Bayesian Predictive Synthesis:
Forecast Calibration and Combination

Matthew C. Johnson and Mike West
1QMS Capital Management
2Department of Statistical Science, Duke University
Durham, NC, USA

Abstract. The combination of forecast densities, whether they result from a set of models, a group of consulted experts, or other sources, is becoming increasingly important in the fields of economics, policy and finance, among others. Requiring methodology that goes beyond standard Bayesian model uncertainty and model mixing—with its well-known limitations based on a clearly proscribed theoretical basis—multiple ‘density combination’ methods have been proposed. While some proposals have demonstrated empirical success, most apparently lack a core philosophical and theoretical foundation. Interesting recent examples generalize the common ‘linear opinion pool’ with flexible mixing weights that depend on the forecast variable itself—i.e., outcome-dependent mixing. Taking a foundational subjective Bayesian perspective, we show that such a density combination scheme is in fact justified as one example of Bayesian agent opinion analysis, or ‘predictive synthesis’. This logically coherent framework clearly delineates the underlying assumptions as well as the theoretical constraints and limitations of many combination ‘rules’, defining a broad class of Bayesian models for the general problem. A number of examples, including an application to a set of predictive densities in foreign exchange, provide illustrations.

Keywords: Bayesian aggregation, Bayesian forecasting, Density forecast combination, Expert opinion, Forecaster calibration, Model combination, Probability forecasts.

1 Introduction

Expert opinion and density forecast combination is becoming increasingly important in economics, policy, and finance (Hall and Mitchell, 2007; Amisano and Giacomini, 2007; Hoogerheide et al., 2010; Kascha and Ravazzolo, 2010; Geweke and Amisano, 2011, 2012; Billio et al., 2012, 2013; Aastveit et al., 2014; Kapetanios et al., 2015; Aastveit et al., 2015; Pettenuzzo and Ravazzolo, 2016), but is also of use in areas as diverse as meteorology, military intelligence, seismic risk, and environmental risk, among others (Clemen, 1989; Clemen and Winkler, 1999; Timmermann, 2004; Clemen and Winkler, 2007; Rufo et al., 2012). The field of agent opinion synthesis, in which a decision maker solicits the opinions of experts in order to create an informed opinion, is applicable in this context. Agent opinion synthesis fits naturally into a Bayesian analysis (e.g. Lindley et al., 1979; West, 1992b), in which a decision maker updates a prior distribution in light
of new information gained from expert opinions (the ‘data’ – we label this information set $H$). We approach generalized density combination within this framework.

Recent developments in economic and forecasting literature have expanded the scope of density combination techniques. Kapetanios et al. (2015) propose a method to combine a set of $J$ forecast densities $h_j(y)$ using outcome-dependent weights $a_j(y)$, resulting in a generalized linear pool $p(y|H)$:

$$p(y|H) = \sum_{j=1:J} a_j(y)h_j(y).$$  \hspace{1cm} (1)

The authors present their method as a generalization of the traditional linear opinion pool using constant weights $a_j$. With more flexible weights, Kapetanios et al. (2015) make substantial improvements over the traditional linear pool; Aastveit et al. (2015) and Pettenuzzo and Ravazzolo (2016) have also demonstrated empirical success with outcome-dependent density pooling. While these techniques make intuitive sense as a generalization and produce good empirical results, their use gives rise to the need for theoretical justification within the framework of Bayesian expert opinion synthesis.

Some Notation: Vectors are denoted using lowercase bold font; matrices are written in uppercase bold font. We will use index notation $1:J$ to represent the sequence $1, 2, \ldots, J$. Thus, a vector $(x_1, x_2, \ldots, x_J)$ is denoted $\mathbf{x}$ or sometimes $x_{1:J}$. To omit the $j$th element from $\mathbf{x}$, we write $\mathbf{x}_{-j} = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_J)$. The notation $N(\mu, \Sigma)$ is used to denote a normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$. The density function of $\mathbf{x}$ may be written, for example when $\mathbf{x} \sim N(\mu, \Sigma)$, as $N(\mathbf{x}|\mu, \Sigma)$. As boldface is used to denote vectors and matrices, $\mu$ represents a vector and $\mu$ a scalar.

2 Background and Foundations

A decision maker $D$ is interested in predicting an uncertain quantity $y$. The decision maker has some opinion of $y$, quantified through a subjective prior density $\pi(y)$. With the aim of becoming better informed, $D$ consults $J$ agents $A_j$ (models, analysts, other forecasters, etc.), each of which provides a forecast density $h_j(y)$, $j = 1:J$. How should $D$ consolidate this information, and ultimately update $\pi(y)$?

Bayesian methods allow for a straightforward approach, in theory. The decision maker defines a prior for $y$, receives information related to $y$, and updates to a posterior upon learning the information set $H = \{h_1(\cdot), \ldots, h_J(\cdot)\}$. Specifying a full prior joint distribution $p(y, H)$ is impractical however, and the theoretically straightforward approach cannot be easily implemented. West (1992a) and West and Crosse (1992) extend the related work of Genest and Schervish (1985) to show that under certain consistency conditions, $D$’s posterior density will have the form

$$\pi(y|H) = \int \alpha(y|x)h(x)dx \quad \text{where} \quad h(x) = \prod_{j=1:J} h_j(x_j).$$  \hspace{1cm} (2)
Here, \( \mathbf{x} \) is a vector of latent agent states, and \( \alpha(y|\mathbf{x}) \) is a conditional density function that synthesizes these states. Setting \( \alpha(y|\mathbf{x}) \) is not straightforward, and is the focus of current research. One requirement of \( \alpha(y|\mathbf{x}) \) is that it must be consistent with \( \mathcal{D} \)'s prior:

\[
\pi(y) = \int \alpha(y|\mathbf{x})m(\mathbf{x})d\mathbf{x} \quad \text{where} \quad m(\mathbf{x}) = \mathbb{E}[h(\mathbf{x})]. \tag{3}
\]

An immediate conclusion then is that if the agent-provided densities match \( \mathcal{D} \)'s expectations, i.e. if \( h(\mathbf{x}) = m(\mathbf{x}) \), then \( \mathcal{D} \)'s posterior and prior will be identical. Here, \( \mathcal{D} \) need only specify an expectation function \( m(\mathbf{x}) \) and the conditional density function \( \alpha(y|\mathbf{x}) \), rather than the entire joint distribution \( p(y, \mathcal{H}) \). Specifying these two density functions allows \( \mathcal{D} \) to incorporate any prior knowledge or opinions of the agents in terms of calibration, biases, relative expertise, and importantly, dependencies.

Updating through the substitution \( h(\mathbf{x}) \) for \( m(\mathbf{x}) \) is done in accordance with (Richard) Jeffrey’s rule rather than a full Bayesian model; in a sense, \( m(\mathbf{x}) \) gives \( \mathcal{D} \)'s prior for the latent agent states, and \( h(\mathbf{x}) \) describes the “true” distribution of the states, while the conditional density \( \alpha(y|\mathbf{x}) \) remains unchanged after learning \( h(\mathbf{x}) \). This method of updating is presented in Jeffrey (1990), and further discussed in Diaconis and Zabell (1982).

