The Degree Sequence of a Scale-Free Random Graph Process with Hard Copying

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Abstract: In this paper we consider a simple model of random graph process with hard copying as follows: At each time step $t$, with probability $0 < \alpha \leq 1$ a new vertex $v_t$ is added and $m$ edges incident with $v_t$ are added in the manner of preferential attachment; or with probability $1 - \alpha$ an existing vertex is copied uniformly at random. In this way, while a vertex with large degree is copied, the number of added edges is its degree and thus the number of added edges is not upper bounded. We prove that, in the case of $\alpha$ being large enough, the model possesses a mean degree sequence as $d_k \sim Ck^{-(1+2\alpha)}$, where $d_k$ is the limit mean proportion of vertices of degree $k$.

AMS classification: 60K 35; 05C 80.

Key words and phrases: degree sequence; power law; hard copying; random graph process

*Supported in part by the Natural Science Foundation of China

†Supported in part by the Natural Science Foundation of China and MicroSoft Research Asia under grant 60633010
1 Introduction and the statement of the main result

Real-world networks such as economic companies, biological oscillators, social networks, and the World Wide Web (internet) etc. can be modeled by random complex graphs [7, 15, 16, 17, 19, 22]. By studying random complex graphs, various topological properties such as degree-distribution [6, 8, 12, 14], diameter [1, 3, 10], clustering [9, 18], stability [1, 5, 11] and spectral gap [2] of these real-world networks have been presented. One of the most basic properties of many real-world networks is concerned with the power law degree distributions. As indicated in [6], the emergence of the power law degree distributions should be a consequence of two generic mechanisms:

1. Evolution: new vertices and edges are added continuously, and

2. Preferential attachment: new vertices are preferentially attached to vertices that are already well connected.

The above mechanisms are referred to as BA mechanisms. Besides the original model proposed in [6], many other models with the BA mechanisms have been introduced and aimed to explain the underlying causes for the emergence of the power law degree distributions. This can be observed in ‘LCD model’ [10], the generalization of ‘LCD model’ due to Buckley and Osthus [8], the very general models defined by Copper and Frieze [13], Copper, Frieze and Vera [14] etc.

Copying is another mechanism that may be observed in real-world networks. The basic idea of copying comes from the fact that a new web page is often made by copying an old one. A kind of copying models was proposed in Kumar et al. [15] to explain the emergence of the degree power laws in the web graphs. These models are parameterized by a copy factor $\alpha \in (0, 1)$ and a constant out-degree $d \geq 1$. At each time step, one vertex $u$ is added and $d$ out-links are generated for $u$ as follows. First, an existing vertex $p$ is chosen uniformly at random; then with probability $1 - \alpha$ the $i^{th}$ out-link of $p$ is taken to be the $i^{th}$ out-link of $u$, and with probability $\alpha$ a vertex is chosen from the existing vertices uniformly at random to be the destination of the $i^{th}$ out-link of $u$. It is proved in [15] that the above copying models possess a power law degree sequence as $d_k \sim C k^{-(2-\alpha)/(1-\alpha)}$. 

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In this paper we will introduce and study a new copying model created by lazy copiers. Our copiers are so lazy that the only thing they want to do is copying. However, the copiers corresponding to the copying action discussed in [15] should be more clever and diligent: for the chosen vertex $p$, they have to distinguish which link be a original out-link of $p$ first and then decide whether or not to copy it.

Let’s consider the following random process $G_t$, $t = 2, 3, \ldots$. Assume that graph $G_t = (V_t, E_t)$ and $t = |V_t|$, $e_t = |E_t|$ (In order to simplify the statement and the proof of our main result, technically, we start our process at time step 2).

Time-Step 2: To begin the process, we start with $G_2$ consisting of vertices $v_1, v_2$ and $2m$ multi-edges between them.

Time-Steps $t \geq 3$:

- With probability $\alpha > 0$ we add a new vertex $v_t$ to $G_{t-1}$ and then add $m$ random edges incident with $v_t$. The $m$ random neighbors $w_1, w_2, \ldots, w_m$ are chosen independently and for any $1 \leq i \leq m$, $w \in V_{t-1}$,
  \[ P(w_i = w) = \frac{d_w(t - 1)}{2e_{t-1}}, \]  
  where $d_w(t - 1)$ denotes the degree of vertex $w$ in $G_{t-1}$. Thus neighbors are chosen by preferential attachment.

