Estimates for parameters and characteristics of the confining SU(3)-gluonic field in charged pions and kaons from leptonic decays and chiral symmetry breaking

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Abstract

The confinement mechanism proposed earlier by the author is employed for to compute the decay constants \( f_P \) corresponding to leptonic decays \( P \rightarrow l^\pm + \nu_l, l = \mu, e \), where \( P \) stands for any meson from \( \pi^\pm, K^\pm \). For this aim the weak axial form factor of \( P \)-meson is nonperturbatively calculated. The study entails estimates for parameters of the confining SU(3)-gluonic field in charged pions and kaons. The corresponding estimates of the gluon concentrations, electric and magnetic colour field strengths are also adduced for the mentioned field at the scales of the mesons under consideration. Further the obtained results are applied to the problem of chiral symmetry breaking in quantum chromodynamics (QCD). It is shown that in chirally symmetric world masses of pions and kaons are fully determined by the confining SU(3)-gluonic field among (massless) \( u, d \) and \( s \) quarks and not equal to zero. Accordingly chiral symmetry is sufficiently rough approximate one holding true only when neglecting the mentioned SU(3)-gluonic field between quarks and no additional mechanism of the spontaneous chiral symmetry breaking connected to the so-called Goldstone bosons is required. Finally, a possible relation of the results obtained with a phenomenological string-like picture of confinement is discussed too.

Key words: Quantum chromodynamics; Confinement; Charged pions and kaons

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1 Introduction

In Refs. [1–3] for the Dirac-Yang-Mills system derived from QCD-Lagrangian an unique family of compatible nonperturbative solutions was found and explored, which could pretend to describing confinement of two quarks. The
applications of the family to the description of both the heavy quarkonia spectra \cite{4,5} and a number of properties of pions, kaons, $\eta$- and $\eta'$-mesons \cite{6-8} showed that the confinement mechanism is qualitatively the same for both light mesons and heavy quarkonia. At this moment it can be described in the following way.

Two main physical reasons underlie linear confinement in the mechanism under discussion. The first one is that gluon exchange between quarks is realized with the propagator different from the photon-like one and existence and form of such a propagator is a direct consequence of the unique confining nonperturbative solutions of the Yang-Mills equations \cite{2,3}. The second reason is that, owing to the structure of mentioned propagator, quarks mainly emit and interchange the soft gluons so the gluon condensate (a classical gluon field) between quarks basically consists of soft gluons (for more details see Refs. \cite{2,3}) but, because of that any gluon also emits gluons (still softer), the corresponding gluon concentrations rapidly become huge and form a linear confining magnetic colour field of enormous strengths, which leads to the confinement of quarks. This is by virtue of the fact that just magnetic part of the mentioned propagator is responsible for a larger portion of gluon concentrations at large distances since the magnetic part has stronger infrared singularities than the electric one. Under the circumstances physically nonlinearity of the Yang-Mills equations effectively vanishes so the latter possess the unique nonperturbative confining solutions of the Abelian-like form (with the values in Cartan subalgebra of SU(3)-Lie algebra) \cite{2,3} which describe the gluon condensate under consideration. Moreover, since the overwhelming majority of gluons is soft they cannot leave hadron (meson) until some gluons obtain additional energy (due to an external reason) to rush out. So we also deal with confinement of gluons.

The approach under discussion equips us with the explicit wave functions for every two quarks (meson or quarkonium). The wave functions are parametrized by a set of real constants $a_j, b_j, B_j$ describing the mentioned nonperturbative confining SU(3)-gluonic field (the gluon condensate), and they are nonperturbative modulo square integrable solutions of the Dirac equation in the above confining SU(3)-field and also depend on $\mu_0$, the reduced mass of the current masses of quarks forming meson. It is clear that under the given approach just constants $a_j, b_j, B_j, \mu_0$ determine all properties of any meson (quarkonium), i. e., the approach directly appeals to quark and gluonic degrees of freedom as should be according to the first principles of QCD. Also it is clear that the constants mentioned should be extracted from experimental data.

Such a program has been to a certain extent advanced in Refs. \cite{4-8}. The aim of the present paper is to continue obtaining estimates for $a_j, b_j, B_j$ for concrete mesons starting from experimental data on spectroscopy of one or another meson. We here consider charged pions and kaons and their leptonic
decays $P \to l^{\pm} + \nu_l$, $l = \mu, e$, where $P$ stands for any meson from $\pi^{\pm}, K^{\pm}$.

Of course, when conducting our considerations we shall rely on the standard quark model (SQM) based on SU(3)-flavor symmetry (see, e. g., Ref. [9]), so in accordance with SQM $\pi^{\pm} = u\bar{d}$, $\bar{u}d$, $K^{\pm} = u\bar{s}$, $\bar{u}s$ respectively.

Under the circumstances Section 2 contains main relations underlying our approach. Section 3 is devoted to computing the electric form factor, the root-mean-square radius $<r>$ and the magnetic moment of the mesons under consideration in an explicit analytic form while Section 4 gives an explicit analytic expression of the weak axial form factor for those mesons. Results of Sections 3 and 4 are used in Section 5 for obtaining estimates for parameters of the confining SU(3)-gluonic field for the mesons under discussion. Section 6 employs the obtained parameters of SU(3)-gluonic field to get the corresponding estimates for such characteristics of the mentioned field as gluon concentrations, electric and magnetic colour field strengths at the scales of the mesons in question. In Section 7 results of previous sections are applied to the problem of chiral symmetry breaking in QCD. Section 8 explores a possible connection of the results obtained with a phenomenological string-like picture of confinement while Section 9 is devoted to discussion and concluding remarks.

At last, Appendices A and B contain the detailed description of main building blocks for meson wave functions in the approach under discussion, respectively: eigenspinors of the Euclidean Dirac operator on 2-sphere $S^2$ and radial parts for the modulo square integrable solutions of Dirac equation in the confining SU(3)-Yang-Mills field.

Further we shall deal with the metric of the flat Minkowski spacetime $M$ that we write down (using the ordinary set of local spherical coordinates $r$, $\vartheta$, $\varphi$ for the spatial part) in the form

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu = dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2).$$ (1)

Besides, we have $|\delta| = |\det(g_{\mu\nu})| = (r^2 \sin \vartheta)^2$ and $0 \leq r < \infty$, $0 \leq \vartheta < \pi$, $0 \leq \varphi < 2\pi$.

Throughout the paper we employ the Heaviside-Lorentz system of units with $\hbar = c = 1$, unless explicitly stated otherwise, so the gauge coupling constant $g$ and the strong coupling constant $\alpha_s$ are connected by relation $g^2/(4\pi) = \alpha_s$. Further, we shall denote by $L_2(F)$ the set of the modulo square integrable complex functions on any manifold $F$ furnished with an integration measure, then $L_2^n(F)$ will be the $n$-fold direct product of $L_2(F)$ endowed with the obvious scalar product while $\dagger$ and $*$ stand, respectively, for Hermitian and complex conjugation. Our choice of Dirac $\gamma$-matrices conforms to the so-called standard representation and is the same as in Ref. [6]. At last $\otimes$ means the tensorial
product of matrices and \( I_n \) is the unit \( n \times n \) matrix so that, e.g., we have

\[
I_3 \otimes \gamma^\mu = \begin{pmatrix}
\gamma^\mu & 0 & 0 \\
0 & \gamma^\mu & 0 \\
0 & 0 & \gamma^\mu
\end{pmatrix}
\]

for any Dirac \( \gamma \)-matrix \( \gamma^\mu \) and so forth.

When calculating we apply the relations

\[
1 \text{ GeV}^{-1} \approx 0.1973269679 \text{ fm}, \quad 1 \text{ s}^{-1} \approx 0.658211915 \times 10^{-24} \text{ GeV},
\]

\[
1 \text{ V/m} \approx 0.2309956375 \times 10^{-23} \text{ GeV}^2, \quad 1 \text{ T} = 4\pi \times 10^{-7} \text{ H/m} \times 1 \text{ A/m} \approx 0.6925075988 \times 10^{-15} \text{ GeV}^2.
\]

Finally, for the necessary estimates we shall employ the \( T_{00} \)-component (volumetric energy density) of the energy-momentum tensor for a SU(3)-Yang-Mills field which should be written in the chosen system of units in the form

\[
T_{\mu\nu} = -F^a_{\mu\alpha} F^a_{\nu\beta} g^{\alpha\beta} + \frac{1}{4} F^a_{\beta\gamma} F^a_{\alpha\delta} g^{\gamma\delta} g^{\mu\nu}.
\]

### 2 Main relations

As was mentioned above, our considerations shall be based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system (derived from QCD-Lagrangian) studied at the whole length in Refs. [1–3]. Referring for more details to those references, let us briefly describe and specify only the relations necessary to us in the present paper.

One part of the mentioned family is presented by the unique nonperturbative confining solution of the SU(3)-Yang-Mills equations for gluonic field \( A = A_\mu dx^\mu = A^a_\alpha dx^\mu \) (\( \lambda_a \) are the known Gell-Mann matrices, \( \mu = t, r, \vartheta, \varphi \), \( a = 1, \ldots, 8 \)) and looks as follows

\[
A_{1t} \equiv A^3_t + \frac{1}{\sqrt{3}} A^8_t = -\frac{a_1}{r} + A_1, \quad A_{2t} \equiv -A^3_t + \frac{1}{\sqrt{3}} A^8_t = -\frac{a_2}{r} + A_2,
\]

\[
A_{3t} \equiv -\frac{2}{\sqrt{3}} A^8_t = \frac{a_1 + a_2}{r} - (A_1 + A_2),
\]

\[
A_{1\varphi} \equiv A^3_\varphi + \frac{1}{\sqrt{3}} A^8_\varphi = b_1 r + B_1, \quad A_{2\varphi} \equiv -A^3_\varphi + \frac{1}{\sqrt{3}} A^8_\varphi = b_2 r + B_2,
\]

\[
A_{3\varphi} \equiv -\frac{2}{\sqrt{3}} A^8_\varphi = -(b_1 + b_2)r - (B_1 + B_2)
\]

with the real constants \( a_j, A_j, b_j, B_j \) parametrizing the family. The word unique should be understood in the strict mathematical sense. In fact in Ref. [2] the following theorem was proved:
The unique exact spherically symmetric (nonperturbative) solutions (i.e. depending only on r) of SU(3)-Yang-Mills equations in Minkowski spacetime consist of the family of (3).

It should be noted that solution (3) was found early in Ref. [1] but its uniqueness was proved just in Ref. [2] (see also Ref. [3]). Besides, in Ref. [2] (see also Ref. [6]) it was shown that the above unique confining solutions (3) satisfy the so-called Wilson confinement criterion [10]. Up to now nobody contested this result so if we want to describe interaction between quarks by spherically symmetric SU(3)-fields then they can be only the ones from the above theorem.

As has been repeatedly explained in Refs. [2–4,6], parameters $A_1, A_2$ of solution (3) are inessential for physics in question and we can consider $A_1 = A_2 = 0$. Obviously we have $\sum_{j=1}^3 A_{jt} = \sum_{j=1}^3 A_{j\varphi} = 0$ which reflects the fact that for any matrix $T$ from SU(3)-Lie algebra we have $\text{Tr} T = 0$. Also, as has been repeatedly discussed by us earlier (see, e.g., Refs. [2,3]), from the above form it is clear that the solution (3) is a configuration describing the electric Coulomb-like colour field (components $A_3^{\varphi}$) and the magnetic colour field linear in $r$ (components $A_3^{\varphi}$) and we wrote down the solution (3) in the combinations that are just needed further to insert into the Dirac equation (4).

