A SHARP REGULARITY ESTIMATE FOR THE SCHRÖDINGER PROPAGATOR ON THE SPHERE

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ABSTRACT. Let $\Delta_{S^n}$ denote the Laplace-Beltrami operator on the $n$-dimensional unit sphere $S^n$. In this paper we show that
\[ \|e^{it\Delta_{S^n}}f\|_{L^4([0,2\pi)\times S^n)} \leq C\|f\|_{W^{\alpha,4}(S^n)} \]
holds provided that $n \geq 2$, $\alpha > (n-2)/4$. The range of $\alpha$ is sharp up to the endpoint. As a consequence, we obtain space-time estimates for the Schrödinger propagator $e^{it\Delta_{S^n}}$ on the $L^p$ spaces for $2 \leq p \leq \infty$. We also prove that for zonal functions on $S^n$, the Schrödinger maximal operator $\sup_{0 \leq t < 2\pi}|e^{it\Delta_{S^n}}f|$ is bounded from $W^{\alpha,2}(S^n)$ to $L^{\frac{6n}{3n-2}}(S^n)$ whenever $\alpha > 1/3$.

1. INTRODUCTION

Let $S^n$ denote the $n$-dimensional unit sphere in $\mathbb{R}^{n+1}$ endowed with the standard metric. Denote by $\Delta_{S^n}$ the Laplace-Beltrami operator on $S^n$. For $k = 0, 1, \ldots$, denote by $H^n_k$ the space of spherical harmonics of degree $k$ (for background on the spherical harmonics, cf. [19] Chapter IV). It is well-known that one has the orthogonal decomposition
\[ L^2(S^n) = \bigoplus_{k=0}^{\infty} H^n_k; \]
moreover,
\[ \Delta_{S^n}Y_k = -k(k+n-1)Y_k, \quad \forall Y_k \in H^n_k, \]
and $H^n_k$ is of dimension $\sim k^{n-1}$. Denote by
\[ P^k : L^2(S^n) \to H^n_k \]
the orthogonal projection from $L^2(S^n)$ to $H^n_k$.

In this paper we study regularity properties of solutions to the Cauchy problem for the Schrödinger equation on $S^n$:
\[ i\partial_t u + \Delta_{S^n} u = 0, \quad u(0, x) = f(x), \tag{1.1} \]
where the unknown $u(t, x)$ is a complex-valued function on $[0, 2\pi) \times S^n$. For convenience, we will write $T = [0, 2\pi)$. By spectral theory, the solution operator for equation (1.1) is

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where the Sobolev space $W^{s,p}$ is defined by

\[
W^{s,p}(\mathbb{S}^n) = \{ f : \| (I - \Delta_{\mathbb{S}^n})^{\frac{s}{2}} f \|_p < \infty \}.
\]

Taking the $L^p(\mathbb{T})$ norm then gives

\[
\| e^{it\Delta_{\mathbb{S}^n}} f \|_{L^p(\mathbb{T} \times \mathbb{S}^n)} \leq C \| f \|_{W^{s,p}(\mathbb{S}^n)}.
\]

Note that estimate (1.4) does not take into account possible gain of provided by the average over $\mathbb{T}$.

In contrast to the fixed time estimate (1.3), it is of interest to seek the minimal $\alpha$ for which the bound

\[
\| e^{it\Delta_{\mathbb{S}^n}} f \|_{L^p(\mathbb{T} \times \mathbb{S}^n)} \leq C \| f \|_{W^{\alpha,p}(\mathbb{S}^n)}
\]

holds. On the circle $\mathbb{S}^1$, it is known that (1.5) holds for $\alpha = 0$ when $2 \leq p \leq 4$, for $\alpha > 0$ when $4 < p \leq 6$, and for $\alpha > 1/2 - 3/p$ when $6 < p < \infty$, by a bound of Zygmund [24]:

\[
\| e^{it\Delta_{\mathbb{S}^1}} f \|_{L^1(\mathbb{T} \times \mathbb{S}^1)} \leq C \| f \|_{L^2(\mathbb{S}^1)},
\]

and the following well-known inequality due to Bourgain [1]:

\[
\| e^{it\Delta_{\mathbb{S}^1}} f \|_{L^6(\mathbb{T} \times \mathbb{S}^1)} \leq C \| f \|_{W^{s,2}(\mathbb{S}^1)}, \quad \forall \varepsilon > 0.
\]

For the sphere $\mathbb{S}^n$, $n \geq 2$, it is remarkable that in [4, Theorem 4], Burq-Gérard-Tzvetkov used the clustering property of the spectrum of the Laplace-Beltrami operator and $L^2$-$L^4$ norm of spectral projections of the Laplace associated to finite intervals of high frequencies to establish the following Strichartz estimates:

\[
\| e^{it\Delta_{\mathbb{S}^n}} f \|_{L^4(\mathbb{T} \times \mathbb{S}^n)} \leq C \| f \|_{W^{\alpha,2}(\mathbb{S}^n)}, \quad \alpha > \alpha(4,n),
\]

where $\alpha(4,n)$ is given by

\[
\alpha(4,n) = \begin{cases} \frac{1}{8}, & \text{if } n = 2; \\ \frac{n-2}{4}, & \text{if } n \geq 3. \end{cases}
\]

The loss of $\alpha$ derivatives in the estimate (1.8) is essentially sharp in the sense that similar estimates fail with $\alpha \leq \alpha(4,n)$ if $n \geq 3$ (resp. $\alpha \leq \alpha(4,2)$ if $n = 2$). From (1.8), one infers that for $p = 4$, estimate (1.5) averaging over time $\mathbb{T}$ yields a gain $3/8$ derivatives for $n = 2$; and a gain $1/2$ derivatives for $n \geq 3$.

Our first goal in this paper is to prove an $L^4$-estimate with a loss of $\varepsilon > 0$ derivative on $\mathbb{S}^2$; and a loss of of $(n - 2)/4$ derivatives on $\mathbb{S}^n$, $n \geq 3$. More precisely, we have the following result.
Theorem 1.1. Let $n \geq 2$. The solution $e^{it\Delta_{\mathbb{S}^n}} f$ of (1.1) satisfies
\begin{equation}
\|e^{it\Delta_{\mathbb{S}^n}} f\|_{L^1_t(N \times \mathbb{S}^n)} \leq C\|f\|_{W^{\alpha,4}(\mathbb{S}^n)}, \quad \alpha > \frac{n-2}{4}.
\end{equation}
Moreover, (1.10) fails when $\alpha < (n-2)/4$.

