When Hom-Lie Structures Form a Jordan Algebra

Pasha Zusmanovich

Abstract. We are concerned with the question when Hom-Lie structures on a Lie algebra are closed with respect to the Jordan product. Somewhat unexpectedly, this leads us to certain questions connected with the Yang–Baxter equation, and with decomposition of a Lie algebra into the sum of subalgebras with given properties.

Introduction

Recall that a Hom-Lie structure on a Lie algebra $L$ is a linear map $\varphi : L \to L$ satisfying the Hom-Jacobi equation:

\[(1) \quad [[x,y], \varphi(z)] + [[z,x], \varphi(y)] + [[y,z], \varphi(x)] = 0\]

for any $x,y,z \in L$. The interest in such structures arose in the more general context of so-called Hom-Lie algebras, which are algebras with multiplication $[\cdot, \cdot]$ and endomorphism $\varphi$ satisfying the Hom-Jacobi equation $(1)$. During the last decade there was a surge of interest in such algebras; for the history and motivations see [M], [MZ1], [MZ2], [XJL], [XL], and references therein.

The set of all Hom-Lie structures on a Lie algebra $L$, denoted by $\text{HomLie}(L)$, obviously forms a vector space. Hom-Lie structures on simple finite-dimensional Lie algebras over a field of zero characteristic, as well as on simple graded Lie algebras of finite growth (which are exhausted by loop algebras, untwisted or twisted; Lie algebras of Cartan type; and the Witt algebra) were described in [XJL] and [XL]. There, an interesting observation was made: on these algebras, the space of all Hom-Lie structures is closed with respect to the anticommutator; that is, for any two Hom-Lie structures $\varphi, \psi \in \text{HomLie}(L)$ on a Lie algebra $L$ from these classes,

\[(2) \quad \frac{1}{2}(\varphi \circ \psi + \psi \circ \varphi) \in \text{HomLie}(L),\]

where $\circ$ denotes the composition of linear maps. In other words, $\text{HomLie}(L)$ with respect to the anticommutator forms a (special) Jordan algebra. This was proved by case-by-case computations: for most of the algebras $L$ from these classes, $\text{HomLie}(L)$ coincides with the one-dimensional space $K\text{id}_L$ consisting of scalar multiples of the identity map $\text{id}_L$, and thus (2) is satisfied trivially; in the nontrivial cases (mostly related to $\mathfrak{sl}(2)$ and the Witt algebra), the validity of (2) was established by direct verification.

In [MZ1] we have provided further examples, in the class of current and Kac-Moody Lie algebras, for which $\text{HomLie}(L)$ forms a Jordan algebra, and other examples for which it does not. A natural question arises: for which Lie algebras this is true? To which degree this is a common phenomenon?

In this note we show that unless the space of Hom-Lie structures is trivial, this phenomenon is rare, at least in the classes of “interesting” (i.e., simple and close to them) Lie algebras. In §2 we show that if the space of Hom-Lie structures is closed with respect to the anticommutator, then it either satisfies, as a Jordan algebra, some restrictive properties, or the underlying Lie algebra satisfies another (Lie-algebraic) restrictive properties, dubbed by us the properties $\diamondsuit$ and $\heartsuit$. We focus mainly on finite-dimensional algebras, so our results do not imply automatically all aforementioned results from [XJL] and [XL]; but at least in the finite-dimensional case, in §4 we sketch a uniform proof without going to case-by-case computations. In passing, in §4 we reformulate the Hom-Jacobi equation in terms of another equation which, in its turn, is related to the classical Yang-Baxter equation; in §3 we discuss Hom-Lie structures on generalized Witt algebras; and in §4 we discuss the properties $\diamondsuit$ and $\heartsuit$; the property $\diamondsuit$ can be considered in the context which attracted a considerable attention in the literature: structure of Lie algebras decomposable into the vector space direct sum of subalgebras with given properties. In the last §5 we speculate about the possibility to replace in the considerations above “Jordan” by “Hom-Jordan”.

Our notation and conventions are mostly standard. The ground field $K$ is assumed to be arbitrary, of characteristic $\neq 2$, unless specified otherwise. By $\overline{K}$ is denoted the algebraic closure of $K$. All Hom’s are understood in the category of vector spaces over $K$. Occasionally we will use the notion of the plus algebra of an algebra $A$, denoted by $A^{(+)}$; this is the algebra defined on the same vector space $A$, with multiplication defined by the anticommutator of the initial multiplication in $A$: $a * b = \frac{1}{2}(ab + ba)$.

