The Stützfunktion and the Cut Function*

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Abstract

I review some standard theory of convex bodies in $\mathbb{R}^3$ and rephrase it in a formalism of Ted Newman to show the relation between the Stützfunktion of the former theory and the cut function introduced by Ted. This leads to a conjectured inequality for space-like two-spheres in Minkowski space that generalises Minkowski’s inequality and is implied by Penrose’s cosmic censorship hypothesis.

1 Introduction

The work described in this paper arises from a problem posed to me by Ted Newman during my first visit to the University of Pittsburgh in the mid-1970s. It turns out that this problem can be solved by Newman-style methods and that it leads on to making interesting connections with other areas of Ted’s work.

The problem is as follows: given a cut $\Sigma$ of the future-null infinity $I^+$ of Minkowski space $M$, how do you reconstruct a space-like 2-surface $S$ inside $M$ such that $\Sigma$ is the intersection with $I^+$ of $\dot{J}(S)$, the boundary of the future of $S$? This is related to a version of the “fuzzy point” idea which was current at that time: if $\Sigma$ is a cut of the $I^+$ of a non-flat but asymptotically-flat space-time $M$ arising from a point $p$ in $M$ (known then as a light-cone cut) then, when transferred to the $I^+$ of $\tilde{M}$, $\Sigma$ will determine a null hypersurface $N$ which does not converge to a point; however $N$ may nearly converge to a point and may determine small 2-surfaces $S$ which are nearly points, or are fuzzy points. If so, then by taking all possible $\Sigma$ for all possible $p$ in $M$, one might obtain a representation of the curved space-time $M$ as fuzzy points in the flat space-time $\tilde{M}$.

The plan of this paper is as follows:

In section 2 I discuss convex bodies in $\mathbb{R}^3$. The theory of convex bodies centres on the Stützfunktion or support-function, which I’ll anglicise as stutzfunction, and I review some of this theory.

In section 3, I turn to Minkowski space and identify the relation between the stutzfunction of a convex body and the cut-function which the boundary of the future of the body defines at $I^+$. This effectively solves the problem posed above, and it also illuminates some of the theory in section 2.

In section 4, I sketch some further developments of the theory of convexity for 2-surfaces in Minkowski space. These include an approach to an inequality found by Gibbons and Penrose [3], as a prediction of the cosmic censorship hypothesis.

It gives me great pleasure to dedicate this paper to Ted Newman in his 60th year, to acknowledge his long-standing and beneficial influence and to record my debt and gratitude to him.

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1Note the previous footnote.
2 Convex bodies in $\mathbb{R}^3$

In this section, I develop some standard theory of convex bodies in $\mathbb{R}^3$ following [1] and [3], but with the kind of formalism that I learned from Ted Newman.

We may define a convex body $B$ in $\mathbb{R}^3$ to be a closed body such that, if $p, q$ are two points of $B$ then the line segment $tp + (1 - t)q$ for $0 \leq t \leq 1$ lies entirely in $B$. Then a convex surface $S$ is the surface of a convex body.

We define the Gauss map in the familiar way: choose an orthonormal triad and parametrise a unit vector $\ell$ by spherical polars as

$$\ell = \ell(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),$$

(1)

in the triad. Thus $\ell$ corresponds to a point on the unit sphere $S^2$. Given a choice of $\ell$, that is a choice of $(\theta, \phi)$, take the plane with normal $\ell$ that is tangent to the convex surface $S$, with $\ell$ the outward normal. If this happens at $p \in S$ then the Gauss map from $S$ to $S^2$ takes $p$ to the point labelled $(\theta, \phi)$ on the unit sphere. For a smooth, strictly convex body, the Gauss map is one-one as we shall see. In that case we have introduced coordinates $(\theta, \phi)$ on $S$. The tangent plane to $S$ at $p$ has the equation

$$x \cdot \ell(\theta, \phi) = H(\theta, \phi),$$

(2)

where $H(\theta, \phi)$ is the perpendicular distance from the origin (which we’ll assume to be inside $S$) to the tangent plane. A knowledge of $H(\theta, \phi)$ determines $S$ as an envelope of tangent planes and $H$ is the Stützfunktion [1] or support function, which we’ll call the stutzfunction.

