Full Poincaré-invariant equation of motion for an extended charged particle in an electromagnetic field

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Abstract

For the rigid, nonrotating motion of an extended charge in an arbitrary electromagnetic field, an equation of motion is derived by Lorentz-invariantly calculating the 4-Lorentz force $\vec{F} = \vec{F}_e + \vec{F}_s = \text{external 4-force} + \text{4-self-force}$, acting upon the charge. For small 4-acceleration $\vec{a}$, the 4-self-force becomes $\vec{F}_s \approx -m_i \vec{a}$, where $m_i c^2$ equals the electrostatic self-energy. The notorious factor $4/3$ does not occur.

The equation of motion is invariant under any full Poincaré transformation. It is a differential-integral equation for the particle’s 4-centre $\vec{z}$ in Minkowski space as a function of proper time $\tau$. The twofold proper time integrated form of the equation of motion is suited to derive the solution $\vec{z}(\tau)$ iteratively if the initial conditions $\vec{z}(0) = \vec{z}_0, \dot{\vec{z}}(0) = \dot{\vec{z}}_0$ and the external electromagnetic field are given and the iteration procedure converges. Diverse approximations for the self-force and with that also for the equation of motion are derived.

The (approximated) equation of motion is solved for two examples: (1) Rectilinear hyperbolic motion in a homogeneous electric field. (2) Spiral (nonrelativistic) motion in a homogeneous magnetic field. In both cases amazingly simple solutions result.

The advantages of the equation against other ones, for example the Lorentz-Dirac equation, are discussed.

Keywords: Equation of motion, extended charged particle, self-force, effective inertial mass.

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1. Introduction

A physicist needs that his equations should be mathematically sound and that in working with his equations he should not neglect quantities unless they are small.  

P.A.M. Dirac [13]

How does a charged particle move in an electromagnetic field under consideration of the reaction force?

Enormous efforts to answer this question have been made for more than a century. The history of these efforts may be looked up in the following selected literature: [1,3,4,6,8,12,13, 14,15,20,21,25,27,30,35,40,44,47,49,50,54,59,62,63,66]. In these works the drawbacks of the various equations of motion and their solutions are described in detail (see also Appendix F). A further trial to answer this question shall be made in the following.

As the assumption of a point charge leads to mathematically and physically unsatisfying consequences [24,43,44,48,63], we suppose an extended charge with spherically symmetric charge density. The particle shall execute a rigid nonrotating motion [9,18,26,41,64,65]. At sufficiently small particle radius (4.9) the spherically symmetric charge density distribution in the rest frame is conserved under the transition to the local rest frame.

To derive the Lorentz-invariant equation of motion (8.2), we first explain the conception of a rigid nonrotating motion. We prefer the compact notation of 4-vectors and tensors against the more frequently used coordinate notation, as the former is more lucid. The compact vectors and tensors (dyadics) are Lorentz-invariant, their coordinate representations are only form invariant (covariant).

The unknown density function \( \rho(r) \) (4.1) of the particle is chosen so that the square \( \tilde{\rho}^2 \) of its Fourier transform (4.6) is as simple as possible. The density \( \rho \) is used as weight factor to calculate the mean values of the 4-current density \( j \), of the retarded 4-self-potential \( A \), and of the electromagnetic 4-self-field strength \( B \) (7.1a) (4-dyadic = compact representation of the electromagnetic field strength tensor). The space part \( B_s \) of \( B \) (7.1c) is linearly connected with the magnetic self-field strength \( B_m \). We prefer the letter \( B \) against the frequently used letter \( F \) to designate the electromagnetic field strength, in analogy to \( j = j^0 e_0 + j^i e_i \) and \( A = A^0 e_0 + A_i e_i \) [cf. (6.7)], and as the letter \( F \) is used to designate forces.

Additionally to the mean electromagnetic self-field strength \( < B_s > \) (7.6), the external electromagnetic field strength \( B_e \) acts on the particle. The sum of both, \( < B > := B_s + < B_s > \), causes the 4-Lorentz force \( \vec{F} \) in the equation of motion (8.1) or (8.2). Twofold proper time integration of (8.2) yields the equivalent differential-integral equation (8.5). This is suited to derive the solution \( \vec{x}(\tau) \) iteratively if the initial conditions (8.4) and the external electromagnetic field strength \( \vec{B}_e(x) \), with \( \vec{x} = t e_0 + x^i e_i + x^i e_i + x^i e_j \) (2.6), are given. The 4-position vector \( \vec{x}(\tau) \) describes the worldline of the particle centre with the proper time as a parameter. Various approximations for the self-force and with it for the equation of motion are derived.

The approximated equation of motion is solved for two examples: (1) The rectilinear (hyperbolic) motion of a charge \( q \) in a homogeneous electric field \( E_e = \text{const.} \). (2) The spiral (nonrelativistic) motion of \( q \) in a homogeneous magnetic field \( B_e = \text{const.} \). In both cases, the solutions \( \vec{x}(\tau) \) satisfy the generalized Newton law \( m a = F_e \).
The generalization consists in that the effective mass $m$ becomes a (generally nonanalytic) function of $|a|$, where $a \equiv 2(\tau)$ is the proper acceleration. In case of $B_\epsilon = \text{const}$, it is advantageous, to assume $m, a$ and $F_\epsilon$ as complex quantities. The notorious factor $4/3$ does not occur.

For simplicity, we set $c = \varepsilon_0 = \mu_0 = 1$. For the rest, we use the rational international quantity system.

The equation of motion (8.2) is invariant under any full Poincaré transformation (= full inhomogeneous Lorentz transformation) of coordinates and base vectors [see Sect. 19]. By each passive transformation [22, 56], the physical processes shall not be transformed, but only their description as to another orthonormal base \{\overrightarrow{e_i}\}. A Lorentz transformation is always a passive transformation [22].

A time inversion $t \rightarrow -t$, \( \overrightarrow{e_0} \rightarrow -\overrightarrow{e_0} \) is a special transformation of the full Poincaré group. It lets each 4-point \( \overrightarrow{x} = x^i \overrightarrow{e_i} \), $i = 0, ..., 3$ (sum convention), invariant. For example, by a passive time inversion an inward spiraling charge $q$ is not transformed into an outward spiraling one. Instead of this, a backward running clock describes the inward spiraling process.

For the spiraling particle, power balances are discussed. General discussions of 4-momentum balances and of the energy-momentum tensor are omitted. For, they are in detail discussed in the textbooks (see, e.g., [20, 22, 28, 34, 42, 54, 56, 57, 60, 65]).

In the Appendix F the Lorentz-Dirac equation is derived as an approximation of the equation of motion (8.2). Unfortunately, the Laurent expansion (F4) of the 4-self-force $\overrightarrow{F}_s$ and the truncation of this in general divergent, asymptotic expansion are inadmissible. These unsound mathematical operations obviously cause the unsound results.

At the end, advantages of the equation of motion (8.2), compared with other ones, are discussed. In the Appendices A to F various proofs and supplements are given.

2. Rigid nonrotating motion of a charged particle

Laxly spoken, we’ll describe the motion of a rigid, nonrotating charged particle in an electromagnetic field. How is this Lorentz-invariantly possible under consideration of the particle’s self-field? For simplicity, we assume that the particle’s charge centre coincides with its mass centre.

The 4-position of the centre may be represented by

$$\overrightarrow{x} = z^i \overrightarrow{e}_i. \quad (2.1)$$

(Sum convention: Over double Latin indices \( i, j, ..., \) is to be summed from 0 to 3, over Greek indices \( \alpha, \beta, ..., \) from 1 to 3; like, e.g., in [34].) The $z^i$ are the contravariant coordinates of $\overrightarrow{x}$ with respect to the (covariant) 4-base \{\overrightarrow{e}_i\} of the lab frame. The orthonormal unit vectors $\overrightarrow{e}_i$ point respectively into the t, x, y, z directions. They define the Lorentz (lab) metric

$$\eta_{ij} := \eta^i_\alpha \eta^j_\beta := \text{diag}(1, -1, -1, -1). \quad (2.2a)$$

(The symbols $\equiv$ and $:\equiv$ mean: equals by definition. The colon stands on the side of the defined quantity.)

The contravariant (dual) base \{\overrightarrow{e}^\alpha\} is defined by the orthonormality relations

$$\overrightarrow{e}^\alpha \cdot \overrightarrow{e}_\beta := \eta^\alpha_\beta := \delta^\alpha_\beta = \text{diag}(1, 1, 1, 1). \quad (2.2b)$$

Hence, $\overrightarrow{e}_0 = \overrightarrow{e}_0$, $\overrightarrow{e}_\alpha = -\overrightarrow{e}_\alpha$.

The 4-vector $\overrightarrow{z}$ depends on its proper time $\tau$. The 4-velocity
\[
\frac{d\vec{z}}{d\tau} = \vec{u}, \quad \text{with} \quad \vec{u}^2 = \vec{u} \cdot \vec{u} = (u^0)^2 - u_a u^a = 1, \quad (2.3)
\]
is a time-like unit vector.

Additionally to the lab base \(\{\vec{e}_i\}\), we introduce the primed base
\[
\{\vec{e}_i'\} \quad \text{with} \quad \vec{e}_i' \cdot \vec{e}_j' = \eta_{ij}, \quad \vec{e}_0' := \vec{u}(\tau) \quad (2.4)
\]
of the local (instantaneous) rest frame (= inertial frame) at a 4-point \(\vec{z}(\tau)\) of a worldline. The 4-vector \(\vec{z}\) may also be represented with respect to this base:
\[
\vec{z} = \vec{z}' \vec{e}_i'. \quad (2.5)
\]
The same holds for an arbitrary 4-point of the particle:
\[
\vec{x} = x^i \vec{e}_i = x'^i \vec{e}_i' = x_i \vec{e}_i' = x_i' \vec{e}_i'. \quad (2.6)
\]

We speak from a \textit{rigid nonrotating motion} \([9,18,26,64,65]\) if \(\vec{x}(\tau)\) is connected with \(\vec{z}(\tau)\) as follows:
\[
\vec{x}(\tau) := \vec{z}(\tau) + r^a \vec{e}_a(\tau) = \vec{z}(\tau) + \vec{r}(\tau), \quad r^a = \text{const}, \quad (2.7a)
\]
\[
\frac{d\vec{x}}{d\tau} =: \dot{\vec{x}} := \dot{\lambda} \vec{z}, \quad \lambda \quad \text{scalar}.
\]
That is, the particle fixed position vector
\[
\vec{r} := \vec{x} - \vec{z} = r^a \vec{e}_a(\tau) \quad (2.8)
\]
lies in the 3-plane \(\{\vec{r} \mid \vec{r} \cdot \vec{z}(\tau) = 0\}\) \((2.9)\).

Fig. 1 \textbf{through } \vec{z} \text{ orthogonal to } \dot{\vec{z}} \text{ (Fig. 1).}

3. Functional determinant

Corresponding to (2.7a), the 4-point \(\vec{x}\) is also determined by the coordinates \(\tau =: r^0, r^1, r^2, r^3\). These are connected with the lab coordinates \(x^i\) by
\[
x^i = x^i(\tau) + r^a \vec{e}_a^i(\tau). \quad (3.1)
\]

The functional determinant of this coordinate transformation reads
\[
D := \det \left( \frac{\partial x^i}{\partial r^j} \right) = \begin{vmatrix}
\vec{e}_0^0 + r^a \vec{e}_0^a & \vec{e}_1^0 & \vec{e}_2^0 & \vec{e}_3^0 \\
\vec{e}_0^1 & \vec{e}_1^1 & \vec{e}_2^1 & \vec{e}_3^1 \\
\vec{e}_0^2 & \vec{e}_1^2 & \vec{e}_2^2 & \vec{e}_3^2 \\
\vec{e}_0^3 & \vec{e}_1^3 & \vec{e}_2^3 & \vec{e}_3^3
\end{vmatrix}. \quad (3.2)
\]

For a local rest base holds \(e_i^0 = e_i'^0 = e_i'^0 = 0\) [see (2.4) and Fig. 1], so that this determinant specializes to
\[
D = 1 + r^a \dot{e}_a^0 = 1 + r^a \vec{e}_a^0 \cdot \dot{\vec{z}} = 1 + \vec{r} \cdot \dot{\vec{z}} = 1 - \vec{r} \cdot \dot{\vec{z}}. \quad (2.8)
\]
(The equation number \(2.8\) above the equality sign refers to the cited equation.) The last identity follows from
\[
\vec{r} \cdot \dot{\vec{z}} = 0 \quad (2.9)
\]
by differentiation with respect to \(\tau\). As the result \(3.3\) is Lorentz-invariant, it also holds with respect to the lab base.
The factor of proportionality $\lambda$ in (2.7b) equals $D$ (3.3). To prove this assertion, we equate (2.7b) with the proper time derivative of (2.7a):
\[ \dot{x} = \lambda \dot{\tilde{x}} = \ddot{r} + \dot{r}. \]
Scalar multiplication with $\tilde{z}$ yields
\[ \lambda = 1 + \tilde{r} \cdot \tilde{z} = D, \quad \text{q.e.d.} \] (3.5)
On condition that
\[ < r > a \ll 1, \quad a := |\tilde{a}| := |\tilde{z}|, \] (3.6)
(cf. (14.13)), the functional determinant can very well be approximated by
\[ D \approx 1. \] (3.7)
Here $< r >$ means the particle radius (4.9). Then the 4-volume elements $d^4\tilde{x}$ and $d^4\bar{r}$ are approximately equal:
\[ d^4\tilde{x} := dx^0dx^1dx^2dx^3 = D d^4\bar{r} := D d\tau dr^1dr^2dr^3 \]
\[ \approx d\tau dr^1dr^2dr^3 := d\tau d\sigma. \] (3.8)
The 3-volume element $d\sigma = dr^1dr^2dr^3$ is identical with the surface element of the 3-plane $\sigma$ (2.9).

4. Density

For simplicity, let us assume that the density
\[ \rho := \text{charge density} / \text{charge} = \rho_q / q \] (4.1)
in the particle’s rest frame is spherically symmetric:
\[ \rho = \rho(r) \quad \text{with} \quad r := |\tilde{r}| := \sqrt{-\tilde{r}^2} = \sqrt{(r^1)^2 + (r^2)^2 + (r^3)^2}. \] (4.2)
On condition (3.6), this also holds in each local rest frame.
In the following it proves favorable to transform the density $\rho(r)$ into its Lorentz-invariant 3-Fourier transform (structure function)
\[ \tilde{\rho}(\kappa) := \int_\sigma d\sigma \exp(\pm i \tilde{\kappa} \cdot \tilde{r}) \rho(r) = \frac{4\pi}{\kappa} \int_0^{\infty} dr r \sin \kappa r \rho(r) = \tilde{\rho}(-\kappa), \] (4.3a)
with
\[ \kappa := \sqrt{(\tilde{\kappa} \cdot \tilde{\kappa})^2 - \tilde{\kappa}^2}, \quad \tilde{\rho}(0) = \int_\sigma d\sigma \rho(r) = 1. \] (4.3b)
In the (primed) local rest frame holds $\tilde{\tilde{z}} = \tilde{e}_0$ (2.4), with which we get
\[ \kappa = \sqrt{(k^0)^2 + (k^1)^2 + (k^3)^2} = |\tilde{\kappa}|. \] (4.3c)
The reversal of (4.3a) is
\[ \rho(r) = \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dk \kappa \exp(ikr)\tilde{\rho}(\kappa) = \frac{1}{4\pi^2 r} \int_{-\infty}^{\infty} d\kappa \kappa \tilde{\rho}(\kappa) \sin \kappa r = \rho(-r). \] (4.4)
For a point charge holds
\[ \rho(r) = \frac{\delta(r)}{2\pi r}, \quad \tilde{\rho} = 1. \] (4.5)
Later we will specialize on the simple Fourier transform
\[ \tilde{\rho}_\ell(\kappa) := (1 + \ell^2 \kappa^2)^{-1/2}. \] (4.6)
The reason for this choice is that \( \tilde{\rho}_1^2(\kappa) \) has only poles of first order at \( \kappa = \pm i/r_1 \) as singularities in the complex \( \kappa \)-plane. The parameter \( r_1 \) has the meaning of a quasi-radius. The comparison with [23,31] suggests identifying \( r_1 \) with half the Compton wavelength:

\[
r_1 := \frac{1}{2} \lambda_c := \frac{\hbar}{2m_e c}.
\]

(4.7)

The insertion of (4.6) into (4.4) yields the density function

\[
\rho_1(r) = \frac{1}{2\pi^2 r_1^2} K_1\left(\frac{r}{r_1}\right)
\]

(4.8)

with

\[
K_1(x) := \int_0^\infty dz \cosh z \exp(-x \cosh z)
\]

and

\[
\kappa := \sqrt{\cosh \left(\frac{\kappa \pi}{2}\right)}.
\]

Fig. 2 (cf. [2,19,29,60] and Fig. 2).