Date: First written September 9, 2021; last minor revision June 4, 2022.
J. Algebra Appl., to appear.
1. Connection with the Classical Yang–Baxter Equation

**Lemma 1.** For any Lie algebra $L$, a linear map $\varphi : L \to L$ is a Hom-Lie structure on $L$ if and only if the bilinear map $F_\varphi : L \times L \to L$ defined by

\[ F_\varphi(x, y) = [\varphi(x), y] + [x, \varphi(y)] \]

satisfies the equation

\[ [F_\varphi(x, y), z] + [F_\varphi(z, x), y] + [F_\varphi(y, z), x] = 0 \]

for any $x, y, z \in L$.

**Proof.** Substituting (3) to (4) and rearranging terms, we get

\[
[[\varphi(x), y], z] + [[z, \varphi(x)], y] + [[\varphi(y), z], x] + [[x, \varphi(y)], z] + [[\varphi(z), x], y] + [[y, \varphi(z)], x] = 0.
\]

Using the Jacobi identity, the latter equality is equivalent to

\[ -[[y, z], \varphi(x)] - [[z, x], \varphi(y)] - [[x, y], \varphi(z)] = 0,\]

which is exactly the Hom-Jacobi equation. 

The equation (4) is remarkable. Recall that a linear map $\varphi : L \to L$ on a Lie algebra $L$ is called an R-matrix if the bracket

\[ [x, y]_R = \frac{1}{2}([\varphi(x), y] + [x, \varphi(y)]) \]

defines a new Lie algebra structure on $L$, i.e., satisfies the Jacobi identity (for the definitions and facts related to R-matrices and Yang–Baxter equations mentioned in this paragraph, we refer to the survey [RS], §2). It is known that $\varphi$ is an R-matrix if and only if the bilinear map

\[ F_\varphi(x, y) = [\varphi(x), y] + [x, \varphi(y)] \]

satisfies the equation (4). In this case the equation (4) is “usually replaced” by the mere $F_\varphi(x, y) = 0$, or by $F_\varphi(x, y) = -[x, y]$, which constitutes, respectively, the classical Yang–Baxter equation, and the modified classical Yang–Baxter equation. Quite surprisingly, the equation (4) in the case of skew-symmetric (or, more generally, arbitrary bilinear) $F$ was not studied systematically on various “interesting” classes of Lie algebras. We suggest that such systematic study may justify that “usual replacement” and, at the same time, generalize all computations of Hom-Lie structures done so far (as, given (4) and knowing $F_\varphi$, it is fairly easy to infer $\varphi$). Probably, it will be relevant also in other situations (the equation (4) also arises in questions related to Hochschild cohomology of the smash product of a symmetric algebra and a group acting on it – see [FK, §4] and references therein). On the other hand, symmetric solutions of the equation (4) were studied in [Be] in the context of yet another, totally unrelated algebraic problem (determination of Lie-admissible third power-associative algebras).

The equation (4) can be also interpreted as a binary extension of the Hom-Lie equation (1), where the univariate “twisting” map is replaced by a two-variate one, fitted into the Jacobi-like identity. Viewed this way, Lemma 1 provides an elementary, but interesting connection between “unary” and “binary” Hom-Lie structures on the same Lie algebra.

**Lemma 2.** For any Lie algebra $L$ the following are equivalent:

(i) HomLie($L$) is closed with respect to the anticommutator;
(ii) For any $\varphi \in$ HomLie($L$), $\varphi^2 \in$ HomLie($L$);
(iii) For any $\varphi \in$ HomLie($L$) and any polynomial $f \in K[t]$, $f(\varphi) \in$ HomLie($L$);
(iv) For any $\varphi, \psi \in$ HomLie($L$), the bilinear map $F_{\varphi, \psi} : L \times L \to L$ defined as

\[ F_{\varphi, \psi}(x, y) = [\varphi(x), \psi(y)] + [\psi(x), \varphi(y)] \]

satisfies the equation (4).
Proof. (i) ⇒ (ii). In the condition (2), set ψ = ϕ.
(ii) ⇒ (i). Linearize: replace ϕ by ϕ + ψ.
(i) ⇒ (iii). Follows from the fact that for any nonnegative integer n, the n-fold anticommutator of a map ϕ with itself coincides with ϕ^n.
(iii) ⇒ (ii). Obvious.
(i) ⇔ (iv). Straightforward computation, like in the proof of Lemma [1]

