1} (-1)^{(l+1)\theta(\alpha)} q_\alpha^{-l} e_{(n+2m-1-l)\delta + \alpha} e_{(-l+n+1)\delta - \alpha} k_{l\delta}^{-1}$$

(3.23)

$$[e_{(n+2m)\delta - \alpha}, e_{-n\delta - \alpha}] = (-1)^{(n+1)\theta(\alpha)}(q_\alpha^2 - 1)$$

$$\times \sum_{l=n}^{n+m-1} q_\alpha^{-l} e_{(n+2m-l)\delta + \alpha} e_{(l-n)\delta - \alpha} k_{n\delta + \alpha}^{-1}$$

$$- (q_\alpha^2 - 1) \sum_{l=1}^{n-1} (-1)^{(l+1)\theta(\alpha)} q_\alpha^{-l} e_{(n+2m-l)\delta + \alpha} e_{(-l+n+1)\delta - \alpha} k_{l\delta}^{-1}$$

$$+ (-1)^{(n+1)\theta(\alpha)}(q_\alpha - 1) q_\alpha^{-m+n+1} e_{m\delta - \alpha}^2 k_{m\delta + \alpha}^{-1}$$

(3.24)

for any integers $n \geq 0$, $m > 0$;

$$[e_{n\delta + \alpha}, e_{m\delta - \alpha}] = e_{(n+m)\delta}$$

$$n \geq 0, m > 0$$

(3.25)

$$[e_{n\delta + \alpha}, e_{-m\delta - \alpha}] = -(-1)^{(m+1)\theta(\alpha)} e_{(n-m)\delta} k_{m\delta + \alpha}^{-1}$$

$$n > m \geq 0$$

(3.26)

$$[e_{-m\delta + \alpha}, e_{n\delta - \alpha}] = -(-1)^{(m+1)\theta(\alpha)} k_{m\delta - \alpha} e_{(n-m)\delta}$$

$$n > m \geq 0$$

(3.27)

$$[e_{n\delta}, e_{m\delta}] = [e_{-n\delta}, e_{-m\delta}] = 0$$

$$n > 0, m > 0$$

(3.28)
\[ [e_{n\delta + \alpha}, e'_{m\delta}] = q_{\alpha}^{-m+1} a e_{(n+m)\delta + \alpha} + (q_{\alpha}^2 - 1) \sum_{l=1}^{m-1} q_{\alpha}^{-l} e_{(n+l)\delta + \alpha} e'_{(m-l)\delta} \] (3.29)

for any integers \( n \geq 0, m > 0; \)

\[ [e'_{m\delta}, e_{n\delta - \alpha}] = q_{\alpha}^{-m+1} a e_{(n+m)\delta - \alpha} + (q_{\alpha}^2 - 1) \sum_{l=1}^{m-1} q_{\alpha}^{-l'} e_{(m-l)\delta} e_{(n+l)\delta - \alpha} \] (3.30)

for any integers \( n, m > 0; \)

\[ [e_{-n\delta + \alpha}, e'_{m\delta}] = -(1)^{(n-1)\theta(\alpha)} q_{\alpha}^{-m+1} a k_{n\delta - \alpha} e_{(m-n)\delta + \alpha} \]
\[- (1)^{(n-1)\theta(\alpha)} (q_{\alpha}^2 - 1) k_{n\delta - \alpha} \sum_{l=n}^{m-1} q_{\alpha}^{-l} e_{(l-n)\delta + \alpha} e'_{(m-l)\delta} \] (3.31)

\[ + (q_{\alpha}^2 - 1) \sum_{l=1}^{n-1} (-1)^{l\theta(\alpha)} q_{\alpha}^{-l} k_{l\delta} e_{(n-l)\delta + \alpha} e'_{(m-l)\delta} \]

for any integers \( m \geq n > 0; \)

\[ [e_{-n\delta + \alpha}, e'_{m\delta}] = -(1)^{m\theta(\alpha)} q_{\alpha}^{-m+1} a k_{m\delta} e_{(-n+m)\delta + \alpha} \]
\[ + (q_{\alpha}^2 - 1) \sum_{l=1}^{m-1} (-1)^{l\theta(\alpha)} q_{\alpha}^{-l} k_{l\delta} e_{(-n+l)\delta + \alpha} e'_{(m-l)\delta} \] (3.32)

for any integers \( n > m > 0; \)