The interpretation of \( \alpha(y|\mathbf{x}) \) is a focus of discussion in West (1992a) and West and Crosse (1992). The first interpretation is that if each agent \( A_j \) were to provide a predictive density degenerate at a point \( x_j \), i.e. \( h_j(y) = \delta_{x_j}(y) \), then \( \mathcal{D} \)'s posterior is given by \( \pi(y|\mathcal{H}) = \alpha(y|\mathbf{x}) \). A second interpretation is that in order to sample \( y^* \sim \pi(y|\mathcal{H}) \), \( \mathcal{D} \) may first sample a vector \( \mathbf{x}^* \) from \( h(\mathbf{x}) \), and then sample \( y^* \sim \alpha(y|\mathbf{x}^*) \). The new methodological developments of this paper are based on an exploration of the theoretical framework of BPS using specific mixture distributional forms for \( \alpha(y|\mathbf{x}) \).

**Example 1.** McAlinn and West (2018) explore BPS examples in which \( \alpha(y|\mathbf{x})m(\mathbf{x}) \) is multivariate normal, or conditionally normal give relevant defining parameters. This example easily and intuitively allows for: (i) ranges of agent biases and mis-calibration, viewed through shifts in means and/or variances of implied conditional distributions of individual conditional distributions \( (x_j|y) \); and (ii) inter-dependencies, reflected in patterns of correlations and other aspects of conditional dependence among the \( x_j \) (West and Crosse, 1992; West and Harrison, 1997, Sect 16.3.2). This has recently been extended to multivariate forecast density synthesis and applied in a detailed macroeconomic study in McAlinn et al. (2017).

In general, it is not a requirement that the agent densities \( h_j(\cdot) \) are predictive densities for \( y \). These could instead be densities for values related to \( y \), and \( \mathcal{D} \) synthesizes this related information using \( \alpha(y|\mathbf{x}) \). For example, suppose \( y \) is tomorrow afternoon’s closing price of the stock for a certain company, and \( \mathcal{D} \) has available a density forecast \( h(\cdot) \) for that company’s quarterly earnings, which will be announced tomorrow sometime before the close. The general framework of Bayesian Predictive Synthesis allows a decision maker to incorporate this information into an updated forecast for the stock index, without specifying an entire joint distribution over the closing price and the earnings announcement. In this paper however, with an aim to justify the generalized linear
pool of Kapetanios et al. (2015), we assume each agent forecasts $y$ directly. We explore certain mixture forms of $\alpha(y|x)$, inspired by the interpretation in which each agent provides an ‘oracle’ prediction $\delta_x(y)$, and focus on the inclusion of agent dependencies and their evolution over time.

3 Mixture Modeling BPS: A Single Agent

We begin our investigation of mixture modeling BPS by examining the case of a single agent, i.e. $J = 1$.

3.1 Synthesis Function

With $J = 1$ so that $x = x$ is univariate, suppose the synthesis function has the form

$$\alpha(y|x) = (1 - q\alpha(x))\pi_0(y) + q\alpha(x)\delta_x(y)$$

where $q$ is a probability, $\pi_0(y)$ a base forecast p.d.f. set by $\mathcal{D}$ and $\alpha(x)$ a non-negative weight function such that $0 < q\alpha(x) < 1$ for all $x$. If the single agent $\mathcal{A}$ provides a point forecast degenerate at $x$, $\mathcal{D}$'s new predictive density would be a mixture between the base p.d.f. $\pi_0(y)$ and the point forecast density $\delta_x(y)$, with the weights in this mixture depending on the value of $x$. The probability $q$ is essentially a starting weight that is adjusted by $\alpha(x)$ depending on $\mathcal{D}$’s view of the agent and the forecast value.

Note that $\pi_0(y)$ is not in general the same as $\mathcal{D}$’s prior density; forcing $\pi_0(y) = \pi(y)$ is a further restriction with some unfavorable implications. With $\mathcal{D}$’s prior on the latent agent state given by $m(x) = \mathbb{E}[h(x)]$, the mixture form of $\alpha(y|x)$ implies that $\mathcal{D}$’s prior is, as in eqn. (3),

$$\pi(y) = (1 - qc)\pi_0(y) + qcp(y)$$

with a p.d.f. $p(y) = \alpha(y)m(y)/c$ and a constant $c = \int_y \alpha(y)m(y)dy$ implying that $0 < qc < 1$. Similarly, $\mathcal{D}$’s posterior after learning the forecast density $h(y)$ is given by

$$\pi(y|\mathcal{H}) = (1 - qc^\mathcal{H})\pi_0(y) + qc^\mathcal{H}p(y|\mathcal{H})$$

where $c^\mathcal{H} = \int_y \alpha(y)h(y)dy$ and $p(y|\mathcal{H}) = \alpha(y)h(y)/c^\mathcal{H}$ is a reweighted $h(y)$.

Example 2. To help set ideas, consider a simple example with $\alpha(x) = 1$. The synthesis function reduces to $\alpha(y|x) = (1 - q)\pi_0(y) + q\delta_x(y)$. The prior density is $\pi(y) = (1 - q)\pi_0(y) + qm(y)$, a simple mixture between the base density and the expectation. The posterior density, after $\mathcal{D}$ learns $h(y)$, is $\pi(y|\mathcal{H}) = (1 - q)\pi_0(y) + qh(y)$.

Example 3. Take an example with $\pi_0(y) = N(y|0,1)$ and $m(x) = N(x|\mu, \sigma^2)$ so that $\mathcal{D}$ expects the agent to generate forecasts that, relative to his or her own base density, are location-biased by a factor $\mu$ and scale-biased through $\sigma^2$. One way to incorporate
outcome-dependent weights is to define $\alpha(x)$ proportional to a normal density centered at $\mu$:

$$\alpha(x) = \exp\{- (x - \mu)^2 / (2r\sigma^2)\}$$

so that the weight on $\delta_x(y)$ is decreased for values of $x$ far from the expectation $\mu$. This allows $D$ to ignore forecasts that are considered unreasonable, with the rate of decrease controlled by the tuning parameter $r$. Suppose the agent now presents $h(y) = N(y|f, s)$ for some point forecast $f$ and variance $s$. The effect of $\alpha(y)$ on $h(y)$ is to downweight the portions of $h(y)$ that are further from $\mu$, resulting in the reweighted agent density $p(y|\mathcal{H})$:

$$p(y|\mathcal{H}) = N(y|w_1\mu + w_2f, w_1w_2(r\sigma^2 + s))$$

where $w_1 = s/(r\sigma^2 + s)$ and $w_2 = r\sigma^2/(r\sigma^2 + s)$. So $p(y|\mathcal{H})$ is a compromise between the forecast that $D$ expects and the forecast that $A$ provides. Finally, the weight on $p(y|\mathcal{H})$ in the mixture form is

$$q_{c\mathcal{H}} = q\sqrt{r\sigma^2/(r\sigma^2 + s)} \exp\{- (f - \mu)^2 / (2(r\sigma^2 + s))\},$$

further emphasizing how less weight is given to the adjusted agent density as $f$ moves away from $\mu$. Figure 1 demonstrates the prior to posterior update for different agent forecasts.