- With probability $1 - \alpha$ we generate vertex $v_t$ by copying a existing vertex $v_i$, $1 \leq i \leq t - 1$ from $V_{t-1}$ uniformly at random. Note that in this case, all neighbors of $v_t$ are those of the copied vertex $v_i$.

As defined above, our copying is executed in a direct and simple way, which is referred to as hard copying here. With hard copying, $e_t$ may increase nonlinearly, this makes bounding $e_t$ a rather hard problem.

Now, Let $D_k(t)$ be the number of vertices with degree $k \geq 0$ in $G_t$ and let $\overline{D}_k(t)$ be the expectation of $D_k(t)$. The main result of this paper follow as:
Theorem 1.1 Assume that \(2m(1 - \alpha) < \alpha\). Then, for all \(k \geq 0\), the limit \(d_k = \lim_{t \to \infty} \frac{\overline{D}_k(t)}{t}\) exists and satisfies
\[
d_k = 0, \quad 0 \leq k < m; \quad d_m = \frac{2\alpha}{m + 2\alpha}; \quad d_k = \prod_{i=m+1}^{k} \left(1 + \frac{1 + 2\alpha}{i + 2\alpha}\right) d_m, \quad \forall \, k > m.
\]

Obviously, \(d_k \sim Ck^{-(1+2\alpha)}\) for some constant \(C\).

We follow the basic procedures in [13] and [14] to prove our main theorem. The rest of the paper is organized as follows. In Section 2, we bound the maximum degree and then bound \(e_t\), the number of edges in \(G_t\). In Section 3, using the estimates given in Section 2, we establish the recurrence for \(\overline{D}_k(t)\). Finally, in section 4, we derive the approximation of \(\overline{D}_k(t)\) by a recurrence with respect to \(k\) and then solve the recurrence in \(k\) to finish the proof of Theorem 1.1.

Here we note that although this paper focuses on the power law degree distributions, other degree distributions including the exponential degree distributions of random graph process have also been observed [3, 7, 16, 22]. Furthermore, phase transition may emerge in the degree distributions of random graph processes [20, 21]. The phase transition problem of the copying model proposed in this paper is left to future investigation.

2 Bounding the degree and the number of edges

In this section, we first bound the maximum degree in \(G_t\) and then bound \(e_t\). Actually, we will give four kinds of estimates to \(e_t\), as will be seen in section 3, the four estimates are all necessary for establishing the recurrence of \(\overline{D}_k(t)\).

For \(t \geq 2\), let \(V_t^o\) be set of original vertices in \(V_t\), namely
\[
V_t^o := \{v \in V_t : v = v_1, v_2 \text{ or } v \text{ is added as a new vertex at some time step } 3 \leq s \leq t\}.
\]

For any times \(s\) and \(t\) with \(3 \leq s \leq t\), if \(v_s \in V_t^o\), then,
\[
d_{v_s}(s) = \frac{1}{2}d_{v_1}(2) = \frac{1}{2}d_{v_2}(2) = m.
\]  
(2.1)
We say an event happens quite surely (qs) if the probability of the complimentary set of the event is $O(t^{-K})$ for any $K > 0$.

We bound the degree in $G_t$ from top as follows

**Lemma 2.1** Assume that $2m(1 - \alpha) < 1$ and $v_s \in V_t^o$. Then

$$d_{v_s}(t) \leq (t/s)^{\alpha/2+m(1-\alpha)} (\log t)^3 \text{ qs.} \quad (2.2)$$

**Proof:** Let $Y$ be the $\{0,1\}$-valued random variable with $\mathbb{P}(Y = 1) = \alpha = 1 - \mathbb{P}(Y = 0)$. Then using the fact that $e_t \geq mt$, we have

$$\mathbb{E}(d_{v_s}(t+1) \mid G_t) \leq d_{v_s}(t) + Y B \left( m, \frac{d_{v_s}(t)}{2mt} \right) + (1-Y)mB \left( 1, \frac{d_{v_s}(t)}{t} \right), \quad (2.3)$$

where $B(\cdot, \cdot)$ be the general Binomial random variable.