As particle radius we define the mean value of \( r \) in the particle’s rest frame:

\[
<r> = \int_0^\infty dr \rho(r) r = 4\pi \int_0^\infty dr r^3 \rho(r).
\]

(4.9)

Expressing \( \rho(r) \) by \( \tilde{\rho}(\kappa) \) (4.4) and integrating over \( r \) gives

\[
<r> = \frac{2}{\pi} \int_\kappa^\infty d\kappa \frac{\tilde{\rho}(\kappa)}{(\kappa+i0)^2} := \lim_{\varepsilon \to 0} \frac{2}{\pi} \int_\kappa^\infty d\kappa \frac{\tilde{\rho}(\kappa)}{(\kappa+i\varepsilon)^2}.
\]

(4.10)

For the special form factor (4.6) one gets

\[
<r> = \frac{4r_1}{\pi}.
\]

(4.11)

5. Mean four-current density

The 4-current density of the particle with the charge \( q \) is

\[
\overline{J} = q \rho(r) \hat{x}(\tau) \approx q \rho \hat{z}(\tau) \Rightarrow \rho = \frac{1}{q} \sqrt{\overline{J}^2}.
\]

(5.1)

The 4-Fourier transform becomes

\[
\overline{J}_\vec{k} = \int d^4\vec{x} \exp(i\vec{k} \cdot \vec{x}) \overline{J}.
\]

(5.2)

The 4-volume integral is to be taken over that domain of spacetime in which \( \overline{J} \neq 0 \). Because \( \rho(r) \) is \( \delta \)-distribution like, only volume elements \( d^4\vec{x} \) within a thin tube of radius \( <r> \) about the worldline \( \vec{x}(\tau) \) essentially contribute to the integral. Therefore, for not too large acceleration \( a \), the inequality (3.6) is fulfilled, and \( d^4\vec{x} \approx d\tau d\sigma \) (3.8) is a very good approximation:

\[
\overline{J}_\vec{k} \approx q \int_{-\infty}^{\infty} d\tau \hat{z} \exp(i\vec{k} \cdot \vec{x}) \int d\sigma \exp[i\vec{k} \cdot (\vec{x} - \vec{\sigma})] \rho(|\vec{x} - \vec{\sigma}|)
\]

\[
\left(\text{4.3a}\right)
\]

\[
= q \int d\vec{\sigma} \exp(i\vec{k} \cdot \vec{\sigma}) \tilde{\rho}(\kappa).
\]

(5.3)

In case (4.5) of a point charge, \( \overline{J}_\vec{k} \) simplifies to the integral

\[
\overline{J}_\vec{k} = \int_{\vec{x}(\tau)}^{\vec{x}(\tau)} d\vec{\sigma} \exp(i\vec{k} \cdot \vec{\sigma}) \quad \text{for} \quad \tilde{\rho} = 1
\]

(5.4)

along the worldline \( \vec{x}(\tau) \) [22].
In analogy to the radius \(< r >\) \((4.9)\), we define the mean 4-current density \(< \overline{J} >\) as the mean value of the 4-current density \(\overline{J}(\overline{x})\), averaged over the 3-plane \(\sigma\) \((2.9)\) with the density \(\rho\) as a weight factor:

\[
< \overline{J} > (\overline{z}) := \int d\sigma \rho \overline{J} = \int d\sigma \rho(\overline{x} - \overline{z}) C^4 \int d^4\overline{k} \exp(-i\overline{k} \cdot \overline{x}) \overline{J}_\sigma \\
= C^4 \int d^4\overline{k} \exp(-i\overline{k} \cdot \overline{z}) \overline{J}_\sigma \int d\sigma \rho(\overline{x} - \overline{z}) \exp[-i\overline{k} \cdot (\overline{x} - \overline{z})]
\]

\[(4.3a)\]

\[
= C^4 \int d^4\overline{k} \exp(-i\overline{k} \cdot \overline{z}) \overline{J}_\sigma \rho(\kappa) = C^4 \int d^4\overline{k} \exp(-i\overline{k} \cdot \overline{z}) < \overline{J} > _\sigma , \quad (5.5a)
\]

with

\[
C := \frac{1}{2\pi}, \quad < \overline{J} > _\sigma := \overline{J}_\sigma \rho(\kappa) \quad (5.5b,c)
\]

and \(\kappa\) corresponding to \((4.3b)\). This simple result signifies: The 4-Fourier transform of \(< \overline{J} >\) equals the 4-Fourier transform of \(\overline{J}\), multiplied by the 3-Fourier transform of the density \(\rho\). The rule \((5.5c)\) also holds for any other quantity (scalar, 4-vector, 4-tensor) instead of \(\overline{J}\).

6. Mean retarded four-self-potential

The 4-current density

\[
\overline{J}(\overline{x}) = C^4 \int d^4\overline{k} \exp(-i\overline{k} \cdot \overline{x}) \overline{J}_\sigma 
\]

of the moving charge generates the Lorenz-gauged [28] 4-self-potential [22,34]

\[
\overline{A}_\sigma(\overline{x}) = \partial_\sigma \overline{J} = C^4 \int d^4\overline{k} \exp(-i\overline{k} \cdot \overline{x}) \frac{\overline{J}_\sigma}{-\overline{k}^2}, \quad \partial_\sigma \cdot \overline{A}_\sigma = \partial_\sigma \cdot \overline{J} = 0 . \quad (6.2)
\]

Here \(\partial_\sigma^2 = \Box^{-1}\) means the inverse wave operator and

\[
\partial_\sigma := \overline{\sigma} \frac{\partial}{\partial x_\sigma} = \overline{\sigma} \partial^{\prime}, \quad \text{with} \quad x_\sigma := \eta, x^{\prime}, \quad (6.3)
\]

is the 4-nabla operator.

The mean retarded 4-self-potential follows analogously to \((5.5a)\), substituting there \(\overline{J}_\sigma\) by \(\overline{A}_\sigma = -\overline{J}_\sigma / \overline{k}^2\):

\[
< \overline{A}_\sigma > (\overline{z}) = C^4 \int d^4\overline{k} \exp(-i\overline{k} \cdot \overline{z}) \frac{\overline{J}_\sigma}{-\overline{k}^2} \rho(\kappa) = \partial_\sigma < \overline{J} > (\overline{z}) . \quad (6.4)
\]

Inserting \((5.3)\) we get

\[
< \overline{A}_\sigma > (\overline{z}) = -q C^4 \int d\tau’ \overline{z}’ \int d^4\overline{k} \exp[i\overline{k} \cdot (\overline{z}’ - \overline{z})] \frac{\hat{\rho} \hat{\rho}’}{\overline{k}^2} , \quad (6.5)
\]

with

\[
\overline{z} := \overline{z}(\tau'), \quad \hat{\overline{z}} := d\overline{z} / d\tau', \\
\hat{\rho} := \hat{\rho}(\kappa) := \hat{\rho}\left(\sqrt{(\overline{k} \cdot \overline{z})^2 - \overline{k}^2}\right), \quad \hat{\rho}’ := \hat{\rho}(\kappa’) := \hat{\rho}\left(\sqrt{(\overline{k} \cdot \overline{z}’)^2 - \overline{k}^2}\right) . \quad (6.6)
\]

We now decompose the 4-wave vector \(\overline{k}\), the particle’s 4-position \(\overline{z}(\tau)\) and \(\overline{z}':=\overline{z}(\tau')\) into (base-dependent) time- and space-components with respect to the lab base \(\{\overline{\sigma}\}\):

\[
\overline{k} = k^0 \overline{\sigma}_0 + \overline{k}, \quad \overline{z} = z^0 \overline{\sigma}_0 + \overline{z}, \quad \overline{z}' = z'^0 \overline{\sigma}_0 + \overline{z}' . \quad (6.7)
\]
We distinguish the *dot product* between 4-vectors of Minkowski space from that between 3-vectors of the spatial Euklidean subspace by \( \cdot \), respectively:

\[
\vec{k} \cdot \vec{z} := k^0 z^0 - \vec{k} \cdot \vec{z} = k^0 z^0 - k^\alpha z^\alpha.
\]

(6.8)

In the manner of speaking we distinguish between the (scalar) time coordinate \( k^0 = \omega \) of \( \vec{k} \) and the (vectorial) time component \( k^0 \alpha \). The *space component* \( \vec{k} \) of \( \vec{k} \) has the *space coordinates* \( k^\alpha \). If \( k^0 = 0 \), then \( \vec{k} = \vec{\omega} \); cf. (10.1). With that, (6.5) reads

\[
\begin{align*}
\langle \vec{A}_i \rangle > & \quad \approx -qC^4 \int \frac{d\tau'}{\omega} \int d\omega d3 \vec{k} \exp \{i \omega (z^0 - z'^0) - \vec{k} \cdot (\vec{z}' - \vec{z})\} \frac{\vec{\rho} \cdot \vec{\rho}'}{\omega^3 - k^2}. \\
\end{align*}
\]

(6.9)

*Fig. 3* The contour \( C_i \) of integration in the complex \( \omega \)-plane is to be chosen like visualized in Fig. 3 so that \( \langle \vec{A}_i \rangle > \) becomes the *retarded 4-potential* belonging to \( \langle \vec{J} \rangle \) \([22,38,54]\). The poles of the integrand at

\[
\omega = \pm |\vec{k}|
\]

and eventual singularities of \( \vec{\rho}' \) (cf. (9.4a,b)) are by-passed above. Consequently, the \( \omega \)-integral vanishes for times \( z^0 > z'^0 \). That is, \( \langle \vec{A}_i \rangle > \) is causal, and the upper border of the proper time integral may be replaced by \( \tau \):

\[
\int_{-\infty}^{\infty} d\tau' \vec{z}' \ldots = \int_{-\infty}^{\infty} dz^0 \frac{dz'}{dz^0} \frac{dz'}{dz^0} \ldots = \int_{\tau(-\infty)}^{\tau(+\infty)} dz' \ldots .
\]

(6.11)

### 7. Mean electromagnetic self-field strength

From the 4-self-potential (6.2), the *electromagnetic 4-self-field strength* follows as the antisymmetric tensor (dyadic) \([38]\)

\[
\begin{align*}
\vec{B}_i := -\vec{\partial}_x \vec{A}_i + \vec{A}_i \vec{\partial}_x = -iC^4 \int d4 \vec{k} \exp(-i\vec{k} \cdot \vec{x}) \frac{\vec{k} \vec{J}_\tau - \vec{J}_\tau \vec{k}}{k^2}. \\
\end{align*}
\]

(7.1a)

Here \( \vec{\partial}_x \) means the 4-nabla differential operator acting upon the left:

\[
\vec{\partial}_x := \frac{\partial}{\partial x_i} \vec{e}_i =: \vec{\partial} \cdot \vec{e}_i .
\]

(7.1b)

The electromagnetic 4-self-field strength \( \vec{B} \) linearly depends on the electric and magnetic self-field strengths \( \vec{E} \) and \( \vec{B} \) \([22]\):

\[
\begin{align*}
\vec{B}_i = \vec{e}_0 \vec{E}_i - \vec{E}_i \vec{e}_0 + \vec{B}_i, & \quad \vec{B}_i := -\vec{\partial}_x \vec{A}_i = -\vec{\partial}_x \vec{A}_i, \\
\vec{B}_i = \vec{e}_0 \vec{E}_i & \quad \vec{E}_i := \vec{e}_0 \vec{B}_i .
\end{align*}
\]

(7.1c)

The contravariant coordinates \( B^{i/j}_s \) of the 4-dyadic \( \vec{B}_s \) with respect to the covariant base vectors \( \vec{e}_i \) are defined by the expansion

\[
\vec{B}_s = B^{i/j}_s \vec{e}_i \vec{e}_j = (-\partial^i A^j + A^j \partial^i) \vec{e}_i \vec{e}_j .
\]

(7.2)

Left-multiplying by \( \vec{e}_0 \) yields the *electric self-field strength*:

\[
\vec{e}_0 \cdot \vec{B}_s = \left(-\frac{\partial}{\partial x_0} A^0 + \frac{\partial}{\partial x_i} A^i\right) \vec{e}_0 =: -\frac{\partial}{\partial t} \vec{A}_s - \vec{e}_0 \frac{\partial}{\partial x_i} A^i =: \vec{E}_s .
\]

(7.3)

The space coordinates \( B_s^\alpha \) belong to the *magnetic self-field strength*. For example is
\[ B_s^{12} = -\frac{\partial}{\partial x_1} A_s^2 + \frac{\partial}{\partial x_2} A_s^1 = \frac{\partial}{\partial x_1} A_s^2 - \frac{\partial}{\partial x_2} A_s^1 = (\nabla \times \vec{A}_s) \cdot \vec{e}_3 = B_s^3. \]  

(7.4)

With that, the matrix representation of the \( B_s^{ij} \) reads

\[
\begin{pmatrix}
0 & E_s^1 & E_s^2 & E_s^3 \\
-E_s^1 & 0 & B_s^3 & -B_s^2 \\
-E_s^2 & -B_s^3 & 0 & B_s^1 \\
-E_s^3 & B_s^2 & B_s^1 & 0
\end{pmatrix}.
\]

(7.5)

The mean electromagnetic self-field strength \( \langle \vec{B}_s \rangle \) follows from \( \vec{B}_s \) (7.1a) just as \( \langle \vec{A}_s \rangle \) (6.9) follows from \( \vec{A}_s \) (6.2):

\[
\langle \vec{B}_s \rangle (\zeta) = i q C^4 \int_{-\infty}^{t} d\tau' \left[ d^4k \exp[i \vec{k} \cdot (\zeta' - \zeta)] \right] \hat{\rho} \hat{\rho'} \frac{\hat{\vec{k}} - \hat{\vec{k}'} \hat{\vec{\kappa}}}{k^2}
\]

\[ = -\partial_\zeta \langle \vec{A}_s \rangle + \langle \vec{A}_s \rangle \partial_\zeta. \]

(7.6)

The total mean electromagnetic field strength \( \langle \vec{B} \rangle \), acting onto the particle, is the sum of the (mean) external electromagnetic field strength \( \langle \vec{B}_e \rangle \approx \vec{B}_e \) and \( \langle \vec{B}_s \rangle \):

\[
\langle \vec{B} \rangle (\zeta) = \langle \vec{B}_e \rangle (\zeta) + \langle \vec{B}_s \rangle (\zeta).
\]

(7.7)

The approximation \( \langle \vec{B}_s \rangle \approx \vec{B}_s \) holds on condition that \( \vec{B}_s(\zeta + \vec{r}) \) changes only very slowly with \( \vec{r} \) for \( |\vec{r}| \leq < r > \), where \( < r > \) (4.9) is the particle radius. The total electromagnetic field strength \( \vec{B} = \vec{B}_e + \vec{B}_s \) causes the Lorentz 4-force (cf. [22,34])

\[
\vec{F} = q < \hat{x} \cdot \vec{B} > \approx q \hat{z} \cdot < \vec{B} > = \langle q(u^0 \vec{e}_0 + \vec{u}) \cdot < \vec{e}_0 \vec{E} - \vec{E} \vec{e}_0 > - \vec{E} \times \vec{B} \rangle
\]

\[ = \langle q(\vec{u} \cdot < \vec{E} > \vec{e}_0 + u^0 < \vec{E} > + \vec{u} \times < \vec{B} > \rangle \rangle = F^0 \vec{e}_0 + \vec{F}. \]

(7.8)

Analogously to (6.7), we have split up the 4-speed \( \vec{u} = \vec{u} \) into the time component \( u^0 \vec{e}_0 \) and the space component \( \vec{u} \). The same holds for \( \vec{F} \). In (7.8), \( < \vec{E} > \) and \( < \vec{B} > \) are the mean electric and magnetic field strengths, respectively.

