Abstract. Let $M$ be a $2d$–dimensional compact connected Riemannian manifold and $\omega$ be a symplectic form on $M$. In this paper, we prove that a symplectic diffeomorphism, with all Lyapunov exponent zero for almost everywhere, can be $C^1$ approximated by one with a positive Lyapunov exponent for a positive-measured subset of $M$. That is, the set

$$\left\{ f \in \text{Sym}^1_\omega(M) \mid \lambda_1(f, x) > 0 \right\}$$

for a positive measure set

is dense in $\text{Sym}^1_\omega(M)$.

1. INTRODUCTION

Lyapunov exponent is a useful tool to describe hyperbolicity of a system. Using this concept, Pesin introduced a weaker form of hyperbolicity, which he termed nonuniform hyperbolicity. A system that admits an invariant measure with nonzero Lyapunov exponents for almost everywhere is nonuniformly hyperbolic. Since uniformly hyperbolic systems are not dense in the set of smooth dynamical systems, Pesin questioned that whether non-uniformly hyperbolic systems are dense or generic in $C^r$ volume-preserving diffeomorphisms. KAM theory gives a negative answer to this conjecture when $r$ is large enough. And later Mañé-Bochi-Viana’s results\[2, 3\] imply that nonuniformly hyperbolic systems can not be generic. Hence Pesin’s question should be whether non-uniformly hyperbolic systems are dense in $C^1$ volume-preserving diffeomorphisms. Thus it is of utmost importance to detect when the zero exponents can be removed by perturbations. Shub-Wilkinson’s example\[6\] builds a conservative perturbation to a skew product of an Anosov diffeomorphism of the torus $T^2$ by rotations and create positive exponents in the center direction for Lebesgue almost every point. Baraviera-Bonatti\[1\] present a local version of Shub-Wilkinson’s argument, allowing one to perturb the sum of all the center integrated Lyapunov exponents of any conservative partially hyperbolic systems. These perturbations can be assumed to be done on some invariant bundles (not all bundles) of a dominated splitting by decreasing the larger integrated Lyapunov exponent and at the same time increasing the smaller one. But the changes should be different, which is exactly the new integrated Lyapunov exponent of the
center bundle. For symplectic diffeomorphisms, one can not do this kind of perturbation since the absolute values of the changes will always be the same. Thus, we want to consider the question that for a system with all the Lyapunov exponents zero, which implies that we can not ‘borrow’ some nonzero exponent from one bundle to perturb zero exponent on the center bundle, wether it can be approximated by a new system with positive Lyapunov exponents. That is the density of systems with positive Lyapunov exponents for a positive-measured set. This paper is to deal with the problem.

Let $M$ be a $2d$-dimensional compact connected Riemannian manifold and $\omega$ be a symplectic form on $M$, i.e. a non-degenerate closed 2-form. Taking $d$ times the wedge product of $\omega$ with itself we obtain a volume form on $M$. A $C^r$ diffeomorphism $f$ of $M$, $r \geq 1$, is called symplectic if it preserves the symplectic form, $f^*\omega = \omega$. Denote by $\text{Sym}^1_\omega(M)$ the set of all $C^1$ symplectic diffeomorphisms on $M$. Our main result is the following theorem.

**Theorem 1.1.** The set $\{ f \in \text{Sym}^1_\omega(M) \mid \text{The largest Lyapunov exponent } \lambda_1(f, x) > 0 \text{ for a positive measure set} \}$ is dense in $\text{Sym}^1_\omega(M)$.

2. PROOF OF THEOREM 1.1

In 1979, Katok[5] gave an example of 2-dimensional nonuniformly hyperbolic system. It is derived from an Anosov map by a big $C^1$ perturbation. We use the product of finitely many Katok’s examples to construct a nonuniformly hyperbolic system of high dimension. In this section, we will prove our main result by pasting this example to neighborhoods of elliptic periodic points. First, let us introduce some notations and an important lemma as the following.

If for a periodic point $p$ of period $k$ the tangent map $Df^k(p)$ has exactly $2l$ simple non-real eigenvalues of norm 1 and the other ones have norm different from 1, then we say that $p$ is an $l$-elliptic periodic point. Sometimes it is also called quasi-elliptic.

**Lemma 2.1.** Let $f \in \text{Sym}^1_\omega(M)$ with all Lyapunov exponents zero. And $p$ be an elliptic periodic point of $f$. Then there is an arbitrarily small perturbation $g \in \text{Sym}^1_\omega(M)$ of $f$ with a positive Lyapunov exponent on a positive-measured set.

**Proof** Let $n$ be the period of the periodic point $p$. By a small perturbation, we can assume that $D^n f$ is a rational rotation. Then there exists a positive integer $N > 0$ and a measurable set $B$ in $M$, such that $f^N|B = id$.

In the following we will construct a perturbation $\hat{h}$ of $id$ and define a perturbation $g$ of $f$ such that $g = f$ outside $\bigcup_{1 \leq i \leq N} f^iB$ and $g = \hat{h} \circ f$ in $\bigcup_{i \leq k} R_{2\pi j/k}D^i$.