Using the fact (2.1) and the relation (2.3), Lemma 2.1 follows from the same argument as used in [13], [14] and [20]. \qed

For any $v \in V_t$, if $v$ is copied at time step $s$ from some vertex $v_r$, $1 \leq r \leq s - 1$, we call $v$ the daughter vertex of $v_r$ and call $v_r$ the mother vertex of $v$. Denote by $D(v, G_t)$ the set of all descendants of $v$ in $G_t$. By the definition of the model, we know that, for any $v_s \in V_t^o$ and $v \in D(v_s, G_t)$, $d_v(t)$ is same distributed as $d_{v_s}(t)$. Now, denote by $\Delta_t$ the maximum degree in $G_t$, then, by Lemma 2.1 and the above analysis, we have

$$\Delta_t \leq t^{\alpha/2+m(1-\alpha)}(\log t)^3, \text{ qs.} \quad (2.4)$$

For any $v_s \in V_t^o$, let $f_{v_s}(t) = |D(v_s, G_t)|$ be the number of all descendants of $v_s$, then, we have

**Lemma 2.2** For any $s \geq 1$, if $v_s$ is a original vertex, i.e., for some $t \geq 2$, $v_s \in V_t^o$, then

$$f_{v_s}(t) \leq (t/s)^{1-\alpha} (\log t)^3, \text{ qs.} \quad (2.5)$$

**Proof:** Let $Y$ be the random variable used in the proof of Lemma 2.1 then,

$$\mathbb{E}(f_{v_s}(t+1) \mid G_t) = f_{v_s}(t) + (1-Y)B \left( 1, \frac{f_{v_s}(t)}{t} \right). \quad (2.6)$$

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The Lemma follows from the relation (2.6) and the same argument as used in Lemma 2.1.

Now we begin to bound $e_t$, the number of edges in $G_t$. Let $a_t$ be the number of edges added at time step $t + 1$, i.e., $e_{t+1} = a_t + e_t$. By the definition of the model, we have $a_t \leq \max\{\Delta_t, m\} = \Delta_t$, $\forall \ t \geq 2$; on the other hand, noticing that the number of multi-edges between any given vertices pair is fewer than $2m$, we have

$$\Delta_2 = 2m, \Delta_{t+1} \leq \Delta_t + 2m, \forall \ t \geq 2.$$ 

This gives the following determined upper bound on $e_t$,

$$e_t = 2m + \sum_{s=2}^{t-1} a_s \leq 2m + \sum_{s=2}^{t-1} 2m(s - 1) = O(t^2). \quad (2.7)$$

For random upper bounds on $e_t$, firstly, we prove a crude one as

$$e_t \leq O\left(t(\log t)^6\right), \quad qs. \quad (2.8)$$

Indeed, we have

$$2e_t = \sum_{s=1}^{t} d_{v_s}(t) = \sum_{v_s \in V^t} \sum_{v \in D(v_s, G_t)} d_v(t).$$

By Lemma 2.1 and Lemma 2.2,

$$\sum_{v_s \in V^t} \sum_{v \in D(v_s, G_t)} d_v(t) \leq \sum_{s=1}^{t} \left[\frac{(t/s)^{\alpha/2 + (m+1)(1-\alpha)}}{(\log t)^6}\right] = O\left(t(\log t)^6\right), \quad qs.$$ 

Note that for the last equality we have used the condition $2m(1 - \alpha) < \alpha$, which is given in the statement of Theorem 1.1.

Secondly, we try to give an estimate to $\mathbb{E}(e_t)$, the expectation of the number of edges in $G_t$. By the definition of the model, we have

$$\mathbb{E}(e_{t+1}|G_t) = e_t + \alpha m + (1 - \alpha) \frac{2e_t}{t}, \quad (2.9)$$

so

$$\mathbb{E}(e_{t+1}) = \mathbb{E}(e_t) \left(1 + \frac{2(1 - \alpha)}{t}\right) + \alpha m. \quad (2.10)$$
Let
\[
\eta_t := e_t - \mu t,
\]
where \( \mu = \frac{\alpha m}{1 - 2(1 - \alpha)} \). Then, (2.10) implies that
\[
E(\eta_{t+1}) = E(\eta_t) \left( 1 + \frac{2(1 - \alpha)}{t} \right).
\]
Thus, \( E(\eta_t) = O(t^{2(1-\alpha)}) \) and we have
\[
E(e_t) = \mu t + O(t^{2(1-\alpha)}). \tag{2.11}
\]
Finally, we have the following probability estimate on \( e_t \) as

**Lemma 2.3** Assume that \( 2m(1 - \alpha) < 1 \). Take \( \varepsilon_0 > 0 \) such that \( 1 + 2\varepsilon_0 + 2m(1 - \alpha) < 2 \), then
\[
P \left( |e_t - \mu t| > t^{1/2 + \varepsilon_0 + m(1 - \alpha)} \right) = O(t^{-\varepsilon_0}). \tag{2.12}
\]