### 8. Equation of motion

The 4-dimensional generalization of Newton’s second law reads

\[
m_\infty \ddot{\vec{u}} = \vec{F} = \vec{F}_e + \vec{F}_s = q \vec{u} \cdot < \vec{B} > = q \vec{u} \cdot (\vec{B}_e + < \vec{B}_s >). \]

(8.1)

\( m_\infty \) is the mechanical (bare) inertial mass. Inserting (7.6) we get

\[
m_\infty \ddot{\vec{u}} = q \vec{u} \cdot \vec{B}_e + i q^2 C^4 \vec{u} \cdot \int_{-\infty}^{t} d\tau' \left[ d^4k \exp[i \vec{k} \cdot (\zeta' - \zeta)] \right] \hat{\rho} \hat{\rho'} \frac{\vec{u} \vec{k} - \vec{k} \vec{u}'}{k^2}. \]

(8.2)

At this mean

\[
C := \frac{1}{2\pi}, \quad \vec{z} = \vec{z}(\tau) = 4\text{-position of the particle centre at proper time } \tau, \quad \vec{z'} := \vec{z}(\tau'), \\
\vec{u} := \vec{u}(\tau) = 4\text{-speed}, \quad \vec{u'} := \vec{u}(\tau'), \quad \vec{k} = 4\text{-wave vector},
\]
22 1/2() : [() ] ,

\[ \rho = \rho(\kappa), \quad \kappa := [(k \cdot u)^2 - k^2]^{1/2}, \quad \rho' = \rho(\kappa'), \quad \kappa' := [(k' \cdot u')^2 - k'^2]^{1/2}, \]

\[ d^4\kappa = d\omega d^3\kappa = \text{volume element in } 4-k \text{-space}, \quad C_r = \text{contour in complex } \omega \text{-plane} \]

[see Fig. 3], \[ \vec{B} = \text{external electromagnetic 4-field strength}, \quad q = \text{charge}. \]

In more conventional coordinate representation, the equation of motion (8.2) reads

\[ m_u \dot{u} = q u B_\nu^\mu + i q^2 C^4 u \int d\tau' \int d\omega \int d^3\kappa \exp[i k_i (z'^i - z^i)] \rho \rho' \left( \frac{u^i k' + k'u'^i}{k''} \right). \]  

(8.3)

The 4-force balance (8.2) is obviously Lorentz-invariant, because all occurring quantities are.

This equation of motion is a differential-integral equation (integrodifferential equation) of second order in the solution \( \vec{z}(\tau) \) sought. The solution is to be determined under the initial conditions

\[ \vec{z}(\tau_0) = \vec{z}_0, \quad \dot{z}(\tau_0) = \dot{z}_0. \]  

(8.4)

Twofold proper time integration of (8.1) yields the equation of motion in the form

\[ \vec{z}(\tau) = \vec{z}_0 + (\tau - \tau_0) \dot{\vec{z}}_0 + \frac{q}{m_\infty} \int_{\tau_0}^{\tau_2} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \cdot \vec{z}(\tau_1) \cdot < \vec{B} > (\tau_1) \]

with \( < \vec{B} > \) corresponding to (7.7) and (7.6). This differential-integral equation implicates the initial conditions (8.4) and is suitable to derive the solution \( \vec{z}(\tau) \) iteratively if the external field \( \vec{B}(\vec{z}) \) is given. In Appendix D an example is calculated.

In Appendix C our results are compared with Nodvik’s [40].

9. Four self-force for very small acceleration. Zeroth approximation

We now want to calculate the 4-self-force \( \vec{F}_s := \frac{q \vec{u} \cdot < \vec{B} >}{m} \). For simplicity, as a reference frame we choose the local (instantaneous) rest frame (= inertial frame) of the particle. As a reference point we choose \( \vec{z}(0) = \vec{z}_0 := 0 \). Let the 4-speed at this 4-point be

\[ \dot{\vec{z}}(0) = \frac{\vec{u}(0)}{m} = \vec{v}_0 \quad \text{at} \quad \vec{z}(0) = 0. \]  

(9.1)

In this case the local rest frame coincides with the lab base \( \{ \vec{e}_i \} \). By means of (7.6) and (6.7), the 4-self-force gets the form [cf. (8.2)]

\[ \vec{F}_s = q \vec{v}_0 \cdot < \vec{B} > = i q^2 C^4 \int d\tau' \int d^3\kappa \int d\omega \exp[i (\omega z'^0 - \kappa \cdot z')] \frac{u^0' k' - \omega u'}{\omega^2 - \kappa^2} \rho \rho' \]  

(9.2)

with \( \kappa := | \vec{k} | \).

If the magnitude of the 4-acceleration \( \vec{a} \) is very small against the reciprocal of the particle radius \( 4.9 \), the 4-self-force \( \vec{F}_s \) is in zeroth approximation proportional to \( \vec{a} \):

\[ \vec{F}_s \approx -m_\infty \vec{a} \quad \text{for} \quad |\vec{a}| \ll < \vec{r} >^{-1}. \]  

(9.3)

The factor of proportionality \( m_\infty \) equals the electrostatic self-energy (A2) of the charged particle.

To prove this claim, we calculate \( \vec{F}_s \) in linear nonrelativistic approximation as to \( \vec{a} \). In (9.2) the following approximations are to be inserted:

\[ z'^0 = z(\tau') \approx \frac{\vec{u}}{2} \tau'^2, \quad \vec{u}' = \vec{z}' = \vec{a} \tau', \quad u'^0 = \sqrt{1 + \vec{u}'^2} \approx 1, \]

(9.4a)

\[ d\tau' = dt' / u'^0 \approx dt', \quad \exp(-i \vec{k} \cdot \vec{z}') \approx 1 - i \vec{k} \cdot \vec{z}', \quad \kappa^2 = (\vec{k} \cdot \vec{v}_0)^2 - \vec{k}^2 = \vec{k}^2, \]
\[ \kappa'^2 := (\vec{k} \cdot \vec{u})^2 - \vec{k}^2 = (\omega u^0 - \vec{k} \cdot \vec{u'})^2 - \omega^2 + \vec{k}^2 \approx -2\omega \vec{k} \cdot \vec{u'} + \vec{k}^2, \]
\[ \tilde{\rho}' := \tilde{\rho}(\kappa') \approx \tilde{\rho}(\kappa) + \left[ \frac{d \tilde{\rho}'}{d\kappa'^2} \frac{d\kappa'^2}{d\kappa'} \right]_{\kappa'=0} \cdot \vec{u}' = \tilde{\rho} - \frac{d\tilde{\rho}}{d\kappa^2} 2\omega \vec{k} \cdot \vec{u'}, \quad (9.4b) \]
\[ \tilde{\rho} \tilde{\rho}' \approx \left(1 - \omega \vec{k} \cdot \vec{u'} \frac{d}{d\kappa^2}\right) \tilde{\rho}^2. \]

Hence, (9.2) reads
\[ \bar{F}_s \approx i q^2 C^4 \int_{-\infty}^{\infty} dt' \int d^3 \vec{k} \int d\omega \, e^{i\omega t'} \left(1 - i\vec{k} \cdot \vec{z'}\right) \tilde{k} - \frac{\omega \vec{u'}}{\omega^2 - \kappa^2} \left(1 - \omega \vec{k} \cdot \vec{u'} \frac{d}{d\kappa^2}\right) \tilde{\rho}^2 \]
\[ \approx -i q^2 C^4 \int_{-\infty}^{\infty} dt' \int d^3 \vec{k} \int d\omega \, e^{i\omega t'} \frac{1}{\omega^2 - \kappa^2} \left\{ \omega \left[ \vec{u'} + \vec{k} \vec{k} \cdot \vec{u'} \frac{d}{d\kappa^2}\right] + i \vec{k} \vec{z'} \right\} \tilde{\rho}^2. \quad (9.4c) \]

In the integrand we have omitted the \( \vec{k} \)-odd term, since for symmetry reasons
\[ \int d^3 \vec{k} \tilde{k} \frac{\tilde{\rho}^2}{\omega^2 - \kappa^2} = 0. \]

The volume element in \( \vec{k} \)-space may be factorized:
\[ d^3 \vec{k} = d\kappa \kappa^2 d\Omega, \]
where \( d\Omega \) is the solid angle element. The directional mean of the dyadic product \( \vec{k} \vec{k} \) is
\[ \frac{1}{4\pi} \int d\Omega \vec{k} \vec{k} = \kappa^2 = N \kappa^2 3 = N \kappa^2 \sigma_\alpha \sigma_\alpha, \]
where \( \bar{1} \) means the 3-unit dyadic. The last but one of these equalities holds for symmetry reasons. The number \( N \) can easily be determined by contraction:
\[ \kappa \vec{k} \vec{k} > \bar{1} = \kappa^2 = N \kappa^2 3 \quad \Rightarrow \quad \kappa \vec{k} \vec{k} > \bar{1} = \frac{\kappa^2}{3} 1. \quad (9.5) \]

With that, (9.4c) goes over into
\[ \bar{F}_s \approx -i q^2 C^4 \int_{-\infty}^{\infty} dt' \int d\kappa \kappa^2 \int d\omega \, e^{i\omega t'} \left\{ \omega \left[ \vec{u'} + \frac{\kappa^2}{3} \vec{u'} \frac{d}{d\kappa^2}\right] + \frac{i}{3} \kappa^2 \vec{z'} \right\} \tilde{\rho}^2. \quad (9.6) \]

Performing the \( \omega \)-integration with the calculus of residues [33] yields
\[ \bar{F}_s \approx -q^2 C^4 \int_{-\infty}^{\infty} dt' \int d\kappa \kappa^2 \left\{ \cos \kappa t' \left[ \vec{u'} + \frac{\kappa}{3} \vec{u'} \frac{d}{d\kappa} \right] - \frac{\kappa}{3} \vec{z'} \sin \kappa t' \right\} \tilde{\rho}^2. \quad (9.7) \]

The time integral with \( \vec{z'} \) can be transformed by partial integration:
\[ \int_{-\infty}^{\infty} dt' \vec{z'} \sin \kappa t' = - \left[ \vec{z}(t') \frac{\cos \kappa t'}{\kappa} \right]_{-\infty}^{0} + \int_{-\infty}^{0} dt' \vec{u'} \frac{\cos \kappa t'}{\kappa}. \]

The term outside the integral vanishes. Consequently, (9.7) reads with \( d/d\kappa = -\vec{d}/d\kappa \) (partial integration)
\[ \bar{F}_s = -q^2 C^4 \int_{-\infty}^{\infty} dt' \vec{u'} \int d\kappa \kappa^2 e^{i\kappa t'} \left\{ \frac{2}{3} - \frac{\kappa}{6} \frac{\vec{d}}{d\kappa} \right\} \tilde{\rho}^2 \left[ 1 + O(\vec{u'^2}) \right] \]
\[ \approx -q^2 \frac{C^4}{24\pi^2} \int_{-\infty}^{\infty} dt' \vec{u'} \int d\kappa (\kappa^2 - i t' \kappa^3) e^{i\kappa t'} \tilde{\rho}^2. \quad (9.8) \]

Inserting
\[ \vec{u'} \approx \vec{a} t' \quad (9.9) \]
and integrating over $t'$, we get

$$\vec{F}_s \approx -\vec{a} \frac{q^2}{8\pi^2} \int d\kappa \, \vec{\rho}^2 \quad \text{(42)}$$

$$= -\vec{a} m_i, \quad \text{q.e.d.}$$

(9.10)

If we had approximated $\vec{\rho}'$ not by (9.4b), but by $\vec{\rho}$, the result would have been

$$\vec{F}_s \approx -(4/3)m_i \vec{a}.$$

The correction term $\vec{\rho}' - \vec{\rho}$ in (9.4b) just eliminates the notorious factor $4/3$, which violates the relativistic invariance [28,54,59]. The result (9.10) coincides with that of Kolbenstvedt [32], who derived it (without Fourier transformation) for a constant linear acceleration.

By proper time integration of the 4-self-force (9.3) one gets (approximately for small acceleration) the 4-momentum of the self-field:

$$-\int d\tau \vec{F}_s = m_i \vec{u}, \quad |\vec{u}| \ll r^{-1}.$$  \hspace{1cm} (9.10a)

The last equation has already been derived by Fermi [16] and Rohrlich [53], yet only for a constant 4-velocity $\vec{u}$. They calculated $\vec{p}_s$ as a Lorentz invariant volume integral over the 4-momentum density of the electromagnetic self-field at constant proper time $\tau$.

If we replace in (9.2)

$$\vec{\rho}' \to \frac{3}{4} \vec{\rho},$$

(9.11)

we get the correct result (9.10).

With (9.10), the equation of motion (8.2) can in zeroth approximation be written

$$m_0 \vec{u} \approx \vec{F}_e, \quad \text{where} \quad m_0 = m_\infty + m_i$$

(9.12)

means the (observable) inertial mass for small acceleration of the particle.

For the structure function $\tilde{\rho}_1(\kappa) = (1 + r_i^2 \kappa^2)^{-1/2}$ (4.6), the electrostatic mass becomes

$$m_i : = \frac{\mu_0 q^2}{8\pi^2} \int d\kappa \, \vec{\rho}^2 = \frac{\mu_0 q^2}{8\pi r_i}.$$  \hspace{1cm} (9.13)

We have completed the right side with the magnetic field constant $\mu_0 := (\varepsilon_0 c^2)^{-1}$. With (4.7) we get for an electron (or any other simply charged particle)

$$m_i = \alpha m_0, \quad \text{where} \quad \alpha = \frac{q^2}{4\pi\varepsilon_0 c h} \approx \frac{1}{137},$$

(9.14)

is the fine structure constant.

10. Four-self-force in second approximation

If we insert the replacement (9.11) in (9.2), we get a second approximation for the 4-self-force:

$$\vec{F}_s(0) \approx \vec{F}_e = \frac{3}{4} i q^2 C^4 \int_0^\infty d\tau' \int d^3 \vec{k} \, \tilde{\rho}^2(\kappa) \int d\omega \exp\left[ i \left( \omega z^0 - \vec{k} \cdot \vec{z}' \right) \right] \frac{u^0 \vec{k} - \omega \vec{u}'}{\omega^2 - \kappa^2}. \quad \text{(10.1)}$$

The $\omega$-integration yields, using the method of residues [33, 38],

$$\vec{F}_s = \frac{3}{4} q^2 C^4 \int d^3 \vec{k} \, \tilde{\rho}^2 \int_0^\infty d\tau' \exp\left( -i \vec{k} \cdot \vec{z}' \right) \left( \frac{i u^0 \vec{k}}{\kappa} \sin \kappa z^0 - \vec{u} \cos \kappa z^0 \right). \quad \text{(10.2)}$$

Integration by parts over the retarded proper time $\tau'$ delivers
\[ \vec{F}_s = \frac{3}{4} q^2 C^3 \int d^3 \vec{k} \vec{P}^2 \left\{ \frac{\vec{k}}{i \kappa^2} \left[ 1 - \exp(-i \vec{k} \cdot \vec{z}) \cos(\kappa z^0(-\infty)) \right] \right. \\
\left. - \int_{-\infty}^{0} d\tau' \exp(-i \vec{k} \cdot \vec{z}') \vec{P}_{\perp k} \cdot \vec{u}' \cos \kappa z'^0 \right\}, \]  
(10.3)

where the projection dyadic
\[ \vec{P}_{\perp k} := \vec{1} - \frac{\vec{k}}{\kappa^2} =: \vec{1} - \vec{e}_k \vec{e}_k \]  
(10.4)

projects a space vector (here \( \vec{u}' \)) onto a plane perpendicular to the wave vector \( \vec{k} \). The term proportional to \( \vec{k} \) can be omitted, because both \( \vec{k} \)-space integrals (with \( \tau' = 0 \) and \( \tau' = -\infty \)) vanish. (The former integral vanishes, because the integrand is odd in \( \vec{k} \); the latter, because the integrand oscillates very rapidly for \( \tau' \rightarrow -\infty \).) With \( d\tau' u' = d\vec{z}' \) the self-force simplifies to

\[ \vec{F}_s = -\frac{3}{4} q^2 C^3 \int d^3 \vec{k} \vec{P}^2 \int_{\vec{z}'(-\infty)}^{0} d\vec{z}' \cdot \vec{P}_{\perp k} \cos \kappa z'^0 \cos \vec{k} \cdot \vec{z}' . \]  
(10.5)

This formula differs from nonrelativistic derivations [8,21,47] by the factor 3/4.