For the sake of completeness one should note that the similar unique confining solutions exist for all semisimple and non-semisimple compact Lie groups, in particular, for SU(N) with $N \geq 2$ and U(N) with $N \geq 1$ [2,3]. Explicit form of solutions, e.g., for SU(N) with $N = 2, 4$ can be found in Ref. [3] but it should be emphasized that components linear in $r$ always represent the magnetic colour field in all the mentioned solutions.

Another part of the compatible nonperturbative solutions for the SU(3)-Dirac-Yang-Mills system is given by the unique nonperturbative modulo square integrable solutions of the Dirac equation in the confining SU(3)-field of (3) $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ with the four-dimensional Dirac spinors $\Psi_j$ representing the $j$th colour component of the meson, so $\Psi$ may describe relative motion (relativistic bound states) of two quarks in mesons and the mentioned Dirac equation looks as follows

$$i \partial_t \Psi \equiv i \begin{pmatrix} \partial_t \Psi_1 \\ \partial_t \Psi_2 \\ \partial_t \Psi_3 \end{pmatrix} = H \Psi \equiv \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} H_1 \Psi_1 \\ H_2 \Psi_2 \\ H_3 \Psi_3 \end{pmatrix}, \quad (4)$$

where Hamiltonian $H_j$ is

$$H_j = \gamma^0 \left[ \mu_0 - i \gamma^1 \partial_r - i \gamma^2 \frac{1}{r} \left( \partial_\varphi + \frac{1}{2} \gamma^1 \gamma^2 \right) - i \gamma^3 \frac{1}{r \sin \vartheta} \left( \partial_\varphi + \frac{1}{2} \sin \vartheta \gamma^1 \gamma^3 + \frac{1}{2} \cos \vartheta \gamma^2 \gamma^3 \right) \right]$$
that yields \( (g\omega_{S\omega}) \) should put \( \omega \) energy, i.e., to rewrite it in the form if we want to interpret (4) as equation for eigenvalues of the relative motion \[
\frac{dr}{dt} = g^0 \left( \gamma_0 A_{j\mu} + \gamma_3 \frac{1}{r \sin \vartheta} A_{j\varphi} \right)
\] with the gauge coupling constant \( g \) while \( \mu_0 \) is a mass parameter and one should consider it to be the reduced mass which is equal, e.g., for quarkonia, to half the current mass of quarks forming a quarkonium.

Then the unique nonperturbative modulo square integrable solutions of (4) are (with Pauli matrix \( \sigma_1 \), for more details see Refs. [1,3])

\[
\Psi_j = e^{-i\omega_j t} \psi_j \equiv e^{-i\omega_j t} r^{-1} \left( \begin{array}{c} F_{j1}(r) \Phi_j(\vartheta, \varphi) \\ F_{j2}(r) \sigma_1 \Phi_j(\vartheta, \varphi) \end{array} \right), j = 1, 2, 3
\]

with the 2D eigenspinor \( \Phi_j = \left( \begin{array}{c} \Phi_{j1} \\ \Phi_{j2} \end{array} \right) \) of the Euclidean Dirac operator \( D_0 \) on the unit sphere \( S^2 \), while the coordinate \( r \) stands for the distance between quarks. The explicit form of \( \Phi_j \) is discussed in Appendix A. We can call the quantity \( \omega_j \) relative energy of \( j \)th colour component of meson (while \( \psi_j \) is wave function of a stationary state for \( j \)th colour component) but we can see that if we want to interpret (4) as equation for eigenvalues of the relative motion energy, i.e., to rewrite it in the form \( H \psi = \omega \psi \) with \( \psi = (\psi_1, \psi_2, \psi_3) \) then we should put \( \omega = \omega_j \) for any \( j \) so that \( H_j \psi_j = \omega_j \psi_j = \omega \psi_j \). Under this situation, if a meson is composed of quarks \( q_{1,2} \) with different flavours then the energy spectrum of the meson will be given by \( \epsilon = m_{q_1} + m_{q_2} + \omega \) with the current quark masses \( m_{q_k} \) (rest energies) of the corresponding quarks. On the other hand for determination of \( \omega_j \) the following quadratic equation can be obtained [1–3]

\[
[g^2 \alpha_j^2 + (n_j + \alpha_j) \omega_j^2] \omega_j^2 - 2(\lambda_j - gB_j)g^2 a_j b_j \omega_j + [(\lambda_j - gB_j)^2 - (n_j + \alpha_j)^2]g^2 b_j^2 - \mu_0^2 (n_j + \alpha_j)^2 = 0,
\] (7)

that yields (at \( g \neq 0 \))

\[
\omega_j = \omega_j(n_j, l_j, \lambda_j) = \frac{\Lambda_j g^2 a_j b_j \pm (n_j + \alpha_j) \sqrt{(n_j^2 + 2n_j \alpha_j + \Lambda_j^2) \mu_0^2 + g^4 b_j^2 (n_j^2 + 2n_j \alpha_j)}}{n_j^2 + 2n_j \alpha_j + \Lambda_j^2}, j = 1, 2, 3,
\] (8)

where \( a_3 = -(a_1 + a_2), b_3 = -(b_1 + b_2), B_3 = -(B_1 + B_2), \Lambda_j = \lambda_j - gB_j, \alpha_j = \sqrt{\Lambda_j^2 - g^2 a_j^2}, n_j = 0, 1, 2, ... \), while \( \lambda_j = \pm (l_j + 1) \) are the eigenvalues of Euclidean Dirac operator \( D_0 \) on unit sphere with \( l_j = 0, 1, 2, ... \). It should be noted that in the papers [1–4,6] we used the ansatz (6) with the factor \( e^{i\omega_j t} \) instead of \( e^{-i\omega_j t} \) but then the Dirac equation (4) would look as \( -i\partial_t \Psi = H \Psi \) and in equation (7) the second summand would have the plus sign while the first summand in numerator of (8) would have the minus sign. In the papers [5,7,8] we returned to the conventional form of writing Dirac equation and this slightly modified the equations (7)–(8). In the given paper we conform to the same prescription as in Refs. [5,7,8].
In line with the above we should have $\omega = \omega_1 = \omega_2 = \omega_3$ in energy spectrum $\epsilon = m_{q_1} + m_{q_2} + \omega$ for any meson (quarkonium) and this at once imposes two conditions on parameters $a_j, b_j, B_j$ when choosing some experimental value for $\epsilon$ at the given current quark masses $m_{q_1}, m_{q_2}$.

The general form of the radial parts of (6) is considered in Appendix B. Within the given paper we need only of the radial parts of (6) at $n_j = 0$ (the ground state) that are [see (B.5)]

$$F_{j1} = C_j P_j r^\alpha j e^{-\beta_j r} \left(1 - \frac{g b_j}{\beta_j}\right), P_j = g b_j + \beta_j,$$

$$F_{j2} = i C_j Q_j r^\alpha j e^{-\beta_j r} \left(1 + \frac{g b_j}{\beta_j}\right), Q_j = \mu_0 - \omega_j$$

with $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2}$ while $C_j$ is determined from the normalization condition $\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2)dr = \frac{1}{3}$. Consequently, we shall gain that $\Psi_j \in L_2^4(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, as a result, the solutions of (6) may describe relativistic bound states (mesons) with the energy (mass) spectrum $\epsilon$.

### 2.1 Nonrelativistic and the weak coupling limits

It is useful to specify the nonrelativistic limit (when $c \to \infty$) for spectrum (8). For that one should replace $g \to g/\sqrt{\hbar c}, a_j \to a_j/\sqrt{\hbar c}, b_j \to b_j/\sqrt{\hbar c}, B_j \to B_j/\sqrt{\hbar c}$ and, expanding (8) in $z = 1/c$, we shall get

$$\omega_j(n_j, l_j, \lambda_j) =$$

$$\pm \mu_0 c^2 \left[1 \mp \frac{g^2 a_j^2}{2 \hbar^2 (n_j + |\lambda_j|)^2} z^2 \right] + \left[\frac{\lambda_j g^2 a_j b_j}{\hbar (n_j + |\lambda_j|)^2} \mp \mu_0 \frac{g^3 B_j a_j^2 f(n_j, \lambda_j)}{\hbar^3 (n_j + |\lambda_j|)^7} z \right] + O(z^2),$$

(10)

where $f(n_j, \lambda_j) = 4 \lambda_j n_j (n_j^2 + \lambda_j^2) + \frac{|\lambda_j|}{\lambda_j} \left(n_j^4 + 6 n_j^2 \lambda_j^2 + \lambda_j^4\right)$.

As is seen from (10), at $c \to \infty$ the contribution of linear magnetic colour field (parameters $b_j, B_j$) to spectrum really vanishes and spectrum in essence becomes the purely nonrelativistic Coulomb one (modulo the rest energy). Also it is clear that when $n_j \to \infty$, $\omega_j \to \pm \sqrt{\mu_0^2 + g^2 b_j^2}$.

At last, one should specify the weak coupling limit of (8), i.e., the case $g \to 0$. As is not complicated to see from (8), $\omega_j \to \pm \mu_0$ when $g \to 0$. But then quantities $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2} \to 0$ and wave functions of (9) cease to be the modulo square integrable ones at $g = 0$, i.e., they cease to describe relativistic bound states. Accordingly, this means that equation (8) does not make physical meaning at $g = 0$.  

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We may seemingly use (8) with various combinations of signs (±) before second summand in numerators of (8) but, due to (10), it is reasonable to take all signs equal to plus which is our choice within the paper. Besides, as is not complicated to see, radial parts in nonrelativistic limit have the behaviour of form \( F_j, F_j \sim r^{l_j+1} \), which allows one to call quantum number \( l_j \) angular momentum for \( j \)th colour component though angular momentum is not conserved in the field (3) [1,3]. So for meson (quarkonium) under consideration we should put all \( l_j = 0 \).

2.2 Eigenspinors with \( \lambda = \pm 1 \)

Finally it should be noted that spectrum (8) is degenerated owing to degeneracy of eigenvalues for the Euclidean Dirac operator \( D_0 \) on the unit sphere \( S^2 \). Namely, each eigenvalue of \( D_0 \) \( \lambda = \pm (l+1), l = 0, 1, 2, \ldots \) has multiplicity \( 2(l+1) \) so we has \( 2(l+1) \) eigenspinors orthogonal to each other. Ad referendum we need eigenspinors corresponding to \( \lambda = \pm 1 \) \( (l = 0) \) so here is their explicit form [see (A.16)]

\[
\lambda = -1 : \Phi = \frac{C}{2} \left( e^{i\frac{\phi}{2}} \right) e^{i\varphi/2}, \quad \text{or} \quad \Phi = \frac{C}{2} \left( -e^{-i\frac{\phi}{2}} \right) e^{-i\varphi/2},
\]

\[
\lambda = 1 : \Phi = \frac{C}{2} \left( e^{-i\frac{\phi}{2}} \right) e^{i\varphi/2}, \quad \text{or} \quad \Phi = \frac{C}{2} \left( -e^{-i\frac{\phi}{2}} \right) e^{-i\varphi/2}
\]

with the coefficient \( C = 1/\sqrt{2\pi} \) (for more details see Appendix A).

