Hairy black holes in the XX-th and XXI-st centuries

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This is a brief summary of the most important hairy black hole solutions in 3+1 spacetime dimensions discovered over the last 25 years. These were first of all the Einstein-Yang-Mills black holes and their various generalizations including the Higgs field, the dilaton and the curvature corrections, and also the Skyrme black holes. More recently, these were black holes supporting a scalar field violating the energy conditions or non-minimally coupled to gravity, and also spinning black holes with massive complex scalar hair. Finally, these were black holes with massive graviton hair.

Keywords: black holes, no-hair conjecture

1. Introduction

The famous no-hair conjecture formulated almost half a century ago is still a hot research topic. It essentially states that all black holes in Nature are of the Kerr-Newman type, and for a long time this statement was broadly considered to be true. Besides, it was supported by a number of mathematical results – the uniqueness theorems in the Einstein-Maxwell theory and the no-hair theorems proven for a number of other field-theory models.

However, 20 years later the first manifest counter-example to the no-hair conjecture was found in the context of the gravity-coupled Yang-Mills theory. This discovery triggered an avalanche of similar findings during the last decade of the past XX-th century, when it became clear that hairy black holes generically exist in systems with non-Abelian fields. As a result, one can say that, strictly speaking, the no-hair conjecture is incorrect. At the same time, in most cases hairy black holes with non-Abelian fields are either microscopically small or unstable and when become large or perturbed they loose their hair. Therefore, the conjecture essentially applies to large black holes that are astrophysically relevant.

A new interest towards hairy black holes has emerged in the current XXI-st century, but the focus is now shifted from gravitating non-Abelian fields towards gravitating scalars. This is a consequence of the discovery of the dark energy, which can probably be modelled by a scalar field. If this field violates the energy conditions or couples non-minimally to gravity, then the standard no-hair theorems do not apply and there could be hairy black holes. Therefore, one is currently interested in black holes with exotic scalar fields, such as phantoms, Galileons, etc. Surprisingly, this has lead to an astonishing discovery also within the conventional model of a complex massive scalar field that was shown to admit spinning hairy black holes. Yet one more theory that has recently become popular, also in connection with the
dark energy problem, is the ghost-free massive gravity, and there are hairy black holes in this theory too.

In view of all this, this text presents an attempt to briefly summarize the most important asymptotically flat hairy black hole solutions in 3+1 spacetime dimensions found after their first discovery more than 25 years ago. This subject is so vast that it is hardly possible to be complete or merely objective, but at least some idea of it should be given by what follows.

2. No-hair conjecture

In 1969 Ruffini and Wheeler summarized the progress in black hole physics of the time in the famous phrase: *black holes have no hair*. This means that

- All stationary black holes are completely characterized by their mass, angular momentum, and electric charge seen from far away in the form of Gaussian fluxes.
- Black holes cannot support hair = any other independent parameters not seen from far away.

Therefore, according to this no-hair conjecture, the only allowed characteristics of stationary black holes are those associated with the Gauss law. The logic behind this is the following. Black holes are formed in the gravitational collapse, which is so violent a process that it breaks all conservation laws not related to the exact symmetries. For example, the chemical content, atomic structure, baryon number, etc. are not conserved during the collapse – the black hole ‘swallows’ and loses all the memory of them. Only few exact local symmetries, such as the local Lorentz or local $U(1)$, can survive the gravitational collapse. Associated to them conserved quantities – the mass $M$, angular momentum $J$, and electric charge $Q$ – cannot be absorbed by the black hole but remain attached to it as parameters. They give rise to the Gaussian fluxes that can be measured at infinity. This implies that black holes are very simple objects – they are characterized by only three parameters, and black holes with the same $M, J, Q$ are identically equal.

The only known in 1969 black holes were the Kerr-Newman solutions. They are characterized by precisely three parameters $M, J, Q$, in perfect agreement with the conjecture. Moreover, a chain of uniqueness theorems initiated by Israel\(^2\)–\(^4\) (see\(^5\) for a review) established that the Kerr-Newman solutions describe all possible stationary *electrovacuum* black holes with a non-degenerate horizon\(^6\). This proves the no-hair conjecture within the Einstein-Maxwell theory.

Of course, not everything can be described by the Einstein-Maxwell theory, and the conjecture does not actually require all black holes in Nature to be necessarily

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\(^2\) Black holes can also carry a magnetic charge, if it exists.

\(^3\) Stationary solutions with zero surface gravity comprise the Israel-Wilson family of multi-black holes.
Kerr-Newman. However, it requires them, whatever they are, to be completely characterized by their charges. Only the non-uniqueness—existence of different black holes with exactly the same Gaussian charges would contradict the conjecture. Therefore, to see if the conjecture applies to other field theories, one should study the corresponding black hole solutions.

Consider a gravity-coupled field or a system of fields of any spin collectively denoted by $\Psi$. The corresponding field equations together with the Einstein equations read schematically

$$
G_{\mu\nu} = \kappa T_{\mu\nu}(\Psi), \quad \Box \Psi = U(\Psi).
$$

One can wonder if these equations admit black hole solutions. According to the conjecture, there should be either no such solutions at all, or only solutions labelled by their charges. In view of this, a number of the no-hair theorems, largely due to Bekenstein\textsuperscript{7–11}, have been proven to confirm the absence of hairy black hole solutions of Eqs.(1) in cases where $\Psi$ denotes a free scalar, spinor, massive vector, etc. field. The common feature in all these cases is that if $\Psi$ does not vanish, then the field equations require that it should diverge at the black hole horizon, where the curvature would diverge too. Notably, this happens for the massive fields. Therefore, to get regular black holes one is bound to set $\Psi = 0$, but then the solution is a vacuum black hole belonging to the Kerr-Newman family. Similar no-go results can be established also for the interacting fields\textsuperscript{12–15}. All this was confirming the non-existence of hairy black holes.

