Split 3-Lie-Rinehart color algebras

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1. Introduction

The notion of $n$–Lie algebra introduced by Filippov in 1985 (see [17]), is a natural generalization of the concept of a Lie algebra to the case where the fundamental multiplication is $n$–ary, $n \geq 2$ (when $n = 2$ the definition agrees with the usual definition of a Lie algebra). In particular 3–Lie algebras have close relationships with many important fields in mathematics and mathematical physics. For example, the study of supersymmetry, Bagger-Lambert theory, Nambu mechanics or gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes (see [5, 18, 21, 24, 26]). The concept of $n$–Lie superalgebras as generalization of $n$–Lie algebras were initially introduced by Daletskii and Kushnirevich in 1996 (see [16]). Moreover, Cantarini and Kac gave a more general concept of $n$–Lie superalgebras again in 2010 (see [11]). $n$–Lie superalgebras are more general structures including $n$–Lie algebras, $n$–ary Nambu-Lie superalgebras, and Lie superalgebras. In 2015, Tao Zhang also introduced the concept of n-Lie colour algebras, which is the generalization of $n$–Lie superalgebras and developed cohomology theory and deformations of 3–Lie colour algebras(see [27]).
Lie-Rinehart algebra was introduced by J. Herz in [19] and developed in [23] by G. Rinehart, is an algebraic generalization of the notion of a Lie algebroid: the space of sections of a vector bundle is replaced by a module over a ring, a vector field by a derivation of the ring [20]. Some generalizations of Lie-Rinehart algebras, such as Lie-Rinehart superalgebras [8] or restricted Lie-Rinehart algebras [15], have been recently studied.

Determining the inner structure of split algebras by the techniques of connection of roots becomes more and more meaningful in the area of research in mathematical physics. It is worth mentioning that these techniques had long been introduced by Calderon, Antonio J, on split Lie algebras with symmetric root systems in [7]. Recently, in [2, 3, 9, 10, 12, 13, 14], the structure of arbitrary split Lie color algebras, Split regular Hom-Lie algebras, split Leibniz algebras, split Lie-Rinehart algebras, split 3−Lie algebras, split 3−Leibniz algebras, and arbitrary split Lie algebras of order 3 have been determined by the techniques of connection of roots.

In this paper, we continue to study the structure of a generalization of Lie-Rinehart superalgebras to the class which are called split 3−Lie-Rinehart color algebras. The definition, basic structures, actions and crossed modules of 3−Lie-Rinehart algebras and also 3−Lie-Rinehart superalgebras can be found in [4, 6]. A 3−Lie-Rinehart color algebra is a triple (L, A, ρ), where L is a 3−Lie color algebra, A is a commutative, associative color algebra, L is an A−module, (A, ρ) is a 3−Lie color algebra L−module and ρ(L, L) ⊂ Der(A). Our goal in this work is to study the inner structure of arbitrary split 3−Lie-Rinehart color algebras by the developing techniques of connections of root systems and weight systems associated to a splitting Cartan subalgebra. The finding of the present paper is an improvement and extension of the works conducted in [2].

We briefly outline the contents of the paper. In Section 2, we begin by recalling the necessary background on split 3−Lie-Rinehart color algebras. Section 3, develops techniques of connections of roots for in the framework of split 3−Lie-Rinehart color algebra (L, A, ρ, ε) and apply, as a first step, all of these techniques to the study of the inner structure of L. In section 4, we get, as a second step, a decomposition of A as the orthogonal direct sums of adequate ideals. In section 5, we show that the decompositions of L and A as direct sum of ideals, given in Sections 2 and 3 respectively, are closely related. Section 6, devoted to show that under certain conditions, the given decomposition of L and A are by means of the family of their corresponding, graded simple ideals.

Throughout this paper, algebras and vector spaces are over a field F of characteristic zero, and A denotes an associative and commutative algebra over F. We also consider an additive abelian group G with identity zero.

2. Preliminaries

Let us begin with some definitions and results concerning graded algebraic structures. For a detailed discussion of this subject, we refer the reader to the literature [25]. Let G be any additive abelian group, a vector space V is said to be G−graded,
if there is a family \( \{ V_g \}_{g \in G} \) vector subspaces such that \( V = \bigoplus_{g \in G} V_g \). An element \( v \in V \) is said to be **homogeneous of degree** \( g \) if \( v \in V_g, \ g \in G \), and in this case, \( g \) is called the **color** of \( v \). As usual, we denote by \( |v| \) the color of an element \( v \in V \). Thus, each homogeneous element \( v \in V \) determines a unique group element \( \|v\| \in G \) by \( v \in V_{\|v\|} \). Fortunately, we can almost drop the symbol \( "\ | \,|" \), since confusion rarely occurs.

Let \( V = \bigoplus_{g \in G} V_g \) and \( W = \bigoplus_{g \in G} W_g \) be two \( G \)-graded vector spaces. A linear mapping \( f : V \to W \) is said to be **homogeneous of degree** \( h \in G \) if

\[
f(V_g) \subset W_{g+h}, \ \forall \ g \in G.
\]

If in addition, \( f \) is homogeneous of degree zero, namely, \( f(V_g) \subset W_g \) holds for any \( g \in G \), then we call \( f \) is **even**.

An algebra \( A \) is said to be \( G \)-graded or color algebra if its underlying vector space is \( G \)-graded, i.e., \( A = \bigoplus_{g \in G} A_g \), and if \( A_g A_h \subset A_{g+h}, \) for \( g, h \in G \). A subalgebra (an ideal) of \( A \) is said to be graded if it is a graded as a subspace of \( A \).

Let \( B \) be another \( G \)-graded algebra. A homomorphism \( \varphi : A \to B \) of \( G \)-graded algebras is a homomorphism of the algebra \( A \) into the algebra \( B \), which is an even mapping.

**Definition 2.1.** Let \( G \) be an abelian group. A map \( \epsilon : G \times G \to \mathbb{K} \setminus \{ 0 \} \) is called a **skew-symmetric bi-character** on \( G \) if for all \( g, h, f \in G \),

\[
\begin{align*}
(i) & \quad \epsilon(g, h)\epsilon(h, g) = 1, \\
(ii) & \quad \epsilon(g + h, f) = \epsilon(g, f)\epsilon(h, f), \\
(iii) & \quad \epsilon(g, h + f) = \epsilon(g, h)\epsilon(g, f).
\end{align*}
\]

The definition above implies that in particular, the following relations hold

\[
\epsilon(g, 0) = 1 = \epsilon(0, g), \ \epsilon(g, g) = 1(\text{or} - 1), \ \forall g \in G.
\]

Throughout this paper, if \( x \) and \( y \) are homogeneous elements of a \( G \)-graded vector space and \( |x| \) and \( |y| \) which are in \( G \) denote their degrees respectively, then for convenience, we write \( \epsilon(x, y) \) instead of \( \epsilon(|x|, |y|) \). It is worth mentioning that, we unless otherwise stated, in the sequel all the graded spaces are over the same abelian group \( G \) and the bi-character will be the same for all structures.

**Definition 2.2.** \cite{27} A 3-Lie colour algebra consists of a \( G \)-graded vector space \( \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}^g \) together with a trilinear map \([\cdot, \cdot, \cdot] : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \to \mathcal{L} \) such that the following conditions are satisfied:

\[
\begin{align*}
(i) & \quad \text{graded condition:} \quad |[x_1, x_2, x_3]| = |x_1| + |x_2| + |x_3|; \\
(ii) & \quad \epsilon\text{-skew symmetry:} \\
& \quad [x_1, x_2, x_3] = -\epsilon(x_1, x_2)[x_2, x_1, x_3], [x_1, x_2, x_3] = -\epsilon(x_2, x_3)[x_1, x_3, x_2]; \\
(iii) & \quad \epsilon\text{-fundamental identity:} \\
(2.3) & \quad [x_1, x_2, [y_1, y_2, y_3]] = [[x_1, x_2, y_1], y_2, y_3] \\
& \quad + \epsilon(x_1 + x_2, y_1)[y_1, [x_1, x_2, y_2], y_3] \\
& \quad + \epsilon(x_1 + x_2, y_1 + y_2)[y_1, y_2, [x_1, x_2, y_3]].
\end{align*}
\]
for all homogeneous elements \(x_1, x_2, y_1, y_2, y_3 \in \mathcal{L}\).

Let \((\mathcal{L}, [[., .], .], \epsilon)\) be a 3–Lie color algebra. A subalgebra \(S\) of \(\mathcal{L}\) is a \(G\)-graded subspace \(S = \bigoplus_{g \in G} S^g\) of \(\mathcal{L}\) such that \([S, S, S] \subseteq S\) and a \(G\)-graded subspace \(I = \bigoplus_{g \in G} I^g\) of \(\mathcal{L}\) is called an ideal if \([I, \mathcal{L}, \mathcal{L}] \subseteq I\). A 3–Lie color algebra \(\mathcal{L}\) is called simple if its triple product is nonzero and its only ideals are \(\{0\}\) and \(\mathcal{L}\). We recall that the Annihilator of a 3–Lie color algebra \(\mathcal{L}\) is defined as the set of elements \(x\) in \(\mathcal{L}\) such that \([x, \mathcal{L}, \mathcal{L}] = 0\). This is an ideal of \(\mathcal{L}\) denoted by \(\text{Ann}(\mathcal{L})\).

**Definition 2.4.** \([4]\) Let \((\mathcal{L}, [[., .], .], \epsilon)\) be a 3–Lie color algebra and \(V\) be a \(G\)-graded vector space and \(\rho : \mathcal{L} \times \mathcal{L} \rightarrow \text{gl}(V)\) be a even linear mapping. Then \((V, \rho)\) is called a representation of \(\mathcal{L}\) or \(V\) is an \(\mathcal{L}\)-module if the following two conditions are satisfied:

\[
(2.5) \quad \rho(x_1, x_2)\rho(x_3, x_4) = \epsilon(x_1 + x_2, x_3 + x_4)\rho(x_3, x_4)\rho(x_1, x_2) = \rho([x_1, x_2, x_3], x_4) - \epsilon(x_3, x_4)\rho([x_1, x_2, x_4], x_3),
\]

and

\[
(2.6) \quad \rho([x_1, x_2, x_3], x_4) = \rho(x_1, x_2)\rho(x_3, x_4) + \epsilon(x_1, x_2 + x_3)\rho(x_2, x_3)\rho(x_1, x_4) + \epsilon(x_1 + x_2, x_3)\rho(x_3, x_1)\rho(x_2, x_4).
\]

Next, define

\[
ad : \mathcal{L} \times \mathcal{L} \rightarrow \text{gl}(\mathcal{L}); \quad ad_{x,y}z = [x, y, z], \quad \forall x, y, z \in \mathcal{L}.
\]

Thanks to \(\epsilon\)-fundamental identity, \((\mathcal{L}, ad)\) is a representation of the 3–Lie color algebra \(\mathcal{L}\), and it is called the adjoint representation of \(\mathcal{L}\). One can see that \(ad(\mathcal{L}, \mathcal{L})\) is a Lie algebra which is called inner derivation of \(\mathcal{L}\). We also have by Eq \([2.5]\),

\[
[ad_{x_1, x_2}, ad_{y_1, y_2}] = ad_{[x_1, y_1, x_2], y_2} + ad_{[x_2, [x_1, y_1, y_2]]}.
\]

**Definition 2.7.** Let \((\mathcal{L}, [[., .], .], \epsilon)\) be a 3–Lie color algebra and \(A\) be an associative commutative color algebra which \((A, \rho)\) is an \(\mathcal{L}\)-module. If \(\rho(\mathcal{L}, \mathcal{L}) \subset \text{Der}(A)\) and

\[
(2.8) \quad [x, y, a\mathcal{L}] = \epsilon(a, x + y)a[x, y, z] + \rho(x, y)a\mathcal{L}, \quad \forall x, y, z \in \mathcal{L}, \quad \forall a \in A,
\]

\[
(2.9) \quad \rho(ax, y) = \epsilon(a, x)\rho(x, ay) = axp(x, y), \quad \forall x, y \in \mathcal{L}, \quad \forall a \in A,
\]

then \((\mathcal{L}, A, [[., .], .], \rho, \epsilon)\) is called a 3–Lie-Rinehart color algebra.

**Example 2.10.**

1. If \(\rho = 0\) then \((\mathcal{L}, A, [[., .], .], \epsilon)\) is a 3–Lie \(A\)-color algebra.
2. Let \((G, [[., .], .], \epsilon)\) be a color Lie algebra and an \(A\)-module. Let \((A, \rho)\) be a \(G\)-module. If \(\rho(G) \subset \text{Der}(A)\) and

\[
[x, ay] = \epsilon(a, x)a[x, y] + \rho(x, y)a\mathcal{L}, \quad \rho(ax) = axp(x), \quad \forall x, y \in G, \quad \forall a \in A,
\]

then \((G, A, [[., .], .], \rho, \epsilon)\) is a Lie-Rinehart color algebra.
3. If \(G = \mathbb{Z}_2\) and \(\epsilon(x, y) = (-1)^{xy}\) for any homogeneous elements \(x, y \in \mathcal{L}\). We then get the notion of 3–Lie-Rinehart superalgebra (see \([4]\)).
Example 2.11. We recall that given a Lie color algebra analogues of color trace one can construct a 3–Lie color algebra. Let \((L, [[, , ]], \rho, \epsilon)\) be a Lie color algebra and \(\tau : L \to F\) an even linear form. We say that \(\tau\) is a color trace of \(L\) if \(\tau([[, , ]]) = 0\). For any \(x_1, x_2, x_3 \in L\), we define the 3–ary bracket by

\[
[x_1, x_2, x_3]_\tau = \tau(x_1)[x_2, x_3] - \epsilon(x_1, x_2)\tau(x_2)[x_1, x_3] + \epsilon(x_3, x_1 + x_2)\tau(x_3)[x_1, x_2].
\]

then \((L, [[, , ]], \tau)\) is a 3–Lie color algebra (see [1] for super algebras). Next, We begin by constructing 3–Lie–Rinehart color algebras starting with a Lie–Rinehart color algebras. Let \((L, A, [ , , ], \rho, \epsilon)\) be a Lie–Rinehart color algebra and \(\tau\) is a color trace. If the condition

\[\tau(ax)y = \tau(x)ay,\]

is satisfied for any \(x, y \in L, a \in A\), then \((L, A, [[, , ]], \rho, \tau)\) is a 3–Lie–Rinehart color algebra, where \([[, , ]], \rho)\) is defined as Eq. (2.12) and \(\rho_\tau\) is defined by

\[\rho_\tau : L \times L \to gl(V); \quad \rho_\tau(x, y) = \tau(x)\rho(y) - \epsilon(x, y)\tau(y)\rho(x), \quad \forall x, y \in L.\]

(see Theorem 2.1 in [1] for super algebras)

Definition 2.13. Let \((L, A, [[, , ]], \rho, \epsilon)\) be a 3–Lie–Rinehart color algebra.

1. If \(S\) is a 3–Lie color subalgebra of \(L\) satisfying \(AS \subset S\), then

\[(S, A, [[, , ]]|_{S \times S}, \rho|_{S \times S}, \epsilon)\]

is a 3–Lie–Rinehart color algebra which is called a subalgebra of the 3–Lie–Rinehart color algebra \((L, A, [[, , ]], \rho, \epsilon)\).