\[ [e_{m\delta} e_{-n\delta - \alpha}] = -(1)^{(n+1)\theta(\alpha)} q_{\alpha}^{-m+1} a e_{(m-n)\delta - \alpha} k_{n\delta + \alpha}^{-1} \]
\[ - (1)^{(n+1)\theta(\alpha)} (q_{\alpha}^2 - 1) \sum_{l=n+1}^{m-1} q_{\alpha}^{-l} e'_{(m-l)\delta} e_{(l-n)\delta - \alpha} k_{l\delta + \alpha}^{-1} \] (3.33)

\[ + (q_{\alpha}^2 - 1) \sum_{l=1}^{n} (-1)^{l\theta(\alpha)} q_{\alpha}^{-l} e'_{(m-l)\delta} e_{(-n+l)\delta - \alpha} k_{l\delta}^{-l} \]

for any integers \( m > n \geq 0; \)

\[ [e'_{m\delta} e_{-n\delta - \alpha}] = -(1)^{m\theta(\alpha)} q_{\alpha}^{-m+1} a e_{(-n+m)\delta - \alpha} k_{m\delta}^{-m} \]
\[ + (q_{\alpha}^2 - 1) \sum_{l=1}^{m-1} (-1)^{l\theta(\alpha)} q_{\alpha}^{-l} e'_{(m-l)\delta} e_{(-n+l)\delta - \alpha} k_{l\delta}^{-l} \] (3.34)

for any integers \( n \geq m > 0. \)
Here in the relations (3.17)-(3.34) and in what follows we denote $q_{\alpha} := (-1)^{\theta(\alpha)}q^{(\alpha, \alpha)}$. The imaginary root vectors $e'_{n\delta}$ do not satisfy the relations of the type (3.15) and therefore we introduce new imaginary roots vectors $e_{\pm n\delta}$ by the following (Schur) relations:

$$e'_{n\delta} = \sum_{p_1+2p_2+...+np_n=n} \frac{((-1)^{\theta(\alpha)}(q-q^{-1}))^{\sum p_i-1}}{p_1!...p_n!} e_{\delta}^{p_1}...e_{n\delta}^{p_n}. \quad (3.35)$$

In terms of generating functions

$$\mathcal{E}'(u) := (q-q^{-1}) \sum_{n \geq 1} e'_{n\delta} u^{-n}, \quad (3.36)$$

$$\mathcal{E}(u) = (q-q^{-1}) \sum_{n \geq 1} e_{n\delta} u^{-n} \quad (3.37)$$

the relation (3.33) may be rewritten in the form

$$\mathcal{E}'(u) = -1 + \exp \mathcal{E}(u) \quad (3.38)$$

or

$$\mathcal{E}(u) = \ln(1 + \mathcal{E}'(u)). \quad (3.39)$$

This provides a formula inverse to (3.33)

$$e_{n\delta} = \sum_{p_1+2p_2+...+np_n=n} \frac{((-1)^{\theta(\alpha)}(q^{-1}-q))^{\sum p_i-1}}{(\sum_{i=1}^{n} p_i-1)!} (e_{\delta}')^{p_1}... (e_{n\delta}')^{p_n}. \quad (3.40)$$

The new root vectors corresponding to negative roots are obtained by the Cartan conjugation ($^*$):

$$e_{-n\delta} = (e_{n\delta})^*. \quad (3.41)$$

**Proposition 3.2** The new root vectors $e_{\pm n\delta}$ satisfy the following commutation relations:

$$[e_{n\delta+a}, e_{m\delta}] = (-1)^{(m-1)\theta(\alpha)}a(m)e_{(n+m)\delta+a} \quad (n \geq 0, m > 0), \quad (3.42)$$

$$[e_{m\delta}, e_{n\delta-a}] = (-1)^{(m-1)\theta(\alpha)}a(m)e_{(n+m)\delta-a} \quad (n, m > 0). \quad (3.43)$$
Applying to these relations the Dynkin involution $\tau$ of the Cartan-Weyl bases corresponding to the 'direct' normal ordering from them by the conjugation describe complete list of the permutation relations of Propositions (3.1), (3.2) together with the ones obtained where

$$\Delta_R^{-1} = a(m) k_n^m e_{(m-n)\delta+\alpha}(n \geq m > 0), \quad (3.44)$$

$$\Delta_R^{m} = a(m) k_n^m e_{(m-n)\delta+\alpha}(n > m > 0), \quad (3.45)$$

$$\Delta_R^{m} = a(m) e_{(m-n)\delta+\alpha} k_n^m(\delta, \theta) \ (m > n \geq 0), \quad (3.46)$$