The \( \vec{k} \)-space integral can be calculated by introduction of spherical coordinates:

\[ d^3 \vec{k} = \kappa^2 \sin \theta d\theta d\varphi, \quad \vartheta = \angle(\vec{k}, \vec{z}'), \]

\[ \vec{F}_s = -\frac{3}{4} q^2 C^3 \int d^3 \vec{k} \vec{P}^2 \int_{\vec{z}'(-\infty)}^{0} d\vec{z}' \cdot \vec{P}_{\perp k} \cos \kappa z'^0 \frac{1}{4\pi} \int d\Omega \left( \vec{1} - \vec{e}_k \vec{e}_k \right) \cos \vec{k} \cdot \vec{z}' . \]  
(10.6)

With
\[ \vec{e}_k := \frac{\vec{k}}{\kappa} = \sin \vartheta \cos \varphi \vec{e}_1 + \sin \vartheta \sin \varphi \vec{e}_2 + \cos \varphi \vec{e}_3, \quad \vec{e}_3' := \frac{\vec{z}'}{|\vec{z}'|}, \]  
(10.7)

the last solid angle integral can easily be evaluated:

\[ M_1 := \frac{1}{4\pi} \int d\Omega \cos \vec{k} \cdot \vec{z}' =: < \cos \vec{k} \cdot \vec{z}' > \]
\[ = \frac{1}{2} \int_{0}^{\pi} d\vartheta \sin \vartheta \cos(\kappa |\vec{z}'| \cos \vartheta) = \frac{\sin \lambda}{\lambda}, \quad \lambda := \kappa |\vec{z}'| ; \]  
(10.8)

\[ \tilde{M}_2 := < \vec{e}_k \vec{e}_k \cos \vec{k} \cdot \vec{z}' > = \frac{1}{2} (1 + d_{\lambda}^2) M_1 \vec{P}_{\perp x} - d_{\lambda}^2 M_1 \vec{P}_x, \quad d_{\lambda} := \frac{d}{d\lambda} ; \]  
(10.9)

\[ \vec{P}_z := \vec{e}_z \vec{e}_z' = \vec{1} - \vec{e}_z' \vec{e}_z', \quad \vec{e}_z' = \vec{1} \perp \vec{z}' \]

\[ \vec{M}(\lambda) = \vec{M}(-\lambda) = M_1 \vec{1} - \tilde{M}_2 ; \]  
(10.10)

\[ = \frac{2}{\lambda^2} (\sin \lambda - \cos \lambda) \vec{P}_z + \frac{1}{\lambda^2} \left[ (\lambda^2 - 1) \sin \lambda + \lambda \cos \lambda \right] \vec{P}_{\perp x} \]
\[ =: M_1 \vec{P}_z + M_{\perp} \vec{P}_{\perp x} \]
\[ = 4 \left( \frac{1}{3!} - \frac{2}{5!} \lambda^3 + \frac{3}{7!} \lambda^5 - \cdots \right) \vec{P}_z + \frac{1}{\lambda^2} \left( (\lambda^2 - 1) \sin \lambda + \lambda \cos \lambda \right) \vec{P}_{\perp x} \]
\[ = 2 \left( 1 \vec{1} - \frac{1}{10} \lambda^2 \vec{P}_z - \frac{1}{5} \lambda^2 \vec{P}_{\perp x} \right) + O(\lambda^4) . \]  
(10.11)
With that, the self-force at $\tilde{z} = \tilde{0}$ reads

$$\tilde{F}_s = -\frac{3q^2}{16\pi} \int_{-\infty}^{\infty} d\kappa \kappa^2 \tilde{\rho}^2 \tilde{z} \int_{z'(-\infty)}^{0} d\tilde{z}' M(\kappa | \tilde{z}') \cos \kappa \tilde{z}'.$$  \hfill (10.12)

Inserting the structure function $\tilde{\rho}(\kappa)$ (4.6) and calculating the $\kappa$-integral with the residues theorem one gets

$$\tilde{F}_s = \frac{m_i}{r_i} \int_{z'(-\infty)}^{0} d\tilde{z}' A\left(\frac{|\tilde{z}'|}{r_i}\right) A\left(\frac{t'}{r_i}\right),$$  \hfill (10.13)

with the electrostatic mass $m_i$ (9.14) and the dyadic

$$\tilde{A}(\alpha) := \frac{3}{2} \tilde{M}(i\alpha) = \frac{z'z'}{|z'|^2} A_\parallel(\alpha) + \left(1 - \frac{z'z'}{|z'|^2}\right) A_\perp(\alpha),$$  \hfill (10.14)

$$A_\parallel(\alpha) := \frac{3}{2} M_\parallel(i\alpha) = \frac{3}{\alpha^2} (\alpha \cosh\alpha - \sinh\alpha)$$

$$= 1 + \frac{\alpha^2}{10} + \frac{\alpha^4}{280} + O(\alpha^6)$$

$$= 6 \sum_{n=0}^{\infty} \frac{1+n}{(3+2n)!} \alpha^{2n},$$  \hfill (10.15)

$$A_\perp(\alpha) := \frac{3}{2} M_\perp(i\alpha) = \frac{3}{2\alpha^2} \left[(1+\alpha^2) \sinh\alpha - \cosh\alpha\right]$$

$$= 1 + \frac{\alpha^2}{5} + \frac{3\alpha^4}{280} + O(\alpha^6)$$

$$= 6 \sum_{n=0}^{\infty} \frac{(1+n)^2}{(3+2n)!} \alpha^{2n}. \hfill (10.16)$$

With $d\tilde{z}' = \tilde{z} d\tau'$, the self-force at $\tilde{z} = \tilde{0}$ can also be written in the form

$$\tilde{F}_s = \frac{m_i}{r_i} \int_{-\infty}^{0} d\tau' \left\{ \tilde{z}' A_\parallel\left(\frac{|\tilde{z}'|}{|\tilde{z}'|^2}\right) - A_\parallel\left(\tilde{z}'\right) + \tilde{z}' A_\perp\left(\frac{|\tilde{z}'|}{|\tilde{z}'|^2}\right) \right\} \exp \left(\frac{t'(\tau')}{r_i}\right).$$  \hfill (10.17)

The choice of the reference point $\tilde{z}(0) = \tilde{0}$ does not mean a restriction of generality, as we can arbitrarily choose the zero of the proper time scale. In each case, $\tilde{z}' := \tilde{z}(\tau')$ means the spatial (vectorial) distance of $\tilde{z}(\tau')$ [as to the local rest frame $\{\tilde{e}_i\}$ (2.4); cf. (6.7)] from the reference point on the worldline. The vector $\tilde{z}'$ lies in the 3-plane $\sigma$ [see (2.9) and Fig. 1] through the reference point.
11. Rectilinear motion

For a rectilinear motion in \( x \)-direction, the 4-position vector of the particle’s centre specializes to the 2-position vector
\[
\vec{z} = t \vec{e}_0 + \vec{z} = t \vec{e}_0 + x \vec{e}_1, \quad x_2 = x_3 = 0. \tag{11.1}
\]
Analogously holds for the external 2-force
\[
\vec{F}_e = F_e \vec{w} \quad \text{with} \quad \vec{w} := u^0 \vec{e}_0 + u^0 \vec{e}_1 =: \frac{\vec{a}}{a}, \quad \vec{w} \cdot \vec{a} = 0. \tag{11.2}
\]
Here \( \vec{w} \) means the spacelike unit vector parallel to the 2-acceleration \( \vec{a} = a \vec{w} \), which is orthogonal to the 2-speed
\[
\vec{u} := \vec{z} = u^0 \vec{e}_0 + u^1 \vec{e}_1 .
\]

We now calculate the self-force (10.13) at \( \vec{z} = x \vec{e}_1 = 0 \). With
\[
\vec{w} = \vec{e}_1, \quad \vec{F}_e = F_e \vec{e}_1, \quad \vec{z}' = x \vec{e}_1, \quad d\vec{z}' \vec{A} \left( \frac{\vec{z}'}{r_1} \right) = dx' A_1 \left( \frac{x'}{r_1} \right) \vec{e}_1, \tag{11.3}
\]
eq. (10.13) specializes for a rectilinear motion to
\[
F_s = \frac{m_s}{r_1^2} \int_x^{t(-\infty)} dx' \exp \frac{t'(x')}{r_1} A_1 \left( \frac{x'}{r_1} \right). \tag{11.4}
\]
The integration runs over the past \( t' = -\infty \ldots 0 \). At time \( t = 0 \), the initial conditions (9.1) shall hold:
\[
x(0) = \frac{dx}{dt}(0) = 0. \tag{11.5}
\]
Then, for
\[
F_e > 0, \quad t' < 0 \quad \text{we have} \quad x'(\infty) = \infty, \quad dx' < 0. \tag{11.6}
\]
That is, as the integrand in (11.4) is positive,
\[
F_e F_s < 0: \tag{11.7}
\]
The self-force is oppositely directed to the external force [cf. Fig. 5 and (13.2)].

From (11.2) and (8.1) follows that \( \vec{F}_e, \vec{F}_s \) and \( \vec{a} \) are parallel 2-vectors. This can also be expressed by the equations
\[
\vec{F}_e = m \vec{a}, \quad \vec{F}_s = -m_s \vec{a}. \tag{11.8a,b}
\]
We call \( m \) effective inertial mass and \( m_s \) (effective) electrodynamic mass. They are connected via \( m_s \vec{a} = \vec{F}_e + \vec{F}_s \) (8.1):
\[
m = m_e + m_s . \tag{11.9}
\]
Approximately holds
\[
m \approx m_0 = m_e + m_i \quad \text{for} \quad |\vec{a}| \ll < r >^{-1}. \tag{11.10}
\]

12. First example: Rectilinear motion in a homogeneous electric field

For a constant external electric field \( E_e = F_e / q \) the position-time law
\[
x = \frac{1}{f} \left( \sqrt{1 + f^2 t^2} - 1 \right) \quad \text{with} \quad f = \frac{F_e}{m} \tag{12.1}
\]
Fig. 4 satisfies, as you know, the equation of motion (11.8a) (with time-independent $m$ ) and the initial conditions (11.5); see Fig. 4 and [5,22,25,54]. The hyperbolic worldline (12.1) may also be derived by iteration of (8.5); see App. D.

The (effective) field strength $f$ depends on the asymptotic field strength
\[ f_\infty := \frac{F_r}{m_\infty}. \]  
(12.2)
To determine the dependence of $f$ on $f_\infty$, we insert the inverse function of (12.1),
\[ t = \text{sign} t \sqrt{x(x+2/f)}, \]  
(12.3)
into (11.4):
\[ F_s = \frac{m_\infty}{r_i} \int_0^x dx' \exp \left[ -\frac{1}{r_i} \sqrt{x'(x'+2f^{-1})} \right] A_\parallel \left( \frac{x'}{r_i} \right). \]  
(12.4)
Introducing this integral representation of the self-force into (8.1) yields the connection between the field strengths $f$ and $f_\infty$ at $\tau = 0$:
\[ f = f_\infty - \frac{r_i}{r_i} \int_0^x dx' \exp \left[ -\frac{1}{r_i} \sqrt{x'(x'+2f^{-1})} \right] A_\parallel \left( \frac{x'}{r_i} \right) =: f_\infty + f_s, \]  
(12.5)
\[ r_\infty := \frac{q^2}{8\pi m_\infty}, \quad f := \frac{qE_e}{m}, \quad f_\infty := \frac{qE_e}{m_\infty}, \quad f_s := \frac{F_r}{m_\infty} = q < E_i >. \]  
(12.6)

Fig. 5 From it we can determine $f_\infty(f)$ or $f(f_\infty)$ (see Fig. 5 and App. B).

To simplify the notation, let us introduce dimensionless variables:
\[ \delta_\infty := r_i f_\infty = \frac{r_i q E_e}{m_\infty}, \quad \delta := r_i f = r_i \dot{x}(0) = r_i a, \quad \alpha = \frac{x'}{r_i}, \]  
(12.7)
\[ \Delta(\delta) := \int_0^1 d\alpha \exp \left[ -\sqrt{\alpha^2 + 2\alpha / \delta} \right] \frac{3}{\alpha^2} (\alpha \cosh \alpha - \sinh \alpha) = -\frac{r_i}{m_i} F_s. \]  
(12.8)

With that, (12.5) reads
\[ \delta_\infty = \delta + \frac{m_i}{m_\infty} \Delta(\delta). \]  
(12.9)

The integral (12.8) was numerically integrated (with Simpson’s rule; see Table 1 and App. E).

Asymptotic approximations are
\[ \Delta(\delta) \approx \delta \quad \text{for} \quad 0 \leq \delta \ll 1, \]  
(12.10a)
\[ = 1.5 \quad \text{for} \quad \delta \gg 1. \]  
(12.10b)
\[ \Delta(\delta) \] is an odd function:
\[ \Delta(-\delta) := -\Delta(\delta). \]  
(12.11)

This function and the relative reactive mass
\[ \mu_r(\delta) := 1 - \frac{\Delta(\delta)}{\delta} = \frac{F_r}{m_i f} =: m_r \]  
(12.12)
are not analytic at $\delta = r_i f = 0$. Therefore, the self-force $F_s[r_i]$ (12.8) and the reaction force
\[ F_r[r_i] := m_i a + F_s \]  
(12.13)
are not developable into Laurent series at $r_i = 0$. We suppose that this statement is not only valid for our special structure function (4.6).

A good approximation for $\mu_r(\delta)$ is
\[ \tilde{\mu}_r(\delta) := \exp \left( -1.5 \frac{\sqrt{\delta} \left( 0.1078 + \sqrt{\delta} \right)}{r} \right), \quad (12.14) \]

see Table 1.

Corresponding to (12.7), (12.8) and (12.12), the self-force \( F_s \) (12.4) may be written in the forms

\[ F_s = -\frac{m_i}{r_i} \Delta(r_i a) = -m_i a [1 - \mu_i (r_i a)]. \quad (12.15) \]

A formal Taylor expansion in powers of the acceleration \( a \) yields

\[ F_s(a) = -m_i a + 0 \cdot a^2 + 0 \cdot a^3 + \cdots. \quad (12.16) \]

This asymptotic expansion at \( a = 0 \), where \( F_s(a) \) is nonanalytic, also holds for the exact self-force (9.2) and for the approximated form with \( \tilde{\mu}_i (r_i a) \) (12.14) instead of \( \mu_i (r_i a) \) (12.12).

For large \( a \), the self-force \( F_s \) (12.15) remains finite:

\[ F_s = -1.5 \frac{m_i}{r_i} \quad \text{for} \quad a \gg r^{-1}, \quad (12.17) \]

as it follows from (12.8) and (12.10b). Hence, the equation of motion asymptotically reads

\[ m_s a = F_r + F_s \approx F_s \quad \text{for} \quad a \gg r^{-1}. \quad (12.18) \]

The finiteness of \( F_s \) (12.17) and the deduction (12.18) also hold for other rectilinear motions in extreme electric fields. For, in this case the particle asymptotically moves with light velocity. That is, in (11.4) we have to insert \( t' = x' \), which yields (12.17), too.

These asymptotic relations have more mathematical than physical interest, since \( a \gg r^{-1} \) passes over the restriction \( a \ll r^{-1} \) (3.6). In this case of extreme acceleration we would have to take into account the nonapproximated functional determinant \( D = 1 - \vec{r} \cdot \vec{a} \) (3.3).