2. If \(I\) is a 3–Lie color ideal of \(L\) satisfying \(AI \subset I\) and \(\rho(I, I)(A) \subset I\), then

\[(I, A, [[, , ]]|_{I \times I}, \rho|_{I \times I}, \epsilon)\]

is a 3–Lie–Rinehart color algebra which is called an ideal of the 3–Lie–Rinehart color algebra \((L, A, [[, , ]], \rho, \epsilon)\). As in [1], Proposition 2.4, \(\ker \rho\) is an ideal of \((L, A, [[, , ]], \rho, \epsilon)\).

3. If a 3–Lie–Rinehart color algebra \((L, A, [[, , ]], \rho, \epsilon)\) cannot be decomposed into the direct sum of two non-zero ideals, then it is called an indecomposable 3–Lie–Rinehart color algebra. We also say that \((L, A, [[, , ]], \rho, \epsilon)\) is simple if \([L, L, L] \neq 0\), \(AA \neq 0, LA \neq 0\) and its only ideals are \(\{0\}, L\) and \(\ker \rho\).

Denotes:

\[Z_L(A) := \{a \in A : ax = 0 \ \forall x \in L\}, \quad Z_\rho(L) := \{x \in L : [x, L, L] = 0, \rho(x, L) = 0\}.\]

Note that \(Z_\rho(L) = \text{Ann}(L) \cap \ker \rho\).

From now on, \((L, A, [[, , ]], \rho, \epsilon)\) denotes a 3–Lie–Rinehart color algebra. We introduce the class of split algebras in the framework of 3–Lie–Rinehart color algebras as in [2].
Definition 2.14. Let \( (\mathcal{L}, A, [\cdot, \cdot], \rho, \epsilon) \) be a 3–Lie-Rinehart color algebra. If there exist a maximal abelian subalgebra \( \mathcal{H} \) of \( \mathcal{L} \) satisfying that
\[
\mathcal{L} = \mathcal{H} \oplus \left( \bigoplus_{\alpha \in \Pi} \mathcal{L}_\alpha \right), \quad \text{where } \pi = \{0 \neq \alpha \in (\mathcal{H} \times \mathcal{H})^* \mid \mathcal{L}_\alpha \neq 0\},
\]

then \( (\mathcal{L}, A, [\cdot, \cdot], \rho, \epsilon) \) is called a split 3–Lie-Rinehart color algebra, and \( \mathcal{H} \) is a splitting Cartan subalgebra of \( \mathcal{L} \). \( \Pi \) and \( \Lambda \) are called the root system of \( \mathcal{L} \), and \( \Lambda \) the weight system of \( A \) associated to \( \mathcal{H} \), respectively. Linear subspaces \( \mathcal{L}_\alpha (\alpha \in \Pi) \) and \( A_\lambda (\lambda \in \Lambda) \) are called the root space of \( \mathcal{L} \) and the weight space of \( A \), respectively.

For convenience, in the rest of the paper, the split 3–Lie-Rinehart color algebra \( (\mathcal{L}, A, [\cdot, \cdot], \rho, \epsilon) \) is simply denoted by \( (\mathcal{L}, A) \), and denote
\[
-\pi = \{-\alpha \mid \alpha \in \pi\}, \quad -\Lambda = \{-\lambda \mid \lambda \in \Lambda\}, \quad \pm \pi = \pi \cup -\pi, \quad \pm \Lambda = \Lambda \cap -\Lambda.
\]

We recall some properties of split 3–Lie-Rinehart color algebra that can be found in [22] for a split regular involutive Hom-Lie color algebras.

Lemma 2.19. Let \( (\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}^g, A = \bigoplus_{g \in G} A_g) \) be a split 3–Lie-Rinehart color algebra, with root space decomposition \( \mathcal{L} = \mathcal{H} \oplus \left( \bigoplus_{\alpha \in \Pi} \mathcal{L}_\alpha \right) \) and weight space decomposition \( A = A_0 \oplus \left( \bigoplus_{\lambda \in \Lambda} A_\lambda \right) \). Then
\begin{enumerate}
\item for any \( \alpha \in \Pi \cup \{0\} \), we have \( \mathcal{L}_\alpha = \bigoplus_{g \in G} \mathcal{L}_\alpha^g \), where \( \mathcal{L}_\alpha^g = \mathcal{L}_\alpha \cap \mathcal{L}^g \), and \( \mathcal{L}_0 = \mathcal{H} \).
\item \( \mathcal{H}^0 = \mathcal{L}_0^0 \). In particular, \( \mathcal{H}^0 = \mathcal{L}_0^0 \).
\item \( \mathcal{L}^0 \) is a split 3–Lie color algebra, with respect to \( \mathcal{H}^0 = \mathcal{L}_0^0 \), with root space decomposition \( \mathcal{L}^0 = \mathcal{H}^0 \oplus \left( \bigoplus_{\alpha \in \Pi} \mathcal{L}_\alpha^0 \right) \).
\item for any \( \lambda \in \Lambda \cup \{0\} \), we have \( A_\lambda = \bigoplus_{g \in G} A_\lambda^g \), where \( A_\lambda^g = A_g \cap A_\lambda \). We also have \( A^g = A_0^g \), in particular \( A^0 = A_0^0 \).
\end{enumerate}

Proof. Similar to Lemma 2.6 in [22].

If \( (\mathcal{L}, A) \) is a split 3–Lie-Rinehart color algebra, with root space decomposition \( \mathcal{L} = \mathcal{H} \oplus \left( \bigoplus_{\alpha \in \Pi} \mathcal{L}_\alpha \right) \) and weight space decomposition \( A = A_0 \oplus \left( \bigoplus_{\lambda \in \Lambda} A_\lambda \right) \), taking into account Lemma 2.19, we then write
\[
\mathcal{L} = \bigoplus_{g \in G} \left( \mathcal{H}^g \oplus \left( \bigoplus_{\alpha \in \Pi} \mathcal{L}_\alpha^g \right) \right) = \mathcal{H}^0 \oplus \left( \bigoplus_{g \in G} \bigoplus_{\alpha \in \Pi} \mathcal{L}_\alpha^g \right),
\]
\[
A = \bigoplus_{g \in G} \left( A^g \oplus \left( \bigoplus_{\lambda \in \Lambda} A_\lambda^g \right) \right) = A^0 \oplus \left( \bigoplus_{g \in G} \bigoplus_{\lambda \in \Lambda} A_\lambda^g \right),
\]
we denote by $\Pi^g := \{ \alpha \in \Pi \mid L^g_\alpha \neq 0 \}$ and $A^g := \{ \lambda \in A \mid A^g_\lambda \neq 0 \}$, for any $g \in G$. Then $\Pi = \bigcup_{g \in G} \Pi^g$ and $A = \bigcup_{g \in G} A^g$.

**Proposition 2.22.** For any $\alpha_1, \alpha_2, \alpha_3 \in \Pi \cup \{0\}$, $\lambda_1, \lambda_2 \in A \cup \{0\}$ and any $g_1, g_2, g_3 \in G$, the following assertions hold.

1. If $[L^g_{\alpha_1}, L^g_{\alpha_2}, L^g_{\alpha_3}] \neq 0$, then $\alpha_1 + \alpha_2 + \alpha_3 \in \Pi^{g_1+g_2+g_3} \cup \{0\}$ and
   
   $[L^g_{\alpha_1}, L^g_{\alpha_2}, L^g_{\alpha_3}] \subset L^{g_1+g_2+g_3}_{\alpha_1+\alpha_2+\alpha_3}$.

2. If $A^{g_1}_{\lambda_1} A^{g_2}_{\lambda_2} \neq 0$, then $\lambda_1 + \lambda_2 \in A^{g_1+g_2} \cup \{0\}$ and $A^{g_1}_{\lambda_1} A^{g_2}_{\lambda_2} \subset A^{g_1+g_2}_{\lambda_1+\lambda_2}$.

3. If $A^{g_1}_{\lambda_1} L^{g_2}_{\alpha_2} \neq 0$, then $\lambda_1 + \alpha_2 \in \Pi^{g_1+g_2} \cup \{0\}$ and $A^{g_1}_{\lambda_1} L^{g_2}_{\alpha_2} \subset L^{g_1+g_2}_{\lambda_1+\alpha_2}$.

4. If $\rho(L^g_{\alpha_1}, L^g_{\alpha_2})(A^{g_3}_{\lambda_3}) \neq 0$, then $\alpha_1 + \alpha_2 + \lambda_3 \in A^{g_1+g_2+g_3} \cup \{0\}$ and
   
   $\rho(L^g_{\alpha_1}, L^g_{\alpha_2})(A^{g_3}_{\lambda_3}) \subset A^{g_1+g_2+g_3}_{\alpha_1+\alpha_2+\lambda_3}$.

**Proof.**

1. For any $h_1, h_2 \in H^0$, $y_i \in L^g_{\alpha_i}$, $i = 1, 2, 3$ from $\epsilon$–fundamental identity, we have

   $[h_1, h_2, [y_1, y_2, y_3]] = \epsilon(h_1, h_2, y_1, y_2, y_3) + \epsilon(0, y_1)[y_1, [h_1, h_2, y_2], y_3]
   + \epsilon(0, y_1 + y_2)[y_1, y_2, [h_1, h_2, y_3]]
   + \epsilon(0, y_1 + y_2)[y_1, y_2, \alpha_3(h_1, h_2)y_3]
   + \epsilon(0, y_1 + y_2)[y_1, y_2, \alpha_3(h_1, h_2)y_3]
   = \epsilon(\alpha_1 + \alpha_2 + \alpha_3)(h_1, h_2)[y_1, y_2, y_3].$

   Therefore, we get $\alpha_1 + \alpha_2 + \alpha_3 \in \Pi^{g_1+g_2+g_3} \cup \{0\}$ and $[L^g_{\alpha_1}, L^g_{\alpha_2}, L^g_{\alpha_3}] \subset L^{g_1+g_2+g_3}_{\alpha_1+\alpha_2+\alpha_3}$.

2. For any $a_1 \in A^{g_1}_{\lambda_1}, a_2 \in A^{g_2}_{\lambda_2}$, since $\rho(h_1, h_2) \in Der(A)$, $\forall h_1, h_2 \in H^0$ we have

   $\rho(h_1, h_2)(a_1 a_2) = \rho(h_1, h_2)(a_1) a_2 + a_1 \rho(h_1, h_2)(a_2)
   = \lambda_1(h_1, h_2)(a_1) a_2 + a_1 \lambda_2(h_1, h_2)(a_2)
   = \lambda_1 + \lambda_2(h_1, h_2)(a_1 a_2)$.

   Thus $\lambda_1 + \lambda_2 \in A^{g_1+g_2} \cup \{0\}$ and $A^{g_1}_{\lambda_1} A^{g_2}_{\lambda_2} \subset A^{g_1+g_2}_{\lambda_1+\lambda_2}$.

3. For any $a \in A^{g_3}_{\lambda_3}$, $x \in L^{g_2}_{\alpha_2}$ and $h_1, h_2 \in H^0$, by Eq. [2.3] we have

   $[h_1, h_2, ax] = \epsilon(a, 0)[h_1, h_2, x] + \rho(h_1, h_2)ax
   = a \epsilon(h_1, h_2)x + \lambda_1(h_1, h_2) ax
   = (a_1 + \lambda_1)(h_1, h_2)(ax)$.

   Therefore, we get $\lambda_1 + \alpha_2 \in \Pi^{g_1+g_2} \cup \{0\}$ and $ax \in L^{g_1+g_2}_{\alpha_1+\alpha_2}$.

4. Since $A$ is an $L$–module, for any $a \in A^{g_3}_{\lambda_3}, x_i \in L^{g_3}_{\alpha_i}(i = 1, 2)$ and $h_1, h_2 \in H^0$, we have
\[
\rho(h_1, h_2)(\rho(x_1, x_2)a) = (\rho(h_1, h_2)\rho(x_1, x_2))(a)
\]
\[
= \epsilon(0, x_1 + x_2)\rho(x_1, x_2)\rho(h_1, h_2)(a) + \rho([h_1, h_2, x_1], x_2)a
\]
\[
- \epsilon(x_1, x_2)\rho([h_1, h_2, x_2], x_1)a
\]
\[
= \rho(x_1, x_2)\lambda_1(h_1, h_2)(a) + \rho(\alpha_1(h_1, h_2)x_1, x_2)a
\]
\[
- \epsilon(x_1, x_2)\rho(\alpha_2(h_1, h_2)x_2, x_1)a
\]
\[
= (a_1 + a_2 + \lambda_1)(h_1, h_2)\rho(x_1, x_2)(a).
\]

Thus \(\alpha_1 + \alpha_2 + \lambda_3 \in \Lambda \oplus g_1 + g_2 \cup \{0\}\) and \(\rho(x_1, x_2)(a) \in A_{\alpha_1 + \alpha_2 + \lambda_3}^g\).

\[\square\]

3. Connections of roots and decompositions

In this section, we begin by developing the techniques of connections of roots in the setting as [2]. Let \((\mathcal{L}, A)\) be a split 3–Lie-Rinehart color algebra, with root space decomposition \(\mathcal{L} = H^0 \oplus (\bigoplus_{g \in G} \bigoplus_{\alpha \in \Pi^g} \mathcal{L}^g_{\alpha})\), and with symmetric root system \(\Pi = \bigcup_{g \in G} \Pi^g\).

**Definition 3.1.** Let \(\alpha, \beta\) be two non zero roots in \(\Pi\). We say that \(\alpha\) is connected to \(\beta\) and denoted by \(\alpha \sim \beta\) if there exists a family

\[\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{2k+1}\} \subset \pm\Pi \cup \pm\Lambda \cup \{0\},\]

satisfying the following conditions;

If \(k = 1:\)

1. \(\beta = \pm \alpha_1.\)

If \(k \geq 2:\)

1. \(\alpha_1 + (\alpha_2 + \alpha_3) \in \pm\Pi,\)
2. \(\alpha_1 + (\alpha_2 + (\alpha_3 + (\alpha_4 + \alpha_5))) \in \pm\Pi,\)
   ... 
3. \(\alpha_1 + \sum_{j=1}^{l} (\alpha_{2j} + \alpha_{2j+1}) \in \pm\Pi, \ 0 < i < k.\)
4. \(\alpha_1 + \sum_{j=1}^{k} (\alpha_{2j} + \alpha_{2j+1}) \in \{\beta, -\beta\}.\)

The family \(\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{2k+1}\}\) is called a connection from \(\alpha\) to \(\beta\).

**Proposition 3.2.** The relation \(\sim\) in \(\Pi\) defined by

\(\alpha \sim \beta\) if and only if \(\alpha\) is connected to \(\beta\),

is an equivalence relation.

**Proof.** The proof is vertically identical to the proof of Proposition 4.5 in [3]. \(\square\)

**Lemma 3.3.** For any \(\gamma, \mu \in \pm\Pi \cup \pm\Lambda \cup \{0\}\), if \(\alpha \sim \beta\) and \(\alpha + \gamma + \mu \in \pi\), then \(\beta \sim \alpha + \gamma + \mu\).

