SYMMETRIC UNIMODAL EXPANSIONS OF EXCEDANCES IN COLORED PERMUTATIONS

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Abstract. We consider several generalizations of the classical $\gamma$-positivity of Eulerian polynomials (and their derangement analogues) using generating functions and combinatorial theory of continued fractions. For the symmetric group, we prove an expansion formula for inversions and excedances as well as a similar expansion for derangements. We also prove the $\gamma$-positivity for Eulerian polynomials for derangements of type $B$. More general expansion formulae are also given for Eulerian polynomials for $r$-colored derangements. Our results answer and generalize several recent open problems in the literature.

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Date: September 5, 2021.
1. Introduction and main results in $\mathfrak{S}_n$

Let $\mathfrak{S}_n$ be the set of permutations of $[n] = \{1, \ldots, n\}$. For any permutation $\sigma \in \mathfrak{S}_n$, written as the word $\sigma = \sigma(1) \ldots \sigma(n)$, the entry $i \in [n]$ is called a descent of $\sigma$ if $i < n$ and $\sigma(i) > \sigma(i+1)$; the entry $i \in [n]$ is called an excedance (resp. drop, fixed-point) of $\sigma$ if $i < \sigma(i)$ (resp. $i > \sigma(i)$, $i = \sigma(i)$). Denote the number of descents, excedances, drops, and fixed-points in $\sigma$ by des $\sigma$, exc $\sigma$, drop $\sigma$, and fix $\sigma$ respectively. It is well known [FS70] that the statistics des, exc and drop have the same distribution on $\mathfrak{S}_n$ and their common enumerative polynomial is the Eulerian polynomial:

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{drop } \sigma}. \quad (1.1)$$

Define the set $\text{DD}_{n,k} := \{\sigma \in \mathfrak{S}_n : \text{des } \sigma = k \text{ and } \text{dd}(\sigma(0)) = 0\}$, where dd($\sigma(0)$) is the number of double descents in the word $\sigma(1) \ldots \sigma(n)0$, i.e., indices $i$, $1 < i \leq n$, such that $\sigma(i-1) > \sigma(i) > \sigma(i+1)$ with $\sigma(n+1) = 0$. For example,

$$\text{DD}_{4,1} = \{1324, 1423, 2314, 2413, 3412, 2134, 3124, 4123\}.$$

Foata and Schützenberger [FS70] proved the following expansion formula

$$A_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} |\text{DD}_{n,k}| t^k (1+t)^{n-1-2k}. \quad (1.2)$$

A finite sequence of positive integers $a_0, \ldots, a_n$ is unimodal if there exists an index $i'$ such that $a_i \leq a_{i'+1}$ if $i < i'$ and $a_i > a_{i'+1}$ otherwise; it is log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i \in [1, n-1]$. A polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with nonnegative coefficients is unimodal (resp. log-concave) if and only if the sequence of its coefficients is unimodal (resp. log-concave). It is well known that a polynomial with nonnegative coefficients and with only real roots is log-concave and that log-concavity implies the unimodality. The set of polynomials whose coefficients are symmetric with center of symmetry $d/2$ is a vector space with a basis given by $\{t^k (1+t)^{d-2k}\}_{k=0}^{\lfloor d/2 \rfloor}$. If a polynomial $p(t)$ has nonnegative coefficients when expanded in this basis, then the coefficients in the standard basis $\{t^k\}_{k=0}^{\infty}$ form a unimodal sequence (being a nonnegative sum of symmetric and unimodal sequences with the same center of symmetry). Hence the expansion (1.2) implies both the symmetry and the unimodality of the Eulerian numbers.

Analognes of (1.2) for types $B$ and $D$ were given by Petersen [Pet07], Stembridge [Ste08] and Chow [Cho08]. In the recent years several $q$-analogues of this formula have been established in [SW10, SZ12, HZ13, LZ14] by combining one of the Eulerian statistics in (1.1) and one of the two classical Mahonian statistics, i.e., inversion numbers and Major index. Recall that for $\sigma \in \mathfrak{S}_n$ the inversion number inv $\sigma$ is the number of pairs $(i, j)$ such that $i < j$ and $\sigma(i) > \sigma(j)$ and the major index maj $\sigma$ is the sum of the descent positions. In [SZ12] we have shown that some symmetric unimodal expansion formulas of Eulerian polynomials follow easily from the combinatorics of the continued fraction expansions of Jacobi type of their ordinary generating functions. In this paper, we shall prove an INV $q$-analogue of (1.2) as well as some expansions for the Eulerian polynomials of type $B$ by
Table 1. The values of $\gamma_{n,k}(q)$ for $1 \leq k \leq n \leq 5$

| $n \setminus k$ | 0   | 1   | 2   |
|-----------------|-----|-----|-----|
| 1               | 1   | 1   |     |
| 2               | 1   | 1   |     |
| 3               | $q(1 + q)$ | $q(1 + q)(q^2 + q + 2)$ |     |
| 4               | $q(1 + q)(q^4 + 2q^3 + 3q^2 + 2q + 3)$ | $q^2(1 + q)^2(q^4 + q^3 + q^2 + 1)$ |     |
| 5               | $q^2(1 + q)^2(q^4 + q^3 + q^2 + 1)$ |     |     |

an analogue study of the correspondence between signed permutations and the weighted lattice paths.

For $\sigma \in S_n$, the statistic $(31-2)\sigma$ (resp. $(13-2)\sigma$) is the number of pairs $(i, j)$ such that $2 \leq i < j \leq n$ and $\sigma(i - 1) > \sigma(j) > \sigma(i)$ (resp. $\sigma(i - 1) < \sigma(j) < \sigma(i)$). Similarly, the statistic $(2-13)\sigma$ (resp. $(2-31)\sigma$) is the number of pairs $(i, j)$ such that $1 \leq i < j \leq n - 1$ and $\sigma(j + 1) > \sigma(i) > \sigma(j)$ (resp. $\sigma(j + 1) < \sigma(i) < \sigma(j)$).

Theorem 1. We have

$$\sum_{\sigma \in S_n} \sigma^{\text{exc}} q^{\text{inv} - \text{exc}} = \sum_{0 \leq k \leq (n-1)/2} \gamma_{n,k}(q)t^k(1 + t)^{n - 2k},$$

where the coefficient $\gamma_{n,k}(q)$ has the following combinatorial interpretation

$$\gamma_{n,k}(q) = \sum_{\sigma \in DD_{n,k}} q^{(2-13)\sigma + (31-2)\sigma}$$

and is divisible by $q^k(1 + q)^k$ for $0 \leq k \leq (n - 1)/2$.

Remark. The first values of $\gamma_{n,k}(q)$ in Table 1 are obtained by replacing $p = q^2$ in [SZ12, Table of $a_{n,k}(p, q)$ in Appendix].

There is a derangement analogue of (1.3). An element $\sigma \in S_n$ is called a derangement if it has no fixed point. Denote by $D_n$ the set of all derangements in $S_n$ and let

$$\text{DE}_{n,k} := \{ \sigma \in D_n : \text{exc} \sigma = k \text{ and } \text{cda}(\sigma) = 0 \},$$

where $\text{cda}(\sigma)$ is the number of cyclic double ascents (or double excedances) of $\sigma$, i.e., indices $i \in [n]$ such that $i < \sigma(i) < \sigma^2(i)$. For example, we have $\text{DE}_{4,1} = \{4123\} = \{(4321)\}$ and $\text{DE}_{4,2} = \{2143, 3412, 4321, 4312, 3421\} = \{(21)(43), (31)(42), (41)(32), (4231), (4132)\}$.

Theorem 2. For $n \geq 1$ we have

$$\sum_{\sigma \in D_n} q^{\text{inv} \sigma} t^{\text{exc} \sigma} = \sum_{0 \leq k \leq n/2} \left( \sum_{\sigma \in \text{DE}_{n,k}} q^{\text{inv} \sigma} \right) t^k(1 + t)^{n - 2k},$$

Remark. The $t = -1$ case of Theorems 1 and 2 were first proved in [SZ10, Theorem 3] as a $q$-analogue of two well-known results connecting Eulerian polynomials (resp. derangement polynomials) to tangent (resp. secant) numbers.
Table 2. The values of $\sum q^{\text{inv} \sigma}$ for $\sigma \in \text{DE}_{n,k}$ and $1 \leq k \leq n \leq 4$

| $n \setminus k$ | 0   | 1   | 2   |
|-----------------|-----|-----|-----|
| 0               | 1   |     |     |
| 1               | 0   |     |     |
| 2               | 0   | $q$ |     |
| 3               | 0   | $q^2$ | $q^3 + q^2 + 2q^4 + q^6$ |

Note that Athanasiadis and Savvidou [AS12] used the $q = 1$ case of (1.4) to prove the $\gamma$-positivity of the local $h$-polynomials of certain simplicial divisions. More recently, Athanasiadis and Savvidou [AS13] and Athanasiadis [Ath13] have extended their results to hyperoctahedral group $B_n$ and colored groups, while Mongelli [Mon13] studied the excedances in affine Weyl groups. Motivated by the aforementioned works, we shall study the similar expansion formula for two kinds of derangement Eulerian polynomials in $B_n$ and wreath product $\mathbb{Z}_r \wr S_n$. We shall introduce the necessary definitions and present the main results for these derangement polynomials in the next section.