To obtain a parametric expression for the surface $S$ we can solve (2) and its derivatives for $(x, y, z)$. To this end, we introduce the Newman-Penrose operator ‘eth’ [12] defined on a spin-weight $s$ function $\eta$ by

$$\partial \eta := \frac{1}{\sqrt{2}}(\sin \theta)^s \left( \frac{\partial}{\partial \theta} + i \frac{\partial}{\sin \theta \partial \phi} \right) (\sin \theta)^{-s} \eta,$$

and define

$$m = \partial \ell = \frac{1}{\sqrt{2}}(\cos \theta \cos \phi - i \sin \phi, \cos \theta \sin \phi + i \cos \phi, - \sin \theta)$$

$$\overline{m} = \overline{\partial \ell} = \frac{1}{\sqrt{2}}(\cos \theta \cos \phi + i \sin \phi, \cos \theta \sin \phi - i \cos \phi, - \sin \theta).$$

These have $s = 1, -1$ respectively and, by differentiating again,

$$\partial m = 0 = \overline{\partial m}, \quad \overline{\partial m} = \partial \overline{m} = -\ell.$$

The positive-definite metric of $\mathbb{R}^3$ can be written

$$\delta = \ell \ell + mm + \overline{m} \overline{m}.$$

(3)

Note also, as usual, that

$$(\partial \overline{\partial} - \overline{\partial} \partial) \eta = -s \eta,$$

(4)

when $\eta$ has spin-weight $s$.

To obtain the convex surface parametrically we must solve (2) simultaneously with

$$x \cdot m = \partial H, \quad x \cdot \overline{m} = \overline{\partial H},$$

which with the aid of (3) can be solved to give

$$x = x(\theta, \phi) = H \ell + \overline{\partial H} m + \partial H \overline{m}.$$
This gives an explicit parametrisation of $S$. Using the standard theory of surfaces in $\mathbb{R}^3$ (see e.g. [2]) we find the area element of $S$ to be

$$dA = \left((H + \partial^2 H)^2 - \partial^2 H \partial^2 H \right) \sin \theta d\theta d\phi.$$  \hspace{1cm} (6)

By general theory, the Jacobian of the Gauss map is the Gauss curvature $k$ so that

$$k^{-1} = R_1 R_2 = (H + \partial^2 H)^2 - \partial^2 H \partial^2 H$$  \hspace{1cm} (7)

in terms of principal radii of curvature $R_1, R_2$. A similar calculation gives the mean curvature $h$:

$$h = \frac{1}{2}(R_1^{-1} + R_2^{-1}) = (H + \partial^2 H)k,$$

so that

$$hdA = (H + \partial^2 H) \sin \theta d\theta d\phi.$$  \hspace{1cm} (8)

For use below note that then

$$\int_S hdA = \int_{S^2} H \sin \theta d\theta d\phi,$$

since $\partial^2 \partial$ is a constant multiple of the 2-sphere Laplacian, so the second term integrates to zero.

For strict convexity we require $R_1, R_2 > 0$ which is equivalent to $h, k > 0$. Since necessarily $h^2 \geq k$ it is sufficient to require $k > 0$ ($h$ will be positive somewhere on $S$ since we are using the outward normal). Finally, since this is the Jacobian of the Gauss map, the Gauss map is one-one and onto precisely for smooth, strictly convex surfaces. This imposes restrictions on $H$ which we shall discuss. First we note how simple the Gauss-Bonnet theorem is in this context:

**Proposition 2.1 The Gauss-Bonnet Theorem**

For a strictly convex surface $S$

$$\int_S kdA = 4\pi.$$  \hspace{1cm} (9)

**Proof**

From (6) and (7), $kdA = \sin \theta d\theta d\phi$.