3 Electric form factor, the root-mean-square radius and magnetic moment

Within the present paper we shall use relations (8) at \( n_j = 0 = l_j \) so energy (mass) of mesons under consideration is given by \( m_P = m_{q_1} + m_{q_2} + \omega \) with \( \omega = \omega_j(0, 0, \lambda_j) \) for any \( j = 1, 2, 3 \) whereas

\[
\omega = \frac{g^2 a_1 b_1}{\Lambda_1} + \frac{\alpha_1 \mu_0}{|\Lambda_1|} + \frac{g^2 a_2 b_2}{\Lambda_2} + \frac{\alpha_2 \mu_0}{|\Lambda_2|} + \frac{g^2 a_3 b_3}{\Lambda_3} + \frac{\alpha_3 \mu_0}{|\Lambda_3|} = \left\{ \begin{array}{l}
\pm m_{\pi} - m_u - m_d, \\
\pm m_{K} - m_u - m_s.
\end{array} \right.
\]

and, as a consequence, the corresponding meson (quarkonium) wave functions of (6) are represented by (9) and (11).
3.1 Choice of quark masses and the gauge coupling constant

It is evident for employing the above relations we have to assign some values to quark masses and gauge coupling constant $g$. In accordance with Ref. [9], at present the current quark masses necessary to us are restricted to intervals $1.5\text{ MeV} \leq m_u \leq 3\text{ MeV}$, $3.0\text{ MeV} \leq m_d \leq 7\text{ MeV}$, $95\text{ MeV} \leq m_s \leq 120\text{ MeV}$, so we take $m_u = (1.5 + 3)/2\text{ MeV} = 2.25\text{ MeV}$, $m_d = (3 + 7)/2\text{ MeV} = 5\text{ MeV}$, $m_s = (95 + 120)/2\text{ MeV} = 107.5\text{ MeV}$. Under the circumstances, the reduced mass $\mu_0$ of (5) will be $m_um_d/(m_u + m_d)$ or $m_um_s/(m_u + m_s)$.

As to gauge coupling constant $g = \sqrt{4\pi\alpha_s}$, it should be noted that recently some attempts have been made to generalize the standard formula for $\alpha_s = \frac{\alpha_s(Q^2)}{[\ln(Q^2/\Lambda^2)](n_f\text{ is number of quark flavours})}$ holding true at the momentum transfer $\sqrt{Q^2} \rightarrow \infty$ to the whole interval $0 \leq \sqrt{Q^2} \leq \infty$. We shall employ one such a generalization used in Refs. [11]. It looks as follows ($x = \sqrt{Q^2}$ in GeV)

$$\alpha(x) = \frac{12\pi}{(33 - 2n_f)\ln \frac{x^2f_2(x)}{\Lambda^2}}$$

with

$$f_1(x) = 1 + \left[\frac{(1 + x)(33 - 2n_f)}{12} \ln \frac{m^2}{\Lambda^2} - 1\right]^{-1} + 0.6x^{1.3}^{-1}, f_2(x) = m^2(1 + 2.8x^2)^{-2},$$

wherefrom one can conclude that $\alpha_s \rightarrow \pi = 3.1415...$ when $x \rightarrow 0$, i. e., $g \rightarrow 2\pi = 6.2831...$. We used (13) at $m = 1\text{ GeV}$, $\Lambda = 0.234\text{ GeV}$, $n_f = 3$, $x = m_{\pi^\pm} = 139.56995\text{ MeV}$ or $x = m_{K^\pm} = 493.677\text{ MeV}$ to obtain $g = 6.091309951$ necessary for our further computations at the mass scale of $\pi^\pm$-mesons or $g = 5.301208569$ at the mass scale of $K^\pm$-mesons.

3.2 Electric form factor

For each meson (quarkonium) with the wave function $\Psi = (\Psi_j)$ of (6) we can define electromagnetic current $J^\mu = \overline{\Psi}(I_3 \otimes \gamma^\mu)\Psi = (\Psi^\dagger\Psi, \Psi^\dagger(I_3 \otimes \alpha)\Psi) = (\rho, J)$, $\alpha = \gamma^0\gamma$. Electric form factor $f(K)$ is the Fourier transform of $\rho$

$$f(K) = \int \Psi^\dagger\Psi e^{-iKr}d^3x = \sum_{j=1}^3 \int \Psi_{j}^\dagger\Psi_{j} e^{-iKr}d^3x = \sum_{j=1}^3 f_j(K) =$$

$$\sum_{j=1}^3 \int (|F_{j1}|^2 + |F_{j2}|^2)\Phi_j^\dagger\Phi_j \frac{e^{-iKr}}{r^2}d^3x, d^3x = r^2 \sin \vartheta dr d\vartheta d\varphi$$

with the momentum transfer $K$. At $n_j = 0 = l_j$, as is easily seen, for any spinor of (11) we have $\Phi_{j}^\dagger\Phi_{j} = 1/(4\pi)$, so the integrand in (14) does
not depend on $\varphi$ and we can consider vector $\mathbf{K}$ to be directed along $z$-axis. Then $\mathbf{K}r = Kr \cos \vartheta$ and with the help of (9) and relations (see Ref. [12]):

$$f_0^\infty r^{-\alpha - 1} e^{-pr} dr = \Gamma(\alpha) p^{-\alpha}, \text{ Re } \alpha, p > 0, f_0^\infty r^{\alpha - 1} e^{-pr} \left( \sin \left( Kr \right) \right) dr = \Gamma(\alpha)(K^2 + p^2)^{-\alpha/2} \left( \frac{\sin \left( \alpha \arctan \left( K/p \right) \right)}{\cos \left( \alpha \arctan \left( K/p \right) \right)} \right), \text{ Re } \alpha > -1, \text{ Re } p > |\text{Im } K|, \Gamma(\alpha + 1) = \alpha \Gamma(\alpha),$$

$$\int_0^\infty r^{\alpha - 1} e^{-pr} \sin \left( Kr \cos \vartheta \right) \sin \vartheta d\vartheta = 2 \sin \left( Kr \right)/(Kr),$$

we shall obtain

$$f(K) = \frac{3}{\sqrt{3}} \sum_{j=1}^3 f_j(K) = \frac{3}{\sqrt{6}} \sum_{j=1}^3 \frac{(2\beta_j)^{2\alpha_j + 1}}{6\alpha_j} \sin \left[ 2\alpha_j \arctan \left( K/(2\beta_j) \right) \right] \sqrt{K(K^2 + 4\beta_j^2)^{\alpha_j}},$$

wherefrom it is clear that $f(K)$ is a function of $K^2$, as should be, and we can determine the root-mean-square radius of meson (quarkonium) in the form

$$<r> = \sqrt{\frac{\sum_{j=1}^3 2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2}}.$$

When calculating (15) also the fact was used that by virtue of the normalization condition for wave functions we have

$$C_j^2\left[ P_j^2(1 - gb_j/\beta_j)^2 + Q_j^2(1 + gb_j/\beta_j)^2 \right] = (2\beta_j)^{2\alpha_j + 1}/[3\Gamma(2\alpha_j + 1)].$$

On the other hand, we can directly calculate $<r>$ in accordance with the standard quantum mechanics rules as

$$<r> = \sqrt{\int r^2 \Psi^\dagger \Psi \, d^3x} = \sqrt{\sum_{j=1}^3 \int r^2 \Psi_j^\dagger \Psi_j \, d^3x},$$

and the result will be the same as in (16). So we should not call $<r>$ of (16) the charge radius of meson (quarkonium)– it is just the radius of meson (quarkonium) determined by the wave functions of (6) (at $n_j = 0 = l_j$) with respect to strong interaction, i.e., radius of confinement. Now we should notice the expression (15) to depend on 3-vector $\mathbf{K}$. To rewrite it in the form holding true for any 4-vector $Q$, let us remind that according to general considerations (see, e.g., Ref. [13]) the relation (15) should correspond to the so-called Breit frame where $Q^2 = -K^2$ [when fixing metric by (1)] so it is not complicated to rewrite (15) for arbitrary $Q$ in the form

$$f(Q^2) = \sum_{j=1}^3 f_j(Q^2) = \frac{3}{\sqrt{6}} \sum_{j=1}^3 \frac{(2\beta_j)^{2\alpha_j + 1}}{6\alpha_j} \frac{\sin \left[ 2\alpha_j \arctan \left( \sqrt{|Q^2|}/(2\beta_j) \right) \right]}{\sqrt{|Q^2|}(4\beta_j^2 - Q^2)^{\alpha_j}},$$

which passes on to (15) in the Breit frame.
3.3 Magnetic moment

We can define the volumetric magnetic moment density by

\[ m = q (r \times J) / 2 = q [(y J_x - z J_y) \hat{i} + (z J_x - x J_z) \hat{j} + (x J_y - y J_x) \hat{k}] / 2 \]

with the meson charge \( q \) and \( J = \Psi^\dagger (I_3 \otimes \alpha) \Psi \). Using (6) we have in the explicit form

\[ J_x = \sum_{j=1}^{3} (F_{j1}^* F_{j2} + F_{j2}^* F_{j1}) \frac{\Phi_j^\dagger \Phi_j}{r^2}, \quad J_y = \sum_{j=1}^{3} (F_{j1}^* F_{j2} - F_{j2}^* F_{j1}) \frac{\Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j}{r^2}, \]

\[ J_z = \sum_{j=1}^{3} (F_{j1}^* F_{j2} - F_{j2}^* F_{j1}) \frac{\Phi_j^\dagger \sigma_3 \sigma_1 \Phi_j}{r^2} \tag{18} \]

with Pauli matrices \( \sigma_{1,2,3} \). Magnetic moment of meson (quarkonium) is

\[ M = \int_V m \, d^3 x, \]

where \( V \) is volume of meson (quarkonium) (the ball of radius \( <r> \)). Then at \( n_j = l_j = 0 \), as is seen from (9), (11), \( F_{j1}^* = F_{j1}, F_{j2}^* = -F_{j2} \), \( \Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j = 0 \) for any spinor of (11) which entails \( J_x = J_y = 0 \), i.e., \( m_z = 0 \) while \( \int_V m_x y \, d^3 x = 0 \) because of turning the integral over \( \varphi \) to zero, which is easy to check. As a result, magnetic moments of mesons (quarkonia) with the wave functions of (6) (at \( l_j = 0 \)) are equal to zero, as should be according to experimental data [9].

Though we can also evaluate magnetic form factor \( F(Q^2) \) of meson (quarkonium) which is also a function of \( Q^2 \) (see Refs. [7,8]) but the latter will not be used in the given paper so we shall not dwell upon it.