2. Excedances and derangements in colored permutations

2.1. Excedances in $B_n$. Let $B_n$ be the set of permutations $\sigma$ of $\{\pm 1, \ldots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for every $i \in [n]$, that is, $B_n = \mathbb{Z}_2 \wr S_n$ with $-i = \bar{i}$. From Steingrímsson [Ste94, Definition 3], an index $i \in [n]$ is called an excedance of $\sigma \in B_n$ if

$$i <_f \sigma(i)$$

in the friends order $<_f$ of $\{\pm 1, \ldots, \pm n\}$:

$$1 <_f -1 <_f 2 <_f -2 <_f \cdots <_f n <_f -n$$

and $\text{exc}(\sigma)$ is defined by the number of excedances of $\sigma \in B_n$. Following Brenti [Bre94, page 431], we say that $i \in [n]$ is a $B$-excedance of $\sigma$ if

$$\sigma(i) < \sigma(|\sigma(i)|) \quad \text{or} \quad \sigma(i) = -i$$

in the natural order $<$ of $\{\pm 1, \ldots, \pm n\}$:

$$-n < \cdots < -2 < -1 < 1 < 2 < \cdots < n.$$ 

and the number of $B$-excedances of $\sigma$ will be denoted by $\text{exc}_B(\sigma)$. We also say that $i \in [0, n - 1]$ is a $B$-descent of $\sigma$ if

$$\sigma(i) > \sigma(i - 1)$$

in the natural order, where $\sigma(0) = 0$, and the number of $B$-descents of $\sigma$ will be denoted by $\text{des}_B(\sigma)$. From [Ste94, Theorem 20] and [Bre94, Eq.(14)] it follows that

$$\sum_{n \geq 0} \sum_{\sigma \in B_n} t^{\text{exc}(\sigma)} \frac{z^n}{n!} = \sum_{n \geq 0} \sum_{\sigma \in B_n} t^{\text{des}_B(\sigma)} \frac{z^n}{n!} = \sum_{n \geq 0} \sum_{\sigma \in B_n} t^{\text{exc}_B(\sigma)} \frac{z^n}{n!} = (1 - t)e^{z(1-t)}/(1 - t e^{2z(1-t)}), \quad (2.1)$$
i.e., the two statistics exc and $\text{exc}_B$ are equidistributed on $\mathcal{B}_n$. In 2004, Fire [Fir04] Definition 6.18] introduced the flag excedance statistic $\text{fexc}$ of an element $\sigma \in \mathcal{B}_n$ by

$$\text{fexc}(\sigma) = \# \{ i \in \{\pm 1, \ldots, \pm n\} : i < \sigma(i) \} = 2 \cdot \# \{ i \in [n] : i < \sigma(i) \} + \# \{ i \in [n] : \sigma(i) < 0 \},$$

where we use the color order $<_c$ of $\{±1, \ldots, ±n\}$:

$$-1 <_c -2 <_c \cdots <_c -n <_c 1 <_c 2 <_c \cdots <_c n.$$

2.2. **Excedances in wreath product** $\mathbb{Z}_r \wr \mathfrak{S}_n$. The wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$ is the set of colored permutations $\left(\begin{array}{c} r \\ z \end{array}\right)$, where $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$ and $z = (z_1, \ldots, z_n) \in [0, r-1]^n$. The number $z_i$ will be considered as the color assigned to $\pi_i$. In this generalized symmetric group $\mathbb{Z}_r \wr \mathfrak{S}_n$, the product is defined by

$$\left( \begin{array}{c} \tau \\ w \end{array} \right) \left( \begin{array}{c} \pi \\ z \end{array} \right) = \left( \begin{array}{c} \tau \pi \\ w(\pi) + z \end{array} \right),$$

where the composition $\tau \pi = \tau \circ \pi$ in $\mathfrak{S}_n$ is composed from right to left,

$$\pi(\omega) = (\omega_{\pi_1}, \omega_{\pi_2}, \ldots, \omega_{\pi_n}),$$

and the addition is summed modulo $r$ in individual coordinate.

From now, for $\sigma = \sigma(1) \ldots \sigma(n) = (\sigma_{z_1} \ldots \sigma_{z_n}) \in \mathbb{Z}_r \wr \mathfrak{S}_n$, we write

$$\sigma(i) = \left( \begin{array}{c} \pi_i \\ z_i \end{array} \right) \text{ as } \pi_i^{[z_i]}, \quad (2.2)$$

while $\pi_i$ (resp. $z_i$) is called the number (resp. the color) of $\pi_i^{[z_i]}$, denote by

$$\pi_i = \left| \pi_i^{[z_i]} \right| \text{ and } z_i = \text{col}(\pi_i^{[z_i]}).$$

Conventionally, we use also the window notation $i = i^{[0]}$, $\bar{i} = i^{[1]}$, and $\bar{\bar{i}} = i^{[2]}$. For an example

$$\sigma = \left( \begin{array}{c} 4 & 7 & 2 & 5 & 1 & 6 & 3 \\ 0 & 1 & 0 & 1 & 2 & 0 & 0 \end{array} \right) = 4 \bar{7} 2 \bar{5} \bar{1} 6 3 \in \mathbb{Z}_3 \wr \mathfrak{S}_7,$$

we have $\left(\begin{array}{c} 1 \\ 2 \end{array}\right) = 1^{[2]} = \bar{1}$, $\left(\begin{array}{c} 7 \\ 5 \end{array}\right) = 7^{[1]} = \bar{7}$, $\bar{5} = 5$, and col($\bar{5}$) = 1. So the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$ could be considered the set of $r$-colored permutations $\sigma$ of the alphabet $\Sigma$ of $rn$ letters

$$\Sigma := \{ 1, \bar{1}, \bar{\bar{1}}, \ldots, 1^{[r-1]}, 2, \bar{2}, \bar{\bar{2}}, \ldots, 2^{[r-1]}, \ldots, n, \bar{n}, \bar{\bar{n}}, \ldots, n^{[r-1]} \},$$

such that $\sigma(\bar{i}) = \sigma(i)$ for all $i \in \Sigma$. From Steingrímsson [Ste94], an index $i \in [n]$ is called an excedance of $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$ if $i <_f \sigma(i)$ in the friends order $<_f$ of $\Sigma$:

$$1 <_f \bar{1} <_f \cdots <_f 1^{[r-1]} <_f 2 <_f \bar{2} <_f \cdots <_f 2^{[r-1]} <_f \cdots <_f n <_f \bar{n} <_f \cdots <_f n^{[r-1]}.$$

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Let $\text{exc}(\sigma)$ be the number of excedances of $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$. Bagno and Garber [BG06, Definition 3.4] define the flag excedance statistic $\text{fexc}$ of an element $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$ by

$$\text{fexc}(\sigma) = \# \{ i \in \Sigma : i <_c \sigma(i) \} = r \cdot \# \{ i \in [n] : i <_c \sigma(i) \} + \sum_{i \in [n]} \text{col}(\sigma(i)),$$

where we use the color order $<_c$ of $\Sigma$:

$$1^{[r-1]} <_c 2^{[r-1]} <_c \cdots <_c n^{[r-1]} <_c \cdots <_c \bar{1} <_c 2 <_c \cdots <_c \bar{n} <_c 1 <_c 2 <_c \cdots <_c n.$$  

2.3. Derangements in $\mathbb{Z}_r \wr \mathfrak{S}_n$. An element $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$ is called a derangement if it has no fixed point, that is, $\sigma(i) \neq i$ for all $i \in \Sigma$. Denote by $\mathcal{D}_n^{(r)}$ the set of all derangements in $\mathbb{Z}_r \wr \mathfrak{S}_n$. The two corresponding derangement Eulerian polynomials on $\mathcal{D}_n^{(r)}$ are defined by

$$D_n^{(r)}(t) = \sum_{\sigma \in \mathcal{D}_n^{(r)}} t^{\text{fexc}(\sigma)} = \sum_{k \geq 0} D_{n,k}^{(r)} t^k \quad (2.3)$$

and

$$d_n^{(r)}(t) = \sum_{\sigma \in \mathcal{D}_n^{(r)}} t^{\text{fexc}(\sigma)} = \sum_{k \geq 0} d_{n,k}^{(r)} t^k. \quad (2.4)$$

By convention we set $D_0^{(2)}(t) = d_0^{(2)}(t) = 1$. The first values of $D_n^{(2)}(t)$ and $d_n^{(2)}(t)$ are as follows:

$$D_1^{(2)}(t) = t, \quad d_1^{(2)}(t) = t,$$

$$D_2^{(2)}(t) = t + 3t^2 + t^3, \quad d_2^{(2)}(t) = 4t + t^2,$$

$$D_3^{(2)}(t) = t + 7t^2 + 13t^3 + 7t^4 + t^5, \quad d_3^{(2)}(t) = 8t + 20t^2 + t^3,$$

$$D_4^{(2)}(t) = t + 15t^2 + 57t^3 + 87t^4 + 57t^5 + 15t^6 + t^7, \quad d_4^{(2)}(t) = 16t + 144t^2 + 72t^3 + t^4.$$

The polynomials $d_n^{(2)}(t)$ were first studied by Chen et al. [CTYZ09] and Chow [Cho09]. In particular, the polynomials $d_n^{(2)}(t)$ have only real roots, see [Cho09]. Recently Mongelli [Mon13] has proved the symmetry $D_{n,j}^{(2)} = D_{n,2n-j}^{(2)}$ for all $0 \leq j \leq n$ and conjectured its unimodality. Note that the polynomials $D_{3}^{(2)}(t)$ has non-real complex roots. We will prove similar expansions for the derangement Eulerian polynomials $D_n^{(r)}(t)$ and $d_n^{(r)}(t)$ when $r \geq 1$. In particular, we prove the $\gamma$-positivity of $D_{n}^{(2)}(t)$, which implies both symmetry and unimodality of its coefficients, see [ZKS].

2.4. Main results in $\mathbb{Z}_r \wr \mathfrak{S}_n$. For any integer $n \geq 0$ define $[n]_q = \frac{1-q^n}{1-q}$ as its $q$-analogue. Let $\gamma_{n,i,j}$ be the number of permutations in $\mathfrak{S}_n$ with exactly $i$ fixed-points, $j$ excedances and without double excedance, i.e.,

$$\gamma_{n,i,j} = |\{ \sigma \in \mathfrak{S}_n : \text{fix } \sigma = i, \text{exc } \sigma = j, \text{cda } \sigma = 0 \}|.$$
Theorem 3. For $r \geq 1$, we have $\gamma_{n,i,j} > 0$ for $1 \leq i + 2j \leq n$,

$$D^{(r)}_n(t) = \sum_{1 \leq i + 2j \leq n} \gamma_{n,i,j} t^{i+j} (1 + t)^{n-i-2j} ([r - 1] t)^{i} ([r] t)^{n-i}, \quad (2.5)$$

and

$$d^{(r)}_n(t) = \sum_{1 \leq i + 2j \leq n} \gamma_{n,i,j} t^{i+j} (1 + t)^{n-i-2j} (r-1)^i r^{n-i}. \quad (2.6)$$

The special $r = 1$ case of (2.5) corresponds to the $q = 1$ case of (1.4). We derive immediately from Theorem 3 the following result.

Corollary 4. For all $r \geq 1$ and $n \geq 1$, the polynomial $D^{(r)}_n(t)$ is strictly unimodal and symmetric, namely, the coefficients $D^{(r)}_{n,k}$ in (2.3) satisfy

$$0 \leq D^{(r)}_{n,0} < D^{(r)}_{n,1} < \cdots < D^{(r)}_{n,\lfloor rn/2 \rfloor}, \quad (2.7)$$

and $D^{(r)}_{n,j} = D^{(r)}_{n,rn-j}$ for all $0 \leq j \leq \lfloor rn/2 \rfloor$.

