ANALYTICITY AND LOSS OF DERIVATIVES

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Abstract. A very recent paper of Kohn studies hypoellipticity for a sum of squares of complex vector fields which exhibit a large loss of derivatives. We prove analytic hypoellipticity for this operator.

1. Introduction

In [1], J.J. Kohn proved the hypoellipticity of the operator

\[ P = LL^* + (\overline{\partial}^k L)^*(\overline{\partial}^k L), \quad L = \frac{\partial}{\partial z} + i\overline{z} \frac{\partial}{\partial t}, \]

for which there is a large loss of derivatives - indeed in the \textit{a priori} estimate one bounds only the Sobolev norm of order \(-\frac{k-1}{2}\).

We show in this note that solutions of \(Pu = f\) with \(f\) real analytic are themselves real analytic in any open set where \(f\) is.

The \textit{a priori} estimate which Kohn established for this operator and with which we will work is

\[ \|Lv\|_0^2 + \|\overline{\partial}^k Lv\|_0^2 + \|v\|_\gamma^2 \lesssim |(Pv, v)_{L^2}| + \|v\|_{-\frac{1}{2}}^2, \quad v \in C_0^\infty. \]

The estimate has two interesting parts. The first two terms on the left exhibit maximal control in \(\overline{\partial}\) and \(\overline{\partial}^k L\), but only these complex directions. Hence in obtaining recursive bounds for derivatives it is essential to keep one of these vector fields available for as long as possible. For this, we will construct a carefully balanced localization of high powers of \(T = -2i\partial/\partial t\). When this becomes no long possible (even with Ehrenpreis-type cut-off functions and the constructions of the second author in [2, 3]), one can only localize a fixed, though arbitrarily high, power of \(T\), one must accept the lack of a ‘good’ derivative (\(\overline{L}\) or \(\overline{\partial}^k L\)) and use the third term on the left of the estimate, introduce a new localizing function, and accept the loss of the (large but finite) number of...
derivatives and start the whole process again, but with only a fraction of the original power of \( T \).

Our first observation is that we know the analyticity of the solution for \( z \) different from 0 from the earlier work of the second author [2], [3] and Treves [4]. Thus, modulo brackets with localizing functions whose derivatives are supported in the known analytic hypoelliptic region, we take all localizing functions independent of \( z \).

Our second observation is that it suffices to bound derivatives measured in terms of high powers of the vector fields \( L \) and \( \mathcal{L} \) in \( L^2 \) norm, by standard arguments, and indeed estimating high powers of \( L \) can be reduced to bounding high powers of \( \mathcal{L} \) and powers of \( T \) of half the order, by repeated integration by parts. Thus our overall scheme will be to start with high powers (order \( 2p \)) of \( L \) or \( \mathcal{L} \), use integration by parts and the a priori estimate repeatedly to reduce to treating \( T^p u \) in a slightly larger set.

And to do this, we introduce a new special localization of \( T^p \) adapted to this problem.

The new localization of \( T^p \) may be written in the form:

\[
(T^{p_1,p_2})_\varphi = \sum_{a \leq p_1, b \leq p_2} \frac{L^a \circ z^a \circ T^{p_1-a} \circ \varphi^{(a+b)} \circ T^{p_2-b} \circ z^b \circ \mathcal{L}^b}{a!b!},
\]

Here by \( \varphi^{(r)} \) we mean \( (-i\partial/\partial t)^r \varphi(t) \) since near \( z = 0 \) we have seen that we may take the localizing function independent of \( z \). Note that the leading term (with \( a+b = 0 \)) is merely \( T^{p_1} \varphi T^{p_2} \) which equals \( T^{p_1+p_2} \) on the initial open set \( \Omega_0 \) where \( \varphi \equiv 1 \).

We have the commutation relations:

\[
[L, (T^{p_1,p_2})_\varphi] = L \circ (T^{p_1-1,p_2})_\varphi',
\]

\[
[\mathcal{L}, (T^{p_1,p_2})_\varphi] = (T^{p_1,p_2-1})_\varphi' \circ \mathcal{L},
\]

\[
[(T^{p_1,p_2})_\varphi, z] = (T^{p_1-1,p_2})_\varphi' \circ z,
\]

and

\[
[(T^{p_1,p_2})_\varphi, \overline{z}] = \overline{z} \circ (T^{p_1,p_2-1})_\varphi'.
\]
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where the \( \equiv \) denotes modulo \( C^{p_1-p'_1+p_2-p'_2} \) terms of the form

\[
\left( p_1 - p'_1 \right)! \left( p_2 - p'_2 \right)!
\]

with either \( p'_1 = 0 \) or \( p'_2 = 0 \), i.e., terms where all free \( T \) derivatives have been eliminated on one side of \( \varphi \) or the other. Thus if we start with \( p_1 = p_2 = p/2 \), and iteratively apply these commutation relations, the number of \( T \) derivatives not necessarily applied to \( \varphi \) is eventually at most \( p/2 \).