2. CONSEQUENCES OF CLOSEDNESS OF HOM–LIE STRUCTURES WITH RESPECT TO THE ANTICOMMUTATOR

For the convenience of the exposition, we will fix from the very beginning the consequences we will arrive at. These are two conditions imposed on a Lie algebra L:

(◇) L is decomposed into the direct sum of vector spaces L = A ⊕ B such that A, B ≠ 0, [[A, A], B] = 0, and [[B, B], A] = 0.
(▼) There are nonzero subspaces A, B of L such that A ⊆ B, dimA + dimB = dimL, [[A, A], B] = 0, and [[B, B], A] = 0.

We failed to find a reference in the literature to the following elementary linear algebraic fact.

Lemma 3. If ϕ is a linear map of a finite-dimensional vector space over a perfect field K, then there is a polynomial f(t) with coefficients in K such that f(ϕ) is an idempotent map of the same rank as ϕ.

Proof. 1. Let ϕ be nondegenerate. Since the free term of the characteristic polynomial χ(t) of ϕ is equal to ±det(ϕ), the required polynomial is \( \frac{1}{\det(\varphi)} \chi(t) ± 1 \).

2. Let ϕ be semisimple. Then the underlying vector space is decomposed as the direct sum Im(ϕ) ⊕ Ker(ϕ), the restriction of ϕ on Im(ϕ) is nondegenerate, and the rank of ϕ is equal to dim(Im(ϕ)). By step 1, there is a polynomial f(t) such that f(ϕ) is the identity map on Im(ϕ). Since f(ϕ) acts trivially on Ker(ϕ), f(t) will be a required polynomial.

3. In the general case, the semisimple component in the Jordan-Chevalley decomposition of ϕ is equal to f(ϕ) for a certain polynomial f(t) ([C Chap. I, §8, Théorème 7]). By step 2, there is a polynomial g(t) such that g(f(ϕ)) is an idempotent map of the same rank as f(ϕ). Since the rank of f(ϕ) is equal to the rank of ϕ, g(f(t)) will be a required polynomial.

Lemma 4. For any finite-dimensional Lie algebra L the following are equivalent:

(i) L has an idempotent Hom-Lie structure, different from zero and from the identity map;
(ii) L satisfies the property ▼.

Proof. As any idempotent map ϕ can be represented in a suitable basis by a diagonal matrix with 1 and 0 on the diagonal, we have a direct sum decomposition L = A ⊕ B such that ϕ is the identity map on A and the zero map on B. Then the conditions [[A, A], B] = 0 and [[B, B], A] = 0 are equivalent to the validity of the Hom-Jacobi equation.

Proposition 1. Let L be a finite-dimensional Lie algebra over a perfect field, such that HomLie(L) is closed with respect to the anticommutator. Then one of the following holds:

(i) HomLie(L) is a Jordan algebra in which every element is either invertible or nilpotent;
(ii) L satisfies the equivalent conditions of Lemma [3].

Moreover, if the ground field K is algebraically closed, then the condition (i) can be replaced by the condition

(i)' HomLie(L) is isomorphic to the semidirect sum of K and a nilpotent algebra.

Proof. Assume that the condition (i) is not satisfied. Pick a Hom-Lie structure ϕ on L which is not invertible and is not nilpotent as an element of the Jordan algebra HomLie(L). Since HomLie(L) is a special Jordan algebra, the invertibility and nilpotency in the Jordan sense coincides, respectively, with invertibility and nilpotency in the associative sense; that is, the rank of ϕ is strictly between 0 and dimL. By Lemma[2], f(ϕ) is a Hom-Lie structure on L for any polynomial f(t), and then by Lemma[3] L has a nontrivial idempotent Hom-Lie structure.

The Jordan algebra HomLie(L) is isomorphic to the semidirect sum of a semisimple algebra and the nilpotent radical. The semisimple part is isomorphic to the direct sum of simples; if the sum contains
more than one summand, then the unit of each summand is an idempotent different from the unit in the whole \( \text{HomLie}(L) \), and then by Lemma [4] \( L \) satisfies the property \( \lozenge \). Therefore, in the condition (i) we may assume that \( L \) is isomorphic to the semidirect sum of a simple and a nilpotent algebra; but if the ground field \( K \) is algebraically closed, any finite-dimensional simple Jordan algebra with all elements either invertible or nilpotent, is isomorphic to \( K \).