**Proof.** Considering the connection \(\{\beta, \gamma, \mu\}\) we get \(\alpha \sim \alpha + \gamma + \mu\). Now, by transitivity \(\beta \sim \alpha + \gamma + \mu\). \(\square\)
By the Proposition 3.2 we can consider the equivalence relation in $\Pi$ by the connection relation $\sim$ in $\Pi$. So we denote by

$$\Pi/\sim := \{[\alpha] : \alpha \in \Pi\},$$

where $[\alpha]$ denotes the set of non zero roots of $L$ which are connected to $\alpha$.

Our next goal in this section is to associate an adequate ideal $I_{[\alpha]}$ of $L$ to any $[\alpha]$. For a fixed $\alpha \in \Pi$, we define

$$I_{0,[\alpha]} := (\sum_{\beta \in [\alpha]} A_{-\beta} L_{\beta}) + (\sum_{\beta, \gamma, \mu \in [\alpha]} [L_{\beta}, L_{\gamma}, L_{\mu}]).$$

In order to graded case,

$$I_{0,[\alpha]} := (\sum_{\beta \in [\alpha]} A^0_{-\beta} L^0_{\beta}) + (\sum_{\beta, \gamma, \mu \in [\alpha]} [L^0_{\beta}, L^0_{\gamma}, L^0_{\mu}]).$$

Next, we define

$$V_{[\alpha]} := \bigoplus_{\beta \in [\alpha]} L_{\beta} = \bigoplus_{g \in G} \bigoplus_{\beta \in [\alpha]} L^0_{\beta}.$$

Thanks to Proposition 2.22, $I_{0,[\alpha]} \subset \mathcal{H}$ and $I_{0,[\alpha]} \cap V_{[\alpha]} = 0$. Finally, we denote by $I_{[\alpha]}$ the direct sum of the two graded subspaces above, that is,

$$I_{[\alpha]} := I_{0,[\alpha]} \oplus V_{[\alpha]}.$$ 

**Proposition 3.7.** For any $[\alpha] \in \Pi/\sim$ the following assertions hold.

1. $[I_{[\alpha]}, I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$.
2. $A I_{[\alpha]} \subset I_{[\alpha]}$
3. $I_{[\alpha]} \subset I_{[\alpha]}$

**Proof.** (1) By the fact $L_0 = \mathcal{H}$ and Eq. (3.4), it is clear that $[I_{0,[\alpha]}, I_{0,[\alpha]}, I_{0,[\alpha]}] \subset [\mathcal{H}, \mathcal{H}, \mathcal{H}] = 0$, and we can write

$$[I_{[\alpha]}, I_{[\alpha]}, I_{[\alpha]}] = [I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\alpha]} \oplus V_{[\alpha]}] \subset [I_{0,[\alpha]}, V_{[\alpha]}, I_{0,[\alpha]}] + [I_{0,[\alpha]}, I_{0,[\alpha]}, V_{[\alpha]}] + [V_{[\alpha]}, V_{[\alpha]}, V_{[\alpha]}].$$

Let us consider the first summand in (3.8). Since $I_{0,[\alpha]} \subset \mathcal{H}$, for $\beta \in [\alpha]$ and $g \in G$, by Proposition 2.22 one gets $[I_{0,[\alpha]}, I_{0,[\alpha]}, L^0_{\beta}] \subset L^0_{\beta}$. Hence,

$$[I_{0,[\alpha]}, V_{[\alpha]}, I_{0,[\alpha]}] \subset V_{[\alpha]}.$$

Consider now the second summand in (3.8). Given $\beta, \gamma \in [\alpha]$ and $g, g' \in G$ such that $[I_{0,[\alpha]}, L^0_{\beta}, L^0_{\gamma}] \neq 0$, then $[I_{0,[\alpha]}, L^0_{\beta}, L^0_{\gamma}] \subset L^0_{\beta+\gamma}$, by Proposition 2.22. If $\beta + \gamma \in \Pi_{g+g'}$, and $\{\beta, \gamma, 0\}$ is a connection from $\beta$ to $\beta + \gamma$, by Lemma 3.3 we have

$$[I_{0,[\alpha]}, L^0_{\beta}, L^0_{\gamma}] \subset V_{[\alpha]} \subset I_{[\alpha]}.$$

Therefore,

$$[I_{0,[\alpha]}, V_{[\alpha]}, I_{0,[\alpha]}] \subset I_{[\alpha]}.$$
Finally, consider the last summand in (3.3). Let $\beta, \gamma, \mu \in [\alpha]$ and $g, g', g'' \in G$ such that $[\mathcal{L}_\beta^g, \mathcal{L}_\gamma^g, \mathcal{L}_\mu^{g''}] \neq 0$, and by Proposition 2.22 we have $[\mathcal{L}_\beta^g, \mathcal{L}_\gamma^g, \mathcal{L}_\mu^{g''}] \subset \mathcal{L}_{\beta+\gamma+\mu}^{g+g'+g''}$. If $\beta + \gamma + \mu = 0$, we get $[\mathcal{L}_\beta^g, \mathcal{L}_\gamma^g, \mathcal{L}_\mu^{g''}] \subset I_{0,[\alpha]} \subset I_{[\alpha]}$.

Suppose $0 \neq \beta + \gamma + \mu$, by Lemma 3.3 one gets $\beta + \gamma + \mu \in \Pi^{g+g'+g''}$, therefore, $\{\beta, \gamma, \mu\}$ is a connection from $\beta$ to $\beta + \gamma + \mu$. The transitivity of $\sim$ gives us $\beta + \gamma + \mu \in [\alpha]$ and so $[\mathcal{L}_\beta^g, \mathcal{L}_\gamma^g, \mathcal{L}_\mu^{g''}] \subset \mathcal{L}_{\beta+\gamma+\mu}^{g+g'+g''} \subset V_{[\alpha]}$. Hence, (3.11)

$$[V_{[\alpha]}, V_{[\alpha]}, V_{[\alpha]}] \subset I_{[\alpha]}.$$ From Eqs. (3.9), (3.10), and (3.11), we conclude that (3.12)

$$[I_{[\alpha]}, I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}.$$

(2) Observe that

$$AI_{[\alpha]} = (A_0 \oplus \bigoplus_{A_0 \subseteq \mathbb{L}} A_{\beta}(\mathbb{L})) (\sum_{\beta, \gamma, \mu \in [\alpha]} A_{\beta}^{g} \mathcal{L}_{\gamma}^{g} + \sum_{\beta, \gamma, \mu \in [\alpha]} \mathcal{L}_{\beta}^{g}, \mathcal{L}_{\gamma}^{g}, \mathcal{L}_{\mu}^{g''}) \oplus \bigoplus_{\beta, \gamma, \mu \in [\alpha]} A_{\beta}^{g} \mathcal{L}_{\gamma}^{g}, \mathcal{L}_{\mu}^{g''}.$$

We discuss it in six cases:

**Case 1.** For any $\beta \in [\alpha]$, $g, g' \in G$, if $-\beta \in \Lambda^h$ for some $h \in G$ then by Proposition 2.22 and the fact that $\mathcal{L}$ is an $A$–module, we have

$$A_0(A_{-\beta}^{g} \mathcal{L}_{\beta}^{g}) = (A_0 A_{-\beta}^{g} \mathcal{L}_{\beta}^{g}) \subset A_{-\beta}^{g} \mathcal{L}_{\beta}^{g} \subset I_{0,[\alpha]}.$$

Therefore,

$$A_0(A_{-\beta}^{g} \mathcal{L}_{\beta}^{g}) \subset I_{[\alpha]}.$$

**Case 2.** For any $\beta, \gamma, \mu \in [\alpha]$, with $\beta + \gamma + \mu = 0$, and $g, g', g'' \in G$ according to Eq. (2.8), for any $a_0 \in A_0$, $x \in \mathcal{L}_{\beta}^{g}$, $y \in \mathcal{L}_{\gamma}^{g'}$ and $z \in \mathcal{L}_{\mu}^{g''}$, we have

$$\epsilon(a_0, x + y)(x, y, z) = [x, y, a_0 z] - \rho(x, y)a_0 z \in \mathcal{L}_{\beta}^{g}, \mathcal{L}_{\gamma}^{g'}, \mathcal{L}_{\mu}^{g''} + \rho(\mathcal{L}_{\beta}^{g}, \mathcal{L}_{\gamma}^{g'}, \mathcal{L}_{\mu}^{g''})(A_0) \mathcal{L}_{\beta}^{g''}.$$ If $A_{-\mu}^{g+g'} \neq 0$ (otherwise is trivial), then Proposition 2.22 gives us $-\mu \in \Lambda^g + g'$, and

$$\epsilon(a_0, x + y)(x, y, z) \in [\mathcal{L}_{\beta}^{g}, \mathcal{L}_{\gamma}^{g'}, \mathcal{L}_{\mu}^{g''}] + A_{-\mu}^{g+g'} \mathcal{L}_{\mu}^{g''} \subset I_{0,[\alpha]}.$$ Therefore,

$$A_0(\mathcal{L}_{\beta}^{g}, \mathcal{L}_{\gamma}^{g'}, \mathcal{L}_{\mu}^{g''}) \subset I_{[\alpha]}.$$

**Case 3.** For any $\beta \in [\alpha]$, $g \in G$, by Proposition 2.22, we have

$$A_0 \mathcal{L}_{\beta}^{g} \subset \mathcal{L}_{\beta}^{g} \subset V_{[\alpha]}.$$

Hence

$$A_0 \mathcal{L}_{\beta}^{g} \subset \mathcal{L}_{\beta}^{g} \subset I_{[\alpha]}.$$

**Case 4.** For $\lambda \in \Lambda$, $\beta \in [\alpha]$, and $g, g', g'' \in G$, if $-\beta \in \Lambda^h$ for some $h \in G$ then

$$A_{\lambda}^{g'}(A_{-\beta}^{g''} \mathcal{L}_{\beta}^{g}) = (A_{\lambda}^{g'} A_{-\beta}^{g''}) \mathcal{L}_{\beta}^{g} \subset A_{-\beta}^{g+g''} \mathcal{L}_{\beta}^{g}.$$
Proof. Since \( \beta, \lambda - \beta, 0 \) is a connection from \( \beta \) to \( \lambda \) and so \( \lambda \in [\alpha] \). Hence,
\[
A^g_\lambda (A^g_{-\beta} \mathcal{L}_\beta) \subset \mathcal{L}^{g_1 + g_2 + g_3} \subset V_{[\alpha]}.
\]
Therefore,
\[
A^g_\lambda (A^g_{-\beta} \mathcal{L}_\beta) \subset I_{[\alpha]}.
\]

**Case 5.** For any \( \beta, \gamma, \mu \in [\alpha] \) with \( \beta + \gamma + \mu = 0 \), and \( g_1, g_2, g_3 \in G \), if \( \lambda \in \Lambda \), \( g \in G \). According to Eq. (2.8), for any \( a \in A^g_\lambda \), \( x \in \mathcal{L}^{g_1}_\beta \), \( y \in \mathcal{L}^{g_2}_\gamma \) and \( z \in \mathcal{L}^{g_3}_\mu \), we have
\[
\epsilon(a, x + y)a[x, y, z] = [x, y, az] - \rho(x, y)az
\]
\[
\in [\mathcal{L}^{g_1}_\beta, \mathcal{L}^{g_2}_\gamma, A^g_\lambda \mathcal{L}^{g_3}_\mu] + \rho(\mathcal{L}^{g_1}_\beta, \mathcal{L}^{g_2}_\gamma)(A^g_\lambda)\mathcal{L}^{g_3}_\mu
\]
\[
\subset [\mathcal{L}^{g_1}_\beta, \mathcal{L}^{g_2}_\gamma, \mathcal{L}^{g_1 + g_2}_\beta + \mathcal{L}^{g_3}_\mu] + A^{g_1 + g_2 + g_3}_\lambda \mathcal{L}^{g_3}_\mu.
\]
If \( \mathcal{L}^{g_1 + g_2 + g_3} \neq 0 \) and \( \mathcal{L}^{g_1 + g_2 + g_3} \neq 0 \) (otherwise is trivial), then \( \lambda \in \Pi^{g_1 + g_2 + g_3} \) and \( \lambda + \mu \in \Pi^{g_1 + g_3} \). By Proposition 2.22, we get \( \lambda \sim \mu \), so \( \lambda \in [\alpha] \). Therefore,
\[
A^g_\lambda [\mathcal{L}^{g_1}_\beta, \mathcal{L}^{g_2}_\gamma, \mathcal{L}^{g_3}_\mu] \subset I_{[\alpha]}.
\]

**Case 6.** For any \( \lambda \in \Lambda \), \( \beta \in [\alpha] \) and \( g, g' \in G \), By Proposition 2.22 we get
\[
A^g_\lambda \mathcal{L}^{g'}_\beta \subset \mathcal{L}^{g_1 + g'}_\beta \subset V_{[\alpha]}.
\]
Hence,
\[
A^g_\lambda \mathcal{L}^{g'}_\beta \subset I_{[\alpha]}.
\]
Now, summarizing a discussion of above six cases, we get the result.

(3) By part (2) and Eq. (2.8) we have
\[
\rho(I_{[\alpha]}, I_{[\alpha]})(A)\mathcal{L} \subset I_{[\alpha]}, I_{[\alpha]}, A\mathcal{L} + A[I_{[\alpha]}, I_{[\alpha]}, \mathcal{L}] \subset I_{[\alpha]}.
\]

**Proposition 3.13.** Let \([\alpha], [\beta], [\gamma] \in \Pi / \sim \) be different from each other. Then
\[
[I_{[\alpha]}, I_{[\beta]}, I_{[\gamma]}] = 0, \text{ and } [I_{[\alpha]}, I_{[\alpha]}, I_{[\beta]}] = 0.
\]

**Proof.** Since \([I_{[\alpha]}, I_{[\beta]}, I_{[\gamma]}] \subset [\mathcal{H}, \mathcal{H}, \mathcal{H}] = 0 \), we have
\[
(I_{[\alpha]}, I_{[\beta]}, I_{[\gamma]}) = \begin{cases} [I_{[\alpha]} \oplus V_{[\alpha]}, I_{[\beta]} \oplus V_{[\beta]}, I_{[\gamma]} \oplus V_{[\gamma]}] \\ \subset [I_{[\alpha]}, I_{[\beta]}, V_{[\gamma]}] + [I_{[\alpha]}, V_{[\beta]}, I_{[\gamma]}] \\ + [V_{[\alpha]}, I_{[\beta]}, I_{[\gamma]}] + [V_{[\alpha]}, V_{[\beta]}, I_{[\gamma]}] \\ + [V_{[\alpha]}, V_{[\beta]}, V_{[\gamma]}]. \end{cases}
\]
First, we consider the item \([V_{[\alpha]}, V_{[\beta]}, V_{[\gamma]}] \) in Eq. (3.14). Suppose that there exist \( \alpha_1 \in [\alpha], \beta_1 \in [\beta], \gamma_1 \in [\gamma] \) and \( g_1, g_2, g_3 \in G \) such that \([\mathcal{L}^{g_1}_\alpha, \mathcal{L}^{g_2}_\beta, \mathcal{L}^{g_3}_\gamma] \neq 0 \). By Proposition 2.22 \( \alpha_1 + \beta_1 + \gamma_1 \in \Pi^{g_1 + g_2 + g_3} \). Since \( \alpha \sim \alpha_1 \), \( \alpha_1 + \beta_1 + \gamma_1 \in \Pi^{g_1 + g_2 + g_3} \), and Lemma 3.3 give us \( \alpha \sim \alpha_1 + \beta_1 + \gamma_1 \). Similarly \( \beta \sim \alpha_1 + \beta_1 + \gamma_1 \). By transitivity, \( \alpha \sim \beta \), which is a contradiction. Hence,
\[
[V_{[\alpha]}, V_{[\beta]}, V_{[\gamma]}] = \{0\}.
\]
Next, we consider the item $[I_{0,[a]}, V_{[\beta]}, V_{[\gamma]}]$ in Eq. (3.14). We have

$$
[I_{0,[a]}, V_{[\beta]}, V_{[\gamma]}] = \left( \sum_{\beta, \gamma \in \mathbb{A}, g, g' \in G} A^{g}_{-\beta} L^{g}_{\beta} \right) + \left( \sum_{\beta, \gamma, \mu \in [a], g, g', \mu \in [a]} [L^{g}_{\beta}, L^{g'}_{\gamma}, L^{\mu}_{\mu}], V_{[\beta]}, V_{[\gamma]} \right).
$$