Take $h = \underbrace{K \times \cdots \times K}_d$ times, where $K$ is Katok’s example in [5]. Then $h$ is volume preserving but not a small perturbation of $id$. Find a flow $\phi_t$, an isotopic to $\mathbb{D}$ satisfying that

$$\phi_t : B \times [0, 1] \to B$$

$$\phi_0 = id, \quad \phi_1 = h.$$

Take $k$ large and define a sequence of small perturbation of $id : \{ h_i \}_{i=1}^k$ such that

$$h_i \circ \cdots \circ h_0 = \phi_{i/k}, \quad \forall i = 0, 1, \ldots, k - 1.$$
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Then $h$ is volume preserving and $h = h_k \circ \cdots \circ h_0$. Since $k$ is large, the rotation $R_{2\pi/k}$ is a small rotation with the angle $2\pi/k$. Find a small disk $D^\ell \subset f^\ell B$, such that

$$R_{2\pi/k} \cap D^\ell = \emptyset, \forall \ell = 0, 1, ..., N - 1.$$ 

Define

$$\hat{h}(x) = \begin{cases} 
h_{j-1} \circ R_{2\pi/k} \circ \cdots \circ R_{2\pi/k} \circ h_1 \circ R_{2\pi/k} & \text{on } R_{2\pi/k} \cap D^\ell \\
\text{id} & \text{outside } f^\ell B 
\end{cases}, \quad i = 1, ..., k.$$ 

A $Df$–invariant splitting $TM = E^1 \oplus \cdots \oplus E^k$ is called a dominated splitting if each $E^i$ is a continuous $Df$–invariant subbundle of $TM$ and if there is some integer $n > 0$ such that, for any $x \in M$, any $i < j$ and any non-zero vectors $u \in E^i(x)$ and $\nu \in E^j(x)$, one has

$$\|Df^n(u)\| < \frac{1}{2} \|Df^n(\nu)\|.$$ 

A map $f$ is called partially hyperbolic if there exists a dominated splitting $TM = E^s \oplus E^c \oplus E^u$, into nonzero bundles such that, for some Riemannian metric $\| \cdot \|$ on $M$, we have

$$\|(Df|E^u(x))^{-1}\|^{-1} > \|Df|E^c(x)\| \geq \|(Df|E^c(x))^{-1}\|^{-1} > \|Df|E^s(x)\|$$

for every $x \in M$. Such a splitting is automatically continuous. Here $E^s$ and $E^u$ denote the strong expanding and contracting invariant bundle, respectively.

If a system is partially hyperbolic, then at least one of the Lyapunov exponents is nonzero. So we consider the systems that are not partially hyperbolic. The existence of an elliptic periodic point is an obstruction for partial hyperbolicity. Saghin-Xia\cite{7} proved that the converse is also true generically, i.e. if a $C^1$–generic symplectic diffeomorphism is not partially hyperbolic, then it has an elliptic periodic point (actually it has a dense set of elliptic periodic points).

**Lemma 2.2** (Theorem 1 in \cite{7}). There exists an open dense subset $U$ of $\mathrm{Sym}^1_\omega(M)$ such that any function in $U$ is either partially hyperbolic or it has an elliptic periodic point. There exists a residual subset $R$ of $\mathrm{Sym}^1_\omega(M)$ such that any function in $R$ is either partially hyperbolic or the set of elliptic periodic points is dense on the manifold.

**Proof of Theorem 1.1.** We will prove by induction. For every $n \in \mathbb{N}$, by Lemma 2.2 we can find finitely many elliptic periodic points such that they form an $\frac{1}{n}$–net of $M$. We can find small neighbourhoods of them such that they are disjoint from each other and also disjoint from all neighbourhoods we find in the preceding steps for integers smaller than $n$. And then we can do the perturbation as Lemma 2.1 in these neighbourhoods. Then we get our result.

By the proof above, we can further obtain the following corollary.

**Corollary 2.3.** Let $M$ be a compact surface without boundary. The set

$$\left\{ f \in \mathrm{Sym}^1_\omega(M) \mid \begin{array}{l} \text{The largest Lyapunov exponent } \lambda_1(f, x) > 0 \\
\text{for an open, dense and positive-measured set} \end{array} \right\}$$

is dense in $\mathrm{Sym}^1_\omega(M)$.\[\square\]
Let $M$ be a compact surface without boundary. The symplectic form $\omega$ is also a volume form. By Oseledets Theorem, on almost all points, the Lyapunov exponents exist and the sum of the two exponents are zero. We say that $f$ is nonuniformly hyperbolic on a point if the two Lyapunov exponent are nonzero along the orbit of this point.

**Corollary 2.4.** Let $M$ be a compact surface without boundary. The set 

$$ \left\{ f \in \text{Sym}^1_\omega(M) \mid \text{f is nonuniformly hyperbolic on an open, dense and positive-measured set} \right\} $$

is dense in $\text{Sym}^1_\omega(M)$.

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