Quantum Disturbance without State Change: Soundness and Locality of Disturbance Measures

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Abstract

It is often supposed that a quantum system is not disturbed without state change. In a recent debate, this assumption is used to claim that the operator-based disturbance measure, a broadly used disturbance measure, has an unphysical property. Here, we show that a quantum system possibly incurs an operationally detectable disturbance without state change to rebut the claim. Moreover, we establish the reliability, formulated as soundness and locality, of the operator-based disturbance measure, which, we show, quantifies the disturbance on an observable that manifests in the time-like correlation even in the case where its probability distribution does not change.

1 Introduction

Heisenberg’s error-disturbance relation (EDR)

$$\varepsilon(A)\eta(B) \geq \frac{1}{2}|<[A, B]>|$$

(1)

for the mean error $\varepsilon(A)$ of a measurement of an observable $A$ in any state and the mean disturbance $\eta(B)$ caused on an observable $B$, originally introduced by the $\gamma$-ray microscope thought experiment [1], has been commonly believed as a dynamical aspect of Heisenberg’s uncertainty principle, which is formally represented by a rigorously proven relation

$$\sigma(A)\sigma(B) \geq \frac{1}{2}|<[A, B]>|$$

(2)

for the indeterminacies, defined as the standard deviations $\sigma(A), \sigma(B)$, of arbitrary observables $A, B$ in any state [1-3]. There have been longstanding research efforts to prove Heisenberg’s EDR [4-8], while the universal validity has not been reached. Instead, a recent study [9,10] revealed a universally valid form of EDR

$$\varepsilon(A)\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{1}{2}|<[A, B]>|,$$

(3)

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where \( \sigma(A) \) and \( \sigma(B) \) are the standard deviations just before the measurement, and made Heisenberg’s EDR testable \([10,11]\) to observe its violations, confirming the new relation as well \([12-14]\). Subsequently, stronger EDRs were derived \([15-18]\), and confirmed experimentally \([19-24]\).

In order to define the error \( \varepsilon(A) \) and disturbance \( \eta(B) \) in Eq. (3), we suppose that the measurement \( M \) of \( A \) is described by an interaction from time \( t = 0 \) to \( t = \tau \) between the system \( S \) in a state \( \psi \) and the probe \( P \) prepared in a fixed state \( \xi \), and that the outcome of the measurement is obtained by the measurement of the meter observable \( M \) in the probe \( P \) at time \( t = \tau \).

In the Heisenberg picture, we shall write \( X(0) = X \otimes I \), \( X(\tau) = U^\dagger X(0)U \), \( Y(0) = I \otimes Y \), \( Y(\tau) = U^\dagger Y(0)U \) for observables \( X \) in \( S \) and \( Y \) in \( P \), where \( U \) is the unitary evolution operator for \( S + P \) from \( t = 0 \) to \( t = \tau \). The error \( \varepsilon(A) = \varepsilon_O(A, M, |\psi\rangle) \) and disturbance \( \eta(B) = \eta_O(B, M, |\psi\rangle) \) in Eq. (3) are defined by

\[
\varepsilon_O(A, M, |\psi\rangle) = \langle \psi, \xi | [M(\tau) - A(0)]^2 |\psi, \xi\rangle^{1/2},
\]

\[
\eta_O(B, M, |\psi\rangle) = \langle \psi, \xi | [B(\tau) - B(0)]^2 |\psi, \xi\rangle^{1/2}.
\]

See Ref. \([10]\) for details. We call \( \varepsilon_O \) and \( \eta_O \) as the operator-based error measure and the operator-based disturbance measure. We shall write \( \varepsilon(O)(A, M, |\psi\rangle) \) and \( \eta(O)(B, M, |\psi\rangle) \) when no confusion may occur.

In the previous work \([18]\), we have investigated the properties of the operator-based error measure (called therein as noise-operator-based quantum root-mean-square error) \( \varepsilon_O \), and we have introduced its completion \( \varepsilon \), the locally uniform quantum root-mean-square error, and subsequently we have experimentally tested \([27]\) the completeness of \( \varepsilon_O \) and \( \varepsilon \) to show how hidden error in \( \varepsilon_O \) manifests in the defining procedure of \( \varepsilon \).

In the present work, we focus on the properties, soundness and locality, of the operator-based disturbance measure \( \eta_O \), where soundness generally requires a disturbance measure to assign the value 0 to “non-disturbing” measurements, and locality generally requires a disturbance measure to assign the value 0 to “non-disturbing” local measurements.

We say that a measurement is distributionally non-disturbing to an observable \( B \) in the system state \( |\psi\rangle \) if \( B(0) \) and \( B(\tau) \) have identical probability distributions in the initial state \( |\psi, \xi\rangle \). Korzekwa, Jennings, and Rudolph \([28]\) criticized the use of the operator-based disturbance measure, based on the following requirement for disturbance measures.

**Distributional requirement (DR) for disturbance measures.** Any disturbance measure should assign the value 0 to distributionally non-disturbing measurements.\(^1\)

KJR \([28]\) called the DR “the commonly accepted and operationally motivated requirement that all physically meaningful notions of disturbance should satisfy”. They claimed that the operator-based disturbance measure does not satisfy the DR and has even an ‘unphysical’ property, since it takes a positive value for a measurement that does not change the state at all. Further, they concluded that state-dependent formulations of EDRs are not tenable.

\(^1\)Note that this general description of a measuring process, also called an indirect measurement model \([10]\), is introduced and proved in Ref. \([25]\) to be equivalent to the most general description using a completely positive instrument, or a so-called quantum instrument, which is a reformulation of the Davies-Lewis instrument \([26]\) with the additional requirement of complete positivity.

\(^2\)Note that KJR \([28]\) called distributionally non-disturbing measurements as “operationally non-disturbing measurements”; see Eq. (2) in KJR \([28]\).
In this paper, we examine the validity of the DR. For this purpose, we consider a more fundamental principle in quantum mechanics, the correspondence principle, stating that if the classical description is available, quantized concepts should be consistent with the classical description. We argue that the DR violates the correspondence principle. We generally show that even if the measurement does not change the state, the disturbance is operationally detectable as long as the operator-based disturbance measure takes a positive value. Thus, the claims made by KJR are groundless. The DR requires that disturbance measures only count the change of the probability distribution in time, but according to the correspondence principle, valid disturbance measures should also count the change of the observable that manifest in the time-like correlation, as the operator-based disturbance measure does.

Moreover, we show that the DR violates another fundamental requirement for no-signaling under local operations, called the locality requirement. Subsequently, we show that disturbance measures satisfying the DR cannot be used to demonstrate the security of quantum cryptography, because they do not properly describe the disturbance caused by the eavesdropper. In contrast, we show that the operator-based disturbance measure satisfies the correspondence principle and the locality requirement. Based on those arguments, we shall conclude that state-dependent formulations of EDRs based on the operator-based disturbance measure reliably represent the originally motivated dynamical aspect of Heisenberg’s uncertainty principle.

2 Correspondence principle

The correspondence principle generally states that quantum theory should be consistent with classical theories in the case where the classical descriptions are also available. In fact, it is a common practice to apply classical descriptions to commuting observables through their joint probability distributions. In Ref. [18] we consider the correspondence principle as a requirement for error measures. Here, we extend the consideration to disturbance measures.