For $\beta_1 \in [\beta], \gamma_1 \in [\gamma]$, and $x_i \in L^{g_i}_{\alpha_i} (i = 1, 2, 3)$, $y_2 \in L^{h}_{\beta}$, $y_3 \in L^{k}_{\gamma}$ by $\epsilon$-fundamental identity,

$$
[[x_1, x_2, x_3], y_2, y_3] = [[x_1, y_2, y_3], x_2, x_3] + \epsilon(x_1 + x_2, y_2)[x_1, [x_2, y_2, y_3], x_3] + \epsilon(x_1 + x_2, y_2 + y_3)[x_1, x_2, [x_3, y_2, y_3]]
$$

$$
\subset [[L^{g_1}_{\alpha_1}, L^{h}_{\beta}, L^{k}_{\gamma}], L^{g_2}_{\alpha_2}, L^{g_3}_{\alpha_3}] + [L^{g_1}_{\alpha_1}, [L^{g_2}_{\alpha_2}, L^{h}_{\beta}, L^{k}_{\gamma}], L^{g_3}_{\alpha_3}] + [L^{g_1}_{\alpha_1}, L^{g_2}_{\alpha_2}, [L^{g_3}_{\alpha_3}, L^{h}_{\beta}, L^{k}_{\gamma}]].
$$

Using Eq. (3.15), we get

$$
[L^{g_1}_{\alpha_1}, L^{h}_{\beta}, L^{k}_{\gamma}] = [L^{g_2}_{\alpha_2}, L^{h}_{\beta}, L^{k}_{\gamma}] = [L^{g_3}_{\alpha_3}, L^{h}_{\beta}, L^{k}_{\gamma}] = 0.
$$

Therefore,

$$
[[\sum_{\beta, \gamma, \mu \in [a], g, g', \mu \in [a]} [L^{g}_{\beta}, L^{g'}_{\gamma}, L^{\mu}_{\mu}], V_{[\beta]}, V_{[\gamma]}] = 0.
$$

Now, if there exist $\beta_1 \in [\beta], \gamma_1 \in [\gamma]$ such that $[A^{g}_{-\alpha_1} L^{g'}_{\alpha_1}, L^{h}_{\beta_1}, L^{k}_{\gamma_1}] \neq 0$. By Eq. (2.22), we have

$$
[A^{g}_{-\alpha_1} L^{g'}_{\alpha_1}, L^{h}_{\beta_1}, L^{k}_{\gamma_1}] = [L^{h}_{\beta_1}, L^{k}_{\gamma_1}, A^{g}_{-\alpha_1} L^{g'}_{\alpha_1}]
$$

$$
\subset A^{g}_{-\alpha_1} [L^{g'}_{\alpha_1}, L^{h}_{\beta_1}, L^{k}_{\gamma_1}] + \rho(L^{h}_{\beta_1}, L^{k}_{\gamma_1})(A^{g}_{-\alpha_1}) L^{g'}_{\alpha_1}.
$$

By Eq. (3.15), we have $[L^{g'}_{\alpha_1}, L^{h}_{\beta_1}, L^{k}_{\gamma_1}] = 0$, and Proposition 2.22 gives us

$$
\rho(L^{h}_{\beta_1}, L^{k}_{\gamma_1})(A^{g}_{-\alpha_1}) L^{g'}_{\alpha_1} \subset A^{g+h+k}_{\beta_1+\gamma_1-\alpha_1} L^{g'}_{\alpha_1} \neq 0,
$$

then $A^{g+h+k}_{\beta_1+\gamma_1-\alpha_1} \neq 0$ and $\beta_1 + \gamma_1 - \alpha_1 \in A^{g+h+k}$. We get that $\{\alpha_1, -\gamma_1, \beta_1 + \gamma_1 - \alpha_1\}$ is a connection from $\alpha_1$ to $\beta_1$ and so $\alpha_1 \sim \beta_1$, which is a contradiction. Hence,

$$
(3.16) \quad \rho(L^{h}_{\beta_1}, L^{k}_{\gamma_1})(A^{g}_{-\alpha_1}) L^{g'}_{\alpha_1} = 0.
$$

It follows

$$
(3.17) \quad [I_{0,[a]}, V_{[\beta]}, V_{[\gamma]}] = 0.
$$

By a similar argument, we also get,

$$
(3.18) \quad [V_{[\alpha]}, I_{0,[\beta]}, V_{[\gamma]}] = [V_{[\alpha]}, \beta, I_{0,[\gamma]}] = 0.
$$
Finally, we consider the summand \([I_{0,[\alpha]}, I_{0,[\beta]}, \mathcal{V}_{[\gamma]}]\) in Eqs. (3.14). One can write,

\[
[I_{0,[\alpha]}, I_{0,[\beta]}, \mathcal{V}_{[\gamma]}] = (\sum_{\alpha_1 \in [\alpha]} A^\alpha_{-\alpha_1} L^\beta_{\alpha_1}) + (\sum_{\alpha_2, \alpha_3 \in [\alpha]} A^\alpha_{-\alpha_2, \alpha_3} L^{\beta_1}_{\alpha_2, \alpha_3}),
\]

Next, by Eq. (3.15) we have

\[
(\sum_{\beta_1 \in [\beta]} A^h_{B_1} L^{h_1}_{\beta_1}) + (\sum_{\beta_2, \beta_3 \in [\beta]} A^h_{B_2, B_3} L^{h_1}_{\beta_2, \beta_3}), \mathcal{V}_{[\gamma]}].
\]

The above statement includes four items which we consider in the following. First, consider the item \([A^\alpha_{-\alpha_1} L^\beta_{\alpha_1}, A^h_{-\beta_1} L^{h_1}_{\beta_1}, \mathcal{L}_{[\gamma]}]\), where \(\gamma_1 \in [\gamma]\). By Eqs. (2.8), (3.15) and (3.16), we have

\[
[A^\alpha_{-\alpha_1} L^\beta_{\alpha_1}, A^h_{-\beta_1} L^{h_1}_{\beta_1}, \mathcal{L}_{[\gamma]}] = [L^\beta_{\gamma_1}, A^\alpha_{-\alpha_1} L^\beta_{\alpha_1}, A^h_{-\beta_1} L^{h_1}_{\beta_1}]
= A^h_{-\beta_1} L^\beta_{\gamma_1}, A^\alpha_{-\alpha_1} L^\beta_{\alpha_1}, L^{h_1}_{\beta_1}]
+ A^\alpha_{-\alpha_1} \rho(L^\beta_{\gamma_1}, L^{g_{[\alpha]}}_{\alpha_1})(A^h_{-\beta_1} L^{g_{[\alpha]}}_{\alpha_1})
= A^h_{-\beta_1} A_{-\alpha_1} \rho(L^\beta_{\gamma_1}, L^{g_{[\alpha]}}_{\alpha_1}, L^{h_1}_{\beta_1})
+ A^\alpha_{-\alpha_1} \rho(L^\beta_{\gamma_1}, L^{g_{[\alpha]}}_{\alpha_1})(A^h_{-\beta_1} L^{g_{[\alpha]}}_{\beta_1})
= 0.
\]

Next, by Eq. (3.15) we have

\[
[A^\alpha_{-\alpha_1} L^{g_{[\alpha]}}_{\alpha_1}, L^{h_1}_{\beta_1}, L^{h_2}_{\beta_2}, L^{h_3}_{\beta_3}, L^{h_4}_{\beta_4}], \mathcal{L}_{[\gamma]}] \subset [\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\beta]}, \mathcal{V}_{[\gamma]}] = 0,
\]
and

\[
[[L^{g_{[\alpha]}}_{\alpha_1}, L^{g_{[\alpha]}}_{\alpha_2}, L^{g_{[\alpha]}}_{\alpha_3}], A^h_{-\beta_1} L^{h_1}_{\beta_1}, \mathcal{L}_{[\gamma]}] \subset [\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\beta]}, \mathcal{V}_{[\gamma]}] = 0,
\]
and

\[
[[L^{g_{[\alpha]}}_{\alpha_1}, L^{g_{[\alpha]}}_{\alpha_2}, L^{g_{[\alpha]}}_{\alpha_3}], [L^{h_1}_{\beta_1}, L^{h_2}_{\beta_2}, L^{h_3}_{\beta_3}], \mathcal{L}_{[\gamma]}] \subset [\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\beta]}, \mathcal{V}_{[\gamma]}] = 0.
\]

Summarizing a discussion of above, we get

\[
[I_{0,[\alpha]}, I_{0,[\beta]}, \mathcal{V}_{[\gamma]}] = 0.
\]

In a similar way we get

\[
[I_{0,[\alpha]}, \mathcal{V}_{[\beta]}, I_{0,[\gamma]}] = 0, [\mathcal{V}_{[\alpha]}, I_{0,[\beta]}, \mathcal{V}_{[\gamma]}] = 0.
\]

Therefore,

\[
[I_{[\alpha]}, I_{[\beta]}, I_{[\gamma]}] = 0.
\]

By a similar argument as above one can prove \([I_{[\alpha]}, I_{[\beta]}, I_{[\gamma]}] = 0. \)

**Theorem 3.19.** The following assertions hold

1. For any \([\alpha] \in \Pi / \sim\), the linear graded subspace

\[
I_{[\alpha]} = I_{0,[\alpha]} \oplus \mathcal{V}_{[\alpha]},
\]

of split 3–Lie-Rinehart color algebra \((\mathcal{L}, A)\) associated to \([\alpha]\) is an ideal of \((\mathcal{L}, A)\).
(2) If $(\mathcal{L}, A)$ is simple, then there exists a connection from $\alpha$ to $\beta$ for any $\alpha, \beta \in \Pi$ and

\[
\mathcal{H} = \left( \sum_{\beta \in \Pi, \beta \in A} A_{\beta}^g \mathcal{L}_{\beta}^{g'} \right) + \left( \sum_{\beta, \gamma, \mu \in \Pi} \mathcal{L}_{\beta}^g \left[ \mathcal{L}_{\gamma}^\mu, \mathcal{L}_{\nu}^\mu \right] \right).
\]

**Proof.**

(1) Since

\[
[I_{[\alpha]}, \mathcal{H}, \mathcal{H}] = [I_{[\alpha]}, \mathcal{L}_0, \mathcal{L}_0] \subset V_{[\alpha]},
\]

By Propositions 3.7 and 3.13, we have $I_{[\alpha]} = I_{[\beta]}$, and following Proposition 3.7, we have $\mathcal{L} \cap \mathcal{H} = \mathcal{L} \cap \mathcal{H}$. Therefore, by Eqs. (3.4), (3.5) and (3.6), we have

\[
\sum_{\beta \in \Pi, \beta \in A} A_{\beta}^g \mathcal{L}_{\beta}^{g'} + \left( \sum_{\beta, \gamma, \mu \in \Pi} \mathcal{L}_{\beta}^g \left[ \mathcal{L}_{\gamma}^\mu, \mathcal{L}_{\nu}^\mu \right] \right).
\]

Then $I_{[\alpha]}$ is a 3-Lie ideal of $\mathcal{L}$. We also have $AI_{[\alpha]} \subset I_{[\alpha]}$, thanks to Proposition 3.17(2), and

\[
\rho(I_{[\alpha]}, \mathcal{L})(A) \subset A[I_{[\alpha]}, \mathcal{L}, \mathcal{L}] + [I_{[\alpha]}, \mathcal{L}, A\mathcal{L}] I_{[\alpha]} \subset I_{[\alpha]}.
\]

From Proposition 3.7, $I_{[\alpha]}$ is an ideal of $(\mathcal{L}, A)$.

(2) The simplicity of $(\mathcal{L}, A)$ implies that $I_{[\alpha]} \in \{\mathcal{L}, \ker \rho\}$. If $I_{[\alpha]} = \mathcal{L}$ for some $\alpha \in \Pi$, we are done. Otherwise, if $I_{[\alpha]} = \ker \rho$ for all $\alpha \in \Pi$ we have $[\alpha] = [eta]$ for any $\beta \in \Pi$ and again $\Pi = [\alpha]$. We conclude that $\mathcal{L}$ has all of its non zero roots connected and therefore, by Eqs. (3.4), (3.5) and (3.6), we have

\[
\mathcal{H} = \left( \sum_{\beta \in \Pi, \beta \in A} A_{\beta}^g \mathcal{L}_{\beta}^{g'} \right) + \left( \sum_{\beta, \gamma, \mu \in \Pi} \mathcal{L}_{\beta}^g \left[ \mathcal{L}_{\gamma}^\mu, \mathcal{L}_{\nu}^\mu \right] \right).
\]

**Theorem 3.20.** For a vector space complement $U$ in $\mathcal{H}$ of

\[
\left( \sum_{\beta \in \Pi, \beta \in A} A_{\beta}^g \mathcal{L}_{\beta}^{g'} \right) + \left( \sum_{\beta, \gamma, \mu \in \Pi} \mathcal{L}_{\beta}^g \left[ \mathcal{L}_{\gamma}^\mu, \mathcal{L}_{\nu}^\mu \right] \right),
\]

we have

\[
\mathcal{L} = U \oplus \sum_{[\alpha] \in \Pi/\sim} I_{[\alpha]},
\]

where any $I_{[\alpha]}$ is one of the graded ideals of $(\mathcal{L}, A)$ described in Theorem 3.19(1), and satisfying the conditions in Proposition 3.13.

**Proof.** Each $I_{[\alpha]}$ is well defined and, by Theorem 3.19(1), a graded ideal of $(\mathcal{L}, A)$. It is clear that

\[
\mathcal{L} = \mathcal{H} \oplus \left( \bigoplus_{\alpha \in \Pi} \mathcal{L}_{\alpha} \right) = U \oplus \sum_{[\alpha] \in \Pi/\sim} I_{[\alpha]}.
\]
Finally, Proposition 3.13 gives us
\[ [I_\alpha, I_\beta, I_\gamma] = 0, \quad \text{and} \quad [I_\alpha, I_\alpha, I_\beta] = 0, \]
whenever \([\alpha, \beta, \gamma] \in \Pi/\sim\) be different from each other. \(\square\)

**Proposition 3.21.** If \(Z_\rho(L) = \{0\}\) and
\[
\mathcal{H} = \left( \sum_{\beta \in \Pi, \gamma \in A} A_{-\beta}^g \mathcal{L}_{\beta}^{g'} \right) + \left( \sum_{\beta, \gamma, \mu \in \Pi} \mathcal{L}_{\beta}^{g}, \mathcal{L}_{\gamma}^{g'}, \mathcal{L}_{\mu}^{g''} \right),
\]
then \(L\) is the direct sum of the graded ideals given in Theorem 3.19 (1),
\[
L = \bigoplus_{[\alpha] \in \Pi/\sim} I_{[\alpha]},
\]
satisfying the conditions in Proposition 3.13.