4 Weak axial form factor

4.1 Preliminaries

Let us now address to leptonic decays \( P \rightarrow l^\pm + \nu_l, l = \mu, e \). According to the standard theory of electroweak interaction the width of those decays is given by (see, e.g., Refs. [9,14])

\[ \Gamma = \frac{G_F^2 |V|^2}{8\pi} f_P^2 m_l^2 m_P \left( 1 - \frac{m_l^2}{m_P^2} \right)^2 \tag{19} \]

with the Fermi coupling constant \( G_F = 1.16637 \times 10^{-5} \text{ GeV}^{-2} \), lepton mass \( m_l \), the charged \( P \)-meson mass \( m_P \), the corresponding element \( V \) of the Cabibbo–Kobayashi–Maskawa mixing matrix \( (V = V_{ud} = \cos \vartheta_c \text{ for } P = \pi^\pm \text{ and } V = V_{us} = \sin \vartheta_c \text{ for } P = K^\pm \text{ with the Cabibbo angle } \vartheta_c \approx 13^\circ) \), while the information about strong interaction of quarks in \( P \)-mesons is encoded
in a decay constant $f_\rho$. Further parametrization in the form $f_\rho = m_\rho \Phi$ entails $\Phi \approx 0.9364479961$ at $f_\rho = f_\pi \approx 130.7$ MeV and $\Phi \approx 0.3236934271$ at $f_\rho = f_K \approx 159.8$ MeV [9]. We can notice that the only invariant which $\Phi$ might depend on is $Q^2 = m_\rho^2$, i.e. we should find such a function $\mathcal{F}(Q^2)$ for that $\mathcal{F}(Q^2 = m_\rho^2) \approx 0.9364479961$ or $0.3236934271$ respectively. It is obvious from physical point of view that $\mathcal{F}$ should be connected with electroweak properties of $P$-mesons.

Let us now remark that, as is well known [9,14], the mentioned leptonic decays of charged pions and kaons are governed by the axial-vector part of the weak charged hadronic current. We can try to construct that part from the $P$-meson wave function of (6) and the wave function $\Psi_0$ of vacuum state which may be chosen from the considerations that it should be similar to (6) in the form but it should not contain any explicit information about the parameters $a_j$, $b_j$, $B_j$, $\mu_0$ describing the $P$-meson properties in our approach and connected directly to quark and gluonic degrees of freedom. This requirement can be satisfied by $\Psi_0$ of the form

$$\Psi_0 = (\Psi_{0j}) \equiv A_0 \left( e^{-i\omega_j t} \epsilon^{-1} \left( \frac{\Phi_0^0(\vartheta,\varphi)}{\sigma_3 \Phi_0^0(\vartheta,\varphi)} \right) \right), \quad j = 1, 2, 3 \tag{20}$$

with the 2D eigenspinor $\Phi_0^0 = \left( \begin{array}{c} \Phi_{01} \\ \Phi_{02} \end{array} \right)$ of the Euclidean Dirac operator $D_0$ on the unit sphere $S^2$ and some real constant $A_0 > 0$ whose value will be fixed below (see Section 5) while the phase factor $e^{-i\omega_j t}$ is left only for convenience and has no influence on the subsequent calculations.

### 4.2 Weak axial form factor

To fix spinor $\Phi_j^0$ let us consider the axial-vector part of the weak charged hadronic current in the form $A^\mu = \bar{\Psi}_0(I_3 \otimes \gamma^\mu \gamma^5)\Psi = (\Psi_0^0(I_3 \otimes \gamma^5)\Psi, \Psi_0^1(I_3 \otimes \alpha \gamma^5)\Psi) = (\rho_A, J_A), \alpha = \gamma^0 \gamma$, with the wave functions $\Psi_0$ of (6) and $\Psi_0$ of (20) and compute $\rho_A$. Inasmuch as in the chosen representation $\gamma^5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\left( \begin{array}{cc} 0 & I_2 \\ I_2 & 0 \end{array} \right)$ then we shall have $\rho_A = -\frac{A_0}{r^2} \sum_{j=1}^3 (F_{j2} - iF_{j1})(\Phi_{0j}^0 \sigma_3 \Phi_j)$. Under this situation for the ground state of $P$-meson described by (9) and (11) it is natural to put $\Phi_j^0 = \sigma_3 \Phi_j$ which entails $\Phi_j^0, \sigma_1 \Phi_j = 1/(4\pi)$ for any spinor of (11). Then we can define the weak axial form factor $f_A(K)$ for $P$-meson (by analogy with electric form factor in Section 3) as the Fourier transform of $\rho_A$

$$f_A(K) = \int \rho_A e^{-iKr} d^3x = -\frac{iA_0}{4\pi} \sum_{j=1}^3 C_j \int r^\alpha e^{-\beta r} \left[ Q_j \left( 1 + \frac{gb_j}{\beta_j} \right) - P_j \left( 1 - \frac{gb_j}{\beta_j} \right) \right] e^{-iKr} r^2 d^3x, \tag{21}$$
so computation along the same lines as for electric form factor of (15) yields the result

\[
f_A(K) = iA_0 \sum_{j=1}^{3} \frac{(2\beta_j)^{\alpha_j+1/2}\Gamma(\alpha_j)}{\sqrt{3\Gamma(2\alpha_j + 1)}} \cdot \frac{\mathcal{P}_j - Q_j}{\sqrt{\mathcal{P}_j^2 + Q_j^2}} \cdot \frac{\sin[\alpha_j \arctan(K/\beta_j)]}{K(K^2 + \beta_j^2)^{\alpha_j/2}} = \]

\[
iA_0 \sum_{j=1}^{3} \frac{(2\beta_j)^{\alpha_j+1/2}\Gamma(\alpha_j)}{\sqrt{3\Gamma(2\alpha_j + 1)}} \cdot \frac{\mathcal{P}_j - Q_j}{\sqrt{\mathcal{P}_j^2 + Q_j^2}} \cdot \frac{\alpha_j}{(\beta_j^2)^{\alpha_j/2}} \left( \frac{1}{\beta_j} - \frac{\alpha_j^2 + 3\alpha_j + 2}{6\beta_j^3}K^2 + O(K^4) \right)
\]

with \( \mathcal{P}_j = P_j(1 - gb_j/\beta_j) \), \( Q_j = O_j(1 + gb_j/\beta_j) \). It is clear from (22) that \( f_A(K) \) is a function of \( K^2 \) and we can rewrite (22) for arbitrary 4-vector \( Q \) as

\[
f_A(Q^2) = iA_0 \sum_{j=1}^{3} \frac{(2\beta_j)^{\alpha_j+1/2}\Gamma(\alpha_j)}{\sqrt{3\Gamma(2\alpha_j + 1)}} \cdot \frac{\mathcal{P}_j - Q_j}{\sqrt{\mathcal{P}_j^2 + Q_j^2}} \cdot \frac{\sin[\alpha_j \arctan(\sqrt{|Q^2|}/\beta_j)]}{\sqrt{|Q^2|/(\beta_j^2 - Q^2)^{\alpha_j/2}}}
\]

which passes on to (22) in the Breit frame where \( Q^2 = -K^2 \).

5 Estimates for parameters of SU(3)-gluonic field in \( P \)-mesons

5.1 Basic equations

Now we are able to estimate parameters \( a_j, b_j, B_j \) of the confining SU(3)-field (3) in \( P \)-mesons. Though we can consider other possible functions of \( Q^2 \) connected with the mentioned above current \( J_A \) (on the analogy of the meson magnetic form factor, see Refs. [7,8]) but, obviously, the most natural function is \( f_A(Q^2) \) of (23). It is reasonable, therefore, to take for the function \( \mathcal{F}(Q^2) \) mentioned in Subsection 4.1 just the function \( |f_A(Q^2)| \), i.e., to put \( \mathcal{F}(Q^2) = |f_A(Q^2)| \) so that we should have \( |f_A(Q^2 = m_P^2)| \approx 0.9364479961 \) or \( 0.3236934271 \) respectively for \( P = \pi^\pm \) and \( P = K^\pm \). Then, denoting the quantities \( m_P/\beta_j = x_j \), we obtain the following equation for parameters of the confining SU(3)-gluonic field in \( P \)-mesons

\[
|f_A(Q^2 = m_P^2)| = A_0 \left| \sum_{j=1}^{3} \frac{2^{\alpha_j+1/2}\Gamma(\alpha_j)}{\sqrt{3\beta_j^2\Gamma(2\alpha_j + 1)}} \cdot \frac{\mathcal{P}_j - Q_j}{\sqrt{\mathcal{P}_j^2 + Q_j^2}} \cdot \frac{\sin[\alpha_j \arctan x_j]}{x_j(1 - x_j^2)^{\alpha_j/2}} \right| \approx \left\{ \begin{array}{ll} 0.9364479961, & \\ 0.3236934271. & \end{array} \right. \]

Finally, equations (12) and (16) should also be added to (24) and the system obtained in such a way should be solved compatibly if taking \( < r > \approx 0.672 \) fm for \( P = \pi^\pm \) and \( < r > \approx 0.560 \) fm for \( P = K^\pm \) [9]. While computing for distinctness we take all eigenvalues \( \lambda_j \) (\( j = 1, 2, 3 \)) of the Euclidean Dirac operator \( \mathcal{D}_0 \) on the unit two-sphere \( S^2 \) equal to 1.
5.2 Numerical results

The results of numerical compatible solving of equations (12), (16), and (24) are adduced in Tables 1–6 (with $\mu = m_P$).

Table 1
Gauge coupling constant, reduced mass $\mu_0$ and parameters of the confining SU(3)-gluonic field for $\pi^\pm$-mesons

| Particle | $g$ | $\mu_0$ (MeV) | $a_1$ | $a_2$ | $b_1$ (GeV) | $b_2$ (GeV) | $B_1$ | $B_2$ |
|----------|-----|---------------|-------|-------|-------------|-------------|-------|-------|
| $\pi^\pm-u\bar{d}, \bar{u}d$ | 6.09131 | 1.55172 | 0.0473002 | 0.0118497 | 0.178915 | -0.119290 | 0.230 | -0.230 |

Table 2
Theoretical and experimental $\pi^\pm$-meson mass and radius

| Particle | Theoret. $\mu$ (MeV) | Experim. $\mu$ (MeV) | Theoret. $<r>$ (fm) | Experim. $<r>$ (fm) |
|----------|---------------------|----------------------|---------------------|---------------------|
| $\pi^\pm-u\bar{d}, \bar{u}d$ | $\mu = m_u + m_d + \omega_j(0, 0, 1) = 139.570$ | 139.56995 | 0.673837 | 0.672 |

Table 3
Theoretical and experimental values of the charged pion weak axial form factor and constant $A_0$

| Particle | Theoret. $|f_A(Q^2 = \mu^2)|$ | Experim. $|f_A(Q^2 = \mu^2)|$ | $A_0$ (MeV$^{1/2}$) |
|----------|-----------------------------|-----------------------------|-----------------|
| $\pi^\pm-u\bar{d}, \bar{u}d$ | 0.9364898401 | 0.9364479961 | 12.72 |

Table 4
Gauge coupling constant, reduced mass $\mu_0$ and parameters of the confining SU(3)-gluonic field for $K^\pm$-mesons

| Particle | $g$ | $\mu_0$ (MeV) | $a_1$ | $a_2$ | $b_1$ (GeV) | $b_2$ (GeV) | $B_1$ | $B_2$ |
|----------|-----|---------------|-------|-------|-------------|-------------|-------|-------|
| $K^\pm-u\bar{s}, \bar{u}s$ | 5.30121 | 2.20387 | 0.167182 | -0.0557501 | 0.120150 | 0.131046 | -0.900 | 0.290 |

Table 5
Theoretical and experimental $K^\pm$-meson mass and radius

| Particle | Theoret. $\mu$ (MeV) | Experim. $\mu$ (MeV) | Theoret. $<r>$ (fm) | Experim. $<r>$ (fm) |
|----------|---------------------|----------------------|---------------------|---------------------|
| $K^\pm-u\bar{s}, \bar{u}s$ | $\mu = m_u + m_s + \omega_j(0, 0, 1) = 493.677$ | 493.677 | 0.544342 | 0.560 |

Table 6
Theoretical and experimental values of the charged kaon weak axial form factor and constant $A_0$

| Particle | Theoret. $|f_A(Q^2 = \mu^2)|$ | Experim. $|f_A(Q^2 = \mu^2)|$ | $A_0$ (MeV$^{1/2}$) |
|----------|-----------------------------|-----------------------------|-----------------|
| $K^\pm-u\bar{s}, \bar{u}s$ | 0.32332652803 | 0.3236934271 | 2.88 |
6 Estimates of gluon concentrations, electric and magnetic colour field strengths

Now let us remind that, according to Refs. [3,6], one can confront the field (3) with \( T_{00} \)-component (volumetric energy density of the SU(3)-gluonic field) of the energy-momentum tensor (2) so that

\[
T_{00} \equiv T_{tt} = \frac{E^2 + H^2}{2} = \frac{1}{2} \left( \frac{a_1^2 + a_1 a_2 + a_2^2}{r^4} + \frac{b_1^2 + b_1 b_2 + b_2^2}{r^2 \sin^2 \vartheta} \right) \equiv \frac{A}{r^4} + \frac{B}{r^2 \sin^2 \vartheta} \tag{25}
\]

with electric \( E \) and magnetic \( H \) colour field strengths and with real \( A > 0, B > 0 \). One can also introduce magnetic colour induction \( B = (4\pi \times 10^{-7} H/m) H \), where \( H \) in A/m.