Figure 1: Posterior updates under different agent forecasts. In both cases, the base density $\pi_0(y)$ and the expected density $m(y)$ are standard normal. In this example, we set $r = 1$ and $q = 0.5$. Figure 1a displays the results when the agent provides a $N(1, 0.1)$ forecast, and figure 1b a $N(2, 0.1)$ forecast. In each case, the agent forecast is relatively precise. When $f = 1$ is near $\mu = 0$, the decision maker takes this as a good sign and places relatively more weight on it. However, when $f = 2$, the agent forecast is viewed as “unreasonable,” and is given less weight despite the high certainty.
Example 4. Alternatively, the decision maker may prefer to give less weight to forecasts near $\mu$, effectively downweighting what $\mathcal{D}$ “already knows.” An $\alpha(\cdot)$ function that provides this behavior takes a form inspired by the Gaussian well distribution (Manolopoulou et al., 2012):

$$\alpha(x) = 1 - d \cdot \exp\{-\frac{(x - \mu)^2}{2r\sigma^2}\}.$$  \hfill (10)

This weight function increases toward 1 for values further from $\mu$, so that the weight on $\delta_x(y)$ increases toward an upper bound of $q$ for unexpected forecasts, with the rate of increase now controlled by the tuning parameter $r$. The second tuning parameter $d \in [0, 1]$ controls how much weight is discounted for forecasts near $\mu$. When $d = 0$, there is no discount; and when $d = 1$, forecast densities are disregarded at $y = \mu$. In fact, when $d = 1$, if the agent were to provide a point forecast degenerate at $\mu$, $\mathcal{D}$’s posterior would simply be $\pi_0(y)$; this also provides an intuitive interpretation for $\pi_0(y)$. Note the difference from the case in which the agent provides the normal density $h(x) = m(x) = N(x|\mu, \sigma^2)$, which would result in $\mathcal{D}$’s posterior matching the prior $\pi(y)$, as given in eqn. (5). We omit further details of this example simply to emphasize that there is no “correct” form for $\alpha(x)$, and that its formulation is up to the decision maker to specify.

3.2 Calibration

Up to this point, we have implicitly assumed a lack of bias in the agent forecast; i.e. if $\mathcal{A}$ forecasts, $y = x$, $\mathcal{D}$ places some amount of weight on $\delta_x(y)$. However, the forecaster may be viewed to be miscalibrated, either in terms of location, uncertainty, or both. One way to adjust for this is to calibrate forecasts before placing them into BPS; another is to adjust the synthesis function to account for bias directly. Suppose $\mathcal{D}$ views $\mathcal{A}$ to have a bias of $\beta$; then, rather than place weight on $\delta_x(y)$, $\mathcal{D}$ can simply place weight on $\delta_{x-\beta}(y)$. Further, suppose that $\mathcal{D}$ believes $\mathcal{A}$ provides forecasts that are overly certain, and would also like to place weight on values near $x - \beta$. To do so, we can generalize the point density $\delta_{x-\beta}(y)$ to a normal density $N(y|x - \beta, v)$, controlling the variation around $x - b$ with the bandwidth parameter $v$. Then the synthesis function generalizes to

$$\alpha(y|x) = (1 - q\alpha(x))\pi_0(y) + q\alpha(x)N(y|x - \beta, v).$$  \hfill (11)

Note that our earlier synthesis function in eqn. (4) is a specific version of eqn. (11) with $\beta = 0$ and $v \to 0$. We do not investigate here the properties of replacing the Dirac delta function with a normal kernel; however, we do later describe a methodology for incorporating the bias parameter.

4 Mixture Modeling: Multiple Agents

We now examine the more common case in which $\mathcal{D}$ has a larger set of forecasts or models to combine, i.e. $J > 1$. 
4.1 Synthesis Function

We write \( \alpha(y|x) \) as a mixture of point masses \( \delta_{x_j}(y) \) with a base prediction \( \pi_0(y) \), where the mixture weights \( q_j \alpha_j(x) \) are functions of each of the latent agent states given in \( x \):

\[
\alpha(y|x) = \alpha_0(x) \pi_0(y) + \sum_{j=1:J} q_j \alpha_j(x) \delta_{x_j}(y)
\]  

(12)

where

\[
\alpha_0(x) = 1 - \sum_{j=1:J} q_j \alpha_j(x).
\]

The weight \( q_j \alpha_j(x) \) on an individual agent’s forecast is a function of the set of every agent’s forecast. This is a crucial piece of the mixture that allows \( D \) to incorporate dependencies between forecasters.

Our construction leads to a new interpretation of standard Bayesian model averaging, and justifies constant or data-dependent weighting in linear pooling of densities. In particular, it provides a formal theoretical basis for outcome-dependent density pooling as in recently successful empirical methods (e.g. Kapetanios et al., 2015; Pettenuzzo and Ravazzolo, 2016), with practical import in that BPS allows for the integration of information about agent inter-dependencies that are neglected or ignored by other approaches.

**Example 5.** Suppose \( D \) takes a simplified view with \( \alpha_j(x) = 1 \) for all \( j > 0 \). Then

\[
\alpha(y|x) = q_0 \pi_0(y) + \sum_{j=1:J} q_j \delta_{x_j}(y), \quad q_0 = 1 - \sum_{j=1:J} q_j
\]

produces a posterior forecast density

\[
\pi(y|H) = q_0 \pi_0(y) + \sum_{j=1:J} q_j h_j(y).
\]

In the special case that \( \alpha_j(x) = 1 \) and the \( q_{1:J} \) vector sums to 1 (so that \( q_0 = 0 \)), this is the traditional linear pool, in which a constant weight is given to each of the \( J \) agent-provided densities.

For general \( \alpha(x) \), \( D \)’s posterior will have the form

\[
\pi(y|H) = a_0(y) \pi_0(y) + \sum_{j=1:J} a_j(y) h_j(y)
\]  

(13)

where

\[
a_j(y) = \begin{cases} 
\int \alpha_j(y, x_{-j}) \prod_{i \neq j} h_i(x_i) dx_i & j > 0 \\
1 - \sum_{j=1:J} a_j(y) & j = 0
\end{cases}
\]  

(14)
using the slight abuse of notation in \( \alpha_j(y, \mathbf{x}_{-j}) \) to denote that \( y \) has taken the place of \( x_j \) in \( \alpha_j(\mathbf{x}) \). Thus the general form given in eqn. (12) justifies the outcome-dependent weights put forth in Kapetanios et al. (2015).