It is well-known that any commuting observables $X, Y$ have their joint probability distribution in any state. Here, for a given state $|\Psi\rangle$, the \textit{joint probability distribution (JPD)} of any two observables $X, Y$ is defined as a 2-dimensional probability distribution $\mu(u, v)$ satisfying

$$
\langle \Psi | f(X, Y) | \Psi \rangle = \sum_{u, v} f(u, v) \mu(u, v) \tag{6}
$$

for every (non-commutative) polynomial $f(X, Y)$ of $X$ and $Y$. In general, two observables $X, Y$ have their JPD in a state $|\Psi\rangle$ if and only if they commute in $|\Psi\rangle$ in the sense that

$$
P_X(u)P_Y(v)|\Psi\rangle = P_Y(v)P_X(u)|\Psi\rangle, \tag{7}
$$

where $P_X(u)$ and $P_Y(v)$ are the spectral projections of $X$ and $Y$ (Ref. [18], Theorem 1). In this case, the JPD $\mu$ is uniquely determined by

$$
\mu(u, v) = \langle \Psi | P_X(u)P_Y(v)| \Psi \rangle. \tag{8}
$$

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3. “The term [the correspondence principle] codifies the idea that a new theory should reproduce under some conditions the results of older well-established theories in those domains where the old theories work [Wikipedia https://en.wikipedia.org/wiki/Correspondence_principle (August 1, 2022)].”
The JPD $\mu$ determines the (classical) root-mean-square deviation $\delta_G(\mu)$ between the classical random variables $u = u$ and $v = v$, the notion originally introduced by Gauss [29], by

$$\delta_G(\mu) = \left(\sum_{u,v} (u - v)^2 \mu(u,v)\right)^{1/2}.$$  \hfill (9)

We say that a disturbance measure $\eta$ satisfies the correspondence principle (CP) if $\eta(B) = \delta_G(\mu)$ provided that $B(\tau)$ and $B(0)$ have their JPD $\mu$ in the initial state $|\psi, \xi\rangle$. An important property of the operator-based disturbance measure $\eta_O$ is that it satisfies the CP, as easily follows from Eq. (6). Similarly, the operator-based error measure $\varepsilon_O$ also satisfies CP in the sense that $\varepsilon_O(A) = \delta_G(\mu)$ provided that $M(\tau)$ and $A(0)$ have their JPD $\mu$ in the initial state $|\psi, \xi\rangle$ as shown in Ref. [18].

If $B(0)$ and $B(\tau)$ have their JPD $\mu$, the correlation between $B(0)$ and $B(\tau)$ has the classical picture described by $\mu$, and the classical notion $\delta_G(\mu)$ of the root-mean-square deviation is applicable to quantifying the disturbance of $B$. In this case, according to the correspondence principle, any quantum definition of a disturbance measure $\eta$ should be consistent with the classical measure $\delta_G$. Thus, we say that a quantum disturbance measure $\eta$ satisfies the correspondence principle if two measures, the quantum $\eta$ and the classical $\delta_G$, are consistent, whenever the classical picture is available, as a desirable property of a quantum disturbance measure. In this sense, the correspondence principle determines the value of the disturbance on $B$, when the joint probability distribution of $B(0)$ and $B(\tau)$ exists, in an analogous way as the probability distribution of $B(0)$ determines its standard deviation $\sigma(B) = \sigma(B(0))$ appearing in Eq. (3).

### 3 Disturbing observables without state change

KJR [28] identified as ‘unphysical’ the property of the operator-based disturbance measure $\eta_O$ that it does not assign the value 0 in a case where the state has not changed at all. In such a case, the probability distribution of every observable has not changed, so that this is a stronger violation of the DR. However, we shall show here that this is not a peculiarity of the operator-based disturbance measure, but a straightforward consequence of the CP.

Consider a qubit measurement. The projective measurement of $A = \sigma_z$ in the state $|0\rangle := |\sigma_z = +1\rangle$ does not change the initial state $|\psi\rangle = |0\rangle$. In this case, it was shown [30] that the operator-based disturbance measure indicates that $B = \sigma_x$ is disturbed by the amount $\eta_O(\sigma_x) = \sqrt{2}$, and this value was actually obtained by a neutron optical experiment [19]. However, according to the DR, every disturbance measure $\eta$ should assign the value 0, and KJR [28] identified the above property of $\eta_O$ as a very unphysical property. In contrast, we shall show that every disturbance measure $\eta$ satisfying the CP assigns the value $\sqrt{2}$.

It is well-known that the projective measurement of $\sigma_z$ is carried out by the controlled-NOT operation

$$U = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \sigma_x$$ \hfill (10)

for the measured qubit $S$ and the probe qubit $P$ prepared in the fixed state $|\xi\rangle = |0\rangle$ from $t = 0$ to $t = \tau$ and by the subsequent meter measurement for $M = \sigma_z$ in $P$ (Figure 1). The
Figure 1: **Distributionally non-disturbing measurements are disturbing according to the correspondence principle.** A projective measurement of $\sigma_z$ in $|\psi\rangle = |0\rangle$ is probability non-disturbing to the observable $\sigma_x$. Thus, the DR requires any disturbance measure to assign the value 0. However, the CP requires any disturbance measure to assign the value $\sqrt{2}$.

Schrödinger time evolution satisfies

$$U(|0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle,$$

$$U(|1\rangle \otimes |0\rangle) = |1\rangle \otimes |1\rangle. \quad (11)$$

For $B = \sigma_x$ the Heisenberg time evolution is given by

$$\sigma_x(0) = \sigma_x \otimes I,$$

$$\sigma_x(\tau) = \sigma_x \otimes \sigma_x. \quad (12)$$

Here, Eq. (14) follows from

$$U^\dagger(\sigma_x \otimes I)U = |1\rangle \langle 1| \sigma_x |0\rangle \langle 0| \otimes \sigma_x + |0\rangle \langle 0| \sigma_x |1\rangle \langle 1| \otimes \sigma_x = \sigma_x \otimes \sigma_x.$$

It follows that $\sigma_x(\tau)$ and $\sigma_x(0)$ commute and they have the JPD $\mu(u, v)$ in the state $|\psi, \xi\rangle = |0, 0\rangle$ as

$$\mu(u, v) = \langle 0, 0| P^{\sigma_x(\tau)} (u) P^{\sigma_x(0)} (v) |0, 0\rangle = \langle 0, 0| P^{\sigma_x(\tau) \otimes \sigma_x} (u) P^{\sigma_x \otimes I} (v) |0, 0\rangle. \quad (15)$$

Then we obtain

$$\mu(u, v) = \frac{1}{4} \quad (16)$$

(cf. Section 9). Thus, if the disturbance measure $\eta$ satisfies the CP, we have

$$\eta(\sigma_x)^2 = \delta_G(\mu)^2 = \sum_{u, v = \pm 1} (u - v)^2 \mu(u, v) = 2. \quad (17)$$
Therefore we conclude $\eta(\sigma_x) = \sqrt{2}$. Thus, the non-zero value $\eta_0(\sigma_x) = \sqrt{2}$ is not a peculiar property of the operator-based disturbance measure.

It will be instructive to compare the above scenario (1) that the system is prepared in the state $|0\rangle$ and then a projective measurement of $\sigma_z$ is performed and another scenario (2) that the system is prepared in the state $|0\rangle$ but no measurement is performed. In both scenarios, the system state $|0\rangle$ is unchanged and the probability distribution of any observable does not change. How can we operationally distinguish the two scenarios. In scenario 1 we have shown that the observable $B = \sigma_x$ is disturbed. For the time $t = 0$ just before the measurement and the time $t = \tau$ just after the measurement, we obtain $B(0) = \sigma_x \otimes I$ and $B(\tau) = \sigma_x \otimes \sigma_x$ (cf. Eqs. (13) and (14)). Their joint probability distribution $p_2(u, v)$ satisfies $p_2(u, v) = 1/4$ for any $u, v$ (cf. Eq. (16)) that leads to $\eta(B) = \sqrt{2}$. On the other hand, scenario 2 is easily analyzed, so that we obtain $B(0) = B(\tau) = \sigma_x \otimes I$, and their joint probability distribution $p_1(u, v)$ satisfies $p_1(u, v) = \delta_{u,v}/2$ that leads to $\eta(B) = 0$. The joint probability distributions can be experimentally obtained by weak measurements and post-selections as proposed by Lund and Wiseman [11]. Thus, we can operationally distinguish between the above two scenarios.

This conclusion might sound counter-intuitive, as the pure state has the “maximal information” about the system. However, unchanged the pure state does not imply unchanging the observable, because the “maximal information” about the system does not include the “maximal information” about an observable, analogously with the fact that the “maximal information” about the whole system does not include the “maximal information” about subsystems.

In fact, according to the available classical description, the conditional probability

$$\Pr\{\sigma_x(\tau) = u | \sigma_x(0) = v\} = \mu(u|v) = \frac{1}{2}$$

shows that the value of $\sigma_x(0)$ has been completely randomized, although their marginals have not changed at all as

$$\Pr\{\sigma_x(\tau) = u\} = \Pr\{\sigma_x(0) = u\} = \frac{1}{2}.$$ 

Thus, the DR neglects the disturbance caused by the randomization by measurement without changing the probability distribution.

### 4 State-dependent formulation for non-disturbing measurements

We have shown that the DR with the notion of probability non-disturbing measurements contradicts the CP. To reconcile the conflict, we shall characterize non-disturbing measurements from the two fundamental requirements: the CP and the operational accessibility.

Consider the following condition.