To estimate the gluon concentrations we can employ (25) and, taking the quantity \( \omega = \Gamma \), the full decay width of a meson, for the characteristic frequency of gluons we obtain the sought characteristic concentration \( n \) in the form

\[
n = \frac{T_{00}}{\Gamma} \tag{26}
\]

so we can rewrite (25) in the form \( T_{00} = T_{00}^\text{coul} + T_{00}^\text{lin} \) conforming to the contributions from the Coulomb and linear parts of the solution (3). This entails the corresponding split of \( n \) from (26) as \( n = n_{\text{coul}} + n_{\text{lin}} \).

The parameters of Tables 1 and 4 were employed when computing and for simplicity we put \( \sin \vartheta = 1 \) in (25). Also there were used the following present-day full decay widths – for \( \pi^\pm \)-mesons: \( \Gamma = 1/\tau \) with the life time \( \tau = 2.6033 \times 10^{-8} \) s and for \( K^\pm \)-mesons: \( \Gamma = 1/\tau \) with the life time \( \tau = 1.2386 \times 10^{-8} \) s, whereas the Bohr radius \( a_0 = 0.529177249 \cdot 10^5 \) fm [9].

Tables 7–8 contain the numerical results for \( n_{\text{coul}}, n_{\text{lin}}, n, E, H, B \) for the mesons under discussion.

Table 7

Gluon concentrations, electric and magnetic colour field strengths in \( \pi^\pm \)-mesons

| \( r \) (fm) | \( n_{\text{coul}} \) (m\(^{-3}\)) | \( n_{\text{lin}} \) (m\(^{-3}\)) | \( n \) (m\(^{-3}\)) | \( E \) (V/m) | \( H \) (A/m) | \( B \) (T) |
|-------------|----------------|----------------|---------------|--------|--------|--------|
| 0.1 \( r_0 \) | \( 0.366804 \times 10^{66} \) | \( 0.141131 \times 10^{65} \) | \( 0.380917 \times 10^{66} \) | \( 0.201234 \times 10^{24} \) | \( 0.530981 \times 10^{21} \) | \( 0.667250 \times 10^{15} \) |
| \( r_0 \) | \( 0.366804 \times 10^{62} \) | \( 0.141131 \times 10^{63} \) | \( 0.177811 \times 10^{63} \) | \( 0.201234 \times 10^{22} \) | \( 0.530981 \times 10^{20} \) | \( 0.667250 \times 10^{14} \) |
| 1.0 | \( 0.756229 \times 10^{61} \) | \( 0.640814 \times 10^{62} \) | \( 0.716437 \times 10^{62} \) | \( 0.913716 \times 10^{21} \) | \( 0.357794 \times 10^{20} \) | \( 0.449618 \times 10^{14} \) |
| 10 \( r_0 \) | \( 0.366804 \times 10^{58} \) | \( 0.141131 \times 10^{61} \) | \( 0.141498 \times 10^{61} \) | \( 0.201234 \times 10^{20} \) | \( 0.530981 \times 10^{19} \) | \( 0.667250 \times 10^{13} \) |
| \( a_0 \) | \( 0.964380 \times 10^{42} \) | \( 0.228839 \times 10^{53} \) | \( 0.228839 \times 10^{53} \) | \( 0.326294 \times 10^{12} \) | \( 0.676133 \times 10^{15} \) | \( 0.849654 \times 10^{9} \) |
Table 8
Gluon concentrations, electric and magnetic colour field strengths in $K^\pm$-mesons

| $r$ (fm) | $n_{\text{coul}}$ (m$^{-3}$) | $n_{\text{lin}}$ (m$^{-3}$) | $n$ (m$^{-5}$) | $E$ (V/m) | $H$ (A/m) | $B$ (T) |
|----------|-------------------------------|-------------------------------|----------------|-----------|--------|-------|
| 0.1r$_0$ | $0.303178 \times 10^{67}$     | $0.195700 \times 10^{65}$     | $0.305135 \times 10^{67}$ | $0.838745 \times 10^{24}$ | $0.966483 \times 10^{21}$ | $0.113912 \times 10^{16}$ |
| $r_0$    | $0.303178 \times 10^{63}$     | $0.195700 \times 10^{63}$     | $0.498878 \times 10^{63}$  | $0.838745 \times 10^{22}$ | $0.966483 \times 10^{20}$ | $0.113912 \times 10^{15}$ |
| 1.0      | $0.266186 \times 10^{62}$     | $0.579876 \times 10^{62}$     | $0.846062 \times 10^{62}$  | $0.248527 \times 10^{22}$ | $0.493437 \times 10^{20}$ | $0.620071 \times 10^{14}$ |
| 10r$_0$  | $0.303178 \times 10^{59}$     | $0.195700 \times 10^{61}$     | $0.198732 \times 10^{61}$  | $0.838745 \times 10^{20}$ | $0.966483 \times 10^{19}$ | $0.113912 \times 10^{14}$ |
| $a_0$    | $0.339454 \times 10^{43}$     | $0.207077 \times 10^{53}$     | $0.207077 \times 10^{53}$  | $0.887506 \times 10^{12}$ | $0.932460 \times 10^{15}$ | $0.117176 \times 10^{10}$ |

7 Chiral symmetry breaking

7.1 Preliminaries

As is known, in the late sixties of XX century, in light meson physics there arose notion of chiral symmetry and so far the latter has been actively exploited in phenomenology (see, e.g., reviews [15]). In its turn, to provide real world with chiral symmetry breaking there was supposed that chiral symmetry had been spontaneously broken. On the other hand, historically in fact simultaneously with notion of chiral symmetry there was created standard model (SM) of electroweak interactions which later quarks were also included in. We should note that SM (with one Higgs doublet) contains some description of chiral symmetry breaking: current masses of quarks (as well as lepton masses) are acquired through Higgs mechanism so for quark masses $m_q$ we obtain (without taking mixings into account) $m_q = f_q v / \sqrt{2}$, where vacuum Higgs condensate $v \approx 246$ GeV. But little is known about coupling constants $f_q$ and much may be elucidated only with discovering Higgs bosons.

As a result, we at present have two postulated mechanisms for spontaneous breaking of (global) chiral symmetry: one is associated with SM and another one should be related with the so-called Goldstone bosons accompanying violation of any global symmetry. It is clear that both mechanisms should be connected in one or another way. As far as is known to us, up to now there has been not generally accepted (if any) recipe for reconciliation between mentioned mechanisms.

After QCD was created in the early seventies of XX century there appeared a hope that chiral symmetry breaking should be explained within framework of QCD and it should closely be related to confinement mechanism (see, e.g., Refs. [16]). Under the situation we can remark that our approach has a well-defined chiral limit with perfectly clear physical meaning and in view of this
it contains some description of chiral symmetry breaking. Let us look into it in more detail.

7.2 Pion and kaon masses and chiral limit

Let us more in detail write out expressions for pion and kaon masses from (12)

\[ m_{\pi \pm} = m_u + m_d + \frac{g^2 a_1 b_1}{1 - g B_1} + \mu_0 \frac{\sqrt{(1 - g B_1)^2 - g^2 a_1^2}}{|1 - g B_1|} = m_u + m_d + \frac{g^2 a_2 b_2}{1 - g B_2} + \mu_0 \frac{\sqrt{(1 - g B_2)^2 - g^2 a_2^2}}{|1 - g B_2|} = m_u + m_d + \frac{g^2(a_1 + a_2)(b_1 + b_2)}{1 + g(B_1 + B_2)} + \mu_0 \frac{\sqrt{[1 + g(B_1 + B_2)]^2 - g^2(a_1 + a_2)^2}}{|1 + g(B_1 + B_2)|}, \]

\[ \mu_0 = \frac{m_u m_d}{m_u + m_d}, \]  
\[ m_{K^\pm} = m_u + m_s + \frac{g^2 a_1 b_1}{1 - g B_1} + \mu_0 \frac{\sqrt{(1 - g B_1)^2 - g^2 a_1^2}}{|1 - g B_1|} = m_u + m_s + \frac{g^2 a_2 b_2}{1 - g B_2} + \mu_0 \frac{\sqrt{(1 - g B_2)^2 - g^2 a_2^2}}{|1 - g B_2|} = m_u + m_s + \frac{g^2(a_1 + a_2)(b_1 + b_2)}{1 + g(B_1 + B_2)} + \mu_0 \frac{\sqrt{[1 + g(B_1 + B_2)]^2 - g^2(a_1 + a_2)^2}}{|1 + g(B_1 + B_2)|}, \]

\[ \mu_0 = \frac{m_u m_s}{m_u + m_s}. \]  

In chiral limit \( m_u, m_d, m_s \to 0 \) we obtain

\[ (m_{\pi \pm})_{\text{chiral}} \approx \frac{g^2 a_1 b_1}{1 - g B_1} \approx \frac{g^2 a_2 b_2}{1 - g B_2} \approx \frac{g^2(a_1 + a_2)(b_1 + b_2)}{1 + g(B_1 + B_2)} \approx 130.8 \text{ MeV} \neq 0, \]  
\[ (m_{K^\pm})_{\text{chiral}} \approx \frac{g^2 a_1 b_1}{1 - g B_1} \approx \frac{g^2 a_2 b_2}{1 - g B_2} \approx \frac{g^2(a_1 + a_2)(b_1 + b_2)}{1 + g(B_1 + B_2)} \approx 382 \text{ MeV} \neq 0 \]

using the parameters \( g, a_j, b_j, B_j \) adduced in Tables 1 and 4 respectively.