A peculiar finding was made in 1972 in the context of the theory with a free conformally-coupled scalar field,

$$
L = \frac{1}{4} R - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{12} R \Phi^2.
$$

This theory admits a static (BBMB) black hole solution\textsuperscript{16,17},

$$
ds^2 = - \left( 1 - \frac{M}{r} \right)^2 dt^2 + \frac{dr^2}{\left( 1 - \frac{M}{r} \right)^2} + r^2 d\Omega^2, \quad \Phi = \sqrt{3} \frac{M}{r},
$$

where $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$. The metric is extreme Reissner-Nordstrom but the scalar field is non-trivial, which looks like hair. However, this is merely secondary hair without own independent parameters, the only free parameter of the solution being the black hole mass $M$. Hence, the uniqueness is preserved and the no-hair conjecture holds.

The first explicit example of black holes whose geometry is not Kerr-Newman was found by Gibbons in 1982\textsuperscript{18} (see also\textsuperscript{19}) in the context of a supergravity model whose simplest version is the Einstein-Maxwell-dilaton theory,

$$
L = \frac{1}{4} R - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{4} e^{2\Phi} F_{\mu\nu} F^{\mu\nu}.
$$

The solution is obtained by setting $F = P \sin \vartheta d\vartheta \wedge d\varphi$ and

$$
ds^2 = -N dt^2 + \frac{dr^2}{N} + e^{2\Phi} r^2 d\Omega^2, \quad N = 1 - \frac{2M}{r}, \quad e^{2\Phi} = 1 - \frac{P^2}{M r},
$$
it describes static black holes with a purely magnetic Maxwell field and a non-trivial scalar field. The independent parameters are the black hole mass $M$ and the magnetic charge $P$, both subject to the Gauss law, while the scalar field carries no extra parameters, hence it is again secondary hair\(^c\).

The first indication of independent primary hair on black holes was found by Luckock and Moss in 1986 in the context of the gravity-coupled Skyrme model\(^a\). This can be viewed as a theory of three real scalars $\Phi^a$ defining a unitary matrix $U = \exp\{i\tau^a\Phi^a\}$ (here $\tau^a$ are the Pauli matrices) with the Lagrangian

$$\mathcal{L} = \frac{1}{2\kappa} R - \frac{\alpha}{2} \text{tr} (\partial_\mu U \partial^\mu U^\dagger) - \frac{\beta}{4} \text{tr} (F_{\mu\nu} F^{\mu\nu})$$

(6)

where $F_{\mu\nu} = [U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]$. If the gravitational coupling $\kappa$ is small then the backreaction of the scalars on the spacetime geometry should also be small, hence the scalars can be obtained by solving the scalar field equations on a fixed Schwarzschild background. Assuming a static and spherically symmetric ansatz $\Phi^a = f(r) n^a$ where $n^a$ is the normal to the sphere, Luckock and Moss found a regular solution for $f(r)$ on the Schwarzschild background. This can be viewed as a perturbative approximation for a hairy black hole solution in the limit where the backreaction is negligible. A numerical evidence for a fully back-reacting solution was reported by Luckock\(^b\), but this work was published only in the conference proceedings and has gone unnoticed.

3. XX-th century hairy black holes

The first broadly recognized example of a manifest violation of the no-hair conjecture was found by Volkov and Gal’tsov in 1989\(^c\). This result, soon confirmed by other groups\(^d\),\(^e\), was obtained within the Einstein-Yang-Mills (EYM) theory with gauge group $SU(2)$ defined by the Lagrangian

$$\mathcal{L} = \frac{1}{4} R - \frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} \quad \text{with} \quad F^{a\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon_{abc} A^b_\mu A^c_\nu.$$

(7)

This theory was natural to consider, since the no-hair conjecture allows black holes to have charges associated with local internal symmetries, but the latter can be non-Abelian. As first observed by Yasskin\(^f\), any electrovacuum black hole can be embedded into the EYM theory via multiplying its $U(1)$ gauge potential by a constant hermitian matrix. This gives a solution of the EYM equations but the geometry is still Kerr-Newman. One can show that the theory admits no other black holes with non-vanishing Yang-Mills charges, at least in the static and spherically symmetric case\(^g\). However, there is still a possibility to have black holes without a Yang-Mills charge, and it is in this way that the new solutions were found.