We have yet to define the weight functions \( \alpha_j(\mathbf{x}) \), and must do so for each agent \( A_j \), for all possible \( \mathbf{x} \). We note that the mixture density in eqn. (12) requires \( 0 \leq q_j \alpha_j(\mathbf{x}) \leq 1 \) for all \( \mathbf{x} \), and \( \sum_{j=1:J} q_j \alpha_j(\mathbf{x}) \leq 1 \). These functions will define the weights placed on each \( A_j \)’s hypothetical point forecast as a function of every agent’s point forecast.

**Example 6.** In setting the weight function, the general way to phrase the question is “How much weight should be placed on \( \delta_{x_j}(y) \), given the entire vector \( \mathbf{x}? \)” A more useful phrasing is to ask “For a given set of \( \mathbf{x}_{-j} \), how much weight should be placed on \( \delta_{x_j}(y) \), as a function of \( x_j \)?” We may wish to ignore a value that disagrees with the others, i.e. it falls outside of a “consensus” of the remaining \( J - 1 \) agents. At the same time, we may wish to decrease the weight on the consensus if we view the agents to be highly dependent, viewing their agreement more as a sign of herding than anything else. Intuitively, we wish to decrease the weight on \( A_j \) when he or she provides little additional information, so as to account for redundancy. One way to do this is to decrease the weight for \( A_j \)’s density around \( \mathbb{E}[y|\mathbf{x}_{-j}] \), which is defined through the density \( m(\mathbf{x}) \). If \( m(\mathbf{x}) \) is specified as a multivariate normal density (given all relevant parameters), we are able to set \( \alpha_j(\mathbf{x}) \) using the kernel from the conditional normal density \( m_{j-1}(x_j|\mathbf{x}_{-j}) \).

Suppose we specify \( m(\mathbf{x}) = N(\mathbf{x} | \mu, \Sigma) \). One way to set \( \alpha_j(\mathbf{x}) \) generalizes the examples from the case of a single agent to allow for both of the above ideas:

\[
\alpha_j(\mathbf{x}) = \exp\left\{ -e_j^2/(2r_1\delta_j) \right\} - d \exp\left\{ -e_j^2/(2r_2\delta_j) \right\}
= \exp\left\{ -e_j^2/(2r_1\delta_j) \right\} [1 - d \exp\left\{ -e_j^2/(2r_2\delta_j) \right\}] (15a)
\]

where \( e_j = x_j - \mu_j - \gamma_j'(\mathbf{x}_{-j} - \mu_{-j}) \), \( r_3 = r_1r_2/(r_1 - r_2) \), and \( \delta_j \) and \( \gamma_j \) represent the conditional variance and regression vector implied by \( \Sigma \). The first term accounts for the consensus by discounting \( x_j \) far from the conditional expectation; the second term accounts for herding behavior by discounting forecasts near the conditional expectation. The decision maker’s views of “near” and “far” are controlled by the specification of \( r_1 \) and \( r_2 \), while the depth of the herding discount is controlled by \( d \). Note that the pure “consensus” formulation is achieved by setting \( d = 0 \), while the pure “herding” formulation is achieved by allowing \( r_1 \to \infty \).

The interpretation of the parameters is best understood by considering the product form in eq. (15b). The first term decreases weight far from weight the conditional expectation (looking for a consensus), while the second decreases weight near the conditional expectation (accounting for herding). In each case setting \( e_j \) to a number \( n \) of conditional standard deviations \( \delta_j \) implies \( \exp\left\{ -e_j^2/(2r\delta_j) \right\} = \exp\left\{ -n^2/(2r) \right\} \) for either \( r \); setting this to a specific weight \( w \) implies \( r = -n^2/(2 \log w) \) and provides the interpretation of the two terms in the product. If \( D \) sets \( r_1 = 18.0337 \), forecasts are discounted by half when they fall 5 conditional standard deviations from the conditional mean. If \( D \) sets \( r_3 = 0.180337 \), forecasts are given weight \( 1 - d \) when they fall at exactly the
conditional expectation and \( 1 - d/2 \) when they are 0.5 conditional standard deviations from the conditional mean. Setting these parameters largely depends on the decision maker and how much (and what type) of an effect he or she desires from the varying weights.

Due to its generality, we focus the remainder of our investigation on the formulation of \( \alpha_j(x) \) given in Equation 15a. Figure 2 illustrates \( D \)'s updated forecasts upon receiving similar forecast densities for varying expected levels of dependency among agent forecasts. The proposed weight functions \( \alpha_j(x) \) have the intuitive and desired effect: when \( D \) expects the agents to agree, less weight is placed on similar agent forecasts. When they are expected to disagree, the forecasts receive relatively more weight. When agent states are viewed as uncorrelated, the resulting posterior falls between the two relative extremes.

![Figure 2: Prior to posterior updates with similar agent forecast densities: a \( N(2,1) \) density and a \( N(2.5,1) \) density. In each case, the expectation is bivariate normal with standard margins, but the correlation varies among \{0.7, 0, -0.7\}. As is desired, less weight is put on similar agent forecast densities when \( D \) expects them to agree, and the weight is increased as the set of forecasts becomes more “unexpected.” This example sets \( r_1 \approx 72.13, r_2 \approx 2.89, \) and \( d = 1. \)]
4.2 Calibration with Multiple Agents

As in the single forecaster case, we can generalize the synthesis function to account for agent miscalibration. The generalized equation for $J$ forecasters is

$$\alpha(y|x) = \alpha_0(x)\pi_0(y) + \sum_{j=1:J} q_j \alpha_j(x)N(y|x_j - \beta_j, v_j),$$

(16)

which places weight on $N(y|x_j - \beta_j, v_j)$ when $A_j$ forecasts $y = x_j$. Again, we do not pursue the use of a normal kernel with $v_j > 0$, but in the next section we develop the use of the bias term $\beta_j$.

5 Time Series Context and Bayesian Computation

In many applications, $D$ receives repeated forecasts from the same set of agents. An obvious example is combining a set of models that each repeatedly provide one-step ahead forecasts—each model provides a new forecast each day, and $D$ should update $(\mu, \Sigma)$ to reflect evolving perceptions of bias and dependence between models. To do so, we first define $\mu = f_0 + \beta$, where $f_0$ is the point estimate of the base density $\pi_0(y)$, indicating that $D$ expects $A_j$ to have a bias of $\beta_j$ relative to $f$. We further expect these parameters to change over time, and so we generalize to $(\beta_t, \Sigma_t)$. The time-varying nature of the problem drives the change of variables from $\mu_t$ to $\beta_t$—while we expect $\mu_t$ to vary with the data, it is more reasonable to assume that $\beta_t$ will be relatively stable over time, or at the very least independent of $f_t$.

Additionally, $D$ may desire to change the $q_j$ weights over time, so we allow the vector $q_t$ to follow a dynamic Dirichlet distribution. Dynamic parameters are implemented with the inclusion of discount factors to increase uncertainty in our distributions moving from time $t$ to time $t + 1$. The tuning parameters $r_1, r_2$ and $d$ are set by the decision maker at the beginning of the analysis.