As a consequence of Theorem 1.1, we deduce the following space-time estimates for the Schrödinger propagator $e^{it\Delta_{\mathbb{S}^n}}$ on the $L^p$ spaces, $2 \leq p \leq \infty$. This result is optimal when $n = 2$ and $2 \leq p \leq \infty$, and when $n \geq 3$ and $4 \leq p \leq \infty$.

Corollary 1.2. Let $n \geq 2$ and $2 \leq p \leq \infty$. Then the following estimate
\begin{equation}
\|e^{it\Delta_{\mathbb{S}^n}} f\|_{L^p_t(N \times \mathbb{S}^n)} \leq C\|f\|_{W^{\alpha,p}(\mathbb{S}^n)}
\end{equation}
holds for $\alpha > \alpha_0(p, n)$, where $\alpha_0(p, 2) = \max\{0, 1 - 4/p\}$ and, for $n \geq 3$,
\begin{equation}
\alpha_0(p, n) = \begin{cases} n\left(\frac{1}{2} - \frac{1}{p}\right) + \frac{2}{p} - 1, & \text{if } 2 \leq p \leq 4; \\ n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2}{p}, & \text{if } 4 \leq p \leq \infty. \end{cases}
\end{equation}
Conversely, if (1.11) holds, then $\alpha \geq \max\{0, n(1/2 - 1/p) - 2/p\}$.

From (1.8) and (1.9), the proof of (1.10) in Theorem 1.1 reduces to show it for the case $n = 2$, whose proof combines several arguments: firstly, using a number-theoretic argument in Burq-Gérard-Tzvetkov [5], we reduce the $L^4_t$ norm (the norm in $t$) to the $L^2_t$ norm. This makes use of the fact that the eigenvalues of $\Delta_{\mathbb{S}^2}^2$ essentially take quadratic values (such an argument goes back to earlier work on $\Lambda(p)$ sets in the 60’s). Secondly, using the spectral information, we apply an almost orthogonality result of Kadec [14] to reduce the $L^2_t$ norm of the Schrödinger evolution $e^{it\Delta_{\mathbb{S}^n}} f$ to that of the half-wave evolution $e^{it\sqrt{-\Delta_{\mathbb{S}^n}} f}$. This allows us to apply an $L^4_t \rightarrow L^4_t L^4_t$ local smoothing estimate for the half-wave group $e^{it\sqrt{-\Delta_{\mathbb{S}^n}}}$ due to Mockenhaupt-Seeger-Sogge [16] to conclude the proof of (1.10) for $n = 2$.

The sharpness of Theorem 1.1 is proved by using a semiclassical dispersion estimate of Burq-Gérard-Tzvetkov [4] as well as ideas from Rogers [18]. We remark that the same argument can be used to show that, on a general compact manifold of dimension $n$, a necessary condition for the analogue of (1.10) to hold is $\alpha \geq \max\{0, n(1/2 - 1/p) - 2/p\}$; in particular, sharp regularity estimates on $T^2$ follow immediately from the Strichartz estimates in Bourgain [1] and Bourgain-Demeter [3] for the Schrödinger equation.

In the second part of the paper, we consider the problem of identifying the values $\alpha$ for which
\begin{equation}
\sup_{0 \leq t < 2\pi} \|e^{it\Delta_{\mathbb{S}^n}} f\|_{L^q(\mathbb{S}^n)} \leq C(\alpha)\|f\|_{W^{\alpha,2}(\mathbb{S}^n)}
\end{equation}
holds for some $2 \leq q < \infty$. Inequality (1.13) has implications on the existence almost everywhere of $\lim_{T \to 0} u(t, x)$ for solutions $u$ of the Schrödinger equation (1.1). On the circle $\mathbb{S}^1$, Moyua and Vega [17] used (1.7) and the Sobolev embedding to show that (1.13) holds with $\alpha > 1/3$ and $q = 6$. They also point out that $\alpha \geq 1/4$ is a necessary condition for (1.13) to be true. For the sphere $\mathbb{S}^n$, $n \geq 2$, Wang and Zhang [23] proved (1.13) with $\alpha > 1/2$ and $q = 2$ by taking full advantage of the spectrum concentration.
The second goal of this paper is to investigate (1.13) for zonal functions on \( S^n \). We will show the following.

**Theorem 1.3.** Let \( n \geq 2 \). For any \( \alpha > 1/3 \), there exists a constant \( C = C(\alpha) > 0 \) such that for any zonal polynomial \( f \),

\[
\left\| \sup_{0 \leq t < 2\pi} |e^{it\Delta_{S^n}} f| \right\|_{L^6(S^n)} \leq C \|f\|_{W^{\alpha,2}(S^n)}.
\]

Moreover, estimate (1.14) fails when \( \alpha < 1/4 \), even if the left-hand size is replaced by the \( L^1 \) norm.

The proof of Theorem 1.3 utilizes asymptotic formulas for the zonal spherical harmonics to reduce the estimates to the setting of \( S^1 \). More precisely, we use an asymptotic formula from Szegö [20] to expand (up to an error term) the zonal function away from the poles into a modulated cosine series. The sufficiency part then follows from a maximal inequality for the Schrödinger equation on \( S^1 \) and the Sobolev embedding (the latter is used to bound the maximal function near the poles). The proof of the necessity part is an adaptation of a counterexample on \( S^1 \) due to Moyua-Vega [17], who used the Gauss sum to show that \( s \geq 1/4 \) is necessary for the Schrödinger maximal function on \( S^1 \) to be bounded.

In our case, by writing the cosine function as a sum of conjugate exponentials, we are led to consider how conjugate initial data are evolved under the Schrödinger equation (in particular, how do they add for specific initial data). By examining the argument of the value of the Gauss sum, we manage to show that Moyua-Vega’s counterexample carries over to \( S^n \), with the blowup occurring on a set of possibly smaller measure.

As a consequence of Theorem 1.3, we have the following result.

**Corollary 1.4.** Let \( n \geq 2 \). For any \( \alpha > 1/3 \), the solution \( u(t, x) \) to equation (1.1) converges pointwise to the initial data \( f \), whenever a zonal function \( f \) belongs to \( W^{\alpha,2}(S^n) \) for \( \alpha > 1/3 \).

We would like to mention that in the Euclidean setting, Carleson [8] proposed the problem of identifying the optimal \( s \) for which

\[
\lim_{t \to 0} e^{it\Delta} f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n
\]

whenever \( f \in H^{\alpha}(\mathbb{R}^n) \). In dimension one, Carleson [8] proved convergence for \( \alpha \geq 1/4 \) and Dahlberg and Kenig [9] showed that this is sharp. In higher dimensions, this problem was recently settled by Du-Guth-Li [10] for \( n = 2 \) and \( \alpha > 1/3 \); and by Du-Zhang [11] for \( n \geq 3 \) and \( \alpha > n/2(n + 1) \). Due to a counterexample by Bourgain [2], up to the endpoint, these two latter results are sharp. In contrast to the case of \( \mathbb{R}^n \), very little is known for the sphere \( S^n \). Even for the case of \( S^1 \), as pointed out by Moyua and Vega [17], the strategy of Carleson [8] gives a worse result than the case on the real line \( \mathbb{R} \), and we do not know whether \( f \in W^{\alpha,2}(S^1) \) for \( \alpha \geq 1/4 \) is sufficient yet. It would be interesting to establish the sharp version of (1.13) to find the optimal value \( \alpha \) for which (1.13) holds.