\(^c\)One can generalize the solution to include a third parameter, the asymptotic value $\Phi(\infty)$, but this parameter would relate not to the black hole itself and rather to the surrounding world.
Making the static, spherically symmetric, and purely magnetic ansatz for the fields,
\[ ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2, \quad A^a_i = \epsilon_{aik} \frac{r^k}{r^2} (1 - w(r)), \]
the field equations reduce to a system of three coupled ODEs for three unknown functions \( \sigma(r), N(r), w(r) \). To solve these equations numerically, one assumes a regular event horizon at a point \( r_h > 0 \) where \( N(r_h) = 0 \) while \( \sigma(r_h) \neq 0 \). It turns out that for a discrete sequence of horizon values \( w(r_h) = w_n (n = 1, 2, \ldots) \) of the Yang-Mills amplitude the solution extends from \( r = r_h \) towards large \( r \) and approaches the Schwarzschild metric as \( r \to \infty \). The Yang-Mills amplitude \( w(r) \) oscillates \( n \) times in the \( r > r_h \) region (see Fig.1), but in the far field zone one has \( w(r) \to (-1)^n \) and \( F^a_{ik} \sim 1/r^3 \), hence the Yang-Mills charge is zero. Therefore, the Yang-Mills field is supported by the black hole but is not visible from far away.

The solutions can be labeled by two parameters: their ADM mass \( M \) and the integer number \( n \) of oscillations of the Yang-Mills field. Only \( M \) is measurable at infinity, but for a given \( M \) there are infinitely many black holes with different \( n \)'s whose structure in the near field zone is different. Therefore, the Yang-Mills hair is primary as it carries an independent parameter not visible from far away, hence the uniqueness is violated.

Profiles of these EYM black holes are shown in Fig.1. Their horizon size \( r_h \) can be arbitrary, and it is interesting to see that for \( r_h \to 0 \) the black hole masses approach finite values corresponding to masses of the lumps – globally regular particle-like objects made of gravitating Yang-Mills fields. This allows one to view the hairy

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**Fig. 1.** Left: profiles \( w(r), \sigma(r), m(r) \) of the EYM black hole solutions with \( r_h = 1 \) (in Planck units); one has \( N(r) = 1 - 2m(r)/r \). Right: the ADM mass \( M = m(\infty) \) against \( r_h \).
black holes as non-linear superpositions of vacuum black holes and the lumps – “horizons inside classical lumps” 29.

The discovery of the EYM hairy black holes triggered an avalanche of similar findings in other models. The procedure is similar to that in the EYM theory: one uses the metric (8) parameterized by $N(r), \sigma(r)$, in addition there are matter field amplitudes. For example, in the Skyrme theory (6) this is the function $f(r)$ in $\Phi^a = n^a f(r)$. One has at the horizon $N(r_h) = 0$ and one adjusts the horizon values $f(r_h), \sigma(r_h)$ in such a way that the numerical solution matches the asymptotic values $N(\infty) = \sigma(\infty) = 1, f(\infty) = 0$. This yields black holes with the Skyrme hair 30–32, or with some other hair type depending on the model considered.

In most cases hairy black holes were found within various extensions of the EYM theory (7). For example, the theory can be generalized to gauge group $SU(N)$, in which case black holes carry a non-zero (chromo)electric charge 33, but there is still a part of the gauge potential that oscillates in the near-horizon region and is not characterized by any charge, thus violating the uniqueness.

One can extend the EYM theory by adding a Higgs field triplet $\Phi^a$ in the adjoint representation,

$$ L = \frac{1}{2\kappa} R - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a - \frac{\lambda}{4} (\Phi^a \Phi^a - v^2)^2, \quad (8) $$

where $D_\mu \Phi^a = \partial_\mu \Phi^a + g f_{abc} A_\mu^b \Phi^c$ and $g, \lambda, v$ are parameters. In flat space limit, $\kappa \to 0$, the theory admits regular soliton solutions – magnetic monopoles. For $\kappa \neq 0$ there are regular gravitating monopoles, but there are also black holes with non-trivial Yang-Mills-Higgs hair violating the uniqueness 34–36. Hairy black holes exist also within a global version of this theory 37. Alternatively, the EYM theory can be extended by adding a doublet Higgs field in the fundamental representation, in which case one also finds gravitating solitons and hairy black holes 38.

Yet another possibility to extend the pure EYM model (7) is to add terms inspired by string theory, for example

$$ L = \frac{1}{2\kappa} R - \frac{1}{2} (\partial \Phi)^2 + e^{\gamma \Phi} \left( - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \alpha G_{GB} \right) - V(\Phi), \quad (9) $$

where $\gamma, \alpha$ are parameters and $G_{GB} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$ is the Gauss-Bonnet invariant. Setting here $\alpha = V(\Phi) = 0$ gives the EYM-dilaton theory, admitting the hairy black holes and regular lumps very similar to those in the pure EYM theory 39. Setting $\alpha = 0, \gamma = 2, V(\Phi) = -(1/8) \exp(-2\Phi)$ gives a sector of the $N = 4$ gauged supergravity whose hairy black hole solutions can be used for a holographic description of confinement 40. One can consistently set the gauge field to zero, and what remains corresponds (if $V(\Phi) = 0$) to the dilaton-Gauss-Bonnet model,

$$ L = \frac{1}{2\kappa} R - \frac{1}{2} (\partial \Phi)^2 + \alpha e^{\gamma \Phi} G_{GB}. \quad (10) $$

This theory also admits hairy black holes, but the scalar hair is secondary and completely determined by the black hole mass 41–44.
Quite a lot has been done to study also more general hairy black holes with lower symmetry. This has revealed surprising facts. For example, although it is natural to believe the static black holes to be always spherically symmetric (if their horizon is non-degenerate), one finds that static EYM(-dilaton) black holes can be only axially symmetric\(^45\) (a similar result holds for the Skyrme black holes\(^46\)). The EYM-dilaton theory admits also stationary spinning black holes, which is not very surprising, however, there are solutions which remain stationary and non-static even when their angular momentum vanishes\(^47\). Stationary spinning black holes have also been constructed in the dilaton-Gauss-Bonnet theory\(^48\), in the Einstein-Skyrme theory\(^49\), and in the conformally-coupled model (2)\(^50\).