Now what conditions do we require $H$ to satisfy for it to be the stutzfunction of a smooth, strictly convex surface? We need $H$ positive and in (7) we want the right-hand-side to be positive. If it fails to be positive then the surface enveloped by $H$ will have cusps. However if $k$ from (7) is not positive then it can be made positive by adding a positive constant to $H$.

The process of adding a positive constant to $H$ is an interesting transformation that changes a convex surface $S$ into another, $S'$, which is parallel to it in the sense of [15]. In that reference, the idea is motivated by imagining rolling a sphere $K$ of constant radius $s$ over the surface $S$. The locus of the centre of $K$ defines $S'$. Equivalently one moves the centre of $K$ over $S$ and takes $S'$ to be envelope swept out by $K$. In this second form, one sees a connection with the idea of Huyghens’ secondary wavelets which will reappear in section 3.

Given a convex surface $S$, one can consider a sequence of surfaces parallel to $S$ with larger and larger separations $s$. In this way one arrives at the following string of theorems.

**Proposition 2.2 Steiner’s Theorem**

Along such a sequence, the area is given by

$$A(s) = A + 2sM + 4\pi s^2,$$  \hspace{1cm} (9)

while the volume contained is

$$V(s) = V + sA + s^2 M + \frac{4}{3} \pi s^3.$$  \hspace{1cm} (10)
Here $A$ and $V$ are the area and volume of $S$ and $M$ is the integral of mean curvature:

$$M = \int_S h dA.$$  \hfill (11)

**Proof**

Clearly the surface parallel to $S$ at distance $s$ has stutzfunction $H + s$. Substitute into (6) and expand in powers of $s$ to obtain (9) (using (8) along the way). Integrate (9) to obtain (10).

Along a sequence of parallel surfaces, the surfaces should become “rounder”. This intuitive feeling is made precise in the following (which I won’t prove):

**Proposition 2.3 The Brunn-Minkowski Theorem**

Define $R(s) = (V(s))^{1/3}$ then $R$ is convex in $s$, in that $\frac{d^2 R}{ds^2} \leq 0$.

Take this to be true and calculate the derivative, then for positive $s$:

$$(6MV - 2A^2) + 2s(12\pi V - AM) + 2s^2(4\pi A - M^2) \leq 0.$$  \hfill (12)

From this we may deduce Minkowski’s inequality

$$M^2 \geq 4\pi A,$$  \hfill (13)

as well as the isoperimetric inequality

$$36\pi V^2 \leq A^3.$$  \hfill (14)

Although (13) follows from (12) there is a straightforward direct proof due to Blaschke and using the stutzfunction (see (1)):

**Blaschke’s proof of Minkowski’s Inequality**

From (6), (11) and integration by parts

$$A = \int (H^2 - \partial H \bar{\partial} H) \sin \theta d\theta d\phi$$

while from (8)

$$M = \int H \sin \theta d\theta d\phi.$$  

Set $H = H_0 + H_1$ where $\int H_1 \sin \theta d\theta d\phi = 0$ and $H_0$ is constant then

$$M = 4\pi H_0, \quad A = 4\pi H_0^2 + \int (H_1^2 - \partial H_1 \bar{\partial} H_1) \sin \theta d\theta d\phi.$$  

It follows by expanding $H_1$ in spherical harmonics that the integral contribution to $A$ is strictly negative unless $H_1$ is a combination of $\ell = 1$ spherical harmonics. This case corresponds to a sphere with a translated origin, so that (13) is proved, with equality only for a round sphere.