**Proposition 2.** Let \( L \) be a finite-dimensional Lie algebra such that \( \text{HomLie}(L) \) is closed with respect to the anticommutator. Then one of the following holds:

(i) \( \text{HomLie}(L) \) is a semisimple Jordan algebra without nonzero nilpotent elements;

(ii) \( L \) satisfies the property \( \lozenge \).

Moreover, if the ground field \( K \) is algebraically closed, then the condition (i) can be replaced by the condition

(i)' \( \text{HomLie}(L) \) is isomorphic to the direct sum of several copies of \( K \).

**Proof.** If \( \text{HomLie}(L) \) does not contain nonzero nilpotent elements, then its radical is zero, and hence it is a semisimple Jordan algebra.

If \( \text{HomLie}(L) \) contains a nonzero nilpotent element, then, raising it to the appropriate power, and utilizing Lemma [2] we can find a nonzero Hom-Lie structure \( \phi \) on \( L \) such that \( \phi^2 = 0 \). Set \( A = \text{Im}(\phi) \) and \( B = \text{Ker}(\phi) \). The equality \( [[B, B], A] = 0 \) follows from the Hom-Jacobi equation. Since \( A \subseteq B \), the latter equality implies also \( [[B, A], A] = 0 \), and by the Jacobi identity we get \( [[A, A], B] = 0 \).

The Jordan algebra \( \text{HomLie}(L) \), being semisimple, is isomorphic to the direct sum of simple algebras, and if the ground field \( K \) is algebraically closed, then each simple finite-dimensional Jordan algebra without nonzero nilpotent elements is isomorphic to \( K \).

**Corollary 1.** Let \( L \) be a finite-dimensional Lie algebra over an algebraically closed field \( K \), such that \( \text{HomLie}(L) \) is closed with respect to the anticommutator. Then one of the following holds:

(i) \( \text{HomLie}(L) \cong K \);

(ii) \( L \) satisfies the property \( \Diamond \);

(iii) \( L \) satisfies the property \( 
\lozenge \).

**Proof.** If \( L \) satisfies neither \( \Diamond \) nor \( \lozenge \), then by Propositions [1] and [2] \( L \) satisfies simultaneously the respective conditions (i)' of those propositions, whence \( \text{HomLie}(L) \cong K \).}

Finally, let us indicate another strong consequence of closedness of Hom-Lie structures with respect to the anticommutator.

Recall that the centroid of a Lie algebra \( L \), denoted by Cent\((L)\), is the space of linear maps \( \varphi : L \to L \) commuting with adjoint maps, i.e., satisfying the condition

\[
\varphi([x, y]) = [\varphi(x), y]
\]

for any \( x, y \in L \). Centroid can be thought as the invariant submodule \( \text{Hom}(L, L)^L \) of the standard \( L \)-module \( \text{Hom}(L, L) \), with the \( L \)-action given by the formula

\[
(y \cdot \varphi)(x) = [\varphi(x), y] - \varphi([x, y]),
\]

where \( x, y \in L \) and \( \varphi \in \text{Hom}(L, L) \).

It is clear that

\[
\text{Cent}(L) \subseteq \text{HomLie}(L).
\]

**Proposition 3.** Let \( L \) be a Lie algebra such that \( \text{HomLie}(L) \) is closed with respect to the anticommutator. Then:

(i) \( L \) is homomorphically mapped to the Lie algebra \( \text{Der}(\text{HomLie}(L)) \);

(ii) \( \text{Aut}(L) \) is homomorphically mapped to the group \( \text{Aut}(\text{HomLie}(L)) \).

Here, in both cases, \( \text{HomLie}(L) \) is considered as a Jordan algebra with respect to the anticommutator.

**Proof.** (i) As noted in [MZ1, Lemma 1], \( \text{HomLie}(L) \) is a submodule of the \( L \)-module \( \text{Hom}(L, L) \). The \( L \)-action (5) is obviously compatible with the product (2), thus \( L \) acts on the Jordan algebra \( \text{HomLie}(L) \) by derivations.
(ii) Similarly, $\text{HomLie}(L)$ is a submodule of the $\text{Aut}(L)$-module $\text{Hom}(L, L)$, where the $\text{Aut}(L)$-action is given by the conjugation $\text{Ad}_\alpha: \varphi \mapsto \alpha^{-1} \circ \varphi \circ \alpha$ for $\varphi \in \text{HomLie}(L)$ and $\alpha \in \text{Aut}(L)$ (MZ1 §1). This action is obviously compatible with the product (2), which yields the action of $\text{Aut}(L)$ on the Jordan algebra $\text{HomLie}(L)$.