Here is a brief summary of some common features of the XX-th century hairy black holes. They exist in generic models with gravity-coupled non-Abelian gauge fields and support primary Yang-Mills hair characterized by radial oscillations. They can support in addition a scalar Higgs and/or dilaton field. Static solutions may or may not be spherically symmetric, while stationary solutions may be spinning but not necessarily. When the black hole horizon shrinks to zero, the solutions reduce not to the vacuum but to globally-regular “lumps” made of the gravitating Yang-Mills(+other) fields. Therefore, the hairy black holes can be viewed as horizons inside lumps.

In models containing apart from Planck’s mass also another mass scale, as for example in the EYM-Higgs theory, the black holes have a finite maximal size above which they loose their hair and reduce to electrovacuum black holes. In models with only one (Planck) mass scale, as in the EYM(-dilaton) theory, hairy black holes can be of any size. On the other hand, solutions in models with only one mass scale are generically unstable\(^51,52\), while those in the two-scale models can be stable – at least they do not show the most dangerous S-mode instability\(^36,53\). The unstable solutions in the pure EYM theory can be stabilized by adding a negative cosmological constant, which introduces a second scale\(^54\). A more detailed account of the XX-th century black holes can be found in the review article\(^55\).

4. XXI-st century black holes – solutions with a scalar field

One discovered at the very end of the last century that our universe is actually accelerating\(^56,57\). Explaining this fact within the GR context requires introducing a dark energy of an unclear origin. Assuming it to be dynamical and not just a cosmological constant, the most popular dark energy models consider a cosmic scalar field. This field couples to gravity, but not necessarily minimally, and its energy may be not necessarily positive. This explains the current interest towards models with gravitating scalar fields, but if one believes that they describe cosmology, then they should describe black holes as well.
4.1. Minimal models violating the energy conditions

The simplest possibility is to consider

\[ \mathcal{L} = \frac{1}{4} R - \frac{1}{2} (\partial \Phi)^2 - V(\Phi). \]  

However, if \( V(\Phi) \geq 0 \) (the strong energy condition) then one can show\(^{15}\) that the only static and spherically symmetric black hole in the model is the vacuum one, with \( \Phi = \Phi_0 \) and \( V(\Phi_0) = 0 \). Therefore, to obtain black holes in this case one should abandon the strong energy condition. At the same time, according to the modern paradigm, the potential \( V(\Phi) \) may be not necessarily positive, and if it is non-positive definite, then one finds indeed asymptotically flat hairy black holes. Such solutions can be constructed numerically for a chosen function \( V(\Phi) \)^{58}, or the potential can be specially adjusted to obtain solutions analytically. For example, choosing^{59-62}

\[ V(\Phi) = 3 \sinh(2\Phi) - 2\Phi [\cosh(2\Phi) + 2] \]

yields, as can be directly checked, an exact solution of the theory,

\[ ds^2 = -N dt^2 + \frac{dr^2}{N} + R^2 d\Omega^2, \quad R^2 = r + 2Q, \]
\[ N = 1 - 4 [Q(Q + r) - R^2 \Phi], \quad e^{2\Phi} = 1 + \frac{2Q}{r}. \]

If \( 4Q^2 > 1 \) then this describes hairy black holes since \( N(r_h) = 0 \) at \( r_h > 0 \) while for \( r \to \infty \) one has \( N = 1 - 8Q^2/(3r) + \ldots \) and \( \Phi = Q/r + \ldots \). The theory also admits the vacuum Schwarzschild solution with \( \Phi = 0, R = r \) and \( N = 1 - 2M/r \). These solutions are completely characterized by their mass \( M \) and the scalar charge \( Q \), since to different \( M, Q \) pairs there correspond different solutions. However, only \( M \) gives rise to the Gaussian flux, and there can be different solutions with the same \( M \). Hence, the no-hair conjecture is violated.

The physical role of these solutions is not very clear, but at least they are simple enough. Other explicitly known solutions^{63,64} obtained via adjusting the potential \( V(\Phi) \) are extremely complicated, although some of them may perhaps be related to a supergravity model^{65}. Unfortunately, all solutions of this type seem to be generically unstable under small fluctuations^{58,66}.

One may violate also the weak energy condition by considering the phantom model with a negative kinetic energy,

\[ \mathcal{L} = \frac{1}{4} R - \frac{1}{2} (\partial \Phi)^2 - V(\Phi). \]

It turns out^{67} that in this case too one needs a non-positive definite potential to get black holes, but these solutions^{68} are unstable as well^{69}. 
4.2. Spinning black holes with scalar hair

The above solutions are rather exotic, but the analysis of the scalar models has lead to an important discovery within the simplest model with a massive complex scalar,

\[ \mathcal{L} = \frac{1}{4} R - |\partial \Phi|^2 - \mu^2 |\Phi|^2. \]  