Proof. Note that each summand at the right-hand side of (2.5) is unimodal and symmetric with the same center of symmetry at

$$(i + j) + \frac{n-i-2j}{2} + \frac{(r-2)i}{2} + \frac{(r-1)(n-i)}{2} = \frac{rn}{2}. \quad \square$$

Let $\gamma^{(2)}_{n,k}$ be the number of permutations in $\mathfrak{S}_n$ with exactly $k$ weak excedances and without double excedance, i.e., $\gamma^{(2)}_{n,k} = \sum_{i+j=k} \gamma_{n,i,j}$. For example, the permutations (written in product of disjoint cycles) in $\mathfrak{S}_4$ with exactly 2 weak excedances and without double excedance are as follows:

$$(1)(432), \ (2)(431), \ (3)(421), \ (4)(432), \ (21)(43), \ (31)(42), \ (32)(41), \ (4132), \ (4231).$$

Thus $\gamma^{(2)}_{4,2} = 9$.

For $r = 2$, Theorem 3 reduces to the following result, of which (2.8) infers the $\gamma$-positivity of $D^{(2)}_n(t)$.

Corollary 5. We have $\gamma^{(2)}_{n,k} > 0$ for $1 \leq k \leq n$,

$$D^{(2)}_n(t) = \sum_{k=1}^{n} \gamma^{(2)}_{n,k} t^k (1 + t)^{2n-2k}, \quad (2.8)$$

and

$$d^{(2)}_n(t) = \sum_{1 \leq i + 2j \leq n} \gamma_{n,i,j} 2^{n-i} t^{i+j} (1 + t)^{n-i-2j}. \quad (2.9)$$
Corollary 7. For all $r \geq 1$, the coefficients $d_{n,k}^{(r)}$ of the polynomials $D_{n,k}^{(r)}(t)$ and $d_n^{(r)}(t)$ (cf. (2.3) and (2.4)) are connected by $d_n^{(r)} = D_{n,0}^{(r)}$ and

$$d_{n,k}^{(r)} = \sum_{j=0}^{r-1} D_{n,rk-j}^{(r)} \quad \text{for all } 1 \leq k \leq n. \quad (2.10)$$

The following theorem generalizes the above spiral property to the sequence $\{d_{n,k}^{(r)}\}_{0 \leq k \leq n}$ for all $r \geq 2$.

Corollary 7. For all $r \geq 2$ and $n \geq 0$, the polynomial $d_n^{(r)}(t)$ has the spiral property, namely,

$$d_{n,k}^{(r)} < d_{n,n-k}^{(r)} < d_{n,k+1}^{(r)} \quad \text{for all } 0 \leq k < \lfloor n/2 \rfloor,$$

and $d_{n,[n/2]}^{(r)} < d_{n,[n/2]}^{(r)}$ if $n$ is odd, where $d_{n,k}^{(r)}$ is the coefficient of $t^k$ in the polynomial $d_n^{(r)}(t)$.

Proof. For $r \geq 1$ and $n \geq 1$, letting $a_{n,k}^{(r)} = D_{n,rk}^{(r)}$ and $b_{n,k}^{(r)} = \sum_{j=1}^{r-1} D_{n,rk-j}^{(r)}$, Corollary 4 gives that these two sequences $\{a_{n,k}^{(r)}\}_{0 \leq k \leq n}$ and $\{b_{n,k}^{(r)}\}_{1 \leq k \leq n}$ are also strict unimodal and symmetric, that is,

$$0 \leq a_{n,0}^{(r)} < a_{n,1}^{(r)} < \cdots < a_{n,[n/2]}^{(r)}, \quad 0 < b_{n,1}^{(r)} < b_{n,2}^{(r)} < \cdots < b_{n,[n/2]}^{(r)}. \quad (2.11)$$
and \( a_{n,k}^{(r)} = a_{n,n-k}^{(r)} \) for \( 0 \leq k \leq \lfloor n/2 \rfloor \) and \( b_{n,k}^{(r)} = b_{n,n+1-k}^{(r)} \) for \( 1 \leq k \leq \lfloor n/2 \rfloor \). By Theorem 6, we have \( d_{n,0}^{(r)} = a_{n,0}^{(r)} \) and

\[
d_{n,k}^{(r)} = a_{n,k}^{(r)} + b_{n,k}^{(r)} \quad \text{for all } 1 \leq k \leq n.
\]

Combining (2.11) and (2.12), the sequence \( \{d_{n,k}^{(r)}\}_{1 \leq k \leq n} \) has a spiral property for all \( r \geq 2 \) as follows.

**Remark.** Let \( a_{n}^{(r)}(t) = \sum_{k=0}^{n} a_{n,k}^{(r)} t^k \) and \( b_{n}^{(r)}(t) = \sum_{k=1}^{n} b_{n,k}^{(r)} t^k \) such that

\[
d_{n}^{(r)}(t) = a_{n}^{(r)}(t) + b_{n}^{(r)}(t).
\]

Observing the equation (4.18) in proof of Theorem 6, we have

\[
\sum_{n \geq 0} a_{n}^{(r)}(t) \frac{z^n}{n!} = \frac{e^{(r-1)tz} - te^{(r-1)tz}}{e^{rtz} - te^{rtz}},
\]

(2.13)

\[
\sum_{n \geq 0} b_{n}^{(r)}(t) \frac{z^n}{n!} = \frac{te^{(r-1)tz} - te^{(r-1)tz}}{e^{rtz} - te^{rtz}}.
\]

(2.14)

Athanasiadis and Savvidou [AS13] proved that there are nonnegative integers \( \xi_{n,k}^+ \) and \( \xi_{n,k}^- \) satisfying

\[
d_{n}^{(2)}(t) = \sum_{k \geq 0} \xi_{n,k}^+ t^k (1 + t)^{n-2k} + \sum_{k \geq 1} \xi_{n,k}^- t^k (1 + t)^{n+1-2k}.
\]

(2.15)

We show that the polynomials \( D_{n}^{(2)}(t) \) have a similar expansion as (2.15) with the same coefficients \( \xi_{n,k}^+ \) and \( \xi_{n,k}^- \) and also derive a combinatorial interpretation from Theorem 5. A permutation \( \sigma \in S_n \) is called a drop-colored permutation if some of the drops of \( \sigma \) are colored by, say, bar or tilde. We denote by \( \text{cdrop}(\sigma) \) the number of colored drops in \( \sigma \). Denote by \( \tilde{\gamma}_{n,k}^{(2)} \) the number of permutations \( \sigma \in S_n \) whose total number of weak excedances and colored drops is \( k \), namely, \( \text{wex}(\sigma) + \text{cdrop}(\sigma) = k \). Let \( Z_n \) be the set of drop-colored permutations without double excedance in \( S_n \). For example, the drop-colored permutations (written in product of disjoint cycles) in \( Z_4 \) whose total number of weak excedances and colored drops is 2 are:

\[
(1)(4321), (2)(431), (3)(421), (4)(321), (21)(43), (31)(42), (41)(32),
\]

\[
(4231), (4132), (4321), (4312), (4321), (4321), (4321).
\]

Thus \( \tilde{\gamma}_{4,2}^{(2)} = 15 \). The first values of \( \tilde{\gamma}_{n,k}^{(2)} \) are given in Table 4.

**Theorem 8.** For \( 0 \leq k \leq n \) we have

\[
\tilde{\gamma}_{n,k}^{(2)} = \sum_{i=0}^{k} \tilde{\gamma}_{n,i}^{(2)} \binom{n-i}{k-i}.
\]

(2.16)
Furthermore the following expansions hold true:

\[
D_n^{(2)}(t) = \sum_{k \geq 0} \hat{\gamma}^{(2)}_{n,k} t^k (1 + t^2)^{n-k},
\]

(2.17)

\[
d_n^{(2)}(t) = \sum_{k \geq 0} \hat{\gamma}^{(2)}_{n,k} t^\lfloor k/2 \rfloor (1 + t)^{n-k},
\]

(2.18)

where \([x]\) denotes the least integer great than or equal to \(x\).

For example, we have

\[
D_4^{(2)}(t) = t(1 + t^2)^3 + 15t^2(1 + t^2)^2 + 54t^3(1 + t^2) + 57t^4,
\]

\[
d_4^{(2)}(t) = t(1 + t)^3 + 15t(1 + t)^2 + 54t^2(1 + t) + 57t^2.
\]

Some remarks are in order:

(i) The two different polynomials \(D_n^{(2)}(t)\) and \(d_n^{(2)}(t)\) have the same coefficients \(\hat{\gamma}^{(2)}_{n,k}\), when they are expanded in the basis \(\{t^k(1 + t^2)^{n-k}\}\) and \(\{t^\lfloor k/2 \rfloor (1 + t)^{n-k}\}\), respectively. It would be interesting to have a combinatorial explanation of this coincidence.

(ii) Comparing (2.15) and (2.18) we derive that

\[
\hat{\gamma}^{(2)}_{n,2k} = \xi_{n,k}^+ \quad \text{for} \quad k \geq 0 \quad \text{and} \quad \hat{\gamma}^{(2)}_{n,2k-1} = \xi_{n,k}^- \quad \text{for} \quad k \geq 1.
\]

(2.19)

(iii) Athanasiadis \[Ath13\] has given an apparently different interpretation of the numbers in (2.19) using linear statistics on permutations. Is it easy to relate these two interpretations?

The rest of this paper is organized as follows. In Section 3, after recalling the necessary results in our previous paper \[SZ12\] we prove Theorems 1 and 2. In Section 4 we prove Theorems 3, 6, and 8. Finally, in Section 5, we prove Lemma 16 after recalling some background for the classical combinatorial theory of continued fractions.

3. Proof of Theorems 1 and 2

3.1. Preliminaries. We need some more definitions. For \(\sigma \in S_n\), let \(\sigma(0) = \sigma(n+1) = 0\). Then any entry \(\sigma(i) \ (i \in [n])\) can be classified according to one of the four cases:

- a peak if \(\sigma(i-1) < \sigma(i)\) and \(\sigma(i) > \sigma(i+1)\);
- a valley if \(\sigma(i-1) > \sigma(i)\) and \(\sigma(i) < \sigma(i+1)\);
- a double ascent if \(\sigma(i-1) < \sigma(i)\) and \(\sigma(i) < \sigma(i+1)\);
- a double descent if \(\sigma(i-1) > \sigma(i)\) and \(\sigma(i) > \sigma(i+1)\).