**Proof:** To get the estimate (2.12), we have to bound \( \text{Var}(e_t) \), the variance of \( e_t \). First of all, we have
\[
\text{Var}(e_{t+1}) = \text{Var}(a_t + e_t) = \text{Var}(e_t) + \text{Var}(a_t) + 2 \left( E(a_t e_t) - E(a_t)E(e_t) \right). \tag{2.13}
\]
By definition, we have
\[
E(a_t^2 \mid G_t) = \alpha m^2 + (1 - \alpha) \sum_{s=1}^{t} \frac{d_v^2(t)}{t}.
\]
Then, by Lemma 2.1 and Lemma 2.2,
\[
E(a_t^2) = \alpha m^2 + \frac{(1 - \alpha)}{t} \sum_{v_t \in V_t^m} \sum_{v \in D(v_t, G_t)} d_v^2(t)
\leq \alpha m^2 + \frac{(1 - \alpha)}{t} \sum_{s=1}^{t} \left[ (t/s)^{\alpha+2m(1-\alpha)(\log t)^6} \right] \left[ (t/s)^{1-\alpha(\log t)^3} \right] + O(t^{-10})
= O \left( t^{2(1-\alpha)}(\log t)^9 \right). \tag{2.14}
\]
In addition, by (2.9) and (2.11), we have

$$E(a_t) = \alpha m + 2(1 - \alpha) \mu + O(t^{2(1-\alpha)-1}). \quad (2.15)$$

Thus

$$\text{Var}(a_t) = O \left(t^{2m(1-\alpha)}(\log t)^9\right). \quad (2.16)$$

For the term $E(a_t e_t)$, using (2.9), it is clear that

$$E(a_t e_t | G_t) = e_t E(a_t | G_t) = e_t \left(m\alpha + 2(1 - \alpha) \frac{e_t}{t}\right),$$

then

$$E(a_t e_t) = m\alpha E(e_t) + \frac{2(1 - \alpha)}{t} E(e_t^2). \quad (2.17)$$

Using (2.9) again, we have

$$E(a_t) E(e_t) = m\alpha E(e_t) + \frac{2(1 - \alpha)}{t} E(e_t^2). \quad (2.18)$$

Substituting (2.16), (2.17) and (2.18) into (2.13), we get

$$\text{Var}(e_{t+1}) = \left(1 + \frac{4(1 - \alpha)}{t}\right) \text{Var}(e_t) + O \left(t^{2m(1-\alpha)}(\log t)^9\right)$$

$$= \left(1 + \frac{4(1 - \alpha)}{t}\right) \text{Var}(e_t) + O \left(t^{2m(1-\alpha)+\epsilon_0}\right), \quad (2.19)$$

where $\epsilon_0 > 0$ is given in the statement of the Lemma. The recurrence (2.19) can be solved directly to get

$$\text{Var}(e_t) = \prod_{s=3}^{t-1} \left(1 + \frac{4(1 - \alpha)}{s}\right) \left(\text{Var}(e_3) + O \left(\sum_{s=3}^{t-1} \frac{s^{2m(1-\alpha)+\epsilon_0}}{\prod_{j=3}^{s} (1 + 4(1 - \alpha)/j)}\right)\right)$$

for large $t$, this implies that

$$\text{Var}(e_t) = O \left(t^{1+2m(1-\alpha)+\epsilon_0}\right). \quad (2.20)$$

The Lemma follows immediately from (2.11), (2.20) and the Chebychev’s inequality. \qed
3 Establishing The Recurrence for $D_k(t)$

Before we establish the recurrence for $D_k(t)$, we have to bound the multi-edges first. For $t \geq 2$, let

$$Z_t = \{ v \in V_t : \exists u \in V_t \text{ s.t. there are multi-edges between } u \text{ and } v \}$$

and $X_t = |Z_t|$, the cardinality of random set $Z_t$. Clearly, the number of multi-edges in $G_t$ is less than $2mX_t$.