There is a no-hair theorem\(^70\) that forbids static and spherically symmetric black holes in this case. Nevertheless, Herdeiro and Radu\(^71\) were able to show that there are stationary spinning black holes with primary scalar hair which do not have the static limit. Their existence can be revealed by the following means. Setting \( \Phi = 0 \), the theory admits the vacuum Kerr solution,

\[ ds^2 = -N dt^2 + \frac{1}{\Delta} (d\varphi + W dt)^2 + R^2 (dr^2 + r^2 d\vartheta^2), \]  

where \( N, \Delta, W, R \) depend on \( r, \vartheta \). The solution is characterized by its mass \( M \) and angular momentum \( J \). Suppose \( \Phi \) is so small that one can neglect its backreaction on the geometry, then it should fulfil the Klein-Gordon equation on the Kerr background. It turns out\(^72\) that this equation admits stationary bound state solutions (scalar clouds) described by

\[ \Phi = F(r, \vartheta) \exp\{i\omega t + im\varphi\} \]  

with \( m = \pm 1, \pm 2, \ldots \) and \( \omega = m\Omega_H \) where \( \Omega_H \) is the event horizon angular velocity. For these solutions one can compute the global Noether charge \( Q \) distributed outside the black hole, and since \( \Phi \) is assumed to be small, \( Q \) is small too. These scalar clouds on the Kerr background can be viewed as an approximation to the fully back-reacting solutions described by three parameters \( M, J, Q \) in the limit where \( Q \) is small. The full solutions for a non-small \( Q \) were constructed by Herdeiro and Radu\(^71\) within the ansatz \((16),(17)\) by starting from the scalar clouds and then iteratively increasing the scalar field amplitude, adjusting at the same time \( \omega \) and the event horizon size to preserve the regularity at the horizon and at infinity.

Here are some properties of these solutions. For a given value of the azimuthal number \( m \) they can be labeled by \( M, J, Q \). Only \( M \) and \( J \) are seen at infinity and there are different solutions with the same \( M, J \), hence the no-hair conjecture is violated. For \( Q \to 0 \) one recovers the scalar clouds. When the event horizon shrinks, the solutions reduce to globally regular gravitating solitons with \( J = mQ \); boson stars\(^73\). For solutions with fixed \( M \) and \( Q \neq 0 \) the angular momentum \( J \) can vary only within a finite range and there is a non-zero lower bound for \( J \), hence the solutions do not admit the static limit \( J \to 0 \). On the other hand, they can have \( J > M^2 \) thus violating the Kerr bound. Their mass \( M \) also varies within a finite range and cannot be too large if \( Q \neq 0 \). Stability of these solutions remains an open issue.
4.3. Non-minimal models

The considered above examples assume the minimal coupling of the scalar to gravity. Some of them can be generalized to the non-minimally coupled case in the standard way, by adding to the Lagrangian the $\xi R \Phi^2$ term. There is, however, a much more general way to approach the problem. All possible gravity-coupled scalar field models with at most second order field equations are contained in the general theory discovered by Horndeski in 1972\textsuperscript{74d}. This theory is described by the Lagrangian

$$ L = L_2 + L_3 + L_4 + L_5 $$

where

$$ L_2 = G_2(X, \Phi), \quad L_3 = G_3(X, \Phi) \Box \Phi, $$
$$ L_4 = G_4(X, \Phi) R + \partial_X G_4(X, \Phi) \delta^{\mu \nu}_{\alpha \beta} \nabla^\alpha_{\mu} \Phi \nabla^\beta_{\nu} \Phi, $$
$$ L_5 = G_5(X, \Phi) G_{\mu \nu} \nabla^{\mu \nu} \Phi - \frac{1}{6} \partial_X G_5(X, \Phi) \delta^{\mu \nu \rho}_{\alpha \beta \gamma} \nabla^\alpha_{\mu} \Phi \nabla^\beta_{\nu} \Phi \nabla^\gamma_{\rho} \Phi, \quad (18) $$

with $X = -\frac{1}{2} (\partial \Phi)^2$ and $G_k(X, \Phi)$ are arbitrary functions. This theory contains all previously studied models with a gravity-coupled scalar field. If the coefficient functions $G_k$ do not depend on $\Phi$, it reduces to the covariant Galileon invariant under shifts $\Phi \rightarrow \Phi + \Phi_0$. Due to this shift symmetry, the equation for $\Phi$ is the conservation condition for the corresponding Noether current, $\nabla^\mu J_\mu = 0$.

The latter fact was used by Hui and Nicolis\textsuperscript{76} to produce a no-hair theorem for Galileons in the static and spherically symmetric case. The argument is simple: the metric can always be chosen in the form (13) while $\Phi = \Phi(r)$, hence the only non-zero component of the Noether current is $J^r$. The scalar field equation is $\left( R^2 J^r \right)' = 0$ hence $J^r = C/R^2$ where $C$ is an integration constant, however, $C$ should be set to zero since otherwise the invariant $J^\mu J_\mu = C^2/(NR^4)$ would diverge at the horizon where $N$ vanishes. This shows that $J^r = 0$ everywhere. Now, assuming the Lagrangian to be a polynomial in $\Phi'$ of at least second order gives

$$ J^r = N \frac{\partial L}{\partial \Phi'} = \Phi' F(\Phi', N, N', R, R', R'') = 0. \quad (19) $$

For asymptotically flat solutions $F$ approaches a constant value as $R \rightarrow \infty$ hence it cannot vanish, therefore one concludes that $\Phi' = 0$.