Let \(\text{peak}^*\sigma\) (resp. \(\text{valley}^*\sigma, \text{da}^*\sigma, \text{dd}^*\sigma\)) denote the number of peaks (resp. valleys, double ascents, double descents) in \(\sigma\). Clearly we have \(\text{peak}^*\sigma = \text{valley}^*\sigma + 1\). Now, it is easy to see that

\[
\text{DD}_{n,k} = \{\sigma \in S_n : \text{valley}^*\sigma = k \quad \text{and} \quad \text{dd}^*\sigma = 0\}.
\]
Define the generalized Eulerian polynomial by
\[
A_n(p, q, t, u, v, w) = \sum_{\sigma \in S_n} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des}} u^{\text{da}^*} v^{\text{dd}^*} w^{\text{valley}^*}\sigma.
\] (3.1)

We need the known formula for the so-called Jacobi-Rogers polynomials [GJ83].

Lemma 9. If the ordinary generating function of the sequence \(\{\mu_n\}_n\) has the continued fraction expansion
\[
\sum_{n \geq 0} \mu_n z^n = \frac{1}{1 - b_0 z - \frac{\lambda_1 z^2}{1 - b_1 z - \frac{\lambda_2 z^2}{\ldots}}},
\]
then
\[
\mu_n = \sum_{h \geq 0} \sum_{\substack{n_0, \ldots, n_{h-1} \geq 1 \\ m_0, \ldots, m_h \geq 0}} b_0^{n_0} \cdots b_h^{n_h} \lambda_1^{n_0} \cdots \lambda_h^{n_{h-1}} \rho(n, m),
\] (3.2)
where \(2 (n_0 + \cdots + n_{h-1}) + (m_0 + \cdots + m_h) = n\) and
\[
\rho(n, m) = \prod_{j=0}^{h-1} \binom{n_j + n_{j+1} - 1}{n_j - 1} \prod_{l=0}^{h} \binom{m_l + n_l + n_{l-1} - 1}{m_l}
\]
with convention \(n_{-1} = 1, n_h = 0, \binom{p}{-1} = \delta_{p, -1}\).

The following result was proved in [SZ12, Theorem 2]. We provide below a new and neat proof by using the Jacobi-Rogers formula.

Lemma 10. We have the expansion formula
\[
A_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q)(tw)^k(u + vt)^{n-1-2k},
\] (3.3)
where
\[
a_{n,k}(p, q) = \sum_{\sigma \in DD_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma}.
\] (3.4)
and, for all \(0 \leq k \leq \lfloor (n-1)/2 \rfloor\), the polynomial \(a_{n,k}(p, q)\) is divisible by \((p + q)^k\).

Proof. By [SZ12, Eq. (28)] we have
\[
\sum_{n \geq 1} A_n(p, q, t, u, v, w)x^{n-1} = \frac{1}{1 - (u + tv)[1]_{p,q}x - \frac{[1]_{p,q}[2]_{p,q}twx^2}{1 - (u + tv)[2]_{p,q}x - \frac{[2]_{p,q}[3]_{p,q}twx^2}{\ldots}}},
\]
where \([n]_{p,q} = \frac{p^n - q^n}{p - q}\). By Lemma 9 there are \(a_{n,k}(p,q) \in \mathbb{N}[p,q]\) such that
\[
A_n(p,q,t,u,v,w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p,q)(u + tv)^{n-1-2k}(tw)^k.
\] (3.5)
Thus
\[
A_n(p,q,1,1,0,w) = \sum_{k \geq 0} a_{n,k}(p,q)w^k,
\]
and we derive the combinatorial interpretation (3.4) for \(a_{n,k}(p,q)\) from (3.1). Finally, as \(p + q \mid [n]_{p,q}[n + 1]_{p,q}\) for all \(n \geq 1\), each \(w\) appears with a factor \((p + q)\) in the expansion of the continued fraction, so the polynomial \(a_{n,k}(p,q)\) is divisible by \((p + q)^k\).

\[\square\]

**Definition 11.** For \(\sigma \in S_n\), a value \(x = \sigma(i)\) \(i \in [n]\) is called
- a cyclic peak if \(i = \sigma^{-1}(x) < x\) and \(x > \sigma(x)\);
- a cyclic valley if \(i = \sigma^{-1}(x) > x\) and \(x < \sigma(x)\);
- a double excedance if \(i = \sigma^{-1}(x) < x\) and \(x < \sigma(x)\);
- a double drop if \(i = \sigma^{-1}(x) > x\) and \(x > \sigma(x)\);
- a fixed point if \(x = \sigma(x)\).

Let cpeak \(\sigma\) (resp. cvalley \(\sigma\), cda \(\sigma\), cdd \(\sigma\), fix \(\sigma\)) be the number of cyclic peaks (resp. valleys, double excedances, double drops, fixed points) in \(\sigma\).

For a permutation \(\sigma \in S_n\) the crossing and nesting numbers are defined by
\[
cros \sigma = \# \{(i,j) \in [n] \times [n] : (i < j \leq \sigma(i) < \sigma(j)) \lor (i > j > \sigma(i) > \sigma(j))\}, \quad (3.6)
nest \sigma = \# \{(i,j) \in [n] \times [n] : (i < j \leq \sigma(j) < \sigma(i)) \lor (i > j > \sigma(j) > \sigma(i))\}. \quad (3.7)
\]
For example, the diagram of \(\sigma = 9\ 3\ 7\ 4\ 6\ 10\ 5\ 8\ 1\ 2\) is as follows:

![Diagram of \(\sigma\)]

It easy to see that \(cros \sigma = 5\) and \(nest \sigma = 10\). Indeed, 5 crossings are \((1,6)\), \((2,3)\), \((3,6)\), \((5,6)\), and \((10,9)\) and 10 nestings are \((1,2)\), \((1,3)\), \((1,4)\), \((1,5)\), \((1,8)\), \((3,4)\), \((3,5)\), \((6,8)\), \((9,7)\), and \((10,7)\).

**Definition 12.** For \(\sigma \in S_n\), let \(\sigma(0) = 0\) and \(\sigma(n+1) = n+1\). The corresponding numbers of peaks, valleys, double ascents, and double descents of permutation \(\sigma\) are denoted by
- peak \(\sigma\), valley \(\sigma\), da \(\sigma\), dd \(\sigma\).
Moreover, a double ascent $\sigma(i)$ of $\sigma$ ($i \in [n]$) is said to be a foremaximum if $\sigma(i)$ is a left-to-right maximum of $\sigma$, i.e., $\sigma(j) < \sigma(i)$ for all $1 \leq i < j$. Let $\text{fmax } \sigma$ denote the number of foremaxima of $\sigma$.

For instance, $\text{fmax}(4 \ 2 \ 1 \ 5 \ 7 \ 3 \ 6 \ 8) = 2$. Note that peak $\sigma = \text{valley } \sigma$ for any $\sigma \in \mathcal{S}_n$.

Recall the following result in [SZ12, Theorem 5].

**Lemma 13.** There is a bijection $\Phi$ on $\mathcal{S}_n$ such that for all $\sigma \in \mathcal{S}_n$ we have

$$(\text{nest } \sigma, \text{cros } \sigma, \text{drop } \sigma, \text{cda } \sigma, \text{cdd } \sigma, \text{valley } \sigma, \text{fix } \sigma) = (2-31, 31-2, \text{des } \sigma, \text{da } \sigma - \text{fmax } \sigma, \text{dd } \sigma, \text{valley } \sigma, \text{fmax } \sigma) \Phi(\sigma).$$

Consider the enumerative polynomial

$$B_n(p, q, t, u, v, w, y) = \sum_{\sigma \in \mathcal{S}_n} p^{\text{nest } \sigma} q^{\text{cros } \sigma} t^{\text{drop } \sigma} u^{\text{cda } \sigma} v^{\text{cdd } \sigma} w^{\text{valley } \sigma} y^{\text{fix } \sigma}. \quad (3.8)$$

By [SZ12, (34)] the ordinary generating function of $B_n(p, q, t, u, v, w, y)$ has the following continued fraction expansion:

$$\sum_{n \geq 0} B_n(p, q, t, u, v, w, y) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\ldots}}} \quad (3.9)$$

where $a_h = tw[h + 1]_{p,q}$, $b_h = yp^h + (qu + tv)[h]_{p,q}$, and $c_h = [h]_{p,q}$.

Let $\mathcal{S}^*_n$ be the subset of $\mathcal{S}_n$ consisting of permutations of which each double ascent is also a foremaxima, and

$$\mathcal{S}_{n,k,j} = \{\sigma \in \mathcal{S}_n : \text{valley } \sigma = k, \text{fix } \sigma = j, \text{cda } \sigma = 0\}, \quad (3.10)$$

$$\mathcal{S}^*_{n,k,j} = \{\sigma \in \mathcal{S}^*_n : \text{valley } \sigma = k, \text{da } \sigma = j\}. \quad (3.11)$$

The following result is derived by combining Corollary 7 and Theorem 8 in [SZ12].

**Lemma 14.** We have

$$B_n(p, q, t, u, v, w, y) = \sum_{j=0}^{n} y^j \sum_{k=0}^{\lfloor(n-j)/2\rfloor} b_{n,k,j}(p, q)(tw)^k(qu + tv)^{n-j-2k},$$

where the coefficient $b_{n,k,j}(p, q)$ is given by

$$b_{n,k,j}(p, q) = \sum_{\sigma \in \mathcal{S}_{n,k,j}} p^{\text{nest } \sigma} q^{\text{cros } \sigma} \sum_{\sigma \in \mathcal{S}^*_{n,k,j}} p^{(2-31) \sigma} q^{(31-2) \sigma}. \quad (3.8)$$

A permutation $\sigma \in \mathcal{S}_n$ is a coderangement if $\text{fmax } \sigma = 0$. Let

$$\mathcal{D}^*_n = \{\sigma \in \mathcal{S}_n : \text{fmax } \sigma = 0\}.$$

For example, we have $\mathcal{D}^*_4 = \{2143, 3142, 3241, 4123, 4132, 4213, 4231, 4312, 4321\}$. Thus, $\mathcal{S}_{n,k,0}$ is the subset of derangements in $\mathcal{D}_n$ with exactly $k$ cyclic valleys, and without double excedance, and $\mathcal{S}^*_{n,k,0}$ is the subset of coderangements in $\mathcal{D}^*_n$ with exactly $k$ valleys and without double ascents.
3.2. Proof of Theorem 1. Let
\[ \tilde{A}_n(p, q, t) = \sum_{\pi \in \mathfrak{S}_n} p^{(13-2)\pi} q^{(31-2)\pi} t^{\text{des} \pi}. \]
Since the triple statistics \(((2-31), (31-2), \text{des})\) and \(((2-13), (31-2), \text{des})\) are equidistributed on \(\mathfrak{S}_n\), see [SZ12, Eq. (39)], Lemma 13 infers that
\[ \tilde{A}_n(p, q, t) = \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest} \sigma} q^{\text{cros} \sigma} t^{\text{drop} \sigma}. \]
As \(\text{inv} \sigma^{-1} = \text{inv} \sigma\) and \(\text{drop} \sigma = \text{exc} \sigma^{-1}\) for any \(\sigma \in \mathfrak{S}_n\), we have
\[ A_n(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv} \sigma} t^{\text{exc} \sigma} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv} \sigma} t^{\text{drop} \sigma}. \]
Invoking the known result (see [SZ10, Eq. (40)])
\[ \text{inv} = \text{drop} + \text{cros} + 2 \text{nest}, \]
we have \(\text{inv} - \text{drop} = \text{cros} + 2 \text{nest}\) and
\[ A_n(q, t/q) = \tilde{A}_n(q^2, q, t). \]
The result follows then from Lemma 10 with \(\gamma_{n,k}(q) := a_{n,k}(q^2, q)\).