**Proof.** From Theorem 3.20 and
\[
\mathcal{H} = \left( \sum_{\beta \in \Pi, \gamma \in A} A_{-\beta}^g \mathcal{L}_{\beta}^{g'} \right) + \left( \sum_{\beta, \gamma, \mu \in \Pi} \mathcal{L}_{\beta}^{g}, \mathcal{L}_{\gamma}^{g'}, \mathcal{L}_{\mu}^{g''} \right),
\]
it is clear that \(L = \sum_{[\alpha] \in \Pi/\sim} I_{[\alpha]}\). For the direct character, take some \(x \in I_{[\alpha]} \cap \sum_{[\alpha] \in \Pi/\sim, [\beta] \neq [\alpha]} I_{[\beta]}\). By Proposition 3.13 we have
\[
[x, \mathcal{L}, \mathcal{L}] \subset [x, I_{[\alpha]}, I_{[\alpha]}] + [x, I_{[\alpha]}, \sum_{\beta \not\in [\alpha]} I_{[\beta]}] + [x, \sum_{\beta \in [\alpha]} I_{[\beta]}, \sum_{\beta \not\in [\alpha]} I_{[\beta]}] = 0.
\]
That is, \(x \in \text{Ann}(L)\). We also have \(\rho(x, \mathcal{L}) = 0\), and so \(x \in Z_\rho(L) = \{0\}\). Hence, \(L = \bigoplus_{[\alpha] \in \Pi/\sim} I_{[\alpha]}\), as by Proposition 3.13
\[
[I_{[\alpha]}, I_{[\beta]}, I_{[\gamma]}] = 0, \quad \text{and} \quad [I_{[\alpha]}, I_{[\alpha]}, I_{[\beta]}] = 0,
\]
whenever \([\alpha, [\beta, [\gamma] \in \Pi/\sim\) be different from each other. \(\square\)

4. **Connections in the Weight System of \(A\).**

**Decomposition of \(A\)**

We begin this section by introducing an adequate notion of connection among the weights in \(\Lambda\). For a split regular 3–Lie–Rinehart color algebra \((\mathcal{L}, A)\), since \(A\) is a commutative associative color algebra, then the decomposition of \(A\) is similar to [2] and omit the proof of some results.

**Definition 4.1.** Let \(\lambda, \mu \in \Lambda\), we say that \(\lambda\) is connected to \(\mu\) and denoted by \(\lambda \approx \mu\), if either \(\lambda = \pm \mu\), or there exists a family
\[
\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k\} \subset \pm \Pi \cup \pm \Lambda,
\]
with \( k \geq 2 \), such that satisfying the following conditions:

(1) \( \lambda_1 = \lambda \).
(2) \( \lambda_1 + \lambda_2 \in \pm \Lambda \),
    \( \lambda_1 + \lambda_2 + \lambda_3 \in \pm \Lambda \),
    ...
    \( \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_{k-1} \in \pm \Lambda \).
(3) \( \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_k \in \{ \mu, -\mu \} \).

The family \( \{ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k \} \) is called a connection from \( \lambda \) to \( \mu \).

As in the previous section we can prove the next result;

**Proposition 4.2.** The relation \( \approx \) in \( \Lambda \) defined by

\[ \lambda \approx \mu \quad \text{if and only if} \quad \lambda \text{ is connected to } \mu, \]

is an equivalence relation.

**Remark 4.3.** Let \( \lambda, \mu \in \Lambda \) such that \( \lambda \approx \mu \). If \( \lambda + \eta \in \Lambda \), for \( \eta \in \Pi \cup \Lambda \) then \( \lambda \approx \mu + \eta \). Considering the connection \( \{ \mu, \eta \} \) we get \( \mu \approx \mu + \eta \) and by transitivity \( \lambda \approx \mu + \eta \).

By the Proposition 4.2 we can consider the equivalence relation in \( \Lambda \) by the connection relation \( \approx \) in \( \Lambda \). So we denote by \( \Lambda / \approx := \{ [\lambda] : \lambda \in \Lambda \} \), where \([\lambda]\) denotes the set of non zero weights which are connected to \( \lambda \).

Our next goal in this section is to associate an adequate ideal \( A_{[\lambda]} \) of \( A \) to any \([\lambda]\). For a fixed \( \lambda \in \Lambda \), we define

\[
A_{0,[\lambda]} := \left( \sum_{\mu \in [\lambda]} A_{-\mu} A_{\mu} \right) + \left( \sum_{\alpha, \beta \in \Pi, \mu \in [\lambda]} \rho(\mathcal{L}_\alpha, \mathcal{L}_\beta) A_{\mu} \right) \subset A_0.
\]

That is

\[
(4.4) \quad A_{0,[\lambda]} := \left( \sum_{\mu \in [\lambda]} A_{g^\mu}^0 A_{\mu}^g \right) + \left( \sum_{\alpha, \beta \in \Pi, \mu \in [\lambda]} \rho(\mathcal{L}_{\alpha}^g, \mathcal{L}_{\beta}^{g'}) A_{\mu}^{g''} \right) \subset A_0^{g+g'+g''}.
\]

Next, we define

\[
(4.5) \quad A_{[\lambda]} := \bigoplus_{\mu \in [\lambda]} A_{\mu} = \bigoplus_{g \in G} \bigoplus_{\mu \in [\lambda]} A_{g}^\mu.
\]

Finally, we denote by \( A_{[\lambda]} \) the direct sum of the two graded subspaces above, that is,

\[
(4.6) \quad A_{[\lambda]} := A_{0,[\lambda]} \oplus A_{[\lambda]}.
\]

**Proposition 4.7.** For any \( \lambda, \mu \in \Lambda \), the following assertions hold.

(1) \( A_{[\lambda]} A_{[\mu]} \subset A_{[\lambda]} \).
(2) If \( [\lambda] \neq [\mu] \), then \( A_{[\lambda]} A_{[\mu]} = 0 \).
Proof. We only prove (1) and similar for (2). By Eq. (4.9) and comutativity of $A$ we have

$$A_{\lambda,[\lambda]} A_{[\lambda]} = (A_{0,[\lambda]} \oplus A_{[\lambda]}) (A_{0,[\lambda]} \oplus A_{[\lambda]}) + A_{0,[\lambda]} A_{[\lambda]} + A_{[\lambda]} A_{[\lambda]}.$$  

Consider the second summand in Eq. (4.8). Given $\mu \in [\lambda], g \in G$ we have

$$A_{0,[\lambda]} A_{q}^{g} \subset A_{0} A_{0}^{g} \subset A_{0}.$$ 

Therefore

(4.9) $$A_{0,[\lambda]} A_{[\lambda]} \subset A_{[\lambda]}.$$ 

Let us consider the third summand in Eq. (4.8). Given $\mu, \eta \in [\lambda], g, g' \in G$ such that $A_{\mu}^{g} A_{\eta}^{g'} \neq 0$. Then $A_{\mu}^{g} A_{\eta}^{g'} \subset A_{\mu+\eta}^{g+g'}$. If $\mu + \eta = 0$, by Eq. (4.4) we have $A_{\mu}^{g} A_{\mu}^{g'} \subset A_{0,[\lambda]}$. Suppose $\mu + \eta \in A_{\mu}^{g} A_{\mu}^{g'}$, then thanks to Remark 4.3 we have $\mu + \eta \in [\lambda]$ and so $A_{\mu}^{g} A_{\eta}^{g'} \subset A_{\mu+\eta}^{g+g'} \subset A_{[\lambda]}$. Hence,

$$\left( \bigoplus_{g \in G} A_{\mu}^{g} \right) \left( \bigoplus_{\eta \in G} A_{\eta}^{g'} \right) \subset A_{0,[\lambda]} \oplus A_{[\lambda]},$$

and so

(4.10) $$A_{[\lambda]} A_{[\lambda]} \subset A_{[\lambda]}.$$ 

Finally, consider the first summand in Eq. (4.8). Given $\mu, \eta, \in \Lambda, \alpha, \beta, i \in \Pi, i = 1, 2$ and $g, g_1, g_1', g_2, h, h_2, h_2', h_2'' \in G$ such that

$$0 \neq (A_{-\mu_1}^{g} A_{\mu_1}^{g}) (A_{-\mu_2}^{g} A_{\mu_2}^{g}) (A_{-h_1}^{g} A_{h_1}^{g}) (A_{-h_2}^{g} A_{h_2}^{g}) (A_{-h_2'}^{g} A_{h_2'}^{g}) (A_{-h_2''}^{g} A_{h_2''}^{g})$$

(4.11) $$+ (A_{-\mu_1}^{g} A_{\mu_1}^{g}) (A_{-h_1}^{g} A_{h_1}^{g}) (A_{-h_2}^{g} A_{h_2}^{g}) (A_{-h_2'}^{g} A_{h_2'}^{g}) (A_{-h_2''}^{g} A_{h_2''}^{g}) (A_{-\mu_2}^{g} A_{\mu_2}^{g})$$

We are going to consider all summands in Eq. (4.11), in four cases;

**Case 1.** For the first summand, if $\mu_1 + \mu_2 \neq 0$, by Remark 4.3 and the fact that $A$ is a commutative and associative algebra, we have

(4.12) $$A_{-\mu_1}^{g} A_{\mu_1}^{g} (A_{-h_1}^{g} A_{h_1}^{g}) (A_{-h_2}^{g} A_{h_2}^{g}) (A_{-h_2'}^{g} A_{h_2'}^{g}) \subset A_{0,[\lambda].}$$

If $\mu_1 + \mu_2 = 0$, it follows

(4.13) $$A_{-\mu_1}^{g} A_{\mu_1}^{g} (A_{-h_1}^{g} A_{h_1}^{g}) (A_{-h_2}^{g} A_{h_2}^{g}) (A_{-h_2'}^{g} A_{h_2'}^{g}) \subset A_{0,[\lambda]}.$$ 

From Eqs. (4.12) and (4.13) we get

(4.14) $$A_{-\mu_1}^{g} A_{\mu_1}^{g} (A_{-h_2}^{g} A_{h_2}^{g}) \subset A_{[\lambda]}.$$ 

**Case 2.** For the second summand, since $\rho(L_{\alpha_1}^{g_1}, L_{\beta_1}^{g_1})$ is a derivation in $A$ we have

$$\rho(L_{\alpha_1}^{g_1}, L_{\beta_1}^{g_1}) (A_{-h_1}^{g_1} A_{h_1}^{g_1}) (A_{-h_2}^{g_1} A_{h_2}^{g_1}) (A_{-h_2'}^{g_1} A_{h_2'}^{g_1}) \subset \rho(L_{\alpha_1}^{g_1}, L_{\beta_1}^{g_1}) (A_{-h_1}^{g_1} A_{h_1}^{g_1})$$

(4.15) $$+ A_{-h_2}^{g_1} \rho(L_{\alpha_1}^{g_1}, L_{\beta_1}^{g_1}) (A_{-h_2}^{g_1} A_{h_2}^{g_1}).$$
Thanks to Proposition 2.22
\[ \rho(\mathcal{L}_{\alpha_1}, \mathcal{L}_{\beta_1}^g, (A_{h_1}^g (A_{-\mu_2}^h A_{\mu_2}^h))) \subset \rho(\mathcal{L}_{\alpha_1}, \mathcal{L}_{\beta_1}^g, (A_{\alpha_1, 1}^{g_1} + g_1^\prime + g_1^\prime)) \]
and
\[ A_{\alpha_1, 1}^{g_1} \rho(\mathcal{L}_{\alpha_1}, \mathcal{L}_{\beta_1}^g, (A_{-\mu_2}^h A_{\mu_2}^h)) \subset A_{\alpha_1, 1}^{g_1} + g_1^\prime \subset A_{\alpha_1, 1}^{g_1} + g_1^\prime + g_1^\prime. \]
Therefore, we get
(4.15)
\[ \rho(\mathcal{L}_{\alpha_1}, \mathcal{L}_{\beta_1}^g, (A_{-\mu_2}^h A_{\mu_2}^h)) \subset \mathcal{A}[\lambda]. \]

Case 3. Consider the third summand in Eq. (4.11), by the commutativity of \( A \), we have
\[ (A_{-\mu_1}^g A_{\mu_1}^g) \rho(\mathcal{L}_{\alpha_2}, \mathcal{L}_{\beta_2}^g, (A_{\alpha_2}^{h_2}, A_{\beta_2}^{h_2}, (A_{\beta_2}^{h_2}))) \subset A_{\alpha_2, 2}^{h_2 + h_2 + h_2} \subset A_{\lambda}. \]

Case 4. For the last summand in Eq. (4.11), since \( \rho(\mathcal{L}_{\alpha_1}, \mathcal{L}_{\beta_1}^g) \) is a derivation in \( A \), we have
\[ \rho(\mathcal{L}_{\alpha_1}, \mathcal{L}_{\beta_1}^g, (A_{h_1}^g, \rho(\mathcal{L}_{\alpha_2}, \mathcal{L}_{\beta_2}^g, (A_{\alpha_2}^{h_2}, A_{\beta_2}^{h_2, (A_{\beta_2}^{h_2})))) \subset \rho(\mathcal{L}_{\alpha_1}, \mathcal{L}_{\beta_1}^g, (A_{\alpha_2}^{h_2 + h_2 + h_2, 1} + g_1^\prime + g_1^\prime)) \subset A_{\alpha_1, 1}^{g_1^\prime} + g_1^\prime + h_1^\prime + h_2^\prime \subset A_{\alpha_1, 1}^{g_1^\prime} + g_1^\prime + h_1^\prime + h_2^\prime \subset A_{\lambda}. \]

Similarly,
\[ A_{\alpha_1, 1}^{g_1^\prime} \rho(\mathcal{L}_{\alpha_1}, \mathcal{L}_{\beta_1}^g, (A_{h_1}^g, \rho(\mathcal{L}_{\alpha_2}, \mathcal{L}_{\beta_2}^g, (A_{\alpha_2}^{h_2}, A_{\beta_2}^{h_2, (A_{\beta_2}^{h_2})))) \subset A_{\alpha_1, 1}^{g_1^\prime} + g_1^\prime + h_1^\prime + h_2^\prime \subset A_{\lambda}. \]

Summarizing a discussion of above four cases, we get
(4.16)
\[ A_{\lambda} \subset A_{\lambda}. \]

From Eqs. (4.12), (4.11) and (4.16) we conclude the result. \( \square \)

We recall that a \( G \)-graded subspace \( I \) of a commutative and associative color algebra \( A \) is called an ideal if \( AI \subseteq I \). We say that \( A \) is simple if \( AA \neq 0 \) and it contains no proper ideals.

**Theorem 4.17.** Let \( A \) be a commutative and associative color algebra associated to a split \( 3 \)-Lie-Rinehart color algebra \((\mathcal{L}, A)\). Then the following assertions hold.