However, Sotiriou and Zhou\textsuperscript{77} pointed out that $L$ can also contain terms linear in $\Phi'$. Specifically, if $G_5(X) = \text{const.} \times \ln(|X|) + \tilde{G}_5(X)$ then the Galileon theory becomes

$$ L = R + X + \alpha \Phi \tilde{G}_{\text{GB}} + \ldots $$

where $\alpha$ is a constant and $\tilde{G}_{\text{GB}} = \nabla_\mu G^\mu$ is the Gauss-Bonnet invariant. The dots denote terms depending on the remaining coefficient functions, but they can be set to zero for simplicity and then the current is $J^\mu = \partial^\mu \Phi + \alpha G^\mu$. Therefore, the zero current condition $J^r = N\Phi' + \alpha G^r = 0$ does not imply that $\Phi' = 0$. As a result, one

\textsuperscript{d} There is an even more general possibility allowing for higher order equations but still keeping only three propagating degrees of freedom\textsuperscript{75}, but we do not consider it here.
can construct asymptotically flat black holes with a regular horizon and a non-trivial scalar field\textsuperscript{77}. These solutions are very similar to those\textsuperscript{41–44} previously studied in the dilaton-Gauss-Bonnet model (10) (also a Horndeski theory), for example there is a non-trivial lower bound for their mass. Their scalar charge is not an independent parameter but determined by the black holes mass, hence the hair is only secondary.

Sotiriou and Zhou also noticed\textsuperscript{77} that symmetries of the problem allow for a more general choice of $\Phi$ since actually only its gradient needs to be static and spherically symmetric. This idea has been implemented by Babichev and Charmousis\textsuperscript{78} within the context of a particular Galileon model,

$$L = \mu R - (\sigma G_{\mu\nu} + \varepsilon g_{\mu\nu})\nabla^{\mu}\Phi\nabla^{\nu}\Phi - 2\Lambda.$$  

(21)

Specifically, choosing $\Phi = Q t + \phi(r)$ and setting $\mu = \varepsilon = \Lambda = 0$ gives an exact solution,

$$ds^2 = -N dt^2 + \frac{dr^2}{N} + r^2 d\Omega^2, \quad \Phi = Q t \pm Q \int \frac{\sqrt{1 - N}}{N} dr,$$  

(22)

with $N = 1 - 2M/r$. This is called “stealth Schwarzschild” because, although the full backreaction problem is solved, the metric is pure Schwarzschild and the scalar field effectively does not back-react, even though it diverges at the (past or future) event horizon and at infinity. It is interesting to see how the no-go argument is circumvented. The Noether current $J^a$ has now two components, $J^0$ and $J^r$, but $J^a J_a$ is still finite at the horizon. The scalar field equation still has the form (19) and is fulfilled by setting $F = 0$ and not $\Phi' = 0$, which is possible because $\partial_r \Phi \neq 0$.

Notice that $\Phi$ for this solution is non-trivial in the far field zone, which breaks the asymptotic Lorentz invariance. It is possible\textsuperscript{77} that there could be other similar solutions if one allows the Lagrangian to contain negative powers of $\Phi'$. Similar solutions with the linear time-dependence of $\Phi$ have been obtained within other subclasses of the general covariant Galileon model\textsuperscript{79}, as well as in the $F(R)$ gravity\textsuperscript{80} (also a Horndeski theory). However, such solutions seem to be generically unstable\textsuperscript{81}.

The model (21) also admits an exact solution\textsuperscript{82–84} for generic values of its parameters $\mu, \varepsilon, \sigma, \Lambda$ and with $\Phi = \Phi(r)$,

$$ds^2 = -N dt^2 + \frac{dr^2}{H} + r^2 d\Omega^2, \quad H = \left(\frac{\eta r^2 + 1}{rN}\right), \quad \Phi'^2 = \frac{\eta (\lambda + \mu) r^2}{\sigma (\eta r^2 + 1) H},$$

$$N = \frac{\eta (\mu - \lambda) r^2 + 3(\mu - \lambda)(3\mu + \lambda) + 3(\lambda + \mu)^2}{\sqrt{\eta} r} \arctan(\sqrt{\eta} r) - \frac{2M}{r},$$  

(23)

where $\eta = -\varepsilon/\sigma$ and $\lambda = \Lambda/\eta$. One can adjust the parameters $\mu, \eta, \lambda$ and the integration constant $M$ such that $N, H$ vanish at $r = r_h$ and are positive for $r > r_h$, which corresponds to a black hole. This solution fulfills Eq.(19) with $F = 0$, which is possible because the geometry is not asymptotically flat. However, there is a problem, since if one adjusts the parameter $\sigma$ such that $\Phi'^2 > 0$ outside the horizon, then one will have $\Phi'^2 < 0$ inside hence $\Phi$ becomes complex-valued. This renders
unclear the status of the solution. More information on black holes with scalar hair can be found in the recent review\textsuperscript{85}.

5. XXI-st century black holes – solutions with massive gravitons

Recently becoming popular theories of massive gravity provide an alternative explanation for the cosmic self-acceleration. The basic idea is simple – if gravitons are massive, then the Newton potential is replaced by the Yukawa potential, hence the gravitational attraction is weaker at large distances and the universe expands faster.

Models with massive gravitons have been known for a long time but attracted a serious attention only recently, after the discovery of the special massive gravity theory\textsuperscript{86} circumventing the long standing problem of the ghost – an unphysical mode in the spectrum rendering everything unstable (see\textsuperscript{87,88} for a review).