13. Connections between forces, masses and field strengths

From Fig. 5 is evident: The reaction force

\[ F_r := m_r a - F_s = m_r a + F_s = m_0 f \]

has the direction of the external force \( F_e := m f = m a \). The self-force \( F_s := m_s f_s = -m_s f \) has the opposite direction [see (11.7)]. These forces are connected as follows:

\[ F_e = m_e a - F_s = m_e a - F_r = ma = \frac{m}{m_s} F_s + \frac{m}{m_e} F_r. \quad (13.2) \]

Between the positive definite masses \( m_e, m_i, m_0, m_r, m_s, m_r \) hold the relations

\[ m_e + m_i = m_0 = \lim_{|a| \to 0} m(|a|) > m = m_0 - m_r = m_e + m_s > m_e = \lim_{|a| \to \infty} m(|a|). \quad (13.3) \]

With \( f_0 = F_r / m_0 \) instead of \( f_e = F_e / m_e \) and with the relative reactive mass \( \mu_r := m_r / m_i \) (12.12), equation (12.9) may also be written in the form

\[ f_0 = f \left( 1 - \mu_i \mu_r \left( \frac{f r_0}{\mu_i} \right) \right), \quad \mu_i := \frac{m_i}{m_0}, \quad r_0 := \frac{q^2}{8 \pi m_0}. \quad (13.4) \]

Fig. 6 This function \( f_0 (f) \) is visualized in Fig. 6.
As long as the relative electrodynamic mass \( \mu \) is smaller than 1, the acceleration \( a = f \) tends with the field strength \( f_0 := F_e / m_0 \) to infinity. For \( \mu = 1 \) this happens already at \( f_0 = 1.5 / r_0 \). This unphysical behaviour is only to exclude if the inequality

\[
\mu = \frac{m}{m_0} = \frac{r}{r_0} < 1
\]

holds. That is, \( r_1 \) must be larger than \( r_0 \). Hence, the hypothesis \( m_1 = m_0 \) of Abraham and Lorentz \([1,35]\) and electric point charges \((r_1 = 0 \ [12,43,54])\) are excluded.

14. Effective inertial mass

In analogy to \( m_\infty = qE_e / f_\infty = F_e / f_\infty \) and \( m_0 = F_e / f_0 \), we have already defined in (12.1) by means of the effective field strength \( f \) an effective inertial mass

\[
m := \frac{F_e}{f} = \frac{qE_e}{f}.
\]

Dividing eq. (13.4) by \( f \), we get the relative effective mass

\[
\mu := \frac{m}{m_0} = \frac{f_0}{f} = 1 - \mu_1 \mu_2 \left( \frac{r_0 f}{\mu} \right).
\]

Fig. 7

In Fig. 7 it is displayed as a function of \( \delta_0 = r_0 f = r_0 \dot{x}(0) = r_0 a \) with the relative electrodynamic mass \( \mu_1 = m_1 / m_0 \) as a parameter. This function \( \mu(\delta_0) \) decreases from 1 to \( \mu_\infty = 1 - \mu_1 \) if \( |\delta_0| \) increases from 0 to \( \infty \). Hence, the effective mass \( m = \mu m_0 \) decreases from \( m_0 \) to \( m_\infty \) with increasing magnitude of acceleration \( |a| = |\delta_0| / r_0 \):

\[
m_0 > m > m_\infty \quad \text{for} \quad 0 < |a| < \infty.
\]

With \( f = a \), eq. (14.1) may be written in the form of Newton’s second law:

\[
\frac{\mu m_0}{qE_e} a = 1.
\]

It is a matter of discretion, whether one combines \( \mu \) with \( m_0 \) to the effective inertial mass \( m := \mu m_0 \) (14.2) or with the acceleration \( a \) to the effective acceleration \( a_{eff} := \mu a \) (14.5) or with \( E_e \) to the effective external electric field strength

\[
E_{e,eff} := \mu^{-1} E_e = \frac{m_0}{m_\infty} \left( E_e + < E_s > \right)
\]

(14.6) or with the external force \( F_e = q E_e \) to the effective external force \( F_{e,eff} := \mu^{-1} F_e \) (14.7) or with the charge \( q \) to the effective charge \( q_{eff} := \mu^{-1} q \) (14.8).

A test for the quality of the classical theory exposed would be to check the equation of motion

\[
m \ddot{a} = \bar{F}_e
\]

(14.9)
experimentally. Then principally from the run of $\mu(\delta)(14.2)$, the relative electrodynamic mass $\mu_0 := m_0/m_0$ could be determined. Certainly, for extreme electric field strengths $E_e$, quantum mechanical effects superpose themselves [61].

To estimate the possibility of an experimental test of the equation of motion (14.9), we calculate the ratio $F_r/F_e = \text{reaction force/external force}$. By means of (12.12) and (14.1), we get

$$\frac{F_r}{F_e} = \frac{m_0 \mu(\delta)}{m}.$$  (14.10)

For

$$\delta := \frac{r_i F_e}{m c^2} = 0.1$$  (14.11)

the ratio (14.10) becomes $F_r/F_e \approx 10^{-5} \mu_i$. Assuming $r_i = \chi_c/2$ (4.7) and $m \approx m_0$, one gets from (14.11) the critical electric field strength (cf. [35])

$$E_{0.1} = \frac{F_e}{q} := \frac{0.2 m_0 c^2}{q \chi_c} = \frac{0.2 m_0^2 c^3}{q \hbar}.$$  (14.12)

For a positron it becomes $E_{0.1} = 2.6 \times 10^{17}$ V m$^{-1}$. The corresponding critical acceleration is

$$a_{0.1} = \frac{q E_{0.1}}{m_0} = 4.6 \times 10^{28} m s^{-2}.$$  (14.13)

Below this gigantic electric field strength and acceleration, the reaction force $F_r$ is practically always negligible against $F_e = q E_e$. Therefore, in case of a homogeneous electric field, our calculations have mainly theoretical interest in the consistency of classical electrodynamics. The equation of motion $m_0 \ddot{\alpha} \approx \bar{F}_e$ is a very good approximation at $|F_e| = \text{const.}$

A better approximation than $\bar{\alpha} = \bar{\alpha} = F_e/m \approx \bar{F}_0 := \bar{F}_e/m_0$ is [cf. (14.10) and (12.14)]

$$\bar{\alpha} = \bar{\alpha} = \bar{\alpha} + \alpha_r \approx \left[ 1 + \mu_i \exp \left( -\frac{14 c}{\sqrt{r_i} |\bar{F}_0|} \right) \right] \bar{F}_0, \quad |\bar{F}_0| \ll c^2/r_i.$$  (14.14)

This relativistically invariant equation of rectilinear motion also approximately holds for external field strengths $\bar{F}_0 := q \tilde{\alpha} \cdot \bar{B}_e(\bar{z})/m_0$, slowly depending on the particle’s 2-position $\bar{z} = t \bar{e}_0 + x \bar{e}_1$ and its proper time derivative $\dot{\bar{z}}$. At given $\bar{F}_0(\bar{z}, \dot{\bar{z}})$, eq. (14.14) is a differential equation of second order for $\bar{z}(\tau)$. Its solution will in general more easily to be found than that of the corresponding exact differential-integral equation (8.2).

15. Four-self-force in first approximation

Corresponding to (9.4a), (10.8) and (10.11) holds for $|\bar{\alpha}| \ll |r|^{-1}$:

$$\ddot{\bar{z}} \approx \frac{\bar{\alpha}}{2} t^2, \quad \cos \tilde{\kappa} \cdot \ddot{z} \approx 1, \quad \tilde{M}(\tilde{\lambda}) \approx \tilde{M} \left( \kappa \frac{|\bar{\alpha}|}{2} r^2 \right) \approx \frac{2}{3} \tilde{\lambda}.$$  (15.1)

With that, the second approximation (10.12) goes over into the less accurate first approximation for the self-force at $\bar{z} = t \bar{e}_0 + \tilde{\bar{z}} = 0$:

$$\bar{F}_s = \tilde{\bar{F}}_s = - \frac{q^2}{8 \pi^2} \int_0^\infty dt' \frac{d\tilde{\bar{z}}^*}{dt'} \int_0^\infty d\kappa \kappa^2 \cos \kappa t' \tilde{\rho}^2.$$  (15.2)
Twofold partial integration of the time integral gives

\[
\mathcal{F}_s \approx -\frac{q^2}{8\pi^2} \int_{-\infty}^{0} d\kappa \hat{\rho}^2 \left( \ddot{z}(0) - \int_{-\infty}^{0} dt' \frac{d^2z'}{dt'^3} \cos \kappa t' \right)
\]

\[
= -m_1\dddot{z}(0) + \int_{-\infty}^{0} dt' \frac{d^3z'}{dt'^3} Q(t'), \quad \dddot{z}(0) := \frac{d^2\ddot{z}}{dt^2}(0) = \frac{d^2\ddot{z}}{dt^2}(0).
\]  

(15.3)

The function

\[
Q(t) := \frac{q^2}{8\pi^2} \int_{-\infty}^{0} d\kappa \hat{\rho}^2 \cos \kappa t \quad \text{with} \quad Q(0) = m_1
\]

(15.4)
is called memory function \[20\]. For our special structure function \( \tilde{\rho}_1 = (1 + r_1^2 \kappa^2)^{-1/2} \) (4.6), we get

\[
Q(t) = m_1 \exp \left( -\frac{|t|}{r_1} \right).
\]

With that, the first approximation (15.2) of the 4-self-force at \( \tau = 0 \) reads

\[
\mathcal{F}_s \approx -m_1\dddot{z}(0) + m_1 \int_{-\infty}^{0} dt' \frac{d^3z(t')}{dt'^3} \exp \left( -\frac{|t'|}{r_1} \right).
\]

(15.6)

16. Second example: Spiraling of a charged particle in a homogeneous magnetic field

As another example for the (approximate) solution of the equation of motion (8.2), we consider the plane motion of a charged particle in a homogeneous magnetic field. The constant external magnetic field strength (induction) may have the negative 3-direction:

\[
\vec{B}_e = -B_e \hat{e}_3.
\]

(16.1)

To simplify the calculation, we first decompose [analogously to (6.7)] the vectors \( \vec{z}, \vec{u}, \vec{u}, \vec{F}_e, \) and \( \vec{F}_s, \) occurring in the equation of motion (8.2), into time and space components:

\[
\vec{z} = z^0 \vec{e}_0 + \vec{z}, \quad \vec{z}(\tau) = \vec{u} = u^0 \vec{e}_0 + \vec{u}, \quad \vec{u} = u^0 \vec{e}_0 + \vec{u},
\]

(16.2a)

\[
\vec{F}_e = q \vec{u} \times \vec{B}_e = \vec{F}_e, \quad \vec{F}_s = F_s^0 \vec{e}_0 + \vec{F}_s.
\]

(16.2b)

With that, the equation of motion (8.2) splits up into a time and a space part:

\[
m_s \vec{u} = \vec{F}_s^0,
\]

(16.3a)

\[
m_s \ddot{u} = q \vec{u} \times \vec{B}_e + \vec{F}_s.
\]

(16.3b)

The first equation expresses the power balance (see e.g. [22]). It follows from the second by scalar multiplication of the latter with \( \vec{u} / u^0 = d\vec{z} / dt \).

If we choose the initial conditions

\[
\vec{z}(0) = 0, \quad \dot{\vec{z}}(0) = \vec{u}(0) =: \vec{u}_0 = u_0 \vec{e}_1,
\]

(16.4)

the sought solution of (16.3b),

\[
\vec{z}(\tau) = x(\tau) \vec{e}_1 + y(\tau) \vec{e}_2,
\]

(16.5)

has no 3-component.
For simplicity, we restrict the following calculation onto the nonrelativistic case \( u_0 \ll 1 \). Then we need not distinguish between time \( t \) and proper time \( \tau \). With (15.6), the equation of motion (16.3b) approximately reads at \( \tau = 0 \)

\[
m_0 \ddot{u} = q \dot{u} \times \dot{B}_e + m_0 \int_\infty^0 d\tau' \dddot{u}(\tau') \exp \left( -\frac{r'}{r_1} \right), \quad \dot{\dddot{u}}_0 := \dddot{u}(0), \quad m_0 = m_e + m_1. \tag{16.6}
\]

The calculation takes the simplest form in complex notation:

\[
z(\tau) = x(\tau) + iy(\tau), \quad u(\tau) := \dot{z}(\tau) = \dot{x} + i \dot{y}. \tag{16.7}
\]

Then, the initial conditions (16.4) read

\[
z(0) = 0, \quad \dot{z}(0) = u_0. \tag{16.8}
\]

**Fig. 8**

We’ll show that a positively charged particle spirals inward around the asymptotic position \( z_\infty := z(\infty) \); see Fig. 8.

The pointers of the complex Lorentz force \( F_e = i q u B_e \) and of the proper acceleration \( \dddot{u} \) lie in the \( x, iy \)-plane. Therefore, \( F_e \) is a complex multiple \( m \) of \( \dddot{u} \):

\[
m \dddot{u} = F_e = i q u B_e. \tag{16.9}
\]

We call \( m \) the complex (effective) mass of the particle. The solution of (16.9) is (for \( \tau \)-independent \( m \))

\[
u = u_0 e^{i \omega \tau} \quad \text{with} \quad \omega := \frac{q B_e}{m_0} = \omega_r + i \omega_i = \frac{q B_e}{m + i m_i}. \tag{16.10}
\]

For an inward spiraling particle, the complex angular frequency \( \omega \) must have a positive imaginary part. That is, the imaginary part of \( m \) must be negative:

\[
\omega_i > 0 \iff m_i < 0. \tag{16.11}
\]

The initial (proper) acceleration is

\[
a_0 := \dddot{u}(0) = i \omega u_0 = \frac{i q u_0 B_e}{m}. \tag{16.12}
\]

Asymptotic cases are [see (11.10a,b)]

\[
|a_0| \ll r^{-1} : \quad \omega \approx \omega_0 := \frac{q B_e}{m_0}; \tag{16.13a}
\]

\[
|a_0| \gg r^{-1} : \quad \omega \approx \omega_\infty := \frac{q B_e}{m_\infty}. \tag{16.13b}
\]

In both borderline cases the spiral degenerates to a circle with radius \( R_0 := u_0 / \omega_0 \) and \( R_\infty := u_0 / \omega_\infty \), respectively.

### 17. Complex angular frequency and mass

To calculate the complex angular frequency \( \omega \), we have to insert the solution (16.10) of the differential equation (16.9) into the complex form of the equation of motion (16.3b):

\[
m_0 \dddot{u} = i q u B_e + F_e. \tag{17.1}
\]

Then this equation has to be solved for the single unknown \( \omega \) (or \( m = q B_e / \omega \)). The advantage of this procedure is that we reduce the solution of the differential-integral equation (8.2) to the solution of the algebraic equation (17.1) for \( \omega \).

To simplify the calculation, let us restrict to small acceleration:
\[
\frac{1}{c} |\dot{u}| = \left| \frac{q u B}{mc} \right| \ll < r >^{-1}.
\]

(17.2)

With \(< r >\) corresponding to (4.11) and (4.7), one gets for electrons and a magnetic field strength \(B_e = 1 \, T = 1 \, V/s \, m^{-2}\) the restraint

\[
|u| \ll 7 \times 10^7.
\]

(17.3)

If the inequality (17.2) is fulfilled, the equation of motion (16.6) is valid at \(\tau = 0\). Its complex form reads

\[
m \ddot{u}_0 = i q u_0 B_e + m \int_0^\tau d \tau' \dddot{u}(\tau) \exp \left[ -\frac{|\tau'|}{r_i} \right], \quad \ddot{u}_0 := \dddot{u}(0).
\]

(17.4)

We now insert (16.10) in (17.4), calculate the integral and get (after some elementary transformations) the equation

\[
(1 - \mu_i) \Omega^2 - \left( 1 + \frac{i \mu_i}{b} \right) \Omega + \frac{i \mu_i}{b} = 0,
\]

with

\[
\mu_i := \frac{m_i}{m} = \frac{q^2}{8 \pi \varepsilon_0 m_0} \frac{r_0}{r_i}, \quad b := r_i \omega_0 = \frac{r_0 q B_e}{m_0}.
\]

(17.5a)

\[
\Omega := \frac{\omega}{\omega_0} = \frac{m_i}{m} = \Omega_1 + i \Omega_\omega.
\]

(17.5b)

For \(\mu_i \to 0\), the quadratic equation (17.5a) yields two limits: \(\Omega \to 0\) and \(\Omega \to 1\). Only the last is consistent with (17.5c). It gives \(m := m_i / \Omega \to m_0\). The corresponding solution of (17.5a) reads for \(\mu_i < 0.5\)

\[
\Omega(\mu_i, b) = \frac{1}{2(1 - \mu_i)} \left\{ 1 + \sqrt{\frac{w - \alpha}{2}} + i \left[ \frac{\mu_i}{b} - \text{sgn} b \sqrt{\frac{w + \alpha}{2}} \right] \right\},
\]

(16.6a)

where

\[
\alpha := \frac{\mu_i^2}{b^2} - 1 = \frac{1}{r_i^2 \omega_0^2} - 1 = \left( \frac{m_0}{r_0 q B_e} \right)^2 - 1, \quad \beta := 2 \mu_i - 1, \quad w := \sqrt{\alpha^2 + 4 \mu_i^2 \beta^2 / b^2}.
\]

(16.7)

Fig. 9

In Fig. 9 some polar diagrams of \(\Omega(\mu_i, b)\) are drawn for \(\mu_i = \text{const}\) or \(b = \text{const}\).