1. For any \([\lambda] \in \Lambda/\approx\), the linear graded subspace
\[ A_{[\lambda]} = A_{0,[\lambda]} \oplus A_{[\lambda]}, \]
of color algebra \( A \) associated to \([\lambda]\) is an ideal of \( A \).

2. If \( A \) is simple then all weights of \( \Lambda \) are connected. Furthermore,
\[ A_0 = \sum_{\mu \in \Lambda} A_{\mu}^g A_{-\mu}^g + \sum_{\alpha, \beta, \mu, \rho, g, g', g'' \in G} \rho(\mathcal{L}_{\alpha}, \mathcal{L}_{\beta}^g, A_{\mu}^{g''}). \]
Proof. (1) By Eq. (2.21), we have
\[ \mathcal{A}_{[\lambda]}A = \mathcal{A}_{[\lambda]}(A^0 \oplus \bigoplus_{g \in G} (\bigoplus_{\mu \in [\lambda]^g} A^g_{\mu} + \bigoplus_{\mu \in [\lambda]^g} A^g_{\mu})). \]

Now, by associativity of \( A \), we have \( \mathcal{A}_{[\lambda]}A^0 \subset \mathcal{A}_{[\lambda]} \), and by Proposition 4.7-(2), if \( \lambda \)
\[ \text{By Eq. (2.21), we have} \]
\[ \text{Proof.} \]
\[ \text{By Theorem 4.17-(1),} \]
\[ \text{Hence,} \]
\[ \mathcal{A}_{[\lambda]}A \subset \mathcal{A}_{[\lambda]} \].
That is, \( \mathcal{A}_{[\lambda]} \) is an ideal of \( A \).

(2) The simplicity of \( A \) implies \( \mathcal{A}_{[\lambda]} = A \), for any \( \lambda \in \Lambda \). From here it is clear that \( \Lambda = [\lambda] \), and so \( A_0 = (\sum_{\mu \in \Lambda} A^g_{\mu}A^{-g}_{\mu}) + (\sum_{\alpha, \beta, \mu \in \Lambda \mid \alpha + \beta + \mu = 0, g, g', g'' \in G} \rho(L^g_{\alpha}, L^{g'}_{\beta})A^g_{\mu}) \). \[ \square \]

Theorem 4.18. Let \( A \) be a commutative and associative algebra associated to a split 3–Lie-Rinehart color algebra \((L, A)\). Then
\[ A = V + \sum_{[\lambda] \in \Lambda/\approx} \mathcal{A}_{[\lambda]}, \]
where \( V \) is a graded linear complement in \( A^0 \) of
\[ \left( \sum_{\mu \in \Lambda} A^g_{\mu}A^{-g}_{\mu} \right) + \left( \sum_{\alpha, \beta, \mu \in \Lambda \mid \alpha + \beta + \mu = 0, g, g', g'' \in G} \rho(L^g_{\alpha}, L^{g'}_{\beta})A^g_{\mu} \right), \]
and any \( \mathcal{A}_{[\lambda]} \) is one of ideals of \( A \) described in Theorem 4.17-(1). Furthermore \( \mathcal{A}_{[\lambda]}A_{[\mu]} = 0 \), when \( [\lambda] \neq [\mu] \).

Proof. By Theorem 4.17-(1), \( \mathcal{A}_{[\lambda]} \) is a well defined ideal of \( A \), being clear that
\[ A = \bigoplus_{g \in G} (A^g \oplus \bigoplus_{\lambda \in \Lambda} A^g_{\lambda}) = A^0 \oplus \bigoplus_{g \in G, \lambda \in \Lambda^g} A^g_{\lambda} = V + \sum_{[\lambda] \in \Lambda/\approx} \mathcal{A}_{[\lambda]}, \]
and by Proposition 4.7-(2), if \([\lambda] \neq [\mu]\) then \( \mathcal{A}_{[\lambda]}A_{[\mu]} = 0 \). \[ \square \]

Let us denote by \( \text{Ann}(A) := \{ a \in A : aA = 0 \} \) the annihilator of the commutative and associative algebra \( A \).

Corollary 4.19. Let \((L, A)\) be a 3–Lie-Rinehart color algebra. If \( \text{Ann}(A) = 0 \) and
\[ A_0 = (\sum_{\mu \in \Lambda} A^g_{\mu}A^{-g}_{\mu}) + \left( \sum_{\alpha, \beta, \mu \in \Lambda \mid \alpha + \beta + \mu = 0, g, g', g'' \in G} \rho(L^g_{\alpha}, L^{g'}_{\beta})A^g_{\mu} \right), \]
then \( A \) is the direct sum of the ideals given in Theorem 4.17-(1),
\[ A = \bigoplus_{[\lambda] \in \Lambda/\approx} \mathcal{A}_{[\lambda]}. \]
Furthermore, $A_{[\lambda]} A_{[\mu]} = 0$, when $[\lambda] \neq [\mu]$.

**Proof.** This can be proved analogously to Corollary 3.8 in [2].

5. RELATING THE DECOMPOSITIONS OF $\mathcal{L}$ AND $\mathcal{A}$

In this section, we will show that the decompositions of $\mathcal{L}$ and $\mathcal{A}$ as direct sum of ideals, given in Sections 2 and 3 respectively, are closely related.

**Lemma 5.1.** Let $(\mathcal{L}, A)$ be a split 3–Lie-Rinehart color algebra and $I$ an ideal of $\mathcal{L}$. Then $I = (I \cap \mathcal{H}) \oplus (\bigoplus_{\alpha \in \Pi}(I \cap \mathcal{L}_{\alpha}))$

**Proof.** Since $(\mathcal{L}, A)$ is split, we get $\mathcal{L} = \mathcal{H} \oplus (\bigoplus_{\alpha \in \Pi} \mathcal{L}_{\alpha})$. By the assumption that $I$ is an ideal of $\mathcal{L}$, it is clear that $I$ is a submodule of $\mathcal{L}$. Thus $I$ is a weight module and therefore $I = (I \cap \mathcal{H}) \oplus (\bigoplus_{\alpha \in \Pi}(I \cap \mathcal{L}_{\alpha}))$.

Observe that for a split 3–Lie-Rinehart color algebra, by grading, let us assert that given any non-zero graded ideal $I$ of $\mathcal{L}$ we can write

$$I = \bigoplus_{g \in G} ((I \cap\mathcal{H}^g) \oplus (\bigoplus_{\alpha \in \Pi^g}(I \cap \mathcal{L}_{\alpha}^g)))$$

where $\Pi^g := \{\alpha \in \Pi : I \cap \mathcal{L}_{\alpha}^g \neq 0\}$ for each $g \in G$.

**Lemma 5.3.** Let $(\mathcal{L}, A)$ be a split 3–Lie-Rinehart color algebra with $Z_\rho(\mathcal{L}) = 0$ and $I$ an ideal of $\mathcal{L}$. If $I \subseteq \mathcal{H}$, then $I = \{0\}$.

**Proof.** By Proposition 5.21 and $I \subseteq \mathcal{H}$, we have $I \cap \mathcal{L}_{\alpha} = 0$, for any $\alpha \in \Pi$. So $[I, \mathcal{H}, \mathcal{L}_{\alpha}] = 0$, $[I, \mathcal{L}_{\alpha}, \mathcal{L}_{\beta}] = 0$, $\forall \alpha, \beta \in \Pi$.

Therefore,

$$[I, \mathcal{L}, \mathcal{L}] = [I, \mathcal{H} \bigoplus \bigoplus_{\alpha \in \Pi} \mathcal{L}_{\alpha}, \mathcal{H} \bigoplus \bigoplus_{\beta \in \Pi} \mathcal{L}_{\beta}] \subseteq [I, \mathcal{H}, \bigoplus_{\beta \in \Pi} \mathcal{L}_{\beta}] + [I, \bigoplus_{\alpha \in \Pi} \mathcal{L}_{\alpha}, \bigoplus_{\beta \in \Pi} \mathcal{L}_{\beta}] = 0.$$ 

So $I \subseteq Z_\rho(\mathcal{L}) = 0$.

**Proposition 5.4.** If a split 3–Lie-Rinehart color algebra $(\mathcal{L}, A)$ can be decomposed into a direct sum of finite ideals $\mathcal{L}_i, 1 \leq i \leq s$, and $Z_\rho(\mathcal{A}) = 0$, then $(\mathcal{L}, A)$ is a split 3–Lie-Rinehart color algebra with a splitting Cartan subalgebra $\mathcal{H}$ if and only if $(\mathcal{L}_i, A, \rho|_{\mathcal{L}_i \times \mathcal{L}_i}), 1 \leq i \leq s$ are split 3–Lie-Rinehart color algebras with a splitting Cartan subalgebra $\mathcal{H}_i$, respectively, such that $\mathcal{H} = \bigoplus_{i=1}^s \mathcal{H}_i$, $\Pi = \bigcup_{i=1}^s \Pi_i$, and $\Lambda = \bigcup_{i=1}^s \Lambda_i$, where $\Pi_i$ are root systems of $\mathcal{L}_i$ and $\Lambda_i$ are weight systems of $A$ associated to $\mathcal{H}_i$, respectively.

**Proof.** Suppose $(\mathcal{L}, A)$ is a split 3–Lie-Rinehart color algebra with Cartan subalgebra $\mathcal{H}$ and root system $\Pi$. Let $\mathcal{L} = \bigoplus_{i=1}^s \mathcal{L}_i$, where $\mathcal{L}_i$ be non-zero ideals of $\mathcal{L}$. By Eq. (5.2),

$$\mathcal{L}_i = \bigoplus_{g \in G} ((\mathcal{L}_i \cap \mathcal{H}^g) \oplus (\bigoplus_{\alpha \in \Pi}(\mathcal{L}_i \cap \mathcal{L}_{\alpha}^g))), \ i = 1, 2, 3, ..., s.$$
If \( \mathcal{L}_i \cap \mathcal{L}_g^g \neq 0 \), \( \forall g \in G \) then \( \alpha|_{\mathcal{H}_i \times \mathcal{H}_i} \neq 0 \), where \( \mathcal{H}_i = \bigoplus_{g \in G} (\mathcal{L}_i \cap \mathcal{H}^g) = \bigoplus_{g \in G} \mathcal{H}_i^g \). In fact, if \( \alpha|_{\mathcal{H}_i \times \mathcal{H}_i} = 0 \). For any \( 0 \neq x_i \in \mathcal{L}_i \cap \mathcal{L}_g^g \), from \( \mathcal{H} = \bigoplus_{i=1}^{s} \mathcal{H}_i = \bigoplus_{i=1}^{s} \bigoplus_{g \in G} \mathcal{H}_i^g \), we have

\[
[\mathcal{H}_i, \mathcal{H}_i, x_i] = \sum_{i=1}^{s} (\sum_{g \in G} [\mathcal{H}_i^g, \mathcal{H}_i^g, x_i]) = \sum_{i=1}^{s} (\sum_{g \in G} \alpha(\mathcal{H}_i^g, \mathcal{H}_i^g)) x_i = 0.
\]

Then \( x_i \in \mathcal{H}_i \), which is a contradiction. Thus

\[
\mathcal{L}_i = \mathcal{H}_i \oplus \bigoplus_{g \in \mathcal{G}_i} (\bigoplus_{\alpha \in \Pi_i} \mathcal{L}_i^{g, \alpha}) = \mathcal{H}_i \oplus \bigoplus_{\alpha \in \Pi_i} \mathcal{L}_{\alpha, i},
\]

where

\[
(5.6) \Pi_i^g = \{ \alpha|_{\mathcal{H}_i \times \mathcal{H}_i} : \mathcal{L}_i^{g, \alpha} \cap \mathcal{L}_g^g \neq 0 \}, \quad \Pi_i = \bigcup_{g \in \mathcal{G}_i} \Pi_i^g.
\]

Note that \( \Pi = \bigcup_{i=1}^{s} \Pi_i \). By Definition 2.3, we have \( \rho(\mathcal{L}_i, \mathcal{L})(A) \subset \mathcal{L}_i \) and \( \rho(\mathcal{L}_i, \mathcal{L}_j)(A) \subset \mathcal{L}_i, i = 1, 2, 3, ..., s \). Since \( Z_\mathcal{L}(A) = 0 \), we also have \( \rho(\mathcal{L}_i, \mathcal{L}_j)A = 0 \) when \( i \neq j \). Now, we can be written as

\[
A_i = \bigoplus_{g \in \mathcal{G}_i} (A_{i, i}^{g} \oplus (\bigoplus_{\lambda \in \Lambda_i^g} A_{\lambda, i}^{g})),
\]

where for any \( g \in \mathcal{G}_i \),

\[
A_{i, i}^{g} = \{ a \in A^g : \rho(h, h')a = 0 \},
\]

\[
A_{\lambda, i}^{g} = \{ a \in A^g : \rho(h, h')a = \lambda(h, h')a, \forall h, h' \in \mathcal{H}_i^g, \lambda \in \Lambda, \lambda(\mathcal{H}_i^g, \mathcal{H}_i^g) \neq 0 \},
\]

and

\[
(5.7) \Lambda_i^g = \{ \lambda|_{\mathcal{H}_i^g \times \mathcal{H}_i^g} : \lambda \in \Lambda, \lambda(\mathcal{H}_i^g, \mathcal{H}_i^g) \neq 0 \}, \quad \Lambda_i = \bigcup_{g \in \mathcal{G}_i} \Lambda_i^g.
\]

Note that \( \Lambda = \bigcup_{i=1}^{s} \Lambda_i \). Therefore, \( (\mathcal{L}_i, A, \rho|_{\mathcal{L}_i \times \mathcal{L}_i}) \) is a split 3–Lie-Rinehart color algebra with a splitting Cartan subalgebra \( \mathcal{H}_i \) and the root system \( \Pi_i \) defined by \( 5.6 \), and the weight system \( \Lambda_i \) defined by \( 5.7 \) associated to \( \mathcal{H}_i \).

Conversely, suppose that \( (\mathcal{L}_i, A, \rho|_{\mathcal{L}_i \times \mathcal{L}_i}), 1 \leq i \leq s \) are split 3–Lie-Rinehart color algebras, and \( \mathcal{L}_i = \mathcal{H}_i \oplus \bigoplus_{g \in \mathcal{G}_i} (\bigoplus_{\alpha \in \Pi_i} \mathcal{L}_i^{g, \alpha}) = \mathcal{H}_i \oplus \bigoplus_{\alpha \in \Pi_i} \mathcal{L}_{\alpha, i} \), with a splitting Cartan subalgebra \( \mathcal{H}_i \), the root system \( \Pi_i \), and the weight system \( \Lambda_i \) associated to \( \mathcal{H}_i \), respectively. By taking \( \mathcal{H} = \bigoplus_{i=1}^{s} \mathcal{H}_i \) being an abelian subalgebra of \( \mathcal{L} = \bigoplus_{i=1}^{s} \mathcal{L}_i \). Now, for any \( \alpha \in \Pi_i \) with \( \alpha|_{\mathcal{H}_i \times \mathcal{H}_i} \neq 0 \), we extend \( \alpha \) to \( \alpha : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{K} \) by

\[
\alpha(h, h') := \begin{cases} 
0 & : h \notin \mathcal{H}_i \text{ or } h' \notin \mathcal{H}_i \\
\alpha(h, h') & : h, h' \in \mathcal{H}_i.
\end{cases}
\]

We get that \( \mathcal{L}_\alpha = \mathcal{L}_{\alpha, i} \neq 0 \) and \( \alpha \in (\mathcal{H} \times \mathcal{H})^* \neq 0 \). Therefore, \( \mathcal{H} = \bigoplus_{i=1}^{s} \mathcal{H}_i \) is a splitting Cartan subalgebra of \( \mathcal{L} \) with root system \( \Pi = \bigcup_{i=1}^{s} \Pi_i \) and \( \mathcal{L} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Pi} \mathcal{L}_\alpha \). Thanks to Definition 2.3 and \( Z_\mathcal{L}(A) = 0, \rho(\mathcal{L}_i, \mathcal{L}_j)A = 0 \) for \( i \neq j \), and \( \rho(\mathcal{L}_i, \mathcal{L}_j)(A) \subset \mathcal{L}_i, 1 \leq i \leq s \). Therefore, \( \rho(\mathcal{H}_i, \mathcal{H}_i)A \mathcal{L}_i = 0, 1 \leq i, j \leq s \).