Corollary 2. Let $L$ be a simple Lie algebra such that $\text{HomLie}(L)$ is closed with respect to the anticommutator. Then either $\text{HomLie}(L) = \text{Cent}(L)$, or $L$ is isomorphic to a subalgebra of $\text{Der}(\text{HomLie}(L))$.

Proof. Since $L$ is simple, the kernel of the homomorphism from Proposition[3]i) either coincides with the whole $L$, or is zero. In the first case, $L$ acts on $\text{HomLie}(L)$ trivially, i.e., $\text{HomLie}(L) \subseteq \text{Cent}(L)$, and due to (6), the equality holds. In the second case, the homomorphism is an embedding.

3. WITT ALGEBRAS, FINITE- AND INFINITE-DIMENSIONAL

To see that the proofs of Propositions[1] and[2] do not work in the infinite-dimensional case, let us turn to (generalized) Witt algebras. Let $G$ be a subgroup of the additive group of the ground field $K$, and $W_G$ is the Lie algebra linearly spanned by elements $e_\alpha$, $\alpha \in G$, with the bracket

\[ [e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha + \beta}. \]

Specializing $G$ to various particular cases, we get various instances of Witt algebras. Thus, if $K$ is of characteristic zero, and $G = \mathbb{Z}$, we get the (two-sided) infinite-dimensional Witt algebra, and if $K$ is of characteristic $p > 0$, containing the field $G = \mathbb{GF}(p^n)$ (isomorphic, as an additive group, to $([\mathbb{Z}/p\mathbb{Z}]^n)$, we get the Zassenhaus algebra $W_1(n)$. The latter algebra has another realization as a $\mathbb{Z}$-graded Lie algebra, with a basis $\{e_{-1}, e_0, e_1, \ldots, e_{p^n-2}\}$, and multiplication

\[ [e_i, e_j] = \left( \binom{i+j+1}{j} - \binom{i+j+1}{i} \right) e_{i+j}. \]

Theorem 1. $\text{HomLie}(W_G) \simeq K[G]$. A basis of $\text{HomLie}(W_G)$ may be chosen to consist of “shifts” $e_\alpha \mapsto e_{\alpha + \sigma}$ for a fixed $\sigma \in G$.

Proof. A verbatim repetition of reasonings in the proof of [XL, Theorem 3.2], which treats the case $G = \mathbb{Z}$.

Therefore, the Hom-Lie structures on $W_G$ are closed with respect to composition (and thus with respect to the anticommutator), and form the commutative associative algebra isomorphic to the group algebra $K[G]$ (in the case $G = \mathbb{Z}$ this was already noted in [XL]).

Proposition 4. If $K$ is of characteristic zero, then the algebra $W_G$ satisfies neither the property $\Diamond$, nor the property $\heartsuit$.

Proof. Assume the contrary. Any nonzero abelian subalgebra of $W_G$ is one-dimensional (see, e.g., [Ku, Corollary(a)])). Hence the condition $[[A, A], B] = 0$ implies either $[A, A] = 0$, and hence $A$ is one-dimensional, or $[A, A] = B$ is one-dimensional. As in [Ku], using the fact that $G$ can be ordered, it is easy to see that if $A$ is a subspace of $W_G$ such that $[A, A]$ is one-dimensional, then $A$ is two-dimensional. Therefore, in any case $\dim A \leq 2$. Similarly, $[[B, B], A] = 0$ implies $\dim B \leq 2$, and hence $\dim W_G \leq 4$, a contradiction.

By contrast with the infinite-dimensional characteristic zero case, we have

Proposition 5. If $G$ is finite, then the algebra $W_G$ does not satisfy the property $\Diamond$, and satisfies the property $\heartsuit$.

Proof. If $G$ is finite, then $K$ is necessarily of characteristic $p > 0$, $G$ is the additive group of $\mathbb{GF}(p^n)$ for some $n$, $W_G \simeq W_1(n)$, and by Theorem[1] $\text{HomLie}(W_G)$ is isomorphic to the reduced polynomial algebra $K[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$. The latter algebra does not contain nontrivial idempotents, and by Lemma[4] $W_G$ does not satisfy the property $\Diamond$.