(S) $B(\tau)$ and $B(0)$ have their JPD $\mu$ in $|\psi, \xi\rangle$ satisfying that $\mu(u, v) = 0$ if $u \neq v$.

From the point of view of the CP, if condition (S) holds, we should conclude that the measurement $M$ does not disturb $B$ in $|\psi\rangle$. Thus, condition (S) is considered as a sufficient condition for a proper definition of non-disturbing measurements.

On the other hand, from the point of view of operational accessibility, it is convenient to consider the weak joint distribution (WJD) $\nu(u, v)$ of $B(\tau)$ and $B(0)$ in $|\psi, \xi\rangle$ defined by

$$\nu(u, v) = \langle \psi, \xi | P^{B(\tau)}(u) P^{B(0)}(v) | \psi, \xi \rangle.$$ 

(cf. Eqs. (13) and (14)). Their joint probability distribution $p_2(u, v)$ satisfies $p_2(u, v) = 1/4$ for any $u, v$ (cf. Eq. (16)) that leads to $\eta(B) = \sqrt{2}$. On the other hand, scenario 2 is easily analyzed, so that we obtain $B(0) = B(\tau) = \sigma_x \otimes I$, and their joint probability distribution $p_1(u, v)$ satisfies $p_1(u, v) = \delta_{u,v}/2$ that leads to $\eta(B) = 0$. The joint probability distributions can be experimentally obtained by weak measurements and post-selections as proposed by Lund and Wiseman [11]. Thus, we can operationally distinguish between the above two scenarios.

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(cf. Eqs. (13) and (14)). Their joint probability distribution $p_2(u, v)$ satisfies $p_2(u, v) = 1/4$ for any $u, v$ (cf. Eq. (16)) that leads to $\eta(B) = \sqrt{2}$. On the other hand, scenario 2 is easily analyzed, so that we obtain $B(0) = B(\tau) = \sigma_x \otimes I$, and their joint probability distribution $p_1(u, v)$ satisfies $p_1(u, v) = \delta_{u,v}/2$ that leads to $\eta(B) = 0$. The joint probability distributions can be experimentally obtained by weak measurements and post-selections as proposed by Lund and Wiseman [11]. Thus, we can operationally distinguish between the above two scenarios.

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$$\Pr\{\sigma_x(\tau) = u\} = \Pr\{\sigma_x(0) = u\} = \frac{1}{2}.$$ 

Thus, the DR neglects the disturbance caused by the randomization by measurement without changing the probability distribution.
The WJD always exists, though possibly takes negative or complex values, and is operationally accessible by weak measurement and post-selection \cite{31-33}; see also Ref. \cite{34} for a short survey. Then it is natural to consider the following condition.

(W) The WJD of $B(\tau)$ and $B(0)$ in $|\psi, \xi\rangle$ satisfies that $\nu(u, v) = 0$ if $u \neq v$.

If the measurement $M$ does not disturb the observable $B$ in $|\psi\rangle$, any operational tests for witnessing the disturbance should fail. Since measuring WJD is one of such operational tests for which the disturbance is detected if $\nu(u, v) \neq 0$ for some $u \neq v$ \cite{35, 36}, condition (W) is considered as a necessary condition for a proper definition of non-disturbing measurements.

Obviously, (W) is logically weaker than or equivalent to (S). However, Theorem 1 (Section 10) shows that both conditions are actually equivalent. In fact, according to the theory of quantum perfect correlations \cite{37, 38}, both conditions (S) and (W) equivalently require that $B(\tau)$ and $B(0)$ are perfectly correlated in the state $|\psi, \xi\rangle$ \cite{34}. Thus, the above argument justifies the following definition of non-disturbing measurements. We say that the measurement $M$ is properly non-disturbing to an observable $B$ in $|\psi\rangle$ if one of the conditions (S) or (W) is satisfied. Since the WJD is operationally accessible, this definition is also operationally accessible.

5 Reliability of the operator-based disturbance measure

To consider the reliability of the operator-based disturbance measure, we examine the following requirements: (i) the CP, (ii) soundness, (iii) operational accessibility, and (iv) completeness.

We have already shown that the operator-based disturbance measure $\eta_O$ satisfies the CP, i.e., $\eta_O(B) = \delta_G(\mu)$ if $B(\tau)$ and $B(0)$ have the JPD $\mu$. We introduce the soundness requirement: Any disturbance measure $\eta$ should assign the value 0 to any properly non-disturbing measurements. It is interesting to see that the CP implies soundness. To show this, suppose that the measurement is properly non-disturbing to $B$ in $|\psi\rangle$. Then $B(\tau)$ and $B(0)$ have the JPD $\mu$ satisfying that $\mu(u, v) = 0$ if $u \neq v$. It follows that $\varepsilon_G(\mu) = 0$ and by the CP we have $\eta(B) = \varepsilon_G(\mu) = 0$. Accordingly, the operator-based disturbance measure $\eta_O$ satisfies the soundness requirement. We conclude, therefore, that even if the measurement does not change the state, the disturbance can be operationally detected as long as the operator-based disturbance measure takes a positive value.

It has been known that the operator-based disturbance measure $\eta_O$ is operationally accessible in the two ways: (i) the tomographic three state method, proposed by Ozawa \cite{10} and experimentally realized by Erhalt et al. \cite{12} and others \cite{14, 19, 23} and (ii) the weak measurement method, proposed by Lund and Wiseman \cite{11} and experimentally realized by Rozema et al. \cite{13} and others \cite{20, 22}.

As the converse of soundness, a disturbance measure $\eta$ is said to be complete if $\eta$ assigns the value 0 only to properly non-disturbing measurements. There is an example in which $\eta_O$ does not satisfy completeness (Ref. \cite{38}, p. 750). However, it is known that $\eta_O$ satisfies completeness if (i) (commutative case) $B(\tau)$ and $B(0)$ commute in $|\psi, \xi\rangle$ or if (ii) (dichotomic case) $B^2 = I$ (Ref. \cite{18}, Theorem 3).

We have seen that the operator-based disturbance measure satisfies all requirements (i)–(iii), and partially satisfies requirement (iv) above.

Analogously from an argument for the operator-based error measure $\varepsilon_O$ in Ref. \cite{18}, it follows that $\eta_O$ can be modified to satisfy completeness by defining the operator-based locally...
uniform disturbance measure $\eta$ as

$$\eta(B, M, |\psi\rangle) = \sup_{t \in \mathbb{R}} \eta_O\left(B, M, e^{-itB}|\psi\rangle\right).$$  \hspace{1cm} (21)$$

Then the error measure $\eta$ satisfies requirements (i) – (iv) and also (v) (Dominating property) $\eta_O(B, M, |\psi\rangle) \leq \eta(B, M, |\psi\rangle)$ for any $|\psi\rangle$, and (vi) (Conservation property for dichotomic measurements) $\eta(B) = \eta_O(B)$ if $B^2 = I$. Thus, all the EDRs for $\eta_O$ also holds for $\eta$; see analogous discussions for the operator-based error measure in Ref. [18].

In the following we shall discuss another requirement on locality, which the operator-based disturbance measure satisfies, but contradicts the DR.

6 Locality of disturbance

We have argued that state-dependent formulations of error-disturbance relations are well-founded by the operator-based disturbance measure, which is a sound disturbance measure according to the notion of properly non-disturbing measurement that is supported by the CP and operational accessibility, in contrast to KJR’s claim that the operator-based disturbance measure is not sound under the notion of distributionally non-disturbing measurements, which we have shown to contradict the CP.

Yet, there is a prevailing view that only probability distributions of outcomes of measurements can be operationally compared [39], despite the fact that the new experimental techniques enable us to operationally detect the change of an observable in time: (i) the tomographic three state method [10, 12, 14, 19, 23] and (ii) the weak measurement method [11, 13, 20–22].

In what follows, we shall show below another drawback of the DR that the notion of probability non-disturbing measurements violates a locality requirement to be posed below.

Consider a composite system $S_1 + S_2$ in a state $|\Psi\rangle$. Since any local measurement of $S_2$ does not interact with the system $S_1$, we naturally take it for granted that any local measurement of $S_2$ non-disturbing to an observable $B_2$ in $S_2$ should be non-disturbing to the observable $B_1 \otimes B_2$ for any observable $B_1$ in $S_1$. We call this requirement the locality requirement for a definition of disturbing measurements. We shall show that the definition of distributionally non-disturbing measurements does not satisfy this requirement, whereas the definition of properly non-disturbing measurements does satisfy the requirement as shown in Theorem 2 (Section 11).