To conclude this section, I shall record another way of obtaining the surface $S$ from the stutzfunction $H$. Define

$$\hat{H}(r, \theta, \phi) = rH(\theta, \phi) = F(\hat{x}, \hat{y}, \hat{z}),$$

where $\hat{x}, \hat{y}, \hat{z}$ are expressed in terms of spherical polar coordinates $r, \theta, \phi$ in the usual way. Then (5) is equivalent to the parametrisation given by

$$x = \frac{\partial F}{\partial \hat{x}}, \quad y = \frac{\partial F}{\partial \hat{y}}, \quad z = \frac{\partial F}{\partial \hat{z}},$$

where, after the differentiation, $\hat{x}, \hat{y}, \hat{z}$ are again eliminated.
3 Stutzfunction and cut function

In Minkowski space $\mathbb{M}$ we introduce the null tetrad

$$\ell^a = (1, \ell), \quad m^a = \partial \ell^a = (0, m), \quad m^a = (0, \overline{m}), \quad n^a = \frac{1}{2}(1, -\ell),$$

with $\ell, m, \partial$ as in section 2. The Minkowski metric can then be written

$$\eta^{ab} = 2\ell^a\ell^b - 2m^a\overline{m}^b.$$ 

Note that

$$\overline{m}^a = (0, -\ell) = -\frac{1}{2}\ell^a + n^a.$$ 

Define the unit time-like vector

$$t^a = (1, 0)$$

so that also $\eta_{ab}t^at^b = 1$, and introduce advanced null polar coordinates $(u, r, \theta, \phi)$ by

$$x^a = ut^a + r\ell^a(\theta, \phi)$$

(see e.g. [7]). Then $(u, \theta, \phi)$ are coordinates on $\mathcal{I}^+$ which is located at $r = \infty$.

A cut of $\mathcal{I}^+$ is defined by a function $u = V(\theta, \phi)$ where $V$ can conveniently be called the cut function for the cut. If we choose an arbitrary point $p$ with coordinates $x_0^a$ and a null-vector $\ell^a(\theta_0, \phi_0)$ at $p$ then the null geodesic from $p$ in the direction of $\ell^a(\theta_0, \phi_0)$ meets $\mathcal{I}^+$ at

$$u = x_0^a\ell_a(\theta_0, \phi_0), \quad \theta = \theta_0, \quad \phi = \phi_0.$$ 

(16)

Now suppose we are given a convex surface $S$ in the form

$$x^a = (0, x(\theta, \phi))$$

with $x(\theta, \phi)$ determined by a stutzfunction $H$ according to (5). The boundary of the future of $S$, $\mathcal{J}(S)$, is ruled by null geodesics that meet $S$ orthogonally. From the definitions so far made, the outward null normal at the point $p$ of $S$ labelled by $(\theta_0, \phi_0)$ is $\ell^a(\theta_0, \phi_0)$ and the null geodesic from $p$ in this direction meets $\mathcal{I}^+$ at $u$ given by (16). As $p$ runs over $S$, the cut $\Sigma = \mathcal{J}(S) \cap \mathcal{I}^+$ is generated with the cut function

$$u = x^a\ell_a(\theta, \phi) = -x \cdot \ell(\theta, \phi).$$ 

(17)

Comparing (17) with (2) we conclude that the cut function is minus the stutzfunction.

Conversely, if we are given the cut $\Sigma$ and its cut function $V$ then (17) determines a null hypersurface $\mathcal{N}$

$$x^a\ell_a(\theta, \phi) = V(\theta, \phi),$$

which, with its angular derivatives, we can solve for $\mathcal{N}$ parametrically as

$$x^a = x^a(\lambda, \theta, \phi) = -Hn^a + \partial Hm^a + \overline{\partial Hm}^a + \lambda \ell^a$$

for arbitrary real $\lambda$. Now intersecting $\mathcal{N}$ with hypersurfaces of constant $t$, i.e. hypersurfaces orthogonal to $t^a$ in (15), gives a sequence of parallel surfaces in the sense of section 2. This shows how the parallel-surface idea is related to Huyghen’s secondary wavelets: if a convex surface is
momentarily lit up then the resulting expanding (out-going) wave-front traces out a sequence of surfaces parallel to the first.

Of course this converse is incomplete in the following sense: if what we are given is just the cut function then we can define the null surface $N$ but we cannot fix a unique value of $\lambda$ to represent the 2-surface $S$ without some extra input. If we are trying to make precise the fuzzy-point idea then we might want to pick out an instant of minimum volume or of best focus. This could also involve boosting the cut function or considering a different set of constant-time hypersurfaces.