Special cases of (16.6a) are

1. \(\omega_0 = \frac{q B_e}{m_0} = 0 \Rightarrow \omega = \omega_0 \Omega = 0, \quad u = u_0\).

(16.10)

2. \(\mu_i = \frac{m_i}{m_0} \ll 1 \Rightarrow \Omega := \frac{\omega}{\omega_0} := \frac{m_0}{m} = 1 + \left( 1 + i \frac{\mu_i}{b} \right) \mu_i + O(\mu_i^2)\).

(16.11)

3. \(b := \frac{q^3 B_e^2}{8 \pi \varepsilon_0 m_0^2} \ll 1 \Rightarrow \Omega = 1 + i b + \frac{1 - 2 \mu_i}{\mu_i} b^2 + O(b^3)\).

(16.12)

4. \(b \to \infty \Rightarrow \Omega \to \frac{1}{1 - \mu_i} = \frac{m_0}{m_\infty} = \frac{1}{\mu_i}\).

(16.13)

The last limit is also correct if the inequality (17.2) is no more fulfilled [cf. (16.13b)]. Therefore, we suppose that (16.6a) is also a good approximation for arbitrary accelerations \(\dddot{u}\).

From the particle’s speed \(u = \dot{z}(\tau)\) (16.10), its position \(z(\tau)\) immediately follows by (proper) time integration:
\[
\begin{align*}
    z(\tau) &= \int_0^\tau d\tau' z(\tau') = \frac{u_0}{i\omega} e^{i\omega\tau} - 1, \quad \text{with} \quad \omega = \frac{qB}{m_0} \Omega. \quad (17.8)
\end{align*}
\]

The asymptotic point of this spiral is (see Fig. 8)
\[
    z_e := z(\infty) = \frac{i u_0}{\omega} = \frac{i u_0 m_0}{qB_0 \Omega(\mu_1, \mu_2)}. \quad (17.9)
\]

In the relativistic case (\(u_0\) arbitrary), the trajectory (17.8) has the same form. But the complex angular frequency \(\omega\) may depend on \(u_0^2\).

That we had chosen \(\tau = 0\) in (17.4) for the calculation of \(\Omega\) did not mean a restriction of generality. The generalization of (17.4) for \(\tau \neq 0\) is the equation of motion
\[
    m_0 \ddot{u} \approx i q u B_c + m_1 \int_{-\infty}^{\infty} d\tau' i u(\tau') \exp \left(-\frac{1}{r_1^2} \right) \exp -\left|\tau - \tau'\right| / r_1. \quad (17.10)
\]

\[
    m_0 \ddot{u} \approx i q u B_c - m_1 \int_{-\infty}^{\infty} d\tau' i u(\tau') \exp -\left|\tau - \tau'\right| / r_1. \quad (17.11)
\]

Insertion of (16.10) into (17.10) or (17.11) leads to the same equation (17.5a) for \(\Omega\).

The differential-integral equations (17.10) and (17.11) are linear homogeneous in \(u\).

Therefore, the initial speed \(u_0\) drops out, so that \(\Omega\) is independent of \(u_0\).

Applying the differential operator
\[
e^{-\tau/i} \frac{d}{d\tau} e^{\tau/i} \quad (17.12)
\]
on to the differential-integral equation (17.10) transfers it into the differential equation
\[
    m_0 \ddot{u} - i q u B_c u = -r_1 \frac{d}{d\tau} (m_0 \ddot{u} - i q u B_c u). \quad (17.13)
\]

From it we can also derive the \(\Omega\)-determining equation (17.5a) by insertion of (16.10).

18 Forces, masses and powers

Let us summarize the results of the preceding sections. We had approximately solved the complex equation of motion (17.1) under the restraint (17.2). The approximate complex equations of motion (17.10) and (17.11) are satisfied by the solution \(u = u_0 e^{i\omega\tau}\) (16.10) with the complex angular frequency \(\omega = \omega_0 \Omega\), where \(\Omega\) is given by (17.6a). The proper time derivative of the complex proper speed \(u\) gives the complex proper acceleration
\[
    a = \dot{u} = i \omega u. \quad (18.1)
\]

The equation of motion (17.1) may also be written in form of Newton’s law \(F = ma\) (16.9) with the complex mass
\[
    m = m_0 / \Omega = m_0 \omega_0 / \omega = qB_c / \omega. \quad (18.2)
\]

The complex self-force is
\[
    F_s := m_0 a F_c = m_1 \int_{-\infty}^{\infty} d\tau' \dot{u}(\tau') \exp -\left|\tau - \tau'\right| / r_1. \quad (18.3)
\]

Reiterated integration by parts gives
\[
    F_s = -m_1 \left( a - r_1 \ddot{a} + r_1^2 \dddot{a} - \cdots \right) = - \frac{m_1}{1 + r_1 d / d\tau} a = - \frac{m_1}{1 + r_1 i \omega} a =: -m_0 a \quad (18.4a)
\]

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with the complex electrodynamic mass
\[ m_s = \frac{m_i}{1 + ir_i \omega}, \quad m_i := m_0 - m_\infty. \]  

(18.4b)

From this the complex reactive force immediately follows:
\[ F'_r := F_r + m_\infty a \Rightarrow m_\infty a, \]

(18.5)

with the complex reactive mass
\[ m_r = m_i - m_\infty = \frac{ir_i \omega m_i}{1 + ir_i \omega}. \]

(18.6)

The complex mass \( m \) is connected with \( m_s \):
\[ m = m_\infty + m_s. \]

(18.7)

Inserting (18.2) and (18.4b), we again get eq. (17.5a) for the determination of the complex angular frequency \( \omega = \omega_r + i \omega_i = \omega_0 \Omega \).

The total energy of the spiraling mass \( m_0 \) is
\[ W_0(\tau) = m_0 \gamma := m_0 \left(1 + \left| u^0 \right|^2 \right)^{1/2}, \quad \left| u \right| = u_0 \exp(-\omega \tau), \]

(18.8)

with \( \omega_i := \omega_0 \Omega_i = (q B / m_0) \Omega_i \) and \( \Omega_i \) corresponding to (17.6a). This result is easier and physically more plausible than the approximation in [34, p. 235].

19. Invariance of the equation of motion (8.2) under any full Poincaré transformation

We postulate (in concordance with Tamm and Markov [36]) that all fundamental equations of electrodynamics (in compact form) shall be invariant under any full Poincaré transformation. That is, especially, they shall be invariant under any full (also nonorthochronous) Lorentz transformation. This implies the invariance under a time inversion. Let us interpret these transformations as passive [56]: The same physical process is described with respect to different reference frames of the Poincaré group. For the Maxwell equations this postulate is, as you know, fulfilled. It is also fulfilled for the equation of motion (9.12), valid for small magnitude \(| \alpha | \) of acceleration.

The group of full Poincaré transformations (inhomogeneous Lorentz transformations) may be defined by the postulate: The 4-differential
\[ d\vec{x} = dx' \vec{e}' = dx' / \vec{e}' = d\vec{x} \]

of the 4-position vector \( \vec{x} \) be invariant under any transformation of the group.

A time inversion \( T \) [11,28,57,58] is a special group element. It may be defined by the transformation law for the coordinates \( x' \) of the \( T \)-invariant 4-position vector
\[ \vec{x} = x' \vec{e}' = x_i \vec{e}' =: x'' \vec{e}' = \vec{x}'; \]

(19.1)

\[ T x' =: x'' = -\eta_{ij} x^j = -x_i, \]

(19.2)

with the metric
\[ (\eta_{ij}) := (\vec{e}' \cdot \vec{e}') = \text{diag}(1, -1, -1, -1) = (\vec{e}' \cdot \vec{e}') =: (\eta'^{ij}). \]

From (19.1) and (19.2) follows the transformation law for the base vectors:
\[ \vec{e}'_i = -\vec{e}'^i, \quad \text{i.e.} \quad \vec{e}'_0 = -\vec{e}'_0, \quad \vec{e}'_1 = \vec{e}'_1, \quad \vec{e}'_2 = \vec{e}'_2, \quad \vec{e}'_3 = \vec{e}'_3. \]

(19.3)

Contrarily follows (19.2) from (19.3) and (19.1), or (19.1) from (19.2) and (19.3).

The passive time inversion, used here, differs from a (more frequently used) active one in that not only the time coordinate \( x^0 \) is inverted, but also the corresponding base vector \( \vec{e}_0 \). All
spacetime events $\bar{x}$ and worldlines $\bar{x}(\tau)$ remain invariant under the passive $T$-inversion. Natural processes are described with backward running clocks. A natural process is not inverted, but its timely description.

The time differential $dt$ and the proper time differential $d\tau := \sqrt{(d\bar{x}/dt)^2}$ have the same $T$-eigenvalue $-1$:

$$ T dt = T \bar{x}^0 = - \bar{x}^0 = - dt, \quad T d\tau = - d\tau. $$

The same holds for all odd proper time derivatives of $\bar{x}$:

$$ T \bar{u}(\bar{x}) := T \frac{d\bar{x}}{d\tau} = T \dot{x} = \frac{d\bar{x}}{-d\tau} = - \bar{u}(\bar{x}) = - \dot{x}, $$

$$ T \bar{x} = - \bar{x}, \ldots, $$

whereas even derivatives of $\bar{x}$ have the $T$-eigenvalue $+1$:

$$ T \ddot{x} = \ddot{x}, \quad T \frac{d^4}{d\tau^4} \bar{x} = \frac{d^4}{d\tau^4} \bar{x}, \quad \ldots. $$

The 4-current density $\vec{J} := \rho_{\text{el}} \bar{u}$ is odd like the 4-velocity $\bar{u}$:

$$ T \vec{J} = - \vec{J}, $$

because the proper charge density $\rho_{\text{el}} = (\vec{J}^2)^{1/2}$ is $T$-even.

Application of the above postulate onto the wave equation

$$ \dot{\bar{\sigma}} \bar{A} = \bar{J} $$

yields that the 4-Potential $\bar{A}$ must have the $T$-eigenvalue $-1$, like $\vec{J}$:

$$ T \bar{A} = - \bar{A}. $$

The same is valid for the electromagnetic field strength $\bar{B}$:

$$ T \bar{B} = T(-\dot{\bar{\sigma}} \bar{A} + \bar{A} \ddot{\bar{\sigma}}) = - \bar{B}. $$

Hence, the 4-force density $\vec{f}_\sigma := \rho_{\text{el}} \bar{u} \cdot \bar{B}$ is even:

$$ T \vec{f}_\sigma = \vec{f}_\sigma. \quad (19.4) $$

The integration over the $T$-invariant 3-plane $\sigma = \{ \bar{x} | \bar{x}^0 = 0 \}$ does not change the $T$-eigenvalue $+1$ of $\vec{f}_\sigma$:

$$ T \int d\sigma \vec{f}_\sigma = T \int dq \bar{u} \cdot \bar{B} = T \bar{F} = \bar{F}. $$

That is, the 4-force $\bar{F} = \bar{F}_e + \bar{F}_s$ is even, too, like also the external 4-force $\bar{F}_e$ and the self-force $\bar{F}_s$. [In (8.2), the following quantities are $T$-odd: $\bar{k}$, $i$, $d^4 \bar{k}$. This results, e.g., from

$$ \bar{p} = m \bar{u} = \hbar \bar{k} \hat{\tau} = - i \hbar \partial / \partial \bar{x}. $$

Hence, the equation of motion (8.2) is full Lorentz invariant.

This does not hold for the Lorentz-Dirac equation [31,52,54]

$$ 0 = \bar{F}_e, \quad (19.6a) $$

with

$$ \bar{F}_e := \frac{q^2}{6\pi} \left( \bar{u} \cdot \bar{u} \right) \bar{u}. \quad (19.6b) $$

The left side of (19.6a) is $T$-even, the right is odd. From the $T$-invariance postulate would follow $\bar{F}_e = 0$. For a homogeneous electric field this is a well-known (unphysical) result [45,54,60].

The opposite conclusion in [54] and [55] is caused by other definitions of the time-inverted fields and forces, and by the renunciation of the postulate that all forces must be $T$-even. Especially, eq. (9.20) in [55] is not compatible with our $T$-odd 4-potential (19.7).
The $T$-invariance postulate is not only fulfilled by our relativistic equation of motion (8.2), but also by its nonrelativistic approximation in [7] and [36] and by its approximation (16.6) for small acceleration.

That the 4-force density $\mathbf{f}_\sigma$ must be $T$-even is also provable in the following way: The equation of motion for a (charged) continuum of mechanical mass density $\mu_\infty$ reads

$$\mu_\infty \ddot{u} = \mathbf{f}_\sigma.$$  

As the left side is $T$-even, the right must likewise be even. From this immediately follows, that $\mathbf{B}$ must be odd:

$$\mathbf{f}_\sigma - T \mathbf{f}_\sigma = (1 - T) \rho_\sigma \mathbf{B} = 0 \Rightarrow T \mathbf{B} = -\mathbf{B}.$$  

The Liénard-Wiechert field $\mathbf{B}$ [56,57] of an accelerated point charge $q$ is a well-known example: Considering that

$$T R := T \mathbf{u} \cdot \mathbf{r} = -\mathbf{u} \cdot \mathbf{r} = R,$$

one gets the $T$-odd retarded 4-potential [54]

$$\mathbf{A}(\mathbf{x}) = \frac{q \mathbf{u}}{4\pi |R|},$$  

from which the odd retarded electromagnetic field strength

$$\mathbf{B}(\mathbf{x}) := -\partial_\tau \mathbf{A} + \mathbf{A} \partial_\tau$$

follows. The 4-velocity $\mathbf{u}$ and the 4-acceleration $\dot{\mathbf{u}} := \ddot{\mathbf{u}}$ are to be taken at the retarded 4-point $\mathbf{x}$. This event is full Lorentz invariant just as the 4-point $\mathbf{x}$ of observation.

In a proper (orthochronous) Lorentz frame, the retarded 4-point $\mathbf{x}$ is defined as the first penetration point of the particle’s worldline $\mathbf{x}(\tau)$ through the light cone $(\mathbf{x} - \mathbf{z})^2 = 0$ (with its vertex at $\mathbf{x}$). For it holds $R := \mathbf{u} \cdot (\mathbf{x} - \mathbf{z}) > 0$.

The retarded 4-potential $\mathbf{A}$ and the retarded electromagnetic field strength $\mathbf{B}$ depend both explicitly and implicitly on $\mathbf{x}$, the latter because of $\overrightarrow{\mathbf{u}} = \mathbf{u}(\mathbf{x}), \overrightarrow{\mathbf{a}} = \mathbf{a}(\mathbf{x})$ and $(\mathbf{x} - \mathbf{z})^2 = 0$. (In [10] and [34] this fact is ignored.)

The self-field $\mathbf{B}_s$ can be represented as superposition of elementary waves (19.8), emitted by retarded charge elements $dq'$:

$$\mathbf{B}_s = \int dq' \mathbf{b}.$$  

Thus, the mean electromagnetic self-field strength is

$$\langle \mathbf{B}_s \rangle := \frac{1}{q} \int dq \int dq' \mathbf{b}.$$  

Therefore, $\langle \mathbf{B}_s \rangle$ is odd and the self-force $\mathbf{F}_s := q \mathbf{u} \cdot \langle \mathbf{B}_s \rangle$ even. The former also follows from (7.6).

To guarantee the full Lorentz invariance of electrodynamics, we postulate:

Each (compact) field tensor $F$ (of rank 0, 1, 2, ... , i.e. scalar, 4-vector, 4-dyadic, ...) shall be an eigentensor of the eigenvalue equation

$$L F = \pm F$$  

with the eigenvalue $+1$ or $-1$, where $L$ is any full (homogeneous) Lorentz transformation.
This demand is equivalent to the postulate: *Each field tensor $F$ shall commute or anticommute with the Lorentz operator $L$*:

$$LF \pm FL = 0.$$  \hfill (19.11)

In our paper, we preferred the *compact representation* of physical quantities against their coordinate representation, otherwise more frequently used, because the former is *simpler* and more lucid. For example, the electromagnetic field strength in compact notation, $\vec{B}$, is $T$-odd. On the contrary, in contravariant coordinate notation, $(\vec{B}')^\alpha$ (7.5) is no eigenmatrix of the time inversion operator $T$.