3.3. Proof of Theorem 2. Let
\[ D_n(q, t) := \sum_{\sigma \in \mathfrak{D}_n} q^{\text{inv} \sigma} t^{\text{exc} \sigma} = \sum_{\sigma \in \mathfrak{D}_n} q^{\text{inv} \sigma} t^{\text{drop} \sigma}. \]
By (3.8) we have \(D_n(q, t) = B_n(q^2, q, qt, 1, 1, 1, 0)\). Hence, setting \(y = 0, p = q^2\) and \(u = v = w = 1\) in Lemma 14 and then replacing \(t\) by \(qt\) we obtain
\[ D_n(q, t) = \sum_{\sigma \in \mathfrak{D}_n} q^{\text{inv} \sigma} t^{\text{drop} \sigma} = \sum_{k=0}^{[n/2]} b_{n,0,k}(q^2, q) q^{n-k} t^k (1 + t)^{n-2k}, \]
where
\[ b_{n,0,k}(q^2, q) q^{n-k} = \sum_{\sigma \in \mathfrak{S}_{n,k,0}} q^{2 \text{nest} \sigma + \text{cros} \sigma + n - \text{cvalley} \sigma}. \]
Since \(n - \text{cvalley} \sigma = n - \text{cpeak} \sigma = \text{drop} \sigma\) for any \(\sigma \in \mathfrak{D}_n\) and
\[ \mathfrak{S}_{n,k,0} = \{ \sigma \in \mathfrak{D}_n : \text{cvalley} \sigma) = k \} = \{ \sigma \in \mathfrak{D}_n : \text{exc} \sigma) = k \} = \mathfrak{D}_{n,k}, \]
the result follows from (3.12).
4. Proofs of Theorems 3, 6 and 8

4.1. Preliminaries. Recall that an integer \( i \) is called \( \text{fixed point} \) (resp. \( A\text{-excedance}, A\text{-weak excedance} \)) of \( \sigma \in \mathbb{Z}_r \wr S_n \) if \( \sigma(i) = i \) (resp. \( \sigma(i) > c_i \), \( \sigma(i) \geq c_i \)). Let

\[
\begin{align*}
\text{fix} \sigma &= \# \{ i \in [n] : i = \sigma(i) \}, \\
\text{exc}_A \sigma &= \# \{ i \in [n] : i < c_i \sigma(i) \}, \\
\text{wex}_A \sigma &= \# \{ i \in [n] : i \leq c_i \sigma(i) \} = \text{exc}_A \sigma + \text{fix} \sigma.
\end{align*}
\]

Also, we define two statistics about colors:

\[
\begin{align*}
\text{wex}_c \sigma &= \# \{ i \in [n] : i \leq |\sigma(i)| \text{ and } \text{col}(\sigma(i)) > 0 \}, \\
\text{csum} \sigma &= \sum_{i \in [n]} \text{col}(\sigma(i)).
\end{align*}
\]

We generalize the crossings of permutations in \( B_n \) (see [CKJV13]) to colored permutations.

**Definition 15** (Crossings). For a permutation \( \sigma = \sigma(1) \ldots \sigma(n) = (\pi_1 \ldots \pi_n) \in \mathbb{Z}_r \wr S_n \) a crossing is a pair \((i, j)\) such that

\[
\begin{align*}
&\bullet \ z_i = z_j = 0 \text{ and } i < j \leq \pi_i < \pi_j; \text{ or } \\
&\bullet \ z_i = z_j = 0 \text{ and } \pi_i < \pi_j < i < j; \text{ or } \\
&\bullet \ z_i > 0, z_j = 0, \text{ and } j \leq \pi_i < \pi_j; \text{ or } \\
&\bullet \ z_i > 0, z_j = 0, \text{ and } \pi_j < i < j; \text{ or } \\
&\bullet \ z_i > 0, z_j > 0, i < j, \text{ and } \pi_j < \pi_i.
\end{align*}
\]

Let \( \text{cros} \sigma \) be the number of crossings in \( \sigma \in \mathbb{Z}_r \wr S_n \).

**Lemma 16.** We have

\[
\sum_{n \geq 0} \sum_{\sigma \in \mathbb{Z}_r \wr S_n} q^{\text{cros} \sigma} t^{\text{wex}_A \sigma} w^{\text{wex}_c \sigma} x^{\text{fix} \sigma} y^{\text{csum} \sigma} z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{1 - b_2 z - \cdots}}}, \tag{4.1}
\]

where the coefficients \( a_h, b_h \ (h \geq 0) \) and \( c_h \ (h \geq 1) \) are given by

\[
\begin{align*}
a_h &= (t + wy[r - 1]q^h)(1 + y[r - 1]q^{h+1}), \\
b_h &= (1 + y[r - 1]q^h)[h]_q + t(x + q[h]_q) + wy[r - 1]q^h[h + 1]_q, \\
c_h &= [h]_q^2.
\end{align*}
\]
Remark. Corteel et al [CKJV13] have proved (4.1) in the case of \( r = 2 \) and \( x = 1 \). Actually we shall prove a refined version of (4.1). See (5.4) in Section 5.

**Lemma 17.** We have

\[
\sum_{n \geq 0} D_n^{(r)}(t) z^n = \frac{1}{1 - b_0 z - \frac{\lambda_1 z^2}{1 - b_1 z - \frac{\lambda_2 z^2}{\ldots}}},
\]

(4.2)

where \( b_h = t[r - 1]_t + h(1 + t)[r]_t \) (\( h \geq 0 \)) and \( \lambda_h = t[r]^2 h^2 \) (\( h \geq 1 \)) and

\[
\sum_{n \geq 0} d_n^{(r)}(t) z^n = \frac{1}{1 - b'_0 z - \frac{\lambda'_1 z^2}{1 - b'_1 z - \frac{\lambda'_2 z^2}{\ldots}}},
\]

(4.3)

where \( b'_h = (r - 1)t + rh(1 + t) \) (\( h \geq 0 \)) and \( \lambda'_h = (rh)^2 t \) (\( h \geq 1 \)).

**Proof.** We can derive (4.2) and (4.3) from Lemma 16 as

\[
\begin{align*}
\text{fexc} &= r(wex_A - \text{fix}) + csum, \\
\text{exc} &= (wex_A - \text{fix}) + wex_c.
\end{align*}
\]

(4.4)

(4.5)

Letting \( q = 1 \), \( t \leftarrow t' \), \( w = 1 \), \( x = 0 \) and \( y \leftarrow t \) in (4.1), we obtain (4.2). Also, letting \( q = 1 \), \( t \leftarrow t \), \( w \leftarrow t \), \( x = 0 \) and \( y = 1 \) in (4.1), we obtain (4.3). \( \square \)

Remark. It is also possible to derive (4.2) and (4.3) from the known generating functions.

(i) Let \( B_n^{(r)}(t, x) = \sum_{\sigma \in \mathcal{S}_n} t^{\text{fexc}} x^{\text{fix}}. \) Since \( D_n^{(r)}(t) = B_n^{(r)}(t, 0) \), from the known generating function formula (see [Lin14, (2.23)] and [FH11, (1.13)] or [FH09, (1.9)] for \( r = 2 \))

\[
\sum_{n \geq 0} D_n^{(r)}(t) \frac{z^n}{n!} = \frac{1 - t}{e^{t/r} - te^z},
\]

(4.6)

we derive that

\[
\sum_{n \geq 0} D_n^{(r)}(t) \frac{z^n}{(r^t)_t n!} = e^{-r/t} \frac{1 - t}{1 - te^{(1-t)z}}.
\]

(4.7)

Also, it is known (see [CTZ09, Cho09]) that

\[
\sum_{n \geq 0} d_n^{(r)}(t) \frac{z^n}{r_n} \frac{1}{n!} = e^{-tz/r} \frac{1 - t}{1 - te^{(1-t)z}}.
\]

(4.8)

Applying the addition formula of Rogers-Stieltjes (see [GJS83, Chap. 5] and [Zen93]) to (4.7) and (4.8) we also obtain the continued fraction expansions (4.2) and (4.3).

(ii) Note that \( \frac{1 - t}{1 - te^{(1-t)z}} \) is the exponential generating function of Eulerian polynomials for symmetric groups.
(iii) In (4.1) if we set \( r = 2 \) and do the following substitution
\[
q \leftarrow 1, \quad w \leftarrow t, \quad x \leftarrow t^{-1}, \quad \text{and} \quad y \leftarrow 1,
\]
then we obtain \( a_h = 4t, \ b_h = (2h + 1)(t + 1) \) and \( c_h = h^2 \). These are the same coefficients obtained when we specialize (3.9) as follows:
\[
p = q = t = u = v = y = 1, \quad w \leftarrow \frac{4t}{(1 + t)^2}, \quad \text{and} \quad z \leftarrow (1 + t)z.
\]
Combining Lemmas 14 and 16, we have
\[
\sum_{\sigma \in \mathfrak{S}_n} \mathfrak{t}^{\text{exc} \sigma} = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( \sum_{j=0}^{n-2k} 2^{(n-2k-j)} \mathfrak{b}_{n,k,j}(1, 1) \right) (4t)^k (1 + t)^{n-2k}, \tag{4.9}
\]
where \( n \geq 2k + j \) and \( \mathfrak{b}_{n,k,j}(1, 1) \) is the cardinality of the set \( \mathfrak{S}_{n,k,j} \), which is defined by (3.10). Note that this result is comparable with the \( \gamma \)-positivity result of Chow [Cho08, Theorem 4.7].