4 Further developments

In this last section I describe some attempts to carry over other parts of the theory of convex 2-surfaces into Minkowski space. I shall work with the GHP formalism \[4\] and omit proofs.

A space-like 2-surface $S$ in $\mathbb{M}$ defines a pair of future-pointing null normals $\ell^a, n^a$ (where we shall take $\ell^a$ to be the outward normal and $n^a$ the inward normal) or equivalently a normalised spinor dyad $(\alpha^A, \beta^A)$. The second fundamental form of $S$ is coded by the GHP formalism into weighted scalars $(\rho, \rho', \sigma, \sigma')$ (see e.g. \[16\] for an account of this). In terms of these, I shall say that $S$ is

- future convex iff $\rho < 0$, $\rho^2 - \sigma \sigma' > 0$
- past convex iff $\rho' > 0$, $\rho'^2 - \sigma' \sigma' > 0$.

We recall that the Gauss curvature of $S$ is twice the real part of the complex curvature $Q = -\rho \rho' + \sigma \sigma' \[14\].

**Proposition 4.1** If $S$ is future and past convex then the Gauss curvature of $S$ is everywhere positive.

**Proof:** This is elementary since

$$k = -2\rho \rho' + \sigma \sigma' + \overline{\sigma} \overline{\sigma'} \geq -2\rho \rho' - 2|\sigma \sigma'|$$

and

$$(-\rho \rho' - |\sigma \sigma'|)^2 = (\rho^2 - \overline{\sigma} \sigma)(\rho'^2 - \sigma' \overline{\sigma'}) + (\rho |\sigma'| + \rho' |\sigma|)^2.$$ 

The quantities occurring in the above definitions of convexity arise in the various Gauss maps that can be defined for $S$. If we choose and fix a constant normalised spinor dyad $(\alpha^A, \beta^A)$ then we can define a future Gauss map by

$$f : S \to \mathbb{CP}^1; \quad p \mapsto \zeta = \frac{o_A \alpha^A}{o_A \beta^A}$$

and a past Gauss map by

$$f : S \to \mathbb{CP}^1; \quad p \mapsto \eta = \frac{\nu_A \alpha^A}{\nu_A \beta^A}.$$

Equivalently, these express $o^A$ and $\nu^A$ in terms of $(\alpha^A, \beta^A)$ as

$$o^A = \lambda \frac{(\alpha^A + \zeta \beta^A)}{(1 + \zeta \overline{\zeta})^{1/2}}, \quad \nu^A = \mu \frac{(\alpha^A + \eta \beta^A)}{(1 + \eta \overline{\eta})^{1/2}},$$

where $\lambda, \mu$ are not fixed by the Gauss maps, but note that

$$t^a \ell_a = \frac{1}{\sqrt{2}} \lambda \overline{\lambda}, \quad t^a n_a = \frac{1}{\sqrt{2}} \mu \overline{\mu},$$
where \( t^{AA'} = \frac{1}{\sqrt{2}} (\alpha^A \alpha^{A'} + \beta^A \beta^{A'}) \), so that \( t^a \) is a unit time-like vector determined by the chosen tetrad. (These Gauss maps are similar to but different from those defined by Kossowski [8].)

There is a third Gauss map, conveniently called the complex Gauss map which can be defined by

\[
2o^{(A, B)} = \zeta \alpha^A \alpha^B + 2\eta \alpha^A \beta^B + \xi \beta^A \beta^B
\]

where this \( \zeta, \eta \) are to be distinguished from the previous. This maps \( S \) to the complex quadric \( Q \) defined by

\[
\zeta \xi - \eta^2 = 1
\]
in \( \mathbb{C}^3 \) (this Gauss map has also been considered by Roger Penrose).