Looking at the realization (7) of $W_1(n)$, and setting $A = \langle e_{p^n-2} \rangle$ and $B = \langle e_0, e_1, \ldots, e_{p^n-2} \rangle$, we see that $W_1(n)$ satisfies the property $\heartsuit$.
We believe that the algebra $W_G$ does not satisfy the property ♦ in all cases, but the proof of this is lacking (infinite-dimensional Witt algebras over fields of positive characteristic seem to be more tricky).

Proposition 4 shows that $W_G$, in the case of zero characteristic, provides an infinite-dimensional counterexample to Propositions 1 and 2. Indeed, using the fact that $G$ is ordered, it is easy to see that $K[G]$ does contain neither nontrivial idempotents, nor nontrivial nilpotent elements. Moreover, $K[G]$ satisfies neither the condition (i) of Proposition 1 nor the condition (i)' of Proposition 2 (but satisfies the condition (ii) of Proposition 2).

4. MORE ALGEBRAS SATISFYING THE PROPERTIES ♦ AND ♣

As an illustration of an application of Propositions 1, 3 let us sketch the proof of the following

**Theorem 2.** If $g$ is a central simple finite-dimensional Lie algebra over a field $K$ of characteristic zero such that HomLie$(g)$ is closed with respect to the commutator, then either HomLie$(g) \simeq K$, or $g$ is 3-dimensional.

We give merely a sketch of the proof as, first, as noted in the introduction, this result is not new, and follows from computation of Hom-Lie structures on these algebras in [XJL] and [XL] (see also [MZ1, Theorem 1]), and, second, the proof – modulo the results of §2 – is fairly elementary. However, we want to demonstrate how one can achieve such sort of results without elaborate computations of Hom-Lie structures. Also, in our approach the exceptional 3-dimensional case emerges in a quite interesting way.

Since the Hom-Jacobi equation (1) is linear in $\varphi$, for any Lie algebra $L$ over a field $K$ we have

$$\text{HomLie}(L \otimes_K \overline{K}) \simeq \text{HomLie}(L) \otimes_K \overline{K}.$$  

Further, since the condition (2) of closedness of Hom-Lie structures with respect to the anticommutator is bilinear in $\varphi, \psi$, a Lie $K$-algebra satisfies this condition if and only if the Lie $\overline{K}$-algebra $L \otimes_K \overline{K}$ does. This allows us to use the “Weyl’s unitary trick” (see, e.g., [J, Chap. IV, §7]), and reduce the proof to the case of compact Lie algebras.

Compact Lie algebras possess many peculiar properties (they are closed with respect to subalgebras, and have no nilpotent subalgebras and elements), and it is fairly easy to prove – using, for example, induction by dimension – that if a simple compact Lie algebra satisfies one of the properties ♦, ♣, then it is isomorphic to the 3-dimensional algebra $su(2)$. Now, we cannot use Corollary 1 as we are not over an algebraically closed field, but over $\mathbb{R}$, but we can use similar reasonings valid in the real case. Namely, by Propositions 1 and 2 if $g \not\simeq su(2)$, then HomLie$(g)$ is a semisimple Jordan algebra in which every nonzero element is invertible. Then HomLie$(g)$ does not have nontrivial idempotents, and hence is simple. Inspection of the list of simple real Jordan algebras (available, for example, in [BK, Appendix A]) reveals that the only simple Jordan algebras with the required property are either the 1-dimensional algebra $\mathbb{R}$, or the 4-dimensional plus algebra $\mathbb{Q}^+(\cdot)$ of the real quaternion division algebra $\mathbb{Q}$. In the latter case, by Corollary 2, $g$ is isomorphic to a subalgebra of Der($\mathbb{Q}^+(\cdot)$) $\simeq su(2)$, and hence is isomorphic to $su(2)$.

What can be said about Lie algebras satisfying the property ♦ or ♣ in general? It seems that the exact description in the general case could be difficult. However, the condition ♦ can be interpreted as a generalization of the following condition: a Lie algebra is a vector space sum (not necessarily direct) of two abelian subalgebras. It is easy to prove that such Lie algebras are metabelian (see, for example, [Ko, Proposition 1.5]). There is a vast body of literature devoted to generalizations and extensions of this situation: what happens when we impose various restrictions on summands (nilpotency, simplicity, etc.); see, for example, [Bu] and references therein. The property ♦ can be thought as a generalization in another direction: in the decomposition $L = A \oplus B$ we no longer assume $[A, A] = 0$ and $[B, B] = 0$, but impose the weaker conditions $[[A, A], B] = 0$ and $[[B, B], A] = 0$, without assuming that $A, B$ are necessarily subalgebras.