4.2. Proof of Theorem 3. Let
\[
\mathfrak{B}_n(1, 1, 1, 0, 1, w, y) = \sum_{\sigma \in \mathfrak{S}_n} w^{\text{exc} \sigma} y^{\text{fix} \sigma} = \sum_{i,j} \gamma_{n,i,j} w^i y^j, \tag{4.10}
\]
where \( \gamma_{n,i,j} = |\{ \sigma \in \mathfrak{S}_n : \text{cda } \sigma = 0, \ \text{fix } \sigma = i, \ \text{exc } \sigma = j \}|. \) Now, specializing the formula (3.9) yields
\[
\sum_{n \geq 0} \mathfrak{B}_n(1, 1, 1, 0, 1, w, y) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\ldots}}}, \tag{4.11}
\]
where \( a_h = w(h + 1) \ (h \geq 0), \ b_h = y + h \ (h \geq 0), \) and \( c_h = h \ (h \geq 1) \). Comparing the right-hand sides of (4.12) and (4.11), we derive
\[
\mathfrak{D}_n^{(r)}(t) = (1 + t)^n r^n \mathfrak{B}_n(1, 1, 1, 0, 1, w, y) \tag{4.12}
\]
with
\[
w = \frac{t}{(1 + t)^2} \quad \text{and} \quad y = \frac{t[r - 1]_t}{(1 + t)[r]_t}.
\]
Since \( \text{valley}(\sigma) = \text{exc}(\sigma) \) for any \( \sigma \in \mathfrak{S}_n \), it follows from (4.10) and (4.12) that
\[
\mathfrak{D}_n^{(r)}(t) = \sum_{1 \leq i + 2j \leq n} \gamma_{n,i,j} t^{i+j}(1 + t)^{n-i-2j}(r - 1)_j [r]_t [r]_t^{n-i}. \tag{4.13}
\]
In the same vein, comparing the right-hand sides of (4.3) and (4.11), we derive
\[
\mathfrak{d}_n^{(r)}(t) = (1 + t)^n r^n \mathfrak{B}_n(1, 1, 1, 0, 1, w, y) \tag{4.14}
\]
with
\[
w = \frac{t}{r(1 + t)^2} \quad \text{and} \quad y = \frac{t(r - 1)}{r(1 + t)}.
\]
Therefore, by (4.10) and (4.14) we have
\[ d_n^{(r)}(t) = \sum_{1 \leq i+2j \leq n} \gamma_{n,i,j} t^{i+j} (1 + t)^{n-i-2j} (r-1)^i t^{n-i}. \] (4.15)

This completes the proof of Theorem 3. \( \square \)

4.3. **Proof of Theorem 6.** Since any primitive \( r \)th root \( \omega = e^{2\pi i/r} \) of unity satisfies
\[ \sum_{i=0}^{r-1} \omega^i = \begin{cases} 0 & \text{if } r \nmid k, \\ r & \text{if } r \mid k, \end{cases} \]
for integer \( j = 0, \ldots, r-1 \), we have
\[ \frac{1}{r} \sum_{i=0}^{r-1} \frac{\omega^i}{X - \omega^i Y} = \begin{cases} \frac{X^{r-1}}{X^r - Y^r} & \text{if } j = 0, \\ \frac{X^{r-1} Y^{-j}}{X^r - Y^r} & \text{if } 1 \leq j < r. \end{cases} \] (4.16)

It follows that for all \( 0 \leq j < r \),
\[ \sum_{i=0}^{r-1} \omega^i D_n^{(r)}(\omega^i t) = \sum_{k \geq 0} D_n^{(r)} \left( \sum_{i=0}^{r-1} \omega^i (\omega^i t)^k \right) = \sum_{k \geq 0} D_n^{(r)} \left( \sum_{i=0}^{r-1} (\omega^{k+j})^i \right) t^k = r \sum_{k \geq [j/r]} D_{n,k-r}^{(r)} t^{r-k-j}. \] (4.17)

From (4.17) and (4.6), for all \( 0 \leq j < r \), we derive that
\[ \sum_{n \geq 0} \sum_{k \geq [j/r]} D_n^{(r)} t^{r-k-j} \frac{x^n}{n!} = \sum_{n \geq 0} \frac{1}{r} \sum_{i=0}^{r-1} \omega^i D_n^{(r)}(\omega^i t) \frac{x^n}{n!} = \frac{1}{r} \sum_{i=0}^{r-1} \frac{\omega^{i} - \omega^{i(j+1)}}{e^{t x} - \omega^i t e^{x}} \]
\[ = \begin{cases} \frac{e^{(r-1) t x} - t e^{(r-1) t x}}{e^{t x} - t e^{x}} & \text{if } j = 0, \\ \frac{e^{t x} - t e^{x}}{e^{t x} - t e^{x}} \frac{e^{(r-1) t x} - t e^{(r-1) t x}}{e^{t x} - t e^{x}} & \text{if } 1 \leq j < r. \end{cases} \] (4.18)

Summing (4.18) multiplied by \( t^j \) over all \( 0 \leq j < r \) yields
\[ \sum_{n \geq 0} \sum_{k \geq 0} D_n^{(r)} t^{k/r} \frac{x^n}{n!} = (1 - t^r)^{e^{(r-1) t x}} \frac{e^{(r-1) t x}}{e^{t x} - t e^{x}}. \] (4.19)

Comparing (4.19) with (4.8) we see that
\[ d_n^{(r)}(t) = \sum_{\sigma \in D_n^{(r)}} t^{[\text{exc}(\sigma)/r]}. \] (4.20)
Since $\lceil \text{fexc}(\sigma)/r \rceil = k$ if and only if $\text{fexc}(\sigma) = rk - j$ for $j = 0, \ldots, r - 1$, we obtain (2.10).

4.4. Proof of Theorem 8. By Theorem 5 there are $\gamma_{n,i}^{(2)}$ permutations in $S_n$ with $i$ weak excedances with $0 \leq i \leq k$, for any such permutation, there are $\binom{n-i}{k-i}$ ways to choose $k-i$ drops among $n-i$ drops and $2^{k-i}$ ways to color the chosen drops by either tilde or bar. Summing over $i$ we see that the number of permutations in $S_n$ whose total number of weak excedances and colored drops is $k$ is given by

$$\sum_{i=0}^{k} \gamma_{n,i}^{(2)} 2^{k-i} \binom{n-i}{k-i},$$

which is (2.16). Now, writing $(1 + t)^{2n-2k} = (1 + 2t + t^2)^{n-k} = \sum_{j \geq 0} \binom{n-k}{j} (2t)^j (1 + t^2)^{n-k-j}$ and substituting this into (2.8) we get

$$D_n^{(2)}(t) = \sum_{k=1}^{n} \sum_{j \geq 0} \gamma_{n,k}^{(2)} 2^j \binom{n-k}{j} t^{k+j} (1 + t^2)^{n-k-j}$$

$$= \sum_{k=1}^{n} \sum_{i \geq 0} \gamma_{n,i}^{(2)} 2^{k-i} \binom{n-k}{k-i} t^k (1 + t^2)^{n-k},$$

which is (2.17) in view of (2.16). Finally, we derive (2.18) from Theorem 6.

5. A continued fraction expansion

5.1. $r$-colored Laguerre histories. A Motzkin path of length $n$ is a sequence of points $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ in the plan $\mathbb{N} \times \mathbb{N}$ with $\gamma_i = (x_i, y_i)$ ($0 \leq i \leq n$) such that $\gamma_0 = (0, 0)$ and $\gamma_n = (n, 0)$; and each step $(\gamma_{i-1}, \gamma_i)$ satisfies $\gamma_i - \gamma_{i-1} = (1, 1), (1, 0)$ or $(1, -1)$, and is called East, North-East, and South-East, respectively. The height $h_i$ of the $i$th step $(\gamma_{i-1}, \gamma_i)$ is the ordinate of $\gamma_{i-1}$. We can depict a Motzkin path $\gamma$ by drawing a line connecting $\gamma_i$ and $\gamma_{i+1}$ as follows:

![Motzkin Path Diagram]

Given a Motzkin path $\gamma$, we weight each step $(x_{i+1}, y_{i+1}) - (x_i, y_i)$ of height $i$ by $a_i$ (resp. $b_i$ and $c_i$) and define the weight of $\gamma$ as the product of its step weights. For example, the weight of the above Motzkin path is $w(\gamma) = a_0 a_1^2 a_2 b_1 b_2 b_1 c_1 c_2^2 c_3$. Denote by $M_n$ the set
of Motzkin paths of length \( n \geq 1 \). It is folklore (see [GJ83]) that

\[
1 + \sum_{n \geq 1} \sum_{\gamma \in \mathcal{M}_n} w(\gamma) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\ldots}}}. \tag{5.1}
\]

Our starting point is the following \( r \)-dilatation of Laguerre’s continued fraction:

\[
\sum_{n=0}^{\infty} (n! r^n) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\ldots}}}, \tag{5.2}
\]

where \( a_h = r^2, b_h = (2h + 1)r \) (\( h \geq 0 \)), \( c_h = h^2 \) (\( h \geq 1 \)). Indeed, this formula is related to the moment sequence of the simple orthogonal Laguerre polynomials when \( r = 1 \). Clearly \( r^n n! \) is the cardinality of \( \mathbb{Z}_r \wr \mathcal{S}_n \). In order to make the counterpart of \( \mathbb{Z}_r \wr \mathcal{S}_n \) in terms of Motzkin paths, we need to generalize the notion of Laguerre history [dMV94].

**Definition 18** \( (r \)-colored Laguerre history). An \( r \)-colored Laguerre history of length \( n \) is a couple \( h = (\gamma, \xi) \), where \( \gamma \) is a Motzkin path of length \( n \) and \( \xi = ((p_1, q_1), \ldots, (p_n, q_n)) \) is an integer-pair sequence (as labels of steps) satisfying

(i) if the \( i \)th step is North-East, then \( (p_i, q_i) \in [-r, 0]^2 \);

(ii) if the \( i \)th step is East, then \( (p_i, q_i) \in [1, h_i] \times [-r, 0] \cup [-r, 0] \times [1, h_i + 1] \);

(iii) if the \( i \)th step is South-East, then \( (p_i, q_i) \in [1, h_i]^2 \);

where \( h_i \) is the height of the \( i \)th step of \( \gamma \).

Counting the number of possible pairs \( (p, q) \) for each step of height \( h \), the number of \( r \)-colored Laguerre histories associated to a Motzkin path \( \gamma \) is equal to the weight \( w(\gamma) \). Hence, according to (5.2), the number of \( r \)-colored Laguerre histories of length \( n \) is \( n! r^n \).