The *equation of motion* (8.2) and its equivalent (8.5) are not only invariant under any full Lorentz transformation, but also under any 4-translation in spacetime. Hence, they are *invariant under any full Poincaré transformation*, too.

The full Poincaré invariance also holds, as you know, for the Maxwell equations. The claim that this invariance shall be conserved, if one adds to the electric charges hypothetical *magnetic charges* (see, e.g., [60]) with $T$-even density $\rho_m$, yields $\rho_m = 0$. The contrary assumption that a (time independent) magnetic charge $q_m$ is $T$-odd is physically implausible.

20. **Properties of the solutions of the equation of motion**

From the equation of motion (8.2) or its integrated form (8.5), the following properties of the solutions can be inferred:

1. **A preacceleration does not occur.**

   That is: If the external 4-force $\vec{F}_e = \vec{0}$ for $\tau \leq 0$ and is $\neq \vec{0}$ for $\tau > 0$, then this also holds for the 4-acceleration $\vec{a}$ and the 4-self-force $\vec{F}_s$ (9.2), which only depends on the past (see App. D for the special case $\vec{f} := qE_e/m = 0$). Hence,

   $$\vec{F}_e = q \vec{a} \cdot \vec{B}_e = \vec{0} \quad \text{for} \quad \tau \leq 0 \quad \Rightarrow \quad \vec{a} = \dot{\vec{u}} = \vec{0} \quad \text{for} \quad \tau \leq 0.$$  \hfill (20.1)

2. **A postacceleration does occur.**

   That is: If the external 4-force $\vec{F}_e \neq \vec{0}$ for $\tau \leq 0$ and is $= \vec{0}$ for $\tau > 0$, then the latter does not hold for the 4-acceleration $\vec{a}$. This follows from

   $$\vec{a}(\tau = 0) = \frac{1}{m_c} \vec{F}_s(0) \neq \vec{0},$$  \hfill (20.2)

because the self-force is a continuous quantity of process, depending on the past.

21. **Discussion**

As advantages of the equation of motion (8.2) or its equivalent (8.5) against other ones (cf., e.g., [3,35,39,42,43,46,47,54,56,61,62,63]) we want to name (see also App. C):

1. For the derivation of these differential-integral equations only Maxwell’s continuum electrodynamics has been used. The Maxwell equations were not changed (in contrast, e.g., to [17]). Expansions of nonanalytic functions in powers of a small parameter (for example the pseudo radius $r$) were not applied.

2. The electromagnetic 4-self-field strength $\vec{F}_s$, used for the calculation of the 4-self-force $\vec{F}_s$, is the retarded solution of Maxwell’s equations belonging to the 4-current density $\vec{J}$. With that the causality principle is preserved.
3. It was dispensed with the divergent, mathematically and physically unacceptable mass
renormalization usual for point charges. That is, Dirac’s postulate (see preamble) is satisfied.
4. It was also dispensed with additional ad hoc boundary conditions at infinity, like \( \ell := 0 \) for \( t = \pm \infty \), to exclude runaway solutions. The latter, just as noncausal preacceleration [21], do not occur. Instead of that, a postacceleration occurs [see Sect. 20].
5. The magnitudes \(| F_\epsilon | \) and \(| \alpha | \) of the external 4-force and the 4-acceleration need not be very small. Of course, quantum mechanical effects, like pair creation, are not described by our classical theory. The contrary assumption in [18] is contestable.
6. For the homogeneous electric field holds: If one of the 4-vectors \( \alpha_\epsilon, F_\epsilon, \bar{F}_\epsilon \) vanishes, this also holds for the other three [cf. (13.2) and Fig. 5]. This inference from our solution of the equation of motion (8.2) [approximated by (9.11)] is physically more plausible than that in case of the Lorentz-Dirac equation (G6a,b), where \( \bar{F}_\epsilon := m_\epsilon \alpha_\epsilon - F_\epsilon = \bar{0} \) also holds for \( | F_\epsilon | = \text{const} \neq 0 \). The notorious discrepancy, that a uniformly accelerated charged particle radiates and simultaneously the reaction force vanishes, disappears [37, 45]. Instead of this, our equation (8.2) yields the physically more plausible result that \( \bar{F}_\epsilon \neq \bar{0} \) for the hyperbolic motion.
7. The differential-integral equation (8.2) is causal. That is, the solution \( \bar{z}(\tau) \) only depends on the past. The equation (8.2) simplifies for small \( | \alpha | \) and \( | \bar{F}_\epsilon | \) to \( m_\epsilon \alpha_\epsilon \approx \bar{F}_\epsilon \), for large \( | \alpha_\epsilon | \) and \( | \bar{F}_\epsilon | \) to \( m_\epsilon \alpha_\epsilon \approx \bar{F}_\epsilon \). Our conclusions are consistent with those of Markov [36], who nonrelativistically calculated the response of an extended charge to an oscillating external force.
8. The asymptotic equations of motion are especially fulfilled by our solutions for the motion of the particle in a homogeneous electric or magnetic field.
9. Like Maxwell’s equations and the asymptotic equations \( m_0 \alpha_\epsilon \approx \bar{F}_\epsilon \) and \( m_\epsilon \alpha_\epsilon \approx \bar{F}_\epsilon \), the equations of motion (8.2) and (8.5) are invariant under any full Poincaré transformation (not only under any orthochronous one, see Sect. 19). This does not hold for the Lorentz-Dirac equation and its generalizations in [17] and [18]. Nor it holds for the Caldirola-Yaghjian equation [55,66] and the equations, proposed by Eliezer, Mo and Papas, Herrera, and Bonnor ([43], p. 211).
10. The equation of motion (8.2), written in compact notation, is not only form invariant (= covariant) under full Poincaré transformations, but also invariant, because both sides are.
11. For our examples with
\[
\bar{B}_\epsilon = E_\epsilon (\bar{e}_0 \bar{e}_2 - \bar{e}_0 \bar{e}_2) = \text{const}
\]
or
\[
\bar{B}_\epsilon = B_\epsilon (\bar{e}_2 \bar{e}_2 - \bar{e}_2 \bar{e}_2) = \text{const}
\]
amazingly simple and physically reasonable solutions result. It is to conjecture that this also holds for other external electromagnetic field strengths \( \bar{B}_\epsilon(\bar{x}) \).
12. For rectilinear motions follows from \( \pi \cdot \bar{F}_\epsilon = 0 \) and \( \pi \cdot \alpha = 0 \) that the equation of motion (8.2) may be written in the form \( m \bar{\alpha} = \bar{F}_\epsilon \). The effective inertial mass \( m = m_\epsilon - \bar{F}_\epsilon / \alpha \) is in general a functional of \( \bar{x}(t') < t \) [cf. (8.2)]. In the case of a homogeneous electric field treated above, \( m \) is a time-independent nonanalytic function of \( | \alpha_\epsilon | \).
13. If it would be possible to measure the dependence \( m(\alpha_\epsilon) = \bar{F}_\epsilon / \alpha \), one could derive from this function the density function \( \rho(t') \) (4.4).
14. If the particle moves in a plane, it is advantageous to go over to complex quantities \( z(\tau), \alpha = \bar{z}(\tau), F_\epsilon, m_\epsilon, \ldots \). Then, the complex equation of motion \( m \alpha = \bar{F}_\epsilon \) is valid additionally to (8.2). With this method, the spiraling particle in a homogeneous magnetic field \( B_\epsilon \) was treated. In this case, \( m(B_\epsilon) \) (18.2) is time-independent.
The relativistic equation of motion $m_0 \ddot{u} \approx q \dot{u} \cdot \mathbf{B}$ holds for a small, extended charge $q$ of inertial mass $m_0 = m_e + m_i$, moving in a weak external electromagnetic field $\mathbf{B}$. The 4-self-force $\mathbf{F}_s \approx -m_i \ddot{u}$ may be combined with the mechanical inertial 4-force $-m_e \ddot{u}$ to the total inertial 4-force $-m_e \ddot{u}$. For a very strong electromagnetic field $\mathbf{B}$, in the above weak field equation, $m_0$ has to be replaced by $m_i$.

The accelerated heavy mass part $m_h$ of $m_0 = m_h + m_g$ is a source of a gravitational wave effecting a gravitational 4-self-force $-m_g \ddot{u}$, which is a tiny part of the mechanical inertial 4-force $-m_e \ddot{u}$.

If in the local rest frame $R^*$ acts, additionally to the external electric field strength $\mathbf{E}^*$, a small external gravitational field strength $\mathbf{g}^*$, we can consider this by the ansatz $\mathbf{F}_e^* = q \mathbf{E}^* + m_e \mathbf{g}^*$.

That the extended charge does not explode, although enormous electromagnetic repulsive forces act between the charge elements, is explainable by the assumption that nonelectromagnetic forces compensate these repulsive forces. With this (Poincaré) model conception, the notorious factor $4/3$ in (9.10) is explainable, too [39,59].

The product $r_1 \mathbf{F}(r_1)$ of the particle’s quasi-radius with the 4-self-force is nonanalytic at $r_1 = 0$ [cf. (12.8)]. This is an essential cause for the fact that the equation of motion (8.2) does not go over into the Lorentz-Dirac equation [47,54] for $\mathbf{F}(r) \approx \mathbf{F} (9.12)$. This claim is in general not fulfilled by other equations of motion, e.g. the Lorentz-Dirac equation, if $|\mathbf{a}| \neq 0$.

Any equation of motion should for $|\mathbf{a}| \rightarrow 0$ go over into $m_0 \mathbf{a} \approx \mathbf{F}$ (9.12). This claim is in general not fulfilled by other equations of motion, e.g. the Lorentz-Dirac equation, if $|\mathbf{a}| \neq 0$.

The consideration of the reaction force in quantum theory is possible by substitution of the external 4-force $\mathbf{F}_e$ by the effective force $\mathbf{F}_{e,\text{eff}}(\mathbf{F}_e)$ [cf. (14.7)].

Acknowledgment

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Appendices

A. Electrostatic self-energy of the charged particle

To prove the claim after (9.3), we calculate the electrostatic self-energy of the charged particle in spherical coordinates:

\[
m_1 = \frac{q^2}{8\pi} \int d^3\vec{r}' \int_0^\infty dr' \int_0^\pi d\theta' \int_0^{\pi/2} d\varphi' \sin \theta' \rho \left( r'^2 + r^2 - 2rr' \cos \theta \right)^{-1/2}
\]

\[
= \frac{q^2}{8\pi} \int dr' 4\pi r'^2 \rho' \int_0^\pi dr 4\pi r^2 \rho \left[ \frac{\Theta(r'-r)}{r'} + \frac{\Theta(r'-r')}{r} \right]
\]

\[
= \frac{1}{8\pi} \int dq' dq \frac{r}{r'},
\]

where

\[
j = \angle(\vec{r}, \vec{r}'), \quad \rho = \rho(r), \quad \rho' = \rho(r'), \quad \Theta(x) = \text{unit step function} = \int_{-\infty}^x dy \delta(y),
\]

\[
dq = q\rho 4\pi r^2 dr, \quad dq' = q\rho' 4\pi r'^2 dr', \quad r_\approx = \text{Max}(r, r').
\]

The integral over \( dr \) may be transformed with the help of (4.4):

\[
\int_0^\infty dr' = \frac{4\pi}{r'} \int_0^\infty r'^2 \rho + 4\pi \int_0^\infty dr r \rho
\]

\[
= \frac{1}{i\pi} \int_{-\infty}^\infty d\kappa \kappa \bar{\rho}(\kappa) \left[ \int_{r'}^\infty dr r e^{i\kappa r} + \int_{r'}^\infty dr e^{i\kappa r} \right] = \frac{1}{\pi r'} \int_0^\infty d\kappa \bar{\rho} \frac{\sin \kappa r'}{\kappa}.
\]

Insertion in (A1) yields

\[
m_1 = \frac{q^2}{8\pi} \int_0^\infty dr' 4\pi r'^2 \frac{1}{4\pi^2 r'} \int_{-\infty}^\infty d\kappa' \kappa' \bar{\rho}' \sin \kappa' r' \frac{1}{\pi r'} \int_{-\infty}^\infty d\kappa \bar{\rho} \sin \kappa r
\]

\[
= \frac{q^2}{8\pi^3} \int d\kappa' \int_0^\kappa \frac{\kappa'}{\kappa} \bar{\rho}' \bar{\rho} \int dr \sin \kappa' r \sin \kappa r
\]

\[
= \frac{q^2}{8\pi^3} \int d\kappa' d\kappa \frac{\kappa'}{\kappa} \bar{\rho}' \bar{\rho} \int_{-\infty}^\infty \left[ \delta(\kappa'-\kappa) - \delta(\kappa'+\kappa) \right]
\]

\[
= \frac{q^2}{8\pi^2} \int d\kappa \bar{\rho}^2,
\]

in agreement with (9.10).

The calculated interaction energy between the charge elements equals the electrostatic self-field energy (see, e.g., [28,34]).
B. Proof that the field strength balance \((12.5)\) holds for an arbitrary reference point of the particle’s worldline

For the derivation of the field strength balance \((12.5)\), we had supposed as a reference point \(\overline{z} = 0\). We now want to show that the same balance results for an arbitrary reference point \(z_i\) of the particle’s worldline \((12.1)\), too.

For the position vector \(\overline{z}\) of an arbitrary point on the particle’s worldline holds [see \((D6)\)]

\[
\overline{z} = z^0 \overline{e}_0 + z^1 \overline{e}_1 = t \overline{e}_0 + x \overline{e}_1 = \frac{\sinh f \tau}{f} \overline{e}_0 + \frac{\cosh f \tau - 1}{f} \overline{e}_1 .
\]  

(B1)

At this, \(\overline{e}_0\) and \(\overline{e}_1\) are the orthonormal base vectors of the lab frame, pointing into the positive directions of the \(t\) and \(x\) axis, respectively, with the origin at \(\overline{z} = 0\) (Fig. 10). The proper 2-velocity at a fixed point \(z_i := \overline{z}(\tau_i)\) is

\[
\ddot{z}(\tau_i) := \ddot{U} = \overline{e}_0 U^0 + \overline{e}_1 U^1 = \overline{e}_0 \cosh f \tau_1 + \overline{e}_1 \sinh f \tau_1 .
\]  

(B2)

Fig. 10

Additionally to the lab frame, we introduce the local rest frame with its origin at \(\overline{z}_i\). Its base vectors \(\overline{e}_0' = \overline{U}\) and \(\overline{e}_1' = \overline{e}_0 U^1 + \overline{e}_1 U^0\) point into the positive directions of the \(t'\) and \(x'\) coordinate axes. The inversion of this Lorentz transformation of the base vectors reads

\[
\overline{e}_0 = U^0 \overline{e}_0' - U^1 \overline{e}_1' ,
\]

\[
\overline{e}_1 = - U^1 \overline{e}_0' + U^0 \overline{e}_1' .
\]

With that, one gets

\[
\overline{z}' := \overline{z} - z_i = (t - t_i) \overline{e}_0 + (x - x_i) \overline{e}_1
\]

\[
= f^{-1} \left[ (\sinh f \tau - \sinh f \tau_1)(\overline{e}_0 \cosh f \tau_1 - \overline{e}_1 \sinh f \tau_1) \right.
\]

\[
+ (\cosh f \tau - \cosh f \tau_1)(- \overline{e}_0' \sinh f \tau_1 + \overline{e}_1' \cosh f \tau_1) \right]
\]

\[
= \frac{\sinh f \tau'}{f} \overline{e}_0' + \frac{\cosh f \tau' - 1}{f} \overline{e}_1' ,
\]

where \(\tau' := \tau - \tau_i\).  

(B3)

This 2-position vector has in all local rest frames the same form: It is form invariant under any proper-time translation \(\tau \rightarrow \tau - \tau_i\). Equation \((B1)\) is the special case for \(\overline{z}_i = 0\). This specialization means, therefore, no restriction for the general validity of \((12.5)\).

Taking into account the Poincaré invariance of the equation of motion \((8.2)\), the above result implies the conclusion: The solution \((B1)\) fulfils this equation not only at \(\overline{z} = 0\), but also at an arbitrary trajectory point \(\overline{z} = \overline{z}_i\).