The property ♣ seems to be more tricky; we conjecture that it is related to existence of subalgebras of codimension 1 (recall that a simple finite-dimensional Lie algebra with a subalgebra of codimension 1 is a form either of $\mathfrak{sl}(2)$, or of Zassenhaus algebra, see [E] and references therein).

**Conjecture.** Let $L$ be a finite-dimensional Lie algebra over an algebraically closed field of characteristic $\neq 2$, and $\text{Rad}(L)$ its solvable radical.

(i) If $L$ satisfies the property ♦, then $L/\text{Rad}(L)$ is isomorphic to the direct sum of several copies of $\mathfrak{sl}(2)$.

(ii) If $L$ satisfies the property ♣, then $L/\text{Rad}(L)$ is isomorphic to the direct sum of several copies of $\mathfrak{sl}(2)$ and $W_1(n)$.
As the properties ◊ and ⊙ are preserved under field extensions, the condition that the ground field is algebraically closed is immaterial here, and is added merely to avoid cumbersome formulations related to forms of not necessarily central semisimple algebras.

5. WHEN HOM-LIE STRUCTURES FORM A HOM-JORDAN ALGEBRA?

Here we briefly discuss the question posed by Sergei Silvestrov: if we are dealing with Hom-algebras, wouldn’t it be natural to replace in the question we dealt with in this paper, “Jordan” by “Hom-Jordan”? To properly interpret this question, we should replace the ordinary anticommutator (2) in the associative algebra $\text{End}(L)$ of all linear maps of $L$ as a vector space, by its Hom-version; and for this, we need the notion of both Hom-associative and Hom-Jordan algebra.

According to the general idea, to get Hom-versions of identities from their standard counterparts, one should to “twist” them by a linear map, similarly how the identity (1) is obtained from the Jacobi identity. In this way, a *Hom-associative algebra* is an algebra $A$ with a binary multiplication $\cdot$, and a twisting linear map $\phi : A \to A$, satisfying the Hom-version of the associative identity:

$$(x \cdot y) \cdot \phi(z) = \phi(x) \cdot (y \cdot z).$$

As for *Hom-Jordan algebras*, there are two versions of them in the literature: Makhlouf in [M] defines them as commutative algebras satisfying the identity

$$(y \cdot x^2) \cdot \phi^2(x) = (y \cdot \phi(x)) \cdot \phi(x^2),$$

while Yau in [Y] defines them as commutative algebras satisfying the identity

$$(\phi(y) \cdot x^2) \cdot \phi^2(x) = (\phi(y) \cdot \phi(x)) \cdot \phi(x^2).$$

These two definitions are different, and Yau argues, not without reason, that his definition is more “correct”, as the plus algebra of a Hom-alternative algebra (whatever it is), is always Hom-Jordan in his sense, but not in Makhlouf’s sense. For us, however, this difference is immaterial, as we are concerned here exclusively with Hom-Jordan algebras of the form $A^{(\pm)}$ for Hom-associative algebras $A$. As shown respectively in [M] and [Y] (and is easy to see), the plus algebra of a Hom-associative algebra is Hom-Jordan in both senses.

Now, as explained in [MZ2, §2], the proper Hom-analog of the associative algebra $\text{End}(V)$ of all linear maps on the vector space $V$, is the Hom-associative algebra $\text{End}(V)_{\alpha}$, with multiplication

$$\phi \cdot_\alpha \psi = \alpha^{-1} \circ \phi \circ \alpha \circ \psi \circ \alpha$$

and the twisting map $\text{Ad}_{\alpha} : \phi \mapsto \alpha^{-1} \circ \phi \circ \alpha$, where $\phi, \psi \in \text{End}(V)$, and $\alpha : V \to V$ is a fixed invertible linear map. Thus, the multiplication in the Hom-Jordan algebra $\text{End}(V)^{(+)}_{\alpha}$ is defined by the anticommutator (8)

$$\phi \ast_\alpha \psi = \frac{1}{2} \alpha^{-1} \circ (\phi \circ \alpha \circ \psi + \psi \circ \alpha \circ \phi) \circ \alpha = \frac{1}{2} \text{Ad}_{\alpha}(\phi \circ \alpha \circ \psi + \psi \circ \alpha \circ \phi).$$

Therefore, the question can be formulated as follows: for which Lie algebras $L$ and an invertible linear map $\alpha : L \to L$, for any two Hom-Lie structures $\phi, \psi \in \text{HomLie}(L)$, their $\alpha$-“twisted” anticommutator, as defined in (8), is also a Hom-Lie structure on $L$?