We can see that in chiral limit the pion and kaon masses are completely determined only by the parameters \( a_j, b_j, B_j \) of SU(3)-gluonic field between quarks, i.e. by interaction between quarks, and those masses have purely gluonic nature! As a consequence, if neglecting gluon field we exactly obtain \( (m_P)_{\text{chiral}} = 0 \). Analogously, the root-mean-square radii of pions (kaons) of (16) are well-defined in chiral limit and \( < r >_{\text{chiral}} \approx 0.673069 \text{ fm or 0.543223} \text{ fm with parameters } a_j, b_j, B_j \) of Tables 1 and 4 respectively and those values
only slightly differ from 0.673837 fm or 0.544342 fm of Tables 2 and 5 accordingly at \( m_u, m_d, m_s \neq 0 \). The same holds true for decay constants of (19) for leptonic decays \( f_P = m_P |f_A(Q^2 = m_P^2)| \) with \( |f_A(Q^2 = m_P^2)| \) of (24). So, even in chirally symmetric world the pions and kaons would have nonzero masses, the root-mean-square radii and decay constants \( f_P \) for leptonic decays and all of those quantities would be determined only by SU(3)-gluonic interaction between massless quarks, i.e. they would have a purely gluonic nature. It should be emphasized that we can neglect neither Coulomb electric colour field of solution (3) (parameters \( a_1, a_2 \)) nor magnetic colour field (parameters \( b_1, b_2 \)) or else effect does vanish according to (29)–(30), i.e. both parts of SU(3)-gluonic field of (3) are important for confinement and mass generation in chiral limit. Moreover, since gluons are verily relativistic particles then the most part of mass for light mesons is conditioned by relativistic effects, as is seen from (29) and (30).

7.3 Quark condensate and chiral perturbation expansion

Thus, our approach neatly says that chiral symmetry is rather rough approximation holding true more or less only when neglecting SU(3)-gluonic field between quarks. But we historically know that notion of chiral symmetry arose when no indications to existence of gluons were. So this was a reasonable approach for its time. However, after discovering gluons in the late seventies of XX century, main constructions developed within chiral symmetry approach have not been essentially changed to take into account the fact of existence of gluons and such a situation is continuing up to now (see, e.g. reviews [15]).

Let us consider, for example, how one should interpret one of the key relations of chiral symmetry approach - the Gell-Mann-Oakes-Renner relation connecting the so-called quark condensate \( < 0|\bar{u}u + \bar{d}d|0 > \approx -2(240 \text{ MeV})^3 \) (an order parameter characterizing chiral symmetry breaking) with \( f_\pi, m_u, m_d, m_\pi \) (for more details see Refs. [15])

\[
 f_\pi^2 m_\pi^2 = -(m_u + m_d) < 0|\bar{u}u + \bar{d}d|0 > .
\]  

(31)

Then, under usual interpretation, since \( f_\pi \neq 0 \) (experimental fact !) in chiral limit one should conclude that \( m_\pi \sim 0 \) and identify pion with a Goldstone boson. It is clear that absence of gluons was tacitly supposed because when deriving (31) no gluons were known about.

As has been said above, however, presence of gluons between quarks entails that in chiral limit the left-hand side of (31) is well-defined and not equal to 0. Under the circumstances

\[
 < 0|\bar{u}u + \bar{d}d|0 > = -\left( \frac{f_\pi^2 m_\pi^2}{m_u + m_d} \right)_{\text{chiral}} \to -\infty ,
\]  

(32)
i.e., $<0|\bar{u}u + \bar{d}d|0>$ becomes unphysical, unobserved parameter. Similar
analogy: when the speed of light $c \to \infty$ relativistic mechanics passes on
to Newtonian one where $c$ is unobservable and no relations of Newtonian
mechanics depend on $c$. Accordingly, the quark condensate is rather crude
effective parameter that exists only when neglecting the fact of existence for
gluons. We can, of course, try to amend (31) by changing $m_u + m_d \to m_u +
m_d + (m_\pi)_{\text{chiral}}$ with $(m_\pi)_{\text{chiral}}$ of (29) but all the same $m_\pi$ will be $\sim 0$ only if
parameters of SU(3)-gluonic field $a_j, b_j$ are very small.

Also we can make a comment on the so-called chiral perturbation theory (for
more details see reviews [15]). Within the latter approach, for example, mass
square of $P$-meson ($P = \pi^\pm, K^\pm$) is sought in the form of chiral perturbation
expansion

$$m_P^2 = xA_P + yB_P + O(x^2, y^2)$$

(33)

with current masses $x, y$ of quarks composing $P$-meson while coefficients $A_P, B_P$
are calculated in accordance with the rather hazy rules [15]. We can, however,
see from (27)–(28) that actual form of $m_P^2$ is

$$m_P^2 = \left(A + x + y + \frac{xy}{x + y}B\right)^2 = f(x, y),$$

(34)

where $A, B$ depend only on gluonic field. But then standard differential cal-
culus says to us that function $f(x, y)$ (though being continuous at $x = y = 0$)
is not differentiable at $x = y = 0$ and, accordingly, $f(x, y)$ does not possess
the Taylor expansion of the form (33) at the point $x = y = 0$. As a result, the
expansion (33) is incorrect.

We should, however, be somewhat careful: from nowhere it follows that pa-
rameters $a_j, b_j, B_j$ of the confining SU(3)-gluonic field in chirally symmetric
world for pions and kaons should be the same as ones at present – the latter
were evaluated in accordance with (12) at nonzero $m_u, m_d, m_s$. Under the
situation we can at least describe two scenarios of chiral symmetry breaking.

7.4 First scenario

In chirally symmetric world (e.g., in early stages of universe evolution) massless
quarks interchange with gluons which generate nonzero masses of hadrons of
purely gluonic nature, as was discussed above. After spontaneous breaking
of symmetry in SM quarks acquire current masses through Higgs mechanism
which entails additional contributions to hadron masses but parameters of
gluon field between quarks remain the same. I.e., we suppose massless quarks
to emit gluons in the same proportion as massive ones. As a result, parameters
of gluon field in hadrons at present are the same as in chirally symmetric world.
7.5 Second scenario

After spontaneous breaking of symmetry in SM massive quarks emit gluons in other proportion than massless ones and parameters of gluon field in hadrons at present are different from those in chirally symmetric world. But for to evaluate the latter we should know, for example, pion mass in chirally symmetric world that is not equal to zero due to gluons as was discussed above.

7.6 Concluding remarks

At any rate, in either scenarios no additional mechanism of spontaneous symmetry breaking connected with Goldstone bosons is required. Another matter that massless quarks differ from each other only by their flavours and we come to the problem of origin for flavours. It is clear, however, the latter problem cannot be resolved within QCD which just takes flavours as given from outside. So problem of origin of flavours requires coming out from QCD-framework. By the way, one possible solution of this problem from cosmological positions was proposed by us a long time ago [17].

To summarize, our confinement mechanism gives a physically reasonable approach to problem of chiral symmetry breaking without any additional mechanism of spontaneous symmetry breaking connected with Goldstone bosons.

8 A possible relation with a phenomenological string-like picture of confinement

8.1 The confining potential and string tension

The results obtained in Sections 5–6 allow us to shed some light on one more problem which has been touched upon in Ref. [6]. As is known, during a long time up to now there exists the so-called string-like picture of quark confinement but only at qualitative phenomenological level (see, e. g., Ref. [18]). Up to now, however, it is unknown as such a picture might be warranted from the point of view of QCD. Let us in short outline how our results (based on and derived from QCD-Lagrangian directly) naturally lead to possible justification of the mentioned construction. Thereto we note that one can calculate energy $E$ of gluon condensate conforming to solution (3) in a volume $V$ through relation $E = \int_V T_{00} r^2 \sin \vartheta drd\vartheta d\varphi$ with $T_{00}$ of (25) but one should take into account that classical $T_{00}$ has a singularity along $z$-axis ($\vartheta = 0, \pi$) and we...
Fig. 1. Vertical projection of region with the gluon condensate energy between quarks

Fig. 2. Horizontal projection of region with the gluon condensate energy between quarks

have to introduce some angle $\vartheta_0$ so $\vartheta_0 \leq \vartheta \leq \pi - \vartheta_0$. As well as in Ref. [2], we may consider $\vartheta_0$ to be a parameter determining some cone $\vartheta = \vartheta_0$ so the quark emits gluons outside of the cone. Now if there are two quarks $Q_1, Q_2$ and each of them emits gluons outside of its own cone $\vartheta = \vartheta_{1,2}$ (see Figs. 1, 2) then we have soft gluons (as mentioned in Section 1) in regions I, II and between quarks.

Accordingly, we shall have some region $V$ with gluon condensate between quarks $Q_1, Q_2$ and its vertical projection is shown in Fig. 1. Another projection of $V$ onto a plane perpendicular to the one of Fig. 1 is sketched out in Fig. 2.

Then, as is clear from Fig. 1, for distance $R$ between quarks we have $R = R_1 \sin \vartheta_1 + R_2 \sin \vartheta_2$ and gluonic energy between quarks will be equal to

$$
\mathcal{V}(R) = \int_V T_{00} r^2 \sin \vartheta dr d\vartheta d\varphi = \int_{\vartheta_1}^{\pi - \vartheta_1} \int_{-\varphi_1}^{\varphi_1} \int_{r_1}^{r_2} \left( \frac{A}{r^2} + \frac{B}{\sin^2 \vartheta} \right) \sin \vartheta dr d\vartheta d\varphi + \int_{\vartheta_2}^{\pi - \vartheta_2} \int_{-\varphi_2}^{\varphi_2} \int_{r_2}^{r_1} \left( \frac{A}{r^2} + \frac{B}{\sin^2 \vartheta} \right) \sin \vartheta dr d\vartheta d\varphi
$$

(35)

with constants $A, B$ defined in (25).

To clarify a physical meaning of the quantities $r_{1,2}$ in Figs. 1, 2, let us recall an analogy with classical electrodynamics where is well known (see e.g. Ref. [19]) that the notion of classical electromagnetic field (a photon condensate) generated by a charged particle is applicable only at distances much greater than the Compton wavelength $\lambda_c = 1/m$ for the given particle with mass $m$. Within the QCD framework the parameter $\Lambda_{QCD}$ plays a similar part (see, e.g., Ref. [9,16]). Namely, the notion of classical SU(3)-gluonic field (a gluon condensate) is not applicable at the distances much less than $1/\Lambda_{QCD}$. In accordance with Section 3.1 we took $\Lambda_{QCD} = \Lambda = 0.234$ GeV which entails $1/\Lambda \sim 0.8433$ fm so one may consider $r_{1,2} \sim 0.1r_0$ where $r_0$ is adduced in Tables 7–8 for charged pions and kaons.

Under the circumstances, performing a simple integration in (35) with employing the relations $\int d\vartheta/\sin \vartheta = \ln \tan \vartheta/2$, $\tan \vartheta/2 = \sin \vartheta/(1 + \cos \vartheta) = (1 - \cos \vartheta)/\sin \vartheta$, we shall without going into details (see also Ref. [2]) obtain
\[
\mathcal{V}(R_1, R_2) = \mathcal{V}_0 - \sum_{i=1}^{2} \frac{4\varphi_i A \cos \vartheta_i}{R_i} + \sum_{i=1}^{2} 2\varphi_i B R_i \ln \frac{1 + \cos \vartheta_i}{1 - \cos \vartheta_i},
\]

where \(\mathcal{V}_0 = \sum_{i=1}^{2} \mathcal{V}_{0i} = \sum_{i=1}^{2} \left( \frac{4\varphi_i A \cos \vartheta_i}{R_i} - 2\varphi_i B R_i \ln \frac{1 + \cos \vartheta_i}{1 - \cos \vartheta_i} \right)\).