**Lemma 3.1** For any $\epsilon > 0$, we have

$$\mathbb{E}(X_t) = O \left( t^{\alpha/2+m(1-\alpha)+\epsilon} \right). \quad (3.1)$$

**Proof:** By the definition of the model, we have

$$\mathbb{E}(X_{t+1} \mid G_t) \leq X_t + (1 - \alpha) \frac{X_t}{t} + \alpha \left( \frac{m}{2} \right) \frac{\Delta_t}{e_t}. $$

Taking expectation and then using (2.4) and the fact that $e_t \geq mt$, we have

$$\mathbb{E}(X_{t+1}) \leq \left( 1 + \frac{1 - \alpha}{t} \right) \mathbb{E}(X_t) + O \left( t^{\alpha/2+m(1-\alpha)-1} \left( \log t \right)^3 \right)$$

$$= \left( 1 + \frac{1 - \alpha}{t} \right) \mathbb{E}(X_t) + O \left( t^{\alpha/2+m(1-\alpha)-1+\epsilon} \right). \quad (3.2)$$

Using the argument between (2.19) and (2.20), the Lemma follows immediately from (3.2).

□

Now, we try to establish the recurrence for $D_k(t)$. Put $D_k(t) = 0, 0 \leq k < m$, for all $t \geq 2$. For $k \geq m$, we have

$$D_k(t + 1) = D_k(t) + \alpha m \mathbb{E} \left( -\frac{kD_k(t)}{2e_t} + \frac{(k - 1)D_{k-1}(t)}{2e_t} - O \left( \frac{\Delta_t}{e_t} \right) \right)$$

$$+(1 - \alpha)(k - 1)\mathbb{E} \left( -\frac{D_k(t)}{t} + \frac{D_{k-1}(t)}{t} - O \left( \frac{X_t}{t} \right) \right) + \alpha I_{k=m}. \quad (3.3)$$

The terms $O \left( \frac{\Delta_t}{e_t} \right)$ and $O \left( \frac{X_t}{t} \right)$ account for the probabilities that we create more than one degree changes due to new vertex addition and vertex copying from $Z_t$ respectively.
By Lemma 2.3 the term $E\left(\frac{kD_k(t)}{2e_t}\right)$ can be expressed as

$$E\left(\frac{kD_k(t)}{2e_t} \mid |e_t - \mu t| \leq t^{1/2+\varepsilon_0+m(1-\alpha)}\right) \mathbb{P}\left(|e_t - \mu t| \leq t^{1/2+\varepsilon_0+m(1-\alpha)}\right)$$

$$+ E\left(\frac{kD_k(t)}{2e_t} \mid |e_t - \mu t| > t^{1/2+\varepsilon_0+m(1-\alpha)}\right) \mathbb{P}\left(|e_t - \mu t| > t^{1/2+\varepsilon_0+m(1-\alpha)}\right)$$

$$= \frac{E\left(kD_k(t) \mid |e_t - \mu t| \leq t^{1/2+\varepsilon_0+m(1-\alpha)}\right) \mathbb{P}\left(|e_t - \mu t| \leq t^{1/2+\varepsilon_0+m(1-\alpha)}\right)}{2\mu t} \times (1 + O\left(t^{-1/2+\varepsilon_0+m(1-\alpha)}\right)) + O(t^{-\varepsilon_0}),$$

(3.4)

where we used the fact that $kD_k(t) \leq 2e_t$ to hand the second term. In addition, we have

$$E\left(kD_k(t) \mid |e_t - \mu t| \leq t^{1/2+\varepsilon_0+m(1-\alpha)}\right) \mathbb{P}\left(|e_t - \mu t| \leq t^{1/2+\varepsilon_0+m(1-\alpha)}\right)$$

$$= kD_k(t) - E(kD_k(t) ; |e_t - \mu t| > t^{1/2+\varepsilon_0+m(1-\alpha)}),$$

(3.5)

and

$$E(kD_k(t) ; |e_t - \mu t| > t^{1/2+\varepsilon_0+m(1-\alpha)})$$

$$= E(kD_k(t) ; |e_t - \mu t| > t^{1/2+\varepsilon_0+m(1-\alpha)}, e_t \leq O(t(\log t)^6))$$

$$+ E(kD_k(t) ; |e_t - \mu t| > t^{1/2+\varepsilon_0+m(1-\alpha)}, e_t > O(t(\log t)^6))$$

$$\leq O(t(\log t)^6) \mathbb{P}(|e_t - \mu t| > t^{1/2+\varepsilon_0+m(1-\alpha)})$$

$$+ O(t^2) \mathbb{P}(e_t > O(t(\log t)^6))$$

$$\leq O(t^{1-\varepsilon_0}(\log t)^6) + O(t^{-\beta_0}) = O(t^{1-\varepsilon_0}(\log t)^6).$$

(3.6)

Note that to get (3.6), we used the fact that $kD_k(t) \leq 2e_t$ and the bounds on $e_t$ given in (2.7) and (2.8).