$$\Delta_R^{m} = a(m) e_{(m-n)\delta+\alpha} k_n^m(\delta, \theta) \ (n \geq m > 0), \quad (3.47)$$

$$\Delta_R^{m} = \delta_{mn} a(m) \frac{k_n^m - k_{n+m}^m}{q-q^{-1}} \ (n, m > 0), \quad (3.48)$$

where

$$a(m) := \frac{q^{m(n+1)} - q^{-m(n+1)}}{m(q-q^{-1})} \quad (3.49)$$

All the relations of Propositions (3.1), (3.2) together with the ones obtained from them by the conjugation describe complete list of the permutation relations of the Cartan-Weyl bases corresponding to the 'inverse' normal ordering (3.2). Applying these relations the Dynkin involution $\tau$, it is easy to obtain these results for the 'inverse' normal ordering (3.2).

## 4 Universal $R$-matrix for $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$

Any quantum (super)algebra $U_q(g)$ is a non-cocommutative Hopf (super)algebra which has the intertwining operator called the universal $R$-matrix. By definition [2], the universal $R$-matrix for the Hopf (super)algebra $U_q(g)$ is an invertible element $R$ of some extension $U_q(g) \otimes U_q(g)$, satisfying the equations

$$\Delta_R(a) = R \Delta_R(a) R^{-1} \quad \forall \ a \in U_q(g), \quad (4.1)$$

$$(\Delta R \otimes \text{id}) R = R^{13} R^{23}, \quad (\text{id} \otimes \Delta_R) R = R^{13} R^{12}, \quad (4.2)$$

where $\Delta_R$ is the opposite comultiplication: $\Delta_R^\alpha = \sigma \Delta_R^\alpha$, $\sigma(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a$ for all homogeneous elements $a, b \in U_q(g)$. In the relation (4.2) we use the standard notations $R^{12} = \sum a_i \otimes b_i \otimes \text{id}$, $R^{13} = \sum a_i \otimes \text{id} \otimes b_i$, $R^{23} = \sum \text{id} \otimes a_i \otimes b_i$ if $R$ has the form $R = \sum a_i \otimes b_i$. We employ the following standard notation for the $q$-exponential:

$$\exp_q(x) := 1 + x + \frac{x^2}{(2)_q^2} + \ldots + \frac{x^n}{(n)_q^n} + \ldots = \sum_{n \geq 0} \frac{x^n}{(n)_q^n}, \quad (4.3)$$
where

$$(n)_q := \frac{q^n - 1}{q - 1}.$$  \hfill (4.4)

An explicit expression of the universal $R$-matrix $R$ for our case $U_q(g)$ can be presented as follows:

$$R = R_+ R_0 R_- K.$$  \hfill (4.5)

Here the factors $K$ and $R_\pm$ have the following form

$$K = \frac{1}{q^{\langle \alpha, \alpha \rangle}} h_{\alpha} \otimes h_{\alpha} + h_\delta \otimes h_\delta + h_\delta \otimes h_\delta ,$$  \hfill (4.6)

$$R_+ = \prod_{n \geq 0} R_{n\delta + \alpha}, \quad R_- = \prod_{n \geq 1} R_{n\delta - \alpha}.$$  \hfill (4.7)

were the elements $R_\gamma$ are given by the formula

$$R_\gamma = \exp_{q_{\gamma}} \left(A(\gamma)(q - q^{-1})(e_\gamma \otimes e_{-\gamma})\right),$$  \hfill (4.8)

where

$$q_{\gamma} = (-1)^{\theta(\gamma)} q^{\langle \gamma, \gamma \rangle},$$  \hfill (4.9)

$$A(\gamma) = \begin{cases} (-1)^{n\theta(\alpha)} & \text{if } \gamma = n\delta + \alpha, \\ (-1)^{(n-1)\theta(\alpha)} & \text{if } \gamma = n\delta - \alpha. \end{cases}$$  \hfill (4.10)

Finally, the factor $R_0$ is defined as follows

$$R_0 = \exp \left((q - q^{-1}) \sum_{n > 0} d(n)e_{n\delta} \otimes e_{-n\delta}\right),$$  \hfill (4.11)

where $d(n)$ is the inverse to $a(n)$, i.e.

$$d(n) = \frac{n(q^{-1} - q)}{q^{n(\alpha, \alpha)} - q^{-n(\alpha, \alpha)}}.$$  \hfill (4.12)

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