The paper is organized as follows. In Section 2 we first prove the \( L^4 \)-estimate stated in Theorem 1.1. Then, by analytic interpolation, we obtain the \( L^p(\mathbb{T} \times S^n) \)-estimates of \( e^{it\Delta_{S^n}} f \) stated in Corollary 1.2 for \( 2 \leq p \leq \infty \). The proof of Theorem 1.3 is given in Section 3.
2. Proof of Theorem 1.1

To prove (1.10) in Theorem 1.1 from (1.8) and (1.9) it suffices to show it for $n = 2$, whose proof is based on the following Lemma 2.1 and Lemma 2.2.

Lemma 2.1. For any $\alpha > 0$, there exists a constant $C = C(\alpha)$ independent of $f$ such that

(2.1) \[ \|e^{it\Delta_2} f\|_{L^4(T)} \leq C \left( \sum_{k=0}^{\infty} \left(1 - \Delta_2^2\right)^{\alpha/2} \mathbb{P}^2_k(f) \right)^{1/2}. \]

Proof. The proof of Lemma 2.1 is inspired by the result of Burq-Gérard-Tzvetkov [5, Proposition 3.1] (with $u_0 = v_0$). From (1.2), we write

(2.1) \[ \|e^{it\Delta_2} f\|_{L^4(T)}^4 = \left\| \sum_{s=0}^{\infty} e^{-it s} \mathbb{P}^2_k(f) \mathbb{P}^2_k(f) \right\|_{L^2(T)}^2. \]

By the Parseval identity, we get

(2.2) \[ \left\| e^{it\Delta_2} f \right\|_{L^4(T)}^4 = 2\pi \sum_{s=0}^{\infty} r(s) \left( \sum_{k(k+1) + \ell(\ell+1) = s} \mathbb{P}^2_k(f) \mathbb{P}^2_k(f) \right)^2, \]

where

\[ r(s) = \#\{ (k, \ell) : k(k+1) + \ell(\ell+1) = s \}. \]

Notice that

\[ k(k+1) + \ell(\ell+1) = s \iff (2k+1)^2 + (2\ell+1)^2 = 4s + 2. \]

It follows from classical results (see [13, Theorem 278]) on the sum of squares function that for any $\alpha > 0$, there exists a constant $C = C(\alpha) > 0$ such that

\[ r(s) \leq C(1 + s)^{\alpha}. \]

Consequently, one can bound (2.2) by

\[ \sum_{s=0}^{\infty} (1 + s)^{\alpha} \sum_{k(k+1) + \ell(\ell+1) = s} \mathbb{P}^2_k(f) \mathbb{P}^2_k(f). \]

Since $s = k(k+1) + \ell(\ell+1)$, this can be bounded by

\[ \sum_{s=0}^{\infty} k(k+1) + \ell(\ell+1) = s \left(1 + k(k+1)\right)^{\alpha} \left(1 + \ell(\ell+1)\right)^{\alpha} \mathbb{P}^2_k(f) \mathbb{P}^2_k(f), \]

which equals

\[ \left( \sum_{k=0}^{\infty} (1 - \Delta_2^2)^{\alpha/2} \mathbb{P}^2_k(f)^2 \right) \left( \sum_{\ell} (1 - \Delta_2^2)^{\alpha/2} \mathbb{P}^2_\ell(f)^2 \right). \]
Therefore,
\[ \| e^{it\Delta_{S^2}} f \|_{L^4(T)}^4 \leq C \left( \sum_{k=0}^{\infty} |(I - \Delta_{S^2})^{\alpha/2} \mathbb{P}_k^2(f)|^2 \right)^2. \]
This proves Lemma 2.1.

Recall that the half-wave group on \( L^2(S^2) \) is defined by
\[ e^{it\sqrt{-\Delta_{S^2}}} (f) = \sum_{k=0}^{\infty} e^{ik(k+1)} \mathbb{P}_k^2(f). \]
Lemma 2.1 provides a useful way to relate the Schrödinger group and the half-wave group as in the following.

**Lemma 2.2.** We have
\[ \left( \sum_{k=0}^{\infty} |\mathbb{P}_k^2(f)|^2 \right)^{1/2} \approx \left\| e^{it\sqrt{-\Delta_{S^2}}} (f) - \mathbb{P}_0^2(f) \right\|_{L^2(T)} + |\mathbb{P}_0^2(f)|. \]

**Proof.** Recall that an exponential system \( \{ e^{i\lambda_k t} \} \) (\( \lambda_k \in \mathbb{R} \)) is said to be a Riesz sequence in \( L^2(T) \) if for any coefficients \( \{ c_k \} \),
\[ \left\| \sum_{k=0}^{\infty} c_k e^{i\lambda_k t} \right\|_{L^2(T)} \approx \left( \sum_{k=0}^{\infty} |c_k|^2 \right)^{1/2}. \]
A celebrated theorem of Kadec [14] implies that \( \{ e^{i\lambda_k t} \} \) forms a Riesz sequence in \( L^2(T) \) provided
\[ \sup_k |\lambda_k - k| < \frac{1}{4}. \]
By modulation, it is easy to see that the same conclusion holds if
\[ \sup_k \left| \lambda_k - \frac{1}{2} \right| < \frac{1}{4}. \]
Take \( \lambda_k = \sqrt{k(k+1)}, k \geq 1 \). By direct checking, we see that
\[ \left| \sqrt{k(k+1)} - k - \frac{1}{2} \right| = \frac{\frac{1}{8}}{k(k+1) + k + \frac{1}{2}} \leq \frac{1}{8} < \frac{1}{4}. \]
Thus the last condition is satisfied, and so
\[ \left\| \sum_{k \geq 1} c_k e^{it\sqrt{k(k+1)}} \right\|_{L^2(T)} \approx \left( \sum_{k \geq 1} |c_k|^2 \right)^{1/2}. \]
With \( c_k = \mathbb{P}_k^2(f) \), we obtain
\[ \left( \sum_{k=0}^{\infty} |\mathbb{P}_k^2(f)|^2 \right)^{1/2} \approx \left( \sum_{k \geq 1} |\mathbb{P}_k^2(f)|^2 \right)^{1/2} + |\mathbb{P}_0^2(f)| \]
\[ \approx \left\| e^{it\sqrt{-\Delta_{S^2}}} (f) - \mathbb{P}_0^2(f) \right\|_{L^2(T)} + |\mathbb{P}_0^2(f)|. \]
This proves Lemma 2.2.