For this purpose, we consider a maximally entangled two-qubit system $S_1 + S_2$ in the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Since $|\Phi^+\rangle = (|0_x0_x\rangle + |1_x1_x\rangle)/\sqrt{2}$, the outcomes of the joint local measurements of the observables $\sigma_x^{(1)} = \sigma_x \otimes I$ and $\sigma_x^{(2)} = I \otimes \sigma_x$ show a perfect correlation. From Theorem 4 (Section 11), measurements properly non-disturbing to $\sigma_x^{(2)}$ does not change the JPD of $\sigma_x^{(1)}$ and $\sigma_x^{(2)}$, so that the perfect correlation between $\sigma_x^{(1)}$ and $\sigma_x^{(2)}$ is not disturbed. However, we shall show that a probability non-disturbing measurement breaks the perfect correlation, and this concludes that the definition of probability non-disturbing measurements does not satisfy the locality requirement, according to Theorem 3 (Section 11).
Figure 2: Definition of distributionally non-disturbing measurements violates the locality requirement. (i) The projective $\sigma^{(2)}_z$ measurement in $|\Phi^+\rangle$ is probability non-disturbing to $\sigma^{(2)}_x$, but disturbs the JPD $\mu$ between $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$ in $|\Phi^+\rangle$. The perfect correlation between $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$ at time 0, i.e., $\delta_G(\mu_0) = 0$, is disturbed by the amount $\delta_G(\mu_\tau) = \eta_O(\sigma^{(2)}_x) = \sqrt{2}$. (ii) The projective $\sigma^{(2)}_\theta$ measurement in $|\Phi^+\rangle$ is distributionally non-disturbing to $\sigma^{(2)}_x$, but disturbs the JPD $\mu$ between $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$ in $|\Phi^+\rangle$ for $0 \leq \theta < \pi/2$. The perfect correlation between $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$, i.e., $\delta_G(\mu_0) = 0$, is disturbed by the amount $\delta_G(\mu_\tau) = \eta_O(\sigma^{(2)}_x) = \sqrt{2}\cos\theta$. (iii) An arbitrary local measurement of $S_2$ in $|\Phi^+\rangle$ with the disturbance $\eta_O(\sigma^{(2)}_x)$ disturbs the perfect correlation between $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$, i.e., $\delta_G(\mu_0) = 0$, by the amount $\delta_G(\mu_\tau) = \eta_O(\sigma^{(2)}_x)$. This relation leads to a security tradeoff relation for the E91 quantum cryptography protocol [40].

(i) Projective $\sigma^{(2)}_z$ measurement.

Suppose that the observer makes a projective $\sigma^{(2)}_z$ measurement just before the joint local measurements of $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$ (Figure 2(i)). The measuring interaction is given by

$$U = I \otimes |0\rangle \langle 0| \otimes I + I \otimes |1\rangle \langle 1| \otimes \sigma_x,$$

(22)

turned on from $t = 0$ to $t = \tau$ between $S_1 + S_2$ and the probe $P = S_3$ prepared in $|\xi\rangle = |0\rangle$ with the meter $M = \sigma^{(3)}_z$. The time evolutions of relevant observables are given by

$$\sigma^{(1)}_x(0) = \sigma_x \otimes I \otimes I,$$

(23)

$$\sigma^{(1)}_x(\tau) = \sigma_x \otimes I \otimes I,$$

(24)

$$\sigma^{(2)}_x(0) = I \otimes \sigma_x \otimes I,$$

(25)

$$\sigma^{(2)}_x(\tau) = I \otimes \sigma_x \otimes \sigma_x.$$ 

(26)
Then we shall see that the projective $\sigma^{(2)}_x$ measurement is distributionally non-disturbing to $\sigma^{(2)}_x$, but it disturbs the perfect correlation between $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$. To show that, let $\mu_t$ be the JPD of $\sigma^{(1)}_x(t)$ and $\sigma^{(2)}_x(t)$ for $t = 0, \tau$. Then we have

$$\mu_0(u, v) = \frac{1}{2}\delta_{u,v}, \quad \mu_\tau(u, v) = \frac{1}{4}$$

for any $u, v = \pm 1$ (cf. Section 12.1). Since the marginal probability for $\sigma^{(2)}_x$ does not change, the projective $\sigma^{(2)}_x$ measurement is distributionally non-disturbing to $\sigma^{(2)}_x$. However, the perfect correlation between $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$ at time $t = 0$ has been disturbed. The amount of the disturbance of the perfect correlation, i.e., $\delta_G(\mu_0) = 0$, is measured by the classical root-mean-square deviation $\delta_G(\mu_\tau)$, and we have

$$\delta_G(\mu_\tau) = \eta_0(\sigma^{(2)}_x) = \sqrt{2}$$

(cf. Section 12.1). Thus, the projective $\sigma^{(2)}_x$ measurement is distributionally disturbing to $B_1 \otimes \sigma^{(2)}_x$ for some observable $B_1$ of $S_1$ by Theorem 3 (Section 11). Therefore, we conclude that the definition of distributionally non-disturbing measurements does not satisfy the locality requirement.

Since all observables $\sigma^{(1)}_x(0), \sigma^{(1)}_x(\tau), \sigma^{(2)}_x(0), \sigma^{(2)}_x(\tau)$ are mutually commuting, we have their joint probability distribution. The relation $\sigma^{(2)}_x(0) = \sigma^{(1)}_x(0)$ holds with probability one by entanglement, and $\sigma^{(1)}_x(\tau) = \sigma^{(1)}_x(0)$ holds by locality of the measurement. Thus, we have

$$\Pr\{\sigma^{(1)}_x(\tau) = \sigma^{(2)}_x(0) = \sigma^{(1)}_x(0)\} = 1.$$

(29)

From this and the relation $\mu_\tau(u, v) = 1/4$ above, we obtain

$$\Pr\{\sigma^{(2)}_x(\tau) = v', \sigma^{(1)}_x(\tau) = u', \sigma^{(2)}_x(0) = v, \sigma^{(1)}_x(0) = u\} = \frac{1}{4}\delta_{u,v}\delta_{u,u'}.$$

(30)

Thus, we have the conditional probability

$$\Pr\{\sigma^{(2)}_x(\tau) = v' | \sigma^{(2)}_x(0) = v\} = \frac{1}{2}$$

(31)

showing that $\sigma^{(2)}_x$ is completely randomized by the measuring interaction, whereas the DR neglects this randomization manifest in the joint probability $\mu_\tau(u, v)$ of outcomes of local measurements of $\sigma^{(1)}_x(\tau)$ and $\sigma^{(2)}_x(\tau)$.

(ii) **Projective $\sigma^{(2)}_\theta$ measurement.**

For quantitative considerations, suppose that the observer makes a projective $\sigma^{(2)}_\theta$ measurement just before the joint local measurements of $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$, where $\sigma_\theta = \cos \theta \sigma_z + \sin \theta \sigma_x$ for $0 \leq \theta < \pi/2$ (Figure 2(ii)). Then the projective $\sigma_\theta$ measurement is distributionally non-disturbing to the observable $\sigma^{(2)}_x$ (cf. Section 12.2). However, they disturb the perfect correlation between $\sigma^{(1)}_x$ and $\sigma^{(2)}_x$. In fact, the JPD $\mu_\tau$ of $\sigma^{(1)}_x(\tau)$ and $\sigma^{(2)}_x(\tau)$ is given by

$$\mu_\tau(u, v) = \frac{1}{4}\delta_{u,v}(1 + \sin^2 \theta) + \frac{1}{4}(1 - \delta_{u,v})\cos^2 \theta$$

(32)
and the classical root-mean-square deviation $\delta_G(\mu_\tau)$ and the disturbance $\eta_O(\sigma_x^{(2)})$ are given by

$$\delta_G(\mu_\tau) = \eta_O(\sigma_x^{(2)}) = \sqrt{2} \cos \theta$$

(cf. Section 12.2). Thus, the joint probability distribution of the outcomes of joint local measurements of $\sigma_x^{(1)}(\tau)$ and $\sigma_x^{(2)}(\tau)$ favors the non-zero value $\eta_O(\sigma_x^{(2)}) = \sqrt{2} \cos \theta$, in contrast to the DR requiring $\eta_O(\sigma_x^{(2)}) = 0$.

(iii) Arbitrary local measurements.