C. Comparison with Nodvik’s work

In a fundamental work, Nodvik [40] derived an equation of motion for an extended charge in an electromagnetic field. He did not restrict himself to a nonrotating charge with not too large an acceleration. The complexity of his differential-integral equation may be the reason that (to the author’s knowledge) no nontrivial solutions have been found up to now.
To simplify the problem, we restricted ourselves from the beginning to a nonrotating charge and to \( a \ll <r>^{-1} \) (3.6). A further simplification was the passing over to Fourier transformed quantities, like usual in quantumelectrodynamics.

Independently of Nodvik’s derivation, we found the equation of motion (8.2), valid under the above-mentioned conditions. For this equation, nontrivial (approximate) solutions were found. The asymptotic forms for small and large acceleration of the differential-integral equation and its solutions were discussed.

If we had not approximated the functional determinant \( D = 1 - \vec{r} \cdot \vec{u} \) (3.3) by 1 and \( \rho' \) by \((3/4)\rho\) (9.11), our corresponding generalization of (8.2) should accord with Nodvik’s equation of motion, specialized on a nonrotating charge.

D. Example for the iterative solution of the equation of motion (8.5)

For a homogeneous electric field \( \vec{E}_e = E_e \vec{e}_t \), we get from (8.1) and (11.8a)

\[
\vec{u} = \frac{q}{m_e} \vec{u} \cdot <\vec{B}> = \frac{q}{m} \vec{u} \cdot \vec{B}_e \quad \text{or} \\
\frac{q}{m_e} <\vec{B}> = \frac{q}{m} \vec{B}_e = f(\vec{e}_0 \vec{e}_1 - \vec{e}_1 \vec{e}_0) =: \vec{d} = \vec{f}, \quad f := \frac{qE_e}{m}.
\]

In the following we show that in this case the iteration procedure converges against the unique solution (D4).

Assuming the initial conditions
\[
\vec{\pi}(0) = \vec{0}, \quad \vec{\pi}(0) = \vec{e}_0
\]
and beginning with the zeroth approximation
\[
\vec{\pi}(0) = \vec{0},
\]
the differential-integral equation (8.5) successively yields (with \( \vec{e}_0 \cdot \vec{e}_0 = - \vec{e}_1 \cdot \vec{e}_1 = 1 \), \( \vec{e}_0 \cdot \vec{e}_1 = 0 \))
\[
\vec{\pi}(1) = \tau \vec{e}_0, \\
\vec{\pi}(2) = \tau \vec{e}_0 + \frac{1}{2} f \tau^2 \vec{e}_1, \\
\vec{\pi}(3) = \left( \tau + \frac{1}{6} f \tau^3 \right) \vec{e}_0 + \frac{1}{2} f \tau^2 \vec{e}_1, \\
\vec{\pi}(4) = \left( \tau + \frac{1}{3!} f \tau^3 \right) \vec{e}_0 + \left( \frac{1}{2!} f \tau^2 + \frac{1}{4!} f^3 \tau^4 \right) \vec{e}_1.
\]

The \( n \) th approximation \( \vec{\pi}^{(n)} \) converges for \( n \to \infty \) against the well-known hyperbola
\[
\vec{\pi}(\tau, f) = \lim_{n \to \infty} \vec{\pi}^{(n)} = \frac{\sinh f \tau}{f} \vec{e}_0 + \frac{\cosh f \tau - 1}{f} \vec{e}_1 = t \vec{e}_0 + \tau \vec{e}_1.
\]

From this follows (12.1):
\[
x = \frac{1}{f} \left( \cosh f \tau - 1 \right) = \frac{1}{f} \left( \sqrt{1 + \sinh^2 f \tau} - 1 \right) = \frac{1}{f} \left( \sqrt{1 + f^2 \tau^2} - 1 \right).
\]

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The solution (D4) of (8.5) may also be written in the form (see [51])

\[
\varpi(\tau, f) = \int_0^\tau d\tau' \bar{\varpi}(\tau', f) = \bar{\varepsilon}_0 \cdot \frac{\exp(\tau f) - \bar{1}}{f} = \varepsilon_0 \left[ \tau \bar{1} + \frac{f \tau^2}{2!} (\bar{\varepsilon}_0 \bar{\varepsilon}_1 - \bar{\varepsilon}_1 \bar{\varepsilon}_0) + \frac{f^2 \tau^3}{3!} (\bar{\varepsilon}_0 \bar{\varepsilon}_1 \bar{\varepsilon}_1 - \bar{\varepsilon}_1 \bar{\varepsilon}_0 \bar{\varepsilon}_0) + \cdots \right]
\]

\[
= \varepsilon_0 \sinh f \tau + \bar{\varepsilon}_1 \cosh f \tau - 1.
\]

The mean field strength dyadic \( \bar{f} \) (D2) is the generator [22] of the Lorentz transformation dyadic (= \( \tau \)-translation)

\[
\bar{f} = \exp(\tau \bar{d}) = \exp(\tau f \bar{d}) := \bar{1} + \tau f \bar{d} + \frac{1}{2} \tau^2 f^2 \bar{d}^2 + \cdots, \tag{D7}
\]

mapping the initial 4-speed \( \hat{\varpi}(0, f) = \varepsilon_0 \) onto the speed at proper time \( \tau \):

\[
\hat{\varpi}(\tau, f) = \hat{\varpi}(0, f) \cdot \bar{f} = \varepsilon_0 \exp(\tau \bar{f}) = \varepsilon_0 \cosh f \tau + \varepsilon_1 \sinh f \tau. \tag{D8}
\]

At this we have used the identities

\[
\bar{d}^{2+n} = \bar{d} := \varepsilon_0 \bar{\varepsilon}_1 - \bar{\varepsilon}_1 \bar{\varepsilon}_0, \quad n = 0, 1, 2, \ldots ;
\]

\[
\bar{d}^{2+n+2} = \bar{d}^* = \bar{\varepsilon}_0 \bar{\varepsilon}_0 - \bar{\varepsilon}_1 \bar{\varepsilon}_1 = \bar{1}, \quad \bar{d}_0 \cdot \bar{d}^{2+n} = \bar{1} \tag{D9}
\]

\( \bar{1} \) is the unit dyadic in the two-dimensional 0, 1 Minkowski space.

The field strength \( f := qE / m \) may be determined as a function of \( f_\infty := qE / m_\infty \) like in Sect.12.

E. Calculation of \( \Delta(\delta) \) (12.8) with generalized factorial functions

The function \( \Delta(\delta) \) (12.8) may be calculated, e.g., as follows.

Insertion of the power series (10.15) for \( A_\lambda(\alpha) \) into (12.8) yields

\[
\Delta(\delta) = \sum_{n=0,1,\ldots}^{\infty} c_{2n} f_{2n}(\delta), \tag{E1}
\]

with

\[
c_{2n} := 6^{1+n} (3+2n)!; \quad \Delta(\infty) = \sum_{n=0}^{\infty} c_{2n} f_{2n}(\infty) = \sum_{n=0}^{\infty} c_{2n} (2n)! = 1.5 \tag{E2}
\]

and for \( \delta \geq 0 \):

\[
f_n(\delta) := \frac{\int_0^\infty dt' \exp(-\sqrt{t^2 + 2t / \delta})}{\delta^{n+1}} = \frac{2^n}{\delta^{n+1}} \int_0^\infty dx \sinh x \sinh^{2n} \frac{x}{2} \exp\left(-\frac{\sinh x}{\delta}\right). \tag{E3}
\]

These generalized factorial functions increase with \( \delta \) from \( f_n(0) = 0 \) to \( f_n(\infty) = n! \). Their asymptotic behavior is
\[ f_n(\delta) \approx \frac{(2n)!}{2^n} \delta^{1+n} \quad \text{for} \quad \delta \ll 1, \]
\[ \approx n! (1 - \delta^{-1} + \cdots) \quad \text{for} \quad \delta \gg 1. \]  
(E4)

\( f_n(\delta) \) resembles the incomplete factorial function \((n, \delta)! = \int_0^\delta dt t^n e^{-t}\) [27].

F. The Lorentz-Dirac equation as an inadmissible approximation of the equation of motion (8.2)

The Lorentz-Dirac equation can be derived as an approximation of (8.2). To show this, we start from eq. (9.2) for the self-force. With the approximation
\[ \tilde{\rho} \approx \hat{\rho} \]  
(F1)

instead of (9.4b), one gets, analogously to the (nonrelativistic) derivation of (9.10), instead of (9.8)
\[ \tilde{F}_s \approx -\frac{q^2}{6\pi^2} \int_0^0 dt' \tilde{u}' \int d\kappa \kappa^2 \exp(i\kappa t') \tilde{\rho}^2. \]  
(F2)

Inserting
\[ \tilde{u}' \approx \bar{u}' + \hat{\dot{u}} t'^2 \]  
(F3)
gives with (A2) the nonrelativistic self-force at the origin \( z = \bar{0} \) of a local rest frame, where \( \bar{u} = \bar{v}_0 \):
\[ \tilde{F}_s \approx -\frac{4}{3} m_i \bar{a} + \frac{q^2}{6\pi} \hat{a}, \quad \bar{u} \cdot \tilde{F}_s = 0. \]  
(F4)

For \( z \neq \bar{0} \) and \( \bar{u} = \bar{v} \neq \bar{v}_0 \), the relativistic generalization of this formula obviously reads
\[ \tilde{F}_s \approx -\frac{4}{3} m_i \bar{a} + \frac{q^2}{6\pi} \hat{a}_\sigma \quad \text{with} \quad \hat{a}_\sigma := \tilde{F}_\sigma \cdot \hat{\bar{a}} := (\bar{1} - \bar{u} \bar{u}) \cdot \hat{\bar{a}}, \quad \bar{u} \cdot \tilde{F}_s = 0. \]  
(F5)

The projection dyadic
\[ \tilde{\bar{F}}_\sigma := \bar{1} - \bar{u} \bar{u} \]  
(F6)
projects a 4-vector, here \( \hat{\bar{a}} \), onto the 3-plane \( \sigma \perp \bar{u} \) [see (2.9) and Fig. 1]. The approximate 4-self-force \( \tilde{F}_s \) (F5) yields the Lorentz-Dirac equation (19.6a,b),
\[ \left( m_i + \frac{4}{3} \frac{m_i}{m_\infty} \right) \bar{a} = \tilde{F}_e + \frac{q^2}{6\pi} \tilde{F}_\sigma \cdot \hat{\bar{a}}. \]  
(F7)

Unfortunately, this “derivation” of the Lorentz-Dirac equation as an approximation of the equation of motion (8.2) is mathematically and physically inadmissible, and this because of the following reasons:

1. The product \( <r> \tilde{F}_s \) is in general a nonanalytic function of the radius \( <r> \) [cf. (12.8)].

Therefore, \( \tilde{F}_s \) is not developable into a Laurent expansion in \( <r> \). The approximation (F5) may be considered as a truncated (divergent asymptotic) Laurent series with \( m_i \propto <r^{-1}> \) [cf. (9.14)].
(2) The limit $\lim_{\gamma \to 0} m_\gamma = \infty$. That means: The transition to a point charge is to exclude. The so-called mass renormalization with the ad hoc assumption $m_\gamma \to -\infty$, so that the “observable mass” $m_\infty + 4m_\gamma / 3 = -\infty + \infty \ll \infty$, is mathematically and physically unsound and contradicts Dirac’s postulate (see preamble).

(3) The Lorentz-Dirac equation contains the relativistically incorrect “electrodynamic mass” $4m_\gamma / 3$, caused by the incorrect approximation (F1).

(4) The equation of motion (8.2) yields [cf. (9.10)]: Decreasing acceleration implies decreasing reaction force,

$$\vec{a} \to 0 \Rightarrow \vec{F}_r := m_\gamma \vec{a} + \vec{F}_e := m_0 \vec{a} - \vec{F}_e \to \vec{0}. \quad \text{(F8)}$$

This physically necessary property of (8.2) is destroyed at the transition to the Lorentz-Dirac equation.

(5) The equation (8.2) also implies the inversion

$$\vec{F}_r \to \vec{0} \Rightarrow \vec{a} \to \vec{0} \quad \text{(F9)}$$

[cf. (13.2) and Fig. 5]. This physically senseful connection is destroyed, too. A well-known example is the hyperbolic motion, where in the Lorentz-Dirac equation $\vec{F}_r = \vec{0}$, although $\vec{a} \neq \vec{0}$, and although electromagnetic radiation is emitted.

Consequently, the Lorentz-Dirac equation (F5) cannot be considered as an admissible approximation of (8.2). Similar objections can be rendered to other derivations of the Lorentz-Dirac equation, for example to Dirac’s [12]. His self-force is not the Lorentz force acted upon the particle’s charge by the retarded electromagnetic self-field at proper time $\tau$.

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Legends

Fig. 1. Parallel worldlines of the particle centre $\bar{z}(\tau)$ and a particle-fixed point $\bar{x}(\tau)$. For simplicity, we have omitted the bars, indicating 4-vectors.

Fig. 2. Radial density $\rho_\nu := \frac{dq}{dr} = 4\pi r^2 r_\nu = \frac{2}{\sqrt{\pi} r_{\nu}} K_1 \left( \frac{r}{r_{\nu}} \right)$ as a function of the reduced radius $r / r_{\nu}$.

Fig. 3. The contour $C_r$ is a semicircle in the upper complex $\omega$ half plane with arbitrarily large radius $R$.

Fig. 4. Hyperbolic worldlines (12.1), belonging to the external field strengths $f_0 := F_\nu / m_0$, $f = F_\nu / m$ and $f_\nu = F_\nu / m_\nu$.

Fig. 5. The connections (12.5) and (13.1) between the field strengths $f = a = F_\nu / m$, $f_\nu = F_\nu / m_\nu$, $f_0 = F_\nu / m_0$ and $f_r = F_r / m_0$: $f = f_0 + f_r$. Example: $m_0 / m_\nu = 1.5$.

Fig. 6. $f_\nu(f)$ with $\mu_i := m_i / m_0 = r_\nu / r_i$ as a parameter.

Fig. 7. Relative effective mass $\mu = m / m_0$ (14.2) as a function of $\delta_0 := r_0 f$ with the relative electrostatic mass $\mu_i := m_i / m_0$ as a parameter.

Fig. 8. Spiral $z(\tau)$ (17.8) of $q > 0$ in a homogeneous external magnetic field $\vec{B}_e$.

Fig. 9. The complex relative frequency $\Omega(\mu_i, b) = \Omega_\tau + i \Omega_\nu$ (17.6a) for $\mu_i = 0.05, 0.1, ..., 0.5$ (17.5b) and $b = 0.5, 1, ..., 3$, respectively.

Fig. 10. The $t, x$ and $t', x'$ coordinate axes span the local rest frames with the origins at $\bar{z} = \bar{0}$ and $\bar{x}_1$.

Table 1. The functions $\Delta(\delta)$ (12.8), $\mu_i(\delta) := 1 - \Delta / \delta$ (12.12) and $\bar{\mu}_i(\delta)$ (12.14)

| $\delta$ | 0  | 0.1 | 0.2 | 0.5 | 1   | 5   | 10  | 100 | $\infty$ |
|----------|----|-----|-----|-----|-----|-----|-----|-----|---------|
| $\Delta$ | 0  | 0.0099999 | 0.1995 | 0.4624 | 0.7418 | 1.2577 | 1.3667 | 1.4719 | 1.5 |
| $\mu_r$ | 0  | 0.000010 | 0.00236 | 0.0752 | 0.2582 | 0.7485 | 0.8633 | 0.98528 | 1 |
| $\bar{\mu}_r$ | 0  | 0.000014 | 0.00237 | 0.0740 | 0.2582 | 0.7511 | 0.8650 | 0.98527 | 1 |
\[ \tau = \text{const} \]
H. Stöckel: Fig. 4
H. Stöckel: Fig. 5

\[ f \]

\[ 10 \frac{c^2}{r_1} \]

\[ f_{\infty} \]

\[ f_0 \]

\[ f_r \]

\[ f_s \]

\[ f_\infty(f) \]

\[ f \rightarrow 10 \frac{c^2}{r_1} \]
H. Stöckel: Fig. 6

Graph showing the relationship between $f_0$, $10/r_0$, $1.5/r_0$, and $f$ with lines for different values of $\mu_1$. The graph includes the following values for $\mu_1$: 0, 0.2, 0.4, 0.6, 0.8, and 1.
$\mu_1 = \mu - \delta_0$
H. Stöckel: Fig. 9
H. Stöckel: Fig. 10