Now we start to prove our main result, Theorem 1.1.
Proof of Theorem 1.1. To prove (1.10), from (1.8) and (1.9) it suffices to show it for \( n = 2 \).

In this case, an essential observation is to reduce to the following local smoothing result by Mockenhaupt-Seeger-Sogge [16, Theorem 6.2] (with \( X = Y = S^2 \), \( F = e^{it\sqrt{-\Delta_{S^2}}} (1 - \Delta_{S^2})^{-\alpha/2}, p = q = 4 \)): For any \( \alpha > 0 \), there exists a constant \( C = C(\alpha) \)
\[
(2.4) \quad \left\| e^{it\sqrt{-\Delta_{S^2}}} f \right\|_{L^4(S^2; L^2(T))} \leq C \left\| (1 - \Delta_{S^2})^{\alpha/2} f \right\|_{L^4(S^2)}.
\]

Indeed, we apply Lemma 2.2 with \( (1 - \Delta_{S^2})^{\alpha/2} \) replaced by \( (1 - \Delta_{S^2})^{2\alpha/2} \) to obtain
\[
\left( \sum_{k=0}^{\infty} \left| (1 - \Delta_{S^2})^{\alpha/2} \mathbb{P}_k(f) \right|^2 \right)^{1/2} \approx \left\| e^{it\sqrt{-\Delta_{S^2}}} (1 - \Delta_{S^2})^{\alpha/2} f - \mathbb{P}_0(f) \right\|_{L^2(T)} + \left| \mathbb{P}_0(f) \right| \leq C \left\| e^{it\sqrt{-\Delta_{S^2}}} (1 - \Delta_{S^2})^{\alpha/2} f \right\|_{L^2(T)} + C \left| \mathbb{P}_0(f) \right|.
\]

This, in combination with Lemma 2.1 yields
\[
\left\| e^{it\Delta_{S^2}} f \right\|_{L^4(T \times S^2)} \leq C \left\| e^{it\sqrt{-\Delta_{S^2}}} (1 - \Delta_{S^2})^{\alpha/2} f \right\|_{L^4(S^2; L^2(T))} + C \left\| \mathbb{P}_0(f) \right\|_{L^4(S^2)}.
\]

Estimate (1.10) then follows readily from (2.4). This proves the sufficiency part of Theorem 1.1. \( \square \)

Remark 2.3. Note that if the Sobolev norm on the right-hand side of (2.4) is based on \( L^2 \), Cardona and Esquivel [7] obtained sharp \( W^{\alpha,2} \rightarrow L^q L^2 \) estimate on \( S^n \) with the critical exponent \( q = 2(n + 1)/(n - 1) \) by using purely the \( L^2 \rightarrow L^p \) spectral estimates for the operator norm of the spectral projections associated to the spherical harmonics proved in [15]. Using the method in the proof of Theorem 1.1 above, their result can be deduced from a special case of [16] Theorem 3.2) regarding the corresponding estimate for the half-wave equation, and vice versa.

The sharpness of the range \( \alpha \) in (1.10) in Theorem 1.1 is a special case of the following more general proposition.

Proposition 2.4. Let \( n \geq 1 \) and let \( \Delta_{S^n} \) be the Laplace-Beltrami operator on \( S^n \). Suppose \( p \geq 2 \) and for some number \( \alpha \) it holds that
\[
(2.5) \quad \left\| e^{it\Delta_{S^n}} f \right\|_{L^p(T \times S^n)} \leq C \left\| f \right\|_{W^{\alpha,p}(S^n)}.
\]

Then
\[
\alpha \geq \max \left\{ 0, n \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{2}{p} \right\}.
\]

Proof. The necessity of \( \alpha \geq 0 \) can be easily seen by taking \( f = Z_k \), \( k \rightarrow \infty \) and noting that \( \left\| e^{it\Delta_{S^n}} f \right\|_{L^p(S^n)} = \left\| f \right\|_{L^p(S^n)} \) in this case. The necessity of \( \alpha \geq n \left( 1/2 - 1/p \right) - 2/p \) will be shown below using a semiclassical dispersion estimate of Burq-Gérard-Tzvetkov [4] and ideas from Rogers [13] Section 2]. More precisely, Lemma 2.5 (with \( M = S^n \) of [4] implies that there exists a bump function \( 0 \leq \varphi \in C_0^\infty(\mathbb{R}) \), such that for all sufficiently small \( h > 0 \),
\[
\left\| e^{-ih\Delta_{S^n}} \varphi(h\sqrt{-\Delta_{S^n}}) \right\|_{L^\infty(S^n)} \leq Ch^{-n/2}.
\]

Fix \( x \in S^n \), and let
\[
f(y) = (1 - \Delta_{S^n})^{-\alpha/2} e^{-ih\Delta_{S^n}} \varphi(h\sqrt{-\Delta_{S^n}})(x, y).
\]
Then
\[\|(1 - \Delta_{\mathbb{S}^n})^{\alpha/2} f\|_{L^p(\mathbb{S}^n)} = \|e^{-ih\Delta_{\mathbb{S}^n}} \varphi(h\sqrt{-\Delta_{\mathbb{S}^n}})\|_{L^p(\mathbb{S}^n)} \leq C\|e^{-ih\Delta_{\mathbb{S}^n}} \varphi(h\sqrt{-\Delta_{\mathbb{S}^n}})\|_{L^\infty(\mathbb{S}^n)} \leq C h^{-n/2}.\] (2.6)

On the other hand, we have
\[e^{it\Delta_{\mathbb{S}^n}} f(y) = e^{i(t-h)\Delta_{\mathbb{S}^n}} (1 - \Delta_{\mathbb{S}^n})^{-\alpha/2} \varphi(h\sqrt{-\Delta_{\mathbb{S}^n}})(x,y) = \sum_{k \geq 0} e^{-i(t-h)k(k+n-1)} \frac{\varphi(h\sqrt{k(k+n-1)})}{(1 + k(k+n-1))^{n/2}} Z_k^n(x,y).\]

Due to the support property of \(\varphi\), the sum above is over the \(k\)'s with \(h\sqrt{k(k+n-1)} < 1\). Consequently, we can find a small constant \(c > 0\) so that
\[|t-h| \leq ch^2 \implies \Re(e^{-i(t-h)k(k+n-1)}) \geq 1/2.\]