Suppose that the observer makes an arbitrary local measurement $M$ of $S_2$ from $t = 0$ to $t = \tau$ with the probe prepared in $|\xi\rangle$ just before the joint local measurements of $\sigma_x^{(1)}$ and $\sigma_x^{(2)}$ (Figure 2 (iii)). Then the JPD $\mu_0$ and the classical root-mean-square deviation $\delta_G(\mu_0)$ satisfy $\mu_0(u, v) = \delta_{u,v}/2$ and $\delta_G(\mu_0) = 0$. From Theorem 5 (ii) (Section 12), the relation

$$\delta_G(\mu_\tau) = \eta_O(\sigma_x^{(2)})$$

holds for any local measurement $M$ of $S_2$. Since $\eta_O(\sigma_x^{(2)}) = 0$ if and only if $M$ is properly non-disturbing to $\sigma_x^{(2)}$ from Theorem 5 (iii) (Section 12), we conclude $\delta_G(\mu_\tau) = 0$ if and only if $M$ is properly non-disturbing to $\sigma_x^{(2)}$.

Since Eq. (34) holds for an arbitrary local measurement, it has an interesting application to quantum cryptography protocol E91 [40]. Suppose that Alice and Bob share a maximally entangled pair $S_1 + S_2$ in $|\Phi^+\rangle$ and that Eve measures $S_2$ for eavesdropping the shared key. Suppose that Alice and Bob share a key encoded in $\sigma_x^{(1)}$ and $\sigma_x^{(2)}$. To estimate how much information leaks to Eve, cooperative Alice and Bob measure the error probability $P_e^{AB}$ defined by $P_e^{AB} = \delta_G(\mu)^2/4$. Let $\epsilon_O(\sigma_x^{(2)})$ be Eve’s error for $\sigma_x^{(2)}$ measurement and let $\eta_O(\sigma_x^{(2)})$ be Eve’s disturbance caused on $\sigma_x^{(2)}$. Then Eq. (34) serves as a bridge between $P_e^{AB}$ and the disturbance $\eta_O(\sigma_x^{(2)})$, and the error-disturbance relation further relates $P_e^{AB}$ with Eve’s error probability $P_e^E$ for eavesdropping on the key defined by $P_e^E = \epsilon_O(\sigma_x^{(2)})^2/4$, as follows. Recall that the tight EDR

$$(\epsilon_O(\sigma_x^{(2)})^2 - 2)^2 + (\eta_O(\sigma_x^{(2)})^2 - 2)^2 \leq 4$$

holds for $\epsilon_O(\sigma_x^{(2)})$ and $\eta_O(\sigma_x^{(2)})$ (Ref. [17], Eq. (28)). Then this optimizes Eve’s error probability $P_e^E$ as

$$P_e^E(\text{optimal}) = \frac{1}{2} - \sqrt{\frac{1}{4} - \left( P_e^{AB} - \frac{1}{2} \right)^2}. \quad (36)$$

Thus, if the entanglement is not disturbed, i.e., $P_e^{AB} = 0$, then Alice and Bob conclude $P_e^E(\text{optimal}) = 1/2$ to ensure that no information has leaked to Eve. On the other hand, if Eve makes the projective measurement of $\sigma_x^{(2)}$ with $\epsilon_O(\sigma_x^{(2)}) = 0$, then she has the complete information $P_e^E = P_e^E(\text{optimal}) = 0$ but this is detected by Alice and Bob as $P_e^{AB} = 1/2$ and $\eta_O(\sigma_x^{(2)}) = \sqrt{2}$. However, the DR forces any disturbance measure $\eta$ to assign $\eta(\sigma_x^{(2)}) = 0$. How does the DR work to analyze the security of quantum communication?
7 Defense of state-dependent formulations

In order to examine the reliability of the operator-based disturbance measure, KJR [28] introduced the following definition. A state $|\psi\rangle$ is called a zero-noise zero-disturbance (ZNZD) state with respect to observables $A$ and $B$ if the projective measurement of $A$ in the state $|\psi\rangle$, which always satisfies $\varepsilon(A) = 0$, is distributionally non-disturbing to $B$. Then they proved that for every pair of non-commuting observables $A$ and $B$, there exists a ZNZD state $|\psi\rangle$ such that $|\langle\psi|[A,B]|\psi\rangle| \neq 0$. Thus, if the disturbance measure $\eta$ satisfies the DR, any relation of the form

$$\sum_{m,n=0}^{\infty} f_{mn}(A,B) \varepsilon(A,\rho)^m \eta(B,\eta)^n \geq |\langle[A,B]\rangle|,$$

(37)

where $f_{00}(A,B) = 0$, must be violated. From this, KJR [28] concluded that any state-dependent EDR, based on the expectation value of the commutator as a lower bound, is not tenable, and that state-independent formulations are inevitable.

We have two objections to their claims. First of all, the universally valid relation (3) with $\varepsilon_O(A) = 0$ leads to the relation

$$\eta_O(B) \geq \frac{|\langle[A,B]\rangle|}{2\sigma(A)} > 0$$

(38)

for any projective measurement of $A$ in any ZNZD state such that $|\langle\psi|[A,B]|\psi\rangle| \neq 0$. Thus, the measurement is properly disturbing to $B$ by the soundness of $\eta_O$, and consequently the disturbance is operationally detectable by the operational accessibility of the definition of properly non-disturbing measurements, so that the assumption by KJR [28] that $\eta(B) = 0$ in any ZNZD state is unfounded.

Secondly, they concluded that state-independent formulations are inevitable for alternative formulations. However, currently proposed state-independent formulations of EDRs [39,41,42] do not appear to capture the essence of Heisenberg’s original idea. Recall that Heisenberg derived his EDR by the $\gamma$-ray microscope thought experiment, in which the EDR is derived from the relation between the resolution power and the Compton recoil, reciprocally relating to the wave length of the incident light. Since the wave length is independent of the state of the object, the above formulation might be considered as state-independent. However, the analysis is valid only state-dependently, since the resolution power of the microscope can be defined by the wave length only in the limited situation in which the object is properly placed in the scope of the microscope. Thus, we can adequately define the error of the $\gamma$-ray microscope only state-dependently. In the state-independent formulations, currently one defines the state-independent error as the worst case of the state-dependent error, which must diverge to infinity as the object wave function spreads out of, or moves far apart from, the scope of the microscope. Such state-independent definitions would facilitate to reproduce the form of Heisenberg’s original formulation, but do not keep the physics underlying it. Thus, state-dependent formulations are inevitable to represent Heisenberg’s original idea underlying the uncertainty principle.

8 Discussion

In this paper, we have given a definition of non-disturbing measurement from the point of view of the correspondence principle and operational accessibility. Subsequently, we have
established the reliability of the operator-based disturbance measure. We have already discussed the reliability of the operator-based error measure in our previous work [13]. Both accounts ensure that universally valid EDRs [9,15,17] reliably represent a dynamical aspect of Heisenberg’s uncertainty principle besides the well-established relation for the indeterminacy in quantum states representing a kinetic aspect of the principle. Thus, the objections to state-dependent formulations of EDRs shown in [28,39] are unfounded, although those views appear to still prevail in the literature [43,44]. We conclude that the theory [9,11,15,18] and experiments [12,14,19,21] for state-dependent formulations of EDRs are reliable and that state-dependent formulations are inevitable to represent Heisenberg’s original idea underlying the uncertainty principle.

The new quantitative methods developed in this paper for universally valid EDRs with the well-defined operator-based disturbance measure incorporating with the methods of weak values and weak measurements will provide new quantitative methods to understand the change, transfer, or disturbance of observables in time, which does not manifest in the change of the probability distribution, but which does manifest in the time-like correlation. This quantity will be useful and even inevitable for exploring foundational problems in quantum physics including the long-lasting controversy over the roles of uncertainty principle in which-way measurements and experiments [12–14,19,21] for state-dependent formulations of EDRs are reliable and that state-dependent formulations are inevitable to represent Heisenberg’s original idea underlying the uncertainty principle.