By a complete similar discussion to the above, for any \( \lambda \in \Lambda_i \), extending \( \lambda \) to \( \lambda : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{K} \), we get that \( A_\lambda = A_i \neq 0 \), and \( \lambda \in (\mathcal{H} \times \mathcal{H})^* \neq 0 \). Therefore,
\[ \Lambda = \bigcup_{i=1}^{s} A_i \subset (\mathcal{H} \times \mathcal{H})^* \neq 0 \text{ and } A = A_0 \oplus \bigoplus_{\lambda \in \Lambda} A_\lambda, \text{ where } A_0 = \bigcap_{i=1}^{s} A_{0,i}. \] We get the result.

**Definition 5.8.** A split 3–Lie-Rinehart color algebra \((\mathcal{L}, A)\) is tight if \(Z_\rho(\mathcal{L}) = Ann(A) = 0, AA = A, \mathcal{A}\mathcal{L} = \mathcal{L}\) and

\[
\mathcal{H} = \left( \sum_{\beta \in \Pi, \beta \in \Lambda} A_\beta^g L_\beta^g \right) + \left( \sum_{\beta, \gamma, \mu \in \Pi, \beta + \gamma + \mu = 0} [L_\beta^g, L_\gamma^\rho, L_\mu^\rho''] \right), \\
A_0 = \left( \sum_{\mu \in \Lambda, g \in G} A_\mu^g A_\mu^{-g} \right) + \left( \sum_{\alpha, \beta, \gamma, \mu \in \Lambda, \alpha + \beta + \gamma + \mu = 0} \rho(\mathcal{L}_\alpha^g, \mathcal{L}_\beta^\rho, \mathcal{L}_\gamma^\rho') A_\mu^g \right).
\]

**Remark 5.9.** If \((\mathcal{L}, A)\) is a tight split 3–Lie-Rinehart color algebra then it follows from Proposition 5.2 and Corollary 4.19 that

\[
\mathcal{L} = \bigoplus_{[\alpha] \in \Pi/\sim} I_{[\alpha]}, \quad A = \bigoplus_{[\lambda] \in \Lambda/\autoequivalence} A_{[\lambda]},
\]

with any \(I_{[\alpha]}\) an ideal of \(\mathcal{L}\) satisfying

\[
[I_{[\alpha]}, I_{[\beta]}, I_{[\gamma]}] = 0, \quad [I_{[\alpha]}, I_{[\alpha]}, I_{[\beta]}] = 0,
\]

whenever \([\alpha], [\beta], [\gamma] \in \Pi/\sim\) be different from each other, and any \(A_{[\lambda]}\) an ideal of \(A\) satisfying \(A_{[\lambda]} A_{[\mu]} = 0\), when \([\lambda] \neq [\mu]\).

**Proposition 5.10.** Let \((\mathcal{L}, A)\) be a tight split 3–Lie-Rinehart color algebra. Then for any \([\alpha] \in \Pi/\sim\) there exists a unique \([\lambda] \in \Lambda/\autoequivalence\) such that \(A_{[\lambda]} I_{[\alpha]} \neq 0\).

**Proof.** It can be analogous Proposition 4.2 in [2].

**Theorem 5.11.** Let \((\mathcal{L}, A)\) be a tight split 3–Lie-Rinehart color algebra. Then

\[
\mathcal{L} = \bigoplus_{g \in G} \bigoplus_{i \in I} L_i^g = \bigoplus_{i \in I} \mathcal{L}_i, \quad A = \bigoplus_{h \in G} \bigoplus_{j \in J} A_j^h = \bigoplus_{j \in J} A_j,
\]

where any \(\mathcal{L}_i = \bigoplus_{g \in G} L_i^g\) is a non-zero graded ideal of \(\mathcal{L}\) satisfying \([L_i^{g_1}, L_i^{g_2}, L_i^{g_3}] = 0\), when \(i, i_2, i_3 \in I, g_1, g_2, g_3 \in G\) be different from each other, and any \(A_j = \bigoplus_{h \in G} A_j^h\) is a graded ideal of \(A\) such that \(A_j^{h_1} A_j^{h_2} = 0\) when \((j_1, h_1) \neq (j_2, h_2)\).

Moreover, both decompositions satisfy that for any \((i, g) \in I \times G\) there exists a unique \((j, h) \in J \times G\) such that

\[
A_j^h L_i^g \neq 0.
\]

**Proof.** Proposition 5.10 shows that \(\mathcal{L}_i\) is an \(A_j\)–module. Hence we can get the results of theorem. For the final result of theorem see Proposition 5.4.
6. The Simple Components of Split 3-Lie-Rinehart Color Algebras

In this section we focus on the simplicity of split 3-Lie-Rinehart color algebra \((\mathcal{L}, A)\) by centering our attention in those of maximal length. From now on, we will suppose \(\Pi\) and \(\Lambda\) are symmetric.

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split 3-Lie-Rinehart color algebra, in a similar way to the ones for split Lie-Rinehart algebra in [7].

**Definition 6.1.** A split 3-Lie-Rinehart color algebra \((\mathcal{L}, A)\) is called *root-multiplicative* if for any \(\alpha, \beta, \gamma \in \Pi\) and \(\lambda, \mu \in \Lambda\) the following conditions hold

- If \(\alpha + \beta + \gamma \in \Pi^g, g \in G\) then \([\mathcal{L}_\beta^g, \mathcal{L}_\gamma^g, \mathcal{L}_\alpha^g] \neq 0\).
- If \(\lambda + \alpha \in \Pi^g, g \in G\) then \(A_\lambda^g \mathcal{L}_\alpha^g \neq 0\).
- If \(\lambda + \mu \in \Lambda^g, g \in G\) then \(A_\lambda^g A_\mu^g \neq 0\).

**Definition 6.2.** A split 3-Lie-Rinehart color algebra \((\mathcal{L}, A)\) is called of *maximal length* if for any \(\alpha \in \Pi^g, g \in G\) and \(\lambda \in \Lambda^h, h \in G\) we have \(\dim \mathcal{L}_\alpha^g = \dim A_\lambda^h = 1\).

**Remark 6.3.** If \((\mathcal{L}, A)\) is a split 3-Lie-Rinehart color algebra such that \(\mathcal{L}\) and \(A\) are simple algebras then \(Z_\rho(\mathcal{L}) = \text{Ann}(A) = \{0\}\). Also as consequence of Theorem 5.19 (2) and Theorem 4.17 (2) we get that all of the non-zero roots in \(\Pi\) are connected, that all of the non-zero weights in \(\Lambda\) are also connected and that

\[
\mathcal{H} = (\sum_{\beta \in \Pi \cap \Lambda} A_\beta^g \mathcal{L}_\beta^g) + (\sum_{\beta, \gamma, \mu \in \Pi, \beta + \gamma + \mu = 0, g, g', g'' \in G} [\mathcal{L}_\beta^g, \mathcal{L}_\gamma^g, \mathcal{L}_\mu^g]),
\]

\[
A_0 = (\sum_{\mu \in \Lambda} A_\mu^g A_-\mu^g) + (\sum_{\alpha, \beta, \mu \in \Lambda, \alpha + \beta + \mu = 0, g, g', g'' \in G} \rho(\mathcal{L}_\alpha^g, \mathcal{L}_\beta^g) A_\mu^g).
\]

From here, the conditions for \((\mathcal{L}, A)\) of being tight together with the ones of having \(\Pi\) and \(\Lambda\) all of their elements connected, are necessary conditions to get a characterization of the simplicity of the algebras \(\mathcal{L}\) and \(A\). Actually, we are going to show that under the hypothesis of being \((\mathcal{L}, A)\) of maximal length and root-multiplicative, these are also sufficient conditions.

**Proposition 6.4.** Let \((\mathcal{L}, A)\) be a tight split 3-Lie-Rinehart color algebra of maximal length and root-multiplicative. If \(\mathcal{L}\) has all of its nonzero roots connected, then either \(\mathcal{L}\) is simple or \(\mathcal{L} = I \oplus I'\) where \(I\) and \(I'\) are simple ideals of \(\mathcal{L}\).

**Proof.** Consider any nonzero ideal \(I\) of \(\mathcal{L}\), by Lemma 5.13 \(I \not\subset \mathcal{H}^g, \forall g \in G\) and by Eq. (5.24) we can write

\[
I = \bigoplus_{g \in G} ((I \cap \mathcal{H}^g) \oplus (\bigoplus_{\alpha \in \Pi_\alpha^g} (I \cap \mathcal{L}_\alpha^g))),
\]

where \(\Pi_\alpha^g := \{\alpha \in \Pi : I_\alpha^g = I \cap \mathcal{L}_\alpha^g \neq 0\} \subset \Pi^g\) for all \(g \in G\) and some \(\Pi_\alpha^g \neq \emptyset\). Then we can write \(I = \bigoplus_{g \in G} ((I \cap \mathcal{H}^g) \oplus (\bigoplus_{\alpha \in \Pi_\alpha^g} I_\alpha^g))\). Let us distinguish two cases.
Case 1. Suppose there exists \( \alpha_0 \in \Pi_f^\beta \) such that \( -\alpha_0 \in \Pi_f^\beta \) for all \( g \in G \). Then \( 0 \neq I_{\alpha_0}^g \) and by the maximal length of \((L, A)\) that
\[
0 \neq \mathcal{L}_{\alpha_0}^g \subset I.
\]

Now, let us take some \( \beta \in \Pi \) satisfying \( \beta \notin \{\alpha_0, -\alpha_0\} \). Since \( \alpha_0 \) and \( \beta \) are connected, we have a connection \( \{\beta_1, \beta_2, ..., \beta_{2k+1}\}, k \geq 2 \) from \( \alpha_0 \) to \( \beta \) satisfying the following conditions:

1. \( \beta_1 = \pm \alpha_0 \).
2. \( \beta_1 + (\beta_2 + \beta_3) \in \pm \pi \),
   \[ \beta_1 + (\beta_2 + (\beta_3 + (\beta_4 + \beta_5))) \in \pm \Pi, \]
   ... 
   \[ \beta_1 + \sum_{j=1}^k (\beta_2j + \beta_2j+1) \in \pm \Pi, \ 0 < i < k. \]
3. \( \beta_1 + \sum_{j=1}^k (\beta_2j + \beta_2j+1) \in \{\beta, -\beta\} \).

Consider \( \beta_1, \beta_2, \beta_3 \in \Pi \cup \Lambda \) and \( \beta_2 + \beta_3 \in \Pi \), since \( \beta_1 \in \Pi \) there exist \( g_1 \in \Lambda \) such that \( \mathcal{L}_{\beta_1}^{g_1} \neq 0 \). From here, the root-multiplicativity and maximal length of \( \mathcal{L} \) allow us to get
\[
0 \neq [\mathcal{L}_{\beta_1}^{g_1}, \mathcal{L}_{\beta_2}^{g_2}, \mathcal{L}_{\beta_3}^{g_3}] = \mathcal{L}_{\beta_1+\beta_2+\beta_3}^{g_1+g_2+g_3}.
\]

Since \( 0 \neq \mathcal{L}_{\beta_1}^{g_1} \subset I \) as a consequence of Eq. (??) we have
\[
0 \neq \mathcal{L}_{\beta_1+\beta_2+\beta_3}^{g_1+g_2+g_3} \subset I.
\]

We can argue in a similar way from the \( \beta_1 + (\beta_2 + \beta_3), \beta_4 \) and \( \beta_1 + ((\beta_2 + \beta_3) + \beta_4) \in \Pi \) to get
\[
0 \neq L_{\beta_1+\beta_2+\beta_3+\beta_4}^{g_1+g_2+g_3} \subset I, \text{ for some } g_1 \in G.
\]

We can follow this process with the connection \( \{\beta_1, \beta_2, ..., \beta_{2k+1}\} \) we obtain that
\[
0 \neq L_{\beta_1+\beta_2+...+\beta_{2k+1}}^h \subset I, \text{ for some } h \in G.
\]

Thus we have shown that
\[
(6.6) \quad \text{for any } \beta \in \Pi, \text{ we have that } 0 \neq \mathcal{L}_{\varepsilon \beta}^h \subset I \text{ for some } \varepsilon \in \{\pm 1\}.
\]

Since \( -\beta \in \Pi_f^\beta \) we have \( \{-\beta_1, -\beta_2, ..., -\beta_{2k+1}\} \) is a connection from \( -\beta \) to \( \alpha_0 \) satisfying
\[
\{-\beta_1, -\beta_2, ..., -\beta_{2k+1}\} \in \{\beta, -\beta\}.
\]

By arguing as above we get,
\[
0 \neq \mathcal{L}_{-\varepsilon \beta}^g \subset I,
\]
and so \( \bigcup_{g \in G} \Pi_f^\beta = \Pi \). From the fact
\[
\mathcal{H} = \left( \sum_{\beta \in \Pi \cap \Lambda} A_{-\beta}^g L_{\beta}^g \right) + \left( \sum_{\beta, \gamma, \mu \in \Pi} [L_{\beta}^g, L_{\gamma}^g, L_{\mu}^g] \right),
\]

imply that
\[
(6.8) \quad \mathcal{H} \subset I.
\]