With a possibly smaller \(c\), we also have
\[d(x,y) \leq ck^{-1} \implies Z_k^n(x,y) \geq ck^{n-1}.\]

Therefore, when \(|t-h| \leq ch^2\) and \(d(x,y) \leq ch\) (change \(c\) again if necessary),
\[\Re(e^{it\Delta_{\mathbb{S}^n}} f) \geq ch^{-(n-\alpha)}.\]

It follows that
\[\|e^{it\Delta_{\mathbb{S}^n}} f\|_{L^p(\mathbb{T} \times \mathbb{S}^n)}^p \geq ch^2 h^n h^{-(n-\alpha)p}.\] (2.7)

Combining (2.6) and (2.7), we see that for (1.10) to hold, we must have
\[\frac{n+2}{p} - (n-\alpha) \geq -\frac{n}{2},\]
that is, \(\alpha \geq n\left(1/2 - 1/p\right) - 2/p\). \(\square\)

By the Sobolev embedding (see for example, [22, (9), p.315]), we have
\[L^\infty(\mathbb{T}, W^{\frac{n}{p}+\varepsilon, 2}(\mathbb{S}^n)) \hookrightarrow L^\infty(\mathbb{T} \times \mathbb{S}^n), \quad \forall \varepsilon > 0.\]

To show Corollary 1.2 we need the following result.

Lemma 2.5. Let \(n \geq 1\) and let \(\Delta_{\mathbb{S}^n}\) be the Laplace-Beltrami operator on \(\mathbb{S}^n\). For every \(y \in \mathbb{R}\) and \(\varepsilon > 0\), there exists a constant \(C = C(\varepsilon, n)\) independent of \(y\) such that
\[\|e^{it\Delta_{\mathbb{S}^n}} f\|_{L^\infty(\mathbb{T} \times \mathbb{S}^n)} \leq C \|(1 - \Delta_{\mathbb{S}^n})^{iy + \frac{\alpha}{2} + \varepsilon} f\|_{L^\infty(\mathbb{S}^n)}.\] (2.8)

Proof. It suffices to show that the operator \(e^{it\Delta_{\mathbb{S}^n}} (1 - \Delta_{\mathbb{S}^n})^{-iy - \frac{\alpha}{2} - \varepsilon}\) is uniformly bounded on \(L^\infty(\mathbb{S}^n)\). To show it, let us first recall some properties of the zonal spherical harmonic functions (see for example [19, 20]). Let \(C_k^\lambda(t)\) be the Gegenbauer polynomial of degree \(k\) and index \(\lambda\), i.e.,
\[C_k^\lambda(t) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(k + 2\lambda)}{\Gamma(k + \lambda + \frac{1}{2})} P^\lambda_{\frac{1}{2}, \frac{1}{2}}(t),\]

where \(P^\lambda_{\frac{1}{2}, \frac{1}{2}}(t)\) are the associated Legendre polynomials.
where \( P_{n}^{a,b} \) is the Jacobi polynomial of degree \( k \) (see [20] p. 80). Denote by \(|x - y| \in [0, \pi]\) the great-circle distance between \( x \) and \( y \) on \( S^{n} \). It is a standard fact that

\[
\mathbb{P}^n_k f(x) = \int_{S^n} Z^\alpha_k(x, y) f(y) dy,
\]

where \( Z^\alpha_k(x, y) \) is the zonal spherical harmonic function of degree \( k \), given by

\[
Z^\alpha_k(x, y) = \frac{k + \lambda}{\lambda} C^\lambda_k(\cos |x - y|).
\]

Note that \( \cos |x - y| \) represents the inner product in \( \mathbb{R}^{n+1} \).

Denote by \( Z^n_h(x, y) \) the spherical harmonic of degree \( k \). We can write

\[
e^{it\Delta_{S^n}} = \sum_{k=0}^\infty e^{-it(k+n-1)} \left( 1 + k(k + n - 1) \right)^{i\epsilon + \frac{1}{2}} Z^n_k(x, y).
\]

For any fixed \( x \), we have

\[
\left\| \left( 1 - \Delta_{S^n} \right)^{i\epsilon + \frac{1}{2}} \right\|_{L^2(S^n)}^2 = \sum_{k=0}^\infty \frac{\| Z^n_k \|^2_{L^2(S^n)}}{\left( 1 + k(k + n - 1) \right)^{\frac{1}{2} + 2\epsilon}}.
\]

Since \( \| Z^n_k \|^2_{L^2(S^n)} \leq Ck^{n-1} \) ([19] p.140), it follows that the kernel of the operator \( e^{it\Delta_{S^n}} (1 - \Delta_{S^n})^{-i\epsilon - \frac{1}{2}} \) is uniformly bounded in \( L^2(S^n) \subset L^1(S^n) \), thus defines a uniformly bounded operator on \( L^\infty(S^n) \).

\[ \square \]

**Remark 2.6.** In [21], M. Taylor studied the Schrödinger equation on the spheres at times that are rational multiples of \( \pi \) to that for all \( 1 < p < \infty \) and all \( s \in \mathbb{R} \). It is shown that

\[
e^{-\pi i (m/k) \Delta_{S^n}} : W^{s,p}(S^n) \to W^{s-(n-1)(\frac{1}{2} - \frac{1}{p}),p}(S^n)
\]

extends to a bounded operator. Such estimates also hold in the endpoint cases \( p = 1, \infty \), with \( L^1 \) replaced by the local Hardy space and \( L^\infty \) replaced by \( \text{bmo} \). For the sharpness of this estimate (2.9), we refer the reader to Taylor [21] page 148.

In the end of this section, we give a proof of Corollary 1.2.

**Proof of Corollary 1.2.** In view of Proposition 2.4 the necessity part of Corollary 1.2 follows readily.

For (1.1), let us first prove it for \( n \geq 3 \). The case \( 4 < p < \infty \) is treated by interpolating between (1.10) and the case \( p = \infty \). Given any \( \epsilon > 0 \), we consider the analytic family of operators

\[
T^\epsilon = e^\frac{-1}{2} (1 - \Delta_{S^n})^{\frac{-n-2+4\epsilon + (n+1)\epsilon}{4}} e^{it\Delta_{S^n}}, \quad 0 < \text{Re} \epsilon \leq 1.
\]