The images of the future and past Gauss maps carry volume forms \( 4d\zeta d\zeta (1 + \zeta \zeta)^{-2} \) and \( 4d\eta d\eta (1 + \eta \eta)^{-2} \) while \( Q \) admits the holomorphic 2-form

\[
\frac{d\zeta \wedge d\xi}{\eta} = 2 \frac{d\zeta \wedge d\eta}{\zeta} = 2 \frac{d\eta \wedge d\xi}{\xi},
\]

so that we can calculate the Jacobians for the Gauss maps as in section 2.

**Proposition 4.2** For the future Gauss map we find

\[
\frac{4d\zeta d\zeta}{(1 + \zeta \zeta)^2} = \frac{1}{(t^a \ell_a)^2} (\rho^2 - \sigma \sigma) dA,
\]

for the past Gauss map

\[
\frac{4d\eta d\eta}{(1 + \eta \eta)^2} = \frac{1}{(t^a n_a)^2} (\rho'\rho' - \sigma' \sigma') dA,
\]

and for the complex Gauss map

\[
\frac{d\zeta \wedge d\xi}{\eta} = (-\rho \rho' + \sigma \sigma') dA.
\]

As in section 2, we integrate these expressions over \( S \):

**Proposition 4.3** Generalised Gauss-Bonnet Theorem

For the three cases treated above integration gives:

\[
\int_S \frac{1}{(t^a \ell_a)^2} (\rho^2 - \sigma \sigma) dA = \int_S \frac{1}{(t^a n_a)^2} (\rho'\rho' - \sigma' \sigma') dA = \frac{4\pi}{t^a t_a}
\]

\[
\int_S (-\rho \rho' + \sigma \sigma') dA = 2\pi.
\]

In (18) I have included the term \( t^a t_a \) explicitly both to give a slightly more general formula (valid when \( t^a \) is any constant time-like vector) and to point up the resemblance to Newman’s expression for the \( H \)-space metric [10].

Next we turn to consideration of possible generalisations of the notion of parallel bodies and Propositions 2.2-2.4. For this we need to write down and solve the Sachs equations which are the NP spin-coefficient equations for the evolution of \( \rho, \sigma \) and \( \rho', \sigma' \) [11]. Given \( S \) we consider the null hypersurface \( \mathcal{N} \) generated by the outgoing null normals to \( S \). We scale \( \ell^a \) to be affinely parametrised:

\[
D\ell^a \equiv \ell^b \nabla_b \ell^a = 0,
\]

and choose an affine parameter \( s \) with

\[
Ds = 1, \quad s = 0 \text{ at } S.
\]
Then the Sachs equations are

\[ D\rho = \rho^2 + \sigma\sigma, \quad D\sigma = 2\rho\sigma, \]

while the area element is carried along \( \ell^a \) according to

\[ D(dA) = -2\rho dA. \]

Then the Sachs equations can be solved explicitly as

\[ \rho(s) = \Delta^{-1}(\rho_0 - s(\rho_0^2 - \sigma_0\sigma_0)), \quad \sigma(s) = \Delta^{-1}\sigma_0 \]

with

\[ \Delta = 1 - 2s\rho_0 + s^2(\rho_0^2 - \sigma_0\sigma_0) \]

and \( \rho_0, \sigma_0 \) are the values at \( S \). For the area element we similarly find

\[ dA(s) = \Delta dA_0. \]

We deduce at once the following proposition:

**Proposition 4.4** For a future convex surface, the outgoing null hypersurface encounters no caustics to the future.

**Proof:** Caustics to the future are signalled by singularities in \( \rho(s) \), or equivalently zeroes in \( \Delta \), for positive \( s \) but from the definition of future convex \( \Delta \) is positive definite in this range.

There is a corresponding statement for past convex.