9 Projective $\sigma_z$ measurement

Here, we shall give a derivation of Eq. (16). The JPD $\mu$ of $\sigma_z(\tau) = \sigma_z \otimes \sigma_x$ and $\sigma_z(0) = \sigma_z \otimes I$ in the state $|0,0\rangle$ is given by

$$\mu(u, v) = \langle 0, 0 | P^{\sigma_z \otimes \sigma_x}(u) P^{\sigma_z \otimes I}(v) | 0, 0 \rangle.$$ 

We have

$$P^{\sigma_z \otimes \sigma_x}(+1) P^{\sigma_z \otimes I}(\pm 1) = [P^{\sigma_z}(+1) \otimes P^{\sigma_x}(+1) + P^{\sigma_z}(-1) \otimes P^{\sigma_x}(-1)](P^{\sigma_z}(\pm 1) \otimes I) = P^{\sigma_z}(\pm 1) \otimes P^{\sigma_x}(\pm 1),$$

$$P^{\sigma_z \otimes \sigma_x}(-1) P^{\sigma_z \otimes I}(\pm 1) = [P^{\sigma_z}(+1) \otimes P^{\sigma_x}(-1) + P^{\sigma_z}(-1) \otimes P^{\sigma_x}(+1)](P^{\sigma_z}(\pm 1) \otimes I) = P^{\sigma_z}(\pm 1) \otimes P^{\sigma_x}(\mp 1).$$

Consequently,

$$\mu(+1, \pm 1) = \langle 0, 0 | P^{\sigma_z}(\pm 1) \otimes P^{\sigma_x}(\pm 1) | 0, 0 \rangle = \langle 0 | P^{\sigma_z}(\pm 1) | 0 \rangle \langle 0 | P^{\sigma_x}(\pm 1) | 0 \rangle = \frac{1}{4},$$

$$\mu(-1, \pm 1) = \langle 0, 0 | P^{\sigma_z}(\pm 1) \otimes P^{\sigma_x}(\mp 1) | 0, 0 \rangle = \langle 0 | P^{\sigma_z}(\pm 1) | 0 \rangle \langle 0 | P^{\sigma_x}(\mp 1) | 0 \rangle = \frac{1}{4}.$$
Therefore, we obtain Eq. (16), i.e.,
\[ \mu(u, v) = \frac{1}{4}. \]

10 Equivalence for properly non-disturbing measurements

**Theorem 1.** Let \( M \) be a measurement of a system \( S \) in a state \( |\psi\rangle \) carried out by a measuring interaction with a probe \( P \) prepared in a fixed state \( |\xi\rangle \) from \( t = 0 \) to \( t = \tau \). Then for any observable \( B \) in \( S \), the following conditions are equivalent.

(i) Condition (W): The WJD \( \nu(u, v) \) of \( B(\tau) \) and \( B(0) \) in \( |\psi, \xi\rangle \) satisfies that \( \nu(u, v) = 0 \) if \( u \neq v \).

(ii) The relation
\[ P^{B(\tau)}(u)|\psi, \xi\rangle = P^{B(0)}(u)|\psi, \xi\rangle \]
holds for any \( u \).

(iii) Condition (S): \( B(\tau) \) and \( B(0) \) have their JPD \( \mu(u, v) \) in \( |\psi, \xi\rangle \) satisfying that \( \mu(u, v) = 0 \) if \( u \neq v \).

**Proof.** The assertion was generally proved in Refs. [37, 38] after a lengthy argument. We give a direct proof for the present context.

(i) ⇒ (ii): Suppose (i) holds. Then the WJD \( \nu(u, v) \) of \( B(\tau) \) and \( B(0) \) in \( |\psi, \xi\rangle \) satisfies \( \nu(u, v) = 0 \) if \( u \neq v \). It follows that \( \nu(u, v) = \sum_v \nu(u, v) \). Thus,
\[ \langle \psi, \xi|P^{B(\tau)}(u)P^{B(0)}(0)|\psi, \xi\rangle = \langle \psi, \xi|P^{B(0)}(0)|\psi, \xi\rangle, \]
\[ \langle \psi, \xi|P^{B(0)}(0)P^{B(\tau)}(0)|\psi, \xi\rangle = \langle \psi, \xi|P^{B(\tau)}(0)|\psi, \xi\rangle. \]
Consequently,
\[ \|P^{B(\tau)}(u)|\psi, \xi\rangle - P^{B(0)}(0)|\psi, \xi\rangle\|^2 = 0, \]
and
\[ P^{B(\tau)}(u)|\psi, \xi\rangle = P^{B(0)}(0)|\psi, \xi\rangle. \]
Thus, condition (ii) holds and the implication (i) ⇒ (ii) follows.

(ii) ⇒ (iii): Suppose (ii) holds. Then
\[ P^{B(0)}(0)P^{B(\tau)}(\tau)|\psi, \xi\rangle = \delta_{u,v}P^{B(0)}(0)|\psi, \xi\rangle, \]
\[ P^{B(\tau)}(\tau)P^{B(0)}(0)|\psi, \xi\rangle = \delta_{u,v}P^{B(0)}(0)|\psi, \xi\rangle. \]
Consequently,
\[ P^{B(0)}(0)P^{B(\tau)}(\tau)|\psi, \xi\rangle = P^{B(\tau)}(\tau)P^{B(0)}(0)|\psi, \xi\rangle. \]
It follows that \( B(0) \) and \( B(\tau) \) commute in \( |\psi, \xi\rangle \) and condition (S) holds. Thus the implication (ii) ⇒ (iii) follows.

Since the implication (iii) ⇒ (i) holds obviously, all conditions (i) – (iii) are equivalent. \( \Box \)
11 Locality of properly non-disturbing measurements

**Theorem 2.** The definition of properly disturbing measurements satisfies the locality requirement.

*Proof.* Let M be a local measurement of $S_2$ in a composite system $S_1 + S_2$ in a state $|\Psi\rangle$. Without any loss of generality that M is carried out by a measuring interaction $U$ with a probe $P$ prepared in a state $|\xi\rangle$ from time $t = 0$ to $t = \tau$. Suppose that M is properly non-disturbing to an observable $B_2$ in $S_2$. Let $g(v)$ be a polynomial in $v$. From Theorem 2(i) (Section 10) we have

$$g(B_2(\tau))|\Psi, \xi\rangle = g(B(0))|\Psi, \xi\rangle.$$  

Let $B_1$ be an observable in $S_1$. Let $f(u)$ be a polynomial in $u$. Since $B_1(0) = B_1(\tau)$ by the locality of M we have

$$f(B_1(\tau))g(B_2(\tau))|\Psi, \xi\rangle = f(B_1(0))g(B_2(0))|\Psi, \xi\rangle.$$  

It follows from linearity that

$$h(B_1(\tau), B_2(\tau))|\Psi, \xi\rangle = h(B_1(0), B_2(0))|\Psi, \xi\rangle$$

for any polynomial $h(u, v)$ in $(u, v)$, and in particular we have

$$P^{B_1(\tau)B_2(\tau)}(v)|\Psi, \xi\rangle = P^{B_1(0)B_2(0)}(v)|\Psi, \xi\rangle.$$  

Thus, M does not disturb $B_1 \otimes B_2$ for any $B_1$ in $S_1$ by Theorem 2(ii) (Section 10). Therefore, the definition of properly disturbing measurements satisfies the locality requirement. \hfill \Box

**Theorem 3.** Let $S_1 + S_2$ be a composite system in a state $|\Psi\rangle$. Let $B_1$ be an observable in $S_2$. Any local measurement M of $S_2$ distributionally non-disturbing to $B_1 \otimes B_2$ for any $B_1$ in $S_1$ does not change the JPD of observable $B_1$ and $B_2$ for any observable $B_1$ in $S_1$.

*Proof.* Let $B_1$ be an observable in $S_1$ and $A_1 = P^{B_1}(u)$. By assumption, M does not change the probability distribution of $A_1 \otimes B_2 = P^{B_1}(u) \otimes B_2$, so that all moments of $A_1 \otimes B_2$ are unchanged as

$$\langle \Psi, \xi | P^{B_1(\tau)}(u)B_2(\tau)^n | \Psi, \xi \rangle = \langle \Psi, \xi | P^{B_1(0)}(u)B_2(0)^n | \Psi, \xi \rangle$$

for all $n$. By linearity, we have

$$\langle \Psi, \xi | P^{B_1(\tau)}(u)f(B_2(\tau)) | \Psi, \xi \rangle = \langle \Psi, \xi | P^{B_1(0)}(u)f(B_2(\tau)) | \Psi, \xi \rangle$$

for any polynomial $f(w)$ in $w$. It follows that

$$\langle \Psi, \xi | P^{B_1(\tau)}(u)P^{B_2(\tau)}(v) | \Psi, \xi \rangle = \langle \Psi, \xi | P^{B_1(0)}(u)P^{B_2(0)}(v) | \Psi, \xi \rangle,$$

and hence M does not change the JPD of $B_1$ and $B_2$ for any observable $B_1$ in $S_1$. \hfill \Box

**Theorem 4.** Let $S_1 + S_2$ be a composite system in a state $|\Psi\rangle$. Any local measurement M of $S_2$ properly non-disturbing to $B_2$ in $S_2$ does not change the JPD of observable $B_1$ and $B_2$ for any observable $B_1$ in $S_1$.