For \( z = iy, \text{Re} z = 0 \), we apply (1.10) of Theorem 1.1 to obtain that the operators

\[
T^\epsilon = e^\frac{-1}{2} (1 - \Delta_{S^n})^{\frac{-n-2+4\epsilon + (n+1)\epsilon}{4}} e^{it\Delta_{S^n}}
\]

are bounded from \( L^4(S^n) \) into \( L^4(S^n) \) and there exists a constant \( C \) independent of \( y \) such that

\[
\| T^\epsilon \|_{L^4(T \times S^n)} = e^{-\frac{1}{2} \epsilon} \left( 1 - \Delta_{S^n} \right)^{\frac{-n-2+4\epsilon + (n+1)\epsilon}{4}} e^{it\Delta_{S^n}} \| f \|_{L^4(T \times S^n)} \leq Ce^{-\frac{1}{2} \epsilon} \left( 1 - \Delta_{S^n} \right)^{\frac{-n-2+4\epsilon + (n+1)\epsilon}{4}} f \|_{L^4(S^n)}
\]

for \( n \geq 3 \).
\[ \leq C\|f\|_{L^4(\mathbb{R}^n)} \]

since \[ \| (1 - \Delta_{S^m})^{-i(\alpha+1)/2} \|_{L^4(\mathbb{R}^n)} \leq C(1 + |\alpha|)^{1/2} \] (see [12, Theorem 3.1]). On the other hand, we apply Lemma 2.5 to get

\[ \| T_{1+iy}f \|_{L^\infty(T \times \mathbb{R}^n)} = e^{-y^2} \| (1 - \Delta_{S^m})^{-\frac{i\alpha+n+1}{4}} e^{it\Delta_{S^m}} f \|_{L^\infty(T \times \mathbb{R}^n)} \leq C\|f\|_{L^\infty(\mathbb{R}^n)} \]

with \( C \) independent of \( y \). Then by the complex interpolation theorem (see [6, Theorem 3.4, pp. 151-152]),

\[ (2.11) \quad \| T_{\theta}f \|_{L^p(T \times \mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \]

where \( 0 \leq \theta \leq 1 \) and \( 1/p = (1 - \theta)/4 \). This gives (2.11) for all \( 4 \leq p < \infty \).

Now for \( 2 \leq p \leq 4 \), we consider the analytic family of operators \( T_z = e^{z^2(1 - \Delta_{S^m})} e^{i(tz + 4z^4/4)e^{it\Delta_{S^m}}}, 0 < \text{Re} z \leq 1 \). Interpolating between (1.10) of Theorem 1.1 and (1.4) (with \( p = 2 \)) yields (1.11) for all \( 2 \leq p < 4 \) by making a minor modification to the proof of (2.11). This proves (1.11) for the case \( n = 3 \).

The proof of (1.11) for the case \( n = 2 \) is similar to that of the case \( n \geq 3 \), and we omit the detail here. This completes the proof of Corollary 1.2.

\[ \Box \]

Remark 2.7. In [7, Theorem 1.1], Cardona and Esquivel showed that for \( n \geq 2 \) and \( p, q \) satisfying \( 2 \leq p \leq \infty \) and \( 2 \leq q < \infty \), the following estimate

\[ (2.12) \quad \| e^{it\Delta_{S^m}} f \|_{L^p(\mathbb{R}^n), L^q(T)} \leq C\|f\|_{W^{\alpha,2}(\mathbb{R}^n)} \]

holds for \( \alpha > \mathfrak{N}(p, q, n) \), where

\[ (2.13) \quad \mathfrak{N}(p, q, n) = \begin{cases} \frac{n-1}{2} (\frac{1}{2} - \frac{1}{p}) + \left( \frac{1}{2} - \frac{1}{q} \right), & \text{if } 2 \leq p \leq \frac{2(n+1)}{n-1}; \\ n (\frac{1}{p} - \frac{1}{2}) - \frac{1}{q}, & \text{if } p > \frac{2(n+1)}{n-1}. \end{cases} \]

The regularity order \( \mathfrak{N}(p, 2, n) \) is sharp in any dimension \( n \), in the sense that (2.12) does not hold for all \( \alpha < \mathfrak{N}(p, 2, n) \).

From (1.12) and (2.13), we see that when \( n \geq 6 \) and \( 2 < p < 3 \), improvement on Corollary 1.2 can be obtained from the \( W^{\alpha,2}_{\mathbb{R}^n} \to L^p_{\mathbb{R}^n} L^q_{\mathbb{T}} \) estimate as in (2.12) above. That is, (1.11) holds provided that \( n \geq 6 \), \( \alpha > \mathfrak{N}(p, p, n) \) and \( 2 < p < 3 \).

3. PROOF OF THEOREM 1.3

To prove (1.14) in Theorem 1.3, we need a slight variant of a Strichartz estimate on \( T \) due to Bourgain [11, Proposition 2.36].

Lemma 3.1. For any \( \varepsilon > 0 \), integer \( N \geq 1 \), and numerical sequence \( a = \{a_k\} \), there exists a constant \( C = C(\varepsilon) \) independent of \( N \) and \( a \) such that

\[ (3.1) \quad \left\| \sum_{k=0}^{N-1} a_k e^{-i(k+n-1)x} e^{\pm ik\theta} \right\|_{L^6(T \times \mathbb{T})} \leq C(\varepsilon) N^\varepsilon \|a\|_{\ell^2}. \]
exists a constant $C$. 

Corollary 3.2. By the Cauchy-Schwarz inequality, (3.2) can be bounded by

$$\| \sum_{k=0}^{N-1} a_k e^{-itk(k+m)} e^{-itk} \|_{L^6(\mathbb{T}^2)}^6 = c \sum_{u\neq v} a_u a_v a_{u+v}^2,$$

where $c$ is an absolute constant. Note that since $0 < j, k, \ell < N$, in the last sum $u \leq N, v \lesssim N^2$. Denote

$$r_{u,v}^{(m)} = \# \{ j, k, \ell : j + k + \ell = u, j(j+m) + k(k+m) + \ell(\ell+m) = v \}.$$

By the Cauchy-Schwarz inequality, (3.3) can be bounded by

$$\sum_{u \neq v} r_{u,v}^{(m)} \sum_{j \neq k} |a_j|^2 |a_k|^2 |a_\ell|^2.$$

Since

$$\sum_{u \neq v} \sum_{j \neq k} |a_j|^2 |a_k|^2 |a_\ell|^2 = \| a \|_{L^2}^6,$$

to prove (3.3) it suffices to show

$$r_{u,v}^{(m)} \leq C(\varepsilon) N^{3\varepsilon}, \forall \varepsilon > 0.$$

When $m = 0$, (3.3) has been shown to hold in the proof of Proposition 2.36. On the other hand, by definition, $r_{u,v}^{(m)} = r_{u,v}^{(0)}$. It follows immediately that (3.3) also holds for $m \geq 1$. This completes the proof of the lemma. \hfill \square

By a standard argument (see Moyua-Vega \cite[Proposition 1]{MV}), (3.1) implies the following maximal inequality.