Thus, combining (3.4), (3.5) and (3.6),

$$E\left(\frac{kD_k(t)}{2e_t}\right) = \frac{kD_k(t)}{2\mu t} \left(1 + O\left(t^{-1/2+\varepsilon_0+m(1-\alpha)}\right)\right) + O(t^{-\varepsilon_0}(\log t)^6)$$

$$\leq \frac{kD_k(t)}{2\mu t} + \frac{E(2e_t)}{2\mu t} O\left(t^{-1/2+\varepsilon_0+m(1-\alpha)}\right) + O(t^{-\varepsilon_0}(\log t)^6),$$

(3.6)
using (2.11), we have for $k \geq m$
\[
E \left( \frac{kD_k(t)}{2e_t} \right) = \frac{kD_k(t)}{2\mu t} + O \left( t^{-1/2+\varepsilon_0+m(1-\alpha)} \right) + O(t^{-\varepsilon_0}(\log t)^0). \tag{3.7}
\]

On the other hand, by inequality (2.4) and Lemma 3.1 for any fixed $\varepsilon \in (0, 1-\alpha/2 - m(1-\alpha))$, we have
\[
E \left( \Delta_t e_t \right), \ E \left( \frac{X_t}{t} \right) = O(t^{-1+\alpha/2+m(1-\alpha)+\varepsilon}). \tag{3.8}
\]

Let
\[
\varepsilon_1 = \frac{1}{2} \min \{\varepsilon_0, 1-\alpha/2 - m(1-\alpha), 1/2 - \varepsilon_0 - m(1-\alpha)\}. \tag{3.9}
\]

Now, substitute (3.7) and (3.8) into (3.3), we get the recurrence for $D_k(t)$ as
\[
\frac{D_k(t+1)}{t} = \frac{D_k(t)}{t} - \frac{(k-1)}{2} \frac{D_{k-1}(t)}{t} + O(t^{-\varepsilon_1}) + \alpha I_{k=m}, \ \forall \ k \geq m. \tag{3.10}
\]

Note that the hidden constant, denote by $L$, in term $O(t^{-\varepsilon_1})$ is independent of $k$.

4 Solving (3.10) and The Proof of Theorem 1.1

In recurrence (3.10), if we heuristically put $\bar{d}_k = \frac{D_k(t)}{t}$ and assume it is a constant, we get
\[
\frac{(k+2\alpha)}{2} \bar{d}_k = \frac{(k-1)}{2} \bar{d}_{k-1} + O(t^{-\varepsilon_1}) + \alpha I_{k=m}.
\]

This leads to the consideration of the following recurrence in $k$:
\[
\left\{ \begin{array}{l}
\frac{(k+2\alpha)}{2} d_k = \frac{(k-1)}{2} d_{k-1} + \alpha I_{k=m}, \ k \geq m; \\
d_k = 0, \ 0 \leq k < m.
\end{array} \right. \tag{4.1}
\]

The following Lemma shows that (4.1) is a good approximation to (3.10).
Lemma 4.1 Suppose that \( \{d_k : k \geq 0\} \) be the solution of (4.1), then there exists a constant \( M > 0 \) such that

\[
|D_k(t) - td_k| \leq Mt^{1-\varepsilon_1},
\]

for all \( t \geq 1 \) and \( k \geq 0 \), where \( \varepsilon_1 \) is given in (3.9).

Proof: The recurrence can be solved directly as: \( d_k = 0, 0 \leq k < m \); \( d_m = \frac{2\alpha}{m + 2\alpha} \) and

\[
d_k = \prod_{i=m+1}^{k} \left( 1 + \frac{1 + 2\alpha}{i + 2\alpha} \right) d_m, \quad \forall k > m.
\]

Obviously, \( d_k \) decay as \( k^{-(1+2\alpha)} \), consequently, for some constant \( C \),

\[
d_k \leq C/k \text{ for all } k \geq 1.
\]

Using (4.4) and the degree estimate given in Lemma 2.1, the Lemma follows from a standard argument which can be found in [14] (see Lemma 5.1) and [20] (see Lemma 3.1). □

Proof of Theorem 1.1: Theorem 1.1 follows immediately from (4.2) and (4.3). □

Acknowledgements

The authors thank Prof. Zhao Dong and Prof. Ke Liu for useful advice and discussion.

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