*Proof.* Any local measurement M of $S_2$ properly non-disturbing to $B_2$ in $S_2$ is properly non-disturbing to $B_1 \otimes B_2$ for any $B_1$ in $S_1$ by Theorem 2 (Section 11), and hence it is distributionally non-disturbing to $B_1 \otimes B_2$ for any $B_1$ in $S_1$. Consequently, the assertion follows from Theorem 3. \hfill \Box
12 Operator-based disturbance measure and disturbance of entanglement

Theorem 5. Let $S_1 + S_2$ be a composite system in a state $|\Psi\rangle$. Let $M$ be a local measurement of the system $S_2$ carried out by a measuring interaction with a probe $P$ prepared in a fixed state $|\xi\rangle$ from $t = 0$ to $t = \tau$. Let $\mu_{t}(u, v)$ be the JPD of an observable $B_1(t)$ in $S_1$ and an observable $B_2(t)$ in $S_2$ for $t = 0, \tau$. Let $\eta_O(B_2)$ be the operator-based disturbance of $M$ for $B_2$. Let $\delta_G(\mu_t)$ be the classical root-mean-square deviation determined by $\mu_t$. Then we have the following.

(i) The relation

$$|\delta_G(\mu_\tau) - \delta_G(\mu_0)| \leq \eta_O(B_2) \leq \delta_G(\mu_\tau) + \delta_G(\mu_0).$$

holds.

(ii) If $\delta_G(\mu_0) = 0$ then $\delta_G(\mu_\tau) = \eta_O(B_2)$.

(iii) If $\delta_G(\mu_0) = 0$ and $B_2^2 = I$ then $M$ is properly non-disturbing to $B_2$ if and only $\delta_G(\mu_\tau) = 0$

Proof. (i) We have the relations

$$\delta_G(\mu_t) = \|B_1(t)|\Psi, \xi\rangle - B_2(t)|\Psi, \xi\rangle\|,$$

$$\eta_O(B_2) = \|B_2(\tau)|\Psi, \xi\rangle - B_2(0)|\Psi, \xi\rangle\|,$$

and hence assertion (i) follows from repeated uses of the triangular inequality.

(ii) Follows by substituting $\delta_G(\mu_0) = 0$ in (i).

(iii) Follows from (ii) and the completeness of $\eta_O$ for dichotomic observables.

12.1 Projective $\sigma_z^{(2)}$ measurement

Consider the projective measurement of $A = \sigma_z^{(2)}$ in $S_1 + S_2$ carried out by the measuring interaction

$$U = I \otimes |0\rangle\langle 0| \otimes I + I \otimes |1\rangle\langle 1| \otimes \sigma_x$$

turned on from $t = 0$ to $t = \tau$ between $S_1 + S_2$ and the probe $P = S_3$ prepared in $|\xi\rangle = |0\rangle$ with the meter observable $M = \sigma_z^{(3)}$. Consider the Heisenberg operators $\sigma_x^{(2)}(0) = \sigma_x^{(2)} \otimes I$ and $\sigma_x^{(2)}(\tau) = U^\dagger(\sigma_x^{(2)} \otimes I)U$ for $B = \sigma_x^{(2)}$. From Eq. (14) we have

$$\sigma_x^{(1)}(0) = \sigma_x \otimes I \otimes I,$$

$$\sigma_x^{(1)}(\tau) = \sigma_x \otimes I \otimes I,$$

$$\sigma_x^{(2)}(0) = I \otimes \sigma_x \otimes I,$$

$$\sigma_x^{(2)}(\tau) = I \otimes \sigma_x \otimes \sigma_x.$$

Let $\mu_t$ be the JPD of $\sigma_x^{(1)}(t)$ and $\sigma_x^{(2)}(t)$ in the state $|\Phi^+, 0\rangle = |\Phi^+\rangle \otimes |0\rangle$. We shall show
(i) \( \mu_0(u, v) = \frac{1}{2} \delta_{u,v} \),

(ii) \( \mu_r(u, v) = \frac{1}{4} \),

(iii) \( \delta_C(\mu_r) = \eta_O(\sigma_x^{(2)}) = \sqrt{2} \).

We have

\[
\mu_0(u, v) = \langle \Phi^+, 0 | P^{(0)}_{\sigma_z}(u) P^{(2)}_{\sigma_z}(v) | \Phi^+, 0 \rangle = \langle \Phi^+ | P^{\sigma_z} (u) \otimes P^{\sigma_z} (v) | \Phi^+ \rangle
\]

\[
= \frac{1}{2} \langle 0_\sigma 0_\sigma | P^{\sigma_z} (u) \otimes P^{\sigma_z} (v) | 0_\sigma 0_\sigma \rangle + \frac{1}{2} \langle 1_\sigma 1_\sigma | P^{\sigma_z} (u) \otimes P^{\sigma_z} (v) | 0_\sigma 0_\sigma \rangle
\]

\[
+ \frac{1}{2} \langle 0_\sigma 0_\sigma | P^{\sigma_z} (u) \otimes P^{\sigma_z} (v) | 1_\sigma 1_\sigma \rangle + \frac{1}{2} \langle 1_\sigma 1_\sigma | P^{\sigma_z} (u) \otimes P^{\sigma_z} (v) | 1_\sigma 1_\sigma \rangle
\]

\[
= \frac{1}{2} \delta_{u,v},
\]

and (i) follows.

We have

\[
P^{\sigma_z \otimes I \otimes I}(\pm 1) P^{I \otimes \sigma_z \otimes \sigma_z}(+1)
\]

\[
= (P^{\sigma_z}(\pm 1) \otimes I \otimes I) [I \otimes P^{\sigma_z}(+1) \otimes P^{\sigma_z}(1) + I \otimes P^{\sigma_z}(-1) \otimes P^{\sigma_z}(-1)]
\]

\[
= P^{\sigma_z}(\pm 1) \otimes P^{\sigma_z}(+1) \otimes P^{\sigma_z}(+1) + P^{\sigma_z}(\pm 1) \otimes P^{\sigma_z}(-1) \otimes P^{\sigma_z}(-1),
\]

\[
P^{\sigma_z \otimes I \otimes I}(\pm 1) P^{I \otimes \sigma_z \otimes \sigma_z}(-1)
\]

\[
= (P^{\sigma_z}(\pm 1) \otimes I \otimes I) [I \otimes P^{\sigma_z}(+1) \otimes P^{\sigma_z}(-1) + I \otimes P^{\sigma_z}(-1) \otimes P^{\sigma_z}(1)]
\]

\[
= P^{\sigma_z}(\pm 1) \otimes P^{\sigma_z}(+1) \otimes P^{\sigma_z}(-1) + P^{\sigma_z}(\pm 1) \otimes P^{\sigma_z}(-1) \otimes P^{\sigma_z}(1).
\]

Consequently,

\[
\sqrt{2} P^{\sigma_z \otimes I \otimes I}(+1) P^{I \otimes \sigma_z \otimes \sigma_z}(+1) | \Phi^+, 0 \rangle
\]

\[
= P^{\sigma_z}(+1) \otimes P^{\sigma_z}(+1) \otimes P^{\sigma_z}(+1) | 0_\sigma 0_\sigma 0_\sigma \rangle + P^{\sigma_z}(+1) \otimes P^{\sigma_z}(+1) \otimes P^{\sigma_z}(+1) | 1_\sigma 1_\sigma 1_\sigma \rangle
\]

\[
+ P^{\sigma_z}(+1) \otimes P^{\sigma_z}(-1) \otimes P^{\sigma_z}(-1) | 0_\sigma 0_\sigma 0_\sigma \rangle + P^{\sigma_z}(+1) \otimes P^{\sigma_z}(-1) \otimes P^{\sigma_z}(-1) | 1_\sigma 1_\sigma 1_\sigma \rangle
\]

\[
= | 0_\sigma 0_\sigma 0_\sigma \rangle \otimes P^{\sigma_z}(+1) | 0 \rangle.
\]