For the analogue of Proposition 2.2, Steiner’s Theorem, we need to integrate \( dA(s) \). However there is a problem of weights in the GHP sense: at this point we have the freedom to rescale \( \ell^a \) at \( S \) by

\[ \ell^a \rightarrow \Omega(\theta, \phi)\ell^a, \]

and under this transformation

\[ s \rightarrow \Omega^{-1}s, \quad \rho_0 \rightarrow \Omega\rho_0, \quad \sigma_0 \rightarrow \Omega\sigma_0, \]

so that we would not get a formula like \( [9] \) by simply integrating \( dA(s) \). The simplest way to resolve this difficulty is to choose a constant unit time-like vector \( t^a \) and define

\[ \hat{s} = st^a\ell_a, \]

as \( \hat{s} \) is unchanged by \( [20] \) and now we can integrate \( dA(s) \):

**Proposition 4.5** Generalised Steiner Theorem

\[ A(\hat{s}) = A_0 + 2\hat{s}\hat{M} + 4\pi\hat{s}^2, \]

where

\[ \hat{M} = -\int_S \frac{\rho_0}{t^a\ell_a} dA \]

and we have used Proposition 4.3.
Now we might hope to prove a counterpart of the Brunn-Minkowski Theorem, Proposition 2.3, or of Minkowski’s Inequality, Proposition 2.4. However this is impossible since with $\hat{M}$ as in (21) examples can be found to show that it is not the case that $\hat{M}^2 \geq 4\pi A$. In fact the Isoperimetric Inequality (14) can also be violated in Minkowski space in the following sense: given a space-like 2-surface $S$ one may be able to find space-like 3-surfaces spanning $S$ on which the volume $V$ enclosed by $S$ and the area $A$ of $S$ have

$$36\pi V^2 > A^3.$$ 

The correct inequality to generalise (13) would seem to be one proposed by Gibbons and Penrose in an investigation of Cosmic Censorship, [13, 5]. This may be phrased as follows: consider the vector

$$P^a = \frac{1}{2} \int_S (\rho' \ell^a - \rho n^a) dA = \int_S p^a dA,$$  

(22)

with $p^a = \frac{1}{2}(\rho' \ell^a - \rho n^a)$. If $S$ is future convex with $\rho' > 0$ as well, or past convex with $\rho < 0$ as well, then one conjectures the inequality:

$$P_a P^a \geq 2\pi A,$$  

(23)

where $A$ is the area of $S$. (This is not the form in which the inequality is stated by Gibbons and Penrose but I believe it to be equivalent.)

Note that the mean curvature vector $H_a$ of $S$, equivalently the trace of the second fundamental form, is

$$H_a = \frac{1}{2}(\rho' \ell_a + \rho n_a).$$  

(24)

The significance of $H_a$ is that, given a vector field $X^a$ on $S$, the rate of change of the area of $S$ under displacement along $X^a$ is

$$\dot{A} = \int_S H_a X^a dA.$$ 

We see that the vector $p^a$ in (22) lies in the normal 2-plane to $S$ and is orthogonal to $H_a$ so it defines the direction in which $dA$ does not change (to first order). Also, by taking $X^a$ to be a constant translation, under which the area will not change, it is clear that

$$\int_S H_a dA = 0$$

and so

$$\int_S \rho' \ell^a dA = - \int_S \rho n^a dA.$$

Thus we can write $P^a$ as

$$P^a = \int_S (-\rho)n^a dA = \int_S \rho' \ell^a dA.$$ 

In this form it is clear that $P^a$ is time-like and future pointing for past or future convex $S$.

As partial confirmation of (23) we note that if $S$ lies in a flat space-like 3-surface with unit time-like normal $t^a$ then

$$P^a = \frac{1}{\sqrt{2}} M t^a,$$

with $M$ as in (11). Thus in this case (23) is Minkowski’s inequality. Further, if $S$ lies in an in- or out-going null cone then (23) can be established directly, as it reduces to an inequality for functions on the unit sphere, [13, 5], which can be proved [16, 17]. Finally one can verify (23) for surfaces infinitesimally close to a round sphere in a flat hyperplane. What is still lacking is a proof of (23) in full generality, subject only to the conditions of convexity.

\footnote{The claimed proof in [6] is defective: see e.g. [9].}
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