5.1 Inference and Variational Bayes

We construct a Gibbs sampler at each time step in order to sample from the full conditional distribution the relevant parameters. Much of the sampler is done using accept/reject sampling; notably, however, there are some cases that allow for exact sampling. Details for the sampler can be found in Appendix B.

The primary parameters of interest in our model are $\beta_t$ and $\Sigma_t$, and we assign to these parameters a normal inverse-Wishart (NIW) prior distributions. Then if $D$ expects $m(x) = N(x|f_0 + \beta_t, \Sigma_t)$ conditional on $\beta_t$ and $\Sigma_t$, $D$’s marginal expectation is a multivariate $t$ density. While a conjugate update is desirable in the time series context, the time $t$ posterior densities for $\beta_t$ and $\Sigma_t$ will not be exactly NIW. However, because the data does not directly inform on $(\mu_t, \Sigma_t)$, the posterior will be fairly similar to the prior. For these reasons, we choose to approximate a NIW distribution to the posterior samples of $(\beta_t, \Sigma_t|y)$ using Variational Bayes techniques, similar to the approach in...
Gruber and West (2016). Specifically, after learning each $y_t$, we minimize the Kullback-Leibler divergence between the posterior samples and the approximating distributions. A similar approach is taken to approximate a Dirichlet distribution to the posterior samples of $q_t$. See Appendix C for details.

6 Time Series Example and Forward Filtering Forecasts

In this section we apply Mixture BPS to a series of one-step ahead forecasts from a set of dynamic linear models.

6.1 Data and Models

Our data consists of log-prices for the Euro currency, priced in US dollars, over the final 100 trading days of 2016, 8/15/2016 – 12/30/2016 (notably, this time period includes the U.S. presidential election, which caused some quick movements in the Euro). The base density results from a TVAR(1) model, while the agent densities in this case are provided by a TVAR(2), TVAR(5), and a linear growth DLM. Details of these models may be found in Prado and West (2010). At each time point, each model provides a one-step ahead forecast for the next day’s closing (log) price, and we dynamically combine these forecasts using BPS according to our views of the agents at that time. We compare our results from Bayesian Predictive Synthesis (BPS) to those from Bayesian Model Averaging (BMA) and an equally-weighted linear pool (POOL). The data and model point forecasts are shown in Figure 3 along with the BPS-combined point forecasts.

Figure 3: Euro log-price daily data, synthesized one step ahead point forecasts, and individual model point forecasts. The base density is a TVAR(1) model, and agent densities are TVAR(2), TVAR(5), and a linear growth DLM with low discount factors, allowing for quick adaptation to new levels and trends in the time series. Each model picks up on a varying amount of “momentum” in the daily data.
6.2 Priors and Tuning Parameters

We first consider the normal inverse Wishart prior for \((\beta_t, \Sigma_t)\), where the expected product of the time \(t\) agent densities is given by a normal \(N(x|f_0 + \beta_t, \Sigma_t)\) density. As the data do not directly inform on \((\beta_t, \Sigma_t)\), \(D\)’s subjective prior can be especially important. We begin with a somewhat vague inverse Wishart prior on \(\Sigma_t\), with \(n_0 = 15\) degrees of freedom and point estimate \(S_0\); \(S_0\) has diagonal elements equal to \(s_{0,1}\), the base density scale parameter, and pairwise correlations of 0.5 throughout. In terms of bias calibration, we assume no bias initially, so that \(b_{j0} = 0\) for all \(j\). We set \(c_0 = 1\), implying a prior variance for the latent states \(x\) given \(\Sigma\) alone that is double that of the conditional variance given both \(\Sigma\) and \(\beta\). In effect, larger values of \(c_t\) imply that we expect the latent agent states to be near each other, we just do not know their location—hence the inflated variance on the bias. The discount factors are fairly high, indicating fairly stable bias and dependence, and also controlling for the amount of uncertainty in forecasts. We set the discount factor on \(\Sigma_t\) to 0.99, and that on \((\beta_t|\Sigma_t)\) to 0.975. The base mixture weights \(q_t\) are initially assigned a flat Dirichlet prior, with each Dirichlet parameter set to \(u_{j0} = 1\). The BPS-specific tuning parameters are set at \(r_1 = 18.0337\), \(d = 0.5\) and \(r_3 = 0.180337\) (see Example 6 of Section 4.1 for interpretation of these particular values).

6.3 Discussion and Interpretation

Our main interest is in learning about bias and dependence among agents over time. In this case, we expect the agents to perform fairly similarly for most time periods, with time-varying biases, correlations, and scales. Figure 4 displays the correlations implied by the matrix \(S_t\) at each time point, representing the pairwise dependence between agents. Despite the high discount factor and the fact that the likelihood does not inform directly on \(\beta\) and \(\Sigma\), we see some learning over time. The correlations increase from the initial value of 0.5, but begin to break down in the weeks leading up to the U.S. presidential election. There is a second turning point after the election in which the pairwise correlations begin to rise once more.

Also of interest is the vector of bias parameters \(\beta_t\). While the use of TVAR and a highly adaptable linear growth DLM prevents any obvious bias, we still see some filtering of point forecasts, evident in Figure 5. Despite the fact that the likelihood does not inform directly on \(\beta\), we see an ability to filter the apparent bias in agent-provided densities.

A final point of interest is the trajectory of the weights \(q_t\). The dynamic means of the Dirichlet distribution are shown in Figure 6a and contrast with the BMA weights in Figure 6b. As is typical of BMA, the combination weights converge on a single model, while BPS decreases the weights on some but does not “pick” a single agent. Importantly, the implication of BPS is not that there is a single correct model; it simply adjusts the Dirichlet distribution in order to aid with a subjective decision maker’s forecasts.

To assess model performance, we compare our results from BPS to two standard methods of model combination: Bayesian model averaging (BMA) and the equally-weighted linear pool. For BMA, each of the four models is initially given equal weight,
Figure 4: Pairwise correlations implied by the matrix $S_t$ (the point estimate of $\Sigma$) at each time point, representing the pairwise dependence between agents. Despite the high discount factor of 0.99 and the fact that the likelihood does not inform directly on $\Sigma$, we see some learning over time. The correlations increase from the initial value of 0.5, but begin to break down in the weeks leading up to the U.S. presidential election. There is a second turning point after the election in which the pairwise correlations begin to rise once more.

Figure 5: Difference between agent point forecasts $f_{ij}$ and data $y_t$, compared with the bias mean $b_t$. Despite the fact that the likelihood does not inform directly on $\beta$, we see an ability to filter the apparent bias in agent-provided densities.

so that the weight at a given time $t$ is proportional to the likelihood of the data under each model up to that time point. For the equally-weighted linear pool, the combined forecast density is simply the mean of the base density and the three agent-provided
(a) Dirichlet means, $\mathbb{E}[q_j]$ for $j = 0 : 3$. 

(b) BMA density combination weights.