The following is an example of 3-colored Laguerre history of length 11, where the label on the \( i \)th step stands for \((p_i, q_i)\).

![Diagram of a 3-colored Laguerre history](diagram.png)

When \( r = 1 \), there are two well-known bijections between \( \mathcal{S}_n \) and the Laguerre histories of length \( n \) due to Françon-Viennot and Foata-Zeilberger, which correspond to the two interpretations of Eulerian polynomials in \( \mathcal{S}_n \): one using linear statistics and the other...
using cyclic statistics. The connection between these two bijections is explained in [CSZ97] (see also [dMV94, SZ12]).

The pignose diagram representation for a permutation was introduced in [dMV94, CKJV13]. This is a useful device to illustrate the crossings of a permutation by the intersecting points of two arcs. Actually, we can associate a pignose diagram to any \( r \)-colored permutation \( \sigma = \pi \in \mathbb{Z}_r \wr S_n \) (see §2.2) as follows: A pair of two vertices positioned side by side enclosed by an ellipse is called a pignose. We arrange \( n \) pignoses labeled with \( 1, 2, \ldots, n \) from left to right on an horizontal line. For each \( i \in [n] \), we connect the left vertex of \( i \)th pignose and the right vertex of \( \pi_i \)th pignose with an arc:

- if \( i \leq \pi_i \) and \( z_i = 0 \), draw an arc above the horizontal line;
- if \( \pi_i < i \) and \( z_i = 0 \), draw an arc below the horizontal line;
- if \( z_i > 0 \), draw an arc starting from the left vertex of \( i \)th pignose below the horizontal line to the right vertex of \( \pi_i \)th pignose above the horizontal line clockwise (like a spiral) and label this spiral arc with \( z_i \);
- optionally, if \( i \leq \pi_i \), use a blue color in drawing arcs.

Note that \( \text{cros} \sigma \) is equal to the number of crossing points of two arcs in the pignose diagram of \( \sigma \). For example, the pignose diagram of

\[
\sigma = \begin{pmatrix} 4 & 7 & 2 & 5 & 1 & 6 & 3 \\ 0 & 1 & 0 & 1 & 2 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_4 \wr S_7
\]

is depicted as follows:

and \( \text{cros}(\sigma) = 6 \).

5.2. Bijection \( \Phi \). We construct a bijection \( \Phi \) between \( \mathbb{Z}_r \wr S_n \) and the \( r \)-colored Laguerre histories of size \( n \) by generalizing the Foata-Zeilberger bijection. Given an \( r \)-colored permutation \( \sigma = \sigma(1) \ldots \sigma(n) \in \mathbb{Z}_r \wr S_n \), for \( k = 1, \ldots, n \), we build successively the partial pignose diagrams of the restrictions of \( \sigma \) on \([i]\) by drawing the pignoses labeled with \( 1, \ldots, k \) along with a half-arc connecting with each vertex in these pignoses. Denote by \( P^{(k)} \) the partial pignose diagram on \([k]\). Let \( P^{(0)} \) denote the \( n \) pignoses on the horizontal line. Also \( \gamma^{(0)} = ((0, 0)) \) and \( \xi^{(0)} = \emptyset \). For \( 1 \leq k \leq n \), assume that the mappings...
\[ P^{(k-1)} \mapsto (\gamma^{(k-1)}, \xi^{(k-1)}), \]
where
\[ \gamma^{(k-1)} = (\gamma_0, \ldots, \gamma_{k-1}) \]
and \[ \xi^{(k-1)} = (\xi_1, \ldots, \xi_{k-1}) \],
are already defined from \( \sigma \). We show how to extend this to \( k \). First we draw a half-arc starting from the left vertex of the \( k \)th pignose in \( P^{(k-1)} \) in one of the three possible ways:

For the first case let
\[ p_k = \# \{ j : z_k = z_j = 0, \pi_k < \pi_j < k < j \} + \# \{ j : z_k > 0, z_j = 0, \pi_j < k < j \} + 1, \]
which is positive, for the second case let \( p_k = 0 \), and for the third case let \( p_k = -z_k \), which is negative. If \( p_k \) is positive, then \( (p_k - 1) \) means the number of new crossing points by connecting two vertices by a new arc into a partial pignose diagram, otherwise \( -p_k \) means the color of new arc drawn into a partial pignose diagram.

Then we get the partial diagram \( P^{(k)} \) by drawing a half-arc connecting the right vertex in \( k \)th pignose in the above diagram, for the index \( \ell \) such that \( \pi_\ell = k \), as one of the followings.

For the first case let
\[ q_k = \# \{ j : z_\ell = z_j = 0, \ell < j \leq \pi_\ell < \pi_j \} + \# \{ j : z_\ell > 0, z_j = 0, j \leq \pi_\ell < \pi_j \} + 1, \]
which is positive, for the second case let \( q_k = 0 \), and for the third case let \( q_k = -z_\ell \), which is negative. Similarly, if \( q_k \) is positive, \( (q_k - 1) \) means the number of new crossing points by connecting two vertices by a new arc into a partial pignose diagram, otherwise \( -q_k \) means the color of new arc drawn into a partial pignose diagram.

Lastly, when \( \gamma_{k-1} = (k-1, h_{k-1}) \), we get \( \gamma_k = (k, h_k) \) by defining
\begin{enumerate}
\item \( h_k = h_{k-1} - 1 \) (South-East) if both of \( p_i \) and \( q_i \) are positive,
\item \( h_k = h_{k-1} \) (East) if exactly one of \( p_i \) and \( q_i \) is positive, and
\item \( h_k = h_{k-1} + 1 \) (North-East) if none of \( p_i \) and \( q_i \) are positive.
\end{enumerate}
Actually, \( h_k \) means half of the number of open half-arcs, which do not become complete-arcs, in the partial pignose diagram \( P^{(k)} \).
By taking the above process inductively for \( k = 1, \ldots, n \), we obtain the pignose diagram \( P = P^{(n)} \) of \( \sigma \). Since there is no half-arcs in \( P^{(n)} \), we have \( h_n = 0 \) and \( \gamma^{(n)} \) is a Motzkin path of length \( n \). As the \( r \)-colored Laguerre history corresponding to an \( r \)-colored permutation \( \sigma \), let \( \Phi(\sigma) = (\gamma^{(n)}, \xi^{(n)}) \).

Note that for \( r = 1 \), the above bijection is the same as the Foata-Zeilberger bijection. In Figure 1, we run the above algorithm for the permutation \( \sigma \) of length \( n \). As the \( r \)-colored Laguerre history \( \Phi(\sigma) \) of \( \gamma^{(n)} \) and \( \xi^{(n)} \) is a Motzkin path, we have

\[
\begin{align*}
\text{fix} \sigma &= \# \{i \in [n] : i = \pi_i \text{ and } z_i = 0\}, \\
\text{fix}_c \sigma &= \# \{i \in [n] : i = \pi_i \text{ and } z_i > 0\}, \\
\text{wex}_A \sigma &= \# \{i \in [n] : i \leq \pi_i \text{ and } z_i = 0\}, \\
\text{wex}_c \sigma &= \# \{i \in [n] : i \leq \pi_i \text{ and } z_i > 0\}, \\
\text{drop}_A \sigma &= \# \{i \in [n] : \pi_i < i \text{ and } z_i = 0\}, \\
\text{drop}_c \sigma &= \# \{i \in [n] : \pi_i < i \text{ and } z_i > 0\}, \\
\text{csum}_w \sigma &= \sum_{1 \leq i \leq \pi_n \leq n} z_i, \\
\text{csum}_d \sigma &= \sum_{1 \leq \pi_i < i \leq n} z_i.
\end{align*}
\]

For the above example, we have

\[
\begin{align*}
\text{fix} \sigma &= 1, \\
\text{wex}_A \sigma &= 2, \\
\text{wex}_c \sigma &= 2, \\
\text{drop}_A \sigma &= 2, \\
\text{drop}_c \sigma &= 1,
\end{align*}
\]

also \( \text{csum}_w \sigma = 2 \) and \( \text{csum}_d \sigma = 2 \).

For \( n \geq 1 \) define the weight of a permutation \( \sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n \) by

\[
\mathbf{w}(\sigma) := q^{\text{cros}_A \sigma} \mathbf{wex}_A \sigma \mathbf{w} \mathbf{wex}_c \sigma \mathbf{w} \mathbf{drop}_A \sigma \mathbf{w} \mathbf{drop}_c \sigma \mathbf{w} \text{fix}_A \sigma \mathbf{w} \text{fix}_c \sigma \mathbf{w} \text{csum}_w \sigma \mathbf{w} \text{csum}_d \sigma.
\]

Then, we have the following refined version of Lemma 16:

\[
1 + \sum_{n \geq 1} \sum_{\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n} \mathbf{w}(\sigma) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\ldots}}},
\]

where

\[
\begin{align*}
a_h &= (t + w y[r - 1] y^h)(\bar{t} + \bar{w} y[r - 1] y^h), \\
c_h &= [h]^2, \\
b_h &= (\bar{t} + \bar{w} y[r - 1] y^h) [h] q + t (x + q [h] q) + w y[r - 1] y^h ([h] q + \bar{x} q^h).
\end{align*}
\]

Clearly, the formula (5.4) becomes Lemma 16 for \( \bar{y} = y \) and \( \bar{t} = \bar{w} = \bar{x} = 1 \).