A particular, but, perhaps, more attractive variant of this question assumes $\alpha \in \text{Aut}(L)$. In this case, by a result from [MZ1, §1], already mentioned in the proof of Proposition[3] HomLie($L$) is invariant under $\text{Ad}_{\alpha}$, and hence $\text{Ad}_{\alpha}$ in the formula (8) can be dropped. Thus this version of the question reads: for which Lie algebras $L$ and $\alpha \in \text{Aut}(L)$, for any two Hom-Lie structures $\phi, \psi \in \text{HomLie}(L)$, the map

$$\frac{1}{2}(\phi \circ \alpha \circ \psi + \psi \circ \alpha \circ \phi)$$

is also a Hom-Lie structure on $L$? Note that this imposes a strong restriction on the automorphism $\alpha$: in particular, it itself should be a Hom-Lie structure on $L$.

ACKNOWLEDGEMENT

Thanks are due to Sergei Silvestrov for interesting discussions. This work was supported by the grant AP08855944 of the Ministry of Education and Science of the Republic of Kazakhstan.
REFERENCES

[BCK] S. Ben Saïd, J.-L. Clerc, K. Koufany, *Conformally covariant bi-differential operators on a simple real Jordan algebra*, Intern. Math. Res. Notices 2020, no.8, 2287–2351.

[Be] G.M. Benkart, *Power-associative Lie-admissible algebras*, J. Algebra 90 (1984), no.1, 37–58.

[Bu] D. Burde, *Derived length and nildecomposable Lie algebras*, Scientific Bulletin of the “Politehnica” University of Timisoara 58 (2013), 15–24.

[C] C. Chevalley, *Théorie des groupes de Lie. Tome II. Groupes algébriques*, Hermann, Paris, 1951.

[E] A. Elduque, *On Lie algebras with a subalgebra of codimension one*, Lie Algebras, Madison 1987 (ed. G. Benkart, J.M. Osborn), Lect. Notes Math. 1373 (1989), 58–66.

[FK] B. Foster-Greenwood, C. Kriloff, *Drinfeld orbifold algebras for symmetric groups*, J. Algebra 491 (2017), 573–610.

[J] N. Jacobson, *Lie Algebras*, Interscience Publ., 1962; reprinted by Dover, 1979.

[Ko] B. Kolman, *Semi-modular Lie algebras*, J. Sci. Hiroshima Univ. Ser. A-I 29 (1965), no.2, 149–163.

[Ku] F. Kubo, *A note on Witt algebras*, Hiroshima Math. J. 7 (1977), no.2, 473–477.

[M] A. Makhlouf, *Hom-alternative algebras and Hom-Jordan algebras*, Intern. Electr. J. Algebra 8 (2010), 177–190.

[MZ1] A. Makhlouf, P. Zusmanovich, *Hom-Lie structures on Kac–Moody algebras*, J. Algebra 515 (2018), 278–297.

[MZ2] A. Makhlouf, P. Zusmanovich, *Ado theorem for nilpotent Hom-Lie algebras*, Intern. J. Algebra Comp. 29 (2019), no.7, 1343–1365.

[RS] A.G. Reyman, M.A. Semenov-Tian-Shansky, *Group-theoretical methods in the theory of finite-dimensional integrable systems*, Dinamicheskie Sistemy - 7, VINITI, 1987, 119–194 (in Russian); Dynamical Systems VII, Encyclopaedia of Mathematical Sciences, Vol. 16, Springer-Verlag, 1994, 116–225 (English translation).

[XJL] W. Xie, Q. Jin, W. Liu, *Hom-structures on semi-simple Lie algebras*, Open Math. 13 (2015), 617–630.

[XL] W. Xie, W. Liu, *Hom-structures on simple graded Lie algebras of finite growth*, J. Algebra Appl. 16 (2017), no.8, 1750154.

[Y] D. Yau, *Hom-Maltsev, Hom-alternative, and Hom-Jordan algebras*, Intern. Electr. J. Algebra 11 (2012), 177–217.

UNIVERSITY OF OSTRAVA, CZECH REPUBLIC

Email address: pasha.zusmanovich@osu.cz