Figure 6: Base combination weights for BPS compared to combination weights for Bayesian Model Averaging (BMA). Figure 6a displays the time $t$ mean components of the Dirichlet distribution over $q_t$, while Figure 6b displays the trajectory of the weights resulting from BMA. As is typical of BMA, the combination weights converge on a single model, while BPS decreases the weights on some but does not “pick” a single agent.

densities. We see that BPS performs comparably to both standard methods in terms of both root mean square error (RMSE) and predictive log score.

| Method | RMSE  | Log Score |
|--------|-------|-----------|
| BPS    | 1.000 | 1.000     |
| BMA    | 0.986 | 1.010     |
| POOL   | 0.998 | 1.001     |
| TVAR(1)| 1.002 | 1.006     |
| TVAR(2)| 0.979 | 1.014     |
| TVAR(5)| 1.066 | 0.996     |
| DLM    | 1.350 | 0.890     |

Table 1: Model combination results, normalized to BPS. The root mean square error (RMSE) and mean log score of the data for BPS are comparable to those of standard combination techniques such as BMA or an equally weighted pool using all four models.
7 Summary Comments

This paper provides an investigation into the meanings and implications of outcome-dependent mixture weighting. By placing this decision process into the Bayesian Predictive Synthesis framework, we are able to define a logically coherent method for model combination, allowing us to clearly delineate an intuitive process for the decision maker’s thought process. The BPS framework, in addition to allowing a decision maker to incorporate beliefs about specific forecasting agents, allows practitioners to understand the assumptions of commonly used methods for forecast combination. Placing this framework into a time series setting allows the decision maker to sequentially infer biases and dependencies among different agents. Of note is that setting the conditional synthesis density $\alpha(y|x)$ is the task of the decision maker. The forms we highlight here were chosen for their intuitive appeal and their computational properties; however, we make no claim to have found an “ideal” solution. Indeed, different subjective Bayesian decision makers will have different “ideal” forms.

References

Aastveit, K. A., Gerdrup, K. R., Jore, A. S., and Thorsrud, L. A. (2014). “Nowcasting GDP in real time: A density combination approach.” *Journal of Business & Economic Statistics*, 32: 48–68. 1

Aastveit, K. A., Ravazzolo, F., and van Dijk, H. K. (2015). “Combined density Nowcasting in an uncertain economic environment.” *Journal of Business & Economic Statistics*, -. To appear. 1, 2

Amisano, G. G. and Giacomini, R. (2007). “Comparing density forecasts via weighted likelihood ratio tests.” *Journal of Business & Economic Statistics*, 25: 177–190. 1

Billio, M., Casarin, R., Ravazzolo, F., and van Dijk, H. K. (2012). “Combination schemes for turning point predictions.” *Quarterly Review of Finance and Economics*, 52: 402–412. 1

— (2013). “Time-varying combinations of predictive densities using nonlinear filtering.” *Journal of Econometrics*, 177: 213–232. 1

Clemen, R. T. (1989). “Combining forecasts: A review and annotated bibliography.” *International Journal of Forecasting*, 5: 559–583. 1

Clemen, R. T. and Winkler, R. L. (1999). “Combining probability distributions from experts in risk analysis.” *Risk Analysis*, 19: 187–203. 1

— (2007). “Aggregating probability distributions.” In W. Edwards, R. M. and von Winterfeldt, D. (eds.), *Advances in Decision Analysis: From Foundations to Applications*, chapter 9, 154–176. Cambridge University Press. 1

Diaconis, P. and Zabell, S. L. (1982). “Updating subjective probability.” *Journal of the American Statistical Association*, 77: 822–830. 3

Genest, C. and Schervish, M. J. (1985). “Modelling expert judgements for Bayesian updating.” *Annals of Statistics*, 13: 1198–1212. 2
Geweke, J. and Amisano, G. G. (2012). “Prediction with misspecified models.” *The American Economic Review*, 102: 482–486. 1

Geweke, J. F. and Amisano, G. G. (2011). “Optimal prediction pools.”* Journal of Econometrics*, 164: 130–141. 1

Gruber, L. F. and West, M. (2016). “GPU-accelerated Bayesian learning in simultaneous graphical dynamic linear models.”* Bayesian Analysis*, 11: 125–149. 11, 21

Hall, S. G. and Mitchell, J. (2007). “Combining density forecasts.”* International Journal of Forecasting*, 23: 1–13. 1

Hoogerheide, L., Kleijn, R., Ravazzolo, F., Van Dijk, H. K., and Verbeek, M. (2010). “Forecast accuracy and economic gains from Bayesian model averaging using time-varying weights.”* Journal of Forecasting*, 29: 251–269. 1

Jeffrey, R. C. (1990). *The Logic of Decision*. University of Chicago Press, 2nd edition. 3

Kapetanios, G., Mitchell, J., Price, S., and Fawcett, N. (2015). “Generalised density forecast combinations.”* Journal of Econometrics*, 188: 150–165. 1, 2, 4, 7, 8

Kascha, C. and Ravazzolo, F. (2010). “Combining inflation density forecasts.”* Journal of Forecasting*, 29: 231–250. 1

Lindley, D. V., Tversky, A., and Brown, R. V. (1979). “On the reconciliation of probability assessments.”* Journal of the Royal Statistical Society (Series A: General)*, 142: 146–180. 1

Manolopoulou, I., Kepler, T. B., and Merl, D. M. (2012). “Mixtures of Gaussian wells: theory, computation, and application.”* Comput. Stat. Data Anal.*, 56(12): 3809–3820. 6

McAlinn, K., Aastveit, K. A., Nakajima, J., and West, M. (2017). “Multivariate Bayesian predictive synthesis in macrroeconomic forecasting.” Submitted for publication. ArXiv:1711.01667. 3

McAlinn, K. and West, M. (2018). “Dynamic Bayesian predictive synthesis in time series forecasting.”* Journal of Econometrics*, in press. ArXiv:1601.07463. 3

Pettenuzzo, D. and Ravazzolo, F. (2016). “Optimal portfolio choice under decision-based model combinations.”* Journal of Applied Econometrics*, -: –. Forthcoming. 1, 2, 7

Prado, R. and West, M. (2010). *Time Series: Modelling, Computation & Inference*. Chapman & Hall/CRC Press.

URL  http://www.stat.duke.edu/~mw/PradoWestBook/  11

Rufo, M. J., Martín, J., and Prez, C. J. (2012). “Log-linear pool to combine prior distributions: A suggestion for a calibration-based approach.”* Bayesian Anal.*, 7(2): 411–438. 1

Timmermann, A. (2004). “Forecast combinations.” In G. Elliott, C. G. and Timmer-
mann, A. (eds.), *Handbook of Economic Forecasting*, volume 1, chapter 4, 135–196. North Holland.

West, M. (1992a). “Modelling agent forecast distributions.” *Journal of the Royal Statistical Society (Ser. B)*, 54: 553–567.

— (1992b). “Modelling with mixtures (with discussion).” In Bernardo, J. M., Berger, J. O., Dawid, A. P., and Smith, A. F. M. (eds.), *Bayesian Statistics 4*, 503–524. Oxford University Press.