From Eqs. (6.5), (6.8), we obtain \( \mathcal{L} \subset I \), and so \( \mathcal{L} \) is simple.
Case 2. In the second case, suppose that for any \( \alpha_0 \in \Pi_2^g \) we have that \(-\alpha_0 \notin \Pi_2^g\), for all \( g \in G \). Observe that by arguing as in the previous case we can write
\[
\Pi^g = \Pi_1^g \cup -\Pi_2^g, \quad \forall g \in G,
\]
where \(-\Pi_2^g = \{-\alpha : \alpha \in \Pi_2^g\}\). Denote by
\[
I' := \bigoplus_{g \in G} ( \sum_{\alpha \in \Lambda^g} A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha} \oplus ( \bigoplus_{-\alpha \in -\Pi_1^g} \mathcal{L}^{-g}_{-\alpha} ) )
\]
We are going to show that \( I' \) is a 3-Lie color ideal of \( \mathcal{L} \). First, we will show that \( I' \) is a Lie ideal of \( \mathcal{L} \). Taking into account Eq. \((6.10)\) and the fact that \( \mathcal{H} \) is abelian we have
\[
[\mathcal{L}, \mathcal{L}, I'] = \left[ \mathcal{H} \oplus \bigoplus_{h \in \mathcal{H}} \mathcal{L}^{\alpha}_{\beta}, \mathcal{H} \oplus \bigoplus_{\gamma \in \Pi_2^g} \mathcal{L}^{\gamma}_{\gamma}, \sum_{\alpha \in \Lambda^g} A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha} \oplus ( \bigoplus_{-\alpha \in -\Pi_1^g} \mathcal{L}^{-g}_{-\alpha} ) \right]
\]
\[
\subset \bigoplus_{-\alpha \in -\Pi_1^g} \mathcal{L}^{-g}_{-\alpha} + \left[ \bigoplus_{h \in \mathcal{H}} \mathcal{L}^{\alpha}_{\beta}, \mathcal{H}, \sum_{\alpha \in \Lambda^g} A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha} \right]
\]
\[
+ \left[ \bigoplus_{h \in \mathcal{H}} \mathcal{L}^{\alpha}_{\beta}, \mathcal{H}, \bigoplus_{\gamma \in \Pi_2^g} \mathcal{L}^{\gamma}_{\gamma} \right] + \left[ \bigoplus_{h \in \mathcal{H}} \mathcal{L}^{\alpha}_{\beta}, \bigoplus_{\gamma \in \Pi_2^g} \mathcal{L}^{\gamma}_{\gamma}, \sum_{\alpha \in \Lambda^g} A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha} \right]
\]
\[
+ \left[ \bigoplus_{h \in \mathcal{H}} \mathcal{L}^{\alpha}_{\beta}, \bigoplus_{\gamma \in \Pi_2^g} \mathcal{L}^{\gamma}_{\gamma}, \bigoplus_{-\alpha \in -\Lambda^g} \mathcal{L}^{-g}_{-\alpha} \right].
\]

Consider the second summand in Eq. \((6.10)\). If some \([\mathcal{L}^{\alpha}_{\beta}, \mathcal{H}, A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}] \neq 0\) and \( \beta = -\alpha \), we have that \([\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}] \subset \mathcal{L}^{h+I} \subset I' \). In case \( \beta = \alpha \), since \( I \) is an ideal of \( \mathcal{L} \), we have \(-\alpha \notin \Pi_2^g\). Imply \([\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}] = 0\), by symmetry of \( \Pi \) and maximal length of \( \mathcal{L} \), we have \([\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}] = 0\). Suppose that \( \beta \notin \{\alpha, -\alpha\} \). As \([\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}] \neq 0\), By Eq. \((6.8)\), either \( A^g_{\alpha} [\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}] \neq 0\) or \( \rho(\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}) \neq 0\). By the maximal length of \( \mathcal{L} \), either \( A^g_{\alpha} [\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}] \neq 0\) or \( \rho(\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}) \neq 0\). In both cases, since \( \alpha \in \Pi_2^g \), by root-multiplicativity, we have \( \mathcal{L}^{-\beta} \subset I \) and therefore \(-\beta \in \Pi_2^g\), that is \( \mathcal{L}^{h+I} \subset I' \). Thus
\[
[\bigoplus_{h \in \mathcal{H}} \mathcal{L}^{\alpha}_{\beta}, \mathcal{H}, \sum_{\alpha \in \Lambda^g} A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}] \subset I'.
\]
A similar discussion as above, one can show that
\[
[\bigoplus_{h \in \mathcal{H}} \mathcal{L}^{\beta}_{\gamma}, \bigoplus_{\gamma \in \Pi_2^g} \mathcal{L}^{\gamma}_{\gamma}, \sum_{\alpha \in \Lambda^g} A^g_{\alpha} \mathcal{L}^{-g}_{-\alpha}] \subset I'.
\]
Now, if we consider the third summand in Eq. \((6.10)\) and some \([\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, \mathcal{L}^{-g}_{-\alpha}] \neq 0\). Then we have \([\mathcal{L}^{\alpha}_{-\beta}, \mathcal{H}, \mathcal{L}^{-g}_{-\alpha}] = \mathcal{L}^{-g+h+I}_{-\alpha+\beta} \). If \( \beta \neq \alpha \), thanks to \( \alpha \in \Pi_2^g \) and
the root-multiplicativity give us $[\mathcal{L}_\alpha^g, \mathcal{H}^{-l}, \mathcal{L}^{-h}_\beta] = \mathcal{L}^{-\alpha + \beta - l}_{\alpha + \beta} \subset I$. Hence $-\alpha + \beta \in -\Pi_I^{-g + h + l}$ and then $\mathcal{L}^{-g + h + l}_{-\alpha + \beta} \subset I'$. Now, if $\beta = \alpha$, in case $[\mathcal{L}_\alpha^h, \mathcal{H}^l, \mathcal{L}^{-g}_0] \neq 0$, we get $[\mathcal{L}_\alpha^l, \mathcal{H}_\alpha^l, \mathcal{L}^{-g}_{-\alpha}] \subset I$, because of $\alpha \in \Pi_I^g$. Thus

$$\mathcal{L}^{-g}_{-\alpha} = [[\mathcal{L}_\alpha^h, \mathcal{H}^l, \mathcal{L}^{-g}_0], \mathcal{H}^{-l}, \mathcal{L}^{-h}_{-\alpha}] \subset I.$$  

From here, we deduce $-\alpha, \alpha \in \Pi_I^g$, a contradiction with Eq. (6.3). Thus

$$\mathcal{L}^{-g}_{-\alpha} = \left[ \bigoplus_{\beta \in \Pi_I^h} \mathcal{L}_\beta^h, \bigoplus_{\gamma \in \Pi_I^l} \mathcal{L}_\gamma^l, \bigoplus_{-\alpha \in \Lambda} \mathcal{L}_{-\alpha}^g \right] \subset I'. \tag{6.13}$$

Again a similar discussion as above, one can show that

$$\mathcal{L}^{-g}_{-\alpha} = \left[ \bigoplus_{\beta \in \Pi_I^h} \mathcal{L}_\beta^g, \bigoplus_{\gamma \in \Pi_I^l} \mathcal{L}_\gamma^g, \bigoplus_{-\alpha \in \Lambda} \mathcal{L}_{-\alpha}^g \right] \subset I'. \tag{6.14}$$

From Eqs. (6.11)-(6.14), we conclude that $I'$ is a 3-Lie color ideal of $\mathcal{L}$. Second, we will check $AI' \subset I'$. Taking into account Eq. (2.21), we have

$$AI' = (A^0 \oplus \bigoplus_{h \in G} \left( \bigoplus_{\lambda \in \Lambda^h} A_\lambda^h \right)) \left( \bigoplus_{g \in G} \left( \sum_{-\alpha \in \Lambda^g} A_\alpha^g \mathcal{L}^{-g}_{-\alpha} \oplus \bigoplus_{-\alpha \in \Pi_I^g} \mathcal{L}^{-g}_{-\alpha} \right) \right)$$

$$\subset I' + \bigoplus_{h \in G} \left( \bigoplus_{\lambda \in \Lambda^h} A_\lambda^h \right) \left( \bigoplus_{-\alpha \in \Pi_I^g} \mathcal{L}^{-g}_{-\alpha} \right) + \bigoplus_{h \in G} \left( \bigoplus_{\lambda \in \Lambda^h} A_\lambda^h \right) \left( \bigoplus_{-\alpha \in \Pi_I^g} \mathcal{L}^{-g}_{-\alpha} \right) \tag{6.15}$$

Consider the third summand in (6.15) and suppose that $A_\lambda^h \mathcal{L}^{-g}_{-\alpha} \neq 0$ for some $\lambda \in \Lambda^h$, $-\alpha \in \Pi_I^g$. If $\alpha - \lambda \in \Pi_I^{-g + h}$, then by the root-multiplicativity of $\mathcal{L}$ we get $A_{-\alpha}^- \mathcal{L}^{-g}_{-\alpha} \neq 0$. Now by the maximal length of $\mathcal{L}$ and the fact $\alpha \in \Pi_I^g$, we conclude that $A_{-\alpha}^- \mathcal{L}^{-g}_{-\alpha} \subset I$. Therefore $-\alpha + \lambda \in \Pi_I^{-g + h}$ which is a contradiction. Hence $\alpha - \lambda \in \Pi_I^{-g + h}$, and so $A_{\lambda}^h \mathcal{L}^{-g}_{-\alpha} \subset I'$. Therefore,

$$\bigoplus_{\lambda \in \Lambda^h} A_\lambda^h \left( \bigoplus_{-\alpha \in \Pi_I^g} \mathcal{L}^{-g}_{-\alpha} \right) \subset I'. \tag{6.16}$$

We can argue as above with the second summand in (6.15), so as to conclude that

$$\bigoplus_{\lambda \in \Lambda^h} A_\lambda^h \left( \bigoplus_{-\alpha \in \Pi_I^g} \mathcal{L}^{-g}_{-\alpha} \right) \subset I'. \tag{6.17}$$

From Eqs. (6.16) and (6.17) we get $AI' \subset I'$. Finally, let us check $\rho(I', I')(A)\mathcal{L} \subset I'$. In fact by Eq. (2.8) we have

$$\rho(I', I')(A)\mathcal{L} \subset [I', I', A\mathcal{L}] + A[I', I', \mathcal{L}]$$

Tanks to $I'$ is a 3-Lie color ideal we get the result.

Summarizing a discussion of above, we conclude that $I'$ is an ideal of the split 3-Lie-Rinehart color algebra $(\mathcal{L}, A)$. 

Next, by Eq. \( (6.9) \) we get \( \sum_{\beta, \gamma, \mu \in \Pi} \left[ L_{\mu}^{g} \otimes L_{\gamma}^{h} \otimes L_{\eta}^{i} \right] = 0 \), so by hypothesis must have
\[
H = \sum_{\alpha \in \Pi} A^{-\alpha} L^{\alpha} \oplus \sum_{\alpha \in \Pi} A_{\alpha}^{-\alpha} L_{\alpha}.
\]
For direct character, take
\[
0 \neq h, h' \in \sum_{\alpha \in \Pi} A^{-\alpha} L^{\alpha} \cap \sum_{\alpha \in \Pi} A_{\alpha}^{-\alpha} L_{\alpha}.
\]
Taking into account \( Z_{\rho}(L) = \{0\} \) and \( L \) is split, there exists \( 0 \neq x \in L_{b}^{h}, \beta \in \Pi^{h} \), such that \( [h, h', x] \neq 0 \), being then \( x \in I \cap I' = \{0\} \), a contradiction. Hence the sum is direct. Taking into account the above observation and Eq. \( (6.9) \) we have
\[
L = I \oplus I'.
\]
Finally, we can proceed with \( I \) and \( I' \) as we did for \( L \) in the first case of the proof to conclude that \( I \) and \( I' \) are simple ideals, which completes the proof of the theorem.

In a similar way to Proposition \( 6.4 \) one can prove the next result;

**Proposition 6.18.** Let \((L, A)\) be a tight split 3-Lie-Rinehart color algebra of maximal length and root-multiplicative. If \( A \) has all of its nonzero root weights connected, then either \( A \) is simple or \( A = J \oplus J' \) where \( J \) and \( J' \) are simple ideals of \( A \).

Now, we are ready to state our main result:

**Theorem 6.19.** Let \((L, A)\) be a tight split 3–Lie-Rinehart color algebra of maximal length, root-multiplicative, with symmetric roots and weight systems in such a way that the root system \( \Pi \) has all its elements connected and the weight system \( \Lambda \) has all its elements connected. Then
\[
L = \bigoplus_{g \in G} L^{g} = \bigoplus_{i \in I} L_{i}, \quad A = \bigoplus_{h \in G} A_{h}^{j} = \bigoplus_{j \in J} A_{j},
\]
where any \( L_{i} = \bigoplus_{g \in G} L_{i}^{g} \) is a simple graded ideal of \( L \) satisfying \( [L_{i}^{g_{1}}, L_{i}^{g_{2}}, L_{i}^{g_{3}}] = 0 \), when \( i_{1}, i_{2}, i_{3} \in I, g_{1}, g_{2}, g_{3} \in G \) be different from each other, and any \( A_{j} = \bigoplus_{h \in G} A_{j}^{h} \) is a simple graded ideal of \( A \) such that \( A_{j_{1}}^{h_{1}} A_{j_{2}}^{h_{2}} = 0 \) when \( (j_{1}, h_{1}) \neq (j_{2}, h_{2}) \). Moreover, both decompositions satisfy that for any \((i, g) \in I \times G \) there exists a unique \((j, h) \in J \times G \) such that
\[
A_{j}^{h} L_{i}^{g} \neq 0.
\]
Furthermore, any \((L_{i} = \bigoplus_{g \in G} L_{i}^{g}, A_{j} = \bigoplus_{h \in G} A_{j}^{h}, \rho|_{I \times L_{i}} \) is a split 3–Lie-Rinehart color algebra.

**Proof.** By Theorem \( 5.11 \) we can write
\[
L = \bigoplus_{[\alpha] \in \Pi/\sim} I_{[\alpha]},
\]
with any \( I_\alpha \) an ideal of \( L \), being each \( I_\alpha \) a split 3–Lie-Rinehart color algebra having as root system \( [\alpha] \). Also we can write \( A \) as the direct sum of the ideals

\[
A = \bigoplus_{[\lambda] \in \Lambda/\sim} A_{[\lambda]},
\]

in such a way that any \( A_{[\lambda]} \) has as weight system \( [\lambda] \), for any \( [\alpha] \in \Pi/\sim \) there exists a unique \( [\lambda] \in \Lambda/\sim \) such that \( A_{[\lambda]} I_\alpha \neq 0 \) and being \( (I_\alpha, A_{[\lambda]}) \) a split 3–Lie-Rinehart color algebra.

In order to apply Proposition 6.4 and Proposition 6.18 to each \( (I_\alpha, A_{[\lambda]}) \), we previously have to observe that the root-multiplicativity of \( (I_\alpha, A_{[\lambda]}) \), Proposition 3.13 and Theorem 3.20 show that \( [\alpha] \) and \( [\lambda] \) have, respectively, all of their elements \( [\alpha], [\lambda] \)–connected. That is, connected through connections contained in \( [\alpha] \) and \( [\lambda] \). Any of the \( (I_\alpha, A_{[\lambda]}) \) is root-multiplicative as consequence of the root-multiplicativity of \( (L, A) \). Clearly \( (I_\alpha, A_{[\lambda]}) \) is of maximal length and tight, last fact consequence of tightness of \( (L, A) \), Proposition 6.4 and Proposition 6.18. So we can apply Proposition 6.4 and Proposition 6.18 to each \( (I_\alpha, A_{[\lambda]}) \) so as to conclude that any \( I_\alpha \) is either simple or the direct sum of simple ideals \( I_\alpha = J \oplus J' \), and that any \( A_{[\lambda]} \) is either simple or the direct sum of simple ideals \( A_{[\lambda]} = B \oplus B' \). From here, it is clear that by writing \( I_i = J \oplus J' \) and \( A_j = B \oplus B' \) if \( I_i \) or \( A_j \) are not, respectively, simple, then Theorem 5.11 allows as to assert that the resulting decomposition satisfies the assertions of the theorem.

\[
\square
\]

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