For the sake of simplicity let us put \(R_1 = R_2, \vartheta_1 = \vartheta_2 = \vartheta_0, \varphi_1 = \varphi_2 = \varphi_0\). Then \(R_1 = R_2 = R/(2 \sin \vartheta_0)\) and from (36) it follows

\[
\mathcal{V}(R) = \mathcal{V}_0 + \frac{a}{R} + kR
\]

with \(a = -8\varphi_0 A \sin 2\vartheta_0, k = 2\varphi_0 \frac{B}{\sin \vartheta_0} \ln \frac{1 + \cos \vartheta_0}{1 - \cos \vartheta_0}\).

We recognize the modeling confining potential in (37) which is often used when applying to meson and heavy quarkonia physics (see, e.g., Refs. [20,21]). We can, however, see that phenomenological parameters \(a, k, \mathcal{V}_0\) of potential (37) are expressed through more fundamental parameters \(a_j, b_j\) connected with the unique exact solution (3) of Yang-Mills equations describing confinement. One can notice that the quantity \(k\) (string tension) is usually related to the so-called Regge slope \(\alpha' = 1/(2\pi k)\) and in many if not all of the papers using potential approach it is accepted \(k \approx 0.18 \text{ GeV}^2\) (see, e.g., Refs. [20,21]).

### 8.2 Estimates of \(\vartheta_0, \varphi_0\) for charged pions and kaons

Under the situation we have the equation

\[
k = 2\varphi_0 \frac{B}{\sin \vartheta_0} \ln \frac{1 + \cos \vartheta_0}{1 - \cos \vartheta_0} \approx 0.18 \text{ GeV}^2
\]

with \(B = (b_1^2 + b_1 b_2 + b_2^2)/2\), so let us employ (38) to estimate \(\vartheta_0, \varphi_0\) if using the results obtained in Table 1 and 4 for charged pions and kaons and also for the ground state of toponium \(\eta_t\) for that we use the parametrization from Ref. [5] with the values \(a_1 = 0.361253, a_2 = 0.339442, b_1 = 48.9402 \text{ GeV}, b_2 = 76.7974 \text{ GeV}\) for the parameters of solution (3). Results of computations are presented in Table 9.

If taking into account that only the values of \(\vartheta_0, \varphi_0\) between 0 and 90° are of physical meaning and, according to Figs. 1, 2, the corresponding region \(V\) between quarks will be similar to a string-like one under the condition \(\vartheta_0 \to \pi/2, \varphi_0 \to 0\), then we can see from Table 9 that the characteristic transverse sizes \(D_{1,2}\) of the gluon condensate between quarks in fact tend to zero only in the case of heavy quarks, i.e., only for heavy quarks the gluon configuration between them might practically transform into a string. As a result, there arises the string-like picture of quark confinement but the latter seems to be warranted enough only for heavy quarks. It should be emphasized
Table 9
Angular parameters determining the gluon condensate between quarks for charged pions, kaons and toponium ground state.

| Particle      | $\theta_0$ | $\varphi_0$  |
|---------------|------------|--------------|
| $\pi^\pm$—ud, $\bar{u}d$ | 10° | 14.76° |
|               | 30° | 78.63° |
|               | 45° | 166.16° |
|               | 60° | 326.53° |
| $K^\pm$—u$s$, $\bar{s}u$ | 10° | 7.76° |
|               | 30° | 41.34° |
|               | 45° | 87.36° |
|               | 60° | 171.68° |
| $\eta_t$—tt | 10° | $(0.305 \times 10^{-4})^\circ$ |
|               | 30° | $(0.162 \times 10^{-3})^\circ$ |
|               | 45° | $(0.343 \times 10^{-3})^\circ$ |
|               | 60° | $(0.675 \times 10^{-3})^\circ$ |
|               | 80° | $(0.240 \times 10^{-2})^\circ$ |
|               | 88° | $(0.123 \times 10^{-1})^\circ$ |

that string tension $k$ of (38) is determined just by parameters $b_{1,2}$ of linear magnetic colour field from solution (3) which indirectly confirms the dominant role of the mentioned field for confinement.

We cannot, however, speak about potential $\mathcal{V}(R)$ of (37) as describing some gluon configuration between quarks. It would be possible if the mentioned potential were a solution of Yang-Mills equations directly derived from QCD-Lagrangian since, from the QCD-point of view, any gluonic field should be a solution of Yang-Mills equations (as well as any electromagnetic field is by definition always a solution of Maxwell equations).

In reality, as was shown in Refs. [2,3] (see also Appendix C in Ref. [5]), potential of form (37) cannot be a solution of the Yang-Mills equations if simultaneously $a \neq 0, k \neq 0$. Therefore, it is impossible to obtain compatible solutions of the Yang-Mills-Dirac (Pauli, Schrödinger) system when inserting potential of form (37) into Dirac (Pauli, Schrödinger) equation. So, we draw the conclusion (mentioned as far back as in Refs. [4] and elaborated more in detail in Ref. [5]) that the potential approach seems to be inconsistent: it is not based on compatible nonperturbative solutions for the Dirac-Yang-Mills system derived from QCD-Lagrangian in contrast to our confinement.
mechanism. Actually potential approach for heavy quarkonia has been historically modeled on positronium theory. In the latter case, however, one uses the unique modulo square integrable solutions of Dirac (Schrödinger) equation in the Coulomb field [condensate of huge number of (virtual) photons], i.e., one employs the unique compatible nonperturbative solutions of the Maxwell-Dirac (Schrödinger) system directly derived from QED-Lagrangian to describe positronium (or hydrogen atom) spectrum.

To summarize, from the point of view of our approach both potential and string-like pictures of confinement arise only as some effective models derived in a certain way from the more fundamental theory based on exact solution (3) of SU(3)-Yang-Mills equations. This conclusion is in concordance with the preliminary one obtained in Ref. [6].

9 Discussion and concluding remarks

9.1 Discussion

As is seen from Tables 7–8, at the characteristic scales of charged pions and kaons the gluon concentrations are huge and the corresponding fields (electric and magnetic colour ones) can be considered to be the classical ones with enormous strengths. The part $n_{\text{coul}}$ of gluon concentration $n$ connected with the Coulomb electric colour field is decreasing faster than $n_{\text{lin}}$, the part of $n$ related to the linear magnetic colour field, and at large distances $n_{\text{lin}}$ becomes dominant. It should be emphasized that in fact the gluon concentrations are much greater than the estimates given in Tables 7–8 because the latter are the estimates for maximal possible gluon frequencies, i.e. for maximal possible gluon impulses (under the concrete situation of charged pions and kaons). As was mentioned in Section 1, the overwhelming majority of gluons between quarks is soft, i.e., with frequencies much less than $\Gamma = 1/\tau \approx 0.253 \times 10^{-7}$ eV, for example, in the case of pions, so the corresponding concentrations are much greater than those in Tables 7–8. The given picture is in concordance with the one obtained in Refs. [4–8]. As a result, the confinement mechanism developed in Refs. [1–3] and described early in Section 1 is also confirmed by the considerations of the present paper.

It should be noted, however, that our results are of a preliminary character which is readily apparent, for example, from that the current quark masses (as well as the gauge coupling constant $g$) used in computation are known only within the certain limits and we can expect similar limits for the magnitudes discussed in the paper so it is necessary further specification of the parameters for the confining SU(3)-gluonic field in charged pions and kaons which can be
obtained, for instance, by calculating the width of decay $\pi^\pm \to \pi^0 + e^\pm + \nu_e (\bar{\nu}_e)$ with the help of wave function of $\pi^0$-meson discussed in Ref. [7] and so on. We hope to continue analysing the given problems elsewhere.

9.2 Concluding remarks

The results of present paper as well as the ones of Refs. [4–8] allow one to speak about that the confinement mechanism elaborated in Refs. [1–3] gives new possibilities for considering many old problems of hadronic (meson) physics (such as nonperturbative computation of decay constants, masses and radii of mesons, chiral symmetry breaking and so forth) from the first principles of QCD immediately appealing to the quark and gluonic degrees of freedom. This is possible because the given confinement mechanism is based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system directly derived from QCD-Lagrangian and, as a result, the approach is itself nonperturbative, relativistic from the outset, admits self-consistent nonrelativistic limit and may be employed for any meson (quarkonium).

The given paper to a certain degree summarizes studying nonet of pseudoscalar mesons realized in Refs. [6–8] within the framework of our approach and we can ascertain the fact that, on the whole, this nonet can be described from the unified point of view of our confinement mechanism. In line with the above, obviously, one should now pass on to vector mesons ($\rho, \phi, \omega...$) and also to the light scalar mesons whose nature has been controversial over 30 years [22]. As is clear from Section 2 and Appendices A, B, there exists a large number of relativistic bound states in the confining SU(3)-gluonic field (3) so all the mentioned mesons can probably correspond to some of those states and be described by their own sets of parameters $a_j, b_j, B_j$ of solution (3). Finally, one should think about possible ways to extend the approach over baryons, in particular, over nucleons.

Appendix A

We here represent some results about eigenspinors of the Euclidean Dirac operator on two-sphere $S^2$ employed in the main part of the paper.

When separating variables in the Dirac equation (4) there naturally arises the Euclidean Dirac operator $D_0$ on the unit two-dimensional sphere $S^2$ and we should know its eigenvalues with the corresponding eigenspinors. Such a problem also arises in the black hole theory while describing the so-called twisted spinors on Schwarzschild and Reissner-Nordström black holes and it
was analysed in Refs. [3,23], so we can use the results obtained therein for our aims. Let us adduce the necessary relations.

The eigenvalue equation for corresponding spinors \( \Phi \) may look as follows

\[
D_0 \Phi = \lambda \Phi. \tag{A.1}
\]

As was discussed in Refs. [23], the natural form of \( D_0 \) (arising within applications) in local coordinates \( \vartheta, \varphi \) on the unit sphere \( S^2 \) looks as

\[
D_0 = -i \sigma_1 \left[ i \sigma_2 \partial_{\vartheta} + i \sigma_3 \frac{1}{\sin \vartheta} \left( \partial_{\varphi} - \frac{1}{2} \sigma_2 \cos \vartheta \right) \right] = 
\sigma_1 \sigma_2 \partial_{\vartheta} + \frac{1}{\sin \vartheta} \sigma_1 \sigma_3 \partial_{\varphi} - \frac{\cot \vartheta}{2} \sigma_1 \sigma_2 \tag{A.2}
\]

with the ordinary Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

so that \( \sigma_1 D_0 = -D_0 \sigma_1 \).