West, M. and Crosse, J. (1992). “Modelling of probabilistic agent opinion.” *Journal of the Royal Statistical Society (Ser. B)*, 54: 285–299.

West, M. and Harrison, P. J. (1997). *Bayesian Forecasting & Dynamic Models*. Springer Verlag, 2nd edition.

**About the Authors**

Matt Johnson is a PhD student in the Department of Statistical Science at Duke University. His research focuses on Bayesian methods for predictive forecast combination (Bayesian Predictive Synthesis), and other interests include time series, stochastic processes, dynamic models, forecasting and Bayesian statistics.

Mike West is the Arts & Sciences Professor of Statistical Science in the Department of Statistical Science at Duke University. Mike's research and teaching activities in Bayesian statistics involve stochastic modelling in higher-dimensional problems; theory and methods of dynamic models in time series analysis, multivariate analysis, latent structure, high-dimensional inference and computation, stochastic computational methods, parallel/GPU computing and computation, among others.

**Acknowledgments**

We are grateful for useful discussions with colleagues that helped with perspective in developing this work, including Kenichiro McAllinn (Duke University) and the organizers and participants of the 5th European Seminar on Bayesian Econometrics (ESOBE) held at the Study Center Gerzensee, Switzerland, October 29-30th 2015.
Bayesian Predictive Synthesis for Probabilistic Forecast Calibration and Combination
SUPPLEMENT

Appendix A: Derivations of $\alpha(x)$ results

Our general form of $\alpha_j(x)$ as defined in eqn. (15a) is given by

$$\alpha_j(x) \propto \exp\{-e_j^2/(2r_1\delta_j)\} - d \exp\{-e_j^2/(2r_2\delta_j)\}$$  \hspace{1cm} (17)

where $e_j = (x_j - \mu_j - \gamma_j(x_{-j} - \mu_{-j})), and \delta_j$ and $\gamma_j$ represent the conditional variance and regression vector implied by $\Sigma$. In this section, we are concerned with the properties of $\alpha_j(x)$ as a function of $x_j$ for a given $x_{-j}$. Then for the sake of readability and clarity, we will without loss of generality allow $J = 1$, so there is a single agent. Then define $\alpha(x)$ as

$$\alpha(x) = \exp\left\{-\frac{(x - \mu)^2}{2r_1\sigma^2}\right\} - d \exp\left\{-\frac{(x - \mu)^2}{2r_2\sigma^2}\right\}.$$  \hspace{1cm} (18)

Note that in order to guarantee $\alpha(x) > 0$ for all $x$, we must have $r_1 > r_2$. Of interest is the maximum value of $\alpha(x)$, as well as the $x$ values that achieve the maximum. Equating to zero the derivative with respect to $x$ gives

$$\frac{\partial \alpha(x)}{\partial x} = - \exp\left\{-\frac{(x - \mu)^2}{2r_1\sigma^2}\right\} \cdot \frac{x - \mu}{r_1\sigma^2} + d \exp\left\{-\frac{(x - \mu)^2}{2r_2\sigma^2}\right\} \cdot \frac{x - \mu}{r_2\sigma^2} = 0.$$  \hspace{1cm} (19)

The first solution is when $x = \mu$. This critical point gives the local minimum (the bottom of the well), or the global maximum depending on the values of $r_1$, $r_2$, and $d$ (if there is no well, see below). When $x \neq \mu$, the above equation reduces to

$$\log(dr_1/r_2) = \frac{(x - \mu)^2(r_1 - r_2)}{2r_1r_2\delta}.$$  \hspace{1cm} (20)

In order to satisfy the positivity of $\alpha(x)$, it was necessary to have $r_1 > r_2$, meaning the RHS of the above is necessarily positive (keeping in mind that we have assumed $x \neq \mu$ at this point). This implies that if there is a critical point other than $x = \mu$, then it must be true that $\log(dr_1/r_2) > 0$, meaning $dr_1 > r_2$. That is, in order to have the desired bimodal Gaussian well shape, the tuning parameters must be set such that $dr_1 > r_2$. In this case, the solutions to the above are

$$x = \mu \pm \sqrt{\frac{2r_1r_2\delta}{r_1 - r_2} \log(dr_1/r_2)},$$  \hspace{1cm} (21)

at which points the above expression takes the maximum value

$$\max \alpha(x) = (dr_1/r_2)^{-\frac{r_2}{r_1 - r_2}}(1 - r_2/r_1),$$  \hspace{1cm} (22)

which depends only on $(r_1, r_2, d)$. Note that a full derivation includes taking a second derivative and checking for minimum compared to a maximum, but in the context of the problem, the results are clear, so we omit this portion.
Appendix B: Gibbs Sampler

In this section, we detail the Monte Carlo methods used to sample from the posterior distribution for \((q_t, x_t, \beta_t, \Sigma_t | y_t)\). For readability, we omit the \(t\) subscript, understanding that sampling takes place within a single point in time after observing \(y_t\).

Our Gibbs sampler has some complications due to the discrete nature of our synthesis mixture model (12). To aid in computation, we augment with a latent variable \(z \in (0 : J)\) that denotes the component of the mixture, with

\[
P(z = j | q, x, \beta, \Sigma) = \begin{cases} 
1 - \sum_{j=1}^{J} q_j \alpha_j(x, \beta, \Sigma) & j = 0 \\
q_j \alpha_j(x, \beta, \Sigma) & j > 0
\end{cases}
\]  
(23)

where we have made it explicit that \(\alpha_j(\cdot)\) depends on all three parameters, and conditional likelihood

\[
\alpha(y | q, x, z = j, \mu, \Sigma) = \begin{cases} 
\pi_0(y) & j = 0 \\
\delta_{x_j - \beta_j}(y) & j > 0.
\end{cases}
\]  
(24)

We see that \((y \perp \perp q x_{-j}, \beta_{-j}, \Sigma | x_j, z = j, \beta_j)\), so that for \(j > 0\), \(\alpha(y | q, x, z = j, \beta, \Sigma) = \alpha(y | x_j, z = j, \beta_j)\), while \(\alpha(y | q, x, z = 0, \beta, \Sigma) = \alpha(y | z = 0)\). This is partially evident from the construction of the directed graph of the vector parameters and the associated undirected graph:

The joint density is then

\[
p(q, x, y, z = j, \mu, \Sigma) = \alpha(y | x_j, z = j, \beta_j) P(z = j | q, x, \beta, \Sigma) p(q) h(x) p(\beta, \Sigma),
\]  
(25)

where \(h(x)\) is the product of agent forecast densities.