The massive gravity theory is described by two metrics, $g_{\mu\nu}$ and $f_{\mu\nu}$, of which the first one is dynamical and the second one is fixed. The symmetry between the two metrics is restored in the ghost-free bigravity theory where they are both dynamical\textsuperscript{89}. This theory describes two interacting gravitons, one massive and one massless. In the limit where the massless graviton decouples, the theory reduces to the massive gravity. Alongside with the self-accelerating cosmologies\textsuperscript{90}, this theory admits hairy black holes.

The action of the ghost-free bigravity is

\[
\frac{m^2}{M_{Pl}^2} S = \frac{1}{2\kappa_1} \int R(g)\sqrt{-g} \, d^4 x + \frac{1}{2\kappa_2} \int R(f)\sqrt{-f} \, d^4 x - \int U\sqrt{-g} \, d^4 x, \tag{24}
\]

where all quantities on the right are rescaled to be dimensionless, the length scale being the inverse graviton mass $1/m$, and $\kappa_1 + \kappa_2 = 1$. The interaction between the metrics is determined by the scalar potential

\[
U = b_0 + b_1 \sum \lambda_a + b_2 \sum_{a<b} \lambda_a \lambda_b + b_3 \sum_{a<b<c} \lambda_a \lambda_b \lambda_c + b_4 \lambda_0 \lambda_1 \lambda_2 \lambda_3, \tag{25}
\]

where $\lambda_a$ are eigenvalues of $\gamma_{\mu\nu} = \sqrt{g^{\mu\sigma}f_{\sigma\nu}}$ with the square root understood in the sense that $\gamma^\mu_{\alpha}\gamma^\alpha_{\nu} = g^{\mu\nu}f_{\alpha\nu}$. The five parameters $b_k$ in (25) can be arbitrary, but if one requires the Minkowski space $g_{\mu\nu} = f_{\mu\nu} = \eta_{\mu\nu}$ to be a solution of the theory, then $b_k = b_k(c_3,c_4)$ where $c_3,c_4$ are independent. Varying the action with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$ gives two coupled sets of equations.

All known bigravity black holes\textsuperscript{91,92} can be divided into three types.

I. Proportional solutions: $f_{\mu\nu} = \lambda^2 g_{\mu\nu}$ with $G_{\mu\nu}(g) + \Lambda(\lambda)g_{\mu\nu} = 0$ and

\[
\Lambda(\lambda) = \kappa_1 [P_0(\lambda) + \lambda P_1(\lambda)] = \frac{k_2}{\lambda} [P_1(\lambda) + \lambda P_2(\lambda)], \tag{26}
\]

where $P_k(\lambda) \equiv b_k + 2b_{k+1}\lambda + b_{k+2}\lambda^2$. The second equality here implies that $\lambda$ is a root of the algebraic equation, and depending on its choice $\Lambda(\lambda)$ can be positive, negative, or zero. Therefore, solutions are the ordinary GR black holes, for example Schwarzschild or Schwarzschild-(anti)-de Sitter. If $b_k = b_k(c_3,c_4)$ then $\lambda = 1$ is
always a root and $\Lambda(1) = 0$, hence setting $g_{\mu\nu} = f_{\mu\nu}$ reduces the bigravity to the vacuum GR.

II. Non-diagonal solutions:

\[ ds_g^2 = -\Sigma dt^2 + \frac{dr^2}{\Sigma} + r^2 d\Omega^2, \]
\[ ds_f^2 = -\Delta dT^2 + \frac{dr^2}{\Delta} + r^2 d\Omega^2, \]

with $P_1(\lambda) = 0$, $\Lambda_g = \kappa_1 P_0(\lambda)$, and $\Lambda_f = \kappa_2 P_2(\lambda)$. Both metrics are Schwarzschild-(anti)-de Sitter but with different cosmological terms which do not necessarily have the same sign. The time $T$ for the second metric is not the coordinate time $t$ but a function $T(t, r)$ subject to

\[ \frac{\Delta}{\Sigma} \left( \partial_t T \right)^2 + \frac{\Delta \Sigma}{\Delta - \Sigma} \left( \partial_r T \right)^2 = 1. \]  

(28)

A simple solution of this equation can be obtained by separating the variables,

\[ T = t + \int \frac{dr}{\Sigma} - \int \frac{dr}{\Delta} \equiv t + r_{\Sigma}^* - r_{\Delta}^*, \]  

(29)

and using the light-like coordinate $V = t + r_{\Sigma}^* = T + r_{\Delta}^*$ both metrics can be written in the Eddington-Finkelstein form,

\[ ds_g^2 = -\Sigma dV^2 + 2dV dr + r^2 d\Omega^2, \]
\[ ds_f^2 = -\Delta dV^2 + 2dV dr + r^2 d\Omega^2. \]  

(30)

Other functions $T(t, r)$ subject to (28) give rise to physically inequivalent solutions (27). Since the f-metric becomes flat for $\kappa_2 \to 0$ (if $M_f = 0$), the solutions describe in this limit black holes in the massive gravity theory with flat reference metric. These and their stationary generalizations exhaust all known massive gravity black holes (more solutions exist within the extended versions of the theory).