In order to interpret \( \mathbf{w}(\sigma) \) in (5.3) as a weight of \( r \)-colored Laguerre history \( \Phi(\sigma) \), we define the weight of \( k \)-th step \((\gamma_{k-1}, \gamma_k)\) of height \( h_{k-1} \) with its label \((p_k, q_k)\) of Laguerre
Figure 1. An illustration of the bijection $\Phi$
history \((\gamma, \xi)\) by
\[
\begin{align*}
\mathbf{w}(\gamma_k, \xi_k, (p_k, q_k)) = & \begin{cases} 
\mathbf{w}_l(p_k, h_{k-1}) \cdot \mathbf{w}_r(q_k, h_{k-1}) & \text{if } p_k > 0, \\
x \mathbf{w}_l(p_k, h_{k-1}) \cdot \mathbf{w}_r(-q_k, h_{k-1} + 1) & \text{if } p_k = 0 \text{ and } q_k = 1, \\
x \mathbf{w}_l(p_k, h_{k-1}) \cdot \mathbf{w}_r(q_k, h_{k-1} + 1) & \text{if } p_k < 0 \text{ and } q_k = h_{k-1} + 1, \\
x \mathbf{w}_l(p_k, h_{k-1}) \cdot \mathbf{w}_r(q_k, h_{k-1} + 1) & \text{otherwise}, 
\end{cases}
\end{align*}
\]
where
\[
\begin{align*}
\mathbf{w}_l(z, h) = & \begin{cases} 
q^{z-1} & \text{if } z > 0, \\
t & \text{if } z = 0, \\
wy^{-z}q^h & \text{if } z < 0,
\end{cases} \\
\mathbf{w}_r(z, h) = & \begin{cases} 
q^{z-1} & \text{if } z > 0, \\
t & \text{if } z = 0, \\
wy^{-z}q^h & \text{if } z < 0.
\end{cases}
\end{align*}
\]

We define the weight of \(r\)-colored Laguerre history \((\gamma, \xi)\) of length \(n\) as
\[
\mathbf{w}(\gamma, \xi) = \prod_{k=1}^{n} \mathbf{w}(\gamma_k, \xi_k, (p_k, q_k)).
\]

It is easy to check the followings.
\begin{enumerate}
\item[(i)] \(k = \pi_k, \ z_k = 0\) if and only if \( (p_k, q_k) = (0, 1) \). (for the statistic fix)
\item[(ii)] \(k = \pi_k, \ z_k > 0\) if and only if \( (p_k, q_k) = (-z_k, h_{k-1} + 1) \). (for the statistic fix_c)
\item[(iii)] \(k > \pi_k\) if and only if \( p_k > 0 \) and we make a full arc, by connecting two half-arcs, which meets \( (p_k - 1) \) open half-arcs. (for the statistic cros).
\item[(iv)] \(k \leq \pi_k, \ z_k = 0\) if and only if \( p_k = 0 \). (for the statistic \( \text{wex}_A \))
\item[(v)] \(k \leq \pi_k, \ z_k > 0\) if and only if \( p_k = -z_k \). (for two statistics \( \text{wex}_c \) and \( \text{csun}_w \))
\item[(vi)] \(k = \pi_\ell \geq \ell\) if and only if \( q_k > 0 \) and we make a full arc, by connecting two half-arcs, which meets \( (q_k - 1) \) open half-arcs. (for the statistic \( \text{cros} \))
\item[(vii)] \(k = \pi_\ell < \ell, \ z_\ell = 0\) if and only if \( q_k = 0 \). (for the statistic \( \text{drop}_A \))
\item[(viii)] \(k = \pi_\ell < \ell, \ z_\ell > 0\) if and only if \( q_k = -z_\ell \). (for two statistics \( \text{drop}_c \) and \( \text{csun}_d \))
\item[(ix)] if \( z_\ell > 0 \), we draw a spiral arc, which meets \( h \) open half-arcs. (for the statistic \( \text{cros} \))
\end{enumerate}

According to the above conditions, given an \(r\)-colored permutation \( \sigma \in \mathbb{Z}_r \setminus \mathcal{G}_n \), we have \( \mathbf{w}(\sigma) = \mathbf{w}(\gamma, \xi) \), where \( \Phi(\sigma) = (\gamma, \xi) \).

Summing the weight of possible pairs \((p, q)\) for each step of height \(h\), the weight sum of \(r\)-colored Laguerre histories associated to a Motzkin path \(\gamma\) is equal to the weight \(\mathbf{w}(\gamma)\) with
\[
\begin{align*}
\alpha_h &= \sum_{p=1-r}^{0} \sum_{q=1-r}^{0} \mathbf{w}(\gamma, (p, q)), \\
\chi_h &= \sum_{p=1}^{h} \sum_{q=1}^{h} \mathbf{w}(\gamma, (p, q)), \\
\beta_h &= \sum_{p=1}^{h} \sum_{q=1}^{h} \mathbf{w}(\gamma, (p, q)) + \sum_{p=1-r}^{0} \sum_{q=1}^{h+1} \mathbf{w}(\gamma, (p, q)),
\end{align*}
\]
where
\[
\mathbf{w}(\gamma, (p, q)) = \begin{cases} 
\mathbf{w}_l(p, h) \cdot \mathbf{w}_r(q, h) & \text{if } p > 0, \\
x \mathbf{w}_l(p, h) \cdot \mathbf{w}_r(q, h + 1) & \text{if } (p, q) = (0, 0), \\
x \mathbf{w}_l(p, h) \cdot \mathbf{w}_r(q, h + 1) & \text{if } (p, q) = (0, h + 1), \\
x \mathbf{w}_l(p, h) \cdot \mathbf{w}_r(q, h + 1) & \text{otherwise}.
\end{cases}
\]
We derive (5.5) from (5.6).

Remark. It is instructive to look at the weight of each step from the corresponding pignose diagram. For $x = \tilde{x} = 1$, every vertex is connected to only one half-arc with its weight in one of following six ways:

- $[h]_q \circ t$
- $t \circ [h]_q$
- $i \circ [h]_q$
- $\tilde{w} \circ [r - 1]_q$
- $\tilde{w} \circ [r - 1]_q$
- $q^h \circ [r - 1]_q$

Here the parameter $h$ stands for the number of open half-arcs, which do not become complete-arcs, above horizontal line in the partial pignose diagram.

Acknowledgements

The research of the first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (NRF-2012R1A1A1014154) and INHA UNIVERSITY Research Grant.

References

[AS12] Christos A Athanasiadis and Christina Savvidou. The local $h$-vector of the cluster subdivision of a simplex. Sém. Lothar. Combin. 66 (2011/12), Art. B66c, 21 pp.
[AS13] Christos A Athanasiadis and Christina Savvidou. A symmetric unimodal decomposition of the derangement polynomial of type $B$. arXiv preprint arXiv:1305.3202, 2013.
[Ath13] Christos A Athanasiadis. Edgewise subdivisions, local $h$-polynomials and excedances in the wreath product $\mathbb{Z}_r \wr S_n$, SIAM J. Discrete Math., Vol. 28, No. 3, pp. 1479–1492.
[BG06] Eli Bagno and David Garber. On the excedance number of colored permutation groups. Sém. Lothar. Combin., 53:Art. B53f, 17 pp. (electronic), 2006.
[Brä08] Petter Brändén. Actions on permutations and unimodality of descent polynomials. European J. Combin., 29(2):514–531, 2008.
[Bre94] Francesco Brenti. $q$-Eulerian polynomials arising from Coxeter groups. European J. Combin., 15(5):417–441, 1994.
[CTZ09] William Y. C. Chen, Robert L. Tang, and Alina F. Y. Zhao. Derangement polynomials and excedances of type $B$. Electron. J. Combin., 16(2, Special volume in honor of Anders Bjorner):Research Paper 15, 16, 2009.
[Cho08] Chak-On Chow. On certain combinatorial expansions of the Eulerian polynomials. Adv. in Appl. Math., 41 (2008), no. 2, 133–157.
[Cho09] Chak-On Chow. On derangement polynomials of type $B$. II. J. Combin. Theory Ser. A, 116(4):816–830, 2009.
[CSZ97] Robert J. Clarke, Einar Steingrímsson, and Jiang Zeng. New Euler-Mahonian statistics on permutations and words. Adv. in Appl. Math., 18(3):237–270, 1997.

[CKJV13] Sylvie Corteel, Jang Soo Kim, and Matthieu Josuat-Vergès. Crossings of signed permutations and q-eulerian numbers of type B. Journal of Combinatorics, 4(2):191–228, 2013.

[DPS09] Dilks, Kevin; Petersen, T. Kyle; Stembridge, John R. Affine descents and the Steinberg torus. Adv. in Appl. Math., 42 (2009), no. 4, 423–444.

[dMV94] Anne de Médicis and Xavier G. Viennot. Moments des q-polynômes de Laguerre et la bijection de Foata-Zeilberger. Adv. in Appl. Math., 15(3):262–304, 1994.

[FH09] Dominique Foata and Guo-Niu Han. Signed words and permutations. V. A sextuple distribution. Ramanujan J., 19(1):29–52, 2009.

[FH11] Dominique Foata and Guo-Niu Han. The decrease value theorem with an application to permutation statistics. Adv. in Appl. Math. 46 (2011), no. 1-4, 296–311.

[Fir04] Michael Fire. Statistics on Wreath Products. ArXiv Mathematics e-prints, sep 2004.

[FS70] Dominique Foata and Marcel-P. Schützenberger. Théorie géométrique des polynômes eulériens. Lecture Notes in Mathematics, Vol. 138. Springer-Verlag, Berlin, 1970.

[GJ83] I. P. Goulden and D. M. Jackson. Combinatorial enumeration. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1983. With a foreword by Gian-Carlo Rota, Wiley-Interscience Series in Discrete Mathematics.

[HZ13] Guoniu Han, Frédéric Jouhet, and Jiang Zeng. Two new triangles of q-integers via q-Eulerian polynomials of type A and B. Ramanujan J., 31(1-2):115–127, 2013.

[Lin14] Zhicong Lin. Eulerian calculus arising from permutation statistics. Ph.D. thesis, https://tel.archives-ouvertes.fr/tel-00996105, Université Claude Bernard - Lyon I, 2014.

[LZ14] Zhicong Lin and Jiang Zeng. The γ-positivity of basic eulerian polynomials via group actions. J. Combin. Theory Ser. A, 135:112–129, 2015.

[Mon13] Pietro Mongelli. Excedances in classical and affine Weyl groups. J. Combin. Theory Ser. A, 120(6):1216–1234, 2013.

[Pet07] T. Kyle Petersen. Enriched P-partitions and peak algebras. Adv. Math., 209 (2007), no. 2, 561–610.

[Ste94] Einar Steingrímsson. Permutation statistics of indexed permutations. European J. Combin., 15(2):187–205, 1994.

[Ste08] John R. Stembridge. Coxeter cones and their h-vectors. Adv. Math. 217 (2008), no. 5, 1935–1961.

[SW10] John Shareshian and Michelle L. Wachs. Eulerian quasisymmetric functions. Adv. Math., 225(6):2921–2966, 2010.

[SZ10] Heesung Shin and Jiang Zeng. The q-tangent and q-secant numbers via continued fractions. European J. Combin., 31(7):1689–1705, 2010.

[SZ12] Heesung Shin and Jiang Zeng. The symmetric and unimodal expansion of Eulerian polynomials via continued fractions. European J. Combin., 33(2):111–127, 2012.

[Zen93] Jiang Zeng. Énumérations de permutations et J-fractions continues. European J. Combin., 14(4):373–382, 1993.

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