Similarly,

\[
\sqrt{2} P^{\sigma_z \otimes I \otimes I}(-1) P^{I \otimes \sigma_z \otimes \sigma_z}(+1) | \Phi^+, 0 \rangle = | 1_\sigma 1_\sigma 1_\sigma \rangle \otimes P^{\sigma_z}(-1) | 0 \rangle,
\]

\[
\sqrt{2} P^{\sigma_z \otimes I \otimes I}(+1) P^{I \otimes \sigma_z \otimes \sigma_z}(-1) | \Phi^+, 0 \rangle = | 0_\sigma 0_\sigma 0_\sigma \rangle \otimes P^{\sigma_z}(-1) | 0 \rangle,
\]

\[
\sqrt{2} P^{\sigma_z \otimes I \otimes I}(-1) P^{I \otimes \sigma_z \otimes \sigma_z}(-1) | \Phi^+, 0 \rangle = | 1_\sigma 1_\sigma 1_\sigma \rangle \otimes P^{\sigma_z}(+1) | 0 \rangle.
\]

Thus, we have

\[
\mu_r(u, v) = \| P^{\sigma_z \otimes I \otimes I}(u) P^{I \otimes \sigma_z \otimes \sigma_z} (v) | \Phi^+, 0 \rangle \|^2 = \frac{1}{4},
\]

for any \( u, v = \pm 1 \), and (ii) follows. Thus, Eq. \ref{27} is obtained.

From (ii), \( \delta_C(\mu_r) = \sqrt{2} \) follows. From (i), \( \delta_C(\mu_0) = 0 \), so that it follows from Theorem \ref{5} (ii) that \( \eta_O(\sigma_x^{(2)}) = \delta_C(\mu_r) = \sqrt{2} \). Thus, (iii) follows and Eq. \ref{28} is obtained.
12.2 Projective $σ_θ^{(2)}$ measurement

Suppose that the observer makes a measurement $M(θ)$ of $A = σ_θ^{(2)}$ in $S_1 + S_2$, carried out by the measuring interaction

$$U = I \otimes P^{σ_θ}(+1) \otimes I + I \otimes P^{σ_θ}(-1) \otimes σ_x$$

turned on from $t = 0$ to $t = τ$ and by the subsequent measurement of the meter observable $M = σ_2^{(3)}$ of $P = S_3$ prepared in $|ξ⟩ = |0⟩$. This realizes the projective measurement of $A = σ_θ^{(2)}$ as

$$U(|α⟩ \otimes |0θ⟩ \otimes |0⟩) = |α⟩ \otimes |0θ⟩ \otimes |0⟩,$$
$$U(|α⟩ \otimes |1θ⟩ \otimes |0⟩) = |α⟩ \otimes |1θ⟩ \otimes |1⟩,$$

for $α = 0, 1$, where $|0θ⟩ := |σ_θ = +1⟩$ and $|1θ⟩ := |σ_θ = -1⟩$. Consider the Heisenberg operators $B(0) = σ_x^{(2)}(0) = σ_x^{(2)} \otimes I$ and $B(τ) = σ_x^{(2)}(τ) = U^†(σ_x^{(2)} \otimes I)U$ for $B = σ_θ^{(2)}$. We have

$$σ_x^{(1)}(0) = σ_x \otimes I \otimes I,$$
$$σ_x^{(1)}(τ) = σ_x \otimes I \otimes I,$$
$$σ_x^{(2)}(0) = I \otimes σ_x \otimes I,$$
$$σ_x^{(2)}(τ) = I \otimes \sin θσ_θ \otimes I + I \otimes \cos θσ_{-θ} \otimes σ_x.$$

Let $μ_τ$ for $t = 0, τ$ be the JPD of $σ_x^{(1)}(t)$ and $σ_x^{(2)}(t)$ in the state $|Φ^+, 0⟩$, i.e.,

$$μ(u, v) = ⟨Φ^+, 0| P^{σ_x^{(1)}(t)}(u) P^{σ_x^{(2)}(t)}(v)|Φ^+, 0⟩.$$  

Then we have

$$μ_0(u, v) = \frac{1}{2} δ_{u,v}.$$  

For $u = \pm 1$ we have

$$|P^{σ_θ^{(2)}(τ)}(u)|Φ^+, 0⟩|^2$$

$$= |P^{σ_2^{(2)}(0)}(u)U|Φ^+, 0⟩|^2$$

$$= \frac{1}{2}(|P^{σ_z}(u)P^{σ_θ}(-1)|0⟩|^2 + |P^{σ_z}(u)P^{σ_θ}(-1)|1⟩|^2 + |P^{σ_z}(u)P^{σ_θ}(+1)|1⟩|^2$$

$$= \frac{1}{4}(|P^{σ_z}(u)|0⟩|^2 + |P^{σ_z}(u)σ_θ|0⟩|^2 + |P^{σ_z}(u)σ_θ|1⟩|^2 + |P^{σ_z}(u)|1⟩|^2)$$

$$= \frac{1}{4}(|P^{σ_z}(u)|0⟩|^2 + |P^{σ_z}(u)|1⟩|^2 + |P^{σ_z}(u)σ_θ|0_x⟩|^2 + |P^{σ_z}(u)σ_θ|1_x⟩|^2)$$

$$= \frac{1}{2}.$$

We used the parallelogram law twice in the third last and the penultimate equalities. It follows that

$$\sum_u μ_τ(u, v) = \frac{1}{2}.$$
Thus, with \( \sum_{u} \mu_{0}(u, v) = 1/2 \), the projective measurement of \( \sigma_{\theta}^{(2)} \) is distributionally non-disturbing to \( \sigma_{x}^{(2)} \).

Let \( \delta_{G}(\mu_{t}) \) be the classical root-mean-square deviation for \( \mu_{t} \). We have \( \delta_{G}(\mu_{0}) = 0 \). By Theorem 5(ii) we have \( \delta_{G}(\mu_{t}) = \eta_{O}(\sigma_{x}^{(2)}) \). Then from (Ref. [12], Eq. (6)) we have

\[
\eta_{O}(\sigma_{x}^{(2)})^2 = \sum_{y} \| [P_{\sigma_{\theta}^{(2)}}^{(2)}(v), \sigma_{x}^{(2)}] |\Phi^{+}\rangle \|^2 = 2 \| [\sigma_{\theta}^{(2)}/2, \sigma_{x}^{(2)}] |\Phi^{+}\rangle \|^2 = 2 \cos^2 \theta.
\]

Thus, we obtain Eq. (33), i.e.,

\[
\delta_{G}(\mu_{t}) = \eta_{O}(\sigma_{x}^{(2)}) = \sqrt{2} \cos \theta.
\]

In what follows we will determine \( \mu_{t} \) without tedious calculations on relevant projections. We have

\[
\sum_{u, v: u \neq v} \mu_{t}(u, v) = \frac{1}{4} \delta_{G}(\mu_{t}) = \frac{1}{2} \cos^2 \theta.
\]

Since \( \sigma_{x}^{(1)}(0) = \sigma_{x}^{(1)}(\tau) \) and \( M(\theta) \) is distributionally non-disturbing to \( \sigma_{x}^{(2)} \), we have

\[
\sum_{u} \mu_{t}(u, v) = \sum_{u} \mu_{0}(u, v) = \frac{1}{2},
\]

\[
\sum_{u} \mu_{t}(u, v) = \sum_{u} \mu_{0}(u, v) = \frac{1}{2}.
\]

Since,

\[
\sum_{u, v: u \neq v} \mu_{t}(u, v) + \sum_{v} \mu_{t}(+1, v) = \mu_{t}(+1, -1) + \mu_{t}(-1, +1) + \mu_{t}(+1, -1) + \mu_{t}(+1, +1)
\]

\[
= 2 \mu_{t}(+1, -1) + \sum_{u} \mu_{t}(u, +1),
\]

we obtain

\[
\mu_{t}(+1, -1) = \frac{1}{2} \sum_{u, v: u \neq v} \mu_{t}(u, v) = \frac{1}{4} \cos^2 \theta.
\]

It follows that

\[
\mu_{t}(+1, -1) = \mu_{t}(-1, +1) = \frac{1}{4} \cos^2 \theta,
\]

\[
\mu_{t}(+1, +1) = \mu_{t}(-1, -1) = \frac{1}{4} (1 + \sin^2 \theta).
\]

Therefore, we have derived Eq. (32).
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