Corollary 3.2. For any $\varepsilon > 0$, integer $N \geq 1$, and numerical sequence $a = \{a_k\}$, there exists a constant $C(\varepsilon)$ such that

$$\left\| \sup_{t \in \mathbb{T}} \sum_{k=0}^{N-1} a_k e^{-itk(k+n-1)} e^{-itk} \right\|_{L^6(\mathbb{T})} \leq C(\varepsilon) N^{\frac{3\varepsilon}{2}} \| a \|_{L^2}.$$
for some dyadic $N \geq 1$ and coefficients $a_k \in \mathbb{C}$. By considering a dyadic decomposition, we may (and will) further assume that

$$f(y) = \sum_{k=0}^{2N-1} a_k \tilde{Z}_k(y).$$

Write $q = \frac{6n}{3n-2}$ and

$$\langle x_0, y \rangle_{\mathbb{R}^{n+1}} = \cos \theta, \ 0 \leq \theta \leq \pi.$$

Since $Mf := \sup_{0 \leq t < 2\pi} |e^{it\Delta}f|$ is also zonal with respect to $x_0$, we can write

$$\|Mf\|_{L^q(\mathbb{R}^n)} \approx \left(\int_0^\pi |Mf(\theta)|^q (\sin \theta)^{n-1} d\theta\right)^{1/q}.$$ 

By the Sobolev embedding, we have

$$\|Mf\|_{L^\infty} \leq CN^\frac{n}{2} \|a\|_{l^2}.$$ 

Therefore, for any fixed $c > 0$,

$$\left(\int_0^{cN^{-1}} + \int_{\pi-cN^{-1}}^\pi \right) |Mf(\theta)|^q (\sin \theta)^{n-1} d\theta \leq CN^\frac{n}{2} \|a\|_{l^2}.$$ 

In the region $cN^{-1} \leq \theta \leq \pi - cN^{-1}$, by Theorem 8.21.13 of Szegő [20], we have the uniform estimate

$$\tilde{Z}_k(\theta) = \frac{c_k}{(\sin \theta)^{n/2}} \cos \left( k + \frac{n-1}{2} \right) \theta - \frac{n-1}{4} \pi + O(1)$$

where $c_k > 0$ is a constant bounded above and below. Correspondingly, we can bound

$$\sup_{0 \leq t < 2\pi} |e^{it\Delta} f(\theta)| \leq M^{(0)} f(\theta) + M^{(1)} f(\theta),$$

where $M^{(0)} f(\theta)$ is given by

$$\frac{1}{(\sin \theta)^{n/2}} \sup_{0 \leq t < 2\pi} \left| \sum_{k=0}^{2N-1} c_k a_k e^{-it(k+n-1)} \cos \left( k + \frac{n-1}{2} \right) \theta - \frac{n-1}{4} \pi \right|$$

and, after applying the Cauchy-Schwarz inequality,

$$M^{(1)} f(\theta) \leq C \frac{\|a\|_{l^2}}{\sqrt{N(\sin \theta)^{n/4}}}.$$ 

From (3.7) it follows immediately that

$$\int_{cN^{-1}}^{\pi-cN^{-1}} |M^{(1)} f(\theta)|^q (\sin \theta)^{n-1} d\theta \leq CN^\frac{n}{2} \|a\|_{l^2}^q.$$ 

On the other hand, writing $\cos x = \frac{1}{2} \sum \pm e^{\pm ix}$, we can bound

$$M^{(0)} f(\theta) \leq \frac{1}{2(\sin \theta)^{n/2}} \sup_{0 \leq t < 2\pi} \left| \sum_{k=0}^{2N-1} c_k a_k e^{-it(k+n-1)} e^{\pm ik\theta} \right|.$$ 

Thus, by Hölder’s inequality, we have

$$\int_{cN^{-1}}^{\pi-cN^{-1}} |M^{(0)} f(\theta)|^q (\sin \theta)^{n-1} d\theta$$
the uniform estimate 
Therefore, when
where
Using zonal polynomials. The proof is based on a counterexample on
This completes the proof of the sufficiency part of Theorem 1.3.

Vega [17]. Fix \( x \in \mathbb{R}^n \). Let \( N \geq 1 \) be a large integer and consider
\[
 f_N(y) = \sum_{k=0}^{N-1} \frac{Z_k(y)}{c_k},
\]
where \( Z_k(y) \) and \( c_k \) are as above (with \( c_0 := 1 \)). Since \( \|f_N\|_{H^s(\mathbb{S}^n)} \lesssim N^{\frac{1}{2}+s} \), it suffices to show that
\[
 Mf_N \geq C N^{\frac{1}{2}}
\]
holds on a set \( E_N \) of measure \( \mu(E_N) \geq C > 0 \).

As before, write \( (x_0, y) \in \mathbb{R}^{n+1} = \cos \theta, \ 0 \leq \theta \leq \pi \). Fix a small \( \varepsilon > 0 \). By (3.6), we have the uniform estimate

\[
\frac{Z_k(y)}{c_k} = \frac{1}{(\sin \theta)^{\frac{n}{2}}} \cos \left( \left( k + \frac{n-1}{2} \right) \theta - \frac{n-1}{4} \pi \right) + O(1) \theta \leq \pi - \varepsilon.
\]
Therefore, when \( \varepsilon \leq \theta \leq \pi - \varepsilon \),

\[
e^{i\Delta} f_N(\theta) = \frac{1}{(\sin \theta)^{\frac{n}{2}}} \sum_{k=0}^{N-1} e^{-ik(k+n-1)} \cos (k\theta + \phi_n(\theta)) + O(\log N),
\]
where \( \phi_n(\theta) := \frac{n-1}{2} \theta - \frac{n-1}{4} \pi \). Denote the sum above by \( S(t, \theta) \), i.e.

\[
S(t, \theta) = \sum_{k=0}^{N-1} e^{-ik(k+n-1)} \cos (k\theta + \phi_n(\theta)).
\]

By writing \( \cos x = \frac{1}{2} \sum_{\pm} e^{\pm ix} \), we have

\[
S(t, \theta) = \frac{1}{2} \sum_{\pm} \left( \sum_{k=0}^{N-1} e^{-ik(k+n-1)} e^{\pm ik\theta} \right) e^{\pm i\phi_n(\theta)}
\]

(3.10)
Suppose $t = \frac{2\pi}{q}$ with $q \approx \sqrt{N}$ being an odd integer. Then

$$S_+ \left( \frac{2\pi}{q}, \theta \right) = \sum_{k=0}^{N-1} e^{-2\pi i \frac{k(k+n-1)}{q}} e^{ik\theta}.$$ 