III. Hairy black holes. Black holes of the previous two types are described by the known metrics. New results are obtained when the two metrics are simultaneously diagonal but not necessarily proportional.

\[ ds_g^2 = N^2 dt^2 - \frac{dr^2}{N^2} - r^2 d\Omega^2, \]
\[ ds_f^2 = A^2 dt^2 - \frac{U^2 dr^2}{A^2} - U^2 d\Omega^2, \]

(31)

where $N, \Delta, Y, U, A$ depend on $r$ and fulfill a system of differential equations. The simplest solutions are the proportional black holes of type I. More general solutions are obtained by numerically integrating the equations. It turns out that the equations for the three amplitudes $\Delta, Y, U$ comprise a closed subsystem whose local solution near the horizon

\[ \Delta^2 = \sum_{n \geq 1} a_n (r - r_h)^n, \quad Y^2 = \sum_{n \geq 1} b_n (r - r_h)^n, \quad U = ur_h + \sum_{n \geq 1} c_n (r - r_h)^n, \]

contains only one free parameter $u = U(r_h)/r_h$, that is the ratio of the horizon size measured by $f_{\mu\nu}$ to that measured by $g_{\mu\nu}$. The horizon is common for both metrics and its surface gravities and temperatures determined with respect to both metrics.
are the same. Choosing a value of $u$ and integrating numerically the equations from $r = r_h$ towards large $r$, one generically finds solutions approaching the proportional AdS metrics. These solutions describe black holes with massive graviton hair; their typical profile is shown in Fig. 2. The hair is localized in the horizon vicinity, while far from the horizon the solutions rapidly approach the AdS. For a fixed horizon size $r_h$ these solutions comprise a one-parameter family labeled by $u$. In the $r_h \to 0$ limit the solutions reduce to regular lumps – localized objects made of self-gravitating massive graviton modes. If on the contrary $r_h$ is very large and of the order of the inverse graviton mass, $r_h \sim 1$, then there are discrete values of $u$ for which the solutions are not asymptotically AdS but asymptotically flat; however this is only possible for cosmologically large black holes.

Stability has been studied for solutions of types I and II. The simplest solution of type I is the Schwarzschild black hole $g_{\mu\nu} = f_{\mu\nu} = g^{(0)}_{\mu\nu}$; it is linearly stable in GR. However, when perturbed, $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$, $f_{\mu\nu} = f^{(0)}_{\mu\nu} + \delta f_{\mu\nu}$, the perturbations $\delta g_{\mu\nu}$ and $\delta f_{\mu\nu}$ need not to coincide to each other, hence the GR result will not be necessarily recovered. It turns out that the massive graviton described by the linear combination $h_{\mu\nu} = \delta g_{\mu\nu} + \sqrt{\kappa_2/\kappa_1} \delta f_{\mu\nu}$ fulfills the same equations as those describing the Gregory-Laflamme instability,

\[ \Box h_{\mu\nu} + 2 R^{(0)}_{\mu\nu\alpha\beta} h^{\alpha\beta} = h_{\mu\nu}, \quad \nabla_\mu h^{\mu}_\nu = 0, \quad h^{\mu}_\mu = 0. \]  

(32)

Setting $h_{\mu\nu} = e^{i\omega t} H_{\mu\nu}(r, \vartheta, \varphi)$ they admit a bound state solution with $\omega^2 < 0$. It follows that small black holes are unstable since $\omega$ is imaginary and the perturbations grow in time. The condition on $r_h$ is not crucial since the lengthscale is of the order of the Hubble radius, hence

\[ \text{It is also possible that only one of the two metrics is asymptotically AdS}. \]
all ordinary black holes are small enough to be unstable. On the other hand, the instability is very mild and takes a Hubble time to develop. Therefore, if real astrophysical black holes were described by the bigravity theory, their instability would be largely irrelevant and they would actually be stable for all practical purposes over a cosmologically long period of time.

Additional information on black holes in theories with massive gravitons can be found in the review articles 98, 99.

6. Concluding remarks

Above is only a very brief description of stationary and asymptotically flat hairy black holes in 3+1 spacetime dimensions. One finds many more of them if one abandons the asymptotic flatness condition, if one considers different horizon topologies, if one goes to higher spacetime dimensions, etc. In view of this variety of hairy black holes one can wonder if there is still any sense in the no-hair conjecture?

Surprisingly, the answer seems to be affirmative for the astrophysically relevant, which means stable and macroscopically large black holes. This simply follows from the fact that Einstein-Maxwell theory is definitely valid at the macroscopic scale, but within this theory the no-hair conjecture is proven. The situation is different at the microscopic scale where other theories apply, as for example the non-Abelian models. There are stable hairy black holes in that case, for example those supporting the Yang-Mills-Higgs or Skyrme hair. However, they are always microscopically small and loose their hair when grow beyond a certain size.

If there exists a cosmic scalar field accounting for the dark energy, this might perhaps modify the macroscopic black hole structure, however, most of the known solutions with scalar hair are unstable. Black holes in models violating the energy conditions are unstable, while the known non-minimally coupled Horndeski solutions (22) and (23) are either unstable or non-asymptotically flat. However, the Horndeski theory is quite general and it is not excluded that it may admit stable hairy black holes. The possible candidates are black holes 41–44 in the dilaton-Gauss-Bonnet theory (9) (and maybe their cousins 77 in the Galileon theory (20)) because they do not have the most dangerous S and P sector instabilities 42, 100. Notice however that their hair is only secondary.

If theories with massive gravitons indeed describe the world, then the black holes are almost Kerr-Newman, apart from a narrow region in the horizon vicinity where a slow accretion of massive graviton modes takes place.

In summary, it seems that all stationary black holes that are astrophysically relevant indeed obey the no-hair conjecture, at least no explicit counter-examples are known.

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