B.1 Sampling \(z\)

Directly sampling \(z\) from its full conditional distribution leads to a nonergodic chain--for a given \(y, \beta, \) and \(x\), if \(y = x_j - \beta_j\), then \(P(z = j | q, x, y, \beta, \Sigma) = 1\), and \(p(x | q, y, z = j, \beta, \Sigma)\) always sets \(x_j = y + \beta_j\). We avoid this issue by first sampling \((z | q, y, \beta, \Sigma)\) (with \(x\) marginalized out), and then \((x | y, z = j, \beta, \Sigma)\) as detailed below. We have

\[
p(z = j | q, y, \beta, \Sigma) \propto p(z = j | q, \beta, \Sigma)p(y | q, z = j, \beta, \Sigma).
\]  
(26)
We estimate each term of this product using Monte Carlo averages. The first is
\[ p(z = j|q, \beta, \Sigma) = \int P(z = j|q, x, \beta, \Sigma)h(x)dx \]
\[ = \begin{cases} 1 - \sum_{j=1:J} q_j \int \alpha_j(x, \beta, \Sigma)h(x)dx & j = 0 \\ q_j \int \alpha_j(x, \beta, \Sigma)h(x)dx & j > 0. \end{cases} \]

Then the first term results from the mean values of \( \alpha_j(x, \beta, \Sigma) \) under \( h(x) \). The second term takes a closed form:
\[ p(y|q, z = j, \beta, \Sigma) = \int \alpha(y|q, x, z = j, \beta, \Sigma)h(x)dx \]
\[ = \begin{cases} \pi_0(y) & j = 0 \\ b_j(y + \beta_j)dx_j & j > 0. \end{cases} \]

Normalizing these products gives the desired probabilities.

**B.2 Sampling \( x \)**

The full conditional density for \( (x|y, z = j, \beta, \Sigma) \) breaks down into two cases, depending on the value of \( z \). When \( z = j > 0 \), the likelihood for \( y \) depends on \( x_j \):
\[ p(x|q, y, z = j, \beta, \Sigma) \propto \delta_{x_j - \beta_j}(y) \cdot \alpha_j(x, \beta, \Sigma) \cdot h(x). \]
In this case, we have \( x_j = y + \beta_j \), and sample the remainder of the \( x \) vector via rejection sampling with acceptance rate \( \alpha_j(y, \beta_j, x_{-j}, \beta, \Sigma) \). When \( z = 0 \),
\[ p(x|y, z = 0, \beta, \Sigma) \propto [1 - \sum_{j=1:J} q_j \alpha_j(x, \beta, \Sigma)] \cdot h(x). \]
We again performing rejection sampling, this time with acceptance rate
\[ \alpha_0(x) = 1 - \sum_{j=1:J} q_j \alpha_j(x, \beta, \Sigma). \]

**B.3 Sampling \( (\beta, \Sigma) \)**

The full conditional density for \( (\beta, \Sigma) \) is given by
\[ p(\beta, \Sigma|q, x, y, z = j) \propto P(z = j|x, \beta, \Sigma)p(\beta, \Sigma). \]
Rejection sampling is straightforward. We again split the sampler into case \( z = j > 0 \) and \( z = 0 \). When \( z = j > 0 \), the acceptance probability is \( \alpha_j(x, \beta, \Sigma) \), and when \( z = 0 \), the acceptance probability is again
\[ \alpha_0(x) = 1 - \sum_{j=1:J} q_j \alpha_j(x, \beta, \Sigma). \]
The full conditional density for \((q|x, y, z = j, \beta, \Sigma)\) also breaks down into two cases. The only relevant terms are

\[ p(q|x, y, z, \beta, \Sigma) \propto q_j p(q). \]  

(34)

If \(p(q)\) is a Dirichlet density with parameters \((u_1, \ldots, u_J)\), this allows for a conjugate update and exact sampling with \(u_j \rightarrow u_j + 1\). When \(z = j > 0\),

\[ p(q|x, y, z, \beta, \Sigma) \propto \alpha_0(q, x, \beta, \Sigma)p(q). \]  

(35)

where

\[ \alpha_0(q, x, \beta, \Sigma) = 1 - \sum_{j=1}^{J} q_j \alpha_j(x, \beta, \Sigma). \]  

(36)

Rejection sampling is again used with acceptance probability \(\alpha_0(q, x, \beta, \Sigma)\).

### Appendix C: Variational Bayes

In the time series context with sequential forecasts, we require specific prior forms for \(\mu_t, \Sigma_t, \) and \(q_t\) at each time step \(t\), but the posterior from the previous step does take the desired form. We account for this by respectively fitting normal, inverse Wishart, and Dirichlet distributions to the posterior samples using Variational Bayes techniques. To do so, we find parameters that minimize the Kullback-Liebler (KL) divergence of the approximating distribution from that of the posterior samples. This method is similar to that of Gruber and West (2016), in which the authors fit a normal-inverse-gamma distribution to posterior samples at each time point by minimizing the KL divergence.

Speaking generally, we approximate a density \(g(\theta)\) to samples of \(\theta \sim p(\theta)\). We can write the divergence of \(g(\cdot)\) from \(p(\cdot)\) as

\[ k - \frac{1}{2} \int f(\theta)p(\theta)d\theta \]  

(37)

where \(k\) does not depend on \(\theta\) and \(f(\theta)\) is some function relating to the known probability density function \(g(\theta)\). Then we simply minimize the KL divergence by maximizing \(\mathbb{E}_{p}[f(\theta)]\), the expectation of \(f(\cdot)\) under \(p(\cdot)\). Analytical solutions are available for many of the optimal parameters; others are solved trivially using the Newton-Raphson method within an arbitrary tolerance.

### C.1 Normal Inverse Wishart Approximation

We write the joint NIW distribution such that \((\beta|\Sigma) \sim N(b, c\Sigma)\) and \(\Sigma \sim IW(n, S)\). Using this notation, the optimal parameters are given by

1. \(b = E[\Sigma]^{-1}E[\Sigma^{-1}\beta]\)
2. \( c = \mathbb{E}[(\beta - \mathbf{b})^{\prime} \Sigma^{-1}(\beta - \mathbf{b})]/J \);

3. \( n \) satisfies

\[
\mathbb{E}[\log(|\Sigma|)] + \log(|\mathbb{E}[\Sigma^{-1}]|) - J \log((n + J - 1)/2) + \sum_{j=1:J} \psi((n + j - 1)/2) = 0,
\]

where \( \psi(\cdot) \) denotes the digamma function \( \psi(x) = \Gamma'(x)/\Gamma(x) \);

4. \( S = \mathbb{E}[\Sigma^{-1}]^{-1}(n + J - 1)/n. \)

C.2 Dirichlet Approximation

For a \( \text{Dir}(u_1, \ldots, u_J) \) approximation to a set of samples of \( J \)-vector \( \mathbf{q} \) on the simplex, the parameters \( u_1, \ldots, u_J \) satisfy

\[
\psi\left( \sum_{j=1:J} u_j \right) + \psi(u_i) - \sum_{j=1:J} \mathbb{E}[\log(q_j)] = 0 \tag{38}
\]

where \( \psi(\cdot) \) again denotes the digamma function \( \psi(x) = \Gamma'(x)/\Gamma(x) \). An analytical solution is not available, but an approximate solution is again trivially solved using the multivariate Newton-Raphson method, solving the above equation within an arbitrary tolerance.