The equation (A.1) was explored in Refs. [23]. Spectrum of \( D_0 \) consists of the numbers \( \lambda = \pm (l+1) \) with multiplicity \( 2(l+1) \) of each one, where \( l = 0, 1, 2, \ldots \). Let us introduce the number \( m \) such that \(-l \leq m \leq l+1\) and the corresponding number \( m' = m - 1/2 \) so \(|m'| \leq l + 1/2\). Then the conforming eigenspinors of operator \( D_0 \) are

\[
\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \Phi_{\pm \lambda} = \frac{C}{2} \left( P_{m' - 1/2}^k \pm P_{m' + 1/2}^k \right) e^{-im' \varphi} \tag{A.3}
\]

with the coefficient \( C = \sqrt{\frac{l+1}{2\pi}} \) and \( k = l + 1/2 \). These spinors form an orthonormal basis in \( L^2_2(S^2) \) and are subject to the normalization condition

\[
\int_{S^2} \Phi^\dagger \Phi d\Omega = \int_0^{2\pi} \int_0^{\pi} \left( |\Phi_1|^2 + |\Phi_2|^2 \right) \sin \vartheta d\vartheta d\varphi = 1 . \tag{A.4}
\]

Further, owing to the relation \( \sigma_1 D_0 = -D_0 \sigma_1 \) we, obviously, have

\[
\sigma_1 \Phi_{\mp \lambda} = \Phi_{\pm \lambda} . \tag{A.5}
\]

As to functions \( P^k_{m', n'}(\cos \vartheta) \equiv P^k_{m', n'}(\cos \vartheta) \) then they can be chosen by miscellaneous ways, for instance, as follows (see, e. g., Ref. [24])

\[
P^k_{m', n'}(\cos \vartheta) = i^{-m' - n'} \sqrt{\frac{(k - m')!(k - n')!}{(k + m')!(k + n')!}} \left( \frac{1 + \cos \vartheta}{1 - \cos \vartheta} \right)^{m' - n' / 2} \times
\]

26
\[
\sum_{j=\max(m',n')}^{k} \frac{(k+j)!i^{2j}}{(k-j)!(j-m')!(j-n')!} \left( \frac{1 - \cos \varrho}{2} \right)^j
\]

(A.6)

with the orthogonality relation at \(m', n'\) fixed

\[
\int_0^\pi P_{m'n'}^k(\cos \vartheta) P_{m'n'}^{k'}(\cos \vartheta) \sin \vartheta d\vartheta = \frac{2}{2k+1} \delta_{k,k'}.
\]

(A.7)

It should be noted that square of \(D_0\) is

\[
D_0^2 = -\Delta g^2 I_2 + \sigma_2 \sigma_3 \frac{\cos \vartheta}{\sin^2 \vartheta} \partial_\varphi + \frac{1}{4 \sin^2 \vartheta} + \frac{1}{4},
\]

(A.8)

while laplacian on the unit sphere is

\[
\Delta g^2 = \frac{1}{\sin \vartheta} \partial_\vartheta \sin \vartheta \partial_\vartheta + \frac{1}{\sin^2 \vartheta} \partial_\varphi^2 = \partial_\vartheta^2 + \cot \vartheta \partial_\vartheta + \frac{1}{\sin^2 \vartheta} \partial_\varphi^2,
\]

(A.9)

so the relation (A.8) is a particular case of the so-called Weitzenböck-Lichnerowicz formulas (see Refs. [25]). Then from (A.1) it follows \(D_0^2 \Phi = \lambda^2 \Phi\) and, when using the ansatz \(\Phi = P(\vartheta)e^{-im'\varphi}\), \(P_{1,2} = P_{1,2}(\vartheta)\), the equation \(D_0^2 \Phi = \lambda^2 \Phi\) turns into

\[
\left(-\partial_\vartheta^2 - \cot \vartheta \partial_\vartheta + \frac{m^2 + \frac{1}{4}}{\sin^2 \vartheta} + \frac{m' \cos \vartheta}{\sin^2 \vartheta} - \sigma_1\right) P = \left(\lambda^2 - \frac{1}{4}\right) P,
\]

(A.10)

wherefrom all the above results concerning spectrum of \(D_0\) can be derived [23].

When calculating the functions \(P_{m'n'}^k(\cos \vartheta)\) directly, to our mind, it is the most convenient to use the integral expression [24]

\[
P_{m'n'}^k(\cos \vartheta) = \frac{1}{2\pi} \sqrt{\frac{(k-m')!(k+m')!}{(k-n')!(k+n')!}} \int_0^{2\pi} \left( e^{i\varphi/2} \cos \frac{\vartheta}{2} + ie^{-i\varphi/2} \sin \frac{\vartheta}{2} \right)^{k-n'} \times \left( ie^{i\varphi/2} \sin \frac{\vartheta}{2} + e^{-i\varphi/2} \cos \frac{\vartheta}{2} \right)^{k+n'} e^{im'\varphi} d\varphi
\]

(A.11)

and the symmetry relations \((z = \cos \vartheta)\)

\[
P_{m'n'}^k(z) = P_{n'm'}^k(z), \quad P_{m',-n'}^k(z) = P_{-m',n'}^k(z), \quad P_{m'n'}^k(z) = P_{-m',n'}^k(z), \quad P_{m'n'}^k(-z) = i^{2k-2m'-2n'} P_{m',n'}^k(z).
\]

(A.12)
In particular

\[ P_{kk}(z) = \cos^{2k}(\vartheta/2), \quad P_{k,-k}(z) = i^{2k}\sin^{2k}(\vartheta/2), \quad P_{k0}(z) = \frac{i^{k}\sqrt{(2k)!}}{2k!} \sin^{k}\vartheta, \]

\[ P_{kn}(z) = i^{k-n'}\sqrt{\frac{(2k)!}{(k-n'!)(k+n')!}} \sin^{k-n'}(\vartheta/2) \cos^{k+n'}(\vartheta/2). \quad (A.13) \]

**Eigenspinors with \( \lambda = \pm 1, \pm 2 \)**

If \( \lambda = \pm(l+1) = \pm 1 \) then \( l = 0 \) and from (A.3) it follows that \( k = l+1/2 = 1/2, |m'| \leq 1/2 \) and we need the functions \( P^{1/2}_{m',\pm 1/2} \) that are easily evaluated with the help of (A.11)–(A.13) so the eigenspinors for \( \lambda = -1 \) are

\[ \Phi = \frac{C}{2} \left( \cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \right) e^{i\varphi/2}, \quad \Phi = \frac{C}{2} \left( -\cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \right) e^{-i\varphi/2}, \quad (A.14) \]

while for \( \lambda = 1 \) the conforming spinors are

\[ \Phi = \frac{C}{2} \left( \cos \frac{\vartheta}{2} - i \sin \frac{\vartheta}{2} \right) e^{i\varphi/2}, \quad \Phi = \frac{C}{2} \left( -\cos \frac{\vartheta}{2} - i \sin \frac{\vartheta}{2} \right) e^{-i\varphi/2} \quad (A.15) \]

with the coefficient \( C = \sqrt{1/(2\pi)} \).

It is clear that (A.14)–(A.15) can be rewritten in the form

\[ \lambda = -1 : \Phi = \frac{C}{2} \left( e^{i\varphi/2} e^{-i\varphi/2}, \quad or \quad \Phi = \frac{C}{2} \left( -e^{i\varphi/2} \right) e^{-i\varphi/2} \right. \]

\[ \lambda = 1 : \Phi = \frac{C}{2} \left( -e^{-i\varphi/2} e^{i\varphi/2}, \quad or \quad \Phi = \frac{C}{2} \left( e^{-i\varphi/2} \right) e^{i\varphi/2} \right. \quad (A.16) \]

so the relation (A.5) is easily verified at \( \lambda = \pm 1 \).

In studying vector mesons and excited states of heavy quarkonia eigenspinors with \( \lambda = \pm 2 \) may also be useful. Then \( k = l+1/2 = 3/2, |m'| \leq 3/2 \) and we need the functions \( P^{3/2}_{m',\pm 1/2} \) that can be evaluated with the help of (A.11)–(A.13). Computation gives rise to

\[ P^{3/2}_{3/2,-1/2} = -\frac{\sqrt{3}}{2} \sin \vartheta \sin \frac{\vartheta}{2} = P^{3/2}_{-3/2,1/2}; \]

\[ P^{3/2}_{3/2,1/2} = \frac{i\sqrt{3}}{2} \sin \vartheta \cos \frac{\vartheta}{2} = P^{3/2}_{-3/2,-1/2}; \]

\[ P^{3/2}_{1/2,-1/2} = -\frac{i}{4} \left( \sin \frac{\vartheta}{2} - 3 \sin \frac{3}{2} \vartheta \right) = P^{3/2}_{-1/2,1/2}; \]
and according to (A.3) this entails eigenspinors with \( \lambda = 2 \) in the form

\[
\frac{C}{2} \frac{\sqrt{3}}{2} \sin \theta \left( e^{-i \theta/2} e^{i \varphi/2} \right) e^{i \varphi/2}, \quad \frac{C}{8} \frac{3ei^{3\theta/2} + e^{i \theta/2}}{3ei^{3\theta/2} + e^{-i \theta/2}} e^{i \varphi/2},
\]

with \( C = 1/\sqrt{\pi} \), while eigenspinors with \( \lambda = -2 \) are obtained in accordance with relation (A.5).

**Appendix B**

We here adduce the explicit form for the radial parts of meson wave functions from (6). At \( n_j = 0 \) they are given by

\[
F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left( 1 - \frac{Y_j}{Z_j} \right), \quad F_{j2} = i C_j Q_j r^{\alpha_j} e^{-\beta_j r} \left( 1 + \frac{Y_j}{Z_j} \right),
\]

while at \( n_j > 0 \) by

\[
F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left[ \left( 1 - \frac{Y_j}{Z_j} \right) L_{n_j}^{\alpha_j}(r_j) + \frac{P_j Q_j}{Z_j} r_j L_{n_j-1}^{\alpha_j+1}(r_j) \right],
\]

\[
F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r} \left[ \left( 1 + \frac{Y_j}{Z_j} \right) L_{n_j}^{\alpha_j}(r_j) - \frac{P_j Q_j}{Z_j} r_j L_{n_j-1}^{\alpha_j+1}(r_j) \right]
\]

with the Laguerre polynomials \( L_{n_j}^{\alpha_j}(r_j) \), \( r_j = 2\beta_j r \), \( \beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2b_j^2} \) at \( j = 1, 2, 3 \) with \( b_3 = -(b_1 + b_2) \), \( P_j = gb_j + \beta_j \), \( Q_j = \mu_0 - \omega_j \), \( Y_j = P_j Q_j \alpha_j + (P_j^2 - Q_j^2) g a_j/2 \), \( Z_j = P_j Q_j \lambda_j + (P_j^2 + Q_j^2) g a_j/2 \) with \( a_3 = -(a_1 + a_2) \), \( \lambda_j = \lambda_j - g B_j \) with \( B_3 = -(B_1 + B_2) \), \( \alpha_j = \sqrt{\lambda_j^2 - g^2 a_j^2} \), while \( \lambda_j = \pm (l_j + 1) \) are the eigenvalues of Euclidean Dirac operator \( D_0 \) on unit two-sphere with \( l_j = 0, 1, 2, ... \) (see Appendix A) and quantum numbers \( n_j = 0, 1, 2, ... \) are defined by the relations

\[
n_j = \frac{gb_j Z_j - \beta_j Y_j}{\beta_j P_j Q_j},
\]

which entails the quadratic equation (7) and spectrum (8). Further, \( C_j \) of (B.1)–(B.2) should be determined from the normalization condition

\[
\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{3}.
\]
As a consequence, we shall gain that in (6) \( \Psi_j \in L^4_2(\mathbb{R}^3) \) at any \( t \in \mathbb{R} \) and, accordingly, \( \Psi = (\Psi_1, \Psi_2, \Psi_3) \) may describe relativistic bound states in the field (3) with the energy spectrum (8). As is clear from (B.3) at \( n_j = 0 \) we have \( gb_j/\beta_j = Y_j/Z_j \) so the radial parts of (B.1) can be rewritten as

\[
F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left( 1 - \frac{gb_j}{\beta_j} \right), F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r} \left( 1 + \frac{gb_j}{\beta_j} \right). \quad (B.5)
\]

More details can be found in Refs. [1,3].

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