So we insert first \( v = (T^{\mathbf{\nabla}^{k}_{z}})_{\varphi} u \) in the \emph{a priori} inequality, then bring \((T^{\mathbf{\nabla}^{k}_{z}})_{\varphi}\) to the left of \( P = -\mathcal{L} - \mathcal{L}z^{k}z^{k}L \) since \( Pu \) is known and analytic. We have, omitting for now the ‘subelliptic’ term,

\[
\| \mathcal{L}(T^{\mathbf{\nabla}^{k}_{z}})_{\varphi} u \|_{0}^{2} + \| z^{k}L(T^{\mathbf{\nabla}^{k}_{z}})_{\varphi} u \|_{0}^{2} \lessapprox |P(T^{\mathbf{\nabla}^{k}_{z}})_{\varphi} u, (T^{\mathbf{\nabla}^{k}_{z}})_{\varphi} u)_{L^{2}}| \leq |P(T^{\mathbf{\nabla}^{k}_{z}})_{\varphi} u, (T^{\mathbf{\nabla}^{k}_{z}})_{\varphi} u)_{L^{2}}| \]

and, by the above bracket relations,

\[
|P(T^{\mathbf{\nabla}^{k}_{z}})_{\varphi} u, (T^{\mathbf{\nabla}^{k}_{z}})_{\varphi} u) |
\]

with the same meaning for \( \equiv \) as above. In every term, no powers of \( z \) or \( \nabla \) have been lost, though some may need to be brought to the left of the \((T^{q_{1},q_{2}})_{\varphi}\) with again no loss of powers of \( z \) or \( \nabla \) and a further reduction in order, every bracket reduces the order of the sum of the two indices \( p_1 \) and \( p_2 \) by one (here we started with \( p_1 = p_2 = p/2 \)), pick up one derivative on \( \varphi \), and leave the vector fields over which we have maximal control in the estimate intact and in the correct order. Thus we may bring either \( \mathcal{L}z^{k} \) or \( L \) to the right as \( \nabla L \) or \( L \), and use a weighted Schwarz inequality on the result to take maximal advantage of the \emph{a priori} inequality. Iterations of all of this continue until there
remain at most $p/2$ free $T$ derivatives (i.e., the $T$ derivatives on at least one side of $\varphi$ are all ‘corrected’ by good vector fields) and perhaps as many as $p/2$ $L$ or $\overline{L}$ derivatives, and we may continue further until, at worst, the remaining $L$ and $\overline{L}$ derivatives bracket two at a time to produce more $T$’s, one at a time. After all of this, there will be at most $T^{3p/4}$ remaining.

It is here that the final term on the left of the a priori inequality is used, in order to bring the localizing function out of the norm after creating another balanced localization of $T^{3p/4}$ with a new localizing function of Ehrenpreis type with slightly larger support, geared to $3p/4$ instead of to $p$.

This means that the entire process, which reduced the order from $p$ to $3p/4$, or more precisely to $3p/4 + (k - 1)/2$, is repeated, over and over, each time essentially reducing the order by a factor of $3/4$. After on the order of $\log_{4/3} p$ such iterations we are reduced to a bounded number of derivatives, and, as in [2] and [3], all of these nested open sets may be chosen to fit in the one open set $\Omega_1$ where $Pu$ is known to be analytic, and all constants chosen independent of $p$ (but depending on $Pu$). The fact that in those works one full iteration reduced the order by half played no essential role - a factor of $3/4$ would have worked just as well.

The final estimate, as in those works, is that for all $\alpha$ with $|\alpha| \leq p$,

$$|D^{[\alpha]}u|_{L^\infty(\Omega_0)} \leq CC'p^p \sim C'C'^p p!$$

in $\Omega_0$ with $C$ independent of $p$, which proves the analyticity of the solution in $\Omega_0$.

References

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