Since $e^{-2\pi i \frac{k(k+n-1)}{q}}$ is $q$-periodic in $k$, we can write

$$S_+ \left( \frac{2\pi}{q}, \theta \right) = \left( \sum_{k=0}^{\left\lfloor \frac{N}{q} \right\rfloor} \sum_{k=q}^{N-1} \right) e^{-2\pi i \frac{k(k+n-1)}{q}} e^{ik\theta}$$

$$= \left( \sum_{\ell=0}^{\left\lfloor \frac{N}{q} \right\rfloor-1} e^{i\ell q\theta} \right) \left( \sum_{k=0}^{q-1} e^{-2\pi i \frac{k(k+n-1)}{q}} e^{ik\theta} \right) + O(q)$$

$$= \left( \sum_{\ell=0}^{\left\lfloor \frac{N}{q} \right\rfloor-1} e^{i\ell q\theta} \right) s_+ \left( \frac{2\pi}{q}, \theta \right) + O(q).$$

If $\theta = \frac{2\pi p}{q}$ for some integer $p$, then

$$s_+ \left( \frac{2\pi}{q}, \frac{2\pi p}{q} \right) = \sum_{k=0}^{q-1} e^{-2\pi i \frac{k(k+n-1)}{q}} e^{2\pi i \frac{kp}{q}}$$

$$= \sum_{k=0}^{q-1} e^{-2\pi i \frac{k^2 + (n-1)kp}{q}}.$$ 

The last sum is a Gauss sum and can be evaluated explicitly to give

$$s_+ \left( \frac{2\pi}{q}, \frac{2\pi p}{q} \right) = \omega_q \sqrt{q} e^{2\pi i \frac{r(n-1)p^2}{q}},$$

where $r$ is an integer such that $4r \equiv 1 \pmod{q}$, and

$$\omega_q = \begin{cases} 1 & \text{if } q \equiv 1 \pmod{4}, \\ -i & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

If $p$ is even, we can further write

$$s_+ \left( \frac{2\pi}{q}, \frac{2\pi p}{q} \right) = \omega_q \sqrt{q} e^{2\pi i \frac{r(n-1)^2 + p^2}{q}} e^{-2\pi i \frac{2p(n-1)p}{q}}$$

$$= \omega_q \sqrt{q} e^{2\pi i \frac{(n-1)^2 + p^2}{q}} e^{-i \frac{\pi (n-1)p}{q}}.$$ 

Suppose $\theta = \frac{2\pi p}{q} + \eta$ with $p$ even and $|\eta| \leq \frac{\pi}{2\sqrt{N}}$. Then

$$\left| s_+ \left( \frac{2\pi}{q}, \theta \right) - s_+ \left( \frac{2\pi}{q}, \frac{2\pi p}{q} \right) \right| = \sum_{k=0}^{q-1} e^{-2\pi i \frac{k(k+n-1)}{q}} e^{i\frac{2\pi p}{q}} (e^{i\eta} - 1)$$

$$\leq \sum_{k=0}^{q-1} |e^{i\eta} - 1| \leq \sum_{k=0}^{q-1} k|\eta| \leq \frac{q^2}{N}.$$
Thus,

\[ s_+ \left( \frac{2\pi}{q}, \theta \right) = s_+ \left( \frac{2\pi}{q}, \frac{2\pi p}{q} \right) + O \left( \frac{q^2}{N} \right). \]

Combining this with

\[ e^{i\phi_n(\theta)} = e^{i\phi_n \left( \frac{2\pi p}{q} \right)} + O \left( \frac{1}{N} \right), \]

we see that

\[
S_+ \left( \frac{2\pi}{q}, \theta \right) e^{i\phi_n(\theta)} = \left( \sum_{\ell=0}^{\left\lfloor \frac{N}{q} \right\rfloor - 1} e^{i\ell q\theta} \right) \left( \frac{2\pi}{q} \right) \left( \frac{2\pi p}{q} \right) e^{i\phi_n(\frac{2\pi p}{q})} + O(q)
\]

\[
= \left( \sum_{\ell=0}^{\left\lfloor \frac{N}{q} \right\rfloor - 1} e^{i\ell q\eta} \right) \left( \omega q \sqrt{q} e^{2\pi i \frac{r(n-1)^2 + rp^2}{q}} e^{-i \frac{\pi(n-1)p}{q}} e^{-i \frac{\pi(n-1)p}{q}} + O(q) \right)
\]

\[
= \omega q \sqrt{q} e^{2\pi i \frac{r(n-1)^2 + rp^2}{q}} \left( \sum_{\ell=0}^{\left\lfloor \frac{N}{q} \right\rfloor - 1} e^{i\ell q\eta} \right) e^{-i \frac{\pi(n-1)p}{q}} + O(q).
\]

A similar argument shows that, for the same \( \theta \),

\[ S_- \left( \frac{2\pi}{q}, \theta \right) e^{-i\phi_n(\theta)} = \omega q \sqrt{q} e^{2\pi i \frac{r(n-1)^2 + rp^2}{q}} \left( \sum_{\ell=0}^{\left\lfloor \frac{N}{q} \right\rfloor - 1} e^{-i\ell q\eta} \right) e^{i \frac{n-1}{4} \pi} + O(q).
\]

Thus, by (3.10),

\[
\left| S \left( \frac{2\pi}{q}, \theta \right) \right| = \sqrt{q} \left| \sum_{\ell=0}^{\left\lfloor \frac{N}{q} \right\rfloor - 1} \cos \left( \frac{n-1}{4} \pi - \ell q\eta \right) \right| + O(\sqrt{N}).
\]

Since \( |\ell q\eta| \leq \pi \), the cosine’s in the sum are of the same sign and satisfy

\[
\left| \cos \left( \frac{n-1}{4} \pi - \ell q\eta \right) \right| \geq C \begin{cases} |\ell q| & \text{if } n \equiv 3 \text{ or } 7 \text{ (mod 8)}, \\ 1 & \text{otherwise}. \end{cases}
\]

It follows that if \( |\eta| \geq \frac{\pi}{16N} \), then

\[
\left| S \left( \frac{2\pi}{q}, \theta \right) \right| \geq C \sqrt{q} \cdot \frac{N}{q} \approx N^{3/2},
\]

and, consequently, \( M f_N(\theta) \geq C N^{3/2} \).

Now consider the set

\[
E_N = \bigcup_{q \text{ odd}, \sqrt{2} \leq q \leq \sqrt{N}} \left( \frac{2\pi p}{q} + \frac{\pi}{16N}, \frac{2\pi p}{q} + \frac{\pi}{8N} \right).
\]

By the argument above,

\[ M f_N(\theta) \geq C N^{3/2}, \forall \theta \in E_N. \]
Since any two intervals in the definition of \( E_N \) are either disjoint or identical, by a counting argument (see [17]), it follows that \( \| E_N \| \geq C \). This proves the proof of the necessity part of Theorem 1.3.

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